NEW TYPE OF SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER-NEWTON SYSTEM

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ABSTRACT. The nonlinear Schrödinger-Newton system
\[
\begin{align*}
\Delta u - V(|x|)u + \Psi u &= 0, & x &\in \mathbb{R}^3, \\
\Delta \Psi + \frac{1}{2}u^2 &= 0, & x &\in \mathbb{R}^3,
\end{align*}
\]

is a nonlinear system obtained by coupling the linear Schrödinger equation of quantum mechanics with the gravitation law of Newtonian mechanics. Wei and Yan in (Calc. Var. Partial Differential Equations 37 (2010), 423–439) proved that the Schrödinger equation has infinitely many positive solutions in \(\mathbb{R}^N\) and these solutions have polygonal symmetry in the \((y_1, y_2)\) plane and they are radially symmetric in the other variables. Duan et al. in (arXiv:2006.16125v1) extended the results got by Wei and Yan and they proved that the Schrödinger equation has infinitely many positive solutions in \(\mathbb{R}^N\) and these solutions have polygonal symmetry in the \((y_1, y_2)\) plane and they are even in \(y_2\) with one more more parameter in the expression of the solutions. Hu et al. also extended the results got by Wei and Yan. Under the appropriate assumption on the potential function \(V\), Hu et al. in (arXiv: 2106.04288v1) constructed infinitely many non-radial positive solutions for the Schrödinger-Newton system and these positive solutions have polygonal symmetry in the \((y_1, y_2)\) plane and they are even in \(y_2\). Assuming that \(V(r)\) has the following character
\[
V(r) = V_1 + \frac{b}{r^q} + O\left(\frac{1}{r^{q+\sigma}}\right), \quad \text{as } r \to \infty,
\]

where \(\frac{1}{2} \leq q < 1\) and \(b, V_1, \sigma\) are some positive constants, \(V(y) \geq V_1 > 0\), we construct infinitely many non-radial positive solutions which have polygonal symmetry in the \((y_1, y_2)\) plane and are even in \(y_2\) for the Schrödinger-Newton system by the Lyapunov-Schmidt reduction method. We extend the results got by Duan et al. in (arXiv:2006.16125v1) to the nonlinear Schrödinger-Newton system. Meanwhile, the solutions constructed by us are different from what constructed by Hu et al. as there is one more parameter in the expression of the solutions constructed by Hu et al.

KEYWORD: Nonlinear Schrödinger-Newton system; Infinitely many solutions; New solutions; Finite dimensional reduction.

AMS Subject Classifications: 35B25 · 35J20 · 35J60

1. INTRODUCTION

The nonlinear Schrödinger-Newton system is a nonlinear system obtained by coupling the linear Schrödinger equation of quantum mechanics with the gravitation law of Newtonian mechanics, where the wave function \(u\) means a stationary solution for a quantum system describing a nonlinear modification of the Schrödinger equation with a Newtonian gravitational potential representing the interaction of the particle with its own gravitational field. Our aim is to construct solutions of the following nonlinear Schrödinger-Newton equation
\[
\begin{align*}
\Delta u - V(|x|)u + \Psi u &= 0, & x &\in \mathbb{R}^3, \\
\Delta \Psi + \frac{1}{2}u^2 &= 0, & x &\in \mathbb{R}^3,
\end{align*}
\]

where \(u \in H^1(\mathbb{R}^3)\) and \(V\) is a given external potential and a bounded radially symmetric potential with \(V(y) \geq V_1 > 0\), \(\Psi\) is the Newtonian gravitational potential. The latter model was proposed...
in [15]. We can see that the second equation of (1.1) (see [3]) has a unique positive solution \( \Psi \in D^{1,2}(\mathbb{R}^3) \) which has the form as follows

\[
\Psi_u(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy.
\]  

(1.2)

So, the system (1.1) is equivalent to the single nonlocal equation as follows

\[
-\Delta u + V(x)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy \right) u, \quad x \in \mathbb{R}^3.
\]  

(1.3)

Obviously, \((u, \Psi_u)\) is a solution of the system (1.1) if and only if \(u\) is a solution of the equation (1.3). From [7, 13, 8], we can make the equation (1.3) as a special case of the Choquard equation:

\[
-\Delta u + V(x)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{A_\alpha}{|x-y|^{N-2}} u^p(y) dy \right) |u|^{p-2} u, \quad x \in \mathbb{R}^N,
\]

which describes an electron trapped in its own hole, where

\[
A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}} \alpha}. 
\]

We can also get

\[
-\Delta u + V(x)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy \right) u, \quad x \in \mathbb{R}^3,
\]

with \(N = 3\), \(p = 2\), \(\alpha = 2\).

The study of standing waves of nonlinear Hartree equations:

\[
i\varepsilon \frac{\partial \varphi}{\partial t} = -\varepsilon^2 \Delta_x \varphi + (V(x) + E)\varphi - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{\varphi^2(y)}{|x-y|} \, dy \right) \varphi, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+,
\]

can come down to the study of the equation (1.3), with \(\varphi(x,t) = e^{-iEt/\varepsilon}u(x)\), where \(i\) is the imaginary unit and \(\varepsilon\) is the Planck constant.

The existence and the uniqueness of ground state solution to the latter problem of the system:

\[
\begin{align*}
\Delta u - u + \Psi u &= 0, \quad x \in \mathbb{R}^3, \\
\Delta \Psi + \frac{1}{2} u^2 &= 0, \quad x \in \mathbb{R}^3,
\end{align*}
\]  

(1.4)

have been proven in [9, 10] and [14]. The nondegeneracy of the ground state also has been proven in [16]. By [14] and [16], we can know that the following equation:

\[
\begin{align*}
-\Delta u + u - \Psi u &= 0, \quad x \in \mathbb{R}^3, \\
\Delta \Psi + \frac{1}{2} u^2 &= 0, \quad x \in \mathbb{R}^3, \\
u, \Psi > 0, \quad u(0) = \max_{x \in \mathbb{R}^3} u(x)
\end{align*}
\]  

(1.5)

has a unique radial solution \((U, \Psi)\) with

\[
U(x) \to 0, \quad \Psi(x) \to 0, \quad \text{as } |x| \to \infty.
\]

Furthermore, \(U\) decreases strictly and

\[
\lim_{|x| \to \infty} U(x)|x|e^{|x|} = \lambda_2, \quad \lim_{|x| \to \infty} U'(x) = -1,
\]  

(1.6)

and

\[
\lim_{|x| \to \infty} \Psi(x)|x| = \lambda_3,
\]  

(1.7)

where \(\lambda_2\), \(\lambda_3\) are positive constants.

Kang and Wei proved the existence of positive \(K\)–bump solutions to the nonlinear Schrödinger equation:

\[
h^2 \Delta u - V(x)u + u^p = 0, \quad u > 0, \quad x \in \mathbb{R}^N,
\]  

(1.8)
concentrating at local maximum points of \( V \) as \( \varepsilon \to 0 \) in [6]. Furthermore, each pair of bumps has a strong interaction. Wei and Winter extended the results to the following singularly perturbed Schrödinger-Newton problem

\[
-\varepsilon^2 \Delta u + V(x)u = \frac{1}{8\pi \varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy \right) u, \quad x \in \mathbb{R}^3, \tag{1.9}
\]

where \( \varepsilon > 0 \) is a parameter and \( \inf_{\mathbb{R}^3} V > 0 \). Also, they proved the existence of positive \( K \)-bump solutions to (1.9) concentrating at local maximum (minimum) or nondegenerate critical points of \( V \) as \( \varepsilon \to 0 \). Furthermore, it is indicated that there is a strong interacting between each pair of bumps. Luo and Peng et al. have shown the solutions to (1.1) are distributed along the vertices of a regular \( \star \)-polygon with high energy. From [14] and [16], we can know the equation (1.5) has the unique radial solution with large radius \( r \). Hu and Jevnikar et al. in [5] produce infinitely many non-radial solutions to (1.1) concentrating at degenerate critical points of the potential \( V \). Wei and Winter extended the results to the following singularly perturbed Schrödinger-Newton problem with high energy. From [14] and [16], we can know the equation (1.5) has the unique radial solution (U, \( \Psi \)), so the equation (1.11) has the unique radial solution with \( V(x) \equiv 1 \). For any large integer \( m \) they constructed a solution \( u_m \) looking like a sum of \( m \) bumps \( U(x-x^*_i) \),

\[
U^*_m(y) \sim \sum_{i=1}^{m} U(y-x^*_i), \tag{1.13}
\]

where the location points \( x^*_i \) are distributed along the vertices of a regular \( m \)-polygon

\[
x^*_i = r \left( \cos \frac{2(i-1)\pi}{m}, \sin \frac{2(i-1)\pi}{m}, 0 \right), \quad \text{for } i = 1, \ldots, m
\]

with large radius \( r \sim (m \ln m)^{\frac{1}{1-q}} \) as \( m \to \infty \). These solutions have polygonal symmetry in the \((y_1, y_2)\) plane, which are even in \( y_2 \) and \( y_3 \) and have \( m \) bumps. Very recently, in [1], Duan and Musso used a new way to construct a new family of solutions of

\[
\begin{cases}
-\Delta u + V(y)u = u^p, \quad u > 0, \text{ in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\]

These solutions have polygonal symmetry in the \((y_1, y_2)\) plane, which are even in \( y_2 \) and have \( 2m \) bumps.
Recently, Gao and Yang in \cite{2} proved that the nonlinear Choquard equation:

\[
\begin{aligned}
-\Delta u + V(|x|)u &= \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \, dy \right) u, & x \in \mathbb{R}^3, \\
u &\in H^1(\mathbb{R}^3),
\end{aligned}
\]  

(1.14)

has infinitely many non-radial positive solutions, where the potential \( V \) satisfies:

\[
V(r) = V_1 + \frac{b}{r^q} + O\left( \frac{1}{r^{q+\sigma}} \right), \quad \text{as } r \to \infty,
\]  

(1.15)

where \( b, \sigma, V_1 > 0 \) and \( q \geq 3 \).

In the paper, the main result is that we extend the result got by Duan et al. in \cite{1} to the nonlinear Schrödinger-Newton system. As in \cite{1}, let \( m \) be an integer and define the points \( \bar{x}_i = \bar{x}_i(r, t, m) \) and \( \bar{z}_i = \bar{z}_i(r, t, m) \):

\[
\begin{aligned}
\bar{x}_i &= r \left( \sqrt{1-t^2} \cos \frac{2(i-1)\pi}{m}, \sqrt{1-t^2} \sin \frac{2(i-1)\pi}{m}, t \right), \quad i = 1, \ldots, m, \\
\bar{z}_i &= r \left( \sqrt{1-t^2} \cos \frac{2(i-1)\pi}{m}, \sqrt{1-t^2} \sin \frac{2(i-1)\pi}{m}, -t \right), \quad i = 1, \ldots, m.
\end{aligned}
\]  

(1.16)

The parameters \( t \) and \( r \) are positive numbers and the range of them are chosen as follows

\[
t \in \left[ \alpha_1 (\ln m)^{-\frac{2}{3}}, \alpha_2 (\ln m)^{-\frac{2}{3}} \right], \quad r \in \left[ 3, b, \sigma, V \right]
\]  

for \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) fixed positive constants, independent of \( m \).

Define the approximate solution as

\[
W_{r,t}(y) = \sum_{i=1}^{m} U_{\bar{x}_i}(y) + \sum_{i=1}^{m} U_{\bar{z}_i}(y),
\]  

(1.18)

where \( U_{\bar{x}_i}(y) = U(y - \bar{x}_i), U_{\bar{z}_i}(y) = U(y - \bar{z}_i) \) and \( m \) is large enough.

In the paper, we will prove that for any \( m \) sufficiently large problem \((1.1)\) has a new type of solutions \( u_m \) with the form

\[
u_m(y) \sim W_{r,t}(y),
\]  

(1.19)

as \( m \to \infty \). The solutions will have polygonal symmetry in the \((y_1, y_2)\)-plane and be even in the \( y_3 \) direction. Particularly, they do not belong to the same class of symmetry as the solutions built in \cite{5}, because we have one more parameter. In fact, if we take \( t = 0 \) in \((1.16)\), we have \( \bar{x}_i = \bar{z}_i \) for any \( i \) and the two constructions \((1.13)\) and \((1.18)-(1.19)\) are the same. Assuming that the range of the parameters \( t \) and \( r \) is the same as \((1.17)\), the two constructions \((1.13)\) and \((1.18)-(1.19)\) are different. The main distinction between the present construction and the one in \cite{5} is that there exist two parameters \( r, t \) to choose in the locations \( \bar{x}_i, \bar{z}_i \) of the bumps in \((1.18)\). Although, we use the method in \cite{1}, the non-local term brings some new difficulties, which involves many complex and technical estimates. We would like to stress that the energy estimate is different from \cite{1} because of the appearance of the non-local term. And we have to choose different range of \( r \) and \( t \) to ensure the mapping is contractible in the proof of theorem \((1.1)\). Compared with \cite{1}, we also check that the mapping is contractible by a different method.

In the following subsection, we will discuss our result in details.

In this paper, we use \( C, C_i, \phi, \theta_i, \mu, \mu_i \), \( i = 0, 1, \ldots \) to denote fixed constants. Moreover, we also use the common notation by writing \( O_m(F(r, t)), o_m(F(r, t)) \) for the functions satisfying

\[
\text{if } G(r, t) \in O_m(F(r, t)) \quad \text{then} \quad \lim_{m \to +\infty} \left| \frac{G(r, h)}{F(r, t)} \right| \leq C < +\infty,
\]

and

\[
\text{if } G(r, t) \in o_m(F(r, t)) \quad \text{then} \quad \lim_{m \to +\infty} \frac{G(r, t)}{F(r, t)} = 0.
\]
1.1. Main result and scheme of the proof.

