PREPOTENTIALS FOR (2,2) SUPERGRAVITY

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ABSTRACT

We present a complete solution of the constraints for two-dimensional, N=2 supergravity in N=2 superspace. We obtain explicit expressions for the covariant derivatives in terms of the vector superfield $H^m$ and, for the two versions of minimal (2,2) supergravity, a chiral or twisted chiral scalar superfield $\phi$.

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1 Introduction

In 1978 Siegel\textsuperscript{1} presented the solution of the Wess-Zumino constraints\textsuperscript{2} for four-dimensional $N = 1$ superspace supergravity. Ever since, for any supergravity theory, one of the main endeavours of superspace practitioners has been to solve constraints on superspace covariant derivatives in terms of unconstrained (pre)potentials. Solutions generally exist whenever a complete (component) set of auxiliary fields exists.

In two dimensions, where a number of $(p, q)$ supergravities can be constructed, the solution to the constraints is fairly straightforward for $(1, 0)^3$, $(1, 1)^4$, and $(p, 0)^5$ supergravity. The situation for $(2, 2)$ i.e. $N = 2$ supergravity is more complicated. In principle the solution can be obtained by direct dimensional reduction from four-dimensional $N = 1$ supergravity as described in \textit{Superspace}\textsuperscript{6}, pp. 469-472 but in practice this has not been carried out. The solution can be obtained easily in conformal gauge\textsuperscript{7} or in light-cone gauge, but a general, fully covariant answer has not been worked out. To the best of our knowledge the only published attempt is by Alnowaiser\textsuperscript{8}. However, his work is incomplete and some of its aspects are questionable.

We have reanalysed the problem of obtaining prepotentials for $(2, 2)$ supergravity in $N = 2$ superspace. We present a complete solution in a form which is suitable for some applications. We achieve this, in part, by simply working in a spinor, light-cone basis, rather than using $\gamma$-matrices. Trivial as this may seem, it allows for a more transparent treatment. In particular, it becomes obvious that at a certain stage one has to solve a quadratic equation rather than the quartic equation that appears in ref. 8; complete, relatively simple results follow. We obtain explicit forms for the covariant derivatives and the vielbein superdeterminant.

Besides contributing to the program of finding prepotentials for all two-dimensional supergravities, there are some additional advantages to a fully covariant solution of the constraints. First, it may allow the study of some situations with nontrivial (super-)Riemann surface topology where light-cone gauge is not accessible and conformal gauge may not be convenient. Second, it may facilitate the understanding of some issues in induced $(2, 2)$ supergravity and shed some light on its higher-loop quantum properties. For this purpose the development of a fully covariant background-field formalism is desirable, and possible\textsuperscript{9}. Finally, we hope that by being slightly pedagogical, this work provides a useful review of techniques used in four dimensions\textsuperscript{6}.

It is known that minimal $(2, 2)$ supergravity comes in two versions, depending whether one gauges, in addition to the Lorentz symmetry, an axial $U(1)$ or a vector $U(1)$ tangent space symmetry\textsuperscript{10,11}. We will point out however that the solution of the constraints for one of the theories can be obtained easily from the solution for
the other, and we will concentrate on the axial version, which is related to four-dimensional minimal \((n = -\frac{1}{3})\) supergravity by dimensional reduction\(^6\).

We use the following notation. Flat N=2 superspace is described by bosonic coordinates \(x^\pm\) and \(x^\mp\), and fermionic coordinates \(\theta^+, \theta^-\), and their complex conjugates \(\theta^+\) and \(\theta^-\). The spinorial derivatives satisfy the anticommutation relations

\[
\{D_+, D_+\} = i\partial_+ , \quad \{D_-, D_-\} = i\partial_- \tag{1.1}
\]

with all others equal to zero. We define Lorentz, \(U_V(1)\) and \(U_A(1)\) tangent space generators \(\Lambda, \tilde{\Lambda}\) and \(N\) by their action on spinors:

\[
[\Lambda, \psi_\pm] = \pm \frac{1}{2} \psi_\pm , \quad [\Lambda, \psi_\pm] = \pm \frac{1}{2} \psi_\pm,
\]

\[
[\tilde{\Lambda}, \psi_\pm] = \mp \frac{i}{2} \psi_\pm , \quad [\tilde{\Lambda}, \psi_\pm] = \mp \frac{i}{2} \psi_\pm,
\]

\[
[N, \psi_\pm] = - \frac{i}{2} \psi_\pm , \quad [N, \psi_\pm] = + \frac{i}{2} \psi_\pm. \tag{1.2}
\]

It will prove convenient however to define combinations of operators which act only on undotted or dotted variables:

\[
M = \frac{1}{2}(\Lambda + i\tilde{\Lambda}) , \quad \overline{M} = \frac{1}{2}(\Lambda - i\tilde{\Lambda}) . \tag{1.3}
\]

Then

\[
[M, \psi_\pm] = \pm \frac{1}{2} \psi_\pm , \quad [M, \psi_\pm] = 0
\]

\[
[\overline{M}, \psi_\pm] = \pm \frac{1}{2} \psi_\pm , \quad [\overline{M}, \psi_\pm] = 0. \tag{1.4}
\]

The action of the operators on all other quantities can be readily deduced from these.

