A weighted $L_p$-theory for second-order elliptic and parabolic partial differential systems on a half space

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- Dedicated to 70th birthday of N.V. Krylov

Abstract

In this paper we develop a Fefferman-Stein theorem, a Hardy-Littlewood theorem and sharp function estimations in weighted Sobolev spaces. We also provide uniqueness and existence results for second-order elliptic and parabolic partial differential systems in weighed Sobolev spaces.

Keywords: Fefferman-Stein theorem, Hardy-Littlewood theorem, Weighted Sobolev spaces, Sharp function estimations, $L_p$-theory, Elliptic partial differential systems, Parabolic partial differential systems.

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1 Introduction

In this article we consider the elliptic system

$$
\sum_{i,j=1}^{d} \sum_{r=1}^{d_1} a^{ij}_{kr}(x) u^r_{x_i x_j}(x) = f^k(x), \quad (k = 1, 2, \cdots, d_1)
$$

(1.1)

and the parabolic system

$$
a^k_t(t,x) = \sum_{i,j=1}^{d} \sum_{r=1}^{d_1} a^{ij}_{kr}(t) u^r_{x_i x_j}(t,x) + f^k(t,x), \quad (k = 1, 2, \cdots, d_1)
$$

(1.2)

defined for $t > 0$ and $x \in \mathbb{R}^d_+$.

In the study of partial differential equations (PDEs) or of partial differential systems (PDSs) regularity theory play the key role of describing essential relations between input data and the unknown solutions; the sharper the theory is, the more understanding of the relations we get.

The primary goals of this article are to introduce some new mathematical tools and ideas which are useful in the study of systems in $L_p$-spaces involving weights and to provide another nice regularity theory for these systems.
In this article we use weighted Sobolev spaces for the unknown function $u = (u_1, \cdots, u_d)$ and the inputs $f^k$. The need to introduce weights comes from, for instance, the theory of stochastic partial differential equations (SPDEs) or stochastic partial differential systems (SPDSs), where a Hölder space approach does not allow us to obtain results of reasonable generality and Sobolev spaces without weights are trivially inappropriate (see [14] for details). To study such stochastic systems one has to develop a nice regularity theory for the corresponding deterministic systems in advance. Also Sobolev spaces with weights are very useful in treating degenerate elliptic and parabolic equations (see, for instance, [16]) and in studying equations defined on non-smooth domains such as domains with wedges (see, for instance, [5, 10, 18]).

In principle there are three main methods for $L_p$-theory: multiplier theory, Calderón-Zygmund theory and the pointwise estimate using sharp functions. Multiplier theory fits well when the principal operator is almost Laplacian and the equation under consideration is defined on the entire space, and Calderón-Zygmund theory works well when there exists an integral representation of solutions and the integral is taken over $\mathbb{R}^n$ for some $n$. However, these two methods do not fit our case since we are dealing with weighted $L_p$-theories for systems (1.1) and (1.1) defined on a half space. Thus we use an approach based on pointwise estimates of the sharp function of second order derivatives, but unlike the standard theory (for instance, [13]) we need to use the weighted version. The elaboration of this approach is one of our main results.

We also mention that if $d_1 = 1$ then weighted $L_p$-theories for single equations defined on a half space can be constructed based on integration by parts without relying on sharp function estimations (see the proof of Lemma 4.8 and Lemma 6.3 of [10]). However it seems that the arguments in the proof of Lemma 4.8 and Lemma 6.3 of [10] cannot be reproduced for $L_p$-theory of systems unless $p = 2$ and some stronger algebraic conditions on $A^{ij}$ are additionally assumed.

Interestingly, we discovered some very useful tools in the perspective of linear Partial differential equations/systems theory. Even though, in this article, we only consider the systems with coefficients independent of $x$, the sharp function estimates and the tools used to derive them will naturally lead to many subsequent works studying, for instance, elliptic and parabolic equations and systems with discontinuous coefficients defined in an arbitrary domain $U$ of $\mathbb{R}^d$. In this context, we refer the readers to very extensive literature [13] and recent articles [1, 2, 3, 7, 6] (also see the references therein), where (standard) $L_p$-theories are constructed for single equations with VMO (or small BMO)-coefficients.

The article is organized as follows. In section [2] we prove the Fefferman-Stein theorem and Hardy-Littletwood theorem with our special weights; the proofs are quite elementary. In section [3] we introduce weighted Sobolev spaces and formulate our regularity results for the systems, Theorem [3.10] and Theorem [3.13]. The useful tools and ideas for proving Theorem [3.10] and Theorem [3.13] are in section [4] and [5], the local estimations and the sharp function estimations. Finally Theorem [3.10] and Theorem [3.13] are proved in section [6].

As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$ and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. 

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For \( i = 1, \ldots, d \), multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i \in \{0, 1, 2, \ldots\} \), and functions \( u(x) \) we set
\[
    u_{\alpha} = \frac{\partial u}{\partial x^\alpha} = D_i u, \quad D^n u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.
\]
By \( \delta^{kr} \) we denote the Kronecker delta on the indices \( k, r \). If we write \( N = N(\cdots) \), this means that the constant \( N \) depends only on what are in parenthesis.

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## 2 F-S and H-L theorems in weighted \( L_p \)-spaces

Denote
\[
    \Omega = \mathbb{R} \times \mathbb{R}^d_+ := \{(t, x) = (t, x^1, x^2, \ldots, x^d) : x^1 > 0\}.
\]
Also, by \( \mathcal{B}(\mathbb{R}^d_+) \) and \( \mathcal{B}(\Omega) \) we denote the Borel \( \sigma \)-algebra on \( \mathbb{R}^d_+ \) and \( \Omega \) respectively. Fix \( \alpha \in (-1, \infty) \) and define the weighted measures
\[
    \nu(dx) = \nu_{\alpha}(dx) = (x^1)^\alpha dx, \quad d\mu = \mu_{\alpha}(dt dx) := \nu_{\alpha}(dx)dt.
\]
Then \( (\mathbb{R}^d_+, \mathcal{B}(\mathbb{R}^d_+), \nu) \) and \( (\Omega, \mathcal{B}(\Omega), \mu) \) are measure spaces with \( \nu(\mathbb{R}^d_+) = \mu(\Omega) = \infty \). Let \( p \in [1, \infty) \) and \( L_p(\Omega, \mu) = L_p(\Omega, \mu; \mathbb{R}^d) \) \( (L_p(\mathbb{R}^d_+, \nu) \) resp.) be the collection of Borel-measurable functions \( u = (u^1, \ldots, u^d) \) defined on \( \Omega \) (on \( \mathbb{R}^d_+ \) resp.) satisfying
\[
    \|u\|_{L^p(\Omega, \mu)} := \int_{\Omega} |u|^p d\mu < \infty, \quad \|u\|_{L^p(\mathbb{R}^d_+, \nu)} := \int_{\mathbb{R}^d_+} |u|^p \nu(dx) < \infty, \text{ respectively}.
\]
Denote
\[
    \mathcal{B}^0(\Omega) := \{ C \in \mathcal{B}(\Omega) : |C| := \mu(C) < \infty \}, \quad \mathcal{B}^0(\mathbb{R}^d_+) := \{ D \in \mathcal{B}(\mathbb{R}^d_+) : |D| := \nu(D) < \infty \}.
\]
We say \( f \in L_{1,\text{loc}}(\Omega, \mu; \mathbb{R}^d) \) if \( f I_C \in L_1(\Omega, \mu) \) for any \( C \in \mathcal{B}^0(\Omega) \), where \( I_C \) is the indicator function of \( C \). For \( f = (f^1, \ldots, f^d) \in L_1(\Omega, \mu; \mathbb{R}^d) \) and \( C \in \mathcal{B}^0(\Omega) \) we define
\[
    f_C := \frac{1}{|C|} \int_C f d\mu = \int_C f_1 d\mu, \ldots, \int_C f_d d\mu.
\]
Similarly write \( h \in L_{1,\text{loc}}(\mathbb{R}^d_+, \nu; \mathbb{R}^d) \) if \( h I_D \in L_1(\mathbb{R}^d_+, \nu) \) for any \( D \in \mathcal{B}^0(\mathbb{R}^d_+) \), and define
\[
    h_D := \frac{1}{|D|} \int_D h d\nu = \int_D h^1 d\nu, \ldots, \int_D h^d d\nu.
\]
Let \( (\mathbb{C}_n, n \in \mathbb{Z}) \) denote the filtration of the partitions of \( \Omega \) defined by
\[
    \mathbb{C}_n = \left\{ \left[ i_0 \frac{1}{4^n}, \frac{i_0 + 1}{4^n} \right) \times \left[ i_1 \frac{1}{2^n}, \frac{i_1 + 1}{2^n} \right) \times \cdots \times \left[ i_d \frac{1}{2^n}, \frac{i_d + 1}{2^n} \right) : i_0, i_1, \ldots, i_d \in \mathbb{Z}, \ i_1 \in \{0\} \cup \mathbb{N} \right\},
\]
and \( (\mathbb{D}_n, n \in \mathbb{Z}) \) be the corresponding filtration of the partitions of \( \mathbb{R}^d_+ \), that is,
\[
    \mathbb{D}_n := \left\{ \left[ i_0 \frac{1}{2^n}, \frac{i_0 + 1}{2^n} \right) \times \cdots \times \left[ i_d \frac{1}{2^n}, \frac{i_d + 1}{2^n} \right) : i_0, i_1, \ldots, i_d \in \mathbb{Z}, \ i_1 \in \{0\} \cup \mathbb{N} \right\}.
\]
For any \((t,x) \in \Omega\), by \(C_n(t,x)\) \((D_n(x)\) resp.) we denote the unique cube in \(C_n\) (in \(D_n\) resp.) containing \((t,x)\) \((x\) respectively). Let \(L = L(\Omega)\) (resp. \(L(\mathbb{R}^d_+)\)) denote the set of \(\mathbb{R}^d\)-valued continuous functions with compact support in \(\Omega\) (in \(\mathbb{R}^d_+\) respectively).

**Lemma 2.1.** (i) We have \(\inf_{C \in C_n} |C| \to \infty\) as \(n \to -\infty\) and, for any \(f \in L(\Omega)\), \(\lim_{n \to \infty} f_{C_n(t,x)} = f(t,x)\) holds for any \((t,x) \in \Omega\).

(ii) We have \(\inf_{D \in D_n} |D| \to \infty\) as \(n \to -\infty\) and, for any \(f \in L(\mathbb{R}^d_+)\), \(\lim_{n \to \infty} f_{D_n(x)} = f(x)\) holds for any \(x \in \mathbb{R}^d_+\).

**Proof.** It is obvious since \(f\) is continuous. \(\square\)

**Lemma 2.2.** (i) For any \(C \in C_n\) there exists a unique \(C' \in C_{n-1}\) such that \(C \subset C'\) and

\[
\frac{|C'|}{|C|} \leq N(\alpha) < \infty.
\]

(ii) For any \(D \in D_n\) there exists a unique \(D' \in D_{n-1}\) such that \(D \subset D'\) and

\[
\frac{|D'|}{|D|} \leq N(\alpha) < \infty.
\]

**Proof.** We only prove (i). Since \(C_{n-1}\) is a partition of \(\Omega\), only one member of it contains \(C\); we call it \(C'\). Let

\[
C' = \left[ \frac{i_0}{4^n-1}, \frac{i_0+1}{4^n-1} \right] \times \left[ \frac{i_1}{2^n-1}, \frac{i_1+1}{2^n-1} \right] \times \cdots \times \left[ \frac{i_d}{2^n-1}, \frac{i_d+1}{2^n-1} \right].
\]

Then we have

\[
|C'| = \mu(C') = \frac{1}{2(d+1)(n-1)} \int_{\frac{i_0}{4^n-1}}^{\frac{i_0+1}{4^n-1}} (x^1)^\alpha \, dx^1 = \frac{1}{2(d+1)(n-1)} \cdot \frac{1}{\alpha+1} \left[ \left( \frac{i_1+1}{2^n-1} \right)^{\alpha+1} - \left( \frac{i_1}{2^n-1} \right)^{\alpha+1} \right].
\]

Note that \(C\) is one of \(4 \cdot 2^d\) cubes belonging to \(C_n\) inside \(C'\) and by the location of \(C\) we have either

\[
|C| = \frac{1}{2(d+1)n} \cdot \frac{1}{\alpha+1} \left[ \left( \frac{i_1+1}{2^n-1} \right)^{\alpha+1} - \left( \frac{i_1}{2^n-1} \right)^{\alpha+1} \right] \quad (2.1)
\]

or

\[
|C| = \frac{1}{2(d+1)n} \cdot \frac{1}{\alpha+1} \left[ \left( \frac{i_1+1}{2^n-1} - \frac{1}{2^n} \right)^{\alpha+1} - \left( \frac{i_1}{2^n-1} \right)^{\alpha+1} \right]. \quad (2.2)
\]

**Case 1:** Let \(i_1 \geq 1\) and \(\alpha \geq 0\). Denoting

\[
a = \frac{i_1+1}{2^n-1}, \quad b = \frac{i_1}{2^n-1}, \quad c = \frac{i_1+1}{2^n-1} - \frac{1}{2^n}, \quad \phi(x) = x^{\alpha+1},
\]

...
we get

\[
\frac{|C'|}{|C|} = 2^{d+1} \frac{\phi(a) - \phi(b)}{\phi(a) - \phi(c)} \text{ or } 2^{d+1} \frac{\phi(a) - \phi(b)}{\phi(c) - \phi(b)}
\]

\[
= 2^{d+1} \left(1 + \frac{\phi(c) - \phi(b)}{\phi(a) - \phi(c)}\right) \text{ or } 2^{d+1} \left(1 + \frac{\phi(a) - \phi(c)}{\phi(c) - \phi(b)}\right)
\]

\[
= 2^{d+1} \left(1 + \frac{\phi'(\beta)}{\phi'(\alpha)}\right) \text{ or } 2^{d+1} \left(1 + \frac{\phi'(\alpha)}{\phi'(\beta)}\right),
\]

(2.3)

where \(\alpha, \beta\) are some numbers satisfying \(b < \beta < c < \alpha < a\); we used mean value theorem. Since \(\alpha + 1 > 1\), the function \(\phi\) is convex and increasing on \((0, \infty)\). Hence, we have

\[
\frac{\phi'(\beta)}{\phi'(\alpha)} \leq 1, \quad \frac{\phi'(\alpha)}{\phi'(\beta)} = \frac{\alpha}{b^\alpha} = \left(\frac{i_1 + 1}{i_1}\right) \leq 2^\alpha,
\]

and therefore

\[
\frac{|C'|}{|C|} \leq 2^{d+1}(1 + 2^\alpha) \leq 2^{\alpha+d+2}.
\]

**Case 2:** Assume \(i_1 = 0\) and \(\alpha \geq 0\). By similar but simpler calculation we obtain

\[
\frac{|C'|}{|C|} \leq 2^{\alpha+d+2}.
\]

**Case 3:** Assume \(\alpha \in (-1, 0)\). If \(|C|\) is given as in \([2.2]\), then since \(\phi(x)\) is concave,

\[
\frac{(i_{1+1} + 1)^{\alpha + 1} - (i_{1+1})^{\alpha + 1}}{(i_{1+1} + 1)^{\alpha + 1} - (i_{1+1} + \frac{1}{2^\alpha})^{\alpha + 1}} \leq 2.
\]

Let \(|C|\) be given as in \([2.1]\). If \(i_1 = 0\), then

\[
\frac{(i_1 + 1 + \frac{1}{2^\alpha})^{\alpha + 1} - (i_1 + 1)^{\alpha + 1}}{(i_1 + 1 + \frac{1}{2^\alpha} - \frac{1}{2^\alpha})^{\alpha + 1} - (i_1 + 1 - \frac{1}{2^\alpha})^{\alpha + 1}} = \frac{2^{\alpha+1}}{2^{\alpha+1} - 1},
\]

and if \(i_1 \geq 1\) then since \(\phi\) is concave and \(\phi'\) is positive on \((0, \infty)\)

\[
\frac{(i_1 + 1 + \frac{1}{2^\alpha})^{\alpha + 1} - (i_1 + 1)^{\alpha + 1}}{(i_1 + 1 + \frac{1}{2^\alpha} - \frac{1}{2^\alpha})^{\alpha + 1} - (i_1 + 1 - \frac{1}{2^\alpha})^{\alpha + 1}} \leq \frac{2^{\alpha+1}}{2^{\alpha+1} - 1} \leq 2^{1-\alpha}.
\]

The lemma is proved. \(\square\)

**Remark 2.3.** (i) By Lemma \([2.1]\), Lemma \([2.2]\), and the outline of Section 3.1, 3.2 of \([13]\), we get Lemma \([2.5]\) Theorem \([2.7]\) and Theorem \([2.8]\) below for free.

(ii) if \(C_n \subseteq \mathbb{C}_n\) and \(C_m \subseteq \mathbb{C}_m\) with \(n \leq m\), then \(C_n \cap C_m = C_m\) or \(\emptyset\).

