MONOIDS OF SELF-MAPS OF TOPOLOGICAL SPHERICAL SPACE FORMS

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Abstract. A topological spherical space form is the quotient of a sphere by a free action of a finite group. In general, their homotopy types depend on specific actions of a group. We show that the monoid of homotopy classes of self-maps of a topological spherical space form is determined by the acting group and the dimension of the sphere, not depending on a specific action.

1. Introduction

Let $X$ be a pointed space, and let $M(X)$ denote the pointed homotopy set $[X, X]$. Then $M(X)$ is a monoid under the composition of maps. The monoid $M(X)$ is obviously fundamental for understanding the space $X$. Invertible elements of $M(X)$ form a group, which is the group of self-homotopy equivalences of $X$, denoted by $E(X)$. The groups of self-homotopy equivalences have been intensely studied so that there are a lot of results on them. There is a comprehensive survey on them [8]. However, despite its importance, not much is known about the monoids of self-maps $M(X)$, and in particular, there are only two cases that we know an explicit description of $M(X)$: the case $X$ is a sphere or a complex projective spaces. Notice that $M(X)$ has not been determined even in the case $X$ is a real projective space or a lens space.

A topological spherical space form is, by definition, the quotient space of a sphere by a free action of a finite group. Then real projective spaces and lens spaces are typical examples of such. We refer to [4] for details about topological spherical space forms. The purpose of this paper is to determine the monoids of self-maps of topological spherical space forms.

We recall basic facts about free actions of finite groups on spheres. Let $G$ be a finite group acting freely on $S^n$. Then it is well known that

$$H^{2n+2}(BG; \mathbb{Z}) \cong \mathbb{Z}/|G|.$$  

If $n$ is even, then $G$ must be a cyclic group of order 2, and so $S^n/G$ is homotopy equivalent to $\mathbb{R}P^n$. Suppose $n$ is odd. Then every orientation reversing self-map of $S^n$ has a fixed point.
by the Lefschetz fixed-point theorem, and so the action of $G$ on $S^n$ is orientation-preserving. Hence $S^n/G$ is an oriented compact connected manifold.

First, we state the main theorem in odd dimension. Let $G$ be a finite group acting freely on $S^{2n+1}$. We introduce a new monoid out of a finite group $G$. Let $\alpha \in \text{End}(G)$. By (1.1), the induced map $\alpha_* : H^{2n+2}(BG; \mathbb{Z}) \to H^{2n+2}(BG; \mathbb{Z})$ is identified with an element of $\mathbb{Z}/|G|$, which gives rise to a monoid homomorphism

$$d : \text{End}(G) \to (\mathbb{Z}/|G|)_\times$$

where $(\mathbb{Z}/m)_\times$ denotes the monoid of integers mod $m$ under multiplication. Let $M_\alpha$ be a subset $d(\alpha) + |G|\mathbb{Z}$ of $\mathbb{Z}$. Then we can define a new monoid by

$$M(G, n) = \bigsqcup_{\alpha \in \text{End}(G)} M_\alpha$$

such that the product of $x \in M_\alpha$ and $y \in M_\beta$ is $xy \in M_{\alpha\beta}$. Clearly, the identity element of $M(G, n)$ is $1 \in M_1$.

Now we are ready to state the main theorem in odd dimension.

**Theorem 1.1.** Let $G$ be a finite group acting freely on $S^{2n+1}$. Then there is an isomorphism

$$M(S^{2n+1}/G) \cong M(G, n).$$

Here is an important remark. It is well known that lens spaces of the same dimension with the same $\pi_1$ can have different homotopy types. See [6, Theorem VI]. Then different free actions of the same finite group $G$ on the same sphere $S^{2n+1}$ can produce topological spherical sphere forms of different homotopy types. However, Theorem 1.1 implies that the monoid of self-maps does not distinguish actions.

When $G$ is abelian, the map $d$ in (1.2) is explicitly given in terms of the order of $G$ and the integer $n$, where $G$ acts freely on $S^{2n+1}$. Then a more precise description of $M(S^{2n+1}/G)$ is available, which will be shown in Section 3. For example, one gets:

**Corollary 1.2.** $M(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}_\times$.

Next, we state the main theorem in even dimension. As mentioned above, each of topological spherical forms of dimension $2n$ is of the homotopy type of $\mathbb{RP}^{2n}$. Then the even dimensional case is covered by the following.

