ON THE CLASSIFICATION OF FIFTH-ORDER INTEGRABLE SYSTEMS

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Abstract. In this work we classify the fifth-order integrable symmetrically coupled systems of weight 0 that possess seventh-order symmetry. We obtained 2 new integrable systems that related bi-Hamiltonian formulations are constructed too.

1. Introduction

Many approaches have been advocated to the classification of integrable evolution equations, all of which have their benefit but the most successful ones are the generalized symmetry method and the conservation law method. In recent years studies on fifth-order systems of two-component nonlinear evolution equations have received considerable attention. Some completely integrable equations of this type were found by Mikhailov, Novikov and Wang [15, 14] and Talati [19]. It is inevitable to consider limited classes of this type equations for doing a complete classification. It is useful in classifying symmetric generalization of scalar evolution equations. A milestone in this direction is the work of Foursov on the classification of third-order systems with two-components [8]. Many integrable systems have been classified using this direct classification. In this paper we extend this approach to the case of fifth-order evolution systems of weight 0 and have found all coupled integrable equations of this class and have demonstrated that these equations possess infinitely higher generalized symmetries and conservation laws.

Coupled two component integrable equations of form

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
A_1[u,v] \\
B_1[u,v]
\end{pmatrix}
\]

possess a hierarchy of higher symmetries

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
A_i[u,v] \\
B_i[u,v]
\end{pmatrix}
\]

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
A_i[u,v] \\
B_i[u,v]
\end{pmatrix}
\]

i = 2, 3, 4, ...

and this property can be taken as a definition of integrability. System [2] is said to be generalized symmetry of System [1] if

\[
\begin{pmatrix}
D_{A_1}^u & D_{A_1}^v \\
D_{B_1}^u & D_{B_1}^v
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
D_{A_i}^u & D_{A_i}^v \\
D_{B_i}^u & D_{B_i}^v
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

= 0.

Here \(D_F\) denotes the Frechet derivative defined by \(D_F^u[l, u, v]H[u, v] = \frac{d}{d\epsilon}F[u + \epsilon H[u, v], v]|_{\epsilon=0}\) and \(D_F^v[l, u, v]H[u, v] = \frac{d}{d\epsilon}F[u, v + \epsilon H[u, v]]|_{\epsilon=0}\). In all known cases the existence of one higher order symmetry seems to be sufficient for the existence of infinitely generalized symmetry.
In case $B[u, v] = A[v, u]$, system (1) become
\[ \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} A[u, v] \\ A[v, u] \end{pmatrix} \] (3)
that is said symmetricaly coupled integrable system. The well-known Sasa-Satsuma system
\[ \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_3 + u^2 v_x + 3uvu_x \\ v_3 + v^2 u_x + 3uvv_x \end{pmatrix} \] (4)
is one example of this type systems. It is easy to see that the system (4) is homogeneous if we assign weightings of 1 to the dependent variables, while $x$- and $t$-differentiation have weights 1 and 4, respectively. It is known that the majority of integrable systems have infinitely conservation gradient. To prove the integrability of an equation suspected to be bi-Hamiltonian, one need to find an appropriate compatible pair of Hamilton operator $J$ and $K$ such that the Magri scheme
\[ \begin{align*}
  u_t &= F_i[u] = KG_i[u] = JG_{i+1}[u], \\
  &\quad i = -1, 0, 1, 2, 3, \ldots
\end{align*} \]
constructed by the operators contains the equation in hand. Here $F_i[u]$ are characteristics of symmetries and $G_i$ are the conserved gradients. A system of evolution equations,
\[ G = \begin{pmatrix} M[u, v] \\ N[u, v] \end{pmatrix} \] (5)
is said to be a generalized conserved gradient (whose Frechet derivative is self-adjoint $D_G^* = D_G$) of system (1) if and only if
\[ \begin{pmatrix} D^u_A & D^v_A \\ D^u_B & D^v_B \end{pmatrix}^* \begin{pmatrix} M \\ N \end{pmatrix} + \begin{pmatrix} D^u_M & D^v_M \\ D^u_N & D^v_N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \]

Let $\rho_i$ be a conserved density of two-component evolution system. The relationship between conserved gradient and conserved density of evolution system can be written as
\[ \begin{pmatrix} M_i[u, v] \\ N_i[u, v] \end{pmatrix} = \left( \sum_j (-D)^j \frac{\partial \rho_i}{\partial x_j} \right). \]

