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Stability Estimates for an Inverse Hyperbolic Initial Boundary Value Problem with Unknown Boundaries

Sergio Vessella*

Abstract

In this paper we consider an inverse initial boundary value problem for the anisotropic wave equation. We prove that the unknown portion of the boundary of a domain in $\mathbb{R}^n$ depends logarithmically from the overdetermined boundary data.

Mathematics Subject Classification (2010) Primary 35R30, 35A25 Secondary 35R25, 35L

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1 Introduction

The reconstruction of obstacles from scattering waves has been widely investigated [Col-Kr]. This approach requires enough information on the scattered amplitude and generally infinitely many boundary measurements. In many practical situations these data are not available, for example in physical situations where only transient waves are detectable and one measurement in a finite time observation is obtainable.

A typical example is described by the following problem: consider the wave equation in a domain $\Omega$ in $\mathbb{R}^n$ ($n \geq 2$) whose boundary, which we

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assume sufficiently smooth, consists of two non overlapping portions $\Gamma^{(a)}$ (accessible portion) and $\Gamma^{(i)}$ (inaccessible portion) where $\Gamma^{(i)}$ is an unknown obstacle.

In the case where $\Gamma^{(i)}$ is a soft obstacle the mathematical problem is represented by the following initial boundary value problem (the direct problem). Given a nontrivial function $\psi$ on $\partial \Omega \times [0, T]$, $0 < T < +\infty$, such that

$$\psi = 0 \quad \text{on } \Gamma^{(i)} \times (0, T),$$

let $u$ be the (weak) solution to the following problem

(1.1) \[
\begin{cases}
\partial_t^2 u - \text{div} \left( A(x) \nabla_x u \right) = 0, & \text{in } \Omega \times [0, T], \\
u|_{\partial \Omega \times [0, T]} = \psi, & \text{on } \partial \Omega \times [0, T], \\
u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega,
\end{cases}
\]

(div := $\sum_{j=1}^n \partial_{x_j}$) where $A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ denotes a known symmetric matrix which satisfies a hypothesis of uniform ellipticity and some smoothness conditions that we will specify in the sequel of the paper. The inverse problem, we are interested in, is to determine $\Gamma^{(i)}$ from the knowledge of

(1.2) $$A(x) \nabla_x u(x, t) \cdot \nu, \quad \text{on } \Sigma \times (0, T),$$

where $\Sigma \subset \Gamma^{(a)}$ and $\nu$ denotes the exterior unit normal to $\Omega$.

The uniqueness for the above inverse problem has been proved in [Is2], however, contrary to the analogue problems for elliptic equations or systems [Al-B-Ro-Ve], [Be-Ve], [Che-H-Y], [M-R1], [M-R2], [M-R-V2] and parabolic equations [C-Ro-Ve1], [C-Ro-Ve2], [De-R-Ve], [Ve1], [Ve2], the stability issue in the hyperbolic context is much less studied. This is due to the lack of a complete and well understood analysis of the unique continuation property for the wave equation in (1.1) (more generally, for equations with partial analytic coefficients), we refer to [Ro-Zu], [Ta], and the corresponding quantitative estimates of unique continuation. In this paper we are interested in the stability issue for the above inverse problem. More precisely we are interested in the continuous dependence of $\Gamma^{(i)}$ from the Cauchy data $u$, $A \nabla_x u \cdot \nu$ on $\Sigma \times (0, T)$. We prove a logarithmic stability estimate under some a priori information on the domain $\Omega$, on $\Gamma^{(i)}$, on $\psi$ and whenever $T$ is large enough, but finite and independent by the errors on the Cauchy data. In view of John’s counterexample [Jo] it is reasonable to expect that the logarithmic rate of stability is the optimal one. We are currently working on this topic.

We now describe briefly the main tools used in order to prove our main stability result.
(a) Stability Estimates for Cauchy Problem and Smallness Propagation Estimates. In order to determine the unknown portion of boundary $\Gamma^{(i)}$ it seems to be necessary to determine the values of $u$ from Cauchy data on $\Sigma \times (0, T)$ up to $\Gamma^{(i)} \times (0, T')$ for suitable $T' < T$. More precisely, let $\Omega_1$ and $\Omega_2$ be two domains whose boundary agree on $\Gamma^{(a)}$ and and let $u_j$ be the solutions of (1.1) for $\Omega = \Omega_j$, $j = 1, 2$. Denote by $G$ the connected component of $\Omega_1 \setminus \Omega_2$ that contains $\Gamma^{(a)}$, we need to estimate $u_1 - u_2$ in $G \times (0, T')$ in terms of the error on the Cauchy data on $\Sigma \times (0, T)$. In order to establish such estimates we use the method introduced by Robbiano in [Ro1] and [Ro2] based on the Fourier Bros Iagolnitzer (FBI) transform defined by

$$U(x, y) := \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{i}{2}(iy+\tau-t)^2} (u_1 - u_2) \, dt, \quad \text{for every } (x, y) \in G \times \mathbb{R},$$

where $\mu$ be a positive number and $\tau \in (0, T)$.

By applying such a FBI transform the wave equation is transformed in a second order elliptic equation in $G \times \mathbb{R}$ in the new unknown function $U$ with a nonhomogeneous term $f$ depending on the final values $(u_1 - u_2)(\cdot, T), \partial_t(u_1 - u_2)(\cdot, T)$ and on $\mu$. Since, roughly speaking, $f$ is small when $\mu$ and $T$ are large and $U(\cdot, 0)$ is close to $(u_1 - u_2)(\cdot, \tau)$ for large $\mu$, we can apply the estimates for the Cauchy problem for elliptic equations proved in [Al-R-Ro-Ve] obtaining useful estimates of $u_1 - u_2$ in $G \times (0, T')$. We wish to stress that here, differently to [Ro1] and [Ro2], we have the additional difficulty that the boundary of $G$ might not be smooth.

(b) Quantitative estimates of strong unique continuation for wave equations. For our proof it is crucial to know that the vanishing rate of $u$ near the unknown boundary $\Gamma^{(i)}$ is of polynomial type. Namely, we need quantitative estimates of strong unique continuation in the interior and at the boundary. Such estimates have been proved in [Ve3] (in the present paper, Theorems 4.1 and 4.2). It is exactly this property that allows us to obtain a sharp estimate of the Hausdorff distance, $d_H(\overline{\Omega}_1, \overline{\Omega}_2)$, of the unknown domains $\Omega_1, \Omega_2$ in terms of the error on the Cauchy data (Corollary 5.5). The use of quantitative estimate of strong unique continuation is not new in inverse problem with unknown boundaries. The first paper in which quantitative estimates have been successfully used is, in the elliptic context, [Al-B-Ro-Ve]. Afterwards, quantitative estimates of strong unique continuation have been proved and used also for parabolic problems, we refer to the papers mentioned above and the review paper [Ve2]. To the authors knowledge the quantitative estimate of strong unique continuation was never used before in the framework of hyperbolic inverse problems.

(c) Lemma of relative graphs and sharp three sphere inequality. At a first stage the estimate of $d_H(\overline{\Omega}_1, \overline{\Omega}_2)$ is worse than logarithmic and, in addition,
the observation time $T$ for which such estimate is available may depend on the error on the Cauchy data. In order to obtain the logarithmic stability estimate for $T$ finite and independent of the errors on the Cauchy data we combine the geometric Lemma of relative graphs (Lemma 5.3) and a three sphere inequality for elliptic equations whose exponent is sharply evaluated when the radii of the three balls are close to each other (Theorem 4.6). This point is the most delicate part of the proof and is developed in Section 5.3.

The plan of the paper is as follows.

In Section 2 we will introduce the main notation and definition.

In Section 3 we will state the main Theorem 3.2.

The Section 4 contains some preliminary results concerning the quantitative estimates of strong unique continuation (Subsection 4.1), a regularity result for hyperbolic equation (Subsection 4.2), some elementary estimates for the FBI transform (Subsection 4.3) and a sharp form of the three sphere inequality for elliptic equations (Subsection 4.4).

In Section 5 we prove the main Theorem 3.2.

In the Appendix (Section 6) we prove some results of Section 4.

2 Notation and Definition

Let $n \in \mathbb{N}$, $n \geq 2$. For any $x \in \mathbb{R}^n$, we will denote $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$. Given $x \in \mathbb{R}^n$, $r > 0$, we will use the following notation for balls and cylinders.

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}, \quad B_r = B_r(0),$$

$$B'_r(x') = \{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\}, \quad B'_r = B'_r(0),$$

$$Q_{a,b}(x) = \{y = (y', y_n) \in \mathbb{R}^n : |y' - x'| < a, |y_n - x_n| < b\}, \quad Q_{a,b} = Q_{a,b}(0).$$

For any $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$ and any $r > 0$ we denote by $\bar{x} \in \mathbb{R}^{n+1}$ the point $\bar{x} = (x_1, \ldots, x_n, 0)$, or shortly $\bar{x} = (x, 0)$ and by $\bar{B}_r(\bar{x})$ the ball of $\mathbb{R}^{n+1}$ of radius $r$ centered at $\bar{x}$. For any open set $\Omega \subset \mathbb{R}^n$ and any function (smooth enough) $u$ we denote by $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ the gradient of $u$. Also, for the gradient of $u$ we use the notation $D_x$. If $j = 0, 1, 2$ we denote by $D^j_x u$ the set of the derivatives of $u$ of order $j$, so that $D^0_x u = u$, $D^1_x u = \nabla_x u$ and $D^2_x$ is the hessian matrix $\{\partial_{x_i x_j} u\}_{i,j=1}^{n}$. Similar notation are used whenever other variables occur and $\Omega$ is an open subset of $\mathbb{R}^{n-1}$ or a subset $\mathbb{R}^{n+1}$. By $H^\ell(\Omega)$, $\ell = 0, 1, 2$ we denote the usual Sobolev spaces of order $\ell$, in particular we have $H^0(\Omega) = L^2(\Omega)$.
For any interval \( J \subset \mathbb{R} \) and \( \Omega \) as above we denote by
\[
W(J; \Omega) = \{ u \in C^0(J; H^2(\Omega)) : \partial^\ell u \in C^0(J; H^{2-\ell}(\Omega)), \ell = 1, 2 \}.
\]

**Definition 2.1 (\( C^{k,1} \) regularity of a domain).** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Given \( k \in \mathbb{N} \cup \{ 0 \} \), we say that a portion \( S \) of \( \partial \Omega \) is of class \( C^{k,1} \) with constants \( \rho_0, E > 0 \), if, for any \( P \in S \), there exists a rigid transformation of coordinates under which we have \( P = 0 \) and
\[
\Omega \cap Q_{\frac{\rho_0}{2},\rho_0} = \{ x = (x',x_n) \in Q_{\rho_0,\rho_0} \mid x_n > \varphi(x') \},
\]
where \( \varphi \) is a \( C^{k,1} \) function on \( B_{\rho_0}' \) satisfying
\[
\| \varphi \|_{C^{k,1}(B_{\rho_0}')} \leq E \rho_0,
\]
\[
\varphi(0) = 0,
\]
and, whenever \( k \geq 1 \),
\[
\nabla_{x'} \varphi(0) = 0.
\]

When \( \partial \Omega \) is of class \( C^{k,1} \) with constants \( \rho_0, E > 0 \) we also say that \( \Omega \) is of class \( C^{k,1} \) with constants \( \rho_0, E > 0 \). Moreover, when \( k = 0 \) we also say that \( S \) is of Lipschitz class with constants \( \rho_0, E \).

**Remark 2.2.** We use the convention of normalizing all norms in such a way that all their terms are dimensionally homogeneous. For example:
\[
\| \varphi \|_{C^{0,1}(B_{\rho_0})} = \| \varphi \|_{L^\infty(B_{\rho_0})} + r_0 \| \nabla_{x'} \varphi \|_{L^\infty(B_{\rho_0})}.
\]
Similarly, if \( u \in H^m(\Omega) \), where \( \Omega \) is a domain of \( \mathbb{R}^n \) of class \( C^{k,1} \) with constants \( \rho_0, E \), denoting by \( D^j u \) the vector which components are the derivatives of order \( j \) of the function \( u \),
\[
\| u \|_{H^m(\Omega)} = \rho_0^{-n/2} \left( \sum_{j=0}^m \rho_0^{2j} \int_{\Omega} |D^j u|^2 \right)^{1/2},
\]
\[
\| u \|_{C^k(\Omega)} = \sum_{i=0}^k r_0^i \| D^i u \|_{L^\infty(\Omega)}.
\]
Definition 2.3. (relative graphs). We shall say that two bounded domains \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{R}^n \) of class \( C^{1,1} \) with constants \( \rho_0, E \) are relative graphs if for any \( P \in \partial \Omega_1 \) there exists a rigid transformation of coordinates under which we have \( P \equiv 0 \) and there exist \( \varphi_{P,1}, \varphi_{P,2} \in C^{1,1}(B'_{\rho_0}(0)) \), where \( \frac{\rho_2}{\rho_0} \leq 1 \) depends on \( E \) only, satisfying the following conditions

\[
\begin{align*}
(2.1a) & \quad \varphi_{P,1}(0) = |\nabla_{x'} \varphi_{P,1}(0)| = 0 , \quad |\varphi_{P,2}(0)| \leq \frac{\rho_0}{2}, \\
(2.1b) & \quad \|\varphi_{P,1}\|_{C^{1,1}(B'_{\rho_0}(0))} \leq E \rho_0, \\
(2.1c) & \quad \Omega_1 \cap B_{\rho_0}(0) = \{ x \in B_{\rho_0}(0) : x_n > \varphi_{P,i}(x') \}, \ i = 1, 2.
\end{align*}
\]

We will denote

\[
\gamma_0 (\Omega_1, \Omega_2) = \sup_{P \in \partial \Omega_1} \| \varphi_{P,1} - \varphi_{P,2} \|_{L^\infty(B'_{\rho_0}(0))}
\]

and, for any \( \alpha \in (0, 1] \),

\[
\gamma_{1,\alpha} (\Omega_1, \Omega_2) = \sup_{P \in \partial \Omega_1} \| \varphi_{P,1} - \varphi_{P,2} \|_{C^{1,\alpha}(B'_{\rho_0}(0))}.
\]

Definition 2.4. (Hausdorff distance). Let \( \Omega_1 \) and \( \Omega_2 \) be bounded domains in \( \mathbb{R}^n \). We call Hausdorff distance between \( \Omega_1 \) and \( \Omega_2 \) the number

\[
(2.4) \quad d_H(\overline{\Omega}_1, \overline{\Omega}_2) = \max \left\{ \sup_{x \in \Omega_1} \text{dist} (x, \overline{\Omega}_2) , \sup_{x \in \Omega_2} \text{dist} (x, \overline{\Omega}_1) \right\}.
\]

Definition 2.5. (modified distance). Let \( \Omega_1 \) and \( \Omega_2 \) be bounded domains in \( \mathbb{R}^n \). We call modified distance between \( \Omega_1 \) and \( \Omega_2 \) the number

\[
(2.5) \quad d_m(\overline{\Omega}_1, \overline{\Omega}_2) = \max \left\{ \sup_{x \in \Omega_1} \text{dist} (x, \overline{\Omega}_2) , \sup_{x \in \Omega_2} \text{dist} (x, \overline{\Omega}_1) \right\}.
\]

For any open set \( \Omega \subset \mathbb{R}^n \) and \( r > 0 \), we will denote

\[
\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}.
\]

We will use the the letters \( C \) to denote constants larger or equal than 1. Sometime, for special constants or to emphasize the role that it have in the proof, we will use the notation \( C_0, C_1, \ldots \). The value of the constants may change from line to line, but we shall specified their dependence everywhere they appear.
3 The Inverse Problem: The Main Theorem

i) A priori information on the domain.

Given $\rho_0$, $M > 0$, $E \geq 1$ we assume

\begin{align}
(3.1a) & \quad |\Omega| \leq M \rho_0^n, \\
(3.1b) & \quad \partial \Omega \quad \text{of class } C^{1,1} \text{ with constants } \rho_0 \text{ and } E,
\end{align}

here, and in the sequel, $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

Let $\Gamma^{(a)}$ be a nonempty closed proper subset of $\partial \Omega$ and assume that the closure of the interior part of $\Gamma^{(a)}$ in the relative topology in $\partial \Omega$ is equal to $\Gamma^{(a)}$. In addition we assume that

\begin{equation}
\text{Int}_{\partial \Omega} \left( \Gamma^{(a)} \right) \quad \text{is connected},
\end{equation}

and we set

\begin{equation}
\Gamma^{(i)} = \partial \Omega \setminus \text{Int}_{\partial \Omega} \left( \Gamma^{(a)} \right),
\end{equation}

here and in the sequel, $\text{Int}_{\partial \Omega} \left( \Gamma^{(a)} \right)$ denotes the interior part of $\Gamma^{(a)}$ in the relative topology in $\partial \Omega$. In the sequel we will refer to $\Gamma^{(a)}$ and $\Gamma^{(i)}$ as the accessible and inaccessible part of $\partial \Omega$ respectively.

