An example concerning the Theory of Levels for codimension-one foliations

Andrés Navas*

An important aspect of foliations concerns the existence of local minimal sets. Recall that a foliated manifold has the LMS property if, for every open, saturated set $W$ and every leaf $L \subset W$, the relative closure $\bar{L} \cap W$ contains a minimal set of $F|_W$. A fundamental result (due to Cantwell-Conlon [2] and Duminy-Hector [5]) establishes the LMS property for codimension-one foliations that are transversely of class $C^{1+\text{Lipschitz}}$. This is the basic tool of the so-called Theory of Levels.

A classical example due to Hector (which corresponds to the suspension of a group action on the interval) shows that the LMS property is no longer true for codimension-one foliations which transversely are only continuous (see [1, Example 8.1.13]). Despite of this, in recent years, the possibility of extending some of the results of the Theory of Levels to smoothness smaller than $C^{1+\text{Lipschitz}}$ has been naturally addressed [3, 4]. In this Note we will show that, however, analogues of Hector’s example appear in class $C^1$ (and actually in class $C^{1+\alpha}$ for small values of $\alpha$).

1 A General Construction

Let $(a_n)_{n \in \mathbb{Z}}$ be a sequence such that $a_{n+1} < a_n$ for all $n \in \mathbb{Z}$, $a_n \to 0$ as $n \to \infty$, and $a_n \to 1$ as $n \to -\infty$. Let $(n_k)$ be a strictly increasing sequence of positive integers, and let $f: [0, 1] \to [0, 1]$ be a homeomorphism such that $f(a_{n+1}) = a_n$ for all $n \in \mathbb{Z}$. For each $k > 0$, we let $u_k, v_k, b_k, c_k$ be such that $a_{n_k+1} < b_k < u_k < v_k < c_k < a_{n_k}$. For each $i \in \{0, \ldots, n_{k+1} - n_k\}$, we set $u_k^i := f^i(u_k)$ and $v_k^i := f^i(v_k)$. Notice that

$f^i([u_k^{0}, v_k^{0}]) = [u_k^{i}, v_k^{i}] \subset f^i([a_{1+n_{k+1}}, a_{n_k+1}]) = [a_{n_k-1+i}, a_{n_{k+1}-i}]$.

Now, we let $g: [0, 1] \to [0, 1]$ be a homeomorphism such that:
- $g = \text{Id}$ on $[a_{n+1}, a_n]$ for each $n < 0$, as well as each $n > 0$ such that $n \neq n_k$ for every $k$;
- $g = \text{Id}$ on $[a_{1+n_k}, b_k] \cup [c_k, a_{n_k}]$, $g(u_k^{0}) = v_k^{0}$, and $g$ has no fixed point on $]b_k, c_k[.$

Main assumption: In order that $f, g$ generate a group of homeomorphisms of $[0, 1]$ whose associated suspension does not have the LMS property, we assume that (see Figure 1)

\[ u_k^{n_{k+1}-n_k} = b_k \quad \text{and} \quad v_k^{n_{k+1}-n_k} = c_k. \]

With these general notations, Hector’s example corresponds to the choice $n_k = k$. We will show that, by taking $n_k = 2^k$, one may perform this construction in such a way the

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resulting maps \( f \) and \( g \) are diffeomorphisms of class \( C^1 \) (actually, of class \( C^{1+\alpha} \) for any \( \alpha < (\sqrt{5} - 1)/2 \)). It is quite possible that slightly improving our method, one can smooth the action up to the class \( C^{2-\delta} \) for any \( \delta > 0 \). (Compare [7], where for a similar construction, T. Tsuboi deals with the \( C^{3/2-\delta} \) case before the \( C^{2-\delta} \) case due to technical difficulties.)

![Figure 1](image)

2 The length of the intervals and some estimates

We let \( \|u_{k+1}, v_{k+1}\| := \lambda_k \|u_{k+1}, v_{k+1}\| \), where the constant \( \lambda_k > 1 \) satisfies the compatibility relation

\[
\lambda_k^2 = \frac{\|u_{k+1}, v_{k+1}\|}{\|u_{k+1}, v_{k+1}\|} = \frac{\|b_k, c_k\|}{\|a_{k+1}, v_{k+1}\|}.
\] (1)

Let \( \varepsilon > 0 \) be very small (to be fixed in a while). We set:

- \( \|a_{n+1}, a_n\| := \frac{\varepsilon}{(1+|a_k|)k^{2\eta}} \), where \( c_k \) is chosen so that \( \sum_{n\in\mathbb{Z}} \|a_{n+1}, a_n\| = 1 \);
- \( \|b_k, c_k\| := \frac{1}{2} \|a_{2k+1}, a_{2k}\| = \frac{\varepsilon}{2(1+2^{k+\eta})} \), where \( k > 0 \);
- \( \|u_k, v_k\| := \|b_k, c_k\|^{1+\eta} \).

