Robust EM kernel-based methods for linear system identification

Giulio Bottegal\textsuperscript{a} Aleksandr Y. Aravkin\textsuperscript{b} Häkan Hjalmarsson\textsuperscript{a} Gianluigi Pillonetto\textsuperscript{c}

\textsuperscript{a}ACCESS Linnaeus Centre, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden (e-mail: \{bottegal; hjalmars\}@kth.se)

\textsuperscript{b}IBM T.J. Watson Research Center, Yorktown Heights, NY, USA (e-mail: saravkin@us.ibm.com)

\textsuperscript{c}Department of Information Engineering, University of Padova, Padova, Italy (e-mail: giapt@dei.unipd.it)

Abstract

Recent developments in system identification have brought attention to regularized kernel-based methods. This type of approach has been proven to compare favorably with classic parametric methods. However, current formulations are not robust with respect to outliers. In this paper, we introduce a novel method to robustify kernel-based system identification methods. To this end, we model the output measurement noise using random variables with heavy-tailed probability density functions (pdfs), focusing on the Laplacian and the Student’s t distributions. Exploiting the representation of these pdfs as scale mixture of Gaussians, we cast our system identification problem into a Gaussian process regression framework, which requires estimating a number of hyperparameters of the data size order. To overcome this difficulty, we design a new maximum a posteriori (MAP) estimator of the hyperparameters based on an Expectation-Maximization (EM) method. In presence of outliers, numerical experiments show a substantial performance improvement compared to currently used kernel-based methods for linear system identification.

Key words: System identification; kernel-based methods; outliers; MAP estimate; EM method

1 Introduction

Regularization techniques for linear regression have a very long history in statistics and data analysis [34], [27], [39]. Recently, in a series of papers, new regularization strategies have been proposed for linear dynamic system identification [24], [23], [12], [26]. The basic idea is to define a nonparametric estimator of the impulse response of the system. As compared to classic parametric methods such as the prediction error method (PEM) [16], [33], the main motivation for this alternative approach is to avoid the model order selection step, which is usually required in parametric methods. If no information about the structure of the system is given, in order to establish the number of parameters required to describe the dynamics of the system, one has to rely on complexity criteria such as AIC and BIC [1], [32] or cross validation [16]. However, results of these criteria may be not satisfactory when short data sets are available [25].

To circumvent model order selection issues, one can use regularized least-squares that avoid high variance in the estimates using a regularization matrix, related to the so-called kernels introduced in the machine learning literature. In the context of system identification, several kernels have been proposed, e.g. the tuned/correlated (TC), diagonal/correlated (DC) kernels [12], [11], and the family of stable spline kernels [24,22]. Stable spline kernels have also been employed for estimating autocorrelation functions of stationary stochastic processes [8].

In order to guarantee flexibility of regularized kernel-based methods, the kernels structure always depends on a few parameters (in this context usually called hyperparameters), which are selected using available data; this process can be seen as the counterpart to model order selection in parametric approaches. An effective technique for hyperparameter selection is based on the Em-
pirical Bayes method [18]. Exploiting the Bayesian interpretation of regularization [39], the impulse response is modeled as a Gaussian process whose covariance matrix corresponds to the kernel. Hyperparameters are chosen by maximizing the marginal likelihood of the output data, obtained by integrating out the dependence on the impulse response, see e.g. [10,35] for algorithms for marginal likelihood optimization in system identification and Gaussian regression contexts. Then, the unknown impulse response is retrieved by computing its minimum mean square error Bayesian estimate. However, this approach, by relying on Gaussian noise assumptions, uses a quadratic loss to measure adherence to experimental data. As a result, it can be non-robust when outliers corrupt the output data [4], as described in the following example.

1.1 A motivating example

Suppose we want to estimate the impulse response of a linear system fed by white noise using the kernel-based method proposed in [24]. We consider two different situations, depicted in Figure 1. In the first one, 100 samples of the output signal are measured with a low-variance Gaussian additive noise (left panel); note that the estimated impulse response is very close to the truth (right panel). In the second situation we introduce 5 outliers in the measured output, obtaining a much poorer estimate of the same impulse response. This suggests that outliers may have a significant detrimental effect if kernel-based methods are not adequately robustified.

1.2 Statement of contribution and organization of the paper

We derive a novel outlier-robust system identification algorithm for linear dynamic systems. The starting point of our approach is to establish a Bayesian setting where the impulse response is modeled as a Gaussian random vector. The covariance matrix of such a vector is given by the stable spline kernel, which encodes information on the BIBO stability of the unknown system.

To handle outliers, we model noise using independent identically distributed random variables with heavy-tailed probability densities. Specifically, we make use of the Laplacian and the Student’s t distributions. In order to obtain an efficient system identification procedure, we exploit the representation of these noise distributions as scale mixtures of Gaussians [3]. Each noise sample is seen as a Gaussian variable whose variance is unknown but has a prior distribution that depends on the choice of the noise distribution. The variance of each noise sample needs to be estimated from data. To accomplish this task, we propose a novel maximum a posteriori (MAP) estimator able to determine noise variances and kernel hyperparameters simultaneously. Making use of the Expectation-Maximization (EM) method [14,6], we derive a new iterative identification procedure based on a very efficient joint update of all the optimization variables. The performance of the proposed algorithm is evaluated by numerical experiments. When outliers corrupt the output data, results show that there is a clear advantage of the new method compared to the kernel-based methods proposed in [24] and [12].

It is worth stressing that robust estimation is a classic and well studied topic in applied statistics and data analysis. Popular methods for robust regression hinge on the so-called M-estimators (such as the Huber estimator) [15] or on outlier diagnostics techniques [30]. In the context of Gaussian regression, recent contributions that exploit Student’s t noise models can be found also in [36,38]. In the system identification context, some outlier robust methods have been developed in recent years [20,29,4,31]. In particular, [13,7] use non-Gaussian descriptions of noise, while [5] describes a computational
framework based on interior point methods. In comparison with all these papers, the novelty of this work is to combine kernel-based approaches, noise mixture representations and EM techniques, to derive a new efficient estimator of the impulse response and kernel/noise hyperparameters. In particular, we will show that the MAP estimator of the hyperparameters can be implemented solving a sequence of one-dimensional optimization problems, defined by two key quantities which will be called a posteriori total residual energy and a posteriori differential impulse response energy. All of these subproblems, except one involving a parameter connected with the dominant pole of the system, can be solved in parallel and admit a closed-form solution.

The organization of the paper is as follows. In Section 2, we introduce the problem of linear dynamic system identification. In Section 3 we give our Bayesian description of the problem. In Section 4, we introduce the MAP-based approach to the problem and describe how to efficiently handle it. Numerical tests to evaluate the proposed approach are in Section 5. Some conclusions end the paper while the Appendix gathers the proof of the main results.

2 Problem statement

We consider a SISO linear time-invariant discrete-time linear dynamic system (see Figure 2)

\[ y_t = \sum_{i=0}^{+\infty} g_i u_{t-i} + v_t, \]

where \( \{g_i\}_{i=0}^{+\infty} \) is a strictly causal transfer function (i.e., \( g_0 = 0 \)) representing the dynamics of the system, driven by the input \( u_t \). The measurements of the output \( y_t \) are corrupted by the process \( v_t \), which is zero-mean white noise with variance \( \sigma^2 \). For the sake of simplicity, we will also hereby assume that the system is at rest until \( t = 0 \).

We assume that \( N \) samples of the input and output measurements are collected, and denote them by \( \{u_t\}_{t=0}^{N-1} \), \( \{y_t\}_{t=1}^{N} \). Our system identification problem is to obtain an estimate of the impulse response \( g_t \) for \( n \) time instants, namely \( \{g_t\}_{t=1}^{n} \) (\( g_0 = 0 \)). Recall that, by choosing \( n \) sufficiently large, these samples can be used to approximate \( g_t \) with arbitrary accuracy [17].