Moreover denoting

$$
\Gamma^{(a)}_{\rho} = \{ x \in \Gamma^{(a)} : \text{dist}(x, \Gamma^{(i)}) \geq \rho \},
$$

we assume that, for any $\rho \in (0, \rho_0]$, $\Gamma^{(a)}_{\rho}$ is a nonempty and connected set and we assume that we can select a portion $\Sigma$ satisfying for some $P_0 \in \Sigma$

\begin{equation}
\partial \Omega \cap B_{\rho_0}(P_0) \subset \Sigma \subset \Gamma^{(a)}_{\rho_0}.
\end{equation}

Remark 3.1. Observe that (3.1b) automatically implies a lower bound on the diameter of every connected component of $\partial \Omega$. Moreover, by combining (3.1a) with (3.1b), an upper bound on the diameter of $\Omega$ can also be obtained. Note also that (3.1a), (3.1b) implicitly comprise an a priori upper bound on the number of connected components of $\partial \Omega$. Finally observe that the hypotheses (3.1)-(3.4) are satisfied in the case $\Omega = \hat{\Omega} \setminus D$, where $\hat{\Omega}$ and $D$ are two open domains in $\mathbb{R}^n$ whose boundaries, $\partial \hat{\Omega}$ and $\partial D$, are connected, $D \subset \hat{\Omega}$, $\text{dist}(D, \partial \Omega) \geq 2\rho_0$ and $\Omega, D$ satisfy condition (3.1). In addition $\Gamma^{a} = \partial \hat{\Omega}$, $\Gamma^{i} = \partial D$ and $\Sigma$ is a portion of $\partial \hat{\Omega}$ satisfying, for some $P_0 \in \Sigma$, the condition $\partial \hat{\Omega} \cap B_{\rho_0}(P_0) \subset \Sigma$. 

7
ii) Assumptions about the boundary data.

Let $m := \left\lceil \frac{n+2}{4} \right\rceil$. Assume that $\psi$ is a function on $\partial \Omega \times [0, +\infty)$ which satisfies the following conditions

\[(3.5a) \quad \partial_t^j \psi(\cdot, t) \in C^{1,1}(\partial \Omega), \quad \text{for } j \in \{0, \cdots, 2m+4\}, \text{ and } t \in [0, +\infty),\]

\[(3.5b) \quad \partial_t^j \psi(\cdot, 0) = 0, \quad \text{for } j \in \{0, \cdots, 2m+4\}, \text{ and } t \in [0, +\infty).\]

Denote, for $t \in [0, +\infty)$

\[(3.6) \quad H(t) = \sum_{j=0}^{2m+4} \rho_j^0 \sup_{\xi \in [0,1]} \| \partial_t^j \psi(\cdot, \xi) \|_{C^{1,1}(\partial \Omega)}.\]

Let $t_1 \geq \rho_0$ and assume

\[(3.7) \quad \frac{H(t_1)}{\| \psi \|_{L^\infty(\Gamma^{(a)} \times [0,t_1])}} \leq F.\]

iii) Assumptions about the matrix $A$.

$A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ is assumed to be a real-valued symmetric $n \times n$ matrix whose the entries are measurable function and satisfying the following conditions for given constants $\lambda \in (0, 1]$, $\Lambda > 0$,

\[(3.8a) \quad \lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \text{for every } x, \xi \in \mathbb{R}^n,\]

\[(3.8b) \quad |A(x) - A(y)| \leq \frac{\Lambda}{\rho_0} |x - y|, \quad \text{for every } x, y \in \mathbb{R}^n.\]

**Theorem 3.2.** Let $\Omega_1, \Omega_2$ be two domains satisfying (3.1). Let $\Gamma^{(a)}_j, \Gamma^{(i)}_j = \partial \Omega_j \setminus \text{Int}_\partial \Omega_j(\Gamma^{(a)}_j)$, $j = 1, 2$, be the corresponding accessible and inaccessible parts of their boundaries. Let us assume $\Gamma^{(a)}_1 = \Gamma^{(a)}_2 = \Gamma^{(a)}$, $\Omega_1, \Omega_2$ lie on the same side of $\Gamma^{(a)}$ and that (3.2), (3.3) and (3.4) are satisfied.

Then there exists a constant $C$ depending on $\lambda, \Lambda, E, M$ and $F$ only such that if $T = \max\{C\rho_0, 2t_1\}$ then the following holds true.

Let $u_j \in \mathcal{W}([0,T]; \Omega)$ be the solution to (1.1) when $\Omega = \Omega_j$, $j = 1, 2$, and if, for a given $\varepsilon \in (0, e^{-1})$, we have

\[(3.9) \quad \int_0^T \int_{\Sigma} |A(x)\nabla u_1 \cdot \nu - A(x)\nabla u_2 \cdot \nu|^2 dS dt \leq T \rho_0^{n-3} \varepsilon^2,\]
where $dS$ is the surface element in dimension $n - 1$, then we have

$$(3.10) \quad d_H(\overline{\Omega}_1, \overline{\Omega}_2) \leq C_* \rho_0 |\log \varepsilon|^{-1/C_*},$$

where $C_*$ depends on $\lambda, \Lambda, E, M, F$ and the ratio $\frac{H(T)}{H(t_1)}$.

We prove this Theorem in Section 5.

4 Preliminary results

4.1 Quantitative estimates of strong unique continuation.

The theorems presented in this subsection are crucial to prove Theorem 3.2. They are analogs of the quantitative estimates of strong unique continuation (doubling inequalities, three sphere inequality, three cylinders inequality, two-sphere one cylinder inequality at the interior and at the boundary) which are well known in the elliptic [Ga-Li], [La], [A-E] and in the parabolic context [Es-Fe-Ve], [Es-Ve]. Theorem 4.1 is basically the quantitative version of the strong unique continuation property for the self-adjoint hyperbolic equation proved by Lebeau in [Le]. Theorems 4.1 and 4.2 have been proved in [Ve3].

Let $u \in \mathcal{W}([-\lambda \rho_0, \lambda \rho_0]; B_{\rho_0})$ be a weak solution to

$$\partial_t^2 u - \text{div} (A(x) \nabla_x u) = 0, \quad \text{in } B_{\rho_0} \times (-\lambda \rho_0, \lambda \rho_0).$$

Let $r_0 \in (0, \rho_0]$ and denote by

$$(4.1) \quad \varepsilon_0 := \sup_{t \in (-\lambda \rho_0, \lambda \rho_0)} \left( \rho_0^{-n} \int_{B_{t_0}} u^2(x,t) dx \right)^{1/2}$$

and

$$(4.2) \quad H_0 := \left( \sum_{j=0}^{2} r_0^{j-n} \int_{B_{\rho_0}} |D_x^j u(x,0)|^2 dx \right)^{1/2}.$$

**Theorem 4.1.** Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8) and let $u \in \mathcal{W}([-\lambda \rho_0, \lambda \rho_0]; B_{\rho_0})$ be a weak solution to (4.1). Then there
exist constants $s_0 \in (0, 1)$ and $C \geq 1$ depending on $\lambda$ and $\Lambda$ only such that for every $r_0$ and $\rho$ satisfying $0 < r_0 \leq \rho \leq s_0 \rho_0$ the following inequality holds true

$$
\|u(\cdot, 0)\|_{L^2(B_\rho)} \leq C \left( \rho_0 \rho^{-1} \right)^C \left( H_0 + e^{\varepsilon_0} \right) \left( \theta \log \left( \frac{H_0 + e^{\varepsilon_0}}{\varepsilon_0} \right) \right)^{1/6},
$$

where

$$
\theta = \frac{\log(\rho_0/C \rho)}{\log(\rho_0/r_0)}.
$$

In order to state Theorem 4.2 below let us introduce some notation. Let $\varphi$ be a function belonging to $C^{1,1}(B'_0)$ that satisfies

$$
\varphi(0) = |\nabla_{x'}\varphi(0)| = 0
$$

and

$$
\|\varphi\|_{C^{1,1}(B'_0)} \leq E \rho_0,
$$

where

$$
\|\varphi\|_{C^{1,1}(B'_0)} = \|\varphi\|_{L^\infty(B'_0)} + \rho_0 \|\nabla_{x'}\varphi\|_{L^\infty(B'_0)} + \rho_0^2 \|D^2_{x'}\varphi\|_{L^\infty(B'_0)}.
$$

For any $r \in (0, \rho_0]$ denote by

$$
K_r := \{(x', x_n) \in B_r : x_n > \varphi(x')\}
$$

and

$$
S_{\rho_0} := \{(x', \varphi(x')) : x' \in B'_0\}.
$$

Let $u \in \mathcal{W}([-\lambda \rho_0, \lambda \rho_0]; K_{\rho_0})$ be a solution to

$$
\partial_t^2 u - \text{div} (A(x)\nabla_x u) = 0, \quad \text{in} \ K_{\rho_0} \times (-\lambda \rho_0, \lambda \rho_0),
$$

satisfying one of the following conditions

$$
u = 0, \quad \text{on} \ S_{\rho_0} \times (-\lambda \rho_0, \lambda \rho_0),$$
(4.10) \[ A \nabla_x u \cdot \nu = 0, \quad \text{on } S_{\rho_0} \times (-\lambda \rho_0, \lambda \rho_0), \]

where \( \nu \) denotes the outer unit normal to \( S_{\rho_0} \).

Let \( r_0 \in (0, \rho_0] \) and denote by

\[
\varepsilon_0 := \sup_{t \in (-\lambda \rho_0, \lambda \rho_0)} \left( \int_{K_{r_0}} u^2(x, t) \, dx \right)^{1/2}
\]

and

\[
H_0 := \left( \sum_{j=0}^2 \rho_0^{2-j} \int_{K_{r_0}} |D^1_x u(x, 0)|^2 \, dx \right)^{1/2}.
\]

**Theorem 4.2.** Let (3.8) be satisfied. Let \( u \in \mathcal{W}([-\lambda \rho_0, \lambda \rho_0]; K_{\rho_0}) \) be a solution to (4.8) satisfying (4.11) and (4.12). Assume that \( u \) satisfies either (4.9) or (4.10). There exist constants \( \bar{s}_0 \in (0, 1) \) and \( C \geq 1 \) depending on \( \lambda, \Lambda \) and \( E \) only such that for every \( r_0 \) and \( \rho \) satisfying \( 0 < r_0 \leq \rho \leq \bar{s}_0 \rho_0 \) the following inequality holds true

\[
\| u(\cdot, 0) \|_{L^2(K_{\rho})} \leq \frac{C (\rho_0 \rho^{-1})^C (H_0 + \varepsilon_0)}{\bar{\theta} \log \left( \frac{H_0 + \varepsilon_0}{\varepsilon_0} \right)^{1/6}},
\]

where

\[
\bar{\theta} = \frac{\log(\rho_0/C \rho)}{\log(\rho_0/r_0)}.
\]

**4.2 A regularity result for hyperbolic equation**

The next Theorem is a mere simplified version of a regularity result proved in [Co]. For the reader convenience we give a sketch of the proof of such a result in the Appendix, Subsection 6.1.

**Theorem 4.3.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) that satisfies (3.1). Let \( A(x) \) be a real-valued symmetric \( n \times n \) matrix satisfying (3.8). Let \( m := \left\lceil \frac{n+2}{4} \right\rceil \). Assume that \( \psi \) is a function on \( \partial \Omega \times [0, T] \) which satisfies the condition (3.5).
Let $u \in W([0, T]; \Omega)$ be the solution to the problem

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div}(A(x) \nabla u) = 0, & \text{in } \Omega \times [0, T], \\
u = \psi, & \text{on } \partial \Omega \times [0, T], \\
u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases} \tag{4.15}
$$

Then for every $\alpha \in (0, 1)$ and $t \in [0, T]$ we have $\partial_t^2 u(\cdot, t) \in L^\infty(\Omega)$, $u(\cdot, t) \in C^{1,\alpha}(\Omega)$ and the following inequalities hold true

$$
\begin{align*}
\sup_{t \in [0, T]} \|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)} &\leq C \rho_0^{-2}(T \rho_0^{-1} + 1) H(T), \\
\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^2(\Omega)} &\leq C(T \rho_0^{-1} + 1) H(T), \\
\sup_{t \in [0, T]} \|u(\cdot, t)\|_{C^{1,\alpha}(\Omega)} &\leq C(T \rho_0^{-1} + 1) H(T),
\end{align*}
$$

where $H(T)$ is defined by (3.6) and $C$ depends on $\alpha, n, E, M, \lambda$ and $\Lambda$ only.

**4.3 Elementary estimates for the FBI transform**

For the convenience of the the reader, we collect in this section some well known elementary properties of the FBI transform see also [Che-D-Y], [Che-P-Y], [Ro1], [Ro2], [Ro-Zu]. Let $\Omega$ be a domain of $\mathbb{R}^n$ and $T$ a positive number. Let $u \in W([0, T]; \Omega)$ satisfy

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div}(A(x) \nabla u) = 0, & \text{in } \Omega \times [0, T], \\
u(\cdot, 0) = 0, & \text{in } \Omega, \\
\partial_t u(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases} \tag{4.17}
$$

Let $\mu$ be a positive number. For a fixed $\tau \in (0, T/2]$ we denote by $U_\mu(\tau)$ the FBI transform of $u$ defined by

$$
U_\mu(\tau)(x, y) := \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{1}{2}(iy + \tau t)^2} u(x, t) dt, \quad \text{for every } (x, y) \in \Omega \times \mathbb{R}.
$$

Observe that $U_\mu(\tau)$, as a function of $y$, is a $C^\infty(\mathbb{R})$ with values in $H^2(\Omega)$.

The following propositions holds true.
Proposition 4.4. We have

$$\left| D^j_x U^{(\tau)}_\mu(x, y) \right| \leq c \mu^{1/4} e^{2y^2} \left( \int_0^T \left| D^j_x u(x, t) \right|^2 \, dt \right)^{1/2}, \quad \text{for a.e. } x \in \Omega, \quad \text{and } 0 \leq j \leq 2,$$

and

$$\left| U^{(\tau)}_\mu(x, 0) - u(x, \tau) \right| \leq c \mu^{-1/2} \| \partial_t u(x, \cdot) \|_{L^\infty([0, T])},$$

where $c$ is an absolute constant.

**Proof.** See Subsection 6.2.

Proposition 4.5. Let $u \in W([0, T]; \Omega)$ satisfy (4.17) and let $U^{(\tau)}_\mu$ be defined by (4.18). Then $U^{(\tau)}_\mu$ satisfies the equation

$$\partial^2_y U^{(\tau)}_\mu + \text{div} \left( A(x) \nabla_x U^{(\tau)}_\mu \right) = f^{(\tau)}_\mu(x, y), \quad \text{in } \Omega \times \mathbb{R},$$

where

$$f^{(\tau)}_\mu(x, y) = \sqrt{\frac{\mu}{2\pi}} e^{-\frac{\mu}{2}(iy + \tau - T)^2} \left( \partial_t u(x, T) - \mu(iy + \tau - T)u(x, T) \right).$$

**Proof.** See Subsection 6.2.

### 4.4 A sharp three sphere inequality for elliptic equations

In the following theorem we give a three sphere inequality for elliptic equations in which we take care to evaluate the exponent of such an inequality whenever the radii of the three balls are close to each other. Except for this feature the following Theorem is quite standard and, for the convenience of the reader, we will prove it in the Appendix (Subsection 6.3).

