Factorization of Graded Traces on Nichols Algebras

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Abstract

We study the factorization of the Hilbert series and more general graded traces of a Nichols algebra into cyclotomic polynomials. Nichols algebras play the role of Borel subalgebras of finite-dimensional quantum groups, so this observation can be viewed as an analog to the factorization of the order of finite Lie groups into cyclotomic polynomials. We prove results on this factorization and give a table of many examples of rank 1 over nonabelian groups.

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1 Introduction

A Nichols algebra $\mathcal{B}(M)$ over a finite group is a graded algebra and in particular a braided Hopf algebra, that plays the role of a Borel subalgebra of a finite-dimensional quantum group. For example, the Borel subalgebras $u_q^\pm(\mathfrak{g})$ of the Frobenius-Lusztig kernels $u_q(\mathfrak{g})$ are finite-dimensional Nichols algebras over quotients of the root lattice of $\mathfrak{g}$ as an abelian group.

In this article we discuss the following curious phenomenon, that is apparent throughout the ongoing classification of finite-dimensional Nichols algebra: The graded dimension or Hilbert series of any finite-dimensional Nichols algebra known so far factorizes as a polynomial in one variable into the product of cyclotomic polynomials. For Nichols algebras over abelian groups, this factorization can be explained (see below), but this mechanism cannot explain the empirically observed complete factorization of the Hilbert series of Nichols algebras over nonabelian groups.

However, it has been shown in a series of joint papers of the second author ([10], [13], [14]) that the existence of such a complete factorization into cyclotomic polynomials implies strong bounds on the number of relations in low degree in the Nichols algebra. On the other hand, the first author has obtained in [19] new families of Nichols algebras over nonabelian groups, where such a complete factorization appears by construction.

A similar factorization into cyclotomic polynomials appears for the order of any finite simple group of Lie type, which is due to the action of
the Weyl group on the space of polynomial functions on the root lattice ([6], Sec. 2.9). Moreover, the existence of such a factorization continues to characters of unipotent representations ([6], Sec 13.7–13.9). This observation may be less surprising, due to the fact that Lusztig has related the representation theory of $u_q(\mathfrak{g})$ over $\mathbb{C}$ to the representation theory of the corresponding Lie group over a finite field. The precise connection seems however unclear at this point. At least we shall see below that the Weyl-group theory of Nichols algebras indeed generally implies a factorization of the Hilbert series and also of graded characters, but for nonabelian groups, this is not the finest factorization we observe.

More concretely, a key observation in this article is that in the available examples not only the Hilbert series itself (the graded trace of the identity), but arbitrary graded characters of the group acting on the Nichols algebras factorize into cyclotomic polynomials. The aim of the present article is to gather and prove basic facts about this factorization, calculate a comprehensive list of examples, and start with providing systematic factorization mechanisms.

The content of this article is as follows:

In Section 2 we define graded traces and prove the relevant basic facts about graded traces on finite-dimensional Nichols algebras, including additivity and multiplicativity with respect to the representations, rationality, and especially Poincaré duality. Most of these contents have appeared in literature in some form, we gather them here for convenience. We also give an example for a graded representation, where the graded characters factor in a nice way which is not reflected by the representation itself. This phenomenon will occur when we study the characters of Nichols algebras over non-abelian groups.

In Section 3 we study factorization mechanisms for Nichols algebras, and hence for their graded traces and especially their Hilbert series. The root system $\Delta^+$ of a Nichols algebra $\mathcal{B}(M)$ in the sense of [4] directly presents a factorization of $\mathcal{B}(M)$ as graded vectorspace: This completely explains the factorization of the graded trace of an endomorphism $Q$ that respects the root system grading.

For Nichols algebras over abelian groups, we use the theory of Lyndon words to significantly weaken the assumption on $Q$. On the other hand, we give an example of an endomorphism (the outer automorphism of $A_2$ containing a loop) where this mechanism fails; a factorization of the graded trace is nevertheless observed and can be tracked to the surprising existence of an alternative “symmetrized” PBW-basis. The formulae appearing involve the orbits of the roots under $Q$ and are in resemblance to the formulae
given in [6] Sec. 13.7 for finite Lie groups.

For Nichols algebras over nonabelian groups, the root system factorization discussed above still applies, but is too crude in general to explain the full observed factorization into cyclotomic polynomials. We give however a family of examples constructed as covering Nichols algebras by the first author in [20], where the complete factorization can be tracked back to a finer PBW-basis. The key ingredient is an additional root system in a symplectic vector space over $\mathbb{F}_2$ (see [21]), a structure appearing as well in the representation theory of finite Lie groups, see [6] Sec. 13.8.

Finally we give results on the divisibility of the Hilbert series derived by the second author in [22]. By the freeness of the Nichols algebra over a sub-Nichols algebra ([25], [8]) one can derive a divisibility of the graded trace $\text{tr}^g$ by that of a sub-Nichols algebra. Moreover, in many cases there is a shift-operator $\xi_x$ for some $x \in G$, which can be used to prove that there is an additional cyclotomic divisor of the graded trace. This holds in particular for $x \in G$ commuting with $g$.

Section 4 finally displays a table of graded characters for all known examples of finite-dimensional Nichols algebras of non-abelian group type and rank 1, which were computationally accessible to us. With respect to the previous discussion, the rank 1 case is particularly challenging, as the Weyl group is trivial. We observe that again all graded characters factorize in these examples.

## 2 Hilbert Series and Graded Traces

### 2.1 The Graded Trace $\text{tr}^Q_V$

In the following we suppose $V = \bigoplus_{n \geq 0} V_n$ to be a graded vector space with finite-dimensional layers $V_n$. We frequently call a linear map $Q : V \to V$ an operator $Q$. We denote the identity operator by $1_V = \bigoplus_{n \geq 0} 1_{V_n}$ and the projectors to each $V_n$ by $P_n$. An operator $Q$ is called graded if one of the following equivalent conditions is fulfilled:

- $Q$ is commuting with all projections $P_n$.
- $Q$ preserves all layers $V_n$.

We denote the restriction of a graded operator $Q : V \to V$ to each $V_n$ by $Q_n \in \text{End}(V_n)$. An operator $Q$ is called algebra operator resp. Hopf algebra operator if $V$ is a graded algebra resp. Hopf algebra and $Q$ is an algebra resp. Hopf algebra morphism.
Definition 2.1. For a graded vector space $V$ with finite-dimensional layers define

- the Hamilton operator $E \in \text{End}(V)$ by $E|_{V_n} = n$, and
- the Boltzmann operator $t^E \in \text{End}(V)[[t]]$ as $\text{End}(V)$-valued series $t^E = \sum_{n \geq 0} t^n \cdot P_n$, i.e. $t^E|_{V_n} = t^n$.

Remark 2.2. Let $V$ be a graded algebra, then the Boltzmann operator is an automorphism of the graded algebra $V[[t]]$.

Note that the trace of the Boltzmann operator is the Hilbert series $\text{tr}(t^E) = \sum_{n \geq 0} \dim(V_n) t^n =: \mathcal{H}(t)$. If $V$ is finite-dimensional, $\mathcal{H}(t)$ is a polynomial and has a well-defined value at $t = 1$, which turns out to be the dimension:

$$\text{tr}(t^E)|_{t=1} = \text{tr}(1^E) = \text{tr}(1_V) = \dim(V)$$

Definition 2.3. For $V$ a graded vector space with finite-dimensional layers and $Q$ a graded operator, define the graded trace of $Q$ as the series $\text{tr}_V^Q(t) := \text{tr}(t^E \cdot Q) = \sum_{n \geq 0} t^n \cdot \text{tr}(Q_n)$ in the variable $t$.

Example 2.4. In particular the identity $Q = 1_V$ has as its graded trace $\text{tr}_V^1(t)$ again the Hilbert series $\text{tr}(t^E) = \mathcal{H}(t)$. More generally, for $\lambda \in k^\times$, the scaling operator $Q = \lambda 1_V$ has graded trace $\text{tr}_V^\lambda(t) = \text{tr}(t^E \cdot \lambda 1_V) = \lambda \cdot \text{tr}(t^E) = \lambda \cdot \text{tr}_V^1(t) = \lambda \cdot \mathcal{H}(t)$.

Remark 2.5. Assume $V$ is finite-dimensional. Then only finitely many layers $V_n$ are non-trivial. Thus $\text{tr}_V^Q(t)$ is a polynomial in the formal variable $t$ with $\text{tr}_V^Q(1) = \text{tr}(Q)$. If $A$ is an infinite-dimensional algebra, it is proven in [18] that $\text{tr}_A^Q$ is a rational function if either

- $A$ is commutative and finitely generated,
- $A$ is right noetherian with finite global dimension, or
- $A$ is regular.

Lemma 2.6. The graded trace $\text{tr}_V^Q(t)$ is linear in $Q$.

Lemma 2.7. For graded vector spaces $V, W$ with finite-dimensional layers, the sum $V \oplus W$ and product $V \otimes W$ again are graded vector spaces with finite-dimensional layers. The respective codiagonal grading by definition implies

$$t^{E_{V \oplus W}} = t^{E_V} \oplus t^{E_W} \quad \text{and} \quad t^{E_{V \otimes W}} = t^{E_V} \otimes t^{E_W}.$$
Then the following properties for the graded trace hold immediately from the respective properties of the trace:

$$\text{tr}_{V \oplus W}^{Q \oplus R}(t) = \text{tr}_{V}^{Q}(t) + \text{tr}_{W}^{R}(t) \quad \text{and} \quad \text{tr}_{V \otimes W}^{Q \otimes R}(t) = \text{tr}_{V}^{Q}(t) \cdot \text{tr}_{W}^{R}(t).$$

**Remark 2.8.** The Boltzmann operator $t^E$ is used in a thermodynamic ensemble to stochastically average over eigenspaces of the Hamiltonian $E$ weighted by the eigenvalue of $E$ (“energy”) and depending on a free parameter $t$ (for $t = e^{-\beta}$ with $\beta$ the “inverse temperature”), see [24] Chp. 3, especially 3.4.50–3.4.53.

Equivalently, after a so-called Wick rotation, the Boltzmann operator is the time-development operator in a quantum field theory, see [24] 2.6.21:

$$U(T_1, T_2) = U(T_2 - T_1) = t^E, \quad t = e^{-i\Delta T/\hbar}$$

The trace of $t^E$ is called partition function $Z$ and yields the Hilbert series is in our situation. To yield a probability distribution one normalizes the Boltzmann operator $t^E$ by the partition function to obtain trace $1$ and the result is known as the quantum density operator $\rho := t^E / \text{tr}(t^E)$. Finally, then up to the factor $Z$, the graded trace is the quantum expectation value

$$\langle Q \rangle := \text{tr}(\rho Q) = \frac{\text{tr}(t^E Q)}{\text{tr}(t^E)} = \frac{\text{tr}_{V}^{Q}(t)}{\mathcal{H}(t)}.$$

### 2.2 Poincaré Duality

The Hilbert series of a Nichols algebra exhibits a Poincaré Duality, see e.g. [23] Rem. 2.2.4. We generalize this approach to the calculation of arbitrary algebra operator traces in Nichols algebras:

**Lemma 2.9.** Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra with top degree $L$ and integral $\Lambda$. Let $Y$ be an arbitrary algebra automorphism of $\mathfrak{B}(M)$ and the scalar $\lambda_{Y} \in k^\times$ such that $Y \Lambda = \lambda_{Y} \cdot \Lambda$. Then $\text{tr}(Y) = \lambda_{Y} \cdot \text{tr}(Y^{-1})$.

