Exact Renormalization of Massless QED$_2$

Rodolfo Casana S.$^1$ and Sebastião Alves Dias$^{2,3}$

$^1$Instituto de Física Teórica-UNESP
Rua Pamplona, 145, 01405-900, São Paulo, SP, Brazil
$^2$Centro Brasileiro de Pesquisas Físicas, Departamento de
Campos e Partículas
Rua Xavier Sigaud, 150, 22290-180, Rio de Janeiro, RJ, Brazil
$^3$Pontifícia Universidade Católica do Rio de
Janeiro, Departamento de Física
Rua Marquês de São Vicente, 225, 22543-900, Rio de
Janeiro, RJ, Brazil
(January 27, 2022)

Typeset using REVTeX

$^\ast$casana@ift.unesp.br

$^{\dagger}$tiao@cbpf.br
Abstract

We perform the exact renormalization of two-dimensional massless gauge theories. Using these exact results we discuss the cluster property and confinement in both the anomalous and chiral Schwinger models.

1. INTRODUCTION

The gauge principle has been the most important tool to build the models that describe successfully most of the fundamental interactions. The possibility of a perturbative renormalization of gauge theories made this description practical. Quantum gauge invariance was fundamental in producing Ward or Slavnov-Taylor identities that related renormalization constants and allowed cancellation of otherwise intractable divergences. Gauge anomalies apparently destroy quantum gauge invariance and invalidate the perturbative renormalization program. As almost everything that is accessible from these theories is obtained on perturbative grounds, at least in four dimensions, this turned cancellation of gauge anomalies almost into a new physical principle.

In two dimensions, however, it is possible to examine in much more detail the renormalization of anomalous gauge theories. Jackiw and Rajaraman showed the quantum consistency of an anomalous gauge theory (two dimensional massless QED with chiral fermions, also called chiral Schwinger model (CSM)) [1]. Other studies ([2], [3], [4]) showed that the gauge anomaly, at least in this context, far from being a sign of inconsistency, was a source of dynamic richness. The model was shown to exhibit the dynamical mass generation phenomenon for gauge bosons, without fermion screening or confinement, a highly desirable characteristic in a realistic theory of weak interactions. More recently, the conventional Schwinger model (two dimensional massless QED with Dirac fermions) regularized in a non-gauge invariant way [5] (we call it anomalous Schwinger model (ASM)) was considered as a possibly non-confining model with dynamical mass generation as well [6]. In both cases, the greatest emphasis was put in the formal consistency of the theory, in the structure of the Hilbert space, constraints, etc. No questions about renormalization have been raised in these papers, as it was not necessary to consider it in detail for most of these discussions. They concentrated in aspects related to the gauge boson propagator, and bosonic correlation functions are finite (up to regularization of the photon self-energy to one loop).

This does not mean that the renormalization problem was absent. The need for a fermion wave function renormalization has been recognized long time ago both in the CSM [7] [8] and in the ASM [9]. The general structure of this renormalization was studied and clarified in [10]. It is now clear how it can be performed, in a semi-perturbative regime [11], and what is the precise structure and origin of the divergence [12]. But would it be possible that the exact renormalization of these models could be performed?

There are crucial aspects of the models (like the precise infrared behavior of the renormalized correlation functions) that may, in principle, depend on the details of the exact (here taken as the opposite of perturbative) renormalization. These aspects are fundamental to establish physical properties like confinement, for example. Also, one could ask: why should these models be called exactly solvable? In principle, because one could compute
any correlation function, bosonic or fermionic, with an arbitrary number of external legs, exactly, or in other words, to all orders of the coupling constant. For the bosonic ones this is true, but this does not happen for the fermionic ones. After all, thanks to the need of renormalization, we could only compute correlation functions semi-perturbatively \[10\], and this is just a little more than what we can obtain in four dimensional, not exactly solvable theories. Not answering these questions is really not knowing the fermionic sector of the theory (or knowing it very poorly, within the context of a rough approximation).

In this paper we perform the exact renormalization of massless QED$_2$. This means the exact computation of the renormalization constant and the definition of exactly renormalized fermion amplitudes. We use the two-point fermion amplitude in ASM and CSM to investigate the possible existence of asymptotic fermions and the cluster property, thus addressing the question of confinement in these theories. Because the calculations are quite similar for both the CSM and ASM, we will detail our calculations in the ASM, in section 2, and only state the results for the CSM, in section 3. We do this because, as we will show, the ASM has much less trivial physical properties than the CSM, in the sense that two regimes appear after renormalization, one for non-confined and one for eventually confined fermions. In section 4 we present our conclusions.

