Black hole creation in 2+1 dimensions

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Abstract
When two point particles, coupled to three dimensional gravity with a negative cosmological constant, approach each other with a sufficiently large center of mass energy, then a BTZ black hole is created. An explicit solution to the Einstein equations is presented, describing the collapse of two massless particles into a non-rotating black hole. Some general arguments imply that massive particles can be used as well, and the creation of a rotating black hole is also possible.

Outline
The three dimensional black hole of Banados, Teitelboim and Zanelli [1, 2] has turned out to be a useful toy model to study various aspects of black hole quantum physics and thermodynamics. It is a solution to the vacuum Einstein equations with a negative cosmological constant. In its maximally extended version, its global structure is very similar to the maximally extended Schwarzschild black hole, or wormhole solution to Einstein gravity in four dimensions. Spacetime splits into four regions, the interiors of a black hole and a white hole, and two causally disconnected external regions. They are asymptotically flat in the Schwarzschild case, whereas for the BTZ black hole they are anti-de-Sitter spaces.

For the Schwarzschild black hole it is well know that it can be created by, for example, a star collapse. What is necessary for such a collapse is that a sufficiently large amount of matter
is concentrated inside a small region of space. If the black hole is created in this way, then only two of the four regions of spacetime exist: one exterior region, which is asymptotically flat and contains the initial matter configuration, and the interior of the black hole, which is separated from the exterior by a future horizon. There is no white hole and no second asymptotically flat region. In this sense, the star collapse is more realistic than the wormhole solution, because it can evolve from a singularity free initial condition.

Another way to create a Schwarzschild black hole is to start from a collapsing spherically symmetric dust shell, which is somewhat easier to deal with than a star, because there are no matter interactions other than the gravitational ones. The analog solution to the three dimensional Einstein equations, describing a circular dust shell collapsing into a BTZ black hole, has in fact been found shortly after the discovery of the BTZ black hole itself [3]. Here, I would like to present another way to create a BTZ black hole, starting from a very different initial condition. Instead of a dust shell, which can be considered as a special arrangement of infinitely many particles, it is sufficient to consider just two particles, which approach each other such that at some time they collide. The four dimensional analog of this process would be the collision of two stars, with sufficient masses and center of mass energy to create a black hole.

The situation is however much simpler in three dimensions, because the particles can be taken to be pointlike, and we can even choose them to be massless, which simplifies the construction of an explicit solution to the Einstein equations even further. This is because pointlike particles in three dimensional gravity are very easy to deal with. Unlike in higher dimensions, they do not themselves form black holes. Their gravitational fields are conical singularities located on their world lines. Outside the matter sources, spacetime is just flat, respectively constantly curved for a non-vanishing cosmological constant [4]. Spacetimes containing only such point particles as matter sources can be constructed by cutting out special subsets, sometimes called wedges, from Minkowski space, and then identifying the boundaries of these subsets in a certain way [5, 6].

After setting up the notation and summarizing some basic features of anti-de-Sitter space in, I will give a brief description of this cutting and gluing procedure and its generalization to anti-de-Sitter space. It is then straightforward to consider a special process where two particles collide and join into a single particle. It turns out that, depending on the energy of the incoming particles, the joint object is either a massive particle or a black hole. More precise, if the center of mass energy of the incoming particles lies beyond a certain threshold, then the object that is created after the collision is not a massive particle moving on a timelike geodesic, but some other kind of singular object, which is located on a spacelike geodesic.

A closer analysis of the causal structure of the resulting spacetime shows that this object is the future singularity inside a black hole. The black hole has all the typical features such as, for example, an interior region which is causally disconnected from spatial infinity, and there is also a horizon, whose size is a function of the amount of matter that has fallen in. Finally, reconsidering the same process in a different coordinate system will show that the black hole created by the collapse is indeed the BTZ black hole.
1 Anti-de-Sitter Space

Three dimensional anti-de-Sitter space $S$ can be covered by a global, cylindrical coordinate system $(t, \chi, \varphi)$, with a real time coordinate $t$, a radial coordinate $\chi \geq 0$, and an angular coordinate $\varphi$ with period $2\pi$, which is redundant at $\chi = 0$. The metric is
\[
    ds^2 = d\chi^2 + \sinh^2 \chi \, d\varphi^2 - \cosh^2 \chi \, dt^2. \tag{1.1}
\]
It is useful to replace the radial coordinate $\chi$ by $r = \tanh(\chi/2)$, which ranges from zero to one only. Anti-de-Sitter space is then represented by an infinitely long cylinder of radius one in $\mathbb{R}^3$. Expressed in terms of the coordinates $(t, r, \varphi)$, the metric becomes
\[
    ds^2 = \left(\frac{2}{1-r^2}\right)^2 (dr^2 + r^2 \, d\varphi^2) - \left(\frac{1+r^2}{1-r^2}\right)^2 \, dt^2. \tag{1.2}
\]

The time $t$ will be considered as an ADM-like coordinate time, providing a foliation of anti-de-Sitter space. The hyperbolic geometry of a spatial surface of constant $t$ is that of the Poincaré disc, which is conformally isometric to a disc of radius one in flat $\mathbb{R}^2$. Hence, anti-de-Sitter space can be considered as a Poincaré disc evolving in time. The time evolution is however not homogeneous. The lapse function, that is, the factor in front of the $dt$-term in the metric, which relates the physical time to the coordinate time $t$, depends on $r$. It diverges at the boundary of the disc, indicating that the physical time is running infinitely fast there.

### Geodesics

The Poincaré disc has some nice properties, which allow a convenient visualization of the constructions made in this article. The geodesics on the disc are circle segments intersecting the boundary at $r = 1$ perpendicularly. Figure 1 shows the construction of such a geodesic. It is determined by two points $A$ and $B$ on the boundary. If the angular coordinates of $A$ and $B$ are $\alpha \pm \beta$, with $0 < \beta < \pi$, we call the circle segment $APB$ the geodesic centered at $\alpha$, with radius $\beta$. To derive an equation for the geodesic in terms of the coordinates $r$ and $\varphi$, consider the points in the figure as complex numbers, such that $A = e^{i(\alpha+\beta)}$ and $B = e^{i(\alpha-\beta)}$. It then follows that the center of the circle segment $APB$ is at $C = e^{i\theta}/\cos \beta$, and that its radius is $\tan \beta$. Using this, it is not difficult to show that, for a point $P = r \, e^{i\varphi}$ on the geodesic, we have
\[
    \frac{2r}{1+r^2} \cos(\varphi - \alpha) = \cos \beta. \tag{1.3}
\]
This also applies for $\beta = \pi/2$, where the circle segment becomes a diameter of the disc. In this case, the solution to (1.3) is $r = 0$ or $\varphi = \alpha \pm \pi/2$, which is fulfilled on the diameter orthogonal to the direction $\alpha$.

Another special class of geodesics are spacetime geodesics passing through the origin of the coordinate system at $r = 0$ and $t = 0$. They can be parametrized by an angular direction $\theta$,
Figure 1: Construction of a geodesic on the Poincaré disc

and a velocity $0 \leq \xi \leq \infty$. The equation specifying such a geodesic in terms of the cylindrical coordinates $(t, r, \varphi)$ is

$$\frac{2r}{1 + r^2} = \xi \sin t,$$

(1.4)

and $\varphi$ has to be equal to $\theta$. Let us consider this as a definition of $r$ as a function of $t$, describing the motion of a test particle in anti-de-Sitter space. To see that $\xi$ is its velocity when it passes the origin, we have to differentiate (1.4) at $t = 0$ and $r = 0$. This gives $2 \, dr = \xi \, dt$. As there is also a relative factor of 2 between $dr$ and $dt$ in the metric (1.2), the physical velocity is indeed equal to $\xi$. For $0 \leq \xi < 1$, the geodesic is timelike. The radial coordinate $r$ is then oscillating with a period of $2\pi$ in $t$ between two extremals $-\rho \leq r \leq \rho$, determined by

$$\frac{2\rho}{1 + \rho^2} = \xi.$$

(1.5)

The test particle starts off from the center of the disc at $t = 0$, moving into the direction $\varphi = \theta$ with velocity $\xi$. At $t = \pi/2$, it reaches the maximal distance, and it returns to the center at $t = \pi$. Then it moves into the opposite direction, returns at $t = 2\pi$, and so on. To keep the equations describing this kind motion as simple as possible, we have to allow negative values of $r$, with the obvious identification of the point $(t, r, \varphi)$ with $(t, -r, \varphi \pm \pi)$.

For a lightlike geodesic with $\xi = 1$, the relation (1.4) between $r$ and $t$ simplifies to $r = \tan(t/2)$, which holds for $-\pi/2 < t < \pi/2$. At the ends of this time interval, $r$ becomes equal to one, which means that the geodesic reaches the boundary of the disc. Unlike a timelike test particle, a light ray is not oscillating forth and back. It travels once through the whole Poincaré disc, and the amount of time that it takes to travel from one side to the other is $\pi$. The existence of lightlike geodesics like this implies that the discs of constant $t$ are not Cauchy surfaces of anti-de-Sitter space. Causal curves enter at any moment of time from the boundary, and they disappear there as well.

Being the origin and destination of light rays, the cylindrical boundary of spacetime at $r = 1$ is called $\mathcal{J}$. To be precise, it is the boundary of the conformal compactification of anti-de-
Sitter space, which is obtained by multiplying the metric with the conformal factor $\frac{1}{4}(1 - r^2)^2$, which makes all coefficients of (1.2) regular at $r = 1$, and then including the boundary into the manifold. In contrast to asymptotically flat spacetimes, we cannot separate the origin of light rays $J_-$ from their destination $J_+$. There are, for example, light rays emerging from $J$ after other light rays have arrived there.

Moreover, as a consequence of this unusual feature, $J$ itself has a causal structure. The future light cone of a point on $J$ is the set of all points where light rays emerging from that point arrive back on $J$. To see how the light cones look like, it is sufficient to consider the pullback of the conformal metric on the boundary at $r = 1$, which is given by $d\varphi^2 - dt^2$. The light rays on $J$ are thus the left and right moving lines with $dt = \pm d\varphi$. These are in fact limits of light rays inside anti-de-Sitter space. In accordance with the special light ray considered above, which passes through the origin of anti-de-Sitter space, the amount of coordinate time that it takes for light to travel from some point on $J$ to the opposite point is $\pi$.

