Properties of the Scalar Mesons $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$

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Abstract

In the three-state mixing framework, considering the possible glueball components of $\eta$ and $\eta'$, we investigate the hadronic decays of $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ into two pseudoscalar mesons. The quarkonia-glueball content of the three states is determined from the fit to the new data presented by the WA102 Collaboration. We find that these data are insensitive to the possible glueball components of $\eta$ and $\eta'$. Furthermore, we discuss some properties of the mass matrix describing the mixing of the isoscalar scalar mesons.

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1 Introduction

Recently, based on the mass matrix motivated by Ref.\cite{1}, Ref.\cite{2} has investigated the implications of the new data presented by the WA102 Collaboration\cite{3} for the glueball-quarkonia content of $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ ($f_1$, $f_2$ and $f_3$ respectively stand for $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ below). We propose that some points can be improved on. First, in the reduced partial width $\Gamma(f_i \to \eta \eta')$ in Ref.\cite{2}, the sign of the contribution of the diagram iv) of Fig. 1 was flipped, it should be negative. The flipped sign actually arose from a typo in the equation (A5) of Ref.\cite{4} where the $\lambda$ should all read $1/\lambda^0$. Second, the mixing angle of $\eta$ and $\eta'$ was determined to be a very small value of $-5 \pm 4^\circ$\cite{2} which is inconsistent with the value of $-15.5 \pm 1.5^\circ$ determined from a rather exhaustive and up-to-date analysis of data including strong decays of tensor and higher spin mesons, electromagnetic decays of vector and pseudoscalar mesons, and the decays of $J/\psi$\cite{5}. Also, the possibility that the glueball components exist in $\eta$ and $\eta'$ was not considered in Ref.\cite{2}. Refs.\cite{6} already suggested that the $\eta$ and $\eta'$ wave functions need glueball components.

In this work, instead of the mass matrix in which one should confront with the confused mass level order about the masses of the bare states $(u\bar{u} + d\bar{d})/\sqrt{2}, s\bar{s}$ and glueball\cite{1, 2, 3}, we shall adopt another mixing scheme which can be related to the mass matrix to describe the mixing of $f_1$, $f_2$ and $f_3$, then we can discuss some properties of the mass matrix based on our preferred results. In addition, we shall consider the possibility that the glueball components exist in $\eta$ and $\eta'$ when we investigate the hadronic decays of $f_1$, $f_2$ and $f_3$ into two pseudoscalar mesons, and check whether these new data are sensitive to the possible glueball components of $\eta$ and $\eta'$ or not.

\footnote{We wish to thank F.E. Close and A. Kirk for useful discussions on this matter.}
2 Mixing scheme and decays

Based on three Euler angles $\theta_1$, $\theta_2$ and $\theta_3$, the mixing of $f_1$, $f_2$ and $f_3$ can be described as:

$$
\begin{pmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{pmatrix}
= \begin{pmatrix}
    a_8 & a_1 & a_g \\
    b_8 & b_1 & b_g \\
    c_8 & c_1 & c_g
\end{pmatrix}
\begin{pmatrix}
|8\rangle \\
|1\rangle \\
|G\rangle
\end{pmatrix}
= \begin{pmatrix}
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2 \\
    x_3 & y_3 & z_3
\end{pmatrix}
\begin{pmatrix}
|N\rangle \\
|S\rangle \\
|G\rangle
\end{pmatrix}
$$

(1)

with

$$
\begin{pmatrix}
    a_8 & a_1 & a_g \\
    b_8 & b_1 & b_g \\
    c_8 & c_1 & c_g
\end{pmatrix}
= \begin{pmatrix}
    c_1 c_2 c_3 - s_1 s_3 & -c_1 c_2 s_3 - s_1 c_3 & c_1 s_2 \\
    s_1 c_2 c_3 + s_1 s_3 & -s_1 c_2 s_3 + c_1 c_3 & s_1 s_2 \\
    -s_2 c_3 & s_2 s_3 & c_2
\end{pmatrix}
$$

(2)

$$
\begin{pmatrix}
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2 \\
    x_3 & y_3 & z_3
\end{pmatrix}
= \begin{pmatrix}
    a_8 & a_1 & a_g \\
    b_8 & b_1 & b_g \\
    c_8 & c_1 & c_g
\end{pmatrix}
\begin{pmatrix}
    \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & 0 \\
    \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & 0 \\
    0 & 0 & 1
\end{pmatrix}
$$

