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GENERALIZED MULLINEUX INVOLUTION AND PERVERSE EQUivalences

THOMAS GERBER, NICOLAS JACON, AND EMILY NORTON

Abstract. We define a generalization of the Mullineux involution on multipartitions using the theory of crystals for higher level Fock spaces. Our generalized Mullineux involution turns up in representation theory via two important derived functors on cyclotomic Cherednik category $O$: Losev’s “$\kappa = 0$” wall-crossing, and the Ringel duality.

Introduction

It has been known since the foundational work of Frobenius that partitions of $n$ naturally label the complex irreducible representations of the symmetric group $S_n$. If we take an irreducible representation labeled by a partition $\lambda$ and tensor it with the sign representation, we obtain an irreducible representation labeled by the transpose of $\lambda$. The story in positive characteristic is more subtle: the irreducible representations of $S_n$ over a field of characteristic $p > 0$ are labeled by the $p$-regular partitions (partitions in which each non-zero part occurs at most $p-1$ times). Tensoring such a representation with the sign representation still yields an irreducible representation, but the resulting involution on $p$-regular partitions lacks such a simple description as taking the transpose.

In 1979, Mullineux defined a combinatorial algorithm producing an involution on $p$-regular partitions (now called the Mullineux involution), and he conjectured that this involution describes the result of tensoring an irreducible representation in characteristic $p$ [32].

In 1995, Kleshchev came up with a surprising algorithm to compute the Mullineux involution [26]. In fact, whereas Mullineux’s algorithm involved repeated operations with strips of boxes in the rim of the Young diagram, it has been understood later that Kleshchev’s algorithm can be interpreted in terms of the Kashiwara crystal of an irreducible highest weight module of level 1 for the quantum group of affine type $A_{p-1}$ [27]: the Mullineux involution is the automorphism of oriented $\mathbb{Z}/p\mathbb{Z}$-colored graphs which switches the sign of each arrow. This algorithm led to Ford and Kleschev’s proof of the Mullineux Conjecture [13]; different proofs were given later by Bessenrodt and Olsson [2] and by Brundan and Kujawa [5].

The Mullineux involution can be generalized to various extents. First, one can look at the Hecke algebra of $S_n$ with parameter specialized to an $e$-th root of 1, $e \in \mathbb{Z}_{\geq 2}$ (see [4]). An involution on the set of $e$-regular partitions (which parametrize the associated irreducible representations) can then be defined using crystals as above, see [27] Section 7]. Next, Fayers defined a Mullineux involution for the Hecke algebra of the complex reflection group $G(\ell,1,n)$ (the Ariki-Koike algebra) [11]. Fayers’ involution can also be computed using crystal graphs (now for irreducible highest weight modules of level $\ell$) or via a combinatorial algorithm generalizing Mullineux’s original procedure [21]. The Ariki-Koike algebra has cell modules labeled by all $\ell$-partitions, but simples labeled only by the “$e$-regular” ones. However, its module category is a quotient of a highest weight category $O_{\kappa,s}$ where

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every \(\ell\)-partition labels a simple module, raising the question whether the Mullineux involution admits a further meaningful extension to that bigger category.

Namely, consider the category \(\mathcal{O}_{n,s}(n)\) of the Cherednik algebra of \(G(\ell, 1, n)\). This category depends on parameters \(\kappa \in \mathbb{Q}^\times\) and \(s \in \mathbb{Q}_\ell\) and its Grothendieck group has a basis consisting of \(\ell\)-partitions of \(n\). In order to relate categories depending on different parameters, Losev introduced derived equivalences called wall-crossing functors \([29]\). Each wall-crossing can be thought of as a partial version of a duality functor called the Ringel duality. The wall-crossing functors and the Ringel duality are examples of a special kind of derived equivalence called a perverse equivalence \([6], [28]\), and consequently they effect a permutation of the set of simple objects, that is, a permutation of \(\ell\)-partitions. A natural question is what exactly these maps do to an \(\ell\)-partition.

We now summarize the main results of this paper. In Theorem 2.9 we define a generalization of the Mullineux involution on all multipartitions. The proof uses the result of \([16]\) that the \(\widehat{\mathfrak{sl}}_\kappa\), \(\mathfrak{sl}_\infty\), and \(\widehat{\mathfrak{sl}}_\ell\)-crystals on the level \(\ell\) Fock space all commute. Our involution \(\Phi\) is compatible with both Fayers’ and Losev’s involutions, recovering Fayers’ in the case of Uglov multipartitions. The next question is the representation-theoretic meaning of \(\Phi\). In Section 3 we study the combinatorics of perverse equivalences on module categories of Cherednik algebras. Theorems 3.6 and 3.8 give some formulas for the \(\kappa = 0\) wall-crossing in terms of \(\ell\) copies of the level 1 Mullineux involution; we recover \([29, Corollary 5.7]\) when \(\ell = 1\). Next, we look for a duality functor which produces the involution \(\Phi\), and we find in Theorem 3.16 that \(\Phi\) arises from the Ringel duality. Here the perspective of diagrammatic Cherednik algebras \([39]\) is crucial, especially \([39, Corollary 5.11]\). In Section 4, we define a refinement of \(\Phi\) with a speculative eye towards the Alvis-Curtis duality, a perverse equivalence for finite groups of Lie type which still lacks a combinatorial description outside type \(A\). This generalizes Dudas and the second author’s definition of a generalized Mullineux involution in the case \(\ell = 1\) \([9]\) by refining the \(\mathfrak{sl}_\infty\)-crystal with respect to an integer parameter \(d\).

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1. The Mullineux involution for cyclotomic Hecke algebras

We here give a quick review of the definition of the Mullineux involution for cyclotomic Hecke algebras and its crystal interpretation \([11], [21]\). This generalizes the usual notion of Mullineux involution.

1.1. Definition. Let \(\ell \in \mathbb{Z}_{\geq 1}\) and \(n \in \mathbb{Z}_{\geq 0}\). Denote \(W_{\ell,n}\) be the complex reflection group \(G(\ell, 1, n) = \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n\). Let \(R\) be a field of arbitrary characteristic and let \(v \in R^\times\) and let \((s_1, s_2, ..., s_\ell)\) be an \(\ell\)-tuple of integers.

The cyclotomic Hecke algebra (also called Ariki-Koike algebra) \(H_{R,n}^\kappa = H(v; s_1, \ldots, s_\ell)\) over \(R\) is the unital associative \(R\)-algebra with a presentation by:

- generators: \(T_0, T_1, ..., T_{n-1}\),
- relations:

\[
\begin{align*}
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad (i = 1, ..., n-2), \\
T_iT_j &= T_jT_i \quad (|j-i| > 1), \\
(T_0 - v^{s_1})(T_0 - v^{s_2})... (T_0 - v^{s_\ell}) &= 0, \\
(T_i - v)(T_i + 1) &= 0 \quad (i = 1, ..., n-1).
\end{align*}
\]
It can be seen as a deformation of the group algebra of $W_{\ell,n}$. In particular $\ell = 1$, it is the usual Hecke algebra of type $A$ and if moreover $v = 1$, we obtain the group algebra $R\mathcal{S}_n$ of the symmetric group. We denote by:

- $\Pi^\ell$ the set of all $\ell$-partitions, that is, the set of all $\ell$-tuples $(\lambda^1, \ldots, \lambda^\ell)$ of partitions.
- $\Pi = \Pi^1$ the set of all partitions.

For any subset $\mathcal{E}$ of $\Pi^\ell$ and any $n \in \mathbb{Z}_{\geq 0}$, we denote $\mathcal{E}(n)$ the set of $\ell$-partitions in $\mathcal{E}$ with total size $n$.

We let $e$ the multiplicative order of $v$ in $R$. We assume that $v \neq 1$ so that we have $e \in \{2, 3, \ldots\} \cup \{\infty\}$. We now recall several facts on the representation theory of cyclotomic Hecke algebras. We refer to [14, Chapter 5] for details. For each $\lambda \in \Pi^\ell(n)$, there is a $H_{R,n}^\lambda$-module, known as the Specht module $S^\lambda$. There exists a natural bilinear form, $H_{R,n}^\lambda$-invariant, on each of these modules and an associated radical such that the quotients $D^\lambda := S^\lambda/\rad(S^\lambda)$ are 0 or irreducible. The non-zero $D^\lambda$ give then a complete set of non isomorphic simple $H_{R,n}^\lambda$-modules.

The set $K_{l,e,s}(n)$ of $\ell$-partitions such that $D^\lambda$ is non-zero is known as the set of Kleshchev $\ell$-partitions. It is originally defined using the notion of crystal (see below) but there is another description of them, which is independant of this notion, see [20].

**Remark 1.1.** Let $e \geq 2$, $s = (s_1, \ldots, s_t)$ and $t = (t_1, \ldots, t_\ell)$ such that $t_i = s_i$ mod $e$ for all $i = 1, \ldots, \ell$. Observe that for any $n \in \mathbb{Z}_{\geq 0}$, $H_{R,n}^\lambda = H_{R,n}^{\epsilon, e}$. In particular, we have $K_{l,e,s}(n) = K_{l,e,t}(n)$.

Set $\tilde{H}_{R,n}^\lambda := H_{R,n}(v^{-1}; t_\ell, \ldots, s_1)$ and denote by $\tilde{T}_0, \ldots, \tilde{T}_{\ell - 1}$ the associated standard generators. For each $\lambda \in \Pi^\ell(n)$, denote by $\tilde{S}^\lambda$ the associated Specht module on $\tilde{H}_{R,n}$. By [11], the simple modules are this time labeled by the set $K_{l,e,-s_{\text{ev}}}(n)$ where $-s_{\text{ev}} = (-s_{t_\ell}, \ldots, -s_1) \in (\mathbb{Z}/e\mathbb{Z})^\ell$. Thus, for each $\lambda \in K_{l,e,-s_{\text{ev}}}(n)$, we have an associated simple $\tilde{H}_{R,n}$-module $\tilde{D}^\lambda$. We have an isomorphism $\theta : H_{R,n} \rightarrow \tilde{H}_{R,n}$ given by

$$T_0 \mapsto \tilde{T}_0 \quad T_i \mapsto -v\tilde{T}_i \quad (i = 1, \ldots, n - 1).$$

Then, $\theta$ induces a functor $F$ from the category of $\tilde{H}_{R,n}$-modules to the category of $H_{R,n}$-modules. As a consequence, we obtain a bijective map

$$m_{e,s} : K_{l,e,s}(n) \rightarrow K_{l,e,-s_{\text{ev}}}(n),$$

satisfying

$$F(\tilde{D}^{m_{e,s}(\lambda)}) \simeq D^\lambda,$$

for all $\lambda \in K_{l,e,s}$.