Finally, the geodesic equation (1.4) also includes spacelike geodesics. For $\xi > 1$, it has a solution for $r$ only within an even smaller time interval $-\tau < t < \tau$, where $\tau$ is determined by $\sin \tau = 1/\xi$. At the boundary of this interval, $r$ becomes equal to one. In the limit $\xi \to \infty$, we also recover the diameters of the Poincaré disc. Hence, a spacelike geodesics passes through the whole disc even faster than a lightlike one, and it also starts and ends on $J$. However, in contrast to the lightlike geodesic, which has a vanishing physical length, its total length is infinite. In this sense, the boundary $J$ can also be considered as spacelike infinity.

The group $\text{SL}(2)$

What we didn’t show so far is that all these curves are actually geodesics. To do this, it is convenient to use a different representation of anti-de-Sitter space. It is isometric to the covering of the group manifold $\text{SL}(2)$, consisting of real $2 \times 2$ matrices with unit determinant. As a basis of $2 \times 2$ matrices, we introduce the unit matrix $1$, and the gamma matrices

$$
\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}
$$

They form an $\mathfrak{sl}(2)$ Clifford algebra,

$$
\gamma_a \gamma_b = \eta_{ab} 1 - \varepsilon_{abc} \gamma_c, \tag{1.7}
$$

where the indices $a, b, \ldots$ run from 0 to 2, $\eta_{ab} = \text{diag}(-1, 1, 1)$ is the three dimensional Minkowski metric, which is used to raise and lower indices, and $\varepsilon^{abc}$ is the Levi Civita symbol with $\varepsilon^{012} = 1$. Expanding a generic matrix $x$ in this basis,

$$
x = x_3 1 + x^a \gamma_a, \quad x_3 = \frac{1}{2} \text{Tr}(x), \quad x^a = \frac{1}{2} \text{Tr}(x \gamma^a), \tag{1.8}
$$

we obtain a scalar $x_3$ and a vector $x^a$. The condition for the determinant to be one is

$$
x_3^2 - x^a x_a = x_3^2 + x_0^2 - x_1^2 - x_2^2 = 1. \tag{1.9}
$$
This defines a unit hyperboloid in $\mathbb{R}^{(2,2)}$. It is not simply connected, because there is a non-contractible loop in the $(x_3, x_0)$ plane. To see that anti-de-Sitter space is the covering thereof, we define a projection $S \to \text{SL}(2)$. In terms of the coordinates $(t, \chi, \varphi)$ it is essentially the Euler angle parametrization of $\text{SL}(2)$,

$$
\mathbf{x} = e^{\frac{1}{2}(t+\varphi)} \gamma_0 e^{\chi} \gamma_1 e^{\frac{1}{2}(t-\varphi)} \gamma_0
= \cosh \chi (\cos t \mathbf{1} + \sin t \gamma_0) + \sinh \chi (\cos \varphi \gamma_1 + \sin \varphi \gamma_2).
$$

(1.10)

The projection is locally one-to-one, but not globally. The right hand side of (1.10) is obviously periodic in $t$. The time coordinate of anti-de-Sitter space is wound up on the group manifold, with a period of $2\pi$. To check that the projection is an isometry, one can show, by straightforward calculation, that the anti-de-Sitter metric given above is equal to the pullback of the Cartan Killing metric on $\text{SL}(2)$, which is the same as the induced metric obtained by embedding $\text{SL}(2)$ into $\mathbb{R}^{(2,2)}$,

$$
ds^2 = \frac{1}{2} \text{Tr}(x^{-1} dx x^{-1} dx) = dx_1^2 + dx_2^2 - dx_0^2 - dx_3^2.
$$

(1.11)

A compact way to write the projection in terms of the alternative coordinates $(t, r, \varphi)$ is

$$
\mathbf{x} = \frac{1 + r^2}{1 - r^2} \omega(t) + \frac{2r}{1 - r^2} \gamma(\varphi),
$$

(1.12)

where

$$
\gamma(\alpha) = \cos \alpha \gamma_1 + \sin \alpha \gamma_2, \quad \omega(\alpha) = \cos \alpha \mathbf{1} + \sin \alpha \gamma_0.
$$

(1.13)

Together with their derivatives,

$$
\gamma'(\alpha) = \cos \alpha \gamma_2 - \sin \alpha \gamma_1, \quad \omega'(\alpha) = \cos \alpha \gamma_0 - \sin \alpha \mathbf{1},
$$

(1.14)

they provide a rotated basis of $2 \times 2$ matrices. Some formulas for products are

$$
\gamma(\alpha) \gamma(\beta) = \omega(\alpha - \beta), \quad \gamma(\alpha) \omega(\beta) = \gamma(\alpha - \beta),
\omega(\alpha) \omega(\beta) = \omega(\alpha + \beta), \quad \omega(\alpha) \gamma(\beta) = \gamma(\alpha + \beta).
$$

(1.15)

It is now straightforward to show that the curves considered above are geodesics of anti-de-Sitter space. What we have to show is that their projections are geodesics on the group manifold. Let us first consider the geodesics passing through the origin, that is, through the unit element $\mathbf{1} \in \text{SL}(2)$. On the group manifold, such geodesics are one-dimensional subgroups, consisting of the elements $\mathbf{x}(s) = e^{sn}$, $s \in \mathbb{R}$, where $\mathbf{n} = \gamma^a \gamma_a$ is some vector in the Lie algebra $\mathfrak{sl}(2)$. Let us take $\mathbf{n} = \gamma_0 + \xi \gamma(\theta)$, and work out the exponential explicitly. This gives

$$
\mathbf{x}(s) = e^{sn} = \cos s \mathbf{1} + \sin s (\gamma_0 + \xi \gamma(\theta)),
$$

(1.16)
where the real and analytic functions \( sn \) and \( cs \) are defined such that

\[
\begin{align*}
\text{sn } s &= \frac{\sinh(s \sqrt{\xi^2 - 1})}{s \sqrt{\xi^2 - 1}} = \frac{\sin(s \sqrt{1 - \xi^2})}{s \sqrt{1 - \xi^2}}, \\
\text{cs } s &= \cosh(s \sqrt{\xi^2 - 1}) = \cos(s \sqrt{1 - \xi^2}).
\end{align*}
\]  

Comparing this to (1.12), we find the following relation between the coordinates and the curve parameter \( s \),

\[
\frac{1 + r^2}{1 - r^2} \cos t = \text{cs } s, \quad \frac{1 + r^2}{1 - r^2} \sin t = \text{sn } s, \quad \frac{2r}{1 - r^2} = \xi \text{ sn } s,
\]  

and \( \varphi \) has to be equal to \( \theta \). One of the three relations between \( r \) and \( t \) turns out to be redundant, and eliminating the curve parameter \( s \), for example by taking the quotient of the last two equations, we recover the relation (1.4).

**Isometries**

To show that the circle segments (1.3) are geodesics as well, we can exploit the fact that, being a group manifold, anti-de-Sitter space is maximally symmetric. Every geodesic can be obtained by acting with an isometry on a geodesic passing through a special point. On a simple group manifold, isometries are arbitrary combinations of left and right multiplications with constants,

\[ x \mapsto g^{-1} x h, \quad g, h \in \text{SL}(2). \]  

(1.19)

Except for time and space inversion, every isometry of \( \text{SL}(2) \) can be written in this way, and the corresponding isometry of anti-de-Sitter space is then determined up to a time shift \( t \mapsto t + 2\pi z \), \( z \in \mathbb{Z} \). It follows that every geodesic on \( \text{SL}(2) \) can be written as

\[ x(s) = g^{-1} e^s n h, \quad g, h \in \text{SL}(2), \quad n \in \mathfrak{sl}(2). \]  

(1.20)

The parametrization is somewhat redundant, but for our purposes it is sufficient. Choosing

\[ g = e^{-\frac{1}{2} \zeta} \gamma(\alpha) e^{-\frac{1}{2} \tau \gamma_0}, \quad h = e^{\frac{1}{2} \zeta} \gamma(\alpha) e^{\frac{1}{2} \tau \gamma_0}, \quad n = \gamma'(\alpha), \]  

(1.21)

a straightforward calculation gives

\[ x(s) = \cosh \zeta \cosh s \omega(\tau) + \sinh \zeta \cosh s \gamma(\alpha) + \sinh s \gamma'(\alpha). \]  

(1.22)

If we compare this to (1.12), we find that this is a curve that entirely lies inside the Poincaré disc at \( t = \tau \), with the following relations between the coordinates \( r \) and \( \varphi \) and the curve parameter \( s \),

\[
\begin{align*}
\cosh \zeta \cosh s &= \frac{1 + r^2}{1 - r^2}, \quad \sinh s = \frac{2r}{1 - r^2} \sin(\varphi - \alpha), \\
\sinh \zeta \cosh s &= \frac{2r}{1 - r^2} \cos(\varphi - \alpha).
\end{align*}
\]  

(1.23)
Again, one of these equations is redundant, and after eliminating the curve parameter $s$, we are left with a single relation between $r$ and $\varphi$,

$$\frac{2r}{1 + r^2} \cos(\varphi - \alpha) = \tanh \zeta,$$

which is the same as (1.3), defining a geodesic centered at $\alpha$ with radius $\beta$, if we choose $\zeta$ such that $\cos \beta = \tanh \zeta$.

## 2 Point Particles

So far, we only considered empty anti-de-Sitter space, which is the unique solution to the vacuum Einstein equations with a negative cosmological constant, provided that the topology of spacetime is that of $\mathbb{R}^3$. Let us now include a matter source, in form of a pointlike particle. The effect of such a particle is almost the same with or without a cosmological constant. The most convenient way to construct a spacetime containing a point particle is by cutting and gluing [5, 6].

