BRS COHOMOLOGY IN TOPOLOGICAL STRING THEORY AND INTEGRABLE SYSTEMS

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Abstract

In cohomological field theory we can obtain topological invariants as correlation functions of BRS cohomology classes. A proper understanding of BRS cohomology which gives non-trivial results requires the equivariant cohomology theory. Both topological Yang-Mills theory and topological string theory are typical examples of this fact. After reviewing the role of the equivariant cohomology in topological Yang-Mills theory, we show in purely algebraic framework how the $U(1)$ equivariant cohomology in topological string theory gives the gravitational descendants. The free energy gives a generating function of topological correlation functions and leads us to consider a deformation family of cohomological field theories. In topological strings such a family is controlled by the theory of integrable system. This is most easily seen in the Landau-Ginzburg approach by looking at the contact term interactions between topological observables.

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1 Introduction

The principle of BRS quantization tells us that a BRS exact term has no contribution to the physical quantities, for example, the physical $S$ matrix elements. The states annihilated by the BRS charge are called physical. The observables should commute with the BRS charge. The expectation values of observables with respect to the physical states only depend on the BRS cohomology classes they define. Hence, the BRS cohomology plays a key role both in conceptual analysis and in practical computations. In cohomological quantum field theory the action has topological BRS symmetry and the notion of BRS cohomology becomes more essential. However, the observables of our interest (even any BRS closed operators) are often formally BRS exact and the naive argument of decoupling of BRS exact operators prevents us from obtaining topological invariants we expect. This fact is closely related to the nature of cohomological field theory sometimes claimed as no propagating degrees of freedom in the theory. One of ways, in our opinion, to manage this subtle business is to employ the equivariant cohomology theory \cite{1, 30, 22, 34}. (See also an excellent lecture note \cite{6}.) The main aim of this article is to show how the equivariant cohomology in topological string gives a rich spectrum of the gravitational descendants and how the contact term interactions between them naturally give rise to the integrable system which governs the theory.

Let us begin with reviewing some basic structures of cohomological quantum field theories \cite{40}. The property which characterizes cohomological field theories is that the energy momentum tensor is BRS exact;

\[ T_{\mu\nu} = \{ Q_B, \Lambda_{\mu\nu} \} , \]

where $Q_B$ is the topological BRS charge. The observables $O_I$ are defined to be $Q_B$ cohomology classes as usual. If the vacuum is annihilated by the BRS charge, we can see by the standard argument in BRS quantization procedure the vacuum expectation value of topological observables $\langle O_{I_1} O_{I_2} \cdots O_{I_n} \rangle$ defines a (possible) topological invariant in the sense that it is independent of the background metric. Here we have used the fact that the variation of correlation functions with respect to the background metric is obtained by the insertion of the energy momentum tensor. It is crucial in this argument that the
topological BRS symmetry is not spontaneously broken. We require that
\[ \langle 0 | [Q_B, \chi] | 0 \rangle = 0. \] (2)

But we have to be careful about what kind of the operator ‘\( \chi \)’ in (2) is “admissible”, when we use it. This is the point which is answered by the equivariant cohomology.

Introducing a parameter \( t_I \) for each observable \( O_I \), we can define a generating function of topological correlation functions;
\[
F[t_I] = \left< \exp(\sum_I t_I O_I) \right>,
\]
\[
\langle O_{I_1} O_{I_2} \cdots O_{I_n} \rangle = \frac{\partial^n F}{\partial t_{I_1} \partial t_{I_2} \cdots \partial t_{I_n}}. \] (3)

Explicit examples are discussed below. The generating function \( F[t_I] \) organizes the topological invariants in a nice way. For example, by deriving recursion relations among topological correlation functions one may get a deep insight which is hard to see for those who look at each invariant separately. Such topological recursion relations or topological Ward identities are expressed in terms of differential equations for \( F[t_I] \) which sometimes allow geometrical interpretation. Furthermore, it is possible to think of the generating function as the path integral by the action with local deformations of the form \( \mathcal{L} \rightarrow \mathcal{L} + \sum t_I O_I \). This means that topological invariants are generated by the deformations of local Lagrangian. The fact that the topological invariants have local density is quite important, since we will make full use of the machinery of local quantum field theory in the path integral computation of topological invariants. The most intriguing aspect of cohomological quantum field theory is that we have a description of global topological quantities in the framework of local quantum field theories. In this way, we are naturally led to the geometry of deformation family of cohomological field theories. One of the reasons the BRS approach is so natural and powerful in cohomological field theory is that we are looking at the deformation theory of local quantum field theories. Historically, the BRS approach enjoyed its first success in the renormalization of non-abelian gauge theory. The viewpoint that the theory of renormalization is a kind of the deformation theory suggests that one of the common aspects in BRS approach to local quantum field theory is the idea of deformation, where the mathematical concept of cohomology is useful. A nice
and typical example in mathematics is the Kodaira-Spencer theory of the deformation of complex structure.

