Uniqueness of the non-singular family and characterisation of cosmological models

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Abstract

We prove that for an orthogonal spacetime metric separable in space and time in comoving coordinates, the requirements of perfect fluid and non-singularity single out the unique family of singularity free cosmological models. Further homogeneous models could only be Bianchi I or FLRW while inhomogeneous ones can be with or without a singularity.

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The characteristics of the relativistic cosmology are the expansion and the big-bang singular origin of the Universe. The imprint of the latter is believed to be seen in the cosmic microwave background radiation observations [1,2]. The theoretical model imbibing these features is an exact solution of Einstein’s equations known as the Friedman-Lemaitre-Robertson-Walker (FLRW) model. It represents a homogeneous and isotropic Universe filled with perfect fluid. This is the generally accepted current model of the Universe. Despite its success, there are some questions of principles that deserved to be addressed to. The model is highly specialised for it is homogeneous and isotropic. These properties could by no means be considered generic enough features of the Universe. The Universe should in fact have more general initial conditions. Secondly, inhomogeneity may be quite appropriate for evolution of large scale structures in the Universe. Above all even at the present epoch, the question of homogeneity

It is therefore important to find solutions of Einstein’s equations when homogeneity and isotropy could be dropped. The first step in this direction came with the Bianchi models that are homogeneous but anisotropic. For a long time only homogeneous models were studied. Inhomogeneity came in through what is known as orthogonally transitive $G_2$ cosmologies [3-5]. These are the simplest inhomogeneous models that admit two spacelike Killing vectors which are mutually as well as hypersurface orthogonal. The class of solutions of Einstein’s equations of inhomogeneous family was first obtained by Wainwright and Goode [6] and subsequently by others [7-9]. Amongst them was a remarkable solution obtained by Senovilla [8] which was free of the big-bang (or of any other kind) singularity. It
satisfied all the physical reasonableness conditions and had an acceptable equation of state \( \rho = 3p > 0 \). Until then on the strength of the singularity theorems [10] it was generally believed that the occurrence of big-bang is an inevitable feature of general relativity (GR). The conflict was soon resolved [11] when it was shown that the solution in question does not obey the assumption of existence of compact trapped surfaces and hence the singularity theorems become inapplicable. All prior attempts to construct singularity free models had either to ascribe physically unacceptable behaviour for matter leading to violation of energy and causality conditions or to invoke quantum effects or modification of GR [12,13]. Senovilla’s [8] was the first singularity free solution, true to GR, conforming to energy and causality conditions.

The most interesting feature of inhomogeneous cosmology is the non-occurrence of singularity in the models. Hence it has no beginning and no end. This is aesthetically very attractive and appealing feature, which was first propounded by the steady state theory. Ruiz and Senovilla [14] have identified a large family of singularity free cosmological models. This family is unique for cylindrically symmetric metric separable in space and time in the comoving coordinates (i.e. fluid lines are orthogonal to \( t = \text{const.} \) surface). In this note we wish to establish the uniqueness of the non-singular family by dropping the assumption of cylindrical symmetry and in the process we succeed in characterising the perfect fluid models.

Before we go any further let us note a general result arising out of the following two relations [15],

\[
\theta, \alpha = \frac{3}{2} \left[ (\sigma^i + \omega^i)_{,i} - (\sigma_{\alpha i} + w_{\alpha i}) \dot{u}^i \right]
\] (1)
\[
\theta \dot{u}_\alpha + \frac{1}{\sqrt{g_{00}}} \left( \ln \sqrt{|g/g_{00}|} \right)_{,0\alpha}
\]

where \( \theta, \sigma, \omega, \dot{u}_\alpha \) are the kinematic parameters; expansion, shear, rotation and acceleration, \( \dot{u}_\alpha = u_{\alpha;i}u^i \). We have however assumed \( u_i = \sqrt{g_{00}}\delta_i^0 \).

We infer from the above relations:

**Lemma**: In the absence of shear and vorticity, the expansion of fluid is constant over the 3-space orthogonal to the fluid congruence and further, the acceleration also vanishes when the quantity \( g/g_{00} \) is a separable function of space and time in the comoving coordinates.

**Corollary**: For the vorticity free spacetime with the separability (as is the case for the metric (3)), acceleration can be non-zero only if shear is non-zero.