### A point particle in Minkowski space

For a vanishing cosmological constant, the method works as follows. One starts from flat Minkowski space, which is isomorphic to the Lie algebra $\mathfrak{sl}(2)$, that is, the spinor representation of the three dimensional Lorentz algebra. This is quite useful, because to adapt the procedure to anti-de-Sitter space, we only need to replace the Lie algebra by the group manifold. Let us introduce orthonormal coordinates $(t, x, y)$ on Minkowski space, and write a general vector as

$$z = t \gamma_0 + x \gamma_1 + y \gamma_2.$$

Now, consider a Lorentz transformation acting on $z$. It can be written as the adjoint action of some group element $u$,

$$z \mapsto u^{-1} z u, \quad u \in \text{SL}(2).$$

The fixed points of this map lie on the *axis* of the Lorentz transformation, consisting of all matrices $z$ that commute with $u$. If we expand $u$ in terms of the gamma matrices, and define a vector $p$ as

$$u = u \mathbf{1} + p^a \gamma_a, \quad p = p^a \gamma_a,$$

then the direction of $p$ specifies the axis. Let us assume that $p$ is timelike or lightlike. The axis can then be interpreted as the world line of a particle, with momentum vector $p$. In three dimensional Einstein gravity, the effect of such a particle as a matter source is that it produces a conical singularity, whose *holonomy* is $u$ [6]. The resulting spacetime is everywhere flat, except
on the world line, and transporting a vector once around the world line results in the Lorentz transformation (2.2).

The complete spacetime can be constructed by cutting out a wedge from Minkowski space. The wedge is bounded by two half planes emerging from the world line, such that one of them is mapped onto the other by the given Lorentz transformation. Note that this requires the points on the world line to be fixed. If we identify the two faces according to the Lorentz transformation, we obtain a spacetime that is locally flat, because the map that provides the identification is an isometry of Minkowski space. There is however a curvature singularity on the world line. By construction, it has the required property. Transporting a vector around the particle results in a Lorentz transformation, which is the same as the one that defines the identification.

A convenient way to visualize this construction is to use an ADM-like foliation of spacetime. At a given moment of time \( t \), the space manifold is a plane with coordinates \( x \) and \( y \). From this plane, we cut out a wedge, which is bounded by two half lines \( w_\pm \), such that \( w_+ \) is the Lorentz transformed image of \( w_- \). It is not immediately clear that the wedge can be chosen like this. It is only possible if the identification takes place within the planes of constant \( t \). It turns out that this can be achieved by choosing the wedge to lie symmetrically in front of or behind the particle. For simplicity, let us consider the following example. We choose a massless particle with a lightlike momentum vector pointing into the \( x \)-direction. Its holonomy is

\[
\mathbf{u} = 1 + \tan \epsilon \left( \gamma_0 + \gamma_1 \right), \quad 0 < \epsilon < \pi/2. \quad (2.4)
\]

The corresponding isometry of Minkowski space is a lightlike, or parabolic Lorentz transformation. The fixed points are at \( x = t \) at \( y = 0 \). Hence, the particle is moving with the velocity of light from the left to the right. To construct the wedge, we have to find a curve \( w_- \) in the \( t \)-plane, such that its image \( w_+ \) also lies in this plane. Let us make the following symmetric ansatz. The world line is invariant under vertical reflections, \( y \mapsto -y \). So, we assume that the wedge has this symmetry as well. A point \( (t, x, y) \in w_+ \) then corresponds to the point \( (t, x, -y) \in w_- \). The matrix representations of these points are

\[
\mathbf{w}_\pm = t \gamma_0 + x \gamma_1 \pm y \gamma_2, \quad (2.5)
\]

and for them to be mapped onto each other, we must have \( \mathbf{u} \mathbf{w}_+ = \mathbf{w}_- \mathbf{u} \). Inserting the expressions for \( \mathbf{w}_\pm \) and \( \mathbf{u} \), we find that this is fulfilled if and only if \( y = (t - x) \tan \epsilon \), which means that the faces \( w_\pm \) of the wedge are determined by

\[
w_+: \quad y = (t - x) \tan \epsilon, \quad w_-: \quad y = -(t - x) \tan \epsilon. \quad (2.6)
\]

For a given value of \( t \), these are two straight lines in the \((x, y)\)-plane, with angular directions \( \pm \epsilon \). They intersect at the fixed point, which is the position of the particle at time \( t \). Figure 2 shows the lines \( w_+ \) and \( w_- \) for three different times \( t \). The dot indicates the position of the particle, and the cross represents that origin of the spatial plane at \( x = 0 \) and \( y = 0 \).
We can now cut out the wedge between the two lines, either in front of or behind the particle. Both choices lead to the same spacetime manifold, but covered with different coordinates. Let us choose the wedge behind the particle. The space manifold is then the shaded region shown in the figure, with the boundaries marked by the double strokes identified. The opening angle \( \epsilon \) of the wedge, which is half of the \textit{deficit angle} of the conical space surrounding the particle, can be considered as the total energy of the particle together with its gravitational field, in units where Newton’s constant is \( G = 1/4\pi \) \([6]\). It is bounded from below by zero and from above by \( \pi/2 \). Note that this energy is smaller than the energy of the particle itself, that is, the zero component of its momentum vector, which is \( \tan \epsilon \).

The same construction can be made for massive particles. The holonomy \( u \) is then a timelike, or elliptic Lorentz transformation. It represents a rotation of space around a timelike axis, which becomes the world line of the particle. The wedge can be arranged in the same symmetric way, and its opening angle is also equal to the total energy. It is then bounded from below by the rest mass \( m \), which is equal to half of the angle of rotation. The angle of rotation coincides with the deficit angle if the particle is at rest. The angle of rotation and thus the rest mass of the particle can also be read off directly from the holonomy \( u \), using the \textit{mass shell} relation \([6]\).

\[
\frac{1}{2} \text{Tr}(u) = \cos m, \quad 0 \leq m \leq \pi.
\] (2.7)

**A point particle in anti-de-Sitter space**

To construct a spacetime containing the same kind of point particle, but with a negative cosmological constant, let us repeat the same procedure, step by step, in anti-de-Sitter space. The first step was to specify the world line as the set of fixed points of an isometry. We chose the isometry to be a Lorentz transformation, which has the special property that it leaves the origin of Minkowski space fixed. In principle, we could weaken this restriction, but on the other hand we can always choose coordinates such that the particle is passing through the origin. It is therefore sufficient to consider those isometries of anti-de-Sitter space which leave the origin fixed. We know already that a general isometry of the group manifold SL(2) can be written as \([1.19]\). The
unit element 1 is a fixed point if and only if \( g = h \). Hence, the isometry has to be of the form

\[
x \mapsto u^{-1} xu, \quad u \in \text{SL}(2).
\]  

(2.8)

So, the relevant isometry group is again \( \text{SL}(2) \). Indeed, \( u \) can be considered as the holonomy of the particle, in the same way as in Minkowski space above. If we are in a neighbourhood of the origin, which is small compared to the curvature radius of anti-de-Sitter space, we can expand \( x = 1 + z + \ldots \) with \( z \in \mathfrak{sl}(2) \). On the Minkowski vector \( z \), the map (2.8) acts exactly like the one defined by (2.2) above. We can therefore expect that, in the neighbourhood of the particle, spacetime will have the same conical structure. Also in analogy with Minkowski space, the fixed points of the given isometry are those elements of \( \text{SL}(2) \) that commute with \( u \). They can be found in the same way. We expand \( u \) in terms of the gamma matrices, and define a momentum vector \( p \),

\[
u = u 1 + p^a \gamma_a, \quad p = p^a \gamma_a.
\]  

(2.9)

The only difference is that now the fixed points are not the vectors proportional to \( \mathfrak{p} \), but the elements of the one dimensional subgroup generated by \( \mathfrak{p} \), consisting of the matrices

\[
x(s) = e^{sp}, \quad s \in \mathbb{R}.
\]  

(2.10)

This is a geodesic on the group manifold. Assuming that \( \mathfrak{p} \) is timelike or lightlike, it is the projection of a world line of a massive, respectively massless particle in anti-de-Sitter space. As an example, we consider the same massless particle once again, with holonomy (2.4),

\[
u = 1 + \tan \epsilon (\gamma_0 + \gamma_1).
\]  

(2.11)

The fixed points lie on a lightlike world line, with \( r = \tan(t/2) \) and \( \varphi = 0 \). To construct the wedge that the particle cuts out from anti-de-Sitter space, we proceed in the same way as before. First, we switch to an ADM point of view, so that anti-de-Sitter space becomes a space manifold, the Poincaré disc, evolving in time. Then, we look for a pair of lines \( w_\pm \) on the disc of constant time \( t \), which are mapped onto each other by the given isometry. Finally, we cut out the wedge between these lines, and identify the faces according to the isometry.

There is however one crucial difference to the Minkowski space example considered above. It only takes a finite amount of time for the particle to travel through the whole disc. It enters at \( t = -\pi/2 \), and it leaves again at \( t = \pi/2 \). Before and after that, there is no matter present, and therefore spacetime is expected to be empty anti-de-Sitter space, with no wedge or whatsoever cut out. Only for \( -\pi/2 < t < \pi/2 \) the particle is present, and we expect the space manifold to be a Poincaré disc with a wedge cut out. For the shape of this wedge, we make the same symmetric ansatz as in Minkowski space. The world line of the particle is invariant under reflections of the vertical axis. In cylindrical coordinates, this is the transformation \( \varphi \mapsto -\varphi \).

So, we assume that a point \( (t, r, -\varphi) \in w_- \) is mapped onto \( (t, r, \varphi) \in w_+ \). The matrices
representing these points on the group manifold are given by (1.12),

\[ w_\pm = \frac{1 + r^2}{1 - r^2} \omega(t) + \frac{2r}{1 - r^2} \gamma(\pm \varphi). \] (2.12)

Evaluating the equation \( u w_+ = w_- u \), we find that the faces \( w_+ \) and \( w_- \) are uniquely determined by the following coordinate relations,

\[ w_\pm : 2r \frac{\sin(\epsilon \pm \varphi)}{1 + r^2} = \sin t \sin \epsilon. \] (2.13)

If we define a parameter \( \alpha = \pi/2 - \epsilon \), such that \( \sin \epsilon = \cos \alpha \), and compare this to (1.3), we find that \( w_+ \) is a geodesic centered at \( \alpha \), whose radius \( \beta \) is given by \( \cos \beta = \sin t \cos \alpha \), and \( w_- \) is a geodesic with the same radius, centered at \( -\alpha \). As a function of \( t \), the radius decreases from \( \beta = \pi - \alpha \) at \( t = -\pi/2 \) to \( \beta = \alpha \) at \( t = \pi/2 \).