![Diagram](attachment:system_identification_scenario.png)

Figure 2. Block scheme of the system identification scenario.

Introducing the vector notation

\[
y := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad g := \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}, \quad v := \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}
\]

\[
U = \begin{bmatrix}
    u_0 & 0 & \ldots & 0 \\
    u_1 & u_0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{N-2} & u_{N-3} & \ldots & u_{N-n+1} & 0 \\
    u_{N-1} & u_{N-2} & \ldots & u_{N-n} & u_N \\
\end{bmatrix}
\in \mathbb{R}^{N \times n},
\]

the input-output relation for the available samples can be written

\[ y = Ug + v, \quad (2) \]

so that our estimation problem can be cast as a linear regression problem.

Remark 1 The identification method we propose in this paper can be derived also in the continuous-time setting, using the same arguments as in [24]. However, for ease of exposition, here we focus only on the discrete-time case.

Often, in the system identification framework the distribution of the noise samples is assumed to be Gaussian. We consider instead the following two models for the noise:

- the Laplacian distribution, where the probability density function (pdf) is given by

\[
p(v_t | \sigma) = \frac{1}{\sqrt{2 \sigma}} e^{-\frac{|v_t|}{\sigma}}, \quad (3)
\]

with \( \mathbb{E}[v_t^2] = \sigma^2 \);

- the Student’s t distribution, where the pdf is given by

\[
p(v_t | \rho, \nu) = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\pi \nu \rho} \Gamma \left( \frac{\nu}{2} \right)} \left( 1 + \frac{v_t^2}{\nu \rho} \right)^{-\frac{\nu+1}{2}}, \quad (4)
\]

The above equation represents a family of densities parameterized by \( \nu \), which is usually called degrees of freedom of the distribution. The parameter \( \rho \) is a scale factor and is related to the variance of the random variable through the relation \( \mathbb{E}[v_t^2] := \sigma^2 = \rho \frac{\nu}{\nu-2} \).

Note that the variance of the Student’s t distribution is defined only for \( \nu > 2 \).

Independently of the model employed in our identification scheme, we shall assume that the noise variance \( \sigma^2 \) is known or that it has been estimated using the consistent estimator defined by the following steps:
In particular, when the Student’s t is adopted as model, so-called first-order stable spline kernel kernels [24], [23]. In particular we shall make use of the $Kg$ pulse responses.

In this context, the noise $v$ is Gaussian, defined as $Kg$ is usually called a key idea is to represent the noise samples $\{v_t\}_{t=1}^N$, defined as a zero-mean Gaussian, an efficient method to choose $\hat{\lambda}$ and $\hat{\beta}$ is given by maximization of the marginal likelihood [21], which is obtained by integrating out $g$ from the joint probability density of $(y, g)$. Then we have

$$
(\hat{\lambda}, \hat{\beta}) = \arg \max_{\lambda, \beta} p(y | \lambda, \beta) = \arg \min_{\lambda, \beta} \log \det(\Sigma_y) + y^T \Sigma^{-1} y,
$$

where $\Sigma_y = \lambda UKgU^T + \Sigma_v$ is the variance of the vector $y$.

3.2 Modeling noise as a scale mixture of Gaussians

The assumptions on the noise distribution adopted in this paper imply that the joint probabilistic description of $g$ and $y$ is not Gaussian, and so the method briefly described in the previous section does not apply. In this section, we show how to deal with this problem. The key idea is to represent the noise samples $\{v_t\}_{t=1}^N$ as a scale mixture of normals [3]. Specifically, with the noise models adopted in this paper, the pdf of each variable $v_t$ can always be expressed as

$$
p(v_t) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi\tau_t} e^{-\frac{\tau^2}{2\tau_t}} \pi(\tau_t)d\tau_t},
$$

where $\pi(\tau_t)$ is a proper pdf for $\tau_t$. Since

$$
p(v_t) = \int_0^{+\infty} p(v_t, \tau_t)d\tau_t = \int_0^{+\infty} p(v_t | \tau_t)\pi(\tau_t)d\tau_t,
$$

each sample $v_t$ can be thought of as generated in two steps.

(1) A random variable $\tau_t$ is drawn from the pdf $\pi(\tau_t)$

(2) $v_t$ is drawn from a Gaussian distribution with variance $\tau_t$.

The distribution of $\tau_t$ depends on the model of $v_t$.

(1) When $v_t$ is Laplacian, we have

$$
\pi(\tau_t) = \frac{1}{\sigma^2}e^{-\frac{\tau_t}{\sigma^2}}, \tau_t \geq 0,
$$

In (11), $\Sigma_v$ represents the covariance matrix of the noise vector $v$; in this case we have $\Sigma_v = \sigma^2 I_N$. Equation (10) is instrumental to derive our impulse response estimator, which can be obtained as its minimum mean square error (MSE) (or Bayesian) estimate [2]

$$
\hat{g} = \mathbb{E}[g | y, \lambda, \beta] = Cy.
$$

The above equation depends on hyperparameters $\lambda$ and $\beta$. Estimates of these parameters, denoted by $\hat{\lambda}$ and $\hat{\beta}$, can be obtained by exploiting the Bayesian problem formulation. More precisely, since $y$ and $g$ are jointly Gaussian, an efficient method to choose $\hat{\lambda}$ and $\hat{\beta}$ is given by maximization of the marginal likelihood [21], which is obtained by integrating out $g$ from the joint probability density of $(y, g)$. Then we have

$$
(\hat{\lambda}, \hat{\beta}) = \arg \max_{\lambda, \beta} p(y | \lambda, \beta) = \arg \min_{\lambda, \beta} \log \det(\Sigma_y) + y^T \Sigma^{-1} y,
$$

where $\Sigma_y = \lambda UKgU^T + \Sigma_v$ is the variance of the vector $y$.

3 Bayesian modeling of system and noise

In this section we describe the probabilistic models adopted for the quantities of interest in the problem.

3.1 The stable spline kernel and system identification under Gaussian noise assumptions

We first focus on setting a prior on $g$. Following a Gaussian regression approach [28], we model $g$ as a zero-mean Gaussian random vector, i.e.

$$
p(g) \sim \mathcal{N}(0, \lambda K_{\beta}),
$$

where $K_{\beta}$ is a covariance matrix whose structure depends on the parameter $\beta$, and $\lambda \geq 0$ is a scaling factor. In this context, $K_{\beta}$ is usually called a kernel and determines the properties of the realizations of $g$. In this paper, we choose $K_{\beta}$ from the class of the stable spline kernels [24], [23]. In particular we shall make use of the so-called first-order stable spline kernel (or TC kernel in [12]), defined as

$$
\{K_{\beta}\}_{i,j} := \beta^{\max(i,j)}.
$$

In the above equation, $\beta$ is a scalar in the interval $[0, 1)$ and regulates the decaying velocity of the generated impulse responses.

Let us assume that the noise $v$ is Gaussian (see e.g. [24], [12], [26]). In this case, the joint distribution of the vectors $y$ and $g$, given values of $\lambda$ and $\beta$, is jointly Gaussian. It follows that the posterior of $g$, given $y$ (and values of $\lambda$ and $\beta$) is Gaussian, namely

$$
p(g | y, \lambda, \beta) = \mathcal{N}(Cy, P),
$$

where

$$
P = (U^T \Sigma_v^{-1} U + (\lambda K_{\beta})^{-1})^{-1}
$$

$$
C = PU^T \Sigma_v^{-1}.
$$

In (11), $\Sigma_v$ represents the covariance matrix of the noise vector $v$; in this case we have $\Sigma_v = \sigma^2 I_N$. Equation (10) is instrumental to derive our impulse response estimator, which can be obtained as its minimum mean square error (MSE) (or Bayesian) estimate [2]

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The above equation depends on hyperparameters $\lambda$ and $\beta$. Estimates of these parameters, denoted by $\hat{\lambda}$ and $\hat{\beta}$, can be obtained by exploiting the Bayesian problem formulation. More precisely, since $y$ and $g$ are jointly Gaussian, an efficient method to choose $\hat{\lambda}$ and $\hat{\beta}$ is given by maximization of the marginal likelihood [21], which is obtained by integrating out $g$ from the joint probability density of $(y, g)$. Then we have

$$
(\hat{\lambda}, \hat{\beta}) = \arg \max_{\lambda, \beta} p(y | \lambda, \beta) = \arg \min_{\lambda, \beta} \log \det(\Sigma_y) + y^T \Sigma^{-1} y,
$$

where $\Sigma_y = \lambda UKgU^T + \Sigma_v$ is the variance of the vector $y$.
i.e., $\tau_i$ is distributed as an exponential random variable with parameter $\sigma^2$.