(3)

where $|8\rangle = |u\pi + d\bar{\pi} - 2s\Sigma|/\sqrt{6}$, $|1\rangle = |u\pi + d\bar{\pi} + s\Sigma|/\sqrt{3}$, $|N\rangle = |u\pi + d\bar{\pi}|/\sqrt{2}$, $|S\rangle = |s\Sigma|$, $|G\rangle = |gg\rangle$; $c_1$ ($c_2$, $c_3$) $\equiv \cos \theta_1$ ($\cos \theta_2$, $\cos \theta_3$), $s_1$ ($s_2$, $s_3$) $\equiv \sin \theta_1$ ($\sin \theta_2$, $\sin \theta_3$), and $-180^\circ \leq \theta_1 \leq 180^\circ$, $0^\circ \leq \theta_2 \leq 180^\circ$, $-180^\circ \leq \theta_3 \leq 180^\circ$. One advantage of this mixing model is only 3 unknown parameters with the definite changing ranges.

Considering that the glueball components possibly exist in the final isoscalar pseudoscalar mesons $|G\rangle$, for the hadronic decays of $f_i$ (here and below, $i = 1, 2, 3$) into pseudoscalar meson pairs, we consider the following coupling modes as indicated in Fig. 1: i) the coupling of the $q\bar{q}$ components of $f_i$ to the final pseudoscalar meson pairs, ii) the coupling of the glueball components of $f_i$ to the final pseudoscalar meson pairs via $qq\bar{q}\bar{q}$ intermediate states, iii) the coupling of the glueball components of $f_i$ to the glueball components of the final isoscalar pseudoscalar mesons, and iv) the coupling of the glueball components of $f_i$ to the $q\bar{q}$ components of the final isoscalar pseudoscalar meson pairs. Based on these coupling modes, the effective Hamiltonian describing the hadronic decays of $f_i$ into two pseudoscalar mesons can be written.
as \[H_{ef} = g_1 \text{Tr}(fF P_F P_F) + g_2 f_G \text{Tr}(P_F P_F) + g_3 f_G P_G + g_4 f_G \text{Tr}(P_F) \text{Tr}(P_F),\] (4)

where \(g_1, g_2, g_3\) and \(g_4\) describe the effective coupling strengths of the coupling modes i), ii), iii) and iv), respectively. \(f_G\) and \(P_G\) are SU(3) flavor singlets describing the glueball components of \(f_i\) and the final isoscalar pseudoscalar mesons, respectively. \(f_F\) and \(P_F\) can be given by

\[f_G = \sum_i z_i f_i, \quad P_G = \sum_j z_j j,\] (5)

where \(z_j\) denotes the glueball content of \(j\) (here and below \(j = \eta, \eta'\)). \(f_F\) and \(P_F\) are \(3 \times 3\) flavor matrixes describing the \(q\bar{q}\) components of \(f_i\) and the final pseudoscalar mesons, respectively. \(f_F\) can be written as

\[f_F = \begin{pmatrix}
\sum_i \frac{x_i}{\sqrt{2}} f_i & 0 & 0 \\
0 & \sum_i \frac{x_i}{\sqrt{2}} f_i & 0 \\
0 & 0 & \sum_i y_i f_i
\end{pmatrix},\] (6)

\(P_F\) can be written as

\[P_F = \begin{pmatrix}
\frac{1}{\sqrt{2}} \pi^0 + \sum_j \frac{x_j}{\sqrt{2}} j & \pi^+ & K^+ \\
\pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \sum_j \frac{x_j}{\sqrt{2}} j & K^0 \\
K^- & \bar{K}^0 & \sum_j y_j j
\end{pmatrix},\] (7)

where \(x_j\) and \(y_j\) denote the \((u\bar{u} + d\bar{d})/\sqrt{2}, s\bar{s}\) contents of \(j\), respectively, and they satisfy \(x_j^2 + y_j^2 + z_j^2 = 1\).

