HAMILTONIAN SYSTEMS ON ALMOST COSYMPLECTIC MANIFOLDS

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ABSTRACT. We determine the Hamiltonian vector field on an odd dimensional manifold endowed with almost cosymplectic structure. This is a generalization of the corresponding Hamiltonian vector field on manifolds with almost transitive contact structures, which extends the contact Hamiltonian systems. Applications are presented to the equations of motion on a particular five-dimensional manifold, the extended Siegel-Jacobi upper-half plane $\tilde{X}_1^J$. The $\tilde{X}_1^J$ manifold is endowed with a generalized transitive almost cosymplectic structure, an almost cosymplectic structure, more general than transitive almost contact structure and cosymplectic structure. The equations of motion on $\tilde{X}_1^J$ extend the Riccati equations of motion on the four-dimensional Siegel-Jacobi manifold $X_1^J$ attached to a linear Hamiltonian in the generators of the real Jacobi group $G_1^J(\mathbb{R})$.

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1. INTRODUCTION

The semi-direct product $G^n_1(\mathbb{R})$ of $\text{Sp}(n, \mathbb{R})$ with the $(2n + 1)$-dimensional Heisenberg group $H_n(\mathbb{R})$ is called real Jacobi group of degree $n$, while the isomorph group $H_n \ltimes \text{Sp}(n, \mathbb{R})_C$ is denoted $G_1^n$. Both Jacobi groups $G_1^n(\mathbb{R})$ and $G_1^n$ are intensively studied in

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Mathematics, Mathematical Physics and Theoretical Physics \[7, 11, 12, 17, 18, 24, 35, 54-57\].

The \(G^J_n(R)\)-homogeneous space \(X^J_n := G^J_n(R) \times \mathbb{R}\) is called Siegel-Jacobi upper half space and \(X^J_n \approx X_n \times \mathbb{R}^{2n}\), where \(X_n\) denotes the Siegel upper half space \[11, 12, 54, 56\]. Similarly, the Siegel-Jacobi ball is defined as \(D^J_n := G^J_n(U(n)) \times \mathbb{R} \approx D_n \times \mathbb{C}^n\) [8], where \(D_n\) denotes the Siegel (open) ball of degree \(n\) [40].

Systems of coherent states (CS) based on the Siegel-Jacobi ball have applications in quantum mechanics, geometric quantization, dequantization, quantum optics, squeezed states, quantum teleportation, quantum tomography, nuclear structure, signal processing, Vlasov kinetic equation, see references in \[3, 10, 12\].

In order to construct invariant metric on homogeneous spaces attached to the real Jacobi group of degree one, in \[12\] we gave up CS technique inspired by Berezin [19]-[21] and we applied moving frame method \[36\], initiated by Cartan [31, 32]. In \[12\] we got invariant metrics on \(X^J_1\) and \(\tilde{X}^J_1\), while in \[13\] the results were generalized to extended Siegel-Jacobi upper half space \(\tilde{X}^J_n \approx X^J_n \times \mathbb{R}\).

It raises the question: how can we distinguish between the different invariant metrics obtained in \[12\] on \(X^J_1\) and \(\tilde{X}^J_1\)? In the present paper we give an answer to this problem, underlining the differences between equations of motion on \(X^J_1\) and \(\tilde{X}^J_1\).

We developed a technique to calculate equations of classical and quantum motion on a homogenous manifold \(M = G/H\) attached to a linear Hamiltonian in the generators of the group \(G\), simultaneously with the determination of the Berry phase \[6-9\]. The method works on the so called CS manifolds, i.e. Kähler manifolds for which the generators of group \(G\) admits a realisation as first order holomorphic differential operators with polynomial coefficients \[6, 15, 16\].

In this paper we are interested in a similar problem: find the equations of motion on odd dimensional manifolds \(M_{2n+1}\) with almost cosymplectic structure in the meaning of \[45\]. For an almost cosymplectic manifold \((M_{2n+1}, \theta, \Omega)\), where \(\theta\) (\(\Omega\)) is a one (respectively, two)-form, we determine the equations of motion attached to a Hamiltonian function \(H\).

We determined the invariant metric to the action of the Jacobi group \(G^J_1(R)\) on \(\tilde{X}^J_1\) \[12\]. We pointed out that the extended Siegel–Jacobi manifold does not admit a Sasaki structure in the parametrization used in \[12\]. In the present paper we introduce on \(\tilde{X}^J_1\) an almost cosymplectic structure in the sense of Libermann \[45\], and moreover, we consider the particular case when \(d\Omega = 0\). Such a manifold is called generalized transitive almost cosymplectic space (GTACOS). The place of the GTACOS structure in the set of geometric structures with which a manifold of odd dimension can be endowed is underlined in the Table in the Appendix: GTACOS is an almost cosymplectic structure more general than the transive almost contact structure and contact structure.

We also endow \(\tilde{X}^J_1\) with a contact structure in the sens of \[33\].

The paper is laid out as follows. In Section \[2\] are collected several previous results extracted from \[3, 12, 14\] used in this paper. Proposition \[1\] presents the invariant metrics on five \(G^J_1(R)\)-homogeneous manifolds. Proposition \[2\] expresses the Kähler two-form
and the invariant metric on $D_j^1$ and $X_j^1$ in several sets of variables. Proposition \ref{prop:invmetric1} recalls the invariant metric on $X_j^1$ in the S-variables $(x, y, p, q, \kappa)$ \cite{24, 35}. In Section \ref{section:Hamiltonian} are calculated the Hamiltonian vector field $X_H$ and $\text{grad} H$ associated with the Hamiltonian function $H$ and the corresponding equations of motion on almost cosymplectic manifolds in the variables $(q^i, p_i, \kappa)$, $i = 1, \ldots, n$, which appear in the one-form $\theta$, while $\Omega$ is expressed in Darboux coordinates (Theorem \ref{thm:1}). The new results on almost cosymplectic manifolds are compared with the corresponding results for transitive almost contact manifolds \cite{1} (Remark \ref{rem:1}). Section \ref{section:equations} particularizes the results of Section \ref{section:Hamiltonian} to the extended Siegel-Jacobi upper half-plane $\tilde{X}_j^1$ organized as GTACOS manifold. Proposition \ref{prop:equations1} displays the equations of motion on $\tilde{X}_j^1$ in the variables $(x, y, p, q, \kappa)$ organized as a GTACOS manifold for a Hamiltonian function $H$ comparatively to the equations of motion on $X_j^1$ in the variables $(x, y, p, q)$. Proposition \ref{prop:noncontact1} underlines that $\tilde{X}_j^1$ does not admit an almost contact structure with the invariant metric $g_{\tilde{X}_j^1}$ constructed in \cite{12}, but $\tilde{X}_j^1$ can be equipped with an almost contact metric structure with a metric different of the metric $g_{\tilde{X}_j^1}$. We underline that the parametrization introduced in \cite{37} for Sasaki potential on Sasaki manifolds is not appropriate for $\tilde{X}_j^1$ (Remark \ref{rem:3}). Proposition \ref{prop:equations2} of Section \ref{section:equations2} shows the equations of motion on $\tilde{X}_j^1$ organized as contact Hamiltonian system. Proposition \ref{prop:equations3} in Section \ref{section:equations3} presents the equations of motion on $\tilde{X}_j^1$ organized as GTACOS manifold generated by a Hamiltonian linear in generators of the real Jacobi group comparatively with the corresponding equations on $X_j^1$. Finally, the Appendix summarizes the definitions of different mathematical concepts used in the paper.

The new results of the paper are contained in Lemma \ref{lem:1}, the six Remarks, Propositions \ref{prop:equations1} – \ref{prop:equations3} and Theorem \ref{thm:1} which presents the equations of motion on odd dimensional manifolds endowed with almost cosymplectic structure. The Remark \ref{rem:4} emphasizes the differences between the equations of motion on $\tilde{X}_j^1$ endowed with almost cosymplectic structure $(\tilde{X}_j^1, \theta, \omega)$ and contact structure $(\tilde{X}_j^1, \eta_0)$, which in principle could be experimentally highlighted. Although we are not able to determine effectively the almost contact structure associated with the concrete realization of the contact structure $(\tilde{X}_j^1, \eta_0)$, we report the partial results obtained at Proposition \ref{prop:noncontact1} b) in order to emphasize the difficulty of the problem.

**Notation** We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. We denote the imaginary unit $\sqrt{-1}$ by $i$, the real and imaginary parts of a complex number $z \in \mathbb{C}$ by $\text{Re} z$ and $\text{Im} z$ respectively, and the complex conjugate of $z$ by $\bar{z}$. We denote by $d$ the differential. We use Einstein’s summation convention, i.e. repeated indices are implicitly summed over. The set of vector fields (1-forms) is denoted by $\mathcal{D}_1$ (respectively $\mathcal{D}_1$). If we denote with Roman capital letters the Lie groups, then their associated Lie algebras are denoted with the corresponding lower-case letters. The interior product $i_X \omega$ (interior multiplication or contraction) of the differential form $\omega$ with $X \in \mathcal{D}_1$ is denoted $X \lrcorner \omega$. We denote by $\mathcal{M}(n, m, \mathbb{F})$ the set of $n \times m$ matrices with elements in the field $\mathbb{F}$. If $X \in \mathcal{M}(n, m, \mathbb{F})$, then $X^t$ denotes the transpose of $X$. 
2. Preparation: Invariant metrics on $G^J_1(\mathbb{R})$-homogenous spaces