II. EXACT RENORMALIZATION OF THE ANOMALOUS SCHWINGER MODEL

The anomalous Schwinger model is defined by the following regularized Lagrangian density \[11\],

$$
\mathcal{L}_v^\Lambda [\psi, \bar{\psi}, A] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2 (a_v - 1)}{4\pi\Lambda^2} (\partial \cdot A)^2 + \bar{\psi} i\gamma_k + eA \psi, \tag{1}
$$

where the parameter $\Lambda$ has the dimension of a mass, that goes to infinity at the end of the computations, $\psi$ is a Dirac fermion in two dimensions and $a_v$ is the Jackiw-Rajaraman parameter, representing ambiguities in the regularization of the two point photon function, which is exactly calculated, up to this regularization \[12\] \[13\]. The fermion propagator can be exactly computed and is given by

$$
G_v^\Lambda (x - y) = i \exp \left( i e^2 \int \frac{d^2 k}{(2\pi)^2} f_v^\Lambda (k) \left[ 1 - e^{i k \cdot (x-y)} \right] \right) G_F (x - y), \tag{2}
$$

where $G_F (x - y)$ is the free fermion propagator and

$$
f_v^\Lambda (k) = \frac{2\pi\Lambda^2}{e^2 (a_v - 1) k^2 (k^2 - \Lambda^2)} - \frac{1}{k^2 (k^2 - m_v^2)}, \tag{3}
$$

$m_v$ being the dynamically generated mass for the photon,

$$
m_v = \frac{e^2}{2\pi} (a_v + 1). \tag{4}
$$

One can easily see that, when $\Lambda$ goes to infinity, the function $f_v^\Lambda (k)$ (which, for $k \to \infty$ behaves as $k^{-4}$) goes to
part of which behaves as $k^{-2}$, when $k \to \infty$. This induces a logarithmic divergence in
expression (2). The same divergence appears in arbitrary ($n$ point) fermionic correlation functions
[9]. Treating this divergence in the two point fermionic function is then enough to
define completely the theory.

We begin by noticing that the two point function satisfies a Schwinger-Dyson equation

$$\left(i\partial + ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \not{k} e^{ik \cdot (x-y)}\right) G^\Lambda_v (x-y) = i\delta (x-y).$$

This can be written, in momentum space, as

$$\tilde{G}^\Lambda_v (p) = \frac{i}{p} - ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \frac{1}{p} \not{k} \tilde{G}^\Lambda_v (p-k).$$

Iterating this equation, we obtain an expression for $\tilde{G}$ as a series in $f^\Lambda_v$ that is the semiper-
turbative expansion obtained in [10],

$$\tilde{G}^\Lambda_v (p) = \frac{i}{p} + \sum_{n=1}^{\infty} \tilde{G}^\Lambda_v^{(n)} (p),$$

where we are considering

$$\tilde{G}^\Lambda_v^{(0)} (p) = \frac{i}{p},$$

and $\tilde{G}^\Lambda_v^{(n)} (p)$ is the $n$-loop contribution to the fermion propagator, obtained with the use
of the exact photon propagator. We can use equation (8) together with (7) to obtain a
recurrence relation for the different contributions to $\tilde{G}^\Lambda_v (p),$

$$\tilde{G}^\Lambda_v^{(n+1)} (p) = -ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \frac{1}{p} \not{k} \tilde{G}^\Lambda_v^{(n)} (p-k).$$

To first order, we have the following identity

$$\tilde{G}^\Lambda_v^{(1)} (p) = -ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \frac{1}{p} \not{k} \tilde{G}^\Lambda_v^{(0)} (p-k)$$

$$= -ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \frac{1}{p} \not{k} \left( \frac{i}{p} - \frac{i}{p-k} \right)$$

$$= ie^2 \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v (k) \left( \tilde{G}^\Lambda_v^{(0)} (p) - \tilde{G}^\Lambda_v^{(0)} (p-k) \right),$$

that we can easily generalize, by induction, to all orders.

\[ f_v (k) = -\frac{2\pi}{e^2 (a_v - 1) k^2} - \frac{1}{k^2 (k^2 - m_v^2)}, \]  

(5)
\[ G^{\Lambda(n+1)}_v(p) = \frac{ie^2}{n+1} \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v(k) \left( \tilde{G}^{\Lambda(n)}_v(p) - \tilde{G}^{\Lambda(n)}_v(p-k) \right). \]  

(12)

This expression is the basis of the solution for the Fourier transform of the propagator. It is analogous to the one obtained in [14] in the case \( a_v = 1 \) (in which the ASM reduces to the conventional Schwinger model, thanks to preservation of intermediate gauge invariance). It has its origin in the fact that breaking gauge symmetry in intermediate steps (we mean, by regularization) does not affect the purely fermionic Ward identities.

