Inhomogeneous Cosmology, Inflation and Late-Time Accelerating Universe

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Abstract

An exact inhomogeneous solution of Einstein’s field equations is shown to be able to inflate in a non-uniform way in the early universe and explain anomalies in the WMAP power spectrum data. It is also possible for the model to explain the accelerated expansion of the universe by late-time inhomogeneous structure.

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1 Introduction

In the following, we shall investigate an exact inhomogeneous cosmological solution of Einstein’s field equations obtained by Szekeres [1], for an irrotational dust dominated universe and subsequently generalized by Szafron [2] and Szafron and Wainright [3, 4] to the case when the pressure \( p \) is non-zero. The cosmological model contains the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology and the spherically symmetric Lemaître-Tolman-Bondi (LTB) model [5, 6, 7] as special solutions. The Szafron model describes a more general inhomogeneous spacetime with no prescribed initial symmetry [8].

We will show that the Szafron inhomogeneous cosmology in the early universe near the Planck time can lead to an inhomogeneous inflationary period, which can predict that the primordial power spectrum is not uniform across the sky. Recent investigations have revealed that there appears to be a lack of power in the CMB power spectrum above \( \sim 60^\circ \) and anisotropy in hot and cold spots on the sky [9, 10, 11, 12, 13]. Moreover, there has been the claim that there is a peculiar alignment between the quadrupole and octopole moments called the “axis of evil” [13]. These anomalies do not agree with the standard homogeneous and isotropic inflationary
models [14, 15, 16, 17]. The anomalies in the WMAP data could be explained by the inhomogeneous solution presented in the following. Several possible explanations for the anomalies in the WMAP data have been proposed including an exact planar solution of Einstein’s field equations [18], the violation of Lorentz symmetry and rotational invariance [19, 20, 21, 22] and a cutoff of inflation that leads to a loss of primordial power spectrum [23].

The problem of explaining the acceleration of the universe as determined by supernovae data and the cosmic microwave background (CMB) data is one of the most significant outstanding problems in modern physics and cosmology [24, 25, 26, 27]. The standard explanation is either based on postulating a cosmological constant \( \Lambda \) or assuming that some form of uniform dark energy with negative pressure exists in the universe [28]. The explanation as to why the cosmological constant is zero or very small has led to a crisis in physics and cosmology.

The LTB model has been used to explain the non-Gaussian behavior observed in the WMAP data [26, 27] and give a possible explanation for the late-time acceleration of the universe [29, 30, 31, 32]. A criticism of the LTB model is that it assumes a spherically symmetric universe with one spatial degree of inhomogeneity, requiring a center of the universe and that observers be located not too far from the center to avoid undetected large anisotropy. With this restriction the LTB solution has provided a toy-model description of the late-time, non-linear regime with voids and collapsing matter.

2 Inhomogeneous Cosmological Solution

The metric takes the form

\[
ds^2 = dt^2 - R^2(x, t)(dx^2 + dy^2) - S^2(x, t)dz^2,
\]

where \( R(x, t) \) and \( S(x, t) \) are to be determined from Einstein’s field equations:

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} + \Lambda g_{\mu\nu},
\]

where \( T_{\mu\nu} \) denotes the perfect fluid energy-momentum tensor

\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},
\]

and \( \Lambda \) is the cosmological constant. Moreover, \( \rho \) denotes the energy-density of matter, \( p \) the pressure and \( u_\mu \) the velocity field of the fluid which is normalized to \( u^\mu u_\mu = 1 \). The coordinates are assumed to be comoving so that \( u^\mu = \delta^\mu_0 \) and \( \dot{u}^\mu = 0 \) where \( \dot{u}^\mu = du^\mu/dt \). Let us introduce the notation

\[
R(x, t) = \exp(\beta(x, t)),
\]

\[1\] We use units with the speed of light \( c=1 \).
and
\[ S(x, t) = \exp(\alpha(x, t)). \] (5)