(2) When $v_i$ is modeled using the Student’s $t$-distribution, it follows that

$$
\pi(\tau_i) = \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu}{\pi \tau_i^2}\right)^{\frac{\nu+1}{2}} e^{-\frac{\nu}{2\tau_i^2}}, \quad \tau_i > 0,
$$

which is the probability density of an Inverse Gamma of parameters $\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$.

Independently of the noise model, if the value of $\tau_i$ is given, one has

$$
p(v_i | \tau_i) = \frac{1}{\sqrt{2\pi\tau_i}} e^{-\frac{v_i^2}{2\tau_i}},
$$

which means that the distribution of the noise samples given their variance is Gaussian.

Let

$$
\theta := [\lambda, \beta, \tau_1, \ldots, \tau_N] \in \mathbb{R}^{N+2}.
$$

Due to the representation of noise as scale mixture of Gaussians, the joint distribution of the vectors $y$ and $g$, given values of $\theta$, is jointly Gaussian. Furthermore, by rewriting the noise covariance matrix as follows

$$
\Sigma_y = \text{diag}\{\tau_1, \ldots, \tau_N\},
$$

Equations (10) and (11) hold, so that our impulse response estimate can still be written as the Bayesian estimate of $g$, i.e.

$$
\hat{g} = \mathbb{E}[g|y, \theta] = Cy.
$$

Note that this estimator, compared to the estimator in the Gaussian noise case (12), depends on the $(N+2)$-dimensional vector $\theta$, which we shall call the hyperparameter vector. Hence, our Bayesian system identification algorithm consists of the following steps.

(1) Compute an estimate of the hyperparameter vector $\theta$.

(2) Obtain $\hat{g}$ by computing (21).

In the remainder of the paper, we shall discuss how to efficiently compute the first step of the algorithm.

4 MAP estimate of hyperparameters via Expectation Maximization

4.1 MAP estimation of the hyperparameters

In the previous section we have seen that the joint description of the output and the impulse response is parameterized by the vector $\theta$. In this section, we discuss how to estimate it from data. First, we give a Bayesian interpretation to the constraints on the kernel hyperparameters. To this end, let us denote by $\chi_S(\cdot)$ the indicator function with support $S$ and introduce the following notation

$$
p(\lambda) \propto \chi_{\mathbb{R}^+}(\lambda),
$$

which represents a flat (improper) prior accounting of the positivity of the scaling factor $\lambda$. Similarly, we define

$$
p(\beta) = \chi_{[0,1]}(\beta),
$$

according to the constraint $\beta \in [0, 1)$. Hyperparameters $\lambda, \beta$ are then assumed independent of each other and of all the variances $\tau_i$.

A natural approach to choose the hyperparameter vector $\theta$ is given by its maximum a posteriori (MAP) estimate, which is obtained by solving

$$
\hat{\theta} = \arg\max_{\theta} \log (p(y|\theta)p(\theta)),
$$

where $p(\theta)$ is the prior distribution of the hyperparameter vector. In view of the stated assumptions, recalling also that all the $\tau_i$ are independent identically distributed, we have

$$
p(\theta) = p(\lambda)p(\beta) \prod_{i=1}^{N} \pi(\tau_i).
$$

Solving (24) in that form can be hard, because it is a nonlinear and non-convex problem involving $N+2$ decision variables. For this reason, we propose an iterative solution scheme using the EM method. To this end, we introduce the complete likelihood

$$
\log \left( p(y, g|\theta)p(\theta) \right)
$$

the solution to (24) is obtained by iteratively marginalizing over $g$, which plays the role of missing data or latent variable. Each iteration of the EM scheme provides an estimate of the hyperparameter vector, which we denote by $\hat{\theta}^{(k)}$ (where $k$ indicates the iteration index). The main advantage of employing the EM method is that convergence to a (local or global) maximum of the objective function is guaranteed [19], [37].

4.2 A posteriori total residual and differential impulse response energy

As it will be seen in the following subsection, our EM scheme for robust system identification procedure relies on two key quantities which will be defined below and called a posteriori total residual energy and a posteriori differential impulse response energy.

Assume that, at iteration $k+1$ of the EM scheme, the
estimate \( \hat{\theta}^{(k)} \) of \( \theta \) is available. Using the current estimate of the hyperparameter vector, we construct the matrices \( \hat{C}^{(k)} \) and \( \hat{P}^{(k)} \) using (11) and, accordingly, we denote by \( \hat{g}^{(k)} \) the estimate of \( g \) computed using (21), i.e. \( \hat{g}^{(k)} = \hat{C}^{(k)} y \). The linear predictor of \( y \) is

\[
\hat{y}^{(k)} = U \hat{C}^{(k)} y ,
\]

with covariance matrix

\[
\hat{S}^{(k)} = U \hat{P}^{(k)} U^T .
\]

Then, we define the a posteriori total residual energy at time instant \( t \) as

\[
\hat{\varepsilon}_t^{(k)} := (y_t - \hat{y}_t)^2 + \hat{s}_{tt}^{(k)} ,
\]

where \( \hat{s}_{tt}^{(k)} \) is the \( t \)-th diagonal element of \( \hat{S}^{(k)} \). Note that \( \hat{\varepsilon}_t^{(k)} \) is the sum of a component related to the adherence to data and a term accounting of the model uncertainty (represented by \( \hat{S}^{(k)} \)).

Now, let

\[
\Delta := \begin{bmatrix} 1 & -1 & 0 \\ 0 & \ddots & 1 \\ 1 & \ddots & -1 \\ 0 & \cdots & 1 \end{bmatrix} ,
\]

and define

\[
\hat{v}^{(k)} := \Delta \hat{g}^{(k)}
\]

and

\[
\hat{H}^{(k)} := \Delta \hat{P}^{(k)} \Delta^T .
\]

Note that \( \Delta \) acts as a discrete derivator, so that \( \hat{v}^{(k)} \) represents the estimate of the discrete derivative of the impulse response \( g \) (using the current hyperparameter vector \( \hat{\theta}^{(k)} \)). The matrix \( \hat{H}^{(k)} \) is its a posteriori covariance. Then, denoting by \( \hat{h}^{(k)} \) the \( i \)-th diagonal element of \( \hat{H}^{(k)} \), we define the a posteriori differential impulse response energy at \( i \) as

\[
\hat{d}_i^{(k)} := (\hat{v}_i^{(k)})^2 + \hat{h}^{(k)} .
\]

Note that this quantity is the sum of the energy of the estimated impulse response derivative and a term accounting for its uncertainty.

### 4.3 Robust EM kernel-based system identification procedure

The following theorem states how to solve (24) using the EM method.

