Positivity-preserving numerical scheme for hyperbolic systems with \( \delta \)-shock solutions and its convergence analysis

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Abstract. In this article, numerical schemes are proposed for approximating the solutions, possibly measure-valued with concentration (delta shocks), for a class of nonstrictly hyperbolic systems. These systems are known to model physical phenomena such as the collision of clouds and dynamics of sticky particles, for example. The scheme is constructed by extending the theory of discontinuous flux for scalar conservation laws, to capture measure-valued solutions with concentration. The numerical approximations are analytically shown to be entropy stable in the framework of Bouchut (Adv Kinet Theory Comput 22:171–190, 1994), satisfy the physical properties of the state variables, and converge to the weak solution. The construction allows natural extensions of the scheme to its higher-order and multi-dimensional versions. The scheme is also extended for some more classes of fluxes, which admit delta shocks and are also known to model physical phenomena. Various physical systems are simulated both in one dimension and multi-dimensions to display the performance of the numerical scheme, and comparisons are made with the test problems available in the literature.

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1. Introduction

This paper studies the following \( 2 \times 2 \) hyperbolic system

\[
\begin{align*}
\rho_t + F(\rho, w)_x &= 0, \\
w_t + G(\rho, w)_x &= S(\rho, w),
\end{align*}
\] (1.1, 1.2)

where \( F, G \) and \( S \) are sufficiently smooth real-valued functions. The numerical approximation of such systems has been of interest, in the past decades, and the proposed numerical schemes have been primarily based on the eigenstructure of the system. The article aims to develop relatively simpler convergent numerical schemes for the system using the techniques for scalar conservation laws. If we assume that \( w(x, t) \) is known for all \((x, t) \in \mathbb{R} \times \mathbb{R}^+\), Eq. (1.1) can be viewed as a scalar conservation law in \( \rho \), whose flux function \( F(\rho, w(x, t)) \) may be discontinuous in the space variable \( x \), and vice versa for second Eq. (1.2), but unfortunately, this technique may not work in general. However, in this article, we propose numerical methods based on this type of technique for certain class of \( F \) and \( G \) and also prove their convergence under some additional assumptions on the fluxes. From now on, we restrict ourselves to the following:

\[
F(\rho, w) = \rho g \left( \frac{w}{\rho} \right), \quad G(\rho, w) = wg \left( \frac{w}{\rho} \right) + P \left( \rho, \frac{w}{\rho} \right).
\]

In particular, system (1.1)–(1.2) can be rewritten as:

\[
\begin{align*}
\rho_t + \langle \rho g(u) \rangle_x &= 0, \\
w_t + \langle wg(u) + P(\rho, u) \rangle_x &= S(\rho, w),
\end{align*}
\] (1.3, 1.4)
where \( w = \rho u \). This class finds numerous physical applications that depend on the nature of the functions \( g, S \) and \( P \), and may admit nonclassical shocks, which are called as \( \delta \)-shocks. The \( \delta \)-shock wave is a generalization of a classical shock wave and is a kind of discontinuity, on which the state variables of system (1.3)–(1.4) develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. Physically, the delta-shock wave represents the process where the mass is concentrated and may be interpreted as the galaxies in the universe. This generalization was introduced in the Ph.D. thesis of [18], post which it has been explored extensively in the literature, see for example, [13, 23, 24, 28] and references therein.

These kinds of systems have not only been of mathematical interest due to the admission of \( \delta \)-shock solution, but they are also known to model physically important phenomena. Some of the important ones are generalized pressureless gas dynamics (GPGD) system, where \( g \) is a nondecreasing function, \( S = 0 = P \), and isentropic Euler equations for modified Chaplygin gas dynamics (CGD) system, where \( g(u) = u, S = s\rho^{-\alpha} \) and \( P = 0 \). (GPGD) system has been studied in [9, 16, 21]. With \( g(u) = u \), the system is called as pressureless gas dynamics (PGD) system and can be used to describe the process of the motion of free particles sticking under collision. When (PGD) system is augmented with a Coulomb friction source term \( S = \beta \rho \), see [25], it can be used to model the sticky particle dynamics with interaction.

We will call it as (PGDS) in this paper. (CGD) system was studied in [12, 29] and was shown to work as a suitable mathematical approximation to calculate the lift on a wing of an airplane in aerodynamics. The system also finds presence in cosmology [31] and is also used as a possible model for dark energy [15].

There have been various numerical studies in the past for (PGD) system. To name a few, schemes were proposed in [9, 10, 19] which used the analytical expression for the shock location derived in [26] to determine the numerical flux in the case, \( u_l > 0 > u_r \). Generalized eigenvectors obtained from Jordan canonical form were used to construct flux difference splitting- based numerical schemes in [14]. First and second-order relaxation and kinetic schemes were proposed in [8, 10], where the authors established that the solutions preserve both physical properties of the system and discrete entropy inequality proposed in [9]. Discontinuous Galerkin-based higher-order schemes were proposed in [30], and the authors showed that the solutions preserve physical properties. Additionally, semi-discrete central-upwind scheme of [11] and nonoscillatory central difference scheme of [22] were used for approximating (CGD) in [29], respectively.

This paper aims to construct efficient numerical schemes to approximate these systems and capture \( \delta \)-shock solutions by suitably treating the system through, two interdependent scalar conservation laws with spatially dependent discontinuous flux. Godunov-type schemes will be constructed by solving appropriate local Riemann problems for each of them at each numerical interface, and the set of two schemes thus obtained will be taken as a scheme for the system. The scheme will be called as (DDF) scheme, “Decoupled Discontinuous Flux Scheme” in the paper. In (GPGD), it can be noted that for a given \( u(x, t) \), the first equation

\[
\rho_t + (\rho g(u(x, t)))_x = 0 \tag{1.5}
\]

is a linear conservation law in \( \rho \) with a possibly discontinuous variable coefficient \( g(u(x, t)) \). Similarly, given a \( \rho(x, t) > 0 \), the second equation

\[
w_t + \left( wg\left( \frac{w}{\rho(x, t)} \right) \right)_x = 0 \tag{1.6}
\]

can be treated as a nonlinear scalar conservation laws with discontinuous flux, if we assume the following

\[
g(0) = 0, \quad w \mapsto wg\left( \frac{w}{\rho(x, t)} \right) \text{ is a function with only one local minimum.} \tag{1.7}
\]

The above condition implies that \( w \mapsto wg\left( \frac{w}{\rho(x, t)} \right) \) has a minimum at \( w = 0 \), independent of \( \rho(\cdot, \cdot) \).

Two Eqs. (1.5) and (1.6) behave differently as (1.6) is a nonlinear conservation law with discontinuous
convex flux, which admits bounded solutions, while (1.5) is a conservation law with a linear advective flux with sign changing coefficient, which may admit measure-valued solutions. Both the conservation laws admit infinitely many solutions. It is thus necessary to choose appropriate individual entropy setups so that they are compatible to converge to the right expected physical solution of (1.3)–(1.4). We will choose the entropy setup of [6] for (1.6), while, for (1.5), the current setups of discontinuous flux will be appropriately modified to capture measure-valued solutions. In particular, let us consider (1.5) with \( g(u) = u \) . To invoke the theory of nonlinear conservation laws with discontinuous flux, Eq. (1.5) for a fixed \( u(x,t) \) will be perturbed through a vanishing parameter \( \epsilon \) resulting in the following nonlinear conservation law with discontinuous flux

\[
\rho_t + (\rho u(x,t) - \epsilon u(x,t) \rho^2)_x = 0.
\]

(1.8)

Physically, this perturbation will not allow any fluid flow across the interface with high concentration, i.e., overcompressive interface with \( u_l > 0 > u_r \), as will be seen through the Riemann problem solutions in Sect. 2. More precisely, in the notations of Sect. 2,

\[
g_\epsilon \left( \frac{1}{\epsilon} \right) = f_\epsilon \left( \frac{1}{\epsilon} \right) = 0,
\]

where \( \frac{1}{\epsilon} \) represents the symmetric concentration around the overcompressive interface and produces \( \delta \)-shock in vanishing \( \epsilon \)-limit. This behavior is similar to the behavior exhibited by (1.5) for a given \( u(x,t) \), which indicates that this perturbation is a good physical approximation for (1.5). From the system point of view, perturbation (1.8) when combined with momentum Eq. (1.4), which gives the following strictly hyperbolic system

\[
\rho_t + (\rho u - \epsilon u \rho^2)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+,
\]

(1.9)

\[
(\rho u)_t + (\rho u^2)_x = 0 \quad \text{for } x \in \mathbb{R},
\]

(1.10)

whose solution depends on the sign of velocity \( u \) and develops concentration when \( u_l > 0 > u_r \) as \( \epsilon \) goes to zero. Physically, this system satisfies the conservation of mass and the second law of thermodynamics as it is entropy stable (can be proven as in Theorem 4.3 in the paper). This behavior is similar to the solution of pressureless system (1.3)–(1.4) with \( P = 0 = S \) in the vanishing \( \epsilon \) limit and can be verified by numerical experiments. This motivates us to take the numerical flux for (1.3)–(1.4) as the vanishing \( \epsilon \) limit of the numerical flux for perturbed system (1.9)–(1.10), and choose this specific perturbation (see Sects. 2 and 3.1). Mathematically, the choice of the term \(-\epsilon u(x,t) \rho^2\) in perturbation (1.8) is special because of its dependence on:

i. the sign of \( u(x,t) \), to have a similar behavior like (1.3)–(1.4) with \( P = 0 = S \) and \( g(u) = u \), whose concentration solutions depend on sign of \( u \), and

ii. \( \rho^2 \), to make the flux nonlinear, so as to invoke theory of nonlinear conservation laws with discontinuous flux.

Hence, the perturbation presented in this article will motivate to coin the numerical flux for first Eq. (1.3). This choice will not only help in invoking the theory of nonlinear conservation laws with discontinuous flux but will also enable to prove mathematically that the numerical approximations satisfy physical properties and the entropy inequality (see Sect. 3.3).

In this article, the schemes are constructed for (1.3)–(1.4) in absence of pressure and source and then are adapted to (PGDS) which has nonzero \( S \). The scheme is also extended to capture the \( \delta \)-shock solutions of strictly hyperbolic systems like (CGD) which has nonzero \( P \). Since the scheme is based on theory of scalar conservation laws, it is easily extended to higher order, using appropriate limiters, and to multi-dimensions using the dimension splitting techniques. The scheme is tested with various initial data, modeling physical applications in both, one and multi-dimensions. For (GPGD) system, with some additional assumptions on \( g \), and in particular, for (PGD) system, the numerical solutions are shown to preserve the physical properties of the system analytically. The limit of the numerical approximations is
shown to satisfy the entropy inequality, introduced in [9]. Though the proofs are based on the idea of writing the scheme in the incremental form, the estimates on the incremental coefficients are nontrivial and more involved unlike in the case of decoupled systems.

The paper has been organized as follows: in Sect. 2, we propose the numerical scheme for the PDEs of type (1.5) with the help of perturbation (1.8) discussed in introduction. In Sect. 3, we propose the (DDF) scheme to approximate (GPGD). The scheme is also extended to its higher-order version. In Sect. 4, various analytical aspects of the scheme are discussed such as convergence of the scheme, the entropy stability and preservation of the physical properties of the system, under certain assumptions on the fluxes. In Sect. 5, the efficiency of the scheme, along with its extensions, is displayed, by comparing their performance with the existing literature. The schemes are also tested for hyperbolic systems that admit δ-shocks in presence of pressure and source terms such as (PGDS) and (CGD) and are also extended to multi-dimensions, using dimensional splitting.

