The finite one–dimensional wire problem

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Abstract

We discuss an elementary problem in electrostatics: What does the charge distribution look like for a free charge on a strictly one-dimensional wire of finite length? To the best of our knowledge this question has so far not been discussed anywhere. One notices that a solution of this problem is not as simple as it might appear at first sight.
I Introduction

Some time ago one of the authors of this paper was confronted with the following question by a student: What does the charge distribution look like for a free charge on a strictly one-dimensional wire of finite length, if we consider the usual Coulomb repulsion law? Despite its rather trivial appearance this problem led to controversial discussions. The immediate “obvious” answer by most considering the problem for the first time is to suggest a charge distribution like the one in Fig. 1: That is one intuitively expects an accumulation of charge at the ends of the wire since there is no repelling charge outside. We urge the reader to make up his mind too before reading on.

Let us remark that one-dimensional systems of electrons interacting with long-range Coulomb forces have attracted much attention recently since it has become possible to realize them experimentally as one-dimensional semiconductor structures\(^3\). To avoid misunderstandings we want to emphasize that this paper does not aim at contributing in this direction of research, though this serves as an interesting background. Quantum effects and the fermionic nature of electrons are not taken into account here. We rather want to introduce a nice exercise in classical electrostatics for students or anybody else interested in elementary problems. Only very elementary tools of mathematics and physics will be used, still at first sight the answer might appear counter-intuitive and surprising.

Let us make one more remark in order to avoid confusion. We do not discuss a problem in a one-dimensional world, but rather a one-dimensional problem defined in three-dimensional space. Therefore we use the three-dimensional Coulomb law \(V(r) \propto |r|^{-1}\). Notice that starting from the one-dimensional Laplace equation the interaction potential has the different form \(V(r) \propto |r|\). This then defines the well-investigated one-dimensional Coulomb gas model that has been solved exactly in the thermodynamic limit independently by Prager\(^4\) and Lenard\(^5\). Unfortunately the techniques used in these solutions cannot be carried over to our problem\(^6\) and we have to rely on other tools in the sequel.

II Regularization prescriptions

Once one starts analyzing the problem, one immediately notices that it is ill-defined in its original formulation. For simplicity we will assume that the wire has unit length
throughout this work: The charge distribution $\rho(x)$ is defined on the interval $[0, 1]$ of the x–axis. In order to solve our problem we want to minimize the energy functional

$$E[\rho] = \frac{1}{2} \int_0^1 dx \int_0^1 dy \frac{\rho(x) \rho(y)}{|x - y|}$$

(1)

under the constraints of a fixed total charge $Q$ here set to one

$$\int_0^1 dx \, \rho(x) = 1$$

(2)

and

$$\rho(x) \geq 0.$$  

(3)

The integral (1) diverges since

$$\left( \int_{x-\epsilon}^{x+\epsilon} dy \right) \frac{1}{|x - y|} \sim 2 \ln \epsilon.$$  

(4)

We will discuss two obvious possibilities to make the problem well–defined, i.e. ways to regularize the energy functional (1). It is not immediately clear that both give the same answer.

(i) Define the one–dimensional wire to be the limit of ellipsoids in three–dimensional space when two semiprincipal axes of the ellipsoid shrink to zero. This way we use a regularization by going to a well–defined problem in the embedding three–dimensional space, thereby avoiding the singular one–dimensional problem. This approach is discussed in Sec. III.

(ii) A physically appealing approach is to put $n$ equally charged classical particles in equilibrium on the wire. Each particle has an individual charge $q_i = 1/n$. This defines the looked for charge density in the limit $n \to \infty$. Although this regularization procedure requires more effort than (i), it is more convincing in the sense of being a “microscopic” approach. This makes up the main part of our paper and is worked out in Sec. IV.

Of course other regularization procedures are also possible. A very natural choice would be

$$V_d(r) = \frac{1}{\sqrt{r^2 + d^2}} \xrightarrow{d \to 0} V(r) = \frac{1}{|r|}$$

(5)

since this is usually used for one–dimensional systems interacting with long–range Coulomb forces. The main advantage is that the one–dimensional Fourier transform of $V_d(r)$ exists for finite $d$. However, no exact solution seems possible for finite $d$ and it is therefore difficult to investigate the limit $d \to 0$. 

