On the complete synchronization of a time–fractional reaction–diffusion system with the Newton–Leipnik nonlinearity

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Abstract

In this paper, we consider a time–fractional reaction–diffusion system with the same nonlinearities of the Newton–Leipnik chaotic system. Through analytical tools and numerical results, we derive sufficient conditions for the asymptotic stability of the proposed model and show the existence of chaos. We also propose a nonlinear synchronization controller for a pair of systems and establish the local and global asymptotic convergence of the trajectories by means of fractional stability theory and the Lyapunov method.

Keywords: Newton–Leipnik, fractional chaotic system, reaction diffusion, chaos synchronization, complete synchronization.

1. Introduction

Chaos has become a very common term in a number of scientific and engineering disciplines. Over the last three decades the amount of research publications dealing with chaotic dynamical systems and their applications is in the thousands. One of the main reasons behind this interest is the chaotic nature of several physical phenomena such as the weather or the turbulent flow of fluids. Another important reason is that in certain fields
such as secure communications and data encryption, the random nature of the chaotic system’s states is a desirable property [1, 2, 3, 4, 5]. Chaotic systems are generally characterized by their high sensitivity to variations in the initial conditions, which can be attributed to the system having at least one positive Lyapunov exponent. If two identical systems with the exact same parameters start from very close points in phase space, they end up following trajectories that move away from one another at an exponential rate. As a result, the trajectories of such systems seem random but are completely deterministic and can be easily replicated if the exact initial conditions are known. Perhaps, the usefulness of chaos became more apparent once their synchronization became possible. Synchronization refers to the control of one chaotic system to follow the exact same trajectory of another through adaptive rules. Among the first studies that realized the possibility of such a controller are [6, 7, 8, 9].

Originally, chaotic dynamical systems were considered with integer differentiation orders. However, researchers quickly realized that fractional calculus can improve the modeling of natural phenomena and lead to a wider range of system dynamics and attractor types. Fractional calculus is an old topic as it goes back to the seventeenth century. However, its importance in modeling was only recently observed. Fractional–order systems come with some added implementation complexity but at the same time offer a higher level of flexibility and a wider range of chaotic trajectories. This is mainly due to the fact that unlike integer–order differentiation, its fractional counterpart comes with an infinite memory and thus takes into consideration all previous states. The synchronization of fractional chaotic systems is widely considered, see for instance [10, 11, 12].

Another piece of background that is important to us in this paper is that of chaotic reaction–diffusion systems. It is easy to see that the vast majority of studies dealing with chaos consider an ODE system where the dependent variables represent the time evolution of certain physical quantities. However, in [13], the authors pointed out that chaos is particularly important in modeling and understanding the laminar and turbulent flow of fluids. They argued that since fluid consists of a continuum of hydrodynamic modes, it is more suitable to describe its dynamics by means of a spatially extended system of differential equations, i.e. a reaction–diffusion system. They studied the dynamics of the complex Ginzburg–Landau and Kuramoto–Sivashinsky equations. Recently after that, the authors of [14] showed that the general chaotic behavior of a reaction–diffusion system is similar to the ODE case.
in the sense that the system is extremely sensitive to changes in the initial states as well as the system’s bifurcation parameters. Parekh et al. [15] studied the control of an autocatalytic reaction–diffusion system. They devised a synchronization scheme and established the convergence of the error by means of appropriate Lyapunov functionals. An interesting summary of chaos in reaction–diffusion systems is given in [16]. Several studies have been published recently dealing with the stabilization and synchronization of spatio–temporal chaotic systems. For instance, it has been shown that neural networks can exhibit chaotic dynamics [17, 18, 19]. Other types of systems that may under certain circumstances be chaotic include predator–prey models [20] and the FitzHugh–Nagumo model [21, 22].

This paper is concerned with the chaotic dynamics and synchronization of a reaction–diffusion system that assumes the same nonlinearities of the Newton–Leipnik system first proposed in [23] as a model of the rigid body motion through linear feedback (LFRBM). The dynamics of the original system as well as its control were studied in [24, 25, 26, 28]. In [27], the authors examine a reaction–diffusion version of the system. Kang et al. proposed a fractional version of the system in [29]. They investigated the fractional model numerically over wide parameter ranges and commented on the impact of parameters on the system dynamics. The same system was examined further in [30] and shown to exhibit complex dynamics including fixed points, periodic motions, chaotic motions, and transient chaos. In [31], the authors develop a disturbance observer based adaptive sliding mode hybrid projective synchronization scheme for the fractional system. In our work, we consider a combination of the above mentioned properties in a single system. The proposed fractional–time reaction–diffusion Newton–Leipnik system is investigated analytically and experimentally in terms of its dynamics and synchronization.