**Proof.** Let $\{b_{j}\}_{j \in J}$ and $\{b_{j}^{*}\}_{j \in J}$ be two bases of $\mathfrak{B}(M)$ with $b_{j}^{*} b_{j} = \delta_{ij} \cdot \Lambda$ for all $i,j \in J$. Then holds

$$\text{tr}(Y) \cdot \Lambda = \sum_{j \in J} \langle Y b_{j}^{*} \rangle b_{j} = Y \sum_{j \in J} b_{j}^{*} (Y^{-1} b_{j}) = Y \text{tr}(Y^{-1}) \cdot \Lambda$$

$$= \text{tr}(Y^{-1}) \cdot Y \Lambda = \lambda_{Y} \cdot \text{tr}(Y^{-1}) \cdot \Lambda.$$
Corollary 2.10. Let $\mathcal{B}(M)$ be a finite-dimensional Nichols algebra with top degree $L$ and integral $\Lambda$. Let $Q$ be an arbitrary algebra automorphism of $\mathcal{B}(M)$ and the scalar $\lambda_Q \in \mathbb{k}^\times$ such that $Q\Lambda = \lambda_Q \cdot \Lambda$. Then 
\[
tr^Q_{\mathcal{B}(M)}(t) = \lambda_Q \cdot t^L \cdot \tr^{-1}_{\mathcal{B}(M)}(t^{-1})
\]

Proof. We apply Lemma 2.9 to $Y = t^E Q$ and $\lambda_Y = t^L \lambda_Q$:
\[
\begin{align*}
\tr^Q_{\mathcal{B}(M)}(t) &= \tr(t^E Q) = t^L \lambda_Q \cdot \tr\left(\left(t^E Q\right)^{-1}\right) = t^L \lambda_Q \cdot \tr\left(Q^{-1} \left(t^{-1}\right)^E\right) \\
&= t^L \lambda_Q \cdot \tr\left(\left(t^{-1}\right)^E Q^{-1}\right) = \lambda_Q \cdot t^L \cdot \tr^{-1}_{\mathcal{B}(M)}(t^{-1})
\end{align*}
\]

The special case $Q = Q^{-1} = 1_M$ recovers the Poincaré duality $\mathcal{H}(t) = t^L \cdot \mathcal{H}(t^{-1})$ of the Hilbert series, and therefore $\dim \mathcal{B}(M)_l = \dim \mathcal{B}(M)_{L-l}$ for all $l$.

2.3 An Example for Factorization only in the Trace

We want to present a type of graded representations which exhibits a seemingly paradoxical property: Their graded characters factor nicely, whereas the representations themselves do not. We will encounter this property in the case of Nichols algebras of non-abelian group type of rank 1.

Theorem 2.11. Let $G$ be a finite group and let $\pi : G \to A$ be some epimorphism into an abelian group $A$. Let $A = \bigoplus_{j \in J} A_j$ be some decomposition of $A$ into cyclic groups $A_j \cong \mathbb{Z}/n_j \mathbb{Z}$. Let $f_j : A_j \to \mathbb{Z}$ be the set-theoretic sections with $f_j(0) = 0$ and $f_j(x+1) = f_j(x) + 1$ for all $x \in A_j \setminus \{-1\}$. Set $f := \sum_{j \in J} f_j : A \to \mathbb{Z}$. Then $\pi$ defines $A$- and $A_j$-gradings of $kG$ as algebra and $f \circ \pi$ defines a $\mathbb{Z}$-grading of $kG$ as vector space. Let $G$ act on $kG$ by conjugation. This action respects the grading and $kG$ becomes a $\mathbb{Z}$-graded representation. Then $\tr^g_{kG}(t)$ is an integer multiple of $\prod_{j \in J} \left(\frac{n_j}{m_j}\right)^{m_j}$ for any $g \in G$ and suitable $m_j \in \mathbb{N}$ (depending on $g$). (Note that this depends on the chosen decomposition of $A$.)

To prove Theorem 2.11 we will need a short lemma.

Lemma 2.12. Let $\pi : G \to A$ be a homomorphism of finite groups and $g \in G$ arbitrary. Denote the centralizer of $g$ with $Z(g)$. Then $\#(Z(g) \cap \pi^{-1}(a))$ is either zero or $\#(Z(g) \cap \pi^{-1}(e))$, and $A' := \{a \in A : Z(g) \cap \pi^{-1}(a) \neq \emptyset\}$ is a subgroup of $A$. 

Proof. Set $Z_a := Z(g) \cap \pi^{-1}(a)$. Let $a, b \in A$ be arbitrary, such that $Z_a, Z_b \neq \emptyset$. Choose $x_0 \in Z_a$ and $y_0 \in Z_b$. For any $y \in Z_b$, we find that $x_0y^{-1}y_0 \in Z_a$, because $Z(g)$ is a subgroup and $\pi(x_0y^{-1}y_0) = a$. Furthermore, letting $y$ run, the $x_0y^{-1}y_0$ are $#Z_b$ pairwise different elements, thus $#Z_b \leq #Z_a$, and by symmetry, we find $#Z_a = #Z_b$. Clearly, $e \in Z_e$, so all non-empty $Z_a$ have the same cardinality $#Z_e$.

We have $e \in Z_e$, thus $e \in A'$. Now let $a, b \in A'$ be arbitrary and choose $x \in Z_a, y \in Z_b$. Then $xy \in Z_{ab}$ and $x^{-1} \in Z_{a^{-1}}$, thus $ab, a^{-1} \in A'$, and $A'$ must be a subgroup of $A$.

Proof. [Theorem 2.11] Denote with $i_j : A \rightarrow A_j$ the canonical projection and set $\pi_j := i_j \circ \pi$. Let $\chi_j(g)$ be the graded character of the adjoint representation of $G$ on $kG$, where the grading is induced by $\pi_j$. The adjoint representation is a permutation representation, and thus its character is given by the number of fixed points:

$$\chi_j(g) = \sum_{a \in A_j} \#(Z(g) \cap \pi_j^{-1}(a)) \cdot t^{f_j(a)}$$

We denote $Z_a^{(j)} := Z(g) \cap \pi_j^{-1}(a)$ and $Z_a := Z(g) \cap \pi^{-1}(a)$ for $a \in A_j$ and $a \in A$, respectively. According to Lemma 2.12, there is a subgroup $A_j'$ of $A_j$, such that $\chi_j(g) = \#Z_e^{(j)} \cdot \sum_{a \in A_j'} t^{f_j(a)}$. The subgroup of a cyclic subgroup is cyclic, so there is some $m_j \in \mathbb{N}$ with $A_j' = m_j A_j$ and

$$\chi_j(g) = \#Z_e^{(j)} \cdot \sum_{b=1}^{\#A_j'} t^{bm_j} = \#Z_e^{(j)} \cdot \left( \frac{n_j}{m_j} \right) t^{m_j}.$$

Set $k := \#J$. By definition,

$$\text{tr}_{kG}^g(t) = \sum_{a \in A} \#Z_a \cdot t^{f(a)} = \sum_{a_1 \in A_1} \cdots \sum_{a_k \in A_k} \#Z_{a_1 + \cdots + a_k} t^{f_1(a_1)} \cdots t^{f_k(a_k)}. \quad (1)$$

Now $Z_{a_1 + \cdots + a_k} = Z_{a_1}^{(1)} \cap \cdots \cap Z_{a_k}^{(k)}$. By Lemma 2.12, we can normalize their cardinalities to lie within $\{0, 1\}$, replace $\cap$ by multiplication, and find

$$\frac{\#Z_{a_1 + \cdots + a_k}}{\#Z_e} = \prod_{j \in J} \frac{\#Z_{a_j}^{(j)}}{\#Z_e^{(j)}}.$$
Inserting this into Equation 1 yields

\[
\text{tr}^d_{kG}(t) = \sum_{a_1 \in A_1} \cdots \sum_{a_k \in A_k} Z_e \cdot \prod_{j \in J} \frac{Z_e^{(j)}}{Z_e^{(j)}} \cdot \sum_{a_j \in A_j} \prod_{j \in J} Z_e^{(j)} t f_j(a_j)
\]

\[
= \frac{Z_e}{Z_e^{(j)}} \cdot \sum_{a_1 \in A_1} \cdots \sum_{a_k \in A_k} \prod_{j \in J} Z_e^{(j)} t f_j(a_j)
\]

\[
= \frac{Z_e}{Z_e^{(j)}} \cdot \prod_{j \in J} \sum_{a_j \in A_j} Z_e^{(j)} t f_j(a_j)
\]

\[
= \frac{Z_e}{Z_e^{(j)}} \cdot \prod_{j \in J} \chi_j(g) = \frac{Z_e}{Z_e^{(j)}} \cdot \prod_{j \in J} \left( \frac{n_j}{m_j} \right) t^{m_j} .
\]

Let \( \pi_1 : G_1 \to A_1 \) be as in Theorem 2.11 and set \( G_2 := \ker \pi_1 \). As a subgroup of \( G_1 \), \( G_2 \) acts on \( G_1 \) by conjugation, thus \( kG_1 \) is a \( \mathbb{Z} \)-graded \( G_2 \)-representation with \( kG_2 \) in degree zero. Assume there is an epimorphism \( \pi_2 : G_2 \to A_2 \) as in the theorem. Then \( kG_2 \) becomes a \( \mathbb{Z} \)-graded \( G_2 \)-representation. A transversal \( R \) of \( G_2 \) in \( G_1 \) now defines a new \( \mathbb{Z} \)-grading on \( kG_1 \) by \( \deg(gr) := \deg(g) + \deg(r) \) for all \( g \in G_2 \) and \( r \in R \), which makes \( kG_1 \) into a \( G_2 \)-representation with a \( [G_2 : A_2] \)-dimensional degree-zero-subspace. If \( G_1 \) is solvable, we may use induction until the final group \( G_k \) is abelian by itself, so \( kG_1 \) becomes a \( G_k \)-representation with one-dimensional degree-zero-subspace. The grading however still depends on the chosen subnormal series.

**Example 2.13.** Let \( G \) be the dihedral group \( G = \mathbb{D}_4 = \langle a, b : a^4 = b^2 = e, ab = ba^3 \rangle \), acting on its group algebra by conjugation, and let \( A \) be its abelianization. The corresponding \( \mathbb{Z} \)-gradation of \( kG \) is

\[
kG = \text{Lin}_k(e, a^2) \oplus \text{Lin}_k(a, b, a^3, a^2 b) \cdot t \oplus \text{Lin}_k(ab, a^3 b) \cdot t^2
\]

with graded characters \( \chi(e) = \chi(a^2) = 2(1+t)^2, \chi(a) = \chi(b) = 2(1+t), \) and \( \chi(ab) = 2(1 + t^2) \). Denote with \( T \) the trivial irreducible \( G \)-representation, and with \( X \) and \( Y \) certain one-dimensional \( G \)-representations with \( X \otimes X = Y \otimes Y = T \). Then \( kG \) is isomorphic to

\[
(T \oplus T) \oplus (T \oplus T \oplus X \oplus Y) \cdot t \oplus (T \oplus X \otimes Y) \cdot t^2
\]

as a graded \( G \)-representation. While the graded characters factor nicely, the representation itself does not.
3 Graded Traces and Hilbert Series over Nichols Algebras

In this section we study factorization mechanisms for Nichols algebras, and hence for their graded traces and especially their Hilbert series.

The root system $\Delta^+$ of a Nichols algebra $\mathcal{B}(M)$, introduced in Subsection 3.1, directly presents a factorization of $\mathcal{B}(M)$ as graded vector space:

\[ \bigotimes_{\alpha \in \Delta^+} \mathcal{B}(M_{\alpha}) \sim \rightarrow \mathcal{B}(M) \]

Note however that over nonabelian groups, the root system factorization is too crude in general to explain the full observed factorization into cyclotomic polynomials; especially for the rank 1 cases in the next section the factorization obtained this way is trivial.

Nevertheless, we will start in Subsection 3.2 by demonstrating a factorization of the graded trace of an endomorphism $Q$ that stabilizes a given axiomatized Nichols algebra factorization, such as the root system above:

**Corollary 3.1.** Let $\mathcal{B}(M)$ be a Nichols algebra with factorization $W_\alpha, \alpha \in \Delta^+$ and $Q$ an algebra operator that stabilizes this factorization. Then

\[ \text{tr}^Q_{\mathcal{B}(M)}(t) = \prod_{\alpha \in \Delta^+} \text{tr}^{Q_\alpha}_{\mathcal{B}(M_{\alpha})}(t). \]

In Subsection 3.3 we focus on Nichols algebras over abelian groups. The preceding corollary immediately gives an explicit trace product formula for endomorphisms $Q$ stabilizing the root system, in terms of cyclotomic polynomials. In particular, it shows the complete factorization of their Hilbert series.