The integral of the first term in (12) can be performed easily. Defining it as \( I^\Lambda_v \), we obtain

\[ I^\Lambda_v = \int \frac{d^2k}{(2\pi)^2} f^\Lambda_v(k) = i \frac{e^2}{2c^2(a_v - 1)} \ln \left( \frac{\Lambda^2}{-i\varepsilon} \right) - i \frac{e^2}{2c^2(a_v + 1)} \ln \left( \frac{m^2}{-ie} \right). \]  

(13)

It is then easy to obtain, again by induction, the general expression for the \( n \)-loop contribution for the fermion propagator in terms of the 0-loop order,

\[ \tilde{G}^{\Lambda(n)}_v(p) = \left( ie^2 \right)^n \sum_{j=0}^{n} \frac{(-1)^j \left( I^\Lambda_v \right)^{n-j}}{j!(n-j)!} \int \frac{d^2k_1}{(2\pi)^2} f^\Lambda_v(k_1) \int \frac{d^2k_2}{(2\pi)^2} f^\Lambda_v(k_2) \ldots \int \frac{d^2k_j}{(2\pi)^2} f^\Lambda_v(k_j) \tilde{G}^{(0)}_v(p-k_1-k_2-\ldots-k_j). \]  

(14)

Inserting this formula in the expression for the complete fermionic propagator, we obtain

\[ \tilde{G}^\Lambda_v = \exp \left( ie^2 I^\Lambda_v \right) \sum_{n=0}^{\infty} \frac{(-ie^2)^n}{n!} \int \frac{d^2k_1}{(2\pi)^2} f^\Lambda_v(k_1) \int \frac{d^2k_2}{(2\pi)^2} f^\Lambda_v(k_2) \ldots \int \frac{d^2k_n}{(2\pi)^2} f^\Lambda_v(k_n) \tilde{G}^{(0)}_v(p-k_1-k_2-\ldots-k_n). \]  

(15)

Now we perform the \( k_n \) integrations. To do this, we use Schwinger parametrization for the functions \( f^\Lambda_v(k) \),

\[ f^\Lambda_v(k) = \frac{2\pi i}{e^2} \int_0^\infty d\alpha \tilde{f}^\Lambda_v(\alpha) e^{iak^2-\varepsilon\alpha}, \]  

(16)

with

\[ \tilde{f}^\Lambda_v(\alpha) = \frac{1 - e^{-i\alpha\Lambda^2}}{a_v - 1} - \frac{1 - e^{-i\alpha m_v^2}}{a_v + 1}, \]  

(17)

and for the free propagator

\[ \tilde{G}^{(0)}_v(p-k) = \frac{i}{p-k} = \frac{1}{2i} \gamma^\mu \frac{\partial}{\partial p^\mu} \int_0^\infty \frac{d\beta}{\beta} e^{i\beta(p-k)^2-\varepsilon\beta}. \]  

(18)

Integrating over the \( k_n \), we obtain
\( \tilde{G}_v^\Lambda = \frac{i}{p} \exp \left( i e^2 I_v^\Lambda \right) \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \int_0^\infty \frac{d\alpha_1}{\alpha_1} \tilde{f}_v^\Lambda (\alpha_1) \int_0^\infty \frac{d\alpha_2}{\alpha_2} \tilde{f}_v^\Lambda (\alpha_2) \ldots \)

\[ \ldots \int_0^\infty \frac{d\alpha_n}{\alpha_n} \tilde{f}_v^\Lambda (\alpha_n) \int_0^\infty d\beta \left( \frac{1}{\beta} + \sum_{j=1}^{n} \frac{1}{\alpha_j} \right)^{-2} \times \]

\[ \times \exp \left( \frac{i p^2}{\beta} + \sum_{j=1}^{n} \frac{i}{\alpha_j} - \varepsilon \beta - \sum_{j=1}^{n} \alpha_j \right) , \] (19)

The \( \beta \)-integration can be done exactly,

\[ \int_0^\infty \frac{d\beta}{\beta^2} \exp \left( \frac{i p^2}{\beta} + \sum_{j=1}^{n} \frac{i}{\alpha_j} - \varepsilon \beta - \sum_{j=1}^{n} \alpha_j \right) = \frac{i}{p^2} \left[ 1 - \exp \left( \frac{i p^2}{\sum_{j=1}^{n} \frac{1}{\alpha_j}} \right) \right] , \] (20)

so that, inserting it back in (19) and using the identity, valid for \( z > 0 \),

\[ 1 - e^{-1/4z} = \int_0^\infty d\eta J_1 (\eta) e^{-\eta^2} , \] (21)

where \( J_1 (\eta) \) is the first class Bessel function, we obtain

\[ \tilde{G}_v^\Lambda (p) = \frac{i}{p} \exp \left( i e^2 I_v^\Lambda \right) \int_0^\infty d\eta J_1 (\eta) \exp \left[ \frac{1}{2} \int_0^\infty \frac{d\alpha}{\alpha} \tilde{f}_v^\Lambda (\alpha) e^{-\varepsilon \alpha - \eta^2/4 \alpha^2} \right] . \] (22)