In figure 3 these curves are shown for different values of \( t \). Let us first consider the pictures (a–d), for the time between \( t = -\pi/2 \) and \( t = \pi/2 \). What we see is that the lines \( w_+ \) and \( w_- \) are moving upwards, respectively downwards, such that their intersection, the fixed point of the isometry at \( r = \tan(t/2) \), moves from the left to the right. The space manifold is obtained by cutting out the wedge behind the particle and identifying the boundaries marked by the double strokes. The resulting spacetime manifold has a constant curvature everywhere, except on the
world line. The reason is the same as before. The map that provides the identification is an
isometry of anti-de-Sitter space, and therefore there is no extra curvature introduced by gluing
together the two faces of the wedge.

To see that the matter source is the same as before, it is, as already mentioned, sufficient to
consider a small neighbourhood of the world line, where the curvature of anti-de-Sitter space can
be neglected. Indeed, if we enlarge the region around the center of the disc in figure 3, it looks
exactly like the one shown in figure 2. So, what we have so far is a piece of spacetime between
\( t = -\pi/2 \) and \( t = \pi/2 \). The continuation has to be a solution to the vacuum Einstein equations,
because there is no matter present outside this time interval. This is not in contradiction with
causality, because the foliation of anti-de-Sitter spacetime by discs of constant \( t \) is not a foliation
by Cauchy surfaces. Matter consisting of massless particles can appear and disappear at the
boundary at any time.

For \( t < -\pi/2 \), it is more or less obvious how to continue. At \( t = -\pi/2 \), the space manifold
is a complete Poincaré disc, with no matter inside. At this time the particle is still at \( r = 1 \),
which is outside the open disc. Only a small time later the particle is actually there. If we
assume that for all earlier times the space manifold is a complete disc as well, then we obtain
a continuous solution to the Einstein equations, which is matter free for all \( t \leq -\pi/2 \). What
is not so obvious is what happens at \( t = \pi/2 \), after the particle has left. The shaded region of
figure 3(d) does not at all look like a complete disc. Let us consider the further evolution of this
space manifold.

If we want to stick to \( t \) as a global time coordinate, and the vacuum Einstein equations to be
fulfilled everywhere, then the time evolution of the boundaries of the shaded regions is uniquely
determined. This is because the curves \( w_{\pm} \) defined by \( \sqrt{s} \) are the only curves inside the discs
of constant \( t \) which are mapped onto each other by the given isometry. For times \( t > \pi/2 \), the
curves start to move backwards, as shown in figure 3(e–f), and then they oscillate between two
extremals. The shaded regions start to overlap, but this is just a coordinate effect. One has to
consider them as two separate charts covering the space manifold. The only identification takes
place along the boundaries \( w_{+} \) and \( w_{-} \). As this is always defined by the same isometry, we
obtain a continuous solution to the Einstein equations for all \( t \), which is matter free outside the
time interval \( -\pi/2 < t < \pi/2 \).

Now it seems that after the particle has left, spacetime looks very different from what is was
before, although we know that anti-de-Sitter space is the only matter free solution to the Einstein
equations on a topologically trivial spacetime manifold. But this is also just a coordinate effect.
In fact, the foliation of spacetime by the space manifolds shown in figure 3(e–f) is a somewhat
skew foliation of anti-de-Sitter space. To see this, let us take a three dimensional point of view.
The two shaded regions, evolving in time, then define two subsets of anti-de-Sitter space, whose
boundaries are the surfaces \( w_{\pm} \). By definition, these two surfaces are mapped onto each other
by an isometry, which is continuous and one-to-one. Therefore, one of the subsets is isometric
to the complement of the other, and thus both together form a complete anti-de-Sitter space.

So, finally we see that the whole situation is time symmetric. The spacetime looks like
empty anti-de-Sitter space before and after the particle is there. The asymmetry in the pictures is only due to the fact that it is not possible to cover the whole manifold symmetrically with a single coordinate chart, which locally looks like the standard chart of anti-de-Sitter space. We can reverse the picture if we cut out the wedge in front of the particle instead of the one behind. We then obtain the same spacetime, covered with different coordinates, providing the standard foliation of anti-de-Sitter space after the particle has left, but the skew one before it enters.

3 Colliding Particles

Let us now describe the process of two particles colliding, and thereby joining and forming a single particle. The basic idea is as follows. Consider two relativistic point particles in flat Minkowski space, with no gravitational interaction. If they collide, that is, if their world lines intersect at some point in spacetime, then we assume that from that moment on they form a single particle, whose momentum vector is given by the sum of the momenta of the incoming particles. This is consistent with energy momentum conservation, and it is a deterministic classical process, although not time-reversible. All properties of the joint particle can be deduced from the incoming particles. In particular, for a scalar particle the momentum vector is the only relevant quantity.

When gravity is taken into account, the situation changes slightly. The process is still deterministic, but it is not the sum of the momentum vectors that is preserved. Instead, it is the total holonomy, which is the product of the two holonomies of the incoming particles, and which becomes the holonomy of the joint particle. This has some strange consequences. For example, unlike the sum of two timelike or lightlike vectors, the product of two timelike or lightlike holonomies is not necessarily timelike. The joint particle can, for example, become a tachyon. To understand this, it is again most convenient to study the process in Minkowski space first, and then apply the same methods to anti-de-Sitter space.

Joining particles in Minkowski space

Let us first consider the collision of two massless particles graphically. We can always choose a coordinate system in Minkowski space such that the collision takes place at the origin. The world lines of both incoming particles, and also that of the outgoing particle, are then passing through the origin, and we can apply the methods of the previous section. Furthermore, we can choose a center of mass reference frame, such that the particles come from opposite spatial directions and have the same energy. Hence, without loss of generality, we can assume that the holonomies of the incoming particles are given by

\[ u_1 = 1 + \tan \epsilon (\gamma_0 + \gamma_1), \quad u_2 = 1 + \tan \epsilon (\gamma_0 - \gamma_1). \]  

(3.1)
The first particle is then the same as the one considered in the previous section. The wedge that it cuts out from Minkowski space is bounded by the faces

$$w_{1+}: \quad y = (t - x) \tan \epsilon, \quad w_{1-}: \quad y = -(t - x) \tan \epsilon.$$  \hfill (3.2)

The second particle has the same properties, except that it is moving into the opposite direction. The wedge is found by rotating that of the first particle by 180 degrees,

$$w_{2+}: \quad y = -(t + x) \tan \epsilon, \quad w_{2-}: \quad y = (t + x) \tan \epsilon.$$  \hfill (3.3)

For times $t < 0$, before the collision, the space manifold is the shaded region of figure 4(a). It is a plane with two wedges cut out behind the two particles, both with opening angle $\epsilon$. The identification along the boundaries, indicated by the double and triple strokes, is again such that points with the same $x$-coordinate correspond to each other. The resulting space manifold looks like a double cone, that is, a cone with two tips, moving towards each other with the velocity of light.

At $t = 0$, when the particles collide, the space manifold becomes a simple cone with a single tip. Assuming that this is also the case at any later time, the further evolution of spacetime is uniquely determined by the Einstein equations, without making any additional assumptions about the joint particle itself. The argument is very similar to the one used in the end of the previous section. It is sufficient to know that spacetime is everywhere flat, except at one point in space, which is the position of the joint particle. If we want to stick to the foliation of spacetime by surfaces of constant $t$, then there is only one way how the lines $w_{1\pm}$ and $w_{2\pm}$ can evolve. They must be given by (3.7) and (3.8) for all times, because these are the only lines inside the surface of constant $t$, which are mapped onto each other by the given isometries.

In figure 4(c), we can see what they look like after the collision. If we require that there is only one point in space where the curvature is non-zero, then this can only be the point where $w_{1+}$ and $w_{2-}$, respectively $w_{2+}$ and $w_{1-}$ intersect. Note that due to the identification, these two points in the picture represent the same physical point in space. The space manifold is then covered by two charts, the two separate shaded regions, glued together along their boundaries.
The positions of the joint particle in the two charts can be found as the intersections of the lines \( w_1 \pm \) and \( w_2 \pm \). They lie on the \( y \)-axis, at \( y = \pm t \tan \epsilon \). Hence, in the upper chart the particle is moving upwards with a velocity of \( \tan \epsilon \), and in the lower chart it is moving downwards with the same velocity.

**Energy momentum conservation**

As we should have expected, the velocity of the joint particle depends on the energy of the incoming particles. What is however somewhat peculiar is that, for sufficient high energies, the velocity becomes bigger than one and thus the outgoing particle is moving faster than the speed of light. In other words, two incoming massless particles with sufficient high energy can form a tachyon. This is impossible for relativistic point particles in Minkowski space, without gravitational interaction. The sum of two lightlike momentum vectors is always a timelike vector.

