The Synchrotron Low-energy Spectrum Arising from the Cooling of Electrons in Gamma-Ray Bursts

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Abstract

This work is a continuation of a previous effort (Panaitescu) to study the cooling of relativistic electrons through radiation (synchrotron and self-Compton) emission and adiabatic losses, with application to the spectra and light curves of the synchrotron gamma-ray burst (GRB) produced by such cooling electrons. Here, we derive the low-energy slope $\beta_{LE}$ of a GRB pulse-integrated spectrum and quantify the implications of the measured distribution of $\beta_{LE}$. Radiative processes that produce soft integrated spectra can accommodate the harder slopes measured by CGRO/BATSE and Fermi/GBM only if the magnetic field lifetime $t_B$ is shorter than the time during which the typical GRB electrons cool to radiate below 1–10 keV, which is less than (at most) 10 radiative cooling timescales $t_{rad}$ of the typical GRB electron. In this case, there is a one-to-one correspondence between $t_B$ and $\beta_{LE}$. To account for low-energy slopes $\beta_{LE} > -3/4$, the adiabatic electron-cooling requires a similar restriction on $t_B$. In this case, the diversity of slopes arises mostly from how the electron-injection rate varies with time (temporal power-law injection rates yield power-law low-energy GRB spectra) and not from the magnetic field timescale.

Unified Astronomy Thesaurus concepts: Gamma-ray bursts (629); Non-thermal radiation sources (1119); Radiative processes (2055); Plasma astrophysics (1261)

1. Introduction

1.1. GRB Pulse Temporal Properties

Gamma-ray burst (GRB) observations (e.g., Fenimore et al. 1995; Norris et al. 1996; Lee et al. 2000) have established some essential/basic features of GRB pulses:

1. they peak earlier at higher energies;
2. they are time-asymmetric, rising faster than they fall, with a rise-to-fall time ratio $t_r/t_f$ in the range (0.1–0.9);
3. their temporal asymmetry is on average energy-independent; and
4. they last longer at lower energies, having a pulse-duration–energy dependence $\delta t_r \sim \epsilon^{-0.4}$.

Figures 5 and 6 of Panaitescu (2019, hereafter P19) provide a limited assessment of the ability of adiabatic and synchrotron (SY) electron cooling to account for the above pulse features:

1. peaks occurring earlier at higher energies are a trivial consequence for any electron-cooling process;
2. both cooling processes yield pulses that are more time-symmetric at higher energies, in conflict with observations of most GRB pulses; and
3. if the pulse duration $\delta t_r$ dependence on energy arises from only electron cooling, then, for a constant magnetic field, adiabatic-cooling yields $\delta t_r \sim \epsilon^{-0.4}$ (weaker than expected analytically) and synchrotron cooling leads to $\delta t_r \sim \epsilon^{-0.5}$ (as expected), both being compatible with GRB observations.

The geometrical curvature of the emitting surface leads to a spread in emission angles over the spherical surface of the GRB ejecta, increasing all observer-frame timescales by $\sim$50%. Additionally, it delays the arrival-time of a photon emitted (toward the observer) from the fluid moving at a larger angle (relative to its radial direction of motion) due to a longer path to observer and reduces its energy (due to a lower relativistic boost). Therefore, the integration of emission over the angle of the fluid motion softens continuously the received emission by delaying the arrival of photons of lesser energy.

Numerical calculations (P19) of GRB pulses show that the angular integration associated with the geometrical curvature of the emitting surface has the following effects on the pulse properties:

1. contributes to pulses peaking earlier at higher energies (which is the continuous emission softening described above);
2. mitigates the wrong trend of pulses to be more time-symmetric at higher energies when synchrotron cooling is dominant, because, in that case, the synchrotron-cooling timescale $t_{SY}$, being shorter than the adiabatic-cooling timescale $t_{ad} = 3 t_{ang}$ (see Section 3.2.2), is also (likely) smaller than the angular time-spread $t_{ang}$; thus, the pulse rise and fall timescales $t_r$ and $t_f$ are set by the angular integration, which does not induce an energy dependence of the ratio $t_r/t_f$;
3. is unable to compensate for pulses being more time-symmetric at higher energy when adiabatic-cooling is dominant because the angular time-spread $t_{ang}$ is smaller than the adiabatic-cooling timescale $t_{ad}$; thus, the pulse rise and fall timescales are not changed much by the angular integration; and
4. leads to pulses lasting longer at lower energies (owing to the progressive softening of the received emission) and induces a pulse-duration energy dependence $\delta t_r(\epsilon) \sim \epsilon^{-0.4}$ that is similar to that produced by each cooling process for a constant magnetic field.
The Astrophysical Journal, obtained from a decreasing with an emphasis on the diverse low-energy slopes that can be
extends to an angle larger than \( \Gamma^{-1} \) above. Therefore, the above evaluation of the pulse properties resulting when electron cooling is synchrotron-dominated applies only to GRB pulses that arise from bright-spots. However, given that the angular integration has little effect on the pulse properties when the electron cooling is adiabatic, the previous evaluation of those pulse properties is correct for both a bright-spot and an uniformly bright surface.

Consequently, if the trend of numerically calculated pulses being more symmetric at higher energies is firmly established, then its incompatibility with observations (for either electron-cooling process) favors the hypothesis that GRB pulses arise from a uniformly bright surface and that the electron cooling is synchrotron-dominated, i.e., disfavors a bright-spot origin for GRB pulses and an adiabatic-dominated electron cooling.

However, the pulse timescales and properties depend on the evolution of the electron injection rate \( R(t) \) and of the magnetic field \( B \) (the effect of monotonically varying such quantities is illustrated by the pulse shapes and durations shown in Figures 5 and 6 of P19); thus, a comprehensive numerical study of the pulse properties expected for various electron-cooling processes might (not guaranteed) identify evolving injection rates \( R(t) \) and magnetic fields \( B(t) \) that accommodate all of the basic GRB pulse features.

This work shows the effect of a power-law evolving injection rate \( R(t) \sim t^n \) on the GRB pulse-integrated spectrum, with an emphasis on the diverse low-energy slopes that can be obtained from a decreasing \( R(t) \) in the case of adiabatic electron cooling. A decreasing magnetic field \( B(t) \) is important for reconciling with observations the pulse-duration dependence on energy resulting when the electron cooling is dominated by scatterings at the Thomson–Klein–Nishina (T–KN) transition of the synchrotron photons below the peak energy \( E_\gamma \) of the GRB spectrum.

1.2. GRB Low-energy Spectrum

The GRB low-energy slope \( \beta_{\text{LE}} \) (of the energy spectrum below its peak-energy \( E_\gamma \)) is measured by fitting the GRB count spectrum with various empirical functions:

1. a pure power law;
2. a power law with an exponential cutoff (COMP), which is the Band function with a large high-energy spectral slope;
3. the Band function, which is a broken power law with a fixed width for the transition between the asymptotic power laws; and
4. a smoothly broken power law (SBPL), which has a free parameter for the width of the transition between the low- and high-energy power laws.

Preece et al. (2000) have analyzed 5500 pulse-integrated spectra at 25 keV–2 MeV of the 156 brightest (in peak flux or fluence) CGRO/BATSE GRBs, with 80% of bursts being fit with the Band and the SBPL functions, and have found a distribution for the low-energy slope of the pulse-peak spectra that is approximately a Gaussian

\[
F(\epsilon < E_\gamma) \sim e^{-\beta_{\text{LE}} \epsilon} P(\beta_{\text{LE}}) \sim \exp\left\{ -\frac{(\beta_{\text{LE}} - \beta_0)^2}{2\sigma^2} \right\}
\]

peaking at \( \beta_0 = 0.0 \) and with a dispersion \( \sigma \approx 0.40 \) (half-width at half-maximum of 0.45).

The adopted “parameter error” criterion by Poolakkil et al. (2021) for selecting the fitting function for Fermi/GBM peak-flux spectra at 10 keV–1 MeV leads to a bimodal distribution for the low-energy spectral slope \( \beta_{\text{LE}} \). The COMP, Band, and SBPL functions were used to fit the 1 s peak-flux spectra (for most pulses, they should be instantaneous spectra) of 1897 bursts, leading to an average spectral slope \( \beta_{\text{LE}} \) of \( 0.32 \pm 0.17 \). Power-law fits were used for the peak-flux spectra of 2287 bursts, leading to a median slope \( \beta_{\text{LE}} \) of \( -0.50 \pm 0.18 \).

That bimodal distribution of low-energy indices persists at the same soft and hard average values when Poolakkil et al. (2021) selected only one fitting model for each peak-flux spectrum based on the difference in the four models’ C-statistic (which assesses if a more complex model is warranted). That model selection was done with the aim to obtain the best estimate of the observed GBM spectral properties and resulted in fitting a larger fraction of the harder low-energy spectra to the COMP model than to the Band and SBPL models.

With a larger sample of 8459 time-resolved spectra (which can be either instantaneous or pulse-integrated, depending on the pulse duration) from 350 bright BATSE bursts, Kaneko et al. (2006) found a distribution of the low-energy slope \( P(\beta_{\text{LE}}) \) similar to that of Preece et al. (2000), displaying a peak at the same \( \beta_0 = 0.0 \) and a smaller dispersion \( \sigma = 0.27 \), and lacking the bimodality of the GBM distribution. The analysis of Kaneko et al. (2006) is similar to that of Poolakkil et al. (2021), as they both retain only fitting models that lead to lower parameter errors (the GOOD sample) and which have a higher statistical significance (the BEST sample). In contrast to the analysis of GBM bursts, most peak-flux spectra of BATSE bursts are fit with the COMP, Band, and SBPL, and not with the power-law model, for both the GOOD and BEST samples.

Poolakkil et al. (2021) attribute this bimodality to that many GBM peak-flux spectra have lower fluences, which makes the power-law model sufficient for spectral fitting (probably because the break to a softer spectrum above the peak energy \( E_\gamma \) was “lost” in the noise, which leads to a softer best-fit slope for the spectrum over the entire GBM window). Given that the bimodality of the \( P(\beta_{\text{LE}}) \) distribution for GBM bursts is compromised by the use of single power-law fits, we will make further use of that distribution for BATSE bursts, and we will forget (and forgive) that the peaks at \( \beta_0^{\text{hard}} = 1/3 \) and \( \beta_0^{\text{soft}} = -1/2 \) of the GBM bimodal distribution are exactly at the values expected for synchrotron emission from uncooled electrons and from a cooling-tail, respectively.

P19 presents an analytical derivation of the low-energy instantaneous synchrotron spectrum resulting for adiabatic, synchrotron, and inverse-Compton electron cooling. This work presents a numerical calculation of the instantaneous GRB spectrum for synchrotron and adiabatic electron cooling, and an analytical derivation of the low-energy pulse-integrated spectrum.

Here, we focus on the GRB low-energy slope distribution measured for the BATSE bursts and use the compatibility of
the resulting pulse-integrated low-energy spectrum slope and observations (Equation (1)) to set upper limits on the lifetime $t_B$ of the magnetic field, when the electron-cooling stops (if it is radiation-dominated) and when the production of synchrotron emission ends, both factors also having an effect on the brightness of the counterpart emission (Panaitescu & Vestrand 2022).

### 1.3. Limitations of the Standard Synchrotron Model

#### 1.3.1. The Low-energy Spectral Slope

An important shortcoming of the basic synchrotron model for the GRB emission is that it cannot account for low-energy slopes harder than the $\beta_{LE} = 1/3$ displayed by about:

1. one-third of CGRO/BATSE 25 keV–2 MeV time-resolved spectra (Preece et al. 2000);
2. one-tenth of the 30 time-integrated 2–20 keV spectra of X-ray flashes and GRBs observed by BSAX/WFC (Kippen et al. 2004) and BATSE; and
3. one-fourth of Fermi/GBM peak-flux 10 keV–1 MeV spectra of the BEST sample (Poolakkil et al. 2021).

Thus, if an as-of-yet unidentified large systematic error $\sigma(\beta_{LE}) \approx 0.3$ does not explain away the low-energy spectral slopes harder than $\beta_{LE} = 1/3$, then the following formalism for studying the effects of electron cooling on the GRB synchrotron emission is relevant for a majority of GRBs, but a deviation from that model (or another emission process) is needed for a substantial fraction of bursts.

The shortest departures from that model harden the low-energy slope to $\beta_{LE} = 1$ by relying on a very small electron pitch-angle $\alpha < \gamma^{-1}$ (with $\gamma$ being the electron Lorentz factor), i.e., a pitch-angle less than the opening of the cone into which the cyclotron emission is relativistically beamed, as proposed by Lloyd & Petrosian (2000), or on a very small length-scale for the magnetic field, $\lambda_B < \mu_L / \gamma$ (with $\mu_L$ being the electron gyration radius), so that electrons are deflected by angles less than $\gamma^{-1}$ and produce a “jitter” radiation, as proposed by Medvedev (2000). A hard slope $\beta_{LE} = 1$ is obtained if the GRB emission is the upscattering of self-absorbed lower-energy synchrotron photons (Panaitescu & Mészáros 2000), but the $e\gamma$ spectrum of the upscattered emission may be too broad compared to real GRB spectra.

In addition to these models that employ synchrotron emission and explain measured low-energy slopes harder than $\beta_{LE} = 1/3$, a photospheric blackbody component (proposed by, e.g., Mészáros & Rees 2000, used to account for most of the spectrum of GRB 090902B by Ryde et al. 2010, but being, in general, a subdominant component; e.g., Axelsson et al. 2012 for GRB 110721A) can yield low-energy spectra as hard as $\beta_{LE} = 2$, while a combination of synchrotron and thermal emission can lead to intermediate low-energy slopes $\beta_{LE} \in (1/3, 2)$ if the photospheric plus synchrotron GRB spectrum is fit with just the Band function. The issue of some measured low-energy slopes being too hard for the synchrotron model may be also alleviated by the addition of a power-law component to the Band (strongest component) plus thermal (weakest component) decomposition (e.g., Guiriec et al. 2015), although that has been proven for only a small number of bursts.

### 1.3.2. Deficiency of Our Treatment

A limitation of the following treatment of GRB pulses as synchrotron emission from a population of cooling relativistic electrons is that the effect of electron cooling on the pulse spectral evolution is calculated assuming that the typical energy $\gamma_I$ of the injected electrons is constant during the GRB pulse. Another default assumption (occasionally relaxed) is that the magnetic field $B$ is also constant. These assumptions are needed for an easier calculation of electron-cooling electrons, but they imply that the peak-energy $E_\gamma$ of the $e\gamma$ instantaneous spectrum is constant and so will be the peak-energy of the integrated spectrum, if the low-energy slope is harder than $\beta_{LE} = -1$.

However, measurements of the pulse spectral evolution (e.g., Crider et al. 1997; Ghirlanda et al. 2003) show that the peak-energy $E_\gamma$ decreases monotonically throughout the pulse.

Consequently, the following description of the spectral evolution due to electron cooling for a constant typical electron energy $\gamma_I$ and a constant magnetic field $B$ is representative for real GRBs displaying a decreasing peak-energy $E_\gamma$, only if that decrease of the best-fit $E_\gamma$ value is the artifact of fitting the curvature below $E_\gamma$ of real instantaneous spectra with an empirical function of free or fixed smoothness for the transition between the two (low- and high-energy) power laws.

#### 1.4. Magnetic Field Lifetime and Duration of Electron Injection

The GRB low-energy slope and the GRB pulse duration (as well as the GRB-to-counterpart relative brightness and counterpart pulse duration) depend on the magnetic field lifetime $t_B$ (real or apparent) and the duration over which relativistic electrons are injected into the region with a magnetic field.

For first-order Fermi acceleration at relativistic shocks, the duration $t_I$ of particle injection in the downstream region is the sum of the shock lifetime $t_{sh}$ (the time it takes the shock to cross the ejecta shell) and the duration it takes for a given particle to be accelerated, i.e., the time for it to diffuse (for a magnetic field perpendicular to the shock front) or to gyrate (for a magnetic field parallel to the shock surface) many times in the upstream and downstream regions and undergo multiple shock-crossings.

For magnetic fields generated by turbulence or two-stream instability (Medvedev & Loeb 1999) at relativistic shocks, which decay in the downstream region, the magnetic field intrinsic lifetime $t_B$ would be the shock lifetime $t_{sh}$. However, if the particle injection is impulsive (shorter-lived) relative to the shock life and lasts $t_I < t_B$, then the apparent magnetic field lifetime $t_B$ that a particle spends in the magnetic field region would be the time that it takes a particle to cross the downstream region where there is a magnetic field.

The above suggests that the durations $t_B$ and $t_I$ may be correlated if particles are accelerated and if magnetic fields are produced at relativistic shocks. For generality (i.e., to include other mechanisms that produce magnetic fields and relativistic particles, such as magnetic reconnection; Zhang & Huirong 2011; Granot 2016), we consider the two parameters $t_I$ and $t_B$ to be independent.

If electrons are re-accelerated (Kumar & McMahon 2008), the magnetic field lifetime $t_B$ used here can be seen as a surrogate for the re-acceleration timescale, as particles are
Electron Energy

| Symbol | Description |
|--------|-------------|
| $\gamma_i$ | typical energy of injected electrons |
| $\gamma_p$ | energy of electrons radiating at $\varepsilon_p$ |
| $\gamma_{\text{cr}}$ | critical electron energy, where $t_p(\gamma_{\text{cr}}) = t_{\text{ad}}$ |
| $\gamma_{\text{min}}$ | lowest energy of cooled electrons |

Spectral Quantities

| Symbol | Description |
|--------|-------------|
| $\beta_{\text{LE}}$ | GRB low-energy slope (below $E_\gamma$) |
| $E_\gamma$ | peak energy of GRB $\nu F_\nu$ spectrum |
| $\varepsilon$ | observing energy |
| $f_\varepsilon$ | spectral flux density |
| $\varepsilon_p$ | peak energy of the $F_\varepsilon$ SY spectrum |
| $E_{\gamma \text{m}}$ | SY energy for the $\gamma_m$ electrons |
| $\beta_{\text{cr}}$ | optical-to-gamma effective spectral slope |
| $F_\gamma$ | SY characteristic energy |
| $f_\nu$ | pulse-integrated spectral flux density |
| $f_{\text{FLUX}}$ | SY flux at $\varepsilon_p$ |
| $f_m$ | SY flux at $\varepsilon_m$ |

Electron Timescales

| Symbol | Description |
|--------|-------------|
| $t_{\text{ad}}$ | AD-cooling timescale |
| $t_{\text{sy}}$ | SY-cooling timescale |
| $t_{\gamma,i}$ | SY-cooling timescale for $\gamma_i$ electrons |
| $t_{\gamma,j}$ | transit-time from GRB $E_\gamma$ energy to $\varepsilon$ |
| $t_{\gamma,10k}$ | transit-time from GRBs to mid-X-rays (10 keV) |
| $t_{\gamma,10k}$ | epoch when $t_p(\gamma_{\text{cr}}) = t_{\text{ad}}$ |

Other Timescales

| Symbol | Description |
|--------|-------------|
| $t_B$ | magnetic field lifetime |
| $t_p$ | pulse peak epoch |
| $t_{\text{iang}}$ | angular spread in photon arrival-time |
| $\delta t_\epsilon$ | pulse duration at energy $\varepsilon$ |

allowed to cool only for that re-acceleration timescale; thus, our assumption that synchrotron emission stops at $t_B$ does not affect the following results about the GRB low-energy spectral slope (or the brightness of the prompt counterpart). However, if the GRB pulse duration is set by the magnetic field lifetime $t_B$, electron re-acceleration on a timescale $t_{\text{re-acc}} > t_B$ could lead to GRB pulses longer than $t_B$; thus, a finite magnetic field lifetime is not completely equivalent to the electron re-acceleration.

Table 1 lists the notations used most frequently here.

### 2. The Electron-cooling Law

For any cooling process, conservation of particles during their flow in energy can be written as

$$\frac{\partial N}{\partial \tau} + \frac{\partial}{\partial \gamma}(N \frac{d\gamma}{dt}) = N_i$$

(2)

where $N(\gamma) = dN/d\gamma$ is the particle distribution with energy, $N_i(\gamma_i < \gamma) \sim \gamma^{-p}$

(3)

is the distribution of the injected electrons, set to zero below a typical/lowest electron energy $\gamma_i$, and

$$-\frac{d\gamma}{dt} = Q(t) \gamma^n$$

(4)

is the electron cooling law for the corresponding cooling process. For ADiabatic cooling, $Q$ is a constant; for synchrotron cooling, $Q \sim B^2$, where $B$ is the magnetic field, because the SY power is proportional to the energy density of the virtual photons that are upscattered to SY photons. For self-Compton cooling, $Q \sim B^2 \tau$, where $\tau$ is the optical-thickness to electron scattering, because the inverse-Compton cooling power is proportional to the energy density of SY photons.