According to the Raychaudhuri equation [16], in the absence of vorticity acceleration is necessary for halting the collapse to avoid singularity which in our case can only exist if shear is non-zero. Thus non-singular solutions represented by the metric (3) will always have to be both inhomogeneous and anisotropic.

In cosmology in general and \( G_2 \) cosmologies in particular the metric is generally taken to be orthogonal and the comoving coordinates are employed. That is the fluid velocity vector is vorticity free and is orthogonal to \( t = \text{const.} \) hypersurface. Further as is common for most cosmological models, we take the metric to be separable in space and time coordinates. Hence we write the metric in the form

\[
ds^2 = Ddt^2 - A dx_1^2 - B dx_2^2 - C dx_3^2
\]
where by separability we mean $A = A(t)A(x_\alpha)$ and so on. The velocity field of the fluid is $U_i = \sqrt{D}\delta^0_i$.

It can be easily seen that the invariant characterisation of this metric is given by (i) $\theta, \alpha = \theta \dot{u}_\alpha$ and (ii) $(\sigma/\theta), \alpha = 0$, i.e. the anisotropy parameter is constant on the 3-hypersurface orthogonal to the fluid flow [17]. The condition (i) follows from eqn (2) while (ii) can be verified easily by writing $\sigma/\theta$ for the metric (3).

Let us couch the general result as a theorem as follows:

**Theorem:** Let the metric (3) be a perfect fluid cosmological solution of Einstein’s equations then it can only be

(a) if homogeneous, $G_3$ (homogeneity) - Bianchi I or $G_6$ (both homogeneity and isotropy) - FLRW model

(b) if inhomogeneous, $G_2$ (admitting only two spacelike Killing vectors) models with or without the big-bang singularity.

Further the singularity free family as already identified in [14] is unique and is cylindrically symmetric.

**Proof:** The theorem characterises all the perfect fluid models that the metric (3) can represent. But for the separability and orthogonality of the metric we make no assumptions. The fluid conditions alone will impose symmetries on the metric.

The perfect fluid distribution will imply the conditions; $T_{0\alpha} = 0, T_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $T^1_1 = T^2_2 = T^3_3$.

To go any further we need the explicit expressions for $T^k_i$ [18] which look quite frightening and formidable. Fortunately, we have discovered an underlying order
in them that allows us to write the rest of them from the given two (one each of
diagonal and off diagonal) by prescribing the appropriate permutation rules. We
begin by

\[-32\pi AT^1_0 = -2\left(\frac{B_0}{B} + \frac{C_0}{C}\right)_1 + \frac{A_0}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) + \frac{B_0}{B}\left(-\frac{B_1}{B} + \frac{D_1}{D}\right) \]
\[+ \frac{C_0}{C}\left(-\frac{C_1}{C} + \frac{D_1}{D}\right)\]

\[\text{(4)}\]

\[-32\pi T^1_1 = \frac{1}{A}\left[\frac{B_1 C_1}{BC} + \frac{D_1}{D}\left(\frac{B_1}{B} + \frac{C_1}{C}\right)\right]
+ \frac{1}{B}\left[2\left(\frac{C_2}{C} + \frac{D_2}{D}\right) + \frac{C_2}{C}\left(-\frac{B_2}{B} + \frac{C_2}{C}\right) + \frac{D_2}{D}\left(-\frac{B_2}{B} + \frac{C_2}{C} + \frac{D_2}{D}\right)\right]
+ \frac{1}{C}\left[2\left(\frac{B_3}{B} + \frac{D_3}{D}\right) + \frac{B_3}{B}\left(\frac{B_3}{B} - \frac{C_3}{C}\right)\right]
+ \frac{D_3}{D}\left(\frac{B_3}{B} - \frac{C_3}{C}\right) + \frac{D_3}{D}\left(\frac{B_3}{B} - \frac{C_3}{C} + \frac{D_3}{D}\right)
+ \frac{1}{D}\left[-2\left(\frac{B_0}{B} + \frac{C_0}{C}\right)_0 - \frac{B_0}{B}\left(\frac{B_0}{B} + \frac{C_0}{C} - \frac{D_0}{D}\right) - \frac{C_0}{C}\left(\frac{C_0}{C} - \frac{D_0}{D}\right)\right]\]

\[\text{(5)}\]

where a subscript denotes partial differentiation and here the assumption of
separability is not effected.