Now, using the identity below, involving the second class modified Bessel function \( K_0 \),

\[ \int_0^\infty \frac{d\alpha}{\alpha} \left( 1 - e^{-i \alpha m^2} \right) e^{-\varepsilon \alpha - \eta^2/4 \alpha} = 2K_0 \left( \sqrt{i \varepsilon \eta^2} \right) - 2K_0 \left( \sqrt{-m^2 \eta^2} \right) , \] (23)

we get

\[ \tilde{G}_v^\Lambda (p^2) = \frac{i}{p} \left( \frac{\Lambda^2}{m_v^2} \right)^{1/\left(1-a_v^2\right)} \tilde{G}_v (p^2) , \] (24)

with

\[ \tilde{G}_v (p^2) = \int_0^\infty d\eta J_1 (\eta) \left[ -\frac{\bar{m}_v^2 \eta^2}{p^2} \right]^{1/\left(1-a_v^2\right)} \exp \left\{ \frac{1}{1+a_v} K_0 \left( \sqrt{-\frac{m_v^2 \eta^2}{p^2}} \right) \right\} , \] (25)

where \( \bar{m}_v^2 = (m_v^2 e^{2\gamma_E})/4 \). This expression is enough to compute the exact fermion wave function renormalization needed to renormalize the theory. Proceeding as is usual, we impose a suitable renormalization condition on the 1PI renormalized two point function

\[ \tilde{\Gamma}_v^R (p) \mid_{p=\mu} = 0 , \quad \tilde{\Gamma}_v^R (p) \tilde{G}_v (p) = i , \quad \tilde{G}_v^R (p) = Z_{\psi}^{-1} \tilde{G}_v^\Lambda (p) . \] (26)

This shows that
\[ Z_\psi^v = \left( \frac{\Lambda^4}{m_v^2} \right)^{1/2(1-a_v)} \tilde{G}_v \left( \mu^2 \right). \]  

(27)

So, the expression for the exact renormalized two point function (and its 1PI counterpart) is

\[ \tilde{G}_v^R(p) = \frac{i}{\tilde{p}} \frac{\tilde{G}_v(p^2)}{G_v(\mu^2)}, \quad \tilde{\Gamma}_v^R(p) = \frac{\tilde{p}}{\tilde{p}} \frac{\tilde{G}_v(\mu^2)}{G_v(p^2)}. \]

(28)

Now it is easy to obtain, taking into account the properties of \( K_0(\eta) \) and \( J_1(\eta) \), when \( \eta \to 0 \), the exact behavior of the fermion propagator, when \( p \to 0 \),

\[ \tilde{G}_v^R(p) \approx \frac{i}{\tilde{p}} \left( \frac{p^2}{\tilde{m}_v^2} \right)^{1/(a_v^2-1)}. \]

(29)

For \( a > 1 \), the propagator has, at most, a pole with null residue in \( p^2 = 0 \), what indicates that there are no single particle states, as it happens in the conventional Schwinger model [15], in the Thirring model [16] and in the Schröer model [17]. In particular we stress that this behavior is the same as that of the exact propagator of the Thirring model, if we perform a suitable correspondence between the coupling constants. The Thirring model is characterized by the following Lagrangian density

\[ \mathcal{L}_{Th} = \bar{\psi}i\gamma^\mu \partial_\mu \psi - \frac{g^2}{2} \left( \bar{\psi} \gamma^\mu \psi \right)^2, \]

(30)

with a dimensionless coupling constant \( g > 0 \). It is possible, by using the same techniques developed in this section, to find the exact renormalized propagator of this model

\[ \tilde{G}_{Th}^R(p) = \frac{i}{\tilde{p}} \left( \frac{p^2}{\mu^2} \right)^{g^2/(1+2g)}, \quad \bar{g} = g/2\pi. \]

(31)

If we choose \( a_v > 1 \), such that

\[ a_v = 1 + \frac{1}{\bar{g}}, \]

(32)

we see that the behavior at low momentum of the ASM propagator is the same as that of the exact Thirring model propagator.

In the interval \(-1 \leq a_v < 1\), there are no simple poles at \( p^2 = 0 \), what means that there are no asymptotic one particle states in this case too. However, it exhibits poles of order greater than one at \( p^2 = 0 \), a symptom of infrared slavery [18], what can mean that fermions are confined in this sector. The value of \( a_v \), in this case, determines the order of the pole.

For \( a_v = 1 \), the infrared behavior is

\[ \tilde{G}_{a_v=1}(p) \approx \frac{i}{\tilde{p}} \left( \frac{\bar{c}^2/\pi}{p^2} \right)^{1/4}, \]

(33)

which is a well known result [14] [15]. One can see directly that the residue of the pole at \( p^2 = 0 \) is null, which means the absence of asymptotic fermionic states. The question of
whether this means confinement or not is more subtle, and requires analysis of other kinds of correlation functions, including the bosonic ones.