Introducing \(g_2/g_1 = r_1, g_3/g_1 = r_2, g_4/g_1 = r_3\), from Eqs. (4)\textasciitilde(7), one can obtain

\[\Gamma(f_i \to \pi\pi) = 3g_1^2 q_{i\pi\pi} |x_i + \sqrt{2} z_i r_1|^2,\] (8)

\[\Gamma(f_i \to K\bar{K}) = g_1^2 q_{iK\bar{K}} |x_i + \sqrt{2} y_i + 2\sqrt{2} z_i r_1|^2,\] (9)

\[\Gamma(f_i \to \eta\eta) = g_1^2 q_{i\eta\eta} |x_i^2 x_i + \sqrt{2} y_i^2 y_i + \sqrt{2}(x_i^2 + y_i^2) z_i r_1 + \sqrt{2} z_i^2 z_i r_2 + (2\sqrt{2} x_i^2 + \sqrt{2} y_i^2 + 4x_i y_i) z_i r_3|^2,\] (10)

\[\Gamma(f_i \to \eta\eta') = g_1^2 q_{i\eta\eta'} |\sqrt{2} x_i y_i x_i + 2 y_i y_i y_i + 2(x_i x_i' + y_i y_i') z_i r_1 + 2z_i z_i' z_i r_2 + 2(2x_i x_i' + \sqrt{2} x_i y_i' + \sqrt{2} x_i' y_i + y_i y_i') z_i r_3|^2,\] (11)
where \(q_{iP_1P_2}\) is the decay momentum for the decay mode \(f_i \rightarrow P_1P_2\),
\[
q_{iP_1P_2} = \sqrt{[M_i^2 - (M_{P_1} + M_{P_2})^2][M_i^2 - (M_{P_1} - M_{P_2})^2]/2M_i},
\tag{12}
\]

\(M_i\) is the mass of \(f_i\), \(M_{P_1}\) and \(M_{P_2}\) are the masses of the final pseudoscalar mesons \(P_1\) and \(P_2\), respectively, and we take \(M_K = \sqrt{(M_{K^0}^2 + M_{K^+}^2)}/2\).

For \(\Gamma(f_i \rightarrow \eta\eta)\) and \(\Gamma(f_i \rightarrow \eta\eta')\), the contribution of the coupling mode iv) given in our present work differs from that given in Ref.\[2\] since we don’t adopt the assumption employed by Ref.\[2\] that the coupling of the glueball components of \(f_i\) to the \(q\bar{q}\) components of the isoscalar pseudoscalar mesons occurs dominantly through their \(s\bar{s}\) content in chiral symmetry. In addition, even under this assumption (i.e., \(x_j\) in the terms containing \(r_3\) is set to be zero), for \(\Gamma(f_i \rightarrow \eta\eta')\), the contribution of the mode iv) should be proportional to \(+y_\eta y_\eta'\) but not \(+2\alpha\beta = -y_\eta y_\eta'\) given by Ref.\[2\].

3 Fit results

Before performing the fit to determine the glueball-quarkonia content of \(f_i\), we should first determine the parameters \(x_j\), \(y_j\) and \(z_j\). We will adopt the mixing scheme mentioned above to discuss the mixing of \(\eta, \eta'\) and \(\eta(1410)\). Recently, the mixing of the three states based on a mass matrix has been discussed in Ref.[10]. Based on the equations (22)−(29) in Appendix A, the \(\theta_1, \theta_2\) and \(\theta_3\) are determined as \(\theta_1 = -98^\circ, \theta_2 = 30^\circ\) and \(\theta_3 = -95^\circ\), and \(x_j, y_j\) and \(z_j\) are determined as
\[
x_\eta = -0.731, \ y_\eta = 0.679, \ z_\eta = -0.069, \ x_{\eta'} = -0.566, \ y_{\eta'} = -0.660, \ z_{\eta'} = -0.495,\tag{13}
\]
with \(\chi^2 = 1.64\), which is consistent with the results given by Refs.[10]. If we set \(\theta_2\) and \(\theta_3\) to be zero, i.e., we do not consider the possible glueball components of \(j\), the mixing angle of \(\eta\) and \(\eta'\) is determined to be the value of \(-15^\circ\) which is in good agreement with the value of \(-15.5 \pm 1.5^\circ\) given by[3], and \(x_j\) and \(y_j\) are determined as
\[
x_\eta = y_{\eta'} = [\cos(-15^\circ) - \sqrt{2} \sin(-15^\circ)]/\sqrt{3},
\tag{14}
\]
\[
x_{\eta'} = -y_\eta = [\sin(-15^\circ) + \sqrt{2} \cos(-15^\circ)]/\sqrt{3},
\tag{15}
\]
with $\chi^2 = 9.19$. The $\chi^2$ implies that the $\eta$ and $\eta'$ wave functions need the additional glueball components. The predicted and measured results are shown in Table I.

