A NON-LOCAL ONE-PHASE FREE BOUNDARY PROBLEM FROM OBSTACLE TO CAVITATION

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ABSTRACT. We consider a one-phase free boundary problem of the minimizer of the energy
\[ J_\gamma(u) = \frac{1}{2} \int_{(B^{n+1}_1)_+} y^{1-2s} |\nabla u(x, y)|^2 dxdy + \int_{B^n_1 \times \{y = 0\}} u^\gamma dx, \]
with constants 0 < s, γ < 1. It is an intermediate case of the fractional cavitation problem (as γ = 0) and the fractional obstacle problem (as γ = 1). We prove that the blow-up near every free boundary point is homogeneous of degree \( \beta = \frac{2s}{2-\gamma} \), and flat free boundary is \( C^{1,\theta} \) when γ is close to 0.

1. INTRODUCTION

In this paper, we prove free boundary properties for minimizers of the following energy
\[ J_\gamma(u) = \frac{1}{2} \int_{(B^{n+1}_1)_+} y^{1-2s} |\nabla u(x, y)|^2 dxdy + \int_{B^n_1 \times \{y = 0\}} u^\gamma dx, \]
with 0 < s, γ < 1, subject to \( u \geq 0 \). The first part of the energy is related to the extension of the fractional Laplacian operator, and the second one is considered as a penalty for the function \( u \) being greater than 0. The set \( \{u = 0\} \) only lies on \( \{y = 0\} \), and is non-trivial if \( u \) is small enough on \( \partial B^{n+1}_1 \cap \{y > 0\} \). The boundary of the set \( \{u > 0\} \) in the topology of \( \mathbb{R}^n \) is called the free boundary. And there is one important number \( \beta = \frac{2s}{2-\gamma} \), which is the critical exponent in the scaling of the energy.

This problem is a non-local analogue of the problem introduced in [1] by Alt and Philips, in which a free boundary problem of the energy functional \( \int_{B^n_1(0)} |\nabla u|^2 + |\max(u, 0)|^\gamma \) is discussed. We are now considering the case for the fractional Laplacian operator instead of Laplacian, and this is an intersection of one-phase free boundary problems and non-local integrodifferential operators. Heuristically, two limiting classical problems, one as \( \gamma \to 0 \) is the Bornuelli type one-phase free boundary problem from the minimization of \( J_0(u) = \frac{1}{2} \int |\nabla u|^2 + \chi_{\{u > 0\}} \), discussed by Caffarelli and Salsa in [2]; and the other one as \( \gamma \to 1 \) is the obstacle problem from the minimization of \( J_1(u) = \frac{1}{2} \int |\nabla u|^2 + \max(u, 0) \), discussed by Caffarelli in [3]. Analogues of both problems in the fractional cases are also discussed in [4][8][9][10] for the
Bornuelli type problems, and in [5, 7] for the thin obstacle problems. These are the inspirations for our minimization problem, which is an intermediate case of the fractional one-phase cavitation problem and obstacle problem.

There are some previous results on the properties of the minimizers of the energy $J_\gamma(u)$. In [11] by Ray Yang, optimal regularity is proved, that the minimizer is $C^\beta$ continuous if $\beta < 1$ and is $C^\alpha$ continuous for any $\alpha < 1$, if $\beta \geq 1$. And the minimizer along the set $\{y = 0\}$ is $C^\beta$ continuous if $\beta < 1$ and is $C^{1,\beta-1}$ continuous if $\beta \geq 1$. Non-degeneracy of the minimizer is also proved, that $\sup_{x \in B_{r}(x_0)} u(x,0) \geq C(n,s,\gamma)r^\beta$ if $x_0$ is a free boundary point.

This paper is divided into two parts. In the first part, we use Weiss type monotonicity formula introduced in [13] to prove blow-up profiles are homogeneous of degree $\beta = \frac{2s}{n-2s}$, the critical exponent, and the blow-up limit is unique regardless of subsequences, using Monneau type monotonicity formula introduced in [14]. We also prove that the half-plane solution is unique up to rotation. The other part is to prove there exists a small constant $\gamma_0 > 0$, such that for each $0 < \gamma < \gamma_0$, flatness condition of the free boundary implies $C^{1,\theta}$ regularity, applying the method introduced in [8] by De Silva, Savin and Sire.

We define the scaling of the minimizer near a free boundary point $(x_0,0)$,

$$u_R(x,y) = \frac{u(R(x-x_0)+x_0,Ry)}{R^\beta},$$

and the blow-up of the minimizer at a point $(x_0,0)$ on the free boundary is the limit of $u_R$ as $R \to 0$.

The fractional Laplacian is a non-local integral operator defined as

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}dy,$$

$$C_{n,s} = \frac{4^s \Gamma(n/2+s)}{\pi^{n/2} \Gamma(-s)},$$

with a corresponding nonlocal energy

$$E(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}dydx$$

which is hard to handle. So an extension of the function to one extra dimension is introduced by Caffarelli and Silvestre in [6], transforming a non-local equation on $\mathbb{R}^n$ to an elliptic equation on the upper half space $\mathbb{R}^n \times \mathbb{R}^+$ with a Neumann boundary condition. Consider a fractional Laplacian equation $(-\Delta)^s u(x) = f(x)$ in $\mathbb{R}^n$, and $u \in H^s(\mathbb{R}^n)$. Define an extension $U(x,y)$ in $\mathbb{R}^n \times \mathbb{R}^+$ by a Poisson kernel in Section 2.4 in [6], such that $U(x,0) = u(x)$.
and the extension $U(x, y)$ satisfies the following equations with Neumann boundary condition,

\begin{equation}
\text{div}(y^{1-2s} \nabla U(x, y)) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+
\end{equation}

and

\begin{equation}
\lim_{y \to 0^+} y^{1-2s} \partial_y U(x, y) = -C_{n,s}(-\Delta)^su(x) \quad \text{in} \quad \mathbb{R}^n.
\end{equation}

And there is a natural energy

\[ E(U) = \int_{\mathbb{R}^n \times \mathbb{R}^+} y^{1-2s}|\nabla U(x, y)|^2 \, dx \, dy \]

corresponding to the Euler-Lagrange equation (1.1).

From the Euler-Lagrange equation of the energy

\[ J_\gamma(u) = \frac{1}{2} \int_{(B^{n+1}_R)^+ \times \{y=0\}} y^{1-2s}|\nabla u(x, y)|^2 \, dx \, dy + \int_{B^n \times \{y=0\}} u^\gamma \, dx, \]

the minimizer satisfies a second order PDE,

\[ \text{div}(y^\alpha \nabla u) = 0 \]

in the upper half ball $(B^{n+1}_1)^+$ in a distributional sense, and

\[ \lim_{y \to 0^+} y^\alpha \partial_y u(x, y) = \gamma u^{\gamma-1}(x, 0) \]

on $\{u > 0\} \cap \{y = 0\}$. In the paper we denote $\alpha = 1 - 2s$.

2. Preliminaries

Throughout this paper, we have the following notations. A point in the upper half space is $X = (x, y) \in (\mathbb{R}^{n+1})^+ = \mathbb{R}^n \times \mathbb{R}^+$; the upper half ball of radius $R$ centered at $0$ is $(B^{n+1}_R)^+ = \{(x, y) \in (\mathbb{R}^{n+1})^+, |(x, y)| < R, y > 0\}$, its boundary in $\{y > 0\}$ is $(\partial B^{n+1}_R)^+ = \{(x, y) \in (\mathbb{R}^{n+1})^+, |(x, y)| = R, y > 0\}$, and its boundary in $\{y = 0\}$ is $B^n_R = \{(x, y) \in (\mathbb{R}^{n+1})^+, |x| < R, y = 0\}$. Sometimes, we denote $B^n_1$ as $(B^{n+1}_1)^+$ for simplification.

We define $\alpha = 1 - 2s$ with $s \in (0, 1)$ the order of fractional Laplacian, and $\beta = \frac{2s}{2-\gamma}$ is the critical scaling exponent with $0 < \gamma < 1$.

We denote the energy

\[ J_\gamma(u) = J_\gamma(u) = \frac{1}{2} \int_{(B^{n+1}_1)^+ \times \{y=0\}} y^{1-2s}|\nabla u(x, y)|^2 \, dx \, dy + \int_{B^n \times \{y=0\}} u^\gamma \, dx. \]

The set $\{u = 0\}$ which necessarily lies on $\{y = 0\}$ is called the contact set of $u$, and we denote the free boundary $F(u)$ as the interface between the set $\{u > 0\} \cap \{y = 0\}$ and the contact set.
2.1. **Scaling of the problem.** Define $u(\lambda X) = \lambda^{-\beta} u(X)$, $X = (x, y) \in (B^{n+1}_{\lambda^{-1}})^+$, then by the change of variables,

$J(B_{\lambda^{-1}}, u_\lambda) = \frac{1}{2} \int_{(B^{n+1}_{\lambda^{-1}})^+} \lambda^{-2\beta} |\nabla u(\lambda x, \lambda y)|^2 dxdy$

$+ \int_{(B^{n+1}_{\lambda^{-1}})^+ \times \{y=0\}} \lambda^{-\beta\gamma} u(\lambda x) dx$

$= \frac{1}{2} \lambda^{-n+2-2\beta-\alpha} \int_{(B^{n+1}_1)^+} y^\alpha |\nabla u(x, y)|^2 dxdy$

$+ \lambda^{-n+1-\beta\gamma} \int_{B_1^+ \times \{y=0\}} u^\gamma dx.$

We require two equal exponents of $\lambda$, and this leads to

$\beta = \frac{2s}{2 - \gamma} = \frac{1 - \alpha}{2 - \gamma},$

and thus

$J((B^{n+1}_{\lambda^{-1}})^+, u_\lambda) = \lambda^{-n+1-\beta\gamma} J((B^{n+1}_1)^+, u).$

So if $u$ is a minimizer for the energy in $(B^{n+1}_1)^+$, then $u_\lambda$ is a minimizer in $(B^{n+1}_{\lambda^{-1}})^+$.

2.2. **Function space.** We are considering minimizers of energy

$J_\gamma(u) = \frac{1}{2} \int_{(B^{n+1}_1)^+} y^\alpha |\nabla u(x, y)|^2 dxdy + \int_{B_1^+ \times \{y=0\}} u^\gamma dx$

in the space $H^1(y^\alpha, B^+_1)$, which is a weighted $H^1$ space, with norm

$||u||_{H^1(y^\alpha, B^+_1)} = \left( \int_{(B^{n+1}_1)^+} y^\alpha (|\nabla u|^2 + u^2) dxdy \right)^{1/2},$

and seminorm

$[u]_{H^1(y^\alpha, B^+_1)} = \left( \int_{(B^{n+1}_1)^+} y^\alpha |\nabla u|^2 dxdy \right)^{1/2}.$

From the extension theorem of Caffarelli and Silverstre in [8], trace of any $H^1(y^\alpha, B^+_1)$ function lies in $H^s(B^n_1(0))$, and Sobolev embedding makes sure it also lies in $L^2(B^n_1(0))$.

3. **Blow-ups are homogeneous of degree $\beta$**

In this section, we will use Weiss type monotonicity formula to prove the blow-up of the energy minimizer at every free boundary point is homogeneous of degree $\beta$.

If $u$ is a minimizer of the energy $J(u)$, then it satisfies

(3.1) \hspace{1cm} \text{div}(y^\alpha \nabla u) = 0 \hspace{0.5cm} \text{in} \hspace{0.5cm} (B^{n+1}_1)^+,$
(3.2) \[ \lim_{y \to 0} y^\alpha \partial_y u(x, y) = \gamma u^{\gamma - 1}(x, 0) \text{ on } B^n_1 \cap \{u > 0\}. \]

Here we introduce a boundary adjusted energy and define

\[
W(R, u) = R^{-(n-1+\frac{2-\alpha}{2-\gamma})} \int_{(B^{n+1}_R)^+} y^\alpha |\nabla u|^2 dxdy + 2R^{-(n-1+\frac{2-\alpha}{2-\gamma})} \int_{B^n_R} u^\gamma dx - \beta R^{-(n+\frac{2-\alpha}{2-\gamma})} \int_{(\partial B^{n+1}_R)^+} y^\alpha u^2 d\sigma.
\]

This energy is invariant under scaling,

\[
W(R\rho, u) = W(\rho, u_R),
\]

where

\[ u_R(x, y) = \frac{u(Rx, Ry)}{R^\beta}. \]

**Theorem 3.1 (Weiss type monotonicity formula).** If \( u \) is a minimizer of \( J(u) \) and 0 is a free boundary point, then the boundary adjusted energy \( W(R, u) \) satisfies the monotonicity formula

\[
\frac{d}{dR} W(R, u) = 2R^{-(n-1+\frac{2-\alpha}{2-\gamma})} \int_{(B^{n+1}_R)^+} y^\alpha (u_\nu - \beta u R^\beta)^2 d\sigma.
\]

Moreover, when \( \frac{d}{dR} W(R, u) = 0 \), it is equivalent that

\[ 0 = <(x, y), \nabla u(x, y) > - \beta u(x, y) = \frac{d}{d\rho}|_{\rho=1} \frac{u(\rho x, \rho y)}{\rho^\beta} \]

a.e. on \((\partial B^{n+1}_1)^+\), which means \( u \) is homogeneous of degree \( \beta \).

**Proof of Weiss type monotonicity formula.** If \( u \) is a minimizer of the energy \( J(u) \), then it satisfies \( \text{div}(y^\alpha \nabla u) = 0 \), \( \text{div}(y^\alpha u \nabla u) = y^\alpha |\nabla u|^2 \) in \((B^{n+1}_1)^+\) and \( \lim_{y \to 0} y^\alpha \partial_y u(x, y) = \gamma u^{\gamma - 1}(x, 0) \) on \( B^n_1 \cap \{u > 0\} \). And the following equalities are obtained:

\[
\int_{(B^{n+1}_R)^+} y^\gamma |\nabla u|^2 dxdy = \int_{(\partial B^{n+1}_R)^+} y^\alpha uu_\nu d\sigma - \gamma \int_{B^n_R} u^\gamma dx;
\]

\[
\int_{B^n_R} <x, \nabla u^\gamma > dx = R \int_{\partial B^n_R} u^\gamma d\sigma - n \int_{B^n_R} u^\gamma dx;
\]

\[
\frac{d}{dR} \left( \int_{(\partial B^{n+1}_R)^+} y^\alpha u^2 d\sigma \right) = \frac{n + \alpha}{R} \int_{(\partial B^{n+1}_R)^+} y^\alpha u^2 d\sigma + 2 \int_{(\partial B^{n+1}_R)^+} y^\alpha uu_\nu d\sigma;
\]
(n + α - 1) \int_{(B^n_{R} + 1)^+} y^\alpha |\nabla u|^2 dx dy \\
(3.9) = R \int_{(\partial B^n_{R} + 1)^+} y^\alpha (|\nabla u|^2 - 2u_\nu^2) d\sigma + 2 \int_{B^n_R} <x, \nabla u^\gamma > dx \\
= R \int_{(\partial B^n_{R} + 1)^+} y^\alpha (|\nabla u|^2 - 2u_\nu^2) d\sigma + 2R \int_{\partial B^n_{R}} u^\gamma d\sigma - 2n \int_{B^n_{R}} u^\gamma dx.