Let $\widetilde{A}(X) = \{ \widetilde{a}^{ij}(x) \}_{i,j=1}^N$, $N \geq 2$ be a real-valued symmetric $N \times N$ matrix. Assume that the entries of matrix $\widetilde{A}$ are measurable function and it satisfies

$$\lambda_0 |\xi|^2 \leq \widetilde{A}(X) \xi \cdot \xi \leq \lambda_0^{-1} |\xi|^2, \quad \text{for every } X, \xi \in \mathbb{R}^N,$$

where $\lambda_0 \in (0, 1)$.
Theorem 4.6 (Three sphere inequality). Let $\tilde{r}_3$ and $\Lambda_0$ be positive numbers. Assume that $\tilde{A}$ satisfies (4.23) and

$$(4.24) \quad \left| \tilde{A}(X) - \tilde{A}(Y) \right| \leq \frac{\Lambda_0}{\tilde{r}_3} |X - Y|, \quad \text{for every } X, Y \in B_{\tilde{r}_3}.$$  

Let $\tilde{f} \in L^2(B_{\tilde{r}_3})$ and let $u \in H^1(B_{\tilde{r}_3})$ be a solution to

$$(4.25) \quad Pu := \text{div} (\tilde{A} \nabla u) = \tilde{f}, \quad \text{in } B_{\tilde{r}_3}.$$  

Let $\tilde{r}_1, \tilde{r}_2, \tilde{r}_3$ be such that $0 < \tilde{r}_1 \leq \tilde{r}_2 < \tilde{r}_3$. Let $\delta$ be such that

$$(4.26) \quad 0 < \delta \leq \frac{\tilde{r}_3 - \tilde{r}_2}{2\tilde{r}_3}.$$  

Denote by

$$(4.27) \quad \vartheta_0 = \frac{\tilde{r}_2^{-\beta} - [(1 - \delta)\tilde{r}_3]^{-\beta}}{[(1 - 2\delta)\tilde{r}_1]^{-\beta} - [(1 - \delta)\tilde{r}_3]^{-\beta}}.$$  

and

$$(4.28) \quad C_0 = \frac{e^{C[(\tilde{r}_2\tilde{r}_1)^{-\beta} - (1 - \delta)^{-\beta} \right]}}{\delta^4};$$

where $C$ depends on $\lambda_0, \Lambda_0$.

There exists $\beta_1 \geq 1$ depending on $\lambda_0, \Lambda_0$ only such that if $\beta \geq \beta_1$ then the following inequality holds true

$$(4.29) \quad \int_{B_{\tilde{r}_2}} |u|^2 \leq C_0 \left( \int_{B_{\tilde{r}_1}} |u|^2 + \tilde{r}_3^2 \int_{B_{\tilde{r}_3}} |\tilde{f}|^2 \right)^{\vartheta_0} \left( \int_{B_{\tilde{r}_1}} |u|^2 + \tilde{r}_3^2 \int_{B_{\tilde{r}_3}} |\tilde{f}|^2 \right)^{1-\vartheta_0}.$$  

5 Proof of the Main Theorem

In order to prove Theorem 3.2 we proceed in the following way.

Set

$G$ the connected component of $\Omega_1 \cap \Omega_2$ whose closure contains $\Gamma^{(a)}$. 
First step In Proposition 5.1 we prove that for a given \( t_0 > 0 \) there exists \( T(\varepsilon) > 2t_0 \) such that if (3.9) is satisfied for \( T = T(\varepsilon) \) and \( u_j \in W([0, T(\varepsilon)]; \Omega) \) are the solutions to (1.1) when \( \Omega = \Omega_j, j = 1, 2 \) then

\[
\sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq C \omega(\varepsilon, t_0), \text{ for } j = 1, 2,
\]

where

\[
\lim_{\varepsilon \to 0} \omega(\varepsilon, t_0) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} T(\varepsilon) = +\infty.
\]

Second step First we prove (Proposition 5.2) an estimate from below, in terms of the a priori information and boundary data, of the quantity

\[
\sup_{t \in [0, t_0]} \|u(\cdot, t)\|_{L^2(B_{\varrho(y_0)})}
\]

where the sup is taken for \( t \geq \overline{t} \), \( \overline{t} \) is large enough, \( B_{\varrho(y_0)} \subseteq \Omega \) and \( \varrho \in (0, \rho_0/2E] \). Afterwards (Proposition 5.4) we prove that if \( t_0 \) is large enough and

\[
\sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega_j \setminus G} u_j^2(x, t) dx \right) \leq \eta^2
\]

then

\[
d_H(\Omega_1, \Omega_2) \leq C \rho_0 \eta^\alpha,
\]

for suitable constant \( C \geq 1 \) and \( \alpha \in (0, 1) \).

Third step We conclude the proof of Theorem 3.2.

5.1 Step 1

Proposition 5.1. There exist \( C \geq 1 \) and \( \overline{\varrho}, \sigma, \varrho_2 \in (0, 1] \) depending on \( E, M, \lambda \) and \( \Lambda \) only such that the following holds true.

Denoting

\[
T_\sigma := \max \left\{ 2t_0, \sqrt{10} \rho_0 \varrho_2^{-\frac{1}{2}} \sigma^{-(n+1)} \right\},
\]

\[
\Phi(\sigma) = \sigma^{-\left(\frac{n+1}{2}\right)}(T_\sigma^{-1}\rho_0^{-1})^{11/2}(H(T_\sigma) + 1)^2.
\]

Let us define for any \( \varepsilon \in (0, \overline{\varrho}] \)

\[
T(\varepsilon) := T_{\sigma(\varepsilon)},
\]

where

\[
\sigma(\varepsilon) := \inf\{\sigma \in (0, \overline{\varrho}] : \Phi(\sigma) \leq |\log \varepsilon|^{\frac{1}{2}}\}.
\]
Let \( u_j \in \mathcal{W}([0, T(\varepsilon)]; \Omega) \) be the solution to (1.1) (when \( T = T(\varepsilon) \)) and \( \Omega = \Omega_j, j = 1, 2 \).

If, for a given \( \varepsilon \in (0, \varepsilon] \), we have

\[
\text{(5.5)} \quad \frac{1}{T(\varepsilon)\rho_0^{-3}} \int_0^{T(\varepsilon)} \int_\Sigma |A(x)\nabla u_1 \cdot \nu - A(x)\nabla u_2 \cdot \nu|^2 \, dS \, dt \leq \varepsilon^2
\]

then for every \( t_0 \in (0, T(\varepsilon)/2] \) we have

\[
\text{(5.6)} \quad \sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega_j \cap G} u_j^2(x, t) \, dx \right) \leq C\omega(\varepsilon, t_0), \quad \text{for } j = 1, 2,
\]

where

\[
\text{(5.7)} \quad \omega(\varepsilon, t_0) = (t_0\rho_0^{-1})^6 (H(t_0))^2 (\sigma(\varepsilon))^{1/4} + |\log \varepsilon|^{-1/8}.
\]

**Proof of Proposition 5.1.** Let \( t_0 > 0 \). We begin by assuming only that \( T \geq 2t_0 \). Let \( u_j \in \mathcal{W}([0, T]; \Omega) \) be the solution to (1.1) when \( \Omega = \Omega_j, j = 1, 2 \). Let \( u = u_1 - u_2 \), in \( G \times [0, T] \) and for any positive number \( \mu \) such that \( \mu T^2 \geq 1 \) and \( \tau \in (0, T/2] \) denote by \( U_\mu^{(\tau)} \) the FBI transform of \( u \) defined by

\[
\text{(5.8)} \quad U_\mu^{(\tau)}(x, y) = \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2}(iy + \tau - t)^2} u(x, t) \, dt, \quad \text{for every } (x, y) \in G \times \mathbb{R}.
\]

By (1.1), (3.9) and Proposition 4.5 we have

\[
\text{(5.9)} \quad \begin{cases}
\partial_t^{(\tau + \tau)} + \text{div} \left( A(x)\nabla U_\mu^{(\tau)} \right) = f_\mu^{(\tau)}(x, y), & \text{in } G \times \mathbb{R}, \\
U_\mu^{(\tau)}(x, y) = 0, & \text{for } (x, y) \in \Sigma \times \mathbb{R}, \\
\int_\Sigma |A(x)\nabla U_\mu^{(\tau)}(x, y) \cdot \nu|^2 \, dS \leq C\mu^{1/2}T\rho_0^{-3}e^{\mu y^2}\varepsilon^2, & \text{for } y \in \mathbb{R},
\end{cases}
\]

and \( C \) is an absolute constant.

By (3.5), Proposition 4.4, Proposition 4.5, by Theorem 4.3 and by the elementary inequality \( s^{3/2}e^{-s^2/8} \leq ce^{-s^2/10} \) we have, for every \( R > 0 \)

\[
\text{(5.10)} \quad \|f_\mu\|_{L^\infty(G \times (-R, R))} \leq CT\rho_0^{-3}H(T)e^{\mu(R^2/2 - T^2/10)},
\]
and

\[(5.11) \quad \|U^{(r)}\|_{L^\infty(G \times (-R,R))} \leq CT \rho_0^{-1} H(T) e^{\mu R^2/2},\]

where \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only. Here and in the sequel, we fix \( \alpha = \frac{1}{2} \) in Theorem 4.3.

Now denote by \( P_1 = P_0 - \frac{e_0}{2r} \nu \), \( \tilde{P}_1 = (P_1, 0) \), \( \rho_1 = \sigma_1 \rho_0 \), where \( \sigma_1 = \frac{1}{4E \sqrt{1 + E^2}} \) and denote by

\[(5.12) \quad \varepsilon_1 = \frac{(\mu T^2)^{1/4} \varepsilon}{(H(T) + 1) T \rho_0^{-1}}.\]

By (5.9), (5.10) and by applying [Al-R-Ro-Ve, Theorem 1.7] we have

\[(5.13) \quad \|U^{(r)}\|_{L^2(\bar{B}_{\rho_1}(P_1))} \leq CT \rho_0^{-1} H(T) e^{\mu \rho_0^{3/2}} (e^{-\mu T^2/10} + \varepsilon_1)^{\vartheta_1},\]

where \( \vartheta_1, \vartheta_1 \in (0, 1) \), and \( C \) depend on \( E, M, \lambda \) and \( \Lambda \) only.

Let \( \sigma \in (0, \sigma_1) \) and denote by \( r = \rho_0 \sigma \). Let \( V_r \) be the connected component of \( \Omega_{1,r} \cap \Omega_{2,r} \) whose closure contains \( B_{\rho_1}(P_1) \). Moreover denote by \( \omega_r = \Omega_{1,r} \setminus V_r \). We have

\[(5.14a) \quad \Omega_1 \setminus G \subset [(\Omega_1 \setminus \Omega_{1,r}) \setminus G] \cup \omega_r,\]

\[(5.14b) \quad \partial \omega_r = \Gamma_{1,r} \cup \Gamma_{2,r},\]

where

\[\Gamma_{1,r} \subset \partial \Omega_{1,r}, \quad \Gamma_{2,r} \subset \partial \Omega_{2,r} \cap \partial V_r.\]

Let \( z \in \Gamma_{2,r} \) be fixed. Since \( V_r \) is connected, \( \Gamma_{2,r} \subset \partial V_r \) and \( P_1 \in V_r \), there exists a continuous path \( \gamma : [0,1] \to V_r \) such that \( \gamma(0) = P_1, \gamma(1) = z \).

Let us define \( 0 = s_0 < s_1 < \ldots < s_N = 1 \), according to the following rule. We set \( s_{k+1} = \max\{s \mid |\gamma(s) - x_k| = \frac{r}{2}\} \) if \( |x_k - z| > \frac{r}{2} \), otherwise we stop the process and set \( N = k + 1, s_N = 1 \). By (3.1a) we have \( N \leq c_n M \sigma^{-n} \) where \( c_n \) depends on \( n \) only. Let \( x_k = \gamma(s_k) \) and \( \tilde{x}_k = (x_k, 0) \). The balls (of \( \mathbb{R}^{n+1} \)) \( \bar{B}_{r/4}(\tilde{x}_k) \) are pairwise disjoint for \( k = 0, \ldots, N - 1 \) and \( |\tilde{x}_{k+1} - \tilde{x}_k| = \frac{r}{2} \). We have that \( \bar{B}_{r/4}(\tilde{x}_{k+1}) \subset \bar{B}_{3r/4}(\tilde{x}_k) \) and \( \bar{B}_{r}(\tilde{x}_k) \subset G \times (-r, r) \) and therefore, by the three sphere inequality (4.29), we have

\[(5.15) \quad \|U^{(r)}\|_{L^2(\bar{B}_{r/4}(\tilde{x}_{k+1}))} \leq \|U^{(r)}\|_{L^2(\bar{B}_{3r/4}(\tilde{x}_k))} \leq \]

\[\leq C \left( \|U^{(r)}\|_{L^2(\bar{B}_{r/4}(\tilde{x}_k))} + \|f^{(r)}\|_{L^2(\bar{B}_{r}(\tilde{x}_k))} \right)^{\vartheta_1} \left( \|U^{(r)}\|_{L^2(\bar{B}_{r}(\tilde{x}_k))} + \|f^{(r)}\|_{L^2(\bar{B}_{r}(\tilde{x}_k))} \right)^{1-\vartheta_1},\]

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where $C$ and $\vartheta_*$, $0 < \vartheta_* < 1$, depend on $E, \lambda$ and $\Lambda$ only.

Now, we denote by

$$\alpha_k = \frac{\|U_{\mu}^{(\sigma)}\|_{L^2(\tilde{B}_{1/4}(\tilde{x}))} e^{-\mu r^2/2}}{Tp_0^{-1}(H(T) + 1)} + e^{-\mu T^2/10} \quad \text{for } k = 0, \ldots, N,$$

and by (5.15), (5.10) and (5.11) we have

$$\alpha_{k+1} \leq C\alpha_k^{\vartheta_*} \quad \text{for } k = 0, \ldots, N - 1,$$

where $C$ and $\vartheta_*$, $0 < \vartheta_* < 1$, depend on $E, \lambda$ and $\Lambda$ only. By iterating (5.17) we get

$$\alpha_N \leq C^{1/1-\vartheta_*}\alpha_0^{\vartheta_*}.$$

Now let us denote by $\vartheta_2 = \min \{ \vartheta_1, \vartheta_* M \}$. By (5.16) and (5.18) we have

$$\left\| U_{\mu}^{(\sigma)}(z, 0) \right\|_{L^1(\tilde{B}_{1/4}(\tilde{x}))} \leq CT \rho_0^{-1}H(T)e^{\mu r^2/2} \times$$

$$\times \left( \frac{\|U_{\mu}^{(\sigma)}\|_{L^2(\tilde{B}_{1/4}(\tilde{x}))} e^{-\mu r^2/2}}{Tp_0^{-1}(H(T) + 1)} + e^{-\mu T^2/10} \right)^{\vartheta_2^{-n}},$$

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only. Moreover, by applying [G-T, Theorem 8.17] and by using (5.10), (5.13) and (5.19) we have

$$\left\| U_{\mu}^{(\sigma)}(z, 0) \right\|_{L^1(\tilde{B}_{1/4}(\tilde{x}))} \leq CT \rho_0^{-1}(H(T) + 1)e^{\mu r^2/2}\varepsilon_2,$$

where

$$\varepsilon_2 = \sigma^{-\frac{n+1}{2}} \left( e^{-\mu T^2/10} + e^{\mu \rho_0^2/2} \left( e^{-\mu T^2/10} + \varepsilon_1 \right)^{\vartheta_2} \right)^{\vartheta_2^{-n}}$$

and $C$ depends on $E, M, \lambda$ and $\Lambda$ only.

By (5.20), (4.16) and (4.20) we have

$$\left\| U_{\mu}^{(\sigma)}(\cdot, 0) \right\|_{L^\infty(\Gamma_{2,r} \times [0,t_0])} \leq C(\mu T^2)^{-1/2} \left( \rho_0^{-1}H(T) + e^{\mu T^2/10} + \varepsilon_1 \right)^{\vartheta_2} \vartheta_2^{-n}$$

and

$$\left\| u \right\|_{L^\infty(\Gamma_{2,r} \times [0,t_0])} \leq C(\mu T^2)^{-1/2} \left( \rho_0^{-1}H(T) + e^{\mu T^2/10} + \varepsilon_1 \right)^{\vartheta_2} \vartheta_2^{-n}$$

and

$$\left\| U_{\mu}^{(\sigma)}(\cdot, 0) \right\|_{L^\infty(\Gamma_{2,r} \times [0,t_0])} \leq C\left( \rho_0^{-1}H(T) + 1 \right)\varepsilon_3,$$
where
\[(5.23) \quad \varepsilon_3 = (\mu T^2)^{-1/2} + e^{\alpha^2/2} \varepsilon_2\]
and \(C\) depends on \(E, M, \lambda\) and \(\Lambda\) only.