Our paper is organized as follows: In Section 2 we present the \((2,2)\) geometry and the constraints for an extended theory where one gauges both \(U(1)\) tangent space symmetries. In Section 3 we obtain their solution. In Section 4 we obtain the prepotentials for minimal axial supergravity by restriction to the axial \(U(1)\) tangent space symmetry. Finally, in Section 5 we discuss superWeyl invariance and construct the vielbein superdeterminant.
2 The N=2 geometry

The geometry of two-dimensional N=2 supergravity has been described by Howe and Papadopoulos\(^\text{10}\) and is also discussed by Gates et al\(^\text{11}\). The theory is described by suitably defined covariant derivatives which include the action of the tangent space generators introduced in the previous section. Suitable constraints on the torsions and curvatures lead to minimal supergravity multiplets. It is convenient, to begin with, to keep the theory fully locally invariant under all the tangent space generators. The minimal multiplets are obtained, however, by removing either the \(U_V(1)\) connection, which leads to what one would obtain directly by dimensional reduction from four-dimensional supergravity using a chiral compensator, or the \(U_A(1)\) connection, leading to a theory which uses a twisted chiral compensator. The introduction of an extra \(U(1)\) symmetry and its subsequent degauging is similar to the procedure used in the four-dimensional situation\(^\text{6}\).

The spinorial covariant derivatives are defined by

\[
\nabla_\alpha = E_\alpha + \Phi_\alpha \Lambda + \tilde{\Phi}_\alpha \tilde{\Lambda} + \Sigma_\alpha N
\]

\[
= E_\alpha + \Omega_\alpha M + \Gamma_\alpha \bar{M} + \Sigma_\alpha N
\]  

(2.1)

with \(\alpha = \pm\), and corresponding expressions for the complex conjugate spinorial derivatives as well as the vectorial derivatives. The vielbein is given by

\[
E_A = E_A^M \partial_M .
\]  

(2.2)

Torsions and curvatures are defined as usual by

\[
\{\nabla_A, \nabla_B\} = T_{AC}^B \nabla_C + R_{AB}^M + \bar{R}_{AB} \bar{M} + F_{AB} N .
\]  

(2.3)

They satisfy constraints which can be described by the following anticommutators\(^\text{10,11}\).

\[
\{\nabla_+, \nabla_+\} = 0 , \quad \{\nabla_-, \nabla_-\} = 0
\]

\[
\{\nabla_+, \nabla_-\} = i\nabla_+ , \quad \{\nabla_-, \nabla_\pm\} = i\nabla_\pm
\]

\[
\{\nabla_+, \nabla_-\} = -\frac{R}{2}(\Lambda - i\tilde{\Lambda}) = -\bar{R}\bar{M}
\]

\[
\{\nabla_+, \nabla_\pm\} = F(\Lambda - iN) = F(M + \bar{M} - iN)
\]  

(2.4)

as well as their complex conjugates. Additional constraints follow from the use of Bianchi identities. Furthermore, for the minimal supergravities one restricts the gauge group so that either \(F = 0\) for the \(U_A(1)\) version, or \(R = 0\) for the \(U_V(1)\) version, by setting either \(\Sigma_\alpha = 0\) or \(\tilde{\Phi}_\alpha = \Omega_\alpha - \Gamma_\alpha = 0\)^\text{10}.

We solve the constraints in (2.4) by expressing the covariant derivatives in terms of two (pre)potentials, a real vector superfield \(H^m\) and a (scale compensator) complex scalar superfield \(S\). This leads to a description of nonminimal \(U_V(1) \times U_A(1)\)
supergravity. In doing so, we implicitly make certain supersymmetric gauge choices which remove a large number of irrelevant superfields by means of algebraic (ghost-nongenerating) gauge transformations. We also use educated guesses based on four-dimensional experience to partially determine the dependence on the prepotentials.

We make a useful observation: the minimal axial version of the theory is obtained by setting $\Sigma^{2} = F = 0$ and this additional constraint implies that $S$ satisfies a condition which eventually expresses it in terms of a chiral compensator $\phi$. Therefore, axial supergravity is described in terms of the two prepotentials $H$ and $\phi$. However, the constraints in (2.4) and their solution are invariant under the interchange $\nabla_{-} \leftrightarrow \nabla_{\pm}$ together with $\tilde{\Lambda} \leftrightarrow N$ and $R \leftrightarrow -2F$. Therefore, the minimal vector version of (2,2) supergravity, with $\tilde{\Phi} = R = 0$ is obtained directly from the axial vector version by the above substitutions, which imply in particular that the chiral scalar compensator is replaced by a twisted chiral compensator.

3 Solving the N=2 constraints

3.1 Determining the connections

We begin by defining the “hat” differential operators

$$\hat{E}_{\pm} = e^{-H}D_{\pm}e^{H}, \quad H = H^{m}i\partial_{m}. \quad (3.1)$$

We note the explicit forms $\hat{E}_{\pm} = D_{\pm} + iH^{m}\partial_{m}$, where $H^{m}$ is a function of $H^{m}$ and its derivatives. Additional features are $\{\hat{E}_{+}, \hat{E}_{-}\} = 0$ while $\{\hat{E}_{+}, \hat{E}_{\pm}\} \equiv i\hat{E}_{\pm} = i\partial_{\pm} + \cdots$. Thus $\hat{E}_{\alpha}, \hat{E}_{\dot{\alpha}}$ and $\hat{E}_{a}$ form a linearly independent basis of derivative operators.

