ON SINGULAR EQUATIONS WITH CRITICAL AND SUPERCRITICAL EXPONENTS

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Abstract. We study the problem
\[
(I_\varepsilon) \begin{cases}
-\Delta u - \frac{\mu u}{|x|^2} = u^p - \varepsilon u^q & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u \in H^1_0(\Omega) \cap L^{q+1}(\Omega),
\end{cases}
\]
where \( q > p \geq 2^* - 1, \varepsilon > 0, \Omega \subseteq \mathbb{R}^N \) is a bounded domain with smooth boundary, \( 0 \in \Omega, N \geq 3 \) and \( 0 < \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2 \). We completely classify the singularity of solution at 0 in the supercritical case. Using the transformation \( v = |x|\nu u \), we reduce the problem \((I_\varepsilon)\) to \((J_\varepsilon)\)
\[
(J_\varepsilon) \begin{cases}
-\text{div}(|x|^{-2\nu} \nabla v) = |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v \in H^1_0(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}),
\end{cases}
\]
and then formulating a variational problem for \((J_\varepsilon)\), we establish the existence of a variational solution \( v_\varepsilon \) and characterise the asymptotic behaviour of \( v_\varepsilon \) as \( \varepsilon \to 0 \) by variational arguments and when \( p = 2^* - 1 \).

This is the first paper where the results have been established with super critical exponents for \( \mu > 0 \).

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1. Introduction

In this paper, we consider the following family of singular problems:

\begin{align}
-\Delta u - \frac{\mu u}{|x|^2} &= u^p - \varepsilon u^q \quad \text{in } \Omega,
\end{align}

(1.1)

\begin{align}
&u > 0 \quad \text{in } \Omega,
\end{align}

and

\begin{align}
-\text{div}(|x|^{-2\nu} \nabla v) &= |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q \quad \text{in } \Omega,
\end{align}

(1.2)

\begin{align}
v > 0 \quad \text{in } \Omega,
\end{align}

where \( q > p \geq 2^* - 1 = \frac{N+2}{N-2} \), \( \varepsilon > 0 \) is a parameter, \( \Omega \subseteq \mathbb{R}^N \) is a star-shaped bounded domain with smooth boundary, \( 0 \in \Omega \), \( N \geq 3 \) and \( 0 < \mu < \bar{\mu} := \left( \frac{N-2}{2} \right)^2 \) and \( \nu \in (0, \frac{N-2}{2}) \). By the Pohozaev’s identity, we know that when \( \varepsilon = \mu = \nu = 0 \) and \( \Omega \) is star shaped, (1.1) and (1.2) have no solutions. In this paper, we are mainly concerned with two issues. One of them is to classify the nature of singularity to the solutions of Eq.(1.1) and the other one is to study the asymptotic behavior of solutions of the problem (1.2) as \( \varepsilon \to 0 \). When \( \mu = 0 \), the asymptotic behaviour of this class of equation with supercritical exponent was studied by Merle and Peletier in \([18, 19]\). Also see Mcleod et.al. \([17]\) for the uniqueness proof for the entire solution in the supercritical case, Han \([14]\) and Brezis–Peletier \([2]\) for the subcritical blow up. As per our knowledge, there is no existing result with supercritical exponents for \( \mu > 0 \).

We assume that

\begin{align}
v_\varepsilon(0) &= \max_{\Omega} v_\varepsilon(x).
\end{align}

(1.3)

If we look for radial solutions of Eq. (1.1), we would expect \( u \) as a function of the radial variable \( r \) to behave like \( Ar^{-m} \) near 0, where \( A \) and \( m \) satisfy

\begin{align}
A \left[ -m(m+1) + m(N-1) - \mu \right] r^{-m-2} &= -(1 + o(1)) A q r^{-mq},
\end{align}

(1.4)

so that either

\begin{align}
m(m-N+2) + \mu > 0, \quad m+2 = mq &\implies m = \frac{2}{q-1} \quad \text{and} \quad q > \frac{\mu + 2
u'}{\mu},
\end{align}

(1.5)

or

\begin{align}
m+2 < mq, \quad m(m+1) - m(N-1) + \mu = 0 &\implies m = \nu \text{ or } \nu',
\end{align}

(1.6)

for

\begin{align}
\nu := \sqrt{\mu} - \sqrt{\mu - \bar{\mu}}; \quad \nu' := \sqrt{\mu} + \sqrt{\mu - \bar{\mu}}.
\end{align}

(1.7)

Note that \( \nu < \nu' \). Also, one can readily check \( \frac{\mu + 2
u'}{\mu} = \frac{2 + \nu}{\nu} \). In the region where \( q < \frac{2 + \nu}{\nu} \), we have \( \nu < \frac{2}{q-1} \). Therefore in this region

\begin{align}
r^{-\nu} < c \min\{r^{-\frac{2}{q-1}}, r^{-\nu'}\}.
\end{align}

(1.8)

On the other hand, in the region where \( q \geq \frac{2 + \nu}{\nu} \), we have

\begin{align}
r^{-\frac{2}{q-1}} \leq c \min\{r^{-\nu}, r^{-\nu'}\}.
\end{align}

(1.9)
It is easy to check from (1.2) that for the blow up with exponent \( \frac{2}{q-1} \) (see (1.3)) the constant \( A \) would be determined, whereas for the second type of blow up it would appear to be free.

In Section 3 we prove that near 0, any solution \( u \) of Eq. (1.1) satisfies

\[
C_1|x|^{-\nu} \leq u(x) \leq C_2|x|^{-\nu} \quad \text{if} \quad 2^* - 1 \leq p < q < \frac{2+\nu}{\nu},
\]

and

\[
C_3|x|^{-\frac{2\nu}{q-1}} \leq u(x) \leq C_4|x|^{-\frac{2\nu}{q-1}} \quad \text{if} \quad q > \max\left\{ p, \frac{2+\nu}{\nu} \right\},
\]

for some positive constants \( C_1, C_2, C_3 \) and \( C_4 \). Moreover when \( u(x) = u(|x|) \) and \( q = \frac{2\nu}{p-1} \),

\[
u(|x|) \sim |x|^{-\nu} \log |x|^{-\frac{1}{2}}, \quad \text{as} \quad |x| \to 0.
\]

More precisely, if

\[-\Delta u - \frac{\mu u}{|x|^2} = f_i(u), \quad i = 1, 2,\]

we can classify the singularity of \( u(x) \) near the origin with the nonlinearities \( f_1(u) = u^p + \varepsilon u^q \) where \( 1 \leq q < p = 2^* - 1 \) or \( f_2(u) = u^p - \varepsilon u^q \) where \( 2^* - 1 \leq p < q \) in the following way.

| Range of \((p, q)\) | Singularity at 0 |
|----------------|-----------------|
| \(1 \leq q < p = 2^* - 1\) | \(1 \leq q < p < \frac{\nu}{2} + 1\) |
| \(2^* - 1 \leq p < q < \frac{\nu}{2} + 1\) | \(q > \max\left\{ p, \frac{\nu}{2} + 1 \right\}\)
| \(q > \max\left\{ p, \frac{\nu}{2} + 1 \right\}\) | \(C_1 \leq |x|^{\nu} u(x) \leq C_2\) |

for some \( C_1 > 0, C_2 > 0 \). For the subcritical case see [13].

Near 0, Eq. (1.1) can be written as \(-\Delta u - \frac{\mu u}{|x|^2} = -(1 + o(1))u^q\). Therefore, if \( u \) is radial then by setting \( v(r) = u^\frac{1}{\alpha} \), the above equation reduces to

(1.10)

\[v'' + \frac{N - 1 - 2\nu}{r}v' = Ar^{-(q-1)\nu} v^q \quad \text{in} \quad (0, a),\]

for some \( a > 0 \) and \( 1 - \delta < A < 1 + \delta \), for some \( \delta > 0 \) small. Using the Emden-Fowler transformation \( t = (\frac{r}{a})^\alpha \) and \( y(t) = \alpha^{-\nu} v(r) \), where \( \alpha = N - 2 - 2\nu \), (1.10) reduces to the so-called Emden-Fowler type equation

(1.11)

\[y''(t) = A t^{-(2\alpha+2)(q-1)\nu} y^q, \quad t \geq R,\]

for some \( R > 0 \) large. These type of equations have several interesting applications in mathematical physics. It appears in astrophysics in the form of Emden equation and in atomic physics in the form of Thomas-Fermi statistical model of atoms. Emden-Fowler type equations appears in modelling the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. For more details see [11 10 15 21].

Recently, a great deal of attention is given to the mathematical study of following class of semilinear elliptic problem

(1.11)

\[\begin{cases}
\Delta u + \frac{\mu}{|x|^2} u + f(u) = 0 & \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial\Omega,
\end{cases}\]

where \( f \) is a super-linear function; \( 0 \in \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( 0 \leq \mu < \frac{(N-2)^2}{4} \) and \( N \geq 3 \). This class of problems is of particular interest.
as this arises in mathematical models related to reaction diffusion equations and celestial mechanics. We recall the classical Hardy’s inequality: if \( u \in H^1_0(\Omega) \), then

\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \tilde{\mu} \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx,
\]

where \( \tilde{\mu} \) is never achieved by any \( u \in H^1_0(\Omega) \).

We denote by \( D^{1,2}(\mathbb{R}^N) \), the closure of \( C_0^{\infty} (\mathbb{R}^N) \) with respect to the norm \(( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx )^{\frac{1}{2}} \). When \( 0 < \mu < \tilde{\mu} \), it is easy to check that the following is an equivalent norm for \( D^{1,2}(\mathbb{R}^N) \):

\[
\| u \|_{D^{1,2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \right)^{\frac{1}{2}}.
\]

When \( \mu = 0 \), define

\[
S(v) := \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}}}, \quad S_N = \inf_{v \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, dx}{\left( \int_{\mathbb{R}^N} v^2 \, dx \right)^{\frac{1}{2}}}.
\]

For \( N \geq 3 \) and \( \nu = 0 \), Merle–Peletier \cite{19} proved:

**Theorem (Merle–Peletier, \cite{19})**

(i) There exist \( \varepsilon \) and \( \theta_\varepsilon \) with \( \varepsilon \to 0 \) and \( \theta_\varepsilon \) is uniformly above and away from 0, such that there exists a solution \( u_\varepsilon \) of Eq. \((\text{1.2})\) with \( \nu = 0 \) and

Furthermore, if \( p = 2^* - 1 \), then \( S(\theta_\varepsilon u_\varepsilon) \to S_N \) as \( \varepsilon \to 0 \) and there exists constants \( A, B \) such that \( A < \int_{\Omega} u_\varepsilon^{p+1} < B \).

and when \( x \neq 0 \)

\[
\varepsilon^{\frac{1}{q-p+2}} u_\varepsilon(x) \to \left[ (N(N-2))^{\frac{2}{q-2}} \frac{G(x, x_0)}{A(q, N)R(x_0)} \right] \quad \text{as} \quad \varepsilon \to 0.
\]

where

\[
A(q, N) = \left[ \frac{N^2 c(q, N)}{|N(N-2)|^{\frac{2}{q}}} B \left( \frac{N(N-2)}{2} - 1 \right) \right]^2, \quad c(q, N) = \frac{(N-2)q-(N+2)}{2(q+1)}
\]
and $B(a, b)$ denotes the Beta function [19] defined by

\begin{equation}
B(a, b) = \int_0^\infty t^{a-1}(1 + t)^{-a-b}.\end{equation}

$G$ is the Green function and $x_0$ is the critical point of the Robin function, see (1.27) with $\nu = 0$. Moreover, if $p > 2^* - 1$, then

\[ \varepsilon^{\frac{1}{p'}} \|u\|_{\infty} \sim c^* \quad \text{as} \quad \varepsilon \to 0, \]

and when $x \neq 0$,

\begin{equation}
\varepsilon^{-\theta} u_\varepsilon(x) \to (c^*)^{-\theta}(J_p - c^* J_q)G(x, x_0), \quad \text{as} \quad \varepsilon \to 0,
\end{equation}

where

\[ \theta = \frac{(N - 2)p - N}{2(q - p)}; \quad J_p = \int_{\mathbb{R}^N} V^p dx \]

and $(c^*, V)$ is the unique solution of

\[ \begin{cases}
-\Delta V = V^p - c^* V^q \quad \text{in} \quad \mathbb{R}^N, \\
V \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N).
\end{cases} \]

In [16, 20] the following problem with critical exponent and Hardy potential was studied:

\begin{equation}
\begin{cases}
-\Delta u - \frac{\mu}{|x|^2} u = u^{2^*-1} + \varepsilon u \quad \text{in} \quad \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial\Omega,
\end{cases}
\end{equation}

where $0 \in \Omega \subset \mathbb{R}^N$; $N \geq 3$, $0 < \mu < \mu_1$ and $\varepsilon > 0$ is a parameter. Jannelli [16] proved the following: If $0 \leq \mu < \mu_1 - 1$, then (1.17) has a positive solution $u \in H_0^1(\Omega)$ for all $0 < \varepsilon < \lambda_1$. Furthermore, he proved that if $\mu_1 - 1 < \mu < \mu_1$, Eq. (1.17) has a positive solution $u \in H_0^1(\Omega)$ if and only if $\varepsilon \in (\lambda^*, \lambda_1)$ for some $\lambda^* \in (0, \lambda_1)$, when $\Omega$ is the ball then Eq. (1.17) has no positive solution for all $\varepsilon \leq \lambda^*$. Cao-Peng [4] studied problem similar to Eq. (1.17) for the almost critical case. Cao-Peng [4] and Ramaswamy-Santra [20] used the radial nature of the positive solution to obtain the global uniqueness and blow-up profile as $\varepsilon \to 0$. It was proved in [20], when $N \geq 5$ and $v_\varepsilon \in H_0^1(\Omega, |x|^{-2\nu})$ is a solution of Eq. (1.17) satisfying

\[ S = \lim_{\varepsilon \to 0} \frac{\|x|^{-\nu} |\nabla v_\varepsilon|_{L^2(\Omega)}^2}{\|x|^{-\nu} v_\varepsilon\|_{L^{2\nu}(\Omega)}^2}, \quad \mu < \mu_1 - 1, \]

then along a subsequence

\begin{equation}
\lim_{\varepsilon \to 0} \varepsilon \|v_\varepsilon\|_{L^{2(N-2\nu+4)}(\Omega)} = \frac{(N - 2)^2}{2N^2(N - 2 - 2\nu)} b_n S(\mu) - \omega_N |R(0)|
\end{equation}

where $b_n = \int_0^\infty \frac{t^{N-2\nu-1}}{(1 + t^{\frac{N-2\nu}{2}})^N} dt$; and when $x \neq 0$

\[ \lim_{\varepsilon \to 0} \varepsilon v_\varepsilon(x) \|v_\varepsilon\|_{\infty} = \frac{N - 2}{N(N - 2 - 2\nu)} \omega_N G(x, 0), \]

where $R(0)$ and $G(x, 0)$ are as defined in (1.27) and (1.24) respectively.
Define,
\[ S = \inf_{u \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |x|^{-2\nu} u^2 \, dx \right)^{\frac{1}{2}}}, \]
where \( \nu \in \left[0, \frac{N-2}{2}\right) \). Thanks to Caffarelli-Kohn-Nirenberg inequality \[3\], we have \( S > 0 \). It is also well-known that \( S \) in the above expression is same as
\[ \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \, dx}{\left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}}} \]
and independent of the domain \( \Omega \), where \( \mu \in (0, \bar{\mu}) \). In the above two expression of \( S \), the parameters \( \mu \) and \( \nu \) are related by \( \nu = \sqrt{\bar{\mu}} - \sqrt{\mu - \bar{\mu}} \). From \[5\], we know
\[ S = S_N \left( 1 - \frac{4\mu}{(N-2)^2} \right)^{-\frac{N+2}{N-2}}, \]
where \( S_N \) is the usual Sobolev constant. Moreover, Catrina-Wang \[5, 9\] proved that \( S \) is achieved by
\[ U(x) = \left( \frac{N\alpha^2}{N-2} \right)^{\frac{N-2}{N}} \left( 1 + \frac{\alpha}{N-2} \right)^{-\frac{N+2}{N-2}}, \]
where
\[ \alpha = N - 2 - 2\nu. \]
Furthermore, by \[22\], \( U \) is the unique solution (dilation invariant) of the following entire problem:
\[ \begin{cases} -\nabla(|x|^{-2\nu} \nabla U) = |x|^{-2\nu} U^{2^*-1} & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N \setminus \{0\}, \\ U \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}). \end{cases} \]
Define the Green’s function \( G \) as
\[ H(x, y) = G(x, y) + F(x, y), \]
where \( G(x, y) \) is defined by
\[ \begin{cases} \nabla_x(|x|^{-2\nu} \nabla_x G(x, y)) = \delta_y & \text{in } \Omega, \\ G(x, y) = 0 & \text{on } \partial\Omega, \end{cases} \]
and \( H(x, y) \) is the regular part of the Green function
\[ \begin{cases} \nabla_x(|x|^{-2\nu} \nabla_x H(x, y)) = 0 & \text{in } \Omega, \\ H(x, y) = F(x, y) & \text{on } \partial\Omega, \end{cases} \]
for any fixed \( y \in \Omega \) and
\[ F(x, y) = -\frac{1}{(N-2-2\nu)\omega_N |x-y|^{N-2\nu-2}}. \]
is the fundamental solution of the non-degenerate elliptic operator $\nabla(|x|^{-2\nu} \nabla)$. By construction, $H(x,0)$ is negative and Hölder continuous near the origin \[6\]. Define the Robin function as

$$R(x) = H(x,x).$$

Hence $R$ is continuous at the origin and we can write

$$\lim_{|x| \to 0} R(x) = R(0).$$

For the supercritical case ($p > 2^* - 1$), we define the functional

$$F(v, \Omega) = \frac{1}{2} \int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx + \frac{1}{q+1} \left( \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx \right)^{\frac{1}{l}},$$

where

$$l = \frac{2(q+1) - N(p-1)}{2(p+1) - N(p-1)},$$

and $v \in H^1_0(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu})$.

Also define,

$$K := \inf \left\{ F(v, \mathbb{R}^N) : v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}), \int_{\mathbb{R}^N} |x|^{(p+1)\nu} |v|^{p+1} = 1 \right\}.$$

For the critical case ($p = 2^* - 1$), we consider the usual functional

$$S(v) = \left( \int_{\Omega} |x|^{-2\nu} |\nabla v|^2 dx \right)^{\frac{1}{p+1}},$$

where $v \in H^1_0(\Omega, |x|^{-2\nu}) \cap L^{p+1}(\Omega, |x|^{-(p+1)\nu})$.

We turn now to a brief description of the results presented below. The first result concerns the non-existence result when $p = 2^* - 1$.

**Theorem 1.1.** Let $0 \leq \mu < \bar{\mu}$ and

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = u^{2^*-1} - u^q & \text{in } \mathbb{R}^N, \\ u > 0, \\ u \in D^{1,2}(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N), \end{cases}$$

where $q > 2^* - 1$. Then Eq. (1.32) does not admit any solution.

The proof of this theorem is based on the Pohozaev identity. The difficulty in applying this identity comes from the fact that the solution blows up at origin (see Section 3).
where

Theorem 1.4. point theorem and comparison principle. (1.36)

exists) of
data and which is bounded away from 0.
to show the existence of positive solution of this equation with suitable boundary

Standard methods of finding lower estimate, e.g. the methods of

there exists a constant

1.1]) and

Remark 1.2. Since

Theorem 1.2. Let

D

achieved by a radially decreasing function in

u

Setting the transformation

v = |x|^p u

in

in

with

λ > 0

there exists a constant

K := inf \{ F(u, |x|^p u) : u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{p+1} = 1 \}.

there exists a constant λ > 0

\begin{align}
-\Delta u - \mu \frac{u}{|x|^2} &= \lambda u^p - u^q \quad \text{in} \quad \mathbb{R}^N, \\
\text{u > 0} \quad \text{in} \quad \mathbb{R}^N, \\
\text{u ∈ D}^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N). 
\end{align}

Theorem 1.3. Assume

with 0 < \mu < \bar{\mu} and \Omega be any smooth domain (bounded or unbounded). Then there exists \( r_0 > 0 \) (small) and \( C_1 > 0 \) \((r_0 \text{ and } C_1 \text{ independent of } u)\) such that \( u(x) \geq C_1 |x|^{-\nu} \quad \forall \ x \in B_{r_0}(0) \setminus \{0\}. \)

Remark 1.1. Standard methods of finding lower estimate, e.g. the methods of [14] do not work here. In Section 3, we have shown that to get the estimate

\( |u(x)| \geq C |x|^{-\nu}, \) it is enough to show that solution of the following equation is bounded away from 0,

\[-\text{div}(|x|^{-2\nu} \nabla w) + |x|^{-(q+1)\nu} w^{q} = 0.\]

To show the existence of positive solution of this equation with suitable boundary data and which is bounded away from 0, we have used ODE technique, Banach fixed point theorem and comparison principle.

Theorem 1.4. (i) If \( p = 2^* - 1 \), then any solution \( u \) of Eq. (1.36) satisfies

\[ u(x) \leq C |x|^{-\nu} \quad \forall \ x \in B_{\rho_0}(0) \setminus \{0\}, \]

where \( \rho_0 > 0 \) is sufficiently small.

(ii) If \( p > 2^* - 1 \) and \( q > (p-1)\frac{N}{2} - 1 \) then the same conclusion holds as in (i).

Remark 1.2. Since \( p = 2^* - 1 \) implies \( (p-1)\frac{N}{2} - 1 = 2^* - 1 \), the condition \( q > (p-1)\frac{N}{2} - 1 \) is readily satisfied in the case \( p = 2^* - 1 \) as \( q \) is supercritical.
Theorem 1.5. Let \( \mu \in (0,\bar{\mu}) \) and \( q > \max\{p, \frac{2+\nu}{p}\} \). Then any solution of Eq. (1.36) satisfies
\[
u(x) \leq C|x|^{-\frac{2}{q-1}} \quad \forall \ x \in B_\rho(0) \setminus \{0\},
\]
where \( \rho > 0 \) is sufficiently small.

