Dirichlet is not just Bad
and Singular

Victor Beresnevich  Lifan Guan  Antoine Marnat
(York)  (Gottingen)  (TU Graz)
Felipe Ramirez  Sanju Velani
(Wesleyan)  (York)

Abstract

It is well known that in one dimension the set of Dirichlet improvable real numbers consists precisely of badly approximable and singular numbers. We show that in higher dimensions this is not the case by proving that there exist uncountably many Dirichlet improvable vectors that are neither badly approximable nor singular. This is a consequence of a stronger statement concerning well approximable sets.

1 Introduction

1.1 Background and motivation

The goal of this note is to investigate the relation between three basic sets arising from Dirichlet’s fundamental theorem in the classical theory of Diophantine approximation. It is therefore natural to start with the statement of the theorem and in turn describe the associated sets. For $x \in \mathbb{R}$, let $\langle x \rangle := \min\{|x - m| : m \in \mathbb{Z}\}$ denote the distance from $x$ to the nearest integer and for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let

$$\langle qx \rangle := \max_{1 \leq i \leq n} \langle qx_i \rangle .$$

Theorem 1.1 (Dirichlet). For any $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$, there exists $q \in \mathbb{Z}$ such that

$$\langle qx \rangle < N^{-\frac{1}{n}} \quad \text{and} \quad 1 \leq q \leq N . \quad (1.1)$$

An important consequence of Dirichlet’s theorem is the following statement.

Corollary 1.1 (Dirichlet). For any $x \in \mathbb{R}^n$, there exists infinitely many $q \in \mathbb{Z}$ such that

$$\langle qx \rangle < q^{-\frac{1}{n}} . \quad (1.2)$$

The above foundational theorem from the theory of simultaneous Diophantine approximation prompts the natural question:

Question I. Can Dirichlet’s theorem be improved?
The following notions help make the question more precise. Following Davenport & Schmidt \[9\], for a particular $x \in \mathbb{R}^n$ we say that improvement in Dirichlet’s theorem is possible if there exists a constant $\varepsilon \in (0, 1)$ such that, for all sufficiently large $N$ there exists $q \in \mathbb{Z}$ such that

$$\langle qx \rangle < \varepsilon N^{-\frac{1}{n}} \quad \text{and} \quad 1 \leq q \leq N.$$ \hfill (1.3)

For obvious reasons such an $x$ is referred to as \textit{Dirichlet improvable} and we let $\text{DI}_n$ denote the set of Dirichlet improvable points in $\mathbb{R}^n$. Furthermore, we say that $x \in \mathbb{R}^n$ is \textit{singular} if it is Dirichlet improvable with $\varepsilon > 0$ arbitrarily small; that is, for any $\varepsilon > 0$ for all sufficiently large $N$ there exists $q \in \mathbb{Z}$ satisfying (1.3). We let $\text{Sing}_n$ denote the set of singular points in $\mathbb{R}^n$. By definition, we clearly have that

$$\text{Sing}_n \subseteq \text{DI}_n$$

and it is easily verified that $\text{Sing}_n$ contains every rational hyperplane in $\mathbb{R}^n$. Thus

$$n - 1 \leq \dim \text{Sing}_n \leq n.$$ 

Here and throughout, $\dim X$ will denote the Hausdorff dimension of a subset $X$ of $\mathbb{R}^n$. In the case $n = 1$, a nifty argument due to Khintchine \[15\] dating back to the twenties shows that a real number is singular if and only if it is rational; that is

$$\text{Sing}_1 = \mathbb{Q}.$$ \hfill (1.4)

Recently, Cheung and Chevallier \[3\], building on the spectacular $n = 2$ work of Cheung \[4\], have shown that

$$\dim \text{Sing}_n = \frac{n^2}{n + 1} \quad (n \geq 2).$$

Note that since $\frac{n^2}{n + 1} > n - 1$, this immediately implies that in higher dimensions $\text{Sing}_n$ does not simply correspond to rationally dependent $x \in \mathbb{R}^n$ as in the one-dimensional case – the theory is much richer.

In \[9\], Davenport & Schmidt established various results concerning the set $\text{DI}_n$ of Dirichlet improvable points and the set $\text{Bad}_n$ of simultaneously badly approximable points. In particular they showed (see \[9\, Theorem 2\]) that

$$\text{Bad}_n \subseteq \text{DI}_n.$$ \hfill (1.5)

Recall, $x \in \mathbb{R}^n$ is said to be \textit{badly approximable} if there exists a constant $\varepsilon = \varepsilon(x) \in (0, 1)$ so that

$$\langle qx \rangle > \varepsilon q^{-\frac{1}{2}} \quad \forall \quad q \in \mathbb{Z}\setminus\{0\}.$$ \hfill (1.6)

In other words, $\text{Bad}_n$ corresponds to those $x \in \mathbb{R}^n$ for which the right hand side of the inequality appearing in Dirichlet’s corollary (namely (1.2)) cannot be improved by $\varepsilon > 0$ arbitrarily small. By definition, we clearly have that $\text{Bad}_n \cap \text{Sing}_n = \emptyset$. It is worth mentioning the well known fact that $\text{Bad}_n$ is a set of $n$-dimensional Lebesgue measure zero but of full dimension; i.e.

$$\dim \text{Bad}_n = n.$$ 

In view of (1.5) it thus follows that

$$\dim \text{DI}_n = n.$$
In a follow-up paper [10], Davenport & Schmidt showed that $\text{DI}_n$ is a set of $n$-dimensional Lebesgue measure zero and thus in terms of measure and dimension it has the same properties as the set $\text{Bad}_n$. In the case $n = 1$, much more is true: any irrational $x \in \mathbb{R}$ is Dirichlet improvable if and only if it is badly approximable. This for example follows directly from [9, Theorem 1] and together with (1.4) implies that

$$\text{DI}_1 = \text{Bad}_1 \cup \text{Sing}_1.$$  \hfill (1.7)

To the best of our knowledge, the above discussion essentially sums up the current state of knowledge centered around the problem of improving Dirichlet’s theorem. Clearly, in view of (1.7) we have a complete characterisation in dimension one. In higher dimension, we know that

$$\text{DI}_n \supseteq \text{Bad}_n \cup \text{Sing}_n$$

but surprisingly it seems unknown whether or not equality is possible. In other words, the answer to the following basic problem seems unknown. To the best of our knowledge, it first appeared in print in Fabian Süss’ beautifully written PhD thesis [24, Section 4.1].

**Problem 1.1.** *Is the set $\text{DI}_n \setminus (\text{Bad}_n \cup \text{Sing}_n)$ empty when $n \geq 2$?*

The purpose of this note is to show that it is not. Maybe it is “folklore” that in higher dimensions there exist Dirichlet improvable points that are neither badly approximable nor singular. However, we would like to stress that we are unaware of any such a statement.

**Theorem 1.2.** *For $n \geq 2$, the set

$$\text{FS}_n := \text{DI}_n \setminus (\text{Bad}_n \cup \text{Sing}_n)$$

is uncountable.*

We suspect that our theorem is far from the truth. Indeed, it may well be the case that for $n \geq 2$

$$\dim \text{FS}_n = n.$$  

As we shall see in the next section, we actually prove a more general and effective version of Theorem 1.2. Unfortunately, it sheds no light on the dimension of $\text{FS}_n$.

### 1.2 The setup, further background and main results

Recall that from the classical point of view there are two forms of Diophantine approximation in $\mathbb{R}^n$: one corresponding to (simultaneous) approximation by rational points as considered in the previous section and the other corresponding to (dual) approximation by rational hyperplanes. Concerning the latter, the dual version of Dirichlet’s theorem states that for any $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$ there exists $q \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle q \cdot x \rangle \leq N^{-n} \quad \text{and} \quad \|q\| \leq N.$$  \hfill (1.8)

Here and elsewhere $q \cdot x := q_1 x_1 + \ldots + q_n x_n$ is the standard inner product and $\|q\| := \max\{|q_1|, \ldots, |q_n|\}$ is the maximal norm. In [10], Davenport & Schmidt proved that the
denote the minimal distance between \( x \) we approximate points \( x \in \mathbb{R}^n \) such that, for all \( N > N_0(x, \varepsilon) \) sufficiently large there exists \( q \in \mathbb{Z}^n \setminus \{0\} \) such that

\[
\langle q \cdot x \rangle \leq \varepsilon N^{-n} \quad \text{and} \quad \|q\| \leq N.
\]

(1.9)

Hopefully it is clear that the dual form of Dirichlet’s theorem is said to be improvable for a particular \( x \in \mathbb{R}^n \) if there exists a constant \( \varepsilon \in (0, 1) \) such that \( x \in \text{DI}_n(\varepsilon) \). However, in view of Davenport & Schmidt [10, Theorem 2], it makes no difference which form of Dirichlet’s theorem we take to define the set \( \text{DI}_n \). Indeed, the same is true when considering the set of singular points \( \text{Sing}_n \). However, this dual versus simultaneous equivalence for singular points (and indeed badly approximable points) holds in a much wider context. This we now describe since it will be the setting of our main result.

Let \( d \) be integer satisfying \( 0 \leq d \leq n - 1 \). The setup we now consider is one in which we approximate points \( x \in \mathbb{R}^n \) by \( d \)-dimensional rational affine subspaces \( L \subset \mathbb{R}^n \). With this in mind, we let

\[
d(x, L) := \min_{y \in L} \|x - y\| = \min_{y \in L} \max_{1 \leq i \leq n} |x_i - y_i|
\]

(1.10)

denote the minimal distance between \( x \) and \( L \). We also let \( H(L) \) denote the height of \( L \). In short, \( H(L) \) is the volume of the sub-lattice \( \mathbb{Z}^{n+1} \cap L_0 \) where \( L_0 \) is the unique \((d + 1)\)-dimensional subspace of \( \mathbb{R}^{n+1} \) containing the \( d \)-dimensional embedding \( \{(y_1, \ldots, y_n, 1) : y \in L\} \) of \( L \) into \( \mathbb{R}^{n+1} \). This notion of height is relatively standard and is usually referred to as the projective or Weil height of \( L \) – see [7, 17, 21] for more details. Note that when \( d = 0 \), \( L \) corresponds to a rational point \( \mathbf{p}_q := (\frac{p_1}{q}, \ldots, \frac{p_n}{q}) \) for some \((p, q) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}\).

In turn, we have that

\[
H(L) \asymp \max\{\|p\|, |q|\} \quad \text{and} \quad d(x, L) = \max_{1 \leq i \leq n} \frac{|qx_i - p_i|}{|q|}.
\]

Also note that when \( d = n - 1 \), \( L \) corresponds to a rational affine hyperplane \( \{y \in \mathbb{R}^n : q \cdot y = p\} \) for some \((q, p) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z} \). In turn, we have that

\[
H(L) \asymp \max\{|p|, \|q\|\} \quad \text{and} \quad d(x, L) \asymp \frac{|q_1 x_1 + \cdots + q_n x_n - p|}{\|q\|}.
\]

To simplify notation the symbols \( \ll \) and \( \gg \) will be used to indicate an inequality with an unspecified positive multiplicative constant. If \( a \ll b \) and \( a \gg b \) we write \( a \asymp b \), and say that the quantities \( a \) and \( b \) are *comparable*. In the above, the implied ‘comparability’ constants are dependent on \( n \). Thus, up to some multiplicative constants, the extreme cases \( d = 0 \) and \( d = n - 1 \) correspond to the standard simultaneous and dual forms of Diophantine approximation. We now consider the natural analogues of the sets \( \text{Bad}_n \) and \( \text{Sing}_n \) introduced within the framework of simultaneous Diophantine approximation in §1.1. With this in mind, we start by stating a Dirichlet type theorem for approximation by \( d \)-dimensional rational subspaces. Throughout, given \( n \in \mathbb{N} \) and \( d \in \{0, 1, \ldots, n - 1\} \), we let

\[
\omega_d := \frac{d + 1}{n - d}.
\]
Theorem 1.3. Let \( n \in \mathbb{N} \) and \( d \) be integer satisfying \( 0 \leq d \leq n - 1 \). Then for any \( x \in \mathbb{R}^n \) there exists a constant \( c = c(x,n) > 0 \), such that for any \( N \in \mathbb{N} \) there exists a \( d \)-dimensional rational affine subspace \( L \subset \mathbb{R}^n \), such that
\[
d(x,L) \leq c H(L)^{-1} N^{-\omega_d} \quad \text{and} \quad H(L) \leq N.
\] (1.11)

The above statement is a consequence of standard tools from the geometry of numbers such as Minkowski’s second convex body theorem and Mahler’s theory for compound bodies. For completeness, the details are given in the appendix (more precisely, Proposition A.1 in §A.2). In turn, the theorem gives rise to the following statement.

Corollary 1.2. Let \( n \in \mathbb{N} \) and \( d \) be integer satisfying \( 0 \leq d \leq n - 1 \). Then for any \( x \in \mathbb{R}^n \) with at least \((d + 1)\)-rationally independent coordinates, there exists a constant \( c = c(x,n) > 0 \) and infinitely many \( d \)-dimensional rational affine subspaces \( L \subset \mathbb{R}^n \) such that
\[
d(x,L) \leq c H(L)^{-1-\omega_d}.
\] (1.12)

Remark 1.1. Observe that if we restrict \( x \) to a bounded subset of \( \mathbb{R}^n \), then the constant \( c = c(x,n) \) appearing in the above results can be made to be independent of \( x \). In particular, if \( x \in [0,1]^n \) then in the simultaneous (resp. dual) case we can replace \( H(L) \) by \( |q| \) (resp. \( \|q\| \)) in the theorem and corollary, and the inequalities corresponding to (1.11) and (1.12) remain valid if we translate \( x \) by an integer vector. Thus, up to a constant dependent only on the dimension \( n \), Theorem 1.3 and its corollary coincide with the classical simultaneous and dual forms of Dirichlet theorem and its corollary.

Taking our lead from the classical simultaneous and dual settings, we say that a point \( x \in \mathbb{R}^n \) is \( d \)-singular if for any given \( \varepsilon \in (0,1) \) and \( N > N_0(x,\varepsilon,d) \) sufficiently large, there exists a \( d \)-dimensional rational affine subspaces \( L \subset \mathbb{R}^n \) such that
\[
d(x,L) \leq \varepsilon H(L)^{-1} N^{-\omega_d} \quad \text{and} \quad H(L) \leq N.
\] (1.13)

On the other hand, we say that a point \( x \in \mathbb{R}^n \) is \( d \)-badly approximable if there exists a constant \( \varepsilon = \varepsilon(x) \in (0,1) \) so that
\[
d(x,L) > \varepsilon H(L)^{-1-\omega_d}
\] (1.14)
for all \( d \)-dimensional rational affine subspaces \( L \subset \mathbb{R}^n \). Finally, we let \( \text{Sing}_n^d \) (resp. \( \text{Bad}_n^d \)) denote the set of \( d \)-singular (resp. \( d \)-badly approximable) points in \( \mathbb{R}^n \).

The following shows that the well known classical equivalence between the simultaneous and dual singular points (and indeed badly approximable points) holds in the general context of approximation by \( d \)-dimensional rational affine subspaces. We provide a proof in §2.2.

