Picard groups of Hilbert schemes of Curves

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Abstract: We calculate the Picard group, over the integers, of the Hilbert scheme of smooth, irreducible, non-degenerate curves of degree \(d\) and genus \(g \geq 4\) in \(\mathbb{P}^r\), in the case when \(d \geq 2g + 1\) and \(r \leq d - g\). We express the classes of the generators in terms of some “natural” divisor classes.

Notation and conventions

\(\mathcal{M}_g\) : Moduli space of smooth, irreducible curves of genus \(g \geq 4\), without automorphisms.
\(\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g\) : Universal curve over \(\mathcal{M}_g\).
\(J^d(C)\) : Jacobian variety which parametrizes line bundles of degree \(d\) on the curve \(C\).
\(\psi : \mathcal{J}^d \rightarrow \mathcal{M}_g\) : Universal Jacobian variety; the fiber over \([C] \in \mathcal{M}_g\) is \(J^d(C)\).
\(\mathcal{H}_{d,g,r}\) : Hilbert scheme of smooth, irreducible, non-degenerate curves of degree \(d\) and genus \(g\) in \(\mathbb{P}^r\).
\(q : \mathcal{C}_g \rightarrow \mathcal{H}_{d,g,r}\) : Universal curve over \(\mathcal{H}_{d,g,r}\).
\(\mathcal{F}\) : Tautological line bundle over \(\mathcal{C}_g\).
\(\mathbb{P}(V)\) : denotes the space of one dimensional subspaces of \(V\).

1 Introduction

Given positive integers \(d, g, r\), let \(\mathcal{H}_{d,g,r}\) denote the Hilbert scheme of smooth, irreducible, non-degenerate curves of degree \(d\) and genus \(g\) in \(\mathbb{P}^r\). In general, the geometric structure of \(\mathcal{H}_{d,g,r}\) is difficult to describe and in most of the cases is unknown but for \(d \geq 2g + 1\) and

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$r \leq d - g$, it turns out that $\mathcal{H}_{d,g,r}$ is smooth and irreducible, see $\mathfrak{H}$, $\mathfrak{H}$. The natural forgetful map $\mathcal{H}_{d,g,r} \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of smooth curves of genus $g \geq 4$, is onto. The purpose of this paper is to describe the Picard group of $\mathcal{H}_{d,g,r}$ over the integers when $d \geq 2g + 1$ and $r \leq d - g$. From now on, we are going to exclude from the Hilbert scheme those points which represent curves with automorphisms. By doing so, the description of the Picard group is not effected, since the locus of such points is of big codimension when $g \geq 4$.

Let $\mathcal{M}_g^0$ denote the moduli space of smooth, irreducible curves of genus $g \geq 4$ without automorphisms. Over $\mathcal{M}_g^0$ we have the universal family $\pi : \mathcal{C}_g \to \mathcal{M}_g^0$. To that we can associate the family $\psi : \mathcal{J}^d \to \mathcal{M}_g^0$, the universal Picard variety of degree $d$, whose fiber over $[C] \in \mathcal{M}_g^0$ is $J^d(C)$. Over $\mathcal{J}^d$ one can construct a projective fibration $\phi : \mathcal{P}_d \to \mathcal{J}^d$ whose fiber over a point $[L] \in J^d(C)$ is $\mathbb{P}(C^{r+1} \otimes H^0(C,L))$. Note that for $r = 0$, this is just the universal symmetric product $\mathcal{E}_g^{(d)} \to \mathcal{J}^d$ of degree $d$. The existence of such a bundle is based on the existence of the bundle $\mathcal{E}_g^{(d)} \to \mathcal{J}^d$ and the existence of a local section (in the analytic topology) of the map $\pi : \mathcal{C}_g \to \mathcal{M}_g^0$.

The variety $\mathcal{H}_{d,g,r}$ can be included in $\mathcal{P}_d$ as follows. An element $h$ in $\mathcal{H}_{d,g,r}$ corresponds to a smooth, irreducible, non-degenerate curve $C$ of degree $d$ and genus $g$ in $\mathbb{P}^r$. Let $H_i$, $i = 1, \ldots, r + 1$, denote the hyperplane section $X_i = 0$. Then, define $L \overset{\text{def}}{=} O(1)|_C \in J^d(C)$ and $s_i \overset{\text{def}}{=} H_i|_C \in H^0(C,L)$. We then correspond to $h$ the point $[C, L, < s_1, \ldots, s_{r+1} >] \in \mathcal{P}_d$ and the map is obviously one to one. By doing so, one can factor the canonical map $\mathcal{H}_{d,g,r} \to \mathcal{M}_g^0$ as

$$\mathcal{H}_{d,g,r} \overset{i}{\hookrightarrow} \mathcal{P}_d \xrightarrow{\phi} \mathcal{J}^d \xrightarrow{\psi} \mathcal{M}_g^0.$$  