2. Preliminaries

This section aims to discuss the notion of solution for the PDE of type (1.5), which for a given $u(x, t)$ is a transport equation with possible spatial discontinuities. Hence, in particular, we study the following transport equation with one spatial discontinuity at $x = 0$,

$$
\begin{align*}
\rho_t + F(x, \rho)x &= 0, \\
\rho(x, 0) &= \rho_0(x), \\
(x, t) &\in \mathbb{R} \times \mathbb{R}^+,
\end{align*}
$$

(2.1)

where

$$
F(x, \rho) = H(x)b\rho + (1 - H(x))a\rho, \quad a, b, x \in \mathbb{R},
$$

and further use it to propose numerical flux for PDE of type (1.5). For Riemann data

$$
\rho_0(x) = \rho_l\chi_{\{x < 0\}} + \rho_r\chi_{\{x > 0\}}, \quad x \in \mathbb{R}
$$

all cases except for $a \geq 0, b \leq 0$, can be handled by [4,6]. The overcompressive pair $a \geq 0, b \leq 0$, was recently studied in [7], where measure-valued solutions were proposed, while bounded solutions were proposed in [20]. In the overcompressive case, characteristics overlap each other at the interface $x = 0$ and cases may arise when there may not exist a weak solution satisfying:

$$
\int_0^\infty \int_{-\infty}^\infty (\rho(x, t)\phi_t(x, t) + F(x, \rho(x, t))\phi_x(x, t)) \, dx \, dt = 0, \forall \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+).
$$

To this end, we concentrate on the case, $a \geq 0, b \leq 0$ and for each $\epsilon > 0$, we consider a nonlinear approximation of (2.1):

$$
\rho_t + (F_{\epsilon}(x, \rho))_x = 0, \quad F_{\epsilon}(x, \rho) = H(x)f_{\epsilon}(\rho) + (1 - H(x))g_{\epsilon}(\rho)
$$

(2.2)

with initial data as Riemann data $(\rho_l, \rho_r)$ and with

$$
g_{\epsilon}(\rho) = (a\rho - a\epsilon\rho^2)\chi_{\{\rho \geq 0\}} + (a\rho + a\epsilon\rho^2)\chi_{\{\rho < 0\}},
$$

(2.3)

$$
f_{\epsilon}(\rho) = (b\rho - b\epsilon\rho^2)\chi_{\{\rho \geq 0\}} + (b\rho + b\epsilon\rho^2)\chi_{\{\rho < 0\}}.
$$

(2.4)

It can be noted that the above nonlinearification is not the same as the one proposed in [7] and depends on $a$ and $b$ linearly. This choice is crucial as pointed out in Sect. 1 and will also be seen in Sect. 3.

The flux structure for Eq. (2.2) is displayed in Fig. 1. Various cases arise and can be handled by theory of discontinuous flux, detailed in [4–6,20], which are briefly exhibited below for completeness:
Fig. 1. Flux structure

(a) $\rho_l, \rho_r \leq 0$: Note that $g_\epsilon, f_\epsilon$ are convex and concave, respectively. The solution $\rho^\epsilon(x, t)$ is given in Fig. 2, where

$$
\left(\rho_l - \frac{1}{\epsilon}\right) s_g = g_\epsilon(\rho_l), \left(\rho_r - \frac{1}{\epsilon}\right) s_f = f_\epsilon(\rho_r).
$$

(b) $\rho_l, \rho_r \geq 0$: Note that $g_\epsilon, f_\epsilon$ are concave and convex, respectively. The solution $\rho^\epsilon(x, t)$ is given in Fig. 3, where

$$
\left(\rho_l - \frac{1}{\epsilon}\right) s_g = g_\epsilon(\rho_l), \left(\rho_r - \frac{1}{\epsilon}\right) s_f = f_\epsilon(\rho_r).
$$

(c) $\rho_l < 0, \rho_r > 0$: Note that $g_\epsilon, f_\epsilon$ are convex and convex, respectively. The flux structure and the solution $\rho^\epsilon(x, t)$ are given in Fig. 4. The second, third and fourth figures in Fig. 4 represent the cases $g_\epsilon(\rho_l) = f_\epsilon(\rho_r)$, $g_\epsilon(\rho_l) < f_\epsilon(\rho_r)$ and $g_\epsilon(\rho_l) > f_\epsilon(\rho_r)$, respectively, where

$$
g_\epsilon(\rho^-) = f_\epsilon(\rho_r), (\rho_l - \rho^-) s_g = g_\epsilon(\rho_l) - g_\epsilon(\rho^-)
$$

and

$$
g_\epsilon(\rho_l) = f_\epsilon(\rho^+), (\rho_r - \rho^+) s_f = f_\epsilon(\rho_r) - f_\epsilon(\rho^+).
$$

(d) $\rho_l \geq 0, \rho_r \leq 0$: Note that $g_\epsilon, f_\epsilon$ are concave and concave, respectively. The flux structure and the solution $\rho^\epsilon(x, t)$ are given in Fig. 5. The second, third and fourth figures in Fig. 5 represent the cases $g_\epsilon(\rho_l) = f_\epsilon(\rho_r)$, $g_\epsilon(\rho_l) > f_\epsilon(\rho_r)$ and $g_\epsilon(\rho_l) < f_\epsilon(\rho_r)$, respectively, where

$$
g_\epsilon(\rho^-) = f_\epsilon(\rho_r), (\rho_l - \rho^-) s_g = g_\epsilon(\rho_l) - g_\epsilon(\rho^-),
$$

and

$$
g_\epsilon(\rho_l) = f_\epsilon(\rho^+), (\rho_r - \rho^+) s_f = f_\epsilon(\rho_r) - f_\epsilon(\rho^+).
$$

It can be deduced that the pointwise limit

$$
\lim_{\epsilon \to 0} \rho^\epsilon(x, t) = \overline{p}(x, t) := \rho_l \chi_{\{x<0\}} + \rho_r \chi_{\{x>0\}}
$$

(2.5)
is, in fact, the solution proposed in [4]. It is worthwhile noting that the pointwise limit in (2.5) does not respect the conservation of mass in the interval \([\alpha, \beta]\), \(\alpha < 0 < \beta\) as

\[
0 = \frac{d}{dt} \int_{\alpha}^{\beta} \rho(x, t) dx = \frac{d}{dt} \int_{\alpha}^{\beta} \overline{\rho}(x, t) dx \neq -b \rho_r + a \rho_l,
\]

which is expected by any physical solution \(\rho\) of conservation law (2.1) owing to the conservation of mass. Instead, the weak convergence of \(\{\rho_\epsilon\}_{\epsilon > 0} \in L^1_{\text{loc}}(\mathbb{R})\) in the space of signed Radon measures gives

\[
\rho(x, t) := \overline{\rho}(x, t) + t(a \rho_l - b \rho_r) \delta_{\{x=0\}}, \tag{2.6}
\]

which takes care of the missing mass by concentrating it at the point \(x = 0\), through the term \(t(a \rho_l - b \rho_r) \delta_{\{x=0\}}\). This motivates to look for solutions of the type

\[
\overline{\rho}(x, t) + w(t) \delta_0 \tag{2.7}
\]
which solves problem (2.1) in the following sense: \( \forall \phi \in C_c^\infty(\mathbb{R} \times (0, \infty)) \),

\[
\int_0^\infty \int_{-\infty}^{\infty} (\bar{p}(x, t)\phi_t(x, t) + F(x, \bar{p}(x, t))\phi_x(x, t)) \, dx \, dt \\
+ \int_0^\infty w_\delta(t)\phi_t(0,t) \, ds = 0.
\] (2.8)

Since the solution of Riemann problem (2.1) is measure valued at the interface \( x = 0 \), the Godunov flux at \( x = 0 \) for the numerical scheme for (2.1) cannot be evaluated in the usual way.

Other cases except \( a \geq 0, b \leq 0 \), can also be obtained with \( g_\epsilon(\rho) = a\rho - a\epsilon \rho^2 \), \( f_\epsilon(\rho) = b\rho - b\epsilon \rho^2 \). Also, \( \lim_{\epsilon \to 0} g_\epsilon(\rho) = a\rho \), \( \lim_{\epsilon \to 0} f_\epsilon(\rho) = b\rho \), \( \rho(x, t) = \lim_{\epsilon \to 0} \rho_\epsilon(x, t) \). The solution for Riemann Problem (2.2) is known, and hence, the flux at the interface \( x = 0 \) for (2.2) is given by

\[ F_{\epsilon, 0}(a, b, \rho_l, \rho_r) := g_\epsilon(\rho^-_{\epsilon}) = f_\epsilon(\rho^+_{\epsilon}), \]
where $\rho_e^- = \lim_{x \to 0^-} \rho_e(x, t)$, $\rho_e^+ = \lim_{x \to 0^+} \rho_e(x, t)$. It can be derived using the results in [3,4] that $F_{\epsilon, 0}(a, b, \rho_l, \rho_r) =$

\[
\begin{align*}
&\begin{cases}
\min \left( g_{\epsilon} \left( \min(\rho_l, \frac{1}{2\epsilon}) \right), f_{\epsilon} \left( \max(\rho_r, \frac{1}{2\epsilon}) \right) \right) & \text{if } a \geq 0, b > 0, \\
\max \left( g_{\epsilon} \left( \max(\rho_l, \frac{1}{2\epsilon}) \right), f_{\epsilon} \left( \min(\rho_r, \frac{1}{2\epsilon}) \right) \right) & \text{if } a < 0, b \leq 0, \\
\max \left( g_{\epsilon}(\rho_l), f_{\epsilon}(\rho_r) \right) & \text{if } \rho_l < 0, \rho_r > 0, \\
\min \left( g_{\epsilon}(\rho_l), f_{\epsilon}(\rho_r) \right) & \text{if } \rho_l > 0, \rho_r < 0, \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

(2.9)

Owing to the behavior of the solutions and the fluxes of (2.1) and (2.2) as $\epsilon \to 0$, we define the flux at the interface $x = 0$ for (2.1) as

\[
F_0(a, b, \rho_l, \rho_r) := \lim_{\epsilon \to 0} g_{\epsilon}(\rho^-) = \lim_{\epsilon \to 0} f_{\epsilon}(\rho^+) = \lim_{\epsilon \to 0} F_{\epsilon, 0}(a, b, \rho_l, \rho_r),
\]
which implies that $F_0(a, b, \rho_l, \rho_r) = \begin{cases} 
 a \rho_l & \text{if } a \geq 0, b > 0, 
 b \rho_r & \text{if } a < 0, b \leq 0, 
 \max(a \rho_l, b \rho_r) & \text{if } a < 0, b > 0, 
 \min(a \rho_l, b \rho_r) & \text{if } a > 0, b < 0, 
 0 & \text{otherwise}. \end{cases} (2.10)$

Knowing the flux at the interface $x = 0$, a finite volume scheme for (2.1) can now be proposed with general initial data $\rho_0(x)$. For $h > 0$, let the space grid points as $x_{i+\frac{1}{2}} = ih$, $i \in \mathbb{Z}$ such that $x_{\frac{1}{2}} = 0$. For $\Delta t > 0$, define the time discretization points $t_n = n\Delta t$ for non-negative integer $n$, and $\lambda = \Delta t / h$. Define $\rho_i^n = \frac{1}{h} \int_{C_i} \rho(x, t^n) \, dx$, as the approximation for $\rho(x, t)$ in the cell $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at time $t_n$. Then, the finite volume scheme is given by

$$\rho_i^{n+1} = \rho_i^n - \lambda \left( \hat{F}(a_i, a_{i+1}, \rho_i^n, \rho_{i+1}^n) - \hat{F}(a_{i-1}, a_i, \rho_{i-1}^n, \rho_i^n) \right), (2.11)$$

where

$$a_i = \begin{cases} 
 a & \text{if } i \leq 0, 
 b & \text{if } i > 0, \end{cases}, \quad \hat{F}(a_i, a_{i+1}, \rho_i^n, \rho_{i+1}^n) = F_0(a_i, a_{i+1}, \rho_i^n, \rho_{i+1}^n),$$

where $\hat{F}(a_i, a_{i+1}, \rho_i^n, \rho_{i+1}^n)$ is the numerical flux associated with the flux $F(x, \rho)$ at the interface $x_{i+\frac{1}{2}}$ at the time $t^n$, with $F_0$ given by (2.10). Since $F(x, \rho) = a\rho$ if $x < 0$ and $b\rho$ for $x > 0$, with $a \geq 0$, $b \leq 0$, the flux at any point away from the point $x_{\frac{1}{2}} = 0$ is the usual upwind flux for the linear transport equation. At the interface $x_{\frac{1}{2}} = 0$, though the solution is measure-valued and a Godunov flux cannot be calculated in the usual way, we have, however, the flux, owing to the nonlinearification in the previous section and is given by $F_0(a, b, \rho_i^n, \rho_{i+1}^n)$, whose expression is given by (2.10).

