A prediction for three neutrino masses and mixings and similarity between quark and lepton mixings

Saul Barshay and Patrick Heiliger
III. Physikalisches Institut (A), RWTH Aachen
D–52056 Aachen, Germany

Abstract

If neutrinos have mass, we give reasons for a possible pattern of three (squared) mass eigenvalues: \( m_1^2 \simeq (2.8 - 5.8) \text{ (eV)}^2 \), \( m_2^2 \simeq 0.01 \text{ (eV)}^2 \), \( m_3^2 \simeq (1.5 - 1) \times 10^{-4} \text{ (eV)}^2 \). The flavor states \( \nu_{\mu} \) and \( \nu_e \) are mixtures of the eigenstates with \( m_2 \) and \( m_3 \) with a significant mixing, corresponding to an effective mixing angle of about 0.45. The \( \nu_\tau \) is nearly the state with \( m_1 \); the other two effective mixing angles are about an order of magnitude smaller than 0.45. There is a marked similarity to mixing in the quark sector.

*Present address: Union Investment, Westendstr. 1, D-60325 Frankfurt, Germany
The purpose of this note is to give reasons for the following speculative prediction for three (Dirac) neutrino (squared) mass eigenvalues: 

\[ m_1^2 \simeq 2.8 \text{ (eV)}^2, \]
\[ m_2^2 \simeq 0.01 \text{ (eV)}^2, \]
\[ m_3^2 \simeq 1.5 \times 10^{-4} \text{ (eV)}^2. \]

In addition one predicts that \( \nu_\mu \) and \( \nu_e \) are mixtures of the eigenstates with \( m_1 \) (predominantly in \( \nu_\mu \) via a \( \cos \Theta \) factor) and \( m_3 \) (predominantly in \( \nu_e \)) with a significant mixing, corresponding to an effective mixing angle of about \( \Theta = 0.45 \). The \( \nu_\tau \) is nearly the state with \( m_1 \); the other two effective mixing angles are about an order of magnitude smaller than 0.45. We show that this situation with effective mixing angles in the lepton sector bears a strong resemblance to that in the quark sector where \( d \) and \( s \) mix significantly via a Cabbibo angle \( \Theta_C \simeq 0.22 \), while the other two effective mixing angles bringing in \( b \) are much smaller. One can ask the question: why is one mixing angle significant and the others much smaller, and is there a relation between the sizes?

We answer this question in both the lepton and quark sectors by arguing that the significant effective mixing angle is the sum of the quantities involving mass ratios, whereas the much smaller angles involves the difference between such quantities of comparable size. One of our main purposes here is thus to explicitly exhibit the potential numerical similarity between all mixings in the lepton and quark sectors.

We summarize first the empirical bases for our argument concerning a possible set of connected values for three neutrino masses and their effective mixings in three states of flavor. Central to the considerations in this paper is the apparent difference between measured and predicted (muon/electron) flavor composition in the atmospheric neutrino flux [1], [2], [3]. This difference is not yet seen in some experiments [4], [5], but if it exists and is due to neutrino oscillations, then two recent detailed analyses in particular [6], [7], have shown that there must exist a squared mass difference of about \( 10^{-2} \text{ (eV)}^2 \) and a significant mixing, corresponding to an effective mixing angle \( \Theta \geq 0.35 \). On the other hand, there exist definite indications from analyses [8], [9] of different data, that two of the three effective mixing angles
are much smaller. We take as a second basis the assumption that there exists a squared mass difference of about \(3 \text{ (eV)}^2\). This takes cosmological arguments for the existence of some hot dark matter and assumes that at least one neutrino mass contributes to this.

To define our notation, we give two approximate forms of the matrix which gives the lepton flavor states in terms of the mass eigenstates. The full matrix is that of the standard form \([10]\) (eq. (28.3) in \([10]\), with \(\delta_{13} = 0\)). We have \([F1]\)

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
\approx
\begin{pmatrix}
0 & c_{13} & s_{13} \\
-c_{23} & -s_{23}s_{13} & s_{23}c_{13} \\
s_{23} & c_{23}s_{13} & c_{23}c_{13}
\end{pmatrix}
\begin{pmatrix}
\nu_1(m_1) \\
\nu_2(m_2) \\
\nu_3(m_3)
\end{pmatrix}
\approx
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-c_{12}s_{13} & -s_{12}s_{13} & c_{13} \\
s_{12} & -c_{12} & 0
\end{pmatrix}
\begin{pmatrix}
\nu_1(m_1) \\
\nu_2(m_2) \\
\nu_3(m_3)
\end{pmatrix}
\]