**Theorem 1.3.** Let $M = \mathbb{Z}_\times / \sim$ where $x \sim y$ for $x, y \in \mathbb{Z}_\times$ if $x \equiv y \equiv 0$ or $2 \mod 4$. Then

$$M(\mathbb{RP}^{2n}) \cong M.$$
Finally, we present two corollaries of Theorem 1.1. The monoid of self-maps is not abelian in general. See [1] for instance. But by Theorem 1.3, \( M(\mathbb{R}P^{2n}) \) is abelian. Moreover, if \( G \) is abelian, implying \( G \) is cyclic, then \( M(G, n) \) is so, hence \( M(S^{2n+1}/G) \) by Theorem 1.1. Thus one gets:

**Corollary 1.5.** Let \( G \) be a finite abelian group acting freely on \( S^{2n+1} \). Then \( M(S^{2n+1}/G) \) is abelian.

Let \( E(G, n) \) denote the group of invertible elements of \( M(G, n) \). Then by Theorem 1.1, 
\[
E(S^{2n+1}/G) \cong E(G, n).
\]
Clearly, \( E(G, n) \) consists of \( \pm 1 \) in \( M_\alpha \) for \( \alpha \in \text{Aut}(G) \), that is,
\[
E(G, n) = \prod_{\alpha \in \text{Aut}(G)} M_\alpha \cap \{\pm 1\}.
\]
If \( |G| > 2 \), then \( |M_\alpha \cap \{\pm 1\}| \leq 1 \), implying \( E(G, n) \cong \{ \alpha \in \text{Aut}(G) \mid d(\alpha) = \pm 1 \} \). On the other hand, for \( |G| \leq 2 \), \( E(G, n) = M_1 = \{\pm 1\} = C_2 \). Thus we obtain the following, which reproves the result of Smallen [9] and Plotnick [7], where their results seem to exclude the case \( |G| \leq 2 \).

**Corollary 1.6.** Let \( G \) be a finite group acting freely on \( S^{2n+1} \). Then
\[
E(S^{2n+1}/G) \cong \begin{cases} \{ \alpha \in \text{Aut}(G) \mid d(\alpha) = \pm 1 \} & |G| \geq 3 \\ C_2 & |G| \leq 2 \end{cases}
\]

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2. Mapping degree

In this section, we characterize self-maps of odd dimensional topological spherical space forms in terms of mapping degrees and the induced maps on the fundamental groups.

Recall that we can define the mapping degree of a map \( f : X \to Y \), denoted by \( \deg(f) \), in the following cases:

1. \( X \) is an \( n \)-dimensional CW-complex with \( H_n(X; \mathbb{Z}) \cong \mathbb{Z} \) and \( Y \) is an \( (n - 1) \)-connected space with \( \pi_n(Y) \cong \mathbb{Z} \).
2. \( X \) and \( Y \) are oriented compact connected manifolds of dimension \( n \).

In particular, we can define the mapping degrees of self-maps of topological spherical space forms of odd dimension.

Let \( G \) be a group, and let \( X, Y \) be \( G \)-spaces. We denote the set of \( G \)-equivariant homotopy classes of \( G \)-equivariant maps from \( X \) to \( Y \) by \( [X, Y]_G \). The following can be easily deduced from the equivariant Hopf degree theorem [10, Theorem 8.4.1].
Lemma 2.1. Let $G$ be a finite group. Let $X$ be a free $G$-complex of dimension $n$ such that $H_n(X; \mathbb{Z}) \cong \mathbb{Z}$, and let $Y$ be an $(n-1)$-connected $G$-space with $\pi_n(Y) \cong \mathbb{Z}$. Then the map

$$[X, Y]_G \to \mathbb{Z}, \quad [f] \mapsto \deg(f)$$

is injective.

The following lemma is proved by Olum [6, Theorem IIIc] in the case that $|G|$ is odd, where we impose nothing on $|G|$.

Lemma 2.2. Let $G$ be a finite group acting freely on $S^{2n+1}$. Then for $f, g: S^{2n+1}/G \to S^{2n+1}/G$, the following are equivalent:

1. $f$ and $g$ are homotopic;
2. $\pi_1(f) = \pi_1(g)$ and $\deg(f) = \deg(g)$.