In order to investigate whether the new data given by Ref.[3] are sensitive to the possible glueball components of $\eta$ and $\eta'$ or not, we perform the fit to the data presented in Table III in two cases: a) $z_j \neq 0$ and b) $z_j = 0$. In the fit procedure, we take $M_1 = 1.312$ GeV, $M_2 = 1.502$ GeV, $M_3 = 1.727$ GeV[3], and use the average value of 194 MeV for the decay momentum $q_{\eta\eta'}[4]$ since $f_2$ lies very near to the threshold in the $\eta\eta'$ decay mode[3]. In fit a) the parameters $x_j$, $y_j$ and $z_j$ are taken from Eq. (13) and in fit b) $x_j$, $y_j$ are taken from Eqs. (14) and (15). The parameters $\theta_1$, $\theta_2$, $\theta_3$, $r_1$, $r_2$ and $r_3$ in two fits are determined as shown in Table II and the predicted and the measured results are shown in Table III. Comparing fit a) with fit b), we find that three Euler angles and the predicted results are not much altered, and that the $\chi^2$ of the two fits are nearly equal, which shows that the new data on the hadronic decays of $f_i$ into two pseudoscalar mesons are insensitive to the possible glueball components of $\eta$ and $\eta'$.

Based on the parameters with the lowest $\chi^2$, the physical states $|f_1\rangle$, $|f_2\rangle$ and $|f_3\rangle$ can be given by

$$|f_1\rangle = -0.599|N\rangle + 0.326|S\rangle - 0.732|G\rangle,$$

$$|f_2\rangle = 0.795|N\rangle + 0.350|S\rangle - 0.495|G\rangle;$$

$$|f_3\rangle = 0.095|N\rangle - 0.878|S\rangle - 0.469|G\rangle. \tag{16}$$

From Eq. (16), one also can obtain

$$\Gamma(f_1 \to \gamma\gamma) : \Gamma(f_2 \to \gamma\gamma) : \Gamma(f_3 \to \gamma\gamma) =$$

$$M_1^3(5x_1 + \sqrt{2}y_1)^2 : M_2^3(5x_2 + \sqrt{2}y_2)^2 : M_3^3(5x_3 + \sqrt{2}y_3)^2 = 14.50 : 67.75 : 3.02. \tag{17}$$

This prediction can provide a test for the consistency of our results.

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2In this paper, the values of the masses of other mesons are taken from Ref.[11]
4 Discussions

Now we wish to discuss the properties of the mass matrix which can be used to describe the mixing of the scalar mesons based on our preferred results. In the $|N\rangle$, $|S\rangle$ and $|G\rangle$ basis, the general form of the mass matrix $M$ describing the mixing of the quarkonia and a glueball can be written as

$$M = \begin{pmatrix}
M_N + 2A_1 & \sqrt{2}A_2 & \sqrt{2}B_1 \\
\sqrt{2}A_2 & M_S + A_3 & B_2 \\
\sqrt{2}B_1 & B_2 & M_G
\end{pmatrix},$$

(18)

where $M_N$, $M_S$ and $M_G$ represent the masses of the bare states $|N\rangle$, $|S\rangle$ and $|G\rangle$, respectively; $A_1$ ($A_3$) is the amplitude of $|N\rangle$ ($|S\rangle$) annihilation and reconstruction via intermediate gluons states; $A_2$ is the amplitude of the transition between $|N\rangle$ and $|S\rangle$; $B_1$ ($B_2$) is the amplitude of the transition between $|N\rangle$ ($|S\rangle$) and $|G\rangle$. If $A_1$, $A_2$ and $A_3$ are set to be zero, and $B_1$ is assumed to be equal to $B_2$, Eq. (16) would reduced to the form employed in Ref. [1].

