Contributions of the kaon pair from $\rho(770)$ for the three-body decays $B \to D K \bar{K}$

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We study the contributions of the kaon pair originating from the resonance $\rho(770)$ for the three-body decays $B \to D K \bar{K}$ by employing the perturbative QCD approach. According to the predictions in this work, the contributions from the intermediate state $\rho(770)^0$ are relative small for the three-body decays such as $B^0 \to \bar{D}^0 K^+ K^-$, $B^0_s \to \bar{D}^0 K^+ K^-$ and $B^+ \to D^+_s K^+ K^-$, while a percent at about 20% of the total three-body branching fraction for $B^+ \to \bar{D}^0 K^+ K^0$ could possibly come from the subprocess $\rho(770)^+ \to K^+ \bar{K}^0$. We also estimate the branching fractions for $\rho(770)^\pm$ decay into kaon pair to be about one percent and that for the neutral $\rho(770)$ into $K^+ K^-$ or $K^0 \bar{K}^0$ to be about 0.5%, which will be tested by the future experiments.

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I. INTRODUCTION

Three-body hadronic $B$ meson decays are much more complicated than the two-body cases partly due to the entangled resonant and nonresonant contributions, while these decay processes provide us many advantages for the study of the spectroscopy, the test of the factorization and the extraction of the CKM angles from the $CP$ asymmetries [1]. Attempts have been made to describe the whole region of the Dalitz plot for the three-body $B$ decays [2–4], but more attentions have been put on the resonance contributions originating from the low energy scalar, vector and tensor intermediate states in the subprocesses of the three-body hadronic $B$ meson decays within different methods, such as the QCD factorization (QCDF) [5–20] and the perturbative QCD (PQCD) approach [21–37]. In addition, there are many works within the symmetries one can find in [38–49] dedicated to the relevant decay modes.

The decays of the $B$ meson into a charmed $D$ meson plus kaon pair, offering rich opportunities to study the resonant components in the $DK$ or $KK$ system, have been measured in the past two decades [50–55]. The analysis of the $B \to D^{(*)} K^- K^{0(*)}$ decays was performed for the first time by Belle Collaboration with the detailed investigation of the invariant mass and the polarization distributions of the $K^- K^{0(*)}$ pair [50]. In Ref. [51], BaBar Collaboration reported their measurement for the process $B^- \to D_s^- K^- K^-$. While in the later study [52] by Belle, a significant deviation from the simple phase-space model in the $D_s K$ invariant mass distribution was found. In the recent works by LHCb Collaboration, the observations of the decays $B^0 \to \bar{D}^0 K^+ K^-$ [53], $B^0_s \to \bar{D}^0 K^+ K^-$ [54] and $B^+ \to D^+_s K^+ K^-$ [55] together with the measurements of corresponding branching fractions were presented respectively. Moreover, the studies on the $B \to D \phi(1020)$ decays, where the $\phi(1020)$ meson was reconstructed through its decay to a $K^+ K^-$ pair, were performed in [55–60] by CLEO, BaBar and LHCb Collaborations.

The vector state $\phi(1020)$ in the $K \bar{K}$ invariant-mass spectrum for the three-body hadronic $B$ meson decays has attracted much attentions [55, 61–64], while one should note that the $P$-wave resonance contributions of the kaon pair can also come from $\rho(770)$, $\omega(782)$ and their excited states [65–67]. Besides, the charged $\rho(770)$ and its excited states are the only possible sources of vector intermediate states for the $K^+ K^0$ or $K^- K^0$ system in the three-body $B$ decays. Although the pole mass of $\rho(770)$ is below the threshold of kaon pair, the virtual contribution [68–70] from the Breit-Wigner (BW) [71] tail of $\rho(770)$ for the $K \bar{K}$ was found indispensable for specific processes, such as $\pi^- p(n) \to K^- K^+ n(p)$ [72, 73], $e^+ e^- \to K^+ K^-$ [74–78] and $\pi \pi \to K \bar{K}$ scattering [79]. Recently, the component $\rho(1450)^0 \to K^+ K^-$ in the decays $B^+ \to \pi^+ K^+ K^-$ was reported by LHCb to be 30% of the total fit fraction and much larger than the fit fraction 0.3% from $\phi(1020)$ [64]. The subprocess $\rho(1450)^0 \to K^+ K^-$ and the related topics for the decays $B^+ \to \pi^+ K^+ K^-$ have been analysed in [16, 80, 81] recently, and contribution in these decays for $K^+ K^-$ from $\rho(770)^0$ which has been ignored in the experimental and theoretical studies was found to be the same order of that from $\rho(1450)^0$ in Ref. [80].

In the previous works [70, 82–89], the resonance contributions from various intermediate states for the three-body
decays $B \to D h_1 h_2$ ($h_{1,2}$ stands for pion or kaon) have been studied within the PQCD approach based on the $k_T$ factorization theorem [90–93]. In this work, we shall focus on the contributions of the subprocesses $\rho(770) \to K\bar{K}$ for the three-body decays $B \to D K \bar{K}$, where the symbol $\bar{K}$ means the kaons $K^+$ and $K^0$ and the symbol $K$ means the kaons $K^-$ and $K^0$. In view of the narrow decay width of $\omega(782)$ and the gap between its pole mass and the threshold of kaon pair, the branching fractions for the decays with the subprocess $\omega(782) \to K\bar{K}$ are small and negligible comparing with the contribution from $\rho(770) \to K\bar{K}$ in the same decay mode [80]. Meanwhile, there are still disparities between the fitted coefficients of the time-like form factors for kaons from currently known experimental results [65, 66, 94], we will leave the possible subprocesses with those excited states of $\rho(770)$ and $\omega(782)$ decay into $K\bar{K}$ to the future study.

The rest of this paper is organized as follows. In Sec. II, we give a brief review of the PQCD framework for the concerned decay processes. The numerical results and the phenomenological analyses are given in Sec. III. The summary of this work is presented in Sec. IV. The relevant quasi-two-body decay amplitudes are collected in the Appendix.

