Linkage on arithmetically Cohen-Macaulay schemes with application to the classification of curves of maximal genus.

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1 Introduction

In algebraic geometry and commutative algebra the notion of linkage by a complete intersection, which we will here call classical linkage, has been for a long time an interesting and active topic. In this note we provide a generalization of classical linkage in a different context. Namely we will look at residuals in the scheme theoretic intersections of arithmetically Cohen-Macaulay schemes (briefly aCM schemes) of dimension $r$ (resp. $r + 1$) with $r$ hypersurfaces of degree $a_1, \ldots, a_r$ (a c.i. of type $(a_1, \ldots, a_r)$ on the aCM scheme, see Def. 2.15). When the aCM scheme is singular a c.i. on it may not be Gorenstein, i.e. its dualizing sheaf may not be invertible. If this is the case, classical linkage, even if suitably generalized, does not apply.

The main purpose of this article is to prove some results related to the invariance of the deficiency module under such linkage. In the last part of the paper we show how to apply these results and techniques to the classification of curves $C$ in $\mathbb{P}^n$ of degree $d$ and maximal genus $G(d, n, s)$ among those not contained in surfaces of degree less than a certain fixed one $s$. This was the original motivation of this work. A complete classification theorem has been given for $n = 3$ by L. Gruson and C. Peskine in [GP], for $n = 4$ by L. Chiantini and C. Ciliberto in [CC] and for $n = 5$ by the author in [F3]. For $n = 3$ and $n = 4$ the respective classification Theorems have been proven with techniques of classical linkage but for $n \geq 5$ this is no longer possible. For $n \geq 5$ and $s \geq 2n - 1$ the classification procedure consists in the precise description of the linked curve to $C$ by a certain c.i. on a rational normal 3-fold $X$. In Example 4.8 we describe this linked curve in the easiest case, i.e. when it is a plane curve. In Example 4.10 we construct examples of smooth curves of maximal genus $G(d, n, s)$ for every $d$ and $s$ in the range of Example 4.8.
Turning to a detailed presentation of the results, our first one is the following (see Cor. 2.11):

**Theorem** Let $Y_1$, $Y_2$, $Y$ be projective locally Cohen-Macaulay schemes. If $Y_1$, $Y_2$ are geometrically linked by $Y$, or if $Y_1$, $Y_2$ are algebraically linked by $Y$ and $Y$ is Gorenstein, then:

$$
\mathcal{I}_{Y_2/Y} \otimes \omega_Y \cong \omega_{Y_1}
$$

$$
\mathcal{I}_{Y_1/Y} \otimes \omega_Y \cong \omega_{Y_2}.
$$

Let now $W \subset \mathbb{P}^{n-1}$ and $X \subset \mathbb{P}^n$ be aCM schemes of dimension $r$ and $r + 1$ respectively; throughout the article $W$ will be often a general hyperplane section of $X$. Let $Z_1$ and $Z_2$ (resp. $Y_1$ and $Y_2$) be the two linked schemes by a c.i. of type $(a_1, \ldots, a_r)$ on $W$ (resp. $X$). Our two main results are the following isomorphisms of cohomology groups (see Prop. 3.1, Th. 3.3, Prop. 3.6 and Th. 3.11):

**Theorem**

$$
H^0(\mathcal{I}_{Z_2/W} \otimes \omega_W(i + ch_W)) \cong H^1(\mathcal{I}_{Z_1/W} \otimes \omega_W(c - ch_W - i))^\vee
$$

for $i < \min_j \{a_j\}$, and

$$
H^1(\mathcal{I}_{Y_2/X} \otimes \omega_X((i + ch_X)) \cong H^1(\mathcal{I}_{Y_1/X} \otimes \omega_X((c - ch_X - i))^\vee
$$

for every $i$.

Here $c = a_1 + \cdots + a_r$, $\omega_W$ (resp. $\omega_X$) is the dualizing sheaf of $W$ (resp. of $X$) and $ch_W$ (resp. $ch_X$) is the smaller integer $k$ such that $\omega_W(k)$ (resp. $\omega_W(k)$) has sections (see Def. 2.7 of canonical characteristic).

The first isomorphism above allows us to compute $h^0(\mathcal{I}_{Z_2/W} \otimes \omega_W((i + ch_W))$ for low values of $i$ in terms of the Hilbert function $h_{Z_1}(c - ch_W - i)$ of the residual scheme $Z_1$. If $Y_1$ is arithmetically Cohen Macaulay, the second isomorphism implies that $H^1(\mathcal{I}_{Z_2/X} \otimes \omega_X((i + ch_X)) = 0$ for every $i$, and therefore the restriction map $H^0(\mathcal{I}_{Z_2/X} \otimes \omega_X((i + ch_X)) \to H^0(\mathcal{I}_{Z_2/W} \otimes \omega_W((i + ch_W))$ is surjective for every $i$ (see Cor. 3.10 and Cor. 3.12). This means we can lift curves on $W$ linearly equivalent to $(i + ch_W)H + K_W$ and passing through a general hyperplane section $Z_2$ of $Y_2$ to surfaces on $X$ linearly equivalent to $(i + ch_X)H + K_X$ passing through $Y_2$. Here $H$ denotes the divisor of a hyperplane section and $K_W$ (resp. $K_X$) is the canonical divisor of $W$ (resp. of $X$).

The technique used to prove the above results allows us to prove also a formula (see Proposition 3.14) which relates the arithmetic genera of the curves $Y_1$ and $Y_2$, linked by a c.i. $Y$ on the aCM scheme $X$, in the case that $Y$ has no components contained in the locus where $\omega_X$ is not invertible (the jump locus of $X$, see Def. 2.4):

**Proposition**

$$
p_a(Y_2) = p_a(Y_1) - p_a(Y) + \deg K_{Y|Y_2} + 1.
$$
In a first draft of this paper the above results were proved in a -somewhat weaker version- for linkage by c.i. on rational normal surfaces and 3-folds. In the smooth case the results were proven using a straightforward generalization of classical linkage (in particular of Prop. 2.5 of [PS]). The author warmly thanks the referee who suggests the actual proofs. The key point is that Theorem 1 (see Cor. 2.11) holds also in the singular case, this permits us to easily generalize Prop. 3.1 and Prop. 3.6 to the singular case (Th. 3.5 and Th. 3.9).

The linkage results presented in this paper, limited to the case of c.i. on rational normal 3-folds in \( \mathbb{P}^5 \) and on rational normal surfaces in \( \mathbb{P}^4 \), and the classification for curves of maximal genus \( G(d, n, s) \) in case \( n = 5 \) appeared as part of my doctoral dissertation [F1]. The author thanks her advisor Ciro Ciliberto.

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2 Preliminaries

In this section, we collect the definitions and notation to be used in this paper, and state some of the basic results of linkage theory. We introduce the definitions of geometric and algebraic linkage by a projective scheme \( Y \), without supposing \( Y \) to be a complete intersection. Moreover we will briefly introduce rational normal scrolls, in particular what we need about Weil divisors on them, including linkage.

**Definition 2.1** Let \( Y_1, Y_2, Y \) be subschemes of a projective space \( \mathbb{P} \), then \( Y_1 \) and \( Y_2 \) are geometrically linked by \( Y \) if:

1. \( Y_1 \) and \( Y_2 \) are equidimensional, have no embedded components and have no common components
2. \( Y_1 \cup Y_2 = Y \), scheme theoretically.

The following Proposition is essentially Prop. 1.1 of [PS].

**Proposition 2.2** Let \( Y_1 \) and \( Y_2 \) be closed subschemes of \( \mathbb{P} \) geometrically linked by \( Y \), then:

\[
\mathcal{I}_{Y_1/Y} \cong \mathcal{H}om_Y(\mathcal{O}_{Y_2}, \mathcal{O}_Y) \\
\mathcal{I}_{Y_2/Y} \cong \mathcal{H}om_Y(\mathcal{O}_{Y_1}, \mathcal{O}_Y).
\]

**Proof.** By [PS] Prop. 1.1 we have that \( \mathcal{I}_{Y_1/Y} \cong \mathcal{H}om_Y(\mathcal{O}_{Y_2}, \mathcal{O}_Y) \) and \( \mathcal{I}_{Y_2/Y} \cong \mathcal{H}om_Y(\mathcal{O}_{Y_1}, \mathcal{O}_Y) \). Since \( Y_1 \) and \( Y_2 \) are both subschemes of \( Y \subset \mathbb{P} \) these isomorphisms can be rewritten as in the statement.

**Definition 2.3** Let \( Y_1, Y_2 \) be projective schemes, then \( Y_1 \) and \( Y_2 \) are algebraically linked by a projective scheme \( Y \) containing them, if:
1. $Y_1$ and $Y_2$ are equidimensional and have no embedded components

2.

$$\mathcal{I}_{Y_1/Y} \cong \text{Hom}_Y(\mathcal{O}_{Y_2}, \mathcal{O}_Y)$$

$$\mathcal{I}_{Y_2/Y} \cong \text{Hom}_Y(\mathcal{O}_{Y_1}, \mathcal{O}_Y).$$

**Remarks 2.4** If $Y_1$ and $Y_2$ are geometrically linked by $Y$, then by Prop. 2.2 they are also algebraically linked. Moreover if $Y_1$ and $Y_2$ are algebraically linked by $Y$ and have no common components, then they are geometrically linked. See [M] Prop. 5.2.2 (c).

**Definition 2.5** Let $Y$ be a projective scheme. The dualizing sheaf of $Y$ is

$$\omega_Y := \mathcal{E}xt^c_P(\mathcal{O}_Y, \omega_P)$$

where $Y \hookrightarrow \mathbb{P}$ is an embedding of $Y$ in some projective space $\mathbb{P}$ and $c = \text{codim}(Y, \mathbb{P})$.

**Definition 2.6** Let $Y$ be a projective locally CM scheme and let $\mathcal{F}$ be a coherent sheaf in $Y$ of dimension $> 0$. We define the jump locus of $\mathcal{F}$ the closed subscheme $\text{Jump}(\mathcal{F})$ of $Y$ where $\mathcal{F}$ is not locally free.