More information can be obtained if we come back to coordinate space, where we will see that there is an even more explicit expression for the renormalized fermion propagator. In doing that, first we continue the propagator to Euclidean space making

\[ p_0 = ip_2, \quad \gamma_0 = i\gamma_2. \]

Then, in expression (25) after renormalization, we perform the following change of variables, remembering that \( p = \sqrt{p_\mu p^\mu}, \)

\[ \eta = xp, \]  

and use the identity

\[ \gamma_\mu \frac{\partial}{\partial p_\mu} J_0 (xp) = -x \frac{p}{p} J_1 (xp), \]

to obtain

\[
\tilde{G}^R_{\nu E} (p) = \frac{i}{G_v (\mu^2)} \gamma_\mu \frac{\partial}{\partial p_\mu} \int_0^\infty dx \frac{J_0 (xp)}{x} \left[ \frac{m_v^2 x^2}{x^2} \right]^{1/(1-a_v^2)} 
\times \exp \left\{ \frac{1}{1 + a_v} K_0 \left( \sqrt{\frac{m_v^2 x^2}{x^2}} \right) \right\}. \]

Using now the following representation for \( J_0 \)

\[ J_0 (xp) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-ixp \cos (\theta - \alpha)}, \]

with \( \alpha = \arctan (p_2/p_1) \), and defining \( x_1 = x \cos \theta \) and \( x_2 = x \sin \theta \), we see that \( xp \cos (\theta - \alpha) = x \cdot p \). So,

\[
\tilde{G}^R_{\nu E} (p) = \frac{i/2\pi}{G_v (\mu^2)} \int_0^\infty d^2x e^{-ixp} \frac{\hat{f}}{x^2} \left[ \frac{m_v^2 x^2}{x^2} \right]^{1/(1-a_v^2)} 
\times \exp \left\{ \frac{1}{1 + a_v} K_0 \left( \sqrt{\frac{m_v^2 x^2}{x^2}} \right) \right\}. \]

Continuing back to Minkowski space,

\[
\tilde{G}^R_{\nu} (p) = -\frac{i/2\pi}{G_v (\mu^2)} \int_0^\infty d^2x e^{ixp} \frac{\hat{f}}{x^2} \left[ -\frac{m_v^2 x^2}{x^2} \right]^{1/(1-a_v^2)} 
\times \exp \left\{ \frac{1}{1 + a_v} K_0 \left( \sqrt{-\frac{m_v^2 x^2}{x^2}} \right) \right\}, \]

we can now clearly read the expression for the renormalized propagator in coordinate space, for spacelike \( x^\mu \),

\[ G^R_{\nu} (x) = \frac{i}{G_v (\mu^2)} \exp \left\{ \frac{1}{1 - a_v^2} \ln (-m_v^2 x^2) + \frac{1}{1 + a_v} K_0 \left( \sqrt{-m_v^2 x^2} \right) \right\} G_F (x). \]

It is easy to obtain the expression valid for timelike \( x^\mu \), again by analytic continuation, remembering the continuations of \( K_0 \) and of the logarithm,
\[ K_0 \left( \sqrt{-m_v^2} \right) \rightarrow \frac{i\pi}{2} H_0^{(1)} \left( \sqrt{m_v^2 x^2} \right), \quad x^2 > 0 \]

\[ \ln \left( -\tilde{m}_v^2 x^2 \right) \rightarrow -i\pi + \ln \left( \tilde{m}_v^2 x^2 \right), \quad x^2 > 0 \]

where \( H_0^{(1)} \) is the first class Hankel function. The expression below allows us to analyze the cluster property of the propagator explicitly:

\[ G^R_v(x, y) = \frac{i}{G_v(\mu^2)} \left[ -\tilde{m}_v^2 (x - y)^2 \right]^{1/(1-a_v^2)} \]

\[ \times \exp \left\{ \frac{1}{1 + a_v} K_0 \left( \sqrt{-\tilde{m}_v^2 (x - y)^2} \right) \right\} G_F(x - y). \]

Doing \( x \rightarrow x + \lambda \eta \), while keeping \( y \) fixed, and then \( \lambda \rightarrow +\infty \) we obtain

\[ G^R_v(x + \lambda \eta, y) \rightarrow \frac{i}{G_v(\mu^2)} \left[ -\tilde{m}_v^2 \eta^2 \right]^{1/(1-a_v^2)} G_F(\eta), \]

which means

\[ G^R_v(x + \lambda \eta, y) \rightarrow \begin{cases} \infty, & a_v < 1 \\ 0, & a_v > 1 \end{cases} \]

So, in the interval \(-1 < a_v < 1\) the cluster property is lost, while it is apparently maintained for \( a_v > 1 \) (the decision about this last fact depends obviously on the analysis of the behavior of arbitrary correlation functions). This result, and all the others in this section, would be impossible to be guessed without the detailed analysis of the limits, that became possible thanks to the exact expressions furnished after renormalization. It can be easily seen also that the previous analysis in momentum space is completely consistent with the results obtained in coordinate space.