2. STATEMENT OF THE PROBLEM

The right-hand side of all integrable systems of evolution equations is a homogeneous differential polinomial under a suitable weighting scheme. Let us consider a symmetric system of two equations
\[ \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} A[u_x, v_x] \\ A[v_x, u_x] \end{pmatrix}. \] (6)
Here $A[u_x, v_x] = A(u_x, v_x, u_{xx}, v_{xx}, \ldots)$ denotes a differential polynomial function of $x$-derivatives of $u$ and $v$. The system of differential equation (6) is said to be $\lambda$-homogeneous if it admits the group of scaling symmetries [7]
\[ (x, u, v) \rightarrow (a^{-1}x, a^{\lambda}u, a^{\lambda}v), \quad a \in \mathbb{R}^+. \]
In this work we are restricting our attention to $\lambda = 0$. We determine all equations of the form (6) with

$$A = \gamma_1 u_{5x} + \gamma_2 v_{5x} + \alpha_1 u_2 u_{4x} + \alpha_2 v_2 u_{4x} + \alpha_3 u_3 u_{4x} + \alpha_4 v_4 u_{4x} + \alpha_5 u_{5x} u_{3x} + \alpha_6 v_2 u_{3x} + \alpha_7 u^2_2 u_{3x} + \alpha_8 u_3 v_3 u_{3x} + \alpha_9 v^2_2 u_{3x} + \alpha_{10} u_{5x} v_5 x + \gamma_{11} u_4 v_4 x + \alpha_{12} u^2_2 v_{3x} + \alpha_{13} u_3 u_{5x} + \alpha_{14} v_4^2 u_{3x} + \alpha_{15} u_5 u_{3x} + \alpha_{16} v_4 v_{4x} + \alpha_{17} u_4 u_{3x} u_{3x} + \alpha_{18} u_{5x} u_{xx} x + \alpha_{19} u_4^2 v_3 u_{3x} + \alpha_{20} u_4 v^2_2 u_{3x} + \alpha_{21} u_5 v^2_2 u_{3x} + \alpha_{22} v^2_2 u_{3x} x + \alpha_{23} u_{5x} v^2_2 x + \alpha_{24} u_4 v^2_2 x + \alpha_{25} u^2_2 v_{3x} + \alpha_{26} u_4^2 v_3 v_{3x} + \alpha_{27} u_5 v^2_2 v_{3x} + \alpha_{28} v^2_2 v_{3x} + \gamma_{29} u^5 x + \gamma_{30} u^4_2 v_x + \gamma_{31} u^3_2 v^3 x + \gamma_{32} u^2_2 v^3 + \gamma_{33} u_4 v^4 x + \gamma_{34} u^4 v^3 x;$$

possessing an admissible generator of form (6) with

$$A = \gamma_1 u_{7x} + \gamma_2 v_{7x} + \beta_1 u_2 u_{6x} + \beta_2 v_2 u_{6x} + \beta_3 u_3 u_{6x} + \beta_4 v_4 u_{6x} + \beta_5 u_{7x} u_{5x} + \beta_6 v_2 u_{5x} + \beta_7 u^2_2 u_{5x} + \beta_8 u_3 v_3 u_{5x} + \beta_9 v^2_2 u_{5x} + \beta_{10} u_{7x} v_5 x + \beta_{11} u_5 v_{7x} + \beta_{12} u_{5x} v^2_2 x + \beta_{13} u_3 u_{5x} v_{5x} + \beta_{14} v^2_2 u_{5x} + \beta_{15} u_3 u_{3x} + \beta_{16} v^2_2 u_{3x} + \beta_{17} u_{7x} v_{xx} x + \beta_{18} u_2 v_{xx} x + \beta_{19} u_3 v_{xx} x + \beta_{20} v_{xx} v_{xx} x + \beta_{21} u^2_2 x + \beta_{22} u_4 u_{4x} + \beta_{23} u^2_2 u_{4x} + \beta_{24} u^3 v_{4x} + \beta_{25} u_{3x} v_{4x} + \beta_{26} v_{3x} v_{4x} + \beta_{27} u_2 v_{xxx} x + \beta_{28} u_5 v_{xxx} x + \beta_{29} u_{5x} u_{xxx} x + \beta_{30} u^2_2 v_{xxx} x + \beta_{31} u^2_2 v_{xx} x + \beta_{32} u_3 v_{xx} x + \beta_{33} u^2_2 v_{xx} x + \beta_{34} v^2_2 u_{xx} x + \beta_{35} u_{4x} u_{4x} + \beta_{36} v_{4x} u_{4x} + \beta_{37} u_3 u_{4x} + \beta_{38} v_{3x} u_{4x} + \beta_{39} u^2_2 v_{4x} + \beta_{40} u_3 v_{4x} + \beta_{41} u^2_2 u_{4x} + \beta_{42} u_4 v_{4x} + \beta_{43} u^2_2 u_{4x} + \beta_{44} u^2_2 v_{4x} x + \beta_{45} u^2_2 v_{xx} x + \beta_{46} u_3 v_{xx} x;$$

