Scaling of acceleration in locally isotropic turbulence

Reginald J. Hill
NOAA/Environmental Technology Laboratory, 325 Broadway, Boulder CO 80305, U. S. A.
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The variances of the fluid-particle acceleration and of the pressure-gradient and viscous force are given. The scaling parameters for these variances are velocity statistics measurable with a single-wire anemometer. For both high and low Reynolds numbers, asymptotic scaling formulas are given; these agree quantitatively with DNS data. Thus, the scaling can be presumed known for all Reynolds numbers. Fluid-particle acceleration variance does not obey K41 scaling at any Reynolds number; this is consistent with recent experimental data. The non-dimensional pressure-gradient variance named \( \lambda_T/\lambda_P \) is shown to be obsolete.

\textbf{I. INTRODUCTION}

Accelerations in turbulent flow are violent and are important to many types of studies (La Porta et al., 2001). The accelerations discussed here are caused by viscosity, \( \nu \nabla^2 u_i \), the pressure gradient, \( -\partial_x p \), and the fluid-particle acceleration, \( a_i \), which are related by the Navier–Stokes equation:

\[
a_i \equiv Du_i/Dt = -\partial_x p + \nu \partial_{x_n} \partial_{x_n} u_i,
\]

where \( Du_i/Dt \) denotes the time derivative following the motion of the fluid particle, and \( u_i \) is velocity. Here, \( \nu \) is the kinematic viscosity, \( \nabla^2 \equiv \partial_{x_n} \partial_{x_n} \) is the Laplacian operator, \( \partial \) denotes differentiation with respect to the subscript variable, summation is implied by repeated indices, \( \partial_x \) is pressure divided by fluid density; density is constant. \( \varepsilon \) is energy dissipation rate per unit mass of fluid.

Kolmogorov’s (1941) scaling (K41 scaling) uses \( \varepsilon \) and \( \nu \) as parameters, and is based on the assumption that local isotropy is accurate for a given distance \( r \) between two points of measurements, \( \mathbf{x} \) and \( \mathbf{x}' \equiv \mathbf{x} + \mathbf{r} \), \( \mathbf{r} \equiv [r] \). Refinement of K41 to include the effects of turbulence intermittency (Kolmogorov, 1962) leads to quantification of the Reynolds number dependence of the deviation from K41 scaling. The choice of Reynolds number has been \( R_\lambda \equiv u_{rms} \lambda_T/\nu \), based on Taylor’s (1935) scale \( \lambda_T \equiv \langle (\partial_{x_1} u_1)^2 \rangle^{1/2} \) and \( u_{rms} \equiv \langle u_1^2 \rangle^{1/2} \), where subscript 1 denotes components along the axis parallel to \( \mathbf{r} \); also, from \( \varepsilon = 15\nu \langle (\partial_{x_1} u_1)^2 \rangle \), \( R_\lambda = u_{rms}^2 / (\varepsilon \nu/15)^{1/2} \). Quantities \( \mathbf{x}' \) are denoted \( u'_i \), \( p' \), and \( \Delta u_i \equiv u_i - u'_i \), \( \Delta p \equiv p - p' \), etc.

During 1948-1951 [Heisenberg (1948), Obukhov & Yaglom (1951), Batchelor (1951)], the pressure-gradient correlation, \( \langle \partial_{x_1} p(\mathbf{x},t) \partial_{x_1} p(\mathbf{x}',t) \rangle \), mean-squared pressure gradient, \( \langle \partial_{x_1} p \partial_{x_1} p \rangle \), and, closely related to those, the pressure structure function, \( D_p(r) \equiv \langle (\Delta p)^2 \rangle \) were related to the velocity structure function \( D_{11}(r) = \langle (\Delta u_1)^2 \rangle \) by means of the assumption that \( u_i \) and \( u'_i \) are joint Gaussian random fields. In fact, the essential approximation, now known to be poor, is that \( \Delta u_i \) must be Gaussian (Hill, 1994). One result of that joint Gaussian theory is that pressure-gradient acceleration has as unobservably small effect from intermittency as does \( D_{11} \). In fact, pressure-gradient statistics are strongly affected by intermittency [Hill & Wilczak (1995, hereafter HW), Hill & Thorodden (1997, hereafter HT), Hill & Boratav (1997, hereafter HB), Vedula & Yeung (1999, hereafter VY), Gotoh & Rogallo (1999, hereafter GR), Gotoh & Fukayama (2001, hereafter GF), Nelkin & Chen (1998), Antonia et al. (1999), La Porta et al. (2001)].

