Liquid Transport through Packings of Elastic Shells

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Fluid transport in porous materials is mostly studied in geological samples (soil, sediments etc.) or idealized systems. But the fluid flow through compacted granular materials, consisting of substantially strained granules, remains relatively unexplored. As a step towards filling this gap, we study a model of liquid transport in packings of compacted elastic shells using Finite Element and Lattice-Boltzmann methods. We find that the fluid flow abruptly vanishes as the porosity of the material falls below a critical value. Our results suggest that the fluid flow obstruction exhibits features of a percolation transition that occurs at a porosity about $\psi^* = 0.035$. We further show that the fluid flow can be captured by a simplified permeability model in which the complex porous material is replaced by a collection of disordered capillaries, which are distributed and shaped by the percolation transition. To that end, we numerically explore the divergence of tortuosity $\tau_H$ and a decrease of a hydraulic radius $R_h$ as the percolation threshold is approached. We interpret our results in terms of scaling predictions derived from the percolation theory applied to random packings of spheres.

I. INTRODUCTION

The physics of fluid flow through disordered porous media is of a fundamental importance to a wide range of engineering and scientific fields including enhanced oil recovery, carbon capture and storage, contamination migration in ground-water, water transport, and nutrients transport in tissues and microbial colonies [1–5]. This has led to a substantial effort in looking for relationships between the effective physical transport properties and the structural properties of porous materials. In spite of an extensive work that has been done up to date, a full description in a broad range of material parameters is elusive [6]. Experimental studies, especially in 3D systems, are limited because imaging material samples and resolving fluid streamlines is a challenging task [7–9]. Numerical studies are most often tackled in 2D due to high computational burden [10–16]. Even though the broad range of material porosities in 2D systems has been covered, a drawback of these studies is that for disordered materials the flow can not exist at the onset of material mechanical stability [17]. For the 3D systems, simulations are commonly performed for an idealized model of randomly distributed, inter-penetrating objects like cubes or spheres [18–21]. These systems are good prototypes to study critical phenomena, but the liquid transport in complex geometries strongly depends on boundary condition details, thus the relevance of these models for the real materials is not clear [22]. There is also work done on fluid transport in geometries obtained from the microtomography of a collected material, however these studies are performed for a small number of samples and at relatively high porosity [23, 24].

In this work we numerically study a single-phase viscous flow through a compacted granular material in Darcy’s regime, i.e. laminar flow with a linear relation between volumetric flow and pressure gradient. We consider packings in a very broad range of porosities, from the point the packings start to be mechanically stable (jamming transition [25]), down to the porosities where the liquid transport ceases to exist (percolation transition [17]). Particles in our granular material are modeled as elastic, spherical membranes that hold a constant volume upon deformation.

In this work, we focus on a capillary model by Kozeny & Carman, the classical permeability-porosity framework. First, we briefly introduce the Kozeny-Carman model. Later on, we present how the key features of Kozeny-Carman model can be physically grounded in a percolation theory. Finally we present numerical evidences on how different structural features of granular porous material contribute to the fluid transport in granular porous media.

Kozeny-Carman Model

Darcy’s law defines the permeability $\kappa$ as a proportionality constant in the relation between fluid volumetric discharge per unit area $U$ (in units of length/time), and a pressure gradient:

$$U = \frac{-\kappa \nabla P}{\eta}$$

where $\eta$ is the dynamic viscosity of the fluid, and $P^*(r)$ is the pressure at the location $r$. This phenomenological relation is valid at low Reynolds numbers when the flow is laminar. For small gradients we can further assume $\nabla P = \Delta P / L$, where $L$ is a linear size of the system.
For low Reynolds number flow in a straight, circular capillary channel we have the Poiseuille equation:

\[ U_{\text{capillary}} = -\beta \frac{R^2 \Delta P}{\eta L} \]  \hspace{1cm} (2)

where \( R \) is the radius of a capillary, \( \beta \) is a numerical factor, and \( L \) is the length of the capillary. If a capillary occupies only a fraction of the material, the liquid discharge per area unit is correspondingly lower. Assuming that capillaries are homogeneously distributed in a material, the scaling factor is the amount of the void space in the material, called a porosity \( \psi \):

\[ U = -\beta \psi \frac{R^2 \Delta P}{\eta L} \]  \hspace{1cm} (3)

For the capillaries that are not straight, Kozeny pointed out that owning to the tortuous character of the flow, the length of the equivalent channels should be \( \langle \lambda \rangle = \tau_H \cdot L \), where \( \tau_H \) is called hydraulic tortuosity, and the fluid discharge needs to be scaled down by \( \tau_H \) \cite{29}. Carman further reasoned that it takes \( \tau_H \) more time to discharge the same amount of fluid through porous media than it takes for straight capillaries (in a macroscopic direction of the flow). Thus, the discharge rate should additionally be \( \tau_H \) times smaller \cite{27}. Capillaries are not limited only to the circular cross-sections. For the general shape of the capillary we can define a hydraulic radius \( R_h \) \cite{28}, as the ratio of the cross-sectional area normal to flow to the wetted perimeter of the flow channels. Thus from Equation 3 the final relation for the capillary flow in a porous material is \cite{27}:

\[ U = -\psi \frac{R^2_h \cdot \Delta P}{\eta \cdot \tau_H \cdot L} = -\frac{\beta \psi R^2_h}{\tau^2_H \cdot \eta} \Delta P \]  \hspace{1cm} (4)

Comparing Equation 1 with Equation 4 a general formula for permeability reads

\[ \kappa = \frac{\beta \psi R^2_h}{\tau^2_H} \]  \hspace{1cm} (5)

and is called Kozeny-Carman equation. Despite being semi-empirical, Equation 3 is commonly used as a simple model for the permeability of in porous materials.

II. METHODS

A. Elastic Shells Packings

1. Generating Compressed Packings

The initial packings of the shells has been generated using a standard jamming algorithm \cite{23}. Mechanically stable packings are generated with periodic boundary conditions. Starting from these jammed packings, more compacted packings are generated by changing a linear dimension of the simulation box. The changes of the box size are minute, and in terms of the absolute values, at each step the box size changes by less than 0.4% of the linear size any elastic shell. After every box size change, the mechanical stresses are relaxed using FIRE algorithm \cite{25}, see Section VI A for more details.