Theorem 1.6. Let \( \mu \in (0,\bar{\mu}) \) and \( q > \max\{p, \frac{2+\nu}{p}\} \). Then any solution of Eq. (1.36) satisfies
\[
u(x) \geq C|x|^{-\frac{2}{q-1}} \quad \forall \ x \in B_R(0) \setminus \{0\},
\]
where \( R > 0 \) is sufficiently small.

Theorem 1.7. Let \( \mu \in (0,\bar{\mu}) \) and \( q = \frac{2+\nu}{p} \). Then any radial solution \( u \) of Eq. (1.36) satisfies
\[
\lim_{|x| \to 0} \frac{u(x)}{|x|^{-\nu} \log |x|} = \left( \frac{C}{2} \right) ^{2}. \]
where \( \alpha = N - 2 - 2\nu \).

Theorem 1.8. Let \( 2^*-1 \leq p < (p-1)\frac{N}{2} - 1 < q \) and \( \tilde{\rho} := \frac{1}{2} \min\{\rho_0, \rho\} \), where \( \rho_0 \) and \( \rho \) be as in Theorem 1.3 and Theorem 1.5 respectively. Then there exists \( \mu^* = \mu^*(N,q) > 0 \) and a constant \( C \) depending on \( N,p,q,\mu \) such that any solution \( u \) of Eq. (1.36) satisfies
\[
\|\nabla u(x)\| \leq \begin{cases} C|x|^{-(\nu+1)} & \text{if } \mu \in [0,\mu^*), \\ C|x|^{-(\frac{2+\nu}{p})} & \text{if } \mu \in [\mu^*,\bar{\mu}]. \end{cases}
\]
for \( 0 < |x| < \tilde{\rho} \).

Remark 1.3. In the above theorem \( \mu^* = \left( \frac{N-2}{2} \right)^2 - \left( \frac{N-2}{2} - \frac{2}{q-1} \right)^2 \). It’s easy to note that \( \mu < \mu^* \iff q < \frac{2+\nu}{p} \). From Theorem 1.3 and Theorem 1.4 it follows any solution \( u \) has singularity of the order \( \nu \) when \( q < \frac{2+\nu}{p} \). Therefore in this region of \( q \), it is anticipated that \( \|\nabla u\| \leq C|x|^{-(\nu+1)} \) near 0. On the other hand when \( q > \frac{2+\nu}{p} \), from Theorem 1.5 and Theorem 1.6, we have singularity of \( u \) at 0 is of order \( \frac{2}{q-1} \). Consequently in this region we expect \( \|\nabla u\| \leq C|x|^{-(\frac{2+\nu}{p})} \).

Theorem 1.9. Let \( 2^*-1 \leq p < (p-1)\frac{N}{2} - 1 < q \) and \( 0 \leq \mu < \left( \frac{N-2}{2} \right)^2 \). Then any positive solution \( u \in D^{1/2}(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N) \) of Eq. (1.35) is radially symmetric with respect to origin and radially decreasing.

To discuss the asymptotic behaviour of problem (1.2) for general domain, we first formulate a variational problem for (1.2). Then we establish existence of variational solution \( v_\varepsilon \) for small positive values of \( \varepsilon \) and finally we derive the asymptotic behavior of \( v_\varepsilon \) as \( \varepsilon \to 0 \), using variational arguments again. This is the first result for the problem with critical and supercritical exponent in the singular case and the case appears to be more complicated than the smooth case.

Theorem 1.10. There exists \( \varepsilon_n > 0 \) and \( \lambda_n > 0 \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \) and \( \lambda_n \) uniformly bounded above and away from zero, such that
(i) there exists a solution \( u_n \) to Eq. (1.2) corresponding to \( \varepsilon = \varepsilon_n \); 
(ii) if \( p > 2^*-1 \), then \( F(\lambda_n u_n) \to \mathcal{K} \) and \( \int_\Omega |x|^{-(p+1)\nu} u_n^{p+1} dx \to 0 \) as \( n \to \infty \);
(iii) if \( p = 2^* - 1 \), then \( S(\lambda_n u_n) \to S \) as \( n \to \infty \) and there exist constants \( A, B > 0 \) such that for all \( n \geq 1 \), it holds \( A < \int_{\Omega} |x|^{-(p+1)\nu} u_n^{p+1} \, dx < B \), where \( F(\cdot), K \) and \( S(\cdot), S \) are defined as in (1.28), (1.30) and (1.31), (1.19) respectively.

**Theorem 1.11.** Let \( \nu \in \left(0, \frac{N-2}{4}\right) \), \( 2^* - 1 = p < q < \frac{1+\nu}{\nu} \) and \( v_\varepsilon \in H^1_0(\Omega, |x|^{-2\nu}) \) be a solution of Eq. (1.2) such that

\[
S(\lambda_\varepsilon v_\varepsilon) \to S \quad \text{and} \quad A < \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} \, dx < B,
\]

where \( S(\cdot), S \) are as in (1.31) and (1.19) respectively. Moreover, assume (1.3) is satisfied. Then along a subsequence

\[
\lim_{\varepsilon \to 0} \varepsilon \|v_\varepsilon\|_\infty = \omega_N \left( N-2-2\nu \right) \frac{q(N-2) - (N+2) + 2\alpha}{\alpha} \times B\left( \frac{N-2}{2\alpha}(N-(q+1)\nu), \frac{N-2}{2\alpha} \{ q(N-2-\nu) - (2+\nu) \} \right)^{-1},
\]

where

\[
(1.37) \quad C_{q,N} = \frac{(N-2)q - (N+2)}{2(q+1)},
\]

\( R(\varepsilon) \) and \( B(a,b) \) are as defined in (1.27) and (1.15) respectively. Furthermore, for \( x \neq 0 \),

\[
(1.38) \quad \lim_{\varepsilon \to 0} v_\varepsilon(x)\|v_\varepsilon\|_\infty = \omega_N (N-2-2\nu)^{N-1} \left( \frac{N}{N-2} \right)^{\frac{N-2}{2}} G(x,0),
\]

where \( G(x,0) \) is the Green function as defined in (1.24).

**Remark 1.4.** Now we point out the difference between the supercritical and subcritical case. First we notice there is a critical exponent \( q^* := \frac{1+\nu}{\nu} \) which plays a huge role in determining the singularity of solution (1.1). This implies that there is some competition between the \( \mu \) and \( q \) (or equivalently between \( \nu \) and \( q \)) which never arise in the subcritical case.

**Remark 1.5.** In a forthcoming paper, we show this phenomena holds for the fractional laplacian case with \( \mu = 0 \).

**Notation:** Throughout this paper \( C \) denotes the generic constants which may vary from line to line. Below are few notations which we use throughout the paper:

- \( \bar{\mu} := \left( \frac{N-2}{2} \right)^2 \)
- \( \nu := \sqrt{\mu - \sqrt{\mu - \mu}} \)
- \( \alpha := N-2-2\nu \)
- \( \omega_N := \text{surface measure of unit ball} \).
2. Existence and non-existence of entire solution

In this section, we will study the existence and non-existence result of entire problem with critical and supercritical exponents. We first establish the general Pohozaev identity which will also be used in the next sections.

Proposition 2.1. Let \( \Omega \) be a smooth domain, \( 0 \in \Omega \), \( 0 \leq \mu < \bar{\mu} \), \( N \geq 3 \), \( 2^* - 1 \leq p < q \) and \( u \) be a solution of

\[
\begin{align*}
-\Delta u - \frac{u}{|x|^2} &= u^p - \varepsilon u^q \quad \text{in} \quad \Omega, \\
u > 0, \\
u &\in D^{1,2}(\Omega) \cap L^{q+1}(\Omega),
\end{align*}
\]

Then \( u \) satisfies:

\[
\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x,n) dS + \frac{N-2}{2} \int_{\partial\Omega} u \frac{\partial u}{\partial n} dS + \frac{\mu}{2} \int_{\partial\Omega} \frac{u^2}{|x|^2} (x,n) dS = \varepsilon \left( \frac{N}{q+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{q+1} dx
\]

(2.2) \quad + \left( \frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{p+1} dx - \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} (x,n) dS.

In particular, if \( u = 0 \) on \( \partial\Omega \) we have

\[
\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x,n) dS = \left( \frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{p+1} dx + \varepsilon \left( \frac{N}{q+1} - \frac{N}{q+1} \right) \int_{\Omega} u^{q+1} dx.
\]

(2.3)

Proof. We multiply Eq. (2.1) by a suitable test function and to make the test function smooth we introduce cut-off functions and then pass to the limit.

For \( \delta > 0 \) and \( R > 0 \), we define \( \phi_{\delta,R}(x) = \phi_{\delta}(x)\psi_{R}(x) \) where \( \phi_{\delta}(x) = \phi\left(\frac{|x|}{\delta}\right) \) and \( \psi_{R}(x) = \psi\left(\frac{|x|}{R}\right) \), \( \phi \) and \( \psi \) are smooth functions in \( \mathbb{R} \) with the properties \( 0 \leq \phi, \psi \leq 1 \), with supports of \( \phi \) and \( \psi \) in \( (1, \infty) \) and \( (-\infty, 2) \) respectively and \( \phi(t) = 1 \) for \( t \geq 2 \), and \( \psi(t) = 1 \) for \( t \leq 1 \).

Let \( u \) be a solution of Eq. (2.1). Then \( u \) is smooth away from the origin and hence \( (x \cdot \nabla u)\phi_{\delta,R} \in C^2_c(\Omega) \). Multiplying Eq. (2.1) by this test function and integrating by parts, we obtain

\[
\int_{\Omega} \nabla u \cdot \nabla (x \cdot \nabla u) \phi_{\delta,R} dx - \mu \int_{\Omega} \frac{u(x \cdot \nabla u) \phi_{\delta,R}}{|x|^2} dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} (x \cdot \nabla u) \phi_{\delta,R} dS
\]

(2.4) \quad = \int_{\Omega} (u^p - \varepsilon u^q)(x \cdot \nabla u) \phi_{\delta,R} dx.

Now the RHS of (2.4) can be simplified as

\[
RHS = -\frac{N}{p+1} \int_{\Omega} u^{p+1} \phi_{\delta,R} dx - \frac{1}{p+1} \int_{\Omega} u^{p+1} [x \cdot (\psi_R \nabla \phi_{\delta} + \phi_{\delta} \nabla \psi_R)] dx
\]
\[+ \varepsilon \left( \frac{N}{q+1} - \frac{N}{q+1} \right) \int_{\Omega} u^{q+1} \phi_{\delta,R} dx + \frac{1}{q+1} \int_{\Omega} u^{q+1} [x \cdot (\psi_R \nabla \phi_{\delta} + \phi_{\delta} \nabla \psi_R)] dx
\]
\[+ \frac{1}{p+1} \int_{\partial\Omega} u^{p+1} (x,n) \phi_{\delta,R} dS - \frac{\varepsilon}{q+1} \int_{\partial\Omega} u^{q+1} (x,n) \phi_{\delta,R} dS.
\]
Note that \(|x \cdot (\psi_R \nabla \varphi_\delta + \varphi_\delta \nabla \psi_R)| \leq C\) and hence using the dominated convergence theorem we get,

\[
\lim_{R \to \infty} \left[ \lim_{\delta \to 0} \text{RHS} \right] = -\frac{N}{p+1} \int_\Omega u^{p+1} dx + \frac{N \varepsilon}{q+1} \int_\Omega u^{q+1} dx \\
+ \frac{1}{p+1} \int_{\partial \Omega} u^{p+1} (x, n) dS - \frac{\varepsilon}{q+1} \int_{\partial \Omega} u^{q+1} (x, n) dS.
\]

(2.5)

By a direct calculation and integration by parts, LHS of (2.4) simplifies as,

\[
\text{LHS} = \int_\Omega |\nabla u|^2 \varphi_{\delta, R} + \sum_{i=1}^n \sum_{j=1}^n \frac{1}{2} \int_\Omega \left( (u_{x_i})^2 \right)_{x_j} \varphi_{\delta, R} + \int_\Omega (x \cdot \nabla u)(\nabla u \cdot \nabla \varphi_{\delta, R}) + \mu \int_\Omega \frac{u^2}{|x|^2} \varphi_{\delta, R} dx \\
+ \frac{\mu N}{2} \int_\Omega \frac{u^2}{|x|^2} \varphi_{\delta, R} dx + \mu \int_\Omega \frac{u^2}{|x|^2} (x \cdot \nabla \varphi_{\delta, R}) dx - \mu \int_\Omega \frac{u^2}{|x|^2} \varphi_{\delta, R} dx \\
- \int_{\partial \Omega} u^2 (x, n) \phi_{\delta, R} dS - \frac{\mu}{2} \int_\Omega \frac{u^2}{|x|^2} \varphi_{\delta, R} dx \\
= -\frac{N-2}{2} \left( \int_\Omega |\nabla u|^2 \varphi_{\delta, R} - \mu \int_\Omega \frac{u^2}{|x|^2} \varphi_{\delta, R} \right) dx - \frac{1}{2} \int_\Omega |\nabla u|^2 (x, n) \phi_{\delta, R} dS \\
- \frac{1}{2} \int_\Omega \left( |\varphi_{\delta, R}| - \mu \frac{u^2}{|x|^2} \right) \left[ (x \cdot \nabla \varphi_\delta) \psi_R + (x \cdot \nabla \psi_R) \varphi_\delta \right] dx
\]

(2.6)

Also we note that,

\[
\lim_{R \to \infty} \lim_{\delta \to 0} \left| \int_{\mathbb{R}^N} (x \cdot \nabla u)(\nabla u \cdot \nabla \varphi_\delta) \psi_R | dx \leq C \lim_{\delta \to 0} \int_{\delta \leq |x| \leq 2\delta} |\nabla u|^2 \frac{|x|}{\delta} dx \\
\leq 2C \lim_{\delta \to 0} \int_{|x| \leq 2\delta} |\nabla u|^2 dx = 0.
\]

Similarly

\[
\lim_{R \to \infty} \lim_{\delta \to 0} \left| \int_{\mathbb{R}^N} (x \cdot \nabla u)(\nabla u \cdot \nabla \psi_R) \varphi_\delta | dx \leq C \lim_{R \to \infty} \int_{R \leq |x| \leq 2R} |\nabla u|^2 \frac{|x|}{R} dx = 0.
\]

Using the above estimates and taking the limit using dominated convergence theorem and using the fact \(|x \cdot (\psi_R \nabla \varphi_\delta + \varphi_\delta \nabla \psi_R)| \leq C\), we get from (2.6),

\[
\lim_{R \to \infty} \left[ \lim_{\delta \to 0} \text{LHS} \right] = -\frac{N-2}{2} \left( \int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx \\
- \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 (x, n) dS - \frac{\mu}{2} \int_{\partial \Omega} \frac{u^2}{|x|^2} (x, n) dS.
\]

(2.7)

Moreover, multiplying the Eq. (2.1) by \(u\), we have

\[
\int_\Omega |\nabla u|^2 dx - \int_{\partial \Omega} u (\nabla u \cdot n) dS - \mu \int_\Omega \frac{u^2}{|x|^2} dx = \int_\Omega (u^{p+1} - \varepsilon u^{q+1}) dx
\]

(2.8)
Substituting (2.5) and (2.7) in (2.4) and using (2.8) we get
\[
- \frac{N-2}{2} \left( \int_{\Omega} u^{p+1} dx - \varepsilon \int_{\Omega} u^{q+1} dx + \int_{\partial \Omega} u \frac{\partial u}{\partial n} dS \right) - \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 (x, n) dS \\
- \frac{\mu}{2} \int_{\Omega} \frac{u^2}{|x|^2} (x, n) dS = - \frac{N}{p+1} \int_{\Omega} u^{p+1} dx + \frac{N\varepsilon}{q+1} \int_{\Omega} u^{q+1} dx
\]
(2.9)+ \quad \frac{1}{p+1} \int_{\partial \Omega} u^{p+1} (x, n) dS - \frac{\varepsilon}{q+1} \int_{\partial \Omega} u^{q+1} (x, n) dS.

This implies (2.2). If \( u = 0 \) on \( \partial \Omega \), it is easy to see that (2.3) follows from (2.2). \( \square \)

**Proof of Theorem 1.1.** If \( u \) is a solution of Eq. (1.32), then it follows from Proposition 2.1 that
\[
\left( \frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} u^{q+1} dx = 0,
\]
which is a contradiction as \( q > 2^* - 1 \) and \( u > 0 \). This proves the theorem. \( \square \)

**Proof of Theorem 1.2.** We are going to work on the manifold
\[
N = \left\{ u \in D^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^{p+1} dx = 1 \right\}.
\]
Then \( F \) reduces to
\[
F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \frac{1}{q+1} \int_{\mathbb{R}^N} u^{q+1} dx.
\]
Let
\[
(2.10) \quad \mathcal{K} = \inf_{N} F(u).
\]
Let \( u_n \) be a minimizing sequence in \( N \) such that
\[
F(u_n) \to \mathcal{K} \text{ with } \int_{\mathbb{R}^N} u_n^{p+1} dx = 1.
\]
As \( \mu < \frac{4}{N} \) implies \( ||u|| := \left( \int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right)^{\frac{1}{2}} \) is an equivalent norm in \( D^{1,2}(\mathbb{R}^N) \), we have \( \{u_n\} \) is a bounded sequence in \( D^{1,2}(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \). Therefore there exists \( u \in D^{1,2}(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \) such that \( u_n \to u \) in \( D^{1,2}(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \). Consequently \( u_n \to u \) pointwise almost everywhere.

Using symmetric rearrangement technique, without loss of generality we can assume that \( u_n \) is radially symmetric. Hence \( u_n(x) = u_n(r) \), where \( r = |x| \), and we can write
\[
u_n(r) = - \int_{-\infty}^{r} u'_n(s) ds.
\]
Using a standard argument it can be shown that \( u_n \) satisfies Strauss type uniform estimate
\[
(2.11) \quad |u_n(r)| \leq C r^{-\frac{N-2}{2}}
\]
for some \( C > 0 \). We claim that \( u_n \to u \) in \( L^{p+1}(\mathbb{R}^N) \).

To see the claim, we note that \( u_n^{p+1} \to u^{p+1} \) pointwise almost everywhere. Since \( \{u_n\} \) is uniformly bounded in \( L^{q+1}(\mathbb{R}^N) \), using Vitali’s convergence theorem, it is
easy to check that \( \int_K u_n^{p+1} dx \to \int_K u^{p+1} dx \) for any compact set \( K \) in \( \mathbb{R}^N \) containing the origin. Furthermore, \( \int_{\mathbb{R}^N \setminus K} u_n^{p+1} dx \) is very small by (2.11) and hence we have strong convergence. Moreover, \( \int_{\mathbb{R}^N} u_n^{p+1} dx = 1 \) implies \( \int_{\mathbb{R}^N} u^{p+1} dx = 1 \).

Now we show that \( K = F(u) \).

We note that \( u \to ||u||^2 \) is weakly lower semicontinuous. Therefore using Fatou’s lemma we can write

\[
K = \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^2} dx + \frac{1}{q + 1} \int_{\mathbb{R}^N} u_n^{q+1} dx \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2} ||u_n||^2 + \frac{1}{q + 1} \int_{\mathbb{R}^N} u_n^{q+1} dx \right]
\]

\[
\geq \frac{1}{2} ||u||^2 + \frac{1}{q + 1} \int_{\mathbb{R}^N} u^{q+1} dx
\]

\[
\geq F(u).
\]

This proves \( F(u) = K \). Moreover, using the Schwartz symmetrisation method via Poly-a-Szego inequality, it is easy to check that \( u \) is radially symmetric and radially decreasing. Applying the Lagrange multiplier rule, we obtain \( u \) satisfies

\[ -\Delta u - \mu \frac{u}{|x|^2} + u^q = \lambda u^p, \]

for some \( \lambda > 0 \). This in turn implies

\[ -\Delta u - \mu \frac{u}{|x|^2} = \lambda u^p - u^q \text{ in } \mathbb{R}^N. \]

3. Classification of singularity near 0

3.1. Lower and upper estimate of solution. In this subsection, we study the asymptotic behavior of solutions (whenever exists) at origin of Eq. (1.36).

Following Lemma 3.1 and 3.2 are crucially used to prove Theorem 1.3.

**Lemma 3.1.** Let \( q < \frac{2 + \nu}{2} \) and \( \nu \in (0, \frac{N-2}{2}) \). Then there exists \( l > 0 \) (can be chosen small) such that the following problem

\[
\begin{cases}
-\Delta w + \mu \frac{w}{|x|^2} + |x|^{-(q+1)\nu} w^q = 0 & \text{in } B_l(0) \\
w > 0 & \text{in } B_l(0) \\
w \in H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu})
\end{cases}
\]

has a continuous radial solution \( w_1 \) such that \( w_1(0) = 1 \).

**Proof.** To prove this lemma, it is enough to show that the following ODE has a unique solution \( w_1 \) in \( (0, l) \) for some \( l > 0 \) and \( w_1 \) is a solution of Eq. (3.1),

\[
\begin{cases}
w'' + \frac{N - 1 - 2\nu}{r} w'(r) = r^{-(q-1)\nu} w^q & \text{in } (0, 1) \\
w > 0 & \text{in } (0, 1) \\
w(0) = 1
\end{cases}
\]
We can write a solution of the above ODE as
\begin{equation}
(3.3) \quad w(r) = 1 + \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w^q(t) dt ds.
\end{equation}
Since \( q < \frac{2+\nu}{\nu} \), using Banach fixed point theorem, it is easy to check that solution of the integral equation \((3.3)\) exists and unique in \((0, l)\) for some \( l > 0 \). From \((3.3)\), it follows \( w \) is continuous in \([0, l]\) and
\begin{equation}
(3.4) \quad w'(r) = r^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w^q(t) dt \quad \text{for} \quad r > 0.
\end{equation}
Therefore by a straight forward computation it follows
\begin{equation}
(3.5) \quad \int_0^l w'(r)^2 r^{N-1-2\nu} dr < \infty \quad \text{and} \quad \int_0^l w^{q+1}(r) r^{N-1-(q+1)\nu} < \infty
\end{equation}
as \( q < \frac{2+\nu}{\nu} \) and \( \nu < \frac{N-2}{2} \). Define \( w_1(x) := w(r) \), where \( r = |x| \).

Claim: \( w_1 \) is a weak solution of Eq. (3.1).