A advanced theory (HW) relates \( D_p(r) \), \( \langle \partial_{x_1} p(\mathbf{x},t) \partial_{x_1} p(\mathbf{x}',t) \rangle \), \( \langle \partial_{x_1} p \partial_{x_1} p \rangle \), and the pressure spectrum to the fourth-order velocity structure function: \( \mathbf{D}_{ijkl}(r) = \langle \Delta u_i \Delta u_j \Delta u_k \Delta u_l \rangle \). The theory allows calculation of such pressure-gradient-related statistics from components of \( \mathbf{D}_{ijkl} \) and therefore by means of hot-wire anemometry, as in HT. This theory is valid for all Reynolds numbers and is based on local isotropy without further assumptions. The degree to which the theory’s predictions are accurate must depend on how anisotropic the large scales are and how large the Reynolds number is; isotropy must be approached as Reynolds number becomes small. A advanced theory for the correlation \( \langle \nu \nabla^2 u_i \nu \nabla^2 u'_j \rangle \) is given by HT; HT’s relationship of this correlation to the third-order velocity structure function \( \mathbf{D}_{ijk}(r) = \langle \Delta u_i \Delta u_j \Delta u_k \rangle \) has advantages (HT) over the 1948-1951 theory that related \( \langle \nu \nabla^2 u_i \nu \nabla^2 u'_j \rangle \) to \( D_{11} \). The advanced theories have been used to compare \( D_p(r) \) calculated from \( \mathbf{D}_{ijkl}(r) \) with \( D_p(r) \) calculated from DNS pressure fields (figure 1 of HB), as well as the corresponding calculations for \( \langle \partial_{x_1} p(\mathbf{x},t) \partial_{x_1} p(\mathbf{x}',t) \rangle \) and
\[ \langle \nabla_x^2 u \nabla_x^2 u \rangle \] (figures 12, 13 of VY). Since the advanced theories use only the Navier–Stokes equation, incompressibility, and local isotropy, comparisons of data with the theory give a measure of the local anisotropy of the data, of numerical limitations, or of inaccuracy of Taylor’s hypothesis (when used).

Most studies of turbulent acceleration use the traditional approach of determining the \( R_\lambda \) dependence that results from use of K41 scaling of acceleration statistics [e.g., VY, GR, GF, Antonia et al. (1999), La Porta et al. (2001)]. The resultant deviation from K41 scaling, i.e. the \( R_\lambda \) dependence, is often called ‘anomalous’. There is no anomaly when the advanced theory is employed. Because \( R_\lambda \) contains \( u_{rms} \), it is affected by the large scales where anisotropy is possible, and \( R_\lambda \) is therefore not a parameter of the advanced theory. However, to compare the advanced theory with the existing body of empirical knowledge, the advanced-theory’s scales must be expressed in terms of K41 scales, thereby producing dependence on \( R_\lambda \) and \( \varepsilon \). However, K41 scaling parameters and \( R_\lambda \) are not the scales within the advanced theory.

II. SCALING OF MEAN-SQUARED PRESSURE GRADIENT

For locally isotropic turbulence, HW gave the relationship between the mean-squared pressure gradient and the fourth-order velocity structure function:

\[ \langle \partial_x \partial_x p \rangle = \chi = 4 \int_0^\infty r^{-3} [D_{1111}(r) + D_{\alpha\alpha\alpha\alpha}(r) - 6D_{1\beta\beta\beta}(r)] \, dr, \]  

(1)

where \( \chi \) is shorthand for the mean-squared pressure gradient for the case of local isotropy. In (1), \( D_{1111}(r) \), \( D_{\alpha\alpha\alpha\alpha}(r) \), and \( D_{1\beta\beta\beta}(r) \) are components of \( D_{ijkl}(r) \); \( \alpha \) and \( \beta \) denote the Cartesian axes perpendicular to \( r \), and the 1-axis is parallel to \( r \). Repeated Greek indices do not imply summation. The result (1) applies for all Reynolds numbers and without approximation other than local isotropy.

