Sznajd model with synchronous updating on complex networks

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We analyze the evolution of Sznajd Model with synchronous updating in several complex networks. Similar to the model on square lattice, we have found a transition between the state with no-consensus and the state with complete consensus in several complex networks. Furthermore, by adjusting the network parameters, we find that a large clustering coefficient favors development of a consensus. In particular, in the limit of large system size with the initial concentration \( p = 0.5 \) of opinion +1, a consensus seems to be never reached for the Watts-Strogatz small-world network, when we fix the connectivity \( k \) and the rewiring probability \( p_w \); nor for the scale-free network, when we fix the minimum node degree \( m \) and the triad formation step probability \( p_t \).

Keywords: opinion dynamics, Sznajd model, small-world networks, synchronous updating, computer simulation.

1. Introduction

Recently, Sznajd-Weron proposed a consensus model1,2 (it’s now called Sznajd model), which is a successful Ising spin model describing a simple mechanism of making up decisions in a closed community: A pair of nearest neighbors convinces its neighbors of the pair opinion if and only if both members of the pair have the same opinion; otherwise the pair and its neighbors do not change opinion. The consensus model of Sznajd has rapidly acquired importance in the new field of computational socio-physics3,4.

In Sznajd model there are two ways of updating the system states: Asynchronous and Synchronous updating. While the asynchronous updating has been already analyzed considering the Sznajd model on an one-dimensional lattice, on a square lattice5, on a triangular lattice5, the dilute6, on a three-dimensional cubic lattice7 and even on some networks8,9,10, the synchronous updating has been only studied on a square lattice11,12,13,14.

Additionally, it is meaningful that consensus models are set up in complex
networks. However, the statistical properties of real-world social networks vary strongly, for example, the degree distribution can be single-scale, broad-scale or scale-free\textsuperscript{15,16}. Due to the lack of a single model encompassing the topological features of social networks, we consider a few established network models aiming to unveil the effect to different aspects of the topology: a small-world network (i.e., Watts-Strogatz model\textsuperscript{17}) is generated by rewiring with a probability the links of a regular lattice by long-distance random links\textsuperscript{17}; scale-free networks (i.e., Barabási-Albert model\textsuperscript{18}) are characterized by a fat-tailed (power-law) degree distribution and usually modelled by growing networks considering a preferential attachment of links; by adding a triad formation step on the Barabási-Albert prescription, a scale-free model with tunable clustering can be obtained (we call it the triad scale-free model)\textsuperscript{19}.

Therefore, the aim of our paper is to discuss the original Sznajd model\textsuperscript{11} with synchronous updating on different complex networks: Watts-Strogatz, Barabási-Albert and triad networks.

2. The Model

On the system lattice, each site \(i (i = 1, 2, \ldots, N)\), where \(N\) is the total number of sites) carries a spin \(s_i\), which has two possible directions: \(s_i = +1\) or \(s_i = -1\). It can be considered like an individual that can take one of two possible opinions: \(s_i = +1\) represents a positive opinion and \(s_i = -1\), a negative one. Initially the opinions are distributed randomly, \(+1\) with probability \(p\) and \(-1\) with probability \(1-p\).

The synchronous updating: the system state at time step \(t+1\) is decided by its state at time step \(t\). At every time step \(t > 0\), we go systemically through the lattice to find the first member of a pair, then the second member of the pair is randomly selected from the neighbors of the first one. In this way, one time step means that on average every lattice node is selected once as the first member of the pair. The pair persuades all its neighbors to assume its (pair) actual opinion at the next time-step \(t+1\), if and only if at the time-step \(t\) the pair shares the same opinion; otherwise, the neighbor opinions are not affected. In fact, after going through the whole lattice once, the time-step \(t\) is completed and in the beginning of the next time-step \((t+1)\), the state of each site is updated according to all results of persuasion.

It is important to emphasize that in the synchronous updating, some sites may feel frustrated and cannot decide the opinion at the next time-step. This phenomenon is called as \textbf{frustration}. There are two reasons for occurring frustration: (1) When at the same time-step \(t\), an individual is persuaded by different pairs to follow different opinions; (2) If an individual \(i\) is selected as member of a pair which persuades the others to follow its opinion, it intends to keep its opinion unchanged at the next time-step \(t+1\) (\(s_{i,t+1} = s_{i,t}\)). However, if at the same time-step \(t\), the individual \(i\) is persuaded by other pairs to assume an opinion \(s\) different from its
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actual one \( s_{i,t} \), then at the next time-step \( t + 1 \) the individual \( i \) is considered to be an frustrated one.

In our paper, frustrated sites do nothing, i.e., they stay at the time-step \( t + 1 \) with the opinion of previous time-step \( t \) \( (s_{i,t+1} = s_{i,t}) \).