**Theorem 2** Let \( \hat{\theta}^{(k)} \) be the estimate of the hyperparameter vector at the \( k \)-th iteration of the EM method, employed to solve (24). Then, the estimate \( \hat{\theta}^{(k+1)} \) is obtained with the following update rules:

- Depending on the noise model, for any \( \tau_t, t = 1, \ldots, N \) we have
  1. In the case of Laplacian distribution,
     \[
     \hat{\varepsilon}_t^{(k+1)} = \frac{\sigma^2}{4} \left( 1 + \frac{8\hat{\varepsilon}_t^{(k)}}{\sigma^2} - 1 \right) ;
     \]
  2. In the case of Student’s t-distribution,
     \[
     \hat{\varepsilon}_t^{(k+1)} = \frac{\hat{\varepsilon}_t^{(k)} + (\nu - 2)\sigma^2}{\nu + 3} ;
     \]

- The hyperparameter \( \beta \) is obtained solving
  \[
  \hat{\beta}^{(k+1)} = \arg \min_{\beta \in (0,1)} Q(\beta) ,
  \]
  where
  \[
  Q(\beta) := n \log f(\beta) + \frac{n(n - 1)}{2} \log \beta - \log(1 - \beta) ,
  \]
  and
  \[
  f(\beta) := \sum_{i=1}^{n-1} d_i^{(k)} \beta^{-i} + d_n^{(k)} (1 - \beta)^{1-n} ;
  \]

- The hyperparameter \( \lambda \) is obtained computing
  \[
  \hat{\lambda}^{(k+1)} = \frac{1}{n} \sum_{i=1}^{n} d_i^{(k)} w_\beta^{(k+1),i}
  \]
  where
  \[
  w_\beta := \frac{1}{\beta - \beta^2} \left[ 1 \ 1 \frac{1}{\beta} \ldots \frac{1}{\beta^{n-1}} \right] ^T
  \]
  and \( w_\beta^{(k+1),i} \) are the elements of \( w_\beta \) when \( \beta = \hat{\beta}^{(k+1)} \).

The result of the above theorem is remarkable. It establishes that, if one employs the EM method to solve (24), at each iteration of the EM the estimate of the hyperparameter vector \( \theta \) can be obtained by solving a sequence of simple scalar optimization problems. All of them crucially depend on the a posteriori total residual and differential impulse response energy, as defined in (29) and (33). Furthermore, the update of any \( \tau \) admits a closed-form expression which depends on the adopted noise model. As for the kernel hyperparameters, \( \beta \) needs to be updated by solving a problem which, at least in
principle, does not admit a closed-form solution. However, the objective function (37) can be evaluated using only “simple” operations and, since $\beta$ is constrained into the interval $[0, 1]$, its minimum can be quickly achieved using a grid search. Once $\beta$ is chosen, the updated value of $\lambda$ is available in closed-form.

Algorithm 1 summarizes our outlier robust system identification algorithm. Here, the initial value of the hyperparameter vector $\hat{\theta}^{(0)}$ can be set to

$$\hat{\theta}^{(0)} = \begin{bmatrix} \hat{\lambda}_{ML} & \hat{\beta}_{ML} & \sigma^2 & \ldots & \sigma^2 \end{bmatrix},$$

(41)

where $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$ are obtained from (13), i.e. by maximizing the marginal likelihood related to the Gaussian noise case, while the choice $\sigma^2$ is motivated by the fact that $E[\tau_i] = \sigma^2$ in both the Laplacian and Student’s t noise models.

Algorithm 1: EM-based outlier robust system identification

Input: $\{y(t)\}_{i=1}^{N}$, $\{u(t)\}_{i=1}^{N}$  
Output: $\{\hat{y}\}_{i=1}^{N}$

1. Initialization: Set $\hat{\theta}^{(0)}$;
2. Repeat until $\frac{||\hat{\theta}^{(k+1)} - \hat{\theta}^{(k)}||}{||\hat{\theta}^{(k)}||}$ is below a given threshold or a prescribed number of iterations is reached:
   a. Update the a posteriori total residual $\hat{\varepsilon}_t^{(k)}$ and the differential impulse response energy $\hat{d}_t^{(k)}$ according to (29) and (33), respectively;
   b. Update $\hat{\tau}_t^{(k+1)}$ using (34) or (35);
   c. Update $\hat{\beta}^{(k+1)}$ solving (36);
   d. Update $\hat{\lambda}^{(k+1)}$ computing (39);
3. Compute $\hat{y}$ as in (21).

Remark 3 Note that the iteration complexity depends also on the number of operations required to update the a posteriori total residual $\hat{\varepsilon}_t^{(k)}$ and the differential impulse response energy $\hat{d}_t^{(k)}$. In view of the definitions (29) and (33), this is related to the computation of the posterior covariance $P^{(k)}$ which, recalling (11), requires the inversion of a $n \times n$ matrix. Since $n$ is the number of unknown impulse response coefficients, this result is also computational appealing since, in system identification, typically one has $n \ll N$, where $N$ is the data set size.

Remark 4 Regarding the noise model induced by the Student’s t distribution, when the parameter $\nu$ is set to 2, the prior (17) becomes flat, i.e. $\pi(\tau_i) \propto \chi^2(\tau_i)$, $t = 1, \ldots, N$. Hence, the update rule (35) becomes

$$\hat{\tau}_t^{(k+1)} = \hat{\tau}_t^{(k)}$$

(42)

i.e., no information on the noise variance is used and the values of the $\tau_t$ are completely estimated from the data. This is in accordance with the fact that the Student’s t-distribution has infinite variance for $\nu = 2$, which means that the prior on noise is not carrying information (from the second order moments point of view). Conversely, when $\nu = +\infty$ Equation (35) becomes

$$\hat{\tau}_t^{(k+1)} = \sigma^2,$$

(43)

which means that all the noise samples must have the same variance equal $\sigma^2$. This reflects the fact that a Student’s t-distribution with $\nu = +\infty$ is in fact a Gaussian distribution (in this case with variance equal to $\sigma^2$), so that no outliers are expected by this noise model.

4.4 Reducing the number of hyperparameters in the identification process

As seen in Section 3, the unknown vector $\theta$ contains $N + 2$ hyperparameters to be estimated. If the data set size is large, e.g. $N \sim 10^4 - 10^5$, even if all of these updates are decoupled and consist of one simple operation, it might be desirable to have an even faster identification process. This could be obtained by reducing the size of $\theta$, that is, by constraining groups of $\tau_t$ to assume the same value. Given an integer $p$, let us assume that $m := N/p$ is integer. Then, it is possible to readapt Theorem 2 in order to impose the constraint

$$\Upsilon_1 := \tau_1 = \ldots = \tau_m,$$

$$\Upsilon_2 := \tau_{m+1} = \ldots = \tau_{2m},$$

$$\vdots$$

$$\Upsilon_p := \tau_{(p-1)m+1} = \ldots = \tau_N,$$

(44)

so that the new hyperparameter vector

$$\theta = \begin{bmatrix} \lambda & \beta & \Upsilon_1 & \ldots & \Upsilon_p \end{bmatrix} \in \mathbb{R}^{p+2}$$

consists in $p + 2$ components (with possibly $p \ll N$).

To this aim, let us introduce the following partition of the matrix $\hat{S}^{(k)}$ introduced in (28)

$$\hat{S}^{(k)} = \begin{bmatrix} \hat{S}_{11}^{(k)} & \ldots & \hat{S}_{1p}^{(k)} \\ \vdots & \ddots & \vdots \\ \hat{S}_{p1}^{(k)} & \ldots & \hat{S}_{pp}^{(k)} \end{bmatrix}, \hat{S}_{ij}^{(k)} \in \mathbb{R}^{m \times m}$$

(45)
and, similarly,

\[ y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}, \quad \hat{y}^{(k)} = \begin{bmatrix} \hat{Y}_1^{(k)} \\ \vdots \\ \hat{Y}_p^{(k)} \end{bmatrix}, \quad Y_i, \hat{Y}_i^{(k)} \in \mathbb{R}^m. \]

The following result then holds.