Defining \( H_\chi \) as the ratio of the integral in (1) to its first term, HW wrote (1) as

\[ \chi = 4H_\chi \int_0^\infty r^{-3} D_{1111}(r) \, dr. \]  

(2)

Equivalently, (2) defines \( H_\chi \). The purpose of (2) as stated in HW is that if the Reynolds number variation of \( H_\chi \) is known, then (3) enables evaluation of \( \chi \) by calculating the integral in (2) using data from a single hot-wire anemometer. Further, HW argued that \( H_\chi \) is a constant at large Reynolds numbers. VY evaluated \( H_\chi \) by means of DNS data and found that it is constant at a value of about 0.65 for \( 80 < R_\lambda < 230 \), 230 being their maximum \( R_\lambda \). Their \( H_\chi \) only decreased to about 0.55 at \( R_\lambda = 20 \). Hill (1994) gives \( H_\chi \to 0.36 \) as \( R_\lambda \to 0 \) [on the basis that the joint Gaussian assumption can be used in this limit and by use of a formula for the velocity correlation for \( R_\lambda \to 0 \) given by Batchelor (1956)].

It is useful to express the integral in (2) in terms of quantities that have been measured in the past. On the basis of empirical data described in Appendix A, the approximation for high Reynolds numbers is

\[ \chi \simeq 3.1H_\chi \varepsilon^{3/2} \nu^{-1/2} F^{0.79} \simeq 3.9H_\chi \varepsilon^{3/2} \nu^{-1/2} R_\lambda^{0.25} \]  

for \( R_\lambda \gtrsim 400 \),

(3)

where \( F \equiv \langle (\partial_x u_1)^4 \rangle / \langle (\partial_x u_1)^2 \rangle^2 \) is a velocity-derivative flatness. For \( H_\chi = 0.65 \), (3) agrees quantitatively with the DNS data in Table 1 of GF for \( R_\lambda \geq 387 \); thereby, the estimated limitation supported in Appendix A, i.e., \( R_\lambda \gtrsim 10^4 \), seems too conservative. That is why the limitation \( R_\lambda \gtrsim 400 \) is given in (3).

The case of low Reynolds numbers is in Appendix B where Taylor’s scaling and data from VY are used; the result is

\[ \chi \simeq 0.11 \varepsilon^{3/2} \nu^{-1/2} R_\lambda \]  

for \( R_\lambda \lesssim 20 \),

(4)

which is shown in figure 1 of VY. Neither (3) nor (4) is K41 scaling because of their \( R_\lambda \) dependence.

The data of Pearson & Antonia (2001) reveal how the approximation (3) is approached as \( R_\lambda \) increases. The inner scale of \( D_{1111}(r) \) is denoted by \( \ell \) and is defined in Appendix A as the intersection of viscous- and inertial-range asymptotic formulas for \( D_{1111}(r) \). Thus, \( \ell \) is a length scale in the dissipation range. In figures 4 and 5 of Pearson & Antonia (2001), the scaled components of \( D_{1111}(r) \) increase most rapidly at \( r > \ell \) as \( R_\lambda \) is increased until an inertial range is attained. This implies that the integral in (2) will approach the asymptote (3) from below. That
has been observed, as shown in figure 1, wherein the DNS data of VY and GF are plotted with the asymptotic formulas (3) and (4); those asymptotes are graphed to \( R_\lambda = 10^3 \) and 40, respectively. Because \( H_\chi = 0.65 \) was used in (3) to obtain figure 1, and because VY found \( H_\chi \sim 0.65 \) for \( 80 < R_\lambda < 230 \), it appears that \( H_\chi \) remains constant at about 0.65 for \( R_\lambda > 80 \). There was no adjustment of (3) to cause agreement with the DNS. The agreement is surprising because the empirical data used in Appendix A suggests at least a 15% uncertainty of the coefficients in (3). As a reminder of this fortunate circumstance, a ±15% error bar is shown at \( R_\lambda = 10^3 \) in figure 1. Previously, \( \langle \partial_{x_i} p \partial_{x_j} p \rangle / \varepsilon^{3/2} \nu^{-1/2} \propto R_\lambda^{1/2} \) has been reported (VY, GR); this is a good fit to the data of VY in the range \( 130 < R_\lambda < 300 \), and \( \langle \partial_{x_i} p \partial_{x_j} p \rangle / \varepsilon^{3/2} \nu^{-1/2} \propto R_\lambda^{0.62} \) is a good fit to the data of GR in the range \( 39 < R_\lambda < 170 \).