2. On Fractional Calculus and Stability

Since the reader may not be very familiar with some of the notation and terminology used throughout this paper with regards to fractional calculus, it seems suitable to provide the following definitions and lemma along with helpful references.
Definition 1. [22] The Liouville fractional derivative of order $\delta$ of an integrable function $f(t)$ is defined as
\begin{equation}
t_0 D_t^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\delta}} d\tau.
\end{equation}
(2.1)

Definition 2. [22] The Caputo fractional derivative of order $\delta > 0$ of a function $f$ of class $C^n$ for $t > t_0$ is defined as
\begin{equation}
C_{t_0} D_t^\delta f(t) = \frac{1}{\Gamma(n - \delta)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\delta-n-1}} d\tau,
\end{equation}
(2.2)
with $n = \min \{ k \in \mathbb{N} | k > \delta \}$ and $\Gamma$ representing the gamma function.

Definition 3. [33] The constant $x_0$ is considered to be an equilibrium for the Caputo fractional nonautonomous dynamic system
\begin{equation}
C_{t_0} D_t^\delta x(t) = f(t, x),
\end{equation}
(2.3)
if and only if
\begin{equation}
f(t, x_0) = 0.
\end{equation}
(2.4)

Lemma 1. [34] Let $x(t)$ be a continuous and differentiable real function. For any time instant $t \geq t_0$,
\begin{equation}
C_{t_0} D_t^{\delta} x^2(t) \leq 2 x(t)C_{t_0} D_t^{\delta} x(t),
\end{equation}
(2.5)
with $\delta \in (0, 1)$.

We say that the constant $(u^*, v^*)$ is an equilibrium for the Caputo fractional non–autonomous dynamic system
\begin{equation}
\begin{cases}
C_{t_0} D_t^{\delta_1} u = F(u, v), & \text{in } \mathbb{R}^+, \\
C_{t_0} D_t^{\delta_2} v = G(u, v), & \text{in } \mathbb{R}^+,
\end{cases}
\end{equation}
(2.6)
if and only if
\begin{equation}
F(u^*, v^*) = G(u^*, v^*) = 0.
\end{equation}
(2.7)
Assuming a fractional order system comprising of two differential equations with an equilibrium $(u^*, v^*)$ and Jacobian matrix $J(u^*, v^*)$ evaluated at
(u\*, v\*)}, the following local and global asymptotic stability results are important.

**Lemma 2.** [35] Assuming \( \delta_1 = \delta_2 = \delta \), the equilibrium point \((u\*, v\*)\) is locally asymptotically stable iff

\[
|\arg (\lambda_i)| > \frac{\delta \pi}{2}, \quad i = 1, 2,
\]

where \(\lambda_i\) are the eigenvalues of \(J (u\*, v\*)\) and \(\arg (\cdot)\) denotes the argument of a complex number.

**Lemma 3.** [36] Assuming \( \delta_i = \frac{l_i}{m_i}, \) \( i = 1, 2, \) with \((l_i, m_i) = 1\) and \(l_i, m_i \in \mathbb{N}\), the equilibrium point \((u\*, v\*)\) is locally asymptotically stable iff all the roots \(\lambda\) of the characteristic equation

\[
\det \left( \text{diag} \left( \lambda^{m_1}, \lambda^{m_2} \right) - J (u\*, v\*) \right) = 0,
\]

satisfy

\[
|\arg (\lambda)| > \frac{\pi}{2m},
\]

where \(m\) is the least common multiple of the denominators \(m_i\).

**Lemma 4.** If there exists a positive definite Lyapunov function

\[
V(U) = \frac{1}{2} U^T (t) U (t)
\]

such that

\[
D_t \tilde{V} (U) < 0
\]

for all \(t \geq t_0\), then the trivial solution of system

\[
D_t \tilde{X} = F (U),
\]

where \(F : \mathbb{R}^n \to \mathbb{R}^n, U = (u_1, u_2, \ldots, u_n), \tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n), \) and \(0 < \delta_i \leq 1\), is globally asymptotically stable.

