ABSENCE OF WANDERING DOMAINS
FOR SOME REAL ENTIRE FUNCTIONS
WITH BOUNDED SINGULAR SETS

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Abstract. Let \( f \) be a real entire function whose set \( S(f) \) of singular values is real and bounded. We show that, if \( f \) satisfies a certain function-theoretic condition (the “sector condition”), then \( f \) has no wandering domains. Our result includes all maps of the form \( z \mapsto \lambda \sinh(z) + a \) with \( \lambda > 0 \) and \( a \in \mathbb{R} \).

We also show the absence of wandering domains for certain non-real entire functions for which \( S(f) \) is bounded and \( f^n|_{S(f)} \to \infty \) uniformly.

As a special case of our theorem, we give a short, elementary and non-technical proof that the Julia set of the exponential map \( f(z) = e^z \) is the entire complex plane.

Furthermore, we apply similar methods to extend a result of Bergweiler, concerning Baker domains of entire functions and their relation to the postsingular set, to the case of meromorphic functions.

1. Introduction

Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Recall that the Fatou set \( \mathcal{F}(f) \) consists of those points near which the family \((f^n)\) of iterates of \( f \) is equicontinuous with respect to the spherical metric. A wandering domain of \( f \) is a component \( U \) of \( \mathcal{F}(f) \) such that \( f^n(U) \cap f^m(U) = \emptyset \) whenever \( n \neq m \).

Problems concerning wandering domains tend to be difficult. The question whether polynomials (and rational maps) can have wandering domains remained open for the best part of the twentieth century, until Sullivan [Sul85] gave his celebrated negative answer using quasiconformal deformation theory. This proof generalizes to the class \( \mathcal{S} \) of transcendental entire functions \( f \) for which the set \( S(f) \) of singular values is finite [EL92, Theorem 3]. (See the end of this section for definitions.) Functions with infinitely many singular values may well have wandering domains; elementary examples are given by \( f(z) = z - 1 + e^{-z} + 2\pi i, \ f(z) = z + 2\pi \sin(z) \) or \( f(z) = z + \lambda \sin(2\pi z) + 1 \) for suitable \( \lambda \); see [Ber93, p. 168].

Quasiconformal deformation theory still appears to be essentially the only known method of proving the absence of wandering domains for general complex families of entire functions. This method cannot be applied in most situations where \( S(f) \) is infinite, and hence many problems are open in this setting. For example, it is not known whether an entire function can have a wandering domain with a bounded orbit [BH89, Problems 2.77, 2.87]. Indeed, for all explicit examples of wandering domains \( U \), the iterates in

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U converge to infinity locally uniformly, although Eremenko and Lyubich \cite{EL87} used approximation theory to construct examples with a finite accumulation point.

When the dynamical behaviour of the maps in question is restricted, other methods can be brought to bear. Indeed, it is known that every limit point of an orbit in a wandering domain is a non-isolated point of the postsingular set $P(f)$ \cite[Theorem]{BHK+93}. Furthermore, Eremenko and Lyubich \cite[Theorem 1]{EL92} showed that a function in the class

$$\mathcal{B} := \{ f : \mathbb{C} \to \mathbb{C} \text{ transcendental entire : } S(f) \text{ is bounded} \}$$

cannot have a wandering domain in which the iterates tend to infinity.

Nonetheless, the settings in which these results can be applied to rule out wandering domains altogether tend to be rather restrictive, and many natural questions remain. For example, it is not known whether a function in the Eremenko-Lyubich class $\mathcal{B}$ can have wandering domains at all.

We propose the following, apparently simpler, question.

1.1. Question.
Let $f \in \mathcal{B}$, and suppose that the singular values of $f$ tend to infinity uniformly under iteration, that is, $\lim_{n \to \infty} \inf_{s \in S(f)} |f^n(s)| = \infty$. Can $f$ have a wandering domain?

A function satisfying these hypotheses is given by $f(z) = \sinh(z)/z + a$, for $a$ sufficiently large. The question seems to have remained open until now even for this simple example. The following result allows us to solve the problem in this case and many others. (Here and throughout, dist denotes Euclidean distance.)

1.2. Theorem. Let $f \in \mathcal{B}$ be a function for which $f^n|_{S(f)} \to \infty$ uniformly. Let $A \subset \mathbb{C}$ be a closed set with $(S(f) \cup f(A)) \subset A$ such that all connected components of $A$ are unbounded.

Suppose that there exist $\varepsilon > 0$ and $c \in (0, 1)$ with the following property: if $z \in A$ is sufficiently large and $w \in \mathbb{C}$ satisfies $|w - z| < c|z|$, then $\text{dist}(f(w), S(f)) > \varepsilon$.

Then $f$ has no wandering domains.

Remark 1. For the function $f(z) = \sinh(z)/z + a$, the hypotheses are satisfied for $A = [a, \infty)$.

Remark 2. Usually, we think of the set $A$ as a union of one or finitely many curves to infinity ("hairs" or "dynamic rays"). Assuming the existence of such a set of curves, the condition in the theorem then reduces essentially to the function-theoretic requirement that the tracts of the function $f$ over infinity are sufficiently thick along $A$. This condition tends to be automatically satisfied by explicit entire functions of finite order.

If $f \in \mathcal{B}$ is real—i.e., $f(\mathbb{R}) \subset \mathbb{R}$—with only real singular values, then it follows from the recent solution of the rigidity problem for real-analytic maps of the interval \cite{KSvS07, vS09} that $f$ cannot have a wandering domain with a bounded orbit. (Compare \cite{RvS10}.) We can combine this with our method to obtain a more complete picture for a large class of such functions.
1.3. Theorem. Let \( f \in \mathcal{B} \) such that \( S(f) \cup f(\mathbb{R}) \subset \mathbb{R} \). Furthermore, assume that there are constants \( r, K > 0 \) such that
\[
\frac{|f'(x)|}{|f(x)|} \leq K \cdot \frac{\log |f(x)|}{|x|}
\]
whenever \( x \in \mathbb{R} \) with \( |x| > r \) and \( |f(x)| > r \). Then \( f \) does not have a wandering domain.

Remark. The analytic condition on \( f \) is simply a reformulation of a geometric condition similar to the one in Theorem 1.2; compare Theorem 6.1.

1.4. Corollary. Let \( \lambda, a \in \mathbb{R} \) with \( \lambda \neq 0 \). Then the function \( f(z) = \lambda \sinh(z)/z + a \) does not have wandering domains.

Note that our results give, in particular, a new proof of the fact that the Julia set of the exponential map \( f(z) = \exp(z) \) is the entire complex plane. This celebrated conjecture of Fatou was proved by Misiurewicz [Mis81] in 1981, and can also be established by using Sullivan’s argument (see [BR84, EL92]) or by appealing to the results of [BHK+93]. We will present the argument in this special case separately in Section 2 to illustrate the basic idea underlying our proof.

Our results rely on a very general fact that relates to the comparison between hyperbolic metrics; see Theorem 4.1 for the precise statement. We note that this argument is somewhat similar in spirit to other applications of hyperbolic geometry in transcendental dynamics: These include some of Bergweiler’s results [Ber95] on the relation between the postsingular set and Baker domains, and Zheng’s generalization of the results of [BHK+93] to meromorphic and more general functions. For completeness, we discuss generalizations of the results of Bergweiler and Zheng in Section 7. In particular, we prove the following result, which can sometimes be used to exclude the existence of Baker domains of meromorphic functions.

1.5. Theorem (Baker domains of meromorphic functions). Let \( f : \mathbb{C} \to \hat{\mathbb{C}} \) be a transcendental meromorphic function and suppose that \( W \) is a Baker domain of \( f \) with \( f^{\pm k}|_W \to \infty \) locally uniformly for some \( k \geq 1 \). Then there exists a sequence \( p_n \) of points in the postsingular set of \( f \) such that
\[
|p_n| \to \infty, \quad \sup \frac{|p_{n+1}|}{|p_n|} < \infty \quad \text{and} \quad \frac{\text{dist}(p_n, W)}{|p_n|} \to 0.
\]

We also note that Theorem 1.1 could be used to exclude the presence of wandering domains also in situations quite different from those formulated in Theorems 1.2 and 1.3. In particular, it also lends itself to applications to suitable families of transcendental meromorphic functions. However, our techniques do not allow us to answer Question 1.1 in general.

Notation. The complex plane and the unit disk are denoted by \( \mathbb{C} \) and \( \mathbb{D} \), respectively; we use \( \mathbb{H}_{>0} \) and \( \mathbb{H}_{\geq 0} \) to denote the open resp. closed upper half plane. The Euclidean length of a rectifiable curve \( \gamma \subset \mathbb{C} \) is denoted by \( \text{len}(\gamma) \); for the Euclidean distance between a point \( z \) and a set \( A \), we write \( \text{dist}(z, A) \). We denote by \( D_r(z) \) the Euclidean disk of radius \( r \) centred at the point \( z \).
One of the main tools in our proofs is given by plane hyperbolic geometry; compare e.g. [BM07, KL07]. A Riemann surface \( U \) is called hyperbolic if there exists a holomorphic universal covering map \( \pi : \mathbb{D} \to U \). Our main focus will be on hyperbolic domains in \( \mathbb{C} \); a domain \( U \subset \mathbb{C} \) is hyperbolic if and only if \( \mathbb{C} \setminus U \) contains at least two points. The hyperbolic metric on a hyperbolic surface \( U \) is defined to be the unique complete conformal Riemannian metric on \( U \) of constant curvature \(-1\), or, equivalently, as the image of the hyperbolic metric on \( \mathbb{D} \) under the covering \( \pi \). (This is well-defined because the hyperbolic metric on \( \mathbb{D} \) is invariant under Möbius transformations.) For plane domains, this metric will be denoted by \( \rho_U(z) |dz| \); we can also do so for hyperbolic surfaces, but here the function \( \rho_U \) will depend on the choice of a local coordinate \( z \). If \( U \) is not connected and every component of \( U \) is hyperbolic, then by the hyperbolic metric on \( U \) we mean the hyperbolic metric defined componentwise on the components of \( U \).