### III. SCALING OF \( \nu^2 \left\langle \left| \nabla^2 u_i \right|^2 \right\rangle \)

Derivation of \( \nu^2 \left\langle \left| \nabla^2 u_i \right|^2 \right\rangle \) from the spatial correlation of \( \nu \nabla^2 u_i \) is given by Hill (2001). It suffices here to state the various equivalent formulas:

\[
V_{ii} (0) = \nu^2 \left\langle \left| \nabla^2 u_i \right|^2 \right\rangle = 12 \nu^2 \left( \langle \partial_{x_i} u_j \rangle \right)^2 = 35 \nu^2 \left( \langle \partial_{x} u_1 \rangle \right)^2, \tag{5}
\]

where \( \partial_{x_i} \equiv \partial / \partial x_i^2 \), and for local stationarity,

\[
V_{ii} (0) = \nu \langle \omega_i \omega_j s_{ij} \rangle = -\frac{4}{3} \nu \langle s_{ij} s_{jk} s_{ki} \rangle - \frac{105}{4} \nu \langle \partial_{x_j} u_j \rangle^2 \partial_{x_i} u_1 \rangle = -\frac{35}{2} \nu \langle \partial_{x_i} u_1 \rangle^3 \rangle = 0.3 \nu^{3/2} \nu^{-1/2} |S|, \tag{7}
\]

where \( \omega_i \) is vorticity and \( s_{ij} \) is the rate of strain, and \( S \equiv \left( \langle \partial_{x_i} u_1 \rangle^3 \right) / \left( \langle \partial_{x_i} u_1 \rangle^2 \right)^{3/2} \) is the velocity-derivative skewness. \(|S|\) is known (Sreenivasan & Antonia, 1997) to increase with increasing \( R_\lambda \). Thus, all statistics in (3) and (4) have the same increase with \( R_\lambda \) when they are nondimensionalized using K41 scaling. On the other hand, \( S \) is approximately constant over the range of about \( 20 < R_\lambda < 400 \) (Sreenivasan & Antonia, 1997) such that \( V_{ii} (0) \) approximately follows K41 scaling in that range.

The most carefully selected data for \( S \) and \( F \) at high Reynolds numbers are those of Antonia et al. (1981), which data are in agreement with data at \( R_\lambda = 10^3 \) by Kolmynsky, Tsinober & Yorish (2001). The data of Antonia et al. (1981) are used for \( F \) in (3) and give \(|S| \simeq 0.5 (R_\lambda/400)^{0.11} \) for \( R_\lambda > 400 \). Substituting \(|S| \simeq 0.5 (R_\lambda/400)^{0.11} \) in (7) gives

\[
V_{ii} (0) \simeq 0.08 R_\lambda^{0.11} \quad \text{for} \quad R_\lambda > 400. \tag{8}
\]

For \( R_\lambda < 20 \), Tavoularis, Bennett & Corrsin (1978), Herring & Kerr (1982) and Kerr (1985) show that \(|S|\) decreases and does so more rapidly as \( R_\lambda \to 0 \). The data of Herring and Kerr (1982) suggest that \(|S| \sim R_\lambda / 5 \) for \( R_\lambda < 1 \). Although stronger empirical evidence would be helpful, \(|S| \sim R_\lambda / 5 \) will serve as the asymptotic formula for \( R_\lambda < 1 \), in which case (3) becomes

\[
V_{ii} (0) \simeq 0.06 \nu^{3/2} \nu^{-1/2} R_\lambda \quad \text{for} \quad R_\lambda < 1. \tag{9}
\]
IV. SCALING OF THE MEAN-SQUARED FLUID-PARTICLE ACCELERATION

For locally isotropic turbulence, the mean-squared fluid-particle acceleration $A_{ii}(0)$ is $\chi + V_{ii}(0)$ because the correlation of $\nabla^2 \chi' u'_i$ and $\partial x_j p$ vanishes by local isotropy (Obukhov & Yaglom, 1951). Thus, for any Reynolds number for which local isotropy is valid

$$ A_{ii}(0) = 4H_\chi \int_0^\infty r^{-3} D_{1111}(r) \, dr - \frac{35}{2} \nu \langle (\partial x_i u_1)^3 \rangle. \quad (10) $$

Therefore, $A_{ii}(0)$ scales with the sum of two terms that behave differently with Reynolds number.

