On Distributed Graph Coloring with Iterative Recoloring

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Abstract

Identifying the sets of operations that can be executed simultaneously is an important problem appearing in many parallel applications. By modeling the operations and their interactions as a graph, one can identify the independent operations by solving a graph coloring problem. Many efficient sequential algorithms are known for this NP-Complete problem, but they are typically unsuitable when the operations and their interactions are distributed in the memory of large parallel computers. On top of an existing distributed-memory graph coloring algorithm, we investigate two compatible techniques in this paper for fast and scalable distributed-memory graph coloring. First, we introduce an improvement for the distributed post-processing operation, called recoloring, which drastically improves the number of colors. We propose a novel and efficient communication scheme for recoloring which enables it to scale gracefully. Recoloring must be seeded with an existing coloring of the graph. Our second contribution is to introduce a randomized color selection strategy for initial coloring which quickly produces solutions of modest quality. We extensively evaluate the impact of our new techniques on existing distributed algorithms and show the time-quality tradeoffs. We show that combining an initial randomized coloring with multiple recoloring iterations yields better quality solutions with the smaller runtime at large scale.

1 Introduction

In parallel computing, the problem of organizing computations so that no two concurrent procedures access shared resources simultaneously appears often. This problem might be solved by ordering them explicitly so that concurrent accesses can not happen, using concurrency control mechanisms such as locks, lock-free data structures or transactional memory. Even when using concurrency controls mechanisms, it is important to minimize the access to the lock or page of the transactional memory. The problem can be modeled as a graph coloring problem where the vertices of the graphs are operations of the problem and edges represent concurrent accesses to a resource. A coloring of the graph is a partition of the vertices of the graph in a number of independent sets. Minimizing the number of colors (i.e., the number of independent sets)

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reduces the number of synchronization points in the computation and enhances the efficiency of the parallel computers.

Graph coloring appears in many other applications, including, but not limited to, optimization [6], efficient computation of sparse Jacobian and Hessian matrices [18], preconditioners [25], iterative solution of sparse linear systems [18], sparse tiling [27], printed circuit testing [12], eigenvalue computation [21], frequency assignment [11], parallel numerical computation [11] and register allocation [5] areas.

There exist different types of the graph coloring problem. For instance, the distance-$k$ coloring problem requires that vertices separated by less than or equal to $k$ edges to have different colors. In this paper, we are only interested in distance-1 coloring; however, we believe that all the techniques and results presented in this paper can be extended to the other variants of the graph coloring problem.

Let $G = (V, E)$ be a graph with $|V|$ vertices and $|E|$ edges. Set of neighbors of a vertex $v$ is $\text{adj}(v)$; its cardinality, also called the degree of $v$, is $\delta_v$. The degree of the vertex having the most neighbors is $\Delta = \max_v \delta_v$. A distance-1 coloring $C : V \to \mathbb{N}$ is a function that maps each vertex of the graph to a color (represented by an integer), such that two adjacent vertices get different colors, i.e., $\forall (u, v) \in E, C(u) \neq C(v)$. Without loss of generality, the number of colors used is $\max_u C(u)$. Finding a coloring with as few colors as possible is an optimization problem. The problem of deciding whether a graph can be colored with less than $k$ colors is known to be NP-Complete for arbitrary graphs [22]. Therefore, finding the minimal number of colors a graph can be colored with (also called the chromatic number of the graph) is NP-Hard. Recently, it has been shown that, for all $\epsilon > 0$, it is NP-Hard to approximate the graph coloring problem within $|V|^{1-\epsilon}$ [31]. Hence, we are focusing on heuristics which work well in practice on many graphs.

We are studying the distributed-memory graph coloring problem in this paper. We focus on obtaining high quality solutions for large-scale scientific parallel applications. In those applications, the computational model (the graph) is already distributed to the nodes of the parallel machine. If the graph is sufficiently small, with a naive approach, one can aggregate it in the memory of a single node and color it there. It would actually be better to take advantage of partitioning in this case, i.e., first coloring the interior vertices (vertices for which all their neighbors are local) in parallel and then color the other, boundary vertices (vertices that have at least one non-local neighbor), sequentially, by aggregating the graph induced by them in the memory of a single processor. However, even if one implements such optimizations, one can still achieve significantly faster solutions by coloring in parallel [3]. If the graph is too large to fit in the memory of a single computer, coloring in distributed memory is inevitable. Also, people usually use coloring as a tool in contexts where good and quick solutions are desired. Therefore, one cannot afford repartitioning the graph for the sole purpose of coloring, since repartitioning has a higher computational complexity than coloring itself.

