Self-adjointness of generalized MIC-Kepler system

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September 1, 2018

Abstract
We have studied the self-adjointness of generalized MIC-Kepler Hamiltonian, obtained from the formally self-adjoint generalized MIC-Kepler Hamiltonian. We have shown that for \( \tilde{l} = 0 \), the system admits a 1-parameter family of self-adjoint extensions and for \( \tilde{l} \neq 0 \) but \( \tilde{l} < \frac{1}{2} \), it has also a 1-parameter family of self-adjoint extensions.

Keywords: MIC-Kepler system; Self-adjoint extensions ; Bound states
PACS numbers: 03.65.Db, 02.30.Ik, 03.65.-w

1 Introduction

Exactly solvable models are extremely important in both theoretical and mathematical physics because with these model one could get an idea of the the different physics which could arise in reality. The non-relativistic Coulomb problem is one such example, which has immense importance due to its symmetry under specific transformation. It can be shown that the over-complete symmetry gives rise to the degeneracy in energy spectrum, separability of variables in some coordinate systems. The Hamiltonian with only Coulomb potential has \( O(4) \) symmetry for bound states and \( O(1,3) \) symmetry for scattering states. But if the particles of the Coulomb problem have magnetic charges \( g_1 \) and \( g_2 \) in addition to their electric charges \( e_1 \) and \( e_2 \), then usually the symmetry is destroyed. It was shown [1] that this symmetry can be restored if an additional scalar potential of the form \( \sim \frac{1}{r^4} \), where \( r \) is the radial distance of the center of mass from the origin, is added to the Hamiltonian. This system is known as MIC-Kepler system and is discussed in Ref. [2] also. Apart from \( O(4) \) symmetry for bound state and \( O(1,3) \) symmetry for continuum state, there are two more features which are preserved for MIC-Kepler system: (1) the differential cross section remains the same, both classically and quantum mechanically and (2) if there are three nonrelativistic particles with both electric and magnetic charges with pair wise interaction of the form \( \sim \frac{1}{r^4} \), then if two particles are held fixed, the Hamiltonian of the third particle is separable in ellipsoidal coordinates.

One can obtain MIC-Kepler system from the four dimensional isotropic oscillator by Kustaanheimo-Stiefel transformation [3, 4]. Similarly from two and eight dimensional isotropic oscillator one can get two [5, 6] and five [7, 8] dimensional analog of MIC-Kepler system. It has been generalised from different perspective. For example, we can have generalization of MIC-Kepler system on a class of three dimensional spaces having conic singularities from quantum mechanical oscillator defined on Kähler conifold [9]. There is another generalization [10], by adding Smorodinsky-Winternitz type potential [11, 12, 13] to the kinetic energy term of the Hamiltonian. The Hamiltonian \( H(s,c_1,c_2) \) [10] dependents on the parameters \( s, c_1, c_2 \).

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The Smorodinsky-Winternitz type potentials where studied in Ref. [1] [15] [10]. This potential can be reduced to Hartmann potential [17] [18] [19], which has been used for describing axially symmetric systems like ring-shaped molecules and investigated from different point of view [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32]. This system has rich features, i.e., exact solvability, separability of variables in few coordinate systems. It is solved in various coordinate systems [33], for example; spherical, parabolic and prolate spheroidal coordinate systems. Interesting feature of this generalized system \((H(s,c_1,c_2))\) is that it can be reduced to many interesting known quantum mechanical systems. For \(c_1 = c_2 = 0\), it reduces to MIC-Kepler system [1] [2]. For \(s = c_1 = c_2 = 0\) in generalized MIC-Kepler system we get the Hydrogen atom Hamiltonian [34] [35] [36]. Generalized Kepler-Coulomb system [37] can be found by making \(s \neq 0, c_i \neq 0\) \((i = 1, 2)\) in generalized MIC-Kepler Hamiltonian \(H(s,c_1,c_2)\) [10]. We can get Hartmann system [38], when \(s = 0, c_1 = c_2 = 0\) and also charge-dyon system [39] when \(s \neq 0, c_1 = c_2 = 0\).

