ON HADAMARD TYPE INTEGRAL INEQUALITIES FOR NONCONVEX FUNCTIONS

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Abstract. In this paper, we extend some estimates of the right and left side of a Hermite-Hadamard type inequality for nonconvex functions whose derivatives absolute values are $\varphi$-convex and quasi-$\varphi$-convex was introduced by Noor in [3].

1. Introduction

It is well known that if $f$ is a convex function on the interval $I = [a, b]$ and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$  

which is known as the Hermite-Hadamard inequality for the convex functions.

Both inequalities hold in the reversed direction if $f$ is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1], [2], [8]-[10], [12], [14]-[20]).

In [8] some inequalities of Hermite-Hadamard type for differentiable convex mappings connected with the left part of (1.1) were proved by using the following lemma:

Lemma 1.1. Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on $I^o$, $a, b \in I^o$ ($I^o$ is the interior of $I$) with $a < b$. If $f' \in L ([a, b])$, then we have

$$\frac{1}{b-a} \int_{a}^{b} f(x) \, dx - f\left(\frac{a+b}{2}\right) = (b-a) \left[ \int_{0}^{\frac{b}{2}} tf'(ta + (1-t)b) \, dt + \int_{\frac{b}{2}}^{1} (t-1) f'(ta + (1-t)b) \, dt \right].$$  

\hspace{1cm}
One more general result related to \[1,2\] was established in \[3\]. The main result in \[8\] is as follows:

**Theorem 1.1.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \), be a differentiable mapping on \( I^0 \), \( a, b \in I \) with \( a < b \). If the mapping \( |f'| \) is convex on \( [a, b] \), then

\[
\left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)| + |f'(b)|}{2} \right).
\]

In \[1\], Dragomir and Agarwal established the following results connected with the right part of \[\text{(1.1)}\] as well as to apply them for some elementary inequalities for real numbers and numerical integration:

**Theorem 1.2.** Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \), and \( f' \in L(a, b) \). If the mapping \( |f'| \) is convex on \( [a, b] \), then the following inequality holds:

\[
\left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) dx \leq (b-a) \left( \frac{|f'(a)| + |f'(b)|}{8} \right).
\]

In \[12\], Pearce and Pečarić proved the following theorem:

**Theorem 1.3.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \), be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If the mapping \( |f'|^q \) is convex on \( [a, b] \) for some \( q \geq 1 \), then

\[
\left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) dx \leq (b-a) \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.
\]

and

\[
\left| \frac{1}{b-a} \int_a^b f(x) dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.
\]

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function \( f : [a, b] \subset \mathbb{R} \to \mathbb{R} \) is said quasi-convex on \( [a, b] \) if

\[
f(tx + (1-t)y) \leq \sup \{ f(x), f(y) \}
\]

for all \( x, y \in [a, b] \) and \( t \in [0, 1] \). Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see \[11\]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function \( f : [a, b] \to \mathbb{R} \). Ion in \[11\] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some quasi-convex functions are involved. The main results of \[11\] are given by the following theorems.

**Theorem 1.4.** Assume \( a, b \in \mathbb{R} \) with \( a < b \) and \( f : [a, b] \to \mathbb{R} \) is a differentiable function on \( (a, b) \). If \( |f'| \) is quasi-convex on \( [a, b] \), then the following inequality holds

\[
\left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{(b-a)}{4} \sup \{ |f'(a)|, |f'(b)| \}.
\]
Theorem 1.5. Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on $(a, b)$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is quasi-convex on $[a, b]$, then the following inequality holds
\[
\left| f(a) + f(b) \right| - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \left( \frac{b-a}{2(p+1)^{1/p}} \sup \left\{ |f'(a)|^{p/p-1}, |f'(b)|^{p/p-1} \right\} \right)^{1/p}.
\]

Convexity plays a central and fundamental role in mathematical finance, economics, engineering, management sciences and optimization theory. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of $\varphi$-convex functions introduced by Noor in [3]. In [3] and [7], the authors have studied the basic properties of the $\varphi$-convex functions. It is well-known that the $\varphi$-convex functions and $\varphi$-sets may not be convex functions and convex sets. This class of nonconvex functions include the classical convex functions and its various classes as special cases. For some recent results related to this nonconvex functions, see the papers 3-7.