In [12, (4.10), (5.15), (5.17)] we have introduced 6 invariant one-forms $\lambda_1, \ldots, \lambda_6$ in the $S$-coordinates $(x, y, \theta, p, q, \kappa)$ [24] associated with the real Jacobi group $G^J_1(\mathbb{R})$

\begin{align*}
(2.1a) & \quad \lambda_1 = \sqrt{\alpha} \left( \cos 2\theta \, dx + \sin 2\theta \, dy \right), \quad \alpha > 0, \\
(2.1b) & \quad \lambda_2 = \sqrt{\alpha} \left( -\sin 2\theta \, dx + \cos 2\theta \, dy \right), \\
(2.1c) & \quad \lambda_3 = \sqrt{\beta} \left( \frac{dx}{y} + 2 \, d\theta \right), \quad \beta > 0, \\
(2.1d) & \quad \lambda_4 = \sqrt{\gamma} \left[ -y^{\frac{1}{2}} \sin \theta \, dq + \left( y^{\frac{1}{2}} \cos \theta - x y^{-\frac{1}{2}} \sin \theta \right) \, dp \right], \quad \gamma > 0, \\
(2.1e) & \quad \lambda_5 = \sqrt{\gamma} \left[ y^{\frac{1}{2}} \cos \theta \, dq + \left( y^{\frac{1}{2}} \sin \theta + x y^{-\frac{1}{2}} \cos \theta \right) \, dp \right], \\
(2.1f) & \quad \lambda_6 = \sqrt{\delta} \left( d\kappa - p \, dq + q \, dp \right), \quad \delta > 0,
\end{align*}

and we have expressed the invariant metric on several homogenous spaces associated with $G^J_1(\mathbb{R})$ [12, Theorem 5.7]:

**Proposition 1.** The four-parameter left-invariant metric on the real Jacobi group $G^J_1(\mathbb{R})$ in the $S$-coordinates $(x, y, \theta, p, q, \kappa)$ is

\begin{equation}
 ds^2_{G^J_1(\mathbb{R})} = \sum_{i=1}^{6} \lambda_i^2 = \alpha \frac{dx^2 + dy^2}{y^2} + \beta \left( \frac{dx}{y} + 2 \, d\theta \right)^2, \\
 + \frac{\gamma}{y} \left( dq^2 + S dp^2 + 2 xdqd \right) + \frac{\delta}{2} \left( d\kappa - p dq + q dp \right)^2, \quad S := x^2 + y^2.
\end{equation}

Depending on the parameter values $\alpha, \beta, \gamma, \delta \geq 0$ in the metric (2.2), we have invariant metric on the following $G^J_1(\mathbb{R})$-homogeneous manifolds:

1) the Siegel upper half-plane $X_1$ if $\beta, \gamma, \delta = 0$, 
2) the group $SL(2, \mathbb{R})$ if $\gamma, \delta = 0$, $\alpha \beta \neq 0$, 
3) the Siegel–Jacobi half-plane $X^J_1$ if $\beta, \delta = 0$, 
4) the extended Siegel–Jacobi half-plane $\tilde{X}^J_1$ if $\beta = 0$, 
5) the Jacobi group $G^J_1$ if $\alpha \beta \gamma \delta \neq 0$.

The action of Jacobi group on several homogeneous spaces associated with it is presented in [3, Lemma 1], [12, Lemma 5.1]. We have studied the geometry of the Siegel–Jacobi upper half-plane $X^J_1$ in [3, 7, 9, 12, 14].

The parts a), b), c) of Proposition 2 below are extracted from [12, Proposition 2.1 in], [3, Proposition 1]. Below $(w, z) \in (D_1, \mathbb{C})$, $(v, u) \in (X_1, \mathbb{C})$, and the parameters $k$ and $\nu$ come from representation theory of the Jacobi group: $k$ indexes the positive discrete series of SU(1, 1), $2k \in \mathbb{N}$, while $\nu > 0$ indexes the representations of the Heisenberg group $H_1$. See also [7, 11]. By the two-form of Berndt–Kähler we mean the two-parameter invariant Kähler two-form on $X^J_1$ determined in [22, 23, 41, 42], see also [3, 12]. Part d) is taken from [14, Proposition 3]. Part e) was proved in [12, Proposition 5.4, see (5.21b)] and appears also in [14, Proposition 7, see (4.18), (4.23)].
Proposition 2. a) The Kähler two-form on $\mathcal{D}_1^J$, invariant to the action of $G_0^J = \text{SU}(1,1) \ltimes \mathbb{C}$, is

$$
- i \omega_{\mathcal{D}_1^J}(w, z) = \frac{2k}{p^2} \, d \bar{w} \wedge d w + \nu A \wedge \bar{A}, \quad P := 1 - |w|^2, \quad A = A(w, z) := d z + \bar{\eta} \, d w.
$$

We have the change of variables $FC : (w, z) \rightarrow (w, \eta)$

$$
FC: \quad z = \eta - w \bar{\eta}, \quad FC^{-1}: \quad \eta = \frac{z + \bar{z} w}{P},
$$

and

$$
FC: \quad A(w, z) \rightarrow d \eta - w d \bar{\eta}.
$$

b) Using the partial Cayley transform $\Phi : (w, z) \rightarrow (v, u)$ and its inverse

$$
\Phi : w = \frac{v - i}{v + i}, \quad z = 2i \frac{u}{v + i}, \quad v, u \in \mathbb{C}, \quad \text{Im} \, v > 0,
$$

$$
\Phi^{-1}: v = i \frac{1 + w}{1 - w}, \quad u = \frac{z}{1 - w}, \quad w, z \in \mathbb{C}, \quad |w| < 1,
$$

we obtain

$$
A \left( \frac{v - i}{v + i}, \frac{2i u}{v + i} \right) = \frac{2i}{v + i} B(v, u),
$$

where

$$
B(v, u) := du - r dv, \quad r := \frac{u - \bar{u}}{v - \bar{v}}.
$$

The Berndt–Kähler’s two-form, invariant to the action of $G^J(\mathbb{R})_0 = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{C}$, is

$$
- i \omega_{\mathcal{X}_1^J}(v, u) = - \frac{2k}{(\bar{v} - v)^2} \, d v \wedge d \bar{v} + \frac{2\nu}{i(\bar{v} - v)} B \wedge \bar{B}.
$$

We have the change of variables $FC_1 : (v, u) \rightarrow (v, \eta)$

$$
FC_1: \quad 2i u = (v + i) \eta - (v - i) \bar{\eta}, \quad FC_1^{-1}: \quad \eta = \frac{\bar{w} \bar{v} - w v}{\bar{v} - v} + i r.
$$

c) If we apply the change of coordinates $\mathcal{D}_1^J \ni (v, u) \rightarrow (x, y, p, q) \in \mathcal{X}_1^J$

$$
\mathbb{C} \ni u := p v + q, \quad p, q \in \mathbb{R}, \quad \mathbb{C} \ni v := x + i y, \quad x, y \in \mathbb{R}, \quad y > 0,
$$

then

$$
r = p, \quad B(v, u) = d u - p d v,
$$

$$
B(v, u) = B(x, y, p, q) := F d t = v d p + d q = (x + i y) d p + d q, \quad F := \dot{p} v + \dot{q}.
$$

d) The second partial Cayley transform $\Phi_1 : \mathcal{D}_1^J \rightarrow \mathcal{X}_1^J$

$$
\Phi_1 := FC_1 \circ \Phi : (w, z) \rightarrow (v = x + i y, \eta = q + i p)
$$

is given by

$$
\Phi_1 : w = \frac{v - i}{v + i}, \quad z = 2i \frac{p v + q}{v + i},
$$

$$
\Phi_1^{-1}: v = i \frac{1 + w}{1 - w}, \quad \eta = \frac{(1 + i \bar{v})(z - \bar{z}) + v(\bar{v} - i)(z + \bar{z})}{2 i(\bar{v} - v)} = \frac{z + \bar{z} w}{P}.
$$
e) The two-parameter balanced metric on the Siegel–Jacobi upper half-plane $\mathcal{X}_1^J$, the particular case of (2.2) corresponding to item 3) in Proposition 1 associated to the Kähler two-form (2.6), (2.7), is
\begin{equation}
\text{d} s^2_{\mathcal{X}_1^J}(x, y, p, q) = \alpha \frac{\text{d} x^2 + \text{d} y^2}{y^2} + \frac{\gamma}{y} (S \text{d} p^2 + \text{d} q^2 + 2x \text{d} p \text{d} q),
\end{equation}
where $S$ was defined in (2.2) and
\begin{equation}
\alpha := \frac{k}{2}, \quad \gamma := \nu.
\end{equation}
The metric matrix associated with (2.9) is
\begin{equation}
g_{\mathcal{X}_1^J} = \begin{pmatrix}
g_{xx} & 0 & 0 & 0 \\
0 & g_{yy} & 0 & 0 \\
0 & 0 & g_{pp} & g_{pq} \\
0 & 0 & g_{qp} & g_{qq}
\end{pmatrix},
\end{equation}
\begin{align*}
g_{xx} &= \frac{\alpha}{y^2}, \\
g_{yy} &= \frac{\alpha}{y^2}, \\
g_{pp} &= \frac{\gamma}{y}, \\
g_{pq} &= \frac{\gamma S}{y}, \\
g_{qp} &= \frac{\gamma}{y}, \\
g_{qq} &= \frac{\gamma}{y}.
\end{align*}
We have also obtained invariant metric to the action of the Jacobi group $G_1^J(\mathbb{R})$ on the extended Siegel-Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$ [12, Proposition 5.6, (5.25), (5.26)], see also [14, Theorem 1, (5.1) (5.5)]

