Mode contributions to the Casimir effect

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Applying a sum-over-modes approach to the Casimir interaction between two plates with finite conductivity, we isolate and study the contributions of surface plasmons and Foucault (eddy current) modes. We show in particular that for the TE-polarization eddy currents provide a repulsive force that cancels, at high temperatures, the Casimir free energy calculated with the plasma model.

Keywords: Mode contributions, surface plasmons, eddy currents.

1. Introduction

Intense theoretical effort is currently devoted to the understanding of the Casimir effect for real experimental setups. This involves the impact of temperature, finite conductivity, engineered materials, and may identify routes to design the final Casimir pressure. Almost all analyses rely on the Lifshitz formula\(^1,2\) where the physical properties of the material are encoded in the scattering amplitudes (i.e., reflection coefficients in planar geometries). Their evaluation at imaginary frequencies obscures, however, how the material objects modify the modes of the electromagnetic field. A ‘sum over modes’ approach is nevertheless possible, even if the eigenfrequencies \(\omega_m\) are complex (due to material absorption, for example). For two objects at distance \(L\) the Casimir energy at zero temperature can be written as\(^3\)

\[
E = \frac{\hbar}{2} \sum_{p,k}^\prime \text{Re} \left[ \sum_m (\omega_m - \frac{2i\omega_m}{\pi} \ln \frac{\omega_m}{\Lambda}) \right]_L, \quad \text{Im} \left[ \sum_{p,k,m}^\prime \omega_m \right]_\infty = 0 \quad (1)
\]

where the prime indicates that purely imaginary eigenfrequencies are weighted with 1/2. Eq. (1) generalizes Casimir’s formula for the vacuum en-

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ergy between two perfect reflectors\(^4\) and is valid for generic (causal) mirrors with arbitrary thickness. Note that one does not simply take real parts of the complex eigenfrequencies, as suggested some time ago\(^5\) (see also Ref.\(^6\)).  

The logarithmic correction in Eq.(1) is consistent with the ‘system+bath’ paradigm that describes the thermodynamics of quantum dissipative systems.\(^7\) In this context, the frequency scale \(\Lambda\) is interpreted as the cutoff frequency of the bath spectral density. The Casimir energy does not depend on this constant because of the sum rule in (1).

The sum-over-modes approach provides an ‘anatomic view’ of the Casimir effect where contributions from different modes are clearly identified. This is useful to understand unusual behaviours and may suggest new ways to tailor the Casimir force.\(^8\)–\(^10\) In the following, we illustrate Eq.(1) with the help of a few examples.

### 2. Dissipative Plasmons at short distance

One of the most interesting contributions to the Casimir force originates from surface modes bound to the vacuum/medium interface.\(^11\) These modes have a dispersion relation that splits in two branches, \(\omega = \Omega_\pm(k)\), as two surfaces are approached. Substituting these frequencies in Eq.(1), we get a plasmonic contribution to the Casimir energy (\(A\): surface area)

\[
E_{pl} = \frac{\hbar A}{2} \int \frac{k dk}{2\pi} \text{Re} \left[ \sum_{i=\pm} \left( \Omega_i(k) - \frac{2i\Omega_i(k)}{\pi} \ln \frac{\Omega_i(k)}{\Lambda} \right) \right]_\infty^L 
\]

Consider the case of two metals at a distance smaller than the plasma wavelength \(\lambda_{pl} = \frac{2\pi c}{\omega_{pl}}\). We are then in the quasi-electrostatic regime, and the surface plasmon modes are given by\(^12\) (red and blue points in Fig.1)

\[
\Omega_\pm = \sqrt{\frac{\omega_{pl}^2}{4} - \frac{\gamma^2}{4}} - i\frac{\gamma}{2}, \quad \omega_{pl}^2 = \frac{\omega_{pl}^2}{2} (1 \pm e^{-kL}) 
\]

where \(\gamma\) is the damping rate in a Drude description of the metal. One can easily check that the sum rule in Eq.(1) is automatically satisfied. To leading order in \(\gamma \ll \omega_{pl}\) (good conductors) Eq.(2) yields

\[
E_{pl} \approx -\frac{\pi^2 \hbar c A}{720 L^3} \frac{3}{2} \left( \alpha L \frac{15\zeta(3)}{\pi^4} \frac{\gamma L}{c} \right), \quad \alpha = 1.193\ldots
\]

where \(\zeta(3) \approx 1.202\) is a Zeta function. This corresponds exactly to the total Casimir force calculated in Ref.13, including the dissipative correction. In fact, in this short distance limit, the Casimir energy is completely dominated by the plasmonic contribution.\(^13\)–\(^15\) Eq.(2) is valid also beyond the
good conductor limit, however, and could be used, e.g., to analyze semiconductors where surface plasmons appear in a different frequency range and can have much stronger damping.

3. Eddy currents

As a second example, consider the contribution from eddy current modes. They are connected with low-frequency currents that satisfy a diffusion equation in the conducting metal and are completely absent within the lossless description of the so-called plasma model.