We assume that the center of \( [a_{2k+1}, a_{2k}] \) coincides with the center of \( [b_k, c_k] \) and with that of \( [u_k, v_k] \). Furthermore, we assume that for each \( i \in \{0, \ldots, 2^k\} \), the centers of \( [u_{k+1}, v_{k+1}] \) and \( [a_{2k+1-i}, a_{2k+1-i}] \) coincide.

For the estimates concerning regularity, we will strongly use the following lemma from [6].

**Technical Lemma.** Let \( \omega : [0, \eta] \to [0, \omega(\eta)] \) be a function (modulus of continuity) such that \( s \mapsto s/\omega(s) \) is non-increasing. If \( I, J \) are closed non-degenerate intervals such that \( 1/2 \leq |I|/|J| \leq 2 \) and

\[
\left| \frac{|J|}{|I|} - 1 \right| \leq \frac{1}{\omega(|I|)} \leq M,
\]

then there exists a \( C^{1+\omega} \) diffeomorphism \( f : I \to J \) that is tangent to the identity at the endpoints and whose derivative has \( \omega \)-norm bounded from above by \( 6\pi M \).

Actually, for \( I := [a, b] \) and \( J := [a', b'] \), one may take \( f = \varphi_{a', b'}^{-1} \circ \varphi_{a, b} \), where \( \varphi_{a,b} \) is defined
Therefore, \( \varphi_{a,b}(x) = \frac{1}{b-a} \cotg \left( \frac{x}{b-a} \right) \).

The condition on the derivative at the endpoints allows us to fit together the maps in order to create a diffeomorphism of a larger interval. Actually, if all of the involved sub-intervals of type \( I, J \) satisfy the hypothesis of the lemma above for the same constant \( M \), then the \( \omega \)-norm of the derivative of the induced diffeomorphism is bounded from above by \( 12\pi M \).

In what follows, we will deal with the modulus of continuity \( \omega(s) = s^\alpha \) for the derivative, where \( \alpha > 0 \). A constant depending on the three parameters \( \alpha, \theta, \varepsilon \), and whose value is irrelevant for our purposes, will be generically denoted by \( M \).

**Estimates for \( f \):** The diffeomorphism \( f \) is constructed by fitting together the maps provided by the Technical Lemma sending (see Figure 2):

(i) \([u_{k+1}^i, v_{k+1}^i]\) into \([u_{k+1}^{i+1}, v_{k+1}^{i+1}]\),

(ii) \([a_{2k+1-i}, u_{k+1}^i]\) into \([a_{2k+1-i-1}, u_{k+1}^{i+1}]\),

(iii) \([v_{k+1}, a_{2k+1-i-2}]\) into \([v_{k+1}^{i+1}, a_{2k+1-i-1}]\).

For (i), we have

\[
\left| \frac{[u_{k+1}^{i+1}, v_{k+1}^{i+1}]}{[u_{k+1}^i, v_{k+1}^i]} - 1 \right| \frac{1}{[u_{k+1}^i, v_{k+1}^i]} = \left| \lambda_k - 1 \right| \frac{1}{(\lambda_k [u_{k+1}^i, v_{k+1}^i])^\alpha} \leq \frac{1}{[b_{k+1}, c_{k+1}]^{(1+\theta)\alpha}}.
\]

Now from (1) one obtains

\[
\lambda_k^{2k} = \frac{2^{(1+2^k+\varepsilon)}}{(2(1+2^k+\varepsilon)+\varepsilon)} \leq M \left( \frac{1 + 2^{k+1} \varepsilon}{1 + 2^k} \right)^{1+\varepsilon} \leq M 2^k (1+\varepsilon).
\]

From the inequality \( |2^\alpha - 1| \leq \alpha \) (which holds for \( \alpha \) positive and small) one concludes that

\[
|\lambda_k - 1| \leq M \frac{k}{2^k}.
\]

On the other hand,

\[
\frac{1}{[b_{k+1}, c_{k+1}]} \leq M (1 + 2^{k+1})^{1+\varepsilon} \leq M 2^{k(1+\varepsilon)}.
\]