Using the theory of Lyndon words, we are able to weaken the assumptions on $Q$ to only normalize the root system, i.e. acting on it by permutation. We give such examples where $Q$ interchanges two disconnected subalgebras in the Nichols algebra, as well as the outer automorphism of a Nichols algebra of type $A_3$. The authors expect that a more systematic treatment via root vectors will carry over to endomorphisms normalizing the root system of a non-abelian Nichols algebra as well.

In Subsection 3.4 we present an example of a Nichols algebra of type $A_2$ and an endomorphism $Q$ induced by its outer automorphism which fails the normalizing condition on the non-simple root. Note that in contrast to the $A_3$-example above, there is an edge flipped by the automorphism, which is called a “loop” in literature (e.g. [7], p. 47ff). Nevertheless one
observes a factorization of the graded trace of $Q$, and in this example this can be traced back to a surprising and apparently new “symmetrized” PBW-basis.

In Subsection 3.5, we start approaching Nichols algebras over nonabelian groups, where one observes astonishingly also factorization of graded traces into cyclotomic polynomials. This cannot be explained by the root system alone and might indicate the existence of a finer root system, which is not at the level of Yetter-Drinfel’d modules.

We can indeed give a family of example constructed as covering Nichols algebras by the first author ([20]). By construction, these Nichols algebras possess indeed such a finer root system of different type (e.g. $E_6 \to F_4$). In these examples, the root systems lead to a complete factorization, but this mechanism does not seem to easily carry over to the general case.

3.1 Nichols Algebras over Groups

The following notions are standard. We summarize them to fix notation and refer to [11] for a detailed account.

**Definition 3.2.** A Yetter-Drinfel’d module $M$ over a group $G$ is a $G$-graded vector space over $k$ denoted by layers $M = \bigoplus_{g \in G} M_g$ with a $G$-action on $M$ such that $g.M_h = M_{ghg^{-1}}$. To exclude trivial cases, we call $M$ indecomposable iff the support $\{ g \mid M_g \neq 0 \}$ generates all $G$ and faithful iff the action is.

Note that for abelian groups, the compatibility condition is just the stability of the layers $M_g$ under the action of $G$.

The notion of a Yetter-Drinfel’d module automatically brings with it a braiding $\tau$ on $M$—in fact, each group $G$ defines an entire braided category of $G$-Yetter-Drinfel’d modules with graded module homomorphisms as morphisms (e.g. [3], Def. 1.1.15)

**Lemma 3.3.** Consider $\tau : M \otimes M \to M \otimes M$, $v \otimes w \mapsto g.w \otimes v \in M_{ghg^{-1}} \otimes M_g$ for all $v \in M_g$ and $w \in M_h$. Then $\tau$ fulfills the Yang-Baxter-equation

$$(\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau) = (\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id})$$

turning $M$ into a braided vector space.

In the non-modular case, the structure of Yetter-Drinfel’d modules is well understood ([3] Section 3.1) and can be summarized in the following three lemmata:
Lemma 3.4. Let $G$ be a finite group and let $\mathbb{k}$ be an algebraically closed field whose characteristic does not divide $\#G$. Then any finite-dimensional Yetter-Drinfel’d module $M$ over $G$ is semisimple, i.e. decomposes into simple Yetter-Drinfel’d modules (the number is called rank of $M$): $M = \bigoplus_i M_i$.

Lemma 3.5. Let $G$ be a finite group, $g \in G$ arbitrary and $\chi : G \to \mathbb{k}$ the character of an irreducible representation $V$ of the centralizer subgroup $\text{Cent}(g) = \{ h \in G \mid gh = hg \}$. Define the Yetter-Drinfel’d module $\mathcal{O}_g^\chi$ to be the induced $G$-representation $\mathbb{k}G \otimes_{\mathbb{k}\text{Cent}(g)} V$ with $G$-grading ($\mathbb{k}G$-coaction) $\delta(h \otimes v) := hgh^{-1} \otimes (h \otimes v) \in \mathbb{k}G \otimes (\mathbb{k}G \otimes_{\mathbb{k}\text{Cent}(g)} V)$ for all $h \in G$, $v \in V$. It can be constructed as follows:

- Define the $G$-graded vector space by
  
  $$\mathcal{O}_g^\chi = \bigoplus_{h \in G} (\mathcal{O}_g^\chi)_h \quad \text{with} \quad (\mathcal{O}_g^\chi)_h := \begin{cases} V & \text{for } g\text{-conjugates } h \in [g], \\ \{0\} & \text{else.} \end{cases}$$

- Choose a set $S = \{s_1, \ldots, s_n\}$ of representatives for the left $\text{Cent}(g)$-cosets $G = \bigcup_k s_k \text{Cent}(g)$. Then for any $g$-conjugate $h \in [g]$ there is precisely one $s_k$ with $h = s_k gs_k^{-1}$.

- For the action of any $t \in G$ on any $v_h \in (\mathcal{O}_g^\chi)_h$ for $h \in [g]$ determine the unique $s_i, s_j$, such that $s_i gs_i^{-1} = h$ and $s_j gs_j^{-1} = tht^{-1}$. Then $s_j^{-1} ts_i \in \text{Cent}(g)$ and using the given $\text{Cent}(g)$-action on $V$ we may define $t.v_h := (s_j^{-1} ts_i.v)_{th^{-1}}$.

Then $\mathcal{O}_g^\chi$ is simple as Yetter-Drinfel’d module and $\mathcal{O}_g^\chi$ and $\mathcal{O}_g^{\chi'}$ are isomorphic if and only if $g$ and $g'$ are conjugate and $\chi$ and $\chi'$ are isomorphic.

If $\chi$ is the character of a one-dimensional representation $(V, \rho)$, we will identify $\chi$ with the action $\rho$. In positive characteristic, we will restrict to $\text{dim } V = 1$, where the character determines its representation.

Lemma 3.6. Let $G$ be a finite group and let $\mathbb{k}$ be an algebraically closed field whose characteristic does not divide $\#G$. Then any simple Yetter-Drinfel’d module $M$ over $G$ is isomorphic to some $\mathcal{O}_g^\chi$ for some $g \in G$ and a character $\chi : G \to \mathbb{k}$ of an irreducible representation $V$ of the centralizer subgroup $\text{Cent}(g)$.

Example 3.7. For finite and abelian $G$ over algebraically closed $\mathbb{k}$ with $\text{char } K \nmid \#G$ we have 1-dimensional simple Yetter-Drinfel’d modules $M_i = \mathcal{O}_g^{\chi_i} = x_i \mathbb{k}$ and hence the braiding is diagonal (i.e. $\tau(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$) with braiding matrix $q_{ij} := \chi_j(g_i)$.
Definition 3.8. Consider the tensor algebra $\Sigma M$, i.e. for any homogeneous basis $x_i \in M_g$, the algebra of words in all $x_i$. We may uniquely define skew derivations on this algebra, i.e. maps $\partial_i : \Sigma M \to \Sigma M$ by $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}1$, and $\partial_i(ab) = \partial_i(a)b + (g_i.a)\partial_i(b)$.

Definition 3.9. The Nichols algebra $\mathfrak{B}(M)$ is the quotient of $\Sigma M$ by the largest homogeneous ideal $\mathfrak{I}$ invariant under all $\partial_i$, such that $M \cap \mathfrak{I} = \{0\}$.

In specific instances, the Nichols algebra may be finite-dimensional. This is a remarkable phenomenon (and the direct reason for the finite-dimensional truncations of $U_q(g)$ for $q$ a root of unity):

Example 3.10. Take $G = \mathbb{Z}_2$ and $M = M_e \oplus M_g$ the Yetter-Drinfel’d module with dimensions $0 + 1$ i.e. $q_{11} = -1$, then $x^2 \in \mathfrak{I}$ and hence the Nichols algebra $\mathfrak{B}(M) = k[x]/(x^2)$ has dimension 2.

More generally a 1-dimensional Yetter-Drinfel’d module with $q_{ii} \in k_n$ a primitive $n$-th root of unity has Nichols algebra $\mathfrak{B}(M) = k[x]/(x^n)$.

Example 3.11. Take $G = \mathbb{Z}_2$ and $M = M_e \oplus M_g$ the Yetter-Drinfel’d module with dimensions $0 + 2$ i.e. $q_{11} = q_{22} = q_{12} = q_{21} = -1$ then

$$\mathfrak{B}(M) = k(x,y)/(x^2,y^2,xy + yx) = \wedge M$$

In the abelian case, Heckenberger (e.g [12]) introduced $q$-decorated diagrams, with each node corresponding to a simple Yetter-Drinfel’d module decorated by $q_{ii}$, and each edge decorated by $\tau^2 = q_{ij}q_{ji}$ and edges are drawn if the decoration is $\neq 1$; it turns out that this data is all that is needed to determine the respective Nichols algebra.

Theorem 3.12. [16] Let $\mathfrak{B}(M)$ be a Nichols algebra of finite dimension over an arbitrary group $G$, then there exists a rootsystem $\Delta \subset \mathbb{Z}^N$ with positive roots $\Delta^+$ and a truncated basis of monomials in $x_\alpha \in \mathfrak{B}(M)_{\alpha}$. Namely, the multiplication in $\mathfrak{B}(M)$ is an isomorphism of graded vectorspaces $\mathfrak{B}(M) \cong \bigotimes_{\alpha \in \Delta^+} \mathfrak{B}(M_\alpha)$.

Example 3.13. Assume $q_{11} = q_{22} = q_{12}q_{21} = -1$, then the diagram is:

Some calculations show that $x_3 := [x_1, x_2]_\tau \neq 0 (\not\in M)$, but $[x_2, [x_1, x_2]_\tau]_\tau = [x_1, [x_1, x_2]_\tau]_\tau = 0$. Hence $\mathfrak{B}(M)$ corresponds to the Borel part of $A_2 = \mathfrak{sl}_3$. 

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As \( q_{11} = q_{22} = -1 \) as well as \( q_{33} = (q_{11} q_{12})(q_{22} q_{21}) = -1 \), all three Nichols algebras \( \mathfrak{B}(x_i k) \) of rank 1 are 2-dimensional. The Nichols algebra \( \mathfrak{B}(x_1 k \oplus x_2 k) \) itself is 8-dimensional with PBW-Basis \( x_1^i x_2^j x_3^k \) with \( i, j, k \in \{0, 1\} \), i.e. multiplication in \( \mathfrak{B}(M) \) yields a vector space bijection:

\[
\mathfrak{B}(M) \cong \mathfrak{B}(x_1 k) \otimes \mathfrak{B}(x_2 k) \otimes \mathfrak{B}(x_3 k) = \mathbb{k}[x_1]/(x_1^2) \otimes \mathbb{k}[x_2]/(x_2^2) \otimes \mathbb{k}[x_3]/(x_3^2)
\]

In the same sense, over abelian \( G \) for \( a_{ij} \) any proper Cartan matrix of a semisimple Lie algebra is realized for braiding matrix \( q_{ij} q_{ji} = q_{ii}^{-a_{ij}} \).

However, several additional exotic examples of finite-dimensional Nichols algebras exist, that possess unfamiliar Dynkin diagrams, such as a multiply-laced triangle, and where Weyl reflections may connect different diagrams (yielding a Weyl groupoid). Heckenberger completely classified all Nichols algebras over abelian \( G \) in [12].

Over nonabelian groups, still much is open, but progress is made: Andruskiewitsch, Heckenberger, and Schneider studied the Weyl groupoid in this setting as well and established a root system and a PBW-basis for finite-dimensional Nichols algebras in [4].

- By detecting certain “defect” subconfigurations (so-called type \( D \)) most higher symmetric and all alternating groups and later many especially sporadic groups were totally discarded (Andruskiewitsch et. al. [5], [2] etc.).

- Only few finite-dimensional indecomposable examples are known so far, namely \( D_4 \) of type \( A_2 \) and \( S_3, S_4, S_5 \) of type \( A_1 \) (Schneider et. al. [23]), higher analogues of \( D_4 \) ([15]), and some rank 1 examples over metacyclic groups [9], as well as infinite families of Lie type over extraspecial groups [20].