### II. FRAMEWORK

In the light-cone coordinates, the momenta $p_B$, $p$ and $p_3$ for the $B$ meson, the resonance $\rho$ and the final state $D$, respectively, are chosen as

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, 0_T), \quad p = \frac{m_B}{\sqrt{2}}(1 - r^2, \eta, 0_T), \quad p_3 = \frac{m_B}{\sqrt{2}}(r^2, 1 - \eta, 0_T),$$

where $m_B$ denotes the mass for $B$ meson, the variable $\eta$ is defined as $\eta = s/(m_D^2 - m_B^2)$, the invariant mass square $s = p^2 = m_{\bar{K}K}^2$ for the kaon pair and the mass ratio $r = m_D/m_B$. The momentum of the light quark in the $B$ meson, $\rho$ and $D$ meson are denoted as $k_B$, $k$ and $k_3$ with

$$k_B = (0, x_B \frac{m_B}{\sqrt{2}}, k_{BT}), \quad k = (z \frac{(1-r^2)m_B}{\sqrt{2}}, 0, k_T), \quad k_3 = (0, x_3 \frac{(1-\eta)m_B}{\sqrt{2}}, k_{3T}),$$

where the momentum fraction $x_B$, $z$ and $x_3$ run between zero and unity.

The decay amplitude $A$ for the quasi-two-body processes $B \to D\rho(770) \to D K \bar{K}$ in the PQCD approach can be expressed as the convolution of a hard kernel $H$ containing one hard gluon exchange with the relevant hadron distribution amplitudes [21, 95]

$$A = \Phi_B \otimes H \otimes \Phi_D \otimes \Phi_{KK},$$

where $\Phi_B$, $\Phi_D$, $\Phi_{KK}$ are the distribution amplitudes of the $B$, $D$, $K\bar{K}$ mesons respectively.
where the distribution amplitudes $\Phi_B$, $\Phi_D$ and $\Phi_{KK}$ for the initial and final state mesons absorb the nonperturbative dynamics. In this work, we employ the same distribution amplitudes for $B$ and $D$ mesons as those widely adopted in the studies of the hadronic $B$ meson decays in the PQCD approach, one can find their explicit expressions and parameters in the Ref. [82] and the references therein.

The $P$-wave $KK$ system distribution amplitudes along with the subprocesses $\rho(770) \rightarrow KK$ are defined as [37, 80]

$$\phi_{KK}^{P\text{-wave}}(z, s) = \frac{-1}{\sqrt{2N_c}} \left[ \sqrt{s} f_L \phi^0(z, s) + f_L \phi^t(z, s) + \sqrt{s} \phi^s(z, s) \right] ,$$

(4)

where $z$ is the momentum fraction for the spectator quark, $s$ is the squared invariant mass of the kaon pair, $\epsilon_L$ and $\rho$ are the longitudinal polarization vector and momentum for the resonance. The twist-2 and twist-3 distribution amplitudes $\phi^0$, $\phi^s$ and $\phi^t$ are parameterized as [80]

$$\phi^0(z, s) = \frac{3 F_{K^0}(s)}{\sqrt{a N_c}} z(1-z) \left[ 1 + a_0^0 C_2^{3/2}(1-2z) \right] ,$$

(5)

$$\phi^s(z, s) = \frac{3 F_{K^0}(s)}{2 \sqrt{a N_c}} (1-2z) \left[ 1 + a_2^s (1-10z+12z^2) \right] ,$$

(6)

$$\phi^t(z, s) = \frac{3 F_{K^0}(s)}{2 \sqrt{a N_c}} (1-2z)^2 \left[ 1 + a_2^t C_2^{3/2}(1-2z) \right] ,$$

(7)

with the Gegenbauer polynomial $C_2^{3/2}(t) = 3(5t^2 - 1)/2$, $F_{K^0}^u(s) \approx (f_T^e/f_T^m) F_{KK}^u(s)$ [24], $a = 1$ for $\rho(770)^0$ and $a = 2$ for $\rho(770)^\pm$. In the numerical calculation, we adopt $f_\rho = 0.216$ GeV [96, 97] and $f_{\rho}^T = 0.184$ GeV [98]. The Gegenbauer moments $a_2^0,s,t$ are the same as they in the distribution amplitudes for the intermediate state $\rho(770)$ in Refs. [24, 80]. The vector time-like form factors for kaons are written as [12]

$$F_{K^+K^-}^u = F_{K^0\bar{K}^0}^d = F_\rho + 3F_\omega,$$

(8)

$$F_{K^+K^-}^d = F_{K^0\bar{K}^0}^u = -F_\rho + 3F_\omega,$$

(9)

$$F_{K^+K^-}^s = F_{K^0\bar{K}^0}^s = -3F_\phi,$$

(10)

where $F_\rho$, $F_\omega$ and $F_\phi$ come from the definitions of the electromagnetic form factors for the charged and neutral kaon [65, 66]

$$F_{K^+K^-}^{l=1}(s) = \frac{1}{2} \sum_{\rho} c^K_{\rho} \text{BW}_{\rho}(s) + \frac{1}{6} \sum_{\omega} c^K_{\omega} \text{BW}_{\omega}(s) + \frac{1}{3} \sum_{\phi} c^K_{\phi} \text{BW}_{\phi}(s) = F_\rho + F_\omega + F_\phi,$$

(11)

$$F_{K^0\bar{K}^0}^{l=1}(s) = \frac{1}{2} \sum_{\rho} c^K_{\rho} \text{BW}_{\rho}(s) + \frac{1}{6} \sum_{\omega} c^K_{\omega} \text{BW}_{\omega}(s) + \frac{1}{3} \sum_{\phi} c^K_{\phi} \text{BW}_{\phi}(s) = -F_\rho + F_\omega + F_\phi.$$
BW formula can be found in the Refs. [65, 66, 94]. It is not difficult to find that the corresponding coefficients $c_i^s$ for $\rho(770)$, $\omega(782)$ or $\phi(1020)$ are close to each other in [65, 66, 94], while those coefficients for the excited states have significant differences by comparing the fitted parameters in the Table 2 in Refs. [65, 66] and Table 1 in [94]. In this work, we concern only the $\rho(770)$ components of the vector kaon time-like form factors and the fitted values for the coefficients $C_{\rho(770)}^K$ in the kaon form factors collected from the Refs. [65, 66, 94] have been listed in the TABLE I. The “Fit(1)”, “Fit(2)” and “Model I”, “Model II” represent the values parameterized with different constraints in each work. Due to the closeness of the coefficients $C_{\rho(770)}^K$ in Refs. [65, 66, 94], we choose the value of “Fit(1)” in the Ref. [65] in our numerical calculation. The resonance shape for $\rho(770)$ is described by the KS version of the BW formula [65, 99]

