Edge-Matching Graph Contractions and their Interlacing Properties

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Abstract

Interlacing is a property comparing the entire spectra of two graph matrices. For a given graph $\mathcal{G}$ of order $n$ with $m$ edges and a real symmetric graph matrix $M(\mathcal{G}) \in \mathbb{R}^{n \times n}$, the interlacing graph reduction problem is to find a graph $\mathcal{G}_r$ of order $r < n$ such that the eigenvalues of $M(\mathcal{G}_r)$ interlace the eigenvalues of $M(\mathcal{G})$. Graph contractions over partitions of the vertices are widely used as a combinatorial graph reduction tool. In this study, we define a class of edge-matching graph contractions and show how two types of edge-matching contractions provide Laplacian and normalized Laplacian interlacing. An $O(mn)$ algorithm is provided for finding a normalized Laplacian interlacing contraction and an $O(n^2 + nm)$ algorithm is provided for finding a Laplacian interlacing contraction.

Keywords: Spectral clustering, Laplacian interlacing, Graph contractions

1. Introduction

The effect of combinatorial operations on graph spectra is an evolving branch of graph theory, linking together combinatorial graph theory with the spectral analysis of the algebraic structures of graphs. In general, there is an interest to understand how certain graph reduction operations relate to the spectral and combinatorial properties. Of particular interest are reductions that satisfy an

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interlacing property between algebraic graph representations. Interlacing properties of algebraic structures of graphs have been shown to have combinatorial interpretations. Haemers has used the adjacency and Laplacian interlacing to provide combinatorical results on the chromatic number and spectral bounds [7]. The neighborhood reassignment operation has been shown to provide an interlacing of the normalized Laplacian [15], and Chen et al. provide an interlacing result on contracted normalized Laplacians [2].

Partitioning the vertices of a graph is a combinatorial operation extensively studied in graph theory in the context of graph clustering [12] and network communities [9], and for spectral clustering methods [10]. Partitioning combined with node and edge contractions along those partitions lead to reduced order graphs. In this direction, we define edge-matching contractions as a class of graph contractions with one-to-one correspondence of a subset of edges in the full order graph to those in the contracted graph. We then explore two types of edge-matching contractions, cycle invariant contractions and node-removal equivalent contractions. Cycle-invariant contractions preserve the cycle structure of the graph in the contracted graph, and node-removal equivalent contractions are cases where a contraction can be obtained also from a node-removal operation. We show how these contraction types provide interlacing of normalized-Laplacian and Laplacian graph matrices. Two algorithms of complexity $O(mn)$ and $O(n^2 + nm)$ are then provided for finding a cycle-invariant contraction and a node-removal equivalent contraction respectively, if exists, for a given graph with $n$ vertices and $m$ edges.

The remaining sections of this paper are as follows. In Section 2 the interlacing graph reduction problem is presented. In Section 3 we formulate the graph contraction operation for simple undirected graphs, and introduce the class of edge-matching graph contractions and two sub-classes of cycle-invariant and node-removal equivalent graph contractions. In Section 4 the interlacing graph reduction problem is solved for these two classes for the Laplacian and normalized-Laplacian matrices, and Section 5 provides case studies of the interlacing methods.
Preliminaries. The integer set \( \{1, \ldots, n\} \) is denoted as \([1, n]\). An undirected graph \( G = (V, E) \) consists of a vertex set \( V(G) \), and an edge set \( E(G) = \{e_1, \ldots, e_{|E|}\} \) with \( e_k \in V^2 \). The order of the graph is the number of vertices \(|V(G)|\). Two nodes \( u, v \in V(G) \) are adjacent if they are the endpoints of an edge, and we denote this by \( u \sim v \). The neighborhood \( N_v(G) \) is the set of all nodes adjacent to \( v \) in \( G \). The degree of a node \( v \), denoted \( d_v(G) \), is the number of nodes adjacent to it, \( d_v(G) = |N_v(G)| \). A path in a graph is a sequence of distinct adjacent nodes. A simple cycle is a path with an additional edge such that the first and last vertices are repeated. A graph \( G \) is connected if we can find a path between any pair of nodes. A simple graph does not include self-loops or duplicate edges. A multi-graph is a graph that may include duplicate edges. We denote \( G \backslash V_R \) as the graph obtained from \( G \) by removing all nodes \( v \in V_R \subseteq V \) from \( V(G) \) and removing all edges in \( E(G) \) adjacent to \( v \). We denote \( G \backslash E_R \) as a graph obtained from \( G \) by removing all edges \( e \in E_R \) from \( E(G) \). A subgraph \( G_S = (V_S, E_S) \) of a graph \( G = (V, E) \), denoted as \( G_S \subseteq G \), is any graph such that \( V_S \subseteq V \) and \( E_S \subseteq E \cap V_S^2 \). An induced subgraph \( G[V_S] \) is a subgraph \( G_S \subseteq G \) such that \( E_S = E_G \cap V_S^2 \). An induced subgraph \( G[V_S] \) is a connected component of \( G \) if it is connected and no node in \( V_S \) is adjacent to a node in \( V(G) \backslash V_S \). The set \( \mathbb{T}(G) \) denotes all spanning trees of a connected graph \( G \). For \( T \in \mathbb{T}(G) \), the co-tree graph \( G \backslash E(T) \) is denoted as \( C(T) \).

2. Interlacing Graph Reductions

Graph matrices are algebraic representations of graphs, and the spectral and algebraic properties of these matrices can provide insights about combinatorial properties of the underlying graph, e.g., Fiedler’s seminal results on the Laplacian algebraic connectivity \([4]\). The interlacing property of matrices has been extensively studied with classic algebraic results such as the Poincare separation theorem \([1] \text{ p. 119}\), and matrix combinatorial results such as the relation of equitable partitions with tight interlacing \([6]\). Here we study what types of reduced graphs have interlacing graph matrices.
The spectrum of a real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is the set of eigenvalues \( \{\lambda_k(A)\}_{k=1}^n \) where \( \lambda_k(A) \) is the \( k \)th eigenvalue of \( A \) in ascending order. Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{r \times r}$ be real symmetric matrices with $0 < r < n$. Then the eigenvalues of $B$ interlace the eigenvalues of $A$, denoted $B \preceq A$, if $\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{n-r+k}(A)$ for $k = 1, 2, \ldots, r$. The interlacing is tight if $\lambda_k(A) = \lambda_k(B)$ or $\lambda_k(B) = \lambda_{n-r+k}(A)$ for $k = 1, 2, \ldots, r$. It is straightforward to show that interlacing is a transitive property.

**Proposition 1.** Let $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$ and $A_3 \in \mathbb{R}^{n_3 \times n_3}$ be real symmetric matrices with $0 < n_3 < n_2 < n_1$. If $A_3 \preceq A_2$ and $A_2 \preceq A_1$, then $A_3 \preceq A_1$.

*Proof.* From $A_3 \preceq A_2$ and $A_2 \preceq A_1$ we have $\lambda_k(A_2) \leq \lambda_k(A_3) \leq \lambda_{n_2-n_3+k}(A_2)$ for $k = 1, 2, \ldots, n_3$ and $\lambda_l(A_1) \leq \lambda_l(A_2) \leq \lambda_{n_1-n_2+l}(A_1)$ for $l = 1, 2, \ldots, n_2$. From $l = k$ we get $\lambda_k(A_1) \leq \lambda_k(A_2) \leq \lambda_k(A_3)$, and from $l = n_2 - n_3 + k$ we get $\lambda_k(A_3) \leq \lambda_{n_2-n_3+k}(A_2) \leq \lambda_{n_1-n_2+l}(A_1)$, such that $\lambda_k(A_1) \leq \lambda_k(A_3) \leq \lambda_{n_1-n_3+k}(A_1)$ for $k = 1, 2, \ldots, n_3$ and we obtain that $A_3 \preceq A_1$. \qed

The most commonly studied matrices in algebraic graph theory are the adjacency matrix $A(G) \in \mathbb{R}^{|V| \times |V|}$, the Laplacian matrix $L(G) \in \mathbb{R}^{|V| \times |V|}$ and the normalized Laplacian matrix $\mathcal{L}(G) \in \mathbb{R}^{|V| \times |V|}$, all of which are real symmetric matrices. They are defined below, where each row and column is indexed by a vertex in the graph $G$.

\[
[A(G)]_{uv} = \begin{cases} 1, & u \sim v \\ 0, & \text{otherwise} \end{cases},
\]

\[
[L(G)]_{uv} = \begin{cases} d_u(G), & u = v \\ -1 & u \sim v \\ 0, & \text{otherwise} \end{cases},
\]

and

\[
[L(G)]_{uv} = \begin{cases} 1, & u = v \\ -\left(\sqrt{d_u(G)d_v(G)}\right)^{-1} & u \sim v \\ 0, & \text{otherwise} \end{cases}.
\]
We now extend the notion of spectral interlacing properties to graphs.

**Definition 1** (interlacing graphs). Consider two graphs $G_n$ and $G_r$ of order $n$ and $r$ respectively, with $n > r$, and let $M(G) \in \mathbb{R}^{n \times n}$ be any real symmetric matrix associated with the graph $G$. We say that the two graphs are $M$-interlacing if $M(G_r) \preceq M(G_n)$, and denote the property by $G_r \preceq_M G_n$.

The problem arising naturally from the definition of interlacing graphs is the interlacing graph reduction problem.

**Problem 1** (interlacing graph reduction). Consider a graph $G_n$ of order $n$ and let $M(G) \in \mathbb{R}^{n \times n}$ be any real symmetric matrix associated with the graph $G$. Find a graph $G_r$ of a given order $r < n$ such that $G_r \preceq_M G_n$.