Indeed by \((3.4)\), \( u_1 \in H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu}) \). Choose \( 0 < \eta < l \) and define \( \chi_\eta \in C_0^\infty(B_l(0)) \) such that \( \chi_\eta = 1 \) for \( |x| \leq \frac{\eta}{2} \), \( \chi_\eta = 0 \) for \( |x| > \eta \) and \( |\nabla \chi_\eta| \leq \frac{4}{\eta} \). Let \( \phi \in C_0^\infty(B_l(0)) \) be arbitrarily chosen. Set \( D_\eta := B_l(0) \setminus B_\eta(0) \).

Therefore,
\begin{align*}
&\int_{B_l(0)} |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \int_{B_l(0)} |x|^{-(q+1)\nu} w_1^q \phi dx \\
&= \lim_{\eta \to 0} \int_{B_l(0)} \chi_\eta |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \lim_{\eta \to 0} \int_{B_l(0)} \chi_\eta |x|^{-(q+1)\nu} w_1^q \phi dx \\
&\quad + \lim_{\eta \to 0} \int_{B_l(0)} (1 - \chi_\eta) (|x|^{-2\nu} \nabla w_1 \nabla \phi + |x|^{-(q+1)\nu} w_1^q \phi) dx \\
&\quad - \lim_{\eta \to 0} \int_{D_\eta} \nabla (1 - \chi_\eta) \nabla w_1 |x|^{-2\nu} \phi \\
&\quad - \lim_{\eta \to 0} \int_{D_\eta} (1 - \chi_\eta) \left( \nabla \cdot (|x|^{-2\nu} \nabla w_1) - |x|^{-(q+1)\nu} w_1^q \right) \phi dx
\end{align*}
(3.5)
Since \( w_1 \) is a solution of the ODE \((3.2)\) in \((0, l)\) and it is \( C^1 \) away from 0, it easily follows that \( w_1 \) is a \( C^1 \) solution of Eq. \((3.1)\) in \( D_\eta \), for every \( \eta > 0 \). Thus the last integral in \((3.5)\) equals 0. Furthermore,
\begin{equation}
| \lim_{\eta \to 0} \int_{D_\eta} \nabla (1 - \chi_\eta) \nabla |x|^{-2\nu} w_1 \phi | \leq \lim_{\eta \to 0} C \eta^{N+1} \cdot \eta^{-2\nu+2\eta+1-(q+1)\nu} = 0.
\end{equation}
Hence \((3.5)\) yields
\begin{equation}
\int_{B_l(0)} |x|^{-2\nu} \nabla w_1 \nabla \phi dx + \int_{B_l(0)} |x|^{-(q+1)\nu} w_1^q \phi dx = 0,
\end{equation}
which in turn proves the claim. This completes the proof of the lemma. \( \square \)

Remark 3.1. It is easy to see that \((3.3)\) is related to a 2nd order ODE and solving this ODE requires two initial/boundary conditions. In our case it is natural to have initial values on \( u(0) \) and \( u'(0) \). But it is not hard to see from \((3.3)\) that \( u''(0) \) is not defined for \( \frac{1+\nu}{\nu} \leq q < \frac{2+\nu}{\nu} \). Therefore a standard ODE technique does not give
existence of solution here. Moreover, as the solution of the integral equation (3.3) is not differentiable at 0, it does not directly follow that \( w \) is a solution of the given PDE (3.1).

**Lemma 3.2.** Let \( m > 0 \), \( q < \frac{2+\nu}{N} \) and \( \nu \in (0, \frac{N-2}{2}) \). Then for some \( \delta \in (0,1) \), there exists a radial continuous solution \( w_3 \) of Eq. (3.1) in \( B_1(0) \), where \( l \) is as in Lemma 3.1 with the property that \( w_3(0) = \delta \) and \( w_3|_{\partial B_1(0)} < m \) and \( \int_0^l |w_3'(r)|^2 r^{N-1-2\nu} \, dr < \infty \).

**Proof.** Given \( \delta > 0 \), let \( w_3 \) be the solution of

\[
\begin{align*}
\frac{d^2w}{dr^2} + \frac{N-1-2\nu}{r} w' &= r^{-q-1}u^q \quad \text{in} \quad (0,l_3), \\
w(0) &= \delta,
\end{align*}
\]

(3.6)

where \( [0,l_3] \) is maximum neighbourhood of 0 where the solution exists. Due to local existence, we have \( l_3 > 0 \) (see for instance, Lemma 3.1). Moreover, we can write the solution as

\[
w_3(r) = \delta + \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w_3^q(t) \, dt \, ds
\]

(3.7)

To see the claim, let \( 0 < \delta_1 < \delta_2 \leq 1 \), then \( w_{\delta_1} \leq w_{\delta_2} \leq w_1 \) in \([0,l] \), where \( w_1 \) and \( l \) are as in Lemma 3.1.

To see the claim, let \( 0 < \delta_1 < \delta_2 \leq 1 \). Since \( w_{\delta_1}(0) < w_{\delta_2}(0) \), there exits \( r_0 > 0 \) such that \( w_{\delta_1} < w_{\delta_2} \) in \([0,r_0] \). Define

\[ S := \{ s \in [0,l] : w_{\delta_1}(s) > w_{\delta_2}(s) \}. \]

If \( S = \emptyset \), then we are done. Suppose \( S \neq \emptyset \). We define

\[ \hat{r}_0 := \inf S. \]

Clearly \( \hat{r}_0 > 0 \). We show that \( \hat{r}_0 \not\leq l \). Indeed, from (3.7), we have

\[
w_{\delta_1}'(r) - w_{\delta_2}'(r) = r^{2\nu+1-N} \int_0^r t^{N-1-(q+1)\nu} [w_{\delta_1}^q(t) - w_{\delta_2}^q(t)] \, dt.
\]

Therefore \( (w_{\delta_1} - w_{\delta_2})'(r) < 0 \) for \( r \in [0,\hat{r}_0] \). This implies \( w_{\delta_1}(\hat{r}_0) < w_{\delta_2}(\hat{r}_0) \), which is a contradiction to the definition of \( \hat{r}_0 \). Hence the claim follows.

**Claim 2.** \( w_3 \to 0 \) uniformly in \([0,l] \), as \( \delta \to 0 \). Note that \( \lim_{\delta \to 0} w_3 \) exists, since \( w_3 > 0 \) and Claim 1 holds. Let \( w := \lim_{\delta \to 0} w_3 \). Using monotone convergence theorem, we pass the limit in (3.7) to obtain

\[
w(r) = \int_0^r s^{2\nu+1-N} \int_0^s t^{N-1-(q+1)\nu} w^q(t) \, dt \, ds.
\]

Solution of this integral equation uniquely exists in \((0,l) \) (see for instance Lemma 3.1). Therefore \( w = 0 \). Hence the claim follows by Dini’s theorem.

Combining Claim 1 and Claim 2, the lemma follows. \( \square \)
Proof of Theorem 1.3. Define, \( v = |x|^\nu u \). Then it follows from [13 Theorem 1.1] that \( v \in H^1_0(\Omega, |x|^{-2\nu}) \) and \( v \) satisfies the following equation:

\[
\begin{cases}
\text{div}(|x|^{-2\nu}\nabla v) = |x|^{-(p+1)\nu}v^p - |x|^{-(q+1)\nu}v^q & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v & \in H^1_0(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}).
\end{cases}
\]  

(3.8)

By elliptic regularity theory \( v \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\}) \) (see [12, 13]). It is easy to see that \( v \) is a super-solution of the following problem

\[
\begin{align*}
-\text{div}(|x|^{-2\nu}\nabla w) + |x|^{-(q+1)\nu}w^q & = 0 & \text{in } & B_l(0), \\
w & = m & \text{on } & \partial B_l(0), \\
w & > 0 & \text{in } & B_l(0), \\
w & \in & H^1(B_l(0), |x|^{-2\nu}) \cap L^{q+1}(B_l(0), |x|^{-(q+1)\nu}),
\end{align*}
\]

(3.9)

where \( l > 0 \) is as in Lemma 3.1 and \( 0 < m < m_l = \min_{|x|=1} v \).

Claim: If \( w \) is any solution of (3.9), then \( v \geq w \) in \( B_l(0) \).

To see the claim, we note that \((v - w)\) satisfies

\[
-\text{div}(|x|^{-2\nu}\nabla (v - w)) \geq -|x|^{-(q+1)\nu} A(x)(v - w) \quad \text{in } \quad B_l(0),
\]

where \( A(x) := \frac{\nu^2(x) - w^q(x)}{v^q(x) - w^q(x)} \leq q \max[v(x), w(x)]^{q-1} \). Moreover, \( w \leq v \) on \( \partial B_l(0) \). Thus taking \((v - w)^-\) as the test function we obtain

\[
\int_{B_l(0)} |x|^{-2\nu}\nabla (v - w)^- \cdot \nabla dx + \int_{B_l(0)} |x|^{-(q+1)\nu} A(x)|v - w^-|^2 dx \leq 0,
\]

which implies \( v \geq w \) in \( B_l(0) \).

By Lemma 3.2 it follows that Eq. (3.9) admits a solution \( w_\delta \) with \( w_\delta(0) = \delta > 0 \). As a result \( \lim_{x \to 0} v(x) \geq \delta \), which in turn implies

\[
u(x) \geq c|x|^{-\nu}, \quad x \in B_{r_0}(0) \setminus \{0\},
\]

for some \( r_0 > 0 \) small.

Proof of Theorem 1.4. We prove this theorem in the spirit of [13]. Define,

\[
u(x) = |x|^\nu u(x) \quad \text{and} \quad f(x, u) = u^p - u^q.
\]

(3.10)

Then Eq. (1.30) reduces to

\[
-\text{div}(|x|^{-2\nu}\nabla v) = |x|^{-(p+1)\nu}v^p - |x|^{-(q+1)\nu}v^q \quad \forall \quad x \in \Omega \setminus \{0\}.
\]

(3.11)

By elliptic regularity theory \( v \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\}) \) (see [12, 13]). Let \( \rho > 0 \) small enough such that \( B_{\rho}(0) \subset \Omega \). For \( s, l > 1 \), we choose the test function \( \varphi \) as follows:

\[
\varphi = \eta^2 v l^{2(s-1)} \in H^1_0(\Omega, |x|^{-2\nu} dx),
\]

\[
v_l = \min\{v, l\}, \quad \eta \in C_0^\infty (B_{\rho}(0)),
\]

with the properties \( 0 \leq \eta \leq 1, \eta = 1 \) in \( B_{r}(0), \quad r < \rho \) and \( |\nabla \eta| \leq \frac{1}{\rho - r} \). Using this test function \( \varphi \), we obtain from (3.11),

\[
\int_{\Omega} |x|^{-2\nu}\nabla v \nabla \varphi dx = \int_{\Omega} (|x|^{-(p+1)\nu}v^p - |x|^{-(q+1)\nu}v^q) \varphi dx.
\]

(3.12)
Substituting the function $f$, RHS of (3.12) can be simplified as below

$$\text{(3.13)} \quad \text{RHS} = \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^2 \, dx - \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} \, dx.$$

After doing a standard computation, the LHS of (3.12) can be rewritten as:

$$\int_{\Omega} |x|^{-2\nu} \left( 2\eta v \nabla \nabla v + \eta^2 v^2 \nabla |v|^2 + 2(s-1)\eta^2 v^{2(s-1)} |\nabla v|^2 \right) \, dx.$$

Using Cauchy-Schwartz inequality, for any $\epsilon > 0$ we have,

$$|2 \int_{\Omega} |x|^{-2\nu} \eta v \nabla v \, dx| \leq \epsilon \int_{\Omega} |x|^{-2\nu} \eta^2 v^{2(s-1)} |\nabla v|^2 \, dx + C(\epsilon) \int_{\Omega} |x|^{-2\nu} \eta^2 v^{2(s-1)} \, dx.$$

Combining (3.14) and (3.15) we obtain,

$$\int_{\Omega} |x|^{-2\nu} \left( \eta^2 v^{2(s-1)} |\nabla v|^2 + 2(s-1)\eta^2 v^{2(s-1)} |\nabla v|^2 \right) \, dx$$

$$\leq C \int_{\Omega} |x|^{-2\nu} \eta^2 v^{2(s-1)} \, dx + \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} \, dx$$

$$- \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} \, dx.$$

We recall here Caffarelli-Kohn-Nirenberg inequality (see [3]):

$$\left( \int_{\Omega} |x|^{-b} |w|^r \, dx \right)^{\frac{2}{r}} \leq C_{a,b} \int_{\Omega} |x|^{-2a} |\nabla w|^2 \, dx \quad \forall \ w \in H^1_0(\Omega, |x|^{-2a} \, dx),$$

where $-\infty < a < \frac{N-2}{2}$, $a \leq b \leq a+1$, $r = \frac{2N}{N-2+2(b-a)}$ and $C_{a,b}$ is a positive constant.

Let $w = \eta v v^{s-1}$ and $a = b = \nu < \frac{N-2}{2}$ in (3.17). Then $r = 2^*$. Consequently we get from (3.17),

$$\left( \int_{\Omega} |x|^{-2\nu} |\eta v v^{s-1}|^2 \, dx \right)^{\frac{1}{2}} \leq C_{a,a} \int_{\Omega} |x|^{-2\nu} |\nabla (\eta v v^{s-1})|^2 \, dx.$$

Using (3.16), we simplify the RHS of (3.18) as in [13], i.e.,

$$\text{RHS} \leq 2C_{a,a} \int_{\Omega} |x|^{-2\nu} \times \left( |\nabla \eta|^2 \eta^2 v^{2(s-1)} + \eta^2 v^{2(s-1)} |\nabla v|^2 + (s-1)^2 \eta^2 v^{2(s-1)} |\nabla v|^2 \right) \, dx$$

$$\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 \eta^2 v^{2(s-1)} + Cs \int_{\Omega} |x|^{-(p+1)\nu} \eta^2 v^{p+1} \, dx$$

$$- \int_{\Omega} |x|^{-(q+1)\nu} \eta^2 v^{q+1} \, dx.$$
For $p \geq 2^* - 1$, choose $t > 1$ as follows:

$$
\frac{N}{2} < t < \frac{q + 1}{p - 1}.
$$

(3.20)

Note that for $p = 2^* - 1$ the interval $(\frac{N}{2}, \frac{q + 1}{p - 1})$ is always a nonempty set. On the other hand, $(\frac{N}{2}, \frac{q + 1}{p - 1}) \neq \emptyset$, since $q > (p - 1) \frac{N}{2} - 1$. From (3.20) we have,

$$(p - 1)t < q + 1 \quad \text{and} \quad 2 < \frac{2t}{t - 1} < 2^*.$$

Consequently

$$
\int_{\Omega} |x|^{-(p+1)\nu} \eta^2 \nu v_i^{2(s-1)} dx = \int_{\Omega} \eta^2 \nu v_i^{2(s-1)} dx
$$

$$
= \int_{\Omega} |\eta \nu v_i^{s-1}|^2 u^{p-1}|x|^{-2p} dx
$$

$$\leq \left[ u^{p-1}_{L^{p-1}}(\Omega) \| x \|^{-\nu} \eta \nu v_i^{s-1} \right]_{L^p(\Omega)}^2
$$

$$\leq C\left( \epsilon \| x \|^{-\nu} \eta \nu v_i^{s-1} \right)_{L^p(\Omega)}^2
$$

$$+ C(\epsilon) \left( \int_{\Omega} |x|^{-2^*\nu} |\eta \nu v_i^{s-1}|^2 dx \right)^\frac{2}{2^*}
$$

$$+ C\epsilon \left( \int_{\Omega} |x|^{-2\nu} |\eta \nu v_i^{s-1}|^2 dx \right)^\frac{2}{2\nu}.
$$

(3.21)

Plugging (3.21) into (3.19) and then (3.19) into (3.18), we have

$$
\left( \int_{\Omega} |x|^{-2^*\nu} |\eta \nu v_i^{s-1}|^2 dx \right)^\frac{2}{2^*}
\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_i^{2(s-1)} dx
$$

$$+ Cse^2 \left( \int_{\Omega} |x|^{-2^*\nu} |\eta \nu v_i^{s-1}|^2 dx \right)^\frac{2}{2^*}
$$

$$+ Cse^{-\frac{2t}{t-1}} \int_{\Omega} |x|^{-2\nu} |\eta \nu v_i^{s-1}|^2 dx.
$$

(3.22)

By choosing $\epsilon = \frac{1}{\sqrt{2t s}}$, we obtain from (3.22)

$$
\left( \int_{\Omega} |x|^{-2^*\nu} |\eta \nu v_i^{s-1}|^2 dx \right)^\frac{2}{2^*}
\leq Cs \int_{\Omega} |x|^{-2\nu} |\nabla \eta|^2 v^2 v_i^{2(s-1)} dx
$$

$$+ Cse^{-\frac{2t}{t-1}} \int_{\Omega} |x|^{-2\nu} |\eta \nu v_i^{s-1}|^2 dx
$$

$$\leq Cse^{-\frac{2t}{t-1}} \int_{\Omega} |x|^{-2\nu} (\eta^2 + |\nabla \eta|^2) v^2 v_i^{2(s-1)} dx,
$$

(3.23)

where $\alpha = \frac{2t}{2^*-1}$. Moreover, it is not difficult to check that

$$
\int_{\Omega} |x|^{-2^*\nu} |\eta \nu v_i^{s-1}|^2 dx \geq \int_{\Omega} |x|^{-2^*\nu} |\eta|^2 v^2 v_i^{2s-2} dx.
$$
Consequently as in [13], we have
\[
\left( \int_{\Omega} |x|^{-2\nu} |\eta|^2 v^2 v_1^{2s-2} \, dx \right)^{\frac{1}{2}} \leq C s^\alpha \int_{\Omega} |x|^{-2\nu} (\eta^2 + |\nabla \eta|^2) v^2 v_1^{2(s-1)} \, dx
\]
(3.24)
\[
\leq C s^\alpha \int_{\Omega} |x|^{-2\nu} (\eta^2 + |\nabla \eta|^2) v^2 v_1^{2(s-1)} \, dx.
\]
Substituting \( \eta \) and \( \nabla \eta \) we deduce
\[
\left( \int_{B_r(0)} |x|^{-2\nu} v^2 v_1^{2s-2} \, dx \right)^{\frac{1}{2}} \leq \frac{C s^\alpha}{(\rho - r)^2} \left( \int_{B_r(0)} |x|^{-2\nu} v^2 v_1^{2s-2} \, dx \right)^{\frac{1}{2}}.
\]
(3.25)
Set \( s^* \) and \( s_j \) as follows:
\[
\frac{N}{N - 2} < s^* < \frac{q + 1}{2} \quad \text{and} \quad s_j = s^* \left( \frac{2^*}{2} \right)^{j}, \quad j = 1, 2, \ldots.
\]
If we take \( s = s_j \) in (3.25), a straight forward computation yields:
\[
\left( \int_{B_{r_j+1}(0)} |x|^{-2\nu} v^2 v_1^{2s_{j+1}-2} \, dx \right)^{\frac{1}{2}} \leq \left( \frac{C}{(1 - \rho_0)\rho_0} \right)^{\frac{1}{2}} \left( \int_{B_{r_j}(0)} |x|^{-2\nu} v^2 v_1^{2s_{j}-2} \, dx \right)^{\frac{1}{2}} \times
\]
(3.27)
\[
\prod_{j=0}^{\infty} s_j^{\frac{1}{2}} \left( \int_{B_{r_0}(0)} |x|^{-2\nu} v^2 v_1^{2s_{j}-2} \, dx \right)^{\frac{1}{2}}.
\]
By standard computation it follows that (see [13])
\[
\sum_{j=0}^{\infty} \frac{1}{2s_j} \leq C, \quad \sum_{j=0}^{\infty} \frac{j}{2s_j} \leq C \quad \text{and} \quad \prod_{j=0}^{\infty} s_j^{\frac{1}{2}} \leq C.
\]
Since \( 2^* < 2s^* < q + 1 \), after a straight forward computation as in [13], we obtain
\[
\int_{B_{r_0}(0)} |x|^{-2\nu} v^2 v_1^{2s_{j+1}-2} \, dx \leq (\text{diam } \Omega)^{(2s_{j+1}-2)\nu} \int_{\Omega} u^{2s_{j+1}} \, dx \leq C.
\]
As a result, from (3.27) we have
\[
\left( \int_{B_{r_j+1}(0)} |x|^{-2\nu} v^2 v_1^{2s_{j+1}-2} \, dx \right)^{\frac{1}{2}} \leq C.
\]
Moreover,
\[
\text{LHS of (3.28)} \geq \left( \int_{B_{r_j+1}(0)} |x|^{-2\nu} v_1^{2s_{j+1}} \, dx \right)^{\frac{1}{2}} \geq (\text{diam } \Omega)^{\frac{s_{j+1}}{2s_{j+1}}} |v_1|_{L^{2s_{j+1}}(B_{r_0}(0))}
\]
(3.29)
Combining (3.29) with (3.28), we obtain
\[ |v_l|_{L^{2^*_{j+1}}(B_{\rho_0}(0))} \leq C(\text{diam } \Omega)^{2^*_{j+1}}. \]

Note that \( s_j \rightarrow \infty \) as \( j \rightarrow \infty \). Hence
\[ |v_l|_{L^{\infty}(B_{\rho_0}(0))} \leq C. \]
Finally letting \( l \rightarrow \infty \) we have
\[ u(x) \leq C|x|^{-\nu} \quad \forall \ x \in B_{\rho_0}(0) \setminus \{0\}. \]

\[ \square \]

**Proof of Theorem 1.5.** We use an idea from [11]. If \( u \) is a positive solution of Eq. (1.36), then
\[ -\Delta u - \mu u|x|^2 = -(1 + o(1))u^q, \quad \text{in } B_R(0), \]
for some \( R > 0 \) small. Using the transformation \( v = |x|^{\nu}u \), we get \( v \) satisfies
\[ -\text{div}(|x|^{-2\nu}\nabla v) = -(1 + o(1))|x|^{-(q+1)^\nu}v^q, \quad \text{in } B_R(0). \]
Therefore we can write
\[ -\text{div}(|x|^{-2\nu}\nabla v) = -A|x|^{-(q+1)^\nu}v^q, \quad \text{in } B_R(0), \]
where \( 1 - \delta < A < 1 + \delta \), for some \( \delta > 0 \).