$$m_p^2 - s - i\sqrt{\Gamma_{tot}(s)}$$

where the effective $s$-dependent width is given by

$$\Gamma_{tot}(s) \approx \Gamma_{\rho \to 2\pi}(s) = \frac{m_{\rho}^2}{s} \left( \frac{\beta(s, m_{\pi})}{\beta(m_{\rho}^2, m_{\pi})} \right)^3$$

with $\beta(s, m) = \sqrt{1 - 4m^2/s}$. In addition, one has the time-like form factor for $K^+\bar{K}^0$ and $K^-\bar{K}^0$ from the relation [65, 67]

$$F_{K^+\bar{K}^0}(s) = -F_{K^-\bar{K}^0}(s) = 2F_{K^+}^{I=1}(s)$$

and keep only the $\rho$ resonance contributions with isospin symmetry.

| TABLE I: The fitted results for the coefficients $C_{\rho(770)}^K$ of the kaon form factors. |
|---------------------------------|---|---|---|---|---|---|---|
| $C_{\rho(770)}^K$               | Fit(1) [65] | Fit(2) [65] | Fit(1) [66] | Fit(2) [66] | Model I [94] | Model II [94] |
| $1.195 \pm 0.009$              | $1.139 \pm 0.010$ | $1.138 \pm 0.011$ | $1.120 \pm 0.007$ | $1.162 \pm 0.005$ | $1.067 \pm 0.041$ |

For the decays $B(s) \to \bar{D}(s)\rho(770) \to \bar{D}(s)K\bar{K}$ and the CKM-suppressed decays $B(s) \to D(s)\rho(770) \to D(s)K\bar{K}$, the effective Hamiltonian $H_{eff}$ can be expressed as

$$H_{eff} = \left\{ \begin{array}{ll}
\frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} \left[ C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu) \right], & \text{for } B(s) \to \bar{D}(s)\rho(770) \to \bar{D}(s)K\bar{K} \text{ decays}, \\
\frac{G_F}{\sqrt{2}} V_{cb}^* V_{cd} \left[ C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu) \right], & \text{for } B(s) \to D(s)\rho(770) \to D(s)K\bar{K} \text{ decays},
\end{array} \right.$$  

(16)

where $G_F = 1.16639 \times 10^{-5}$ GeV$^{-2}$, $V_{ij}$ are the CKM matrix elements, $C_{1,2}(\mu)$ represent the Wilson coefficients at the renormalization scale $\mu$ and $O_{1,2}$ are the local four-quark operators. According to the typical Feynman diagrams for the concerned decays as shown in Figs. 1 and 2, the decay amplitudes for $B(s) \to \bar{D}(s)\rho(770)$ with the subprocesses $\rho(770)^0 \to K^+K^-K_0^0\bar{K}^0$ and $\rho(770)^+ \to K^+K^0\bar{K}_0^0\bar{K}$ are given as follows:

$$A(B^+ \to \bar{D}^0\rho^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} \left[ a_1 F_{\ell\ell}^{LL} + C_2 M_{\ell\ell}^{LL} + a_2 F_{\ell\ell}^{LD} + C_1 M_{\ell\ell}^{LD} \right],$$

(17)

$$A(B^0 \to D^-\rho^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} \left[ a_1 F_{\ell\ell}^{LL} + C_2 M_{\ell\ell}^{LL} + a_2 F_{\ell\ell}^{LD} + C_1 M_{\ell\ell}^{LD} \right],$$

(18)

$$A(B^0 \to \bar{D}^0\rho^0) = \frac{G_F}{2} V_{cb}^* V_{ud} \left[ a_1 (-F_{\ell\ell}^{LL} + F_{\ell\ell}^{LL}) + C_2 (-M_{\ell\ell}^{LL} + M_{\ell\ell}^{LL}) \right],$$

(19)

$$A(B_s^+ \to D^-\rho^+) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} \left[ a_1 F_{\ell\ell}^{LL} + C_2 M_{\ell\ell}^{LL} \right],$$

(20)

$$A(B_s^0 \to \bar{D}^0\rho^0) = \frac{G_F}{2} V_{cb}^* V_{us} \left[ a_1 F_{\ell\ell}^{LL} + C_2 M_{\ell\ell}^{LL} \right],$$

(21)

$$A(B_s^0 \to D^-\rho^0) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{us} \left[ a_2 F_{\ell\ell}^{LL} + C_1 M_{\ell\ell}^{LL} \right],$$

(22)
while the decay amplitudes for \( B_s \to D_s \rho(770) \) with the subprocesses \( \rho(770)^0 \to K^+K^-/K^0\bar{K}^0 \), \( \rho(770)^+ \to K^+\bar{K}^0 \) and \( \rho(770)^- \to K^-K^0 \) can be written as:

\[
\mathcal{A}(B^+ \to D^0 \rho^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}[a_1 F_{LL}^{LL} + C_2 M_{LL}^{LL} + a_2 F_{LL}^{LL} + C_1 M_{LL}^{LL}],
\]

\[
\mathcal{A}(B^+ \to D^0 \rho^-) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}[a_2 (F_{LL}^{L} - F_{LL}^{L}) + C_1 (M_{LL}^{L} - M_{LL}^{L})],
\]

\[
\mathcal{A}(B^+ \to D_s^0 \rho^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}[a_2 F_{LL}^{LL} + C_1 M_{LL}^{LL}],
\]

\[
\mathcal{A}(B^+ \to D_s^0 \rho^-) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}[a_2 (-F_{LL}^{L} + F_{LL}^{L}) + C_2 (-M_{LL}^{L} + M_{LL}^{L})],
\]

\[
\mathcal{A}(B^0 \to D^0 \rho^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}[a_1 (F_{LL}^{L} - F_{LL}^{L}) + C_1 (M_{LL}^{L} - M_{LL}^{L})],
\]

\[
\mathcal{A}(B^0 \to D^0 \rho^-) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd}[a_2 F_{LL}^{LL} + C_1 M_{LL}^{LL}],
\]

\[
\mathcal{A}(B^0 \to D_s^0 \rho^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}[a_1 F_{LL}^{LL} + C_2 M_{LL}^{LL}],
\]

\[
\mathcal{A}(B_s^0 \to D^0 \rho^+) = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cs}[a_1 F_{LL}^{LL} + C_1 M_{LL}^{LL}],
\]

with the Wilson coefficients \( a_1 = C_1 + C_2/3 \) and \( a_2 = C_2 + C_1/3 \). The explicit expressions of individual amplitude \( F \) and \( M \) for the factorizable and nonfactorizable Feynman diagrams can be found in Appendix.