What makes the situation different if gravity is present, is, as already mentioned, that it is not the sum of the momentum vectors of the particles which is preserved, but the product of their holonomies. In particular, the holonomy of the outgoing particle is given by the product of the two holonomies of the incoming particles. To see this, consider once again figure [c]. If we transport a vector once around the joint particle, then we have to pass once over the left wedge and once over the right wedge. The result is that we have to act on the vector first with the Lorentz transformation represented by \( u_1 \), and then with the one represented by \( u_2 \). The holonomy of the joint particle is therefore the product of two individual holonomies of the incoming particles. We can multiply them in two different ways,

\[
\begin{align*}
\mathbf{u}_+ &= \mathbf{u}_2 \mathbf{u}_1 = (1 - 2 \tan^2 \epsilon) \mathbf{1} + 2 \tan \epsilon (\gamma_0 + \tan \epsilon \gamma_2), \\
\mathbf{u}_- &= \mathbf{u}_1 \mathbf{u}_2 = (1 - 2 \tan^2 \epsilon) \mathbf{1} + 2 \tan \epsilon (\gamma_0 - \tan \epsilon \gamma_2).
\end{align*}
\]

(3.4)

There are two representations of the holonomy of the joint particle, because spacetime is covered by two coordinate charts, the upper and lower shaded region in the figure. Each expression represents the holonomy in one of the charts. To find out which is which, consider the momentum vectors

\[
\begin{align*}
\mathbf{p}_+ &= 2 \tan \epsilon (\gamma_0 + \tan \epsilon \gamma_2), \\
\mathbf{p}_- &= 2 \tan \epsilon (\gamma_0 - \tan \epsilon \gamma_2).
\end{align*}
\]

(3.5)

Obviously, \( \mathbf{p}_+ \) describes a particle that is moving upwards with a velocity of \( \tan \epsilon \), and \( \mathbf{p}_- \) corresponds to a particle moving downwards with the same velocity. This implies that the quantities with a plus index are the ones corresponding to the upper chart. We can now also see is that the joint momentum vector is timelike for \( 0 < \epsilon < \pi/4 \). Only in this case, the resulting joint object can be considered as a massive particle. Its mass is given by the formula (2.7). Inserting either \( \mathbf{u}_+ \) or \( \mathbf{u}_- \), we find that

\[
\sin(m/2) = \tan \epsilon.
\]

(3.6)
For small energies, this becomes $m \approx 2\epsilon$, which is the naive expression that applies when gravity is switched off. For higher energies, it deviates from this flat space relation. At $\epsilon = \pi/4$, the mass reaches its maximum $m = \pi$, and for $\epsilon > \pi/4$, the formal solution for $m$ becomes imaginary, indicating once again that the joint particle is a tachyon. What the existence of such a spacelike conical singularity in spacetime means depends rather crucially on its global structure. We shall therefore not go into more details regarding the tachyon in Minkowski space, where it can be considered as a kind of big bang or big crunch singularity \cite{7}. A more interesting situation arises in anti-de-Sitter space, where the spacelike singularity becomes the future singularity of a black hole.

**Joining particles in anti-de-Sitter space**

Before coming to this, let us first consider the colliding and joining of two particles in anti-de-Sitter space, with the individual energies being below the threshold, $0 < \epsilon < \pi/4$. The construction is the same as in Minkowski space, we only have to adapt the pictures and the formulas. The holonomies of the incoming particles are given by the same expressions \cite{3}. Both world lines are lightlike geodesics passing through the origin, entering from $\mathcal{J}$ at $t = -\pi/2$. Before that, the space manifold is a complete Poincaré disc. Afterwards, and until the particles collide at $t = 0$, it is a disc with two wedges cut out, as shown in figure 5(a). The
positions of the particles are on the horizontal axis, at \( r = \tan(t/2) \). The wedge of the first particle is bounded by the curves (2.13),

\[
    w_{1\pm} : \quad \frac{2r}{1 + r^2} \sin(\epsilon \pm \varphi) = \sin t \sin \epsilon.
\]

and that of the second particle is obtained by rotation, \( \varphi \mapsto \varphi + \pi \),

\[
    w_{2\pm} : \quad \frac{2r}{1 + r^2} \sin(\epsilon \pm \varphi) = -\sin t \sin \epsilon.
\]

Locally, the process at \( t = 0 \) is exactly the same as in Minkowski space. If we enlarge a small neighbourhood of the origin of the disc at \( t \approx 0 \), everything looks the same as in figure 4. When the particles collide in figure 5(b), the two conical singularities form a single one. A short time later, the two shaded regions of figure 5(c) separate from each other. Space is then covered by two coordinate charts, and the positions of the joint particle in these charts can be found as the intersection of the curves \( w_{1+} \) and \( w_{2-} \), respectively \( w_{1-} \) and \( w_{2+} \). They lie on the vertical diameter of the disc, with the radial coordinate given by

\[
    \frac{2r}{1 + r^2} = \sin t \tan \epsilon.
\]

This is a geodesic of the form (1.4). Whether it is timelike, lightlike, or spacelike, and thus whether the joint particle is slower or faster than the speed of light, depends on the size of \( \epsilon \). In fact, the relation between the velocity of the outgoing particle and the energy of the incoming ones is the same as in flat space. In accordance with the properties of the holonomy of the joint particle (3.4), it is timelike for \( 0 < \epsilon < \pi/4 \), lightlike for \( \epsilon = \pi/4 \), and spacelike for \( \pi/4 < \epsilon < \pi/2 \).

In case of a timelike particle, the further time evolution looks as follows. The radial coordinate of the joint particle reaches a maximum and the size of the shaded region has a minimum in figure 5(d), at \( t = \pi/2 \). After that, the shaded regions of figure 5(e) are growing again and moving towards each other. At \( t = \pi \), the particle is back at \( r = 0 \), and thereafter the shaded regions start to overlap, as can be seen in figure 5(f). This is however, once again, only a coordinate effect. The space manifold is always covered by two separate charts, with the only identifications taking place along the boundaries. The maximal size of the space manifold is reached at \( t = 3\pi/2 \), then it starts to shrink again, and finally it oscillates between the two extremals with a period of \( 2\pi \) in \( t \). In contrast to a massless particle, the massive particle does not leave the disc. This is because only lightlike and spacelike geodesics can reach the boundary \( J \), but not timelike ones.

What is not so obvious is that the final state is indeed a spacetime containing a single particle whose mass is given by (3.6). In the rest frame of such a particle, the complete spacetime would be an anti-de-Sitter space, with the particle sitting in the center of the Poincaré disc, from where a wedge with opening angle \( m \) is cut out. With a similar argument as in the end of section 3.
one can show that the foliation obtained here is again a somewhat skew foliation of the standard spacetime where the particle is at rest. We are not going to show this explicitly, because we are more interested in the case where the energy of the infalling particles is beyond the threshold, and the joint particle becomes a tachyon.

4 Black Hole Creation

For $\pi/4 < \epsilon < \pi/2$, the process of colliding and joining of the particles, at least before and shortly after the collision, looks almost the same as in figure 3. Only the parameter $\epsilon$, and thus the opening angle of the two wedges, is somewhat bigger. The space manifolds for three different times before the collision are shown in figure 6(a–c), the collision takes place in figure 6(d), and shortly after that the two shaded regions separate from each other. So far, nothing new is happening. However, the dots in figure 6(e), representing the joint object after the collision, are now moving upwards, respectively downwards, on a spacelike geodesic, which is defined by

$$\frac{2r}{1 + r^2} = \sin t \tan \epsilon.$$  \hspace{1cm} (4.1)

As we already know from the general consideration of spacelike geodesics, this equation has a solution only within a finite time interval. The geodesic reaches the boundary of the Poincaré disc at some time $t = \tau$, which is in this case given by

$$\sin \tau = \cot \epsilon, \quad 0 < \tau < \pi/2.$$  \hspace{1cm} (4.2)

So far, this is not a problem. The massless particle in figure 3 also reaches the boundary of the disc and disappears at some time, but nevertheless it was possible to continue spacetime afterwards. But now the situation is different. If we look at figure 6(f), we see that, together with the joint particle, the whole space manifold has disappeared at $t = \tau$. There is nothing to be continued after that moment. It is not immediately clear whether this is again just a coordinate effect or something more relevant, but let us for the moment assume that it is indeed not possible to continue. Then we have to conclude that the spacelike world line of the joint object is a future singularity, that is, a line on which time ends.

The horizon

If there is such a singularity in spacetime, then it is, in addition to $\mathcal{J}$, another possible destination of causal curves. We can then ask the question whether there is a region of spacetime, from where every causal curve ends on the singularity. Or, in other words, a region which is causally disconnected from $\mathcal{J}$, in the sense that no information can be passed from there to spatial infinity. The boundary of such a region is usually called a horizon, and what is beyond the horizon is the interior of a black hole.
Before showing that there is such a horizon, let us first consider $\mathcal{J}$ itself. In figure 6, it is that part of the boundary of the disc, which is also a boundary of the shaded region. For $t \leq -\pi/2$, this is the full boundary of the Poincaré disc. For $-\pi/2 < t < \tau$, it consists of two parts, one between the curves $w_{1+}$ and $w_{2-}$, and one between $w_{2+}$ and $w_{1-}$. Because the end points of the two parts are identified, $\mathcal{J}$ has always the shape of a closed circle. However, the circumference of this circle shrinks, and it goes to zero at $t = \tau$. Let us call the point which is represented by the two dots in figure 6(f) the last point on $\mathcal{J}$. Note again that the identification is such that both dots represent the same physical point on $\mathcal{J}$.

It is not difficult to show that every point on $\mathcal{J}$ is causally connected to the last point. This is because the end points of the curves $w_{1\pm}$ and $w_{2\pm}$ on $\mathcal{J}$ are moving slower than the speed of light. The amount of time they need to traverse a quarter of the circumference of the disc is $\tau + \pi/2$, whereas a light ray only needs $\pi/2$ to travel over the same distance. From this we conclude that the horizon is the backward light cone of the last point on $\mathcal{J}$. To construct this light cone, we make use of some general properties of light cones emerging from $\mathcal{J}$. First of all, such a light cone is a geodesic surface. If we intersect it with another geodesic surface, say, a disc of constant $t$, then the intersection is a geodesic. Moreover, we know that light travels on $\mathcal{J}$ with a velocity of $d\phi/dt = 1$.

Using all this, it follows that the backward light cone of the upper dot in figure 6(f) is, at some time $t$ in the past, a geodesic centered at $\pi/2$, with radius $\tau - t$. The backward light cone
emerging from the lower dot is centered at \(-\pi/2\), and has the same radius. These two curves are shown as the dashed lines inside the shaded regions of figure 3. Due to the identifications along the boundaries of the wedges, they form a closed circle. From the area that is enclosed by this circle, no signal can be sent along a causal curve to \(\mathcal{I}\), even if we make use of the identifications and jump over the wedges. On the other hand, from every point outside the circle, we can find a causal curve either to the upper or to the lower part of \(\mathcal{I}\). This is also possible for every point in space before \(t = \tau - \pi/2\), which is the moment when the two dashed lines meet in the middle of the disc.