**Definition 2.7** Let $Y$ be a projective scheme, we define the canonical characteristic of $Y$ the smallest integer $\text{ch}_Y$ such that $h^0(\omega_Y(\text{ch}_Y)) > 0$.

For a proof of the following Theorem the reader may consult [F] Th. 21.15 or for more details [H] Cor. 1.2.3.

**Theorem 2.8** Let $Y$ and $X$ be two equidimensional projective locally Cohen-Macaulay schemes. Suppose $Y \subset X$ and let $c'$ be $\text{codim}(Y, X)$. Let $\mathcal{F}$ be a sheaf in $\text{Mod}(Y)$. Then, for every $j \geq 0$:

$$\mathcal{E}xt^j_Y(\mathcal{F}, \omega_Y) \cong \mathcal{E}xt^{c'+j}_X(\mathcal{F}, \omega_X).$$

In particular:

$$\omega_Y \cong \mathcal{E}xt^{c'}_X(\mathcal{O}_Y, \omega_X).$$

As a corollary of the previous Theorem we show that, supposing $X$ normal, $\omega_X$ is the divisorial sheaf associated to the canonical divisor $K_X$ of $X$. For this purpose we briefly recall the notion of divisorial sheaves on a normal scheme $X$, for details and for a more general point of view the reader may consult the paper of Hartshorne on generalized divisors, [H], §2. On a normal scheme, generalized divisors and Weil divisors are the same (see [H], Prop. 2.7).

**Definition 2.9** Let $X$ be a normal scheme. Let $D$ be a Weil divisor on $X$. If $K(X)$ denotes the function field of $X$, then the sheaf $\mathcal{O}_X(D)$ defined for every open set $U \subset X$ as

$$\Gamma(U, \mathcal{O}_X(D)) = \{ f \in K(X) \mid \text{div} f + D \geq 0 \text{ on } U \}.$$
It is known (see [H1] Prop. 2.7 and Prop 2.8) that the group $\text{Div}(X)$ of divisorial sheaves on $X$ is naturally isomorphic to the group $\text{Cl}(X)$ of Weil divisors modulo linear equivalence. Moreover there is an equivalence between reflexive sheaves of rank one and divisorial sheaves.

The following result is known (see e.g. [KoMo], Prop. 5.75). However for sake of self-containedness we give a direct proof which uses the ideas underlying the present work.

**Corollary 2.10** If $X$ is an irreducible normal locally Cohen-Macaulay projective scheme of positive dimension $r$, then the dualizing sheaf $\omega_X$ is a twist of the ideal sheaf of a divisor in $X$. In particular $\omega_X$ is the divisorial sheaf $\mathcal{O}_X(K_X)$, associated to the canonical divisor $K_X$.

**Proof.** We can always find a complete intersection of dimension $r$ and of certain multi-degree containing $X$. Let $A$ be a generic such complete intersection and let $\mathcal{O}_A(f)$ be the dualizing sheaf of $A$. Let $B$ be the residual scheme to $X$ by $A$; by [H1] Prop. 4.1. we see that $X$ and $B$ are algebraically linked by $A$. Therefore by Theorem 2.8 we have that:

$$\omega_X \cong \text{Hom}_A(\mathcal{O}_X, \omega_A) \cong \text{Hom}_A(\mathcal{O}_X, \mathcal{O}_A)(f) \cong \mathcal{I}_{B/A}(f).$$

Let $X_B$ be the scheme theoretic intersection $X \cap B$. Since $\mathcal{I}_{B/A}(f) \cong \omega_X$ is supported on $X$, tensoring by $\mathcal{O}_X$ do not affect the inclusion $\mathcal{I}_{B/A}(f) \hookrightarrow \mathcal{O}_A(f)$. Therefore the exact sequence $0 \rightarrow \mathcal{I}_{B/A}(f) \rightarrow \mathcal{O}_A(f) \rightarrow \mathcal{O}_B(f)$ tensorized by $\mathcal{O}_X$ stays exact, and we find that $\omega_X$ is the ideal sheaf of $X_B$ twisted by $\mathcal{O}_X(f)$. Hartshorne’s Connectedness Theorem ([E] Th. 18.12) implies that $X_B$ is pure of codimension 1 in $X$, i.e. $X_B$ is a Weil divisor of $X$. Therefore $\omega_X$ is the divisorial sheaf $\mathcal{O}_X(fH - X_B)$ (where $H$ is a hyperplane section). Since divisorial sheaves and Weil divisors do not depend on closed subschemes of codimension $\geq 2$, and on the smooth part $X_S$ of $X$ the sheaf $\omega_X|_{X_S}$ is associated to the canonical divisor $K_{X_S}$ we have that $\omega_X = \mathcal{O}_X(K_X)$. □

**Corollary 2.11** If $Y_1, Y_2, Y$ are projective locally Cohen-Macaulay schemes such that $Y_1, Y_2$ are algebraically linked by $Y$, then there are the following exact sequences:

$$0 \rightarrow K_1 \rightarrow \mathcal{I}_{Y_2/Y} \otimes \omega_Y \rightarrow \omega_{Y_1} \rightarrow C_1 \rightarrow 0$$
$$0 \rightarrow K_2 \rightarrow \mathcal{I}_{Y_1/Y} \otimes \omega_Y \rightarrow \omega_{Y_2} \rightarrow C_2 \rightarrow 0$$

where $K_1, K_2, C_1$ and $C_2$ are coherent sheaves on $Y$ with supports contained in $\text{Jump}(\omega_Y) \cap Y_1 \cap Y_2$. If $Y_1$ and $Y_2$ are geometrically linked by $Y$, or if $Y$ is Gorenstein, then we have the isomorphisms:

$$\mathcal{I}_{Y_2/Y} \otimes \omega_Y \cong \omega_{Y_1}$$
$$\mathcal{I}_{Y_1/Y} \otimes \omega_Y \cong \omega_{Y_2}.$$
Proof. Let $Y_1$ and $Y_2$ be algebraically linked by $Y$. It is sufficient to prove only the first of the two exact sequences. Since $I_{Y_2/Y} \cong \text{Hom}_Y(O_{Y_1}, O_Y)$ by Definition 2.13 and $\text{Hom}_Y(O_{Y_1}, \omega_Y) \cong \text{Ext}^1_Y(O_{Y_1}, \omega_Y) \cong \omega_Y$ by Theorem 2.8 (where $Y \hookrightarrow \mathbb{P}$ is an embedding of $Y$ in some projective space $\mathbb{P}$ and $c = \text{codim}(Y, \mathbb{P})$), we have a natural map

$$I_{Y_2/Y} \otimes \omega_Y \to \omega_{Y_1}. \quad (2.12)$$

If $\omega_Y$ is locally free we have $\text{Hom}_Y(O_{Y_1}, O_Y) \otimes \omega_Y \cong \text{Hom}_Y(O_{Y_1}, \omega_Y)$, i.e. map (2.12) is an isomorphism (which proves the statement if $Y$ is Gorenstein); therefore the kernel $K_1$ and the cokernel $C_1$ of (2.12) have their supports contained in $Y_1 \cap \text{Jump}(\omega_Y)$. The map (2.12) can fits into the following commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathcal{K} & \to & I_{Y_2/Y} \otimes \omega_Y \to \omega_Y \to \omega_{Y_2} \to 0 \\
\downarrow & & \downarrow \\
0 & \to & \omega_{Y_1} \to \omega_Y \to \text{Hom}(I_{Y_1/Y}, \omega_Y) \to 0.
\end{array} \quad (2.13)$$

The top row of the diagram is obtained from the exact short sequence $0 \to I_{Y_2/Y} \to O_Y \to O_{Y_2} \to 0$ by tensoring with $\omega_Y$. The kernel $\mathcal{K}$ of the map $I_{Y_2/Y} \otimes \omega_Y \to O_Y \otimes \omega_Y$ is $\text{Tor}(O_{Y_1}, \omega_Y)$, which is supported on $\text{Jump}(\omega_Y) \cap Y_2$, therefore $K_1 \cong \mathcal{K}$ has support contained in $\text{Jump} Y \cap Y_1 \cap Y_2$. The bottom row of the diagram is obtained from the exact short sequence $0 \to I_{Y_1/Y} \to O_Y \to O_{Y_1} \to 0$ by applying the functor $\text{Hom}(\cdot, \omega_Y)$. By Snake’s Lemma $C_1 \cong \ker \{ \omega_{Y_2} \to \text{Hom}(I_{Y_1/Y}, \omega_Y) \}$, therefore also $C_1$ is supported in $\text{Jump} Y \cap Y_1 \cap Y_2$.

Let $Y_1$ and $Y_2$ be geometrically linked by $Y$. It is sufficient to prove only the first of the two isomorphisms. We use induction on the dimension $k$ of $Y$. If $\dim Y = 0$, then $Y_1$ and $Y_2$ are disjoint, therefore by the first part of this proof we get $K_1 = C_1 = 0$. Let us now suppose that the statement holds when the dimension is $k - 1$. In particular Cor. 2.11 holds for the generic hyperplane sections $Z_1, Z_2, Z$ of $Y_1, Y_2, Y$ respectively. Let us consider, for every integer $\alpha$, the following commutative diagram:

$$\begin{array}{ccc}
I_{Y_2/Y} \otimes \omega_Y(\alpha - 1) & \to & I_{Y_2/Y} \otimes \omega_Y(\alpha) \\
\downarrow & & \downarrow \\
\omega_Y(\alpha - 1) & \to & \omega_Y(\alpha) \\
\downarrow & & \downarrow \\
\omega_Z(\alpha - 1) & \to & \omega_Z(\alpha).
\end{array}$$

The cokernels of the three vertical maps are $\omega_{Y_2}(\alpha - 1)$, $\omega_{Y_2}(\alpha)$ and $\omega_{Z}(\alpha - 1)$ respectively. We want to prove that the kernels are zero. By induction $I_{Z_2/Z} \otimes \omega_Z(\alpha - 1)$ is isomorphic to $\omega_Z(\alpha - 1)$, therefore the third vertical map of the diagram is injective (the kernel $K_Z(\alpha - 1)$ is isomorphic to the kernel $K_1(\alpha - 1)$ of map (2.12), that is zero by induction). By Snake’s Lemma we have that $K(\alpha - 1)$ maps surjectively on $K(\alpha)$ for every $\alpha$, where $K$ is the kernel of the map $I_{Y_2/Y} \otimes \omega_Y \to \omega_Y$. This implies that $h^0(K(\alpha - 1)) \geq h^0(K(\alpha))$ for every $\alpha$ and this is possible if and only if $h^0(K(\alpha)) = 0$ for every $\alpha$ or if $K$ is supported over a zero-dimensional scheme. The former case clearly yields $K = 0$. We show
now that \( \mathcal{K} = 0 \) also if we suppose that \( \mathcal{K} \) is supported on a zero-dimensional scheme. For this purpose it is enough to prove that \( h^0(\mathcal{I}_{Y_2/Y} \otimes \omega_Y(\alpha)) = 0 \) for some \( \alpha \). Let \( \alpha < ch_Y \), so that \( h^0(\omega_Y(\alpha)) = h^0(\omega_Y(\alpha - 1)) = 0 \). Therefore: \( h^0(\mathcal{I}_{Y_2/Y} \otimes \omega_Y(\alpha - 1)) = h^0(\mathcal{K}(\alpha - 1)) = h^0(\mathcal{I}_{Y_2/Y} \otimes \omega_Y(\alpha)) \). This implies \( h^0(\mathcal{I}_{Y_2/Y} \otimes \omega_Y(\alpha)) = 0 \) for \( \alpha < ch_Y \). Since \( \mathcal{K} \cong \mathcal{K}_1 \) we have proven that the natural map \((2.12)\) is injective.