The differential branching fractions \( (B) \) for the quasi-two-body decays \( B \to D\rho(770) \to DK\bar{K} \) can be written as \([7, 37, 80]\)

\[
\frac{d\mathcal{B}}{dt_B} = \tau_B \frac{q^3 q_D^3}{12\pi^4 m_B^3} |\mathcal{A}|^2.
\]

The magnitudes of the momenta for \( K \) and \( D \) in the center-of-mass frame of the kaon pair are written as

\[
q = \frac{1}{2} \sqrt{s - 4m_K^2}, \quad q_D = \frac{1}{2} \sqrt{\left(m_B^2 - m_D^2\right)^2 - 2\left(m_B^2 + m_D^2\right)s + s^2}.
\]

III. RESULTS

In the numerical calculations, the input parameters, such as masses and decay constants (in units of GeV) and \( B \) meson lifetimes (in units of ps), are adopted as follows \([100]\):

\[
\begin{align*}
    m_{B^\pm} &= 5.279, & m_{B^0} &= 5.280, & m_{B_s^0} &= 5.367, & m_{D^\pm} &= 1.870, & m_{D^0} &= 1.865, \\
    m_{D_s^\pm} &= 1.968, & m_{K^\pm} &= 0.494, & m_{K^0} &= 0.498, & m_c &= 1.27, & m_{\pi^\pm} &= 0.140, \\
    m_{\pi^0} &= 0.135, & f_B &= 0.189, & f_{B_s} &= 0.231, & f_D &= 0.2126, & f_{D_s} &= 0.2499, \\
    \tau_{B^\pm} &= 1.638, & \tau_{B^0} &= 1.519, & \tau_{B_s^0} &= 1.515.
\end{align*}
\]

For the Wolfenstein parameters of the CKM mixing matrix, we use the values \( A = 0.790^{+0.012}_{-0.012}, \lambda = 0.22650 \pm 0.00048, \bar{\rho} = 0.141^{+0.016}_{-0.017}, \bar{\eta} = 0.357 \pm 0.011 \) as listed in Ref. \([100]\).

In Tables II and III, we list our numerical results for the branching fractions of the \( B_s \to D_s \rho(770) \to \bar{D}_s \rho(770) \) and the CKM-suppressed \( B_s \to D_s \rho(770) \to D_s \rho K \bar{K} \) decays. The first error of these branching fractions comes from the uncertainty of the \( B_s \) meson shape parameter \( \omega_B = 0.40 \pm 0.04 \) or \( \omega_{B_s} = 0.50 \pm 0.05 \), the second error is induced by the uncertainties of the Gegenbauer moments \( a_0^2 = 0.25 \pm 0.10, a_2^2 = 0.75 \pm 0.25 \) and \( a_2^2 = -0.60 \pm 0.20 \) in the kaon-kaon distribution amplitudes, the last one is due to \( C_D = 0.5 \pm 0.1 \) or \( C_{D_s} = 0.4 \pm 0.1 \) for \( D_s \) meson wave function. The errors come from the uncertainties of other parameters are small and have been neglected. Since the concerned decay modes occur only through the tree-level quark diagrams, there are no direct \( CP \) asymmetries for these decays in the standard model.
TABLE II: The PQCD predictions of the branching fractions for the $B_{(s)} \to D_{(s)} \rho(770) \to D_{(s)} K\bar{K}$ decays. The decay mode with the subprocess $\rho(770)^0 \to K^-K^0$ has the same branching fraction of its corresponding mode with $\rho(770)^0 \to K^+K^-$.  

| Decay modes | Unit | Quasi-two-body results |
|-------------|------|------------------------|
| $B^+ \to D^0\rho(770)^+ \to D^0 K^+K^0$ | $(10^{-4})$ | $1.18^{+0.62}_{-0.40} \cdot 1.02 (a_0 + a_2 + a_4)^{+0.02}_{-0.07}$ $(C_D)$ |
| $B^0 \to D^-\rho(770)^0 \to D^- K^+K^0$ | $(10^{-5})$ | $7.93_{-0.36}^{+0.30} (a_0 + a_2 + a_4)^{+0.03}_{-0.03}$ $(C_D)$ |
| $B^0 \to D^0\rho(770)^0 \to D^0 K^-K^-$ | $(10^{-6})$ | $1.07_{-0.37}^{+0.46} (a_0 + a_2 + a_4)^{+0.01}_{-0.01}$ $(C_D)$ |
| $B^0_s \to D^-\rho(770)^0 \to D^- K^+K^0$ | $(10^{-8})$ | $4.22_{-0.67}^{+0.58} (a_0 + a_2 + a_4)^{+0.40}_{-0.30}$ $(C_D)$ |
| $B^0_s \to D^0\rho(770)^0 \to D^0 K^-K^-$ | $(10^{-8})$ | $1.05_{-0.17}^{+0.15} (a_0 + a_2 + a_4)^{+0.07}_{-0.07}$ $(C_D)$ |
| $B^0 \to D^+\rho(770)^- \to D^+ K^0K^0$ | $(10^{-9})$ | $6.06_{-0.26}^{+0.37} (a_0 + a_2 + a_4)^{+0.47}_{-0.47}$ $(C_D)$ |

TABLE III: The PQCD predictions of the branching fractions for the CKM-suppressed $B_{(s)} \to D_{(s)} \rho(770) \to D_{(s)} K\bar{K}$ decays. The decay mode with the subprocess $\rho(770)^0 \to K^-K^0$ has the same branching fraction of its corresponding decay with $\rho(770)^0 \to K^+K^-$.  

