Optimal behavior of weighted Hardy operators on rearrangement-invariant spaces

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Abstract
The behavior of certain weighted Hardy-type operators on rearrangement-invariant function spaces is thoroughly studied. Emphasis is put on the optimality of the obtained results. First, the optimal rearrangement-invariant function spaces guaranteeing the boundedness of the operators from/to a given rearrangement-invariant function space are described. Second, the optimal rearrangement-invariant function norms being sometimes complicated, the question of whether and how they can be simplified to more manageable expressions is addressed. Next, the relation between optimal rearrangement-invariant function spaces and interpolation spaces is investigated. Last, iterated weighted Hardy-type operators are also studied.

KEYWORDS
iterated operators, optimal spaces, rearrangement-invariant spaces, supremum operators, weighted Hardy operators

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1 INTRODUCTION

In this paper, we thoroughly study the behavior of Hardy-type operators $R_{u,v,\nu}$ and $H_{u,v,\nu}$ on rearrangement-invariant function spaces, focusing on the optimality of our results. The Hardy-type operators are defined for measurable functions $g$ on $(0, L)$, $L \in (0, \infty)$, as

$$R_{u,v,\nu} g(t) = v(t) \int_0^{\nu(t)} |g(s)| u(s) \, ds, \quad t \in (0, L),$$

(1.1)

and

$$H_{u,v,\nu} g(t) = u(t) \int_{\nu(t)}^L |g(s)| v(s) \, ds, \quad t \in (0, L).$$

(1.2)

Here, $u, \nu$ are nonnegative nonincreasing functions on $(0, L)$ and $\nu$ is an increasing bijection of the interval $(0, L)$ onto itself. Recall that rearrangement-invariant function spaces are, loosely speaking, Banach spaces of functions whose norms are invariant with respect to measure-preserving rearrangements/transpositions of functions. Rearrangement-invariant function spaces constitute a broad class of function spaces. Some classical examples of rearrangement-invariant function...
spaces are Lebesgue spaces, Orlicz spaces, or Lorentz (–Zygmund) spaces to name a few. Precise definitions as well as some preliminary results and notations used in this paper are presented in Section 2.

First, let $T$ be either of the operators and $X(0, L)$ a rearrangement-invariant function space over the interval $(0, L)$. We characterize the optimal domain and the optimal target rearrangement-invariant function space $Y(0, L)$ for $X(0, L)$ and $T$. By that we mean the following. We describe the weakest rearrangement-invariant function norm $\| \cdot \|_{Y(0, L)}$ for which there is a positive constant $C$ such that $\| Tf \|_{X(0, L)} \leq C \| f \|_{Y(0, L)}$ for every $f \in X(0, L)$. We also describe the strongest rearrangement-invariant function norm $\| \cdot \|_{Y(0, L)}$ for which there is a positive constant $C$ such that $\| Tf \|_{Y(0, L)} \leq C \| f \|_{X(0, L)}$ for every $f \in X(0, L)$. In other words, we characterize the largest and the smallest rearrangement-invariant function space $Y(0, L)$ such that $T$ is bounded from $Y(0, L)$ to $X(0, L)$ and from $X(0, L)$ to $Y(0, L)$, respectively. This is the content of Section 3. As a simple corollary, we also obtain a description of the optimal rearrangement-invariant function spaces for a sum of the two operators, each with possibly different functions $u, v, \nu$. The description is less explicit than it could be if we studied directly the sum, though. Next, in Section 4, we take a close look at how to simplify the description of these optimal rearrangement-invariant function norms and whether it is possible at all. The motivation behind this is simple: The simpler and more manageable description we have at our disposal, the more useful it is. It turns out that this problem is more complex than it may appear at first glance. It leads us to studying a certain supremum operator, and it is closely related to the notion of interpolation spaces. Next, in Section 5, we investigate the optimal behavior of iterated Hardy-type operators—namely, $R_{u_1, v_1, \nu_1} \circ R_{u_2, v_2, \nu_2}$ and $H_{u_1, v_1, \nu_1} \circ H_{u_2, v_2, \nu_2}$—on rearrangement-invariant function spaces. These iterated operators naturally arise when one studies the question of whether iteration of optimal function spaces leads to an optimal function space. Last, in Section 6, we present some concrete examples of optimal rearrangement-invariant function spaces when $X(0, L)$ is a Lorentz–Zygmund space.

In considerably less general settings, the questions mentioned in the preceding paragraph were already studied, see [20–23, 28, 30, 39, 40, 46, 58] and references therein. However, those results are limited to some particular choices of the functions $u, v, \nu$ and $\nu$—namely, $u \equiv 1$ and $v, \nu$ being power functions for the most part, but see [33, 37]. Moreover, they are also scattered and often hidden somewhere between the lines with varying degrees of generality. The aim of this paper is to thoroughly address the questions in a coherent unified way and in considerable generality. Not only do the results obtained here encompass their already-known particular cases, but they also provide a general theory suitable for various future applications. Some are outlined at the end of this introductory section.

General as the results in this paper are, we do usually impose some mild restrictions on the functions $u, v, \nu$ so that we can obtain interesting, strong results. However, the imposed assumptions on the functions are actually not too restrictive for the most part and often exclude only cases being in a way pathological. The assumptions also often reflect the very forms of the Hardy-type operators considered here. In particular, the operators do not involve kernels. Indisputably, Hardy-type operators with kernels are of great importance, too. However, they go beyond the scope of this paper, although to investigate thoroughly their behavior on rearrangement-invariant function spaces would be of interest (e.g., see [1, 22]).

Our motivation behind studying the Hardy-type operators $R_{u, v, \nu}$ and $H_{u, v, \nu}$ is the following. Questions involving considerably more complicated operators can sometimes be reduced to questions concerning these Hardy-type operators for suitable choices of $u, v, \nu$. In turn, the better we control the Hardy-type operators, the better we control the more complicated ones. Arguably the most straightforwardly, this can be illustrated by the following well-known example, which traces back to the 1930s. Consider the question of establishing the boundedness of the Hardy–Littlewood maximal operator $M$ from a rearrangement-invariant function space $X(\mathbb{R}^n)$ to a rearrangement-invariant function space $Y(\mathbb{R}^n)$. It turns out that $M$ is bounded from $X(\mathbb{R}^n)$ to $Y(\mathbb{R}^n)$ if and only if $R_{u, v, \nu}$ with $u \equiv 1, v(t) = t^{-1}, \nu(t) = t, L = \infty$, is bounded from $X(0, \infty)$ to $Y(0, \infty)$. This is a consequence of the famous equivalence

$$C_1 \frac{1}{T} \int_0^T f^+(s) \, ds \leq (Mf)^+(t) \leq C_2 \frac{1}{T} \int_0^T f^+(s) \, ds \quad \text{for every } t \in (0, \infty).$$

Here $f^+$ denotes the nonincreasing rearrangement and $C_1, C_2$ are positive constants depending only on $n$. The upper bound on $(Mf)^+$ was proved by F. Riesz ([54], $n = 1$) and N. Wiener ([61], $n \in \mathbb{N}$). The lower one was proved by C. Hertz ([38], $n = 1$) and by C. Bennett and R. Sharpley ([4], $n \in \mathbb{N}$). There are other important operators of harmonic analysis that sharp inequalities for their nonincreasing rearrangements are known for. For example, the Hilbert transform ([3, Theorem 16.12], [57, Lemma 2.1]), or, more generally, certain singular integral operators with odd kernels ([16, p. 55]). It is easy to show that the boundedness of these operators on rearrangement-invariant function spaces is equivalent to the boundedness of a sum of two Hardy-type operators—namely, $R_{u, v, \nu} + H_{u, v, \nu}$. Here $u, v, \nu$ are the same as those for the Hardy–Littlewood maximal function. Other classical operators whose nonincreasing rearrangements are controlled by $R_{u, v, \nu}$ and/or $H_{u, v, \nu}$
for suitable choices of \( u, v, \) and \( \nu \) are certain convolution operators \([32, 49]\) or the fractional maximal operator and its variants \([24, 29]\). The number of operators that sharp inequalities for their nonincreasing rearrangements are known for is limited. Nevertheless, what is often at our disposal is at least an upper bound on the nonincreasing rearrangement of a given operator. Obviously, the better we control the upper bound, the better we control the given operator. It is also worth noting that inequalities for rearrangements of various maximal operators may actually involve a Hardy-type operator inside a supremum (see \([42]\) and references therein). However, the supremum usually does not cause any trouble (see \([30, \text{Lemma } 4.10]\)). For example, consider the fractional maximal operator \( M_\gamma \) of order \( \gamma \in (0, n) \). It is bounded from \( X(\mathbb{R}^n) \) to \( Y(\mathbb{R}^n) \) if and only if the supremum operator mapping a measurable function \( f \) on \((0, \infty)\) to the function

\[
(0, \infty) \ni t \mapsto \sup_{s \in [t, \infty)} R_{u, v, \nu}(f^*)(s)
\]

is bounded from \( X(0, \infty) \) to \( Y(0, \infty) \). Here, \( u \equiv 1, v(t) = t^{\gamma/n-1} \), and \( \nu(t) = t^{\gamma/n} \). This equivalence follows from the sharp inequality for the nonincreasing rearrangement of \( M_\gamma \) (see \([24, \text{Theorem } 1.1]\) and \([42, \text{Example } 1]\)). Importantly, it turns out that the supremum operator is bounded from \( X(0, \infty) \) to \( Y(0, \infty) \) if and only if the Hardy-type operator \( R_{u, v, \nu} \) itself is. This follows from \([30, \text{Lemma } 4.10]\) combined with the Hardy–Littlewood inequality (see \((2.5)\)). The interested reader can find more information on boundedness of some classical operators of harmonic analysis on rearrangement-invariant function spaces in \([30]\).

Pointwise inequalities for rearrangements are not the only way to reduce complicated questions to simpler ones involving Hardy-type operators. Reductions are also sometimes achieved with the right use of interpolation or by making use of some intrinsic properties of the problem in question. Such approaches have been notably successful in connection with various embeddings of Sobolev-type spaces built upon rearrangement-invariant function spaces into rearrangement-invariant function spaces. There, interpolation techniques, symmetrization principles, and isoperimetric inequalities have been of great use. For a wide variety of such embeddings, either complete characterizations or at least sufficient and/or necessary conditions have been obtained. See \([2, 7, 17, 20–23, 39, 46]\) for complete characterizations and \([18, 19, 23, 47]\) for sufficient and/or necessary conditions. These so-called reduction principles effectively transform the question of whether a certain Sobolev-type embedding is valid to that of whether a Hardy-type operator is bounded. For example, consider the Sobolev-type embedding

\[
W^m X(\Omega) \hookrightarrow Y(\overline{\Omega}, \mu),
\]

which was thoroughly studied in \([23]\). Here, \( W^m X(\Omega) \) is the \( m \)-th order Sobolev space built upon a rearrangement-invariant function space \( X \) over a bounded Lipschitz domain \( \Omega \) in \( \mathbb{R}^n, m < n, m \in \mathbb{N} \), and \( Y \) is a rearrangement-invariant function space over \( \overline{\Omega} \) endowed with a positive \( d \)-upper Ahlfors measure \( \mu \). A \( d \)-upper Ahlfors measure \( \mu \) is a finite Borel measure \( \mu \) on \( \overline{\Omega} \) satisfying

\[
\sup_{x \in \mathbb{R}^n, r > 0} \frac{\mu(B_r(x) \cap \overline{\Omega})}{r^d} < \infty
\]

with \( d \in (0, n] \). Here, \( B_r(x) \) is the open ball centered at \( x \) with radius \( r \). It turns out that, when \( d \in [n - m, n] \), the question of whether \((1.3)\) is valid leads us to the Hardy-type operator \( H_{u_1, v_1, \nu} \) with \( u_1 \equiv 1, v_1(t) = t^{-1+\frac{m}{n}}, \nu(v) = t^{-\frac{\gamma}{n}} \), and \( L = 1 \). When \( d \in (0, n - m) \), the question is more complicated and leads us not only to the same Hardy-type operator \( H_{u_1, v_1, \nu} \) but also to \( R_{u_2, v_2, \nu} \) with \( u_2(t) = t^{-1+\frac{m}{n-d}} \) and \( v_2(t) = t^{-1+\frac{m}{n-d}} \).

We conclude this introductory section by briefly mentioning some new applications that general results obtained in this paper could be useful for. For example, we get under control the optimal behavior of upper bounds for nonincreasing rearrangements of various less standard (nonfractional and fractional) maximal operators (see \([29, 42]\)). Furthermore, we get under control the optimal behavior of upper bounds for some operators that play a role in the a.e. convergence of the partial spherical Fourier integrals or in the solvability of the Dirichlet problem for the Laplacian on planar domains. See \([12–14]\) and references therein for more information on such operators. Another possible application is related to traces of Sobolev functions. There are \( d \)-dimensional sets \( \Omega_d \subseteq \mathbb{R}^n, d \in (0, n] \), that are “unrecognizable” by \( d \)-upper Ahlfors measures \( \mu \), that is, it may happen that \( \mu(\Omega_d) = 0 \) for every \( d \)-upper Ahlfors measure \( \mu \). For instance, this is, with probability 1, the case when \( \Omega_d \) is a Brownian path in \( \mathbb{R}^n, n \geq 2 \). With probability 1, its Hausdorff dimension is 2 but it is unrecognizable by 2-upper Ahlfors measures. To rectify the situation, more general functions than power functions have to be considered.
in (1.4). For more information, see [10, 25, 31]. Inevitably, if one is to generalize the results of [23] to cover such exceptional sets, one will need to deal with general enough Hardy-type operators. In particular, one would need to allow \( \nu \) to have nonpower growth, as is the case with the Hardy-type operators studied in this paper.

2 | PRELIMINARIES

Conventions and notation

1. Throughout the paper, \( L \in (0, \infty] \).
2. We adhere to the convention that \( \frac{1}{\infty} = 0 \cdot \infty = 0 \).
3. We write \( P \lesssim Q \), where \( P, Q \) are nonnegative quantities, when there is a positive constant \( c \) independent of all appropriate quantities appearing in the expressions \( P \) and \( Q \) such that \( P \leq c \cdot Q \). If not stated explicitly, what “the appropriate quantities appearing in the expressions \( P \) and \( Q \)” are should be obvious from the context. At the few places where it is not obvious, we will explicitly specify what the appropriate quantities are. We also write \( P \gtrsim Q \) with the obvious meaning, and \( P \approx Q \) when \( P \lesssim Q \) and \( P \gtrsim Q \) simultaneously.
4. When \( A \subseteq (0, L) \) is a (Lebesgue) measurable set, \( |A| \) stands for its Lebesgue measure.
5. When \( u \) is a nonnegative measurable function defined on \( (0, L) \), we denote by \( U \) the function defined as \( U(t) = \int_0^t u(s) \, ds \), \( t \in (0, L] \). We say that \( u \) is nondegenerate if there is \( t_0 \in (0, L) \) such that \( 0 < U(t_0) < \infty \).

We set

\[
\mathcal{M}(0, L) = \{ f : f \text{ is a measurable function on } (0, L) \text{ with values in } [-\infty, \infty] \},
\]

\[
\mathcal{M}_0(0, L) = \{ f \in \mathcal{M}(0, L) : f \text{ is finite a.e. on } (0, L) \},
\]

and

\[
\mathcal{M}^+(0, L) = \{ f \in \mathcal{M}(0, L) : f \geq 0 \text{ a.e. on } (0, L) \}.
\]

The nonincreasing rearrangement \( f^* : (0, \infty) \to [0, \infty] \) of a function \( f \in \mathcal{M}(0, L) \) is defined as

\[
f^*(t) = \inf \{ \lambda \in (0, \infty) : |\{ s \in (0, L) : |f(s)| > \lambda \}| \leq t \}, \quad t \in (0, \infty).
\]

Note that \( f^*(t) = 0 \) for every \( t \in [L, \infty) \). We say that functions \( f, g \in \mathcal{M}(0, L) \) are equimeasurable, and we write \( f \sim g \), if \( |\{ s \in (0, L) : |f(s)| > \lambda \}| = |\{ s \in (0, L) : |g(s)| > \lambda \}| \) for every \( \lambda \in (0, \infty) \). We always have that \( f \sim f^* \). The relation \( \sim \) is transitive.

The maximal nonincreasing rearrangement \( f^{**} : (0, \infty) \to [0, \infty] \) of a function \( f \in \mathcal{M}(0, L) \) is defined as

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t \in (0, \infty).
\]

The mapping \( f \mapsto f^* \) is monotone in the sense that, for every \( f, g \in \mathcal{M}(0, L) \),

\[
|f| \leq |g| \text{ a.e. on } (0, L) \implies f^* \leq g^* \text{ on } (0, \infty).
\]

The same implication remains true if \( ^* \) is replaced by \( ^{**} \). We have that

\[
f^* \leq f^{**} \text{ for every } f \in \mathcal{M}(0, L).
\] (2.1)

The operation \( f \mapsto f^* \) is neither subadditive nor multiplicative. Although \( f \mapsto f^* \) is not subadditive, the following pointwise inequality is valid for every \( f, g \in \mathcal{M}_0(0, L) \) [5, Chapter 2, Proposition 1.7, (1.16)]:

\[
(f + g)^*(t) \leq f^*(t) + g^*(t) \quad \text{for every } t \in (0, L).
\] (2.2)
Furthermore, the lack of subadditivity of the operation of taking the nonincreasing rearrangement is, up to some extent, compensated by the following fact [5, Chapter 2, (3.10)]. For every \( t \in (0, \infty) \) and \( f, g \in \mathcal{N}_0(0, L) \), we have that

\[
\int_0^t (f + g)^*(s) \, ds \leq \int_0^t f^*(s) \, ds + \int_0^t g^*(s) \, ds.
\tag{2.3}
\]

In other words, the operation \( f \mapsto f^{**} \) is subadditive.