Above $\gamma_{\text{cr}}$, Equation (2) has a broken power-law solution, the cooled-injected electron distribution having a break at the electron energy $\gamma_{\text{cr}}$ where the radiative cooling timescale equals the time elapsed since the beginning of electron injection (AD-cooling does not yield a “cooling-break” because the AD-cooling timescale is slightly larger than the system age). Going to higher energies, the exponent of the cooled-injected distribution decreases by unity at $\gamma_{\text{cr}}$. The cooled-injected electron distribution is of importance for calculating the GRB spectral evolution and pulse shape (e.g., P19), but could also be relevant for the SY emission at lower energies, after electron injection stops and the injected electrons migrate toward lower energies, yielding a pulse decay, provided that $n \leq 1$, because that injected distribution shrinks to quasi-monochromatic for $n > 1$.

For the SY spectrum and pulse shape (light curve) at lower energies (X-ray and optical), we are interested in the cooled electron distribution below $\gamma_i$ (or cooling-tail)

$$N(\gamma_i < \gamma) \sim \gamma^{-m}.$$  

(5)

Substitution of that power-law cooling-tail in the conservation Equation (2) leads to $m = n$: the exponent of the cooling-tail distribution with energy is equal to the exponent at which the electron energy appears in the cooling power for any $n \neq 1$, provided that a certain condition (dependent on the radiative cooling process) is satisfied (P19). Adiabatic-cooling, for which $n = 1$, does not yield $m = n$. A solution-continuity argument (based on the assumption that if the above result $m = n$ is valid for $n > 1$ and $n < 1$, then it should also be valid for $n = 1$) seems reasonable but is wrong.

### 3. Synchrotron Cooling

Synchrotron electron cooling is governed by

$$-\frac{d\gamma}{dt} = \frac{P_\gamma(\gamma)}{m_e c^2} = \frac{\sigma_e}{6\pi m_e c^2} \gamma^2 B^2$$

(6)

where $\sigma_e$ is the cross section for electron scattering, and $B$ is the magnetic field strength. The photon SY characteristic energy at
which an electron radiates is
\[ \epsilon_{\text{sy}} = \frac{3 \hbar e}{16 m_e c^3} B \gamma^2. \] (7)

For a constant magnetic field \( B \), integration of Equation (6) leads to the lowest electron energy
\[ \gamma_m(t) = \gamma_i \left( 1 + \frac{t}{t_{\text{sy},i}} \right)^{-1}, \quad t_{\text{sy},i} \equiv t_{\text{sy}}(\gamma_i) = \frac{\gamma_i m_e c^2}{P_{\gamma}(\gamma_i)} \] (8)
for an initial electron energy \( \gamma_i \), where \( t_{\text{sy},i} \) is the SY-cooling timescale for the \( \gamma_i \) electrons. Then, the SY photon energy \( \epsilon_{\text{sy}} \) at which \( \gamma_m \) electrons radiate and the transit-time \( t_e \) for a \( \gamma_i \) electron radiating at the GRB peak energy \( E_{\gamma} \approx 100 \text{ keV} \) to cool to an energy for which its SY characteristic energy is \( \epsilon_{\text{sy}} = \epsilon \) are
\[ \epsilon_{\text{sy}}(t) = E_{\gamma} \left( 1 + \frac{t}{t_{\text{sy},i}} \right)^{-2} \rightarrow t_{\text{sy},i} \approx \left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} t_{\text{sy},i}. \] (9)

For later use, the SY-cooling law of Equation (6) can be written as
\[ -\left( \frac{d \gamma}{dt} \right)_{\text{sy}} = \frac{1}{t_{\text{sy},i} \gamma_i}, \] (10)
and the SY-cooling timescale for an electron of energy \( \gamma \) radiating at SY energy \( \epsilon \) is
\[ t_{\text{sy}}(\epsilon) = \frac{\gamma}{\left( \frac{d \gamma}{dt} \right)_{\text{sy}}} = \frac{\gamma_i t_{\text{sy},i}}{\epsilon E_{\gamma}} \approx \left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} t_{\text{sy},i}. \] (11)

Thus, the SY-cooling timescale for an electron of energy \( \gamma \) is the transit-time from GRB to the SY characteristic energy \( \epsilon(\gamma) \) at which that electron radiates.

3.1. Cooled Electron Distribution (Cooling-tail)

At \( t < t_{\text{sy},i} \), most electrons are at energies above \( \gamma_i \) and have a distribution with energy that shows the injected one
\[ (t < t_{\text{sy},i}) \quad N(\gamma_i < \gamma) \sim \frac{R_i t_{\text{sy},i}}{\gamma_i} \gamma^{-p} \] (12)
for a constant injection rate \( R_i \). At \( t_{\text{sy},i} < t < t_i \), if the magnetic field \( B \) is also constant, the cooled electron distribution of Equation (13) develops, and its normalization at \( \gamma = \gamma_i \) is constant because the number of electrons above \( \gamma_i \) is that injected in the last cooling timescale \( t_{\text{sy},i} \), \( N(\gamma > \gamma_i) = R_i t_{\text{sy},i} \), which is constant
\[ (t_{\text{sy},i} < t < t_i, \quad R_i \sim B^2): \quad N(\gamma_m < \gamma < \gamma_i) \sim \frac{R_i t_{\text{sy},i}}{\gamma_i} \gamma^{-2} \] (13)
with the lowest electron energy \( \gamma_m = m_e c^2 \) given in Equation (8). The above condition for a power-law cooling-tail is satisfied if the magnetic field energy density \( \sim B^2 \) is a constant fraction of the internal energy of relativistic electrons \( \sim n_e \gamma \) because the comoving-frame density of those electrons should satisfy \( n_e' \sim R_i \).

The growth of the above \( \gamma^{-2} \) cooling-tail is confirmed by numerically tracking the SY-cooling of electrons (Figure 1).

At \( t > t_i > t_{\text{sy},i} \), the electron density at the peak of the cooled electron distribution is
\[ N(\gamma_m) = \frac{R_i t_{\text{sy},i}}{\gamma_i} \gamma^{-2} \approx \frac{R_i (t + t_{\text{sy},i})^2}{\gamma_i t_{\text{sy},i}} \] (14)
where \( \gamma(t) \) is given in Equation (8). Therefore, after electron injection stops, the peak of the cooled electron distribution slides on the same cooling curve \( \gamma^{-2} \) (Figure 1). The width of the cooled electron distribution is
\[ \frac{\Delta \gamma}{\gamma_m} \sim \frac{\gamma_m(t - t_i)}{\gamma_m(t)} - 1 = \frac{t_i}{t + t_{\text{sy},i} - t_i}. \] (15)

Thus, after the end of electron injection, the cooled electron distribution shrinks, becoming asymptotically monoenergetic at energy \( \gamma_m \).

As shown in Figure 2 and in Figure 2 of P19, if the power-law cooling-tail condition \( R_i \sim B^2 \) is not satisfied, then the cooled electron distribution becomes harder if \( R_i \) increases or if \( B \) decreases faster than the power-law condition above. The former case leads to a GRB low-energy slope for the instantaneous spectrum that is harder than \( \beta_{\text{LE}} = -1/2 \), but the latter does not because the decreasing peak energy \( E_{\gamma} \) brings, at \( 10 \text{ keV} \), the high-energy softer SY spectrum. Conversely, if \( R_i \) decreases or if \( B \) increases, the distribution of cooled electrons becomes softer, yielding a spectral slope softer than \( \beta_{\text{LE}} = -1/2 \) for the instantaneous spectrum.

However, the hardening of the low-energy instantaneous spectrum for an increasing \( R_i \) is a transient feature and disappears after several cooling timescales \( t_{\text{sy},i} \), because it depends on the differential/relative time-derivative of the injection rate \( (dR_i/dt)/R_i \sim \gamma/t \) (for a power law \( R_i \sim t^{-1} \)), but lasts longer for faster evolving \( R_i \)'s, as shown by how fast the spectral slopes \( \beta_{\text{LE}} \) given in the legend of Figure 2 approach the asymptotic value \( \beta_{\text{LE}} = -1/2 \). For that spectral hardening to become persistent, the logarithmic derivative of \( R_i \) would have to be constant, which would mean an exponentially increasing electron injection rate \( R_i \).

Nevertheless, for an increasing \( R_i \), the hardening of the instantaneous spectrum lasts for a few/several cooling timescales \( t_{\text{sy},i} \), such as \( R_i \) yields a GRB low-energy slope for the integrated spectrum harder than \( \beta_{\text{LE}} = -1/2 \) if the SY emission is integrated over a duration not much longer than \( 10 t_{\text{sy},i} \).

3.2. Instantaneous Spectrum and Pulse Light Curve

3.2.1. Pulse Rise

The SY spectral peak flux \( f_p \sim B N_e \sim BR \lim_{t \rightarrow t_i} \min(t, t_i) \) at the photon energy \( \epsilon_p \), where the most numerous \( \gamma_m \) electrons radiate
The SY spectrum at a photon energy \( E < E_{\gamma} \) is

\[
I_{\gamma}(\varepsilon) \approx I_{\gamma}(E_{\gamma}) \left( \frac{\varepsilon}{E_{\gamma}} \right)^{1/3} \quad \varepsilon < \varepsilon_{p} (t < t_{\gamma})
\]

where \( t_{\gamma} \) is the epoch when the spectral peak energy \( \varepsilon_{p} \) reaches the observing photon energy \( \varepsilon \) (Equation (9)), and the last branch is due to the exponential cutoff of the SY function.

From the above three equations, it follows that, for an electron injection lasting shorter than the SY-cooling timescale \( t_{I} < t_{sy,i} \), the pulse light curve at \( \varepsilon < E_{\gamma} \) has a very slow rise and is characterized by the electron distributions of Figure 1. From Equation (9), the evolution of the spectral peak energy \( \varepsilon_{p} \) is approximately

\[
\varepsilon_{p}(t) \approx \begin{cases} E_{\gamma} & t < t_{sy,i} \\ E_{\gamma}(t_{sy,i}/t)^2 & t_{sy,i} < t \end{cases}
\]

The SY spectrum at a photon energy \( E < E_{\gamma} \) is

\[
f_{\gamma}(t) \approx \begin{cases} \left( \frac{E_{\gamma}}{\varepsilon_{p}} \right)^{1/3} & \varepsilon < \varepsilon_{p} (t < t_{\gamma}) \\ \left( \frac{E_{\gamma}}{\varepsilon} \right)^{1/2} & \varepsilon_{p} < \varepsilon (t_{\gamma} < t < t_{I}) \\ 0 & \varepsilon_{p} < \varepsilon (t_{I} + t_{\gamma} < t) \end{cases}
\]

The SY spectrum at a photon energy \( E < E_{\gamma} \) is

\[
I_{\gamma}(\varepsilon) \approx I_{\gamma}(E_{\gamma}) \left( \frac{\varepsilon}{E_{\gamma}} \right)^{1/3} \quad \varepsilon < \varepsilon_{p} (t < t_{\gamma})
\]

where \( t_{\gamma} \) is the epoch when the spectral peak energy \( \varepsilon_{p} \) reaches the observing photon energy \( \varepsilon \) (Equation (9)), and the last branch is due to the exponential cutoff of the SY function.

From the above three equations, it follows that, for an electron injection lasting shorter than the SY-cooling timescale \( t_{I} < t_{sy,i} \), the pulse light curve at \( \varepsilon < E_{\gamma} \) has a very slow rise and is characterized by the electron distributions of Figure 1. From Equation (9), the evolution of the spectral peak energy \( \varepsilon_{p} \) is approximately

\[
\varepsilon_{p}(t) \approx \begin{cases} E_{\gamma} & t < t_{sy,i} \\ E_{\gamma}(t_{sy,i}/t)^2 & t_{sy,i} < t \end{cases}
\]

The SY spectrum at a photon energy \( E < E_{\gamma} \) is

\[
f_{\gamma}(t) \approx \begin{cases} \left( \frac{E_{\gamma}}{\varepsilon_{p}} \right)^{1/3} & \varepsilon < \varepsilon_{p} (t < t_{\gamma}) \\ \left( \frac{E_{\gamma}}{\varepsilon} \right)^{1/2} & \varepsilon_{p} < \varepsilon (t_{\gamma} < t < t_{I}) \\ 0 & \varepsilon_{p} < \varepsilon (t_{I} + t_{\gamma} < t) \end{cases}
\]
The pulse should display a very slow rise from the end of electron injection at \( t_f \) and until the electron SY-cooling timescale \( t_{\text{sy},i} \). Most GRB pulses are peaky (resembling a double, rising-and-falling exponential or Gaussian; Norris et al. 1996); thus, the lack of the above slow rise indicates that \( t_f > t_{\text{sy},i} \), unless the magnetic field evolution shapes the pulse rise.

For an electron injection lasting longer than the SY-cooling timescale \((t_f > t_{\text{sy},j})\) or, equivalently, for a sufficiently low observing energy \( \epsilon < \tilde{\epsilon} \equiv E_{\gamma}(t_{\text{sy},i}/t_f)^2 \), the pulse light curve is

\[
\frac{f_{\epsilon}(t_f)}{F_p(t_{\text{sy},i})} = \left( \frac{\epsilon}{E_{\gamma}} \right)^{1/3}
\]

\[
\begin{cases}
\frac{\epsilon}{t_{\text{sy},i}} & t < t_{\text{sy},i} \quad \text{(rise)} \\
\left( \frac{\epsilon}{t_{\text{sy},i}} \right)^{5/3} t_{\text{sy},i} < t < t_f \quad \text{(fast rise)} \\
\frac{\epsilon}{t_{\text{sy},j}} \left( \frac{t_f}{t_{\text{sy},j}} \right)^{2/3} t_f < t < t_{\gamma} \quad \text{(slow rise)}
\end{cases}
\]

Lastly, for an electron injection lasting longer than the transit-time \((t_f > t_{\gamma})\) or for an observing energy \( \epsilon > \tilde{\epsilon} \), the first two rising branches of Equation (21) remain unchanged (with \( t_{\gamma} \) instead of \( t_f \)), and the third rising branch is replaced by a plateau

\[
f_{\epsilon}(t_{\gamma} < t < t_f + t_{\gamma}) = F_p(t_{\text{sy},j})\left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} \quad \text{(plateau)}
\]

for \( t_{\gamma} < t_f \). The constancy of the SY flux at \( t > t_{\gamma} \) is indicated in Figure 1 by the overlapping cooling-tails.

That GRB pulses do not display the plateau expected for \( t_f > t_{\gamma} \) indicates that the electron injection timescale \( t_f \) is not much larger than the transit-time \( t_{\gamma} \) from the spectral peak energy (of the pulse-integrated spectrum) \( E_{\gamma} \approx 100-200 \text{ keV} \) to an observing energy \( \epsilon = 25-100 \text{ keV} \). The constancy of the SY flux at \( t > t_{\gamma} \) is indicated in Figure 1 by the overlapping cooling-tails.

Alternatively, the underlying assumption of a constant magnetic field (or varying on a timescale \( t_{\gamma} > \max \{t_{\text{sy},i}, t_f\} \)) is incorrect. If the magnetic field lifetime \( t_{\gamma} < \min \{t_f, t_{\text{sy},i}\} \), then the pulse shape is determined by the evolution of \( B \), without any relation between the other timescales being implied by GRB observations.

### 3.2.2. Pulse Fall

After the transit-time \( t_{\gamma} \) (for \( t_f < t_{\gamma} \)) or after epoch \( t_f + t_{\gamma} \) (for \( t_f < t_f \)), all electrons radiate below the observing energy \( \epsilon \),
the flux received from the region of angular extent $\Gamma^{-1}$ moving toward the observer (the region of maximal relativistic boost $\Gamma$) is exponentially decreasing, and the flux received becomes dominated by the emission from angles larger than $\Gamma^{-1}$. This “larger-angle emission” (LAE) is progressively less enhanced relativistically, and its decay can be easily calculated if the observer-frame pulse peak-time $t_p$ is shorter than the angular spread in the photon arrival-time $t_{\text{ang}}$. In the case of a sufficiently short-lived emission, there is a one-to-one correspondence between the angle of emission and the photon arrival-time, so that the LAE decay is (Kumar & Panaitescu 2000)

$$f^{(\text{LAE})}_p(t > \tilde{t}_p) = f_{\text{pk}} \left( \frac{t}{\tilde{t}_p} \right)^{-2+\beta(\epsilon)}$$  \hspace{1cm} (fall) \hspace{1cm} (23)

where $\beta(\epsilon)$ is the spectral slope at the higher (and higher) photon energy that gets less (and less) Doppler boosted to the observing energy $\epsilon$,

$$f_{\text{pk}} = f_p(t_{\gamma}) = \left\{ \begin{array}{ll} F_p(t_I) & t_I < t_{\gamma,i} < t_{\gamma}\epsilon \\
F_p(t_{\gamma,i}) \frac{\beta}{t_{\gamma,j}} & t_{\gamma,j} < t_I < t_{\gamma}\epsilon \\
F_p(t_{\gamma,i}) \left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} & (t_{\gamma,i} < \gamma) t_{\gamma} < t_I \end{array} \right. \hspace{1cm} (24)$$

is the pulse peak-flux of Equations (20) and (21), and

$$\tilde{t}_p = t_p + t_{\text{ang}}, \quad t_{\text{ang}} = \frac{R}{2c^2}, \quad t_p \approx t_{\gamma}\epsilon$$  \hspace{1cm} (25)

are, respectively, the comoving-frame pulse peak epoch, after being stretched linearly by the spread in the photon arrival-time over the region of angular opening $\Gamma^{-1}$, the comoving-frame time-interval $t_{\text{ang}}$ corresponding to the observer-frame spread in the photon arrival-time $t_{\text{ang}}^{(\text{obs})} = R \beta^2 / 2 = R / 2c^2 \Gamma^2$, and the pulse peak-time $t_p$, as shown by the pulse light curves given in Equations (20) and (21).

If $t_p > t_I$, i.e., for any epoch well after the beginning of electron injection and of the SY emission), the integration over the spherical surface up to an angle $\theta = \Gamma^{-1}$ (beyond which the relativistic boost $D = 2\Gamma / (1 + \Gamma^2 \theta^2)$ decreases substantially) doubles the photon arrival-time $t_{\text{ang}}^{(\text{obs})}$ corresponding to $\theta = 0$ (i.e, from the fluid moving toward the observer). Thus, well after the initial adiabatic timescale, the angular integration increases $t_{\text{ang}}^{(\text{obs})}$ by 50%, on average, and it can be shown that the integration over the spherical surface of the photon arrival-time weighted by the received flux yields a relative increase by one-third.

For GRB pulses, the peak epoch is $t_p = t_{\gamma,i}$ and the peak flux is $f_{\gamma}(E_{\gamma}, t_{\gamma,i}) = F_p(t_{\gamma,i}) = F_p(t_I)$; thus, Equation (24) relates the low-energy pulse peak-flux $f_{\text{pk}}$ to the flux at the GRB pulse peak $f(E_{\gamma}, t_p)$, which is also the flux at the GRB peak-energy $E_{\gamma}$. The conclusion that the pulse peak-time $t_p$ is comparable to the SY-cooling timescale $t_{\gamma,i}$ is based on the lack of slowly rising and flat-top low-energy GRB pulses expected for a constant magnetic field and a constant electron injection rate. If the evolution of these quantities shapes the pulse light curve, then the pulse-peak epoch is $t_p = \max \{ t_I, t_{\gamma,i} \}$, as shown by Equations (20) and (22).

After noting that the comoving-frame angular timescale $t_{\text{ang}}$ is comparable to the AD-cooling timescale $t_{\text{ad}} = (3/2)R/c_{\gamma} = (3/2)R/(c^2 \Gamma)$, where $t_{\gamma,i}$ is the comoving-frame ejecta age, the condition that the electron cooling is SY-dominated $(t_{\gamma,i} < t_{\text{ad}})$ is equivalent to the angular timescale setting the pulse duration $(t_{\gamma,i} < t_{\text{ang}})$, as long as no other factors (e.g., duration of electron injection and magnetic field lifetime) determine the pulse duration. Thus, the pulse rise $t_r$ and fall $t_f$ timescales should always be comparable to $t_{\text{ang}}$ and GRB pulses should not be too time-asymmetric. Very asymmetric pulses, such as those with a measured ratio $t_r/t_f < 0.2$, require that the emitting surface extends much less than $\Gamma^{-1}$, i.e., the pulse emission arises from a bright-spot, and, as shown by numerically calculated pulses, a short electron injection timescale $t_{\gamma,i} \approx t_{\gamma,i}/10$ or a magnetic field evolving on a timescale $t_B < t_{\gamma,i}$ are responsible for the asymmetric pulse shape.