For \( i = 1, \cdots, m \), we divide \( \mathbb{R}^3 \) into \( m \) parts:
\[
\Omega_i := \left\{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : \langle \frac{(y_1, y_2)}{|y_1, y_2|}, (\cos \frac{2(i-1)\pi}{m}, \sin \frac{2(i-1)\pi}{m}) \rangle \geq \cos \frac{\pi}{m} \right\},
\]
where \( \langle , \rangle_{\mathbb{R}^2} \) denotes the dot product in \( \mathbb{R}^2 \). For \( \Omega_i \), we divide it into two parts:
\[
\Omega_i^+ = \left\{ y = (y_1, y_2, y_3) \in \Omega_i, y_3 \geq 0 \right\},
\]
\[
\Omega_i^- = \left\{ y = (y_1, y_2, y_3) \in \Omega_i, y_3 < 0 \right\}.
\]

We have that
\[
\mathbb{R}^3 = \bigcup_{i=1}^{m} \Omega_i, \quad \Omega_i = \Omega_i^+ \cup \Omega_i^{-}
\]
and the interior of
\[
\Omega_i \cap \Omega_j, \quad \Omega_i^+ \cap \Omega_i^{-}
\]
are empty sets for \( j \neq i \).

Now we define the symmetric Sobolev space:
\[
H = \left\{ u : u \in H^1(\mathbb{R}^3), u \text{ is even in } y_2, u \left( \sqrt{y_1^2 + y_2^2} \cos \theta, \sqrt{y_1^2 + y_2^2} \sin \theta, y_3 \right) \right\}
\]
where \( \theta = \arctan \frac{y_2}{y_1} \).

In the paper, we always assume
\[
(r, t) \in S_m =: \left[ \left( \left( \frac{A_1}{16q(\pi)^2} \right)^{\frac{1}{q-2}} - \beta_0 \right)(m \ln m)^{\frac{1}{q-2}}, \left( \frac{A_1}{16q(\pi)^2} \right)^{\frac{1}{q-2}} + \beta_0 \right](m \ln m)^{\frac{1}{q-2}} \times \left[ (1 - \alpha_0)(\ln m)^{-\frac{1}{2}}, (1 + \alpha_0)(\ln m)^{-\frac{1}{2}} \right],
\]
for some \( \alpha_0, \beta_0 > 0 \) small enough, and independent of \( m \). We can refer to Remark 3.2 to discuss the assumption (1.20) for \( (r, t) \).

The following is our main result.

**Theorem 1.1.** Assume that \( V(|y|) \) satisfies (H$_a$) and the parameters \( (r, t) \) satisfy (1.20). So there exists an integer \( m_0 \), such that for any integer \( m \geq m_0 \), (1.1) has a solution \( u_m \) of the form
\[
u_m = W_{r_m, t_m}(y) + \omega_m(y),
\]
where \( \omega_m \in H, (r_m, s_m) \in S_m \) and \( \omega_m \) satisfies
\[
\int_{\mathbb{R}^3} (|\nabla \omega_m|^2 + V(y)|\omega_m|^2) \to 0, \quad \text{as} \quad m \to \infty.
\]

We want to prove Theorem 1.1 by using the Lyapunov- Schmidt reduction technique adapted to our context as developed in [17]. We sketch the scheme of the proof briefly. The critical point \( u \) of the energy functional \( I \) is defined as follows:
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|)u^2) - \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(y)u^2(x)}{|x - y|},
\]
which corresponds to a solution for (1.21). For any $m$ sufficiently large and for any $(r, t) \in S_m$, assuming that the critical point of the energy functional $I$ has the following form $u = W_{r,t} + \phi$, the solution $u$ will have the form $u = W_{r,t} + \phi$.

Furthermore, assume that $\phi \in H^1(\mathbb{R}^3)$ satisfies the following eigenvalue problem:

$$-\Delta \phi + \phi = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy \right) \phi + \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{U(y)\phi(y)}{|x-y|} dy \right) U.$$

Then, $\phi$ can be spanned by $\{ U/\partial x_1, \partial U/\partial x_2, \partial U/\partial x_3 \}$, that is to say

$$\phi \in \text{span} \left\{ \frac{\partial U}{\partial x_i}, i = 1, 2, 3 \right\}.$$ 

We define the inner product and norm on the space $H^1(\mathbb{R}^3)$ by

$$\langle u, v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(|x|)uv) \quad \text{and} \quad ||u|| = \langle u, u \rangle^{1/2},$$

respectively. Noting that $V$ is bounded, so the norm $|| \cdot ||$ is equivalent to the standard norm. Particularly, by Lemma A.11 we have

$$||\Psi_u||_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \Psi_u u^2 \leq C||u||^4_{L^{4/3}(\mathbb{R}^3)} \leq C||u||^4. \quad (1.23)$$

Throughout the paper, $C$ denotes various positive constants whose exact value is not essential. We define

$$R(\phi) = I(W_{r,t} + \phi), \quad \phi \in \mathcal{E}_1.$$ 

The space $\mathcal{E}_1$ is defined later. For $i = 1, \cdots, m$, we define

$$\mathcal{Z}_{1i} = \frac{\partial U_{x_i}}{\partial r}, \quad \mathcal{Z}_{2i} = \frac{\partial U_{x_i}}{\partial t}, \quad \mathcal{Z}_3 = \frac{\partial U_{x_3}}{\partial t}.$$ 

The constrained space is defined as follows

$$\mathcal{E}_1 = \left\{ v : v \in H_a, \quad \int_{\mathbb{R}^3} T[U_{x_i}^2] \mathcal{Z}_{3i} v + 2T[U_{x_i} \mathcal{Z}_{3i}] U_{x_i} v = 0 \right\}$$

and

$$\int_{\mathbb{R}^3} T[U_{x_i}^2] \mathcal{Z}_{3i} v + 2T[U_{x_i} \mathcal{Z}_{3i}] U_{x_i} v = 0, \quad i = 1, \cdots, m, \quad \ell = 1, 2 \right\}, \quad (1.24)$$

where

$$T[uv](x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(y)v(y)}{|x-y|} dy. \quad (1.25)$$

Expand $R(\phi)$ in the following:

$$R(\phi) = R(0) + 1(\phi) + \frac{1}{2} \langle L\phi, \phi \rangle - N(\phi), \quad \phi \in \mathcal{E}_1,$$

where

$$1(\phi) = \sum_{i=1}^m \int_{\mathbb{R}^3} (V(|y|) - 1)(U_{x_i} + U_{\mathcal{Z}_i}) \phi$$

and

$$N(\phi) = \frac{1}{32\pi} \left( \int_{\mathbb{R}^3} \frac{(W_{r,t} + \phi)^2(y)(W_{r,t} + \phi)^2(x)}{|x-y|} - \int_{\mathbb{R}^3} \frac{(W_{r,t})(y)(W_{r,t})(x)}{|x-y|} \right).$$
Furthermore, $L$ is a linear operator from $E_1$ to $E_1$, which satisfies for all $v_0, v_1 \in E_1,$
\[
\langle L v_0, v_1 \rangle = \int_{\mathbb{R}^3} \nabla v_0 \nabla v_1 + V(|y|) v_0 v_1 - \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)}{|x-y|} dy \right) v_0 v_1
\]
\[
- \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_0}{|x-y|} dy \right) W_{r,t} v_1.
\]
Since $W_{r,t}$ is bounded and has the symmetries of the space $H_a$, it is easy to prove that $L$ is a bounded linear operator from $E_1$ to $E_1$ by Hardy-Littlewood-Sobolev inequality. We are going to prove that $I(\phi)$ is a bounded linear functional in $E_1$. Thus there exists an $l_m \in E_1$, such that
\[
I(\phi) = \langle l_m, \phi \rangle.
\]
Then a critical point of $R(\phi)$ is also a solution to
\[
l_m + L\phi - N'(\phi) = 0.
\]
Thus the function $u$ which has the form $u = W_{r,t} + \phi$ will be a solution to (1.1) if $\phi$ is a solution to (1.27). Firstly the strategy now consists presenting that, for any $(r,t) \in S_m$, there is a function $\phi_{r,t}$ solution to (1.27) in the space $E_1$ (see Proposition 2.2). Secondly, we will reduce the problem to find a critical point $u$ of $I(u)$ to the problem of finding a stable critical point $(r^*, t^*)$ of the function $F(r,t) = I(W_{r,t} + \phi_{r,t})$.

We will present that there is such a critical point in the set $S_m$. Lots of properties of the solutions follows by the construction (see (1.20)).

**Plan of the paper**

We write the paper in the following. In section 2 we do the finite-dimensional reduction. And we are going to prove Theorem 1.1 in section 3. In the Appendix A we give some known results and some technical estimates. Finally, in the Appendix B we make some energy expansions.

**2. Finite-dimensional reduction**

The following lemma proves the existence and boundedness of inverse operator of $L$ in $E_1$.

**Lemma 2.1.** There is a constant $\xi > 0$, independent of $m$, such that for any $(r,t) \in S_m$
\[
\|L v\| \geq \xi \|v\|, \quad v \in E_1.
\]

**Proof.** We prove it by contradiction. Assuming that there are $t_m, r_m \in S_m, v_m \in E_1$ satisfying
\[
\|L v_m\| = o_m(1) \|v_m\|, \quad m \to +\infty.
\]

Then we can easily get
\[
\langle L v_m, \varphi \rangle = o_m(1) \|v_m\| \|\varphi\|, \quad \forall \varphi \in E_1.
\]

Similar to [17], we assume $\|v_m\|^2 = m$.

Firstly, we claim that if $\psi \in E_1$, then $T[U \psi]$ satisfies
\[
T[U \psi] \text{ is even in } y_2,
\]
and
\[
T[U \psi] \left(\sqrt{y_1^2 + y_2^2} \cos \theta, \sqrt{y_1^2 + y_2^2} \sin \theta, y_3\right)
\]

\[
\frac{1}{2} \int_{\mathbb{R}^3} T[W_{r,t}^2 W_{r,t} \phi] - \frac{1}{16\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)}{|x-y|} dy \right) \phi^2
\]
\[
- \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} \phi}{|x-y|} dy \right) W_{r,t} \phi.
\]

(1.26)
\[\begin{aligned}
&= T[U \psi]\left( \sqrt{y_1^2 + y_2^2} \cos \left( \theta + \frac{2\pi i}{m} \right), \sqrt{y_1^2 + y_2^2} \sin \left( \theta + \frac{2\pi i}{m} \right), y_3 \right). \\
\text{Indeed, define } h = (h_{ij}) \in SO(3) \text{ with } h = \text{diag}(1, -1, 1). \text{ Obviously, } U \in E_1, \text{ which implies } U(x) = U(hx), \text{ so is } \psi. \text{ Then, we have}
&T[U \psi](hx) = \frac{1}{4\pi} \int_{\mathbb{R}^3} U(y) \psi(y) \frac{dy}{|hx - y|} = -\frac{1}{4\pi} \int_{\mathbb{R}^3} U(h^{-1}y) \psi(h^{-1}y) \frac{dy}{|x - h^{-1}y|} \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} U(y) \psi(y) \frac{dy}{|x - y|} = T[U \psi](x),
\end{aligned}\]

that is equivalent to (2.1). Similarly, define with the form as follows:

\[ h = \begin{pmatrix}
\cos(2\pi i/m) & -\sin(2\pi i/m) & 0 \\
\sin(2\pi i/m) & \cos(2\pi i/m) & 0 \\
0 & 0 & 1
\end{pmatrix}. \]

So we can get (2.2). With the symmetric property, we can get

\[ (L v_m, \varphi) = \int_{\mathbb{R}^3} \nabla v_m \nabla \varphi + V(|y|) v_m \varphi - \frac{1}{8\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} (W_{r,t})^2(y) \frac{dy}{|y - z|} \right) v_m \varphi \right) \\
- \frac{1}{4\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_m}{|y - z|} \frac{dy}{|y - z|} \right) W_{r,t} \varphi \\
= m \int_{\Omega_1} \nabla v_m \nabla \varphi + V(|y|) v_m \varphi - \frac{1}{8\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} (W_{r,t})^2(y) \frac{dy}{|y - z|} \right) v_m \varphi \\
- \frac{1}{4\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_m}{|y - z|} \frac{dy}{|y - z|} \right) W_{r,t} \varphi \\
= o_m(1) ||v_m|| ||\varphi|| = o(\sqrt{m}) ||\varphi||. \quad (2.3) \]

Moreover we have

\[ \int_{\Omega_1} |\nabla v_m|^2 + V(|y|) v_m^2 = 1. \quad (2.4) \]

Inserting \( \varphi = v_m \) into (2.3), we can get immediately

\[ \int_{\Omega_1} |\nabla v_m|^2 + V(|y|) v_m^2 = \frac{1}{8\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} (W_{r,t})^2(y) \frac{dy}{|y - z|} \right) v_m^2 - \frac{1}{4\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_m}{|y - z|} \frac{dy}{|y - z|} \right) W_{r,t} v_m = o_m(1). \]

Denote

\[ \bar{v}_m(y) = v_m(y + \mathbf{1}). \]

For this sequence \( \bar{v}_m(y) \), we are able to prove that \( \bar{v}_m(y) \) has boundedness in \( H^1_{\text{loc}}(\mathbb{R}^3) \). Actually, for any \( R > 0 \), since \( |\mathbf{1} - \mathbf{1}| = 2r \sqrt{1 - t^2} \sin \frac{\theta}{m} \geq \frac{n}{2} \ln m \), we can choose \( m \) sufficiently large such that \( B_{\mathbf{1}}(\mathbf{1}) \subset \Omega_1 \). Then, we can get

\[ \int_{B_{\mathbf{1}}(0)} (|\nabla \bar{v}_m|^2 + V(|y|) \bar{v}_m^2) = \int_{B_{\mathbf{1}}(\mathbf{1})} (|\nabla v_m|^2 + V(|y - \mathbf{1}|) v_m^2) \leq \int_{\Omega_1} (|\nabla v_m|^2 + V(|y - \mathbf{1}|) v_m^2) \leq 1. \quad (2.5) \]

