Harnessing electroacoustic analogies in designing acoustic topological systems

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Topological acoustics has recently witnessed a spurt in research activity, owing to their unprecedented properties that extend beyond the standard wave dispersion for vibration control. In recent years, the use of coupled arrays of acoustic chambers has gained popularity in designing acoustic topological systems. In their common form, acoustic chambers with relatively large volume are coupled with others via narrow channels. Such configuration generally necessitates modeling in full three-dimensional model and may require extended computational time for simulation their harmonic response. To this end, this paper establishes a comprehensive mathematical treatment of the use of electroacoustic analogies for designing acoustic topological systems. We demonstrate the potential of such analytical approach via two types of topological systems: (1) edge states with quantized winding numbers in an acoustic diatomic lattice and (2) valley Hall transition in an acoustic honeycomb lattice that leads to robust waveguiding. In both cases, the established analytical approach shows an excellent agreement with the full three-dimensional model, whether in dispersion analyses or the response of an acoustic system with finite number of cells. The established framework opens avenues for designing a verity of acoustic topological insulators with simplified analytical formulation and minimal computational costs.

Keywords
Topological Insulators; Acoustic Lattices; Electroacoustic Analogies; Quantum Valley Hall Effect; Edge states.

1 Introduction

The study of topological acoustics has recently witnessed a spurt in research activity, owing to their unprecedented properties including robustness against defects, unidirectionality in wave transmission, and backscattering-immunity have been recently demonstrated [1,2]. Such interesting behaviours have given rise to various applications with inherent topological protection, such as robust wave guiding and one-way signal transport[3,4], logic operations[5], and negative refraction[6].

In recent years, the use of coupled arrays of acoustic chambers has gained popularity in designing acoustic topological insulators [7–13]. In their common form, acoustic chambers with relatively large volume are coupled with their neighboring chambers via narrow channels in a single or multiple directions, which will be referred to as acoustic lattices henceforth. The fascinating part of using these acoustic lattices is their ability to closely mimic Hamiltonians pertaining to a variety of topological system types in analogy to electronic systems. To date, coupled acoustic chambers have been used in inducing robust edge states [7,8], Quantum Valley Hall Effect (QVHE) [9,10], Quantum Spin Hall Effect (QSHE) [10], Hofstadter-butterfly effect in quasi-periodic lattices [11] and, more recently, higher order topological insulators with corner states [12–14]. Owing to the three-dimensional nature of such acoustic lattices, numerical modeling is often achieved via finite-element based procedure, and the effective Hamiltonians are commonly established via parameter fitting of the coupling [7]. In addition, electroacoustic analogies (circuit lumped-parameter model) have also been demonstrated as a suitable methodology for modeling Hamiltonians of an acoustic Kagome lattice [15], among other examples in literature [16–20].

In light of the aforementioned studies, we aim to establish a comprehensive analytical framework for acoustic lattices using electroacoustic analogies to ultimately design non-trivial topological states. The key importance of such approach is two-fold: (i) to provide an analytical insight into the topological protection mechanism in acoustic lattices and (ii) to reduce computational cost for analyzing them. While modeling acoustic lattices via harnessing lumped circuit equivalents is convenient in terms of fast calculation and obtaining analytical models, the approach becomes less accurate as the frequency gets higher [15,17]. Nonetheless, as will be shown, the first few dispersion branches and their corre-
Figure 1: (A) Schematic for a monatomic acoustic lattice and its equivalent electrical model. (B) Unit cell of the monatomic lattice detailing the geometrical properties of the chamber and channel. (C) Fundamental frequency of an isolated unit cell (acting as a Helmholtz resonator) with varying chamber height $H$ and square channel side $d$. The length of the air channel and hexagon’s side remain unchanged and are chosen to be $\ell = 4$ mm and $d_c = 14$ mm, respectively.

The corresponding frequency response in a finite acoustic lattice can be estimated with relatively high accuracy, rendering such a methodology an asset in studying topological acoustic systems with high computational efficiency.