The physical states $|f_1\rangle$, $|f_2\rangle$ and $|f_3\rangle$ are assumed to be the eigenvectors of the mass matrix $M$ with the eigenvalues of $M_1$, $M_2$ and $M_3$, then we can have

$$UMU^\dagger = \begin{pmatrix} M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{pmatrix}, \begin{pmatrix} |f_1\rangle \\
|f_2\rangle \\
|f_3\rangle
\end{pmatrix} = U \begin{pmatrix} |N\rangle \\
|S\rangle \\
|G\rangle
\end{pmatrix}.$$

(19)

Comparing Eq. (16) with Eq. (19), we have

$$U = \begin{pmatrix}
-0.599 & 0.326 & -0.732 \\
0.795 & 0.350 & -0.495 \\
0.095 & -0.878 & -0.469
\end{pmatrix}.$$

(20)

Then the numerical form of the mass matrix can be given by

$$M = U^\dagger \begin{pmatrix} M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{pmatrix} U = \begin{pmatrix} 1.436 & 0.018 & -0.093 \\
0.018 & 1.656 & 0.138 \\
-0.093 & 0.138 & 1.450
\end{pmatrix}.$$

(21)
Eq. (21) shows that $A_2$ is very small. If $A_1$ and $A_3$ also can be expected to be very small, the mass level order of the bare states $|N\rangle$, $|S\rangle$ and $|G\rangle$ would be $M_S > M_G > M_N$, which is consistent with the argument given by Refs.[2, 8] while disagrees with the prediction that the glueball state has a higher mass than the $q\bar{q}$ state[13]. Otherwise, the mass level order of $M_N$, $M_S$ and $M_G$ in scalar sector would remain unclear. In addition, Eq. (21) implies that the mass of the pure scalar glueball is about 1.5 GeV, which is consistent with the lattice QCD prediction[14].

A salient property of Eq. (21) is that $B_1 < 0$ and $B_2 > 0$. This shows that the amplitude of the transition between $|N\rangle$ and $|G\rangle$ is negative while the amplitude of the transition between $|S\rangle$ and $|G\rangle$ is positive, which disagrees with the assumption that $B_1 = B_2$ in the model[1]. In fact, in the scalar sector, $B_1$ and $B_2$ should be nonperturbative effects dominantly, there are not convincing reasons to expect that the relation between $B_1$ and $B_2$ should behave as $B_1 = B_2$.

We note that the values of $r_1$ and $r_2$ are inconsistent with that $r_1$ and $r_2$ should be less than the unit, the prediction given by the perturbative theory. We find that if we restrict that $r_1$, $r_2$ and $r_3$ in the viewpoint of the perturbative theory, i.e., $r_1 < 1$, $r_2 < 1$ and $r_3 < 1$, the $\chi^2$ increases from 2.05 to 3.80, but the results given above are not much altered. However, in the scalar sector, there are not any convincing reasons to expect that the perturbative theory should be valid. The values of $r_1$ and $r_2$ imply that the nonperturbative effects in the scalar sector could be rather large.

5 Summary and conclusions

Using three Euler angles, we introduce a mixing scheme to describing the mixing of the isoscalar scalar mesons. In this mixing framework, considering the four coupling modes as shown in Fig. 1, we construct the effective Hamiltonian to investigate the two-body hadronic decays of $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$. The glueball-quarkonia content of the three states is determined from the fit to the new data about the hadronic decays of the three states presented by the WA102 collaboration. Our conclusions are as follows:
1). The large mixing effect exist in the three states.

1). The new data about the hadronic decays of $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$ are insensitive to the possible glueball components of $\eta$ and $\eta'$.  

3). The nonperturbative effects in the scalar sector are rather large.  

4). Our preferred results don’t support the assumption employed by Weingarten’s mass matrix describing the mixing of the isoscalar scalar states\cite{1} that $B_1 = B_2$.

6 Acknowledgments

This project is supported by the National Natural Science Foundation of China under Grant Nos. 19991487 and 19835060, and the Foundation of Chinese Academy of Sciences under Grant No. LWTZ-1298.