The hyperbolic length of a rectifiable curve \( \gamma \subset U \) will be denoted by \( \ell_U(\gamma) \). For any two points \( z, w \in U \), the hyperbolic distance \( d_U(z, w) \) is the smallest hyperbolic length of a curve connecting \( z \) and \( w \) in \( U \). By Pick’s theorem, also referred to as the Schwarz-Pick lemma, [BM07, Theorem 10.5], a holomorphic map \( f : V \to U \) between two hyperbolic Riemann surfaces does not increase the respective hyperbolic metrics, i.e., \( d_U(z, w) \geq d_U(f(z), f(w)) \) holds for all \( z, w \in V \), or in other words, \( \rho_U(z) \geq \rho_U(f(z)) \cdot |f'(z)| \) holds for all \( z \in V \). In fact, equality holds in the previous formula if and only if \( f \) is a covering map; in this case, we say that \( f \) is a local isometry. Otherwise \( f \) is a strict contraction. In particular, it follows that if \( V \subsetneq U \), then \( \rho_V(z) > \rho_U(z) \) for all \( z \in V \). (If \( U \) and \( V \) are Riemann surfaces, all of the corresponding statements are true when interpreted in local coordinates.)

Let \( U \) and \( V \) be hyperbolic Riemann surfaces, and let \( z \in U \). Suppose that \( f \) is a holomorphic function defined on a neighborhood of \( z \) and taking values in \( V \). Then we denote the hyperbolic derivative of \( f \) with respect to the metrics in \( U \) and \( V \) by

\[
\|Df(z)\|_V := |f'(z)| \cdot \frac{\rho_V(f(z))}{\rho_U(z)}.
\]

(Note that this quantity is indeed independent of local coordinates.) If both \( z \) and \( f(z) \) belong to \( U \), we also abbreviate \( \|Df(z)\|_U := \|Df(z)\|_U^U \).

Let \( f : \mathbb{C} \to \mathbb{C} \) be a transcendental entire function. The set of (finite) singular values \( S(f) \) of \( f \) is the smallest closed subset of \( \mathbb{C} \) such that \( f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f) \) is a covering map. It is well-known that \( S(f) \) agrees with the closure of the set of critical and asymptotic values of \( f \), but we shall not require this fact. The postsingular set of \( f \) is defined by

\[
P(f) = \bigcup_{n \geq 0} f^n(S(f)).
\]

Suppose that \( W \) is a wandering domain. It is elementary to see that every accumulation point \( z_0 \) of \( f^n(w) \), for \( w \in W \), belongs to \( P(f) \cap J(f) \), where \( J(f) = \mathbb{C} \setminus F(f) \) is the Julia set of \( f \). As mentioned in the introduction, \( z_0 \) is even known to be a non-isolated point of \( P(f) \) [BHK+03], but we shall not use this fact.

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2. The Julia set of the exponential

2.1. Theorem (Misiurewicz, 1981). \( J(\exp) = \mathbb{C} \).

Proof. Set \( f := \exp \); we have \( S(f) = \{0\} \). Suppose, by contradiction, that \( F(f) \neq \emptyset \). Let \( w \in F(f) \) and let \( W \) be the connected component of \( F(f) \) containing \( w \).

Preliminary claim. \( W \) is a simply connected wandering domain, and there is a sequence \((n_k)\) with \( f^{n_k}(w) \to 0 \).

Proof. This is classical, and was, at least mostly, already known to Fatou. (He states explicitly that \( f \) does not have any attracting periodic orbits and no domains where the iterates converge to infinity \[\text{Fat26, p. 370}].) We provide a self-contained account for the reader’s convenience. Firstly, every attracting or parabolic basin of an entire function must contain a singular value, but the orbit of 0 tends to infinity. Similarly, there cannot be any rotation domains, since the boundary of such a component would need to be contained in the postsingular set. Hence \( W \) is either a wandering domain, or is eventually mapped to a Baker domain (a periodic component of \( F(f) \) in which the iterates tend to infinity). In particular, every limit function of \( f^n|_{W} \) is constant; i.e., if \( z_0 \in \mathbb{C} \cup \{\infty\} \) and \( f^{n_k}(w) \to z_0 \), then \( f^{n_k} \to z_0 \) locally uniformly on \( W \).

Next we show that we cannot have \( f^n(w) \to \infty \). Indeed, otherwise \( \text{Re} \ f^n(w) \to +\infty \). By replacing \( w \) with its image under a suitable forward iterate, we may assume that \( \text{Re} \ f^n(w) \geq 2 + 2\pi \) for all \( n \geq 0 \). Now \( |f'(z)| = |f(z)| \), and in particular \( |f'(z)| \geq 2 \) if \( |f(z) - f^n(w)| \leq 2\pi \). It follows that, for each \( n \geq 0 \), we can define a branch \( \psi_n \) of \( f^{-n} \), defined on \( D_n := D_{2\pi}(f^n(w)) \), such that \( \psi_n(f^n(w)) = w \) and \( |\psi_n'(z)| \leq 2^n \) for all \( z \). In particular, the size of \( \psi_n(D_n) \) shrinks to zero as \( n \to \infty \). On the other hand, \( f(D_n) = f^{n+1}(\psi_n(D_n)) \) intersects the negative real axis, and hence \( f^2(D_n) \) intersects the unit disk. This contradicts the assumption that \( f^n \) converges to infinity uniformly in a neighborhood of \( w \).

So we have shown that the iterates in \( W \) cannot converge to infinity locally uniformly; hence there is some \( z_0 \in \mathbb{C} \) with \( f^{n_j}(w) \to z_0 \) for a suitable sequence \((n_j)\). By the maximum principle, this implies that \( W \) is simply connected. (Recall that, by Montel’s theorem, there are preimages of the real axis, and hence points whose orbits converge to infinity, arbitrarily close to any point of \( J(f) \).)

Now let us assume that, if \( z_0 = f^j(0) \) for some \( j \), then \( z_0 \) is chosen such that \( j \) is minimal with this property.

We claim that we must have \( j = 0 \). Indeed, otherwise, for small \( \varepsilon \) and all sufficiently large \( k \), we can define a branch \( \psi_k \) of \( f^{-nk} \) on \( D_{2\varepsilon}(z_0) \) that maps \( f^{nk}(w) \) to \( w \). Since \( z_0 \) belongs to the Julia set, we must have \( \text{diam}(\psi_k(D_{\varepsilon}(z_0))) \to 0 \) (this uses Koebe’s distortion theorem). We have obtained a contradiction to the fact that \( f^{nk} \to z_0 \) uniformly on a neighborhood of \( w \). \( \Box \)

We now come to the new part of the argument. Set \( U := \mathbb{C} \setminus [0, \infty) \); then \( f^n(W) \subset U \) for all \( n \). Set \( w_n := f^n(w) \). We study the derivative \( \delta_n \) of \( f^n \) at \( w \) with respect to the hyperbolic metric of \( U \), i.e.

\[
\delta_n := \|Df^n(w)\|_U = |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_U(w)}.
\]
Since \( f^{-n}(U) \subset U \) and \( f^n : f^{-n}(U) \to U \) is a covering map, we have \( \delta_{n+1} \geq \delta_n \) for all \( n \).

On the other hand, set \( \mathcal{W} := \bigcup_{n \geq 0} f^n(W) \). Then \( f(\mathcal{W}) \subseteq \mathcal{W} \), and hence \( f^n \) contracts the hyperbolic metric on \( \mathcal{W} \). So if \( \tilde{\delta}_n \) is the derivative of \( f^n \) at \( w \) with respect to the hyperbolic metric of \( \mathcal{W} \), then \( \tilde{\delta}_{n+1} \leq \tilde{\delta}_n \) for all \( n \). The quantities \( \delta_n \) and \( \tilde{\delta}_n \) are related:

\[
0 < \delta_0 \leq \delta_n = \tilde{\delta}_n \cdot \frac{\rho_\mathcal{W}(w_n)}{\rho_\mathcal{W}(w)} \leq \tilde{\delta}_n \cdot \frac{\rho_\mathcal{W}(w)}{\rho_\mathcal{W}(w)} \leq \delta_0 \cdot \frac{\rho_\mathcal{W}(w)}{\rho_\mathcal{W}(w)} =: \delta_0' < \infty.
\]

Let \( n_k \) be as in the preliminary claim; i.e. \( w_{n_k} \to 0 \), and hence \( \text{Re} \ w_{n_k-1} \to -\infty \). We shall derive a contradiction by showing that, at points with very negative real parts, \( f \) strongly expands the hyperbolic metric of \( U \). To do so, we use two simple estimates on \( \rho_U \):

\[
\rho_U(z) \geq \frac{1}{2|z|} \quad \text{for all } z \in U \quad \text{and} \quad \rho_U(z) \leq \frac{2}{|z|} \quad \text{when } \text{Re} \ z < 0
\]