Szafron [2, 3, 4] solved the Einstein equations with \( p \neq 0 \) and \( \Lambda = 0 \), generalizing the dust solution of Szekeres [1]. There are two classes of solution \( \beta' \neq 0 \) and \( \beta' = 0 \) where \( \beta' = \partial \beta / \partial z \). The more general solution \( \beta' \neq 0 \) is given for \( \Lambda = 0 \) by

\[ R(x, t) \equiv \exp(\beta(x, t)) = a(z, t) \exp(\nu(x)), \] (6)
\[ S(x, t) \equiv \exp(\alpha(x, t)) = h(z) \exp(-\nu(x)) \partial \exp(\beta(x, t)/\partial z), \] (7)

where
\[ \exp(-\nu(x)) = A(z)(x^2 + y^2) + 2B_1(z)x + 2B_2(z)y + C(z). \] (8)
The \( A(z), B_1(z), B_2(z), C(z) \) and \( h(z) \) are arbitrary functions of \( z \). Moreover, \( a(z, t) \) satisfies the equation

\[ \frac{2\dot{a}(z, t)}{a(z, t)} + \frac{\dot{a}^2(z, t)}{a^2(z, t)} + \frac{k(z)}{a^2(z, t)} = -8\pi G p(t), \] (9)

which has the same form as one of the Friedmann equations in FLRW cosmology, except that \( a = a(z, t) \) and \( k = k(z) \). The function \( k(z) \) is determined by

\[ k(z) = 4 \left( A(z)C(z) - B_1^2(z) - B_2^2(z) - \frac{1}{4h^2(z)} \right). \] (10)

The case \( \beta' \to 0 \) is singular. Eq. (9) can be formally integrated once \( p(t) \) is specified. The density equation is given by

\[ \ddot{\alpha}(x, t) + 2\dot{\beta}(x, t) + \dot{\alpha}^2(x, t) + 2\dot{\beta}^2(x, t) = -4\pi G (\rho(x, t) + 3p(t)). \] (11)

An algorithm for generating an exact solution is to specify explicitly \( p = p(t) \) and solve (9) for \( a(z, t) \). The metric is now obtained by solving for \( R(x, t) \) and \( S(x, t) \) from Eqs. (6), (7) and (5). The equation for the density \( \rho(x, t) \) is given by (11).

We shall consider, in the following, the simpler Szafron solution with \( \beta' = 0 \) given by

\[ R(x, t) \equiv \exp(\beta(x, t)) = \frac{a(t)}{1 + \frac{1}{4}k(x^2 + y^2)}, \] (12)
\[ S(x, t) \equiv \exp(\alpha(x, t)) = \lambda(z, t) + a(t)\Sigma(x), \] (13)
\[ \Sigma(x) = \frac{\frac{1}{2}U(z)(x^2 + y^2) + V_1(z)x + V_2(z)y + 2W(z)}{1 + \frac{1}{4}k(x^2 + y^2)}. \] (14)

Now \( k \) is a constant, \( U(z), V_1(z), V_2(z) \) and \( W(z) \) are arbitrary functions of \( z \) and \( a(t) \) is determined by the Friedmann equation

\[ \frac{2\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)} = -8\pi G p(t). \] (15)
We can choose without loss of generality $W(z) = 0$ and $\lambda(z, t)$ is determined by
\[
\ddot{\lambda}(z, t)a(t) + \dot{\lambda}(z, t)\dot{a}(t) + \lambda(z, t)a(t)\ddot{a}(t) = -8\pi G\lambda(z, t)a(t)p(t) + U(z). \tag{16}
\]
The matter density equation is given by
\[
-\frac{2}{3} \left[ \ddot{\lambda}(z, t) - \lambda(z, t)\frac{\ddot{a}(t)}{a(t)} \right] \exp(-\alpha(x)) + H^2(t) + \frac{k}{a^2(t)} = \left( \frac{8\pi G}{3} \right) \rho(x, t), \tag{17}
\]
where $H(t) = \dot{a}(t)/a(t)$. Eqs. (15) and (16) can be solved once the pressure $p = p(t)$ is specified, while (17) determines the density $\rho(x, t)$. The FLRW spacetime is obtained when $\lambda(z, t) = U(z) = 0$. When $U(z) = 0$ and $V_1(z) = V_2(z) = 0$, the model possesses a 3-dimensional symmetry group acting on 2-dimensional orbits. The symmetry is spherical, plane or hyperbolic when $k > 0, k = 0$ or $k < 0$, respectively.

