On the existence and uniqueness of minima and maxima on spheres of the integral functional of the calculus of variations

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Abstract: Given a bounded domain $\Omega \subset \mathbb{R}^n$, we prove that if $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a $C^1$ function whose gradient is Lipschitzian in $\mathbb{R}^{n+1}$ and non-zero at 0, then, for each $r > 0$ small enough, the restriction of the integral functional $u \to \int_\Omega f(u(x), \nabla u(x))dx$ to the sphere $\{u \in H^1(\Omega) : \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)dx = r\}$ has a unique global minimum and a unique global maximum.

Key words: Sobolev space; integral functional; minimum; maximum; sphere; existence; uniqueness.

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Introduction

Here and in the sequel, $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a $C^1$ function whose gradient is non-constant and Lipschitzian (with respect to the Euclidean metric).

We will consider the Sobolev space $H^1(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)dx\right)^{\frac{1}{2}}$$

which is induced by the scalar product

$$\langle u, v \rangle = \int_\Omega (\nabla u(x) \nabla v(x) + u(x)v(x))dx .$$

The linear growth of $\nabla f$ (coming from its Lipschitzianity) implies that the functional

$$u \to J(u) := \int_\Omega f(u(x), \nabla u(x))dx$$

is (well-defined and) $C^1$ on $H^1(\Omega)$, with derivative given by

$$\langle J'(u), v \rangle = \int_\Omega (f_\xi(u(x), \nabla u(x))v(x) + \nabla_\eta f(u(x), \nabla u(x))\nabla v(x))dx$$

for all $u, v \in H^1(\Omega)$ ([2], p. 249).
Let $r > 0$. We are interested in minima and maxima of the restriction of the functional $J$ to the sphere $S_r := \{u \in H^1(\Omega) : \|u\| = r\}$.

In the present setting, there is no evidence of their existence and uniqueness. In fact, with regard to the existence aspect, not only $S_r$ is not weakly compact but also, if $f(\xi, \cdot)$ is neither convex nor concave in $\mathbb{R}^n$, the functional $J$ is neither lower nor upper weakly semicontinuous. But, even when $J$ is sequentially weakly continuous, it may happen that $J$ has no minima and/or maxima on $S_r$.

In this connection, consider the following simple and enlightening situation. Assume that $f$ depends only on the first variable and that has a unique global maximum in $\mathbb{R}$, say $\xi_0$. So, $J(u) = \int_{\Omega} f(u(x))dx$. Then, it is clear that the constant function $x \to \xi_0$ is the unique maximum of the functional $J$. In this case, $J$ turns out to be sequentially weakly continuous, thanks to the Rellich-Kondrachov theorem. Then, by Lemma 2.1 of [1], the function $\rho \to \sup_{S_\rho} J$ is non-decreasing in $]0, +\infty[$. Consequently, if $r > |\xi_0|(\text{meas}(\Omega))^\frac{1}{2}$, $J|_{S_r}$ has no maxima.

Nevertheless, we will show that if $\nabla f(0) \neq 0$ then $J|_{S_r}$ possesses exactly one minimum and exactly one maximum for each $r > 0$ small enough.

**The result**

To shorten the statement of our result, let us introduce some further notations. In the sequel, $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is another $C^1$ function which is non-negative, with $g(0) = 0$, and whose gradient is Lipschitzian, with Lipschitz constant $\nu < 2$. We set

$$I(u) = \int_{\Omega} g(u(x), \nabla u(x))dx$$

for all $u \in H^1(\Omega)$.