There are a large number of inequalities concerning rearrangements (e.g., [36, 41, Chapter II, Section 2]). We state two of them, which shall prove particularly useful for us. The Hardy–Littlewood inequality [5, Chapter 2, Theorem 2.2] tells us that, for every \( t \in (0, \infty) \),

\[
\int_0^t (f + g)^*(s) \, ds \leq \int_0^t f^*(s) \, ds + \int_0^t g^*(s) \, ds.
\tag{2.4}
\]

In particular, by taking \( g = \chi_E \) in (2.4), one obtains that

\[
\int_E |f(t)| \, dt \leq \int_0^t f^*(t) \, dt
\tag{2.5}
\]

for every measurable \( E \subseteq (0, L) \). The Hardy lemma [5, Chapter 2, Proposition 3.6] states that

\[
\text{if } f, g \in \mathcal{M}(0, L) \text{ are such that } \int_0^t f(s) \, ds \leq \int_0^t g(s) \, ds \text{ for all } t \in (0, \infty),
\]

\[
\text{then } \int_0^\infty f(t)h(t) \, dt \leq \int_0^\infty g(t)h(t) \, dt \text{ for every nonincreasing } h \in \mathcal{M}(0, \infty).
\tag{2.6}
\]

A functional \( \| \cdot \|_{X(0, L)} : \mathcal{M}(0, L) \to [0, \infty) \) is called a rearrangement-invariant function norm (on \( (0, L) \)) if, for all \( f, g \) and \( \{f_k\}_{k=1}^\infty \) in \( \mathcal{M}(0, L) \), and every \( \lambda \in [0, \infty) \),

\[
\text{(P1) } \|f\|_{X(0, L)} = 0 \text{ if and only if } f = 0 \text{ a.e. on } (0, L); \|\lambda f\|_{X(0, L)} = \lambda \|f\|_{X(0, L)}; \|f + g\|_{X(0, L)} \leq \|f\|_{X(0, L)} + \|g\|_{X(0, L)};
\]

\[
\text{(P2) } \|f\|_{X(0, L)} \leq \|g\|_{X(0, L)} \text{ if } f \leq g \text{ a.e. on } (0, L);
\]

\[
\text{(P3) } \|f_k\|_{X(0, L)} \not\leq \|f\|_{X(0, L)} \text{ if } f_k \not\leq f \text{ a.e. on } (0, L);
\]

\[
\text{(P4) } \|\chi_E\|_{X(0, L)} < \infty \text{ for every measurable } E \subseteq (0, L) \text{ of finite measure};
\]

\[
\text{(P5) } \text{for every measurable } E \subseteq (0, L) \text{ of finite measure, there is a positive, finite constant } C_{E,X}, \text{ possibly depending on } E \text{ and } \| \cdot \|_{X(0, L)}, \text{ such that } \int_E f(t) \, dt \leq C_{E,X} \|f\|_{X(0, L)};
\]

\[
\text{(P6) } \|f\|_{X(0, L)} = \|g\|_{X(0, L)} \text{ whenever } f \sim g.
\]

The Hardy–Littlewood–Pólya principle [5, Chapter 2, Theorem 4.6] asserts that, for every \( f, g \in \mathcal{M}(0, L) \) and every rearrangement-invariant function norm \( \| \cdot \|_{X(0, L)} \),

\[
\text{if } \int_0^t f^*(s) \, ds \leq \int_0^t g^*(s) \, ds \text{ for all } t \in (0, L), \text{ then } \|f\|_{X(0, L)} \leq \|g\|_{X(0, L)}.
\tag{2.7}
\]

With every rearrangement-invariant function norm \( \| \cdot \|_{X(0, L)} \), we associate another functional \( \| \cdot \|_{X'(0, L)} \) defined as

\[
\|f\|_{X'(0, L)} = \sup_{g \in \mathcal{M}^*(0, L) : \|g\|_{X(0, L)} \leq 1} \int_0^L f(t)g(t) \, dt, \ f \in \mathcal{M}^+(0, L).
\tag{2.8}
\]

The functional \( \| \cdot \|_{X'(0, L)} \) is also a rearrangement-invariant function norm [5, Chapter 2, Proposition 4.2], and it is called the associate function norm of \( \| \cdot \|_{X(0, L)} \). Furthermore, we always have that [5, Chapter 1, Theorem 2.7]

\[
\|f\|_{X(0, L)} = \sup_{g \in \mathcal{M}^+(0, L) : \|g\|_{X'(0, L)} \leq 1} \int_0^L f(t)g(t) \, dt \text{ for every } f \in \mathcal{M}^+(0, L).
\tag{2.9}
\]
that is,

\[
\| \cdot \|_{X'(0,L)} = \| \cdot \|_{X(0,L)}.
\]  

(2.10)

Consequently, statements like “Let \( \| \cdot \|_{X'(0,L)} \) be the rearrangement-invariant function norm whose associate function norm is ...” are well justified. The supremum in (2.9) does not change when the functions involved are replaced with their nonincreasing rearrangements \([5, \text{Chapter 2, Proposition 4.2}]\), that is,

\[
\| f \|_{X'(0,L)} = \sup_{g \in \mathcal{M}^+(0,L)} \int_0^L f^*(t) g^*(t) \, dt \quad \text{for every } f \in \mathcal{M}^+(0,L).
\]  

(2.11)

Given a rearrangement-invariant function norm \( \| \cdot \|_{X(0,L)} \), we extend it from \( \mathcal{M}^+(0,L) \) to \( \mathcal{M}(0,L) \) by \( \| f \|_{X(0,L)} = \| f \|_{X(0,L)} \). The extended functional \( \| \cdot \|_{X(0,L)} \) restricted to the linear set \( X(0,L) \) defined as

\[
X(0,L) = \{ f \in \mathcal{M}(0,L) : \| f \|_{X(0,L)} < \infty \}
\]

is a norm, provided that we identify any two functions from \( \mathcal{M}(0,L) \) coinciding a.e. on \( (0,L) \), as usual. In fact, \( X(0,L) \) endowed with the norm \( \| \cdot \|_{X(0,L)} \) is a Banach space \([5, \text{Chapter 1, Theorem 1.6}]\). We say that \( X(0,L) \) is a rearrangement-invariant function space. Note that \( f \in \mathcal{M}(0,L) \) belongs to \( X(0,L) \) if and only if \( \| f \|_{X(0,L)} < \infty \). We always have that

\[
S(0,L) \subseteq X(0,L) \subseteq \mathcal{M}_0(0,L),
\]  

(2.12)

where \( S(0,L) \) denotes the set of all simple functions on \( (0,L) \). By a simple function, we mean a (finite) linear combination of characteristic functions of measurable sets having finite measure. Moreover, the second inclusion is continuous if the linear set \( \mathcal{M}_0(0,L) \) is endowed with the (metrizable) topology of convergence in measure on sets of finite measure \([5, \text{Chapter 1, Theorem 1.4}]\).

The class of rearrangement-invariant function spaces contains a large number of customary function spaces, such as Lebesgue spaces \( L^p \) \((p \in [1,\infty])\), Lorentz spaces \( L^{p,q} \) \((e.g., [5, pp. 216–220])\), Orlicz spaces \((e.g., [53])\), or Lorentz–Zygmund spaces \((e.g., [3, 50])\), to name a few. Here, we provide definitions of only those rearrangement-invariant function norms that we shall explicitly need. For \( p \in [1,\infty] \), we define the Lebesgue function norm \( \| \cdot \|_{L^p(0,L)} \) as

\[
\| f \|_{L^p(0,L)} = \begin{cases} 
\int_0^L f(t)^p \, dt & \text{if } p \in [1,\infty), \\
\text{ess sup}_{t \in (0,L)} f(t) & \text{if } p = \infty,
\end{cases}
\]

(2.13)

\( f \in \mathcal{M}^+(0,L) \). Given a measurable function \( v : (0,L) \to (0,\infty) \) such that \( V(t) < \infty \) for every \( t \in (0,L) \), we define the functional \( \| \cdot \|_{\Lambda^v_1(0,L)} \) as

\[
\| f \|_{\Lambda^v_1(0,L)} = \int_0^L f^*(s) v(s) \, ds, \quad f \in \mathcal{M}^+(0,L).
\]

Here, \( V(t) = \int_0^t v(s) \, ds, t \in (0,L) \). The functional is equivalent to a rearrangement-invariant function norm if and only if

\[
\frac{V(t)}{t} \leq \frac{V(s)}{s} \quad \text{for every } 0 < s < t < L.
\]  

This follows from \([11, \text{Theorem 2.3}]\) (see also \([59, \text{Proposition 1}]\) with regard to local embedding of \( \Lambda^v_1(0,L) \) in \( L^1(0,L) \)). By the fact that it is equivalent to a rearrangement-invariant function norm, we mean that there is a rearrangement-invariant function norm \( \| \cdot \|_{X(0,L)} \) on \( (0,L) \) such that \( \| f \|_{\Lambda^v_1(0,L)} \approx \| f \|_{X(0,L)} \) for every \( f \in \mathcal{M}^+(0,L) \). Hence, we can treat \( \Lambda^v_1(0,L) \) as a rearrangement-invariant function space whenever (2.13) is satisfied. Let \( \psi : (0,L) \to (0,\infty) \) be a quasiconcave function, that is, a nondecreasing function such that the function \( (0,L) \ni t \mapsto \frac{\psi(t)}{t} \) is nonincreasing. The functional \( \| \cdot \|_{\mathcal{M}_\psi(0,L)} \)
defined as
\[ \|f\|_{M_\psi(0,L)} = \sup_{t \in (0,L)} \psi(t)f^+(t), \quad t \in (0,L), \]
is a rearrangement-invariant function norm [52, Proposition 7.10.2]. We shall also meet Lorentz–Zygmund spaces. They are defined in Section 6, where they are used.

The rearrangement-invariant function space \( X'(0, L) \) built upon the associate function norm \( \| \cdot \|_X(0, L) \) of a rearrangement-invariant function norm \( \| \cdot \|_X(0, L) \) is called the associate function space of \( X(0, L) \). Thanks to (2.10), we have that \( (X')'(0, L) = X(0, L) \). Furthermore, one has that
\[
\int_0^L |f(t)||g(t)| \, dt \leq \|f\|_X(0, L) \|g\|_{X'(0, L)} \quad \text{for every} \; f, g \in \mathfrak{M}(0, L). \tag{2.14}
\]
Inequality (2.14) is a Hölder-type inequality, and we shall refer to it as the Hölder inequality.

Let \( X(0, L) \) and \( Y(0, L) \) be rearrangement-invariant function spaces. We say that \( X(0, L) \) is embedded in \( Y(0, L) \), and we write \( X(0, L) \hookrightarrow Y(0, L) \), if there is a positive constant \( C \) such that \( \|f\|_{Y(0, L)} \leq C\|f\|_{X(0, L)} \) for every \( f \in \mathfrak{M}(0, L) \). If \( X(0, L) \hookrightarrow Y(0, L) \) and \( Y(0, L) \hookrightarrow X(0, L) \) simultaneously, we write that \( X(0, L) = Y(0, L) \). We have that [5, Chapter 1, Theorem 1.8]
\[
X(0, L) \hookrightarrow Y(0, L) \quad \text{if and only if} \quad X(0, L) \subseteq Y(0, L). \tag{2.15}
\]
Furthermore,
\[
X(0, L) \hookrightarrow Y(0, L) \quad \text{if and only if} \quad Y'(0, L) \hookrightarrow X'(0, L) \tag{2.16}
\]
with the same embedding constants.

If \( \| \cdot \|_X(0, L) \) and \( \| \cdot \|_Y(0, L) \) are rearrangement-invariant function norms, then so are \( \| \cdot \|_{X(0, L) \cap Y(0, L)} \) and \( \| \cdot \|_{(X+Y)(0, L)} \) defined as
\[
\|f\|_{X(0, L) \cap Y(0, L)} = \max\{\|f\|_{X(0, L)}, \|f\|_{Y(0, L)}\}, \quad f \in \mathfrak{M}^+(0, L),
\]
and
\[
\|f\|_{(X+Y)(0, L)} = \inf_{f = g + h} (\|g\|_{X(0, L)} + \|h\|_{Y(0, L)}), \quad f \in \mathfrak{M}^+(0, L).
\]
Here, the infimum extends over all possible decompositions \( f = g + h, \; g, h \in \mathfrak{M}^+(0, L) \). Furthermore, we have that ([43, Theorem 3.1], also [27, Lemma 1.12])
\[
(X(0, L) \cap Y(0, L))' = (X' + Y')(0, L) \quad \text{and} \quad (X + Y)'(0, L) = X'(0, L) \cap Y'(0, L) \tag{2.17}
\]
with equality of norms. The K-functional between \( X(0, L) \) and \( Y(0, L) \) is, for every \( f \in (X + Y)(0, L) \) and \( t \in (0, \infty) \), defined as
\[
K(f, t; X, Y) = \inf_{f = g + h} (\|g\|_{X(0, L)} + t\|h\|_{Y(0, L)}).
\]
Here, the infimum extends over all possible decompositions \( f = g + h \) with \( g \in X(0, L) \) and \( h \in Y(0, L) \). For every \( f \in (X + Y)(0, L) \setminus \{0\} \), \( K(f, \cdot; X, Y) \) is a positive increasing concave function on \( (0, \infty) \) [5, Chapter 5, Proposition 1.2].

Let \( X_0(0, L) \) and \( X_1(0, L) \) be rearrangement-invariant function spaces. We say that a rearrangement-invariant function space \( X(0, L) \) is an intermediate space between \( X_0(0, L) \) and \( X_1(0, L) \) if \( X_0(0, L) \cap X_1(0, L) \hookrightarrow X(0, L) \hookrightarrow (X_0 + X_1)(0, L) \). A linear operator \( T \) defined on \( (X_0 + X_1)(0, L) \) having values in \( (X_0 + X_1)(0, L) \) is said to be admissible for the couple \( (X_0(0, L), X_1(0, L)) \) if \( T \) is bounded on both \( X_0(0, L) \) and \( X_1(0, L) \). An intermediate space \( X(0, L) \) between \( X_0(0, L) \) and \( X_1(0, L) \) is an interpolation space with respect to the couple \( (X_0(0, L), X_1(0, L)) \) if every admissible operator for the couple is bounded on \( X(0, L) \). By [9, Theorem 3], \( X(0, L) \) is always an interpolation space with respect to the couple \( (L^1(0, L), L^{\infty}(0, L)) \).
We always have that [5, Chapter 2, Theorem 6.6]

\( L^1(0, L) \cap L^\infty(0, L) \hookrightarrow X(0, L) \hookrightarrow L^1(0, L) + L^\infty(0, L) \).

In particular,

\( L^\infty(0, L) \hookrightarrow X(0, L) \hookrightarrow L^1(0, L) \) \hspace{1cm} (2.18)

provided that \( L < \infty \).

Let \( a > 0 \). The dilation operator \( D_a \) maps a function \( f \in \mathcal{M}(0, L) \) to the function

\[
D_a f(t) = \begin{cases} 
  f\left(\frac{t}{a}\right), & \text{if } L = \infty, \\
  f\left(\frac{t}{a}\right)\chi(0,at)(t), & \text{if } L < \infty.
\end{cases}
\]

Importantly, \( D_a \) is bounded on every rearrangement-invariant function space \( X(0, L) \) (see [5, Chapter 3, Proposition 5.11]). More precisely, we have that

\[
\|D_a f\|_{X(0, L)} \leq \max\{1, a\}\|f\|_{X(0, L)} \quad \text{for every } f \in \mathcal{M}(0, L).
\] \hspace{1cm} (2.19)

\section{OPTIMAL REARRANGEMENT-INVARIANT FUNCTION SPACES}

In this section, we shall investigate optimal mapping properties of the operators \( H_{u,v,\nu} \) and \( R_{u,v,\nu} \). Let \( T \) be either of them. We say that a rearrangement-invariant function space \( Y(0, L) \) is the \emph{optimal domain space} for the operator \( T \) and a rearrangement-invariant function space \( X(0, L) \) if the following two facts are true. \( T : Y(0, L) \to X(0, L) \) is bounded and \( Z(0, L) \hookrightarrow Y(0, L) \) whenever \( Z(0, L) \) is a rearrangement-invariant function space such that \( T : Z(0, L) \to X(0, L) \) is bounded. In other words, \( \| \cdot \|_{Y(0, L)} \) is the weakest domain rearrangement-invariant function norm for \( T \) and \( \| \cdot \|_{X(0, L)} \).

We say that a rearrangement-invariant function space \( Y(0, L) \) is the \emph{optimal target space} for the operator \( T \) and a rearrangement-invariant function space \( X(0, L) \) if the following two facts are true. \( T : X(0, L) \to Y(0, L) \) is bounded and \( Y(0, L) \hookrightarrow Z(0, L) \) whenever \( Z(0, L) \) is a rearrangement-invariant function space such that \( T : X(0, L) \to Z(0, L) \) is bounded. In other words, \( \| \cdot \|_{Y(0, L)} \) is the strongest target rearrangement-invariant function norm for \( T \) and \( \| \cdot \|_{X(0, L)} \).

\subsection*{3.1 Optimal domain spaces}

We start by characterizing when the functional \( \mathcal{M}^+(0, L) \ni f \mapsto \| R_{u,v,\nu}(f^*) \|_{X(0, L)} \) is a rearrangement-invariant function norm. It turns out that it also enables us to characterize optimal domain spaces for \( R_{u,v,\nu} \). In the following subsection, we will also use it to characterize optimal target spaces for \( H_{u,v,\nu} \).

\textbf{Proposition 3.1.} Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

(1) Let \( \nu : (0, L) \to (0, L) \) be an increasing bijection.

(2) Let \( u : (0, L) \to (0, \infty) \) be a nondegenerate nonincreasing function. If \( L < \infty \), assume that \( u(L^-) > 0 \).

(3) Let \( \nu : (0, L) \to (0, \infty) \) be measurable.

Set

\[
\| f \|_{Y(0, L)} = \| R_{u,v,\nu}(f^*) \|_{X(0, L)}, \quad f \in \mathcal{M}^+(0, L),
\]

and

\[
\xi(t) = \begin{cases} 
  \nu(t)U(\nu(t)), & t \in (0, L), \\
  \nu(t)U(\nu(t))\chi(0,1)(t) + \nu(t)\chi(1,\infty)(t), & t \in (0, \infty),
\end{cases} \quad \text{if } L < \infty,
\]

\[
\xi(t) = \begin{cases} 
  \nu(t)U(\nu(t))\chi(0,1)(t) + \nu(t)\chi(1,\infty)(t), & t \in (0, \infty),
\end{cases} \quad \text{if } L = \infty.
\] \hspace{1cm} (3.1)

The functional \( \| \cdot \|_{Y(0, L)} \) is a rearrangement-invariant function norm if and only if \( \xi \in X(0, L) \).
If $\xi \in X(0, L)$, then the rearrangement-invariant function space $Y(0, L)$ is the optimal domain space for the operator $R_{u,v,\nu}$ and $X(0, L)$. If $\xi \notin X(0, L)$, then there is no rearrangement-invariant function space $Z(0, L)$ such that $R_{u,v,\nu} : Z(0, L) \to X(0, L)$ is bounded.

**Proof.** We shall show that $\| \cdot \|_{Y(0, L)}$ is a rearrangement-invariant function norm provided that $\xi \in X(0, L)$. Before we do that, note that, since $u$ is positive and nonincreasing, its nondegeneracy implies that $0 < U(t) < \infty$ for every $t \in (0, L) \cap \mathbb{R}$.