So we have

\[ \bar{v}_m \rightarrow \bar{v} \text{ \ weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \quad (2.6) \]

and

\[ \bar{v}_m \rightarrow \bar{v} \text{ \ strongly in } L^2_{\text{loc}}(\mathbb{R}^3). \quad (2.7) \]
Since \( \overline{v}_m \) is even in \( y_2 \), we know that \( \overline{v} \) is even in \( y_2 \). By the orthogonal conditions of functions in \( E_1 \)
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U_{x_1}}{\partial r} v_m + 2T[U_{x_1}] \frac{\partial U_{x_1}}{\partial r} U_{x_1} v_m = 0,
\]
we know the following identity
\[
\frac{\partial U_{x_1}}{\partial r} = -\sqrt{1 - t^2} \frac{\partial U_{x_1}}{\partial y_1} - t \frac{\partial U_{x_1}}{\partial y_3},
\]
so we can obtain
\[
\int_{\mathbb{R}^3} T[U^2] \left( \sqrt{1 - t^2} \frac{\partial U}{\partial x_1} + t \frac{\partial U}{\partial x_3} \right) \overline{v}_m dx + 2 \int_{\mathbb{R}^3} T[U] \left( \sqrt{1 - t^2} \frac{\partial U}{\partial y_1} + t \frac{\partial U}{\partial y_3} \right) U \overline{v}_m dx = 0. \tag{2.8}
\]
Similarly, combining
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U_{x_1}}{\partial t} v_m + 2T[U_{x_1}] \frac{\partial U_{x_1}}{\partial t} U_{x_1} v_m = 0,
\]
and
\[
\frac{\partial U_{x_1}}{\partial t} = \frac{rt}{\sqrt{1 - t^2}} \frac{\partial U_{x_1}}{\partial y_1} - \frac{rt}{\partial y_3},
\]
we can obtain
\[
\int_{\mathbb{R}^3} T[U^2] \left( \frac{rt}{\sqrt{1 - t^2}} \frac{\partial U}{\partial x_1} - \frac{rt}{\partial x_3} \right) \overline{v}_m dx + 2 \int_{\mathbb{R}^3} T[U] \left( \frac{rt}{\sqrt{1 - t^2}} \frac{\partial U}{\partial y_1} - \frac{rt}{\partial y_3} \right) U \overline{v}_m dx = 0. \tag{2.9}
\]
From (2.8), (2.9), we can get
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U}{\partial x_1} \overline{v}_m dx + 2 \int_{\mathbb{R}^3} T[U] \frac{\partial U}{\partial y_1} U \overline{v}_m dx = 0.
\]
Letting \( m \to +\infty \), we can get
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U}{\partial x_1} \overline{v}_m dx + 2 \int_{\mathbb{R}^3} T[U] \frac{\partial U}{\partial y_1} U \overline{v}_m dx = 0, \tag{2.10}
\]
and
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U}{\partial x_3} \overline{v}_m dx + 2 \int_{\mathbb{R}^3} T[U] \frac{\partial U}{\partial y_3} U \overline{v}_m dx = 0. \tag{2.11}
\]
In the following, we will prove that \( \overline{v} \) can solve
\[
-\Delta \phi + \phi - \frac{1}{2} T[U^2] \phi - T[U \phi] U = 0, \quad \text{in } \mathbb{R}^3. \tag{2.12}
\]
Define the constrained space as follows:
\[
E_1^+ = \left\{ \phi : \phi \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} T[U^2] \frac{\partial U}{\partial x_1} \phi dx + 2 \int_{\mathbb{R}^3} T \left[ \frac{\partial U}{\partial y_1} \right] U \phi dx = 0 \right\}
\]
and
\[
\int_{\mathbb{R}^3} T[U^2] \frac{\partial U}{\partial x_3} \phi dx + 2 \int_{\mathbb{R}^3} T \left[ \frac{\partial U}{\partial y_3} \right] U \phi dx = 0 \right\}.
\]
To prove (2.12), we provide a claim firstly.
Claim 1: \( \overline{v} \) can solve
\[
-\Delta \phi + \phi - \frac{1}{2} T[U^2] \phi - T[U \phi] U = 0, \quad \text{in } E_1^+.
\]
Now we prove the Claim 1.
For any \( R > 0 \), we let \( \phi \in C_0^\infty(B_R(0)) \cap E_1^+ \) which is even in \( y_2 \). Then, we can denote \( \phi_m(y) = \phi(y - \overline{v}) \in C_0^\infty(B_R(\overline{v})) \).
When we insert \( \phi_m(y) = \varphi \) into (2.3) and combine (2.7), (2.17) and Lemma A.2, we can obtain

\[
\int_{\mathbb{R}^3} \left( \nabla \varphi \nabla \phi - \frac{1}{2} T[U^2] \phi \varphi - T[U \phi] U \varphi \right) = 0. \tag{2.13}
\]

Moreover, since \( \varphi \) and \( T[U \phi] \) are even in \( y_2 \), we can get that (2.13) holds for all functions \( \phi \in C_0^\infty(B_R(\Omega_1) \cap \overline{E_1}^+) \) which is odd in \( y_2 \) with the symmetric conditions. So, (2.13) is valid for all functions \( \phi \in C_0^\infty(B_R(\Omega_1) \cap \overline{E_1}^+) \). Density argument indicates that,

\[
\int_{\mathbb{R}^8} \left( \nabla \varphi \nabla \phi - \frac{1}{2} T[U^2] \phi \varphi - T[U \phi] U \varphi \right) = 0, \quad \text{for all } \phi \in \overline{E_1}^+. \tag{2.14}
\]

The Claim 1 is proved.

Putting Claim 1 and the fact that (2.12) is valid for \( \phi = \frac{\partial U}{\partial y_1} \) and \( \phi = \frac{\partial U}{\partial y_3} \) together, then we can obtain

\[
\int_{\mathbb{R}^3} \left( \nabla \varphi \nabla \phi - \frac{1}{2} T[U^2] \phi \varphi - T[U \phi] U \varphi \right) = 0, \quad \text{for all } \phi \in H^1(\mathbb{R}^3), \tag{2.15}
\]

that is (2.12). With the help of the Non-degeneracy result for \( U \) and the fact that \( \varphi \) is even in \( y_2 \), we can derive

\[
\varphi = C_1 \frac{\partial U}{\partial y_1} + C_3 \frac{\partial U}{\partial y_3}, \tag{2.16}
\]

for some constants \( C_1 \) and \( C_3 \). Putting (2.10), (2.16) together, we obtain

\[
C_1 = C_3 = 0.
\]

So we can get

\[
\varphi = 0. \tag{2.17}
\]

From (2.7) and (2.17), we have that

\[
\int_{B_R(\Omega_1)} v_m^2 = o_m(1). \tag{2.18}
\]

From Lemma A.2, we can know \( W_{r,t} \leq C e^{-(1-\delta)|y-\Omega_1|} \).

From Lemma A.2, the symmetry and Hardy-Littlewood-Sobolev inequality, we have

\[
\frac{1}{4\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_m}{|x-y|} dy \right) W_{r,t} v_m = \frac{1}{2\pi} \int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{W_{r,t} v_m}{|x-y|} dy \right) W_{r,t} v_m \leq C \int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{e^{-(1-\delta)|y-\Omega_1|} v_m}{|x-y|} dy \right) e^{-(1-\delta)|x-\Omega_1|} v_m \leq C \int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{e^{-(1-\delta)|y-\Omega_1|} v_m}{|x-y|} dy \right)^\frac{5}{3} \leq C m_{m_0}^z(1) = o(1), \tag{2.19}
\]

and

\[
\frac{1}{8\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)}{|x-y|} dy \right) v_m^2 \leq C \int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{(U_{\Omega_1} + \sum_{i=2}^m U_{\Omega_i})^2(y)}{|x-y|} dy \right) v_m^2. \tag{2.20}
\]

By Lemma A.5 and (2.18), we have

\[
\int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{(U_{\Omega_1} + \sum_{i=2}^m U_{\Omega_i})^2(y)}{|x-y|} dy \right) v_m^2 \leq C \left( e^{-(1-\delta)R} + \frac{1}{R} \right) \int_{\Omega_1} v_m^2(x) dx + C \int_{B_R(\Omega_1)} v_m^2(x) dx
\]

where

\[
U_{\Omega_i} = U|_{\Omega_i}, \quad m_{m_0}^z(1) = o(1)
\]

is proved.
\[
= C \left( e^{-(1-\delta)R} + \frac{1}{R} \right) \int_{\Omega} v_m^2(x) dx + o(1).
\] (2.21)

By (2.19), (2.20), (2.21) and letting \( R \) and \( m \) sufficiently large and inserting \( \varphi = v_m \) in (2.3), we can obtain,

\[
o_m(1) = \int_{\Omega_1} |\nabla v_m|^2 + V(|y|)v_m^2 - \frac{1}{2} \int_{\Omega_1} T[(W_{r,t})^2]v_m^2 - \int_{\Omega_1} T[W_{r,t}v_m]W_{r,t}v_m
\]
\[
= \int_{\Omega_1} |\nabla v_m|^2 + V(|y|)v_m^2 - \frac{1}{8\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)}{|x-y|} dy \right) v_m^2 - \frac{1}{4\pi} \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W_{r,t}v_m W_{r,t}v_m}{|x-y|} dy \right) W_{r,t}v_m
\]
\[
= \int_{\Omega_1} |\nabla v_m|^2 + V(|y|)v_m^2 + o_m(1) + O_m \left( e^{-(1-\delta)R} + \frac{1}{R} \right) \int_{\Omega_1} |v_m|^2
\]
\[
\geq \frac{1}{2} \int_{\Omega_1} |\nabla v_m|^2 + V(|y|)v_m^2 + o_m(1)
\]
\[
= \frac{1}{2} + o_m(1),
\]

which is impossible. The proof of Lemma 2.1 is completed. \( \square \)

Now we prove the following Proposition which is crucial for the sequel.

**Proposition 2.2.** For any \( m \) sufficiently large, there is a \( C^1 \) map \( \Phi : \mathbb{S}_m \to \mathbb{E}_1 \) such that \( \Phi(r,t) = \phi_{r,t}(y) \in \mathbb{E}_1 \), and

\[
R'(\phi_{r,t}) = 0, \quad \text{in} \quad \mathbb{E}_1.
\] (2.22)

Furthermore, there exists a small \( \sigma > 0 \), such that

\[
\|\phi_{r,t}\| \leq \frac{C}{m^{\frac{q-1}{4(1-q)}+\sigma}}.
\]

From Riesz theorem, there exists a \( I_m \in \mathbb{E}_1 \) satisfying

\[
I(\phi) = (I_m, \phi), \quad \forall \phi \in \mathbb{E}_1.
\]

In order to apply the contraction mapping theorem to prove that (2.22) is uniquely solvable, we have to estimate \( I_m \) and \( N(\phi) \) respectively.

**Lemma 2.3.** For all \( (r,t) \in \mathbb{S}_m \), there exists a small \( \sigma > 0 \), such that

\[
\|I_m\| \leq \frac{C}{m^{\frac{q-1}{4(1-q)}+\sigma}}.
\] (2.23)

**Proof.** We know that for any \( \phi \in \mathbb{E}_1 \)

\[
(I_m, \phi) = \sum_{i=1}^{m} \int_{\mathbb{R}^3} \left( V(|y|) - 1 \right) \left( U_{\tau_i} + U_{\bar{\tau}_i} \right) \phi
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{m} T[(U_{\tau_i})^2]U_{\tau_i} + \sum_{i=1}^{m} T[(U_{\bar{\tau}_i})^2]U_{\bar{\tau}_i} - T[W_{r,t}^2]W_{r,t} \right) \phi.
\] (2.24)

With the help of symmetric property of the functions and an elementary calculation, we can get