**Lemma 5.** If \(U (t) \in \mathbb{R}^n, \tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n), \) and \(0 < \delta_i \leq 1\), then

\[
\frac{1}{2} D_t ^{\tilde{\delta}} U^T (t) U (t) \leq U^T (t) D_t ^{\tilde{\delta}} U (t).
\]
Lemma 6. Consider the fractional-order system

\[ D^\beta \phi(t) = f(\phi(t)), \quad (2.15) \]

where \(0 < \delta < 1\), with \(\phi(t) \in \mathbb{R}\) and \(\phi^* = 0\) as its equilibrium. If for any \(\phi(t)\),

\[ \phi(t) f(\phi(t)) \leq 0, \quad (2.16) \]

then \(\phi^*\) is asymptotically stable. Moreover, if for any \(\phi(t) \neq 0\),

\[ \phi(t) f(\phi(t)) < 0, \quad (2.17) \]

then \(\phi^*\) is asymptotically stable.

3. Standard and Fractional Newton–Leipnik Models

The Newton–Leipnik system first proposed in [23] is of the form

\[
\begin{align*}
\frac{du_1}{dt} &= -au_1 + u_2 + 10u_2u_3 := f_1(u_1, u_2, u_3), \\
\frac{du_2}{dt} &= -u_1 - 0.4u_2 + 5u_1u_3 := f_2(u_1, u_2, u_3), \\
\frac{du_3}{dt} &= \alpha u_3 - 5u_1u_2 := f_3(u_1, u_2, u_3),
\end{align*}
\]

\[ (3.1) \]

\(u_i, \ i = 1, 2, 3\), denote the system’s the states and \(a\) and \(\alpha\) are bifurcation parameters. It is easy to show that system (3.1) is dissipative. By taking the divergence of the vector field \(f\) on \(\mathbb{R}^3\), we to obtain (see [23])

\[ \text{div} f = \alpha - a - 0.4. \quad (3.2) \]

Let \(\Omega\) be some region within \(\mathbb{R}^3\) with a smooth boundary \(\partial \Omega\) and let \(\Omega(t) = \Phi_t(\Omega)\), where \(\Phi_t\) is the flow of the vector field \(f\), and \(V(t)\) denote the volume of \(\Omega(t)\). Using Liouville’s theorem, we obtain

\[
\frac{dV(t)}{dt} = \int_{\Omega(t)} (\text{div} f) \, du_1 du_2 du_3 = (\alpha - a - 0.4) V(t). \quad (3.3)
\]

Solving the differential equation yields

\[ V(t) = V(0) e^{(\alpha - a - 0.4)t}. \quad (3.4) \]
It is easy to see that assuming $\alpha - a - 0.4 < 0$, the volume decays to zero asymptotically as $t \to \infty$, which means that the system is a dissipative one. It is well known that a dissipative chaotic system has a strange attractor.

In order to determine the equilibria of system (3.1), we set the time derivatives to zero and solve for $(u_1, u_2, u_3)$. This yields the five equilibria (see [25])

$$
O_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
O_2 = \begin{pmatrix} -0.031549 \\ 0.12238 \\ -0.11031 \end{pmatrix},
O_3 = \begin{pmatrix} 0.031549 \\ -0.12238 \\ -0.11031 \end{pmatrix},
O_4 = \begin{pmatrix} 0.23897 \\ 0.030803 \\ 0.21031 \end{pmatrix},
O_5 = \begin{pmatrix} -0.23897 \\ -0.030803 \\ 0.21031 \end{pmatrix}.
$$

(3.5)

The asymptotic stability of these equilibria can be examined by means of standard stability results. The system has been shown to exhibit a chaotic behavior for specific values of the parameters $(a, \alpha)$. For instance, subject to parameters $(a, \alpha) = (0.4, 0.175)$ and initial conditions $(u_1(0), u_2(0), u_3(0)) = (0.349, 0, -0.3)$, the phase portraits of the system are depicted in Figure 1. The system exhibits a strange attractor with two equilibria.

Figure 1: Phase–space portraits of the standard Newton–Leipnik chaotic system for $(a, \alpha) = (0.4, 0.175)$ and initial conditions $(u_1(0), u_2(0), u_3(0)) = (0.349, 0, -0.3)$. 
The Caputo–type fractional version of the Newton–Leipnik system was formulated in [29] as

\[
\begin{align*}
C_{t_0}^\delta_1 u_1 &= -au_1 + u_2 + 10u_2u_3, \\
C_{t_0}^\delta_2 u_2 &= -u_1 - 0.4u_2 + 5u_1u_3, \\
C_{t_0}^\delta_3 u_3 &= \alpha u_3 - 5u_1u_2,
\end{align*}
\]

where \(0 < \delta_1, \delta_2, \delta_3 \leq 1\) are the fractional differentiation orders and \(C_{t_0}^\delta \frac{d}{dt}\) denotes the Caputo fractional derivative over \((t_0, \infty)\) as defined in (2.2). Assuming identical orders \(\delta_1 = \delta_2 = \delta_3\), it has been shown in [28] that equilibrium \(O_1\) is always unstable whereas \(O_2\) and \(O_3\) are asymptotically stable subject to

\[
\delta < 0.93660,
\]

and \(O_4\) and \(O_5\) are asymptotically stable subject to

\[
\delta < 0.9541.
\]