## 2 Topological Yang-Mills theory

To illustrate the idea of the equivariant cohomology, we first look at 4-dimensional topological Yang-Mills theory briefly. The topological BRS transformation is given as follows [3, 20]:

$$
\begin{align*}
\delta A_\mu & = \psi_\mu - D_\mu c , \\
\delta \psi_\mu & = -D_\mu \phi + [\psi, c] , \\
\delta c & = \phi - \frac{1}{2}[c, c] , \\
\delta \phi & = [\phi, c] ,
\end{align*}
$$

where $A_\mu$ is a gauge field and $c$ is the Faddeev-Popov ghost. The remaining ghost fields $\psi_\mu$ and $\phi$, which are absent in the physical Yang-Mills theory, are topological BRS partners of $A_\mu$ and $c$, respectively, as the transformation law indicates. This transformation law can be identified as the canonical coboundary operator on the Weil algebra of the universal moduli space [23]. From the second Chern class of this Weil algebra we can construct a series of operators;

$$
\begin{align*}
\mathcal{O}^{(0)} & = \frac{1}{8\pi^2} \text{Tr}\phi^2 , \\
\mathcal{O}^{(1)} & = \frac{1}{4\pi^2} \text{Tr}\phi \psi , \\
\mathcal{O}^{(2)} & = \frac{1}{4\pi^2} \text{Tr}(\phi F_\mathcal{A} + \frac{1}{2} \psi \wedge \psi) , \\
\mathcal{O}^{(3)} & = \frac{1}{4\pi^2} \text{Tr}\psi \wedge F_\mathcal{A} , \\
\mathcal{O}^{(4)} & = \frac{1}{8\pi^2} \text{Tr}F_\mathcal{A} \wedge F_\mathcal{A} ,
\end{align*}
$$

which satisfies the descent equation;

$$
\begin{align*}
d\mathcal{O}^{(4)} & = 0 \\
\delta \mathcal{O}^{(n)} + d\mathcal{O}^{(n-1)} & = 0 \quad (1 \leq n \leq 4) \\
\delta \mathcal{O}^{(0)} & = 0 .
\end{align*}
$$

The operator $\mathcal{O}^{(n)}$ is a space-time $n$-form with ghost number $(4 - n)$. For a simply connected four manifold topological observables are $\mathcal{O}^{(0)}(x)$ and

$$
I(\Sigma) = \int_\Sigma \mathcal{O}^{(2)} ,
$$

3
where $\Sigma \in H_2(M,\mathbb{Z})$ is a closed two dimensional surface. The descent equation implies that the topological correlation function $\langle I(\Sigma_1)I(\Sigma_2)\cdots I(\Sigma_n)\rangle$ only depends on the homology class of $\Sigma_i$. It is these correlation functions which give a field theoretical realization of the cerebrated Donaldson polynomials.