- In [17] all finite-dimensional Nichols algebras of Rank 2 have been determined

  - Type \( A_2 \) over a quotient of the group \( \Gamma_2 \), which has been known (e.g. \( D_4 \))
  - Type \( G_2 \) over a rack of type \( T \), between a central element and a group \( A_4 \).
  - Three Nichols algebras over quotients of the group \( \Gamma_3 \) (e.g. \( S_3 \)), not of finite Cartan type and depending on the characteristic of the base field.
3.2 A First Trace Product Formula

In the following we want to use the root system $\Delta^+$ of a Nichols algebra $\mathcal{B}(M)$ to derive a trace product formula.

**Definition 3.14.** A factorization of a Nichols algebra $\mathcal{B}(M)$ of a braided vectorspace is a collection of braided vector spaces $(M_\alpha)_{\alpha \in \Delta^+}$ with some arbitrary index set $\Delta^+$, such that:

- All $M_\alpha$ are grading-homogeneous, braided subspaces of the Nichols algebra $M_\alpha \subset \mathcal{B}(M)$.

- The multiplication in $\mathcal{B}(M)$ induces a graded isomorphism of braided vector spaces $\mu_{\mathcal{B}(M)} : \bigotimes_{\alpha \in \Delta^+} \mathcal{B}(M_\alpha) \xrightarrow{\sim} \mathcal{B}(M)$.

An operator $Q$ on $\mathcal{B}(M)$ is said to stabilize the factorization iff $QM_\alpha \subset M_\alpha$ for all $\alpha \in \Delta^+$. In this case we denote by $Q_\alpha := Q|_{\mathcal{B}(M_\alpha)}$ the restriction. $Q$ is said to normalize the factorization iff for each $\alpha \in \Delta^+$ there is a $\beta \in \Delta^+$ with $QM_\alpha \subset M_\beta$. In this case $Q$ acts on $\Delta^+$ by permutations and we denote the shifting restriction $Q_{\alpha \to \beta} := Q|_{M_\alpha}$ with image in $M_\beta = M_{Q\alpha}$.

**Example 3.15.** The root system $W_\alpha$, $\alpha \in \Delta^+$ of a Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel’d module $M$ (see section 3.1) is the leading example of a factorization.

A factorization of a Nichols algebra can be used to derive a product formula of the trace and graded trace of some operator $Q$. A first application is:

**Lemma 3.16.** Let $\mathcal{B}(M)$ be a Nichols with factorization $M_\alpha$, $\alpha \in \Delta^+$ and $Q$ an algebra operator that stabilizes this factorization. Then the trace of $Q$ is $\text{tr} (Q) = \prod_{\alpha \in \Delta^+} \text{tr} (Q_\alpha)$.

**Proof.** We evaluate the trace in the provided factorization: Let $Q$ act diagonally on the tensor product $\bigotimes_{\alpha \in \Delta^+} \mathcal{B}(M_\alpha)$ by acting on each factor via the restriction $Q_\alpha$, which is possible because $Q$ was assumed to stabilize this factorization. The action of an algebra operator $Q$ commutes with the multiplication $\mu_{\mathcal{B}(M)}$ so the trace of $Q$ acting on both sides coincides. The trace on a tensor product is the product of the respective trace and hence we get

$$\text{tr} (Q|_{\mathcal{B}(M)}) = \text{tr} \left( Q|_{\bigotimes_{\alpha \in \Delta^+} \mathcal{B}(M_\alpha)} \right) = \prod_{\alpha \in \Delta^+} \text{tr} \left( Q|_{\mathcal{B}(M_\alpha)} \right) = \prod_{\alpha \in \Delta^+} \text{tr} (Q_\alpha).$$

$\square$
To calculate the graded trace with the preceeding lemma, first note that $t^E$ is a graded algebra automorphism, so if $Q$ fulfills the conditions of the lemma, so does $Y = t^E Q$. Hence, we find $\text{tr}_{\mathcal{B}(M)}^Q = \text{tr}(t^E Q) = \text{tr}(Y)$:

**Corollary 3.17.** Let $\mathcal{B}(M)$ be a Nichols algebra with factorization $W_\alpha, \alpha \in \Delta^+$ and $Q$ an algebra operator that stabilizes this factorization. Then

$$\text{tr}_{\mathcal{B}(M)}^Q(t) = \prod_{\alpha \in \Delta^+} \text{tr}_{\mathcal{B}(M_\alpha)}^Q(t).$$

**Example 3.18.** Let the braided vector space $M = x_1 k \oplus x_2 k$ be defined by $q_{ij} = \begin{pmatrix} -1 & -1 \\ +1 & -1 \end{pmatrix}$. Then the diagonal Nichols algebra $\mathcal{B}(M)$ is of standard Cartan type $A_2$ and possesses a factorization $\Delta^+ = \{ \alpha_1, \alpha_2, \alpha_{12} \}$ with $M_{\alpha_1} = x_1 k$, $M_{\alpha_2} = x_2 k$, $M_{\alpha_{12}} = x_{12} k$, $x_{12} := [x_1, x_2]_q := x_1 x_2 + x_2 x_1$.

All braidings are $-1$, hence $\mathcal{B}(x_\alpha) \cong k[x_\alpha]/(x_\alpha^2)$. This implies that the multiplication in $\mathcal{B}(M)$ is an isomorphism of graded vector spaces:

$$\mu_{\mathcal{B}(M)} : k[x_1]/(x_1^2) \otimes k[x_2]/(x_2^2) \otimes k[x_{12}]/(x_{12}^2) \cong \mathcal{B}(M)$$

This shows that the Hilbert series $\mathcal{H}(t) = \text{tr}_{\mathcal{B}(M)}^1(t)$ is

$$\text{tr}_{\mathcal{B}(M)}^1(t) = \text{tr}_{\mathcal{B}(M_{\alpha_1})}^1(t) \cdot \text{tr}_{\mathcal{B}(M_{\alpha_2})}^1(t) \cdot \text{tr}_{\mathcal{B}(M_{\alpha_{12}})}^1(t) = (1 + t)(1 + t)(1 + t^2)$$

$$= 1 + 2t + 2t^2 + 2t^3 + t^4$$

**Example 3.19.** In the previous example of a Nichols algebra $\mathcal{B}(M)$, let $Q \in \text{End}(M)$ be defined by $Qx_1 := x_1$ and $Qx_2 := -x_2$. This map preserves the braiding and hence extends uniquely to an algebra automorphism on $\mathcal{B}(M)$, in particular, $Qx_{12} = Q(x_1 x_2 + x_2 x_1) = -(x_1 x_2 + x_2 x_1) = -x_{12}$ holds. A direct calculation yields:

$$\text{tr}_{\mathcal{B}(M)}^Q(t) = \text{tr}(Q|_{\mathcal{B}(M)}) + t \cdot \text{tr}(Q|_{x_1 x_2}) + t^2 \cdot \text{tr}(Q|_{x_1 x_2 x_1})$$

$$+ t^3 \cdot \text{tr}(Q|_{x_1 x_12 x_2 x_12}) + t^4 \cdot \text{tr}(Q|_{x_1 x_2 x_12})$$

$$= \text{tr}(1) + t \cdot \text{tr}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) + t^2 \cdot \text{tr}(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})$$

$$+ t^3 \cdot \text{tr}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) + t^4 \text{tr}(1)$$

$$= 1 - 2t^2 + t^4$$

The product formula returns for the same trace:

$$\text{tr}_{\mathcal{B}(M)}^Q(t) = \text{tr}_{\mathcal{B}(M_{\alpha_1})}^Q(t) \cdot \text{tr}_{\mathcal{B}(M_{\alpha_2})}^Q(t) \cdot \text{tr}_{\mathcal{B}(M_{\alpha_{12}})}^Q(t) = (1 + t)(1 - t)(1 - t^2)$$

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In the next section, we study the special case of a diagonal Nichols algebra, we will also study examples of operators $Q$, that do neither stabilize nor normalize the root system. However, their graded trace is still factorizing, which indicates the existence of alternative PBW-basis. We will construct such for the case $N = 2$, $q = \sqrt{-1}$.

### 3.3 Nichols Algebra Over Abelian Groups

We now restrict our attention to the Nichols algebra $\mathfrak{B}(M)$ of a Yetter-Drinfel’d module $M$ of rank $n$ over an abelian group $G$ and $k = \mathbb{C}$. This means $M$ is diagonal i.e. the sum of 1-dimensional braided vector spaces $x_i k$. According to [12], $\mathfrak{B}(M)$ possesses an arithmetic root system $\Delta^+$ that can be identified with a set of Lyndon words $L$ in $n$ letters, with word length corresponding to the grading in the Nichols algebra. Such a Lyndon word corresponds to iterated $q$-commutators in the letters $x_i$ according to iterated Shirshov decomposition of the word.

For any positive root $\alpha \in \Delta^+$ we denote by $N_\alpha$ the order of the self-braiding $\chi(\alpha, \alpha) = q_{\alpha, \alpha}$. It is known that this determines $\mathfrak{B}(x_\alpha) = k[x_\alpha]/(x_\alpha^N)$ and we denote by $|\alpha|$ the length of the Lyndon word resp. the degree of the root vector $x_\alpha$ in the Nichols algebra grading.

We further denote by $g_\alpha$ the $G$-grading of $x_\alpha$, extending the $G$-grading of $M$ on simple roots. Moreover, we denote the scalar action of any $g \in G$ on $x_\alpha$ by $\alpha(g) = \chi_M(g_\alpha, g) \in k^\times$, extending the $G$-action of $M$ on simple roots. We frequently denote $(N)_t = 1 + t + \cdots + t^N = \frac{1 - t^{N+1}}{1 - t}$.

**Lemma 3.20.** Let $Q$ be an algebra operator on a diagonal Nichols algebra $\mathfrak{B}(M)$ that stabilizes the root system, i.e. for all $\alpha \in \Delta^+$ the root vector $x_\alpha$ is an eigenvector to $Q$ with eigenvalue $\lambda_\alpha \in k$. Then the product formula of Corollary 3.17 reads as follows:

$$\text{tr}^Q_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \Delta^+} (N_\alpha)_{\lambda_\alpha t^{|\alpha|}}$$

Especially for the action of a group element $g \in G$ we get

$$\text{tr}^g_{\mathfrak{B}(M)}(t) = \prod_{\alpha \in \Delta^+} (N_\alpha)_{\alpha(g) t^{|\alpha|}}.$$  

**Proof.** The factorization follows from Corollary 3.17. We yet have to verify the formula on each factor $W_\alpha = \mathfrak{B}(x_\alpha) = k[x_\alpha]/(x_\alpha^N)$. $W_\alpha$ has a basis $x^k_\alpha$ for $0 \leq k < N_\alpha$ with degrees $k|\alpha|$. By assumption $Q$ acts on $x_\alpha$ via the scalar $\lambda_\alpha$ and by multiplicativity on $x^k_\alpha$ by $\lambda_\alpha^k$. Altogether:

$$\text{tr}^Q_{\mathfrak{B}(x_\alpha)} = \sum_{k=0}^{N_\alpha-1} \lambda_\alpha^k \cdot t^{|\alpha|} = \sum_{k=0}^{N_\alpha-1} (\lambda_\alpha \cdot t^{|\alpha|})^k = (N_\alpha)_{\lambda_\alpha |\alpha|}.$$
Example 3.21. Let the braided vector space $M = x_1k \oplus x_2k$ be defined by $q_{ij} = \left( \frac{q^2}{q} \frac{q^{-1}}{q^2} \right)$ with $q$ a primitive $2N$-th root of unity. Then the diagonal Nichols algebra $\mathcal{B}(M)$ is of standard Cartan type $A_2$ and possesses a factorization $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_{12}\}$ with $M_{\alpha_1} = x_1k$, $M_{\alpha_2} = x_2k$, and $M_{\alpha_{12}} = x_{12}k$, where $x_{12} := [x_1, x_2]_q := x_1x_2 - q^{-1}x_2x_1$. This implies that the multiplication in $\mathcal{B}(M)$ is an isomorphism of graded vector spaces:

$$\mu_{\mathcal{B}(M)} : \mathcal{B}(M) \cong \mathcal{B}(M_{\alpha_1}) \otimes \mathcal{B}(M_{\alpha_2}) \otimes \mathcal{B}(M_{\alpha_{12}}) \cong k[x_1]/(x_1^N) \otimes k[x_2]/(x_2^N) \otimes k[x_{12}]/(x_{12}^N).$$

This obviously agrees with the Hilbert series in Lemma 3.20:

$$\mathcal{H}(t) = \text{tr}_{\mathcal{B}(M)}^1 (t) = \prod_{i=1}^3 \text{tr}_{\mathcal{B}(M_{\alpha_i})}^1 (t) = (N)_t(N)_t(N)_{t^2}.$$  

Let us now apply the formula of Lemma 3.20 to calculate the graded trace of the action of group elements (which stabilize the root system): We realize the braided vector space $M$ as a Yetter-Drinfel’d module over $G := \mathbb{Z}_{2N} \times \mathbb{Z}_{2N} = \langle g_1, g_2 \rangle$, such that $x_1$ is $g_1$-graded and $x_2$ is $g_2$ graded, with suitable actions

$$g_1x_1 = q^2x_1 \quad g_1x_1 = q^2x_1 \quad g_2x_2 = q^{-1}x_2 \quad g_2x_1 = q^{-1}x_1.$$  

Then we get for the action of each group element $g_k$:

$$\text{tr}_{\mathcal{B}(M)}(t) = \prod_{\alpha \in \{\alpha_1, \alpha_2, \alpha_{12}\}} \text{tr}_{\mathcal{B}(M_{\alpha})}^g = (N)_{q^2 t} (N)_{q^{-1} t} (N)_{t^2}.$$  

We now look at graded traces of automorphisms $Q$, where $Q$ does not stabilize the root system. We restrict ourselves to diagonal Nichols algebras $\mathcal{B}(M)$, so we may use the theory of Lyndon words.