Use of (3) and (7, 8) in (10) gives for $R_\lambda \gtrsim 400$

$$ A_{ii}(0) \simeq \varepsilon^{3/2} \nu^{-1/2} \left( 2.0 F^{0.79} + 0.3 |S| \right) \simeq \varepsilon^{3/2} \nu^{-1/2} \left( 2.5 R_\lambda^{0.25} + 0.08 R_\lambda^{0.11} \right). \quad (11) $$

At $R_\lambda = 400$, the term from $\chi$ is 70 times greater than the term from $V_{ii}(0)$, and $\chi$ increases much faster than $V_{ii}(0)$ for increasing $R_\lambda$. The theory of 1948-1951 used the joint Gaussian assumption and thereby greatly underestimated $\langle \partial x_i p \partial x_j p \rangle$ (HW, HT). That theory gives equation (3.18) of Obukhov & Yaglom (1951), which is the same as (11) with the exception that $1.1 |S|^{-1}$ appears in place of $2.0 F^{0.79}$. Not only is the magnitude of $1.1 |S|^{-1}$ smaller than $2.0 F^{0.79}$ by a factor of 5 at $R_\lambda = 400$, in addition, $|S|^{-1}$ decreases with further increases of $R_\lambda$ contrary to the increase of $2.0 F^{0.79}$. An empirical result that seems accurate for a variety of flows for $R_\lambda \gtrsim 400$ (Champagne, 1978; Antonia et al., 1981) is $|S| = 0.25 F^{3/8}$, such that (11) can be written as $A_{ii}(0) \simeq \varepsilon^{3/2} \nu^{-1/2} \left( 2.0 F^{0.79} + 0.075 F^{0.375} \right)$ for $R_\lambda \gtrsim 400$.

For low Reynolds numbers, use of (3) in (10) gives

$$ A_{ii}(0) \simeq \varepsilon^{3/2} \nu^{-1/2} \left( 0.11 R_\lambda + 0.3 |S| \right) \quad \text{for} \quad R_\lambda < 20. \quad (12) $$

From figure 8 of Herring & Kerr (1982) one sees that $0.11 R_\lambda > 0.3 |S|$ even at their minimum $R_\lambda$ of 0.5, and that $0.11 R_\lambda$ increases rapidly relative to $0.3 |S|$ as $R_\lambda$ increases. Thus, (12) shows that the term from $\chi$ is the larger contribution to $A_{ii}(0)$ for all $R_\lambda$. The behaviour of $A_{ii}(0)$, $\chi$, and $V_{ii}(0)$ for moderate $R_\lambda$ is shown particularly well in figure 1 of VY.

V. DISCUSSION

A. Obsolescence of $\lambda_T/\lambda_P$

The length scales $\lambda_P \equiv u_{rms}^2 \left( \langle \partial x_i p \rangle \right)^{1/2}$ and $\lambda_T \equiv u_{rms} \left( \langle \partial x_i u_1 \rangle \right)^{1/2}$ were introduced by Taylor (1935) (he included a factor of $\sqrt{2}$ that has historically been dropped from these definitions), and the ratio $\lambda_T/\lambda_P$, and he gave the first evaluation of $\lambda_T/\lambda_P$ from turbulent diffusion measurements. $\lambda_T/\lambda_P$ is a scaled mean-squared pressure gradient; that scaling depends on a large-scale parameter, $u_{rms}$. As such $\lambda_T/\lambda_P$ is not relevant in the advanced theory except in the limit of $R_\lambda \to 0$. Batchelor (1951) obtained, for very large Reynolds numbers a dependence of $\lambda_T/\lambda_P \propto R_\lambda^{-1/2}$, which is not correct because it is based on the joint Gaussian assumption. By attempting evaluation of $\lambda_T/\lambda_P$, Batchelor (1951) was, in effect, attempting to enable determination of $\chi$ from measurements of velocity variance and energy dissipation rate. The advanced theory replaces $\lambda_T/\lambda_P$ with $H_\chi$. Evaluation of $H_\chi$ allows $\chi$ to be determined from measurement of a single velocity component and the simple formula (6). Figure 3 of GR shows $\lambda_T/\lambda_P$ versus $R_\lambda$ and reveals the following: i) the strong $R_\lambda$ dependence of $\lambda_T/\lambda_P$ that $H_\chi$ does not have; ii) the deviation of the $R_\lambda$ dependence of $\lambda_T/\lambda_P$ from that predicted by the joint Gaussian assumption; (this was also found by VY) and iii) $R_\lambda < 20$ is required to approach the low-Reynolds-number asymptote. Whereas $H_\chi$ depends only on the small scales of turbulence, the dependence of $\lambda_T/\lambda_P$ on the large scales via $u_{rms}$ shows that $\lambda_T/\lambda_P$ is not relevant in the advanced theory of local isotropy, except in the limit of $R_\lambda \to 0$; for that limit HW shows that $H_\chi \propto (\lambda_T/\lambda_P)^2$. 
B. Acceleration data