2. Preliminaries

Let $f, \varphi : K \to \mathbb{R}^n$, where $K$ is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. First of all, we recall the following well know results and concepts, which are mainly due to Noor and Noor [3] and Noor [7] as follows:

Definition 2.1. Let $u, v \in K$. Then the set $K$ is said to be $\varphi$-convex at $u$ with respect to $\varphi$, if
\[
u + t\varphi(v - u) \in K, \; \forall u, v \in K, \; t \in [0, 1].
\]

Remark 2.1. We would like to mention that the Definition 2.1 of a $\varphi$-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point $u$ which is contained in $K$ and only if, $\varphi = 0$, and consequently $\varphi$-convexity reduces to convexity. Thus, it is true that every convex set is also a $\varphi$-convex set, but the converse is not necessarily true, see [3, 7] and the references therein.

Definition 2.2. The function $f$ on the $\varphi$-convex set $K$ is said to be $\varphi$-convex with respect to $\varphi$, if
\[
f \left( u + t \varphi(v - u) \right) \leq (1 - t) f(u) + tf(v), \; \forall u, v \in K, \; t \in [0, 1].
\]

The function $f$ is said to be $\varphi$-concave if and only if $-f$ is $\varphi$-convex. Note that every convex function is a $\varphi$-convex function, but the converse is not true. For example, the function $f : \mathbb{R} \to \mathbb{R}, f(x) = -|x|$ is not a convex function, but $f(x) = -|x|$ is a $\varphi$-convex with respect to $\varphi$ where
\[
\varphi(v, u) = \begin{cases} 
2k\pi, & u, v \geq 0, \; k \in \mathbb{Z} \\
 k\pi, & u, v < 0, \; k \in \mathbb{Z}.
\end{cases}
\]

Definition 2.3. The function $f$ on the $\varphi$-convex set $K$ is said to be logarithmic $\varphi$-convex with respect to $\varphi$, such that
\[
f \left( u + t \varphi(v - u) \right) \leq \left( f(u) \right)^{1-t} \left( f(v) \right)^t, \; u, v \in K, \; t \in [0, 1] \]

where $f(.) > 0$. 


Now we define a new definition for quasi-$\varphi$–convex functions as follows:

**Definition 2.4.** The function $f$ on the quasi $\varphi$–convex set $K$ is said to be quasi $\varphi$–convex with respect to $\varphi$, if

$$f\left(u + t e^{i\varphi} (v - u)\right) \leq \max\{f(u), f(v)\}.$$ 

From the above definitions, we have

$$f\left(u + t e^{i\varphi} (v - u)\right) \leq (f(u))^{1-t} (f(v))^t \leq (1-t) f(u) + tf(v) \leq \max\{f(u), f(v)\}.$$ 

In [5], Noor proved the Hermite-Hadamard inequality for the $\varphi$–convex functions as follows:

**Theorem 2.1.** Let $f : K = [a, a + e^{i\varphi} (b - a)] \rightarrow (0, \infty)$ be a $\varphi$–convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K^0$ with $a < a + e^{i\varphi} (b - a)$ and $0 \leq \varphi \leq \frac{\pi}{2}$. Then the following inequality holds:

$$f\left(\frac{2a + e^{i\varphi} (b - a)}{2}\right) \leq \frac{1}{e^{i\varphi} (b - a)} \int_a^{a + e^{i\varphi} (b - a)} f(x) \, dx \leq \frac{f(a) + f\left(a + e^{i\varphi} (b - a)\right)}{2} \leq \frac{f(a) + f(b)}{2}.$$ 

This inequality can easily show that using the $\varphi$–convex function’s definition and $f\left(a + e^{i\varphi} (b - a)\right) < f(b)$.

In this article, using functions whose derivatives absolute values are $\varphi$-convex and quasi-$\varphi$-convex, we obtained new inequalities related to the right and left side of Hermite-Hadamard inequality. In particular if $\varphi = 0$ is taken as, our results obtained reduce to the Hermite-Hadamard type inequality for classical convex functions.