The complement of $\mathcal{H}_{d,g,r}$ in $\mathcal{P}_d$ corresponds to those tuples $[C, L, < s_1, \ldots, s_{r+1} >]$ for which the space of sections $< s_1, \ldots, s_{r+1} >$ has either a base point or does not separate points and tangent directions on the curve or the dimension of the span $< s_1, \ldots, s_{r+1} >$ is $\leq r$. In the first case the map of $C$ to $\mathbb{P}^r$ defined by the above data is of degree $< d$, in the second the map is not an embedding and in the third the map is degenerate.

Over $\mathcal{H}_{d,g,r}$ we have the universal curve $q : \mathcal{C}_g \to \mathcal{H}_{d,g,r}$. By construction $\mathcal{C}_g \subset \mathcal{H}_{d,g,r} \times \mathbb{P}^r$. On $\mathcal{C}_g$, we have the tautological bundle $\mathcal{F}$ which is the pull back by the projection map $\eta : \mathcal{C}_g \to \mathbb{P}^r$ of $O(1)$ on $\mathbb{P}^r$. If $\pi : \mathcal{C}_g \to \mathcal{M}_g^0$ is the universal curve over $\mathcal{M}_g^0$ and $\psi$ the map as above, then we denote by $\mathcal{E}_{g,d}$ the fiber product $\mathcal{E}_{g,d} \overset{\text{def}}{=} \mathcal{C}_g \times_{\mathcal{M}_g^0} \mathcal{J}^d$. Let $\nu$ denote the projection map $\mathcal{E}_{g,d} \to \mathcal{J}^d$. The basic diagram we are going to use is the following:
Let $\omega_q$ denote the relative dualizing sheaf of the map $q : \mathcal{C}_H \to \mathcal{H}_{d,g,r}$. Then we have on $\mathcal{H}_{d,g,r}$ the following three natural divisor classes:

\begin{align*}
A &= q_*(\mathcal{F}^2), \\
B &= q_*(\mathcal{F}\omega_q), \\
C &= q_*(\omega_q^2).
\end{align*}

In this paper we describe the Picard group of $\mathcal{H}_{d,g,r}$ and we express the classes of its generators in terms of the classes $A, B, C$ given above. This is the content of Theorem 4.1.

\section{Intersection calculations.}

We are going to use the following results

1. The Picard group of $\mathcal{M}_0^g$ is freely generated over the integers by the determinant $\lambda$ of the Hodge bundle. In other words, $\lambda = \det \pi_* \omega_\pi$, where $\pi : \mathcal{C}_g \to \mathcal{M}_0^g$ is the universal curve, see [2].

2. The Picard group of the universal Jacobian $\psi : J^d \to \mathcal{M}_0^g$ is freely generated over the integers by the pull back $\psi^* \lambda$ and a line bundle $\mathcal{L}_d$. The later is uniquely defined up to the pull back of line bundles from $\mathcal{M}_0^g$ and has the following characteristic property: its restriction to a fiber $J^d(C)$ has class $k_d \theta$, where $k_d = \frac{2g-2}{\text{gcd}(2g-2,d-g+1)}$ and $\theta$ denotes the class of the theta divisor, see [7].

3. On $J^d \times \mathcal{M}_0^g \mathcal{C}_g$ there is a line bundle $\mathcal{P}_d$ with the property $\mathcal{P}_d|_{L \times C} \cong L^\otimes s_d$, where $s_d = \text{gcd}(2g-2,d+g-1)$ is minimum with this property, see Application in Section 5 of [8].