Now our problem is that $I(\Sigma)$ looks formally BRS exact, due to the identity
\begin{equation}
8\pi^2 O^{(2)} = \delta[\text{Tr}(A \wedge \psi + cdA)] + d[\text{Tr}(c\psi + \phi A - \frac{1}{2}[\phi,\phi]A)] ,
\end{equation}
which follows from the triviality of cohomology of the Weil algebra. Hence, if we believe in the naive decoupling of BRS exact operators, we lose the Donaldson polynomials. However, the idea of the equivariant cohomology saves the situation. In the equivariant theory we restrict the BRS operator on the space of the “basic” cochains which satisfy:
\begin{equation}
\mathcal{L}_V \chi = \iota_V \chi = 0 .
\end{equation}
More precisely, in the $G$-equivariant theory we have two actions of the Lie group $G$, given by the Lie derivative $\mathcal{L}_V$ and the interior product $\iota_V$. The cochain space of the equivariant cohomology is defined to be the fixed points of both actions. Together with the exterior derivative (the BRS operator in our context), $\mathcal{L}_V$ and $\iota_V$ constitute basic operations in Cartan’s differential calculus. In the next section we will see that these operations are naturally accommodated in (topological) string theory. The first term of the right hand side of eq. (8) is not an admissible operator in the equivariant cohomology, because
\begin{equation}
\iota_V \text{Tr}(A \wedge \psi + cdA) = \text{Tr}(VdA) ,
\end{equation}
where $V$ is a vector field along the orbit of the gauge transformation group $G$. We should take the $G$-equivariant cohomology of BRS operator in topological Yang-Mills theory. In eq. (10) we have followed the usual geometrical identification of the Faddeev-Popov ghost and regarded it as a basis of one forms (the Maurer-Cartan forms) along the gauge orbit.

The topological observable $I(\Sigma)$ is non-trivial in the $G$-equivariant cohomology. The generating function $F[t]$ of topological invariants in topological Yang-Mills theory is
\begin{equation}
F[\alpha_a,\lambda] = \langle \exp(\lambda O^{(0)}(x) + \sum_{a=1}^{b_2} \alpha_a I(\Sigma_a)) \rangle ,
\end{equation}
where $b_2$ is the second Betti number and $\{\Sigma_{\alpha}\}$ is a basis of the second homology group of the four manifold. One of the most striking recent developments in topological Yang-Mills theory is that $F[\alpha, \lambda]$ is evaluated exactly for a large class of four manifolds (called simple type) [11]. The expression of $F[\alpha, \lambda]$ involves the classical topological data encoded in the intersection form on $H_2(M, \mathbb{Z})$ and the number of the solutions to the (abelian) monopole equation as a new “quantum” information. To answer the questions from the side of the traditional quantum field theory, for example, the issue of spontaneous symmetry breaking of topological symmetry [2], it is desirable to have a concept of states associated with four manifolds (with boundary). In string theory we know such a construction of topological states for (punctured) Riemann surfaces in terms of the Universal Grassmannian. In the computation of $F[\alpha, \lambda]$ mentioned above there appear distinguished two dimensional homology classes called the Seiberg-Witten class. These classes regarded as two dimensional world sheets are “cosmic strings” embedded in the four manifold [12]. Hence, four manifolds with string world sheets playing the role of generalized punctures may be natural objects in trying to construct topological states in four dimensions. In any case it would be interesting to see if the recent progress in understanding four dimensional topological theory and its cousins gives some clues in this direction.

### 3 Topological string theory

Topological string theory has the following symmetry of topological conformal algebra, which can be obtained by twisting the $N = 2$ superconformal algebra [10, 9];

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} , \\
[L_m, G_n] &= (m - n)G_{m+n} , \\
[L_m, Q_n] &= -nQ_{m+n} , \\
[L_m, J_n] &= -nJ_{m+n} - \frac{d}{2}m(m + 1)\delta_{m+n,0} , \\
[J_m, Q_n] &= Q_{m+n} , \\
[J_m, G_n] &= -G_{m+n} , \\
[J_m, J_n] &= nQ_{m+n} + \frac{d}{2}m(m + 1)\delta_{m+n,0} , \\
[Q_m, G_n] &= L_{m+n} + mJ_{m+n} + \frac{d}{2}m(m + 1)\delta_{m+n,0} .
\end{align*}
\]

We have the same algebra in the anti-holomorphic sector, which will be denoted with bar in the following. The holomorphic sector and the anti-holomorphic sector (anti-)commute each other. The constant $d$ is a $U(1)$ current anomaly and it is the central anomaly anomaly and it is the central anomaly anomaly and it is the central anomaly anomaly and it is the central anomaly anomaly and it is the central anomaly.
extension of the algebra. We identify $Q(z) = \sum Q_{-n}z^{n-1}$ as topological BRS current and $J(z) = \sum J_{-n}z^{n-1}$ as ghost number current. The BRS charge is defined by $Q_B = \oint dQ(z) + \oint d\bar{Q}(\bar{z})$. Now we observe the crucial identity