Calculate derivative of $W(R,u)$ with respect to $R$ and we can get:

$$R^{n+\frac{2-\alpha}{2-\gamma}} dR W(R,u) = -(n - 1 + \frac{2 - \alpha \gamma}{2 - \gamma}) \int_{(B^n_{R} + 1)^+} y^\alpha |\nabla u|^2 dx dy \quad (I_1)$$

$$+ R \int_{(\partial B^n_{R} + 1)^+} y^\alpha |\nabla u|^2 d\sigma \quad (I_2)$$

$$- 2(n + 2 - \alpha \gamma) \int_{B^n_{R}} u^\gamma dx \quad (I_3)$$

$$+ 2R \int_{\partial B^n_{R}} u^\gamma d\sigma \quad (I_4)$$

$$+ \beta(n + \frac{2 - \alpha \gamma}{2 - \gamma})R^{-1} \int_{(\partial B^n_{R} + 1)^+} y^\alpha u^2 d\sigma \quad (I_5)$$

$$- \beta \frac{n + \alpha}{R} \int_{(\partial B^n_{R} + 1)^+} y^\alpha u^2 d\sigma \quad (I_6)$$

$$- 2\beta \int_{(\partial B^n_{R} + 1)^+} y^\alpha u_\nu d\sigma. \quad (I_7)$$

Apply (3.9) and (3.6), then

$$R^{n+\frac{2-\alpha}{2-\gamma}} [(I_1) + (I_2)] = 2R \int_{(\partial B^n_{R} + 1)^+} y^\alpha u_\nu^2 d\sigma$$

$$- 2R \int_{\partial B^n_{R}} u^\gamma + 2n \int_{B^n_{R}} u^\gamma$$

$$- \frac{2 - 2\alpha}{2 - \gamma} \left( \int_{(\partial B^n_{R} + 1)^+} y^\alpha u_\nu d\sigma - \gamma \int_{B^n_{R}} u^\gamma dx \right).$$

After adding $(I_3)$ and $(I_4)$ we obtain:

$$(I_1) + (I_2) + (I_3) + (I_4) = 2R^{-(n-1+\frac{2-\alpha}{2-\gamma})} \int_{(\partial B^n_{R} + 1)^+} y^\alpha u_\nu^2 d\sigma$$

$$- \frac{2 - 2\alpha}{2 - \gamma} R^{-(n+\frac{2-\alpha}{2-\gamma})} \int_{(\partial B^n_{R} + 1)^+} y^\alpha u^2 d\sigma,$$
And adding the last three terms \((I_5), (I_6)\) and \((I_7)\), we can calculate
\[
\frac{d}{dR} W(R, u) = 2R^{-(n-1+\frac{2-\sigma}{2-n})} \int_{\partial B^n_R^+} y^\alpha(u_\nu - \frac{\beta u}{R})^2 d\sigma.
\]

Let \(0 \in \partial\{u > 0\} \cap \{y = 0\}\), and consider the function \(u_r(X) = r^{-\beta} u(rX)\). As \(r_k \to 0\), \(u_{r_k}\) converges to \(u_0\) weakly in \(H^1(y^\alpha, (\mathbb{R}^{n+1})^+)\). Pass to a subsequence (still denoted by \(r_k\)), \(u_{r_k} \to u_0\) in \(L^2_{\text{loc}}(y^\alpha, (\mathbb{R}^{n+1})^+)\), and in \(L^2_{\text{loc}}(\mathbb{R}^n \times \{y = 0\})\). And the blow-up \(u_0\) is a global minimizer of \(J(\Omega, u)\) on any \(\Omega \subset (\mathbb{R}^{n+1})^+\). Thus, \(W(r_k, u)\) is a bounded non-decreasing function of \(r_k\) by Theorem 3.1 if \(u\) is a minimizer. Then with the boundedness of the sequence \(\{u_{r_k}\}\) in \(H^1(y^\alpha, (\mathbb{R}^{n+1})^+)\), we can prove the following corollary.

**Corollary 3.2** (Blow-ups are homogeneous of degree \(\beta\)). If \(u\) is a minimizer of \(J(u)\), then the blow-up limit \(u_0\) at every free boundary point is homogeneous of degree \(\beta\).

**Proof.** Since \(W(pr, u) = W(p, u_r)\) by the scaling property of \(W\), then for any \(R > 0\),
\[
W(R, u_0) = \lim_{k \to \infty} W(R, u_{r_k}) = \lim_{k \to \infty} W(Rr_k, u) = W(0^+, u)
\]
is a constant, since \(W(Rr_k, u)\) is a bounded non-decreasing function of \(r_k\) by Theorem 3.1. Thus
\[
\frac{d}{dR} W(R, u_0) = 0,
\]
and this implies that \(u_0\) is homogeneous of degree \(\beta\). \(\square\)

4. **Uniqueness of the Blow-up Profile Regardless of Subsequences**

We define
\[
u_0(x, y) = \lim_{r_k \to 0} \frac{u(r_k x, r_k y)}{r_k^\beta}
\]
as the blow-up profile of the minimizer \(u\) near a free boundary point 0. But different subsequences may lead to different blow-up profile \(u_0\). In this section, our aim is to prove that the limit is unique regardless of subsequences.

The blow-up \(u_0\) satisfies the same equations \((3.1)\) and \((3.2)\) as \(u\) does. For any function \(p \geq 0\) homogeneous of degree \(\beta\) and satisfying these equations
\begin{equation}
\tag{3.1}
W(R, p) = R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{\partial B^0_R + 1} \gamma^\alpha p (p - \beta \frac{p}{R}) d\sigma
\end{equation}

\begin{equation}
\tag{3.2}
+ (2 - \gamma) R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} p^\gamma d\sigma
\end{equation}

= (2 - \gamma) R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} p^\gamma dx

Now we prove the Monneau type monotonicity formula. Let \( w = u - p \), and define

\[M(R, u, p) = R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} (u - p)^2 d\sigma = R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} w^2 d\sigma.\]

Here \( p \) satisfies the same equations as \( u \) does. Then

\[
\frac{d}{dR} M(R, u, p) = \frac{d}{dR} \int_{\partial B^0_R + 1} \frac{\gamma^\alpha w^2(Rx, Ry)}{R^{2\beta}} d\sigma
\]

\[
= \int_{\partial B^0_R + 1} \gamma^\alpha \frac{2w(RX)(RX \cdot \nabla w(RX) - \beta w(RX))}{R^{2\beta + 1}} d\sigma
\]

\[
= 2R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} \gamma^\alpha w(x, y)(u - \beta \frac{w}{R}) d\sigma.
\]

We have the following equality

\[W(R, u) = W(R, u) - W(R, p) + (2 - \gamma) R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} p^\gamma dx
\]

\[= R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} \gamma^\alpha (|\nabla u|^2 - |\nabla p|^2) dxy
\]

\[- \beta R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} \gamma^\alpha (u^2 - p^2) d\sigma
\]

\[+ 2R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} (u^\gamma - p^\gamma) dx
\]

\[+ (2 - \gamma) R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} p^\gamma dx
\]

\[= R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} \gamma^\alpha |\nabla w|^2 dxy - \beta R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} \gamma^\alpha w^2 d\sigma
\]

\[+ 2R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} \gamma^\alpha \nabla w \cdot \nabla p dxy - 2\beta R^{-\left(n + \frac{2 - \alpha}{2 - \gamma}\right)} \int_{\partial B^0_R + 1} \gamma^\alpha wp d\sigma
\]

\[+ 2R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} u^\gamma dx - \gamma R^{-\left(n + \frac{2 - \alpha}{2 - \gamma} - 1\right)} \int_{B^0_R} p^\gamma dx
\]
And since

\[ \text{div}(y^\alpha w \nabla p) = y^\alpha \nabla w \cdot \nabla p + w \text{div}(y^\alpha \nabla p) = y^\alpha \nabla w \cdot \nabla p, \]

we can see

\[ \int_{(B_R^{n+1})^+} y^\alpha \nabla w \cdot \nabla p dxdy = \int_{(\partial B_R^{n+1})^+} y^\alpha \nabla w d\sigma - \int_{B_R^n} w \gamma p^{\gamma-1} dx. \]

Thus plug in this equation and since \( p \) is homogeneous of degree \( \beta \), we are able to obtain

\[ W(R, u) = R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{(B_R^{n+1})^+} y^\alpha |\nabla w|^2 dxdy - \beta R^{-(n+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha w^2 d\sigma \]

\[ + 2R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{(\partial B_R^{n+1})^+} y^\alpha wp d\sigma - 2\beta R^{-(n+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha wp d\sigma \]

\[ = R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{(B_R^{n+1})^+} y^\alpha |\nabla w|^2 dxdy - \beta R^{-(n+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha w^2 d\sigma \]

\[ + R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{B_R^n} (2u^\gamma - \gamma p^\gamma - 2\gamma wp^{\gamma-1}) dx \]

Also, we can see

\[ \int_{(B_R^{n+1})^+} w \text{div}(y^\alpha \nabla w) dxdy = \int_{(\partial B_R^{n+1})^+} y^\alpha w w d\sigma - \int_{B_R^n} y^\alpha |\nabla w|^2 dx, \]

and

\[ \text{div}(y^\alpha \nabla w) = \text{div}(y^\alpha \nabla u) - \text{div}(y^\alpha \nabla p) = 0. \]

Then we will obtain

\[ W(R, u) = R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{(B_R^{n+1})^+} y^\alpha |\nabla w|^2 dxdy - \beta R^{-(n+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha w^2 d\sigma \]

\[ + R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{B_R^n} (2u^\gamma - \gamma p^\gamma - 2\gamma wp^{\gamma-1}) dx \]

\[ = R \frac{d}{dR} M(R, u, p) + R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{B_R^n} (2u^\gamma - \gamma p^\gamma - 2\gamma wp^{\gamma-1}) dx. \]

Since \( 0 < \gamma < 1 \), so function \( f(x) = x^\gamma \) is concave on \( \mathbb{R}^+ \), and thus

\[ u^\gamma = (w + p)^\gamma \leq p^\gamma + \gamma p^{\gamma-1} w, \]

since \( u, p \geq 0 \). Therefore,

\[ W(R, u) \leq R \frac{d}{dR} M(R, u, p) + (2 - \gamma) R^{-(n+\frac{2-\alpha\gamma}{2-\gamma}-1)} \int_{B_R^n} p^\gamma dx. \]

We know there is a subsequence \( u_{r_j} \) such that

\[ M(0^+, u, u_0) = \lim_{r_j \to 0} M(1, u_{r_j}, u_0) = 0, \]
and we also know
\[
R \frac{d}{dR} M(R, u, u_0) \geq W(R, u) - (2 - \gamma) R^{-(n + \frac{2 - \alpha \gamma}{2 - \gamma} - 1)} \int_{B_R^n} u_0^\gamma dx
\]
\[
\geq W(0^+, u) - (2 - \gamma) R^{-(n + \frac{2 - \alpha \gamma}{2 - \gamma} - 1)} \int_{B_R^n} u_0^\gamma dx
\]
\[
= W(R, u_0) - (2 - \gamma) R^{-(n + \frac{2 - \alpha \gamma}{2 - \gamma} - 1)} \int_{B_R^n} u_0^\gamma dx
\]
\[
= \gamma R^{-(n + \frac{2 - \alpha \gamma}{2 - \gamma}) - 1)} \int_{B_R^n} u_0^\gamma dx
\]
\[
\geq 0.
\]

Therefore,
\[
\lim_{R \to 0} M(R, u, u_0) = \lim_{R \to 0} M(1, u_R, u_0) = 0,
\]
which means the blow-up profile is unique regardless of subsequence.

5. Uniqueness of half-plane solution

In this section, we apply the method introduced in [4] to prove the following theorem.

**Theorem 5.1.** If \( u \) is the minimizer in \((\mathbb{R}^{n+1})^+\), and \( u(x, 0) = A(x_n)_+^\beta \), then
\[
A = \left( \frac{\beta - s}{-\beta A_1} \right)^{1/(2 - \gamma)}
\]
is determined by \( s \) and \( \gamma \), where
\[
A_1 = -\frac{C_{1,s}}{2} \int_{-\infty}^\infty \frac{(1 + y)_+^{\beta} + (1 - y)_+^{\beta} - 2}{|y|^{1+2s}} dy < 0,
\]
with constant
\[
C_{1,s} = \frac{4^s \Gamma(1/2 + s)}{\pi^{1/2} |\Gamma(-s)|}.
\]

**Proof.** First we prove the theorem when \( n = 1 \). Let
\[
J(u) = \int_{B_1^+} y^\alpha |\nabla u|^2 dxdy + \int_{-1}^1 u^\gamma dx,
\]
and consider \( U_0(x, y) \) as the extension of \( u_0(x) = (x)_+^\beta \). Define
\[
u_\epsilon(x) = \frac{(x + \epsilon)_+^\beta}{(1 + \epsilon)^\beta},
\]
and
\[
\tilde{u}_\epsilon = \begin{cases} 
 u_\epsilon(x) & |x| \leq 1 \\
 u_0(x) & |x| > 1.
\end{cases}
\]
And the function $U_\epsilon(x, y)$ satisfies the following equation:

\[
\begin{cases}
\text{div}(y^\alpha \nabla U_\epsilon(x, y)) = 0 & \text{in } (B_1^2)^+ \\
U_\epsilon(x, 0) = u_\epsilon(x) & |x| \leq 1 \\
U_\epsilon(x, y) = U_0(x, y) & \text{on } (\partial B_1^2)^+.
\end{cases}
\]

If $AU_0$ is a local minimizer of $J(u)$, then $J(AU_0) \leq J(AU_\epsilon)$ for any $\epsilon$, that is

\[
A^2 \int_{B_1^+} y^\alpha |\nabla U_0|^2 \, dx + A^\gamma \int_{-1}^1 u_0^\gamma \, dx \leq A^2 \int_{B_1^+} y^\alpha |\nabla U_\epsilon|^2 \, dx + A^\gamma \int_{-1}^1 u_\epsilon^\gamma \, dx.
\]

We can see

\[
\int_{-1}^1 u_\epsilon^\gamma \, dx - \int_{-1}^1 u_0^\gamma \, dx = \frac{1}{(1 + \epsilon)^{\beta \gamma}} \frac{1}{1 + \beta \gamma} (1 + \epsilon)^{\beta \gamma + 1} - \frac{1}{1 + \beta \gamma} \epsilon
\]

and

\[
(1 - 1) \left[ \int y^\alpha |\nabla U_0|^2 - \int y^\alpha |\nabla U_\epsilon|^2 \right] = \int y^\alpha |\nabla(U_0 - U_\epsilon)|^2 + 2 \int y^\alpha \nabla U_0 \nabla (U_\epsilon - U_0) = I_2 + 2I_1.
\]

First let us calculate $I_1$:

\[
I_1 = \int y^\alpha \nabla U_0 \nabla (U_\epsilon - U_0)
\]

\[
= \int_{(B_1^2)^+} \text{div}(y^\alpha \nabla U_0 (U_\epsilon - U_0)) - \int ((U_\epsilon - U_0) \text{div}(y^\alpha \nabla U_0))
\]

\[
= \int_{(\partial B_1^2)^+} y^\alpha (U_0) \nu (U_\epsilon - U_0) - \int_{-1}^1 (\lim_{y \to 0^+} y^\alpha \partial_y U_0)(U_\epsilon - U_0)
\]

\[
= \int_{-1}^1 (-\Delta)^s u_0(x) (u_\epsilon - u_0).
\]

By the homogeneity property of $u_0$, we can calculate that when $x > 0$,

\[
(-\Delta)^s u_0(x) = -\frac{C_{1, s}}{2} \int_{-\infty}^\infty \frac{(x + y)^\beta + (x - y)^\beta - 2x^\beta}{|y|^{1+2s}} \, dy
\]

\[
= \frac{x^{\beta - 2s} - C_{1, s}}{2} \int_{-\infty}^{\infty} \frac{(1 + y)^\beta + (1 - y)^\beta - 2}{|y|^{1+2s}} \, dy
\]

\[
= A_1 x^{\beta - 2s},
\]
and when \( x < 0 \),

\[
(-\Delta)^s u_0(x) = -(-x)^{\beta - 2s} C_{1,s} P.V. \int_1^{\infty} \frac{(y - 1)^{\beta}}{|y|^{1+2s}} dy
\]

\[
= A_2(-x)^{\beta - 2s}.
\]

Notice that \( A_1, A_2 < 0 \), with

\[
A_1 = -\frac{C_{1,s}}{2} \int_{-\infty}^{\infty} \frac{(1+y)^{\beta} + (1-y)^{\beta} - 2}{|y|^{1+2s}} dy,
\]

and

\[
A_2 = -C_{1,s} P.V. \int_1^{\infty} \frac{(y - 1)^{\beta}}{|y|^{1+2s}} dy.
\]

Then we can calculate that

\[
I_1 = \int_{-1}^{1} (-\Delta)^s u_0(x)(u_\epsilon - u_0)
\]

\[
= A_1 \int_0^1 x^{\beta - 2s} \frac{(x + \epsilon)^{\beta}}{(1+\epsilon)^{\beta}} - x^{\beta} dx
\]

\[
+ A_2 \int_{-1}^0 (-x)^{\beta - 2s} \frac{(x + \epsilon)^{\beta}}{(1+\epsilon)^{\beta}} dx
\]

\[
= A_1 \beta \left( 1 - \frac{1}{2\beta - 2s} - \frac{1}{2\beta - 2s + 1} \right) \epsilon + o(\epsilon).
\]

Then we try to calculate \( I_2 \),

\[
I_2 = \int y^\alpha |\nabla(U_0 - U_\epsilon)|^2
\]

\[
= \int y^\alpha (U_\epsilon - U_0) \nabla(U_\epsilon - U_0) - \int (U_\epsilon - U_0) \text{div}(y^\alpha \nabla(U_\epsilon - U_0))
\]

\[
= \int_{(\partial B_1^\epsilon)^+} y^\alpha (U_\epsilon - U_0)(U_\epsilon - U_0)_{\nu} - \int_{-1}^1 \lim_{y \to 0^+} y^\alpha \partial_y(U_\epsilon - U_0)(U_\epsilon - U_0)
\]

\[
= \int_{-1}^1 (-\Delta)^s(\tilde{u}_\epsilon - u_0)(u_\epsilon - u_0)
\]

\[
= \int_{-1}^1 (-\Delta)^s(\tilde{u}_\epsilon - u_\epsilon)(u_\epsilon - u_0) + \int_{-1}^1 (-\Delta)^s(u_\epsilon - u_0)(u_\epsilon - u_0).
\]