To illustrate the concept, we start by studying an acoustic monatomic lattice to establish variables and physical meanings of coupled chambers. Established circuit analogues typically consist of a series of coupled inductors and grounded capacitors, whose inductance and capacitance are estimated based on the geometrical properties. The parameters of the dispersion relations are interpreted based on the renowned Helmholtz resonance, which enable a better estimation of resonance frequencies based on the geometry of acoustic system. Afterwards, and based on the established understanding of acoustic monatomic lattices, we present two types of topological acoustic systems: (1) a diatomic lattice with non-vanishing quantized winding number and edge states, and (2) a honeycomb lattice with QVHE for designing robust waveguides. The developed circuit model for both lattices is compared with the numerical results from a full-scale three-dimensional counterpart in regards of dispersion analysis and a lattice with a finite number of cells. In addition, we show how to correctly account for the boundary conditions and implement the right correction parameters for the equivalent length of air channels at lattice boundaries, resulting in a more accurate prediction of system’s resonances.

2 Acoustic monatomic lattices

2.1 Electroacoustic analogies: Modeling and geometrical dependence

We start by demonstrating the electroacoustic analogy in a chain of identical acoustic chambers (or cavities) coupled via narrow channels with rigid walls to satisfy sound hard boundaries (Fig. 1A,B). We shall refer to this chain as an acoustic monatomic lattice due to its resemblance to a typical monatomic lattice as will be shown shortly. At relatively low frequencies, the acoustic pressure inside each of these chambers is assumed to be constant throughout, thus constituting its only degree of freedom, denoted here as $p_i$ with $i$ symbolizing the chamber’s order in the acoustic chain (Fig. 1A). As a result, electroacoustic analogies can be readily utilized and offer intriguing mapping of acoustic pressure and particle velocity in acoustic medium to voltage and current in circuitry [17]. In fact, narrow channels and acoustic chambers are analogous to electric inductors with inductance $L_0$ and grounded capacitors of capacitance $C_0$, respectively, and their values are estimated based on chamber/ channel geometrical properties [17]:

\[
L_0 = \frac{\rho \ell_e}{S} \quad \text{(1a)}
\]

\[
C_0 = \frac{V}{\rho c^2} \quad \text{(1b)}
\]

Here, $S$ is the cross-sectional area of the narrow channel, $\rho$ is the fluid’s density, $V$ is the chamber’s volume, $c$ is the sound speed in the fluid medium, and $\ell_e$ is the effective length of channels (which is often longer than the channel’s physical length $\ell$).
2.2 Dispersion relation and relevance to Helmholtz resonance

The equivalence of the acoustic monatomic lattice to a circuit model is intended to derive a simplified analytical expression of dispersion relation to avoid analyzing lattice’s full-scale three-dimensional unit cell. From electrical circuit analysis, the electrical current is induced via a change in voltage across an electrical element. Assuming harmonic motion, the current $I$ flowing through inductors (capacitors), in response to a drop in voltage $\Delta V$, is governed by the reciprocal of a frequency-dependent impedance $Z_0 = \frac{i\omega}{L_0}$ ($Z_0 = \frac{1}{i\omega C_0}$), yielding $I = \Delta V / Z_0$ ($i$ is the imaginary unit). Exploiting the mapping of pressure to voltage and knowing that currents entering and exiting a junction sum to zero (per Kirchhoff law), the governing equation of the $i$th acoustic chamber is:

$$
\left(\frac{2}{i\omega L_0} + i\omega C_0\right)p_i - \frac{1}{i\omega L_0}(p_{i+1} + p_{i-1}) = 0
$$  \hspace{1cm} (2)

Applying Bloch theorem $p_{i\pm 1} = p_i e^{\pm i\mu}$ and multiplying by $i\omega L_0$, further symbolic computations yield the dispersion relation:

$$
\omega = 2\omega_0 \left| \sin \left( \frac{\mu}{2} \right) \right|
$$  \hspace{1cm} (3)

where $\mu$ is the non-dimensional wavenumber and:

$$
\omega_0 = \sqrt{\frac{1}{L_0 C_0}}
$$  \hspace{1cm} (4)

Equation (3) is in perfect analogy with the dispersion relation of a typical elastic monatomic lattice, albeit with the definition of $\omega_0$ being the natural frequency of a spring-mass system[21]. For our acoustic lattice, we interpret the quantity $\omega_0$ in Equation (4) as the natural frequency of a stand-alone Helmholtz resonator (an isolated unit cell as seen in Fig. 1B), owing to the fact that a Helmholtz resonator is often mapped to a spring-mass system[22-24]. Consequently,
an acoustic monatomic lattice can be perceived as serially coupled Helmholtz resonators. Implementing this physical understanding, in addition, facilitates a better estimation of acoustic lattice properties. For example, the effective length \( \ell_e \) for a Helmholtz resonator is suggested to be:

\[
\ell_e = \ell (1 + \delta_e)
\]  

