Convective Equations and a Generalized Cole-Hopf Transformation

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Abstract

Differential equations with convective terms such as the Burger’s equation appear in many applications and have been the subject of intense research. In this paper we use a generalized form of Cole-Hopf transformation to relate the solutions of some of these nonlinear equations to the solutions of linear equations. In particular we consider generalized forms of Burger’s equation and second order nonlinear ordinary differential equations with convective terms which can represent steady state one-dimensional convection.
1 Background

Equations with convective terms appear in various applications in applied mathematics and theoretical physics. In particular Burger's equation [1]

\[ \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2}, \quad \nu = \text{constant} \] (1.1)

which serves as a prototype for turbulence modeling, gas dynamics and traffic flow contains such a term. However it was found that this equation can be linearized and reduced to the linear Heat equation

\[ \frac{\partial \psi}{\partial t} = k \frac{\partial^2 \psi}{\partial x^2} \] (1.2)

by Cole-Hopf transformation [2,3]

\[ \psi = \frac{\partial \phi}{\partial x}. \] (1.3)

Since this discovery many attempts were made in the literature to generalize this result to larger class of equations[4-10] or to relate other nonlinear equations to this equation[8].

Recently however, we introduced [11] a generalization of Cole-Hopf transformation and used it to linearize various nonlinear ordinary differential equations e.g Duffing equation. We showed also that many of the special functions of mathematical physics are exact solutions for a class of nonlinear equations.

In this paper we apply in Sec. 2 the same generalization of (1.3) to linearize a class of Burger’s equations with additional quadratic nonlinear terms. Next in Sec. 3 we consider second nonlinear ordinary differential equations(ODEs) with convective term and derive conditions under which they can be linearized by the same transformation. These type of equations represent steady state convection in one dimension [12,13]. We end up in Sec. 4 with some conclusions.

2 Generalized Burger Equation

In this section we consider equations of the form

\[ \frac{\partial \psi}{\partial t} - M(x) \frac{\partial^2 \psi}{\partial x^2} = H(x)\psi \frac{\partial \psi}{\partial x} + V(x)\psi + W(x)\psi^2 \] (2.1)
where \( \psi = \psi(x, t) \). We shall say that the solutions of this equation are related to the solution of the linear equation

\[
\frac{\partial \phi}{\partial t} - M(x) \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{2.2}
\]

if we can find functions \( P(x), Q(x) \) so that

\[
\psi = P(x) + Q(x) \frac{\partial \phi}{\partial x} \tag{2.3}
\]

To classify those equations of the form (2.1) which can be paired to the linear equation (2.2) we substitute (2.3) in (2.1). After some algebra we find that the following equation has to be satisfied

\[
a_3 \left( \frac{\partial \phi}{\partial x} \right)^3 + a_2 \left( \frac{\partial \phi}{\partial x} \right)^2 + a_{21} \left( \frac{\partial \phi}{\partial x^2} \right) + a_1 \frac{\partial \phi}{\partial x} + \frac{a_{11}}{\phi(x)} + a_0 = 0, \tag{2.4}
\]

where

\[
a_{21} = -Q(x) \left[ (Q(x)H(x) - 3M(x)) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial t} \right], \tag{2.5}
\]

\[
a_{11} = -(Q(x)H(x)P(x) + 2M(x)Q(x)') \frac{\partial^2 \phi}{\partial x^2} + Q(x) \frac{\partial^2 \phi}{\partial t \partial x} - M(x)Q(x) \frac{\partial^3 \phi}{\partial x^3}, \tag{2.6}
\]

\[
a_3 = 2M(x) - Q(x)H(x), \tag{2.7}
\]

\[
a_2 = H(x)Q(x)P(x) + (2M(x) - H(x)Q(x))Q(x)' - W(x)Q(x)^2, \tag{2.8}
\]

\[
a_1 = -H(x)Q(x)P(x)' - H(x)P(x)Q(x)' - M(x)Q(x)' - V(x)Q(x) - 2W(x)P(x)Q(x), \tag{2.9}
\]

\[
a_0 = -M(x)P(x)' - H(x)P(x)P(x)' - W(x)P(x)^2 - V(x)P(x), \tag{2.10}
\]

where primes denote differentiation with respect to \( x \). To satisfy (2.4) it is sufficient to let \( a_{21} = a_{11} = 0 \) and \( a_3 = a_2 = a_1 = a_0 = 0 \).