The main matrix of these systems is

$$\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix}.$$

By a linear change of variables, the matrix (9) can be reduced to following canonical Jordan form

$$J_\gamma = \begin{pmatrix} \gamma_1 + \gamma_2 & 0 \\ 0 & \gamma_1 - \gamma_2 \end{pmatrix}.$$
3. Classification of 0-homogenous symmetrically coupled fifth-order integrable systems of two-component evolution equations

**Theorem 1.** A coupled fifth-order system of two-component evolution equations of form (6) and (7) that possesses a seventh-order generalized symmetry of (6) and (8) with $\gamma_1 = \gamma_2 = 1$ have a lower order symmetry or transform by a linear change of variables to one of the following two equations:

$$
\begin{align*}
\mathbf{u}_t (x) &= \begin{pmatrix}
\begin{vmatrix}
5w_1 & w_1 & -w_1 & -w_1 & 2u_1 & -v_1 & -w_1 & 0 & -w_1 \\
-2v_1 & w_1 & -v_1 & -v_1 & 2u_1 & -v_1 & -v_1 & 0 & -v_1 \\
-2u_1 & w_1 & -u_1 & -u_1 & 2v_1 & -u_1 & -u_1 & 0 & -u_1 \\
-w_1 & w_1 & w_1 & w_1 & 2u_1 & -v_1 & -v_1 & 0 & -v_1 \\
0 & v_1 & -v_1 & -v_1 & 2u_1 & 0 & 0 & 0 & 0 \\
-2u_1 & v_1 & -u_1 & -u_1 & 0 & 2v_1 & -w_1 & -w_1 & -w_1 \\
-w_1 & v_1 & w_1 & w_1 & 0 & 0 & 2v_1 & -w_1 & -w_1 \\
0 & v_1 & v_1 & v_1 & 0 & 0 & 0 & 2v_1 & -v_1 \\
-w_1 & w_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 2v_1 \\
\end{vmatrix}
\end{pmatrix} \\
\mathbf{v}_t (x) &= \begin{pmatrix}
\begin{vmatrix}
v_1 & v_1 & -v_1 & -v_1 & 0 & 0 & 0 & 0 & 0 \\
-w_1 & v_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\
0 & v_1 & -v_1 & -v_1 & 0 & 0 & 0 & 0 & 0 \\
w_1 & w_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\
-2u_1 & v_1 & -u_1 & -u_1 & 0 & 0 & 0 & 0 & 0 \\
-w_1 & v_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\
0 & v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 \\
w_1 & w_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\
-w_1 & w_1 & w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\
\end{vmatrix}
\end{pmatrix}
\end{align*}
$$

(11)

**Theorem 2.** Every coupled fifth-order systems of two-component evolution equations of form (6) and (7) that possesses a seventh-order generalized symmetry of (6) and (8) with $\gamma_1 = 1$, $\gamma_2 = 0$ have a lower order symmetry.

Very recently, He and Geng [17] introduced a $3 \times 3$ matrix spectral problem, from which they founded a hierarchy of new nonlinear evolution equations

$$
\begin{pmatrix}
\mathbf{u}
\mathbf{v}
\end{pmatrix}_t = \begin{pmatrix}
w_{xx} + 2w_{wx} - 2w_{w} - 2z_{wx} - 2z_{w} & -2w_{xx} - 2w_{w} - 2z_{wx} + 2z_{w}\end{pmatrix}.
$$

(13)

The potential form of system (13) can be written as

$$
\begin{pmatrix}
\mathbf{u}
\mathbf{v}
\end{pmatrix}_t = \begin{pmatrix}
-u_{xx} + 2v_{xx} - u_x^2 - 2u_x v_x + 2v_x^2 \\
-2u_{xx} + v_{xx} - u_x^2 + 2u_x v_x + v_x^2
\end{pmatrix}.
$$

(14)
and the potential form of second member of the hierarchy is the following symmetrycally coupled
0-homogeneous fifth-order system:

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t =
\begin{pmatrix}
  u_{5x} + 5u_{xx}u_{3x} - 5v_{xx}u_{3x} - 5u_x^2u_{3x} + 5u_xv_xu_{3x} - 5v_x^2u_{3x} - 5u_{xxx}v_{3x} \\
  v_{5x} - 5v_{xx}u_{3x} - 5u_{xx}v_{3x} + 5v_{xx}v_{3x} - 5u_x^2v_{3x} + 5u_xv_xv_{3x} - 5v_x^2v_{3x} + 5u_{xxx}v_{3x}
\end{pmatrix}
\]