Property (P1). The positive homogeneity and positive definiteness of $\| \cdot \|_{Y(0, L)}$ can be readily verified. As for the subadditivity of $\| \cdot \|_{Y(0, L)}$, it follows from (2.3) combined with Hardy's lemma (2.6) that

$$\int_0^L (f + g)^*(s)u(s)\chi_{(0,\nu(t))}(s)ds \leq \int_0^L f^*(s)u(s)\chi_{(0,\nu(t))}(s)ds + \int_0^L g^*(s)u(s)\chi_{(0,\nu(t))}(s)ds$$

for every $f, g \in \mathbb{M}^+(0, L)$ and $t \in (0, L)$ thanks to the fact that $u$ is nonincreasing. Since $\| \cdot \|_{X(0, L)}$ is subadditive, it follows that

$$\|f + g\|_{Y(0, L)} \leq \|f\|_{Y(0, L)} + \|g\|_{Y(0, L)}$$

for every $f, g \in \mathbb{M}^+(0, L)$.

Properties (P2) and (P3). Since $\| \cdot \|_{X(0, L)}$ has these properties, it can be readily verified that $\| \cdot \|_{Y(0, L)}$, too, has them.

Property (P4). First, assume that $L < \infty$. Clearly, $\|X(0,L)\|_{Y(0,L)} < \infty$ if and only if $v(t)U(\nu(t)) \in X(0, L)$. Since $\| \cdot \|_{Y(0, L)}$ has property (P2), $\| \cdot \|_{X(0, L)}$ has property (P4) if and only if $v(t)U(\nu(t)) \in X(0, L)$. Second, assume that $L = \infty$. Let $E \subseteq (0, \infty)$ be a set of finite positive measure. Clearly, $\|X_E\|_{Y(0, \infty)} < \infty$ if and only if $v(t)U(\nu(t)) \chi(0,1)(t) + v(t)\chi(1,\infty)(t) \in X(0, \infty)$. If $\|E\| \leq 1$, then

$$\|v(t)U(\nu(t))\chi(0,|E|)(t) + v(t)\chi(0,\nu^{-1}(|E|))(t)\|_{X(0, \infty)} + \|v(t)\chi(1,\nu^{-1}(|E|))(t)\|_{X(0, \infty)}$$

If $E \geq 1$, we can obtain, in a similar way, that

$$\|v(t)U(\nu(t))\chi(0,|E|)(t) + v(t)\chi(0,\nu^{-1}(|E|))(t)\|_{X(0, \infty)} + \|v(t)\chi(1,\nu^{-1}(|E|))(t)\|_{X(0, \infty)}$$

Either way, we have that $\|X_E\|_{Y(0, \infty)} < \infty$ if and only if

$$v(t)U(\nu(t))\chi(0,1)(t) + v(t)\chi(1,\nu^{-1}(|E|))(t) \in X(0, \infty).$$

Property (P5). Let $E \subseteq (0, L)$ be a set of finite positive measure. Let $f \in \mathbb{M}^+(0, L)$. Note that the function $0 < L \ni t \mapsto \frac{1}{U(\nu(t))} \int_0^{\nu(t)} f^*(s)u(s)ds$ is nonincreasing because it is the integral mean of a nonnegative nonincreasing function over the interval $(0, \nu(t))$ with respect to the measure $u(s)ds$. Thanks to that and the monotonicity of $u$, we obtain that

$$\left\|v(t) \int_0^{\nu(t)} f^*(s)u(s)ds\right\|_{X(0, L)} \geq \left\|v(t)U(\nu(t))\chi(0,\nu^{-1}(|E|))(t) \int_0^{\nu(t)} f^*(s)u(s)ds\right\|_{X(0, L)}$$

$$\geq \left\|v(t)U(\nu(t))\chi(0,\nu^{-1}(|E|))(t)\|_{X(0, L)} \frac{1}{U(|E|)} \int_0^{|E|} f^*(s)u(s)ds$$
\[
\begin{align*}
\geq & \left\| v(t) U(\nu(t)) x_{(0, \nu^{-1}(|E|))}(t) \right\|_{X(0, L)} \frac{u(|E^-|)}{U(|E|)} \int_0^{|E|} f^+(s) \, ds \\
\geq & \left\| v(t) U(\nu(t)) x_{(0, \nu^{-1}(|E|))}(t) \right\|_{X(0, L)} \frac{u(|E^-|)}{U(|E|)} \int_E f(s) \, ds.
\end{align*}
\]

Here, we used (2.5) in the last inequality.

**Property (P6).** Since \( f^+ = g^+ \) when \( f, g \in \mathbb{M}^+(0, L) \) are equimeasurable, this is obvious.

Note that the necessity of \( \xi \in X(0, L) \) for \( \| \cdot \|_{Y(0, L)} \) to be a rearrangement-invariant function norm was already proved in the paragraph devoted to property (P4).

Assume now that \( \xi \in X(0, L) \). Thanks to the Hardy–Littlewood inequality (2.4) and the monotonicity of \( u \), we have that

\[
\| R_{u,v,\nu} f \|_{X(0, L)} \leq \| R_{u,v,\nu}(f^+) \|_{X(0, L)} = \| f \|_{Y(0, L)} \quad \text{for every } f \in \mathbb{M}^+(0, L).
\]

Hence, \( R_{u,v,\nu} : Y(0, L) \to X(0, L) \) is bounded. Next, if \( Z(0, L) \) is a rearrangement-invariant function space such that \( R_{u,v,\nu} : Z(0, L) \to X(0, L) \) is bounded, then we have that

\[
\| f \|_{Z(0, L)} = \| R_{u,v,\nu}(f^+) \|_{X(0, L)} \leq \| f \|_{Z(0, L)} = \| f \|_{Z(0, L)} \quad \text{for every } f \in \mathbb{M}^+(0, L),
\]

and so \( Z(0, L) \hookrightarrow Y(0, L) \). Finally, note that, if \( R_{u,v,\nu} : Z(0, L) \to X(0, L) \) is bounded, then

\[
\| \xi \|_{X(0, L)} \approx \| R_{u,v,\nu}(x_{(0, a)}) \|_{X(0, L)} \leq \| x_{(0, a)} \|_{Z(0, L)} < \infty.
\]

Here,

\[
a = \begin{cases} 
L & \text{if } L < \infty, \\
1 & \text{if } L = \infty.
\end{cases}
\]

(3.2)

Hence, \( \xi \in X(0, L) \).

**Remark 3.2.** Since \( \nu : (0, L) \to (0, L) \) will always be an increasing bijection, let us briefly comment on this assumption. It may seem that the assumption is quite restrictive. However, from the point of view of applications that this paper is motivated by, the assumption is quite natural and not overly restrictive. They tell us that it is reasonable to assume that \( \nu \) is an increasing bijection mapping the interval \( (0, L) \) onto an interval \( (0, \tilde{L}) \), and that \( L \) is finite if and only if \( \tilde{L} \) is. Furthermore, when \( L < \infty \), our assumption that \( L = \tilde{L} \) only makes some computations easier and is not restrictive. The reason is that, if \( \nu : (0, L) \to (0, L) \) is an increasing bijection, then \( \tilde{\nu}(t) = L \nu(t) / \tilde{L} \) is an increasing bijection of the interval \( (0, L) \) onto itself. Although it might be of interest to allow \( \nu \) to map an unbounded interval onto a bounded one or vice versa, that would make this paper considerably more technical. Nevertheless, the interested reader should be able to follow proofs presented in this paper and modify them if needed.

We now turn our attention to \( H_{u,v,\nu} \). It turns out that the situation becomes significantly more complicated. Notably the fact that, unlike with \( R_{u,v,\nu} \), the integration is carried out over intervals away from 0 often causes great difficulties. In particular, the functional \( \mathbb{M}^+(0, L) \ni f \mapsto \| H_{u,v,\nu}(f^+) \|_{X(0, L)} \) is hardly ever subadditive. Instead, in general, we need to consider a more complicated functional (see Proposition 4.1, however). Here and in subsequent sections, we will often need to impose certain mild conditions on \( \nu \).

1. We write \( \nu \in D^0 \) if there is \( \vartheta > 1 \) such that \( \liminf_{t \to 0^+} \frac{\nu(\vartheta t)}{\nu(t)} > 1 \).
2. We write \( \nu \in D^\infty \) if there is \( \vartheta > 1 \) such that \( \liminf_{t \to \infty} \frac{\nu(\vartheta t)}{\nu(t)} > 1 \).
3. We write \( \nu \in \overline{D}^0 \) if there is \( \vartheta > 1 \) such that \( \limsup_{t \to 0^+} \frac{\nu(\vartheta t)}{\nu(t)} < \infty \).
4. We write \( \nu \in \overline{D}^\infty \) if there is \( \vartheta > 1 \) such that \( \limsup_{t \to \infty} \frac{\nu(\vartheta t)}{\nu(t)} < \infty \).

When we need to emphasize the exact value of \( \vartheta \), we will write \( \nu \in D^0_\vartheta \) and so forth.
Proposition 3.3. Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

(1) Let $\nu : (0, L) \to (0, L)$ be an increasing bijection. If $L = \infty$, assume that $\nu \in D^\infty$.
(2) Let $u : (0, L) \to (0, \infty)$ be nonincreasing.
(3) Let $v : (0, L) \to (0, \infty)$ be nonincreasing. If $L < \infty$, assume that $v(L^-) > 0$.

Set
\[
\|f\|_{Y(0,L)} = \sup_{h \sim f} \|H_{u,v,\nu}h\|_{X(0,L)}, \quad f \in \mathfrak{M}^+(0,L),
\] (3.3)
where the supremum extends over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$. The functional $\| \cdot \|_{Y(0,L)}$ is a rearrangement-invariant function norm if and only if
\[
\begin{cases}
  u(t) \int_0^L v(s) ds \in X(0,L) & \text{if } L < \infty, \\
  u(t)X(0,v^{-1}(t)) \int_0^1 v(s) ds \in X(0,\infty) \quad \text{and} \\
  \limsup_{\tau \to \infty} v(\tau)\|uX(0,v^{-1}(\tau))\|_{X(0,\infty)} < \infty & \text{if } L = \infty.
\end{cases}
\] (3.4)

If (3.4) is satisfied, then the rearrangement-invariant function space $Y(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu} : Z(0,L) \to X(0,L)$.

Proof. We shall show that $\| \cdot \|_{Y(0,L)}$ is a rearrangement-invariant function norm provided that (3.4) is satisfied.

Property (P2). Let $f, g \in \mathfrak{M}^+(0,L)$ be such that $f \leq g$ a.e. Consequently, $f^* \leq g^*$. Suppose that $\|f\|_{Y(0,L)} > \|g\|_{Y(0,L)}$. It implies that there is $\tilde{f} \in \mathfrak{M}^+(0,L)$, $\tilde{f} \sim f$, such that
\[
\sup_{h \sim g} \|H_{u,v,\nu}h\|_{X(0,L)} < \|H_{u,v,\nu}\tilde{f}\|_{X(0,L)}.
\] (3.5)
When $L = \infty$, we may assume that $\lim_{t \to \infty} (\tilde{f})^*(t) = \lim_{t \to \infty} f^*(t) = 0$, for we would otherwise approximate $\tilde{f}$ by functions $f_n = \tilde{f}X(0,n)$, $n \in \mathbb{N}$. The monotone convergence theorem and property (P3) of $\| \cdot \|_{X(0,L)}$ would guarantee that the inequality above holds with $\tilde{f}$ replaced by $f_n$ for $n$ large enough. Thanks to [5, Chapter 2, Corollary 7.6] (also [55, Proposition 3]), there is a measure-preserving transformation $\sigma : (0,L) \to (0,L)$ such that $\tilde{f} = f^* \circ \sigma$. For the definition of measure-preserving transformations, see [5, Chapter 2, Definition 7.1]. Since $\sigma$ is measure preserving, we have that $(g^* \circ \sigma) \sim g^* \sim g$ [5, Chapter 2, Proposition 7.2]. Consequently,
\[
\sup_{h \sim g} \|u(t) \int_0^L h(s)v(s) ds\|_{X(0,L)} \geq \|u(t) \int_0^L g^*(\sigma(s))v(s) ds\|_{X(0,L)}
\]
\[
\geq \|u(t) \int_0^L f^*(\sigma(s))v(s) ds\|_{X(0,L)}
\]
\[
= \|u(t) \int_0^L \tilde{f}(s)v(s) ds\|_{X(0,L)}.
\] (3.6)
By combining (3.5) and (3.6), we reach a contradiction. Hence, $\|f\|_{Y(0,L)} \leq \|g\|_{Y(0,L)}$.

Property (P3). Let $f, f_k \in \mathfrak{M}^+(0,L)$, $k \in \mathbb{N}$, be such that $f_k \not\to f$ a.e. Thanks to property (P2) of $\| \cdot \|_{Y(0,L)}$, the limit $\lim_{k \to \infty} \|f_k\|_{Y(0,L)}$ exists and we clearly have that $\lim_{k \to \infty} \|f_k\|_{Y(0,L)} \leq \|f\|_{Y(0,L)}$. The fact that $\lim_{k \to \infty} \|f_k\|_{Y(0,L)} = \|f\|_{Y(0,L)}$ can be proved by contradiction in a similar way to the proof of (P2).

Property (P1). The positive homogeneity and positive definiteness of $\| \cdot \|_{Y(0,L)}$ can be readily verified. As for the subadditivity of $\| \cdot \|_{Y(0,L)}$, let $f, g \in \mathfrak{M}^+(0,L)$ be simple functions. Let $h \in \mathfrak{M}^+(0,L)$ be such that $h \sim f + g$. Being
equimeasurable with \( f + g \), \( h \) is a simple function having the same range as \( f + g \). Furthermore, it is easy to see that \( h \) can be decomposed as \( h = h_1 + h_2 \), where \( h_1, h_2 \in \mathcal{M}^+(0, L) \) are simple functions such that \( h_1 \sim f \) and \( h_2 \sim g \). Using the subadditivity of \( \| \cdot \|_{X(0, L)} \), we obtain that

\[
\| u(t) \int_{\nu(t)}^L h(s)v(s)ds \|_{X(0, L)} \leq \| u(t) \int_{\nu(t)}^L h_1(s)v(s)ds \|_{X(0, L)} + \| u(t) \int_{\nu(t)}^L h_2(s)v(s)ds \|_{X(0, L)}
\]

Hence, \( \| f + g \|_{Y(0, L)} \leq \| f \|_{Y(0, L)} + \| g \|_{Y(0, L)} \). When \( f, g \in \mathcal{M}^+(0, L) \) are general functions, we approximate each of them by a nondecreasing sequence of nonnegative, simple functions and use property (P3) of \( \| \cdot \|_{Y(0, L)} \) to get

\[
\| f + g \|_{Y(0, L)} \leq \| f \|_{Y(0, L)} + \| g \|_{Y(0, L)}.
\]

Property (P4). Assume that \( L < \infty \). Since \( \| \cdot \|_{Y(0, L)} \) has property (P2), \( \| \cdot \|_{Y(0, L)} \) has property (P4) if and only if \( \| \chi_{(0, L)} \|_{Y(0, L)} < \infty \). If \( h \in \mathcal{M}^+(0, L) \) is equimeasurable with \( \chi_{(0, L)} \), then \( h = 1 \) a.e. on \( (0, L) \); therefore,

\[
\| \chi_{(0, L)} \|_{Y(0, L)} = \| H_{u, v, \nu} \chi_{(0, L)} \|_{X(0, L)}.
\]

Hence, \( \| \cdot \|_{Y(0, L)} \) has property (P4) if and only if \( u(t) \int_{\nu(t)}^L v(s)ds \in X(0, L) \). Assume now that \( L = \infty \). Fix \( \theta > 1 \) such that \( v \in D^\infty_\theta \). Let \( E \subseteq (0, \infty) \) be of finite measure. Set \( b = \max \left\{ 1, \nu(1), \frac{\theta|E|}{M-1} \right\} \), where \( M = \inf_{t \in [1, \infty)} \frac{\nu(\theta t)}{\nu(t)} \). Note that \( M > 1 \). Let \( h \in \mathcal{M}^+(0, \infty) \) be equimeasurable with \( \chi_E \). It is easy to see that \( h = \chi_F \) for some measurable \( F \subseteq (0, \infty) \) such that \( |F| = |E| \). Thanks to the (outer) regularity of the Lebesgue measure, there is an open set \( G \supseteq F \) such that \( |G| \leq \theta |F| \). Since \( G \) is an open set on the real line, there is a countable system of mutually disjoint open intervals \( \{(a_k, b_k)\}_k \) such that \( G \cap (b, \infty) = \bigcup_k (a_k, b_k) \). We plainly have that \( F \subseteq (0, b) \cup (G \cap (b, \infty)) \) and \( a_k > b \). Furthermore, we have that \( b_k - a_k \leq \theta |F| \leq (M-1)b < (M-1)a_k \), whence

\[
v^{-1}(b_k) - v^{-1}(a_k) < (\theta - 1)v^{-1}(a_k). (3.7)
\]

We have that

\[
\| u(t) \int_{\nu(t)}^{\infty} \chi_F(s)v(s)ds \|_{X(0, \infty)} \leq \| u(t) \int_{\nu(t)}^{\infty} \left( \chi_{(0, b]}(s) + \sum_k \chi_{(a_k, b_k]}(s) \right)v(s)ds \|_{X(0, \infty)}
\]

\[
\leq \| u(t) \chi_{(0, \nu^{-1}(b))}(t) \int_{\nu(t)}^{b} v(s)ds \|_{X(0, \infty)}
\]

\[
+ \sum_k \| u(t) \chi_{(\nu^{-1}(a_k), \nu^{-1}(b_k))}(t) \int_{\nu(t)}^{b_k} v(s)ds \|_{X(0, \infty)}
\]

\[
+ \sum_k \| u(t) \chi_{(\nu^{-1}(a_k), \nu^{-1}(b_k))}(t) \int_{\nu(t)}^{b_k} v(s)ds \|_{X(0, \infty)}.
\]

Note the assumption

\[
\| u(t) \chi_{(0, \nu^{-1}(1))}(t) \int_{\nu(t)}^{1} v(s)ds \|_{X(0, \infty)} < \infty (3.9)
\]

together with the monotonicity of \( u \) and \( v \) implies that

\[
\| u \chi_{(0, a)} \|_{X(0, \infty)} < \infty \quad \text{for every } a \in (0, \infty).
\]

\[
(3.10)
\]
Indeed, since $u$ is nonincreasing, it is sufficient to show that $\|u \chi_{(0, \nu^{-1}(\frac{1}{2}))}\|_{X(0, \infty)} < \infty$, which follows from

$$
\infty > \left\| u(t) \chi_{(0, \nu^{-1}(1))}(t) \int_{\nu(t)}^{1} v(s) \, ds \right\|_{X(0, \infty)} \geq \left\| u(t) \chi_{(0, \nu^{-1}(\frac{1}{2}))}(t) \int_{\nu(t)}^{1} v(s) \, ds \right\|_{X(0, \infty)} \geq \frac{v(1)}{2} \left\| u \chi_{(0, \nu^{-1}(\frac{1}{2}))}\right\|_{X(0, \infty)}.
$$