Finding a solution to Problem 1 may be numerically intractable for a moderate number of nodes, as the number $c_r$ of simple connected graphs of order $r$ increases exponentially according to the recurrence

$$\sum_k \binom{r}{k} k c_k 2^{r-k} = r 2^{r-1}$$

for $r \geq 1$ [13, p.87], e.g., for $r = 1, \ldots, 6$, $c_r = 1, 1, 4, 38, 728, 26704$.

### 3. Graph Contractions

Graph contractions are a graph reduction method based on partitions of the vertex set. They are a useful algorithmic tool applied to a variety of graph-theoretical problems, e.g., for obtaining the connected components [3] or finding all spanning trees of a graph [8, 14]. We now define several graph operations required for vertex partitions and graph contractions and derive results that will allow us to relate graph contractions and graph interlacing.

For an integer $r$ satisfying $1 \leq r \leq n$, an $r$-partition of a vertex set $\mathcal{V}$ of order $n$, denoted $\pi_r(\mathcal{V})$, is a set of $r$ cells $\{C_i\}_{i=1}^r$ such that $C_i \cap C_j = \emptyset$ and $\bigcup_{i=1}^r C_i = \mathcal{V}$. We denote the $i$th cell of a partition $\pi$ as $C_i(\pi)$, and the cell neighborhood $\mathcal{N}_{C_i}(G)$ is defined as $\mathcal{N}_{C_i} \triangleq \{ \cup_{v \in C_i} \mathcal{N}_v(G) \} \setminus C_i$. For $r = n$, $C_i(\pi_n) = i$ is the identity partition, which contains $n$ singletons (a cell with a single vertex). An atom partition $\pi_{n-1}(\mathcal{V})$ contains $n-2$ singletons and a single 2-vertex cell. The set of all $r$-partitions of $\mathcal{V}$ is denoted by $\Pi_r(\mathcal{V})$, and
the set of all partitions of $V$ is $\Pi(V) \triangleq \cup_{r=1}^{n} \Pi_{r}(V)$. For a graph $G = (V, E)$, we may denote $\pi_{r}(V)$ and $\Pi_{r}(V)$ as $\pi_{r}(G)$ and $\Pi_{r}(G)$. For a graph with $n_{cc}$ connected components, we define the connected components partition $\pi_{cc}(G)$ as the partition $\pi_{cc}(G) = \{C_{i}\}_{i=1}^{n_{cc}}$, such that $G[C_{i}]$ is the $i$th connected components of $G$. Hereafter $G = (V, E)$ is a simple connected graph of order $n$.

**Definition 2** (partition function). For a graph $G$ and $r$-partition $\pi \in \Pi_{r}(G)$, the partition function is a map $f_{\pi}: V(G) \rightarrow [1, r]$ from each node in $V$ to its cell index, i.e., $f_{\pi}(v) \triangleq \{i \in [1, r] | C_{i}(\pi) \cap v \neq \emptyset\}$. More generally, for a subset $V_{S} \subseteq V(G)$ we have $f_{\pi}(V_{S}) \triangleq \{i \in [1, r] | C_{i}(\pi) \cap V_{S} \neq \emptyset\}$.

The quotient of a graph $G$ over a partition $\pi \in \Pi_{r}(G)$, denoted by $G/\pi$, is the multi-graph of order $r$ with an edge $\{u, v\}$ for each edge between nodes in $C_{u}(\pi)$ and $C_{v}(\pi)$, i.e., $G/\pi = \left([1, r], \{\tilde{\epsilon}_{j}\}_{j=1}^{|E|}\right)$ with

$$\tilde{\epsilon}_{j} = \{f_{\pi}(h_{E}(\epsilon_{j})), f_{\pi}(t_{E}(\epsilon_{j}))\}$$

where $\epsilon_{j} \in E(G)$ and $h_{E}(\epsilon), t_{E}(\epsilon): E(G) \rightarrow V(G)$ assign a head and a tail to the end-nodes of each edge. The graph contraction of $G$ over $\pi$ is the simple graph denoted as $G/\sslash_{\pi}$ which is obtained from the quotient $G/\pi$ by removing all self-loops and redundant duplicate edges, $G/\sslash_{\pi} = \left([1, r], E_{r}\right)$ with $E_{r} = \{\tilde{\epsilon} \in [1, r]^{2} | \tilde{\epsilon} \in E(G/\pi), h_{E}(\tilde{\epsilon}) \neq t_{E}(\tilde{\epsilon})\}$. If $\pi$ is an atom partition we call $G/\sslash_{\pi}$ an atom contraction. For example, consider the partition $\pi = \{\{v_{1}\}, \{v_{2}\}, \{v_{3}\}, \{v_{4}, v_{5}\}\}$, for the graph $G$ shown in Figure 1. The quotient $G/\pi$ and contraction $G/\sslash_{\pi}$ of the graph are shown in Figure 1. Notice that this is an example of an atom partition and atom contraction.

Node removal is the simplest graph-reduction method. However, in some cases the same reduced graph can be obtained either from node-removal or from a graph contraction. We define here these contractions as node-removal equivalent contractions.
Definition 3 (node-removal equivalent contraction). For the graph $\mathcal{G}$ and its contraction $\mathcal{G} \sslash \pi$, we say that $\mathcal{G} \sslash \pi$ is node-removal equivalent if there is a subset $\mathcal{V}_S \subset \mathcal{V}(\mathcal{G})$ such that $\mathcal{G} \sslash \pi = \mathcal{G} \setminus \mathcal{V}_S$.

Cycles play an important role in the properties of graphs, and we define a cycle-invariant graph contraction as a contraction that preserves the cycle structure of the full graph.

Definition 4 (cycle-invariant contraction). Consider a graph $\mathcal{G}$ and its contraction $\mathcal{G} \sslash \pi$, and let $\mathcal{S}_{\text{cyc}}(\mathcal{G})$ be the set of all simple cycles of $\mathcal{G}$, i.e., $\mathcal{S}_{\text{cyc}}(\mathcal{G}) = \{\mathcal{E}_{\text{cyc}} \subseteq \mathcal{E}(\mathcal{G}) | \mathcal{E}_{\text{cyc}} \text{ is a simple cycle of } \mathcal{G}\}$. Then we say that the contraction $\mathcal{G} \sslash \pi$ is cycle-invariant if there is one-to-one correspondence between the cycles of the full-order graph and the cycles of the contracted graph, i.e., $\forall \mathcal{E}_{\text{cyc}} \in \mathcal{S}_{\text{cyc}}(\mathcal{G})$, $\exists \tilde{\mathcal{E}}_{\text{cyc}} \in \mathcal{S}_{\text{cyc}}(\mathcal{G} \sslash \pi)$ such that $|\tilde{\mathcal{E}}_{\text{cyc}}| = |\mathcal{E}_{\text{cyc}}|$ and $\forall \tilde{\epsilon} \in \tilde{\mathcal{E}}_{\text{cyc}}$, $\exists \epsilon \in \mathcal{E}_{\text{cyc}}$ such that $\tilde{\epsilon} = \{f_\pi(h_\mathcal{E}(\epsilon)), f_\pi(t_\mathcal{E}(\epsilon))\}$.

For example, consider the partition $\pi = \{\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}\}$ for the graph shown in Figure 2. The resulting contraction over the graph is cycle-invariant (Definition 3) with $\mathcal{S}_{\text{cyc}}(\mathcal{G}) = \{\{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_3\}\}$ and $\mathcal{S}_{\text{cyc}}(\mathcal{G} \sslash \pi) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\}$, and is also node-removal equivalent (Definition 3) with $\mathcal{V}_S = \{v_1, v_2\}$. Notice that if the edge $\{v_1, v_5\}$ were added in Figure 2, the same contraction would not be a cycle-invariant contraction; however, it would still be node-removal equivalent with $\mathcal{V}_S = \{v_1, v_2\}$.
Lemma 1 (subgraph contraction lemma). Consider a graph $G$ and its subgraph $G_R = G \setminus \mathcal{E}_R$ for $\mathcal{E}_R \subseteq \mathcal{E}(G)$. Then for any $\pi \in \Pi(G)$, $G_R / \pi \subseteq G / \pi$.

Proof. For any $\hat{\epsilon} \in \mathcal{E}(G_R / \pi)$ we can find $\epsilon \in \mathcal{E}(G_R)$ such that $\hat{\epsilon} = \{f_\pi(h_\epsilon (\epsilon)), f_\pi(t_\epsilon (\epsilon))\}$. Since $\mathcal{E}(G_R) \subseteq \mathcal{E}(G)$, therefore $\epsilon \in \mathcal{E}(G)$ and $\{f_\pi(t_\epsilon (\epsilon)), f_\pi(h_\epsilon (\epsilon))\} \in \mathcal{E}(G / \pi)$. We conclude that $\mathcal{E}(G_R / \pi) \subseteq \mathcal{E}(G / \pi)$, and since $\mathcal{V}(G_R / \pi) = \mathcal{V}(G / \pi)$ we obtain that $G_R / \pi \subseteq G / \pi$.

Lemma 2. Consider a graph $G$ and its contraction $G / \pi$ for $\pi \in \Pi(G)$. Then $\forall u \in \mathcal{V}(G), \forall \hat{u} \in \mathcal{V}(G / \pi)$, we have $u \in N_{C_{\hat{u}}}(G)$ if and only if $f_\pi(u) \sim \hat{u}$.

Proof. If $u \in N_{C_{\hat{u}}}$ then $\exists v \in C_{\hat{u}}$ such that $u \sim v$ with $\epsilon = \{u, v\} \in \mathcal{E}(G)$, and therefore $\{f_\pi(u), f_\pi(v)\} = \{f_\pi(u), \hat{u}\} \in \mathcal{E}(G / \pi)$ and $f_\pi(u) \sim \hat{u}$. If $f_\pi(u) \sim \hat{u}$, then $\exists v \in C_{\hat{u}}$ such that $u \sim v$ and therefore $u \in N_{C_{\hat{u}}}$.

Lemma 3. If a graph $G$ is connected then its graph contraction $G / \pi$ is connected.

