The cardinality of the set of real numbers

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Abstract

A proof that the set of real numbers is denumerable is given.

1 Introduction

Cantor’s proof that the real set is uncountable is now included in the standard mathematics curriculum. The objections and reservations presented by Poincaré, Kronecker, Weil and Brouwer, for example, are silenced. Apparently we are in the Paradise that Cantor has created. This paper seeks to show that this is not the case.

Each real number of the interval [0, 1] can be represented by an infinite path in a given binary tree. In Section 2 the binary tree is projected on a grid $N \times N$ and it is shown that the set of the infinite paths corresponds one-to-one to the set $N$. The Theorems 2.1 and 2.2 give the first proof and the Theorem 2.3 provides a second proof.

Section 3 contains the 1891 Cantor’s proof and section 4 examines the the inconsistency of this proof.

2 The proof of $|F| = |N|$

Theorem 2.1 Let $B$ be a binary tree such that every node has two children and its depth is equal to $|N|$. Let $A$ be the set of the nodes of the binary tree $B$. The cardinality of $A$ is less than or equal to $|N|$.

Proof:

Let us now consider the Figure 1. The horizontal sequence of finite natural numbers, presented in increasing order from left to right, contains all numbers of $N$; to each natural number of the sequence corresponds a vertical line. The vertical sequence of numbers, presented in increasing order from top to bottom, contains all numbers of $N$; to each natural number of the sequence corresponds a horizontal line. To each node of the grid formed by the horizontal and vertical lines corresponds a pair $(m, n)$, where $m$ belongs to the horizontal sequence of numbers and $n$ belongs to the vertical sequence of numbers.
Let us project the binary tree $B$ on the grid of the Figure 1 in the following way: to the root of the tree corresponds the pair $(0, 0)$; to the 2 nodes of the level 1 correspond the pairs $(0, 1)$ and $(1, 0)$; to the 4 nodes of the level 2 correspond the pairs $(0, 3)$, $(1, 2)$, $(2, 1)$ and $(3, 0)$; to the 8 nodes of the level 3 correspond the pairs $(0, 7)$, $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, $(6, 1)$ and $(7, 0)$ and to the $2^k$ nodes of the level $k$ correspond the pairs

$$(0, 2^k - 1), (1, 2^k - 2), (2, 2^k - 3), \ldots, (2^k - 3, 2), (2^k - 2, 1) \text{ and } (2^k - 1, 0) \quad (1)$$

The Figure 2 shows the binary tree $B$ up to the depth 4. The Figure 3 shows the projection of the binary tree up to the depth 3 on the grid of the Figure 1.

![Figure 1](image)

![Figure 2](image)
It is known that the bijection $f : N \rightarrow N \times N$, where $N$ is the set of finite natural numbers, can be defined applying the diagonal method.

The Figure 4 shows how starting from the pair $(0, 0)$ and following the thick line we can establish the one-to-one correspondence between the set of the pairs $(m, n)$ and $N$, this is,

$$(0,0) \quad (1,0) \quad (0,1) \quad (0,2) \quad (1,1) \quad (2,0) \quad (3,0) \ldots$$

Considering that to any node of the binary tree corresponds a node of the grid and that there are nodes in the grid without corresponding nodes in the binary tree, we conclude that

$$|A| \leq |N|$$

**Theorem 2.2** The cardinality of the set of real numbers of the interval $[0, 1]$ is equal to $|N|$.

**Proof:**

Let us denote by $F$ the set of real numbers of the interval $[0, 1]$. Any element of $F$ can be represented in the binary system by
\[ f_1 \times 2^{-1} + f_2 \times 2^{-2} + f_3 \times 2^{-3} + f_4 \times 2^{-4} + \ldots \] (4)

where
\[ f_i \in \{0, 1\} \quad \text{and} \quad i \in N \quad \text{and} \quad i \neq 0 \] (5)

The representation \((4)\) can be simplified to
\[ .f_1 f_2 f_3 f_4 f_5 \ldots \] (6)

where the first character of the sequence is the point. Each \(f_i\) of \((6)\) is substituted by 0 or 1 to represent a given number. The \(i\)-th 0 or 1 on the right of “.” corresponds to \(f_i\). The set \(F\) can also be represented by a binary tree where each node has two children. Each infinite path on the binary tree, \(q_1 q_2 \ldots q_i \ldots\) with depth equal to \(|N|\), represents an element of \(F\).