**Definition 2.4.** We call \(\tau = \tau(x) \in \mathbb{Z} \cup \{\infty\}\) a stopping time if \(\{x : \tau(x) = n\} = \emptyset\) or union of some elements in \(\mathbb{C}_n\) for each \(n \in \mathbb{Z}\).
For \( f \in L_{1,\text{loc}}(\Omega, \mu; \mathbb{R}^{d_1}) \), \( h \in L_{1,\text{loc}}(\mathbb{R}^{d_1}_+, \nu; \mathbb{R}^{d_1}) \) and \( n \in \mathbb{Z} \) we define
\[
f_{\mid n}(t, x) := \frac{1}{\mu(C_n(t, x))} \int_{C_n(t, x)} f(s, y) \mu(dy) = \int_{C_n(t, x)} f(s, y) \mu(dy),
\]
and
\[
h_{\mid n}(x) := \frac{1}{\nu(D_n(t, x))} \int_{D_n(t, x)} h(y) \nu(dy) = \int_{D_n(t, x)} h(y) \nu(dy),
\]
and
\[
f_{\mid \tau}(t, x) := f_{\mid \tau(t,x)}(t,x) \quad \text{if} \quad \tau(t,x) \neq \infty; \quad f_{\mid \tau}(t, x) := f(t, x) \quad \text{if} \quad \tau(t,x) = \infty.
\]

**Lemma 2.5.** Let \( \{C_n : n \in \mathbb{Z}\} \) be a filtration of partitions of \( \Omega \).

(i) Let \( g \in L_{1,\text{loc}}(\Omega, \mu; \mathbb{R}^1) \), \( g \geq 0 \) and let \( \tau \) be a stopping time. Then
\[
\int_{\Omega} g_{\mid \tau}(t, x) I_{\tau < \infty}(t, x) \mu(dtdx) = \int_{\Omega} g(t, x) I_{\tau < \infty}(t, x) \mu(dtdx),
\]
and
\[
\int_{\Omega} g_{\mid \tau}(t, x) \mu(dtdx) = \int_{\Omega} g(t, x) \mu(dtdx).
\]

(ii) Let \( g \in L_1(\Omega, \mu; \mathbb{R}^1) \), \( g \geq 0 \) and let \( \lambda > 0 \) be a constant. Then
\[
\tau(t, x) := \inf\{n : g_{\mid n}(t, x) > \lambda\} \quad (\inf \emptyset := \infty)
\]
is a stopping time. Furthermore, we have
\[
0 \leq g_{\mid \tau}(t, x) I_{\tau < \infty} \leq N_0 \lambda, \quad \|\{|(t,x) : \tau(t,x) < \infty\}|\| \leq \lambda^{-1} \int_{\Omega} g(t, x) I_{\tau < \infty} \mu(dtdx).
\]

**Remark 2.6.** (Riesz-Calderón-Zygmund decomposition) Any \( g \in L_1(\Omega, \mu; \mathbb{R}^1) \) is decomposed by
\[
g = \xi + \eta,
\]
where \( \xi = g - g_{\mid \tau}, \eta = g_{\mid \tau} = g_{\mid \tau} I_{\tau < \infty} + g_{\mid \tau} I_{\tau = \infty} \). Moreover, we have (i) \( \eta \leq N_0 \lambda \) a.e. (ii) \( \|\{|(t,x) : \xi(t,x) \neq 0\}\| \leq \lambda^{-1} \|g\|_{L_1(\Omega, \mu)} \) (iii) \( \xi_{\mid \tau} = 0 \).

Now, for \( f \in L_{1,\text{loc}}(\Omega, \mu; \mathbb{R}^{d_1}) \) we define the maximal function
\[
Mf(t,x) := \left(\sup_{n < \infty} |f_{\mid n}(t,x)|, \ldots, \sup_{n < \infty} |f_{\mid n}^{d_1}(t,x)|\right)
\]
and the sharp function
\[
f^{\#}(t, x) = \left(\sup_{n < \infty} \int_{C_n(t,x)} |f_{\mid n}(s, y) - f_{\mid n}(s, y)| \mu(dsdy), \ldots, \sup_{n < \infty} \int_{C_n(t,x)} |f_{\mid n}^{d_1}(s, y) - f_{\mid n}^{d_1}(s, y)| \mu(dsdy)\right).
\]
We define \( Mh(x) \) and \( h^{\#}(x) \) similarly for functions \( h = h(x) \in L_{1,\text{loc}}(\mathbb{R}^{d_1}_+, \nu; \mathbb{R}^{d_1}) \).

**Theorem 2.7.** Let \( p \in (1, \infty) \). Then for any \( f \in L_p(\Omega, \mu; \mathbb{R}^{d_1}) \) and \( h \in L_p(\mathbb{R}^{d_1}_+, \nu; \mathbb{R}^{d_1}) \), we have
\[
\|Mf\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})} \leq N\|f\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})}, \quad \|Mh\|_{L_p(\mathbb{R}^{d_1}_+, \nu; \mathbb{R}^{d_1})} \leq N\|h\|_{L_p(\mathbb{R}^{d_1}_+, \nu; \mathbb{R}^{d_1})}
\]
where \( N = N(\theta, p, d, d_1) \).
Theorem 2.8. Let $p \in (1, \infty)$. Then for any $f \in L_p(\Omega,\mu;\mathbb{R}^{d_1})$ and $h \in L_p(\mathbb{R}^d,\nu;\mathbb{R}^{d_1})$ we have

$$
\|f\|_{L_p(\Omega,\mu;\mathbb{R}^{d_1})} \leq N \|f^\#\|_{L_p(\Omega,\mu;\mathbb{R}^{d_1})}, \quad \|h\|_{L_p(\mathbb{R}^d,\nu;\mathbb{R}^{d_1})} \leq N \|h^\#\|_{L_p(\mathbb{R}^d,\nu;\mathbb{R}^{d_1})}
$$

where $N = N(\theta, p, d, d_1)$.

We investigate the relation between our maximal and sharp functions and more general ones. Let $B'_r(x')$ denote the open ball in $\mathbb{R}^{d-1}$ of radius $r$ with center $x'$. For $x = (x^1, x') \in \mathbb{R}^d_\star$ and $t \in \mathbb{R}$, denote

$$
B_r(x) = B_r(x^1, x') = (x^1 - r, x^1 + r) \times B'_r(x'), \quad Q_r(t, x) := (t, t + r^2) \times B_r(x)
$$

and $\mathcal{Q}$ be the collection of all such open sets $Q_r(t, x) \subset \Omega$. For $f \in L_{1,loc}(\Omega,\mu : \mathbb{R}^{d_1})$ we define

$$
f^*_Q = \int_{Q} f^i \, d\mu, \quad M f^i(t, x) = \sup_{(t,x) \in Q} \int_{Q} f^i \, d\mu, \quad (f^i)^\#(t, x) = \sup_{(t,x) \in Q} \int |f^i - f^*_Q| \, d\mu, \quad i = 1, \ldots, d_1,$$

where the supremum is taken for all $Q \in \mathcal{Q}$ containing $(t, x)$. Denote

$$
Mf := (Mf^1, \ldots, Mf^{d_1}), \quad f^i_\sharp := ((f^1)^\sharp, \ldots, (f^{d_1})^\sharp).
$$

For functions $h \in L_{1,loc}(\mathbb{R}^d_\star, \nu, \mathbb{R}^{d_1})$, the functions $Mh(x)$ and $(h)^\sharp(x)$ are defined similarly.

Lemma 2.9. For a scalar function $g = g(t, x)$ and $h = h(x)$ we have

$$
g^\#(t, x) \leq N \ g^\sharp(t, x), \quad h^\#(x) \leq N \ h^\sharp(x)
$$

where $N = N(\theta, p, d)$.

Proof. We only prove the first assertion. For $(t, x) \in \Omega$, denote the corresponding unique cube $C_n(t, x) \subset \mathbb{C}_n$ by

$$
\left[ \frac{i_0}{2^n}, \frac{i_0 + 1}{2^n} \right) \times \left[ \frac{i_1}{2^n}, \frac{i_1 + 1}{2^n} \right) \times \cdots \times \left[ \frac{i_d}{2^n}, \frac{i_d + 1}{2^n} \right)
$$

where $i_0, i_1, \ldots, i_d \in \mathbb{Z}$ and $i_1 \in \{0\} \cup \mathbb{N}$. We define $Q_{(n)}(t, x) := Q_{C_n}(t^*, x^*)$ with $t^* = \frac{i_0}{2^n}$ and $x^* = (\frac{i_1 + 2d}{2^n}, \frac{i_2}{2^n}, \ldots, \frac{i_d}{2^n})$. We have $(t, x) \in C_n(t, x) \subset Q_{(n)}(t, x)$ and

$$
\frac{|Q_{(n)}(t, x)|}{|C_{n}(t, x)|} = N(d) \cdot \frac{(i_1 + 2d)^{\alpha+1} - i_1^{\alpha+1}}{(i_1 + 1)^{\alpha+1} - i_1^{\alpha+1}} \tag{2.4}
$$

by simple calculation. If $i_1 = 0$, (2.4) is $N(d)(2d)^{\alpha+1}$; if $i_1 \geq 1$ and $\alpha \geq 0$ then (2.4) is less than or equal to

$$
N(d) \cdot (2d) \left( \frac{i_1 + 2d}{i_1} \right)^\alpha \leq N(d) \cdot (2d) \cdot (1 + 2d)^\alpha
$$

by mean value theorem. If $\alpha \in (-1, 0)$ then we use the concavity of $x^{\alpha+1}$ to prove that (2.4) is less than $N(d)(2d)^{\alpha+1}$. The lemma is proved. \qed
Lemma 2.9 and Theorem 2.8 imply the following version of Fefferman-Stein theorem:

**Theorem 2.10.** (Fefferman-Stein) Let $p \in (1, \infty)$. Then for any $f \in L_p(\Omega, \mu; \mathbb{R}^{d_1})$ and $h \in L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})$, we have

$$
\|f\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})} \leq N\|f\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})}, \quad \|h\|_{L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})} \leq N\|h\|_{L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})}
$$

where $N = N(\theta, p, d, d_1)$.

The following lemma will be used in the proof of Theorem 2.12 below.

**Lemma 2.11.** Let $\alpha > -1$ and $\phi(x) = x^{\alpha+1}$ on $x > 0$. Then for any $x > 0$ and $r > 0$ we have

$$
\frac{\phi(x+2r) - \phi(x+r)}{\phi(x+r) - \phi(x)} \leq 2^{\alpha+1}.
$$

**Proof.** If $\alpha \in (-1, 0]$ the claim is obvious since $\phi$ is concave.

Assume $\alpha > 0$, fix $r > 0$, and define

$$
f(x) := \frac{\phi(x+2r) - \phi(x+r)}{\phi(x+r) - \phi(x)}.
$$

We show that $f'(x) \leq 0$ for $x > 0$ so that $f(x) \leq f(0) = 2^{\alpha+1} - 1$; note that $f(0)$ does not depend on $r$. A simple calculation shows

$$
f'(x) = r(\alpha + 1) \cdot \frac{2(x+2r)^\alpha x^\alpha - (x+2r)^\alpha (x+r)^\alpha - (x+r)^\alpha x^\alpha}{((x+r)^{\alpha+1} - x^{\alpha+1})^2}.
$$

(2.5)

The numerator in (2.6) is

$$
2 \cdot x^\alpha (x+r)^\alpha (x+2r)^\alpha \cdot \left[ (x+r)^{-\alpha} - \frac{x^{-\alpha} + (x+2r)^{-\alpha}}{2} \right].
$$

(2.6)

Since the function $x^{-\alpha}$ is convex and $x+r$ is the midpoint of $x$ and $x+2r$, the square bracket in (2.6) is non-positive and so is $f'(x)$. The lemma is proved. \qed

**Theorem 2.12.** (Hardy-Littlewood) Let $p \in (1, \infty)$. Then for $f \in L_p(\Omega, \mu; \mathbb{R}^{d_1})$ and $h \in L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})$ we have

$$
\|\mathbb{M}f\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})} \leq N\|f\|_{L_p(\Omega, \mu; \mathbb{R}^{d_1})}, \quad \|\mathbb{M}h\|_{L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})} \leq N\|h\|_{L_p(\mathbb{R}^d_+, \nu; \mathbb{R}^{d_1})}.
$$

**Proof.** Again we only proof the first assertion. We follow the outline for the proof of Theorem 3.3.2 which does not involve a weight in the norm. Without loss of generality we assume $d_1 = 1$ and $g := f \geq 0$.

For $\lambda > 0$, denote $A(\lambda) := \{(t, x) : \mathbb{M}g(t, x) > \lambda\}$. Then since $\mathbb{M}g$ is lower semi-continuous, $A(\lambda)$ is open. To prove the theorem it is enough to show that for any $\lambda > 0$ and compact set $K \subset A(\lambda)$

$$
|K| \leq \frac{N}{\lambda} \int_{\Omega} I_{A(\lambda)}(t,x) g(t,x) \mu(dtdx),
$$
where $N = N(\theta, p, d)$. For the details see the proof of Theorem 3.3.2 of [13].

For any $(t, x) \in K$ there exists $Q$ containing $(t, x)$ such that \( \int_Q g \, d\mu > \lambda|Q| \). Also, we observe that $Q \subset A(\lambda)$ and there exists a finite cover \( \{Q_1, \ldots, Q_n\} \) of $K$ such that

\[
\int_{Q_i} g \, d\mu \geq \lambda|Q_i|.
\]

For $Q = (t - \frac{1}{2}r^2, t + \frac{1}{2}r^2) \times (x^1 - r, x^1 + r) \times B'_r(x') \in \mathcal{Q}$, denote

\[
3Q := (t - \frac{3}{2}r^2, t + \frac{3}{2}r^2) \times (x^1 - 3r, x^1 + 3r) \times B'_r(x').
\]

When $Q$ is close to the boundary of $\Omega$, $3Q$ may not be in $\Omega$. Hence, we define

\[
Q^* = 3Q \cap \Omega.
\]

Using a Vitali covering argument one can find the disjoint subset \( \{\tilde{Q}_1, \ldots, \tilde{Q}_k\} \) of \( \{Q_1, \ldots, Q_n\} \) satisfying $K \subset \bigcup_{j=1}^k \tilde{Q}_j^*$ (see the proof of Theorem 3.3.2 of [13]). To measure $|K|$ we compute the ratio \( \frac{|\tilde{Q}_j^*|}{|Q_j|} \). For $Q_j = (t - \frac{q^2}{2}, t + \frac{q^2}{2}) \times (x^1 - r, x^1 + r) \times B'_r(x')$ we have

\[
\frac{|\tilde{Q}_j^*|}{|Q_j|} = 3^d \cdot \frac{\phi(x + 3r) - \phi((x - 3r) \lor 0)}{\phi(x + r) - \phi(x - r)},
\]

where $\phi(x) = x^{d-d+p+1}$ and $a \lor b := \max\{a, b\}$. We note

\[
\frac{\phi(x + 3r) - \phi((x - 3r) \lor 0)}{\phi(x + r) - \phi(x - r)} = \frac{\phi(x + r) - \phi((x - 3r) \lor 0) + \phi(x + r) - \phi(x - r) + \phi(x + 3r) - \phi(x + r)}{\phi(x + r) - \phi(x - r)} \leq 2 + \frac{\phi(x + 3r) - \phi(x + r)}{\phi(x + r) - \phi(x - r)},
\]

where the last inequality is true since $\phi$ is increasing and convex. Now, Lemma 2.11 with $x - r, 2r$ instead of $x, r$ implies (2.7) is less than or equal to $2 + 2^{\alpha+1}$. Hence, we have

\[
\frac{|\tilde{Q}_j^*|}{|Q_j|} \leq 3^d \cdot (2 + 2^{\alpha+1}), \quad |\tilde{Q}_j^*| \leq 3^d \cdot (2 + 2^{\alpha+1})|\tilde{Q}_j|.
\]

Thus,

\[
|K| \leq \sum_{j=1}^k |\tilde{Q}_j^*| \leq 3^d \cdot (2 + 2^{\alpha+1}) \sum_{j=1}^k |\tilde{Q}_j| \leq 3^d \cdot (2 + 2^{\alpha+1}) \lambda^{-1} \int_{\Omega} g \, d\mu \leq 3^d \cdot (2 + 2^{\alpha+1}) \lambda^{-1} \int_{\Omega} gI_{\Lambda(\lambda)} \, d\mu.
\]

The theorem is proved.
3 A weighted $L_p$-theory for systems in a half space

Let $C_0^\infty (\mathbb{R}^d; \mathbb{R}^{d_1})$ denote the set of all $\mathbb{R}^{d_1}$-valued infinitely differentiable functions with compact support in $\mathbb{R}^d$. By $\mathcal{D}$ we denote the space of $d$-dimensional distributions on $C_0^\infty (\mathbb{R}^d; \mathbb{R}^{d_1})$; precisely, for $u \in \mathcal{D}$ and $\phi \in C_0^\infty (\mathbb{R}^d; \mathbb{R}^{d_1})$ we define $(u, \phi) \in \mathbb{R}^{d_1}$ with components $(u, \phi)^k = (u^k, \phi^k)$, $k = 1, \ldots, d_1$; each $u^k$ is a usual scalar-valued distribution.

For $p \in (1, \infty)$ we define $L_p = L_p(\mathbb{R}^d; \mathbb{R}^{d_1})$ as the space of all $\mathbb{R}^{d_1}$-valued functions $u = (u^1, \ldots, u^{d_1})$ satisfying

$$\|u\|^p_{L_p} := \sum_{k=1}^{d_1} \|u^k\|^p_{L_p} < \infty.$$  

Denote $x = (x^1, \ldots, x^d)$. In this paper we define

$$\|u_x\|^p_{L_p} = \sum_{i=1}^d \|u_{x^i}\|^p_{L_p}, \quad \|u_{xx}\|^p_{L_p} = \sum_{i,j=1}^d \|u_{x^i x^j}\|^p_{L_p}, \quad \text{etc.}$$

For any $\gamma \in \mathbb{R}$, define the space of Bessel potential $H_p^\gamma = H_p^\gamma (\mathbb{R}; \mathbb{R}^{d_1})$ as the space of all distributions $u$ on $\mathbb{R}^d$ such that $(1-\Delta)^{\gamma/2} u \in L_p$, where each component is defined by

$$((1-\Delta)^{\gamma/2} u)^k = (1-\Delta)^{\gamma/2} u^k$$

and the norm is given by

$$\|u\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2} u\|_{L_p}.$$  

Then $H_p^\gamma$ is a Banach space with the given norm and $C_0^\infty (\mathbb{R}^d; \mathbb{R}^{d_1})$ is dense in $H_p^\gamma$ (see \cite{19}). Note that $H_p^\gamma$ is usual Sobolev spaces for $\gamma = 0, 1, 2, \ldots$. It is well known that the first order differentiation operator, $D : H_p^\gamma \to H_p^{-\gamma-1}$, is bounded. On the other hand, if $\text{supp}(u) \subset (a, b)$, where $-\infty < a < b < \infty$, then

$$\|u\|_{H_p^\gamma} \leq c(d, a, b) \|u_x\|_{H_p^{-\gamma-1}}, \quad \text{(3.1)}$$

(see, for instance, Remark 1.13 in \cite{10}).

