Nonlinear frequency response of the multi-resonant ring cavities

VITALII V. VITKO, ANDREY A. NIKITIN, MIKHAIL A. CHERKASSKII, ALEXEY B. USTINOV, AND BORIS A. KALINKOS

St. Petersburg Electrotechnical University 'LETI', Department of Physical Electronics & Technology
St. Petersburg, 197376, Russia
*and.a.nikitin@gmail.com

Abstract: New theoretical approach describing a nonlinear frequency response of the multi-resonant nonlinear ring cavities (RC) to an intense monochromatic wave is developed. The approach closely relates the many-valuednesses of the RC frequency response and the dispersion relation of a waveguide, from which the cavity is produced. Arising of the multistability regime in the nonlinear RC is treated. The threshold and dynamic range of the bistability regime for an optical ring cavity with the Kerr nonlinearity are analytically derived and discussed.

1. Introduction

Resonant ring cavities (RC) belong to the core constructions of modern technology. They are utilized to produce various passive and active devices of optics, microwave electronics, spin-wave electronics, plasmas, and other. As is well known, there are two conditions that are necessary to observe a resonant behavior. The first one is a phase condition. It consists in an in-phase addition of all the waves circulating in the ring. The second one is a dissipative condition. It claims in a small attenuation, which is necessary for multiple addition of the circulating waves. It is the multiple in-phase addition that leads to the resonant enhancement of the signal at a certain frequency, in case the both conditions are satisfied. The resonant enhancement provides a decrease in the nonlinear processes threshold, which makes the RC exceptionally useful constructions to study a variety of nonlinear phenomena.

The Lugiato-Lefever equation (LLE) is widely used to describe the nonlinear phenomena of the ring cavities. This equation was derived using both the mean-field approach and infinite-dimensional map (Ikeda map) [1–4]. In other words, in a representative LLE approach one plugs nonlinearity and integrates dispersion as a function of the modal index, so that despite dispersion is linear, one obtains bistability as a result of nonlinear shift of the resonance. Hence, the LLE is instrumental in modeling of the different nonlinear waveforms [5–11]. Recently the LLE was extended to consider the multiple nonlinear resonances appearing in the optical RC as well as the multi-valued stationary states [12–13]. In parallel, such states were independently investigated with a general Ikeda map [14]. The investigations done beyond the conventional LLE open new possibilities to describe the multistability in the solitary and coupled micro-rings [15–17]. As an example, we mention investigations of a super-cavity soliton formation [14], a soliton formation in the coupled micro-rings [18], and a cavity soliton formation in the micro-rings with active elements [19].

Two types of the nonlinear instabilities are distinguished in the RC studies – the dispersive and absorptive ones [20]. Note that in existing RC literature they are indicated but not entirely viewed. Specifically speaking, the dispersive instability is described using RC frequency response, which does not ponder the wave-number many-valuednesses of the dispersion characteristic. Due to this, the important phase-dependent features of the nonlinear processes are lost. For example, when calculating the RC transmission spectra, its real part is only obtained.
Our theoretical approach, being the field of the extended LLE, does not challenge the existing literature, yet, at the same time, it introduces a new essential feature. This feature consists in a self-consistent unification of the observed frequency response and changing in the dispersion law. The aim of this paper is to propose a new theoretical approach, which bonds the effects of the nonlinear dispersion and frequency response of the multi-resonant nonlinear ring cavities. The approach enables one to relate the many-valuednesses of the RC frequency response and the dispersion characteristic of a waveguide the RC is made of.