Define

\[
g_\epsilon(x) = \tilde{u}_\epsilon(x) - u_\epsilon(x) = \epsilon h(x) = \begin{cases} 
0 & x \leq 1 \\
\epsilon \beta(x^{\beta} - x^{\beta - 1}) + o(\epsilon) & x > 1,
\end{cases}
\]

and

\[
(-\Delta)^s(\tilde{u}_\epsilon - u_\epsilon)(x) = \epsilon C_{1,s} P.V. \int_{-\infty}^{\infty} \frac{h(x + y) - h(x)}{|y|^{1+2s}} dy,
\]
and
\[
\int_{-1}^{1} (-\Delta)^s(\ddot{u}_\epsilon - u_\epsilon)(u_\epsilon - u_0) \leq 2\max|u_\epsilon - u_0|O(\epsilon) = o(\epsilon).
\]

Thus,
\[
I_2 = o(\epsilon) + \int_{-1}^{1} (-\Delta)^s u_\epsilon(u_\epsilon - u_0) - \int_{-1}^{1} (-\Delta)^s u_0(u_\epsilon - u_0) = o(\epsilon) + I_3 - I_1,
\]
where
\[
I_3 = \int_{-1}^{1} (-\Delta)^s u_\epsilon(u_\epsilon - u_0).
\]

Since \(u_\epsilon(x) = \frac{(x+\epsilon)^\beta}{(1+\epsilon)^\beta}\), then
\[
(-\Delta)^s u_\epsilon(x) = \begin{cases} 
\frac{1}{(1+\epsilon)^\beta} A_1(x+\epsilon)^{\beta-2s} & x + \epsilon > 0 \\
\frac{1}{(1+\epsilon)^\beta} A_2(-x-\epsilon)^{\beta-2s} & x + \epsilon > 0,
\end{cases}
\]
and we can calculate \(I_3\) that
\[
I_3 = \int_{-1}^{1} (-\Delta)^s u_\epsilon(u_\epsilon - u_0)
= \int_{0}^{1} \frac{1}{(1+\epsilon)^\beta} A_1(x+\epsilon)^{\beta-2s}(\frac{(x+\epsilon)^\beta}{(1+\epsilon)^\beta} - x^\beta)dx
+ \int_{-\epsilon}^{0} \frac{1}{(1+\epsilon)^\beta} A_1(x+\epsilon)^{\beta-2s}(\frac{(x+\epsilon)^\beta}{(1+\epsilon)^\beta} - x^\beta)dx
= \epsilon A_1(\frac{\beta-2s+1}{2\beta-2s+1} - \frac{\beta-2s}{2\beta-2s}) + o(\epsilon)
= \epsilon A_1\beta(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}) + o(\epsilon).
\]

Therefore,
\[
A^2(I_2 + 2I_1) = (-1)A^2[\int y^\alpha|\nabla U_0|^2 - \int y^\alpha|\nabla U_\epsilon|^2]
= -2\epsilon A^2 A_1\beta(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}) + o(\epsilon),
\]
and
\[
A^\gamma \int_{-1}^{1} u_\gamma^\alpha dx - A^\gamma \int_{-1}^{1} u_0^\alpha dx = A^\gamma \frac{\epsilon}{1+\beta^\gamma},
\]
and since \(AU_0\) is a local minimizer of energy \(J(u)\), it is required that for all \(\epsilon > 0\) and \(\epsilon < 0\),
\[
-2\epsilon A^2 A_1\beta(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}) + o(\epsilon) \leq A^\gamma \frac{\epsilon}{1+\beta^\gamma},
\]
and this means that
\[
A = \left(\frac{\beta - s}{-\beta A_1}\right)^{1/(2-\gamma)},
\]
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and A is determined by $s$ and $\gamma$, where

$$A_1 = -\frac{C_{1,s}}{2} \int_{-\infty}^{\infty} \frac{(1 + y)^{\beta} + (1 - y)^{\beta} - 2}{|y|^{1+2s}} dy < 0.$$  

Just to notice, as $\gamma \to 0$, which is the case of fractional one-phase Bournelli-type problem, the constant $A_1 = O(\beta - s)$ and this ensures the unique half plane minimizer will not go to 0.

Then applying the same proof in Theorem 1.4 in [4], we prove the theorem for general $n$. □

6. POSITIVE DENSITY WHEN $\gamma$ IS SMALL ENOUGH

When $\gamma \to 1$, in the thin obstacle problem [5], near a free boundary point $x_0$, the set $\{u = 0\} \cap B^\gamma_1(x_0)$ does not always have positive density. In this section, we try to prove there exists a positive number $\gamma_0 > 0$, and for each $0 < \gamma < \gamma_0$, the minimizer of energy $J_\gamma(u)$ has positive density of zero set near every free boundary point.

Theorem 6.1. There exists $\gamma_0 = \gamma_0(n,s) > 0$ and $\delta > 0$ such that for each $0 < \gamma < \gamma_0$, if $u_\gamma$ is a minimizer of $J_\gamma(u)$, then

$$L^n(\{u_\gamma = 0\} \cap B^n_1) \geq \delta > 0.$$  

We prove the theorem by the method of compactness. And before the proof, a lemma of non-degeneracy is required.

Lemma 6.2. Assume $u_\gamma$ is a minimizer of the energy $J_\gamma(u)$ and 0 is a free boundary point. There exists a positive constant $C_0 > 0$ independent of $\gamma$, such that for each $x \in B^n_{1/2} \cap \{u > 0\}$,

$$u_\gamma(x, 0) \geq C_0(d(x, \partial \{u_\gamma > 0\}))^\beta.$$  

Proof. Up to rescaling, it is enough to show, if $(x_0, 0)$ is at distance 1 from the free boundary and $u_\gamma(x_0, 0) > 0$, then $\epsilon = u_\gamma(x_0, 0)$ cannot be too small, and $\epsilon$ will not go to 0 as $\gamma \to 0$.

By the Harnack inequality in the upper half space (since $y^\alpha$ belongs to the class of $A_2$ functions defined by Muchenhoupt in [13]) and the variant boundary Harnack inequality proved in Theorem 4.1 in [11], there exists $c', C' > 0$ independent of $\gamma$, such that

$$0 < c' \epsilon \leq u_\gamma(x, y) \leq C' \epsilon.$$  

in \((B^{n+1}_{1/2}(x_0,0))^+\). Take test function \(\phi \in C_c^\infty((B^{n+1}_{1/2}(x_0,0))^+)\), and apply standard Green’s identity to obtain
\[
\int_{\{u_\gamma > 0\} \cap B^{n}_{1/2}(x_0)} u_\gamma (\lim_{y \to 0^+} y^\alpha \partial_y \phi) - \phi (\lim_{y \to 0^+} y^\alpha \partial_y u_\gamma)
= - \int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma \text{div}(y^\alpha \nabla \phi),
\]
using \(\text{div}(y^\alpha \nabla u_\gamma) = 0\) in the formula. Then
\[
| \int_{\{u_\gamma > 0\} \cap B^{n}_{1/2}(x_0)} \gamma \phi (C^\epsilon \epsilon^{\gamma-1}) \leq | \int_{\{u_\gamma > 0\} \cap B^{n}_{1/2}(x_0)} \gamma \phi u_\gamma^{\gamma-1} | 
\leq | \int_{\{u_\gamma > 0\} \cap B^{n}_{1/2}(x_0)} u_\gamma (\lim_{y \to 0^+} y^\alpha \partial_y \phi) | 
+ \int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma \text{div}(y^\alpha \nabla \phi). 
\]
(6.1)

Since \(d(x, \partial \{u_\gamma > 0\}) \leq C\) if \((x,y) \in (B^{n+1}_{1/2}(x_0,0))^+\), then
\[
u(x,y) \leq \tilde{C}
\]
by \(C^2\) estimates of the minimizer. And the test function \(\phi \in C_c^\infty((B^{n+1}_{1/2}(x_0,0))^+)\) is smooth enough, so the integral of \(\lim_{y \to 0^+} y^\alpha \partial_y \phi\) and \(\text{div}(y^\alpha \nabla \phi)\) are both bounded, and therefore by (6.1), \(\epsilon\) cannot be too small.

However, \(\gamma \epsilon^{\gamma-1} < \infty\) cannot ensure \(\epsilon \geq C > 0\) as \(\gamma \to 0\). To prove that \(\epsilon \geq C\) independent of \(\gamma\), we consider a smooth function \(P(x,y) \geq 0\) defined on \((B^{n+1}_{1/2}(x_0,0))^+)\), with \(P(x,y) = 0\) in \((B^{n+1}_{1/4}(x_0,0))^+)\) and \(P(x,y) = 2C^\epsilon\) in \((B^{n+1}_{7/16}(x_0,0))^+) \setminus (B^{n+1}_{3/8}(x_0,0))^+)\). And define a function \(v(x,y) = \min \{u(x,y), \epsilon P(x,y)\}\) on \((B^{n+1}_{1/2}(x_0,0))^+)\). Then \(J(v) \geq J(u)\) since \(u(x,y)\) is the energy minimizer. First we can see
\[
\int_{(B^{n+1}_{1/2}(x_0,0))^+} y^\alpha \nabla v^2 dxdy - \int_{(B^{n+1}_{1/2}(x_0,0))^+} y^\alpha \nabla u^2 dxdy \leq O(\epsilon)
\]
from our definition of the function \(v(x,y)\). (Same as in Section 3.4, proof of Theorem 1.2 in [4]). And
\[
\int_{B^{n}_{1/2}(x_0)} v^\gamma - u^\gamma \leq - \int_{B^{n}_{1/4}(x_0)} u^\gamma,
\]
since \(v = 0\) on \(B^{n}_{1/4}(x_0)\) and \(v \leq u\) on \(B^{n}_{1/2}(x_0)\). Therefore,
\[
J(v) - J(u) \leq O(\epsilon) - \int_{B^{n}_{1/4}(x_0)} u^\gamma.
\]
(6.2) However, \(J(v) \geq J(u)\) since \(u\) is the energy minimizer. Therefore, if \(\epsilon \to 0\) as \(\gamma \to 0\), then (6.2) requires \(\epsilon^\gamma \to 0\) as \(\gamma \to 0\). If not, (6.2) will lead to a
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contradiction of $u$ being the energy minimizer. Therefore, now it is required that, if $\epsilon \to 0$ as $\gamma \to 0$, then

$$\lim_{\gamma \to 0} \epsilon^\gamma = 0$$

and

$$\lim_{\gamma \to 0} \gamma \epsilon^{\gamma - 1} < \infty$$

from (6.2) and (6.1).

The first limit shows $\epsilon = e^{-\frac{1}{\gamma o(\gamma)}}$, and then as $\gamma \to 0$.

$$\gamma \epsilon^{\gamma - 1} = \gamma e^{\frac{1}{\gamma o(\gamma)} - \frac{1}{o(\gamma)}} \to \gamma e^{\frac{1}{\gamma o(\gamma)}} \to \infty$$

Thus $\epsilon$ will not converge to 0 as $\gamma \to 0$, and therefore, $\epsilon \geq C_0$ independent of $\gamma$. □

With the non-degeneracy property of the minimizer, we can prove the theorem by the method of compactness.

**Proof.** If not, then there exists $\gamma_k \to 0$ with $\{u_{\gamma_k}^j\}_{j=1}^\infty$ a sequence of minimizers of $J_{\gamma_k}$, and

$$\lim_{\gamma_k \to 0, j \to \infty} L^n(\{u_{\gamma_k}^j = 0\} \cap B_1^n) = 0. \quad (6.3)$$

Without loss of generality, we assume 0 is a common free boundary point and take blow-up limit at point 0. Let $u_0^j = \lim_{\gamma_k \to 0} u_{\gamma_k}^j$. By the $\Gamma$–convergence of

$$J_{\gamma}(u) \to J_0(u) = \int_{(B_1^{n+1})^+} \gamma^\alpha |\nabla u|^2 + \int_{B_1^n} \chi_{\{u > 0\}},$$

we know $\{u_0^j\}_{j=1}^\infty$ is a sequence of minimizers of $J_0(u)$. Then Lemma 6.2 and (6.3) will show

$$\lim_{j \to \infty} L^n(\{u_0^j = 0\} \cap B_1^n) = 0$$

which leads to contradiction, since in Theorem 1.3 in [4] the authors prove that in the fractional cavitation problem, near every free boundary point, the zero set has positive density. □

7. Flatness to regularity preliminaries and Main Theorem

In the following sections we apply the method introduced in [8] by De Silva, Savin and Sire to prove the regularity of free boundary given flatness condition when $0 < \gamma < \gamma_0$ (Theorem 7.6).
7.1. Preliminaries. First we give definitions and preliminaries of viscosity solutions to the free boundary problem and discuss the half-plane solution.

A point $X \in \mathbb{R}^{n+1}$ will be denoted by $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}$. We also use the notation $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For a function $g$ defined in $(B_1^{n+1})^+ = \{X \in \mathbb{R}^{n+1}, |X| < 1, y > 0\}$, we denote $\Omega^+(g) = \{g(x, 0) > 0\} \cap B_1^n$ as the positive set in $\mathbb{R}^n$, and $F(g) = \partial_{\mathbb{R}^n} \Omega^+(g) \cap B_1^n$ as the free boundary. We denote $\mathcal{G}(u) = \partial\{u > 0\} \cap \partial B_1^n \cap \partial B_1^{n+1}$ which is the boundary of the set $\partial\{u > 0\} \cap \partial B_1^n$ in $\partial B_1^{n+1}$. We consider the free boundary problem

$$
\begin{align*}
\text{div}(y^a \nabla g) &= 0 \quad \text{in} \ (B_1^{n+1})^+, \\
\frac{\partial g}{\partial \nu} &= 1 \quad \text{on} \ F(g), \\
\lim_{y \to 0^+} y^a \partial_y g(x, y) &= \gamma g^γ(x) \quad \text{in} \ \Omega^+(g).
\end{align*}
$$

(7.1)

Here we denote

$$
\frac{\partial g}{\partial U}(x) = \lim_{t \to 0^+} \frac{g(x + tv(x), 0)}{t^β}, x \in F(g),
$$

and $\nu(x)$ is the unit normal to $F(g)$ at $x$ towards the positive set $\Omega^+(g)$, and $U$ is defined as the following.

Consider $U(t, z)$ as the extension of $(t)^3_+$ to upper half plane, which satisfies $U(t, 0) = (t)^3_+$, and $\text{div}(z^a \nabla U(t, z)) = 0$ in $\{t \in \mathbb{R}, z > 0\}$.

Write $U(t, z) = r^β g(\theta), r = \sqrt{t^2 + z^2} > 0, t = r \cos \theta, z = r \sin \theta$, and $\theta \in [0, \pi]$. Then the equation for $g(\theta) \geq 0$ is

$$
g''(\theta) + \alpha \cot \theta g'(\theta) + \beta(\alpha + \beta) g(\theta) = 0
$$

with $g(\pi) = 0, g(0) = 1$, and $g(\theta) = 1 + \gamma(\sin \theta)^{2s} + o((\sin \theta)^{2s})$. The last equation is derived from $\lim_{t \to 0} z^a \partial_z U(t, z) = \gamma U^γ(t, 0)$ when $t > 0$. The $(n+1)$-dimensional function $U(X) = U(x_n, z)$ is a solution with free boundary $\{x_n = 0\}$.

7.2. Viscosity solutions. We now introduce the definition of viscosity solutions to (7.1).

**Definition 7.1.** Given $g, v$ continuous, we say that $v$ touches $g$ by below (resp. above) at $X_0 \in B_1^{n+1}$ if $g(X_0) = v(X_0)$ and

$$
g(X) \geq v(X) \quad \text{(resp. } g(X) \leq v(X)) \quad \text{in a neighborhood } O \text{ of } X_0.
$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that $v$ touches $g$ strictly by below (resp. above).

**Definition 7.2.** We say $v \in C((B_1^{n+1})^+)$ is a (strict) comparison subsolution to (7.1) if $v$ is a non-negative function in $(B_1^{n+1})^+$ which is $C^2$ in the set where it is positive, and it satisfies

(i) $\text{div}(\gamma^a \nabla v) \geq 0 \quad \text{in} \ (B_1^{n+1})^+$. 