$$T(z) = [Q_B, G(z)] , \quad Q(z) = [J(z), Q_B] ,$$

which justifies the name “topological”. The energy momentum tensor $T(z) = \sum L_{-n}z^{n-2}$ is topological BRS “daughter” of the super current $G(z) = \sum G_{-n}z^{n-2}$. It is curious to compare the first relation with the basic relation in the (bosonic) string theory;

$$T^{\text{tot}}(z) = [Q_{Vir}, b(z)] ,$$

for the Virasoro BRS charge $Q_{Vir}$ and the reparametrization anti-ghost $b(z)$, which is not to be confused with the Nakanishi-Lautrup field. The total energy momentum tensor $T^{\text{tot}}$ has a ghost contribution, so that the total Virasoro central charge vanishes. This common algebraic structure leads us to several observations which suggest that any string theory is topological in some sense. Furthermore, as several authors have noticed before this algebraic structure is a stringy version of the Cartan’s differential calculus. We recognize the correspondence;

$$(d, \mathcal{L}_V, \iota_V) \longleftrightarrow (Q_{Vir}, T^{\text{tot}}(z), b(z)) \longleftrightarrow (Q_B, T(z), G(z)) .$$

A model of topological string theory is obtained by taking a topological matter theory (a twisted $N = 2$ superconformal model) coupled to topological gravity. The essential part of topological gravity is the topological Virasoro ghost system $(c, b, \gamma, \beta)$ which determines the “measure” on the moduli space. The commuting ghosts $(\gamma, \beta)$ are topological BRS partners of the Virasoro ghosts $(c, b)$. Concerning the two dimensional metric variables, there are several options of a field theoretical realization. The spectrum of the theory consists of the primaries, which come from the chiral ring of the $N = 2$ theory, and their gravitational descendants. The dressing operator which creates the descendants from the primaries arises from topological gravity sector. It is a field theoretical realization of the non-trivial cohomology class on the moduli space of (punctured) Riemann surface and is an analogue of $I(\Sigma)$ in topological Yang-Mills theory.
which represents the non-trivial cohomology class on the instanton moduli space. After coupling to gravity the topological BRS charge should be

\[ \hat{Q}_B = Q_{\text{Susy}} + Q_{\text{Vir}} , \]
\[ Q_{\text{Vir}} = \oint c(z)\hat{T}(z) - \oint \gamma(z)\hat{G}(z) . \]  

(16)

The gauge part \( Q_{\text{Vir}} \) with \( \hat{T}(z) = T_{\text{matter}} + T_{\text{metric}} + \frac{1}{2} T_{\text{ghost}} \) and \( \hat{G}(z) = G_{\text{matter}} + G_{\text{metric}} + \frac{1}{2} G_{\text{ghost}} \) is the canonical BRS operator for the (super) Virasoro algebra. The super charge \( Q_{\text{Susy}} \) comes from the (twisted) \( N = 2 \) super symmetry which is present in any two dimensional topological theory. Here and below only the holomorphic sector is explicitly written. But the same expressions with bar are applied to the anti-holomorphic sector too.

Using the trick of similarity transformation [25], or constructing an appropriate homotopy operation, we can find a representative of descendants which only contains the matter degrees of freedom [13]. This has an advantage that we do not have to rely on explicit realizations of topological gravity and enables us to use twisted \( N = 2 \) models to investigate topological string. The homotopy operation we employ is

\[ U = \exp\left(-\oint dz c(z)\hat{G}(z)\right) . \]  

(17)

This homotopy transformation simplify the BRS transformation;

\[ U (Q_{\text{Susy}} + Q_{\text{Vir}}) U^{-1} = Q_{\text{Susy}} . \]  

(18)

at the expense of the following shift in the Virasoro anti-ghost \( b(z) \);

\[ UbU^{-1} = b + G_{\text{tot}} . \]  

(19)

The appearance of the super current \( G(z) \) plays a crucial role in the following argument of the equivariant cohomology in string theory. After coupling to topological gravity, we have to look at the equivariance condition. Originally this is imposed on the topological ghost sector and called the semi-relative condition in the closed string theory;

\[ (b_0)^{-1}\text{state} = 0 , \]  