Let us denote by \(B\) the binary tree above mentioned; by \(B_i\) the binary tree of depth \(i\); by \(P\) the set of all paths starting from the root of \(B\); by \(P_i\) the set of all paths from the root of \(B_i\) to its leaves. Each element of \(P_i\) or \(P\) is the set of the nodes (except the root) that form the path. Let the set
\[ \{P_1, P_2, \ldots, P_i, \ldots\} \quad \text{such that} \quad i \in N \] (7)

The set \(\{P_1, P_2, \ldots, P_i\}\) contains all paths from the root of the tree \(B_i\) and the endings of the paths that exist in \(\{P_1, P_2, \ldots, P_i\}\) are all the nodes of \(B_i\), except the root. This indicates that the set in \((7)\) - which corresponds to all the tree \(B\) - contains all paths of \(B\).

Let us consider the set of all infinite paths from the root of \(B\)
\[ Q = \{q_1, q_2, q_1, q_2, \ldots, q_i, \ldots\} \] (8)

The infinite path \(q_\tau\) can be represented by the set
\[ \{q_{\tau 1}, q_{\tau 2}, \ldots, q_{\tau i}, \ldots\} \] (9)

where \(q_{\tau 1}\) is the \(i\)-th node of \(q_\tau\) after the root. Let
\[ Q_\tau = \{\{q_{\tau 1}\}, \{q_{\tau 1}, q_{\tau 2}\}, \ldots, \{q_{\tau 1}, q_{\tau 2}, \ldots, q_{\tau i}\}, \ldots\} \] (10)

The union of all elements of \(Q_\tau\) is equal to \(q_\tau\), that is,
\[ q_\tau = \{q_{\tau 1}\} \cup \{q_{\tau 1}, q_{\tau 2}\} \cup \ldots \cup \{q_{\tau 1}, q_{\tau 2}, \ldots, q_{\tau i}\} \cup \ldots \] (11)

Considering \((11)\) we can obtain from \((8)\)
\[ Q = \{\{q_{\alpha 1}\} \cup \{q_{\alpha 1}, q_{\alpha 2}\} \cup \ldots \cup \{q_{\alpha 1}, q_{\alpha 2}, \ldots, q_{\alpha i}\} \cup \ldots, \{q_{\beta 1}\} \cup \{q_{\beta 1}, q_{\beta 2}\} \cup \ldots \cup \{q_{\beta 1}, q_{\beta 2}, \ldots, q_{\beta i}\} \cup \ldots, \ldots, \{q_{\tau 1}\} \cup \{q_{\tau 1}, q_{\tau 2}\} \cup \ldots \cup \{q_{\tau 1}, q_{\tau 2}, \ldots, q_{\tau i}\} \cup \ldots \} \] (12)

From \((12)\) we have
\[ \left| Q \right| \leq \left| \{ q_{a_1} \} \cup \{ q_{a_1}, q_{a_2} \} \cup \ldots \cup \{ q_{a_1}, q_{a_2}, \ldots, q_{a_1} \} \cup \ldots \right. \]
\[ \left. \cup \{ q_{b_1} \} \cup \{ q_{b_1}, q_{b_2} \} \cup \ldots \cup \{ q_{b_1}, q_{b_2}, \ldots, q_{b_1} \} \cup \ldots \right. \]
\[ \left. \ldots \ldots \ldots \right. \]
\[ \left. \cup \{ q_{r_1} \} \cup \{ q_{r_1}, q_{r_2} \} \cup \ldots \cup \{ q_{r_1}, q_{r_2}, \ldots, q_{r_1} \} \cup \ldots \right. \]
\[ \left. \ldots \ldots \ldots \right. \]
\[ \left. \right| \]  \hspace{1cm} (13)

Since the sets on the right side of (13) with cardinality equal to \(i\) are the elements of \(P_i\), that is,
\[ P_i = \{ q_{a_1}, q_{a_2}, \ldots, q_{a_i} \} \cup \{ q_{b_1}, q_{b_2}, \ldots, q_{b_i} \} \cup \{ q_{r_1}, q_{r_2}, \ldots, q_{r_i} \} \cup \ldots \]  \hspace{1cm} (14)
we conclude that
\[ \left| Q \right| \leq \left| P_1 \right| + \left| P_2 \right| + \left| P_3 \right| + \ldots + \left| P_i \right| + \ldots \]  \hspace{1cm} (15)

The set of the nodes of \(B\) has cardinality less than or equal to \(|N|\) (Theorem 2.1). Since the number of nodes in the level \(i\) of the tree \(B_i\) is equal to \(|P_i|\), we have
\[ |N| \geq |P_1| + |P_2| + |P_3| + \ldots + |P_i| + \ldots \]  \hspace{1cm} (16)
Therefore
\[ \left| Q \right| \leq |N| \]  \hspace{1cm} (17)