(5)

The correction factor \( \delta_e \) is the summation of the inner and outer end corrections, which are chosen here as \( \delta_{\text{in}} = 0.425d_h/\ell \) and \( \delta_{\text{out}} = 0.3d_h/\ell \), respectively, for flanged and unflanged ends\[25\]. Accordingly, the complete correction factor is \( \delta_e = 0.725d_h/\ell \), with \( d_h = 4S/P \) (\( P \) is the channel’s perimeter) is defined herein as the channel’s hydraulic diameter\[26\]. Note that the hydraulic diameter for a square (circular) channel is equal to the channel’s side length (nominal diameter). Making use of Equations (1), (4), and (5), the Helmholtz resonance can be explicitly expressed in terms of the resonator’s geometrical properties in units of Hertz:

\[
f_R = \frac{\omega_0}{2\pi} = \frac{c}{2\pi\ell \sqrt{\frac{V_c/V}{1 + \delta_e}}}
\]  

(6)

where \( V_c \) is the volume of the channel. Equation (6) implies that the larger the ratio of the volumes, the larger the natural frequency, given a constant length \( \ell \) (Fig. 1C). Doubling this quantity provides an estimate for the cutoff frequency (in Hz) of the acoustic monatomic lattice, as inferred from Equation (5).

2.3 Numerical examples

We simulate three different combinations of the acoustic chamber/channel’s geometrical cross-sectional areas: (1) square chambers with square channels, (2) hexagonal chambers with square channels, and, (3) circular chambers with circular channels (Fig. 2). Three-dimensional model simulations are done via COMSOL commercial software and assuming a discretization of the Quadratic Lagrange type. For distinction between the circuit model and the full three-dimensional model, we shall label the dispersion relation or related system dynamics obtained from the circuit model as analytical, while the one obtained from COMSOL simulations as numerical. Calculating the dispersion relation for a swept range of the hydraulic diameter \( d \) and height \( H \), while maintaining the length \( \ell = 4\text{mm} \) and parameter \( d_c = 14\text{mm} \) (from which the area \( S \) is calculated) constant throughout, we observe that all analytical dispersion relations closely resemble their numerical counterparts (Fig. 2). As such, it is now evident that Equation (6) can effectively estimate the natural frequency of the stand-alone Helmholtz resonator with excellent accuracy, and consequently producing the most fitting dispersion relation per Equation (5). This also displays the competence of the circuit model in predicting the dispersion relation while maintaining very low computational cost relative to the three-dimensional model.

3 Acoustic diatomic lattices

3.1 Mathematical formulation and dispersion analysis

Next, we consider an acoustic diatomic lattice constituting an array of equally sized acoustic chambers, coupled via alternating narrow/wide channels with hydraulic diameters \( d_1 \) and \( d_2 \) (Fig. 3). We shall demonstrate the emergence of edge states via analyzing the band topology associated with the unit cell of the diatomic lattice. Analogous to the analysis of the acoustic monatomic lattice, the governing equations of a diatomic unit cell are derived:

\[
\left( \frac{1}{i\omega L_1} + \frac{1}{i\omega L_2} + i\omega C \right) p_i - \left( \frac{1}{i\omega L_1} q_i + \frac{1}{i\omega L_2} q_{i-1} \right) = 0 
\]  

(7a)

\[
\left( \frac{1}{i\omega L_1} + \frac{1}{i\omega L_2} + i\omega C \right) q_i - \left( \frac{1}{i\omega L_1} p_i + \frac{1}{i\omega L_2} p_{i+1} \right) = 0
\]  

(7b)

where \( C \) is the capacitance of chambers and \( L_1 \) (\( L_2 \)) is the inductance of the first (second) Helmholtz resonator of the diatomic unit cell. The pressure in the first (second) chamber of the \( ith \) unit cell is denoted by \( p_i \) (\( q_i \)). Introducing \( \omega_1 = 1/\sqrt{CL_1} \) (\( \omega_2 = 1/\sqrt{CL_2} \)) as the fundamental frequency of the first (second) Helmholtz resonator, we further simplify the governing equations to get:

\[
\left( \omega_1^2 + \omega_2^2 - \omega^2 \right) p_i - \left( \omega_1^2 q_i + \omega_2^2 q_{i-1} \right) = 0
\]  