(20)
where the superscript $-\text{ }$ means taking the difference of the holomorphic sector and the anti-holomorphic sector, e.g. $(b_0)^- = b_0 - \bar{b}_0$. The semi-relative condition is responsible for the non-decoupling of the dilaton vertex operators in the closed string theory [12]. The symmetry in question is a change of the base point of the parametrization of the closed string by $S^1$. It is a $U(1)$ symmetry which is present in any closed string theory. Now if we perform the similarity transformations introduced above, the semi-relative condition is transformed into

$$(b_0 + G_0)^-|\text{state}\rangle = 0 .$$

(21)

For the states depending only on the matter degrees of freedom, we get

$$(G_0)^-|\text{state}\rangle_{\text{matter}} = 0 .$$

(22)

Taking the correspondence (15) into account, we see this is nothing but the condition of “basic” cochains (cf. eq.(9)) for the $U(1)$ equivariant cohomology in string theory.

A convenient realization of topological matter is provided by the Landau-Ginzburg description, which is a powerful tool in a classification of $N = 2$ superconformal models as fixed points of renormalization group flow and also in a construction of superstring vacua [29, 38, 20]. We will use the Landau-Ginzburg model to examine the above idea. The integrable structure is most easily seen in the Landau-Ginzburg approach as we will show below. In superspace the action takes the following form;

$$\mathcal{L} = \int d^2z d^4\theta \ K(X_A, \bar{X}_A) + \int d^2z d^2\theta_+ W(X_A) + \int d^2z d^2\theta_- \bar{W}(\bar{X}_A) ,$$

(23)

where the chiral superfields $X_A(z, \theta_+)$ are treated as the Landau-Ginzburg variables. Due to the non-renormalization theorem, the model is characterized by the superpotential $W(X_A)$. Let us introduce the following notations for the component fields;

$$X_A = \ x_A + \theta_+ \psi_A + \bar{\theta}_+ \bar{\psi}_A + \theta_+ \bar{\theta}_+ F_A ,$$

$$\bar{X}_A = \ x_A + \theta_- \rho_A + \bar{\theta}_- \bar{\rho}_A + \theta_- \bar{\theta}_- \bar{F}_A .$$

(24)

Eliminating the auxiliary fields by the equation of motion;

$$F_A = \partial_A \bar{W} := \frac{\partial W}{\partial \bar{X}_A} , \quad \bar{F}_A = \partial_A W := \frac{\partial W}{\partial X_A} ,$$

(25)
we arrive at the action in component fields;

\[ \mathcal{L} = \int d^2 z \left( |\partial x_A|^2 + \overline{\psi}_A \partial \rho_A + \overline{\psi}_A \partial A \overline{\partial A} + |\partial A W|^2 + (\partial A \partial B W) \psi_A \overline{\psi}_B + (\overline{\partial A} \overline{\partial B} \overline{W}) \rho_A \rho_B \right). \]  \hspace{1cm} (26)

After topological twisting which changes the spin of fermions, \((\psi_A, \overline{\psi}_A)\) are zero forms and \((\rho_A, \overline{\rho}_A)\) are one forms on the world sheet. The topological BRS transformations are \[ \delta x_A = 0, \quad \delta \overline{\partial A} = \psi_A + \overline{\psi}_A, \]
\[ \delta \psi_A = \partial A W, \quad \delta \overline{\psi}_A = -\partial A W, \]
\[ \delta \rho_A = -\partial x_A, \quad \delta \overline{\rho}_A = -\overline{\partial x}_A. \] \hspace{1cm} (27)

Though the action is not BRS exact, the energy-momentum tensor is BRS exact;

\[ T_{zz} = -\delta (\overline{\partial A} \rho_A). \] \hspace{1cm} (28)

Hence the super-current of the Landau-Ginzburg model is identified with

\[ G_{zz} = -\overline{\partial A} \rho_A. \] \hspace{1cm} (29)

We note that the super-current \(G_{zz}\) is independent of the super potential. On the other hand the BRS current does depend on the potential.