Proof. Suppose that (2) holds. Let $\tilde{f}, \tilde{g}: S^{2n+1} \to S^{2n+1}$ be lifts of $f, g$, respectively. Let $X$ be a sphere $S^{2n+1}$ equipped with a $G$-action which is the composite of $\pi_1(f) = \pi_1(g)$ and a given $G$-action on $S^{2n+1}$. Then $\tilde{f}, \tilde{g}$ are $G$-equivariant maps $S^{2n+1} \to X$. Since the projection $S^{2n+1} \to S^{2n+1}/G$ is injective in $H_{2n+1}$, $\deg(\tilde{f}) = \deg(f) = \deg(g) = \deg(\tilde{g})$. Then by applying Lemma 2.1 to $[S^{2n+1}, X]_G$, we obtain that $\tilde{f}$ and $\tilde{g}$ are $G$-equivariantly homotopic. Thus $f$ and $g$ are homotopic, and so (2) implies (1). Clearly, (1) implies (2). Therefore the proof is complete.

3. Proof of Theorem 1.1

Lemma 3.1. Let $S^n \to E \to B$ be a fibration such that $H^{n+1}(B; \mathbb{Z}) \cong \mathbb{Z}/m$ and the transgression $\tau: H^n(S^n; \mathbb{Z}) \to H^{n+1}(B; \mathbb{Z})$ is surjective. If there is a homotopy commutative diagram

$$
\begin{array}{ccc}
S^n & \longrightarrow & E \\
\downarrow f & & \downarrow g \\
S^n & \longrightarrow & E \\
\end{array}
$$

such that $g^* = k: H^{n+1}(B; \mathbb{Z}) \to H^{n+1}(B; \mathbb{Z})$, then

$$\deg(f) \equiv k \mod m.$$

Proof. Let $u$ denote a generator of $H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$. Then by naturality

$$\deg(f)\tau(u) = \tau(\deg(f)u) = \tau(f^*(u)) = g^*(\tau(u)) = k\tau(u)$$

and since $\tau(u)$ is of order $m$, $\deg(f) \equiv k \mod m$, as claimed.
Lemma 3.2. Let $G$ be a finite group acting freely on $S^{2n+1}$, and let $\alpha$ be any endomorphism of $G$. Then for any integer $k$ with $k \equiv d(\alpha) \mod |G|$, there is $f: S^{2n+1}/G \to S^{2n+1}/G$ such that
\[
\deg(f) = k \quad \text{and} \quad \pi_1(f) = \alpha.
\]

Proof. Let $u: BG \to K(\mathbb{Z}, 2n + 2)$ denote a generator of $H^{2n+2}(BG; \mathbb{Z}) \cong \mathbb{Z}/|G|$, and let $F$ denote the homotopy fiber of $u$. Then $F$ is the second stage Postnikov tower of $S^{2n+1}/G$, and so there is a natural map $g: S^{2n+1}/G \to F$. Let $F^{2n+1}$ and $X$ denote the $(2n+1)$-skeleton of $F$ and the homotopy fiber of the canonical map $F^{2n+1} \to BG$, respectively. By considering the Serre spectral sequence associated to a homotopy fibration $X \to F^{2n+1} \to BG$, one gets
\[
H^*(X; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & * = 0, 2n + 1 \\
0 & * \not= 0, 2n + 1.
\end{cases}
\]
Moreover, $X$ is simply connected since the map $F^{2n+1} \to BG$ is an isomorphism in $\pi_1$. Then $X \simeq S^{2n+1}$. Now there is a homotopy commutative diagram
\[
\begin{array}{ccc}
S^{2n+1} & \longrightarrow & S^{2n+1}/G \\
\downarrow & & \downarrow \\
S^{2n+1} & \longrightarrow & F^{2n+1}
\end{array}
\quad
\begin{array}{ccc}
S^{2n+1}/G & \longrightarrow & BG \\
\downarrow & & \downarrow \\
F^{2n+1} & \longrightarrow & BG
\end{array}
\]
where rows are homotopy fibrations. Then $S^{2n+1}/G \simeq F^{2n+1}$, and so the map $g: S^{2n+1}/G \to F$ is identified with the inclusion of the $(2n + 1)$-skeleton.