2. Membrane Mechanics

Every shell is modeled as a membrane using about 5000 triangular finite elements per shell. Ratio of a shell thickness \( t \) to the initial radius \( R_0 \) is \( t/R_0 = 0.04 \), so bending effects can be neglected and the shell material is modeled as an isotropic St. Venant-Kirchhoff membrane \cite{30, 31}. All of the shells are slightly pressurized at the beginning of the simulation, with the initial pressure \( P_0 \). Ratio between \( P_0 \) and Young's modulus \( E \) is equal to \( P_0/E = 0.0025 \). Additionally, shells are filled with an incompressible liquid. Thus any shape deformation leads to the change of the internal pressure, see Section II A for more details. The force due to the shell volume-dependent pressure \( P(V_{\text{shell}}) \) on a vertex \( i \) is calculated as: \( F(r_i) = -\nabla_r (P(V_{\text{shell}}) \cdot V_{\text{shell}}) \) where \( V_{\text{shell}}(r_1, ..., r_{N_{\text{vert}}}) \) is a function of the \( N_{\text{vert}} \) vertices in the meshwork and the volume change for the vertex \( i \) is calculated using the tetrahedral volume defined by the vertex \( i \), its neighboring vertices in the meshwork, and center of the mass \cite{4}.

3. Constant Volume Algorithm

Once the mechanical forces are equilibrated, the constant shell volume constraint is enforced by varying the shells internal pressure. If the volume of a shell is not equal to the preassigned value \( V_0 \), the pressure \( P_{\text{new}} \) is adjusted to the value \( P_{\text{new}} = P_{\text{old}} (1 + (V_0 - V)/V) \). This inevitably drags the system out of mechanical equilibrium and the system needs to be equilibrated again. The protocol continues until the volume of the shells reaches their preassigned volumes within 0.1% of accuracy.

B. Identification of a Percolating Cluster

To identify a percolating cluster of a void space, we project the system onto a 3D lattice, see Figure 1. Every lattice site that contains an element of an elastic shell is considered not permeable to the liquid, see Figure 1. Finally, we look for a percolating cluster using the connected-component labeling algorithm (implemented in scipy.ndimage Python library). The cluster is said to percolate the system if it contains lattice sites on two opposite sides of the simulation box. One of the characteristic length-scales in the system is the initial diameter of a shell, \( D_0 \). We chose to express the lattice sizes, \( \delta \), in units of \( D_0 \). The resolution of the lattices in our study
varies, from a coarse one to a fine one, and it is in the range [0.026, 0.080]. In principle, we would like to generate a lattice with $\delta \to 0$ as we want to estimate a fluid flow in the continuum limit. However, the representation of the elastic shells is finite and below some lattice size $\delta_s \approx 0.025$, which roughly corresponds to the linear size of a finite element of the shell, the lattices penetrate the solid objects (i.e. membrane would be permeable to transported liquid). Below this length threshold, the lattice cannot be used anymore, and the continuum limit has to be estimated from the finite size lattices extrapolation. Finally, percolation clusters identified in this way are used for hydraulic radius and Lattice-Boltzmann calculations.

C. Lattice Boltzmann Simulations

Velocity fields of the fluid flow through the packings of the shells, are solved with the Lattice-Boltzmann method [32] (L-B) using the D3Q19 lattice topology. This method has proven to be successful in studies of liquid flow in porous materials [10–13, 15, 18–21, 23, 24, 33–37]. We use this method to obtain a solution to the Navier-Stokes equation for the flow of incompressible fluids. The L-B method is using velocity distribution function rather than velocity and pressure fields and is numerically more stable than the Finite Element Method at the irregular boundaries that are inevitable in porous materials [32]. To ensure better numerical stability for the complex geometry of the pores, we use multiple relaxation times (MRT) to solve linearized Boltzmann equation with L-B method [38].

Permeability of the packing and the flow field are resolved by setting a small pressure drive: $\Delta P/C_\psi^2 \rho = 0.00015$, where $C_\psi$ is speed of sound and $\rho$ is fluid density, and $\Delta P$ is a pressure difference between two opposite sides of the simulation box. The value of the pressure gradient is sufficiently small to keep the flow in the incompressible and in the laminar regimes, $Ma = \langle u(\mathbf{r}) \rangle/C_\psi < 2 \cdot 10^{-6}$, and $Re = 1.5 \cdot 10^{-5}$ [39]. Every simulation is performed for periodic boundary condition (PBC) in directions perpendicular to the pressure gradient. In a direction of the pressure gradient, the system is open and the boundary conditions are set by pressure difference [23, 24]. No-slip boundary condition has been applied on the solid material boundaries. It has been found [13, 32] that when the channels carrying liquid become very narrow (of the order of one lattice site) L-B simulations become unstable and the evaluation of the streamlines become inaccurate. To deal with this problem we use an approach proposed in [13], where every lattice site on which flow equations are solved, is further refined into $\mathbb{R}^3$ smaller cubic elements (refinement level: R). Strictly speaking $\delta = \delta/R$ is a lattice size of the fluid phase. Due to computational limitations, LB calculations are performed for the lattice constant $\delta = 0.04$ (unless stated otherwise).

The flow fields obtained from L-B simulations for each lattice site, $u(\mathbf{r})$, are further used to calculate the permeability and the tortuosity. Permeability is calculated as $\kappa = \eta \cdot \langle u(\mathbf{r}) \rangle / \nabla P$, and tortuosity as $\tau_H = \langle u^2(\mathbf{r}) \rangle / \langle u(\mathbf{r}) \rangle$ [39], see Section VIB for the formal derivation. All the L-B simulations are performed with PALABOS (http://www.palabos.org), and the source-code is publicly available at https://github.com/pgniewko/porous-LB.

D. Finite Size Effects

The simulated model of the porous material accounts for deformability and the mechanics of the shell membrane using Finite Elements method. Mechanics of the elastic shells is resolved with $\approx 3.75 \cdot 10^7$ degrees of freedom, and some of L-B simulations required up to $\approx 2 \cdot 10^7$ lattice points to resolve the fluid velocity field. In turn, the resolution of the calculation imposes restrictions on the largest system size that we are able to study. Finite size effects for the studied systems may result in small anisotropies in the permeability tensor [14], but recent studies show that transport in complex porous geometries can be quantitatively captured if the size of the system is roughly $\gtrsim 10$ times larger than the pore size [23, 33, 40, 11].