Moreover, $V$ is a closed linear subspace of $H^1(\Omega)$ with the following property: there exists $v_0 \in V$ such that

$$\int_{\Omega} (f_{\xi}(0)v_0(x) + \nabla_\eta f(0)\nabla v_0(x))dx \neq 0.$$

Finally, if $L$ is the Lipschitz constant of $\nabla f$, we denote by $S$ the set (possibly empty) of all global minima of the restriction to $V$ of the functional

$$u \to \|u\|^2 + I(u) + \frac{2-\nu}{L} J(u).$$

Then, with the convention $\inf \emptyset = +\infty$, our result reads as follows:
THEOREM 1. - Under the above assumptions, one has

\[ \delta := \inf_{u \in S} (\|u\|^2 + I(u)) > 0 \]

and, for each \( r \in ]0, \delta[ \), the restriction of the functional \( J \) to the set

\[ C_r := \{ u \in V : \|u\|^2 + I(u) = r \} \]

has a unique global minimum.

PROOF. Let \( \mu \geq 0 \) and let \( u, v, w \in H^1(\Omega) \), with \( \|w\| = 1 \). Using Cauchy-Schwartz and Hölder inequalities, we have

\[ |\langle I'(u) + \mu J'(u) - I'(v) - \mu J'(v), w \rangle| \leq \int_{\Omega} \left| (g_\xi(u, \nabla u) - g_\xi(v, \nabla v))w + (\nabla_\eta g(u, \nabla u) - \nabla_\eta g(v, \nabla v))\nabla w \right| dx + \\
+ \mu \int_{\Omega} \left| (f_\xi(u, \nabla u) - f_\xi(v, \nabla v))w + (\nabla_\eta f(u, \nabla u) - \nabla_\eta f(v, \nabla v))\nabla w \right| dx \leq \\
\leq \int_{\Omega} \left( (|g_\xi(u, \nabla u) - g_\xi(v, \nabla v)|^2 + |\nabla_\eta g(u, \nabla u) - \nabla_\eta g(v, \nabla v)|^2) \right)^{\frac{1}{2}} \left( |w|^2 + |\nabla w|^2 \right)^{\frac{1}{2}} dx + \\
+ \mu \int_{\Omega} \left( (|f_\xi(u, \nabla u) - f_\xi(v, \nabla v)|^2 + |\nabla_\eta f(u, \nabla u) - \nabla_\eta f(v, \nabla v)|^2) \right)^{\frac{1}{2}} \left( |w|^2 + |\nabla w|^2 \right)^{\frac{1}{2}} dx \leq \\
\leq (\nu + \mu \lambda) \|u - v\| .

Hence, the derivative of the functional \( I + \mu J \) is Lipschitzian, with constant \( \nu + \mu \lambda \). As a consequence, if \( 0 \leq \mu < \frac{2-\nu}{\lambda} \), the functional \( u \rightarrow \|u\|^2 + I(u) + \mu J(u) \) is strictly convex and coercive. To see this, it is enough to show that its derivative is strongly monotone ([3], pp. 247-248). Indeed, if \( \Phi(\cdot) := \|\cdot\|^2 \), we have for all \( u, v \in H^1(\Omega) \)

\[ \langle \Phi'(u) + I'(u) + \mu J'(u) - \Phi'(v) - I'(v) - \mu J'(v), u - v \rangle \geq \\
\geq 2\|u - v\|^2 - \|I'(u) - J'(v) + \mu (I'(v) - J'(v))\| \cdot \|u - v\| \geq (2 - \nu - \mu \lambda) \|u - v\|^2 .

Clearly, this shows also the convexity of the functional \( \Phi + I + \frac{2-\nu}{\lambda} J \). Assume \( S \neq \emptyset \). Then, \( S \) is closed and convex, and so there exists a unique \( \hat{u} \in S \) such that

\[ \|\hat{u}\|^2 + I(\hat{u}) = \delta . \]
Observe that \( \|u\|^2 + I(u) > 0 \) for all \( u \in V \setminus \{0\} \). So, \( \delta \geq 0 \). Arguing by contradiction, assume \( \delta = 0 \). Then, it would follow \( \hat{u} = 0 \). Hence, since \( 0 \in S \), we would have

\[
\langle \Phi'(0) + I'(0) + \frac{2 - \nu}{L} J'(0), v \rangle = 0
\]

for all \( u \in V \) and so, since \( \Phi'(0) + I'(0) = 0 \) (being 0 the global minimum of \( \Phi + I \)), it would follow

