Quasinormal modes of a Schwarzschild black hole surrounded by free static spherically symmetric quintessence: Electromagnetic perturbations

Yu Zhang, Yuanxing Gui, and Fei Yu
School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian, 116024, P. R. China

FengLin Li
Department of Materials Science and Engineering, Chungnam National University 220 Gung-dong, Yuseong-gu, Daejeon 305-764, Korea.

In this paper, we evaluated the quasinormal modes of electromagnetic perturbation in a Schwarzschild black hole surrounded by the static spherically symmetric quintessence by using the third-order WKB approximation when the quintessential state parameter \( w_q \) in the range of \(-1/3 < w_q < 0\). Due to the presence of quintessence, Maxwell field damps more slowly. And when at \(-1 < w_q < -1/3\), it is similar to the black hole solution in the \( ds/AdS \) spacetime. The appropriate boundary conditions need to be modified.

PACS numbers: 04.30.Nk, 04.70.Bw
Keywords: Quasinormal modes; Electromagnetic perturbation; WKB approximation.

I. INTRODUCTION

The study of QNMs of black hole has a long history\[1\]-\[12\] since they were first pointed out by Vishveshwara\[13\] in calculations of the scattering of gravitational waves by a black hole. Many techniques such as WKB approximation\[14\]-\[20\], the “potential fit”\[21\], and the method of continued fractions (Leaver,1985\[22\]) have been developed to calculate the QNMs of black holes since Chandrasekhar and Deweiler\[23\] had succeeded in finding some of the Schwarzschild QN frequencies using the technique of integration of the time independent wave equation. In this paper, we use the third-order WKB approximation.

Recently, V.V.Kiselev\[24\] has considered Einstein’s field equations for a black hole surrounded by the static spherically symmetric quintessential matter and obtained a new solution that depends on the state parameter \( w_q \) of the quintessence. And Songbai Chen et al\[25\] evaluated the quasinormal frequencies of massless scalar field perturbation around the black hole which is surrounded by quintessence. The result shows that due to the presence of quintessence, the scalar field damps more rapidly.

In this paper, we discuss electromagnetic perturbation.

II. ELECTROMAGNETIC PERTURBATIONS

The metric for the spacetime of Schwarzschild black hole surrounded by the static spherically-symmetric quintessence is given by\[25\]

\[
 ds^2 = \left(1 - \frac{2M}{r} - \frac{c}{r^{3w_q+1}}\right)dt^2 - \left(1 - \frac{2M}{r} - \frac{c}{r^{3w_q+1}}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

where \( M \) is the black hole mass, \( w_q \) is the quintessential state parameter, \( c \) is the normalization factor related to \( \rho_q = -\frac{3w_q}{r^{3w_q+1}} \), and \( \rho_q \) is the density of quintessence. We consider the evolution of a Maxwell field in this spacetime. And the evolution is given by Maxwell’s equations:

\[
 F_{\mu\nu}^{\mu\nu} = 0, \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}
\]

*Electronic address: zhangyu128@student.dlut.edu.cn
†Electronic address: guiyx@dlut.edu.cn
The vector potential $A_\mu$ can be expanded in four-dimensional vector spherical harmonics (see Ref. [26]):

$$A_\mu(t, r, \theta, \phi) = \sum_{l, m} \left( \begin{array}{c} 0 \\ a_{lm}(t, r) \sin \theta \partial_\theta Y_{lm} \\ 0 \\ -a_{lm}(t, r) \sin \theta \partial_\theta Y_{lm} \end{array} \right) \left[ \begin{array}{c} f^{lm}(t, r) Y_{lm} \\ h^{lm}(t, r) Y_{lm} \\ k^{lm}(t, r) \partial_\theta Y_{lm} \\ k^{lm}(t, r) \partial_\theta Y_{lm} \end{array} \right]$$

(3)

Where $l$ is the angular quantum number, $m$ is the azimuthal number. The first column has parity $(-1)^{l+1}$ and the second $(-1)^l$.

Submit (3) to (2), and define $d_r^2 = (1 - \frac{2M}{r} - \frac{c}{r^{q+1}})^{-1}$, we can get

$$\frac{d^2}{dr^2} \Phi(r) + (\omega^2 - V) \Phi(r) = 0$$

(4)

where $\Phi(r) = a_{lm}$ is for parity $(-1)^{l+1}$ and $\Phi(r) = r^2(l + 1/2) - i\omega h^{lm} - df^{lm}/dr$ is for parity $(-1)^l$ (for further details see Ref. [26]).

