Ricci operator in a Hopf real Hypersurfaces of complex space form

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Abstract. We initiate the study of real hypersurface of a complex space form and it is proved that Lie parallelism of the Ricci operator in the direction of \(\xi\), locally symmetric Ricci operator in a real hypersurface of a complex space form reduce the hypersurface to Hopf hypersurface.

1. Introduction
Let \(M_n(c)\) denote an \(n\)-dimensional Kaehlerian manifold of constant holomorphic sectional curvature \(c\) is called a complex space form. An \(M\) be a real hypersurface in \(M_n(c)\). In the event that \(A\xi = \alpha \xi\) is contented, then the structure vector field \(\xi\) of \(M\) could be a principal vector and \(M\) is named a Hopf hypersurface of \(M_n(c)\), wherever \(A\) means the shape operator of \(M\) and \(\alpha = g(A\xi, \xi)\). Complex space form could be a projective space \(P_n(c)\), a Euclidean space \(c_n\) or a hyperbolic space \(H_n(c)\), according to \(c > 0\), \(c = 0\) or \(c < 0\). Real hypersurfaces of complex space forms are studied widely several authors like [1] - [14] and numerous others. In this paper, we tend to prove conditions in terms of symmetry, Lie parallelism of the Ricci operator for a hypersurface to be a Hopf hypersurface. The paper is organized as follows: The section 2, contains preliminaries of real hypersurface of a complex space form. In section 3, we tend to considered Lie \(\xi\)-parallel, locally symmetric and obtained conditions for real hypersurface to be a Hopf hypersurface. Further we tend to derived expressions for constant holomorphic sectional curvature \(c\).

2. Preliminaries
Let \(M\) be a real hypersurface complex \(n\)-dimensional complex space form \(M_n(c)\). We give basic formulas,

\[
\phi^2 U = -U + \eta(U)\xi, \quad g(\phi U, \phi V) = -g(U, V), \quad \phi \xi = 0, \quad \eta(\phi U) = 0, \quad \eta(\xi) = 1. 
\tag{1}
\]

The Gauss and Weingarten formulas the followings:

\[
\nabla_U \xi = \phi AU. 
\tag{2}
\]

\[
(\nabla_U \phi)V = \eta(V)AU - g(AU, V)\xi. 
\tag{3}
\]

From equations of Gauss and Codazzi:

\[
R(U, V)T = \frac{c}{4} [g(V, T)U - g(U, T)V + g(\phi V, T)\phi U - g(\phi U, T)\phi V - 2g(\phi U, V)\phi T] + g(AV, T)AU - g(AU, T)AV, 
\tag{4}
\]
\[
(\nabla_U A)V - (\nabla_V A)U = \frac{c}{4} \left[ \eta(U)\phi V - \eta(V)\phi U - 2g(\phi U, V)\xi \right],
\]
(5)

\[
S(V, T) = \frac{c}{4} \left[ (2n + 1)g(V, T) \right] + hg(AV, T) - g(AV, AT),
\]
(6)

\[
QV = \frac{c}{4} \left[ (2n + 1)V \right] + hAV - A^2 V,
\]
(7)

where the Ricci operator \(Q\) is characterized by
\[
g(QU, V) = S(U, V).
\]

From (6), we get
\[
r = \frac{c}{4} \left[ (2n)^2 - 1 \right] + h^2 - 2n + 1.
\]
(8)

Let \(\Pi\) be the open subset of \(M\) characterized by
\[
\Pi = \{ p \in M | A\xi - \alpha\xi \neq 0 \},
\]
(9)

where \(\alpha = \eta(A\xi)\). We put
\[
A\xi = \alpha\xi + \mu W,
\]
(10)

where \(W\) is a unit vector orthogonal to \(\xi\) and \(\mu\) does not vanish on \(\Pi\).

3. Hopf real hypersurface of a complex space form
In here consider Lie \(\xi\)-parallel, locally symmetric and obtained conditions for \(M\) to be a Hopf hypersurface. Further we derived expressions for \(c\).

**Theorem 3.1** The hypersurface \(M\) is a Hopf hypersurface for constant \(h = \text{trace}A\) or holomorphic curvature \(c\) satisfies (23).

Suppose \((L_\xi Q)(U) = 0\), for any \(X\) in \(T(M)\) holds in \(M\).