(ii) $F(v)$ is $C^2$ and if $x_0 \in F(v)$ we have

$$v(x, y) = aU((x - x_0) \cdot \nu(x_0), y) + o(|(x - x_0, y)|^\beta), \quad \text{as } (x, y) \to (x_0, 0),$$

with $a \geq 1$, and $\nu(x_0)$ denotes the unit normal at $x_0$ to $F(v)$ towards the positive set $\Omega^+(v)$.

(iii) $\lim_{y \to 0^+} y^\alpha \partial y v(x, y) \geq \gamma v^{\gamma-1}(x)$.

(iv) Either $v$ satisfies (i) and (iii) strictly or $a > 1$.

Similarly one can define a comparison supersolution.

**Definition 7.3.** We say that $g$ is a viscosity solution to (7.1) if $g$ is a continuous non-negative function which satisfies

(i) $g$ is locally $C^{1,1}$ in $(B^{n+1}_1)^+$ and solves (in the viscosity sense)

$$\begin{cases}
\text{div}(y^\alpha \nabla g) = 0 & \text{in } (B^{n+1}_1)^+,
\lim_{y \to 0^+} y^\alpha \partial y g(x, y) = \gamma g^{\gamma-1}(x, 0) & \text{in } \Omega^+(g).
\end{cases}$$

(ii) Any (strict) comparison subsolution (resp. supersolution) cannot touch $g$ by below (resp. by above) at a point $X_0 = (x_0, 0) \in F(g)$.

**7.3. Comparison Principle.** We state the comparison principle for the problem (7.1). The proof is standard and can be found at Lemma 2.6 in [10].

**Lemma 7.4.** Let $g, v_t \in C((B_1^1)^+)$ be respectively a solution and a family of subsolutions to (7.1) with $0 \leq t \leq 1$. Assume that

(i) $v_0 \leq g$ in $(B_1^1)^+$.
(ii) $v_t \leq g$ on $(\partial B^{n+1}_1)^+$ for all $t \in [0, 1]$.
(iii) $v_t < g$ on $\mathcal{G}(v_t) = \partial\{v_t > 0\} \cap \partial B^{n+1}_1 \subset \partial B^{n+1}_1$.
(iv) $v_t(x)$ is continuous in $(x, t) \in (B_1^1)^+ \times [0, 1]$ and $\{v_t > 0\} \cap \overline{B^{n+1}_1}$ is continuous in the Hausdorff metric.

Then

$$v_t \leq g \quad \text{in } (B_1^1)^+ \quad \text{for all } t \in [0, 1].$$

Then as a consequence of the lemma, we introduced the comparison principle used in this paper.

**Corollary 7.5.** Let $g$ be a solution to (7.1) and let $v$ be a subsolution to (7.1) in $(B^{n+1}_2)^+$ which is strictly monotone in the $e_n$-direction in the set $\{v > 0\} \cap B^{n+1}_2 \cap \{y \geq 0\}$. Call

$$v_t(X) = v(X + te_n), X \in B_1^+.$$

Assume that for $-1 \leq t_0 \leq t_1 \leq 1$,

$$v_{t_0} \leq g \quad \text{in } (B^{n+1}_1)^+,$$

and

$$v_{t_1} \leq g \quad \text{on } \partial(B^{n+1}_1)^+, \quad v_{t_1} < g \quad \text{on } \mathcal{G}(v_{t_1}).$$
Then
\[ v_{tt} \leq g \text{ in } (B_{1}^{n+1})^+. \]

7.4. Main Theorem.

**Theorem 7.6** (Main Theorem). There exists \( \gamma_0 > 0 \) such that for each \( 0 < \gamma < \gamma_0 \), there exists a universal constant \( \bar{c} > 0 \), such that if \( g \) is a viscosity solution to \( (7.1) \) satisfying the flatness condition
\[ \{ x \in B_1^n, x_n \leq -\bar{c} \} \subset \{ x \in B_1^n, g(x,0) = 0 \} \subset \{ x \in B_1^n, x_n \leq \bar{c} \}, \]
then \( F(g) \) is \( C^{1,\beta} \) in \( B_{\gamma/2}^n \), with \( \theta > 0 \) depending on \( n, s \) and \( \gamma \).

**Lemma 7.7.** Assume \( g_\gamma \) solves \( (7.1) \), and \( U_\gamma \) is the half-plane solution. There exists \( \gamma_0 > 0 \) such that for each \( 0 < \gamma < \gamma_0 \), given any \( \epsilon > 0 \), there exists \( \bar{c} > 0 \) and \( \delta > 0 \) depending on \( \epsilon \) such that if
\[ \{ x \in B_1^n, x_n \leq -\bar{c} \} \subset \{ x \in B_1^n, g_\gamma(x,0) = 0 \} \subset \{ x \in B_1^n, x_n \leq \bar{c} \}, \]
then the rescaling \( \delta^{-\beta} g_\gamma(\delta X) \) satisfies
\[ U_\gamma(X - \epsilon e_n) \leq \delta^{-\beta} g_\gamma(\delta X) \leq U_\gamma(X + \epsilon e_n) \text{ in } B_1^n. \]

**Proof of Lemma 7.7.** We use the method of compactness since this lemma for case \( \gamma = 0 \) is proved in Lemma 2.10 in [8]. Assume that there exists \( \gamma_k \to 0 \) such that the lemma does not hold for each \( \gamma_k \). Then for each \( \gamma_k \), there exists a sequence \( \{ g^i_{\gamma_k} \}_{i=1}^\infty \), \( g^i_{\gamma_k} \) are solutions of \( (7.1) \) with \( \gamma = \gamma_k \), and a sequence \( \{ e^j_k \}_{j=1}^\infty \) with \( e^j_k \to 0 \) as \( j \to \infty \) for each \( k \), such that \( g^i_{\gamma_k} \) satisfies the following condition with \( e^j_k \to 0 \) as \( j \to \infty \),
\[ \{ x \in B_1^n, x_n \leq -e^j_k \} \subset \{ x \in B_1^n, g^i_{\gamma_k}(x,0) = 0 \} \subset \{ x \in B_1^n, x_n \leq e^j_k \}, \]
but the conclusion does not hold for \( e^j_k \to 0 \) as \( j \to \infty \).

Let \( g^0_j = \lim_{\gamma_k \to 0} g^j_{\gamma_k} \), the limit exists since in \( (7.1) \) the optimal \( C^\beta \) estimates for the minimizers are given, with \( \beta = \frac{2s}{2-s} > s \) and the \( C^\beta \) norm does not blow-up as \( \gamma \to 0 \). And let \( e^0_j = \lim_{j \to \infty} e^j_k \to 0 \) as \( j \to \infty \). The limit \( U_0(X) = \lim_{\gamma_k \to 0} U_{\gamma_k} \) is the half-plane solution for the one-phase cavitation problem. In addition, we proved in Lemma 6.2 that the minimizers are uniformly non-degenerate as \( \gamma \to 0 \). Then \( \{ v^j_0 \}_{j=1}^\infty \) are the solutions of the case \( \gamma = 0 \), and satisfy the flatness assumption with width \( \sup_k e^j_k \to 0 \) as \( j \to \infty \), but the conclusion does not hold, which leads to a contradiction.  

So from now on we may assume that
\[ U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \text{ in } B_1^n. \]

The proof of Theorem 7.6 is organized as follows. In Section 8 we recall the \( \epsilon \)-domain variation of the solutions and the associated linearized equations. In Section 9 we give the proof of a Harnack inequality and then we improve the flatness in Section 10. And in Section 11 the regularity of the solutions.
to the linearized equations are proved and we finish our proof of the main
Theorem in Section [12]. In the Appendix, several useful inequalities of the
half-plane solution $U(t, z)$ are given.

8. LINEARIZED PROBLEM

In this section we recall the $\epsilon-$domain variation of the solution to (7.1)
and state the associated linearized problem, which are introduced in [8].

8.1. The $\epsilon$-domain variations. Let $P = \{X \in \mathbb{R}^{n+1}, x_n \leq 0, y = 0\}$ and
$L = \{X \in \mathbb{R}^{n+1}, x_n = 0, y = 0\}$. To each $X \in \mathbb{R}^{n+1} \cap \{y \geq 0\} \setminus P$ we
associate a set $\tilde{g}_\epsilon(X) \subset \mathbb{R}$ such that

$$U(X) = g(X - \epsilon we_n), \forall w \in \tilde{g}_\epsilon(X).$$

We call $\tilde{g}_\epsilon$ the $\epsilon$-domain variation associated to $g$. And from now on we
write $\tilde{g}_\epsilon(X)$ to denote any of the values in this set, by abuse of notation. We
claim the following: if $g$ satisfies

$$(8.1) \quad U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in} \quad B_{\rho}^{n+1} \cap \{y \geq 0\},$$

then

$$\tilde{g}_\epsilon(X) \in [-1, 1].$$

To prove this, same as in [10], we let

$$Y = X - \tilde{g}_\epsilon(X)e_n, \quad X \in \mathbb{R}^{n+1} \cap \{y \geq 0\} \setminus P,$$

then we can see

$$U(Y - \epsilon e_n) \leq g(Y) = U(Y + \tilde{g}_\epsilon(X)e_n) \leq U(Y + \epsilon e_n),$$

by our definition $U(X) = g(X - \epsilon \tilde{g}_\epsilon(X)e_n) > 0$ and $U$ is strictly monotone
in $e_n$-direction outside of $P$. And by (8.1), for each $X \in B_{\rho - \epsilon}^{n+1} \cap \{y \geq 0\} \setminus P$,
the set $\tilde{g}_\epsilon(X)$ is non-empty and there exists at least one value such that

$$U(X) = g(X - \epsilon c(X)e_n).$$

And our claim follows by the continuity of $g(X - \delta e_n)$, for $\delta \in (-\epsilon, 1]$.

Moreover, if $g$ is strictly monotone in the $e_n$-direction, the $\tilde{g}_\epsilon(X)$ is single-valued.

The following lemma will be useful to obtain a comparison principle.

**Lemma 8.1.** Let $g, v$ be respectively a solution and a subsolution to (7.1) in
$(B_2^{n+1})^+$. Assume that $g$ satisfies the flatness condition (8.1) in $(B_2^{n+1})^+$,
that $v$ is strictly increasing in the $e_n$-direction in $\{v > 0\}$, and that $\tilde{\nu}_\epsilon$ is defined on $B_{2-\epsilon}^{n+1} \cap \{y \geq 0\} \setminus P$ with

$$|\tilde{\nu}_\epsilon| \leq C < \infty.$$

If

$$\tilde{\nu}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{in} \quad B_{\rho_{3/2}}^{n+1} \setminus B_{\rho_{1/2}}^{n+1} \cap \{y \geq 0\} \setminus P,$$

then we have

$$\tilde{\nu}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{on} \quad B_{\rho_{3/2}}^{n+1} \cap \{y \geq 0\} \setminus P.$$
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The proof given in Lemma 3.2 in [10] is still valid since it only involves the comparison principle in Corollary 7.5 and the definition of \( \tilde{g}_\epsilon \).

Given \( \epsilon > 0 \) and a Lipschitz function \( \tilde{\psi} \) defined on \( B_\rho^{n+1}(Y) \cap \{ y \geq 0 \} \) with values in \([-1, 1]\), there exists a unique function \( \psi_\epsilon \) defined on \( B_\rho^{n+1}(Y) \cap \{ y \geq 0 \} \) such that

\[
U(X) = \psi_\epsilon(X - \epsilon \tilde{\psi}(X)e_n), \quad X \in B_\rho^{n+1}(Y) \cap \{ y \geq 0 \}.
\]

And moreover \( \psi_\epsilon \) is increasing in the \( e_n \) direction. Thus, if \( g \) satisfies the flatness condition (8.1) and \( \tilde{\psi}_0 \) and it satisfies (in the viscosity sense)

\[
\psi_\epsilon \leq \tilde{g}_\epsilon \quad \text{in} \quad B_\rho^{n+1}(Y) \cap \{ y \geq 0 \} \setminus P
\]

will lead to

(8.2) \[
\psi_\epsilon \leq g \quad \text{in} \quad B_{\rho-\epsilon}^{n+1}(Y) \cap \{ y \geq 0 \}.
\]

8.2. The linearized problem. We introduce here the linearized problem associated to (7.1). \( U_n \) is the \( \chi_n \)-derivative of the function \( U \). Given \( w \in C((B_1^{n+1})^+) \) and \( X_0 = (x'_0, 0, 0) \), we define

\[
|\nabla_{\chi} w|(X_0) = \lim_{(x, y) \to (0, 0)} \frac{w(x'_0, x_n, y) - w(x'_0, 0, 0)}{r}, \quad r^2 = x_n^2 + y^2.
\]

And the linearized problem associated to (7.1) is

(8.3)

\[
\begin{cases}
\text{div}(y^n \nabla (U_n w)) = 0 \quad \text{in} \quad (B_1^{n+1})^+, \\
|\nabla_{\chi} w|(X_0) = 0 \quad \text{on} \quad B_1^n \cap L, \\
\lim_{y \to 0^+} y^n \partial_y w(x, y) = 0 \quad \text{on} \quad B_1^n \cap \{x_n > 0\}.
\end{cases}
\]

The notion of the viscosity solution for this problem is the following.

Definition 8.2. We say that \( w \) is a solution to (8.3) if \( w \in C_{1,1}^{1,1}((B_1^{n+1})^+) \) and it satisfies (in the viscosity sense)

(i)

\[
\begin{cases}
\text{div}(y^n \nabla (U_n w)) = 0 \quad \text{in} \quad (B_1^{n+1})^+, \\
\lim_{y \to 0^+} y^n \partial_y w(x, y) = 0 \quad \text{on} \quad B_1^n \cap \{x_n > 0\}.
\end{cases}
\]

(ii) Let \( \phi \) be continuous around \( X_0 = (x'_0, 0, 0) \in B_1^n \cap L \) and satisfies

\[
\phi(X) = \phi(X_0) + a(X_0) \cdot (x' - x'_0) + b(X_0)r + O(|x' - x'_0|^2 + r^{1+\theta}),
\]

for some \( \theta > 0 \) and \( b(X_0) \neq 0 \).

If \( b(X_0) > 0 \) then \( \phi \) cannot touch \( w \) by below at \( X_0 \), and if \( b(X_0) < 0 \) then \( \phi \) cannot touch \( w \) by above at \( X_0 \).
9. Harnack Inequality

In this section, we try to prove the following Harnack type inequality for solutions to the free boundary problem (7.1).

**Theorem 9.1 (Harnack Inequality).** There exists \( \bar{\epsilon} > 0 \) such that if \( g \) solves (7.1) and it satisfies

\[
U(X + \epsilon a_0 e_n) \leq g(X) \leq U(X + \epsilon b_0 e_n) \quad \text{in} \quad (B^{n+1}_\rho(X^*))^+,
\]

with \( \epsilon(b_0 - a_0) \leq \bar{\epsilon}\rho \), then

\[
U(X + \epsilon a_1 e_n) \leq g(X) \leq U(X + \epsilon b_1 e_n) \quad \text{in} \quad (B^{n+1}_{\eta\rho}(X^*))^+,
\]

with

\[
a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - \eta)(b_0 - a_0),
\]

for a small universal constant \( \eta \).

Let \( g \) be a solution to (7.1) which satisfies

\[
U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in} \quad (B^{n+1}_1)^+.
\]

Let \( A_\epsilon \) be the set

\[
A_\epsilon = \{(X, \tilde{g}_\epsilon(X)) : X \in (B^{n+1}_{1-\epsilon})^+ \} \subset \mathbb{R}^{n+1} \times [a_0, b_0].
\]

Since \( \tilde{g}_\epsilon \) may be multi-valued, we mean all pairs \( (X, \tilde{g}_\epsilon(X)) \) for all possible values of \( \tilde{g}_\epsilon \). An iterative argument will give the following corollary of Theorem 9.1.

**Corollary 9.2.** If

\[
U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in} \quad (B^{n+1}_1)^+.
\]

with \( \epsilon \leq \bar{\epsilon}/2 \), given \( m_0 > 0 \) such that

\[
2\epsilon(1 - \eta)^{m_0} \eta^{-m_0} \leq \bar{\epsilon},
\]

then the set \( A_\epsilon \cap ((B^{n+1}_{1/2})^+ \times [-1, 1]) \) is above the graph of a function \( y = a_\epsilon(X) \) and is below the graph of a function \( y = b_\epsilon(X) \) with

\[
b_\epsilon - a_\epsilon \leq 2(1 - \eta)^{m_0-1},
\]

and \( a_\epsilon, b_\epsilon \) having a modulus of continuity bounded by the Hölder function \( A_\epsilon B \) with \( A, B \) depending only on \( \eta \).