The purpose of this paper is to study self-adjointness of generalized MIC-Kepler system. Self-adjointness of similar type problems have been studied in many papers. For example, it is shown in Ref. [40] [41] that the radial Hamiltonian \(h_t\) of the Hydrogen atom with effective potential \(V_{eff} = \frac{1}{r} + \frac{l(l+1)}{r^2}\) has one parameter family of self-adjoint extensions only for angular momentum quantum number \(l = 0\) and for all other values of \(l\), since deficiency indices are \(< 0, 0 >\), it is essentially self-adjoint. Actually if one demands the square integrability of the deficiency space solution \(u_l^\pm\) at origin, one should get the condition \(-\frac{1}{2} < l < \frac{1}{2}\). Outside this limit since the solutions \(u_l^\pm\) are not square integrable, the deficiency indices are \(< 0, 0 >\). So Hamiltonian is essentially self-adjoint. Since for Hydrogen atom problem \(l\) can have only nonnegative integral values, it is sufficient to say that for \(l \geq 1\) the radial Hamiltonian is essentially self-adjoint. In [12] it has been shown that the Dirac Hamiltonian of an isospin-1/2 particle in the field of \(SU(2)\) monopole admits a one-parameter family of self-adjoint extensions. In case of Calogero model effective potential of the form \(\frac{\hat{g}}{r}\) appears when written in terms of radial variable \(r\). Self-adjointness of this model has been studied in Ref. [43] [44] and it is shown that for \(-\frac{1}{4} \leq \hat{g} < \frac{1}{4}\), the Hamiltonian has self-adjoint extensions. Outside this domain since the Hamiltonian is essentially self-adjoint, there is no scope for self-adjoint extensions. In our case, the generalized MIC-Kepler system has effective potential \(V_{eff} = -\frac{1}{r} + \frac{l(l+1)}{r^2}\). The difference in radial Hamiltonian between generalized MIC-Kepler system and Hydrogen atom problem is in the coefficient of the potential \(\sim \frac{1}{r^2}\). In case of Hydrogen atom \(l\) can take values 0, 1, 2, 3, ···, but in case of generalized MIC-Kepler system, \(\hat{l}\) can take other positive values also, besides 0, 1, 2, 3, ··· (in fact \(\hat{l}\) can have positive continuous values). It will allow us to get self-adjoint extensions of the radial Hamiltonian for \(\hat{l} \neq 0\) besides \(\hat{l} = 0\), as opposed to Hydrogen atom problem, where only \(l = 0\) has self-adjoint extensions. This is not strange, as we can see for Calogero model we get self-adjoint extensions for \(\hat{g} \neq 0\) [13] [44].

The paper is organized as follows: In Sec. (2), we discuss the generalized MIC-Kepler system briefly. In Sec. (3), we study the self-adjointness issue of this system. We conclude in Sec. (4).

### 2 Generalized MIC-Kepler system

The generalized MIC-Kepler Hamiltonian [10], in \(\hbar = c = e = 1\) unit and with mass taken to be unity is given by,

\[
H(s,c_1,c_2) = \frac{1}{2} (-i\nabla - sA)^2 + \frac{s^2}{2r^2} - \frac{1}{r} + \frac{c_1}{r^2(1 + \cos \theta)} + \frac{c_2}{r^2(1 - \cos \theta)}. \tag{2.1}
\]

Here \(A\) is the magnetic vector potential of the Dirac monopole, given by

\[
A = -\frac{\sin \theta}{r(1 - \cos \theta)} \hat{\phi}, \tag{2.2}
\]

such that \(\text{curl}A = \frac{1}{r^2}\). \(c_1, c_2\) are nonnegative constants and \(s\) takes the values 0, ±\(\frac{1}{2}\), ±1, ···.

We now want to study the eigenvalue problem of the Hamiltonian Eq. (2.1), given by

\[
H\Psi = E\Psi. \tag{2.3}
\]
This equation can be easily separated out in radial and angular part in spherical polar coordinate system, if we consider the wave function $\Psi$ of the form

$$\Psi(r, \theta, \phi) = R(r)\Phi(\theta, \phi).$$

(2.4)

Substituting Eq. (2.4) into Eq. (2.3) we get two decoupled differential equations. The differential equation involving angular co-ordinates is of the form

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{4 \cos^2 \frac{\theta}{2}} \left( \frac{\partial^2 \Phi}{\partial \phi^2} - 4c_1 \right) \Phi + \frac{1}{4 \sin^2 \frac{\theta}{2}} \left[ \left( \frac{\partial}{\partial \phi} + 2is \right)^2 - 4c_2 \right] \Phi = -i(\tilde{l}(\tilde{l} + 1)) \Phi,$$

(2.5)

where $\tilde{l}(\tilde{l} + 1)$ is the separation constant. Solution of Eq. (2.5) is well discussed in Ref. [10], and is of the form

$$\Phi_{jm}^{(s)}(\theta, \phi; \delta_1, \delta_2) = N_{jm}(\delta_1, \delta_2) \left( \cos \frac{\theta}{2} \right)^{m_1} \left( \sin \frac{\theta}{2} \right)^{m_2} P_{jm+(m_1)}^{(m_2, m_1)}(\cos \theta)e^{i(m-s)\phi}.$$