We want to prove now that it is also surjective. Let us look at the exact sequence:

\[
0 \to I_{Y_2/Y} \otimes \omega_Y \to \omega_Y \to \omega_Y|_{Y_2} \to 0.
\]

(2.14)

Following the notation of Cor. 2.10, the sheaf \( \omega_Y \) is \( \mathcal{I}_{Y/Y}(f) \). Therefore \( \mathcal{I}_{Y_2/Y} \otimes \omega_Y \to \mathcal{I}_{Y_2/Y}(f) \to \mathcal{O}_Y(f) \) is the ideal sheaf in \( Y \) of a scheme which certainly contains \( Y_2 \), twisted by \( \mathcal{O}_Y(f) \). Hence by the exact sequence \((2.14)\) we deduce that also the third term \( \omega_Y|_{Y_2} \) is an ideal sheaf in \( Y_2 \) twisted by \( \mathcal{O}_Y(f) \), hence torsion free in \( Y_2 \). Therefore \( \mathcal{C}_1 = \ker\{ \omega_Y|_{Y_2} \to \text{Hom}(\mathcal{I}_{Y_1/Y}, \omega_Y) \} \), which is supported on \( \text{Jump}(\omega_Y) \cap Y_1 \cap Y_2 \), is zero.

**Definition 2.15** Let \( X \) be a projective scheme of dimension \( r \); let \( a_i \in \mathbb{N}^+ \) and let \( 1 \leq k \leq r \). A complete intersection (c.i. for short) on \( X \) of kind \((a_1, \ldots, a_r)\) is an equidimensional projective scheme \( Y \subset X \) such that \( \text{codim}(Y, X) = k \), which is scheme theoretic intersection of Cartier divisors \( D_i \in |\mathcal{O}_X(a_i)| \) for \( i = 1, \ldots, k \).

We want now to fix some notation about rational normal scrolls and point out what we will need in the next sections. A rational normal scroll \( X \subset \mathbb{P} \) of dimension \( r \) and degree \( f \) is the image of a projective bundle \( \mathbb{P}(E) \to \mathbb{P}^1 \) over \( \mathbb{P}^1 \) through the morphism \( j \) defined by the tautological line bundle \( \mathcal{O}_\mathbb{P}(1) \), where \( E \cong \mathcal{O}_2 \oplus \cdots \oplus \mathcal{O}_{2+r} \) with \( 0 \leq a_1 \leq \cdots \leq a_r \) and \( \sum a_i = f = n - r \). If \( a_1 = \cdots = a_l = 0 \), \( 1 \leq l < r \), \( X \) is singular and the vertex \( V \) of \( X \) has dimension \( l - 1 \). Let us denote \( \mathbb{P}(E) = \tilde{X} \). The morphism \( j : \tilde{X} \to X \) is a rational resolution of singularities, i.e. \( X \) is normal and arithmetically Cohen-Macaulay and \( R^{l-1} \mathcal{O}_V = 0 \) for \( j > 0 \). We will call \( \tilde{j} : \tilde{X} \to X \) the canonical resolution of \( X \). It is well known that \( \text{Pic}(\tilde{X}) = \mathbb{Z}[\tilde{H}] \oplus \mathbb{Z}[\tilde{R}] \), where \( [\tilde{H}] = [\mathcal{O}_V(1)] \) is the hyperplane class and \( [\tilde{R}] = [\pi^* \mathcal{O}_V(1)] \) is the class of the fibre of the map \( \pi : \tilde{X} \to \mathbb{P}^1 \). The intersection form on \( \tilde{X} \) is determined by the rule:

\[
\tilde{H}^r = f \quad \tilde{H}^{r-1} \cdot \tilde{R} = 1 \quad \tilde{H}^{r-2} \cdot \tilde{R}^2 = 0.
\]

Let us denote with \( X_S \) the smooth part of \( X \) and with \( \text{Exc}(j) \) the exceptional locus of \( j \). Then \( j : \tilde{X} \setminus \text{Exc}(j) \to X_S \) is an isomorphism. Let \( \tilde{H} \) and \( \tilde{R} \) be the strict images of \( \tilde{H} \) and \( \tilde{R} \) respectively (i.e. the scheme theoretic closure \( j(\tilde{H}_{j^{-1}X_S}) \) and \( j(\tilde{R}_{j^{-1}X_S}) \)). Then we have the following well known result:

**Lemma 2.16** Let \( X \subset \mathbb{P}^n \) be a rational normal scroll of degree \( f \) and let \( j : \tilde{X} \to X \) be its canonical resolution. Let \( \text{Cl}(X) \) be the group of Weil divisors on \( X \) modulo linear equivalence. Then:
1. If \( \text{codim}(V, X) > 2 \), \( \text{Cl}(X) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[R] \);

2. If \( \text{codim}(V, X) = 2 \), \( H \sim fR \) and \( \text{Cl}(X) \cong \mathbb{Z}[R] \).

We recall here from [F2] the definition of proper and (integral) total transform of a Weil divisor in \( X \). In the last section (Example 4.8) we will use proper and integral total transforms together with [F2] Prop. 4.11 to compute the multiplicity of the vertex \( V \) in the intersection scheme of two effective divisors on a rational normal 3-fold \( X \) with \( \text{codim}(V, X) = 2 \).

**Definition 2.17** Given a prime divisor \( D \) on \( X \), the proper transform \( \tilde{D} \) of \( D \) in \( \tilde{X} \) is the scheme theoretic closure \( j^{-1}(D \cap X_S) \). The proper transform of any Weil divisor in \( X \) is then defined by linearity.

**Definition 2.18** Let \( \text{codim}(V, X) = 2 \) and let \( D \sim dR \) be an effective Weil divisor on \( X \), divide \( d - 1 = kf + h \) (\( k \geq -1 \) and \( 0 \leq h < f \)), we define the integral total transform of \( D \) in \( \tilde{X} \) as

\[
\tilde{D}^* \sim (k + 1) \tilde{H} - (f - h - 1) \tilde{R}.
\]

Let us define on \( X \) the following coherent sheaves for \( a, b \in \mathbb{Z} \):

**Definition 2.19**

\[
\mathcal{O}_X(a, b) := j_* \mathcal{O}_{\tilde{X}}(a \tilde{H} + b \tilde{R}).
\]

We will usually write \( \mathcal{O}_X(a) \) instead of \( \mathcal{O}_X(a, 0) \). Moreover for every coherent sheaf \( \mathcal{F} \) on \( X \) we will write \( \mathcal{F}(a, b) \) instead of \( \mathcal{F} \otimes \mathcal{O}_X(a, b) \). If the scroll \( X \) is smooth, then the sheaves \( \mathcal{O}_X(a, b) \) are the invertible sheaves associated to the Cartier divisors \( \sim aH + bR \) while when \( X \) is singular this is no longer true. In this case we have the following Proposition which is proved in [F2] (Cor. 3.10 and Th. 3.17).

**Proposition 2.20** Let \( X \subset \mathbb{P}^n \) be a singular rational normal scroll of degree \( f \), dimension \( r \) and vertex \( V \), then:

1. If \( \text{codim}(V, X) > 2 \) the sheaf \( \mathcal{O}_X(a, b) \) is reflexive for every \( a, b \in \mathbb{Z} \) and it is the divisorial sheaf associated to a Weil divisor \( \sim aH + bR \);

2. If \( \text{codim}(V, X) = 2 \) the sheaf \( \mathcal{O}_X(a, b) \) is reflexive for every \( a, b \in \mathbb{Z} \) such that \( b < f \); in this case the sheaves \( \mathcal{O}_X(a, b) \) with \( a + fb = d \) are all isomorphic to the the divisorial sheaf associated to a Weil divisor \( \sim dR \);

In the hypotheses of Prop. 2.20, the dualizing sheaf \( \omega_X \) of \( X \) is (see (5)):

\[
\omega_X = j_* \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) = \mathcal{O}_X(-r, f - 2).
\]

(2.21)

By Prop. 2.20 we see that \( \omega_X \) is a divisorial sheaf (see also Cor. 2.10), therefore the canonical divisor of \( X \) is \( K_X \sim -rH + (f - 2)R \). The canonical characteristic \( ch_X \) is then:

\[
ch_X = r.
\]

(2.22)

The following result is essentially due to Hartshorne (Linkage of generalized divisors by a complete intersection: [H1], Prop. 4.1). He states it for divisors on a complete intersection but the same proof goes over as well.
Proof. Just for sake of simplicity we put \( \omega \) where the right term is by Cor. 2.11 recursively as \( H \rightarrow I \) → integer defined by the previous identity. Let us consider the exact sequence

\[ \oplus M \to I \to Y \to 0. \]

This implies, in terms of the Hilbert function of \( Z \) tensored with the invertible sheaf \( Z \), then for \( H \rightarrow A \) c.i. on \( W \) a normal scheme is algebraically linked to \( D \) Proposition 2.23 (Linkage of divisors) Let \( D_1 \) be an effective Weil divisor on a normal scheme \( X \subset \mathbb{P}^n \). Let \( F \subset \mathbb{P}^n \) be a hypersurface containing \( D_1 \); let \( D \) be the Cartier divisor on \( X \) defined by \( F \), then the effective divisor \( D_2 = D - D_1 \) is algebraically linked to \( D_1 \) by \( D \).