We have analyzed these modes recently and constructed from the ‘system+bath’ paradigm their quantum thermodynamics. They behave like free Brownian particles, since the eigenfrequencies of bulk eddy currents are purely imaginary \( \omega_m = -i\xi_m \) \((\xi_m > 0)\). From Eq.(1), we get the Casimir energy

\[
E_{\text{eddy}} = -\sum_{p,k} \left[ \sum_m \frac{\hbar \xi_m}{2\pi} \ln \frac{\xi_m}{\Lambda} \right]_\infty^L
\]

For these modes alone, the sum rule [Eq.(1)] is not satisfied, and the eddy current contribution to the Casimir energy depends on the cutoff \(\Lambda\). This is also well-known from quantum Brownian motion where bath modes up to \(\Lambda\) are entangled to the particle.

Mathematically, eddy currents form a mode continuum that can be identified in the complex frequency plane from the branch cut of the root

\[
k_m = \sqrt{\epsilon(\omega)\omega^2/c^2 - k^2}
\]

which describes the propagation of the electromagnetic field inside the medium. For a Drude metal, the cut is located between
\(\omega_m = -i\xi_0(k) \approx -iDk^2 \) (for \(k \ll \omega_{pl}/c\)) and \(\omega_m = -i\gamma\) (see Fig. 1), where \(D = \gamma(\lambda_{pl}/2\pi)^2\) is the electromagnetic diffusion constant. We get the \(L\)-dependent change in the mode density along the branch cut by applying the logarithmic argument theorem to the Green function of the electromagnetic field. Using the contour sketched in Fig. 1(left), it is possible to show that Eq.(5) can be written as

\[
E_{\text{eddy}} = \int_0^\infty \frac{d\xi}{\pi} \sum_{p,k} \partial_\xi \left( \frac{\hbar\xi}{2\pi} \ln \frac{\xi}{\Lambda} \right) \text{Im} \ln \left[ 1 - r_p^2(-i\xi - 0^+)e^{-2\kappa L} \right], \tag{6}
\]

with \(\kappa = \sqrt{\xi^2 + k^2}\) and \(r_p\) the reflection coefficient of the mirrors in polarization \(p = \text{TE}, \text{TM}\). This gives rise to a repulsive Casimir force (Fig. 1 of Ref. 10), provided \(\Lambda\) is sufficiently large, e.g., \(\Lambda \geq \gamma\).

The structure of Eq.(6) allows for an immediate translation to the high-temperature (classical) limit. Replace the zero-point energy with the classical free energy per mode, \(k_B T \ln(h\xi/k_B T)\), and get

\[
\mathcal{F}_{\text{eddy}} \approx -\int_0^\infty \frac{d\xi}{\pi} \sum_{p,k} \frac{k_B T}{\xi} \text{Im} \ln \left[ 1 - r_p^2(-i\xi - 0^+)e^{-2\kappa L} \right], \tag{7}
\]

(A more rigorous proof follows from the representation for the free energy given in Ref. 10.) Eq. (7) is thus the result of the logarithmic argument theorem applied to the high-temperature limit of the free energy. Now the contour around the eddy current continuum can also be interpreted as a contour encircling the whole complex plane, i.e., the surface plasmon and propagating modes [Fig. 1(right)]. This is particularly interesting in the TE-polarization because there are no surface plasmons, and the residue at \(\omega = 0\) vanishes \(|r_{TE}^2(\omega \to 0)| = 0\). This means that eddy currents and propagating modes give, up to a sign, the same Casimir energy at high temperature (or large distance). Since propagating modes are only slightly affected by conduction on the metal (i.e., they behave similarly in the Drude and plasma models), we find the simple relation

\[
\mathcal{F}_{\text{eddy}}^{\text{TE}} \approx -\mathcal{F}_{\text{C}}^{\text{TE}(\text{pl.m.})}, \quad \gamma/\omega_p \ll 1 \tag{8}
\]

where \(\mathcal{F}_{\text{C}}^{\text{TE}(\text{pl.m.})}\) is the Casimir free energy at high temperature calculated within the plasma model.\(^2\) In the Drude model, the two contributions are present and cancel each other when they are both in the high-temperature regime (which happens at different distances, see Fig. 4 of Ref.10).

A different scenario occurs in the TM-polarization. The residue at \(\omega = 0\) does not vanish and corresponds exactly to the high-temperature limit of the plasma model.\(^2\) Indeed, we have checked that eddy currents give only a very small contribution.
4. Conclusions

Using a mode-summation approach, we have isolated and analyzed the contribution of two classes of modes to the Casimir effect, allowing for complex eigenfrequencies of the electromagnetic field. A previous result for the short-distance limit between good conductors has been generalized to any conductivity and distance by considering coupled surface plasmonic modes (for the lossless case, see Refs. 8,9). We also considered eddy currents which are overdamped or diffusive modes in the bulk of a Drude metal, and showed that they contribute a repulsive Casimir interaction, in agreement with Ref.10. At high temperature and for a good conductor, we found in a simple way that their free energy in the TE-polarization differs only slightly from the Casimir free energy within a dissipationless description (the plasma model), but is of the opposite sign. In this way, eddy currents nearly cancel out the attractive Casimir interaction from propagating modes. This explains the strong difference between the Drude and plasma models for the temperature correction of the electromagnetic Casimir effect.2

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