Considering the bijection \(f : F \rightarrow Q\) and \(|F| \geq |N|\), we conclude
\[ |F| = |N| \]  \hspace{1cm} (18)

**Theorem 2.3** The cardinal number of the set of infinite paths of the binary tree \(B\) is equal to \(|N|\).

**Proof:**
Be the infinite path of \(Q\)
\[ q_{r_0}, q_{r_1}, q_{r_2}, \ldots, q_{r_i}, \ldots \]  \hspace{1cm} (19)
where \(q_{r_i} = 1\) for all \(i \in N\) with \(i > 0\), projected in the grid as shows the Figure 3. In the coordinates of the grid, the path (19) corresponds to the pairs
\[ (0,0), (0,1), (0,3), (0,7), (0,15), \ldots, (0,2^k-1), \ldots \]  \hspace{1cm} (20)
Let us notice that in \((0,0)\) all infinite paths begin. By the pair \((0, 1)\) a part of all infinite paths passes, by \((0, 3)\) a part of the infinite paths that passed by \((0, 1)\) passes, by \((0, 7)\) a part of the infinite paths that passed by \((0, 3)\) passes and so forth. When an entire path is accomplished, the path exists. In the examined case, (19) is the accomplished path.

Each pair \((0, 2^k-1)\) of the sequence (20) belongs to the set of pairs given by (21)
\[ (0,2^k-1), (1,2^k-2), (2,2^k-3), \ldots, (2^k-3, 2), (2^k-2, 1) \text{ and } (2^k-1, 0) \]  \hspace{1cm} (21)
Let us denote by \(G_k\) the set of pairs given by (21) for \(k\). For any \(k\) the cardinal number of the set of infinite paths that pass by \((0, 2^k-1)\) is equal to the cardinal number of the set of infinite paths that pass by any other pair of (21). For any \(k\) the distance from \((0,0)\) to \((0, 2^k-1)\) in the grid is equal to \(|G_k|\). When we examine the pairs of (20), from left to right, and we accomplished
the path (19), \( G_k \) becomes the set of the infinite path whose cardinality is equal to the cardinality of the set of the nodes of the path (19).

The cardinality of the set of the infinite paths is \(|F|\) and the cardinality of the set of nodes of any infinite path is \(|N|\). Therefore

\[ |F| = |N| \]  

\[(22)\]

3 The 1891 proof

Theorem. Let \( F \) be the set of real numbers of the interval \([0, 1]\). The set \( F \) is not countable.

Proof. We assume that the set \( F \) is countable. This means, by the definition of countable sets, that \( F \) is finite or denumerable. Let us notice that \(|F|\) is not less than \(|N|\) where \( N \) the set of natural numbers. Be

\[ F = \{a_1, a_2, a_3, \ldots\} \]

where the cardinality of \( \{a_0, a_1, a_2, \ldots\} \) is \(|N|\). We can write their decimal expansions as follows:

\[
\begin{align*}
a_1 &= 0. d_{1,1} d_{1,2} d_{1,3} \ldots \\
a_2 &= 0. d_{2,1} d_{2,2} d_{2,3} \ldots \\
a_3 &= 0. d_{3,1} d_{3,2} d_{3,3} \ldots \\
\vdots & \quad \vdots 
\end{align*}
\]

\[(23)\]

where the \( d \)'s are binary characters 0 and 1. Now we define the number

\[ x = 0.d_1d_2d_3\ldots \]

by selecting \( d_1 \neq d_{1,1}, d_2 \neq d_{2,2}, d_3 \neq d_{3,3}, \ldots \). This gives a number not in the set \( \{a_1, a_2, a_3, \ldots\} \), but \( x \in F \). Therefore, \( F \) is not countable \[1\].

4 The inconsistency of the proof

When (23) is represented as (24) bellow, the proof can be called the written list form of the Cantor's argument of 1891.

\[
\begin{align*}
1 & \leftrightarrow 0. d_{1,1} d_{1,2} d_{1,3} \ldots \\
2 & \leftrightarrow 0. d_{2,1} d_{2,2} d_{2,3} \ldots \\
3 & \leftrightarrow 0. d_{3,1} d_{3,2} d_{3,3} \ldots \\
\vdots & \quad \vdots 
\end{align*}
\]

\[(24)\]

Let be the list
succession of the pattern constituted of $2^n$'s followed by $2^n$ 1's.