These is confirmed by the numerical results in Figure 2, where the same parameters and initial conditions assumed previously were adopted and the fractional order is varied.

As for the incommensurate case where the three orders are non–identical, no exact bound has been found for the asymptotic stability of the system. It was, however, shown in [28] that for lower orders, the system is asymptotically stable and for orders close to one, the system becomes chaotic. For instance, it was shown that for \((\delta_1, \delta_2, \delta_3) = (1, 0.95, 0.975)\), all euilibria are asymptotically unstable, whereas for \((\delta_1, \delta_2, \delta_3) = (0.85, 0.9, 0.8)\), all of \(O_2, O_3, O_4\) and \(O_5\) are asymptotically stable. This, again, may be verified through numerical simulations as depicted in Figure 3.

4. Time–Fractional Reaction–Diffusion Model

In this paper, we are concerned with the time–fractional reaction–diffusion Newton–Leipnik system given by

\[
\begin{align*}
C_{t_0}^\delta_1 u_1 - d_1 \Delta u_1 &= -au_1 + u_2 + 10u_2u_3, \quad \text{in } \mathbb{R}^+ \times \Omega, \\
C_{t_0}^\delta_2 u_2 - d_2 \Delta u_2 &= -u_1 - 0.4u_2 + 5u_1u_3, \quad \text{in } \mathbb{R}^+ \times \Omega, \\
C_{t_0}^\delta_3 u_3 - d_3 \Delta u_3 &= \alpha u_3 - 5u_1u_2, \quad \text{in } \mathbb{R}^+ \times \Omega,
\end{align*}
\]

(4.1)
Figure 2: Phase-space portraits of the fractional commensurate Newton–Leipnik chaotic system for $(a, \alpha) = (0.4, 0.175)$ and initial conditions $(u_1(0), u_2(0), u_3(0)) = (0.349, 0, -0.3)$ with different fractional orders.
Figure 3: Phase–space portraits of the fractional incommensurate Newton–Leipnik chaotic system for \((a, \alpha) = (0.4, 0.175)\) and initial conditions \((u_1(0), u_2(0), u_3(0)) = (0.349, 0, -0.3)\) with different fractional orders.
where \( u_i(x,t), i = 1, 2, 3 \), are the spatio–temporal states of the system, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is the Laplacian operator on \( \Omega \), and \( d_i > 0, i = 1, 2, 3 \), are the diffusivity constants for each of the states. We assume the nonnegative initial conditions

\[
0 \leq u_i(0,x) = u_{i,0}(x), \quad i = 1, 2, 3, \quad \text{in} \ \Omega, \quad (4.2)
\]

with \( u_{i,0} \in C^2(\Omega) \cap C(\overline{\Omega}) \), and impose homogenous Neumann boundary conditions

\[
\frac{\partial u_i}{\partial \nu} = 0, \quad i = 1, 2, 3, \quad \text{on} \quad \mathbb{R}^+ \times \partial \Omega, \quad (4.3)
\]

where \( \nu \) is the unit outer normal to \( \partial \Omega \).

Subject to certain parameter values and fractional orders, system (4.1) can be shown to exhibit spatio–temporal chaos. For instance, choosing parameters \((a, \alpha) = (0.4, 0.175)\) and initial conditions

\[
\begin{align*}
&u_1(x,0) = 0.349 \left[ 1 + 0.3 \cos \left( \frac{x}{2} \right) \right], \\
&u_2(x,0) = 0, \\
&u_3(x,0) = -0.3 \left[ 1 + 0.3 \cos \left( \frac{x}{2} \right) \right],
\end{align*}
\]

yields spatio–temporal chaos as shown in Figure 4(left) for \( \delta = 0.99 \). Figure 4(left) was produced over the ranges \( t \in [0, 50] \) and \( x \in [0, 20] \). Although the chaotic nature of the states is apparent, it always helps to visualize the phase–space portraits of the system. In order to be able to do that, we choose the single spatial point \( x = 10 \) and plot its phase space over time. The result is depicted in Figure 4(right). It is interesting to realize that the fractional order has an impact on the dynamics of the system. Keeping the same parameters and initial conditions and changing the fractional order to \( \delta = 0.99 \) yields the results depicted in Figure 5. The trajectories clearly converge to a closed orbit, which implies an oscillatory behavior. Reducing the order further to \( \delta = 0.90 \) yields an asymptotically stable solution as shown in Figure 6.