The proof of Harnack inequality follows as in the case when \( \gamma = 0 \) in [8]. The key ingredient is the lemma below.

**Lemma 9.3.** There exists \( \bar{\epsilon} > 0 \) such that for all \( 0 < \epsilon < \bar{\epsilon} \), if \( g \) is a solution to (7.1) such that

\[
g(X) \geq U(X) \quad \text{in} \quad (B^{n+1}_{1/2})^+,
\]

and at \( \bar{X} \in (B^{n+1}_{1/8}(\frac{1}{4}e_n))^+ \)

\[
(9.1) \quad g(\bar{X}) \geq U(\bar{X} + \epsilon e_n),
\]
then
\begin{equation}
(9.2) \quad g(X) \geq U(X + \tau \varepsilon_n) \quad \text{in} \quad (B_{\delta}^{n+1})^+,
\end{equation}
for universal constants \(\tau, \delta\). Similarly, if
\[ g(X) \leq U(X) \quad \text{in} \quad (B_{1/2}^{n+1})^+, \]
and
\[ g(\bar{X}) \leq U(\bar{X} - \varepsilon_n), \]
then
\[ g(X) \leq U(X - \tau \varepsilon_n) \quad \text{in} \quad (B_{\delta}^{n+1})^+. \]

There is a preliminary lemma.

**Lemma 9.4.** Let \(g \geq 0\) be the \(C^{1,1}_{\text{loc}}\) in \((B_{2}^{n+1})^+\) and solves
\[ \text{div}(y^\alpha \nabla g) = 0 \quad \text{in} \quad (B_{3/2}^{n+1})^+, \]
and let \(\bar{X} = \frac{3}{2} \varepsilon_n\). Assume that
\[ g \geq U \quad \text{in} \quad (B_{2}^{n+1})^+, \quad g(\bar{X}) - U(\bar{X}) \geq \delta_0 \]
for some \(\delta_0 > 0\). Then
\[ g \geq (1 + c \delta_0)U \quad \text{in} \quad (B_{1}^{n+1})^+, \]
for a small universal constant \(c\). In particular, for any \(0 < \epsilon < 2\),
\[ U(X + \varepsilon_n) \geq (1 + \epsilon \epsilon)U(X) \quad \text{in} \quad (B_{1}^{n+1})^+. \]

The proof is slightly different since the boundary Harnack inequality of \(U\) does not work. So instead we have the following proof.

**Proof.** We do an even extension of \(U\) and \(g\) with respect to \(\{y = 0\}\), and let \(g^* - U\) solves the following equation:
\[
\begin{cases}
\text{div}(y^\alpha \nabla (g^* - U)) = 0 & \text{in} \quad D = (B_{3/2}^{n+1}) \setminus \{x_n < 0, y = 0\}, \\
g^* - U = g - U \geq 0 & \text{on} \quad \partial B_{3/2}^{n+1}, \\
g^* - U = 0 & \text{on} \quad \{x_n < 0, y = 0\}.
\end{cases}
\]

Then \(g^*\) satisfies
\[
\begin{cases}
\text{div}(y^\alpha \nabla g^*) = 0 & \text{in} \quad (B_{3/2}^{n+1})^+, \\
g^* \leq g & \text{on} \quad (\partial B_{3/2}^{n+1})^+, \\
g^* = 0 \leq g & \text{on} \quad \{x_n < 0, y = 0\},
\end{cases}
\]
\[ \lim_{y \to 0} y^\alpha \partial_y g^* \geq \lim_{y \to 0} y^\alpha \partial_y g \quad \text{on} \quad \{x_n > 0, y = 0\}. \]

The last inequality holds since
\[ \lim_{y \to 0} y^\alpha \partial_y g^* = \lim_{y \to 0} y^\alpha \partial_y U = \gamma U^{\gamma - 1} \geq \gamma g^{\gamma - 1} = \lim_{y \to 0} y^\alpha \partial_y g. \]
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By maximum principle, \( g^* \leq g \) in \((B^n_{3/2})^+\). Let \( \bar{X} = \frac{3}{2}e_n \), and \( g(\bar{X}) - U(\bar{X}) \geq \delta_0 \). Then since \( g^* - U \) satisfies Harnack inequality,

\[
g^* - U = g - U \geq \delta_0 \text{ on } (\partial B^n_{3/2})^+ \cap B^n_{1/4}(\bar{X}),
\]

and

\[
g^*(\bar{X}) - U(\bar{X}) \geq C_1\delta_0
\]
at some \( \bar{X} \in B^n_{1} \cap D \). Since \( g^* - U \) satisfies boundary Harnack,

\[
g^*(X) - U(X) \geq C_2 \frac{g^*(\bar{X}) - U(\bar{X})}{V(\bar{X})} \text{ in } (B^n_{1})^+.
\]

Here \( V(X) \) solves

\[
\text{div}(g^o \nabla V) = 0 \text{ in } D,
\]

and \( V(X) = 0 \) on \( \{x_n < 0, y = 0\} \). We can see \( V(X) \) is the extension of \((x_n)^+_\). Here we want to prove

\[
V(X) \geq C U(X) \text{ in } (B^n_{1})^+.
\]

We know

\[
\begin{align*}
V(X) &= |X|^s V\left(\frac{X}{|X|}\right) = |X|^s h(\theta), \\
U(X) &= |X|\beta U\left(\frac{X}{|X|}\right) = |X|\beta g(\theta).
\end{align*}
\]

and \( \beta = \frac{2s}{2-s} > s \). So we want to prove \( \frac{h(\theta)}{g(\theta)} \geq C > 0 \) for \( \theta \in [0, \pi] \). From Section 2.2 in \([5]\), \( h(\theta) = (\cos(\theta/2))^{2s} \). And from Section \([7]\), \( g(\theta) \geq 0 \) solves the ODE

(9.3)

\[
g''(\theta) + \alpha \cot \theta g'(\theta) + \beta(\alpha + \beta) g(\theta) = 0
\]

with \( g(\pi) = 0, g(0) = 1, \) and \( g(\theta) = 1 + \gamma(\sin\theta)^{2s} + o((\sin\theta)^{2s}) \) as \( \theta \to 0 \).

So only problem occurs near \( \theta = \pi \), where \( h(\pi) = g(\pi) = 0 \). So

\[
\lim_{\theta \to \pi} \frac{h(\theta)}{g(\theta)} = \lim_{\theta \to \pi} \frac{\cos(\theta/2)^{2s}}{g(\theta)} = \lim_{\theta \to \pi} \frac{s(\cos(\theta/2)^{2s-1}(-\sin(\theta/2)))}{g'(\theta)} = \lim_{\theta \to \pi} \frac{(-s)\cos(\theta/2)^{2s-1}\sin(\theta/2)\sin(\theta)^{1-2s}}{g'(\theta)(\sin\theta)^{\alpha}} = \lim_{\theta \to \pi} \frac{(-s)2^{1-2s}(\sin(\theta/2))^{2-2s}}{g'(\theta)(\sin\theta)^{\alpha}}.
\]

Our aim is to prove

(9.4)

\[
\gamma \geq g'(\theta)(\sin\theta)^{\alpha} \geq \gamma - C_0(\alpha + \beta) \left\| g \right\|_{L^\infty}.
\]
Since
\[
\lim_{\theta \to 0} g'(\theta)(\sin \theta)^{\alpha} = \lim_{\theta \to 0} \frac{g(\theta) - g(0)}{\theta}(\sin \theta)^{\alpha} = \lim_{\theta \to 0} \frac{\gamma \sin \theta^{2s}}{\theta}(\sin \theta)^{\alpha} = \gamma,
\]
and \(g\) solves the equation (9.3), which is equivalent to
\[
(g'(\theta)(\sin \theta)^{\alpha})' = -\beta(\alpha + \beta)(\sin \theta)^{\alpha} g(\theta),
\]
we can apply fundamental theorem of calculus and get
\[
g'(\theta)(\sin \theta)^{\alpha} = \gamma - \beta(\alpha + \beta) \int_0^\theta (\sin \phi)^{\alpha} g(\phi) d\phi,
\]
so we need to prove \(C_0 = \int_0^{1/2} (\sin \theta)^{1-2s} d\theta > 0\) is a bounded number, which is ensured since \(1 - 2s > -1\). Now it is confirmed that
\[
\gamma \geq g'(\theta)(\sin \theta)^{\alpha} \geq \gamma - C_0.
\]
If \(\tilde{C} \leq g'(\theta)(\sin \theta)^{\alpha} \leq 0\) for some \(\tilde{C} \leq 0\), then the limit will be a positive number (maybe positive infinity) and proof completed. If not, then it will contradict our assumption that \(h(\theta) \geq 0\).

From above we prove that
\[
V(X) \geq C U(X) \text{ in } (B_1^{n+1})^+.
\]
Then the proof follows as
\[
g^*(X) - U(X) \geq C_2 \frac{g^*(\tilde{X}) - U(\tilde{X})}{V(\tilde{X})} V(X) \geq C\delta_0 U(X) \text{ in } (B_1^{n+1})^+,
\]
and
\[
g(X) \geq g^*(X) \geq (1 + C\delta_0) U(X).
\]

Proof of Lemma 9.3 will use the following family of radial subsolutions. Let \(R > 0\) and denote
\[
V_R(t, z) = U(t, z)((n - 1) \frac{t}{R} + 1).
\]
Then set the \((n+1)\)-dimensional function \(v_R\) by rotating function \(V_R\) around \((0, R, z)\).
\[
v_R(X) = V_R(R - \sqrt{|x'|^2 + (x_n - R)^2}, z).
\]

**Proposition 9.5.** If \(R\) is large enough, the function \(v_R\) is a comparison subsolution to (7.1) in \((B_1^{n+1})^+\) which is strictly monotone increasing in the \(e_n\)-direction. Moreover, there exists a function \(\tilde{v}_R\) such that
\[
U(X) = v_R(X - \tilde{v}_R(X)e_n) \text{ in } (B_1^{n+1})^+.
\]
and
\begin{equation}
|\tilde{v}_R(X) - \gamma_R(X)| \leq \frac{C}{R^2} |X|^2, \quad \gamma_R(X) = -\frac{|x' |^2}{2R} + 2(n-1) \frac{x_n r}{R},
\end{equation}
with \( r = \sqrt{x_n^2 + z^2} \) and \( C \) universal.

**Proof.** Step 1. In this part we prove that \( v_R \) is a comparison subsolution and is strictly monotone increasing in the \( e_n \)-direction.

First, we need to prove \( v_R \) is a strict subsolution to (9.9)
\begin{equation}
\text{div}(z^n \nabla v_R) = 0 \quad \text{in} \quad (B_n^{2n+1})^+.
\end{equation}

We can compute that
\begin{equation}
\Delta v_R + \frac{\alpha}{z} \partial_z v_R
= \Delta_{t,z} V_R(R - \rho, z) - \frac{n-1}{\rho} \partial_t V_R(R - \rho, z) + \frac{\alpha}{z} \partial_z V_R(R - \rho, z),
\end{equation}
where \( \rho = \sqrt{|x'|^2 + (x_n - R)^2} \). Then for \((t, z) \in (\mathbb{R}^2)^+\),
\begin{equation}
\Delta_{t,z} v_R(t, z) + \frac{\alpha}{z} \partial_z V_R(t, z)
= (\partial_t + \partial_z z) V_R(t, z) + \frac{\alpha}{z} \partial_z V_R(t, z)
= 2(n-1) \frac{R}{R} \partial_t U + \partial_t U \left( \frac{t(n-1)}{R} + 1 \right) + \partial_z U \left( \frac{t(n-1)}{R} + 1 \right) + \frac{\alpha}{z} \partial_z U \left( \frac{t(n-1)}{R} + 1 \right)
= \frac{2(n-1)}{R} \partial_t U(t, z),
\end{equation}
and
\begin{equation}
\partial_t V_R(t, z) = \partial_t U(t, z) \left( \frac{t(n-1)}{R} + 1 \right) + \frac{n-1}{R} U(t, z).
\end{equation}

To prove \( v_R \) is a subsolution to (9.9) in \((B_n^{2n+1})^+\), we need to show that
\begin{equation}
\frac{2(n-1)}{R} \partial_t U - \frac{n-1}{\rho} \left[ \left( \frac{t(n-1)}{R} + 1 \right) \partial_t U + \frac{n-1}{R} U \right] \geq 0
\end{equation}
evaluated at \((R - \rho, z)\). Set \( t = R - \rho \), the inequality is reduced to
\begin{equation}
\left( R - t \right) - \frac{n-1}{R} \partial_t U - \left( n-1 \right) U \geq 0.
\end{equation}

To prove this, an inequality for function \( U \) is required as
\begin{equation}
\frac{r^2}{U(t, z)} \geq C > 0,
\end{equation}
with \( r^2 = t^2 + z^2 \). The proof of (9.12) is given in Section 13.1 in the Appendix.

Then we can show when \( R \) is large enough, the inequality (9.11) is satisfied.

Next we want to prove that \( v_R \) satisfies the free boundary condition. First observe that
\begin{equation}
F(v_R) = \partial B_{R}^n(Re_n) \cap B_{2}^n(0),
\end{equation}
then we want to show
\[ v_R(x, z) = aU(x_n, z) + o(|(x, z)|^\beta) \text{ as } (x, z) \to (0, 0), \]
with \( a \geq 1 \). By the Hölder continuity of \( U \) with exponent \( \beta \), we can see
\[
|V_R(t, z) - V_R(t_0, z)| \leq C|t - t_0|^{\beta} \text{ for } |t - t_0| \leq 1,
\]
thus for \((x, z) \in B^{n+1}_l \) for small \( l > 0 \),
\[
|v_R(x, z) - V_R(x_n, z)| = |V_R(R - \rho, z) - V_R(x_n, z)| \leq C|R - \rho - x_n|^\beta \leq Cl^{2\beta}.
\]
Here we used
\[
R - \rho - x_n = -\frac{|x'|^2}{R - x_n + \rho}.
\]
Then it follows that
\[
|v_R(x, z) - U(x_n, z)| \leq |v_R(x, z) - V_R(x_n, z)| + |V_R(x_n, z) - U(x_n, z)|
\]
\[
\leq C\tilde{t}^{2\beta} + |U(x_n, z)|(n - 1)\frac{|x_n|}{R}
\]
\[
\leq C\tilde{t}^{2\beta} + \tilde{C}l^{\beta} + 1
\]
\[
\leq \tilde{C}l^{2\beta},
\]
since we can require \( \gamma > 0 \) small enough such that \( \beta = \frac{2s}{2-\gamma} \leq 1 \). And this gives the desired expansion \((9.13)\) with \( a = 1 \).

In the last part, we need to show that
\[
\lim_{z \to 0} z^\alpha \partial_z v_R(x, z) \geq \gamma v_R^{\gamma-1}(x, 0)
\]
for all \( x \in \{v_R(x, 0) > 0\} \cap B^n \). From our definition of \( v_R \), \( x \in \{v_R(x, 0) > 0\} \)
means \( t = R - \rho > 0 \). We prove \((9.14)\) by showing
\[
\lim_{z \to 0} z^\alpha \partial_z v_R(x, z) = \lim_{z \to 0} z^\alpha \partial_z V_R(R - \rho, z)
\]
\[
= \left(\frac{(n-1)t}{R} + 1\right)\lim_{z \to 0} z^\alpha \partial_z U(R - \rho, z)
\]
\[
= \left(\frac{(n-1)t}{R} + 1\right)\gamma U^{\gamma-1}(R - \rho, 0)
\]
\[
= \left(\frac{(n-1)t}{R} + 1\right)^2 \gamma v_R^{\gamma-1}(x, 0)
\]
\[
\geq \gamma v_R^{\gamma-1}(x, 0).
\]
So we complete the proof that \( v_R \) is a comparison subsolution to the equation \((7.1)\).

And now, we show that \( v_R \) is strictly monotone increasing in the \( e_n \)-direction. Since
\[
\partial_{x_n} v_R(x) = -\frac{x_n - R}{\rho} \partial_{t} V_R(R - \rho, z),
\]
so we only need to show $\partial_t V_R(R - \rho, z) > 0$, which follows from (9.10) and (9.12).