(8a)
It is straightforward to compute the eigenvalues from values of the wavenumber in the Brillouin zone, i.e., 

\[ \left( \omega_1^2 + \omega_2^2 - \omega^2 \right) q_i - \left( \omega_1^2 p_i + \omega_2^2 p_{i+1} \right) = 0 \] 

(8b)

Finally, applying the Bloch theorem yields an eigenvalue problem:

\[ H p_i = \omega^2 p_i \] 

(9)

where the unit-cell’s Hamiltonian \( H \) and pressure vector \( p_i \), respectively, are:

\[ p_i^T = \{ p_i, q_i \} \] 

(10a)

\[ H = \begin{bmatrix} \omega_1^2 + \omega_2^2 & -\omega_1^2 - \omega_2^2 e^{-i\mu} \\ -\omega_1^2 - \omega_2^2 e^{i\mu} & \omega_1^2 + \omega_2^2 \end{bmatrix} \] 

(10b)

It is straightforward to compute the eigenvalues from \( |H - \omega^2 I| = 0 \), which results in the dispersion branches:

\[ \omega = \sqrt{\omega_1^2 + \omega_2^2} \pm \sqrt{\omega_1^4 + \omega_2^4 + 2\omega_1^2 \omega_2^2 \cos(\mu)} \] 

(11)

Figure 4A shows the analytical dispersion branches that are computed directly from Equation (11) by sweeping the values of the wavenumber in the Brillouin zone, i.e., \( \mu \in [-\pi, \pi] \), for the cases of (i) \( \omega_2 < \omega_1 \), (ii) \( \omega_2 = \omega_1 \) and (iii) \( \omega_2 > \omega_1 \). Superimposed are the numerically obtained dispersion relation (presented as circles), displaying excellent agreement with their analytical counterpart (presented as lines). We change the values of \( \omega_1,2 \) by introducing the parameterization \( d_{1,2} = d_0 \mp \Delta d / 2 \), where \( \Delta d \) is the difference in channel’s hydraulic diameter and \( d_0 = (d_1 + d_2) / 2 \). In this example, we choose \( \Delta d = -1, 0, +1 \) mm, which give (i) \( d_1 = 3.5 \) mm and \( d_2 = 2.5 \) mm, (ii) \( d_1 = d_2 = 3 \) mm, and (iii) \( d_1 = 2.5 \) mm and \( d_2 = 3.5 \) mm, respectively. Chamber’s height and side length (diameter) are \( H = 20 \) mm and \( d_c = 14 \) mm, respectively, and they remain unchanged throughout. As expected, a bandgap opens only for \( \omega_2 \neq \omega_1 \) and its limits are \( \omega = \sqrt{2} \omega_1 \) and \( \omega = \sqrt{2} \omega_2 \), which are computed from the dispersion relation at the Brillouin zone edge \( \mu = \pm \pi \) (Fig. 4B). Moreover, the first (Fig. 4A(i)) and last (Fig. 4A(iii)) dispersion relations have no apparent difference as they signify a different choice of the unit cell by flipping the order of narrow/wide channels, which warrants an identical dispersion relation as evident from Equation (11). Note that the order of the bandgap limits is contingent on \( \omega_{1,2} \) values and, hence, its bandgap width is \( \Delta \omega = \sqrt{2} |\omega_1 - \omega_2| \).

3.2 Winding number and edge states

Although the dispersion relations in Fig. 4A(i,iii) are identical, a deeper look into their band topology would reveal that they are, in truth, topologically distinct. To attain an insight of the topological properties of the lattice, we re-write the Hamiltonian as a summation of Pauli matrices:

\[ H = - \sum_i h_i \sigma_i \] 

(12)
where \( l = 0, 1, 2, 3 \) and

\[
\begin{align*}
    h_0 &= -(\omega_1^2 + \omega_2^2) \\
    h_1 &= \omega_1^2 + \omega_2^2 \cos(\mu) \\
    h_2 &= \omega_2^2 \sin(\mu) \\
    h_3 &= 0
\end{align*}
\]  