In the corresponding four-dimensional case one can find Lorentz gauges in which the actual spinorial vielbein is proportional to the corresponding “hat” object. In two dimensions however the Lorentz group is more restricted and we postulate instead

$$E_{+} \equiv e^{S}(\hat{E}_{+} + A_{-}\hat{E}_{-}), \quad E_{-} \equiv e^{S}(\hat{E}_{-} + A_{+}\hat{E}_{+}) \quad (3.2)$$

with corresponding expressions for the complex conjugate spinorial vielbein. Here $S$ is, for the time being, an arbitrary scalar (non chiral) superfield, and the $A$’s will be determined later as functions of $H^{m}$.

We begin by imposing the first constraint, $\{\nabla_{+}, \nabla_{+}\} = 0$, which leads to the conditions

$$\{E_{+}, E_{+}\} + (\Omega_{+} - i\Sigma_{+})E_{+} = 0$$
\[
2E_+ \Gamma_+ + (\Omega_+ - i \Sigma_+) \Gamma_+ = 0 \\
2E_+ \Omega_+ - i \Sigma_+ \Omega_+ = 0 \\
2E_+ \Sigma_+ + \Omega_+ \Sigma_+ = 0 .
\] (3.3)

Substituting in the first equation the explicit expressions for the vielbein we find, since the \( \hat{E} \) are linearly independent,

\[
\Omega_+ - i \Sigma_+ = -2e^S (\hat{E}_+ \hat{S} + A_+ \hat{E}_-) 
\] (3.4)

while the coefficient \( A_+ \) must satisfy

\[
\hat{E}_+ A_- + A_- \hat{E}_+ A_+ = 0 .
\] (3.5)

We will show later on that our solution for \( A_+ \) satisfies this equation. Notice that the equation also implies that \( A_+ \) is “linear” in the sense that

\[
\hat{E}_+ \hat{E}_- A_+ = 0 .
\] (3.6)

The remaining conditions in (3.3) can also be verified once we have explicit expressions for the connections.

Similar results are obtained from the \( \{ \nabla_-, \nabla_- \} = 0 \) constraint. In particular we have

\[
\Omega_- + i \Sigma_- = 2e^S (\hat{E}_- \hat{S} + A_- \hat{E}_+) 
\] (3.7)

and we must satisfy

\[
\hat{E}_- A_+ + A_+ \hat{E}_- A_- = 0 .
\] (3.8)

We turn next to the second constraint, \( \{ \nabla_+, \nabla_- \} = -\hat{R}_+ \hat{M} \) which leads to

\[
\{ E_+, E_- \} + \frac{1}{2}(\Omega_- - i \Sigma_-) E_+ - \frac{1}{2}(\Omega_+ + i \Sigma_+) E_- = 0 \\
E_+ \Omega_- + E_- \Omega_+ - \Omega_+ \Omega_- - i \Sigma_+ \Omega_- - i \Sigma_- \Omega_+ = 0 \\
E_+ \Sigma_- + E_- \Sigma_+ - \frac{1}{2} \Omega_+ \Sigma_- + \frac{1}{2} \Omega_- \Sigma_+ = 0
\] (3.9)

as well as

\[
E_+ \Gamma_- + E_- \Gamma_+ - \frac{1}{2}(\Omega_+ + i \Sigma_+) \Gamma_- + \frac{1}{2}(\Omega_- - i \Sigma_-) \Gamma_+ \equiv -\hat{R} .
\] (3.10)

We substitute in the first equation of (3.9) the explicit expressions for the vielbein and connections and after some algebra, making use also of (3.5) and (3.8), we find

\[
\Omega_+ = +e^S (\hat{E}_- A_+ - A_- \hat{E}_+ A_+) \\
\Omega_- = -e^S (\hat{E}_+ A_- - A_- \hat{E}_- A_+). 
\] (3.11)
The other conditions can be verified once we have explicit expressions for \( \Sigma_{\pm} \).

Finally we turn to the anticommutator
\[
\{ \nabla_+ , \nabla_\pm \} = \{ E_+ , E_\pm \} - \frac{1}{2} ( \Gamma_+ - i \Sigma_+ ) E_\pm + \frac{1}{2} ( \Gamma_\pm - i \Sigma_\pm ) E_+
\]
\[
+ [ E_+ \Gamma_\pm + E_\pm \Gamma_+ - \frac{1}{2} \Gamma_+ \Gamma_\pm + \frac{1}{2} \Sigma_\pm \Sigma_\pm + \frac{i}{2} \Sigma_+ \Gamma_\pm - \frac{i}{2} \Gamma_\pm \Sigma_+ ][ M
\]
\[
+ [ E_+ \Omega_\pm + E_\pm \Omega_+ - \frac{1}{2} \Gamma_+ \Omega_\pm + \frac{1}{2} \Gamma_\pm \Omega_+ + \frac{i}{2} \Sigma_+ \Omega_\pm - \frac{i}{2} \Omega_\pm \Sigma_+ ][ M
\]
\[
+ [ E_+ \Sigma_\pm + E_\pm \Sigma_+ - \frac{1}{2} \Gamma_+ \Sigma_\pm + \frac{1}{2} \Gamma_\pm \Sigma_+ + i \Sigma_+ \Sigma_+ ][ N . \quad (3.12)
\]

