Mixed-Mean Inequality for Submatrix

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Abstract. For a \( m \times n \) matrix \( B = (b_{ij})_{m \times n} \) with nonnegative entries \( b_{ij} \) and any \( k \times l \)-submatrix \( B_{ij} \) of \( B \), let \( a_{B_{ij}} \) and \( g_{B_{ij}} \) denote the arithmetic mean and geometric mean of elements of \( B_{ij} \) respectively. It is proved that if \( k \) is an integer in \( (\frac{m}{2}, m] \) and \( l \) is an integer in \( (\frac{n}{2}, n] \) respectively, then

\[
\left( \prod_{i=k, j=l}^{a_{B_{ij}}} B_{ij} \right)^{\frac{1}{C_m \cdot C_n}} \geq \frac{1}{C_m \cdot C_n} \left( \sum_{i=k, j=l}^{g_{B_{ij}}} B_{ij} \right),
\]

with equality if and only if \( b_{ij} \) is a constant for every \( i, j \).

1. Introduction

Let \( x_1, ..., x_n \) be positive real numbers, then the arithmetic-geometric mean inequality is

\[
\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},
\]

with equality if and only if \( x_1 = \cdots = x_n \).

There are many research articles devoted to the classical arithmetic-geometric mean inequality and its generalizations([1],[2],[4],[9],[10]). The mixed-type arithmetic-geometric mean inequalities([3],[5],[6],[7],[8],[11]) are one of the most important branch in these generalizations.

In [6], Kedlaya established the following mixed mean inequality, which was conjectured by F.Holland and was given an inductive proof by T.Matsuda[8]:

The arithmetic mean of the numbers

\[
x_1, \sqrt{x_1x_2}, \sqrt[3]{x_1x_2x_3}, ..., \sqrt[n]{x_1x_2 \cdots x_n}
\]

does not exceed the geometric mean of the numbers

\[
x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

with equality if and only if \( x_1 = \cdots = x_n \).

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In [3], Carlson established the following mixed mean inequality:

Let the arithmetic and geometric means of the real nonnegative numbers \(x_1, \ldots, x_n\) taken \(n-1\) at a time be denoted by

\[
a_i = \frac{x_1 + \cdots + x_n - x_i}{n-1}, \quad g_i = \left(\frac{x_1 \cdots x_n}{x_i}\right)^{\frac{1}{n-1}}.
\]

Then for \(n \geq 3\),

\[
\left(\prod_{|A|=k} a_{A} \right)^{\frac{1}{C_{kn}}} \geq \frac{1}{C_{kn}} \left(\sum_{|A|=k} g_{A}\right),
\]

with equality if and only if \(x_1 = \cdots = x_n\).

In [7], Leng, Si and Zhu generalized the above result to any subsets and established another mixed arithmetic-geometric mean inequality:

Let \(X = \{x_1, \ldots, x_n \mid x_i > 0, \ i = 1, 2, \ldots, n\}\). For \(A \subset X\), let \(a_{A}\) and \(g_{A}\) denote the arithmetic mean and geometric mean of all elements of \(A\), respectively. If \(k\) is an integer in \((\frac{n}{2}, n]\), then

\[
\left(\prod_{|A|=k} a_{A}\right)^{\frac{1}{C_{kn}}} \geq \frac{1}{C_{kn}} \left(\sum_{|A|=k} g_{A}\right),
\]

with equality if and only if \(x_1 = \cdots = x_n\).

In this note, we established a new mixed arithmetic-geometric mean inequality for submatrix, which was an extension of the Carlson inequality and also an extension of (1).

Our main result is the following theorem.

**Theorem** For a \(m \times n\) matrix \(B = (b_{ij})_{m \times n}\) with nonnegative entries \(b_{ij} \geq 0\) and any \(k \times l\)-submatrix \(B_{ij}\) of \(B\), let \(a_{B_{ij}}\) and \(g_{B_{ij}}\) denote the arithmetic mean and geometric mean of elements of \(B_{ij}\) respectively. If \(k\) is an integer in \((\frac{n}{2}, m]\) and \(l\) is an integer in \((\frac{n}{2}, n]\) respectively, then

\[
\left(\prod_{i=k, j=l \atop B_{ij} \subset B} a_{B_{ij}}\right)^{\frac{1}{C_{km} \cdot C_{ln}}} \geq \frac{1}{C_{km} \cdot C_{ln}} \left(\sum_{i=k, j=l \atop B_{ij} \subset B} g_{B_{ij}}\right),
\]

with equality if and only if \(b_{ij}\) is a constant for every \(i, j\).

**Remark 1.** If \(k \leq \lfloor \frac{m}{2} \rfloor\), for the matrix \(B = (b_{ij})_{m \times n}\), taking \(b_{11} = b_{12} = \cdots = b_{1n} = b_{21} = b_{22} = \cdots = b_{2n} = \cdots = b_{k1} = b_{k2} = \cdots = b_{kn} = 1\), \(b_{ij} = 0\) for \(k + 1 \leq i \leq m\), then the right-hand side of (2) equals \(1/(C_{m}^{k} \cdot C_{n}^{l})\), but the left-hand side is zero. If \(l \leq \lfloor \frac{n}{2} \rfloor\), by the same argument, one can get a contradiction. Hence the statement of Theorem fails for \(k \leq \lfloor \frac{m}{2} \rfloor\) or \(l \leq \lfloor \frac{n}{2} \rfloor\).