The successive cyclic permutations \(A \to B \to C \to A\) and \(1 \to 2 \to 3 \to 1\) will
give \(T^2_0, T^3_0\) from \(T^1_0\); \(T^2_3, T^3_1\) from \(T^1_2\); and \(T^2_2, T^3_3\) from \(T^1_1\). To write \(T^1_2\) from
\(T^1_0\), let \(0 \to i2\) (i.e. \(A_0 \to iA_2, T^1_0 \to iT^1_2\)) and \(B \to C \to D \to B\) while \(T^0_0\) follows
from \(T^1_1\) for \(2 \to 3 \to 1 \to i0 \to -2(T^1_1 \to T^0_0)\) and \(A \to D \to B \to C \to A\). Thus
we can write all ten \(T^k_i\), given the two, one each of diagonal and off diagonal.

Let us begin with the general case where the metric is a function of all the
coordinates. Eqns. \(T_{\alpha 0} = 0\), of which \(T_{10} = 0\) reads as
\[
\frac{D_1}{D} = \left( \frac{C_0/C - A_0/A}{B_0/B + C_0/C} \right) \frac{C_1}{C} + \left( \frac{B_0/B - A_0/A}{B_0/B + C_0/C} \right) \frac{B_1}{B}
\]

(6)

and similarly \(D_2/D\) and \(D_3/D\). It is clear if \(A_0/A = B_0/B = C_0/C\) which implies \(\sigma = 0\) and \(\dot{u}_\alpha = 0\) (because \(D_\alpha = 0\)), then the spacetime is both isotropic and homogeneous and it can be no other than the big-bang singular FLRW [19].

If \(A_0/A \neq B_0/B \neq C_0/C\) and since the metric is assumed to be separable in \(t\) and \(x_\alpha\), the above equation will imply

\[
C_0/C - A_0/A = k_1(B_0/B - A_0/A) + n_1(B_0/B + C_0/C)
\]

(7)

and

\[
B_1/B = k_1C_1/C
\]

(8)

where \(k_1\) and \(n_1\) are constants, and similar equations will follow from \(D_2/D\) and \(D_3/D\). We can now set the exact differential for the space dependence of \(D\),

\[
d(lnD) = (lnD)_1dx_1 + (lnD)_2dx_2 + (lnD)_3dx_3
\]

which can be integrated along any path to give the same result. Evaluating it along two different paths, we obtain \(B = C^{k_1}, D = C^{m_1}\) and \(A = C^{k_2}\). Thus the space dependence of the metric is all determined but for the single function \(C(x_\alpha)\). Eqns. \(T_{\alpha\beta} = 0\) for \(\alpha \neq \beta\) determine \(C(x_\alpha) = constant\). Hence the metric can only represent homogeneous and big-bang singular Bianchi I model. When \(A_0/A = B_0/B \neq C_0/C\), eqn (6) and its permutants will imply that either it is Bianchi I or it admits a spacelike Killing vector. This is the case we consider next.
We shall now consider the case of metric (3) admitting a spacelike Killing vector, say $\frac{\partial}{\partial x_3}$. That means the metric depends only on the two space variables $x_1$ and $x_2$.

From (5) it is clear that $T^1_1 = T^2_2 = T^3_3$ will imply the two equations of the type,

$$\frac{f_1}{A(t)} + \frac{f_2}{B(t)} + \frac{f_3}{C(t)} = F(t) \quad (9)$$

where $f_1, f_2, f_3$ are functions of space variables $x_\alpha$ and containing derivatives with respect to $x_1, x_2$ and $x_3$ respectively.

In this case $f_3 = 0$ in (9). Note that $T^\alpha_{00} \equiv 0$ now and $T_{\alpha0} = 0$ will lead to $A(t) = B(t) \neq C(t)$ and $D(x_\alpha) = C^k(x_\alpha), k= \text{const}$. Hence $\sigma$ can be non-zero to give rise to acceleration which in turn can lead to a viable non-singular case. There will be three equations ($T^\alpha_{12} = 0$ plus the two following from (9)) determining the space dependence of the metric.

Since $A(t) = B(t)$, it is possible to perform a coordinate transformation to set $A(x_\alpha) = B(x_\alpha)$, which means $A = B$. A very detailed and involved Lie analysis of the equations [20] leads to the conclusion that space dependence can only arise as a function of $x_1 + x_2$ or $x_1^2 + x_2^2$ or $x_1/x_2$ (i.e. $A(x_\alpha) = A(x_1 + x_2)$ etc). In either of the first two cases, it could be reduced to the case of one-coordinate dependence by coordinate transformations. The last case is obviously singular and would not lead to a viable model.