**Proposition 5**  Let \( \hat{\lambda}^{(k)} \) be the estimate of the hyperparameter vector at the \( k \)-th iteration of the EM method. Define

\[ \tilde{\zeta}_i^{(k)} := \|Y_i - \hat{Y}_i^{(k)}\|^2 + \text{tr}\{\hat{S}_{ii}^{(k)}\}, \quad i = 1, \ldots, p. \]

Then, depending on the noise model adopted, the new estimates \( \hat{\Upsilon}_i^{(k+1)}, \ i = 1, \ldots, p \) are obtained with the following update rules:

1. In the case of Laplacian distribution,

\[ \hat{\Upsilon}_i^{(k+1)} = \frac{m\sigma^2}{4} \left( 1 + \frac{8\tilde{\zeta}_i^{(k)}}{m^2\sigma^2} - 1 \right); \quad (48) \]

2. In the case of Student’s t-distribution,

\[ \hat{\Upsilon}_i^{(k+1)} = \frac{\tilde{\zeta}_i^{(k)} + (\nu - 2)\sigma^2}{\nu + 2 + m}. \quad (49) \]

Note that the update rules for \( \hat{\lambda}^{(k+1)} \) and \( \hat{\beta}^{(k+1)} \) remain the same as in Theorem 2. The value of the \( \hat{\Upsilon}_i \) can be interpreted as an averaging among the \( \tau_i \) “sharing” the same \( \Upsilon_i \). Clearly, the price to pay for reducing the number of hyperparameters is a lower sensitivity of the algorithm to outliers. Furthermore, there is a shrinkage effect that increases as \( p \) decreases.

**Remark 6**  In the limit case \( p = 1 \), i.e. \( m = N \), it is reasonable to expect that, at least for large indices \( k \), \( \tilde{\zeta}_i^{(k)} \approx N\sigma^2 \). Then, if \( N \) is large, from (48) we get

\[ \hat{\Upsilon}_i^{(k+1)} \approx \frac{N\sigma^2}{4} \left( 1 + \frac{8\tilde{\zeta}_i^{(k)}}{N} - 1 \right) \approx \frac{N\sigma^2}{4} \frac{4}{N} = \sigma^2, \quad (50) \]

whereas (49) can be approximated as

\[ \hat{\Upsilon}_i^{(k+1)} \approx \frac{N\sigma^2 + (\nu - 2)\sigma^2}{\nu + 2 + N} \approx \sigma^2. \quad (51) \]

Hence, when \( p = 1 \), i.e. when all the \( \tau_i \) are forced to converge to the nominal noise variance \( \sigma^2 \) and thus the algorithm will behave as in the Gaussian noise case.

## 5 Numerical examples

### 5.1 Monte Carlo studies in presence of outliers

We perform Monte Carlo simulations to assess the performance of the proposed method. Specifically, we set up two groups of numerical experiments of 100 independent runs each. At each Monte Carlo run, we generate random dynamic systems of order 30 as detailed in Section 7.2 of [26]. In order to simulate the presence of outliers in the measurement process, the noise samples \( v(t) \) are drawn from a mixture of two Gaussians, i.e.

\[ v(t) \sim c_1 N(0, \sigma^2) + c_2 N(0, 100\sigma^2), \quad (52) \]

with \( c_1 = 0.7 \) and \( c_2 = 0.3 \). In this way, outliers are measurements with 100 times higher variance, generated with probability 0.3. The value of \( \sigma^2 \) is such that the noise has variance equal to 0.01 the variance of the noiseless output. Two different types of input signals are considered:

1. \( u(t) \) is white noise;
2. \( u(t) \) is obtained by filtering a white noise sequence through a second-order low pass filter randomly generated as described in Section 7.2 of [26].

Random trajectories of input and noise are generated at each run. The data size is \( N = 200 \), while the number of samples of the impulse response to be estimated is \( n = 50 \). The performance of an estimator \( \hat{g} \) is evaluated at any run by computing the fitting score, i.e.

\[ FIT_i = 1 - \frac{\|g_i - \hat{g}_i\|_2}{\|g_i\|_2}, \quad (53) \]

where \( g_i \) and \( \hat{g}_i \) represent, respectively, the true and the estimated impulse responses (truncated at the \( n \)-th sample) obtained at the \( i \)-th Monte Carlo run. In particular, the following estimators \( \hat{g} \) are tested:

1. EM-L: this is the new kernel-based method proposed in this paper adopting a Laplacian model noise. Algorithm 1 is used to estimate hyperparameters. The EM run stops when \( \|\hat{g}^{(k+1)} - \hat{g}^{(k)}\|_2 < 10^{-3} \).
2. EM-S-\( \nu \): the same as before except that a Student’s t-distribution is used as noise model. To assess which choice of degrees of freedom \( \nu \) gives the best performance, we test values of \( \nu \) in the set \{2, 2.25, 2.5, 2.75, 3, 5\}.
3. SS-ML: this is the kernel-based identification method proposed in [22], revisited in [12] and briefly described in Section 3.1. The impulse response is modeled as in (8) and the hyperparameters \( \lambda \) and \( \beta \) are estimated by marginal likelihood maximization (13). Note that this estimator does not attempt to model the presence of outliers.
(4) SS-ML-Oracle: this is the same method described above except that we choose those values of λ and β which maximizes the fit (53). Note that this kind of estimator knows the true impulse response, hence it is not utilizable in practice.

Figures 3 and 4 show the results of the simulations. It can be seen that, in general, accounting for outliers pays off in terms of accuracy. Both the proposed estimators perform better than the oracle-type non-robust estimator which does not account of the possible presence of outliers. Modeling noise using the Student’s t-distribution seems to give better results in terms of fitting. As for the sensitivity of the results to the degrees of freedom, while the value ν = 2 does not give good results, the values 2.25, 2.5 and 2.75 all perform really well. In particular, ν = 2.25 seems to be optimal (among the chosen grid).

5.2 Monte Carlo studies in presence of outliers: estimation of Student’s t degrees of freedom from data

Since Fig. 3 and 4 show that the parameter ν of the Student’s t distribution influences the performance of the estimator, we have also investigated determination of ν from data. A possible way is to still use the MAP estimator (24): the estimate becomes

\[ \hat{\nu} = \arg \max_{\nu} \log(p(y|\hat{\theta})p(\hat{\theta})) , \]  

(54)

where \( \hat{\theta} \) is the estimated value of the hyperparameter vector.

This approach has been applied to the same data generated for the second experiment (filtered white noise as input). The set of candidate values for ν is the same as above, i.e. \{2, 2.25, 2.5, 2.75, 3, 5\}. The value chosen by the estimator (54) is always \( \hat{\nu} = 2.25 \). Table 1 shows a comparison between the estimator EM-S-\( \nu \), with \( \nu \) chosen by (54) (we call this estimator EM-S-\( \hat{\nu} \)) and the esti-
5.3 Monte Carlo studies in presence of outliers: reduction of the number of hyperparameters

We test the performance of the estimator described in Section 4.4, where the size of the hyperparameter vector is reduced, using the same setting of the first Monte Carlo experiment (white noise as input). We compare, by means of 100 Monte Carlo runs, the estimator SS-ML with a new class of estimators, dubbed EM-L-p. These estimators employ the Laplacian model of noise but, instead of attempting the estimation of $N$ distinct values of the noise variances $\tau_t$, they estimate $p$ values, as described in Section 4.4. In this experiment, we choose 4 different values of $p$, namely $p = 1, 20, 40, 200$, so that the same noise variance is shared by 200, 10, 5 and 1 output measurements, respectively. Note that the case $p = 200$ corresponds to the estimator EM-L described above (since $m = N$), while the case $p = 1$ forces all the noise variances to be the same (this thus corresponds to the Gaussian noise case).