For the incommensurate case, we consider the set of fractional constants \((\delta_1, \delta_2, \delta_3) = (0.97, 0.98, 0.99)\). The spatio–temporal states and the phase space at \( x = 10 \) are depicted in Figure 7. The chaotic behavior with a double strange attractor is apparent.
Figure 4: Time evolution of the spatio–temporal states and the phase portraits taken at $x = 10$ for parameters $(a, \alpha) = (0.4, 0.175)$, initial conditions $(4.4)$, and fractional order $\delta = 0.99$. 
Figure 5: Time evolution of the spatio–temporal states and the phase portraits taken at $x = 10$ for parameters $(a, \alpha) = (0.4, 0.175)$, initial conditions (4.4), and fractional order $\delta = 0.95$. 

Figure 6: Time evolution of the spatio–temporal states and the phase portraits taken at $x = 10$ for parameters $(a, \alpha) = (0.4, 0.175)$, initial conditions (4.4), and fractional order $\delta = 0.90$. 
Figure 7: Time evolution of the spatio–temporal states and the phase portraits taken at \( x = 10 \) for parameters \((a, \alpha) = (0.4, 0.175)\), initial conditions \((4.4)\), and fractional order \((\delta_1, \delta_2, \delta_3) = (0.97, 0.98, 0.99)\).
5. Complete Synchronization

The main objective of our paper is to develop an adaptive control scheme to synchronize an identical slave system of the form

\[
\begin{align*}
C_t^\alpha D_t^{\delta_1} v_1 - d_1 \Delta v_1 &= -av_1 + v_2 + 10v_2v_3 + \phi_1, \\
C_t^\alpha D_t^{\delta_2} v_2 - d_2 \Delta v_2 &= -v_1 - 0.4v_2 + 5v_1 v_3 + \phi_2, \\
C_t^\alpha D_t^{\delta_3} v_3 - d_3 \Delta v_3 &= \alpha v_3 - 5v_1v_2 + \phi_3,
\end{align*}
\]

(5.1)

with \( v_i(x,t), i = 1, 2, 3, \) denoting the states of the slave system and \( \phi_i(x,t), i = 1, 2, 3, \) being some control parameters, to the master system given in (4.1).

Before we present the synchronization scheme and assess the local and global asymptotic stability of the zero solution to the error system, let us define the goal of synchronization.

**Definition 4.** System (4.1) and the controlled system (5.1) are said to be asymptotically synchronized if

\[
\lim_{t \to \infty} \|u - v\| = 0
\]

(5.2)

for any \( t > 0, \) where \( u = (u_1, u_2, u_3)^T \in (\mathbb{R}^+ \times \Omega)^3 \) and \( v = (v_1, v_2, v_3)^T \in (\mathbb{R}^+ \times \Omega)^3. \)

We also need to define some necessary notation relating to the eigenvalues and eigenfunctions of the Laplacian operator. We denote the eigenvalues of the elliptic operator \((-\Delta)\) subject to the homogeneous Neumann boundary conditions on \( \Omega \) by

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...
\]

(5.3)

We assume that each eigenvalue \( \lambda_i \) has multiplicity \( m_i \geq 1. \) We also denote the normalized eigenfunctions corresponding to \( \lambda_i \) by \( \Phi_{ij}, 1 \leq j \leq m_i. \) It should be noted that \( \Phi_0 \) is a constant and \( \lambda_i \to \infty \) as \( i \to \infty. \) The eigenfunctions and eigenvalues possess a number of interesting properties including

\[
-\Delta \Phi_{ij} = \lambda_i \Phi_{ij} \quad \text{in} \quad \Omega, \\
\frac{\partial \Phi_{ij}}{\partial n} = 0 \quad \text{on} \quad \partial \Omega, \\
\int_{\Omega} \Phi_{ij}^2(x) dx = 1.
\]

(5.4)

We are ready to present our main result as stated in the following theorem. We assume that the fractional order is identical for all components
of the master and slave systems yielding a commensurate system. The local asymptotic convergence of the synchronization error is established by means of eigenfunction analysis and the global convergence is guaranteed by the Lyapunov method. For the synchronization problem, we will assume \( \delta_1 = \delta_2 = \delta_3 =: \delta \).