Step 2. In this part we state the existence of $\tilde{v}_R$ satisfying (9.7) and (9.8). First we want to show there exists unique $\tilde{t}$ such that

$$U(t, z) = V_R(t + \tilde{t}, z) \text{ in } (B^+_1)^{+}$$

and

$$|\tilde{t} + 2(n - 1)tr_R| \leq \frac{\tilde{C}}{R^2}r^3,$$

with $r^2 = t^2 + z^2$ and universal $\tilde{C}$. Since $V_R$ is strictly increasing in $t-$direction except $\{(t, 0), t \leq 0\}$, so it suffices to show

$$V_R(t - 2(n - 1)tr_R - \tilde{C}r^3) < U(t, z) < V_R(t - 2(n - 1)tr_R + \tilde{C}r^3).$$

To prove this, let

$$\tilde{t} = -\frac{2(n - 1)tr_R}{R} - \frac{\tilde{C}}{R^2}r^3$$

and then

$$V_R(t + \tilde{t}, z) = V_R(t, z) + \tilde{t} \partial_t V_R(t, z) + \frac{1}{2}E|\tilde{t}|^2$$

with

$$|E| \leq |\partial_t V_R(\tau, z)|, t + \tilde{t} < \tau < t.$$ 

Claim that

$$|\partial_{tt} V_R(\tau, z)| \leq \frac{C'}{r^2} U(t, z).$$

And following is the proof.

$$\partial_{tt} V_R(\tau, z) = \frac{n - 1}{R} U_t + \frac{(n - 1)\tau}{R} + 1)U_{tt} + \frac{n - 1}{R} U_t$$

$$= 2 \frac{n - 1}{R} r^{\beta - 1} U_t(\frac{\tau}{r}, \frac{z}{r}) + \frac{(n - 1)\tau}{R} + 1)r^{\beta - 2} U_{tt}(\frac{\tau}{r}, \frac{z}{r}),$$

using $U$ is homogeneous of degree $\beta$. Since $\tau$ is between $t$ and $t + \tilde{t}$, so $(\frac{\tau}{r}, \frac{z}{r}) \in B^+_{3/2}/B^+_{1/2}$. Here we claim that

$$|\partial_{tt} U(\frac{\tau}{r}, \frac{z}{r})| \leq K_1 U(\frac{\tau}{r}, \frac{z}{r}),$$

and

$$|\partial_{tt} U(\frac{\tau}{r}, \frac{z}{r})| \leq K_2 U(\frac{\tau}{r}, \frac{z}{r}).$$

The proofs of these two inequalities are given in Section 13.2 and Section 13.4 in the Appendix. Then

$$|\partial_{tt} V_R(\tau, z)| \leq 2 \frac{n - 1}{R} r^{\beta - 1} K_2 U(\frac{\tau}{r}, \frac{z}{r}) + \frac{(n - 1)\tau}{R} + 1)r^{\beta - 2} K_1 U(\frac{\tau}{r}, \frac{z}{r})$$

$$\leq C r^{\beta - 2} U(\frac{\tau}{r}, \frac{z}{r}).$$
Now what we want to prove is

\[(9.20) \quad U(\frac{\tau}{r}, \frac{z}{r}) \leq KU(\frac{t}{r}, \frac{z}{r}),\]

and then we can show

\[|\partial_{tt}V_R(\tau, z)| \leq \bar{C}r^{-2}U(\frac{\tau}{r}, \frac{z}{r}) \leq \bar{C}Kr^{-2}U(t, z).\]

In Section 13.5 in the Appendix a proof of (9.20) is given, and our claim (9.19) is now proved. Using (9.18) with the claim (9.19), we will be able to prove the lower bound in (9.17) if we prove the following

\[U(t, z) > V_R(t, z) + \bar{t}\partial_t V_R(t, z) + \frac{C'}{2r^2}U(t, z)|\bar{t}|^2,\]

and it is equivalent to prove

\[U(t, z) > U(t, z)(\frac{n-1}{R} + 1) + \bar{t}(\frac{n-1}{R} + 1)U_t(t, z) + \frac{n-1}{R}U(t, z))\]
\[+ \frac{C'}{2r^2}U(t, z)|\bar{t}|^2.\]

Divide both sides by \(U\) and times \(r\), it is equivalent to show

\[\frac{(n-1)t}{R}r + \bar{t}\left(\frac{(n-1)r}{R} + \left[\frac{(n-1)t}{R} + 1\right]U_t(t, z)\right) + \frac{C'}{2r}r^2|\bar{t}|^2 < 0.\]

Plug in \(\bar{t} = -\frac{2(n-1)tr}{R} - C'R^3\), it is equivalent to show

\[\bar{t}\left[\frac{(n-1)r}{R} - \frac{1}{2} + \left(\frac{U_t}{U}\right)\left(\frac{(n-1)t}{R} + 1\right)\right] + \frac{C'}{2r^2}r^2|\bar{t}|^2 < \frac{C}{2R^2}r^3.\]

By what we proved in (9.12), and for \(R\) large enough such that

\[|\bar{t}| \leq Kr^2/R,\]

we can show the above inequality is right for appropriate universal \(\bar{C}\) and \(R\) large enough, thus lower bound in (9.17) is proved.

To conclude, we use \(R - \rho - x_n = -\frac{|x'|^2}{R-x_n+\rho}\) with \(\rho = \sqrt{|x'|^2 + (x_n - R)^2}\) to write

\[v_R(X - \tilde{v}_Re_n) = V_R(R - \rho(\tilde{v}_R), z) = V_R(x_n - \tilde{v}_R, -\frac{|x'|^2}{R-x_n+\tilde{v}_R+\rho(\tilde{v}_R)}), z),\]

with \(\rho(\eta) = \sqrt{|x'|^2 + (x_n - \eta - R)^2}\). In view of (9.15), if there exists \(\tilde{v}_R = \tilde{v}_R(X)\) such that

\[(9.21) \quad -\tilde{v}_R - \frac{|x'|^2}{R-x_n+\tilde{v}_R+\rho(\tilde{v}_R)} = \bar{t},\]

then

\[U(X) = v_R(X - \tilde{v}_Re_n),\]
and by the strict monotonicity of $v_R$ in $e_n$ direction, $\tilde{v}_R$ must be unique. Thus, the proposition will be proved if we show that there exists $\tilde{v}_R$ satisfying (9.21) and such that

$$|\tilde{v}_R(X) - \gamma_R(X)| \leq C\frac{|X|^2}{R^2}.$$ 

To do so, we define

$$f(\eta) = -\eta - \frac{|x'|^2}{R - x_n + \eta + \rho(\eta)}, -1 \leq \eta \leq 1,$$

and we show that

$$f(\gamma_R(X) + C\frac{|X|^2}{R^2}) \leq \bar{t} \leq f(\gamma_R(X) - C\frac{|X|^2}{R^2}),$$

and using (9.16) we only need to prove that

$$f(\gamma_R(X) + C\frac{|X|^2}{R^2}) \leq -2(n-1)x_nR - \tilde{C}r^3 \frac{R^2}{R^2},$$

and

$$f(\gamma_R(X) - C\frac{|X|^2}{R^2}) \geq -2(n-1)x_nR + \tilde{C}r^3 \frac{R^2}{R^2}.$$

To prove the first (the second one follows similarly), we define

$$\bar{\eta} = \gamma_R(X) + C\frac{|X|^2}{R^2},$$

and from the definition of $f$ and $\gamma_R$, it is equivalent to show

$$\frac{|x'|^2}{2R} - C\frac{|X|^2}{R^2} - \frac{|x'|^2}{R - x_n + \bar{\eta} + \rho(\bar{\eta})} \leq -\tilde{C}r^3 \frac{R^2}{R^2}.$$ 

Since $-1 \leq \bar{\eta} \leq 1$, so

$$R - x_n + \bar{\eta} + \rho(\bar{\eta}) \leq 2R + 5$$

and the inequality is reduced to

$$-C\frac{|X|^2}{R^2} + \frac{|x'|^2}{R^2} \leq -\tilde{C}r^3 \frac{R^2}{R^2},$$

which is satisfied as long as $C - \tilde{C} \geq 1$. 

Then we can easily obtain the following Corollary.

**Corollary 9.6.** There exist $\delta, c_0, C_0, C_1$ universal constants such that

(9.22) \hspace{1cm} v_R(X + \frac{c_0}{R}e_n) \leq (1 + \frac{C_0}{R})U(X) \text{ in } (B_1^{n+1})^+ / B_1^{1/4}

with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n)) \cap ((B_1^{n+1})^+ / B_1^{1/4})$, and

(9.23) \hspace{1cm} v_R(X + \frac{c_0}{R}e_n) \geq U(X + \frac{c_0}{2R}e_n) \text{ in } (B_\delta^{n+1})^+,$

(9.24) \hspace{1cm} v_R(X - \frac{C_1}{R}e_n) \leq U(X) \text{ in } (B_1^{n+1})^+.$
And now we will start proving Lemma 9.3. We prove the first statement, and the second one follows similarly.

**Proof.** In view of (9.1),
\[ g(\bar{X}) - U(\bar{X}) \geq U(\bar{X} + \lambda e_n) - U(\bar{X}) = \partial_t U(\bar{X} + \lambda e_n) \epsilon \geq c \epsilon \]
for \( \lambda \in (0, \epsilon) \). From Lemma 9.4, we get
\[ (9.25) \quad g(X) \geq (1 + c' \epsilon)U(X) \text{ in } (B^n_{1/4})^+. \]
Now let \( R = \frac{C_0 c'}{c \epsilon} \), constants in Corollary 9.6. Then for \( \epsilon \) small enough, \( v_R \) is a subsolution to (7.1) in \( (B^n_{2})^+ \) which is monotone increasing in the \( e_n \)-direction and it also satisfies inequalities in Corollary 9.6. We now apply the Comparison Principle stated in Corollary 7.5. Let
\[ v^t_R(X) = v_R(X + te_n) \]
ad according to (9.24),
\[ v^t_0 \leq U \leq g \text{ in } (B^n_{1/4})^+, \]
with \( t_0 = -C_1/R \). Moreover, from (9.22) to (9.25), we get that for our choice of \( R \),
\[ v^t_1 \leq (1 + c' \epsilon)U \leq g \text{ in } \partial(B_{1/4}^{n+1})^+, \]
with \( t_1 = c_0/R \), with strict inequality on \( F(v^t_1) \cap \partial(B_{1/4}^{n+1})^+ \). In particular,
\[ g > 0 \text{ on } G(v^t_1) \cap (B_{1/4}^{n+1})^+. \]
Thus we can apply Comparison Principle to prove
\[ v^t_1 \leq g \text{ in } (B_{1/4}^{n+1})^+. \]
And thus from (9.23) we obtain
\[ U(X + \frac{c_1}{R} e_n) \leq v^t_1(X) \leq g(X) \text{ in } (B_{s}^{n+1})^+, \]
which is desired in (9.2) with \( \tau = \frac{c_0 c'}{c_0} \).

**10. Improvement of flatness**

In this section we will show the proof of the improvement of flatness property for solutions to (7.1).

**Theorem 10.1** (Improvement of flatness). There exists \( \bar{\epsilon} > 0 \) and \( \rho > 0 \) universal constants such that for all \( 0 < \epsilon < \bar{\epsilon} \), if \( g \) solves (7.1) with \( 0 \in F(g) \) and it satisfies
\[ (10.1) \quad U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad \text{in} \quad (B_{1}^{n+1})^+, \]
then
\[ (10.2) \quad U(x - \nu - \epsilon \rho/2, z) \leq g(X) \leq U(x - \nu + \epsilon \rho/2, z) \quad \text{in} \quad (B_{\rho}^{n+1})^+, \]
for some direction \( \nu \in \mathbb{R}^n \), \( |\nu| = 1 \).
The proof of Theorem 10.1 is divided into the next four lemmas.

The following lemma is the same as in Lemma 7.2 in [10] and its proof remained unchanged since it only depend on elementary properties related to the definition of \( \tilde{g}_\epsilon \), and does not depend on the equation satisfied by \( g \).

**Lemma 10.2.** Let \( g \) be a solution to (7.1) with \( 0 \in F(g) \) and satisfying (10.1). Assume that
\[(10.3) \quad a_0 \cdot x' - \rho/4 \leq \tilde{g}_\epsilon(X) \leq a_0 \cdot x' + \rho/4 \quad \text{in} \quad (B_2^{n+1})^+,\]
for some \( a_0 \in \mathbb{R}^{n-1} \). Then if \( \epsilon \leq \bar{\epsilon}(a_0, \rho) \), \( g \) satisfies (10.2) in \( (B_\rho^{n+1})^+ \).

The next lemma follows immediately from Corollary 9.2.

**Lemma 10.3.** Let \( \epsilon_k \to 0 \) and let \( g_k \) be a sequence of solutions to (7.1) with \( 0 \in F(g_k) \) satisfying (10.4)
\[(10.4) \quad U(X - \epsilon_k e_n) \leq g_k(X) \leq U(X + \epsilon_k e_n) \quad \text{in} \quad (B_1^{n+1})^+.\]
Denote by \( \tilde{g}_k \) the \( \epsilon_k \)-domain variation of \( g_k \). Then the sequence of sets
\[A_k := \{(X, \tilde{g}_k(X)) : X \in (B_1^{n+1})^+\},\]
has a subsequence that converges uniformly in Hausdorff distance in \( (B_1^{n+1})^+ \) to the graph
\[A_\infty := \{(X, \tilde{g}_\infty(X)) : X \in (B_{1/2}^{n+1})^+\},\]
where \( \tilde{g}_\infty \) is Hölder continuous.

**Lemma 10.4.** The limiting function satisfies \( \tilde{g}_\infty \in C^{1,1}_{loc}(B_{1/2}^{n+1})^+ \).

**Proof.** We fix a point \( Y \in (B_{1/2}^{n+1})^+ \), and let \( \delta \) be the distance from \( Y \) to \( L = \{x_n = 0, y = 0\} \). It suffices to show that the function \( \tilde{g}_\epsilon \) are uniformly \( C^{1,1} \) in \( B_{\delta/8}^{n+1}(Y) \). Since \( g_\epsilon - U \) solves
\[
\text{div}(y^\alpha \nabla (g_\epsilon - U)) = 0 \quad \text{in} \quad B_{\delta/2}^{n+1}(Y),
\]
we can see
\[
\|g_\epsilon - U\|_{C^{1,1}(B_{\delta/4}^{n+1}(Y))} \leq C\|g_\epsilon - U\|_{L^\infty(B_{\delta/2}^{n+1}(Y))} \leq C\epsilon,
\]
and by implicit function theorem it follows as
\[
\|\tilde{g}_\epsilon\|_{C^{1,1}(B_{\delta/8}^{n+1}(Y))} \leq C,
\]
with constant \( C \) depending on \( Y \) and \( \delta \). \( \Box \)

**Lemma 10.5.** The function \( \tilde{g}_\infty \) solves the linearized problem (8.3) in \( (B_{1/2}^{n+1})^+ \).
Proof. We start by showing that in the sense of viscosity, \( U_n \tilde{g}_\infty \) satisfies
\[
div(z^\alpha \nabla(U_n \tilde{g}_\infty)) = 0 \text{ in } (B_{1/2}^{n+1})^+.
\]
Let \( \tilde{\phi} \) be a \( C^2 \) function touching \( \tilde{g}_\infty \) by below at \( X_0 = (x_0, z_0) \in (B_{1/2}^{n+1})^+ \), and we want to show that
\[
(10.5) \quad \Delta(U_n \tilde{\phi})(X_0) + \alpha \frac{\partial_z(U_n \tilde{\phi})(X_0)}{z_0} \leq 0.
\]
By Lemma [10.3] the sequence \( A_k \) converges uniformly to \( A_\infty \), thus there exists a sequence of constants \( c_k \to 0 \) and a sequence of points \( X_k \to X_0 \) such that
\( \tilde{\phi}_k := \tilde{\phi} + c_k \) touches \( \tilde{g}_k \) by below at \( X_k \) for \( k \) large enough.
Define \( \phi_k \) by below
\[
(10.6) \quad \phi_k(X - \epsilon_k \tilde{\phi}_k(X)e_n) = U(X).
\]
Then according to [8.2], \( \phi_k \) touches \( g_k \) by below at \( Y_k = X_k - \epsilon_k \tilde{\phi}_k(X_k)e_n \), for \( k \) large enough. Thus, since \( g_k \) solves
\[
div(z^\alpha \nabla g_k) = 0 \text{ in } (B_{1}^{n+1})^+,
\]
it follows that
\[
(10.7) \quad \Delta(\phi_k)(Y_k) + \alpha \frac{\partial_{n+1}(\phi_k)(Y_k)}{z_k} \leq 0.
\]
Here we denote \( \partial_{n+1} \) as the \((n+1)\)-th derivative (same as \( \partial_z \)), and \( z_k \) is the \( n+1 \)-th coordinate of \( Y_k \). Now we will compute \( \Delta(\phi_k)(Y_k) \) and \( \partial_{n+1}(\phi_k)(Y_k) \).
Since \( \tilde{\phi} \) is smooth, for any \( Y \) in a neighborhood of \( Y_k \), there exists a unique \( X = X(Y) \) such that
\[
(10.8) \quad Y = X - \epsilon_k \tilde{\phi}_k(X)e_n.
\]
Thus (10.6) reads as
\[
\phi_k(Y) = U(X(Y)),
\]
with \( Y_i = X_i \) if \( i \neq n \) and when \( j \neq n \),
\[
\frac{\partial X_j}{\partial Y_i} = \delta_{ij}.
\]
Then
\[
(10.9) \quad D_X Y = I - \epsilon_k D_X (\tilde{\phi}_k(X)e_n),
\]
and
\[
(10.10) \quad D_Y X = I + \epsilon_k D_X (\tilde{\phi} e_n) + O(\epsilon_k^2),
\]
since
\[
\tilde{\phi}_k = \tilde{\phi} + c_k.
\]
It follows that
\[
(10.11) \quad \frac{\partial X_n}{\partial Y_j} = \delta_{jn} + \epsilon_k \partial_j \tilde{\phi}(X) + O(\epsilon_k^2).
\]
Then we can compute
\begin{equation}
\Delta \phi_k(Y) = U_n(X) \Delta X_n(Y) + \sum_{j \neq n} (U_{jj}(X) + 2U_{jn} \frac{\partial X_n}{\partial Y_j}) + U_{nn}(X) |\nabla X_n|^2(Y).
\end{equation}