(2.6)

Here

$$m_1 = |m - s| + \delta_1 = \sqrt{(m - s)^2 + 4c_1},$$
$$m_2 = |m + s| + \delta_2 = \sqrt{(m + s)^2 + 4c_2},$$

(2.7)

and

$$m_+ = (|m + s| + |m - s|)/2.$$  

(2.8)

$P_n^{(a,b)}$ is a Jacobi polynomial. For a fixed value of $j$ the quantum number $m$ runs through values:

$$m = -j, -j + 1, \ldots, j - 1, j,$$

(2.9)

whereas the allowed values of $j$ are

$$j = |s|, |s| + 1, |s| + 2, \ldots.$$

(2.10)

The separation constant $\tilde{l}$ is of the form

$$\tilde{l} = j + \frac{\delta_1 + \delta_2}{2}.$$  

(2.11)

From Eq. (2.7) and Eq. (2.10) it is clear that $\tilde{l}$ can have only positive values. $N_{jm}(\delta_1, \delta_2)$ is the normalization constant. The radial part of the differential equation obtained from Eq. (2.3), Eq. (2.4) and Eq. (2.5) is of the form

$$H_{\tilde{l}}(r)R(r) = ER(r),$$

(2.12)

where the effective formal radial Hamiltonian $H_{\tilde{l}}(r)$ is

$$H_{\tilde{l}}(r) = -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{r} + \frac{i(\tilde{l}(\tilde{l} + 1))}{2r^2}.$$  

(2.13)

The bound state solution of the eigenvalue problem Eq. (2.3) is previously known [10]. However since the Hamiltonian self-adjointness of Eq. (2.13) is important and the operator Eq. (2.13) is not self-adjoint in
a chosen domain, we will solve the eigenvalue problem by making a self-adjoint extensions of the radial Hamiltonian in the next section.

The generalized MIC-Kepler system can easily be reduced to MIC-Kepler system By making \( c_1 = c_2 = 0 \) and MIC-Kepler system is known to be \( O(4) \) invariant for the bound state. It should therefore be relevant to mention the generators of the \( O(4) \) group for MIC-Kepler system here. The angular momentum and the Runge-Lenz vector are given respectively by

\[
\begin{align*}
\hat{J} &= r \times (-i\nabla - s A) - \frac{r}{r} \\
\hat{I} &= \frac{1}{2} [(-i\nabla - s A) \times \hat{J} - \hat{J} \times (-i\nabla - s A)] + \frac{r}{r}.
\end{align*}
\]

The generators \( \hat{J}, \hat{I} \) together with the Hamiltonian \( H(s, c_1, c_2) \) form the \( o(4) \) algebra. In case of generalized MIC-Kepler system since two extra potentials with coefficients \( c_1 \) and \( c_2 \) are added with the MIC-Kepler Hamiltonian, the constants of motion should be modified. The square of the generalized angular momentum is given by

\[
\mathcal{J}^2 = \hat{J}^2 + \frac{2c_1}{1 + \cos \theta} + \frac{2c_2}{1 - \cos \theta},
\]

with eigenvalue \( (j + \frac{1}{2}, \pm \frac{1}{2}) \) \( (j + \frac{1}{2}, \pm 1) \). The \( z \) component of the angular momentum remains the same

\[
\hat{J}_z = s - i \frac{\partial}{\partial \phi}.
\]

But the Runge-Lenz vector gets modified. The \( z \) component of the modified Runge-Lenz vector is given by

\[
\hat{I}_z = I_z + c_1 \frac{r - z}{r(r + z)} - c_2 \frac{r + z}{r(r - z)}.
\]

### 3 Study of self-adjointness of radial differential Hamiltonian

We now move to the analysis of the effective differential operator Eq. (2.13). For simplicity of calculation we remove the first order derivative term in Eq. (2.13). It can be done by the transformation \( R(r) \to \frac{X(r)}{r} \) in Eq. (2.12) and the resulting radial Hamiltonian becomes

\[
\mathcal{H}_t(r) = -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{r} + \frac{\tilde{\mathcal{L}}(l + 1)}{2r^2},
\]

which is formally self-adjoint [18].