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Appendix A:

Formulae for the electromagnetic decays widths rates involving $\eta$ and $\eta'$

\[
\frac{\Gamma(\eta \to \gamma\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = \frac{1}{9} \left( \frac{M_\eta}{M_{\pi^0}} \right)^3 (5x_\eta + \sqrt{2}y_\eta)^2, \tag{22}
\]

\[
\frac{\Gamma(\rho' \to \gamma\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = \frac{1}{9} \left( \frac{M_{\rho'}}{M_{\pi^0}} \right)^3 (5x_{\rho'} + \sqrt{2}y_{\rho'})^2, \tag{23}
\]

\[
\frac{\Gamma(\rho \to \eta\gamma)}{\Gamma(\pi^0 \to \eta\gamma)} = \left[ \left( \frac{M_\rho^2 - M_{\rho'}^2}{M_\rho^2} \right) \frac{M_\omega^2}{M_\eta^2} \right]^{\frac{1}{2}} x_{\eta'}^2, \tag{24}
\]

\[
\frac{\Gamma(\rho' \to \rho\gamma)}{\Gamma(\pi^0 \to \rho\gamma)} = 3 \left[ \left( \frac{M_\rho^2 - M_{\rho'}^2}{M_\rho^2} \right) \frac{M_\omega^2}{M_{\rho'}^2} \right]^{\frac{1}{2}} x_{\rho'}^2, \tag{25}
\]

\[
\frac{\Gamma(\phi \to \eta\gamma)}{\Gamma(\omega \to \eta\gamma)} = \frac{4}{9} \frac{m_u^2}{m_s^2} \left[ \left( \frac{M_\phi^2 - M_{\rho'}^2}{M_\phi^2} \right) \frac{M_\omega^2}{M_{\rho'}^2} \right]^{\frac{1}{2}} y_{\eta'}^2, \tag{26}
\]

\[
\frac{\Gamma(\phi \to \eta'\gamma)}{\Gamma(\omega \to \eta'\gamma)} = \frac{4}{9} \frac{m_u^2}{m_s^2} \left[ \left( \frac{M_\phi^2 - M_{\rho'}^2}{M_\phi^2} \right) \frac{M_\omega^2}{M_{\rho'}^2} \right]^{\frac{1}{2}} y_{\rho'}^2, \tag{27}
\]

\[
\frac{\Gamma(J/\psi \to \rho\eta)}{\Gamma(J/\psi \to \omega\pi^0)} = \left[ \sqrt{\frac{M_{J/\psi}^2 - (M_\rho + M_{\eta})^2}{M_{J/\psi}^2 - (M_\omega + M_{\pi^0})^2}} \right]^{\frac{1}{2}} \left[ \sqrt{\frac{M_{J/\psi}^2 - (M_{\rho'} + M_{\eta'})^2}{M_{J/\psi}^2 - (M_{\rho'} + M_{\pi^0})^2}} \right]^{\frac{1}{2}} x_{\eta'}^2, \tag{28}
\]

\[
\frac{\Gamma(J/\psi \to \rho\eta')}{\Gamma(J/\psi \to \omega\pi^0)} = \left[ \sqrt{\frac{M_{J/\psi}^2 - (M_\rho + M_{\eta'})^2}{M_{J/\psi}^2 - (M_\omega + M_{\pi^0})^2}} \right]^{\frac{1}{2}} \left[ \sqrt{\frac{M_{J/\psi}^2 - (M_{\rho} + M_{\eta})^2}{M_{J/\psi}^2 - (M_\omega + M_{\pi^0})^2}} \right]^{\frac{1}{2}} x_{\rho'}^2, \tag{29}
\]

where $M_\rho$, $M_\omega$, $M_\phi$ and $M_{J/\psi}$ are the masses of $\rho$, $\omega$, $\phi$ and $J/\psi$, respectively; $m_u$ and $m_s$ are the masses of the constituent quark $u$ and $d$, respectively. Here we take $m_u/m_s = 0.642$ used in Ref. [15].
Fit1                      Fit2

Exp.[1]        $z_j \neq 0$ ($j = \eta, \eta'$)  $z_j = 0$ ($j = \eta, \eta'$)