In this work, we aim to achieve good quality coloring with good runtime in distributed-memory settings. As a starting point, we use the framework presented in [3]; it mainly relies on processor-local greedy coloring techniques with multiple iterations to converge to a valid solution. Traditionally, the problem of improving the quality of coloring obtained from greedy coloring methods is addressed by considering the vertices in an order that have good properties for coloring. We investigated such techniques in [26] and found that processor-local ordering techniques do not yield quality improvement at large scale. Good global ordering techniques exhibit little parallelism and will not yield good runtimes. In this paper, we investigate recoloring, a post-processing operation which refines an existing coloring. We showed in [26] that the quality of the solution the recoloring procedure generates is not affected by the scale of the distributed-memory machine. However, the communications required by the algorithm make it non-scalable in terms of the runtime.

In this paper, we show how recoloring can be made scalable in terms of runtime by improving its communication scheme. In particular, we use piggybacking to reduce the number of communications. With a fast recoloring method, it becomes efficient to run recoloring multiple times. This allows to potentially start
with a faster initial coloring algorithm of lower quality. Pursuing this idea, we investigate the Random-X Fit color selection strategy [15] for generating a first coloring. Random-X Fit leads to a solution of modest quality but with a balanced color distribution which makes it very suitable for recoloring. We show that combining these different techniques allows to reach better time-quality trade-offs than previously existing algorithms. We extensively evaluate all the techniques we propose at different scales in terms of both quality and runtime. This allows us to identify two sets of parameters “speed” and “quality” which enables the user to achieve the tradeoff she is interested in without having to understand the inner working of our coloring framework.

The remaining of the document is organized as follows. Section 2 presents the different coloring algorithms existing for sequential and parallel architectures, including the previous distributed-memory coloring algorithm which is the reference algorithm we use and ordering solutions. Section 3 discusses the techniques we use to improve coloring in distributed-memory architecture. The proposed techniques are experimentally evaluated on real-world graphs and on random graphs in Section 4. Final conclusions and ideas to improve further are given in Section 5.

2 Graph Coloring Algorithms

Graph coloring is one of the well studied problems in the literature [10, 9, 16, 20]. Literature is abundant with many different techniques, such as the one that utilizes greedy coloring [24, 20], cliques [29] and Zykov trees [9]. In the following sections, we will first present a simple greedy sequential coloring algorithm and how it can be improved with vertex visit orderings and recoloring. We then briefly discuss other shared- and distributed-memory parallel coloring algorithms.

2.1 Sequential Coloring

In spite of the existing pessimistic theoretical results and the existence of more complicated algorithms, for many graphs that shows up in practice, solutions that are provably optimal or near optimal can be obtained using a simple greedy algorithm [6]. In this algorithm, the vertices are visited in some order and the smallest permissible color at each iteration is assigned to the vertex. Algorithm 1 gives the pseudocode of this technique.

| Algorithm 1: Sequential greedy coloring. |
|-----------------------------------------|
| Data: $G = (V,E)$                      |
| for each $v \in V$ do                  |
|   for each $w \in \text{adj}(v)$ do   |
|     forbiddenColors[color[w]] ← $v$   |
|     color[v] ← $\min\{i > 0 : \text{forbiddenColors}[i] \neq v\}$ |

Different ways for improving the resulting number of colors are presented in the literature. Changing the color selection strategy has an impact in the quality of coloring. Choosing the smallest permissible color, as stated in Algorithm 1, is known as the First Fit strategy. Selecting a color based on an initial estimate of the number of colors in a distributed-memory setting is proposed in [3] and called Staggered First Fit. The Least Used strategy picks (locally) the least used color so far so that a more even color distribution is achieved. Apart from that, Gebremedhin et al. [15] proposed randomized color selection strategies.
Algorithm 1 has two nice properties. First, for any vertex-visit ordering, it produces a coloring with at most \(1 + \Delta\) colors. Second, for some vertex-visit orderings it will produce an optimal coloring \([16]\). Many heuristics for ordering the vertices have been proposed in the literature \([16]\). These heuristics can be grouped into two categories: static and dynamic orderings. Largest First (LF) and Smallest Last (SL) \([23]\) orderings are static orderings, in the sense that the coloring order is obtained before the coloring starts. Saturation Degree \([4]\) and Incidence Degree orderings are dynamic orderings in which the coloring order of the vertices is obtained while the coloring is done. We refer the reader to \([16]\) for a summary of these ordering techniques. The LF ordering, introduced by Welsh and Powell \([30]\), visits the vertices in non-increasing order of their degree. SL obtains the ordering from backwards via selecting a vertex with the minimum degree to be ordered last and removing it from the graph for the rest of the ordering phase. Then, the next vertex with the minimum degree within the remaining graph is selected to be ordered second-to-last. This procedure is repeated until all vertices are ordered.