For GRB pulses, the slope $\beta$ in Equation (23) is that measured above the peak-energy $E_{\gamma}$. However, for low-energy (optical and X-ray) pulses, for which the pulse peaks at the transit-time $t_{\gamma,i}$, when a quasi-energetic cooled electron distribution “crosses” the observing energy, the above approximation of an infinitesimally short emission implies that, after $t_p = t_{\gamma,i}$, the pulse turns off exponentially. This is because there would not be any cooled electrons to radiate above the observing energy $\epsilon$ and whose emission would be (less and less) relativistically boosted to energy $\epsilon$.

Relaxing the approximation of an infinitesimally short emission, the LAE received after the peak $t_p = t_{\gamma,i}$ (if $t_I < \gamma t_I$) or after the plateau-end at $t_{\text{flat}} = t_{\gamma,i} + t_I$ (if $t_I < t_I$) will be the integral over the ellipsoidal surface of equal arrival-time, with emission from the fluid moving at larger angles relative to the outflow origin—observer direction radiating at earlier epochs, when the quasi-monoenergetic cooling-tail was radiating at a peak energy $E_{\gamma} \approx \epsilon$ (for $t_I < t_I$), and hence, $\beta(\epsilon) = 1/3$, or when the high-energy end of the cooling-tail was radiating at $E_{\gamma} < \epsilon$ (for $t_I < t_I$), and hence $\beta(\epsilon) = -1/2$.

Then, if the entire surface of the ejecta outflow is radiating at a uniform brightness, the LAE is that given in Equation (23) but with peak-time $t_p$ stretched by the angular time-spread $t_{\text{ang}}$:

$$\begin{align*}
\text{(fall)} & \quad f^{(\text{LAE})}_p(t > t_p) = f_{\text{pk}} \left( \frac{t}{t_{\gamma,i}} \right)^{-5/6} \quad t_I < t_{\gamma,i} \\
& = F_p(t_{\gamma,i}) \left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} \left( \frac{t_{\gamma,i}}{t_{\gamma}} \right)^{-5/2} \quad t_{\gamma,i} < t_I \quad (26)
\end{align*}$$

3.2.3. Pulse Light Curve

Equations (20), (21), and (22) provide both the instantaneous spectrum and the pulse rise or light curve at an energy below gamma-rays (a soft X-ray or optical pulse), for a constant electron injection rate $R$ and magnetic field $B$, and in the case of a bright-spot emission. The rise is followed by an

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1 This linearity can be easily proven by calculating the delay in arrival-time between a photon emitted at time $t$ by the fluid moving directly toward the observer and a photon emitted at time $t + \delta t$ by the fluid moving at angle $\Gamma^{-1}$ relative to the direction toward the observer. However, a different recipe for adding timescales will result if times are weighted by the intensity of the emission produced at that time and at a certain angular location.
exponential decay owing to the electron distribution having cooled to a quasi-monoenergetic distribution and to the lack of the LAE. For a surface of uniform brightness, the same equations give the pulse rise light curve if timescales are stretched by the angular time-spread $t_{\text{ang}}$ and Equation (26) gives the pulse power-law decay from the LAE.

The SY pulse light curves for SY-dominated electron cooling are also given in Equations (A5)–(A7) for inverse-Compton (IC)–dominated electron cooling with an inverse-Compton-power of exponent $n > 1$ (Appendix A), if one sets $n = 2$ and replaces the inverse-Compton-cooling timescale $t_{\text{IC}}$ with the SY-cooling timescale $t_{\text{SY,i}}$.

The above pulse light-curve equations show that the optical/X-ray pulse emission (instantaneous spectrum) displays a gradual softening, with the spectral slope 1/3 during the pulse rise evolving to $-1/2$, $-5/6$ after the pulse peak. The low-energy slope of the GRB pulses softens from an initial $\beta_{\text{LE}} = 1/3$ to $\beta_{\text{LE}} = -1/2$ after $(1 - 2)t_{\text{SY,i}}$, which may explain qualitatively the decrease of the count hardness-ratio measured for GRBs pulses (e.g., Bhat et al. 1994; Band et al. 1997).

### 3.3. Pulse-duration Dependence on Energy

If the pulse duration is set by radiative cooling (Equation (11)), then

$$
\delta t_r = t_{\text{SY}}(\epsilon) = \left( \frac{\epsilon}{E_{\gamma}} \right)^{-1/2} t_{\text{SY,i}} = t_{\gamma r} = t_p
$$

with the second-to-last equality following from Equation (9) and the last equality from Equation (25). The equality of the pulse duration with the pulse-peak epoch stands naturally for any pulse whose rise or fall is not too fast or too slow, which is the case of the pulse light curves given in Equations (20) and (21), and is an argument that applies to other cooling processes, not just SY.

Thus, for SY-dominated electron cooling, the pulse duration should decrease with energy, with the expected dependence\(^2\) $\delta t_r \sim \epsilon^{-1/2}$ being close to that observed for GRB pulses $\delta t_r \sim \epsilon^{-0.4}$. However, as discussed above, when the electron cooling is SY-dominated $t_{\text{SY,i}} < t_{\text{ad}}$, the pulse duration may be set by the spread $t_{\text{ang}} = t_{\text{ad}}/3$ in the photon arrival-time caused by the spherical curvature of the emitting surface because $t_{\text{ang}} > t_{\text{SY,i}}$. Thus, an immediate consistency between the pulse-duration dependence on energy $\delta t_r$ given in Equation (27) and GRB observations is readily achieved only if the angular timescale is not dominant, e.g., if the emitting region is a small bright-spot of angular extent much smaller than the “visible” $\Gamma^{-1}$ area moving toward the observer or if the pulse duration is determined by another timescale (duration of electron injection $t_I$ or magnetic field lifetime $t_B$) longer than the angular timespread $t_{\text{ang}}$.

Conversely, for a uniformly bright spherically curved emitting surface and for a radiative electron cooling, the pulse-duration dependence on energy $\delta t_r$ may not be set by the cooling timescale of that radiative process but by the continuous softening of the received emission induced by the differential relativistic boost (photons arriving later have less energy) of the emission from the region of angular opening $\Gamma^{-1}$ moving toward the observer (corresponding to the pulse rise) and of the LAE from the fluid outside that $\Gamma^{-1}$ region (corresponding to the pulse fall).

### 3.4. Pulse-integrated Synchrotron Spectrum

By integrating the above instantaneous spectra over the entire pulse, i.e., past the peak epochs, one obtains the pulse-integrated spectrum. Due to its fast decay, the contribution of the LAE is a small fraction of the integral up to the pulse peak epoch. For $t_r < t_I$, the flat pulse-plateau flux is dominant and trivially sets the slope of the integrated spectrum $F_\gamma \sim \epsilon^{-1/2}$. A more interesting situation occurs for $t_I < t_r$, where

$$
F_\gamma(t > t_r > t_I) = \int_0^t f_\gamma(t') dt' \approx \int_0^{t_r} f_\gamma(t') dt' \\
\approx F_p(t_{\text{SY,i}}) \left( \frac{\epsilon}{E_{\gamma}} \right)^{1/3} \int_0^{t_r} \left( \frac{t'}{t_{\text{SY,i}}} \right)^{2/3} dt'
$$

where $f_\gamma(t_r) = F_p(t_{\text{SY,i}})(t_r/t_{\text{SY,i}}) = f_p$ is equal to the constant flux $f_p(t > t_I)$ at the peak $\epsilon_p$ of the SY spectrum. (Integrating the instantaneous spectrum only until the pulse peak is a good approximation only for the emission from a bright-spot. If the emitting surface is of uniform brightness then, from Equation (26), one can show that the post-peak LAE fluence has the same spectrum $\epsilon^{-1/2}$).

Thus, although the pulse instantaneous spectrum is hard during the pulse rise, $F_\gamma(t < t_r) \sim \epsilon^{1/3}$ a much softer integrated spectrum $F_\gamma(t > t_r) \approx \left\{ f_\gamma(t_r) \right\} \propto \left\{ t_{\text{SY,i}} \right\}$ is obtained because the transit-time $t_{\gamma r} \sim \epsilon^{-1/2}$ over which the flux is integrated increases with a decreasing energy $\epsilon$. Adding that the pulse duration $\delta t_r$ should be comparable to the transit-time $t_{\gamma r}$, the above result suggests that the softness of the integrated spectrum can be seen as arising from the pulse-duration dependence on the observing energy.

Therefore, the pulse-integrated spectrum is $F_\gamma \sim \epsilon^{-1/2}$ irrespective of the ordering of electron injection time $t_I$ and electron transit-time $t_r$. This result was derived assuming a constant electron injection rate $R_e$, but it is valid even for a variable $R_e$, as shown in Figure 2 for a power-law electron injection rate. A spectral slope $\beta_{\text{LE}} = -1/2$ is about half-way on the soft side of the distribution of GRB low-energy slopes.

The SY-cooling-tail shown in Figure 1 illustrates the trivial fact that, for a long-lived electron injection, a GRB low-energy spectral slope harder than $\beta_{\text{LE}} = -1/2$ requires that electron cooling or, equivalently, the SY emission stops before the GRB-to-10 keV transit-time (Equation (11))

$$
t_{\text{SY,i}}(\gamma = 10^{-6}) = 3 E_{\gamma}^{1/2} t_{\text{SY,i}}
$$

if the electron injection rate $R_e$ is constant. On the other hand, Figure 2 suggests that the SY emission integrated up to $\approx 2t_{\gamma r} = 10^{-6}$ should have a slope $\beta_{\text{LE}} > 0$ for a rising $R_e(t)$. The same temporal upper limit on the electron cooling and SY emission is required by $\beta_{\text{LE}} > 0$ when electron injection lasts shorter than the GRB-to-10 keV transit-time as, otherwise, the soft integrated spectrum of Equation (28) holds. That fact is also illustrated by the spectral slopes given in the legend of...
Figure 2 for the \( t_I = 3t_{\text{sys}} \) case, which shows a soft spectrum if it is integrated longer than \( t_I \).

Therefore, a harder GRB low-energy slope \( \beta_{\text{LE}} \) requires that the magnetic field fades on a shorter timescale, and the low-energy slopes of the integrated spectra given in the legend of Figure 2 suggest that

\[
(SY) \beta_{\text{LE}} = \begin{cases} 
1/3 & t_B \lesssim t_{\text{sys}}, \\
(0, 1/3) & t_{\text{sys},i} \lesssim t_B \lesssim t^{(\text{sy})}_{-10k}, \\
(-1/2, 0) & t^{(\text{sy})}_{-10k} \lesssim t_B \lesssim 3 t^{(\text{sy})}_{-10k}, \\
-1/2 & 3 t^{(\text{sy})}_{-10k} \lesssim t_B 
\end{cases}
\]

This anticorrelation between the magnetic field lifetime \( t_B \) and the hardness of the GRB low-energy slope applies to any cooling process because it arises from the softening (decrease of peak energy) of the cooling-tail SY emission.

The above conclusion that harder GRB low-energy spectral slopes \( \beta_{\text{LE}} \) are the result of electrons not cooling below the lowest-energy channel (10–25 keV), offers a way to identify GRB pulses arising from bright-spots extending over much less than the visible region of the ejecta. In the absence of a substantial electron cooling and of a significant spread in photon arrival-time (due to the small angular extent of a bright-spot), the GRB pulse duration would be more time-symmetric at higher energies, and their duration would then be less dependent on energy.

### 4. Inverse-Compton Cooling

For a constant inverse-Compton (iC)-cooling timescale of the GRB \( \gamma_i \)-electrons, iC-cooling is governed by

\[
\frac{d\gamma_i}{dt} = \frac{P_{\text{ic}}(\gamma)}{\gamma_i m_e c^2} = \frac{1}{t_{\text{ic},i}} \gamma_i^{-\alpha \tau}, \quad t_{\text{ic},i} \equiv t_{\text{ic}}(\gamma_i) \tag{31}
\]

where \( t_{\text{ic},i} \) is the iC-cooling timescale of the GRB \( \gamma_i \) electrons. If the \( \gamma_i \) electrons scatter their own photons in the Klein–Nishina regime (\( \gamma_i E_i^\gamma > m_e c^2 \)), i.e., they cool mostly by scattering SY photons \( \epsilon < E_i \), at the T-KN transition, then their cooling begins with \( n = 2/3 \). After \( t_{\text{ic},i} \), the injected \( \gamma_i \) electrons cool by scattering at the T-KN transition photons produced by the cooling-tail, and the exponent changes to \( n = 1 \) (Table 2 of P19).\(^3\) If the \( \gamma_i \) electrons scatter their SY photons in the Thomson regime (\( \gamma_i E_i^\gamma < m_e c^2 \)), their cooling \( n = \min\{(p + 1)/2, 2\} \), which then changes progressively to \( n = \min\{(3p - 1)/4, 2\} \) and \( n = \min\{p, 2\} \) (Table 1 of P19).

The iC-cooled electron distribution (i.e., the solution to Equation (2) for \( N_i = 0 \)) is a power law with the same exponent \(-n\) as that of the iC power in Equation (31), \( N(\gamma < \gamma_i) = a(t) \gamma_{i,-1}^{-\alpha \tau} \), only if \( a(t) \simeq \gamma_{i,-1}^{-1} R_{\text{ic},i} = \text{const}, \) i.e., if \( a(t) \) is time independent. For SY-cooling, this condition becomes \( R_{\text{ic}} \sim B^2 \) (Equation (13)), which may have a good reason to be satisfied. For iC-dominated cooling, the same condition may be expressed as a relation between \( B, R_{\text{ic}}, \) and \( \gamma_i \) and has no obvious rationale.

If the above condition for a power-law cooling-tail is not satisfied, then the cooling-tail should be curved, with the local slope \( n \) depending on the evolutions of the injection rate \( R_{\text{ic}} \) and magnetic field \( B \), which could explain why the measured GRB low-energy spectral slopes \( \beta_{\text{LE}} \) have a smooth distribution encompassing the values for \( \beta_{\text{LE}} = -(n-1)/2 \) listed above.

#### 4.1. Instantaneous and Integrated Spectra

The SY instantaneous spectrum (equal to the pulse light curve) and integrated spectrum for iC-dominated electron cooling are derived in Appendix A, where a constant electron injection rate \( R_i \) and magnetic field \( B \) were assumed. Then, the condition for the growth of a power-law cooled electron distribution, \( t_{\text{ic},i} \sim R_{\text{ic}}^{-1} \), is equivalent to a constant cooling timescale \( t_{\text{ic},i} \) for the typical GRB electron of energy \( \gamma_i \).

Taken together, these three assumptions can easily be incompatible because the cooling timescale \( t_{\text{ic},i} \) depends on the injection rate \( R_i \) and magnetic field \( B \) (this is not an issue for SY-dominated cooling because, in that case, \( t_{\text{ic},i} \) depends only on \( B \)). Given that the iC-cooling timescale is \( t_{\text{ic},i} \sim t_{\text{sys},i}/ Y \sim (B^2 \tau)^{-1} \) with \( Y = P_{\text{ic}}/P_{\text{sys}} \sim \tau \) the Compton parameter and \( \tau(t) \sim \int_0^t R_i(t')dt' \) the electron optical-thickness to photon scattering, a constant \( t_{\text{ic},i} \) requires a decaying magnetic field \( B \sim \tau^{-1/2} \) that diverges at \( t = 0 \), when the electron injection begins and the optical-thickness is \( \tau = 0 \).

It is easy to recalculate the light curves and spectra that account for an evolving magnetic field \( B(t) \), which requires multiplying all break energies and spectral peak-flux densities by a factor \( B \). However, the evolution of the magnetic field that ensures the power-law cooling-tail condition \( R_i t_{\text{ic},i} = \text{const} \) depends on the iC-cooling regime for the \( \gamma_i \) electrons (the exponent \( n \) of the electron-cooling law in Equation (31)); thus, a generalized treatment is not possible. Furthermore, specializing results to a particular \( B(t) \) limits the usefulness (if any!) of the results.

Alternatively, one could assume a constant magnetic field, calculate the time-dependence of the cooling timescale \( t_{\text{ic},i} \sim R_{\text{ic}}^{-1} \) from the evolution of the iC-cooling power \( P_{\text{ic}} \sim \tau \), i.e., from the evolution of the scattering optical-thickness \( \tau \), and integrate the electron-cooling law (Equation (31)). However, the power-law cooling-tail condition \( t_{\text{ic},i} \sim R_{\text{ic}}^{-1} \) will not be satisfied (unless a variable \( B \) is allowed, as discussed above for a constant \( t_{\text{ic},i} \), and the SY spectrum above the lowest break-energy will not be a power law. Further use of that essential feature will lead to inaccurate results.

In conclusion, there is no generalized/comprehensive and accurate way to calculate analytically iC-cooling SY spectra and light curves. We return to all three constancy assumptions (for \( R_i, B, t_{\text{ic},i} \)), and recognize that the analytical results of Appendix A are only illustrative and of limited applicability.

If the power-law cooling-tail condition is satisfied, then the cooling-tail and its SY emission spectrum are:

\[
N(\gamma < \gamma_i) \sim \gamma_i^{-\alpha \tau} f_\epsilon (\epsilon < E_\gamma) = \frac{f_\epsilon}{E_\gamma^{\alpha \tau}} \left(\frac{\epsilon}{E_\gamma}\right)^{-(n-1)/2}, \tag{32}
\]

the latter result holding for \( n > 1/3 \) (if \( n < 1/3 \), the SY emission from the cooled electron distribution is dominated by the highest-energy \( \gamma_i \) electrons and is \( f_\epsilon \sim \epsilon^{1/3} \), but such a hard cooling-tail is not expected to arise).

Therefore, the SY instantaneous spectrum from the cooling-tail has a low-energy slope \( \beta_{\text{LE}} = -(n-1)/2 \). The smallest two values for the exponent \( n \) of the iC-cooling law are obtained if the \( \gamma_i \)-electrons cool weakly through scatterings of sub-GRB peak-energy photons at the T-KN transition. For the

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\(^3\) This \( n = 1 \) cooling-tail arising from scatterings at the T-KN transition has been identified also by Nakar et al. (2009) and Daigne et al. (2011).
The smallest exponent \( n = 2/3 \), the resulting slope \( \beta_{\text{LE}} = 1/6 \) is the hardest instantaneous SY spectrum arising from the cooling-tail and the only slope harder than the peak of the measured distribution \( P(\beta_{\text{LE}}) \). The next exponent \( n = 1 \) allows \( \beta_{\text{LE}} = 0 \), which is at the peak of \( P(\beta_{\text{LE}}) \). All other exponents \( n > 1 \) occur when the \( \gamma \)-electrons cool strongly by scattering photons in the Thomson regime, and yield slopes \( \beta_{\text{LE}} < 0 \), on the softer half of the measured distribution \( P(\beta_{\text{LE}}) \).

Equations (A3) and (A13) give the transit-time

\[
 t_{\gamma\gamma}^{(ic)} \simeq t_{\text{ic},i} \left\{ \begin{array}{ll}
 \frac{\epsilon}{E_\gamma} & n > 1 \\
 1 - \frac{\epsilon}{E_\gamma} & n < 1
 \end{array} \right. 
\]  

(33)

For \( n > 1 \) (electron cooling dominated by iC-scatterings in the Thomson regime), integration of the instantaneous spectrum over the pulse duration leads to an integrated spectrum of a similar low-energy slope \( \beta_{\text{LE}} = -(n-1)/2 \), irrespective of the duration \( t_B \) of electron injection relative to the gamma-to-X-ray transit-time \( t_{\gamma<10^k} \). Therefore, GRB low-energy slopes \( \beta_{\text{LE}} > 0 \) require that electrons do not cool below 10 keV, i.e., a magnetic field lifetime \( t_B \) shorter than the GRB-to-10 keV transit-time \( t_{\gamma<10^k} \approx 3 E_{12}^2 t_{\text{ic},i} \) for \( n = 2 \). The dependence of the integrated spectrum slope \( \beta_{\text{LE}} \) on the magnetic field lifetime is the same as for SY-cooling (Equation (30)) but with \( t_{\text{ic},i} \) instead of \( t_{\text{ic}} \).