(13a) (13b) (13c) (13d)

Note here that the zeroth Pauli matrix \( \sigma_0 \) is a \( 2 \times 2 \) identity matrix. In this class of periodic systems, a quantized winding number, denoted as \( \nu \), is sought to achieve a topologically protected edge state. To guarantee such quantization, the diagonal of the Hamiltonian for the diatomic chain must be constant, suggesting that \( \sigma_3 \) does not play any role in the Hamiltonian [28]. This condition is already satisfied by setting equal capacitance \( C \), which ultimately returns \( h_3 = 0 \) as evident from Equation (13). The meaning of the winding number can be interpreted graphically from plotting \( h_1 \) versus \( h_2 \), showing the corresponding winding number associated with their oriented path, similar to elastic diatomic lattices [27]. (D) Natural frequency distribution of a finite acoustic chain of 40 chambers with open-open boundary conditions, showing the emergence of doubly degenerate edge states when the winding number is non-zero. All simulations show an excellent agreement between analytical and numerical results.

Note that, unlike elastic diatomic lattice that requires equal masses to achieve quantized winding numbers [27, 29, 30], the acoustic diatomic lattice requires constant capacitance (mechanically equivalent to the reciprocal of stiffness) and different inductance (mechanically equivalent to mass) to achieve said quantization. It is also worth noting that elastic diatomic lattice has identical behavior observed in Fig. 4C, albeit that the circle radius and distance are related to the lattice’s spring constants [27].

A quantized winding number dictates the emergence, or the lack thereof, of topological edge states at the boundaries of a chain with a finite number of unit cells, \( n \). As the boundary of a finite chain is well-defined, unlike the infinite case where the structure is theoretically unbounded, the distinction between the cases of \( \omega_2 > \omega_1 \) or \( \omega_2 < \omega_1 \) is
Figure 5: Natural frequency spectrum for three different acoustic diatomic lattices, showing the emergence of edge states only when $\omega_2 > \omega_1$, following the band topology analysis in Fig. 4. The mode shapes of these topological states are also depicted for the full-scale three-dimensional model and compared to the ones obtained from the analytical one.

unambiguous. Therefore, we consider a finite acoustic diatomic lattice that is terminated at channels with hydraulic diameter of $d_2$ from both ends. The boundary conditions at both ends are open, i.e., the pressure of peripheral chambers is connected to a node with zero oscillatory pressure via an $I_2$ inductor. Following a similar parametrization procedure to that in the unit cell analysis, the unforced response of the lattice, in frequency domain, is governed by:

$$[D - \omega^2 I] p(\omega) = 0$$  \hspace{1cm} (14)

where the degrees of freedom are the acoustic pressure in chambers and read:

$$p(\omega) = \{p_1 \quad q_1 \quad p_2 \quad q_2 \quad \ldots \quad p_n \quad q_n\}^T$$  \hspace{1cm} (15)

The matrix $D$ is a tridiagonal 2-Toeplitz matrix of size $2n \times 2n$ that dictates the dynamical characteristics of the acoustic system and is solely a function of $\omega_{1,2}$:

$$D = (\omega_1^2 + \omega_2^2)I_{2n} - \Omega$$  \hspace{1cm} (16)

where $I_{ij}$ is an identity matrix with its size indicated in the subscript and

$$\Omega = \begin{bmatrix} 0 & \omega_1^2 & 0 & \cdots & 0 \\ \omega_1^2 & 0 & \omega_2^2 & \cdots & \vdots \\ 0 & \omega_2^2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \omega_2^2 \\ 0 & \cdots & 0 & \omega_1^2 & 0 \end{bmatrix}$$  \hspace{1cm} (17)
It is worth noting that the structure of $D$ resembles the dynamical matrix of an elastic diatomic lattice with fixed-fixed boundary conditions [31]. Numerically solving the eigenvalues of matrix $D$ for swept values of $\Delta d$ (thus simultaneously changing $\omega_1$ and $\omega_2$) reveal the emergence of edge modes, which are pinned at:

$$\omega = \sqrt{\omega_1^2 + \omega_2^2}$$

only when $d_2 > d_1$ (or $\omega_2 > \omega_1$). The latter is expected in accordance to the non-vanishing winding number from band topology predictions. These analytical results are compared to the numerical ones obtained from COMSOL simulations. To avoid discrepancies in computed natural frequencies in the numerical problem, the length of both open-end channels must be adjusted by considering the effective length $\ell$ rather than the nominal theoretical one $\ell$. Recall that the effective length for a Helmholtz resonator’s neck is achieved from adding two corrections: one for the inner side (connected to the chamber with correction factor $\delta_{in} = 0.425 d_2/\ell$) and one for the outside open side (with correction factor $\delta_{out} = 0.3 d_2/\ell$). As such, the correction for end channels needs to be modified only for the outside (open) end, given that the inner side correction is already accounted for in the three-dimensional model. That is, the open channels at both lattice ends should have an effective length of $\ell (1 + \delta_{out})$. Taking a channel’s hydraulic diameter of $d_2 = 3.5 \text{mm}$ as a case in point, the effective length evaluates to $\ell_{e} = 5.05 \text{mm}$. Setting up the numerical problem as described above, the overall distribution of the natural frequencies from the numerical solution excellently match those from the analytical model (Fig. 4D). This is further emphasized from the mode shapes for the in-gap edge states, which shows wave localization at edges with strong wave attenuation away from them, as expected. Figure 5 also evinces the agreement in the mode shapes of the numerical three-dimensional model with their analytical counterpart. Note that the results in Fig. 5 are obtained using identical parameter set to that in Fig. 4A.

### 4 Acoustic honeycomb lattices

#### 4.1 Dispersion surfaces and unit-cell Hamiltonian

Next, consider an acoustic honeycomb lattice and its unit cell equivalent circuit model (Fig. 6A,B). We assign the chamber pressure of the first and second chambers of the $i, j$ unit cell to be $p_{i,j}$ and $q_{i,j}$, respectively, and the impedance of the channels to be $i \omega L$, where $L$ is the equivalent inductance of the connecting channels and is constant throughout (Fig. 6B). The two chambers of the unit cell, on the other hand, are assigned different heights $H_1$ and $H_2$, yielding distinct equivalent capacitance of $C_1$ and $C_2$, respectively. As such, the governing equations of a unit cell are given by:

\[
\begin{align*}
\left( \frac{3}{i \omega L} + i \omega C_1 \right) p_{i,j} - \frac{1}{i \omega L} \left( q_{i,j} + q_{i,j-1} + q_{i,j-1} \right) &= 0 \quad (19a) \\
\left( \frac{3}{i \omega L} + i \omega C_2 \right) q_{i,j} - \frac{1}{i \omega L} \left( p_{i,j} + p_{i+1,j} + p_{i+1,j} \right) &= 0 \quad (19b)
\end{align*}
\]

Applying Bloch theorem and introducing

\[
\begin{align*}
\omega_{1,2} &= \sqrt{1/(LC_{1,2})} \quad (20a) \\
\varepsilon &= 1 + e^{i \mu_+} + e^{i \mu_-} \quad (20b) \\
\mu_{\pm} &= \frac{1}{2} \left( \sqrt{3} \mu_y \pm \mu_x \right) \quad (20c)
\end{align*}
\]

the degrees of freedom in Equation (19) can be condensed and cast into a similar matrix form to Equation (9) with:

\[
H = \begin{bmatrix} 3 \omega_1^2 & 0 \\ -\omega_2^2 \varepsilon & 3 \omega_2^2 \end{bmatrix}
\]

\[
P_{i,j} = \begin{bmatrix} p_{i,j} \\ q_{i,j} \end{bmatrix}
\]

The dimensionless wavenumbers $\mu_{\pm}$ are functions of $\mu_x$ and $\mu_y$, defined as the wavenumbers in the $x$- and $y$- directions, respectively. By defining $E(\mu_+, \mu_-) = (\cos(\mu_+) + \cos(\mu_-) + \cos(\mu_+ - \mu_-) - 3)$, the associated eigenvalues of $H$ dictating the dispersion surfaces are expressed in the following compact form:

\[
\omega = \sqrt{\frac{3}{2} (\omega_1^2 + \omega_2^2) \pm \sqrt{\frac{9}{4} (\omega_1^2 + \omega_2^2)^2 + 2 \omega_1^2 \omega_2^2 E(\mu_+, \mu_-)}}
\]