We require that the first line vanish. Matching terms, and also using the fact that the anticommutator of \( \hat{E} \)'s does not produce spinorial derivatives, we find the results
\[
\Gamma_+ - i \Sigma_+ = + 2e^S ( \hat{E}_+ S + A_+ \hat{E}_- S )
\]
\[
\Gamma_\pm - i \Sigma_\pm = - 2e^S ( \hat{E}_\pm S + A_\pm \hat{E}_\mp S ) \quad (3.13)
\]
as well as the conditions
\[
\hat{E}_+ A_\pm + A_\pm \hat{E}_+ A_\pm = 0
\]
\[
\hat{E}_\pm A_\pm + A_\pm \hat{E}_\pm A_\pm = 0 \quad (3.14)
\]
and
\[
\{ \hat{E}_+ , \hat{E}_\pm \} + A_\pm \{ \hat{E}_+ , \hat{E}_\pm \} + A_\mp \{ \hat{E}_- , \hat{E}_\pm \} + A_\mp A_\pm \{ \hat{E}_- , \hat{E}_\pm \} = 0 . \quad (3.15)
\]
This last condition determines the \( A \)'s.

At this point all the connections are determined and, provided the conditions on the \( A \)'s hold, it is possible to check that all the other conditions we have encountered are indeed satisfied. In particular one can check that the second and third lines in (3.12) equal \( i \) times the fourth line so that the last constraint in (2.4) is obeyed, with
\[
F = i [ E_+ \Sigma_\pm + E_\pm \Sigma_+ - \frac{1}{2} \Gamma_+ \Sigma_\pm + \frac{1}{2} \Gamma_\pm \Sigma_+ + i \Sigma_+ \Sigma_+ ] . \quad (3.16)
\]

Besides the \( \Omega_\pm \) in (3.11) we have, from (3.4) and (3.13),
\[
\Sigma_+ = - 2ie^S ( \hat{E}_+ S + A_+ \hat{E}_- S ) - ie^S ( \hat{E}_+ A_+ - A_+ \hat{E}_+ A_+ ) \quad (3.17)
\]
\[
\Sigma_- = - 2ie^S ( \hat{E}_- S + A_+ \hat{E}_- S ) - ie^S ( \hat{E}_+ A_+ + A_+ \hat{E}_+ A_+ )
\]
\[
\Gamma_+ = + 2e^S ( \hat{E}_+ S + A_+ \hat{E}_- S ) + 2e^S ( \hat{E}_+ S + A_+ \hat{E}_- S ) + e^S ( \hat{E}_- A_+ - A_+ \hat{E}_+ A_+ )
\]
\[
\Gamma_- = - 2e^S ( \hat{E}_- S + A_+ \hat{E}_- S ) - 2e^S ( \hat{E}_- S + A_+ \hat{E}_- S ) - e^S ( \hat{E}_+ A_+ - A_+ \hat{E}_+ A_+ )
\]
and corresponding expressions for the complex conjugates.
3.2 Determining $A_+^-$ and $A_+^+$

We have defined

$$\hat{E}_+ \equiv -i\{\hat{E}_+, \hat{E}_-\} = \partial_+ + \cdots$$
$$\hat{E}_- \equiv -i\{\hat{E}_-, \hat{E}_+\} = \partial_- + \cdots .$$

(3.18)

From the form of the hatted objects it is clear then that

$$\{\hat{E}_+, \hat{E}_-\} = \hat{C}_+^+ \hat{E}_+ + \hat{C}_-^- \hat{E}_-$$
$$\{\hat{E}_-, \hat{E}_+\} = \hat{C}_+^- \hat{E}_+ + \hat{C}_-^+ \hat{E}_-$$

without any spinor contributions on the right hand side. These equations define the hatted anholonomy coefficients as power series in the field $H^m$ and its derivatives.

With these definitions, (3.15) breaks up into two equations

$$\dot{\hat{C}}_+^+ + iA_+^+ + \dot{\hat{C}}_-^- A_+^+ = 0$$
$$\dot{\hat{C}}_-^- + iA_-^- + \dot{\hat{C}}_+^+ A_-^- = 0 .$$

(3.20)

In a similar manner, from $\{\nabla_-, \nabla_+\}$ one finds

$$\dot{\hat{C}}_-^+ + iA_-^+ + \dot{\hat{C}}_+^- A_-^+ = 0$$
$$\dot{\hat{C}}_+^- + iA_+^- + \dot{\hat{C}}_-^+ A_+^- = 0 .$$

(3.21)

By eliminating one of the unknowns in (3.20) one is led to a quadratic equation, and therefore the $A$’s can be found explicitly:

$$A_+^- = \frac{\hat{C}_+^+ \hat{C}_-^- - \hat{C}_+^- \hat{C}_-^+ + 1 - \sqrt{(\hat{C}_+^+ \hat{C}_-^- \hat{C}_-^+ \hat{C}_+^+ + 1)^2 + \hat{4}\hat{C}_+^+ \hat{C}_-^- \hat{C}_-^+ \hat{C}_+^+}}{2i\hat{C}_-^+}$$
$$A_+^+ = \frac{\hat{C}_-^+ \hat{C}_+^- - \hat{C}_-^- \hat{C}_+^+ + 1 - \sqrt{(\hat{C}_-^+ \hat{C}_+^- \hat{C}_+^+ \hat{C}_-^- + 1)^2 + \hat{4}\hat{C}_-^+ \hat{C}_+^- \hat{C}_+^+ \hat{C}_-^-}}{2i\hat{C}_-^-}$$

(3.22)

We have chosen the signs of the square roots so that the $A$’s vanish when $H^m$ and therefore the $\hat{C}$’s vanish.