When $A_0/A \neq B_0/B \neq C_0/C$, following the above route, eqns. $T_{\alpha0} = 0$ determine
all others in terms of $C(x_\alpha), \alpha = 1, 2; T_{12} = 0$ fixes $C(x_\alpha) = (f(x_1) + g(x_2))^k$ and
then eqns. (9) force $C(x_\alpha)$ to be a constant. We are again led to Bianchi I model.

Finally we come to the case of $G_2$ models which admit two spacelike Killing vectors that are mutually as well as hypersurface orthogonal. The metric depends upon only one space variable.

Now the spacetime is general enough to sustain inhomogeneous perfect fluid. There will occur two kinds of inhomogeneous fluid models, one with singularity and the other without it [14]. The singularity free family possesses cylindrical symmetry and is unique. That means that the already identified singularity free family [14] is unique not only for cylindrical symmetry but for the general metric (3). In this case homogeneous models can however occur but will be Bianchi I only.

Thus is proven the theorem.

We give below the general enough [21] though not the most general [14] inhomogeneous non-singular cosmological models described by

\[ ds^2 = \cosh^{2\alpha}(kt) \cosh^{2\alpha}(mr)(dt^2 - dr^2) - \cosh^{2\beta}(kt) \cosh^{2b}(mr)dz^2 \]
\[ - m^{-2} \sinh^{2}(mr) \cosh^{2\alpha}(kt) \cosh^{2c}(mr)d\phi^2 \]

with $\alpha + \beta = 1$. It admits the only two cases, (i) $b = c$ and (ii) $b + c = 1$ for perfect fluid distribution. The former includes the Senovilla’s solution [8] when $c = b = -1/3, a = 1, \alpha = 2, \beta = -1$ and $k = 3m$; and

\[ \rho = 3p = \frac{15k^2}{8\pi} \cosh^{-4}(kt) \cosh^{-4}(kr/3) > 0. \]

The latter always gives the stiff fluid,
\[ 8\pi\rho = 8\pi p = (b^2 - 4)m^2 \cosh^{-2\alpha-2}(2mt) \cosh^{-2\alpha}(mr) \]

(12)

with \(\alpha = 1 - b/2, a = b(b - 1), c = 1 - b, k = 2m\). Clearly \(b^2 \geq 4\) which means \(b\) must lie outside the interval \(-2 \leq b \leq 2\).

Though \(\rho > 0\) for both \(b > 2\) and \(b < -2\), the geodesic completeness [22] demands \(b < 0\), and hence the latter case is the truly singularity free solution. At the two ends of the range for \(b = \pm 2\), we have the two distinct empty space solutions as matter free limits of the stiff fluid. Here again, \(b = -2\) will be geodesically complete and it can be thought of as representing field of plane gravitational waves. We have also shown [23] elsewhere that a natural inhomogenisation of FLRW open model leads to the non-singular metric (9).

The metric (3) can only describe fluid models of the following kinds; homogenous FLRW and Bianchi I and \(G_2\) inhomogenous with or without singularity. That is inhomogeneity cannot be sustained by any symmetry higher than \(G_2\) [20]. This is an important conclusion; at whatever scale we wish to incorporate inhomogeneity in fluid models represented by an exact solution of Einstein’s equations, the metric has to be \(G_2\). Howsoever much it is not supported by observations, it appears essential for bringing in inhomogeneity, which cannot be ignored even at the present epoch in view of lumpiness in the Universe. The ideal solution to the question is if inhomogeneity and anisotropy were scale and time dependent. Then at larger scales and at late times the model could have evolved to homogeneous and isotropic FLRW. This unfortunately cannot happen for the
metric (3) for $\sigma/\theta$ turns out to be constant for inhomogeneous fluid models.

It may be noted that the metric (3) is general enough for cosmology. For both the assumptions of orthogonality and separability are quite common and shared by most of the cosmological models. It is in this context very important that it can only admit inhomogeneous perfect fluid models that could never isotropise. The only hope is to give up the separability which unfortunately will make the problem mathematically formidable. If one can succeed in finding an inhomogeneous non-singular solution of Einstein’s equations that isotropises to approximate to FLRW as time progresses, it will have very important bearing on our overall cosmological perception of the Universe.

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