**Claim:** \( v(x) \leq C|x|^{\mu - \frac{2}{q+1}} \quad \text{in } B_{\frac{3R}{4}}(0) \setminus \{0\}, \) for some \( C = C(N,q,p,R,\mu) > 0 \).

To see the claim, for \( 0 < r < R \), set
\[ y = \frac{x}{r} \quad \text{and} \quad w(y) = r^{-\nu + \frac{2}{q+1}}v(x). \]
Then \( w \) satisfies Eq. (3.31) in \( B_{\frac{1}{4}}(0) \).

Now define
\[ W(y) := c \left[ \left( \frac{9}{16} - |y|^2 \right) \left( |y|^2 - \frac{1}{16} \right) \right]^{-\beta}, \]
where \( \beta > \frac{2}{q+1} \) and \( c > 0 \) will be chosen later. Clearly
\[ W = \infty \quad \text{on} \quad \partial(B_{\frac{1}{4}}(0) \setminus B_{\frac{1}{4}}(0)). \]
We show that \( \beta \) and \( c \) in the definition of \( W \) can be chosen such that
\[ -\text{div}(|x|^{-2\nu}\nabla W) \geq -A|x|^{-(q+1)^\nu}W^q, \quad \text{in } B_{\frac{3R}{4}}(0) \setminus B_{\frac{1}{4}}(0). \]
Since \( W \) is radial, it is enough to show that
\[ W'' + \frac{N - 1 - 2\nu}{r}W' \leq A r^{-(q-1)^\nu}W^q, \quad \frac{1}{4} < r < \frac{3}{4}. \]
By a direct computation, when $\frac{1}{4} < r < \frac{3}{4}$, we obtain
\[ W'' + \frac{N - 1 - 2\nu}{r} W' = -2W\beta\left[-\left(\frac{9}{16} - r^2\right)^{-1} + \left(r^2 - \frac{1}{16}\right)^{-1}\right](N + 2r^2 - 2\nu) \\
+ 4r^2W\beta\left[\frac{9}{16} - r^2\right]^{-2} + \left(r^2 - \frac{1}{16}\right)^{-2}] \\
\leq CW\beta\left[\frac{9}{16} - r^2\right]^{-2} + \left(r^2 - \frac{1}{16}\right)^{-2}] \\
\leq CW\beta\left[\frac{9}{16} - r^2\right]^{-2}(r^2 - \frac{1}{16})^{-2}] \\
= C\beta W^{1 + \frac{2}{\beta}} \]
Since $\beta > \frac{2}{4 - 1}$ implies $1 + \frac{2}{\beta} < q$, (3.32) follows. Therefore we obtain,
\[-\text{div}\left(|x|^{-2\nu}\nabla (W - w)\right) \geq -A|x|^{-(q+1)\nu} B(x)(W - w) \text{ in } B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0),\]
where $0 \leq B(x) := \frac{W(x) - w(x)}{W(x) - w(x)} \leq q \max\{W(x), w(x)\}^{q-1}$. Moreover, $(W - w)^- = 0$ on $\partial(B_{\frac{1}{4}}(0) \setminus B_{\frac{1}{4}}(0))$. Thus taking $(W - w)^-$ as the test function we obtain
\[
\int_{B_{\frac{1}{4}}(0) \setminus B_{\frac{1}{4}}(0)} |x|^{-2\nu}\nabla (W - w)^{-2}dx + \int_{B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0)} A|x|^{-(q+1)\nu} B(x)(W - w)^{-2}dx \leq 0,
\]
which implies $w \leq W$ in $B_{\frac{1}{4}}(0) \setminus B_{\frac{1}{4}}(0)$. In particular,
\[ w(y) \leq \max_{\frac{1}{4} < |y| < \frac{3}{4}} W(y) \text{ in } B_{\frac{3}{4}}(0) \setminus B_{\frac{1}{4}}(0), \]
which yields
\[ \max_{\frac{1}{4} < |y| < \frac{3}{4}} v(x) \leq C|x|^{\nu - \frac{2}{\nu}}. \]
Since $0 < r < R$ was arbitrarily chosen, the claim follows.

Hence $u(x) \leq C|x|^{-\frac{2}{\nu}}$ in $B_{\frac{3}{4}}(0) \setminus \{0\}$. From Theorem 1.4 it also follows that $u(x) \leq C|x|^{-\nu}$. Since $q > \frac{2+\nu}{\nu}$ implies $|x|^{-\frac{2}{\nu}} \leq C|x|^{-\nu}$, the theorem follows. \( \square \)

**Proof of Theorem 1.6.** If $u$ is a positive solution of Eq. (1.36), then as in the proof of Theorem 1.5 $u$ satisfies
\[-\Delta u - \mu \frac{u}{|x|^2} = -(1 + o(1))u^q, \text{ in } B_R(0), \]
for some $R > 0$ small. Using the transformation $v = |x|^\nu u$, we get $v$ satisfies
\[-\text{div}\left(|x|^{-2\nu}\nabla v\right) = -(1 + o(1))|x|^{-(q+1)\nu} v^q, \text{ in } B_R(0). \]
Given $\delta > 0$, we can write
\[ (3.33) \quad -\text{div}\left(|x|^{-2\nu}\nabla v\right) = -A|x|^{-(q+1)\nu} v^q, \text{ in } B_{R}(0), \]
where $1 - \delta < A < 1 + \delta$. Define
\[ (3.34) \quad V(x) := c|x|^{\nu - \frac{2}{\nu}}, \]
where $c < \min\{c_1, c_2\}$,
\[ c_1 := R^{-\nu + \frac{2}{\nu}} \min v, \]
and $c_2$ is defined in (3.40). Therefore, it is easy to see

\begin{equation}
(3.35) \quad v \geq V \text{ on } \partial B_R(0).
\end{equation}

**Claim:** $-\text{div}(|x|^{-2\nu} \nabla V) \leq -A|x|^{-(q+1)\nu} V^q$ in $B_R(0)$.

To prove the claim, we note that since $V$ is radial, it is enough to show that

\begin{equation}
V'' + \frac{N - 1 - 2\nu}{r} V' \geq A r^{-(q-1)\nu} V^q \quad r \in (0, R).
\end{equation}

Using the Emden-Fowler transformation

\begin{equation}
(3.36) \quad y(t) = \alpha^\nu V(r), \quad t = \left(\frac{\alpha}{r}\right)^\alpha,
\end{equation}

where $\alpha = N - 2 - 2\nu$, it is equivalent to prove that

\begin{equation}
(3.37) \quad y''(t) \geq At^{-(2+2+(q-1)\nu)} y^q(t), \quad t > \left(\frac{\alpha}{R}\right)^\alpha.
\end{equation}

Using (3.34) in (3.36), it is not difficult to see that $y(t) = c \alpha^{-\frac{q}{q-1}} t^{-\frac{1}{q} - \frac{1}{q-1}}$. Consequently, by a straightforward computation we obtain,

\begin{equation}
(3.38) \quad y''(t) = c \alpha^{-\frac{q}{q-1}} \left(\nu - \frac{2}{q-1}\right) \left[\left(\nu - \frac{2}{q-1}\right) \frac{1}{\alpha} + 1\right] t^{-\frac{1}{q} - \frac{1}{q-1}}.
\end{equation}

On the other hand, by direct computation it follows

\begin{equation}
(3.39) \quad At^{-(2+2+(q-1)\nu)} y^q(t) = A c \alpha^{-\frac{q}{q-1}} q t^{-\frac{1}{q} - \frac{1}{q-1}}.
\end{equation}

Define

\begin{equation}
(3.40) \quad c_2 := \left(\frac{1}{A} \left(\nu - \frac{2}{q-1}\right) \left[\left(\nu - \frac{2}{q-1}\right) \frac{1}{\alpha} + 1\right]\right)^{\frac{1}{q-1}}.
\end{equation}

Note that $q > \frac{2+\nu}{q}$ implies $\nu - \frac{2}{q-1} > 0$. Therefore, since $c < \min\{c_1, c_2\}$, comparing (3.38) and (3.39), we conclude (3.37) holds true. Hence the claim follows.

We also note that both $v$ and $V$ are bounded in $B_R(0)$ (for $V$ it follows from Theorem 1.5). Therefore combining the Claim and (3.35) and using comparison principle as in previous theorem, we obtain $v \geq V$ in $B_R(0)$. This in turn implies, $u(x) \geq c|x|^{-\frac{1}{q-1}}$ in $B_R(0) \setminus \{0\}$ which completes the proof. \hfill \Box

3.2. The Critical Case $q = \frac{2+\nu}{\nu}$.

**Proof of Theorem 1.7.** Let $u$ be any radial solution of Eq. (1.36) with $q = \frac{2+\nu}{\nu}$. Note that this implies $\nu = \frac{2}{q-1}$. Then as in the proof of previous theorem, $v = r^\nu u$ satisfies

\begin{equation}
(3.41) \quad v'' + \frac{N - 1 - 2\nu}{r} v' = A r^{-2} v^q, \quad r \geq R,
\end{equation}

where $1 - \delta < A < 1 + \delta$, for some $\delta > 0$ (see 3.33). Using the Emden-Fowler transformation

\begin{equation}
(3.42) \quad y(t) = \alpha^\nu v(r), \quad t = \left(\frac{\alpha}{r}\right)^\alpha,
\end{equation}

where $\alpha = N - 2 - 2\nu$, (3.41) reduces to

\begin{equation}
(3.43) \quad t^2 y''(t) - Ay^q(t) = 0, \quad t > \left(\frac{\alpha}{R}\right)^\alpha.
\end{equation}
Claim: \( y(t) \to 0 \) as \( t \to \infty \).

To see the claim, we note that for large \( t \), \( y' \) is increasing and nonnegative. From Theorem 1.13 we have \( \nu \) is bounded near \( 0 \). Therefore \( y \) is bounded near infinity. Using this fact, it is easy to check that \( \lim_{t \to \infty} y'(t) = 0 \). Consequently, \( y'(t) \leq 0 \) for large \( t \) which implies \( y \) is decreasing for large \( t \). Hence \( \lim_{t \to \infty} y(t) = c < +\infty \).

If \( c \neq 0 \)

\[
y'(\theta) = -\int_0^\infty \frac{y^2}{s^2} ds, \quad \text{implies} \quad y(T) = y(t) + \int_T^\infty \frac{y^2}{s^2} ds d\theta.
\]

Hence we have

\[
y(T) \geq y(t) + \frac{c}{2} y \log \frac{t}{T}.
\]

Since \( y \) is bounded, taking the limit \( t \to \infty \) in the above expression yields a contradiction. Hence \( c = 0 \) and the claim follows.

Setting

\[t = e^s \quad \text{and} \quad x(s) = y(t),\]

3.43 yields

\[x''(s) - x'(s) - Ax^q(s) = 0 \quad s \geq R',\]

where \( R' = \log \frac{\alpha}{q} \). We are only interested in the solutions of (3.45), \( x(s) \to 0 \) as \( s \to \infty \). Following an argument along the same line of [25, Lemma 3.2], it can be shown that

\[x(s) = \left(\frac{1}{(q-1)s}\right)^{\frac{1}{q-1}} \left(1 + \frac{q}{q-1}^{\frac{q}{q-1}} \log \frac{s}{s} (1 + o(1))\right).
\]

Using (3.42) and (3.44) and the fact that \( \nu = \frac{2}{q-1} \), we obtain

\[u(r) = \left(\frac{\alpha}{q-1}\right)^{\frac{q}{q-1}} r^{(-\nu) \left(-\log r\right)^{-\frac{q}{q-1}}} \left[1 + \frac{q}{q-1}^{\frac{q}{q-1}} \log(\alpha \log \frac{q}{q} \log \frac{q}{q}) (1 + o(1))\right].
\]

Therefore

\[\frac{u(r)}{r^{-\nu} \log r} \leq \left(\frac{\alpha}{q-1}\right)^{\frac{q}{q-1}} \left[1 + \frac{q}{q-1}^{\frac{q}{q-1}} \log(\alpha \log \frac{q}{q} \log \frac{q}{q}) (1 + o(1))\right].
\]

Hence it is easy to see that

\[\lim_{|x| \to 0} \frac{u(x)}{|x|^{-\nu} \log |x|} = \left(\frac{\alpha}{2}\right)^{\frac{q}{2}}.
\]

3.3. Gradient estimate. In this subsection we establish gradient estimate of any solution of Eq. (1.36) near origin. More precise we prove Theorem 1.8. Towards this goal, first we need the following two lemmas.

Lemma 3.3. Let \( \rho_0, \rho \) be as in Theorem 1.4 and Theorem 1.5, respectively, \( u \) be a weak solution of Eq. (1.36) and \( p, q \) be as in Theorem 1.8. Then there exists \( \mu^* = \mu^*(N, q) > 0 \) and a constant \( C \) depending on \( N, p, q, \mu \) such that \( u \) satisfies

\[
\int_{B_{\frac{\mu^*}{2}} (x)} |\nabla u(x)|^2 dx \leq \begin{cases}
C|x|^{-2(p+1)} & \text{if } \mu \in [0, \mu^*),
C|x|^{-2(\frac{p+1}{q})} & \text{if } \mu \in [\mu^*, \bar{\mu}),
\end{cases}
\]

for \( 0 < |x| < \frac{1}{2} \min \{\rho_0, \rho\} \).
Proof. Define $\tilde{\rho}_0 := \frac{1}{2} \min\{\rho_0, \rho\}$. Fix $x \in \mathbb{R}^N$ such that $0 < |x| < \tilde{\rho}_0$. Let $B = B_{\tilde{\rho}_0}(x)$ and $2B = B_{|x|}(x)$. Choose $\eta \in C_0^\infty(2B)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B$ and $|\nabla \eta| < \frac{8}{|\xi|}$. Define $\varphi := \eta^2 u$. Using this test function $\varphi$, we obtain from Eq. (1.36)

$$\int_{2B} \nabla u \nabla \varphi dx = \int_{2B} \left( \frac{\mu u^2 \eta^2}{|x|^2} + u^{p+1} \eta^2 - u^{q+1} \eta^2 \right) dx.$$ 

Moreover, by a straightforward computation it follows

$$\mu < \mu^* \quad \text{on} \quad B$$

Define

$$\tilde{\rho}_0 := \frac{1}{2} \min\{\rho_0, \rho\}$$

$q < \frac{2 + \mu}{\nu}$. Applying Theorem 1.4 in (3.48), we obtain

$$\int_B |\nabla u|^2 dx \leq C \int_B \left( u^2 |\nabla \eta|^2 + \frac{\mu u^2 \eta^2}{|x|^2} + u^{p+1} \eta^2 - u^{q+1} \eta^2 \right) dx.$$ 

Define

$$\mu^* = \left( \frac{N - 2}{2} \right)^2 - \left( \frac{N - 2}{2} - \frac{2}{q - 1} \right)^2.$$ 

We observe that $\mu < \mu^* \iff q < \frac{2 + \mu}{\nu}$. 

**Case 1:** $q < \frac{2 + \mu}{\nu}$.

In this case we have $\mu \geq \mu^*$. Applying Theorem 1.5 in (3.50), we obtain

$$\int_B |\nabla u|^2 dx \leq C \left( |x|^{-2\nu + N} + |x|^{-(p+1)\nu + N} \right),$$

for every $x$ satisfying $0 < |x| < \tilde{\rho}_0$. Therefore from (3.50), we have

$$\int_B |\nabla u|^2 dx \leq C|x|^{-2(\nu+2)+N} \quad \text{if} \quad \mu \in (0, \mu^*).$$

**Case 2:** $q \geq \frac{2 + \mu}{\nu}$. 

In this case we have $\mu \geq \mu^*$. Applying Theorem 1.5 in (3.50), we obtain

$$\int_B |\nabla u|^2 dx \leq C \left( |x|^{-2\nu + N} + |x|^{-2(\frac{4+\mu}{\nu+1})+N} \right),$$

for $0 < |x| < \tilde{\rho}$.

Combining (3.51) and (3.52), the lemma follows. □

The next lemma is due to Xiang, see [29] Proposition 2.1.

**Lemma 3.4.** Let $\Omega$ be a domain in $\mathbb{R}^N$, $f \in L^\infty_{loc}(\Omega)$ and $u \in H^1_{loc}(\Omega)$ be a weak solution of the equation

$$-\Delta u = f \quad \text{in} \quad \Omega.$$ 

Then for any $B_2R(x_0) \subseteq \Omega$, it holds

$$\sup_{B_\rho(x_0)} |\nabla u| \leq C \left( \int_{B_R(x_0)} |\nabla u|^2 dx \right)^{\frac{1}{2}} + CR|f|_{L^\infty(B_R(x_0))}.$$
Lemma 4.1. Let $s$ satisfy (1.24) and $t$ be close to $r$. Then we can write $-\Delta u = f(x)$, where
\[ f(x) = \mu \frac{u}{|x|^2} + u^p - u^q. \]

Proof. This is a modification of the theorem by Chanillo-Wheeden [6, pg. 311].

Case 1: $q < \frac{2+\nu}{\nu}$.
In this case by Theorem 1.4 it follows $|f(x)| \leq C(|x|^{-\nu - 2} + |x|^{-\nu p} + |x|^{-\nu q})$. Since $q < \frac{2+\nu}{\nu} \iff \mu < \mu^*$, we have
\[ |f(x)| \leq C|x|^{-\nu - 2} \quad \text{if} \quad \mu \in [0, \mu^*), \]
for $0 < |x| < \hat{\rho}$.

Case 2: $q \geq \frac{2+\nu}{\nu}$.
In this case by Theorem 1.3 we obtain
\[ |f(x)| \leq C(|x|^{-\nu - \frac{2+\nu}{\nu}} + |x|^{-\nu p} \leq C|x|^{-\frac{2+\nu}{\nu}}, \quad \mu \in [\mu^*, \hat{\mu}) \]
for $0 < |x| < \hat{\rho}$.

Consequently, in both Case 1 and Case 2, $f \in L^\infty(B_{\hat{\rho}}(0) \setminus \{0\})$. As a result, for any $x \in B_{\hat{\rho}}(0) \setminus \{0\}$, we apply Lemma 3.1 on the domain $B_{\hat{\rho}}(x)$ to obtain that
\[ \sup_{B_{\hat{\rho}}(x)} |\nabla u| \leq C \left( \int_{B_{\hat{\rho}}(x)} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} + C|x||f|_{L^\infty(B_{\hat{\rho}}(x))}. \]
Combining (3.53), (3.54), and Lemma 3.1 we obtain from the above expression
\[ \sup_{B_{\hat{\rho}}(x)} |\nabla u| \leq \begin{cases} C|x|^{-(\nu+1)} \quad &\text{if} \quad \mu \in [0, \mu^*), \\ C|x|^{-(\frac{\nu+2}{\nu+2})} \quad &\text{if} \quad \mu \in [\mu^*, \hat{\mu}), \end{cases} \]
for every $x$ satisfying $0 < |x| < \hat{\rho}$. This completes the theorem. \hfill $\square$

4. Holder continuity and Green function estimates

Lemma 4.1. Let $R > 0$ be given a small number. Then Green function defined in (4.21) satisfies
\[ \sup_{r/2 < |x-y| < r} G(x, y) \leq C \int_0^R \frac{t^2}{w(B_t(x))} \, dt \]
where $w(B_t(x)) = \int_{|x-y| < t} |y|^{-2\nu} \, dy$ with $N - 2\nu - 2 > 0$ and $r \in (0, \frac{R}{2})$ and $\text{dist}(x, \partial \Omega) > R$, $\text{dist}(y, \partial \Omega) > R$. In fact, we have
\[ G(x, y) \leq \frac{Ct^2}{w(B_t(x))}. \]

Proof. This is a modification of the theorem by Chanillo-Wheeden [6, pg. 311]. Note that the term in (4.11) contribute when $t$ is close to $r$. Define $f(y) = |y|^{-2\nu}$ As $f$ is a Muckenhoupt weight it satisfies the doubling property:
\[ \int_{|x-y| < t} f(y) \, dy \leq \int_{|x-y| < r} f(y) \, dy \leq C \int_{|x-y| < \frac{r}{2}} f(y) \, dy. \]
Using this fact we can cut the RHS dia-dically; we are left simply with the term near \( r \) and the above expression reduces to

\[
\sup_{r/2 < |x-y| < r} G(x, y) \leq C \sum_{k \geq 0} \int_{B_{k+1}(x)} \frac{t^2}{w(B_t(x))} \frac{dt}{t}.
\]

where \( 2^{m+1}r = R \).

Let

\[
I = \sum_{k \geq 0} \frac{2^{2k}r^2}{f(y)dy}.
\]

It follows from Lemma \ref{lem:4.2} (see Appendix B) that

\[
\int_{B_{2k}(x)} f(y)dy \geq C 2^{k(N-2\nu)} \int_{B_r(x)} f(y)dy
\]

where \( C > 0 \) is independent of \( x, k \) and \( r \). Therefore

\[
I \leq \frac{Cr^2}{f(y)dy} \sum_{k \geq 0} 2^{k(2+2\nu-N)} \leq \frac{Cr^2}{f(y)dy} = \frac{Cr^2}{w(B_r(x))}.
\]

Hence the lemma follows. \( \square \)

**Lemma 4.2.** The Green function satisfies the following estimate

\[
G(x, y) \leq C \left( \frac{|x|^{2\nu} + |x-y|^{2\nu}}{|x-y|^{N-2\nu}} \right)
\]

for any \( x \neq y \) and \( x, y \in \Omega \) for some \( C > 0 \) depending on \( \Omega \). Moreover, \( G(x, \cdot) \) is continuous whenever \( x \neq y \).

**Proof.** Since \( \frac{r}{2} < |x-y| < r \), we can write \( r = O(|x-y|) \). Then from (4.2), we have

\[
G(x, y) \leq \frac{C|x-y|^2}{\omega_N^{1/2}}
\]

Now we estimate the denominator \( D = w(B_r(x)) \) in the two cases.

Case 1: \( |x-y| \leq \frac{1}{4}|x| \).

In this case we note that

\[
D = \int_{|x-y| < r} |y|^{-2\nu} dy = \int_{|y| < r} |x-y|^{-2\nu} dy \geq \omega_N \left( \frac{1}{4}|x| \right)^{-2\nu} r^N \geq C|x|^{-2\nu}|x-y|^N,
\]

where \( \omega_N \) is volume of unit ball in \( \mathbb{R}^N \). Therefore \( G(x, y) \leq C \frac{|x|^{2\nu}}{|x-y|^{N-2\nu}} \).