### 3. Numerical scheme

We start by proposing a numerical scheme for (1.3)-(1.4) with $S = P = 0$. Instead of creating a Riemann solver based on the eigenstructure, each equation of the system will be treated separately, assuming that the flux of the equation is a function of the remaining state variable at the previous time step. Consider the system (GPGD)

$$U_t + F(U)_x = 0, \quad U_0(x) = U(x, 0), (3.1)$$

with

$$U = \begin{pmatrix} \rho \\ w \end{pmatrix}, \quad F(U) = \begin{pmatrix} F^\rho \\ F^w \end{pmatrix}, \quad F^\rho(\rho, w) = \rho \rho g \left( \frac{w}{\rho} \right), \quad F^w(\rho, w) = w g \left( \frac{w}{\rho} \right), \quad \rho > 0. (3.2)$$

This system has double eigenvalue $g(u)$, and it has been shown in [21] that it admits $\delta$-shocks. Therefore, as in the case of strictly hyperbolic systems, traditional approximate/exact Riemann solvers cannot be used here.

We now propose the (DDF) scheme.
3.1. (DDF) scheme

With the notations same as before, define

\[ U_i^n = \frac{1}{h} \int_{C_i} U(x, t^n) \, dx, \quad u_i^n = \frac{w_i^n}{\rho_i^n} \]

as the approximation for \( U \) and \( u \) in the cell \( C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \) at time \( t_n \). For any state variable \( z := \rho, w, u \), iteratively, we define a piecewise constant \( z^h(x, t) \) such that

\[ z^h(x, t) = z_i^n \quad \text{for} \quad t \in [t^n, t^{n+1}), \quad x \in C_i, \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}. \]

Let \((U_i^n, U_i^{n+1}) := (\rho_i^n, \rho_i^{n+1}, w_i^n, w_i^{n+1})\). Then, the finite volume scheme for system (3.1) is given by:

\[ U_i^{n+1} = U_i^n - \lambda \left( \hat{F}(U_i^n, U_i^{n+1}) - \hat{F}(U_i^{n-1}, U_i^n) \right), \quad (3.3) \]

where

\[ \hat{F}(U_i^n, U_i^{n+1}) = \left( \hat{F}^\rho(U_i^n, U_i^{n+1}), \hat{F}^w(U_i^n, U_i^{n+1}) \right) \]

(3.4)

where \( \hat{F}^\rho(U_i^n, U_i^{n+1}), \hat{F}^w(U_i^n, U_i^{n+1}) \) and \( \hat{F}(U_i^n, U_i^{n+1}) \) are the numerical fluxes associated with \( F^\rho, F^w \) and \( F \) (defined by (3.2)), at \( x_{i+\frac{1}{2}} \) at time \( t^n \). We will have the following fluxes:

\[ \hat{F}^\rho(U_i^n, U_i^{n+1}) = F_0 \left( g(u_i^n), g(u_{i+1}^n), \rho_i^n, \rho_{i+1}^n \right), \quad (3.5) \]

where \( F_0 \) is given by (2.10) and

\[ \hat{F}^w(U_i^n, U_i^{n+1}) = \max \left( F^w(\rho_i^n, \max(w_i^n, 0)), F^w(\rho_{i+1}^n, \min(w_{i+1}^n, 0)) \right). \]

(3.6)

It is to be noted that \( \hat{F}^\rho(U_i^n, U_i^{n+1}) \) is computationally less expensive than the one proposed in [7].

The flux \( \hat{F}^\rho \) is obtained by solving the local Riemann problems at the interface \( x_{i+\frac{1}{2}} \) as described below:

On each \( C_i \times (t^n, t^{n+1}) \), we look at the conservation law,

\[ \rho_t + (F^\rho(u_i^n, \rho))_x = 0 \quad \text{in} \quad (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t^n, t^{n+1}), \]

with \( F^\rho(u_i^n, \rho) = g(u_i^n)\rho \) and the initial condition \( \rho(x, t^n) = \rho_i^n \) for \( x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \) (Fig. 6).
Hence, the problem reduces to the corresponding local Riemann problem, as in [1,2],

\[ \rho_t + l^p(x, \rho) = 0 \quad \text{in} \quad (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t^n, t^{n+1}), \tag{3.7} \]

where

\[ l(x, \rho) = \begin{cases} \frac{\rho_n}{\rho} u_n \rho & \text{if } x < x_{i+\frac{1}{2}}, \\
\frac{\rho_{n+1}}{\rho} u_{n+1} \rho & \text{if } x > x_{i+\frac{1}{2}}, \end{cases} \]

with the initial data

\[ \rho(x, t^n) = \begin{cases} \rho_n & \text{if } x < x_{i+\frac{1}{2}}, \\
\rho_{n+1} & \text{if } x > x_{i+\frac{1}{2}}. \end{cases} \]

Each local Riemann Problem (3.7) at the interface \( x_{i+\frac{1}{2}} \) is of form (2.1), and the flux at each interface is given by \( F_0((g(u_n^i), g(u_{n+1}^i), \rho_n, \rho_{n+1})) \).

For the second equation

\[ w_t + l^w(x, w) = 0 \quad \text{in} \quad (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t^n, t^{n+1}), \tag{3.9} \]

where

\[ l^w(x, w) = \begin{cases} \frac{w}{\rho} & \text{if } x < x_{i+\frac{1}{2}}, \\
\frac{w}{\rho_{i+1}} & \text{if } x > x_{i+\frac{1}{2}}, \end{cases} \]

with the initial data

\[ w(x, t^n) = \begin{cases} w_n & \text{if } x < x_{i+\frac{1}{2}}, \\
w_{n+1} & \text{if } x > x_{i+\frac{1}{2}}. \end{cases} \]

Using theory of discontinuous flux for convex conservation laws of [1,2], we get the following required flux at each interface \( x_{i+\frac{1}{2}} \),

\[ \tilde{F}^w(U_n^i, U_{n+1}^i) = \max \left( F^w(\rho_n^i, \max(w_n^i, 0)), F^w(\rho_{i+1}^n, \min(w_{n+1}^n, 0)) \right). \]

We now provide a higher-order extension of the scheme.

### 3.2. Higher-order extension of (DDF) scheme

We first consider the forward Euler time discretization. Let \( \overline{U}_i^n \) be the cell average of \( U(x, t) \) in the cell \( C_i \) at time \( t^n \). Let \( U_{i+\frac{1}{2}}^{L, R} \) be the second-order approximations of \( U(x, t^n) \) at the cell interface \( x_{i+\frac{1}{2}} \).
within the cell $C_i$ and $C_{i+1}$, which are reconstructed from the cell average $\overline{U}_i^n$. Let $(U_{i+\frac{1}{2}}, U_{i+\frac{1}{2}}^R) := 
abla \rho_i^{L}, \rho_i^{R}, w_i^{L}, w_i^{R})$ and $u_{i+\frac{1}{2}} = \frac{w_i^{R} - w_i^{L}}{\rho_i^{R} - \rho_i^{L}}$. Finite volume scheme approximating (3.1) is given by:

$$\overline{U}_{i+1} = \overline{U}_i - \lambda \left[ \hat{F} \left( U_{i+\frac{1}{2}}^L, U_{i+\frac{1}{2}}^R \right) - \hat{F} \left( U_{i-\frac{1}{2}}^L, U_{i-\frac{1}{2}}^R \right) \right],$$

(3.10)

where $\hat{F}$ is given by (3.4) and is the first-order numerical flux associated with the flux $F$ at the interface $x_{i+\frac{1}{2}}$ at time $t^n$, which preserves the positivity of $\rho$ and bounds of the velocity $u$. For $z = \rho, u$, let

$$z_{i+\frac{1}{2}} := p_i^z(x_{i+\frac{1}{2}}), \overline{z}_i = \frac{1}{2} \left( z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}} \right),$$

where $p_i^z(x)$ is the linear approximation in the cell $C_i$ for the piecewise constant solution $z_i^n$ such that

$$\frac{1}{h} \int_{C_i} p_i^z(x) dx = \overline{z}_i, \quad p_i^z(x) = \overline{z}_i + \sigma_i^z(x - x_i).$$

The slope $\sigma_i^z$ is controlled by the choice of a suitable limiter ensuring that the physical properties of the system are preserved. Define $p_i^w(x) := p_i^z(x)p_i^w(x)$. For higher-order time discretization, we use second-order strong stability preserving Runge–Kutta method of [27], which we summarize below:

$$U_i^n = U_i^n - \lambda \left[ \hat{F} \left( U_{i+\frac{1}{2}}^L, U_{i+\frac{1}{2}}^R \right) - \hat{F} \left( U_{i-\frac{1}{2}}^L, U_{i-\frac{1}{2}}^R \right) \right],$$

(3.11)

$$U_{i+1}^n = U_i^n - \lambda \left[ \hat{F} \left( U_{i+\frac{1}{2}}^L, U_{i+\frac{1}{2}}^R \right) - \hat{F} \left( U_{i-\frac{1}{2}}^L, U_{i-\frac{1}{2}}^R \right) \right],$$

(3.12)

$$U_{i+1}^{n+1} = \frac{1}{2} (U_i^n + U_{i+1}^n).$$

(3.13)

$U_{i+\frac{1}{2}}$ is reconstructed from $U_i^n$, in the same way as $U_{i+\frac{1}{2}}^R$ is reconstructed from $U_i^n$.

4. Stability and convergence analysis

We now prove the numerical solutions given by the first-order (DDF) scheme are entropy stable in the framework of [9], satisfy the physical properties of the state variables, and converge to the weak solution of (GPGD) in one space dimension.

Let us define the piecewise constant approximate solution to (GPGD), $U_h(x, t) = \left( \rho_h, u_h \right)$ such that $U_h(x, t) = U_i^n = \left( \rho_i^n, w_i^n \right), t \in [t^n, t^{n+1}], x \in C_i, n \in \mathbb{N}, i \in \mathbb{Z}$, where $U_i^n$ is the numerical solution obtained by 3-points algorithm (3.3). Physically, the density $\rho \geq 0$ and the velocity $u$ satisfy the maximum principle, see for example, [30] and references therein. Let

$$S = \{ (\rho, w) : \rho \geq 0, m \rho \leq w \leq M \rho \},$$

where $m = \min_{x \in \mathbb{R}} g(u_0(x)), M = \max_{x \in \mathbb{R}} g(u_0(x))$. We denote $\hat{F}_{i+\frac{1}{2}}^z := \hat{F}^z \left( U_i^n, U_{i+1}^n \right), z = \rho, w$. We then have the following theorem:

**Theorem 4.1.** Under CFL-like condition (4.1), $U_{i+1}^{n+1} \in S$ if $U_i^n \in S$.

The above theorem is a consequence of the following two lemmas:
Lemma 4.2. (Positivity of $\rho^n$) For each $n \in \mathbb{N}$,

$$\rho^n > 0 \implies \rho^{n+1} > 0,$$

under the condition

$$\lambda \max_{i, n} |g(u^n_i)| \leq 1. \quad (4.1)$$

Proof. It can be easily shown that $F^\rho_{i+\frac{1}{2}}$ given by (3.5) is increasing in the variable $\rho^n_i$ and decreasing in the variable $\rho^n_{i+1}$ and hence,

$$1 - \lambda \left( \frac{\partial F^\rho_{i+\frac{1}{2}}}{\partial \rho^n_i} - \frac{\partial F^\rho_{i+\frac{1}{2}}}{\partial \rho^n_{i+1}} \right) = \begin{cases} 
1 - \lambda g(u^n_i) & \text{if } g(u^n_i) > 0, g(u^n_{i+1}) > 0, \\
1 + \lambda g(u^n_i) & \text{if } g(u^n_{i-1}) < 0, g(u^n_{i}) < 0, \\
1 & \text{else,}
\end{cases}$$

which shows that $\frac{\partial \rho^{n+1}}{\partial \rho^n_j} \geq 0 \ \forall i, j \in \{i \pm 1, i\}$ under condition (4.1) which gives the desired result. \qed

Using (3.3), we have the following:

$$\rho^{n+1}_i(u^{n+1}_i - u^n_i) = -\lambda(F^\rho_{i+\frac{1}{2}} - F^\rho_{i-\frac{1}{2}} u^n_i) + \lambda(F^\rho_{i-\frac{1}{2}} - F^\rho_{i+\frac{1}{2}} u^n_i),$$

and hence, we have

$$u^{n+1}_i = u^n_i(1 - C^n_{i+\frac{1}{2}} - D^n_{i-\frac{1}{2}}) + u^n_{i-1} C^n_{i-\frac{1}{2}} + u^n_{i+1} D^n_{i+\frac{1}{2}}, \quad (4.2)$$

where for each $i, n,$

$$C^n_{i-\frac{1}{2}} = \begin{cases} 
-\lambda \frac{F^\rho_{i-\frac{1}{2}} - F^\rho_{i+\frac{1}{2}} u^n_i}{\rho^{n+1}_i(u^n_i - u^n_{i-1})} & \text{if } u^n_i - u^n_{i-1} \neq 0, \\
0 & \text{if } u^n_i - u^n_{i-1} = 0,
\end{cases} \quad (4.3)$$

and

$$D^n_{i+\frac{1}{2}} = \begin{cases} 
-\lambda \frac{F^\rho_{i+\frac{1}{2}} - F^\rho_{i-\frac{1}{2}} u^n_i}{\rho^{n+1}_i(u^n_{i+1} - u^n_i)} & \text{if } u^n_{i+1} - u^n_i \neq 0, \\
0 & \text{if } u^n_{i+1} - u^n_i = 0,
\end{cases} \quad (4.4)$$

Using the above incremental form, we have the following result which will be proved in Lemma 4.9.