There are various ways to construct the bundles $\mathcal{L}_d$ and $\mathcal{P}_d$ in 2. and 3. above. In the following we give a construction which is the most convenient for our purposes.
We start with some notation. On the \( d \)-th symmetric product of a curve we denote by \( \Delta \) the diagonal or its class in the Chow ring. We denote by \( x \) the class of the image of a coordinate plane from the \( d \)-th ordinary product under the canonical map. Let \( u : C^{(d)} \to J^{d}(C) \) be the Abel-Jacobi map and let \( \omega_u \) the relative dualizing sheaf. Given a line bundle \( M \) on \( C \), then we associate to that a line bundle \( L_M \) on the symmetric product \( C^{(d)} \) as follows: Take a meromorphic section of \( M \) written in the form \( D_1 - D_2 \), where \( D_1, D_2 \) are effective divisors which are sums of distinct points. Define on \( C^{(d)} \) the divisors \( X_{D_i} \), \( i = 1, 2 \) with support \( \{ D \in C_d, \text{ s.t. } D \cap D_i \neq \emptyset \} \). Then we define \( L_M \) to be the line bundle \( \mathcal{O}(X_{D_1}) \otimes \mathcal{O}(X_{D_2})^{-1} \). It is easy to see that this is independent from the choice of \( D_1 \) and \( D_2 \).

For \( M = K \), we denote by \( L_K \) the line bundle which corresponds to the canonical divisor \( K \).

**Remark 2.1** For \( d \geq 2g + 1 \), the Abel-Jacobi map \( u : C^{(d)} \to J^{d}(C) \) is a fibration of projective spaces of dimension \( r = d - g \). In that case, if \( A \in J^{d}(C) \), then \( L_M|_{u^{-1}(A)} \cong \mathcal{O}_{\mathbb{P}^r}(\deg M) \).

We describe now the analogue of \( L_K \) for families of curves. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{E}^d_g & \xrightarrow{c} & \mathcal{E}^{(d)}_g \\
\pi_i \downarrow & & \downarrow \chi \\
\mathcal{E}_g & \xrightarrow{\pi} & \mathcal{M}^0_g
\end{array}
\]

where the maps \( c, \chi, \chi', \pi \) are the canonical ones and \( \pi_i \) is the \( i \)-th projection.

For \( l \) big enough, let \( \sigma(l) \) be a generic section of \( H^0(\mathcal{E}_g, \omega_{\pi} \otimes \pi^* \lambda^{\otimes l}) \cong H^0(\mathcal{M}^0_g, \pi_* \omega_{\pi} \otimes \lambda^{\otimes l}) \). Since \( \pi_* \omega_{\pi} \) is a locally free sheaf of rank \( g \), we may assume that the section \( \sigma(l) \) does not vanish identically on the fibers of \( \pi \) over a Zariski open \( U \) of \( \mathcal{M}^0_g \) with complement of codimension \( g \). Then, as above, we can construct over \( U \) a divisor whose restriction to the fiber \( C^{(d)} \) is the divisor \( X_{\sigma(l)|_{C}} \) (following the above notation). The corresponding line bundle extends uniquely to a line bundle on \( \mathcal{E}^{(d)}_g \). We denote that by \( \mathcal{L}_{\sigma(l)} \). On \( \mathcal{E}^d_g \) we define \( \mathcal{L}_K = \otimes_{i=1}^{d} \pi^* \omega_{\pi} \).

Then we have:

**Lemma 2.1** Following the above notation, we have

\[
c^* \mathcal{L}_{\sigma(l)} \cong \mathcal{L}_K \otimes \chi^*(\lambda^{\otimes dl}).
\]
**Proof:** By construction we have that $c^*\mathcal{L}(l) \cong \otimes_{i=1}^d \pi_i^*(\mathcal{O}(\sigma(l))) \cong \otimes_{i=1}^d \pi_i^*(\omega \otimes \pi^*\lambda) \cong \mathcal{L}_K \otimes \chi^*(\lambda^{\otimes d})$.

\[ \square \]

**Definition 2.1** We define on $\mathcal{E}_y^{(d)}$ the line bundle $\mathcal{L}_\omega \overset{\text{def}}{=} \mathcal{L}_\sigma(l) \otimes \chi' \ast (\lambda - dl)$.

**Remark 2.2** The characteristic property of $\mathcal{L}_\omega$ is that $c^*\mathcal{L}_\omega \cong \mathcal{L}_K$. This is an application of the above Lemma 2.1. Note also that the line bundle $\mathcal{L}_\omega$ does not depend on the choice of the number $l$ and that the restriction of $\mathcal{L}_\omega$ to a fiber $C^{(d)}$ is isomorphic to $\mathcal{L}_K$.

Before we continue we need two more things: The first is that the universal symmetric product carries a universal bundle $\mathcal{D}$. Indeed, if $\delta : \mathcal{E}^{(d-1)} \times \mathcal{E}_y \longrightarrow \mathcal{E}_y^{(d)} \times \mathcal{E}_y$ is the map sending the pair $(D, p)$ to $(D + p, p)$, then the line bundle $\mathcal{D}$ corresponds to the divisor which is the image of the above map. The second is the MacDonnald’s formula which expresses the class of the pull back by the Abel-Jacobi map of the theta divisor on the Jacobian in terms of the classes $x$ and $\Delta$. That is, $u^*\theta = (d + g - 1)x - \frac{\Delta}{2}$.