Case 2: \( |x-y| > \frac{1}{4}|x| \).

In this case we can write \( |y| \leq |x-y| + |x| \leq 5|x-y| \). Therefore

\[
D = \int_{|x-y| < r} |y|^{-2\nu} dy \geq \omega_N (5|x-y|)^{-2\nu} r^N = C|x-y|^{-2\nu+N}.
\]

Thus \( G(x, y) \leq C \frac{|x|^{2\nu}}{|x-y|^{N-2\nu}} \). Combining case 1 and case 2, the lemma follows. \( \square \)
**Lemma 4.3.** Consider the problem
\begin{equation}
\begin{cases}
-\text{div}(|x|^{-2\nu} \nabla v) = |x|^{-(p+1)\nu} \nu^p - |x|^{-(q+1)\nu} \nu^q & \text{in } \Omega; \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where \( q < \frac{2 + \nu}{\nu} \). Then \( v \) is Hölder continuous at the origin with Hölder exponent \( \theta = 2 + 2\nu - (q+1)\nu \).

**Proof.** In order to prove this result we use the information on the Green function. We know that \( v \) is bounded at the origin. Also note that near the origin \( |x|^{-(q+1)\nu} \nu^q \) is the dominating term. Then
\begin{equation}
v(x) = \int_\Omega G(x, z)[|z|^{-(p+1)\nu} \nu^p(z) - |z|^{-(q+1)\nu} \nu^q(z)]dz.
\end{equation}
So we have
\begin{align*}
v(x) - v(y) &= \int_\Omega [G(x, z) - G(y, z)][|z|^{-(p+1)\nu} \nu^p(z) - |z|^{-(q+1)\nu} \nu^q(z)]dz.
\end{align*}
By the self adjoint-ness of the Green function, for fixed \( z \) we consider
\begin{align*}
G(x, z) - G(y, z) &= G_z(x) - G_z(y).
\end{align*}
Consider the ball of radius \( |x - y| = \rho \) centered at \( x \). We take \( x = 0 \), then using the fact that \( v \) is bounded and \( |y| = \rho \), we obtain
\begin{align*}
|v(y) - v(0)| &\leq C(A + B),
\end{align*}
where
\begin{align*}
A &= \int_{|z| \leq \rho} |G_z(0) - G_z(y)||z|^{-(q+1)\nu} dz \\
B &= \int_{|z| > \rho} |G_z(0) - G_z(y)||z|^{-(q+1)\nu} dz.
\end{align*}

**Case 1** If \( z \) lies outside the ball of radius \( |y| = \rho \). Then we can think of \( z \) as the pole and we are far away. That is, we can consider a ball \( B \) of radius \( \frac{\rho}{2} \) centered at 0 such that double the ball does not meet \( z \). Then we can use the Moser–Harnack inequality as in Chanillo–Wheeden \( \text{[7]} \) to obtain
\begin{align*}
|G_z(0) - G_z(y)| &\leq C \left( \frac{|y|}{|z|} \right)^\delta \left( \frac{|z|^2}{|y|^2|z|^{2\nu - 2}} \right) = C \left( \frac{|y|}{|z|} \right)^\delta \left( \frac{|z|^2}{|y|^{N-2\nu}} \right),
\end{align*}
where \( \delta > 0 \). This is a bound from the Harnack inequality as the Green’s function is non-negative. Therefore,
\begin{align*}
B &\leq |y|^{\delta - N + 2\nu} \int_{|z| \geq \rho} \frac{|z|^2}{|y|^2|z|^{(q+1)\nu} + 3} dz = O(|y|^\theta).
\end{align*}

**Case 2** In this case, \( z \) lies in the ball of radius \( |y| = \rho \). Here we use the crude bound on the Green function from Lemma 4.2
\begin{align*}
|G_z(0) - G_z(y)| &\leq |G_z(0)| + |G_z(y)| \leq \frac{C}{|z|^{N-2\nu - 2}} + |G_z(0)|.
\end{align*}
Hence we have
\begin{align*}
C \int_{|z| \leq \rho} \frac{|z|^{-(q+1)\nu} dz}{|z|^{N-2\nu - 2}} = O(|y|^{2+2\nu-(q+1)\nu}) = O(|y|^\theta).
\end{align*}
Hence from (4.10), we obtain
\[
|G_z(y)| \leq C \left( \frac{|z - y|^2}{|B_{|z-y|}(y)|} \right)^{-2 \nu}.
\]
As a result we have
\[
|G_z(y)| = O(|y|^{2+2\nu-N}).
\]
Hence we obtain
\[
\int_{|z| \leq |y|/2} |G_z(y)||z|^{-(q+1)\nu}|v|^q dz \leq c|y|^\theta.
\]
We are now left to check what happens in the region $|y|/2 \leq |z| \leq |y| = \rho$. So in this region we have
\[
(4.10) \quad \int_{|y|/2 \leq |z| \leq |y|} |G_z(y)||z|^{-(q+1)\nu}|v|^q dz \leq C|y|^{-(q+1)\nu} \int_{|y|/2 \leq |z| \leq |y|} |G_z(y)| dz.
\]
Now suppose that $|z - y| \leq |y|/2$. This implies if $\xi \in B_{|z-y|}(y)$, then we have $|\xi - y| \leq |z - y| \leq |y|/2$. Consequently, $|\xi| - |y| \leq |y|/2$ which in fact implies that $|\xi| \leq 3|y|/2$. Therefore
\[
\int_{B_{|z-y|}(y)} |\xi|^{-2\nu} d\xi = O(|y|^{-2\nu}|z - y|^N).
\]
Using the above expression, we obtain
\[
|G_z(y)| \leq C \left( \frac{|z - y|^2}{|B_{|z-y|}(y)|} \right)^{-2 \nu} = O(|y|^{2\nu}|z - y|^{-N}).
\]
When $|z - y| \geq |y|/2$, then since $|z| \leq |y|$ we have
\[
|G_z(y)| = O(|y|^{2+2\nu-N}).
\]
Hence from (4.11), we obtain
\[
\int_{|y|/2 \leq |z| \leq |y|} |G_z(y)||z|^{-(q+1)\nu}|v|^q dz \leq |y|^{-(q+1)\nu} \int_{|y|/2 \leq |z| \leq |y|} |G_z(y)| dz
\]
\[
+ |y|^{-(q+1)\nu} \int_{|y|/2 \leq |z| \leq |y|} |G_z(y)| dz
\]
\[
\leq C|y|^{2\nu-(q+1)\nu} \int_{|y|/2 \leq |z| \leq |y|} \frac{1}{|z - y|^{N-2}} dz
\]
\[
+ C|y|^{2+2\nu-(q+1)\nu}
\]
\[
= O(|y|^\theta).
\]
\]
\]

Lemma 4.4. (Weighted Pohozaev Identity) Let $\Omega$ be a smooth bounded domain and $v \in C^1(\Omega \setminus \{0\}) \cap C^0(\Omega)$ be a positive solution of
\[
(4.11) \quad \begin{cases}
-\nabla(|x|^{-2\nu} \nabla v) + \varepsilon |x|^{-(q+1)\nu} v^q = |x|^{-(p+1)\nu} v^p & \text{in } \Omega,
\end{cases}
\]
\[
v \in H^1(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)}),
\]
with $2^* - 1 \leq p < q$, $\nu \in (0, \frac{N-2}{2})$. Then $v$ satisfies
\[
\frac{1}{2} \int_{\partial \Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS + \left(\frac{N - 2 - 2\nu}{2}\right) \int_{\partial \Omega} |x|^{-2\nu} \frac{\partial v}{\partial n} dS
\]
\[
- \frac{\varepsilon}{q + 1} \int_{\partial \Omega} |x|^{-(q+1)\nu} \langle x, n \rangle v^{q+1} dS + \left(\frac{N}{q + 1} - \frac{N - 2}{2}\right) \varepsilon \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx
\]
\[
= (\frac{N}{p + 1} - \frac{N - 2}{2}) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx
\]
\[
(4.12) \quad \frac{1}{p + 1} \int_{\partial \Omega} |x|^{-(p+1)\nu} \langle x, n \rangle v^{p+1} dS.
\]

Moreover, if $v = 0$ on $\partial \Omega$, then
\[
\frac{1}{2} \int_{\partial \Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS = \left(\frac{N}{p + 1} - \frac{N - 2}{2}\right) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx
\]
\[
+ \varepsilon \left(\frac{N - 2}{2} - \frac{N}{q + 1}\right) \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx.
\]

(4.13)

**Proof.** This follows from Proposition 2.1. Note that, $v$ is a solution of (4.11) implies $u(x) = |x|^{-\nu} v(x)$ is a solution of (2.1). Therefore substituting $u(x) = |x|^{-\nu} v(x)$ in (2.2) we obtain,
\[
\frac{1}{2} \int_{\partial \Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla v|^2 dS + \left(\frac{N - 2 - 2\nu}{2}\right) \int_{\partial \Omega} |x|^{-2\nu} \frac{\partial v}{\partial n} dS
\]
\[
+ \frac{\nu^2 - \nu(N - 2) + \mu}{2} \int_{\partial \Omega} |x|^{-2\nu - 2\nu^2} \langle x, n \rangle dS
\]
\[
= \frac{\varepsilon}{q + 1} \int_{\partial \Omega} |x|^{-(q+1)\nu} \langle x, n \rangle v^{q+1} dS - \varepsilon \left(\frac{N}{q + 1} - \frac{N - 2}{2}\right) \int_{\Omega} |x|^{-(q+1)\nu} v^{q+1} dx
\]
\[
+ \left(\frac{N}{p + 1} - \frac{N - 2}{2}\right) \int_{\Omega} |x|^{-(p+1)\nu} v^{p+1} dx - \frac{1}{p + 1} \int_{\partial \Omega} |x|^{-(p+1)\nu} \langle x, n \rangle v^{p+1} dS.
\]

Since $\nu^2 - \nu(N - 2) + \mu = 0$, the above expression reduces to (4.12). Consequently $v = 0$ on $\partial \Omega$ implies (4.13). \hfill \Box

**Lemma 4.5.** Let $\nu \in (0, \frac{N-2}{2})$. Then the Green function $G(x,0)$ satisfies
\[
\int_{\partial \Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x,0)|^2 dS = (N - 2 - 2\nu)|R(0)|.
\]

(4.14)

**Proof.** We apply Pohozaev identity (4.12) to (1.24) on $\Omega \setminus B_r(0)$, for $r$ sufficiently small. Then we have
\[
\int_{\partial \Omega} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x,0)|^2 dS = \int_{\partial B_r} |x|^{-2\nu} \langle x, n \rangle |\nabla G(x,0)|^2 dS
\]
\[
+ (N - 2 - 2\nu) \int_{\partial B_r} |x|^{-2\nu} G(x,0) \frac{\partial G(x,0)}{\partial n} dS
\]
\[
= r \int_{\partial B_r} |x|^{-2\nu} |\nabla G(x,0)|^2 dS
\]
\[
+ (N - 2 - 2\nu) \int_{\partial B_r} |x|^{-2\nu} G(x,0) \frac{\partial G(x,0)}{\partial n} dS
\]
\[
(4.15)
\]
Moreover, from (1.23) and (1.26) we have,
\begin{equation}
G(x, 0) = H(x, 0) + \frac{1}{(N - 2\nu - 2)\omega_N|x - y|^N - 2\nu - 2}
\end{equation}
and hence
\begin{equation}
\nabla G(x, 0) = -\frac{1}{\omega_N)|x|^{(2\nu - N)}x + \nabla H(x, 0).
\end{equation}
Substituting \(G(x, 0)\) and \(\nabla G(x, 0)\) in (4.15), we take the limit \(r \to 0\). After simplifying the terms, we obtain
\begin{equation}
\lim_{r \to 0} \text{RHS of (4.15)} = \lim_{r \to 0} r^{-2\nu - 1}\int_{\partial B_r} |x|^{2\nu - N}x \cdot \nabla H(x, 0)\,dS
- \frac{1}{\omega_N} r^{-N + 1}\int_{\partial B_r} H(x, 0)\,dS
+ \frac{(N - 2\nu)r^{-2\nu - 1}}{\omega_N} \int_{\partial B_r} H(x, 0)|x|^{2\nu - N}x \cdot \nabla H(x, 0)\,dS
\end{equation}
Note that, as \(H(x, 0)\) is Holder continuous at origin [6], it follows \(|x \cdot \nabla H(x, 0)| \to 0\) on \(\partial B_r\) as \(r \to 0\). Therefore a straightforward computation yields
\begin{equation}
\text{RHS of (4.15)} = -\frac{(N - 2\nu)r^{-2\nu - 1}}{\omega_N r^{N - 1}} \int_{\partial B_r} H(x, 0)\,dS.
\end{equation}
Using the mean value theorem,
\begin{equation}
R(0) = H(0, 0) = \frac{1}{\omega_N r^{N - 1}} \int_{\partial B_r} H(x, 0)\,dS.
\end{equation}
Hence the lemma follows.

5. Symmetry and decay properties of entire problem

In this section using moving plane method, we give the proof of Theorem 1.9

**Proof of Theorem 1.9** It is enough to show that \(u\) is symmetric with respect to each coordinate axis. For \(\alpha > 0\), we define
\[
\Omega_\alpha = \{x \in \mathbb{R}^N : x_1 > \alpha\},
\]
and for \(x \in \Omega_\alpha\), let \(x_\alpha\) denote the its reflection to the hyperplane \(x_1 = \alpha\), that is \(x_\alpha = (2\alpha - x_1, x_2, \cdots, x_n)\). Set
\[
u_\alpha(x) := u(x_\alpha), \quad x \in \Omega_\alpha \quad \text{and} \quad w_\alpha = u_\alpha - u.
\]
We note that \(w_\alpha\) is smooth away from the point \((2\alpha, 0, \cdots, 0)\) and \(w_\alpha = 0\) on \(\partial \Omega_\alpha\). It is easy to check that \(w_\alpha \in D^{1,2}(\Omega_\alpha)\).

**Claim 1:** \(w_\alpha \geq 0\) in \(\Omega_\alpha\), if \(\alpha > 0\) is large enough.

To see the claim, we note that \(|x_\alpha| < |x|\) if \(\alpha > 0\). By a straightforward computation it follows that \(w_\alpha\) satisfies the following equation
\begin{equation}
-\Delta w_\alpha - \mu \frac{w_\alpha}{|x|^2} \geq A_1(x)w_\alpha - A_2(x)w_\alpha \quad \text{in} \quad \Omega_\alpha,
\end{equation}
where
\[
0 \leq A_1(x) := \lambda \frac{u_\alpha^p - u^p}{u_\alpha - u} \leq \lambda p \left[\max\{u_\alpha(x), u(x)\}\right]^{p-1}
\]
and

\[ 0 \leq A_2(x) := \frac{u_\alpha^q - u^q}{u_\alpha - u} \leq q \max \{ u_\alpha(x), u(x) \}^{q-1}. \]

Multiplying (5.1) by \( w_\alpha^- \) and integrating by parts over \( \Omega_\alpha \), we obtain

\[
\int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 dx - \mu \frac{|w_\alpha^-|^2}{|x|^2} dx \leq \int_{\Omega_\alpha} (A_1(x) - A_2(x)) |w_\alpha^-|^2 dx
\]

\[
\leq \int_{\Omega_\alpha} A_1(x) |w_\alpha^-|^2 dx
\]

\[
\leq \left( \int_{\Omega_\alpha} |w_\alpha^-|^2 dx \right)^{\frac{N-2}{N}} \left( \int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} A_1^\frac{N}{2} dx \right)^{\frac{2}{N}}.
\]

(5.2)

As \( \mu < (\frac{N-2}{2})^2 \), it is not difficult to check that \( \left( \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 - \mu \frac{|w_\alpha^-|^2}{|x|^2} \right)^{\frac{1}{2}} \) is an equivalent norm to \( D^{1,2} (\mathbb{R}^N) \). Therefore there exists a positive constant \( C_1 \) such that

\[
C_1 \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 \leq \int_{\Omega_\alpha} |\nabla w_\alpha^-|^2 - \mu \frac{|w_\alpha^-|^2}{|x|^2}.
\]

Applying this estimate along with Sobolev inequality, we have from (5.2):

\[
C_1 S \left( \int_{\Omega_\alpha} |w_\alpha^-|^2 dx \right)^{\frac{2}{N}} \leq \left( \int_{\Omega_\alpha} |w_\alpha^-|^2 dx \right)^{\frac{2}{N}} \left( \int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} A_1^\frac{N}{2} dx \right)^{\frac{2}{N}},
\]

(5.3) \( C_1 S \left( \int_{\Omega_\alpha} |w_\alpha^-|^2 dx \right)^{\frac{2}{N}} \leq \left( \int_{\Omega_\alpha} |w_\alpha^-|^2 dx \right)^{\frac{2}{N}} \left( \int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} A_1^\frac{N}{2} dx \right)^{\frac{2}{N}},
\]

where \( S \) is the Sobolev constant. On the other hand, \( u_\alpha < u \) on \( \{ w_\alpha < 0 \} \) implies

\[
\int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} A_1^\frac{N}{2} dx \leq C \int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} u^{(p-1) \frac{N}{2}}.
\]

We know that \( u \in L^{2^*} (\mathbb{R}^N) \cap L^{q+1} (\mathbb{R}^N) \). As by the given assumption

\[
q > (p - 1) \frac{N}{2} - 1 \quad \text{and} \quad p \geq 2^* - 1,
\]

using interpolation theory we can show that \( u \in L^{(p-1) \frac{N}{2}} \). Consequently

\[
\int_{\Omega_\alpha \cap \{ w_\alpha < 0 \}} u^{(p-1) \frac{N}{2}} \rightarrow 0 \quad \text{if} \quad \alpha \quad \text{is large enough.}
\]

Hence from (5.3) we conclude \( w_\alpha^- = 0 \) in \( \Omega_\alpha \) if \( \alpha \) is large enough. This proves the claim.

Let \( \alpha_0 = \inf \{ \alpha > 0 : u_{\alpha'} \geq u \in \Omega_{\alpha'} \forall \alpha' > \alpha \} \).

Claim 2: \( \alpha_0 = 0 \).

We will prove this claim by method of contradiction. Let \( \alpha_0 > 0 \). Define \( w_{\alpha_0} = u_{\alpha_0} - u \). Then \( w_{\alpha_0} \geq 0 \) in \( \Omega_{\alpha_0} \) and

\[-\Delta w_{\alpha_0} + A_2(x) w_{\alpha_0} = \mu \frac{|w_{\alpha_0}|^{q-1}}{|x|^2} + A_1(x) w_{\alpha_0} \geq 0 \text{ in } \Omega_{\alpha_0} \text{ and away from the point } (2\alpha_0, 0, \ldots, 0). \]

As \( A_2 \geq 0 \), by maximum principle we have \( w_{\alpha_0} > 0 \) in this region.

Let \( \epsilon > 0 \). We choose \( R > 0 \) and \( \delta_0 > 0 \) such that

\[
\int_{|x| > R} u^{(p-1) \frac{N}{2}} dx < \frac{\epsilon}{2}.
\]

(5.4)
By the choice of $\alpha$ and $\hat{\nu} < \nu < \frac{2}{N}$, $K$ is a compact set and $w_{\alpha} > 0$ in $K$. Choose $\delta_1 \in (0, \delta_0)$ such that $w_{\alpha} - \delta > 0$ in $K$ for all $\delta \in (0, \delta_1)$. Define $\alpha_1 := \alpha_0 - \delta$. Next we will show that $w_{\alpha_1} \geq u$ in $\Omega_{\alpha_1}$ and this will contradict the definition of $\alpha_0$. Towards this goal, we define $w_{\alpha_1} := u_{\alpha_1} - u$. We proceed as in the case of (5.3) to get

\[
C_1 \mathcal{S} \left( \int_{\Omega_1} |w_{\alpha_1}^-|^{2^*} \, dx \right)^{\frac{\nu}{2}} \leq \left( \int_{\Omega_1} |w_{\alpha_1}^-|^{2^*} \, dx \right)^{\frac{\nu}{2}} \left( \int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} A_1^{\frac{\nu}{2}} \, dx \right)^{\frac{\nu}{2}}.
\]

By the choice of $\alpha_1$, we have $w_{\alpha_1} > 0$ in $K$ and thus by (5.4) and (5.5), we conclude that

\[
\int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} A_1^{\frac{\nu}{2}} \, dx \leq \lambda_p \int_{\Omega_{\alpha_1} \cap \{w_{\alpha_1} < 0\}} u^{(p-1)\frac{\nu}{2}} \, dx < \epsilon.
\]

As $\epsilon > 0$ is arbitrarily chosen, we can conclude that $w_{\alpha_1}^+ = 0$, which contradicts the definition of $\alpha_0$. Hence the claim follows.

Consequently we have

\[
u(-x_1, x_2, \cdots, x_n) \geq u(x_1, x_2, \cdots, x_n) \quad \forall \ x_1 > 0.
\]

Now repeating the same arguments for $\bar{u}(x) = u(-x_1, x_2, \cdots, x_n)$, we can prove that

\[
u(-x_1, x_2, \cdots, x_n) \leq u(x_1, x_2, \cdots, x_n) \quad \forall \ x_1 > 0.
\]

Hence

\[
u(-x_1, x_2, \cdots, x_n) = u(x_1, x_2, \cdots, x_n) \quad \forall \ x_1 > 0.
\]

As a result symmetry follows since we can do the moving plane argument in any direction instead of $x_1$ direction.

**Remark 5.1.** Doing some simple modifications to the proof of Theorem 1.2, it can be shown that $u$ is a radially symmetric solution of (1.1), when $\Omega = B_R(0)$, for any $R > 0$. Therefore $v$ is a radially symmetric solution of (1.2), when $\Omega = B_R(0)$, for any $R > 0$.