4
III Shrinking ellipsoids

Consider the conducting ellipsoid in Fig. 2 with semiprincipal axes $a, b$ and $c$. One can argue that for $b, c \to 0$ the ellipsoid shrinks to a one-dimensional wire of finite length $2a$.

Now the potential problem of a free charge $Q$ on a conducting ellipsoid has been well-known for a long time and is treated in many advanced textbooks on electrostatics, see for example Ref. 1. The explicit solution relies on the fact that the Laplace equation is separable in elliptic coordinates. Let us simply quote the result for the potential

$$V(\xi) = \frac{Q}{8\pi} \int_\xi^\infty \frac{d\lambda}{\sqrt{(\lambda + a^2)(\lambda + b^2)(\lambda + c^2)}}$$

(6)

with $\xi = \xi(x, y, z)$ defined implicitly by

$$\frac{x^2}{a^2 + \xi} + \frac{y^2}{b^2 + \xi} + \frac{z^2}{c^2 + \xi} = 1.$$  
(7)

The surface charge $\sigma$ is given by the normal derivative of the potential at the surface

$$\sigma = -\left( \frac{\partial V}{\partial n} \right)_{\xi=0}.$$  
(8)

In rectangular coordinates this is

$$\sigma = \frac{Q}{4\pi abc} \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$  
(9)

For our purposes we can assume $b = c$. One would argue that the total surface charge in the strip $S$ in Fig. 2 collapses onto the line charge $\tau(x) \, dx$ of the wire at that point $x$

$$\tau(x) = \sigma(x) \cdot S(x),$$  
(10)

where $S(x) \, dx$ is the surface of the strip $S$. It is a simple exercise in geometry to show

$$S(x) \, dx = 2\pi b \sqrt{1 - \frac{x^2}{a^2} \left( 1 - \frac{b^2}{a^2} \right)} \, dx.$$  
(11)

On the other hand, evaluating Eq. (9) at the surface gives

$$\sigma(x) = \frac{Q}{4\pi ab} \frac{1}{\sqrt{1 - \frac{x^2}{a^2} \left( 1 - \frac{b^2}{a^2} \right)}}.$$  
(12)
This implies that independent of \( b \) the line charge density defined like in Eq. (10) is constant along the wire

\[
\tau(x) = \frac{Q}{2a}.
\]

(13)

According to this reasoning there are no finite size effects in our one-dimensional wire problem: There is no accumulation of charge at the ends of the wire! 

The sceptical reader can certainly question the validity of this proof by pointing out that even for infinitesimal \( b \), in some respects the ellipsoid is in no way similar to a wire. E.g. the ends of the ellipsoid are always much thinner than its middle, and the smaller available space may compensate a charge accumulation at the ends of the wire. Therefore we will use a different, more microscopic regularization in the next section.

IV Regularization by discretization

We regularize the singular one-dimensional problem by considering \( n \) equally charged classical particles with individual charges \( 1/n \) in the interval \([0, 1]\). Due to Coulomb repulsion there is a unique equilibrium state with charges at positions \( x_i \). In the limit \( n \to \infty \) this will define the continuum charge distribution on the wire that we are interested in.

IV.1 Upper and lower bounds

First of all we will show that the energy of the discretized problem diverges like \( \ln n \) but can still be determined within bounds of width \( \frac{1}{2} \). This will be done by calculating an upper and a lower bound for the energy of the equilibrium distribution.

An upper bound is given by the energy \( E_{\text{max}} \) of a uniform distribution with charges at positions \( x_i = (i - 1)/(n - 1), \ i = 1 \ldots n \)

\[
E_{\text{max}}(n) = \frac{1}{2} \sum_{i \neq j}^{n} \frac{1}{|x_i - x_j|} = \frac{n - 1}{2n^2} \sum_{j=1}^{n} \left( \sum_{i=1}^{j-1} \frac{1}{i} + \sum_{i=1}^{n-j} \frac{1}{i} \right).
\]

(14)

We use the well-known asymptotic behaviour (see e.g. formula 0.131 in Ref. 2)

\[
\sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + O(1/n),
\]

(15)
where $\gamma = 0.5772\ldots$ is Euler’s constant, and find

$$E_{\text{max}}(n) = \frac{n-1}{2n^2} \left( 2 \sum_{j=1}^{n-1} \ln j + 2n\gamma + O(\ln n) \right)$$

$$= \ln n + \gamma - 1 + O(1/n) \ln n.$$ \hspace{1cm} (16)

In the last step we have used Stirling’s formula.