Pioneering technology for measuring turbulence accelerations is being developed at Cornell. La Porta et al. (2001) report fluid-particle acceleration measured in a cylindrical enclosure containing turbulent water driven by counter-rotating blades. Here, ‘x’ and ‘y’ axes are transverse and parallel to their cylinder axis, respectively. The acceleration’s flatness factor \( \langle a_x^2 \rangle / \langle a_y^2 \rangle \) in their figure 3 reaches a maximum at \( R_\lambda \approx 700 \), and is decreased at their next-higher \( R_\lambda \) value, namely 970, the same is true for their K41-scaled acceleration variances \( \langle a_x^2 \rangle / \varepsilon^3/2 \nu^{-1/2} \) in their figure 4, where \( i = x \) and \( y \). From (11), \( A_{ii} (0) / \varepsilon^{3/2} \nu^{-1/2} = 3 \langle a_x^2 \rangle / \varepsilon^3/2 \nu^{-1/2} \) is monotonic with \( R_\lambda \), unlike in figure 4 of La Porta et al. (2001). This suggests that the cause of the maxima in flatness and variance have the same cause. Their estimates of \( R_\lambda \) and \( \varepsilon \) both depend on the choice of a velocity component; because the turbulence is anisotropic, the choice of another velocity component will shift their data points along both ordinate and abscissa. This nonuniversality of their scaling is implicated by the disappearance of the maximum in the flatness of \( \partial_x u_i \) as presented by Belin et al. (1997) when \( R_\lambda \) is replaced by a universal Reynolds number (Hill, 2001). Belin et al. (1997) measure near one counter-rotating blade whereas LaPorta et al. (2001) measure in the flow’s center. The two cylinders have different aspect ratios. It is nevertheless instructive to substitute the values of \( F \) and \( |S| \) measured by Belin et al. (1997) into (11), divide by 3 to obtain the variance of one component of acceleration, and compare the result with the data of La Porta et al. (2001). This is done in figure 2, wherein the data of VY and GF are shown to agree with the data of Belin et al. (1997). For the Belin et al. data, (11) produces a maximum similar to that of the La Porta et al. (2001) data.

FIG. 2. Figure 2 K41 scaled mean-squared acceleration component. La Porta et al. (2001) data, x-component: asterisks; y-component: squares; Belin et al. (1997) data in equation (11): diamonds; combined DNS data of VY and GF: triangles; eq.(4.3) divided by 3: solid line.

As Belin et al. (1997) point out, the maximum in their data for \( F \) might be specific to the flow between counter-rotating blades. If so, the same is likely true of the data of La Porta et al. (2001) such that their data do not support K41 scaling of acceleration, and therefore do not contradict the advanced theory. Another possibility is that the data of LaPorta et al. (2001) at \( R_\lambda = 970 \) is underestimated for unknown reasons. The conclusion suggested by figure 2 and the above uncertainties in interpretation of the data is that the data supports the scaling given here and that such important acceleration measurements must continue.

VI. CONCLUSION

The asymptotes (3) and (4) combined with the DNS data in figure 1 determine \( \langle \partial_x, p \partial_x, p \rangle \) for all Reynolds numbers. For all Reynolds numbers the advanced-theory scaling is that \( \langle \partial_x, p \partial_x, p \rangle \) scales with the integral in (3). For \( R_\lambda < 20 \), \( \langle \partial_x, p \partial_x, p \rangle \) scales with \( \varepsilon^{3/2} \nu^{-1/2} R_\lambda \). Because \( H_x \) is apparently constant for \( R_\lambda > 80 \), \( \langle \partial_x, p \partial_x, p \rangle \) scales with \( \int_0^\infty r^{-3} D_{111} (r) \, dr \) [i.e., the integral in (3)], and \( \langle \partial_x, p \partial_x, p \rangle \) scales approximately with \( \varepsilon^{3/2} \nu^{-1/2} F^0.79 \) for \( R_\lambda > 400 \), and \( \varepsilon^{3/2} \nu^{-1/2} F^0.79 \) is a good approximation when \( R_\lambda \) as small as 200 (see figure 1). Given that \( H_x \) is constant for \( R_\lambda > 80 \), \( \langle \partial_x, p \partial_x, p \rangle \) could be obtained for \( R_\lambda > 80 \) using data from a single-wire anemometer by evaluating the integral in (3); DNS is not necessary. Using velocity data, it is more accurate to evaluate the integral in (3) than its approximation \( \varepsilon^{3/2} \nu^{-1/2} F^0.79 \) because evaluating \( F \) requires greater spatial resolution than does evaluation of the integral for the same level of accuracy. Evaluating \( H_x \) from (3) using DNS at \( R_\lambda > 230 \) would be useful.

