FRW in cosmological self-creation theory: Hamiltonian approach

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We use the Brans-Dicke theory from the framework of General Relativity (Einstein frame), but now the total energy momentum tensor fulfills the following condition

\[
\frac{1}{8\pi} \left( 8\pi T^\mu_\nu(M) + T^\mu_\nu(\phi) \right) = 0.
\]

We take as a first model the flat FRW metric in the Hamilton-Jacobi scheme and we present the Lagrange-Charpit approach in order to find classical solutions. In the quantum scheme, once we determine the characteristic surfaces, the quantum solution is obtained. These two classes of solutions are found for all values of the barotropic parameter \( \gamma \).

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I. INTRODUCTION

The Scalar-Tensor theories have their origin in the 50’s. Pascual Jordan was intrigued by the appearance of a new scalar field in Kaluza-Klein theories, especially in its possible role as a generalized gravitational constant. In this way appeared the Brans-Dicke theory, with the particularity that each one of the energy momentum tensor satisfy the covariant derivative \( T^\mu_\nu(i) = 0 \), where \( i \) corresponds to the \( i \)-th ingredient of matter content. A few years latter (1982), appeared a new proposal by Barber, known as self-creation cosmology (SCC) [4, 5]. Since the original paper appeared in 1982, more and more authors [6–9] have worked in the different versions of this theory in the classical fashion. By instant, Singh and Singh [10] have studied Raychaudhary-type equations for perfect fluid in self-creation theory. Pimentel [11] and Soleng [12, 13] have studied in detail the cosmological solutions of Barber’s self-creation theories. Reddy [14], Venkateswarlu and Reddy [15], Shri and Singh [16, 17], Mohanty et al. [18], Pradhan and Vishwakarma [19, 20], Sahu and Panigrahi [21], Venkateswarlu and Kumar [22] are some of the authors who have studied various aspects of different cosmological models in self-creation theory, many of them usually adopt a particular ansatz for solve the Einstein field equations. These papers adapted the Brans Dicke theory to create mass out of the universes self contained scalar, gravitational and matter fields in simplest way.

The gravitational theory must be a metric theory, because this is the easiest way to introduce the Equivalence Principle. However always it is possible to put additional terms to the metric tensor, the most obvious proposal is a scalar field \( \phi \). Recently Chirde and Rahate [23] investigated spatially homogeneous isotropic Friedman-Robertson-Walker cosmological model with bulk viscosity and zero-mass scalar field in the framework of Barber’s second self-creation theory and found classical solutions, it is the simplest way to work with this theory because only takes in account the energy-momentum tensor of usual matter and the scalar field \( \phi \).

This work is arranged as follow. In section II we present the method used in general way, where we take the Brans-Dicke Lagrangian density and we consider this in the self-creation theory. In section III, employing the flat FRW metric as a toy model with a barotropic perfect fluid, we found the energy density in dependence of the scalar factor and the

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In Section IV we construct the Lagrangian and Hamiltonian densities for the cosmological model under consideration and it is presented the classical solution using the Hamilton-Jacobi approach. The classical solutions are obtained using the Lagrange-Charpit method \[24\,\text{[26]}\], which structure is such that we find classical solutions for all values of the \(\gamma\) parameter. In this same section we solve the corresponding Wheeler-DeWitt equation for this case. The section V we present the conclusions of the work.

II. SELF CREATION COSMOLOGY IN GR

The Lagrangian density in the Brans-Dicke theory is

\[
L[g,\phi] = \sqrt{-g}\left(R\phi - \frac{\omega}{\phi} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu}\right) + \sqrt{-g} L_{\text{matter}},
\]

where \(L_{\text{matter}} = 16\pi \rho\), so making the corresponding variation to the scalar field and the tensor metric, the field equations in this theory become

\[
R - \frac{\omega}{\phi^2} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} + \frac{2\omega}{\phi} \Box^2 \phi = 0,
\]