On the square lattice, \( L \times L \), the Sznajd model with synchronous updating shows frustration hindering the development of a consensus\(^{11} \). The initial probability \( p = 0.5 \) of opinion +1 makes a complete consensus much more difficult than \( p \neq 0.5 \). When different initial concentrations \( p \) of opinion +1 are considered, the system shows a transition between the non-consensus state and the full consensus state\(^{12} \). Moreover, there is a power-law relation \( p \propto L^{-0.38} \) between the necessary value of \( p \) to get a full consensus in half the cases and the system size \( L \). In the following section, we discuss the evolution of Sznajd model with synchronous updating in the complex networks.

3. Simulations of Sznajd model on Watts-Strogatz small-world network

![Fig. 1. Dependence between \( C \) and \( p_s \) for the WS small-world network.](image)

3.1. Watts-Strogatz small-world network

Social networks are far from being regular or completely random. In the past few years, it has been found that most real-life networks have some common characteristics, the most important of which are called small-world effect and scale-free distribution. The recognition of small-world effect involves two factors: the clustering coefficient (\( C \)) and the average shortest path length\(^{17,20} \); a network is called a small-world network as long as it has small average shortest path length and great clustering coefficient. One of the most well-known small-world models is Watts-Strogatz small-world network (WS model), which can be constructed by the following algorithm: the initial network is a one-dimensional lattice of \( N \) sites, with periodic boundary conditions (i.e., a ring), each site being connected to \( k \) nearest
neighbors. We choose a vertex and the edge that connects it to its nearest neighbor in a clockwise sense. With probability $p_s$, we reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place. This process repeats until one lap is completed and proceeding outward to more distant neighbors after each lap, until each edge in the original lattice has been considered once. As it is shown in Figure 1, we gain the same dependence between $C \sim p_s$ as obtained in Ref. 17, as well as we can see that when $N > 500$ and $k > 8$ the clustering coefficient $(C)$ does not vary very strongly.

![Fig. 2. Dependence between probability without consensus and $p$ for the WS small-world network](image)

### 3.2. The probability of non-consensus as various $p$

Every node in the network is considered to be an individual with an opinion that in the beginning of the simulation is randomly chosen with probability $p$ for opinion $+1$. Once the network has been completely constructed, we start the consensus process of Sznajd. With up to 1000 samples ($N \geq 17000$, 400 samples), similar to the square lattice case, frustration hinders the development of a consensus, which is not found even after 40000 time-steps. Figure 2 shows the relative number of samples which did not reach a consensus as a function of the probability $p$ for initial opinion $+1$. The problem is by definition symmetric about $p = 0.5$, and only $p \leq 0.5$ is thus plotted in our figures. As it can be seen, we have also observed a transition between the non-consensus state and the state with complete consensus as function as $p$. When $p = 0.5$, it is most difficult to find consensus in the system.

We have also measured the system size $N_c$ from which we never find a consensus in any of the samples, i.e, for $N > N_c$ never a full consensus can be found in any of the samples. Figure 3 shows how this critical system size $N_c$ varies with the rewiring probability $p_s$ and with the node degree $k$. With the rewiring probability $p_s$ increasing at the same $k$, the clustering coefficient $C$ decreases (see Figure 1) and $N_c$ increases. Furthermore, there is a power increase relation between $N_c \sim k$, and
the bigger $p_s$ is, the faster $N_c$ increases (see Figure 3b). This behavior indicates that a large clustering coefficient favors the development of a consensus.

### 3.3. The necessary value of $p$ to get a consensus in half the cases

When $N > N_c$, Figure 4 shows how the $p$ needed to get a consensus in half the cases varies for different lattice sizes $N$ and various $p_s$. Since the clustering coefficient $C$ decreases as $p_s$ increases (see Figure 1), for $p_s < 0.5$ and equal-size system $N$, $p$ increases and the slope becomes small. However, if $p_s > 0.5$, the tendency of $C$ decreasing slows down (see Figure 1) and according to Figure 4, the two lines when $p_s = 0.5$ and $p_s = 1$ are so close.

![Diagram](image-url)
4. Simulations of Sznajd model on triad scale-free network

4.1. Triad scale-free network

The small-world networks generated by rewiring links have degree (number of edges) distributions with exponential tails. In contrast, scale-free networks are characterized by a fat-tailed (power-law) degree distribution. The most fashionable network presenting both properties, scale-free and small-world aspects, is the Barabási-Albert scale-free network (BA network). Although the BA network has successfully explained the scale-free nature of many networks, a striking discrepancy between it and real networks is that the value of the clustering coefficient varies fast with the network size $N$ and for large systems is typically several orders of magnitude lower than found empirically (it vanishes in the thermodynamic limit $^{19,21}$). In social networks, for instance, the clustering coefficient distribution $C(k)$ exhibits a power-law behavior, $C(k) \propto k^{-\gamma}$, where $k$ is the node degree (number of neighbors) and $\gamma \approx 1$ (everyone in the network knows each other).