More generally, considering the matrix (26) the procedure to obtain (25) is: the first column of $d$ equal to 1, (ii) $d_2$, $d_5$, $d_6$, $d_9$, $d_2$, $d_{10}$, $d_{12}$, $d_{14}$, $d_{16}$, $d_{18}$ ... are equal to 1, (iii) $d_1$, $d_2$, $d_5$, $d_6$, $d_9$, $d_2$, $d_{10}$, $d_{12}$, $d_{14}$, $d_{16}$, $d_{18}$ ... are equal to 0 and $d_3$, $d_4$, $d_7$, $d_8$, $d_{11}$, $d_{12}$, $d_{15}$, $d_{16}$, $d_{19}$ ... are equal to 1, (iii) $d_1$, $d_2$, $d_3$, $d_4$, $d_7$, $d_8$, $d_{11}$, $d_{12}$, $d_{15}$, $d_{16}$, $d_{19}$ ... are equal to 0 and $d_5$, $d_6$, $d_7$, $d_8$, $d_{13}$, $d_{14}$, $d_{15}$, $d_{16}$, ... are equal to 1, (iv) ... and so on ...

\[
\begin{array}{cccc}
0 & 0.00000 & \ldots \\
1 & 0.10000 & \ldots \\
2 & 0.01000 & \ldots \\
3 & 0.11000 & \ldots \\
4 & 0.00100 & \ldots \\
5 & 0.10100 & \ldots \\
6 & 0.01100 & \ldots \\
7 & 0.11100 & \ldots \\
8 & 0.00010 & \ldots \\
9 & 0.10010 & \ldots \\
10 & 0.01010 & \ldots \\
11 & 0.11010 & \ldots \\
12 & 0.00110 & \ldots \\
13 & 0.10110 & \ldots \\
14 & 0.01110 & \ldots \\
15 & 0.11110 & \ldots \\
16 & 0.00001 & \ldots \\
\vdots & \vdots & \vdots \\
\end{array}
\]

Let us notice that (i) $d_{1,1}$, $d_{3,1}$, $d_{5,1}$, $d_{7,1}$, $d_{9,1}$, $d_{11,1}$, $d_{13,1}$, $d_{15,1}$, $d_{17,1}$ ... are equal to 0 and $d_{2,1}$, $d_{4,1}$, $d_{6,1}$, $d_{8,1}$, $d_{10,1}$, $d_{12,1}$, $d_{14,1}$, $d_{16,1}$, $d_{18,1}$ ... are equal to 1, (ii) $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_9$, $d_{10}$, $d_{12}$, $d_{14}$, $d_{16}$, $d_{18}$, $d_{20}$, $d_{22}$, $d_{24}$, $d_{26}$, $d_{28}$, $d_{30}$, $d_{32}$, $d_{34}$, $d_{36}$, $d_{38}$, $d_{40}$ ... are equal to 0 and $d_2$, $d_4$, $d_6$, $d_8$, $d_{10}$, $d_{12}$, $d_{14}$, $d_{16}$, $d_{18}$, $d_{20}$ ... are equal to 1, (iii) $d_1$, $d_2$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_9$, $d_{10}$, $d_{11}$, $d_{12}$, $d_{13}$, $d_{14}$, $d_{15}$, $d_{16}$, $d_{17}$, $d_{18}$ ... are equal to 0 and $d_3$, $d_4$, $d_5$, $d_6$, $d_7$, $d_8$, $d_{13}$, $d_{14}$, $d_{15}$, $d_{16}$, $d_{17}$, $d_{18}$, $d_{19}$, $d_{20}$ ... are equal to 1, (iv) ... and so on ...

\[
\begin{array}{ccc}
d_{1,1} & d_{1,2} & d_{1,3} \\
d_{2,1} & d_{2,2} & d_{2,3} \\
d_{3,1} & d_{3,2} & d_{3,3} \\
\vdots & \vdots & \vdots \\
\end{array}
\]

More generally, considering the matrix (26) the procedure to obtain (25) is: the first column of the matrix is filled from top to bottom with a succession of the pattern 01. The second column is filled with a succession of the pattern 0011. The $n$-th column is filled from top to bottom with a succession of the pattern constituted of $2^n$ 0's followed by $2^n$ 1's.

When the number of rows in (25) goes to infinity we have that a $x$ equal to

\[
\lim_{n \to \infty} \left(2^{-1} + 2^{-2} + 2^{-3} + \ldots + 2^{-n}\right) = 0.11111111\ldots
\]

belongs to (26). There is no way out of this. All the reals in the interval [0,1] are included in the list. However applying the Cantor’s argument to (25) we found that 0.11111111... does not belongs to (26).

References

[1] G. Cantor, Über eine elementare Frage der Mannigfaltigkeitslehre, Jahresbericht der Deutschen Mathematiker-Vereinigung, 1 (1891), 75-78.