**Remark 1.2.** Assume that $\ell = 1$ and $s \in \mathbb{Z}$. Then the involution $m_{e,s}$ corresponds to the usual Mullineux involution $m_e$ that we have defined in the introduction (it does not depend on the choice of $s$ in this case).

### 1.2. Crystal interpretation

Let us fix some notation. Fix $e, \ell \geq 2$ and $s \in \mathbb{Z}$. Denote

$$\mathbb{Z}^\ell(s) = \left\{ (s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell \mid \sum_{i=1}^\ell s_i = s \right\}.$$

For $s \in \mathbb{Z}^\ell(s)$, we denote by $\Pi^\ell_s$ the set of all symbols of the form $|\lambda, s|$ with $\lambda \in \Pi^\ell$. Further, denote by $\Pi^\ell_s$ the sets of all elements in $\Pi^\ell_s$ where $s \in \mathbb{Z}^\ell(s)$. Let $F_{e,s}$ be the $\mathbb{C}$-vector space with standard basis $\Pi^\ell_s$, i.e. $F_{e,s} = \bigoplus_{\lambda \in \Pi^\ell_s} \mathbb{C}[|\lambda, s|]$, called the Fock space of level $\ell$ and rank $e$ (associated to the charge $s$). Let $F_{e,s} = \bigoplus_{s \in \mathbb{Z}^\ell(s)} F_{e,s}$ and $F_e = \bigoplus_{s \in \mathbb{Z}} F_{e,s}$.

As mentioned in the introduction, an important part of the representation theory of cyclotomic Hecke algebras is controlled by the theory of crystals for Fock spaces. The $\mathfrak{sl}_e$-crystal [21], [19] of the
Moreover, there is a unique isomorphism of \( \widehat{\text{Fix}} \) index irreducible modules of cyclotomic Hecke algebras by certain vertices in Fock spaces crystals. Let \( s \in \mathbb{Z}^e \) and \( \ell \)-partitions, i.e. we can index irreducible modules of cyclotomic Hecke algebras by certain vertices in Fock spaces crystals.

Theorem 1.3 (Ariki). Fix \( n \in \mathbb{Z}_{\geq 0} \). Let \( s = (s_1, \ldots, s_e) \in \mathbb{Z}^e \) and \( t = (t_1, \ldots, t_\ell) \in \mathbb{Z}^\ell \) be such that \( s_i = t_i \mod e \) for all \( i = 1, \ldots, \ell \), and \( t_i - t_{i-1} > n - 1 \) for all \( i = 2, \ldots, \ell \). Then

\[
\text{Ug}_{e, t}(n) = \text{Kl}_{e, t}(n).
\]

Moreover, there is a unique isomorphism of \( \widehat{s}_\ell \)-crystals mapping \( \text{Ug}_{e, s}(n) \) to \( \text{Ug}_{e, t}(n) \).

In other words, Kleshchev \( \ell \)-partitions are a particular case of Uglov \( \ell \)-partitions, i.e. we can index irreducible modules of cyclotomic Hecke algebras by certain vertices in Fock spaces crystals.

Fayers [11, Section 2] proved the following result.

Theorem 1.4. Fix \( n \in \mathbb{Z}_{\geq 0} \). Let \( s \in \mathbb{Z}^\ell(s) \) and \( e \geq 2 \). There exists a unique bijection

\[
\Phi_{e, s} : \text{Kl}_{e, s}(n) \rightarrow \text{Kl}_{e, -s_{rev}}(n)
\]

such that

- \( \Phi_{e, s}(\emptyset) = \emptyset \),
- for all \( 0 \leq i \leq e - 1 \), we have \( \Phi_{e, s} \circ \tilde{f}_i = \tilde{f}_{-i} \circ \Phi_{e, s} \).

This means that for all path

\[
\emptyset \rightarrow \cdots \rightarrow \lambda
\]

in the \( \widehat{s}_\ell \)-crystal on the Fock space \( F_{e, s} \), there exists a corresponding path

\[
\emptyset \rightarrow \cdots \rightarrow \mu
\]

in the \( \widehat{s}_\ell \)-crystal on the Fock space \( F_{e, -s_{rev}} \) from the empty \( \ell \)-partition to an \( \ell \)-partition \( \mu \in \text{Kl}_{e, -s_{rev}} \). Then \( \Phi_{e, s}(\lambda) = \mu \). In [21], it is explained how the map \( \Phi_{e, s} \) can be explicitly computed without constructing the \( \widehat{s}_\ell \)-crystal. Moreover, observe that by Theorem 1.3 this property of crystals also holds when replacing \( \text{Kl}_{e, s}(n) \) by the more general set \( \text{Ug}_{e, s}(n) \), so that we get the following corollary (where we decide to use the same notation for the map appearing).

Corollary 1.5. Let \( s \in \mathbb{Z}^\ell(s) \) and \( e \geq 2 \). There exists a unique bijection

\[
\Phi_{e, s} : \text{Ug}_{e, s} \rightarrow \text{Ug}_{e, -s_{rev}}
\]

such that

- \( \Phi_{e, s}(\emptyset) = \emptyset \),
- for all \( 0 \leq i \leq e - 1 \), we have \( \Phi_{e, s} \circ \tilde{f}_i = \tilde{f}_{-i} \circ \Phi_{e, s} \).
Example 1.6. Take \( s = 4, \ e = 4, \ \ell = 3, \ s = (5, -1, 0) \) (so that \(-s_{\text{rev}} = (0, 1, -5)\)) and \( \lambda = (1, 3, 2, 0) \). One can write for instance \( \lambda = f_1 f_1 f_3 f_0 f_2 f_3 \emptyset \), so that \( \lambda \in U_{g,e,s}(6) \). Therefore, in the crystal of the Fock space \( \mathcal{F}_{e,-s_{\text{rev}}} \), we get

\[
\Phi_{e,s} = \tilde{f}_1 \tilde{f}_1 \tilde{f}_3 \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 \emptyset
\]

\[
= \tilde{f}_3 \tilde{f}_3 \tilde{f}_1 \tilde{f}_0 \tilde{f}_2 \tilde{f}_1 \emptyset
\]

\[
= (2.1, 3, \emptyset).
\]

The following result by Fayers \[11\] gives the desired crystal interpretation of the Mullineux involution for cyclotomic Hecke algebras.

Theorem 1.7 (Fayers). Fix \( n \in \mathbb{Z}_{\geq 0}, \ s \in \mathbb{Z}^{\ell} \) and \( e \geq 2 \). For all \( \lambda \in \text{Kl}_{e,s}(n) \), we have

\[
m_{e,s}(\lambda) = \Phi_{e,s}(\lambda).
\]

To summarize, starting with the usual Mullineux involution \( m_e \) for the symmetric group, we obtain:

- a generalisation of \( m_e \) : the involution \( m_{e,s} \) on the set of Kleshchev \( \ell \)-partitions which label the irreducible representations of cyclotomic Hecke algebras. If \( \ell = 1 \), we have \( m_{e,s} = m_e \).
- a generalisation of \( m_{e,s} \) : the involution \( \Phi_{e,s} \) on the set of Uglov \( \ell \)-partitions. If \( s \) is such that \( s_i - s_{i-1} > n - 1 \) for all \( i = 2, \ldots, \ell \), we have \( \Phi_{e,s} = m_{e,g} \).

2. The generalised Mullineux involution

2.1. Triple crystal structure on higher level Fock spaces. For \( s \in \mathbb{Z} \), we denote

\[
A(s) = \{(s_1, \ldots, s_{\ell}) \in \mathbb{Z}^{\ell}(s) \mid s_1 \leq \ldots \leq s_{\ell} \leq s_1 + e\}
\]

and, in a dual fashion,

\[
\hat{A}(s) = \{(t_1, \ldots, t_e) \in \mathbb{Z}^{e}(s) \mid t_1 \leq \ldots \leq t_e \leq t_1 + \ell\}
\]

Write \( \hat{\emptyset} = (\emptyset, \ldots, \emptyset) \in \Pi^e \). For \( s \in A(s) \), we will denote by \( \text{Reg}_{e,s} \) the subset of \( \Pi^e_s \) consisting of these elements \( |\lambda, s\rangle \) which are FLOTW, see \[14\] Definition 5.7.8. Suppose \( s \in A(s) \). Then we have \( U_{g,e,s} = \text{Reg}_{e,s} \) by \[12\] Theorem 2.10.

We have seen in Section \[12\] that there is an \( \hat{\mathfrak{sl}}_\ell \)-crystal structure on \( \Pi^e_s \), arising from the integrable action of \( \mathcal{U}'(\hat{\mathfrak{sl}}_\ell) \) \[23\] on \( \mathcal{F}_{e,s} \). Moreover, there is an \( \mathfrak{sl}_\ell \)-crystal and \( \mathfrak{sl}_\infty \)-crystal structure on \( \Pi^e_s \). The \( \hat{\mathfrak{sl}}_\ell \)-crystal arises similarly to the \( \hat{\mathfrak{sl}}_\ell \)-crystal via level-rank duality

\[
k : \Pi^e_s \leftrightarrow \Pi^{e-\hat{\ell}}_s
\]

see \[16\] Formula (3.8)] for the definition of \( k \). That is, the action of \( \tilde{f}_j \) on an \( \ell \)-partition is defined using the following diagram

\[
\begin{array}{ccc}
\Pi^e_s & \xrightarrow{k} & \Pi^{e-\hat{\ell}}_s \\
\downarrow & & \downarrow \tilde{f}_j \\
\Pi^e_s & \xleftarrow{k} & \Pi^{e-\hat{\ell}}_s
\end{array}
\]

(2.1)

For \( s \in A(s) \), we will denote \( \hat{s} \) the element of \( \hat{A}(-s) \) such that \( k(\emptyset, s) = (\emptyset, \hat{s}) \). Finally, the \( \mathfrak{sl}_\infty \)-crystal arises from the action of a Heisenberg algebra \[37, 29, 15\]. Its connected components are all isomorphic to the branching graph of the symmetric group in characteristic 0 and thus have

\[3\]In the examples, we use the multiplicative notation for partitions and we forget the brackets around components of a multipartition.
vertices in bijection with \( \Pi \). If \( \lambda_0 \) is a highest weight vertex for the \( \mathfrak{s}_\ell \)-crystal, then any \( \ell \)-partition in the same crystal component as \( \lambda_0 \) is obtained as \( \tilde{a}_\sigma(\lambda_0) \) for a unique \( \sigma \in \Pi \), where \( \tilde{a}_\sigma \) denotes the Heisenberg crystal operator associated to \( \sigma \), see \cite{29,15}.