By (10.11), we can calculate
\[|\nabla X_n|^2(Y) = 1 + 2\epsilon_k \partial_n \tilde{\phi}(X) + O(\epsilon_k^2),\]
and
\begin{equation}
\frac{\partial^2 X_n}{\partial Y_j^2} = \epsilon_k \sum_i \partial_i \tilde{\phi} \frac{\partial X_i}{\partial Y_j} + O(\epsilon_k^2)
= \epsilon_k \sum_{i \neq n} \partial_i \tilde{\phi} \delta_{ij} + \epsilon_k \partial_{jn} \tilde{\phi} \frac{\partial X_n}{\partial Y_j} + O(\epsilon_k^2).
\end{equation}

Then
\begin{equation}
\Delta X_n = \epsilon_k \Delta \tilde{\phi} + O(\epsilon_k^2).
\end{equation}

Using (10.14) and (10.13) in (10.12), we can get
\begin{equation}
\Delta \phi_k(Y) = \Delta U(X) + \epsilon_k U_n \Delta \tilde{\phi} + 2\epsilon_k \nabla \tilde{\phi} \cdot \nabla U_n + O(\epsilon_k^2)(U_n n + 2 \sum_{j \neq n} U_{jn}).
\end{equation}

And we can also calculate that
\begin{equation}
(\phi_k)_{n+1}(Y) = U_n(X) \frac{\partial X_n}{\partial Y_{n+1}} + U_z(X) \frac{\partial X_{n+1}}{\partial Y_{n+1}}
= U_n(X)(\epsilon_k \partial_{n+1} \tilde{\phi}(X) + O(\epsilon_k^2)) + U_z(X).
\end{equation}

Plug in (10.15), and \(\Delta U(X_k) + \frac{a}{z} U_z(X_k) = 0\) to (10.7), we can calculate that
\begin{equation}
\epsilon_k (U_n \Delta \tilde{\phi} + 2\nabla \tilde{\phi} \nabla U_n + \Delta U_n \tilde{\phi} + \frac{a}{z_k} U_n \partial_{n+1} \tilde{\phi} + \frac{a}{z_k} (U_n) z \tilde{\phi}) + O(\epsilon_k^2) \leq 0,
\end{equation}
which means
\[\Delta(U_n \tilde{\phi})(X_k) + \frac{a}{z_k} \partial_z(U_n \tilde{\phi})(X_k) + O(\epsilon_k) \leq 0.
\]
And the desired (10.5) follows as \(k \to \infty\).

The next step is to show that \(\tilde{g}_\infty\) solves
\begin{equation}
\lim_{z \to 0} z^a \partial_z \tilde{g}_\infty = 0 \quad \text{on} \quad \{x_n > 0\} \cap B^n_1.
\end{equation}

Since \(\phi_k\) touches \(g_k\) by below at \(Y_k\) and \(g_k\) solves (10.18), so
\[\lim_{z \to 0} z^a \partial_z \phi_k(Y_k) \geq \gamma \phi_k^{-1}(Y_k),\]
and by the calculation in the previous part,
\[ \partial_z \phi_k(Y_k) = U_n(X)(\epsilon_k \partial_n + 1 \tilde{\phi}(X_k) + O(\epsilon_k^2)) + U_z(X_k), \]
therefore,
\[ \gamma \phi_k^{-1}(Y_k) \leq \partial_z \phi_k(Y_k) = \epsilon_k U_n \partial_n + 1 \tilde{\phi}(X_k) + O(\epsilon_k^2) U_n(X_k) + \partial_z U(X_k). \]  
Since
\[ \phi_k(Y_k) = U(X_k) \]
as defined and \( U \) satisfies
\[ \lim_{z \to 0} \epsilon_k U \partial_z = \gamma U^{-1}, \]
we can show
\[ \epsilon_k U \partial_n + 1 \tilde{\phi}(X_k) + O(\epsilon_k^2) U_n(X_k) \geq 0 \]
and thus
\[ \epsilon_k U \partial_n + 1 \tilde{\phi}(X_k) \geq 0. \]
Here we use \( U_n \) is strictly monotonous increasing in the \( r \)-direction in \( B_1^{n+1} \cap \{ y \geq 0 \} \setminus P \). Since \( \tilde{\phi}_k = \phi + c_k \) touches \( \tilde{g}_k \) by below, letting \( k \to \infty \), we can prove that \( \tilde{g}_\infty \) solves (10.18) on \( \{ x_n > 0 \} \cap B_1^n \).

Then we want to show that \( \tilde{g}_\infty \) solves
\[ |\nabla_r \tilde{g}_\infty|(X_0) = 0, X_0 = (x'_0, 0, 0) \in B_{1/2} \cap L. \]
Assume by contradiction, there exists \( \psi \) touching by below at \( X_0 \) and
\[ \psi(X) = \psi(X_0) + a(X_0)(x' - x'_0) + b(X_0)r + O(|x' - x'_0|^2 + r^{1+l}) \]
for some \( l > 0 \) and \( b(X_0) > 0 \). Then there exists \( \theta, \delta, \bar{r} \) and \( Y = (y'_0, 0, 0) \in B_2 \) depending on \( \psi \) such that
\[ q(X) = \psi(X_0) - \frac{\theta}{2} |x' - y'_0|^2 + 2\theta(n-1)xnr \]
which is a second order polynomial touches \( \psi \) by below at \( X_0 \), in a neighborhood \( N_r = \{ |x' - x'_0| \leq \bar{r}, r \leq \bar{r} \} \) of \( X_0 \). And \( \psi - q \geq \delta > 0 \) on \( N_r \cap N_{\bar{r}/2}. \) Then we can see
\[ \tilde{g}_\infty - q \geq \delta > 0 \text{ on } N_r \setminus N_{\bar{r}/2}, \]
and
\[ \tilde{g}_\infty(X_0) - q(X_0) = 0. \]
In particular,
\[ |\tilde{g}_\infty(X_k) - q(X_k)| \to 0, X_k \in N_r \setminus \{ x_n \leq 0, z = 0 \}, X_k \to X_0. \]
Now choose \( R_k = \frac{1}{\psi_k} \) and and define
\[ w_k(X) = v_{R_k}(X' + \epsilon_k \psi(X_0) e_n), Y = (y'_0, 0, 0), \]
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with \( v_R \) defined in (9.6). Then the \( \epsilon_k \) domain variation of \( w_k \) can be defined by
\[
  w_k(X - \epsilon_k \tilde{w}_k(X)e_n) = U(X),
\]
and since \( U \) is invariant in \( x' \)-direction, this is equivalent to
\[
  v_{R_k}(X - Y' + \epsilon_k \psi(X_0)e_n - \epsilon_k \tilde{w}_k(X)e_n) = U(X - Y').
\]
Proposition 9.5 tells that \( \tilde{v}_{R_k}(X - Y') = \epsilon_k(\tilde{w}_k(X) - \psi(X_0)) \).

Then we can conclude from (9.8) that
\[
  \tilde{w}_k(X) = q(X) + \theta^2 \epsilon_k \mathcal{O}(|X - Y'|^2),
\]
and hence
\[
  |\tilde{w}_k - q| \leq C \epsilon_k \text{ on } N_r \setminus \{x_n \leq 0, z = 0\}
\]
Thus from the uniform convergence of \( A_k \to A_\infty \), we get for \( k \) large enough,
\[
  \tilde{g}_k - \tilde{w}_k \geq \delta/2 \text{ on } (N_r \setminus N_{r/2}) \setminus \{x_n \leq 0, z = 0\}.
\]
And similarly we can get
\[
  \tilde{g}_k(X_k) - \tilde{w}_k(X_k) \leq \delta/4,
\]
for some sequence \( X_k \in N_r \setminus \{x_n \leq 0, z = 0\} \), and \( X_k \to X_0 \).
However from Lemma 8.1 and (10.21), we can see
\[
  \tilde{g}_k - \tilde{w}_k \geq \delta/2 \text{ on } N_{r}/\{x_n \leq 0, z = 0\}
\]
which leads to contradiction.
We complete the proof of Lemma 10.5 that \( \tilde{g}_\infty \) solves the linearized problem (8.3) in \( (B_{1/2}^{n+1})^+ \).

We require regularity of the solutions to the linearized problem (8.3) (in Section 11) to finish the proof of Theorem 10.1 in Section 12, and then the proof of the Main Theorem follows in that section.

11. THE REGULARITY OF LINEARIZED PROBLEM

In this section, our aim is to prove the regularity results for \( w \) solving the linearized equation in the case \( \gamma \) is small enough.

\[
\begin{align*}
  \begin{cases}
  \text{div}(y^\alpha \nabla ((U_\gamma)_n w)) = 0 & \text{in } (B_1^{n+1})^+, \\
  |\nabla_r w|(X_0) = 0 & \text{on } B_1^n \cap L, \\
  \lim_{y \to 0^+} y^\alpha \partial_y w(x, y) = 0 & \text{on } B_1^n \cap \{x_n > 0\}.
  \end{cases}
\end{align*}
\]

Here we denote the function \( U_\gamma \) as the extension of \((x_n)^\beta\) to upper half space \((\mathbb{R}^{n+1})^+)\, and the exponent \( \beta = \frac{2\gamma}{2 - \gamma} \) depends on \( \gamma \).

The following is the main theorem of this section.
Theorem 11.1. There exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, the following regularity results hold.

Given a boundary data $\tilde{h} \in C(\partial B_1^{n+1})^+, |\tilde{h}| \leq 1$, then there exists a unique classical solution $h$ to (11.1) such that $h \in C((B_1^{n+1})^+), h = \tilde{h}$ on $(\partial B_1^{n+1})^+$, and it satisfies

$$|h(X) - h(X_0) - a' \cdot (x' - x'_0)| \leq C(|x' - x'_0|^2 + r^{1+\theta}), X_0 \in B_{1/2}^{n+1} \cap L,$$

for universal constants $C, \theta$ and a vector $a' \in \mathbb{R}^{n-1}$ depending on $X_0$.

A corollary of the theorem above is what we require in the proof of the Theorem 7.6.

Corollary 11.2. There exists a universal constant $C$ such that if $w$ is a viscosity solution to (11.1), with

$$-1 \leq w(X) \leq 1 \text{ in } (B_1^{n+1})^+,$$

then

$$a_0 \cdot x' - C |X|^{1+\theta} \leq w(X) - w(0) \leq a_0 \cdot x' + C |X|^{1+\theta},$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

From Corollary 11.2, there exists $\rho > 0$, if $w$ is a viscosity solution to (11.1), with $w(0) = 0$ and

$$-1 \leq w(X) \leq 1 \text{ in } (B_1^{n+1})^+,$$

then

$$a_0 \cdot x' - \frac{1}{8} \rho \leq w(X) \leq a_0 \cdot x' + \frac{1}{8} \rho, \text{ in } (B_2^{n+1})^+$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

The proof of Theorem 11.1 is based on method of compactness. In paper [8] section 6, Theorem 6.1 states the same results for the linearized problem of the limiting case $\gamma = 0$. In the $\gamma = 0$ case, $w$ solves

$$\begin{cases}
\text{div}(y^\alpha \nabla ((U_0)_n w)) = 0 \text{ in } (B_1^{n+1})^+, \\
|\nabla_r w|(X_0) = 0 \text{ on } B_1^n \cap L, \\
\lim_{y \to 0^+} y^\alpha \partial_{y^i}((U_0)_n w(x, y)) = 0 \text{ on } B_1^n \cap \{x_n > 0\},
\end{cases}$$

with $U_0(X) = U_0(x_n, y) = (r^{1/2} \cos(\theta/2))^2, r^2 = x_n^2 + y^2$. The regularity is stated same in Theorem 11.1. Our aim is to use method of compactness to prove Theorem 11.1 for $0 < \gamma < \gamma_0$ small enough.

Proof. If not, then there exists a sequence $\gamma_k \to 0$ such that given boundary data $\tilde{h}$ and $|\tilde{h}| \leq 1$, $w_k$ solves (11.1) for $\gamma = \gamma_k$ with boundary data $\tilde{h}$, and for any $a' \in \mathbb{R}^{n-1}$, and for any $C > 0, \theta > 0$, there exists $X_k, \tilde{X}_k \in B_{1/2}^{n+1} \cap L$, such that

$$|w_k(\tilde{X}_k) - w_k(X_k) - a'(x'_k - \tilde{x}'_k)| > C(|x'_k - \tilde{x}'_k|^2 + r^{1+\theta}).$$
Consider the limit of its subsequence (denoted as $\gamma_k$, $X_k$, and $\tilde{X}_k$ as well), that $\tilde{X}_k \to \tilde{X}_0$, $X_k \to X_0$, and $w_k \to w_0$. Then $w_0 = \tilde{h}$ on $(\partial B_1^{n+1})^+$ and for any $a' \in \mathbb{R}^{n-1}$, and for any $C > 0$, $\theta > 0$, 

$$|w_0(\tilde{X}_0) - w_0(X_0) - a'(x_0' - \tilde{x}_0')| > C(|x_0' - \tilde{x}_0'|^2 + r^{1+\theta}).$$

Now we want to prove the limit $w_0$ solves (11.4), and then it leads to contradiction. Let 

$$J(w) = \int_{(B_1^{n+1})^+} y^a U_0^2 |\nabla w|^2 dX.$$ 

Then as proved in section 6 in [8], the minimizer of the energy $J$ solves (11.5) 

$$\text{div}(y^a U_0^2 \nabla w) = 0 \quad \text{in} \quad (B_1^{n+1})^+,$$

and (11.5) is equivalent to (11.6) 

$$\text{div}(y^a \nabla (U_0 w)) = 0 \quad \text{in} \quad (B_1^{n+1})^+.$$ 

Moreover, it is proved that if $w$ solves (11.6), and 

$$\lim_{r \to 0^+} w_r(x', x_n, y) = b(x'), \quad \text{on} \quad L \cap B_1^0,$$

then $w$ is a minimizer of $J(w)$ is equivalent to $b = 0$.

Therefore, let $w_k$ be the solution to (11.1) for $\gamma = \gamma_k$. Then $w_k$ is a minimizer of $J_{\gamma_k}(w) = \int_{(B_1^{n+1})^+} y^a (U_{\gamma_k})^2 |\nabla w|^2 dX$, and $w_k$ satisfies 

$$\lim_{y \to 0^+} y^a \partial_y((U_{\gamma_k})_n w_k(x,y)) = w_k(x,0) \lim_{y \to 0^+} y^a \partial_y(U_{\gamma_k})_n.$$ 

This equality is derived from $\lim_{y \to 0^+} y^a \partial_y w_k(x,y) = 0$.

Let 

$$J_0(w) = \int_{(B_1^{n+1})^+} y^a (U_0)^2 |\nabla w|^2 dX.$$ 

Since we have $\lim_{\gamma \to 0^+} U_\gamma = U_0$ in $C((B_1^{n+1})^+)$, thus if $w_k$ is a minimizer of $J_k(w)$, then $w_0 = \lim_{\gamma \to 0^+} w_k$ is a minimizer of $J_0$. And by the convergence, 

$$(w_0)_r(x', x_n, y) = 0$$ 

Moreover, since $w_k \to w_0$, 

$$\lim_{y \to 0^+} y^a \partial_y((U_{\gamma_k})_n w_k(x,y)) = w_k(x,0) \lim_{y \to 0^+} y^a (U_{\gamma_k})_n,$$

and 

$$\lim_{y \to 0^+} y^a \partial_y((U_{\gamma_k})_n) = \lim_{y \to 0^+} y^a \partial_y(U_0)_n = 0,$$

we can prove 

$$\lim_{y \to 0^+} y^a \partial_y((U_0)_n w_0(x,y)) = 0.$$ 

Therefore, we showed that the limit $w_0$ solves (11.4), which leads to contradiction. And Theorem [11.1] is proved and Corollary [11.2] follows. \qed
12. Proof of the Main Theorem

In this section, we apply the regularity results of linearized problem (8.3) to prove Theorem 10.1. And then the proof of Main Theorem simply follows by Theorem 10.1 and Lemma 7.7.