\[
\int_{\Omega} (f_\xi(0)v(x) + \nabla_{\eta} f(0)\nabla v(x))dx = 0
\]

for all \( v \in V \), against one of the hypotheses. Hence, we have proven that \( \delta > 0 \). Now, fix \( r \in [0, \delta] \) and consider the function \( \Psi : V \times [\frac{L}{2 - \nu}, +\infty[ \to \mathbb{R} \) defined by

\[
\Psi(u, \lambda) = J(u) + \lambda(\|u\|^2 + I(u) - r)
\]

for all \( (u, \lambda) \in V \times [\frac{L}{2 - \nu}, +\infty[ \). As we have seen above, \( \Psi(\cdot, \lambda) \) is continuous and convex for all \( \lambda \geq \frac{L}{2 - \nu} \) and coercive for all \( \lambda > \frac{L}{2 - \nu} \), while \( \Psi(u, \cdot) \) is continuous and concave for all \( u \in V \), with \( \lim_{\lambda \to +\infty} \Psi(0, \lambda) = -\infty \). So, we can apply to \( \Psi \) a classical saddle-point theorem ([3], Theorem 49.4) which ensures the existence of \( (u^*, \lambda^*) \in V \times [\frac{L}{2 - \nu}, +\infty[ \) such that

\[
J(u^*) + \lambda^*(\|u^*\|^2 + I(u^*) - r) = \inf_{u \in V} (J(u) + \lambda(\|u\|^2 + I(u) - r)) =
\]

\[
= J(u^*) + \sup_{\lambda \geq \frac{L}{2 - \nu}} \lambda(\|u^*\|^2 + I(u^*) - r) .
\]

Of course, we have \( \|u^*\|^2 + I(u^*) \leq r \), since the sup is finite. But, if it were \( \|u^*\|^2 + I(u^*) < r \), we would have \( \lambda^* = \frac{L}{2 - \nu} \). This, in turn, would imply that \( u^* \in S \), against the fact that \( r < \delta \). Hence, we have \( \|u^*\|^2 + I(u^*) = r \). Consequently

\[
J(u^*) + \lambda^* r = \inf_{u \in V} (J(u) + \lambda^*(\|u\|^2 + I(u))) .
\]

From this, we infer that \( \lambda^* > \frac{L}{2 - \nu} \) (since \( r < \delta \)), that \( u^* \) is a global minimum of \( J_{|C_r} \) and that if each global minimum of \( J_{|C_r} \) is a global minimum in \( V \) of the functional \( u \to \|u\|^2 + I(u) + \lambda^* J(u) \). Since \( \lambda^* > \frac{L}{2 - \nu} \), this functional is strictly convex and so \( u^* \) is its unique global minimum in \( V \). The proof is complete. \( \triangle \)

REMARK 1. It is almost superfluous to remark that the conclusion of Theorem 1 may fail if the assumption that involves \( V \) and \( \nabla f(0) \) is not satisfied. In this connection, consider, for instance, the case \( f(\sigma) = -|\sigma|^2 \), with \( g = 0 \). This assumption, however, serves only to ensure that \( \delta > 0 \). So, it becomes superfluous, in particular, when \( S = \emptyset \).

Now, denote by \( S_1 \) the set (possibly empty) of all global minima of the restriction to \( V \) of the functional

\[
u \to \|u\|^2 + I(u) - \frac{2 - \nu}{L} J(u) .
\]

Clearly, applying Theorem 1 also to \( -f \), we get
THEOREM 2. - Under the assumptions of Theorem 1, one has

$$\delta_1 := \min \left\{ \inf_{u \in S} (\|u\|^2 + I(u)), \inf_{u \in S_1} (\|u\|^2 + I(u)) \right\} > 0$$

and, for each $r \in [0, \delta_1]$, the restriction of the functional $J$ to the set

$$\{ u \in V : \|u\|^2 + I(u) = r \}$$

has a unique global minimum and a unique global maximum.

References

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