For \( n < 1 \) (electron cooling dominated by iC-scatterings at the T-KN transition, with only \( n = 2/3 \) possible), Equations (A19) and (A19) show that the pulse instantaneous spectrum softens progressively but the spectral slope of the integrated spectrum is always that of the pulse rise: one-third if \( t_B < t_{\gamma} \) or one-sixth if \( t_{\gamma} < t_B \). Equation (A21) shows that, even when the soft pulse decay of spectral slope \( -(p-1)/2 \) is at maximal brightness \( (t_B > t_p) \), the pulse emission is from the \( \Gamma^{-1} \) region moving toward the observer, the integrated spectrum still has the harder pre-peak slope of one-sixth. For a magnetic field lifetime \( t_B < t_p \), when the pulse decay is the faster decaying LAE (because emission from the fluid moving at angles larger than \( \Gamma^{-1} \) relative to the observer is less beam formed relativistically), it is quite likely that the soft pulse-decay contribution to the pulse fluence is dominated by the pulse rise. Thus, the expected GRB low-energy spectral slope is

\[
(iC/T - KN: n < 1): \quad \beta_{\text{LE}} = \begin{cases} 
1/3 & t_B < t_{\gamma<10^k} \\
1/6 & t_{\gamma<10^k} < t_B < t_{2/3-1} \gamma_{<10^k} \approx 1/3 t_{\text{ic},i} \\
0 & t_{2/3-1} < t_B
\end{cases}
\]  

(34)

with \( t_{2/3-1} \) given in Equation (A22) and with the middle branch second condition \( (t_B < t_{2/3-1}) \) being effective only if \( t_{2/3-1} < t_B \). More realistically, the low-energy GRB spectral slope spans the range \((0, 1/3)\).

4.2. Pulse Duration and Transit-time

Integration of the iC-cooling law of Equation (31) allows for the calculation of the transit-time to a certain observing energy and of the pulse duration produced by the passage through the observing band of the SY characteristic energy of the electrons that produce the pulse peak. For an iC-cooling of exponent \( n > 1 \) (Appendix A), the pulse peak is set by the passage of the minimal energy of the SY spectrum from the cooling-tail, while for \( n < 1 \) (Appendix A), that epoch is set by the passage of the GRB \( \gamma \)-electrons after the end of electron injection at \( t_B \), provided that the electron-scattering (optical) thickness is approximately constant before \( t_B \) (i.e., for a sufficiently fast decreasing electron injection rate) and that the magnetic field is also constant.

For a constant cooling timescale \( t_{\text{ic},i} \), i.e., in the case of a constant magnetic field \( B \) and a constant electron scattering thickness \( \tau \), the pulse duration resulting from the electron iC-cooling is

\[
\delta t_p = \frac{\gamma \epsilon(n)}{d\epsilon} = \left( \frac{\gamma_i}{\gamma(n)} \right)^{n-1} t_{\text{ic},i} \left( \frac{\epsilon}{E_\gamma} \right)^{(1-n)/2} t_{\text{ic},i}
\]  

(35)

after using Equation (31).

For iC-cooling dominated by Thomson scatterings of \( \epsilon \approx E_\gamma \), SY photons \( n > 1 \), when the rate of electron cooling decreases faster with decreasing electron energy, pulses should last longer at lower energy: \( \delta t_p \sim \epsilon^{-(n-1)/2} \), which is consistent with GRB observations if \( n = 2 \). Thus, the pulse duration \( \delta t_p \) is the same as the transit-time \( t_{\gamma<10^k} \) (first branch of Equation (33)). If the electron injection lasts shorter than the transit-time \( t_B < t_{\gamma<10^k} \), Equation (A8) shows that the pulse peak-time \( t_p \) is equal to the transit-time \( t_{\gamma<10^k} \).

\[
(n > 1): \quad t_p = t_{\gamma<10^k} = \delta t_p \sim \epsilon^{(n-1)/2}.
\]  

(36)

For \( t_B > t_{\gamma<10^k} \), the pulse peak is at either \( t_{\gamma<10^k} \) or \( t_{\gamma<10^k} + t_B \) depending on the evolution of the injection rate and of the magnetic field.

If iC-cooling is dominated by T-KN scatterings \( n < 1 \) of \( \epsilon \ll E_\gamma \), SY photons, then the rate of electron cooling decreases more slowly with decreasing electron energy, and pulses should last shorter at lower energies: \( \delta t_p \sim \epsilon^{(n-1)/2} \approx \epsilon^{1/2} \) (for the one and only \( n = 2/3 \)), which contradicts the GRB observations.\(^4\) The pulse peak-time (Equation (A20)) is set by the transit of the higher-energy break \( \gamma_p \) after the end of electron injection, and the pulse duration \( \delta t_p \) is not equal to the transit-time \( t_{\gamma<10^k} \) (second branch of Equation (33)).

\[
(n < 1): \quad t_p = t_{\gamma<10^k} + t_B, \quad \delta t_p = t_{\text{ic},i} - t_{\gamma<10^k}.
\]  

(37)

Further investigations to identify the conditions under which the iC-dominated electron cooling may explain the observed trend of GRB pulses to last longer at lower energies are presented in Appendix A.

The first conclusion is that an increasing scattering optical-thickness \( \tau(t) \) affords some flexibility to the resulting energy dependence of the pulse duration \( \delta t_p \) for iC-cooling with \( n > 1 \), but for \( n < 1 \) pulses, it should last longer at higher energies, in contradiction with observations.

The second conclusion is that, for an iC-dominated cooling with \( n < 1 \), a decreasing magnetic field \( B \) should lead to a pulse-duration dependence on energy that is compatible with observations. Somewhat surprising, the pulse-duration dependence on energy is independent of how fast \( B(t) \) decreases; although, that result may be an artifact of some approximations.

\(^4\) The integration of emission over the equal arrival-time surface may induce a decreasing pulse duration with observing energy, and could reverse the above expected trend; thus, this limitation of iC-cooling applies mostly to the emission from a bright-spot.
The evolution of \( \tau \) does not play any role; however, how \( B(t) \) and \( \tau(t) \) evolve sets the GRB low-energy slope \( \beta_{\text{LE}} \).

The above conclusions are relevant for the SY emission from bright-spots of angular opening less than that of the “visible” region of angular extent \( \Gamma^{-1} \), when all pulse properties could be determined by the electron iC-cooling:

(i) For electron iC-cooling dominated by scatterings in the Thomson regime \( (n \approx 2) \), the same considerations apply for the pulse time-symmetry and pulse-duration dependence on energy as for SY-dominated electron cooling \( (n = 2) \): the faster pulse rise \( t^{1/3}, t^{1/3} \) implies a rise timescale \( t_{\text{R}} \) that is set by the iC-cooling timescale \( t_{\text{ke}} \) (Equations (A5)–(A7)), while the pulse-fall timescale \( t_{\text{F}} \) is set by the pulse peak-time \( t_{\text{p}} \), which is the transit-time \( t_{\gamma} \). Thus, electron iC-cooling should lead to pulses with a rise-to-fall ratio \( t_{\text{F}}/t_{\text{R}} \) that increases with energy, i.e., to pulses that are more time-symmetric at higher energies if pulses rise faster than they fall \( (t_{\text{R}}/t_{\text{F}} < 1) \), which contradicts the observations. However, the pulse-duration dependence on energy (Equation (35)) is consistent with measurements.

(ii) For iC-cooling dominated by scatterings at the T-KN transition \( (n = 2/3) \), the pulse rise \( (1 - t^{1/2})^{-3}, t, t/(1 - t/t_{\text{ke}}) \) is faster than the pulse-fall \( (1 - t/t_{\text{p}})^{3(p-1)} \) (Equations (A18) and (A19)); thus, a rise-to-fall ratio \( t_{\text{F}}/t_{\text{R}} < 1 \) independent of energy is expected, which is in accord with observations. However, pulses should last longer at higher energies (Equation (35)), which is inconsistent with observations.

Within the bright-spot emission scenario, the above incompatibilities may be solved by an evolving magnetic field; alternatively, those incompatibilities disappear if the GRB emission arises from a spherical surface of uniform brightness (in the lab-frame), in which case all pulse properties are determined by the spread in photon arrival-time and by the emission softening due to the spherical curvature of the emitting surface.

5. Adiabatic Cooling

For a constant radial thickness of the already shocked GRB ejecta, the adiabatic (AD) cooling of relativistic electrons is

\[
\gamma_m(t) = \gamma_0 \left(1 + \frac{t}{t_0}\right)^{-2/3} \rightarrow \varepsilon_m(t) \approx E_0 \left(\frac{t}{t_0}\right)^{-4/3}
\]

where \( \varepsilon_m \) is the SY characteristic energy \( \varepsilon_m(\gamma_m) \) (assuming a constant magnetic field); thus, the AD-cooling law is

\[
\frac{d\gamma}{dt}_{\text{ad}} = \frac{P_{\text{ad}}(\gamma)}{m_e c^2} = \frac{2}{3} \frac{\gamma}{t + t_0},
\]

and the AD-cooling timescale is

\[
t_{\text{ad}} = \frac{\gamma}{\frac{d\gamma}{dt}_{\text{ad}}} = \frac{3}{2} (t + t_0)
\]

for any electron energy. Equation (38) implies that, at the initial time \( t_0 \) (when electron injection begins), the electron transit-time from GRB emission to an observing energy \( \varepsilon \) is

\[
t_{\gamma_{\varepsilon}}^{\text{ad}} = t_0 \left(\frac{\varepsilon}{E_0}\right)^{3/4}
\]

for a constant magnetic field.

Unlike for SY and (most cases of) iC-cooling, for AD-cooling, where \( n = 1 \) \( (P_{\text{ad}} \sim \gamma) \), the conservation of Equation (2) does not determine the \( \gamma \)-exponent of the power-law cooling-tail. Instead, that exponent can be determined from the continuity of the cooling-tail \( N(\gamma < \gamma) \sim a(t) \gamma^{-m} \) and the cooled-injected distribution \( N(\gamma > \gamma) \sim A(t) \gamma^{-p} \) at the typical energy \( \gamma_i \) of the injected electrons, where \( p \) is the exponent of the injected electron distribution: \( N(\gamma > \gamma) \sim R_i \gamma^{-p} \).

Substitution of the above two power-law electron distributions in the conservation of Equation (2) and the use of the AD-cooling law of Equation (39) lead to

\[
ds \frac{dA}{dt} = -2(1 - m) \frac{a(t)}{3(t + t_0)} \rightarrow a(t) \sim (t + t_0)^{2(1 - m)/3}
\]

\[
dA \frac{dt}{dt} + \frac{2(3 - p)}{3(t + t_0)} A \sim R_i(t)
\]

\[
R_i \sim (t + t_0)^{-y} \rightarrow A(t) \sim (t + t_0)^{1-y}
\]

where a power-law injection rate \( R_i \) was assumed, to allow for an easy solving of the differential equation for \( A(t) \). The two functions \( a(t) \) and \( A(t) \) are continuous at \( \gamma_i \) only if they have the same time-dependence, which implies that

\[
m = \frac{1}{2} (3y - 1);
\]

thus, the slope of the cooling-tail depends on the evolution of \( R_i \). The slope of the cooling-tail instantaneous SY spectrum, \( \beta = d \ln f / d \ln \varepsilon = \min[1/3, -(m - 1)/2] \), is

\[
\beta_{\text{LE}} = \begin{cases} 1/3 & (y < 5/9, \ m < 1/3) \\ 3(1 - y)/4 & (y > 5/9, \ m > 1/3) \end{cases}
\]

For \( y > 5/9 \), the cooling-tail SY spectrum becomes softer for a faster-decreasing injection rate \( R_i \); for \( y < 5/9 \), the cooling-tail is harder than \( N(\gamma < \gamma_i) \sim \gamma^{-1/3} \), and its SY emission is outshined by that from the highest-energy \( \gamma \) electrons in the cooling-tail, leading to a hard \( \beta_{\text{LE}} = 1/3 \) spectrum. That is the case for a constant \( R_i \); \( y = 0 \rightarrow m = -1/2 \).

Equations (B6) and (B8) of Appendix B show that the instantaneous spectrum of AD-cooling electrons is harder during the pulse rise than during the pulse fall, with the pulse peak occurring at the time \( t_{\text{pe}} \) (if \( y > 1 \) when the photon energy \( \varepsilon_m = \varepsilon_{m}(\gamma_{im}) \) crosses the observing energy \( \varepsilon \) or at the time \( t_{\gamma} \) (if \( y < 1 \); Equation (B7)) when the higher-break energy \( \varepsilon_{p} \) of the last injected \( \gamma \)-electrons crosses \( \varepsilon \). For GRB spectra at the lowest observing energy \( (10 \text{keV}) \), these crossing epochs are

\[
t_{\gamma_{10k}}^{\text{ad}} \approx 5.6 E_{\gamma_{10k}}^{-3/4}, \quad t_{\gamma_{10k}}^{\text{ad}} = \frac{t_{\gamma_{10k}}^{\text{ad}}}{t_{\gamma_{10k}}^{\text{ad}} > t_{\gamma_{10k}}^{\text{ad}}},
\]

Equation (B10) shows that the SY spectrum integrated over the entire pulse has a soft slope \( \beta_{\text{LE}} = -3/4 \) (if the injected electron distribution has an index \( p > 5/2 \), which is softer than that of the instantaneous spectrum (Equation (46)) for a reason similar to that discussed above for the integrated spectrum from SY-cooling electrons.

Consequently, for the integrated spectrum of AD-cooling electrons to display a hard low-energy slope, the instantaneous spectrum must not be integrated past the crossing epochs \( t_{\gamma_{10k}}^{\text{ad}} \) and \( t_{\gamma_{10k}}^{\text{ad}} \), i.e., the SY emission must stop and the magnetic
field must disappear before the pulse-peak epochs \( t_p \), given in Equation (B9):

\[
(R_i \sim t^{-\gamma}): \quad \beta_{LE} = \frac{1}{3} \rightarrow t_B < \begin{cases} t_{ad}^{(ad)} \gamma > 5/9 \\ t_{10k}^{(ad)} \gamma < 5/9 \end{cases} \quad (48)
\]

The epoch \( t_B \) when the magnetic field fades out is before the natural pulse peak; thus, \( t_B \) becomes the pulse peak-date, after which the LAE emission describes the pulse decay, and the pulse duration \( \delta t \) has a weaker dependence on \( \epsilon \) than given below.

If the magnetic field lives longer than \( t_{10k}^{(ad)} \), then a softer spectrum results after the crossing of the lower-end energy \( \varepsilon_m \) of the cooling-tail SY spectrum

\[
(5/9 < \gamma < 2): \quad \beta_{LE} = \frac{3}{4} (1 - y) \in \left( -\frac{3}{4}, \frac{1}{3} \right) \rightarrow t_{10k}^{(ad)} < t_B < t_{10k}^{(ad)},
\]

and an even softer integrated spectrum is produced by the passage of the higher-end energy \( \varepsilon_p \) of the cooling-tail SY spectrum

\[
\beta_{LE} = -\frac{3}{4} \rightarrow t_B > \begin{cases} t_{10k}^{(ad)} \gamma > 2 \\ t_{10k}^{(ad)} \gamma > 5/2 \end{cases}
\]

\[ (50) \]

where \( p \) is the exponent of the injected electron distribution with energy. For an injected distribution with \( p < 5/2 \), the integrated spectrum is dominated by the emission from the injected electrons, as they cool after the end of electron injection, with AD-cooling preserving the slope of their distribution with energy: \( \Lambda(\varepsilon_p < \gamma) \sim \gamma^{-p} \); thus, \( \beta_{LE} = -(p - 1)/2 \in (-1/2, -3/4) \) for \( p \in (2, 2.5) \) is harder than for the last case above.

If the magnetic field lasts longer than the pulse-peak epoch \( t_p \) given in Equation (B9), then the pulse duration corresponding to the cooling law given in Equation (39) is

\[
\delta t = \frac{\gamma c}{\gamma - 1} t = t_p \sim \frac{3}{2} t_p \sim \left( \frac{\varepsilon}{E_\gamma} \right)^{-3/4} \begin{cases} t_t \quad y < 1 \\ t_o \quad y > 1 \end{cases}. \quad (51)
\]

Thus, AD-dominated electron-cooling should yield pulses whose duration decreases with the observing energy \( \epsilon \), as is observed, but the resulting dependence \( \delta t \sim \epsilon^{-3/4} \) is stronger than measured. However, Figure 5 of P19 shows that the numerically calculated displays a duration dependence on energy that is weaker than that in Equation (51) and consistent with that measured.

That the comoving-frame angular time-spread \( t_{ang} = R/(2c\Gamma) \) over the visible \( \Gamma^{-1} \) region of maximal relativistic boost (by a factor \( \Gamma \)) is always three times smaller than the current comoving-frame adiabatic timescale \( t_{ad} = 1.5 t = 1.5 R/(c\Gamma) \) implies that, for AD-dominated electron cooling, all pulse properties are determined by the electron cooling, and the above pulse-duration dependence on energy is accurate for either a bright-spot emission or a uniform brightness surface.

### 6. Synchrotron and Adiabatic Cooling

Equations (10) and (39) show that the SY and AD-cooling powers are equal at the critical electron energy

\[
\gamma_{cr} = \frac{2t_{sy,i}}{3(t + t_0)} \gamma_i \rightarrow \begin{cases} \gamma < \gamma_{cr}, \quad P_{ad} > P_{sy} \quad (AD - cool) \\ \gamma_{cr} < \gamma, \quad P_{ad} < P_{sy} \quad (SY - cool) \end{cases}
\]

Below the critical electron energy \( \gamma_{cr} \), electrons cool adiabatically, and the slope of the cooling-tail \( \Lambda(\varepsilon_m < \gamma < \gamma_{cr}) \) is determined only by the history of the electron injection rate \( R_i \). Above \( \gamma_{cr} \), electrons cool radiatively and the slope of the cooling-tail \( \Lambda(\gamma_{cr} < \gamma < \gamma) \) is set by the history of the electron injection rate \( R_i \) and of the magnetic field \( B \) (which sets the radiative cooling power).

At \( t_0 \), the typical \( \gamma_i \) electrons cool adiabatically if \( 3t_0 < 2t_{sy,i} \) and radiatively if \( 2t_{sy,i} < 3t_0 \). Appendix C shows that the solution (Equation (C5)) to the AD+SY electron cooling implies that, if the \( \gamma_i \) electrons are initially cooling adiabatically \( (\gamma_i < \gamma_{ad}(t = 0)) \), then their cooling remains adiabatic at all times \( (\gamma_{ad}(t) < \gamma_{ad}(t), \gamma_{ad}(t)(0) = \gamma_i) \), while if the \( \gamma_i \) electrons are initially cooling radiatively \( (\gamma_{ad}(t = 0) < \gamma_i) \), then their cooling switches from radiative to adiabatic after a “critical” time \( t_{cr} \) (Equation (C13)) defined by \( \gamma_{cr}(t_{cr}) = \gamma_{ad}(t_{cr}) \). Thus, in either case, the electrons cool adiabatically eventually, yet the exact electron-cooling law (Equations (C9)–(C11)) is close to (one-third of) that expected for SY-dominated cooling: \( \gamma_{ad}(t) \sim t^{-1} \) (Equation (8)).

It may be surprising that, if SY and AD electron cooling are considered separately, they lead to the opposite conclusion. The timescales for these two cooling processes, given in Equations (11) and (40), indicate that both cooling timescales increase linearly with time, but faster for AD-cooling \( (t_{ad} \approx 1.5t) \) than for SY \( (t_{sy}(\varepsilon) \approx \varepsilon) \). Consequently, if the \( \gamma_i \) electrons begin by cooling radiatively \( (t_{sy,i} < t_{ad}(t = 0) = 1.5 t_0) \), then \( t_{ad}(\varepsilon) < t_{sy}(\varepsilon) \) at any time; thus, the electrons cool radiatively at all times. Conversely, if the \( \gamma_i \) electrons cool adiabatically initially \( (1.5 t_0 < t_{sy,i}) \), then their cooling switches to SY-dominated at a (erroneous) critical time \( t_{cr} \) defined by \( t_{sy,i}(\varepsilon(t_{cr}) = t_{ad}(t_{cr}) \) which leads to \( t_{cr} = 2t_{sy,i} - 3t_0 \), after which \( t_{sy}(\varepsilon(t_0)) < t_{ad} \) and the electron cooling should become SY-dominated.

Thus, if the two electron-cooling processes are treated as acting independently, the electron cooling becomes radiative at late times irrespective of which cooling process was dominant initially, in total contradiction with the expectations from the solution to the double-process cooling, which shows that electron cooling should always become adiabatic eventually. The reason for this discrepancy is the computed (ab)use of the SY-cooling solution (Equation (A11)) in the calculation of the SY-cooling timescale (Equation (11)), which is correct only at early times and only if the electron-cooling begins in the SY-dominated regime, but is incorrect at later times, when the SY and AD-cooling timescales \( t_{sy}(\varepsilon(t)) \) and \( t_{ad}(t) \) are comparable and when the exact electron-cooling law (Equation (C5)) is inaccurately described by the SY-cooling of Equation (8).