**Construction of (normalized) $\mathcal{L}_d$:** Consider the Abel-Jacobi map $u : \mathcal{E}_y^{(d)} \longrightarrow \mathfrak{J}^d$. The bundle $\frac{d+g-1}{s_d} \mathcal{L}_\omega - k_d\frac{\Delta}{2}$ is trivial on the fibers of $u$: Indeed, by Remark 2.1, we have $\mathcal{L}_\omega|_{u^{-1}(L)} \cong \mathcal{O}(2g - 2)$ and, by MacDonnald’s formula, we have $\frac{\Delta}{2}|_{u^{-1}(L)} \cong \mathcal{O}(d + g - 1)$. We thus get by the see saw principle, see [10], that the above bundle descents to a bundle on $\mathfrak{J}^d$ and this is exactly the bundle $\mathcal{L}_d$. That $\mathcal{L}_d$ has class $k_d\theta$, it is again an application of the MacDonnald’s formula.

**Construction of $\mathcal{P}_d$:** Consider the diagram

\[
\begin{array}{ccc}
\mathcal{E}_y^{(d)} \times_{\mathfrak{M}_9^g} \mathcal{E}_y & \xrightarrow{\tilde{u}} & \mathfrak{J}^d \times_{\mathfrak{M}_9^g} \mathcal{E}_y \\
\downarrow p_1 & & \downarrow \nu \\
\mathcal{E}_y^{(d)} & \xrightarrow{u} & \mathfrak{J}^d
\end{array}
\]

On $\mathcal{E}_y^{(d)} \times_{\mathfrak{M}_9^g} \mathcal{E}_y$ consider the line bundle $p_1^!(m\mathcal{L}_\omega - n\frac{\Delta}{2}) + s_d\mathcal{D}$, where $n, m$ are integers satisfying $n(d + g - 1) - m(2g - 2) = s_d$. One can see again that this is trivial on the fibers of $\tilde{u}$ (note that $\mathcal{D}|_{u^{-1}(L) \times \{p\}} \cong \mathcal{O}(1)$) and that its restriction on $\{D\} \times C$ is isomorphic to $D^{\otimes s_d}$. The bundle $\mathcal{P}_d$ is the descent of that one on $\mathfrak{J}^d \times_{\mathfrak{M}_9^g} \mathcal{E}_y$. Note that different choices of...
\( n \), give rise to line bundles \( \mathcal{P}_d \) which differ by the pull back of a line bundle from \( \mathcal{J}^d \). In the following we can fix one such \( n \), for example choose \( n \) to be the smallest positive integer such that the number \( \frac{n(d+g-1)-s_d}{2(g-1)} \) is an integer, or equivalently, such that the number \( \frac{nd-s_d}{g-1} + n \) is an even integer.

We continue with some lemmas about intersections. In the following we denote a line bundle and its first Chern class in the Chow group by the same symbol.

**Lemma 2.2** If \( p_1 : \mathcal{C}^{(d)}_g \times \mathcal{M}_g \to \mathcal{C}^{(d)}_g \) denotes the first projection, then

\[
p_1^* D^2 = -L_\omega + \Delta.
\]

**Proof:** Consider the diagram

\[
\begin{array}{ccc}
\mathcal{C}^{d}_g & \xrightarrow{\delta_{i,d+1}} & \mathcal{C}^{d}_g \\
\downarrow{\text{id}} & & \downarrow{q_1} \\
\mathcal{C}_g & \xrightarrow{\pi_i} & \mathcal{C}^{d}_g
\end{array}
\]

The map \( \delta_{i,d+1} \) is the \( i, d+1 \)-diagonal embedding, and \( c \) the canonical map. The maps \( q_1 \) and \( \pi_i \) are the projections. The map \( c \) is flat and its pull back defines an injection in the intersection rings. By the commutativity and Remark 2.2, it is enough to show that \( q_1^* \tilde{c}^*(D^2) = -L_K + 2\Delta \), where we denote also by \( \Delta \) the sum of the big diagonals in \( \mathcal{C}^{d}_g \). We have

\[
\tilde{c}^*(D^2) = \tilde{c}^*(D)^2 = \left( \sum_{i=1}^{d} \Delta_{i,d+1} \right)^2 = \sum_{1 \leq i < j \leq d} \Delta_{i,d+1} \Delta_{j,d+1}.
\]

There are two cases to consider

- if \( i \neq j \) then \( q_1^*(\Delta_{i,d+1}\Delta_{j,d+1}) = \Delta_{i,j} \)
- if \( i = j \) then \( q_1^*\Delta_{i,d+1}^2 = \delta_{i,d+1}^*\Delta_{i,d+1} = \pi_i^*\omega_{\pi_i}^{-1} \).