2. Theoretical approach to describe nonlinear multi-resonant RC

In order to introduce the approach, let us consider a multi-resonant RC of the length $l$, which includes of a unidirectional coupler between a ring resonator and a waveguide shown in Fig. 1. Assume that a plane monochromatic wave with amplitude $A_0$ is applied to the ring input. Coupling of the RC with the pumping waveguide is described by the drop-in and drop-out power coupling coefficients $\kappa_1$ and $\kappa_2$, respectively. On assumption of the lossless coupling, the amplitude of the wave after the first round trip over the ring is $A_1 = \sqrt{\kappa_1 \kappa_2} A_0 \exp(i\beta l - \alpha l)$ where $\beta$ is a propagation constant and $\alpha$ is a damping decrement. Every single round trip over the ring is described by the wave factor $\sqrt{\kappa_2} \exp(i\beta l - \alpha l)$. Denoting the resulted amplitude of the signal circulating in the ring by $A_c$, we can find it as a superposition of an infinite number of the circulating waves in the following form $A_c = A_0 \sqrt{\kappa_1} \sum_{q=1}^{\infty} (\sqrt{\kappa_2})^q \exp\left[q (i\beta l - \alpha l)\right]$ where $q$ is a summation index representing a number of the wave circulations. As far as damping decrement in a dissipative medium has a positive value ($\alpha > 0$), the series converges and after summation the resulted complex amplitude becomes

$$A_c = \frac{A_0 \sqrt{\kappa_1} \sqrt{\kappa_2} \exp(i\beta l - \alpha l)}{1 - \sqrt{\kappa_2} \exp(i\beta l - \alpha l)}. \tag{1}$$

Defining the complex transmission coefficient as $H = A_c/A_0$, we obtain the power transmission coefficient in the form

$$H_p = |H|^2 = \frac{\kappa_1 \kappa_2}{e^{2\alpha l} - 2e^{\alpha l} \sqrt{\kappa_2 \cos(\beta l)} + \kappa_2} = \frac{I_c}{I_0} \tag{2}$$

where $I_0$ and $I_c$ are the input and intracavity intensities. The derived coefficient [Eq. (2)] demonstrates a multi-resonant behavior because of the constructive interference of the circulating waves having the resonant wave-numbers $\beta_m = 2\pi m/l$ where $m$ is a mode number. The corresponding frequency response $H_p(\omega)$ is found by substitution of the appropriate dispersion relation $\beta(\omega)$ and the coupling coefficients frequency dependencies in [Eq. (2)].

Note that relations akin [Eq. (2)] were used in a number of investigations to describe the frequency response of the optical micro-resonators [21,22], the multiferroic active ring resonators [23], the optoelectronic rings [24], and the spin-wave optoelectronic rings [25–27]. A critical feature of the frequency response [Eq. (2)] is an enhancement of the intracavity field intensity close to the resonant frequencies, being the effect to take into account for the multi-resonant rings. To thoroughly study this effect, one should consider the dependence of an RC performance on the wave sensitivity of the waveguide parameters of which it is made.
In this work, we use an optical wave intensity dependent refractive index as the nonlinear parameter of the waveguide substance. In the case of the Kerr nonlinearity, we write 

\[ n(I) = n_0 + n_2 I \]

where \( n_0 \) is a linear refractive index, \( n_2 \) is a second-order refractive index, and \( I \) is an optical wave intensity. For investigation of the nonlinear RC, we substitute the intracavity intensity \( I_c(\omega) \) obtained from [Eq. (2)] into the expression for refractive index, receiving \( n(\omega, I_0) = n_0 + n_2 I_0 H_p(\omega) \). The latter formula shows that the refractive index is enhanced at all the resonant frequencies. The next step in introducing of our approach should be done utilizing some dispersion law for a regular waveguide fabricating the resonant RC.

For demonstration of a generality and advantages of our approach, we will use below the simplest approximation for the linear dispersion law in a form

\[ \omega = c \beta n \]  

(3)

where \( c \) is the speed of light. Note that this step is valid for the frequency band, in which the dispersion impact on the wave process is relatively weak. Usually it works well for analysis of several neighboring resonant frequencies. Provided it is necessary to study a nonlinear RC in a wide frequency range, one should consider a higher-order dispersion.