The PBW-basis consists of monotonic monomials

$$[u_1]^{n_1} [u_2]^{n_2} \cdots [u_k]^{n_k} \quad \text{with} \quad u_1 > u_2 > \cdots > u_k \quad \forall u_i \in \mathcal{L}.$$  

The PBW-basis carries the lexicographic order $<$ and by Remark after Thm 3.5 this is the same as the lexicographic order of the composed words $u_1^{n_1} u_2^{n_2} \cdots u_k^{n_k}$. For a sequence of Lyndon words $\vec{u} = (u_1, \ldots, u_k)$, not necessarily monotonically sorted, we define $[\vec{u}] := [u_1][u_2] \cdots [u_k]$. In particular, sorted sequences correspond to the PBW-basis. For any sequence of Lyndon words $\vec{u}$, denote by $\vec{u}^\text{sort}$ its monotonic sorting.
Lemma 3.22. a) For any sequence of Lyndon words $\vec{u}$, not necessarily monotonically sorted, we have $[\vec{u}] = q \cdot [\vec{u}^{\text{sort}}] + \text{smaller}$, where $q \neq 0$ and “smaller” denotes linear combinations of PBW-elements lexicographically smaller than the PBW-element $[\vec{u}^{\text{sort}}]$.

b) Let $\sigma \in S_k$ and $u_1 > u_2 > \cdots > u_k \in \mathcal{L}$ then

$$[u_{\sigma(1)}]^{n_{\sigma(1)}}[u_{\sigma(2)}]^{n_{\sigma(2)}} \cdots [u_{\sigma(k)}]^{n_{\sigma(k)}} = q_{\sigma} \cdot [u_1]^{n_1}[u_2]^{n_2} \cdots [u_k]^{n_k} + \text{smaller},$$

where $q_{\sigma}$ is the scalar factor associated to the braid group element $\hat{\sigma} \in B_k$, which is the image of $\sigma$ under the Matsumoto section. Explicitly $q_{(i,i+1)} = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_i n_{i+1}}$ in the notation of [11] and general $q_{\sigma}$ are obtained by multiplying such factors along a reduced expression of $\sigma$.

Proof. Claim a): We perform induction on the multiplicity of the highest appearing Lyndon word: So for a fixed $w \in \mathcal{L}, N \in \mathbb{N}$ suppose that the claim has been proven for all sequences $\vec{u}'$ with $u'_i \leq w$ and strictly less then $N$ indices $i$ with $u'_i = w$.

Consider then a sequence $\vec{u}$ with $u_i \leq w$ and precisely $N$ indices $i$ with $u_i = w$. We perform a second induction on the index $i$ of the leftmost appearing $w = u_i$:

- If $w = u_1$ we may consider the sequence $\vec{u}_1 := (u_2, \ldots, u_k)$ having strictly less $w$-multiplicity. By induction hypothesis, $[\vec{u}_1] = [\vec{u}_1^{\text{sort}}] + \text{smaller}$. Since $\vec{u}^{\text{sort}} = (w, \vec{u}_1^{\text{sort}})$ the assertion then also holds for $\vec{u}$.

- Otherwise, let $u_{i+1} = w$ be the leftmost appearance of $w$, especially $u_i < u_{i+1} = w$. By [11] Prop. 3.9 we then have $[u_i][u_{i+1}] = q \cdot [u_{i+1}][u_i] + \text{smaller}$ where $q \neq 0$ and “smaller” means products of Lyndon words $[v]$ with $u_i < v < u_{i+1} = w$. Thus all products $[u_1] \cdots [u_{i-1}] [u_i] \cdots [u_k]$ contain $w$ with a multiplicity less then $\vec{u}$; by induction hypothesis these are linear combination of PBW-elements lexicographically strictly smaller then $[\vec{u}^{\text{sort}}]$. The remaining summand $[u_1] \cdots [u_{i+1}][u_i] \cdots [u_k]$ has $w = u_{i+1}$ in a leftmost position and the claim follows by the second induction hypothesis.

Claim b): We proceed by induction on the length of $\sigma \in S_k$, which is the length of any reduced expression for $\sigma$. For $\sigma = \text{id}$ we’re done, so assume for some $i$ that $u_{\sigma(i)} < u_{\sigma(i+1)}$, hence $\sigma = (\sigma(i), \sigma(i+1))\sigma'$ with $\sigma'$ shorter. Again by [11] Prop. 3.9 we have:

$$[u_{\sigma(i)}][u_{\sigma(i+1)}] = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}})) \cdot [u_{\sigma(i+1)}][u_{\sigma(i)}] + \text{smaller}.$$

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Moreover, for any sequences of Lyndon words \( \vec{a}, \vec{b} \), claim a) proves that
\([\vec{a}] \cdot \text{smaller} \cdot [\vec{b}]\) is a linear-combination of PBW-elements lexicographically smaller than \([\vec{u}]\). Hence inductively
\[
[u_{\sigma(i)}]^{n_{\sigma(i)}} [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_{\sigma(i)} n_{\sigma(i+1)}} \cdot [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} [u_{\sigma(i)}]^{n_{\sigma(i)}} + \text{smaller}.
\]

By the same argument (again using claim a) we may multiply both sides with the remaining factors:
\[
[u_1]^{n_1} \cdots [u_{\sigma(i)}]^{n_{\sigma(i)}} [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} \cdots [u_k]^{n_k} = \chi(\deg(x_{u_i}), \deg(x_{u_{i+1}}))^{n_{\sigma(i)} n_{\sigma(i+1)}} \cdot [u_1]^{n_1} \cdots [u_{\sigma(i+1)}]^{n_{\sigma(i+1)}} [u_{\sigma(i)}]^{n_{\sigma(i)}} \cdots [u_k]^{n_k} + \text{smaller}.
\]

We may now use the induction hypothesis on \(\sigma'\), which is shorter.

\[\square\]

**Theorem 3.23.** Let \(\mathcal{B}(M)\) be a finite-dimensional Nichols algebra over a Yetter-Drinfel’d module \(M\) over an abelian group \(G\). Let \(Q\) be an automorphism of the graded algebra \(V = \mathcal{B}(M)\) permuting the roots \(QV_{\alpha} = V_{Q\alpha}\) and denote the action on root vectors by
\[
Qx_{\alpha} =: \lambda_Q(\alpha)x_{Q\alpha}\quad \lambda_Q : \Delta^+ \to k.
\]

On any orbit \(A \in \mathcal{O}_Q(\Delta^+)\) all orders \(n_{\alpha}\) of \(x_{\alpha}\) coincide for \(\alpha \in A\) and we denote this value by \(N_A\). Similarly, all degrees \(\alpha\) coincide and we denote the sum over the orbit in slight abuse of notation \(|A| = |\alpha| \cdot \#A\). Then
\[
\text{tr}^Q_{V_A}(t) = \prod_{A \in \mathcal{O}_Q(\Delta^+)} (N_A)_{q_A(Q)} \lambda_Q(A)|t|^{|A|}
\]

with the \(q\)-symbol \((N)_t := 1 + t + \cdots + t^{N-1} = \frac{1 - t^N}{1 - t}\) and \(q(Q) \in k^\times\) the scalar braiding factor of \(Q\) acting as an element of \(\mathbb{B}_{|A|}\) on \(A\), as in the preceding lemma.

**Proof.** We start with the factorization along the rootsystem
\[
\text{tr}^Q_{V}(t) = \prod_{A \in \mathcal{O}_Q(\Delta^+)} \text{tr}^Q_{V_A}(t) \quad V_A := \bigotimes_{\alpha \in A} V_{\alpha}
\]

with \(Qx_{\alpha} =: \lambda_Q(\alpha)x_{Q\alpha}\) as assumed. The action of \(Q\) on monomials \(\bigotimes_{\alpha \in A} x_{\alpha}^{k_{\alpha}} \in V_A\) can be calculated using the previous lemma:
\[
Q \bigotimes_{\alpha \in A} x_{\alpha}^{k_{\alpha}} = \bigotimes_{\alpha \in A} \lambda_Q(\alpha)^{k_{\alpha}}x_{Q\alpha}^{k_{\alpha}} = \prod_{\alpha \in A} \lambda_Q(\alpha)^{k_{\alpha}} : \bigotimes_{\alpha \in A} x_{Q\alpha}^{k_{\alpha}} = q_A(Q)^{k} \bigotimes_{\alpha \in A} x_{\alpha}^{k_{\alpha}} + \text{smaller},
\]

20
A Drinfel’d module with braiding matrix corresponding to $A$ we obtain nontrivial examples, such as $\langle A \rangle$. Thus we can calculate the trace in terms of a basis and the only contributions come from monomials with all $k$-\textit{ponents} of $q$-\textit{below} crucially rely on exceptional behaviour for the normalizing-condition seems to be very restrictive and the examples thus obtained are.

Let Example 3.24. Note that any other off-diagonal entries $a$, $a^{-1}$ would let $Q$ fail to preserve the braiding matrix. We have $\mathcal{B}(M) \cong k[x_1]/(x_1^N) \otimes k[x_1]/(x_1^N)$ and explicitly calculate $Qx_1^i x_2^j = x_2^i x_1^j = (-1)^{ij} x_1^i x_2^j$. Hence all contributions to the graded trace are balanced monomials with $i = j$, yielding $\text{tr}_{\mathcal{B}(M)}^Q(t) = \sum_{i=0}^{N-1} (-1)^i t^{2i} = \sum_{i=0}^{N-1} (-1)^i t^{2i} = (N-1)^2$.

We now give examples where this formula can be applied. Note that the normalizing-condition seems to be very restrictive and the examples below crucially rely on exceptional behaviour for $q = -1$. Nevertheless, we obtain nontrivial examples, such as $A_3^\sim = -1$, and will use the previous formula systematically in the last subsection in conjunction with the finer root system presented in the examples there.