In order to derive a lower bound $E_{\text{min}}$ we first of all notice that the energy of a discrete distribution can be split up as

$$E = E_1 + E_2 + \ldots + E_{n-1},$$ \hspace{1cm} (17)

where $E_k$ is the sum of energies of all particles to the $k$-th neighbours (see Fig. 3)

$$E_k = \frac{1}{n^2} \sum_{i,i+k \leq n} \frac{1}{|x_{i+k} - x_i|}.$$ \hspace{1cm} (18)

It is straightforward to see that each $E_k$ is bounded from below by the energy obtained if a uniform distribution of the $k$-th neighbours is assumed. This gives e.g. in Eq. (18)

$$E_1 \geq \frac{1}{n^2} (n-1)^2.$$ \hspace{1cm} (19)

Furthermore we have for even $n$

$$E_2 \geq \frac{2}{n^2} \left( \frac{n}{2} - 1 \right)^2$$ \hspace{1cm} (20)

and for odd $n$

$$E_2 \geq \frac{1}{n^2} \left( \left( \frac{n+1}{2} - 1 \right)^2 + \left( \frac{n-1}{2} - 1 \right)^2 \right) \geq \frac{2}{n^2} \left( \frac{n}{2} - 1 \right)^2.$$ \hspace{1cm} (21)

These results are easily generalized for arbitrary $k$

$$E_k \geq \frac{k}{n^2} \left( \frac{n}{k} - 1 \right)^2.$$ \hspace{1cm} (22)

A lower energy bound $E_{\text{min}}$ is therefore

$$E_{\text{min}} = \frac{1}{n^2} \sum_{k=1}^{n-1} k \left( \frac{n}{k} - 1 \right)^2 = \sum_{k=1}^{n-1} \left( \frac{1}{k} - \frac{2}{n} + \frac{k}{n^2} \right)$$

$$= \ln n + \gamma - \frac{3}{2} + O(1/n)$$ \hspace{1cm} (23)

where we have again used Eq. (16).

Comparing Eq. (23) with the upper bound from Eq. (16) one notices that we have obtained rather strict limits for the energy of the equilibrium state with minimum energy. Obviously a uniform distribution cannot be too far from the true equilibrium solution with minimum energy. That this is indeed true will be shown in the next section.
IV.2 Rigorous results for the charge distribution

Theorem Suppose the charge distribution $\rho_n(x)$ for $n$ particles converges for $n \to \infty$ towards a continuous distribution $\rho(x)$ defined on $]0, 1[$. Then $\rho(x)$ is constant.

Proof: Let us assume that the theorem is wrong and there exist $y_1, y_2 \in ]0, 1[$ with $\rho(y_1) > \rho(y_2)$, see Fig. 4. It shall be shown that there exists an $N$ so that for $n > N$ transferring a particle from $y_1$ to the middle of two particles at $y_2$ results in a decrease of the energy. Hence the particles were not in equilibrium for $n > N$ and the theorem is proven.

This can be seen as follows: First of all we regard the energy needed to transfer a particle $P$ from $y_1$ to $y_2$ while neglecting the particles in $[y_1 - \varepsilon, y_1 + \varepsilon]$ and in $[y_2 - \varepsilon, y_2 + \varepsilon]$. Here $\varepsilon$ has been chosen so small that both intervals are completely in $]0, 1[$ and do not overlap. The change of energy is

$$n \Delta E_I = \int_0^1 dt \left( \frac{\tilde{\rho}(t)}{|y_1 - t|} - \frac{\tilde{\rho}(t)}{|y_2 - t|} \right) \xrightarrow{n \to \infty} O(\varepsilon), \quad (24)$$

where

$$\tilde{\rho}(t) = \begin{cases} 0 & \text{in } [y_1 - \varepsilon, y_1 + \varepsilon] \text{ and in } [y_2 - \varepsilon, y_2 + \varepsilon] \\ \rho(t) & \text{elsewhere}. \end{cases} \quad (25)$$