It is obvious that the solutions of (3.21) are simply related to those of (3.20). One finds

$$A_+^- = \frac{\hat{C}_-^+ \hat{C}_-^-}{\hat{C}_+^+} A_+^-$$
$$A_+^+ = \frac{\hat{C}_+^- \hat{C}_+^+}{\hat{C}_+^-} A_+^+ .$$

(3.23)
Thus, the coefficients $A$ are determined explicitly as power series in $H^m$ and its derivatives.

We are now in a position to check that (3.5) and the first equation in (3.14) are satisfied. We proceed as follows: We take the commutator of (3.15) first with $\hat{E}_+$ and then with $\hat{E}_-$. Using Jacobi identities for triple commutators we rewrite the results in terms of $\hat{E}_+$ and $\hat{E}_-$ and set their coefficients to zero. We find the following four equations:

\begin{align*}
ix + \hat{C}_{-+}(A_{\pm}^\dagger x + A_{-}^\pm y) &= -A_{+}^-(i\hat{C}_{++} - iA_{\pm}^\dagger \hat{C}_{-+}) \\
iy + \hat{C}_{-+}(A_{\pm}^\dagger x + A_{-}^\pm y) &= -A_{+}^-(i\hat{C}_{++} - iA_{\pm}^\dagger \hat{C}_{-+}) \\
iu + \hat{C}_{-+}(A_{\pm}^\dagger u + A_{-}^\pm v) &= i\hat{C}_{++} - iA_{\pm}^\dagger \hat{C}_{-+} \\
v + \hat{C}_{-+}(A_{\pm}^\dagger u + A_{-}^\pm v) &= i\hat{C}_{++} - iA_{\pm}^\dagger \hat{C}_{-+} 
\end{align*}

(3.24)

where we have temporarily denoted $x = \hat{E}_+ A_{-}^\pm$, $y = \hat{E}_+ A_{-}^\pm$, $u = \hat{E}_- A_{+}^\pm$ and $v = \hat{E}_- A_{+}^\pm$. It is obvious then that the solutions of the last two equations are proportional to the solutions of the first two, with proportionality factor $-A_{+}^-$, and this fact expresses the content of (3.5) and (3.14). Obviously (3.8) and the second equation in (3.14) are checked in a similar manner.

With this, and some additional checks on the equations involving the connections, we have completed the solution of the constraints for $N = 2$, $U_V(1) \times U_A(1)$ supergravity, expressing the covariant derivatives in terms of the unconstrained superfields $H^m$ (a real vector superfield) and $S$ (a complex scalar superfield). The relevant results are contained in (3.1,2,22,23) and (3.11,17). Additional restrictions arise when we reduce the theory by eliminating one of the $U(1)$ connections.

## 4 Degauging

We obtain a minimal supergravity multiplet by imposing an additional constraint which requires that either the curvature $F$ or the curvature $R$ in (2.4) vanish. In the first case this can be achieved by setting the connection $\Sigma_{\pm}$ to zero, while in the second case the condition becomes $\Omega_{\pm} = \Gamma_{\pm}$. We discuss axial supergravity, obtained by eliminating the $U_V(1)$ connection $\Sigma_{\pm}$, but as explained in Section 2 the results for vector supergravity follow immediately.

The connection $\Sigma_{\pm}$ has been given explicitly in (3.17), and setting it to zero implies a differential constraint on the scalar superfield $S$. It is not immediately obvious how this constraint is to be solved and therefore we derive a different, and much more suitable, expression for this connection, following techniques similar to those used in four dimensions as described in ref. 6 (in particular the derivation
of eq. (5.3.25), as well as subsection (5.3.b.4)). However, the manipulations we perform, though elementary, are rather baroque and the impatient reader may wish to skip directly to the final result in eq. (4.21).

It is convenient to define “checked” operators $\hat{E}$ by

$$\hat{E}_\pm \equiv E_\pm , \quad \hat{E}_\mp \equiv -i\{ \hat{E}_+, \hat{E}_- \} , \quad \hat{E}_\mp \equiv -i\{ \hat{E}_-, \hat{E}_+ \} \quad (4.1)$$

and corresponding “checked” anholonomy coefficients. In particular, using Jacobi identities, from

$$\{ \hat{E}_+, \hat{E}_\mp \} = \frac{1}{2}(\Omega_+ - i\Sigma_+)\hat{E}_\mp - \frac{i}{2}[\hat{E}_+(\Omega_+ - i\Sigma_+)]\hat{E}_+$$

$$\{ \hat{E}_+, \hat{E}_- \} = \frac{1}{2}(\Omega_+ + \Gamma_+)\hat{E}_- + \text{spinorial vielbeins} \quad (4.2)$$

we identify

$$\Omega_+ - i\Sigma_+ = -2\check{\mathcal{C}}_+^\mp , \quad \Omega_+ + \Gamma_+ = 2\check{\mathcal{C}}_+^\mp \quad (4.3)$$

so that

$$\Gamma_+ + i\Sigma_+ = 2(\check{\mathcal{C}}_+^\mp + \check{\mathcal{C}}_+^\mp) = 2\check{\mathcal{C}}_+^a \quad (4.4)$$

On the other hand, from the vanishing of the first line in (3.12) it is evident that

$$\Gamma_+ - i\Sigma_+ = 2\check{\mathcal{C}}_+^\mp \quad (4.5)$$

so that

$$\Sigma_+ = i\check{\mathcal{C}}_+^\mp - i\check{\mathcal{C}}_+^a = -i[(-1)^B\check{\mathcal{C}}_B^a + \check{\mathcal{C}}_+^a] \quad (4.6)$$

since obviously, from the definitions in (4.1), $\check{\mathcal{C}}_+^\mp = 0$. In the equation above $(-1)^B$ denotes the usual graded sum.