Let $\alpha$ and $k$ be as in the statement. Then there is a homotopy commutative diagram
\[
\begin{array}{ccc}
F & \longrightarrow & BG \\
\downarrow & & \downarrow \\
F & \longrightarrow & K(\mathbb{Z}, 2n + 2)
\end{array}
\quad
\begin{array}{ccc}
BG & \longrightarrow & K(\mathbb{Z}, 2n + 2) \\
\downarrow & & \downarrow \\
BG & \longrightarrow & K(\mathbb{Z}, 2n + 2)
\end{array}
\]

Then $\tilde{\alpha}^* = k$ on $H^{2n+1}(F; \mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(\tilde{\alpha}) = \alpha$. Thus the restriction of $\tilde{\alpha}$ to $S^{2n+1}/G$, which is identified with the $(2n + 1)$-skeleton of $F$, is the desired map. □

Proof of Theorem 1.1. For $\alpha \in \text{End}(G)$, let
\[
N_\alpha = \{ f \in M(S^{2n+1}/G) \mid \pi_1(f) = \alpha \text{ and } \deg(f) \equiv d(\alpha) \mod |G| \}.
\]
By Lemmas 3.1 and 3.2, $M(S^{2n+1}/G) = \bigsqcup_{\alpha \in \text{End}(G)} N_\alpha$ as a set. For $f \in N_\alpha$ and $g \in N_\beta$, one has
\[
\deg(fg) = \deg(f) \deg(g), \quad \pi_1(fg) = \pi_1(f) \pi_1(g).
\]
By Lemmas 2.2, 3.1 and 3.2, the map
\[
N_\alpha \to M_\alpha, \quad f \mapsto \deg(f)
\]
is well-defined and bijective. Then one gets a bijection \( M(S^{2n+1}/G) \to M(G, n) \). Moreover, this map is a monoid homomorphism by (3.1). Thus the proof is complete. \( \square \)

We describe \( M_\alpha \) in \( M(G, n) \) when \( G \) is cyclic. Let \( C_m \) denote a cyclic group of order \( m \), and consider a free action of \( C_m \) on \( S^{2n+1} \). Since

\[
H^*(BC_m; \mathbb{Z}) = \mathbb{Z}[x]/(mx), \quad |x| = 2,
\]

the map \( d: \text{End}(C_m) \to (\mathbb{Z}/m)_x \) is given by \( d(\alpha) = \alpha^{n+1} \), where \( \alpha \in \text{End}(C_m) \) is assumed to be an element of \( (\mathbb{Z}/m)_x \) through a natural isomorphism \( \text{End}(C_m) \cong (\mathbb{Z}/m)_x \). Then

\[
M_r = r^{n+1} + m\mathbb{Z}
\]
for \( r \in (\mathbb{Z}/m)_x \cong \text{End}(G) \). This gives us, for example, an explicit description of the monoid of self-maps of a lens space. From the description of \( M_r \) above, one can see that there is an isomorphism \( M(C_2, n) \cong \mathbb{Z}_x \), hence Corollary 1.2.

4. Proof of Theorem 1.3

First, we set notation that we are going to use in this section. Let \( p: S^n \to \mathbb{R}P^n \) denote the universal covering, and let \( q: \mathbb{R}P^n \to S^n \) be the pinch map onto the top cell. Let \( j: \mathbb{R}P^{n-1} \to \mathbb{R}P^n \) denote the inclusion. We collect well known facts about real projective spaces that we are going to use. See [2] for the proof.

**Lemma 4.1.** Let \( n \) be an integer \( \geq 2 \).

1. The mod 2 cohomology of \( \mathbb{R}P^n \) is given by

\[
H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[w]/(w^{n+1}), \quad |w| = 1.
\]

2. \( q_*: H_n(\mathbb{R}P^n; \mathbb{Z}/2) \to H_n(S^n; \mathbb{Z}/2) \) is an isomorphism.

3. The composite

\[
S^n \xrightarrow{p} \mathbb{R}P^n \xrightarrow{q} S^n
\]

is of degree \( 1 + (-1)^n+1 \).

4. There is a cofibration

\[
S^{n-1} \xrightarrow{p} \mathbb{R}P^{n-1} \xrightarrow{j} \mathbb{R}P^n.
\]

First, we determine \( M(\mathbb{R}P^{2n}) \) as a set in a way different from [3]. Since \( \pi_1(\mathbb{R}P^{2n}) \cong \mathbb{Z}/2 \) and \( \text{End}(\mathbb{Z}/2) \cong (\mathbb{Z}/2)_x \), we have the following decomposition