III. RESULTS

A. Percolation transition

Following the protocol described in Section IIIB, percolating clusters have been identified for three system sizes $N=16, 32, 50$ elastic shells, and various lattice resolutions. The results for $\delta = 0.04$ are shown in the Figure 2A. As the system gets larger, the transition becomes steeper, like one expects in a first-order transition. The steepness of this transition depends on the system size $L$, and scales as $\sim L^{1/\nu}$, where $\nu$ is a critical exponent of the correlation length. In a continuum percolation, this exponent is approximately equal $\nu \approx 0.88$ [12].

Figure 2B, shows that an abrupt drop in fluid transport capabilities is a feature that occurs for different lattice resolutions and the percolation threshold shifts towards lower porosity values together with the increase of the lattice resolution - the effect anticipated from the studies on idealized models [43]. The finite representation of the finite elements does not allow for the calculations in the continuum limit. It is nevertheless possible to extrapolate a percolation threshold in the continuum limit $\delta \to 0$ limit. In Figure 2C, a power-law dependence of the finite-lattice size and the threshold position is fitted. The threshold in the continuum is found to be $\psi_c^* = 0.035 \pm 0.014$ (see Section VIB for fitting procedure). This result is consistent with the previous models, where the percolation threshold is estimated to be
**FIG. 1. Projecting packings on a lattice:** A. A sketch of a 2D system, projected on a lattice with four different lattice sizes. Every lattice site that contains any part of a particle (red squares) is considered to be occupied and impermeable to liquid (gray squares). It can be noticed that for the lattice resolutions $\delta_1$ and a given configuration, there is no percolating cluster capable of carrying liquid through the packing. For $\delta_2$ and $\delta_3$ despite there is no percolating cluster, there are some unoccupied lattice sites (blue squares). For the lattice size $\delta_4$, there is a percolating cluster (green squares). There are also some unoccupied lattice sites that do not belong to the percolating cluster (blue squares). B. Percolating clusters (in green) and elastic shells (in red) as the compaction of the system progresses. For clarity the smallest system is presented (N=16), with a lattice resolution $\delta = 0.04$. As the system is more and more compacted a percolating cluster gets smaller and more tortuous, and eventually disappears at the critical porosity.

$\psi_c^* = 0.030 \pm 0.002$ for the packings of unequal, overlapping spheres [44], $\psi_c^* = 0.0317 \pm 0.0004$ for the packings of equal, overlapping spheres [45], $\psi_c^* = 0.039$ for sandstone [46], and $\psi_c^* = 0.036 \pm 0.001$ for overlapping, aligned cubes [48]. Due to the finite lattice size, the percolation threshold for a finite $\delta$ is larger than in the continuum limit $\psi_c^*$, and is related to $\psi_c^*$ by a power-law $\psi_c - \psi_c^* \approx \Delta \psi \sim \delta^\beta$. From Figure 2 C, we estimate the lattice-size scaling exponent to be $\beta = 1.1$. This is in a good agreement with a prediction made in a reference [49], where the exponent is estimated to be 1—yielding an approximate relation for the lattice-size dependent percolation threshold that obeys: $\psi_c - \psi_c^* \sim \delta$. Although a relatively small range of the system sizes is studied, the numerical results point to common characteristics between the model studied in this work and previously studied percolation models [43,45,47,48]. Thus, we use the formalism of a percolation theory in the analysis of the fluid flow obstruction in the vicinity of the critical porosity value $\psi_c$.

**B. Decrease of hydraulic radius $R_h$ with the porosity**

The hydraulic radius is defined as a ratio of a cross-section of a liquid carrying channel to its wetted perimeter, see Section VII for more details. Only in relatively simple cases, like a laminar flow inside a pipe, the hydraulic radius can be directly related to the geometry of the system. In practice, finding this value is problematic because it is difficult to accurately predict a channel’s shape along the flow streamlines. The situation gets even more complicated in complex geometries, where percolating channels can merge or branch out. Thus, the hydraulic radius is commonly approximated by the ratio of the volume to the wetted area of a cluster carrying the liquid [49].
FIG. 2. **Percolation transition**: A. Percolation probability for three system sizes: $N=16$, 32 and 50. Dashed lines are sigmoid fits to the numerical data, and binned averages are given by open dots. The lattice-size dependent percolation threshold $\psi_c(N)$ has been estimated as the porosity value for which percolation probability is equal 0.5. The plots represent data for the lattice $\delta = 0.04$. B. Percolation probabilities for the system size $N=50$ and varying lattice sizes: $\delta = \{0.03, 0.04, 0.05, 0.06, 0.07\}$. As the resolution of the lattice increases the percolation threshold shifts toward lower porosity values. C. We extrapolated the percolation threshold in the continuum limit. Dashed line is a power-law fit, where $\psi_c^*$ and the exponent for $\delta$ are two fitting parameters. The fitted percolation threshold is $\psi_c^* = 0.035 \pm 0.014$, and the exponent is equal to $1.1 \pm 0.2$. The relation $\Delta \psi \sim \delta^{1.1}$ agrees well with the work of Koza et al. [43]. Details of a fitting procedure can be found in Section VI F.

Using the percolating clusters identified for the packings of elastic shells, we estimated the hydraulic radii for different lattice resolutions as a ratio of the number of lattice sites belonging to the cluster divided by the number of surface sites [33, 50]. Using a geometric argument adapted from references [51, 52], the hydraulic radius is predicted to vanish linearly at the limit of zero porosity, see Section VI B. Results corroborating this prediction can be found in Figure 3. The results indicate that the hydraulic radius decays like:

$$R_h \propto \psi / (1 - \psi)$$  \hspace{1cm} (6)

as the porosity tends to 0. The hydraulic radius vanishes independently of lattice-size, and its value at the percolation threshold is non-zero, as one would expect from a percolation theory [53].

C. **Tortuosity divergence at the percolation threshold**

Tortuosity underpins the relationship between a transport process and the underlying geometry and topology of the pores [54]. Recently it has been shown numerically and analytically, that the tortuosity depends on material structural properties, and may vary significantly close to the percolation threshold [13, 15, 28, 55–57]. Although percolation ideas have been proposed in the context of tortuosity in 3D porous materials [58], they have not been tested near the percolation threshold. In this contribution, we numerically show a link between the geometry of a percolating cluster and the liquid transport through

FIG. 3. **Hydraulic radius $R_h$**: Hydraulic radius as a function of porosity $\psi$ for the system size $N=50$. For each packing, a percolating cluster has been identified. The hydraulic radius is then calculated as a ratio of the volume $(1 - \alpha)$ of the cluster and the total surface area $(\Sigma)$: $R_h = (1 - \alpha) / \Sigma$. Error bars give one standard deviation. Dashed-lines are the fits to $R_h \propto \psi / (1 - \psi)$. Horizontal dashed-line (gray) gives the value for the hydraulic radius of a straight capillary with a diameter of a single lattice site $(R_h/R_h(\text{capillary}) = 1)$. 