where \(\lambda\) is a coupling constant to be determined from experiments. The measurements of the deflection of light restrict the value of coupling to \(|\lambda| < 10^{-1}\). In the limit \(\lambda \to 0\), the Barber’s second theory approaches the standard general relativity theory in every aspect. \(\Box \phi = \phi,_{\mu} \phi,^{\mu}\) is the invariant D’Alembertian and \(T\) is the trace of the energy momentum tensor that describes all non gravitational and non scalar field matter and energy. Taking the trace of the equation (3) and then substitute in (4), we obtain the following wave equation to \(\phi\)

\[
\Box \phi = 8\pi \lambda T,
\]

comparing both equations (4) and (5), we note that \(\lambda = \frac{2}{3 + 2\omega}\) in the Brans-Dicke theory, where \(\omega\) is a coupling constant. In the Barber’s theory, the \(\lambda\) parameter is a new coupling constant.

Now the Einstein equation can be rewritten as

\[
G^{\alpha\beta} = \frac{8\pi}{\phi} T^{\alpha\beta}(M) + \frac{1}{\phi} T^{\alpha\beta}(\phi) = T^{\alpha\beta}(T),
\]

where

\[
T^{\alpha\beta}(\phi) = \frac{\omega}{\phi} \left(\phi^{\alpha\phi,\beta} - \frac{1}{2} \phi^{,\lambda} \phi^{,\lambda} g^{\alpha\beta}\right) + \left(\phi^{,\alpha\beta} - g^{\alpha\beta} \Box^2 \phi\right).
\]

In Brans-Dicke theory, each one energy momentum tensor satisfy the covariant derivative, \(T^{\mu\nu} :_{\nu} = 0\). In the self creation theory, we introduce that the total energy momentum tensor is which satisfy the covariant derivative, \(T^{\mu\nu(T) :_{\nu}} = 0\), namely;

\[
T^{\alpha\beta(T) :_{\beta}} = 0, \quad \Rightarrow \quad \left[\frac{8\pi}{\phi} T^{\alpha\beta(M)} + \frac{1}{\phi} T^{\alpha\beta(\phi)}\right]_{:_{\beta}} = 0,
\]
which imply that \([Q_{\alpha\beta}(T)]_{\gamma\delta} = \sum_{d} \varphi_{d} [Q_{\alpha\beta}(T)]\), where \(Q_{\alpha\beta}(T) = \phi T_{\alpha\beta}(T)\). This equation is the master equation which gives the name of self creation theory, because the covariant derivative of this tensor have a source of the same tensor multiply by a function of the scalar field \(\phi\).

III. FRW IN SELF CREATION THEORY

We apply the formalism using the geometry of Friedmann-Robertson-Walker

\[
ds^2 = -N^2(t) dt^2 + A^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right],
\]

where \(N\) is the lapse function, \(A\) is the scalar factor. Now we solve the equations (3), (4) and (7), with the aim to find solutions to density \(\rho(t)\), scalar factor \(A(t)\) and the scalar field \(\phi(t)\).

Taking the transformation \(t = \frac{d}{N \kappa} = \frac{d}{\pi} \) and using \(T_{\alpha\beta}\) as a fluid perfect, first compute the classical field equations (3), together with the barotropic equation of state \(P = \gamma \rho\), this equation become

\[
3 \left( \frac{A'}{A} \right)^2 + 3 \frac{A'}{A} \frac{\phi'}{\phi} - \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - 8 \pi \frac{\rho}{\phi} + 3 \frac{\kappa}{A^2} = 0,
\]

(9)

\[
2 \frac{A''}{A} + \frac{A^2}{A^2} + 2 \frac{A'}{A} \frac{\phi'}{\phi} + \omega \left( \frac{\phi'}{\phi} \right)^2 + \frac{\phi''}{\phi} + 8 \pi \frac{\rho}{\phi} + \frac{\kappa}{A^2} = 0,
\]

(10)

the equation (11) become

\[
3 \frac{A'}{A} \frac{\phi'}{\phi} + \frac{\phi''}{\phi} = 4 \pi \lambda (1 - 3 \gamma) \frac{\rho}{\phi}.
\]

(11)

The covariant equation (11) or conservation equation to the total energy momentum take the following form

\[
3 \frac{A''}{A} \frac{\phi'}{\phi} - 3 \frac{A'}{A} \left( \frac{\phi'}{\phi} \right)^2 + \omega \left( \frac{\phi'}{\phi} \right)^3 - 3 \frac{A'}{A} \left( \frac{\phi'}{\phi} \right)^2 - 24 \pi \frac{A'}{A} (1 + \gamma) \frac{\rho}{\phi} - \omega \frac{\phi''}{\phi} \frac{\phi'}{\phi} + 8 \pi \frac{\phi'}{\phi} \frac{\rho}{\phi} - 8 \pi \frac{\rho}{\phi} = 0,
\]

(12)

thus, the system equations to be solved are (10)-(12).