From \( a_3 = 0 \) it follows that

\[
Q(x) = \frac{2M(x)}{H(x)} \tag{2.11}
\]

In view of this relationship \( a_{21} = 0 \) is satisfied in virtue of (2.2). To satisfy \( a_{11} = 0 \) we add and subtract \( M(x) \frac{\partial^2 \phi}{\partial x^2} \) and rewrite this equation in the form

\[
a_{11} = -(Q(x)H(x)P(x) + 2M(x)Q(x)') \frac{\partial^2 \phi}{\partial x^2} + Q(x) \frac{\partial^2 \phi}{\partial t \partial x} - Q(x) \frac{\partial}{\partial x} \left( M \frac{\partial^2 \phi}{\partial x^2} \right) = 0. \tag{2.12}
\]
Letting

\[ P(x) = \frac{2M(x)H(x)'}{H(x)^2} - \frac{M(x)'}{H(x)} \]  \hspace{1cm} (2.13)

we infer that (2.12) is satisfied using (2.2).

We now use \( a_2 = a_1 = 0 \) to express \( W(x) \) and \( V(x) \) in terms of \( M(x), H(x) \). Substituting the expressions for \( Q(x) \) and \( P(x) \) in \( a_2 = 0 \) we obtain

\[ W(x) = -\frac{H(x)M(x)'}{2M(x)} + H(x)' \]  \hspace{1cm} (2.14)

Similarly \( a_1 = 0 \) yields

\[ V(x) = -\frac{M(x)H(x)''}{H(x)} \]  \hspace{1cm} (2.15)

With these results \( a_0 = 0 \) yield a differential equation relating \( M(x) \) and \( H(x) \)

\[ \left[ \frac{H(x)M(x)'}{2M(x)} - H(x)' \right] P(x)^2 + \left[ \frac{M(x)H(x)''}{H(x)} - H(x)P(x)'ight] P(x) - M(x)P(x)^{\prime\prime} = 0 \]  \hspace{1cm} (2.16)

where for brevity we did not substitute for \( P(x) \).

When \( M(x) = A \) where \( A \) is a constant (2.16) becomes

\[ H(x)^2H(x)'' - 5H(x)H(x)'H(x)'' + 4(H(x)')^3 = 0. \]  \hspace{1cm} (2.17)

To solve this equation we introduce \( z(x) = \frac{1}{H(x)} \). The resulting equation can be written as

\[ \frac{d}{dx} \left( \frac{1}{z(x)} \frac{d^2 z(x)}{dx^2} \right) = 0. \]  \hspace{1cm} (2.18)

Hence either

\[ H(x) = \frac{1}{ax + b} \]  \hspace{1cm} (2.19)

where \( a, b \) are constants or

\[ H(x) = \frac{B}{\cos(\omega x + \beta)}, \quad B, \omega, \beta, \text{ constants}, \]  \hspace{1cm} (2.20)

or

\[ H(x) = Ce^{\alpha x}, \quad C, \alpha, \text{ constants}, \]  \hspace{1cm} (2.21)

When \( H(x) = 1 \) eq. (2.16) becomes

\[ \frac{M(x)'}{2M(x)} - M(x)M(x)'' + M(x)M(x)''' = 0. \]  \hspace{1cm} (2.22)
Substituting $M(x) = w(x)^2$ this equation reduces to

$$\frac{d}{dx} \left( w(x) \frac{d^2 w}{dx^2} \right) = 0. \quad (2.23)$$

Hence

$$w(x) \frac{d^2 w}{dx^2} = c$$

where $c$ is a constant. When $c = 0$ it follows that

$$M(x) = (a_1 x + b_1)^2 \quad (2.24)$$

where $a_1, b_1$ are constants. When $c \neq 0$ we can find an implicit expression $w(x)$

$$x = C_2 \pm \int_{w(x)}^{w(x_0)} \frac{ds}{2c \ln(s) - C_1 c},$$

where $C_1, C_2$ are integration constants.