Therefore

\[
q_1^*\tilde{c}^*(D^2) = 2 \sum_{1 \leq i < j \leq d} \Delta_{i,j} + \sum_{i=1}^{d} \omega_{\pi_i}^{-1} = 2\Delta - L_K.
\]

\( \square \)
Lemma 2.3 Following the notation of diagram 3, we have the following (where again we keep the same notation for a line bundle and its first Chern class):

1. \( \nu^* (\mathcal{P}_2^2) = s_d^2 \frac{nd - s_d}{g - 1} \mathcal{L}_d, \)
2. \( \nu^* (\mathcal{P}_d \omega) = s_d n \mathcal{L}_d, \)
3. \( \nu^*(\omega_2^2) = 12 \psi^* \lambda, \)

where \( n \) is the integer defined in the construction of \( \mathcal{P}_d \), see Section 2.

Proof: Again, since \( u \) is flat and its pull back defines an injection in the chow rings, it is enough to prove that \( p_{1*} \tilde{u}^* (\mathcal{P}_2^2) = s_d^2 \frac{nd - s_d}{g - 1} u^* \mathcal{L}_d. \) By the construction of \( \mathcal{P}_d \), we have

\[
\tilde{u}^* (\mathcal{P}_2^2) = \left[ s_d \mathcal{D} + p_1^* (m \mathcal{L}_\omega - n \frac{\Delta}{2}) \right]^2 = s_d^2 \mathcal{D}^2 + 2s_d \mathcal{D} p_1^* (m \mathcal{L}_\omega - n \frac{\Delta}{2}) + p_1^* (m \mathcal{L}_\omega - n \frac{\Delta}{2})^2.
\]

Thus,

\[
p_{1*} \tilde{u}^* \mathcal{P}_2^2 = s_d^2 p_{1*} \mathcal{D}^2 + 2s_d p_{1*} (m \mathcal{L}_\omega - n \frac{\Delta}{2}) = s_d^2 (-\mathcal{L}_\omega + \Delta) + 2ds_d (m \mathcal{L}_\omega - n \frac{\Delta}{2})
\]

The first coefficient can be written as

\[
2dms_d - s_d^2 = \frac{ds_d}{g - 1} (n(d + g - 1) - s_d) - s_d^2 = \frac{ds_d}{g - 1} n(d + g - 1) - \frac{ds_d}{g - 1} - s_d^2
\]

\[
= \frac{ds_d}{g - 1} n(d + g - 1) - \frac{s_d^2}{g - 1} (d + g - 1) = \frac{s_d}{g - 1} (d + g - 1)(dn - s_d).
\]

The second coefficient can be written as

\[
2s_d^2 - 2dns_d = -2s_d (dn - s_d).
\]

We thus have

\[
p_{1*} \tilde{u}^* \mathcal{P}_d^2 = \frac{s_d}{g - 1} (d + g - 1)(dn - s_d) \mathcal{L}_\omega - 2s_d (dn - s_d) \frac{\Delta}{2}
\]

\[
= \frac{s_d^2}{g - 1} (dn - s_d) \frac{d + g - 1}{s_d} \mathcal{L}_\omega - s_d^2 \frac{dn - s_d}{g - 1} k_d \frac{\Delta}{2}
\]

\[
= s_d^2 \frac{dn - s_d}{g - 1} \left( \frac{d + g - 1}{s_d} \mathcal{L}_\omega - k_d \frac{\Delta}{2} \right) = s_d^2 \frac{dn - s_d}{g - 1} u^* \mathcal{L}_d
\]

This proves the first part of the lemma.