By (5.14a) and Schwarz inequality we have, for any \(t \in (0, t_0]\),
\[(5.24) \quad \int_{\Omega_1 \setminus \Omega_{1,r}} u_1^2(x, t) dx = \]
\[= \int_{\Omega_1 \setminus \Omega_{1,r}} \left( \int_0^t \partial_\xi u_1(x, \xi) d\xi \right)^2 \leq t_0 \int_0^{t_0} \int_{\Omega_1 \setminus \Omega_{1,r}} |\partial_\xi u_1(x, \xi)|^2 dxd\xi \leq \]
\[\leq t_0 \int_0^{t_0} \int_{\Omega_{1,r}} |\partial_\xi u_1(x, \xi)|^2 dxd\xi + t_0 \int_0^{t_0} \int_{\Omega_1 \setminus \Omega_{1,r}} |\partial_\xi u_1(x, \xi)|^2 dxd\xi.\]

Now by (3.1) we have
\[(5.25) \quad |\Omega_1 \setminus \Omega_{1,r}| \leq C \rho_0^\alpha \sigma, \]
where \(C\) depends on \(E\) and \(M\) only.

By (3.1a), (5.24), (5.25) and by (4.16a) we have, for any \(t \in (0, t_0]\),
\[(5.26) \quad \rho_0^{-n} \int_{\Omega_1 \setminus \Omega_{1,r}} u_1^2(x, t) dx \leq \]
\[\leq t_0 \rho_0^{-n} \int_0^{t_0} \int_{\omega_r} |\partial_\xi u_1(x, \xi)|^2 dxd\xi + C (t_0 \rho_0^{-1})^6 H(t_0)^2 \sigma,\]
where \(C\) depends on \(E, M, \lambda\) and \(\Lambda\) only.

Now, in order to estimate from above the integral on the right hand side of (5.26) we multiply both the side of the equation \(\partial_\xi^2 u_1 - \text{div}(A(x)\nabla u_1) = 0\) by \(\partial_\xi u_3\) and integrate over \(\omega_r\) and by integration by parts we have, for every \(\xi \in [0, t_0]\),
\[(5.27) \quad \frac{1}{2} \int_{\omega_r} \left( |\partial_\xi u_1(x, \xi)|^2 + A(x)\nabla u_1(x, \xi) \cdot \nabla u_1(x, \xi) \right) dx = \]
\[= \int_0^\xi \int_{\Gamma_{1,r}} (A(x)\nabla u_1(x, t)) \partial_t u_1(x, t) dSdt + \int_0^\xi \int_{\Gamma_{2,r}} (A(x)\nabla u_1(x, t)) \partial_t u_1(x, t) dSdt := J_1 + J_2.\]

**Estimate of \(J_1\).**
By Schwarz inequality and by (3.8a) we have

\begin{equation}
|J_1| \leq \lambda^{-1} \left( \int_{\Gamma_1, r} |\nabla u_1|^2 dS \right)^{1/2} \left( \int_{\Gamma_1, r} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2}.
\end{equation}

By (5.28) and (4.16c) we have, for every \( \xi \in [0, t_0] \),

\begin{equation}
|J_1| \leq C \left( (t_0 \rho_0^{-1} + 1)t_0 \rho_0^{n-3} \right)^{1/2} H(t_0) \left( \int_{\Gamma_1, r} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2},
\end{equation}

where \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

Now, by interpolation inequality we have

\[ \|\partial_t u_1\|_{L^\infty(\Gamma_1, r \times [0, t_0])} \leq C \|u_1\|_{L^1(\Gamma_1, r \times [0, t_0])} \|\partial_t^2 u_1\|_{L^1(\Gamma_1, r \times [0, t_0])}, \]

where \( C \) is an absolute constant, hence by using (5.29), (4.16) and recalling that \( u_1 = 0 \) on \( \Omega_1 \times [0, T] \) we obtain

\begin{equation}
|J_1| \leq C (t_0 \rho_0^{-1})^{5/2} \rho_0^{n-2} (H(t_0))^2 \sigma^{1/4},
\end{equation}

where \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

**Estimate of \( J_2 \).**

By Schwarz inequality, (3.8a) and by (4.16c) we have, for every \( \xi \in [0, t_0] \),

\begin{equation}
|J_2| \leq C \left( t_0 \rho_0^{n-4} \right)^{1/2} H(t_0) \left( \int_{\Gamma_2, r} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2},
\end{equation}

where \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

By the triangle inequality and taking into account that \( u = u_1 - u_2 \) on \( \Gamma_2, r \times [0, T] \) we have, for every \( \xi \in [0, t_0] \),

\begin{equation}
\left( \int_{\Gamma_2, r} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2} \leq \left( \int_{\Gamma_2, r} |\partial_t u_1(x, t)|^2 dS dt \right)^{1/2},
\end{equation}

\begin{equation}
\leq C \left( t_0 \rho_0^{-1} \right)^{1/2} \left( \|\partial_t u\|_{L^\infty(\Gamma_2, r \times [0, t_0])} + \|\partial_t u_2\|_{L^\infty(\Gamma_2, r \times [0, t_0])} \right),
\end{equation}
where $C$ depends on $E$ and $M$ only.

Arguing as in the estimate of $J_1$, by (5.22), (5.31), (5.32) and (4.16a) we have

\begin{equation}
|J_2| \leq C \rho_0^{-2}(t_0 \rho_0^{-1})^{5/2}(H(t_0))^2 \sigma^{1/4} + \\
+ C \rho_0^{-2}(T \rho_0^{-1})^3 (H(T) + 1)^2 \varepsilon_3^{1/2},
\end{equation}

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only.

By (5.26), (5.27), (5.30) and (5.33) we have, for every $t \in (0, t_0]$,

\begin{equation}
\rho_0^{-n} \int_{\Omega \setminus G} u_1^2(x, t) dx \leq C(t_0 \rho_0^{-1})^6 (H(t_0))^2 \sigma^{1/4} + \\
+ C(T \rho_0^{-1})^5 (H(T) + 1)^2 \varepsilon_3^{1/2},
\end{equation}

where $C$ depends on $\alpha, E, M, \lambda$ and $\Lambda$ only. In order to estimate from above the right hand side of (5.34) first we assume

$$
\varepsilon \leq e^{-5}.
$$

Let $\mu$ and $T$ be such that

\begin{equation}
\mu T^2 = \frac{1}{5} |\log \varepsilon|.
\end{equation}

By (5.12), (5.35) we have trivially

\begin{equation}
e^{-\mu T^2/10} + \varepsilon_1 \leq c \varepsilon^{1/2},
\end{equation}

where $c$ is an absolute constant. Hence, taking into account (5.21) and (5.23), we have

\begin{equation}
\varepsilon_3^{1/2} \leq (\mu T^2)^{-1/4} + e^{\mu T^2 (\rho_0 T^{-1})^2 \sigma^2 / 4} \varepsilon_2^{1/2} \leq \\
\leq (\mu T^2)^{-1/4} + C \sigma^{-\frac{1}{2} + 1} e^{\frac{1}{2} \mu K(\sigma, T)},
\end{equation}

where

\begin{equation}
K(\sigma, T) = 2 \rho_0^2 \sigma^2 - \frac{T^2}{5} \vartheta_2^{\frac{1}{2} - n - 1}.
\end{equation}

Let us choose

\begin{equation}
T = T_\sigma = \max \left\{ 2t_0, \sqrt{10} \rho_0 \vartheta_2^{-\frac{1}{2} - \sigma^{-(n+1)}} \right\},
\end{equation}

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and we have $K(\sigma, T_0) \leq -3\rho_0^2$. Hence by (5.37) and (5.39) we have

$$
(5.40) \quad \varepsilon_3^{1/2} \leq C\sigma^{-(\frac{n+1}{2})} (T_0\rho_0^{-1})^{1/2} |\log \varepsilon|^{-1/4},
$$

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only. By (5.34) and (5.40) we have, for every $t \in (0, t_0]$ and $\sigma \in (0, \sigma_1]$,

$$
(5.41) \quad \rho_0^{-n} \int_{\Omega \setminus G} u_1^2(x, t)dx \leq C \left( (t_0\rho_0^{-1})^{6} (H(t_0))^{2\sigma^{1/4} + \Phi(\sigma) |\log \varepsilon|^{-1/4}} \right),
$$

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only and $\Phi(\sigma)$ is defined by (5.2).

Let $\bar{\sigma} = \min\{\sigma_1, (2n|\log \varphi_2|^{1/(n+1)})\}$, by (5.39) we have that $\Phi$ is a decreasing function in $(0, \bar{\sigma})$, so that $\min_{(0, \bar{\sigma})} \Phi = \Phi(\bar{\sigma})$. Now, let us denote by $\bar{\varepsilon} = \min\{e^{-5}, e^{-\Phi(\bar{\sigma})}\}$ and for any $\varepsilon \in (0, \bar{\varepsilon}]$ let us choose $\sigma = \sigma(\varepsilon)$ where $\sigma = \sigma(\varepsilon)$ is defined by (5.4). By (5.41) we have

$$
(5.42) \quad \rho_0^{-n} \int_{\Omega \setminus G} u_1^2(x, t)dx \leq C \omega(\varepsilon, t_0),
$$

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only and $\omega(\varepsilon, t_0)$ is defined by (5.7).

5.2 Step 2

**Proposition 5.2.** Let $\bar{\varrho} \in (0, \rho_0/2E]$ and let $y_0 \in \Omega$ be such that $B_{\bar{\varrho}}(y_0) \subset \Omega$. Assume that $u$ is solution to (1.1). Then there exists a constant $C_F$, $C_F \geq 2$, depending on $E, M, \lambda, \Lambda, \bar{\varrho}, \rho_0^{-1}$ and $F$ only such that if $\bar{t} \geq t_* := \max\{C_F \rho_0, 2t_1\}$ then the following inequality holds true

$$
(5.43) \quad \bar{t}\rho_0^{-1} H(\bar{t}) e^{-\mathcal{F}(\bar{t})} \leq \sup_{t \in [0, \bar{t}]} \|u(\cdot, t)\|_{L^2(B_{\bar{\varrho}(y_0))},
$$

where

$$
(5.44) \quad \mathcal{F}(\bar{t}) = \left( \frac{C_F (\bar{t}\rho_0^{-1})^{3} H(\bar{t})}{H(t_1)} \right)^2.
$$

**Proof.** For any number $\bar{t}$ such that $\bar{t} \geq 2t_1$ let us denote

$$
(5.45) \quad \eta = \sup_{t \in [0, \bar{t}]} \|u(\cdot, t)\|_{L^2(B_{\bar{\varrho}(y_0))}.}
$$

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Let \((x_0, \tau) \in \Gamma^{(a)} \times [0,t_1]\) be such that
\[
\psi(x_0, \tau) = \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0,t_1])}.
\]

Let \(\delta \in (0, \frac{1}{4}]\) be a number that we will choose later and let \(x_\delta = x_0 - 4\delta \bar{\nu}(x_0)\). By Theorem 4.3 and by (5.46) we have
\[
\|\psi\|_{L^\infty(\Gamma^{(a)} \times [0,t_1])} \leq |u(x_0, \tau) - u(x_\delta, \tau)| + |u(x_\delta, \tau)| \leq C_0 t_1 \rho_0^{-2} H(t_1) \delta + |u(x_\delta, \tau)|
\]
where \(C_0\) depends on \(E, M, \lambda\) and \(\Lambda\) only. Now let us choose
\[
\bar{\delta} = \min \left\{ \frac{1}{4}, \frac{\rho_0}{2C_0 t_1 F} \right\}
\]
and by (5.47) we have
\[
\frac{1}{2} \|\psi\|_{L^\infty(\Gamma^{(a)} \times [0,t_1])} \leq \left| u\left(x(\bar{\delta}), \tau\right) \right| .
\]

Now we estimate from above the right hand side of (5.48) in terms of \(\eta\). In order to get such an estimate we proceed similarly to Proposition 5.1. For any positive number \(\eta\) such that \(\mu^2 \geq 1\) and \(\tau \in (0, \bar{\tau}/2]\) denote by \(U^{(\tau)}\) the FBI transform of \(u\) defined by
\[
U^{(\tau)}(x,y) := \sqrt{\frac{\mu}{2\pi}} \int_0^\tau e^{-\frac{1}{2}(iy+\tau-t)^2} u(x,t) dt, \quad \text{for every } (x,y) \in \Omega \times \mathbb{R}.
\]

Denote by \(x_1 = x_0 - \bar{\nu}(x_0)\) where \(\nu(x_0)\) is the exterior unit normal to \(\partial \Omega\) in \(x_0\). Since
\[
\begin{cases}
\partial_y^2 U^{(\tau)} + \text{div} \left( A(x) \nabla U^{(\tau)} \right) = f^{(\tau)}(x,y), & \text{in } \Omega \times \mathbb{R}, \\
\int_{B_{\rho_0}(y_0)} \left| U^{(\tau)}(x,y) \right|^2 dx \leq c \mu^{1/2} t_1 \rho_0^6 \eta^2 \mu^2,
\end{cases}
\]
by arguing as in Proposition 5.1 we get
\[
\|U^{(\tau)}\|_{L^2(B_{\rho_0}(x_1))} \leq C t_1 \rho_0^{-1} H(\bar{\tau}) \mu^2 \times \left( \frac{\|U^{(\tau)}\|_{L^2(B_{\rho_0}(y_0))} e^{-\mu^2/2}}{t_1 \rho_0^{-1} (H(\bar{\tau}) + 1) + e^{-\mu^2/10}} \right)^{\eta \bar{\tau}}.
\]
and

\begin{equation}
\| U_\mu \|_{L^2(B_{\eta/4}(z_1))} \leq C(\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta)e^{\mu\bar{\varrho}^2/2}\left( (\mu\bar{t}^2)^{3/2}e^{-\mu\bar{t}^2/8} + \eta_1 \right)^{\vartheta_1^*},
\end{equation}

where

\begin{equation}
\eta_1 = \frac{(\mu\bar{t}^2)^{1/4}\eta}{H(\bar{t})\bar{t}\rho_0^{-1} + 2\eta},
\end{equation}

\( \vartheta_1^* \in (0, 1) \) depends on \( E, M, \lambda, \Lambda \) and \( \bar{\varrho} \) only and \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

By (4.16), (4.20), (4.22), (5.48) and by applying [G-T, Theorem 8.17] we have

\begin{equation}
\frac{1}{2} \| \psi \|_{L^\infty(\Gamma(\omega) \times [0,t_1])} \leq \left| u(x_{\bar{\varrho}}, \bar{\tau}) - U_\mu^{(\bar{\tau})}(x_{\bar{\varrho}}, 0) \right| + \left| U_\mu^{(\bar{\tau})}(x_{\bar{\varrho}}, 0) \right| \leq C(\mu\bar{t}^2)^{-1/2}(\bar{t}\rho_0^{-1})^3H(\bar{t}) + \| U_\mu^{(\bar{\tau})} \|_{L^\infty(B_{\bar{\varrho}^2/(\bar{\varrho} - 2\bar{\varrho})}(\bar{\varrho}))} \leq C \left( (\mu\bar{t}^2)^{-1/2} + e^{\mu\bar{\varrho}^2/2-\bar{\varrho}^2/10} \right)(\bar{t}\rho_0^{-1})^3H(\bar{t}) + \frac{C'}{\delta^{(n+1)/2}} \| U_\mu^{(\bar{\tau})} \|_{L^2(B_{\bar{\varrho}^2/(\bar{\varrho} - 2\bar{\varrho})}(\bar{\varrho}))},
\end{equation}

where \( C \) depends on \( E, M, \lambda, \Lambda \) and \( \bar{\varrho} \) only and \( \bar{\varrho} \) depends on \( E, M, \lambda, \bar{\varrho}\rho_0^{-1} \) and \( F \) only.

Now let us apply the three sphere inequality (4.29) with \( r_1 = \frac{T}{4}, r_2 = \bar{\varrho}(1 - 2\bar{\varrho}) \) and \( r_3 = \bar{\varrho} \). By (4.22) and (5.51) we have

\begin{equation}
\| U_\mu \|_{L^2(B_{\bar{\varrho}^2/(\bar{\varrho} - 2\bar{\varrho})}(\bar{\varrho}))} \leq C \left( (\bar{t}\rho_0^{-1})^3H(\bar{t}) + 2\eta \right)e^{\mu\bar{\varrho}^2/2}\left( e^{-\mu\bar{t}^2/10} + \eta_1 \right)^{\vartheta^*_2},
\end{equation}

where \( \vartheta^*_2, \vartheta^*_2 \in (0, 1) \), and \( C \) depend on \( E, M, \lambda, \bar{\varrho}, \rho_0^{-1} \) and \( F \) only.