Now we introduce the weighted Sobolev spaces taken from \cite{10} and \cite{17}. Take a nonnegative real-valued function $\zeta(x) = \zeta(x^1) \in C_0^\infty (\mathbb{R}^+)$. Let $\zeta(e^{n+s}) > c > 0$, $\forall s \in \mathbb{R}$, $\quad \text{(3.2)}$

where $c$ is a constant. Note that any nonnegative function $\xi$ with $\xi > 0$ on $[1, e]$ satisfies $\text{(3.2)}$. For $\theta \in \mathbb{R}$, let $H_{p,\theta}^\gamma := H_{p,\theta}^\gamma (\mathbb{R}^d; \mathbb{R}^{d_1})$ denote the set of all $d$-dimensional distributions $u = (u^1, u^2, \ldots, u^{d_1})$ on $\mathbb{R}^d_+$ such that

$$\|u\|_{H_{p,\theta}^\gamma} := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta(\cdot) u(e^{n\cdot})\|_{H_p}^p < \infty.$$  

It is known that for different $\zeta$ satisfying $\text{(3.2)}$, we get the same spaces $H_{p,\theta}^\gamma$ with equivalent norms, and for any $\eta \in C_0^\infty (\mathbb{R}^+; \mathbb{R})$,

$$\sum_{n=-\infty}^{\infty} e^{\eta d} \|\eta(\cdot) u(e^{n\cdot})\|_{H_p}^p \leq N \sum_{n=-\infty}^{\infty} e^{\eta d} \|\zeta(\cdot) u(e^{n\cdot})\|_{H_p}^p, \quad \text{(3.4)}$$

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where $N$ depends only on $\gamma, \theta, p, d_1, \eta, \zeta$. Furthermore, if $\gamma$ is a nonnegative integer, then
\[
\|u\|^\gamma_{H^\gamma_{p,\theta}} \sim \sum_{|\beta| \leq \gamma} \int_{\mathbb{R}^d_+} |(x^\gamma)^\beta D^\beta u(x)|^p (x^\gamma)^{\theta - d} dx.
\] (3.5)

Let $M^\alpha$ be the operator of multiplying by $(x^\gamma)^\alpha$ and $M := M^1$. For $\nu \in (0, 1]$, denote
\[
|u|_C = \sup_{x \in \mathbb{R}^d_+} |u(x)|, \quad [u]_{C^\nu} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\nu}.
\]

Below we collect some other important properties of the spaces $H^\gamma_{p,\theta}$.

**Lemma 3.1.** \((\text{[??]})\) Let $\gamma, \theta \in \mathbb{R}$ and $p \in (1, \infty)$.

(i) $C_0^\infty(\mathbb{R}^d_+: \mathbb{R}^{d_1})$ is dense in $H^\gamma_{p,\theta}$.

(ii) Assume that $\gamma = m + \nu + d/p$ for some $m = 0, 1, \cdots$ and $\nu \in (0, 1]$. Then for any $u \in H^\gamma_{p,\theta}$ and $i \in \{0, 1, \cdots, m\}$, we have
\[
|M^{i+\nu/p} D^i u|_C + [M^{m+\nu/p} D^m u]_{C^\nu} \leq N \|u\|_{H^\gamma_{p,\theta}}.
\] (3.6)

(iii) Let $\alpha \in \mathbb{R}$. Then $M^\alpha H^\gamma_{p,\theta} \oplus = H^{\gamma}_{p,\theta}$ and
\[
\|u\|_{H^\gamma_{p,\theta}} \leq N \|M^{-\alpha} u\|_{H^{\gamma}_{p,\theta} \oplus} \leq N \|u\|_{H^\gamma_{p,\theta}}.
\]

(iv) For any $MD, DM : H^\gamma_{p,\theta} \to H^{\gamma-1}_{p,\theta}$ are bounded linear operators, and
\[
\|u\|_{H^\gamma_{p,\theta}} \leq N \|u\|_{H^{\gamma-1}_{p,\theta}} + N \|Mu_x\|_{H^{\gamma-1}_{p,\theta}} \leq N \|u\|_{H^\gamma_{p,\theta}},
\]
\[
\|u\|_{H^\gamma_{p,\theta}} \leq N \|u\|_{H^{\gamma-1}_{p,\theta}} + N \|(Mu)_x\|_{H^{\gamma-1}_{p,\theta}} \leq N \|u\|_{H^\gamma_{p,\theta}}.
\]

Furthermore, if $\theta \neq d - 1, d + 1 - p$, then
\[
\|u\|_{H^\gamma_{p,\theta}} \leq N \|Mu_x\|_{H^{\gamma-1}_{p,\theta}}, \quad \|u\|_{H^\gamma_{p,\theta}} \leq N \|(Mu)_x\|_{H^{\gamma-1}_{p,\theta}}.
\] (3.7)

(v) For $i = 0, 1$ let $\kappa \in (0, 1], p_i \in (1, \infty), \gamma_i, \theta_i \in \mathbb{R}$ and assume the relations
\[
\gamma = \kappa \gamma_1 + (1 - \kappa) \gamma_0, \quad \frac{1}{p} = \frac{\kappa}{p_1} + \frac{1 - \kappa}{p_0}, \quad \frac{\theta}{p} = \frac{\theta_1 \kappa}{p_1} + \frac{\theta_0 (1 - \kappa)}{p_0}.
\]

Then
\[
\|u\|_{H^\gamma_{p,\theta}} \leq N \|u\|_{H^{\gamma_1}_{p_1,\theta_1}} \|u\|_{H^{\gamma_0}_{p_0,\theta_0}}^{1-\kappa}.
\]

**Remark 3.2.** Let $\theta \in (d - 1, d + 1 - p)$ and $n$ be a nonnegative integer. By Lemma 3.1 (iii), (iv)
\[
\|M^{-n} v\|_{H^\gamma_{p,\theta}} \leq N \|D^n v\|_{H^{\gamma-n}_{p,\theta}}
\] (3.8)

for any $v \in C_0^\infty(\mathbb{R}^d_+: \mathbb{R}^{d_1})$. Indeed, since $\theta + mp \neq d - 1, d + 1 - p$ for any integer $m$
\[
\|M^{-n} v\|_{H^\gamma_{p,\theta}} \leq N \|M^{-1} v\|_{H^{\gamma-(n-1)}_{p,\theta-(n-1)p}} \leq N \|v_x\|_{H^{\gamma-1}_{p,\theta-(n-1)p}},
\]
\[
\leq N \|M^{-1} v_x\|_{H^{\gamma-1}_{p,\theta-(n-2)p}} \leq N \|D^2 v\|_{H^{\gamma-2}_{p,\theta-(n-2)p}} \cdots
\]
For \(-\infty \leq S < T \leq \infty\), we define the Banach spaces:

\[
H_{p,\theta}^\gamma(S,T) := L_p((S,T), H_p^\gamma), \quad \mathbb{H}_{p,\theta}^\gamma(T) := \mathbb{H}_{p,\theta}^\gamma(0,T), \quad \mathbb{L}_{p,\theta}(S,T) := H_{p,\theta}^0(S,T), \quad \mathbb{L}_{p,\theta}^\gamma(T) := L_{p,\theta}^\gamma(0,T)
\]

with norms given by

\[
\|u\|_{H_{p,\theta}^\gamma(S,T)}^p = \int_S^T \|u(t)\|_{H_{p,\theta}^\gamma}^p \, dt.
\]

**Lemma 3.3.** For \(\phi, \psi \in C_0^\infty((S,T) \times \mathbb{R}^d)\), define \((\phi, \psi) = \int_S^T \int_{\mathbb{R}^d} \phi(s,x)\psi(t,x) \, dt \, dx\). For \(p \in (1, \infty)\) and \(\gamma, \theta \in \mathbb{R}\), define \(\gamma', p', \theta'\) so that

\[
\gamma' = -\gamma, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{\theta}{p} + \frac{\theta'}{p'} = d.
\]

Then for any \(\phi \in C_0^\infty((S,T) \times \mathbb{R}^d)\)

\[
\|\phi\|_{\mathbb{H}_{p,\theta}^\gamma(S,T)} \leq N \sup_{\psi \in C_0^\infty((S,T) \times \mathbb{R}^d)} \frac{\|(\phi, \psi)\|_{\mathbb{H}_{p,\theta}^{\gamma'}(S,T)}}{\|\psi\|_{\mathbb{H}_{p,\theta}^{\gamma'}(S,T)}} \leq N \|\phi\|_{\mathbb{H}_{p,\theta}^\gamma(S,T)},
\]

where \(N\) is independent of \(\phi\). Moreover the relation \((\phi, \psi)\) can be extended by continuity on all \(\phi \in \mathbb{H}_{p,\theta}^\gamma(S,T)\) and \(\psi \in \mathbb{H}_{p,\theta}^{\gamma'}(S,T)\), and then it identifies the dual to \(\mathbb{H}_{p,\theta}^\gamma(S,T)\) with \(\mathbb{H}_{p,\theta}^{\gamma'}(S,T)\).

**Proof.** See Theorem 2.5 of [II]; this actually proves the duality between \(H_{p,\theta}^\gamma\) and \(H_{p,\theta}^{\gamma'}\), but the proof of our claim is essentially the same. The only difference is that one has to consider integrations on the time variable, too. \(\Box\)

Finally, we set \(U_{p,\theta}^\gamma := M^{1-2/p} H_{p,\theta}^{\gamma-2/p}\), meaning that any \(u \in U_{p,\theta}^\gamma\) has the form \(u = M^{1-2/p} \cdot v\) with \(v \in H_{p,\theta}^{\gamma-2/p}\) and \(\|u\|_{U_{p,\theta}^\gamma} = \|M^{-1+2/p} u\|_{H_{p,\theta}^{\gamma-2/p}} = \|v\|_{H_{p,\theta}^{\gamma-2/p}}\). Using these spaces, we define our solution spaces.

**Definition 3.4.** We write \(u \in \mathcal{S}_{p,\theta}^{\gamma+2}(S,T)\) if \(u \in M \mathbb{H}_{p,\theta}^{\gamma+2}(S,T)\), \(u(S,\cdot) \in U_{p,\theta}^{\gamma+2}\) \((u(-\infty,\cdot) := 0 \text{ if } S = -\infty)\), and for some \(\tilde{f} \in M^{-1} \mathbb{H}_{p,\theta}^\gamma(T)\) it holds \(u_t = \tilde{f}\) in the sense of distributions, that is for any \(\phi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d+1})\) the equality

\[
(u(t,\cdot), \phi) = (u(S,\cdot), \phi) + \int_S^t (\tilde{f}(s,\cdot), \phi) \, ds
\]

holds for all \(t \in (S,T)\). In this case we write \(u_t = \tilde{f}\). The norm in \(\mathcal{S}_{p,\theta}^{\gamma+2}(S,T)\) is defined by

\[
\|u\|_{\mathcal{S}_{p,\theta}^{\gamma+2}(S,T)} = \|M^{-1} u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(S,T)} + \|M u_t\|_{\mathbb{H}_{p,\theta}^\gamma(S,T)} + \|u(S,\cdot)\|_{U_{p,\theta}^{\gamma+2}}.
\]

Define \(\mathcal{S}_{p,\theta}^{\gamma+2}(T) := \mathcal{S}_{p,\theta}^{\gamma+2}(0,T)\) and \(\mathcal{S}_{p,\theta}^{\gamma+2} := \mathcal{S}_{p,\theta}^{\gamma+2}(0,\infty)\).

**Theorem 3.5.** (i) The space \(\mathcal{S}_{p,\theta}^{\gamma+2}(S,T)\) is a Banach space.

(ii) Let \(0 < T < \infty\). Then for any \(u \in \mathcal{S}_{p,\theta}^{\gamma+2}(T)\),

\[
\sup_{t \leq T} \|u(t)\|_{\mathcal{H}^{\gamma+1}_{p,\theta}} \leq N(d, p, \theta, T) \|u\|_{\mathcal{S}_{p,\theta}^{\gamma+2}(T)}.
\]
(iii) Let $0 < T < \infty$. For any nonnegative integer $n \geq \gamma + 2$, the set

$$\mathcal{H}^{\gamma+2}_{p,\theta}(T) \cap \bigcup_{k=1}^{\infty} C([0, T], C^0_n(G_k))$$

where $G_k = (1/k, k) \times \{|x| < k\}$ is dense in $\mathcal{H}^{\gamma+2}_{p,\theta}(T)$.

**Proof.** See Theorem 2.9 and Theorem 2.11 of [14].

Here are some interior Hölder estimates of functions in the space $\mathcal{H}^{\gamma+2}_{p,\theta}(T)$.

**Theorem 3.6.** Let $p > 2$ and assume

$$2/p < \alpha < \beta \leq 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon,$$

where $k \in \{0, 1, 2, \cdots\}$ and $\varepsilon \in (0, 1]$. Denote $\sigma = \beta - 1 + \theta/p$. Then for any $u \in \mathcal{H}^{\gamma+2}_{p,\theta}(T)$ and multi-indices $i, j$ such that $|i| \leq j$ and $|j| = k$,

(i) the functions $D^i u(t, x)$ are continuous in $[0, T] \times \mathbb{R}_+^d$ and

$$M^{\sigma + |i|} D^i u(t, \cdot) - M^{\sigma + |i|} D^i u(0, \cdot) \in C^{\alpha/2 - 1/p}([0, T], C(\mathbb{R}_+^d));$$

(ii) there exists a constant $N = N(p, d, \alpha, \beta)$ so that

$$\sup_{t, s \leq T} \left( \frac{|M^{\sigma + |i|} D^i (u(t) - u(s))|_{C(\mathbb{R}_+^d)}}{|t - s|^{\alpha/2 - 1/p}} + \frac{|M^{\sigma + |j| + \varepsilon} D^j (u(t) - u(s))|_{C(\mathbb{R}_+^d)}}{|t - s|^{\alpha/2 - 1/p}} \right)$$

$$\leq NT^{(\beta - \alpha)/2} \|u\|_{\mathcal{H}^{\gamma+2}_{p,\theta}(T)},$$

(3.10)

**Proof.** See Theorem 4.7 of [8].

**Remark 3.7.** (see Remark 4.8 of [8] for details) For instance, if $\theta = d$, $\gamma \geq -1$ and $\kappa_0 = 1 - 2 + d/p > 0$, then for any $\kappa \in (0, \kappa_0)$ and $u \in \mathcal{H}^1_{p,\theta}(T)$ with $u(0) = 0$,

$$\sup_{t \leq T} \sup_{x, y \in \mathbb{R}_+^d} \frac{|u(t, x) - u(t, y)|}{|x - y|^{\kappa}} < \infty.$$  \hspace{1cm} (3.11)

$$\sup_{x \in \mathbb{R}_+^d} \sup_{s, t \leq T} \frac{|u(t, x) - u(s, x)|}{|t - s|^{\kappa/2}} < \infty.$$  \hspace{1cm} (3.12)

Indeed, for (3.11) take $j = 0, \beta = \kappa_0 - \kappa + 2/p$ and $\varepsilon = 1 - \beta - d/p = \kappa = -\sigma$, then $\sigma + |j| + \varepsilon = 0$ and (3.10) yields (3.11). Also for (3.12), take $i = 0, \alpha = \kappa + 2/p, \beta = 1 - d/p$ then $\sigma + |i| = 0, 2/p < \alpha < \beta < 1$ and $\alpha/2 - 1/p = \kappa/2$.

For any $d_1 \times d_1$ matrix $C = (c_{kr})$ we let

$$|C| := \sqrt{\sum_{k, r} (c_{kr})^2}.$$  \hspace{1cm} (3.11)

We set $A^{ij} = (a^{ij}_{kr})_{k, r = 1, \ldots, d_1}$ for each $i, j = 1, \ldots, d$. Throughout the article we assume the followings.
Assumption 3.8. For each $i$ and $j$, $A^{ij}$ depends only on $t$ and there exist finite constants $\delta, K > 0$ so that
\[
\delta |\xi|^2 \leq \sum_{i,j=1}^{d} (\xi^i)^* A^{ij} \xi^i
\] (3.13)
for all (real valued) $d_1 \times d$-matrix $\xi$, where $\xi^i$ denotes the $i$-th column of $\xi$. Also, there exists a constant $K < \infty$ such that
\[
|A^{ij}| \leq K, \quad \forall \; i, j = 1, \ldots, d,
\] (3.14)
where $*$ means matrix transposition.

We recall (1.2) and write it as
\[
\frac{\partial u}{\partial t} = a^{ij}_{kr}(t) \frac{\partial^2 u}{\partial x^i \partial x^j} + f_k, \quad u^k(S) = u_0^k, \quad k = 1, 2, \ldots, d_1,
\] (3.15)
assuming the summation convention on indices $i, j, r$; such convention will be used throughout the article. In short, we will write (3.15) as
\[
\frac{\partial u}{\partial t} = A^{ij}(t) \frac{\partial^2 u}{\partial x^i \partial x^j} + f, \quad u(S) = u_0,
\] (3.16)
where we regard $u, u_0, f$ as $d_1 \times 1$ matrix-valued functions.