**Theorem 1.** For the general fractional orders \( 0 < \delta \leq 1 \), the master–slave pair (4.1–5.1) is globally synchronized subject to the nonlinear control laws

\[
\begin{aligned}
\phi_1 &= -10 (e_2 e_3 + u_2 e_3 + e_2 u_3), \\
\phi_2 &= -5 (e_1 e_2 + u_1 e_3 + e_1 u_3), \\
\phi_3 &= 5 (e_1 e_2 + u_1 e_2 + e_1 u_2) - (\alpha + 0.4) e_3,
\end{aligned}
\]

if for all eigenvalues \( \lambda_i \) satisfying

\[
\lambda_i < \frac{2}{|d_1 - d_2|}, \quad d_1 \neq d_2,
\]

condition

\[
|\arg (\xi_{1,2})| > \frac{\delta \pi}{2}
\]

is fulfilled, where

\[
\xi_{1,2} = \frac{1}{2} \left[ (- (d_1 + d_2) \lambda_i - 0.8) \pm i \sqrt{4 - (d_1 - d_2)^2 \lambda_i^2} \right].
\]

**Proof 1. Part I:** In the first part of our proof, we show that the zero solution of the error system is locally asymptotically stable in the diffusion free case. The synchronization errors can be given by

\[
\begin{aligned}
D_1^\delta e_1 - d_1 \Delta e_1 &= -0.4 e_1 + e_2 + 10 (v_2 v_3 - u_2 u_3) + \phi_1, \\
D_2^\delta e_2 - d_2 \Delta e_2 &= -e_1 - 0.4 e_2 + 5 (v_1 v_3 - u_1 u_3) + \phi_2, \\
D_3^\delta e_3 - d_3 \Delta e_3 &= \alpha e_3 - 5 (v_1 v_2 - u_1 u_2) + \phi_3.
\end{aligned}
\]

This can be rewritten in the more compact form

\[
\begin{aligned}
D_1^\delta e_1 - d_1 \Delta e_1 &= -0.4 e_1 + e_2 + 10 (e_2 e_3 + u_2 e_3 + e_2 u_3) + \phi_1, \\
D_2^\delta e_2 - d_2 \Delta e_2 &= -e_1 - 0.4 e_2 + 5 (e_1 e_2 + u_1 e_3 + e_1 u_3) + \phi_2, \\
D_3^\delta e_3 - d_3 \Delta e_3 &= \alpha e_3 - 5 (e_1 e_2 + u_1 e_2 + e_1 u_2) + \phi_3.
\end{aligned}
\]
Subsituting the controls (5.5) in (5.10) yields the error dynamics

\[
\begin{align*}
D_t^\delta e_1 - d_1 \Delta e_1 &= -0.4e_1 + e_2, \quad \text{in } \mathbb{R}^+ \times \Omega, \\
D_t^\delta e_2 - d_2 \Delta e_2 &= -e_1 - 0.4e_2, \quad \text{in } \mathbb{R}^+ \times \Omega, \\
D_t^\delta e_3 - d_3 \Delta e_3 &= -0.4e_3, \quad \text{in } \mathbb{R}^+ \times \Omega,
\end{align*}
\]

(5.11)

with the Jacobian matrix

\[
J_e = \begin{pmatrix}
-0.4 & 1 & 0 \\
-1 & -0.4 & 0 \\
0 & 0 & -0.4
\end{pmatrix}.
\]

(5.12)

The eigenvalues of \(J_e\) are simply \(-0.4 + 1.0i\), \(-0.4 - 1.0i\), and \(-0.4\). We see that

\[|\arg(-0.4 \pm i)| = 1.9513,\]

and

\[|\arg(-0.4)| = \pi.\]

Selecting \(\delta < 1.2422\) guarantees asymptotic stability. Since is assumed to lie in the interval \(0 < \delta \leq 1\), local asymptotic stability of the zero solution to (5.10) in the diffusion free case is evident.