5.4 Monte Carlo studies in absence of outliers

We also test our method in a case where no outliers are present in the data, i.e. when $c_1 = 1$ and $c_2 = 0$ in (52). We assume the input to be white noise and we test the same estimators introduced above. From Figure 6, it can be seen that the performance of the outlier robust methods are slightly worse than the non-robust methods. This is explained by the fact that a non-Gaussian noise model is assumed. However, it can be seen that, in this case, higher values of $\nu$ increase the accuracy of the proposed methods when the Student’s t-distribution is adopted as noise model, due to the fact that the Student’s t-distribution gets closer and closer to a Gaussian as $\nu$ increases.

5.5 Robust solution of the introductory example

We apply the proposed algorithm also to solve the motivating example shown in Section 1.1. Results are visible in Figure 7: in comparison with the estimates reported in Figure 1, the quality of the reconstruction is clearly improved. This can be appreciated also by inspecting Table 2 which reports the fitting scores of the proposed methods, also comparing them with the non-robust identification method SS-ML applied also in absence of outliers. Table 3 shows the values of each of $\tau_t$ corresponding to those time instants where the outliers are added (i.e., $t = 24, 39, 46, 82, 94$), comparing them to the median value of all the estimated $\tau_t$. It is clear that the proposed method is able to identify the abnormal measurements. Note that, as $\nu$ increases, the median value of the $\tau_t$ tends to increase as well, while the variance of the outliers tend to shrink.

6 Conclusions

We have proposed a novel regularized identification scheme robust against outliers. The recently proposed nonparametric kernel-based methods [24], [12] constitute our starting point. These methods use particular kernel matrices, e.g. derived by stable spline kernels, and Gaussian noise assumptions. This can be a limitation if outliers corrupt the output measurements. In this paper, we instead model the noise using heavy-tailed pdf’s, in particular Laplacian and Student’s t. Exploiting their representation as scale mixture of Gaussians, we have shown that joint estimation of noise and kernel hyperparameters can be performed efficiently. In particular, our robust kernel based method for linear system identification relies on EM iterations which are

| Method | Median | Mean |
|--------|--------|------|
| EM-S-$\nu$ | 0.7819 | 0.7461 |
| EM-S-opt | 0.7831 | 0.7484 |

Table 1

Monte Carlo in presence of outliers with degrees of freedom of the Student’s t estimated from data (subsection 5.2, filtered white noise as input): average fit of the estimator EM-S-$\nu$ selecting $\nu$ by (54) and of the estimator which has access at any run to the optimal value of $\nu$. Note that, in accordance with Remark 6, when $p = 1$ the estimator gives exactly the same performance of SS-ML.
Figure 6. **Monte Carlo in absence of outliers (subsection 5.4, white noise as input):** box plots of the fits achieved by different estimators.

| Method   | Median | $t=24$ | $t=39$ | $t=46$ | $t=82$ | $t=94$ |
|----------|--------|--------|--------|--------|--------|--------|
| EM-L     | 8.4 × 10^{-4} | 1.8771 | 2.4853 | 6.5494 | 6.4728 | 3.1748 |
| EM-S-2   | 1.93 × 10^{-4} | 2.7637 | 1.4461 | 16.1567 | 3.9345 | 15.0338 |
| EM-S-2.25| 0.0649 | 2.3916 | 1.3245 | 14.9122 | 3.5927 | 13.9444 |
| EM-S-2.5 | 0.1211 | 2.2679 | 1.3341 | 14.2223 | 3.4646 | 13.4026 |
| EM-S-2.75| 0.1715 | 2.1711 | 1.3371 | 13.6122 | 3.3527 | 12.8957 |
| EM-S-3   | 0.2177 | 2.0894 | 1.3379 | 13.0696 | 3.2520 | 12.4235 |
| EM-S-5   | 0.4801 | 1.7077 | 1.3285 | 10.0031 | 2.6959 | 9.6513 |

Table 3
**Robust solution of the introductory example (subsection 5.5):** capability of detecting outliers variances in the introductory example. The table reports the median values of all the estimates of the variances $\tau_t$ and the single estimates of $\tau_t$ associated to the $t$ where outliers are present.

| Method      | Fit % |
|-------------|-------|
| EM-S-2.25   | 94.24 |
| EM-S-2.5    | 94.09 |
| EM-S-2.75   | 93.24 |
| EM-S-3      | 92.19 |
| SS-ML (no outliers) | 92.13  |
| EM-L        | 90.97 |
| EM-S-5      | 84.93 |
| EM-S-2      | 84.06 |
| SS-ML (with outliers) | 51.23  |

Table 2
**Robust solution of the introductory example (subsection 5.5):** fitting scores of several estimators ranked w.r.t. their performance.

essentially all available in closed form. Numerical results show the effectiveness of the proposed method in contrasting outliers.
A Appendix

A.1 Preliminaries

The starting point of the derivation of our EM scheme is (26). Let us define

\[ L_p(y, g|\theta) := \log p(y, g|\theta), \]  

(A.1)

where \( g \) here is the missing data. For ease of notation, we also define

\[ L_\pi(\tau_i) := \log \pi(\tau_i). \]  

(A.2)

Then, the EM method provides \( \hat{\theta} \) by iterating the following steps:

(E-step) Given an estimate \( \hat{\theta}^{(k)} \), compute

\[ Q(\theta, \hat{\theta}^{(k)}) := E_{p(g|y, \hat{\theta}^{(k)})} \left[ L_p(y, g|\theta) + \sum_{i=1}^{N} L_\pi(\tau_i) \right]; \]  

(A.3)

(M-step) Compute

\[ \hat{\theta}^{(k+1)} = \arg \max_{\theta} Q(\theta, \hat{\theta}^{(k)}). \]  

(A.4)

First, we show how to compute the E-step of the EM scheme, i.e., which form \( Q(\theta, \hat{\theta}^{(k)}) \) assumes in our problem. The computation of the M-step corresponds to the Proof of Theorem 2 and is given in the next section.

Lemma 7 Let \( \hat{C}^{(k)} \) and \( \hat{P}^{(k)} \) be defined as in (11), computed using the vector \( \hat{\theta}^{(k)} \). Then,

\[ Q(\theta, \hat{\theta}^{(k)}) = -\frac{1}{2} \left( y^T M(\theta, \hat{\theta}^{(k)}) y + \log \det \lambda K_\beta \right) \]

\[ + \operatorname{tr} \left\{ (U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1}) \hat{P}^{(k)} \right\} \]

\[ + \sum_{i=1}^{N} \log \tau_i - 2 \sum_{i=1}^{N} L_\pi(\tau_i), \]  

(A.5)

where

\[ M(\theta, \hat{\theta}^{(k)}) := \Sigma_v^{-1} (I - 2U \hat{C}^{(k)}) \]

\[ + \hat{C}^{(k)T} (U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1}) \hat{C}^{(k)}. \]  

(A.6)

Proof: First, note that \( p(y, g|\theta) = p(y|g, \theta)p(g|\theta), \)

where \( p(y|g, \theta) \sim \mathcal{N}(Ug, \Sigma_v) \)  

(A.7)

and \( p(g|\theta) \) is given by (8). Hence

\[ L_p(y, g|\theta) + \sum_{i=1}^{N} L_\pi(\tau_i) = \]  

\[ = \log p(y|g, \theta) + \log p(g|\theta) + \sum_{i=1}^{N} L_\pi(\tau_i) \]

\[ = -\frac{1}{2} \sum_{i=1}^{N} \log \tau_i - \frac{1}{2} (y - U \hat{g})^T \Sigma_v^{-1} (y - U \hat{g}) \]

\[ - \frac{1}{2} \log \det \lambda K_\beta - \frac{1}{2} y^T (\lambda K_\beta)^{-1} y + \sum_{i=1}^{N} L_\pi(\tau_i) \]

\[ = -\frac{1}{2} \sum_{i=1}^{N} \log \tau_i - \frac{1}{2} y^T \Sigma_v^{-1} y + y^T \Sigma_v^{-1} Ug \]

\[ - \frac{1}{2} y^T \left( U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1} \right) \hat{g} \]

\[ - \frac{1}{2} \log \det \lambda K_\beta + \sum_{i=1}^{N} L_\pi(\tau_i) \]

\[ := A + B + C + D + E + F. \]  