Lemma 4.3. (Bounds on $g(u^n)$) Under CFL-like condition (4.1), $C^n_{i-\frac{1}{2}}, D^n_{i+\frac{1}{2}} \geq 0$ and $C^n_{i-\frac{1}{2}} + D^n_{i+\frac{1}{2}} \leq 1.$ Also, if

$$g^{-1}(m) \rho^n_i \leq w^n_i \leq g^{-1}(M) \rho^n_i, \ m = \min_i g(u^n_i), \ M = \max_i g(u^n_i),$$

then

$$g^{-1}(m) \rho^{n+1}_i \leq w^{n+1}_i \leq g^{-1}(M) \rho^{n+1}_i.$$

The following lemma is an easy consequence of the above lemmas and establishes conservative property of $U$.

Lemma 4.4. (L^1 Stability of $\rho, w, \rho g(u), w u, w g(u)$) For any time $t^n, n \geq 0$, the following holds true:

$$\|z^n\|_{L^1(\mathbb{R})} \leq K_z \|z^0\|_{L^1(\mathbb{R})},$$

where $z$ can be $\rho, w, \rho g(u), w u, w g(u)$ and the constant $K_z$ depends on $\|u^0\|_{L^\infty(\mathbb{R})}$. 
\textbf{Theorem 4.5.} For every \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \), and for \( z = \rho, w \), we have
\[
\lim_{h \to 0} \int_{\mathbb{R} \times \mathbb{R}^+} z_h \phi_t + z_h g(u_h) \phi_x = 0,
\]
Proof. Let \( i \in \mathbb{Z}, n \in \mathbb{N}, \phi^n_i = \phi(x_i, t_n) \) and
\[
\phi^n_{x, i} = \frac{\phi(x_i, t^n) - \phi(x_{i-1}, t^n)}{h}, \forall \, i, n.
\]
To prove theorem, it is enough to show that for \( z = \rho, w \),
\[
\lim_{h \to 0} A(h) = 0 \quad \text{with} \quad A(h) = -h \sum_{i \in \mathbb{Z}, \, n \in \mathbb{N}} \left[ z^{n+1}_i - z^n_i + \lambda \left( z^n_i g(u^n_i) - z^n_{i-1} g(u^n_{i-1}) \right) \right] \phi^n_i,
\]
By the definition of the scheme,
\[
A(h) = -h \sum_{i, n} \left[ -\lambda(F^z_{i+\frac{1}{2}} - F^z_{i-\frac{1}{2}}) + \lambda\left( z^n_i g(u^n_i) - z^n_{i-1} g(u^n_{i-1}) \right) \right] \phi^n_i,
\]
which, on rearranging the terms, gives,
\[
A(h) = \lambda h \sum_{i, n} \left[ F^z_{i+\frac{1}{2}} - z^n_i g(u^n_i) \right] - \left( F^z_{i-\frac{1}{2}} - z^n_{i-1} g(u^n_{i-1}) \right) \phi^n_i.
\]
Now, applying summation by parts, we get
\[
A(h) = -\lambda h^2 \sum_{i, n} \left[ F^z_{i+\frac{1}{2}} - z^n_i g(u^n_i) \right] \phi^n_{x, i}, \tag{4.5}
\]
where \( F^\phi_{i-\frac{1}{2}} \) and \( F^{w, n}_{i-\frac{1}{2}} \) are given by (3.5) and (3.6), respectively. Note that the possible expressions for \( F^z_{i+\frac{1}{2}} \) are \( z^n_{i-1} g(u^n_{i-1}) \), \( z^n_i g(u^n_i) \) and 0. Let \( j \in \mathbb{Z} \). For a fixed \( n \), the terms containing \( g(u^n_j) \) in expression (4.5) are
\[
-\lambda h^2 z^n_j g(u^n_j) \left[ \phi^n_{x, j} - \phi^n_{x, j+1} \right] \leq L_\phi \lambda h^2 \| z_0 g(u_0) \|_{L^1}, \quad L_\phi = K \| \phi \|_{L\infty(\mathbb{R} \times \mathbb{R}^+)}
\]
using the fact that the scheme is a 3–point scheme and \( z g(u) \) is \( L^1 \) stable. Summing over \( n \in \mathbb{N} \), we get
\[
-\lambda h^2 \sum_{i, n} \left[ F^z_{i+\frac{1}{2}} - z^n_i g(u^n_i) \right] \phi^n_{x, i} \leq \sum_n L_\phi \lambda h^2 \| z_0 g(u_0) \|_{L^1} + L_\phi T h \| z_0 g(u_0) \|_{L^1},
\]
where \( T \) is the final time. This shows that \( A(h) = O(h) \), which proves the claim. \hfill \Box

The above theorem shows that the distribution limit \( U \) of the approximate solution \( U_h \) is a solution of (GPDG) in the sense of distributions. It has been pointed in the seminal paper of [9] that the solutions of the system (PGD) satisfy the following inequality for any convex real-valued function \( S \)
\[
\left( \rho S(u) \right)_t + \left( \rho u S(u) \right)_x \leq 0,
\]
which can be extended to (GPDG) and reads as:
\[
(\rho S(u))_t + (\rho g(u) S(u))_x \leq 0.
\]

In the following theorem, we show that the limit of numerical approximation \( U^n_1 \) satisfies the entropy inequality.

\textbf{Theorem 4.6.} For every \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \), and for \( z = \rho S(u) \), we have
\[
\lim_{h \to 0} \int_{\mathbb{R} \times \mathbb{R}^+} z_h \phi_t + z_h g(u_h) \phi_x \geq 0.
\]
Lemma 4.7. (Discrete entropy inequality) For all convex functions $S$,

$$
\rho^n_{i+1} S(u^n_{i+1}) - \rho^n_i S(u^n_i) + \lambda \left( G(\rho^n_i, \rho^n_{i+1}, u^n_i, u^n_{i+1}) - G(\rho^n_{i-1}, \rho^n_i, u^n_{i-1}, u^n_i) \right) \leq 0, \quad (4.6)
$$

where $G(\cdot, \cdot, \cdot)$ is the numerical entropy flux, consistent in the following sense

$$
G(\rho, \rho, u, u) = S(u) \rho g(u).
$$

Proof. Using Lemma 4.3, the convexity of $S$ and Jensen’s inequality on incremental form (4.2), one gets,

$$
S(u^n_{i+1}) \leq S(u^n_i)(1 - C^n_{i-\frac{1}{2}} - D^n_{i+\frac{1}{2}}) + S(u^n_{i-1})C^n_{i-\frac{1}{2}} + S(u^n_{i+1})D^n_{i+\frac{1}{2}},
$$

which implies

$$
S(u^n_{i+1}) - S(u^n_i) \leq -(S(u^n_i) - S(u^n_{i-1}))C^n_{i-\frac{1}{2}} + (S(u^n_{i+1}) - S(u^n_i))D^n_{i+\frac{1}{2}},
$$

$$
\rho^n_{i+1} S(u^n_{i+1}) - \rho^n_i S(u^n_i) \leq -\lambda \left( S(u^n_{i-1}) - S(u^n_i) \right) C^n_{i-\frac{1}{2}} - \lambda \left( S(u^n_{i+1}) - S(u^n_i) \right) D^n_{i+\frac{1}{2}}.
$$

Substituting the value of $\rho^n_{i+1}$, in the last inequality, we have,

$$
-\lambda \left( S(u^n_{i-1}) - S(u^n_i) \right) C^n_{i-\frac{1}{2}} - \lambda \left( S(u^n_{i+1}) - S(u^n_i) \right) D^n_{i+\frac{1}{2}} \geq \rho^n_{i+1} S(u^n_{i+1}) - \rho^n_i S(u^n_i) \leq -\lambda \left( F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i-\frac{1}{2}} \right) S(u^n_i) + \lambda \left( S(u^n_i) - S(u^n_{i-1}) \right) C^n_{i-\frac{1}{2}}
$$

which on further rearrangement,

$$
\rho^n_{i+1} S(u^n_{i+1}) - \rho^n_i S(u^n_i) \leq -\lambda \left( F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i-\frac{1}{2}} \right) S(u^n_i) + \lambda \left( S(u^n_i) - S(u^n_{i-1}) \right) C^n_{i-\frac{1}{2}}
$$

Now, for the case $u^n_{i+1} \neq u^n_i$, the term

$$
F^{\rho, n}_{i+\frac{1}{2}} S(u^n_i) + \left( S(u^n_{i+1}) - S(u^n_i) \right) D^n_{i+\frac{1}{2}}
$$

$$
= \frac{(u^n_{i+1} - u^n_i) F^{\rho, n}_{i+\frac{1}{2}} S(u^n_i) + (S(u^n_{i+1}) - S(u^n_i))(F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i+\frac{1}{2}} u^n_i)}{(u^n_{i+1} - u^n_i)}
$$

$$
= \frac{(u^n_{i+1} F^{\rho, n}_{i+\frac{1}{2}} S(u^n_i) + S(u^n_{i+1})(F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i+\frac{1}{2}} u^n_i))}{(u^n_{i+1} - u^n_i)}
$$

$$
= \frac{(u^n_{i+1} F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i+\frac{1}{2}} u^n_i) S(u^n_i) + S(u^n_{i+1})(F^{\rho, n}_{i+\frac{1}{2}} - F^{\rho, n}_{i+\frac{1}{2}} u^n_i)}{(u^n_{i+1} - u^n_i)}.
$$

and for the case $u^n_i = u^n_{i+1}$, the term

$$
F^{\rho, n}_{i+\frac{1}{2}} S(u^n_i) + \left( S(u^n_{i+1}) - S(u^n_i) \right) D^n_{i+\frac{1}{2}} = F^{\rho, n}_{i+\frac{1}{2}} S(u^n_i),
$$
since \( D_{i+\frac{1}{2}}^n = 0 \). Similarly, for the case \( u_i^n \neq 6u_{i-1}^n \), the term
\[
F_{i+\frac{1}{2}}^{\phi, n} S(u^n_i) + \left( S(u^n_i) - S(u^n_{i-1}) \right) \frac{F_{i-\frac{1}{2}}^{w, n} - F_{i-\frac{1}{2}}^{\phi, n} u^n_i}{(u^n_i - u^n_{i-1})}
\]
\[= \frac{(u_i^n - u_{i-1}^n) F_{i+\frac{1}{2}}^{\phi, n} S(u^n_i) + \left( S(u^n_i) - S(u^n_{i-1}) \right) (F_{i-\frac{1}{2}}^{w, n} - F_{i-\frac{1}{2}}^{\phi, n} u^n_i)}{(u_i^n - u_{i-1}^n)},
\]
and for the case \( u_i^n = u_{i-1}^n \), the term
\[
F_{i+\frac{1}{2}}^{\phi, n} S(u^n_i) + \left( S(u^n_i) - S(u^n_{i-1}) \right) C_{i+\frac{1}{2}}^n = F_{i+\frac{1}{2}}^{\phi, n} S(u^n_i),
\]

since \( C_{i+\frac{1}{2}}^n = 0 \). This implies that
\[
\rho_i^{n+1} S(u_i^{n+1}) - \rho_i^n S(u_i^n) + \lambda \left( F_{i+\frac{1}{2}}^{\phi S(u), n} - F_{i-\frac{1}{2}}^{\phi S(u), n} \right) \leq 0,
\]
(4.7)

where
\[
F_{i+\frac{1}{2}}^{\phi S(u), n} = \begin{cases} 
\frac{(u_{i+1}^{n+1} F_{i+\frac{1}{2}}^{\phi, n} - F_{i+\frac{1}{2}}^{\phi, n} u^n_i) S(u^n_i) + (u^{n+1}_{i+1} - u^n_i) (F_{i+\frac{1}{2}}^{w, n} - F_{i+\frac{1}{2}}^{\phi, n} u^n_i)}{(u^n_{i+1} - u^n_i)} & \text{if } u_{i+1}^n - u_i^n \neq 0, \\
F_{i+\frac{1}{2}}^{\phi, n} S(u^n_i) & \text{if } u_i^n - u_{i+1}^n = 0.
\end{cases}
\]
(4.8)