(1)

The first form corresponds to setting \(c_{12} = 0, s_{12} = 1\); the second corresponds to setting \(c_{23} = 0, s_{23} = 1\). Defining \(\Theta_{12} = \left(\frac{\pi}{2} - \tilde{\Theta}_{12}\right)\) and \(\Theta_{23} = \left(\frac{\pi}{2} - \tilde{\Theta}_{23}\right)\), this corresponds to effective mixing angles \(\tilde{\Theta}_{12}\) and \(\tilde{\Theta}_{23}\) tending to zero, respectively. The reason for the interchange of the usual roles of \(\sin \Theta_{ij}\) (\(\cos \Theta_{ij}\)) close to zero (unity) is simply because we have placed the largest–mass state \(\nu_1(m_1) = \nu_1\) uppermost in the column and the smallest–mass state \(\nu_3(m_3) = \nu_3\) lowest. We do this in order to conform to the conventions utilized in the recent analysis of data by Minakata \([F2]\); then our predicted mass hierarchy corresponds to the allowed case \(b\) following eq. (15) in \([F2]\). We also define \(\Theta_{13} = \left(\frac{\pi}{2} - \tilde{\Theta}_{13}\right)\). It is our result below, \(\tilde{\Theta}_{13} \simeq 0.45\), which gives rise to a significant mixing of \(\nu(m_2) = \nu_2\) and \(\nu(m_3) = \nu_3\) in the states \(\nu_\mu\) and \(\nu_e\). The theoretical speculation which we utilize is that the significant effective mixing angle \(\tilde{\Theta}_{13}\), and the much smaller angles \(\tilde{\Theta}_{12}, \tilde{\Theta}_{23}\) are given by the following...
equations, respectively, in terms of mass ratios.

\[ \tilde{\Theta}_{13} \simeq \left\{ \frac{m_u}{m_c} \right\}^{1/2} + \left( \frac{m_3}{m_2} \right)^{1/2} \simeq \{0.07+0.38\} \simeq 0.45 \] (2a)

\[ \tilde{\Theta}_{12} \simeq \tilde{\Theta}_{23} \simeq \left\{ \frac{m_u}{m_e} \right\}^{1/2} - \left( \frac{m_2}{m_1} \right)^{1/2} \simeq \{0.245 - 0.245\} \simeq 0 \] (2b)

Starting with the indication from the atmospheric neutrino anomaly, \( m_2 \simeq \sqrt{0.01} \) eV in eq. (2b), \( m_1 \sim \sqrt{2.8} \) eV is estimated by taking \( \tilde{\Theta}_{12}, \tilde{\Theta}_{23} \) as tending to zero. This is, of course, an idealization. A small value for these angles \( \leq 0.04 \) arises for \( m_1 \leq \sqrt{5.8} \) eV. Then, \( m_3 \simeq \sqrt{1.5 \times 10^{-4}} \) eV in eq. (2a) is estimated by taking the ratio of ratios \( \left( \frac{m_2/m_1}{m_3/m_2} \right)^{1/2} \approx 0.65 \), for orientation. This is the same number that occurs in the quark sector (note the ratio \( (0.146/0.224) \) in eqs. (4a, b) below). With this number, for \( m_1 \leq \sqrt{5.8} \) eV, we have \( m_3 \geq \sqrt{10^{-4}} \) eV. Its is noteworthy that if the minus sign between the two terms in eq. (2b) were to be changed to a plus sign, the sum would be \( \sim 0.49 \), nearly the same number as in eq. (2a). With the minus sign, a near cancellation of two comparable terms occurs.