From Theorem 1.2, we know $K$ is achieved by a radial function $v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu})$. Furthermore, there exists a constant $\lambda > 0$ such that $v$ satisfies the following problem:

\[
\left\{
\begin{array}{ll}
-\text{div}(|x|^{-2\nu} \nabla v) = \lambda |x|^{-(p+1)\nu} v^p - |x|^{-(q+1)\nu} v^q & \text{in } \mathbb{R}^N, \\
v > 0 & \text{in } \mathbb{R}^N, \\
v \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}).
\end{array}
\right.
\]

**Lemma 5.1.** Define $\alpha = N - 2 - 2\nu$. Let $2^* - 1 < p$ and $q > (p-1)\frac{\nu}{2} - 1$, $0 < \nu < \frac{\alpha}{2} - \frac{\nu}{2}$. Suppose $v$ is a solution of (5.6). Then $v(x) = v(|x|) = v(r)$, where $|x| = r$. Moreover,

\[
\int_0^\infty (v(r))^2 r^{N-1-2\nu} \, dr < +\infty.
\]

Furthermore, if $v(r) \leq Cr^{-\alpha}$ for $r > 1$ then $v \sim r^{-\alpha}$ as $r \to \infty$. 

Proof: If \( v \) is any solution of (5.6), then \( u = |x|^{-\beta} v \) is a solution of (1.35). By Theorem 1.9 \( u \) is radially symmetric. Therefore \( v(x) = v(|x|) = v(r) \) and \( v \) satisfies the following ode

\[
\begin{align*}
\frac{\nu r}{r} + \frac{N - 2\nu - 1}{r} v_r + \lambda r^{-(p-1)\nu} v^p - r^{-(q-1)\nu} v^q &= 0 \quad \text{in } (0, \infty), \\
v(r) &> 0 \quad \text{in } (0, \infty), \\
\int_0^\infty (v'(r))^2 r^{N-2\nu} dr < +\infty.
\end{align*}
\tag{5.7}
\]

Applying Sobolev embedding theorem it follows \( \int_0^\infty (v(r))^2 r^{N-2\nu} dr < +\infty. \)

To prove the last assertion, we first show that \( v(r) \geq C r^{-\alpha} \) for some \( C > 0 \) and \( r \gg 1. \)

To see this, let \( w = |x|^{-\alpha}. \) Then \( \text{div}(x|^{-2\nu} \nabla w) = 0 \) in \( \mathbb{R}^N \setminus B_R(0). \) Hence by standard method using comparison principle it follows that

\[
v \geq C w = C |x|^{-\alpha} \quad \text{in } \mathbb{R}^N \setminus B_R(0).
\]

Hence there exist \( C_1, C_2 > 0 \) such that

\[
C_1 \leq v(r) r^{\alpha} \leq C_2 \quad \text{for } r \gg 1.
\]

But from (5.7) we have

\[
- (r^{N-2\nu} v_r)_r = v^{p} r^{N-2(p+1)\nu (1 + o(1))} \quad \text{for } r \gg 1.
\]

Since \( v \) satisfies (5.8) and \( p > 2^* - 1, \) the RHS of the above expression is integrable in \((s, \infty)\) and positive. This implies that

\[
\lim_{r \to \infty} v_r (r^{N-2\nu}) = -c.
\]

for some \( c > 0. \) This in fact implies that \( v_r \sim -r^{-(N-2\nu)}. \) Integrating this expression from \((s, \infty)\) we obtain,

\[
\lim_{r \to \infty} r^{N-2\nu} v = a \in (0, +\infty).
\]

6. Proof of Theorem 1.10

6.1. Auxiliary results. Define

\[
\hat{F}(w) = \frac{1}{2} \int |x|^{-2p} |\nabla w|^2 dx + \frac{1}{q+1} \int |x|^{-(q+1)\nu} w^{q+1} dx,
\]

where \( \nu \in (0, \frac{N-2}{2}), \) \( q > p \geq 2^* - 1. \) For \( \rho > 0, \) set

\[
N_\rho = \left\{ w \in H^1_0(\rho \Omega, |x|^{-2\nu}) \cap L^{q+1}(\rho \Omega, |x|^{-(q+1)\nu}) : \int_{\rho \Omega} |x|^{-(p+1)\nu} w^{p+1} dx = 1 \right\}.
\]

Define

\[
S_\rho := \inf_{w \in N_\rho} \hat{F}(w).
\]

Theorem 6.1. Let \( p = 2^* - 1. \) Then \( S_\rho \to \frac{\rho}{2} \) as \( \rho \to \infty, \) where \( S \) is as defined in (1.19).
Proof. **Step 1**: \( \lim_{\rho \to \infty} S_\rho \leq \frac{S}{2} \)

To see this, let \( U(x) \) be as in \((1.20)\). We know from \([5]\) that, \( U \) is an extremal of \( S \), with \( \int_{\mathbb{R}^N} |x|^{2^*} U^{2^*}(x) dx = 1 \) and \( U \) is a ground state solution of \((1.22)\). It is easy to check that \( \mu^\frac{2}{2^*} U(\frac{x}{\mu}) \) is also a solution of \((1.22)\), for any \( \mu > 0 \), where \( \alpha = N - 2 - 2\nu \).

Set \( \rho := \mu^2 \). Define

\[
U_\rho(x) := \mu^{-\frac{\alpha}{2^*}} U(\frac{x}{\mu}) \quad \text{and} \quad \phi_\rho(x) = \phi(\frac{x}{\rho}),
\]

where \( \phi \in C^\infty_c(\mathbb{R}^N) \) such that \( \text{supp} \phi \subset \Omega, \phi = 1 \) in \( \Omega \), \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq \frac{2}{\rho} \) and \( d = \text{diam}(\Omega) \). We set

\[
v_\rho(x) := U_\rho(x) \phi_\rho(x) \quad \text{and} \quad \nabla v_\rho := \frac{v_\rho}{|x|^{-\nu} v_\rho|_{L^2(\rho \Omega)}}.
\]

Then \( \nabla v_\rho \in N_\rho \).

\[
\lim_{\rho \to \infty} \int_{\rho \Omega} |x|^{-2^*} v_\rho^2 dx = \lim_{\rho \to \infty} \mu^{-\frac{\alpha}{2^*}} \int_{\mathbb{R}^N} |x|^{-2^*} U^{2^*}(\frac{x}{\mu}) \phi_\rho^2(\frac{x}{\rho}) dx
\]

\[
= \lim_{\rho \to \infty} \rho^{-\frac{\alpha}{2^*} + N - 2^*} \int_{\mathbb{R}^N} |x|^{-2^*} U^{2^*}(x) \phi_\rho^2(\frac{x}{\sqrt{\rho}}) dx
\]

\[
= \lim_{\rho \to \infty} \int_{\mathbb{R}^N} |x|^{-2^*} U^{2^*}(x) \phi_\rho^2(\frac{x}{\sqrt{\rho}}) dx
\]

\[
(6.2)
\]

Similarly we see that

\[
\int_{\rho \Omega} |x|^{-(q+1)\nu} \nabla v_\rho^{q+1} dx = \frac{\rho^{-\frac{\alpha}{2^*} + N - (q+1)\nu}}{|x|^{-\nu} v_\rho|_{L^2(\rho \Omega)}} \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) \phi^{q+1}(\frac{x}{\sqrt{\rho}}) dx.
\]

As before

\[
\lim_{\rho \to \infty} \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) \phi^{q+1}(\frac{x}{\sqrt{\rho}}) dx = \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} U^{q+1}(x) dx.
\]

Moreover \( q > 2^* - 1 \) implies \( -\frac{\alpha}{2}(q+1) + N - (q+1)\nu < 0 \). Hence by \((6.2)\), we have

\[
\lim_{\rho \to \infty} \int_{\rho \Omega} |x|^{-(q+1)\nu} \nabla v_\rho^{q+1} dx = 0.
\]

\[
(6.3)
\]

\[
\int_{\rho \Omega} |x|^{-2^*} |\nabla v_\rho|^2 dx = \int_{\rho \Omega} |x|^{-2^*} |\nabla v_\rho|^2 dx = I_\rho^1 + I_\rho^2 + I_\rho^3,
\]

where

\[
I_\rho^1 = \mu^{-(\alpha+2)} \int_{\mathbb{R}^N} |x|^{-2^*} |\nabla U(\frac{x}{\mu})|^2 \phi^2(\frac{x}{\rho});
\]

\[
I_\rho^2 = \mu^{-(\alpha+4)} \int_{\rho \Omega \setminus \rho \frac{1}{4}} |x|^{-2^*} U^2(\frac{x}{\mu}) |\nabla \phi(\frac{x}{\rho})|^2 dx;
\]

\[
I_\rho^3 = 2 \mu^{-(\alpha+3)} \int_{\rho \Omega \setminus \rho \frac{1}{4}} |x|^{-2^*} U(\frac{x}{\mu}) \phi(\frac{x}{\rho}) |\nabla U(\frac{x}{\mu}) \nabla \phi(\frac{x}{\rho})| dx.
\]
By straightforward computation we see that

\[ \lim_{\rho \to \infty} I^1_{\rho} = \lim_{\rho \to \infty} \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 \phi^2(\frac{x}{\rho}) dx = \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 dx = S. \]

\[ \lim_{\rho \to \infty} I^2_{\rho} \leq \lim_{\rho \to \infty} 4 \mu^{-\alpha+1} \int_{\mathbb{R} \setminus \mathbb{R}^N} |x|^{-2\nu} U^2(\frac{x}{\rho}) dx \]

\[ = \lim_{\rho \to \infty} 4 \mu^{-\alpha+1+2\nu-N} \int_{\mathbb{R} \setminus \mathbb{R}^N} |x|^{-2\nu} U^2(x) dx \]

\[ \leq \lim_{\rho \to \infty} 4 \mu^{-\alpha-2\nu+N} \left( \int_{\mathbb{R}^N} |x|^{-2\nu} U^2(x) dx \right)^{\frac{2}{\nu}} \left( \int_{\mathbb{R}^N} |x|^{-2\nu} U^2(x) dx \right)^{\frac{1}{\nu}} \]

\[ \leq \lim_{\rho \to \infty} C \mu^{-\alpha-2\nu+N+1} = \lim_{\rho \to \infty} C \mu^{-1} = 0. \]

Similarly,

\[ \lim_{\rho \to \infty} I^3_{\rho} \leq \lim_{\rho \to \infty} \mu^{-\alpha+3} \left( \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 \phi^2(\frac{x}{\rho}) dx \right)^{\frac{1}{2}} \times \left( \int_{\mathbb{R}^N} |x|^{-2\nu} |\nabla U(x)|^2 \phi^2(\frac{x}{\rho}) dx \right)^{\frac{1}{2}}. \]

Consequently,

\[ \lim_{\rho \to \infty} I^3_{\rho} \leq \lim_{\rho \to \infty} C \mu^{-\frac{\alpha}{2}} = 0. \]

Combining (6.6), (6.3), (6.5), (6.7) and (6.2) we obtain

\[ S_{\rho} \leq F(v) \quad \text{and} \quad F(v) \to S \quad \text{as} \quad \rho \to \infty. \]

Hence

\[ \lim_{\rho \to \infty} S_{\rho} \leq \frac{S}{2}. \]

**Step 2:** \( \frac{S}{2} \leq \lim_{\rho \to \infty} S_{\rho} \).

This is standard to prove. Therefore we just give here a sketch of the proof. Let \( \varepsilon > 0 \). Then there exists \( u_{\rho, \varepsilon} \in N_\rho \) such that

\[ \tilde{F}(u_{\rho, \varepsilon}) < S_{\rho} + \varepsilon. \]

Extend \( u_{\rho, \varepsilon} \) by 0 outside \( \rho \Omega \) and we denote it by \( u_{\rho, \varepsilon} \) too. Let \( \eta(x) = C \exp(-\frac{1}{|x|^2}) \) if \( |x| < 1 \) and 0 otherwise. Set \( \eta_{\delta}(x) = \delta^{-N} \eta(\frac{x}{\delta}). \)

Define \( u_{\rho, \varepsilon}^\delta := u_{\rho, \varepsilon} * \eta_{\delta} \) and \( v_{\rho, \varepsilon}^\delta = \frac{u_{\rho, \varepsilon}^\delta}{\|u_{\rho, \varepsilon}^\delta\|_{L^2(\mathbb{R}^N)}}. \) Thus \( v_{\rho, \varepsilon}^\delta \in C_0(\mathbb{R}^N) \cap N \), where

\[ N := \left\{ w \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu} dx) : w \in L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu} dx), \int_{\mathbb{R}^N} |x|^{-2\nu} w^2 dx = 1 \right\}. \]

Moreover,

\[ v_{\rho, \varepsilon}^\delta \to u_{\rho, \varepsilon} \quad \text{in} \quad D^{1,2}(\mathbb{R}^N, |x|^{-2\nu} dx) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu} dx) \quad \text{as} \quad \delta \to 0. \]

Hence

\[ \frac{S}{2} \leq \tilde{F}(v_{\rho, \varepsilon}^\delta) \to \tilde{F}(u_{\rho, \varepsilon}) \quad \text{as} \quad \delta \to 0. \]
Combing this with (6.13), we conclude \( \frac{S}{2} < S_p + \varepsilon \). As \( \varepsilon > 0 \) is arbitrary, this proves Step 2.

Combining Step 1 and Step 2, theorem follows.

**Theorem 6.2.** Let \( p > 2^* - 1 \). Then \( S_p \to K \) as \( \rho \to \infty \), where \( K \) is as defined in (1.28).

**Proof.** Let \( w \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}dx) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}dx) \) be a minimizer of \( K \) (which exists by Theorem 1.2) with \( \int_{\mathbb{R}^N} |x|^{-(p+1)\nu}w^{q+1}dx = 1 \). Define \( \phi_\rho \) as in Step 1 of the proof of Theorem 6.1. Set \( w_\rho = w \phi_\rho \) and \( \hat{w}_\rho = \frac{w_\rho}{\|x|^{-\nu}w_\rho \|_{L^{p+1}(\mathbb{R}^N)}} \).

Then \( \hat{w}_\rho \in N_\rho \) and consequently \( S_p \leq F(\hat{w}_\rho) \). Proceeding the same way as in Step 1 of Theorem 6.1, we obtain \( F(\hat{w}_\rho) \to K \) as \( \rho \to \infty \). Hence \( \lim_{\rho \to \infty} S_p \leq K \). To get the other sided inequality we use the same idea as in Step 2 of Theorem 6.2. This completes the proof.

### 6.2. Asymptotic Behavior

For \( v \in H^1_0(\Omega, |x|^{-2\nu}) \cap L^{q+1}(\Omega, |x|^{-(q+1)\nu}) \), we recall the definition of the functional \( F(., \Omega) \) from (1.28) for \( p > 2^* - 1 \) and \( S(., \Omega) \) from (1.31) for \( p = 2^* - 1 \):

\[
F(v, \Omega) = \frac{1}{2} \int_\Omega |x|^{-2\nu} |\nabla v|^2 dx + \frac{1}{q+1} \int_\Omega |x|^{-(q+1)\nu} v^{q+1} dx
\]

where

\[
l = \frac{2(q+1) - N(p-1)}{2(p+1) - N(p-1)}, \quad q > p > 2^* - 1.
\]

\[
S(v) = \frac{\int_\Omega |x|^{-2\nu} |\nabla v|^2 dx}{\left( \int_\Omega |x|^{-(p+1)\nu} v^{p+1} dx \right)^{\frac{p}{p+1}}}, \quad p = 2^* - 1.
\]

Using the transform

\[
v(x) = \varepsilon^{-\frac{2\nu-2(q+1)\nu}{2(p+1)-N(p-1)}} w(\varepsilon^{-\frac{\nu}{2(p-1)}} x),
\]

Eq. (1.2) reduces to

\[
\begin{cases}
-\text{div}(|x|^{-2\nu} \nabla w) = |x|^{-(p+1)\nu} w^p - |x|^{-(q+1)\nu} w^q & \text{in } \Omega_\varepsilon, \\
w > 0 & \text{in } \Omega_\varepsilon, \\
w(x) = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]

where \( \Omega_\varepsilon = \frac{\Omega}{\varepsilon^{\frac{2(q+1)\nu}{2(p+1)-N(p-1)}}} \). Clearly \( \Omega_\varepsilon \mapsto \mathbb{R}^N \) as \( \varepsilon \to 0 \).

**Proposition 6.1.** Let \( 2^* - 1 < p < q \) and \( \nu \in (0, \frac{N-2}{2}) \). Then there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), the problem

\[
\begin{cases}
-\text{div}(|x|^{-2\nu} \nabla v) = \lambda \varepsilon |x|^{-(p+1)\nu} v^p - \varepsilon |x|^{-(q+1)\nu} v^q & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
admits a solution \( v_\varepsilon \), with the property that

\[
A < \lambda_\varepsilon < B,
\]

for some constants \( A, B > 0 \), independent of \( n \). In addition

(i) if \( p > 2^* - 1 \), then \( F(v_\varepsilon) \to K \) and \( \int_\Omega |x|^{-(p+1)\nu}v_\varepsilon^{p+1}dx \to 0 \) as \( \varepsilon \to 0 \);

(ii) if \( p = 2^* - 1 \), then \( S(v_\varepsilon) \to S \) as \( \varepsilon \to 0 \) and \( \int_\Omega |x|^{-(p+1)\nu}v_\varepsilon^{p+1}dx = 1 \),

where \( K \) and \( S \) are defined as in \((1.30) \) and \((1.19) \) respectively.

Proof. Let \( \Omega_\varepsilon = \frac{\Omega}{\varepsilon^{2\nu - p}} \). We are going to work on the manifold

\[
N_\varepsilon = \left\{ w \in H^1_0(\Omega_\varepsilon, |x|^{-2\nu}) \cap L^{q+1}(\Omega_\varepsilon, |x|^{-(q+1)\nu}) : \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu}w_\varepsilon^{p+1} = 1 \right\}.
\]

Then \( F \) on \( N_\varepsilon \) reduces to \( \tilde{F} \) (defined as in Subsection 6.1)

\[
F(w) = \frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu}|\nabla w|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu}w_\varepsilon^{q+1} dx = \tilde{F}(w).
\]

For every \( p \geq 2^* - 1 \), let

\[(6.13) \quad S_\varepsilon = \inf_{N_\varepsilon} \tilde{F}(w) = \inf_{N_\varepsilon} F(w).\]

Let \( \{w_{n,\varepsilon}\} \) be a minimising sequence in \( N_\varepsilon \) such that

\[
\tilde{F}(w_{n,\varepsilon}) \to S_\varepsilon \text{ with } \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu}w_{n,\varepsilon}^{p+1} dx = 1.
\]

Thus \( \{w_{n,\varepsilon}\} \) is bounded in \( H^1_0(\Omega_\varepsilon, |x|^{-2\nu}) \cap L^{q+1}(\Omega_\varepsilon, |x|^{-(q+1)\nu}) \). Hence \( w_{n,\varepsilon} \to w_\varepsilon \) in \( H^1_0(\Omega_\varepsilon, |x|^{-2\nu}) \) and \( w_{n,\varepsilon} \to w_\varepsilon \) in \( L^2(\Omega_\varepsilon, |x|^{-2\nu}) \). As a result, \( w_{n,\varepsilon} \to w_\varepsilon \) pointwise almost everywhere. By the interpolation inequality, we have \( w_{n,\varepsilon} \to w_\varepsilon \) on \( L^{q+1}(\Omega_\varepsilon, |x|^{-(p+1)\nu}) \). Consequently

\[
\int_{\Omega_\varepsilon} |x|^{-2\nu}w_\varepsilon^{p+1} dx = 1.
\]

Now we show that \( S_\varepsilon = \tilde{F}(w_\varepsilon) \). Clearly \( S_\varepsilon \leq \tilde{F}(w_\varepsilon) \). Furthermore, applying Fatou’s Lemma and the fact that \( w \mapsto ||w||^2_{H^1_0(\Omega_\varepsilon, |x|^{-2\nu} dx) \) is weakly lower semicontinuous, we have

\[
S_\varepsilon = \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu}|\nabla w_{n,\varepsilon}|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu}w_{n,\varepsilon}^{q+1} dx \right] \geq \frac{1}{2} \int_{\Omega_\varepsilon} |x|^{-2\nu}|\nabla w_\varepsilon|^2 dx + \frac{1}{q+1} \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu}w_\varepsilon^{q+1} dx \geq \tilde{F}(w_\varepsilon).
\]

Hence \( S_\varepsilon \) is achieved by \( w_\varepsilon \).

Using the Lagrange multiplier rule, we obtain \( w_\varepsilon \) satisfies

\[(6.14) \quad -\text{div}(|x|^{-2\nu}\nabla w_\varepsilon) = \lambda_\varepsilon |x|^{-(p+1)\nu}w_\varepsilon^p - |x|^{-(q+1)\nu}w_\varepsilon^q \text{ in } \Omega_\varepsilon,
\]

where \( \lambda_\varepsilon = \lambda(\varepsilon) \). Moreover,

\[
\int_{\Omega_\varepsilon} |x|^{-2\nu}|\nabla w_\varepsilon|^2 dx = \lambda_\varepsilon \int_{\Omega_\varepsilon} |x|^{-(p+1)\nu}w_\varepsilon^{p+1} dx - \int_{\Omega_\varepsilon} |x|^{-(q+1)\nu}w_\varepsilon^{q+1} dx,
\]
which implies that
\[ \lambda \varepsilon = \int_{\Omega} |x|^{-2\nu} |\nabla w_\varepsilon|^2 \, dx + \int_{\Omega} |x|^{-(q+1)\nu} w_\varepsilon^{q+1} \, dx. \]

This fact along with \( \hat{F}(w_\varepsilon) = S_\varepsilon \) implies
\[ 2S_\varepsilon < \lambda \varepsilon < (g+1)S_\varepsilon. \]

In Theorem 6.1 and 6.2, if we take \( \rho = \varepsilon^{\frac{2\nu}{2(q-\nu)}}, \) then \( N_\rho \) and \( S_\rho \) of those theorems reduces to \( N_\varepsilon \) and \( S_\varepsilon \) defined as above. Therefore taking the limit \( \varepsilon \to 0, \) it follows from Theorem 6.1 and 6.2 that
\[ (6.15) \quad S_\varepsilon \to K \quad \text{if } p > 2^* - 1 \quad \text{and} \quad S_\varepsilon \to \frac{S}{2} \quad \text{if } p = 2^* - 1. \]

Hence there exist constants \( \varepsilon_0 > 0 \) and \( A, B > 0 \) such that
\[ A < \lambda \varepsilon < B \quad \forall \quad \varepsilon \in (0, \varepsilon_0). \]

Using the transformation (6.10), we obtain from (6.14) that \( v_\varepsilon \) is a solution of (6.12).