We can also compute the short distance behavior of the propagator

\[ G^R_v(x) \rightarrow x^2 \rightarrow 0 \left[ -\tilde{m}_v^2 x^2 \right]^{1/2(1-a_v)} G_F(x) \approx |x|^{a_v/(1-a_v)}, \]

which shows explicitly the anomalous dimension under scaling (anomalous in the sense of being distinct from the scaling dimension of the free propagator). Only the limit \( a_v \rightarrow +\infty \) produces a scaling behavior similar to the free fermion propagator (\(|x|^{-1}\)).

At this point it is worth commenting on a denomination usually given in the literature \[6\] for the ASM: it is called also non-confining Schwinger model. Mitra and Rahaman recognized the need for a renormalization, but did not performed it in detail. In the light of the previous discussion, we see that there are reasons to believe that there is fermion confinement (thanks to infrared slavery and violation of cluster property, both indicated by our results) when the parameter \( a_v \) lies in the range \([-1, 1)\). It is this behavior that makes this terminology inadequate for all values of \( a_v \). Again, before a detailed consideration of renormalization, it is not possible to say that the model is “confining” or “non-confining”.
III. EXACT RENORMALIZATION OF THE CHIRAL SCHWINGER MODEL

As we said in the introduction the renormalization procedure is quite similar to that of the ASM. So, we will present the modifications that appear thanks to the particular structure of the CSM and directly discuss the implications of exact renormalization.

The regularized Lagrangian is given by

\[ \mathcal{L}^\Lambda_v [\psi, \bar{\psi}, A] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2 (a_c - 1)}{8\pi\Lambda^2} (\partial \cdot A)^2 + \bar{\psi} (i \partial + e A^+ \gamma_5) \psi, \tag{47} \]

where \( P_+ = (1 + \gamma_5)/2 \), and we work with Dirac fermions in order to have the eigenvalue problem for the determinant of the Dirac operator well defined. The regularized fermion propagator is

\[ G^\Lambda_v (x - y) = i \exp \left( i e^2 \int \frac{d^2 k}{(2\pi)^2} f_c^\Lambda (k) \left[ 1 - e^{ik \cdot (x - y)} \right] \right) P_+ G_F (x - y) + i P_- G_F (x - y), \tag{48} \]

where now,

\[ f_c^\Lambda (k) = -\frac{(\Lambda^2 - \omega_m^2)}{\omega_m^2} \frac{1}{(k^2 - \Lambda^2)(k^2 - \omega_m^2)} + \frac{(\Lambda^2 - \omega_m^2)}{\omega_m^2} \frac{1}{(k^2 - \Lambda^2)(k^2 - \omega_m^2)} \tag{49} \]

with \( \omega_\Lambda \) and \( \omega_m \) satisfying the equations

\[ \omega_\Lambda^2 + \omega_m^2 = \Lambda^2 + \frac{e^2 (a_c + 1)}{4\pi}, \]
\[ \omega_\Lambda^2 \omega_m^2 = \Lambda^2 m_c^2. \tag{50} \]

The parameter \( m_c \) is given by

\[ m_c^2 = \frac{e^2}{4\pi} \frac{a_c^2}{a_c - 1}, \tag{51} \]

and is the usual dynamically generated mass for the photon in this model \[1\]. The Dyson-Schwinger equation satisfied by the right piece of the propagator \( G_{c^+}^\Lambda (x) = P_+ G_c^\Lambda (x) \) is, in coordinate space,

\[ \left( i \partial + ie^2 \int \frac{d^2 k}{(2\pi)^2} f_c^\Lambda (k) \gamma_5 e^{ik \cdot (x - y)} \right) G_{c^+}^\Lambda (x - y) = iP_- \delta (x - y), \tag{52} \]

and, in momentum space

\[ \tilde{G}_{c^+}^\Lambda (p) = P_+ \frac{i}{\not{p} - ie^2 \int \frac{d^2 k}{(2\pi)^2} f_c^\Lambda (k) \gamma_5 \not{k} G_{c^+}^\Lambda (p - k)}. \tag{53} \]
Again, this equation can be solved recursively in the same way that we did in Section 2. We arrive at an expression similar to equation (15), with an \( \bar{f}^\Lambda (\alpha) \) given by

\[
\bar{f}^\Lambda (\alpha) = \frac{e^\Lambda (\frac{\Lambda^2}{\omega_c^2} - \omega^2)}{2\pi \omega_c^2 \omega_m^2} \frac{\Lambda^2 - \omega_c^2}{\Lambda^2 - \omega_m^2} \left( e^{-i\alpha\Lambda^2} - e^{-i\alpha\omega_m^2} \right) \left( \frac{\Lambda^2}{\omega_m^2} \right), \tag{54}
\]