Throughout this study, we always assume that $K = [a, a + e^{i\varphi} (b - a)]$ and $0 \leq \varphi \leq \frac{\pi}{2}$ the interval, unless otherwise specified.

We shall start with the following refinements of the Hermite-Hadamard inequality for $\varphi$–convex functions. Firstly, we give the following results connected with the right part of (2.1):

**Theorem 2.2.** Let $f : K \rightarrow (0, \infty)$ be a differentiable mapping on $K^0$. If $|f'|$ is $\varphi$–convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K$ with $a < a + e^{i\varphi} (b - a)$. Then, the following inequality holds:

$$\left|\frac{1}{e^{i\varphi} (b - a)} \int_a^{a + e^{i\varphi} (b - a)} f(x) \, dx - \frac{f(a) + f\left(a + e^{i\varphi} (b - a)\right)}{2}\right|$$

$$\leq \frac{e^{i\varphi} (b - a)}{8} \left[|f'(a)| + |f'(b)|\right].$$
Proof. Since $K$ is $\varphi-$convex with respect to $\varphi$, for every $t \in [0, 1]$, we have $a + te^{i\varphi}(b - a) \in K$. Integrating by parts implies that

$$
\int_0^1 (1 - 2t)f'(a + te^{i\varphi}(b - a))dt
$$

(2.3)

$$
= \left[ \frac{(1 - 2t)f(a + te^{i\varphi}(b - a))}{e^{i\varphi}(b - a)} \right]_0^1 + \frac{2}{e^{i\varphi}(b - a)} \int_0^1 f(a + te^{i\varphi}(b - a))dt
$$

$$
= -f(a) + f(a + e^{i\varphi}(b - a)) + \frac{2}{e^{2i\varphi}(b - a)^2} \int_a^{a + e^{i\varphi}(b - a)} f(x)dx
$$

By $\varphi-$convexity of $|f'|$ and (2.3), we have

$$
\left| \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x)dx - \frac{f(a) + f(a + e^{i\varphi}(b - a))}{2} \right|
$$

$$
= \frac{e^{i\varphi}(b - a)}{2} \left| \int_0^1 (1 - 2t)f'(a + te^{i\varphi}(b - a))dt \right|
$$

$$
\leq \frac{e^{i\varphi}(b - a)}{2} \int_0^1 |1 - 2t| |f'(a) + t|f'(b)| dt
$$

$$
= \frac{e^{i\varphi}(b - a)}{8} \left( |f'(a)| + |f'(b)| \right).
$$

which completes the proof. □

Theorem 2.3. Let $f : K \rightarrow (0, \infty)$ be a differentiable mapping on $K^0$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/(p-1)}$ is $\varphi-$convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. Then, the following inequality holds:

$$
\left| \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x)dx - \frac{f(a) + f(a + e^{i\varphi}(b - a))}{2} \right|
$$

(2.4)

$$
\leq \frac{e^{i\varphi}(b - a)}{2(p + 1)^{p-1}2^p} \left( \frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}}$$
Proof. From Hölder’s inequality and by using (2.3) in the proof of Theorem 2.2, we have

\[
\left| \frac{1}{e^{i\varphi(b-a)}} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(a) + f(a + e^{i\varphi}(b-a)) \right| \leq \frac{e^{i\varphi}(b-a)}{2} \int_0^1 |1 - 2t| |f'(a + te^{i\varphi}(b-a))| dt
\]

\[
\leq \frac{e^{i\varphi}(b-a)}{2} \left( \int_0^1 (1 - 2t)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(a + te^{i\varphi}(b-a))|^{\frac{2}{p}} dt \right)^{\frac{p-1}{p}}
\]

\[
\leq \frac{e^{i\varphi}(b-a)}{2} \left( \int_0^1 (1 - 2t)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \left( \int_0^1 [(1 - t) |f'(a)|^{\frac{2}{p}} + t |f'(b)|^{\frac{2}{p}}] dt \right)^{\frac{p-1}{p}}
\]

\[
= \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{2}}} \left( \frac{|f'(a)|^{\frac{2}{p}} + |f'(b)|^{\frac{2}{p}}}{2} \right)^{\frac{p-1}{p}}.
\]