**Proposition 3.** The three-parameter metric of the extended Siegel-Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$ expressed in the $S$-coordinates $(x, y, p, q, \kappa)$, left-invariant with respect to the action of the Jacobi group $G_1^J(\mathbb{R})$, is given by item 4) in Proposition 1 as
\begin{equation}
\text{d}s^2_{\tilde{\mathcal{X}}_1^J}(x, y, p, q, \kappa) = \text{d}s^2_{\mathcal{X}_1^J}(x, y, p, q) + \lambda_6^2(p, q, \kappa)
\end{equation}
\begin{align*}
&= \frac{\alpha}{y^2}(\text{d} x^2 + \text{d} y^2) + \left(\frac{\gamma}{y} S + \delta q^2\right) \text{d} p^2 + \left(\frac{\gamma}{y} + \delta p^2\right) \text{d} q^2 + \delta \kappa^2 \\
&+ 2 \left(\frac{\gamma}{y} x - \delta pq\right) \text{d} p \text{d} q + 2 \delta(q \text{d} p \text{d} \kappa - p \text{d} q \text{d} \kappa).
\end{align*}
The metric matrix associated to the metric (2.11) is
\begin{align}
g_{\tilde{\mathcal{X}}_1^J} &= \begin{pmatrix}
g_{xx} & 0 & 0 & 0 & 0 \\
0 & g_{yy} & 0 & 0 & 0 \\
0 & 0 & g_{pp} & g_{pq} & g_{pp} \\
0 & 0 & g_{qp} & g_{qq} & g_{qc} \\
0 & 0 & g_{kp} & g_{kq} & g_{kk}
\end{pmatrix},
\end{align}
\begin{align*}
g_{xx} &= \frac{\alpha}{y^2}, \\
g_{yy} &= \frac{\alpha}{y^2}, \\
g_{pp} &= \frac{\gamma}{y}, \\
g_{pq} &= \frac{\gamma S}{y}, \\
g_{qp} &= \frac{\gamma}{y}, \\
g_{qq} &= \frac{\gamma}{y}, \\
g_{pp} &= \frac{\gamma}{y}, \\
g_{pq} &= \frac{\gamma S}{y}, \\
g_{qp} &= \frac{\gamma}{y}, \\
g_{qq} &= \frac{\gamma}{y}, \\
g_{kp} &= \frac{\gamma}{y}, \\
g_{kq} &= \frac{\gamma}{y}, \\
g_{kk} &= \delta.
\end{align*}
The extended Siegel–Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$ does not admit an almost contact structure $(\Phi, \xi, \eta)$ in the sense of Definition 1 with a contact form $\eta = \lambda_6$ and Reeb vector $\xi = \text{Ker}(\eta)$.

### 3. Equations of motion on almost cosymplectic manifolds

In the present paper we adopt the terminology of Libermann [15] recalled at ACOS in Appendix. We consider the almost cosymplectic manifold $(M_{2n+1}, \theta, \Omega)$ such that condition (5.5) is satisfied for the real one-form $\theta$
\begin{equation}
\theta = a_i \text{d} q^i + b_i \text{d} p_i + c \kappa, \quad a_i, b_i \in \mathbb{R}, \quad i = 1, \ldots, n, \quad c \neq 0,
\end{equation}
while for the two-form Ω we use the Darboux parametrization given in (5.2).

We determine the Hamiltonian vector field \( X_H \) verifying a relation similar to (5.15) for the contact manifold \( (M, \eta) \) for the one-form \( \theta \) (3.1) and two-form \( \Omega \) (5.2):

**Theorem 1.** Let \( (M_{2n+1}, \theta, \Omega) \) be an almost cosymplectic manifold and let \( H \in C^\infty(M) \) be a smooth Hamiltonian. Then the coefficients that define the Hamiltonian vector field \( X_H \) attached the Hamiltonian \( H \)

\[
X_H = A_i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial p_i} + C \frac{\partial}{\partial \kappa},
\]

are solution of the equation

\[
\flat(X_H) = \phi H - (\Ree H + H)\theta,
\]

where the Reeb vector which is the solution of the equation (5.6) is

\[
\Ree = \frac{1}{c} \frac{\partial}{\partial \kappa},
\]

and the isomorphism \( \flat \) is defined in (5.7).

It is obtained

\[
A_i = \frac{\partial H}{\partial p^i} - b_i R(H), \quad B_i = -\frac{\partial H}{\partial q^i} + a_i R(H), \quad C = \frac{1}{c}(-a_i \frac{\partial H}{\partial p^i} + b_i \frac{\partial H}{\partial q^i} - H),
\]

and then

\[
X_H = \left( \frac{\partial H}{\partial p^i} - b_i R(H) \right) \frac{\partial}{\partial q^i} + \left( -\frac{\partial H}{\partial q^i} + a_i R(H) \right) \frac{\partial}{\partial p_i} + \left( -a_i \frac{\partial H}{\partial p^i} + b_i \frac{\partial H}{\partial q^i} - H \right) R.
\]

The vector field \( \grad H \), i.e. the solution of the equation

\[
(\grad H) = \phi H
\]

has the expression

\[
\grad H = \left( \frac{\partial H}{\partial p_i} - b_i R(H) \right) \frac{\partial}{\partial q^i} + \left( -\frac{\partial H}{\partial q^i} + a_i R(H) \right) \frac{\partial}{\partial p_i} + \left( -a_i \frac{\partial H}{\partial p^i} + b_i \frac{\partial H}{\partial q^i} + R(H) \right) R.
\]

The vector fields \( X_H \) (3.4) and \( \grad H \) (3.5) are related by the relation

\[
X_H = \grad H - (H + R(H)) R,
\]

which is similar with (5.15) but with Reeb vector (3.2) instead of (5.14).

The equations of motion on the almost cosymplectic manifold \( (M, \theta, \Omega) \) are

\[
\frac{dq^i}{dt} - A_i = 0, \quad \frac{dp_i}{dt} - B_i = 0, \quad \frac{d\kappa}{dt} - C = 0, \quad i = 1, \ldots, n,
\]

where \( (A_i, B_i, C) \) have the expressions given in (3.3).

In particular, we find

\[
\grad q^i = -\frac{\partial}{\partial p_i} + b_i R,
\]

\[
\grad p_i = \frac{\partial}{\partial q^i} - a_i R,
\]
\[ \text{(3.8c)} \quad \text{grad } \kappa = \frac{1}{c}(-b_i \frac{\partial}{\partial q^i} + a_i \frac{\partial}{\partial p_i} + R), \]

\[ \text{(3.8d)} \quad X_{q^i} = -\frac{\partial}{\partial p_i} + (b_i - q^i)R, \]

\[ \text{(3.8e)} \quad X_{p_i} = \frac{\partial}{\partial q^i} - (a_i + p_i)R, \]

\[ \text{(3.8f)} \quad X_\kappa = \frac{1}{c}(-b_i \frac{\partial}{\partial q^i} + a_i \frac{\partial}{\partial p_i} - \kappa \frac{\partial}{\partial \kappa}). \]

**Proof.** The proof is elementary. We indicate the main steps.

In order to apply (5.7) we calculate

\[ X_H \theta = A_i a_i + B_i b_i + C, \quad X_H \Omega = -B_i d q^i + A_i d p_i, \]

and we find

\[ (3.9) \quad X_H = [-B_i + a_i(A_j a_j + B_j b_j + C)] d q^i + [A_i b_i(A_j a_j + B_j b_j + C)] d p_i + (A_j a_j + B_j b_j + C) c d \kappa. \]

We calculate

\[ (3.10) \quad d H - (R_H + H) \theta = \left[ \frac{\partial H}{\partial q^i} - a_i(R(H) + H) \right] d q^i + \left[ \frac{\partial H}{\partial p_i} - b_i(R(H) + H) \right] d p_i - c H d \kappa \]

and (3.3) follows from the identification of (3.9) with (3.10).

The calculation of \( \text{grad } H \) is done similarly and from (3.4) and (3.5) it follows (3.6).

□

Now we compare our results in Theorem I with the results of Albert II.

**Remark 1.** The transitive almost contact structure considered by Albert II at TACS in the Appendix corresponds in our case in (3.1) to

\[ a_i = \epsilon p_i, \quad b_i = 0, \quad c = 1, \quad i = 1, \ldots, n, \]

and the Reeb vector is

\[ R(H) = \frac{\partial H}{\partial \kappa}. \]

\[ (3.4) \]

becomes

\[ X_H = \frac{\partial H}{\partial \varphi^i} \frac{\partial}{\partial q^i} + \left( -\frac{\partial H}{\partial q^i} + \epsilon p_i \frac{\partial H}{\partial \kappa} \right) \frac{\partial}{\partial p_i} - (\epsilon p_i \frac{\partial H}{\partial \varphi^i} + H) \frac{\partial}{\partial \kappa}, \]

and \( X_H \) verifies the relations

\[ (3.11) \quad X_H \theta = -H, \quad X_H \Omega = d H - R(H) \theta. \]

**Equation of grad f becomes**

\[ \text{grad } f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} + (\epsilon p_i \frac{\partial f}{\partial \kappa} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} + (\frac{\partial f}{\partial \kappa} - \epsilon p_i \frac{\partial f}{\partial p_i} \frac{\partial}{\partial \kappa}. \]

**Equations (3.8) become**

\[ (3.12a) \quad \text{grad } q^i = -\frac{\partial}{\partial p_i}. \]
grad \ p_i = \frac{\partial}{\partial q^i} - \epsilon p_i R, \hspace{1cm} (3.12b)

grad \ \kappa = \epsilon p_i \frac{\partial}{\partial p_i} + \frac{\partial}{\partial \kappa}, \hspace{1cm} (3.12c)

X_{q^i} = -\frac{\partial}{\partial p_i} - q^i \frac{\partial}{\partial \kappa}, \hspace{1cm} (3.12d)

X_{p_i} = \frac{\partial}{\partial q_i} - (\epsilon + 1)p_i \frac{\partial}{\partial \kappa}, \hspace{1cm} (3.12e)

X_{\kappa} = +\epsilon p_i \frac{\partial}{\partial p_i} - \kappa \frac{\partial}{\partial \kappa}. \hspace{1cm} (3.12f)

Equations of the vector field \( X_f \) at \([1, \text{p} \ 636]\) are in agreement with our (3.3), except the following two differences:

- the coefficient \( \epsilon(f - p_i \frac{\partial}{\partial t}) \) of \( \frac{\partial}{\partial t} \) should be replaced with
  \[ -\epsilon p_i \frac{\partial f}{\partial p_i} - f. \]

This comes from the fact that:
- in formulas \([1, (3)]\) instead of
  \[ X_f \Theta = \epsilon f, \]
  we should have
  \[ X_f \Theta = -f \]
as in (3.11). With these correction, Albert’s formulas for \( X_f \) on \( \text{p} \ 636 \) are particular cases of our formula (3.3) and, if \( \epsilon = -1 \), corrected equation (2.12) in \([33]\) is find again. Similarly, if instead of (3.11) we consider (5.12), equations of motion \([1, (2.13), (2.14), (2.15)]\) in \([33]\) are particular cases of ours equations (3.7).