To study the self-adjointness of this operator we first need to associate this formal differential operator \( \mathcal{H}_t(r) \) to a closed symmetric operator \( T \{ \mathcal{H}_t(r) \} \) whose domain \( D(T) \) has to be chosen suitably. The operator \( T \{ \mathcal{H}_t(r) \} \) in the symmetric domain \( D(T) \), takes the same form of Eq. (2.3). To find out the domain \( D(T) \) we need to consider the symmetry condition

\[
\int_0^\infty \phi_1(r)^T T \{ \mathcal{H}_t(r) \} \phi_2(r) dr = \int_0^\infty [ T \{ \mathcal{H}_t(r) \} \phi_1(r) ]^T \phi_2(r) dr,
\]

which can be satisfied if the functions \( \phi_1(r), \phi_2(r) \) satisfy the condition

\[
(\phi_1(r) \phi_2'(r))_{r=0} - (\phi_1'(r) \phi_2(r))_{r=0} = 0.
\]

This can be easily satisfied if \( \phi_1(r), \phi_2(r) \) belong to the domain

\[
D(T) = \{ \phi(r) : \phi(r = 0) = \phi'(r = 0) = 0, \text{ absolutely continuous, square integrable on the half line with measure } dr \}.
\]
Here $\phi'(r)$ is the derivative of $\phi(r)$ with respect to $r$. In order to ensure that the function $T \{ \mathcal{H}_f(r) \} \phi(r)$ is square integrable at $r = 0$, we need $\phi''(r)$ (second order derivative of $\phi(r)$) to be square integrable at $r = 0$. Which means near the origin the function should behave as $\phi(r) \sim r^k$, where $k > \frac{3}{2}$. Note that square integrability of $\phi''(r)$ at $r = 0$ automatically ensure the condition $\phi(r = 0) = \phi'(r = 0) = 0$, which has been used in the above Eq. (3.4). We can easily check that $T \{ \mathcal{H}_f(r) \} \phi(r)$ is indeed square integrable at $r = 0$ from

\[
\left( \frac{\phi''(r)}{r^2} + \frac{\phi(r)}{r^4} + \frac{\phi'(r)}{r^3} + \frac{\phi(r)}{r^2} + \frac{\phi''(r)}{r} \right) dr \sim 0.
\]

at $r = 0$, using the above mentioned condition $\phi(r) \sim r^k$, where $k > \frac{3}{2}$.

To study the self-adjointness of $T\{ H_f(r) \}$ with domain $D(T)$, we have to look for the solution of the equation

\[
T^\dagger \{ H_f(r) \} \phi^\pm = \pm i \phi^\pm,
\]

where $T^\dagger \{ H_f(r) \}$ is the adjoint of $T\{ H_f(r) \}$. The differential form of the adjoint operator $T^\dagger \{ H_f(r) \}$ remains the same as Eq. (3.1). This can be easily checked from the Green’s formula, which relate operator with its adjoint. The domain of the adjoint operator $T^\dagger \{ H_f(r) \}$ can be calculated from

\[
\int_0^\infty \phi(r)^\dagger T \{ H_f(r) \} \zeta(r) dr = \int_0^\infty [T^\dagger \{ H_f(r) \} \phi(r)]^\dagger \zeta(r) dr \quad \forall \zeta \in D(T).
\]

The relation Eq. (3.7) will be satisfied if $\phi(r)$ ( $\phi(r)$ should belong to the adjoint domain $D(T^\dagger)$, which we need to find out) and $\zeta(r) \in D(T)$ satisfy the condition

\[
(\phi(r)\zeta'(r))_{r=0} - (\phi'(r)\zeta(r))_{r=0} = 0.
\]

From Eq. (3.7) and Eq. (3.8) we can see that $\phi(r)$ should belong to the most general domain

\[
D(T^\dagger) = \{ \phi(r) : \text{absolutely continuous, square integrable on the half line} \}
\]

with measure $dr$, in order to satisfy Eq. (3.7). Obviously $T \{ H_f(r) \}$ is not self adjoint, because

\[
D(T) \neq D(T^\dagger).
\]