To our best knowledge the linearization of the ”modified Burger’s equations” represented by (2.19)-(2.21) and (2.24) did not appear in the literature so far.

Example: For $M(x) = 1, H(x) = Ce^{\alpha x}$ we obtain from (2.14) (2.15) respectively that

$$W(x) = C \alpha e^{\alpha x}, \quad V(x) = -\alpha^2.$$

Hence the solutions of (2.1) with these coefficients is related to the solutions of the Heat equation (2.2) by the transformation (2.3) with

$$Q(x) = \frac{2 e^{-\alpha x}}{C}, \quad P(x) = \frac{2 \alpha e^{-\alpha x}}{C},$$

and this fact can be verified by direct substitution.

### 3 Second Order Convective ODEs

We shall say that the solutions of the equations

$$\psi(x)'' = S(x) + [V(x) + F(x)\psi(x)']\psi(x) + W(x)\psi(x)^2 \quad (3.1)$$

and

$$\phi(x)'' = U(x)\phi(x) \quad (3.2)$$
are related if we can find functions \( P(x) \) and \( Q(x) \) so that
\[
\psi(x) = P(x) + Q(x) \frac{\phi(x)'}{\phi(x)}.
\] (3.3)

Furthermore we observe that (3.1) can take the more general form
\[
\psi(x)'' = S(x) + (V(x) + F(x)\psi(x)')\psi(x) + V_1(x)\psi(x)' + W(x)\psi(x)^2.
\] (3.4)

In this case we can find \( p(x) \) so that \( V_1(x) = -2\frac{p'(x)}{p(x)} \). Introducing \( \xi(x) = p(x)\psi(x) \), (3.4) becomes
\[
\xi(x)'' = p(x)S(x) + \left[ V(x) + \frac{p(x)''}{p(x)} + \frac{F(x)}{p(x)}\xi(x)'ight]\xi(x) + \left[ \frac{W(x)}{p(x)} - \frac{F(x)p(x)'}{p(x)^2} \right] \xi(x)^2. \] (3.5)

which has the same form as (3.1).

To classify those nonlinear equations (3.1) which can be ”paired” with a linear equation of the form (3.2), we differentiate (3.3) twice and in each step replace the second order derivative of \( \phi(x) \) by \( U(x)\phi(x) \). We then use (3.1) to eliminate \( \psi(x)'' \). As a result we find that the following equation must hold;
\[
a_3(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^3 + a_2(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^2 + a_1(x) \frac{\phi(x)'}{\phi(x)} + a_0(x) = 0 \] (3.6)

where
\[
a_3(x) = (Q(x)F(x)+2), \quad a_2(x) = Q(x)(F(x)P(x) - W(x)Q(x)) - (2+F(x)Q(x))Q(x)', \quad (3.7)
\]
\[
a_1(x) = Q(x)'' - F(x)P(x)Q(x)' - U(x)F(x)Q(x)^2 - (2U(x) + V(x) + F(x)P(x)' + 2W(x)P(x))Q(x), \] (3.8)
\[
a_0(x) = P(x)'' - W(x)P(x)^2 - (F(x)Q(x)U(x) + V(x) + F(x)P(x)')P(x) + Q(x)U(x)' + 2U(x)Q(x)' - S(x). \] (3.9)

To satisfy (3.6) it is therefore sufficient to let \( a_i(x) = 0, \ i = 0, 1, 2, 3 \).

One can use these conditions in two ways. The first is to assume that a nonlinear equation (3.1) is given and try to determine the appropriate \( P, Q, U \) (if they exist) that
relates it to (3.3). Otherwise one may fix the functions $P, Q, U$ and classify those nonlinear equations of the form (3.1) which are related to (3.3) by the transformation (3.2). In the following we provide separate solutions for these two possibilities.