| Decay modes | Unit | Quasi-two-body results |
|-------------|------|------------------------|
| $B^+ \to D^0\rho(770)^+ \to D^0 K^+K^0$ | $(10^{-13})$ | $5.27_{-0.59}^{+0.23} (a_0 + a_2 + a_4)^{+0.00}_{-0.03}$ $(C_D)$ |
| $B^+ \to D^+\rho(770)^0 \to D^+ K^+K^-$ | $(10^{-9})$ | $3.22_{-0.66}^{+0.52} (a_0 + a_2 + a_4)^{+0.01}_{-0.01}$ $(C_D)$ |
| $B^+ \to D^0\rho(770)^0 \to D^0 K^-K^-$ | $(10^{-8})$ | $6.62_{-1.30}^{+1.69} (a_0 + a_2 + a_4)^{+0.03}_{-0.03}$ $(C_D)$ |
| $B^0 \to D^0\rho(770)^0 \to D^0 K^0K^-$ | $(10^{-11})$ | $7.79_{-2.36}^{+2.02} (a_0 + a_2 + a_4)^{+0.01}_{-0.01}$ $(C_D)$ |
| $B^0_s \to D^+\rho(770)^- \to D^+ K^0K^0$ | $(10^{-9})$ | $6.87_{-2.00}^{+2.05} (a_0 + a_2 + a_4)^{+0.06}_{-0.06}$ $(C_D)$ |
| $B^0_s \to D^0\rho(770)^0 \to D^0 K^-K^-$ | $(10^{-7})$ | $2.32_{-0.00}^{+0.48} (a_0 + a_2 + a_4)^{+0.01}_{-0.01}$ $(C_D)$ |
| $B^0_s \to D^0\rho(770)^0 \to D^0 K^-K^-$ | $(10^{-9})$ | $1.89_{-0.32}^{+0.36} (a_0 + a_2 + a_4)^{+0.08}_{-0.08}$ $(C_D)$ |
| $B^0_s \to D^+\rho(770)^- \to D^+ K^0K^0$ | $(10^{-9})$ | $7.47_{-0.32}^{+1.49} (a_0 + a_2 + a_4)^{+0.10}_{-0.10}$ $(C_D)$ |

The predictions for the branching fractions of the decays $B_{(s)} \to D_{(s)} \rho(770) \to D_{(s)} K\bar{K}$ in Table II are generally smaller than the corresponding results for the $B_{(s)} \to D_{(s)} \rho(770) \to D_{(s)} K\bar{K}$ decays in Table III due to the strong CKM suppression factor $|V_{ub}V_{cb}^{*}|^2$ or $|V_{us}V_{cs}^{*}|^2$ as discussed in Ref. [82]. The central value for the PQCD predicted branching fractions of the decays $B^0 \to D^0\rho(770)^0 \to D^0 K^+K^-$ and $B^0_s \to D^0\rho(770)^0 \to D^0 K^+K^-$ are 0.18% and 0.019% of the experimental measurements $B(B^0 \to D^0 K^+K^-) = (5.9 \pm 0.5) \times 10^{-4}$ and $B(B^0_s \to D^0 K^+K^-) = (5.5 \pm 0.8) \times 10^{-5}$ respectively in the Review of Particle Physics [100], which have been averaged from the results in Refs. [53, 54] presented by LHCb. However, with the branching ratio $B(B^+ \to D^0 K^0K^0) = (5.5 \pm 1.4 \pm 0.8) \times 10^{-4}$ presented by Belle Collaboration [50], one has a sizable percent at 21.45% of the total branching fraction for the quasi-two-body decay $B^+ \to D^0\rho(770)^+ \to D^0 K^0K^0$. This tells us that the contributions from $\rho(770)^0 \to K\bar{K}$ could be considerable large in the relevant three-body $B$ meson decays.

In Ref. [55], LHCb presented the first observation of the decay $B^+ \to D^+ K^+K^-$ and the branching fraction was determined to be $(7.1 \pm 0.5 \pm 0.6 \pm 0.7) \times 10^{-6}$. Utilize our prediction $B(B^+ \to D^+_s \rho(770)^0 \to D^+_s K^+K^-) = (6.26_{-1.69}^{+1.30}) \times 10^{-7}$, where the individual errors have been added in quadrature, we obtain the ratio $B(B^+ \to D^+_s \rho(770)^0 \to D^+_s K^+K^-) = 0.88_{-0.27}^{+0.26} \%$ which is quite small as expected. In addition, they also gave a branching ratio for $B^+ \to D^+_s \phi(1020)$ decay of $(1.2_{-1.4}^{+1.6} \pm 0.8 \pm 0.1) \times 10^{-7}$ and set an upper limit as $4.9(4.2) \times 10^{-7}$ at 95%(90%) confidence level, which is roughly one order smaller than their previous result in [58]. By adopting $B(\phi(1020) \to K^-K^-) = 0.492$ [100] and the relation between the quasi-body decay and corresponding two-body decay

$$B(B \to DR \to Dh_1h_2) \approx B(B \to DR) \cdot B(R \to h_1h_2),$$

we find that $B(B^+ \to D^+_s \rho(770)^0 \to D^+_s K^+K^-)$ predicted in this work has the same magnitude with the branching ratio for $B^+ \to D^+_s \phi(1020)$ measured by LHCb within large uncertainties, while $B(B^+ \to D^+_s \phi(1020) \to D^+_s K^+K^-)$ was predicted to be $(1.53 \pm 0.23) \times 10^{-7}$ within the PQCD approach in Ref. [88].

In Fig. 3, we show the differential branching fraction of the decay mode $B(B^+ \to D^+_s \rho(770)^0 \to D^+_s K^+K^-)$ with the invariant mass in the range of $[2m_K, 3$ GeV]. The bump in the curve is caused by the strong depression of the phase space factors $q$ and $q_0$ in Eqs. (32) and (33) near the $K^- K^-$ threshold. It is this depression near the threshold along with the similar mass between $K^-$ and $K^0$ and $K^0$ make the decay channel with the subprocess $\rho(770)^0 \to K^+K^-$. 

has the same branching fraction of its corresponding decay mode with $\rho(770)^0 \to K^+K^-$. 