Now let us consider the most simple example of \(A_k\) type super potential;

\[ W = \frac{1}{k+2} X^{k+2}, \quad (\partial W = X^{k+1}), \] \hspace{1cm} (30)

which has a single Landau-Ginzburg variable \(X\) and describes the deformations of topological minimal models. (For multi-variable case, our understanding of integrable structure is still poor.) The primary fields, which coincide with the chiral ring of the \(N = 2\) theory, are

\[ \mathcal{R} = \{1, X, X^2, \ldots, X^k\}. \] \hspace{1cm} (31)

After coupling to topological gravity we have to look at the equivariance condition \[ \hspace{1cm} (22). \] Since the super current \(G_{zz}\) has conformal weight two, its zero mode is

\[ G_0 = \oint \! dzz (\overline{\partial x}_A \rho_A). \] \hspace{1cm} (32)

\footnote{There is another type of BRS transformations in topological Landau-Ginzburg model which looks better in some respects. I am grateful to F. De Jonghe to point it out in the symposium.}
For any polynomial $P(x)$ in $x$, we see

$$\left[G_0, P(x)\right] = 0,$$  \hspace{1cm} (33)

since we have only simple pole in operator product expansion. If the polynomial has the form $P(x) = \partial W \cdot Q(x)$, then it is BRS exact;

$$P(x) = \left[Q_B, (\psi - \bar{\psi})Q(x)\right].$$  \hspace{1cm} (34)

In fact this is the reason we get the chiral ring (31) before coupling to gravity. Now we have

$$\left[G_0, (\psi - \bar{\psi})Q(x)\right] \neq 0,$$  \hspace{1cm} (35)

since the double pole is created in the operator product expansion in this case. The additional contribution comes from $\phi - \rho$ contraction. Thus, the higher order polynomials are not BRS exact in the $U(1)$ equivariant cohomology in topological string. We can check they actually define the same cohomology class as the gravitational descendants. In the picture we have introduced above the coupling to topological gravity is effectively achieved by imposing the equivariance condition on topological matter sector. Comparing eq.(10) in topological Yang-Mills theory with eq.(35) above, we see that the observable $I(\Sigma)$ for the Donaldson polynomials and the gravitational descendants, typically the dilaton operator in string theory, have the common feature.

### 4 Integrable Deformation

We have seen that the $U(1)$ equivariant cohomology in topological string theory gives us a rich spectrum of the gravitational descendants. For each primary field $\phi_\alpha$, ($\alpha = 1, 2, \cdots, \text{dim.R}$), we have a tower of its descendants denoted by $\sigma_n(\phi_\alpha), \ (n = 1, 2, \cdots)$. In the Landau-Ginzburg description of the minimal models these descendants correspond to higher order polynomials in the Landau-Ginzburg variable. We will show more precise identification below. Following a general prescription, we introduce the free energy of topological string;

$$F[t_{\alpha,n}] = \exp\left(\sum_{\alpha,n} t_{\alpha,n}\sigma_n(\phi_\alpha)\right),$$  \hspace{1cm} (36)
where $n = 0$ part stands for the primaries. A standard way of associating an integrable structure with topological string is to claim that $F[t_{\alpha,n}]$ is the logarithm of a tau function of KP or Toda lattice hierarchy \[7\]. The basic generator of the $U(1)$ equivariant cohomology allows us to introduce a spectral parameter and the gravitational descendants are identified with the Hamiltonians of higher integrable flows. In the Landau-Ginzburg approach the super potential is naturally promoted to (the dispersionless limit of) the Lax operator. This observation makes the Landau-Ginzburg approach extremely useful. From such a correspondence we can easily recognize the ADE type reductions of KP hierarchy as the integrable structure of topological strings with the ADE type potential \[9\]. In a similar manner, though they are a little bit involved, we can identify certain reductions of Toda lattice hierarchy as the integrable structure of the $c = 1$ string theory \[8, 19, 21, 35, 14\] and the topological $CP^1$ string theory together with its generalizations \[17, 24\].

In the language of local quantum field theory what we are looking at is the deformation of topological action;

$$
\mathcal{L}(t) = \mathcal{L}_0 + \sum_\alpha t_\alpha \int_\Sigma \phi_\alpha^{(2)} + \sum_{\alpha,n} t_{\alpha,n} \int_\Sigma \sigma_n(\phi_\alpha)^{(2)},
$$

(37)

where $\phi_\alpha^{(2)}$ and $\sigma_n(\phi_\alpha)^{(2)}$ are two form observables related to the original zero form ones $\phi_\alpha^{(0)} \equiv \dot{\phi}_\alpha$ and $\sigma_n(\phi_\alpha)^{(0)} \equiv \sigma_n(\phi_\alpha)$ by the descent equation;

$$
d\phi_\alpha^{(0)} = \left[ Q_B, \phi_\alpha^{(1)} \right],
$$

$$
d\phi_\alpha^{(1)} = \left[ Q_B, \phi_\alpha^{(2)} \right].
$$

(38)