Following the same steps indicated before we arrive at

\[
\tilde{G}_c^\Lambda^+ (p) = P_+ \frac{i}{p} \left( \frac{\Lambda^2}{m_c^2} \right)^{1/(1-a_c)} \tilde{G}_c (p^2), \tag{55}
\]

with

\[
\tilde{G}_c (p^2) = \int_0^\infty d\eta J_1 (\eta) \exp \left\{ \frac{2}{a_c - 1} K_0 \left( \sqrt{-m_c^2 \eta^2} \right) \right\}. \tag{56}
\]

Now we use the relation between renormalized and bare quantities (\( \tilde{G}_c^R^+ (p) = Z_\psi^{-1} \tilde{G}_c^\Lambda^+ (p); \tilde{\Gamma}_c^R^+ (p) = Z_\psi \tilde{\Gamma}_c^\Lambda^+ (p) \)) and the renormalization condition,

\[
\tilde{\Gamma}_c^R^+ (p) \bigg|_{p = \mu} = \mu P_+, \tag{57}
\]

to fix \( Z_\psi \),

\[
Z_\psi = \left( \frac{\Lambda^2}{m_c^2} \right)^{1/(1-a_c)} \tilde{G}_c (\mu^2). \tag{58}
\]

This gives us immediately the renormalized two-point functions

\[
\tilde{G}_c^R^+ (p) = P_+ \frac{i}{p} \tilde{G}_c (p^2), \quad \tilde{\Gamma}_c^R^+ (p) = \mu P_+ \tilde{G}_c (\mu^2). \tag{59}
\]

Here, also, it is possible to come back to coordinate space to obtain

\[
G_c^R^+ (x) = \frac{i}{G_c (\mu^2)} \exp \left\{ \frac{2}{a_c - 1} K_0 \left( \sqrt{-m_c^2 x^2} \right) \right\} P_+ G_F (x), \tag{60}
\]

where \( x^\mu \) is spacelike.

This propagator has the following behavior for \( x^2 \to \infty \),

\[
G_c^R^+ (x) \xrightarrow{x^2 \to \infty} P_+ G_F (x), \tag{61}
\]

as is well known in the conventional analysis of the CSM \[7]. The cluster property is preserved and there exist asymptotic fermions. There are not two different regimes, as in the case of the ASM, and the parameter \( a_c \) represents only an ambiguity in the mass of the photon. Our results confirm the ones known in literature, but with a much higher degree of rigor.

A final remark concerns the short distance behavior of the propagator,

\[
G_c^R^+ (x) \xrightarrow{x^2 \to 0} \left( -\frac{m_c^2 x^2}{2} \right)^{1/(1-a_c)} P_+ G_F (x), \tag{62}
\]

that again (as in the case of the ASM) reveals an anomalous scaling dimension. This, however, does not prevent the existence of asymptotic fermions.
IV. CONCLUSIONS AND PERSPECTIVES

Both CSM and ASM have an extremely similar divergence structure and can be regularized by the introduction of a term that is formally similar to a gauge fixing term with an infinite gauge parameter (in fact it has its origin in a Pauli-Villars regularization of the propagator of the longitudinal part of the gauge field [10]). The origin of the divergence is the lack of intermediate gauge invariance, which forces one to take into account the longitudinal degree of freedom or, in the gauge invariant formalism, the Wess-Zumino field. Both, when exactly integrated, give rise to a divergence in fermionic correlation functions, thanks to their interaction with singular currents. If one uses the complete photon propagator in the loopwise expansion of the fermionic correlation functions, then the divergence is under control and can be renormalized in a quite conventional way, defining what we called a semi-perturbative approach.

The sensation that the exact renormalization of both ASM and CSM was possible remained. The divergence could be explicitly computed, in its regularized form, non-perturbatively. The main obstacle to remove it is that the renormalization procedure in coordinate space is not well known. Usually, one goes to momentum space, where it is clear what are the renormalization conditions to be imposed in order to fix systematically the form of the renormalization constants. The point is that bosonization (necessary to establish the precise form of the divergence and ultimately to exactly solve the model) is only known in coordinate space. One way to go to momentum space would be in a semi-perturbative way, as we have shown [10], but then one looses the complete information about the divergence, present in coordinate space.

In this paper we performed the exact renormalization of some gauge theories by performing exactly the necessary Fourier transform. Although exact renormalizations have been already achieved, within the context and using specific features of supersymmetric theories [21], this is the first time (to our knowledge) that this is done outside this context. We see that the result for the ASM depends strongly on the value assumed by the Jackiw-Rajaraman parameter $a_v$. The precise form of this dependence could not be guessed perturbatively and the value $a_v = 1$ seems to be critical to characterize confinement or not.

To answer this question completely we should compute generalized fermionic correlation functions and verify the cluster property in connection to the appearance of $\theta$-vacua. A analysis of the Wilson loop, for the renormalized theory, can help to clarify this.