**Remark 2.** Taking \(k = m, l = n\) in Theorem, inequality (2) is just the classical arithmetic-geometric mean inequality. For \(m = 1\) or \(n = 1\) in Theorem, inequality (2) is just the inequality (1).
Remark 3. The condition of our theorem is weaker than the ones of (1), because the infimum of \( k \times l \) is \( \frac{m}{2} \times \frac{n}{2} \), which is less than \( \frac{1}{2} \times m \times n \), the half of the element number of the matrix \( B = (b_{ij})_{m \times n} \).

2. Proof of Main Results

Let \( X \) denote the finite set with positive real numbers \( x_1, x_2, ..., x_n \) and let \( X_i \) denote the subset of \( X \) with \( k \) elements \( x_{i_1}, ..., x_{i_k} \). The \( r \) power mean of the elements of \( X_i \) is denoted by

\[
m_r(X_i) = \left[ \frac{1}{k} (x_{i_1}^r + ... + x_{i_k}^r) \right]^{\frac{1}{r}},
\]

where \( r > 0 \) is some real number. If \( r = 1 \) in the above equality, we get \( a_{X_i} \), the arithmetic mean of the elements of \( X_i \). Let \( r \to 0 \) in (3), we get \( g_{X_i} \), the geometric mean of the elements of \( X_i \).

We first established the following lemma.

**Lemma** For any \( m \times n \) matrix \( X = (x_{ij})_{m \times n} \) with nonnegative entries \( x_{ij} \geq 0 \). Denote by \( X_1, ..., X_{C_m \cdot C_n} \) its all \( k \times l \)-submatrix \( X_{ij} \) of \( X \). If \( k \) is an integer in \( (\frac{m}{2}, m] \) and \( l \) is an integer in \( (\frac{n}{2}, n] \) respectively, then

\[
m_r(X_i) = \left[ \frac{1}{C_m \cdot C_n} (m_r(X_i \cap X_{1}))^r + (m_r(X_i \cap X_2))^r + ... + (m_r(X_i \cap X_{C_m \cdot C_n}))^r \right]^{\frac{1}{r}}. \tag{4}
\]

Here the intersection of \( X_i \cap X_j, j = 1, ..., C_m \cdot C_n \), is just the general set intersection.

**Proof.** It suffices to prove the equality

\[
(m_r(X_i))^r = \frac{1}{C_m \cdot C_n} \left[ (m_r(X_i \cap X_1))^r + (m_r(X_i \cap X_2))^r + ... + (m_r(X_i \cap X_{C_m \cdot C_n}))^r \right]. \tag{5}
\]

Assume that \( X_i = \{x_{i_1}, ..., x_{i_{k \times l}}\} \), then the left hand of (5) is that

\[
(m_r(X_i))^r = \frac{1}{k \times l} (x_{i_1}^r + ... + x_{i_{k \times l}}^r).
\]

Now we need show that the right hand of (5) is also the mean of \( \{x_{i_1}^r, ..., x_{i_{k \times l}}^r\} \) with the same coefficient \( \frac{1}{k \times l} \).

Assume that the right hand of (5) is \( c_{i_1} x_{i_1}^r + ... + c_{i_{k \times l}} x_{i_{k \times l}}^r \), by the arbitrariness of \( X_i \), we get that \( c_{i_1} = ... = c_{i_{k \times l}} \).

Since \( k > \frac{m}{2} \) and \( l > \frac{n}{2} \), we have

\[
X_i \cap X_j \neq \Phi, \ j = 1, 2, ..., C_m \cdot C_n.
\]
Then the sum of all coefficients in \( (m_r(X_i \cap X_j))^r \) is 1, \( j = 1, 2, \ldots, C_m^k \cdot C_n^l \). As a result, the sum of all coefficients in the right hand of (5) is \[
\frac{1}{C_m^k \cdot C_n^l} \left( \frac{1 + 1 + \cdots + 1}{C_m^k \cdot C_n^l} \right) = 1,
\]
i.e., \( c_{i_1} + \cdots + c_{i_{k \times l}} = 1 \). Therefore \( c_{i_1} = \cdots = c_{i_{k \times l}} = \frac{1}{k \times l} \).

**Remark 1.** (4) is the generalization of related result in [7]. But the condition in Lemma is weaker, because the infimum of \( k \times l \) is \( \frac{m^2}{2} \times \frac{n^2}{2} \), which is less than \( \frac{m}{2} \times m \times n \), the half of the element number of the matrix \( X = (x_{ij})_{m \times n} \).