Furthermore, note that (3.10) guarantees that

$$
\limsup_{\tau \to \infty} v(\tau) \|u \chi_{(0, \nu^{-1}(\tau))}\|_{X(0, \infty)} < \infty
$$

if and only if

$$
\sup_{\tau \in [1, \infty)} v(\tau) \|u \chi_{(0, \nu^{-1}(\tau))}\|_{X(0, \infty)} < \infty. \quad (3.12)
$$

Now, as for the first term on the right-hand side of (3.8), we have that

$$
\left\| u(t) \chi_{(0, \nu^{-1}(b))}(t) \int_{\nu(t)}^{b} v(s) \, ds \right\|_{X(0, \infty)} \leq \left\| u(t) \chi_{(0, \nu^{-1}(1))}(t) \int_{\nu(t)}^{1} v(s) \, ds \right\|_{X(0, \infty)} + v(1)(b - 1) \|u \chi_{(0, \nu^{-1}(1))}\|_{X(0, \infty)} \leq A < \infty. \quad (3.13)
$$

Here,

$$
A = \left\| u(t) \chi_{(0, \nu^{-1}(1))}(t) \int_{\nu(t)}^{1} v(s) \, ds \right\|_{X(0, \infty)} + v(1)(b - 1) \|u \chi_{(0, \nu^{-1}(1))}\|_{X(0, \infty)} + v(1)(b - 1) \|u \chi_{(0, \nu^{-1}(b))}\|_{X(0, \infty)}.
$$

As for the second term on the right-hand side of (3.8), we have that

$$
\left\| u(t) \chi_{(0, \nu^{-1}(a_k))}(t) \int_{a_k}^{b_k} v(s) \, ds \right\|_{X(0, \infty)} \leq v(a_k) (b_k - a_k) \|u \chi_{(0, \nu^{-1}(a_k))}\|_{X(0, \infty)} \leq B(b_k - a_k). \quad (3.14)
$$

where $B$ is the supremum in (3.12), which is independent of $k$. Next,

$$
\left\| u(t) \chi_{(\nu^{-1}(a_k), \nu^{-1}(b_k))}(t) \int_{a_k}^{b_k} v(s) \, ds \right\|_{X(0, \infty)} \leq \int_{a_k}^{b_k} v(s) \, ds \|u \chi_{(\nu^{-1}(a_k), \nu^{-1}(b_k))}\|_{X(0, \infty)} \leq v(a_k) (b_k - a_k) \|u \chi_{(\nu^{-1}(a_k), \nu^{-1}(b_k))}\|_{X(0, \infty)} \leq v(a_k) (b_k - a_k) \|u \chi_{(0, \nu^{-1}(a_k))}\|_{X(0, \infty)} \leq \left| \theta - 1 \right| v(a_k) (b_k - a_k) \|u \chi_{(0, \nu^{-1}(a_k))}\|_{X(0, \infty)} \leq \left| \theta - 1 \right| B(b_k - a_k). \quad (3.15)
$$
Here, we used the monotonicity of $u$ and $v$ in the second inequality, (3.7) in the third one, and the monotonicity of $u$ in the fourth one. By combining (3.8) with (3.13), (3.14), and (3.15), we obtain that

$$
\| u(t) \int_{\nu(t)}^{\infty} h(s)v(s) \, ds \|_{X(0,\infty)} \leq A + [\theta] B \sum_{k} (b_k - a_k) \leq A + [\theta] \theta B |E| < \infty.
$$

Hence, $\| X_{E} \|_{Y(0,L)} < \infty$ provided that (3.9) and (3.11) are satisfied. The necessity of (3.9) is obvious because we have that

$$
\| u(\tau) \chi(0,\nu^{-1}(\tau))(t) \int_{\nu(t)}^{\infty} \nu(s)v(s) \, ds \|_{X(0,\infty)} < \infty.
$$

As for the necessity of (3.11), suppose that $\limsup_{\tau \to \infty} \nu(\tau) \| u(\tau) \chi(0,\nu^{-1}(\tau)) \|_{X(0,\infty)} = \infty$. It follows that there is a sequence $\tau_k \to \infty$, $k \to \infty$, such that

$$
\lim_{k \to \infty} \nu(\tau_k) \| u(\tau_k) \chi(0,\nu^{-1}(\tau_k)) \|_{X(0,\infty)} = \infty.
$$

Since $\inf_{t \in [1,\infty)} \frac{\nu(\theta t)}{\nu(t)} > 1$, we can find an $\varepsilon > 0$ such that $\frac{\nu(\theta t)}{\nu(t)} \geq 1 + \varepsilon$ for every $t \in [1,\infty)$. Moreover, we may clearly assume that $\tau_k \geq \nu(1) + 1$ and $\frac{\tau_k}{\tau_k - 1} \leq 1 + \varepsilon$. Hence,

$$
\nu^{-1}(\tau_k) - \nu^{-1}(\tau_k - 1) \leq (\theta - 1) \nu^{-1}(\tau_k - 1)
$$

(3.16)
inasmuch as $\frac{\nu(\theta \nu^{-1}(\tau_k - 1))}{\nu(\nu^{-1}(\tau_k - 1))} \geq 1 + \varepsilon$. Using (3.16) and the fact that $u$ is nonincreasing, we obtain that

$$
\| u(\tau_k) \chi(0,\nu^{-1}(\tau_k)) \|_{X(0,\infty)} \leq \| u(\tau_k) \chi(0,\nu^{-1}(\tau_k - 1)) \|_{X(0,\infty)} + \| u(\tau_k) \chi(\nu^{-1}(\tau_k - 1),\nu^{-1}(\tau_k)) \|_{X(0,\infty)}
$$

$$
\leq \| u(\tau_k) \chi(0,\nu^{-1}(\tau_k - 1)) \|_{X(0,\infty)} + \| u(\tau_k) \chi(0,\nu^{-1}(\tau_k - 1)) \|_{X(0,\infty)}
$$

which tends to $\infty$ as $k \to \infty$. Hence, $\| \chi(0,1) \|_{Y(0,\infty)} = \infty$, and so $\| \cdot \|_{Y(0,\infty)}$ does not have property (P4).

Property (P5). Assume that $L < \infty$. Note that (3.4) together with $\nu(L) > 0$ implies that $\| u \|_{X(0,L)} < \infty$. Let $f \in \mathbb{M}^{+}(0,L)$. Since $f^*$ is nonincreasing, we have that $\int_{0}^{L} f^*(s) \, ds \leq 2 \int_{0}^{L} f^*(s) \, ds$. Since the function $(0,L) \ni t \mapsto f^*(L-t)$ is equimeasurable with $f$, we have that

$$
\| f \|_{Y(0,L)} \geq \left\| u(t) \int_{\nu(t)}^{L} f^*(L-s) v(s) \, ds \right\|_{X(0,L)}
$$

$$
\geq \nu(L) \left\| u(t) \chi(0,\nu^{-1}(\frac{L}{2}))(t) \int_{\nu(t)}^{L} f^*(L-s) \, ds \right\|_{X(0,L)}
$$

$$
= \nu(L) \left\| u(t) \chi(0,\nu^{-1}(\frac{L}{2}))(t) \int_{0}^{L-\nu(t)} f^*(s) \, ds \right\|_{X(0,L)}
$$
Here, we used \((2.5)\) in the last inequality. Since \( \frac{v(L^-)}{2} \|u_{(0,\nu^{-1}(\frac{L}{2}))}\|_{X(0,L)} \in (0, \infty) \) does not depend on \( f \), property (P5) follows. Assume now that \( L = \infty \). Recall that (3.10) is satisfied provided that (3.9) is satisfied. Let \( f \in \mathcal{M}^+(0, \infty) \) and \( E \subseteq (0, \infty) \) be of finite measure. The function \((0, \infty) \ni t \mapsto f^*(t - |E|) \chi_{[|E|, \infty)}(t)\) is equimeasurable with \( f \). By arguing similarly to the case \( L < \infty \), we obtain that
\[
\|f\|_{Y(0, \infty)} \geq v(2|E|) \|u_{(0,\nu^{-1}(|E|))}\|_{X(0, \infty)} \int_E f(s) \, ds,
\]
whence property (P5) follows.

Property (P6). Since the relation \( \sim \) is transitive, it plainly follows that \( \| \cdot \|_{Y(0,L)} \) has property (P6).

Note that the necessity of (3.4) for \( \| \cdot \|_{Y(0,L)} \) to be a rearrangement-invariant function norm was already proved in the paragraph devoted to property (P4).

Assume now that (3.4) is satisfied. We plainly have that
\[\|H_{u,v,\nu} f\|_{X(0,L)} \leq \|f\|_{Y(0,L)} \quad \text{for every } f \in \mathcal{M}^+(0,L),\]
and so \( H_{u,v,\nu} : Y(0,L) \to X(0,L) \) is bounded. Next, let \( Z(0,L) \) be a rearrangement-invariant function space such that \( H_{u,v,\nu} : Z(0,L) \to X(0,L) \) is bounded. For every \( f \in \mathcal{M}^+(0, \infty) \) and each \( h \in \mathcal{M}^+(0, \infty) \) equimeasurable with \( f \), we have that
\[\|H_{u,v,\nu} h\|_{X(0,L)} \leq \|h\|_{Z(0,L)} = \|f\|_{Z(0,L)}.\]
Therefore,
\[\|f\|_{Y(0,L)} \leq \|f\|_{Z(0,L)} \quad \text{for every } f \in \mathcal{M}^+(0,L).\]

Hence, \( Z(0,L) \preceq Y(0,L) \). Finally, we claim that (3.4) needs to be satisfied if there is any rearrangement-invariant function space \( Z(0,L) \) such that \( H_{u,v,\nu} : Z(0,L) \to X(0,L) \) is bounded. If \( L < \infty \), we plainly have that
\[\left\| \int_{\mathcal{M}(f)} u(t) \int_0^L v(s) \, ds \right\|_{X(0,L)} = \|H_{u,v,\nu} X(0,L)\|_{X(0,L)} \leq \|X(0,L)\|_{Z(0,L)} < \infty.\]
If \( L = \infty \), we can argue as in the paragraph devoted to property (P4) to show that, if (3.4) is not satisfied, then
\[\sup_{h \sim \chi_{(0,1)}} \|H_{u,v,\nu} h\|_{X(0,\infty)} = \infty.\]
However, this implies, thanks to the boundedness of \( H_{u,v,\nu} : Z(0, \infty) \to X(0, \infty) \),
\[\infty = \sup_{h \sim \chi_{(0,1)}} \|H_{u,v,\nu} h\|_{X(0,\infty)} \leq \|\chi_{(0,1)}\|_{Z(0,\infty)} < \infty,\]
which would be a contradiction. \(\square\)
Remark 3.4.

(1) The assumption \( \nu \in D^\infty \) is not overly restrictive. For example, it is satisfied whenever \( \nu \) is equivalent to \( t \mapsto t^\alpha b(t) \) near \( \infty \) for some \( \alpha > 0 \) and a slowly varying function \( b \) (cf. [35, Proposition 2.2]). On the other hand, \( \nu(t) = \log^2(t) \) near \( \infty \), where \( \alpha > 0 \), is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption \( \nu \in D^0 \), which will appear in Proposition 5.4.

(2) When \( u \equiv 1 \), (3.11) is equivalent to
\[
\limsup_{t \to \infty} \nu(t) \|X(0,t)\|_{X(0,\infty)} < \infty.
\]

(3) The functional (3.3) is quite complicated; however, we shall see in Section 4 that it can often be significantly simplified.

(4) Let \( Y_1(0,L) \) and \( Y_2(0,L) \) be the optimal domain spaces for \( R_{u_1,v_1,\nu_1} \) and \( H_{u_2,v_2,\nu_2} \), respectively. Note that \((R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}) : Z(0,L) \to X(0,L)\) is bounded if and only if both \( R_{u_1,v_1,\nu_1} \) and \( H_{u_2,v_2,\nu_2} \) are bounded from \( Z(0,L) \) to \( X(0,L) \). Consequently, \( Y_1(0,L) \cap Y_2(0,L) \) is the optimal domain space for \( R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2} \) and \( X(0,L) \).

3.2 Optimal target spaces

We start with an easy but useful observation concerning the Hardy-type operators defined by (1.1) and (1.2). Let \( u, v : (0,L) \to (0,\infty) \) be measurable functions. Let \( \nu : (0,L) \to (0,L) \) be an increasing bijection. The operators \( R_{u,v,\nu} \) and \( H_{u,v,\nu^{-1}} \) are in a sense dual to each other. More precisely, by using the Fubini theorem, one can easily verify that
\[
\int_0^L f(t)R_{u,v,\nu}g(t)dt = \int_0^L g(t)H_{u,v,\nu^{-1}}f(t)dt \quad \text{for every } f, g \in \mathcal{M}^+(0,L).
\]

(3.17)

Here, \( \nu^{-1} \) is the inverse function to \( \nu \).

The validity of (3.17) has an unsurprising, well-known consequence, which we state here for future reference (see also Corollary 4.9).

Proposition 3.5. Let \( \| \cdot \|_{X(0,L)} \), \( \| \cdot \|_{Y(0,L)} \) be rearrangement-invariant function norms.

(1) Let \( \nu : (0,L) \to (0,L) \) be an increasing bijection.

(2) Let \( u, v : (0,L) \to (0,\infty) \) be measurable.

We have that
\[
\sup_{\|f\|_{X(0,L)} \leq 1} \|R_{u,v,\nu}f\|_{Y(0,L)} = \sup_{\|g\|_{\nu^{-1}(0,L)} \leq 1} \|H_{u,v,\nu^{-1}}g\|_{X(0,L)}.
\]

(3.18)

In particular,
\[
R_{u,v,\nu} : X(0,L) \to Y(0,L) \quad \text{is bounded if and only if}
\]
\[
H_{u,v,\nu^{-1}} : Y'(0,L) \to X'(0,L) \quad \text{is bounded.}
\]

(3.19)

Proof. We have that
\[
\sup_{\|f\|_{X(0,L)} \leq 1} \|R_{u,v,\nu}f\|_{Y(0,L)} = \sup_{\|f\|_{X(0,L)} \leq 1} \sup_{\|g\|_{\nu^{-1}(0,L)} \leq 1} \int_0^L R_{u,v,\nu}f(t)|g(t)|dt
\]
\[
= \sup_{\|f\|_{X(0,L)} \leq 1} \sup_{\|g\|_{\nu^{-1}(0,L)} \leq 1} \int_0^L |f(t)||H_{u,v,\nu^{-1}}g(t)|dt
\]
\[
= \sup_{\|g\|_{\nu^{-1}(0,L)} \leq 1} \|H_{u,v,\nu^{-1}}g\|_{X'(0,L)}
\]

thanks to (2.9), (3.17), and (2.8).
Remark 3.6. Thanks to $(3.19)$ and $(2.10)$, $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and $X(0,L)$ if and only if $Y'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0,L)$. Similarly, $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ if and only if $Y'(0,L)$ is the optimal domain space for the operator $H_{u,v,\nu^{-1}}$ and $X'(0,L)$.

As immediate corollaries of Remark 3.6 combined with Proposition 3.1 and Proposition 3.3, we obtain the following descriptions of the optimal target spaces for the operators $H_{u,v,\nu}$ and $R_{u,v,\nu}$, respectively.

**Proposition 3.7.** Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

- Let $v,u,v$ be as in Proposition 3.1.

Assume that $\xi \in X'(0,L)$, where $\xi$ is defined by $(3.1)$ with $\nu$ replaced by $\nu^{-1}$. Let $\| \cdot \|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\| \cdot \|_{Y'(0,L)}$ is defined as

$$\| f \|_{Y'(0,L)} = \| R_{u,v,\nu^{-1}}(f^+) \|_{X'(0,L)}, \quad f \in \mathfrak{M}^+(0,L).$$

The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $H_{u,v,\nu}$ and $X(0,L)$. Moreover, if $\xi \notin X'(0,L)$, then there is no rearrangement-invariant function space $Z(0,L)$ such that $H_{u,v,\nu} : X(0,L) \to Z(0,L)$ is bounded.

**Proposition 3.8.** Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

- Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. If $L = \infty$, assume that $\nu^{-1} \in \mathcal{D}_\infty$.
- Let $u,v$ be as in Proposition 3.3.

Assume that

$$\begin{cases}
u(t) \int_0^L u(s) \, ds \in X'(0,L) & \text{if } L < \infty, \\
u(t) \chi(0,\nu(1))(t) \int_0^{L} u(s) \, ds \in X'(0,\infty) & \text{if } L = \infty, \\
 \limsup_{\tau \to \infty} v(\tau) \| u \chi(0,\nu(\tau)) \|_{X'(0,\infty)} < \infty
\end{cases} \quad (3.20)$$

Let $\| \cdot \|_{Y(0,L)}$ be the rearrangement-invariant function norm whose associate function norm $\| \cdot \|_{Y'(0,L)}$ is defined as

$$\| f \|_{Y'(0,L)} = \sup_{h \sim f} \| H_{u,v,\nu^{-1}} h \|_{X'(0,L)}, \quad f \in \mathfrak{M}^+(0,L).$$

(3.21)

Here, the supremum extends over all $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$. The rearrangement-invariant function space $Y(0,L)$ is the optimal target space for the operator $R_{u,v,\nu}$ and $X(0,L)$. Moreover, if $(3.20)$ is not satisfied, then there is no rearrangement-invariant function space $Z(0,L)$ such that $R_{u,v,\nu} : X(0,L) \to Z(0,L)$ is bounded.

**Remark 3.9.** Let $Y_1(0,L)$ and $Y_2(0,L)$ be the optimal target spaces for $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$, respectively. Note that $(R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}) : X(0,L) \to Z(0,L)$ is bounded if and only if both $R_{u_1,v_1,\nu_1}$ and $H_{u_2,v_2,\nu_2}$ are bounded from $X(0,L)$ to $Z(0,L)$. It follows that $(Y_1 + Y_2)(0,L)$ is the optimal target space for $R_{u_1,v_1,\nu_1} + H_{u_2,v_2,\nu_2}$ and $X(0,L)$.

## 4 SIMPLIFICATION OF OPTIMAL FUNCTION NORMS AND THEIR CONNECTION WITH INTERPOLATION

### 4.1 Simplification of optimal function norms

The description of the optimal domain spaces for the operator $H_{u,v,\nu}$ is complicated by the fact that the functional $\mathfrak{M}^+(0,L) \ni f \mapsto \| H_{u,v,\nu}(f^+) \|_{X(0,L)}$ is usually not a rearrangement-invariant function norm. However, it actually is a rearrangement-invariant function norm when $u$, $v$, and $\nu$ are related to each other in such a way that the function...
$R_{u,v,\nu^{-1}}(g^*)$ is nonincreasing for every $g \in \mathcal{M}^+(0,L)$. This fact is the content of the following proposition, in which we omit its obvious consequences for optimal spaces. In the light of Proposition 3.8, the situation is similar for the optimal target spaces for the operator $R_{u,v,\nu}$.