\[
1 \left| x - \tau_i \right|^t = \frac{1}{|\tau_i|^t} \left( 1 + O \left( \frac{|x|}{|\tau_i|} \right) \right), \quad x \in B_{\frac{1}{2} \tau_i}(0), \quad \forall t > 0.
\] (2.25)
Assuming that $m$ is even, we claim that
\[ |y - \overline{\mathbf{r}}_1| \geq (4 - \tau) \frac{r}{m} \pi \quad \text{for} \quad 5 \leq l \leq \frac{m}{2}, \quad y \in \Omega_1^+, \]
for some small $\tau$ and sufficiently large $m$. In fact, we know
\[
|y - \overline{\mathbf{r}}_1| \geq |\overline{\mathbf{r}}_1 - \overline{\mathbf{r}}_l| - |y - \overline{\mathbf{r}}_l| \geq 2r\sqrt{1 - l^2} \sin \left(\frac{(l - 1)\pi}{m}\right) - 4 \frac{r}{m} \pi \geq (4 - \tau) \frac{r}{m} \pi, \quad \text{if} \quad |y - \overline{\mathbf{r}}_l| \leq \frac{4r}{m} \pi,
\]
and
\[
|y - \overline{\mathbf{r}}_1| \geq |y - \overline{\mathbf{r}}_l| \geq 4 \frac{r}{m} \pi, \quad \text{if} \quad |y - \overline{\mathbf{r}}_l| \geq 4 \frac{r}{m} \pi.
\]
By the symmetric property of the functions, we can get
\[
\sum_{i=1}^{m} \int_{\mathbb{R}^3} (V(|y|) - 1) (U_{\mathbf{r}_i} + U_{\mathbf{x}_i}) \phi = 2m \int_{\mathbb{R}^3} (V(|y|) - 1) U_{\mathbf{r}_1} \phi
\]
\[
= 2m \left( \int_{\Omega_1^+} - \int_{\Omega_1^-} \right) + \int_{\bigcup_{4 \leq l \leq \frac{m}{2}} \Omega_i^+} + \int_{\Omega_1^+} + \int_{\bigcup_{\frac{m}{2} < l \leq m} \Omega_i^+} \right) (V(|y|) - 1) U_{\mathbf{r}_1} \phi \]
\[
\leq Cm \left( \int_{\Omega_1^+} + \int_{\bigcup_{4 \leq l \leq \frac{m}{2}} \Omega_i^+} \right) (V(|y|) - 1) U_{\mathbf{r}_1} \phi \]
\[
\leq Cm \left( \int_{\Omega_1^+} + \int_{\bigcup_{4 \leq l \leq \frac{m}{2}} \Omega_i^+} \right) (V(|y|) - 1) U_{\mathbf{r}_1} \phi \]
\[
+ Cm \int_{\bigcup_{4 \leq l \leq \frac{m}{2}} \Omega_i^+} (V(|y|) - 1) U_{\mathbf{r}_1} \phi, \quad \mathbf{(2.26)}
\]
where $\delta_1 > 0$ is some small enough constant. There exists a positive constant $C$ such that
\[
(V(|y|) - 1) U_{\mathbf{r}_1} \leq \frac{C}{|y - \overline{\mathbf{r}}_1|} e^{-|y - \overline{\mathbf{r}}_1|}
\]
\[
\leq \frac{C}{r^q} e^{-|y - \overline{\mathbf{r}}_1|}, \quad \text{for} \quad y \in \bigcup_{l=1}^{4} \Omega_l^+ \cap B_{\delta_1 \mathbf{r}_1} (\overline{\mathbf{r}}_1), \quad \mathbf{(2.27)}
\]
\[
(V(|y|) - 1) U_{\mathbf{r}_1} \leq C e^{-|y - \overline{\mathbf{r}}_1|}
\]
\[
\leq \frac{C}{r^q} e^{-\delta_1 r}, \quad \text{for} \quad y \in \bigcup_{l=1}^{4} \Omega_l^+ \setminus B_{\delta_1 \mathbf{r}_1} (\overline{\mathbf{r}}_1), \quad \mathbf{(2.28)}
\]
\[
(V(|y|) - 1) U_{\mathbf{r}_1} \leq C e^{-|y - \overline{\mathbf{r}}_1|}
\]
\[
\leq e^{-(4 - \tau) \frac{r}{m} \pi} \leq \frac{C}{m^{\frac{1+q}{2}}}, \quad \text{for} \quad y \in \bigcup_{4 \leq l \leq \frac{m}{2}} \Omega_i^+. \quad \mathbf{(2.29)}
\]
Then by symmetry, Hölder inequality and combining $\mathbf{(2.26)}$, $\mathbf{(2.29)}$ together, we can have
\[
\sum_{i=1}^{m} \int_{\mathbb{R}^3} (V(|y|) - 1) (U_{\mathbf{r}_i} + U_{\mathbf{x}_i}) \phi \leq C \left( \frac{1}{(m \ln m)^{1-q}} + \frac{1}{m^{\frac{q}{1-q}}} \right) ||\phi|| \leq \frac{C}{m^{\frac{1+q}{2}}} ||\phi||. \quad \mathbf{(2.30)}
\]
The last inequality is true as $\frac{1}{2} \leq n < 1.$

We know that

\[ U_{\mathcal{T}_i} + U_{\mathcal{Z}_i} \geq U_{\mathcal{T}_i} + U_{\mathcal{Z}_j} \quad \text{for} \quad y \in \Omega_1, \]

\[ U_{\mathcal{T}_j} \geq U_{\mathcal{Z}_j} \quad \text{for} \quad y \in \Omega_1^+. \]

From the second term in (2.21), we can get

\[
\frac{1}{2} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{m} T((U_{\mathcal{T}_i}))^2 U_{\mathcal{T}_i} + \sum_{i=1}^{m} T((U_{\mathcal{Z}_i}))^2 U_{\mathcal{Z}_i} - T[W_{r,t}^2]W_{r,t} \right) \phi
\]

\[ \begin{aligned}
&= \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \sum_{i=1}^{m} \frac{U_{\mathcal{T}_i}^2(y)}{|x-y|} U_{\mathcal{T}_i} \phi dx dy + \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \sum_{i=1}^{m} \frac{U_{\mathcal{Z}_i}^2(y)}{|x-y|} U_{\mathcal{Z}_i} \phi dx dy \right) \right)
\end{aligned}
\]

\[ \begin{aligned}
&- \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \sum_{i,j=1}^{m} \frac{(U_{\mathcal{T}_i} + U_{\mathcal{Z}_i})(U_{\mathcal{T}_j} + U_{\mathcal{Z}_j})(y)}{|x-y|} \phi dx dy \right) \left( \sum_{i=1}^{m} (U_{\mathcal{T}_i} + U_{\mathcal{Z}_i}) \phi \right)
\end{aligned}
\]

\[ \begin{aligned}
&= \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{T}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{T}_i} \phi \right) - \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{Z}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{Z}_i} \phi \right)
\end{aligned}
\]

\[ \begin{aligned}
&+ O \left( \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \sum_{i,j=1}^{m} (U_{\mathcal{T}_i} U_{\mathcal{T}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j})(y) \phi dx \right) \right)
\end{aligned}
\]

\[ \begin{aligned}
&= \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{T}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{T}_i} \phi \right) - \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{Z}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{Z}_i} \phi \right)
\end{aligned}
\]

\[ \begin{aligned}
&+ O \left( \frac{1}{8\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \sum_{i,j=1}^{m} (U_{\mathcal{T}_i} U_{\mathcal{T}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j} + U_{\mathcal{Z}_i} U_{\mathcal{Z}_j})(y) \phi dx \right) \right)
\end{aligned}
\]

\[ \begin{aligned}
&= \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{T}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{T}_i} \phi \right) - \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{Z}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{Z}_i} \phi \right)
\end{aligned}
\]

\[ \begin{aligned}
&+ O \left( \sum_{i \neq j} \left[ e^{-(1-\delta)|\mathcal{T}_i - \mathcal{T}_j|} + e^{-(1-\delta)|\mathcal{T}_i - \mathcal{Z}_j|} \right] \phi \phi \left[ e^{-(1-\delta)|\mathcal{Z}_i - \mathcal{T}_j|} + e^{-(1-\delta)|\mathcal{Z}_i - \mathcal{Z}_j|} \right] \right)
\end{aligned}
\]

\[ \begin{aligned}
&= \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{T}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{T}_i} \phi \right) - \frac{m}{4\pi} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{(U_{\mathcal{Z}_i})^2(y)}{|x-y|} \phi dx \right) \left( \sum_{i=1}^{m} U_{\mathcal{Z}_i} \phi \right)
\end{aligned}
\]

\[ \begin{aligned}
&+ O \left( \sum_{i \neq j} \left[ e^{-(1-\delta)|\mathcal{T}_i - \mathcal{T}_j|} + e^{-(1-\delta)|\mathcal{T}_i - \mathcal{Z}_j|} \right] \phi \phi \left[ e^{-(1-\delta)|\mathcal{Z}_i - \mathcal{T}_j|} + e^{-(1-\delta)|\mathcal{Z}_i - \mathcal{Z}_j|} \right] \right)
\end{aligned}
\]
From (1.6) and (2.32), we can get that for any
c which insures (2.32) if

\[
\frac{1}{|x-y|} \int \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=1}^{m} U_{x_i} \right) \phi - \frac{1}{|x-y|} \int \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=1}^{m} U_{x_i} \right) \phi
\]

\[+ O \left( \sum_{i \neq j} e^{-(1-\delta)|x_i-x_j|} + e^{-(1-\delta)|x_i-x_j|} \right)\]

\[= - \frac{m}{4\pi} D_i + O \left( e^{-2\pi(1-\delta)\sqrt{1-\epsilon} \frac{r^2}{m}} + e^{-2(1-\delta)rt} \right),\]

where

\[D_i = \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=1}^{m} U_{x_i} \right) \phi.\]

Next, we will estimate \(D_i\). We easily know that for any \(y \in \Omega_j^+,\)

\[|y - x_i| \geq \frac{1}{2}|x_j - x_i|,\]

(2.32)

In fact, we can easily verify that

\[|y - x_j| \leq |y - x_i|,\]

which insures (2.32) if \(|y - x_j| \geq \frac{1}{2}|x_j - x_i|\). On the other hand, we have

\[|y - x_i| \geq |x_j - x_i| - |y - x_j| \geq \frac{1}{2}|x_j - x_i|.

From (1.6) and (2.32), we can get that for any \(y \in \Omega_j^+,\)

\[U_{x_i} \leq C e^{-(1-\delta)|y-x_i|} \leq C e^{-\frac{1-\delta}{2}|x_j-x_i|},\]

(2.33)

where \(\delta\) is any constant small enough.

By \(A.6\), \(A.10\), \(2.33\) and Hölder inequality, we can get

\[D_i \leq C \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + C \sum_{j=2}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy U_{x_1} \phi
\]

\[+ C \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + C \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy U_{x_1} \phi
\]

\[= C \left[ \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + \sum_{j=2}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=1}^{m} U_{x_i} \right) \phi \right]
\]

\[+ C \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + C \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy U_{x_1} \phi
\]

\[+ C \left( \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy \left( \sum_{i=2}^{m} U_{x_i} \right) \phi + \sum_{j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy U_{x_1} \phi \right)
\]

\[\leq C \left( \sum_{i=2}^{m} e^{-\frac{1-\delta}{2}|x_i-x_1|} + \sum_{i,j=1}^{m} e^{-\frac{1-\delta}{2}|x_i-x_j|} + \sum_{i,j=1}^{m} \int_{\Omega_i} \int_{\Omega_i} \frac{(U_{x_1})^2(y)}{|x-y|} dy U_{x_1} \phi \right).
\]
\begin{align*}
&+ C \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi \\
&+ C \left( \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi + \sum_{i,j=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_2})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&\leq C \left( \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&+ C \left( \sum_{i,j=1}^{m} e^{-\frac{1}{2}\delta \|x_i-x_j\|} + \sum_{i,j=1}^{m} e^{-\frac{1}{2}\delta \|x_i-x_j\|} \right) \\
&\leq C \left( \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&+ C \left( \sum_{i,j=1}^{m} e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \sin \frac{\|x_i-x_j\|}{m}} + \sum_{i,j=1}^{m} e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \sin \frac{\|x_i-x_j\|}{m}} \right) \\
&\leq C \left( \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&+ C m \left( \sum_{i=1}^{m-1} e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} + e^{-(1-\delta)r} \right) \\
&\leq C \left( \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&+ C m \left( e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} + e^{-(1-\delta)r} \right) \\
&= C \left( \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_1} \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy U_{m_2} \phi \right) \\
&+ C \left( m e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} + m e^{-\frac{1}{2}\delta 2r e^{-\frac{1}{2}\delta 2r}} \right) \\
&= C \left[ \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy (U_{m_1}) \phi + \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{\Omega_+^i} \frac{(U_{m_1})^2(y)}{|x-y|} dy (U_{m_2}) \phi \right] \\
&+ O \left( e^{-\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} + e^{-\frac{1}{2}\delta 2r} \right) \\
&\leq C \sum_{i=2}^{m} \int_{\Omega_+^i} \int_{B_{\frac{|x_i-x_j|}{\delta} \cap \Omega_+^i} \cap \Omega_+^i} e^{-(1-\delta)(2|y-x_i|+|x-x_j|)} \frac{\phi}{|x-y|} dy dx \\
&+ O \left( e^{\frac{1}{2}\delta 2r \sqrt{1-t^2} \frac{x_i}{m}} + e^{\frac{1}{2}\delta 2r} \right) \\
&+ C \sum_{i=1}^{m} \int_{\Omega_+^i} \int_{B_{\frac{|x_i-x_j|}{\delta} \cap \Omega_+^i} \cap \Omega_+^i} e^{-(1-\delta)(2|y-x_i|+|x-x_j|)} \frac{\phi}{|x-y|} dy dx
\end{align*}
\begin{align*}
&\leq C \sum_{i=2}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (W_{r,t} + \phi)(y)\phi(y)(W_{r,t} + \phi)^2(x) \frac{\phi}{|x-y|} dy dx + \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (W_{r,t} + \phi)^2(y)(W_{r,t} + \phi)(x) \frac{\phi}{|x-y|} dx dy \\
&\quad + \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} (W_{r,t} + \phi)(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} (W_{r,t} + \phi)(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy \\
&\quad - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} W_{r,t}(y) v(y) dy) W_{r,t}(x) \phi(x) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} W_{r,t}(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy
\end{align*}

With \( \phi(y) = o_r(1) \) when \( |y| \geq \frac{r}{2} \), from (2.24), (2.30), (2.31) and (2.34), we can obtain that

\[ |I(\phi)| \leq \frac{C}{m^{\frac{dq-1}{2(1-q)}}} \| \phi \|, \]

because of \( \frac{1}{2} \leq q < 1 \). The proof is completed. \( \square \)

**Lemma 2.4.** There exists a positive constant \( C \) satisfying

\[ \| N'(\phi) \| \leq C \| \phi \|^2 \text{ and } \| N''(\phi) \| \leq C \| \phi \|. \]

**Proof.** From (1.26) and by direct calculation, there exists \( v \in H_0^1 \) satisfying

\[ N'(\phi) \]

\[ = \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (W_{r,t} + \phi)(y)\phi(y)(W_{r,t} + \phi)^2(x) \frac{\phi}{|x-y|} dy dx + \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (W_{r,t} + \phi)^2(y)(W_{r,t} + \phi)(x) \frac{\phi}{|x-y|} dx dy \\
\quad + \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} (W_{r,t} + \phi)(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} (W_{r,t} + \phi)(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy \\
\quad - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} W_{r,t}(y) v(y) dy) W_{r,t}(x) \phi(x) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} W_{r,t}(y) \phi(y) dy) W_{r,t}(x) v(x) dx dy. \]
is invertible. Moreover, we can write (1.27) as follows:

\[
\frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{r,t}(y)v(y)\phi^2(x)}{|x-y|} + \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)v(y)\phi^2(x)}{|x-y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)v(y)W_{r,t}(x)\phi(x)}{|x-y|},
\]

(2.35)

and

\[
N''(\phi) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(y)(W_{r,t} + \phi)^2(x)}{|x-y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r,t} + \phi)(y)v(y)(W_{r,t} + \phi)(x)v(x)}{|x-y|} \\
- \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)v^2(x)}{|x-y|} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{r,t}(y)v(y)\phi(x)v(x) dy}{|x-y|} \\
+ \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)v(y)\phi(x)v(x)}{|x-y|} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)v(y)\phi(x)v(x)}{|x-y|}.
\]

(2.36)

Then, by Hardy-Littlewood-Sobolev inequality, Hölder inequality and Sobolev inequality, we can have

\[\|N'(\phi)\| \leq C(\|\phi\|^2 + \|\phi\|^3) \leq C\|\phi\|^2, \quad \|N''(\phi)\| \leq C(\|\phi\|^2 + \|\phi\|) \leq C\|\phi\|.\]

\[\square\]

Now we are in a position to prove Proposition 2.2.