3 Linkage by complete intersection on aCM schemes

In this section \( W \) is an aCM scheme of dimension \( r \) in \( \mathbb{P}^{n-1} \) and \( X \) is an aCM scheme of dimension \( r + 1 \) in \( \mathbb{P}^n \). When \( W \) is smooth we consider algebraic linkage, where the subschemes need not have distinct components. In the singular case we consider geometric linkage, where Cor. 2.11 holds. Let us start with the smooth case and prove the following.

Proposition 3.1 Let \( W \subset \mathbb{P}^{n-1} \) be a smooth aCM scheme of dimension \( r \). Let \( Z_1 \subset W \) be a projective 0-dimensional locally Cohen-Macaulay scheme. Let \( Z_2 \subset W \) be a projective 0-dimensional scheme. Let \( Z = W \cap F_1 \cap \cdots \cap F_r \) be a c.i. on \( W \) of type \( (a_1, \ldots, a_r) \) and let \( c = a_1 + \cdots + a_r \). Assume that \( Z_1 \) and \( Z_2 \) are algebraically linked by \( Z \). Let \( ch \) be the canonical characteristic of \( W \), then for \( i < \min \{a_j\} \):

\[ H^0(I_{Z_2}/W \otimes \omega_W(i + ch)) \cong H^1(I_{Z_2}/W(c - ch - i))^\vee. \]

This implies, in terms of the Hilbert function of \( Z_1 \):

\[ h^0(I_{Z_2}/W \otimes \omega_W(i + ch)) = \deg Z_1 - h_{Z_2}(c - ch_W - i). \]

Proof. Just for sake of simplicity we put \( i + ch_W = c - v \), where \( v \) is an integer defined by the previous identity. Let us consider the exact sequence

\[ 0 \to I_{Z_2}/W \to I_{Z_2}/W \to I_{Z_2}/Z \to 0 \]

tensored by \( \omega_W(c - v) \):

\[ 0 \to I_{Z_2}/W \otimes \omega_W(c - v) \to I_{Z_2}/W \otimes \omega_W(c - v) \to I_{Z_2}/Z \otimes \omega_W(c - v) \to 0, \]

where the right term is by Cor. 2.11 \( \omega_Z(-v) \) (since \( \omega_Z = \omega_W(c)_{|Z} \)). We want to prove that for \( v > \max_j \{c - a_j - ch_W\} \):

\[ H^0(I_{Z_2}/W \otimes \omega_W(c - v)) \cong \ker \{H^0(\omega_Z(-v)) \to H^1(I_{Z_2}/W \otimes \omega_W(c - v))\}, \]

which is equivalent to prove that \( H^0(I_{Z_2}/W \otimes \omega_W(c - v)) = 0 \) for \( v > \max_j \{c - a_j - ch_W\} \). For this purpose let us consider the Koszul resolution of \( I_{Z_2}/W \) in \( \mathcal{M}(W) \):

\[ 0 \to E_r \to E_{r-1} \to E_{r-2} \to \cdots \to E_2 \to E_1 \to I_{Z_2}/W \to 0, \]

where the \( E_i \)'s are finite direct sum \( \oplus O_W(\alpha_i) \) with \( \alpha_i \in Z \); in particular \( E_1 = \oplus_j O_W(-a_j) \) and \( E_r = O_W(-c) \). Let \( H_1 = \ker \{E_1 \to I_{Z_2}/W\} \) and \( H_i \) be defined recursively as \( H_i = \ker \{E_i \to H_{i-1}\}, i = 2, \ldots, r - 2 \). Let us look now at (3.3) tensored with the invertible sheaf \( \omega_W(c - v) \) and let us denote the sheaf.
\[ F \otimes \omega_W(c-v), \text{for every sheaf } F \text{ in } \mathcal{W}, \text{with } \tilde{\mathcal{F}}. \] Since \( W \) is aCM and \( E_i = \oplus \mathcal{O}_W(\alpha_i) \), we find that \( H^1(H_1) \cong H^{r-2}(H_{r-2}) \cong H^{r-1}(E_r) = 0. \) Therefore from the short exact sequence \( 0 \to H_1 \to E_1 \to \mathcal{I}_{Z/W} \to 0 \) we deduce that \( H^{0}(\mathcal{I}_{Z/W}) = 0 \) when \( H^{0}(\tilde{E}_1) = \oplus_j H^{0}(\omega_W(c - a_j - v)) = 0, \) and this happens for \( c - a_j - v < h_W \) for all \( j \) (i.e. \( v > \max \{c - a_j - ch_W\} \)), as we claimed.

At this point we want to prove that

\[ \text{Ext}^{r-1}_W(\mathcal{I}_{Z_1/W}(v), \omega_W) \cong \ker \{ H^0(\omega_Z(-v)) \to H^r(\omega_W(-v)) \}. \] (3.4)

Let us consider now exact sequence

\[ 0 \to \mathcal{I}_{Z_1/W}(v) \to \mathcal{O}_W(v) \to \mathcal{O}_{Z_1}(v) \to 0. \]

We apply the functor Hom(, , \omega_W):

\[ \ldots \text{Ext}^{r-1}_W(\mathcal{O}_W(v), \omega_W) \to \text{Ext}^{r-1}_W(\mathcal{I}_{Z_1/W}(v), \omega_W) \to \text{Ext}^{r}_W(\mathcal{O}_{Z_1}(v), \omega_W) \to \text{Ext}^{r}_W(\mathcal{O}_W(v), \omega_W) \ldots \]

By Serre’s duality \( \text{Ext}^{r-1}_W(\mathcal{O}_W(v), \omega_W) \cong H^1(\mathcal{O}_W(v)) \cong 0 \), since \( W \) is aCM.

Then note that \( \text{Ext}^{r}_W(\mathcal{O}_{Z_1}(v), \omega_W) \cong H^0(\mathcal{O}_{Z_1}(v)) \cong H^0(\omega_{Z_1}(-v)) \), by Serre’s duality on \( W \) for the first isomorphism and on \( Z_1 \) for the second one. By [H2, III, 6.3] \( \text{Ext}^{r}_W(\mathcal{O}_W(v), \omega_W) \cong H^r(\omega_W(-v)) \), and we have proven (3.4).

Since \( H^r(\omega_W(-v)) \) “functorially” contains \( H^1(\mathcal{I}_{Z_1/W} \otimes \omega_W(c-v)) \), as one can easily check by looking at the Koszul complex \([4, 3]\) tensored by \( \omega_W(c-v) \), we deduce that the kernels \([4, 2]\) and \([4, 3]\) are isomorphic. Therefore, applying Serre’s duality to \( \text{Ext}^{r-1}_W(\mathcal{I}_{Z_1/W}(v), \omega_W) \), it follows that for \( v > \max \{c - a_j - ch_W\} \):

\[ H^0(\mathcal{I}_{Z_2/W} \otimes \omega_W(c-v)) \cong H^1(\mathcal{I}_{Z_1/W}(v)) \]

which proves the first part of the statement after the substitution \( v = c - i - ch_W \).

Moreover, since \( W \) is aCM, from the exact sequence

\[ 0 \to \mathcal{I}_{W/P} \to \mathcal{I}_{Z_1/P} \to \mathcal{I}_{Z_1/W} \to 0 \]

we get \( H^1(\mathcal{I}_{Z_1/W}(k)) \cong H^1(\mathcal{I}_{Z_1/P}(k)) \) for every \( k \). Moreover \( h^1(\mathcal{I}_{Z_1/P}(k)) = h^0(\mathcal{O}_{Z_1}(k)) - h_{Z_1}(k) = \deg(Z_1) - h_{Z_1}(k) \). This proves the second part of the statement.

We note that Proposition \([3, 1]\) can be easily proven using a straightforward generalization of classical linkage (in particular of Prop. 2.5 of [PS]). Namely the construction through the mapping cone of a locally free resolution of \( \mathcal{O}_{Z_2} \) in \( \mathcal{M}od(W) \) from a locally free resolution of \( \mathcal{O}_{Z_1} \). Nevertheless we prefer the proof we have given because it can be generalized to the singular case:

**Theorem 3.5** Let \( W \subset \mathbb{P}^{n-1} \) be a singular aCM scheme of dimension \( r \). Let \( Z_1, Z_2, Z \subset W \) be as in the hypotheses of Prop. \([3, 4]\) and assume moreover that \( Z_1 \) and \( Z_2 \) are geometrically linked by \( Z \), or that \( Z_1 \) (equivalently \( Z_2 \)) is contained in the smooth part of \( W \). Let \( ch_W \) be the canonical characteristic of \( W \), then for \( i < \min_j \{a_j\} \):

\[ H^0(\mathcal{I}_{Z_2/W} \otimes \omega_W(i + ch_W)) = H^1(\mathcal{I}_{Z_1/W}(c - ch_W - i)) \.]
This implies, in terms of the Hilbert function of $Z_1$:

$$h^0(\mathcal{I}_{Z_2\cap W} \otimes \omega_W(i + ch_W)) = \deg Z_1 - h_{Z_1}(c - ch_W - i).$$