For the second part, consider the diagram
The map $\sigma$ in the diagram is the addition map. We have

$$\tilde{u}^*(\mathcal{P}_d\omega_\nu) = (p_1^*(mL_\omega - n\frac{\Delta}{2}) + s_dD)\omega_{p_1} = p_1^*(mL_\omega - n\frac{\Delta}{2})p_2^*\omega_\pi + s_dDp_2^*\omega_\pi.$$ 

By applying $p_1*$ and using the definition of $D$, we get

$$p_1*\tilde{u}^*(\mathcal{P}_d\omega_\nu) = (2g - 2)(mL_\omega - n\frac{\Delta}{2}) + s_d\sigma^*\delta^*p_2^*\omega_\pi.$$ 

Since $p_2 \circ \delta = \pi_2$, we have that $\sigma^*\delta^*p_2^*\omega_\pi = \sigma_2^*\pi_2^*\omega_\pi$. By the definition of $L_\omega$ and by some diagram chasing it is easy to see that $\sigma^*\pi_2^*\omega_\pi = L_\omega$. We thus get

$$p_1*\tilde{u}^*(\mathcal{P}_d\omega_\nu) = (n(d + g - 1)L_\omega - n(2g - 2)\frac{\Delta}{2}) = ns_du^*L_d.$$ 

This proves the second part.

The third part is an immediate consequence of the fact $\pi^*_*(\omega_\pi^2) = 12\lambda$, see [1].

\hfill \Box

3 The Hilbert scheme

Let $C$ be a smooth curve of genus $g \geq 4$ and $d > 2g - 2$, $r \leq d - g$ given integers. We give now estimates for the dimension of the complement of $\mathcal{S}_{d,g,r}$ in $\mathcal{P}_d$. Let $L$ denote a line bundle of degree $d$ on $C$. The complement of $\mathcal{S}_{d,g,r}$ in the fiber $\mathcal{P}_d = \mathbb{P}(C^r + 1 \otimes H^0(C, L))$ of $\mathcal{S}_d \to J^d$ over $L$, consists of two (maybe overlapping) loci $\mathcal{U}_{\text{deg}}^L$ and $\mathcal{U}_{\text{emb}}^L$. The first contains those points which define degenerate maps and the second those for which the corresponding map is not an embedding of degree $d$. We have the following lemmas:

**Lemma 3.1** Following the above notation, we have that $\text{codim}_{\mathcal{P}_d} \mathcal{U}_{\text{deg}}^L = d - g - r + 1$. In particular, if $3 \leq r < d - g$, then $\text{codim}_{\mathcal{P}_d} \mathcal{U}_{\text{deg}}^L \geq 2$ and if $r = d - g$, then $\text{codim}_{\mathcal{P}_d} \mathcal{U}_{\text{deg}}^L = 1$. In the later case, $\mathcal{U}_{\text{deg}}^L$ is an irreducible divisor of degree $d - g + 1$ in the projective space $\mathcal{P}_d^L$. 

8
Proof: The above locus $\mathcal{U}_{\text{deg}}$ corresponds to those tuples $<s_1, \ldots, s_{r+1}> \in \mathbb{C}^{r+1} \otimes H^0(C, L)$ which span a space of dimension $\leq r$. Since $\dim H^0(C, L) = d - g + 1$, this proves the Lemma.

Remark 3.1 For $r = d - g$, the assumption that $d \geq 2g + 1$, implies that $\mathcal{U}_{\text{nemb}} \subset \mathcal{U}_{\text{deg}}$ with codimension $\geq 1$.

Lemma 3.2 For $4 \leq r < d - g$, we have that $\text{codim}_{\mathcal{P}^d} \mathcal{U}_{\text{nemb}} \geq 2$. For $3 = r < d - g$, the locus $\mathcal{U}_{\text{nemb}}$ is an irreducible divisor of degree $2(d-1)(d-2) - 4g$ in the projective space $\mathcal{P}^d$.

Proof: The space $\mathcal{P}^d \setminus \mathcal{U}_{\text{deg}}$ maps in a natural way to the Grassmanian $\text{Gr}(r+1, H^0(C, L))$ which parametrizes linear series $g_d^r$ of $L$ on $C$. Thus, for $r < d - g$, there is a rational map $\alpha : \mathcal{P}^d \rightarrow \text{Gr}(r+1, H^0(C, L))$ which is not defined in a codimension $\geq 2$ locus, see Lemma 3.1. The fiber of $\alpha$ is isomorphic to $\text{PGL}(r+1)$. The locus $\mathcal{U}_{\text{nemb}} \setminus \mathcal{U}_{\text{deg}}$ is the pull back of the correspondent locus in $\text{Gr}(r+1, H^0(C, L))$.