Proposition 1.1. Let \( n \in \mathbb{N} \) and \( d \) be integer satisfying \( 0 \leq d \leq n - 1 \). Then,
\[
\text{Sing}_n := \text{Sing}_n^0 = \text{Sing}_n^d \quad \text{and} \quad \text{Bad}_n := \text{Bad}_n^0 = \text{Bad}_n^d.
\]
Remark 1.2. Note that for the purpose of defining $d$-singular and $d$-badly approximable points it makes no difference whether the minimal distance $d(x, L)$ is defined via the maximum norm (as in (1.10)) or some other norm (such as the Euclidean norm). The point is that these notions are not sensitive to the actual value of the constant $c = c(x, n)$ appearing in Theorem 1.3 and its corollary. Thus, most importantly, the set $\text{Sing}_n^d$ (resp. $\text{Bad}_n^d$) coincides with the classical simultaneous singular (resp. badly approximable) set when $d = 0$ and the dual singular (resp. badly approximable) set when $d = n-1$. However, when it comes to defining the ‘right’ notion of $d$-Dirichlet improvable it is paramount that $c$ is optimal and that Theorem 1.3 coincides with the classical simultaneous and dual forms of Dirichlet theorem. Clearly, in its current form it fails to do so. The fact that it is possible to establish such a version of Theorem 1.3 by appropriately defining the height $H(L)$ and the minimal distance $d(x, L)$, is the main subject of the appendix. Although it is not particularly relevant within the context of our main result, we hope the appendix is of independent interest. Indeed, having defined the sets $\text{DI}_n^d$ ($0 \leq d \leq n-1$) of $d$-Dirichlet improvable points, we show that any two corresponding to the pair $(d, n-1-d)$ are equivalent. Note that when $d = 0$, this equivalence is in line with the aforementioned statement of Davenport & Schmidt; namely, that the dual form of Dirichlet’s theorem is improvable if and only if the simultaneous form of Dirichlet theorem is improvable.

In order to state our main result, it is convenient to introduce the notion of exponents of Diophantine approximation.

**Definition 1.1.** Let $d$ be an integer with $0 \leq d \leq n-1$ and let $x \in \mathbb{R}^n$. We define the $d^{th}$ ordinary exponent $\omega_d(x)$ (resp. the $d^{th}$ uniform exponent $\hat{\omega}_d(x)$) as the supremum of the real numbers $\omega$ for which there exist $d$-dimensional rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$d(x, L) \leq H(L)^{-1}N^{-\omega} \quad \text{and} \quad H(L) \leq N.$$

for arbitrarily large real numbers $N$ (resp. for every sufficiently large real number $N$).

**Remark 1.3.** By definition, whenever $\omega_d(x)$ is finite, there exists infinitely many $d$-dimensional rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$d(x, L) \leq H(L)^{-1-\omega}$$

if $\omega < \omega_d(x)$, and if $\omega > \omega_d(x)$ there are at most finitely many such subspaces $L \subset \mathbb{R}^n$.

**Remark 1.4.** In [8] a point $x \in \mathbb{R}^n$ satisfying $\hat{\omega}_0(x) > 1/n$ (equivalently $\hat{\omega}_{n-1}(x) > n$) is called very singular and the set of such points is denoted by $\text{VSing}_n$. In the context of approximation by $d$-dimensional rational affine subspaces, it is natural to define the notion of $d$-very singular points as points in the set

$$\text{VSing}_n^d := \{x \in \mathbb{R}^n : \hat{\omega}_d(x) > \omega_d\}.$$

As is the case of badly approximable and singular sets, it turns out that the sets $\text{VSing}_n^d$ ($0 \leq d \leq n-1$) are the same – see Remark 1.5 below. By definition, a $d$-very singular point is $d$-singular and since both notions are independent of $d$ we can simply write $\text{VSing}_n \subseteq \text{Sing}_n$. 

Within the classical simultaneous and dual forms of Diophantine approximation, the above exponents were introduced by Khintchine [15, 16] and Jarník [14] in the nineteen twenties and thirties. For \( n \geq 3 \), the intermediate exponents (i.e., those corresponding to \( 1 \leq d \leq n - 2 \)) were formally introduced by Laurent [17] in 2009 but had implicitly been studied by Schmidt [21] some fifty years earlier. Clearly, for any \( x \in \mathbb{R}^n \) we have that \( \omega_d(x) \geq \hat{\omega}_d(x) \) and a direct consequence of Theorem 1.3 is that
\[
\omega_d(x) \geq \hat{\omega}_d(x) \geq \omega_{d+1} := \frac{d + 1}{n - d}.
\]

Observe that \( \omega_0 = \frac{1}{n} \) and \( \omega_{n-1} = n \). Thus, when \( d = 0 \) (resp. \( d = n - 1 \)) the quantity \( \omega_d \) coincides with the exponent appearing in the classical simultaneous (resp. dual) form of Dirichlet’s theorem. Another reasonably straightforward consequence, this time of the Borel-Cantelli lemma from probability theory, is that
\[
\omega_d(x) = \omega_d \quad \text{for almost all} \quad x \in \mathbb{R}^n.
\]

The following elegant transference principle enables us to transfer information between the ordinary Diophantine exponents \( \omega_d(x) \) associated with approximating points in \( x \in \mathbb{R}^n \) by \( d \)-dimensional rational subspaces of \( \mathbb{R}^n \). It makes sense to include the statement at this point since one of the conditions turns up in the statement of our main theorem.

**Theorem 1.4** (Laurent & Roy). Let \( n \geq 2 \). For any point \( x \in \mathbb{R}^n \) with \( 1, x_1, \ldots, x_n \) linearly independent over \( \mathbb{Q} \), we have that \( \omega_0(x) \geq \omega_0 \) and
\[
\frac{d \omega_d(x)}{\omega_d(x) + d + 1} \leq \omega_{d-1}(x) \leq \frac{(n - d) \omega_d(x) - 1}{n - d + 1} \quad (1 \leq d \leq n - 1).
\]

If \( \omega_d(x) = \infty \), the left hand side in (1.16) is replace by \( d \). Furthermore, given any \( n \)-tuple of real numbers \( \tau_0, \ldots, \tau_{n-1} \in [0, \infty] \) with \( \tau_0 \geq \omega_0 \) and
\[
\frac{d \tau_d}{\tau_d + d + 1} \leq \tau_{d-1} \leq \frac{(n - d) \tau_d - 1}{n - d + 1} \quad (1 \leq d \leq n - 1),
\]

there exists a point \( x \in \mathbb{R}^n \) with \( 1, x_1, \ldots, x_n \) linearly independent over \( \mathbb{Q} \) such that \( \omega_d(x) = \tau_d \) and \( \hat{\omega}_d(x) = \omega_d \) for \( 0 \leq d \leq n - 1 \).

The transference inequalities (1.16) are due to Laurent [17]. Equivalently, they can be re-written in the language of Schmidt [21] as the **Going-up transfer**
\[
\omega_{d+1}(x) \geq \frac{(n - d) \omega_d(x) + 1}{n - d - 1} \quad (0 \leq d \leq n - 2)
\]

and the **Going-down transfer**
\[
\omega_{d-1}(x) \geq \frac{d \omega_d(x)}{\omega_d(x) + d + 1} \quad (1 \leq d \leq n - 1).
\]

As pointed in [17], on iterating (1.18) and (1.19) we obtain Khintchine’s classical transference principle [15]:
\[
\frac{\omega_{n-1}(x)}{(n - 1)\omega_{n-1}(x) + n} \leq \omega_0(x) \leq \frac{\omega_{n-1}(x) - n + 1}{n}.
\]
Thus the transference inequalities (1.16) of Laurent naturally split those of Khintchine relating the simultaneous and dual exponents $\omega_0(x)$ and $\omega_{n-1}(x)$. The furthermore part of Theorem 1.4 shows that transference inequalities of Laurent are optimal and was proved by Roy [20]. It extends the classical work of Jarník [14] showing that Khintchine’s transference principle is optimal.

**Remark 1.5.** The Laurent transference inequalities (1.16) are equally valid for the uniform exponents. Indeed, Laurent’s proof for the ordinary exponents can be naturally adapted to the uniform setting – see for example [11]. With (1.16) for uniform exponents at hand, it is easily seen that for any $x \in \mathbb{R}^n$ and $1 \leq d \leq n$, the statement that $\hat{\omega}_d(x) = \omega_d$ is equivalent to $\hat{\omega}_{d-1}(x) = \omega_{d-1}$. Hence, it follows that the very singular sets $V_{Sing}^d (0 \leq d \leq n-1)$ discussed within Remark 1.4 are equivalent.

As usual let $d$ be an integer with $0 \leq d \leq n-1$. Then given a real number $\tau \geq 0$, consider the Diophantine sets

$$W^d_n(\tau) := \{ x \in \mathbb{R}^n : \omega_d(x) \geq \tau \}$$

and

$$E^d_n(\tau) := \{ x \in \mathbb{R}^n : \omega_d(x) = \tau \} .$$

In dimension one, the latter corresponds to the exact order sets first studied by Güting within the context of Mahler’s classification of transcendental numbers – see [1] and references within for further details. Note that by definition, for any $0 \leq d \leq n-1$ we have that

$$W^d_n(\tau) = \mathbb{R}^n \quad \text{if} \quad \tau \leq \omega_d$$

and

$$\text{Bad}_n \cap W^d_n(\tau) = \emptyset \quad \text{if} \quad \tau > \omega_d .$$

Note that in view of (1.15), the set $E^d_n(\omega_d)$ is of full $n$-dimensional Lebesgue measure and since $E^d_n(\tau) \subseteq W^d_n(\tau)$, it follows that $\text{Bad}_n \cap E^d_n(\tau) = \emptyset$ if $\tau > \omega_d$.

Using the parametric geometry of numbers a la Schmidt & Summerer [22, 23] and Roy [19], we prove the following theorem. It constitutes our main result.

**Theorem 1.5.** Let $n \geq 2$ and $\varepsilon \in (0, 1)$. Then, given any $n$-tuple of real numbers $\tau_0, \ldots, \tau_{n-1} \in [0, \infty]$ with $\tau_0 \geq \omega_0$ and $\tau_d$ ($1 \leq d \leq n-1$) satisfying (1.17), the set

$$\left( \bigcap_{d=0}^{n-1} E^d_n(\tau_d) \cap \left( DI_n(\varepsilon) \setminus DI_n(\varepsilon e^{-10(n+1)^2(n+10)}) \right) \right) \setminus (\text{Bad}_n \cup \text{Sing}_n)$$

is uncountable. In particular, for any $\tau \geq \omega_d$, the set $(DI_n \cap E^d_n(\tau)) \setminus (\text{Bad}_n \cup \text{Sing}_n)$ is uncountable.

Note that on taking $\tau = \omega_d$ in the ‘in particular’ part of the Theorem 1.5, we immediately obtain the statement of Theorem 1.2.
2 Preliminaries

In this section we start by recalling aspects of the theory of parametric geometry of numbers that will be used in establishing Theorem \[1.5\]. We then use this to essentially reformulate the Diophantine sets appearing in the statement of Theorem \[1.5\] in terms of successive minima. Moreover, we will see that the proof of Proposition \[1.1\] is a pretty straightforward application of this reformulation.

2.1 The parametric geometry of numbers

Fix \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^n \). For each real number \( t \geq 0 \), consider the convex body

\[
\mathcal{C}_x(e^t) := \left\{ y \in \mathbb{R}^{n+1} : |y_i| \leq 1 \ (1 \leq i \leq n), \quad \sum_{i=1}^{n} y_i x_i + y_{n+1} \leq e^{-t} \right\}. \tag{2.1}
\]

Then, for each \( i = 1, \ldots, n+1 \), let \( \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_x(e^t)) \) denotes the \( i \)-th successive minima of the convex body \( \mathcal{C}_x(e^t) \) with respect to the lattice \( \mathbb{Z}^{n+1} \). In other words, \( \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_x(e^t)) \) is the smallest real number \( \lambda \) such that the rescaled convex body \( \lambda \mathcal{C}_x(e^t) \) contains at least \( i \) linearly independent points of \( \mathbb{Z}^{n+1} \). In turn, following Schmidt & Summerer \[23\], we let

\[
L_{x,i}(t) := \log \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_x(e^t)) \quad (t \geq 0, \ 1 \leq i \leq n+1) \tag{2.2}
\]

and consider the map

\[
L_x : [0, \infty) \to \mathbb{R}^{n+1} : t \to L_x(t) := (L_{x,1}(t), \ldots, L_{x,n+1}(t)). \tag{2.3}
\]

The following notion was introduced by Roy in \[20\] Definition 4.5]. It generalises the \((n+1)\)-systems of Schmidt & Summerer \[23\]. In short, these ‘systems’ incorporate desirable behavior of the maps \( L_x \) that in turn lead to desirable approximation results.

**Definition 2.1.** Let \( I \) be an subinterval of \([0, \infty)\) with non-empty interior. A Roy \((n+1)\)-system on \( I \) is a continuous piecewise linear map \( P = (P_1, \ldots, P_{n+1}) : I \to \mathbb{R}^{n+1} \) with the following properties:

- For each \( t \in I \), we have \( 0 \leq P_1(t) \leq \cdots \leq P_{n+1}(t) \) and \( P_1(t) + \cdots + P_{n+1}(t) = t \).
- If \( I' \subset I \) is a nonempty open subinterval on which \( P \) is differentiable, then there are integers \( r_1, r_2 \) with \( 1 \leq r_1 \leq r_2 \leq n+1 \) such that the functions \( P_{r_1}, P_{r+1}, \ldots, P_{r_2} \) coincide on the whole interval \( I' \) and have slope \( 1/(r_2-r_1+1) \) on \( I' \), while all other components \( P_i \) of \( P \) have slope 0 on \( I' \).
- If \( t \) is an interior point of \( I \) at which \( P \) is not differentiable and if \( r_1, r_2, s_1, s_2 \) are integers for which

\[
P_i(t^-) = \frac{1}{r_2 - r_1 + 1} \quad (r_1 \leq i \leq r_2) \quad \text{and} \quad P_i(t^+) = \frac{1}{s_2 - s_1 + 1} \quad (s_1 \leq i \leq s_2),
\]

and if \( r_1 \leq s_2 \), then we have that \( P_{r_1}(t) = P_{r+1}(t) = \cdots = P_{s_2}(t) \).
Note that, for any piecewise linear function $F : \mathbb{R} \to \mathbb{R}$, the left derivative $F'(t^-)$ and the right derivative $F'(t^+)$ always exist and the points at which $F$ is not differentiable are just the points with different left and right derivatives.

**Remark 2.1.** The $(n + 1)$–systems of Schmidt & Summerer correspond to taking $r_1 = r_2$ and $s_1 = s_2$ in Definition 2.1.

A Roy $(n + 1)$–system has the following useful approximation property. It essentially represents an amalgamation of [19] Theorems 1.3 & 1.8 and [20] Corollary 4.7 adapted for our purposes.