We will make repeated use of the following important theorem, proved in \cite{16} Theorems 6.17 and 6.19):

**Theorem 2.2.**

(1) The three crystals pairwise commute.

(2) For every \( \lambda \in \Pi^\ell \), we can decompose \( |\lambda, s\rangle \in \mathcal{F}_{e,s} \) as follows

\[
|\lambda, s\rangle = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r\rangle
\]

for a unique \( r \in A(s) \), \( \sigma \in \Pi \), \( p, r \in \mathbb{Z}_{\geq 0} \) and for some \( i_p, \ldots, i_1 \in \{0, 1, \ldots, e - 1\} \) and \( j_r, \ldots, j_1 \in \{0, 1, \ldots, \ell - 1\} \).

We deduce the following way of indexing the crystal of Fock spaces:

**Corollary 2.3.** With the notations of Theorem 2.2, we get a bijection

\[
\beta: \quad \Pi^\ell \longrightarrow \prod_{r \in A(s)} \text{Reg}_{e,r} \times \prod \times \text{Reg}_{\ell,r}
\]

\[
|\lambda, s\rangle \longrightarrow (\tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r\rangle, \sigma, \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} |\emptyset, r\rangle).
\]

**Proof.** Let \( \lambda \in \Pi^\ell \). By Theorem 2.2 (2), there exist \( r \in A(s) \), \( \sigma \in \Pi \), \( p, r \in \mathbb{Z}_{\geq 0} \) and elements \( i_p, \ldots, i_1 \in \{0, 1, \ldots, e - 1\} \) and \( j_r, \ldots, j_1 \in \{0, 1, \ldots, \ell - 1\} \) such that

\[
|\lambda, s\rangle = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r\rangle
\]

Assume that we have:

\[
\tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r'\rangle = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r\rangle
\]

for \( r' \in A(s) \), \( \sigma' \in \Pi \), \( p', r' \in \mathbb{Z}_{\geq 0} \) and elements \( i_{p'}, \ldots, i_1 \in \{0, 1, \ldots, e - 1\} \) and \( j_{r'}, \ldots, j_1 \in \{0, 1, \ldots, \ell - 1\} \). Then the elements \( |\mu', r'\rangle := \tilde{f}_{j_{r'}} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma |\emptyset, r'\rangle \) and \( |\mu, t\rangle := \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma |\emptyset, r\rangle \) are both highest weight vertices in the \( \mathfrak{s}_\ell \)-crystal. As we have \( \tilde{f}_{i_{p'}} \ldots \tilde{f}_{i_1} |\mu', r'\rangle = \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\mu, r\rangle \), these two elements are in the same connected component of the \( \mathfrak{s}_\ell \)-crystal so they must be equal. We deduce in the same way that \( \tilde{a}_\sigma |\emptyset, r'\rangle = \tilde{a}_\sigma |\emptyset, r\rangle \). By the description of the \( \mathfrak{s}_\ell \)-crystal operators \cite{29,16}, we obtain \( \sigma = \sigma' \) and \( r' = r \). We deduce that \( \tilde{f}_{j_{r'}} \ldots \tilde{f}_{j_1} |\emptyset, r\rangle = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} |\emptyset, r\rangle \) and thus that our map is well defined.

Let \( r \in A(s) \), take now \( (\lambda, \sigma, \mu) \in \text{Reg}_{e,r} \times \Pi \times \text{Reg}_{\ell,r} \). Again by \cite{12} Theorem 2.10, we know that there exist \( i_1, \ldots, i_p \in \{0, 1, \ldots, e - 1\} \) and \( j_1, \ldots, j_r \in \{0, 1, \ldots, \ell - 1\} \) such that

\[
\tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r\rangle = |\lambda, r\rangle \quad \text{and} \quad \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} |\emptyset, r\rangle = |\mu, r\rangle
\]

and this shows that the map is bijective.

\[\square\]

**Example 2.4.** Take \( s = 1 \), \( e = 3 \), \( \ell = 4 \), \( s = (-3, 2, 1, 1) \) and \( \lambda = (0, 3, 2^2, 0, 3) \). Then

\[
\beta(|\lambda, s\rangle) = ( |(0, 0, 0, 3), (-1, 0, 0, 2) \rangle, (2), |(2^2, 2.1, 0), (-1, -1, 1) \rangle )
\]

\[= ( \tilde{f}_1 \tilde{f}_2 |(0, 0, 0, 3), (-1, 0, 0, 2) \rangle, (2), \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 \tilde{f}_4 |(0, 0, 0, 3), (-1, -1, 1) \rangle ) \).

**Remark 2.5.** Note that for \( \ell = 1 \), Corollary 2.3 reduces to a very simple bijection. Indeed, there is no \( \mathfrak{s}_1 \)-crystal (and no level-rank duality) in this case, and the bijection associates to any partition \( \lambda \) a pair \((\rho, \sigma)\) where \( \rho \) is \( e \)-regular and \( \sigma \in \Pi \). Namely, \((\rho, \sigma)\) are determined by the “euclidean
division” of λ by e. More precisely, given two partitions μ and μ’, we denote by μ □ μ’ the partition obtained by concatenation of the two partitions and by reordering the parts to obtain a partition (see for instance [3] Section 3.1). Then we can uniquely write

\[ \lambda = (\sigma)^e \sqcup \rho. \]

where ρ is a e-regular partition.

Example 2.6. Choose e = 3 and \( \lambda = (4^4.3^2.2.1^8) \). Then \( \lambda = (4.1^2)^3 \sqcup (4.3^2.2.1^2) \).

2.2. The generalised Mullineux map. Now, let us define several maps. For \((m_1, m_2) \in \mathbb{Z}_0^e\), we denote

\[ \theta_{m_1,m_2} : \mathbb{Q}^{m_1}(s) \rightarrow \mathbb{Q}^{m_1}(m_2) \]

\[ (s_1, \ldots, s_{m_1}) \mapsto (m_2 - s_1 + s_{m_1}, s_1 - s_2, \ldots, s_{m_1} - s_{m_1}) \]

This is a bijection with inverse map:

\[ \theta_{m_1,m_2}^{-1} : \mathbb{Q}^{m_1}(m_2) \rightarrow \mathbb{Q}^{m_1}(s) \]

\[ (a_1, \ldots, a_{m_1}) \mapsto (s_1, \ldots, s_{m_1}) \]

where we have for all \( 1 \leq i \leq m_1 \):

\[ s_i = \frac{1}{m_1} (s - \sum_{1 \leq j \leq m_1-1} ja_{j+1} + \sum_{i+1 \leq j \leq m_1} a_j) \]

We have the following result which comes from [10] Proposition 2.12.

Proposition 2.7. Let \( t \in \mathbb{Z}_e^s(s) \) and let \( \tilde{w} \in \mathcal{P}(s) \) be a weight for \( \mathcal{F}_{t, s} \). Then there exists a unique \( s \in \mathbb{Z}_e^s(s) \) and a unique \( w \in \mathcal{P}(s) \) such that the \( w \)-weight space of \( \mathcal{F}_{e,s} \) is equal to the \( \tilde{w} \)-weight space of \( \mathcal{F}_{t, s} \). If we write \( \tilde{w} = d\delta + \sum_{0 \leq i \leq e-1} a_i \Lambda_i \) with \( (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell \) we have

\[ s = \theta_{e, \ell}^{-1}(a_1, \ldots, a_\ell) \]

Moreover the associated weight is

\[ w = d\delta + \sum_{0 \leq i < e} (s_i - s_{i+1}) \Lambda_i \]

Lemma 2.8. Keeping the above notation, assume that \( s = \theta_{e, \ell}^{-1}(a_1, \ldots, a_\ell) \) then we have \(-s_{\text{rev}} = \theta_{e, \ell}^{-1}(a_1, a_\ell, \ldots, a_2)\)

Proof. Write \( s = (s_1, \ldots, s_\ell) \) and \( v = \theta_{e, \ell}^{-1}(a_1, a_\ell, \ldots, a_2) \). We have for all \( j = 1, \ldots, \ell \):

\[ s_j + v_{\ell-j+1} = \frac{1}{\ell} (s - \ell (a_2 + \ldots + a_\ell)) + a_{j+1} + \ldots + a_\ell + a_j + \ldots + a_2 = 0 \]

and the result follows.

We are now ready to prove the first main result of this paper. Recall the generalised Mullineux map \( \Phi_{e, s} \) on Uglov \( \ell \)-partitions of Corollary [5]. By level-rank duality, we have a “dual” Mullineux map \( \Phi_{t, s} \) for all \( t \in \mathbb{Z}_e^s(s) \).

Theorem 2.9.

(1) There exists a unique bijection

\[ \Phi : \Pi_s^\ell \rightarrow \Pi_{s^{-}}^\ell \]

\[ |\lambda, s\rangle \mapsto |\mu, -s_{\text{rev}}\rangle \]

such that for all \( 0 \leq i \leq e \), \( \sigma \in \Pi \) and \( 0 \leq j \leq \ell - 1 \),

(a) \( \Phi \circ \tilde{f}_i = \tilde{f}_{\ell-i} \circ \Phi \)

(b) \( \Phi \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^t} \circ \Phi \)
(c) \( \Phi \circ \tilde{f}_j = \tilde{f}_{-j} \circ \Phi \).