It is enough to prove the Lemma on the “level” of $\text{Gr}(r+1, H^0(C, L))$. Consider the curve $C$ embedded in $\mathbb{P}(H^0(C, L)^\vee)$ by the complete linear system of $L$. Maps defined by the $g_d^r$'s, correspond to projections from $(d-g-r-1)$-dimensional projective planes in the above space. Maps which are not embeddings of degree $d$, correspond to projections from those planes which intersect the secant variety of $C$. The later is a locus in the dual of the Grassmanian $\text{Gr}(r+1, H^0(C, L))$ of codimension equal to $r - 2$ (the dimension of the secant variety is 3). Thus, if $r \geq 4$, then the codimension of $\mathcal{U}_{\text{nemb}}$ is $\geq 2$.

We turn now to the case $r = 3$. Observe first that the pull back by $\alpha$ of the generator $\mathcal{O}_{\text{Gr}}(1)$ of the Picard group of the Grassmanian to $\mathcal{P}^d$ is isomorphic to $\mathcal{O}_{\mathcal{P}^d}(r + 1)$. By the above discussion, one can see that $\mathcal{U}_{\text{nemb}}$ is an irreducible divisor; the formula for its degree is a consequence of the previous observation and of the fact that the degree of the secant variety is $(d-1)(d-2) - g$.

We have the following lemmas. The proof of the first is an immediate consequence of Lemmas 3.1 and 3.2 above.
**Lemma 3.3** For $d \geq 2g + 1$ and $3 < r < d - g$, we have an isomorphism of Picard groups $\text{Pic} \mathcal{H}_{d,g,r} \cong \text{Pic} \mathcal{P}_d$.

**Lemma 3.4** The bundle $\mathcal{P}_d \to \mathcal{J}$ admits a line bundle whose restriction to a fiber is isomorphic to $\mathcal{O}(s_d)$. The number $s_d$ is minimum with this property.

**Proof:** Let's consider first the same question for the bundle $u : \mathcal{C}^{(d)}_g \to \mathcal{J}$. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}^{(d)}_g$ whose restriction on the fiber $\mathcal{P}_H(C, L)$ is isomorphic to $\mathcal{O}(t)$. By \cite{7}, pg. 844, the class of its restriction is of the form $(2g - 2)m + n\frac{1}{d}$, where $m, n$ are integers. Since the restrictions of $x$ and $\frac{1}{2}$ on a fiber of the Abel-Jacobi map have classes $c_1 \mathcal{O}(1)$ and $c_1 \mathcal{O}(d + g - 1)$ respectively, we conclude that $s_d | t$. One can see that the same is true for the bundle $\mathcal{P}_d$. The proof is similar to that of Lemma 4, in \cite{8}.

On the other hand, one can construct a line bundle $\mathcal{R}$ on $\mathcal{H}_{d,g,r}$ whose restriction on the fibers is isomorphic to $\mathcal{O}(s_d)$ as follows. We turn back to diagram \cite{9}. The pull back by the map $\eta$ of $\mathcal{O}(1)$ to $\mathcal{C}_g$, is the tautological bundle $\mathcal{F}$. By the see-saw principle we have that there exists a bundle $\mathcal{R}$ on $\mathcal{H}_{d,g,r}$ such that

$$q^* \mathcal{R} \cong \mathcal{F}^{s_d} \otimes \alpha^* \mathcal{P}_d^{-1}.$$

It is easy to see that the restriction of $\mathcal{R}$ to the fibers of the map $\phi$ is $\mathcal{O}(s_d)$.

The above constructed line bundle $\mathcal{R}$ is the generator of the relative Picard group of $\mathcal{H}_{d,g,r}$ over $\mathcal{J}^d$. To summarize, we have that the Picard group of $\mathcal{H}_{d,g,r}$, when $d \geq 2g + 1$ and $3 < r < d - g$, is freely generated over the integers by the line bundles $\psi^* \phi^* \lambda$, $\phi^* \mathcal{L}_d$ and $\mathcal{R}$.

### 4 The classes of the generators

In this section we express the classes of the above generators of $\text{Pic} \mathcal{H}_{d,g,r}$, in terms of the naturally defined classes $A$, $B$, $C$ of Section 1. For the notation, see diagram 1.

**Lemma 4.1** If $A = q_s(\mathcal{F}^2)$, $B = q_s(\mathcal{F} \omega_q)$ and $C = q_s(\omega_q^2)$, we have

1. $A = \frac{nd - s_d}{g - 1} \phi^* \mathcal{L}_d + 2 \frac{d}{s_d} \mathcal{R}$,
2. $B = n \phi^* \mathcal{L}_d + \frac{2g - 2}{s_d} \mathcal{R}$,
3. $C = 12 \phi^* \psi^* \lambda$,

where $n$ is the integer defined in the construction of $P_d$ in Section 2.

