A NOTE ON 2-DISTANT NONCROSSING PARTITIONS AND WEIGHTED MOTZKIN PATHS

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Abstract. We prove a conjecture of Drake and Kim: the number of 2-distant noncrossing partitions of \{1, 2, \ldots, n\} is equal to the sum of weights of Motzkin paths of length \(n\), where the weight of a Motzkin path is a product of certain fractions involving Fibonacci numbers. We provide two proofs of their conjecture: one uses continued fractions and the other is combinatorial.

1. Introduction

A Motzkin path of length \(n\) is a lattice path from \((0, 0)\) to \((n, 0)\) consisting of up steps \(U = (1, 1)\), down steps \(D = (1, -1)\) and horizontal steps \(H = (1, 0)\) that never goes below the \(x\)-axis. The height of a step in a Motzkin path is the \(y\) coordinate of the ending point.

Given two sequences \(b = (b_0, b_1, \ldots)\) and \(\lambda = (\lambda_0, \lambda_1, \ldots)\), the weight of a Motzkin path with respect to \((b, \lambda)\) is the product of \(b_i\) and \(\lambda_i\) for each horizontal step and down step of height \(i\) respectively, see Figure 1. Let \(\text{Mot}_n(b, \lambda)\) denote the sum of weights of Motzkin paths of length \(n\) with respect to \((b, \lambda)\). This sum is closely related to orthogonal polynomials; see [5, 6].

Drake and Kim [1] defined the set \(\text{NC}_k(n)\) of \(k\)-distant noncrossing partitions of \([n] = \{1, 2, \ldots, n\}\). For \(k \geq 0\), a \(k\)-distant noncrossing partition is a set partition of \([n]\) without two arcs \((a, c)\) and \((b, d)\) satisfying \(a < b \leq c < d\) and \(c - b \geq k\), where an arc is a pair \((i, j)\) of integers contained in the same block which does not contain any integer between them. For example, \(\pi = \{\{1, 5, 7\}, \{2, 3, 6\}, \{4\}\}\) is a 3-distant noncrossing partition but not a 2-distant noncrossing partition because \(\pi\) has two arcs \((1, 5)\) and \((3, 6)\) with \(5 - 3 \geq 2\). Note that the 1-distant noncrossing partitions are the ordinary noncrossing partitions, which implies that \(# \text{NC}_1(n)\) is equal to the Catalan number \(\frac{1}{n+1} \binom{2n}{n}\). It is not difficult to see that \(\text{NC}_0(n)\) is in bijection

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Figure 1. A Motzkin path and the weights of its steps with respect to \((b, \lambda)\).
with the set of Motzkin paths of length $n$. In the same paper, they proved that

$$\sum_{n \geq 0} \# NC_2(n)x^n = \frac{3}{2} - \frac{1}{2} \sqrt{1 - \frac{5x}{1 - x}}$$

The number $\# NC_2(n)$ also counts many combinatorial objects: Schröder paths with no peaks at even levels, etc; see [2, 4, 7].

There are simple expressions of $\# NC_k(n)$ using Motzkin paths for $k = 0, 1, 3$:

$$\# NC_0(n) = \text{Mot}_n((1, 1, \ldots), (1, 1, \ldots)),$$
$$\# NC_1(n) = \text{Mot}_n((1, 2, 2, \ldots), (1, 1, \ldots)),$$
$$\# NC_3(n) = \text{Mot}_n((1, 2, 3, 3, \ldots), (1, 2, 2, \ldots)),$$

where the second equation is well known and the third one was first conjectured by Drake and Kim [1] and proved by Kim [3]. The main purpose of this paper is to prove the following theorem which was also conjectured by Drake and Kim [1].

**Theorem 1.1.** Let $b = (b_0, b_1, \ldots)$ and $\lambda = (\lambda_0, \lambda_1, \ldots)$ be the sequences with $b_0 = \lambda_0 = 1$ and for $n \geq 1$,

$$b_n = 3 - \frac{1}{F_{2n-1}F_{2n-3}} \quad \text{and} \quad \lambda_n = 1 + \frac{1}{F_{2n-1}},$$

where $F_m$ is the Fibonacci number defined by $F_0 = 0, F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for all $m$ (so $F_{-1} = 1$). Then we have

$$\# NC_2(n) = \text{Mot}_n(b, \lambda).$$

Theorem 1.1 is very interesting because it is not even obvious that $\text{Mot}_n(b, \lambda)$ is an integer. In this paper, we give two proofs of Theorem 1.1: one uses continued fractions and the other is combinatorial.