This implies inequality (2.4). □

Now, we give the following results connected with the left part of (2.1):

**Theorem 2.4.** Under the assumptions of Theorem 2.2, then the following inequality holds:

\[
\left| \frac{1}{e^{i\varphi(b-a)}} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(a) + f(a + e^{i\varphi}(b-a)) \right| \leq \frac{e^{i\varphi}(b-a)}{8} \left[ |f'(a)| + |f'(b)| \right].
\]

**Proof.** Since \( K \) is \( \varphi \)-convex with respect to \( \varphi \), for every \( t \in [0, 1] \), we have \( a + te^{i\varphi}(b-a) \in K \). Integrating by parts implies that

\[
\int_0^t tf'(a + te^{i\varphi}(b-a))dt + \int_{\frac{1}{2}}^1 (t - 1)f'(a + te^{i\varphi}(b-a))dt
\]

\[
= \left[ \frac{tf(a + te^{i\varphi}(b-a))}{e^{i\varphi(b-a)}} \right]_0^t + \left[ \frac{(t-1)f(a + te^{i\varphi}(b-a))}{e^{i\varphi(b-a)}} \right]_{\frac{1}{2}}^1
\]

\[
- \frac{1}{e^{i\varphi(b-a)}} \int_0^1 f(a + te^{i\varphi}(b-a))dt
\]

\[
= \frac{1}{e^{i\varphi(b-a)}} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{1}{e^{i\varphi(b-a)^2}} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx.
\]
By $\varphi$–convexity of $|f'|$, we have
\[
\left| \frac{1}{e^{i\varphi(b-a)}} \int_{a}^{a+e^{i\varphi(b-a)}} f(x)dx - f\left( \frac{2a + e^{i\varphi(b-a)}}{2} \right) \right| \\
\leq e^{i\varphi(b-a)} \left[ \int_{0}^{1/2} t |f'(a + te^{i\varphi}(b-a))| dt + \int_{1/2}^{1} (1-t) |f'(a + te^{i\varphi}(b-a))| dt \right] \\
\leq e^{i\varphi(b-a)} \left[ \int_{0}^{1/2} t [(1-t) |f'(a)| + t |f'(b)|] dt + \int_{1/2}^{1} (1-t) [(1-t) |f'(a)| + t |f'(b)|] dt \right] \\
\leq e^{i\varphi(b-a)} \left[ \frac{|f'(a)| + |f'(b)|}{8} \right].
\]
The proof is completed. \qed

**Theorem 2.5.** Under the assumptions of Theorem 2.3, then the following inequality holds:
\[
\left| \frac{1}{e^{i\varphi(b-a)}} \int_{a}^{a+e^{i\varphi(b-a)}} f(x)dx - f\left( \frac{2a + e^{i\varphi(b-a)}}{2} \right) \right| \\
\leq \frac{e^{i\varphi(b-a)}}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} \left[ \left( 3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left( |f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right].
\]

**Proof.** From Hölder’s inequality and by using (2.6), we have
\[
\left| \frac{1}{e^{i\varphi(b-a)}} \int_{a}^{a+e^{i\varphi(b-a)}} f(x)dx - f\left( \frac{2a + e^{i\varphi(b-a)}}{2} \right) \right| \\
\leq e^{i\varphi(b-a)} \left[ \int_{0}^{1/2} t |f'(a + te^{i\varphi}(b-a))| dt + \int_{1/2}^{1} (1-t) |f'(a + te^{i\varphi}(b-a))| dt \right] \\
\leq e^{i\varphi(b-a)} \left( \int_{0}^{1/2} t^{p} dt \right)^{\frac{1}{p}} \left( \int_{0}^{1/2} |f'(a + te^{i\varphi}(b-a))|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\
+ e^{i\varphi(b-a)} \left( \int_{1/2}^{1} (1-t)^{p} dt \right)^{\frac{1}{p}} \left( \int_{1/2}^{1} |f'(a + te^{i\varphi}(b-a))|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\
\leq \frac{e^{i\varphi(b-a)}}{2^{1+\frac{p}{2}}(p+1)^{\frac{1}{p}}} \left[ \int_{0}^{1/2} \left[ (1-t) |f'(a)|^{\frac{p}{p-1}} + t |f'(b)|^{\frac{p}{p-1}} \right] dt \right]^{\frac{p-1}{p}} \\
+ \frac{e^{i\varphi(b-a)}}{2^{1+\frac{p}{2}}(p+1)^{\frac{1}{p}}} \left[ \int_{1/2}^{1} \left[ (1-t) |f'(a)|^{\frac{p}{p-1}} + t |f'(b)|^{\frac{p}{p-1}} \right] dt \right]^{\frac{p-1}{p}} \\
= \frac{e^{i\varphi(b-a)}}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} \left[ \left( 3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left( |f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right]
\]
which completes the proof. \qed
Theorem 2.6. Under the assumptions of Theorem 2.3. Then, the following inequality holds:

\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right|
\]

(2.8)

\[ \leq \frac{e^{i\varphi}(b-a)}{4} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} \left( |f'(a)| + |f'(b)| \right). \]

Proof. We consider the inequality (2.7) i.e

\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right|
\]

(2.9)

\[ \leq \frac{e^{i\varphi}(b-a)}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} \left( 3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left( |f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}. \]

Let \( a_1 = 3 |f'(a)|^{\frac{p}{p-1}}, b_1 = |f'(b)|^{\frac{p}{p-1}}, a_2 = |f'(a)|^{\frac{p}{p-1}}, b_2 = 3 |f'(b)|^{\frac{p}{p-1}} \). Here

0 < (p - 1)/p < 1, for p > 1. Using the fact that,

\[
\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s,
\]

for \( 0 \leq s < 1 \), \( a_1, a_2, ..., a_n \geq 0, b_1, b_2, ..., b_n \geq 0 \), we obtain

\[
\left| \frac{e^{i\varphi}(b-a)}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} \left( 3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \left( |f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right|
\]

\[ \leq \frac{e^{i\varphi}(b-a)}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} (3 + 1) \left( |f'(a)| + |f'(b)| \right) \]

\[ \leq \frac{e^{i\varphi}(b-a)}{16} \left( \frac{4}{p+1} \right)^{\frac{1}{p}} 4 \left( |f'(a)| + |f'(b)| \right)
\]

which completed proof. \( \square \)

Theorem 2.7. Let \( f : \rightarrow (0, \infty) \) be a differentiable mapping on \( K^0 \). Assume \( q \in \mathbb{R} \) with \( q \geq 1 \). If \( |f'|^q \) is \( \varphi \)-convex function on the interval of real numbers \( K^0 \) (the interior of \( K \)) and \( a, b \in K \) with \( a < a + e^{i\varphi}(b - a) \). Then the following inequality holds:

\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right|
\]

\[ \leq \frac{e^{i\varphi}(b-a)}{8} \left[ \left( \frac{2|f'(a)|^q + |f'(b)|^q}{3} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 2|f'(b)|^q}{3} \right)^{\frac{1}{q}} \right]. \]
Proof. From Hölder’s inequality and by using (2.6), we have
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right| \\
\leq e^{i\varphi}(b-a) \left[ \int_0^{\frac{1}{2}} t |f'(a + te^{i\varphi}(b-a))| dt + \int_{\frac{1}{2}}^1 (1-t) |f'(a + te^{i\varphi}(b-a))| dt \right] \\
\leq e^{i\varphi}(b-a) \left( \int_0^{\frac{1}{2}} t \left( \int_0^{\frac{1}{2}} t |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right) \\
+ e^{i\varphi}(b-a) \left( \int_{\frac{1}{2}}^1 (1-t) dt \left( \int_{\frac{1}{2}}^1 (1-t) |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right) \\
\leq \frac{e^{i\varphi}(b-a)}{8^{\frac{1}{q}}} \left[ \int_0^{\frac{1}{2}} t \left[(1-t) |f'(a)|^q + t |f'(b)|^q \right] dt \right]^{\frac{1}{q}} \\
+ \frac{e^{i\varphi}(b-a)}{8^{\frac{1}{q}}} \left[ \int_{\frac{1}{2}}^1 (1-t) \left[(1-t) |f'(a)|^q + t |f'(b)|^q \right] dt \right]^{\frac{1}{q}} \\
= \frac{e^{i\varphi}(b-a)}{8} \left[ \left( \frac{2 |f'(a)|^q + |f'(b)|^q}{3} \right)^{\frac{1}{q}} + \left( |f'(a)|^q + 2 |f'(b)|^q \right)^{\frac{1}{q}} \right].
\]