Proof of Proposition 2.2: From Riesz theorem, there exists a \(1_m \in E_1\) satisfying

\[I(\phi) = (1_m, \phi), \quad \forall \phi \in E_1, \quad \text{and} \quad \|1_m\| = \|\phi\|.
\]

So, \(\phi\) which is a critical point of \(R\) in \(E_1\) is a solution to (1.27). From Lemma 2.1, we know that \(L\) is invertible. Moreover, we can write (1.27) as follows:

\[\phi = M(\phi) := -L^{-1}(1_m - N'(\phi)).\]

Define

\[B := \left\{ \phi \in E_1 : \|\phi\| \leq C\left(\frac{1}{m}\right)^{\frac{2q-1}{2(1-q)}}\right\}.
\]

We can claim that \(M\) is a contraction map from \(B\) to \(B\). In fact, from Lemma 2.4, we can obtain

\[\|N'(\phi)\| \leq C\|\phi\|^2 \text{ and } \|N''(\phi)\| \leq C\|\phi\|.
\]

From Lemmas 2.4-2.3 we can get that

\[\|M(\phi)\| \leq C\|1_m\| + C\|\phi\|^2 \leq C\left(\frac{1}{m}\right)^{\frac{2q-1}{2(1-q)}} + C\|\phi\|^2.
\]

Moreover, we have

\[\|M(\phi_1) - M(\phi_2)\| \leq C\|N'(\phi_1) - N'\phi_2)\| \leq C\left(\frac{1}{m}\right)^{\frac{2q-1}{2(1-q)}} \|\phi_1 - \phi_2\| \leq \frac{1}{2} \|\phi_1 - \phi_2\|,
\]

for \(m\) large enough. We complete the proof.
3. Proof of Theorem 1.1

In this section, we mainly prove Theorem 1.1. In order to prove Theorem 1.1, the following proposition is necessary.

**Proposition 3.1.** Assume that \( \Phi(r, t) = \phi_{r, t}(y) \) with \( \Phi(r, t) \) be the map which is defined in Proposition 2.2. Define

\[
\mathbf{F}(r, t) = I(W_{r, t} + \phi_{r, t}(y)), \quad \forall \ (r, t) \in S_m.
\]

If \((r, t)\) is a critical point of \( \mathbf{F}(r, t) \), then

\[
u = W_{r, t} + \phi_{r, t}(y),
\]

is a critical point of \( I(u) \) in \( H^1(\mathbb{R}^3) \). \( \square \)

**Proof.** From (1.22), we can know that

\[
\mathbf{F}(r, t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla(W_{r, t} + \phi_{r, t})|^2 + V(|x|)(W_{r, t} + \phi_{r, t})^2
\]

\[
- \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)(W_{r, t} + \phi_{r, t})^2(x)}{|x - y|} dx dy.
\]

The direct calculation leads to

\[
\mathbf{F}_r = \int_{\mathbb{R}^3} -\Delta (W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial r} + \frac{\partial \phi}{\partial r} \right) + V(|x|)(W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial r} + \frac{\partial \phi}{\partial r} \right) dx
\]

\[
- \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)(W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial r} + \frac{\partial \phi}{\partial r} \right)(x)}{|x - y|} dx dy.
\]

Similarly,

\[
\mathbf{F}_t = \int_{\mathbb{R}^3} -\Delta (W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial t} + \frac{\partial \phi}{\partial t} \right) + V(|x|)(W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial t} + \frac{\partial \phi}{\partial t} \right) dx
\]

\[
- \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)(W_{r, t} + \phi_{r, t}) \left( \frac{\partial W_{r, t}}{\partial t} + \frac{\partial \phi}{\partial t} \right)(x)}{|x - y|} dx dy.
\]

From (1.24), we know that

\[
E_{j}^* = \text{span} \left\{ \frac{\partial U_{x_j}}{\partial r}, \frac{\partial U_{x_j}}{\partial t}, \frac{\partial U_{x_j}}{\partial t}, \frac{\partial U_{x_j}}{\partial t}, j = 1, 2, 3 \right\}.
\]

Since \( R'(\phi_{r, t}) = 0 \) in \( E_1 \) and \( \phi_{r, t} \in E_1 \), we have

\[
\langle R', \phi_{r, t} \rangle = \int_{\mathbb{R}^3} [-\Delta (W_{r, t} + \phi_{r, t}) + V(|x|)(W_{r, t} + \phi_{r, t})] \phi_{r, t} dx
\]

\[
- \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)(W_{r, t} + \phi_{r, t}) \phi_{r, t}(x)}{|x - y|} dy dx = 0.
\]

So, we can get that

\[
-\Delta (W_{r, t} + \phi_{r, t}) + V(|x|)(W_{r, t} + \phi_{r, t}) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)}{|x - y|} dy (W_{r, t} + \phi_{r, t}) \in E_1^*.
\]

Then, there exists some constants \( C_1, C_2, C_3, C_4 \), satisfying

\[
-\Delta (W_{r, t} + \phi_{r, t}) + V(|x|)(W_{r, t} + \phi_{r, t}) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{(W_{r, t} + \phi_{r, t})^2(y)}{|x - y|} dy (W_{r, t} + \phi_{r, t})
\]
which is a critical point of
Similarly, we can prove that the others have the same estimations. So we can get
which implies that
Now we will prove the Theorem 1.1.
\[
\sum_{i=1}^{m} C_i \frac{\partial U_{x_i}}{\partial r} + \sum_{j=1}^{m} C_j \frac{\partial U_{z_j}}{\partial r} + \sum_{p=1}^{m} C_p \frac{\partial U_{x_p}}{\partial t} + \sum_{q=1}^{m} C_q \frac{\partial U_{z_q}}{\partial t}.
\]
(3.4)
So we have
\[
\mathcal{F}_r + \mathcal{F}_t = \left\langle \sum_{i=1}^{m} C_i \frac{\partial U_{x_i}}{\partial r}, \frac{\partial \phi}{\partial r} \right\rangle + \left\langle \sum_{j=1}^{m} C_j \frac{\partial U_{z_j}}{\partial r}, \frac{\partial \phi}{\partial r} \right\rangle + \left\langle \sum_{p=1}^{m} C_p \frac{\partial U_{x_p}}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle + \left\langle \sum_{q=1}^{m} C_q \frac{\partial U_{z_q}}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle
\]
= 0.
Noting that \( \phi_{r,t} \in \mathbb{E}_1 \), we have
\[
\left\langle \frac{\partial U_{x_i}}{\partial r}, \phi \right\rangle = \left\langle \frac{\partial U_{z_j}}{\partial r}, \phi \right\rangle = \left\langle \frac{\partial U_{x_p}}{\partial t}, \phi \right\rangle = \left\langle \frac{\partial U_{z_q}}{\partial t}, \phi \right\rangle = 0.
\]
By the direct calculations, we have
\[
\left\langle \frac{\partial U_{x_i}}{\partial r}, \frac{\partial \phi}{\partial r} \right\rangle = -\left\langle \frac{\partial^2 U_{x_i}}{\partial r^2}, \phi \right\rangle, \quad \left\langle \frac{\partial U_{z_j}}{\partial r}, \frac{\partial \phi}{\partial r} \right\rangle = -\left\langle \frac{\partial^2 U_{z_j}}{\partial r^2}, \phi \right\rangle,
\]
\[
\left\langle \frac{\partial U_{x_i}}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle = -\left\langle \frac{\partial^2 U_{x_i}}{\partial t^2}, \phi \right\rangle, \quad \left\langle \frac{\partial U_{z_j}}{\partial t}, \frac{\partial \phi}{\partial t} \right\rangle = -\left\langle \frac{\partial^2 U_{z_j}}{\partial t^2}, \phi \right\rangle,
\]
which implies that
\[
\left| \left\langle \frac{\partial U_{x_i}}{\partial r}, \frac{\partial \phi}{\partial r} \right\rangle \right| = \left| -\left\langle \frac{\partial^2 U_{x_i}}{\partial r^2}, \phi \right\rangle \right| = O(\|\phi\|) = O\left(\left(\frac{1}{m}\right)^{\frac{2n-1}{2(1-n)+\sigma}}\right).
\]
Similarly, we can prove that the others have the same estimations. So we can get
\[
\mathcal{F}_r + \mathcal{F}_t = \sum_{i=1}^{m} C_i \left(\frac{\partial U_{x_i}}{\partial r}\right)^2 + \sum_{j=1}^{m} C_j \left(\frac{\partial U_{z_j}}{\partial r}\right)^2 + \sum_{p=1}^{m} C_p \left(\frac{\partial U_{x_p}}{\partial t}\right)^2 + \sum_{q=1}^{m} C_q \left(\frac{\partial U_{z_q}}{\partial t}\right)^2 + O\left(\left(\frac{1}{m}\right)^{\frac{2n-1}{2(1-n)+\sigma}}\right)
\]
= 0.
So, we can get \( C_i, C_j, C_p, C_q = 0 \). Consequently, \( I'(W_{r,t} + \phi_{r,t}) = 0 \) in \( H^1(\mathbb{R}^3) \) which shows that \( W_{r,t} + \phi_{r,t}(y) \) is a critical point of \( I(u) \) in \( H^1(\mathbb{R}^3) \). We complete the proof.

Now we will prove the Theorem 1.1.

Proof of Theorem 1.1 : By Proposition 1.1, we need to prove that there exists \((r_m, t_m) \in \mathbb{S}_m\), which is a critical point of \( F(r, t) \).
Indeed, from Proposition B.1 we have

\[ F(r, t) = I(W_{r,t}) + 1(\phi_{r,t}) + \frac{1}{2} \langle L\phi_{r,t}, \phi_{r,t} \rangle - N(\phi_{r,t}) \]

= \( I(W_{r,t}) + O_m(\|1_m\|\|\phi_{r,t}\| + \|\phi_{r,t}\|^2) = I(W_{r,t}) + O_m\left( \frac{1}{m^{\tau-\gamma+\sigma}} \right) \)

= \( m\left( \frac{A_2}{16\pi} + \frac{A_1}{r^q} - \frac{A_1^2}{16\pi^2} \frac{m \ln m}{r} - \frac{m}{16\pi^2} \frac{1}{r^2 - t^2} \right) \left( \frac{1}{r^q + \tau} \right) + mO_m\left( \frac{1}{m^{\tau-\gamma+\sigma}} \right) \)

= \( m\left( \frac{m^2}{r^2 (1-t^2)} \right) + m\left( \frac{m^2}{r^2 - t^2} \right) \ln \frac{\pi}{t} O_m\left( t^2 \ln t \right) + O_m\left( \frac{1}{m^{\tau-\gamma+\sigma}} \right) \),

where \( A_1, A_2 \) are constants in \([3.12]\).

Define

\[ F_1(r, t) = \frac{A_2}{16\pi} + \frac{A_1}{r^q} - \frac{A_1^2}{16\pi^2} \frac{m \ln m}{r} - \frac{m}{16\pi^2} \frac{1}{r^2 - t^2} \ln \frac{\pi}{t} A_1^2. \]

So, we consider the following system

\[
\begin{aligned}
&-A_1 \frac{q}{r^{q+\tau}} + \frac{A_1^2}{16\pi^2} \frac{m \ln m}{r^{1-t^2}} + \frac{A_1^2}{16\pi^2} \frac{m}{r^{1-t^2}} \ln \frac{\pi}{t} = 0, \\
&-\frac{tm \ln m}{r^{1-t^2}} - \frac{m}{r(1-t^2)} \ln \frac{\pi}{t} = 0,
\end{aligned}
\]

ie,

\[
\begin{aligned}
&-A_1 \frac{q}{r^{q+\tau}} + \frac{A_1^2}{16\pi^2} \frac{m \ln m}{r^{1-t^2}} + \frac{A_1^2}{16\pi^2} \frac{m}{r^{1-t^2}} \ln \frac{\pi}{t} = 0, \\
&-\frac{tm \ln m}{r^{1-t^2}} - \frac{m}{r(1-t^2)} \ln \frac{\pi}{t} = 0,
\end{aligned}
\]

ie,

\[
\begin{aligned}
&-A_1 \frac{q}{r^{q+\tau}} + \frac{A_1^2}{16\pi^2} \frac{m \ln m}{r^{1-t^2}} + \frac{A_1^2}{16\pi^2} \frac{m}{r^{1-t^2}} \ln \frac{\pi}{t} = 0, \\
&-\frac{tm \ln m}{r^{1-t^2}} - \frac{m}{r(1-t^2)} \ln \frac{\pi}{t} + \frac{1}{\tau} = 0.
\end{aligned}
\]

The problem is equivalent to the following fixed point

\[ A(t) = \frac{\ln m}{(1-t^2)} + \frac{1}{(1-t^2)} \ln \frac{\pi}{t}, \text{ where } A(t) := \frac{1}{t^2}. \]

Then for any \((r, t) \in \mathbb{S}_m\), define

\[ a(t) = A^{-1}\left( \frac{\ln m}{(1-t^2)} + \frac{1}{(1-t^2)} \ln \frac{\pi}{t} \right) \]

= \( \frac{1}{(\ln m + \ln \pi - \ln t)^{\frac{1}{2}}} \)

= \( \frac{1}{(1-t^2)^{\frac{1}{2}}} \)

= \( \frac{1}{(\ln m + \ln \pi)^{\frac{1}{2}} (1 - \frac{\ln t}{\ln m + \ln \pi})^{\frac{1}{2}}} \)
By the direct calculations, we have
\[
|a(t_1) - a(t_2)| = \left| \frac{(1 - t_1^2)^{\frac{1}{2}}}{\ln m + \ln \pi} \left(1 - \frac{1}{\ln m + \ln \pi} t_1^2\right) - \frac{(1 - t_2^2)^{\frac{1}{2}}}{\ln m + \ln \pi} \left(1 - \frac{1}{\ln m + \ln \pi} t_2^2\right) \right| = \frac{1 + o(1)}{\ln m + \ln \pi} \left| (1 - t_1^2)^{\frac{1}{2}} - (1 - t_2^2)^{\frac{1}{2}} \right| = \frac{1 + o(1)}{\ln m + \ln \pi} |t_1 - t_2| O(t) = O\left(\frac{1}{\ln m}\right)|t_1 - t_2| = \theta |t_1 - t_2|,
\]
(3.13)
where \(0 < \theta < 1\), and we have \(t \sim (\ln m)^{-\frac{1}{2}}\) by Remark 3.2. With the help of the Contraction Mapping principle, we can have that there is a fixed point \((\bar{r}_m, \bar{t}_m) \in \bar{S}_m\). In other words, \(F_1(r, t)\) have a critical point \((\bar{r}_m, \bar{t}_m) \in \bar{S}_m\).