When $k = 0$ the line element becomes
\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - S^2(x, t)dz^2. \tag{18}
\]
A cosmological solution is obtained for $a(t)$ and $S(x, t)$ when $p(t)$ and $\rho(x, t)$ are specified.

### 3 Coarse-Grained Spatial Averaging of Inhomogeneous Model

For our exact inhomogeneous cosmological model, we are required to carry out a volume averaging of physical quantities. We define for a scalar quantity $\Psi$ a coarse-grained spatial smoothing \[33, 34, 35\]:
\[
\langle \Psi(x, t) \rangle_D = \frac{1}{V_D} \int_D d^3 x \sqrt{\gamma} \Psi(x, t), \tag{19}
\]
where
\[
V_D = \int_D d^3 x \sqrt{\gamma} \tag{20}
\]
is the volume of the simply-connected domain, $D$, in a hypersurface. We can define effective scale-factors for our spatially averaged cosmological model:
\[
R_D(t) = \langle R(x, t) \rangle_D = \left( \frac{V(t)_{RD}}{V_{iD}} \right)^{1/3}, \tag{21}
\]
\[
S_D(t) = \langle S(x, t) \rangle_D = \left( \frac{V(t)_{SD}}{V_{iD}} \right)^{1/3}, \tag{22}
\]
where $V_{iD}$ is the initial spatial volume.
We define the spatially averaged Hubble parameters

\[ H_{RD}(t) = \frac{\dot{R}_D(t)}{R_D(t)}, \quad H_{SD}(t) = \frac{\dot{S}_D(t)}{S_D(t)} \]  

and the effective Hubble expansion parameter

\[ H_{\text{eff}}(t) = \frac{1}{3}(2H_{RD}(t) + H_{SD}(t)). \]  

Consider a congruence of curves with a time-like unit vector \( V^\mu \) with \( g_{\mu\nu}V^\mu V^\nu = 1 \) and \( dV^\mu/ds = \nabla_\nu V^\mu V^\nu \) is the acceleration of the flow lines. Here, \( \nabla_\mu \) and \( s \) denote the covariant derivative with respect to \( g_{\mu\nu} \) and the proper time, respectively. The metric tensor \( h_{\mu\nu} \) is given by

\[ h_{\mu\nu} = \delta_{\mu\nu} + V^\mu V^\nu \]  

and describes the metric that projects a vector into its components in the subspace of the vector tangent space that is orthogonal to \( V \).

We define the vorticity tensor

\[ \omega_{\mu\nu} = \nabla_{[\nu} V_{\mu]} \]  

and the shear tensor

\[ \sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} \theta h_{\mu\nu}, \]  

where

\[ \theta_{\mu\nu} = \nabla_{(\nu} V_{\mu)}. \]  

The covariant derivative of \( V \) can be expressed as

\[ \nabla_\nu V_\mu = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} h_{\mu\nu} \theta - \frac{dV_\nu}{ds} V_\mu. \]  

The volume expansion is given by

\[ \theta \equiv h^{\mu\nu} \theta_{\mu\nu} = \nabla_\mu V^\mu. \]  

The volume averaging of the scalar \( \Psi \) does not commute with its time evolution \[33\,34\,35\]:

\[ \langle \dot{\Psi}(x, t) \rangle_D - \partial_t \langle \Psi(x, t) \rangle_D = \langle \dot{\Psi}(x, t) \rangle_D \langle \theta(x, t) \rangle_D - \langle \dot{\Psi}(x, t) \theta(x, t) \rangle_D. \]  