**Lemma 4.2.** For \( r = 0, 1 \), let \( M_r = \{ f \in M(\mathbb{R}P^{2n}) \mid \pi_1(f) = r \} \). Then

\[
M(\mathbb{R}P^{2n}) = M_0 \cup M_1.
\]
Let \( f \in M_0 \). Then \( f \) lifts to \( S^{2n} \), implying that \( f \) factors as the composite

\[
\mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{h} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n}
\]

for some integer \( k \in \mathbb{Z} \). Thus \( M_0 = p_* \circ q^*(\pi_{2n}(S^{2n})) \). There is an exact sequence of pointed sets

\[
[\mathbb{R}P^{2n}, C_2] \rightarrow [\mathbb{R}P^{2n}, S^{2n}] \xrightarrow{P} \mathcal{M}(\mathbb{R}P^{2n})
\]

induced from the covering \( C_2 \rightarrow S^{2n} \xrightarrow{p} \mathbb{R}P^{2n} \). Since \([\mathbb{R}P^{2n}, C_2] = *\), one sees that \( p_*: [\mathbb{R}P^{2n}, S^{2n}] \rightarrow \mathcal{M}(\mathbb{R}P^{2n})\) is injective by considering the action of \([\mathbb{R}P^{2n}, C_2]\) on \([\mathbb{R}P^{2n}, S^{2n}]\). On the other hand, there is a diagram

\[
\begin{array}{cc}
\pi_{2n}(S^{2n}) & \\
\downarrow{\Sigma q^*} & \\
[\Sigma\mathbb{R}P^{2n-1}, S^{2n}] \xrightarrow{\Sigma p^*} \pi_{2n}(S^{2n}) \xrightarrow{q^*} [\mathbb{R}P^{2n}, S^{2n}] & \\
\downarrow{\Sigma j^*} & \\
[\Sigma\mathbb{R}P^{2n-2}, S^{2n}] & 
\end{array}
\]

in which the column and the row are exact sequences of pointed sets induced from the cofibration in Lemma 4.1 (4). Since \([\Sigma\mathbb{R}P^{2n-2}, S^{2n}] = *\), the map \( \Sigma q^*: \pi_{2n}(S^{2n}) \rightarrow [\Sigma\mathbb{R}P^{2n-1}, S^{2n}] \) is surjective. For \( k \in \mathbb{Z} \), let \( a_{2k} \in \mathcal{M}(\mathbb{R}P^{2n}) \) be the composite (4.1). Then one gets the following by Lemma 4.1 (3).

**Proposition 4.3.** \( M_0 = \{a_0, a_2\} \) such that \( a_{4k} = a_0 \) and \( a_{4k+2} = a_2 \) for each \( k \in \mathbb{Z} \).

**Lemma 4.4.** For each \( l \in \mathbb{Z} \), there is a unique \( b_{2l+1} \in M_1 \) which lifts to a map \( S^{2n} \rightarrow S^{2n} \) of degree \( 2l+1 \).

**Proof.** First, we reproduce the construction of the map \( b_{2l+1} \) in [3]. Let \( S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \). Consider the antipodal action of \( C_2 \) on \( S^1 \) and the canonical free action of \( C_2 \) on \( S^0 \). Then the diagonal \( C_2 \)-action on \( S^1 \ast \underbrace{S^0 \ast \cdots \ast S^0}_{2n-1} \) is identified with the antipodal action on \( S^{2n} \), where \( S^1 \ast \underbrace{S^0 \ast \cdots \ast S^0}_{2n-1} = S^{2n} \). Define \( f_1: S^1 \rightarrow S^1 \) by \( f_1(z) = z^{2l+1} \) for \( z \in S^1 \). Since \( f_1 \) is a \( C_2 \)-map of degree \( 2l+1 \), the map

\[
f_1 \ast \underbrace{1 \ast \cdots \ast 1}_{2n-1}: S^1 \ast \underbrace{S^0 \ast \cdots \ast S^0}_{2n-1} = S^{2n} \rightarrow S^1 \ast \underbrace{S^0 \ast \cdots \ast S^0}_{2n-1} = S^{2n}
\]

is a \( C_2 \)-map of degree \( 2l+1 \). Then we get \( b_{2l+1} \in M_1 \).

Next, we show the uniqueness of \( b_{2l+1} \). Let \( b'_{2l+1} \in M_1 \) be a map which lifts to a map \( S^{2n} \rightarrow S^{2n} \) of degree \( 2l+1 \). Clearly, this lift is a \( C_2 \)-map. Then the lifts of \( b_{2l+1} \) and \( b'_{2l+1} \) are \( C_2 \)-maps \( S^{2n} \rightarrow S^{2n} \) of the same degree \( 2l+1 \), and so by Lemma 2.1, these lifts are \( C_2 \)-equivariantly homotopic. Thus \( b_{2l+1} \) and \( b'_{2l+1} \) are homotopic, completing the proof. \( \square \)
Now we are ready to determine $M_1$.