Apart from the different pre-ordering techniques, Culberson \([7]\) introduces a coloring algorithm, called Iterated Greedy (IG), where starting with an initial greedy coloring, vertices are iteratively recolored based on the coloring of the previous iteration. Culberson shows that, if the vertices belonging to the same color class (i.e., the vertices of the same color) at the end of initial coloring are colored consecutively, then the number of colors will either decrease or stay the same. Several different permutations of color classes are considered based on the colors of the vertices in the previous iteration, the number of vertices, degree of the vertices, randomness and combination of them. Culberson suggests that a hybrid approach, which changes the permutation strategy at each recoloring iteration is more effective than applying the same permutation strategy at each round. He proposed using random permutations of the colors to break cycles where number of colors stays the same after some number of iterations that appear when deterministic methods are used exclusively. The first work evaluating the effects of the recoloring scheme on parallel graph coloring is \([13]\) where the focus is on shared-memory computers. Also, Goldberg et al. suggest to use recoloring in multigraph edge-coloring \([17]\).

### 2.2 Distributed-Memory Parallel Coloring

In the literature, Bozdağ et al. \([3]\) introduced the first scalable distributed-memory parallel graph coloring framework. We believe that, their work is the only distributed-memory coloring algorithm that provides a parallel speedup, therefore our work is based on their algorithm, which we present here briefly.

It is assumed that the graph is distributed onto the distributed memory of computers. Each vertex belongs to a single processing unit. At each processing unit, information of the edges connected to any owned vertex is kept. In other words, for an edge \((u, v)\), if both \(u\) and \(v\) belong to a single processing unit, then only this processor has the information of that edge. Given the edge \((u, v)\), if \(u\) and \(v\) belong to two different processing units, then \((u, v)\) is only know by these two processing units. Each processing unit colors the vertices it has. If a vertex \(u\) has a neighbor vertex \(v\), owned by another processing unit, then \(u\) and \(v\) are boundary vertices. On the other hand, if all neighbors of a vertex \(u\) belong to same processing unit with \(u\), then \(u\) is called as internal vertex.

The coloring procedure is performed in multiple rounds. All the colorless vertices are tentatively colored by the greedy coloring algorithm at each round. Two neighbor vertices, belonging to two different processing units, may end up having the same color at the end of this phase, i.e., a conflict can occur. Such conflicts are independently detected by each processing unit. To resolve a conflict, one of the vertices will keep its color whereas the other one is scheduled to be colored again in the next round. Ties are broken based on a random total ordering, which is obtained beforehand. The algorithm continues to iterate in multiple rounds until there is no conflict.
Each round of the coloring procedure is executed in supersteps to reduce the number of conflicts. Each processing unit colors a specified number of owned vertices at each superstep. After that, colors of the boundary vertices (colored in that super step) are exchanged among neighbor processing units. There are two types of coloring procedures based on the communication mechanism between processing units. Coloring is called as synchronous, if a processing unit waits for its neighbors to complete their super step before beginning the next one. Synchronous coloring guarantees that two vertices can be in conflict only if they are colored in the same superstep. The other option, in which no waiting mechanism is enforced, is named as asynchronous. The superstep size matters for the coloring procedure, since a larger size decreases the number of exchanged messages with likely high number of conflicts, whereas the smaller size increases the exchanged message number with expectedly low number of conflicts.

2.2.1 Vertex-visit Ordering

The orderings considered in [3] only consider the partitioning of the graph and not the properties of the graph itself. Three orderings were investigated in their work, coloring internal vertices first, boundary vertices first and coloring the vertices in the order they are stored in the memory (which was called unordered in [3], here we will call it Natural ordering).

In [26], we investigated two more successful ordering heuristics, namely Largest First (LF) and Smallest Last (SL) techniques. The LF ordering can be computed in $O(|V|)$ time [30]. Using a carefully designed bucket data structure allows to implement SL with a complexity of $O(|E|)$ [23]. In distributed memory, we let each processor compute an ordering of the graph based on the knowledge it has and therefore it is not guaranteed that the coloring computed on a single processor and on multiple processors will be the same. We also verified this fact by scalability experiments in [26]. We showed that LF ordering provides less number of colors than sequential Natural ordering on less than 8 processors. SL obtains much fewer number of colors. However, when the number of processors increases, the advantages of LF and SL orderings disappear. On 512 processors, the choice of vertex-visit ordering does not yield a significant difference in number of colors. So, vertex-visit orderings are not beneficial for quality at large scale since the number of internal vertices decreases when the scale increases and the ordering is done locally at each processor.

3 Distributed-Memory Iterative Recoloring

Culberson [7] presented the use of iterative recoloring for improving the number of colors in a sequential algorithm. In [26], we extended this work to distributed-memory graph coloring. The recoloring idea naturally fits distributed-memory graph coloring. Independents sets obtained at the end of the initial coloring are used to efficiently color the graph. Our recoloring algorithm (RC) proceeds in as many steps as the number of colors in the initial solution. All the vertices in the same color class (i.e., having the same color in the previous coloring round) are colored at the same step in the recoloring process. Then the processors exchange the color information with their neighboring processors at the end of the step. Notice that, although a processor does not have any vertices with that color, it will wait for other processors to finish coloring at that step. This procedure guarantees that no conflict is created by the end of recoloring. Therefore, recoloring in sequential and in distributed memory lead to the same solution, if the initial coloring is same, making the recoloring scalable in terms of number of colors.