(22)
It is observed that the Hamiltonian in Equation (21a) is not symmetric, thus cannot be expressed in terms of Pauli matrices. To achieve such a symmetric Hamiltonian and properly interpret topological properties, we introduce a new basis $\hat{p} = Qp_{i,j}$ parallel to the methodology presented in Ref. [30], with the definition of the transformation matrix $Q$ being:

$$Q = \text{diag} \left[ 1/\omega_1, 1/\omega_2 \right]$$

As such, we arrive at the following eigenvalue problem,

$$\hat{H}\hat{p} = \omega^2\hat{p}$$

where

$$\hat{H} = \begin{bmatrix} 3\omega_1^2 & -\omega_1\omega_2\varepsilon \varepsilon^* \\ -\omega_1\omega_2\varepsilon & 3\omega_2^2 \end{bmatrix}$$

The Hamiltonian in Equation (25) can be now expressed as a summation of Pauli matrices, i.e., Equation (12), with the following parameters:

$$h_0 = -3(\omega_1^2 + \omega_2^2)/2$$

$$h_1 = \omega_1\omega_2 (1 + \cos(\mu_+) + \cos(\mu_-))$$

$$h_2 = \omega_1\omega_2 (\sin(\mu_+) + \sin(\mu_-))$$

$$h_3 = -3(\omega_1^2 - \omega_2^2)/2$$

Figure 6: (A,B) Schematics of an acoustic honeycomb lattice, the unit cell definition, and the equivalent circuit model of the unit cell. Note that the change in the chamber height occurs symmetrically from both z-directions, such that the channels remain always centered with respect to the chambers. (C) Dispersion diagrams for $\Delta H = H_2 - H_1 = \pm 4.0$ mm showing that valley Hall transition, dictated by flipping the sign of the valley Chern number (blue circles are $+1/2$ and black circles are $-1/2$). The transition occurs exactly at $\Delta H = 0$ and a Dirac cone is generated. (D) The corresponding linearized Hamiltonian at the corners of the Brilloiun zone gives perfect cones at K (K') points.
A key parameter here is $h_3$ and its presence results in breaking the inversion symmetry $[32]$. The latter forces a frequency bandgap to open (when $\omega_1 \neq \omega_2$) and its limits are found from the solutions of dispersion relation at K point (or equivalently K' point), i.e., $(\mu^K_x, \mu^K_y) = (4\pi/3, 0)$, resulting in $\omega = \sqrt{3}\omega_{1,2}$ (Fig. 6C).

### 4.2 Valley Chern number

Breaking inversion symmetry and maintaining a third order rotational symmetry give rise to QVHE $[30, 32]$. The topological invariant quantifying topological transition is the valley Chern number $C_v$, which is calculated based on a linearized version of the Hamiltonian in Equation (25) near a Dirac cone. The latter is simply achieved by expanding the function $\epsilon$ via a multi-variable Taylor series near the K point,

$$\delta\epsilon = \epsilon + \frac{\partial\epsilon}{\partial\mu_x} \delta\mu_x + \frac{\partial\epsilon}{\partial\mu_y} \delta\mu_y$$

where $\delta\mu_x (\delta\mu_y)$ are wavenumbers measured from $\mu^K_x (\mu^K_y)$. Analogous procedure can be followed for the complex conjugate $\epsilon^\dagger$. Evaluating the expression at the K point and knowing that $\epsilon(\mu^K_x, \mu^K_y) = 0$, Equation (27) boils down to:

$$\delta\epsilon = -\frac{\sqrt{3}}{2} (\tau\delta\mu_x + \delta\mu_y)$$

where $\tau = +1 (-1)$ for the K(K') point. Subsequently, the linearized Hamiltonian, while ignoring the constant diagonal entries (i.e., $h_0\sigma_0$), can be written as $[33]$: 

$$\delta H = v_D (\tau\delta\mu_x \sigma_1 + \delta\mu_y \sigma_2) + m_D v_D^2 \sigma_3$$

where $v_D$ and $m_D$ are the Dirac velocity and effective mass, respectively. For our specific configuration, $v_D$ and $m_D$ are given by:

$$v_D = \frac{\sqrt{3}}{2} \omega_1 \omega_2$$
$$m_D = \frac{2(\omega_1^2 - \omega_2^2)}{(\omega_1 \omega_2)^2}$$

(30a)  
(30b)