**Proposition 4.1.** Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

1. Let $\nu : (0,L) \to (0,L)$ be an increasing bijection.
2. Let $u : (0,L) \to (0,\infty)$ be a nondegenerate nonincreasing function.
3. Let $v : (0,L) \to (0,\infty)$ be defined by
   \[
   \frac{1}{v(t)} = \int_0^{\nu^{-1}(t)} u(s) \, ds \quad \text{for every } t \in (0,L).
   \]

Set
\[
\| f \|_{Y(0,L)} = \| H_{u,v,\nu}(f^*) \|_{X(0,L)}, \quad f \in \mathcal{M}^+(0,L).
\]

The functional $\| \cdot \|_{Y(0,L)}$ is a rearrangement-invariant function norm if and only if
\[
\left\| u(t) \chi(0,\nu^{-1}(a)) \int_{\nu(t)}^a \frac{1}{U(\nu^{-1}(s))} \, ds \right\|_{X(0,L)} < \infty,
\]
where $a$ is defined by (3.2).

**Proof.** We only sketch the proof, which is significantly easier than that of Proposition 3.3. The functional $\| \cdot \|_{Y(0,L)}$ plainly possesses properties (P2), (P3), and (P6). It is easy to see that $\| \cdot \|_{Y(0,L)}$ has property (P4) if and only if (4.1) is satisfied. To this end, note that (4.1) implies (3.10). As for property (P1), only the subadditivity needs a comment. The key observation is that $(0,L \ni t \mapsto R_{u,v,\nu^{-1}}(h^*)(t))$ is nonincreasing for every $h \in \mathcal{M}^+(0,L)$ inasmuch as it is the integral mean of a non-negative nonincreasing function over the interval $(0,\nu^{-1}(t))$ with respect to the measure $u(s) \, ds$. Hence, thanks to (2.11), (3.17), and (2.3) combined with the Hardy lemma (2.6), we have that
\[
\| f + g \|_{Y(0,L)} = \| H_{u,v,\nu}((f + g)^*) \|_{X(0,L)} = \sup_{h \in \mathcal{M}^+(0,L), \|h\|_{X(0,L)} \leq 1} \int_0^L H_{u,v,\nu}((f + g)^*)(t)h^*(t) \, dt,
\]

\[
\leq \sup_{h \in \mathcal{M}^+(0,L), \|h\|_{X(0,L)} \leq 1} \int_0^L f^*(t)R_{u,v,\nu^{-1}}(h^*)(t) \, dt + \sup_{h \in \mathcal{M}^+(0,L), \|h\|_{X(0,L)} \leq 1} \int_0^L g^*(t)R_{u,v,\nu^{-1}}(h^*)(t) \, dt
\]
\[
\leq \sup_{h \in \mathcal{M}^+(0,L), \|h\|_{X(0,L)} \leq 1} \int_0^L H_{u,v,\nu}(f^*)(t)h^*(t) \, dt + \sup_{h \in \mathcal{M}^+(0,L), \|h\|_{X(0,L)} \leq 1} \int_0^L H_{u,v,\nu}(g^*)(t)h^*(t) \, dt
\]
\[
= \| f \|_{Y(0,L)} + \| g \|_{Y(0,L)}
\]
for every $f, g \in \mathcal{M}^+(0, L)$. Finally, as for the validity of property (P5), owing to (2.11), (3.17), the monotonicity of $R_{u, v, \nu^{-1}}(g^*)$, and the Hardy–Littlewood inequality (2.4), we have that

$$
\|f\|_{Y(0, L)} \geq \|f\|_{Y(0, L)} = \sup_{g \in \mathcal{M}^+(0, L), \|g\|_{X'(0, L)} \leq 1} \int_0^L (f \chi_E)^*(t) R_{u, v, \nu^{-1}}(g^*)(t) \, dt
$$

$$
\geq \int_0^{|E|} (f \chi_E)^*(t) \, dt \sup_{g \in \mathcal{M}^+(0, L), \|g\|_{X'(0, L)} \leq 1} R_{u, v, \nu^{-1}}(g^*)(|E|)
$$

$$
\sup_{g \in \mathcal{M}^+(0, L), \|g\|_{X'(0, L)} \leq 1} \int_0^{|E|} (f \chi_E)^*(t) \, dt \geq R_{u, v, \nu^{-1}}\left(\frac{X(0,|E|)}{\|X(0,|E|)\|_{X'(0, L)}}\right)(|E|) \int_0^{|E|} (f \chi_E)^*(t) \, dt
$$

for every $f \in \mathcal{M}^+(0, L)$ and $E \subseteq (0, L)$ having finite measure.

In general, when the functions $u$, $v$, and $\nu$ are not related to each other in the particular way as in Proposition 4.1, we have to live with the complicated functional (3.3). Nevertheless, we shall see that the functional is often equivalent to a significantly more manageable functional (cf. [30, Theorem 4.2]). To this end, we need to introduce a supremum operator. For a fixed measurable function $\varphi : (0, L) \rightarrow (0, \infty)$, we define the operator $T_{\varphi}$ as

$$
T_{\varphi}f(t) = \frac{1}{\varphi(t)} \sup_{s \in [t, L]} \varphi(s)f^*(s), \quad t \in (0, L), \quad f \in \mathcal{M}(0, L).
$$

(4.2)

Note that $T_{\varphi}f(t) = \frac{1}{\varphi(t)} \sup_{s \in [t, L]} \varphi(s)f^*(s)$ for every $t \in (0, L)$ provided that $\varphi$ is nondecreasing and/or right-continuous. If $\varphi$ is nonincreasing, we have that $T_{\varphi}f(t) = \frac{f^*(t)}{\varphi(t)} \varphi(t^+)$ for every $t \in (0, L)$, and so $T_{\varphi}f = f^+$ possibly up to a countably many points.

**Proposition 4.2.** Let $\|\cdot\|_{X(0, L)}$ be a rearrangement-invariant function norm.

1. Let $\nu : (0, L) \rightarrow (0, L)$ be an increasing bijection. If $L = \infty$, assume that $\nu \in D^\infty$.
2. Let $u : (0, L) \rightarrow (0, \infty)$ be nonincreasing.
3. Let $v : (0, L) \rightarrow (0, \infty)$ be defined by

$$
\frac{1}{v(t)} = \int_0^{\nu^{-1}(t)} \xi(s) \, ds \quad \text{for every } t \in (0, L),
$$

(4.3)

where $\xi : (0, L) \rightarrow (0, \infty)$ is a measurable function. If $L < \infty$, assume that $v(L^-) > 0$. Furthermore, assume that the operator $T_\varphi$ defined by (4.2) with $\varphi = u/\xi$ is bounded on $X'(0, L)$.

Assume that

$$
\left\|u(t)X(0, \nu^{-1}(a))(t) \int_a^b v(s) \, ds\right\|_{X(0, L)} < \infty,
$$

for every $f, g \in \mathcal{M}^+(0, L)$. Finally, as for the validity of property (P5), owing to (2.11), (3.17), the monotonicity of $R_{u, v, \nu^{-1}}(g^*)$, and the Hardy–Littlewood inequality (2.4), we have that
where $a$ is defined by (3.2). Let $\| \cdot \|_{Y(0,L)}$ be the functional defined by (3.3) and set

$$
\|f\|_{Z(0,L)} = \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L f^+(t) v(t) \int_0^{\nu^{-1}(t)} T_\varphi g(s) u(s) \, ds \, dt, \quad f \in \mathfrak{M}^+(0,L).
$$

The functionals $\| \cdot \|_{Y(0,L)}$ and $\| \cdot \|_{Z(0,L)}$ are rearrangement-invariant function norms. Furthermore, we have that

$$
\|H_{u,v,\nu}(f^+)\|_{X'(0,L)} \leq \sup_{h \sim f} \|H_{u,v,\nu}h\|_{X'(0,L)} \leq \|f\|_{Z(0,L)}
$$

for every $f \in \mathfrak{M}^+(0,L)$. Here, $\|T_\varphi\|_{X'(0,L)}$ stands for the operator norm of $T_\varphi$ on $X'(0,L)$. In particular, the rearrangement-invariant function norms $\| \cdot \|_{Y(0,L)}$ and $\| \cdot \|_{Z(0,L)}$ are equivalent.

Proof. Since $f \sim f^+$ for every $f \in \mathfrak{M}^+(0,L)$, the first inequality in (4.4) plainly holds. As for the second inequality, note that the function $(0,L) \ni t \mapsto R_{u,v,\nu^{-1}} T_\varphi g(t)$ is nonincreasing for every $g \in \mathfrak{M}^+(0,L)$. Indeed, it is the integral mean of the nonincreasing function $(0,L) \ni s \mapsto \operatorname{esssup}_{r \in [s,L]} \varphi(\tau) g^*(\tau)$ over the interval $(0,\nu^{-1}(t))$ with respect to the measure $\xi(s) \, ds$. Consequently, for every $f \in \mathfrak{M}^+(0,L)$ and every $h \in \mathfrak{M}^+(0,L)$ equimeasurable with $f$, we have that

$$
\|H_{u,v,\nu}h\|_{X'(0,L)} = \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L h(t) R_{u,v,\nu^{-1}}(g^*) \, dt
\leq \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L h(t) R_{u,v,\nu^{-1}}(T_\varphi g) \, dt
\leq \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L h^*(t) R_{u,v,\nu^{-1}}(T_\varphi g) \, dt
= \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L f^*(t) R_{u,v,\nu^{-1}}(T_\varphi g) \, dt
= \|f\|_{Z(0,L)}.
$$

Here, we used (2.11) together with (3.17) in the first equality, the pointwise estimate $g^*(t) \leq T_\varphi g(t)$ for a.e. $t \in (0,L)$ in the first inequality, the Hardy–Littlewood inequality (2.4) in the second inequality, and the equimeasurability of $f$ and $h$ in the last inequality. Hence, the second inequality in (4.4) follows from (4.5). As for the third inequality in (4.4), we have that

$$
\sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L f^*(t) R_{u,v,\nu^{-1}}(T_\varphi g) \, dt = \sup_{g \in \mathfrak{M}^+(0,L)} \int_0^L T_\varphi g(t) H_{u,v,\nu}(f^*) \, dt
\leq \|H_{u,v,\nu}(f^*)\|_{X(0,L)} \sup_{g \in \mathfrak{M}^+(0,L)} \|T_\varphi g\|_{X'(0,L)}
= \|T_\varphi\|_{X'(0,L)} \|H_{u,v,\nu}(f^*)\|_{X(0,L)}
$$

for every $f \in \mathfrak{M}^+(0,L)$ thanks to (3.17) and the Hölder inequality (2.14).
Second, we shall prove that the functional \( \| \cdot \|_{Y(0,L)} \), defined by (3.3), is a rearrangement-invariant function norm. If \( L < \infty \), this follows immediately from Proposition 3.3. If \( L = \infty \), owing to Proposition 3.3 again, we only need to verify that (3.11) is satisfied. To this end, it follows from the proof of property (P4) of \( \| \cdot \|_{Y(0,L)} \) that, if (3.11) did not hold, then we would have

\[
\sup_{h \sim \chi(0,1)} \| H_{u,v,h} \|_{X(0,\infty)} = \infty.
\]

However, thanks to (4.4), we have that

\[
\sup_{h \sim \chi(0,1)} \| H_{u,v,h} \|_{X(0,\infty)} \approx \| H_{u,v,\chi(0,1)} \|_{X(0,\infty)}
\]

\[
= \left\| u(t) \chi(0,\nu^{-1}(1))(t) \int_0^1 \nu(s) ds \right\|_{X(0,\infty)} < \infty.
\]

Therefore, (3.11) is satisfied.

Finally, now that we know that the functionals \( \| \cdot \|_{Y(0,L)} \) and \( \| \cdot \|_{Z(0,L)} \) are equivalent and the former is a rearrangement-invariant function norm, it readily follows that \( \| \cdot \|_{Z(0,L)} \), too, is a rearrangement-invariant function norm once we observe that \( \| \cdot \|_{Z(0,L)} \) is subadditive. The subadditivity follows from

\[
\| f + g \|_{Z(0,L)} = \sup_{h \in \mathcal{M}^+(0,L)} \int_0^L (f + g) \ast (T_{\nu} h)(t) \, dt
\]

\[
\leq \sup_{h \in \mathcal{M}^+(0,L)} \int_0^L f^\ast (t) R_{u,v,\nu^{-1}} (T_{\nu} h)(t) \, dt
\]

\[
+ \sup_{h \in \mathcal{M}^+(0,L)} \int_0^L g^\ast (t) R_{u,v,\nu^{-1}} (T_{\nu} h)(t) \, dt
\]

\[
= \| f \|_{Z(0,L)} + \| g \|_{Z(0,L)} \quad \text{for every } f, g \in \mathcal{M}^+(0,L).
\]

Here, we used (2.3) together with the Hardy lemma (2.6) (recall that the function \( R_{u,v,\nu^{-1}} (T_{\nu} h) \) is nonincreasing for every \( h \in \mathcal{M}^+(0,L) \)).

**Remark 4.3.**

(i) If \( \varphi = u/\xi \) is (equivalent to) a nonincreasing function, \( T_{\varphi} f(t) \) is (equivalent to) \( f^\ast (t) \) for a.e. \( t \in (0,L) \); hence \( T_{\varphi} \) is bounded on every rearrangement-invariant function space in this case. Furthermore, when \( \varphi = u/\xi \) is nonincreasing, the norm of \( T_{\varphi} \) on every rearrangement-invariant function space is equal to 1; therefore, all the inequalities in (4.4) are actually equalities (cf. Proposition 4.1) in this case.

(ii) The boundedness of \( T_{\varphi} \) on a large number of rearrangement-invariant function spaces is characterized by [34, Theorem 3.2].

By combining Proposition 4.2 and Proposition 3.8, we obtain the following proposition. It tells us that the optimal target space for the operator \( R_{u,v,\nu} \) and a rearrangement-invariant function space \( X(0,L) \) often has a much more manageable description than that given by Proposition 3.8. This is the case if the supremum operator \( T_{\varphi} \) defined by (4.2) with an appropriate function \( \varphi \) is bounded on \( X(0,L) \).

**Proposition 4.4.** Let \( \| \cdot \|_{X(0,L)} \) be a rearrangement-invariant function norm.

(1) Let \( \nu : (0,L) \to (0,L) \) be an increasing bijection. If \( L = \infty \), assume that \( \nu^{-1} \in D_\infty \).

(2) Let \( u : (0,L) \to (0,\infty) \) be nonincreasing.
(3) Let \( v : (0, L) \to (0, \infty) \) be defined by
\[
\frac{1}{v(t)} = \int_0^{\nu(t)} \xi(s) \, ds \quad \text{for every } t \in (0, L),
\]
where \( \xi : (0, L) \to (0, \infty) \) is a measurable function. If \( L < \infty \), assume that \( v(L^-) > 0 \). Furthermore, assume that the operator \( T_\varphi \) defined by (4.2) with \( \varphi = u/\xi \) is bounded on \( X(0, L) \).

Assume that
\[
\left\| u(t) \right\|_{X(0, \nu(a))} \int_a^{\nu^{-1}(t)} \int_0^\nu v(s) \, ds \right\|_{X'(0, L)} < \infty,
\]
where \( a \) is defined by (3.2). Let \( \| \cdot \|_{Y(0, L)} \) be the rearrangement-invariant function norm whose associate function norm \( \| \cdot \|_{Y'(0, L)} \) is defined as
\[
\| f \|_{Y'(0, L)} = \sup_{g \in \mathcal{M}^+(0, L), \| g \|_{X(0, L)} \leq 1} \int_0^L f^+(s) v(t) \int_0^{\nu(t)} T_\varphi g(s) u(s) \, ds \, dt, \quad f \in \mathcal{M}^+(0, L).
\]
The rearrangement-invariant function space \( Y(0, L) \) is the optimal target space for the operator \( R_{u, v, \nu} \) and \( X(0, L) \). Moreover,
\[
\| H_{u, v, \nu^{-1}}(f^+) \|_{X'(0, L)} \leq \| f \|_{Y'(0, L)} \leq \| T_\varphi \|_{X(0, L)} \| H_{u, v, \nu^{-1}}(f^+) \|_{X'(0, L)}
\]
for every \( f \in \mathcal{M}^+(0, L) \). Here, \( \| \cdot \|_{X(0, L)} \) stands for the operator norm of \( T_\varphi \) on \( X(0, L) \).

**Remark 4.5.** Owing to Remark 3.6, Proposition 4.4 can also be used to get a simpler description of optimal domain spaces for the operator \( H_{u, v, \nu} \).

A great deal of our effort has been devoted to describing optimal rearrangement-invariant function spaces. A natural, somewhat related question is, can every rearrangement-invariant function space be an optimal space? Suppose that \( Z(0, L) \) is the optimal domain space for \( H_{u, v, \nu} \) and \( X(0, L) \), and denote by \( W(0, L) \) the optimal target space for \( H_{u, v, \nu} \) and \( Z(0, L) \). Owing to the optimality of \( W(0, L) \), we immediately see that \( W(0, L) \preceq X(0, L) \). What is not obvious, however, is whether the opposite embedding, too, (i.e.) (can be) true. This leads us to the following theorem, which shows that the notion of being an optimal function space is related to the question of whether the complicated functional (3.3) can be simplified.

**Theorem 4.6.** Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

(1) Let \( \nu : (0, L) \to (0, L) \) be an increasing bijection. If \( L = \infty \), assume that \( \nu \in D^\infty \).
(2) Let \( u : (0, L) \to (0, \infty) \) be a nondegenerate nonincreasing function. If \( L < \infty \), assume that \( u(L^-) > 0 \).
(3) Let \( v : (0, L) \to (0, \infty) \) be a nonincreasing function. If \( L < \infty \), assume that \( v(L^-) > 0 \).

Let \( \| \cdot \|_{Y(0, L)} \) be the functional defined by (3.3). The following three statements are equivalent.

(i) The space \( X(0, L) \) is the optimal target space for the operator \( H_{u, v, \nu} \) and some rearrangement-invariant function space.
(ii) The space \( X'(0, L) \) is the optimal domain space for the operator \( R_{u, v, \nu^{-1}} \) and some rearrangement-invariant function space.
(iii) We have that
\[
\| f \|_{X'(0, L)} \approx \sup_{g \in \mathcal{M}^+(0, L), \| g \|_{Y(0, L)} \leq 1} \int_0^L g(t) R_{u, v, \nu^{-1}}(f^+)(t) \, dt \quad \text{for every } f \in \mathcal{M}^+(0, L).
\] (4.6)
Finally, assume, in addition, that

(1) $v$ is defined by (4.3) with $\xi$ satisfying

$$u(t) \int_0^t \xi(s) ds \leq \xi(t) \text{ for a.e. } t \in (0,L),$$

(4.7)

(2) the function $\varphi \circ \nu^{-1}$, where $\varphi = u/\xi$, is equivalent to a quasiconcave function.