- in formula (3.12) Albert gives the different values for \( X_{q^i}, X_{p_i}, \) and \( X_{\kappa} \) in (3.12d), (3.12e), respectively (3.12f)

\[ X_{q^i} = -\frac{\partial}{\partial p_i} + \epsilon q^i \frac{\partial}{\partial t}, \quad X_{p_i} = \frac{\partial}{\partial q_i}, \quad X_{\kappa} = \epsilon(t \frac{\partial}{\partial t} + p_i \frac{\partial}{\partial p_i}). \]

4. Equations of motion on \( \tilde{X}_1 \)

4.1. Generalized transitive almost cosymplectic Hamiltonian systems. In the standard approach of CS, the \( G \)-invariant Kähler two-form on a \( 2n \)-dimensional homogeneous manifold \( M = G/H \) is obtained from the Kähler potential \( f \) via the recipe

\[ -i \omega_M = \partial \bar{\partial} f, \quad f(z, \bar{z}) = log \mathcal{K}(z, \bar{z}), \quad \mathcal{K}(z, \bar{z}) := (e_z, e_{\bar{z}}), \]
\[ \omega_M(z, \bar{z}) = i \sum_{\alpha, \beta} h_{\alpha \beta} \, dz_\alpha \wedge d\bar{z}_\beta, \quad h_{\alpha \beta} = \frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta}, \quad h_{\alpha \bar{\beta}} = \bar{h}_{\bar{\beta} \alpha}, \quad \alpha, \beta = 1, \ldots, n, \]

where \( K(z, \bar{z}) \) is the scalar product of two un-normalized Perelomov’s CS-vectors \( e_z \) at \( z \in M \) [8, 11, 48].

In accord with [4, p 42], [39, p 28], [12, Appendix B], the Riemannian metric associated with the Hermitian metric on the manifold \( M \) in local coordinates is

\[ (4.2) \quad ds^2_M(z, \bar{z}) = \sum_{\alpha, \beta=1}^n h_{\alpha \beta} \, dz_\alpha \otimes d\bar{z}_\beta. \]

From the Kähler two-form (2.3) obtained via CS in [8] we get with (4.2) the balanced metric on the Siegel-Jacobi disk \( D^J_1 \). If in \( ds^2(w, z)_{D^J_1} \) we apply the second partial Cayley transform (2.8), then we obtain the invariant metric (2.9) on \( X^J_1 \).

We endow \( \tilde{X}^J_1 \) with a generalized transitive almost cosymplectic structure, i.e. an almost cosymplectic structure \((M, \theta, \Omega)\) such that

\[ d\Omega = 0. \]

**Lemma 1.** If we introduce into the Kähler two-form \( \omega_{D^J_1}(w, z) \) (2.3) the second partial Cayley transform (2.8a) \( (w, z) \rightarrow (x, y, q, p), \, y > 0 \) we get the symplectic two-form

\[ (4.3) \quad \omega_{\tilde{X}^J_1}(x, y, q, p) = \frac{k}{y^2} \, dx \wedge dy + 2\nu \, dq \wedge dp, \quad y > 0. \]

In the notation of [12], we introduce on the extended 5-dimensional Siegel-Jacobi half-plane \( \tilde{X}^J_1 \) parametrized in \( (x, y, p, q, \kappa) \) the almost cosymplectic structure \((\tilde{X}^J_1, \theta, \omega)\), where \( \theta = \lambda_{\delta} \) and \( \omega \) is (4.3), i.e.

\[ (4.4a) \quad \theta = \sqrt{\delta}(d\kappa - p \, dq + q \, dp), \quad \delta > 0, \]

\[ (4.4b) \quad \omega = \frac{k}{y^2} \, dx \wedge dy + 2\nu \, dq \wedge dp, \quad y > 0. \]

We have

\[ (4.5) \quad d\omega = 0, \quad \theta \wedge \omega^2 = 2\frac{k\nu\sqrt{\delta}}{y^2} \, dx \wedge dy \wedge dq \wedge dp \wedge d\kappa, \]

and \((\tilde{X}^J_1, \theta, \omega)\) verifies the condition (5.5) of an almost cosymplectic manifold.

(4.4a) corresponds in (3.1) to the choice:

\[ (4.6) \quad a_1 = b_1 = 0, \quad a_2 = -\sqrt{\delta} \, p, \quad b_2 = \sqrt{\delta} \, q, \quad c = \sqrt{\delta}, \quad n = 2, \]

and (4.4b) corresponds in (5.2) to the choice:

\[ (4.7) \quad q^1 = kx, \quad p_1 = -\frac{1}{y}, \quad q^2 = 2\nu q, \quad p_2 = p, \quad n = 2. \]

In Darboux coordinates we have a particular almost cosymplectic manifold \((\tilde{X}^J_1, \theta, \omega)\) verifying (5.5) and in addition the condition

\[ d\omega = 0. \]
(\tilde{\mathcal{X}}_1^J, \theta, \omega) is a generalized transitive almost cosymplectic manifold.

Introducing (3.3) in Theorem 1 we get

**Proposition 4.** The coefficients (3.3) of vector field \(X_H\) (3.4) on the generalized transitive almost cosymplectic manifold \((\tilde{\mathcal{X}}_1^J, \theta, \omega)\) have the expressions

\[
A_1 = y^2 \frac{\partial}{\partial y}, \quad A_2 = \frac{\partial H}{\partial p} - q \frac{\partial H}{\partial \kappa},
\]

\[
B_1 = -\frac{1}{k} \frac{\partial H}{\partial x}, \quad B_2 = -\frac{1}{2\nu} \left( \frac{\partial H}{\partial q} + p \frac{\partial H}{\partial \kappa} \right),
\]

\[
C = \frac{1}{2\nu} \left( p \frac{\partial H}{\partial q} + q \frac{\partial H}{\partial p} \right) - H.
\]

Equations of motion (3.7) on the 5-dimensional extended Siegel-Jacobi half-plane \(\tilde{\mathcal{X}}_1^J\) are

\[
\dot{x} = \frac{y^2}{k} \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{y^2}{k} \frac{\partial H}{\partial x},
\]

\[
\dot{q} = \frac{1}{2\nu} \left( \frac{\partial H}{\partial p} - q \frac{\partial H}{\partial \kappa} \right), \quad \dot{p} = -\frac{1}{2\nu} \left( \frac{\partial H}{\partial q} + p \frac{\partial H}{\partial \kappa} \right),
\]

\[
\dot{\kappa} = \frac{1}{2\nu} \left( p \frac{\partial H}{\partial q} + q \frac{\partial H}{\partial p} \right) - H.
\]

**4.2. Contact Hamiltonian systems.** We recall that in [12] we noticed that for \(\eta = \lambda_6\) we have \(d\eta^2 = 0\) and if we want to introduce a contact structure on \(\tilde{\mathcal{X}}_1^J\), we have to choose \(\eta \neq \lambda_6\).

Now we organize \(\tilde{\mathcal{X}}_1^J\) as a contact Hamiltonian system in the sense of [33, § 2.3]. Then we apply Proposition 9 to determine the associated almost contact metric structure:

**Proposition 5.** Let \(\eta_0\) be such that \(d\eta_0 = \omega_{\tilde{\mathcal{X}}_1^J}\), where \(\omega_{\tilde{\mathcal{X}}_1^J}\) is given by (1.3). We chose

\[
\eta_0 = \sqrt{\delta} \, d\kappa + \frac{k}{y} \, d\, x + \nu (-p \, d\, q + q \, d\, p).
\]

\(\eta_0 \wedge d\eta_0^2\) has the same nonzero value as the volume form given by the second equation (4.5), (5.11) is verified, and \((\tilde{\mathcal{X}}_1^J, \eta_0)\) is a contact manifold.

a) For (4.10) chosen as \(\eta_0\) and \(g_{\tilde{\mathcal{X}}_1^J}\) given by (2.12), there is no \(\Phi\) so that \((\tilde{\mathcal{X}}_1^J, \Phi, \xi, \eta_0, g_{\tilde{\mathcal{X}}_1^J})\) is an almost contact metric structure, where \(\xi\) is given by (4.15), (4.19).

b) The contact manifold \((\tilde{\mathcal{X}}_1^J, \eta_0)\) can be endowed with an almost contact metric structure \((\tilde{\mathcal{X}}_1^J, \Phi, \xi, \eta_0, g'_{\tilde{\mathcal{X}}_1^J})\). The six components \(\Phi_{xx}, \Phi_{xy}, \Phi_{xq}, \Phi_{xp}, \Phi_{yx}, \Phi_{qq}\) of \(\Phi\) in (4.25) can be expressed as function of the four remaining independent variables \(\Phi_{yy}, \Phi_{yp}, \Phi_{np}, \Phi_{pq}\), while the rest of the components of \(\Phi\) are obtained with (4.20), (4.21), (4.26). The fundamental quadratic form associated to the metric tensor \(g'\) (4.27) must be positive definite.
Proof. a) We use (5.29) for \( \hat{\Phi} \) corresponding to \( \eta_0 \) and (2.12) written as

\[
g_{\tilde{\chi}_i} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A = \begin{pmatrix} g_{xx} & 0 \\ 0 & g_{yy} \end{pmatrix}, \quad B = \begin{pmatrix} g_{qq} & g_{qp} & g_{qk} \\ g_{pq} & g_{pp} & g_{p\kappa} \\ g_{q\kappa} & g_{p\kappa} & g_{\kappa\kappa} \end{pmatrix}.
\]

We write down \( \Phi \) as

\[
\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix}, \quad \Phi_1 \in M(2,2,\mathbb{R}), \quad \Phi_4 \in M(3,3,\mathbb{R}), \quad \Phi_2, \ \Phi_3 \in M(2,3,\mathbb{R}).
\]

With (4.11) and (4.12) we get

\[
g_{\tilde{\chi}_i} \Phi = \begin{pmatrix} A \Phi_1 & A \Phi_2 \\ B \Phi_3 & B \Phi_4 \end{pmatrix}.
\]