Example 3.24. Let $q$ be a primitive $2N$-th root of unity and $M$ a Yetter-Drinfel’d module with braiding matrix corresponding to $A_1 \cup A_1$ and $q_{ij} = \begin{pmatrix} q^2 & -1 \\ -1 & q^2 \end{pmatrix}$. Extend $Q : x_1 \leftrightarrow x_2$ to an algebra automorphism of $\mathcal{B}(M)$. Note that any other off-diagonal entries $a$, $a^{-1}$ would let $Q$ fail to preserve the braiding matrix. We have $\mathcal{B}(M) \cong k[x_1]/(x_1^N) \otimes k[x_1]/(x_1^N)$ and explicitly calculate $Qx_1^i x_2^j = x_2^i x_1^j = (-1)^{ij} x_1^i x_2^j$. Hence all contributions to the graded trace are balanced monomials with $i = j$, yielding $\text{tr}_{\mathcal{B}(M)}^Q(t) = \sum_{i=0}^{N-1} (-1)^i t^{2i} = \sum_{i=0}^{N-1} (-1)^i t^{2i} = (N-1)^2$. 

\[Q \otimes x_{\alpha}^k = q_A(Q)^k \prod_{\alpha \in A} \lambda_Q(\alpha)^k \cdot \otimes_{\alpha \in A} x_{Q\alpha}^k = \lambda_Q(A)^k \cdot \otimes_{\alpha \in A} x_{Q\alpha}^k.\]
Example 3.25. Consider again the $A_2$ example $M = x_1 \mathbb{k} \oplus x_2 \mathbb{k}$ with $q_{ij} = \begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix}$ and $q$ a primitive $2N$-th root of unity. Consider in this case the diagram automorphism $Q : x_1 \leftrightarrow x_2$ again. In the theory of Lie algebra foldings, the flipped edge $x_1x_2$ is called a loop. We easily calculate that in such cases the standard root system is never normalized by $Q$, since $q \neq -1$: $Qx_{12} = Q(x_1x_2 - q^{-1}x_2x_1) = x_2x_1 - q^{-1}x_1x_2 = -q^{-1}(x_1x_2 - qx_2x_1) \neq x_{12}$. We will discuss this example and its factorization in the next subsection.

Example 3.26. Consider the braiding matrix

$$q_{ij} = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

which gives rise to a Nichols algebra of type $A_3$ and hence a root system

$$\mathfrak{B}(M) \cong \mathbb{k}[x_1]/(x_1^2) \otimes \mathbb{k}[x_2]/(x_2^2) \otimes \mathbb{k}[x_2]/(x_2^3)$$

$$\otimes \mathbb{k}[x_{12}]/(x_{12}^2) \otimes \mathbb{k}[x_{32}]/(x_{32}^2) \otimes \mathbb{k}[x_{1(32)}]/(x_{1(32)}^2)$$

(these are choices) with

$$x_{12} = x_1x_2 - ix_2x_1, \quad x_{32} = x_3x_2 - ix_2x_3, \quad x_{123} = x_1x_{32} + ix_3x_{12}.$$ Notice that for the specific choice of $q^2 = -1$, by chance, we also have

$$x_{3(12)} := x_3x_{12} + ix_3x_{32} = x_3x_1x_2 - ix_3x_2x_1 + ix_3x_2x_3 + x_2x_1x_3$$

$$= -x_1x_3x_2 - ix_3x_2x_1 + ix_1x_2x_3 - x_2x_3x_1$$

$$= -(x_1x_3x_2 - ix_1x_2x_3 + ix_3x_2x_1 + x_2x_3x_1) = -x_{1(32)}.$$ Consider the diagram automorphism $Q : x_1 \leftrightarrow x_3$ that preserves the braiding matrix and hence gives rise to an algebra automorphism of $\mathfrak{B}(M)$. We show that it normalizes the chosen factorization:

$$Qx_{12} = Q(x_1x_2 - ix_2x_1) = x_3x_2 - ix_2x_3 = x_{32};$$

$$Qx_{32} = x_{12};$$

$$Qx_{1(32)} = Q(x_1x_{32} + ix_{32}x_1) = x_3x_{12} + ix_{12}x_3 = x_{3(12)} = -x_{1(32)}.$$ Hence our product formula yields for the graded trace of $Q$:

$$\text{tr}^Q_{\mathfrak{B}(M)}(t) = \text{tr}^Q_{(x_2)}(t) \cdot \text{tr}^Q_{(x_1x_3)}(t) \cdot \text{tr}^Q_{(x_{12}x_{32})}(t) \cdot \text{tr}^Q_{(x_{1(32)})}(t)$$

$$= (2)_t \cdot (2)_{-t^2} \cdot (2)_{-t^3} \cdot (2)_{-t^3}.$$
3.4 A Non-Normalizing Example with Alternative PBW-Basis

Consider Example 3.25 in the previous subsection, which is not normalized, for $q = -i$. We first calculate the graded trace directly on the basis $x_1^i x_2^j x_1^k$ with $i, j, k \in \{0, 1\}$:

$$
\text{tr}^Q_{\mathcal{B}(M)}(t) = \text{tr}(Q|_{\mathcal{B}(M)}) + t \cdot \text{tr}(Q|_{x_1}) + t^2 \cdot \text{tr}(Q|_{x_1 x_2, x_1^{12}}) + t^3 \cdot \text{tr}(Q|_{x_1 x_1^{12}, x_2^{12}}) + t^4 \cdot \text{tr}(Q|_{x_1 x_2 x_1^{12}})
$$

$$
= \text{tr}(1) + t \cdot \text{tr}(0 \cdot 1 \cdot 0) + t^2 \cdot \text{tr}(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) + t^3 \cdot \text{tr}(0 \cdot -i \\ i \cdot 0) + t^4 \cdot \text{tr}(1) = 1 + t^4
$$

We observe that in this case we have the following symmetric analogon to a PBW-basis which explains this graded trace: Denote $x^+ := x_1 + x_2$ and $x^- := x_1 - x_2$. Then these elements have a common power:

$$
y := x^+ = x^- = x_1 x_2 + x_2 x_1 \quad z := x^4_+ = x^4_- = 2x_1 x_2 x_1 x_2
$$

Moreover, we have the relation $r := x_+ x_- y = (x_1 x_2 + x_2 x_1)(x_1 x_2 + x_2 x_1) = -x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 = 0$ and up to $r$ the elements $x^i_+ x^j_- y^k z^l$ form a basis. More precisely, we have an alternative presentation for the Hilbert series

$$
H(t) = \frac{(2)_t (2)_t (2)_t (2)_t}{(2)_t} = 1 + 2t + 2t^2 + 2t^3 + t^4
$$

that could be reformulated on the level of graded vector spaces:

$$
\mathcal{B}(rk) \to \mathcal{B}(M) \to \mathcal{B}(x^+ k) \otimes \mathcal{B}(x^- k) \otimes \mathcal{B}(y k) \otimes \mathcal{B}(z k).
$$

This factorization with relation is stabilized by the action of $Q$ ($x^+ , y , z$ even and $x^- , r$ odd), from which we conclude with the formula of Lemma 3.20:

$$
\text{tr}^Q_{\mathcal{B}(M)} = \frac{(2)_t (2)_t (2)_t (2)_t}{(2)_t} = 1 + t^4.
$$

3.5 Factorization Mechanisms In Examples Over Nonabelian Groups

We shall finally consider a family of examples over nonabelian groups obtained by the first author in [20]: For a finite-dimensional semisimple
simply-laced Lie algebra \( \mathfrak{g} \) with a diagram automorphism \( \sigma \), consider the diagonal Nichols algebra \( \mathfrak{N}(M) \) of type \( \mathfrak{g} \). We define a covering Nichols algebra \( \mathfrak{N}(M) \) over a nonabelian group \( G \) (an extraspecial 2-group) and with folded Dynkin diagram \( g^\ast \). The covering Nichols algebra is isomorphic to \( \mathfrak{N}(M) \) as an algebra; however, there exist nondiagonal Doi twists, which leave the Hilbert series invariant.

The root system of \( \mathfrak{N}(M) \) is \( g^\ast \), but because the root spaces \( M_\alpha \) are mostly 2-dimensional, this cannot explain the full factorization of the Hilbert series. The old factorization along the \( \mathfrak{g} \)-root system is not a factorization into sub-Yetter-Drinfel’d modules, but nevertheless still shows the full factorization.

Example 3.27. Let \( G = D_4 = \langle a, b \mid a^4 = b^2 = 1 \rangle \) the dihedral group and consider the Nichols algebra \( \mathfrak{N}(O_b^\chi \oplus O_b^\psi) \) where the centralizer characters are \( \chi(b) = -1 \), \( \psi(ab) = -1 \), and \( \chi(a^2) = \psi(a^2) = 1 \) (respectively \( \chi(a^2) = \psi(a^2) = -1 \) for the nondiagonal Doi twist). This was the first known example for a finite-dimensional Nichols algebra in [23] and it is known to be of type \( A_2 \). We have \( \mathfrak{N}(O_b^\chi \oplus O_b^\psi) \cong \mathfrak{N}(O_b^\chi) \otimes \mathfrak{N}(O_b^\psi) \otimes \mathfrak{N}([O_b^\chi, O_b^\psi]) \) with respective Hilbert series \( H(t) = (2)_t^2 \cdot (2)_t^2 \cdot (2)_t^2 \). From the root system we can only explain the factorization into three factors. However, this Nichols algebra is the covering Nichols algebra of a diagonal Nichols algebra of type \( A_2 \cup A_2 \) and from this presentation we may read off the full factorization in an inhomogeneous PBW-basis:

\[
\mathfrak{N}(O_b^\chi \oplus O_b^\psi) \cong \left( \mathfrak{N}(x_b + x_{a^2b}) \otimes \mathfrak{N}(x_{ab} + x_{a^2b}) \otimes \mathfrak{N}(x_b + x_{a^2b}, x_{ab} + x_{a^2b}) \right) \\
\otimes \left( \mathfrak{N}(x_b - x_{a^2b}) \otimes \mathfrak{N}(x_{ab} - x_{a^2b}) \otimes \mathfrak{N}([x_b - x_{a^2b}, x_{ab} - x_{a^2b}] \right) \\
\Rightarrow \quad H(t) = (2)_t^2 \cdot (2)_t^2 \cdot (2)_t^2 \cdot (2)_t^2 \cdot (2)_t^2 \cdot (2)_t^2
\]

3.6 Factorization by Sub-Nichols-Algebras

During this subsection, let \( k \) be an arbitrary field, \( G \) a finite group and consider a rank-1-Yetter-Drinfel’d module \( M := O_b^\chi \) for some \( g \in G \) and a one-dimensional representation \( \chi \) of \( \text{Cent}(g) \). Denote the conjugacy class of \( g \) in \( G \) with \( X \). Define the enveloping group

\[
\text{Env}(X) := \langle g_x, x \in X \mid g_xg_y = g_{xyx^{-1}}g_x \rangle.
\]

The Nichols algebra \( \mathfrak{N}(M) \) is naturally graded by \( \text{Env}(X) \). Now \( \pi : \text{Env}(X) \to \mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z} \), \( g_x \mapsto 1 \) for all \( x \in X \) establishes \( \mathbb{Z}_k \) as a canonical quotient of \( \text{Env}(X) \) for all \( k \in \mathbb{N} \). The original group \( G \) is another quotient of \( \text{Env}(X) \), and this induces an action of \( \text{Env}(X) \) on \( \mathfrak{N}(M) \).
Denote the generators of the Nichols algebra by \( e_x := x \otimes 1 \) and by \( e_x^* \) the dual base.

Define \( q_{x,y} \) for \( x, y \in X \subset G \) by \( c(e_x \otimes e_y) = q_{x,y} e_{x,y^{-1}} \otimes e_x \) and \( m \in \mathbb{N} \) minimal such that \( 1 + q_{x,x} + q_{x,x}^2 + \cdots + q_{x,x}^{m-1} = 0 \) for all diagonal elements \( q_{x,x} \). As we only consider rank one, \( m \) does not depend on \( x \), and we call \( m = m_x \) the order of \( q \). Throughout this section, assume that each coefficient \( q_{x,y} \) is a (not necessarily primitive) \( m \)-th root of unity.

In \cite{22}, the second author derived divisibility relations for the Hilbert series of Nichols algebras by an analysis of the modified shift

\[
\xi_x : \mathcal{B}(M) \to \mathcal{B}(M), \quad v \mapsto (\partial_x^{op})^{m-1}(v) + e_x v
\]

and similar maps, where \( x \in X \) is arbitrary and \( \partial_x^{op} = (e_x^* \otimes \text{id})\Delta \) is the opposite braided derivation.