It is essential to note that Eq. (24) does not contain any term divergent in $n$, but has a finite and well-defined limit for $n \to \infty$. Let us now calculate the energy $\Delta E_1$ of the particle $P$ with respect to the other particles in the interval $[y_1 - \varepsilon, y_1 + \varepsilon]$. We can assume the distribution to be locally uniform, deviations from that will only contribute in order $\varepsilon^2$. Thus the particles in the interval $[y_1, y_1 + \varepsilon]$ are at positions

$$x_i = y_1 + \frac{\varepsilon}{nq} i, \quad i = 1 \ldots nq, \quad (26)$$

where $q = \varepsilon \rho(y_1)$ is the total charge in $[y_1, y_1 + \varepsilon]$. This gives

$$n \Delta E_1 = \frac{2}{n} \sum_{i=1}^{nq} \frac{1}{|x_i - y_1|} = 2 \sum_{i=1}^{nq} \frac{\rho(y_1)}{i} + O(\varepsilon) = 2 \rho(y_1) \ln(\rho(y_1)\varepsilon n) + O(n^0), \quad (27)$$

where the sum is done like in the previous section. Particle $P$ is placed in the middle between two particles at $y_2$ and this costs the extra energy $\Delta E_2$

$$n \Delta E_2 = 2 \sum_{i=1}^{nq} \frac{\rho(y_2)}{i - \frac{1}{2}} + O(\varepsilon) = 2 \rho(y_2) \ln(\rho(y_2)\varepsilon n) + O(n^0). \quad (28)$$
The total change of energy is

\[ n(\Delta E_I + \Delta E_1 - \Delta E_2) = 2 (\rho(y_1) - \rho(y_2)) \ln n + O(n^0) \].

(29)

For \( n \) large the expression is dominated by the first term on the r.h.s., i.e. the charge distribution is stable only for \( \rho(y_1) = \rho(y_2) \).

Two remarks are to be made:

- The proof does not say anything about the endpoints of the interval. It can be shown that if the charge density at the endpoints is well-defined in the continuum limit, it will obey the inequality \( \rho(0) \leq 2 \rho(0.5) \).

- The proof can be generalized for all potentials \( V(x - y) = |x - y|^{-\alpha} \) with \( \alpha \geq 1 \). Therefore only potentials with longer range forces than the Coulomb potential (i.e. potentials with \( \alpha < 1 \)) can show finite size effects in the wire problem.

IV.3 Some estimates for the discrete charge distribution

After proving that the continuum charge distribution is flat, let us go back to the discretized version of the problem. We want to derive some estimates for the distribution of the charges close to the ends of the wire. In combination with numerical results in the next subsection, this will help us to reconcile the somehow counter-intuitive picture of a flat continuum charge distribution.

We use the same notation as in Sec. IV.1. The distances between particle \( i \) and particle \( i + 1 \) are denoted by \( d_i = x_{i+1} - x_i \). Then force equilibrium for the second, third etc. particle on the wire means that the following set of equations is fulfilled

\[
\frac{1}{d_i^2} = \frac{1}{d_2^2} + \frac{1}{(d_2 + d_3)^2} + \frac{1}{(d_2 + d_3 + d_4)^2} + \ldots
\]

\[
\frac{1}{(d_1 + d_2)^2} + \frac{1}{(d_1 + d_2 + d_3)^2} + \frac{1}{(d_1 + d_2 + d_3 + d_4)^2} + \ldots
\]

(30)

The first and the \( n \)-th particle are trivially at positions \( x_1 = 0 \) and \( x_n = 1 \) and no equilibrium condition can be formulated for them. We sum the first \( l \) of the above equations \( (l \leq n/2) \) and subtract equal terms on both sides

\[
\sum_{m=1}^{l} \left( \frac{1}{\left( \sum_{k=1}^{m} d_k \right)^2} \right) = \sum_{m=1}^{l} \sum_{a=1}^{m} \left( \frac{1}{\left( \sum_{b=1}^{m} d_{i+1+b} \right)^2} \right) + \sum_{m=l+1}^{n-2} \sum_{a=1}^{l} \left( \frac{1}{\left( \sum_{b=1}^{m} d_{a+b} \right)^2} \right).
\]