We also note, from $\nabla_\mp = -i\{ \nabla_+, \nabla_- \} = E_\pm + \text{connections}$, that

$$E_\pm = \hat{E}_\pm - \frac{i}{2}(\Gamma_+ + i\Sigma_+ )E_\pm - \frac{i}{2}(\Gamma_+ - i\Sigma_+ )E_+ \quad (4.7)$$

from which one can deduce that the vielbein superdeterminant equals the “checked” superdeterminant

$$E = \text{sdet}E_A^M = \text{sdet}\hat{E}_A^M \equiv \hat{E} \quad (4.8)$$

Furthermore, we have $(z^M = (x^m, \theta^\mu, \theta^\mu))$

$$(-1)^B\check{\mathcal{C}}_B^B = \hat{E}_M^B[\hat{E}_A, \hat{E}_B]z^M = \hat{E}_M^B\hat{E}_A\hat{E}_B^M - \hat{E}_M^B\hat{E}_B\hat{E}_A^M \quad (4.9)$$

where the summation over repeated indices is graded (but has not been explicitly indicated, for notational simplicity).
Following techniques in subsection (5.3.b.4) of ref. 6 we define adjoint operators such as $\dot{E}_A$ or $\dot{H}$ by, for example,

$$X \dot{E}_A = X \dot{E}_A^M \partial_M = \partial_M (X \dot{E}_A^M)$$

(4.10)

again with appropriate grading. They obey the Leibnitz rule

$$XY \dot{E}_A = X[Y, \dot{E}_A] + X \dot{E}_A Y = X \dot{E}_A Y + X \dot{E}_A Y$$

(4.11)

and are extremely useful for the operations that follow.

Thus we rewrite (4.9) as

$$(-1)^B \dot{C}_{AB}^B = \dot{E}_M^B \dot{E}_A^B - \dot{E}_M^B \dot{E}_A^M \dot{E}_B + \dot{E}_M^B \dot{E}_B \dot{E}_A^M$$

$$= \dot{E}_A \ln \dot{E} - 1 \cdot \dot{E}_A + 0$$

(4.12)

the first term being a standard expression for the derivative of the determinant and the last term vanishing since $\dot{E}_M^B \dot{E}_B = \dot{E}_M^B \dot{E}_M^N \partial_N = 1 \cdot \partial_M = 0$. The right-hand-side can be rewritten, using (4.11), as

$$- E \dot{E}_A E^{-1} - 1 \cdot \dot{E}_A = - E^{-1} \dot{E}_A E$$

(4.13)

so that, from (4.6)

$$\Sigma_+ = iE^{-1} \dot{E}_+ E - i\dot{C}_{+\alpha}^\alpha .$$

(4.14)

Now, from the first equation in (3.3) and (3.9),

$$\dot{C}_{++} = -(\Omega_+ - i\Sigma_+) , \quad \dot{C}_{+-} = \frac{1}{2}(\Omega_+ + i\Sigma_+)$$

(4.15)

and this allows us to write, substituting expressions from (3.11) and (3.17),

$$\Sigma_+ = iE^{-1} \dot{E}_+ E + \frac{i}{2} \Omega_+ + \frac{3}{2} \Sigma_+$$

$$= iE^{-1} e^S [\dot{E}_+ + A_+^- \dot{E}_-^-] E - i e^S (\dot{E}_- A_+^- - A_+^- \dot{E}_+ A_+^+) - 3i E_+ S$$

or

$$-i \Sigma_+ = 1 \cdot [\dot{E}_+ + \dot{E}_- A_+^-] e^S + e^S A_+^- \dot{E}_+ A_+^- - E_+ \ln E - 2E_+ S$$

(4.17)

where the right-hand-side is obtained after a number of intermediate steps using repeatedly the Leibnitz formula for the operators $\dot{E}_\pm$. 

10
We write
\[
1 \cdot \hat{E}_+ = 1 \cdot e^{\hat{H}} \bar{D}_+ e^{-\hat{H}} = 1 \cdot [D_+(1 \cdot e^{\hat{H}})] e^{-\hat{H}} \\
= [1 \cdot e^{-\hat{H}}] e^{-\hat{H}} D_+ e^{H}[1 \cdot e^{-\hat{H}}]^{-1} = [1 \cdot e^{-\hat{H}}] \hat{E}_+ [1 \cdot e^{-\hat{H}}]^{-1} \\
= -\hat{E}_+ [\ln(1 \cdot e^{-\hat{H}})]
\] (4.18)
where in the second line we have used again the Leibnitz rule (in exponential form) as well as the identity \(1 = (1 \cdot e^{\hat{X}}) e^{\hat{X}} = (1 \cdot e^{\hat{X}}) [e^{\hat{X}} (1 \cdot e^{-\hat{X}})]\) (see ref. 6 eq. (5.3.51b)). With a similar expression for \(1 \cdot \hat{E}_-\), the first term in (4.17) becomes
\[-E_+ [\ln(1 \cdot e^{-\hat{H}})].\]