(2) Using the notation of Corollary 2.3, we have
\[
\Phi = \beta^{-1} \circ (\Phi_{e,r}, (.), \Phi_{\ell,t}) \circ \beta.
\]

In other words, writing \( |\lambda, s| = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r \) with \( r \in A(s) \), we have \( \Phi|\lambda, s| = \tilde{f}_{-j_r} \ldots \tilde{f}_{-j_1} \tilde{a}_\sigma \tilde{f}_{-i_p} \ldots \tilde{f}_{-i_1} |\emptyset, -r_{rev} \).

(3) We have \( |\lambda| = |\mu| \).

Proof. Let \( \lambda \in \Pi^\ell \), \( s \in \mathbb{Z}^\ell(s) \) and write \( |\lambda, s| = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \ldots \tilde{f}_{i_1} |\emptyset, r \) according to Theorem 2.2. We need to show that there exists an \( \ell \)-partition \( \mu \) such that
\[
|\mu, -s_{rev}| = \tilde{f}_{-i_p} \ldots \tilde{f}_{-i_1} \tilde{a}_\sigma \tilde{f}_{-j_r} \ldots \tilde{f}_{-j_1} |\emptyset, -r_{rev} \)
\]
To do this, consider the \( e \)-partition \( \lambda_2 \) such that
\[
|\lambda_2, \tilde{r}| = \tilde{f}_{j_r} \ldots \tilde{f}_{j_1} |\emptyset, \tilde{r}|
\]
and the \( e \)-partition \( \mu_2 \) such that
\[
|\mu_2, \tilde{r}_{rev}| = \tilde{f}_{-j_r} \ldots \tilde{f}_{-j_1} |\emptyset, -\tilde{r}_{rev}|
\]
defined thanks to Theorem 1.5 i.e.
\[
|\mu_2, \tilde{r}_{rev}| = \Phi_{\ell,t}(|\lambda_2, \tilde{r}|).
\]

Let \( |\lambda_1, t| = k(|\lambda_2, \tilde{r}|) \) and \( |\mu_1, v| = k(|\mu_2, \tilde{r}_{rev}|) \). By hypothesis, the \( \widehat{sl}_\ell \)-weight \( \hat{w} \) of \( |\lambda_2, \tilde{r}| \) can be written
\[
\hat{w} = \sum_{1 \leq i \leq \ell} \hat{\Lambda}_{r_i} - \sum_{1 \leq j \leq \ell} \alpha_{i_j}
\]
and there exists a sequence of non negative integers \( (a_1, \ldots, a_\ell) \) such that
\[
\hat{w} = \sum_{1 \leq i \leq \ell} a_i \hat{\Lambda}_{i-1}.
\]

By hypothesis, the \( \widehat{sl}_\ell \)-weight \( \hat{w}' \) of \( |\mu_2, \tilde{r}_{rev}| \) can be written
\[
\hat{w}' = \sum_{1 \leq i \leq \ell} \hat{\Lambda}_{-r_i} - \sum_{1 \leq j \leq \ell} \hat{\alpha}_{-i_j}
\]
and thus we have
\[
\hat{w}' = \sum_{1 \leq i \leq \ell} a_i \hat{\Lambda}_{1-i}
\]

Using Proposition 2.7, we see that \( t = \theta_{e,e}^{-1}(a_1, \ldots, a_\ell) \) and \( v = \theta_{e,e}^{-1}(a_1, a_\ell, \ldots, a_2) \). We conclude that \( v = -t_{rev} \) using Lemma 2.8.

Now let us study the \( \widehat{sl}_\ell \)-weight of \( |\lambda_1, t| \) and \( |\mu_1, -t_{rev}^e| \). Again by Prop 2.7, the weight \( w \) of \( |\lambda_1, t| \) is
\[
w = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_1 - r_2)\Lambda_1 + \ldots + (r_{e-1} - r_e)\Lambda_{e-1}
\]
which can be written as:
\[
w = d\delta + \sum_{1 \leq i \leq \ell} \Lambda_{t_i} - \sum_{0 \leq i \leq e-1} m_i \alpha_i
\]
for a sequence of non negative integers \( (m_i)_{i=0,\ldots,e-1} \). Now again, the weight \( w' \) of \( |\mu_1, -t_{rev}^e| \) is
\[
w' = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_{e-1} - r_e)\Lambda_1 + \ldots + (r_1 - r_2)\Lambda_{e-1}
\]
which thus can be written as:

\[ w' = d\delta + \sum_{1 \leq i \leq t} \Lambda_{-t_i} - \sum_{0 \leq i \leq e-1} (m_i-i)\alpha_i. \]

We conclude that \(|\lambda_1| = |\mu_1|\), that is, \(N := \sum_{0 \leq i \leq e-1} m_i\). Now it follows that \(|\mu| = |\lambda| = N + |\sigma|e + r.\]

We may write \(\Phi(\lambda)\) instead of \(\Phi(|\lambda, s|)\) when the charge \(s\) is understood.

**Example 2.10.** Take the same values as in Example 2.4. Denote \(r = (-1, 0, 0, 2)\), so that \(\hat{r} = (-1, -1, 1)\). Then we have

\[ \Phi(|\lambda, s|) = \beta^{-1}(\Phi_{e,r}(\tilde{f}_1\tilde{f}_0\tilde{f}_2 | \emptyset, r)), (2)^t, \Phi_{\ell, \hat{r}}(\tilde{f}_0\tilde{f}_2\tilde{f}_3\tilde{f}_5\tilde{f}_0\tilde{f}_2\tilde{f}_3 | \emptyset, \hat{r})) \]

\[ = \beta^{-1}(\tilde{f}_2\tilde{f}_0\tilde{f}_1 | (\emptyset, \emptyset, \emptyset), (-2, 0, 0, 1)), (2)^t, \tilde{f}_0\tilde{f}_2\tilde{f}_1\tilde{f}_0\tilde{f}_2\tilde{f}_1 | (\emptyset, \emptyset, \emptyset), (-1, 1, 1)) \]

\[ = \beta^{-1}(|(\emptyset, 1, 0, 2), (-2, 0, 0, 1)), (2)^t, |(\emptyset, 2^t, 2.1), (-1, 1, 1)) \]

\[ = |(1, 2.1, 0, 2^3), (-1, -1, -2, 3)) \]

The following corollary shows that \(\Phi\) generalizes the map \(\Phi_{e,s}\) of Section 2.3.

**Corollary 2.11.** Let \(e \geq 2, s \in \mathbb{Z}^t(s), \) and \(n \geq 0\). For all \(\lambda \in \Pi_{e,s}(n)\), we have

\[ \Phi(\lambda) = \Phi_{e,s}(\lambda). \]

**Proof.** Let \(s \in \mathbb{Z}^t(s)\). By Property (3) of Theorem 2.9, we have

\[ \Phi(|\emptyset, s|) = |\emptyset, -s_{\text{rev}}. \]

Now if \(\lambda \in \Pi_{e,s}(n)\) there exists a sequence of Kashiwara operators such that:

\[ \tilde{f}_i \ldots \tilde{f}_1 | \emptyset, s) = |\lambda, s). \]

So we can use Property (1) of Theorem 2.9 to see that

\[ \Phi(\tilde{f}_i \ldots \tilde{f}_1 | \emptyset, s) = \tilde{f}_{-i} \ldots \tilde{f}_{-1} | \emptyset, -s_{\text{rev}}. \]

By definition of \(\Phi_{e,s}\) in Corollary 2.3, we get the result. \(\square\)

### 2.3. More on crystal isomorphisms.

Let \(P_\ell := \mathbb{Z}^t\) be the \(\mathbb{Z}\)-module with standard basis \([y_i | i = 1, \ldots, \ell]\). For \(k = 1, \ldots, \ell - 1\), we denote by \(\sigma_k\) the transposition \((k, k + 1)\) of \(S_\ell\). The extended affine symmetric group \(\widehat{S}_\ell\) is the semidirect product \(P_\ell \rtimes S_\ell\) with the relations given by \(\sigma_i y_j = y_j \sigma_i\) for \(j \neq i, i + 1\) and \(\sigma_i y_i \sigma_i = y_{i+1}\) for \(i = 1, \ldots, \ell - 1\) and \(j = 1, \ldots, \ell\). It acts faithfully on \(\mathbb{Z}^t\) as follows: for any \(s = (s_1, \ldots, s_\ell) \in \mathbb{Z}^t\):

\[ \sigma_c s = (s_1, \ldots, s_{c-1}, s_{c+1}, s_c, s_{c+2}, \ldots, s_\ell) \quad \text{for} \quad c = 1, \ldots, \ell - 1 \quad \text{and} \]

\[ y_i s = (s_1, s_2, \ldots, s_i + e, \ldots, s_\ell) \quad \text{for} \quad i = 1, \ldots, \ell. \]

If \(s\) and \(s'\) are in the same orbit modulo the action of \(\widehat{S}_\ell\), then there is an \(\widehat{S}_\ell\)-crystal isomorphism \(\Psi_{s \rightarrow s'}\) between the Fock spaces \(\mathcal{F}_{e,s}\) and \(\mathcal{F}_{e,s'}\), that is a map:

\[ \Psi_{s \rightarrow s'} : \Pi_{e,s}^{\ell} \rightarrow \Pi_{e,s'}^{\ell} \]

such that:

- \(|\lambda, s)\) is an highest weight vertex in \(\mathcal{F}_{e,s}\) if and only if \(|\Psi_{s \rightarrow s'}(\lambda), s'\) is an highest weight vertex in \(\mathcal{F}_{e,s'}\).

- For all \(\lambda \in \Pi^{\ell}\), we have \(\Psi_{s \rightarrow s'}(\tilde{f}_i | \lambda, s) = \tilde{f}_i \Psi_{s \rightarrow s'}(|\lambda, s))\)
These crystal isomorphisms have been explicitly described in [21] and can be interpreted in the context of the representation theory of rational Cherednik algebras, as wall-crossing functors. Let us now come back to our situation. Assume that \( s \in \mathbb{Z}^{\ell} \) and choose any \( s' \in \mathbb{Z}^{\ell} \) in the orbit of \(-s\) modulo the action of the affine extended symmetric group. This is in particular the case for \(-s_{rev}\). There is a \( \widehat{\mathfrak{sl}}_e \)-crystal isomorphism between the Fock spaces \( \mathcal{F}_{e,-s} \) and \( \mathcal{F}_{e,s'} \). Composing this map with \( \Phi \) thus gives an isomorphism between \( \mathcal{F}_{e,s} \) and \( \mathcal{F}_{e,s'} \).