However, the algorithm is fairly synchronous since a processor cannot start the $i$-th step before its neighbors finish their $(i-1)$-th step. Moreover, there is no guarantee that two processors will have a similar number of vertices in each color, thus potentially leading to a load imbalance. To prevent some
potential load imbalance due to synchronous execution of RC, we also propose a second recoloring approach, named \textit{asynchronous recoloring} (aRC). In this approach, each processor computes their vertex-visit orderings independently using the initial coloring, and apply a second parallel coloring with this new vertex-visit ordering using the algorithm recalled in Section \ref{sec:recoloring}. Note that, conflicts can happen at the end of this procedure, so this second parallel coloring step proceeds in conflict resolution rounds. We expect that this approach will not be as good as synchronous recoloring in terms of number of colors, but it might be beneficial for trading the quality for runtime.

In the recoloring process, all the vertices in the same color class must be colored in a consecutive manner, but one can choose any permutation of the color classes and this permutation affects the number of colors well. Therefore different permutations of the color classes should be taken into account for sequential and parallel recoloring. We considered four permutations of color classes: \textit{Reverse order} (RV) of colors, \textit{Non-Increasing} number of vertices (NI), where the color classes are ordered in the non-increasing order of their vertex counts, \textit{Non-Decreasing number} of vertices (ND), which is similarly derived, and \textit{Random} order of color classes where the color classes are ordered randomly using the Knuth shuffling procedure in linear time \cite{7}. We compute global NI and ND permutations by communicating between processors before coloring to exchange the number of vertices at each color class.

### 3.1 Improvements on the Communication Scheme of Synchronous Recoloring

The communication scheme in \textit{synchronous recoloring} algorithm results in fine-grained communications where the number of messages are high but each of these messages is small. This behavior lowers network performance and results in slower execution. To improve this fine-grained communication scheme, we apply piggybacking techniques.

In \textit{synchronous recoloring}, vertices belonging to same color class are colored in the same step and colors are communicated at the end of that step immediately. However, immediate communication is not necessary for two boundary vertices that are not colored in consecutive steps. For example, for two neighbor boundary vertices $a$ and $b$, belonging to different processors, if $\text{color\_order}(a) > \text{color\_order}(b)$ (in other words if $a$ will be colored later than $b$), then the coloring of $b$ does not depend on the coloring of $a$. Of course $b$ will need the new color of $a$ during the next recoloring iteration, so sending the color of $a$ to $b$’s processor can be deferred to the end of the recoloring iteration. For the reverse case, a similar technique can be used. Assume again that $\text{color\_order}(a) > \text{color\_order}(b)$, then $a$ needs to know the new color of $b$ before being colored. So, sending the color of $b$ can happen at any time between step $\text{color\_order}(a)$ and step $\text{color\_order}(b)$. A processor $P_1$ accumulates the color information in a buffer to send to a processor $P_2$. $P_1$ only sends the whole buffer at the color step before the step where $P_2$ needs any of the information contained in the whole buffer. This way, the color information of vertices are piggybacked and sent in a minimum number of messages which improves the performance of the communication subsystem.

In order to be able to apply the piggybacking technique, each processor needs to have the knowledge of when and from whom to receive messages for each iteration. For this purpose, there is a need for pre-communication at the beginning of each recoloring iteration among processors so that each get the information of from whom to receive at which step. The only information processors should send to each other is the list of color classes when they will communicate with each other. If $P_1$ is to send something to $P_2$ at color class $c$, then $P_2$ will know this at the end of the pre-communication and it will wait an incoming message from $P_1$ at color class $c$. As expected, there will be an overhead for this process. We investigate the gained time by piggybacking as well as the overhead time incurred by pre-communication in the experiments section.

Figure\ref{fig:communication} presents the situation between two processors with a total of six boundary vertices after initial
Piggybacking example. Initial colors are written in the vertex.

Figure 1: Piggybacking example. Initial colors are written in the vertex.

coloring. Vertices that are not on the boundary between processors $P_1$ and $P_2$ are not depicted. Without the use of piggybacking, $P_1$ sends three messages to $P_2$ containing the information of a single vertex at the end of steps 1, 3 and 12. It also needs to send empty messages during the other steps. $P_2$ needs to send similar three messages. Using piggybacking, $P_1$ sends a message to $P_2$ at the end of step 4; the message contains both the color information of vertices $v_B$ and $v_C$. Then $P_2$ sends the color information of vertex $v_D$. Finally, $P_1$ will send the color information of vertex $v_A$ at the end of the recoloring iteration and $P_2$ will send the color information of both $v_E$ and $v_F$. Notice that in this setting no empty messages are exchanged. Piggybacking is able to remove all empty messages and furthermore reduces the total number of non-empty messages from 6 to 4 between $P_1$ and $P_2$.