Moreover
\[ \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} \, dx = 1 \quad \text{imply} \quad \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} \, dx = \varepsilon^{\frac{p(N-2) - (N+2)}{2(q-p)}}. \]

Hence
\[ \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} \, dx = 1 \quad \text{when} \quad p = 2^* - 1 \]

and
\[ \int_{\Omega} |x|^{-(p+1)\nu} v_\varepsilon^{p+1} \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{when} \quad p > 2^* - 1. \]

By a straight forward computation we see that
\[ F(w_\varepsilon) = \hat{F}(w_\varepsilon) = F(v_\varepsilon), \quad \text{when} \quad p > 2^* - 1 \]
where \( F \) and \( \hat{F} \) are as in (1.28) and (6.1) respectively. This along with (6.15) and the fact that \( F(w_\varepsilon) = S_\varepsilon \) implies
\[ F(v_\varepsilon) \to K \quad \text{if} \quad p > 2^* - 1 \]

Moreover when \( p = 2^* - 1, \)
\[ S \leq S(v_\varepsilon) \leq 2\hat{F}(v_\varepsilon, \Omega) = 2\hat{F}(w_\varepsilon, \Omega) = 2S_\varepsilon \to S. \]

Hence
\[ S(v_\varepsilon) \to S \quad \text{if} \quad p = 2^* - 1. \]

This completes the proof. \( \square \)

**Proof of Theorem 1.10** Let \( v_\varepsilon \) and \( \lambda_\varepsilon \) be as in Proposition 6.1. Setting \( u_\varepsilon = \lambda_\varepsilon^{\frac{1}{q-\nu}} v_\varepsilon, \) we find \( u_\varepsilon \) satisfies
\[ -\text{div}(|x|^{-2\nu} \nabla u_\varepsilon) = |x|^{-(p+1)\nu} u_\varepsilon^p - \varepsilon^{\frac{2\nu}{2(q-\nu)}} |x|^{-(q+1)\nu} u_\varepsilon^q \quad \text{in} \quad \Omega. \]

Using the bounds on \( \lambda_\varepsilon \) from Proposition 6.1, we can conclude that there exist solutions \( u_n \) of Problem (1.2), along a sequence \( \{\varepsilon_n\} \) of values of \( \varepsilon \) which tends to zero as \( n \) tends to infinity. By setting \( \lambda_n := \lambda_\varepsilon^{\frac{1}{q-\nu}}, \) theorem follows from Proposition 6.1. \( \square \)
7. The case $p = 2^* - 1$ and proof of Theorem 1.11

Lemma 7.1. Let $v_\varepsilon$ be as in Theorem 1.11. Then $\|v_\varepsilon\|_\infty \to +\infty$ as $\varepsilon \to 0$.

Proof. We have

$$\left(7.1\right) \int_\Omega |x|^{-2^* \nu} v_\varepsilon^{2^* - 1} dx = c,$$

where $c \in (A, B)$. If possible, let $\|v_\varepsilon\|_\infty$ be uniformly bounded. Hence by the Schauder estimate $v_\varepsilon \to v$ in $C^2_{\text{loc}}(\Omega \setminus \{0\})$, where $v$ satisfies

$$\left(7.2\right) \begin{cases} -\nabla(|x|^{-2^* \nu} \nabla v) = |x|^{-2^* \nu} v^{2^* - 1} & \text{in } \Omega, \\ v \not\equiv 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial \Omega. \end{cases}$$

Moreover, by the dominated convergence theorem we have

$$\left(7.3\right) A < \int_\Omega |x|^{-2^* \nu} v^{2^*} dx < B.$$

As $A > 0$, the above expression implies $v$ is nontrivial in a star-shaped domain which is a contradiction. \qed

Define

$$\left(7.4\right) \gamma_\varepsilon := \|v_\varepsilon\|^{-\frac{1}{2^*}}_\infty.$$

Therefore $\|v_\varepsilon\|_\infty = \gamma_\varepsilon^{-\frac{1}{2^*}}$ and $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. Define

$$\left(7.5\right) z_\varepsilon(x) = \gamma_\varepsilon^\frac{q}{2^*} v_\varepsilon(\gamma_\varepsilon x).$$

Then $\|z_\varepsilon\|_\infty = 1$ and satisfies

$$\left(7.6\right) \begin{cases} -\nabla(|x|^{-2^* \nu} \nabla z_\varepsilon) = |x|^{-2^* \nu} z_\varepsilon^{2^* - 1} - \varepsilon \gamma_\varepsilon \frac{(N + 2) - q(N - 2)}{2} |x|^{-(q + 1)\nu} z_\varepsilon^q & \text{in } \Omega_\varepsilon, \\ z_\varepsilon > 0 & \text{in } \Omega_\varepsilon, \\ z_\varepsilon = 0 & \text{in } \partial \Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \gamma_\varepsilon^{-1} \Omega$.

Lemma 7.2. Suppose $z_\varepsilon$ is as in (7.5), $0 < \nu < \frac{N - 2}{4}$, $\frac{N + 2}{N - 2} < q < \frac{1 + \nu}{\nu}$ and (1.3) holds. Then

(i) $\lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon \frac{(N + 2) - q(N - 2)}{2} = 0$

(ii) There exists $Z \in D^{1,2}(\mathbb{R}^N, |x|^{-2^* \nu})$ such that $z_\varepsilon \to Z$ in $C^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap L^\infty(\mathbb{R}^N)$ as $\varepsilon \to 0$.

(iii) $Z$ satisfies Eq. (1.22) and given by (1.20).

Remark 7.1. The upper bound of $q$ comes from the fact that limit of $\varepsilon \gamma_\varepsilon \frac{(N + 2) - q(N - 2)}{2}$ can be $\infty$ as $q$ is supercritical. To exclude this option we need to put this restriction on $q$. Note that, when $q$ is critical or subcritical, the above limit is always 0. Therefore in the subcritical case no extra restriction on the upper bound of $q$ appears.
Moreover, by Fatou’s lemma
\[ \int_{\Omega} |x|^{-2\nu} \nabla \zeta \nabla \phi = \int_{\Omega} |x|^{-2\nu} \zeta^2 - 1 \phi - \varepsilon \gamma_{\varepsilon} \frac{(N+2)-(q(N-2))}{2} \int_{\Omega} |x|^{-(q+1)\nu} \zeta^q \phi. \]

**Case 1:** \( \varepsilon \gamma_{\varepsilon} \frac{(N+2)-(q(N-2))}{2} \) is bounded.

Therefore there exists \( c \geq 0 \) such that \( \varepsilon \gamma_{\varepsilon} \frac{(N+2)-(q(N-2))}{2} \to c \) (along a subsequence). Furthermore, by the elliptic regularity theorem it follows that \( \zeta \to Z \) in \( C^{2}_0(\mathbb{R}^N \setminus \{0\}) \).

Suppose \( c > 0 \). Since \( \zeta \to Z \) a.e and \( ||\zeta||_{L^\infty} = 1 \), by dominated convergence theorem it follows
\[ \lim_{\varepsilon \to 0} \int_{\Omega} |x|^{-2\nu} \zeta^2 - 1 \phi = \int_{\mathbb{R}^N} |x|^{-2\nu} Z^2 - 1 \phi. \]

**Claim:** \( |||x|^{-\nu} \nabla \zeta||_{L^2(\Omega)} \) is uniformly bounded with respect to \( \varepsilon \).

Assuming the claim,
\[ \lim_{\varepsilon \to 0} \int_{\Omega} |x|^{-2\nu} \nabla \zeta \nabla \phi = \int_{\mathbb{R}^N} |x|^{-2\nu} Z \nabla \phi, \]

follows from Vitali’s convergence theorem, since \( \nabla \zeta \to \nabla Z \) a.e. in \( \mathbb{R}^N \). To prove the claim, we see
\[ \int_{\Omega} |x|^{-2\nu} |\nabla \zeta|^2 = \int_{\Omega} |x|^{-2\nu} \frac{(N+2)-(q(N-2))}{2} \int_{\Omega} |x|^{-(q+1)\nu} \zeta^q + 1 \]
\[ \leq \int_{\Omega} |x|^{-2\nu} Z^2 = 1. \]

Combining (7.8)-(7.10), we have
\[ \int_{\mathbb{R}^N} |x|^{-2\nu} Z^2 - 1 = c \int_{\mathbb{R}^N} |x|^{-2\nu} Z^q + 1 \text{ in } \mathbb{R}^N. \]

Moreover, by Fatou’s lemma
\[ c \int_{\mathbb{R}^N} |x|^{-2\nu} Z^q + 1 dx \leq \liminf_{\varepsilon \to 0} \varepsilon \gamma_{\varepsilon} \frac{(N+2)-(q(N-2))}{2} \int_{\Omega} |x|^{-(q+1)\nu} \zeta^q + 1 dx \]
\[ = \liminf_{\varepsilon \to 0} \left[ \int_{\Omega} |x|^{-2\nu} \zeta^2 dx - \int_{\Omega} |x|^{-2\nu} |\nabla \zeta|^2 dx \right] \]
\[ \leq 1. \]

Since \( c > 0 \) and \( \zeta \to Z \) in \( C^{2}_0(\mathbb{R}^N \setminus \{0\}) \), from (7.11) and (7.13), it follows that \( Z \in D^{1,2}(\mathbb{R}^N, |x|^{-2\nu}) \cap L^{q+1}(\mathbb{R}^N, |x|^{-(q+1)\nu}) \). Therefore using Pohoaev identity (see (4.13)), we have
\[ c \left( \frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^q + 1 dx = 0, \]

which is a contradiction as \( |Z|_{L^\infty} = 1 \). Therefore, \( c = 0 \). Consequently, (7.12) yields \( Z \) satisfies (7.22), which is bounded.

**Case 2:** \( \lim_{\varepsilon \to 0} \varepsilon \gamma_{\varepsilon} \frac{(N+2)-(q(N-2))}{2} = \infty. \)
Set, $\lambda_\varepsilon := \varepsilon \gamma \frac{(N+2)-(q(N-2))}{2}$ and define $\tilde{z}_\varepsilon(x) := z_\varepsilon \left( \frac{x}{\lambda_\varepsilon} \right)$, where $m = \frac{1}{2+(q-1)\nu}$.

A straightforward computation yields, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$
\int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-2\nu} \nabla \tilde{z}_\varepsilon(x) \nabla \psi(x) \, dx = \lambda_\varepsilon^{2\nu + 2 - 2\nu} \int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-\nu} \tilde{z}_\varepsilon^{2\nu - 1}(x) \psi(x) \, dx - \int_{\lambda_\varepsilon^m \Omega_\varepsilon} |x|^{-(q+1)\nu} \tilde{z}_\varepsilon^q(x) \psi(x) \, dx.
$$

(7.14)

Since $\lambda_\varepsilon \to \infty$ as $\varepsilon \to 0$, we obtain $\lambda_\varepsilon^m \Omega_\varepsilon \to \mathbb{R}^N$ and $\lambda_\varepsilon^{2\nu + 2 - 2\nu} \to 0$. Using elliptic regularity theory we can argue as before that there exists $\tilde{Z}$ such that $\tilde{z}_\varepsilon \to \tilde{Z}$ in $C^2_{loc}(\mathbb{R}^N \setminus \{0\})$. Moreover, $\|z_\varepsilon\|_{L^\infty} = 1$ implies $\|\tilde{z}_\varepsilon\|_{L^\infty} = 1$. Therefore arguing as in Case 1, we can prove that $\tilde{Z}$ satisfies the following equation:

$$
- \text{div}(|x|^{-2\nu} \nabla \tilde{Z}) + |x|^{-(q+1)\nu} \tilde{Z}q = 0 \quad \text{in} \quad \mathbb{R}^N.
$$

(7.15)

From Theorem A.1 (see Appendix A), it follows that $\tilde{Z} = 0$. This is a contradiction as $\|\tilde{z}_\varepsilon\|_{L^\infty} = 1$ implies $\|\tilde{Z}\|_{L^\infty} = 1$. Hence Case 2 can not occur. Therefore from Case 1 we conclude (i) holds and $z_\varepsilon \to Z$ in $C^2_{loc}(\mathbb{R}^N \setminus \{0\})$.

Since $Z$ satisfies (1.22), $Z$ must be of the form $\mu^{-\frac{1}{2}} U(\frac{x}{\mu})$, where $U$ is as in (1.20) for some $\mu > 0$. By (1.3), it follows that max $z_\varepsilon = z_\varepsilon(0) = 1$. This implies $Z(0) = 1$ and $0 \leq Z \leq 1$. From this it follows $z_\varepsilon \to Z$ in $L^\infty_{loc}(\mathbb{R}^N)$ and $\mu = \left( \alpha \sqrt{\frac{N}{N-2}} \right)^{\frac{N-2}{2}}$.

From this, direct calculation yields that $Z(x) = \left(1 + \frac{|x|^{\frac{2}{N-2}}}{\mu} \right)^{-\frac{N-2}{2}}$.

We know the local behavior of $z_\varepsilon$. Now we need to check the behavior of $z_\varepsilon$ near $\infty$. Hence define the Kelvin transform of $z_\varepsilon$ as

$$
\hat{z}_\varepsilon(x) = |x|^{-\alpha} z_\varepsilon \left( \frac{x}{|x|^2} \right) \quad \text{in} \quad \Omega^*_\varepsilon \setminus \{0\}.
$$

(7.16)

Then from (7.6), $\hat{z}_\varepsilon$ satisfies

$$
- \text{div}(|x|^{-2\nu} \nabla \hat{z}_\varepsilon) = |x|^{-\nu} \hat{z}_\varepsilon^{2\nu - 1} \gamma \frac{(N+2)-(q(N-2))}{2 |x|^2} \left| x \right|^{-(q+1)\nu + \alpha(q-2)\nu + 1} \hat{z}_\varepsilon^q \quad \text{in} \quad \Omega^*_\varepsilon
$$

(7.17)

$$
\hat{z}_\varepsilon = 0 \quad \text{on} \quad \partial \Omega^*_\varepsilon.
$$

where $\Omega^*_\varepsilon$ is the image $\Omega_\varepsilon$ under the Kelvin transform. Hence the behavior of $z_\varepsilon$ near $\infty$ amounts to behavior of $\hat{z}_\varepsilon$ near 0.

**Lemma 7.3.** There exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that any solution of (7.17) satisfy

$$
\|\hat{z}_\varepsilon\|_{L^\infty(B_R)} \leq C \left( \int_{B_R} |x|^{-2\nu} \hat{z}_\varepsilon^{2\nu} \, dx \right)^{\frac{1}{2\nu}}.
$$

(7.18)

**Proof.** The proof of the above lemma follows along the same line of arguments as in Theorem 1.4 (i) with a suitable modification and we skip the proof. \(\square\)

**Remark 7.2.** There exists $C > 0$ independent of $\varepsilon > 0$ such that $z_\varepsilon \leq CZ(x)$ for all $x \in \Omega_\varepsilon$. For this, note that $\|z_\varepsilon\|_{L^\infty} = 1$, this implies that $z_\varepsilon \leq CZ(x)$ locally.
From (7.3) we have
\[ A < \int_{\Omega_{\epsilon}} |x|^{-2^*} z_{\epsilon}^2 \, dx < B. \]

But this implies that
\[ \int_{B_R} |x|^{-2^*} z_{\epsilon}^2 \, dx \leq \int_{\Omega_{\epsilon}} |x|^{-2^*} z_{\epsilon}^2 \, dx < B \]

and since at infinity $Z$ decays as $|x|^{-\alpha}$, we have $z_{\epsilon} \leq CZ(x)$ near infinity. Hence, we have $z_{\epsilon} \leq CZ(x)$ for all $x \in \Omega_{\epsilon}$. As a conclusion, from (7.5) we obtain that there exists $C > 0$ independent of $\epsilon$ such that
\[ v_{\epsilon}(x) \leq C \gamma_{\epsilon}^{-\frac{q}{2}} Z \left( \frac{x}{\gamma_{\epsilon}} \right). \]

Define $w_{\epsilon}(x) = \| v_{\epsilon} \|_{\infty} v_{\epsilon}(x) = \gamma_{\epsilon}^{-\frac{q}{2}} v_{\epsilon}(x)$. Then $w_{\epsilon}$ satisfies
\[ \begin{cases} -\text{div}(\gamma_{\epsilon}^{-\frac{q}{2}} |x|^{-2^*} v_{\epsilon}^{2^*-1} - \epsilon |x|^{-\frac{(q+1)}{2}} v_{\epsilon}^q) = 0 & \text{in } \Omega, \\ w_{\epsilon} = 0 & \text{on } \partial \Omega. \end{cases} \]

**Lemma 7.4.** Let $\nu$ and $q$ be as in Lemma 7.2 and $w_{\epsilon}$ be as in (7.20). Then there exists a constant $C > 0$ such that
\[ \|w_{\epsilon}\|_{L^\infty(K)} + \|\nabla w_{\epsilon}\|_{L^\infty(K)} \leq C, \]
for every compact subset $K$ of $\Omega \setminus \{0\}$.

**Proof.** Using the Green kernel’s representation and Lemma 4.2, we have
\[
|w_{\epsilon}(x)| = \gamma_{\epsilon}^{-\frac{q}{2}} \left| \int_{\Omega} G(x, y)(|y|^{-2^*} v_{\epsilon}^{2^*-1} - \epsilon |y|^{-(q+1)^{1/2}} v_{\epsilon}^q) \, dy \right|
\leq C \gamma_{\epsilon}^{-\frac{q}{2}} |x|^{2^*} \int_{\Omega} |x - y|^{2-N} |y|^{-2^*} v_{\epsilon}^{2^*-1} \, dy \\
+ C \epsilon \gamma_{\epsilon}^{-\frac{q}{2}} |x|^{2^*} \int_{\Omega} |x - y|^{2+2^*-N} |y|^{-2^*} v_{\epsilon}^{2^*-1} \, dy \\
+ C \epsilon |x|^{2^*} \int_{\Omega} |x - y|^{2-N} |y|^{-(q+1)^{1/2}} v_{\epsilon}^q \, dy \\
=: I_1 + I_2 + I_3 + I_4.
\]

Moreover,
\[
I_1 := C |x|^{2^*} \int_{\Omega \cap B_{\frac{1}{\gamma_{\epsilon}}}^{(0)}} |x - y|^{2-N} |y|^{-2^*} v_{\epsilon}^{2^*-1} \, dy \\
= C |x|^{2^*} \int_{\Omega \cap B_{\frac{1}{\gamma_{\epsilon}}}^{(0)}} |x - y|^{2-N} |y|^{-2^*} v_{\epsilon}^{2^*-1} \, dy \\
+ C |x|^{2^*} \int_{\Omega \setminus B_{\frac{1}{\gamma_{\epsilon}}}^{(0)}} |x - y|^{2-N} |y|^{-2^*} v_{\epsilon}^{2^*-1} \, dy \\
=: I_{11} + I_{12}.
\]
Using (7.19) along with the facts that $Z(x) \sim |x|^{-\alpha}$ at infinity and $\gamma_\varepsilon \to 0$, we find

$$
\gamma_\varepsilon^{-\frac{q}{2}} |y|^{-2^*\nu} v_\varepsilon^{2^*-1}(y) \leq \frac{C}{|y|^{(N+2)-\frac{4\nu}{N-2}}} \quad \text{in } \Omega \setminus B_{\frac{R}{2}}(0),
$$

(7.21)

$$
\gamma_\varepsilon^{-\frac{q-1}{2}} |y|^{-(q+1)\nu} v_\varepsilon^q(y) \leq \frac{C\gamma_\varepsilon^{(q-1)\frac{q}{2}}}{|y|^{(N-2)\nu-(q-1)}} \quad \text{in } \Omega \setminus B_{\frac{R}{2}}(0).
$$

(7.22)

Hence

$$
I_{12} \leq C|x|^{2\nu} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} \frac{1}{|x-y|^{N-2}|y|^{(N+2)-\frac{4\nu}{N-2}}} dy
$$

$$
\leq \frac{C|x|^{2\nu}}{|x|^{(N+2)-\frac{4\nu}{N-2}}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |x-y|^{-2^\nu} dy.
$$

(7.23)

When $y \in \Omega \cap B_{\frac{R}{2}}(0)$, we have $|x-y| \geq |x|-|y| \geq \frac{R}{2}|x|$. Therefore using (7.19), we get

$$
I_{11} \leq \frac{C|x|^{2\nu} \gamma_\varepsilon^{-\frac{q}{2}}}{|x|^{N-2}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |y|^{-2^\nu} v_\varepsilon^{2^*-1}(y) dy
$$

$$
\leq \frac{C\gamma_\varepsilon^{-\frac{q}{2}}}{|x|^q} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |y|^{-2^\nu} Z \left( \frac{y}{\gamma_\varepsilon} \right)^{2^*-1} dy
$$

$$
\leq \frac{C}{|x|^q} \gamma_\varepsilon^{-\frac{q}{2} - 2^\nu N} \int_{\mathbb{R}^N} |y|^{-2^\nu} Z(y)^2 + 1 dy
$$

$$
= \frac{C}{|x|^q} \int_{\mathbb{R}^N} |y|^{-2^\nu} Z(y)^2 + 1 dy
$$

$$
= \frac{C}{|x|^q} \omega_N \alpha^{N-1} \left( \frac{N}{N-2} \right)^{N-2},
$$

where the last integral can be computed as in (7.32) in Lemma 7.6. Similarly $I_2$ can be written as

$$
I_2 = C \gamma_\varepsilon^{-\frac{q}{2}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |x-y|^{2^\nu-N} |y|^{-2^\nu} v_\varepsilon^{2^*-1} dy
$$

$$
+ C \gamma_\varepsilon^{-\frac{q}{2}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |x-y|^{2^\nu-N} |y|^{-2^\nu} v_\varepsilon^{2^*-1} dy
$$

(7.24)

$$
= I_{21} + I_{22}.
$$

Proceeding similarly as we did for $I_{12}$ and $I_{11}$, we have

$$
I_{22} \leq \frac{C}{|x|^{(N+2)-\frac{4\nu}{N-2}}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} \frac{1}{|x-y|^{2^\nu-N} |y|^{(N+2)-\frac{4\nu}{N-2}}} dy
$$

$$
\leq \frac{C}{|x|^{(N+2)-\frac{4\nu}{N-2}}} \int_{\Omega \setminus B_{\frac{R}{2}}(0)} |x-y|^{2^\nu-N} dy;
$$

(7.25)

$$
I_{21} \leq \frac{C}{|x|^q} \omega_N \alpha^{N-1} \left( \frac{N}{N-2} \right)^{N-2}.
$$
Similarly to compute $I_4$ and $I_4$, we break those integral into two parts, namely in $\Omega \cap B_{\varepsilon}(0)$ and $\Omega \setminus B_{\varepsilon}(0)$. Using (7.22), integral in $\Omega \setminus B_{\varepsilon}(0)$ can be computed as before. Proceeding as in (7.23), we have

$$\varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}} |x|^{2\nu} \int_{\Omega \cap B_{\varepsilon}(0)} |x - y|^{2 - N} |y|^{-(q+1)\nu} v_{\varepsilon}^q dy$$

$$\leq \frac{C \varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}}}{|x|^{\alpha}} \int_{\Omega \cap B_{\varepsilon}(0)} |y|^{-(q+1)\nu} v_{\varepsilon}^q(y) dy$$

$$\leq \frac{C \varepsilon \gamma_{\varepsilon}^{-\frac{2}{p} - (N-2)}}{|x|^{\alpha}} \int_{\Omega \cap B_{\varepsilon}(0)} |y|^{-(q+1)\nu} Z^q(y) dy$$

By a straight forward computation using the expression of $Z$ from Lemma 7.2, it can be concluded that for any compact set $K \subset \Omega \setminus \{0\}$, we have $\|w_{\varepsilon}\|_{L^\infty(K)} \leq C$ and by the regularity $\|\nabla w_{\varepsilon}\|_{L^\infty(K)} \leq C$. 