$\chi^2 = 1.64$  $\chi^2 = 9.19$

| Decay                      | Fit1         | Fit2         |
|----------------------------|--------------|--------------|
| $\Gamma(\eta \rightarrow \gamma \gamma) / \Gamma(\pi^0 \rightarrow \gamma \gamma)$ | 58.46 ± 9.03 | 53.76        |
| $\Gamma(\eta' \rightarrow \gamma \gamma) / \Gamma(\pi^0 \rightarrow \gamma \gamma)$ | 540.78 ± 104.44 | 561.33       |
| $\Gamma(\rho \rightarrow \eta \eta) / \Gamma(\omega \rightarrow \pi^0 \eta)$ | 0.051 ± 0.023 | 0.066        |
| $\Gamma(\rho \rightarrow \eta \eta) / \Gamma(\omega \rightarrow \pi^0 \eta)$ | 0.086 ± 0.016 | 0.086        |
| $\Gamma(\phi \rightarrow \eta \eta) / \Gamma(\omega \rightarrow \pi^0 \eta)$ | 0.078 ± 0.010 | 0.074        |
| $\Gamma(\phi \rightarrow \eta' \gamma) / \Gamma(\omega \rightarrow \pi^0 \eta)$ | 0.0007 ± 0.0005 | 0.0003       |
| $\Gamma(J/\psi \rightarrow \rho \eta) / \Gamma(J/\psi \rightarrow \omega \pi^0)$ | 0.460 ± 0.120 | 0.482        |
| $\Gamma(J/\psi \rightarrow \rho \eta) / \Gamma(J/\psi \rightarrow \omega \pi^0)$ | 0.250 ± 0.079 | 0.223        |

Table 1: The predicted and measured results of electromagnetic decays involving $\eta, \eta'$.

| Decay                      | $\chi^2$ | $r_1$ | $r_2$ | $r_3$ | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|----------------------------|----------|-------|-------|-------|-------------|-------------|-------------|
| Fit a)                     | 2.05     | 1.0   | 3.4   | 0.33  | $-146^\circ$| 118^\circ   | $-151^\circ$|
| Fit b)                     | 2.15     | 1.0   | 0.7   | 0.7   | $-148^\circ$| 115^\circ   | $-146^\circ$|

Table 2: The parameters determined from the fit.
\[ \chi^2 = 2.05 \]
\[ \chi^2 = 2.15 \]

|                        | Exp. | Fit a) | Fit b) |
|------------------------|------|--------|--------|
| \( \Gamma(f_0(1370) \to \pi\pi) / \Gamma(f_0(1370) \to KK) \) | 2.17 ± 0.90 | 2.453 | 2.397 |
| \( \Gamma(f_0(1370) \to \eta\eta) / \Gamma(f_0(1370) \to KK) \) | 0.35 ± 0.30 | 0.248 | 0.314 |
| \( \Gamma(f_0(1500) \to \pi\pi) / \Gamma(f_0(1500) \to \eta\eta) \) | 5.56 ± 0.93 | 5.581 | 5.853 |
| \( \Gamma(f_0(1500) \to KK) / \Gamma(f_0(1500) \to \pi\pi) \) | 0.33 ± 0.07 | 0.335 | 0.308 |
| \( \Gamma(f_0(1500) \to \eta\eta') / \Gamma(f_0(1500) \to \eta\eta) \) | 0.53 ± 0.23 | 0.528 | 0.484 |
| \( \Gamma(f_0(1710) \to \pi\pi) / \Gamma(f_0(1710) \to KK) \) | 0.20 ± 0.03 | 0.191 | 0.200 |
| \( \Gamma(f_0(1710) \to \eta\eta) / \Gamma(f_0(1710) \to KK) \) | 0.48 ± 0.19 | 0.230 | 0.223 |
| \( \Gamma(f_0(1710) \to \eta\eta') / \Gamma(f_0(1710) \to KK) \) | < 0.04 (90% CL) | 0.035 | 0.021 |

Table 3: The predict and measured results of the hadronic decays of \( f_0 \).
Figure 1: The coupling modes considered in this work. i) The coupling of the quarkonia components of $f_i$ to the final pseudoscalar meson pairs, ii) the coupling of the glueball components of $f_i$ to the final pseudoscalar meson pairs via $qqar{q}ar{q}$ intermediate states, iii) the coupling of the glueball components of $f_i$ to the glueball components of the final isoscalar pseudoscalar mesons, and iv) the coupling of the glueball components of $f_i$ to the quarkonia of the final isoscalar pseudoscalar meson pairs.