Then, each of the three equivalent statements above implies that

(iv) the operator $T_\varphi$, defined by (4.2), is bounded on $X'(0,L)$.

Proof. We start off by observing that each of the three equivalent statements implies that the functional $\| \cdot \|_{Y(0,L)}$ is actually a rearrangement-invariant function norm. Statements (i) and (ii) imply it thanks to Proposition 3.3 and Proposition 3.8, respectively. If we assume (iii), then, in particular, the set $\{ g \in \mathcal{M}^+(0,L) : \| g \|_{Y(0,L)} \leq 1 \}$ needs to contain a function $g \in \mathcal{M}^+(0,L)$ not equal to 0 a.e. Thanks to Proposition 3.3 and its proof, $\| g \|_{Y(0,L)} = \infty$ for every $g \in \mathcal{M}^+(0,L)$ not equal to 0 a.e. if $\varphi$ fails to be a rearrangement-invariant function norm. Hence, $\| \cdot \|_{Y(0,L)}$ is a rearrangement-invariant function norm if (iii) is assumed. Therefore, in all of the cases, we are entitled to denote the corresponding rearrangement-invariant function space over $(0,L)$ by $Y(0,L)$. Moreover, note that (4.6) actually reads as

$$\| f \|_{X'(0,L)} \approx \| R_{u,v,\nu^{-1}}(f^*) \|_{Y'(0,L)} \text{ for every } f \in \mathcal{M}^+(0,L).$$

(4.8)

Second, statements (i) and (ii) are clearly equivalent to each other owing to Remark 3.6.

Next, the proof of the fact that (ii) implies (iii) is based on the following important observation. If $X'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and a rearrangement-invariant function space $Z(0,L)$, then, in particular, $R_{u,v,\nu^{-1}} : X'(0,L) \to Z(0,L)$ is bounded. Consequently, by virtue of Proposition 3.8, the rearrangement-invariant function space whose associate function norm is $\| \cdot \|_{Y(0,L)}$ is the optimal target space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0,L)$. By (2.10), this optimal target space is actually the space $Y'(0,L)$. Owing to Proposition 3.1, the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0,L)$ exists, and we denote it by $W(0,L)$. Moreover,

$$\| f \|_{W(0,L)} \approx \| R_{u,v,\nu^{-1}}(f^*) \|_{Y'(0,L)} \text{ for every } f \in \mathcal{M}^+(0,L).$$

(4.9)

The crucial observation is that we have, in fact, that $X'(0,L) = W(0,L)$. The embedding $X'(0,L) \hookrightarrow W(0,L)$ is valid because $R_{u,v,\nu^{-1}} : X'(0,L) \to Y'(0,L)$ is bounded and $W(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0,L)$. The validity of the opposite embedding is slightly more complicated. Since $R_{u,v,\nu^{-1}} : X'(0,L) \to Z(0,L)$ is bounded and $Y'(0,L)$ is the optimal target space for the operator $R_{u,v,\nu^{-1}}$ and $X'(0,L)$, we have that $Y'(0,L) \hookrightarrow Z(0,L)$. Consequently, since $R_{u,v,\nu^{-1}} : W(0,L) \to Y'(0,L)$ is bounded, so is $R_{u,v,\nu^{-1}} : W(0,L) \to Z(0,L)$. Using the fact that $X'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Z(0,L)$, we obtain that $W(0,L) \hookrightarrow X'(0,L)$. Now that we know that $X'(0,L) = W(0,L)$, (4.8) follows from (4.9).

Next, note that (iii) implies (ii). Indeed, (4.8) coupled with Proposition 3.1 tells us that $X'(0,L)$ is the optimal domain space for the operator $R_{u,v,\nu^{-1}}$ and $Y'(0,L)$.

Finally, it only remains to prove that (iii) implies (iv) under the additional assumptions. By (4.8), we have that

$$\| T_\varphi f \|_{X'(0,L)} \approx \| R_{u,v,\nu^{-1}}((T_\varphi f)^*) \|_{Y'(0,L)} \approx \| R_{u,v,\nu^{-1}}(T_\varphi f) \|_{Y'(0,L)}$$

$$= \left\| u(t) \int_0^{v^{-1}(t)} \xi(s) \sup_{\tau \in [s,L]} \varphi(\tau) f^*(\tau) ds \right\|_{Y'(0,L)}$$

$$\leq \left\| u(t) \int_0^{v^{-1}(t)} \xi(s) \sup_{\tau \in [s,v^{-1}(t)]} \varphi(\tau) f^*(\tau) ds \right\|_{Y'(0,L)}$$
Here, we used the fact that $T_{\varphi}f$ is equivalent to a nonincreasing function for every $f \in \mathcal{M}^+(0, L)$, and the multiplicative constants in this equivalence are independent of $f$. Since $\varphi$ is equivalent to a continuous nondecreasing function and $\xi$ satisfies (4.7), it follows from [34, Theorem 3.2] that

$$\int_0^{\nu^{-1}(t)} \xi(s) \sup_{\tau \in [s, \nu^{-1}(t)]} \varphi(\tau)f^*(\tau) ds \lesssim \int_0^{\nu^{-1}(t)} f^*(s)u(s) ds \quad \text{for every } t \in (0, L).$$

Hence,

$$\left\| \nu(t) \int_0^{\nu^{-1}(t)} \xi(s) \sup_{\tau \in [s, \nu^{-1}(t)]} \varphi(\tau)f^*(\tau) ds \right\|_{Y'(0, L)} \lesssim \left\| R_{u, v, \nu^{-1}}(f^*) \right\|_{Y'(0, L)}. \tag{4.11}$$

Since the function $\varphi \circ \nu^{-1}$ is equivalent to a quasiconcave function, it follows from [30, Lemma 4.10] that

$$\left\| \sup_{\tau \in [\nu^{-1}(t), L]} \varphi(\tau)f^*(\tau) \right\|_{Y'(0, L)} \lesssim \left\| \varphi(\nu^{-1}(t))f^*(\nu^{-1}(t)) \right\|_{Y'(0, L)}. \tag{4.12}$$

We note that, although [30, Lemma 4.10] deals only with the case $L = \infty$, its proof translates verbatim to the case of $L \in (0, \infty)$. Furthermore, we have that

$$\left\| \varphi(\nu^{-1}(t))f^*(\nu^{-1}(t)) \right\|_{Y'(0, L)} \leq \frac{U(\nu^{-1}(t))}{\int_0^{\nu^{-1}(t)} \xi(s) ds} \left\| f^*(\nu^{-1}(t)) \right\|_{Y'(0, L)} \tag{4.12}$$

Here, we used the fact that $\xi$ satisfies (4.7) in the first inequality and the monotonicity of $f^*$ in the second one. By combining (4.10) with (4.11) and (4.12) and using (4.8), we obtain that

$$\left\| T_{\varphi}f \right\|_{X'(0, L)} \lesssim \left\| R_{u, v, \nu^{-1}}(f^*) \right\|_{Y'(0, L)} \approx \left\| f \right\|_{X'(0, L)} \quad \text{for every } f \in \mathcal{M}^+(0, L);$$

hence $T_{\varphi}$ is bounded on $X'(0, L)$. \[\square\]

Remark 4.7.

(i) If $X'(0, L)$ is the optimal domain space for $R_{u, v, \nu^{-1}}$ and some rearrangement-invariant function space $Y(0, L)$, then $X'(0, L)$ is actually the optimal domain space for $R_{u, v, \nu^{-1}}$ and its own optimal target space. This follows from the following. Thanks to Proposition 3.8 and Proposition 3.1, we are entitled to denote by $Z(0, L)$ the optimal target space for $R_{u, v, \nu^{-1}}$ and $X'(0, L)$ and by $W(0, L)$ the optimal domain space for $R_{u, v, \nu^{-1}}$ and $Z(0, L)$. We need to show that $X'(0, L) = W(0, L)$. On the one hand, since $R_{u, v, \nu^{-1}} : X'(0, L) \to Z(0, L)$ is bounded and $W(0, L)$ is the optimal domain space for $R_{u, v, \nu^{-1}}$ and $Z(0, L)$, we have that $X'(0, L) \hookrightarrow W(0, L)$. On the other hand, since $R_{u, v, \nu^{-1}} : X'(0, L) \to Y(0, L)$...
is bounded and \( Z(0,L) \) is the optimal target space for \( R_{u,v,\nu^{-1}} \) and \( X'(0,L) \). We have that \( Z(0,L) \的权利 Y(0,L) \). Consequently, \( R_{u,v,\nu^{-1}} : W(0,L) \rightarrow Y(0,L) \) is bounded. Finally, since \( X'(0,L) \) is the optimal domain space for \( R_{u,v,\nu^{-1}} \) and \( Y(0,L) \), we obtain that \( W(0,L) \rightsquigarrow X'(0,L) \). Furthermore, by combining this observation with Remark 3.6, we also obtain the following fact. If \( X(0,L) \) is the optimal target space for \( H_{u,v,\nu} \) and some rearrangement-invariant function space \( Y(0,L) \), then \( X(0,L) \) is actually the optimal target space for \( H_{u,v,\nu} \) and its own rearrangement-invariant function space.

(ii) If \( \xi \) satisfies the averaging condition (4.24), then (4.7) is satisfied for every nonincreasing function \( u \) inasmuch as \( tu(t) \leq U(t) \) for every \( t \in (0,L) \).

(iii) When \( u(t) = t^{-1+\alpha}, v(t) = t^{-1+\beta}, \) and \( \nu(t) = t^\gamma, t \in (0,L) \), the additional assumptions of Theorem 4.6 are satisfied if \( \alpha \in (0,1], \beta \in [0,1), \gamma > 0, \) and \( 1 \leq \frac{\alpha}{\gamma} + \beta \leq 2 \).

We conclude this subsection with a result that is somewhat unrelated to the rest but of independent interest. It shows that, to verify the boundedness of \( H_{u,v,\nu} \) between a pair of rearrangement-invariant function spaces, it is sufficient to verify it on nonincreasing functions. It is an easy consequence of Hardy–Littlewood inequality (2.4) that this is the case for the operator \( R_{u,v,\nu} \), provided that \( u \) is nonincreasing. However, the validity of such a result for the operator \( H_{u,v,\nu} \) is far from being obvious because this time the integration is not carried out over a right-neighborhood of 0. Such a result was first obtained by Cianchi, Pick, and Slavíková in [22, Corollary 9.8] for \( u \equiv 1, v = \text{id}, \) and \( L < \infty \). Later, Peša generalized their result to cover also the case \( L = \infty \) in [51, Theorem 3.10]. In [15], we needed such a result for \( \nu(t) = t^\alpha, \alpha > 0, \) and \( u \not\equiv 1 \), and, while we felt certain that their proofs would carry over to the needed setting, we still had to carefully check them because there is plenty of fine analysis involved. The following proposition extends the result to the generality considered in this paper. It turns out that their proofs can easily be adapted for our setting. Our proof is actually in a way simpler because they considered operators with kernels.

**Proposition 4.8.** Let \( \| \cdot \|_{X(0,L)} \) and \( \| \cdot \|_{Y(0,L)} \) be rearrangement-invariant function norms.

(1) Let \( \nu : (0,L) \rightarrow (0,L) \) be an increasing bijection. Assume that \( \nu^{-1} \in \overline{D}_\varnothing^\varnothing \) for some \( \varnothing > 1 \). If \( L = \infty \), assume that \( \nu^{-1} \in \overline{D}_\varnothing^{\infty} \).

(2) Let \( u, v : (0,L) \rightarrow (0,\infty) \) be nonincreasing.

The following two statements are equivalent.

(i) There is a positive constant \( C \) such that

\[
\| H_{u,v,\nu} f \|_{Y(0,L)} \leq C \| f \|_{X(0,L)} \tag{4.13}
\]

for every \( f \in \mathfrak{M}(0,L) \).

(ii) There is a positive constant \( C \) such that

\[
\| H_{u,v,\nu} (f^*) \|_{Y(0,L)} \leq C \| f \|_{X(0,L)} \tag{4.14}
\]

for every \( f \in \mathfrak{M}(0,L) \).

Moreover, if (4.14) holds with a constant \( C \), then (4.13) holds with the constant \( C \frac{\varnothing}{\varnothing - 1} \sup_{t \in (0,L)} \frac{\nu^{-1}(t)}{\nu^{-1}(\frac{t}{\varnothing})} \).

**Proof.** Since (i) plainly implies (ii), we only need to prove that (ii) implies (i). Since the quantities in (4.13) and (4.14) do not change when the function \( \nu \) is redefined on a countable set, we may assume that \( \nu \) is left continuous. Note that \( H_{u,v,\nu} f \) is nonincreasing for every \( f \in \mathfrak{M}(0,L) \). Hence, thanks to (2.11) and (3.17), in order to prove that (ii) implies (i), we need to show that

\[
\sup_{f \in \mathfrak{M}(0,L)} \sup_{g \in \mathfrak{M}(0,L)} \int_0^L f(s) R_{u,v,\nu^{-1}}(g^*)(s) ds \leq \sup_{f \in \mathfrak{M}(0,L)} \sup_{g \in \mathfrak{M}(0,L)} \int_0^L f^*(s) R_{u,v,\nu^{-1}}(g^*)(s) ds. \tag{4.15}
\]
We define the operator $G$ as

$$Gg(t) = \sup_{\tau \in [t, L)} R_{u,v,\nu^{-1}}(g^\ast)(\tau), \, t \in (0, L),$$

for every $g \in \mathcal{M}^+(0,L)$. Note that $Gg$ is nonincreasing for every $g \in \mathcal{M}^+(0,L)$. Fix $g \in \mathcal{M}^+(0,L)$ such that $|[t \in (0, L) : g(t) > 0]| < \infty$, and set

$$E = \{ t \in (0, L) : R_{u,v,\nu^{-1}}(g^\ast)(t) < Gg(t) \}.$$

It can be shown that there is a countable system $\{(a_k, b_k)\}_{k \in I}$ of mutually disjoint, bounded intervals in $(0, L)$ such that

$$E = \bigcup_{k \in I} (a_k, b_k), \quad (4.16)$$

$$Gg(t) = R_{u,v,\nu^{-1}}(g^\ast)(t) \text{ if } t \in (0, L) \setminus E, \quad (4.17)$$

$$Gg(t) = R_{u,v,\nu^{-1}}(g^\ast)(b_k) \text{ if } t \in (a_k, b_k) \text{ for } k \in I. \quad (4.18)$$

This was proved in [22, Proposition 9.3] for $L < \infty$ and in [51, Lemma 3.9] for $L = \infty$. Their proofs are for $u \equiv 1$ and $v = \text{id}$, but the fact that $g^u$ is nonincreasing and $R_{u,v,\nu^{-1}}(g^\ast)$ is upper semicontinuous remains valid in our situation, and so it can be readily seen that their proofs carry over verbatim to our setting.

Note that $M = \sup_{t \in (0,L)} \frac{\nu^{-1}(t)}{\nu^{-1}(\frac{t}{\nu})} < \infty$. Set $\sigma = \frac{\vartheta}{\vartheta - 1} \in (1, \infty)$. Since $v$ and $g^u$ are nonincreasing, we have that, for every $k \in I,$

$$(b_k - a_k)R_{u,v,\nu^{-1}}(g^\ast)(b_k) = \sigma \int_{a_k + (\sigma - 1)b_k}^{b_k} R_{u,v,\nu^{-1}}(g^\ast)(b_k) \, dt$$

$$= \sigma \int_{a_k + (\sigma - 1)b_k}^{b_k} \frac{v(b_k)}{\nu^{-1}(b_k)} \nu^{-1}(b_k) \int_0^{\nu^{-1}(b_k)} g^\ast(s)u(s) \, ds \, dt$$

$$\leq \sigma \int_{a_k + (\sigma - 1)b_k}^{b_k} \frac{v(t)}{\nu^{-1}(t)} \nu^{-1}(t) \int_0^{\nu^{-1}(t)} g^\ast(s)u(s) \, ds \, dt$$

$$\leq \sigma \frac{\nu^{-1}(b_k)}{\nu^{-1}(a_k + (\sigma - 1)b_k)} \int_{a_k + (\sigma - 1)b_k}^{b_k} R_{u,v,\nu^{-1}}(g^\ast)(t) \, dt$$

$$\leq \sigma \int_{a_k + (\sigma - 1)b_k}^{b_k} R_{u,v,\nu^{-1}}(g^\ast)(t) \, dt$$

$$\leq \sigma M \int_{a_k + (\sigma - 1)b_k}^{b_k} R_{u,v,\nu^{-1}}(g^\ast)(t) \, dt$$

$$\leq \sigma M \int_{a_k}^{b_k} R_{u,v,\nu^{-1}}(g^\ast)(t) \, dt. \quad (4.19)$$

Here, we used the fact that $v$ and $(g^u)^{**}$ are nonincreasing in the first inequality.