Definition 3.9. A $d$-dimensional distribution-valued function $u$ defined on $\mathbb{S}, T$ is a solution of (3.16) in $H^{\gamma+2, p, \theta}(\mathbb{S}, T)$ if $u \in M_{H^{\gamma+2, p, \theta}(\mathbb{S}, T)}$, $u(S) \in U^{\gamma+2, p, \theta}_{p, \theta}(\mathbb{S}, T)$ holds in the sense of distributions, that is (3.9) holds with $\tilde{f} = \tilde{f}^{ij} u^{i,j} x^i x^j + f$.

The following is our $L_p$-theory for the parabolic system (3.16). The proof is given in section 6.

Theorem 3.10. Let $p \in (1, \infty)$ and $\gamma \geq 0$. Assume $\theta \in (d+1-p, \infty)$ if $p \in (1, 2]$ and $\theta \in (d-1, \infty)$ if $p \in (2, \infty)$. Then for any $f \in M^{-\gamma}_{p, \theta}(T)$ and $u_0 \in U^{\gamma+2}_{p, \theta}$ system (3.16) admits a unique solution $u \in H^{\gamma+2}_{p, \theta}(T)$, and for this solution we have
\[
\|u\|_{H^{\gamma+2}_{p, \theta}(T)} \leq N \left(\|M f\|_{M^{\gamma}_{p, \theta}(T)} + \|u_0\|_{U^{\gamma+2}_{p, \theta}}\right),
\] (3.17)
where $N = N(\gamma, p, \theta, \delta, K)$.

Remark 3.11. Various interior Hölder estimates of the solution in Theorem 3.10 can be obtained according to Theorem 3.6. Also see Lemma 4.11 and Lemma 4.14.

Remark 3.12. (i) The proof of Theorem 3.10 is based on a sharp function estimate (Lemma 5.7). If $d_1 = 1$, then Lemma 5.7 can be proved for any $\theta \in (d-1, \infty)$ as long as $p > 1$; we will prove this in a subsequent article for parabolic equations with (weighted) BMO-coefficients.

(ii) It is known (see Remark 3.6 of [14]) that if $\theta \not\in (d-1, d-1+p)$, then Theorem 3.10 is not true even for the heat equation $u_t = \Delta u + f$.

Now we present our $L_p$-theory for the elliptic system (1.1). The proof is given in section 6.
Theorem 3.13. Let \( p \in (1, \infty) \), \( \gamma \geq 0 \) and \( A^{ij} \) be independent of \( t \). Assume \( \theta \in (d+1-p,d+p-1) \) if \( p \in [1, 2] \) and \( \theta \in (d-1,d+1) \) if \( p \in (2, \infty) \). Then for any \( f = (f^1, f^2, \cdots, f^{d_1}) \in M^{-1}H^\gamma_{p,\theta} \) the system \((4,1)\) admits a unique solution \( u \in MH^\gamma_{p,\theta} \), and for this solution we have

\[
\|M^{-1}u\|_{H^\gamma_{p,\theta}} \leq N\|f\|_{H^\gamma_{p,\theta}},
\]

where \( N = N(\gamma, p, \theta, \delta, K) \).

Remark 3.14. Theorem 3.10 and Theorem 3.13 hold not only for \( \gamma \geq 0 \) but also for any \( \gamma < 0 \). This can be easily proved by using the results for \( \gamma \geq 0 \) and repeating the arguments used for single equations (see the proof of Theorem 5.6 of [10]).

4 Preliminary estimates : Some local estimates of solutions

In this section we prove a version of Theorem 3.10 for \( \theta = d \). This result is used to derive some local estimates of \( D^\alpha u \) for any multi-index \( \alpha \), where \( u \) is a solution of (3.10).

First, we introduce some results for systems defined on the entire space. For \(-\infty < S < T < \infty\) we denote \( \mathbb{H}^\gamma_p(S,T) := L_p((S,T), H^\gamma_p) \) and \( \mathbb{H}^\gamma_p(T) := \mathbb{H}^\gamma_p(0,T) \).

Theorem 4.1. Let \( \gamma \in \mathbb{R} \) and \(-\infty < S < T < \infty\). Let \( f \in \mathbb{H}^\gamma_p(S,T) \) and \( u \in H^\gamma_{p,\theta}(S,T) \) satisfy

\[
 u_t = A^{ij}(t)u_{x^ix^j} + f, \quad t > S, x \in \mathbb{R}^d.
\]

Additionally assume \( u(S, \cdot) = 0 \) if \( S > -\infty \). Then

\[
\|u_{xx}\|_{H^\gamma_{p,\theta}(S,T)} \leq N(d, p, \delta, K)\|f\|_{H^\gamma_p(S,T)},
\]

(4.1)

Also if \(-\infty < S < T < \infty\), then

\[
\|u\|_{H^\gamma_{p,\theta}(S,T)} \leq N(d, p, \delta, K, S,T)\|f\|_{H^\gamma_p(S,T)}.
\]

Proof. This is a classical result. See, for instance, Theorem 1.1 of [15]. Actually in [15] the theorem is proved only when \( \gamma = 0 \), but the general case follows by the fact the operator \((1 - \Delta)^{\mu/2} : \mathbb{H}^\gamma_p(S,T) \to \mathbb{H}^\gamma_{p,\theta}(S,T)\) is an isometry.

Theorem 4.1 yields the following result.

Corollary 4.2. Let \( u \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^{d_1}) \). Then

\[
\|u_{xx}\|_{H^\gamma_p(-\infty, T)} \leq N(d, p, \delta, K) \|u_t - A^{ij}u_{x^ix^j}\|_{H^\gamma_p(\mathbb{R}^d)}.
\]

(4.2)

Corollary 4.3. Let \( 0 < T < \infty \), \( f^t \in L_p(T) \), and \( u \in H^\gamma_{p,\theta}(T) \) satisfies

\[
 u_t = A^{ij}(t)u_{x^ix^j} + f^t, \quad t \in (0,T), x \in \mathbb{R}^d
\]

with zero initial condition \( u(0) = 0 \). Then

\[
\|u_x\|_{L_p(T)} \leq N(d, p, \delta, K)\|f\|_{L_p(T)},
\]

\[
\|u\|_{H^\gamma_p(T)} \leq N(d, p, \delta, K,T)\|f\|_{L_p(T)}.
\]

(4.3)
Now we get (4.3) by taking $c$

Thus for this function (4.4) with $c$

By (4.1) with $c$

Let $c$

Proof. Remember

$$
\|f^i\|_{H^{p-1}_p} \leq N\|f^i\|_{L^p}, \quad \|u_x\|_{L^p} \leq N(\|u_{x_2}\|_{H^{p-1}_p} + \|u\|_{L^p(T)}).
$$

By (4.1) with $\gamma = -1$,

$$
\|u_x\|_{L^p(T)} \leq N(\|f^i\|_{L^p(T)} + \|u\|_{L^p(T)}).
$$

(4.4)

Notice that, for any constant $c > 0$, the function $u^c(t, x) := u(c^2t, cx)$ satisfies

$$
u^c_i = A^{ij}(c^2t)u^c_{x_i x_j} + (c_i f^j(c^2t, cx))_{x_j}.
$$

Thus for this function (4.4) with $c^{-2}T$ in place of $T$ becomes

$$
\|u_x\|_{L^p(T)} \leq N(\|f^i\|_{L^p(T)} + c^{-1}\|u\|_{L^p}).
$$

Now we get (4.3) by taking $c \to \infty$.

**Corollary 4.4.** Let $u \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^{d_1})$ and $A^{ij}$ be independent of $t$. Then

$$
\|u_{xx}\|_{H^\gamma_p}^p \leq N(d, p, \delta, K) \|A^{ij}u_{x_i x_j}\|_{H^\gamma_p}^p.
$$

(4.5)

Proof. Take a nonnegative smooth function $\eta(t) \in C^\infty_0(-1,1)$ so that $\int_\mathbb{R} \eta^p(t)\,dt = 1$. For each $n = 1, 2, \ldots$, define $\eta_n(t) = n^{-1/p}\eta(t/n)$. Then applying (4.2) for $v_n(t, x) := \eta_n(t)u(x)$,

$$
\|u_{xx}\|_{H^\gamma_p}^p \leq N(\|A^{ij}u_{x_i x_j}\|_{H^\gamma_p}^p + N\|u\|_{H^\gamma_p}^p \int_\mathbb{R} |\eta_n'|^p\,dt
$$

Now it is enough to let $n \to \infty$. The corollary is proved.

Remember that for any $t \in \mathbb{R}$, $(x^1, x') \in \mathbb{R}^d$, we defined

$$
B_r(x) = (x^1 - r, x^1 + r) \times B'_r(x'), \quad Q_r(t, x) = (t, t + r^2) \times B_r(x),
$$

where $B'_r(x')$ is the open ball in $\mathbb{R}^{d-1}$ of radius $r$ with center $x'$. By $C^\infty_{\text{loc}}(\mathbb{R}^{d+1}; \mathbb{R}^{d_1})$ we denote the set of $\mathbb{R}^{d_1}$-valued functions $u$ defined on $\mathbb{R}^{d+1}$ and such that $\zeta u \in C^\infty_0(\mathbb{R}^{d+1}; \mathbb{R}^{d_1})$ for any $\zeta \in C^\infty_0(\mathbb{R}^{d+1}; \mathbb{R})$.

**Theorem 4.5.** Let $q \in (1, \infty)$ and $(t, x) \in \mathbb{R}^{d+1}$. Then there exists a constant $N$, depending only on $q, d, d_1, \delta$ and $K$ so that for any $\lambda \geq 4, r > 0$ and $u \in C^\infty_{\text{loc}}(\mathbb{R}^{d+1}; \mathbb{R}^{d_1})$, we have

$$
\int_{Q_r(t, x)} \int_{Q_r(t, x)} |u_{xx}(s, y) - u_{xx}(r, z)|^q \,ds \,dy \,dr \,dz
\leq N\lambda^{-q} \int_{Q_r(t, x)} |u_{xx}|^q \,ds + N\lambda^{d+2} \int_{Q_r(t, x)} |u_t + A^{ij}u_{x_1 x_j}|^q \,ds \,dy.
$$

Proof. See Theorem 6.1.2 of [13]. Actually this theorem is proved when $d_1 = 1$, and the proof is based on Theorem 4.1. Since Theorem 4.1 holds for any $d_1 = 1, 2, \ldots$, the theorem can be proved by repeating the proof of Theorem 6.1.2 of [13] word for word.
Corollary 4.6. Let \( u = u(x) \in C^\infty_{loc}(\mathbb{R}^d; \mathbb{R}^d) \) and \( A^{ij} \) be independent of \( t \). Then for any \( x \in \mathbb{R}^d \), \( \lambda \geq 4 \) and \( r > 0 \),
\[
\int_{B_r(x)} \int_{B_r(x)} |u_x(x) - u_x(z)|^q dy dz \leq N \lambda^{-q} \int_{B_r(x)} |u_x(x)|^q dy + N \lambda^{d+2} \int_{B_r(x)} |A^{ij} u_{x,x}^{ij}|^q dy.
\]

From now on we consider systems defined on a half space. Remember \( H^\gamma_{\mu,\eta}(S,T) := L_p((S,T), H^\gamma_{\mu,\eta}) \), \( \|u\|_{H^\gamma_{\mu,\eta}(S,T)} := \int_T^T \|u(t,\cdot)\|_{H^\gamma_{\mu,\eta}} dt \).

Lemma 4.7. Let \( \gamma, \theta \in \mathbb{R} \) and \( p \in (1, \infty) \).

(i) Let \( -\infty \leq S < T \leq \infty \) and suppose \( u(t,x) \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d) \) satisfies
\[
u_t + A^{ij}(t)u_{x,x}^{ij} = f, \quad (t,x) \in (S,T) \times \mathbb{R}^d
\]
and assume \( u(T,\cdot) = 0 \) if \( T < \infty \).

\[
\|M^{-1}u\|^p_{H^\gamma_{\mu,\eta}^{-1}(S,T)} \leq N(p,d,\theta,\delta,K) \left( \|M^{-1}u\|^p_{H^\gamma_{\mu,\eta}^{-1}(S,T)} + \|Mf\|^p_{H^\gamma_{\mu,\eta}(S,T)} \right). \tag{4.6}
\]

(ii) If \( u(x) \in C^\infty_{loc}(\mathbb{R}^d; \mathbb{R}^d) \) and \( A^{ij} \) is independent of \( t \), then
\[
\|M^{-1}u\|^p_{H^\gamma_{\mu,\eta}^{-1}(S,T)} \leq N(p,d,\theta,\delta,K) \left( \|M^{-1}u\|^p_{H^\gamma_{\mu,\eta}^{-1}} + \|MA^{ij} u_{x,x}^{ij}\|^p_{H^\gamma_{\mu,\eta}^{-1}} \right). \tag{4.7}
\]

Proof. (i) We proceed as in the proof of Lemma 5.8 of [10]. Denote \( S_n = e^{-2n}S \) and \( T_n = e^{-2n}T \).

By Lemma 3.1(iii) and (3.3),
\[
\|M^{-1}u\|^p_{H^\gamma_{\mu,\eta}^{-1}(S,T)} \leq N \sum_{n=-\infty}^{\infty} e^{n(\theta-p)} \|\zeta(x)u(t,e^n x)\|^p_{H^\gamma_{\mu,\eta}^{-1}(S,T)} \]
\[
= N \sum_{n=-\infty}^{\infty} e^{n(2+\theta-p)} \|\zeta(x)u(e^{2n} t, e^n x)\|^p_{H^\gamma_{\mu,\eta}^{-1}(S_n,T_n)} \]
\[
\leq N \sum_{n=-\infty}^{\infty} e^{n(2+\theta-p)} \|\zeta(x)u(e^{2n} t, e^n x)\|^p_{H^\gamma_{\mu,\eta}^{-1}(S_n,T_n)}, \tag{4.8}
\]

where the last inequality is due to (3.1). Denote \( v^n(t,x) = \zeta(x)u(e^{2n} t, e^n x) \), then it satisfies
\[
v^n_t + A^{ij}(e^{2n} t)u^n_{x,x}^{ij} = e^{2n} \zeta(x)f(e^{2n} t, e^n x) + 2e^n A^{ij}(e^{2n} t)\zeta_x u_{x}^{ij}(e^{2n} t, e^n x) + A^{11}(e^{2n} t)\zeta_x u(e^{2n} t, e^n x)
\]
for \( (t,x) \in (S_n,T_n) \times \mathbb{R}^d \). By (3.1),
\[
\|v^n_{x,x}\|^p_{H^\gamma_{\mu,\eta}(S_n,T_n)} \leq Ne^{2np} \|\zeta(x)f(e^{2n} t, e^n x)\|^p_{H^\gamma_{\mu,\eta}(S_n,T_n)} + Ne^{np} \|\zeta_x u_{x}^{ij}(e^{2n} t, e^n x)\|^p_{H^\gamma_{\mu,\eta}(S_n,T_n)} + N\|\zeta_x u(e^{2n} t, e^n x)\|^p_{H^\gamma_{\mu,\eta}(S_n,T_n)}.
\]

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where $N$ is independent of $n$. Plugging this into (3.8) one gets
\[
\|M^{-1}u\|_{H_{p,q}^γ(S,T)}^p \leq N \sum_{n=-∞}^{∞} e^{n(θ+p)}\|ξ_n f(t,e^nx)\|_{L_{p,q}^γ(S,T)}^p
\]
\[+ N \sum_{n=-∞}^{∞} e^{θn}\|ξ_n u(t,e^nx)\|_{H_{p,q}^γ(S,T)}^p + N \sum_{n=-∞}^{∞} e^{n(θ-p)}\|ξ_n u(t,e^nx)\|_{H_{p,q}^γ(S,T)}^p.
\]
This, (3.4) and Lemma 3.1 easily lead us to (4.6). Indeed, for instance, by (3.4)
\[
\sum_{n=-∞}^{∞} e^{θn}\|ξ_n u(t,e^nx)\|_{H_{p,q}^γ(S,T)}^p \leq N\|u_x\|_{H_{p,q}^γ(S,T)}^p
\]
and by Lemma 3.1(iv) applied to $M^{-1}u$ in place of $u$,
\[
\|u_x\|_{H_{p,q}^γ(S,T)} = \|DM(M^{-1}u)\|_{H_{p,q}^γ(S,T)} \leq N\|M^{-1}u\|_{H_{p,q}^{γ+1}(S,T)}.
\]

(ii) This is proved similarly based on (4.5). The lemma is proved.

\[\square\]

Remark 4.8. Let $γ ≥ 0$. By iterating (4.6), one gets
\[
\|M^{-1}u\|_{H_{p,q}^γ(S,T)}^p \leq N\|M^{-1}u\|_{L_{p,q}^γ(S,T)}^p + N\|Mf\|_{L_{p,q}^γ(S,T)}^p \leq N\|Mu_{xx}\|_{L_{p,q}^γ(S,T)}^p + N\|Mf\|_{H_{p,q}^γ(S,T)}^p,
\]
where for the second inequality we use (3.7) twice. We use both inequalities later to estimate $\|M^{-1}u\|_{H_{p,q}^{γ+1}(S,T)}^p$.