The proof is completed. \(\square\)

**Theorem 2.8.** Under the assumptions of Theorem 2.7. Then the following inequality holds:
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right| \\
\leq \frac{e^{i\varphi}(b-a)}{8} \left( \frac{2^{\frac{1}{q}} + 1}{3^{\frac{1}{q}}} \right) [||f'(a)|| + ||f'(b)||].
\]

**Proof.** We consider the inequality (2.7) i.e
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) \right| \\
\leq \frac{e^{i\varphi}(b-a)}{8} \left[ \left( \frac{2 |f'(a)|^q + |f'(b)|^q}{3} \right)^{\frac{1}{q}} + \left( |f'(a)|^q + 2 |f'(b)|^q \right)^{\frac{1}{q}} \right].
\]

Let \(a_1 = 2 |f'(a)|^q / 3, b_1 = |f'(b)|^q / 3, a_2 = |f'(a)|^q / 3, b_2 = 2 |f'(b)|^q / 3.\) Here \(0 < 1/q < 1,\) for \(q \geq 1.\) Using the fact that
\[
\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s,
\]

for \((0 \leq s < 1), a_1, a_2, ..., a_n \geq 0, b_1, b_2, ..., b_n \geq 0,\) we obtain
\[
\frac{e^{i\varphi}(b-a)}{8} \left[ \left( \frac{2|f'(a)|^q + |f'(b)|^q}{3} \right)^{\frac{1}{q}} + \left( \frac{|f'(a)|^q + 2|f'(b)|^q}{3} \right)^{\frac{1}{q}} \right] \\
\leq \frac{e^{i\varphi}(b-a)}{8} \left[ \left( \frac{2^{\frac{1}{q}} + 1}{3^{\frac{1}{q}}} \right) |f'(a)| + |f'(b)| \right].
\]
This concludes the proof. \(\square\)

3. HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI \(\varphi\)-CONVEX FUNCTIONS

In this section, we prove some new inequalities of Hermite-Hadamard for quasi \(\varphi\)-convex function as follows:

**Theorem 3.1.** Let \(f : K \to (0, \infty)\) be a differentiable mapping on \(K^0\). If \(|f'|\) is quasi \(\varphi\)-convex function on the interval of real numbers \(K^0\) (the interior of \(K\)) and \(a, b \in K\) with \(a < a + e^{i\varphi}(b - a)\). Then the following inequality holds:
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2} \right| \\
\leq \frac{e^{i\varphi}(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}.
\]

**Proof.** By quasi \(\varphi\)-convexity of \(|f'|\) and by using \((2.3)\), we have
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2} \right| \\
\leq \frac{e^{i\varphi}(b-a)}{2} \int_0^1 |(1 - 2t)| |f'(a + te^{i\varphi}(b-a))| dt \\
\leq \frac{e^{i\varphi}(b-a)}{2} \max\{|f'(a)|, |f'(b)|\} \int_0^1 |1 - 2t| dt \\
\leq \frac{e^{i\varphi}(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}
\]
which completes the proof. \(\square\)

**Theorem 3.2.** Let \(f : K \to (0, \infty)\) be a differentiable mapping on \(K^0\). Assume \(p \in \mathbb{R}\) with \(p > 1\). If \(|f'|^p/p-1\) is quasi \(\varphi\)-convex function on the interval of real numbers \(K^0\) (the interior of \(K\)) and \(a, b \in K\) with \(a < e^{i\varphi}(b - a)\). Then the following inequality holds:
\[
\left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2} \right| \\
\leq \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{p-1}{p}}} \left[ \max\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\} \right]^{\frac{p-1}{p}}.
\]
Proof. By quasi $\varphi-$convexity of $|f'|^{p/p-1}$ and by using (2.3), we have