Now consider the function \( F_{\rho S(u)} : \mathbb{R}^4 \rightarrow \mathbb{R} \), such that
\[
F_{\rho S(u)}(\rho_1, \rho_r, w_1, w_r) = \begin{cases} 
\frac{(u_i F_{\rho S(u) (\rho_1, \rho_r, w_1, w_r)} - F_{\rho S(u) (\rho_1, \rho_r, w_1, w_r)} u^n_i) S(u^n_i)}{(u_r - u^n_i) (u_{r+1} - u^n_i)} & \text{if } u_r - u_i \neq 0, \\
F_{\rho S(u) (\rho_1, \rho_r, w_1, w_r)} S(u^n_i) & \text{if } u_r - u_i = 0.
\end{cases}
\]
(4.9)

where \( u_1 = \frac{w_1}{\rho_1}, u_r = \frac{w_r}{\rho_r} \). Now consistency of \( F_{\rho} \) and \( F_{w} \) implies the consistency of \( F_{\rho S(u)} \) in the sense that,
\[
F_{\rho S(u)}(\rho, \rho, u, u) = \rho g(u) S(u).
\]
which gives the consistency of \( F_{\rho S(u)} \). This indicates to choose \( G = F_{\rho S(u)} \) in Eq. (4.6). \( \square \)

Finally, we give the proof of Theorem 4.6.

**Proof.** Let \( i \in \mathbb{Z}, n \in \mathbb{N}, \phi_i^n = \phi(x_i, t_n) \) and \( \phi^n_{x, i} = \frac{\phi(x_i + t^n) - \phi(x_i - t^n)}{h}, \forall \ i, n \). To prove theorem, it is enough to show that, for \( z = \rho S(u) \),
\[
\lim_{h \to 0} A(h) \geq 0, \text{ where, } A(h) = -h \sum_{i, n} \left[ z_i^{n+1} - z_i^n + \lambda \left( z_i^n g(u^n_i) - z_{i+1}^n g(u^n_{i+1}) \right) \phi_i^n \right].
\]

By equations (4.7) and (4.8), we have
\[
\rho_i^{n+1} S(u_i^{n+1}) - \rho_i^n S(u_i^n) + \lambda \left( F_{i+\frac{1}{2}}^{\rho S(u), n} - F_{i-\frac{1}{2}}^{\rho S(u), n} \right) \leq 0,
\]
where \( F_{i+\frac{1}{2}} \) is given by (4.8) and can be rewritten as

\[
F_{i+\frac{1}{2}}^{\rho S(u)} = \begin{cases} 
S(u_i^n)\rho_i^n g(u_i^n) & \text{if } g(u_i^n) \geq 0, g(u_{i+1}^n) > 0, \\
S(u_{i+1}^n)\rho_{i+1}^n g(u_{i+1}^n) & \text{if } g(u_i^n) < 0, g(u_{i+1}^n) \leq 0, \\
0 & \text{if } g(u_i^n) < 0, g(u_{i+1}^n) > 0, \\
F_{i+\frac{1}{2}}^{w, u} S(u_{i+1}^n) - S(u_i^n) & \text{if } g(u_i^n) \geq 0, g(u_{i+1}^n) \leq 0.
\end{cases}
\]

This implies that with \( z = \rho S(u) \),

\[-(z_i^{n+1} - z_i^n) \geq \lambda (F_{i+\frac{1}{2}}^{\rho, n} - F_{i-\frac{1}{2}}^{\rho, n}).\]

This further implies

\[A(h) \geq h \sum_{i, n} \left[ \lambda (F_{i+\frac{1}{2}}^{\rho, n} - F_{i-\frac{1}{2}}^{\rho, n}) - \lambda (z_i^n g(u_i^n) - z_i^{n-1} g(u_{i-1}^n)) \right] \phi_i^n,
\]

which, on rearranging the terms, gives,

\[A(h) \geq \lambda h \sum_{i, n} \left( (F_{i+\frac{1}{2}}^{\rho, n} - z_i^n g(u_i^n)) - (F_{i-\frac{1}{2}}^{\rho, n} - z_i^{n-1} g(u_{i-1}^n)) \right) \phi_i^n.
\]

Now, applying summation by parts, we get

\[A(h) \geq -\lambda h^2 \sum_{i, n} \left[ F_{i-\frac{1}{2}}^{\rho, n} - z_i^{n-1} g(u_{i-1}^n) \right] \phi_i^n, i.
\]

Now, the proof follows by similar argument as in Theorem 4.5 and by using, in addition, the Lipschitz continuity of the function \( S \). \( \square \)

We now prove that the solutions of the higher-order scheme also preserve the physical properties of the system. Denote

\[
\hat{F}^z \left( U_{i+\frac{1}{2}}^L, U_{i+\frac{1}{2}}^R \right) = \hat{F}^z_{i+\frac{1}{2}}, z = \rho, w.
\]

**Lemma 4.8.** (Positivity) Under the CFL-like condition,

\[\lambda \max_{i, n} |g(\bar{u}_i^n)| \leq \text{cfl}, \quad (4.10)\]

if \( \rho_{i-\frac{1}{2}}^R, \rho_{i+\frac{1}{2}}^L > 0 \) and \( \text{cfl} = \frac{1}{2} \), then \( \bar{p}_i^{n+1} > 0 \).

**Proof.** Using (3.10) for \( \rho^n \), we have

\[
\bar{p}_i^{n+1} = \bar{p}_i^n - \lambda \left[ \hat{F}^\rho_{i+\frac{1}{2}} - \hat{F}^\rho_{i-\frac{1}{2}} \right] = \frac{1}{2} \left[ \rho_{i-\frac{1}{2}}^R + \rho_{i+\frac{1}{2}}^L \right] - \lambda \left[ \hat{F}^\rho_{i+\frac{1}{2}} - \hat{F}^\rho_{i-\frac{1}{2}} \right]
\]

\[:= H(\rho_{i-\frac{1}{2}}^R, \hat{\rho}_{i-\frac{1}{2}}^L, \rho_{i+\frac{1}{2}}^R, \hat{\rho}_{i+\frac{1}{2}}^L, u_{i-\frac{1}{2}}^L, u_{i+\frac{1}{2}}^L, u_{i-\frac{1}{2}}^R, u_{i+\frac{1}{2}}^R).
\]

Repeating the arguments in Lemma 4.2, \( H \) is nondecreasing in its first four arguments under CFL-like condition (4.10). Since \( H(0, 0, 0, 0, \cdots, \cdot \cdot \cdot) = 0 \), the result follows. \( \square \)

The same result is true for the first-order scheme with \( \text{cfl} = 1 \), see Lemma 4.2. \( \rho_{i+\frac{1}{2}}^L, \rho_{i+\frac{1}{2}}^R \) can now be made positive by appropriate choice of \( \sigma_i^\rho \).

**Lemma 4.9.** (Bounds on momentum) Under CFL-like condition \( (4.10) \) with \( \text{cfl} = \frac{1}{2} \), if

\[g^{-1}(m) \bar{\omega}_i^n \leq \bar{u}_i^n \leq g^{-1}(M) \bar{\omega}_i^n, \quad m = \min_i g(\bar{u}_i^n), \quad M = \max_i g(\bar{u}_i^n), \]

then

\[g^{-1}(m) \bar{p}_i^{n+1} \leq \bar{\omega}_i^{n+1} \leq g^{-1}(M) \bar{p}_i^{n+1}.\]
Proof. The strategy to prove the lemma is to write finite volume scheme (3.10) in an incremental form for \( \tilde{\omega}_{i+1}^{n+1} \). For simplicity, we show the results for Minmod limiter.

Case 1. We start with the case when neither \( u^L_{i+\frac{1}{2}} \geq 0 \) and \( u^R_{i+\frac{1}{2}} \leq 0 \) nor \( u^L_{i-\frac{1}{2}} \geq 0 \) and \( u^R_{i-\frac{1}{2}} \leq 0 \). Then, we have

\[
\dot{\tilde{\omega}}_{i+\frac{1}{2}}^w = u^L_{i+\frac{1}{2}} \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + u^R_{i+\frac{1}{2}} \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0).
\]

Using finite volume scheme (3.10), we get

\[
\bar{p}_{i+1}^{n+1} = \bar{p}_i^n - \lambda \left[ \dot{\tilde{\omega}}_{i+\frac{1}{2}}^w - \dot{\tilde{\omega}}_{i-\frac{1}{2}}^w \right],
\]

\[
\tilde{\omega}_{i+1}^{n+1} = \bar{p}_{i+1}^{n+1} \tilde{\omega}_{i+1}^w - \lambda \left[ \dot{\tilde{\omega}}_{i+\frac{1}{2}}^w - \dot{\tilde{\omega}}_{i-\frac{1}{2}}^w \right].
\]

Multiplying first equation by \( \tilde{\omega}_i^n \), we get,

\[
\bar{p}_{i+1}^{n+1} (\tilde{\omega}_{i+1}^{n+1} - \tilde{\omega}_i^n) = \lambda \left[ \dot{\tilde{\omega}}_{i+\frac{1}{2}}^w - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o \right] - \lambda \left[ \dot{\tilde{\omega}}_{i+\frac{1}{2}}^w - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o \right].
\]

Putting the values of \( u^L_{i+\frac{1}{2}} \) and \( u^R_{i+\frac{1}{2}} \) and \( \dot{\tilde{\omega}}_{i+\frac{1}{2}}^w, \bar{p}_{i+1}^{n+1} (\tilde{\omega}_{i+1}^{n+1} - \tilde{\omega}_i^n) \) is equal to

\[
\lambda \left[ u^L_{i+\frac{1}{2}} \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + u^R_{i+\frac{1}{2}} \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) - \tilde{\omega}_i^n \left( \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) \right) \right]
\]

\[
- \lambda \left[ u^L_{i+\frac{1}{2}} \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + u^R_{i+\frac{1}{2}} \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) - \tilde{\omega}_i^n \left( \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) \right) \right]
\]

\[
= \lambda \left[ (u^L_{i+\frac{1}{2}} - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o) \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + (u^R_{i+\frac{1}{2}} - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o) \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) \right]
\]

\[
- \lambda \left[ (u^L_{i+\frac{1}{2}} - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o) \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + (u^R_{i+\frac{1}{2}} - \tilde{\omega}_i^n \dot{\tilde{\omega}}_{i+\frac{1}{2}}^o) \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) \right].
\]