### 2. Continued Fractions

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_0, \beta_1, \beta_2, \ldots)$, and $c = (c_0, c_1, c_2, \ldots)$ be sequences of numbers.

Let $J(x; \alpha_0, \beta_0; \alpha_1, \beta_1; \alpha_2, \beta_2; \ldots) = J(x; \alpha, \beta)$ denote the $J$-fraction

$$\frac{1}{1 - \alpha_0 x - \frac{\beta_0 x^2}{1 - \alpha_1 x - \frac{\beta_1 x^2}{1 - \alpha_2 x - \ldots}}}$$

and let $S(x; c_0, c_1, \ldots) = S(x; c)$ denote the $S$-fraction

$$\frac{1}{1 - \frac{c_0 x}{1 - \frac{c_1 x}{1 - \ldots}}}$$

A Dyck path of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$ that never goes below the $x$-axis. The height of a step in a Dyck path is the $y$ coordinate of the ending point. The weight of a Dyck path with respect to $c$ is the product of $c_i$ for each down step of height $i$, see Figure 2. Let $\text{Dyck}_n(c)$ denote the sum of weights of Dyck paths of length $2n$ with respect to $c$. 
It is well known that
\[ \sum_{n \geq 0} \text{Mot}_n(\alpha, \beta)x^n = J(x; \alpha, \beta) \quad \text{and} \quad \sum_{n \geq 0} \text{Dyck}_n(c)x^n = S(x; c). \]

The following proposition is easy to see.

**Proposition 2.1.** If \( \alpha_n = c_{2n-1} + c_{2n} \) and \( \beta_n = c_{2n}c_{2n+1} \) for all \( n \geq 0 \), with \( c_{-1} = 0 \), then \( S(x; c) = J(x; \alpha, \beta) \).

One can prove Proposition 2.1 by the following observation: a Motzkin path may be obtained from a Dyck path by taking steps two at a time and changing \( U \) to \( DU \), \( D \) to \( DU \) and \( DD \), respectively, to \( U \), \( H \), \( H \) and \( D \). For example, the Motzkin path in Figure 1 is obtained from the Dyck path in Figure 2 in this way.

Let \( d = (d_0, d_1, d_2, \ldots) \) be the sequence with \( d_0 = 1 \) and for \( n \geq 1 \),

\[ d_{2n-1} = \frac{F_{2n-1}}{F_{2n-3}}, \quad d_{2n} = \frac{1}{d_{2n-1}}. \]

Recall the two sequences \( b = (b_0, b_1, \ldots) \) and \( \lambda = (\lambda_0, \lambda_1, \ldots) \) defined in (2).

**Lemma 2.2.** We have the following.

1. \( b_n = d_{2n-1} + d_{2n} \) for all \( n \geq 0 \), where \( d_{-1} = 0 \).
2. \( \lambda_n = d_{2n}d_{2n+1} \) for all \( n \geq 0 \).
3. \( 1/d_{2n-1} + d_{2n+1} = 3 \) for all \( n \geq 1 \).

**Proof.** We will use two cases of the well-known Catalan identity for Fibonacci numbers, \( F_m^2 - F_{m+i}F_{m-i} = (-1)^{m-i}F_i^2 \).

1. This is true for \( n = 0 \). For \( n \geq 1 \) we have

\[
d_{2n-1} + d_{2n} = \frac{F_{2n-1}}{F_{2n-3}} + \frac{F_{2n-3}}{F_{2n-1}} = \frac{F_{2n-1} + F_{2n-3}}{F_{2n-1}F_{2n-3}} = \frac{2F_{2n-1}F_{2n-3} + (F_{2n-1} - F_{2n-3})^2}{F_{2n-1}F_{2n-3}} = 2 + \frac{F_{2n-2}^2}{F_{2n-1}F_{2n-3}} = 3 + \frac{F_{2n-2}^2 - F_{2n-1}F_{2n-3}}{F_{2n-1}F_{2n-3}} = 3 - \frac{1}{F_{2n-1}F_{2n-3}} = b_n.
\]