(15)

that can be reduced to the potential kupershmidt equation with \( v = 0 \). By a linear change of
variables, one can show that the system (13) is not new, it is related to system (21) listed in
[13]

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t =
\begin{pmatrix}
  u_{xx} + vv_x \\
  -v_{xx} + uu_x
\end{pmatrix}
\]

(16)

it is easy to see that by change of variables

\[
\begin{align*}
  w & \rightarrow \frac{1}{5}(v - u), \\
  z & \rightarrow \frac{1}{5}((31/2i - 1)u + (31/2i + 1)v), \\
  \tau & \rightarrow -3^{-1/2}it.
\end{align*}
\]

(17)

system (13) can be reduced into the system (16). It is obvios that this transformation changes
the system (18) in [17] into the fifth order symety of system (16) as well. The system (16) is
the well known modified Boussinesq system [1] that is the member of class of following systems
of evolution equations

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t =
\begin{pmatrix}
  u_{xx} + A[u, v] \\
  -v_{xx} + B[u, v]
\end{pmatrix}
\]

(18)

All systems (18) possessing higher conservation laws were classified by Mikhailov, Shabat and
Yamilov [2, 3].

3.1. Integrability of systems (11).

By change of dependent variables

\[
\begin{align*}
  u & \rightarrow \frac{1}{2}\int(w - z)dx \\
  v & \rightarrow \frac{1}{2}\int(w + z)dx
\end{align*}
\]
sistem(11) can be written in its canonical form

\[
\begin{pmatrix}
  u \\
  v \\
  t
\end{pmatrix}
= 
\begin{pmatrix}
  w_{5x} - 2z_{4x} - 10w_x w_{4x} - 20w^2 w_{4x} - 2z^2 w_{3x} - 8zw_{3x} - 8wz_{3x} \\
-10w^2 - 80ww_{x} w_{xx} - 8zz_{x} w_{xx} - 6z^2 - 12w_z z_{xx} - 24w_{zw} z_{xx} \\
+ 8w^2 z_{xx} + 4z^3 z_{xx} - 20w^2 - 12w_z z_{2x} + 16w_z w_x z_x + 80w^4 w_x \\
+ 48w^2 z^2 w_x + 4z^4 w_x + 8w^2 z^2 + 12z^2 z_{x}^2 + 32 w^3 z_x + 16 w z^3 z_x \\
4z w_{4x} + 4z_x w_{3x} - 16w_z w_{3x} - 8z^2 z_{3x} - 40w_x w_{xx} - 16w_z w_{xx} \\
- 16w^2 z_{xx} - 8z^3 w_{xx} - 32z z_x z_{xx} - 12w^2 z_x - 32w z w^2 - 16 w^2 w_x z_x \\
- 24w z_x z_x + 64w^4 z_{w} + 32w z^3 w_x - 8z^3 + 16w^4 z_x + 48w^2 z^2 z_x \\
+ 20z^4 z_x
\end{pmatrix}
\]

(19)

**proposition** The infinite hierarchy of system (19) can be write in two different way

\[
\begin{pmatrix}
  w_t \\
  z_t
\end{pmatrix}
= J \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix}
\int \rho_1 \, dx = K \begin{pmatrix}
  \delta_w \\
  \delta_z
\end{pmatrix}
\int \rho_0 \, dx
\]

(20)

with the compatible pair of Hamiltonian operators

\[
J = \begin{pmatrix}
  D_x & 0 \\
  0 & 2D_x
\end{pmatrix}, \quad K = \begin{pmatrix}
  K_1 & K_1' \\
  K_3 & K_4
\end{pmatrix}
\]

where

\[
K_1 = D_x^7 + \omega_1 D_x^5 + D_x^5 \omega_1 + \omega_2 D_x^3 + D_x^5 \omega_2 + \omega_3 D_x + D_x \omega_3 + 8w_x D_x^{-1} w_x + 8w_1 D_x^{-1} w_x
\]

\[
K_2 = D_x^6 \omega_4 + D_x^5 \omega_5 + D_x^4 \omega_6 + D_x^3 \omega_7 + 8w_x D_x^{-1} z_t + 8w_1 D_x^{-1} z_x
\]

\[
K_3 = -\omega_4 D_x^4 + \omega_3 D_x^5 - \omega_5 D_x^4 + \omega_7 D_x^3 - \omega_8 D_x^2 - \omega_9 D_x - \omega_{10} + 8z_x D_x^{-1} w_x
\]

\[
k_4 = \omega_{11} D_x^5 + D_x^5 \omega_{11} + \omega_{12} D_x^3 + D_x^5 \omega_{12} + \omega_{13} D_x + D_x \omega_{13} + 8z_x D_x^{-1} z_t + 8z_{1} D_x^{-1} z_x
\]