Note that

\[
u^L_{i+\frac{1}{2}} = p^u_i(x_{i+\frac{1}{2}}) = \tilde{\omega}_i^n + \sigma_i^n, \quad u^R_{i+\frac{1}{2}} = p^u_i(x_{i-\frac{1}{2}}) = \tilde{\omega}_i^n - \sigma_i^n,
\]

where \( \sigma_i^n := \frac{h}{2} \sigma_i^n = \minmod(\tilde{\omega}^n_i - \tilde{\omega}^n_{i-1}, \tilde{\omega}^n_{i+1} - \tilde{\omega}^n_i) \). Hence, we have

\[
\bar{p}_{i+1}^{n+1} (\tilde{\omega}_{i+1}^{n+1} - \tilde{\omega}_i^n) = -\lambda \left[ (\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n \sigma_{i-1}^n) \max(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0) + \sigma_{i-1}^n \min(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0) \right]
\]

\[
- \lambda \left[ \sigma_i^n \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + (\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n - \sigma_i^n) \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) \right]
\]

which implies that

\[
\tilde{\omega}_{i+1}^{n+1} = \tilde{\omega}_i^n - \lambda \frac{(\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n)}{\bar{p}_{i+1}^{n+1}} \left[ 1 - \frac{\sigma_{i-1}^n}{\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n} \right] \max(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0) - \lambda \frac{\sigma_i^n}{\bar{p}_{i+1}^{n+1}} \min(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0)
\]

\[
- \lambda \frac{(\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n)}{\bar{p}_{i+1}^{n+1}} \left[ 1 - \frac{\sigma_{i+1}^n}{\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n} \right] \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) - \lambda \frac{\sigma_i^n}{\bar{p}_{i+1}^{n+1}} \max(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0)
\]

\[
= \tilde{\omega}_i^n - (\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n) \frac{\lambda}{\bar{p}_{i+1}^{n+1}} \left[ 1 - \frac{\sigma_{i-1}^n}{\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n} \right] \max(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0) + \frac{\sigma_i^n}{(\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n)} \max(\dot{\tilde{\omega}}_{i-\frac{1}{2}}^o, 0)
\]

\[
- (\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n) \frac{\lambda}{\bar{p}_{i+1}^{n+1}} \left[ 1 - \frac{\sigma_{i+1}^n}{\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n} \right] \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0) + \frac{\sigma_i^n}{(\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n)} \min(\dot{\tilde{\omega}}_{i+\frac{1}{2}}^o, 0)
\]

\[
= \tilde{\omega}_i^n - (\tilde{\omega}_i^n - \tilde{\omega}_{i-1}^n) \bar{C}_{i-\frac{1}{2}} + (\tilde{\omega}_{i+1}^n - \tilde{\omega}_i^n) \bar{D}_{i+\frac{1}{2}},
\]
and hence
\[ u_i^{n+1} = u_i^n (1 - \tilde{C}_{i-\frac{1}{2}} - \tilde{D}_{i+\frac{1}{2}}) + u_{i-1}^n \tilde{C}_{i-\frac{1}{2}} + u_{i+1}^n \tilde{D}_{i+\frac{1}{2}}, \tag{4.12} \]
where
\[ \tilde{D}_{i+\frac{1}{2}} = -\frac{\lambda}{p_i^{n+1}} \left[ (1 - \frac{\tilde{\sigma}_i^{n+1}}{u_{i+1}^n - u_i^n}) \min(\hat{F}_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n}), 0 \right] + \frac{\tilde{\sigma}_i^u \min(F_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n})}{(u_{i+1}^n - u_i^n)}, \]
\[ \tilde{C}_{i-\frac{1}{2}} = \frac{\lambda}{p_i^{n+1}} \left[ (1 - \frac{\tilde{\sigma}_{i-1}^u}{u_i^n - u_{i-1}^n}) \max(\hat{F}_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n}), 0 \right] + \frac{\tilde{\sigma}_i^u \max(F_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n})}{(u_i^n - u_{i-1}^n)}. \]

Since for any \( k, 0 \leq \frac{\tilde{\sigma}_k^l}{u_k^n - u_{k-1}^n}, \frac{\tilde{\sigma}_k^u}{u_k^n - u_{k-1}^n} \leq \frac{1}{2} \), hence \( \tilde{C}_{i-\frac{1}{2}}, \tilde{D}_{i+\frac{1}{2}} \geq 0 \). We now prove that \( p_i^{n+1} (1 - \tilde{C}_{i-\frac{1}{2}} - \tilde{D}_{i+\frac{1}{2}}) \geq 0 \). We introduce some notations:
\[ u_{i+\frac{1}{2}}^L := \max(u_{i+\frac{1}{2}}^L, 0), u_{i+\frac{1}{2}}^R := \min(u_{i+\frac{1}{2}}^R, 0). \]

(i) When \( \hat{F}_{i-\frac{1}{2}}^{\rho} + \hat{F}_{i+\frac{1}{2}}^{\rho} \geq 0 \)

Then, \( \hat{F}_{i+\frac{1}{2}}^{\rho} = u_{i+\frac{1}{2}}^{L} + \rho_{i+\frac{1}{2}}^{L} \). We have \( \tilde{D}_{i+\frac{1}{2}} = 0 \) and
\[ \tilde{C}_{i-\frac{1}{2}} = \frac{\lambda}{p_i^{n+1}} \left[ (1 - \frac{\tilde{\sigma}_{i-1}^u}{u_i^n - u_{i-1}^n}) \max(\hat{F}_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n}), 0 \right] + \frac{\tilde{\sigma}_i^u \max(F_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n})}{(u_i^n - u_{i-1}^n)}, \]
which implies that
\[ p_i^{n+1} (1 - \tilde{C}_{i-\frac{1}{2}}) \geq p_i^{n+1} - \lambda \hat{F}_{i-\frac{1}{2}}^{\rho} - \frac{1}{2} \lambda \hat{F}_{i+\frac{1}{2}}^{\rho} = p_i^n - \frac{3}{2} \lambda \hat{F}_{i+\frac{1}{2}}^{\rho} = \frac{1}{2} \rho_{i+\frac{1}{2}}^{L} \left(1 - 3\lambda g(u_{i+\frac{1}{2}}^L) \right) + \frac{1}{2} \rho_{i-\frac{1}{2}}^{R} \geq 0, \]
under CFL-like condition (4.10) with \( cfl = \frac{1}{3} \).

(ii) When \( \hat{F}_{i-\frac{1}{2}}^{\rho} + \hat{F}_{i+\frac{1}{2}}^{\rho} \leq 0 \)

Then, \( \hat{F}_{i+\frac{1}{2}}^{\rho} = u_{i+\frac{1}{2}}^{R} + \rho_{i+\frac{1}{2}}^{R} \). We have \( \tilde{C}_{i-\frac{1}{2}} = 0 \) and
\[ \tilde{D}_{i+\frac{1}{2}} = -\frac{\lambda}{p_i^{n+1}} \left[ (1 - \frac{\tilde{\sigma}_{i+1}^u}{u_{i+1}^n - u_i^n}) \min(\hat{F}_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n}), 0 \right] + \frac{\tilde{\sigma}_i^u \min(F_i^{\rho} + \frac{\tilde{\sigma}_i^u}{u_i^n})}{(u_{i+1}^n - u_i^n)}, \]
The proof is similar to previous case.

(iii) When \( \hat{F}_{i-\frac{1}{2}}^{\rho} \geq 0, \hat{F}_{i+\frac{1}{2}}^{\rho} \leq 0 \)

Then, \( \hat{F}_{i-\frac{1}{2}}^{\rho} = u_{i-\frac{1}{2}}^{L} + \rho_{i-\frac{1}{2}}^{L} \), \( \hat{F}_{i+\frac{1}{2}}^{\rho} = u_{i+\frac{1}{2}}^{R} - \rho_{i+\frac{1}{2}}^{R} \).
\[ \tilde{D}_{i+\frac{1}{2}} = -\frac{\lambda}{p_i^{n+1}} \left(1 - \frac{\tilde{\sigma}_{i+1}^u}{u_{i+1}^n - u_i^n} \right) \hat{F}_i^{\rho} \leq -\frac{\lambda \hat{F}_{i+\frac{1}{2}}^{\rho}}{p_i^{n+1}}, \]
\[ \tilde{C}_{i-\frac{1}{2}} = \frac{\lambda}{p_i^{n+1}} \left(1 - \frac{\tilde{\sigma}_{i-1}^u}{u_i^n - u_{i-1}^n} \right) \hat{F}_i^{\rho} \leq \frac{\lambda \hat{F}_{i-\frac{1}{2}}^{\rho}}{p_i^{n+1}}, \]
which implies that
\[
\tilde{p}^{n+1}_i (1 - \tilde{C}_{i - \frac{1}{2}} - \tilde{D}_{i + \frac{1}{2}}) \geq \tilde{p}^{n+1}_i - \lambda \tilde{F}^\rho_{i - \frac{1}{2}} + \lambda \tilde{F}^\rho_{i + \frac{1}{2}} = \tilde{p}^n_i \geq 0
\]

(iv) When \( \tilde{F}^\rho_{i - \frac{1}{2}} \leq 0, \tilde{F}^\rho_{i + \frac{1}{2}} \geq 0 \)

Then, \( \tilde{F}^\rho_{i - \frac{1}{2}} = u_{i - \frac{1}{2}}^R - \rho_{i - \frac{1}{2}}^R, \tilde{F}^\rho_{i + \frac{1}{2}} = u_{i + \frac{1}{2}}^L + \rho_{i + \frac{1}{2}}^L. \) We have
\[
\tilde{D}_{i + \frac{1}{2}} = -\frac{\lambda}{\tilde{p}^{n+1}_i} \frac{\sigma^u_i \min(\tilde{F}^\rho_{i - \frac{1}{2}}, 0)}{(\tilde{u}^n_i - \tilde{u}^n_{i-1})} \leq -\frac{\lambda \tilde{F}^\rho_{i + \frac{1}{2}}}{2\tilde{p}^{n+1}_i},
\]
and
\[
\tilde{C}_{i - \frac{1}{2}} = \frac{\lambda}{\tilde{p}^{n+1}_i} \frac{\sigma^u_i \max(\tilde{F}^\rho_{i + \frac{1}{2}}, 0)}{(\tilde{u}^n_i - \tilde{u}^n_{i-1})} \leq \frac{\lambda \tilde{F}^\rho_{i + \frac{1}{2}}}{2\tilde{p}^{n+1}_i}.
\]

Now,
\[
\tilde{p}^{n+1}_i (1 - \tilde{C}_{i - \frac{1}{2}} - \tilde{D}_{i + \frac{1}{2}}) \geq \tilde{p}^{n+1}_i - \lambda \tilde{F}^\rho_{i - \frac{1}{2}} + \lambda \tilde{F}^\rho_{i + \frac{1}{2}}
\]
\[
= \tilde{p}^n_i - \frac{3}{2} \lambda \tilde{F}^\rho_{i + \frac{1}{2}} + \frac{3}{2} \lambda \tilde{F}^\rho_{i - \frac{1}{2}}
\]
\[
= \frac{1}{2} \tilde{\rho}^L_{i + \frac{1}{2}} (1 - 3\lambda u_{i + \frac{1}{2}}) + \frac{1}{2} \tilde{\rho}^R_{i - \frac{1}{2}} (1 + 3\lambda u_{i - \frac{1}{2}}) \geq 0
\]
under CFL-like condition (4.10) with \( \text{cfl} = \frac{1}{3}. \)

Case 2. For the remaining cases on \( u_{i \pm \frac{1}{2}} \), using (4.11), we have
\[
\tilde{u}^{n+1}_i = \tilde{u}^n_i (1 - \tilde{C}_{i - \frac{1}{2}} - \tilde{D}_{i + \frac{1}{2}}) + \tilde{u}^n_{i-1} \tilde{C}_{i - \frac{1}{2}} + \tilde{u}^n_{i+1} \tilde{D}_{i + \frac{1}{2}}
\]
(4.13)
where
\[
\tilde{C}_{i - \frac{1}{2}} = -\frac{\tilde{F}^w_{i - \frac{1}{2}} - \tilde{F}^\rho_{i - \frac{1}{2}} \tilde{u}^n_i}{\tilde{p}^{n+1}_i (\tilde{u}^n_i - \tilde{u}^n_{i-1})}, \quad \tilde{D}_{i + \frac{1}{2}} = -\frac{\tilde{F}^w_{i + \frac{1}{2}} - \tilde{F}^\rho_{i + \frac{1}{2}} \tilde{u}^n_i}{\tilde{p}^{n+1}_i (\tilde{u}^n_i - \tilde{u}^n_{i-1})}.
\]

We prove for the case when \( \tilde{u}^n_i \geq \max(\tilde{u}^n_{i-1}, \tilde{u}^n_{i+1}) \). Since \( \tilde{u}^n_i \geq \max(\tilde{u}^n_{i-1}, \tilde{u}^n_{i+1}) \), we have \( \tilde{u}^n_i = 0 \) and hence \( \tilde{u}^n_i = u_{i + \frac{1}{2}}^L = u_{i - \frac{1}{2}}^R \geq 0 \). Let us further assume that \( u_{i - \frac{1}{2}}^L \geq 0, u_{i + \frac{1}{2}}^L = u_{i - \frac{1}{2}}^R = \tilde{u}^n_i, \) and \( u_{i + \frac{1}{2}}^R \geq 0 \). Then, using assumption (1.7), we have \( g(u_{i - \frac{1}{2}}^L) \geq 0, g(u_{i - \frac{1}{2}}^R) = g(u_{i - \frac{1}{2}}) = g(\tilde{u}^n_i) = 0 \) and \( g(u_{i + \frac{1}{2}}^R) \leq 0 \).