Now let us return to the Eq. (3.6). There are two solutions for each differential equation Eq. (3.6). But taking square integrability (at infinity) into account we have to discard one solution. The remaining one, apart from normalization, is found to be

\[
\phi^\pm (2\varepsilon^\pm r) = (2\varepsilon^\pm r)^{\frac{3}{2}} e^{-\varepsilon^\pm r} U(a^\pm; b; 2\varepsilon^\pm r),
\]

where $\varepsilon^\pm = \sqrt{\pm 2i}$ (we should take $\varepsilon^\pm = \sqrt{2}\varepsilon^r e^\pm i\pi/4$ here, so that $\phi^\pm(2\varepsilon^\pm r)$ of Eq. (3.11) does not blow up at spatial infinity), $a^\pm = j + 1 + \frac{\delta_1 - \delta_2}{2} - \frac{1}{4\varepsilon}$ and $b = 2j + \delta_1 + \delta_2 + 2$. $U$ is the Kummer’s second function and its form is given as:

\[
U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left[ \frac{M(a, b, z)}{\Gamma(1 + a - b) \Gamma(b)} - z^{1-b} M(1 + a - b, 2 - b, z) \frac{\Gamma(2-b)}{\Gamma(a) \Gamma(2-b)} \right],
\]

where $M(a, b, z)$ is the Kummer’s first function.

To check square integrability of Eq. (3.11) at infinity we have to consider asymptotic value of the function $U(2\varepsilon^\pm r)$, which is in the limit $r \to \infty$ given by

\[
U(a, b, 2\varepsilon^\pm r) \to (2\varepsilon^\pm r)^{-a} [1 + \mathcal{O}((2\varepsilon^\pm r)^{-1})] \quad \text{for} \quad \Re(2\varepsilon^\pm r) > 0.
\]
So, solutions of Eq. (3.11) go to zero as \( r \) goes to infinity due to the decaying exponential terms \( e^{-\varepsilon^\pm r} \).

Now in the limit \( r \to 0 \), \( M(a, b, z) \to 1 \). Together with Eq. (3.11) and Eq. (3.12), this implies that in the limit \( r \to 0 \),

\[
|\phi(2\varepsilon^\pm r)|^2 dr \to [C_1 r^{2-b} + C_2 r + C_3 r^b + h.o.] dr,
\]

where \( C_1, C_2, C_3 \) are constants and \( h.o \) is higher order (in powers of \( r \) terms. Note that the nontrivial coefficient \( C_1 \) is nonzero here. From Eq. (3.14) it is clear that we can get square integrable solutions Eq. (3.11) only for \(-1 < b < 3\), which implies:

\[
-\frac{3}{2} < \tilde{l} = j + \frac{\delta_1 + \delta_2}{2} < +\frac{1}{2}.
\]

But since \( \tilde{l} \) can take only positive values, the effective range for square integrability becomes

\[
0 \leq \tilde{l} = j + \frac{\delta_1 + \delta_2}{2} < +\frac{1}{2},
\]

which is what we get in case of Hydrogen atom problem. The only difference is that, in case of Hydrogen atom problem \( \tilde{l} \) in Eq. (3.10) should be replaced by angular momentum quantum number \( l \), which can take non negative integral values only. Obviously for Hydrogen atom problem only \( l = 0 \) states belong to the specified range Eq. (3.10) and we can perform self-adjoint extensions for \( l = 0 \) states only.

Beyond the range Eq. (3.10) there is no square integrable solution of Eq. (3.6). The existence of these complex eigenvalues in the range Eq. (3.16) of \( T^\dagger \) signifies that \( T \) is not self-adjoint in that range. The solution \( \phi^\pm \) belong to the null space \( D^\pm \) of \( T^\dagger \) and \( D^\pm \subset D(T^\dagger) \). The dimension of \( D^\pm \) are known as deficiency indices \( n^\pm \) and is defined by

\[
n^\pm = dim(D^\pm).
\]

Depending upon \( n^\pm \), \( T \) is classified as [47]:

1) \( T \) is self-adjoint if \( n^+ = n^- = 0 \).

2) \( T \) has a \( n^2 \)-parameter (real) family of self-adjoint extensions if \( n^+ = n^- = n \neq 0 \).

3) \( T \) has no self-adjoint extensions if \( n^+ \neq n^- \).

Since in the range Eq. (3.10), the deficiency indices \( n^+ = n^- = 1 \), which is evident from Eq. (3.11), we can have a 1-parameter family of self-adjoint extensions of \( T \). The self-adjoint extensions of \( T \) is given by \( T^\omega \) with domain \( D(T^\omega) \), where

\[
D(T^\omega) = \{ \psi(r) = \phi(r) + \phi^+(r) + e^{i\varphi}(r) : \phi(r) \in D(T), \varphi \in \mathbb{R}(\text{mod}2\pi) \}.
\]

Beyond the range Eq. (3.10) the operator \( T \) is self-adjoint. The analysis of the spectrum is same as discussed in many papers [10, 33, 53].