To study the nonlinear effects, we substitute the intensity dependent refractive index \( n(\omega, I_c) \) instead of the linear one in the linear dispersion law [Eq. (3)], obtaining:

\[ \omega(\beta, I_0) = \frac{c \beta}{n_0 + \frac{n_2 \kappa \kappa_j I_0}{\exp(2aI) - 2\exp(aI)\sqrt{\kappa_j \cos(\beta I) + \kappa_j}}}. \]  

(4)

Thus, we arrived at the nonlinear dispersive relation. Using this relation, it is straightforward finding the nonlinear transmission coefficient/response of the RC \( H_p(\omega, I_0) \). We find it through substitution of a solution of [Eq. (4)] into [Eq. (2)]. It is important to underline that the obtained nonlinear relations for \( \beta(\omega, I_0) \) and \( H_p(\omega, I_0) \) are the functions of two-variables. For their demonstrative description, it is instructive to introduce the dispersion surfaces. Here and after, we will name them as the nonlinear dispersion and the nonlinear response surfaces. Fig. 2.a and 2.b show the color-mapped images of the both surfaces.
To construct the figures, we introduce notations that are a dimensionless frequency \( \Omega = (\omega - \omega_m)/\Delta \Omega \) where \( \omega_m \) is a resonant frequency with number \( m \), \( \Delta \Omega/2\pi \) is the free spectral range of the ring, a dimensionless wave-number \( B = (\beta - \beta_m)/\beta_m \) where \( \beta_m = 2\pi m/l \) is a resonant wave-number, a dimensionless frequency response \( T = H_p(\omega, I_0)/H_p(\omega, 0) \) where \( H_p(\omega, 0) \) is a linear frequency response at the resonant frequency, which defines the maximum of the transfer function. Analyzing the obtained data, it is possible to do valuable conclusions on the nonlinear resonant RC characteristics. Below we discuss some of them only.

One can see a fascinating behavior of the nonlinear RC characteristics that manifest themselves with increasing in the input optical intensity \( I_0 \). The green lines calculated for the low input intensity \( (I_0 = 0.1 I_m) \) show the linear dispersion characteristic and transfer function. An increase of the input intensity provides increasing in the refractive index, which produces downshift of all the resonant frequencies. This effect is enhanced due to the constructive interference of the circulating waves and an increase of the intracavity intensity in vicinity of the RC resonant frequencies. Moreover, our approach allows one to definitely relate the nonlinear dispersion shift and the power of the signal circulating in the ring. It complements other approaches such as Ikeda map and LLE.

The orange lines in Fig. 2.a and 2.b present the characteristics calculated for the threshold intensity \( (I_0 = I_m) \). The threshold intensity is defined as a maximum intensity of the input signal, for which a single solution of the two equations [Eq. (2)] and [Eq. (4)] exists. Expression for the threshold intensity will be obtained and discussed below. Note that an increase in the input intensity \( I_0 \) higher than \( I_m \) provides appearance of the region with two stable and one unstable state of the intracavity intensity, which corresponds to the bistability phenomenon. The magenta lines in the both figures show the dispersion characteristics and the transmission coefficient frequency response for the intensities higher than the threshold value \( (I_0 = 15I_m) \). Further increase in the input intensity \( (I_0 = 100I_m) \) extends the frequency band of the instable behavior. Provided this band covers the frequency range located between two neighboring resonant frequencies, then additional stable and unstable states appear.

Our calculations show that the new unstable states can develop progressively with increasing of the circulating power. Such a behavior corresponds to the multistability regime (see blue lines in Fig 2.a and 2.b). So, for the nonlinear frequency shift \( (I_0 = 100I_m) \), which is more than two free spectral ranges \( (\Delta \Omega > 2) \), four stable and three unstable values of the intracavity intensity and corresponded wave-number multivaluedness appear. Furthermore, more and more unstable states may advance with increasing of the circulating power.