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porous materials with a complex geometry of pores at the percolation threshold.

Scaling arguments from references [6, 58, 59] suggest that the tortuosity scales with the distance to the percolation threshold according to \( \tau_H \sim \delta \psi^{\nu(1-D)} \), where \( \nu \) is a critical exponent of the correlation length (\( \nu \approx 0.88 \) for the continuum percolation model in 3D), \( D \) is the fractal dimension of the cluster through which the liquid is transported, and \( \delta \psi = \psi - \psi_c \). It was found that the fractal dimension for the most probable path through which liquid is transported is approximately \( D \approx 1.43 \) [60, 63], implying:

\[
\tau_H \sim \delta \psi^{-0.38}
\]  

(7)

To test this dependence we evaluated the tortuosity from the velocity field as described in the Section II C—and the results are presented in Figure 4. Close to the jamming threshold, \( \delta \psi \approx 0.25 \), we find that tortuosity is \( \tau_H \approx 1.4 \). This result agrees very well with experimental measurements for packings of glass beads \( \tau_H \approx \sqrt{2} \) [49]. For porosities close to jamming, all three lattice refinements overlap and agree very well with the volume-averaged analytic prediction for mono-dispersed spheres [56, 57], cf. magenta dashed-line in Figure 4. For the porosities close to the percolation threshold, \( \delta \psi \approx 0.0 \), we can see that numerical simulations are consistent with the predicted divergence for the hydraulic tortuosity. Moreover the power-law dependence is a good approximation up to a porosity of about \( \delta \psi \approx 0.25 \).

The increase of tortuosity at the percolation threshold is caused by the complex geometry of the percolating cluster rather than just due to numerical artifacts resulted from the increased resolution of the lattice, cf. Figure S3 in Appendix. Interestingly, the optimal fit to the data provides a slightly smaller exponent \( \tau_H \sim \delta \psi^{-0.3} \) (see Section VI F for a fitting procedure). This is remarkably close to the exponent that was found for some porous media: \( \tau_H \sim \psi^{-0.27} \) [23]. To reconcile this discrepancy, simulations of much larger systems are required in order to obtain values of \( \tau_H \) that span many orders of magnitude.

D. Capillary model of permeability

By construction in Kozeny-Carman model the liquid transport thorough the material is ensured down to the porosity \( \psi = 0 \). However this is not the case for granular porous materials. To account for that in Equation 3 the porosity \( \psi \) is replaced by a distance to the percolation threshold \( \psi \rightarrow \delta \psi = (\psi - \psi_c) \). Exponent \( \gamma \) is sometimes taken ad hoc to be equal to \( \gamma = 1 \) in references [43, 54], however there is no firm argument supporting this particular choice. Since this exponent is yet unknown, we try to estimate \( \gamma \) from a fit to the numerical data. Knowing \( \gamma \) is not crucial for highly porous materials, for which \( \delta \psi \approx \psi \), but it is essential for lower porosities, where the factor \( \delta \psi^\gamma \) contributes to the vanishing permeability \( \kappa \) at the percolation threshold, \( \delta \psi \rightarrow 0 \).

In Section III A we found numerically, that consistently with finite-size scaling, the percolation threshold depends on the resolution of the used lattice. Moreover, in Section III B we found that the hydraulic radius reaches 0 at the porosity \( \psi = 0 \). Finally, in Section III C we found that the tortuosity of flow streamlines diverges at the percolation, consistent with the prediction \( \tau_H \sim \delta \psi^{-0.38} \). Using Equations 5–7 we can put together a relationship between material porosity and permeability \( \kappa \), that
where $C$ is a constant. A fit of this model is presented in Figure S3 (black dashed-line). Results are given for the lattice resolution $\delta = 0.04$, for which the tortuosity diverges and the flow ceases at porosity $\psi_c \approx 0.15$. We can see in Figure S3 that Equation 8 captures quite accurately the change of the material permeability $\kappa$ in a broad range of porosities—from the onset of the jamming up to the percolation threshold, and regardless of the model fitting method, cf. Figure S3 and Figure S5.

Depending on the fitting procedure, the value of the exponent $\gamma$ varies slightly, with the average (over four different fitting procedures) value $\gamma = 0.89 \pm 0.15$. This is quite close to the value used ad hoc, $\gamma = 1.0$. In the limit of the large porosities, i.e. where $\psi > \psi_c$, we can approximate $\delta \psi^\gamma \approx \psi^\gamma$, what reduces Equation 8 to a simpler form $\kappa \sim \psi^{3.39}/(1 - \psi)^2$ (with $\gamma \approx 0.83$). Interestingly this approximate form, with a fractional power close to 3.00, is in a good agreement with recent experimental and numerical work, where this exponent has been estimated to be 3.7 (for porosities such that $\psi - \psi_c \approx \psi$.23 24.

We compare Equation 8 to a scaling ansatz $\kappa \sim \delta \psi^\varepsilon$, which is a reasonable guess for the transport properties, close to the critical point 47. Halperin et al. 65 66 showed that there are several universality classes of porous media, where the scaling $\varepsilon$ depends on the model’s details. For example, in so called Swiss-cheese model $\varepsilon \approx 4.4-4.5$, whereas for the Inverted Swiss-cheese model $\varepsilon \approx 2.4-2.5$. The relation $\kappa \sim \delta \psi^{38}$ fits the data in a broad range of porosities, yellow dashed-lines in Figure 5 and Figure S5. However, the fitted exponent values depend on the fitting procedure and vary in the range of [2.72,3.88], with an average value $\varepsilon \approx 3.4$. Moreover, estimated percolation threshold ($\psi_c$) differs noticeably from the estimations made in Figure 2 C. Despite the fact that the power-law scalings are often very useful, it is unclear how they are accounting for the connectedness of the pores and the tortuosity of the flow 47. Additionally, in Figures S5 A and B, we compare our numerical data to the standard Kozeny-Carman model, $\kappa \sim \psi^3/(1 - \psi)^2$, 51 52 55 61 67. This classical model has been successfully applied to many porous materials 2,19 52 68, for which $\psi - \psi_c \approx \psi$, but in our work, we look at permeabilities close to the percolation threshold, so this models is expected to perform with much less success, cf. green dashed-line in Figures S5 A and B.