3. COMBINATORICS OF PERVERSE EQUIVALENCES FOR CYCLOTOMIC CHEREDNIK CATEGORY \( \mathcal{O} \)

In Section 1 we studied the Mullineux involution in the context of representations of cyclotomic Hecke algebras. In this section, we use the results of Section 2 to study the Mullineux involution in the context of representations of cyclotomic rational Cherednik algebras. The goal is to realize the generalized Mullineux involution as the permutation of \( \Pi^e \) induced by certain perverse equivalences. We follow Losev’s approach [30], [29].

3.1. Representations of cyclotomic rational Cherednik algebras. We can deform \( \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \times \mathbb{C}W_{\ell,n} \) to obtain an algebra \( H_{\kappa,s}(n) \) called the cyclotomic rational Cherednik algebra [10]; \( H_{\kappa,s}(n) \) depends on parameters \( \kappa \in \mathbb{Q}^* \) and \( s = (s_1, \ldots, s_\ell) \in \mathbb{Q}^\ell \). The charge \( s \) is identified with \( s + \alpha(1, 1, \ldots, 1) \) for any scalar \( \alpha \), thus the parameter space is \( \ell \)-dimensional. As a \( \mathbb{C} \)-vector space, \( H_{\kappa,s}(n) = \mathbb{C}[y_1, \ldots, y_n] \otimes \mathbb{C}[W_{\ell,n}] \otimes \mathbb{C}[x_1, \ldots, x_n] \). This makes it possible to define a category \( \mathcal{O} \) for \( H_{\kappa,s}(n) \) as the full subcategory of \( H_{\kappa,s}(n) \text{-mod} \) consisting of finitely generated \( H_{\kappa,s}(n) \)-modules which are locally nilpotent for the action of \( \mathbb{C}[y_1, \ldots, y_n] \) [17]. Let \( \mathcal{O}_{\kappa,s}(n) \) denote the category \( \mathcal{O} \) of \( H_{\kappa,s}(n) \), and \( \mathcal{O}_{\kappa,s} = \bigoplus_{n \geq 0} \mathcal{O}_{\kappa,s}(n) \). \( \mathcal{O}_{\kappa,s}(n) \) is a highest weight category whose simple objects are indexed by \( \text{Irr}W_{\ell,n} \), the irreducible representations of the underlying group algebra \( \mathbb{C}W_{\ell,n} \) over complex numbers [17], thus by \( \Pi^e(n) \). Furthermore, the highest weight structure of \( \mathcal{O}_{\kappa,s}(n) \) depends on a partial order on \( \Pi^e(n) \) which is determined by the parameter \((\kappa, s)\).

3.2. Branching rules and crystals. In the rest of this section we restrict to the case of integer parameters, that is

\[ \kappa = \pm 1/e \text{ for some } e \in \mathbb{Z}_{\geq 2} \quad \text{and} \quad s \in \mathbb{Z}^\ell. \]

The (complexified) Grothendieck group of \( \mathcal{O}_{\kappa,s} \) is a level \( \ell \) Fock space. Recall from Section 1.2 that the parameters \( e, s \) for the Fock space come from its \( \mathcal{U}'(\widehat{\mathfrak{sl}}_e) \)-module structure. Shan [35] has shown that if \( \kappa = 1/e \), there is a notion of branching rule arising from Bezrukavnikov and Etingof’s parabolic induction functors [3], which categorifies the \( \widehat{\mathfrak{sl}}_e \)-crystal of the Fock space \( \mathcal{F}_{e,s} \) [35, Theorem 6.3]. Moreover, there is a categorical Heisenberg action on \( \mathcal{O}_{\kappa,s} \) giving rise to the \( \widehat{\mathfrak{sl}}_e \)-crystal on \( \mathcal{F}_{e,s} \) [37], [29]. We have:

**Theorem 3.1.** (Shan-Vasserot [37, Proposition 5.18]) The following are equivalent:

1. \( \lambda \) is a highest weight vertex for both the \( \widehat{\mathfrak{sl}}_e \)- and \( \widehat{\mathfrak{sl}}_\infty \)-crystals on \( \mathcal{F}_{e,s} \),
2. \( L(\lambda) \) is killed by the categorical Heisenberg and \( \widehat{\mathfrak{sl}}_e \) annihilation operators,
3. \( L(\lambda) \) is finite-dimensional.

Thus finite-dimensional simples are labeled by the source vertices of the \( \widehat{\mathfrak{sl}}_e \)- and \( \widehat{\mathfrak{sl}}_\infty \)-crystals.

3.3. Perverse equivalences. Perverse equivalences are a special kind of derived equivalence introduced by Chuang-Rouquier [6], [7] which are well-suited for combinatorial applications. Let \( \mathcal{A}, \mathcal{A}' \) be abelian categories with finitely many simple objects and in which every object has finite length. Let \( S, S' \) be the sets of isomorphism classes of simple objects of \( \mathcal{A}, \mathcal{A}' \) respectively. Let \( 0 \subset S_0 \subset S_1 \subset \cdots \subset S_r = S \) be a filtration of \( S \) and \( 0 \subset S'_0 \subset S'_1 \subset \cdots \subset S'_{r'} = S' \) a filtration of \( S' \). Let \( \mathcal{A}_i \subset \mathcal{A}, \mathcal{A}'_i \subset \mathcal{A}' \) be the Serre subcategories generated by \( S_i, S'_i \) respectively and let \( \pi : \{0, 1, \ldots, r\} \to \mathbb{Z} \) be a function.
Definition 3.2 (Chuang-Rouquier). A derived equivalence \( F : D^b(A) \to D^b(A') \) is perverse if for all \( i \geq 0 \) and all \( s \in S_i \setminus S_{i-1} \), the complex \( F(s) \) satisfies:

1. for \( j \neq \pi(i) \) all composition factors of \( H^j(F(s)) \) are in \( A'_{i-1} \),
2. all the composition factors of \( H^{\pi(i)}(F(s)) \) are in \( A'_i \) except for a unique one in \( A'_i \setminus A'_{i-1} \).

A perverse equivalence \( F : D^b(A) \to D^b(A') \) therefore gives rise to a canonical bijection \( f : S \to S' \) sending \( s \in S_i \setminus S_{i-1} \) to \( f(s) := F(s)[-\pi(i)] \mod A'_{i-1} \). In the following sections 3.4 and 3.6, we show how the generalized Mullineux involution arises from the perverse equivalences given by wall-crossing functors and the Ringel duality.

3.4. Wall-crossing functors. Let us recall Losev’s construction of wall-crossing functors in [29]. These are derived equivalences between category \( O_{\gamma}(n) \)'s for parameters which differ by a perturbation of the partial order on \( \Pi^f(n) \). Fix a parameter \((\kappa, s) \in Q^x \times Q^f \). Denote by \( c_{Z} \subset Q^x \times Q^f \) the \( \ell \)-dimensional lattice in the parameter space consisting of those parameters \((\kappa', s')\) such that \( \kappa' - \kappa \in Z \) and \( \kappa'(s'_i - s'_j) - \kappa(s_i - s_j) \in Z \) for all \( 1 \leq i < j \leq \ell \). There is a finite set of hyperplanes in \( Q \otimes_Z c_Z \), called walls, divvying up \( c_Z \) into open cones called chambers. These hyperplanes are defined as follows. For each \( \lambda = ((\lambda_1^1, \lambda_2^1, \ldots), \ldots, (\lambda_1^\ell, \lambda_2^\ell, \ldots)) \in \Pi^f(n) \), let

\[ [\lambda] := \{(a, b, c) \mid 1 \leq a, 1 \leq b \leq \lambda_0^i, 1 \leq c \leq \ell\} \]

be the Young diagram of \( \lambda \). For each \( (a, b, c) \in [\lambda] \), we define

\[ co(\gamma) = \kappa \ell (b - a) + \ell h_c \]

called the \( c \)-function; the formula in our cyclotomic case was given in [18]. By definition, the walls are the hyperplanes \( \Pi_{\lambda, \mu} \) given by \( c_{X} = c_{\mu} \). Among these walls, we will be interested in the so called “essential walls” which are the following ones:

(a) \( \kappa = 0 \) for parameters \((\kappa, s)\) such that the denominator \( e \) of \( \kappa \) satisfies \( 2 \leq e \leq n \), in terms of the above definition they are of the form \( \Pi_{\lambda, \mu} \) where \( [\lambda] = [\mu] \cup \{\gamma\} \) and \( [\lambda'] = [\mu] \cup \{\gamma'\} \) for a multipartition \( \mu \) and two nodes \( \gamma = (a, b, c) \) and \( \gamma' = (a', b', c') \) with \( c = c' \).

(b) \( h_i - h_j = km \) with \( i \neq j \) and \( |m| < n \) for parameters such that \( s_i - s_j - m \in \kappa^{-1}Z \). in terms of the above definition they are of the form \( \Pi_{\lambda, \lambda'} \) where \( [\lambda] = [\mu] \cup \{\gamma\} \) and \( [\lambda'] = [\mu'] \cup \{\gamma'\} \) for a multipartition \( \mu \) and two nodes \( \gamma = (a, b, i) \) and \( \gamma' = (a', b', j) \).

Two categories whose parameters lie in the same chamber are equivalent as highest weight categories [29] Proposition 2.8]. On the other hand, the bounded derived categories of \( O_{\lambda}(n) \) and \( O_{\lambda-\psi}(n) \) are derived equivalent when \( c := (\kappa', s') \) is obtained from \( c - \psi := (\kappa'', s'') \) by crossing a wall to an adjacent chamber. Two chambers are separated by the wall \( \Pi_{\lambda, \lambda'} \) if and only if the sign of \( c_{\lambda} - c_{\lambda'} \) in a chamber is opposite to \( c_{\lambda} - c_{\lambda'} \) in the adjacent one. The derived equivalences \( WC_{c-\psi+c} : D^b(O_c) \to D^b(O'_{c-\psi}) \) are called wall-crossing functors; see [29] Section 2.8] for the construction of these functors.