(21)
where the coefficients satisfy

\[
\begin{align*}
\omega_1 &= -6 w_x - 12 w^2 - 2 z^2 \\
\omega_2 &= 16 w_{3x} + 40 w w_{xx} + 8 z z_{xx} + 58 w^2_x + 24 w^2 w_x + 12 z^2 w_x + 8 z^2 + 16 w z z_x + 72 w^4 \\
&
+ 40 w^2 z + 18 z^4 \\
\omega_3 &= -10 w_{5x} - 24 w w_{4x} - 4 z z_{4x} - 100 w w_{3x} - 24 w^2 w_{3x} - 12 z^2 w_{3x} - 16 z^2 z_{3x} - 84 w^2_{xx} \\
&- 64 w w_{w_{xx}} - 64 z z_{w_{xx}} - 128 w^3 w_{xx} - 64 z^2 w_{xx} + 12 z^2 z_{xx} - 56 w w_{xx} \\
&- 16 w^2 z_{xx} - 152 z^3 z_{xx} - 48 w^3 + 704 w^2 w_x^2 - 80 z^2 w_x^2 - 56 w w_x z_x^2 - 288 w w w_{xx} \\
&- 96 z^4 w_x - 16 w^2 z_x^2 - 216 z^2 z_x^2 - 64 w z^3 z_x + 96 w z^3 z_x - 128 w^6 - 128 w^4 z^2 \\
&- 96 w^2 z^4 - 16 z^6 \\
\omega_4 &= -4 z \\
\omega_5 &= 4 z - 16 w z \\
\omega_6 &= -72 w_{xx} - 32 w w_{xx} - 160 w z w_x - 32 w^2 z_x + 96 z^2 z_x + 128 w^3 z \\
\omega_7 &= +40 w_{w_{xx}} + 40 z w_{w_{xx}} + 192 w_{w_{xx} x} + 208 w_{w_{xx}}^2 + 96 w w_{w_{xx} x} - 576 w^2 z w_x - 320 z z_x^2 \\
&- 128 w^3 z_x - 64 w^4 z \\
\omega_8 &= -8 w_{w_{4x}} - 128 w w_{3x} - 104 z^2 z_{3x} - 240 z w_{w_{xx}} - 96 w w_{w_{xx}} + 480 w^2 z w_{xx} - 96 w^3 w_{xx} \\
&- 152 z_{w_{3x}} + 576 w^2 z_{w_{xx}} - 112 w_{w_{xx}} + 640 w z w_{w_{xx}} + 192 w^2 w_{w_{xx} x} - 192 w^2 w_{w_{xx}} - 92 w^3 w_{w_{xx}} \\
&+ 384 w^2 w_{w_{xx}}^2 + 384 w w_{w_{xx}}^2 + 160 w_z z_x^2 + 448 w z z_x^2 + 64 w^4 z_x - 768 w^2 z_x^2 - 96 z^4 z_x \\
&- 256 w^3 z_x \\
\omega_9 &= +8 z w_{w_{4x}} + 32 w w_{w_{4x}} + 40 z z_{w_{4x}} + 32 w w_{w_{3x}} + 128 w^2 z w_{3x} + 48 z^2 w_{3x} + 224 z w z_{3x} \\
&- 288 w^2 z_{3x} - 80 w w_{z_{xx}} + 320 w w_{w_{xx}} - 288 w^2 z w_{xx} - 128 w^3 w_{xx} + 192 w^2 w_{xx} \\
&+ 304 w^2 z_{xx} + 120 z_{w_{2x}} + 64 z^2 w_{w_{xx}} + 192 z^2 w_{w_{xx}} + 1376 w z_{w_{xx}} + 384 w^2 z_{w_{xx}} \\
&+ 48 z x_{w_{xx}} - 256 w w z_{w_{xx}} - 256 w w z_{w_{xx}} + 256 w^2 w_{2x} + 64 z^3 w_x + 160 w w z_{2x} - 128 w^2 w_x z_x \\
&- 192 w^2 w_x z_x + 512 w^4 w_x - 512 w^4 w_x - 272 z^3 z_x + 256 w^5 z_x \\
\omega_{10} &= -12 z^2 \\
\omega_{11} &= +100 z z_{2x} - 72 z^2 w_x + 80 z^2 x - 96 w z z_x + 144 w^2 z^2 + 36 z^4 \\
\omega_{12} &= -68 z z_{3x} + 8 z^2 w_{3x} - 292 z z_{3x} + 176 w z z_{3x} + 232 z z_{w_{xx}} + 64 w^2 w_{xx} - 264 z^2 w_{xx} \\
&+ 512 w w z_{xx} + 816 w z z_{xx} - 896 z^2 z_{xx} + 256 z_{w_{xx}} - 642 z^2 w_{xx} - 280 w w z_x^2 \\
&- 1056 w w z_x z_x + 96 z^2 w_x + 752 w^2 z_x^2 - 380 z^2 z_x^2 + 768 w^3 z z_x + 96 w z^3 z_x - 384 w^4 z^2 \\
&- 192 w^2 z^4 - 32 z^6 \\
\end{align*}
\]