IV. DISCUSSION & CONCLUSIONS

Our results support a simple picture of the fluid transport retardation in deformable granular materials, compressed from the onset of mechanical stability at the jamming point down to the percolation threshold. The model essentially describes a porous material as a collection of tortuous and randomly placed capillaries, where close to the percolation threshold, tortuosity and capillaries dilution dominate liquid transport. We have shown that upon compaction, the void space between pressurized, elastic shells undergoes a sharp, system-size dependent transition. We also find that the hydraulic radius vanishes in a lattice-resolution independent manner as the porosity vanishes. Next, using L-B simulations we have shown that tortuosity of the flow streamlines abruptly increases at the percolation threshold. In Equation 5 the effects of the capillaries’ density and tortuosity are factorized, and this has motivated a substantial work devoted only to tortuosity 49 59. Combined with a percolation scaling theory, we were able to explain the fractional dependence of tortuosity on the porosity of the sample. Our work underscores that at higher porosities, where the fluid flow is not tortuous, the major determinants of the flow obstruction are a hydraulic radius $R_h$ and the amount of fluid accessible void space. In turn, upon the approach to the percolation threshold, the complex geometry of liquid transporting channels ultimately leads

\[
\kappa = C \times \frac{\delta \psi^{\gamma + 0.76} \psi^2}{(1 - \psi)^2}
\]

FIG. 5. Permeability of elastic shells packing in Darcy’s regime: Permeability obtained from Lattice-Boltzmann simulations for the system size $N=50$, and lattice resolution $\delta = 0.04$ and $\delta = \delta/3$ for the solid and fluid phase, respectively. Blue crosses represent permeability for individual simulations. Black open circles represent binned averages, and red stars correspond to medians. Dashed lines correspond to three different capillary models: i) $R_h \sim \psi/(\psi - 1)$, $\tau_H \sim \delta \psi^{-0.38}$, and the exponent $\gamma$ being a fitting parameter; ii) $R_h \sim \psi/(\psi - 1)$, $\gamma = 1.76$, and $\tau_H \sim \delta \psi^{-0.38}$; iii) power-law ansatz $\kappa \sim \delta \psi^3$. Fitting details can be found in Section VI F.
to the flow obstruction. Nonetheless, the dilution of the capillaries upon the approach to the percolation threshold, described by the $\gamma$ exponent, remains elusive. We found numerically that $\gamma \approx 0.89 \pm 0.15$, which is close to the ad hoc value $\gamma = 1.0$ [29, 30], but this value does not have a firm grounding in the percolation theory. In Section VI we present a simple argument from the percolation theory, that suggests this exponent to be $\gamma = 1.76$. If tortuosity could be neglected, this would explain our numerical data very well. However, when the tortuosity contribution is included, this leads to the decay of the permeability at the percolation with the exponent close to 2.5, i.e. $\kappa \sim \delta^{2-\gamma}$. Despite the fact that this is very close to the Inverted Swiss-cheese model exponent ($\bar{\delta} \approx 2.4 - 2.5$), this model does not explain our numerical data well, cf. Figure 5 and Figure S5. This intriguing result motivates further research on the capillary model in the proximity of the percolation threshold within a framework of the percolation theory.

Finally, in our work we considered only packings of identical shells. In Section VI we can see that polydispersity seems to contribute only a constant factor in the relation for $R_\text{th}$, without changing its functional dependence on the porosity $\psi$. Furthermore in 3D packings of unequal spheres, polydispersity has only a minor impact on the percolating clusters [41, 69]. Therefore Equation 8 may be applicable to other disordered and compacted systems made of deformable particles.

V. ACKNOWLEDGMENTS

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VI. APPENDIX

A. Generating jammed packings

Generation of jammed packings begins by choosing random particle positions in a cubic box, with periodic boundary conditions. The initial radii of the spherical particles are set such that the initial volume fraction is about $\phi_0 = 0.01$. Next, we successively increase or decrease the radii of the particles, with every particles inflation or deflation followed by the energy minimization using FIRE algorithm [29] with velocity-verlet integrator [70]. The parameters used in the FIRE algorithm are: $dt_{\text{FIRE}} = 0.1$, $dt_{\text{FIRE}} = 1.5$, $\alpha_{\text{FIRE}} = 0.1$, $N_{\text{min}} = 5$, $f_a = 0.99$, $f_{\text{inc}} = 1.1$, $f_{\text{dec}} = 0.25$. Termination condition for the FIRE algorithm is: $\max_i | f_i | \leq 10^{-15}$.

Initially, for each inflation step, the particle’s radius is increased following the rule: $r_{\text{new}} = r_{\text{old}} \cdot (1 + \epsilon_r)$, where initially $\epsilon_r = 0.01$. The forces between particles (in this protocol) are Hertzian: $F(R) = -\frac{1}{2}E^* \sqrt{R^2 \delta^{3/2}}$, where $\delta$ is an overlap between particles, $R$ is a unit vector along $R$, $E^* = E/2(1 - \nu^2)$ is an effective Young’s modulus, $R^* = 0.5 \cdot r$ is an effective radius, $r$ being a radius of a particle. For the particles in the initial packing generation algorithm: $E = 1$, $\nu = 0.5$. The pressure in the simulation box is calculated as: $P = -\frac{1}{V} \sum_i \sigma_{\alpha \alpha}$, where the stress tensor $\sigma_{\alpha \beta}$ is obtained from the virial formula: $\sigma_{\alpha \beta} = -\frac{1}{V} \sum_i \sum_{i>j} r_{ij}^\beta F_{ij}^\beta$, where $r_{ij}^\beta$ is $\beta^{th}$ component of the vector pointing from the center of a particle $j$ to $i$, and $F_{ij}^\beta$ is $\beta^{th}$ component of the contact force between particles $i$ and $j$.