It is clear physically that in order to have multistability, it is necessary to observe the conditions, in which the nonlinear frequency shift is more than the distance between two adjacent frequencies. In case of a single ring, this requires a sufficiently high input power, which in the real-life resonant ring cavities cannot be achieved due to a nonlinear damping. However, the threshold for the multistability observation can be significantly reduced in some cases like coupled resonator systems [28]. But this effect is out of the paper scope.
4. Optical ring bistability threshold

As an example, we consider the bistability phenomenon in an optical ring cavity. As was already mentioned, one of the distinctive features of the bistability behavior is an appearance of the two-valuedness in the relationship between the intracavity and input intensities. Fig. 3 shows the nonlinear dispersion characteristics and the frequency responses calculated with the developed technique using [Eq. (4)] and [Eq. (2)] for the various input intensities $I_0$. The green solid lines calculated for zero input intensity represent the linear dependences. An increase in the intensity up to the bistability threshold causes appearance of the kinks $\partial \omega(\beta, I_0)/\partial \beta = 0$ and $\partial H_p(\omega, I_0)/\partial \omega = 0$ on the dispersion and transmission characteristics (see the orange dotted line in Fig. 3.a and 3.b). As help for eyes, these points are shown on the curves with the small black squares.

Due to the nonlinearity, the dispersion relations $\omega(\beta)$ and the transmission coefficients $H_p(\omega)$ become multi-valued slightly above the threshold value (up to $I_0 = 3I_0$). The bistability region corresponds to the frequency range bounded by the extremums shown in Fig. 3 by the orange and blue open cycles. Note that these extremums correspond to the group velocity zeros for the wave under consideration. As is seen from Fig. 3.a, the nonlinear dispersion characteristics coincide with the linear one in the frequency range situated far enough from the given resonance.
Turn now to analytical description of the bistability phenomenon. For this purpose, we introduce a nonlinear frequency shift \( \sigma = \omega - \omega_m \), which characterizes the offset of the resonant frequency \( \omega_m \) due the input intensity change. Hence the condition for the bistability threshold reads as \( \partial \sigma(\beta, I) / \partial \beta = 0 \). For definiteness, let us consider the critically coupled RC. In such mode, the drop-out coupling coefficient is equal to the power losses for the single round trip over the ring, i.e. \( \kappa_2 = \exp(-2\alpha l) \). Note that \( \kappa_1 = 1 - \kappa_2 = 1 - \exp(-2\alpha l) \) takes place due to the reciprocity of the coupling coefficients. Thus, according to the developed approach the nonlinear frequency shift is

\[
\sigma(\beta, I) = \frac{c(\beta - \beta_m)}{n_f(1 - \exp(\alpha l))I_0}.
\]

Here \( \alpha = 2\alpha \) is a generalized damping decrement for the critically coupled RC. Using the Maclaurin’s expansions for \( \cos(\beta l) \) and \( \exp(\alpha l) \), one could write the equation \( \partial \sigma(\beta, I) / \partial \beta = 0 \) in the following form

\[
\beta^4 + C_2 \beta^2 + C_1 \beta = C_0
\]

where \( C_2 = \frac{\alpha}{n_f} (2\alpha n_f n_0 + 3I_0 n_2) \), \( C_1 = 2 \frac{\alpha}{n_f} \beta n_f n_2 \), and \( C_0 = -\frac{\alpha^2}{n_f^2} (\alpha n_f I_0 n_2) \).

The last equation is equivalent to used in the cusp catastrophe [29]. The number of the real roots of [Eq. (6)] is defined by the value of the input intensity \( I_0 \). [Eq. (6)] has a single root in the case when its discriminant vanishing \( D = 0 \). As in the catastrophe theory, the threshold is found from the conditions \( \partial \sigma(\beta, I) / \partial \beta = 0 \) and \( D = 0 \). These conditions give the threshold value, which reads as

\[
I_{th} = \frac{4\sqrt{5} \pi n_0^2 l}{9 Q^2 n_2 \lambda}
\]

where \( Q = \beta / 2\alpha \) is a quality factor of the critically coupled RC in the linear regime, \( \lambda \) is a wavelength of the laser radiation in free-space.
Substitution [Eq. (7)] in [Eq. (5)] obtains the following threshold value of the nonlinear frequency shift \( \sigma_{th} = -\sqrt{3} \Delta \omega / 2 \) where \( \Delta \omega = 2 \alpha_0 / n \) is a half-power bandwidth. It is clear that if a frequency shift exceeds the threshold value \( \sigma > \sigma_{th} \) the bistability should appear. We note that under the physically clear simplifications, the derived expressions for the threshold intensity [Eq. (7)] and the nonlinear frequency shift coincide with the results obtained in a series of investigations devoted to the Kerr nonlinearity [11,30,31].