For comparison, we apply these considerations to the quark sector, using \[ [F3] \] \( m_u \simeq 2 \) MeV, \( m_d \simeq 5 \) MeV, \( m_c \simeq 1.55 \) GeV, \( m_s \simeq 100 \) MeV, \( m_t \simeq 180 \) GeV, \( m_b \simeq 4.7 \) GeV. Using the standard mixing matrix \[ [10] \], with \( s_{13} \rightarrow 0 \), the flavor states (primed) are

\[
\begin{pmatrix}
  d' \\
  s' \\
  b'
\end{pmatrix} \simeq
\begin{pmatrix}
  c_{12} & s_{12} & 0 \\
  -c_{23}s_{12} & c_{23}c_{12} & s_{23} \\
  s_{23}s_{12} & -s_{23}c_{12} & c_{23}
\end{pmatrix}
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix}
\] (3)

The equations for effective mixing angles analogous to eqs. (2a, b) are

\[ \Theta_{12} = \left\{ \frac{m_u}{m_c} \right\}^{1/2} + \left( \frac{m_d}{m_s} \right)^{1/2} \simeq \{0.036+0.224\} \simeq 0.26 \] (4a)

\[ \Theta_{23} = -\left( \frac{m_c}{m_t} \right)^{1/2} + \left( \frac{m_s}{m_b} \right)^{1/2} \simeq \{-0.093+0.146\} \simeq 0.05 \] (4b)
The closeness of the effective mixing angle in eq. (4a) (specifically, \( (m_u/m_s)^{1/2} \approx 0.224 \)) to the empirical Cabibbo angle is, of course, known [11]. However, the near cancellation of two comparable terms in eq. (4b) results in a significantly smaller angle. This is close to the empirical value (note eq. (28.2) in [10]); it is closer than the quantity \( (m_s/m_b)^{1/2} \) alone. If the top mass were nearer to the intermediate vector–boson masses, \( \approx 90 \text{ GeV} \), eq. (2b) would give a mixing angle which approaches zero. Again it is noteworthy that if the minus sign in eq. (4b) were to be changed to a plus sign, the sum would be 0.24, nearly the same number as in eq. (4a). Moreover, there is a marked similarity between the overall pattern of the four numbers, square roots of mass ratios, involved in the addition and subtraction in eqs. (4a, b) and the hypothetical pattern of the four numbers involved in the addition and subtraction in eqs. (2a, b). This is reflected in the scale factor of only \( \sim 1.7 \) difference between the sum in eq. (2a) from that in eq. (4a), and in the similar smallness of the differences in eq. (2b) and eq. (4b). In this sense, the mixing of mass eigenstates in the lepton and quark sector are not as different as has often seemed.

These numerical similarities are interesting to observe, even in the absence of a detailed theoretical model. The origin of relations like those in eqs. (2a,b) and eqs. (4a,b) can be in dynamical models in which the lower masses are generated in second order of a Higgs–type or a \( \sigma \)–model type [12] mixing interaction to the mass above, that is \( m_2 \sim g_{21}^2 m_1, m_3 \sim g_{32}^2 m_2 \). The sign of the couplings \( g_{21} \) and \( g_{32} \) are physically relevant in determining the addition or subtraction for calculating the effective mixing angles. Of course, the “starting” mass value, i.e. the extremely small value of \( m_{\nu_e} \sim 1.7 \text{ eV} \) relative to \( m_\tau \approx 1.8 \text{ GeV} \) is not explained. (The mass ratio \( \sim 10^9 \) is maintained for \( m_{\nu_\mu} \sim 0.1 \text{ eV} , m_\mu \sim 100 \text{ MeV} \).) Explanations of this (i.e. a “see–saw” type model) usually introduce a new mass scale of about \( 10^9 \text{ GeV} \) (see eq. 11.19 in [13]). However, one can note that in the neutrino mass pattern postulated here, the ratios \((m_\mu/(m_2 \cdot 10^6)), (m_\tau/(m_1 \cdot 10^6)) \sim 10^3 \) have increased
by a factor of $\sim 25$ over the first ratio $(m_e/(m_3 \cdot 10^6)) \simeq 40$. In the quark sector, the ratios $(m_c/m_u) \simeq 15.5$ and $(m_t/m_b) \simeq 38.5$ represent increase factors which are comparable to 25, over the ratio $(m_u/m_d)$ taken as $\sim 1$. The relevant factor of $10^{-9}$ might, for example, be associated with a small admixture $r^2 \geq 10^{-9}$ of right-handed coupling giving $m_3 \sim r^2 m_\tau$ as a weak radiative effect.