**Theorem 2.1.** Let $n \in \mathbb{N}$ and $t_0 \geq 0$. For each $x \in \mathbb{R}^n$, there exists a Roy $(n + 1)$–system $P : [t_0, \infty) \to \mathbb{R}^{n+1}$ such that the function $L_x - P$ is bounded on $[t_0, \infty)$. Conversely, for each Roy $(n + 1)$–system $P : [t_0, \infty) \to \mathbb{R}^{n+1}$, there exists $x \in \mathbb{R}^n$ such that the function $L_x - P$ is bounded on $[t_0, \infty)$. In particular, for each $t \geq t_0$

$$\|L_x(t) - P(t)\| \leq 5(n + 1)^2(n + 10).$$

**Proof of Theorem 2.1.** As already mentioned, Theorem 2.1 draws upon the works of Roy and it is important to note that there, the convex body is defined slightly differently from the one given by (2.1). Indeed, for a fixed $u \in \mathbb{R}^{n+1} \setminus \{0\}$ and each real number $t \geq 0$, Roy works with the convex body

$$\tilde{C}_u(e^t) := \{y \in \mathbb{R}^{n+1} : \|y\| \leq 1, \quad |y \cdot u| \leq e^{-t}\}.$$

Now for any fixed $x \in \mathbb{R}^n$, let $x' := (x, 1) \in \mathbb{R}^{n+1}$ and so by definition

$$\tilde{C}_{x'}(e^t) = \left\{y \in \mathbb{R}^{n+1} : \|y\| \leq 1, \quad \left| \sum_{i=1}^{n} y_i x_i + y_{n+1} \right| \leq e^{-t} \right\}.$$

Furthermore, let $\tilde{L}_{x'}$ denote the function corresponding to (2.3) with $C_x(e^t)$ replaced by $\tilde{C}_{x'}(e^t)$ within (2.2). It is not difficult to see that our convex body and the associated map, which are convenient for what we have in mind, are closely related to those of Roy and indeed Schmidt & Summerer: for any fixed $x \in \mathbb{R}^n$ and any $t \geq 0$

$$\tilde{C}_{x'}(e^t) \subset C_x(e^t) \subset (n\|x\| + 1) \tilde{C}_{x'}(e^t)$$

and thus it follows that

$$\|L_x(t) - \tilde{L}_{x'}(t)\| \leq \log(n\|x\| + 1).$$

We now proceed with establishing the theorem. On combining [19] Theorem 1.3 and [19] Lemma 2.10, we find that for any $x \in \mathbb{R}^n$ there exists a $(n + 1)$–system (see Remark 2.1) $P$ such that the function $\tilde{L}_{u_{x'}} - P$ is bounded for all $t \geq t_0$. Here $u_{x'}$ denotes the unit vector associated to $x' \in \mathbb{R}^{n+1}$. Note that $\tilde{L}_{u_{x'}}$ and $\tilde{L}_{x'}$ only differ by a constant and so the first part of the theorem follows on using (2.4) and the fact that by definition any $(n + 1)$–system is a Roy $(n + 1)$–system. Regarding the converse part, it follows via [20].
Corollary 4.7] that for any given Roy \((n + 1)\)-system \(P : [t_0, \infty) \to \mathbb{R}^{n+1}\) and any \(\varepsilon > 0\), there is a \((n + 1)\)-system \(\bar{P}\) such that

\[ \|P(t) - \bar{P}(t)\| \leq \varepsilon \quad \text{for all } t \geq t_0. \] (2.5)

In view of [19, Theorem 8.1], there exists a unit vector \(u \in \mathbb{R}^{n+1}\) such that

\[ \|\bar{P}(t) - \bar{L}u(t)\| \leq 3(n + 1)^2(n + 10) \quad \text{for all } t \geq t_0. \] (2.6)

Without loss of generality, we can assume that \(u_{n+1} = \|u\| := \max\{|u_1|, \ldots, |u_{n+1}|\}\) and so \((n + 1)^{-1/2} \leq |u_{n+1}| \leq 1\). Now let

\[ x := (u_1u_{n+1}^{-1}, \ldots, u_nu_{n+1}^{-1}) \in \mathbb{R}^n. \]

Then, \(\|x\| \leq 1\) and

\[ \|\bar{L}x(t) - \bar{P}(t)\| \leq \frac{1}{2} \log(n + 1) \quad \text{for all } t \geq t_0. \] (2.7)

The upshot is that on using (2.5) with \(\varepsilon := \log(n + 1)/2\), \(2.4\), \(2.7\) and \(2.6\) in that order, we obtain that

\[ \|L_x(t) - P(t)\| \leq \frac{1}{2} \log(n + 1) + \log(n + 1) + \frac{1}{2} \log(n + 1) + 3(n + 1)^2(n + 10) \]

\[ < 5(n + 1)^2(n + 10) \quad \text{for all } t \geq t_0. \] (2.8)

This completes the proof of Theorem 2.1. 

The following notion of non-equivalent systems will prove to be useful.

**Definition 2.2.** Two Roy \((n + 1)\)-systems \(P_1\) and \(P_2\) defined on the same subinterval \(I\) of \([0, \infty)\) are said to be non-equivalent if there exists some \(t \in I\) such that

\[ |P_1(t) - P_2(t)| > 10(n + 1)^2(n + 10). \]

By definition, it follows that no point in \(\mathbb{R}^n\) can be close (in the sense of Theorem 2.1) to two non-equivalent Roy \((n + 1)\)-systems defined on \([t_0, \infty)\) at the same time.

### 2.2 Expressing Diophantine sets via successive minima

We give a reformulation of the Diophantine sets associated with Theorem 1.5 in terms of the function \(L_x\). This is at the heart of its proof – it brings into play the parametric geometry of numbers. Also, we shall see that the equivalence of the \(d\)-badly approximable sets \(\text{Bad}^d_n\) (resp. the \(d\)-singular sets \(\text{Sing}^d_n\)) is in essence a direct consequence of the reformulation. Indeed, we start with this in mind.

Let \(n \geq 2\) and \(0 \leq d \leq n - 1\). It can be verified, by using the lemma appearing in [3, Section 4] and appropriately adapting the proof of the proposition in [3, Section 4], that

- \(x \in \text{Bad}^d_n\) if and only if there exists a constant \(\delta > 0\) such that for all sufficiently large \(t\)

\[ \frac{(n - d)t}{n + 1} - (L_{x,1}(t) + \cdots + L_{x,n-d}(t)) \leq \delta. \] (2.9)
• $x \in \text{Sing}^d_{n-1}$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$

$$\frac{(n - d)t}{n + 1} - (L_{x,1}(t) + \cdots + L_{x,n-d}(t)) \geq \delta.$$

(2.10)

For the sake of completeness, in the appendix, we will provide the details of how these equivalences follow from [3, Section 4]. We can now swiftly show that

$$\text{Bad}^d_n = \text{Bad}^{n-1}_n \quad \text{and} \quad \text{Sing}^d_n = \text{Sing}^{n-1}_n \quad (0 \leq d \leq n - 2);$$

that is to say that any $d$–badly approximable set (resp. $d$–singular set) is equivalent to the dual set. This will of course establish Proposition 1.1.

Proof of Proposition 1.1. For simplicity, given $x \in \mathbb{R}^n$ we let

$$g_{x,i}(t) := \frac{t}{n + 1} - L_{x,i}(t) \quad (0 \leq i \leq n + 1).$$

By definition the quantity $L_{x,i}$ is increasing with $i$ and so it follows that

$$g_{x,1}(t) \geq g_{x,2}(t) \geq \cdots \geq g_{x,n+1}(t).$$

(2.11)

In view of Minkowski’s second convex body theorem, for any $x \in \mathbb{R}^n$ we have that

$$L_{x,1}(t) + L_{x,2}(t) + \cdots + L_{x,n+1}(t) = t + O(1).$$

Thus, there exists a positive constant $c = c(n) > 0$ depending only on $n$ such that

$$g_{x,1}(t) + g_{x,2}(t) + \cdots + g_{x,n+1}(t) \geq -c.$$

(2.12)

Now suppose $x \in \text{Bad}^d_n$. Then in view of (2.9) and (2.12), it follows that

$$\sum_{i=n-d+1}^{n+1} g_{x,i}(t) \geq -\delta - c$$

which together with (2.11) implies that

$$g_{x,i}(t) \geq \frac{1}{d + 1} \sum_{j=n-d+1}^{d+1} g_{x,j}(t) \geq -\frac{\delta + c}{d + 1} \quad (1 \leq i \leq n - d).$$

In turn, on using (2.11) again, we find that

$$g_{x,1}(t) \leq \sum_{i=1}^{n-d} g_{x,i}(t) - \sum_{i=2}^{n-d} g_{x,i}(t) \leq \delta + (n - d - 1)\frac{\delta + c}{d + 1} < \frac{n}{d + 1}(\delta + c).$$

(2.13)

In other words, (2.13) holds with $d = n - 1$ and so $x \in \text{Bad}^{n-1}_n$. For the converse, simply observe that if (2.9) holds with $d = n - 1$ then for any other $0 \leq d \leq n - 2$

$$\sum_{i=1}^{n-d} g_{x,i}(t) \leq (n - d) g_{x,1}(t) \leq (n - d)\delta.$$

12
In other words, \( x \in \text{Bad}_{n}^{d} \) and this thereby completes the proof of the badly approximable part of the proposition. The proof of the singular part is similar with the most obvious modifications (namely, using (2.10) instead of (2.9)) and will be left for the reader.

\[ \square \]

**Remark 2.2.** In the appendix, apart from providing details of the statements associated with (2.9) and (2.10), we give a ‘dynamical’ proof of Proposition 1.1. In addition to providing an alternative insight, it has the advantage of being self-contained in that it avoids appealing to (2.9) and (2.10) which rely on the lemma and the arguments appearing in [3, Section 4].

The following statement summarises the above findings concerning the badly approximable and singular sets and deals with the other remaining Diophantine sets associated with Theorem 1.5.

**Lemma 2.1.** Let \( x \in \mathbb{R}^{n} \). Then

1. \( x \in \text{DI}_{n}(\varepsilon) \) if and only if for all sufficiently large \( t \)
   \[
   \frac{t}{n + 1} - L_{x,1}(t) \geq -\frac{\log \varepsilon}{n + 1}.
   \]

2. \( x \in \text{Bad}_{n} \) if and only if there exists \( \delta > 0 \) such that
   \[
   \limsup_{t \to \infty} \left( \frac{t}{n + 1} - L_{x,1}(t) \right) \leq \delta.
   \]

3. \( x \in \text{Sing}_{n} \) if and only if for any \( \delta > 0 \)
   \[
   \liminf_{t \to \infty} \left( \frac{t}{n + 1} - L_{x,1}(t) \right) \geq \delta.
   \]

4. \( x \in \text{W}_{n}^{d}(\tau) \) if and only if
   \[
   \liminf_{t \to \infty} \frac{L_{x,1}(t) + \cdots + L_{x,n-d}(t)}{t} \leq \frac{1}{1 + \tau} \quad (0 \leq d \leq n-1).
   \]

5. \( x \in \text{E}_{n}^{d}(\tau) \) if and only if
   \[
   \liminf_{t \to \infty} \frac{L_{x,1}(t) + \cdots + L_{x,n-d}(t)}{t} = \frac{1}{1 + \tau} \quad (0 \leq d \leq n-1).
   \]

**Proof of Lemma 2.1.** Parts 4) and 5) are a direct consequence of [20, Proposition 3.1]. The proof of parts 2) and 3) are a direct consequence of (2.9) and (2.10) respectively together with Proposition 1.1. It remains to prove part 1). Thus, let \( x \in \text{DI}_{n}(\varepsilon) \) for some \( \varepsilon \in (0,1) \). Then by definition, for all sufficiently large \( t \)

\[
|x \cdot q - p| \leq \varepsilon e^{-nt} \quad \text{and} \quad \|q\| \leq e^{t}
\]

always has a solution \( (p, q) \in \mathbb{Z} \times (\mathbb{Z}^{n} \setminus \{0\}) \). This is equivalent to saying that for all sufficiently large \( t \)

\[
\lambda_{1}(\mathbb{Z}^{n+1}, C_{x}(e^{t'})) \leq e^{t}, \quad \text{where} \quad t' = (n + 1)t - \log \varepsilon.
\]
Hence, for all sufficiently $t$
\[
\frac{t}{n+1} - L_{x,1}(t) \geq -\frac{\log \varepsilon}{n+1}
\]
as desired. \hfill \square

**Remark 2.3.** It is relatively straightforward to see that the proof of part 1) given above can be easily adapted to establish (2.9) and (2.10) when $d = n - 1$.

## 3 Proof of Theorem 1.5

Let $n \geq 2$, $\varepsilon \in (0,1)$ and $\tau_0, \ldots, \tau_{n-1}$ be as in Theorem 1.5 and let
\[
\gamma := -\frac{\log \varepsilon}{n+1} + C_n \quad \text{where} \quad C_n := 5(n+1)^2(n+10). \quad (3.1)
\]
Thus, $C_n$ is simply the right hand side of inequality appearing in Theorem 2.1. Then, on making use of Theorem 2.1 and Lemma 2.1, it is easily verified that the proof of Theorem 1.5 is reduced to constructing appropriate Roy $(n+1)$–systems given by the following statement.

**Lemma 3.1.** There exists uncountable many mutually non-equivalent Roy $(n+1)$–systems $P : [0, \infty) \to \mathbb{R}^{n+1}$, such that
\[
\liminf_{t \to \infty} \left( \frac{t}{n+1} - P_1(t) \right) = \gamma, \quad (3.2)
\]
\[
\limsup_{t \to \infty} \left( \frac{t}{n+1} - P_1(t) \right) = \infty, \quad (3.3)
\]
and
\[
\liminf_{t \to \infty} \frac{P_1(t) + \cdots + P_d(t)}{t} = \frac{1}{1 + \tau_{n-d}} \quad (1 \leq d \leq n). \quad (3.4)
\]

**Proof of Theorem 1.5 modulo Lemma 3.1.** Let us assume Lemma 3.1 and let $P : [0, \infty) \to \mathbb{R}^{n+1}$ be a Roy $(n+1)$–system coming from the lemma. In view of the converse part of Theorem 1.1, there exists a point $x \in \mathbb{R}^n$ such that
\[
\|L_x(t) - P(t)\| \leq C_n.
\]
Then this together with Lemma 2.1 and
\begin{itemize}
  \item (3.2) implies $x \in DI_n(\varepsilon) \setminus DI_n(\varepsilon e^{-2C_n})$ and $x \notin Sing_n$,
  \item (3.3) implies $x \notin Bad_n$,
  \item (3.4) implies $x \in \cap_{d=0}^{n-1} E_n^d(\tau_d).
\end{itemize}
The upshot of this is that

\[
x \in \left( \bigcap_{d=0}^{n-1} E_{n}^{d}(\tau_{d}) \cap \left( DI_{n}(\varepsilon) \setminus DI_{n}(\varepsilon e^{-2C_{n}}) \right) \right) \setminus \left( \text{Bad}_{n} \cup \text{Sing}_{n} \right).
\]

Furthermore, Lemma 3.1 implies the existence of uncountably many such Roy \((n + 1)\)–systems that are mutually non-equivalent. Thus, in view of the latter (see Definition 2.2), each such system gives rise to a different point \(x \in \mathbb{R}^{n}\) and this completes the proof of Theorem 1.5.