Proof. If $G$ is connected then $\forall u, v \in \mathcal{V}$, there is a path $wu_1u_2 \ldots u_pv$. For any $\hat{u}, \hat{v} \in \mathcal{V}(G / \pi)$ we can find $u, v \in \mathcal{V}$ such that $f_\pi(u) = \hat{u}$ and $f_\pi(v) = \hat{v}$. If we
then apply the partition function on the path $uu_1u_2\ldots u_pv$ we obtain a walk (including self loops) in $G/\pi$, $\hat{u}f_\pi(u_1)f_\pi(u_2)\ldots f_\pi(u_p)\hat{v}$, therefore, $G/\pi$ is a connected graph.

The following result relates the degree of a node in a contracted graph to its cell-neighborhood.

**Proposition 2** (degree-contraction). Consider a graph $G$ and its contraction $G/\pi$ for $\pi \in \Pi(G)$. Then $\forall \hat{v} \in \mathcal{V}(G/\pi)$, $d_\hat{v}(G/\pi) = |f_\pi(N_{C\hat{v}}(G))|$.

**Proof.** From Definition 2 we have $f_\pi(N_{C\hat{v}}(G)) = \{i \in [1,r] \mid C_i(\pi) \cap N_{C\hat{v}}(G) \neq \emptyset\}$, and from Lemma 2 we obtain that $\forall \hat{u}, \hat{v} \in \mathcal{V}(G/\pi)$, $\hat{v} \sim \hat{u}$ if and only if $\hat{u} \in f_\pi(N_{C\hat{v}})$ such that $f_\pi(N_{C\hat{v}}) = N_{\hat{v}}(G/\pi)$, and therefore, $d_\hat{v}(G/\pi) = |f_\pi(N_{C\hat{v}})|$.

3.1. Graph Contraction Posets

Partially-ordered sets (posets) are an essential set-theoretical concept. Chains are totally-ordered subsets of the posets and are a useful tool for proving set-theoretical results. Here we show how graph contractions fall under the definition of a poset and will then establish contraction chains and their corresponding contraction sequences as a basis for proving cases of graph matrices interlacing.

Two partitions $\pi_{r_1}, \pi_{r_2} \in \Pi(\mathcal{V})$ may comply with a refinement relation.

**Definition 5** (refinement). Consider two partitions $\pi_{r_1}, \pi_{r_2} \in \Pi(\mathcal{V})$ of a vertex set $\mathcal{V}$ where $r_1 \leq r_2 \leq |\mathcal{V}|$. Then we say $\pi_{r_2}$ is a refinement of $\pi_{r_1}$ if $\forall j \in \{1, 2, \ldots, r_2\}$ we can find $i \in \{1, 2, \ldots, r_1\}$ such that $C_j(\pi_{r_2}) \subseteq C_i(\pi_{r_1})$, and we denote $\pi_{r_2} \leq \pi_{r_1}$. If $\pi_{r_2} \leq \pi_{r_1}$ and $r_1 < r_2$ we denote $\pi_{r_2} < \pi_{r_1}$. An $N$-chain is a partition set $\chi(\mathcal{V}) = \{\pi_{r_1}\}_{i=1}^N \subseteq \Pi(\mathcal{V})$ such that $\pi_{r_1} < \pi_{r_2} < \ldots < \pi_{r_N}$.

If two partitions $\pi_{r_1}, \pi_{r_2} \in \Pi(\mathcal{V})$ comply with the refinement relation, we can construct the coarsening partition $\delta(\pi_{r_2}, \pi_{r_1}) \in \Pi_{r_1}(\mathcal{V}_{r_2})$ with $C_j(\delta(\pi_{r_2}, \pi_{r_1})) = \{k \in \{1, 2, \ldots, r_2\} \mid C_k(\pi_{r_2}) \subseteq C_j(\pi_{r_1})\}$. We can then define the coarsening sequence
Definition 6 (coarsening sequence). Consider a vertex set \( V \) and its \( N \)-chain \( \chi (V) \subseteq \Pi (V) \). Then we define the coarsening sequence \( \Delta (\chi) = \{ \delta_i \}_{i=1}^{N-1} \) with \\
\[ \delta_i \triangleq \delta (\pi_{r_{i+1}}, \pi_{r_i}) \]

The refinement relation is reflexive, anti-symmetric and transitive, therefore, the set of partitions together with the refinement relation, \( (\Pi (V), \leq) \), falls under the definition of a finite partial-ordered set (poset). Let \( G = (V, E) \), we define the contraction set \( G // \Pi \triangleq \{ G // \pi | \pi \in \Pi (V) \} \), and define the contraction binary relation \( G // \pi_{r_1} \leq G // \pi_{r_2} \) if \( \pi_{r_1} \leq \pi_{r_2} \). Since there is a one-to-one correspondence between \( (G // \Pi, \leq) \) and \( (\Pi (V), \leq) \), the contraction set with the contraction binary relation, \( (G // \Pi, \leq) \), is also a poset, and for each \( N \)-chain \( \chi \subseteq \Pi (V) \) there is a corresponding contraction chain \( G // \chi = \{ G // \pi_{r_i} \}_{i=1}^{N} \subseteq G // \Pi \).

For each coarsening sequence \( \Delta (\chi) \) we can then define a corresponding contraction sequence, a series of graphs where each graph in the series is a graph contraction of the former graph over the coarsening partition in the coarsening sequence.

Definition 7 (contraction sequence). Consider a graph \( G \) and an \( N \)-chain \( \chi (V) \subseteq \Pi (\mathcal{V} (G)) \) with coarsening sequence \( \Delta (\chi) = \{ \delta_i \}_{i=1}^{N-1} \). Then we define the contraction sequence \( G // \Delta (\chi) \triangleq \{ G_i \}_{i=0}^{N-1} \) with \( G_i = G_{i-1} // \delta_{N-i} \) and \( G_0 = G // \pi_{r_N} \).

Proposition 3. Consider a graph \( G \) and its partition \( \pi \in \Pi (G) \), and let \( \chi = \{ \pi_{r_i} \}_{i=1}^{N} \subseteq \Pi (V) \) be a chain with \( \pi_{r_1} = \pi \) and corresponding contraction sequence \( G // \Delta (\chi) = \{ G_i \}_{i=0}^{N-1} \). Then \( G_{N-1} = G // \pi \).

Proof. It is sufficient to prove for any two-chain \( \pi = \pi_{r_1} < \pi_{r_2} \) with \( \Delta (\chi) = \delta (\pi_{r_2}, \pi_{r_1}) \), i.e., \( G // \pi = (G // \pi_{r_2}) // \delta (\pi_{r_2}, \pi_{r_1}) \), and extend by induction for \( N > 2 \). The order of \( G_0 = G // \pi_{r_2} \) is \( r_2 \) and from the coarsening sequence (Definition 6) we get that the order of \( G_1 = (G // \pi_{r_2}) // \delta (\pi_{r_2}, \pi_{r_1}) \) is \( r_1 = |\pi| \), therefore, \( V (G_1) = V (G // \pi) \). It is left to show that \( E (G_1) = E (G // \pi) \). Let \( \epsilon \in E (G // \pi) \) then \( \exists \xi \in E_G \) such that \( \epsilon = f_\pi (\xi) \). Now let \( \epsilon_1 = f_{\pi_{r_2}} (\epsilon) \) and \( \epsilon_2 = f_\xi (\epsilon_1) \), from the coarsening sequence (Definition 6) we then obtain that the end nodes of \( \epsilon_2 \) are the end nodes of \( \epsilon \), therefore, \( E (G_1) = E (G // \pi) \).
Corollary 1 (atom-contraction sequence). Consider a graph $G$ and its partition $\pi \in \Pi_r (G)$ for $r < n$. Then there exists a chain $\chi (V) = \{\pi_i\}_{i=1}^{n-r+1} \subseteq \Pi (V_n)$ such that $G / \Delta (\chi) = \{G_i\}_{i=0}^{n-r}$ is an atom contraction sequence, i.e., $\delta (\pi_{r+1}, \pi_r)$ is an atom-partition.

Proof. Choose $\pi_{r+1} = \pi (V_n)$, and then construct $\pi_r$ by extracting a singleton from a non-singleton cell of $\pi$. Continue to extract singleton cells until all cells are singletons, i.e., $\pi_N = \pi_n (V_n)$. The number of singleton extractions of non-singleton cells in an $r$-partition is $n-r$, therefore, $N = n-r+1$.

For example, consider the 2-chain $\chi (V) = \{\pi_2, \pi_3\}$ with

$$\pi_2 (V) = \left\{ \begin{array}{c} \{v_1, v_2, v_3\} \subseteq C_1 \\ \{v_4, v_5\} \subseteq C_2 \end{array} \right\}, \quad \text{and} \quad \pi_3 (V) = \left\{ \begin{array}{c} \{v_1, v_2\} \subseteq C_1 \\ \{v_3\} \subseteq C_2 \\ \{v_4, v_5\} \subseteq C_3 \end{array} \right\}.$$

We have $C_1 (\pi_3), C_2 (\pi_3) \subseteq C_1 (\pi_2)$ and $C_3 (\pi_3) \subseteq C_2 (\pi_2)$, therefore, $\pi_3 < \pi_2$. We can then construct the coarsening sequence $\Delta (\chi) = \delta (\pi_3, \pi_2)$ with $\delta (\pi_3, \pi_2) = \{\{1, 2\}, \{3\}\}$. The resulting graph contraction sequence is presented in Figure 3.
3.2. Edge Contractions

Graph contractions are defined over vertex partitions. However, there is also an edge-based approach to perform graph contractions.