We write the term \(\left( \frac{A'}{A} \right)^2 + \frac{\kappa}{A^2}\), in equation (9), using equations (10) and (11), and after some algebra we have

\[
3 \frac{A''}{A} - 3 \frac{A'}{A} \frac{\phi'}{\phi} + \omega \left( \frac{\phi'}{\phi} \right)^2 = 2 \pi [3 \lambda (3 \gamma - 1) - 2(1 + 3 \gamma)] \frac{\rho}{\phi}.
\]

(13)

Equation (12) can be rewritten as

\[
\left( 3 \frac{A''}{A} - 3 \frac{A'}{A} \frac{\phi'}{\phi} + \omega \left( \frac{\phi'}{\phi} \right) \right) \frac{\phi'}{\phi} - \left( 3 \frac{A'}{A} \frac{\phi'}{\phi} + \frac{\phi''}{\phi} \right) \frac{\phi'}{\phi} - 24 \pi (1 + \gamma) \frac{\rho}{\phi} \frac{A'}{A} - 8 \pi \left( \frac{\rho}{\phi} \right)' = 0,
\]

(14)

and using the equations (11) and (13) in (14), then we have the master equation for solve the energy density of the model, as

\[
2 \left[ \lambda (3 \gamma - 1)(3 + 2 \omega) - 2(1 + 3 \gamma) \right] \frac{\rho}{\phi} \frac{A'}{A} - 24 (1 + \gamma) \frac{\rho}{\phi} \frac{A'}{A} - 8 \left( \frac{\rho}{\phi} \right)' = 0,
\]

(15)

defining the function \(F = \frac{\rho}{\phi}\), we have

\[
\frac{d}{d\tau} \ln \left[ FA^{3(1+\gamma)} \phi^{4 \lambda (3 \gamma - 1)(2 \omega + 3) - 2(1 + 3 \gamma)} \right] = 0,
\]
who solution is

$$\rho = M_\gamma A^{-3(1+\gamma)} \phi^\beta, \quad \beta = \frac{(1-3\gamma)}{4} [2 - \lambda a_0], \quad a_0 = 3 + 2\omega. \quad (16)$$

This equation is equivalent to General Relativity expression with the addition the last factor representing the self creation cosmology. Note that for a photon gas $$\gamma = \frac{1}{3}$$, so $$\beta = 0$$ and equation (16) reduce to its General Relativity expression

$$\rho = \rho_0 (A/A_0)^{-4}$$ which is consistent, because in the radiation epoch there was not interaction between photon and the scalar field.

IV. LAGRANGIAN AND HAMILTONIAN DENSITIES IN SCC

We will use the classical Hamilton-Jacobi approach to find solutions to ($$\rho, A, \phi$$). The corresponding Lagrangian density using (8) into (1) as follows

$$\mathcal{L} = \frac{6 A \dot{\phi} \dot{A}}{N} + \frac{6 A^2 \dot{A} \dot{\phi}}{N} - \frac{\omega A^3 \dot{\phi}^2}{N} + 16\pi A^3 \rho N, \quad (17)$$

the momenta are ($$\Pi_q = \frac{\partial S}{\partial q}$$)

$$\Pi_A = \frac{12 A \dot{\phi} \dot{A}}{N} + \frac{6 A^2 \dot{A} \dot{\phi}}{N}, \quad \Pi_\phi = \frac{6 A^2 \dot{A}}{N} - 2 \frac{\omega A^3 \dot{\phi}}{N},$$