For inhomogeneous cosmology, the smoothing due to averaging of the Einstein field equations does not commute with the time evolution of the non-linear field equations. This leads to extra contributions in the effective, averaged Einstein field equations, which do not satisfy the usual energy conditions even though they are satisfied by the original energy-momentum tensor. It is the lack of commutativity of the time evolution of the expansion of the universe in a local patch inside our Hubble horizon, that circumvents the no-go theorem based on the local Raychaudhuri equation \[36\,37\], namely, that the expansion of the universe cannot accelerate when the weak and strong energy conditions: \( \rho > 0 \) and \( \rho + 3p > 0 \) are satisfied.
4 Inhomogeneous Inflationary Model and Microwave Background

Let us now consider the very early universe and for simplicity \( k = 0 \). We choose

\[
p = -Z = \text{const.,}
\]

where \( Z > 0 \). Then, a de Sitter solution of Eq.\( (15) \) is given by

\[
a(t) = \exp(Ha t), \quad Ha = \sqrt{\frac{8\pi G Z}{3}}.
\]

We have from Eqs.\( (16) \) and \( (17) \):

\[
\ddot{\lambda}(z, t) + \dot{\lambda}(z, t)H_a + \lambda(z, t)H_a^2 = 8\pi G Z \lambda(z, t) + \exp(-Ha t)U(z),
\]

and

\[
\lambda(z, t)H_a^2 - \ddot{\lambda}(z, t) + \frac{3}{2}H_a^2 \exp(\alpha(x)) = 4\pi G \rho(x, t) \exp(\alpha(x)).
\]

We now obtain from \( (12) \) and \( (13) \) the metric

\[
ds^2 = dt^2 - \exp(2Ha t)(dx^2 + dy^2) - [\lambda(z, t) + \exp(Ha t)\Sigma(x)]^2 dz^2.
\]

We see that as \( t \to \infty \) the universe inflates in the \( x-y \) plane but could for \( \Sigma \sim 0 \) be non-inflating along the \( z \) direction. For \( \lambda = 0 \) and \( \Sigma = 1 \), we regain the inflating de Sitter spacetime metric

\[
ds^2 = dt^2 - \exp(2Ha t)(dx^2 + dy^2 + dz^2).
\]

When inflation ends, the metric in the early radiation dominated universe should approach a homogeneous and isotropic FLRW metric as \( t \) increases with \( R(x, t) \sim a(t) \) and the equation of state \( p(t) = \rho(t)/3 \). After decoupling the FLRW spacetime is the matter dominated solution with zero pressure, \( p(t) = 0 \).

Let us investigate how the inhomogeneous solution can give rise to a non-uniform power spectrum. The power spectrum for homogeneous and isotropic inflation is given by

\[
\langle \delta(k), \delta(q) \rangle = P(k)\delta^3(k - q),
\]

where \( \delta(k) \) denotes the primordial density contrast and the translational invariance of the inflationary epoch leads to the modes with different wave numbers being uncoupled. Let us assume that there exists a vector in the anisotropic direction of unit vector \( n \). We assume parity symmetry \( k \to -k \) and denote by \( \tilde{P}(k) \) the power spectrum caused by the anisotropy of primordial spacetime. We have \( 19 \):

\[
\tilde{P}(k) = P(k)\left(1 + f(k)(\hat{k} \cdot n)^2\right),
\]
where $k = |\mathbf{k}|$, $\hat{\mathbf{k}}$ is the unit vector along the direction of $\mathbf{k}$ and we have kept only the lowest power of $\hat{\mathbf{k}} \cdot \mathbf{n}$. We require that the power spectrum be scale invariant, $P(k) \sim 1/k^3$, with $f(k)$ being independent of $k$, so that $f(k) \sim \mathcal{A}$ where $\mathcal{A}$ is a constant. We now have

$$\tilde{P}(k) = P(k) \left(1 + \mathcal{A}(\hat{\mathbf{k}} \cdot \mathbf{n})^2\right). \quad (40)$$