The proof of Lemma 3.1 will occupy the rest of this section. It will comprise of three steps. We start by constructing Roy \((n + 1)\)–systems on certain finite intervals which will serve as building blocks for the construction of the desired systems associated with Lemma 3.1.

3.1 Building Blocks.

Let \([T_{-}, T_{+}]\) be a subinterval of \([0, \infty)\) with non-empty interior and let \((a_{-}^{j})_{1 \leq j \leq n+1}, (a_{+}^{j})_{1 \leq j \leq n+1}\) be sequences of increasing positive numbers satisfying:

\[
a_{+}^{1} T_{+} = a_{-}^{n+1} T_{-} - (n + 1) \gamma
\]

\[
\sum_{1 \leq j \leq n+1} a_{+}^{j} = 1
\]

\[
(a_{+}^{j+1} - a_{+}^{j}) T_{*} \geq 4n^{2} \gamma \quad \forall \quad 1 \leq j \leq n,
\]

where \(\gamma\) is as in (3.1) and throughout

\(* := - \text{ or } + .\)

We now construct a Roy \((n + 1)\)–system

\[P = (P_{1}, \ldots, P_{n+1}) : [T_{-}, T_{+}] \to \mathbb{R}^{n+1}\]

on \([T_{-}, T_{+}]\) associated with the sequences \((a_{-}^{j})\) and \((a_{+}^{j})\). With this in mind, let

\[R_{d} := \left( a_{-}^{n+1} + \cdots + a_{-}^{d+1} + da_{-}^{d} \right) T_{-} \quad \forall \quad 1 \leq d \leq n\]

\[R_{n+1} := (n + 1) a_{-}^{n+1} T_{-} - n(n + 1) \gamma\]

\[R_{n+2} := (n + 1) a_{-}^{n+1} T_{-} - (n + 1) \gamma\]

\[S_{0} := (n + 1) a_{+}^{1} T_{+} + (n^{2} + n) \gamma\]

\[S_{d} := \left( a_{+}^{1} + \cdots + a_{+}^{d} + (n + 1 - d)a_{+}^{d+1} \right) T_{+} \quad \forall \quad 1 \leq d \leq n.\]
In view of (3.5), it is easily seen that $S_0 = R_{n+2}$. Also, (3.6) ensures that $T_- = R_1$ and

$S_n = T_+$ while (3.7) gives that $R_{n+1} \geq R_n$ and $S_1 \geq S_0$. Since $(a^j_d)$ is strictly increasing, it thus follows that

$$T_- = R_1 < R_2 < \cdots < R_{n+2} = S_0 < S_1 < \cdots < S_n = T_+.$$

Now set

$$P_j(T_-) := a^j_1 T_- \quad \forall \quad 1 \leq j \leq n + 1.$$

For $1 \leq d \leq n - 1$, on the interval $[R_d, R_{d+1}]$, let the $d$ components $P_1, \ldots, P_d$ coincide and have slope $1/d$ while the components $P_{d+1}, \ldots, P_{n+1}$ have slope 0 and

$$P_j(R_{d+1}) := \begin{cases} a^{d+1}_d T_- & \text{if } 1 \leq j \leq d \\ a^j_1 T_- & \text{if } d + 1 \leq j \leq n + 1. \end{cases}$$

On the interval $[R_n, R_{n+1}]$, let the $n$ components $P_1, \ldots, P_n$ coincide and have slope $1/n$ while the component $P_{n+1}$ has slope 0 and

$$P_j(R_{n+1}) := \begin{cases} a^1_1 T_+ & \text{if } 1 \leq j \leq n \\ a^n_{n+1} T_- & \text{if } j = n + 1. \end{cases}$$

On the interval $[R_{n+1}, R_{n+2}]$, let the $n - 1$ components $P_2, \ldots, P_n$ coincide and have slope $1/(n-1)$ while the components $P_1, P_{n+1}$ have slope 0 and

$$P_j(R_{n+2}) := \begin{cases} a^1_1 T_+ & \text{if } j = 1 \\ a^n_{n+1} T_- & \text{if } 2 \leq j \leq n + 1. \end{cases}$$

On the interval $[S_0, S_1]$, let the $n$ components $P_2, \ldots, P_{n+1}$ coincide and have slope $1/n$ while the component $P_1$ has slope 0 and

$$P_j(S_1) := \begin{cases} a^1_1 T_+ & \text{if } j = 1 \\ a^j_2 T_+ & \text{if } 2 \leq j \leq n + 1. \end{cases}$$

Finally, for $1 \leq d \leq n - 1$, on the interval $[S_d, S_{d+1}]$, let the $n - d$ components $P_{d+2}, \ldots, P_{n+1}$ coincide and have slope $1/(n - d)$ while the components $P_1, \ldots, P_{d+1}$ have slope 0 and

$$P_j(S_{d+1}) := \begin{cases} a^j_1 T_+ & \text{if } 1 \leq j \leq d + 1 \\ a^d_{d+2} T_+ & \text{if } d + 2 \leq j \leq n + 1. \end{cases}$$

In particular, since $S_n = T_+$ it follows that

$$P_j(T_+) = a^j_1 T_+ \quad \forall \quad 1 \leq j \leq n + 1.$$

Figure 11 below represents the combined graph of the functions $P_1, \ldots, P_{n+1}$ over the interval $[T_-, T_+]$. Note that if we set $\gamma = 0$ and $a^j_+ = a^j_-$ for $j = 1, \ldots, n + 1$, our
construction reduces to Roy’s construction in [20, Section 5] – in particular, see [20, Figure 5].

We conclude this section with the following statement. It provides keys estimates for $P(t)$ with $t \in [T_-, T_+]$.

Lemma 3.2. The Roy $(n + 1)$–system $P : [T_-, T_+] \to \mathbb{R}^{n+1}$ constructed above satisfies:

$$
\min_{t \in [T_-, T_+]} \left( \frac{t}{n+1} - P_1(t) \right) = \gamma \tag{3.8}
$$

$$
\max_{t \in [T_-, T_+]} \left( \frac{t}{n+1} - P_1(t) \right) \geq \left( \frac{1}{n+1} - a_+^1 \right) T_+ \tag{3.9}
$$

$$
\min_{t \in [T_-, T_+]} \frac{P_1(t) + \cdots + P_d(t)}{t} = \min \left\{ \sum_{j=1}^{d} a_{j-}^2, \sum_{j=1}^{d} a_{j+}^2 \right\} \quad (1 \leq d \leq n). \tag{3.10}
$$

Proof. By construction the derivative of the function $P_1$ is strictly greater than $1/(n+1)$ on the interval $[T_-, R_{n+1}]$ and is 0 on the interval $[R_{n+1}, T_+]$. Here and throughout, by the derivative of a piecewise linear function on a given interval, we mean the derivative on the union of subintervals on which the derivative exists. It follows that on the interval $[T_-, T_+]$, the local minimum of the function $f : t \to f(t) := t/(n+1) - P_1(t)$ is achieved
at } t = R_{n+1} \text{. In other words, the minimum of } f(t) \text{ on } [T_-, T_+] \text{ is equal to }

\frac{R_{n+1}}{n+1} - P_1(R_{n+1}) = a_1^{n+1}T_+ - n\gamma - a_1^1T_+ = \gamma \tag{3.5}

This shows that } P \text{ satisfies (3.5). On the other hand, it is easily seen that the maximum of } f(t) \text{ on } [T_-, T_+] \text{ is achieved at either } t = T_- \text{ or } t = T_+. \text{ Thus, }

\text{l.h.s. of (3.5) } \geq \frac{T_+}{n+1} - P_1(T_+) = \frac{T_+}{n+1} - a_1^1 T_+ \text{ and this shows that } P \text{ satisfies (3.5). It remains to prove (3.10). For simplicity, we let}

\begin{equation}
Q_d := P_1 + \cdots + P_d \quad (1 \leq d \leq n)
\end{equation}

and note that to determine when } Q_d(t)/t \text{ attains its minimum on } [T_-, T_+], \text{ it suffices to study the function}

\begin{equation}
D : t \to D(t) := Q'_d(t)t - Q_d(t)
\end{equation}

On each connected open interval where } P \text{ is differentiable, it is easily verified that the derivative } D'(t) = 0 \text{ and so } D(t) \text{ is constant. Hence, it suffices to study the quantities } D^*(R_j) \text{ and } D^*(S_j) \text{ for } 1 \leq j \leq n. \text{ This we now do systematically. Recall that for a piecewise continuous function } D,

\begin{equation}
D^+(t) := \lim_{s \to t, s > t} D(s), \quad D^-(t) := \lim_{s \to t, s < t} D(s).
\end{equation}

\begin{itemize}
\item For } 1 \leq j \leq d, \text{ on the interval } (R_j, R_{j+1}), \text{ the derivative of } Q_d \text{ equals 1. Hence } D(t) \text{ is positive on this interval.}
\item For } d + 1 \leq j \leq n, \text{ on the interval } (R_j, R_{j+1}), \text{ the derivative of } Q_d \text{ equals } d/j. \text{ A direct computation shows that }

\begin{equation}
jD^+(R_j) = dR_j - jQ_d(R_j)
= d \left( a_1^{n+1} + \cdots + a_1^j + ja_1^j \right) T_+ - jda_1^1 T_+
\end{equation}

Hence } D(t) \text{ is positive on this interval.}
\item For } 0 \leq j \leq d - 1, \text{ on the interval } (S_j, S_{j+1}), \text{ the derivative of } Q_d \text{ equals } (d - j - 1)/(n - j). \text{ A direct computation shows that }

\begin{equation}
(n - j)D^-(S_{j+1}) = (d - j - 1)S_{j+1} - (n - j)Q_d(S_{j+1})
= (d - j - 1) \left( a_1^1 + \cdots + a_1^j + (n - j)a_1^{j+2} \right) T_+
- (n - j) \left( a_1^1 + \cdots + a_1^j + (d - j - 1)a_1^{j+2} \right) T_+
= (d - n - 1) \left( a_1^1 + \cdots + a_1^j \right) T_+
< 0
\end{equation}

Hence } D(t) \text{ is negative on this interval.}
\end{itemize}
• For \( d \leq j \leq n - 1 \), on the interval \((S_j, S_{j+1})\), the derivative of \(Q_d\) equals 0. Hence \(D(t)\) is negative on this interval.

In conclusion, the function \(Q_d(t)/t\) increases on the interval \((R_1, R_{n+1})\) and decrease on the interval \((S_0, S_n)\). As \(Q_d(t)/t\) is monotonic on \((R_{n+1}, S_0)\), the minimum is thus attained at either \(t = T_-\) or \(t = T_+\). Hence,

\[
\text{l.h.s. of (3.10)} = \min \left\{ \frac{Q_d(T_-)}{T_-}, \frac{Q_d(T_+)}{T_+} \right\} = \min \left\{ \sum_{j=1}^{d} \alpha_j, \sum_{j=1}^{d} \alpha_j \right\}
\]

and this shows that \(P\) satisfies (3.10) which in turn completes the proof of the lemma. \(\square\)

### 3.2 The local construction on blocks.

In this section, will exploit the generic construction presented in §3.1 to essentially prove a ‘local’ version of Lemma 3.1. More precisely, we will construct a family of Roy \((n + 1)\)-systems \(P^\delta\) on certain subintervals \(I\) of \([0, \infty)\) all satisfying the properties of Lemma 3.2. Here \(\delta \in [0, 1/(2n^2)]\) is a parameter and for \(\delta' \neq \delta\), we show that the intervals \(I\) can be chosen so that the Roy \((n + 1)\)-systems \(P^\delta\) and \(P^\delta'\) on \(I\) are mutually non-equivalent. The construction consists of five short steps. Throughout, \(n \geq 2\) and \(\tau_0, \ldots, \tau_{n-1} \in [0, \infty]\) are the real numbers appearing in Theorem 1.3 satisfying (1.17).

**Step 1.** For \(1 \leq i \leq n - 1\) and \(1 \leq j \leq n + 1\), let

\[
\alpha^{i,j} := \begin{cases} 
    i^{-1}(1 + \tau_{n-i})^{-1} & \text{if } j \leq i \\
    (1 + \tau_{n-i-1})^{-1} - (1 + \tau_{n-i})^{-1} & \text{if } j = i + 1 \\
    (n-i)^{-1}\tau_{n-i-1}(1 + \tau_{n-i})^{-1} & \text{if } j > i + 1
\end{cases}
\]

where \((1 + \tau_j)^{-1} = 0\) and \(\tau_j(1 + \tau_j)^{-1} = 1\) if \(\tau_j = \infty\). The following statement summarises useful properties of the associated sequence \((\alpha^{i,j})\) that we shall later exploit.

**Lemma 3.3.** Let \((\alpha^{i,j})\) be given as above, then

(a) for any \(1 \leq i \leq n - 1\), \(\sum_{1 \leq j \leq n+1} \alpha^{i,j} = 1\),

(b) for any \(1 \leq i \leq n - 1\) and \(1 \leq j \leq j' \leq n + 1\), \(\alpha^{i,j} \leq \alpha^{i,j'}\),

(c) \(\alpha^{i,1} + \cdots + \alpha^{i,j} \geq (1 + \tau_{n-j})^{-1}\) with equality holds when \(j = i, i + 1\).

**Proof.** Part (a) follows directly from the definition. To prove the other parts, for \(1 \leq i \leq n\) let

\[
\theta_i = (1 + \tau_{n-i})^{-1}.
\]

Then it follows that (1.17) is equivalent to

\[
\frac{(n-d+1)\theta_{n-d}}{n-d} \leq \theta_{n-d+1} \leq \frac{1 + d\theta_{n-d}}{d+1} \quad \forall \quad 1 \leq d \leq n - 1,
\]

19
which in turn is equivalent to
\[
\frac{\theta_i}{i} \leq \frac{\theta_{i+1}}{i+1} \quad \text{and} \quad \frac{1-\theta_i}{n+1-i} \leq \frac{1-\theta_{i+1}}{n-i} \quad \forall \ 1 \leq i \leq n-1. \tag{3.11}
\]

To prove part (b), it suffices to show that
\[
\frac{\theta_i}{i} \leq \theta_{i+1} - \theta_i \leq \frac{1-\theta_{i+1}}{n-i} \quad \forall \ 1 \leq i \leq n-1.
\]
This follows directly from (3.11). It remains to part (c). When \( j \leq i \), on appropriately iterating the first inequality of (3.11), it follows that
\[
\alpha^{i,1} + \cdots + \alpha^{i,j} = j \frac{\theta_i}{i} \geq \theta_j.
\]
When \( j = i \) or \( j = i + 1 \), the statement with equality is easily checked. When \( j > i + 1 \), on appropriately iterating the second inequality of (3.11), it follows that
\[
\alpha^{i,1} + \cdots + \alpha^{i,j} = \theta_{i+1} + \frac{(j-i-1)(1-\theta_{i+1})}{n-i} = 1 - \frac{(n+1-j)(1-\theta_{i+1})}{n-i} \\
\geq 1 - (1-\theta_j) = \theta_j.
\]
This completes the proof of the lemma. \( \Box \)

**Step 2.** Having chosen the sequence \((\alpha^{i,j})\) as above, the second step involves choosing a sequence of positive real numbers
\[
\{ \beta^{i,j}_k : 1 \leq i \leq n-1, \ 1 \leq j \leq n+1, \ k \geq 1 \} \tag{3.12}
\]
such that for any \( k \geq 1 \) and \( 1 \leq i \leq n-1 \):
\[
\sum_{1 \leq j \leq n+1} \beta^{i,j}_k = 1 \tag{3.13}
\]
\[
\beta^{i,j+1}_k - \beta^{i,j}_k \geq \frac{1}{4n^2k} \quad \text{for all} \ 1 \leq j \leq n, \tag{3.14}
\]
\[
\beta^{i,1}_k \in \left[ \frac{1}{(k+1)(n+1)}, \frac{k}{(k+1)(n+1)} \right], \tag{3.15}
\]
\[
\beta^{i,n+1}_k \in \left[ \frac{k+3}{(k+1)(n+1)}, 1 - \frac{1}{(k+1)(n+1)} \right], \tag{3.16}
\]
\[
\lim_{k \to \infty} \beta^{i,j}_k = \alpha^{i,j} \quad \forall \ 1 \leq j \leq n+1. \tag{3.17}
\]
Note that parts (a) and (b) of Lemma 3.3 guarantees the existence of such a sequence. For instance, they imply that for any $1 \leq i \leq n - 1$

$$\alpha^{i,1} \leq 1/n \quad \text{and} \quad \alpha^{i,n+1} \geq 1/n.$$  

Thus the conditions (3.15), (3.16) and (3.17) are compatible.