These follow from the standard estimate on the hyperbolic metric in a simply connected domain (Proposition 3.1), or alternatively directly from the explicit formula

\[
\rho_U(z) = \frac{1}{2|z| \cdot \sin(\text{arg}(z)/2)}.
\]

Now suppose that \( z \in \mathbb{C} \) with \( \text{Re} \ z < 0 \). Then we have

\[
\|Df(z)\|_U = |f'(z)| \cdot \rho_U(f(z)) \geq \frac{1}{4} \frac{|f'(z)|}{|f(z)|} \cdot |z| = \frac{|z|}{4} \geq \frac{|\text{Re} \ z|}{4}.
\]

Suppose that \( k \) is sufficiently large that \( \text{Re} \ w_{n_k-1} < -4\delta_0'/\delta_0 \). Then

\[
\delta_{n_k} = \delta_{n_k-1} \cdot \|Df(w_{n_k-1})\|_U \geq \delta_0 \cdot \frac{|\text{Re} \ w_{n_k-1}|}{4} > \delta_0'.
\]

This is the desired contradiction. \( \blacksquare \)

Remark. We have presented the proof in the present form to emphasize the ideas that will occur in our more general results. It is in fact possible to simplify the argument yet further, eliminating the need for the preliminary claim and yielding a proof that altogether avoids the classification of periodic Fatou components, the Riemann mapping theorem, Montel’s theorem and Koebe’s theorem. All that is needed are formulas for the hyperbolic metric on a strip and a slit plane, together with Pick’s theorem (i.e., the Schwarz lemma together with the invariance of the hyperbolic metric under automorphisms of the disk).

The proof can be sketched as follows. Suppose, by contradiction, that there is a point \( w \) at which the family of iterates is equicontinuous. First consider the case where \( w \) does not tend to infinity under iteration. Then there is a small open disk \( D \) around \( w \) consisting of nonescaping points, and hence satisfying \( f^n(D) \subset U := \mathbb{C} \setminus [0, \infty) \) for all \( n \). Considering the increasing sequence of hyperbolic derivatives \( \|f^n(w)\|_D \leq 1 \), we see first that no limit point of the orbit of \( w \) can belong to \( U \). Just as above, it then follows that there can be no limit point in \( [0, \infty) \) either, a contradiction.

If, on the other hand, \( f^n(w) \to \infty \), then we can consider \( U' := \mathbb{C} \setminus (-\infty, 0] \), and derive a contradiction in analogous fashion.
3. Estimates on the hyperbolic metric

In this section, we present some elementary facts about the hyperbolic metric on a plane domain. Firstly, recall that the hyperbolic metrics on the unit disk, on the upper half plane $\mathbb{H}_{>0}$ and on the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ are given by

$$
\rho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2},
$$

$$
\rho_{\mathbb{H}_{>0}} = \frac{1}{\text{Im}(z)}
$$

and

$$
\rho_{\mathbb{D}^*}(z) = \frac{1}{|z| \cdot |\log |z||}.
$$

The following estimates are useful for general domains; they follow from the Schwarz lemma and Koebe’s theorem, respectively.

3.1. Proposition. ([BM07, Theorems 8.2 and 8.6]) Let $U \subset \mathbb{C}$ be a hyperbolic domain. Then

$$
\rho_U(z) \leq \frac{2}{\text{dist}(z, \partial U)}.
$$

If $U$ is simply connected, then also

$$
\rho_U(z) \geq \frac{1}{2 \text{dist}(z, \partial U)}.
$$

Now suppose that $U \subset V$ are hyperbolic domains; then the inclusion $\iota$ from $V$ into $U$ is contracting: $\rho_V(z) > \rho_U(z)$ for all $z \in V$. We now state and prove a proposition that relates the strength or weakness of this expansion to the hyperbolic distance between $z$ and the boundary of $V$. (Compare also [SZ03, Lemma 3.1].) In order to state the result, let us introduce the following notation. If $V \subset U$ are hyperbolic Riemann surfaces, we denote by $\rho_{UV}(z)$ the density of the hyperbolic metric of $V$ with respect to the hyperbolic metric of $U$. In other words,

$$
\rho_{UV}(z) = \frac{1}{\|D\iota(z)\|_V},
$$

where $\iota : V \rightarrow U$ is again the inclusion map; i.e. $\iota(z) = z$. When $U$ and $V$ are plane domains (the case of most interest for us), we have

$$
\rho_{UV}(z) = \frac{\rho_V(z)}{\rho_U(z)}.
$$

3.2. Proposition. Let $V$ and $U$ be hyperbolic Riemann surfaces with $V \subset U$. Let $z \in V$ and set $R := d_U(z, U \setminus V)$. Then

$$
1 < \frac{2e^R}{(e^{2R} - 1) \cdot \log \frac{e^{R+1}}{e^R - 1}} \leq \rho_{UV}(z) \leq 1 + \frac{2}{e^R - 1}.
$$

Remark. The exact dependence of the bounds (which are sharp, as the proof will show) on the number $R$ is not relevant for our purposes. What is important is that they depend only on $R$ and that the upper bound tends to $1$ as $R \to \infty$, while the lower bound tends to infinity as $R \to 0$. 
Proof. Let \( \pi : \mathbb{D} \to U \) be a universal covering map with \( \pi(0) = z \). Then \( \pi \) is a local isometry with respect to the metrics in \( \mathbb{D} \) and \( U \). In particular, \( d_{\mathbb{D}}(x, y) \geq d_U(\pi(x), \pi(y)) \) for all \( x, y \in \mathbb{D} \), and furthermore, for all \( x \in \mathbb{D} \) and all \( \tilde{y} \in U \), there is \( y \in \mathbb{D} \) with \( \pi(y) = \tilde{y} \) and \( d_{\mathbb{D}}(x, y) = d_U(\pi(x), \pi(y)) \). (This is the case because geodesics in \( U \) lift to geodesics in \( \mathbb{D} \).)

By assumption, \( V \) contains a hyperbolic disk (in the metric of \( U \)) at \( z \) of radius \( R \), and there is a point of \( \partial V \) on the boundary of this disk. Let \( \tilde{V} \) be the component of \( \pi^{-1}(V) \) containing 0. Then, by the above, \( \tilde{V} \) contains a hyperbolic disk of radius \( R \) at \( 0 \), measured with respect to the hyperbolic metric on \( \mathbb{D} \), and there is a point of \( \partial \tilde{V} \) that has hyperbolic distance \( R \) from 0. By precomposing with a rotation we may normalize \( \pi \) such that this boundary point \( \tilde{R} \) is real and positive.

Hyperbolic disks in \( \mathbb{D} \) centered at 0 are Euclidean disks centered at 0, hence we have

\[
D_{\tilde{R}}(0) \subset \tilde{V} \subset \mathbb{D} \setminus \{\tilde{R}\}.
\]

Thus we obtain upper and lower bounds:

\[
\rho^U_V(z) = \frac{\rho_{\tilde{V}}(0)}{\rho_{\mathbb{D}}(0)} \leq \frac{\rho_{D_{\tilde{R}}(0)}(0)}{\rho_{\mathbb{D}}(0)} = \frac{1}{\tilde{R}}
\]

and likewise

\[
\rho^U_V(z) \geq \frac{\rho_{\mathbb{D}\setminus\{\tilde{R}\}}(0)}{\rho_{\mathbb{D}}(0)} = \frac{\rho_{D^*(\tilde{R})}}{\rho_{\mathbb{D}}(\tilde{R})}.
\]

So overall we see that

\[
(3.1) \quad 1 < \frac{\rho_{D^*(\tilde{R})}}{\rho_{\mathbb{D}}(\tilde{R})} \leq \rho^U_V(z) \leq \frac{1}{\tilde{R}},
\]

and the bounds are sharp. As mentioned in the remark above, these inequalities would be sufficient for the purposes of our article, together with the observation that \( \tilde{R} \to 1 \) as \( R \to \infty \).

To obtain the explicit bounds as stated, we compute the values in (3.1). First note that the hyperbolic distance in \( \mathbb{D} \) between 0 and \( z \in \mathbb{D} \) is given by \( d_{\mathbb{D}}(z) = \log \frac{1+|z|}{1-|z|} \), and thus \( \tilde{R} = \frac{e^R-1}{e^R+1} \). Hence \( 1/\tilde{R} = 1 + 2/(e^R - 1) \), proving the upper bound.

For the lower bound, we insert \( \tilde{R} \) into the explicit formulas for the hyperbolic densities of \( \mathbb{D} \) and \( \mathbb{D}^* \):

\[
\frac{\rho_{D^*(\tilde{R})}}{\rho_{\mathbb{D}}(\tilde{R})} = \frac{1 - \tilde{R}^2}{2\tilde{R} \cdot |\log \tilde{R}|} = \frac{2e^R}{(e^R+1)^2} \cdot \frac{e^R+1}{e^R-1} \cdot \frac{1}{\log \frac{1}{e^R}} = \frac{2e^R}{(e^{2R} - 1) \cdot \log e^{R+1} - 1},
\]

as claimed.