Based on the linearized Hamiltonian $\delta H$, we examine the following eigenvalue problem:

$$\delta H \delta p = \delta\omega^2 \delta p$$

(31)

with the following eigenpair:

$$\delta\omega = \pm v_D \sqrt{(m_Dv_D)^2 + \delta\mu_x^2 + \delta\mu_y^2}$$
$$\delta p = \left\{ m_Dv_D^2 \pm |\delta\omega|^2 \right\} \frac{v_D (\tau\delta\mu_x + i\delta\mu_y)}{2(\omega_1 \omega_2)^2}$$

(32a)  
(32b)

It is important to point out that if $m_D = 0$, i.e., $\omega_1^2 = \omega_2^2 = \omega_0^2$, the eigenvalues in Equation (32a) describe perfect cones (Fig. 6D):

$$\delta\omega = \pm \frac{1}{2} \omega_0 \sqrt{3(\delta\mu_x^2 + \delta\mu_y^2)}$$

(33)

which has been similarly established for an in-plane Kagome elastic lattice $[27]$. The eigenvectors in Equation (32b) are crucial to calculate the Berry curvature, which, after normalizing $\delta p_{\pm}$, can be shown to be $[34]$: 

$$\mathcal{F}_\pm = \frac{\mp \tau m_Dv_D}{2(m_Dv_D^2 + \delta\mu_x^2 + \delta\mu_y^2)^{3/2}}$$

(34)

Finally, the integration of Berry curvature near a single valley yields a quantized Valley Chern number:

$$C_v = \frac{1}{2} \text{sgn}[m_D]$$

(35)
4.3 Finite lattice dynamics and waveguide design

One of the implications of having topological transition dictated by the sign of $C_v$ is to design robust interface modes and waveguides. This can be demonstrated by performing standard supercell dispersion analysis on a single strip of the honeycomb lattice with flipped order of the lattice chambers at a midway interface with a second dimension being infinite (Fig. 7A,B). This configuration results in a lattice with two parts having opposite signs of $C_v$, and, consequently, there should exist a single in-gap interface mode $^{[35]}$. Performing such analysis in a chain of 50 chambers and $\Delta H = H_1 - H_2 = \pm 4$mm, we observe an interface mode (depicted in orange in Fig. 7A) in both the analytical and numerical (three-dimensional) models, with an overall agreement. The nature of such interface modes is further confirmed by examining the mode shape; for example, Fig. 7C shows the absolute pressure of an in-gap mode at a reduced wavenumber of $0.6\pi$, which, again, shows an excellent agreement between the analytical and numerical models. Note here that the left/right boundaries are terminated at an open channel on both lattice’s ends and the length of peripheral channels are adjusted in a similar way to that of the diatomic lattice in Fig. 5.

Based on the supercell analysis, we now devise waveguides with L- and Z- shapes in a parallelogram-like lattice (Fig. 8). The acoustic lattice is built from $n = 25$ rows of diatomic lattice strips (as in Fig. 7B), with each strip having an $n$ unit cells, to form a parallelogram-like honeycomb lattice with a total of $n_l = 2n^2 = 1250$ chambers. We simulate two
waveguides of L- and Z-shapes (Fig. 8A,B) and compute the system response to an arbitrary excitation with a frequency of 2kHz at the right end of the waveguide. It is evident from the results that the pressure is localized at the waveguide, precisely as predicted from the supercell analysis in Fig. 7C. Both the numerical and analytical models are in perfect agreement, which demonstrates the power of the analytical model, while having a much smaller number of degrees of freedom, all without any apparent difference in the overall response.

5 Conclusions

This paper has established the use of electroacoustic analogies for designing acoustic topological lattices. The physical meaning of the parameters established from the effective Hamiltonian is related to Helmholtz resonance, allowing for a better estimation of its frequency based on geometrical properties. The benefit of the established platform is its less demand of numerical computation, in comparison to a full-scale model that typically increases the computation cost several folds. We show the potential and effectiveness of such approach via two examples of topological systems: (i) An acoustic diatomic lattice with edge states and quantized winding numbers. (ii) An acoustic honeycomb lattice with embedded topological waveguides emanating from QVHE. Both cases show excellent agreement between the analytical circuit model and the full-scale three-dimensional one, whether in finite lattice frequency response or unit-cell based analyses. The simplicity and computational efficiency of the established framework can be invaluable for designing various acoustic topological lattices in the future.

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