**Definition 8** (edge contraction partition). Consider a graph $G$ and an edge contraction set $E_{cs} \subset E(G)$ with $|E_{cs}| = n - r$. Then we define the edge contraction partition $\pi_c (G, E_{cs})$ as the connected components partition of the graph $G_c (G, E_{cs}) = (V(G), E_{cs})$, i.e., $\pi_c (G, E_{cs}) = \pi_{cc} (G_c (G, E_{cs}))$. The set of all edge contraction sets of cardinality $p$ is defined as $\Xi_p (G) \triangleq \{ E_{cs} \subset E(G) | |E_{cs}| = p \}$.

With the edge contraction partition definition we can define an edge-based graph contraction.

**Definition 9** (edge-based graph contraction). Consider a graph $G$ and an edge contraction set $E_{cs} \in \Xi_{n-r} (G)$ for $r < n$. Then the edge-based contraction is defined as the contraction over the edge contraction partition, i.e., $G \sslash E_{cs} = G \sslash \pi_c (G, E_{cs})$.

In this work we find that a class of edge-matching contractions has interlacing properties.

**Definition 10** (edge-matching contraction). Consider a graph $G$ and an edge contraction set $E_{cs} \in \Xi_{n-r} (G)$ for $r < n$. Then $G \sslash E_{cs}$ is an edge-matching contraction if there is one-to-one correspondence between $E (G) \setminus E_{cs}$ and $E (G \sslash E_{cs})$.

A graph contraction cannot create new edges, therefore, edge-matching (Definition 10) is equivalent to $|E (G) \setminus E_{cs}| = |E (G \sslash E_{cs})|$.

**Proposition 4.** Consider a graph $G$ and an edge contraction set $E_{cs} \in \Xi_{n-r} (G)$. Then if $G \sslash E_{cs}$ is cycle-invariant (Definition 4) it is also edge-matching (Definition 10).

**Proof.** If $G \sslash E_{cs}$ is cycle-invariant then from Definition 4 the edges in $E_{cs}$ are not part of any cycle of $G$. Therefore, the contraction does not map any two edges in $E (G) \setminus E_{cs}$ to a single edge in $E (G \sslash E_{cs})$, otherwise they would have been part of a cycle with an edge in $E_{cs}$, and we obtain that $|E (G) \setminus E_{cs}| = |E (G \sslash E_{cs})|$. \qed
Proposition 5. Consider a graph $G$ and a node $v \in V(G)$, and let $\pi_{cc}(G \setminus v)$ be the connected component partition of $G \setminus v$, then for $C_i \in \pi_{cc}(G \setminus v)$ and $E_{cs} = E(G [C_i \cup v])$, the contraction $G \parallel E_{cs}$ is node-removal equivalent (Definition 3) with $V_S = C_i$, and is also edge-matching (Definition 10).

Proof. Since $C_i$ is a connected component of $G \setminus E(G [N_v \cup v])$ then $v$ is the only node in any path between $C_i$ and $V(G) \setminus \{C_i \cup v\}$, therefore, by choosing $V_S = C_i$ the graph $G \setminus C_i$ removes all edges $E(G [C_i])$ and all edges connecting $C_i$ to $V(G) \setminus C_i$ which are the edges between $C_i$ and $v$ and we obtain that $G \setminus C_i = G \parallel E(G [C_i \cup v])$, i.e., the contraction $G \parallel E_{cs}$ is node-removal equivalent (Definition 3). Furthermore, contracting all edges $E(G [C_i \cup v])$ does not effect any other edges in $G$ such that $|E(G) \setminus E_{cs}| = |E(G \parallel E_{cs})|$ and we obtain that $G \parallel E_{cs}$ is edge-matching. \[\square\]

We can choose a subset of tree edges to create a tree-based contraction of a graph.

Definition 11 (tree-based contraction). Consider a graph $G$ and its spanning tree $T \in T(G)$ with an edge contraction set $E_{cs} \in \Xi_{n-r}(T)$. Then $G \parallel E_{cs}$ is a tree-based contraction.

For example, the graph contraction $G \parallel \pi$ presented in Figure 2 can also be performed as an edge-based contraction $G \parallel E_{cs}$ with $E_{cs} = \{\{v_1, v_3\}, \{v_2, v_3\}\}$ and a tree-based contraction (Definition 11).

If the contraction edge set is a subset of the edges of a spanning tree, then the contracted tree edges will form a spanning tree of the contracted graph.

Proposition 6. Consider a graph $G$ and its spanning tree $T \in T(G)$ with an edge contraction set $E_{cs} \in \Xi_{n-r}(T)$. Then $T \parallel E_{cs} \in T(G \parallel E_{cs})$, i.e., $T \parallel E_{cs}$ is a tree of order $r$ of the contracted graph.

Proof. A tree of order $n$ has $n - 1$ edges, and by contracting $n - r$ tree edges we are left with $(n - 1) - (n - r)$ edges, such that $|E(T \parallel E_{cs})| = r - 1$. It is left to show that $T \parallel E_{cs}(T) \subseteq G \parallel E_{cs}(T)$. From Lemma 2 we obtain that
\( \mathcal{T} \parallel \mathcal{E}_{cs} \) is connected, therefore, \( \mathcal{T} \parallel \mathcal{E}_{cs} \) is a connected graph of order \( r \) with \( r - 1 \) edges, which is a tree of order \( r \). Since \( \mathcal{E}_{cs}(\mathcal{T}) \subseteq \mathcal{E}(\mathcal{G}) \) we have \( \pi_c(\mathcal{T}, \mathcal{E}_{cs}(\mathcal{T})) = \pi_c(\mathcal{G}, \mathcal{E}_{cs}(\mathcal{T})) \), and since \( \mathcal{T} = \mathcal{G} \setminus \mathcal{E}(\mathcal{C}) \) we obtain from the subgraph contraction lemma (Lemma \( \text{I} \)) that \( \mathcal{T} \parallel \pi_c(\mathcal{T}, \mathcal{E}_{cs}(\mathcal{T})) \subseteq \mathcal{G} \parallel \pi_c(\mathcal{T}, \mathcal{E}_{cs}(\mathcal{T})) \) and conclude that \( \mathcal{T} \parallel \mathcal{E}_{cs}(\mathcal{T}) \subseteq \mathcal{G} \parallel \mathcal{E}_{cs}(\mathcal{T}) \), and therefore, \( \mathcal{T} \parallel \mathcal{E}_{cs}(\mathcal{T}) \in \mathbb{T}(\mathcal{G} \parallel \mathcal{E}_{cs}) \). \( \square \)

**Proposition 7.** Consider a graph \( \mathcal{G} \) and an edge contraction set \( \mathcal{E}_{cs} \in \Xi_{n-r}(\mathcal{G}) \). Then \( \forall \tilde{\nu} \in \mathcal{V}(\mathcal{G} \parallel \mathcal{E}_{cs}) \)

\[
d_{\tilde{\nu}}(\mathcal{G} \parallel \mathcal{E}_{cs}) \leq \left( \sum_{v \in C_{\tilde{\nu}}(\pi)} d_v(\mathcal{G}) \right) - 2 (|C_{\tilde{\nu}}(\pi)| - 1),
\]

where \( \pi = \pi_c(\mathcal{G}, \mathcal{E}_{cs}) \).

**Proof.** From Proposition \( \text{II} \) we obtain that \( d_{\tilde{\nu}}(\mathcal{G} \parallel \pi) = |f_\pi(\mathcal{N}_{\mathcal{C}})| \). We have \( |f_\pi(\mathcal{N}_{\mathcal{C}})| \leq |\mathcal{N}_{\mathcal{C}}| \) and since \( C_{\tilde{\nu}}(\pi) \in \pi_c \) is a connected component of \( \mathcal{G} \) we get

\[
|\mathcal{N}_{\mathcal{C}}| \leq \left( \sum_{v \in C_{\tilde{\nu}}(\pi)} d_v(\mathcal{G}) \right) - 2 |\mathcal{E}(\mathcal{G}[C_{\tilde{\nu}}(\pi)])|.
\]

The number of edges in the cell \( |\mathcal{E}(\mathcal{G}[C_{\tilde{\nu}}(\pi)])| \) is at least the number of spanning tree edges, therefore, \( |\mathcal{E}(\mathcal{G}[C_{\tilde{\nu}}(\pi)])| \geq |C_{\tilde{\nu}}(\pi)| - 1 \), and we obtain that

\[
d_{\tilde{\nu}}(\mathcal{G} \parallel \mathcal{E}_{cs}) \leq \left( \sum_{v \in C_{\tilde{\nu}}(\pi)} d_v(\mathcal{G}) \right) - 2 (|C_{\tilde{\nu}}(\pi)| - 1),
\]

completing the proof. \( \square \)

**Corollary 2.** Consider a graph \( \mathcal{G} \) and an edge contraction set \( \mathcal{E}_{cs} \in \Xi_{n-r}(\mathcal{G}) \) for \( r < n \). Then if \( \mathcal{G} \parallel \mathcal{E}_{cs} \) is cycle-invariant (Definition \( \text{III} \)) then \( \forall \tilde{\nu} \in \mathcal{V}(\mathcal{G} \parallel \mathcal{E}_{cs}) \),

\[
d_{\tilde{\nu}}(\mathcal{G} \parallel \mathcal{E}_{cs}) = \left( \sum_{v \in C_{\tilde{\nu}}(\pi)} d_v(\mathcal{G}) \right) - 2 (|C_{\tilde{\nu}}(\pi)| - 1),
\]

where \( \pi = \pi_c(\mathcal{G}, \mathcal{E}_{cs}) \).