### 3.2 Random X-Fit

We expect that synchronous recoloring will be scalable in terms of number of colors. Piggybacking will make recoloring scalable in terms of runtime, especially if the number of vertices per color is roughly balanced.

So, we investigate a fast coloring alternative for the initial coloring. In [3], Bozdağ et al. proposed two different color selection strategies; First Fit and Staggered First Fit. We propose another color selection strategy, Random-X Fit, which uniformly selects a random color from the first $X$ available colors [15]. We expect that Random-X Fit will significantly reduce the number of conflicts in the coloring procedure leading to a fast coloring. Moreover, it should balance the number of vertices in each color which should help the recoloring procedure being fast.

Random-X Fit is expected to give solutions with high number of colors. But, we believe that recoloring is strong enough to fix any bad number of colors obtained in the initial coloring. And the coordinated use of Random-X Fit and recoloring should make the procedure fast.

### 4 Experiments

In the experiments we study the effectiveness of the recoloring procedure in Section 4.2 and prove three points:
• recoloring can significantly improve the solution of low quality in a few iterations (Section 4.2.1)
• our piggybacking techniques make recoloring scale much better (Section 4.2.2)
• and in a distributed memory setting, recoloring improves significantly the quality of the solution and incurs a low runtime cost allowing to use multiple recoloring iterations (Section 4.2.3).

Then we show in Section 4.3 that using a Random X-Fit coloring leads to a low quality solution but, when coupled with recoloring, it leads to better quality solution in a smaller amount of time than using a first fit coloring. This allows us to study the time-quality trade-off of the coloring problem and we exhibit two sets of parameters that can be used to obtain either a fast solution “speed” or a good solution “quality”.

Before showing experimental results, we present how the experiments are conducted in Section 4.1.

4.1 Experimental Setting

We implemented all the algorithms in Zoltan [2], an MPI-based C library for parallel partitioning, load balancing, coloring and data management services for distributed-memory systems. For the graphs, we partitioned them by ParMETIS [19] version 3.1.1 or with a simple block partitioning onto the parallel platform.

All of our experiments are performed on an in-house cluster with 64 computing nodes. There are two Intel Xeon E5520 (quad-core clocked at 2.27GHz) processors at each node along with 48GB of main memory, and 500 GB of local hard disk. Interconnection between the nodes are done through 20Gbps DDR oversubscribed InfiniBand. Nodes run CentOS with the Linux kernel 2.6.32. The C code is compiled with GCC 4.1.2 using the -O2 optimization flag. We tried different MPI implementations and MVAPICH2 in version 1.6 is chosen to utilize the InfiniBand interconnect efficiently.

Experiments are performed on number of processors which are powers of 2 from 1 to 512 processors. Each physical machine has 8 cores. When we do the allocation of processors on the cluster, we first use processors on different nodes to highlight distributed memory issue. That means, when 64 processors are used, each allocated processor (core) is on a different machine. Using 128 processors allocates 2 cores per machine and an allocation of 512 processors uses the 8 cores of the 64 machines. We have shown in [26] that this setting is the fastest configuration.

We run the experiments on six real-world application graphs and three synthetically generated graphs. The real-world graphs are from different application areas including linear car analysis, finite element, structural engineering and automotive industry [14, 28]. We obtained them from the University of Florida Sparse Matrix Collection [1], and the Parasol project. Table 1 gives the list of the graphs and their main properties. We also listed the number of colors obtained with a sequential run of the three vertex-visit orderings in the table. Lastly, runtime of the Natural coloring in sequential is given in table. One important thing to note is that the biggest real-world graph takes less than half a second to color sequentially using a Natural ordering, which means that the distributed coloring of all the graphs very challenging.

The synthetically generated graphs are RMAT graphs, introduced by Chakrabarti et al. [8]. When generating the RMAT graphs, the adjacency matrix of the graph is subdivided in 4 equal parts and edges are distributed to these parts with a given probabilities. Specifying different probabilities allows the generation of different classes of random graphs. We generated three graphs, RMAT-ER, RMAT-Good and RMAT-Bad. Their degree distributions for the four parts are (0.25, 0.25, 0.25, 0.25), (0.45, 0.15, 0.15, 0.25) and (0.55, 0.15, 0.15, 0.15), respectively. We generate these three graphs to create variety of challenges for

1 http://www.cise.ufl.edu/research/sparse/matrices/
### Table 1: Properties of real-world graphs.