When the pressure of the packing is greater than $P > 2 \cdot P_{\text{min}} = 2 \cdot 10^{-8}$, the parameter $\epsilon_r$ is halved, and the particles’ sizes are deflated according the rule: $r_{\text{new}} = r_{\text{old}} \cdot (1 - \epsilon_r)$. When the pressure drops below $P_{\text{min}}$, the $\epsilon_r$ is again halved and the particles are inflated. The process continues until the pressure settles at the value $P_{\text{min}} < P < 2 \cdot P_{\text{min}}$. If the final packing contains any rattler, the configuration is rejected and a procedure is repeated. The final configuration provides positions of soft-spheres particles that are next replaced by Finite Elements. The packings generated using the described algorithm have been tested in terms of the number of contacts (Figure S1) and the finite size effects on the volume fraction at jamming point (Figure S2) [25].

B. Hydraulic radius: a geometric argument [51, 52]

For packed bed of spherical particles with a size distribution $n(D_p)$, the $i^{th}$ moment of the particle size distribution is:

$$M_i = \int_0^\infty D_p^i n(D_p) dD_p$$

(9)

If a horizontal cut is made across the bed, one obtains circular disks of the size $x$, projected on the sectional plane. The size distributions of these disks is:

$$f(x) = \int_0^\infty P(x|D_p)P(D_p) dD_p$$

(10)

Here $P(D_p)$ is the pdf of $D_p$:

$$P(D_p) = \frac{n(D_p)}{\int n(D_p) dD_p} = \frac{n(D_p)}{M_0}$$

(11)

$P(x|D_p)$ is a conditional probability density function that given a sphere diameter $D_p$, the diameter of a given disk in a plane cut ranges between $x$ and $x + dx$. Note
FIG. S1. **Average contact number:** A mechanically stable system must have a force balance on each particle. For \( N \) spheres in \( d \) dimensions, the number of constraints that has to be satisfied by the inter-particle forces is \( d \times N \). In the system with periodic boundaries this number is \( d \times N - d \). Additionally, there is one more degree of freedom, a volume fraction at the jamming, that has to be constrained. Thus counting argument provides the number of constraining equations to be equal to: \( N_c = d \times N - d + 1 \). According Maxwell’s criterion, the number of inter-particle contacts, \( \langle Z \rangle /2 \), must be at least equal to the number of equations \( N_c \). It turned out that for frictionless spheres the packing at the jamming point has exactly this number of contacts: \( \langle Z \rangle = 2d - 2(d - 1)/N \); for 3D average number of contact per particle is \( \langle Z \rangle = 6 - 4/N \) [25]. The results in Figure S1 are in the range of \( N \) that is meaningful for the present study: \( (10, 100) \). The range for \( N \) is dictated by the need of rattlers free packings. When the system gets larger the probability of generating a packing with no rattlers quickly drops [25]. For each \( N \), 100 different packing are generated. Error bars give one standard deviation.

It has been shown that a plane cut through a random spheres packing, provides a distribution of disks on a plane that follows [51, 52]:

\[
P(x|D_p) = \frac{x}{D_p \sqrt{D_p^2 - x^2}} [1 - \Theta(x - D_p)]
\]  

(12)

where \( \Theta(\cdot) \) is the Heaviside function. Substituting Equation 12 into Equation 10 we get:

\[
f(x) = \int_0^\infty \frac{n(D_p)}{M_0} \frac{x}{D_p \sqrt{D_p^2 - x^2}} [1 - \Theta(x - D_p)] \, dD_p
\]  

(13)

that the disks of the same size can originate from spheres of different size because the disc size depends on the position at which a sphere is cut.

Thus, for a given plane cut, the surface occupied by the disks on that plane is given as:

\[
\alpha = N_c \frac{\pi}{4} \int_0^\infty x^2 f(x) \, dx = N_c \frac{\pi}{6} \frac{M_2}{M_0}
\]  

(14)

FIG. S2. **Distribution of the volume fraction at the jamming threshold:** The position of the maximum of the jamming volume fraction distribution exhibits finite-size scaling: \( \phi^* - \phi_0 = \delta_0 N^{-1/d
u} \), where \( \delta_0 = 0.12 \pm 0.03 \), \( d = 3 \), \( \nu = 0.71 \pm 0.08 \), and \( \phi^* = 0.639 \pm 0.001 \) [25]. The asymptotic value is plotted as a shaded area in Figure S2. Since the distribution of the volume fraction at jamming is quite symmetric (with the exact symmetry in a thermodynamic limit: \( N \to \infty \)), the position of the peak can be taken as a good approximation for the average value of a volume fraction at the jamming(\( \phi_j \)). The green dots in the plot are the average values calculated from 100 independent simulation. Error bars give one standard deviation.

When integrated over the whole body, we obtain the volume of the solid material: \( V = \alpha L^3 \), where \( L \) is a linear dimension of a body. We can see that \( \alpha \) is proportional to the volume fraction \( \phi = V/L^3 \), and finally \( N_c \propto \phi = 1 - \psi \), where \( \psi \) is a sample porosity. Analogically, the wetted perimeter per unit area of bed, \( \Sigma \), can be obtained from:

\[
\Sigma = N_c \int_0^\infty x f(x) \, dx = N_c \frac{\pi}{4} \frac{M_2}{M_0}
\]  

(15)

thus \( \Sigma \sim N_c \propto 1 - \psi \).

Finally the hydraulic radius \( R_h \) is:

\[
R_h = \frac{1 - \alpha}{\Sigma} = \frac{2}{3\pi} \frac{\psi}{1 - \psi} \frac{M_2}{M_1}
\]  

(16)
C. Hydraulic tortuosity: continuum percolation argument

The evolution of the void region between overlapping, randomly located spheres undergoes a percolation transition \([44] [45]\). This transition exhibits a critical behavior and falls into a continuum percolation universality class \([44] [45] [48]\). In the context of porous materials, a structural parameter, a porosity \(\psi\), acts like the percolation probability in a classical percolation theory. Above a certain porosity threshold \(\psi_c\), there exists a channel that spans the whole system and facilitates fluid transport. This idea has been leveraged to connect tortuosity with material porosity \([4] [55] [59]\). Here we present an equivalent but simpler argument.