The self-adjoint radial Hamiltonian \( T^\omega \{ \mathcal{H}(r) \} \) is given by the right hand side of Eq. (3.1) and the domain is given by Eq. (3.18). Bound state solution of this operator \( T^\omega \) is given by, apart from normalization,

\[
\chi(r) = (2\varepsilon r)^{\frac{3}{2}} e^{-\varepsilon r} U(a; b; 2\varepsilon r),
\]

where \( \varepsilon = \sqrt{-2E} \), \( a = j + 1 + \frac{\delta_1 + \delta_2}{2} - 1 \) and \( b = 2j + \delta_1 + \delta_2 + 2 \). To find out the eigenvalue we have to match the function \( \chi(r) \) with the domain Eq. (3.18) at \( r \to 0 \). In the limit \( r \to 0 \),

\[
\chi(r) \to C \left[ -\frac{(2\varepsilon)^{-i}}{\Gamma(a)\Gamma(2-b)} (r)^{-i} + \frac{(2\varepsilon)^{1-i}}{\Gamma(a)\Gamma(2-b)} \left( \frac{1}{2-b} - \frac{1}{2-b} \right) (r)^{1-i} \right.
\]

\[
+ \frac{(2\varepsilon)^{i+1}}{\Gamma(1+a-b)\Gamma(b)} (r)^{i+1} + \mathcal{O}(r^{2-i}) \left. \right],
\]

(3.20)
and

\[ \phi(r) + \phi^+(r) + e^{i\omega} \phi^-(r) \rightarrow D \left[ \frac{1}{\Gamma(2-b)} \left( \frac{(2\varepsilon^+)^{-i}}{\Gamma(\varepsilon^+)} + \frac{e^{i\omega}(2\varepsilon^-)^{-i}}{\Gamma(\varepsilon^-)} \right) (r)^{-i} \right. \\
+ \left. \frac{1}{\Gamma(2-b)} \left( \frac{(2\varepsilon^+)^{-1-i}}{\Gamma(\varepsilon^+)} \left( \varepsilon^+ - \frac{1+a^+ - b}{2-b} \right) + \frac{e^{i\omega}(2\varepsilon^-)^{-1-i}}{\Gamma(\varepsilon^-)} \left( \varepsilon^- - \frac{1+a^- - b}{2-b} \right) \right) (r)^{1-i} \right] + O(r^{2-i}). \]

(3.21)

It should be noted that, in Eq. (3.21) there is no contribution from \( \phi(r) \), because if we make Taylor series expansion of the function \( \phi(r) \) near \( r = 0 \) and use the definition of the domain Eq. (3.14) we get the the leading order term \( \phi(r) = O(r^2) \), which has been ignored in the right hand side in Eq. (3.21). Equating coefficients of same powers of \( r \) in Eq. (3.20) and Eq. (3.21) we get

\[ C(2\varepsilon)^{-i} \Gamma(a) = D \left[ \frac{(2\varepsilon^+)^{-i}}{\Gamma(\varepsilon^+)} \right], \]

(3.22)

\[ C(2\varepsilon)^{-1-i} \Gamma(a) \left( \varepsilon - \frac{1+a-b}{2-b} \right) = D \left[ \frac{(2\varepsilon^+)^{-1-i}}{\Gamma(\varepsilon^+)} \left( \varepsilon^+ - \frac{1+a^+ - b}{2-b} \right) + \frac{e^{i\omega}(2\varepsilon^-)^{-1-i}}{\Gamma(\varepsilon^-)} \left( \varepsilon^- - \frac{1+a^- - b}{2-b} \right) \right], \]

(3.23)

\[ C((2\varepsilon)^{-1} \Gamma(a) = D \left[ \frac{(2\varepsilon^+)^{-1}}{\Gamma(\varepsilon^+)} \left( \varepsilon^+ - \frac{1+a^+ - b}{2-b} \right) + \frac{e^{i\omega}(2\varepsilon^-)^{-1}}{\Gamma(\varepsilon^-)} \left( \varepsilon^- - \frac{1+a^- - b}{2-b} \right) \right]. \]

(3.24)

Now we should mention one point regarding Eq. (3.22), Eq. (3.23) and Eq. (3.24) here. If we would have written the Hamiltonian Eq. (3.1) near the origin like

\[ H(r) = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{i(\hat{l} + 1)}{2r^2}, \]

(3.25)

neglecting the \( \frac{1}{r} \) term compared to \( \frac{i(\hat{l} + 1)}{2r^2} \) which is usually done in case of Hydrogen atom problem to find out the behavior of the wave function near \( r = 0 \), we would get the solution of the form

\[ \chi(2\varepsilon r) \sim A(2\varepsilon)^{-i} + B(2\varepsilon)^{1+i}. \]