The following is an immediate consequence of Proposition 3.2.
3.3. Corollary. Let $X$ be a Riemann surface, let $V \subset U \subset X$ be hyperbolic. Let $z_0 \in X \cap \partial U$ and suppose that $z_0$ has a neighborhood (in $X$) that does not intersect $\partial V \setminus \partial U$. If $(z_n) \subset V$ is a sequence of points that converges to $z_0$, then

$$\rho_V^{U}(z_n) \searrow 1 \text{ as } n \to \infty.$$ 

4. Statement and proof of the main result

We now state our main technical result, which allows us to exclude the existence of wandering domains in many cases. Although we mainly apply the theorem in the case of an entire function, we shall state it generally for holomorphic maps on arbitrary Riemann surfaces.

4.1. Theorem. Let $X$ be a Riemann surface, let $U' \subset U \subset X$ be hyperbolic open subsets of $X$, and let $f: U' \to U$ be a holomorphic covering map.

Assume that there is an open connected set $W \subset X$ such that $f^n(W) \subset U'$ for all $n \geq 0$. Let $w \in W$ and let $z_0 \in X$ be an accumulation point of the orbit of $w$; say $f^{n_k}(w) \to z_0$.

Let $D$ be an open neighborhood of $z_0$ in $X$ and set $V := f^{-1}(D \cap U)$. Then

(4.1) $$d_U(f^{n_k-1}(w), U \setminus V) \to \infty$$

and

(4.2) $$\liminf_{n \to \infty} d_U(f^n(w), U \setminus f^n(W)) > 0.$$ 

Remark 1. The statement is a little bit technical, so the reader may wish to connect it with our argument in the case of the exponential map. Here $X = \hat{\mathbb{C}}, U = \mathbb{C} \setminus [0, \infty)$, $f(z) = e^z$ and $U' = f^{-1}(U)$.

If $z_0 = 0$ and $D = D_{\varepsilon}(0)$, then $V$ is contained in the left half plane $\{\text{Re } z < \ln(\varepsilon)\}$, and the hyperbolic distance in $U$ between any point of this half plane and the boundary $\{\text{Re } z = \ln(\varepsilon)\}$ is easily seen to be bounded from above.

Hence (4.1) implies that there are no wandering domains whose orbit accumulates on $0$ (and thus no wandering domains at all).

Remark 2. The inequality (4.1) will be the main ingredient in the proof of our main results. Inequality (4.2) will not be required in the following, but arises naturally from our considerations; we state it here mainly to highlight the connection with previous work. Indeed, (4.2) appears implicitly (in a less general setting, but with the same idea in the proof) in work of Zheng [Zhe05, Proof of Theorem 2.1], and indeed we can generalize [Zhe05, Theorem 2.1] to a larger class of functions using (4.2); see Proposition 7.3. Furthermore, in the case where $W$ is chosen maximal with the given properties (e.g. if $W$ is a Fatou component of a transcendental entire or meromorphic function), the distance in (4.2) actually tends to infinity. This is the type of argument used by Bergweiler in [Ber95, Lemma 3], and will be used in our proof of Theorem 1.5. Since these extensions of Theorem 4.1 depart from our main line of inquiry, we shall defer their discussion to Section 7.

Remark 3. We have stated the conclusions in terms of the hyperbolic distance as this seems more natural. By Proposition 3.2 we can equivalently phrase them as results
on the density of the hyperbolic metric of $V$ resp. $f^n(W)$ when compared with the hyperbolic metric on $U$:

\begin{align}
\rho_V^{(f^n(W))}(f^{nk}(w)) & \to 1 \\
\limsup_{n \to \infty} \rho_V^{(f^n(W))}(f^n(w)) & < \infty.
\end{align}

Indeed, these are the conditions our proof will establish.

**Proof.** The proof begins just as in the case of the exponential map. We define $w_n := f^n(w)$, $W_n := f^n(W)$ and study the sequences

\[ \delta_n := \| Df^n(w) \|_U \geq 1 \quad \text{and} \quad \tilde{\delta}_n := \| Df^n(w) \|^W_{W_n} \leq 1. \]

Because $f$ is a covering map, the first of these sequences is nondecreasing, while the second is nonincreasing by Pick’s theorem. The sequences satisfy

\[ 0 < \delta_0 \leq \delta_n \leq C \cdot \tilde{\delta}_n \leq C \cdot \tilde{\delta}_0 =: \delta'_0, \]

where $C = \rho_V^{(f^n(W))}$. (I.e., $1/C$ is the hyperbolic derivative at $w$ of the inclusion of $W$ into $U$.)

In particular, we have

\[ \rho_{W_n}(w_n) = C \cdot \frac{\delta'_n}{\delta_n} \leq C. \]

As noted in Remark 3, this implies (4.2) by Proposition 3.2.

Let us set $\eta_n := \| Df(w_n) \|_U$. Then $\delta_{n+1} = \eta_n \cdot \delta_n$, and thus $\eta_n \to 1$ as $n \to \infty$. Now let $z_0$ and $\eta_k$ be as in the statement of the theorem, and set $\tilde{D} := D \cap U$. By disregarding finitely many entries, we may assume that $w_{nk} \in \tilde{D}$, and hence $w_{nk-1} \in V := f^{-1}(\tilde{D})$, for all $k \geq 0$. We write

\[ \eta_{nk-1} = \| Df(w_{nk-1}) \|_U = \frac{\rho_V^{(f(w_{nk-1}))}}{\rho_D^{(w_{nk})}} \cdot \| Df(w_{nk-1}) \|_D = \frac{\rho_V^{(w_{nk-1})}}{\rho_D^{(w_{nk})}} \]

where we used the fact that $f : V \to \tilde{D}$ is a covering map, and hence a local isometry. As $k \to \infty$, we have $d_U(w_{nk}, \partial \tilde{D}) \to \infty$; hence $\rho_D^{(w_{nk})} \to 1$ by Corollary 3.3. Thus we see that

\[ \rho_V^{(w_{nk-1})} = \eta_{nk-1} \cdot \rho_D^{(w_{nk})} \to 1. \]

Again, by Proposition 3.2, this statement is equivalent to (4.1).

In the next section, we will apply the preceding result in the following setting:

- $X = \mathbb{C}$,
- $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function,
- $U = \mathbb{C} \setminus A$, where $A$ is a closed forward-invariant set that contains $S(f)$,
- $W$ is a wandering domain of $f$ whose orbit is disjoint from $A$,
- $z_0 \in \mathbb{C}$ is a finite limit function of the sequence $f^n|_W$.

We note that, in particular, we can let $A$ be the postsingular set of $f$. Since $z_0$ belongs to the Julia set of $f$, the estimate (4.2) then implies that $z_0 \in P(f)$. Furthermore, $W$ must be simply connected, and it is easy to deduce from this and (4.2) that $z_0$ cannot be an isolated point of $P(f)$; compare the claim in the proof of Theorem 7.1. This fact was
originally proved in \cite{BHK93} for entire functions using a different method; our argument is in essence the same as that given by Zheng \cite{Zhe05} for functions meromorphic outside a small set.

5. Application to entire functions

We now use the result of the previous section to deduce Theorem 1.2 and a slightly different form of Theorem 1.3. (In the next section, we shall see that this form implies the original formulation.)

Accumulation at singular values through unbounded sets. We begin by showing that, in the situations we consider, wandering domains must accumulate on singular values.

5.1. Lemma. Let \( f \in \mathcal{B} \). Assume that there is a number \( R > 0 \) such that the iterates of \( f \) tend to infinity uniformly on the set \( \{ z \in P(f) : |z| \geq R \} \).

Let \( W \) be a wandering domain of \( f \) for which \( \infty \) is a limit function, and let \( w \in W \). Then there is \( s \in S(f) \) and a sequence \( (n_k) \) such that

\[
 f^{n_k}(w) \to s \quad \text{and} \quad f^{n_k-1}(w) \to \infty.
\]

Proof. We may assume without loss of generality that \( R \) is chosen sufficiently large that \( |s| < R \) for all \( s \in S(f) \).

Claim 1. For every \( K > R \), there exists \( N \) such that

\[
 f^{-n}(D_K(0)) \cap P(f) \subset D_R(0)
\]

for all \( n \geq N \). In particular, the set

\[
 P(f) \cap \bigcup_{m=0}^{\infty} f^{-m}(D_K(0))
\]

is bounded.

Proof. Let us fix some \( K > R \). By assumption, there is \( N \) such that

\[
 f^n(P(f) \setminus D_R(0)) \cap \overline{D_K(0)} = \emptyset
\]

for \( n \geq N \). This proves the first statement.

To deduce the second statement, set

\[
 M := \max \{|f^j(z)| : |z| \leq R, 0 \leq j \leq N\},
\]

let \( v_0 \in P(f) \) with \( |v_0| > M \) and let \( m \geq 0 \). We will show that \( |f^m(v_0)| \geq K \). Note that this is true by construction for \( m \geq N \), so we may suppose that \( m < N \).

Since \( P(f) \) is the closure of the union of iterated forward images of singular values of \( f \), there is a sequence \( v_k \in P(f) \) with \( v_k \to v_0 \), \( |v_k| > M \) and \( v_k = f^{n_k}(s_k) \), where \( s_k \in S(f) \). Recall that \( |s_k| < R \), and hence we must have \( n_k > N \) and \( |f^{n_k-N+m}(s_k)| > R \) for all \( k \), by the choice of \( M \). Thus it follows from the first statement that

\[
 |f^m(v_k)| = |f^N(f^{n_k-N+m}(s_k))| > K.
\]

By continuity, we see that \( |f^m(v_0)| \geq K \), as claimed. \( \triangle \)

Now let \( W \) and \( w \) be as in the statement of the lemma.
Claim 2. There is a sequence $n_k \to \infty$ such that $f^{n_k-1}(w) \to \infty$ and $f^{n_k}(w) \to s$ for some $s \in P(f)$.