Next we get
\[
\nabla_\xi Q(U) - \nabla Q(\xi) = Q(\nabla_\xi U) - Q(\nabla_U \xi).
\]
(11)

Making use of (7), (2) in (11), we get
\[
(\xi h)AU + h(\nabla_\xi A)U - (\nabla_\xi A)AU = 0,
\]
(12)

where \(h\) is a constant. Then we have
\[
h(\nabla_\xi A)U - (\nabla_\xi A)AU = 0.
\]
(13)

We take the covariant differentiation of (10) and using (2), we obtain
\[
(\nabla_U A)\xi = \alpha \phi AU - A\phi AU + \mu \alpha \phi W,
\]
(14)

for any \(U\) on \(M\). Substituting (14) in (5), we get
\[
(\nabla_\xi A)U = \frac{c}{4} \phi U + \alpha \phi AU - A\phi AU + \mu \alpha \phi W.
\]
(15)

Using (15) in (13), we have
\[
h \left[ \frac{c}{4} \phi U + \alpha \phi AU + \mu \alpha \phi W - A\phi AU \right] - \frac{c}{4} \phi AU - \alpha \phi A^2 U - \mu \alpha \phi AW + A\phi A^2 U = 0.
\]
(16)

Taking \(U = \xi\) in (16) and using (10), we get
\[
\mu \left\{ \alpha(2h - \alpha) - \frac{c}{4} \phi W - [h - \alpha(\alpha - 1)]A\phi W - 2\alpha \phi A^2 W \right\} = 0.
\]
(17)
Applying \( \phi \) in (17) and contracting with \( U \), we get
\[
\mu \left\{ \left[ \alpha(2h - \alpha) - \frac{c}{4} \right] g(W, U) + [h - \alpha(\alpha - 1)] g(AW, U) + 2\alpha g(A^2 W, U) \right\} = 0. \tag{18}
\]
Putting \( U \) by \( \xi \) in (18), by repeated application of (10), we obtain
\[
\mu [h + \alpha(\alpha + 2\gamma + 1)] = 0, \tag{19}
\]
where \( \gamma = g(AW, W) \).

If \( \mu = 0 \) then from (10), we have \( A\xi = \alpha \xi \), i.e., \( M \) is a Hopf hypersurface of \( M_n(c) \).
Now taking \( U = W \) in (18), we obtain
\[
\mu \left\{ \alpha(2h - \alpha) - \frac{c}{4} + [h - \alpha(\alpha - 1)] \gamma + 2\alpha \|AW\|^2 \right\} = 0. \tag{20}
\]
If \( \mu \neq 0 \) then from (19) and (20), we get
\[
h + \alpha(\alpha + 2\gamma + 1) = 0. \tag{21}
\]
\[
c = 4 \left[ \alpha(2h - \alpha) + [h - \alpha(\alpha - 1)] \gamma + 2\alpha \|AW\|^2 \right]. \tag{22}
\]
Using (21) in (22), we obtain
\[
c = 4 \left[ 2\alpha(\|AW\|^2 - \alpha - 1) - ((4 + \alpha)\alpha + \gamma + 1)\gamma \right]. \tag{23}
\]

**Theorem 3.2** If in the hypersurface \( M \), \( QL_{\xi} U + L_{\xi} QU = 0 \) holds then \( M \) is a Hopf hypersurface in \( M_n(c) \) provided \( \alpha + \gamma \neq 0 \).

The \( L_{\xi} \) operator is defined by \( \phi A - A\phi = L_{\xi} \). Suppose Ricci operator \( Q \) satisfies \( QL_{\xi} + L_{\xi} Q = 0 \) and \( \Omega \) is non empty. Then the above condition together with (7) and (10) gives
\[
\mu \left\{ \left\{ (2n + 1)c + 4h\alpha - 4\alpha^2 \right\} \phi W + \left\{ 2h - \alpha \right\} \phi AW - \phi A^2 W - A^2 \phi W \right\} = 0. \tag{24}
\]
Applying \( \phi \) in (24) and contracting with \( U \), by repeated application of (10), we obtain
\[
\mu \left\{ - \left\{ (2n + 1)c + 4h\alpha - 4\alpha^2 \right\} g(W, U) - (2h - \alpha) g(AW, U) + 2\mu(h - \alpha) g(\xi, U) + 2g(A^2 W, U) - \mu \gamma g(\xi, U) \right\} = 0. \tag{25}
\]
Taking \( U \) by \( \xi \) in (25), we get
\[
\mu^2 [\alpha + \gamma] = 0. \tag{26}
\]

**Theorem 3.3** An \( M \) be a real hypersurface of a complex space form \( M_n(c) \), satisfying \( QL_{\xi} U + L_{\xi} QU = 0 \) with trace \( A = \alpha \), then complex space form \( M_n(c) \) is a projective space.