Despite these fundamental differences in the expectations for the single- and double-process cooling, the asymptotic SY solution at late times overestimates the exact electron energy by only up to a factor of three. Thus, if one makes the mistake of using the SY-cooling solution whenever that process appears
dominant, the resulting error is an overestimation by up to an order of magnitude of the corresponding spectral break energies and by up to a factor of three of the corresponding transit-times.

The upper limits on the magnetic field lifetime \( t_B \) given in Equation (48) are valid if the cooling of the lowest-energy \( \gamma_m \) electrons (for \( \gamma > 5/3 \)) or of the GRB \( \gamma_i \) electrons after the end of electron injection (for \( \gamma < 5/3 \)), is described by the AD-cooling solution of Equation (38) until the corresponding transit-times in Equation (47). If the electron cooling is AD-dominated initially \( (t_i < t_{xy,i}) \), then it remains so at any later time. However, the AD-cooling law of Equation (38) remains valid only until the switch-time \( \tilde{t} \) defined in Equation (C12), after which the electron cooling is described by the one-third-SY solution, even though the electron cooling is AD-dominated. Thus, the results for the GRB low-energy slope \( \beta_{LE} \) of Section 5 are applicable if the crossing-times \( \tilde{\tau}_{\sim 10^4}^{(ad)} \) and \( \tilde{\tau}_{\sim 10^5}^{(ad)} \) are shorter than the switch-time \( \tilde{t} \), which lead to the same restriction: \( t_i, \ t_f < 0.2 \ t_{xy,i} \).

Therefore, AD-cooling sets alone the GRB pulse light curve and integrated spectrum if the radiative (SY) cooling timescale is at least an order of magnitude larger than the initial ejecta age \( t_e \) and the duration \( t_f \) of electron injection. The evolution of the electron distribution undergoing AD and SY-cooling, with the strength of AD-cooling increasing from \( t_i = 0.1 t_{xy,i} \) to \( t_f = 0.01 t_{xy,i} \), is shown in Figure 3 and supports the above conclusion.

The low-energy slope \( \beta \) of the GRB instantaneous spectrum depends on the location of the SY characteristic energy \( \gamma_{cr} \) relative to the lowest-energy channel (10 keV) of GRB measurements or, equivalently, the location of the electron critical energy \( \gamma_{cr} \) relative to the energy \( \gamma_X \sim \gamma_{cr} / 3E_{12}^{1/2} \) of the electrons that radiate at 10 keV. From Equation (52), it follows that, if the \( \gamma_{cr} \)-electron cooling begins AD-dominated \( (t_i \ll t_{xy,i}) \), then their cooling remains AD-dominated (i.e., \( \gamma_c > \gamma_c \)) until epoch \( t = (2/3) t_{xy,i} \) and the cooling of \( \gamma_{cr} \)-electrons remains AD-dominated until epoch \( t \approx 2 E_{12}^{-1/2} t_{xy,i} \). For \( t > 2 t_{xy,i} \), SY-cooling sets the cooling-tail \( N(\gamma_X < \gamma < \gamma_{cr}) \) energy distribution below the GRB peak energy \( E_X \), leading to a softening of the SY emission to the expected asymptotic slope \( \beta_{LE} = -1/2 \).

The legend of Figure 3 shows the expected gradual softening of the instantaneous spectrum at 10 keV, asymptotically reaching the expected slope \( \beta_{LE} = -1/2 \). Some of that softening is captured in the integrated spectrum slopes \( \beta_{LE} \) listed in the legend. Indicated photon energies are for \( z = 1 \).
7. Discussion

7.1. GRB Low-energy Slope $\beta_{LE}$ for SY Cooling

Equation (30) and numerical calculations (Figure 2) show that a low-energy (10 keV) GRB spectral slope $\beta_{LE} < 1/3$ of the pulse-integrated spectrum results from an incomplete/partial electron cooling due to the magnetic field lifetime $t_B$ being comparable to the GRB-to-10 keV transit-time $t_{\gamma,10keV}$ (Equation (29)) that it takes the typical GRB electron (radiating initially at the GRB spectral peak energy $E_* \sim 100$ keV) to cool to an energy for which the corresponding SY characteristic photon energy is 10 keV. More exactly, a slope $\beta_{LE} \leq 1/3$ results for $t_B \gtrsim 2 t_{\gamma,10keV}$, $\beta_{LE} \lessgtr 0$ requires that $t_B \sim (3-5) t_{\gamma,10keV}$, and $\beta_{LE} \lessgtr -1/2$ is obtained for $10 t_{\gamma,10keV} \lesssim t_B$.

For a constant electron injection rate $R_i$ and magnetic field $B$, SY-cooling over a duration longer than $3 t_{\gamma,10keV}$ leads to a soft slope $\beta_{LE} = -1/2$, irrespective of the duration $t_I$ over which electrons are injected:

(i) For $t_{\gamma,10keV} < t_I \lesssim t_B$, the electron distribution develops a cooling-tail with energy distribution $N(\gamma < \gamma_i) \sim \gamma^{-2}$ at $t > t_{\gamma,10keV}$ for which the SY instantaneous spectrum is $f_\gamma \sim \gamma^{-2}$ and the integrated spectrum is the same. That GRB pulses do not have a flat plateau at their peak, starting at the transit-time $t_{\gamma,10keV}$ and until the end of electron injection at $t_I$, which indicates that either $R_i$ or $B$ are not constant; and

(ii) For $t_I < t_{\gamma,10keV} < t_B$, a power-law cooled electron distribution does not develop; instead, that distribution shrinks to a monoenergetic one after $t_{\gamma,10keV}$. Integration of the SY instantaneous spectrum $f_\gamma(\gamma < \gamma_i) \sim \gamma^{-2}$ until the SY characteristic energy $\varepsilon_p$ (at which the cooled GRB electrons radiate) decreases below 10 keV leads to an integrated spectrum with the same low-energy slope $\beta_{LE} = -1/2$ as for a cooling-tail.

This coincidence arises from the fact that a cooling law $d\gamma_i/dt \sim \gamma^{-n}$ yields (i) a cooling-tail $N_i \sim \gamma^{-n}$ whose SY spectral slope is $\beta = -(n-1)/2$, and (ii) an electron-cooling $\gamma_i \sim t^{-(n-1)/2}$ for $n > 1$, a transit-time $t_{\gamma,10keV} \sim \varepsilon_p^{-(n-1)/2}$, and an integrated spectrum $F_\gamma \sim f_\gamma(t_{\gamma,10keV}) t_{\gamma,10keV} \sim t_{\gamma,10keV}^{-(n-1)/2}$.

SY electron cooling can yield cooling-tails harder (softer) than $N(\gamma < \gamma_i) \sim \gamma^{-2}$ and corresponding SY spectra harder (softer) than $\beta_{LE} = -1/2$ if electrons are injected at an increasing (decreasing) rate $R_i$. Because the hardness of the 10–100 keV SY spectrum is set by the electrons injected during the last few cooling timescales ($t_{\gamma,10keV} \sim 3 t_{\gamma,10keV}$), a variable electron injection rate $R_i$ can change the resulting cooling-tail only if the injection rate variability timescale is shorter than the transit-time $t_{\gamma,10keV}$. This means that an electron injection rate $R_i$ that varies as a power law in time, and which has a variability timescale equal to the current time, can alter the cooling-tail index only over a duration comparable to transit-time $t_{\gamma,10keV}$. Conversely, an electron injection rate $R_i$ that is a power law in time does not change significantly over the second $t_{\gamma,10keV}$ and leads to the standard slope $\beta_{LE} = -1/2$. Consequently, a variable electron injection rate $R_i$ can change the above magnetic field lifetimes $t_B$ by up to a factor of two.

Harder (softer) cooling-tails can also be obtained if the magnetic field $B$ decreases (increases), but a change in the low-energy slope $\beta_{LE}$ of the pulse-integrated spectrum is less feasible because a decreasing $B$ leads to a decreasing SY spectrum peak-energy $E_*$, which compensates for the effect that the decreasing magnetic field has on the hardness of the cooling-tail, while an increasing $B$ could lead to an increasing peak-energy $E_*$ that contradicts observations.

Thus, there is a direct mapping between the distribution of the GRB low-energy slope $P(\beta_{LE})$ and that of the magnetic field lifetime $P(t_B)$. The peak of the slope distribution $P(\beta_{LE})$ at $\beta_{LE} = 0$ implies that the lifetime distribution $P(t_B)$ peaks at $t_B \approx 3 t_{\gamma,10keV}$, which means that the generation of magnetic fields in GRB ejecta is tied to the cooling of the relativistic electrons.

The puzzling feature of the $\beta_{LE} - t_B$ correlation is that the distribution of slopes $\beta_{LE}$ does not exhibit peaks at the extreme values $\beta_{LE} = 1/3$ (corresponding to $t_B < t_{\gamma,10keV}$) and $\beta_{LE} = -1/2$ (corresponding to $t_B > 10 t_{\gamma,10keV}$), which may be explained in part by the statistical uncertainty $\sigma(\beta_{LE}) \approx 0.1$ of measuring the GRB low-energy slope $\beta_{LE}$.

7.2. GRB Low-energy Slope $\beta_{LE}$ for AD-Cooling

For AD-cooling, the cooling-tail distribution is determined by the only factor at play, the electron injection rate, assumed here to be a power law in time $R_i \sim t^{-\gamma}$, which provides all of the flexibility needed, but using only one parameter.

In contrast to SY-dominated electron cooling, where the dependence on the injection rate $R_i$ of the cooling-tail distribution is a transient feature, lasting for a few SY-cooling timescales $t_{\gamma,10keV}$, the power-law cooling-tail resulting for AD-cooling is a persistent feature because a substantial change in the rate $R_i$ is guaranteed to occur during an AD-cooling timescale, given that both timescales are the same (the current time). Similar to SY-cooling, for AD-dominated electron cooling, the passage of the peak energy of the SY spectrum from the cooling-tail leads to a softer integrated spectrum with $\beta_{LE} = -3/4$.

Consequently, AD-cooling allows for a range of spectral slopes for the instantaneous spectrum more easily than SY-cooling. That diversity is imprinted on the integrated spectrum if the cooling-tail contribution is dominant (which requires $\gamma \lesssim 2$) and if the magnetic field has a lifetime $t_B$ between the transit-times $t_{\gamma,10keV}$ (Equation (41)) and $t_B$ (Equation (B7)) corresponding to the low- and high-energy ends $\varepsilon_m$ and $\varepsilon_p$, respectively, of the cooling-tail spectrum crossing the observing energy.

Equation (49) for the GRB low-energy slope $\beta_{LE}(\gamma)$ shows that the measured distribution of the GRB slope $\beta_{LE} \in (-3/4, 1/3)$ (which is most of the range of GRB slopes) maps directly to the distribution of the exponent $\gamma \in (5/9, 2)$ of the electron injection rate $R_i \sim t^{-\gamma}$. $P(\beta_{LE})$ for a magnetic field lifetime $t_B \in (t_{\gamma,10keV,10keV})$ (Equation (47)). For a value of $t_B$ outside the above range, the GRB low-energy slope can be a hard $\beta_{LE} = 1/3$ (Equation (48)) or a soft $\beta_{LE} = -3/4$ (Equation (50)), with even softer slopes $\beta_{LE} < -3/4$ occurring if the integrated spectrum is dominated by the SY emission from GRB electrons of energy above $\gamma_i$.

As for SY-dominated electron cooling, this conclusion comes with two puzzles: it implies a correlation between the magnetic field lifetime $t_B$ and the cooling of the lowest- and highest-energy electrons in the cooling-tail via the GRB-to-10 keV transit-times (Equation (47)), and peaks in the $P(\beta_{LE})$ distribution at $\beta_{LE} = 1/3$ and $\beta_{LE} = -3/2$.

7.3. GRB Low-energy Slope $\beta_{LE}$ for Inverse-Compton Cooling

If the typical GRB electrons of energy $\gamma_i$ cool through scatterings (of SY photons produced same electrons) in the Thomson regime ($\gamma_i E_i \lesssim m_{e^-} c^2$), when the cooling-power exponent is $n \gtrsim 2$, the integrated spectrum shows the same
features and dependence on the magnetic field lifetime $t_B$ as for
SY-dominated electron cooling (for which $n = 2$): (i) crossing of the lowest-energy of the cooling-tail SY spectrum softens the integrated spectrum to the slope $\beta_{LE} = -(n-1)/2$, whether or not the electron injection lasts longer than the GRB-to-10 keV transit-time $t_{LE,10k}\lesssim 3 \, t_{ic}$, i.e., whether the cooling-tail develops down to an energy for which the SY characteristic energy is below 10 keV or shrinks to a monoenergetic distribution before reaching the observing energy; and (ii) hard GRB low-energy spectra require an incomplete electron cooling due to a short-lived magnetic field, lasting about the transit-time $t_{LE,10k}$ (Equation (A3)), and there should be a one-to-one correspondence between the GRB low-energy slope $\beta_{LE}$ and the magnetic field lifetime $t_B$, modulo a possible variation of the electron injection rate $R_t$, whose effect lasts only for about $t_{ic}^{(\gamma)}$, with $t_B \approx t_{ic}^{(\gamma)}$ accounting for the peak of the measured $P(\beta_{LE})$ distribution at $\beta_{LE} = 0$.

The IC-cooling of GRB electrons through scatterings at the T-KN transition ($\gamma(E,') > m_e c^2$), when $n = 2/3$, has (i) a similarity with the AD-dominated electron cooling ($n = 1$) in that an energy-wide cooling-tail persists after the end of electron injection; (ii) a similarity with the SY-dominated electron cooling ($n = 2$) in that the crossing of either end of the cooling-tail (at $t_{ic}^{(\gamma)}$ or at the pulse-peak epoch $t_p = t_{ic}^{(\gamma)} + t_I$) yields an integrated spectrum with the same slope $\beta_{LE} = 1/6$ as for the SY emission from the cooling-tail; and (iii) a unique feature in that, after the time $t_{2/3} = t_{ic}^{(\gamma)}$ of Equation (A22), the cooling-tail of exponent $n = 2/3$ is replaced by one with $n = 1$, provided that the electron injection lasts $t_I > t_{2/3} = t_{ic}^{(\gamma)}$, which leads to an instantaneous SY spectrum of slope $\beta = 0$ that yields an integrated spectrum of slope $\beta_{LE} = 0$ if the magnetic field lifetime satisfies $t_B > t_{2/3} = t_{ic}^{(\gamma)}$.

IC-dominated electron cooling with $n = 2/3, 1$ cannot lead to integrated spectra with a low-energy slope $\beta < 0$, because the contribution to the integrated spectrum from the GRB electrons above $\gamma_{ic}$ is smaller than that from the cooling-tail after the end of electron injection. Thus, one important feature of electron-cooling dominated by IC-scatterings at the T-KN transition (IC-dominated with $n \leq 1$) is that, without the diversity in slopes $\beta_{LE}$ allowed by a variable electron injection rate, it can explain only the harder half of the measured distribution of GRB low-energy slopes, with $\beta_{LE} \geq 0$ (Equation (34)). The hardest slope $\beta_{LE} = 1/3$ requires that $t_B > t_{ic}^{(\gamma)}$, and the slope $\beta_{LE} = 0$ at the peak of the $P(\beta_{LE})$ distribution requires that $(t_B, t_I) > t_{2/3} = t_{ic}^{(\gamma)}$.

8. Conclusions

The aim of this work is to examine the implications of the low-energy slopes $\beta_{LE}$ measured for GRBs by CGRO/BATSE and Fermi/GBM within a simple model where relativistic electrons (of typical energy $\gamma_{ic}m_ec^2$) in a magnetic field (B) produce SY emission in a relativistic source (of Lorentz factor $\Gamma$) and at some radius ($R$).

Low-energy slope of instantaneous SY spectrum. Slope depends on the dominant electron-cooling process (synchrotron, adiabatic, IC-scatterings) and on how much electrons cool during the magnetic field lifetime $t_B$. For electron cooling dominated by radiative processes (SY and IC), the timescale $t_B$ sets how long electrons cool and radiate. For AD electron cooling, the timescale $t_B$ determines only how long electrons radiate; they cool after $t_B$, but that is irrelevant if no emission is produced.

In addition to the dominant electron-cooling process, the energy distribution of the cooling GRB electrons (the cooling-tail) that sets the GRB low-energy spectral slope $\beta_{LE}$ also depends on the history of the electron injection rate $R_t$ and of the magnetic field $B$. Furthermore, $R_t(t)$ and $B(t)$ also determine the GRB pulse duration and shape. The initial assumption was that both quantities are constant until a certain time, $t_P$ and $t_f$, respectively. This simplification does not change much the ability of radiative processes with a cooling-power $P(\gamma) \sim \gamma^n$ of exponent $n \geq 2$ to account for the GRB low-energy slope $\beta_{LE}$. However, a variable injection rate $R(t)$ is essential for allowing the AD-dominated electron cooling to account for more than two values for the slope $\beta_{LE} = (1/3 \& -3/4)$ and for IC-dominated cooling through scatterings at the T-KN transition of the synchrotron photons of energy below the GRB peak-energy $E_p(n \leq 1)$ to accommodate GRB low-energy slopes softer than $\beta_{LE} = 0$.

Hardest low-energy slope. If GRB electrons do not cool well below their initial energy $\gamma_t$ or do not radiate SY emission while they cool below $\gamma_t$ (either being due to a magnetic field lifetime $t_B$ shorter than the initial electron-cooling timescale $t_{rad}$), the resulting slope $\beta_{LE} = 1/3$ of the instantaneous spectrum is the hardest that SY emission (not self-absorbed, sic!) can produce, which is a trivial fact.

Intermediate low-energy slope. Longer-lived magnetic fields yield softer slopes $\beta_{LE}$ for the integrated spectrum, with an anticorrelation between lifetime $t_B$ and slope $\beta_{LE}$ (longer lifetimes lead to softer slopes) existing for $t_B \in (1, 10) t_{rad} = (1/3, 3) t_{ic}^{(\gamma)}$, where $t_{ic}^{(\gamma)} \approx 3 \, t_{rad}$ is the transit-time for electrons to migrate from emitting SY radiation at $E_p \approx 100$ keV (the GRB peak-energy) to 10 keV.

Softest low-energy slope. For longer magnetic field lifetimes $t_B > t_{rad}$, the slope $\beta_{LE}$ of the instantaneous spectrum settles at an asymptotic value that depends on the dominant electron-cooling process: for radiative cooling with a cooling-power exponent $n$, the resulting slope is $\beta_{LE} = -(n-1)/2$ (for SY-cooling with $n = 2$, the slope $\beta_{LE} = -1/2$ is a textbook result), for AD-cooling, $\beta_{LE} = 0.75(1 - y)$ where $y$ is the exponent of the power-law electron injection rate $R_t \sim \gamma^{-y}$, provided that $5/9 < y < 2$ ($\beta_{LE} = 1/3$ for $y < 5/9$ and $\beta_{LE} = -3/4$ for $y > 2$).

Pulse-integrated spectrum. If electron injection lasts $t_I > t_{ic}^{(\gamma)}$, then the pulse-integrated spectrum has the same slope as the instantaneous spectrum for all radiative processes (another trivial fact), with a possible change from a cooling-tail with $n = 2/3$ to one with $n = 1$ for IC-cooling dominated by scatterings at the T-KN transition. For AD-cooling, the crossing of the lowest- or highest-energy electrons in the cooling-tail below the observing energy leads to a soft slope $\beta_{LE} = -3/4$.

If the electron injection lasts $t_I < t_{ic}^{(\gamma)}$, then, for radiative processes with $n \geq 2$, the cooling-tail width shrinks after the end of electron injection at $t_B$ and the passage of the quasi-monochromatic cooling-tail below the observing energy leads to a GRB pulse-integrated spectrum with the same low-energy spectral slope $\beta_{LE} = -(n-1)/2$ as for a long-lived electron injection. For IC-scatterings at the T-KN transition ($n \leq 1$) and AD-cooling, the cooling-tail width increases or remains constant, respectively, in log(energy), and the previous results for the integrated spectrum for a longer-lived electron injection remain unchanged.
Summarizing the above, a magnetic field lifetime \( t_B \in (1, 10) \) maps the GRB low-energy slope \( \beta_{LE} \in [-1/2, 1/3] \) if SY-cooling is dominant, \( \beta_{LE} \in \{ - (n - 1)/2, 1/3 \} \) if iC-cooling in the Thomson regime is dominant, \( \beta_{LE} \in [0, 1/3] \) if iC-cooling at the T-KN transition is dominant, and \( \beta_{LE} \in [-3/4, 1/3] \) if AD-cooling is dominant, with the softest values for the first two cooling processes applying to short-lived \( (t_I < t_{rad}) \) electron injections, and all softest values applying to long-lived \( (t_I > 10 t_{rad}) \) injections.