Also,
\[
u_{i + \frac{1}{2}}^R \leq 0 \implies \tilde{u}^n_i \leq 0,
\]
which we prove for completeness. Let us assume to the contrary that \( \tilde{u}^n_i > 0 \). Then, since \( u_{i + \frac{1}{2}}^R = \tilde{u}^n_i = \tilde{u}^n_i - \tilde{u}^n_i = 0 \), \( \tilde{u}^n_i \) should necessarily be positive. Now, if \( \tilde{u}^n_{i+2} \geq \tilde{u}^n_i, \) then \( \tilde{u}^n_{i+1} = 0 \) and if \( \tilde{u}^n_{i+2} < \tilde{u}^n_i, \) then \( \tilde{u}^n_{i+1} < 0 \). Both are contradictions to the fact that \( \tilde{u}^n_{i+1} > 0 \).

Hence, \( \tilde{u}^n_{i+1} \leq 0 \) Hence
\[
\tilde{F}^\rho_{i + \frac{1}{2}} = 0, \tilde{F}^w_{i + \frac{1}{2}} = \max \left( u_{i + \frac{1}{2}}^L + g(u_{i + \frac{1}{2}}) \rho_{i + \frac{1}{2}}^L, u_{i + \frac{1}{2}}^L - g(u_{i + \frac{1}{2}}) \rho_{i + \frac{1}{2}}^R \right),
\]
and
\[
\tilde{F}^\rho_{i - \frac{1}{2}} = g(u_{i + \frac{1}{2}}) u_{i - \frac{1}{2}}^L \rho_{i + \frac{1}{2}}^L, \tilde{F}^w_{i - \frac{1}{2}} = u_{i - \frac{1}{2}}^L \tilde{F}^\rho_{i - \frac{1}{2}}^L.
\]
Now, keeping these inferences on $\pi^n_i, \pi^n_{i+1}$ in mind, we prove that $\bar{C}_{i-\frac{1}{2}}, \bar{D}_{i+\frac{1}{2}}$ are non-negative and $\pi^{n+1}_i (1 - \bar{C}_{i-\frac{1}{2}} - \bar{D}_{i+\frac{1}{2}}) \geq 0$. Let us first consider $\bar{C}_{i-\frac{1}{2}}$.

$$
\bar{C}_{i-\frac{1}{2}} = -\lambda \bar{F}^{\rho}_{i-\frac{1}{2}} \frac{\bar{u}^{L}_{i-\frac{1}{2}} - \pi^n_i}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = -\lambda \bar{F}^{\rho}_{i-\frac{1}{2}} \frac{\pi^n_i + \sigma^n_{i-1} - \pi^n_i}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = \lambda \frac{\bar{F}^{\rho}_{i-\frac{1}{2}}}{\bar{\rho}^{n+1}_i} \left( 1 - \frac{\sigma^n_{i-1}}{\bar{u}^n_i - \bar{u}^n_{i-1}} \right),
$$

which is positive under the condition $K = 1 - \frac{\sigma^n_{i-1}}{\bar{u}^n_i - \bar{u}^n_{i-1}} \geq 0$, which is always true. Also,

$$
\bar{D}_{i+\frac{1}{2}} = -\lambda \bar{F}^{\rho}_{i+\frac{1}{2}} \frac{\bar{u}^{L}_{i+\frac{1}{2}} - \pi^n_i}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = -\lambda \frac{\bar{F}^{\rho}_{i+\frac{1}{2}}}{\bar{\rho}^{n+1}_i} \geq 0
$$

since $\bar{u}^{n+1}_i \leq 0, \bar{u}^n_i > 0$ and $\bar{F}^{\rho}_{i+\frac{1}{2}} \geq 0$. Finally, we prove that $\bar{C}_{i-\frac{1}{2}} + \bar{D}_{i+\frac{1}{2}} \leq 1$.

$$
\bar{\rho}^{n+1}_i = \bar{\rho}^n_i - \lambda (\bar{F}^{\rho}_{i-\frac{1}{2}} - \bar{F}^{\rho}_{i+\frac{1}{2}}) = \bar{\rho}^n_i - \lambda (-\bar{F}^{\rho}_{i-\frac{1}{2}}) = \bar{\rho}^n_i + \bar{C}_{i-\frac{1}{2}} \bar{\rho}^{n+1}_i
$$

Adding $-\bar{D}_{i+\frac{1}{2}} \bar{\rho}^{n+1}_i$ on both sides, we have

$$
\bar{\rho}^{n+1}_i \left( 1 - \bar{C}_{i-\frac{1}{2}} \bar{D}_{i+\frac{1}{2}} \right) = \bar{\rho}^n_i + \lambda \frac{\bar{F}^{\rho}_{i+\frac{1}{2}} - \bar{F}^{\rho}_{i-\frac{1}{2}}}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = \bar{\rho}^n_i + \lambda \frac{\bar{F}^{\rho}_{i+\frac{1}{2}}}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})}
$$

Now, let us assume that

$$
\bar{F}^{\rho}_{i+\frac{1}{2}} = u^{L}_{i+\frac{1}{2}} g(u^{L}_{i+\frac{1}{2}}) \rho_{i+\frac{1}{2}} \rho_{i+\frac{1}{2}} = \pi^n_i g(\pi^n_i) \rho_{i+\frac{1}{2}}
$$

Then,

$$
\bar{\rho}^{n+1}_i \left( 1 - \bar{C}_{i-\frac{1}{2}} \bar{D}_{i+\frac{1}{2}} \right) = \bar{\rho}^n_i + \lambda \frac{\bar{F}^{\rho}_{i+\frac{1}{2}}}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = \bar{\rho}^n_i + \lambda \frac{\pi^n_i g(\pi^n_i) \rho_{i+\frac{1}{2}}}{\bar{\rho}^{n+1}_i (\bar{u}^n_i - \bar{u}^n_{i-1})} = \frac{1}{2} \rho^{L}_{i+\frac{1}{2}} \left( 1 - 2 \lambda g(\pi^n_i) \frac{\pi^n_i}{\bar{u}^n_i - \bar{u}^n_{i-1}} \right) + \frac{1}{2} \rho^{R}_{i+\frac{1}{2}} \geq 0
$$

under CFL-like condition (4.10) and using the fact that $\frac{\pi^n_i}{\bar{u}^n_i - \bar{u}^n_{i+1}} \leq 1 \iff \pi^n_{i+1} \leq 0$. Hence, we get

$$
\bar{\rho}^{n+1}_i \left( 1 - \bar{C}_{i-\frac{1}{2}} \bar{D}_{i+\frac{1}{2}} \right) > 0,
$$

and since $0 < K < 1$, we have

$$
\bar{\rho}^{n+1}_i \left( 1 - \bar{C}_{i-\frac{1}{2}} \bar{D}_{i+\frac{1}{2}} \right) > \bar{\rho}^{n+1}_i \left( 1 - \bar{C}_{i-\frac{1}{2}} \bar{D}_{i+\frac{1}{2}} \right) > 0.
$$

Other cases follow similarly.
We have now obtained that $\bar{\rho}_i^{n+1} \left(1 - \tilde{C}_{i-\frac{1}{2}} - \tilde{D}_{i+\frac{1}{2}} \right) \geq 0$, $\tilde{C}_{i-\frac{1}{2}} \geq 0$ and $\tilde{D}_{i+\frac{1}{2}} \geq 0$. Since $\bar{\rho}_i^{n+1} > 0$ by previous lemma under CFL-condition (4.10), we have $0 \leq \tilde{C}_{i-\frac{1}{2}} + \tilde{D}_{i+\frac{1}{2}} \leq 1$. Then, using incremental form (4.12) and (4.13), we get that $\pi_i^{n+1}$ is bounded by bounds of $\pi_i^n$. This completes the proof. □

When $\pi_k^n = 0$ for all $k$, scheme (3.10) reduces to first-order scheme (3.3) and hence the proof of Lemma 4.9 gives the proof of Lemma 4.3 under CFL-like condition (4.10) with $\text{cfl} = 1$. Since (3.11)–(3.13) is a convex combination of Euler forward, hence it also preserves the properties of the system. We now present some numerical experiments to show the performance of the scheme.

5. Numerical experiments

The following experiments display the performance of first and higher-order (DDF) scheme for capturing the solutions of Euler system with varying pressure and sources, both in one and two dimensions. The scheme will be tested on the nontrivial test cases from the existing literature, and it will be seen that it can capture shocks, both classical and $\delta$-shocks as well as rarefactions efficiently.

5.1. One-dimensional experiments

Experiment 1: Moving shocks of (PGD): The first experiment is to compare the performance of (DDF) scheme with the schemes proposed in [8,19,30] for the data producing moving $\delta$-shock. We take the initial data as Riemann data with $(\rho_l, \rho_r) = (1, 0.25)$, $(u_l, u_r) = (1, 0)$ with the domain $[-0.5, 0.5]$, $T = 0.4998$, $h = 0.025$, $\text{cfl} = 0.5$. Limiters such as Minmod and Superbee have been used for reconstruction of $\rho$, $u$ for constructing the higher-order version. It can be observed from Fig. 7 that both the limiters capture the right location of

Fig. 7. (PGD); comparison of first- and second-order (DDF) scheme: first-order numerical solution $U$ (blue dashed curve), higher-order numerical solution $U$ with Minmod Limiter (asterisk); higher-order numerical solution $U$ with Superbee Limiter (green dashed curve); exact solution $u$ (solid curve) (color figure online)
$\delta$-shock, though Superbee limiter performs slightly better than the others. As expected, the first-order scheme produces $\delta$-shock with less height and more smearing around the location of $\delta$- shock as compared to the higher order. Figure 8 shows the graph of the primitive of the approximate solution $\rho_h(T)$ and indicates that the scheme captures the expected weight of the $\delta$-shock, i.e., $u_\delta T = 0.3325$.

Experiment 2: Mixed Type Solutions of (PGD) system: This experiment will show the performance of the first-order and higher-order (DDF) schemes to capture the solutions of (PGD) system, which can have both shocks and vacuum. The first initial data under consideration are given by:

$$
\rho_0(x) = 0.5, \, u_0(x) = \begin{cases} 
-0.5 & \text{if } x < -0.5, \\
0.4 & \text{if } -0.5 < x < 0, \\
0.4 - x & \text{if } 0 < x < 0.5, \\
-0.4 & \text{if } x > 0.
\end{cases}
$$

Let the domain be $[-1, 1]$, $T = 0.4998$, $h = 0.025$. It can be observed that the first-order (DDF) scheme produces a hump in the solution in $\rho$, which is resolved by the higher-order version by all the limiters, though Superbee limiter, as in the case of $\delta$- shock data, gives the best performance of all. The other schemes in literature, for example, the ones in [8, 10] also exhibit same behavior (Fig. 9).