| Name  | $|V|$   | $|E|$   | $\Delta$ | NAT | LF | SL | seq. time |
|-------|--------|--------|----------|------|----|----|-----------|
| auto  | 448,695 | 3,314,611 | 37       | 13  | 12 | 10 | 0.1103s   |
| bmw3.2 | 227,362 | 5,530,634 | 335      | 48  | 48 | 37 | 0.0836s   |
| hood | 220,542 | 4,837,440 | 76       | 40  | 39 | 34 | 0.0752s   |
| ldoor | 952,203 | 20,770,807 | 76      | 42  | 42 | 34 | 0.3307s   |
| msdoor | 415,863 | 9,378,650 | 76      | 42  | 42 | 35 | 0.1458s   |
| pwtk | 217,918 | 5,653,257 | 179     | 48  | 42 | 33 | 0.0820s   |

### Table 2: Properties of synthetic graphs.

| Name  | $|V|$   | $|E|$   | $\Delta$ | NAT | LF | SL |
|-------|--------|--------|----------|------|----|----|
| ER    | 16,777,216 | 134,217,624 | 42       | 12  | 10 | 10 |
| Good  | 16,777,216 | 134,181,065 | 1,278    | 28  | 15 | 14 |
| Bad   | 16,777,216 | 133,658,199 | 38,143   | 146 | 89 | 88 |

For all the experiments we will present both the number of colors obtained and the runtime of the method when the number of processors varies. The real-world graphs all show the same trends and their results are aggregated in the following manner. Each value (number of colors and runtime) is first normalized with respect to the value obtained by the Natural ordering of the same graph on one processor. Then the normalized value for different graphs are aggregated using a geometric mean. The three randomly generated graphs are presented independently for number of colors and they will be aggregated for runtime results, which are normalized with respect to Natural ordering on 4 processors, since they show the same behavior in terms of runtime.

### 4.2 Recoloring

#### 4.2.1 Sequential Setting

In Figure 2, we present the effect of vertex-visit orderings and multiple iterations of recoloring on the number of colors in sequential settings, which was also presented in [26]. Different vertex-visit orderings are multiplexed with different color permutation strategies in each chart. For example, NAT+RC-ND gives the Natural ordering coupled with recoloring with the Non-Decreasing number of vertex order of the color classes. Note that the charts begin with 0 iteration which shows the quality of the vertex-visit ordering only.

The first important result in those charts is that the LF vertex ordering results in lower number of colors than the Natural ordering with 0.96 normalized number of colors without recoloring and SL is the best vertex ordering strategy with 0.78 normalized number of colors. The second result is that three tested permutations of the color classes result in a decrease of the number of colors. Among them, the NI permutation leads to smallest improvement whereas the ND permutation gives the smallest number of colors by obtaining 0.8 normalized number of colors after 20 iterations for all three orderings. Lastly, the best number of colors is obtained by combining the SL vertex ordering with ND color permutation (SL+RC-ND). ND permutation
leads to lower number of colors due to the selection of color classes. Success of a permutation can be measured by its ability to remove as many color classes as possible. In ND permutation strategy, color classes with fewer vertices are selected first so that the classes with larger number of vertices can merge with them.

In this paper, we investigate the effect of randomness of the color class permutations in the sequential setting to get out of the local plateaus. For this purpose, we run experiments on real-world graphs with the three vertex-visit orderings, namely Natural, Largest First and Smallest Last. We choose the ND permutation as a reference point since it gives the best number of colors. The color permutation that picks a random permutation uniformly is denoted RAND. Using the RAND permutation at every $x$ iterations and using ND for the other iterations is denoted (ND-RAND%$x$). For instance using RAND every five iterations is denoted (ND-RAND%5) while using it every 10 iterations is denoted (ND-RAND%10). We also investigate the variant we call (ND-RAND%2$^i$) which uses random permutation only at iterations which are power of two; in other words the ND permutation is used at each iteration except at iterations 2, 4, 8, 16, ... where the permutation is RAND.

Figure 3 presents the results for this experiment. Results are obtained by running all the tests 10 times and taking the average over all runs. For the NAT vertex-visit ordering, randomness helps reducing the number of colors. The RAND permutation is better than the ND permutation for all number of iterations. However, rarefied randomness outperforms pure randomness. As the frequency of randomness decreases, the number of colors decreases as well. For example, ND-RAND%10 is better than ND-RAND%5. Two main conclusions can be drawn from this experiment. First, randomization helps getting ND out of local plateau as is strongly suggested by the sudden decrease in number of colors at the 10th iteration of ND-RAND%10. Second, ND might need numerous iterations to refine a coloring: the normalized number of colors does not change between iteration 32 and 60 of ND. The ND-RAND%2$^i$ manages to successfully use these two properties by trying to get out of local plateau aggressively during the first iterations and then, by letting ND to slowly refine its coloring.