**Remark 4.1** Note that the coefficient matrix has determinant of absolute value $24$.

**Proof:** For the first one:

\[
s_d^2 F^2 = (q^* \mathcal{R} + \alpha^* P_d)^2
= q^* R^2 + 2 q^* \mathcal{R} \alpha^* P_d + \alpha^* P_d^2.
\]

Therefore by applying $q_*$, we get

\[
s_d^2 q_* F^2 = 2 q_* \alpha^* P_d R + q_* \alpha^* P_d^2
= 2 ds_d R + \phi^* \nu^* P_d^2
\overset{\text{Lem 2.3}}{=} 2 ds_d R + s_d^2 \frac{nd - s_d d}{g - 1} \phi^* L_d.
\]

and this proves the first part.

For the second part: By the definition of $\mathcal{R}$ and by multiplying by $\omega_q$ we have

\[F \omega_q = \frac{1}{s_d} q^* R \omega_q + \frac{1}{s_d} \alpha^* P_d \omega_q,\]

and so,

\[q_*(F \omega_q) = \frac{1}{s_d} \mathcal{R} q_* \omega_q + \frac{1}{s_d} q_* \alpha^* (P_d \omega_\nu)
= \frac{2g - 2}{s_d} \mathcal{R} + \frac{1}{s_d} \phi^* \nu_*(P_d \omega_\nu)
\overset{\text{Lem 2.3}}{=} \frac{2g - 2}{s_d} \mathcal{R} + n \phi^* L_d.
\]

The third part is an immediate consequence of Lemma 2.3.

\[\blacksquare\]

The above Lemma 4.1 gives the expression of the generators in terms of the classes $A, B, C$. We thus have

**Theorem 4.1** For $d \geq 2g + 1$ and $3 < r < d - g$, the Picard group of $\mathcal{S}_{d,g,r}$ is freely generated over $\mathbb{Z}$ by the three line bundles $\psi^* \phi^* \lambda$, $\phi^* L_d$ and $\mathcal{R}$. Their classes can be expressed as...
1. \( R = \frac{1}{2} n A - \frac{1}{2} \frac{nd-s_d}{g-1} B, \)

2. \( \phi^* L_d = -\frac{g-1}{s_d} A + \frac{d}{s_d} B, \)

3. \( \phi^* \psi^* \lambda = \frac{1}{12} C, \)

where \( n \) is the integer defined in the construction of \( P_d \), see Section 2.

**Remark 4.2** Note that in the above theorem the numbers \( \frac{g-1}{s_d} \) and \( \frac{d}{s_d} \) are either integers or half integers. The number \( \frac{nd-s_d}{g-1} \) is an integer by its definition.

**Remark 4.3** The cases \( r = 3 < d - g \) and \( r = d - g \) have to be treated separately: For \( r = 3 < d - g \) the Picard group of \( S_{d,g,3} \) fits in an exact sequence

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}^{\oplus 3} \xrightarrow{p} \text{Pic} S_{d,3,g} \longrightarrow 0,
\]

where the map \( \mu \) is the multiplication by \( 2(d-1)(d-2) - 4g \) in the first factor. The case is similar for \( r = d - g \). Here \( \mu \) is the multiplication by \( d - g + 1 \). We leave to the reader to find in this case the analogue of Theorem 4.1.

**Remark 4.4** One can pursue the above discussion further and calculate the Picard groups over the integers of the Severi varieties and the Hurwitz schemes in the case when the degree \( d \) is big with respect to the genus \( g \). For the Severi varieties, the results of Diaz-Harris, see [4], [5], imply that the union of the Severi variety \( V_{d,g} \) with the three irreducible, independent divisors \( CU, TN, TR \) of \( \Psi_d \) which are defined in the above mentioned papers, has complement of codimension \( \geq 2 \) in \( \Psi_d \). Since \( \text{Pic}\Psi_d \) is isomorphic to \( \mathbb{Z}^{\oplus 3} \), this implies that \( \text{Pic} V_{d,g} \) is torsion. Furthermore, by using the expressions of \( CU, TN, TR \) in terms of \( A, B, C \) given in [1] and the above Lemma 4.1, one can calculate that torsion group. A similar result should be obtained for the Hurwitz scheme by using the analogues results of Mockizuki, see [3]. In the case of the Severi varieties, the calculations of the author lead to rather messy formulas.

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