For each \( x \in X \), \( \xi_x \) is a linear isomorphism, leaving \( \ker \partial_y \) invariant for all \( y \in X \setminus \{ x \} \), where \( \partial_y = (\text{id} \otimes e_y^*)\Delta \) is the braided derivation of Definition \ref{def:braided_derivation}. If \( \Xi \) is the group generated by all \( \xi_x, \ x \in X \), the orbit of \( 1 \in \mathcal{B}(M) \) under \( \Xi \) linearly spans \( \mathcal{B}(M) \). Finally, let \( \pi : \text{Env} X \to H \) be some group epimorphism, such that \( \pi(g_x)^m = e \). Then \( \xi_x \) maps the \( \mathcal{B}(M) \)-layer of degree \( h \) to the layer of degree \( \pi(g_x)h \) for all \( h \in H \), see \cite{22}, Proposition 9. For us, the relevant quotient \( H \) of \( \text{Env} X \) will not be \( G \), but \( \mathbb{Z}_m \) as chosen above.

In addition, for each \( x, y \in X \), \( \xi_x \) satisfies \( g_y \circ \xi_x = q_{y,x} \cdot \xi_{y \circ x} \circ g_y \), where we identify \( g_y \) with the action of \( g_y \) on \( \mathcal{B}(M) \):

\[
g_y(e_x \cdot v) = q_{y,x} e_{y \circ x} \cdot (g_y \cdot v)
\]

and

\[
g_y(\partial_x^{op}(v)) = (e_x^* \otimes g_y)\Delta(v) = q_{y,x}^{-1}(e_x^* \otimes \text{id})(g_y \otimes g_y)\Delta(v) = q_{y,x}^{-1} \cdot \partial_{y \circ x}^{op}(g_y \cdot v),
\]

thus

\[
g_y \circ (\partial_x^{op})^{m-1} = q_{y,x}^{(m-1)} \cdot (\partial_{y \circ x}^{op})^{m-1} \circ g_y = q_{y,x} \cdot (\partial_{y \circ x}^{op})^{m-1} \circ g_y
\]

for all \( v \in \mathcal{B}(M) \) (the second equality is due to \( e_x^* (g_y \cdot e_z) = q_{y,z} \delta_{y \circ x,y \circ z} = q_{y,z} e_x^*(e_z) \)).

**Lemma 3.28.** Let \( \lambda \) be a \( k \)-th root of unity (not necessarily primitive). Let \( M \) be some finite-dimensional graded vector space and \( Q \in \text{Aut} M \). Consider \( \mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} \) as a quotient of \( \mathbb{Z} \), then \( M \) has a \( \mathbb{Z}_k \)-grading. With respect to this grading, \( \text{tr} Q |_{M([j+1]_k)} = \lambda \text{tr} Q |_{M([j]_k)} \) for all \( j \in \mathbb{Z} \) if and only if the \( \mathbb{Z} \)-graded character \( \text{tr} Q |_{M}(t) \) is divisible by \( (k)_{\lambda} \).

**Proof.** This is a straight-forward generalization of Lemma 6 of \cite{22}: Given a polynomial \( p \), denote with \( p_j \) the coefficient of \( t^j \) in \( p(t) \) (or zero if \( j < 0 \)). Set \( b_j := \text{tr} Q |_{M(j)} \).
“⇒”: Choose \( p_j := 0 \) for each \( j \in \mathbb{Z}_{<0} \) and \( p_j := b_j - \sum_{i=1}^{k-1} \lambda^i p_{j-i} \), hence \( b_j - \lambda b_{j-1} = p_j - \lambda^k p_{j-k} = p_j - p_{j-k} \). Let \( d \in \mathbb{N}_0 \) be such that \( d \cdot k \) is larger than the top degree of \( M \). Summation of the previous equation then yields for each \( 0 \leq l \leq k - 1 \) the telescoping sum

\[
\sum_{0 \leq j \leq d} b_{j+k+l} - \lambda \cdot \sum_{0 \leq j \leq d} b_{j+k+l-1} = -p_{l-k} + p_{d+k+l}.
\]

The two sums on the left hand side sum to \( \text{tr} Q|_{M([l])} \) and \( \text{tr} Q|_{M([l-1])} \), respectively, so by assumption, the left hand side is zero. \( p_{l-k} \) is zero by definition \( (l-k < 0) \), hence \( p_{d+k+l} \) is zero. This proves that \( p \) is a polynomial, and from \( b_j = \sum_{i=0}^{k-1} \lambda^i p_{j-i} \) follows \( \text{tr}_M(t) = (k)_M \cdot p(t) \).

“⇐”: Let \( \text{tr}_M(t) = (k)_M \cdot p(t) \) for some polynomial \( p \). We have \( b_j = \sum_{i=0}^{k-1} \lambda^i p_{j-i} \) and therefore for each \([l]_k \in \mathbb{Z}_k\)

\[
\text{tr} Q|_{M([l])} = \sum_{j \in \mathbb{N}_0, j \equiv l \pmod{k}} \text{tr} Q|_{M(j)} = \sum_{j \in \mathbb{N}_0} \sum_{i=0}^{k-1} \lambda^i p_{j-i} = \sum_{j \in \mathbb{N}_0} \lambda^l j p_j,
\]

from which follows \( \text{tr} Q|_{M([l+1])} = \lambda \cdot \text{tr} Q|_{M([l])} \).

\[\square\]

**Theorem 3.29.** Let \( G \) be a finite group, \( G' \subset G \) a proper subgroup, and \( h \in G' \) be arbitrary. Let \( \chi \) be a one-dimensional representation of \( \text{Cent}_G(h) \), and let \( \chi' \) be its restriction to \( \text{Cent}_{G'}(h) = \text{Cent}_G(h) \cap G' \). Set \( M := \mathcal{O}_h \) and \( M' := \mathcal{O}_{h'} \). Set \( X \) and \( X' \) to be the conjugacy classes of \( h \) in \( G \) and \( G' \), respectively. Let \( g \in \text{Env}(X') \) be arbitrary and identify \( g \) with its actions on \( \mathcal{B}(M) \) and \( \mathcal{B}(M') \).

1) Then \( \text{tr}_{\mathcal{B}(M)}^g(t) \) is divisible by \( \text{tr}_{\mathcal{B}(M')}^g(t) \).

2) Assume there is some \( x \in X \), such that \( g \circ \xi_x = \lambda \cdot \xi_x \circ g \) for some \( m \)-th root of unity \( \lambda \), where \( m \) is the order of \( q \). Then \( \text{tr}_{\mathcal{B}(M)}^g(t) \) is divisible by \((m)_M \cdot \text{tr}_{\mathcal{B}(M')}^g(t)\).

**Proof.** Set \( K := \bigcap_{x \in X} \ker \partial_x \).

1) \( \mathcal{B}(M) \) is free as a \( \mathcal{B}(M') \)-module, so there is a linear isomorphism \( \mathcal{B}(M) \cong K \otimes \mathcal{B}(M') \) mediated by multiplication (e.g. [25], [8]). \( K \) and \( \mathcal{B}(M') \) are both closed under the action of \( \text{Env}(X') \) (\( K \) is closed because \( X' \) is closed under conjugation). Therefore, \( \mathcal{B}(M) \cong K \otimes \mathcal{B}(M') \) as \( \text{Env}(X') \)-representations and \( \text{tr}_{\mathcal{B}(M)}^g(t) = \text{tr}_K^g(t) \cdot \text{tr}_{\mathcal{B}(M')}^g(t) \).

2) We show that \( \text{tr}_K^g(t) \) is divisible by \((m)_M \). Set \( K_j := K \cap \mathcal{B}(M)_j \) (layer \( j \) of \( \mathcal{B}(M) \) with \( j \in \mathbb{Z}_m \)). The modified shift operator \( \xi_x \) establishes
a linear isomorphism between $K_j$ and $K_{j+1}$ for each $j \in \mathbb{Z}_m$. Let $B$ be a basis for $K_j$ and $B' := \xi_x(B)$, and denote with $v^*$ the dual basis element corresponding to $v \in B$ for the basis $B$ and $v \in B'$ for the basis $B'$, respectively. Then

$$\text{tr}_{K_{j+1}} g = \sum_{v \in B'} v^*(g.v) = \sum_{b \in B} b^*(\xi_x^{-1}g\xi_x(b)) = \lambda \sum_{b \in B} b^*(g.b) = \lambda \text{tr}_B g$$

holds. Apply Lemma 3.28.

The condition $g \circ \xi_x = \lambda \cdot \xi_x \circ g$ of part (2) of Theorem 3.29 is fulfilled for $gx = xg$ and $\lambda = q_{y_1,x} \cdots q_{y_s,x}$ with $g = g_{y_1} \cdots g_{y_s}$, $y_1, \ldots, y_s \in X$.

**Example 3.30.** Choose $G' = S_3 \subset G = S_4$ and $h \in G'$ a transposition, so $X$ and $X'$ are the conjugacy classes of transpositions. Choose $\chi$ and $\chi'$ to be the alternating representations of $G$ and $G'$. Their Nichols algebras will appear again in Subsections 4.6 and 4.1, respectively. Choose $g = (1 2)$ and $x = (3 4)$. Then $g$ and $x$ commute, and Theorem 3.29 explains why $\text{tr}_{\mathcal{B}(M)}^g(t) = (2)^4t(3)^2t(2)t^4$ contains the factor $(2)^{-t}(3)^t$.

## 4 Calculations for Small Rank-1 Nichols Algebras

The following results have been calculated with the help of GAP ([26]) in a straight-forward way: First, calculate a linear basis for the given Nichols algebra, then generate the representing matrix of the action of each element of the conjugacy class $X$, which also generates $G$, and then calculate the graded traces of all conjugacy classes.

The Nichols algebra of Subsection 4.4 admits a large dimension of 5,184. For this size, it was not possible for us to calculate all matrices we needed. In this special case, we made use of Corollary 2.10, so we could restrict our matrix calculations to the lower half of grades and compute the full graded trace by Poincaré duality.

The Nichols algebras of dimensions 326,592 and 8,294,400 are computationally not yet accessible with this method.

### 4.1 dim $M = 3$, dim $\mathcal{B}(M) = 12$

Let $G = S_3$ and $g = (1 2)$, then the centralizer of $g$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and we choose $\chi$ to be its alternating representation. Then $\mathcal{B}(O_g^3)$ is a 12-dimensional Nichols algebra on which $G$ acts faithfully with the following graded characters:

$$\text{tr}^{(12)}_{\mathcal{B}(M)}(t) = (2)^2t(3)^2t \quad \text{tr}^{(123)}_{\mathcal{B}(M)}(t) = (2)^{-t}(3)^2t \quad \text{tr}^{(123)}_{\mathcal{B}(M)}(t) = (2)^2t(2)^2t$$
From this, we can calculate the decomposition into irreducible $G$-representations in each degree. If we denote the trivial irreducible, alternating, and standard $G$-representations with $T$, $A$, and $S$, respectively, we find

\[
\mathcal{B}(O_g^X) \cong T \oplus (A \oplus S)t \oplus 2St^2 \oplus (A \oplus S)t^3 \oplus Tt^4
\]

\[
\cong (T \oplus At) \otimes (T \oplus St \oplus St^2 \oplus At^3)
\]

as $G$-representation. The factorization in line 2 results from a certain sub-Nichols-algebra (see [3.6]) and implies the factorizations

\[
\text{tr}^e_{\mathcal{B}(M)}(t) = (2)_t \cdot (2)_t(3)_t
\]

\[
\text{tr}^{(12)}_{\mathcal{B}(M)}(t) = (2)_{-t} \cdot (2)_{-t}(3)_t
\]

\[
\text{tr}^{(123)}_{\mathcal{B}(M)}(t) = (2)_t \cdot (2)^2_{-t}(2)_t
\]

of the graded characters. To understand the factorization of the remaining terms $(2)_t(3)_t$, $(2)_{-t}(3)_t$, and $(2)^2_{-t}(2)_t$, this line of argument, however, fails, because $T \oplus St \oplus St^2 \oplus At^3$ does not factor into a tensor product of $G$-representations (we have already met this phenomenon in another situation in Section 2.3). For $g = e$ and $g = (12)$, we may apply Theorem [3.29](2) to explain the additional factors $(2)_t$ and $(2)_{-t}$, respectively, but this neither helps in the case $g = (123)$, nor to understand the origin of the factors $(3)_t$ for $g \in \{e, (12), (123)\}$.