$$\left|\frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2}\right|$$

$$\leq \frac{e^{i\varphi}(b-a)}{2} \int_{0}^{1} |(1-2t)| \left|f'(a + te^{i\varphi}(b-a))\right| dt$$

$$\leq \frac{e^{i\varphi}(b-a)}{2} \left(\int_{0}^{1} |(1-2t)| t dt\right)^{\frac{p}{2}} \left(\int_{0}^{1} |f'(a + te^{i\varphi}(b-a))| \frac{p}{2} dt\right)^{\frac{2}{p}}$$

$$\leq \frac{e^{i\varphi}(b-a)}{2} \left(\int_{0}^{1} |(1-2t)| t dt\right)^{\frac{p}{2}} \left(\int_{0}^{1} \max\{|f'(a)|^{\frac{p}{2}}, |f'(b)|^{\frac{p}{2}}\} dt\right)^{\frac{2}{p}}$$

$$\leq \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{p}{2}}} \sqrt[p]{\max\{|f'(a)|^{\frac{p}{2}}, |f'(b)|^{\frac{p}{2}}\}}$$

which completes the proof.

Theorem 3.3. Let $f : K \to (0, \infty)$ be a differentiable mapping on $K^0$. If $|f'|$ is quasi $\varphi-$convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then the following inequality holds:

$$\left|\frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2}\right|$$

$$\leq \frac{e^{i\varphi}(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}.$$ 

Proof. By quasi $\varphi-$convexity of $|f'|$ and by using (2.4), we have

$$\left|\frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{2a + e^{i\varphi}(b-a)}{2}\right|$$

$$\leq e^{i\varphi}(b-a) \left[\int_{0}^{\frac{1}{2}} t |f'(a + te^{i\varphi}(b-a))| dt + \int_{0}^{1} (1-t) |f'(a + te^{i\varphi}(b-a))| dt\right]$$

$$\leq e^{i\varphi}(b-a) \max\{|f'(a)|, |f'(b)|\} \left[\int_{0}^{\frac{1}{2}} t dt + \int_{\frac{1}{2}}^{1} (1-t) dt\right]$$

$$\leq \frac{e^{i\varphi}(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}.$$ 

This concludes the proof.

Theorem 3.4. Let $f : K \to (0, \infty)$ be a differentiable mapping on $K^0$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is quasi $\varphi-$convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K$ with $a < e^{i\varphi}(b-a)$. Then the following inequality holds:

$$\left|\frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a) + f(a + te^{i\varphi}(b-a))}{2}\right|$$

$$\leq \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{p}{2}}} \sqrt[p]{\max\{|f'(a)|^{\frac{p}{2}}, |f'(b)|^{\frac{p}{2}}\}^{\frac{p-1}{p}}}.$$
Proof. By quasi $\varphi-$convexity of $|f'|^{p/p-1}$ and by using (2.3), we have
\[
\frac{1}{e^{i\varphi}(b-a)} \int_{a}^{a+e^{i\varphi}(b-a)} f(x)dx = \frac{f(a + te^{i\varphi}(b-a))}{2}
\]
\[
\leq e^{i\varphi}(b-a) \left[ \int_{0}^{\frac{1}{q}} t \left| f'(a + te^{i\varphi}(b-a)) \right| dt + \int_{\frac{1}{q}}^{1} (1-t) \left| f'(a + te^{i\varphi}(b-a)) \right| dt \right]
\]
\[
\leq e^{i\varphi}(b-a) \left( \int_{0}^{\frac{1}{q}} t^{p} dt \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{1}{q}} \left| f'(a + te^{i\varphi}(b-a)) \right|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}}
\]
\[
+ e^{i\varphi}(b-a) \left( \int_{\frac{1}{q}}^{1} (1-t)^{p} dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{q}}^{1} \left| f'(a + te^{i\varphi}(b-a)) \right|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}}
\]
\[
\leq \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}} \left[ \max\{\left| f'(a)\right|^{\frac{p}{p-1}}, \left| f'(b)\right|^{\frac{p}{p-1}}\} \right]^{\frac{p-1}{p}}
\]
which completes the proof. \qed

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