By (5.53) and (5.54) we have

\begin{equation}
\| \psi \|_{L^\infty(\Gamma(\omega) \times [0,t_1])} \leq C(\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta) \left( (\mu\bar{t}^2)^{-1/2} + e^{\mu\bar{\varrho}^2/2-\bar{\varrho}^2/10} \right)^{\vartheta^*_2},
\end{equation}

where \( C \) depends on \( E, M, \lambda, \bar{\varrho} \) and \( F \) only. Now if \( \bar{t} \geq \max \{ \sqrt{\bar{t}}(\vartheta^*_2)^{-1/2}\bar{\varrho}, 2\rho_0, 2t_1 \} \) then (5.52) and (5.55) give

\begin{equation}
\| \psi \|_{L^\infty(\Gamma(\omega) \times [0,t_1])} \leq C \left( (\bar{t}\rho_0^{-1})^3H(\bar{t}) + 2\eta \right) \times \left( (\mu\bar{t}^2)^{-1/2} + (\mu\bar{t}^2)^{\vartheta^*_2/4}e^{\mu\bar{\varrho}^2/20} \left( \frac{\eta}{\bar{t}\rho_0^{-1}H(\bar{t}) + 2\eta} \right)^{\vartheta^*_2} \right),
\end{equation}

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where $C$ depends on $E, M, \lambda, \Lambda, \varrho_0^{-1}$ and $F$ only.

Now let us choose
\[
\mu = \frac{10}{\ell^2} \left| \log \left( \frac{\eta}{\ell \varrho_0^{-1} H(\ell)} + 2 \eta \right) \right|
\]
and by (5.56), taking into account that $\eta \leq CH(\ell)$, we get
\[\|\psi\|_{L^\infty(\Gamma^{(0)} \times [0, t_1])} \leq C(\ell \varrho_0^{-1})^2 H(\ell) \left| \log \left( \frac{\eta}{CT \varrho_0^{-1} H(\ell)} \right) \right|^{-1/2},\]
where $C$ depends on $E, M, \lambda, \Lambda, \varrho_0^{-1}$ and $F$ only. By (5.57) the thesis follows. $\square$

Now we recall the following Lemma that was proved in [Al-B-Ro-Ve, Lemma 8.1].

**Lemma 5.3 (relative graphs).** Let $\Omega_1$ and $\Omega_2$ be bounded domains in $\mathbb{R}^n$ of class $C^{1,1}$ with constants $\rho_0, E$ and satisfying $|\Omega_j| \leq M \rho_0^n$, $j = 1, 2$. There exist numbers $d_0, \rho_0 \in (0, \rho_0]$ such that $\frac{d}{\rho}$ and $\frac{\rho}{d}$ depend on $E$ only, and such that if we have
\[d_H(\Omega_1, \Omega_2) \leq d_0,\]
then the following facts hold true

i) $\Omega_1$ and $\Omega_2$ are relative graphs and
\[\gamma_0 (\Omega_1, \Omega_2) \leq C d_H(\Omega_1, \Omega_2),\]
where $C$ depends $E$ only,

\[\gamma_{1,\alpha} (\Omega_1, \Omega_2) \leq C \rho_0^{\frac{1-\alpha}{2}} (d_H(\Omega_1, \Omega_2))^{\frac{1}{2}} \left( d_H(\Omega_1, \Omega_2) \right)^{\frac{1-\alpha}{2}}, \text{ for every } \alpha \in (0, 1),\]
where $C$ depends $E$ and $\alpha$ only,

ii) there exists an absolute positive constant $c$ such that
\[d_H(\Omega_1, \Omega_2) \leq c d_m(\Omega_1, \Omega_2),\]

iii) $\Omega_1 \cap \Omega_2$ is a domain of Lipschitz class with constants $\rho_0, L$, where $\rho_0$ is as above and $L > 0$ depends on $E$ only.
Proposition 5.4. There exist constants $C_F$ and $C$ depending on $E, M, \lambda, \Lambda$ and $F$ only and on $E, M, \lambda, \Lambda$ only respectively, such that if $t_0 \geq t_* + \lambda \rho_0$ where $t_*$ is introduced in Proposition 5.2 and if

\[(5.62) \quad \sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega \setminus G} u_2^2(x, t) dx \right) \leq \eta^2\]

then

\[(5.63) \quad d_H(\overline{\Omega_1}, \overline{\Omega_2}) \leq C \rho_0 \left( \frac{\eta}{\bar{t}_0 \rho_0^{-1} H(\bar{t}_0)} \right)^{1/K_0},\]

where

\[(5.64) \quad K_0 = \left( \frac{H(t_0)}{H(\bar{t}_0)} \right)^6 e^{C \mathcal{F}(\bar{t}_0)},\]

and $\mathcal{F}(\bar{t}_0)$ is defined by (5.44).

Proof. First we prove the following inequality

\[(5.65) \quad d_m(\overline{\Omega_1}, \overline{\Omega_2}) \leq C \rho_0 \left( \frac{\eta}{\bar{t}_0 \rho_0^{-1} H(\bar{t}_0)} \right)^{1/K_0},\]

where $d_m(\overline{\Omega_1}, \overline{\Omega_2})$ is the quantity introduced in Definition 2.5.

For the sake of brevity let us denote $d_m = d_m(\overline{\Omega_1}, \overline{\Omega_2})$ Let us assume, with no loss of generality, that there exists $x_0 \in \Gamma_1^{(i)} \subset \partial \Omega_1$ such that $\text{dist}(x_0, \Omega_2) = d_m$.

By (5.62) we have trivially

\[(5.66) \quad \sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega_1 \cap B_{d_m}(x_0)} u_2^2(x, t) dx \right) \leq \eta^2\]

let us distinguish the following two cases

i) $d_m \leq \frac{1}{2} \tau_0 \rho_0$,

ii) $d_m > \frac{1}{2} \tau_0 \rho_0$,

where $\tau_0, \tau_0 \in (0, 1)$, is defined in Theorem 4.2 and depends on $E, \lambda$, and $\Lambda$ only.

In case i), by applying Theorem 4.2 with $r_0 = d_m$ and $\rho = \frac{\tau_0 \rho_0}{2}$ we have
\begin{equation}
\sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{s_0\rho_0/2}(x_0) \cap \Omega_1)} \leq C \left( \rho_0^{-1} t_0 H(t_0) \right) \left( \theta_1 \log \left( \frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right)^{-1/6},
\end{equation}

where

\begin{equation}
\theta_1 = \frac{1}{C \log(\rho_0/d_m)}.
\end{equation}

and \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

Now let us introduce the following notation: \( s^* = \min \left\{ \frac{\tau_0}{\pi}, \frac{1}{\pi} \right\} \) and \( y_0 = x_0 - s^* \rho_0 \nu(x_0), \bar{t}_0 = t_0 - \lambda \rho_0 \). We have \( B_{s^*\rho_0/2}(y_0) \subset B_{s_0\rho_0/2}(x_0) \cap \Omega_1 \). Let us assume that \( t_0 \geq \max\{2C_F\rho_0, 2t_1\} \) where \( C_F \) is defined in Proposition 5.2. By (5.67) and Proposition 5.2 we have

\begin{equation}
\bar{t}_0 \rho_0^{-1} H(\bar{t}_0) e^{-F(\bar{t}_0)} \leq C \left( \rho_0^{-1} t_0 H(t_0) \right) \left( \theta_1 \log \left( \frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right)^{-1/6},
\end{equation}

where \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only and \( F(\bar{t}_0) \) is defined by (5.44). Now, taking into account that \( t_0 / \bar{t}_0 \leq 2 \), by (5.68) and (5.69) we have

\[
\frac{1}{K_0} \log \left( \frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \leq \log(\rho_0/d_m),
\]

so that we have

\begin{equation}
d_m \leq \rho_0 \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0},
\end{equation}

where \( K_0 \) is defined in (5.64).

Consider now case ii). Since we have \( B_{s^*\rho_0/2}(y_0) \subset B_{s_0\rho_0/2}(x_0) \cap \Omega_1 \) we get

\begin{equation}
\bar{t}_0 \rho_0^{-1} H(\bar{t}_0) e^{-F(\bar{t}_0)} \leq \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{s_0\rho_0/2}(x_0))} \leq \sup_{t \in [0, t_0 - \lambda \rho_0]} \|u_1(\cdot, t)\|_{L^2(B_{s_0\rho_0/2}(x_0) \cap \Omega_1)} \leq \eta.
\end{equation}
Hence

\begin{equation}
1 \leq \frac{e^{2(t_0) H}}{t_0 \rho_0^4 H(t_0)}.
\end{equation}

Now by the a priori information we have \( d_m \leq C \rho_0 \) where \( C \) depends on \( E \) and \( M \) only. Therefore by (5.72) we have trivially

\begin{equation}
d_m \leq C \rho_0 \leq C \rho_0 \left( \frac{e^{2(t_0) H}}{t_0 \rho_0^4 H(t_0)} \right)^{1/K_0}.
\end{equation}

Therefore in both the cases we have (5.65).

Now we prove (5.63). Let us denote by \( d = d_H(\overline{\Omega}_1, \overline{\Omega}_2) \). With no loss of generality, let \( \overline{y} \in \overline{\Omega}_1 \setminus \overline{\Omega}_2 \) be such that \( \text{dist}(\overline{y}, \overline{\Omega}_2) = d \). Since in general \( \overline{y} \) needs not to belong to \( \partial \Omega_1 \), [Al-B-Ro-Ve] it is necessary to analyze various different cases separately. Denoting by \( h = \text{dist}(\overline{y}, \partial \Omega_1) \), let us distinguish the following three cases:

i) \( h \leq \frac{d}{2} \),

ii) \( h > \frac{d}{2}, h > \frac{d_0}{2} \),

iii) \( h > \frac{d}{2}, h \leq \frac{d_0}{2} \),

where \( d_0 \) is the number introduced in Proposition 5.3.

If case i) occurs, taking \( \overline{z} \in \partial \Omega_1 \) such that \( |\overline{y} - \overline{z}| = h \), we have that \( \text{dist}(\overline{z}, \overline{\Omega}_2) \geq d - h \geq \frac{d}{2} \), so that \( d \leq 2d_m \) and (5.63) follows by (5.65).

Let us now consider case ii). Let us denote

\begin{equation}
d_1 = \min \left\{ \frac{d}{2}, \frac{s_0 d_0}{4} \right\}.
\end{equation}

where \( s_0, s_0 \in (0, 1) \), is defined in Theorem 4.1 and depends on \( \lambda \) and \( \Lambda \) only.

We have that

\begin{equation}
B_{d_0/2}(\overline{y}) \subset \Omega_1 \quad \text{and} \quad B_{d_1}(\overline{y}) \subset \Omega_1 \setminus \overline{\Omega}_2.
\end{equation}

Now by applying Theorem 4.1 with \( r_0 = d_1 \) and \( \rho = \frac{s_0 \rho_0}{2} \) we have

\[
\sup_{t \in [0, t_0 - \lambda \rho_0]} \| u_1(\cdot, t) \|_{L^2(B_{d_1}(\overline{y}))} \leq C \left( \rho_0^{-1} t_0 H(t_0) \right)^{-1/6} \left( \theta_2 \log \left( \frac{\rho_0^{-1} t_0 H(t_0)}{\eta} \right) \right),
\]

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where
\[ \theta_2 = \frac{1}{C \log(\rho_0/d_m)}, \]
and \( C \) depends on \( \lambda \) and \( \Lambda \) only.

Now proceeding exactly as in the proof of (5.70) we have

(5.76)
\[ d_1 \leq \rho_0 \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0}, \]
where \( K_0 \) is defined by (5.64) (perhaps with a different value of constant \( C \)).

Now, if
\[ \rho_0 \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0} < \frac{s_0 d_0}{4}, \]
then by (5.76) we have \( d_1 < \frac{s_0 d_0}{4} \), hence \( d_1 = \frac{d}{2} \). Therefore we get

(5.77)
\[ d = 2d_1 \leq 2\rho_0 \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0}. \]

If, instead, we have
\[ \rho_0 \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0} \geq \frac{s_0 d_0}{4}, \]
we have trivially

(5.78)
\[ d \leq C \rho_0 \leq \frac{4C \rho_0^2}{s_0 d_0} \left( \frac{\eta}{\rho_0^{-1} t_0 H(t_0)} \right)^{1/K_0}, \]
where \( C \) depends on \( E \) and \( M \) only.

If case iii) occurs we have in particular that \( d < d_0 \), hence by Proposition 5.3 we have \( d \leq cd_m \) and by (5.65) the thesis follows again.

Corollary 5.5. Let \( t_* \) be defined in Proposition 5.4 and let \( t_0 \geq t_* \) be fixed. We have for every \( \varepsilon \in (0, \bar{\varepsilon}] \), \( \bar{\varepsilon} \) is defined in Proposition 5.1,

(5.79)
\[ d_H(\overline{\Omega}_1, \overline{\Omega}_2) \leq \rho_0 \omega_1(\varepsilon, t_0), \]
where

(5.80)
\[ \omega_1(\varepsilon, t_0) := C \left( \frac{\omega(\varepsilon, t_0)}{t_0 \rho_0^{-1} H(t_0)} \right)^{1/K_0}, \]
\( \omega(\varepsilon, t_0) \) is defined by (5.7) and \( C \) on \( E, M, \lambda, \Lambda \) and \( F \) only and \( K_0 \) is defined in (5.64).

Proof. Inequality (5.79) is an immediate consequence of Proposition 5.1 and Proposition 5.4.
5.3 Step 3

Now we conclude the proof of the main Theorem.

Let \( t_0 \geq t_* \) be fixed and let \( d_0 \) be defined in Proposition 5.3 and let \( s \in (0, \frac{d_0}{\rho_0}] \) be a number that we will choose later. Denote by

\[
\epsilon(s) = \sup \{ \epsilon \in (0, \bar{\epsilon}] : \omega_1(\epsilon, t_0) \leq s \}.
\]

By Proposition 5.3 we have that, for every \( s \in (0, \frac{d_0}{\rho_0}] \) and every \( \epsilon \in (0, \epsilon(s)] \), \( \partial \Omega_1 \) and \( \partial \Omega_2 \) are relative graphs, moreover \( G \) it is equal to \( \Omega_1 \cap \Omega_2 \) and is a domain of Lipschitz class with constants \( CE \) and \( \rho_0/C \) where \( C \geq 1 \) depends on \( E \) only.

We have

\[
\partial (\Omega_1 \setminus G) \subset \Gamma_1^{(i)} \cup \left( \Gamma_2^{(i)} \cap \partial G \right),
\]

Denote by \( u = u_1 - u_2 \). By Schwarz inequality, energy inequality, (4.16a), (4.16c) and recalling that \( u_2 = 0 \) on \( \Gamma_2^{(i)} \) we have, for any \( t \in (0, t_0] \),

\[
(5.81) \quad \rho_0^{-n} \int_{\Omega_1 \setminus G} u_1^2(x, t) dx \leq t_0 \rho_0^{-n} \int_0^{t_0} \int_{\Omega_1 \setminus G} |\partial_\xi u_1(x, \xi)|^2 \, dxd\xi \leq C(t_0 \rho_0^{-1})^{5/2} \left( H(t_0) \right)^{3/2} \|u\|^{1/2}_{L^\infty((\Gamma_2^{(i)} \cap \partial G) \times [0, t_0])},
\]

where \( C \) depends on \( \alpha, E, M, \lambda \) and \( \Lambda \) only.