Using the identity (3.8) and the explicit expression for \(E_+\) we also rewrite
\[e^S \hat{E}_+ A_-^+ = \frac{1}{1 - A_-^- A_-^+} E_+ A_-^+ \] (4.19)
and since, by (3.5), \(E_+ A_-^- = 0\),
\[e^S A_-^- \hat{E}_+ A_-^+ = \frac{E_+ (A_-^- A_-^+)}{1 - A_-^- A_-^+} = -E_+ \ln(1 - A_-^- A_-^+) \] (4.20)

Substituting (4.18) and (4.20) into (4.17), and using similar manipulations for \(\Sigma_-\) leads us to the final form for the \(U_V(1)\) connections:
\[i \Sigma_\pm = E_\pm \ln \left[ (1 \cdot e^{\hat{H}}) E e^{2S} (1 - A_-^- A_-^+) \right] \] (4.21)
with similar expressions for the complex conjugates.

The degauging of the \(U_V(1)\) symmetry takes place by simply requiring the connections \(\Sigma_\pm\) and their complex conjugates to vanish. Setting the expression in (4.21) to zero implies that the quantity in the square bracket is a (covariantly) antichiral scalar superfield. We write it as \(e^{2\phi}\) where the superfield \(\bar{\sigma}\) satisfies the condition
\[\hat{E}_+ \bar{\sigma} = \hat{E}_- \bar{\sigma} = 0\] (4.22)
i.e.
\[\bar{\sigma} = e^{-H} \bar{\phi} , \quad D_\pm \bar{\phi} = 0 \] (4.23)
so that \(\phi\) is an ordinary chiral field. Solving for the compensator \(S\) we have then
\[e^S = e^\phi \frac{[1 \cdot e^{-\hat{H}}]^{-\frac{1}{2}}}{[1 - A_-^- A_-^+]^\frac{1}{2}} E^{-\frac{1}{2}} \] (4.24)
and substituting back into the vielbein gives the solution of the constraints for minimal axial $(2, 2)$ supergravity in terms of the superfields $H^m$ and $\sigma$ (or $\phi$).

In view of the remarks at the end of Section 2, the solution for the vector version is immediate. One performs the interchange $- \leftrightarrow \bar{\sigma}$ everywhere, starting with the definitions in (3.1). One has exactly the same form for the solution, except that the compensator satisfies

$$\hat{E}_+ \bar{\sigma} = \hat{E}_- \bar{\sigma} = 0 \quad (4.25)$$

so that the corresponding $\bar{\phi}$ satisfies $D_+ \bar{\phi} = D_- \bar{\phi} = 0$, i.e. $\phi$ is an ordinary twisted chiral superfield.

This completes the solution of the constraints.

5 SuperWeyl scaling and the vielbein superdeterminant

As is well-known, the constraints in (2.4) have an additional invariance under scaling with an arbitrary scalar superfield $L$ (which can be restricted to be real since imaginary scale transformations can be absorbed into the $U_V(1)$ transformations). The scaling can be implemented as a shift in the scale compensator, $S \rightarrow S + L$. In particular we have then the following transformation properties:

$$E_\pm \rightarrow e^L E_\pm \quad \Omega_\pm \rightarrow e^L \Omega_\pm \quad \Sigma_\pm \rightarrow e^L (\Sigma_\pm - 2iE_\pm L) \quad \Gamma_\pm \rightarrow e^L (\Gamma_\pm \pm 4E_\pm L) \quad (5.1)$$

and it is possible to check explicitly the invariance of the constraints with, in particular

$$\bar{R} \rightarrow e^{2L}(\bar{R} + 4[\nabla_-, \nabla_+]L) \quad F \rightarrow e^{2L}(F - 2i[\nabla_-, \nabla_+]L) \quad (5.2)$$

However, after degauging, the scale parameter must be restricted.

We note that under combined infinitesimal scale and $U_V(1)$ gauge transformation with parameter $\nu$ (i.e. $\delta \nabla_\pm = [\nu N, \nabla_\pm]$), the connection $\Sigma_\pm$, in addition to the overall rescaling, shifts:

$$\Sigma_\pm \rightarrow (1 + L - \frac{i}{2} \nu) \Sigma_\pm - 2iE_\pm (L - \frac{i}{2} \nu) \quad (5.3)$$
Therefore, after degauging, the axial version of the theory will only be invariant under (combined) transformations with parameter \( \hat{\lambda} = L - \frac{1}{4} \nu \) which maintain the vanishing of \( \Sigma_{\pm} \), i.e. complex scale transformations with covariantly (anti)chiral parameter \( \hat{\lambda} \) satisfying

\[
E_\pm \hat{\lambda} = 0 \tag{5.4}
\]

It is easy to see, from (4.24), that these are implemented by the shift of the chiral compensator,

\[
\sigma \to \sigma + \lambda . \tag{5.5}
\]

Obviously, in the vector version of the theory the corresponding superWeyl parameter is twisted chiral\(^{10}\).