**Proof.** Since \( \forall \tilde{\nu} \in \mathcal{V}(\mathcal{G} \parallel \mathcal{E}_{cs}) \) \( C_{\tilde{\nu}}(\pi) \) is a connected component of \( \mathcal{G} \), and \( \mathcal{G} \parallel \mathcal{E}_{cs} \) is cycle-invariant then \( |f_\pi(\mathcal{N}_{\mathcal{C}})| = |\mathcal{N}_{\mathcal{C}}| \) and \( \mathcal{G}[C_{\tilde{\nu}}(\pi)] \) is a tree of order
\[ |C_\pi(v)|, \text{ such that from Proposition} \ref{prop2} \text{ we obtain that} \]

\[ d_\pi(G/\sslash E_{cs}) = \left( \sum_{v \in C_\pi(v)} d_\pi(G) \right) - 2 \left( |C_\pi(v)| - 1 \right). \]

**Corollary 3.** If a graph \( G \) is a tree then \( G/\sslash E_{cs} \) is edge-matching for any \( E_{cs} \in \Xi_{n-r}(G) \).

**Proof.** If \( G \) is a tree then \( G/\sslash E_{cs} \) is cycle-invariant for any \( E_{cs} \in \Xi_{n-r}(G) \) and from Proposition \ref{prop4} we obtain that \( G/\sslash E_{cs} \) is edge-matching. \( \square \)

Trees and cycle-completing edges are the building blocks of any connected graph, and this tree and co-tree structure is described by the Tucker representation \[11, \text{ p.113}\].

**Definition 12** (Tucker representation). Consider a graph \( G \) and its spanning tree \( T \in \mathcal{T}(G) \), with arbitrary head and tail assigned to the end-nodes of each edge in \( E(G) \). For each edge \( \epsilon_j \in E(C) \) there is a path from head to tail in \( T \), and we define a corresponding signed path vector \( t_j \in \mathbb{R}^{\lvert E(T) \rvert} \), \([t_j]_k = 1\) if \( \epsilon_k(T) \) (with the assigned head and tail) is along the path, \([t_j]_k = -1\) if \( \epsilon_k(T) \) is opposite to the path, and \([t_j]_k = 0\) otherwise. The Tucker representation of the co-tree is then the matrix \( T_{(T,C)} \in \mathbb{R}^{\lvert E(T) \rvert \times \lvert E(C) \rvert} \) where the \( j \)th column of \( T_{(T,C)} \) is the signed path vector \( t_j \in \mathbb{R}^{\lvert E(T) \rvert} \) of the corresponding edge \( \epsilon_j \in E(C) \).

**Proposition 8.** Consider a graph \( G \) and an edge contraction set \( E_{cs} \in \Xi_{n-r}(G) \) for \( r < n \), and let \( T \in \mathcal{T}(G) \). Then \( G/\sslash E_{cs} \) is cycle-invariant (Definition \[4\]) if and only if \( E_{cs} \subseteq E(T) \) and the corresponding rows of \( T_{(T,C)} \) are all zeros.

**Proof.** If \( G/\sslash E_{cs} \) is cycle-invariant then from Definition \[4\] the edges in \( E_{cs} \) are not part of any cycle of \( G \), therefore, \( E_{cs} \subseteq E(T) \) for any \( T \in \mathcal{T}(G) \). If \( \epsilon \in E(T) \) is not part of any cycle in \( G \) then form the Tucker representation (Definition \[12\]) we get that the corresponding row of \( T_{(T,C)} \) is all zeros.
If $E_{cs} \subseteq E(T)$ and the corresponding rows of $T(T,E)$ are all zeros, then the edges in $E_{cs}$ are not part of any cycle in $G$, such that the tree-based contraction (Definition 11) $G / \sslash E_{cs}$ is cycle-invariant.

4. Interlacing Graph Contractions

The general interlacing graph reduction problem (Problem 1) is combinatorial hard. If we restrict the class of reduced-order graphs to graph contractions then we get the following interlacing graph contraction problem.

**Problem 2** (interlacing graph contraction). Consider a graph $G$ and a real symmetric graph matrix $M(G) \in \mathbb{R}^{n \times n}$. Then given $r < n$ find $\pi \in \Pi_r(G)$ such that $G / \sslash \pi \propto M$. The number of $r$-partitions is $|\Pi_r(G)| = S(n,r)$ where

$$S(n,r) = \sum_{k=1}^{r} (-1)^{r-k} \frac{k^n}{k!(r-k)!}$$

(3)

is the Stirling number of the second kind [13, p.18], which for $r \ll n$ is asymptotically $S(n,r) \sim \frac{n^r}{r!}$. If we restrict the problem to edge-based contractions then the number of partitions is the number of $n-r$ edge contractions is $|\Xi_{n-r}(G)| = \binom{m}{n-r}$ where $m = |E(G)|$. Finding an interlacing contraction is, therefore, combinatorial hard and in the following section we show how cycle-invariant and node-removal equivalent contractions lead to interlacing.

A powerful tool for proving interlacing results is the Courant-Fischer theorem [2].

**Theorem 1** (Courant-Fischer). Consider a real symmetric matrix $M \in \mathbb{R}^{n \times n}$, then for $k = 1, 2, \ldots, n$

$$\lambda_k(M) = \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} R(M,x) \right\}$$

(4)

and

$$\lambda_k(M) = \max_{\mathcal{F}^{(n-k+1)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}^{(n-k+1)} \setminus \{0\}} R(M,x) \right\},$$

(5)
where $\mathcal{F}^{(k)}$ is a $k$-dimensional subspace of $\mathbb{R}^n$, and where $R(M,x) \triangleq \frac{x^T M x}{x^T x}$ is the Rayleigh quotient.

The following min-max properties will be useful in the derivation of interlacing results.

**Proposition 9.** Consider a $k$-dimensional subspace $\mathcal{F}^{(k)}$ of $\mathbb{R}^n$, and a subset $B \subseteq \mathbb{R}^n$, and let $f_1(x), f_2(x) : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions that attain a minimum and a maximum on $\mathbb{R}^n \setminus \{0\}$, with $f_1(x) \leq f_2(x) \ \forall x \in \mathbb{R}^n \setminus \{0\}$. Then the following holds:

i) $\min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \} \leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \max_{x \in \{ \mathcal{F}^{(k)} \setminus \{0\} \} \cap B} f_1(x) \},$

ii) $\max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \} \geq \max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \min_{x \in \{ \mathcal{F}^{(k)} \setminus \{0\} \} \cap B} f_1(x) \},$

iii) $\max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \} \leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_2(x) \}$

and,

iv) $\max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \} \leq \max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_2(x) \}.$

**Proof.** We have

$$\min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\} =$$

$$\min \left\{ \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\}, \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in \{ \mathcal{F}^{(k)} \setminus \{0\} \} \cap B} f_1(x) \right\} \right\},$$

$$\leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\},$$

and

$$\max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\} =$$

$$\min \left\{ \max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\}, \max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \{ \mathcal{F}^{(k)} \setminus \{0\} \} \cap B} f_1(x) \right\} \right\},$$

$$\geq \max_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}^{(k)} \setminus \{0\}} f_1(x) \right\}.$$

For all $\mathcal{F}^{(k)} \subseteq \mathbb{R}^n$ we have

$$\max_{x \in \mathcal{F}^{(k)}} f_1(x) \leq \max_{x \in \mathcal{F}^{(k)}} f_2(x).$$
and
\[
\min_{x \in F^{(k)}} f_1(x) \leq \min_{x \in F^{(k)}} f_2(x),
\]
therefore,
\[
\min_{F^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in F^{(k)} \setminus \{0\}} f_1(x) \right\} \leq \min_{F^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in F^{(k)} \setminus \{0\}} f_2(x) \right\}
\]
and
\[
\max_{F^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in F^{(k)} \setminus \{0\}} f_1(x) \right\} \leq \max_{F^{(k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in F^{(k)} \setminus \{0\}} f_2(x) \right\}.
\]

A well known algebraic result is that a symmetric matrix and a principle submatrix of that matrix interlace [7], which leads to an adjacency interlacing theorem for node-removal graph reductions:

**Theorem 2** (Adjacency interlacing node-removal). Consider a graph \( G \) and a node subset \( V_S \subset V(G) \). Then \( G \setminus V_S \propto_A G_n \).

**Proof.** The matrix \( A(G \setminus V_S) \) is a principle submatrix of \( A(G) \), therefore, \( G \setminus V_S \propto_A G_n \).

The following two theorems are the main contributions of this work and provide Laplacian interlacing for node-removal equivalent edge-matching contractions and normalized Laplacian interlacing for cycle-invariant contractions.

**Theorem 3** (Laplacian interlacing node-removal equivalent contraction). Consider a graph \( G \) and an edge contraction set \( E_{cs} \in \Xi_{n-r}(G) \) for \( r < n \). If \( G/sslash E_{cs} \) is edge-matching (Definition 10) and node-removal equivalent (Definition 3) then \( G_r \propto_L G_n \).