All together we get the following scenario. At \(t = -\pi/2\), the particles enter from opposite sides, approaching each other with the speed of light. When they are close enough, at \(t = \tau - \pi/2\), the horizon emerges, and at the same time the particles fall behind it. The actual collision of the particles, and the creation of the future singularity, takes place behind the horizon at \(t = 0\). This is very similar to the creation of a Schwarzschild black hole in a star collapse. When the star becomes small enough, the horizon emerges from some kind of cusp, and the singularity is created behind the horizon. The differences are mainly due to the different symmetries of the process. For example, the horizon does not emerge from a point in the center of the star, but from the geodesic that connects the two particle at the moment when they fall behind it.

Another and more crucial difference seems to be that time also ends at \(t = \tau\) in the exterior region, which is not the case for a Schwarzschild black hole. However, it is only the coordinate time \(t\) that ends at this point. The physical lifetime of an observer staying outside the horizon can be infinite. This is because the lapse factor in front of the \(dt\)-term in the metric (1.2) diverges at the boundary of the disc. Consider an observer moving on a timelike curve which stays outside the horizon. There is no such geodesic, which is another difference to the Schwarzschild black hole, but there are timelike curves approaching the last point, or any other point on \(\mathcal{I}\). In general, such a curve has an infinite physical length, which means that the outside observer can live infinitely long. This is not the case for a curve ending on the future singularity. The world line of an observer falling into the black hole always ends after a finite physical time.

The size of the black hole

All these features justify the notion of a black hole for the object generated by the two particles. We can then ask the question how big this black hole is. As we are in two space dimensions, the size is specified by the circumference of the horizon. It is constant in time, because there is no more matter falling into the black hole after it has been created and the two particles have fallen in. We shall see this more explicitly in the next section. At \(t = \tau - \pi/2\), the horizon length is twice the physical distance between the two particles. To calculate this explicitly, let us introduce a new parameter \(\mu > 0\), which is essentially the analytic continuation of the rest mass \(m\) of the massive particle, which was given by (3.6) for \(\epsilon < \pi/4\). For \(\epsilon > \pi/4\), we define \(\mu\) such that

\[ \cosh(\mu/2) = \tan \epsilon. \] (4.3)
Now, let $\rho$ be the radial coordinate of the particles at the moment when the horizon is created,

$$\rho = \tan(\tau/2 - \pi/4).$$

(4.4)

The time $\tau$ was given by (4.2). Using this and some trigonometric identities, we obtain

$$\rho = \tanh(\mu/4).$$

(4.5)

To find the physical distance between the particles, we have to integrate the $d\tau$-component of the metric,

$$\int_{-\rho}^{\rho} \frac{2}{1 - r^2} dr = 4 \arctanh \rho = \mu.$$

(4.6)

It follows that the length of the horizon is $2\mu$. Assuming that there is some kind of no hair theorem, this should be the only parameter describing a non-rotating black hole. However, we already considered another quantity to specify this particular solution to the Einstein equations, namely the total holonomy (3.4). So, there should be a relation between the holonomy and the length of the horizon. From (4.3), we infer that

$$\cosh \mu = \tan^2 \epsilon - 1.$$

(4.7)

This is, up to a sign, the contribution proportional to the unit matrix of the holonomy $u_+$ or $u_-$ in (3.4). They both lie in the same conjugacy class of $\text{SL}(2)$, which is specified by

$$\frac{1}{2} \text{Tr}(u) = -\cosh \mu.$$

(4.8)

Hence, the size of the black hole can be read off directly from its holonomy. The remaining information contained in the holonomy $u$ is not a property of the black hole itself. It describes its motion relative to the reference frame. We should also note that the relation (4.8) between the holonomy and the size of the black hole is essentially the analytic continuation of the mass shell relation (2.7). Geometrically, the mass was defined to be half of the angle of rotation of the Lorentz transformation obtained by transporting a vector around the particle. Here, the Lorentz transformation represented by the holonomy $u$ is not a rotation, but a boost, and $\mu$ is half of the hyperbolic angle that parametrizes this boost. Using this analogy, we shall call $\mu$ the mass of the black hole, which is equal to half of the horizon length.

**Extremal black holes**

We have now studied the collision of two particles with energies below and above the threshold of $\epsilon = \pi/4$, but we didn’t consider the case $\epsilon = \pi/4$ so far. What happens in figure 3 in this limit? The joint object is no longer moving on a spacelike, but on a lightlike geodesic. The time when it reaches the boundary of the disc becomes $\tau = \pi/2$. As a consequence, the horizon emerges at $t = 0$, at the moment when the particles collide. As both the horizon and the joint
particle move with the speed of light, there is actually no interior region. This is in accordance with (4.3), which says that for \( \epsilon = \pi/4 \) we have \( \mu = 0 \), and therefore the size of the black hole becomes zero. So, what we get in the lightlike limit is a kind of extremal black hole, whose inside region consists of only the singularity, located on a lightlike geodesic.

But what is the difference between this and a massless particle, which is also moving on a lightlike geodesic? Both objects have a lightlike holonomy, representing a parabolic Lorentz transformation, but nevertheless there is a crucial difference. For the lightlike particle considered in section 2, the moment when it reaches \( \mathcal{J} \) is not the end of time. Here it is, because even in the lightlike limit, there is no space manifold left at \( t = \tau \). To understand the difference, let us also consider the lightlike case as a limit of the timelike one, which is shown in figure 5. The reason why this could be extended for all times in the future was that, when the radial distance of the joint particle from the origin reached its maximum at \( t = \pi/2 \), there was still some space left which could evolve further.

This is no longer the case when the joint particle becomes lightlike, because then it reaches the boundary at \( t = \pi/2 \). The lightlike limit of figure 5 is thus the same as the lightlike limit of figure 3. But we can also see is that, when considered as a limit of the timelike situation, the extremal black hole is not the limit where \( m \to 0 \). Instead, it is the limit \( m \to \pi \). This is the upper bound for the rest mass of a particle in three dimensional Einstein gravity, because the maximal deficit angle of a conical singularity is \( 2\pi \). A peculiar feature of this upper bound is that the world line of a particle with \( m = \pi \) is lightlike, but the structure of the conical singularity on the world line is very different from the one generated by a particle with \( m = 0 \).

These exotic lightlike particles also exist in flat spacetime, with vanishing cosmological constant, and they also have some peculiar features, which are very similar to extremal black holes [6, 7]. The difference between massless and exotic lightlike particles can also be seen at the level of the holonomies. There are two different classes of group elements \( u \in SL(2) \) representing parabolic Lorentz transformations, namely those with \( \frac{1}{2} \text{Tr}(u) = 1 \), and those with \( \frac{1}{2} \text{Tr}(u) = -1 \). The former correspond to massless particle, and the latter are the extremal black holes.

**Massive particles**

As a possible generalization, let us ask the question whether it is really necessary to start with massless particles, or can the black hole also be created by two colliding massive particles? The answer is that it can, but then the global structure of spacetime changes, at least if we do not allow any other interaction between the particles than the gravitational one. If we want the same black hole to be created from two infalling massive particles, then the only constraint is that the product of their holonomies is the same, or at least that it lies in the same conjugacy class of \( SL(2) \). This can be achieved for any given pair of masses. The structure of spacetime after the collision is then exactly the same as the one shown in figure 3(d–f). What is also similar is that the horizon is created and the particles fall behind it at some time \( t = \tau - \pi/2 \), where \( \tau \) is the
time coordinate of the last point on \( J \).

However, the situation looks very different before that. Unlike the massless particles, the massive ones cannot enter from \( J \). They must have been there for all times in the past. At \( t = -\pi/2 \), the timelike geodesics on which the particles are moving reach their maximal distance from the center. If we follow their world lines back into the past, then they meet again in the center at \( t = -\pi \). So, something special is happening there as well. In fact, the whole situation is symmetric with respect to time reversion \( t \mapsto -\pi/2 - t \). Using the same arguments as above for the collision of the particles, we have to conclude that the only reasonable way to continue further to the past is to assume that the particles emerge from the decay of some other object. This joint object must be a past singularity, that is, a spacelike curve on which time begins and causal curves cannot be extended to the past. It reaches \( J \) at the time \( t = -\pi - \tau \), and there is no extension of spacetime before that.

A foliation of this spacetime looks like the one shown in figure 3, but starting not from a full Poincaré disc for \( t \leq -\pi/2 \), but from the singular situation (f) at \( t = -\pi - \tau \), then going backwards to (a) at \( t = -\pi/2 \), and ending up again with the last picture (f) at \( t = \tau \). There is then also a first point in \( J \), and the light cone emerging from this point is the past horizon of a white hole. An outside observer in this spacetime sees two particles coming out of a white hole, reaching a maximal distance, approaching each other again, and finally falling into a black hole. This was the actual reason for considering the collapse of two massless particles first. With that kind of matter it is possible to give an exact solution to the Einstein equations, describing the creation of a black hole, without the need of any other exotic structure like the white hole.

We thereby exploited the global causal structure of anti-de-Sitter space, and in particular that of the boundary \( \partial J \), which is very different from that of asymptotically flat spacetimes. An alternative way to avoid the white hole, but nevertheless create a black hole using massive particles, might be to allow another kind of interaction, which prevents the particles from approaching each other in the past. It is not immediately clear whether this can be achieved, in particular because, for a consistent treatment, one has to describe the second force as a field which also interacts with gravity, and this makes it much harder to find an exact solution. The only way to avoid the white hole without introducing other forces is to create the massive particles themselves, or at least one of them, out of two massless particles, by the process described in section 2.