Consider the averaging operator $A$ defined as

$$Af = f^\ast \chi_{(0,L) \setminus E} + \sum_{k \in I} \left( \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f^\ast(s) \, ds \right) \chi(a_k, b_k), \, f \in \mathcal{M}^+(0,L).$$
Note that $Af$ is a nonincreasing function for every $f \in \mathcal{M}^+(0, L)$. Furthermore, it is known [5, Chapter 2, Theorem 4.8] that
\[
\|Af\|_{X(0, L)} \leq \|f\|_{X(0, L)} \text{ for every } f \in \mathcal{M}^+(0, L).
\] (4.20)

For every $f \in \mathcal{M}^+(0, L)$, we have that
\[
\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt \leq \int_0^L f(t) G g(t) \, dt \leq \int_0^L f^*(t) G g(t) \, dt
\]
\[
= \int_{(0,L)\setminus E} f^*(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt + \sum_{k \in I} \int_{a_k}^{b_k} f^*(t) R_{u,v,\nu^{-1}}(g^*)(b_k) \, dt
\]
\[
\leq \int_0^L f^*(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt
\]
\[
+ \sigma M \sum_{k \in I} \left( \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f^*(t) \, dt \right) \left( \int_{a_k}^{b_k} R_{u,v,\nu^{-1}}(g^*)(t) \, dt \right)
\]
\[
\leq \sigma M \int_0^L Af(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt.
\] (4.21)

Here, we used the Hardy–Littlewood inequality (2.4), (4.16), (4.17), (4.18), and (4.19). If $L = \infty$ and $g \in \mathcal{M}^+(0, \infty)$ is positive on a set of infinite measure, we consider $g \chi_{(0,n)} \nearrow g$, $n \to \infty$, and obtain (4.21) even for such functions $g$, thanks to the monotone convergence theorem. Hence, we have proved that
\[
\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt \leq \sigma M \int_0^L Af(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt \text{ for every } f, g \in \mathcal{M}^+(0, L).
\] (4.22)

By combining (4.20) and (4.22), we obtain that
\[
\int_0^L f(t) R_{u,v,\nu^{-1}}(g^*)(t) \, dt \leq \sigma M \sup_{\|\cdot\|_{X(0,L)} \leq 1} \|R_{u,v,\nu} f\|_{Y(0,L)}
\]
for every $f \in \mathcal{M}^+(0, L)$, $\|f\|_{X(0,L)} \leq 1$, and $g \in \mathcal{M}^+(0, L)$. Note that here we used the fact that $Af$ is nonincreasing for every $f \in \mathcal{M}^+(0, L)$. By taking the supremum over all $f, g \in \mathcal{M}^+(0, L)$ from the closed unit balls of $X(0,L)$ and $Y'(0,L)$, respectively, we obtain (4.15) with the multiplicative constant equal to $\sigma M$. □

**Proposition 4.8** together with **Proposition 3.5** has the following important corollary. Note that the first equality is just a consequence of the Hardy–Littlewood inequality (2.4) combined with the obvious inequality
\[
\sup_{\|\cdot\|_{X(0,L)} \leq 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} \leq \sup_{\|\cdot\|_{X(0,L)} \leq 1} \|R_{u,v,\nu} f\|_{Y(0,L)}.
\]

**Corollary 4.9.** Let $\| \cdot \|_{X(0,L)}$ and $\| \cdot \|_{Y(0,L)}$ be rearrangement-invariant function norms.

(1) Let $\nu : (0, L) \to (0, L)$ be an increasing bijection. Assume that $\nu \in D^0$. If $L = \infty$, assume that $\nu \in D^\infty$.
(2) Let $u, v : (0, L) \to (0, \infty)$ be nonincreasing.

We have that
\[
\sup_{\|f\|_{X(0,L)} \leq 1} \|R_{u,v,\nu}(f^*)\|_{Y(0,L)} = \sup_{\|f\|_{X(0,L)} \leq 1} \|R_{u,v,\nu} f\|_{Y(0,L)} = \sup_{\|g\|_{Y'(0,L)} \leq 1} \|H_{u,v,\nu^{-1}} g\|_{X'(0,L)}
\]
\[
\approx \sup_{\|g\|_{Y'(0,L)} \leq 1} \|H_{u,v,\nu^{-1}}(g^*)\|_{X'(0,L)}.
\]
Remark 4.10. The assumption $\nu \in D^0$ is not overly restrictive. For example, it is satisfied whenever $\nu$ is equivalent to $t \mapsto t^\alpha \ell_1(t)^{\beta_1} \cdots \ell_k(t)^{\beta_k}$ near 0 for any $\alpha > 0$, $k \in \mathbb{N}_0$, and $\beta_j \in \mathbb{R}$, $j = 1, 2, \ldots, k$. Here, the functions $\ell_j$ are $j$-times iterated logarithmic functions defined as

$$\ell_j(t) = \begin{cases} 1 + |\log t| & \text{if } j = 1, \\ 1 + \log \ell_{j-1}(t) & \text{if } j > 1, \end{cases}$$

(4.23)

for $t \in (0, L)$. On the other hand, $\nu(t) = \exp(-t^2)$ near 0, where $\alpha < 0$, is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption $\nu \in D^\infty$.

4.2 Optimal function norms and their connection with interpolation

We already know that a sufficient condition for simplification of the complicated function norm (3.21) is boundedness of a certain supremum operator. Furthermore, we have also already seen that the supremum operator is often bounded on optimal function spaces. We shall also soon see that the boundedness of the supremum operator goes hand in hand with a certain interpolation property of the rearrangement-invariant function space on which the supremum operator acts. In other words, the question of whether the supremum in the function norm (3.21) can be “simplified,” the notion of being an optimal function space, and interpolation are all closely related to each other.

As the following theorem shows, there is a connection between a rearrangement-invariant function space $X(0, L)$ being an interpolation space with respect to a certain pair of endpoint spaces and the boundedness of $T_\varphi$ on the associate space of $X(0, L)$ (cf. [39, Theorem 3.12]). We say that a measurable a.e. positive function on $(0, L)$ satisfies the averaging condition (4.24) (cf. [60, Lemma 2.3]) if

$$\text{ess sup}_{t \in (0, L)} \frac{1}{w(t)} \int_0^t w(s) \, ds < \infty.$$  \hspace{1cm} (4.24)

Here, $w$ temporarily denotes the function in question. The value of the essential supremum will be referred to as the averaging constant of the function.

**Theorem 4.11.** Let $\| \cdot \|_{X(0, L)}$ be a rearrangement-invariant function norm. Let $\varphi : (0, L) \to (0, \infty)$ be a measurable function that is equivalent to a continuous nondecreasing function. Set $\xi = 1/\varphi$. Assume that $\xi$ satisfies the averaging condition (4.24). Set $\psi(t) = t / \int_0^t \xi(s) \, ds$, $t \in (0, L)$. Consider the following three statements.

(i) The operator $T_\varphi$, defined by (4.2), is bounded on $X'(0, L)$.

(ii) $X(0, L) \in \text{Int} \left( \Lambda^1_\xi(0, L), L^\infty(0, L) \right)$.

(iii) $X'(0, L) \in \text{Int} \left( L^1(0, L), M_\psi(0, L) \right)$.

If $L < \infty$, then the three statements are equivalent to each other. If $L = \infty$, then (i) implies (ii), and (iii) implies (i).

**Proof.** We start off by noting that we may without loss of generality assume that $\varphi$ is continuous and nondecreasing. Furthermore, $(\Lambda^1_\xi)'(0, L) = M_\psi(0, L)$ [52, Theorem 10.4.1] and

$$\psi \approx \varphi \quad \text{on } (0, L),$$

(4.25)

thanks to the fact that $\xi$ satisfies the averaging condition (4.24) and is (equivalent to) a nonincreasing function. We shall show that (i) implies (ii), whether $L$ is finite or infinite. First, we observe that $X(0, L)$ is an intermediate space between $\Lambda^1_\xi(0, L)$ and $L^\infty(0, L)$. Set $\Xi = \int_0^L \xi(s) \, ds \in (0, \infty]$, and note that $\Xi < \infty$ if $L < \infty$. Let $\Xi^{-1} : (0, \Xi) \to (0, L)$ be the increasing bijection that is inverse to the function $(0, L) \ni t \mapsto \int_0^t \xi(s) \, ds$. By [56, Lemma 6.8], we have that

$$K(f, t; \Lambda^1_\xi, L^\infty) \approx \int_0^{\Xi^{-1}(t)} f^+(s) \xi(s) \, ds \quad \text{for every } f \in \mathfrak{H}^+(0, L) \text{ and } t \in (0, \Xi).$$

(4.26)
Let $a$ be defined by (3.2). The embedding $X(0, L) \hookrightarrow (\Lambda^1_\xi + L^\infty)(0, L)$ follows from

$$
\|f\|_{(\Lambda^1_\xi + L^\infty)(0, L)} = K(f, 1; \Lambda^1_\xi, L^\infty) \leq \max \left\{ 1, \frac{1}{\int_0^a \xi(s) \, ds} \right\} K(f, \int_0^a \xi(s) \, ds; \Lambda^1_\xi, L^\infty)
$$

$$
\approx \int_0^a f^*(t) \xi(t) \, dt \leq \frac{1}{\psi(a)} \int_0^a f^*(t) T_\varphi(X\{0,\omega\})(t) \, dt
$$

$$
\leq \|f\|_{X(0, L)} \|T_\varphi(X\{0,\omega\})\|_{X'(0, L)} \leq \|f\|_{X(0, L)}.
$$

Here, we used Hölder’s inequality (2.14) in the last but one inequality and the boundedness of $T_\varphi$ on $X'(0, L)$ in the last one. We now turn our attention to the embedding $\Lambda^1_\xi(0, L) \cap L^\infty(0, L) \hookrightarrow X(0, L)$. If $L < \infty$, the embedding is plainly true owing to (2.18). If $L = \infty$, it is sufficient to observe that, for every $f \in X'(0, \infty)$, $f^* = g + h$ for some functions $g \in L^1(0, \infty)$ and $h \in M_\varphi(0, \infty)$, thanks to (2.16), the fact that $(\Lambda^1_\xi(0, \infty) \cap L^\infty(0, \infty))' = L^1(0, \infty) + M_\varphi(0, \infty)$ by (2.17), and (2.15). Set $g = f^* X\{0,1\}$ and $h = f^* X\{1,\infty\}$. Clearly, $g \in L^1(0, \infty)$ thanks to property (P5) of $\| \cdot \|_{X'(0, L)}$. Furthermore,

$$
\|h\|_{M_\varphi(0, \infty)} \approx \sup_{t \in (0, \infty)} \psi(t) (f^* X\{1,\infty\})^+(t) \lesssim \sup_{t \in (0, \infty)} \psi(t) f^+(t + 1) = \psi(1) \sup_{t \in [1, \infty)} \psi(t) f^+(t)
$$

Here, we used Hölder’s inequality (2.14) in the first equivalence (cf. [48, Lemma 2.1]) and (4.25) in the last one. Note that $T_\varphi f(1)$ is finite owing to (2.12) inasmuch as $T_\varphi f \in X'(0, \infty)$ and it is a nonincreasing function. Hence, $X(0, L)$ is an intermediate space between $\Lambda^1_\xi(0, L)$ and $L^\infty(0, L)$. Next, in order to prove that (i) implies (ii), it remains to show that every admissible operator $S$ for the couple $(\Lambda^1_\xi(0, L), L^\infty(0, L))$ is bounded on $X(0, L)$. Let $S$ be such an operator. Since $S$ is linear and bounded on both $\Lambda^1_\xi(0, L)$ and $L^\infty(0, L)$, it follows that (see [5, Chapter 5, Theorem 1.11])

$$
K(Sf, t; \Lambda^1_\xi, L^\infty) \lesssim K(f, t; \Lambda^1_\xi, L^\infty) \quad \text{for every } f \in (\Lambda^1_\xi + L^\infty)(0, L) \text{ and } t \in (0, L).
$$

By combining (4.26) and (4.27), we obtain that

$$
\int_0^t (Sf)^+(s) \xi(s) \, ds \lesssim \int_0^t f^+(s) \xi(s) \, ds \quad \text{for every } f \in (\Lambda^1_\xi + L^\infty)(0, L) \text{ and } t \in (0, L).
$$

Since the function $(0, L) \ni t \mapsto \sup_{s \leq L} \varphi(s) g^*(s)$ is nonincreasing for every $g \in \mathcal{M}^+(0, L)$, the Hardy lemma (2.6) together with (4.28) implies that

$$
\int_0^L (Sf)^+(t) T_\varphi g(t) \, dt \lesssim \int_0^L f^+(t) T_\varphi g(t) \, dt \quad \text{for every } f \in (\Lambda^1_\xi + L^\infty)(0, L) \text{ and } g \in \mathcal{M}^+(0, L).
$$

Therefore,

$$
\|Sf\|_{X(0, L)} = \sup_{\|g\|_{X'(0, L)} \leq 1} \int_0^L (Sf)^+(t) T_\varphi g(t) \, dt \lesssim \sup_{\|g\|_{X'(0, L)} \leq 1} \int_0^L (Sf)^+(t) T_\varphi g(t) \, dt
$$

$$
\lesssim \sup_{\|g\|_{X'(0, L)} \leq 1} \int_0^L f^+(t) T_\varphi g(t) \, dt \leq \|f\|_{X(0, L)} \sup_{\|g\|_{X'(0, L)} \leq 1} \|T_\varphi g\|_{X'(0, L)}
$$

$$
\lesssim \|f\|_{X(0, L)}
$$

for every $f \in X(0, L)$. Here, we used (2.11) in the equality, Hölder’s inequality (2.14) in the last but one inequality, and the boundedness of $T_\varphi$ on $X'(0, L)$ in the last one. Hence, $S$ is bounded on $X(0, L)$. 
We shall now prove that (iii) implies (i), whether \( L \) is finite or infinite. Since \( \xi \) is nonincreasing and satisfies the averaging condition (4.24), it follows from [34, Theorem 3.2] (cf. [48, Lemma 3.1]) that \( T_\varphi \) is bounded on \( L^1(0, L) \). Furthermore, \( T_\varphi \) is also bounded on \( M_\varphi(0, L) \), for
\[
\|T_\varphi f\|_{M_\varphi(0, L)} = \sup_{t \in (0, L)} (T_\varphi f)^*(t) \psi(t) = \sup_{t \in (0, L)} \frac{1}{\int_0^t \xi(s) ds} \int_0^t \xi(s) \sup_{\tau \in [s, L)} \varphi(\tau) f(\tau) ds
\leq \sup_{t \in (0, L)} \varphi(t) f^*(t) \leq \|f\|_{M_\varphi(0, L)}.
\]
Here, we used (4.25) and (2.1) in the last inequality. Fix \( f \in (L^1 + M_\varphi)(0, L) \). We claim that
\[
K(T_\varphi f, t; L^1, M_\varphi) \leq K(f, t; L^1, M_\varphi) \quad \text{for every } t \in (0, \infty)
\tag{4.29}
\]
with a multiplicative constant independent of \( f \). Let \( f = g + h \) with \( g \in L^1(0, L) \) and \( h \in M_\varphi(0, L) \) be a decomposition of \( f \). Note that the fact that \( \xi \) is nonincreasing and satisfies the averaging condition (4.24) implies that
\[
\varphi(s) \lesssim \varphi\left(\frac{s}{2}\right) \quad \text{for every } s \in (0, L).
\]
Thanks to this and (2.2), we have that
\[
T_\varphi f(s) \leq \frac{1}{\varphi(s)} \left( \sup_{\tau \in [s, L)} \varphi(\tau) g^\ast\left(\frac{\tau}{2}\right) + \sup_{\tau \in [s, L)} \varphi(\tau) h^\ast\left(\frac{\tau}{2}\right) \right)
\leq T_\varphi g\left(\frac{s}{2}\right) + T_\varphi h\left(\frac{s}{2}\right)
\tag{4.30}
\]
for every \( s \in (0, L) \). By combining (4.30) and the boundedness of the dilation operator \( D_2 \) (see (2.19)) with the fact that \( T_\varphi \) is bounded on both \( L^1(0, L) \) and \( M_\varphi(0, L) \), we obtain that (cf. [8, p. 497])
\[
K(T_\varphi f, t; L^1, M_\varphi) \leq K\left(T_\varphi g\left(\frac{s}{2}\right), t; L^1, M_\varphi\right) + K\left(T_\varphi h\left(\frac{s}{2}\right), t; L^1, M_\varphi\right)
\leq \|T_\varphi g\|_{L^1(0, L)} + t \|T_\varphi h\|_{M_\varphi(0, L)} \leq \|g\|_{L^1(0, L)} + t \|h\|_{M_\varphi(0, L)}
\]
for every \( t \in (0, \infty) \). Here, the multiplicative constants are independent of \( f, g, h \), and \( t \). Hence (4.29) is true. Now, since we have (4.29) at our disposal, there is a linear operator \( S \) bounded on both \( L^1(0, L) \) and \( M_\varphi(0, L) \) with norms that can be bounded from above by a constant independent of \( f \) such that \( Sf = T_\varphi f \). This follows from [26, Theorem 2]. Owing to (iii), \( S \) is also bounded on \( X'(0, L) \); moreover, its norm on \( X'(0, L) \) can be bounded from above by a constant independent of \( f \) [5, Chapter 3, Proposition 1.11]. Therefore,
\[
\|T_\varphi f\|_{X'(0, L)} = \|Sf\|_{X'(0, L)} \lesssim \|f\|_{X'(0, L)}
\]
in which the multiplicative constant is independent of \( f \). Hence, \( T_\varphi \) is bounded on \( X'(0, L) \).

Finally, if \( L < \infty \), then (ii) is equivalent to (iii); hence, the three statements are equivalent to each other in this case. Indeed, since \( (\Lambda_1^1 + L^\infty)(0, L) = L_1^1(0, L) \) and \( (L^1 + M_\varphi)(0, L) = L^1(0, L) \) owing to (2.18), the equivalence of (ii) and (iii) follows from [44, Corollary 3.6]. Here, we used the fact that both \( \Lambda_1^1(0, L) \) and \( L^1(0, L) \) have absolutely continuous norm (in the sense of [5, Chapter 1, Definition 3.1]).

\[\Box\]

4.3 More on the case \( u \equiv 1 \)

The remainder of this section is devoted to the particular but important case \( u \equiv 1 \). We shall see that the connection between the supremum operator and the various notions that we have met is even tighter in this case.

First, we need to equip ourselves with the following auxiliary result, which generalizes [30, Lemma 4.9] and whose immediate corollary for \( u \equiv 1 \) is of independent interest.
Proposition 4.12. Let \( \| \cdot \|_{X(0,L)} \) be a rearrangement-invariant function norm.

1. Let \( \nu : (0, L) \to (0, L) \) be an increasing bijection. Assume that \( \nu^{-1} \in \overline{D^0} \). If \( L = \infty \), assume that \( \nu^{-1} \in \overline{D^\infty} \).
2. Let \( u : (0, L) \to (0, \infty) \) be nonincreasing.
3. Let \( v : (0, L) \to (0, \infty) \) be nonincreasing. Assume that \( v \) satisfies the averaging condition (4.24), and denote its averaging constant by \( C \).

Set \( f = \sum_{i=1}^{N} c_i \chi_{(0,a_i)}, \) where \( c_i \in (0,\infty), i = 1, \ldots, N, \) and \( 0 < a_1 < \cdots < a_N < L \). We have that

\[
\| H_{u,v,f} \|_{X(0,L)} \approx \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) \right\|_{X(0,L)},
\]

in which the multiplicative constants depend only on \( \nu \) and \( C \).

Proof. First, observe that \( \inf_{t \in (0, L)} \frac{\nu^{-1}(t)}{t} \in (0,1) \), where \( \theta > 1 \) is such that \( \nu^{-1} \in \overline{D^0} \theta \) and, if \( L = \infty \), also \( \nu^{-1} \in \overline{D^\infty} \theta \). We denote the infimum by \( M \).

Second, we have that

\[
\left\| u(t) \int_{\nu(t)}^{L} f(s)v(s) \, ds \right\|_{X(0,L)} = \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) \int_{\nu(t)}^{a_i} v(s) \, ds \right\|_{X(0,L)}
\]

\[
\geq \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) (a_i - \nu(t)) \right\|_{X(0,L)}
\]

\[
\geq \frac{\theta - 1}{\theta} \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)}
\]

\[
\geq M \frac{\theta - 1}{\theta} \left\| u(Mt) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)}
\]

\[
\geq \frac{\theta - 1}{\theta} \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) v(a_i) a_i \right\|_{X(0,L)}
\]

thanks to the fact that \( u \) and \( v \) are nonincreasing and the boundedness of \( D_{1/M} \) (see (2.19)).