2. This is true for \( n = 0 \). For \( n \geq 1 \) we have

\[
d_{2n}d_{2n+1} = \frac{F_{2n-3}F_{2n+1}}{F_{2n-1}F_{2n-1}} = \frac{F_{2n-1}^2 + (F_{2n-3}F_{2n+1} - F_{2n-1}^2)}{F_{2n-1}^2} = 1 + \frac{1}{F_{2n-1}^2} = \lambda_n.
\]

3. We have

\[
\frac{1}{d_{2n-1}} + d_{2n+1} = \frac{F_{2n-3}}{F_{2n-1}} + \frac{F_{2n+1}}{F_{2n-1}} = \frac{(F_{2n-1} - F_{2n-2}) + (F_{2n} + F_{2n-1})}{F_{2n-1}} = 2 + \frac{F_n - F_{n-2}}{F_{2n-1}} = 3.
\]
By Proposition 2.1 and Lemma 2.2 we obtain the following.

**Corollary 2.3.** For the sequences $b, \lambda$ and $d$ defined in (2) and (3), we have

$$\text{Dyck}_n(d) = \text{Mot}_n(b, \lambda).$$

Now we can prove the following $S$-fraction formula for the generating function (1) for $\# \text{NC}_2(n)$.

**Theorem 2.4.** We have

$$\frac{3}{2} - \frac{1}{2} \sqrt{\frac{1-5x}{1-x}} = S(x; 1, 1, 1, 2, \frac{1}{2}, 2, \frac{5}{2}, 5, \frac{89}{15}, \frac{13}{5}, \frac{233}{34}, \frac{89}{34}, \frac{89}{233}, \frac{610}{233}, \frac{610}{610}, \ldots).$$

To prove Theorem 2.4 we define $R_n$ for $n \geq -1$ by

$$\begin{align*}
R_{-1} &= \frac{3}{2} - \frac{1}{2} \sqrt{\frac{1-5x}{1-x}}, \\
R_{2n+1} &= d_{2n+1} + \frac{1 - 3x - \sqrt{(1-x)(1-5x)}}{2x}, \quad n \geq 0, \\
R_{2n} &= \frac{d_{2n}}{1 - x R_{2n+1}}, \quad n \geq 0.
\end{align*}$$

One can easily check that $R_n$ is a power series in $x$ with constant term $d_m$ (with $d_{-1} = 1$), though this will follow from Lemma 2.5.

**Lemma 2.5.** For $m \geq -1$, we have

$$R_m = \frac{d_m}{1 - x R_{m+1}},$$

where $d_{-1} = 1$.

**Proof.** By definition, this is true if $m$ is even. Thus it is enough to prove that for $n \geq 0$,

$$R_{2n} = \frac{d_{2n-1}}{1 - x R_{2n+1}},$$

which is equivalent to

$$R_{2n+1} = \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{1}{R_{2n-1} - d_{2n-1}}.$$

We can check (3) directly for $n = 0$. Assume $n \geq 1$. Then the right-hand side of (1) is equal to

$$\begin{align*}
\frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{2x}{1 - 3x - \sqrt{(1-x)(1-5x)}} &= \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{2x \left(1 - 3x + \sqrt{(1-x)(1-5x)}\right)}{(1 - 3x)^2 - (1 - 6x + 5x^2)} \\
&= \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{1 - 3x + \sqrt{(1-x)(1-5x)}}{2x} \\
&= 3 - \frac{1}{d_{2n-1}} + \frac{1 - 3x - \sqrt{(1-x)(1-5x)}}{2x}.
\end{align*}$$

Since $3 - 1/d_{2n-1} = d_{2n+1}$ by Lemma 2.2, we are done. \qed
Proof of Theorem 2.4. It follows from Lemma 2.5 that
\[
\frac{3}{2} \cdot \sqrt{\frac{1 - 5x}{1 - x}} = \frac{1}{1 - xR_0} = \frac{1}{1 - \frac{d_0x}{1 - xR_1}} = \frac{1}{1 - \frac{d_1x}{1 - xR_2}} = \cdots.
\]
Continuing, and taking a limit, gives the S-fraction for \( R_{-1} \).
\[\square\]

By (1), Theorem 2.4 and Corollary 2.3, we obtain the following which proves Theorem 1.1.
\[
\sum_{n \geq 0} \# NC_2(n)x^n = \frac{3}{2} \cdot \sqrt{\frac{1 - 5x}{1 - x}} = \sum_{n \geq 0} \text{Dyck}_n(d)x^n = \sum_{n \geq 0} \text{Mot}_n(b, \lambda)x^n
\]

3. A COMBINATORIAL PROOF

Let \( b, \lambda \) and \( d \) be the sequences defined in (2) and (3).