**Proof.** As in the proof of Prop. 3.1 we want to prove the isomorphism (3.2), for $v > \max_j \{c - a_j - ch_W\}$. Let us tensor the exact sequence $0 \to \mathcal{I}_{Z/W} \to \mathcal{I}_{Z_2/W} \to \mathcal{I}_{Z_2/Z} \to 0$ with $\omega_W(c - v)$ and obtain the exact sequence:

$$0 \to \mathcal{K} \to \mathcal{I}_{Z/W} \otimes \omega_W(c - v) \to \mathcal{I}_{Z_2/W} \otimes \omega_W(c - v) \to \mathcal{I}_{Z_2/Z} \otimes \omega_W(c - v) \to 0,$$

where $\mathcal{K}$, which is a quotient of $\text{Tor}^1(\mathcal{I}_{Z_2/Z}, \omega_W(c - v))$, has support contained in $\text{Jump}(\omega_W) \cap \text{Jump}(\mathcal{I}_{Z_2/Z}) \subset \text{Jump}(\omega_W) \cap Z_2$. Let $\mathcal{A}$ be the kernel of the map $\mathcal{I}_{Z_2/W} \otimes \omega_W(c - v) \to \mathcal{I}_{Z_2/Z} \otimes \omega_W(c - v)$. Since $H^1(\mathcal{K}) = H^2(\mathcal{K}) = 0$ we have that $H^1(\mathcal{I}_{Z/W} \otimes \omega_W(c - v)) \cong H^1(\mathcal{A})$, hence we consider $\ker \{H^0(\mathcal{I}_{Z_2/Z} \otimes \omega_W(c - v)) \to H^1(\mathcal{A}) \cong H^1(\mathcal{I}_{Z/W} \otimes \omega_W(c - v))\}$. By Cor. 2.11 if $Z_1$ and $Z_2$ are geometrically linked, or if $Z_1$ or $Z_2$ is contained in the smooth part of $W$, we have that $\mathcal{I}_{Z_2/Z} \otimes \omega_W(c - v) \cong \omega_{Z_1}(c - v)$. Since $H^0(\mathcal{I}_{Z/W} \otimes \omega_W(c - v)) = 0$ implies $H^0(\mathcal{A}) = 0$, we prove (3.3) if and only if we prove that $H^0(\mathcal{I}_{Z/W} \otimes \omega_W(c - v)) = 0$ for $v > \max_i \{c - a_i - ch_W\}$. For this purpose let us consider the Koszul resolution (3.3) of $\mathcal{I}_{Z/W}$ in $\text{Mod}(W)$ and note that $\text{Jump}(H_i)$ is contained in $Z$, for all $i = 1, \ldots, r - 2$. Resolution (3.3) tensored with $\omega_W(c - v)$ splits in the following diagrams of exact sequences, where we denote $H_i = H_i \otimes \omega_W(c - v)$ and $E_i = E_i \otimes \omega_W(c - v)$:

\[
\begin{array}{cccccccc}
0 & \to & K_1 & \to & \tilde{H}_1 & \to & \tilde{E}_1 & \to & \mathcal{I}_{Z/W} \otimes \omega_W(c - v) & \to & 0 \\
& & \uparrow C_1 & & \uparrow & & \uparrow & & \uparrow & \\
& & 0 & & 0 & & 0 & & 0 & \\
0 & \to & K_i & \to & \tilde{H}_i & \to & \tilde{E}_i & \to & \tilde{H}_{i-1} & \to & 0 \\
& & \uparrow C_i & & \uparrow & & \uparrow & & \uparrow & \\
& & 0 & & 0 & & 0 & & 0 & \\
0 & \to & K_{r-1} & \to & \tilde{E}_r & \to & \tilde{E}_{r-1} & \to & \tilde{H}_{r-2} & \to & 0 \\
& & \uparrow C_{r-1} & & \uparrow & & \uparrow & & \uparrow & \\
& & 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

for $i = 2, \ldots, r - 2$, and

The supports of the kernels $K_i$ are contained in the intersection of $\text{Jump}(\omega_W)$ with $\text{Jump}(H_i)$, i.e. Supp $K_i \subset \text{Jump}(\omega_W) \cap Z$. Therefore looking at the exact sequences above we deduce that:

$$H^1(C_1) \cong H^1(\tilde{H}_1) \cong H^2(C_2) \cdots \cong H^{r-2}(\tilde{H}_{r-2}) \cong H^{r-1}(C_{r-1}) = 0.$$
Therefore as in the smooth case we have that if $H^0(\tilde{E}_1) = \oplus_j H^0(\omega_W(c - a_j - \nu)) = 0$, and this happens for $c - a_j - \nu < h_W$ for all $j$ (i.e. $\nu > \max_j\{c - a_j - ch_W\}$), then $H^0(\mathcal{I}_{Z/W} \otimes O_W(c - \nu)) = 0$ and we have proven (3.2).

As in the proof of Prop. 3.1 one proves the isomorphism:

$$H^1(\mathcal{I}_{Z/W}(v))^\vee \cong \ker\{H^0(\omega_Z(-v)) \to H^r(\omega_W(-v))\},$$

since it does not depend on the smoothness of $W$. Looking again at the resolution (3.3) tensored by $\omega_W(c - v)$, one can easily check, like in the smooth case, that $H^r(\omega_W(-v))$ "functorially" contains $H^1(\mathcal{I}_{Z/W} \otimes \omega_W(c - v))$ and deduce the first part of the statement. Like in the proof of Prop. 3.1 one deduces the second one. \hfill \Box

In the next proposition we consider the case of a c.i. of type $(a_1, \ldots, a_r)$ on a smooth aCM scheme of dimension $r + 1$. The proof is similar to the proof of Prop. 3.1 and therefore we give just a brief sketch of it.

**Proposition 3.6** Let $X \subset \mathbb{P}^n$ be a smooth aCM scheme of dimension $r + 1$. Let $Y_1, Y_2 \subset X$ be projective equidimensional schemes of dimension 1. Assume that $Y_1$ is locally Cohen-Macaulay. Let $Y \subset X$ be a c.i. of type $(a_1, \ldots, a_r)$ on $X$ and let $c = a_1 + \cdots + a_r$. Assume that $Y_1$ and $Y_2$ are algebraically linked by $Y$. Let $\text{ch}_X$ be the canonical characteristic of $X$, then for every $i$:

$$H^1(\mathcal{I}_{Y_2/X} \otimes \omega_X(i + \text{ch}_X)) \cong H^1(\mathcal{I}_{Y_1/X}(c - \text{ch}_X - i))^\vee.$$

**Proof.** We put again, as in the proof of the previous Proposition, $i + ch_W = c - v$. First we want to prove that:

$$H^1(\mathcal{I}_{Y_2/X} \otimes \omega_X(c - v)) \cong \ker\{H^1(\omega_{Y_1}(-v)) \to H^2(\mathcal{I}_{Y/X} \otimes \omega_X(c - v))\}. \quad (3.7)$$

Looking at the exact sequence:

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{I}_{Y_2/X} \to \mathcal{I}_{Y_2/Y} \to 0$$

tensored by $\omega_X(c - v)$ one see, using Cor. 2.11, that (3.7) is equivalent to $H^1(\mathcal{I}_{Y/X} \otimes \omega_X(c - v)) = 0$ and this is easy to prove using the resolution of $\mathcal{I}_{Y/X}$ in $\text{Mod}(X)$. Similarly to the proof of (3.4) one proves that

$$\text{Ext}_X^1(\mathcal{I}_{Y_1/X}(v), \omega_X) \cong \ker\{H^1(\omega_{Y_1}(-v)) \to H^{r+1}(\omega_X(-v))\}. \quad (3.8)$$

Since $H^{r+1}(\omega_X(-v))$ "functorially" contains $H^2(\mathcal{I}_{Y/X} \otimes \omega_X(c - v))$, one deduces that the kernels (3.7) and (3.8) are isomorphic. Therefore applying Serre’s duality to $\text{Ext}_X^1(\mathcal{I}_{Y_1/X}(v), \omega_X)$ it follows that:

$$H^1(\mathcal{I}_{Y_2/X} \otimes \omega_X(c - v)) \cong H^1(\mathcal{I}_{Y_1/X}(v))^\vee.$$

and we are done by substituting $v = c - i - \text{ch}_X$. \hfill \Box

Let us consider now the singular case and prove the following:
Theorem 3.9 Let $X \subset \mathbb{P}^n$ be a singular aCM scheme of dimension $r + 1$. Let $Y_1, Y_2, Y \subset X$ be as in the hypotheses of Prop. 3.6 and assume moreover that $Y_1$ and $Y_2$ are geometrically linked by $Y$. Let $\omega_X$ be the canonical characteristic of $X$, then for every $i$:

$$H^1(I_{Y_2}/X \otimes \omega_X(i + c\chi_X)) \cong H^1(I_{Y_1}/X(c - ch_X - i))^\vee.$$ 

Proof. Following the proof of Prop. 3.6 step by step, we want first to prove the isomorphism (3.7). We start from the exact sequence:

$$0 \to K \to I_{Y/X} \otimes \omega_X(c - v) \to I_{Y_2/X} \otimes \omega_X(c - v) \to I_{Y_2/Y} \otimes \omega_X(c - v) \to 0,$$

where $K$ has support contained in $\text{Jump}(\omega_X) \cap Y$. Using Cor. 2.11 and the same techniques used in the proof of Th. 3.3, one see that (3.7) is equivalent to $H^1(I_{Y/X} \otimes \omega_X(c - v)) = 0$. Using the resolution of $I_{Y/X}$ in $\text{Mod}(X)$ tensored with $\omega_X(c - v)$ one proves that $H^1(I_{Y/X} \otimes \omega_X(c - v)) = 0$, as in the smooth case. The isomorphism (3.8) does not depend on the smoothness of $X$, therefore it can be proven exactly as in Prop. 3.6. Using again the resolution of $I_{Y/X}$ in $\text{Mod}(X)$ tensored with $\omega_X(c - v)$, one proves also in this case that $H^{r+1}(\omega_X(-v))$ ”functorially” contains $H^2(I_{Y/X} \otimes \omega_X(c - v))$. Therefore the kernels (3.7) and (3.8) are isomorphic. We conclude as in the proof of Prop. 3.6. □

From Prop. 3.6 it follows easily that if we suppose $Y_1$ arithmetically Cohen-Macaulay, then we can lift ”canonical” divisors $\sim K_W + (i + ch_W)H$ on a general hyperplane section $W \subset \mathbb{P}^{n-1}$ of $X$ containing the general hyperplane section $Z_2$ of $Y_2$ to divisors $\sim K_X + (i + ch_X)H$ on $X$ containing $Y_2$. Namely we have the following Corollary:

Corollary 3.10 In the hypotheses of Proposition 3.6, if we suppose $Y_1$ to be arithmetically Cohen-Macaulay, then the map

$$H^0(I_{Y_2/X} \otimes \omega_X(i + ch_X)) \to H^0(I_{Z_2/W} \otimes \omega_W(i + ch_W))$$

is surjective for every $i$.