(A.9)

Now, we have to take the expectation of the above terms with respect to \( p(g|y, \hat{\theta}^{(k)}) \), which is given by (10). We have the following results (we make use of the symbol \( \operatorname{E}[] \) to denote such an expectation).

\[ \operatorname{E}[A] = -\frac{1}{2} \sum_{i=1}^{N} \log \tau_i \]

\[ \operatorname{E}[B] = -\frac{1}{2} y^T \Sigma_v^{-1} y \]

\[ \operatorname{E}[C] = y^T \Sigma_v^{-1} U \operatorname{E}[\hat{g}] = y^T \Sigma_v^{-1} U \hat{C}^{(k)} y \]

\[ \operatorname{E}[D] = -\frac{1}{2} \operatorname{E} \left[ \operatorname{tr} \left\{ (U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1}) \hat{P}^{(k)} \right\} \right] \]

\[ = -\frac{1}{2} \operatorname{tr} \left\{ (U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1}) \hat{P}^{(k)} \right\} \]

\[ - \frac{1}{2} y^T \hat{C}^{(k)T} (U^T \Sigma_v^{-1} U + (\lambda K_\beta)^{-1}) \hat{C}^{(k)} y \]

\[ \operatorname{E}[E] = -\frac{1}{2} \log \det \lambda K_\beta \]

\[ \operatorname{E}[F] = \sum_{i=1}^{N} L_\pi(\tau_i). \]  

(A.10)

Summing up all those elements we eventually get (A.5).

\[ \square \]

A.2 Proof of Theorem 2

The proof is composed of two part:

1. We rewrite \( Q(\theta, \hat{\theta}^{(k)}) \) in a convenient form;
(2) We compute the vector $\theta$ maximizing $Q(\theta, \hat{\theta}^{(k)})$.

Part (1): We show that the function $Q(\theta, \hat{\theta}^{(k)})$ can be rewritten

$$Q(\theta, \hat{\theta}^{(k)}) = -\frac{1}{2} \left( Q(\lambda, \beta, \hat{\theta}^{(k)}) + \sum_{t=1}^{N} Q_t(\tau_t, \hat{\theta}^{(k)}) \right) + c,$$

where $c$ is a constant,

$$Q_0(\lambda, \beta, \hat{\theta}^{(k)}) := y^T \hat{C}^{(k)T}(\lambda K_\beta)^{-1}\hat{C}^{(k)} y + \log \det \lambda K_\beta + \text{tr} \{ (\lambda K_\beta)^{-1} \hat{P}^{(k)} \},$$

is a function of $\lambda$ and $\beta$ only and, for $t = 1, \ldots, N$,

$$Q_t(\tau_t, \hat{\theta}^{(k)}) := \hat{\varepsilon}^{(k)} \tau_t^{-1} + \log \tau_t - 2 L_\pi(\tau_t),$$

are $t$ distinct functions, each depending only on $\tau_t$, $t = 1, \ldots, N$.

For convenience, we rewrite the matrix in (A.6) as $M(\theta, \hat{\theta}^{(k)}) = M_H(\hat{\theta}^{(k)}) + M_r(\hat{\theta}^{(k)})$, where

$$M_H(\hat{\theta}^{(k)}) := \hat{C}^{(k)T} (\lambda K_\beta)^{-1} \hat{C}^{(k)}$$

(A.14)

does not depend on the $\tau_t$ and, conversely,

$$M_r(\hat{\theta}^{(k)}) := \Sigma^{-1}(I - 2U\hat{C}^{(k)}) + \hat{C}^{(k)T}U^T \Sigma^{-1}U\hat{C}^{(k)}$$

(A.15)

depends only on the $\tau_t$. Rearranging (A.5), it follows that

$$Q(\theta, \hat{\theta}^{(k)}) = -\frac{1}{2} \left( y^T M_H(\hat{\theta}^{(k)}) y + \log \det \lambda K_\beta + \text{tr} \{ (\lambda K_\beta)^{-1} \hat{P}^{(k)} \} + \text{tr} \{ U^T \Sigma^{-1} U \hat{P}^{(k)} \} \right) + \sum_{t=1}^{N} \log \tau_t - 2 \sum_{t=1}^{N} L_\pi(\tau_t).$$

(A.16)

where the first part corresponds to $Q_0(\lambda, \beta, \hat{\theta}^{(k)})$, defined in (A.12).

Now, let $\hat{S}^{(k)}$ be as in (28); then

$$\text{tr} \{ U^T \Sigma^{-1} U \hat{P}^{(k)} \} = \text{tr} \{ \hat{S}^{(k)} \Sigma^{-1} \} = \sum_{t=1}^{N} \hat{s}^{(k)}_{tt} \tau_t^{-1},$$

(A.17)

where $\hat{s}^{(k)}_{tt}$ denotes the element of $\hat{S}^{(k)}$ in position $(t, t)$. Recall also that $\hat{y}^{(k)} = UC\hat{C}^{(k)} y$; thus

$$y^T \hat{C}^{(k)T} U^T \Sigma^{-1} U \hat{C}^{(k)} y = \sum_{t=1}^{N} \hat{y}^{(k)}_t \tau_t^{-1};$$

(A.18)

furthermore, since

$$y^T \Sigma^{-1} y = \sum_{t=1}^{N} y_t^2 \tau_t^{-1}$$

(A.19)

and

$$-2y^T \Sigma^{-1} U \hat{C}^{(k)} y = -2 \sum_{t=1}^{N} \hat{y}^{(k)}_t y_t \tau_t^{-1},$$

(A.20)

it follows that

$$y^T M_r(\hat{\theta}^{(k)}) y = \sum_{t=1}^{N} \left( y_t - \hat{y}^{(k)}_t \right)^2 \tau_t^{-1}.$$  

(A.21)

Recalling that $\hat{\varepsilon}^{(k)} = (y_t - \hat{y}^{(k)}_t)^2 + \hat{s}^{(k)}_t$, (A.13) and (A.11) follow.

Part (2): First, we proceed with the minimization of each $Q_t(\tau_t, \hat{\theta}^{(k)})$, $t = 1, \ldots, N$; then, we show how to deal with (A.12).

Depending on the noise model adopted, $Q_t(\tau_t, \hat{\theta}^{(k)})$, $t = 1, \ldots, N$, has two different forms.

(1) If the noise is modeled with the Laplacian density, from (16) we have $L_\pi(\tau_t) = -\frac{1}{\sigma^2} \tau_t + c$, where $c$ is constant so that, for every $t = 1, \ldots, N$,

$$Q_t(\tau_t, \hat{\theta}^{(k)}) := \hat{\varepsilon}^{(k)} \tau_t^{-1} + \log \tau_t + \frac{2}{\sigma^2} \tau_t,$$  

(A.22)

which is minimized by (34).

(2) If noise is modeled with the Student’s t-distribution, from (17) we have $L_\pi(\tau_t) = -\left( \frac{\nu + 1}{2} \right) \log \tau_t - \frac{\nu \hat{\tau}^2_t}{2} + c$, where $c$ is constant so that, for every $t = 1, \ldots, N$,

$$Q_t(\tau_t, \hat{\theta}^{(k)}) := \left( \hat{\varepsilon}^{(k)} + \nu \rho \right) \tau_t^{-1} + (\nu + 3) \log \tau_t.$$  

(A.23)

Recalling the relation between $\rho$ and $\sigma^2$ given by (7), the minimizer (35) easily follows.