Losev proved that \( WC_{c-\psi+c} : D^b(O_c) \to D^b(O'_{c-\psi}) \) is a perverse equivalence with respect to the filtration of simple modules by their supports (the function \( \pi \) then picks out the dimension of the support) [29] Proposition 2.12]. Therefore \( WC_{c-\psi+c} : D^b(O_c) \to D^b(O'_{c-\psi}) \) induces a canonical bijection \( WC_{c-\psi+c} \) on \( \Pi^f \), called the combinatorial wall-crossing. For walls of type (b), the combinatorial wall-crossings have been studied in [29], [22]. In particular, they are given by the crystal isomorphisms of Section [23] for the appropriate parameters [22] Theorem 11]. We are here interested in the walls of type (a). First we need to see for which types of parameters they are defined.
We will denote by \( wc_{--} \) the combinatorial wall-crossing from \( \kappa > 0 \) to \( \kappa' < 0 \) corresponding to the wall of type (a).

**Proposition 3.3.** Assume that \( c := (\kappa, s) \) and \( c' := (\kappa - 1, s') \) are two sets of parameters in \( \mathfrak{c}_\mathbb{Z} \), with \( \kappa > 0 \) and \( \kappa' < 0 \) lying in two different chambers separated by a unique wall of type (a) (and no wall of type (b)). Then there exists \( \pi \in \mathfrak{S}_\ell \) such that

\[
(3.4) \quad s_{\pi(i)} - s_{\pi(i+1)} > n \quad \text{and} \quad s'_{\pi(i+1)} - s'_{\pi(i)} > n
\]

for all \( i = 1, \ldots, \ell \). Conversely, if \( c \) and \( c' \) are as above, they are separated by a unique wall of type (a) and no wall of type (b).

**Proof.** Without loss of generality, we can assume that \( \ell = 2 \). Assume that \( 0 \leq s_1 - s_2 \leq n \). If we have \( s'_1 < s'_2 \) then there exist a bipartition \( \mu \) of \( n - 1 \) and two addable nodes of this bipartition \( \gamma_1 = (a_1, b_1, 1) \) and \( \gamma_2 = (a_2, b_2, c_2) \) such that \( \text{co}(\gamma_1) < \text{co}(\gamma_2) \) for the parameter \( c \) and such that \( \text{co}(\gamma_1) \neq \text{co}(\gamma_2) \) for \( c' \) which implies that a wall of type (b) is crossed.

Assume now that \( s'_1 > s'_2 \) then the condition \( 0 \leq s_1 - s_2 \leq n \) implies that there exist a bipartition \( \mu \) of \( n - 1 \) and two addable nodes of this bipartition \( \gamma_1 = (a_1, b_1, 1) \) and \( \gamma_2 = (a_2, b_2, c_2) \) such that \( \text{co}(\gamma_1) > \text{co}(\gamma_2) \) for the parameter \( c \) and such that \( \text{co}(\gamma_1) \neq \text{co}(\gamma_2) \) for \( c' \). This implies now that that a wall of type (b) must be crossed, this implies that the parameter \( c' \) must satisfy \( s'_2 - s'_1 > n \).

A charge \( s \) verifying the condition (3.4) as in Proposition 3.3 is called \( \pi \)-asymptotic. We call a charge asymptotic if it is \( \pi \)-asymptotic for some \( \pi \in \mathfrak{S}_\ell \). We denote \( s_{\text{opp}} := s' \), so that \( s \) is \( \pi \)-asymptotic if and only if \( s_{\text{opp}} \) is \( \pi_{\text{opp}} \)-asymptotic, where \( \pi_{\text{opp}} \in \mathfrak{S}_\ell \) is defined by \( \pi_{\text{opp}}(i) = \pi(\ell - i + 1) \).

This is a slight abuse of notation because \( s' \) is not unique in general. However, two \( \pi \)-asymptotic parameters lie in the same chamber. Thus, the associated categories are equivalent as highest weight categories, and we can identify these two parameters.

**Proposition 3.5.** [29] Proposition 5.6]

(1) The \( \widehat{sl}_\ell \)-crystal commutes with \( wc_{--} \).

(2) The \( \widehat{sl}_\infty \)-crystal commutes with \( wc_{--} \) up to taking the transpose, i.e. \( wc_{--} \circ \tilde{a}_\sigma = \tilde{a}_{\sigma'} \circ wc_{--} \) for all \( \sigma \in \Pi \).

**Theorem 3.6.** Suppose the denominator \( e \) of \( \kappa \) satisfies \( 2 \leq e \leq n \), so that crossing the \( \kappa = 0 \) wall is defined. Let \( s \) be an asymptotic parameter. Assume that \( \lambda = (\lambda^1, \ldots, \lambda^\ell) \) is in the connected component of empty multipartition for the \( \widehat{sl}_\ell \)-crystal and the \( \widehat{sl}_\infty \)-crystal. Then we have:

\[
\Psi_{-s_{\text{rev}}} \circ \Phi(\lambda) = wc_{--} \circ \Phi(\lambda)^\ell
\]

**Proof.** By hypothesis, there exist \( \sigma \in \Pi \) and \( (i_1, \ldots, i_p) \in (\mathbb{Z}/e\mathbb{Z})^p \) such that

\[
\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma(\emptyset, s) = |\lambda, s|
\]

From Theorem 2.9 (1) together with the definition of the crystal isomorphism in Section 2.3, we have that:

\[
\Psi_{-s_{\text{rev}}} \circ \Phi(\tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_\sigma(\emptyset, s)) = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \Psi_{-s_{\text{rev}}} \circ \Phi(\tilde{a}_\sigma(\emptyset, s))
\]

Again, we can use Theorem 2.9 (1) and (3) to deduce that \( \Phi(\tilde{a}_\sigma(\emptyset, s)) = \tilde{a}_{\sigma'}(\emptyset, -s_{\text{rev}}) \). Using the explicit formulae of the crystal isomorphism together with the explicit formula of the action of \( \tilde{a}_\sigma \) [15 Section 5], we have that \( \Psi_{-s_{\text{rev}}} \circ \Phi(\tilde{a}_{\sigma'}(\emptyset, -s_{\text{rev}})) = \tilde{a}_{\sigma'} \Psi_{-s_{\text{rev}}} \circ \Phi(\emptyset, -s_{\text{rev}}) \). We thus deduce that:

\[
\Psi_{-s_{\text{rev}}} \circ \Phi(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma(\emptyset, s)) = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \Psi_{-s_{\text{rev}}} \circ \Phi(\emptyset, -s_{\text{rev}})
\]
On the other hand, by Proposition 3.5 we also get:
\[ \text{wc}_{-\leftrightarrow}(\tilde{f}_{i_{p}} \cdots \tilde{f}_{i_{1}} \alpha_{|\emptyset, s|}) = \tilde{f}_{-i_{p}} \cdots \tilde{f}_{-i_{1}} \Psi_{s_{\text{rev}} \to -s_{\text{opp}}} \tilde{a}_{\sigma}(0, -s_{\text{opp}}) \]
and the result follows.

This Theorem is in fact a generalization of a result by Losev in level 1 [29 Corollary 5.7]. Using the notation of Remark 2.5 denote
\[ M_{e}: \Pi \xrightarrow{\lambda = (\sigma^{e}) \cup \rho} (\sigma^{t})^{e} \cup m_{e}(\rho) \]

**Corollary 3.7 (Losev).** Suppose \( \ell = 1 \). Then for all \( \lambda \in \Pi \),
\[ \text{wc}_{-\leftrightarrow}(\lambda) = (M_{e}(\lambda))^{t}. \]

**Proof.** In the case \( \ell = 1 \), the charge is irrelevant (see also Remark 1.2), thus so is \( \Psi_{s_{\text{rev}} \to -s_{\text{opp}}} \).

By Remark 2.5, \( \lambda = (\sigma^{e}) \cup \rho \) is the level 1 analogue of the decomposition of Corollary 2.3 used to define \( \Phi \), and we can identify \( M_{e}(\lambda) \) with \((m_{e}(\rho), (\sigma^{t})^{e}) \). Thus by Theorem 2.9 (2), we have
\[ \Phi(\lambda) = (m_{e}(\rho), (\sigma^{t})^{e}) = M_{e}(\lambda), \]
and we conclude using Theorem 3.6.

We are able to partially generalize the level 1 statement to level \( \ell \):

**Theorem 3.8.** Suppose the denominator \( e \) of \( \kappa \) satisfies \( 2 \leq e \leq n \), so that crossing the \( \kappa = 0 \) wall is defined. Let \( s \) be an asymptotic parameter. Let \( j \in \{1, \ldots, \ell\} \) be such that \( s_{j} < s_{k} \) for all \( k \neq j \). Assume that \( \lambda = (\lambda^{1}, \ldots, \lambda^{\ell}) \) is such that \( \lambda^{j} \) is \( e \)-regular for all \( j \in \{1, \ldots, \ell - 1\} \setminus \{k\} \) and \( \lambda^{k} \) is arbitrary. The combinatorial wall-crossing \( \text{wc}_{-\leftrightarrow}(\lambda) \) is then given by the formula:
\[ \text{wc}_{-\leftrightarrow}(\lambda) = (m_{e}(\lambda^{1}), \ldots, M_{e}(\lambda^{k}), \ldots, m_{e}(\lambda^{\ell}))^{t}. \]

**Proof.** Let us first assume that \( \lambda^{j} \) is \( e \)-regular for all \( j = 1, \ldots, \ell \). By [29 Prop. 3.1], we know that the wall-crossing is independent of the choice of a Weil generic parameter. In our situation this means that we can assume that for each \( \lambda \in \Pi^{t}(n) \), if two nodes have the same residues then they are in the same component. We thus have an action of a Kac-Moody algebra as a tensor product of \( \ell \) copies of \( \tilde{\mathfrak{sl}}_{\kappa} \), one for each component of the multipartition. By Proposition 3.5, the associated Kashiwara operators commute with \( \text{wc}_{-\leftrightarrow} \). Moreover, we know that \( \text{wc}_{-\leftrightarrow} \) sends the empty multipartition to the empty multipartition and that for each \( e \)-regular partition \( \lambda \), there exists a sequence of Kashiwara operators sending \( \emptyset \) to \( \lambda \). The result follows.