Let $(w^1_t, w^2_t, \cdots, w^d_t)$ be a $d$-dimensional Wiener process defined on a probability space $(Ω', F, P)$. Denote
\[
ξ_t = w^1_t √2 + 2t, \quad η_t = (√2 \int_0^t e^{ξ_s} dw^2_s, \cdots, √2 \int_0^t e^{ξ_s} dw^d_s)
\]
and define $d × d$ matrix-valued process $σ_t$ so that $(σ_t x)^1 = e^{ξ_t} x^1$ and $(σ_t x)' = x' + x^1 η_t$. It is easy to check (see [10], p.1628) that $x_t(x) := σ_t x$ is the unique solution of the stochastic differential equation
\[
dx_t = √2 x^1_t dw^1_t + 3x^1_t e_1 dt, \quad x_0(x) = x,
\]
where $e_1 = (1, 0, \cdots, 0)$. For any $f ∈ C_0^∞(R^d)$ and $x ∈ R^d$, define
\[
Ef(x) = E\int_0^∞ f(σ_t x) dt := \int_Ω \int_0^∞ f(σ_t x) dtdP.
\]
(See below for the convergence of this integral). Note that if $x^1 ≤ 0$ then $(σ_t x)^1 ≤ 0$ and thus $Ef(x) = 0$. Denote
\[
Łu := M^2 Δ u + 3MD_1 u = \sum_{i=1}^{d} (MD_i)^2 + 2MD_1.
\]
Lemma 4.9. Let $f \in C_0^\infty(\mathbb{R}^d)$.

(i) $\mathcal{E} f \in L_p(\mathbb{R}^d)$ and $f = \mathcal{L}(\mathcal{E} f)$ in the sense of distributions on $\mathbb{R}^d$.

(ii) There exist $f_1, f_2, \cdots, f_d \in L_p(\mathbb{R}^d)$ so that $f = MD_i f^i$ in the sense of distributions on $\mathbb{R}^d$, and

$$\sum_{i=1}^d \|f^i\|_{L_p(\mathbb{R}^d)} \leq N\|f\|_{L_p(\mathbb{R}^d)}.$$ 

Proof. By Theorem 2.11 of [10] (with $\theta = d$ and $b = 3$ there), the map $\mathcal{L}$ is a bounded one-to-one operator from $H^2_{p,d}$ onto $L_{p,d}$, and its inverse ($:= \mathcal{L}^{-1}$) is also bounded. Denote $u := \mathcal{L}^{-1} f \in H^2_{p,d}$. By Lemma 5.1(i), there exists a sequence $u_n \in C_0^\infty(\mathbb{R}^d)$ so that $u_n \to u$ in $H^2_{p,d}$. Denote $f_n(x) := \mathcal{L} u_n(x)$ for each $x \in \mathbb{R}^d$. Then

$$\mathcal{L} u_n \to \mathcal{L} u = (f) \quad \text{in} \quad L_{p,d} \quad \text{and} \quad \|u_n - u_m\|_{H^2_{p,d}} \leq N\|f_n - f_m\|_{L_{p,d}}. \quad (4.9)$$

Obviously $u_n(x) = f_n(\sigma_t x) = 0$ if $x^i \leq 0$. By Itô’s formula (see (2.10) in [10] for details), we get

$$u_n(x) = \mathbb{E} \int_0^\infty f_n(\sigma_t x) dt, \quad \forall x \in \mathbb{R}^d.$$ 

The convergence of this improper integral is discussed in the proof of Theorem 2.11 of [10]. Actually there it is shown that for any $h \in C_0^\infty(\mathbb{R}^d_+)$ (here, $\theta = d$ and $b = 3$ in our case),

$$\mathbb{E} \int_0^\infty \|h(\sigma_t x)\|_{L_{p,d}} dt \leq N\|h\|_{L_{p,d}} \int_0^\infty e^{-(\theta-d+1)(b-1)t+(\theta-d+1)^2 t} dt = N\|h\|_{L_{p,d}}, \quad (4.10)$$

which also implies

$$\|u_n - \mathcal{E} f\|_{L_{p,d}} = \|\mathbb{E} \int_0^\infty f_n(\sigma_t x) dt - \mathbb{E} \int_0^\infty f(\sigma_t x) dt\|_{L_{p,d}} \leq N\|f_n - f\|_{L_{p,d}} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Note $L_{p,d} = L_0(\mathbb{R}^d_+)$. Since $u_n(x), f_n(x), f(x)$ and $\mathcal{E} f$ vanish if $x^i \leq 0$, it follows that

$$\|u_n - \mathcal{E} f\|_{L_p(\mathbb{R}^d)} \to 0, \quad \|f_n - f\|_{L_p(\mathbb{R}^d)} \to 0 \quad (4.11)$$

as $n \to \infty$. Also (4.9) and fact $\|u_n\|_{H^2_{p,d}} = \|\mathcal{L}^{-1} f\|_{H^2_{p,d}} \leq N\|f_n\|_{L_{p,d}}$ show that $\{MDu_n : n = 1, 2, \cdots\}$ is a Cauchy sequence in $L_p(\mathbb{R}^d)$. Indeed, since each $u_n$ has compact support in $\mathbb{R}^d_+$,

$$\|MDu_n - MDu_m\|_{L_p(\mathbb{R}^d)} = \|MDu_n - MDu_m\|_{L_{p,d}} \leq N\|u_n - u_m\|_{H^2_{p,d}} \leq N\|f_n - f_m\|_{L_{p,d}}.$$ 

Let $\mathcal{L}^*$ denote the adjoint operator of $\mathcal{L}$. For any $\phi \in C_0^\infty(\mathbb{R}^d)$, by (4.11),

$$(f, \phi) = \lim_{n \to \infty} (f_n, \phi) = \lim_{n \to \infty} (\mathcal{L} u_n, \phi) = \lim_{n \to \infty} (u_n, \mathcal{L}^* \phi) = (\mathcal{E} f, \mathcal{L}^* \phi) = (\mathcal{L}(\mathcal{E} f), \phi).$$

Thus $f = \mathcal{L}(\mathcal{E} f)$ in the sense of distributions on $\mathbb{R}^d$. Also since $u_n \to \mathcal{E} f$ in $L_p(\mathbb{R}^d)$ and $\{MDu_n\}$ is a Cauchy sequence in $L_p(\mathbb{R}^d)$, we have $MD\mathcal{E} f \in L_p(\mathbb{R}^d)$. Consequently,

$$f = \mathcal{L}(\mathcal{E} f) = MD_1 (MD_1 \mathcal{E} f + 2 \mathcal{E} f) + \sum_{j=2}^d MD_j \mathcal{E} f =: \sum_{i=1}^d MD_i f^i,$$
As usual we only need to prove that the estimate (4.12) holds given that a solution
exists. Furthermore we may assume

\[ u(T) = 0 \]

and by (4.11),

\[ \sum_{i} \| f_{i} \|_{L_{p}(\mathbb{R}^{d})} = \lim_{n \to \infty} (\| u_{n} \|_{L_{p}} + \| M D u_{n} \|_{L_{p}}) \leq \lim_{n \to \infty} \| u_{n} \|_{H_{p,d}^{2}} \leq N \| f_{n} \|_{L_{p,d}} = N \| f \|_{L_{p,d}}. \]

The lemma is proved.  \[ \square \]

Now we prove a version of Theorem 3.10 for \( \theta = d \).

**Lemma 4.10.** Let \(-\infty < S < T < \infty, p \in (1, \infty)\) and \( n = 0, 1, 2, \ldots \). For any \( f \in M^{-1}H_{p,d}^{n}(S, T) \), the equation

\[ u_{t} + A^{ij}(t) u_{x_{i}x_{j}} = f, \quad (t, x) \in (S, T) \times \mathbb{R}^{d} \]

with the condition \( u(T) = 0 \) has a unique solution \( u \in H_{p,d}^{n+2}(S, T) \), and for this solution

\[ \| M^{-1}u \|_{H_{p,d}^{n+2}(S, T)} \leq N(p, d, \delta, K) \| Mf \|_{H_{p,d}^{n}(S, T)}. \]  \[ (4.12) \]

**Proof.** As usual we only need to prove that the estimate (4.12) holds given that a solution \( u \) already exists. Furthermore we may assume \( u(t, x) \in C_{c}^{\infty}(\mathbb{R} \times \mathbb{R}^{d}; \mathbb{R}^{d}) \). Due to Remark 4.8 and the inequality \( \| M^{-1}u \|_{L_{p,d}} \leq N(p, d) \| u \|_{L_{p,d}} \) (see Lemma 3.1(iv)), we only need to prove

\[ \| u_{x} \|_{L_{p,d}(S, T)} \leq N \| Mf \|_{L_{p,d}(S, T)}. \]  \[ (4.13) \]

By Lemma 4.9 we can write \( Mf = MD_{i}f_{i}^{*} \) on \( \mathbb{R}^{d} \) (thus \( f_{i}^{*} = D_{i}f_{i} \)), where \( f_{i}^{*} = (f_{i1}^{*}, \ldots, f_{id_{i}}^{*}) \), so that \( f_{i}^{*} \in L_{p}(S, T) \) (not only in \( L_{p,d}(S, T) \)) and

\[ \sum_{i=1}^{d} \| f_{i}^{*} \|_{L_{p}(S, T)} \leq N \| Mf \|_{L_{p,d}(S, T)}. \]

Thus by Corollary 4.3

\[ \| u_{x} \|_{L_{p,d}(S, T)} = \| u_{x} \|_{L_{p}(S, T)} \leq N \| f_{i}^{*} \|_{L_{p}(S, T)} \leq N \| Mf \|_{L_{p,d}(S, T)}. \]

The lemma is proved.  \[ \square \]

For \( r, a > 0 \), denote

\[ Q_{r}(a) = Q_{r}(0, a, 0) = (0, r^{2}) \times (a - r, a + r) \times B_{r}'(0), \quad U_{r} = (-r^{2}, r^{2}) \times (-2r, 2r) \times B_{r}'(0). \]

**Lemma 4.11.** Let \( 0 < s < r < \infty, u(t, x) \in C_{c}^{\infty}(\mathbb{R} \times \mathbb{R}^{d}_{+}; \mathbb{R}^{d}) \) and

\[ u_{t} + A^{ij}(t) u_{x_{i}x_{j}} = 0 \quad \text{for} \quad (t, x) \in Q_{r}(r). \]

Then for any multi-index \( \beta = (\beta^{1}, \ldots, \beta^{d}) \) there exists a constant \( N = N(p, |\beta|) \) so that the inequality

\[ \int_{Q_{r}(r)} \left( |M^{-1}D^{\beta}u|^{p} + |D^{\beta}u_{x}|^{p} + |MD^{\beta}u_{xx}|^{p} \right) (x^{1})^{\theta-d} \, dx \, dt \]

\[ \leq N(1 + r)^{|\beta|p} \cdot (1 + (r - s)^{-2})^{(|\beta|+1)p} \int_{Q_{r}(r)} |Mu(t, x)|^{p} (x^{1})^{\theta-d} \, dx \, dt \]  \[ (4.14) \]

holds for \( \theta = d \).
Proof. To prove (4.14) we use induction on $|\beta|$. Firstly, consider the case $|\beta| = 0$. We modify the proof of Lemma 2.4.4 of [13]. Denote $r_0 = s$ and $r_m = s + (r - s) \sum_{j=1}^{m} 2^{-j}$ for $m = 1, 2, \ldots$. Choose a smooth function $\zeta_m$ so that $0 \leq \zeta_m \leq 1$,

$$
\zeta_m = 1 \text{ on } U_{r_m}, \quad \zeta_m = 0 \text{ on } \Omega \setminus U_{r_{m+1}},
$$

$$
|\zeta_{mx}| \leq N(r - s)^{-1}2^m, \quad |\zeta_{mxx}| \leq N(r - s)^{-2}2^m, \quad |\zeta_{ml}| \leq N(r - s)^{-2}2^m.
$$

Note that $(u\zeta_m)(r^2, x) = 0$ on $\mathbb{R}^d$, and it satisfies

$$(u\zeta_m)_t + A^{ij}(u\zeta_m)_{x^i x^j} = \zeta_m u + 2A^{ij}(u\zeta_{m+1})_{x^i x^j} + A^{ij}u\zeta_{m+1} =: f_m, \quad (t, x) \in (0, r^2) \times \mathbb{R}^d.$$ 

By Lemma 4.10 for $\gamma = 0$,

$$A_m := \|M^{-1}u\zeta_m\|_{H^2_p(r^2)} \leq N\|Mf_m\|_{L^p_{r^2}}.$$ 

Denote $B := \left( \int_{Q_r(r)} |Mu|^p dx \right)^{1/p}$. Then

$$\|\zeta_{mi}Mu + A^{ij}Mu\zeta_{m+1}x^i x^j\|_{L^p_{r^2}} \leq N(r - s)^{-2}2^m\left( \int_{Q_r(r)} |Mu|^p dx \right)^{1/p} = N(r - s)^{-2}2^mB,$$

$$\|A_{mxx}M(u\zeta_{m+1})_{x^i} x^j\|_{L^p_{r^2}} \leq N(r - s)^{-1}2^m\|M(u\zeta_{m+1})_{x^i} x^j\|_{L^p_{r^2}} \leq N(r - s)^{-1}2^m\|u\zeta_{m+1}\|_{H^2_p(r^2)},$$

and by Lemma 3.1 (v) (take $p_0 = p_1 = p, \gamma = 1, \gamma_0 = 0, \gamma_1 = 2, \theta = d, \theta_0 = d + p, \theta_1 = d - p$ and $\kappa = 1/2$) for any $\epsilon > 0$

$$(r - s)^{-1}2^m\|u\zeta_{m+1}\|_{H^2_p(r^2)} \leq \epsilon A_{m+1} + \epsilon^{-1}(r - s)^{-2}2^mB.$$

It follows (with $\epsilon$ different from the one above),

$$A_m \leq \epsilon A_{m+1} + N(1 + \epsilon^{-1})(r - s)^{-2}2^mB.$$

We take $\epsilon = \frac{1}{11}$ and get

$$\epsilon^mA_m \leq \epsilon^{m+1}A_{m+1} + N\epsilon^m(1 + \epsilon^{-1})2^m(r - s)^{-2}B,$$

$$A_0 + \sum_{m=1}^{\infty} \epsilon^mA_m \leq \sum_{m=1}^{\infty} \epsilon^mA_m + N(r - s)^{-2}B.$$ 

Note that the series $\sum_{m=1}^{\infty} \epsilon^mA_m$ converges because $A_m \leq N2^m\|M^{-1}u\|_{H^2_p(r^2)}$. By Lemma 3.1 (iii), for any $M^{-1}w \in H^2_{p, \theta}$,

$$\|M^{-1}w\|_{H^2_p, \theta} \sim (\|M^{-1}w\|_{L^p_{r^2}} + \|w_{x}\|_{L_{r^2}} + \|Mu_{xx}\|_{L^p_{r^2}}). \quad (4.15)$$

Therefore,

$$\int_{Q_r(s)} (|M^{-1}u|^p + |u_x|^p + |Mu_{xx}|^p) dx dt \leq NA_0^p \leq N(r - s)^{-2p} \int_{Q_r(r)} |u(t, x)|^p (x^1)^p dx dt.$$ 

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Next assume that \(4.14\) holds whenever \(s < r\) and \(|\beta'| = k\), that is
\[
\int_{Q_r(s)} \left( |M^{-1} D^\beta u|^p + |D^\beta u_x|^p + |MD^\beta u_{xx}|^p \right) (x^1)^{\theta - d} dx dt
\leq N(1 + r)^{bp} \cdot (1 + (r - s)^{-2})^{(k+1)p} \int_{Q_r(s)} |Mu(t, x)|^p (x^1)^{\theta - d} dx dt
\]
Let \(|\beta| = k + 1\) and \(D^\beta = D_i D^\beta_i\) for some \(i\) and \(\beta'\) with \(|\beta'| = k\). Fix a smooth function \(\eta\) so that \\
\(\eta = 1\) on \(U_s\), \(\eta = 0\) on \(\Omega \setminus U_{(r+s)/2}\), \(|\eta_x| \leq N(r - s)^{-1}\), \(|\eta_{xx}| \leq N(r-s)^{-2}\) and \(|\eta| \leq N(r - s)^{-2}\).
Note that \(v := \eta D^\beta u\) satisfies \(v(r^2, \cdot) = 0\) and
\[
v_t + A^\beta v_{x^i} = f := \eta D^\beta u + 2 A^\beta \eta_x D^\beta u_x + A^\beta \eta_{xx} D^\beta u, \quad (t, x) \in (0, r^2) \times \mathbb{R}^d_d.
\]
By Lemma 4.10 for \(\gamma = 0\) (also note that \(x^1 \leq r\) on the support of \(\eta\) and \((r-s)^{-1} \leq 1 + (r - s)^{-2}\)),
\[
\|M^{-1} v\|^p_{L^p_p(d^2, r^2)} \leq N \|M \eta D^\beta u + 2 A^\beta \eta_x MD^\beta u_x + M A^\beta \eta_{xx} D^\beta u\|^p_{L^p_p(d^2, r^2)}
\leq N(1 + r)^p \cdot (1 + (r - s)^{-2})^p \int_{Q_r(s)/2} \left( |D^\beta u|^p + |MD^\beta u_x|^p \right) dx dt
\leq N(1 + r)^p \cdot (1 + (r - s)^{-2})^p \int_{Q_r(s)/2} \left( |D^\beta u_x|^p + |MD^\beta u_{xx}|^p \right) dx dt.
\]
This and \(4.15\) show that the induction goes through, and hence the lemma is proved.

**Remark 4.12.** The proof of Lemma 4.11 mainly depends on Lemma 4.10 and it can be easily checked that the assertion of Lemma 4.11 holds for \(\theta = \theta_0\) whenever Lemma 4.10 is true for \(\theta = \theta_0\). Thus due to Theorem 3.10 (which will be proved in section 6), Lemma 4.11 holds for \(\theta \in (d+1-p, d+p-1)\) if \(p \in (1, 2)\) and \(\theta \in (d-1, d+1)\) if \(p \in (2, \infty)\).