$$\dot{A} = \frac{N}{6(3 + 2\omega)\phi A^2} \left(3 \phi \Pi_\phi + \omega A \Pi_A \right), \quad (18)$$

$$\dot{\phi} = \frac{N}{2(3 + 2\omega)A^3} \left(A \Pi_A - 2 \phi \Pi_\phi \right), \quad (19)$$

when we write the canonical Lagrangian density $$\mathcal{L}_{\text{canonical}} = \Pi_q \dot{q} - N \mathcal{H}$$, we obtain the corresponding Hamiltonian density as

$$\mathcal{H} = \frac{A^{-3}}{12a_0 \phi} \left[-6 \phi^2 \Pi_\phi^2 + \omega A^2 \Pi_A^2 + 6 A \phi \Pi_A \Pi_\phi - 192\pi a_0 A^6 \phi \rho \right], \quad a_0 = (3 + 2\omega). \quad (20)$$

A. Classical scheme: Hamilton-Jacobi equation

Using the gauge $$N = 12a_0 \phi A^3$$, and the transformation in the momenta $$\Pi_q = \frac{\partial S}{\partial q}$$, where S is known as the superpotential function, with this, the Hamiltonian density is written as (we include the equation (16) in this last equation)

$$-6 \phi^2 \left(\frac{\partial S}{\partial \phi} \right)^2 + \omega A^2 \left(\frac{\partial S}{\partial A} \right)^2 + 6 A \phi \left(\frac{\partial S}{\partial A} \right) \left(\frac{\partial S}{\partial \phi} \right) - \eta_\gamma A^{-3(\gamma-1)} \phi^{(\beta+1)} = 0, \quad (21)$$

with $$\eta_\gamma = 192\pi a_0 M_\gamma$$. In the following we obtain classical solutions for any value in the barotropic parameter $$p = \gamma \rho$$ using the Lagrange-Sharpit approach [24–26].

1. Lagrange-Sharpit approach

For apply this method, we begin using the equation (21), where we introduce the following transformations

$$\phi = e^b, \quad A = e^{\Omega}, \quad (22)$$
\[ -6 \left( \frac{\partial S}{\partial \Phi} \right)^2 + \omega \left( \frac{\partial S}{\partial \Omega} \right)^2 + 6 \left( \frac{\partial S}{\partial \Omega} \frac{\partial S}{\partial \Phi} \right) - \eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}} = 0. \] (23)

We start defining a function \( F(\Phi, \Omega, p, q, S) = 0 \), where
\[ F(\Phi, \Omega, p, q, S) = -6p^2 + \omega q^2 + 6pq - \eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}}, \] (24)

with the expressions \( p = \frac{\partial S}{\partial \Phi} \) and \( q = \frac{\partial S}{\partial \Omega} \). We build the following set of equations
\[ \begin{align*}
\frac{d\Phi}{F_p} &= \frac{d\Omega}{pF_p + qF_q} = \frac{dS}{F_p + qF_S} = -\frac{dp}{F_\Phi} = -\frac{dq}{F_\Omega} = dt. \tag{25}
\end{align*} \]

calculating the derivatives we have
\[ \begin{align*}
F_p &= -12p + 6q, \\
F_q &= 2\omega q + 6p, \\
F_\Phi &= -(\beta + 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}}, \\
F_\Omega &= 3(\gamma - 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}}, \\
F_S &= 0, \\
F_t &= 0,
\end{align*} \]
equation (25) is read as
\[ \begin{align*}
-12p + 6q &= d\Omega \\
2\omega q + 6p &= dS \\
-3(\gamma - 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}} &= dp \\
(\beta + 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}} &= dt. \tag{26}
\end{align*} \]