TABLE IV: The comparison of the available experimental measurements for the branching fractions of the two-body decay modes

| Decay modes | $B_{exp}$ [100] | $B_{th}$ |
|-------------|-----------------|---------|
| $B^+ \to D^0\rho(770)^+$ | $(1.34 \pm 0.18) \times 10^{-2}$ | $(1.18^{+0.63}_{-0.43}) \times 10^{-3}$ |
| $B^+ \to D^+_\rho(770)^0$ | $< 3.0 \times 10^{-4}$ | $(6.26^{+1.18}_{-1.55}) \times 10^{-8}$ |
| $B^0 \to D^-\rho(770)^+$ | $(7.6 \pm 1.2) \times 10^{-3}$ | $(7.93^{+0.95}_{-1.01}) \times 10^{-5}$ |
| $B^0 \to D^+_\rho(770)^-$ | $< 2.4 \times 10^{-5}$ | $(2.32^{+1.80}_{-1.15}) \times 10^{-7}$ |
| $B^0 \to D^0\rho(770)^0$ | $(3.21 \pm 0.21) \times 10^{-4}$ | $(1.0^{+0.92}_{-0.65}) \times 10^{-6}$ |
| $B^0_\rho \to D^-\rho(770)^+$ | $(6.9 \pm 1.4) \times 10^{-3}$ | $(6.06^{+3.50}_{-2.14}) \times 10^{-5}$ |

For comparison, we list the available experimental measurements for the branching fractions of the $B \to D\rho(770)$ decays and the PQCD predictions for the branching ratios of the relevant decay modes with the subprocess $\rho(770) \to K\bar{K}$ shown in Tables II and III. The ratios between the relevant branching fractions are

$$
R_1 = \frac{B(B^+ \to D^0\rho(770)^+ \to D^0K^+\bar{K}^0)}{B(B^+ \to D^0\rho(770)^+)} = 0.0088^{+0.0048}_{-0.0034}, \\
R_2 = \frac{B(B^0 \to D^-\rho(770)^+ \to D^-K^+\bar{K}^0)}{B(B^0 \to D^-\rho(770)^+)} = 0.010^{+0.007}_{-0.004}, \\
R_3 = \frac{B(B^0 \to D^0\rho(770)^0 \to D^0K^+\bar{K}^-)}{B(B^0 \to D^0\rho(770)^0)} = 0.0033^{+0.0029}_{-0.0022}, \\
R_4 = \frac{B(B^0_\rho \to D^-\rho(770)^+ \to D^-K^+\bar{K}^0)}{B(B^0_\rho \to D^-\rho(770)^+)} = 0.0088^{+0.0054}_{-0.0035}.
$$

Due to the suppression from the phase space, the predicted branching fractions of the quasi-two-decays $B^+ \to D^0\rho(770)^+ \to D^0K^+K^0$, $B^0 \to D^-\rho(770)^+ \to D^-K^+K^0$ and $B^0_\rho \to D^-\rho(770)^+ \to D^-K^+K^0$ are around 0.9% of the experimental data for the corresponding two-body cases, while a ratio near 0.4% for $B^0 \to D^0\rho(770)^0 \to D^0K^+K^-$ is found.

With the relations [37, 65]

$$
|c_{\rho^+}| \approx \frac{f_{\rho(770)}|g_{\rho(770)^0}K^+K^-|}{\sqrt{2}m_{\rho(770)}}, \quad |c_{\rho^0}| \approx \frac{f_{\rho(770)}|g_{\rho(770)^0}K^+\bar{K}^0|}{m_{\rho(770)}}, \quad |c_{\rho^-}| \approx \frac{f_{\rho(770)}|g_{\rho(770)^-}K^0K^-|}{m_{\rho(770)}}
$$

and Eq. (15), one can obtain the relation between the strong couplings $|g_{\rho(770)^0}K^+K^-| = |g_{\rho(770)^-}K^0K^-| \approx \sqrt{2}|g_{\rho(770)^0}K^+K^-|$ which leads to $\Gamma_{\rho(770)^0K^+K^-} = \Gamma_{\rho(770)^-K^0K^-} \approx 2\Gamma_{\rho(770)^0K^+K^-}$. When considering $\Gamma_{\rho(770)^0\pi^+\pi^-} = \ldots$
\[ \begin{align*} \Gamma_{\rho(770)^0} & \text{ and the relation in Eq. (35)}, \text{ we have} \quad \frac{\mathcal{B}(B \to D\rho(770)^0 \to DK^+K^-)}{\mathcal{B}(B \to D\rho(770)^0 \to \pi^+\pi^0)} = \frac{\mathcal{B}(B \to D\rho(770)^- \to DK^-K^0)}{\mathcal{B}(B \to D\rho(770)^- \to \pi^-\pi^0)} \approx 2 \frac{\mathcal{B}(B \to D\rho(770)^0 \to DK^+K^-)}{\mathcal{B}(B \to D\rho(770)^0 \to \pi^+\pi^-)}. \end{align*} \]

Obviously, the above theoretical analysis is consistent with the numerical results basing on the fact that most of the experimental data were measured by assuming \( \mathcal{B}(\rho(770) \to \pi\pi) \approx 100\% \). For the branching fractions of decays \( B^+ \to D_s^+\rho(770)^0 \) and \( B^0 \to D_s^+\rho(770)^- \), no specific values but upper limits of \( 3.0 \times 10^{-4} \) and \( 2.4 \times 10^{-5} \) at 90\% confidence level were given by the CLEO and BABAR Collaborations [101, 102]. Utilize the PQCD predictions \( \mathcal{B}(B^+ \to D_s^+\rho(770)^0) = 1.52 \times 10^{-5} \) and \( \mathcal{B}(B^0 \to D_s^+\rho(770)^-) = 2.82 \times 10^{-5} \) taken from our previous work in Ref. [82], and \( \mathcal{B}(B^+ \to D_s^+\rho(770)^0 \to D_s^+K^+K^-) = 6.26 \times 10^{-8} \) and \( \mathcal{B}(B^0 \to D_s^+\rho(770)^- \to D_s^+K^0K^-) = 2.32 \times 10^{-7} \) in this work, ratios around 0.5\% and 1\%, respectively, can be obtained. Also, from the comparison of the results in [82] and this work, we can find the similar ratios for other decay channels. Thus, we estimate the branching fractions \( \mathcal{B}(\rho(770)^+ \to K^+K^0) = \mathcal{B}(\rho(770)^- \to K^-K^0) \approx 1\% \) and \( \mathcal{B}(\rho(770)^0 \to K^+K^-) = \mathcal{B}(\rho(770)^0 \to K^0K^0) \approx 0.5\% \).