Now, \( V_{ii} (0) \), does scale with any of the derivative moments in (3). It does not scale as in the K41 prediction (i.e., \( \varepsilon^{3/2} \nu^{-1/2} \)) except for those \( R_\lambda \) at which \( S \) is constant. The statement in VY that \( V_{ii} (0) \) does obey K41 scaling is based on their data, which are within the \( R_3 \) range where \( S \) is constant.

Fluid-particle acceleration variance, \( A_{ii} (0) \), does not scale as in the K41 prediction (i.e., \( \varepsilon^{3/2} \nu^{-1/2} \)) at large Reynolds numbers because of the factor \( (2.5 R_\lambda^{0.25} + 0.08 R_\lambda^{0.11}) \) in (11). \( A_{ii} (0) \) does not approach K41 scaling as \( R_\lambda \to 0 \) because (4) and (12) give \( A_{ii} (0) \approx 0.17 \varepsilon^{3/2} \nu^{-1/2} R_\lambda \) for \( R_\lambda < 1 \). For all Reynolds numbers, fluid particle acceleration does scale with the sum of velocity statistics that appears on the right-hand side of (10).

The advanced theory is devoid of statistics of the large scales. It seems paradoxical that \( R_\lambda \), which depends on the large scales through \( u_{rms} \), is used above to delineate asymptotic regimes. However, an alternative Reynolds number that depends on small scales (Hill, 2001) makes the advanced theory self-contained. Use of existing phenomenology caused both \( \varepsilon \) and \( R_\lambda \) to appear in this paper. However, practical applications result. Turbulent acceleration-induced coalescence of droplets might be key to understanding rain initiation from liquid-water clouds.
Thus, power-law. The data of Pearson & Antonia (2001) show such an extent of the power law as below; such a constant value is a reasonable criterion for the integral in (1) to be proportional to \( D_{1111} \). Viscous-range formulas are used; they are: \( D_{1111} (r) = \left( \langle \partial_x u_1 \rangle^4 \right) r^4 \) and \( D_{11} (r) = \left( \langle \partial_x u_1 \rangle^2 \right) r^2 = (\varepsilon/15\nu)c \). The inner scale of \( D_{1111} (r) \), named \( \ell \), is defined by equating the inertial-range formula with the viscous-range formulas at \( r = \ell \). In the integrand in (2), \( D_{1111} (r) \) can be scaled by \( D_{1111} (\ell) \) and \( r \) by \( \ell \). Doing so, the integral equals \((3/2)\ell^2 \left( \langle \partial_x u_1 \rangle^4 \right) \) [HW showed that the remaining dimensionless integral has a value of 3/2 for large Reynolds numbers; this is based on use of an equation for \( D_{1111} (r) \) that is the same as equation (12) by Stolovitzky, Sreenivasan & Juneja (1993), who demonstrate its empirical basis]. Next, the otherwise irrelevant Taylor’s scale \( \lambda_T \) is introduced to make use of published empirical data. The definition of \( \ell \) and the inertial range formulas are used to obtain \((\ell/\eta)^{4-q} = (15C^2 \left( \lambda_T^2/\eta L \right) \nu^{q/2} F(\lambda_T)/F; F(\lambda_T) \equiv D_{1111} (\lambda_T)/[D_{11} (\lambda_T)][2] \). The essential approximation is that \( \lambda_T \) is in the inertial range (Antonia et al. 1982; Pearson & Antonia, 2001). Then, 
\[
(3/2)\ell^2 \left( \langle \partial_x u_1 \rangle^4 \right) = (3/2)C^{4/(4-q)} 15^{2(q-2)/(4-q)} \left[ \left( \lambda_T^2/\eta L \right) \nu^{q/2} F(\lambda_T)/F \right]^{2/(4-q)} F^{2/3}_\varepsilon \nu^{-1/3}.
\]
For \( 10^2 < \lambda_T < 5 \times 10^3 \), Zocchi et al. (1994) have \( \lambda_T^2/L \eta = 30\lambda_T^{-1} \) such that \( (\lambda_T^2/L \eta)^{(2q-2)/(4-q)} = (R_\lambda/900)^{-2q/(9-4q)} \), the exponent of \( \lambda_T^2/L \eta \) is about 0.04; so \( R_\lambda \) can vary greatly but \( (\lambda_T^2/L \eta)^{0.04} \) remains near unity. The most consistent data for \( F \) at high Reynolds numbers are those of Antonia et al. (1981), which give \( F \approx 1.36R_\lambda^{0.31} \); the same data (Antonia et al., 1982) give \( F(\lambda_T) \propto \lambda_T^2 \) and \( F(\lambda_T) \approx 7.1 \) at \( R_\lambda = 9400 \) so \( F(\lambda_T) \approx 0.72R_\lambda^{0.31} \). Thus, 
\[
(3/2)\ell^2 \left( \langle \partial_x u_1 \rangle^4 \right) \approx 0.714R_\lambda^{0.064} F^{2/3}_\varepsilon \nu^{-1/3} = 0.76F^{0.79}\varepsilon^{2/3}_\nu^{-1/2} = 0.97 R_\lambda^{0.25} \varepsilon^{2/3}_\nu^{-1/2}, \text{ i.e.,}
\]
\[
\int_0^\infty r^{-3} D_{1111} (r) dr \approx 0.76 F^{0.79}\varepsilon^{2/3}_\nu^{-1/2} \approx 0.97 R_\lambda^{0.25} \varepsilon^{2/3}_\nu^{-1/2}.
\]
Given that \( \mu \approx 0.25 \), (A1) is insensitive of the value of \( \mu \) and is therefore also insensitive to the choice of intermittency model. Shaw & Oncley (2001) used data from the atmospheric surface layer at \( R_\lambda = 1500 \) to obtain that (A1) balances to within the accuracy of their value of \( F \), i.e., about 15%.