Let \( P \in \partial G \), without restriction we may assume that \( P \equiv 0 \). By (5.79) and Proposition 5.3 we have that if \( s \in (0, \frac{d_0}{\rho_0}] \) and \( \epsilon \in (0, \epsilon(s)] \) then there exist \( \varphi_1, \varphi_2 \in C^{1,1}(B_{r_0}^0(0)) \), where \( \frac{d_0}{\rho_0} \leq 1 \) depends on \( E \) only, satisfying the following conditions

\[
(5.82a) \quad \|\varphi_i\|_{C^{1,1}(B_{r_0}^0(0))} \leq E \rho_0,
\]

(5.82b) \[ \Omega_i \cap B_{r_0}^0(0) = \{ x \in B_{r_0}^0(0) : x_n > \varphi_i(x') \}, \ i = 1, 2. \]

It is not restrictive to assume that

\[
(5.83) \quad \varphi_1(0) = |\nabla x' \varphi_1(0)| = 0, \ \varphi_2(0) \leq 0.
\]

Now, let us denote by \( \varphi = \max\{\varphi_1, \varphi_2\} \) and by \( d_1 = \min\{d_0, r_0\} \) By (5.79) and (5.60) we have (we fix \( \alpha = 1/2 \), for every \( s \in (0, \frac{d_0}{\rho_0}] \) and every \( \epsilon \in (0, \epsilon(s)] \),

\[ 30 \]
\[(5.84) \quad \| \nabla x' \varphi \|_{L^\infty(B_{r_0}')} \leq L_s \equiv C_s s^{1/4}, \]

where \( C_s \) depends on \( E \) only.

For any \( s \in (0, \frac{d_1}{\rho_0}] \) let us introduce the following notation

\[(5.85) \quad T_s := \max \{ T(\varepsilon(s)), 2t_0 \}, \]

where \( T(\varepsilon) \) is defined in (5.3),

\[(5.86) \quad \gamma = \arctan \frac{1}{L_s}. \]

Moreover let \( \gamma_1, \gamma_2 \) be two numbers such that \( 0 < \gamma_1 < \gamma_2 < \gamma < \frac{\pi}{2} \) that we will choose later and let

\[(5.87a) \quad \chi = \frac{1 - \sin \gamma_2}{1 - \sin \gamma_1}, \]

\[(5.87b) \quad l_1 = \frac{sL_5\rho_0/2}{1 + \sin \gamma}, \]

\[(5.87c) \quad l_k = \chi^{k-1}l_1, \quad k \in \mathbb{N}, \]

\[(5.87d) \quad w_k = l_k e_n, \quad k \in \mathbb{N}, \]

\[(5.87e) \quad R_k = l_k \sin \gamma, \quad \rho_k = l_k \sin \gamma_2, \quad r_k = l_k \sin \gamma_1, \quad k \in \mathbb{N}. \]

It is easy to check that, denoting by \( \mathcal{C} \) the cone

\[ \mathcal{C} = \left\{ x \in \mathbb{R}^n : L \cdot x' \leq x_n \leq \frac{sL_5\rho_0}{2} \right\}, \]

we have

\[(5.88) \quad B_{r_{k+1}}(w_{k+1}) \subset B_{\rho_k}(w_k) \subset B_{R_k}(w_k) \subset \mathcal{C} \subset G, \quad \text{for every } k \in \mathbb{N} \]

and

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(5.89) \[ \text{dist} (B_{r_1}(w_1), \partial G) \geq \frac{1}{2} \rho_0 h, \]

where

(5.90) \[ h = \frac{\sin \gamma - \sin \gamma_1}{1 + \sin \gamma}. \]

Let \( T \geq T_0 \) be a number that we will choose. For any positive number \( \mu \) such that \( \mu T^2 \geq 1 \) and \( \tau \in (0, T/2] \) denote by \( U^{(\tau)}_\mu \) the FBI transform of \( u \) defined by

(5.91) \[ U^{(\tau)}_\mu (x, y) = \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\tau}{2}(y+y')^2} u(x, t) dt, \quad \text{for } (x, y) \in G \times \mathbb{R}. \]

Let \( \kappa_0 \leq 1 \) be such that \( G_r \) is connected for every \( r \in (0, \kappa_0 \rho_0) \) [Al-Ro-Ve]. Let \( \kappa_1 = \min\{\kappa_0, \kappa_0\} \). Arguing as in Proposition 5.1 we have by (3.9), for every \( s \in (0, \kappa_1] \) and every \( \epsilon \in (0, \epsilon(s)] \),

(5.92) \[ \|U^{(\tau)}_\mu\|_{L^2 (\overline{B}_{r_1}(\bar{w}_1))} \leq \]

\[ \leq CT\rho_1^{-1} (H(T) + 1) e^{\mu(s\rho)^2/2} \left( e^{2\rho_0^2} \left( e^{-\mu T^2/10} + \epsilon_1 \right)^{\vartheta_2 (h_{s/2})^{-n}} \right), \]

where \( \vartheta_2 \in (0, 1) \) is the same exponent of inequality (5.19), \( \vartheta_2, C \) depend on \( E, M, \lambda \) and \( \Lambda \) only and

(5.93) \[ \epsilon_1 = \frac{(\mu T^2)^{1/4} \vartheta}{(H(T) + 1) T \rho_0^{1/4}}. \]

Now we apply inequality (4.29) when \( \tilde{r}_1 = r_k, \tilde{r}_2 = \rho_k, \tilde{r}_3 = R_k \) and \( x_0 = w_k, k \in \mathbb{N}. \)

Let us denote by

(5.94) \[ \alpha_k = e^{-\mu T^2/10} + \frac{e^{-\mu R_k^2/2} \|U^{(\tau)}_\mu\|_{L^2 (\overline{B}_{r_k} (\bar{w}_k))}}{(H(T) + 1) T \rho_0^{1/4}}. \]
Taking into account (5.88) we have

\begin{equation}
\alpha_{k+1} \leq C_0 E^{\frac{a}{2}} (R_{k+1}^2 - R_k^2) \tilde{\varphi}_0^k, \text{ for every } k \in \mathbb{N},
\end{equation}

where

\begin{equation}
\tilde{\varphi}_0 = \frac{\rho_1^{\beta_1} - [(1 - \delta)R_1]^{-\beta_1}}{[(1 - 2\delta)R_1]^{-\beta_1} - [(1 - \delta)R_1]^{-\beta_1}},
\end{equation}

\begin{equation}
0 < \delta \leq \frac{R_1 - \rho_1}{2R_1},
\end{equation}

\begin{equation}
\tilde{C}_0 = C \frac{C^{(\beta_1 - (1 - \delta) - \beta_1)}}{\delta^4},
\end{equation}

\beta_1 has been introduced in Theorem 4.6 and \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

Notice that

\[ R_k^2 - R_{k+1}^2 = \chi^{2k} R_1^2 (\chi^{-2} - 1). \]

By iterating (5.95) we get

\begin{equation}
\alpha_{k+1} \leq (C\tilde{C}_0)^{1/(1 - \tilde{\varphi}_0)} \left( e^{\mu R_k^2 A_k / 2 \alpha_1} \right) \tilde{\varphi}_0^k, \quad k \in \mathbb{N},
\end{equation}

where

\begin{equation}
A_k = (\chi^{-2} - 1) (\chi^{2 \tilde{\varphi}_0^{-1}}) \frac{1}{1 - (\chi^{2 \tilde{\varphi}_0^{-1}})}, \quad k \in \mathbb{N}.
\end{equation}

Let \( \kappa_2 = \min\{\kappa_1, 2 (\log_4 \tilde{\varphi}_0)^{1/n}\} \) and taking into account that, by (5.90), \( h \leq 1 \), from (5.92) and (5.99) we get that for every \( s \leq \kappa_2 \) the following inequality holds true

\begin{equation}
\left\| U_{\mu}^{(r)} \right\|_{L^2(\bar{B}_{r+1}(\tilde{\varphi}_{k+1})^2)} \leq \frac{1}{(H(T) + 1)} \left( e^{\mu A_k^{(1)} \bar{\varphi}_1^{(s/2)^{-n}}} + e^{\mu A_k^{(2)}} \right) \tilde{\varphi}_0^k, \quad k \in \mathbb{N},
\end{equation}
where

\begin{equation}
A_{s,k}^{(1)} = \frac{1}{2} \left( A_k + (\chi^2 \tilde{\vartheta}_0^{-1}) \right) R_1^2 + \rho_0^2, \quad k \in \mathbb{N},
\end{equation}

\begin{equation}
A_{s,k}^{(2)} = A_{s,k}^{(1)} - \frac{1}{10} T^2 \varphi_2^{1+(sh/2)-n}, \quad k \in \mathbb{N},
\end{equation}

and \( C \) depends on \( E, M, \lambda \) and \( \Lambda \) only.

Since we need that \( A_{s,k}^{(1)} \) is bounded for \( k \in \mathbb{N} \) we search for which \( s \in (0, \kappa_2] \) we have

\begin{equation}
\chi^2 \tilde{\vartheta}_0^{-1} < 1.
\end{equation}

Let \( \varsigma, a, b, q \in (0, 1) \) three numbers that we will fix later on and let

\begin{equation}
\sin \gamma_1 = 1 - \varsigma, \quad \sin \gamma_2 = 1 - a \varsigma, \quad \sin \gamma = 1 - ab \varsigma
\end{equation}

and

\begin{equation}
\delta = q \left( \frac{R_1 - \rho_1}{2R_1} \right) = \frac{q a (1-b) \varsigma}{2 \ (1-ab)},
\end{equation}

by (5.87) and (5.96) we have respectively

\begin{equation}
\chi = a,
\end{equation}

and

\begin{equation}
\tilde{\vartheta}_0 = \frac{a(1-b)(1-q/2)}{1-ab+qa(1-b)/2} + o(1), \quad \text{as } \varsigma \to 0.
\end{equation}

In order that (5.104) is satisfied it is enough that

\begin{equation}
a^2 < \frac{a(1-b)(1-q/2)}{1-ab+qa(1-b)/2},
\end{equation}

for instance if we choose

\begin{equation}
q = \frac{1}{2}, \quad a = \frac{1}{4}, \quad b = \frac{1}{3}
\end{equation}
then (5.109) is satisfied and we have

\begin{equation}
\chi^2 \vec{y}_0^{-1} = \frac{23}{48} + o(1), \quad \text{as } \varsigma \to 0.
\end{equation}

By (5.111) we have that there exists $\varsigma_0 > 0$ such that if $0 < \varsigma \leq \varsigma_0$ then

\begin{equation}
\chi^2 \vec{y}_0^{-1} \leq \frac{1}{2}.
\end{equation}

Let

$$\varsigma_1 = \left(1 - \left(1 + C_* \zeta^2 \right)^{-1/2}\right)^{1/2},$$

where $C_*$ is defined in (5.84) and depends on $E$ only.

Now, let us fix $\varsigma = \overline{\varsigma} := \min \{\varsigma_0, \varsigma_1, \frac{1}{4}\}$ and denote by $\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}$ the numbers belonging to $(0, \frac{\pi}{2})$ such that

\begin{equation}
\sin \overline{\tau}_1 = 1 - \varsigma, \quad \sin \overline{\tau}_2 = 1 - \frac{1}{4} \varsigma, \quad \sin \overline{\tau} = 1 - \frac{1}{12} \varsigma
\end{equation}

and denote by

\begin{equation}
\overline{\tau} = \frac{1}{C_*^4} \left( (1 - \varsigma/12)^{-4} - 1 \right).
\end{equation}

Notice that (5.84), (5.114) and the third equality of (5.113) imply that equality (5.86) is satisfied. Namely we have

$$\overline{\tau} = \arctan \frac{1}{L_{\overline{\tau}}}.$$

Now for any quantity $g$ introduced in (5.87), (5.90), (5.96) and (5.106) we denote by $\overline{g}$ the value of such a quantity when $s = \overline{\tau}$ or, equivalently, when $\varsigma = \overline{\varsigma}$. In particular we have

\begin{equation}
\overline{\delta} = \frac{\varsigma}{24 - 2\overline{\varsigma}},
\end{equation}

\begin{equation}
\overline{h} = \frac{\sin \overline{\tau} - \sin \overline{\tau}_1}{1 + \sin \overline{\tau}} = \frac{2\overline{\tau}}{24 - 3\overline{\varsigma}},
\end{equation}

\begin{equation}
\overline{\vartheta}_0 = \frac{\left( \frac{\sin \overline{\tau}_2}{\sin \overline{\tau}} \right)^{-\beta_1} - (1 - \overline{\delta})^{-\beta_1}}{\left( 1 - 2\overline{\delta} \frac{\sin \overline{\tau}_1}{\sin \overline{\tau}} \right)^{-\beta_1} - (1 - \overline{\delta})^{-\beta_1}}.
\end{equation}
By (5.100), (5.112), (5.102) and (5.114) we have

$$A_{\pi,k}^{(1)} \leq 2\rho_0^2, \quad k \in \mathbb{N}. \tag{5.118}$$

Let

$$\tilde{T} = \max\left\{\left(40\vartheta_2^{-1-(\tau_\pi/2)-\nu}\right)^{1/2}\rho_0, T_\pi\right\}. \tag{5.119}$$

By (5.103), (5.118) and (5.119) we have, for every $T \geq \tilde{T}$,

$$A_{\pi,k}^{(2)} \leq -\frac{1}{20}T^2\vartheta_2^{1+(\tau_\pi/2)-\nu}, \quad k \in \mathbb{N}. \tag{5.120}$$

By (5.118), (5.120) we have, for every $T \geq \tilde{T}$,

$$\|U^{(r)}\|_{L^2(\tilde{\theta}_{k+1}(\pi_k+1))} \leq \overline{C}_0(H(T) + 1)\rho_0^{-1}\left(e^{2\rho_0^2\varepsilon_1^3} + e^{-\frac{1}{\pi}\mu T^2\delta_3}\right)^{\delta_0}, \quad k \in \mathbb{N}, \tag{5.121}$$

where

$$\overline{C}_0 = (C\bar{C}_0)^{1/(1-\nu_0)},$$

$$\varepsilon_1 = \frac{(\mu T^2)^{1/4}\varepsilon}{(H(T) + 1)(\rho_0^{-1} + 1)},$$

$$\delta_3 = \vartheta_2^{1+(\tau_\pi/2)-\nu}$$

and $C$ depends on $E, M, \lambda$ and $\Lambda$ only.

Denote by

$$d_1 = \bar{t}_1 \left(1 - \sin \pi_1\right), \quad d_k = \chi^{k-1}d_1, \quad \text{for every } k \in \mathbb{N}$$

here, we recall that by (5.107) and (5.110) we have $\chi = \frac{1}{4}$ and, by (5.87b)

$$\bar{t}_1 = \frac{C^2\pi/2\rho_0/2}{1+\sin \pi_1}. \tag{5.122}$$

Let $r \in (0, d_1]$ be a number that we will choose later on. Let us denote by $\varrho = rd_1^{-1}$,

$$k_0 = \min\{k \in \mathbb{N} : d_k \leq r\}$$

and

$$\delta_4 = \left|\log_4 \vartheta_0\right|. \tag{5.123}$$

We have

$$|\log_4(\varrho/4)| \leq k_0 < |\log_4(\varrho/16)|$$
and

\[(5.123) \quad \overline{\partial}_0^2 \varrho^\delta_1 \leq \overline{\partial}_0 \leq \overline{\partial}_0^2 \varrho^\delta_1.\]

Now by applying [G-T, Theorem 8.17], (4.16), (4.20) and (4.22) we have, for every \(t \in (0, t_0]\) and every \(T \geq \overline{T},\)

\[(5.124) \quad |u(0, \tau)| \leq |u(0, \tau) - u(\overline{w}_{k_0+1}, \tau)| + |u(\overline{w}_{k_0+1}, \tau) - U^{(\tau)}_{\mu} \overline{w}_{k_0+1}| + \left| U^{(\tau)}_{\mu} \overline{w}_{k_0+1} \right| \leq C t_0 \rho_0^{-1} H(t_0) \varrho + C \left( T \rho_0^{-1} \right)^2 H(T) \left( T^2 \mu \right)^{-1/2} + C \varrho^{-\left( \frac{n+1}{2} \right)} (H(T) + 1) T \rho_0^{-1} \left( e^{2 \mu \rho_0^2 \varepsilon^\delta_1} + e^{-\frac{1}{20} \mu T^2 \delta_5} \right) \overline{\varrho}_0 \varrho^\delta_5 \]

where \(C\) depends on \(E, M, \lambda, \Lambda\) only. Now we have trivially

\[(5.125) \quad e^{2 \mu \rho_0^2 \varepsilon^\delta_1} + e^{-\frac{1}{20} \mu T^2 \delta_5} \leq e^{2 \mu T^2 \varepsilon^\delta_3} + e^{-\frac{1}{20} \mu T^2 \delta_5}.\]

Hence, if

\[\varepsilon \leq e^{-\left(2/\delta_3 + 1/20\right)}\]

then we choose

\[\mu = \frac{1}{T^2} \frac{\delta_3 \log \varepsilon}{2 + \delta_3/20}\]

and by (5.124) and (5.125) we have

\[(5.126) \quad |u(0, \tau)| \leq C t_0 \rho_0^{-1} H(t_0) \varrho + C \left( T \rho_0^{-1} \right)^2 H(T) |\log \varepsilon|^{-1/2} + C \varrho^{-\left( \frac{n+1}{2} \right)} (H(T) + 1) T \rho_0^{-1} \varepsilon^\delta_5 \varrho^\delta_4 \]

where \(C\) depends on \(E, M, \lambda, \Lambda\) only and

\[\delta_5 = \frac{\delta_3 \varrho^2}{40 + \delta_3}.\]

Now let us choose

\[\varrho = |\log \varepsilon|^{-1/(2 \delta_4)}\]

and by (5.126) we have

\[(5.127) \quad |u(0, \tau)| \leq C \left( T \rho_0^{-1} \right)^2 (H(T) + 1) |\log \varepsilon|^{-1/2}\]

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where $C$ depends on $E, M, \lambda$ and $\Lambda$ only. 
Otherwise, if

$$\varepsilon \geq e^{-(2/\delta_3+1/20)}$$

then by (4.16c) we have trivially

(5.128) \hspace{1cm} |u(0, \tau)| \leq C t_0 \rho_0^{-1} H(t_0) \leq C e^{(2/\delta_3+1/20)} t_0 \rho_0^{-1} H(t_0) \varepsilon.