Percussion theory predicts that a mean distance \(\chi\) between any two sites on a cluster is given by a scaling law \([2]\):

\[
\chi \sim |\psi - \psi_c|^{-\nu} \tag{17}
\]

where \(\nu\) is a critical exponent of the correlation length. The total length of a walk \(\lambda\) constructed on that cluster has a fractal dimension \(D\) and reads \(\lambda \sim \chi^D\) \([71]\). At the percolation threshold, the correlation length-scale \(\chi\) diverges and is the same as the system size. From the definition of a tortuosity \(\tau\), we have then (close to the percolation threshold):

\[
\tau = \frac{\lambda}{\chi} \sim \chi^{D-1} \sim |\psi - \psi_c|^{(1-D)} \equiv \delta \psi^{\nu(1-D)} \tag{18}
\]

For a finite system, there is an additional finite-size correction that accounts for the shift of the percolation transition. Taking this into account the scaling reads:

\[
\tau_H \sim |\psi - \psi_c| + C \cdot L^{-\nu}|\psi^{\nu(1-D)} \tag{19}
\]

where \(C\) is a constant.

It has been shown that the most probable traveling length of an incompressible flow on a percolating cluster falls into the same universality class as the optimal path in strongly disordered media and the shortest path in the invasion percolation with trapping \([60] [61]\) for which the fractal dimension is \(D \approx 1.43\) \([62] [63]\). Finally, taking the exponent \(\nu \approx 0.88\), one gets a scaling law for tortuosity \((L \to \infty)\):

\[
\tau_H \sim |\psi - \psi_c|^{-0.38} \equiv \delta \psi^{-0.38} \tag{20}
\]

A similar scaling argument was numerically tested for 2D overlapping squares on Cartesian lattice \([15]\). Via finite-size scaling analysis it was shown, that the tortuosity in the neighborhood of percolation transition is controlled by the fractal geometry of a percolating channel.

D. Tortuosity calculation

For the fluid flow, hydraulic tortuosity \(\tau_H\) is defined as:

\[
\tau_H = \frac{\langle \lambda \rangle}{L} \geq 1 \tag{20}
\]

where \(\langle \lambda \rangle\) is the mean length of fluid particles path and \(L\) is the distance through the medium in the direction of a macroscopic flow. Despite this simple definition, tortuosity is not easy to measure experimentally and computationally. In real porous media, flow streams are complicated, as the fluid fluxes continuously change their sectional area, shape and orientation, or they branch and rejoin. It is also not clear how the average in Equation \([20]\) should be calculated: over the whole volume, over the planar cross-section, and if so, what is the most proper cross-section to do this? This leads to the feeling that tortuosity can be defined only in relatively simple models. Regardless of the difficulties, it has been concluded that the proper hydraulic tortuosity should be calculated as an average in which streamlines are weighted with fluid fluxes \([15] [20] [19]\). Thus tortuosity can be calculated from the formula:

\[
\tau_H = \frac{\sum_i \lambda_i \omega_i}{\sum_i \omega_i} \tag{21}
\]

where \(i\) enumerates discrete streamlines, \(\lambda_i = \lambda_i / L\), \(\lambda_i\) is the length of the \(i^{th}\) streamline with the weight \(\omega_i = 1 / t_i\), where \(t_i\) is a time in which fluid particles move along the \(i^{th}\) streamline \([15]\). The rationale behind \(\omega_i\) factor is to weight each streamline proportionally to the volumetric flow associated with a streamline. For the incompressible flow, \(t_i\) tells how long it takes for the particles in a given streamline to travel a distance \(L\) in a macroscopic flow direction. Thus, the average component of the velocity for that streamline in a direction of the flow is proportional to the weight factor: \(\langle v_x \rangle_i \sim \omega_i\). Extending this idea in the continuous limit, for a cross-section perpendicular to the macroscopic flow the hydraulic tortuosity can be formulated as:

\[
\tau_H = \frac{\int_A u_x(r) \bar{\lambda}(r) \, d\sigma}{\int_A u_x(r) \, d\sigma} \tag{22}
\]

where \(A\) is a cross-section perpendicular to the axis \(x\), both integrals are taken over the surface element \(d\sigma \in A\), \(\bar{\lambda}(r)\) is the length of a streamline cutting \(A\) at \(r\) (normalized by \(L\)), and \(u_x(r)\) is the component of the velocity field at \(r \in A\) normal to \(A\). For the incompressible flow, the stream of fluid between any pair of streamlines is constant, both integrals in Equation \([22]\) are independent on the choice of cross-section \([13]\). Moreover, it was shown that the cut can be done not necessarily in a direction of the macroscopic flow but in principle in any direction \([15]\). Even though there is a freedom in the location of where the cut can be done, both integrals are still difficult to calculate numerically \([15]\).
This numerical problem can be bypassed by noticing that [13]:

\[
\tau_H = \frac{\int_A u_{\perp}(\mathbf{r})\lambda(\mathbf{r})d\sigma}{\int_A u_{\perp}(\mathbf{r})d\sigma} = \frac{\int_V u(\mathbf{r})d\nu}{\int_V u_x(\mathbf{r})d\nu}
\]  

(23)

and the r.h.s. can be further simplified as [13]:

\[
\tau_H = \frac{\langle u \rangle}{\langle u_x \rangle}
\]  

(24)

This form of tortuosity is particularly handy in numerical analysis since it requires only solving the flow field without struggling with resolving streamlines [13, 20]. Some inaccuracies may occur in Equation 25 if the eddies exist in the flow. Although it cannot be assured that such structures do not occur in complex porous materials, the contribution from eddies to Equation 22 is negligible at low Reynolds numbers [13].

Finally the velocity field is found with Lattice-Boltzmann simulations. Then \(\tau_H\) can be calculated from the values of the flow at each node in the lattice:

\[
\tau_H = \frac{\sum_r u(\mathbf{r})}{\sum_r u_x(\mathbf{r})}
\]  

(25)

where \(\mathbf{r}\) runs over all lattice nodes [20].

E. Scaling argument for \(\gamma\) exponent

Taking a planar cut through the porous material, we observe \(n_c\) capillaries distributed over the area of the cut. If the material is isotropic, a direction of the cut does not matter, and we can assume, that the cut is made perpendicularly to the direction of fluid transport. This plane-cut would obviously contain cross-sections of all the capillaries that are responsible for the liquid transport thorough the material in the given direction. Close to the percolation threshold, we expect to have a single capillary in the area that is proportional to \(\sim \xi^2\), where \(\xi\) is the correlation length. If that is the case, the expected number of capillaries penetrating thorough the material is \(n_c \sim L^2/\xi^2\), where \(L\) is a linear size of the body. \(\xi\) is related to the exponent of the correlation length \(\nu \approx 0.88\) as \(\xi \sim \delta \psi^{-\nu}\). Therefore, we have a power-law relation between the number of capillaries and \(\delta\psi\) that reads \(n_c \sim \delta \psi^{2\nu} \approx \delta \psi^{1.76}\).