We now deal with $Q_0(\lambda, \beta, \hat{\theta}^{(k)})$. Its derivative with respect to $\lambda$ is

$$\frac{\partial Q_0}{\partial \lambda} = -\frac{1}{\lambda^2} \left( y^T \hat{C}^{(k)T} K_\beta^{-1} \hat{C}^{(k)} y + \text{tr} \{ K_\beta^{-1} \hat{P}^{(k)} \} \right) + \frac{n}{\lambda},$$

(A.24)

which is equal to zero for

$$\lambda^* = \frac{1}{n} \left( y^T \hat{C}^{(k)T} K_\beta^{-1} \hat{C}^{(k)} y + \text{tr} \{ K_\beta^{-1} \hat{P}^{(k)} \} \right).$$

(A.25)
Plugging back such value into \( Q_0(\lambda, \beta, \hat{\theta}^{(k)}) \), one obtains
\[
Q_0(\lambda^*, \beta, \hat{\theta}^{(k)}) = n \log \left( \left( y^T \hat{C}_i^{(k)} T K_{\beta}^{-1} \hat{C}_i \right)^T y \right) + \text{tr} \left\{ K_{\beta}^{-1} \hat{P}^{(k)} \right\} + \log \det K_{\beta} + c, \tag{A.26}
\]
where \( c \) is a constant. Now consider the following factorization of the first order stable kernel \([9]\)
\[
K_{\beta} = \Delta^{-1} W_{\beta} \Delta^{-T}, \tag{A.27}
\]
where \( \Delta \) is defined in (30) and
\[
W_{\beta} := (\beta - \beta^2) \text{diag} \left\{ 1, \beta, \ldots, \beta^{n-2}, \frac{\beta^{n-1}}{1 - \beta} \right\}. \tag{A.28}
\]
Note that the nonzero elements of \( W_{\beta} \) correspond to the inverse of those of (40). According to (31) and (32), let \( \hat{v}^{(k)} := \Delta \hat{C}_i \) and \( \hat{h}^{(k)} := \Delta \hat{C}_i T_{\beta} \Delta^{-T} \). Then it can be seen that
\[
Q_0(\lambda^*, \beta, \hat{\theta}^{(k)}) = n \log \left( \sum_{i=1}^n \left( \hat{h}^{(k)}_{ii} + \hat{v}^{(k)2}_{ii} \right) w_{\beta,ii}^{-1} \right)
+ \sum_{i=1}^n \log w_{\beta,ii} + c, \tag{A.29}
\]
where \( \hat{h}^{(k)}_{ii} \) and \( w_{\beta,ii} \) are the \( i \)-th diagonal elements of \( \hat{H}^{(k)} \) and \( W_{\beta} \), respectively, while \( \hat{v}^{(k)} \) is the \( i \)-th entry of \( \hat{v}^{(k)} \). A further rewriting of (29) yields
\[
Q(\beta) := Q_0(\lambda^*, \beta, \hat{\theta}^{(k)}) = n \log f(\beta) + \frac{n(n-1)}{2} \log \beta
- \log(1 - \beta) + c, \tag{A.30}
\]
where
\[
f(\beta) = \sum_{i=1}^{n-1} \left( \hat{h}^{(k)}_{ii} + \hat{v}^{(k)2}_{ii} \right) \beta^{-i} + (\hat{h}^{(k)}_{nn} + \hat{v}^{(k)2}_{nn})(1 - \beta)^{1-n}, \tag{A.31}
\]
so that (37) and (38) are obtained. Using similar arguments, one can see that (A.25) can be rewritten as (39).

### A.3 Proof of Proposition 5

Due to (44), the noise covariance matrix \( \Sigma_v \) has the form
\[
\Sigma_v = \text{diag} \{ \Upsilon_1 I_m, \ldots, \Upsilon_p I_m \}, \tag{A.32}
\]
and the prior on the hyperparameter vector is
\[
p(\theta) = p(\lambda) p(\beta) \prod_{i=1}^p \pi(\Upsilon_i). \tag{A.33}
\]
Consequently, the function \( Q(\theta, \hat{\theta}^{(k)}) \), as compared to (A.16), has to be rewritten as follows
\[
Q(\theta, \hat{\theta}^{(k)}) = -\frac{1}{2} \left( Q_0(\lambda, \beta, \hat{\theta}^{(k)}) + \text{tr} \left\{ U^T \Sigma_v^{-1} U \hat{P}^{(k)} \right\} + y^T M_r(\hat{\theta}^{(k)}) y + \sum_{i=1}^p \log \tau_i - 2 \sum_{i=1}^p L_\pi(\tau_i) \right), \tag{A.34}
\]
where \( Q_0(\lambda, \beta, \hat{\theta}^{(k)}) \) does not depend on the \( \Upsilon_i \) and is minimized as in Theorem 2.

Now recall the partitioning of \( \hat{S}^{(k)} \) and \( y^{(k)} \) introduced in (45) and (46); then
\[
\text{tr} \left\{ U^T \Sigma_v^{-1} U \hat{P}^{(k)} \right\} = \text{tr} \left\{ \hat{S}^{(k)} \Sigma_v^{-1} \right\} = \sum_{i=1}^p \hat{\zeta}^{(k)}_{ii} \Upsilon_i^{-1}, \tag{A.35}
\]
\[
y^T \hat{C}_i^{(k)T} U^T \Sigma_v^{-1} U \hat{C}_i^{(k)} y = \sum_{i=1}^p \| \hat{Y}_i^{(k)} \|^2 \Upsilon_i^{-1}, \tag{A.36}
\]
\[
y^T \Sigma_v^{-1} y = \sum_{i=1}^p \| \hat{Y}_i \|^2 \Upsilon_i^{-1}, \tag{A.37}
\]
\[
-2y^T \Sigma_v^{-1} U \hat{C}_i^{(k)} y = -2 \sum_{i=1}^p \hat{Y}_i^{(k)T} \Upsilon_i^{-1} \hat{Y}_i^{(k)} \Upsilon_i^{-1}. \tag{A.38}
\]
Thus, it follows that
\[
y^T M_r(\hat{\theta}^{(k)}) y = \sum_{i=1}^p \| \hat{Y}_i - \hat{Y}_i^{(k)} \|^2 \Upsilon_i^{-1}. \tag{A.39}
\]
and similarly to Theorem 2,
\[
Q(\theta, \hat{\theta}^{(k)}) = -\frac{1}{2} \left( Q_0(\lambda, \beta, \hat{\theta}^{(k)}) + \sum_{i=1}^p Q_i(\Upsilon_i, \hat{\theta}^{(k)}) \right). \tag{A.40}
\]
Depending on the noise model adopted, \( Q_i(\Upsilon_i, \hat{\theta}^{(k)}) \), \( i = 1, \ldots, p \), has two different forms.

1. If the noise is modeled with the Laplacian density, from (16) we have \( L_\pi(\Upsilon_i) = -\frac{1}{\sigma^2} \Upsilon_i + c \), where \( c \) is constant so that, for every \( i = 1, \ldots, p \)
\[
Q_i(\Upsilon_i, \hat{\theta}^{(k)}) := \hat{\zeta}^{(k)}_{ii} \Upsilon_i^{-1} + m \log \tau_i + \frac{2 \sigma^2}{\sigma^2} \Upsilon_i, \tag{A.41}
\]
where \( \hat{\zeta}_i \) is defined in (47). The minimizer corresponds to (48).

2. If noise is modeled with the Student’s t-distribution, from (17) we have \( L_\pi(\Upsilon_i) = -\left( \frac{\nu}{\nu + 1} \right) \log \Upsilon_i - \frac{\nu}{\nu + 1} \Upsilon_i + c \), where \( c \) is constant so that, for every
\[ i = 1, \ldots, p \]
\[
Q_i(\hat{Y}_i, \hat{\theta}(k)) := \left( \hat{\zeta}_i(k) + \nu \right) \hat{Y}_i^{-1} + \left( \nu + 2 + m \right) \log \hat{Y}_i, \tag{A.42}
\]
which is minimized by (49).

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