**Lemma 4.13.** Let \(u(t, x) \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d_+; \mathbb{R}^d)\). Then for any \(T > 0\), \(p > 1\) and \(n = 0, 1, 2, \ldots\),
\[
\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^n_p} \leq N(\|u\|_{L^p_p(\mathbb{R}^d)} + \|u_t\|_{L^p_p(\mathbb{R}^d)})
\]

**Proof.** See p. 66 of [13]; actually in this book, weights are not used and hence we give an outline of the proof. First of all, it is easy to check that for any \(\phi = \phi(t) \in W^1_p((0, T))\) (cf. p.32 of [13])
\[
\sup_{0 \leq t \leq T} |\phi(t)|^p \leq N \int_0^T (|\phi|^p + |\phi'(t)|^p) dt.
\]
Thus it suffices to prove
\[
\phi(t) := \|u(t, \cdot)\|_{H^n_p} \in W^1_p((0, T)), \quad |\phi'(t)| \leq \|u_t(t, \cdot)\|_{H^n_p}.
\]
One can prove \(4.16\) by repeating the proof of Exercise 2.4.8 on p.71 of [13]. It is enough to replace \(H^n_p\) there by \(H^1_{p,0}\).
Lemma 4.14. Let \( \theta \leq d, p > 1, s \in (0, r) \) and \( u \in C^\infty_{loc}(\Omega; \mathbb{R}^d) \) satisfies \( u_t + A^i(t)u_{x^i} = 0 \) for \( (t, x) \in Q_s(r) \). Then for any multi-index \( \beta = (\beta_1, \beta_2, \ldots, \beta_d) \),

\[
\max_{(t, x) \in Q_s(r)} |D^\beta u| \leq N \int_{Q_s(r)} |u|^p(x^1)^{\theta - d + p} dx dt,
\]

where \( N = N(s, r, \beta, p, \delta, K) \).

Proof. Choose the smallest integer \( n \) so that \( np > d \). Let \( v \in C^\infty_0(\mathbb{R}^d_+) \) satisfy \( v(x) = 0 \) for \( x^1 \geq 2r \).

The by Lemma 3.1 (ii) with \( \gamma = n, i = 0, \theta = d \) and \( u = M^{-n}v \),

\[
\sup_x |v(x)| \leq N(r) \sup_x |M^{d/p}M^{-n}v(x)| \leq N\|M^{-n}v\|_{L^p} \leq N(r, p, n)\|D^n v\|_{L^p_{r, d}}, \tag{4.17}
\]

where for the last inequality we use Remark 4.2.

Fix \( \kappa \in (s, r) \). Let \( \psi \) be a smooth function so that \( \psi(x) = 1 \) for \( (t, x) \in Q_s(s) \) and \( \psi = 0 \) for \( (t, x) \notin U_\kappa \). It follows from (4.17) and Lemma 4.13 that

\[
\max_{Q_s(r)} \left( |D^\beta u_x| + |D^\beta u_t| \right) \leq N \max_{(t, x) \in Q_s(r)} \left( |D^\beta \psi u|_x \right) \leq N \max_{t \in [0, r^2]} \|D^n (D^\beta \psi u)|_{L^p_{r, d}} \leq N \left( \|D^n (D^\beta \psi u)|_{L^p_{r, d}} + \|D^n (D^\beta \psi u)|_{L^p_{r, d}} \right)
\]

\[
\leq N \sum_{|\alpha| \leq n + |\beta| + 4} \int_{Q_s(r)} |D^n u|^p dx dt \leq N \int_{Q_s(r)} |u|^p(x^1)^{\theta - d + p} dx dt,
\]

where the last inequality is due to the fact that \( 1 \leq N(r)(x^1)^{\theta - d} \) for \( x^1 \leq 2r \). The lemma is proved. \( \Box \)

Remark 4.15. Actually by inspecting the proof of Lemma 4.14 it can be easily shown that if Lemma 4.11 holds for some \( \theta_0 \in (d - 1, d - 1 + p) \) then Lemma 4.14 holds for any \( \theta \in (d - 1, \theta_0) \).

5 Main estimates : Sharp function estimations

Remember that we denote

\[
\nu_\alpha(dx) = \nu_\alpha(dx^1)dx' := (x^1)^p dx^1 dx'.
\]

The following is a weighted version of Poincaré’s inequality.

Lemma 5.1. Let \( \alpha \geq 0, p \in [1, \infty), D_r(a) := (a - r, a + r) \times B_r'(0) \subset \mathbb{R}^d_+ \), and \( u \in C^\infty_{loc}(\mathbb{R}^d_+; \mathbb{R}^d) \). Then

\[
\int_{D_r(a)} \int_{D_r(a)} |u(x) - u(y)|^p \nu_\alpha(dx) \nu_\alpha(dy) \leq 2^{\alpha + 2}(2r)^p |D_r(a)| \int_{D_r(a)} |u_x(x)|^p \nu_\alpha(dx), \tag{5.1}
\]

where \( |D_r(a)| := \nu_\alpha(D_r(a)) \) and we define

\[
\int_A |f(x)|^p \nu_\alpha(dx) = \sum_{k=1}^{d_1} \int_A |f^k(x)|^p \nu_\alpha(dx)
\]

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for $\mathbb{R}^{d_1}$-valued function $f$ and $A \subset \Omega$.

**Proof.** We use the outline of the proof of Theorem 10.2.5 of [13]. Without loss of generality we may assume $d_1 = 1$. For $x, y \in D_r(a)$ we have

$$|u(x) - u(y)|^p \leq (2r)^p \int_0^1 |u_x(tx + (1 - t)y)|^p dt$$

and the left-hand side of (5.1) is less than

$$(2r)^p \int_0^1 I(t) dt = 2(2r)^p \int_{1/2}^1 I(t) dt,$$

where

$$I(t) := \int_{D_r(a)} \int_{D_r(a)} |u_x(tx + (1 - t)y)|^p \nu_\alpha(dx) \nu_\alpha(dy)$$

and $I$ satisfies $I(t) = I(1 - t)$. For each $t \in [1/2, 1]$ and $y$, substituting $w = tx + (1 - t)y$ and noticing $x^1 = (w^1 - (1 - t)y^1)/t \leq w^1/t$ since $y^1 \geq 0$, we get

$$I(t) \leq t^{-\alpha - 1} \int_{D_r(a)} \left( \int_{D_r(a)} |u_x(w)|^p \nu_\alpha(dw) \right) \nu_\alpha(dy)$$

$$\leq 2^{\alpha+1} \int_{D_r(a)} \left( \int_{D_r(a)} |u_x(x)|^p \nu_\alpha(dx) \right) \nu_\alpha(dy)$$

$$= 2^{\alpha+1} |D_r(a)| \int_{D_r(a)} |u_x(x)|^p \nu_\alpha(dx)$$

with the observation $tD_r(a) + (1 - t)y = \{tz + (1 - t)y : z \in D_r(a) \} \subset D_r(a)$. Now, (5.1) follows. ☐

**Lemma 5.2.** Let $\alpha > -1$. Recall $\nu_\alpha^1(dx^1) = (x^1)^\alpha dx^1$. For any $B^1_r(a) \subset \mathbb{R}^+$ we have a non-negative function $\zeta \in C_0^\infty(\mathbb{R}^+; \mathbb{R})$ such that

$$\text{supp}(\zeta) \in B^1_{r/2}(a), \quad \int_{B^1_{r/2}(a)} \zeta(x^1) \nu_\alpha^1(dx^1) = 1, \quad \sup \zeta \cdot |B^1_r(a)| \leq N, \quad \sup \zeta_{x^1} \cdot |B^1_r(a)| \leq \frac{N}{r}, \quad (5.2)$$

where $N = N(\alpha)$ and $|B^1_r(a)| = \nu_\alpha^1(B^1_r(a))$.

**Proof.** Choose a nonnegative smooth function $\psi = \psi(x^1) \in C_0^\infty(B^1_{1/2}(0))$ so that $\int_\mathbb{R} \psi(x^1) dx^1 = 1$. Define

$$\zeta(x^1) = \frac{(x^1)^{-\alpha}}{\psi}\psi\left(\frac{x^1 - a}{r}\right).$$

Then the first and the second of (5.2) are obvious.

**Case 1:** Let $\alpha \geq 0$. Since $r \leq a$ and $(a + r)^{\alpha + 1} - (a - r)^{\alpha + 1} \leq 2r(\alpha + 1)(2a)^\alpha$, the third follows:

$$\sup |\zeta \cdot |B^1_r(a)| \leq N \sup_{|z^1 - a| \leq r/2} \frac{(x^1)^{-\alpha}}{r} \cdot ((a + r)^{\alpha + 1} - (a - r)^{\alpha + 1})$$

$$\leq N \frac{(a/2)^{-\alpha}}{r} \cdot ((a + r)^{\alpha + 1} - (a - r)^{\alpha + 1}) \leq N.$$
Similarly, the last inequality holds by
\[
\sup_{|x^1-a| \leq r/2} |\zeta_{x^1}| \cdot |B^1_{r'}(a)| \leq N \sup_{|x^1-a| \leq r/2} \left( \frac{(x^1)^{-\alpha}}{r^2} + \frac{(x^1)^{-\alpha-1}}{r} \right) \cdot ((a + r)^{\alpha+1} - (a - r)^{\alpha+1}) \\
\leq \frac{N}{r} \left( 1 + \frac{(2a)^{\alpha+1}}{(a/2)^{\alpha+1}} \right) \leq \frac{N}{r}.
\]

**Case 2:** Let \( \alpha \in (-1, 0) \). First assume \( r \leq a/2 \). Then by mean value theorem \((a + r)^{\alpha+1} - (a - r)^{\alpha+1} \leq 2r(a + 1)(a/2)^\alpha\) and thus the right term of (5.3) is bounded by a constant \( N \). If \( r \in [a/2, a] \), then
\[
\sup_{|x^1-a| \leq r/2} \frac{(x^1)^{-\alpha}}{r} \cdot ((a + r)^{\alpha+1} - (a - r)^{\alpha+1}) \leq \frac{(2a)^{-\alpha}}{a/2} (2a)^{\alpha+1} \leq N.
\]
One can handle \( \sup_{|x^1|} |B^1_{r'}(a)| \) similarly. The lemma is proved. \( \square \)

Now we consider the system
\[
u_t + A^{ij} u_{x^j} = f_i, \quad (t, x) \in \Omega = \mathbb{R} \times \mathbb{R}^d_+; \quad f^i = (f^{1i}, \ldots, f^{d_1 i}),
\]
i.e.,
\[
u_t^k + a_{kr}^{ij} u_{x^j} = f^k_i, \quad k = 1, 2, \ldots, d_1.
\]

Recall that for \( t \in \mathbb{R}, a \in \mathbb{R}_+ \) and \( x' \in \mathbb{R}^{d-1} \)
\[
Q_r(t, a, x') := (t + r^2) \times (a - r, a + r) \times B^1_{r'}(x'), \quad Q_r(a) := Q_r(0, a, 0).
\]

By \( C^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d_1}) \) we denote the set of \( \mathbb{R}^{d_1} \)-valued functions \( \zeta \) defined on \( \Omega \) and such that \( \zeta \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d_1}) \) for any \( \zeta \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}) \).

**Lemma 5.3.** Let \( \alpha \geq 0, p \in [1, \infty), f^i, g \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d_1}) \). Assume that \( u \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}^{d_1}) \) satisfies (5.4) on \( Q_r(a) \subset \Omega \). Then
\[
\int_{Q_r(a)} |u(t, x) - u_{Q_r(a)}|^p \mu_\alpha(\rho d\rho dx) \leq N r^p \int_{Q_r(a)} (|u(x, t)|^p + |f(t, x)|^p + r^p |g(t, x)|^p) \mu_\alpha(\rho d\rho dx),
\]
where \( N = N(\theta, \alpha, p, d, d_1, K) \).

**Proof.** We follow the outline of the proof of Theorem 4.2.1 in [13]. We take the scalar function \( \zeta \) corresponding to \( B^1_{r'}(a) \) and \( \alpha \) from Lemma 5.2 and take a nonnegative function \( \phi = \phi(x') \in C^\infty_{\text{loc}}(B^1_1(0)) \) with unit integral. Denote \( \eta(x') = r^{-d+1} \phi(\frac{x'}{r}) \), \( D_r(a) := (a - r, a + r) \times B^1_r(0) \) as before, and for \( t \in (0, r^2) \) set
\[
\tilde{u}(t) := \int_{D_r(a)} \zeta(y') \eta(y') u(t, y) \nu_\alpha(dy).
\]
Then by Jensen’s inequality and the weighted version of Poincaré’s inequality (Lemma 5.2),

\[
\int_{D_r(a)} |u(t, x) - \bar{u}(t)|^p \nu_\alpha(dx) \\
= \int_{D_r(a)} \left| \int_{D_r(a)} (u(t, x) - u(t, y)) \zeta(y^1) \eta(y^1) \nu_\alpha(dy) \right|^p \nu_\alpha(dx) \\
\leq \int_{D_r(a)} \left( \int_{D_r(a)} |u(t, x) - u(t, y)|^p \zeta(y^1) \eta(y^1) \nu_\alpha(dy) \right) \nu_\alpha(dx) \\
\leq \sup \zeta \cdot \sup \eta \int_{D_r(a)} \int_{D_r(a)} |u(t, x) - u(t, y)|^p \nu_\alpha(dx) \nu_\alpha(dy) \\
\leq N r^{-d+1} \sup \zeta \cdot \nu_\alpha(D_r(a)) \int_{D_r(a)} |u_x(t, x)|^p \nu_\alpha(dx) \\
\leq N r^{-d+1} \sup \zeta \cdot \nu_\alpha^p(B^1_r(a)) \int_{D_r(a)} |u_x(t, x)|^p \nu_\alpha(dx) \\
\leq N \int_{D_r(a)} |u_x(t, x)|^p \nu_\alpha(dx). \tag{5.6}
\]

We observe that for any constant vector \( c \in \mathbb{R}^d \) the left-hand side of (5.6) is less than \( 2 \times 2^p \) times

\[
\int_{Q_r(a)} |u(t, x) - c|^p \mu_\alpha(dt dx) \leq 2^p \int_{Q_r(a)} |u(t, x) - \bar{u}(t)|^p \mu_\alpha(dt dx) + 2^p \nu_\alpha(D_r(a)) \int_0^r |\bar{u}(t) - c|^p dt.
\]

By (5.6) the first term is less than (5.5). To estimate the second term, we take

\[
c = \frac{1}{r^2} \int_0^r \bar{u}(t) dt.
\]

Then by Poincaré’s inequality without a weight in variable \( t \) we have

\[
\nu_\alpha(D_r(a)) \int_0^r |\bar{u}(t) - c|^p dt \\
\leq N \nu_\alpha(D_r(a)) (r^2)^p \int_0^r \left| \int_{D_r(a)} \zeta(x^1) \eta(x^1) u(t, x) \nu_\alpha(dx) \right|^p dt. \tag{5.7}
\]

Remember \( u_t = -A^j(t) u_{x^j} + f^i + g \). We show that (5.7) is less than (5.5). In fact, for handling the integral with \( g \), using Jensen’s inequality and taking the supremum out of the integral, we have

\[
\nu_\alpha(D_r(a)) r^{2p} \int_0^r \left( \int_{D_r(a)} \zeta(x^1) \eta(x^1) g(t, x) \nu_\alpha(dx) \right)^p dt \\
\leq \nu_\alpha(D_r(a)) r^{2p} \sup \zeta \cdot \nu_\alpha^p(D_r(a)) \int_0^r \int_{D_r(a)} |g(t, x)|^p \nu_\alpha(dx) dt \\
\leq N \nu_\alpha^p(B^1_r(a)) r^{-d+1} \ \ r^{2p} \sup \zeta \cdot \nu_\alpha^p(D_r(a)) \int_0^r \int_{D_r(a)} |g(t, x)|^p \nu_\alpha(dx) dt \\
\leq N (\theta, p, d) r^{2p} \int_{Q_r(a)} |g(t, x)|^p \mu_\alpha(dt dx),
\]

where we used \( \sup \zeta \cdot \nu_\alpha^p(B^1_r(a)) \leq N \) (Lemma 5.2).
Next, we handle the integral with \(-A^{ij}u_{x^i x^j}\). Fix \(i, j\). Firstly, assume either \(i\) or \(j\) is 1; say \(j = 1\). We use integration by parts and observe

\[
\nu_\alpha(D_r(a)) \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta(x') A^{ij}(t) u_{x^i x^j}(t, x) \nu_\alpha(dx) \right|^p dt
\]

\[
\leq \nu_\alpha(D_r(a)) r^{2p} \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta(x') A^{ij}(t) u_{x^i x^j}(t, x) \nu_\alpha(dx) \right|^p dt
\]

\[
+ \nu_\alpha(D_r(a)) r^{2p} |r|^p \int_0^2 \left| \int_{D_r(a)} \frac{1}{x} \zeta(x') \eta(x') A(t) u_{x^i x^j}(t, x) \nu_\alpha(dx) \right|^p dt
\]

\[= I_1 + I_2.\]

For \(I_2\) we use the fact \(|A^{ij}u_{x^i x^j}| \leq |A^{ij}| |u_{x^i}| \leq K|u_x|\) and \(1/x \leq 2/r\) on the support of \(\zeta\). The argument handling the case of \(\int D_r a\) is similar. For \(I_1\) we use Hölder’s inequality and get

\[
\nu_\alpha(D_r(a)) \sup_{\int D_r a} \zeta \eta A^{ij} u_{x^i x^j} \nu_\alpha(dx) \leq N(K, \theta, p, d) r^p \int_{Q_r(a)} |u_x(t, x)|^p \mu_\alpha(dt dx).
\]

Since \(\nu_\alpha(B_r^1(a)) \sup \zeta \eta \leq N/r\), it easily follows that

\[
I_1 \leq N(K, \theta, p, d) r^p \int_{Q_r(a)} |u_x(t, x)|^p \mu_\alpha(dt dx).
\]

Secondly, if \(i, j \neq 1\), by integration by parts, Hölder’s inequality and the inequality \(\sup |\eta_{x^i}| \leq N r^{-d}\),

\[
\nu_\alpha(D_r(a)) r^{2p} \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta(x') \left[ -A^{ij}(t) u_{x^i x^j}(t, x) \right] \nu_\alpha(dx) \right|^p dt
\]

\[
= \nu_\alpha(D_r(a)) r^{2p} \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta_{x^j}(x') A^{ij}(t) u_{x^i x^j}(t, x) \nu_\alpha(dx) \right|^p dt
\]

\[
\leq \nu_\alpha(D_r(a)) r^{2p} \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta_{x^j}(x') A^{ij}(t) u_{x^i x^j}(t, x) \right|^p \nu_\alpha(dt dx)
\]

\[
\leq N \nu_\alpha(D_r(a)) r^{2p} r^{-dp} \int_0^2 \sup_{\int D_r a} |\zeta| \eta_{x^j} \sup_{\int D_r a} \left| A^{ij}(t) u_{x^i x^j}(t, x) \right| \nu_\alpha(dx) dt
\]

\[
\leq N r^p \int_{Q_r(a)} |u_x|^p \mu_\alpha(dx dt).
\]

For the integral with \(f_{x^i}\), we use a similar calculation to the one of \(-A^{ij}u_{x^i x^j}\) and get for each \(i\)

\[
\nu_\alpha(D_r(a)) r^{2p} \int_0^2 \left| \int_{D_r(a)} \zeta(x') \eta(x') f_{x^i}(t, x) \nu_\alpha(dx) \right|^p dt
\]

\[
\leq N(K, \theta, p, d) r^p \int_{Q_r(a)} |f(t, x)|^p \mu_\alpha(dt dx).
\]

The lemma is proved. \(\Box\)
Lemma 5.4. Let \( \alpha \geq 0, p \in [1, \infty), 0 < r \leq a \) and \( u \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}^d) \).