On the other hand, the LF and SL orderings show different behaviors than the NAT ordering. Randomness helps reducing the number of colors during the first iterations (up to 20 iterations in our experiments). But the ND permutation outperforms all of the other permutations on large number of iterations. This picture suggests that insisting on the ND permutation for the LF and SL vertex-visit orderings gives the best results if one is interested on high number of iterations. Overall, it can be said that randomness provides a sharp decrease in number of colors at the beginning of the process, but then the number of colors converges rapidly. Since we start from a lower number of colors in LF and SL orderings, it is expected to converge quicker and this is the case in LF and SL orderings with high number iterations of random permutations.
Figure 3: Sequential study of recoloring with color class permutation randomness on the real-world graphs.

Figure 2 showed that the SL+RC-ND combination outperformed all other vertex-visit orderings and color class permutations presented in the sequential case. We will then focus on comparing the distributed synchronous recoloring (RC) and the asynchronous one (aRC) using the non recolored Smallest Last ordering as a reference. However, before diving into more comparison, we need to investigate the effect of the piggybacking technique on the communication scheme of synchronous recoloring procedure.

4.2.2 Improvements on Communication Scheme of Recoloring

We show here the runtime improvement obtained on recoloring by applying the piggybacking techniques discussed in Section 3.1. We run the experiments with 8 processors per node configuration to see the impact easily. Figure 4 presents a comparison between a base implementation of recoloring (without piggybacking presented in [26]) and the improved implementation using piggybacking with detailed timings. Using piggybacking needs a preparation phase, but it takes at most 12% of the improved total coloring time in the worst case. On the other hand, piggybacking provides a huge improvement in recoloring time. Our experiments show that piggybacking provides 80% less number of messages in average, when compared to base implementation. Improved coloring with piggybacking gives 20% to 70% improvement over base total coloring time. These results indicate that accumulating messages into bigger chunks provides a better communication for recoloring and we will then use the improved recoloring technique when we are making runtime comparisons.

4.2.3 Distributed Setting

Figure 5 presents the performance of recoloring on the real-world graphs. We compare the combination of First-Fit color selection, Smallest Last ordering and synchronous communication (denoted FSS as in, [26]), with additional synchronous and asynchronous recoloring. As expected, the synchronous recoloring reduces the normalized number of colors significantly providing to keep the normalized number of colors below the one obtained using the sequential Largest First even on 256 processors, bringing a 18% improvement in the number of colors to FSS. However the synchronous recoloring takes more time than the FSS coloring, reaching a normalized runtime of 2.01 while FSS has a normalized runtime of 0.60 on 256 processors. Asynchronous recoloring provides a middle ground by allowing to obtain a better quality coloring than FSS is able to achieve. However, asynchronous recoloring gives almost the same runtime with synchronous recoloring. It shows a different picture than the one in [26] since the piggybacking technique in synchronous recoloring brings significant improvement. Synchronous recoloring is as fast as the asynchronous recoloring, and leads
The impact of recoloring for number of colors is presented on the randomly generated graphs in Figures 6(a), 6(b), and 6(c). Figure 6(d) shows the aggregated runtime results for RMAT graphs. The number of colors of FSS was mainly given by the number of conflicts it generated. The vertex-visit ordering computed by the asynchronous recoloring does not avoid the majority of these conflicts and the improvement in number of colors compared to FSS is less than 10%.

Synchronous recoloring shows a different picture. While FSS used many colors because of a high number of conflicts on RMAT-Good and RMAT-Bad, the synchronous recoloring does not yield any conflict. Therefore, it obtains a much better number of colors, close to the sequential Largest First and Smallest Last orderings with at most 50% improvement compared to FSS. In terms of runtime, the absence of conflict and the size of the graphs make the synchronous recoloring procedure very scalable, inducing a very low overhead compared to the initial coloring when the number of processors is high.

Given the benefit of the recoloring procedure, it is of the interest to catch the sequential improvement on the number of colors obtained by performing the recoloring procedure multiple times. The effect of multiple recoloring iterations in distributed-memory settings on the real-world graphs is given in Figure 7. Single iteration of recoloring provides significant gain over no recoloring by staying below sequential Largest First when number of processors is 512. On the other hand, running 10 iterations of the recoloring procedure on 512 processors provides very good number of colors, which is close to the sequential Smallest Last vertex-visit ordering.