The measured distribution \( P(\beta_{LE}) \) for the low-energy slopes of the pulse-integrated spectra does not have peaks at the above extreme values: the hard \( \beta_{LE} = 1/3 \) and the soft \( \beta_{LE} = -1/2, -3/4 \). That discrepancy is alleviated in part by the typically reported statistical uncertainty \( \sigma(\beta_{LE}) \approx 0.1 \) in measuring the low-energy slope. Still, it is unlikely that spreading a multi-modal distribution of low-energy slopes with a kernel of dispersion 0.1 could lead to a smooth distribution \( P(\beta_{LE}) \) peaking at \( \beta_{LE} = 0 \), particularly on its soft side with \( \beta_{LE} < 0 \), displaying the largest gap being between the preferred values \( \beta_{LE} = 0 \) and \( \beta_{LE} = -1/2 \). Thus, absent some more substantial systematic errors in measuring the low-energy slope, the observed quasi-Gaussian \( P(\beta_{LE}) \) distribution requires that the distribution of magnetic field lifetimes \( (t_B) \) among GRB pulses is restricted to mostly \( t_B \in (1, 5) \) and peaks at \( t_B \approx 3 t_{rad} \) (which yields the peak of \( P(\beta_{LE}) \) at \( \beta_{LE} \approx 0 \), without a substantial fractions of pulses with \( t_B < t_{rad} \) or \( t_B > 5 t_{rad} \).

At this point, such a correlation between the magnetic field lifetime \( t_B \) and the cooling timescale of GRB electrons \( t_{rad} \) is unwarranted and puzzling.

\[
\frac{\gamma_m}{\gamma_m} = \frac{\gamma_m(t - t_I)}{\gamma_m(t)} = \left( 1 - \frac{t_I}{t_{ic,i} + t} \right)^{-1/(n - 1)} \approx 1 + \frac{1}{n - 1} \frac{t_I}{t} \quad (A2)
\]

The conclusion that intermediate GRB slopes \( \beta_{LE} \approx 0 \) require an incomplete electron cooling (meaning that electrons cool for a time \( t_B \) that ranges from less than one cooling timescale \( t_{rad} \) of the typical GRB electron to at most \( 10 t_{rad} \) is also suggested by the work of Kumar & McMahon (2008), who analyzed the five-dimensional model parameter space for the hard \( \beta_{LE} = 1/3 \) and soft \( \beta_{LE} = -1/2 \) GRB low-energy slopes, but considering that the electron cooling stops after a re-acceleration timescale, which has the same effect on electron cooling as the disappearance of the magnetic field used here. At first sight, none of the possible mechanisms for particle acceleration and magnetic field generation (at shocks, by instabilities, through magnetic reconnection) offer a reason for a correlation between partial electron cooling on a timescale \( t_B \) and the electron cooling timescale \( t_{rad} \).

Appendix A
Spectra and Light Curves of Synchrotron Emission from Inverse-Compton Cooling Electrons

Inverse-Compton cooling comes in two flavors:
(1) Strong/fast cooling with an exponent \( n > 1 \), similar to SY-dominated cooling, where (1a) the electron energy decreases like a power law in time, (1b) the cooled electron distribution shrinks to quasi-monoenergetic after electron injection ends (the higher the exponent \( n \), the faster the cooling-tail shrinks), and (1c) the passage of the peak energy of the SY spectrum of the cooling-tail through the observing band softens the integrated spectrum; and
(2) Weak/slow cooling with \( n < 1 \), similar to AD-dominated cooling \( (n = 1) \) in that the cooling-tail persists after the end of electron injection and the width (in log space) of that cooling-tail is practically constant. However, it is different from an AD-cooling-tail in that (2a) the electron cooling is slower than a power-law in time, and (2b) the passage of the cooling-tail through the observing band does not lead to an integrated spectrum significantly softer than the instantaneous spectrum.

A.1. Strong iC-Cooling \( (n > 1) \) through Thomson Scattering

This case is a generalization of SY-dominated cooling \( (n = 2) \) and is relevant for inverse-Compton cooling when the \( \gamma_e \)-electrons scatter their own SY photons in the Thomson regime \( (n \geq 2) \). Integrating the iC-cooling law of Equation (31), one obtains

\[
\gamma_m(t) = \gamma_i \left( 1 + \frac{t}{t_{ic,i}} \right)^{-1/(n - 1)}, \quad t_{ic,i} \equiv \frac{\gamma_i m_e c^2}{P_{ic}(\gamma_i)} \sim \gamma_i^{-1 - n} \quad (A1)
\]

if the iC-cooling timescale \( t_{ic,i} \) of the \( \gamma_i \) electrons is time independent.

After the end of electron injection \( (t > t_I > t_{ic,i}) \), Equation (A1) shows that the bounds of the cooling-tail \( (\gamma_m(t_I) \) for the electrons injected initially and \( \gamma_m(t_I - t_I) \) for the electrons injected at \( t_I \) and its width evolutions are

\[
\frac{\Delta \gamma}{\gamma_m} = \frac{\gamma_m - \gamma_m}{\gamma_m} \sim \frac{t_I}{t} \ll 1, \quad (A2)
\]

meaning that the cooling-tail becomes quasi-monoenergetic.

The SY peak-energy for the lowest-energy electrons \( \gamma_m \) and the transit-time from GRB peak-energy to an observing energy \( \varepsilon \) are

\[
\varepsilon_m(t > t_{ic,i}) \sim E_{\gamma} \left( \frac{t_{ic,i}}{t} \right)^{2/(n - 1)}, \quad t_{ic,i} \sim t_{ic,i} \left( \frac{\varepsilon}{E_{\gamma}} \right)^{-(n - 1)/2} \quad (A3)
\]

while the SY peak flux at \( \varepsilon_m \) is

\[
\begin{align*}
F_p(t_{ic,i}) & = F_p(t_I) \quad t_{ic,i} < t_I < t_{ic,i} \varepsilon_m \sim F_p(t_{ic,i}) \quad t_I < t \quad \varepsilon_m \sim F_p(t_{ic,i}) \quad t_{ic,i} < t_I < t_I (A4)
\end{align*}
\]

where \( F_p(t_I) \) and \( F_p(t_{ic,i}) \) are the GRB peak flux (the energy density at the spectral peak or at the pulse peak). The first and third branches show the linear increase of the number of electrons radiating at the GRB peak energy \( E_\gamma \) (for a constant electron injection rate), the second and fifth branches arise from...
the constant number of electrons radiating at the peak energy $\varepsilon_m$ after the end of electron injection, and the fourth branch arises from the linear increase of the number of electrons radiating at $\varepsilon_m$, with the flux being independent of the exponent $n$ of the cooling power, all cases assuming a constant magnetic field.

Adding the SY spectrum of Equation (19), but with the slope $\beta = -(n - 1)/2$ above the peak-energy $\varepsilon_m$, and the LAE emerging at $t > t_{\gamma e}$, $t_{\gamma e} + t_f$ (owing to the exponential cutoff of the synchrotron function and to the quasi-monoenergetic cooling-tail), the resulting instantaneous spectrum and pulse light curve at an observing energy $\epsilon < \varepsilon_s$ (below the GRB spectrum peak energy) are

$$f_{\epsilon}^\text{(LAE)}(t) = f_{\epsilon}^\text{FLA}(t) \times \begin{cases} \frac{1}{E_s} t_{\gamma \epsilon} & t < t_{\epsilon, i} \\ \frac{1}{E_s} t_{\gamma \epsilon}^{-(n-1)/6} & t_{\epsilon, i} < t < t_{\gamma \epsilon} \\ \frac{1}{E_s} t_{\gamma \epsilon}^{-(n-1)/6} & t > t_{\gamma \epsilon} \end{cases}$$

The LAE flux above was calculated by assuming that its asymptotic flux decay $f_{\epsilon}^\text{(LAE)} \sim t^{-2+\beta}$ is continuous at the pulse peak of Equation (A8)

$$f_{\epsilon}^\text{(LAE)}(t > t_p) = f_{\epsilon}^\text{FLA}(t_p) \times \begin{cases} \frac{1}{E_s} t_{\gamma \epsilon}^{-(n-1)/6} & t < t_{\gamma \epsilon} \\ \frac{1}{E_s} t_{\gamma \epsilon}^{-(n-1)/6} & t > t_{\gamma \epsilon} \end{cases}$$

From Equations (A5) and (A6), for a sufficiently short electron injection ($t_f < t_{\gamma e}$) or a sufficiently low observing energy $\epsilon$, the SY spectrum integrated until after the pulse peak-time $t_{\gamma e}$ isafter using Equation (A3), with the pre-pulse peak and post-pulse peak (LAE) fluxes having comparable contributions to the pulse fluence.
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\[ \mathcal{F}_c(t > t_{\gamma e} > t_l) \simeq t_p f_{pk} \simeq t_{\gamma e} \times \left[ \frac{\epsilon}{E_{\gamma}} \right]^{(n-1)/2} \frac{(t_{\gamma e})^{2/(3n-3)}}{t_{ic,i}} \]

Thus, for \( t_l < t_{\gamma e} \), the addition of instantaneous \( f_c \sim \epsilon^{1/3} \) hard spectra until the transit-time \( t_{\gamma e} \), when the typical energy \( \epsilon_m \) of the SY emission from the quasi-monoenergetic cooling-tail crosses the observing energy \( \epsilon \), leads to a much softer integrated spectrum \( \mathcal{F}_c(t < t_{\gamma e}) \sim \epsilon^{-(n-1)/2} \). Furthermore, it can be shown that the addition of instantaneous \( f_c \sim \epsilon^{-5(n-1)/6} \) LAE soft spectra after the transit-time \( t_{\gamma e} \) leads to a harder integrated spectrum with the same spectral slope \( -(n-1)/2 \).

Thus, the passage of the peak energy \( \epsilon_m \) of the cooling-tail SY emission softens the contribution of the pre-pulse-peak emission to the integrated spectrum, and hardens the contribution of the post-pulse-peak emission, bringing them to the same slope as the instantaneous spectrum because the former is the integrated spectrum with the same spectral slope as the instantaneous SY spectrum while electrons were injected at \( t < t_l \). This coincidence, which is also obtained for a weaker cooling process of exponent \( n < 1 \) (next section), arises from the correlation between the evolution of the peak-energy \( \epsilon_m \) (Equation (A12)) and the SY spectral slope \( \beta(>\epsilon_m) \), both of which are set by the exponent \( n \) of the electron-cooling law.

In the case of a sufficiently long electron injection \( (t_l > t_{\gamma e}) \) or a sufficiently high observing energy \( \epsilon \), Equation (A7) gives

\[ f_m(t < t_l) = F_p(t)\left(\frac{\epsilon_m}{E_{\gamma}}\right)^{1/6}, \quad F_p(t < t_l) = F_p(t)\frac{t}{t_l} \rightarrow f_m(t < t_l) = F_p(t)\frac{t}{t_l} \left(1 - \frac{t}{t_{ic,i}}\right) \] (A14)

for the SY spectrum integrated until after the pulse plateau,

\[ \mathcal{F}_c(t > t_l > t_{\gamma e}) \simeq t_p f_{pk} = t_l F_p(t_{ic,i}) \left(\frac{\epsilon}{E_{\gamma}}\right)^{-(n-1)/2} \] (A11)
as for the above \( t_l < t_{\gamma e} \) case. The integrated spectrum has the same slope as the instantaneous spectrum because the former is dominated by the emission at the pulse plateau \( (t_{\gamma e}, t_{\gamma e} + t_l) \), whose duration \( t_l \) is independent of the observing energy \( \epsilon \), so that the plateau flux \( f_{pk} \) (third branch of Equation (A8)) imprints its energy dependence on the integrated spectrum.

The initial assumption of a constant cooling timescale \( t_{ic,i} \) for the typical GRB electron of energy \( \gamma_l \) allows all of the results

\[ \gamma_p(t > t_l) = \gamma_m(t - t_l) = \gamma_l \left(1 - \frac{t - t_l}{t_{ic,i}}\right)^3 \rightarrow \frac{\gamma_p(t)}{\gamma_m(t)} = \left(1 + \frac{t_l}{t_{ic,i} - t}\right)^3 \] (A15)

assuming a constant magnetic field \( B \) and a constant iC-cooling timescale \( t_{ic,i} \equiv t_{ic}(\gamma_l) \).

In contrast with the \( n > 1 \) case, for \( n < 1 \), the observing energy \( \epsilon \) is “reached” before the cooling timescale \( t_{ic,i} \) of the \( \gamma_l \) electrons, and the spectrum \( f_c \sim \epsilon^{1/6} \) of the SY emission from the cooled electron distribution rises (instead of falling), peaking at \( \epsilon_p = E_{\gamma} \), the GRB energy peak. Thus, the flux at \( \epsilon_m \) iswhere the evolution of the GRB peak flux \( F_p(t < t_l) \) stands for a constant electron injection rate \( R_l \) and a constant magnetic field \( B \).

The \( \gamma_{\epsilon} \)-electrons injected at \( t_l \) cool following

for SY-dominated electron cooling shown in Section 3 to be recovered by setting the cooling-power exponent \( n = 2 \).

A.2. Weak iC-cooling \( (n < 1) \) through Scatterings at the Thomson–Klein–Nishina Transition of Low-energy Photons

The only case with \( n < 1 \) is that of the electron-cooling dominated by iC-scatterings if the \( \gamma \) electrons scatter their SY photons in the KN regime, and at times before the iC-cooling timescale \( t_{ic,i} \) of the \( \gamma_l \) electrons. In this case, the iC-cooling power \( P_{ic}(\gamma) \sim \gamma^n \) has an exponent \( n \approx 2/3 \).

From the iC-cooling law of Equation (31), the lowest electron energy \( \gamma_m \), its SY photon energy \( \epsilon_m \), and the transit-time \( t_{\gamma e} \) from emission at gamma-ray energy \( E_{\gamma} \) to the observing energy \( \epsilon \) are:

\[ \gamma_m(t < t_{ic,i}) = \gamma_l \left(1 - \frac{t}{t_{ic,i}}\right)^3, \quad t_{ic,i} \equiv \frac{3\gamma_l m_e c^2}{P_{ic}(\gamma_l)} \] (A12)

\[ \epsilon_m(t < t_{ic,i}) = E_{\gamma} \left(1 - \frac{t}{t_{ic,i}}\right)^{\beta}, \quad t_{ic,i} = t_{\gamma e} \left[1 - \left(\frac{\epsilon}{E_{\gamma}}\right)^{1/6}\right] \] (A13)

\[ \gamma_p(t > t_l) = \gamma_m(t - t_l) = \gamma_l \left(1 - \frac{t - t_l}{t_{ic,i}}\right)^3 \rightarrow \frac{\gamma_p(t)}{\gamma_m(t)} = \left(1 + \frac{t_l}{t_{ic,i} - t}\right)^3 \] (A15)
using Equation (A12); thus, the width of the cooling-tail increases slowly at \( t_f < t < t_{ic,i} \), implying that the cooling-tail is stretched and becomes harder. That hardening being slow, we will ignore it and assume that the cooling-tail remains a power law of exponent \(-n\). The SY spectrum peaks at the characteristic SY energy of the \( \gamma_p \)-electrons, \( \varepsilon_p = \varepsilon_{sy} \gamma_p \), and the peak flux at that energy is approximately constant (if \( B = \text{const} \))

\[
    f_p(t > t_f) = F_p(t_f), \quad \varepsilon_p(t) = E_r \left( \frac{\gamma_p}{\gamma_i} \right)^2 ,
\]

for most electrons (of constant number) radiate at \( \varepsilon_p \).

From the SY spectrum corresponding to the broken power-law electron distribution

\[
    f_p(t) = \begin{cases} 
        f_m(\varepsilon/\varepsilon_m)^{1/3} & (\varepsilon < \varepsilon_m) \\
        f_m(\varepsilon/\varepsilon_m)^{1/6} = f_p(\varepsilon/\varepsilon_p)^{1/6} & (\varepsilon_m < \varepsilon < \varepsilon_p) \\
        f_p(\varepsilon/\varepsilon_p)^{-(p-1)/2} & (\varepsilon_p < \varepsilon)
    \end{cases}
\]

and the above evolutions of break energies \( \varepsilon_m \) and \( \varepsilon_p \) and of the corresponding fluxes \( f_m \) and \( f_p \), one can calculate the instantaneous spectrum and pulse light curve at observing energy \( \varepsilon \)

\[
    \frac{\rho}{\varepsilon} \left( \frac{\gamma_i}{\gamma_f} \right)^2 \left( 1 - \frac{t - t_i}{t_{ic,i}} \right)^{-1} \left( 1 - \frac{t - t_f}{t_{ic,i}} \right)^{-1} \left( 1 - \frac{t - t_{ic,i}}{t_{ic,i}} \right) \left( 1 - \frac{t - t_{ic,i}}{t_{ic,i}} \right)^{3(p-1)}
\]

yielding a contribution with the same spectral slope \( 1/6 \): \( F_p(t > t_f) = F_p(t = t_p) = e^{-6/9} \), after using Equation (A13).

This shows that pulse peak epoch and peak flux are

\[
    t_p = t_{ic} + t_f, \quad F_{pk} = F_p(t_p) = F_p(t_f)
\]

where \( F_p(t_f) \) is the GRB pulse peak flux (or the GRB peak spectral energy). The peak epoch \( t_p \) corresponds to the passage of the high-energy end \( \varepsilon_p \) of the cooling-tail, after the end of electron injection.

Equations (A18) and (A19) show that the SY spectrum integrated until the transit-time \( t_{ic} \) (when the lowest-energy \( \varepsilon_m \) of the power-law SY spectrum from the cooling-tail crosses the observing energy \( \varepsilon \)) is \( F_{ic} \sim e^{1/3} \) and that, after \( t_{ic} \), it is \( F_{ic} \sim e^{-1/6} \), with the \( \varepsilon_p \) crossing at the peak-time \( t_p = t_{ic} + t_f \)
The same two equations show that the slow pre-peak rise and post-peak fall after max(t_{\gamma}, t_f) are always dominant over the preceding fluence, the integrated flux being

\[ \mathcal{F}_e(t \gg t_\nu) \gtrsim F_\nu(t) t_{\nu,i,d} \left( \frac{E}{E_\gamma} \right)^{1/6} \left[ \frac{1}{6} \ln \frac{E_\gamma}{\epsilon} + \frac{1}{3p-2} \right] \left[ \frac{1}{6} \ln \left( 1 + \frac{4}{3p-2} \frac{E}{E_\gamma} \right)^{1/6} \right] + \frac{1}{3p-2} \left( \frac{t}{t_{\nu,i,d}} \right)^{1/6} \left( \frac{t}{t_{\nu,i,d}} \right)^{1/3} \]

with the last term representing the contribution from the pulse fall, which can be dominant over the pre-peak contribution depending on the observing energy \( \epsilon \) and on the index \( p \) of the injected electron distribution.