Therefore,

\[
\left| \frac{[u_{k+1}^{i+1}, v_{k+1}^{i+1}]}{[u_{k+1}^i, v_{k+1}^i]} - 1 \right| \frac{1}{[u_{k+1}^i, v_{k+1}^i]} \leq M \frac{k}{2^k} 2^{k(1+\varepsilon)(1+\theta)\alpha}.
\]
Now, for (ii), set \( A := [(a_{k+1}^i, v_{k+1}^i)], B := [(a_{2k+1-i, a_{2k+1-i}}), C := [(u_{k+1}^i, v_{k+1}^i)], and D := [(a_{2k+1-i, a_{2k+1-i}})]. Then

\[
\left| \frac{[a_{2k+1-i, t}^i, u_{k+1}^i]}{[a_{2k+1-i, t}^i, u_{k+1}^i]} - 1 \right| \frac{1}{[a_{2k+1-i, t}^i, u_{k+1}^i]} = \frac{D - C}{B - A} - 1 \frac{2^\alpha}{(B - A)^\alpha}.
\]

Moreover, since \( A \leq B/2 \) and \( C = \lambda_k A \),

\[
\left| \frac{D - C}{B - A} - 1 \right| \leq \left| \frac{D - B}{B - A} \right| + \left| \frac{C - A}{B - A} \right| \leq 2 \left| \frac{D - B}{B} \right| + |\lambda_k - 1|
\]

\[
= \frac{M}{B} \left[ \frac{1}{(2^{k+1} - i - 2)^{1+\varepsilon}} - \frac{1}{(2^{k+1} - i - 1)^{1+\varepsilon}} \right] + M \frac{k}{2^k}
\]

\[
\leq MB \left[ (2^{k+1} - i - 1)^{1+\varepsilon} - (2^{k+1} - i - 2)^{1+\varepsilon} \right] + M \frac{k}{2^k}
\]

\[
\leq \frac{M}{2k(1+\varepsilon)} 2^{k}\varepsilon + M \frac{k}{2^{k}}
\]

\[
\leq M \frac{k}{2^{k}}
\]

Therefore,

\[
\left| \frac{D - C}{B - A} - 1 \right| \frac{2^\alpha}{(B - A)^\alpha} \leq M \frac{k}{2^{k}} 2^{k(1+\varepsilon)}\alpha,
\]

hence

\[
\left| \frac{[a_{2k+1-i, t}^i, u_{k+1}^i]}{[a_{2k+1-i, t}^i, u_{k+1}^i]} - 1 \right| \frac{1}{[a_{2k+1-i, t}^i, u_{k+1}^i]} \leq M \frac{k}{2^{k(1+\varepsilon)}} 2^{k(1+\varepsilon)}\alpha.
\]

Finally, notice that by construction, the estimates for (iii) are the same as those for (ii).

**Estimates for \( g \):** The diffeomorphism \( g \) is obtained by fitting together the maps provided by the Technical Lemma sending:

(i) \([b_k, u_k^0]\) into \([b_k, v_k^0]\),

(ii) \([u_k^0, c_k]\) into \([v_k^0, c_k]\),

(iii) \([a_{2k+1}, b_k]\) and \([c_k, a_{2k}]\) into themselves as the identity.

For (i), notice that

\[
\left| \frac{[b_k, v_k^0]}{[b_k, u_k^0]} - 1 \right| \frac{1}{[b_k, u_k^0]} \leq \frac{1}{2^{1+\varepsilon}} \frac{1}{[b_k, u_k^0]^{1+\varepsilon}} \leq \frac{2^{1+\varepsilon}}{[b_k, c_k]^{1+\varepsilon}} \frac{1}{[b_k, c_k]^{1+\varepsilon}} = \frac{2^{1+\varepsilon}}{[b_k, c_k]^{1+\varepsilon}} \frac{1}{[b_k, c_k]^{1+\varepsilon}}
\]

thus

\[
\left| \frac{[b_k, v_k^0]}{[b_k, u_k^0]} - 1 \right| \frac{1}{[b_k, u_k^0]} \leq M [b_k, c_k]^{\theta - \alpha}.
\]

The estimates for (ii) are similar to those for (i) and we leave them to the reader.

**The choice of the parameters:** According to our Technical Lemma, and due to (2), (3), and (4), sufficient conditions for the \( C^{1+\alpha} \) smoothness of \( f, g \) are:
\[-(1 + \varepsilon)(1 + \theta)\alpha < 1,\]
\[-\frac{1}{1 + \varepsilon} > \alpha\]
\[-\theta > \alpha.\]

Now, for \(0 < \alpha < (\sqrt{5} - 1)/2\), one easily checks that these conditions are satisfied for \(\theta := \alpha + \varepsilon\), where \(\varepsilon > 0\) is small enough so that \((1 + \varepsilon)(1 + \alpha + \varepsilon)\alpha < 1\).

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Andrés Navas
Univ. de Santiago de Santiago
Alameda 3363, Santiago, Chile
email: andres.navas@usach.cl