(i) There is a constant \( N = N(K, \theta, \alpha, p, d, d_1) \) such that for any \( i = 1, \ldots, d \) we have
\[
\int_{Q_r(a)} |u_x(t, x) - (u_x)_{Q_r(a)}|^p \mu_\alpha(dtdx) \leq N r^p \int_{Q_r(a)} |u_{xx}(t, x)|^p + |u_t(t, x)|^p \mu_\alpha(dtdx) \tag{5.8}
\]

(ii) Denote \( \kappa_0 = \kappa_0(r, a) := (\nu_\alpha^1(B^1_r(a))^{-1} \cdot \int_{u^{-r}}^{u^{+r}} x^1 v_\alpha^1(dx^1) \). Then
\[
\int_{Q_r(a)} |u(t, x) - u_{Q_r(a)} + \kappa_0(u_x)_{Q_r(a)} - \sum_i x^i (u_x)_{Q_r(a)}|^p \mu_\alpha(dtdx) \leq N r^p \int_{Q_r(a)} (|u_x(t, x)| - (u_x)_{Q_r(a)}|^p + r^p |u_t(t, x)|^p + r^p |u_{xx}(t, x)|^p) \mu_\alpha(dtdx) \leq N r^{2p} \int_{Q_r(a)} (|u_x(t, x)|^p + |u_t(t, x)|^p) \mu_\alpha(dtdx) \tag{5.9}
\]

Proof. (i) For (5.8) we use the fact that for \( \Theta = \text{Theorem 5.5} \).

Let \( \Lambda = \text{Lemma 5.4} \).

Now it is enough to use Lemma 5.3 and (5.8). The lemma is proved.

From this point on we fix \( \alpha := \theta - d + p \) (note \( \theta > 0 \)) and denote \( v := \nu \), \( \nu^1 := \nu^1_\alpha \), \( \mu(dxdt) = \nu(dx)dt = (x^1)^{\theta-d+p}dxdt \).

Theorem 5.5. Let \( \theta \in (d-1, d) \), \( 0 < r \leq a \) and \( \lambda r/a \geq 2 \).

(i) Assume that \( u \in C^\infty_{\text{loc}}(\Omega; \mathbb{R}^d) \) satisfies \( u_t + A^{ij}(t)u_{x^i x^j} = 0 \) in \( Q_{\lambda r}(t_0, a, x_0') \cap \Omega \). Then there is a constant \( N = N(K, \delta, \theta, p, d, d_1) \) such that

\[
\int_{Q_{r}(t_0, a, x_0')} |u_{xx}(t, x) - (u_{xx})_{Q_r(t_0, a, x_0')}|^p \mu(dtdx) \leq \frac{N}{(1 + \lambda r/a)^p} \int_{Q_{\lambda r}(t_0, a, x_0') \cap \Omega} |u_{xx}(t, x)|^p \mu(dtdx). \tag{5.10}
\]

(ii) If \( u \in C^\infty_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \), \( A^{ij} \) is independent of \( t \) and \( A^{ij}u_{x^i x^j} = 0 \) in \( B_{\lambda r}(a, x_0' \cap \mathbb{R}^d) \), then

\[
\int_{B_{r}(a, x_0')} |u_{xx}(x) - (u_{xx})_{B_r(a, x_0')}|^p \nu(dx) \leq \frac{N}{(1 + \lambda r/a)^p} \int_{B_{\lambda r}(a, x_0' \cap \mathbb{R}^d)} |u_{xx}(x)|^p \nu(dx). \tag{5.11}
\]

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Thus, $Q_β(β) ⊂ Q_{λr}(1) ∩ Ω$, $Q_{r/β}(β^{-1}) ⊂ Q_{2/3}(2/3)$.

Denote $w(t, x) = u(β^2t, βx)$, then obviously

$$w_t + A^{ij}(β^2t)w_{x^ix^j} = 0, \quad \text{for} \quad (t, x) ∈ Q_1(1)$$

and

$$\int_{Q_{r}(1)} |u_{xx}(t, x) − (u_{xx})Q_r(1)|^p(x_1)^{θ−d+p}dxdt ≤ N(d)sup_{Q_r(1)}(|u_{xx}|^p + |u_{xxt}|^p)$$

$$≤ N(d)β^{-3p}sup_{Q_{r/β}(β^{-1})}(|w_{xx}|^p + |w_{xxt}|^p)$$

$$≤ N(d)β^{-3p}sup_{Q_{2/3}(2/3)}(|w_{xx}|^p + |w_{xxt}|^p).$$

Applying Lemma 5.14 to $v(t, x) = w(t, x) − w_{Q_1(1)} + κ_0(w_{x^i})Q_{1(1)} − \sum_{i=1}^d x^i(w_{x^i})Q_{1(1)}$, and then using Lemma 5.4

$$β^{-3p}sup_{Q_{2/3}(2/3)}(|w_{xx}|^p + |w_{xxt}|^p) ≤ Nβ^{-3p}\int_{Q_1(1)} |v|^p(x_1)^{θ−d+p}dxdt$$

$$≤ Nβ^{-3p}\int_{Q_1(1)} |w_{xx}|^p(x_1)^{θ−d+p}dxdt$$

$$= Nβ^{-2p−2−θ}\int_{Q_1(1)} |w_{xx}|^p(x_1)^{θ−d+p}dxdt.$$

This leads to (6.10) since $|Q_{r/α}(1) ∩ Ω| ∼ β^{p+θ+2}$.

**Step 2.** Let $α ≠ 1$. Define $v(t, x) := u(α^2t, αx)$. Then $v_t + A^{ij}(α^2t)v_{x^ix^j} = 0$ in $Q_{r/α}(1) ∩ Ω$. As easy to check,

$|Q_{r/α}(1)| = α^{−θ−p−2}|Q_r(α)|$, $(v_{xx})_{Q_{r/α}(1)} = α^2(u_{xx})_{Q_r(α)}$, $|Q_{r/α}(1) ∩ Ω| = α^{−θ−p−2}|Q_r(α) ∩ Ω|$, and consequently

$$\int_{Q_{r/α}(1)} |v_{xx}(t, x) − (v_{xx})_{Q_{r/α}(1)}|^p(x_1)^{θ−d+p}dxdt = α^{2p}\int_{Q_r(α)} |u_{xx}(t, x) − (u_{xx})_{Q_r(α)}|^p(x_1)^{θ−d+p}dxdt,$$

$$\int_{Q_{r/α}(1) ∩ Ω} |v_{xx}(t, x)|^p(x_1)^{θ−d+p}dxdt = α^{2p}\int_{Q_r(α) ∩ Ω} |u_{xx}(t, x)|^p(x_1)^{θ−d+p}dxdt.$$
It follows
\[ \int_{Q_r(a)} |u_{xx}(t, x) - (u_{xx})_{Q_r(a)}|^{\theta d p} \, dx \, dt = a^{-2p} \int_{Q_{r/a}(1) \cap \Omega} |v_{xx}(t, x)|^{\theta d p} \, dx \, dt \]
\[ \leq a^{-2p} \cdot \frac{N}{(1 + \lambda r/a)^p} \int_{Q_{r/a}(1) \cap \Omega} |v_{xx}(t, x)|^{\theta d p} \, dx \, dt \]
\[ = \frac{N}{(1 + \lambda r/a)^p} \int_{Q_{r/a}(1) \cap \Omega} |u_{xx}(t, x)|^{\theta d p} \, dx \, dt. \]

The theorem is proved.

Remark 5.6. Note that Theorem 5.5 is based on Lemma 1.14. It follows from Remark 4.12 and Remark 5.10 that if \( p \geq 2 \) then Theorem 5.5 holds for any \( \theta \in (d-1, d+1) \) (not only for \( \theta \in (d-1, d) \)). Obviously we cannot use this result yet since Remark 4.12 is valid only after we prove Theorem 5.10.

Lemma 5.7. Assume \( \theta \in (d-1, d] \) if \( p \in (2, \infty) \) and \( \theta \in (d- p + 1, d] \) if \( p \in (1, 2] \). Denote \( q := \theta - d + p \) which is in \((1, p]\).

(i) Let \( u \in C_0^\infty(\mathbb{R} \times \mathbb{R}_d^+ \) and \( f := u + A^{ij}(t)u_{x_i x_j} \). Suppose that \( A^{ij}(t) \) is infinitely differentiable and has bounded derivatives. Then for any \( \varepsilon > 0 \), \( Q_r(t_0, a, x'_0) \subset \Omega \) and \((t, x) \in Q_r(t_0, a, x'_0)\)
\[ \int_{Q_r(t_0, a, x'_0)} |u_{xx} - (u_{xx})_{Q_r(t_0, a, x'_0)}|^q \mu(\,dy, ds) \leq \varepsilon M(|u_{xx}|^q)(t, x) + NM(|f|^q)(t, x), \tag{5.12} \]
where \( N = N(\varepsilon, \theta, q, d, d_1, \delta, K) \).

(ii) Furthermore, if \( u \in C_0^\infty(\mathbb{R}_d^+ \) and \( A^{ij} \) is independent of \( t \), then for any \( \varepsilon > 0 \), \( B_r(a, x'_0) \subset \mathbb{R}_d^+ \) and \( x \in B_r(a, x'_0)\)
\[ \int_{B_r(a, x'_0)} |u_{xx} - (u_{xx})_{B_r(a, x'_0)}|^q \nu(\,dy) \leq \varepsilon M(|u_{xx}|^q)(x) + NM(|A^{ij}u_{x_i x_j}|^q)(x), \tag{5.13} \]
where \( N = N(\varepsilon, \theta, q, d, d_1, \delta, K) \).

Proof. (i) Without loss of generality we may take \( t_0 = 0 \) and \( x'_0 = 0 \); \( Q_r(t_0, a, x'_0) = Q_r(a) \). In fact, for other cases it is enough to consider the function \( v(t, x) := u(t_0 + t, x_1, x'_0 + x') \) in place of \( u(t, x_1, x') \).

**Step 1.** We prove that there exists \( \kappa = \kappa(\varepsilon) \in (0, 1) \) so that (5.12) holds if \( (r/a) \leq \kappa \).

Let \( m \) denote the Lebesgue measure on \( \mathbb{R}_d^+ \). Assume \( \lambda \geq 4 \) and \( \lambda r \leq a/4 \). Then \( (3a/4) \leq x^1 \leq (5a/4) \) if \( x^1 \in B_{\lambda r}(a) \), and therefore

\[ \int_{Q_r(a)} \frac{dt \, dx}{m(Q_r(a))} \leq \mu(\,dt, dx) = \frac{(3/5)^{p+\theta-d}}{m(Q_r(a))} \int_{Q_r(a)} \frac{dt \, dx}{m(Q_r(a))} \] on \( Q_r(a) \),
\[ \int_{Q_{r/4}(a)} \frac{dt \, dx}{m(Q_{r/4}(a))} \leq \mu(\,dt, dx) = \frac{(3/5)^{p+\theta-d}}{m(Q_{r/4}(a))} \int_{Q_{r/4}(a)} \frac{dt \, dx}{m(Q_{r/4}(a))} \] on \( Q_{r/4}(a) \).
Denote \( c_0 := (5/3)^{p+\theta-d} \). By Theorem 4.13,
\[
\int_{Q_r(a)} |u_{xx} - (u_{xx})_{Q_r(a)}|^q \mu(d\tau d\xi) \\
\leq \int_{Q_r(a)} \int_{Q_r(a)} |u_{xx}(s, y) - u_{xx}(\tau, \xi)|^q \mu(d\tau d\xi) \frac{\mu(d\tau d\xi)}{|Q_r(a)|} \\
\leq \frac{c_0^2}{\lambda} \int_{Q_{\lambda r}(a)} |u_{xx}(s, y) - u_{xx}(\tau, \xi)|^q d\tau d\xi \\
\leq N c_0^2 \lambda^{d+2} \int_{Q_{\lambda r}(a)} |f|^q \frac{dyds}{m(Q_{\lambda r}(a))} + N c_0^2 \lambda^{-q} \int_{Q_{\lambda r}(a)} |u_{xx}|^q \frac{dyds}{m(Q_{\lambda r}(a))} \\
\leq N c_0^3 \lambda^{d+2} \int_{Q_{\lambda r}(a)} |f|^q \mu(dyds) + N c_0^3 \lambda^{-q} \int_{Q_{\lambda r}(a)} |u_{xx}|^q \mu(dyds) \\
\leq N \lambda^{d+2} \mu([|f|^q](t, x) + \lambda^{-q} \mu([|u_{xx}|^q](t, x),
\]
where \( N \) depends only on \( d, d_1, p, \theta, \delta, K \). Note that the above inequality holds as long as \( r\lambda/a \leq 1/4 \).

Now we fix \( \lambda \) so that \( N \lambda^{-q} = \varepsilon/2 \), i.e. \( \lambda = (2N/\varepsilon)^{1/q} \) and define \( \kappa = 1/(4\lambda) = 1/4 \cdot (2N/\varepsilon)^{-1/q} \). Then whenever \( r/a \leq \kappa \) we have \( (r/a)\lambda \leq 1/4 \) and thus (5.12) follows.

**Step 2.** For given \( \varepsilon \), take \( \kappa = \varepsilon \) from Step 1. Assume \( r/a \geq \kappa \). Choose \( \lambda \), which will be specified later, so that \( r\lambda > 4a \); this \( \lambda \) is different from the one in Step 1. Take a \( \zeta \in C^\infty_0(\mathbb{R}^{d+1}) \) so that \( \zeta(t, x) = 1 \) for \((t, x) \in Q_{\lambda r/2}(a) \cap \Omega \) and \( \zeta(t, x) = 0 \) if \((t, x) \notin (-\lambda^2 r^2, \lambda^2 r^2) \times (-a, a + \lambda r) \times B_{\lambda r} \).

Denote
\[
g = f \zeta, \quad h = f(1 - \zeta).
\]

Take a large \( T \) so that \( u(t, x) = 0 \) if \( t \geq T \). By Lemma 4.10, we can define \( v \) as the solution of
\[
v_t + A^{ij} v_{x^i x^j} = h, \quad t \in (S, T), \quad v(T, \cdot) = 0 \tag{5.14}
\]
so that \( v \in \mathcal{S}^n_{p,d}(S, T) \) for any \( n \) and \( S > -\infty \). Also let \( \tilde{v} \in \mathcal{S}^n_{p,d}(S, T + 1) \) be the solution of
\[
\tilde{v}_t + A^{ij} \tilde{v}_{x^i x^j} = h, \quad t \in (S, T + 1), \quad \tilde{v}(T + 1, \cdot) = 0.
\]

Then by considering the equation for \( \tilde{v} \) on \((T, T + 1)\), since \( h(t) = 0 \) for \( t \geq T \), we conclude \( \tilde{v}(t) = 0 \) for \( t \in [T, T + 1] \). Thus \( \tilde{v} \) also satisfies (5.14) and \( v = \tilde{v} \). It follows from (5.6) that \( v \) is infinitely differentiable in \( x \) (and hence in \( t \)) in \( \Omega \). By applying Theorem 5.5 with \( \tilde{p} = q, \tilde{\theta} = d \) and \( \lambda/2 \) in places of \( p, \theta \) and \( \lambda \) respectively,
\[
\int_{Q_{\lambda r}(a)} |v_{xx}(t, x) - (v_{xx})_{Q_{\lambda r}(a)}|^q \frac{dyds}{m(Q_{\lambda r}(a))} \leq N \frac{1}{(1 + \lambda r/2a)^q} \int_{Q_{\lambda r/2}(a) \cap \Omega} |v_{xx}(t, x)|^q \frac{dyds}{m(Q_{\lambda r}(a))} \\
\leq N \frac{1}{(1 + \lambda r/a)^q} \int_{Q_{\lambda r}(a) \cap \Omega} |v_{xx}(t, x)|^q \frac{dyds}{m(Q_{\lambda r}(a))}, \tag{5.15}
\]
where \( \tilde{\mu}(dyds) := (y^1)^{\tilde{d} - d + \tilde{\theta}} dyds = (y^1)^d dyds = \mu(dyds) \). On the other hand, \( w := u - v \) satisfies \( w(T, \cdot) = 0 \) and
\[
w_t + A^{ij} w_{x^i x^j} = g, \quad t \in (0, T).
\]
By Lemma 4.10,
\[
\int_{Q_\lambda(a)} |w_{yy}|^q(y^3)^q dyds \leq \int_{Q_\lambda(a) \cap \Omega} |w_{yy}|^q(y^3)^q dyds \leq N \int_{Q_\lambda(a) \cap \Omega} |f|^q(y^3)^q dyds,
\]
\[
\int_{Q_\lambda(a)} |w_{yy}|^q \mu(dyds) \leq N \frac{\lambda^d(1 + \lambda r/a)^{p+\theta-d+1}}{(1 + r/a)^{p+\theta-d+1} - (1 - r/a)^{p+\theta-d+1}} \int_{Q_\lambda(a) \cap \Omega} |f|^q \mu(dyds)
\]
\[
\leq N(\kappa) \lambda^d(1 + \lambda r/a)^{p+\theta-d+1} \int_{Q_\lambda(a) \cap \Omega} |f|^q \mu(dyds), \tag{5.16}
\]
where for the second inequality we use \((1 + r/a)^{p+\theta-d+1} - (1 - r/a)^{p+\theta-d+1} \geq (1 + \kappa)^{p+\theta-d+1} - 1\).