Part II: In this second part, we want to include diffusion and assess the local stability of the zero solution. In the presence of diffusion, the steady state solution satisfies the following system

\[
\begin{align*}
-d_1 \Delta e_1 &= -0.4e_1 + e_2, \\
-d_2 \Delta e_2 &= -e_1 - 0.4e_2, \\
-d_3 \Delta e_3 &= -0.4e_3,
\end{align*}
\]

subject to the homogeneous Neumann boundary conditions

\[
\frac{\partial e_1}{\partial \nu} = \frac{\partial e_2}{\partial \nu} = \frac{\partial e_3}{\partial \nu} = 0 \quad \text{for all } x \in \partial \Omega.
\]
Consider the linearization operator

\[ L = \begin{pmatrix}
-d_1 \Delta - 0.4 & 1 & 0 \\
-1 & -d_2 \Delta - 0.4 & 0 \\
0 & 0 & -d_3 \Delta - 0.4 \\
\end{pmatrix}. \]

Let \((\phi(x), \psi(x), \Upsilon(x))\) be an eigenfunction of \(L\) corresponding to the eigenvalue \(\xi\), i.e. the pair satisfies

\[ L(\phi(x), \psi(x), \Upsilon(x))^t = \xi (\phi(x), \psi(x), \Upsilon(x))^t. \]

Alternatively, we can write

\[ [L - \xi I] (\phi(x), \psi(x))^t = (0, 0, 0)^t, \]

leading to

\[ \begin{pmatrix}
-d_1 \Delta - 0.4 - \xi & 1 & 0 \\
-1 & -d_2 \Delta - 0.4 - \xi & 0 \\
0 & 0 & -d_3 \Delta - 0.4 - \xi \\
\end{pmatrix} \begin{pmatrix}
\phi \\
\psi \\
\Upsilon \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}. \]

Using the factorizations

\[ \phi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} a_{ij} \Phi_{ij}, \quad \psi = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} b_{ij} \Phi_{ij}, \quad \text{and} \quad \Upsilon = \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} c_{ij} \Phi_{ij}, \]

the matrix equation can be formulated as

\[ \sum_{0 \leq i \leq \infty, 1 \leq j \leq m_i} \begin{pmatrix}
-d_1 \lambda_i - 0.4 - \xi & 1 & 0 \\
-1 & -d_2 \lambda_i - 0.4 - \xi & 0 \\
0 & 0 & -d_3 \lambda_i - 0.4 - \xi \\
\end{pmatrix} \begin{pmatrix}
a_{ij} \\
b_{ij} \\
c_{ij} \\
\end{pmatrix} \Phi_{ij} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}. \]

Disregarding the term \(\xi\), the stability of the steady state solution relies on the eigenvalues of

\[ A_i = \begin{pmatrix}
-d_1 \lambda_i - 0.4 & 1 & 0 \\
-1 & -d_2 \lambda_i - 0.4 & 0 \\
0 & 0 & -d_3 \lambda_i - 0.4 \\
\end{pmatrix}. \]

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whose characteristic polynomial is
\[ ((d_1 \lambda_i + 0.4 + \xi) (d_2 \lambda_i + 0.4 + \xi) + 1) (d_3 \lambda_i + 0.4 + \xi) = 0. \]

Clearly, one of the eigenvalues is
\[ \xi_3 = -d_3 \lambda_i - 0.4. \]

The remaining eigenvalues are the solutions of
\[
\left| \begin{array}{cc}
-d_1 \lambda_i - 0.4 - \xi & 1 \\
-1 & -d_2 \lambda_i - 0.4 - \xi \\
\end{array} \right| = 0,
\]
or more compactly
\[ \xi^2 - ((d_1 + d_2) \lambda_i - 0.8) \xi + \left( \lambda_i^2 d_1 d_2 + 0.4 (d_1 + d_2) \lambda_i + 1.16 \right) = 0. \]

The discriminant of this quadratic polynomial is
\[
\Delta = ((d_1 + d_2)^2 \lambda_i - 4) - 4 (\lambda_i^2 d_1 d_2 + 0.4 (d_1 + d_2) \lambda_i + 1.16) \\
= ((d_1 + d_2)^2 - 4d_1 d_2) \lambda_i^2 - 4.0 \\
= (d_1 - d_2)^2 \lambda_i^2 - 4.
\]

Depending on the sign of \( \Delta \), we may end up with different scenarios:

1. First, if \( \Delta = (d_1 - d_2)^2 \lambda_i^2 - 4 \geq 0 \), then the remaining two eigenvalues are both real. It helps to consider the trace
\[
tr \left( \begin{array}{cc}
-d_1 \lambda_i - 0.4 & 1 \\
-1 & -d_2 \lambda_i - 0.4 \\
\end{array} \right),
\]
which is clearly strictly negative for all \( i \geq 0 \), and the determinant
\[
det \left( \begin{array}{cc}
-d_1 \lambda_i - 0.4 & 1 \\
-1 & -d_2 \lambda_i - 0.4 \\
\end{array} \right),
\]
which is clearly strictly positive for all \( i \geq 0 \). Hence, \( \xi_{1,2} \in \mathbb{R}^- \).
• If $\Delta = (d_1 - d_2)^2 \lambda_i^2 - 4 < 0$, then

$$\lambda_i < \frac{2}{|d_1 - d_2|}, \; d_1 \neq d_2.$$ 

Hence, the two eigenvalues $\xi_{1,2}$ are complex and may be given by (5.8).