**Step 3.** Now, the third step is to let

$$T_1 := 128n^4\gamma \tag{3.18}$$

and then define inductively $T^i_k$ for $k \geq 1$ and $1 \leq i \leq n - 1$ as follows:

$$T^1_k := T_k, \quad \beta^{i+1,1}_k T^{i+1}_k := \beta^{i,n+1}_k T^i_k - (n + 1)\gamma, \quad T^n_k := T_{k+1}, \tag{3.19}$$

where we set

$$\beta^{n,j}_k = \beta^{1,j}_{k+1} \quad \forall \ k \geq 1 \quad \text{and} \quad 1 \leq j \leq n + 1. \tag{3.20}$$

Observe that for any $k \geq 1$ and $1 \leq i \leq n - 1$, it follows via (3.15), (3.16) and (3.19) that

$$T^{i+1}_k = (\beta^{i+1,1}_k)^{-1} \left( \beta^{i,n+1}_k T^i_k - (n + 1)\gamma \right)$$

$$\geq \frac{(k + 2)(k + 3)}{(k + 1)^2} T^i_k - \frac{(n + 1)^2(k + 2)\gamma}{k + 1}$$

$$> \frac{(k + 2)^2}{(k + 1)^2} T^i_k + \left( \frac{T^i_k - (n + 1)^2(k + 2)\gamma}{k + 1} \right). \tag{3.21}$$

In turn, on arguing by induction, it follows that for any $k \geq 1$:

$$T^{i+1}_k > T^i_k \quad \forall \ 1 \leq i \leq n - 1 \tag{3.22}$$

and

$$T_k \geq 32n^4(k + 1)^2\gamma. \tag{3.23}$$

Indeed, let $k = 1$. Then (3.23) holds in view of (3.18). To prove (3.22) we use induction on $i$. With this and (3.21) in mind, when $i = 1$ it follows via (3.23) and the fact that $T^1_1 := T_1$, that

$$T^1_1 - (n + 1)^23\gamma \geq 128n^4\gamma - 3(n + 1)^2\gamma > 0. \tag{3.24}$$

Hence, (3.21) implies that $T^2_1 > T^1_1$. In other words, (3.22) holds for $i = 1$. So suppose (3.22) holds for $i$ with $i \leq n - 2$. Then, it follows via (3.24) that

$$T^{i+1}_1 - (n + 1)^23\gamma > T^1_1 - (n + 1)^23\gamma > 0$$

and so (3.21) implies that $T^{i+2}_1 > T^{i+1}_1$. This shows that (3.22) holds with $k = 1$. Now assume that (3.22) and (3.23) holds for $k$. Then, it follows via (3.21) with $i = n - 1$ and
the fact that $T^n_k := T_{k+1}$ and $T^1_k := T_k$, that
\[
T_{k+1} > \frac{(k+2)^2}{(k+1)^2} T_k^{-1} + \left( \frac{T_k^{-1} - (n+1)^2(k+2)\gamma}{k+1} \right) \\
> \frac{(k+2)^2}{(k+1)^2} T_k + \left( \frac{T_k - (n+1)^2(k+2)\gamma}{k+1} \right) \\
> 32n^4(k+2)^2\gamma.
\]

This shows that (3.23) holds for $k+1$ and we now use this to show that (3.22) holds for $k+1$. With this in mind, when $i=1$ it follows that
\[
T^1_{k+1} - (n+1)^2(k+3)\gamma > 32n^4(k+2)^2\gamma - (n+1)^2(k+3)\gamma > 0 \quad (3.25)
\]
and so (3.24) implies that $T^2_{k+1} > T^1_{k+1} := T_{k+1}$. In other words, (3.22) holds for $k+1$ with $i = 1$. So suppose (3.22) holds for $k+1$ with $i \leq n - 2$. Then it follows via (3.25) that
\[
T^{i+1}_{k+1} - (n+1)^23\gamma > T^1_{k+1} - (n+1)^2(k+3)\gamma > 0
\]
and so (3.21) implies that $T^{i+2}_{k+1} > T^{i+1}_{k+1}$. This thereby completes the inductive step and hence establishes (3.22) and (3.23) for all $k \geq 1$.

Now, with the generic construction of §3.1 in mind, for $k \geq 1$, $1 \leq i \leq n - 1$, let
\[
T^-_i = T^i_k \quad \text{and} \quad T^+_i = T^{i+1}_k,
\]
and for $1 \leq j \leq n + 1$, let
\[
a^j_- = \beta^i_{k,j} \quad \text{and} \quad a^j_+ = \beta^{i+1}_{k,j}.
\]

Then, it is readily verified on using (3.13), (3.14), (3.17), (3.19), (3.22) and (3.23) that conditions (3.5), (3.6) and (3.7) are satisfied. The upshot is that the construction described within §3.1 is applicable and gives rise to a Roy $(n+1)$–system $P : [T^i_k, T^{i+1}_k] \rightarrow \mathbb{R}^{n+1}$ associated with the sequences $(\beta^i_{k,j})$ defined via (3.12) and $(T^i_k)$ defined via (3.19). Moreover, for each $k \geq 1$, $1 \leq i \leq n - 1$, the Roy $(n+1)$–system $P$ on the interval $[T^i_k, T^{i+1}_k]$ satisfies Lemma 3.2.

**Remark 3.1.** As we shall see in the next section, it is not difficult to extend this local statement to a Roy $(n+1)$–system $P$ on the interval $[0, \infty)$ that satisfies Lemma 3.1. Note that this would suffice if all we wanted to show was that the sets appearing in Theorem 1.5 are non-empty rather than uncountable.

**Step 4.** The fourth step involves perturbing the above construction of the Roy $(n+1)$–system $P$ on $[T^i_k, T^{i+1}_k]$ by a parameter $\delta$ in such a way that:

- the properties of Lemma 3.2 are satisfied for the perturbed Roy $(n+1)$–system $P^\delta : [T^i_k, T^{i+1}_k] \rightarrow \mathbb{R}^{n+1}$, and
- for $\delta' \neq \delta$, the perturbed Roy $(n+1)$–systems $P^\delta$ and $P^{\delta'}$ are mutually non-equivalent (see Definition 2.2).
With this in mind, let \((\beta_{k}^i)\) be the sequence given by (3.12) and \((T_k^i)\) be the sequences given by (3.19) respectively, and let \\
\[\delta \in \left[0, 1/32n^2\right).\] 
(3.26)

Now define the new sequence \\
\[
\left\{ \beta_{k}^i_{j}(\delta) : 1 \leq i \leq n - 1, \ 1 \leq j \leq n + 1, \ k \geq 1 \right\}
\]
(3.27)

by setting, for any \(k \geq 1\) and \(1 \leq i \leq n - 1:\)
\\n\[\beta_{k}^i_{1}(\delta) := \beta_{1}^i_{1},\] 
(3.28)

\[\beta_{k}^i_{n+1}(\delta) := \beta_{k}^i_{n+1} + \frac{\delta}{k},\] 
(3.29)

\[\beta_{k}^i_{i+1}(\delta)T_{k}^{i+1} := \beta_{k}^i_{n+1}(\delta)T_{k}^{i} - (n + 1)\gamma,\] 
(3.30)

\[\beta_{k}^i_{1}(\delta) + \beta_{k}^i_{2}(\delta) + \beta_{k}^i_{n+1}(\delta) := \beta_{k}^i_{1} + \beta_{k}^i_{2} + \beta_{k}^i_{n+1},\] 
(3.31)

\[\beta_{k}^i_{j}(\delta) := \beta_{k}^i_{j} \quad \forall \ 3 \leq j \leq n.\] 
(3.32)

Also, in line with (3.20), we let
\\n\[\beta_{k}^n_{j}(\delta) := \beta_{k+1}^j_{1}(\delta) \quad \forall \ k \geq 1 \quad \text{and} \quad 1 \leq j \leq n + 1.\] 
(3.33)

Clearly, the sequences \((\beta_{k}^i_{j}(\delta))\) and \((\beta_{k}^i_{j})\) coincide when \(\delta = 0\). An immediate consequence of (3.13), (3.31) and (3.32) is that
\\n\[\sum_{j=1}^{n+1} \beta_{k}^i_{j}(\delta) = 1.\] 
(3.34)

Also note that in view of (3.19), (3.29) and (3.30), we have that
\\n\[
\left( \beta_{k}^{i+1,1}(\delta) - \beta_{k}^{i,1+1},1 \right) T_{k}^{i+1} = \left( \beta_{k}^{i,n+1}(\delta) - \beta_{k}^{i,1},n+1 \right) T_{k}^{i} = \frac{\delta}{k}T_{k}^{i},
\]
from which it follows that
\\n\[\beta_{k}^{i,1} \leq \beta_{k}^{i,1}(\delta) \leq \beta_{k}^{i,1} + \frac{\delta}{k} \quad \forall \ 2 \leq i \leq n - 1\]
and
\\n\[\beta_{k+1}^{1,1} \leq \beta_{k+1}^{1,1}(\delta) \leq \beta_{k+1}^{1,1} + \frac{\delta}{k}.\] 
(3.35)

To sum up, for all \(k \geq 1\) and \(1 \leq i \leq n - 1\) we have that
\\n\[\beta_{k}^{i,1} \leq \beta_{k}^{i,1}(\delta) \leq \beta_{k}^{i,1} + \frac{2\delta}{k}.\] 
(3.36)

Combining (3.35) and (3.31), we get for all \(k \geq 1\) and \(1 \leq i \leq n - 1\),
\\n\[\beta_{k}^{i,2} - \frac{3\delta}{k} \leq \beta_{k}^{i,2}(\delta) \leq \beta_{k}^{i,2}.\] 
(3.37)
Now, with the generic construction of \(3.1\) in mind, for \(k \geq 1, 1 \leq i \leq n - 1\), let
\[
T_+ = T^i_k \quad \text{and} \quad T_- = T^{i+1}_k,
\]
and for \(1 \leq j \leq n + 1\), let
\[
a^i_- = \beta^i_j(\delta) \quad \text{and} \quad a^i_+ = \beta^{i+1}_j(\delta).
\]
Then, it is readily verified that condition (3.5) follows from (3.30) and that condition (3.6) follows from (3.17) and (3.33). To show (3.7), first note that for all \(k \geq 1, 1 \leq i \leq n - 1\) and \(j \neq 1\), on using (3.29), (3.32) and (3.36), it follows that
\[
\left(\beta^{i+1}_k(\delta) - \beta^i_j(\delta)\right) T_k \geq \left(\beta^{i+1}_k - \beta^i_j\right) T_k.
\]
We have already shown that the right-hand side satisfies (3.7). When \(j = 1\), it is readily verified, on using (3.14), (3.23), (3.26), (3.29) and (3.36), that for all \(k \geq 1, 1 \leq i \leq n - 1\)
\[
\left(\beta^{i+1}_k(\delta) - \beta^{i+1}_k\right) T_k \geq \left(\beta^{i+1}_k - \beta^{i+1}_k\right) T_k \geq \frac{T_k}{8n^2} \geq 4n^2 \gamma.
\]
The upshot is that the construction described within (3.1) is applicable and gives rise to a Roy \((n + 1)\)-system \(P^\delta : [T^i_k, T^{i+1}_k] \to \mathbb{R}^{n+1}\) associated with the constant \(\delta\) satisfying (3.26) and sequences \((\beta^i_k(\delta))\) defined via (3.27) and \((T^i_k)\) defined via (3.19). Moreover, for each \(k \geq 1, 1 \leq i \leq n - 1\), the Roy \((n + 1)\)-system \(P^\delta\) on the interval \([T^i_k, T^{i+1}_k]\) satisfies Lemma 3.2.

**Step 5.** It remains to show that the Roy \((n + 1)\)-systems constructed in Step 4 are mutually non-equivalent. This is easily done. Let \(\delta\) and \(\delta'\) satisfy (3.26) and suppose \(\delta' \neq \delta\). Then it is readily verified, that
\[
\left|P'_{n+1}(T_k) - P_{n+1}(T_k)\right| \geq \left|\frac{\delta' - \delta}{T_k}\right| \geq \frac{32n^4k\gamma}{2C_n}
\]
for all \(k > k_0\) sufficiently large. By definition, this implies that for any \(k > k_0\), the Roy \((n + 1)\)-systems \(P^\delta\) and \(P'^{\delta}\) on the interval \([T^i_k, T^{i+1}_k]\) are mutually non-equivalent.