Choosing the equation
\[ \begin{align*}
\frac{dq}{-3(\gamma - 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}}} &= \frac{dp}{(\beta + 1)\eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}}}, \tag{27}
\end{align*} \]
we find the following relation between the functions \((p, q)\),
\[ \begin{align*}
(\beta + 1)q + 3(\gamma - 1)p &= c_\gamma = \text{constant}, \tag{28}
\end{align*} \]

with this and from (24), substituting the term \( \eta \gamma e^{-3(\gamma - 1)\Omega_0 e^{(\beta + 1)\Phi}} \), we write an equation which involves to dependence of time,
\[ \frac{(\beta + 1)dp}{a_1p^2 + a_2p + a_3} = dt, \]
where the solution is
\[ \begin{align*}
p(t) &= \frac{a_2}{2a_1} - \frac{\sqrt{a_2^2 - 4a_1a_3}}{2a_1} \coth \left[ \frac{c_\gamma \sqrt{3a_0t}}{a_1} \right], \tag{29}
\end{align*} \]

and the constants \( a_i \) are
\[ \begin{align*}
a_1 &= -6(\beta + 1)^2 + 9\omega(1 - \gamma)^2 + 18(1 - \gamma)(\beta + 1), \\
a_2 &= 6c_\gamma \left[ \omega(1 - \gamma) + \beta + 1 \right], \\
a_3 &= \omega c_\gamma^2.
\end{align*} \]

From (28) we obtain the corresponding solution for the function \( q(t) \) as
\[ \begin{align*}
q(t) &= \frac{c_\gamma}{\beta + 1} + \frac{3(1 - \gamma)}{\beta + 1} p(t). \tag{30}
\end{align*} \]

Now we found the solution for the functions \((p, q)\), then we use the following equation in order to obtain the function \( \Phi(t) \),
\[ \begin{align*}
-12p + 6q &= dt, \tag{31}
\end{align*} \]
\[ \Phi(t) = a_4 t + a_5 \ln \left[ \sinh(c_\gamma \sqrt{3\lambda a_0} t) \right] + \Phi_0, \]  
where \( \Phi_0 \) is an integration constant, and the constants

\[ a_4 = \frac{6c_\gamma}{\beta + 1} + \frac{3a_2}{\beta + 1} (3\gamma + 2\beta - 1), \quad a_5 = \frac{6(3\gamma + 2\beta - 1)}{a_1}. \]

For obtain the solution for the function \( \Omega \) we use the equation

\[ dt = \frac{d\Omega}{2\omega q + 6p}, \]

after integration, we have

\[ \Omega(t) = a_6 t - a_7 \ln \left[ \sinh(c_0 \sqrt{3\lambda a_0} t) \right] + \Omega_0, \]

where \( \Omega_0 \) is an integration constant, and the constants

\[ a_6 = \frac{1}{\beta + 1} \left[ 2\omega c_\gamma - \frac{3a_2}{a_1} (1 + \beta + \omega(1 - \gamma)) \right], \quad a_7 = \frac{6(1 + \beta + \omega(1 - \gamma))}{a_1}. \]

Now, taking in account the transformation law (22), we have the solution for the scale factor \( A \) and the scalar field \( \phi \) as

\[ A(t) = A_0 e^{a_0 t \sinh(c_\gamma \sqrt{3\lambda a_0} t)}, \quad \phi(t) = \phi_0 e^{a_0 t \sinh(c_\gamma \sqrt{3\lambda a_0} t)}. \]

By introducing these solutions in the equation (31), the value of the constant \( c_\gamma \) is equal to

\[ c_\gamma = \sqrt{64\pi a_1 M_7 A_0^3 \left( \frac{3(3 - \gamma)}{\beta + 1} \right) \frac{\beta + 1}{\phi_0}}, \]

The corresponding value for all constants that appear in the classical calculation are presented in table I.