4.3 Using Random-X Fit with Recoloring

We investigate the effect of Random-5 Fit, Random-10 Fit and Random-50 Fit within our coloring framework with different number of recoloring iterations. We believe that Random-X Fit can provide an initial solution that is fast and that balances the number of vertex per color which is very suitable for recoloring. Various number of processors were tried, but they lead to similar results, so we only present results using 32 processes on 32 nodes.
Firstly, we investigate the original coloring with Random-X Fit color selection strategy. We run experiments with varying and combining all of the parameters we have: superstep size (500, 1000, 5000 and 10000), vertex-visit ordering (Internal First and Smallest Last), synchronous and asynchronous communication patterns for original coloring and color selection strategies (First Fit, Random-5 Fit, Random-10 Fit and Random-50 Fit). Quality and runtime results are presented in Figure 8. Note that, superstep size and communication pattern does not have a significant impact on quality-runtime trade-off, so we clustered the nearer points and tagged them with respect to their color selection strategy and vertex-visit ordering. For example, R5Ix stands for Random-5 Fit with Internal First vertex-visit ordering where superstep size and communication pattern vary. As expected, Internal First ordering gives better runtime results than SL ordering while the situation is the other way around for the number of colors. As the X factor in Random-X Fit strategy increases, the number of colors degrades since the selection is done from a larger set.

Next, we investigate the use of Random-X color selection strategy in the context of recoloring. Results for one recoloring iteration are shown in Figure 9(a). All Random-X color selection strategies give better number of colors than First Fit with one recoloring iteration. It means that although Random-X color selection strategies start recoloring with higher number of colors than First Fit, they result in less colors at the end of the recoloring. In our opinion, this result is very important and shows that random color selection strategies provide perfect ground for recoloring. However, Random-X color selection strategies are not as good as First Fit in terms of runtime. Since the runtime of recoloring is correlated with the number of colors at the beginning of recoloring procedure, the recoloring procedure after Random-X Fit strategy takes longer. Despite this fact, R(5—10)Ixx combinations give better number of colors and better runtime than FSxx combinations, which were suggested as the best combination for recoloring in [26]. We also investigate the impact of Random-X color selection strategies with two recoloring iterations. Figure 9(b) presents the results for this experiment. Results are similar to the ones presented in Figure 9(a).

The trade-off between number of colors and runtime is presented in Figure 10 where results with 0,1 and 2 recoloring iterations are combined. Notice that, in Figure 10, R(5—10)IxxND1 is better than FIxxND2 and FSxxND2 both in terms of number of colors and runtime. In summary, the random color selection strategy compensates the impact of vertex-visit ordering and the best combination to be suggested becomes R(5—10)IxxND1, which is using the Random-5 or Random-10 color selection strategy with Internal First
Ordering and one Non-Decreasing recoloring iteration. This allows us to identify two interesting sets of parameters. This makes R(5—10)IxND1 a good candidate for having a solution of good quality and we call this set of parameters “quality”.

For a obtaining a solution quickly, it is better not to use recoloring. The random strategies tend to significantly hinders the quality of the solution in this case and using First Fit is leading to a higher quality solution at no runtime overhead. The local ordering of the vertices makes little difference in quality because of the distributed execution of algorithm; therefore the fastest ordering (Internal first) is preferred. This makes FIxND0 a good candidate for obtaining quickly a solution of moderate quality and we call this set of parameters “speed”.

Figure 6: Impact of recoloring on RMAT graphs.
Figure 7: Impact of the number of recoloring iterations on real-world graphs in distributed memory [26].

5 Conclusion

In this paper we investigated different ways of improving the number of colors in distributed-memory graph coloring algorithms. We investigated recoloring by utilizing the independent sets of vertices resulted by an existing solution to color large number of vertices independently with little synchronization or with little conflicts. We showed that graphs that result in a larger number of conflicts benefit significantly from synchronous recoloring in terms of the number of colors. The runtime overhead is being small in such cases, therefore multiple iterations of recoloring can be used to obtain even fewer colors. We also showed that synchronous recoloring can be implemented using piggybacking to improve network access patterns; making synchronous recoloring as fast as the previously proposed asynchronous recoloring. Overall, recoloring proved to be a scalable concept for the number of colors when number of processors increases, thanks to the equivalence to a sequential procedure. Last but may be most interestingly, we showed that the Random-X Fit color selection strategy works quite well with recoloring and provides better number of colors and runtime results than the vertex-visit ordering solutions.

Overall, we suggest two combinations of the parameters; if the user is interested in a quick coloring with decent quality, then the “speed” policy FIxxND0 (First Fit, Internal first, no recoloring) is suitable; if the user needs better coloring, then the “quality” R(5—10)IxND1 combination of Random-X Fit, with Internal First ordering and some Non-Decreasing recoloring iterations is best for large number of processors.

As a future work, more thorough study of Random-X Fit strategy could lead to a better understanding of the coloring problem and could suggest ways to derandomize this color selection strategy.

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Figure 8: Comparison of different combinations of parameters on real-world graphs in terms of number of colors and runtime without recoloring.

Figure 9: Comparison of different combinations of parameters on real-world graphs in terms of number of colors and runtime with Non-Decreasing permutation recoloring.

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Figure 10: Comparison of different combination of parameters up to two Non-Decreasing permutation recoloring iterations.

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