Proof. Since both $Y_1$ and $X$ are arithmetically Cohen-Macaulay we have that $h^1(I_{Y_1/X}(k)) = h^1(I_{Y_2/Y}(k)) = 0$ for every $k$. By Prop. 3.6 we have $h^1(I_{Y_2/X} \otimes \omega_X(i + ch_X)) = 0$ for every $i$; the statement follows now from the exact sequence

$$0 \to I_{Y_2/X} \otimes \omega_X(i - 1 + ch_X) \to I_{Y_2/X} \otimes \omega_X(i + ch_X) \to I_{Z_2/W} \otimes \omega_X(i + ch_X) \to 0$$

since $\omega_W = \omega_X(1)|_W$ and $ch_X = ch_{W} + 1$. □

In the next Proposition we prove that even in the singular case $H^0(I_{Y_2/X} \otimes \omega_X(\alpha))$ represents geometrically Weil divisors of $X$ linearly equivalent to $K_X + \alpha H$ containing $Y_2$. The proof is similar to the one we have used in Cor. 2.11 to prove the exact sequence (2.14), therefore we omit it.
Proposition 3.11. In the hypotheses of Theorem 3.9 we have the following exact sequence:

\[ 0 \rightarrow \mathcal{I}_{Y_2/X} \otimes \omega_X \rightarrow \omega_X \rightarrow \omega_X|_{Y_2} \rightarrow 0. \]

With this in mind we can state Cor. 3.10 also in the singular case:

Corollary 3.12. In the hypotheses of Theorem 3.9, if we suppose \( Y_1 \) to be arithmetically Cohen-Macaulay, then the map

\[ H^0(\mathcal{I}_{Y_2/X} \otimes \omega_X(i + ch_X)) \rightarrow H^0(\mathcal{I}_{Z_2/W} \otimes \omega_W(i + ch_W)) \]

is surjective for every \( i \).

Proof. Following the proof of Cor. 3.10 one have to consider the following exact sequence:

\[ 0 \rightarrow K \rightarrow \mathcal{I}_{Y_2/X} \otimes \omega_X(-1) \rightarrow \mathcal{I}_{Y_2/X} \otimes \omega_X \rightarrow \mathcal{I}_{Z_2/W} \otimes \omega_X \rightarrow 0 \]

where kernel \( K \) has support contained in \( \text{Jump}(\omega_X) \cap Z_2 \). Let \( H \) be the kernel of the map \( \mathcal{I}_{Y_2/X} \otimes \omega_X \rightarrow \mathcal{I}_{Z_2/W} \otimes \omega_X \). Since \( h^1(H(i + ch_X)) = h^1(\mathcal{I}_{Y_2/X} \otimes \omega_X(i - 1 + ch_X)) = 0 \) for all \( i \), we are done. \( \square \)

The next result is a formula which relates the arithmetic genera of the curves \( Y_1, Y_2 \) and \( Y \).

Proposition 3.13. In the hypotheses of Prop. 3.6 or of Theorem 3.9, if we suppose that \( Y \) has no components contained in \( \text{Jump}(\omega_X) \), we have the following formula, relating the arithmetic genera of the linked curves:

\[ p_a(Y_2) = p_a(Y_1) - p_a(Y) + \deg(K_Y|_{Y_2}) + 1. \]  

(3.14)

Proof. First we tensor by \( \omega_Y \) the exact sequence

\[ 0 \rightarrow \mathcal{I}_{Y_2/Y} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y_2} \rightarrow 0. \]

By Cor. 2.11 this gives the exact sequence (see exact sequence (2.14)):

\[ 0 \rightarrow \omega_Y \rightarrow \omega_Y \rightarrow \omega_Y|_{Y_2} \rightarrow 0. \]

(3.15)

With the notation of Cor. 2.10 we know that \( \omega_Y \) is \( \mathcal{I}_{Y_B/Y}(f + c) \), where \( Y_B \) is, in this case, the intersection of \( X_B \) with the \( r \) hypersurfaces which cut \( Y \) in \( X \). Let us remark that \( X_B \) passes through the jump locus of \( X \), otherwise \( X \) would be there locally complete intersection, hence Gorenstein, and this can not happen by definition of jump locus. If some components of \( Y \) are contained in \( \text{Jump}(\omega_X) \), then \( Y_B \) contains such components and it is clearly not a divisor of \( Y \). On the other hand, since we can always arrange the complete intersection \( A \) of Cor. 2.10 in such a way that \( X_B \) does not contain any components of \( Y \) outside \( \text{Jump}(\omega_X) \), if \( Y \) has no components contained in \( \text{Jump}(\omega_X) \) the scheme \( Y_B = X_B \cap Y \) is a divisor in \( Y \). In this case \( \omega_Y \) is the divisorial sheaf in \( Y \) associated to \( K_Y \) and \( \omega_Y|_{Y_2} \) is a divisorial sheaf in \( Y_2 \), associated to the divisor
Proposition 4.1

If \(G\) is a genus developed in the previous sections to the classification of curves of maximal genus \(G\), then the difference \(\Delta\) follows now by (3.16) and (3.15). \(\square\)

4 An application to the classification of curves of maximal genus

In this section we will show some examples of application of the techniques developed in the previous sections to the classification of curves of maximal genus \(G(d,n,s)\) in \(\mathbb{P}^n\). Let us first summarize some results of [CCD] and some other preliminary facts useful to introduce the problem.

From now on, let \(C\) be an integral, nondegenerate curve of degree \(d\) and arithmetic genus \(p_a(C)\) in \(\mathbb{P}^n\), with \(d > \frac{2^n}{n-2} \Pi_{i=1}^{n-2} ((n-1)!)^{n-2}\) and \(s \geq n-1\) (later we will assume \(s \geq 2n - 1\)). Assume \(C\) is not contained on surfaces of degree \(s\) and define \(m, \epsilon, w, v, k, \delta\) as follows:

- divide \(d - 1 = sm + \epsilon, 0 \leq \epsilon \leq s - 1\) and \(s - 1 = (n-2)w + v, v = 0, \ldots, n-3\);
- if \(\epsilon < w(n-1-v)\), divide \(\epsilon = kw + \delta, 0 \leq \delta < w\);  
- if \(\epsilon \geq w(n-1-v)\), divide \(\epsilon + n - 2 - v = k(w+1) + \delta, 0 \leq \delta < w + 1\).

It is a result of [CCD] (section 5) that the genus \(p_a(C)\) is bounded by the function:

\[
G(d,n,s) = 1 + \frac{d}{2}(m + w - 2) - \frac{m + 1}{2}(w - 3) + \frac{vm}{2}(w + 1) + \rho
\]

where \(\rho = \frac{d}{2}(w - \delta)\) if \(\epsilon < w(n-1-v)\) and \(\rho = \frac{d}{2} - \frac{w}{2}(n - 2 - v) - \frac{d}{2}(w - \delta + 1)\) if \(\epsilon \geq w(n-1-v)\).

If \(Z\) is a general hyperplane section of \(C\) and \(h_Z\) is the Hilbert function of \(Z\), then the difference \(\Delta h_Z\) must be bigger than the function \(\Delta h\) defined by:

\[
\Delta h(r) = \begin{cases} 
0 & \text{if } r < 0 \\
(n-2)r + 1 & \text{if } 0 \leq r \leq w \\
s & \text{if } w < r \leq m \\
s + k - (n-2)(r - m) & \text{if } m < r \leq m + \delta \\
s + k - (n-2)(r - m) - 1 & \text{if } m + \delta < r \leq m + w + e \\
0 & \text{if } r > m + w + e
\end{cases}
\]

where \(e = 0\) if \(\epsilon < w(n-1-v)\) and \(e = 1\) otherwise ([CCD] Prop. 0.1).

**Proposition 4.1** If \(p_a(C) = G(d,n,s)\), then \(C\) is arithmetically Cohen-Macaulay and \(\Delta h_Z(r) = \Delta h(r)\) for all \(r\). Moreover \(Z\) is contained on a reduced curve \(\Gamma\) of degree \(s\) and maximal genus \(G(s,n-1) = \binom{s}{2} + wv\) in \(\mathbb{P}^{n-1}\) (Castelnuovo curve). \(\Gamma\) is unique and, when we move the hyperplane, all these curves \(\Gamma\)'s patch together giving a surface \(S \subset \mathbb{P}^n\) of degree \(s\) through \(C\) (Castelnuovo surface).
\textbf{Proof.} See [CCO] Prop. 6.1, Prop. 6.2 and Cor. 6.3. \hfill \Box

We recall that a \textit{Castelnuovo curve} in $\mathbb{P}^n$ is a nondegenerate reduced and irreducible curve of degree $d$ and maximal arithmetic genus $G(d,n)$. Castelnuovo in 1893 found the bound $G(d,n)$ and he went on to give a complete geometric description of those curves which achieved his bound, they lie on surfaces of minimal degree. The reader may consult [Ha2] for all the details. By Castelnuovo surface we mean a nondegenerate reduced and irreducible surface in $\mathbb{P}^n$ whose general hyperplane section is a Castelnuovo curve in $\mathbb{P}^{n-1}$ (see [Ha2]).

\textbf{Proposition 4.2} The surface $S$ of Prop. 4.1 is irreducible and if $s \geq 2n - 1$ it lies on a rational normal 3-fold $X \subset \mathbb{P}^n$. As a divisor on $X$ the surface $S$ is linearly equivalent to $(w+1)H - (n-3-v)R$ or $wH + R$ if $v = 0$). If $n = 6$ and if $s$ is even there is the further possibility that the surface $S$ lies in a cone over the Veronese surface in $\mathbb{P}^5$ and is the complete intersection with a hypersurface not containing the vertex.