| \( \beta \) | \( \gamma = -1 \) | \( \gamma = \frac{1}{3} \) | \( \gamma = 1 \) | \( \gamma = 0 \) |
|----------------|----------------|----------------|----------------|----------------|
| \( a_1 \) | \( 6(9 + 6\omega - \lambda^2 a_0^4) \) | \( 2a_0 \) | \( -\frac{3}{2} \lambda^2 a_0^2 \) | \( \frac{2}{9}(36 + 24\omega - \lambda a_0^4) \) |
| \( a_4 \) | \( \frac{2c_\gamma}{2 - 3a_0} \left[ 1 + (2\omega + 3 - \lambda a_0) \ell_0 \right] \) | \( 6c_\gamma/3 \) | \( 0 \) | \( \frac{3c_\gamma}{8 - 3a_0} \left[ 8 + (4\omega + 6 - \lambda a_0) \ell_1 \right] \) |
| \( a_5 \) | \( 2\ell_0 \) | \( 0 \) | \( \frac{4c_\gamma}{4 + 6\omega - \lambda a_0^4} \ell_1 \) | \( \ell_1 \) |
| \( a_6 \) | \( \frac{-\frac{4c_\gamma}{9 - 6\omega - \lambda a_0^4}}{3 - \lambda a_0} \left[ 2\omega - \frac{3(1 - \lambda^2 a_0^2)}{9 + 6\omega - \lambda a_0^4} \right] \) | \( -3c_1/3 \) | \( \frac{4c_\gamma}{4 - 3\lambda a_0} \left( \frac{4\omega - 6c_\gamma}{4 - 3\lambda a_0} \right) \) | \( \frac{4c_\gamma}{4 - 3\lambda a_0} \left( \frac{4\omega - 6c_\gamma}{4 - 3\lambda a_0} \right) \) |
| \( a_7 \) | \( \frac{\ell_0 - \lambda a_0}{9 + 6\omega - \lambda a_0^4} \left[ 1 - \frac{4c_\gamma}{4 - 3\lambda a_0} \right] \) | \( 1 \) | \( \frac{-2}{\lambda a_0} \) | \( \frac{4c_\gamma}{4 + 6\omega - \lambda a_0^4} \ell_1 \) |
| \( c_\gamma \) | \( \sqrt{384\pi (9 + 6\omega - \lambda^2 a_0^4)} M_7 A_0^3 \phi_0^{\frac{3(3 - \gamma)}{2}} \) | \( \sqrt{128\pi a_0 M_7 A_0^3 \phi_0^2} \) | \( \sqrt{-96\pi \lambda^2 a_0^2 M_7 A_0^3 \phi_0^{\frac{3(3 - \gamma)}{2}}} \) | \( \sqrt{24\pi (36 + 24\omega - \lambda a_0^4)} M_7 A_0^3 \phi_0^{\frac{3(3 - \gamma)}{2}} \) |

Table I: All constants that appear in the classical solutions for various values of \( \gamma \), with the following definitions \( \ell_0 = \frac{-\lambda a_0}{9 + 6\omega - \lambda a_0^4} \), \( \ell_1 = \frac{-8\lambda a_0}{36 + 24\omega - \lambda a_0^4} \).

When we calculate the deceleration parameter

\[ q = -\frac{\dot{A}A}{A^2} \]
where the sign of \( q \) indicated whether the model inflates or not. The positive sign of \( q \) i.e. \( q > 0 \) correspond to standard decelerating model, whereas the negative sign \( q < 0 \) indicates acelerate expansion.

Using (35), this parameter have the following form

\[
q = -1 - \frac{3a_7a_0c_7^2}{[\sqrt{3a_0a_7c_7} \cosh (c_7\sqrt{3a_0t}) - a_6 \sinh (c_7\sqrt{3a_0t})]^2},
\]

and checking the table I over the possible value to the constants \( a_0, a_7 \) and \( c_7 \), we observed that when \( \omega > \frac{\gamma}{2} \), this deceleration parameter always is negative for any value in the barotropic parameter \( \gamma \), so, the universe expand always in this theory.

### B. Quantum scheme

Imposing the quantization condition and applying the Hamiltonian density (20) to the wave function \( \Psi \), we obtain the WDW equation for these models in the minisuperspace by the usual identification \( F_{\mu \nu} = -i\partial_{\mu} \) in (20), with \( q^\mu = (A, \phi) \), and following Hartle and Hawking [27] we consider a semi-general factor ordering which gives (we include the equation (16) in this last equation) (Also we could use the loop quantum cosmology in this approach [28, 29])

\[
\hat{H}_\Psi = \left[ 6\phi^2 \frac{\partial^2}{\partial \phi^2} - 6r \phi \frac{\partial}{\partial \phi} - \omega A^2 \frac{\partial^2}{\partial A^2} + \omega QA \frac{\partial}{\partial A} - 6A \frac{\partial}{\partial A} \phi \frac{\partial}{\partial \phi} - \eta\gamma A^{-3(\gamma - 1)} \phi^{\beta + 1} \right] \Psi = 0,
\]

where \( Q \) and \( r \) are real constants that measures the ambiguity in the factor ordering between the scalar functions \((A, \phi)\) and its corresponding momenta.