The above derivations pertain to the case when the lowest-energy \( \gamma_m \) electrons in the cooling-tail cool mostly by scattering \( \gamma \)-rays photons of energy \( m_e c^2/\gamma_m < \epsilon \) at the T-KN limit. Those photons have an \( f, \sim \epsilon^{1/3} \) distribution with energy, leading to \( P_{\nu,\gamma}(\gamma) \sim \gamma^{-2/3} \) and to a cooling-tail distribution \( N(\gamma_m < \gamma < \gamma_c) \sim \gamma^{-2/3} \), with \( \gamma_m \) given in Equation (A12) for \( n = 2/3 \). The iC-cooling power of the \( \gamma_m \) electrons switches exponent from \( n = 2/3 \) to \( n = 1 \) when the \( \gamma_m \) electrons scatter their own SY photons of energy \( e_{\gamma} \gamma_m \) at the T-KN transition (P19), i.e., when \( E_{\gamma} \gamma_m / \epsilon = m_e c^2 / \gamma_m \) where \( E_{\gamma} \) is the GRB spectral peak energy in the comoving frame. After that epoch, a softer distribution \( N(\gamma_m < \gamma) \sim \gamma^{-1} \) grows above the low-energy end of the cooling-tail, up to an electron energy that increases in time, i.e., the harder distribution \( N(\gamma < \gamma_c) \sim \gamma^{-2/3} \) of the cooling-tail below the high-energy end \( \gamma_c \) shrinks progressively. When the \( \gamma \) electrons scatter the lowest-energy \( \gamma_m \) SY photons at the T-KN transition, i.e., when \( \epsilon_{\gamma_m} = m_e c^2 / \gamma_m \), the entire cooling-tail becomes \( N(\gamma_m < \gamma < \gamma_c) \sim \gamma^{-1} \). Adding that, for \( n = 1 \), the cooling of the \( \gamma_m \) electrons is an exponential in time (with timescale \( t_{\nu,i} \)) that continues after the modified power-law cooling given in Equation (A12), it can be shown that the \( n = 2/3 \) initial iC-cooling-tail is completely replaced by a softer \( n = 1 \) cooling-tail at epoch

\[ t_{2/3 \rightarrow 1} = t_{\nu,i} \left[ 1 - \frac{m_e c^2}{\gamma_i E_\gamma} \right]^{1/2} \left( \frac{1}{6} \ln \frac{E_\gamma}{\epsilon} \right) \]

For the \( n = 1 \) cooling-tail to develop up to the GRB typical electron energy \( \gamma_i \), electron injection must last longer than the iC switch-time \( t_{2/3 \rightarrow 1} \): \( t_f > t_{2/3 \rightarrow 1} \) (first condition). For the \( n = 1 \) cooling-tail SY emission to dominate the integrated spectrum, the iC switch-time \( t_{2/3 \rightarrow 1} \) must occur before the peak-rise epoch \( t_p = t_f + t_{\nu,i} \) with the transit-time \( t_{\nu,i} \) for iC-cooling with \( n = 2/3 \): \( t_f > t_{2/3 \rightarrow 1} - t_{\nu,i} \) (second condition). The second condition is satisfied if the first one is fulfilled; thus, the \( n = 1 \) cooling-tail yields an integrated spectrum of slope \( \beta_{LE} = 0 \) only if \( t_f > t_{2/3 \rightarrow 1} \).

### A.3. Pulse Duration and Transit-time

#### A.3.1. Constant B and Increasing \( \tau(\gamma) \) (Decreasing \( t_{\nu,i}(\gamma) \))

To assess the robustness of the above result regarding GRBs with a hard low-energy slope \( \beta_{LE} > 0 \) arising from iC-dominated electron cooling with \( n < 1 \), we consider next the case when the scattering optical-thickness \( \tau \) is not constant. For an electron injection rate \( R_t \sim t^3 \), the above case of a constant \( \tau \)

(leading to a constant iC-cooling timescale \( t_{\nu,i}(\gamma) \)) corresponds to \( y < -1 \). For \( y > -1 \), when \( \tau \sim t^{\nu+1} \) increases and the iC-cooling timescale \( t_{\nu,i} \sim \tau^{-1} \) decreases, the iC-cooling of a

\[ (t_{\nu,i} < t_f < t_{\nu,i,t}) \quad \bar{\epsilon} \approx \epsilon \]

Equation (31) becomes

\[ \frac{d\gamma}{dt} = \frac{1}{t_{\nu,i,t}(t)} \left( \frac{t}{t_{\nu,i}} \right)^{n-1} \gamma^{n-1} \]

with the cooling timescale \( t_{\nu,i,t} \) at the end of electron injection containing all relevant and unspecified quantities: magnetic field \( B \) and optical-thickness \( \tau(\gamma) \).

The above equation can be integrated to derive \( \gamma(t) \) and, by using \( \epsilon / E_\gamma = [\gamma(t_{\nu,i}) / \gamma_i]^2 \) as definition for the transit-time \( t_{\nu,i,t} \), one obtains

\[ t_{\nu,i,t} = t_f \left[ \frac{y + 2}{n-1} \left( \frac{\epsilon}{E_\gamma} \right)^{(1-n)/2} - 1 \right]^{1/(y+2)} \]

The pulse duration can be calculated via the same method as in

Equation (35):

\[ \delta t_c = \left( \frac{\epsilon}{E_\gamma} \right)^{(1-n)/2} \left( \frac{t_{\nu,i}}{t_{\nu,i,t}} \right) \]

For \( y = -1 \), Equations (A24) and (A25) give timescales for a constant scattering optical-thickness \( \tau \). For an increasing \( \tau(t) \), the transit-time \( t_{\nu,i} \) has a weaker dependence on the observing energy \( \epsilon \) than for a constant \( \tau \) (Equation (33)), because the exponent \( 1/(y + 2) < 1 \), while the pulse duration \( \delta t_c \) picks up an energy-dependent factor \( (t_f / t_{\nu,i})^{1+y} \).

For iC-cooling dominated by scatterings in the Thomson regime \((n > 1) \), Equations (A24) and (A25) lead to

\[ \delta t_c = \frac{n - 1}{y + 2} t_{\nu,i} \sim \left( \frac{\epsilon}{E_\gamma} \right)^{(n-1)/(2y+4)} \]

thus, the trend of pulses to last shorter at higher energies still stands even when the scattering optical-thickness \( \tau \) increases. The measured energy dependence of the pulse duration, \( \delta t_c \sim \epsilon^{0.4} \), has an exponent between the values \(-0.25 \) and \(-0.50 \) expected for iC-dominated electron cooling in the Thomson regime \((n = 2) \), for a constant electron injection rate \( R_t \) \((y = 0) \) or a constant optical-thickness \( \tau \) \((y = -1) \), respectively.

For iC-cooling dominated by scatterings at the T-KN transition \((n < 1) \), if the electron injection rate \( R_t \) decreases sufficiently fast (faster than \( 1/t \)), then the cooling-tail will be curved downward, with most of the flux being produced by the lowest-energy \( \gamma_m \) electrons of the tail. Thus, the pulse peak-time will be the transit-time that it takes the \( \gamma_m \) electrons to cool to an SY-emitting energy equal to the observing energy \( \epsilon \), and
the pulse duration will be set by their cooling rate when their SY characteristic energy $\varepsilon_m$ drops to $\epsilon$. The exact evolution of the injection rate $R_i$ is not relevant for the electron iC-cooling because the scattering optical-thickness $\tau$ is practically constant. Equations (A24) and (A25) still apply, but with $y=-1$, for which they reduce to Equations (33) and (35). Thus, in the limit $\epsilon \ll E_\gamma$, the transit-time $t_{\gamma \epsilon}$ is very weakly dependent on the observing energy $\epsilon$, and the pulse duration $\delta t_{\epsilon}$ is $\sim \epsilon^{(1-n)/2}$ increases with $\epsilon$, remaining at odds with GRB observations. For $\epsilon \lesssim E_\gamma$ (just below the GRB peak energy), the term $t_{\gamma \epsilon}$ in the denominator of Equation (A25) makes $\delta t_{\epsilon}$ increase with $\epsilon$ even more strongly than $\epsilon^{(1-n)/2}$, thus, the incompatibility with observations becomes more severe.

If $R_i$ does not decrease faster than $1/t$, then the cooling-tail will be close to a power law or will be curved upward, with most of the flux arising from the highest-energy $\gamma_{p}$-electrons of the tail. Thus, the pulse peak epoch will be the transit-time for the $\gamma_{p}$-electrons to “reach” the observing energy $\epsilon$ after the end of electron injection at $t_i$, and the pulse duration $\delta t_{\epsilon}$ will be set by the cooling rate of the $\gamma_{p}$-electrons when their SY characteristic $\varepsilon_p$ reaches $\epsilon$, at $t_{\gamma \epsilon}^{(\omega)} > t_i$. Because the scattering optical-thickness is constant after the end of electron injection, the iC-cooling is independent of the history of the electron injection $R_i(t < t_i)$, and so are the transit-time $t_{\gamma \epsilon}$ and pulse duration $\delta t_{\epsilon}$, arising from the cooling of the $\gamma_{p}$-electrons after $t_i$. Consequently, the pulse duration is still as given in Equation (35), and the incompatibility with observations remains unchanged.

### A.3.2. Decreasing Magnetic Field B(t)

An evolving magnetic field $B(t)$ affects the transit-time $t_{\gamma \epsilon}$ and the pulse duration $\delta t_{\epsilon}$ in two ways: first, it determines the energy density of the seed SY photons to be upscattered and to iC-cool the electrons (thus, it determines the electron cooling), and second, it determines the evolution of the ends of the SY spectrum from the cooling-tail, whose passage through the observing band defines $t_{\gamma \epsilon}$ and $\delta t_{\epsilon}$.

The case of electron iC-cooling occurring mostly through scatterings in the Thomson regime ($n > 1$) will not be considered further because, as shown above, it can account for the observed trend of GRB pulse duration to decrease with observing energy, provided that electrons cool below the observing energy, and we focus on iC-cooling dominated by scatterings of the sub-GRB SY photons at the T-KN transition ($n < 1$), a case that fails to accommodate that observational feature for a constant magnetic field.

For iC-cooling with $n < 1$, the history of the electron injection rate $R_i(t)$ is irrelevant for the cooling of the electrons that determine the transit-time and the pulse duration. Thus, these two quantities depend only on the evolution of the magnetic field, which we will assume to be a power-law $B(t) = B_i(t/t_i)^x$, normalized at the end of electron injection. It can be shown that the iC-cooling power satisfies $P_{ic}(\gamma) \sim \tau(B\gamma)^x$, either for $n < 1$ (Equation (45) of P19) or for $n > 1$ (Equation (41) of P19); thus, for the above $B(t)$, the iC-cooling law is

$$- \frac{d\gamma}{dt} = \frac{1}{t_{ic}(t_i)} \left( \frac{t_i}{t} \right)^{\alpha x} \gamma^{\alpha x - 1},$$

(A27)

which can be integrated (from $t = 0$ to $t = t_{\gamma \epsilon}$ if $\tau$ is constant and from $t = t_i$ to $t = t_i + t_{\gamma \epsilon}$ if $\tau$ increases) and, after using

$$\frac{\epsilon}{E_\gamma} = \left( \frac{\gamma(t_{\gamma \epsilon})}{\gamma_i} \right)^2 \left( \frac{t_{\gamma \epsilon}}{t_i} \right)^x$$

will lead to an algebraic equation for the transit-time $t_{\gamma \epsilon}$, which can be solved in asymptotic regimes:

$$(n<1, \ B \sim t^{x}) \ t_{\gamma \epsilon}^{(\omega)} \simeq t_i \times \left( \frac{\epsilon}{E_\gamma} \right)^{1/x} \left( \frac{t_{\omega\epsilon}}{t_i} \right)^{1/(nx+1)} \epsilon \ll \epsilon < E_\gamma \ (x<0) \quad \tau \simeq E_\gamma \left( \frac{t_{\omega\epsilon}}{t_i} \right)^{x/(nx+1)}.$$

(A29)

The second branch of Equation (A29) shows that, for a sufficiently low observing energy $\epsilon$, the transit-time is independent of $\epsilon$. In the limit $x \to 0$ (constant $B$), one has $\tau(x = 0) = E_\gamma$; thus, the first branch of Equation (A29) disappears and the second branch reduces to the transit-time given in Equation (33) in the limit $\epsilon \ll E_\gamma$.

The condition $x < 0$ arises from requiring that the transit-time decreases with observing energy $\epsilon$; in the opposite case ($x > 0$), an increasing magnetic field would compensate for the electron cooling and lead to break energies (at either end of the cooling-tail’s SY spectrum) that increase, and there is no transit of the break energies to an observing energy $\epsilon < E_\gamma$. The working condition $x > -1/n$ for the second branch leads to simple temporal power-law dependence for the cooling-equation solution $\gamma(t)$. For $x < -1/n$, the corresponding equation for $t_{\gamma \epsilon}$ becomes even more complicated; however, the result given in the first branch stands for $x < -1/n$ as well.

Once the transit-time $t_{\gamma \epsilon}$ is known, the pulse duration can be calculated by using the cooling law of Equation (A27):

$$\delta t_{\epsilon} = \left( \frac{\epsilon}{E_\gamma} \right)^{(1-n)/2} \left( \frac{t_i}{t_{\gamma \epsilon}} \right)^{(x+1)/2} \left( \frac{t_{\omega\epsilon}}{t_i} \right)$$

(A30)
leading to

\[
\delta t_e \simeq t_{\text{ke},i}(t_I) \times \left\{ \left( \frac{\epsilon}{E_{\gamma}} \right)^{-(n-2/3)} \left( \frac{t_I}{t_{\text{ke},i}} \right)^{(n+1)/(2n+2)} \right\}^{(1-n)/2=1/6} \quad \bar{\epsilon} < \epsilon < E_{\gamma} \quad (x<0)
\]

\[
\epsilon < \bar{\epsilon} \quad (-3/2 = -1/n < x < 0)
\]

For a sufficiently low observing energy (second branch), the pulse duration still increases with energy; however, for energies just below the GRB spectral peak \(E_{\gamma}\), a decreasing magnetic field “opens” the first branch above, for which \(\delta t_e \sim \epsilon^{-n} \sim \epsilon^{-2/3}\) is consistent with (or stronger than) the observed trend of pulse duration to decrease with energy.

**Appendix B**

**Spectra and Light Curves of Synchrotron Emission from Adiabatically Cooling Electrons**

During electron injection (at \(t < t_I\)) at the power-law rate \(R_t \sim (t + t_o)^{-\gamma}\), the cooling-tail extends from the lowest-energy \(\gamma_m\) given in Equation (38) to the minimal electron injection energy \(\gamma_i\). Thus, the cooling-tail SY emission extends from photon energies \(\varepsilon_{m}(t)\) (Equation (38)) to \(\varepsilon_{p}(t < t_I) = E_{\gamma}\), and has the slope \(\beta\) given in Equation (46). For \(y > 1\), \(m = (1 - 3y)/2 < -1\) and most cooled electrons are at \(\gamma_m\); for \(y < 1\), \(m > -1\) and most cooled electrons are at \(\gamma_i\), irrespective of \(y\).

From \(\int_{\infty}^{t_I} d\gamma N(\gamma) \simeq \int_{0}^{t_I} dt' R_t(t')\), where \(N(\gamma) = dN/d\gamma\) is the electron distribution with energy, it can be shown that

\[
\gamma_m N(\gamma_m) = \text{const} \rightarrow N(\gamma_m) \sim \gamma_m^{-1} \sim t^{2/3}, \quad \gamma_i N(\gamma_i) \sim (t + t_o)^{1-y} \rightarrow N(\gamma) \sim t^{1-y}.
\]

(B1)

Thus, for a constant magnetic field \(B\), the flux densities \(f_{\epsilon} \sim B(\epsilon) N(\gamma)\) at the ends of the cooling-tail’s characteristic synchrotron energies \(\varepsilon_m\) and \(\varepsilon_p = E_{\gamma}\) are

\[
f_{\epsilon_m}(\varepsilon_m) = F_p(t_i) \left(1 + \frac{t_I}{t_o}\right)^{1/2} \quad \text{const.,} \quad f_{\epsilon_p}(\varepsilon_p, t < t_I) = F_p(t_i) \left(\frac{t_I + t_o}{t_I}\right)^{1-y} \simeq F_p(t_i) \left(\frac{t_I}{t_I}\right)^{1-y} \quad (t_o \ll t, t_I)
\]

(B2)

where \(F_p(t_i)\) is the flux density at the GRB peak-energy \(E_{\gamma}\) when electron injection ends. For \(y < 1\), i.e., for low-energy slopes \(\beta_{AD}(t < t_I) > 0\), the epoch \(t_I\) also marks the GRB pulse peak; thus, \(F_p(t_I)\) is also the GRB pulse peak flux (or pulse peak flux) for GRBs with a harder low-energy slope during the pulse rise.

After electron injection ends (at \(t > t_I\)), the lowest-energy electrons \(\gamma_m\) continue to cool adiabatically according to Equation (38); thus, \(\varepsilon_m\) evolves as in Equation (38), while the \(\gamma_i\) electrons begin cooling adiabatically at \(t_I\) following the same law but with time measured since \(t_I\) and for the current system age \(t_I + t_o:\)

\[
\gamma_p(t) = \gamma_i \left(1 + \frac{t - t_I}{t_I + t_o}\right)^{2/3} \simeq \gamma_i \left(\frac{t}{t_I}\right)^{2/3} \rightarrow \varepsilon_p(t > t_I) = E_{\gamma} \left(\frac{t}{t_I}\right)^{4/3}, \quad \gamma_m(t) = \left(\frac{t_I}{t_o}\right)^{2/3} \quad (t_o \ll t_I < t).
\]

(B3)

Thus the width of the cooling-tail is constant. From the electron AD-cooling law \(\gamma(t > t_I) \sim t^{-2/3}\), it can be shown that the cooling-tail slope \(-m\) remains unchanged at \(t > t_I\). From \(\int_{\infty}^{t_I} d\gamma N(\gamma) = \text{const at} t > t_I\), it can be shown that the flux densities at the ends \(\varepsilon_m\) and \(\varepsilon_p\) of the cooling-tail remain constant after \(t_I\). Thus, the flux density at the lowest energy of the cooling-tail is the same as that in Equation (B2), while the flux density at the higher-energy break \(\varepsilon_p\) is

\[
f_{\epsilon_p}(\varepsilon_p, t > t_I) = f_{\epsilon_p}(\varepsilon_p, t_I) \simeq F_p(t_I)
\]

(B4)

after using Equation (B2).

Adding the SY spectrum from a broken power-law electron distribution consisting of a cooling-tail \(N(\gamma_m < \gamma < \gamma_p) \sim \gamma^{(1-3y)/2}\) and a cooled-electron distribution \(N(\gamma_p < \gamma) \sim \gamma^{-p}\) is

\[
f_{\epsilon} \sim \begin{cases} 
 f_{\epsilon_m}(\epsilon/\varepsilon_m)^{1/3} & \epsilon < \varepsilon_p \quad (y<5/9) \\
 f_{\epsilon_m}(\epsilon/\varepsilon_m)^{1/3} & \epsilon > \varepsilon_m \quad (y>5/9) \\
 f_{\epsilon_m}(\epsilon/\varepsilon_m)^{3(1-y)/4} & \varepsilon_m < \epsilon < \varepsilon_p \quad (y>5/9), \\
 f_{\epsilon_p}(\epsilon/\varepsilon_p)^{-p(1/2)} & \varepsilon_p < \epsilon
\end{cases}
\]

(B5)
and the instantaneous spectrum (pulse light curve) at lower energies can be calculated:

\[
(R_t \sim t^{-\gamma}, \ y < 5/9) \quad f_p(t) = \mathcal{F}_p(t_I) \times \begin{cases} \left( \frac{\epsilon}{E_\gamma} \right)^{1/3} \left( \frac{t}{t_I} \right)^{1-y} & t < t_I \quad \text{(rise)} \\ \left( \frac{\epsilon}{E_\gamma} \right)^{1/3} \left( \frac{t}{t_I} \right)^{4/9} & t_I < t < \hat{t}_{\gamma} \quad \text{(slow rise)} \\ 1 & t = \hat{t}_{\gamma} \quad \text{(peak)} \\ \left( \frac{\epsilon}{E_\gamma} \right)^{-(p-1)/2} \left( \frac{t}{t_I} \right)^{-2(p-1)/3} & \hat{t}_{\gamma} < t \quad \text{(fall)} \end{cases}
\]  

(B6)

where \(t_{\gamma}\) is the epoch when the lowest-energy electrons \(\varepsilon_m\) radiate at the observing energy \(\epsilon\) (Equation (41)), and \(\hat{t}_{\gamma}\) is the epoch when the higher spectral break at \(\varepsilon_p\) crosses the observing energy \(\epsilon\):

\[
\hat{t}_{\gamma} = \left( \frac{\epsilon}{E_\gamma} \right)^{-3/4} = \frac{t_I}{t_0}
\]

(B7)

and

\[
(R_t \sim t^{-\gamma}, \ 5/9 < y) \quad f_p(t) = \mathcal{F}_p(t_I) \times \begin{cases} \left( \frac{\epsilon}{E_\gamma} \right)^{1/3} \left( \frac{t_I}{t_0} \right)^{y-1} \left( \frac{t}{t_I} \right)^{4/9} & t < t_{\gamma} \quad \text{(slow rise)} \\ \left( \frac{t_I}{t_0} \right)^{y-1} & t = t_{\gamma} \quad \text{(peak if } y > 1) \\ \left( \frac{\epsilon}{E_\gamma} \right)^{3(1-y)/4} \left( \frac{t}{t_I} \right)^{1-y} & t_{\gamma} < t < \hat{t}_{\gamma} \quad (y<1) \text{ slow rise; } y > 1 \text{ fall} \\ 1 & t = \hat{t}_{\gamma} \quad \text{(peak if } y < 1) \\ \left( \frac{\epsilon}{E_\gamma} \right)^{-(p-1)/2} \left( \frac{t}{t_I} \right)^{-2(p-1)/3} & \hat{t}_{\gamma} < t \quad \text{(fall)} \end{cases}
\]

(B8)

For any \(y\), \(f_p(\hat{t}_{\gamma}) = \mathcal{F}_p(t_I)\) follows from Equation (B4) because \(\hat{t}_{\gamma} > t_I\) (Equation (B7)), i.e., the high-energy break \(\varepsilon_p\) decreases below the observing energy \(\epsilon\) only after electron injection stops.