Next, assume that \( \lambda = (\lambda^{1}, \ldots, \lambda^{\ell}) \) is such that \( \lambda^{j} \) is \( e \)-regular for all \( j \in \{1, \ldots, \ell\} \setminus \{k\} \) and \( \lambda^{k} \) is arbitrary. We know that there exists \( \sigma \in \mathcal{P}^{1} \) and \( \mu' = (\mu^{1}, \ldots, \mu^{\ell}) \) such that \( \mu^{i} = \lambda^{i} \) if \( i \neq k \), \( \mu^{k} \) is \( e \)-regular, and \( \tilde{a}_{\sigma} \mu = \lambda \). The result then follows from the definition of the action of the Heisenberg crystal operators \( \tilde{a}_{\sigma} \) together with Proposition 3.5.

We obtain the following interesting corollary which was not immediate from the crystal graph perspective.

**Corollary 3.9.** Let \( s = (s_{1}, \ldots, s_{\ell}) \) be an asymptotic parameter and assume that \( \lambda = (\lambda^{1}, \ldots, \lambda^{\ell}) \) is a highest weight vertex such that each \( \lambda^{j} \) is \( e \)-regular. Then \( \text{wc}_{-\leftrightarrow}(\lambda) = (m_{e}(\lambda^{1}), \ldots, m_{e}(\lambda^{\ell})) \) is a highest weight vertex for the opposite asymptotic parameter.

**Example 3.10.** Take \( \ell = 2 \) and \( s \in \mathbb{Z} \) such that \( s > n - 1 \) so that \( s = (0, s) \) is an asymptotic 2-charge. The bipartitions \( (\lambda^{1}, \lambda^{2}) \) which are both highest weight vertex for the \( \tilde{\mathfrak{sl}}_{\infty} \)-crystal and the \( \tilde{\mathfrak{sl}}_{e} \)-crystal are exactly the ones satisfying \( \lambda^{2} = \emptyset \) and one of the following condition:

- \( \lambda^{1} = \mu^{e} \) for a partition \( \mu \).
- \( \lambda^{1} \) has exactly one good removable node of residue \( s(\text{mod } e) \).
Let us consider the second case and assume that $\lambda^1$ is $e$-regular. Then our theorem asserts that $\text{wc}_{c,-c}(\lambda) = (m_e(\lambda^1), \emptyset)$. This is consistent with the fact that it must be both a highest weight vertex for the $\mathfrak{sl}_\infty$-crystal and the $\mathfrak{sl}_c$-crystal because $m_e(\lambda^1)$ has exactly one removable node of residue $-s(mod \, e)$.

3.5. Ringel duality for highest weight categories. A treatment of Ringel duality for highest weight categories can be found in [3] Appendix]. Let $\mathcal{C}$ be a highest weight category with poset $\Lambda$. Thus for each $\tau \in \Lambda$ we have six highest weight objects indexed by $\tau$ which play an important role in the structure of $\mathcal{C}$: the simple $L(\tau)$, the standard $\Delta(\tau)$, the co-standard $\nabla(\tau)$, the projective $P(\tau)$, the injective $I(\tau)$, and the tilting $T(\tau)$ (recall that a module in a highest weight category is called tilting if it is both $\nabla$-filtered and $\Delta$-filtered). Let $T := \bigoplus_{\tau \in \Lambda} T(\tau)$.

**Definition 3.11.** The Ringel duality is the functor

$$D : D^b(\mathcal{C}) \longrightarrow D^b(\text{End}(T)^{\text{opp}}\text{-mod})$$

given on objects by $D(M) = \text{RHom}(M, T)$. We refer to the category $\mathcal{C}_r := \text{End}(T)^{\text{opp}}\text{-mod}$ as the Ringel dual of $\mathcal{C}$.

We note the following important properties of the Ringel duality:

- The poset for the highest weight category $\mathcal{C}$ is given by $\Lambda$ with the opposite order.
- The Ringel duality is only a derived equivalence. However, $D$ restricts to an equivalence $D : C^\Delta \longrightarrow \mathcal{C}_r^\nabla$ between the full subcategories of $\Delta$-filtered and $\nabla$-filtered objects, respectively; and $D(\Delta(\tau)) = \nabla(\tau)$ for each $\tau \in \Lambda$.
- Moreover, $D(P(\tau)) = \nabla T(\tau)$ and $D(T(\tau)) = \nabla I(\tau)$. However, there is no such formula for the image of an arbitrary simple module; in general, $D$ sends a simple $L(\tau)$ to a complex of modules, not to a single module.

Our goal in the next section is to gain some understanding of what the Ringel duality does to a simple module when $\mathcal{C} = \mathcal{O}_{\kappa, s}$.

3.6. Ringel duality for $\mathcal{O}_{\kappa, s}$, and the diagrammatic approach. Now take $\mathcal{C} = \mathcal{O}_{\kappa, s}$ to be a cyclotomic Cherednik category $\mathcal{O}$ for parameters $\kappa = \pm 1/e$ and $s \in \mathbb{Z}^\ell$ and let $\mathcal{O}_{\kappa, s}^{\text{opp}}$ be its Ringel dual. As with the case of wall-crossings, Losev proved that the Ringel duality is a perverse equivalence with respect to the filtration by support [28]. As discussed in Section 3.3, this means we can pick out a unique composition factor $L(\mu)$ in the homology of the complex $D(L(\lambda))$ such that $\mu$ has maximal bidepth in the $\mathfrak{sl}_c$ and $\mathfrak{sl}_\infty$-crystals. As $\mathcal{O}_{\kappa, s}^{\text{opp}} \cong \mathcal{O}_{\kappa', s'}^{\text{opp}}$ for some parameters $\kappa', s'$ of the same rank and level, the Grothendieck group of $\mathcal{O}_{\kappa, s}$ is again a level $\ell$ and rank $e$ Fock space. The Ringel duality therefore induces an isomorphism of the $\mathfrak{sl}_c$- and $\mathfrak{sl}_\infty$-crystals.

An order on $\Pi^\ell$ with respect to which $\mathcal{O}_{\kappa, s}$ is a highest weight category is given by the $c$-function [17], see the formula in Section 3.4. Unfortunately, nothing nice happens to the Fock space parameters by sending the $c$-function to minus itself. However, if we change our setting from $\mathcal{O}_{\kappa, s}$ to the Morita equivalent category $\mathcal{S}^{\kappa_s, \theta_s}$ [39] Theorem 4.8, the category of finite-dimensional modules over the diagrammatic Cherednik algebra defined by Webster [39], then the combinatorics runs smoothly. Here, $\theta_s^\pm$ corresponds to $\kappa = \pm 1/e$ and $s \in \mathbb{Z}^\ell$ is the charge for the Fock space. Webster inserts a crucial sign into his parameter $\theta$ and hence into the $c$-function as follows: where Gordon-Losev have $h_j = \kappa s_j - j/\ell$, Webster has $\theta_j = \kappa s_j - j(e\kappa)/\ell$. The $c$-functions are then defined identically using $h_j$ and $\theta_j$ respectively. Thus the Gordon-Losev and Webster $c$-functions agree when $\kappa > 0$, but when $\kappa < 0$, they are different; in particular, the Webster $c$-function is divisible by $\kappa$, and so is sent to minus itself by changing $\kappa$ to $-\kappa$. This difference makes our combinatorics agree with Webster’s algebra, as we explain next.
Webster proves that for the category $S_{\ell}^{\kappa}$, taking the Ringel dual corresponds to sending $\kappa$ to $-\kappa$ and keeping the charge $s$ fixed [39, Corollary 5.11]. Next, there is a natural isomorphism of diagram algebras, which is given by replacing the label $i \in \mathbb{Z}/e\mathbb{Z}$ with $-i$ on black strands and the label $j \in \mathbb{Z}/\ell\mathbb{Z}$ with $-j$ on red strands [39, Proposition 4.5]. This isomorphism induces an equivalence

$$* : S_{\ell}^{\kappa} \rightarrow S_{\ell}^{\kappa}$$

which evidently satisfies $\tilde{f}_{-i} \circ * = * \circ \tilde{f}_{i}$ and $\tilde{f}_{-j} \circ * = * \circ \tilde{f}_{j}$. Composing $*$ with Ringel duality for the diagrammatic Cherednik algebra then gives an equivalence

$$* \circ D : D^b(S_{\ell}^{\kappa}) \rightarrow D^b(S_{\ell}^{\kappa})$$

which is perverse, being the composition of an abelian equivalence with a perverse equivalence [7].

### 3.7. The combinatorial Ringel duality

We have an equivalence of highest weight categories $S_{\ell}^{\kappa} \simeq O_{e,s}$, and so on the level of Grothendieck groups, $* \circ D$ yields an involutive isomorphism of Fock spaces

$$d : \mathcal{F}_{e,s} \rightarrow \mathcal{F}_{e,-s}$$

which we call combinatorial Ringel duality.

In order to understand $d$, we consider some properties of the Ringel duality proper.