Next, we consider an example with the vacuum solutions. Consider (PGD) on the domain $[-0.5, 0.5], T = 0.5, M = 200$ with

$$
\rho_0(x) = 0.5, \, u_0(x) = \begin{cases} 
-0.5 & \text{if } x < 0, \\
0.4 & \text{if } x > 0.
\end{cases}
$$
Fig. 9. (PGD): comparison of first- and second-order (DDF) scheme: first-order numerical solution $U$ (blue dashed curve), higher-order numerical solution $U$ with Minmod Limiter (asterisk); higher-order numerical solution $U$ with Superbee Limiter (green dashed curve); exact Solution $u$ (solid curve) (color figure online)

The exact solution for the density at $T = 0.5$ is given by:

$$\rho(x, T) = \begin{cases} 
0.5 & \text{if } x < -0.25, \\
0 & -0.25 < x < 0.2, \\
0.5 & \text{else.}
\end{cases}$$

Figure 10 shows the logarithm of minimal density captured by the first-order and higher-order (DDF) scheme over time, which are of the order $-24$ and $-128$, respectively, and indicates that the scheme is able to capture the vacuum well. Since the exact solution is known, we now verify the convergence rates of the scheme. Figure 10 shows that the slopes of the graphs of the logarithm of $L^1$ error versus log($h$) are around 0.72 and 1 for first- and higher-order (DDF) schemes, respectively, which indicate that the convergence rates are near the expected convergence rates for discontinuous solutions.
Fig. 10. Logarithmic of minimal density for vacuum solutions (left); logarithmic $L^1$ error versus logarithmic space discretization $h$ in vacuum experiment (right); first order (blue asterisk), second order (red asterisk) (color figure online)

Experiment 3: Interaction of two singular shocks for (PGD) : This experiment is to show the performance of (DDF) scheme for the piecewise constant initial data which develop $\delta$-shock as time progresses. These data have been used to describe (PGD) system as a model of collision of two semi-infinite clouds of dust from the moment of impact onward in [19], where the emerged $\delta$-shock is called as delta double-rarefaction when both clouds have been fully accreted. We demonstrate the performance of the scheme to capture this
phenomenon by numerically solving (PGD) system subject to the following initial data

\[ \rho(x, -1) = \begin{cases} 
2 & \text{if } -2 < x < -1, \\
1 & \text{if } 1 < x < 5, \\
0 & \text{otherwise}, 
\end{cases} \quad u(x, -1) = \begin{cases} 
1 & \text{if } -2 < x < -1, \\
-1 & \text{if } 1 < x < 5, \\
0 & \text{otherwise}, 
\end{cases} \]

where the initial data can be seen as collision of 2 dust clouds which are of length 1 and 4, which move in opposite directions. Let the domain be \([-2.5, 5.5]\), \(h = 0.01\). Figure 11 plots the numerical densities at times \(T = -1, -0.5, 0.5, 1.5, 3.5\), and it can be seen that the scheme captures the shocks at the expected locations. All the figures have been plotted on the same vertical scale with peak values of the computed density clipped near in the region of the delta functions. Moreover, the weights of the \(\delta\)-shocks captured by scheme described in [19] at \(T = 0.5, 1.5, 3.5\) and \(T = 6\) are 20.2, 58.2, 51.6 and 29.8, respectively, while for first-order (DDF) scheme, they are 73.9, 301.2, 93.14 and 40.3 and for higher-order (DDF), they are 88.2, 331.1, 157 and 99.7, respectively. The number of the cells, which have not been shown to display the clipped densities, is similar to those in [19] and around 4 and 11 on either side of the \(\delta\)-shock at times \(T = 3.5\) and \(T = 6\).

Experiment 4: Moving Shocks of (GPGD): This experiment is to show the performance of the first-order (DDF) scheme to capture entropic \(\delta\)-shocks for moving shocks of (GPGD). With \(g(u) = u^3\) and initial data as Riemann data \((\rho_l, \rho_r) = (1, 0.25), (u_l, u_r) = (1, 0)\) on the domain \([-0.5, 0.5]\), with \(T = 0.4988\), \(h = 0.025\), cfl = 0.5, it can be seen in Fig. 12 that the scheme captures the entropic \(\delta\)-shock as the shock location lies between \(g(0)\) and \(g(1)\), which fits the results of [16,21].
Experiment 5: System (1.3)–(1.4) with $P = 0$, $S = \beta \rho$; This experiment is to display the performance of the first-order (DDF) scheme for capturing the solutions of (1.3)–(1.4) with zero pressure and nonzero source term $\beta \rho$. Like (PGD), this system is also nonstrictly hyperbolic and has double eigenvalue $u + \beta t$. It has been established in [25] that whenever $u_l > 0 > u_r$, it admits $\delta$-shock at $x(t) = v_\delta t + \frac{1}{2} \beta t^2$ with the weight $w(t) = w_\delta(t)$, where

$$v_\delta = \frac{\sqrt{\rho_l} u_l + \sqrt{\rho_r} u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}, \quad w_\delta = \sqrt{\rho_l \rho_r} [u],$$

and when $u_l < u_r$, it admits the following vacuum solution

$$(\rho, v)(x, t) = \begin{cases} 
(\rho_l, u_l) & \text{if } x < u_l t + \frac{1}{2} \beta t^2, \\
(\rho_r, u_r) & \text{if } x > u_r t + \frac{1}{2} \beta t^2 \\
\text{Vacuum} & \text{Otherwise.}
\end{cases}$$

To extend the first-order (DDF) scheme for this system, we follow the approach of [25] and write the system as

$$\rho_t + (\rho(v + \beta t))_x = 0,$$

$$w_t + (w(v + \beta t))_x = 0,$$

where $w = \rho v$. The first equation can be treated like (1.5) for given $v(x, t) + \beta t$. For given $\rho(x, t) > 0$, the function

$$w \mapsto \frac{w^2}{\rho(x, t)} + w \beta t,$$

is a convex function with the minimum at $-0.5 \rho \beta t$ and hence, the second equation can be treated in a similar way to (3.8). Hence, the numerical fluxes can be defined in the following way:

$$F_{i+\frac{1}{2}}^n = \left(F_{i+\frac{1}{2}}^{\rho n}, \max\left(q(\rho^n_i, \max(w^a_i, G^a_i)), q(\rho^n_{i+1}, \min(w^a_{i+1}, G^a_{i+1}))\right)\right)^T$$
where \( q(\rho, w) = \frac{w^2}{\rho}, \quad G_i^n = -0.5\rho_i^n \beta t^n \). We consider the domain \([-1.2, 1.2]\) with \( T = 0.4983 \), \( M = 500 \) and \( \beta = 0.5 \). It can be seen in Fig. 13 that the scheme is able to capture the expected weight and location of \( \delta \)-shock efficiently.

The performance of the scheme to capture vacuum solutions is displayed in Fig. 14, with \( \rho_l = \rho_r = 1, u_l = -2 \) and \( u_r = 1 \). It is clear that the scheme can capture the vacuum solutions well, where the minimal density is of the order \( 10\epsilon - 233 \).
Experiment 6: System (1.3)–(1.4) with $P \neq 0$, $S = 0$: We now extend (DDF) to capture the $\delta$-shock-type solutions of (1.3)–(1.4) where zero source, but with nonzero pressure. Since the first equation does not change, there is no change in its approximation. For the second equation, since the pressure term $P(\rho)$ is only a function of $\rho$, hence the second equation can still be considered as a convex-convex discontinuous flux for a given $\rho(x, t^n)$ and the numerical flux is given by:

$$F_{i+\frac{1}{2}}^{w, n} = \max \left( \frac{(\max(w^n_i, 0))^2}{\rho^n_i} + P(\rho^n_i), \frac{(\min(w^n_{i+1}, 0))^2}{\rho^n_{i+1}} + P(\rho^n_{i+1}) \right). \quad (5.1)$$

We now use this extension of (DDF) scheme to compute the numerical solutions of (CGD), which have nonzero $P$. We now exhibit the performance of the extended scheme for the physical systems described in the beginning of the article.

(a) $\delta$-shocks of (CGD): This system is of type (1.3) with $S = 0$, $P = s\rho^{-\alpha}$. It has been pointed out in [29] that whenever $u_l \geq u_r$, $0 < \alpha < 1$, this system admits $\delta$-shocks at the location $x(t) = u_\delta t$, where

$$w_\delta = \sqrt{\rho_l \rho_r [u]^2 - |\rho|^2 [P]}, u_\delta = \frac{[w]t + w_\delta}{|\rho|}.$$

Figure 15 shows the performance of extension of (DDF) scheme with $\alpha = 0.5$, $s = 5$, $M = 1000$, with initial data as Riemann data $\rho_l = 3$, $\rho_r = 1$, $u_l = 4$, $u_r = -4$ on the domain $[-2, 2]$ and at times $T_1 = 0.05$, $T_2 = 0.1996$.

It can be seen that the (DDF) scheme can capture the expected shock location for $\rho$ in contrast to the solutions displayed in [29, Fig. 4.1] where the shocks lag behind a few units. Also, the height of $\delta$-shock is more than those presented in [29, Fig. 4.1].

To further evaluate the efficiency of our scheme, we compare the numerical weight of the $\delta$-shock with the expected weight $w_\delta T$, where $w_\delta = 14.0081$ for the given data. Expected weights.
Table 1. (CGD): Location of δ-shock by (DDF) scheme

| Time | Exact shock location | Observed shock location |
|------|----------------------|-------------------------|
| $T_1$ | 0.0538               | 0.54                    |
| $T_2$ | 0.2170               | 0.226                   |

are hence given by 0.7004 and 2.7960. It can be seen in Figure 15 that the numerical weight matches well with the expected weight. The locations for numerical δ-shocks at $T_1$ and $T_2$ are compared with their exact locations in Table 1.

5.2. Two-dimensional experiments

We consider (PGD) system in multi-dimensions with different initial conditions and show the performance of higher-order (DDF) scheme with Superbee Limiter, used for $\rho, u$ and $v$. The first-order (DDF) scheme behaves in a similar manner, with some expected diffusion.

Experiment 1: Vacuum Solutions: We first consider the following test data, considered in [30]:

$$\rho(x, y, 0) = 0.5, (u, v)(x, y, 0) = \begin{cases} 
(0.3, 0.4) & \text{if } x > 0, y > 0, \\
(-0.4, 0.3) & \text{if } x < 0, y > 0, \\
(-0.3, -0.4) & \text{if } x < 0, y < 0, \\
(0.4, -0.3) & \text{if } x > 0, y < 0.
\end{cases}$$

Figure 16 shows the numerical density and the velocity vector field at the time $T = 0.1$. We can see that the vacuum solutions have been captured well and maintain non-negative density. The minimal density achieved by the higher-order (DDF) scheme is of the order of $10^{-4}$.

![Fig. 16. Numerical density (left) and velocity field (right) at $T = 0.1$]
Experiment 2: δ - shock formation due to particles moving toward center: We consider the following initial data considered in [30]:

\[
\rho(x, y, 0) = \frac{1}{100}, \ (u(x, y, 0), v(x, y, 0)) = -\frac{1}{10} (\cos(\theta), \sin(\theta)),
\]

where \(\theta\) is the polar angle.

It is clear from Figure 17 that a single δ-shock of height over 1 is formed at the origin, toward which all the particles are moving.

Experiment 3: δ - shock formation due to cloud collision: As in one dimension, we consider an example to capture the phenomenon of cloud collision in two dimensions. The initial conditions are taken such that \(v = 0, u \neq 0\) so as not to have any directional impact on the solutions, i.e., the particles do not change their directions and move on their own way.

For this purpose, initial data are taken to model two clouds with the same speed and density but having preassigned velocity to move in opposite directions. In particular, we consider the following initial data:

\[
(\rho(x, y, 0), u(x, y, 0), v(x, y, 0)) = \begin{cases} 
(1, 0.5, 0) & \text{if } x \in [-0.3, -0.2], y \in [-0.15, 0.05], \\
(1, -0.5, 0) & \text{if } x \in [0.2, 0.3], y \in [0.05, 0.15], \\
(0.1, 0, 0) & \text{otherwise.}
\end{cases}
\]

A part of each cloud is physically expected to entirely merge at \(T \approx 0.52\), and another part of the cloud moves further on its own way post this time. The results are compared with [17, Fig 2], where the highest densities achieved are 1.4, 8 and 10 at times \(T = 0.2, 0.52\) and 0.8. The results given by (DDF) scheme in Figs. 18 and 19 show that the density concentrations are well captured by the scheme with slightly higher concentrations than those captured by the relaxation schemes in [17]. As expected physically, there are no fluctuations in the \(y\)- direction since \(v = 0\). It can be seen from the velocity plot at time \(T = 0.8\), that a strong δ-shock is formed at the center due to collision of...
Fig. 18. Density contour plots (left) and velocity vector plots (right) at times $T = 0.2, 0.52, 0.8$
Fig. 19. Density three-dimensional plots $T = 0, 0.2, 0.52, 0.8$
clouds, but the remaining part of the clouds, keep their velocity as it is and move in their own directions.

Next, we plot the density and velocity profiles in Fig. 20 at $T = 0.52$ and $T = 0.8$ at the section $x = 0$ to see the behavior around the $\delta$-shock. It can be seen that the density and velocity are stable and do not have any nonphysical fluctuations.

6. Conclusion

The article proposes efficient numerical scheme for certain class of $2 \times 2$ hyperbolic systems, based on theory of scalar conservation laws with discontinuous flux, in one and multiple dimensions. The numerical approximations are analytically shown to preserve the expected physical properties of the system. In comparison with its counterparts, the scheme is relatively simpler, as it is based on Godunov-type approximation for each of the equations of the system. However, this decoupling strategy may not work for any general $2 \times 2$ hyperbolic system and its success may require a suitable compatibility between the two scalar conservation laws, namely (1.3) with a given $u(x,t)$ and (1.4) with a given $\rho(x,t)$, which we aim to explore in future.

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