Proof. As mentioned in the introduction, Eremenko and Lyubich [EL92] proved that functions in $\mathcal{B}$ cannot have Fatou components in which the iterates tend to infinity. Hence we may choose $K \geq R + 1$ sufficiently large that $\liminf_{n \to \infty} |f^n(w)| < K$. Recall also that, by assumption, $\limsup_{n \to \infty} |f^n(w)| = \infty$.

By Claim 1, there is $M > K$ such that $|z| < M$ whenever $z \in P(f)$ with $|f^j(z)| \leq K$ for some $j \geq 0$.

There is a sequence $m_k \to \infty$ such that $|f^{m_k}(w)| > M$. For every $k$, choose $p_k \geq m_k$ minimal with $|f^{p_k}(w)| \leq K$, and let $n_k \in \{m_k+1, \ldots, p_k\}$ be minimal with $|f^{n_k}(w)| \leq M$ for $r_k \leq j \leq p_k$.

We first claim that $p_k - n_k \leq N$ for sufficiently large $k$, where $N$ is as in Claim 1. Indeed, otherwise we may assume, passing to a subsequence, if necessary, that $|f^{n_k}(w)| \to \infty$. For every $k$, choose $m_k$ minimal with $|f^{m_k}(w)| \leq K$, and let $n_k \in \{m_k+1, \ldots, p_k\}$ be minimal with $|f^{n_k}(w)| \leq M$ for $r_k \leq j \leq p_k$.

We claim that $f^{n_k-1}(w) \to \infty$. Indeed, otherwise let $v_2 \in P(f)$ be a finite accumulation point of this sequence. Since $|f^{n_k-1}(w)| > M$, we have $|v_2| > M$. By continuity, $|f^{m_k}(v_2)| \leq K$ for some $m \leq N$. But this is a contradiction to the choice of $M$.

The claim follows, replacing $(n_k)$ by a subsequence for which $f^{n_k}(w)$ is convergent, if necessary. △

Claim 3. If $s$ is as in Claim 2, then $s \in S(f)$.

Proof. Let $\varepsilon > 0$ be sufficiently small. Then it follows from Claim 1 that the union of all components of $V := f^{-1}(D_\varepsilon(s))$ that intersect $P(f)$ is bounded. In particular, for sufficiently large $k$, the component of $V$ containing $f^{n_k-1}(w)$ does not intersect $P(f)$.

If $s$ is not a singular value, it follows that we can define a branch of $f^{-n_k}$ on $D_\varepsilon(0)$ that maps $f^{n_k}(w)$ to $w$. This leads to a contradiction since $s$ belongs to the Julia set. △

Proof of Theorem 1.2 Assume that the hypotheses of Theorem 1.2 are satisfied. That is, $f \in \mathcal{B}$ is a function for which the singular values escape to infinity uniformly and $A \subset \mathbb{C}$ is a closed set with $(S(f) \cup f(A)) \subset A$ all of whose connected components are unbounded. Furthermore, there are $\varepsilon > 0$, $c \in (0, 1)$ such that the following holds: If $z \in A$ with $|z| \geq R$ and $w \in \mathbb{C}$ with $|w - z| \leq c|z|$, then $\text{dist}(f(w), S(f)) > \varepsilon$. We set $U := \mathbb{C} \setminus A$; then $U$ is simply connected.

Assume, by contradiction, that $f$ has a wandering domain $W$ and let $w \in W$. Then every finite accumulation point of $f^n(w)$ is contained in $P(f)$, and in particular $f^n(w)$ accumulates at $\infty$. By the preceding lemma, there is a singular value $s \in S(f)$ and a sequence $n_k$ such that $f^{n_k}(w) \to s$ and $f^{n_k-1}(w) \to \infty$. Disregarding finitely many entries, we may assume that $f^{n_k}(w) \in D_\varepsilon(s)$ for all $k$.

In particular, the hypotheses imply that $w \notin A$; since $w$ was arbitrary, this means that $W \subset U$.
Claim. Set $V := f^{-1}(D_\varepsilon(s) \cap U)$, and let $z_0 \in V$ with $|z_0| \geq 2R$. Then $\text{dist}(z_0, \partial U) \geq c|z_0|/2$. In particular, $d_U(z_0, \partial V) \leq 8\pi/c$.

Proof. Let $u \in \partial U$. If $|u| < |z_0|/2$, then $|z_0 - u| \geq |z_0|/2 \geq c|z_0|/2$. Otherwise, we have $|u| \geq |z_0|/2 \geq R$. Since $\text{dist}(f(z_0), S(f)) < \varepsilon$, the hypotheses of the theorem imply that $|z_0 - u| \geq c \cdot |u| \geq c|z_0|/2$.

For the second statement, simply let $\gamma$ be an arc of the circle $\partial D_{|z_0|}(0)$ that connects $z_0$ to $\partial V$. (Note that every component of $V$ is simply connected because $A$ is connected, and hence such an arc must exist.) By the previous estimate, $\text{dist}(z, \partial U) \geq c|z|/2 = c|z_0|/2$ for all $z \in \gamma$. So we obtain

$$\rho_U(z) \leq \frac{2}{\text{dist}(z, \partial U)} \leq \frac{4}{c|z_0|}$$

for all $z \in \gamma$. Hence

$$d_U(z_0, \partial V) \leq \ell_U(\gamma) \leq 2\pi|z_0| \cdot \frac{4}{c|z_0|} = \frac{8\pi}{c}.$$  

In particular, we see that $d_U(f^{n-1}(w), \partial V)$ stays bounded as $k \to \infty$. This contradicts Theorem 4.1. △

The case of real functions. We now prove a version of Theorem 1.3 by combining our method with rigidity results for real functions, as discussed in [RvS10, Section 3].

5.2. Definition. We denote by $B_{\text{real}}$ the set of all real transcendental entire functions with bounded sets of singular values; The set $B_{\text{real}}^*$ consists of all maps in $B_{\text{real}}$ with real singular values.

The central rigidity result that we will use for the class $B_{\text{real}}$ is the following:

5.3. Proposition ([RvS10, Theorem 3.6]). Let $f \in B_{\text{real}}$. Let $A_{\text{odd}}$ be the set of points $z \in \mathbb{C}$ whose $\omega$-limit set is a compact subset of the real line and which are not contained in attracting or parabolic basins. Then $A_{\text{odd}}$ has empty interior.

In particular, if $f \in B_{\text{real}}^*$, then $f$ has no wandering domain whose set of limit functions is bounded.

We now state a geometric condition on a function $f \in B_{\text{real}}^*$ that will allow us to prove the absence of wandering domains. As we shall see in the next section, this condition holds whenever $f$ satisfies the hypotheses of Theorem 1.3.

5.4. Definition (Sector condition). For $f \in B_{\text{real}}$, we set

$$\Sigma_0(f) := \{\sigma \in \{+, -\} : \lim_{x \to +\infty} |f(\sigma x)| = \infty\}.$$ 

For $\sigma \in \Sigma_0(f)$ we say that $f$ satisfies the (real) sector condition at $\sigma\infty$ if, for all sufficiently large $R > 0$, there are $\vartheta > 0$ and $R' > 0$ such that

$$|f(\sigma x + y)| > R$$
whenever \( x > R' \) and \( |y| < \vartheta x \).

Let us also define
\[
\Sigma(f) := \{ \sigma \in \{+, -\} : \exists s \in S(f) \cap \mathbb{R}, n_j \to \infty : f^{n_j}(s) \to \sigma \infty \} \subset \Sigma_0(f).
\]

We say that \( f \) satisfies the \textit{(real) sector condition} if \( f \) satisfies the sector condition at \( \sigma \infty \) for all \( \sigma \in \Sigma(f) \).

It is well-known (and easy to see) that the sector condition does not change if we require it to only hold for some \( R \), provided \( R \) is chosen sufficiently large that \( |s| < R \) for all \( s \in S(f) \). (This will also follow from the proof of Theorem 6.1).

We also note the following standard fact.

5.5. Lemma. Let \( f \in \mathcal{B}_{\text{real}} \), and let \( M > 0 \) be sufficiently large. Then the set
\[
A := \bigcup_{\sigma \in \Sigma(f)} \sigma \cdot [M, \infty)
\]
satisfies \( A \subset I(f) \) and \( f(A) \subset A \).

Proof. This follows e.g. from the Ahlfors distortion theorem, which implies that
\[
\liminf_{x \to +\infty} \frac{\log \log |f(\sigma x)|}{\log x} \geq \frac{1}{2}
\]
whenever \( \sigma \in \Sigma_0(f) \). \( \square \)

5.6. Theorem. If \( f \in \mathcal{B}_{\text{real}}^* \) satisfies the real sector condition, then \( f \) has no wandering domains.

Proof. Assume that \( f \) has a wandering domain, say \( W \). Since \( I(f) \subset J(f) \) [EL92 Theorem 1], it follows that \( f^n(W) \cap \mathbb{R} \cap I(f) = \emptyset \) for all \( n \in \mathbb{N} \). Furthermore, Proposition 4.3 shows that the set of limit functions in \( W \) cannot be bounded. Thus Lemmas 5.1 and 5.5 show that there is \( s \in S(f) \) and a sequence \((n_k)\) such that \( f^{n_k} \to s \) and \( f^{n_k-1} \to \infty \).