To complete this work, we compute the superdeterminant \( E = \text{sdet} E_A^M \), where \( E_A = E_A^M \partial_M \). As already stated in (4.8) we have \( E = \hat{E} \). It is convenient to write

\[
\hat{E}_A = \hat{E}_A^B \hat{E}_B , \quad \hat{E}_B = \hat{E}_B^M \partial_M \tag{5.6}
\]

so that

\[
E = \hat{E} = \text{sdet} \hat{E}_A^B \cdot \text{sdet} \hat{E}_B^M . \tag{5.7}
\]

Writing

\[
\hat{E}_\pm = e^{-H} D_\pm e^H = D_\pm + iH_\pm^m \partial_m \\
\hat{E}_\mp = e^H D_\pm e^{-H} = D_\pm - iH_\pm^m \partial_m , \tag{5.8}
\]

and also

\[
\hat{E}_+ = -i\{\hat{E}_+, \hat{E}_+\} = \partial_+ + [-D_+ H_-^m + D_+ H_+^m - iH_+^m \partial_m H_-^n - iH_-^m \partial_m H_+^n] \partial_n \\
\hat{E}_- = -i\{\hat{E}_-, \hat{E}_-\} = \partial_- + [-D_- H_-^m + D_- H_+^m - iH_-^m \partial_m H_-^n - iH_+^m \partial_m H_+^n] \partial_n \tag{5.9}
\]

we note that \( \hat{E}_B^M \) is block triangular and therefore

\[
-\text{sdet} \hat{E}_B^M = \left| \begin{array}{c}
1 - D_+ H_-^\mp + D_+ H_+^\mp - iH_+^m \partial_m H_-^\mp + iH_-^m \partial_m H_+^\mp \\
- D_- H_-^\mp + D_- H_+^\mp - iH_-^m \partial_m H_-^\mp + iH_+^m \partial_m H_+^\mp
\end{array} \right| \tag{5.10}
\]

which can be computed to any order in powers of \( H^m \) and its derivatives.

Turning to the computation of \( \text{sdet} \hat{E}_A^B \), with \( E_\pm, E_\pm \) given in (3.2) and \( \hat{E}_+, \hat{E}_- \) defined in (4.1), we note again that \( \hat{E}_A^B \) is block triangular so that in \( \hat{E}_+, \hat{E}_- \) we only need the pieces proportional to \( \hat{E}_+, \hat{E}_- \), e.g.

\[
\hat{E}_+ = -i\{E_+, E_+\} = \{e^5(\hat{E}_+ + A_+^+ \hat{E}_-), e^8(\hat{E}_+ + A_+^- \hat{E}_-)\} \tag{5.11}
\]

\[
= -ie^{(s+5)} \left[ \{\hat{E}_+, \hat{E}_+\} + A_+^+ \{\hat{E}_-, \hat{E}_+\} + A_+^- \{\hat{E}_+, \hat{E}_-\} + A_-^+ \{\hat{E}_-, \hat{E}_-\}, \ldots \right] \\
= -ie^{(s+5)} \left[ (i + A_+^- \hat{C}_-^- \hat{\cdot} + A_-^+ \hat{C}_+^+) \hat{E}_+ + (iA_+^- A_+^- + A_-^+ \hat{C}_+^+ \hat{\cdot} + A_-^+ \hat{C}_+^+) \hat{E}_- + \ldots \right]
\]
and the superdeterminant works out to be

\[
\text{sdet} \hat{E}_A^B = \left| \begin{array}{ccc}
   i + A_+^+ \hat{C}_{-+} + A_+^\dagger \hat{C}_{++} & iA_+^- A_+^\dagger + A_+^\dagger \hat{C}_{-+} + A_+^\dagger \hat{C}_{++} \\
   iA_+^- A_+^\dagger + A_+^\dagger \hat{C}_{-+} + A_+^\dagger \hat{C}_{++} & i + A_+^+ \hat{C}_{-+} + A_+^\dagger \hat{C}_{++} \\
   \end{array} \right| \\
\frac{[1 - A_+^- A_+^\dagger][1 - A_+^\dagger A_+^\dagger]}{[1 - A_+^- A_+^\dagger][1 - A_+^\dagger A_+^\dagger]}
\]

The numerator can be simplified by using (3.20,21,23). After some algebra we obtain as a final form

\[
\text{sdet} \hat{E}_A^B = \left[ (1 + \hat{C}_{++} + \hat{C}_{-+})^2 - 4 \hat{C}_{++} \hat{C}_{-+} \right] \frac{1 - A_+^- A_+^\dagger A_+^\dagger}{(1 - A_+^- A_+^\dagger)(1 - A_+^\dagger A_+^\dagger)}
\]

and again this can be computed to any order in powers of $H^m$ and its derivatives. Note that, as expected, these superdeterminants have no dependence on the scale compensators, reflecting the classical scale invariance of the theory described by the action $\int d^2 x d^4 \theta E^{-1}$.

### 6 Conclusions

We have presented in this work the solution to the constraints of two-dimensional, $(2,2)$ supergravity, expressing the constrained covariant derivatives (vielbein and connections) in terms of two unconstrained prepotentials, $H^m$ and $\phi$, where $H^m$ is a real vector superfield and $\phi$ is an ordinary chiral, or twisted chiral, scalar superfield. Our results are contained in eqs. (3.1,2,22,23) and (4.24), and (3.11,17). The results are not as simple as they are in $(1,0)^3$ or $(1,1)^4$ supergravity, but we hope they will be useful for some studies where a fully covariant $N = 2$ formalism may prove advantageous. These might include on one hand topics concerning $N = 2$ superRiemann surfaces, and on the other issues dealing with the quantization of induced $(2,2)$ supergravity and its interaction with $(2,2)$ matter. Further topics, such as chiral representation, behavior under supercoordinate transformations, the background field method and quantization issues will be presented elsewhere.

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