Recall that in the previous section we have shown that \( \text{Dyck}_n(d) = \text{Mot}_n(b, \lambda) \) by changing a Dyck path of length \( 2n \) to a Motzkin path of length \( n \). We can do the same thing after deleting the first and the last steps of a Dyck path. More precisely, for a Dyck path of length \( 2n \), we delete the first and the last steps, take two steps at a time in the remaining \( 2n - 2 \) steps, and change \( \text{UU}, \text{UD}, \text{DU}, \) and \( \text{DD} \), respectively, to \( \text{U}, \text{H}, \text{H} \) and \( \text{D} \). Then we obtain a Motzkin path of length \( n - 1 \). This argument shows that
\[
\text{Dyck}_n(d) = d_0 \cdot \text{Mot}_{n-1}(\alpha, \beta) = \text{Mot}_{n-1}(\alpha, \beta),
\]
where \( \alpha_n = d_{2n} + d_{2n+1} \) and \( \beta_n = d_{2n+1}d_{2n+2} \). By (3) and Lemma 2.2, we have \( \alpha = (2, 3, 3, \ldots) \) and \( \beta = (1, 1, \ldots) \). Note that we can also prove Theorem 2.4 using (5).

To find a connection between \( \text{Mot}_{n-1}(\alpha, \beta) \) and \( \text{NC}_2(n) \) we need the following definition.

A Schröder path of length \( 2n \) is a lattice path from \((0, 0)\) to \((2n, 0)\) consisting of up steps \( \text{U} = (1, 1) \), down steps \( \text{D} = (1, -1) \) and double horizontal steps \( \text{H}^2 = (2, 0) \) that never goes below the \( x \)-axis. Let \( \text{SCH}_{\text{even}}(n) \) denote the set of Schröder paths of length \( 2n \) such that all horizontal steps have even height.

Proposition 3.1. Let \( \alpha = (2, 3, 3, \ldots) \) and \( \beta = (1, 1, \ldots) \). Then, for \( n \geq 1 \), we have
\[
\text{Mot}_n(\alpha, \beta) = \# \text{SCH}_{\text{even}}(n).
\]

Proof. From a Motzkin path of length \( n \) we obtain a Schröder path of length \( 2n \) as follows. Change \( \text{U} \) and \( \text{D} \) to \( \text{UU} \) and \( \text{DD} \) respectively. For a horizontal step \( \text{H} \), if its height is 0, we change it to either \( \text{UD} \) or \( \text{HH} \), and if its height is greater than 0, we change it to either \( \text{UD}, \text{DU}, \) or \( \text{HH} \). Then we get an element of \( \text{SCH}_{\text{even}}(n) \). Since the weight of a horizontal step \( \text{H} \) in the Motzkin path is equal to the number of choices, the theorem follows.
\[\square\]

Remark 1. The definition of \( \text{SCH}_{\text{even}}(n) \) in [2] is the set of Schröder paths of length \( 2n \) which have no peaks at even height. From such a path, by changing all the horizontal steps at odd height to peaks, we get a Schröder path whose horizontal steps are all at even height, and this transformation is easily seen to be a bijection.
Kim [2] found a bijection between $\text{NC}_2(n)$ and $\text{SCH}_{\text{even}}(n - 1)$. Using Kim's bijection in [2], Proposition 3.1, (5) and Corollary 2.3 we finally get the following sequence of identities which implies Theorem 1.1:

$$\# \text{NC}_2(n) = \# \text{SCH}_{\text{even}}(n - 1) = \text{Mot}_{n-1}(\alpha, \beta) = \text{Dyck}_n(d) = \text{Mot}_n(b, \lambda).$$

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