If $\mathcal{B}(O_g^X)$ does not factor as a $G$-representation, one might think that it may still factor as an $\langle h \rangle_G$-representation for each $h \in G$, which would explain the factorization of the graded characters just as well. This, however, is wrong: Take $h = (123)$, which is of order 3. Let $T$ be the trivial irreducible representation, $B$ one of the non-trivial irreducible representations, and set $C := B \otimes B$. Then $\mathcal{B}(O_g^X)$ is

\[
\mathcal{B}(O_g^X) \cong T \oplus (T \oplus B \oplus C)t \oplus (2B \oplus 2C)t^2 \oplus (T \oplus B \oplus C)t^3 \oplus Tt^4
\]

\[
\cong (T \oplus Tt) \otimes (T \oplus (B \oplus C)t \oplus (B \oplus C)t^2 \oplus Tt^3)
\]

as an $\langle h \rangle_G$-representation and $T \oplus (B \oplus C)t \oplus (B \oplus C)t^2 \oplus Tt^3$ does not factor further.

4.2 dim $M = 3$, dim $\mathcal{B}(M) = 432$

Assume char $k = 2$ and $k$ admits a primitive third root of unity $\zeta$. Choose

\[
G = \langle g_1, g_2 : g_1^6, g_2^6, (g_1g_2)^3, g_1^2g_2^{-2} \rangle \cong \mathbb{Z}_3 \times S_3
\]
and \( g = g_1 \). The centralizer of \( g \) is \( \langle g_1 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \). Choose \( \chi(g_1) = \zeta \). Then \( \mathfrak{B}(\mathcal{O}_3) \) is a faithful \( G \)-representation of dimension 432. \( G \) has nine conjugacy classes, and we choose \( \{ e, g_1, g_1^2, g_1^3, g_1^4, g_1g_2, g_1^2g_2, g_1^5g_2 \} \) as their representatives. Then the graded characters of \( \mathfrak{B}(\mathcal{O}_3) \) (not Brauer characters, but with values in \( \mathbb{Z}_2 \)) are:

\[
\begin{align*}
\text{tr}_{\mathfrak{B}(M)}^e(t) &= (2)_t^0(3)_t^7 \\
\text{tr}_{\mathfrak{B}(M)}^{g_1}(t) &= (2)_t^7(3)_t^6(2)_t^{6\zeta} \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^2}(t) &= (2)_t^7(3)_t^6(2)_t^{6\zeta^2} \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^3}(t) &= (2)_t^{10}(2)_t^{10\zeta} \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^4}(t) &= (2)_t^{10}(2)_t^{10\zeta^2} \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^5}(t) &= (3)_t^{10}
\end{align*}
\]

4.3 \( \dim M = 4, \dim \mathfrak{B}(M) = 36 \) or 72

Consider char \( k = 2 \), \( G = \mathbb{A}_4 \) and \( g = (123) \). The centralizer of \( g \) is isomorphic to \( \mathbb{Z}_3 \), choose \( \chi \) to be the trivial irreducible representation. Then \( \mathfrak{B}(\mathcal{O}_3) \) is 36-dimensional with graded characters (not Brauer characters):

\[
\begin{align*}
\text{tr}_{\mathfrak{B}(M)}^e(t) &= (2)_t^2(3)_t^2 \\
\text{tr}_{\mathfrak{B}(M)}^{(12)}(t) &= (2)_t^2(3)_t^2 \\
\text{tr}_{\mathfrak{B}(M)}^{(34)}(t) &= (2)_t^2(3)_t^2
\end{align*}
\]

In characteristic \( \not\equiv 2 \), there is a very similar Nichols algebra of dimension 72: Assume char \( k \not\equiv 2 \) and

\[
G = \langle g_1, g_2 : g_1^6, g_2^6, [g_1, g_2], (g_1g_2)^3, (g_1g_2)^2 \rangle \cong \mathbb{A}_4 \times \mathbb{Z}_2.
\]

Choose \( g = g_1 \), then the centralizer is \( \langle g_1 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \), to which we choose the representation \( \chi(g_1) := -1 \). Then the graded characters of \( \mathfrak{B}(\mathcal{O}_3) \) are:

\[
\begin{align*}
\text{tr}_{\mathfrak{B}(M)}^e(t) &= (2)_t^3(3)_t^{-t}(3)_t^2 \\
\text{tr}_{\mathfrak{B}(M)}^{g_1}(t) &= (2)_t^3(2)_t^3(3)_t^{-t}(3)_t \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^2}(t) &= (2)_t^3(2)_t^3(3)_t^{-t}(3)_t \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^3}(t) &= (2)_t^3(3)_t^{-t}(3)_t \\
\text{tr}_{\mathfrak{B}(M)}^{g_1^4}(t) &= (2)_t^3(3)_t^{-t}
\end{align*}
\]

Consider the subgroup \( H := \langle g_1^2, g_1^3 \rangle \cong \mathbb{A}_4 \) of \( G \). The graded characters of the \( H \)-action on \( \mathfrak{B}(\mathcal{O}_3) \) are exactly those of the left column in the above list. If considered in characteristic 2, these polynomials are divisible by the corresponding graded characters of the 36-dimensional Nichols algebra, with \( (2)_t(3)_t \) as common quotient.

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4.4 \ dim M = 4, \ dim \mathcal{B}(M) = 5184

Assume \text{char} \mathbb{k} \neq 2 and that \mathbb{k} admits a primitive third root of unity \zeta. Choose

\[ G := \langle a, b : a^3 = b^3 = (ab)^2 \rangle \cong \text{SL}(2, 3) \]

and \( g = a^4 \). The centralizer of \( g \) is \( \langle a \rangle \cong \mathbb{Z}_6 \). Choose the representation \( \chi(a) := -\zeta \). This leads to the following graded characters of \( \mathcal{B}(O_3^+) \):

\[
\begin{align*}
\text{tr}_{\mathcal{B}(M)}^0(t) &= (2)_1^4(2)_2^2(3)_2^4 \\
\text{tr}_{\mathcal{B}(M)}^1(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2 \\
\text{tr}_{\mathcal{B}(M)}^2(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2 \\
\text{tr}_{\mathcal{B}(M)}^3(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2 \\
\text{tr}_{\mathcal{B}(M)}^4(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2 \\
\text{tr}_{\mathcal{B}(M)}^5(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2 \\
\text{tr}_{\mathcal{B}(M)}^{ab}(t) &= (2)_1^4(2)_1^4(2)_2^2(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2\zeta(2)_2^2
\end{align*}
\]

In characteristic 2, \( G = \mathbb{A}_4 \) yields a Nichols algebra with the same Hilbert series. Some of the above conjugacy classes merge in this case, because \( \text{SL}(2, 3) \) is a \( \mathbb{Z}_2 \)-extension of \( \mathbb{A}_4 \), but apart from that, the resulting graded characters are the same as above.

4.5 \ dim M = 5, \ dim \mathcal{B}(M) = 1280

Choose

\[ G := \langle a, b : a^4, b^4, a^3b^2 \rangle \]

and \( g := a \). \( G \) is isomorphic to the GAP’s small group number 3 of size 20 (\[26\]), a semi-direct product of \( \mathbb{Z}_5 \) and \( \mathbb{Z}_4 \). The centralizer of \( g \) is \( \langle a \rangle \cong \mathbb{Z}_4 \). Choose the representation \( \chi(a) := -1 \). Then \( \mathcal{B}(O_5) \) is a faithful \( G \)-representation of dimension 1280 with the following graded characters:

\[
\begin{align*}
\text{tr}_{\mathcal{B}(M)}^0(t) &= (2)_1^4(2)_2^4(5)_2^3 \\
\text{tr}_{\mathcal{B}(M)}^a(t) &= (2)_1^4(2)_1^4(2)_2^4(2)_2^4(2)_2^4(2)_2^4(5)_2^3 \\
\text{tr}_{\mathcal{B}(M)}^{a^2}(t) &= (2)_1^4(2)_1^4(2)_2^4(2)_2^4(5)_2^3 \\
\text{tr}_{\mathcal{B}(M)}^{a^3}(t) &= (2)_1^4(2)_1^4(2)_2^4(2)_2^4(5)_2^3 \\
\text{tr}_{\mathcal{B}(M)}^{ab}(t) &= (2)_1^4(2)_1^4(2)_2^4
\end{align*}
\]

There appears a second, non-isomorphic (but dual) Nichols algebra if one chooses \( g := a^3 \), \( \chi(a) := -1 \) instead (see Example 2.1 in \[1\]). It features the same graded characters as \( \mathcal{B}(O_5) \) above.
There are three pairwise non-isomorphic cases to consider with \( \dim M = 6 \) and \( \dim \mathcal{B}(M) = 576 \).

First, choose \( G = S_4 \) and \( g := (1\,2) \). The centralizer of \( g \) is \( \langle (1\,2), (3\,4) \rangle \cong \mathbb{Z}_2/\mathbb{Z}_2 \). Choose the representation with \( \chi((1\,2)) = -1 \) and \( \chi((3\,4)) = -1 \). The graded characters of \( \mathcal{B}(\mathcal{O}_g^\chi) \) are:

\[
\begin{align*}
\text{tr}^e_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^2(3)_t^2 \\
\text{tr}^{(1\,2)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(3)_t^2 \\
\text{tr}^{(1\,2)(3\,4)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(2)_t^2(3)_t^2
\end{align*}
\]

Now choose the representation \( \chi((1\,2)) = -1, \chi((3\,4)) = 1 \) instead. Then the graded characters of \( \mathcal{B}(\mathcal{O}_g^\chi) \) are:

\[
\begin{align*}
\text{tr}^e_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^2(3)_t^2 \\
\text{tr}^{(1\,2)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(2)_t^2(3)_t^2 \\
\text{tr}^{(1\,2)(3\,4)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(2)_t^2(3)_t^2
\end{align*}
\]

Third, choose \( G = S_4 \) and \( g := (1\,2\,3\,4) \). The centralizer of \( g \) is \( \langle (1\,2\,3\,4) \rangle \cong \mathbb{Z}_4 \). Choose the representation \( \chi((1\,2\,3\,4)) = -1 \). Then \( \mathcal{B}(\mathcal{O}_g^\chi) \) has the following graded characters:

\[
\begin{align*}
\text{tr}^e_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^2(3)_t^2 \\
\text{tr}^{(1\,2)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(2)_t^2(3)_t^2 \\
\text{tr}^{(1\,2)(3\,4)}_{\mathcal{B}(M)}(t) &= (2)_t^4(2)_t^4(2)_t^2
\end{align*}
\]

Note how the graded characters differ pairwise for these three cases, a simple way to see that the three Nichols algebras obtained are non-isomorphic, not even as \( S_4 \)-representations, although the first and the second case are twist-equivalent to each other (\([27]\)).

### 4.7 Observations

From the examples of the previous sections, we derive the following observations, which may help us in understanding the factorization of the Hilbert series and graded characters of any Nichols algebra. A theory of the representations coming from a Nichols algebra should be able to explain all of them.
1. The zeros of the graded characters of all examples above are $n$-th roots of unity, where $n$ most of the time is a divisor of $\#G$, but not always: In Subsection 4.4, ninth roots of unity appear though $\#G = 24$; and in Subsection 4.5, we have $\#G = 20$, but $tr^q_{B(M)}(t)$ has an eighth root of unity. A deeper understanding why there are only roots of unity and which roots appear how often is eligible.

2. Each of the characters $tr^q_{B(M)}(t)$ with $g \neq e$ includes a factor $1 - t$ (or $1 + t$ in characteristic 2), therefore the non-graded character $tr^q_{B(M)}(t)(1)$ vanishes. From this follows that all of the above Nichols algebras are (seen as their respective $G$-representations) multiples of the regular $G$-representation. The only exceptions to this are the 432-dimensional and the 72-dimensional Nichols algebras of Subsections 4.2 and 4.3, each of which admits a single non-trivial conjugacy class with non-vanishing character.

3. The smallest common multiple $p$ of the graded characters of a single Nichols algebra has a surprisingly small degree. We want to point out that the quotient $p/ tr^q_{B(M)}(t)$ typically is a polynomial whose roots have the same order as $g$ has in $G$.

4. Although all of the characters factor nicely (see point (1)), there is no corresponding factorization of the respective representations; we showed this in Subsection 4.1.

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