**Proof.** From the Courant–Fischer theorem (Theorem 1) we have
\[
\lambda_k(L(G)) = \min_{F^{(k)} \subseteq \mathbb{R}^n} \left\{ \max_{x \in F^{(k)} \setminus \{0\}} R(L(G), x) \right\},
\]
and the Rayleigh quotients of the Laplacian takes the form [2]
\[
R(L(G), x) = \sum_{(u,v) \in E(G)} (x_v - x_u)^2 \sum_{v \in V(G)} x_v^2.
\]

(6)
Separating the edges to $\mathcal{E}_{cs}$ and $\mathcal{E}\setminus\mathcal{E}_{cs}$, the sum $\sum_{\{u,v\}\in\mathcal{E}(\mathcal{G})}(x_v - x_u)^2$ can be written as

$$\sum_{\{u,v\}\in\mathcal{E}}(x_v - x_u)^2 = \sum_{\{u,v\}\in\mathcal{E}(\mathcal{G})\setminus\mathcal{E}_{cs}}(x_u - x_v)^2 + \sum_{\{u,v\}\in\mathcal{E}_{cs}}(x_u - x_v)^2. \quad (7)$$

We define the partition space $\mathcal{F}_\pi \subseteq \mathbb{R}^n$ such that for $x \in \mathcal{F}_\pi$ all vector variables with indexes in the same partition cell are equal,

$$\mathcal{F}_\pi \triangleq \{ x \in \mathbb{R}^n | x_j = x_k, \forall j, k \in C_i(\pi), \forall i \in [1, r] \}. \quad (8)$$

Therefore, if $x \in \mathcal{F}_\pi$ and $\{u, v\} \in \mathcal{E}_{cs}$ then

$$\sum_{\{u,v\}\in\mathcal{E}_{cs}}(x_u - x_v)^2 = 0 \quad (9)$$

and

$$\sum_{\{u,v\}\in\mathcal{E}}(x_v - x_u)^2 = \sum_{\{u,v\}\in\mathcal{E}(\mathcal{G})\setminus\mathcal{E}_{cs}}(x_u - x_v)^2. \quad (10)$$

Since $\mathcal{G} \parallel \mathcal{E}_{cs}$ is edge-matching (Definition 10) there is one-to-one correspondence between $\mathcal{E}(\mathcal{G})\setminus\mathcal{E}_{cs}$ and $\mathcal{E}(\mathcal{G})/\mathcal{E}_{cs}$ (Proposition 4), such that for $x \in \mathcal{F}_\pi$

$$\sum_{\{u,v\}\in\mathcal{E}\setminus\mathcal{E}_{cs}}(x_u - x_v)^2 = \sum_{(i,j)\in\mathcal{E}(\mathcal{G}_{\pi})}(\tilde{x}_i - \tilde{x}_j)^2, \quad (11)$$

where $\tilde{x}_i \in \mathbb{R}$ is a value assigned to all $x_v$ with $v \in C_i(\pi)$.

The sum $\sum_{v\in\mathcal{V}}x_v^2$ can be rearranged as the sum over the vertices of each partition cell

$$\sum_{v\in\mathcal{V}}x_v^2 = \sum_{i=1}^r \sum_{v\in C_i(\pi)}x_v^2, \quad (12)$$

and for $x \in \mathcal{F}_\pi$ we get

$$\sum_{v\in\mathcal{V}}x_v^2 = \sum_{i=1}^r \tilde{x}_i^2 |C_i(\pi)|. \quad (13)$$

From the min-max properties (Proposition 4) with $\mathcal{B} \equiv \mathcal{F}_\pi$ we then obtain
\[ \lambda_k(L(G)) = \min_{\mathcal{F}(k) \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}(k) \setminus \{0\}} \frac{\sum_{\{u,v\} \in \mathcal{E}(G)} (x_v - x_u)^2}{\sum_{v \in V(G)} x_v^2} \right\} \]

\[ \leq \min_{\mathcal{F}(k) \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}(k) \setminus \{0\}} \frac{\sum_{\{i,j\} \in \mathcal{E}(r_i)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i=1}^r \tilde{x}_i^2} \right\} \]

\[ \leq \min_{\mathcal{F}(k) \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}(k) \setminus \{0\}} \frac{\sum_{\{i,j\} \in \mathcal{E}(r_i)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i=1}^r \tilde{x}_i^2} \right\} \]

\[ = \lambda_k(L(G_r)), \]

and we obtain that \( \lambda_k(L(G)) \leq \lambda_k(L(G_r)) \) for \( k = 1, 2, \ldots, n \).

In order to complete the interlacing proof it is left to show that \( \lambda_k(L(G_r)) \leq \lambda_{n-r+k}(L(G)) \) for \( k = 1, 2, \ldots, r \). From the Courant–Fischer theorem (Theorem 1) we get

\[ \lambda_{n-r+k}(L(G)) = \max_{\mathcal{F}'(r-k+1) \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}'(r-k+1) \setminus \{0\}} R(L(G), x) \right\} \quad (14) \]

and expanding \( R(L(G), x) \) with Eq. (7) we have

\[ \lambda_{n-r+k}(L(G)) \geq \max_{\mathcal{F}'(r-k+1) \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}'(r-k+1) \setminus \{0\}} \frac{\sum_{\{u,v\} \in \mathcal{E}(G)} (x_u - x_v)^2 + \sum_{\{u,v\} \in \mathcal{E}_{cs}} (x_u - x_v)^2}{\sum_{v \in V(G)} x_v^2} \right\} \]

From the min-max properties (Proposition 9) with

\[ f_2(x) = \frac{\sum_{\{u,v\} \in \mathcal{E}\setminus\mathcal{E}_{cs}} (x_u - x_v)^2 + \sum_{\{u,v\} \in \mathcal{E}_{cs}} (x_u - x_v)^2}{\sum_{v \in V(G)} x_v^2} \]

and

\[ f_1(x) = \frac{\sum_{\{u,v\} \in \mathcal{E}\setminus\mathcal{E}_{cs}} (x_u - x_v)^2}{\sum_{v \in V(G)} x_v^2} \]

we get

\[ \lambda_{n-r+k}(L(G)) \geq \max_{\mathcal{F}'(r-k+1) \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}'(r-k+1) \setminus \{0\}} \frac{\sum_{\{u,v\} \in \mathcal{E}\setminus\mathcal{E}_{cs}} (x_u - x_v)^2}{\sum_{v \in V(G)} x_v^2} \right\} \].

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Since \( G \parallel E_{cs} \) is edge-matching (Definition 10) there is one-to-one correspondence between \( E((G) \setminus E_{cs}) \) and \( E(G \parallel E_{cs}) \) (Proposition 4), such that
\[
\sum_{\{u,v\} \in E \setminus E_{cs}} (x_u - x_v)^2 = \sum_{\{i,j\} \in E(G_r)} (\tilde{x}_i - \tilde{x}_j)^2,
\]
for \( \tilde{x} \in \mathbb{R}^r \). Since \( G \parallel E_{cs} \) is node-removal equivalent (Definition 3) there is a subset \( V_S \subset V(G) \) such that \( G \parallel E_{cs} = G \backslash V_S \), and we define the node-removal space, \( \mathcal{F}_{V_S} \subseteq \mathbb{R}^n \), as
\[
\mathcal{F}_{V_S} \equiv \{ x \in \mathbb{R}^n : x_i = 0, i \in V_S \}.
\]
(15)
For \( x \in \mathcal{F}_{V_S} \), the sum \( \sum_{v \in V(G)} x_v^2 \) can be written as \( \sum_{v \in V(G)} x_v^2 = \sum_{i \in V(G_r)} \tilde{x}_i^2 \), and from the min-max properties (Proposition 9) with \( B \equiv \mathcal{F}_{V_S} \) we then have
\[
\lambda_{n-r+k}(L(G)) \geq \max_{\mathcal{F}(r-k+1) \subseteq \mathbb{R}^n} \left\{ \min_{x \in \mathcal{F}(r-k+1) \setminus \{0\}} \frac{\sum_{\{u,v\} \in E \setminus E_{cs}} (x_u - x_v)^2}{\sum_{v=1}^{r} x_v^2} \right\}
= \min_{\mathcal{F}(r-k+1) \subseteq \mathbb{R}^r} \left\{ \max_{x \in \mathcal{F}(r-k+1) \setminus \{0\}} \frac{\sum_{i,j} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i} \tilde{x}_i^2} \right\}
= \lambda_k(L(G_r))
\]
and we obtain that \( \lambda_k(L(G)) \leq \lambda_{n-r+k}(L(G_r)) \) for \( k = 1, 2, \ldots, r \).

**Theorem 4** (normalized-Laplacian interlacing cycle-invariant contraction). Consider a graph \( G \) and an edge contraction set \( E_{cs} \in \Xi_{n-r}(G) \) for \( r < n \). Then if \( G_r = G \parallel E_{cs} \) is cycle-invariant (Definition 3), \( G_r \propto L(G_n) \).

**Proof.** In order for \( G \) and \( G_r \) to be normalized-Laplacian interlacing we must prove that \( \lambda_k(L(G)) \leq \lambda_k(L(G_r)) \leq \lambda_{n-r+k}(L(G)) \) for \( k = 1, 2, \ldots, r \). From the Courant–Fischer theorem (Theorem 1) we have
\[
\lambda_k(L(G)) = \min_{\mathcal{F}(k) \subseteq \mathbb{R}^n} \left\{ \max_{x \in \mathcal{F}(k) \setminus \{0\}} R(L(G), x) \right\},
\]
and the Rayleigh quotient of the normalized-Laplacian takes the form
\[
R(L(G), x) = \frac{\sum_{\{u,v\} \in E(G)} (x_v - x_u)^2}{\sum_{v \in V(G)} x_v^2 d_v(G)}.
\]
(16)
Separating the edges to $\mathcal{E}_{cs}$ and $\mathcal{E}\backslash\mathcal{E}_{cs}$ as in Eq. 11 and rearranging the sum

$$\sum_{v \in V(\mathcal{G})} x^2_v d(v)$$

as in Eq. 12, we obtain

$$\sum_{v \in V(\mathcal{G})} x^2_v d(v) = \sum_{i=1}^{r} \sum_{v \in C_i(\pi)} x^2_v d(v), \quad (17)$$

we obtain

$$\lambda_k (\mathcal{L}(\mathcal{G})) = \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \frac{\max_{\{u,v\} \in \mathcal{E} \backslash \mathcal{E}_{cs}} (x_u - x_v)^2 + \max_{\{u,v\} \in \mathcal{E}_{cs}} (x_u - x_v)^2}{\sum_{i=1}^{r} \sum_{v \in C_i(\pi)} x^2_v d_v (\mathcal{G})} \right\}.$$ 