(31)
For simplicity we assume that \( n \) is even. Then obviously the largest distance between particles is \( d_{\text{max}} = d_{n/2} \) and we have the following inequality
\[
\sum_{m=1}^{l} \frac{1}{\left( \sum_{k=1}^{m} d_{k} \right)^2} \geq \frac{1}{d_{\text{max}}^2} \left( \sum_{m=1}^{l} \frac{1}{m} + l \sum_{m=l+1}^{n-2} \frac{1}{m^2} \right). \tag{32}
\]
The first sum is like Eq. (15). The second sum can be evaluated using the integral approximation
\[
l \sum_{m=l+1}^{n-2} \frac{1}{m^2} \approx l \int_{l+1/2}^{n-3/2} \frac{1}{m^2} \, dm = \frac{l}{l + 1/2} - \frac{l}{n - 3/2}. \tag{33}
\]
One then finds
\[
\sum_{m=1}^{l} \frac{1}{\left( \sum_{k=1}^{m} d_{k} \right)^2} \geq \frac{1}{d_{\text{max}}^2} \left( \ln l + \gamma + 1 - \frac{l}{n} \right) \tag{34}
\]
for large values of \( l, n \). For \( l = n/2 \) one has \( d_{1} < d_{2} < \ldots < d_{n/2} \), therefore
\[
\sum_{m=1}^{n/2} \frac{1}{\left( \sum_{k=1}^{m} d_{k} \right)^2} \leq \frac{1}{d_{1}^2} \sum_{m=1}^{n/2} \frac{1}{m^2} \leq \frac{1}{d_{1}^2} \frac{\pi^2}{6}. \tag{35}
\]
We finally get
\[
d_{1}^2 \leq d_{\text{max}}^2 \frac{\pi^2}{6 \ln n - \ln 2 + \gamma + \frac{1}{2}}. \tag{36}
\]
Lower bounds can be given too using the first equilibrium condition in Eq. (30)
\[
\frac{1}{d_{1}^2} = \frac{1}{d_{2}^2} + \frac{1}{(d_{2} + d_{3})^2} + \frac{1}{(d_{2} + d_{3} + d_{4})^2} + \ldots \leq \frac{1}{d_{2}^2} \sum_{m=1}^{n-2} \frac{1}{m^2} \leq \frac{1}{d_{2}^2} \frac{\pi^2}{6} \tag{37}
\]
\[
\Rightarrow \quad d_{1}^2 \geq \frac{6}{\pi^2} d_{2}^2. \tag{38}
\]
Eq. (38) says that for large \( n \), the distance of the first two particles on the wire is arbitrarily smaller than the distance of two particles in the middle of the wire. Numerical calculations in the next subsection will show that the equality is nearly realized in Eq. (38). Thus the particles close to the ends of the wire show strong finite size
effects. However, these effects vanish in the continuum limit as we have proved in the last subsection. This discrepancy is resolved by noticing that the finite size region observed in the discretized problem shrinks for \( n \to \infty \) as will be demonstrated in the next subsection.

### IV.4 Numerical results

In order to gain a better understanding of the problem, we have also used numerical methods to find the equilibrium position of the \( n \) particles sitting on the line of unit length. We have employed the Hybrid Monte–Carlo algorithm that updates the positions of the individual particles until they find their equilibrium positions: The molecular dynamics part of the algorithm moves all particles according to the electrostatic forces, which are computed only once for all particles at each step. The Monte–Carlo update scheme ensures that the algorithm is exact, i.e. that equilibrium is reached. During the simulation, the “temperature” was reduced as the charges moved more and more closely to their equilibrium positions. Convergence was verified by checking that the ground state energy did not decrease any more for additional sweeps within a given numerical precision. In general, convergence was very quick since the starting point of the simulation with all particles equidistant is quite “close” to the equilibrium position for the reasons explained in Sec. IV.2. Simulations with up to \( n = 8193 \) particles have been performed.

The first interesting quantity investigated numerically is the ground state energy \( E(n) \) as a function of the number of particles \( n \) plotted in Fig. 5. Also drawn are the upper and lower bounds derived in Eqs. (16) and (23). One notices that the measured energies seem to settle on the upper bound with good precision for \( n \to \infty \). This raises the interesting (and so far unanswered) question whether the upper bound (16) becomes the asymptotically exact result for the ground state energy.

Let us also emphasize that the data in Fig. 5 pedagogically demonstrate the danger of extrapolating to \( n \to \infty \) based on simulations for finite \( n \), at least for systems with long range forces: Extrapolating on the basis of the data for \( n < 1000 \) particles the upper bound would eventually be violated (see the curve in Fig. 5)!

Only for more than 1000 particles the curve for \( E(n) \) bends over and the extrapolation respects the exact results.