Define
\[
B_2(r, t) = \begin{pmatrix} F_{1,rr}(r, t) & F_{1,rt}(r, t) \\ F_{1,rt}(r, t) & F_{1,tt}(r, t) \end{pmatrix}.
\]
By computing, we can get
\[
F_{1,rr}(r, t) = (\bar{r}_m, \bar{t}_m) > 0, \quad F_{1,rt}(r, t) = (\bar{r}_m, \bar{t}_m) < 0, \quad F_{1,rt}(r, t) = (\bar{r}_m, \bar{t}_m) > 0,
\]
and
\[
F_{1,rr} \times F_{1,tt} - F_{1,rt}^2 < 0.
\]
Consequently, we can obtain that \((\bar{r}_m, \bar{t}_m)\) is a maximum point of \(F_1(r, t)\). So the maximum of \(F_1(r, t)\) in \(S_m\) can be achieved.

So, for the function \(F(r, t)\), we can seek a maximum point \((\bar{r}_m, \bar{t}_m)\) that is an interior point of \(S_m\). Thus, \((\bar{r}_m, \bar{t}_m)\) is a critical point of \(F(r, t)\). Then
\[
W_{r_m, t_m} + \phi_{r_m, t_m}(y),
\]
is a critical point of \(I(u)\). The proof of Theorem 1.1 is completed. \(\square\)

**Remark 3.2.** By (3.5), we have
\[
F_{1,r}(r, t) = -A_1 \frac{q}{r^{q+1}} + \frac{A_1^2}{16\pi^2(1 - t^2)} \frac{m \ln m}{r^2} + \frac{mA_1^2}{16\pi^2 \sqrt{1 - t^2}} \ln \frac{\pi}{t} = 0.
\]
Since the second term is much bigger than the third term, we only keep the first term and the second term. By (3.5), we also have
\[
F_{1,t}(r, t) = -\frac{tA_1^2}{16\pi^2(1 - t^2)} \frac{m \ln m}{r} - \frac{tA_1^2}{16\pi^2(1 - t^2)} \frac{m \ln \frac{\pi}{t} + mA_1^2}{r} + \frac{mA_1^2}{16\pi^2 \sqrt{1 - t^2}} \frac{1}{t}.
\]
For the simplicity of getting \(t\), we only keep the first term and the third term. Then, we consider the following system
\[
\begin{align*}
F_{1,r}(r, t) &= -A_1 \frac{q}{r^{q+1}} + \frac{A_1^2}{16\pi^2 \sqrt{1 - t^2}} \frac{m \ln m}{r^2} \approx 0, \\
F_{1,t}(r, t) &= -\frac{tA_1^2}{16\pi^2(1 - t^2)} \frac{m \ln m}{r} + \frac{mA_1^2}{16\pi^2 \sqrt{1 - t^2}} \frac{1}{t} \approx 0.
\end{align*}
\]
(3.14)
And we can get
\[ r = \left( \frac{A_1}{16q(\pi)^2} \right)^{1/7} (1 + o(1)) (m \ln m)^{-1/7}, t = \left( 1 + o(1) \right) (\ln m)^{-1/4}. \]

So we assume that \((r, t)\) satisfies \((\ref{1.20})\).

**APPENDIX A. SOME KNOWN RESULTS AND TECHNICAL ESTIMATES**

Firstly, we provide some essential estimates.

**Lemma A.1.** For any \(u, v, \mu, \varphi \in H^1(\mathbb{R}^3)\), we have
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(y)v(y)\omega(x)v(x)}{|x-y|} \leq C \|u\|\|v\|\|\omega\|\|v\|. \tag{A.1}
\]

**Proof.** By Hardy-Littlewood-Sobolev inequality with \(\frac{1}{s} + \frac{1}{t} + \frac{1}{3} = 2\), Hölder inequality and Sobolev inequality, we can get
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(y)v(y)\omega(x)v(x)}{|x-y|} \leq \|uv\|_{L^s(\mathbb{R}^3)} \|\omega v\|_{L^t(\mathbb{R}^3)} \leq C \|u\|\|v\|\|\omega\|\|v\|. \tag{A.2}
\]

**Lemma A.2.** For \(r, t\) being the parameters in \((\ref{1.16})\) and any \(\delta \in (0, 1]\), there exists \(C > 0\) such that
\[
\sum_{i=2}^{m} U_{x_i}(y) \leq C e^{-\delta \sqrt{1-t^2} \frac{m}{8} e^{-\frac{1-\delta}{2}|y-x_1|}}, \quad \text{for all } y \in \Omega_1^+,
\]
\[
\sum_{i=2}^{m} U_{y_i}(y) \leq C e^{-\delta \sqrt{1-t^2} \frac{m}{8} e^{-\frac{1-\delta}{2}|y-x_1|}}, \quad \text{for all } y \in \Omega_1^+,
\]
and
\[
U_{x_1}(y) \leq C e^{-\delta \sqrt{t} e^{-\frac{1-\delta}{2}|y-x_1|}}, \quad \text{for all } y \in \Omega_1^+.
\]

**Proof.** Since the proof is just similar to Lemma A.1 in \((\ref{17})\), here we omit it. \(\square\)

In this appendices, we assume \((r, t) \in S_m\), where \(S_m\) is defined in \((\ref{1.20})\).

**Lemma A.3.** The following expansion is valid
\[
\sum_{i=2}^{m} \int_{\mathbb{R}^3} \Psi_{r, x_i} U_{x_1}^2 \leq \frac{A_1^2}{8\pi^2 r(1-t^2)} \left( \frac{m \ln m}{r^2(1-t^2)} + O\left( \frac{m^2}{r^2(1-t^2)} \right) \right), \tag{A.3}
\]
where \(A_1\) is defined in \((\ref{3.2})\).

**Proof.** The direct calculation shows that
\[
\sum_{i=2}^{m} \int_{\mathbb{R}^3} \Psi_{r, x_i} U_{x_1}^2 \leq \frac{1}{8\pi} \sum_{i=2}^{m} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_{x_i}^2(y)U_{x_1}(x)}{|x-y|}
\]
\[
= \frac{1}{8\pi} \sum_{i=2}^{m} \int_{\mathbb{R}^3} U^2(x) \int_{\mathbb{R}^3} U^2(y) \frac{1}{|x-y + (x_1 - x_i)|}
\]
\[
= \frac{1}{8\pi} \sum_{i=2}^{m} \int_{\mathbb{R}^3} U^2(x) \int_{\mathbb{R}^3} U^2(y) + O\left( \frac{1}{|x_1 - x_i|} \right). \tag{A.4}
\]
Observe that

\[
\sum_{i=2}^{m} \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|} = \frac{1}{2r \sqrt{1 - t^2}} \sum_{i=1}^{m-1} \frac{1}{\sin \frac{i\pi}{m}}. \tag{A.5}
\]

It has been checked that

\[
\int_{\frac{1}{2}}^{\frac{m-\frac{3}{2}}{2}} \frac{1}{\sin \frac{m\pi}{m}} \leq \sum_{i=1}^{m-1} \frac{1}{\sin \frac{i\pi}{m}} \leq \int_{\frac{1}{2}}^{\frac{m-\frac{1}{2}}{2}} \frac{1}{\sin \frac{m\pi}{m}},
\]

and

\[
\lim_{m \to \infty} \frac{1}{m \ln m} \int_{\frac{1}{2}}^{\frac{m-\frac{3}{2}}{2}} \frac{1}{\sin \frac{m\pi}{m}} = \lim_{m \to \infty} \frac{1}{m \log m} \int_{\frac{1}{2}}^{\frac{m-\frac{1}{2}}{2}} \frac{1}{\sin \frac{m\pi}{m}} = \frac{2}{\pi}.
\]

So, we can get

\[
\sum_{i=2}^{m} \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|} = \frac{m \ln m}{\pi r \sqrt{1 - t^2}} + o_m(1). \tag{A.6}
\]

From the definitions \(\mathbf{x}_j, \mathbf{\bar{x}}_j\), we have

\[
|\mathbf{x}_i - \mathbf{x}_1|^2 = 4r^2(1 - t^2) \sin^2 \frac{(i - 1)\pi}{m},
\]

Similarly, we obtain

\[
O \left( \sum_{i=2}^{m} \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|^2} \right) = O \left( \frac{m^2}{r^2(1 - t^2)} \right). \tag{A.7}
\]

From (A.4), (A.6) and (A.7), we can get (A.3). □

**Lemma A.4.** There is a small enough constant \(\delta > 0, \sigma > 0\) such that the following expansions are valid

\[
\int_{\mathbb{R}^3} U_{\mathbf{\bar{x}}_1} U_{\mathbf{x}_1} = O_m(e^{-2(1-\delta)rt}), \tag{A.8}
\]

\[
\sum_{i=2}^{m} \int_{\mathbb{R}^3} U_{\mathbf{\bar{x}}_i} U_{\mathbf{x}_1} = O_m(e^{-2\pi(1-\delta)\sqrt{1-t^2} \frac{r}{m}} + e^{-2(1-\delta)(1+\sigma)\pi \sqrt{1-t^2} \frac{r}{m}}), \tag{A.9}
\]

and

\[
\sum_{i=1}^{m} \int_{\mathbb{R}^3} U_{\mathbf{\bar{x}}_i} U_{\mathbf{x}_1} = O_m \left( e^{-2(1-\delta)rt} + e^{-2\pi(1-\delta)(1+\sigma)\sqrt{1-t^2} \frac{r}{m}} \right). \tag{A.10}
\]

**Proof.** Recalling the definitions \(\mathbf{x}_j, \mathbf{\bar{x}}_j\), we can know

\[
|\mathbf{x}_i - \mathbf{x}_1|^2 = 4r^2(1 - t^2) \sin^2 \frac{(i - 1)\pi}{m},
\]

\[
|\mathbf{x}_1 - \mathbf{\bar{x}}_1|^2 = 4r^2 t^2,
\]

and

\[
|\mathbf{x}_i - \mathbf{\bar{x}}_1|^2 = 4r^2 \left[ (1 - t^2) \sin^2 \frac{(i - 1)\pi}{m} + t^2 \right].
\]

From the property (1.6) of \(U\), we have

\[
\int_{\mathbb{R}^3} U_{\mathbf{\bar{x}}_1} U_{\mathbf{x}_1} = O_m(e^{-(1-\delta)|\mathbf{x}_1 - \mathbf{\bar{x}}_1|}) = O_m \left( e^{-2(1-\delta)rt} \right). \tag{A.11}
\]
Next, we calculate
\[
\sum_{i=2}^{m} \int_{\mathbb{R}^3} U_{\mathbf{x}_1} U_{\mathbf{x}_i} = O_m \left( \sum_{i=2}^{m} e^{-(1-\delta)|\mathbf{x}_1-\mathbf{x}_i|} \right).
\] (A.12)

Generally, we can suppose that \( m \) is even. Then
\[
\sum_{i=2}^{m} e^{-|\mathbf{x}_1-\mathbf{x}_i|} = \sum_{i=3}^{m/2} e^{-2r\sqrt{1-t^2} \sin \frac{(i-1)\pi}{m}}
+ \sum_{i=m/2+1}^{m-1} e^{-2r\sqrt{1-t^2} \sin \frac{(i-1)\pi}{m}} + 2e^{-2r\sqrt{1-t^2} \sin \frac{\pi}{m}}.
\] (A.13)

Consider
\[
c_0 \frac{(i-1)\pi}{m} \leq \sin \frac{(i-1)\pi}{m} \leq c_1 \frac{(i-1)\pi}{m}, \quad \text{for } i \in \{3, \ldots, m/2\},
\]
with \( \frac{1}{2} < c_0 \leq c_1 \leq 1 \). Then we can derive
\[
\sum_{i=3}^{m/2} e^{-2r\sqrt{1-t^2} \sin \frac{(i-1)\pi}{m}} \leq \sum_{i=3}^{m/2} e^{-2r\sqrt{1-t^2} \sin \frac{\pi}{m}}
= e^{-4r\sqrt{1-t^2} \sin \frac{\pi}{m}} - e^{-2r\sqrt{1-t^2} \sin \frac{\pi}{m}}
= (1 - e^{-2r\sqrt{1-t^2} \sin \frac{\pi}{m}}),
\] (A.14)

With the help of symmetry of function \( \sin x \), we have
\[
\sum_{i=m/2+1}^{m-1} e^{-2r\sqrt{1-t^2} \sin \frac{(i-1)\pi}{m}} = O_m \left( e^{-2(1+\sigma)\pi \sqrt{1-t^2} \frac{\pi}{m}} \right),
\]
in the same manner as (A.14). In the following, we can get
\[
e^{-2r\sqrt{1-t^2} \sin \frac{\pi}{m}} = e^{-2r\sqrt{1-t^2} \left( \frac{\pi}{m} + O_m \left( \frac{\pi^3}{m^3} \right) \right)}
= e^{-2r\sqrt{1-t^2} \frac{\pi}{m}} e^{-2r\sqrt{1-t^2} O_m \left( \frac{\pi^3}{m^3} \right)}
= e^{-2r\sqrt{1-t^2} \frac{\pi}{m}} + O_m \left( e^{-2(1+\sigma)\pi \sqrt{1-t^2} \frac{\pi}{m}} \right).
\] (A.15)

So,
\[
\sum_{i=2}^{m} e^{-(1-\delta)|\mathbf{x}_1-\mathbf{x}_i|} = O_m \left( e^{-2\pi(1-\delta)\sqrt{1-t^2} \frac{\pi}{m}} + e^{-2(1-\delta)(1+\sigma)\pi \sqrt{1-t^2} \frac{\pi}{m}} \right).
\] (A.16)

Following from (A.12) to (A.15), we have
\[
\sum_{i=2}^{m} \int_{\mathbb{R}^3} U_{\mathbf{x}_1} U_{\mathbf{x}_i} = O_m \left( e^{-2\pi(1-\delta)\sqrt{1-t^2} \frac{\pi}{m}} + e^{-2(1-\delta)(1+\sigma)\pi \sqrt{1-t^2} \frac{\pi}{m}} \right).
\] (A.17)

Finally, we estimate
\[
\sum_{i=1}^{m} \int_{\mathbb{R}^3} U_{\mathbf{x}} U_{\mathbf{x}_i} = O_m \left( e^{-(1-\delta)|\mathbf{x}_1-\mathbf{x}_i|} + \sum_{i=2}^{m} e^{-(1-\delta)|\mathbf{x}_1-\mathbf{x}_i|} \right).
\]
The proof of Lemma A.4 is completed.