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only. By (5.127) and (5.128) we have, for $0 < \varepsilon < e^{-1}$ and every $T \geq \tilde{T}$

(5.129) \hspace{1cm} \|u\|_{L^\infty((T^0 \cap \partial G) \times [0, t_0])} \leq C (T \rho_0^{-1})^2 (H(T) + 1) \log \varepsilon^{-1/2}

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only. By (5.129) and (5.81) we have

(5.130) \hspace{1cm} \sup_{t \in [0, t_0]} \left( \rho_0^{-n} \int_{\Omega \setminus G} u_j^2(x, t) dx \right) \leq C \left[ t \rho_0^{-1} + 1 \right]^{5/2} (H(t_0))^{3/2} \left( T \rho_0^{-1} \right) (H(T) + 1)^{1/2} \log \varepsilon^{-1/4}

where $C$ depends on $E, M, \lambda$ and $\Lambda$ only.

Now we fix $t_0 = t_\ast + \lambda \rho_0$ and $T = \tilde{T}$ and by (5.130) and Proposition 5.4 we have

(5.131) \hspace{1cm} d_H(\overline{\Omega}_1, \overline{\Omega}_2) \leq K_1 \rho_0 |\log \varepsilon|^{-1/(8K_0)},

where

$$K_0 = \left( \frac{H(t_0)}{H(t_\ast)} \right)^6 e^{C \mathcal{F}(t_\ast)},$$

$$K_1 = C \left( \frac{H(\tilde{T})}{H(t_\ast)} \right)^{1/(8K_0)},$$

where $\mathcal{F}(t_\ast)$ is defined by (5.44) and $C$ depends on $E, M, \lambda$ and $\Lambda$ only. $\square$

6 Appendix

6.1 Proof of Theorem 4.3

Theorem 4.3 is a straightforward consequence of Theorem 6.1 below and of standard results concerning the extension of function
Theorem 6.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ that satisfies (3.1). Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8). Let $m := \lceil \frac{n+2}{4} \rceil$. Assume that $\partial_k^k F \in L^\infty(\Omega \times (0, T))$ for every $k \in \{0, \cdots, 2m+2\}$ and let $u \in \mathcal{W}(\{0, T\}; \Omega)$ be the solution to the problem

\[
\begin{cases}
\partial_t^2 u - \text{div}(A(x) \nabla x u) = F(x, t), & \text{in } \Omega \times [0, T], \\
u = 0, & \text{on } \partial \Omega \times [0, T], \\
u(\cdot, 0) = \partial_t \nu(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases}
\] (6.1)

Let $\alpha \in (0, 1)$. Then for every $t \in [0, T]$ we have $u(\cdot, t) \in C^{1,\alpha}(\Omega)$ and the following inequalities hold true

\[
\begin{align}
\|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)} & \leq C \rho_0^{-2} \left( \rho_0^{2m+3} TF_{2m+2} + \sum_{j=0}^{m} \rho_0^{2j+2} F_j \right), \\
\|u(\cdot, t)\|_{C^{1,\alpha}(\Omega)} & \leq C \left( \rho_0^{2m+3} TF_{2m+2} + \sum_{j=0}^{m} \rho_0^{2j+2} F_j \right),
\end{align}
\] (6.2a, 6.2b)

where $F_j := \|\partial_t^j F\|_{L^\infty(\Omega \times [0, T])}$ for every $j \in \mathbb{N} \cup \{0\}$ and $C$ depends on $\alpha, n, E, M$ and $\lambda$ only.

In order to prove Theorem 6.1 we use propositions 6.2, 6.3 given below.

Proposition 6.2. Assume that $\Omega$ and $A(x)$ are the same of Theorem 6.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ that satisfies (3.1a) and whose boundary is of class $C^{1,1}$. Let $A(x)$ be a real-valued symmetric $n \times n$ matrix satisfying (3.8). If $f \in L^p(\Omega)$, $p \in (1, \infty)$, then the solution $v$ to the Dirichlet problem

\[
\begin{cases}
\text{div}(A(x) \nabla v) = f, & \text{in } \Omega, \\
v \in H_0^1(\Omega),
\end{cases}
\] (6.3)

belongs to $W^{2,p}(\Omega)$ and the following estimate holds true

\[
\|v\|_{W^{2,p}(\Omega)} \leq C \rho_0^2 \|f\|_{L^p(\Omega)},
\] (6.4)

where $C$ depends on $\lambda, \Lambda, E, M$ and $p$ only.

Proof. The Proposition is an immediate consequence of [G-T, Theorem 9.15] and [G-T, Lemma 9.17].
Proposition 6.3. Assume that $\Omega$ and $A(x)$ are the same of Theorem 6.1. Let $F \in L^2(\Omega \times (0, T))$ and let $u \in W([0, T]; \Omega)$ be the solution to the problem

$$
\begin{aligned}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div}(A(x)\nabla u) = F, & \text{in } \Omega \times [0, T], \\
u = 0 & \text{on } \partial \Omega \times [0, T], \\
u(\cdot, 0) = \partial_t u(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases}
\end{aligned}
$$

Then the following inequality holds true

$$
\|u(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_{p_0} T \|F\|_{L^\infty(\Omega \times (0, T))}, \quad \text{for every } t \in (0, T),
$$

where $p_0$ is the Sobolev imbedding exponent, namely

$$
p_0 = \frac{2n}{n-2}, \quad \text{for } n > 2,
$$

$$
p_0 \text{ is an arbitrary number of } [2, +\infty), \quad \text{for } n = 2
$$

and $C$ depends on $n, E, M$ and $\lambda$ only.

Proof. Let $\tau \in (0, T]$. By multiplying both the sides of the first equation in (6.5) by $\partial_t u$ and by integrating over $\Omega \times (0, \tau)$ we get

$$
\int_0^\tau \int_\Omega F \partial_t u \, dx \, dt = \frac{1}{2} \int_0^\tau \int_\Omega (A(x) \nabla u \cdot \nabla u + (\partial_t u)^2) \, dx \, dt = \frac{1}{2} \int_\Omega (A(x) \nabla u \cdot \nabla u + (\partial_t u)^2) \, dx.
$$

Hence, denoting by

$$
K(\tau) = \int_\Omega (A(x) \nabla u(x, t) \cdot \nabla u(x, t) + (\partial_t u(x, t))^2) \, dx,
$$

we get

$$
K(\tau) \leq 2 \int_0^\tau \int_\Omega |F| |\partial_t u| \, dx \, dt \leq T \int_0^\tau \int_\Omega F^2 \, dx \, dt + \frac{1}{T} \int_0^\tau \int_\Omega (\partial_t u)^2 \, dx \, dt \leq T \int_0^\tau \int_\Omega F^2 \, dx \, dt + \frac{1}{T} \int_0^\tau K(t) \, dt.
$$

By Gronwall inequality we derive the energy inequality

$$
K(\tau) \leq eT \int_0^\tau \int_\Omega F^2 \, dx \, dt.
$$
In particular (6.8) gives

\[(6.9) \quad \int_\Omega |\nabla u(x,t)|^2 \, dx \leq e\lambda^{-1} T \int_0^T \int_\Omega F^2 \, dx \, dt.\]

Since \(u(\cdot, t) \in H^1_0(\Omega)\), by (6.9) and the Poincaré inequality we have

\[(6.10) \quad \|u(\cdot, t)\|_{H^1(\Omega)} \leq C \rho_0 T \|F\|_{L^\infty(\Omega \times (0,T))}.\]

Finally by the imbedding Sobolev theorem the thesis follows. \(\Box\)

**Sketch of the proof of Theorem 6.1.**

In this sketch of the proof we skip on the question of regularity of the solution for which we refer to [Co] and we focus on the proof of inequality (6.2).

In order to estimate \(\|\partial_t^2 u(\cdot, t)\|_{L^\infty(\Omega)}\), for every \(t \in (0, T)\) we distinguish two cases: (a) \(n\) is not of the type \(4h + 2, \, h \in \mathbb{N} \cup \{0\}\), (b) \(n\) is of the type \(4h + 2, \, h \in \mathbb{N} \cup \{0\}\).

**Case (a).** Denote by \(p_k, \, k \in \mathbb{N} \cup \{0\}\), the sequence such that

\[\frac{1}{p_k} = \frac{1}{p_0} - \frac{2k}{n}, \quad \text{for } k \in \mathbb{N} \cup \{0\}.\]

Notice that

\[\frac{1}{p_k} = \frac{1}{p_{k-1}} - \frac{2}{n}, \quad \text{for } k \in \mathbb{N}\]

and that

\[\frac{1}{p_{m-1}} - \frac{2}{n} > 0, \quad \text{and } \frac{1}{p_m} - \frac{2}{n} < 0.\]

Let us denote

\[u^{(j)} := \partial_t^j u \quad \text{for every } j \in \{0, \ldots, 2m + 2\}.\]

By (6.1) we have, for every \(j \in \{0, \ldots, 2m + 2\}, \)

\[(6.11) \quad \begin{cases} \partial_t^2 u^{(j)} - \text{div} \left( A(x) \nabla_x u^{(j)} \right) = \partial_t^j F, & \text{in } \Omega \times [0, T], \\ u^{(j)} = 0, & \text{on } \partial \Omega \times [0, T], \\ u^{(j)}(\cdot, 0) = \partial_t u^{(j)}(\cdot, 0) = 0, & \text{in } \Omega. \end{cases}\]

Observe that, since \(u^{(2j+2)} = \partial_t^2 u^{(2j)}\), by (6.11) we have that \(u^{(2j)}\) is the solution to the following Dirichlet elliptic problem

\[(6.12) \quad \begin{cases} \text{div} \left( A(x) \nabla_x u^{(2j)} \right) = u^{(2j+2)} - \partial_t^{2j} F, & \text{in } \Omega, \\ u^{(2j)} \in H^1_0(\Omega). \end{cases}\]
hence by Proposition 6.2 we have, for every \( j \in \{1, \cdots, m\} \) and \( t \in (0, T) \),

\[
\|u^{(2j)}(\cdot, t)\|_{W^{2,p_{m-j}}(\Omega)} \leq C\rho_0^2 \left( \|u^{(2j+2)}(\cdot, t)\|_{L^{p_{m-j}}(\Omega)} + F_{2j} \right),
\]

where \( C \) depends on \( \lambda, \Lambda, E \) and \( M \) only. Hence by Sobolev imbedding theorem we get, for every \( j \in \{2, \cdots, m\} \) and \( t \in (0, T) \),

\[
\|u^{(2j)}(\cdot, t)\|_{L^{p_{m-j+1}}(\Omega)} \leq C_0\rho_0^2 \left( \|u^{(2j+2)}(\cdot, t)\|_{L^{p_{m-j}}(\Omega)} + F_{2j} \right),
\]

and

\[
\|u^{(2)}(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_0\rho_0^2 \left( \|u^{(4)}(\cdot, t)\|_{L^{p_{m-2}}(\Omega)} + F_2 \right),
\]

where \( C_0 \geq 1 \) depends on \( n, \lambda, \Lambda, E \) and \( M \) only. Now by applying Proposition 6.3 to \( u^{(2m+2)} \) we have, for every \( t \in (0, T) \),

\[
\|u^{(2m+2)}(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C_1 \rho_0 TF_{2m+2},
\]

where \( C_1 \geq 1 \) depends on \( n, \lambda, E \) and \( M \) only. Therefore, by iterating (6.14) and by (6.15) and (6.16) we get, for every \( t \in (0, T) \),

\[
\|u^{(2)}(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_1 C_0^m \left( \rho_0^{2m+1} TF_{2m+2} + \sum_{j=0}^{m} \rho_0^{2j} F_{2j} \right),
\]

Now since \( u = u^{(0)} \), by (6.12) and by [G-T, Theorem 8.33] we get, for every \( t \in (0, T) \),

\[
\|u(\cdot, t)\|_{C^{1,\alpha}(\Omega)} \leq C \rho_0^2 \left( \|u^{(2)}(\cdot, t)\|_{L^{\infty}(\Omega)} + F_0 \right),
\]

where \( C \) depends on \( \alpha, n, E, M, \lambda \) and \( \Lambda \) only. By (6.18) and (6.17) we obtain (6.2) in the case (a).

**Case (b)** We consider only the case \( n > 2 \), because if \( n = 2 \) we can proceed similarly.

If \( n \) is of the type \( 4h + 2, h \in \mathbb{N} \cup \{0\} \) then inequality (6.13) still holds, but by Sobolev imbedding Theorem, instead of inequality (6.15) we have, for every \( q \in [2, \infty) \),

\[
\|u^{(4)}(\cdot, t)\|_{L^{q}(\Omega)} \leq C_2 \rho_0^2 \left( \|u^{(6)}(\cdot, t)\|_{L^{p_{m-1}}(\Omega)} + F_4 \right),
\]

where \( C_2 \geq 1 \) depends on \( n, \lambda, \Lambda, E, M \) and \( q \) only.
Let us choose $q > \frac{n}{2}$, by applying [G-T, Theorem 8.29] to $u^{(2)}(\cdot, t)$ we have

$$
\|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 \rho_0^2 \left( \|u^{(4)}(\cdot, t)\|_{L^q(\Omega)} + F_2 \right),
$$

where $C_3 \geq 1$ depends on $n, \lambda, E, M$ and $q$.

Now, by iterating (6.13) and by using (6.19) and (6.20) we get

$$
\|u^{(2)}(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2 C_3 C_0^{m-1} \left( \rho_0^{2m+1} T F_{2m+2} + \sum_{j=0}^{m} \rho_0^{2j} F_{2j} \right)
$$

and arguing as in the case (a) the thesis follows. \(\Box\)

### 6.2 Proof of Propositions 4.4, 4.5

**Proof of Proposition 4.4**

We prove (4.19) for $j = 0$, the proof for $j > 0$ being the same.