**Proposition 4.5.** $M_1 = \{b_{2l+1} \mid l \in \mathbb{Z}\}$.

**Proof.** The inclusion $M_1 \supset \{b_{2l+1} \mid l \in \mathbb{Z}\}$ follows from Lemma 4.4. Let $f \in M_1$. Then $f$ lifts to a map $g: S^{2n} \to S^{2n}$. Since $\pi_1(f) = 1$, $f^* = 1$ on $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Then there is a homotopy commutative diagram

$$
in which columns are homotopy fibrations. Since the action of $\pi_1(\mathbb{R}P^\infty)$ on $H^*(S^{2n}; \mathbb{Z}/2)$ is trivial and the transgression $\tau: H^{2n}(S^{2n}; \mathbb{Z}/2) \to H^{2n+1}(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is an isomorphism, we can apply a mod 2 cohomology analog of Lemma 3.1 to get that $g$ is of odd degree. Thus we obtain the inclusion $M_1 \supset \{b_{2l+1} \mid l \in \mathbb{Z}\}$, completing the proof. \hfill \Box

Next, we determine the monoid structure of $M(\mathbb{R}P^{2n})$.

**Lemma 4.6.** In $M(\mathbb{R}P^{2n})$,

$$a_{2k}a_{2k'} = a_0, \quad a_{2k}b_{2l+1} = b_{2l+1}a_{2k} = a_{2k(2l+1)}, \quad b_{2l+1}b_{2l'+1} = b_{(2l+1)(2l'+1)}.$$  

**Proof.** By definition, $a_{2k}a_{2k'}$ is the composite

$$\mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{k'} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{k} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n}.$$  

Then the first equality follows from Lemma 4.1 (3).

There is a homotopy commutative diagram:

$$\begin{array}{ccc}
S^{2n} & \xrightarrow{2l+1} & S^{2n} \\
p & & p \\
\mathbb{R}P^{2n} & \xrightarrow{b_{2l+1}} & \mathbb{R}P^{2n} \\
\end{array}$$  

The composite of the bottom maps is $b_{2l+1}a_{2k}$ and the composite around the upper perimeter is $a_{2k(2l+1)}$. Then one gets $b_{2l+1}a_{2k} = a_{2k(2l+1)}$.

As in the proof of Lemma 4.4, one can construct a map $b_{2l+1}: \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ which lifts to a map $S^{2n-1} \to S^{2n-1}$ of degree $2l + 1$ and is a restriction of $b_{2l+1}: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$.


Then by Lemma 4.1 (4), there is a homotopy commutative diagram:

\[
\begin{array}{c}
S^{2n-1} \xrightarrow{p} \mathbb{R}P^{2n-1} \xrightarrow{j} \mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \\
\downarrow^{2l+1} \quad \downarrow^{b_{2l+1}} \quad \downarrow^{b_{2l+1}} \quad \downarrow^{2l+1} \\
S^{2n-1} \xrightarrow{p} \mathbb{R}P^{2n-1} \xrightarrow{j} \mathbb{R}P^{2n} \xrightarrow{q} S^{2n}
\end{array}
\]

Thus \( a_{2k}b_{2l+1} = p \circ k \circ q \circ b_{2l+1} = p \circ k \circ (2l+1) \circ q = a_{2k(2l+1)} \).

Clearly, \( b_{2l+1}b_{2l'+1} \) belongs to \( M_1 \) and lifts to a map \( S^{2n} \to S^{2n} \) of degree \((2l+1)(2l'+1)\). Then the third equality holds.

**Proof of Theorem 1.3.** By Propositions 4.3 and 4.5 and Lemma 4.6, one gets \( M(\mathbb{R}P^{2n}) = \{a_{2k}, b_{2l+1} \mid k = 0, 1 \text{ and } l \in \mathbb{Z}\} \) such that for \( i, j = 0, 2 \),

\[
a_i a_j = a_0, \quad a_i b_{2l+1} = b_{2l+1} a_i = a_i, \quad b_{2l+1} b_{2l'+1} = b_{(2l+1)(2l'+1)}.
\]

Clearly, the map

\[
f : M(\mathbb{R}P^{2n}) \to M, \quad f(a_{2i}) = 2i \quad (i = 0, 1), \quad f(b_{2l+1}) = 2l + 1 \quad (l \in \mathbb{Z})
\]

is well defined. Furthermore, it is bijective and a monoid homomorphism. Thus the proof is complete. \( \square \)

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