Proof of Theorem 10.1. Let \( \rho \) be the universal constant in (11.3), and assume by contradiction that there exists \( \epsilon_k \to 0 \) and a sequence of solutions \( g_k \) to (7.1) such that

\[
(12.1) \quad U(X - \epsilon_k e_n) \leq g_k(X) \leq U(X + \epsilon_k e_n) \quad \text{in} \quad (B_1^{n+1})^+,
\]

but it does not satisfies the conclusion of the Theorem 10.1. Denote \( \tilde{g}_k \) be the \( \epsilon_k \)-domain variation of \( g_k \). Then by Lemma 10.3 the sequence of sets

\[
A_k := \{(X, \tilde{g}_k(X)) : X \in (B_1^{n+1})^+\},
\]

converges uniformly to

\[
A_\infty = \{(X, \tilde{g}_\infty(X)) : X \in (B_1^{n+1})^+\},
\]

where \( \tilde{g}_\infty \) is a H"older continuous function. By Lemma 10.5, the function \( \tilde{g}_\infty \) solves the linearized equation (8.3), and hence by Corollary 11.2,

\[
a_0 \cdot x' - \rho/8 \leq \tilde{g}_\infty \leq a_0 \cdot x' + \rho/8 \quad \text{in} \quad (B_1^{2\rho})^+,
\]

with \( a_0 \in \mathbb{R}^{n-1} \). From the uniform convergence of \( A_k \) to \( A_\infty \), we get that for all \( k \) large enough,

\[
a_0 \cdot x' - \rho/4 \leq \tilde{g}_\infty \leq a_0 \cdot x' + \rho/4 \quad \text{in} \quad (B_1^{2\rho})^+,
\]

and by Lemma 10.2, \( g_k \) satisfies (10.2), which leads to a contradiction. \( \square \)

13. Appendix

Let \( U(t, z) = r^\beta g(\theta) \geq 0, r = \sqrt{t^2 + z^2}, t = r \cos \theta \) and \( z = r \sin \theta \), with \( \theta \in [0, \pi] \). Since \( \text{div}(z^\alpha \nabla U) = 0 \), and \( \lim_{z \to 0} z^\alpha \partial_z U(t, z) = \gamma U^{\gamma - 1}(t, 0) \), so \( g(\theta) \) solves the ODE

\[
(13.1) \quad g''(\theta) + \alpha \cot \theta g'(\theta) + \beta(\alpha + \beta)g(\theta) = 0,
\]

with \( g(\pi) = 0, g(0) = 1 \), and \( g(\theta) = 1 + \gamma (\sin \theta)^{2s} + o((\sin \theta)^{2s}) \) as \( \theta \to 0 \).

13.1. In the first part, we try to prove the following inequality:

\[
r \frac{\partial_t U(t, z)}{U(t, z)} \geq C > 0.
\]

Calculate that

\[
\frac{U_t}{U} = \frac{1}{r} (\beta \cos \theta - \frac{g'(\theta) \sin \theta}{g(\theta)}) = \frac{1}{r} f(\theta).
\]
We define
\[(13.2) \quad f(\theta) = \beta \cos \theta - \frac{g'(\theta) \sin \theta}{g(\theta)},\]
and then
\[(13.3) \quad f'(\theta) = \frac{1}{\sin \theta}[(f(\theta) - (\beta - s) \cos \theta)^2 + (\beta - s)^2 \sin^2 \theta - s^2].\]
We can calculate \(f(0) = \beta\) since
\[(13.4) \quad \lim_{\theta \to 0} g'(\theta) \sin \theta = \lim_{\theta \to 0} \frac{g(\theta) - g(0)}{g(0) + \gamma (\sin \theta)^2 s} = 0,\]
and \(f(\pi) = 2s - \beta > 0\) since
\[(13.5) \quad \lim_{\theta \to \pi} g'(\theta) (\sin \theta)^\alpha \lim_{\theta \to \pi} \frac{2s(\sin \theta)^{2s-1} \cos \theta}{g'(\theta)} = -2s.\]
And to notice, \(g'(\theta)(\sin \theta)^\alpha\) is bounded and proof is given in (9.4). Also, we can calculate that \(f'(0) = 0\) and \(f'(\pi) = 0\) by
\[
\lim_{\theta \to 0} f'(\theta) = \lim_{\theta \to 0} \frac{2f f' - 2(\beta - s) \cos \theta f' + 2(\beta - s) \sin \theta f}{\cos \theta} = \lim_{\theta \to 0} 2sf'(\theta),
\]
and similarly
\[
\lim_{\theta \to \pi} f'(\theta) = \lim_{\theta \to 0} -2sf'(\theta).
\]
Now we want to prove that \(f(\theta) \geq C > 0\) for \(\theta \in [0, \pi]\). If not, then with the information of \(f\) and \(f'\) at the end points, there exists at least one \(\theta_0 \in (0, \pi)\) such that
\[
\begin{cases}
  f'(\theta_0) = 0, \\
  f(\theta_0) \leq 0, \\
  f''(\theta_0) > 0.
\end{cases}
\]
Since \(f'(\theta_0) = 0\),
\[
f(\theta_0)^2 - 2(\beta - s) \cos \theta_0 f(\theta_0) + (\beta - s)^2 - s^2 = 0,
\]
and thus
\[
f(\theta_0) = (\beta - s) \cos \theta_0 \pm \sqrt{s^2 - (\beta - s)^2 \sin^2 \theta_0}.
\]
If is the plus sign, then
\[
f(\theta_0) > (\beta - s) \cos \theta_0 + (\beta - s) |\cos \theta_0| \geq 0
\]
which is not right. Thus
\[
f(\theta_0) = (\beta - s) \cos \theta_0 - \sqrt{s^2 - (\beta - s)^2 \sin^2 \theta_0}.
\]
Then we can calculate $f''(0)$ at $\theta_0$, that

$$f''(0) = \frac{(2f f' - 2(\beta - s) \cos(\theta) f' + 2(\beta - s) \sin(\theta) f) \sin(\theta) - (f' \sin(\theta)) \cos(\theta)}{\sin^2(\theta)}.$$ 

And when $\theta = \theta_0$,

$$0 < \sin^2(\theta) f''(\theta_0) = 2(\beta - s) \sin^2(\theta_0) f(\theta_0) < 0,$$

which leads to a contradiction.

13.2. In this section we try to prove

$$|U(t, z)| \leq K_2 U(t, z),$$

with $(\frac{t}{r}, \frac{z}{r}) \in B^+_\beta / B^+_2$. Let $\theta = \arctan(\frac{z}{t}) \in [0, \pi]$. Since $U$ is homogenous of degree $\beta$, we can see

$$U(t, z) = \frac{r}{\sqrt{t^2 + z^2}} (\beta \cos(\theta) - \frac{g'(\theta) \sin(\theta)}{g(\theta)}) \leq 2f(\theta)$$

with

$$f(\theta) = \beta \cos(\theta) - \frac{g'(\theta) \sin(\theta)}{g(\theta)},$$

which is the same definition as in (13.2). As calculated in the previous section, $f(0) = \beta$, $f(\pi) = 2s - \beta < \beta$, $f(\theta) \geq C > 0$, and

$$f'(\theta) = f'(\theta) = \frac{1}{\sin(\theta)} [(f(\theta) - (\beta - s) \cos(\theta))^2 + (\beta - s)^2 \sin^2(\theta) - s^2].$$

Then if there exists $\theta_0$ such that $f(\theta_0) = +\infty$, then $f' = +\infty$ and will never be negative infinity at such $\theta_0$, which will lead to a contradiction of $f(0) = \beta$, $f(\pi) = 2s - \beta < \beta$ and $\theta \in [0, \pi]$ which is a bounded interval. Therefore, there must exists an upper bound for $f(\theta)$ and then we can prove

$$|U(t, z)| \leq K_2.$$

13.3. We try to prove

$$|U(t, z)| \leq \frac{C(s, \gamma)}{r^2}.$$ 

Write $U(t, z) = r^\beta g(\theta)$, where $t = r \cos(\theta), z = r \sin(\theta)$ and $r = \sqrt{t^2 + z^2}$. Then

$$U_t = r^{\beta - 2} (\beta g(\theta) t - g'(\theta) z),$$

and

$$U_{tt} = r^{\beta - 4} ((\beta^2 - \beta) t^2 + \beta z^2) g(\theta) + (2 - 2\beta) tz g'(\theta) + z^2 g''(\theta)).$$

Then

$$r^2 \frac{U_{tt}}{U} = (\beta^2 - \beta) \cos^2(\theta) + \beta \sin^2(\theta) + \frac{g'(\theta)}{g(\theta)} (2 - 2\beta) \sin(\theta) \cos(\theta) + \frac{g''(\theta)}{g(\theta)} \sin^2(\theta) =: F(\theta).$$
Since $\text{div}(z^\alpha \nabla U) = 0$, so $g(\theta)$ solves

$$g''(\theta) + \alpha \cot \theta g'(\theta) + \beta(\alpha + \beta)g(\theta) = 0.$$  

Then we can replace $g''(\theta)$ in $F(\theta)$ and calculate (13.7)

$$F(\theta) = (\beta^2 - \beta) \cos^2 \theta + \beta \sin^2 \theta + \frac{g'(\theta)}{g(\theta)} (2 - 2\beta) \sin \theta \cos \theta$$

$$- \frac{\alpha \cot \theta g'(\theta) + \beta(\alpha + \beta)g(\theta)}{g(\theta)} \sin^2 \theta$$

$$= (\beta^2 - \beta) \cos^2 \theta + \beta(1 - \alpha - \beta) \sin^2 \theta + (2 - 2\beta - \alpha) \sin \theta \cos \theta \frac{g'(\theta)}{g(\theta)}.$$  

First,

$$F(0) = \beta^2 - \beta + \frac{g'(0)}{g(0)} \sin \theta(2 - \alpha - 2\beta) = \beta^2 - \beta$$

since $\frac{g'(0)}{g(0)} \sin \theta = 0$. And

$$F(\pi) = \beta^2 - \beta - \lim_{\theta \to \pi} \frac{g'(\theta)}{g(\theta)} \sin \theta(2 - \alpha - 2\beta) = \beta^2 - \beta + 2s(2 - \alpha - 2\beta) = (2s - \beta)(2s - \beta + 1).$$

by

$$\lim_{\theta \to \pi} \frac{g'(\theta)}{g(\theta)} \sin \theta = -2s.$$  

Notice that we require $\gamma > 0$ small enough such that $\beta = \frac{2s}{2-\gamma} \leq 1$ in the proof of (13.8). So $F(0) \leq 0$ and $F(\pi) > 0$. Then we calculate $F'(\theta)$:

$$F'(\theta) = \beta(2 - \alpha - 2\beta) \sin 2\theta + \frac{1}{2} (2 - \alpha - 2\beta) gg'' \sin 2\theta + 2gg' \cos 2\theta - (g')^2 \sin 2\theta$$

$$= \frac{\beta}{2} (2 - \alpha - 2\beta)(2 - \alpha - \beta) \sin 2\theta + \frac{1}{2} (2 - \alpha - 2\beta) \frac{g'}{g} (-2 + (2 - 2\alpha) \cos^2 \theta)$$

$$- \frac{1}{2} (2 - \alpha - 2\beta) (\frac{g'}{g})^2 \sin 2\theta.$$  

When $F'(\theta) = 0$,

$$\sin 2\theta (\frac{g'}{g})^2 + (2 - (2 - 2\alpha) \cos^2 \theta) \frac{g'}{g} - \beta(2 - \alpha - \beta) \sin 2\theta = 0.$$  

Then

(13.8)

$$\frac{g'}{g} = -\frac{(1 - (1 - \alpha) \cos^2 \theta) \pm \sqrt{(1 - (1 - \alpha) \cos^2 \theta)^2 + \beta(2 - \alpha - \beta) \sin^2 2\theta}}{\sin 2\theta}$$

$$= -\frac{(1 - (1 - \alpha) \cos^2 \theta) \pm \sqrt{L(\theta)}}{\sin 2\theta}.$$
And also, by (13.7), we can calculate that

\[(13.9) \quad \frac{g'}{g} = 2 \frac{F(\theta) - (\beta^2 - \beta) \cos^2 \theta - \beta (2 - \alpha - 2\beta) \sin^2 \theta}{(2 - \alpha - 2\beta) \sin 2\theta}.\]

Compare (13.8) and (13.9), we can calculate that if

\[F'(\theta) = 0 \text{ at some } \theta_0 \in (0, \pi), \text{ then at } \theta_0,\]

\[F(\theta) = (\beta^2 - \beta) \cos^2 \theta + \beta (2 - \alpha - 2\beta) \sin^2 \theta + \frac{1}{2} (2 - \alpha - 2\beta) [-(-1 - (1 - \alpha) \cos^2 \theta) \pm \sqrt{L(\theta)}],\]

is a bounded number. With the conditions that \(F(0) = \beta^2 - \beta\) and \(F(\pi) = (2s - \beta)(2s - \beta + 1)\), we can prove that

\[|F(\theta)| \leq C(s, \gamma),\]

which is equivalent to

\[|U_{tt}| \leq \frac{C(s, \gamma)}{r^2}.\]

13.4. In this section, we try to prove

\[|U_{tt}(\frac{\tau}{r}, \frac{z}{r})| \leq K_1.\]

Let \(\theta = \arctan \frac{z}{\tau} \in [0, \pi]\), and we know \((\frac{\tau}{r}, \frac{z}{r}) \in B_{3/2}^-/B_{1/2}^+.\) Since \(U\) is homogeneous of degree \(\beta\), we can see

\[|U_{tt}(\frac{\tau}{r}, \frac{z}{r})| = \left(\frac{r}{\sqrt{\tau^2 + z^2}}\right)^2 |F(\theta)| \leq 4|F(\theta)|\]

with \(F(\theta)\) defined in (13.7). Then using the results in the last section,

\[|U_{tt}(\frac{\tau}{r}, \frac{z}{r})| \leq 4|F(\theta)| \leq 4C(s, r) = K_1.\]

13.5. We try to prove if \(\tau\) is between \(t + \bar{t}\) and \(t\), with

\[\bar{t} = -\frac{2(n - 1)tr}{R} - \frac{\bar{C}}{R^2} r^3 < 0,\]

then

\[U(\frac{\tau}{r}, \frac{z}{r}) \leq KU(\frac{t}{r}, \frac{z}{r}).\]

Let \(\theta_1 = \arccos\left(\frac{\tau}{\sqrt{\tau^2 + z^2}}\right)\) and \(\theta_2 = \arccos\left(\frac{t}{r}\right).\) Since \(g(\theta) \geq 0\) and \(g(\theta) = 0\) only when \(\theta = \pi\), we only need to prove the inequality near \(\theta_2 = \pi.\) Since \((\frac{\tau}{r}, \frac{z}{r}) \in B_{3/2}^+/B_{1/2}^+; t + \bar{t} \leq \tau \leq t,\) and near \(\theta_2 = \pi, t < 0,\) we can see

\[0 < \pi - \theta_1 \leq \pi - \theta_2.\]

As calculated in (13.5),

\[\lim_{\theta \to \pi} \frac{g'(\theta) \sin \theta}{g(\theta)} = -2s < 0\]

with \(g \geq 0\) and \(\sin \theta \geq 0,\) we can see

\[g'(\theta) \leq 0\]
as $\theta \to \pi$. Therefore when $\theta_1, \theta_2$ are close to $\pi$

$$g(\theta_1) \leq g(\theta_2),$$

and thus there exists $\bar{K} > 0$ such that

$$g(\theta_1) \leq \bar{K} g(\theta_2)$$

for $\theta_1 = \arccos \left( \frac{\tau}{\sqrt{\tau^2 + z^2}} \right)$ and $\theta_2 = \arccos \left( \frac{t}{r} \right)$. And thus there exists $K > 0$ such that

$$U(\frac{\tau}{r}, \frac{z}{r}) \leq \left( \frac{3}{2} \right)^{\beta} g(\theta_1) \leq \left( \frac{3}{2} \right)^{\beta} \bar{K} g(\theta_2) = K U(\frac{t}{r}, \frac{z}{r}).$$

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