\textbf{Proof.} $S$ is irreducible since $C$ is irreducible and is not contained on surfaces of degree $< s$. The rest of the statement follows using the characterization of Castelnuovo surfaces given in [Ha2]. \hfill \Box

\textbf{Proposition 4.3} There exists a hypersurface $F_{m+1}$ of degree $m + 1$, passing through $C$ and not containing $S$.

\textbf{Proof.} For a general hyperplane section $\Gamma$ of $S$, the Hilbert function $h_{\Gamma}$ is known (see e.g. [Ha1] Th. 3.7): in particular we have $\Delta h_{\Gamma}(r) = \Delta h_Z(r)$ when $0 \leq r \leq m$ and hence $h^0(\mathcal{I}_{C/P}(r)) = h^0(\mathcal{I}_{S/P}(r))$ when $0 \leq r \leq m$. For $r = m+1$ one shows $\Delta h_{\Gamma}(m+1) < \Delta h_Z(m+1)$ and this implies $h^0(\mathcal{I}_{C/P}(m+1)) > h^0(\mathcal{I}_{S/P}(m+1))$. \hfill \Box

Let us suppose $s \geq 2n - 1$. By the Prop. 4.2 a curve $C \subset \mathbb{P}^n$ of maximal genus $G(d,n,s)$ lies then on a rational normal 3-fold $X$ (except in the case where $S$ lies in a cone over a Veronese surface, which we do not intend to go through). Let $F_{w+1}$ be a hypersurface of degree $w+1$ cutting out on $X$ the surface $S$. By Prop. 4.3 we can consider on $S$ the curve $C'$ residual to $C$ by the intersection with the hypersurface $F_{m+1}$. Since $\deg(C') < \deg(C)$, then $C'$ does not contain $C$. Choosing in $X$ a sufficiently general divisor $D \sim (n-3-v)R$ (or $D \sim H - R$ in case $S \sim wH + R$) linked to $S$ by $X \cap F_{w+1}$, then the residual scheme on $X$ to $C$ by the c.i. $X \cap F_{w+1} \cap F_{m+1}$ is a curve which we call $C''$. When $v = n-3$ then $S = X \cap F_{w+1}$ and of course $C' = C''$; otherwise $C''$ is the union of $C'$ with a curve $C_D$ contained in $D$, therefore $C_D$ is formed by $n-3-v$ distinct plane curves of degree $m+1$ or, in case $S \sim wH + R$, $C_D$ is the complete intersection on $D \sim H - R$ by a hypersurface of degree $m+1$. Letting $Z', Z'' \subset W$ be general hyperplane sections of $C'$ and $C''$ respectively, we have the following Lemma:
Lemma 4.4 Let \( C'' \subset X \) be as in the previous notation. Then for \( i \leq w, m \)

\[
h^0(I_{C''/X}(i, n-4)) \geq h^0(I_{Z''/W}(i, n-4)) = \sum_{r=m+w+1}^{\infty} \Delta h(r).
\]

Moreover if \( h^0(I_{Z''/W}(i-1, n-4)) = 0 \) and \( h^0(I_{Z''/W}(i, n-4)) = h > 0 \), then \( h^0(I_{C''/X}(i-1, n-4)) = 0 \) and \( h^0(I_{C''/X}(i, n-4)) = h \).

Proof. \( C \) and \( C'' \) are geometrically linked by \( Y = X \cap F_{w+1} \cap F_{m+1} \) since they are equidimensional, have no common components (\( C \) is irreducible and \( C'' \) does not contain \( C \)) and no embedded components (\( Y \) is arithmetically Cohen Macaulay). By Prop. \( 3.1 \) (if \( X \) is smooth) or Th. \( 3.9 \) (if \( X \) is singular) we know that

\[
h^0(I_{Z''/W}(i, n-4)) = d - h_Z(m + w - i).
\]

Then note that for every \( k \) we have

\[
d - h_Z(k) = d + \Delta h_Z(k - 1) - h_Z(k) = d + \sum_{r=k+1}^{t} \Delta h_Z(r) - h_Z(t) = \sum_{r=k+1}^{\infty} \Delta h_Z(r) \text{ because for } t \text{ big we have } h_Z(t) = d, \text{ and that, by Prop. } 4.1, \Delta h_Z(r) = \Delta h(r) \text{ for all } r. \]

Since \( \omega_X \cong O_X(-3, n-4) \) and \( \omega_W \cong O_W(-2, n-4) \), by Cor. \( 3.10 \) we have that

\[
h^0(I_{C''/X}(i, n-4)) \geq h^0(I_{Z''/W}(i, n-4)). \tag{4.5}
\]

Let us consider the exact sequence

\[
0 \to K(k+3) \to I_{C''/X}(k-1, n-4) \to I_{C''/X}(k, n-4) \to I_{Z''/W}(k, n-4) \to 0,
\]

where \( K \) is the kernel of the map \( I_{C''/X}(-4, n-4) \to I_{C''/X}(-3, n-4) \) and is supported on \( \text{Sing}(X) \cap Z'' \). If, for \( k = i-1 \), we have \( h^0(I_{Z''/W}(i-1, n-4)) = 0 \), since \( h^1(K(i+2)) = h^0 \) we have that \( h^0(I_{C''/X}(i-2, n-4)) \geq h^0(I_{C''/X}(i-1, n-4)) = 0 \). In this hypothesis, for \( k = i \), we have an injection \( h^0(I_{C''/X}(i, n-4)) \hookrightarrow h^0(I_{Z''/W}(i, n-4)) \) and therefore by \( 4.5 \) \( h^0(I_{C''/X}(i, n-4)) = h^0(I_{Z''/W}(i, n-4)) \). \( \square \)

We rewrite now the genus formula \( 3.14 \) for linked curves contained in a rational normal three-fold \( X \):

**Proposition 4.6** Let \( Y_1 \) and \( Y_2 \) be two 1-dimensional projective schemes contained in a rational normal 3-fold \( X \subset \mathbb{P}^n \) which is smooth or whose vertex is a point. Let \( Y_1 \) be locally Cohen-Macaulay. Assume that \( Y_1 \) and \( Y_2 \) are geometrically linked by a complete intersection \( Y = X \cap F_a \cap F_b \) of type \((a, b)\) on \( X \) (if \( X \) is smooth it is enough to suppose \( Y_1 \) and \( Y_2 \) algebraically linked by \( Y \) ). Then:

\[
p_a(Y_2) = p_a(Y_1) - p_a(Y) + (a + b - 3) \cdot \deg(Y_2) + (n - 4) \cdot \deg(R_{Y_2}) + 1. \tag{4.7}
\]

**Proof.** Since \( \text{Jump}(\omega_X) \) is a point, we can apply Proposition \( 3.13 \). Formula \( 3.14 \) follows from formula \( 3.14 \) since \( K_{Y_1/Y_2} \sim (a + b - 3)H_{Y_2} + (n - 4)R_{Y_2} \).

\( \square \)

The strategy is to classify all the curves of maximal genus in \( \mathbb{P}^n \) for arbitrary \( n \) by classifying the linked curves \( C'' \)’s. A complete classification Theorem when
n = 4 is proved in [CC] and when n = 5 in [F1] (and in the forthcoming work [E]). Depending on the numerical parameters (ε, w, v, k) associated to C and on the type of the scroll X the analysis goes on case by case. In the following example we want to show the simplest non trivial case in the classification procedure, when C′ is a plane curve (the trivial case is C′ = ∅). It should be remarked that while in P^3 the curve C′ is always degenerate this is no longer true for n ≥ 4 (see [CC] and [F1] for n = 4, 5).

**Example 4.8** Let s ≥ 2n − 1 and let d > \frac{2s}{d − 1} \Pi_{i=1}^{n-2}((n − 1)!)^{−1}; divide s − 1 = (n − 2)w + v, v = 0, . . . , n − 3 and divide d − 1 = sm + ε, 0 ≤ ε ≤ s − 1. Suppose s − 2 − w ≤ ε ≤ s − 2. Let C ⊂ P^n be a curve of maximal genus G(d, n, s). Then the linked curve C′ is a plane curve of degree s − ε − 1. In case that the vertex of X is a line, C′ will not contain this line as a component.

Here we suppose for the sake of simplicity that v = n − 3 (the result can be proved with similar arguments for every v), i.e. we put ourselves in the simplest case C′ = C′′, with this assumption we always have ε = 1, i.e. ε ≥ w(n − 1 − v) = (n − 3)(w + 1), hence we write ε + 1 = k(w + 1) + δ with k = n − 3 and δ ≤ w. In this case C and C′ are (geometrically) linked by a c.i. Y = X ∩ F_{w+1} ∩ F_{m+1} on X. Applying Lemma 4.4 for i = 0 (of course h^0(\mathcal{I}_{C′/X}(-1, n − 4)) = 0) we compute:

\[ h^0(\mathcal{I}_{C′/X}(0, n − 4)) = n − 4. \]

Let us exclude for now the case X singular along a line. The linear system |\mathcal{O}_X(0, n − 4)| is composed with a rational pencil, i.e. we have π : X → P^1 and |\mathcal{I}_{C′/X}(0, n − 4)| = π∗G, where G is a linear subsystem of |\mathcal{O}_{P^1}(n − 4)|. Since h^0(\mathcal{O}_{P^1}(n − 4)) = n − 3, this implies that |\mathcal{I}_{C′/X}(0, n − 4)| has a fixed part; in this case, since h^0(\mathcal{O}_X(0, a)) = a + 1 for every a ≥ 0, the fixed part of |\mathcal{I}_{C′}(0, n − 4)| is ~ R and the moving part is equal to the whole |\mathcal{O}_X(0, n − 5)|. Therefore we conclude that C′ is contained in a plane π ~ R.