By straightforward calculation It is easily seen that the functional trivector of linear combination $K + \lambda J$ with constant $\lambda$, vanishes independently from the value of $\lambda$. The first few conserved densities of the hierarchy are listed below.

\[
\begin{align*}
\rho_0 &= \alpha \\
\rho_1 &= 2 w^2 + z^2 \\
\rho_2 &= +3 w w_{4x} + 10 w^2 w_{3x} + 6 z^2 w_{3x} - 20 w^3 w_{xx} - 6 z w^2 w_{xx} - 18 w^2 z z_{xx} - 4 z^3 z_{xx} - 18 w^2 z_x^2 + 16 w^4 z_{xx} \\
&+ 24 w^3 z_{xx} + 16 w^6 + 24 w^4 z^2 + 12 z^2 z^4 + 2 z^6 \\
\rho &= -45 w w_{w_{xx}} - 126 w^2 w_{5x} - 90 z^2 w_{5x} + 210 w^3 w_{4x} + 270 w^2 z z_{4x} - 270 w^2 z w_{4x} - 30 z^3 z_x \\
&+ 360 w z z_{3x} + 280 w w_{w_{xx}} + 840 w^2 z_{w_{xx}} + 180 w^4 z_{w_{xx}} + 1080 w^2 z_{w_{xx}} + 640 w^3 z_{3x} \\
&- 360 w z^3 z_{3x} + 630 w^2 w_{xx} + 1440 w^2 z_{w_{xx}} - 2016 w w_{w_{xx}} - 1920 w^2 z^2 w_{xx} - 360 w^3 w_{xx} \\
&+ 810 w^2 z_{w_{xx}} + 270 z^2 z_{w_{xx}} - 192 w^2 z_x z_{xx} - 1080 w^2 z^2 z_{xx} - 1200 w^4 z z_{xx} - 1440 w^2 z^3 z_{xx} \\
&- 216 z^5 z_{xx} - 1200 w^2 z_x^2 + 2160 w^2 z_{xx}^2 + 1152 z^2 z_{xx} + 1920 w^3 z^3 z_x + 1440 w^5 z_{xx} \\
&+ 960 w^8 + 1920 w^6 z^2 + 1440 w^4 z^4 + 480 w^2 z^6 + 60 z^8 \\
&
\end{align*}
\]
These densities suffice to write two magri schemes with same hamiltonian operators that one of them contains the new system proving integrability of the system \(19\).

### 3.2. Integrability of systems \((12)\)

By change of dependent variables

\[
\begin{align*}
u &\rightarrow \frac{1}{2} \int (w - z) \, dx \\
v &\rightarrow \frac{1}{2} \int (w + z) \, dx
\end{align*}
\]

system \((12)\) can be written in its canonical form

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix}
(3w_5 x - z z_4 x + 10 w_x w_3 x - 20 w^2 w_3 x - 8 z^2 w_3 x - 4 z_x z_3 x - 2 w z z_3 x \\
+ 10 w^2_{xx} - 80 w w_x w_{xx} - 32 z z_x w_{xx} - 3 z_x^2 - 8 w_x w_{xx} + 6 w_x z_{xx} \\
+ 16 w^2 z z_{xx} + 83 z_{xx} - 20 w^3 - 18 w_x z_x^2 + 32 w w_3 z_x + 80 w^4 w_x \\
+ 48 w^2 z^2 w_x + 4 z^4 w_x + 16 w^2 z_x^2 + 24 z z_x^2 + 32 w^3 z_x + 16 w^3 z_x
\end{pmatrix}
\]

(22)

**Proposition** The infinite hierarchy of system \((22)\) can be write in not just one but two different way

\[
\begin{pmatrix} w_t \\ z_t \end{pmatrix} = J \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_1 \, dx = K \begin{pmatrix} \delta_w \\ \delta_z \end{pmatrix} \int \rho_1 \, dx
\]