Assuming that one starts from (3.1) i.e the functions $V(x), W(x), F(x)$ and $S(x)$ (with $F(x) \neq 0$) are given it follows from (3.6) that

$$Q(x) = -\frac{2}{F(x)}. \quad (3.10)$$

Substituting this result in (3.7) and solving for $P(x)$ it follows that

$$P(x) = -\frac{2W(x)}{F(x)^2}. \quad (3.11)$$

Using (3.10), (3.11) and (3.9) we obtain a linear first order differential equation for $U(x)$

$$U(x)' + \frac{F(x)S(x)}{2} + \frac{W(x)'' - V(x)W(x) + (2W(x) - 2F(x)')U(x)}{F(x)} = 0 \quad (3.12)$$

Finally eq. (3.8) provides (after using (3.10) and (3.11)) an intrinsic constraint on the functions $V(x), W(x), F(x)$ and $S(x)$ which have to be satisfied for the relationship between (3.1) and (3.3) to exist.

$$V(x) + \frac{F(x)'' - 2W(x)'}{F(x)} + \frac{6W(x)F(x)' - 2(F(x)')^2 - 4W(x)^2}{F(x)^2} = 0. \quad (3.13)$$

We observe that when $F(x) = 0$ the algorithm can be implemented in the same way by adding a term $R(x)\psi(x)^3$ to eq. (3.1).

For the reverse procedure where one elects the functions $U(x), P(x)$ and $Q(x)$ and attempts to evaluate the corresponding nonlinear equation (3.1) we have from (3.9)

$$F(x) = -\frac{2}{Q(x)}, \quad W(x) = -\frac{2P(x)}{Q(x)^2}. \quad (3.14)$$

Substituting these results in (3.9) and solving for $V(x)$ yields

$$V(x) = \frac{Q(x)Q(x)'' + 4P(x)^2 + 2(P(x)Q(x))'}{Q(x)^2}. \quad (3.15)$$

Finally from (3.10) we derive an expression for $S(x)$

$$S(x) = P(x)'' + Q(x)U(x)' + 2(Q(x)' + P(x))U(x) - \frac{P(x)Q(x)''}{Q}(x) - \frac{2P(x)^2(Q(x)' + P(x))}{Q(x)^2}. \quad (3.16)$$
Example: In the differential equation

\[ \psi''(x) = [4a^2 + \psi(x)']\psi(x) + a\psi(x)^2, \] (3.17)

where \( a \) is a constant we have \( F(x) = 1, W(x) = a, V(x) = 4a^2 \) and \( S(x) = 0 \). The equation satisfies the constraint \( \text{(3.13)} \) and using \( \text{(3.10)}-\text{(3.12)} \) we find that

\[ Q(x) = -2, \; P(x) = -2a. \]

From \( \text{(3.12)} \) we find that the general solution for \( U(x) \) is

\[ U(x) = Ce^{-2ax} + a^2. \]

With this \( U(x) \) the general solution of \( \text{(3.2)} \) is

\[ \phi(x) = C_1 \text{BesselI}(1, z) + C_2 \text{BesselY}(1, iz) \]

where \( z = \sqrt{Ce^{-ax}/a} \) and \( \text{BesselI}, \text{BesselY} \) are the modified Bessel functions of the first and second kind. Letting \( C_2 = 0 \) (real solution) it is straightforward to verify that

\[ \psi(x) = -2a - 2\frac{\phi(x)'}{\phi(x)} \]

is a solution of \( \text{(3.17)} \).

4 Conclusions

We demonstrated in this paper that a generalized form of Cole-Hopf transformation \( \text{(1.3)} \) can be used to relate the solutions of some nonlinear equation with convective terms with the solutions of linear differential equations. The algorithm is straightforward and we applied it to a generalized form of Burger’s equation and second order nonlinear ODEs with convective terms.

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