We write down \( \hat{\Phi} \) as

\[
\hat{\Phi} = \begin{pmatrix} \hat{\Phi}_1 \\ 0 \\ 0 \\ \hat{\Phi}_4 \end{pmatrix}, \quad \hat{\Phi}_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}, \quad \hat{\Phi}_4 = \begin{pmatrix} 0 & \sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau = \frac{k}{y^2}, \quad \sigma = 2\nu.
\]

With (5.29) we find the relations

\[
\Phi_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}, \quad \Phi_2 = 0, \quad \Phi_3 = 0, \quad \Phi_4 = \begin{pmatrix} g_{qq} & g_{qp} & 0 \\ g_{pq} & g_{pp} & 0 \\ g_{q\kappa} & g_{p\kappa} & 0 \end{pmatrix}.
\]

With (4.14), the first condition (5.27) implies

\[
\xi^t = (0, 0, 0, 0, \xi_\kappa),
\]

while the second condition (5.27) gives

\[
\xi = \frac{1}{\sqrt{\delta}} \frac{\partial}{\partial \kappa}.
\]

The second condition (5.27) implies

\[
(4.16a) \quad 0 = \frac{k}{y} \Phi_x^x - \nu\Phi_x^x + \nu\Phi_x^x + \sqrt{\delta}\Phi_x^x,

(4.16b) \quad 0 = \frac{k\tau}{gg_{xx}},

(4.16c) \quad 0 = \ldots.
\]

But (4.16b) can’t be satisfied and a) is proved.

b) With (5.26), (5.27), (5.28), for that the contact manifold \((M_{2n+1}, \eta)\) to have an almost contact metric structure \((M, \Phi, \xi, \eta, g)\), the following equations must be satisfied

\[
(4.17a) \quad \text{rank } \Phi = 2n,

(4.17b) \quad \eta \circ \xi = 1, \quad \xi^t, \eta \in M(1,n,\mathbb{R}),

(4.17c) \quad \Phi \xi = 0,

(4.17d) \quad \eta \Phi = 0,
\]
(4.17e) \[ \Phi^2 = -1_n + \xi \otimes \eta, \]
(4.17f) \[ \eta = g\xi, \]
(4.17g) \[ g = \eta \otimes \eta - \hat{\Phi}\Phi, \]
(4.17h) \[ \hat{\Phi}\xi = 0. \]

We recall that \( \hat{\Phi} \) is defined in (5.29) and is connected with \( d\eta \) as in (5.31). Equations (4.17a)-(4.17g) appear in [49] (and also in [51]), where it is underlined that the first 5 relations are not independent. Equation (4.17h) is a consequence of (4.17d), (5.29). See also Theorem 3.1 in [50].

Below we denote the components \( \Phi_{ij} \) of the \((1,1)\)-tensor field \( \Phi \) by \( \Phi_{ij} \), i.e.

(4.18) \[ \Phi = (\Phi)_{ij}, \quad i, j = x, y, q, p, \kappa. \]

1) With \( \eta_0 \) given by (4.10) and (4.17b), we get for \( \xi = (\xi_x, \xi_y, \xi_q, \xi_p, \xi_\kappa) \)

\[ \frac{k}{y}\xi_x - \nu p\xi_q + \nu q\xi_p + \sqrt{\delta}\xi_\kappa = 1. \]

With (4.17h) for \( \hat{\Phi} \) given by (4.13) we get again for \( \xi \) the form (4.15)

(4.19) \[ \xi^t = (0, 0, 0, 0, \frac{1}{\sqrt{\delta}}). \]

2) With (4.17c) and (4.19) we get the relations

(4.20) \[ \Phi_{z\kappa} = 0, \quad z = x, y, q, p, \kappa. \]

3) With (4.17d), we get

(4.21) \[ \Phi_{z\kappa} = -\frac{1}{\sqrt{\delta}}\left(\frac{k^2}{y} - \nu p\Phi_{qz} + \nu q\Phi_{pz}\right), \quad z = x, y, q, p. \]

4) With the values (4.10) for \( \eta \) and (4.19) for \( \xi \) introduced in (4.17f), we get

(4.22) \[ (g_{xx}, g_{yx}, g_{qy}; g_{px}, g_{p\kappa}) = \left(\frac{k\sqrt{\delta}}{y}, 0, -\nu\sqrt{\delta}p, \nu\sqrt{\delta}q, \delta\right). \]

5) Introducing (4.13) in (4.17g), with (4.20), (4.22) we get

\[
g = \begin{pmatrix}
g_{xx} & g_{xy} & g_{xq} & g_{xp} & g_{xk} 
g_{yx} & g_{yy} & g_{yq} & g_{yp} & g_{yk} 
g_{qy} & g_{qq} & g_{qp} & g_{qq} & g_{qk} 
g_{py} & g_{yp} & g_{pp} & g_{pp} & g_{pk} 
g_{kx} & g_{kx} & g_{kq} & g_{kp} & g_{k\kappa}
\end{pmatrix},
\]

(4.23) \[
g = \begin{pmatrix}
k^2/y - \tau\Phi_{xx} & \tau\Phi_{xy} & -\nu^2 p - \tau\Phi_{yy} & -\nu^2 q - \tau\Phi_{yp} & -\nu\sqrt{\delta}/y 
\tau\Phi_{xy} & \tau\Phi_{yy} & -\nu^2 p - \tau\Phi_{xy} & -\nu\sqrt{\delta}/y & 0 
-\nu^2 p - \tau\Phi_{yx} & -\nu^2 q - \tau\Phi_{xp} & -\nu\sqrt{\delta}/y & 0 & 0 

\end{pmatrix},
\]

where we have not written down the matrix elements under the diagonal of the symmetric matrix \( g \).
6) But the symmetry of the matrix \(g\) \[\text{(4.23)}\] imposes the following restrictions on components of the \((1,1)\)-tensor \(\Phi\):

\[(4.24a)\] \[\Phi_{yy} = -\Phi_{xx},\]

\[(4.24b)\] \[\Phi_{qx} = -\zeta \Phi_{yp}, \quad \zeta := \frac{\tau}{\sigma},\]

\[(4.24c)\] \[\Phi_{qy} = \zeta \Phi_{xp},\]

\[(4.24d)\] \[\Phi_{px} = \zeta \Phi_{yp},\]

\[(4.24e)\] \[\Phi_{pp} = -\Phi_{qq},\]

\[(4.24f)\] \[\Phi_{py} = -\zeta \Phi_{xp}.\]

We choose as independent components of tensor field \(\Phi\) the components \(\Phi\) minus the l.h.s. components of \[\text{(4.24)},\] i.e. the submatrix of \(\Phi\) with the following 10 components

\[(4.25)\] \[
\begin{pmatrix}
\Phi_{xx} & \Phi_{xy} & \Phi_{xq} & \Phi_{xp} \\
\Phi_{yx} & \Phi_{yy} & \Phi_{yp} & \Phi_{xp} \\
\Phi_{qy} & \Phi_{xp} & \Phi_{qq} & \Phi_{qp} \\
\Phi_{px} & \Phi_{yp} & \Phi_{qp} & \Phi_{pq}
\end{pmatrix}.
\]

With \[\text{(4.24)},\] the relations \[\text{(4.21)}\] became

\[(4.26a)\] \[\Phi_{xx} = -\frac{k}{\sqrt{\delta}}(\Phi_{xx} + \nu \zeta q \Phi_{yy}),\]

\[(4.26b)\] \[\Phi_{xy} = -\frac{k}{\sqrt{\delta}}(\Phi_{xy} - \nu \zeta p \Phi_{xp}),\]

\[(4.26c)\] \[\Phi_{qy} = -\frac{k}{\sqrt{\delta}}(\Phi_{xq} - \nu \zeta p \Phi_{pp}),\]

\[(4.26d)\] \[\Phi_{xp} = -\frac{k}{\sqrt{\delta}}(\Phi_{xp} - \nu \zeta q \Phi_{pq}).\]

With \[\text{(4.24)},\] equation \[\text{(4.23)},\] becomes

\[(4.27)\] \[
g' = \begin{pmatrix}
\frac{k^2}{\sqrt{\sigma}} - \tau \Phi_{xx} & -\tau \Phi_{yy} & -\nu \zeta p \Phi_{yy} & \nu \zeta q \Phi_{yy} & \frac{k\sqrt{\delta}}{y} \\
\tau \Phi_{xy} & \frac{k^2}{\sqrt{\sigma}} - \tau \Phi_{yy} & \nu \zeta q \Phi_{xy} & \nu \zeta p \Phi_{xx} & 0 \\
\nu^2 p^2 - \sigma \Phi_{pq} & \nu^2 q^2 - \nu^2 pq - \sigma \Phi_{qq} & 0 & \nu^2 q^2 + \sigma \Phi_{qp} & \nu \sqrt{\delta q} \\
\end{pmatrix}.
\]

Taking into account \[\text{(4.20)}, \text{(4.21)}, \text{(4.24)}, \text{(4.26)},\] the 25 equations \[\text{(4.17c)}\] for the tensor \(\Phi\) in the convention \[\text{(4.18)}\] expressed as function of the 10 independent components \[\text{(4.25)}\] are reduced only to the following 6 independent equations

\[(4.28a)\] \[\Phi_{xx} + \Phi_{xy} \Phi_{yx} + \zeta (\Phi_{xp} \Phi_{yp} - \Phi_{xq} \Phi_{qp}) = -1,\]

\[(4.28b)\] \[\Phi_{qy} (\Phi_{xx} + \Phi_{qq}) + \Phi_{xy} \Phi_{yy} + \Phi_{xp} \Phi_{pq} = 0,\]

\[(4.28c)\] \[\Phi_{xp} (\Phi_{xx} - \Phi_{qq}) + \Phi_{xy} \Phi_{yp} + \Phi_{xq} \Phi_{qp} = 0,\]
In (4.28) the first 3 (respectively, next 2, last, no, no) equations are the only independent equations of the product of the first (respectively, second, third, fourth, fifth) line of $\Phi$ with $\Phi$ in (4.17e).

7) The ten independent variables $\Phi$ (4.25) verify the 6 independent equations (4.28).