The data of Pearson & Antonia (2001) show \( D_{bg} (\lambda_T)/D_{1111} (\lambda_T) \) becoming constant as \( \lambda_T \rightarrow 0 \) from below; such a constant value is a reasonable criterion for the integral in (3) to be proportional to \( \int_0^\infty r^{-3} D_{1111} (r) dr \) as in (2). They show the variation of \( D_{1111} (r) \) and \( D_{bg} (r) \) as \( R_\lambda \) varies from 38 to 1200 such that an inertial range appears at the larger \( R_\lambda \). The above approximation (A1) requires that the integral in (A1) approximately converges at its upper limit for the inertial range, for which a reasonable criterion is that there be about one decade of the power-law. The data of Pearson & Antonia (2001) show such an extent of the power law as \( R_\lambda \approx 10^3 \) is attained. Thus, \( R_\lambda \approx 10^3 \) is a well-supported lower bound for the high-Reynolds-number approximation (A1).

Appendix B: Low-Reynolds-number asymptote

The joint Gaussian assumption is not used here. Taylor’s (Taylor, 1935) scaling is used; i.e., scales \( \lambda_T \) and \( u_{rms} \) are used when \( R_\lambda \rightarrow 0 \). Taylor’s scaling gives \( D_{1111} (r) \propto u_{rms}^4, \) \( H_X \) approaches a constant as \( R_\lambda \rightarrow 0 \). Let \( x = r/\lambda_T \). Then (2) can be written as \( \chi \propto u_{rms}^4 \lambda_T^2 \int_0^\infty x^{-3} D_{1111} (x)/u_{rms}^4 \, dx \). The dimensionless integral is a number, so \( \chi \propto u_{rms}^4 \lambda_T^2 \); hence \( \chi/\left( \varepsilon^{3/2} \nu^{-1/2} \right) \propto R_\lambda \). This behaviour is shown in figure 1 of VY where it appears to become accurate between \( R_\lambda = 20 \) and 40. In their Table II, \( \chi/\left( \varepsilon^{3/2} \nu^{-1/2} \right) = 0.74 \) at \( R_\lambda = 21 \). Thus, \( \chi = 0.106 \varepsilon^{3/2} \nu^{-1/2} R_\lambda \) for \( R_\lambda < 20 \). Compare 0.106 with the prediction of the joint Gaussian assumption: \( 6/15^{1/2} = 0.103 \) (Hill, 1994).

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