(23)

with the compatible pair of Hamiltonian operators

\[
J = \begin{pmatrix} D_x & 0 \\ 0 & 2D_x \end{pmatrix}, \quad K^2 = \begin{pmatrix} K_1^2 & K_2^3 \\ K_3^3 & K_4^4 \end{pmatrix}
\]

where

\[
\begin{align*}
K_1^2 &= D_x^7 + \psi_1 D_x^5 + D_x^3 + \psi_2 D_x^2 D_x + \psi_3 D_x + D_x \psi_3 + 8 w_x D_x^{-1} w_x + 8 w D_x^{-1} w \\
K_2^3 &= D_x^6 \psi_4 + D_x^5 \psi_5 + D_x^4 \psi_6 + D_x^3 \psi_7 + D_x^2 \psi_8 + D_x \psi_9 + \psi_{10} + 8 w_x D_x^{-1} z_t + 8 w D_x^{-1} z_x \\
K_3^3 &= -\psi_4 D_x^6 + \psi_5 D_x^5 - \psi_6 D_x^4 + D_x^3 \psi_7 + D_x \psi_8 - \psi_9 D_x + \psi_{10} + 8 z_x D_x^{-1} w_t + 8 z D_x^{-1} w_x \\
K_4^4 &= \psi_{11} D_x^5 + D_x^4 \psi_5 + \psi_{12} D_x^3 + D_x^2 \psi_{12} + \psi_{13} D_x + D_x \psi_{13} + 8 z_x D_x^{-1} z_t + 8 z D_x^{-1} z_x
\end{align*}
\]

(24)

where the coefficients satisfy

\[
\begin{align*}
\psi_1 &= 6 w_x - 12 w^2 - 5 z^2 \\
\psi_2 &= -16 w_{3x} + 40 w w_{xw} + 26 z z_{xx} + 58 w^2_x - 24 w^2 w_x - 12 z^2 w_x + 26 z^2 + 20 w z z_x \\
&+ 72 w^4 + 52 w^3 z^2 + 18 z^4
\end{align*}
\]
ψ₃ = 10w₅x - 24ww₄x - 16zzz₄x - 100wₓw₃x + 24w²w₃x + 12z²w₃x - 64zxz₃x - 12wzz₃x - 84w²z₃x + 64wₓwₓwₓ + 28zzzₓwₓ - 128w₃wₓwₓ - 40wz²wₓwₓ - 48z²wₓwₓ - 16wₓzₓwₓ - 36wₓzₓzₓ - 88w²zzzₓ - 134z²z₃zₓ + 48w³ - 704w²wₓ - 104z²wₓ - 16wₓzₓ - 288w zwzₓzₓ + 24²wₓ - 88w²z²ₓ - 216z²z₂ₓ - 128w₃zzzₓ - 24wz³zₓ - 128w⁶ - 128w⁴z² - 96w²z⁴ - 16z⁶
ψ₄ = -2z
ψ₅ = 2zₙ - 4wz
ψ₆ = -24wₓ + 4wₓz + 40w²z
ψ₇ = +36wₓwₓ + 28wₓzₓ + 200wₓwₓ - 40wₓwₓ + 120z²zₓ + 80w³z
ψ₈ = -20zw₃x - 8zₓwₓ + 192wzwₓ + 104zw³zₓ + 120wₓwₓzₓ - 144w²zₓwₓ - 80w³zₓ - 424z²zₓ - 128w⁴z
ψ₉ = +4zw₄x + 12zₓw₃x - 88wₓw₃x - 154z²z₃x - 120wₓwₓwₓ - 72wₓzₓwₓ + 96w²zₓwₓ + 48z³wₓwₓ + 360w²zₓwₓ + 16w²zₓwₓ - 128wₓwₓwₓ - 96w²zₓwₓ + 312z²wₓzₓ + 768w²zₓwₓ - 96w³zₓwₓ + 224z³zₓ + 232wz²zₓ + 128w⁴zₓ - 672wz²zₓ - 192z⁵zₓ - 256w⁵z
ψ₁₀ = +8zw₄x + 16wₓwₓwₓ + 62z²z₄x + 16wₓw₃x - 32w²w₃x - 24z³w₃x - 180wz⁵z₃x + 364z²z₃x + 80wₓzₓwₓ + 160wₓwₓwₓ - 296z²zₓwₓ - 256w³zₓwₓ - 192z⁵wₓwₓ + 48z³wₓwₓ + 186z⁵zₓwₓ - 176z²zₓwₓ + 348z⁵zₓwₓ - 812wzₓwₓwₓ + 336w²z²zₓwₓ + 96z⁴zₓwₓ - 64wz²wₓwₓ + 128z³wₓwₓ - 656wzₓwₓwₓ - 256w⁵wₓwₓ + 512w⁴zₓwₓ + 912wz²wₓwₓ - 272wz⁴zₓ + 1136w²z²zₓ + 544z⁵z² + 256w⁵zₓ
ψ₁₁ = -12z²
ψ₁₂ = +190zₓwₓwₓ + 144z²wₓ + 74z₄x - 120wzₓwₓ + 144w²z² + 36z⁴
ψ₁₃ = -146z⁵x + 88z²w₃x - 514zₓz₃x + 244wzₓwₓ + 560zₓzₓwₓ - 224wz²wₓwₓ - 480zₓwₓ + 904zxzₓwₓ + 1092wₓzₓwₓ - 824wz₂zₓwₓ - 160z⁴zₓwₓ + 512wz⁵zₓwₓ - 520z⁵wₓwₓ - 1632wzₓwₓzₓ - 864w²z²zₓwₓ + 192z⁴wₓzₓ - 632wz⁵zₓwₓ - 464z⁵z²zₓ + 672w³zₓzₓ + 192wz³zₓ + 384w⁴z² - 192w²z⁴ - 32z⁶