The inhomogeneous spacetime gives rise to correlations between multipole moments that normally are zero for homogeneous and isotropic inflation. Deviations from homogeneous and isotropic inflation can be parameterized by

$$\epsilon_I = \left(\frac{a(t) - R_D(t)}{K(t)}\right), \quad (41)$$

where $a(t) \sim \exp(H_0 t)$, $R_D(t)$ is the smoothed-out scale factor $\text{(21)}$ and $K(t)$ is the averaged cosmic scale:

$$K(t) = \frac{1}{3}(2R_D(t) + a(t)). \quad (42)$$

5. **Acceleration of the Late-Time Matter Dominated Universe**

Let us now consider the matter dominated solution with zero pressure, $p = 0$. We have

$$2 \frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)} = 0. \quad (43)$$

This equation has the standard Friedmann solution for $k = 0$:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}. \quad (44)$$

The matter equation is given by $\text{(17)}$ where

$$\exp(-\alpha(x)) = \left[\lambda(z, t) + \left(\frac{t}{t_0}\right)^{2/3} \Sigma(x)\right]^{-1}. \quad (45)$$

Let us define

$$\Omega_m(x, t) = \frac{8\pi G \rho(x, t)}{3H^2(t)}, \quad (46)$$

and for a spatially flat universe with $k = 0$:

$$\Omega_x(x, t) = \frac{2}[\ddot{\lambda}(z, t) - \lambda(z, t)\frac{\ddot{a}(t)}{a(t)} \exp(-\alpha(x))] \exp(-\alpha(x)). \quad (47)$$
We have
\[ \Omega_m(x, t) + \Omega_X(x, t) = 1. \] (48)

We define spatially averaged \( \Omega_m \) and \( \Omega_X \):
\[
\Omega_{mD}(t) = \langle \Omega_m(x, t) \rangle_D = \frac{1}{V_D} \int_D d^3x \sqrt{\gamma} \Omega_m(x, t),
\]
\[
\Omega_{XD}(t) = \langle \Omega_X(x, t) \rangle_D = \frac{1}{V_D} \int_D d^3x \sqrt{\gamma} \Omega_X(x, t),
\]
(49) (50)

It follows from (48) that
\[ \Omega_{mD}(t) + \Omega_{XD}(t) = 1. \] (51)

We obtain when \( \lambda = 0 \), the standard FLRW density relation for an Einstein-de Sitter universe:
\[ \Omega_m(t) = 1, \] (52)

where
\[ \Omega_m(t) = \frac{8\pi G}{3}\rho(t). \] (53)

By choosing the parameterization:
\[ \Omega_{mD}^0 = 0.28, \quad \Omega_{XD}^0 = 0.72, \] (54)

where \( \Omega_{mD}^0 \) and \( \Omega_{XD}^0 \) denote the present matter and inhomogeneity densities, we can fit the WMAP data [27] and the supernovae data [24, 25].

6 Conclusions

We have derived a primordial inhomogeneous inflationary model from an exact inhomogeneous solution of Einstein’s field equations. Because the model has a preferred direction of anisotropic inflation, it is possible to explain detected anomalies in the WMAP power spectrum such as the apparent alignment of the CMB multipoles on very large scales and the loss of power for \( \theta \geq 60^\circ \). The observed CMB temperature anisotropies can give a window on the primordial inflationary era. We have given generic predictions expected from the existence of a preferred inhomogeneous direction during inflation. The inhomogeneous inflation should reveal itself in a scale-invariant manner, and lead to predictions for the primordial fluctuation correlations determined by a single amplitude \( \mathcal{A} \) and a unit vector \( \mathbf{n} \) signifying a preferred anisotropic direction on the sky.

A possible explanation of the accelerated expansion of the universe at late times and the WMAP data is attributed to the observed late-time non-linear, inhomogeneous galaxy and void structure. Our exact inhomogeneous solution avoids the anti-Copernican assumption of a center to the universe, and it also avoids the postulates of an ill-understood cosmological constant and “dark energy”.
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