**Remark 2.** In (4), if \( r = 1 \), we have
\[
a_{X_i} = \frac{1}{C_m^k \cdot C_n^l} (a_{X_i \cap X_1} + a_{X_i \cap X_2} + \ldots + a_{X_i \cap X_{C_m^k \cdot C_n^l}}).
\]
In (4), if \( r \to 0 \), we have
\[
g_{X_i} = \left( g_{X_i \cap X_1} \cdot g_{X_i \cap X_2} \cdots \cdot g_{X_i \cap X_{C_m^k \cdot C_n^l}} \right)^{\frac{1}{C_m^k \cdot C_n^l}}.
\]

**Proof of Theorem.**

For a \( m \times n \) matrix \( B = (b_{ij})_{m \times n} \) with nonnegative entries \( b_{ij} \geq 0 \) and any \( k \times l \)–submatrix \( B_{ij} \) of \( B \), which is composed by the lines \((i_1, i_2, \ldots, i_k)\) of \( B \) and by the rows \((j_1, j_2, \ldots, j_l)\) of \( B \), by the partial order of \((i_1, i_2, \ldots, i_k)\) and \((j_1, j_2, \ldots, j_l)\), we get a subset chain \( B_1, B_2, \ldots, B_{C_m^k \cdot C_n^l} \).

In Lemma, let \( X \) be \( B \) and \( X_i, j = 1, 2, \ldots, C_m^k \cdot C_n^l \), be corresponding to \( B_1, B_2, \ldots, B_{C_m^k \cdot C_n^l} \), then (6) is
\[
a_{B_i} = \frac{1}{C_m^k \cdot C_n^l} (a_{B_i \cap B_1} + a_{B_i \cap B_2} + \ldots + a_{B_i \cap B_{C_m^k \cdot C_n^l}}),
\]
and (7) is
\[
g_{B_i} = \left( g_{B_i \cap B_1} \cdot g_{B_i \cap B_2} \cdots \cdot g_{B_i \cap B_{C_m^k \cdot C_n^l}} \right)^{\frac{1}{C_m^k \cdot C_n^l}}.
\]

By the arithmetic-geometric inequality, it follows that
\[
a_{B_i \cap B_j} \geq g_{B_i \cap B_j}, \quad j = 1, 2, \ldots, C_m^k \cdot C_n^l.
\]

From (8) and (10), we infer that
\[
a_{B_i} \geq \frac{1}{C_m^k \cdot C_n^l} (g_{B_i \cap B_1} + g_{B_i \cap B_2} + \ldots + g_{B_i \cap B_{C_m^k \cdot C_n^l}}).
\]
Therefore
\[
\left( \prod_{i=k, j=l}^{B_{ij} \subset B} a_{B_{ij}} \right)^{\frac{1}{C_m^k \cdot C_n^l}} = \left( \prod_{i=1}^{C_m^k \cdot C_n^l} a_{B_{ij}} \right)^{\frac{1}{C_m^k \cdot C_n^l}}
\]
\[
\geq \frac{1}{C_m^k \cdot C_n^l} \left( \prod_{i=1}^{C_m^k \cdot C_n^l} \left( \sum_{j=1}^{g_{B_i \cap B_j}} \right) \right)^{\frac{1}{C_m^k \cdot C_n^l}}.
\]

(11)

On the other hand, using the discrete case of Hölder’s inequality in the form
\[
\left( \sum_{k=1}^{n} \left( \prod_{j=1}^{m} x_{jk} \right)^{\frac{1}{m}} \right)^{\frac{1}{m}} \leq \left( \prod_{j=1}^{m} \left( \sum_{k=1}^{n} x_{jk} \right)^{\frac{1}{m}} \right)^{\frac{1}{m}},
\]
where \( n, m \) are positive integers and \( x_{jk} > 0(j, k = 1, 2, ..., m) \), we obtain
\[
\left( \prod_{i=1}^{C_m^k \cdot C_n^l} \left( \sum_{j=1}^{g_{B_i \cap B_j}} \right) \right)^{\frac{1}{C_m^k \cdot C_n^l}} \geq \sum_{i=1}^{C_m^k \cdot C_n^l} \left( \prod_{j=1}^{g_{B_i \cap B_j}} \right)^{\frac{1}{C_m^k \cdot C_n^l}}.
\]

(12)

Combining (9), (11) and (12), it follows that
\[
\left( \prod_{i=k, j=l}^{B_{ij} \subset B} a_{B_{ij}} \right)^{\frac{1}{C_m^k \cdot C_n^l}} \geq \frac{1}{C_m^k \cdot C_n^l} \sum_{i=1}^{C_m^k \cdot C_n^l} \left( \prod_{j=1}^{g_{B_i \cap B_j}} \right)^{\frac{1}{C_m^k \cdot C_n^l}}
\]
\[
= \frac{1}{C_m^k \cdot C_n^l} \sum_{i=1}^{C_m^k \cdot C_n^l} g_{B_i} = \frac{1}{C_m^k \cdot C_n^l} \left( \sum_{i=k, j=l}^{B_{ij} \subset B} g_{B_{ij}} \right)
\]
which is just the inequality (2). □

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