For \(y < 5/9\), the cooling-tail emission is dimmer than the \(1/3\) low-energy SY flux produced by the \(\gamma_p\)-electrons, and the entire spectrum is as if the cooling-tail did not exist; thus, the epoch \(t_{\gamma}\) when \(\varepsilon_m\) crosses the observing band is irrelevant. For \(y > 5/9\), the light curve depends on the evolution of the lowest-energy characteristics \((\varepsilon_m, m_f)\), which are unchanged across \(t_I\); thus, there is no light-curve break at the epoch \(t_I\) when electron injection stops, and the epoch \(t_I\) is irrelevant.

Equations (B6) and (B8) show that the pulse rise is harder (\(\beta_y \geq 0\)) than its decay (\(\beta_y \leq 0\)), and that the pulse peak epoch and peak flux are

\[
t_p = \begin{cases} \frac{t_{(ad)}}{t_{(ad)}} & y < 1 = \left( \frac{\epsilon}{E_\gamma} \right)^{-3/4} \frac{t_I}{t_0} \left( \frac{t}{t_I} \right)^{y-1} & y < 1 \\ \frac{t_{(ad)}}{t_{(ad)}} & y > 1 = \left( \frac{\epsilon}{E_\gamma} \right)^{-3/4} \frac{t_I}{t_0} \left( \frac{t}{t_I} \right)^{y-1} & y > 1 \end{cases}
\]

(B9)

The pulse-integrated spectrum is dominated by the emission prior to when the flux decay becomes faster than \(t^{-1}\). That happens at \(t_{\gamma}\) if \(y > 2\) (leading to \(\mathcal{F}_x \sim \epsilon^{-3/4}\); see below), at \(\hat{t}_{\gamma}\) if \(y < 2\) (leading to \(\mathcal{F}_x \sim \epsilon^{-3/4}\); see below), and if the injected electron distribution has an exponent \(p > 5/2\), but continues through the pulse decay at \(t > \hat{t}_{\gamma}\) if \(p < 5/2\) (leading to \(\mathcal{F}_x \sim \epsilon^{-(p-1)/2}\)). After some calculations, the integrated spectrum is found to be

\[
(R_t \sim t^{-\gamma}) \quad \mathcal{F}_x(t > \hat{t}_{\gamma}) = \mathcal{F}_p(t_I) t_I \times \begin{cases} \left( \frac{\epsilon}{E_\gamma} \right)^{-3/4} & p > 5/2 \\ \left( \frac{\epsilon}{E_\gamma} \right)^{-(p-1)/2} \left( \frac{t}{t_I} \right)^{(5-2p)/3} & p < 5/2 \end{cases}
\]

(B10)

The above slope \(\beta_{\text{int}} = -3/4\) of the integrated spectrum arises from the passage of the higher spectral break \(\varepsilon_p\), because the fluence from the pulse fall is most often dominant, yielding \(\mathcal{F}_x \simeq t_{\gamma} f_p(t_{\gamma}) \sim \varepsilon_p (\epsilon^{1/3} t_{\gamma}^{4/9}) \sim \epsilon^{-3/4}\). However, even when the pulse fluence is dominated by the crossing of the lower spectral break \(\varepsilon_m\) (i.e., for \(y > 3.4\)), the slope of the integrated spectrum would be the same.
\(\beta_{BE} = -3/4\) because the transit-times \(t_\gamma\) (Equation (41)) and \(t_\tau\) (Equation (B7)) have the same dependence on the observing energy \(\epsilon\). As for SY-cooling, the softness of the pulse-integrated spectrum is a consequence of the cooling-time increase with observing energy: \(t_\gamma^{\text{ad}} \sim \epsilon^{-3/4}\).

The last branch above simply states that, if the injected electron distribution is sufficiently hard \((p < 5/2)\), then the pulse fluence is mostly from the cooled-injected distribution, and not from the cooling-tail. In this case, the integrated spectrum will have the slope \(\beta_{BE} = -(p - 1)/2\) of the cooled-injected distribution, which has the slope of the injected power-law energy distribution because AD-cooling shifts distributions to lower energies while preserving their slopes.

Appendix C
Synchrotron and Adiabatic Cooling

Equations (10) and (39) lead to

\[
\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)_{\text{SY}} - \left(\frac{d\gamma}{dt}\right)_{\text{AD}} = \frac{\gamma^2}{\gamma_0 t_{\text{sy}}(\gamma_0)} + \frac{2}{3} \frac{\gamma}{t + t_o}.
\] (C1)

This cooling law applies to any electron of initial energy \(\gamma_0\), which does not have to be the typical electron energy \(\gamma_i\) that appears in the SY-cooling term above. In above equation, \(t_{\text{sy}}(\gamma_0) = 7.7 \times 10^3 / (\gamma_0 B^2)\) (Equation (8)) is the SY-cooling timescale for the \(\gamma_0\) electrons. With the substitution \(\gamma = g^x\), the electron-cooling equation becomes

\[
x g^{x-1} \frac{dg}{dt} + \frac{g^{2x}}{\gamma_0 t_{\text{sy}}(\gamma_0)} + \frac{2}{3} \frac{g^x}{t + t_o} = 0,
\] (C2)

which is a first-order linear differential equation (LDE) of the form

\[
\frac{dg}{dt} + a(t) g = b(t) \quad \Rightarrow \quad g(t) = \frac{1}{\mu(t)} \left[ \text{const} + \int \mu(t) b(t) dt \right], \quad \mu(t) = \exp \left\{ \int a(t) dt \right\}
\] (C3)

only if \(x = -1:\)

\[
\gamma \equiv 1/g \rightarrow \frac{dg}{dt} - \frac{2}{3} \frac{g}{t + t_o} = \frac{1}{\gamma_0 t_{\text{sy}}(\gamma_0)} \quad \rightarrow \quad \mu(t) = (t + t_o)^{-2/3}, \quad g(t) = \text{const}(t + t_o)^{2/3} + \frac{3(t + t_o)}{\gamma_0 t_{\text{sy}}(\gamma_0)}
\] (C4)

where the constant can be determined from the initial condition \(\gamma(t = 0) = \gamma_0\). The solution to the SY and AD-cooling is

\[
(B = \text{const}): \quad \gamma(t) = \frac{\gamma_0}{\left(1 + \frac{Z}{t_o}\right)^{2/3}} \left[1 + 3 \frac{t}{t_o} \left(1 + \frac{Z}{t_o}\right)^{1/3} - 1\right],
\] (C5)

which can be written as

\[
(AD + SY): \quad \gamma(t) = \frac{\gamma_{\text{ad}}(t)}{1 + 2 X(t) - \frac{t}{Z}}, \quad \gamma_{\text{ad}}(t) = \frac{\gamma_0}{X(t)}, \quad X(t) = \left(1 + \frac{t}{t_o}\right)^{1/3}, \quad Z = \frac{2 t_{\text{sy}}(\gamma_0)}{3 t_o} = \frac{t_{\text{ad}}(t = 0)}{t_{\text{ad}}(t = 0)}
\] (C6)

where \(\gamma_{\text{ad}}\) is the solution (Equation (38)) to the AD-cooling law. Thus, the solution to the SY- and AD-cooling is a modified AD-cooling solution, with the denominator containing information about both AD- and SY-cooling. The reason for this structure for the cooling solution is that the linear term of the first-order LDE for electron cooling (Equation (C4)) contains only AD-cooling. In the limit \(t < t_o\), the denominator loses dependence on AD-cooling and the solution becomes

\[
(t \ll t_o): \quad \gamma(t) = \frac{\gamma_0}{\left(1 + \frac{t}{t_o}\right)^{2/3}} \left(1 + \frac{t}{t_{\text{sy}}(\gamma_0)}\right) \\lesssim \gamma_{\text{ad}}(t) = \frac{\gamma_0}{1 + \frac{t}{t_{\text{ad}}(t)}}
\] (C7)

which shows SY-cooling (Equation (8)) and AD-cooling (Equation (38)) operating independently.

The inverse-Compton cooling law (Equation (31)) can be added to the AD- and SY-cooling terms of Equation (C1), but an approximate calculation of the inverse-Compton power is possible only until the overall cooling timescale \(t_{\text{ad}}\) of the typical \(\gamma_i\) electron (i.e., before the cooling-tail develops), because, in the opposite case \(t > t_{\text{ad}}\), the Compton parameter \(Z\) depends on the minimal electron energy \(\gamma_m\) of the cooling-tail, whose evolution is not known in advance (unless iC-cooling is weaker than SY- and AD-cooling, and Equation (C5) can be used as an approximation of \(\gamma_m\)). For \(t < t_{\text{ad}}\), the iC power depends on the electron energy as \(P_{iC} \sim \gamma^{2/3}\) for an electron that scatters the SY photons \(E_\gamma\) produced by the typical \(\gamma_i\)-electron in the Klein–Nishina regime \((\gamma E_\gamma > m c^2)\). Then, iC-cooling introduces a \(t^{2/3}\) term in Equation (C1), with time \(t\) entering through the proportionality of the iC power on the electron scattering-optical-thickness \(\tau\).

In this case, the substitution \(\gamma = 1/g\) that is required for the SY-cooling term of Equation (C1) to yield a first-order LDE brings an iC-cooling term proportional to \(t_{\gamma}^{4/3}\), which cannot be combined with the AD-cooling term \(g/(t + t_o)\) to lead to the term \(a(t)g\) of the
LDE in Equation (C1). However, for electrons that scatter $E_\gamma$ photons in the Thomson regime ($\gamma E'_\gamma < m c^2$), the iC power has the same $P_{ic} \sim \gamma^2$ dependence as the SY power, and the substitution $\gamma = 1/g$ used above leads to a cooling law of the form given in Equation (C3) but with a slightly more complex term $b(t) = \text{const} + kt$.

Then, the first integral given in Equation (C3) can be calculated analytically, and the electron cooling subject to all three processes is

$$(\text{AD + SY + iC/Th}): \frac{\gamma_{ad}}{\gamma(t)} = 1 + 2 \frac{X - 1}{Z} \frac{t_{ic}(\gamma_o)}{t_I} (X^4 - 4X + 3), \quad t_{ic}(\gamma_o) \equiv \frac{t_{sy}(\gamma_o)}{\gamma^4 I_I}$$

where $t_{ic}(\gamma_o)$ is the iC-cooling timescale at the epoch $t_I$ when the electron injection ends and the scattering optical-thickness $\tau$ is maximal.

Again, the solution (Equation (C8)) to the full electron-cooling law (as in Equation (C1) but with an extra term for iC-cooling in the Thomson regime: $- (d\gamma/dt)_{ic} = \gamma^2/|\gamma_{dco}(\gamma_o)|$) is a modified AD-cooling solution, with the decrease of the electron energy $\gamma$ expedited by an SY and an iC-cooling term, each expressed as their strength (1/$t_{sy}$ or 1/$t_{ic}$) relative to that of AD-cooling (1/$t_o$) at the beginning of electron injection.

Because Equation (C8) has a limited applicability, as it describes electron cooling only before the cooling-tail develops significantly, we will not investigate it any further. Instead, we return to Equation (C6) for AD- and SY-cooling, whose asymptotic solutions can be derived in three regimes: (i) $t \ll t_o \rightarrow X - 1 \approx t/(3t_o) \ll 1$ leading to the SY solution, (ii) $t \gg t_o \rightarrow X \approx (t/t_o)^{1/3} \gg 1$ leading to the AD-solution and the one-third-SY solution, and (iii) $X - 1 \ll X/2$, leading to the AD-solution. Depending on the relative strength of the AD and SY losses at $t = 0$, quantified by $Z$, the solution given in Equation (C6) for AD + SY-cooling has the following asymptotic regimes:

$$(Z < 2(2^{1/3} \approx 0.52)): \quad \gamma(t) \simeq \begin{cases} \gamma_{sy} t < t_o & (\text{early SY solution}) \\ \frac{1}{3} \gamma_{sy} t_o \ll t & (\text{late 1/3 - SY solution}) \end{cases}$$

$$(0.52 < Z < 1): \quad \gamma(t) \simeq \begin{cases} \gamma_{ad} t < t_o & (\text{early SY solution}) \\ \gamma_{ad} t_o \ll t < \bar{t} \in \left(1, \frac{19}{8}\right) & (\text{transient AD solution}) \\ \frac{1}{3} \gamma_{sy} \bar{t} \ll t & (\text{late 1/3 - SY solution}) \end{cases}$$

$\gamma_{ad}$ represents the AD loss rate at the injection epoch $t_I$. The dashed green line in Figure 4 shows the epoch $t_{cr}$, the electron cooling eventually turns to the one-third-SY solution, at a time $\max\{t_o, \bar{t}\}$, and that transition occurs even when the electron cooling is AD-dominated after that time. The dotted (black) line shows the epoch when AD- and SY-cooling powers are equal, at $t = t_{cr}$ (Equation (C13)), but that epoch is irrelevant for the electron-cooling, which is (almost) always the one-third-SY solution across $t_{cr}$.
(1 < Z): \[
\gamma(t) \simeq \begin{cases} 
\gamma_{ad} & t \ll \bar{t} \\
\frac{1}{3} \gamma_{sy} \left( \frac{19}{8} t_0 \right)^2 & \bar{t} \ll t 
\end{cases} 
\] (early AD solution) \tag{C11}
where
\[
\bar{t} \equiv t_0 \left( \frac{Z \gamma}{2 + 1} \right)^{3 - 1} 
\] (C12)

is the AD-SY solution switch-time, when \( X - 1 = Z/2 \). The above electron cooling through the three asymptotic regimes is indicated in Figure 4 by horizontal arrows (increasing time toward the right). Which asymptotic solutions are encountered depends on the parameter \( Z \). The AD and SY solutions are separated by the \( Z = 2(X - 1) \) line corresponding to \( t = \bar{t} \).

The above asymptotic solutions have some interesting features:

(i) Electron cooling is asymptotically described at early times by the synchrotron solution \( \gamma_{sy} \) only if \( Z < 1 \), which means \( t_\gamma(\gamma_{ad}) < t_\text{ad}(t = 0) \), i.e., only if the electron cooling is initially SY-dominated (obviously). Furthermore, the SY-cooling solution is accurate only at times \( t < t_\text{cr} \), when \( X - 1 \approx t/3t_\text{cr} \), for which the right-hand-side term of Equation (C5) shows an AD-cooling term smaller than the SY-cooling term, owing to \( Z < 1 \).

(ii) Electron cooling is described by the adiabatic solution \( \gamma_{ad} \) only for \( t < \bar{t} \), i.e., only for \( 2(X - 1) < Z \), when the right-hand side of Equation (C6) is unity. This condition is sufficient for an asymptotic AD-solution at early times if \( Z > 1 \), i.e., if the electron cooling is initially AD-dominated (obviously), but is not sufficient if \( Z < 1 \), i.e., if the electron cooling is SY-dominated. In the latter case, competition between the adiabatic term \( X^2 \) and the mixed term \( 1 + 2(X - 1)/Z \) appearing in Equation (C6) allows the AD-solution to set in at \( t \approx t_\text{cr} \).

(iii) As can be seen in Figure 4, irrespective of which cooling process is dominant initially, electron cooling is asymptotically described at late times \( t > \text{max}(t_\text{cr}, \bar{t}) \) by the one-third-SY solution. Condition \( t > \bar{t} \) implies \( X > 1 \), which implies that \( X^2(1 + 2(X - 1)/Z) \approx X^2(1 + 2X/Z) \), and condition \( t > \bar{t} \) implies \( X > Z/2 \), which leads to \( X^2(1 + 2X/Z) \approx 2X^2/Z \approx 2t/Z_\text{cr} = 3t/\gamma_{ad}(t_\text{ad}) \), thus, \( \gamma = (1/3)\gamma_{ad}(t_\text{ad})/t \approx \gamma_{ad}/3 \). In other words, for sufficiently late times, the product of the AD-cooling term \( X^2 \) and the modified SY-cooling term \( 1 + 2(X - 1)/Z \) is proportional to the “pure” SY-cooling term, leading to a SY-cooling solution despite that, at late times, the electron cooling is guaranteed to be AD-dominated, as shown below.

From Equations (11) and (C6), the SY-cooling timescale of the SY+AD-cooling electron is \( t_\text{sy}(\gamma) = t_\text{sy}(\gamma_0) \gamma_0 / \gamma = 1.5Z t_\text{sy}(\gamma_0) / \gamma = 3t + t_\text{sy}(\gamma_0)(t/\gamma_0)^{2/3} \) at \( t > t_\text{cr} \), while the AD-cooling timescale (Equation (40)) \( t_\text{ad} = 1.5(t + t_\text{cr}) \) is independent of the electron energy. Thus, \( t_\text{sy} > 3t \) and \( t_\text{sy} > t_\text{ad} \) guarantees that, after some time, \( t_\text{sy} > t_\text{ad} \) and the electron cooling will be eventually AD-dominated even if it started in an SY-dominated regime \( t_\text{sy}(\gamma_0) < t_\text{ad}(t_\text{ad}) \). The condition \( t_\text{sy}(\gamma) = t_\text{ad} \) implies \( X = 2 - Z \), which defines a critical time and a critical electron energy

\[
(Z < 1): \quad t_\text{cr} = t_0[(Z - 1)^3 - 1] \in (0, 7)t_0, \quad \gamma(t_\text{cr}) \equiv \gamma_\text{cr} = \frac{Z}{X^3} \gamma_0 = \frac{2t_\text{sy}(\gamma_0)}{3(t + t_\text{ad})} \gamma < \gamma_0, \quad \frac{t_\text{sy}(\gamma)}{t_\text{ad}} = \frac{\gamma_\text{cr}}{\gamma} \quad . \tag{C13}
\]

For \( Z < 1 \), the electron cooling is SY-dominated until \( t_\text{cr} \); at \( t_\text{cr} \), the electron energy is \( \gamma_\text{cr} \) and the powers of the two cooling processes are equal; after \( t_\text{cr} \), AD-cooling is dominant. For \( Z > 1 \), when the electron cooling is AD-dominated at \( t = 0 \), it can be shown using Equation (C6) that \( t_\text{sy}(\gamma) > t_\text{ad} \) at any time; thus, the electron cools adiabatically at all times.

Depending on which cooling process is dominant initially (SY-cooling if \( Z < 1 \), AD-cooling for \( Z > 1 \)), the dominance at later times is established as follows:

\[
\begin{cases} 
\text{if } Z < 1 \text{ then } & \text{SY-cooling: } t_\text{sy}(\gamma(t)) < t_\text{ad}(t) \text{ for } \gamma > \gamma_\text{cr}, \quad t < t_\text{cr}, \quad X < 2 - Z \\
\text{if } Z < 1 \text{ then } & \text{AD-cooling: } t_\text{sy}(\gamma) > t_\text{ad} \text{ for } \gamma < \gamma_\text{cr}, \quad t > t_\text{cr}, \quad X > 2 - Z \\
\text{if } Z > 1 \text{ then } & \text{AD-cooling: } t_\text{sy}(\gamma) > t_\text{ad} \text{ for } \gamma > \gamma_\text{cr}, \quad \text{any } X
\end{cases}
\]

Comparing these expectations with the expanded solution (Equations (C9)–(C11)) for electron cooling, we note that the condition for the SY-cooling solution to be asymptotically displayed at early times, \( t \ll t_0 \), is more restrictive than the condition for SY-cooling to be dominant: \( X < 2 - Z \). Similarly, the conditions for the AD-cooling solution to be asymptotically manifested at early times, \( t < \bar{t} \) (or \( Z < 2 + 1 \)) and \( Z > 1 \), are more restrictive than the condition for AD-cooling to be dominant, \( X > 2 - Z \).

Furthermore, the expanded solution in Equations (C9)–(C11) shows that a change in the evolution of the electron energy is not tied to the competition between the two cooling processes, which defines the critical electron energy \( \gamma_\text{cr} \) (Equation (C13)) that is crossed by the cooling electron at the critical time \( t_\text{cr} \), but instead by the interplay between the SY and AD terms in the solution (Equation (C6)) to the two-process cooling equation. Finally, at late times, \( t \gg \text{max}(t, t_\text{cr}) \), when AD-cooling is dominant \( (X > 2 - Z \text{ for } t > t_\text{cr}) \), the solution to electron cooling is one-third of the SY-cooling solution \( (X \gg Z/2 + 1 \text{ for } t \gg t) \) and not the AD-cooling solution.
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