Using the same transformation as the classical scheme, (22), the equation (38) is read as

\[
\left[ 6 \frac{\partial^2}{\partial \phi^2} - 6(\gamma + 1) \frac{\partial}{\partial \phi} - \omega A^2 \frac{\partial^2}{\partial A^2} + \omega (Q + 1) \frac{\partial}{\partial A} - 6 \frac{\partial^2}{\partial A \partial \phi} - \eta\gamma A^{-3(\gamma - 1)} \phi^{\beta + 1} \right] \Psi = 0,
\]

re-written this equation in the following form,

\[
6 \frac{\partial^2 \Psi}{\partial \phi^2} - 6 \frac{\partial^2 \Psi}{\partial \Omega \partial \phi} - \omega \frac{\partial^2 \Psi}{\partial \Omega^2} = \omega(Q + 1) \frac{\partial \Psi}{\partial \Omega} + \eta\gamma A^{-3(\gamma - 1)} \phi^{\beta + 1} \Psi + 6(\gamma + 1) \frac{\partial \Psi}{\partial \phi},
\]

we can build the following \( \sigma \) parameter

\[
\sigma^\pm = \frac{1}{2} \pm \sqrt{\frac{3a_0}{6}},
\]

which is used to construct a linear partial differential equation of first order of the characteristics surfaces for the functions \((\Phi, \Omega)\), given by

\[
\frac{\partial \theta}{\partial \Phi} + \sigma^\pm \frac{\partial \theta}{\partial \Omega} = 0,
\]

where \( \theta \) is the envelope of the characteristic surfaces, then we have

\[
d\Phi = \frac{d\Omega}{\sigma^\pm} = \frac{d\theta}{0},
\]

once we integrate this result gives us the following relation between these functions \( \sigma^\pm \Phi - \Omega = b_0 \).

Let \( \Omega = \sigma^\pm \Phi - b_0 \), then introducing this in the equation (39), we have

\[
6 \frac{\partial^2 \Psi}{\partial \phi^2} - 6(\gamma + 1) \frac{\partial \Psi}{\partial \phi} - \omega \frac{\partial^2 \Psi}{\partial \Omega} = \omega(Q + 1) \frac{\partial \Psi}{\partial \Omega} - 6 \frac{\partial^2 \Psi}{\partial \Omega \partial \phi} - \eta\gamma A^{-3(\gamma - 1)} \phi^{\beta + 1 - 3(\gamma - 1)} \sigma^\pm \Psi = 0.
\]

Assuming that the wavefunction is separable \( \Psi(\Omega, \Phi) = \psi_1(\Omega)\psi_2(\Phi) \) we have

\[
\frac{6 d^2 \psi_2}{d\Phi^2} - \frac{6(\gamma + 1) d\psi_2}{d\Phi} - \omega \frac{d^2 \psi_1}{d\Omega^2} + \frac{\omega(Q + 1) d\psi_1}{d\Omega} - \frac{1}{6} \frac{d\psi_1}{d\Omega} \frac{1}{d\psi_2} - \frac{d\psi_2}{d\phi} - \eta\gamma A^{-3(\gamma - 1)} \phi^{\beta + 1 - 3(\gamma - 1)} \sigma^\pm \Psi = 0,
\]
and considering the particular ansatz for solving this equation, \( \frac{d^2 \psi}{d \Phi^2} = \mu^2 \), which solution become \( \psi_1 = b_1 e^{-\mu_0 \Omega} \), and \( b_1 \) is a real constant. We choose only the minus sign, because we need to keep a decreasing wavefunction depending to scale factor. With this, the last equation is written as

\[
\frac{d^2 \psi_2}{d \Phi^2} + b_2 \frac{d \psi_2}{d \Phi} - (b_3 + b_4 e^{b_5 \Phi}) \psi_2 = 0,
\]