We shall choose 6 independent variables in the set (4.25) and express the four remaining variables using six independent equations (4.28).

From equations (4.28a) and (4.28c), we get if $\Phi_{xq} \Phi_{xp} \neq 0$

$$2\Phi_{xx} = \frac{1}{\Phi_{xq}}(\Phi_{xy} \Phi_{yq} + \Phi_{pq} + \Phi_{qp}),$$

(4.31a)

$$2\Phi_{qq} = \frac{1}{\Phi_{xq}}(\Phi_{xy} \Phi_{yp} + \Phi_{pq} + \Phi_{qp}),$$

(4.31b)

But from (4.28a), (4.28f) we get

$$\zeta(\Phi_{xp} \Phi_{yq} - \Phi_{xy} \Phi_{xp}) + 1 = -\Phi_{xx} - \Phi_{xy} \Phi_{yx} = -\Phi_{qq} - \Phi_{xp} \Phi_{pq}.$$

Introducing (4.31) into (4.28e), we get

$$\Phi_{xy}[(\Phi_{xq} \Phi_{xp} + \Phi_{pq} \Phi_{yq}) + \Phi_{yp}(\Phi_{xp} \Phi_{pq} + \Phi_{yq} \Phi_{xy})] = 0.$$

If $\Phi_{xy} \neq 0$, we get

$$\Phi_{yx} = -\frac{1}{\Phi_{xq} \Phi_{xp}}[\Phi_{yq}(\Phi_{xy} \Phi_{xp} + \Phi_{pq} \Phi_{yq}) + \Phi_{yp}(\Phi_{xp} \Phi_{pq} + \Phi_{yq} \Phi_{xy})].$$

(4.33)

But comparing (4.30) with (4.33), we get

$$\Phi_{xp} = \Phi_{xq}.$$

(4.34)

With (4.31), (4.34), equations (4.28a), (4.28f) became

$$2\Phi_{xx} = -\frac{1}{\Phi_{xq} \Phi_{xp}}[\Phi_{xy}(\Phi_{yq} \Phi_{xp} + \Phi_{pq} \Phi_{yq}) + \Phi_{yp}(\Phi_{xp} \Phi_{pq} + \Phi_{yq} \Phi_{xy})],$$

(4.35a)

$$2\Phi_{qq} = \frac{1}{\Phi_{xq} \Phi_{xp}}[\Phi_{xy}(-\Phi_{yq} \Phi_{xp} + \Phi_{pq} \Phi_{yq}) + \Phi_{xp}(-\Phi_{pq} + \Phi_{xp})].$$

(4.35b)
With (4.34), equations (4.32) became

\begin{align}
\Phi_{xx}^2 + \zeta \Phi_{xq}(\Phi_{yq} - \Phi_{yp}) + 1 &= -\Phi_{xy} \Phi_{yx}, \\
\Phi_{qq}^2 + \zeta \Phi_{xq}(\Phi_{yq} - \Phi_{yp}) + 1 &= -\Phi_{qp} \Phi_{pq}.
\end{align}

From (4.36a) ((4.36b)) we get with (4.35a), (4.33) ((4.35b)) the value \( \Phi_{xy} \) (respectively \( \Phi_{xq} \)).

We have shown that the six components \( \Phi_{xx}, \Phi_{xy}, \Phi_{xq}, \Phi_{xp}, \Phi_{yx}, \Phi_{qq} \) of \( \Phi \) in (4.25) can be expressed as function of the remaining independent variables \( \Phi_{yq}, \Phi_{yp}, \Phi_{qp}, \Phi_{pq} \), while the rest of the components of \( \Phi \) are obtained with (4.24).

Once the \((1,1)\)-tensor \( \Phi \) is known, it should be verified that the condition (4.17a) is fulfilled. Then the metric tensor \( g' \) is determined with (4.27) and we have to impose the condition that the fundamental quadratic form associated to the tensor \( g' \) must be positive definite.

Remark 2. In order to compare the metric matrices (2.12) and (4.27) on \( \tilde{X}_1^J \) we have to make the replacement \( k, \nu \rightarrow \sqrt{k}, \sqrt{\nu} \) in (2.12) due to the parametrization (2.1) of the one-forms in (2.10), and we get instead of (2.12)

\[
ds^2_{\tilde{X}_1^J}(x, y, p, q, \kappa) = \nu(p \, dq^2 + q \, dp^2 - 2pq \, dq \, dp) + \delta \, d\kappa^2 + 2 \frac{\sqrt{k\delta}}{y} \, dx \, d\kappa
\]

in the convention (2.10).

Also it would be interesting to check that the metric \( g_{\tilde{X}_1^J}(x, y, p, q, \kappa) \) (4.27) cannot be obtained from a potential type (5.36).

Proposition 6. \((\tilde{X}_1^J, \eta_0)\) is a contact Hamiltonian system as in Appendix, C.

Using the same parametrization (4.6) and (4.7) in Section 4.1 we get for \( X_H \) (5.16) the expression

\[
X_H = \frac{y^2}{k} \frac{\partial H}{\partial y} \frac{\partial}{\partial x} + \frac{1}{2\nu} \frac{\partial H}{\partial p} \frac{\partial}{\partial q} + \left( -\frac{y^2}{k} \frac{\partial H}{\partial x} + y \frac{\partial H}{\partial \kappa}\right) \frac{\partial}{\partial y} \\
- \left( \frac{1}{2\nu} \frac{\partial H}{\partial q} + p \frac{\partial H}{\partial \kappa}\right) \frac{\partial}{\partial p} + \left( -y \frac{\partial H}{\partial y} + p \frac{\partial H}{\partial \kappa} - H\right) \frac{\partial}{\partial \kappa}.
\]

The equations of motion associated to the vector field \( X_H \) (4.38) on the extended Sigel–Jacobi upper half-plane organized as the contact manifold \((\tilde{X}_1^J, \iota_0)\) are

\begin{align}
\dot{x} &= \frac{y^2}{k} \frac{\partial H}{\partial y}, & \dot{y} &= -\frac{y^2}{k} \frac{\partial H}{\partial x} + y \frac{\partial H}{\partial \kappa}, \\
\dot{q} &= \frac{1}{2\nu} \frac{\partial H}{\partial p}, & \dot{p} &= -\left( \frac{1}{2\nu} \frac{\partial H}{\partial q} + p \frac{\partial H}{\partial \kappa}\right), \\
\dot{\kappa} &= \left( -y \frac{\partial H}{\partial y} + p \frac{\partial H}{\partial \kappa} - H\right) \frac{\partial H}{\partial \kappa}.
\end{align}
The Jacobi bracket (5.25) on $\tilde{X}_J^1$ is

$$\{f, g\} = \{f, g\}_P + f \frac{\partial g}{\partial \kappa} - g \frac{\partial f}{\partial \kappa},$$

where the Poisson bracket (5.4) with the convention (4.7) reads

$$\{f, g\}_P = \frac{1}{k} \frac{y^2 + 1}{y^2} \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right) + \frac{1}{2\nu} \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p} + \frac{1}{y^2} \left( \frac{\partial f}{\partial q} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial y} \right).$$

while the Euler operator (5.23) is

$$f_\epsilon = f + \frac{1}{y^3} \frac{\partial f}{\partial y} - p \frac{\partial f}{\partial p}.$$

Remark 3. If in the equations of motion expressed in the $S$-variables $(x, y, p, q, \kappa)$ in (4.9) (respectively (4.39)) on the generalized transitive almost cosymplectic manifold $(\tilde{X}_J^1, \theta, \omega)$ (respectively, on the contact manifold $(\tilde{X}_J^1, \eta_0)$) we ignore the “red” (respectively “green”) parts we get the equations of motion on $X_J^1$ in $(x, y, p, q)$.

4.3. Linear Hamiltonian in the generators of the Jacobi group $G_J^1(R)$. In [9, (4.7)] we have considered a linear Hermitian Hamiltonian $H$ in the generators of the Jacobi group $G_J^1$

$$H = \epsilon_a a + \bar{\epsilon}_a a^\dagger + \epsilon_0 K_0 + \epsilon_+ K_+ + \epsilon_- K_-, \quad \bar{\epsilon}_+ = \epsilon_-, \quad \bar{\epsilon}_0 = \epsilon_0.$$  

We use the notation introduced in [9, § 4.1.3.]

$$\epsilon_a := a + i b, \quad \epsilon_+ := m - i n, \quad \epsilon_0 := 2c, \quad a, b, c, m, n \in \mathbb{R}.$$  

We have proved in [9, (4.29) in § 4.3] that the energy function $\mathcal{H}$ associated to the linear Hamiltonian (4.40) expressed in the variables $(\eta, v)$ splits into the sum of two independent functions

$$\mathcal{H}(\eta, v) = \mathcal{H}(\eta) + \mathcal{H}(v), \quad v = x + i y, \quad y > 0, \quad \eta = q + i p,$$

where

$$\mathcal{H}(q, p) = \nu [(m + c)q^2 + (c - m)p^2 + 2nqp + 2(aq + bp)],$$

$$\mathcal{H}(x, y) = k \left\{ \frac{1}{y} [(m + c)(x^2 + y^2) - 2(nx + cy)] + 3c - m \right\}.$$  

We have underlined in [3, Remark 1], [14, Proposition 3] the connection of parameter $\eta = q + i p$ which appear in the FC-transform with the $S$-variables $(p, q)$ which parametrize the Jacobi group $G_J^1(R)$.