Following from \((r, t) \in \mathbb{S}_m\), we can get
\[
    e^{-2r\left(1-t^2\sin^2\frac{\pi}{m}+t^2\right)^{\frac{1}{2}}} = e^{-2r(1-t^2)^{\frac{1}{2}}\sin \frac{\pi}{m}\left[1+\frac{t^2}{1-t^2}\sin^2\frac{\pi}{m}\right]} = O_m(e^{-2(1+\sigma)r(1-t^2)^{\frac{1}{2}}\frac{\pi}{m}}).
\]

Then combining (A.18) and (A.19) together, we have
\[
    \sum_{i=1}^{m} \int_{\mathbb{R}^3} U_{\Sigma} U_{\Sigma_i} = O_m(e^{-2(1-\delta)rt} + e^{-2\pi(1-\delta)(1+\sigma)\sqrt{1-t^2}\frac{\pi}{m}}).
\]

The proof of Lemma A.4 is completed. \(\square\)

**Lemma A.5.** There exists a positive constant \(C\) such that the following estimate is valid
\[
    \int_{\Omega_k^+} \left( \int_{\mathbb{R}^3} \frac{(U_{\Sigma} + \sum_{i=2}^{m} U_{\Sigma_i})^2(y)}{|x-y|} dy \right) v_m^2 \leq C \left( e^{-(1-\delta)R} + \frac{1}{R} \right) \int_{\Omega_1} v_m^2(x) dx + C \int_{B_R(\Sigma_1)} v_m^2(x) dx.
\]

**Proof.** Firstly, letting \(d = \frac{|x-(\Sigma_1, \Sigma_1)|}{2}\), if \(y \in B_d(x)\), we have
\[
    \int_{B_d(x)} \sum_{i=2}^{m} U_i^2(y-(\Sigma_i, \Sigma_1)) \frac{dy}{|x-y|} \leq C \sum_{i=2}^{m} \int_{B_d(x)} \frac{e^{-|y-(\Sigma_i, \Sigma_1)|}}{|x-y|} dy
\]
\[
    \leq C \sum_{i=2}^{m} \int_{B_d(x)} \frac{e^{-|x-(\Sigma_i, \Sigma_1)|}}{|x-y|} dy = C \sum_{i=2}^{m} e^{-|x-(\Sigma_i, \Sigma_1)|} \int_{B_d(x)} \frac{1}{|x-y|} dy
\]
\[
    = C \sum_{i=2}^{m} e^{-|x-(\Sigma_i, \Sigma_1)||x-(\Sigma_i, \Sigma_1)|^2}.\]

If \(y \in B_d(\Sigma_i, \Sigma_1)\), we have
\[
    \int_{B_d(\Sigma_i, \Sigma_1)} \sum_{i=2}^{m} U_i^2(y-(\Sigma_i, \Sigma_1)) \frac{dy}{|x-y|} \leq \sum_{i=2}^{m} \frac{C}{|x-(\Sigma_i, \Sigma_1)|} \int_{B_d(\Sigma_i, \Sigma_1)} e^{-2|y-(\Sigma_i, \Sigma_1)|} dy
\]
\[
    \leq \sum_{i=2}^{m} \frac{C}{|x-(\Sigma_i, \Sigma_1)|} \left[ e^{-|x-(\Sigma_i, \Sigma_1)||x-(\Sigma_i, \Sigma_1)|^2} + e^{-|x-(\Sigma_i, \Sigma_1)||x-(\Sigma_i, \Sigma_1)|} \right] + 1
\]
\[
    \leq C \sum_{i=2}^{m} \left[ e^{-|x-(\Sigma_i, \Sigma_1)||x-(\Sigma_i, \Sigma_1)|} \sum_{i=1}^{\infty} \frac{1}{|x-(\Sigma_i, \Sigma_1)|} \right].\]

If \(y \in \mathbb{R}^3 \setminus B_d(x) \cup B_d(\Sigma_i, \Sigma_1)\), we have \(|y-x| \geq |x-(\Sigma_1, \Sigma_1)|\) and \(|y-(\Sigma_i, \Sigma_1)| \geq |x-(\Sigma_1, \Sigma_1)|^2/2\).

If \(|y-(\Sigma_i, \Sigma_1)| \geq 2|x-(\Sigma_i, \Sigma_1)|\), then \(|x-y| \geq \frac{1}{4}|x-(\Sigma_i, \Sigma_1)|\).

If \(|y-(\Sigma_i, \Sigma_1)| \leq 2|x-(\Sigma_i, \Sigma_1)|\), then \(|y-x| \geq \frac{1}{4}|x-(\Sigma_i, \Sigma_1)|\).

In summary \(|y-(\Sigma_i, \Sigma_1)| \geq \frac{1}{4}|x-(\Sigma_i, \Sigma_1)|\).
Therefore,
\[
\int_{\mathbb{R}^3 \setminus B_d(x) \cup B_d(x_i - x_1)} \frac{\sum_{i=2}^{m} U^2(y - (x_i - x_1))}{|x - y|} dy \leq C \sum_{i=2}^{m} \int_{\mathbb{R}^3 \setminus B_d(x) \cup B_d(x_i - x_1)} \frac{e^{-2|y - (x_i - x_1)|}}{|x - (x_i - x_1)|} dy
\]
\[
\leq \sum_{i=2}^{m} \frac{C}{|x - (x_i - x_1)|} \int_{\mathbb{R}^3 \setminus B_d(x_i - x_1)} e^{-2|y - (x_i - x_1)|} dy
\]
\[
\leq \sum_{i=2}^{m} \frac{C}{|x - (x_i - x_1)|} \left[ e^{-|x - (x_i - x_1)|} |x - (x_i - x_1)|^2 + e^{-|x - (x_i - x_1)|} |x - (x_i - x_1)| + e^{-|x - (x_i - x_1)|} \right]
\]
\[
\leq C \sum_{i=2}^{m} \left[ e^{-|x - (x_i - x_1)|} \sum_{l=-1}^{2} |x - (x_i - x_1)|^l \right].
\]

Hence we have
\[
\int_{\mathbb{R}^3} \frac{\sum_{i=2}^{m} U^2(y - (x_i - x_1))}{|x - y|} dy \leq C \sum_{i=2}^{m} \left[ e^{-|x - (x_i - x_1)|} \sum_{l=-1}^{2} |x - (x_i - x_1)|^l + \frac{1}{|x - (x_i - x_1)|} \right].
\]

Therefore we get
\[
\int_{\Omega_1^+} \left( \int_{\mathbb{R}^3} \frac{(U_{x_1} + \sum_{i=2}^{m} U_{x_i})^2(y)}{|x - y|} dy \right) v_m^2 = \int_{\Omega_1^+} \left( <\int_{\mathbb{R}^3} \left[ U(y) + \sum_{i=2}^{m} U(y - (x_i - x_1)) \right]^2(y) \frac{dy}{|x - y|} \right) v_m^2(x + x_1)
\]
\[
\leq C \left( \int_{\Omega_1^+ - x_1} \left( \int_{\mathbb{R}^3} \frac{U^2(y)}{|x - y|} dy \right) v_m^2(x + x_1) + \int_{\Omega_1^+ - x_1} \left( \int_{\mathbb{R}^3} \frac{\sum_{i=2}^{m} U(y - (x_i - x_1))^2(y) \frac{dy}{|x - y|}}{\sum_{i=2}^{m} |x - (x_i - x_1)|} \right) v_m^2(x + x_1) \right)
\]
\[
\leq C \int_{\Omega_1^+ - x_1} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] + \sum_{i=2}^{m} \left[ e^{-|x - (x_i - x_1)|} \sum_{l=-1}^{2} |x - (x_i - x_1)|^l + \frac{1}{|x - (x_i - x_1)|} \right] v_m^2(x + x_1)
\]
\[
= C \int_{\Omega_1^+ - x_1} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] + \sum_{i=2}^{m} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] v_m^2(x)
\]
\[
\leq C \int_{\Omega_1^+} \sum_{i=1}^{m} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] v_m^2(x)
\]
\[
+ C \int_{\Omega_1^+ \cap B_R(x_1)} \sum_{i=1}^{m} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] v_m^2(x) + C \int_{B_R(x_1)} v_m^2(x) dx.
\]

Indeed, if \(i = 1\) and \(x \in \Omega_1^+ \cap B_R(x_1)\), we can get
\[
\sum_{i=1}^{m} \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] = \left[ e^{-|x - x_1|} \sum_{l=-1}^{2} |x - x_1|^l + \frac{1}{|x - x_1|} \right] = o(1);
\]
if \( i \neq 1 \) and \( x \in \Omega_i^+ \cap B_R(\overline{x}_1) \), we can get
\[
\sum_{i=2}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right]
\leq C \sum_{i=2}^{m} \left[ e^{-r\sqrt{1-t^2} \sin \frac{|i-1|\pi}{m}} \sum_{l=-1}^{2} |r\sqrt{1-t^2} \sin \frac{|i-1|\pi}{m}|^l + \frac{1}{|r\sqrt{1-t^2} \sin \frac{|i-1|\pi}{m}|} \right]
\leq C \sum_{i=2}^{m} \left[ e^{-\frac{r\sqrt{1-t^2} |i-1|\pi}{m}} \sum_{l=-1}^{2} |r\sqrt{1-t^2} \sin \frac{|i-1|\pi}{m}|^l + \frac{1}{|r\sqrt{1-t^2} \sin \frac{|i-1|\pi}{m}|} \right]
= o(1).
\]
So, we have
\[
\int_{\Omega_i^+ \cap B_R(\overline{x}_1)} \sum_{i=1}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right] v_m^2(x) \leq C \int_{B_R(\overline{x}_1)} v_m^2(x)dx.
\]
Similarly, if \( i = 1 \) and \( x \in \Omega_i^+ \setminus B_R(\overline{x}_1) \), we can get
\[
\sum_{i=1}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right]
= \left[ e^{-|x-x_1|} \sum_{l=-1}^{2} |x-x_1|^l + \frac{1}{|x-x_1|} \right] \leq C \left( e^{-\frac{1}{(1-\delta)R}} + \frac{1}{R} \right),
\]
if \( i \neq 1 \) and \( x \in \Omega_i^+ \setminus B_R(\overline{x}_1) \), we can get
\[
\sum_{i=2}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right]
= \sum_{i=2}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right]
\leq C \sum_{i=2}^{m} \left[ e^{-|x-x_i|} \left( \frac{1}{|x_i-x_1|} + 1 + |x-x_i| + |x-x_i|^2 \right) + \frac{1}{|x_i-x_1|} \right]
\leq C \sum_{i=2}^{m} e^{-\frac{r\sqrt{1-t^2} |x-x_i|^2}{4m}} + C \frac{1}{R} \leq C \left( e^{-\frac{1}{(1-\delta)R}} + \frac{1}{R} \right) + C \frac{1}{R}.
\]
So, we have
\[
\int_{\Omega_i^+ \setminus B_R(\overline{x}_1)} \sum_{i=1}^{m} \left[ e^{-|x-x_i|} \sum_{l=-1}^{2} |x-x_i|^l + \frac{1}{|x-x_i|} \right] v_m^2(x) \leq C \left( e^{-\frac{1}{(1-\delta)R}} + \frac{1}{R} \right) \int_{\Omega_i} v_m^2(x)dx.
\]
Proposition B.1. For all \((r, t) \in S_m\), there exists a small constant \(\tau > 0\) such that

\[
I(W_{r,t}) = m \left( \frac{A_2}{16\pi} + \frac{A_1}{r^9} - \frac{A_1^2}{16\pi^2 \sqrt{1 - t^2}} m \ln m - \frac{m}{16\pi^2 r \sqrt{1 - t^2}} \frac{1}{t} \ln \frac{\pi}{t} A_1^2 \right) \\
+ mO_m\left(\frac{1}{r^9 r^+}\right) + mO_m\left(\frac{m^2}{r^2 (1 - t^2)}\right) + \frac{m^2}{r \sqrt{1 - t^2}} \ln \frac{\pi}{t} O_m\left(t^2 |\ln t|^{-1}\right),
\]

where

\[
A_1 = \int_{\mathbb{R}^3} U^2, \quad A_2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(y) U^2(x)}{|x - y|}.
\]