Since the set \( \text{Lim}(f^n, W) \) is contained in the postsingular set, it follows that \( \Sigma(f) \neq \emptyset \).

Let us define \( U := \mathbb{C} \setminus (A \cup P(f)) \), where \( A \) is as in Lemma 5.5. Note that the orbit of \( W \) is contained in \( U \), since \( A \) is contained in \( I(f) \) and every point in \( P(f) \) has either an escaping or bounded orbit.

Now we are in a position to apply Theorem 4.1. Let \( D = D_\varepsilon(s) \) and fix \( R > |s| + \varepsilon \). The sector condition implies that there is \( R' > 0 \) and \( c \in (0, 1) \) such that \( |f(z)| > R \) whenever \( x \geq R' \), \( \sigma \in \Sigma(f) \) and \( |z - \sigma x| \leq c \cdot x \).

Set \( V := f^{-1}(D \cap U) \). Exactly as in the proof of Theorem 1.2 it follows that
\[
\limsup_{n \to \infty} d_U(f^{n-1}(w), \partial V) < \infty,
\]
which contradicts Theorem 4.1. \( \square \)

6. Functions that satisfy the real sector condition

Let us now formulate some conditions under which a function satisfies the real sector condition. Our first result shows that this property can be expressed in terms of an estimate on the logarithmic derivative.
6.1. Theorem. Let $f \in \mathcal{B}_{\text{real}}$ and $\sigma \in \Sigma_0(f)$. Then $f$ satisfies a sector condition at $\sigma_\infty$ if and only if there exist constants $r, K > 0$ with

$$\frac{|f'(\sigma x)|}{|f(\sigma x)|} \leq K \cdot \frac{\log |f(\sigma x)|}{x}$$

for all $x \geq r$.

Remark 1. This will conclude the proof of Theorem 1.3.

Remark 2. We note that the opposite inequality holds for every function $f \in \mathcal{B}$ and a suitable constant $K$; this is an immediate consequence of the expansion property of logarithmic lifts of maps in the class $\mathcal{B}$ (compare e.g. [EL92, Lemma 1]) and also follows from (6.2) below. If $f \in \mathcal{B}$ has finite order of growth, then $f$ satisfies (6.1) for all $x$ outside a set of finite logarithmic measure; this can also be deduced from (6.2).

Proof. Let $R \geq 1 + \max_{s \in S(f)} |s|$ and set $D^* := \mathbb{C} \setminus D_R(0)$. Denote by $T$ the component of $f^{-1}\{D^*\}$ that contains $\sigma x$ for sufficiently large $x$. ($T$ is called a tract of $f$.) Since $f : T \to D^*$ is a covering map and $f$ is transcendental, $T$ is simply connected. Recall from Proposition 3.1 that

$$\frac{1}{2 \text{ dist}(z, \partial T)} \leq \rho_T(z) \leq \frac{2}{\text{ dist}(z, \partial T)}.$$

Since $D^*$ is mapped conformally to the punctured unit disk by $z \mapsto R/z$, we have

$$\rho_D^*(z) = \frac{1}{|z| \log \frac{|z|}{R}}$$

for all $z \in D^*$. In particular,

$$\frac{1}{|z| \cdot \log |z|} \leq \rho_{D^*}(z) \leq \frac{2}{|z| \cdot \log |z|}$$

for $|z| > R^2$. Since $f|_T$ is a covering map, we have $\rho_T(z) = \rho_{D^*}(f(z)) \cdot |f'(z)|$. Combining this with the above estimates, we see that

$$\frac{1}{4 \cdot \text{ dist}(z, \partial T)} \leq \frac{|f'(z)|}{|f(z)| \cdot \log |f(z)|} \leq \frac{2}{\text{ dist}(z, \partial T)}$$

when $|z| > R^2$. So we see that

$$\text{ dist}(\sigma x, \partial T) \geq \varepsilon x$$

holds for some $\varepsilon > 0$ and all sufficiently large $x$ if and only if (6.1) is satisfied for some $K > 0$ and all sufficiently large $x$. Now $f$ satisfies the sector condition at $\sigma_\infty$ if and only if (6.3) holds. The claim follows. (Note that, since (6.1) is independent of $R$, the sector condition is also independent of $R$, provided $R$ is chosen to be of size at least $1 + \max_{s \in S(f)} |s|$.)

The real sector condition is preserved under precomposition with appropriate polynomials.

6.2. Corollary. Assume that $f \in \mathcal{B}_{\text{real}}$ satisfies the real sector condition at $\sigma_\infty$ for all $\sigma \in \Sigma_0(f)$. If $p$ is a real polynomial, then $g := f \circ p$ also satisfies the real sector condition at $\sigma_\infty$ for every $\sigma \in \Sigma_0(g)$. 

Proof. This follows from the fact that the preimage of a sector under any polynomial \( p \) again contains a sector. Indeed, we have \( \arg(p(z)) \approx d \arg z \), where \( d \) is the degree of \( p \).

Alternatively, it is easy to verify (6.1) for \( g \) from the corresponding condition for \( f \). ■

To give some explicit examples, let us consider a transcendental entire function of the form

\[
 f(z) := \prod_{n=1}^{\infty} \left( 1 + \frac{z}{x_n} \right), \quad 0 < x_n \leq x_{n+1}.
\]

(6.4)

So \( f \) is a real function of order \( \rho(f) < 1 \) with only negative real zeros, but infinitely many of these.

If \( f \) is chosen such that \( \sup |f(x)| < \infty \) for \( x < 0 \) then \( f \) belongs to the class \( \mathcal{B} \). Indeed, by definition, \( f \) is the locally uniform limit of polynomials with only real zeros, (i.e., \( f \) belongs to the Laguerre-Polya class). Since all critical points of the approximating polynomials belong to the negative real axis, the same is true of \( f \); hence the set of critical values of \( f \) is bounded. On the other hand, the set of asymptotic values of \( f \) is finite by the Denjoy-Carleman-Ahlfors theorem.

We have

\[
 x \cdot \frac{f'(x)}{f(x)} = x \cdot \sum_{n=1}^{\infty} \frac{1}{x_n + x} = \sum_{n=1}^{\infty} \left( 1 - \frac{x_n}{x_n + x} \right) = \sum_{n=1}^{\infty} \left( 1 - \frac{1}{w_n(x)} \right),
\]

where \( w_n(x) := (x_n + x)/x_n = 1 + x/x_n \). Note that for a fixed \( x \), the sequence \( \{w_n\} = (w_n(x)) \) is decreasing and converges to 1. Since \( \log y > 1 - 1/y \) for \( y > 1 \), we obtain that

\[
 \frac{f'(x)}{f(x)} \leq \sum_{n=1}^{\infty} \log w_n = \log \prod_{n=1}^{\infty} \left( 1 + \frac{x}{x_n} \right) = \log f(x).
\]

Thus Theorem 6.1 implies that \( f \) satisfies the real sector condition. (Alternatively, this is also easy to check from the definition of \( f \) and the original statement of the sector condition.)

Many functions can be written as \( f \circ p \), where \( f \) is as in (6.4) and \( p \) is a polynomial as in Corollary 6.2, e.g.

\[
 F_{2k}(z) := \int_0^{\infty} \exp(-t^{2k}) \cosh(tz) dt, \quad k > 0, \quad \text{or} \quad I_0(z) := 1 + \sum_{k=0}^{\infty} \left( \frac{z^k}{2k k!} \right)^2.
\]

These maps occur for \( p(z) = z^2 \) and \( f \) of order \( 1/2 \), converging to 0 along the negative real axis [Tit39 8.4, 8.63]. This representation is shared by the function \( \sinh z/z \). Hence Corollary 1.4 is a consequence of Theorem 1.3. (Of course it is also very simple to check either (6.1) or the sector condition explicitly for these functions.)

Finally, we remark that the sector condition can also be phrased by locating the preimages of certain compact sets. To do so, let us define truncated sectors

\[
 S_{\sigma,\vartheta} := \{ \sigma x + iy \in \mathbb{C} : x > R, |y| < \vartheta x \},
\]

where \( \sigma \in \{+, -\} \), \( \vartheta > 0 \) and \( R > 0 \).

6.3. Theorem. Let \( f \in \mathcal{B}_{\text{real}} \) and let \( \sigma \in \Sigma_0(f) \). Then the following statements are equivalent:
(i) $f$ satisfies the sector condition at $\sigma_\infty$.
(ii) If $K \subset \mathbb{C}$ is compact, then $f^{-1}(K)$ omits some truncated sector $S_{\sigma,0,R}$.
(iii) There exists a point $w$ in the unbounded component of $\mathbb{C} \setminus S(f)$ such that $f^{-1}(w)$ omits a truncated sector $S_{\sigma,0',R'}$.

**Proof.** Let us assume that $\sigma = +$ and that $f(x) \to +\infty$ as $x \to \infty$; the other cases then follow by pre- resp. post-composing $f$ with the map $x \mapsto -x$.

Obviously (ii) implies (iii). Also, (i) is clearly equivalent to (ii). (The sector condition just states that, for every $R > 0$, the set $f^{-1}(D_R(0))$ omits a truncated sector.)