We have not explained the magnitude of the apparent deficiency of solar neutrinos with our hypothetical neutrino mass and mixing pattern. However, vacuum oscillations for the relatively large relevant $\Delta m^2 \sim 10^{-2}(eV)^2$ do lead to some deficiency, at the 30% level. Other possibilities (i.e. MSW mixing) are summarized in the vertical lines in figs. (4a, b) and figs. (5a, b) of [14], which also discusses the effect of a cooler sun.

We note that some speculations [6], [15] have concentrated upon a hypothesis for the existence of two nearly degenerate neutrino mass eigenstates with mass $\sim (6.5 - 3.5)$ eV and a degeneracy at the level of $10^{-3}$ eV. These must mix significantly [F4] to form $\nu_\mu$ and $\nu_\tau$ [F5]. Then the atmospheric neutrino anomaly involves $\nu_\mu \rightarrow \nu_\tau$ rather than $\nu_\mu \rightarrow \nu_e$ as in the mass pattern given in this paper. There is no explanation of the apparent solar neutrino deficiency with only the three known neutrinos.

There are also recent speculations [18], [19] that the lightest neutrino is also nearly degenerate with the two: with a degree of degeneracy of about [18] $10^{-6}$ eV, or with a remarkable degree of degeneracy of about [19] $10^{-11}$ eV. Such degrees of degeneracy are postulated in order to interpret all of the experiments on the solar neutrino deficiency: in the former case [18] via matter–enhanced oscillations with a very small effective mixing angle, and in the latter case [19] via vacuum oscillations with a large mixing angle.

In conclusion, we note the implications of the present hypothesis for certain
experiments which are continuing, and put them in the context of recent analyses. There is, in principle a mixing effect in the LSND experiment for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ [20]. This occurs effectively via a “virtual $\nu_\tau$”; this effect was analyzed by Minakata [6] and has also been discussed by others [21]. However, with $\tilde{\Theta}_{12}, \tilde{\Theta}_{23} < 0.07$, the probability is more than an order of magnitude below the possible effect in the first paper of [20]. The $(\Delta m^2 - \sin^2 2\tilde{\Theta}_{13})$ values discussed here lie just inside [F 6] the region allowed by reactor experiments [22], [23]; this is best seen by observing the allowed region relevant to the possibility of atmospheric $\nu_\mu - \nu_e$ mixing in the fig. 3 of the recent analysis [24] of atmospheric neutrino oscillations [F 7]. Thus, an effect should emerge in the on–going reactor experiments, and in the up–coming super–Kamiokande experiment [24].

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Footnotes

[F 1 ] The flavor states are thus transparently given by

\[ |\nu_e > = c_{13} |\nu_2 > + s_{13} |\nu_3 > \]
\[ |\nu_\mu > = -c_{23} |\nu_1 > + s_{23} |\nu'_\mu > \]
\[ |\nu_\tau > = s_{23} |\nu_1 > + c_{23} |\nu'_\mu > \]

with \[ |\nu'_\mu > = -s_{13} |\nu_2 > + c_{13} |\nu_3 > . \]

[F 2 ] Minakata emphasizes that the analysis of data in [8] does not depend upon the relative magnitude of \( m_2 \) and \( m_3 \); it depends upon the difference in squared mass.

[F 3 ] Consistently, we use the lower values for \( m_u, m_d \) and \( m_s \), with \( (m_u/m_d) \) and \( (m_s/m_d) \) in the middle of their allowed range (see pages 1437, 1438 in [10]).

[F 4 ] One may note that a relation like eq. (2b) can accommodate a maximal effective mixing angle for \( (m_2/m_1) \sim 1 \), namely

\[ \Theta \simeq \left\{ -\left( \frac{m_\mu}{m_\tau} \right)^{1/2} + \left( \frac{m_2}{m_1} \right)^{1/2} \right\} \simeq \{-0.25 + 1\} \simeq 0.75 \]

[F 5 ] The hypothetical mass degeneracy avoids the stringent limits set for \( \nu_\mu \rightarrow \nu_\tau \) mixing when \( \Delta m^2 \) is greater than about 3(eV)^2 by the experiments in [16]. (It has been stated that such mass degeneracy might be “natural” for Majorana neutrinos [17].)

[F 6 ] S. B. thanks José Valle for emphasizing this point to him.

[F 7 ] Note that the index 3 in [24] is 2 in the present context. In [24], the heaviest mass (denoted by 3) was, in effect, assumed to be no greater than a few tenths of an electron-volt. Thus making no significant contribution to dark matter.
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