Proof. By direct computations, we have

\[
I(W_{r,t}) = \frac{1}{2} \int_{\mathbb{R}^3} \left\{ |\nabla W_{r,t}|^2 + V(|y|) W_{r,t}^2 \right\} - \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)(W_{r,t})^2(x)}{|x - y|} \\
= \frac{1}{2} \int_{\mathbb{R}^3} \left\{ |\nabla W_{r,t}|^2 + |W_{r,t}|^2 \right\} + \frac{1}{2} \int_{\mathbb{R}^3} (V(|y|) - 1)|W_{r,t}|^2 \\
- \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)(W_{r,t})^2(x)}{|x - y|} \\
= \mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3.
\]

With the help of the symmetry and the equation for \(U\), we can derive

\[
\mathbb{I}_1 = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla W_{r,t}|^2 + |W_{r,t}|^2 \right) = \frac{1}{2} \int_{\mathbb{R}^3} \left( -\Delta W_{r,t} + W_{r,t} \right) W_{r,t} \\
= \frac{1}{2} \int_{\mathbb{R}^3} \sum_{j=1}^m \left( -\Delta U_{\tau_j} + U_{\tau_j} - \Delta U_{\omega_j} + U_{\omega_j} \right) \cdot \sum_{i=1}^m (U_{\tau_i} + U_{\omega_i}) \\
= \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^m \int_{\mathbb{R}^3} \left( \Psi_{U_{\tau_j}} U_{\tau_j} + \Psi_{U_{\omega_j}} U_{\omega_j} \right) \cdot (U_{\tau_i} + U_{\omega_i}) \\
= \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^m \int_{\mathbb{R}^3} \left( \Psi_{U_{\tau_j}} U_{\tau_j} U_{\tau_i} + \Psi_{U_{\omega_j}} U_{\omega_j} U_{\tau_i} + \Psi_{U_{\tau_j}} U_{\tau_j} U_{\omega_i} + \Psi_{U_{\omega_j}} U_{\omega_j} U_{\omega_i} \right) \\
= \frac{m}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(y) U^2(x)}{|x - y|} + \frac{m}{8\pi} \sum_{i=2}^m \int_{\mathbb{R}^3} \frac{U_{\tau_i}^2}{|x - y|} U_{\tau_i} U_{\omega_i} + \frac{m}{8\pi} \sum_{i=1}^m \int_{\mathbb{R}^3} \frac{U_{\tau_i}^2}{|x - y|} U_{\tau_i} U_{\omega_i}. \tag{B.4}
\]

An elementary calculation is as follows:

\[
\frac{1}{|x - \tau_i|^t} = \frac{1}{|\tau_i|^t} \left( 1 + O\left(\frac{|x|}{|\tau_i|}\right) \right), \quad x \in B_{\frac{|\tau_i|}{\tau}}(0), \forall \tau > 0,
\]

With symmetry and Lemma [A.2], we can calculate \(\mathbb{I}_2:\)

\[
\mathbb{I}_2 = \frac{1}{2} \int_{\mathbb{R}^3} (V(|y|) - 1)|W_{r,t}|^2 \\
= m \int_{\Omega^1_t} (V(|y|) - 1) \left( U_{\tau_1} + U_{\omega_1} + \sum_{j=2}^m U_{\tau_j} + \sum_{j=2}^m U_{\omega_j} \right)^2.
\]

Appendix B. Energy expansion

In this section, we will estimate \(I(W_{r,t})\).
By (B.7), (B.8), (B.9), and Lemma A.2, we have

\[ m \int_{\Omega_1} (V(|y|) - 1) \left( U_{\|} + O_m \left( e^{-\frac{1}{2}tr}e^{-\frac{1}{2}|y-\|} \right) + e^{-\frac{1}{2}\sqrt{1-t^2}r_m e^{-\frac{1}{2}|y-\|}} + e^{-\frac{1}{2}\sqrt{1-t^2}r \frac{n}{m} e^{-\frac{1}{2}|y-\|}} \right)^2 \]

Moreover, we have

\[ m \int_{\Omega_1} (V(|y|) - 1) U_{\|}^2 + mO_m \left( \int_{\Omega_1} (V(|y|) - 1) e^{-\frac{1}{2}|y-\|} U_{\|} \right) \]

\[ = m \left( \frac{A_1}{r^q} + O_m \left( \frac{1}{r^{n+r}} \right) \right), \quad (B.6) \]

where \( A_1 \) is defined in (B.2), and the last equality is valid because of the asymptotic expression of \( V(y) \) and (B.5). Moreover, for any \( (r, t) \in S_m \), and \( y \in \Omega_1^r \) we can derive

\[ U_{\|} U_{\|} \leq C e^{-|\| - \|} = C e^{-2tr}, \quad (B.7) \]

\[ U_{\|} \sum_{j=2}^{m} U_{\|} \leq C \sum_{j=2}^{m} e^{-|\| - \|} \leq C e^{-2\sqrt{1-t^2}r} + C e^{-2(1+r)\pi \sqrt{1-t^2} r}, \quad (B.8) \]

and

\[ U_{\|} \sum_{j=2}^{m} U_{\|} \leq C \sum_{j=2}^{m} e^{-|\| - \|} \leq C e^{-2\sqrt{1-t^2} r_m}, \quad (B.9) \]

By (B.7), (B.8), (B.9), and Lemma A.2, we have

\[ \mathbb{I}_3 = \frac{1}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(W_{r,t})^2(y)(W_{r,t})^2(x)}{|x-y|} \]

\[ = \frac{1}{4} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} (W_{r,t})^2 dx \]

\[ = \frac{m}{2} \int_{\Omega_1^r} \Psi_{W_{r,t}} \left( U_{\|} + U_{\|} + \sum_{j=2}^{m} U_{\|} + \sum_{i=2}^{m} U_{\|} \right)^2 dx \]

\[ = \frac{m}{2} \int_{\Omega_1^r} \Psi_{W_{r,t}} \left( U_{\|}^2 + U_{\|}^2 + \left( \sum_{j=2}^{m} U_{\|} \right)^2 + \left( \sum_{i=2}^{m} U_{\|} \right)^2 + 2U_{\|} U_{\|} + \sum_{j=2}^{m} \sum_{i=2}^{m} U_{\|} U_{\|} \right) dx \]

\[ \leq \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} U_{\|}^2 + mO_m \left( e^{-tr} + e^{-\sqrt{1-t^2} r_m} \right). \]

Moreover, we have

\[ \frac{m}{2} \int_{\Omega_1^r} \Psi_{W_{r,t}} U_{\|}^2 \leq \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} U_{\|}^2 - \frac{m}{2} \int_{\mathbb{R}^3 \setminus \Omega_1^r} \Psi_{W_{r,t}} U_{\|}^2 \]

\[ \leq \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} U_{\|}^2 - \frac{m}{2} \int_{\mathbb{R}^3 \setminus B_{\| - \|} (\Omega_1^r)} \Psi_{W_{r,t}} U_{\|}^2 \]

\[ \leq \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} U_{\|}^2 + O_m \left( e^{-\sqrt{1-t^2} r_m} \right). \]
We calculate the first term as follows:

\[
\frac{m}{2} \int_{\mathbb{R}^3} \Psi_{W_{r,t}} U_{\varphi_1}^2 = \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} W_{r,t}^2
\]

\[
= \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U^2 + m \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U_{\varphi_1} \sum_{i=2}^{m} U_{\varphi_i} + m \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U_{\varphi_1} \sum_{i=1}^{m} U_{\varphi_i}
\]

\[
+ \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( \sum_{i=2}^{m} U_{\varphi_i} \right)^2 + \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( \sum_{i=1}^{m} U_{\varphi_i} \right)^2
\]

\[
+ m \sum_{j=2}^{m} \sum_{i=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U_{\varphi_j} U_{\varphi_i}.
\]

From Lemma A.2, Lemma A.3 and (A.16), we have

\[
\frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( \sum_{i=2}^{m} U_{\varphi_i} \right)^2 = \frac{m}{2} \sum_{i=2}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( U_{\varphi_i} \right)^2 + \frac{m}{2} \sum_{i,j=2}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U_{\varphi_i} U_{\varphi_j}
\]

\[
= \frac{mA_2^2}{16\pi^2} \frac{m \ln m}{\sqrt{1-t^2}} + mO_m \left( \frac{m^2}{r^2 (1-t^2)} \right) + mO_m \left( \sum_{i,j=2}^{m} e^{-(1-r)|\varphi_i - \varphi_j|} \right)
\]

\[
= \frac{mA_2^2}{16\pi^2} \frac{m \ln m}{\sqrt{1-t^2}} + mO_m \left( \frac{m^2}{r^2 (1-t^2)} \right) + mO_m \left( \sum_{i=2}^{m} e^{-(1-r)|\varphi_i - \varphi_1|} \right)
\]

\[
= \frac{mA_2^2}{16\pi^2} \frac{m \ln m}{\sqrt{1-t^2}} + mO_m \left( \frac{m^2}{r^2 (1-t^2)} \right) + mO_m \left( e^{-r \sqrt{1-t^2} \frac{\pi}{m}} \right).
\]

Also, from the results of Medina and Musso in [12], we know

\[
\sum_{i=1}^{m} \frac{1}{|x_1 - x|} = \frac{1}{rt} \left( \sum_{i=1}^{m} \frac{1}{(1-t^2)(i-1)^2 \pi^2 + 1 + \frac{2}{m}} \right) + O \left( |\ln t|^{-1} \right)
\]

\[
= \frac{m}{r \pi \sqrt{1-t^2}} \ln \frac{\pi}{t} \left( 1 + O(1^2 |\ln t|^{-1}) \right),
\]

so we have

\[
\frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( \sum_{i=1}^{m} U_{\varphi_i} \right)^2 = \frac{m}{2} \sum_{i=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( U_{\varphi_i} \right)^2 + \frac{m}{2} \sum_{i,j=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} U_{\varphi_i} U_{\varphi_j}
\]

\[
= \frac{m}{2} \sum_{i=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( U_{\varphi_i} \right)^2 + mO \left( \sum_{i,j=1}^{m} e^{-(1-r)|\varphi_i - \varphi_j|} \right)
\]

\[
= \frac{m}{2} \sum_{i=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{\varphi_1}} \left( U_{\varphi_i} \right)^2 + mO \left( \sum_{i=1}^{m} e^{-(1-r)|\varphi_i - \varphi_1|} \right)
\]

\[
= \frac{m}{16\pi} \sum_{i=1}^{m} \frac{1}{|x_1 - x_i|^2} \int_{\mathbb{R}^3} U^2(x) \int_{\mathbb{R}^3} U^2(y)
\]

\[
+ mO \left( \sum_{i=1}^{m} \frac{1}{|x_1 - x_i|^2} \right) + mO \left( \sum_{i=1}^{m} e^{-(1-r)|\varphi_i - \varphi_1|} \right)
\]
where $A_{0319}$.

(No.12071169) and the Fundamental Research Funds for the Central Universities (No.KJ02072020-

Combining (B.3), (B.4), (B.6), (B.10), we have

$$
\begin{align*}
&= \frac{m}{16\pi} \left[ \frac{m}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} \left( 1 + O_m \left( t^2 |\ln t|^{-1} \right) \right) \right] \int_{\mathbb{R}^3} U^2(x) \int_{\mathbb{R}^3} U^2(y) \\
&+ mO \left( \frac{1}{|x_1 - x_i|} \right) + mO_m \left( e^{-\sqrt{1-t^2}r \pi m} \right) \\
&= \frac{m}{16\pi} \frac{m}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} A_1^2 + \frac{m^2}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} O_m \left( t^2 |\ln t|^{-1} \right) \\
&+ mO_m \left( \frac{m^2}{r^2(1-t^2)} \right) + mO_m \left( e^{-\sqrt{1-t^2}r \pi m} \right),
\end{align*}
$$

and

$$
m \sum_{j=2}^{m} \sum_{i=1}^{m} \int_{\mathbb{R}^3} \Psi_{U_{x_j}} U_{x_j} U_{x_i} = m \sum_{j=2}^{m} \sum_{i=1}^{m} O_m \left( e^{-\left(1-\delta\right)|x_i - x_j|} \right) = mO_m \left( e^{-\sqrt{1-t^2}r \pi m} \right).
$$

Hence we have

$$
\begin{align*}
\mathbb{I}_3 &= \frac{m}{2} \int_{\mathbb{R}^3} \Psi_{U^2} U^2 + m \int_{\mathbb{R}^3} \Psi_{U_{x_j}} U_{x_j} \sum_{i=1}^{m} U_{x_i} + m \int_{\mathbb{R}^3} \Psi_{U_{x_j}} U_{x_j} \sum_{i=1}^{m} U_{x_i} \\
&+ \frac{mA_1^2}{16\pi^2} \frac{m}{r\sqrt{1-t^2}} + \frac{m}{16\pi} \frac{m}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} A_1^2 + \frac{m^2}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} O_m \left( t^2 |\ln t|^{-1} \right) \\
&+ mO_m \left( \frac{m^2}{r^2(1-t^2)} + e^{-\sqrt{1-t^2}r \pi m} \right).
\end{align*}
$$

Combining (B.3), (B.4), (B.6), (B.10), we have

$$
I(W_{r,t}) = \mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3
$$

$$
= \frac{mA_2}{16\pi} + \frac{A_1}{r\pi^2} - \frac{mA_1^2}{16\pi^2} \frac{m}{r\sqrt{1-t^2}} - \frac{m^2}{16\pi^2} \frac{1}{r\sqrt{1-t^2}} \ln \frac{\pi}{t} A_1^2 \\
+ O_m \left( \frac{1}{r\pi^2} \right) + mO_m \left( \frac{m^2}{r^2(1-t^2)} \right) - \frac{m^2}{r\pi\sqrt{1-t^2}} \ln \frac{\pi}{t} O_m \left( t^2 |\ln t|^{-1} \right),
$$

where $A_1, A_2$ are defined in (B.2). We complete the proof of Proposition B.1.

Acknowledgment. The authors would like to thank Chunhua Wang from Central China Normal University for the helpful discussion with her. This paper was supported by NSFC grants (No.12071169) and the Fundamental Research Funds for the Central Universities (No.KJ02072020-0319).

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