So, it remains to show that (iii) implies (i). We show the contrapositive, so assume that there is a sequence $(z_k)_{k \geq 0}$, $z_k = x_k + iy_k$, such that $x_k \to +\infty$, $y_k/x_k \to 0$ and $\limsup |f(z_k)| < \infty$. By continuity of $f$, we may assume that $|f(z_k)| = R$ is constant and $R > \max_{s \in S(f)} |s|$.

Let $w \in \mathbb{C} \setminus S(f)$; we will show that $f^{-1}(w)$ does not omit any truncated sector; more precisely, we shall show that there is a sequence $(w_k) \subset f^{-1}(w)$ with $|w_k - x_k|/x_k \to 0$.

Let $K$ be a full subset of $\mathbb{C}$ that contains $S(f)$ but does not contain $w$ or any point of modulus $R$, and define $U := \mathbb{C} \setminus K$. (Here full means that $K$ is compact and the complement of $K$ is connected.) Let $T$ be the component of $f^{-1}(U)$ that contains $x$ for all sufficiently large real $x$.

Note first that, since $f(x_k) \to \infty$, we have $d_T(f(x_k), f(z_k)) \to \infty$. By Pick’s theorem, $f|_T : T \to U$ is locally a Poincaré isometry, hence $d_T(x_k, z_k) \to \infty$ as well. Recall that $y_k/x_k \to \infty$, so Proposition 3.1 implies that

\[ \text{dist}(x_k, \partial T)/x_k \to 0, \]

and, in particular,

\[ \text{dist}(z_k, \partial T)/x_k \to 0. \]

Let $\delta$ be the maximal hyperbolic distance in $U$ between $w$ and a point of $\partial D_R(0)$. For every $k$, we can then find some $w_k \in f^{-1}(w) \cap T$ such that $d_T(z_k, w_k) \leq \delta$. Again using Proposition 3.1 we must have

\[ |w_k - z_k|/x_k \to 0, \]

and hence $|w_k - x_k|/x_k \to 0$, as claimed. \hfill $\blacksquare$

We conclude this section with an observation concerning the perturbation of a function with the real sector property within the so-called Eremenko-Lyubich parameter spaces (see [EL92, Rem09b]).

**6.4. Corollary.** Let $f \in B_{\text{real}}$ and let $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ be quasiconformal homeomorphisms fixing the real line such that

\[ g := \psi^{-1} \circ f \circ \varphi \]

is an entire function. Then $f$ satisfies the real sector condition if and only if $g$ does.

**Proof.** The preimage of a truncated sector around the real axis under the quasiconformal map $\varphi$ again contains a sector; this follows e.g. from the geometric characterization of quasicircles. \hfill $\blacksquare$
7. Further results

An extension of Theorem 4.1. We now return to the general setting of Theorem 4.1 and prove two further statements in a similar spirit. The first strengthens (4.2), provided that we choose the domain W to be maximal, and is based on similar ideas as [Ber95, Lemma 3]. The second shows that superattracting basins are the only domains of normality for which an isolated point of the postsingular set can be a constant limit function. This is based on the same ideas that were applied by Zheng in [Zhe05].

7.1. Theorem. Let X be a Riemann surface, let $U' \subset U \subset X$ be hyperbolic open subsets of $X$, and let $f : U' \to U$ be a holomorphic covering map.

Let $W \subset U'$ be the set of all points in $U'$ which have a neighborhood $W$ such that $f^n(W) \subset U'$ for all $n$. Then

$$\lim_{n \to \infty} d_U(f^n(w), U \setminus W) = \infty$$

for all $w \in W$.

Furthermore, let $W$ be the component of $W$ containing $w$, and suppose that $f^{n_k}|W \to z_0$ locally uniformly for some sequence $n_k \to \infty$ and some $z_0 \in X$. Then $z_0 \in X \setminus U$. If $z_0$ is an isolated point of $X \setminus U$, then for sufficiently large $k$, $f^{n_k}(W) \cup \{z_0\}$ is simply connected, and $f^n$ extends holomorphically to $z_0$ with $f^n(z_0) = z_0$ and $(f^n)'(z_0) = 0$ for some $p$. In particular, $f^{n_k+np}|_W \to z_0$ locally uniformly as $n \to \infty$.

Proof. To prove the first statement, we establish the contrapositive. That is, suppose that $W \subset W$ is open and forward-invariant with $w \in W$, and suppose that

$$d_U(f^{n_k}(w), U \setminus W) \leq C < \infty$$

for some constant $C$ and some sequence $n_k \to \infty$. We shall show that $\partial W$ contains a point of $W$. Let us set $U_k := f^{-n_k}(U)$.

Indeed, for each $k$ let $\zeta_k$ be a point of $\partial W$ that is closest to $f^{n_k}(w)$ in the hyperbolic metric of $U$, and hence has hyperbolic distance at most $C$ from $f^{n_k}(w)$. Then the hyperbolic geodesic of $U$ connecting $f^{n_k}(w)$ and $\zeta_k$ belongs to $W$ apart from the endpoint $\zeta_k$. Pulling back this curve under $f^{n_k}$, we obtain a curve in $W \cap U_k$ that connects $w$ to some point $z_k \in \partial W$. Since $f^{n_k} : U_k \to U$ is a covering map, we see that $d_{U_k}(w, z_k) \leq C$.

Let $z_0$ be a limit point of the sequence $z_k$ (note that such a limit point exists and belongs to $U$, since the points $z_k$ have uniformly bounded hyperbolic distance from $w$). Recall that the points $z_k$ have uniformly bounded hyperbolic distance from $w$ in $U_k$, and that $d_{U_k}(w, \partial U_k) > d_{U_k}(w, \partial W) > 0$ for all $k$. It follows that $d_{U_k}(z_k, \partial U_k)$, and thus $d_{U_k}(z_0, \partial U_k)$, is uniformly bounded from below. Hence $z_0$ indeed has a neighborhood on which all iterates of $f$ are defined; i.e. $z_0 \in W$ as desired.

Now let us prove the second claim, so assume that $f^{n_k}|_W \to z_0$ uniformly, as in the statement of the theorem. Firstly we see that $z_0 \in X \setminus U$. Indeed, otherwise the assumption that $f^{n_k} \to z_0$ uniformly on a neighborhood of $w$ would imply that $\|Df^{n_k}(w)\|_U \to 0$. This contradicts the fact that $\dot{f}$ does not contract the hyperbolic metric of $U$.

Now suppose that $z_0$ is an isolated point of $X \setminus U$. Let $D$ be a small simply connected neighborhood of $z_0$ with $D \setminus \{z_0\} \subset U$ and set $D^* := D \setminus \{z_0\}$.


Claim. For sufficiently large $k$, the set $f^{n_k}(W)$ contains a simple closed curve $\gamma \subset D^*$ that surrounds $z_0$ (in particular, $f^{n_k}(W)$ is multiply connected).

Proof. Let $\varphi : D \to \mathbb{D}$ be a conformal isomorphism with $\varphi(z_0) = 0$ and set $r_k := |\varphi(f^{n_k}(w))|$ (for sufficiently large $k$). If we define $\gamma_k := \varphi^{-1}(\{z \in \mathbb{D} : |z| = r_k\})$, then the hyperbolic length of $\gamma_k$ in $D^*$, and hence in $U$, tends to zero as $k \to \infty$.

By (4.2), or the first part of this theorem, it follows that $\gamma_k \subset f^{n_k}(W)$ for sufficiently large $k$, proving the claim. $\triangle$

By replacing $W$ with $f^{n_k}(W)$, for $n_k$ sufficiently large, we can assume that $W$ itself contains a curve $\gamma$ as in the claim. Note that the component of $U \backslash \gamma$ that is not contained in $D^*$ cannot be simply connected or conformally equivalent to a punctured disk, since $U$ is a hyperbolic domain. Let us fix $n = n_k$ sufficiently large to ensure that $\gamma$ surrounds $f^n(\gamma)$.

Let $V^* \subset U$ be the component of $f^{-n}(D^*)$ that contains $\gamma$. Since $V^*$ contains $\gamma$, $V^*$ must be multiply connected. Since $f^n : V^* \to D^*$ is a covering map, it follows that $V^*_k$ is a punctured disk, and thus $V^* = V \backslash \{z_0\}$, for some simply connected domain $V$. Thus $f^n$ extends holomorphically to $z_0$ with $f^n(z_0) = z_0$.

Let $G_0$ be the complementary component of $\gamma$ that contains $z_0$, and set $G^*_0 := G_0 \backslash \{z_0\}$. For $m \geq 0$, let $G^*_m$ be the component of $f^{-mn}(G^*_0)$ that contains $\gamma$. Then, as above, $G^*_m$ is conformally equivalent to a punctured disk, and $G_m := G^*_m \cup \{z_0\}$ is simply connected. Let us set

$$G := \bigcup_{m \geq 0} G_m \quad \text{and} \quad G^* := G \backslash \{z_0\}.$$ 

Then $G$ is simply-connected, and $f^m : G^* \to G^*$ is a covering map. It follows that $G^* = W$, and that

$$F : G \to G; z \mapsto \begin{cases} f^n(z) & \text{if } z \in G^* \\ z_0 & \text{if } z = z_0 \end{cases}$$

is an analytic self-map of $G$, and in fact a branched covering map, with the only possible branch point at $z_0$.