Now we particularize equations (4.9) to the linear Hamiltonian (4.41) to which we add a term $h(\kappa)$

$$H = \mathcal{H}(p, q) + \mathcal{H}(x, y) + h(\kappa),$$

and we get
Proposition 7. The equations of motion (3.7) on the extended Siegel-Jacobi upper half-plane organized as generalized transitive almost cosymplectic manifold \((\tilde{X}_1^J, \theta, \omega)\) corresponding to the energy function (4.43) are

\begin{align}
\dot{x} &= (c + m)(-x^2 + y^2) + mx - c + m, \\
\dot{y} &= -2(c + m)y^2 + 2ny,
\end{align}

\begin{align}
\dot{q} &= - (m + c)q - np - a - \frac{q}{2\nu} \frac{\partial h}{\partial \kappa}, \\
\dot{p} &= qn + (c - m)p + b - \frac{p}{2\nu} \frac{\partial h}{\partial \kappa},
\end{align}

\begin{align}
\dot{\kappa} &= (c + m)qa^2 + (-c + m)p^2 + (m - n)pq + nq + bp - \frac{1}{\sqrt{\delta}} h.
\end{align}

Remark 4. If in the equations of motion (4.44) on \(\tilde{X}_1^J\) generated by the linear Hamiltonian (4.43) we ignore the “red parts”, the well-known matrix Riccati equation (4.44a) in \((x, y)\) and the linear system of differential equations in \((p, q)\) generated by the linear Hamiltonian (4.41), (4.42) on \(X_1^J\) found in [9] are obtained.

5. Appendix – a breviar of terminology

In this section are collected definitions of the main geometric structures used in our paper. We use the following abbreviations:

SH: symplectic Hamiltonian; COS: cosymplectic; ACOS: almost cosymplectic, GTACOS: generalized transitive almost cosymplectic, TACS: transitive almost contact structure, CH: contact Hamiltonian, C: contact, ACM: almost contact metric, SAC: (strict) almost contact, N: normal, SAS: Sasakian, ACOK: almost coKähler.

| COS \(\subset\) GTACOS \(\supset\) TACS \(\cap\) ACOS \(\cup\) CH \(\subset\) C \(\subset\) ACM \(\subset\) SAC \(\supset\) N=SAS \(\cap\) ACOK |

In [16], Chapter III, Symplectic manifolds and Poisson manifolds: a symplectic Hamiltonian system – is the pair \((M, \Omega)\), where \(\dim M = 2n, n \in \mathbb{N}\), \(\Omega\) is a closed non-degenerate two-form, and

\[\Omega^n \neq 0.\]

If \(H : M \to \mathbb{R}\) is a Hamiltonian function, then the Hamiltonian vector field \(X_H\), \(\text{grad} H\), is the solution of the equation

\[\flat(X_H) = dH, \quad \text{where} \quad \flat : TM \to T^*M, \quad \flat(X) = X \omega.\]

In canonical Darboux coordinates \((q^i, p_i)\), \(i = 1, \ldots, n\), we have

\[\Omega = d q^i \wedge d p_i,\]

\[X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.\]
and Hamilton equations of motion are
\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = \{p_i, H\}_P, \]
where the Poisson bracket \( \{f, g\}_P \) in Darboux coordinates is
\[ \{f, g\}_P = \Omega(X_f, X_g) = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i}, \quad f, g \in C^\infty(M). \]

Note that if in (5.4) we make the change of coordinates \((q, p) \rightarrow (p, q)\), then \( \{f, g\}_P \rightarrow -\{f, g\}_P \).

ACOS
In [45]: almost cosymplectic manifold – is the triplet \((M, \theta, \Omega)\), where \(M\) is a \((2n+1)\)-
dimensional manifold, \(\theta \in \mathfrak{D}^1\), \(\Omega\) is a 2-form with \(\text{rank}(\Omega) = 2n\), and
(5.5) \( \theta \wedge \Omega^n \neq 0. \)

The Reeb vector \(R \in \mathfrak{D}^1\) is defined by the equations
(5.6) \( R \cdot \Omega = 0, \quad R \cdot \theta = 1. \)

In [1]: the almost cosymplectic manifold \((M, \theta, \Omega)\) of [45] is called almost contact manifold.
It is proved in [1, Proposition 1] that the application \(\flat : TM \rightarrow TM^*\)
(5.7) \( X \rightarrow X^\flat = X \cdot \Omega + (X \cdot \theta)\theta \)
is a vector bundle isomorphism.

COS
In [45]: cosymplectic manifold – is the triplet \((M, \theta, \Omega)\), defined in ACOS, where
\( d\theta = 0, \quad d\Omega = 0. \)

In [33]: the isomorphism \(\flat\) for the cosymplectic manifold \((M, \eta, \Omega)\) is defined as in
(5.7).
The Darboux coordinates are \((z, q^i, p_i)\), \(i = 1, \ldots, n\), \(\Omega\) is defined in (5.2), and
\( \eta = dz. \)
The gradient vector field \(\text{grad} H\), the Hamiltonian vector field \(X_H\) and the evolution
vector field \(\mathcal{E}_H\) attached to the function \(H\) are defined as solutions of the equations:
(5.8a) \( \flat(\text{grad} H) = dH, \)
(5.8b) \( X_H = \text{grad} H - R(H)R, \)
(5.8c) \( \mathcal{E}_H = X_H + R. \)

TACS
In [1, 29]: transitive almost contact structure - is an almost contact manifold \((M, \theta, \Omega)\)
defined at ACOS with
\( d\Omega = 0, \)
and around every point of \( M \) there is a neighbourhood where there are local Darboux coordinates \((\kappa, q^1, \ldots, q^n, p_1, \ldots, p_n)\) such that

\[
\theta = d\kappa + \epsilon p_i \, dq^i, \quad i = 1, \ldots, n, \; \epsilon \in \mathbb{R}.
\]

To a function \( f \in C^\infty(M) \) it is associated the Hamiltonian vector field \( X_f \) defined by \[1\] (3)] as solution of the equations

\[
\begin{align*}
X_f \cdot \theta &= \epsilon f, \\
X_f \cdot \Omega &= df - (Rf) \theta.
\end{align*}
\]

**CH**

In \[33, 34\]: contact Hamiltonian system \((M, \eta)\)– is defined as the almost cosymplectic manifold \((M, \eta, d\eta)\) and (5.5) verified

\[
\eta \wedge d\eta^n \neq 0.
\]

Apparently the denomination contact structure for a manifold \((M, \eta)\) verifying the condition (5.11) was used firstly by Gray \[38\].

\( \theta \) defined in (5.9) with \( \epsilon = -1 \) corresponds to Darboux coordinates and in accord with [25, Theorem, page 1] \( \eta \) can be taken

\[
\eta = d\kappa - p_i \, dq^i, \quad d\eta = dq^i \wedge dp_i.
\]

The conditions (5.6) defining the Reeb vector became

\[
R \cdot d\eta = 0, \quad R \cdot \eta = 1,
\]

where

\[
R = \frac{\partial}{\partial \kappa}.
\]

The Hamiltonian vector field \( X_H \) attached to the real function \( H \) is defined by \[33\] (2.11)]

\[
\flat(X_H) = dH - (R \cdot H + H)\eta,
\]

where the Reeb vector \( R \) is defined in (5.14) and the application \( \flat \) is defined in (5.7).

The vector field \( X_H \) in Darboux coordinates reads \[33\] (2.12)]

\[
X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} \right) \frac{\partial}{\partial p_i} + p_i \frac{\partial H}{\partial p_i} - H \frac{\partial}{\partial \kappa}.
\]

We determine the vector field \( \text{grad} \, H \) solution of equation (5.8a)

\[
\text{grad} \, H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} \right) \frac{\partial}{\partial p_i} + \frac{\partial H}{\partial \kappa} + p_i \frac{\partial H}{\partial p_i} R,
\]

but instead of (5.8b), we have

\[
X_H = \text{grad} \, H - (H + R(H)) R.
\]

If in equations (5.16) and (5.17) we neglect the “green parts”, we get \( X_H \) \[53\] on the symplectic manifold \((M, \Omega)\).

In order to recall the notion of Jacobi bracket \[33, 47, 52\], we follow \[33\] §4, Contact manifolds as Jacobi structures].
The isomorphism (5.7) becomes

$$X \rightarrow X^\flat = X \lrcorner d\eta + (X \lrcorner \eta)\eta.$$  

If

$$X = A_i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial p^i} + C \frac{\partial}{\partial \kappa},$$

then

$$X^\flat = \alpha_i dq^i + \beta_i dp_i + \gamma d\kappa,$$

where

\begin{align*}
(5.20a) & \quad \flat : \quad \alpha_i = -B_i + p_i(p_j A_j - C), \quad \beta_i = A_i, \quad \gamma = C - p_j A_j, \\
(5.20b) & \quad \sharp = \flat^{-1} : \quad A_i = \beta_i, \quad B_i = -\alpha_i - \gamma p_i, \quad C = p_i \beta_i + \gamma.
\end{align*}

We have to calculate the bracket \{f, g\}_J \text{[33] p. 11}

\begin{equation}
\{f, g\}_J = -d\eta(\sharp d f, \sharp d g) - R(g)f + gR(f).
\end{equation}

With (5.17), (5.19), (5.20), we find

Remark 5. The Jacobi bracket (5.21) has the expression

\begin{equation}
\{f, g\}_J = \{g, f\}_P + \frac{\partial f}{\partial \kappa} g_e - \frac{\partial g}{\partial \kappa} f_e,
\end{equation}

where \(f_e\) denotes Euler's operator \[2\]

\begin{equation}
f_e := f - p_i \frac{\partial f}{\partial p_i}.
\end{equation}

In [46] Chapter V, Contact manifolds] the conventions expressed in (5.11), . . . , (5.14) are used.

Instead of (5.1) it is used [46] Chapter V, Proposition 6.13 page 293]

\begin{equation}
\eta^\flat : X \rightarrow -X \lrcorner d\eta,
\end{equation}

which associates to vector fields semi-basic differential forms [46] pages 56, 68]. Also it is used the notation

$$\eta^\sharp(\gamma) = \iota^\ast \gamma, \quad \gamma \in \mathcal{D}_1.$$ 

Note that in [45] Lemma p 44] instead of (5.24) it is used

$$\eta^\flat : X \rightarrow X \lrcorner d\eta.$$ 

According to [46] Proposition 6.11 p 292]: We have decomposition of the tangent space \(TM\) into the direct sum

$$TM = \text{Ker} \ d\eta \oplus \text{Ker} \ \eta,$$

of the vertical bundle of rank 1 generated by the Reeb vector (5.14) and the horizontal bundle of rank 2n

$$X = (X \lrcorner \eta)R + (X - (X \lrcorner \eta))R.$$ 

According to [46] Proposition 13.1 p 318]: The vector field \(X\) on the contact manifold \((M, \eta)\) is an infinitesimal contact automorphism if and only if there exists a differentiable function \(\rho\) such that

$$\mathcal{L}_X \eta = \rho \eta.$$
According to [46, Theorem 13.3 p 319]: The choice of a contact form $\eta$ on the strictly contact manifold $(M, \eta)$ defines an isomorphism $\Phi : L \to L'$

$$\Phi(X) = X \cdot \eta, \quad X_f = \Phi^{-1}(f) = fR + \eta^i(d f - (R \cdot d f)\eta), \quad \rho = R \cdot d f,$$

where $L$ ($L'$) is the space of the infinitesimal automorphisms of $\eta$ (respectively differentiable real-valued functions on $M$).