Let us consider now the case when the vertex of X is a line. We want to conclude as in the previous case that |\mathcal{I}_{C′/X}(0, n − 4)| has a fixed part. So let us suppose that |\mathcal{I}_{C′/X}(0, n − 4)| has no fixed part, which implies that the support of C′ is the singular line of X. By Bertini’s Theorem the generic divisor in the corresponding linear subsystem G of P^3 is union of n − 4 distinct points in a rational normal curve C_{n−2} of degree n − 2 (a (n−2)-plane section of X), which span a P^{n−5}. Therefore we can choose a basis \{D_1, . . . , D_{n−4}\} in the linear system |\mathcal{I}_{C′/X}(0, n − 4)| such that D_i is union of n − 4 distinct planes of X for every i and such that the linear space spanned by each D_i is < D_i > ∼ P^{n−3} and D_i = X ∩ < D_i >. In this situation the base locus of |\mathcal{I}_{C′/X}(0, n − 4)|, which is equal to D_1 ∩ . . . ∩ D_{n−4} = X ∩ < D_1 > ∩ . . . ∩ < D_{n−4} >, is necessarily the singular line l ∼ P^1 of X counted with multiplicity one, but this is not possible since |\mathcal{I}_{l/X}(0, n − 4)| = |\mathcal{O}_X(0, n − 4)| and we have a contradiction. Therefore, as in the previous case, we conclude that C′ is contained in a plane π ~ R. We claim now that C′ cannot contain the singular line of X as a component. In fact in this case both S and F_{m+1} would pass through it and their proper transforms S and F_{m+1} on the canonical resolution \tilde{X} of X would
be $\tilde{S} \sim (w + 1 - a)\tilde{H} + (n - 2)a\tilde{R}$ and $\tilde{F}_{m+1} \sim (m + 1 - b)\tilde{H} + (n - 2)b\tilde{R}$ with $a, b \geq 1$. In this case, since $C$ is irreducible (therefore it does not contain the singular line), $C'$ would contain the singular line with multiplicity $a$ which we compute using $[F_3]$ Prop. 4.11 as: $\alpha = S^* \cdot F_{m+1}^* \cdot \tilde{H} - \tilde{S} \cdot \tilde{F}_{m+1} \cdot \tilde{H} = (w + 1)\tilde{H} \cdot (m + 1)H^2 - \tilde{S} \cdot 2 \cdot \tilde{F}_{m+1} \cdot \tilde{H} = ab(n - 2)$, where $S^*$ and $F_{m+1}^*$ are respectively the integral total transform of $S$ and $F_{m+1}$ (Def. 2.18). But since $C'$ is contained in a plane $\pi \sim R$ by the same kind of computation we conclude that $C'$ would contain the singular line with multiplicity $\beta = S^* \cdot R^* \cdot \tilde{H} - \tilde{S} \cdot \tilde{R} \cdot \tilde{H} = (w + 1)\tilde{H} \cdot (H - (n - 3)\tilde{R}) \cdot \tilde{H} - ((w + 1 - a)\tilde{H} + (n - 2)a\tilde{R}) \cdot \tilde{R} \cdot H = a$ and this is in contradiction with the previous value.

In the next example we show that in the case of Example 4.8 smooth curves of maximal genus do always exist. Moreover we explicitly construct such curves on a smooth rational normal 3-fold. It is interesting to note that it is not always possible to construct curves of maximal genus on a smooth rational normal 3-fold. There are cases (for some values of $d$ and $s$) where the construction is possible only on a rational normal 3-fold whose vertex is a point and where genus formula (4.7) holds, as showed in $[F_3]$ (Prop. 4.2 case 6) and Theorem 5.2 case $k = v = 1$ for $n = 5$. The existence of curves of maximal genus in $\mathbb{P}^5$ is proved for all cases in $[F_3]$. We state first the following, easy to prove, result (see $[R_3]$ Lemma 1 pg. 133) which we will use later.

Lemma 4.9 Let $X$ be a smooth 3-fold. Let $\Sigma$ be a linear system of surfaces of $X$ and let $\gamma$ be a curve contained in the base locus of $\Sigma$. Suppose that the generic surface of $\Sigma$ is smooth at the generic point of $\gamma$ and that it has at least a singular point which is variable in $\gamma$. Then all the surfaces of $\Sigma$ are tangent along $\gamma$.

Example 4.10 For every $d$ and $s$ in the range of Example 4.8 there exists a smooth curve $C \subset \mathbb{P}^n$ of maximal genus $G(d, n, s)$.

For the sake of simplicity we treat only the cases $v = n - 3, n - 4$, i.e. $s = (n - 2)(w + 1)$ and $s = (n - 2)w + n - 3$. The other cases can be treated in a similar way.

Let us suppose $v = n - 3$. Let $X \subset \mathbb{P}^n$ be a smooth rational normal 3-fold of degree $n - 2$ and let $\pi \sim R$ be a plane contained in $X$. Let $D$ be a smooth curve on $\pi$ of degree $0 \leq \deg D = w + 1 - s + \epsilon + 1 = \epsilon + 1 - (n - 3)(w + 1) \leq w$ (possibly $D = \emptyset$). If we consider the union of $D$ with any plane curve $C' \subset \pi$ of degree $w + 1 - \deg D = s - \epsilon - 1$, then there exists a hypersurface $F_{w+1}$ of degree $w + 1$ cutting out on $\pi$ the union $C' \cup D$. Therefore the linear system $|I_{D/X}(w + 1)|$ of divisors on $X$ cut by hypersurfaces of degree $w + 1$ through $D$ is not empty and cut on $\pi$ the linear system $D + |O_x(s - \epsilon - 1)|$. Moreover the linear system $|I_{D/X}(w + 1)|$ contains the linear subsystem $L + |O_X(w)|$, where $L$ is a fixed hyperplane section containing $\pi$, that has fixed part $L$ and no other base points. This implies that $D$ is the base locus of all $|I_{D/X}(w + 1)|$ and that $|I_{D/X}(w + 1)|$ is not composed with a pencil, because in this case every element
in the system would be a sum of algebraically equivalent divisors, while the divisors in \( L + |\mathcal{O}_X(w)| \) are obviously not of this type. By Bertini’s Theorem we can then conclude that the generic divisor in \(|\mathcal{I}_{D/X}(w + 1)|\) is an irreducible surface \( S \) of degree \((n - 2)(w + 1) = s\) smooth outside \( D \). We claim that \( S \) is in fact smooth at every point \( p \) of \( D \). To see this, by Lemma 4.3 it is enough to prove that, for every \( p \in D \), there exists a surface in \(|\mathcal{I}_{D/X}(w + 1)|\) which is smooth at \( p \), and that for a generic point \( q \in D \), there exist two surfaces in \(|\mathcal{I}_{D/X}(w + 1)|\) with distinct tangent planes at \( q \). In fact, for every \( p \in D \) we can always find a surface \( T \) in the linear system \(|\mathcal{O}_X(w)|\) which does not pass through \( p \), therefore the surface \( L + T \) is smooth at \( p \) with tangent plane \( \pi \). Moreover a generic surface in the linear system \(|\mathcal{I}_{D/X}|\) which cut \( D \) on \( \pi \) has at \( p \) tangent plane \( T_p \neq \pi \).

Let \( C' \subset \pi \) be the linked curve to \( D \) by the intersection \( \pi \cap S \). Let us consider the linear system \(|\mathcal{I}_{C'/S}(m + 1)|\) of divisors cut on \( S \) by the hypersurfaces of degree \( m + 1 \) passing through \( C' \). With the same argument used above we conclude that this linear system is not composed with a pencil, it has \( C' \) as a fixed part and no other base points. Therefore by Bertini’s theorem we deduce that the generic curve \( C = S \cap F_{m+1} - C' \) in the movable part of the linear system is irreducible, smooth and has the required degree \( d = s(m+1) - s + \epsilon + 1 \). By Clebsch formula one computes:

\[
p_a(C') = \frac{1}{2}((n - 2)w + n - 4 - \epsilon)((n - 2)w + n - 5 - \epsilon).
\]

Moreover \( \deg(R \cap C') = 0 \). Substituting these expressions in the genus formula \(|L|\) we find that \( p_a(C) \) has the maximal value \( G(d, n, s) \), therefore \( C \) is the required curve.

We consider now the case \( v = n - 4 \). Let \( \pi \sim R \) and \( p \sim R \) be two distinct planes contained in \( X \). Let \( D \) be a smooth curve on \( \pi \) of degree \( 0 \leq \deg D = \epsilon + 2 - (n - 3)(w + 1) \leq w \) (possibly \( D = \emptyset \)). Let us consider the linear system \(|\mathcal{I}_{D_{wp}/X}(w + 1)|\) of divisors on \( X \) cut by hypersurfaces of degree \( w + 1 \) containing the plane \( p \) and passing through \( D \). This linear system is not empty since hypersurfaces which are union of a hyperplane containing the plane \( p \) and of a hypersurface of degree \( w \) passing through \( D \) cut on \( X \) divisors in the system. From this description one can see that \(|\mathcal{I}_{D_{wp}/X}(w + 1)|\) is not composed with a pencil and that its base locus is \( p \cup D \). By Bertini’s Theorem the generic element in the movable part of the linear system is an irreducible surface \( S \sim (w + 1)H - R \) of degree \( s \), smooth outside \( D \). By the same argument used in the previous case we can prove that \( S \cup p \) is smooth at every point of \( D \), but since \( D \cap p = \emptyset \) this means that \( S \) is smooth at \( D \). Let \( C' \subset \pi \) be the linked curve to \( D \) by the intersection \( S \cap \pi \). Let us consider the linear system \(|\mathcal{I}_{C'/S}(m + 1)|\), which is not empty since \( \deg C' < m + 1 \) and has base locus equal to the curve \( C' \). As in the previous case we deduce that the generic curve \( C = S \cap F_{m+1} - C' \) in the movable part of this linear system is irreducible, smooth and has the required degree \( d = s(m+1) - \deg(C') \). By generality the hypersurface \( F_{m+1} \) does not contain the plane \( p \) and cut on it a curve \( C_1 \) of degree \( m + 1 \). Let \( C'' = C' \cup C_1 \); by construction the curve \( C'' \) is linked to \( C \).
by a c.i. on $X$ of type $(w + 1, m + 1)$

By Noether’s formula one computes:

$$p_a(C'') = \frac{1}{2}((n - 2)w + n - 5 - \epsilon)((n - 2)w + n - 6 - \epsilon) + \frac{1}{2}m(m - 1) - 1.$$ 

Moreover $\deg(R \cap C'') = 0$. Substituting these expressions in the genus formula (4.7) we find that $p_a(C)$ has the maximal value $G(d, n, s)$. Therefore $C$ is the required curve.

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