(3.26)

Keeping Eq (3.20) in mind we will just compare between Eq. (3.22) and Eq. (3.24), and the result is given by

\[ f(E) = \frac{\Gamma(1+a-b)}{\Gamma(a)} \varepsilon^{b+1} = \frac{\chi_1 \cos(\omega/2 - \frac{\pi l}{4} + \beta_1)}{\chi_2 \cos(\omega/2 + \frac{\pi l}{4} + \frac{\pi}{4} + \beta_2)}, \]

(3.27)

where \( \Gamma(a^+) = \chi_2 2^{\frac{l}{2}} e^{\pm i\beta_1} \) and \( \Gamma(1+a^+ - b) = \chi_2 2^{\frac{l}{2}} e^{\pm i\beta_2} \). We can calculate the eigenvalue analytically when the right hand side is \( 0 \) or \( \infty \). When \( R.H.S = 0 \), ie. when \( \omega = \pi + \frac{\pi l}{4} - 2\beta_1 \), we get from left hand side \( \Gamma(a) = \Gamma(j+1 + \frac{\delta_1 + \delta_2}{2} - \frac{1}{2}) = \Gamma(-n) \), which implies

\[ E_n = -\frac{1}{2(n+1+j + \frac{\delta_1 + \delta_2}{2})^2}; n = 0, 1, 2, \ldots \]

(3.28)

On the other hand when \( R.H.S = \infty \), ie. when \( \omega = \pi - \frac{\pi l}{4} + \frac{\pi}{2} - 2\beta_2 \), we get from left hand side \( \Gamma(1+a-b) = \Gamma(-j-1 - \frac{\delta_1 + \delta_2}{2} - \frac{1}{2}) = \Gamma(-n) \), which implies

\[ E_n = -\frac{1}{2(n-1-j - \frac{\delta_1 + \delta_2}{2})^2}; n = 0, 1, 2, \ldots \]

(3.29)

For values other than \( 0 \) and \( \infty \) of the r.h.s of Eq. (3.27) one can calculate the eigenvalues numerically like Ref. 34, 51, 52. But we are not going to plot it in our work. Rather we make the following observations on our self-adjoint extensions of generalized MIC-Kepler system.
3.1 Case 1

Consider the situation $\tilde{l} = j + \frac{\delta_1 + \delta_2}{2} = 0$. This correspond to the effective potential

$$V_0 = -\frac{1}{r}. \quad (3.30)$$

This happens when $j = \delta_1 = \delta_2 = 0$. $\delta_1$ and $\delta_2$ can be made zero by choosing $c_1 = c_2 = 0$ and $j = 0$ can be made by choosing $s = 0$ and considering lowest value of angular momentum, as evident from Eq. (2.7), Eq. (2.10) and Eq. (2.11). But for $s = c_1 = c_2 = 0$, generalized MIC-Kepler system reduces to Hydrogen atom problem. Since $\tilde{l} = 0$ lies inside the range of Eq. (3.16), we can have 1-parameter family of self-adjoint extensions. The above constraint when substituted in Eq. (3.28) gives the well known Bohr level

$$E_n = -\frac{1}{2(n+1)^2}; n = 0, 1, 2, \cdots \quad (3.31)$$

which corresponds to the self-adjoint extension parameter $\omega = \pi + \frac{\pi}{2} - 2\beta_1 (j = \delta_1 = \delta_2 = 0)$. One point should be mentioned here that the condition $j = \delta_1 = \delta_2 = 0$ will not make Eq (3.27) well behaved for negative integral values of $a$, because that makes the l.h.s $= \infty$. So instead of putting $j = \delta_1 = \delta_2 = 0$ directly we will consider the limit $b \to 2$ in Eq. (3.27) to make it well behaved. Any departure from $\omega = \pi + \frac{\pi}{2} - 2\beta_1 (j = \delta_1 = \delta_2 = 0)$ shifts the Bohr levels either to higher energy levels or lower energy levels depending upon what values of $\omega$ are being considered. This shifts from the usual bohr levels is called the Rydberg correction or the quantum defect [40] in atomic physics. The modified Bohr levels can be written in the form

$$E_n = -\frac{1}{2(n+1+\Delta_1)^2}; n = 0, 1, 2, \cdots \quad (3.32)$$

where $\Delta_1$, which plays the role of quantum defect, depends upon the self-adjoint extension parameter $\omega$.