The potential $V$ is given by

$$V(r) = (1 - \frac{2M}{r} - \frac{c}{r^{q+1}}) \frac{l(l + 1)}{r^2}$$

(5)

III. WKB APPROXIMATION AND QUASINORMAL MODES

The equation for perturbations of a black hole can be reduced to a second order differential equation in the form

$$\frac{d^2}{dx^2} \Phi + (\omega^2 - V) \Phi = 0$$

(6)

The coordinate $x$ is a “tortoise coordinate” $r_*$ which ranges from $-\infty$ at the horizon to $+\infty$ at spatial infinity (for asymptotically flat spacetime). And the appropriate boundary conditions defining QNMs are purely ingoing waves at the horizon and purely outgoing waves at infinity.

$$\Phi \sim e^{-i\omega x}, x \rightarrow -\infty$$

(7)

$$\Phi \sim e^{+i\omega x}, x \rightarrow +\infty$$

(8)

Only a discrete set of complex frequencies satisfy these boundary conditions.

B. F. Schutz and C. M. Will [14] presented a new semianalytic technique which is called WKB approximation for determining the quasinormal modes of black hole. Later, S. Iyer and C. M. Will [15] developed the method to the third order and R. A. Konoplya [16] extended it to the sixth order. The accuracy of the WKB formula is better with a larger multipole number $l$ and a smaller overtone $n$. The formula for the complex quasinormal frequencies $\omega$ (for the third order WKB method) is

$$\omega^2 = \left[ V_0 + (-2V''_0)^{1/2} \Lambda \right] - i(n + \frac{1}{2})(-2V''_0)^{1/2}(1 + \Omega)$$

(9)

where

$$\Lambda = \frac{1}{(-2V''_0)^{1/2}} \left\{ \frac{5}{6912} \frac{V''_0}{V_0} \right\} \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V''_0}{V_0} \right)^2 (7 + 60\alpha^2) \right\}$$

$$\Omega = \frac{1}{2\sqrt{V_0}} \left\{ \frac{5}{6912} \frac{V''_0}{V_0} \right\}^4 (77 + 188\alpha^2)$$

(10)
FIG. 1: Variation of the effective potential for the Maxwell field in the Schwarzschild black hole surrounded by quintessence with $r$ for $l = 5$, $c = 0.01$ and $w_q = -1/300, -1/6, -0.33$. 

and 

$$\alpha = n + \frac{1}{2}, V_0^{(n)} = \frac{d^nV}{dr^*_n}\bigg|_{r_*=r_*(r_p)}$$

(11)

For quintessence, $w_q$ keeps in the range of $-1 < w_q < 0$. Looking into the metric of $\mathcal{M}$ we can see that, when $w_q$ is at the range of $-1/3 < w_q < 0$, the spacetime is asymptotically flat. We can calculate the QNMs in the situation by WKB approximation. And when $-1 < w_q < -1/3$, it is not asymptotically flat, which is similar to the ds/Ads spacetime.

Take $M = 1, c = 0.01$ and $M = 1, c = 0$ for our calculation. And $c = 0$ means there is no quintessence. Using the third-order WKB approximation, we can get the solutions as the table 1 and table 2 show, where $l$ is the angular harmonic index, $n$ is the overtone number, $\omega$ is the complex quasinormal frequencies, $w_q$ is the quintessential state parameter.

The variation of the effective potential with $r$ which is respective to the quintessential state $w_q$ parameter for fixed $l = 5$ and $c = 0.01$ is shown in Fig. 1.

**TABLE I:** The quasinormal frequencies of electromagnetic perturbations in the Schwarzschild black hole without quintessence($c=0$).

| $l$ | $n$ | $\omega$ |
|-----|-----|----------|
| 5   | 0   | 0.45713$ - 0.09507i$ |
|     | 1   | 0.43583$ - 0.09507i$  |
|     | 2   | 0.40232$ - 0.09507i$  |
|     | 3   | 0.36050$ - 0.09507i$  |
|     | 4   | 0.32206$ - 0.09507i$  |

| $l$ | $n$ | $\omega$ |
|-----|-----|----------|
| 0   | 0   | 0.85301$ - 0.09507i$  |
|     | 1   | 0.84114$ - 0.09507i$  |
|     | 2   | 0.81956$ - 0.09507i$  |
|     | 3   | 0.79094$ - 0.09507i$  |
|     | 4   | 0.75697$ - 0.09507i$  |

| $l$ | $n$ | $\omega$ |
|-----|-----|----------|
| 5   | 0   | 1.04787$ - 0.09507i$  |
|     | 1   | 1.03815$ - 0.09507i$  |
|     | 2   | 1.01997$ - 0.09507i$  |
|     | 3   | 0.99513$ - 0.09507i$  |
|     | 4   | 0.96518$ - 0.09507i$  |
TABLE II: The quasinormal frequencies of electromagnetic perturbations in the black hole surrounded by quintessence for $l = 2, l = 3, l = 4, l = 5$ and $c = 0.01$.