**Rotating black holes**

Finally, let us very briefly consider another possible generalization. What happens if the particles do not collide, but pass each other at, say, a small distance? An intuitive conclusion would be that they create a rotating black hole. But is there any singularity then? If there is no collision, then there is also no need for the particles to join and to form a single object. They are just scattered, move on along their lightlike geodesics, and disappear to \( \partial J \) at some time in the future, such that in the end, spacetime looks like anti-de-Sitter space again. If we look at this
process in Minkowski space, then we find indeed that, if the particles do not hit each other, they just move on along their world lines extending to infinity.

This is not immediately obvious from figure 4, because the two wedges shown there will in any case overlap from some time on, and then we have to switch to a different coordinate system. But if we give up the condition that spacetime should be foliated by planes of constant \( t \), it is easy to see that, given any two non-intersecting lightlike world lines in three dimensional Minkowski space, one can always choose the wedges such that they do not overlap. If there is no joint object any more, one can also ask the question whether there is still anything special about the threshold \( \epsilon = \pi/4 \), which was previously the minimal energy of the incoming particles to create a spacelike singularity. In the case of colliding particles, it makes a crucial difference whether the energy is below or above the threshold, but it seems that, if the particles are just scattered, this does not play a particular role.

But this is not true. There is a crucial difference between the two situations, even if no joint object is created. The difference is that, if the energy lies above the threshold, then there are closed timelike curves in spacetime. These curves are always present, however large the distance between the particles is, and they are not located in the neighbourhood of the particles, but in a region of spacetime which extends, from outside some area around the particles at the moment of closest approach, to spatial infinity. For a vanishing cosmological constant, this spacetime is known as the Gott universe [8, 9]. There is no singularity like the one for the non-rotating black hole, but instead there is a region in spacetime where closed timelike curves exist.

Such a region of spacetime behaves very much like a future or past singularity. Causal curves can end, respectively start there, because they can wind up in an endless loop. Indeed, the analogy between closed timelike curves and future or past singularities is very close. Remember that above we assumed that it is not possible to continue the black hole spacetime beyond the future singularity. But this is only true if we do not allow closed timelike curves. If we give up this restriction, then there is a possible extension. It is very similar to the Misner universe, which can be obtained by cutting out a piece from two dimensional Minkowski space, and identifying the boundaries according to a hyperbolic Lorentz transformation.

Turning this argument around, we should consider a region of spacetime with closed timelike curves as a future, respectively past singularity as well. Doing so, we find the following scenario of two massless particles, passing each other with sufficiently high energy. They enter the spacetime from \( \mathcal{J} \) at some time, and before that we have an empty anti-de-Sitter space. The particles are then approaching each other, and some time before the moment of closest approach, closed timelike curves arise. They appear at spatial infinity first, and then the boundary of the region where they exist, now interpreted as a singularity in spacetime, approaches the particles. But it never hits them, because there are no closed causal curves intersecting the world lines. There remains a region of space around and between the particles where no closed timelike curves ever occur.

After the moment of closest approach, the particles just move on, the boundary of the region with closed timelike curves moves back to spatial infinity, and the same happens to the particles,
which disappear to \( J \) at some time in the future. After that, spacetime looks again like empty anti-de-Sitter space. We can roughly divide it into three parts, the time before the singularity appears, while it is there, and after it has gone. The boundary \( J \) of this spacetime splits into two disconnected parts, before and after the singularity. Note that, because \( J \) is two dimensional, the existence of already a single closed causal curve, interpreted as a singularity, implies that \( J \) splits into two parts, because the curve cuts it into two pieces. The singularity on \( J \) the same as a last point, respectively a first point on \( J \), depending on which part of \( J \) we are looking at.

The future and past light cones of this singularity define two horizons, which separate the three parts of spacetime from each other. The backward light cone is the future horizon of a black hole sitting in the lower region of spacetime, the forward light cone is the past horizon of a white hole in the upper region, and the region between the two horizons is a wormhole that connects the two exterior regions, which are otherwise completely disconnected. This is the typical structure of a timelike wormhole, except that usually the process is periodic in time. But this can again be achieved by using massive particle instead of massless ones. There will then be a whole series of wormholes connecting a sequence of exterior universes, each with a lifetime of \( \pi \) in the time coordinate \( t \).

All together, it seems that a Gott universe in anti-de-Sitter space is almost the same as a rotating BTZ black hole, and studying a point particle collapse might clarify some still open questions regarding the rotating black hole in three dimensions \([1,2]\). The arguments given here were based on the assumption that the results from Minkowski space can be straightforwardly generalized to anti-de-Sitter space. In particular, we had to assume that the region of spacetime containing closed timelike curves has the same structure. Actually, this is only justified in a neighbourhood of the particles, and if they pass each other at a small distance, as compared to the curvature radius of anti-de-Sitter space. So, a more detailed analysis is necessary to answer the question whether the relation is indeed that close.

## 5 The BTZ Black Hole

The purpose of this last section is to show that the black hole created after the collision of the particles is indeed the same as the BTZ black hole. To do this, we shall consider the same process in a different coordinate system. So far, we used a center of mass frame, in which the sum of the spatial momenta of the two infalling particles was zero. Now, we shall switch to a kind of rest frame of the black hole. This is not the same, again because it is not the sum of the momenta but the product of the holonomies that determines the motion of the black hole relative to the reference frame.
The maximally extended black hole

Before considering the infalling particles, let us briefly describe the maximally extended, matter free BTZ black hole in its rest frame \([1, 11]\). It is a solution to the vacuum Einstein equations with a negative cosmological constant, and it can be obtained by cutting and gluing, very similar to the spacetime containing a point particle in section \([2]\). The only difference is that the holonomy \(u\) has to be spacelike. In the previous section we saw that the holonomy is related to the mass \(\mu\) of the black hole by \((4.8)\), which states that \(u\) has to lie in a special conjugacy class of \(SL(2)\). A simple group element with this property is

\[ u = -e^{-\mu \gamma_1} = -\cosh \mu \mathbf{1} + \sinh \mu \gamma_1. \]  

(5.1)

The first question that arises is, are there any curves \(v_\pm\) inside the discs of constant \(t\), such that one of them is mapped onto the other by the isometry \(x \mapsto u^{-1}xu\)? Such curves do in fact exist. To find them, we can use the same symmetry argument once again. The holonomy \(u\) is an exponential of \(\gamma_1\), which means that it is invariant under vertical reflections \(\gamma_2 \mapsto -\gamma_2\), which is the same as \(\varphi \mapsto -\varphi\) in anti-de-Sitter space. Assuming the same for the curves \(v_+\) and \(v_-\), we make the ansatz that the point \((t, r, \varphi) \in v_+\) is the image of the point \((t, r, -\varphi) \in v_-\). Using the matrix representations \((1.12)\) for these points,

\[ v_\pm = 1 + \frac{r^2}{1 - r^2} \omega(t) + \frac{2r}{1 - r^2} \gamma(\pm \varphi), \]  

(5.2)

and evaluating the equation \(u v_+ = v_- u\), we arrive at the following coordinate relations defining \(v_+\) and \(v_-\),

\[ v_\pm : \frac{2r}{1 + r^2} \sin \varphi = \mp \sin t \tanh \mu. \]  

(5.3)

For \(-\pi \leq t \leq 0\), these curves are shown in figure \([7]\). They are again geodesics on the Poincaré disc, but their behaviour is very different from that of the faces \(w_\pm\) for the lightlike particles. The curves \(v_+\) and \(v_-\) do not intersect, except for \(t = z\pi, z \in \mathbb{Z}\), where they both coincide with the horizontal diameter of the disc. This is the location of the fixed points.

If we define the space manifold to be the region between the two curves, and identify the boundaries according to the isometry, then we obtain a solution to the vacuum Einstein equations with a negative cosmological constant, at least for times \(t\) which are not multiples of \(\pi\). At \(t = z\pi\), space degenerates to a line. As already mentioned in the end of the previous section, it turns out that spacetime can be extended beyond this singularity, but only if closed timelike curves are allowed. If we do not allow them, or consider them as singularities as well, then there is no way to continue spacetime beyond these lines. Let us therefore restrict to a single time interval between two singularities, say, the one with \(-\pi \leq t \leq 0\). There is then a future singularity at \(t = 0\), and a past singularity at \(t = -\pi\), very similar to those in the case of a black and a white hole created by massive particles.
The global structure of the singularities is however somewhat different. They are not located on half lines, emerging from the point where the particles collide, and extending from there to $\mathcal{J}$. Instead, both singularities lie on full spacelike geodesics with two end points on $\mathcal{J}$. As a consequence, there are two distinct last points on $\mathcal{J}$, the left and the right dot in figure 7(f), and also two first points on $\mathcal{J}$, the dots in figure 7(a). Moreover, if we look at the boundary of the disc more closely, we find that $\mathcal{J}$ itself splits into two disconnected parts, the left and the right one, each having its own first and last point. This is because the identification of the curves $v_+$ and $v_-$ is such that the right end of $v_+$ is glued to the right end of $v_-$, and the left end of $v_+$ to the left end of $v_-$. 

Hence, $\mathcal{J}$ consists of two parts, both forming a closed circle. We can also say that there are two spatial infinities. This situation is well known, for example, from the maximally extended Schwarzschild black hole. Indeed, the spacetime shown in figure 7 has the same causal structure. To see this, we have to consider the light cones emerging from the first and the last points on $\mathcal{J}$. They coincide pairwise, because the time distance between the first and the last point is $\pi$, which is exactly the time that a light ray needs to travel from one side of the disc to the other. The actual construction of the light cones is the same as in the previous section. The backward light cone of the last point on, say, the right part $\mathcal{J}$ is, at a time $t$, a geodesic centered at $\alpha = 0$, with radius $\beta = -t$. As we need this later on, let us write down a coordinate equation for this
light cone. It is obtained from (1.4), with $\alpha = 0$ and $\beta = -t$. This gives

$$\frac{2r}{1 + r^2} \cos \varphi = \cos t. \quad (5.4)$$

The backward light cone of the last point on the left part of $J$ is the reflection of this at the vertical axis, $\varphi \mapsto \pi - \varphi$. Both horizons are shown as dashed lines in figure 7. All together, spacetime splits into four regions, and has the same global causal structure as the maximally extended Schwarzschild black hole. We have two external regions, to the left and to the right of both horizons, which are causally completely disconnected. The region between the two horizons for $t < -\pi/2$ is the interior of a white hole, from where signals can be sent to both parts of $J$, and the region between the horizons for $t > -\pi/2$ is the interior of a black hole, from where no signal can be sent to either part of $J$.