By straightforward calculation It is easily seen that the functional trivector of linear combination $\bar{K} + \lambda K$ with constant $\lambda$, vanishes independently from the value of $\lambda$. By constructing trivial compositions

$$
\left( J + \sum_{n=0}^{m} \lambda_n (K_0 J^{-1})^n K_0 \right) J^{-1} \left( J + \sum_{n=0}^{m} \bar{\lambda}_n (K_0 J^{-1})^n K_0 \right), \quad (K_0 J^{-1})^0 = 1,
$$

of the successive partial sums $m = 0, 1, 2, \ldots$ of linear combinations $J + \sum_{n=0}^{m} \lambda_n (K_0 J^{-1})^n K_0$ with arbitrary constants $\lambda_n$, $\bar{\lambda}_n$ and by induction on $m$ that all $K_n$, $n = 0, 1, 2, \ldots$ are mutually compatible HO’s if so is the HO’s $K_0$ and (formally) invertible $J$ [16][18]. The first few conserved densities of system (19) are listed below.

$$
\rho_0 = \alpha \\
\rho_1 = 2w^2 + z^2 \\
\rho_2 = +3ww₄x - 10w²w₃x + 3z²w₃x - 20w³wₓwₓ - 24wz²wₓwₓ + 18w²z₃zₓwₓ - z³z₃zₓ + 18w^²z^₂x + 32w⁷zₙx + 48wz²zₓwₓ + 16w⁶ + 24w⁴z² + 12w²z⁴ + 2z⁶
$$
\[ \rho = -90w_{6x} + 252w^2w_{5x} - 90z^2w_{5x} + 420w^3w_{4x} - 180w^2zw_{4x} + 1080wz^2w_{4x} - 15z^3z_{4x} \\
+ 1440wz^2z_{4x} - 560w^4w_{3x} - 1680w^2z^2w_{3x} + 360z^4w_{3x} - 160w^3zw_{3x} - 720w^2z^2z_{3x} \\
- 720w^3z^2z_{3x} + 1260w^2w^2z_{xx} - 7200w^2z^2w_{xx} - 4032w^3w_{xx} - 5280w^3z^2w_{xx} \\
- 2880w^4w_{xx} - 540w^2z^2w_{xx} + 135z^2z_{xx} - 480w^3z^2w_{xx} - 2160w^2z^2z_{xx} + 480w^4zz_{xx} \\
+ 3600w^2z^3zz_{xx} - 216z^5zz_{xx} + 480w^4z^2z_{xx} + 12960w^2z^2z^2zz_{xx} + 4608w^5z^zz_{xx} + 7680w^3z^3zz_{xx} \\
+ 5760w^5zz_{xx} + 1920w^6z^5 + 3840w^6z^5 + 2880w^4z^4 + 960w^2z^6 + 120z^8 \]

These densities suffice to write two magri schemes with same hamiltonian operators that one of them contains the new system proving integrability of the system [19].

4. Conclusions

It is obvious that it is essential to consider limited classes of equations for doing a complete classification of complicated integrable equations. Most of the classified such systems are generalization of the KdV and Burgers equations or equations related to them (see [9, 5, 6, 12, 8, 11] and references therein). Motivated by existing some examples of bi-Hamiltonian two-component generalization of fifth-order equations, we considered a class of fifth-order two-component systems for integrability. The approach outlined in this paper is to to classify two-component integrable systems of fifth order symmetric evolution equations that depends only to \( x \)-derivatives of dependent variables. Applying this method, we found two integrable systems which suffice to write Magri schemes contains the new systems proving complete integrability of them.

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