| $3w_q + 1$ | $\omega(n = 0)$ | $\omega(n = 1)$ | $\omega(n = 2)$ | $\omega(n = 3)$ | $\omega(n = 4)$ |
|------------|------------------|------------------|------------------|------------------|------------------|
| $l = 2$    |                  |                  |                  |                  |                  |
| 0.99       | 0.65343 - 0.09515i | 0.63825 - 0.28833i | 0.61202 - 0.48758i | 0.57849 - 0.69203i | 0.53889 - 0.89969i |
| 0.8        | 0.65268 - 0.09498i | 0.63753 - 0.28783i | 0.61137 - 0.48673i | 0.57793 - 0.69082i | 0.53842 - 0.89810i |
| 0.6        | 0.65169 - 0.09477i | 0.63660 - 0.28717i | 0.61052 - 0.48560i | 0.57718 - 0.68921i | 0.53779 - 0.89600i |
| 0.4        | 0.65046 - 0.09449i | 0.63543 - 0.28634i | 0.60944 - 0.48418i | 0.57622 - 0.68719i | 0.53698 - 0.89337i |
| 0.2        | 0.64893 - 0.09415i | 0.63396 - 0.28530i | 0.60809 - 0.48242i | 0.57501 - 0.68467i | 0.53503 - 0.89010i |
| 0.01       | 0.64711 - 0.09376i | 0.63222 - 0.28409i | 0.60648 - 0.48036i | 0.57354 - 0.68175i | 0.53464 - 0.88630i |
| $l = 3$    |                  |                  |                  |                  |                  |
| 0.99       | 0.84873 - 0.09538i | 0.83691 - 0.28788i | 0.81544 - 0.48454i | 0.78696 - 0.68574i | 0.75317 - 0.89049i |
| 0.8        | 0.84774 - 0.09522i | 0.83595 - 0.28738i | 0.81454 - 0.48370i | 0.78613 - 0.68454i | 0.75242 - 0.88893i |
| 0.6        | 0.84645 - 0.09500i | 0.83471 - 0.28672i | 0.81335 - 0.48258i | 0.78504 - 0.68295i | 0.75143 - 0.88686i |
| 0.4        | 0.84485 - 0.09472i | 0.83314 - 0.28589i | 0.81187 - 0.48118i | 0.78366 - 0.68095i | 0.75017 - 0.88426i |
| 0.2        | 0.84284 - 0.09438i | 0.83119 - 0.28485i | 0.81002 - 0.47943i | 0.78192 - 0.67847i | 0.74857 - 0.88102i |
| 0.01       | 0.84047 - 0.09398i | 0.82888 - 0.28365i | 0.80781 - 0.47738i | 0.77985 - 0.67557i | 0.74665 - 0.87726i |
| $l = 4$    |                  |                  |                  |                  |                  |
| 0.99       | 1.04260 - 0.09550i | 1.03293 - 0.28765i | 1.01484 - 0.48279i | 0.99913 - 0.68164i | 0.96033 - 0.88391i |
| 0.8        | 1.04138 - 0.09533i | 1.03174 - 0.28716i | 1.01370 - 0.48196i | 0.98905 - 0.68045i | 0.95932 - 0.88236i |
| 0.6        | 1.03980 - 0.09512i | 1.03018 - 0.28650i | 1.01220 - 0.48084i | 0.98763 - 0.67888i | 0.95799 - 0.88031i |
| 0.4        | 1.03782 - 0.09484i | 1.02824 - 0.28567i | 1.01032 - 0.47945i | 0.98585 - 0.67689i | 0.95632 - 0.87773i |
| 0.2        | 1.03534 - 0.09450i | 1.02581 - 0.28463i | 1.00798 - 0.47771i | 0.98360 - 0.67443i | 0.95420 - 0.87453i |
| 0.01       | 1.03243 - 0.09410i | 1.02294 - 0.28343i | 1.00520 - 0.47567i | 0.98094 - 0.67155i | 0.95167 - 0.87079i |

IV. DISCUSSION AND CONCLUSION

The quasinormal modes of a black hole present complex frequencies, the real part of which represents the actual frequencies of the oscillation and the imaginary part represents the damping. From Fig.1 we see that the peak value of potential barrier gets lower as the absolute $w_q$ increases. The data of table I is obtained by using WKB method in Schwarzschild black hole without quintessence, and table II is under the situation that the quintessence exits. Explicitly, we plot the relationship between the real and imaginary parts of quasinormal frequencies with the variation of $w_q$ (for fixed $c = 0.01$), compared with the situation without quintessence. From Fig.2 we can find that for fixed $c$ (unequal to 0) and $l$ the absolute values of the real and imaginary parts decrease as the absolute value of the quintessence state parameter $w_q$ increases. It means that when the absolute value of $w_q$ is bigger, the oscillations damp more slowly. And the absolute values of the real and imaginary parts of quasinormal
modes with quintessence are smaller compared with those with no quintessence for given $l$ and $n$. That is to say, due to the presence of quintessence, the oscillations of the Maxwell fields damp more slowly.