5. Dynamic range of the bistability phenomenon

For analysis of the bistability dynamic range, we use the diagram of the nonlinear behavior [29]. As in the catastrophe theory, such diagram is a curvilinear surface, which demonstrates all real roots of [Eq. (6)] for the intracavity intensity. The diagram of the nonlinear behavior showing dependences of the normalized intracavity intensity \( I_c / I_{th} \) in the RC as a function of the input intensity \( I_0 / I_{th} \) for the various detuning frequencies \( \Delta / \sigma_{th} \) is presented in Fig. 4. Here and after we use the threshold intensity \( I_{th} \) and the frequency shift threshold \( \sigma_{th} \) for normalization. In constructing the diagram, we chose the input intensity \( I_0 \) as a parameter, and then calculated the intracavity intensity \( I_c = I_0 H_p (\omega_0, I_0) \) for the various frequencies of the input signal \( \omega_0 \) detuned from the resonant frequency \( \omega_n \) so that \( \Delta = \omega_0 - \omega_n \).

The calculated diagram could be explained as follows. For \( \Delta / \sigma_{th} < 1 \) the bistability is not observed and the cavity mode is stable regardless of the input intensity. In the case \( \Delta / \sigma_{th} = 1 \) the threshold appears for \( I_0 / I_{th} = 1 \). This case is shown by the black square in Fig. 4 and it corresponds to the square mark in Fig. 3. For the higher values of the detuning \( \Delta / \sigma_{th} > 1 \), the multi-valued range, where the input intensity \( I_0 \) leads to the three possible different intensities of the circulating waves \( I_c \), is observed. The stable values of the intensity are shown in Fig. 4 by solid lines, while the dashed lines represent an intermediate unstable state.

Further increasing of the input intensity leads to extending of the multi-valued range. One can see the positions of the kinks on the characteristics in Fig. 4, marked with the blue and orange circles, that are also shown in Fig. 3. For the illustrative purposes, the extension of the bistability frequency band with increasing of the input intensity is shown by the projection of the nonlinear diagram onto a plane \( (I_0 / I_{th}, \Delta / \sigma_{th}) \) in the bottom of Fig. 4. Here curves \( I_c \) and \( I_0 \) correspond to the bistability band limits that are defined by the frequency positions of the maximum and minimum on the nonlinear dispersion characteristic (see blue and orange open circles in Fig. 3). Note that diagrams akin to Fig. 4 are widely used for the description of bifurcation maps [31,32].
Fig. 4. The intracavity intensity $I_i / I_n$ as a function of the input intensity $I_0 / I_n$ for the various detuning frequencies $\Delta \sigma$. At the bottom plane the dependences of the normalized frequency shift versus the normalized intensity are shown by the blue line for the $I_i$ and the orange line for the $I_0$.

6. Summary

A general theoretical approach uniting the dispersion and the transmission response of the multi-resonant nonlinear ring cavities is proposed for the first time. The approach provides an opportunity to study the nonlinear wave processes in wide frequency range, significantly exceeding the distance between the adjacent resonant RC frequencies. To utilize this, it is enough to know the linear dispersion law and the dependence of material parameters of the ring on the wave amplitude. As an illustration, the main characteristics of the bistability phenomenon in an optical ring with the Kerr nonlinearity are considered. The proposed approach demonstrates the existence of not only two stable and one unstable value of intracavity intensity, but also many-valuedness of stable and unstable wave-numbers.

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