By (4.18) we have

$$
\sqrt{2\pi} \left| U_\mu^{(r)}(x, y) \right| = \left| \sqrt{\mu} \int_0^T e^{-\frac{x^2}{2}(y+\tau-t)^2} u(x, t) dt \right| \leq
$$

$$
\leq \sqrt{\mu e^{\frac{2}{3}y^2}} \int_0^T e^{-\frac{x^2}{2}(\tau-t)^2} |u(x, t)| dt,
$$

hence, the Schwarz inequality yields

$$
\sqrt{2\pi} \left| U_\mu^{(r)}(x, y) \right| \leq \sqrt{\mu e^{\frac{2}{3}y^2}} \left( \int_0^T e^{-\mu(t-\tau)^2} dt \right)^{1/2} \left( \int_0^T |u(x, t)|^2 dt \right)^{1/2} \leq
$$

$$
\leq \sqrt{\mu e^{\frac{2}{3}y^2}} \left( \int_0^T e^{-\mu^2 t} dt \right)^{1/2} \left( \int_0^T |u(x, t)|^2 dt \right)^{1/2} \leq c \mu^{1/4} e^{\frac{2}{3}y^2} \left( \int_0^T |u(x, t)|^2 dt \right)^{1/2}.
$$

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Now we prove (4.20). By the change of variable \( \eta = \sqrt{\mu} (t - \tau) \) we have

\[
\sqrt{2\pi} \left( U_{\mu}(x, 0) - u(x, \tau) \right) = \\
= \sqrt{\mu} \int_0^T e^{-\frac{\mu}{2}(t-\tau)^2} u(x, t) dt - u(x, \tau) \int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta = \\
\int_{-\sqrt{\mu} \tau}^{\sqrt{\mu} (T-\tau)} e^{-\frac{\eta^2}{2}} u \left( x, \tau + \frac{\eta}{\sqrt{\mu}} \right) d\eta - u(x, \tau) \int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta = \\
\int_{-\sqrt{\mu} \tau}^{\sqrt{\mu} (T-\tau)} e^{-\frac{\eta^2}{2}} u \left( x, \tau + \frac{\eta}{\sqrt{\mu}} \right) d\eta - u(x, \tau) \left( \int_{-\sqrt{\mu} \tau}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta + \int_{-\infty}^{-\sqrt{\mu} \tau} e^{-\frac{\eta^2}{2}} d\eta \right) : = I_1 + I_2.
\]

We begin to estimate \( |I_1| \). We have

\[
|I_1| \leq \mu^{-1/2} \| \partial_t u(x, \cdot) \|_{L^\infty[0,T]} \int_{-\infty}^{+\infty} |\eta| e^{-\frac{\eta^2}{2}} d\eta \leq \\
c \mu^{-1/2} \| \partial_t u(x, \cdot) \|_{L^\infty[0,T]},
\]

where \( c \) is an absolute constant.

Now we estimate \( |I_2| \). Taking into account that \( \tau \in (0, T/2) \) we have

\[
|I_2| \leq 2 |u(x, \tau)| \int_{-\sqrt{\mu} \tau}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta \leq \\
\leq 2 |u(x, \tau)| e^{-\frac{\mu}{4} \tau^2} \int_{-\sqrt{\mu} \tau}^{+\infty} e^{-\frac{\eta^2}{2}} d\eta \leq ce^{-\frac{\mu}{4} \tau^2} |u(x, \tau)|,
\]

where \( c \) is an absolute constant.

Now, since \( u(x, 0) = 0 \) we have

\[
e^{-\frac{\mu}{4} \tau^2} |u(x, \tau)| \leq \\
\leq e^{-\frac{\mu}{4} \tau^2} \| \partial_t u(x, \cdot) \|_{L^\infty[0,T]} \leq 2e^{-1} \mu^{-1/2} \| \partial_t u(x, \cdot) \|_{L^\infty[0,T]}.
\]

By (6.22), (6.23), (6.24) and (6.25) we get (4.20). \( \square \)
Proof of Proposition 4.5. We have
\[
\partial_y U_\mu(x, y) = \sqrt{\frac{\mu}{2\pi}} \int_0^T -i\mu(iy + \tau - t)e^{-\frac{\mu}{2} (iy + \tau - t)^2} u(x, t) dt = \\
= \sqrt{\frac{\mu}{2\pi}} \int_0^T -i\partial_t \left( e^{-\frac{\mu}{2} (iy + \tau - t)^2} \right) u(x, t) dt = \\
= -i \sqrt{\frac{\mu}{2\pi}} \left( e^{-\frac{\mu}{2} (iy + \tau - T)^2} u(x, T) - \int_0^T e^{-\frac{\mu}{2} (iy + \tau - t)^2} \partial_t u(x, t) dt \right)
\]
and similarly
\[
\partial_y^2 U_\mu(x, y) = \sqrt{\frac{\mu}{2\pi}} e^{-\frac{\mu}{2} (iy + \tau - T)^2} (\partial_t u(x, T) - \mu(iy + \tau - T) u(x, T)) - \\
- \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2} (iy + \tau - t)^2} \partial_{tt}^2 u(x, t) dt.
\]
On the other side, by (4.17) we have
\[
(6.26) \quad \text{div} \left( A(x) \nabla U_\mu \right) = \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2} (iy + \tau - t)^2} \text{div} \left( A(x) \nabla u \right) dt = \\
= \sqrt{\frac{\mu}{2\pi}} \int_0^T e^{-\frac{\mu}{2} (iy + \tau - t)^2} \partial_{tt}^2 u(x, t) dt.
\]
By (6.26) and (6.27) the thesis follows.□

6.3 Proof of Theorem 4.6

In the sequel, for seek of brevity we omit the tilde over \( r_j \), \( j = 1, 2, 3 \).

First we consider the homogeneous case in which \( \tilde{f} = 0 \) and we assume that \( r_3 = 1 \). In [M-R-V1, Theorem 4.5] it has been proved that there exists a positive number \( \tilde{\beta} \) depending on \( \lambda_0, \Lambda_0 \) only such that if \( \beta > \tilde{\beta} \), then there exist constants \( C, \tau_1 \) and \( r_0, (C \geq 1, \tau_1 \geq 1, 0 < r_0 \leq 1) \) depending only on \( \lambda_0, \Lambda_0 \) and \( \beta \) such that the following estimate holds true
\[
(6.28) \quad \tau \int |X|^{\beta} e^{2r|X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta-2} e^{2r|X|^{-\beta}} |v|^2 \leq \\
\leq C \int |X|^{2\beta + 2} e^{2|X|^{-\beta}} |Pv|^2,
\]
for every \( v \in C_0^\infty (B_{r_0} \setminus \{0\}) \) and for every \( \tau \geq \tau_1 \).

On the other hand it is simple to check that there exists \( \tilde{\beta} \) depending on \( \lambda_0, \Lambda_0 \) only such that if \( \beta \geq \tilde{\beta} \) then \( |X|^{-\beta} \) satisfies the pseudoconvexity
conditions of [Hö, Theorem 8.3.1] in $B_1 \setminus B_{r_0/2}$. Therefore there exist $\tau_2 \geq \tau_1$ and $C$ depending on $\lambda_0, \Lambda_0$ and $\beta$ only such that

\begin{align*}
(6.29) \quad \tau \int e^{2\tau |X|^{-\beta}} |\nabla v|^2 + \tau^3 \int e^{2\tau |X|^{-\beta}} |v|^2 & \leq C \int e^{|X|^{-\alpha}} |Pv|^2, \\
& \text{for every } v \in C^\infty_0 (B_1 \setminus B_{r_0/2}) \text{ and for every } \tau \geq \tau_2.
\end{align*}

Now we have trivially

\begin{align*}
(6.30a) \quad \int e^{2\tau |X|^{-\beta}} |\nabla v|^2 & \geq \int |X|^\beta e^{2\tau |X|^{-\beta}} |\nabla v|^2, \\
(6.30b) \quad \int e^{2\tau |X|^{-\beta}} |v|^2 & \geq (r_0/2)^{\beta + 2} \int |X|^{-\beta - 2} e^{2\tau |X|^{-\beta}} |v|^2, \\
(6.30c) \quad \int e^{|X|^{-\beta}} |Pv|^2 & \leq \frac{1}{(r_0/2)^{2\beta + 2}} \int |X|^{2\beta + 2} e^{|X|^{-\beta}} |Pv|^2,
\end{align*}

for every $v \in C^\infty_0 (B_1 \setminus B_{r_0/2})$. Let $\zeta \in C^\infty_0 (B_{r_0})$ such that $0 \leq \zeta \leq 1, |\nabla \zeta|, |D^2 \zeta| \leq C$ and $\zeta(X) = 1$ for every $X \in B_{r_0/2}$.

Now let us denote by $\beta_1 := \max \{3, \beta, 1\}$ and let $v \in C^\infty_0 (B_1 \setminus \{0\})$. By applying (6.31) and (6.29) to $\zeta v$ and $(1 - \zeta)v$ respectively and taking into account (6.30) we have, for $\beta \geq \beta_1$

\begin{align*}
(6.31) \quad \tau \int |X|^\beta e^{2\tau |X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta - 2} e^{2\tau |X|^{-\beta}} |v|^2 & \leq C \int |X|^{2\beta + 2} e^{|X|^{-\beta}} |Pv|^2 + C \int |X|^{2\beta + 2} e^{|X|^{-\beta}} (|D^2 \zeta|^2 v^2 + |\nabla \zeta|^2 |v|^2).
\end{align*}

Now the second term at the right hand side can be absorbed by the left hand side. Hence there exists $\tau_3 \geq \tau_2$ and $C$ depending on $\lambda_0, \Lambda_0$ and $\beta$ only such that for every $v \in C^\infty_0 (B_1 \setminus \{0\})$ and every $\tau \geq \tau_3$ the following inequality holds true

\begin{align*}
(6.32) \quad \tau \int |X|^\beta e^{2\tau |X|^{-\beta}} |\nabla v|^2 + \tau^3 \int |X|^{-\beta - 2} e^{2\tau |X|^{-\beta}} |u|^2 & \leq C \int |X|^{2\beta + 2} e^{|X|^{-\beta}} |Pv|^2.
\end{align*}

Now we use a standard argument to derive by (6.32) the desired three sphere inequality.
First we observe that, by density, estimate (6.32) holds true for every $v \in H^2_0(B_1 \setminus \{0\})$. Now let $u \in H^1(B_1)$ be a solution to equation $Pu = 0$. By $L^2$ regularity theorem we have that $u \in H^3_{loc}(B_1)$. Let $0 < r_1 \leq r_2 < 1$, $0 < \delta \leq \min \left\{ \frac{1-\rho_2}{2}, \frac{1}{2} \right\}$ and let us consider a cutoff function $\eta \in C^2_0(B_{1-\delta} \setminus \overline{B_{r_1(1-2\delta)}})$ such that $0 \leq \eta \leq 1$ and satisfying the following conditions

$$\eta = 1 \quad \text{in} \quad B_{1-2\delta} \setminus B_{r_1(1-\delta)},$$

$$|\nabla \eta| \leq \frac{c}{\delta r_1^2}, \quad |D^2 \eta| \leq \frac{c}{\delta^2 r_1^2} \quad \text{in} \quad B_{r_1(1-\delta)} \setminus B_{r_1(1-2\delta)}$$

and

$$|\nabla \eta| \leq \frac{c}{\delta}, \quad |D^2 \eta| \leq \frac{c}{\delta^2} \quad \text{in} \quad B_{1-\delta} \setminus B_{1-2\delta},$$

where $c$ is an absolute constant.

By (4.25), since $\bar{f} = 0$ we have

$$|P(\eta u)| \leq C (|\nabla \eta| |\nabla u| + |P \eta| |u|),$$

\begin{equation}
\int_{B_{r_2}} |X|^{2\beta+1} e^{2|X|^{-\beta}} |P(\eta u)|^2 \leq C e^{2(1-2\delta)r_1-\beta} r_1^{2\beta+2} \times
\end{equation}

$$\times \left[ \int_{B_{r_1(1-\delta)} \setminus B_{r_1(1-2\delta)}} ((\delta r_1)^{-2}|\nabla u|^2 + (\delta r_1)^{-4}|u|^2) \right] +
\end{equation}

$$+ C e^{2(1-2\delta)-\beta} \left[ \int_{B_{1-\delta} \setminus B_{1-2\delta}} (\delta^{-2}|\nabla u|^2 + \delta^{-4}|u|^2) \right],$$

where $C$ depends on $\lambda_0, \Lambda_0$ and $\beta$ only.

By applying the Caccioppoli inequality to the right hand side of (6.33) and by (6.32) we have, for every $\tau \geq \tau_3$

\begin{equation}
\int_{B_{r_2}} |X|^{-\beta-2} e^{2\tau |X|^{-\beta}} |\eta u|^2 \leq C e^{2\tau(1-2\delta)r_1-\beta} r_1^{2\beta-2} \delta^{-4} \int_{B_{r_1}} |u|^2 +
\end{equation}

$$+ C e^{2\tau(1-2\delta)-\beta} \delta^{-4} \int_{B_1} |u|^2,$$

where $C$ depends on $\lambda_0, \Lambda_0$ and $\beta$ only.

On the other hand we have trivially

\begin{equation}
\int_{B_{r_2}} |X|^{-\beta-2} e^{2\tau |X|^{-\beta}} |u|^2 \geq r_2^{-\beta-2} e^{2\tau r_2^{-\beta}} \int_{B_{r_2} \setminus B_{r_1}} |u|^2.
\end{equation}

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Now let us denote

\begin{equation}
\epsilon := \left( \int_{B_{r_1}} |u|^2 \right)^{1/2}, \text{ and } K := \left( \int_{B_1} |u|^2 \right)^{1/2}.
\end{equation}

By (6.34) and (6.35) we have for every $\tau \geq \tau_3$

\begin{equation}
\int_{B_{r_2}\setminus B_{r_1}} |u|^2 \leq C\delta^{-4} \left\{ e^{2\tau \left[ ((1 - 2\delta)r_1)^{-\beta} - r_2^{-\beta} \right]} \epsilon^2 + e^{2\tau \left[ ((1 - \delta)^{-\beta} - r_2^{-\beta} \right]} K^2 \right\},
\end{equation}

where $C$ depends on $\lambda_0, \Lambda_0$ and $\beta$ only.

Now we add to both the side of (6.37) the integral $\int_{B_{r_1}} u^2 dX$ and we get

\begin{equation}
\int_{B_{r_2}} |u|^2 \leq C\delta^{-4} \left\{ e^{2\tau \left[ ((1 - 2\delta)r_1)^{-\beta} - r_2^{-\beta} \right]} \epsilon^2 + e^{2\tau \left[ ((1 - \delta)^{-\beta} - r_2^{-\beta} \right]} K^2 \right\},
\end{equation}

where $C$ depends on $\lambda_0, \Lambda_0$ and $\beta$ only.

Now denote by $\tau$ the number

\begin{equation}
\tau = \frac{\log(\epsilon^{-1}K)}{(1 - 2\delta)r_1)^{-\beta} - (1 - \delta)^{-\beta}}
\end{equation}

such a number satisfies the equality

\[ e^{2\tau \left[ ((1 - 2\delta)r_1)^{-\beta} - r_2^{-\beta} \right]} \epsilon^2 = e^{2\tau \left[ ((1 - \delta)^{-\beta} - r_2^{-\beta} \right]} K^2. \]

If $\tau \geq \tau_3$ then we choose $\tau = \tau$ in (6.38) and we obtain

\begin{equation}
\int_{B_{r_2}} |u|^2 \leq C\delta^{-4} K^{2(1 - \beta)} \epsilon^\vartheta,
\end{equation}

where

\[ \vartheta = \frac{r_2^{-\beta} - (1 - \delta)^{-\beta}}{[(1 - 2\delta)r_1)^{-\beta} - (1 - \delta)^{-\beta}}. \]

On the other side if $\tau < \tau_3$ then by (6.39) we have

\begin{equation}
(\epsilon^{-1}K)^{2\beta} < e^{2\tau_3 r_2^{-\beta} - (1 - \delta)^{-\beta}}
\end{equation}
hence we have trivially
\begin{equation}
\int_{B_{r_2}^2} |u|^2 \leq \int_{B_1^2} |u|^2 = K^2 = K^{2(1-\theta)}K^{2^{\theta}} \leq e^{2r_3|\bar{r}_2|^\theta -(1-\delta)^{\theta}}K^{2(1-\theta)}\epsilon^{2\theta}.
\end{equation}
Therefore by (6.43) and (6.42) we have
\begin{equation}
\int_{B_{r_2}^2} |u|^2 \leq C\delta^{-4}e^{2r_3|\bar{r}_2|^\theta -(1-\delta)^{\theta}}K^{2(1-\theta)}\epsilon^{\theta}.
\end{equation}

In the nonhomogeneous case, let $u \in H^1(B_1)$ a solution to $Pu = \tilde{f}$ and let $w$ be the solution to the Dirichlet problem
\begin{equation}
\begin{cases}
Pu = \tilde{f}, & \text{in } B_1, \\
w \in H^1_0(B_1),
\end{cases}
\end{equation}
we have that
\begin{equation}
\int_{B_1} |w|^2 \leq C \int_{B_1} |\tilde{f}|^2,
\end{equation}
where $C$ depends on $\lambda_0$. By applying (6.43) to the function $u - w$ and by (6.44) we obtain inequality (4.29) when $\bar{r}_3 = 1$. Finally, by using the dilation $X \rightarrow r_3X$ the thesis follows easily. □

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