In consideration of the large uncertainties, more precise data from LHCb and Belle-II are expected to test our predictions.

IV. SUMMARY

In this work, we analyzed the contributions for the kaon pair originating from the intermediate state \( \rho(770) \) for the three-body decays \( B \to DKK \) in the PQCD approach. By the numerical evaluations and the phenomenological analyses, we found the following points:

(i) The decay mode of \( B \to D\rho(770)^0 \) with the intermediate state \( \rho(770)^0 \) decays into \( K^0K^0 \) has the same branching fraction of its corresponding mode with the subprocess \( \rho(770)^0 \to K^+K^- \).

(ii) Our predictions for the corresponding branching fractions of the decay modes with the subprocess \( \rho(770)^0 \to K^+K^- \), are much less than the measured branching fractions for the three-body decays \( B^0 \to D^0K^+K^- \), \( B_s^0 \to D^0K^+K^- \), and \( B^+ \to D_s^+K^+K^- \), while the percent at about 20\% of the total three-body branching fraction for the quasi-two-body decay \( B^+ \to D^0\rho(770)^+ \to D^0K^+K^- \) was predicted in this work.

(iii) The branching ratio for the decay \( B^+ \to D_s^+\rho(770)^0 \to D_s^+K^+K^- \) predicted in this work is of the same magnitude as that for \( B^+ \to D_s^+\phi(1020)^0 \to D_s^+K^+K^- \) measured by LHCb within large uncertainties.

(iv) We estimate the branching fractions \( \mathcal{B}(\rho(770)^+ \to K^+K^0) = \mathcal{B}(\rho(770)^- \to K^-K^0) \approx 1\% \) and \( \mathcal{B}(\rho(770)^0 \to K^+K^-) = \mathcal{B}(\rho(770)^0 \to K^0K^0) \approx 0.5\% \) by comparing the available experimental measurements and the PQCD predictions for the branching fractions of the \( B \to D\rho(770) \) decays with the PQCD predicted branching ratios of the relevant decay modes \( B \to D\rho(770) \to DKK \) in this work.

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Appendix A: DECAY AMPLITUDES

The expressions for amplitudes from Figs. 1 (a1-d1) are written as:

\[ F_{\rho 0}^{LL} = 8\pi C_F m_B f_D \int dx_B dz \int b_B b_B b_B \phi_B \{ [-\eta(1 + z) + r^2(1 + 2\eta)z - r^4\eta z]\phi_0 - \sqrt{\eta(1 - r^2)} \]
\[ \times \{ \eta(1 - 2(1 - r^2)z)(\phi_0 + \phi_1) + r^2(\phi_0 - \phi_1) \} E_c(t_u) h_u(x_B, z, b, b_B) S_i(z) - [(1 - r^2)|\eta\phi_0 + r^2(x_B - \eta)|\phi_0 \]
\[ + 2\sqrt{\eta(1 - r^2)}|\eta - r^2(1 - 2\eta + x_B)|\phi_1| E_c(t_u) h_u(x_B, z, b, b_B) S_i(x_B - \eta) \} \}, \] (A1)
\[
M_{\rho \rho}^{LL} = 32\pi C_{F} m_{b}^{4}/\sqrt{6} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ (r^{2}(r^{2} - \eta)) (\bar{\eta}(1 - x_{3}) - x_{B} - \eta z) + r(r_{c}(\bar{\eta} - r^{2}) + \eta r(\bar{\eta} + r^{2})) \phi_{0} + \sqrt{\eta}(1 - r^{2}) \right] (x_{B} + \bar{\eta} x_{3}) (\phi_{s} + \phi_{t}) + \bar{\eta}(1 - r^{2}) (z(\phi_{s} - \phi_{t}) + 2r(\bar{\eta} r - 2r_{c}) \phi_{s}) \right\} E_{n}(t_{d}) h_{d}(x_{B}, z, x_{3}, b, b_{3}), (A2)
\]

The expressions for amplitudes from Figs. 1 \((e_{1}-h_{1})\) are written as:

\[
F_{\nu \nu}^{LL} = 8\pi C_{F} m_{b}^{4}/f_{K} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ \left(1 - \bar{\eta}(1 - r^{2})^{2} z + (1 - 2r_{c})(\bar{\eta} - r^{2}) \phi_{0} + \sqrt{\eta}(1 - r^{2}) \right] (r_{c} \bar{\eta}(\phi_{s} + \phi_{t}) + r(2(1 - r^{2}) z + r_{c}(\phi_{s} - \phi_{t}) - 4r \phi_{s})) \right\} E_{n}(t_{c}) h_{c}(x_{B}, z, x_{3}, b, b_{3}) S_{4}(z) + \left[ (r^{2} - 1)(\bar{\eta} - r^{2}) \eta + \eta^{2} x_{3} \right] \phi_{0} + 2r \sqrt{\eta}(1 - r^{2}) \left[ (1 + x_{3}) + 2\eta - r^{2} \phi_{s}) E_{n}(t_{f}) h_{f}(x_{B}, z, x_{3}, b, b_{3}) S_{4}(|\eta(1 - \bar{\eta} x_{3}) - x_{3} |), (A3)\right\}
\]

\[
M_{\nu \nu}^{LL} = 32\pi C_{F} m_{b}^{4}/\sqrt{6} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ (\bar{\eta} \bar{\eta} + r^{2}(r^{2} - 1) + (\bar{\eta} + r^{2})(\eta - r^{2}) \right] \right\} E_{n}(t_{g}) h_{g}(x_{B}, z, x_{3}, b, b_{B}) + \left[ (\bar{\eta} - r^{2})(\bar{\eta}(1 - z) - (\eta + \bar{\eta} + (1 - z)^{2})) \phi_{0} + r \sqrt{\eta}(1 - r^{2}) \right\} E_{n}(t_{h}) h_{h}(x_{B}, z, x_{3}, b, b_{B}), (A4)\right\}
\]