In fact the topological conformal algebra enables us to solve the descent equation in the following form;

$$
\Phi^{(2)} = G_{-1} \overline{G_{-1}} \Phi^{(0)} d^2 z,
$$

(39)

for any zero form cohomology class $\Phi^{(0)}$. In the Landau-Ginzburg method the deformation of topological string may be described by the perturbed potential. For example, we introduce

$$
W(X, t) = \frac{1}{k + 2} X^{k+2} + \sum u_i(t) X^i,
$$

(40)
in the case of $A_k$ type potential. The perturbed potential is assumed to satisfy the condition \[a\]:

\[
\langle \phi_{I_1} \phi_{I_2} \cdots \phi_{I_n} \rangle_{L(t)} = \langle \phi_{I_1}(X,t) \phi_{I_2}(X,t) \cdots \phi_{I_n}(X,t) \rangle_{W(X,t)} . \tag{41}
\]

The left hand side is topological correlation function in the deformed action $L(t)$. In the right hand side one can compute the correlation function by taking a summation over all the critical points of the potential $W(X,t)$ \[b\]:

\[
\langle \phi_{I_1}(X,t) \cdots \phi_{I_n}(X,t) \rangle_{W(X,t)} = \sum_{\text{critical points}} H^{g-1}(X,t) \phi_{I_1}(X,t) \cdots \phi_{I_n}(X,t) , \tag{42}
\]

where $g$ is the genus and $H$ is the Hessian of $W$. Note that in the Landau-Ginzburg description the observables are defined by

\[
\phi_I(X,t) = \frac{\partial}{\partial t_I} W(X,t) , \tag{43}
\]

and, hence, they are $t$-dependent. This is consistent to the fact that the topological BRS charge is also deformed according to

\[
Q_B(t) = Q_B(0) - \sum t_I \oint \phi^{(1)}_I . \tag{44}
\]

Thus in the Landau-Ginzburg description all the informations of deformation are encoded in the perturbed potential $W(X,t)$. The existence of $W(X,t)$ which satisfies the condition \[c\] is a key to the integrable structure of topological string theory. It seems that the secret of the relation \[d\] is still not well understood yet.

We compute the perturbed potential $W(X,t)$ by assuming a formal power series expansion in the deformation parameters. The coefficients of the expansion are obtained by estimating the (multi-) contact terms. They are a result of contact term interactions between topological observables. At the lowest order\[e\], the second derivatives of $W(X,t)$ are given by the basic contact terms $C(\phi_I, \phi_J)$:

\[
\frac{\partial^2 W}{\partial t_I \partial t_J} = \partial_I \phi_J = \partial_J \phi_I = C(\phi_I, \phi_J) . \tag{45}
\]

\[3\]The first derivatives are the perturbed primaries by definition.
In string theory we can implement the state-operator correspondence by the path integral on the hemisphere with a fixed boundary condition \[5\]. Based on this correspondence, we have the following interpretation of \( \partial_I \phi_J \);

\[
\partial_I \phi_J = \int_D \phi_I^{(2)} | \phi_J \rangle ,
\]

where \( D \) is any small disk around the insertion point of \( \phi_J \). We can think of the result of integration over \( D \) as a result of contact interaction of \( \phi_I \) and \( \phi_J \). The prescription of computing the basic contact term \( C(\phi_I, \phi_J) \) is as follows. We first decompose the product \( \phi_I \cdot \phi_J \) into two parts; the first part is a linear combination of the primary fields and the other part is (formally) BRS exact;

\[
\phi_I \cdot \phi_J = C^\alpha_{IJ} \phi_\alpha + [Q_B, \psi^- \lambda_{IJ}] .
\]

Then the contact term comes from the BRS exact part by the formula \[5, 28\]:

\[
C(\phi_I, \phi_J) = \partial_X \lambda_{IJ} .
\]

Again the BRS exact term plays an important role in the game. We can generalize this method of computation to the multi-contact terms involving more than two operators. The higher derivatives of \( W(X,t) \) are expressed in terms of these contact terms. For example the third derivatives are;

\[
\frac{\partial^3 W}{\partial t_I \partial t_J \partial t_K} = (C(C(\phi_I, \phi_J), \phi_K) + \text{cyclic}) - C^{(3)}(\phi_I, \phi_J, \phi_K) ,
\]

where the last term is the higher contact term of three operators. Thus the contact term interactions between physical operators control the perturbed potential.