(41)

where the constants are defined as

\[
b_2 = \mu_0 - 1 - r, \quad b_3 = \frac{\omega \mu_0}{6} [\mu_0 + 1 + Q], \quad b_4 = \frac{\eta}{6} e^{3(\gamma - 1) \mu_0}, \quad b_5 = \beta + 1 - 3(\gamma - 1) \sigma^\pm,
\]

equation that is similar to (30)

\[
y'' + ay' + (be^{nx} + c)y = 0,
\]

(42)

which solution is

\[
y = e^{-\frac{x^2}{4}} \left[ c_1 J_\nu \left( \frac{2x}{n} \right) + c_2 Y_\nu \left( \frac{2x}{n} \right) \right],
\]

where \( \nu = \frac{1}{n} \sqrt{a^2 - 4c} \), having the following relations between the parameters:

\[
x = \Phi, \quad a = b_2, \quad b = -b_4, \quad n = b_5, \quad c = -b_3,
\]

then substituting this relations and using the transformation equation \( \phi = e^\Phi \), we have the following solution to \( \psi_2 \)

\[
\psi_2 = \phi^{\frac{1+\sigma n}{2}} \left[ c_1 K_\nu \left( \kappa \phi \frac{b_4}{n} \right) + c_2 I_\nu \left( \kappa \phi \frac{b_4}{n} \right) \right], \quad \nu = \frac{1}{b_5} \sqrt{\xi},
\]

(43)

where \( (K_\nu, I_\nu) \) are the modified Bessel functions, \( \kappa = \frac{2^{\frac{1}{2} - \frac{1}{n}}}{b_5} e^{\frac{2n - 3}{6} \mu_0} \) and \( \xi = (1 + r - \mu_0)^2 + \frac{2 \omega \mu_0}{3} [\mu_0 + 1 + Q] \). Then, the wavefunction have the following form

\[
\Psi_\nu = \Lambda^{-\mu_0} \phi^{\frac{1+\sigma n}{2}} \left[ c_1 K_\nu \left( \kappa \phi \frac{b_4}{n} \right) + c_2 I_\nu \left( \kappa \phi \frac{b_4}{n} \right) \right].
\]

(44)

The corresponding value for all constants that appear in the quantum calculation are presented in table I

| \( \gamma \) | 0 | 1 | -1 |
|---|---|---|---|
| \( b_2 \) | \( \frac{2b_4 \lambda_0 + 3\sigma^\pm}{2} \) | \( 1 + 2\sigma^\pm \) | \( \lambda_0 \) |
| \( \nu \) | \( \frac{1}{\lambda_0} \sqrt{\xi} \) | \( \frac{1 + 2\sigma^\pm}{2} \sqrt{\xi} \) | \( \frac{1}{\lambda_0} \sqrt{\xi} \) |
| \( \kappa \) | \( \frac{2}{b_5} \sqrt{\frac{2}{3} \ln e} e^{-\frac{b_4}{n} \lambda_0} \) | \( \frac{2}{1 + 2\sigma^\pm} \sqrt{\frac{3}{2} \ln e} e^{-b_4} \) | \( \frac{2}{b_5} \sqrt{\frac{2}{3} \ln e} e^{-\frac{b_4}{n} \lambda_0} \) |

Table II: All constants that appear in the quantum solutions for various values of \( \gamma \). Here \( \xi = (1 + r - \mu_0)^2 + \frac{2 \omega \mu_0}{3} [\mu_0 + 1 + Q] \)

V. CONCLUSIONS

In this paper we have investigated the flat FRW cosmological model of the universe in the framework of Barber’s second self-creation theory since of point to view of the Hamiltonian systems, where the classical solutions were found under the Hamilton-Jacobi approach combined with the Lagrange-Charpit method for all values of the barotropic parameter in a perfect fluid. By means of the deceleration parameter we obtain that the universe in this theory always suffers an expansion in any epoch of the universe. In the quantum behavior, the solutions were found using the curves characteristics method applied to partial differential equation of second degree, obtaining the solutions, as in the classical scheme, for all values in the \( \gamma \) parameter.
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