3.2 Case 2

Consider the situation $\tilde{l} = j + \frac{\delta_1 + \delta_2}{2} \neq 0$, but $\tilde{l} < \frac{1}{2}$. This correspond to the effective potential

$$V_0 = -\frac{1}{r} + \frac{\tilde{l}(\tilde{l} + 1)}{2}; \quad \tilde{l} < \frac{1}{2}. \quad (3.33)$$

This happens when $j = 0$ and $\delta_1 + \delta_2 < 1$. By appropriately choosing $c_1$ and $c_2$, sum of two delta can be made less that one. $j = 0$ can be made by choosing $s = 0$ and considering lowest value of angular momentum, as evident from Eq. (2.7), Eq. (2.10) and Eq. (2.11). But for $s = 0$ and $c_1, c_2 \neq 0$, generalized MIC-Kepler system reduces to Kepler-Coulomb problem, when $c_1 \neq c_2$. For $s = 0$ and $c_1 = c_2 \neq 0$, generalized MIC-Kepler system reduces to Hartmann system. In these two cases also we can have 1-parameter family of self-adjoint extensions. In these two cases the spectrum for which the r.h.s of Eq. (3.27) is zero is given by

$$E_n = -\frac{1}{2(n+1+\frac{\delta_1 + \delta_2}{2})^2}; n = 0, 1, 2, \cdots \quad (3.34)$$

where $\omega = \pi + \frac{\pi}{2} - 2\beta_1 (j = 0, c_1, c_2 \neq 0)$. For other values of $\omega$ the spectrum should be modified and it can be written as

$$E_n = -\frac{1}{2(n+1+\frac{\delta_1 + \delta_2}{2} + \Delta_2)^2}; n = 0, 1, 2, \cdots \quad (3.35)$$

where $\Delta_2$ is the quantum correction.
3.3 Case 3

Consider the situation $\hat{l} = j + \frac{\delta_1 + \delta_2}{2} \neq 0$, but $\hat{l} > \frac{1}{2}$. This correspond to the effective potential

$$V_{\hat{l}} = -\frac{1}{r} + \frac{\hat{l}(\hat{l} + 1)}{r^2}, \quad \hat{l} > \frac{1}{2}. \quad (3.36)$$

In this case since the deficiency space solutions are not square integrable, deficiency indices become $< 0, 0 >$. So there is no self-adjoint extensions. The system is essentially self-adjoint. For chargedyion case and MIC-Kepler case, $s \neq 0$; and minimum value which $j$ can take is $j = |s|$. So in these cases the corresponding radial Hamiltonian operator is essentially self-adjoint everywhere. For essentially self-adjoint case, the unique spectrum is given by

$$E_n = -\frac{1}{2(n + 1 + j + \frac{\delta_1 + \delta_2}{2})^2}; \quad n = 0, 1, 2, \cdots \quad (3.37)$$

4 Conclusion

In conclusion, self-adjointness of the quantum observables is an important issue. So when an operator is not self-adjoint in a domain one should search for the possible self-adjoint extensions. Because, it not only ensures the eigenvalue to be real but also through this we can explore new spectrum which was not possible in usual analysis. It has been successfully applied in different quantum mechanical models [40, 43, 44, 47, 48, 52]. In our present work we have considered generalized MIC-Kepler system and we have found out a 1-parameter family of self-adjoint extensions of the system for $\hat{l} < \frac{1}{2}$. we have shown that $\hat{l} = 0$ has a 1-parameter family of self-adjoint extensions. $\hat{l} = 0$, corresponds to the orbital angular momentum quantum no $l = 0$ for the Hydrogen atom problem. For all other values of the orbital angular momentum quantum no ($l \neq 0$), Hydrogen atom radial Hamiltonian is essentially self-adjoint.

In case of Kepler-Coulomb system with $j = 0$ and $\delta_1 + \delta_2 < 1$ one gets again a 1-parameter family of self adjoint extensions. For all other values of angular momentum ($j \neq 0$), the system is essentially self adjoint.

Hartmann system with $j = 0$ and $\delta_1 + \delta_2 < 1$ also has a 1-parameter family of self adjoint extensions. For all other values of angular momentum ($j \neq 0$), the system is essentially self adjoint.

Since for Charge-dyon system and MIC-Kepler system, minimum value of $j$ is $\frac{1}{2}$, we don’t have self-adjoint extensions. System is essentially self-adjoint in these cases.

Acknowledgments

We thank B. Basu-Mallick, Kumar S. Gupta and Palash B. Pal for comments on the manuscript and helpful discussions.

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