For the topological minimal models the perturbed superpotential is obtained exactly in the small phase space where only the couplings \( t_\alpha \) to the primaries are turned on \[3\]. The perturbed primary fields are given by

\[
\phi_\alpha(X,t) = \frac{1}{\alpha + 1} \partial_X \left( L^{\alpha+1} \right)_+ , \quad (\alpha = 0, 1, \cdots, k) ,
\]

where \( L \) is defined by the relation

\[
W(X,t) = \frac{1}{k+2} L(X,t)^{k+2} , \quad L(X,t) = X + \cdots ,
\]
and \((\cdot)_+\) means taking the non-negative power part. It is apparent that the Laurent polynomial \(L\) plays the role of the Lax operator in the theory of integrable system. More precise treatment of \(L\) as a Lax operator is given in the theory of dispersionless integrable hierarchy \([13, 36]\). The leading term of \(\phi_\alpha(X, t)\) is \(X^\alpha\) and the lower order terms are developed by deformations. After coupling to gravity the polynomials of higher power serve as the gravitational descendants. Extrapolating \((50)\), we get \([13]\);

\[
\sigma_n(\phi_\alpha) = N_{n, \alpha} \partial_X \left[ L^{(k+2)n+\alpha} \right]_+, \tag{52}
\]

where \(N_{n, \alpha}\) is some normalization constant. With the Lax like operator \(L\), this identification is natural in view of the property that the gravitational descendants generate the higher flows of the integrable hierarchy. Now the gravitational descendants have the following “Hodge decomposition”;

\[
\sigma_n(\phi_\alpha) = \sum_\beta C_{(n, \alpha)}^\beta \phi_\beta + \partial_X W \int_X \sigma_{n-1}(\phi_\alpha), \tag{53}
\]

with the first term regarded as harmonic form. By the formula of the contact terms \([18]\) we immediately see the fundamental recursion relation;

\[
C(\phi_0, \sigma_n(\phi_\alpha)) = \sigma_{n-1}(\phi_\alpha). \tag{54}
\]

We note that the primary field \(\phi_0 = 1\) is identified as the puncture operator \(P\) after coupling to gravity. Upon this identification the contact term \((54)\) implies the puncture equation \([11]\);

\[
\langle P\sigma_{n_1}(\phi_{\alpha_1}) \cdots \sigma_{n_k}(\phi_{\alpha_k}) \rangle = \sum_{\ell=1}^k \langle \sigma_{n_1}(\phi_{\alpha_1}) \cdots \sigma_{n_{\ell-1}}(\phi_{\alpha_{\ell-1}}) \cdots \sigma_{n_k}(\phi_{\alpha_k}) \rangle. \tag{55}
\]

In deriving the puncture equation the first term of the decomposition \((53)\) does not contribute. But the harmonic part is important to obtain the following topological recursion relation at genus zero;

\[
\langle \sigma_n(\phi_\alpha) \phi_\beta \phi_\gamma \rangle = \sum_{\delta=0}^k C_{(n, \alpha)}^\delta \langle \phi_\delta \phi_\beta \phi_\gamma \rangle, \tag{56}
\]

\[
= \sum_{\delta=0}^k \langle \sigma_{n-1}(\phi_\alpha) \phi_\delta \rangle \langle \phi_\delta \phi_\beta \phi_\gamma \rangle.
\]
The topological recursion relation (56) has a clear geometrical meaning on the moduli space of the Riemann sphere with punctures [43]. Thus our formula (52) of the gravitational descendants gives two basic relations among topological correlation functions. It is known that (at genus zero) these recursion relations are equivalent to the Virasoro constraint on the partition functions and enough to fix it uniquely [10, 18]. In this sense the gravitational descendant obtained as the $U(1)$ equivariant cohomology class is the key to the integrable structure. We think that the relation of the equivariant cohomology and the integrability deserves further studies.

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