Since $F(G^*_0) \subset G_0$ by choice of $n$ we see that $z_0$ is an attracting or superattracting fixed point. In particular, $F$ is not a conformal isomorphism. Since the only possible branch point of $F$ is at $z_0$, we see that $z_0$ is a critical point; i.e. $z_0$ is a superattracting fixed point of $F$. This completes the proof. $\blacksquare$

Applications to Ahlfors islands maps. Our results can be applied in a very general framework of one-dimensional holomorphic dynamics, as introduced by Epstein.

7.2. Definition. Let $X$ be a compact Riemann surface, let $V \subset X$ be open and nonempty, and let $f : V \to X$ be holomorphic and nonconstant.

Then the Fatou set of $f$ consists of those points $z \in X$ that have a neighborhood $U \subset X$ such that either the iterates $f^n$ are defined on $U$ for all $n \geq 0$ and form a normal family there, or $f^n(U) \subset X \setminus \overline{V}$ for some $n \geq 0$. The Julia set of $f$ is given by $J(f) := X \setminus F(f)$.

A component $U$ of $F(f)$ is a wandering domain of $f$ if $f^n(U) \cap f^m(U) = \emptyset$ for $n \neq m$. 
The set $S(f)$ of singular values of $f$ is the smallest compact set $S \subset X$ for which $f : f^{-1}(X \setminus S) \to X \setminus S$ is a covering map. The postsingular set of $f$ is given by

$$P(f) := \bigcup_{n \geq 0} f^n(S(f)).$$

Epstein [Eps97] defined a class of holomorphic functions $f : W \to X$ as above, known as Ahlfors islands maps, which includes all rational maps, all transcendental entire functions and all meromorphic functions and their iterates (as well as many more), and for which all the basic results of the standard Julia-Fatou theory remain true. We shall not state or require the formal definition here, which can be found e.g. in [Rem09a], along with some background and references. For further examples of Ahlfors islands maps, compare [RR].

We begin by generalizing the result from [BHK+93] and [Zhe05] regarding limit functions in a wandering domain.

**7.3. Proposition.** Let $f : V \to X$ be holomorphic and nonconstant, where $X$ is a compact Riemann surface and $V \subset X$ is open and nonempty. Suppose that $F(f)$ has a wandering component $W$, and let $w \in W$.

Then every limit point of the sequence $(f^n(w))$ is a non-isolated point of $P(f)$.

**Proof.** Set $U := X \setminus P(f)$. If $U$ is not hyperbolic, then either $f$ is a conformal automorphism of the Riemann sphere or of a torus, or $f$ is a linear endomorphism on a torus, or $X$ is the Riemann sphere and $f$ is conformally conjugate to a power map $z \mapsto z^d$, $d \in \mathbb{Z} \setminus \{-1, 0, 1\}$. None of these examples have wandering domains, and hence we may assume that $U$ is hyperbolic.

If we set $U' := f^{-1}(U)$, then $U' \subset U$ and $f : U' \to U$ is a covering map. If $f^n(W) \cap F(f) \neq \emptyset$ for some $n \geq 0$, then our conclusion holds. (If $f^{n_k}(w) \to a$, then $f^{n_k}|W \to a$ uniformly, and hence $a$ is in the accumulation set of an orbit in $P(f)$.) Otherwise, we can apply Theorem 7.1 to conclude that no limit point of $f^n(w)$ can be an isolated point of $X \setminus U = P(f)$, as claimed. 

We now also apply the first half of Theorem 7.1 in this general setting. To do so, it is useful to also use natural conformal metrics on non-hyperbolic Riemann surfaces. For such a surface $S$, we fix a metric of constant curvature 0 or 1 (the precise normalization in the case of curvature 0 will not matter in our context), and use $d_S$ to denote the distance with respect to this metric.

**7.4. Proposition.** Let $f : V \to X$ be holomorphic and nonconstant, where $X$ is a compact Riemann surface and $V \subset X$ is open and nonempty. Also assume that $f$ does not extend to a conformal automorphism of $X$. Set $U := X \setminus P(f)$, and let $w \in U \cap F(f)$.

Then $d_U(f^n(w), U \cap J(f)) \to \infty$ as $n \to \infty$ (where we use the convention that the distance is infinite if $U \cap J(f) = \emptyset$).

**Proof.** Again, let us begin by handling the case where $U$ is not hyperbolic. If $f$ is an affine toral endomorphism, but not an automorphism of the torus, then $F(f) = \emptyset$, and there is nothing to prove. On the other hand, if $f$ is conformally conjugate to a power map, then $F(f)$ has exactly two components, on which the iterates converge locally uniformly to the two superattracting points in $P(f)$, and the claims follow.
So now suppose that $U$ is hyperbolic. Then the theorem follows directly from Theorem 7.1. 

**Baker domains of meromorphic functions.** We now apply the preceding proposition to prove Theorem 1.5. In fact, we can state a more general result, which includes all meromorphic functions and their iterates, but also e.g. the type of functions studied in [Zhe05].

**7.5. Proposition.** Let $V \subset \mathbb{C}$ be an unbounded domain, and let $f : V \to \hat{\mathbb{C}}$ be a meromorphic function that does not extend analytically to $\infty$. Suppose that $W$ is a component of $F(f)$ with $f(W) \subset W$ and $f^n|_W \to \infty$. Then there exists a sequence $p_n \in P(f) \cap \mathbb{C}$ with

$$|p_n| \to \infty, \quad \sup_{n \geq 1} \frac{|p_{n+1}|}{|p_n|} < \infty, \quad \text{and} \quad \frac{\text{dist}(p_n, W)}{|p_n|} \to 0.$$ 

**Proof.** Let us first suppose that $W \cap P(f) = \emptyset$, and define $U := \hat{\mathbb{C}} \setminus P(f) \supset W$. Fix some point $w \in W$ and set $w_n := f^n(w)$; by the preceding result, we have $d_U(w_n, \partial W) \to \infty$.

By a result of Bonfert [Bon97, Theorem 1.2], there is a constant $C_1$ (depending on $w$) such that

$$\rho_W(w_n) \geq \frac{1}{C_1 \cdot \text{dist}(w_n, \partial W)}.$$ 

This implies that there is a constant $C_2 > 1$ such that, for every $n$, there is a point $\zeta_n \in \partial W$ with $|w_n|/C_2 < |\zeta_n| < C_2|w_n|$. Indeed, let $C > 1$ and consider the annulus

$$A(C, n) := \{ z \in \mathbb{C} : |w_n|/C < |z| < C|w_n| \}.$$ 

Using the universal cover of the annulus $A(C, n)$ given by the exponential map, we can compute the density of the hyperbolic metric in $A(C, n)$ explicitly. In particular, we have

$$\rho_{A(C, n)}(w_n) = \frac{\pi}{2|w_n| \log C}$$ 

[BM07, Section 12.2]. Setting $C_3 := \min\{|z| : z \in \partial W\}$, we have $\text{dist}(w_n, \partial W) \leq |w_n| + C_3$. Hence if $C > 1$ is such that $A(C, n) \subset W$, then

$$\log C \leq \frac{\pi}{2|w_n| \rho_{A(C, n)}(w_n)} \leq \frac{\pi}{2|w_n| \rho_W(w_n)} \leq \frac{\pi C_1 \text{dist}(w_n, \partial W)}{2|w_n|} \leq \pi C_1 \frac{|w_n| + C_3}{2|w_n|}.$$ 

The last expression remains bounded as $n \to \infty$, and the claim follows.

Now let $z_n \in \partial W$ be a point with smallest distance to $w_n$ with $|w_n|/C_2 \leq |z_n| \leq C_2|w_n|$. We can connect $w_n$ and $z_n$ in $A(C_2, n)$ by a curve $\gamma_n$ that has length $\leq C_4 \cdot |w_n|$. Let $p_n$ be a point of $P(f)$ that is closest to this curve $\gamma_n$.

We claim that

$$\frac{\text{dist}(p_n, \gamma)}{|p_n|} \to 0$$

as $n \to \infty$. Indeed, otherwise Proposition 3.1 would imply that the hyperbolic length of the curve $\gamma_n$ in $U$ remains bounded, which contradicts the fact that the hyperbolic distance between $w_n$ and $z_n$ tends to infinity. In particular, there is a constant $C_5$ such that

$$|w_n|/C_5 \leq |p_n| \leq C_5|w_n|.$$
To conclude the proof in the case where \( P(f) \cap W = \emptyset \), we use a result proved by Zheng [Zhe01, Lemma 4] and independently by Rippon [Rip06, Theorem 1], which states that there is a constant \( C_6 > 1 \) such that
\[
|w_{n+1}| \leq C_6 \cdot |w_n|.
\]
(This is established using the result of Bonfert mentioned above.) So we have
\[
|p_{n+1}| \leq C_5 |w_{n+1}| \leq C_5 C_6 |w_n| \leq C_5^2 C_6 |p_n|
\]
and
\[
\frac{\text{dist}(p_n, W)}{|p_n|} \leq C_5 \frac{\text{dist}(p_n, W)}{|w_n|} \to 0.
\]

If \( P(f) \cap W \neq \emptyset \), then the result follows directly from the above-mentioned theorem of Zheng and Rippon by fixing \( p_0 \in P(f) \cap W \) and setting \( p_n := f^n(p_0) \). ■

Remark. In the case of entire functions, Bergweiler shows that the sequence \( p_n \) can even be chosen such that \( |p_{n+1}|/|p_n| \to 1 \). His proof also applies in our setting, provided that the domain \( W \) is simply connected. It is an interesting question whether this stronger result holds for multiply connected domains.

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