**Lemma 7.5.** Let $\nu, q, w_{\varepsilon}$ be as in Lemma 7.4. Then there exists $\gamma_0 > 0$ such that

$$\lim_{\varepsilon \to 0} w_{\varepsilon}(x) = \gamma_0 G(x, 0) \text{ in } C^1_{\text{loc}}(\Omega \setminus \{0\}).$$

**Proof.** Define

$$f_{\varepsilon} := \gamma_{\varepsilon}^{-\frac{2}{p}} |x|^{-2^{\nu} \nu 2^{\nu} - 1} - \varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}} |x|^{-(q+1)\nu} v_{\varepsilon}^q.$$

Choose $R > 0$ such that $\Omega' = \Omega \setminus \overline{B_R}(0)$ is connected. Then $|w_{\varepsilon}| + |\nabla w_{\varepsilon}| \leq C$ for all $x \in \Omega'$. Let $x' \in \partial \Omega \cup \partial \Omega'$, then $|w_{\varepsilon}(x) - w_{\varepsilon}(x')| \leq C$ for all $x \in \Omega'$. But this implies $w_{\varepsilon}$ is uniformly bounded in $\Omega' \cap \Omega$. By the standard regularity, we have $w_{\varepsilon} \to w$ as $\varepsilon \to 0$ in $C^1_{\text{loc}}(\overline{\Omega} \setminus \{0\})$. If $K \subset \Omega \setminus \{0\}$, then for any $x \in K$ and $r > 0$ small, using the fact $\gamma_{\varepsilon} \to 0$, we have

$$w_{\varepsilon}(x) = \int_{\Omega} G(x, y) f_{\varepsilon}(y) dy$$

$$= \int_{B_r(0)} G(x, y) f_{\varepsilon}(y) dy + \int_{\Omega \setminus B_r(0)} G(x, y) f_{\varepsilon}(y) dy$$

$$= \gamma_{\varepsilon}^{-\frac{2}{p}} \int_{B_r(0)} G(x, y) |y|^{-2^{\nu} \nu 2^{\nu} - 1}(y) dy$$

$$+ \gamma_{\varepsilon}^{-\frac{2}{p}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-2^{\nu} \nu 2^{\nu} - 1}(y) dy$$

$$+ \varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}} \int_{B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_{\varepsilon}^q(y) dy$$

$$+ \varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_{\varepsilon}^q(y) dy$$

(7.27)

**Claim:**

(i) $\varepsilon \gamma_{\varepsilon}^{-\frac{2}{p}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-(q+1)\nu} v_{\varepsilon}^q(y) dy = o(1)$ and

(ii) $\gamma_{\varepsilon}^{-\frac{2}{p}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-2^{\nu} \nu 2^{\nu} - 1}(y) dy = o(1)$. 

Similarly (ii) follows. Therefore from (7.30), we obtain
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

To see this,
\[ \lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-\alpha} dy = 0. \]

Moreover from Lemma 7.6 we get
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

Therefore, \( \lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} \int_{\Omega \setminus B_r(0)} G(x, y) |y|^{-\alpha} dy = 0. \)

Similarly (ii) follows. Therefore from (7.27), we obtain
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

Furthermore, \( \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \)

Define
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

By doing a straightforward computation using (7.19), it follows
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

Moreover from Lemma 7.6 we get
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

To see this,
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

Define
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

By doing a straightforward computation using (7.19), it follows
\[ \lim_{\varepsilon \to 0} C \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} = 0. \]

Consequently, a direct computation using Lemma 7.2 yields \( L = o(1) \).

Define
\[ \gamma_0 := \lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} \int_{B_r(0)} |y|^{-\alpha} dy = 0. \]

Then
\[ \lim_{\varepsilon \to 0} w_\varepsilon(x) = \gamma_0 G(x, 0). \]

Moreover from Lemma 7.6 we get
\[ \gamma_0 = \omega_N (N - 2 - 2\nu)^{N-1} \left( \frac{N}{N-2} \right)^{\frac{N-2}{2}}. \]

Using the same procedure as above we can show that
\[ w_\varepsilon \to \gamma_0 G(x, 0) \text{ in } C^1_{\text{loc}}(\Omega \setminus \{x\}). \]

**Lemma 7.6.** Let \( v_\varepsilon \) be as in Theorem 7.11 and \( \gamma_\varepsilon \) be as defined in (7.4). Define
\[ \gamma_0 := \lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \gamma_\varepsilon^{-\frac{q}{2}} \int_{B_r(0)} |y|^{-\alpha} dy = 0. \]

Then
\[ \gamma_0 = \omega_N (N - 2 - 2\nu)^{N-1} \left( \frac{N}{N-2} \right)^{\frac{N-2}{2}}. \]
Proof. We define \( I_{\varepsilon,r} := \sqrt[\alpha]{\frac{2}{\varepsilon}} \int_{B_{1}(0)} |y|^{-2^* - 1} dy \). Since \( v_\varepsilon \) and \( z_\varepsilon \) are related by (\ref{eq:13}), we have \( v_\varepsilon(x) = \sqrt[\alpha]{\frac{2}{\varepsilon}} z_\varepsilon \left( \frac{x}{\varepsilon} \right) \). Thus

\[
I_{\varepsilon,r} = \sqrt[\alpha]{\frac{2}{\varepsilon}} \int_{B_{1}(0)} |x|^{-2^* - 1}(x) dx = \int_{B_{1}(0)} |x|^{-2^* - 1}(x) dx
\]

Since \( \varepsilon \to 0 \) implies \( \gamma_\varepsilon \to 0 \), we have

\[
(7.32) \quad \gamma_0 = \lim_{r \to 0} \lim_{\varepsilon \to 0} I_{\varepsilon,r} = \int_{\mathbb{R}^N} |x|^{-2^*} Z^{2^*-1} dx,
\]

where \( Z \) is as in Lemma \ref{lem:7.2}. Therefore by doing a straightforward computation, we obtain

\[
\gamma_0 = \frac{\omega_N N^\alpha}{2} \left( \frac{N \alpha^2}{N - 2} \right)^{\frac{N-2}{2}} \left( \frac{2}{N} \right)^{\frac{N}{2}},
\]

where \( B(a,b) = \int_0^\infty t^{a-1}(1+t)^{-a-b} dt \) is the Beta function. Recall that \( B(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \). Thus \( B \left( \frac{N}{2}, 1 \right) = \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N}{2} + 1 \right)} = \frac{2}{N} \), the lemma follows. \( \square \)

Proof of Theorem \ref{thm:1.11}. From (\ref{eq:13}) we have

\[
\frac{1}{2} \int_{\partial \Omega} |\nabla v_\varepsilon|^2 dS = \varepsilon \left( \frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx.
\]

Using \( w_\varepsilon = ||v_\varepsilon||_{\infty} v_\varepsilon \) in the above expression, we have

\[
\int_{\partial \Omega} |x|^{-2^*} |\nabla w_\varepsilon|^2 (x,n) dS = 2 \varepsilon \left( \frac{N-2}{2} - \frac{N}{q+1} \right) ||v_\varepsilon||_{\infty}^2 \int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx
\]

\[
= 2 \varepsilon \left( \frac{N-2}{2} - \frac{N}{q+1} \right) ||v_\varepsilon||_{\infty}^2 \int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx.
\]

Since \( z_\varepsilon \to Z \text{ a.e.} \) and \( z_\varepsilon \leq CZ \), by the dominated convergence theorem it follows

\[
\int_{\Omega} |x|^{-(q+1)\nu} v_\varepsilon^{q+1} dx \to \int_{\Omega} |x|^{-(q+1)\nu} Z^q dx.
\]

Therefore taking limit \( \varepsilon \to 0 \) and using (\ref{eq:20}) and Lemma \ref{lem:7.3} we obtain

\[
(7.33) \quad \lim_{\varepsilon \to 0} \varepsilon ||v_\varepsilon||_{\infty}^2 \frac{a(N-2) + (N+2) + 2n}{N} = \frac{(N-2-2\nu)\gamma_0^2 |R(0)|}{2 \left( \frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx}.
\]

From Lemma \ref{lem:7.2} we know \( Z(x) = \left( 1 + \frac{|x|}{\delta N^\alpha} \right)^{-\frac{N-2}{2\alpha}} \). Therefore a straightforward calculation yields

\[
\int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx = \frac{\omega_N N^\alpha}{2} \left( \frac{N \alpha^2}{N - 2} \right)^{\frac{N}{2} - (q+1)\nu} \left( \frac{N}{2} \right)^{-\frac{N+2}{2\alpha}-1}
\]

\[
\times B \left( \frac{N-2}{2\alpha} (N - (q+1)\nu), \frac{N-2}{2\alpha} \{q(N-2-\nu) - (2+\nu)\} \right),
\]

\[
(7.34)
\]
where $B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-a-b} dt$ and $\omega_N$ is the surface measure of unit sphere in $\mathbb{R}^N$. From Lemma \ref{lem:gamma0}, it is known that $\gamma_0 = \omega_N a^{N-1}\left(\frac{N}{n-1}\right)^{N-2}$. Substituting the value of $\gamma_0$, $\alpha$, and $\int_{\mathbb{R}^N} |x|^{-(q+1)\nu} Z^{q+1} dx$ in (\ref{eq:RHS}), we have RHS of (\ref{eq:RHS}) as

$$RHS = \frac{\alpha(q+1)(R(0))}{(N-2)^{(N+2)}} \cdot \frac{\omega_N^{2(N-1)}\left(\frac{N}{n-1}\right)^{N-2}}{\omega_N^{\frac{2}{N}}\left(\frac{N}{n-1}\right)^{N-\nu}} \cdot \frac{\omega_N^{\frac{2}{N}}\left(\frac{N}{n-1}\right)^{N-\nu}}{\omega_N^{\frac{2}{N}}\left(\frac{N}{n-1}\right)^{N-\nu}} \times$$

$$\left[B\left(\frac{N-2}{2\nu}(N-(q+1)\nu), \frac{N-2}{2\nu}(q(N-2-\nu) - (2+\nu))\right)\right]^{-1}$$

$$= \frac{\omega_N^{\frac{2}{N}}\left(\frac{N}{n-1}\right)^{N-\nu}}{C_{q,N} \omega_N^{\frac{2}{N}}\left(\frac{N}{n-1}\right)^{N-\nu}} \times \left[B\left(\frac{N-2}{2\nu}(N-(q+1)\nu), \frac{N-2}{2\nu}(q(N-2-\nu) - (2+\nu))\right)\right]^{-1},$$

where $C_{q,N}$ is as in (\ref{eq:CqN}). Furthermore, (\ref{eq:RHS}) follows from (\ref{eq:7.20}). \hfill \square

**APPENDIX A.**

In this section we consider the following problem:

$$-\text{div}(|x|^{-2\nu}\nabla u) + |x|^{-(q+1)\nu} u^q = 0 \quad \text{in} \quad \mathbb{R}^N,$$

$$u \geq 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\frac{N+2}{N-2} < q < \frac{1+\nu}{\nu}$ and $0 < \nu < \frac{N-2}{4}$.

**Definition A.1.** If $u$ is a solution of (\ref{eq:1.1}) in a domain $\Omega$ such that $u(x) \to \infty$ as $x \to \partial \Omega$, then $u$ is called a large solution.

**Theorem A.1.** Suppose $u$ is any solution of Eq.(\ref{eq:1.1}). Then $u = 0$.

**Proof.** Let $U_1$ be a large solution of (\ref{eq:1.1}) in $B_r(0)$ such that $U_1 \in L_{loc}^\infty(B_1(0))$ (existence of such solution follows from Theorem A.2). Define

$$U_R(x) := R^{n-\nu} u_{1}(\frac{x}{R}).$$

By a straight forward computation, it follows $U_R$ is a large solution of (\ref{eq:1.1}) in $B_r(0)$. Moreover, $U_R \to 0$ as $R \to \infty$. If $u$ is any solution of (\ref{eq:1.1}) in $\mathbb{R}^N$, then $u$ is a solution of (\ref{eq:1.1}) in $B_r(0)$ as well. Consequently,

$$-\text{div}(|x|^{-2\nu}\nabla(u - U_R)) + |x|^{-(q+1)\nu} (u^q - U_R^q) = 0 \quad \text{in} \quad B_r(0).$$

Clearly $u \leq U_R$ on $\partial B_r(0)$. Therefore by taking $(u - U_R)^+$ as a test function for the above equation we obtain

$$\int_{B_r(0)} |x|^{-2\nu} \nabla(u - U_R)^+^2 + \int_{B_r(0)} |x|^{-(q+1)\nu} \frac{(u^q - U_R^q)}{u - U_R}(u - U_R)^+^2 = 0,$$

which in turn implies $(u - U_R)^+ = 0$ in $B_r(0)$. Thus $u \leq U_R$ in $B_r(0)$. Hence by taking limit $R \to \infty$, we have $u \leq 0$ in $\mathbb{R}^N$. Since u is a nonnegative solution, we get $u = 0$ in $\mathbb{R}^N$. \hfill \square
Theorem A.2. There exists a large solution \( u \) to the equation (1.1) in \( B_1(0) \). Moreover, \( u \in L_{\text{loc}}^\infty(B_1(0)) \).

We essentially follow the classical method of Veron-Vazquez \[24\] Lemma 2.1 to prove this result.

**Proof.** We will show that there exists a radial large solution. Towards this goal, let us consider the following ode

\[
\begin{cases}
  u'' + \frac{N - 1 - 2\nu}{r} u'(r) = r^{-(q-1)\nu} u^q & \text{in } (0, 1) \\
  u > 0 & \text{in } (0, 1) \\
  u(0) = 1 \quad u'(0) = 0. 
\end{cases}
\]

(1.2)

Then we can write the solution as

\[
u(t) = 1 + \int_0^t s^{2\nu + 1 - N} \int_0^s t^{N-1-(q+1)\nu} u^q(t) dt ds.
\]

Since \( q < \frac{1 + \nu}{\nu} \) implies \( q < \frac{2 + \nu}{\nu} < \frac{N - \nu}{\nu} \), by the standard existence of ode theory, it follows that solution \( u(r) \) exists in a neighborhood of 0. Moreover, \( q < \frac{1 + \nu}{\nu} \) implies \( u'(0) = 0 \) and \( u \) is \( C^1 \) up to the blow-up time.

**Claim:** There exists a solution \( u \) of the following ode

\[
\begin{cases}
  u'' + \frac{N - 1 - 2\nu}{r} u'(r) = r^{-(q-1)\nu} u^q & \text{in } [0, r^*) \\
\end{cases}
\]

in \([0, r^*)\) such that \( \lim_{r \uparrow r^*} u(r) = +\infty \) for some \( r^* > 0 \).

To see the claim, we use generalised Emden–Fowler transform \( t = (\frac{u}{\alpha})^\nu \) and \( y(t) = \alpha^{-\nu} u(r) \), where \( \alpha = N - 2 - 2\nu \). Therefore we obtain

\[
y''(t) = t^{-(2\nu + 1) + (q-1)\nu} y^q & \text{in } R < t < +\infty. \tag{1.3}
\]

Existence of \( u(r) \) in the neighbourhood of 0 implies, Eq. (1.3) has a solution \( y(t) \) in \((R, \infty)\) for some large \( R > 0 \) with \( y'(\infty) = 0, y(\infty) > 0 \). To prove the claim, it is equivalent to show that there exists a solution \( y \) of (1.3) in \((t^*, \infty)\) such that \( \lim_{t \uparrow t^*} y(t) = \infty \) for some \( t^* \in (0, \infty) \). Suppose this is not true, then \( y(t) \) can be continued as a solution of (1.3) to the left of \( 0 \) till 0, i.e \( y(t) \) can be defined on \((0, \infty)\).

Set \( f(t) := t^{-(2\nu + 1) + (q-1)\nu} \) and let \( 0 < R < R' < \infty \). As \( f \) is continuous and positive we get there exists \( m, M > 0 \) such that \( 0 < m \leq f(t) \leq M \) for \( t \in [R, R'] \). Now consider the ode

\[
v''(t) = M v^q(t) & \text{in } (R, R'); \quad v > 0 \text{ in } (R, R'). \tag{1.4}
\]

Rename the nonlinear term in (1.4) as \( h(v) \), that is \( h(t) := M v^q \). Then

\[
H(t) := \int_0^t h(s) ds, \quad \psi(a) := \int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty,
\]

for any \( a > 0 \). Therefore, applying Vazquez’s classical a-priori estimates \[23\] (also see \[24\] Lemma 2.1) we find a large solution \( v(t) \) of (1.4). That is, \( \lim_{t \uparrow R} v(t) = \)
\[ y(t) \leq v(t) \quad \text{in} \quad (R, R'). \]

From (1.4), it also follows that \( v \) is a convex function. If \( y \) is a solution of (1.3) in \((T, \infty)\) for some large \( T \) with the initial value \( y(\infty) > \min_{R < t < R'} v(t) \) and \( y'(\infty) = 0 \), then graph of \( y \) must lie above all of it's tangent as \( y \) is a convex decreasing function. Consequently, \( y(t) > \min_{R < s < R'} v(s) \) for all \( t < \infty \). Since \( y \) can be extended till \( 0 \), it in turn implies, there exists \( t_1, t_2 \) such that \( R < t_1 < t_2 < R' \) and \( y(t) > v(t) \) in \((t_1, t_2)\). This is a contradiction to (1.3). Hence \( y \) can not be defined to the left of \( \infty \) till \( R \), that is, there must exist \( t^* > R \) such that \( \lim_{t \uparrow t^*} y(t) = \infty \). This proves the claim. Since we have proved existence of a large solution \( u \) of (1.1) in the ball \( B_r(0) \), we use similarity transformation \( T_r \) to get large solution in the unit ball \( B_1(0) \). More precisely, \( U_1(x) := T_r u(x) := r^{\frac{2}{q-1} - \nu} u(rx) \). This completes the proof of the theorem. \( \square \)

**APPENDIX B.**

**Lemma B.1.** Define \( w(B_r(0)) := \int_{|x-y| < t} |y|^{-2\nu} dy. \) Then

\[ w(B_{2^r}(x)) \geq C 2^{k(N-2\nu)} w(B_r(x)) \]

**Proof.** We prove the lemma considering into three cases.

Case 1: Suppose \( r \geq \frac{|x|}{10} \).

Then \( 2^k r \geq \frac{|x|}{10} \). We claim \( B_{2^r}(0) \subset B_{2^{k-1} r}(x) \). Indeed, \( y \in B_{2^r}(0) \) implies \( |y-x| \leq |y| + |x| \leq (2^k r + 10 r) \leq (2^k \cdot 10) r \). Thus the claim follows. Therefore using doubling measure property of \( |y|^{-2\nu} \), we get

\[ w(B_{2^r}(x)) \geq c_1 w(B_{2^{k-1} r}(x)) \geq c_1 w(B_{2^r}(0)), \]

where \( c_1 \) does not depend on \( k, r, x \). As a consequence,

\[ w(B_{2^r}(x)) \geq c_1 \omega_N (2^k r)^{N-2\nu} = c_1 2^{k(N-2\nu)} w(B_r(0)) \geq c_1 2^{k(N-2\nu)} w(B_r(x)). \]

Case 2: \( r < \frac{|x|}{10} \) and \( 2^k r < \frac{|x|}{10} \).

Then \( y \in B_{2^k r}(x) \) implies

\[ |x| \leq |x-y| + |y| \leq 2^k r + |y| \leq \frac{|x|}{10} + |y| \implies \frac{9}{10} |x| \leq |y|. \]

Similarly it follows \( |y| \leq \frac{11}{10} |x| \). Thus \( c_1 |x| \leq |y| \leq c_2 |x| \). If \( y \in B_r(x) \), then using \( r < \frac{|x|}{10} \) and doing the calculation as above we get there exists positive constants \( c_3, c_4 \), independent of \( r, x, k \) such that \( c_3 |x| \leq |y| \leq c_4 |x| \). Consequently,

\[ w(B_{2^k r}(x)) \geq \omega_N c_2^{-2\nu} |x|^{-2\nu} (2^k r)^N \geq \omega_N c_2^{-2\nu} 2^{k(N-2\nu)} |x|^{-2\nu} r^N \]

Moreover,

\[ w(B_r(x)) \leq \omega_N c_3^{-2\nu} |x|^{-2\nu} r^N \]

Hence (2.1) holds with \( C = (\frac{c_3}{c_2})^{-2\nu} \).

Case 3: \( r < \frac{|x|}{10} \) and \( 2^k r \geq \frac{|x|}{10} \).

This case is similar to Case 1 and we skip the proof. \( \square \)
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