Taking \( U = W \) in (25), we get
\[
\left[ 4\alpha^2 - 4h\alpha - (2n + 1)c \right] - \left[ 2h - \alpha \right] g(AW, W) + 2g(W, A^2 W) = 0. \tag{27}
\]
Equivalently,
\[
\left[ 4\alpha^2 - 4h\alpha - (2n + 1)c \right] - \left[ 2h - \alpha \right] \gamma + 2\|AW\|^2 = 0. \tag{28}
\]
This implies
\[
c = \frac{(4\alpha + \gamma)(\alpha - h) + \|AW\|^2}{2n + 1}. \tag{29}
\]
If \( \alpha = h \), then \( c = \frac{\|AW\|^2}{2n + 1} \).
Theorem 3.4 If the Ricci operator $Q$ in $M$ symmetric. Then either $M$ is a Hopf hypersurface for constant $h = \text{trace} A$ or holomorphic curvature $c$ satisfies (38).

Suppose $(\nabla_U Q)V = 0$, for any $U$ and $V$ on $M$. Then differentiating differentiating covariantly with respect to $U$, we have

$$\frac{(2n+1)c}{4}\nabla_U V + (\nabla_U h)AV + h(\nabla_U A)V + hA(\nabla_U V) - (\nabla_U A)AV - A(\nabla_U A)V - A^2(\nabla_U V) = 0.$$  

(30)

Taking $U = V = \xi$ in (30) and using (14), we get

$$\frac{(2n+1)c}{4}\phi A\xi + (\xi h)A\xi + h\alpha \phi A\xi + h\mu \phi \alpha W + hA\phi A\xi - \alpha^2 \phi A\xi + \alpha A\phi A\xi$$

$$- \alpha h^2 \mu \phi W - \alpha A\phi A\xi + hA^2 \phi A\xi - h\mu A\phi W - A^2 \phi A\xi = 0.$$  

(31)

If $(\xi h) = 0$, using (10) in (31), we have

$$\mu \{\frac{nc}{2} - \alpha (\alpha - 2h + 1)\phi W - \alpha \phi AW - \alpha A\phi W + A\phi AW\} = 0.$$  

(32)

Applying $\phi$ in (32) and contracting with $U$, we get

$$\mu \{\alpha (\alpha - 2h + 1) - \frac{nc}{2} g(W, U) + 2\alpha g(AW, U) - \alpha \mu g(\xi, U) - g(A^2W, U)\} = 0.$$  

(33)

Putting $U$ by $\xi$ in (33), we get

$$\mu^2 \alpha - \gamma = 0,$$  

(34)

If $\alpha - \gamma \neq 0$, then $\mu = 0$.

On the other hand we using $U = W$ in (33), and obtain

$$\mu \{\alpha (\alpha - 2h + 1) - \frac{nc}{2} + 2\alpha \gamma - \| AW \|^2\} = 0.$$  

(35)

If $\mu \neq 0$, then from (34), (35), we get

$$\alpha - \gamma = 0.$$  

(36)

This implies

$$c = \frac{2[\alpha (\alpha - 2h + 1) + 2\gamma^2 - \| AW \|^2]}{n}.$$  

(37)

Using (36) in (37), we obtain

$$c = \frac{2[3\gamma^2 - (2n - 1)\gamma - \| AW \|^2]}{n}.$$  

(38)

4. Conclusions

Consider $\xi$-Ricci-semi-symmetric, cyclic parallel, Lie-$\xi$-parallel and locally symmetric hypersurface of a complex space form and obtained conditions for real hypersurface to be a Hopf hypersurface. The constant holomorphic sectional curvature $c$ is given explicitly interns of trace of shape operator.

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