**Lemma 3.12.** Ringel duality commutes with the $\widehat{\mathfrak{sl}}_e$-crystal.

**Proof.** Since the Ringel dual of $O_{\kappa,s}$ can be realized as $O_{\kappa',t}$ for some parameters $\kappa' \in \mathbb{Q}$, $t \in \mathbb{Q}^\ell$ in the same lattice as $\kappa, s$ (as there exists a chamber where the $c$-order is opposite to that given by $(\kappa, s)$), it is possible to reach a category equivalent to the Ringel dual of $O_{\kappa,s}$ via some sequence of wall-crossings. By [29, Proposition 5.6], every wall-crossing commutes with the $\widehat{\mathfrak{sl}}_e$-crystal. □

The symmetrical statement for the $\widehat{\mathfrak{sl}}_\ell$-crystal follows by level-rank duality which switches the roles of $e$ and $\ell$ and commutes with Ringel duality:

**Lemma 3.13.** Ringel duality commutes with the $\widehat{\mathfrak{sl}}_\ell$-crystal.

**Proof.** By [34, Theorem 7.4] and [39], the category $O_{e,s}$ is standard Koszul, which by [31] implies that the Ringel duality commutes with the Koszul duality. The Koszul duality $K$ lifts the level-rank duality $k$, by which the $\widehat{\mathfrak{sl}}_e$-crystal is defined, to the categorical level [34, Theorem 7.4]. Since we need to compare Ringel duality for level $e$ with Ringel duality for level $\ell$, write $D$ for the former and $\tilde{D}$ for the latter; likewise, write $d$ for the former and $\tilde{d}$ for the latter on the level of Grothendieck groups. We have $DK = K\tilde{D}$ and thus $dk = k\tilde{d}$ and $kd = \tilde{d}k$ (since $k$ is an involution). Let $\tilde{f}_j$ be an $\widehat{\mathfrak{sl}}_\ell$ crystal operator. Recall from diagram [2.4] that the action of $\tilde{f}_j$ in level $\ell$ and rank $e$ is defined by $\tilde{f}_j = k\tilde{f}_j k$. By the preceding lemma, in level $e$ and rank $\ell$ we have $\tilde{f}_j \tilde{d} = \tilde{d} \tilde{f}_j$. Therefore, $d\tilde{f}_j = dk\tilde{f}_j k = k\tilde{d}f_j k = k\tilde{f}_j dk = k\tilde{f}_jkd = \tilde{f}_j d$. □

**Corollary 3.14.** For all $i = 0, \ldots, e - 1$ and all $j = 0, \ldots, \ell - 1$ we have

$$d\tilde{f}_i = \tilde{f}_{-i} d \quad \text{and} \quad d\tilde{f}_j = \tilde{f}_{-j} d.$$

**Lemma 3.15.** For all $\sigma \in \Pi$, we have $d\tilde{a}_\sigma = \tilde{a}_\sigma d$.

**Proof.** As in the proof of [29, Proposition 5.6(3)], it suffices to check what happens in level 1. Now we observe that in level 1 the Ringel duality and the wall-crossing across the $\kappa = 0$ wall are the same, and thus the Ringel duality twists the $\mathfrak{sl}_\infty$-crystal operators by their transpose as in [29, Proposition 5.6(3)]. Since $*$ merely relabels the strands it does not affect the $\mathfrak{sl}_\infty$-crystal. □
Theorem 3.16. We have
\[ d = \Phi. \]

Proof. Corollary 3.14 and Lemma 3.15 combined imply that \( d \) and \( \Phi \) satisfy the same commutation relations with the operators for the \( \mathfrak{sl}_n \), \( \widehat{\mathfrak{sl}}_k \), and \( \mathfrak{sl}_\ell \)-crystals. Moreover, both maps send \( |\emptyset, s\rangle \) to \( |\emptyset, -s_{\text{rev}}\rangle \). By Theorem 2.9 (1), \( \Phi \) is the unique involution with this property, so \( d = \Phi \).

Corollary 3.17. Suppose \( L(\lambda) \) is a finite-dimensional irreducible representation of the rational Cherednik algebra of type \( B_n = G(2,1,n) \) for parameters corresponding to Fock space charge \( s = (s_1, s_2) \in \mathbb{Z}^2 \). Then \( \Phi(\lambda) = (\lambda) \), and \( (* \circ D)(L(\lambda)) = L(\lambda) \).

Proof. By Theorem 3.1, \( L(\lambda) \) is finite-dimensional if and only if \( |\lambda, s\rangle \) has depth 0 in both the \( \mathfrak{sl}_\infty \) and \( \widehat{\mathfrak{sl}}_\ell \)-crystals. Recall that \( D \) is perverse with respect to filtration by the depth in the \( \widehat{\mathfrak{sl}}_\ell \)-crystals. Then by Definition 3.2, \( D \) sends \( L(\mu) \) to some \( L(\mu) \) if \( \lambda \) has depth 0 in both crystals, in which case \( (* \circ D)(L(\lambda)) = L(\Phi(\lambda)) \). When \( \ell = 2 \), \( \Phi \) fixes the \( \widehat{\mathfrak{sl}}_\ell \)-crystal operators \( \tilde{f}_j \) since \( j = -j \) mod 2. Moreover, \( -s_{\text{rev}} = (-s_2, -s_1) = (s_1, s_2) - (s_2, s_1) \), i.e. \( -s_{\text{rev}} \) is just an integer shift of \( s \). Now, it is straightforward to see that the action of \( \tilde{f}_j \) on an element of \( \Pi_{s} \) is not affected by an integer shift of the charge, more precisely
\[ \tilde{f}_j |\nu, t\rangle = |\nu, t'\rangle \Rightarrow \tilde{f}_j |\nu, t + k\rangle = |\nu, t' + k\rangle \]
for all \( \nu \in \Pi, t \in \mathbb{Z}(s) \) and \( k \in \mathbb{Z} \). In particular, we get \( \Phi(|\lambda, s\rangle) = |\lambda, -s_{\text{rev}}\rangle \), which proves the claim.

In level 2 we can therefore think of the Ringel duality more or less as a self-equivalence of \( \mathcal{O}_{e,s}(B_n) \) which fixes cuspidal (i.e. finite-dimensional) modules.

Example 3.18. Take \( e = 3 \), \( s = (-1, 3) \) and \( \lambda = (3, 3, 0) \). Then
\[ \beta(|\lambda, s\rangle) = ( |(0, 0), (0, 2)\rangle, \emptyset , |(2, 2, 1), (-1, -2, 1)\rangle ) \]
\[ = ( |(0, 0), (0, 2)\rangle, \emptyset , \tilde{f}_0 \tilde{f}_0 \tilde{f}_0 \tilde{f}_1 |(0, 0, 0), (0, 1, 1)\rangle ) \]
so that \( L(\lambda) \) is finite-dimensional in \( \mathcal{O}_{1/e,s}(6) \). We have
\[ \Phi(|\lambda, s\rangle) = \beta^{-1}( |(0, 0), (-2, 0)\rangle, \emptyset , \tilde{f}_0 \tilde{f}_0 \tilde{f}_0 \tilde{f}_1 |(0, 0, 0), (0, 1, 1)\rangle ) \]
\[ = \beta^{-1}( |(0, 0), (-2, 0)\rangle, \emptyset , |(1, 2, 2), (0, 1, 1)\rangle ) \]
\[ = |(3, 3, 0), (-3, 1)\rangle \]
\[ = |\lambda, -s_{\text{rev}}\rangle. \]

4. Other crystal structures and generalization of Mullineux involutions

We now explain a possible generalization of the Mullineux involution defined in [9] that uses the results of Section 2. We already know that we have three different crystal structures on level \( \ell \) Fock spaces which are pairwise commuting, namely the \( \widehat{\mathfrak{sl}}_\ell \), \( \mathfrak{sl}_\ell \)- and \( \mathfrak{sl}_\ell \)-crystal. We now slightly generalize this by introducing an additional parameter \( d \in \mathbb{Z}_{\geq 0} \) (which plays the role of the parameter \( \ell \) in [9] in level one - that is type A situation). To do this, recall that if \( \lambda \in \Pi \), we can uniquely write the decomposition:
\[ \lambda = \lambda_{(0)} + \lambda_{(1)}^d + \ldots + \lambda_{(n)}^d \]
for \( n \in \mathbb{Z}_{\geq 0} \) and where each \( \lambda_{(i)} \) is \( d \)-regular.

Let \( j \in \mathbb{Z}_{\geq 0} \). One can define an action of a Kashiwara operator \( f_{k,j} \) (for \( k = 0, \ldots, d-1 \)) as follows:
\[ f_{k,j} \lambda = \mu \quad \text{for all } \lambda \in \Pi \]
where $\mu(t) = \lambda(t)$ when $t \neq d$ and $\mu(d) = \tilde{f}_k \lambda(d)$ (where $\tilde{f}_k$ is denoting the usual Kashiwara operator acting on $d$-regular partitions).

Using the decomposition in Theorem 2.2, we get an action of Kashiwara operators on the whole Fock space as follows. Let $|\lambda, s\rangle \in \Pi^\ell_s$ and write

$$\beta(|\lambda, s\rangle) = (|\lambda_1, r\rangle, \sigma, |\lambda_2, \tilde{r}\rangle).$$

Then, for all $j \in \mathbb{Z}_{>0}$ :

$$\tilde{f}_{k,j}(|\lambda, s\rangle) = \beta^{-1}(|\lambda_1, r\rangle, \tilde{f}_{k,j} \sigma, |\lambda_2, \tilde{r}\rangle)$$

For each $j \in \mathbb{Z}_{>0}$, we thus get an $\mathfrak{s}\mathfrak{l}_\ell$-crystal.

It is immediate to see that these actions also commute with the $\mathfrak{s}\mathfrak{l}_\ell$-crystal and the $\mathfrak{s}\mathfrak{l}_\ell$-crystal (it just follows from the existence of the bijection $\beta$). Finally, there is an obvious analogue of the Mullineux involution for this decomposition, which depends on $d$. Namely, for $|\lambda, s\rangle \in \Pi^\ell_s$ and $\beta(|\lambda, s\rangle) = (|\lambda_1, r\rangle, \sigma, |\lambda_2, \tilde{r}\rangle)$, we define:

$$\Phi^{(d)}(|\lambda, s\rangle) = \beta^{-1}(m_{e, r}(\lambda_1), m_e(\sigma(0)) + m_e(\sigma(1))^d + \ldots + m_e(\sigma(n))^d, m_{\ell, \tilde{r}}(\lambda_2)),$$

so that $\Phi^{(d)}$ generalizes simultaneously $\Phi$ and the version of the Mullineux involution of [9] (which we recover by taking $\ell = 1$). As in [9], we believe it would be interesting to look at the case $\ell = 2$ and investigate the relationship between $\Phi^{(d)}$ on the one hand, and the Alvis-Curtis duality for unipotent representations of finite unitary groups in transverse characteristic $d$ on the other hand (or more generally of finite groups of Lie type $B$ and $C$).

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