The length of the horizon can be computed, most conveniently, in the moment when the two horizons coincide in the middle of the disc, which is the situation shown in figure 7(d). The radial coordinate $\rho$ of the points where the horizon intersects with the curves $v_{\pm}$ is $\rho = \tanh(\mu/2)$, which follows from (5.3) with $\varphi = \pm \pi/2$ and $t = -\pi/2$. The physical distance between these points is

$$\int_{-\rho}^{\rho} \frac{2}{1 + r^2} \, dr = 4 \arctanh \rho = 2\mu. \quad (5.5)$$

So, we recover the relation between the horizon length and the mass $\mu$. A very different way to calculate the length of the horizon is the following. Consider any two points in anti-de-Sitter space, and the corresponding matrix representations $x, y \in SL(2)$. If there is a geodesic joining the two points, and only then the distance is defined, then there is also a geodesic on the group manifold. Assume that this geodesic is spacelike. There is then a unit spacelike vector $n \in sl(2)$ such that $y = x e^{s n}$ for some positive $s \in \mathbb{R}$. This $s$ is the proper length of the geodesic joining $x$ and $y$. It is related to the matrices $x$ and $y$ by

$$\frac{1}{2} \text{Tr}(x^{-1} y) = \cosh s. \quad (5.6)$$

For a timelike distance, we have the same formula with $\cosh$ replaced by $\cos$. Now, consider two corresponding points $(t, r, \varphi) \in v_+$ and $(t, r, -\varphi) \in v_-$, and their matrix representations (5.2). Using the formulas (1.13), we find that

$$\frac{1}{2} \text{Tr}(v_{-1} v_+) = 1 + 2 \sin^2 \varphi \left( \frac{2r}{1 - r^2} \right)^2. \quad (5.7)$$

This formula can be used to compute the horizon length at any time $t$. What we need to do is to insert the coordinates of the intersection of the curves (5.3) with the horizon (5.4). The angular coordinate can be found by dividing the two equations and using some trigonometric identities, which gives

$$\tan \varphi = \tan t \tanh \mu \quad \Rightarrow \quad 2 \sin^2 \varphi = \frac{(1 - \cos(2t))}{\cosh(2\mu) + \cos(2t)}. \quad (5.8)$$
For the radial coordinate, we take the sum of the squares of the two equations, and what we get is

\[
\left(\frac{2r}{1+r^2}\right)^2 = \sin^2 t \tanh \mu + \cos^2 t \quad \Rightarrow \quad \left(\frac{2r}{1+r^2}\right)^2 = \frac{\cosh(2\mu) + \cos(2t)}{1 - \cos(2t)}. \tag{5.9}
\]

If we insert this into (5.7), the time dependence drops out, and what remains is \(\cosh(2\mu)\). This implies that the length of the horizon is indeed independent of time, and equal to \(2\mu\).

### Infalling particles

Let us now put in the particles. We can do this once again by cutting and gluing, but this time we start from the BTZ black hole instead of empty anti-de-Sitter space. We should expect that the particles somehow cut away the second exterior region and also the white hole, because these regions of spacetime did not occur in the previous section. If the particles are falling in from, say, the right exterior region, then it must be this region that remains. As an ansatz, let us assume, again motivated by the symmetry of the picture, that one of the particles, say, number two, is moving on the horizontal axis from the right to the left, entering from \(J^+\) at \(t = -\pi/2\).

Its holonomy is then given by

\[
u_2 = 1 + \tan \epsilon_2 (\gamma_0 - \gamma_1). \tag{5.10}\]

Note that this is the same as the holonomy of the second particle in (3.1), so we can later apply some of the results from section 3. The parameter \(\epsilon_2\) is determined by the following condition. The holonomy \(u_1\) of the other particle must be lightlike as well, and the product of the two holonomies must be \(u\). This gives a relation between \(\epsilon_2\) and \(\mu\), namely

\[
\frac{1}{2} \text{Tr}(u_1) = \frac{1}{2} \text{Tr}(u u_2^{-1}) = \sinh \mu \tan \epsilon_2 + \cosh \mu = 1 \quad \Rightarrow \quad \tan \epsilon_2 = \coth(\mu/2). \tag{5.11}\]

There are then two possible choices for \(u_1\), depending again on the order in which the product is taken. If we make the ansatz

\[
u_1^\pm = 1 + \tan \epsilon_1 (\gamma_0 - \gamma(\pm \theta)) \tag{5.12}\]

and then solve the equations

\[
u_1 u_2 = u, \quad u_2 u_1^\pm = u, \tag{5.13}\]

we find that the parameters \(\theta\) and \(\epsilon_1\), specifying the direction and the energy of particle one, have to be chosen such that

\[
\sin \theta = \tanh \mu, \quad \tan \epsilon_1 = \cosh \mu \coth(\mu/2). \tag{5.14}\]

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But what does it mean that there are two possible choices for the holonomy $u_1$? The answer is actually quite simple. Consider the corresponding world line in the BTZ black hole spacetime, for $-\pi/2 \leq t \leq 0$. It is the light ray defined by $r = -\tan(t/2)$ and $\varphi = \pm \theta$, depending on which sign we choose in (5.12). As can be easily checked, these world lines lie entirely inside the surfaces $v_{\pm}$, as defined in (5.3). They represent the same world line, because the two surfaces are identified.

The resulting positions of the two particles as a function of time are shown in figure 8(b–e). Particle one is always sitting at the intersection of the curves $v_{\pm}$ with the diameter of the disc pointing into the angular direction $\pm \theta$. Particle two is moving on the horizontal diameter. Both are at the same distance $r = -\tan(t/2)$ from the center. They both enter from $\mathcal{J}$ at $t = -\pi/2$, and they collide in the center of the disc at $t = 0$. To find the wedges that the particles are cutting out from space, let us first consider particle two, which is the same as the one already considered in section 3. There, we found that the wedge is bounded by the curves (3.8),

$$w_{\pm} : \frac{2r}{1 + r^2} \sin(\epsilon_2 \pm \varphi) = -\sin t \sin \epsilon_2. \quad (5.15)$$

Now, everything fits together very nicely. If we insert $r = -\tan(t/2)$ and $\varphi = \pm \theta$ into these equation, we find that the world line of particle one also lies inside the surfaces $w_{\pm}$. Hence, particle one always sits exactly at the intersection of the curves $w_{\pm}$ and $v_{\pm}$, and $w_{\pm}$ is the geodesic that joins the two particles. If we cut out the wedge which is bounded by $w_+$ and $w_-$, the space manifold becomes the shaded region shown in figure 8. Note that it is now the wedge in front of the particle that is cut out, and not the one behind. Furthermore, it is not necessary to cut out a second wedge emerging from the other particle. This is already done by cutting and gluing along the curves $v_{\pm}$.

As all the identifications are provided by isometries of anti-de-Sitter space, namely by the one represented by $u_2$ along $w_{\pm}$, and by that represented by $u$ along $v_{\pm}$, it follows that the resulting spacetime has a constant negative curvature everywhere, expect on the world lines. One can also check that the holonomies of the particles are the correct ones. For particle two this is obvious, because in its neighbourhood spacetime looks exactly like the one considered previously in figure 3 or 5. To transport a particle around particle one, we have to pass it once from $v_-$ to $v_+$, which gives a Lorentz transformation $u$, and then from $w_+$ to $w_-$, which contributes with a factor $u_2^{-1}$ to the holonomy. The order in which the two factors have to be multiplied depends on whether we start in the neighbourhood of the upper or the lower representation of particle one on figure 8. In any case, we find one of the holonomies $u_{1+}$ or $u_{1-}$, obeying (5.13).

What is then still missing is the continuation of spacetime to the past, for $t < -\pi/2$. At the moment when the particles enter, the space manifold is the shaded region of figure 8(b). This has the same shape as the one in figure 3(d), which remained after a single massless particle has passed through. Here, we have the reverse situation. It is the space manifold before the particles enters. The fact that there are actually two particles entering at the same time does not
make any difference, at least not at this moment, because both particles are still outside the disc. We can continue to the past in the same way as we continued to the future in figure 3. We let the curves $w_\pm$ evolve further, according to (5.15), and define the space manifold to consist of the two partly overlapping shaded regions of figure 8(a), glued together along their boundaries. From section 2 we know that this provides a skew foliation of anti-de-Sitter space.

All together, we have the same scenario as in the previous section, except that the coordinates are slightly different. For $t < -\pi/2$, we have an empty anti-de-Sitter space. At $t = -\pi/2$, two particles enter from $J$. Shortly after that, in figure 8(c), there is still no horizon, because the BTZ horizon lies still outside the shaded region. In figure 8(d), one particle has fallen behind the horizon, but the other particle is still outside. The time $\tau_1$ when the horizon emerges at the position of the first particle can be found by setting $r = -\tan(t/2)$ and $\varphi = \theta$ in (5.4). The result is that $\tan \tau_1 = -\cosh \mu$. After that, the horizon has the shape of a closed loop with a cusp, pointing towards the second particle. It is growing until the second particle also falls behind it at $\tau_2 = -\pi/4$, and thereafter the exterior region shown in figure 8(e) looks exactly like one of the exterior regions of the matter free BTZ black hole in figure 7.

Finally, the particles collide at $t = 0$, inside the black hole, creating the future singularity, which is shown in figure 8(f). It extends from the point of collision in the center of the disc to $\mathcal{J}$. This is what remains from the space manifold. In contrast to the singularity in figure 7(f), it is a half line only. It is the same spacelike geodesic as the one on which the joint particle moves.
in figure 3(d–f). The only difference is that now, in the rest frame of the black hole, it entirely lies inside the disc of constant time at \( t = 0 \). Therefore, we do not see the tachyonic particle moving. Its world line is just there at one moment of time. Nevertheless, the end point of the line on \( J \) is still the last point on \( J \), and the horizon is the backward light cone of this point.

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