Observing that \(u = v + w\),
\[
I : = \int_{Q_\lambda(a)} |u_{yy}(t,x) - (w_{yy})_{Q_\lambda(a)}|^q \mu(dyds)
\]
\[
\leq N(q) \int_{Q_\lambda(a)} |w_{yy}(t,x) - (w_{yy})_{Q_\lambda(a)}|^q \mu(dyds) + N(q) \int_{Q_\lambda(a)} |v_{yy}(t,x) - (v_{yy})_{Q_\lambda(a)}|^q \mu(dyds)
\]
\[
\leq N(q) \int_{Q_\lambda(a)} |w_{yy}(t,x)|^q \mu(dyds) + N(q) \int_{Q_\lambda(a)} |v_{yy}(t,x) - (v_{yy})_{Q_\lambda(a)}|^q \mu(dyds)
\]
and thus by (5.13) and (5.16),
\[
I \leq N \lambda^d(1 + \lambda r/a)^{p+\theta-d+1} \int_{Q_\lambda(a) \cap \Omega} |f|^q \mu(dyds)
\]
\[
+ N \frac{1}{(1 + \lambda r/a)^q} \int_{Q_\lambda(a) \cap \Omega} |v_{yy}(t,x)|^q \mu(dyds)
\]
\[
\leq N \lambda^d(1 + \lambda r/a)^{p+\theta-d+1} \int_{(0,\lambda^2+2) \times (0,a+\lambda r)} |f|^q \mu(dyds)
\]
\[
+ N \frac{1}{(1 + \lambda r/a)^q} \int_{Q_\lambda(a) \cap \Omega} (|u_{yy}(t,x)|^q + |w_{yy}(t,x)|^q) \mu(dyds)
\]
\[
\leq N \lambda^d(1 + \lambda r/a)^{p+\theta-d+1} \int_{Q_\lambda(a) \cap \Omega} |f|^q \mu(dyds)
\]
\[
+ N \frac{1}{(1 + \lambda r/a)^q} \int_{Q_\lambda(a) \cap \Omega} |u_{yy}(t,x)|^q \mu(dyds).
\]

Now to prove the first assertion it is enough to choose \(\lambda\) so large that \(N \frac{1}{(1 + \lambda r/a)^q} \leq \varepsilon\). Also note that since \(r/a \geq \kappa\), we have
\[
N \lambda^d(1 + \lambda r/a)^{p+\theta-d+1} \leq N(\lambda, \kappa).
\]

(ii) The second assertion is proved similarly based on Corollary 4.6 and 5.11 in place Theorem 4.6 and 5.10. The lemma is proved.

\[
\]
Theorem 6.1. Let $p \in (1, \infty)$. Assume $\theta \in (d-1, d+1)$ if $p \in (2, \infty)$, and $\theta \in (d+1-p, d+1-p)$ if $p \in (1, 2]$. Then for any $f \in \mathbb{L}_{p, \theta}(-\infty, \infty)$ the system
\[ u_t + A^{ij}(t)u_{x^i x^j} = f \]
has a unique solution $u$ in $M^{p, \theta}_{p, \theta}(-\infty, \infty)$ and for this solution we have
\[ \|Mu_t\|_{L_{p, \theta}(-\infty, \infty)} + \|M^{-1}u\|_{L_{p, \theta}(-\infty, \infty)} \leq N\|f\|_{L_{p, \theta}(-\infty, \infty)}. \] (6.1)

Proof. If $A^{ij}u_{x^i x^j} = \Delta u = (\Delta u^1, \ldots, \Delta u^d)$, then the theory of single equations is applied and the theorem is true for any $\theta \in (d-1, d-1+p)$; see Theorem 5.6 in [10]. Actually the mentioned theorem is proved for parabolic equations defined on $(0, T) \times \mathbb{R}^d_+$, but one can easily check that the proofs in [10] work for equations defined on $\mathbb{R} \times \mathbb{R}^d_+$.

For $\lambda \in [0, 1]$ and $d_1 \times d_1$ identity matrix $I$ we define
\[ A^{ij}_\lambda = (a^{ij}_{kr, \lambda}) := (1-\lambda)A^{ij} + \delta^{ij}\lambda \delta I. \]

Then for each $\lambda \in [0, 1]$ the coefficient matrices $\{A^{ij}_\lambda : i, j = 1, \ldots, d\}$ satisfy Assumption 3.8 with the same $\delta, K$. Thus due to the method of continuity, we only need to prove that a priori estimate (6.1) holds given that a solution $u$ already exists. Furthermore, since $C^\infty_0(\mathbb{R} \times \mathbb{R}^d_+)$ is dense in $M^{p, \theta}_{p, \theta}(-\infty, \infty)$, we may assume that $u \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d_+)$. By Remark 4.8 we only need to prove the following:
\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^d_+} \left|u_{xx}(t, x)\right|^p \mu(dtdx) \leq N \int_{-\infty}^{\infty} \int_{\mathbb{R}^d_+} \left|f(t, x)\right|^p \mu(dtdx). \] (6.2)

To prove this we certainly may assume that $A^{ij}$ are infinitely differentiable and have bounded derivatives (remember that the constant $N$ in (5.12) do not depend on the regularity of $A^{ij}$).

Case 1. Assume that either (i) $p \in (2, \infty)$ and $\theta \in (d-1, d]$ or (ii) $p \in (1, 2]$ and $\theta \in (d+1-p, d+1]$. Define $q := \theta - d + p$. Recall that the range of $q \in (1, p]$. By Lemma 5.7 if $u \in C^\infty_0(\mathbb{R} \times \mathbb{R}^d_+)$, then for any $\varepsilon > 0$
\[ (u_{xx})^\varepsilon(t, x) \leq \varepsilon M^{1/q}(u_{xx})^\varepsilon(t, x) + N(\varepsilon)\|u_t + A^{ij}u_{x^i x^j}\|_p(t, x). \]

By Theorem 2.10 (Fefferman-Stein) and Theorem 2.12 (Hardy-Littlewood),
\[ \|Mu_{xx}\|_{L_{p, \theta}(-\infty, \infty)} = \|u_{xx}\|_{L_p(\Omega, \mu)} \leq N\|u_{xx}\|^2_{L_p(\Omega, \mu)} \leq N\|u_t + A^{ij}u_{x^i x^j}\|^2_{L_p(\Omega, \mu)} \leq N\varepsilon\|u_t + A^{ij}u_{x^i x^j}\|^2_{L_p(\Omega, \mu)} \leq N\varepsilon\|u_t + A^{ij}u_{x^i x^j}\|^2_{L_{p/q}(\Omega, \mu)} + N \cdot N(\varepsilon)\|u_t + A^{ij}u_{x^i x^j}\|_{L_{p/q}^q(\Omega, \mu)} \]
\[ = N\varepsilon\|u_t + A^{ij}u_{x^i x^j}\|_{L_{p/q}^q(\Omega, \mu)} + N \cdot N(\varepsilon)\|u_t + A^{ij}u_{x^i x^j}\|_{L_{p/q}(\Omega, \mu)} \]
\[ = N\varepsilon\|u_{xx}\|_{L_{p/q}^q(\Omega, \mu)} + N \cdot N(\varepsilon)\|u_t + A^{ij}u_{x^i x^j}\|_{L_{p/q}(\Omega, \mu)} \]
This obviously yields (6.2).
Case 2. Assume that either (i) \( p \in (2, \infty) \) and \( \theta \in [d, d+1) \) or (ii) \( p \in (1, 2] \) and \( \theta \in [d, d+p-1) \). By Remark 3.8 we only need to prove the following:

\[
\int_{-\infty}^{\infty} \int_{R^d_+} |M^{-1}u(t, x)|^p(x^1)^{\theta-d} \, dx \, dt \leq N \int_{-\infty}^{\infty} \int_{R^d_+} |Mf(t, x)|^p(x^1)^{\theta-d} \, dx \, dt.
\]

(6.3)

To prove this, we use a duality (Lemma 3.8). Denote \( p' = p/(p-1) \) and choose \( \theta \) so that \( \theta/p + \theta/p' = d \). Then \( \bar{\theta} \in (d-1, d) \) if \( p' \in (2, \infty) \) and \( \bar{\theta} \in (d-p'+1, d) \) if \( p' \in (1, 2] \).

Changing the variable \( t \to -t \) shows that the result of case 1 is applicable to the operator \( u_t - A^{ij}u_{x_i x_j} \) in place of \( u_t + A^{ij}u_{x_i x_j} \). Therefore for any \( v \in M_H H_{p', \bar{\theta}}^{-1}(\mathbb{R}^d) \), by integration by parts,

\[
\int_{R^d_+} M^{-1}uM(v_t - A^{ij}v_{x_i x_j}) \, dx \, dt = \int_{R^d_+} u(v_t - A^{ij}v_{x_i x_j}) \, dx \, dt
\]

\[
= \int_{R^d_+} M(-u_t - A^{ij}u_{x_i x_j})M^{-1}v \, dx \, dt
\]

\[
\leq \|M(u_t + A^{ij}u_{x_i x_j})\|_{L_{p, \bar{\theta}}(\mathbb{R}^d)} \|M^{-1}v\|_{L_{p', \bar{\theta}}(\mathbb{R}^d)}
\]

\[
\leq N \|M(u_t + A^{ij}u_{x_i x_j})\|_{L_{p, \bar{\theta}}(\mathbb{R}^d)} \|M(v_t - A^{ij}v_{x_i x_j})\|_{L_{p', \bar{\theta}}(\mathbb{R}^d)}.
\]

Since, by Case 1, \( \{v_t - A^{ij}v_{x_i x_j} : v \in M_H H_{p', \bar{\theta}}^{-1}(\mathbb{R}^d)\} \) is dense in \( M_H H_{p', \bar{\theta}}^{-1}(\mathbb{R}^d) \), it follows that

\[
\|M^{-1}u\|_{L_{p, \bar{\theta}}(\mathbb{R}^d)} \leq N \|M(u_t + A^{ij}u_{x_i x_j})\|_{L_{p, \bar{\theta}}(\mathbb{R}^d)}.
\]

The theorem is proved. \( \square \)

**Proof of Theorem 3.10** As usual, we assume \( u_0 = 0 \). For details see the proof of Theorem 5.1 in [9].

**Case 1.** Let \( T = \infty \). As before we only prove the a priori estimate. Suppose \( u \in \mathcal{S}_{p, \theta}^{\gamma+2}(\infty) \) satisfies

\[
u_t = A^{ij}u_{x_i x_j} + f, \quad t \in (0, \infty); \quad u(0, \cdot) = 0.
\]

(6.4)

Define \( v(t, x) = u(t, x)I_{t>0} + f I_{t>0} \), then \( v \in M^{-1}H_{p, \theta}^{-1}(\mathbb{R}^d) \) and \( v \) satisfies (see Definition 3.9)

\[
v_t = A^{ij}v_{x_i x_j} + \bar{f}, \quad (t, x) \in \mathbb{R}^{d+1}_+.
\]

By Theorem 0.31

\[
\|M u_{xx}\|_{L_{p, \theta}(\infty)} \leq N \|M f\|_{L_{p, \theta}(\infty)}.
\]

By Remark 3.8 this certainly proves (6.1).

**Case 2.** Let \( T < \infty \). The existence of the solution in \( \mathcal{S}_{p, \theta}^{\gamma+2}(T) \) is obvious. Now suppose that \( u \in \mathcal{S}_{p, \theta}^{\gamma+2}(T) \) is a solution of (6.4). By the result of Case 1, the system

\[
v_t = \Delta v + (A^{ij}u_{x_i x_j} + f - \Delta u)I_{t \leq T}, \quad t > 0; \quad v(0, \cdot) = 0
\]

(6.5)

has a unique solution \( v \in \mathcal{S}_{p, \theta}^{\gamma+2}(0, \infty) \). Then \( v - u \) satisfies

\[
(v - u)_t = \Delta(v - u), \quad t \in (0, T); \quad (v - u)(0, \cdot) = 0.
\]
If follows from the theory of single equations (see, for instance, Theorem 5.6 in [10]), \( u = v \) for \( t \in [0, T] \). For \( t \geq 0 \), define

\[
A_{ij}^T = (a_{ij}^T)_{kr}, \quad a_{ij}^{T, kr} = a_{ij}^{kr} I_{t \leq T} + \delta^{ij} \delta^{kr} I_{t > T}.
\]

Then (6.5) and the fact \( u = v \) for \( t \in [0, T] \) show that \( v \) satisfies (replace \( u \) by \( v \) for \( t \leq T \) in (6.5))

\[
v_t = A_{ij}^T v_{x^i x^j} + f I_{t < T}, \quad t > 0; \quad v(0, \cdot) = 0.
\]

(6.6)

By Case 1, \( v \in \mathcal{H}_{p, \theta}^{2, \gamma} (\infty) \) is the unique solution of (6.6), and \( u = v \) on \([0, T]\) whenever \( u \) is a solution of (6.4) on \([0, T]\). This obviously yields the uniqueness. The theorem is proved. \( \square \)

**Proof of Theorem 3.13** The proof is very similar to that of the proof of Theorem 3.10 and is based on (5.13). We leave the details to the readers as an exercise.

**References**

[1] M. Bramanti and M.C. Cerutti, \( W^{1, 2}_p \) solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, Comm. Partial Differential Equations, 18 (1993), no. 9-10, 1735-1763.

[2] Sun-Sig Bum, Parabolic equations with BMO coefficients in Lipschitz domains, J. Differential Equations, 209 (2005), no. 2, 229-265.

[3] F. Chiarenza, M. Frasca and P. Longo, \( W^{2, p} \)-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc., 336 (1993), no. 2, 841-853.

[4] D. Kim, Elliptic equations with nonzero boundary conditions in weighted Sobolev spaces, J. Math. Anal. Appl. 337 (2008), 1465-1479.

[5] P. Grisvard, Elliptic Problems in nonsmooth domains, Monographs and Studies in Mathematics 24 (1985), Pittman, Boston-London-Melbourn.

[6] R. Haller-Dintelmann, H. Heck and M. Hieber, \( L^p - L^q \)-estimates for parabolic systems in nondivergence form with VMO coefficients, J. London Math. Soc. (2) 74 (2006), no. 3, 717-736.

[7] Doyoon Kim and N.V. Krylov, Parabolic equations with measurable coefficients, Potential Anal. 26 (2007), no. 4, 345-361.

[8] N.V. Krylov, Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces, Journal of Functional Analysis 183 (2001), 1-41.

[9] N.V. Krylov, An analytic approach to SPDEs, pp. 185-242 in Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, 64 (1999), AMS, Providence, RI.

[10] N.V. Krylov, Weighted Sobolev spaces and Laplace equations and the heat equations in a half space, Comm. in PDEs, 23 (1999), no. 9-10, 1611-1653.

[11] N.V. Krylov, Some properties of weighted Sobolev spaces in \( \mathbb{R}^d \), Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28 (1999), no. 4, 675-693.

[12] N.V. Krylov, A \( W^{2} \)-theory of the Dirichlet problem for SPDEs in general smooth domains, Probab. Theory Relat. Fields 98 (1994), 389-421.

[13] N.V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, American Mathematical Society, Providence, RI, 2008.
[14] N.V. Krylov and S.V. Lototsky, *A Sobolev space theory of SPDEs with constant coefficients in a half space*, SIAM J. on Math. Anal., 31 (1999), no. 1, 19-33.

[15] Kijung Lee, *On a Deterministic Linear Partial Differential System*, Journal of Mathematical Analysis and Applications, 353, (2009), no. 1, 24-42.

[16] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogénes et applications* 1, Dunod, Paris, 1968.

[17] S.V. Lototsky, *Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations*, Methods and Applications of Analysis, 1 (2000), no. 1, 195-204.

[18] V.A. Solonnikov, *Solvability of the classical initial-boundary-value problems for the heat-conduction equations in a dihedral angle*, Zapiski Nauchnykh Seminarov LOMI 138 (1984), 146-180 in Russian; English translation in Journal of Soviet Math. 32 (1986), no. 5, 526-546.

[19] H. Triebel, *Theory of function spaces*, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983.