The expressions for amplitudes from Figs. 2 \((m_{1}-p_{1})\) are written as:

\[
F_{\rho \rho}^{LL} = 8\pi C_{F} m_{b}^{4}/f_{D} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ (1 + r)(-\bar{\eta} - x_{3} + \eta^{2}(r - 1)x_{3} + 2 \eta(r - 1)^{2} x_{3} + r(2x_{3} + r + 3x_{3}]) E_{n}(t_{m}) h_{m}(x_{B}, x_{3}, b, b_{3}) S_{4}(x_{3}) + \left[ \bar{\eta}(r_{c} + \eta x_{B}) + 2r(-\eta x_{B} - \bar{\eta} + r_{c}) \right] \right\} (r^{2} - r_{c}) + 2r^{3}(1 + r_{c} - r^{2}) \right\} E_{n}(t_{n}) h_{n}(x_{B}, x_{3}, b, b_{3}), (A5)\right\}
\]

\[
M_{\rho \rho}^{LL} = 32\pi C_{F} m_{b}^{4}/\sqrt{6} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \phi_{0} \left\{ \left[ - \eta^{2}(1 - x_{B} - z) + r x_{3} + \eta(x_{B} + z - x_{3}) + \bar{\eta}^{2} \right] \right\} E_{n}(t_{b}) h_{b}(x_{B}, z, x_{3}, b, b_{B}) + \left[ (r - 1)(\bar{\eta} + r)(x_{B} + r^{2} - 1) + \bar{\eta}(1 + r - r^{2}) \right] E_{n}(t_{p}) h_{p}(x_{B}, z, x_{3}, b, b_{B}), (A6)\right\}
\]

The expressions for amplitudes from Figs. 2 \((a_{2}-d_{2})\) are written as:

\[
F_{\rho \rho}^{LL} = 8\pi C_{F} m_{b}^{4}/f_{D} \int dx_{B} dz \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ (\bar{\eta}(1 + z) + r^{2}(1 + 2\eta z) - r^{4} \eta z \phi_{0} - \sqrt{\eta}(1 - r^{2}) \right] \right\} E_{n}(t_{a}) h_{a}(x_{B}, z, b, b_{3}) S_{4}(z) - \left[ (1 + r^{2})(\eta \bar{\eta} + r^{2}(x_{B} - \eta) \phi_{0} + 2r \sqrt{\eta}(1 - r^{2}) \bar{\eta} - r^{2}(1 - 2\eta x_{B}) \right\} E_{n}(t_{b}) h_{b}(x_{B}, z, b, b_{3}), (A7)\right\}
\]

\[
M_{\rho \rho}^{LL} = 32\pi C_{F} m_{b}^{4}/\sqrt{6} \int dx_{B} dz dx_{3} \int b_{B} db_{B} b_{3} \phi_{B} \phi_{D} \left\{ \left[ (\bar{\eta} + r^{2})(1 - r^{2}) \right] (x_{B} + \eta z - x_{3}) \right\} E_{n}(t_{c}) h_{c}(x_{B}, z, x_{3}, b, b_{3}) + \left[ (- \bar{\eta} + r^{2}) \right] \phi_{0} + 2r(2r_{c} - \bar{\eta}^{2}) \phi_{s}) E_{n}(t_{d}) h_{d}(x_{B}, z, x_{3}, b, b_{3}), (A8)\right\}
\]

The expressions for amplitudes from Figs. 2 \((e_{2}-h_{2})\) are written as:

\[
F_{\nu \nu}^{LL} = 8\pi C_{F} m_{b}^{4}/f_{B} \int dx_{3} dz \int b_{3} db_{B} b_{3} \phi_{D} \left\{ \left[ (r^{2} - 1)(\bar{\eta} x_{3}) \phi_{0} + 2r \sqrt{\eta}(1 - r^{2}) \right] (1 + \eta + \eta \bar{\eta} x_{3}) \right\} E_{n}(t_{e}) h_{e}(x_{B}, z, x_{3}, b, b_{3}) + \left[ (\bar{\eta} r^{2}) \right] \phi_{0} + \eta \bar{\eta} x_{3} - r^{2} \phi_{s}) E_{n}(t_{f}) h_{f}(x_{B}, z, x_{3}, b, b_{3}) S_{4}(x_{3}), (A9)\right\}
\]
\[ M_{aB}^{LL} = \frac{32 \pi C_F m_b^2}{\sqrt{6}} \int dx_B dz dx_3 \int b_B d^4 b d\phi_B d\phi_D \left\{ \left[ (\bar{\eta} + r^2)\bar{\eta}(r^2(z-x_3)-x_B-z) + r^2 - \eta \right] \phi_0 \\
+ r \sqrt{\eta(1-r^2)} \left[ (z(1-r^2) + x_B)(\phi_s + \phi_t) + \bar{\eta}x_3(\phi_s - \phi_t) + 2\phi_s \right] E_n(t_g) h_g(x_B, z, x_3, b, b_B) - \left[ (\bar{\eta} + r^2) \times \left[ (1-r^2)(i x_3 - \eta z) + x_B \bar{\eta} \phi_0 + r \sqrt{\eta(1-r^2)} \left[ \bar{\eta}x_3(\phi_s + \phi_t) + ((1-r^2)z-x_B)(\phi_s - \phi_t) \right] \right] E_n(t_h) h_h(x_B, z, x_3, b, b_B) \right\} \right] \]

(A10)

In the formulae above, the symbol \( \bar{\eta} = 1 - \eta \), the mass ratio \( r = \frac{m_D}{m_B} \) and \( r_c = \frac{m_c}{m_B} \) are adopted. The \( b_B, b \) and \( b_3 \) are the conjugate variable of the transverse momenta of the light quarks in the \( B \) meson, resonance \( \rho(770) \) and \( D \) meson. The explicit expressions for the hard functions \( h_i \), the evolution factors \( E(t_i) \) and the threshold resummation factor \( S_t \) can be found in Ref. [82].

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