If $x \in \mathcal{F}_r$, we get

$$\sum_{\{u,v\} \in \mathcal{E}(\mathcal{G}) \backslash \mathcal{E}_{cs}} (x_u - x_v)^2 = \sum_{\{i,j\} \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2 \quad (Eq. 11),$$

and

$$\sum_{i=1}^{r} \sum_{v \in C_i(\pi)} x^2_v d_v (\mathcal{G}) = \sum_{i=1}^{r} \tilde{x}_i^2 \sum_{v \in C_i(\pi)} d_v (\mathcal{G})$$

where $\tilde{x} \in \mathbb{R}^r$. From the min-max properties (Proposition 9) with $\mathcal{B} \equiv \mathcal{F}_r$ we then obtain

$$\lambda_k (\mathcal{L}(\mathcal{G})) \leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^n} \left\{ \frac{\max_{\{i,j\} \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{v \in C_i(\pi)} \sum_{i=1}^{r} x^2_v d_v (\mathcal{G})} \right\}.$$ 

The contracted graph $\mathcal{G}_r = \mathcal{G} \parallel \mathcal{E}_{cs}$ is cycle-invariant, therefore, from Proposition 2 we have

$$d_v (\mathcal{G}_r) = \left( \sum_{v \in C_i(\pi)} d_v (\mathcal{G}) \right) - 2 (|C_i (\pi)| - 1).$$

We can then replace $\mathcal{F}^{(k)} \subseteq \mathbb{R}^n$ and $\{ \mathcal{F}^{(k)} \backslash \{0\} \} \cap \mathcal{F}_r \subseteq \mathbb{R}^n$ with $\mathcal{F}^{(k)} \subseteq \mathbb{R}^r$ and $\mathcal{F}^{(k)} \backslash \{0\} \subseteq \mathbb{R}^r$ such that

$$\lambda_k (\mathcal{L}(\mathcal{G})) \leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}^r} \left\{ \frac{\max_{x \in \mathcal{F}^{(k)} \backslash \{0\}} \sum_{\{i,j\} \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i=1}^{r} \tilde{x}_i^2 [d_i (\mathcal{G}_r) + 2 (|C_i (\pi)| - 1)]} \right\}.$$ 

We have $2 (|C_i (\pi)| - 1) \geq 0$, therefore,

$$\frac{\sum_{\{i,j\} \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i=1}^{r} \tilde{x}_i^2 [d_i (\mathcal{G}_r) + 2 (|C_i (\pi)| - 1)]} \leq \frac{\sum_{\{i,j\} \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{v \in V(\mathcal{G}_r)} \tilde{x}_v^2 d_v (\mathcal{G}_r)}.$$
and from the min-max properties (Proposition 9) with
\[ f_1 (\tilde{x}) = \frac{\sum_{(i,j) \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{i=1} x_i^2 [d_i (\mathcal{G}_r) + 2 (|C_i (\pi)| - 1)]} \]

and
\[ f_2 (\tilde{x}) = \frac{\sum_{(i,j) \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{v \in \mathcal{V}} \tilde{x}_v^2 d_v (\mathcal{G}_r)} \]

we get
\[ \lambda_k (\mathcal{L}(\mathcal{G})) \leq \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}} \left\{ \max_{\tilde{x} \in \mathcal{F}^{(k)} \setminus \{0\}} \frac{\sum_{(i,j) \in \mathcal{E}(\mathcal{G}_r)} (\tilde{x}_i - \tilde{x}_j)^2}{\sum_{v \in \mathcal{V}} \tilde{x}_v^2 d_v (\mathcal{G}_r)} \right\} \]
\[ = \min_{\mathcal{F}^{(k)} \subseteq \mathbb{R}} \left\{ \max_{x \in \mathcal{F}^{(k)} \setminus \{0\}} R (\mathcal{L}(\mathcal{G}_r), x) \right\} \]
\[ = \lambda_k (\mathcal{L}(\mathcal{G}_r)) , \]

and we obtain that \( \lambda_k (\mathcal{L}(\mathcal{G})) \leq \lambda_k (\mathcal{L}(\mathcal{G}_r)) \) for \( k = 1, 2, \ldots, r \).

In order to complete the interlacing proof it is left to show that \( \lambda_k (\mathcal{L}(\mathcal{G}_r)) \leq \lambda_{n-r+k} (\mathcal{L}(\mathcal{G})) \) for \( k = 1, 2, \ldots, r \). The graph contraction can be performed by a sequence of atom-contractions (Corollary 1), therefore, it is sufficient to show that the interlacing property holds for a single edge-contraction, i.e., \( \lambda_k (\mathcal{L}(\mathcal{G}_r)) \leq \lambda_{k+1} (\mathcal{L}(\mathcal{G})) \) for \( k = 1, 2, \ldots, n-1 \). The interlacing of the sequence will then follow from Proposition 1. Let \( \pi \in \Pi_{n-1} (\mathcal{G}) \) be an atom-contraction, then from the Courant–Fischer theorem (Theorem 1) we get
\[ \lambda_{k+1} (\mathcal{L}(\mathcal{G})) = \max_{\mathcal{F}^{(n-k)} \subseteq \mathbb{R}^n} \left\{ \min_{\tilde{x} \in \mathcal{F}^{(n-k)} \setminus \{0\}} R (\mathcal{L}(\mathcal{G}), \tilde{x}) \right\} , \]

and expanding \( R (\mathcal{L}(\mathcal{G}), x) \) with Eq.(11) we have
\[ \lambda_{k+1} (\mathcal{L}(\mathcal{G})) = \max_{\mathcal{F}^{(n-k)} \subseteq \mathbb{R}^n} \left\{ \min_{\tilde{x} \in \mathcal{F}^{(n-k)} \setminus \{0\}} \frac{\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{ee}} (x_u - x_v)^2 + \sum_{\{u,v\} \in \mathcal{E}_{ee}} (x_u - x_v)^2}{\sum_{i=1}^r \sum_{v \in \mathcal{C}_i (\pi)} x_v^2 d_v (\mathcal{G})} \right\} . \]

For an atom-contraction there is only one non-singlet cell, and without loss of generality we can choose it to be \( C_{n-1} (\pi) = \{x_{n-1}, x_n\} \) such that the contracted
edge endnodes are \{x_{n-1}, x_n\}, and

\[
\lambda_{k+1}(\mathcal{L}(G)) = \max_{F^{(n-k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in F^{(n-k)} \setminus \{0\}} \frac{\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + (x_{n-1} - x_n)^2}{\sum_{v=1}^{n-2} x_v^2 d_v(G) + x_{n-1}^2 d_{n-1}(G) + x_n^2 d_n(G)} \right\}.
\]

We define the partition null-space \( F^{\perp}_\pi = \{x \in \mathbb{R}^n \mid x^T y = 0, \forall y \in F_\pi\} \), and from the min-max properties (Proposition 9) with \( B \equiv F^{\perp}_\pi \) we then obtain

\[
\lambda_{k+1}(\mathcal{L}(G)) \geq \max_{F^{(n-k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \{F^{(n-k)} \setminus \{0\}} \cap F^{\perp}} \frac{\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + (x_{n-1} - x_n)^2}{\sum_{v=1}^{n-2} x_v^2 d_v(G) + x_{n-1}^2 d_{n-1}(G) + x_n^2 d_n(G)} \right\}.
\]

For the atom-contraction we have \( F_\pi = \{x \in \mathbb{R}^n \mid x_{n-1} = x_n\} \) and \( F^{\perp}_\pi = \{x \in \mathbb{R}^n \mid x_{n-1} = -x_n\} \)

\[
\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + (x_{n-1} - x_n)^2 = \sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + 4x_{n-1}^2.
\]

From Proposition 9, we have \( d_{n-1}(G_r) = d_{n-1}(G) + d_n(G) - 2 \)

\[
\sum_{v=1}^{n-2} x_v^2 d_v(G) + x_{n-1}^2 (d_{n-1}(G) + d_n(G)) = \sum_{v=1}^{n-2} x_v^2 d_v(G) + x_{n-1}^2 (d_{n-1}(G_r) + 2)
\]

and with \( \sum_{v=1}^{n-2} x_v^2 d_v(G) + x_{n-1}^2 (d_{n-1}(G_r) + 2) = \sum_{v=1}^{n-1} x_v^2 d_v(G) + 2x_{n-1} \) we get

\[
\lambda_{k+1}(\mathcal{L}(G)) \geq \max_{F^{(n-k)} \subseteq \mathbb{R}^n} \left\{ \min_{x \in \{F^{(n-k)} \setminus \{0\}} \cap F^{\perp}} \frac{\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + 4x_{n-1}^2}{\sum_{v=1}^{n-1} x_v^2 d_v(G) + 2x_{n-1}^2} \right\}
\]

\[
= \max_{F^{(n-k)} \subseteq \mathbb{R}^{n-1}} \left\{ \min_{x \in \{F^{(n-k)} \setminus \{0\}} \cap F^{\perp}} \frac{\sum_{\{u,v\} \in \mathcal{E} \setminus \mathcal{E}_{cs}} (x_u - x_v)^2 + 4x_{n-1}^2}{\sum_{v=1}^{n-1} x_v^2 d_v(G) + 2x_{n-1}^2} \right\}
\]

\[
= \max_{F^{(n-k)} \subseteq \mathbb{R}^{n-1}} \left\{ \min_{x \in \{F^{(n-k)} \setminus \{0\}} \cap F^{\perp}} \frac{R(\mathcal{L}(G_r), x) 1 + \sum_{\{u,v\} \in \mathcal{E}(G_r)} (x_u - x_v)^2}{\sum_{v=1}^{n-1} x_v^2 d_v(G) + 2x_{n-1}^2} \right\}
\]

\[24\]
where we replace \( F^{(n-k)} \subseteq \mathbb{R}^n \) and \( \{ F^{(n-k)} \setminus \{0\} \} \cap \mathcal{F} \subseteq \mathbb{R}^{n-1} \) and \( \{ F^{(n-k)} \setminus \{0\} \} \), and extract \( R(\mathcal{L}(\mathcal{G}_r), x) \). We have \( R(\mathcal{L}(\mathcal{G}_r), x) \leq 2 \), therefore,

\[
\frac{\sum_{\{u,v\} \in \mathcal{E}(\mathcal{G}_r)} (x_u - x_v)^2}{\sum_{v \in \mathcal{V}(\mathcal{G}_r)} x^2(v) d(v)} \leq 2
\]

and

\[
1 + \frac{4x_{n-1}^2}{\sum_{\{u,v\} \in \mathcal{E}(\mathcal{G}_r)} (x_u - x_v)^2} \geq 1.
\]

From the min-max properties (Proposition 9) we then get

\[
\lambda_{k+1}(\mathcal{L}(\mathcal{G})) \geq \max_{F^{(n-k)} \subseteq \mathbb{R}^{n-1}} \left\{ \min_{x \in \{ F^{(n-k)} \setminus \{0\} \}} R(\mathcal{L}(\mathcal{G}_r), x) \right\}
\]

\[
= \lambda_k(\mathcal{L}(\mathcal{G}_r)),
\]

and we obtain that \( \lambda_k(\mathcal{L}(\mathcal{G}_r)) \leq \lambda_{k+1}(\mathcal{L}(\mathcal{G})) \) for \( k = 1, 2, \ldots, n-1 \). By performing the contraction sequence (Proposition 11) we have \( \lambda_k(\mathcal{L}(\mathcal{G})) \leq \lambda_{n-r+k}(\mathcal{L}(\mathcal{G}_r)) \) for \( k = 1, 2, \ldots, r \).