According to [46, Proposition 14.2 p 325]: The Lie algebra structure of $L'$ is defined by the bracket

\begin{align}
\{f, g\} &= [X_f, X_g] \cdot \eta, \\
&= d \eta(X_f, X_g) + f(R \cdot d g) - g(R \cdot d f), \\
&= \{f, g\}_{P} + f \frac{\partial g}{\partial \kappa} - g \frac{\partial f}{\partial \kappa}.
\end{align}

Remark 6. Note that the expression of the Jacobi bracket (5.25) is minus the expression (5.22)

$$\{f, g\} = -\{f, g\}_{J},$$

and coincides with the expression [1, \{f, g\} on p 636] for $\epsilon = -1$.

SAC

In [12]: following Sasaki [50] and [28, Definition 6.2.5], we used the

Definition 1. The manifold $M_{2n+1}$ has a (strict) almost contact structure $(\Phi, \xi, \eta)$ if there exists a $(1, 1)$-tensor field $\Phi$, the contravariant vector field $\xi \in \mathcal{D}^1$ (Reeb vector field, or characteristic vector field), and the covariant vector field $\eta \in \mathcal{D}_1$ verifying the relations

\begin{align}
\eta \cdot \xi &= 1, \quad \Phi^2 X = -X + \eta(X)\xi.
\end{align}

C

Following [27], in [12] we used the definition of a contact structure $(M_{2n+1}, \eta)$ when $\eta \in \mathcal{D}_1$ satisfies (5.11).

The contact structure can be given by a codimension one subbundle $\mathcal{D}$ of the tangent bundle $TM$ which is as far from being integrable as possible, and $\mathcal{D} := \text{Ker}(\eta)$.

ACM

Sasaki has proved [50, Theorem 1.1] and [51, (5.16)]:

**Proposition 8.** For an almost contact structure $(\Phi, \xi, \eta)$, the following relations hold

\begin{align}
\Phi \xi &= 0, \quad \eta \Phi = 0, \quad \text{Rank}(\Phi^i_j) = 2n, \quad \xi, \eta^i \in M(n, 1, \mathbb{R}).
\end{align}

There exists a positive Riemannian metric $g$ such that

\begin{align}
\eta = g\xi, \quad \Phi^i g \Phi = g - \eta^i \otimes \eta.
\end{align}

If we put

\begin{align}
\hat{\Phi} := g \Phi,
\end{align}

the two-form $\hat{\Phi}$ is antisymmetric.

Sasaki has proved [50, Theorem 3.1], [51], see also [12, (9.12 ), Theorem 14]:
**Proposition 9.** Let \((M, \eta)\) be a contact manifold. Then we can find an almost contact metric structure \((M, \Phi, \xi, \eta, g)\) such that (5.29) is satisfied,
\[
\text{d} \eta(X, Y) = g(X, \Phi(Y)),
\]
and
\[
\text{d} \eta = \frac{1}{2} \sum_{i,j=1}^{2n+1} \tilde{\Phi}_{ij} \, \text{d} x^i \wedge \text{d} x^j = \sum_{1 \leq i < j \leq 2n+1} \tilde{\Phi}_{ij} \, \text{d} x^i \wedge \text{d} x^j,
\]
\(\tilde{\Phi}_{ij} = \partial_i \eta_j - \partial_j \eta_i\).

\(\text{N=SAS}\)

Following [25, p. 47], let us introduce

**Definition 2.** Let \(h\) be a \((1,1)\)-tensor field. Then the \(Nijenhuis torsion\) \([h, h]\) of \(h\) is the tensor field of type \((1,2)\) given by
\[
[h, h](X, Y) = h^2[X, Y] + [hX, hY] - h[hX, Y] - h[X, hY], \quad X, Y \in \mathfrak{D}^1.
\]

Let us define the \((1,2)\)-tensor
\[
N^1 := [\Phi, \Phi] + 2 \, \text{d} \eta \otimes \xi.
\]
According to [28, Theorem 6.5.9]

**Proposition 10.** An almost contact structure \((\xi, \eta, \Phi)\) on \(M\) is normal if and only if \(N^1 = 0\).

A normal contact metric structure \((M, \xi, \eta, \Phi, g)\) is called a \textit{Sasakian} structure, see more details [28, Definitions 6.4.7, 6.5.7, 6.5.13].

Under the conditions (5.26), (5.28), but instead of (5.30) and (5.32) with
\[
\text{d} \eta(X, Y) = g(\Phi(X), Y),
\]
\[
N^1 := [\Phi, \Phi] + \text{d} \eta \otimes \xi,
\]
in local coordinates \((\kappa, z^1, \ldots, z^n)\) on a small neighbourhood \(\mathbb{R} \times \mathbb{C}^k\) of a Sasaki manifold \((M_{2n+1}, \xi, \eta, \Phi, g)\) according with [37, Theorem 1], see also [53], we have
\[
\text{(5.34a)} \quad \xi = \frac{\partial}{\partial \kappa},
\]
\[
\text{(5.34b)} \quad \eta = \text{d} \kappa + i \sum_{j=1}^{n} (K_j \text{d} z^j - K_j \text{d} \bar{z}_j),
\]
\[
\text{(5.34c)} \quad \text{d} \eta = -2 i \sum_{j,k=1}^{n} K_{jk} \text{d} z^j \wedge \text{d} \bar{z}^k,
\]
\[
\text{(5.34d)} \quad g = \eta \otimes \eta + 2 \sum_{j,k=1}^{n} K_{jk} \text{d} z^j \otimes \text{d} \bar{z}^k,
\]
\[
\text{(5.34e)} \quad \Phi = -i \sum_{j=1}^{n} (\partial_j - K_j \partial_\kappa) \otimes \text{d} z^j + i \sum_{j=1}^{n} (\partial_j - K_j \partial_\kappa) \otimes \text{d} \bar{z}^j,
\]
where the \textit{Sasaki potential} \(K\) does not depend on \(\kappa\).
However, using (5.32) and (5.30), equation (5.34d) should be replaced with

\[(5.35)\]
\[g = \eta \otimes \eta - 4 \sum_{j,k=1}^{n} K_{jk} \, dz^{j} \, d\bar{z}^{k}.\]

In the holonomic cobasis \((d y^{\mu}) = (d \kappa, d z^{j}, d \bar{z}^{\bar{j}})\) the covariant components of the Sasakian metric are \([37, (15)]\)

\[(5.36)\]
\[g_{\mu\nu} = \begin{pmatrix} 1 & i K_{j} & -i K_{\bar{j}} \\ i K_{\bar{i}} & -K_{i}K_{\bar{j}} & K_{ij} + K_{\bar{i}}K_{\bar{j}} \\ -i K_{\bar{i}} & K_{ij} - K_{\bar{i}}K_{\bar{j}} \end{pmatrix}.\]

In the conventions used in [12, § 9.3] for the Heisenberg group \(H_1\), we make the

**Remark 7.** The Sasaki potential for the Heisenberg group \((H_1, \xi, \eta, \Phi, g_{H_1})\) has the expression

\[(5.37)\]
\[K_{H_1}(z, \bar{z}) = \frac{1}{8}(z - \bar{z})^2 = -\frac{y^2}{2}, \quad \mathbb{C} \ni z = x + iy, \ x, y \in \mathbb{R},\]

corresponding to the left invariant metric

\[(5.38)\]
\[g_{H_1} = dx^2 + dy^2 + \eta^2, \quad \eta = d\kappa - y \, dx,\]

and

\[\Phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}.\]

The Kähler potential for the manifold \(H_1/\mathbb{R} \approx \mathbb{C}\) corresponding to the scalar product of two coherent state vectors \(e_z\) is

\[(5.39)\]
\[\mathcal{K}_{H_1}(z, \bar{z}) := (e_z, e_z) = \exp(z\bar{z}).\]

**Proof.** We recall also [12, Remark 18, Propositions 25, 26].

We apply (5.36) for the Heisenberg group \(H_1\) with the left invariant metric (5.38) [12, Section 9.3].

We apply formulas (5.34) with (5.34d) replaced with (5.35) to the Sasaki potential (5.37).

Formula (5.39) is extracted from [12, (7.75)]. \(\Box\)

**ACOK**

In [30]: the notion of almost cosymplectic manifold is adopted as in ACOS, the definition of cosymplectic manifold is adopted as in COS, the notion of almost contact structure is introduced as in Definition 1.

An almost coKähler manifold - is an almost contact metric manifold \((M, \Phi, \xi, \eta, g)\) such that the two-form \(\Phi\) (5.29) and \(\eta \in \mathcal{D}_1\) are both closed. If in addition the almost contact structure is normal, \(M\) is called a coKähler manifold.

According to [30, Theorem 3.9]

**Proposition 11.** The almost contact manifold \((M, \Phi, \xi, \eta, g)\) is coKähler if and only if \(\nabla^g \Phi = 0\) or equivalently \(\nabla^g \Phi = 0\).
The same definition of co-symplectic/co-Kähler manifold used by [30] is adopted in the review paper [44].

Apparently the denomination of co-Kähler manifold was used firstly in [5].

In [26]: Blair used the term cosymplectic manifold denoting a cosymplectic manifold in the sense of Libermann [45] endowed with a compatible almost contact metric structure as in $N=SAS$ satisfying the normality condition, i.e. the coKähler manifold of [30].

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