This tells us that if all eigenvalues satisfying (5.6) fulfill (5.7), then the steady state solution is locally asymptotically stable.

**Part III:** Now that we have established sufficient conditions for the local asymptotic stability of the zero solution to (5.10), we move to show that it is globally asymptotically stable. Consider the Lyapunov function

$$V = \frac{1}{2} \int (e_1^2 + e_2^2 + e_3^2) \, dx.$$ 

By taking the $\delta$ fractional derivative and employing Lemma 4, we obtain

$$D_t^\delta V = \frac{1}{2} \int_\Omega \left( D_t^\delta e_1^2 + D_t^\delta e_2^2 + D_t^\delta e_3^2 \right) \, dx$$

$$\leq \int_\Omega e_1 D_t^\delta e_1 + e_2 D_t^\delta e_2 + e_3 D_t^\delta e_3 \, dx$$

$$\leq I + J,$$

where

$$I = -d_1 \int_\Omega |\nabla e_1|^2 \, dx - d_2 \int_\Omega |\nabla e_2|^2 \, dx - d_3 \int_\Omega |\nabla e_3|^2 \, dx < 0,$$

and

$$J = - \int_\Omega 0.4 e_1^2 + 0.4 e_2^2 + [0.4] e_3^2 \, dx < 0,$$

Hence, $D_t^\delta V < 0$ and the zero solution of (5.10) is globally asymptotically stable. The proof is complete.
6. Numerical Results

In order to verify the results of the previous section, we use numerical simulations. We let \((a, \alpha) = (0.4, 0.175)\) and
\[
\begin{align*}
    u_1(x, 0) &= 0.349 \left[1 + 0.3 \cos \left(\frac{x}{2}\right)\right], \\
    u_2(x, 0) &= 0, \\
    u_3(x, 0) &= -0.3 \left[1 + 0.3 \cos \left(\frac{x}{2}\right)\right].
\end{align*}
\]
Assuming \(\delta = 0.99\), Figure 8(left) shows the synchronization error between the master (4.1) and slave (5.1) for \(\Omega \in [0, 20] \times [0, 50]\). Synchronization is achieved by means of the 3D control law (5.5). The errors clearly decay to zero as time progresses indicating successful synchronization. Figure 8(right) shows the master and slave trajectories in phase–space at spatial point \(x = 10\). The same experiment is repeated with the different fractional orders
\[
(\delta_1, \delta_2, \delta_3) = (0.97, 0.98, 0.99).
\]
The results are shown in Figure 9. Again, as shown analytically, the numerical results confirm the successful synchronization of our master–slave pair.

7. Concluding Remarks

In this paper, we have considered a time–fractional spatio–temporal system based on the Newton–Leipnik chaotic system. We started by giving a brief overview of the most important definitions and theory related to fractional dynamical system. Then, we reviewed some important aspects of the standard and fractional Newton–Leipnik systems in the free diffusion scenario. The main result of the paper concerns the global complete synchronization of a master–slave pair of the proposed system. We established sufficient conditions for the asymptotic convergence of the synchronization errors to zero by means of local and global asymptotic stability methods. Throughout the paper, we have used numerical simulations to illustrate the findings of our study.
Figure 8: Time evolution of the spatio–temporal synchronization errors (left) and the phase portraits of the master (blue) and slave (red) taken at $x = 10$ (right) with parameters $(a, \alpha) = (0.4, 0.175)$, initial conditions (4.4), and fractional order $\delta = 0.99$. 
Figure 9: Time evolution of the spatio–temporal synchronization errors (left) and the phase portraits of the master (blue) and slave (red) taken at \(x = 10\) (right) with parameters \((a, \alpha) = (0.4, 0.175)\), initial conditions \((4.4)\), and fractional orders \((\delta_1, \delta_2, \delta_3) = (0.97, 0.98, 0.99)\).
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