**Corollary 4.** Consider a tree \( \mathcal{T} = (\mathcal{V}, \mathcal{E}) \) of order \( n \), and its contraction \( \mathcal{T}_r = \mathcal{T} / \mathcal{E}_{\text{cs}} \) for any \( \mathcal{E}_{\text{cs}} \in \Xi_{n-r}(\mathcal{T}) \). Then \( \mathcal{T}_r \propto \mathcal{T} \).

**Proof.** The contraction \( \mathcal{T} / \mathcal{E}_{\text{cs}} \) is cycle-invariant for any \( \mathcal{E}_{\text{cs}} \in \Xi_{n-r}(\mathcal{T}) \), therefore, from Theorem 4 we obtain that \( \mathcal{T}_r \propto \mathcal{T} \). 

Theorem 3 and Theorem 4 allow us to try and solve the interlacing graph contraction problem (Problem 2) for normalized Laplacian and Laplacian interlacing by finding a cycle-invariant contraction (Problem 3) or a node-removal equivalent and edge matching contraction (Problem 4) respectively.

**Problem 3** (cycle-invariant contraction). For a graph \( \mathcal{G} \) and a given reduction order \( r < n \), find \( \mathcal{E}_{\text{cs}} \in \Xi_{n-r}(\mathcal{G}) \) such that \( \mathcal{G} / \mathcal{E}_{\text{cs}} \) is cycle-invariant (Definition 4).
Problem 4 (node-removal equivalent contraction). For a graph \( G \) and a given reduction order \( r < n \), find \( E_{cs} \in \Xi_{n-r}(G) \) such that \( G \parallel E_{cs} \) is node-removal equivalent (Definition 3) and edge-matching (Definition 10).

From Proposition 8, we can obtain a cycle-invariant contraction, if exists, from the zero rows of the Tucker representation. A Tucker representation \( T(\mathcal{T},\mathcal{C}) \) can be calculated by finding a spanning tree \( \mathcal{T} \in \mathcal{T}(G) \) and then finding the path in \( \mathcal{T} \) between the end-nodes of each edge of \( \mathcal{C}(\mathcal{T}) \) as described in Algorithm 1. Each path finding operation, e.g., with a depth-first search, is of complexity \( O(n) \), and since \( O(|\mathcal{E}(\mathcal{C})|) = O(|\mathcal{E}(G)|) \) the overall complexity of constructing \( T(\mathcal{T},\mathcal{C}) \) is \( O(mn) \), where \( m = |\mathcal{E}(G)| \). Therefore, the cycle-invariant contraction algorithm (Algorithm 1) is of complexity \( O(mn) \).

From Proposition 5, we can obtain a node-removal equivalent and edge-matching contraction, if exists, by first finding for all vertices of \( G \) the connected components partition \( \pi_{cc}(G\setminus v) \) and then constructing \( E_{cs} \) by choosing from all partitions \( \{\pi_{cc}(G\setminus v)\}_{v=1}^{n} \) a subset of cells with a total number of \( n - r \) unique nodes (Algorithm 3). Each connected component finding operation, e.g., with a depth-first search, is of complexity \( O(n + m) \), and repeated \( n \) times, the overall complexity of the algorithm is \( O(n^2 + nm) \).

The feasibility of the cycle-invariant and node-removal equivalent problems requires further study.

Algorithm 1 Cycle-invariant contraction algorithm

\begin{itemize}
  \item \textbf{Input:} graph \( G \) of order \( n \), required reduction order \( r \)
  \item 1. Find a spanning tree \( \mathcal{T} \in \mathcal{T}(G) \) and the co-tree \( \mathcal{C}(\mathcal{T}) \).
  \item 2. Calculate the tucker representation \( T(\mathcal{T},\mathcal{C}) \) (Definition 12).
  \item 3. Choose \( n-r \) cycle-invariant edges from the zero rows of \( T(\mathcal{T},\mathcal{C}) \) and obtain \( E_{cs} \).
\end{itemize}

\textbf{Output:} \( G_r = G \parallel E_{cs} \)
Algorithm 2 Node-removal equivalent contraction algorithm

Input: graph $\mathcal{G}$ of order $n$, required reduction order $r$

1. For $v \in \mathcal{V}(\mathcal{G})$: Calculate $\pi_{cc}(\mathcal{G}\setminus v)$, the connected components partition of $\mathcal{G}\setminus v$.

2. Choose a subset of cells $S \subseteq \{\pi_{cc}(\mathcal{G}\setminus v)\}_{v=1}^{n}$ with a total number of $n - r$ unique nodes.

3. Construct $\mathcal{E}_{cs} = \bigcup_{C_v \in S} \mathcal{E}(\mathcal{G}[C_v \cup v])$.

Output: $\mathcal{G}_r = \mathcal{G} / \mathcal{E}_{cs}$

5. Case Studies

As a small-scale normalized Laplacian interlacing example, we consider a graph of order 6 presented in Figure 4 and we require the reduced graph to be of order $r = 4$. A cycle-invariant graph contraction is then performed with two edges (Figure 4). The resulting reduced graph (Figure 4) has normalized-Laplacian spectra $\{\lambda_k(L(\mathcal{G}_r))\}_{k=1}^{r}$ given in Figure 5 with the upper and lower interlacing bounds $\lambda_k(L(\mathcal{G}))$ and $\lambda_{n-r+k}(L(\mathcal{G}))$. Since $\mathcal{G}/\mathcal{E}_{cs}$ is cycle-invariant, then as according to Theorem 4, we get $\mathcal{G}/\mathcal{E}_{cs} \propto L\mathcal{G}$ and the reduced-order spectra is within the interlacing bounds (Figure 5).

As a small-scale Laplacian interlacing example, we consider a graph of order 6 presented in Figure 6 and require the reduction to be of order $r = 4$. For this case the only node-removal equivalent and edge-matching contraction is with the three edges shown in Figure 6. The resulting reduced graph (Figure 6) has Laplacian spectra given in Figure 7 with the interlacing bounds $\lambda_k(L(\mathcal{G}))$ and $\lambda_{n-r+k}(L(\mathcal{G}))$. Since $\mathcal{G}/\mathcal{E}_{cs}$ is node-removal equivalent and edge-matching, then as according to Theorem 3, we get $\mathcal{G}/\mathcal{E}_{cs} \propto L\mathcal{G}$ and the reduced-order Laplacian spectra is within the interlacing bounds (Figure 7). Notice that for this case there is no cycle-invariant contraction, and for the same choice of $\mathcal{E}_{cs}$
Figure 4: Small scale normalized-Laplacian interlacing graph contraction (contracted edges dashed-red).

Figure 5: Reduced-order normalized-Laplacian spectra (stared-red) and interlacing bounds (circled-blue).
Figure 6: Small scale Laplacian interlacing graph contraction (contracted edges dashed-red).

Figure 6 shows that the reduced-order normalized-Laplacian does not interlace with the full-order normalized-Laplacian as $\lambda_4(L(G_r)) > \lambda_6(L(G))$ (Figure 8).

As a larger and more complicated example, a random tree of order 50 is created and 10 cycle-completing edges are randomly added to it resulting in a graph of order 50 with 59 edges (Figure 9). The required reduction order is $r = 30$. Using the cycle-invariant contraction algorithm (Algorithm 1) an edge-contraction set $E_{cs}$ with $n - r = 20$ edges is chosen from the edges of $G$ (Figure 9), and the graph contraction is performed. As according to Theorem 4, the resulting reduced-order graph $G_r = G \parallel E_{cs}$ is normalized-Laplacian interlacing with $G$ and the reduced spectra is within the interlacing bounds (Figure 10).

Using the node-removal equivalent contraction algorithm (Algorithm 2) a different edge-contraction set $E_{cs}$ with $n - r = 20$ edges is chosen from the
Figure 7: Reduced-order Laplacian spectra (stared-red) and interlacing bounds (circled-blue).

Figure 8: Reduced-order normalized-Laplacian spectra (stared-red) and interlacing bounds (circled-blue).
edges of $\mathcal{G}$ (Figure 11), and the graph contraction is performed. As according to Theorem 3, the resulting reduced order graph $\mathcal{G}_r = \mathcal{G} \parallel E_{cs}$ (Figure 11) is Laplacian interlacing with $\mathcal{G}$ and the reduced spectra is within the interlacing bounds (Figure 12).
Figure 10: Reduced-order normalized-Laplacian spectra (stared-red) and interlacing bounds (circled-blue).

Figure 11: Large scale Laplacian interlacing graph contraction.
Figure 12: Reduced-order Laplacian spectra (starred-red) and interlacing bounds (circled-blue).

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