### 3.3 Proof of Lemma 3.1

The proof of the Lemma 3.1 will follow on extending the local construction of the Roy \((n + 1)\)-systems \(P^\delta\) on the intervals \([T^i_k, T^{i+1}_k]\) presented in \(3.2\) to the interval \([0, \infty)\). With this in mind, for \(\delta\) satisfying (3.26) and \(k \geq 1, 1 \leq i \leq n - 1\), let us denote by \(P^\delta_{k,i}\) the Roy \((n + 1)\)-system on \([T^i_k, T^{i+1}_k]\). Now observe that
\[
[T_1, \infty) = \bigcup_{k \geq 1} \bigcup_{1 \leq i \leq n - 1} [T^i_k, T^{i+1}_k],
\]
where \(T_1\) is given by (3.18). It therefore follows that the continuous piecewise linear map \(P^\delta = (P^\delta_1, \ldots, P^\delta_{n+1}) : [T_1, \infty) \to \mathbb{R}^{n+1}\) given by
\[
P^\delta(t) := P^\delta_{k,i}(t) \quad \text{for} \quad t \in [T_1, \infty),
\]
and for each
is a Roy \((n + 1)\)–system on \([T_1, \infty)\). It remains to extend \(P^\delta\) to the interval \([0, T_1]\). For this let
\[
S_{d+1} := \left( \beta_{1,1}^1(\delta) + \cdots + \beta_{1,d}^1(\delta) + (n + 1 - d)\beta_{1,d+1}^1(\delta) \right) T_1 \quad \forall \quad 0 \leq d \leq n,
\]
and let
\[
P^\delta_j(0) := 0 \quad \forall \quad 1 \leq j \leq n + 1.
\]
On the interval \([0, S_1]\), let the \(n + 1\) components \(P^\delta_1, \ldots, P^\delta_{n+1}\) coincide and have slope \(1/(n + 1)\). It follows that
\[
P^\delta_j(S_1) := \beta_{1,j}^1(\delta) T_1 \quad \forall \quad 1 \leq j \leq n + 1.
\]
For \(1 \leq d \leq n\), on the interval \([S_d, S_{d+1}]\), let the \(n + 1 - d\) components \(P^\delta_{d+1}, \ldots, P^\delta_{n+1}\) coincide and have slope \(1/(n + 1 - d)\) while the components \(P^\delta_1, \ldots, P^\delta_d\) have slope 0 and
\[
P^\delta_j(S_{d+1}) := \begin{cases} 
\beta_{1,j}^1(\delta) T_1 & \text{if } 1 \leq j \leq d \\
\beta_{1,d+1}^1(\delta) T_1 & \text{if } d + 1 \leq j \leq n + 1.
\end{cases}
\]
In particular, by \((3.33)\) we have that \(S_{n+1} = T_1\) and so it follows that
\[
P^\delta_j(T_1) = \beta_{1,j}^1(\delta) T_1 \quad \forall \quad 1 \leq j \leq n + 1.
\]
In short, this coincides with left hand side of Figure 1 with \(T_- = T_1\) and \(a^d_- = \beta_{1,j}^1(\delta)\) \((1 \leq j \leq n + 1)\). The upshot is that the above construction enables us to extend in then obvious manner the Roy \((n + 1)\)–system on \([T_1, \infty)\) to \([0, \infty)\). We now show that the Roy \((n + 1)\)–system \(P^\delta : [0, \infty) \to \mathbb{R}^{n+1}\) satisfies the desired properties of Lemma \(3.1\).

- By construction, \((3.2)\) follows directly from \((3.8)\).
- In view of \((3.9)\), \((3.15)\), \((3.23)\) and \((3.34)\), it follows that
\[
\limsup_{t \to \infty} \left( \frac{t}{n + 1} - P^\delta(t) \right) = \limsup_{k \to \infty} \max_{1 \leq i \leq n} \max_{t \in [T_k^i, T_k^{i+1}]} \left( \frac{t}{n + 1} - P^\delta_1(t) \right)
\]
\[
\geq \limsup_{k \to \infty} \max_{t \in [T_k^n, T_k^{n+1}]} \left( \frac{t}{n + 1} - P^\delta_1(t) \right)
\]
\[
\geq \limsup_{k \to \infty} \left( \frac{1}{n + 1} - \beta_{k,n}^{n,1}(\delta) \right) T_k^n
\]
\[
\geq \limsup_{k \to \infty} \left( \frac{1}{n + 1} - \beta_{k+1,n}^{1,1}(\delta) \right) T_{k+1}
\]
\[
\geq \limsup_{k \to \infty} \left( \frac{1}{n + 1} - \frac{k + 1}{(k + 2)(n + 1) - \frac{1}{32n^2k}} \right) T_{k+1}
\]
\[
\geq \limsup_{k \to \infty} n^3(k + 2)\gamma = \infty.
\]
This shows that $P^\delta$ satisfies (3.3).

- Let $1 \leq d \leq n$. Then, in view of (3.10), (3.17) and part (c) of Lemma 3.3, it follows that

$$
\liminf_{t \to \infty} \frac{P^\delta_1(t) + \cdots + P^\delta_d(t)}{t} = \liminf_{k \to \infty} \min_{1 \leq i \leq n} \min_{t \in [T_k, T_{k+1}]} \left( \frac{P^\delta_1(t) + \cdots + P^\delta_d(t)}{t} \right)
$$

$$
= \liminf_{k \to \infty} \min_{1 \leq i \leq n-1} \min \left\{ \sum_{j=1}^{d} \beta_{k,j}^{i} (\delta), \sum_{j=1}^{d} \beta_{k+1,j}^{i+1} (\delta) \right\}
$$

$$
= \min_{1 \leq i \leq n} \alpha_{i,1} + \cdots + \alpha_{i,d}
$$

$$
= \frac{1}{1 + \tau_{n-d}}.
$$

This shows $P^\delta$ satisfies (3.4).

In the previous section (see Step 5), we have already seen that for any distinct $\delta$ and $\delta'$ satisfying (3.26), the Roy $(n+1)$--systems $P^\delta$ and $P^{\delta'}$ on $[T_k, T_{k+1}]$ are mutually non-equivalent for all $k$ sufficiently large. Consequently, there are uncountable many non-equivalent Roy $(n+1)$--systems $P^\delta$ on $[0, \infty)$ that satisfies the conditions of Lemma 3.1. This completes the proof.

3.4 Final comment

Recall, that once we have constructed in §3.3 the collection of non-equivalent Roy $(n+1)$--systems $P^\delta$ satisfying Lemma 3.1, the key ingredient towards establishing our main result is Theorem 2.1. In short, the latter guarantees that each system in our collection gives rise to a distinct point $x \in \mathbb{R}^n$ such that

$$
\|L_x - P^\delta\| \leq C_n
$$
on $[0, \infty)$. In turn, it is not difficult to show that such $x$ satisfies the desired Diophantine properties associated with Theorem 1.5. Now with this in mind, let $n \in \mathbb{N}$ and $W$ be a collection of Roy $(n+1)$--systems $P : [0, \infty) \to \mathbb{R}^{n+1}$. In [3 Theorem 2.3], it is shown that if $W$ satisfies the so called ‘closed under finite perturbations’ hypothesis, then one is able to compute the Hausdorff dimension of the set

$$
\{ x \in \mathbb{R}^n : \|L_x - P\| < \infty \text{ for some } P \in W \}.
$$
we obtain the Plücker coordinates for dimensional subspace \( V \). In other words, and is independent of the choice of \( X \) given above), respectively. Note that, under this basis, we identify homothety there is a one to one correspondence between decomposable in § 1.2. This will be done in § A.2 via the framework of multilinear algebra. It is worth mentioning that the multilinear algebra framework was the setting of the pioneering works of Laurent [17], Roy [19], Schmidt [21] and Schmidt & Summerer [22] referred to in the main body. In the process of describing the setup leading to the Dirichlet type theorem § A.2 via the framework of multilinear algebra. It is worth highlighting that up to an alternative “dynamical” proof of the proposition that avoids appealing to (2.9) and (2.10). To proceed, we recall notions and results from multilinear algebra. Let \( n \geq 2 \) and \( 0 \leq d \leq n - 1 \). First, we endow the linear space \( \mathbb{R}^{n+1} \) with the usual inner product and let \( \{ e_i \}_{1 \leq i \leq n+1} \) be the standard orthonormal basis. Then the wedge product \( \wedge^d \mathbb{R}^{n+1} \) is also equipped with an inner product with an orthonormal basis given by

\[
\{ e_{i_1} \wedge \cdots \wedge e_{i_{d+1}} : 1 \leq i_1 \leq \cdots \leq i_{d+1} \leq n+1 \}.
\]

Note that, under this basis, we identify \( \wedge^d \mathbb{R}^{n+1} \) with \( \mathbb{R}^{n+1} \). For any \( X \in \wedge^d \mathbb{R}^{n+1} \), we set \( |X| \) and \( ||X|| \) to be the Euclidean norm and maximal norm (with respect to the basis given above), respectively. \( X \in \wedge^d \mathbb{R}^{n+1} \) is called decomposable if and only if there exists \( v_1, \ldots, v_{d+1} \in \mathbb{R}^{n+1} \) such that \( X = v_1 \wedge \cdots \wedge v_{d+1} \). It is worth highlighting that up to an homothety there is a one to one correspondence between decomposable \( X \in \wedge^d \mathbb{R}^{n+1} \) and \( d \)-dimensional rational affine subspaces of \( \mathbb{R}^n \). Indeed, by expressing a \( d \)-dimensional affine subspace \( L \subset \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R}) \) using homogeneous coordinates, we obtain a unique \( (d + 1) \)-dimensional subspace \( V_L \) of \( \mathbb{R}^{n+1} \) satisfying that \( L = \text{P}(V_L) \). Clearly, \( L \) is rational if and only if \( V_L \) has a integer basis \( \{ v_1, \ldots, v_{d+1} \} \subset \mathbb{Z}^{n+1} \). By using the Plücker embedding

\[
\text{Gr}(d, \mathbb{P}^n(\mathbb{R})) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{R}^{n+1}), \quad L \mapsto X_L := v_1 \wedge \cdots \wedge v_{d+1},
\]

we obtain the Plücker coordinates for \( L \). The height \( H(L) \) of \( L \) is the Weil height of \( X_L \). In other words,

\[
H(L) = |X|
\]

where \( X \in \wedge^d \mathbb{Z}^{n+1} \setminus \{0\} \) is an integral vector in the class of \( X_L \) with coprime coordinates and is independent of the choice of \( X \). Given \( x \in \mathbb{R}^n \), we define the projective distance

\[
\{ x \in \mathbb{R}^n : \| L_x - P \| < C \text{ for some } P \in \mathcal{W} \}
\]

for any fixed constant \( C > 0 \). Such a result would potentially enable us to replace uncountable by full Hausdorff dimension in the statement of Theorem 1.2.

A Appendix: Intermediate Diophantine sets revisited

The main goal of this appendix is define the notion of \( d \)-Dirichlet improvable sets \( \text{DI}_n^d \) and investigate the relationship between them. The key ingredient required for achieving this lies in being able to state an appropriate optimal Dirichlet type theorem (see Remark 1.2 in § 1.2). This will be done in § A.2 via the framework of multilinear algebra. It is worth mentioning that the multilinear algebra framework was the setting of the pioneering works of Laurent [17], Roy [19], Schmidt [21] and Schmidt & Summerer [22] referred to in the main body. In the process of describing the setup leading to the Dirichlet type theorem of § A.2 we will take the opportunity to first revisit the intermediate badly approximable and singular sets in order to fill in the details of the arguments (cf. Remark 2.2) leading to (2.9) and (2.10). Recall, that these equivalences are used in the “classical” proof of Proposition 1.1 given in § 2.2 showing that the intermediate \( d \)-badly approximable sets \( \text{Bad}_n^d \) (resp. the \( d \)-singular sets \( \text{Sing}_n^d \)) are equivalent. Moreover, we will provide an alternative “dynamical” proof of the proposition that avoids appealing to (2.9) and (2.10).

To proceed, we recall notions and results from multilinear algebra. Let \( n \geq 2 \) and \( 0 \leq d \leq n - 1 \). First, we endow the linear space \( \mathbb{R}^{n+1} \) with the usual inner product and let \( \{ e_i \}_{1 \leq i \leq n+1} \) be the standard orthonormal basis. Then the wedge product \( \wedge^d \mathbb{R}^{n+1} \) is also equipped with an inner product with an orthonormal basis given by

\[
\{ e_{i_1} \wedge \cdots \wedge e_{i_{d+1}} : 1 \leq i_1 \leq \cdots \leq i_{d+1} \leq n + 1 \}.
\]

Note that, under this basis, we identify \( \wedge^d \mathbb{R}^{n+1} \) with \( \mathbb{R}^{n+1} \). For any \( X \in \wedge^d \mathbb{R}^{n+1} \), we set \( |X| \) and \( ||X|| \) to be the Euclidean norm and maximal norm (with respect to the basis given above), respectively. \( X \in \wedge^d \mathbb{R}^{n+1} \) is called decomposable if and only if there exists \( v_1, \ldots, v_{d+1} \in \mathbb{R}^{n+1} \) such that \( X = v_1 \wedge \cdots \wedge v_{d+1} \). It is worth highlighting that up to an homothety there is a one to one correspondence between decomposable \( X \in \wedge^d \mathbb{R}^{n+1} \) and \( d \)-dimensional rational affine subspaces of \( \mathbb{R}^n \). Indeed, by expressing a \( d \)-dimensional affine subspace \( L \subset \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R}) \) using homogeneous coordinates, we obtain a unique \( (d + 1) \)-dimensional subspace \( V_L \) of \( \mathbb{R}^{n+1} \) satisfying that \( L = \text{P}(V_L) \). Clearly, \( L \) is rational if and only if \( V_L \) has a integer basis \( \{ v_1, \ldots, v_{d+1} \} \subset \mathbb{Z}^{n+1} \). By using the Plücker embedding

\[
\text{Gr}(d, \mathbb{P}^n(\mathbb{R})) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{R}^{n+1}), \quad L \mapsto X_L := v_1 \wedge \cdots \wedge v_{d+1},
\]

we obtain the Plücker coordinates for \( L \). The height \( H(L) \) of \( L \) is the Weil height of \( X_L \). In other words,

\[
H(L) = |X|
\]

where \( X \in \wedge^d \mathbb{Z}^{n+1} \setminus \{0\} \) is an integral vector in the class of \( X_L \) with coprime coordinates and is independent of the choice of \( X \). Given \( x \in \mathbb{R}^n \), we define the projective distance
between \( x \) and \( L \) by
\[
d_p(x, L) := \frac{|x' \wedge X|}{|x^T X|},
\]
where \( x' := (x, 1) \in \mathbb{R}^{n+1} \) and \( X \in \Lambda^{d+1}\mathbb{R}^{n+1} \setminus \{0\} \) is any vector in the class of \( X_L \). Geometrically, it represents the sine of the smallest angle between \( x \) and a non-zero vector of \( L \). The projective distance \( d_p(x, L) \) is easily seen to be locally (depending on \(|x|\)) comparable to the distance \( d(x, L) \) defined by (1.10) in §1.2.

### A.1 Showing \( \text{Sing}_n^d \equiv (2.9) \) and \( \text{Bad}_n^d \equiv (2.10) \)

The following statement provides an algebraic formulation of the sets \( \text{Sing}_n^d \) and \( \text{Bad}_n^d \) as defined via (1.13) and (1.14) in §1.2. Recall, given \( n \in \mathbb{N} \) and \( d \in \{0, 1, \ldots, n-1\} \), we let
\[
\omega_d := \frac{d+1}{n-d}.
\]

**Lemma A.1.** Let \( n \geq 2 \) and \( 0 \leq d \leq n-1 \).

(i) \( x \in \text{Sing}_n^d \) if and only if for any given \( \varepsilon \in (0, 1) \) and \( N > N_0(x, \varepsilon) \) sufficiently large, there exists \( X \in \Lambda^{d+1}\mathbb{Z}^{n+1} \setminus \{0\} \), such that
\[
|X| \leq N \quad \text{and} \quad |x' \wedge X| \leq \varepsilon N^{-\omega_d}. \tag{A.1}
\]

(ii) \( x \in \text{Bad}_n^d \) if and only if there exists a constant \( \varepsilon := \varepsilon(x) \in (0, 1) \) such that for any \( N > N_0(x) \) sufficiently large, there are no solutions \( X \in \Lambda^{d+1}\mathbb{Z}^{n+1} \setminus \{0\} \) to (A.1).