![Graph showing the relation between clustering coefficient $C$ and $N$, $p_t$, $m$ for the triad scale-free network.](image)

However, this problem has been surmounted and scale-free models with high clustering coefficient have been investigated $^{19}$, by adding a triad formation step on the Barabási-Albert prescription. The Barabási-Albert network starts with a small number $m$ of sites all connected with each other. Then a large number $N$ of additional sites is added as follows: First, each new node (node $i$) performs a preferential attachment step, i.e., it is attached randomly to one of the existing nodes (node $j$) with a probability proportional to its degree; then follows a triad formation step with a probability $p_t$: the new node $i$ selects at random a node in the neighborhood of the one linked to in the previous preferential attachment step (node $j$). If all neighbors of $j$ are already connected to $i$, then a preferential attachment step is performed (“friends of friends get friends”). In this model, the original Barabási-Albert network corresponds to the case of $p_t = 0$. It is expected
that a nonzero $p_t$ gives a finite nonzero clustering coefficient as $N$ is increased \cite{19,21}, while the clustering coefficient goes to zero when $p_t = 0$ (the BA scale-free network model), as shown in Figure 5. Indeed, the clustering coefficient increases as the probability $p_t$ and $m$ increase.

![Fig. 6. Dependence between probability without consensus and $p_t$ for the triad scale-free network.](image)

**4.2. The probability of non-consensus as various $p$**

Similar to the WS model, with up to 1000 samples ($N \geq 17000$, 400 samples), a consensus is not found even after 40000 time steps. Figure 6 gives the dependence between the probability of non-consensus and $p_t$. We found a similar transition between the state with no-consensus and the state with complete consensus when various $p_t$ for the BA network (Fig. 6a) and for the triad scale-free network (Fig. 6b).

![Fig. 7. Relation between $N_c$ and $p_t$, $m$ for the triad scale-free network when $p = 0.5$.](image)
In Figure 7, we show the critical system size $N_c$ as a function of various values $p_t$, $m$ and $p = 0.5$. As $p_t$ increases and $m$ decreases, the clustering coefficient $C$ decreases (see Figure 5), thus $N_c$ increases. Similar to the WS network, this behavior indicates that a large clustering coefficient favors development of a consensus. Different from WS network, where $N_c \sim p_t$, now $N_c$ follows a power-law $N_c \propto m^{2.4}$, for different values of $p_t$.

4.3. The necessary value of $p$ to get a consensus in half the cases

When $N > N_c$, Figure 8 shows how the $p$ needed to get a consensus in half the cases varies for different lattice size $N$ and various $p_t$. As $p_t$ decreases, the clustering coefficient $C$ also decreases (see Figure 5), thus for an equal-size system $N$, $p$ increases and the slope becomes smaller (Fig. 8a). Indeed, for a fixed value of $p_t$, $p$ decreases as $N$ increases. Since as $m$ increases, the clustering coefficient decreases (see Fig. 5b), to compare systems with the same clustering coefficient $C = 0.16$ and various $m$, the probability $p_t$ must be also changed according to the straight line in Fig. 5b ($p_t$ increases). For an equal-size system $N$, as $m$ increases $p$ increases and the slope becomes bigger (Fig. 8b). As well as for a fixed $m$, $p$ decreases as the system size $N$ increases.

![Fig. 8. Power law relation between the $p$ needed to get a consensus in half the cases and $N$ for the triad scale-free network.](image)

5. Conclusion

Comparing the results of the Sznajd model on a square lattice with synchronous updating with our results when the Sznajd model is considered on more realistic topologies: Watts-Strogatz small-world network, Barabási-Albert scale-free network and triad scale-free network, we notice the following similar properties: (1) A transition between the state with no-consensus and the state with complete consensus; (2) a power law relation between the initial probability $p$ needed to get a consensus
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in half the cases and the system size $N$. However, it is very interesting to consider the change of these behaviors as we adjust the parameters of the networks: (1) as $C$ decreases, the exponent $\alpha$ of the power-law $p \sim N^{-\alpha}$ decreases; (2) as $C$ decreases, the critical system size $N_c$ increases, which indicates that a large clustering coefficient favors development of a consensus, especially for the triad scale-free network there is a power-law relation: $N_c \sim m^{2.4}$. Moreover, in the limit of very large networks with $p = 0.5$, a consensus seems to be never reached for the WS small-world network when we fix $k$ and $p_s$; nor for the scale-free network, when we fix $m$ and $p_t$.

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