In this paper, we only calculate the QNMs when $w_q$ is at the range of $-1/3 < w_q < 0$. And for $w_q$ at the range of $-1 < w_q < -1/3$, the spacetime is not asymptotically flat, which is similar to the ds/AdS spacetime. And the boundary condition should be modified. An appropriate boundary condition is there are incoming waves at the inner horizon and outgoing waves at the outer horizon. Define the “tortoise coordinate” $r_*$ as $\frac{dr_*}{dr} = (1 - \frac{2M}{r} - \frac{c}{r^{w_q+1}})^{-1}$. The boundary condition can be written as

$$\Phi \sim e^{-i\omega r_*}, r \to r_{\text{inner}}$$

$$\Phi \sim e^{i\omega r_*}, r \to r_{\text{outer}}$$

We notice that when $r = r_{\text{inner}}$ and $r_{\text{outer}}$, $r_*$ is $-\infty$ and $\infty$ respectively. That is to say actually after the variation
of "tortoise coordinate", the boundary conditions for the $-1 < w_q < -1/3$ case and the $-1/3 < w_q < 0$ case are formally the same.

Acknowledgments

Yu Zhang wishes to thank Ph.D LiXin Xu for his helpful discussions. This work is supported by the National Natural Science Foundation of China under Grant No. 10573004.

[1] K. D. Kokkotas and B. G. Schmidt, Living Rev. Rel. 2, 2(1999); H. P. Nollert, Class. Quant. Grav. 16, R159 (1999).
[2] S. Chandrasekhar, Proc. R. Soc. London, Ser. A. 343, 289-298(1975).
[3] R. A. Konoplya, Phys. Rev. D 71, 024038(2005).
[4] V. Cardoso and J. P. S. Lemos Phys. Rev. D 64, 084017 (2001); E. Beri, V. Cardoso and S. Yoshida, Phys. Rev. D69(2004), 124018; V. Cardoso, J. P. S. Lemos, S. Yoshida, Phys. Rev. D70, 12403 (2004).
[5] J. L. Jing, Phys. Rev. D 69, 084009 (2004). S. B. Chen, J. L. Jing, Class. Quant. Grav. 22, 533-540 (2005); S. B. Chen, J. L. Jing, Class. Quant. Grav. 22, 2159-2165 (2005).
[6] H. Liu, Class. Quantum Grav. 12, 543(1995); H. Liu, B. Mashhoon, Class. Quant. Grav.13, 233(1996).
[7] B. Wang, C. Y. Lin and E. Abdalla, Phys. Lett. B 481, 79 (2000). B. Wang, E. Abdalla and R. B. Mann, Phys. Rev. D 65, 084006 (2002); B. Wang, C.Y. Lin and C. Molina, Phys. Rev. D 70, 064025 (2004).
[8] H. T. Cho, Phys.Rev. D73, 024019(2006).
[9] R. A. Konoplya, A. Zhidenko,Phys.Rev. D73, 124040(2006).
[10] A. Lpez-Ortega, Gen.Rel.Grav. 38,743-771(2006).
[11] Ramin G. Daghigh, Gabor Kunstatter, Dave Ostapchuk, Vince Bagnulo, Class.Quant.Grav. 23 5101-5116(2006).
[12] A. J. M. Medved, Damien Martin, Gen.Rel.Grav. 37, 1529-1539(2005).
[13] C. V. Vishveshwara, Nature 227, 936(1970).
[14] B. F. Schutz, and C. M. Will, Astrophys. J.291, L33-L36 (1985).
[15] S. Iyer and C. M. Will, Phys. Rev. D 35, 3621(1987).
[16] R. A. Konoplya, Phys. Rev. D 68, 024018(2003).
[17] S. Iyer, Phys. Rev. D 35, 3632(1987).
[18] K. D. Kokkotas and B. F. Schutz, Phys. Rev. D37, 3378 (1988).
[19] E. Seidel, and S. Iyer, Phys. Rev. D41, 374-382 (1990).
[20] K. D. Kokkotas, Nuovo Cimento B 108, 991-998 (1993).
[21] V. Cardoso and J. P. S. Lemos, Phys. Rev. D 67,084020 (2003).
[22] E. W. Leaver, Proc. R. Soc. A 402, 285(1985).
[23] S. Chandrasekhar, and S. Detweiler, Proc. R. Soc. London, Ser. A. 344, 441-452(1975).
[24] V. V. Kiselev, Class. Quant. Grav. 20, 1187-1197 (2003).
[25] S. B. Chen, J. L. Jing, Class. Quant. Grav. 22, 4651-4657(2005).
[26] R. Ruffini, in Black Hole: les Astres Occlus (Gordon and Breachm New York, 1973).