**Remark A.1.** In view of the discussion at the start of the appendix, it is straightforward to establish the above reformulation of the sets \( \text{Sing}_n^d \) and \( \text{Bad}_n^d \) under the extra assumption that \( X \in \Lambda^{d+1}\mathbb{Z}^{n+1} \setminus \{0\} \) is decomposable. In view of this, the key feature of Lemma A.1 is that it enables us to remove the decomposable assumption.

**Proof.** The ‘only if part’ of both (i) and (ii) follows directly from Remark A.1. Now concentrating on the ‘if part’ of (i), in view of Remark A.1, it suffices to show that for any given \( \varepsilon \in (0, 1) \) and \( N > N_0(x, \varepsilon) \) sufficiently large, there exists a decomposable \( X \in \Lambda^{d+1}\mathbb{Z}^{n+1} \setminus \{0\} \) such that (A.1) holds. With this in mind, let \( \varepsilon \in (0, 1) \), \( N > N_0(x, \varepsilon) \) and \( X \in \Lambda^{d+1}\mathbb{Z}^{n+1} \setminus \{0\} \) satisfying (A.1) be as given. This implies that the first successive minima of the convex body \( C \) defined by
\[
\left\{ X \in \Lambda^{d+1}\mathbb{R}^{n+1} : |X| \leq N, \quad |x' \wedge X| \leq \varepsilon N^{-\omega_d} \right\} \tag{A.2}
\]
is less than 1. Let
\[
U := \varepsilon^{-d/(d+1)} N^{n/(n-d)} \quad \text{and} \quad V := \varepsilon^{(n+d)/(nd+n)} U^{1/n} = \varepsilon^{1/(d+1)} N^{-1/(n-d)}.
\]

Thus, \( N = UV^d \) and \( \varepsilon N^{-\omega_d} = V^{d+1} \). Then, in view of [3, Lemma 3], there exists \( \beta_1 = \beta_1(n, d) > 1 \) such that
\[
\beta_1^{-1} \tilde{C}_{d+1} \subset C \subset \beta_1 \tilde{C}_{d+1}, \tag{A.3}
\]
where $\tilde{C}_{d+1}$ is the $(d+1)$-th compound of the convex body $\tilde{C} \subset \mathbb{R}^{n+1}$ defined as

$$\tilde{C} := \left\{ y \in \mathbb{R}^{n+1} : |y_{n+1}| \leq U, \max_{1 \leq i \leq n} |y_{n+1}x_i - y_i| \leq V \right\}.$$ 

In turn, it follows via Mahler’s theory of compound convex bodies that there exists $\beta_2 = \beta_2(n,d) \geq 1$ such that 

$$X := x_1 \land \cdots \land x_{d+1} \in \beta_2 \lambda_1 \left( \land^{d+1} \mathbb{Z}^{n+1}, \mathcal{C}_x(e^t) \right) \tilde{C}_{d+1}$$

where $x_i \in \mathbb{Z}^{n+1}$ is the integer point at which $\tilde{C}$ attains its $i$-th successive minima. Recall, for a given convex body $C \subset \land^{d+1} \mathbb{R}^{n+1}$ and $i = 1, \ldots, d+1$, we write $\lambda_i(\land^{d+1} \mathbb{Z}^{n+1}, C)$ for the $i$-successive minima of $C$ with respect to the lattice $\land^{d+1} \mathbb{Z}^{n+1}$. The upshot of (A.4) is that $X$ is decomposable and this proves the ‘if part’ of (i). The proof of the ‘if part’ of (ii) is similar and we leave the details to the reader. 

Armed with Lemma A.1, it is relatively straightforward to obtained the sought after statement.

**Lemma A.2.** Let $n \geq 2$ and $0 \leq d \leq n-1$.

(i) $x \in \text{Sing}_n^d$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$ inequality (2.10) holds; that is 

$$\frac{(n-d)t}{n+1} - (L_{x,1}(t) + \cdots + L_{x,n-d}(t)) \geq \delta.$$ 

(ii) $x \in \text{Bad}_n^d$ if and only if there exists a constant $\delta > 0$ such that for all sufficiently large $t$ inequality (2.9) holds; that is 

$$\frac{(n-d)t}{n+1} - (L_{x,1}(t) + \cdots + L_{x,n-d}(t)) \leq \delta.$$ 

**Proof.** Lemma A.1 implies that $x \in \text{Sing}_n^d$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$, 

$$\lambda_1 \left( \land^{d+1} \mathbb{Z}^{n+1}, K_x^{d+1}(e^t) \right) \leq e^{-\delta}$$

where 

$$K_x^{d+1}(e^t) := \left\{ X \in \land^{d+1} \mathbb{R}^{n+1} : |X| \leq e^{\frac{(n-d)t}{n+1}}, |x' \land X| \leq e^{-\frac{(d+1)t}{n+1}} \right\}.$$ 

In view of (A.3), the convex body $K_x^{d+1}(e^t)$ is comparable (with implied constants depending on $n$ and $d$ only) to the $(d+1)$-th compound of the convex body 

$$K_x(e^t) := \left\{ y \in \mathbb{R}^{n+1} : |y_{n+1}| \leq e^{\frac{nt}{n+1}}, \max_{1 \leq i \leq n} |y_{n+1}x_i - y_i| \leq e^{-\frac{t}{n+1}} \right\}.$$ 

Note that 

$$C_x(e^t) = e^{-\frac{t}{n+1}} K_x(e^t)^\vee,$$
where $C_x(e^t)$ is given by (2.1) and $K_x(e^t)^\vee$ denotes the dual of $K_x(e^t)$. Hence, by Minkowski’s second convex body theorem, it follows that

$$\log \lambda_1 \left( \wedge^{d+1} \mathbb{Z}^{n+1}, K_x^{d+1}(e^t) \right) = \sum_{i=1}^{d+1} \log \lambda_i \left( \mathbb{Z}^{n+1}, K_x(e^t) \right) + O(1)$$

$$= \sum_{i=1}^{n-d} -\log \lambda_{n+2-i} \left( \mathbb{Z}^{n+1}, K_x(e^t)^\vee \right) + O(1)$$

$$= \sum_{i=1}^{n-d} \log \lambda_i \left( \mathbb{Z}^{n+1}, K_x(e^t)^\vee \right) + O(1)$$

$$= -\frac{(n-d)t}{n+1} + \sum_{i=1}^{n-d} L_{x,i}(t) + O(1),$$

where the implied constants in the ‘big $O$’ term depend on $n$ and $d$ only and the quantity $L_{x,i}(t)$ is given by (2.2). This thereby completes the proof of first claim made in the lemma. The proof of (ii) is similar and we leave the details to the reader. 

### A.2 An optimal Dirichlet type theorem via multilinear algebra

For obvious reasons, as discussed in Remark 1.2, in order to define Dirichlet improvable sets it is paramount to start with an optimal Dirichlet type theorem. Any such theorem should naturally not only imply Theorem 1.3 concerning the approximation of points by rational subspaces but also coincide with the classical simultaneous and dual forms of Dirichlet theorem. The multilinear algebra framework exploited in the previous section yields the following optimal statement.

**Theorem A.1.** Let $n \in \mathbb{N}$ and $d$ be an integer satisfying $0 \leq d \leq n-1$. Then for any $x \in \mathbb{R}^n$ and $N > 1$, there exist $Z \in \wedge^d \mathbb{Z}^n \setminus \{0\}$ and $Y \in \wedge^{d+1} \mathbb{Z}^n$ such that

$$\|Z\| \leq N \quad \text{and} \quad \|x \wedge Z + Y\| \leq N^{-\omega_d}. \quad (A.5)$$

**Proof.** Consider the linear space $V := \wedge^d \mathbb{R}^n \oplus \wedge^{d+1} \mathbb{R}^n$ with the lattice $L := \wedge^d \mathbb{Z}^n \oplus \wedge^{d+1} \mathbb{Z}^n$. On observing that $\omega_d = \binom{n}{d}/\binom{n}{d+1}$, it is easily verified that for any given $x \in \mathbb{R}^n$ the volume of the convex body given by (A.5) is equal to $2^{\dim V}$. Hence, the statement of the theorem follows as a direct consequence of Minkowski’s convex body theorem.

By taking $d = 0$ and $n-1$ in Theorem A.1, we immediately recover the classical simultaneous and dual forms of Dirichlet’s theorem. We now show that for general $d$ we recover Theorem 1.3.

**Step 1.** We show that Theorem 1.3 has the following equivalent algebraic formulation. Recall, given $x \in \mathbb{R}^n$ we let $x' := (x, 1) \in \mathbb{R}^{n+1}$. 

30
Lemma A.3. Let \( n \in \mathbb{N} \) and \( d \) be integer satisfying \( 0 \leq d \leq n - 1 \). Then for any \( x \in \mathbb{R}^n \) there exists a constant \( c = c(x, n) > 0 \), such that for any \( N \geq 1 \) there exist \( X \in \wedge^{d+1}Z^{n+1} \setminus \{0\} \), such that
\[
|X| \leq N \quad \text{and} \quad |x' \wedge X| \leq cN^{-d}. \tag{A.6}
\]

Proof of equivalence of Theorem 1.3 and Lemma A.3. For the same reasons as outlined in Remark A.1, it is easy to deduce Lemma A.3 from Theorem 1.3. For the converse, we adapt the proof of Lemma A.1. This simply amounts to putting \( \varepsilon = c(x, n) \) when defining the convex body \( C \) given by (A.2). Apart from this the given proof remains unchanged. \( \square \)

Step 2. We show that Theorem A.1 implies Lemma A.3.

Proof that Theorem A.1 implies Lemma A.3. With reference to Theorem A.1, given \( x \in \mathbb{R}^n \) and \( N > \max\{|x|^{-1/\omega_d}, 1\} \) let \( Z \in \wedge^d Z^n \setminus \{0\} \) and \( Y \in \wedge^{d+1}Z^n \) be a solution to (A.5). Then, it follows that
\[
\|Y\| \leq \|x \wedge Z\| + N^{-d} \leq |x \wedge Z| + |x| \leq 2|x||Z|. \tag{A.7}
\]

Now let \( e_{n+1} := (0, \ldots, 0, 1) \in Z^{n+1} \) and identify the set \( \{y = (y_1, \ldots, y_{n+1}) \in Z^{n+1} : y_{n+1} = 0\} \) (resp. \( \{y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\} \)) with \( Z^n \) (resp. \( \mathbb{R}^n \)). Then, we have that
\[
\wedge^{d+1}Z^{n+1} = e_{n+1} \wedge (\wedge^d Z^n) \oplus \wedge^{d+1}Z^n.
\]
Next, let
\[
X := e_{n+1} \wedge Z - Y \in \wedge^{d+1}Z^n.
\]
Then, it follows that
\[
|X| \leq |Z| + |Y| \overset{\text{A.7}}{\leq} (2|x| + 1)|Z| \leq (2|x| + 1)\sqrt{\frac{n}{d}}|Z| \overset{\text{A.6}}{\leq} 2^{\frac{n}{2}}(2|x| + 1)N,
\]
and so
\[
|X| \leq n^{\frac{1}{2}}2^{\frac{n}{2}}(2|x| + 1)N. \tag{A.8}
\]

On the other hand, we have that
\[
x' \wedge X = (e_{n+1} + x) \wedge (e_{n+1} \wedge Z - Y) = -e_{n+1} \wedge Y + x \wedge e_{n+1} \wedge Z - x \wedge Y
= -(e_{n+1} + x) \wedge (x \wedge Z + Y).
\]
Since \( e_{n+1} \) is orthogonal to \( x \), \( e_{n+1} \wedge (x \wedge Z + Y) \) is orthogonal to \( x \wedge (x \wedge Z + Y) \). Thus,
\[
|x \wedge Z + Y| \leq |x' \wedge X| \leq |x'||x \wedge Z + Y| \tag{A.9}
\]
and on using the above right hand side inequality and (A.5), it follows that
\[
|x' \wedge X| \leq 2^{\frac{n}{2}}|x'||x \wedge Z + Y| \leq 2^{\frac{n}{2}}|x'|N^{-d}. \tag{A.10}
\]
Together, (A.8) and (A.10) imply the statement of the lemma. \( \square \)
The upshot of Steps 1 & 2 is the following desired statement.

**Proposition A.1.** Theorem [A.1] \( \implies \) Theorem [1.3]

### A.2.1 Bad, Singular and Dirichlet Improvable in light of Theorem [A.1]

Theorem [A.1] enables us to define an associated notion of \( d \)-Dirichlet improvable vectors as well as alternative “algebraic” notions of \( d \)-singular and \( d \)-badly approximable vectors.

**Definition A.1.** Let \( n \in \mathbb{N} \) and \( d \) be an integer satisfying \( 0 \leq d \leq n - 1 \). Then for \( \varepsilon \in (0,1) \), let

- \( \text{DI}_n^d(\varepsilon) \) be the set of \( x \in \mathbb{R}^n \), such that for all \( N > N_0(x, \varepsilon) \) sufficiently large, there exist \( Z \in \wedge^d \mathbb{Z}^n \setminus \{0\} \) and \( Y \in \wedge^{d+1} \mathbb{Z}^n \) such that
  \[
  \|Z\| \leq N \quad \text{and} \quad \|x \wedge Z + Y\| \leq \varepsilon N^{-\omega_d}. \tag{A.11}
  \]

- \( \text{Bad}_n^d(\varepsilon) \) be the set of \( x \in \mathbb{R}^n \), such that for all \( N > N_0(x, \varepsilon) \) sufficiently large, there are no solutions \( Z \in \wedge^d \mathbb{Z}^n \setminus \{0\} \) and \( Y \in \wedge^{d+1} \mathbb{Z}^n \) to (A.11).

Moreover, let

\[
\text{DI}_n^d := \bigcup_{\varepsilon \in (0,1)} \text{DI}_n^d(\varepsilon), \quad \text{Sing}_n^d := \bigcap_{\varepsilon \in (0,1)} \text{DI}_n^d(\varepsilon), \quad \text{Bad}_n^d := \bigcup_{\varepsilon \in (0,1)} \text{Bad}_n^d(\varepsilon).
\]

For obvious reasons, including consistency, it is vitally important that the following ‘equivalence’ statement is true.

**Lemma A.4.** The definition of \( \text{Sing}_n^d \) (resp. \( \text{Bad}_n^d \)) given by Definition [A.1] is equivalent to that given by (1.13) (resp. (1.14)) in §1.2.