In Fig. 6 the quotient of the smallest (at one end of the wire) to the largest (in the middle of the wire) particle distance is plotted as a function of \( n \). The upper bound
from Eq. (36) is respected, in particular the interparticle distance at the end of the wire becomes much smaller than the maximum distance. On the other hand, the ratio of the distance of the first to the second particle $d_1$ and the second to the third particle $d_2$ approaches a constant nonzero value for $n \to \infty$. This is shown in Fig. 7 together with the lower bound derived in Eq. (38). One clearly sees the strong finite size effects already mentioned in the previous subsection.

However, when one looks at the distribution of all charges along the wire plotted in Fig. 8 for various values of $n$, one notices that the regions with strong finite size effects close to the ends of the wire shrink very slowly with increasing $n$, probably as slow as $1/\sqrt{\ln(n)}$, as suggested by Eq. (36). One can interpret these numerical results as follows: Eventually the continuum charge distribution appears flat, although for a given $n$ the first few particles do never approach equidistant positions. But in the continuum limit $n \to \infty$ these non-equidistant regions eventually vanish as compared to the rest of the wire. This scenario combines the analytical and numerical results established in this work.

V Summary

We have discussed the seemingly trivial, at least conceptually simple problem of a free charge on a one-dimensional wire of finite length. The first thing to notice was that this problem is ill-defined in its original formulation due to the diverging ground state energy. The question how the continuum charge distribution looks like can only be answered after introducing some regularization procedure.

Two particularly intuitive regularizations have been employed in this paper; shrinking ellipsoids in Sec. III and charge discretization in Sec. IV. Both led to the answer that the continuum charge distribution is flat, that is there are no finite size effects! The interested reader will be able to find other regularization prescriptions that lead to the same answer.

A flat charge distribution will also be found for potentials

$$V(x - y) \propto |x - y|^{-\alpha}$$

(39)

with shorter range forces than the Coulomb potential, that is generally for $\alpha \geq 1$. For exponents $\alpha < 1$ the problem is well-defined and finite without the need for regularization. It is easy to show that then there are finite size effects. Thus the
Coulomb law is the limiting case between finite size effects and no finite size effects in a strictly one-dimensional wire of finite length, see Fig. 9.

One interesting problem left open in this respect is the analytic form of the continuum charge distribution for exponents $\alpha < 1$. Another question that we have not been able to answer is whether the ground state energy as a function of the number of particles $n$ in Sec. IV is really given by the upper bound Eq. (16) plus subdominant corrections $O(\ln n/n)$. The numerical results in Fig. 5 seem to indicate this.

Acknowledgments

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10 The observation that an exact bound would be violated based on our simulations for $n < 1000$ particles was the motivation for pushing the simulations to $n = 8193$. This also demonstrates the dangers of using numerical simulations without analytical “guidelines”.

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**Figure captions**

Fig. 1. A frequent first guess for the charge distribution on a finite wire of unit length. Some readers might also suggest divergencies at the ends of the wire.

Fig. 2. A free charge on a conducting ellipsoid.

Fig. 3. Sum of energies to the $k$-th neighbours.

Fig. 4. Proof by contradiction — a nonflat charge distribution cannot be stable in the limit $n \to \infty$.

Fig. 5. Ground state energy $E(n)$ for a system of $n$ particles with individual charges $1/n$ interacting with Coulomb potentials on a wire of unit length.

Fig. 6. The quotient of smallest to largest particle distance as a function of the number of particles $n$.

Fig. 7. The quotient of the distances of particle #1 and #2 and of particle #2 and #3 on the wire as a function of $n$.

Fig. 8. Charge distribution on the wire for various numbers of particles $n$. $x_i$ is the position of particle #i. Nearest neighbor distances $x_{i+1} - x_i$ are plotted as a function of the position along the wire.

Fig. 9. Finite size effects on one-dimensional wires of finite length for potentials $V(x - y) \propto |x - y|^{-\alpha}$. 

\[15\]
Moving particle P to an energetically favourable position

Discrete charges on the wire

Continuous charge distribution $\rho(x)$

Limit $n \to \infty$
Extrapolation on the basis of the data for $n<1000$.

Upper bound from Eq. (16)

Lower bound from Eq. (23)
Upper bound from Eq. (36)
Lower bound from Eq. (38)
\[
\frac{(x_{i+1} - x_i)}{(n-1)}
\]

\[
\frac{(x_{i+1} + x_i)}{2}
\]
Coulomb potential

finite size effects
(longer range forces)

no finite size effects
(shorter range forces)