F. Parameters fitting procedure

Parameters fitting and an estimation of standard deviations is done with a non-linear least squares method from scipy Python library.

1. Extrapolating percolation threshold in the continuum limit

In Figure 2 C, we extrapolate a percolation threshold down to the continuum limit \(\psi_c\), i.e. \(\delta \to 0\). First, we fit a sigmoid function to the percolation probability data in Figure 2 B. Next we take a porosity at which the percolation probability is equal to 1/2 as a percolation threshold (for different \(\delta\)). Finally, we fit a power-law dependence: \(\psi_i - \psi_c = A \cdot \delta^\beta\). The fitting results are in Table I (row: Figure 2 C). Parameters are obtained as a result of minimization of the function: \(\text{Error} = \sum_i (\psi_i^{\text{num}} - \psi_i^{\text{fit}})^2\), where \(\psi_i^{\text{num}}\) is a percolation threshold estimated from the numerical data, and \(\psi_i^{\text{fit}}\) estimated from the power-law dependence for varying \(\psi_c\), \(A\), and \(\beta\).

2. Fitting power-law dependences for tortuosity

We fit two power-law dependencies for tortuosity data obtained from Lattice-Boltzmann simulations. The first relation has a functional form \(\tau = C(\psi - \psi_c)^{-0.38}\), where there are only two fitting parameters: \(\psi_c\) and a constant factor \(C\). Porosity \(\psi\) is a value known from Finite Elements simulations, and the exponent \(-0.38\) is predicted from a percolation theory, see Section V C. In the second fit, this exponent is left out as a free parameter and found from fitting to a relation \(\tau = C(\psi - \psi_c)^{-\alpha}\). In both cases we perform non-linear fitting by minimizing the error function: \(\text{Error} = \sum_i (\tau_i^{\text{num}} - \tau_i^{\text{fit}})^2\), where index \(i\) runs over all experimental samples, \(\tau_i^{\text{num}}\) is a numerical tortuosity from L-B simulations for a system \(i\), whereas \(\tau_i^{\text{fit}}\) is a fit to one of the two power-law dependencies for a given set or parameters. The results are given in the table I (row: Figure 3).

3. Parameters estimation for permeability

Fits are done for three different permeability relations: i) \(\kappa = C(\psi - \psi_c)^{2.52}(1 - \psi)^{-2}\), ii) \(\kappa = C(\psi - \psi_c)^{2.52}(1 - \psi)^{-2}\), and ii) \(\kappa = C(\psi - \psi_c)^{2.52}\). In Figure 5 and Figure S5 A-B, a percolation threshold is a fitting parameter \(\psi_c\), whereas in Figure S5 C-D, the percolation threshold is held fixed and estimated from the equation \(\psi(\delta) = 0.035 + 3.76 \cdot \delta^{1.1}\), where \(\delta = 0.04\) and the numerical parameters are taken from Figure 2 C. Fitting is done for two different error functions i) \(\text{Error} = \sum_i (\log \kappa_i^{\text{num}} - \log \kappa_i^{\text{fit}})^2\) in Figure 5 and Figure S5 A and C, and ii) \(\text{Error} = \sum_i (\kappa_i^{\text{num}} - \kappa_i^{\text{fit}})^2\) in Figure S5 B and D. \(\kappa_i^{\text{num}}\) is permeability value obtained from a L-B simulation for the \(i\)’th packing, whereas \(\kappa_i^{\text{fit}}\) is a fitted value for a given set of parameters. The results of these fits can be found in Table II.
Table I. Fitting parameters for a percolation threshold in the continuum limit $\psi^*_c$, tortuosity $\tau$, and permeability $\kappa$ that are investigated in this paper. In Figure S5 C and Figure S5 D, $\psi_c$ is fixed, so no standard deviations are given.
FIG. S3. The same numerical data as in Figure 4, but with error bars included (only data from Lattice-Boltzmann simulations is shown).
FIG. S4. **Tortuosity for three different $\delta$ and different lattice refinement levels**: L-B simulations are performed on percolating clusters that are detected with three different lattice sizes: $\delta = 0.03, 0.04, 0.05$. The flow fields are resolved on lattices with a size $\tilde{\delta} = \delta/R$, where $R = 1, 2, 3$ are lattice refinement levels [13]. Tortuosity increases as the refinement level increases, consistently with previous studies [13]. The same behavior is observed for all $\delta$. This suggests that the abrupt increase of $\tau_H$ close to the percolation threshold is caused by the fractal geometry of the percolation cluster rather than by the numerical artifacts. Each data-point is an average from about 100 simulations. $\tau_H$ is given as a function of $\delta\psi$, where lattice size dependent percolation threshold was taken from the fit in Figure 2C, and Table I: $\{\psi_c(0.03) = 0.113, \psi_c(0.04) = 0.141, \psi_c(0.05) = 0.171\}$. Error-bars are not shown for better readability.

[68] R. McGregor, J. Soc. Dyers Colour. 81, 429 (1965).
[69] M. Rintoul, Physical Review E 62, 68 (2000).
[70] M. P. Allen and D. J. Tildesley, Computer simulation of liquids (Oxford university press, 1989).
[71] S. Havlin and D. Ben-Avraham, Advances in Physics 36, 695 (1987).
FIG. S5. Permeability obtained from Lattice-Boltzmann simulations for the system size \( N=50 \), the lattice resolution \( \delta = 0.04 \), and the lattice size for the fluid phase \( \bar{\delta} = \delta/3 \). Symbols are the same as in Figure 5; blue crosses represent permeability for individual simulations, black open circles represent binned averages, red stars are median values, and dashed lines correspond to different models. Fitted parameters are given in Table I. In Figure S5 A and B, \( \psi_c \) is a free parameter, whereas in Figure S5 D, \( \psi_c \) is fixed at \( \psi_c = 0.141 \) (see Section VI F 1). Classical Kozeny-Carman model (\( \kappa \sim \psi^3/(1-\psi)^2 \)) is given for a comparison in Figure S5 A and B. Error functions used in a fitting procedure are given in Table I.