**Proof.** Let us write \( \widetilde{\text{Sing}}_n^d \) (resp. \( \widetilde{\text{Bad}}_n^d \)) to denote the set of \( d \)-singular vectors (resp. \( d \)-badly approximable vectors) given by (1.13) (resp. (1.14)). Then, Lemma [A.1] states that

- \( x \in \widetilde{\text{Sing}}_n^d \) if and only if for any given \( \varepsilon \in (0,1) \) and \( N > N_0(x, \varepsilon) \) sufficiently large, there exists \( X \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\} \), such that
  \[
  |X| \leq N \quad \text{and} \quad |x' \wedge X| \leq \varepsilon N^{-\omega_d}. \tag{A.12}
  \]

- \( x \in \widetilde{\text{Bad}}_n^d \) if and only if there exists a constant \( \varepsilon := \varepsilon(x) \in (0,1) \) such that for any \( N > N_0(x) \) sufficiently large, there are no solutions \( X \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\} \) to (A.12).

As in the proof of Step 2 in §A.2 let \( e_{n+1} = (0,\ldots,0,1) \in \mathbb{Z}^{n+1} \) and identify the set \( \{y = (y_1,\ldots,y_{n+1}) \in \mathbb{Z}^{n+1} : y_{n+1} = 0\} \) (resp. \( \{y = (y_1,\ldots,y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\} \)) with \( \mathbb{Z}^n \) (resp. \( \mathbb{R}^n \)). Then, we have that

\[
\wedge^{d+1} \mathbb{Z}^{n+1} = e_{n+1} \wedge (\wedge^d \mathbb{Z}^n) \oplus \wedge^{d+1} \mathbb{Z}^n.
\]
Hence, we can write any \( X \in \mathcal{P}_{d+1}^n \setminus \{0\} \) uniquely as \( e_{n+1} \wedge Z - Y \) with \( Z \in \mathcal{P}_d^n \) and \( Y \in \mathcal{P}_{d+1}^n \) such that either \( Z \) or \( Y \) is non-zero. Then, the same argument leading to (A.9), enables us to conclude that

\[
|X \wedge Z + Y| \leq |x' \wedge X| \leq |x'||x \wedge Z + Y|.
\] (A.13)

Note that we also have that

\[
\max\{|Z|, |Y|\} \leq |X| \leq 2 \max\{|Z|, |Y|\}.
\] (A.14)

Now fix \( \varepsilon \in (0, 1) \) and \( x \in \mathbb{R}^n \). Let \( N > \max\{|x|^{-1/\omega_d}, 1\} \) and suppose that \( Z \in \mathcal{P}_d^n \setminus \{0\} \) and \( Y \in \mathcal{P}_{d+1}^n \) is a solution to (A.11). Let \( X \in \mathcal{P}_{d+1}^{n+1} \) be the corresponding unique point satisfying (A.13) and (A.14). Then, the same argument leading to (A.8) and (A.10) shows that

\[
X \text{ satisfies } |X| \leq n^{1/2} 2^{n/2} (2|x| + 1)N \quad \text{and} \quad |x' \wedge X| \leq \varepsilon N^{-\omega_d}.
\]

Conversely, let \( N > \max\{|x|^{-1/\omega_d}, 1\} \) and suppose that \( X \in \mathcal{P}_{d+1}^{n+1} \setminus \{0\} \) is a solution to (A.12). Let \( Z \in \mathcal{P}_d^n \) and \( Y \in \mathcal{P}_{d+1}^n \) be the corresponding unique points (not both zero) satisfying (A.13) and (A.14). Then it follows that using the left hand side of (A.13) and (A.14), that \( Z \) and \( Y \) satisfy

\[
|Z| \leq N \quad \text{and} \quad ||x \wedge Z + Y|| \leq \varepsilon N^{-\omega_d}.
\]

It remains to show that \( Z \neq 0 \). Suppose it is. Then since the right hand side of the second inequality is strictly less than one, we must have that \( Y = 0 \). This contradicts the assumption that both are not zero. Hence we must have \( Z \in \mathcal{P}_d^n \setminus \{0\} \). This completes the proof.

We now proceed by describing a dynamical reformulation of the intermediate Diophantine sets associated with Definition A.1. This will enable us to provide an alternative proof of Proposition 1.1 that is self-contained in that it avoids appealing to [3, Lemma 3]. Moreover, the dynamical reformulation will enable us to extend the Davenport & Schmidt result [10, Theorem 2] concerning the equivalence of the simultaneous and dual Dirichlet improvable sets to intermediate Dirichlet improvable sets.

For simplicity, given \( n \geq 2 \) and \( 0 \leq d \leq n - 1 \), we write

\[
A_d := \binom{n}{d}, \quad B_d := \binom{n}{d+1} \quad \text{and} \quad N_d := A_d + B_d = \binom{n+1}{d+1}.
\]

For \( x \in \mathbb{R}^n \), let \( H_{d,x} \) to be the linear transformation defined as

\[
H_{d,x} : \mathcal{P}_d^n \to \mathcal{P}_{d+1}^n : Z \mapsto x \wedge Z.
\]

Under the standard basis, \( H_{d,x} \) is given as a matrix \( M_{d,x} \in M_{A_d \times B_d}(\mathbb{R}) \). Recall, a matrix \( M \in M_{A \times B}(\mathbb{R}) \) of \( A \) rows and \( B \) columns, is said to be
• **Dirichlet improvable** if and only if there exists \( \varepsilon = \varepsilon(M) \in (0, 1) \) such that for all \( N \geq N_0(M, \varepsilon) \) sufficiently large, there exists \( Y \in \mathbb{Z}^A \) and \( Z \in \mathbb{Z}^B \setminus \{0\} \) such that

\[
\|Z\| \leq N \quad \text{and} \quad \|MZ + Y\| \leq \varepsilon N^{-\frac{B}{A}}. \tag{A.15}
\]

• **singular** if and only if for any given \( \varepsilon \in (0, 1) \) and \( N \geq N_0(M, \varepsilon) \) sufficiently large, there exists \( Y \in \mathbb{Z}^A \) and \( Z \in \mathbb{Z}^B \setminus \{0\} \) such that (A.15) holds.

• **badly approximable** if and only if there exists \( \varepsilon = \varepsilon(M) \in (0, 1) \) such that for all \( N \geq N_0(M, \varepsilon) \) sufficiently large, there are no solutions \( Y \in \mathbb{Z}^A \) and \( Z \in \mathbb{Z}^B \setminus \{0\} \) to (A.15).

**Lemma A.5.** Let \( x \in \mathbb{R}^n \). Then \( x \) is \( d \)-Dirichlet improvable (resp. \( d \)-badly approximable, \( d \)-singular) if and only if the matrix \( M_{d,x} \) is Dirichlet improvable (resp. badly approximable, singular).

**Proof.** This follows directly on comparing the notions as given in Definition A.1 with the corresponding matrix notions.

Using this Lemma, we can provide another proof of Proposition 1.1.

**Alternative proof of Proposition 1.1.** Let

\[
\rho_d : \text{SL}_{n+1}(\mathbb{R}) \to \text{SL}_{N_d}(\mathbb{R}),
\]

be the homomorphism induced by the natural representation on \( \wedge^{d+1} \mathbb{R}^{n+1} \). It is easily seen that

\[
\rho_d(\text{SL}_{n+1}(\mathbb{Z})) \subset \text{SL}_{N_d}(\mathbb{Z}) \quad \text{and} \quad \rho(g_t) = g_t^d,
\]

where

\[
g_t := \begin{pmatrix} e^{-t}I_n & 0 \\ 0 & e^{nt} \end{pmatrix} \quad \text{and} \quad g_t^d := \begin{pmatrix} e^{-B_d t}I_A & 0 \\ 0 & e^{A_d t}I_B \end{pmatrix}.
\]

According to [18, Theorem 1.13], the induced map

\[
\phi_d : X_{n+1} \to X_{N_d} \quad \text{where} \quad X_m = \text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}).
\]

is proper. This together with Dani’s correspondence [6] and Lemma A.5, implies that

\[
x \text{ is } d \text{-badly approximable } \iff \text{ the orbit } \{g_t^d u(M_{d,x}) : t \geq 0\} \text{ is bounded}
\]

\[
\iff \text{ the orbit } \{g_t u(x) : t \geq 0\} \text{ is bounded}
\]

\[
\iff x \text{ is badly approximable.}
\]

The proof of equivalence in the singular case is similar and we leave the details to the reader.

Concerning the set of intermediate Dirichlet improvable vectors, we are able to prove the following statement.
Proposition A.1. Let \( n \in \mathbb{N} \) and \( d \) be an integer satisfying \( 0 \leq d \leq n - 1 \). Then
\[
\text{DI}_n^d = \text{DI}_{n-1-d}^n \subseteq \text{DI}_n.
\]

Note that when \( d = 0 \) or \( n - 1 \), the proposition reduces to the Davenport & Schmidt [10, Theorem 2] result; namely that
\[
\text{DI}_n^0 = \text{DI}_{n}^n := \text{DI}_n.
\]
The proof of the Proposition A.1 is based on a dynamical reformulation of \( \text{DI}_n^d \), which in turn relies on the following theorem of Hajós [12].

Theorem A.2 (Hajós). Let \( k \in \mathbb{N} \) and \( L \subset \mathbb{R}^k \) be a lattice of covolume 1. Then
\[
L \cap \Pi_k = \emptyset \iff L \subseteq \bigcup_{w \in \text{Sym}_k} w^{-1}U_kw\mathbb{Z}^k,
\]
where \( \Pi_k \) denotes the unit parallelepiped of \( \mathbb{R}^k \), \( \text{Sym}_k \) represents the Weyl group and \( U_k \subset \text{SL}_k(\mathbb{R}) \) denotes the upper triangular group with only 1 on the diagonal.

Proof of Proposition A.1. According to Theorem A.2 and Lemma A.5, \( x \) is \( d \)-Dirichlet improvable if and only if the \( \omega \)-limit set
\[
\{ \Lambda \in X_{N_d} : \text{there exists } (t_k)_{k \geq 0} \text{ with } \lim_{k \to \infty} t_k = \infty \text{ and } \phi_d(g_{t_k}^d\Lambda_x) = \Lambda \}
\]
does not intersect the set
\[
E_d := \bigcup_{w \in \text{Sym}_{N_d}} w^{-1}U_{N_d}w\mathbb{Z}^{N_d}.
\]
Thus, \( x \) is \( d \)-Dirichlet improvable if and only if the \( \omega \)-limit set
\[
\{ \Lambda \in X_{n+1} : \text{there exists } (t_k)_{k \geq 0} \text{ with } \lim_{k \to \infty} t_k = \infty \text{ and } g_{t_k}\Lambda_x = \Lambda \}
\]
does not intersect the set
\[
E'_d := X_{n+1} \cap \phi_d^{-1}(E_d).
\]
For dimensional reasons, \( X_{N_d} = X_{N_{n-1-d}} \). Moreover, we have \( \phi_d = \phi_{n-1-d} \circ \iota \), where \( \iota : X_d \to X_d \) is the map that sends a lattice to its dual. It is easily checked that \( \iota(E_d) = E_d \).
Thus \( \text{DI}_n^d = \text{DI}_{n-1-d}^n \).

On the other hand, for any \( w \in S_{n+1} \),
\[
\rho_d(w^{-1}U_{n+1}w) \subseteq \psi_d(w)^{-1}U_{N_d}\psi_d(w),
\]
where \( \psi_d \) is the natural map from \( \text{Sym}_{n+1} \) to \( \text{Sym}_{N_d} \). Hence, it follows that \( E_1 \subseteq E'_d \) and this implies that \( \text{DI}_n^d \subseteq \text{DI}_n \).

The following is an intriguing problem.
Problem A.1. Let \( n \geq 2 \) and \( 0 \leq d \leq n - 1 \). Do we have \( \text{DI}_n^d = \text{DI}_n \) ?

We feel that the answer is almost certainly yes but a proof eludes us.
References

[1] V. Beresnevich, D. Dickinson and S. Velani : Sets of exact logarithmic order in the theory of Diophantine approximation, Mathematische Annalen, 321 (2), (2001) 253-273

[2] A. S. Besicovitch : Sets of fractional dimensions (IV): On rational approximation to real numbers, J. London Math. Soc. 9 (1934), no. 2, 126–131.

[3] Y. Bugeaud and M. Laurent : On transfer inequalities in Diophantine approximation II, Math. Z., 265:249–262, 2010.

[4] Y. Cheung : Hausdorff dimension of the set of singular pairs, Ann. Math. 173 (2011) 127–167.

[5] Y. Cheung and N. Chevallier : Hausdorff dimension of singular vectors. Duke Math. Jou. 165 (2016) DOI: 10.1215/00127094-3477021.

[6] S.G. Dani : Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, J. Reine Angew. Math. 359 (1985), 55–89.

[7] S. Chow, A. Ghosh, L. Guan, A. Marnat and D. Simmons : Diophantine transference inequalities: weighted, inhomogeneous, and intermediate exponents, Ann. Sc. Norm. Super. Pisa Cl. Sci. 5 to appear. Pre-print: [arXiv:1808.07184].

[8] T. Das, L. Fishman, D. Simmons and M. Urbański : A variational principle in the parametric geometry of numbers, ArXiV preprint 1901.06602.

[9] H. Davenport and W. M. Schmidt : Dirichlet’s Theorem on Diophantine approximation., Symposia mathematica 4 Istituto Natzionale di alta Mathematica (Academic Press, London, 1970) 113-132, II, 829.

[10] H. Davenport and W. M. Schmidt : Dirichlet’s Theorem on Diophantine approximation. II, Acta Arithm., 16 1969/70, 413–424.

[11] O. N. German : Intermediate Diophantine exponents and parametric geometry of numbers, Acta Arithmetica, 154 (2012), 79–101.

[12] G. Hajós : Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, Math. Z., 47(1941), 427–467.

[13] V. Jarník : Zur metrischen Theorie der diophantischen Approximationen, Prace mat. fiz. 36 (1928), 91–106 (German).

[14] V. Jarník : Zum khintchineschen Übertragungssatz, Trav. Inst. Math. Tbilissi, 3:193–212, 1938.

[15] A. Ya. Khinchine : Über eine klasse linearer diophantischer approximationen, Rend. Circ. Mat. Palermo 50, 170–195, 1926.

[16] A. Ya. Khinchine : Zur metrischen theorie der diophantischen approximationen, Math.Z., 24:706–714, 1926.
[17] M. Laurent: *On transfer inequalities in Diophantine approximation*, Analytic number theory, 306–314, 2009.

[18] M. S. Raghunathan: *Discrete subgroups of Lie groups*, Springer. 1972.

[19] D. Roy: *On Schmidt and Summerer parametric geometry of numbers*, Annals of Math. 182 (2015), 739–786.

[20] D. Roy: *Spectrum of the exponents of best rational approximation*, Math. Z., 283:143–155, 2016.

[21] W. M. Schmidt: *On heights of algebraic subspaces and diophantine approximations*, Ann. of Math. (2), 85:430–472, 1967.

[22] W. M. Schmidt and L. Summerer: *Parametric geometry of numbers and applications*, Acta Arithmetica, 140(1): pp. 67–91 (2009).

[23] W. M. Schmidt and L. Summerer: *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math 169:1, pp. 51 – 104 (2013).

[24] F. Süss: *Simultaneous Diophantine approximation on affine subspaces and Dirichlet improvability*, PhD Thesis, University of York, 2017, 113 pages. [arXiv:1711.08288](https://arxiv.org/abs/1711.08288)