The chiral case seems to be much simpler in this respect. There, we found that the Jackiw-Rajaraman parameter $a_c$ does not play any distinctive role in characterizing different regimes (what is consistent with several other results in literature, that say, for example, that non-trivial topology sectors do not contribute to the correlation functions in the CSM [20]). The role of chiral gauge symmetry in this phenomenon deserves more investigation.

Could we arrive at similar conclusions without performing the exact renormalization? In the chiral model, the formal elimination of the UV divergence allows one to get the correct information about the short (UV) and long (IR) distances behavior. So, one would be tempted to say that, to obtain this kind of information, renormalization is superfluous. This is not true. Let us briefly comment on a previous paper on the ASM where renormalization is not performed explicitly [3]. The fermionic propagator does not show the $Z_\psi$ wavefunction renormalization constant (although the authors mention it). The formal elimination of the
UV divergence does not allow one to obtain information on the short (UV) and long (IR) distances behavior because the massless boson propagator $D_F(x)$ has still a dependence in the infrared cut-off. So, it is premature to speak of the model as “non-confining”. Preliminary analysis, carried by us, suggests two quite different phases for the anomalous Schwinger model, according to the condition $a < 1$ or $a > 1$. If this analysis can be carried somehow to four dimensions, it can give quite different perspectives to the problem of confinement. This is being actively investigated by us and will be reported elsewhere.

We are finally in position of computing “physical” (two dimensional) amplitudes, like Compton scattering, for example. This would be very important, to finally decide about the correct degree of arbitrariness represented by the Jackiw-Rajaraman parameter $a$. To see the complete effect of $a$ in the physical results we should be able to produce the analog of a Källen-Lehmann representation for the propagator of a theory with dynamical mass generation, in order to produce a LSZ formula.

We could say that finally the denomination “exactly soluble model” makes full sense. Only after the exact renormalization one can know, in principle, all correlation functions of the model. For a more physical point of view, a Hilbert space analysis of the ASM should be conducted in detail (the corresponding analysis already exists for the CSM \cite{8}). This analysis would identify physical operators and the different sectors present in the Hilbert space of the model, stating clearly what symmetries are present and what are violated at quantum level. This and the other questions mentioned in earlier paragraphs are actually under active investigation, and results will be reported elsewhere.

Acknowledgment: This work is part of the doctorate thesis of R. Casana at CBPF. He is now supported by a CNPq post-doctoral fellowship at IFT. We would like to dedicate this paper to the memory of our teacher and friend Prof. Juan Alberto Mignaco.
REFERENCES

[1] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54, 1219 (1985).
[2] L. D. Faddeev and S. L. Shatashvili, Phys. Lett. B 167, 225 (1986).
[3] O. Babelon, F. A. Schaposnik and C. M. Viallet, Phys. Lett. B 177, 385 (1986).
[4] K. Harada and I. Tsutsui, Phys. Lett. B 183, 311 (1987).
[5] R. Jackiw and K. Johnson, Phys. Rev. 182, 1459 (1969).
[6] P. Mitra and A. Rahaman, Ann. Phys. 249, 34 (1996).
[7] H. O. Girotti, H. J. Rothe and K. D. Rothe, Phys. Rev. D 34, 592 (1986).
[8] D. Boyanovsky, Nucl. Phys. B 294, 223 (1987).
[9] Z. Jian-Ge, D. Qing-Hai and L. Yao-Yang, Phys. Rev. D 43, 613 (1991).
[10] R. Casana and S. A. Dias, Int. Journ. Mod. Phys. A, 15, 4603 (2000).
[11] R. Casana and S. A. Dias, Journ. Phys. G: Nucl. Part. Phys. 27, 1501 (2001).
[12] R. Jackiw, Topological investigations of Quantized Gauge Theories, in Relativity, Groups and Topology II (Les Houches 1983), eds. B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).
[13] S. A. Dias and C. A. Linhares, Phys. Rev. D, 45, 2162 (1992).
[14] T. Radozycki, Eur. Phys. Jour. C6, 549 (1999).
[15] J. Schwinger, Phys. Rev. 125, 397 (1962).
[16] W. Thirring, Ann. Phys. 3, 91 (1958); K. Johnson, Nuovo Cim. 20, 773 (1961).
[17] B. Schroer, Forts. Phys. 11, 1 (1963).
[18] G. Morchio and F. Strocchi, Ann. Inst. Henri Poincaré 33, 251 (1980).
[19] K. Stam, J. Phys. G: Nucl. Phys. 9, L229 (1983).
[20] S. A. Dias and C. A. Linhares, Phys. Rev. D 47 1672 (1993).
[21] M. Shifman and A. Vainshtein, ITEP Lectures in Particle Physics and Field Theory, Vol. 2, pp. 485-648, ed. M. Shifman, Singapore, World Scientific, 1999; hep-th/9902018.