Last, using the fact that \( v \) satisfies the averaging condition (4.24), we obtain that

\[
\left\| u(t) \int_{\nu(t)}^{L} f(s)v(s) \, ds \right\|_{X(0,L)} = \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) \int_{\nu(t)}^{a_i} v(s) \, ds \right\|_{X(0,L)}
\]

\[
\leq C \left\| u(t) \sum_{i=1}^{N} c_i \chi_{(0,\nu^{-1}(a_i))}(t) a_i v(a_i) \right\|_{X(0,L)}.
\]

By combining (4.33) and (4.32), we obtain (4.31).

Since every nonnegative, nonincreasing function on \( (0,L) \) is the pointwise limit of a nondecreasing sequence of nonnegative, nonincreasing simple functions, Proposition 4.12 with \( u \equiv 1 \) has the following important corollary.
Corollary 4.13. Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

(1) Let $\nu, v$ be as in Proposition 4.12.

Let $f \in \mathfrak{M}^+(0,L)$. There is a nondecreasing sequence $\{f_k\}_{k=1}^\infty$ of nonnegative, nonincreasing simple functions on $(0,L)$ such that

$$\lim_{k \to \infty} \|H_{1,\nu,v}(f_k)\|_{X(0,L)} \approx \|f^*\|_{X(0,L)} = \|f\|_{X(0,L)}.$$ 

Here, the multiplicative constants depend only on $\nu$ and the averaging constant of $v$.

Remark 4.14. The assumption $\nu^{-1} \in D_0$ is not overly restrictive. For example, it is satisfied whenever $\nu$ is equivalent to $t \mapsto t^\alpha \ell_1(t)^{\beta_1} \cdots \ell_k(t)^{\beta_k}$ near 0 for any $\alpha > 0$, $k \in \mathbb{N}_0$ and $\beta_j \in \mathbb{R}$, $j = 1, 2, \ldots, k$. Here, the functions $\ell_j$ are iterated logarithmic functions defined by (4.23). In this case, $\nu^{-1}$ is equivalent to $t \mapsto t^{1/\alpha} \ell_1(t)^{-\beta_1/\alpha} \cdots \ell_k(t)^{-\beta_k/\alpha}$ near 0 (cf. [6, Appendix 5]). On the other hand, $\nu(t) = \log \frac{1}{t}$ near 0, where $\alpha < 0$, is a typical example of a function not satisfying the assumption. The same remark (with the obvious modifications) is true for the assumption $\nu^{-1} \in D_\infty$.

While Proposition 4.2 provides a sufficient condition for simplification of (3.3), the following proposition provides a necessary one.

Proposition 4.15. Let $\| \cdot \|_{X(0,L)}$ be a rearrangement-invariant function norm.

(1) Let $\nu : (0,L) \to (0,L)$ be an increasing bijection. Assume that $\nu^{-1} \in D^0$. If $L = \infty$, assume that $\nu^{-1} \in D^\infty$ and $\nu \in D^\infty$.

(2) Let $v : (0,L) \to (0,\infty)$ be a nonincreasing function satisfying the averaging condition (4.24).

If there is a positive constant $C$ such that

$$\sup_{h \sim f} \|H_{1,\nu,v}h\|_{X(0,L)} \leq C \|H_{1,\nu,v}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L),$$

(4.34)

then the three equivalent statements from Theorem 4.6 with $u \equiv 1$ are satisfied.

Proof. Let $a$ be defined by (3.2). Since $\nu$ is integrable over $(0,a)$, for it satisfies the averaging condition (4.24), we have that

$$\left\|X_{(0,\nu^{-1}(a))}(t) \int_0^a v(s) \, ds \right\|_{X(0,L)} \leq \int_0^a v(s) \, ds \left\|X_{(0,\nu^{-1}(a))}\right\|_{X(0,L)} < \infty.$$

(4.35)

Furthermore, if $L = \infty$, then $\limsup_{\tau \to \infty} v(\tau) \|X_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} < \infty$. Indeed, suppose that $\limsup_{\tau \to \infty} v(\tau) \|X_{(0,\nu^{-1}(\tau))}\|_{X(0,\infty)} = \infty$. It follows from the proof of Proposition 3.3 that

$$\sup_{h \sim X_{(0,1)}} \|H_{1,\nu,v}h\|_{X(0,\infty)} = \infty.$$ 

(4.36)

However, since

$$\sup_{h \sim X_{(0,1)}} \|H_{1,\nu,v}h\|_{X(0,\infty)} \approx \|H_{1,\nu,v}X_{(0,1)}\|_{X(0,\infty)} = \left\|X_{(0,\nu^{-1}(1))}(t) \int_0^1 v(s) \, ds \right\|_{X(0,\infty)} < \infty,$$

thanks to (4.34) and (4.35), (4.36) is not possible. Hence, Proposition 3.3 guarantees that the optimal domain space for $H_{1,\nu,v}$ and $X(0,L)$ exists. Moreover, if we denote it by $Z(0,L)$, then (4.34) implies that

$$\|f\|_{Z(0,L)} \approx \|H_{1,\nu,v}(f^*)\|_{X(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).$$

(4.37)
Now, we finally turn our attention to proving that (4.34) implies statement (iii) from Theorem 4.6. Let \( Y(0, L) \) be the optimal target space for the operator \( H_{1,v,\nu} \) and \( Z(0, L) \). Its existence is guaranteed by Proposition 3.7, and we have that
\[
\|f\|_{Y(0, L)} = \|R_{1,v,\nu^{-1}}(f^*)\|_{Z'(0, L)} \quad \text{for every } f \in \mathcal{M}^+(0, L).
\] (4.38)
Using the optimality of \( Y(0, L) \) combined with the fact that \( H_{1,v,\nu} : Z(0, L) \to X(0, L) \) is bounded, and (4.37), we obtain that
\[
\|H_{1,v,\nu}(f^*)\|_{X(0, L)} \leq \|H_{1,v,\nu}(f^*)\|_{Y(0, L)} \leq \|f\|_{Z(0, L)} \approx \|H_{1,v,\nu}(f^*)\|_{X(0, L)}
\]
for every \( f \in \mathcal{M}^+(0, L) \). Hence,
\[
\|H_{1,v,\nu}(f^*)\|_{X(0, L)} \approx \|H_{1,v,\nu}(f^*)\|_{Y(0, L)} \quad \text{for every } f \in \mathcal{M}^+(0, L).
\]
In particular, we have that
\[
\|H_{1,v,\nu}h\|_{X(0, L)} \approx \|H_{1,v,\nu}h\|_{Y(0, L)} \quad \text{(4.39)}
\]
for every nonincreasing simple function \( h \in \mathcal{M}^+(0, L) \). By combining (4.39) with Corollary 4.13, we obtain that
\[
\|f^*\|_{X(0, L)} \approx \|f^*\|_{Y(0, L)} \quad \text{for every } f \in \mathcal{M}^+(0, L).
\]
Owing to the rearrangement invariance of both function norms, it follows that \( X(0, L) = Y(0, L) \). Hence, (4.6) with \( u \equiv 1 \) follows from (4.38) combined with (2.8).

\[\square\]

We obtain the final result of this subsection by combining Theorem 4.6, Proposition 4.15, Proposition 4.2, and Theorem 4.11.

**Theorem 4.16.** Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

(1) Let \( \nu : (0, L) \to (0, L) \) be an increasing bijection. Assume that \( \nu^{-1} \in \overline{D}^0 \). If \( L = \infty \), assume that \( \nu^{-1} \in \overline{D}^\infty \) and \( \nu \in D^\infty \).

(2) Let \( \nu : (0, L) \to (0, \infty) \) be defined by (4.3) with \( \xi : (0, L) \to (0, \infty) \) satisfying the averaging condition (4.24). Assume that \( \nu \), too, satisfies the averaging condition (4.24). Furthermore, assume that the function \( \varphi \circ \nu^{-1} \) is equivalent to a quasiconcave function, where \( \varphi = 1/\xi \).

Let \( \| \cdot \|_{Y(0, L)} \) be the functional defined by (3.3) with \( u \equiv 1 \). The following five statements are equivalent.

(i) The operator \( T_{\nu^o} \), defined by (4.2), is bounded on \( X'(0, L) \).

(ii) There is a positive constant \( C \) such that
\[
\sup_{h \sim f} \|H_{1,v,\nu}h\|_{X(0, L)} \leq C \|H_{1,v,\nu}(f^*)\|_{X(0, L)} \quad \text{for every } f \in \mathcal{M}^+(0, L).
\]

(iii) The space \( X(0, L) \) is the optimal target space for the operator \( H_{1,v,\nu} \) and some rearrangement-invariant function space.

(iv) The space \( X'(0, L) \) is the optimal domain space for the operator \( R_{1,v,\nu^{-1}} \) and some rearrangement-invariant function space.

(v) We have that
\[
\|f\|_{X'(0, L)} \approx \sup_{g \in \mathcal{M}^+(0, L)} \int_0^L g(t)R_{1,v,\nu^{-1}}(f^*)(t) \, dt \quad \text{for every } f \in \mathcal{M}^+(0, L).
\]

If \( L < \infty \), these five statements are also equivalent to

(vi) \( X(0, L) \in \text{Int} \left( \Lambda_\xi^1(0, L), L^\infty(0, L) \right) \).
Remark 4.17.

1. The assumption that \( v \) satisfies the averaging condition (4.24) is natural. It forbids weights \( v \) for which the question of whether \( X(0, L) \) (or \( X'(0, L) \)) is the optimal target (or domain) space for \( H_{1,v',\nu} \) (or \( R_{1,v,\nu^{-1}} \)) and some rearrangement-invariant function space cannot be decided by the boundedness of the corresponding supremum operator \( T_\varphi \). This can be illustrated by a very simple example. Consider \( v = 1 \) and \( \nu \equiv 1 \). Since \( T_\varphi f = f^* \), \( T_\varphi \) is bounded on any \( X'(0, L) \). However, \( R_{1,v,\nu^{-1}} f(t) = \int_0^t |f(s)| ds/t \) clearly need not be bounded from \( X'(0, L) \) to \( (L^1 + L^\infty)(0, L) \), which is the largest rearrangement-invariant function space. To this end, consider, for example, \( X(0, L) = L^\infty(0, \infty) \) (cf. [58, Proposition 4.1]).

2. When \( v(t) = t^{-1+\beta} \) and \( \nu(t) = t^\gamma \), \( t \in (0, L) \), the assumptions of Theorem 4.16 are satisfied if \( \beta \in (0, 1) \), \( \gamma > 0 \), and \( 1 \leq \frac{1}{\gamma} + \beta \leq 2 \).

5 | ITERATION OF OPTIMAL FUNCTION NORMS

This section is devoted to so-called sharp iteration principles for the operators \( R_{u,v,\nu} \) and \( H_{u,v,\nu} \). To illustrate their meaning and importance, suppose that \( Y_1(0, L) \) is the optimal target space for \( H_{u_1,v_1,\nu_1} \) and a rearrangement-invariant function space \( X(0, L) \). Let us now go one step further and suppose that \( Y_2(0, L) \) is the optimal target space for \( H_{u_2,v_2,\nu_2} \) and \( Y_1(0, L) \). In the light of Proposition 3.7, the associate function norm of \( \| \cdot \|_{Y_2(0, L)} \) is equal to \( \| f \|_{Y_1'(0, L)} = \| R_{u_1,v_1,\nu_1^{-1}}((R_{u_2,v_2,\nu_2^{-1}}(f^*))^*) \|_{X'(0, L)} \). We immediately see that there is an inevitable difficulty that we face if we wish to understand the iterated norm. This difficulty is caused by the fact that the function \( R_{u_2,v_2,\nu_2^{-1}}(f^*) \) is hardly ever (equivalent to a nonincreasing function, unless \( u_2, v_2 \) and \( \nu_2 \) are related to each other in a very specific way (see Proposition 4.1)). Therefore, we cannot just readily “delete” the outer star. Nevertheless, with some substantial effort, we shall be able to equivalently express the iterated norm as a noniterated one under suitable assumptions. The suitable assumptions are such that the iteration does not lead to the presence of kernels, which would go beyond the scope of this paper (see [22, section 8] in that regard). It should be noted that such an iteration is not artificial. For example, it is an essential tool for establishing sharp iteration principles for various Sobolev embeddings. Roughly speaking, they ensure that the optimal rearrangement-invariant target space in a Sobolev embedding of \((k + l)\)-th order is the same as that obtained by composing the optimal Sobolev embedding of order \( k \) with the optimal Sobolev embedding of order \( l \) (see [21, 23, 46] and references therein). Another possible application is description of optimal rearrangement-invariant function norms for compositions of some operators of harmonic analysis (see [30] and references therein for optimal behavior of some classical operators on rearrangement-invariant function spaces). Finally, the motivation behind studying function norms induced by \( H_{u_1,v_1,\nu_1} \circ H_{u_1,v_1,\nu_1} \) is similar.

5.1 | Iteration principle for \( R_{u,v,\nu} \)

The following proposition is the first step toward the sharp iteration principle for \( R_{u,v,\nu} \).

Proposition 5.1. Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

1. Let \( v_1, v_2 : (0, L) \to (0, L) \) be increasing bijections. Assume that \( \nu_2 \in \overline{D}^0 \). If \( L = \infty \), assume that \( \nu_2 \in \overline{D}^\infty \).

2. Let \( u_1, u_2 : (0, L) \to (0, \infty) \) be nonincreasing.

3. Let \( v_1 : (0, L) \to (0, \infty) \) be measurable. Let \( v_2 : (0, L) \to (0, \infty) \) be a nonincreasing function satisfying the averaging condition (4.24).

Set \( \nu = \nu_2 \circ \nu_1 \) and

\[
v(t) = u_1(\nu_1(t))v_1(t)v_2(\nu_1(t)), \quad t \in (0, L).
\]

We have that

\[
\| R_{u_1,v_1,\nu_1}((R_{u_2,v_2,\nu_2^{-1}}(f^*))^*) \|_{X(0, L)} \geq \| R_{u_2,v,\nu^{-1}}(f^*) \|_{X(0, L)}
\]

for every \( f \in M^+(0, L) \), in which the multiplicative constant depends only on \( \nu_2 \) and the averaging constant of \( v_2 \).
Proof. Note that \( \inf_{t \in (0, L)} \frac{\nu_2(t)}{\nu_1(t)} > 0 \), where \( \theta > 1 \) is such that \( \nu_2 \in \overline{D}_\theta \) and, if \( L = \infty \), also \( \nu_2 \in \overline{D}_\infty \). Consequently, there is \( N \in \mathbb{N} \), such that \( \nu_2(t) \leq N \nu_2(t) \) for every \( t \in (0, L) \). Hence, for every \( f \in \mathcal{M}^+(0, L) \), we have that

\[
\int_0^{\nu_2(t)} f^*(s) u_2(s) \, ds \leq N \int_0^{\nu_2(t)} f^*(s) u_2(s) \, ds \quad \text{for every } t \in (0, L)
\]

(5.2)

owing to the fact that \( f^* u_2 \) is nonincreasing. Thanks to the monotonicity of \( u_1 \) and \( v_2 \), the fact that \( v_2 \) satisfies the averaging condition (4.24) and the inequality (5.2), we have that

\[
\left\| u_1(t) u_1(v_1(t)) v_1(t) v_2(v_1(t)) \int_0^{v_1(t)} f^*(s) u_2(s) \, ds \right\|_{X(0, L)} \leq \left\| u_1(t) u_1(v_1(t)) \int_0^{v_2(t)} u_2(s) \, ds \int_0^{v_1(t)} f^*(s) u_2(s) \, ds \right\|_{X(0, L)}
\]

\[
\leq \left\| u_1(t) u_1(v_1(t)) \int_0^{v_2(t)} u_2(s) u_1(s) \, ds \int_0^{v_1(t)} f^*(s) u_2(s) \, ds \right\|_{X(0, L)}
\]

\[
\leq \left\| u_1(t) \int_0^{v_1(t)} u_2(s) u_1(s) \, ds \int_0^{v_2(t)} f^*(s) u_2(s) \, ds \right\|_{X(0, L)}
\]

\[
\leq \left\| u_1(t) \int_0^{v_1(t)} \left( \int_0^{v_2(t)} f^*(s) u_2(s) \, ds \right) u_1(s) \, ds \right\|_{X(0, L)}
\]

\[
\leq \left\| R_{u_1,v_1,v_2} ((R_{u_2,v_2,f^*})^*) \right\|_{X(0, L)}
\]

for every \( f \in \mathcal{M}^+(0, L) \). Here, we used the Hardy–Littlewood inequality (2.4) in the last inequality. \( \square \)

We are now in a position to establish the sharp iteration principle for \( R_{u,v,v} \).

**Theorem 5.2.** Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

1. Let \( \nu_1, \nu_2, u_1, u_2 \) be as in Proposition 5.1.
2. Let \( v_1 : (0, L) \to (0, \infty) \) be a continuous function. Let \( v_2 : (0, L) \to (0, \infty) \) be defined by

\[
\frac{1}{v_2(t)} = \int_0^{v_2(t)} \xi(s) \, ds, \quad t \in (0, L),
\]

where \( \xi : (0, L) \to (0, \infty) \) is a measurable function. Assume that the function \( u_1 v_2 \) satisfies the averaging condition (4.24).

Let \( v \) be the function defined by (5.1). Set \( v = \nu_2 \circ v_1 \) and

\[
\eta(t) = \frac{1}{U_2(t) v(t)}, \quad t \in (0, L).
\]
Assume that \( \eta \) and \( \eta / \xi \) are equivalent to nonincreasing functions. Furthermore, assume that there are positive constants \( C_1 \) and \( C_2 \) such that

\[
\int_0^t \eta(s) u_2(s) \, ds \leq C_1 U_2(t) \eta(t) \quad \text{for a.e. } t \in (0, L) \quad (5.3)
\]

and

\[
\frac{1}{t} \int_0^t U_2(\nu(s)) \psi(s) \, ds \geq C_2 U_2(\nu(t)) \psi(t) \quad \text{for a.e. } t \in (0, L). \quad (5.4)
\]

We have that

\[
\| R_{u_1, v_1, \nu_1} ((R_{u_2, v_2, \nu_2}(f^*))^+) \|_{X(0,L)} \approx \| R_{u_2, v, \nu}(f^*) \|_{X(0,L)}
\]

for every \( f \in \mathcal{M}^+(0,L) \). Here, the multiplicative constants depend only on \( v_1, v_2, C_1, C_2, \) the averaging constant of \( u_1 v_2 \) and the multiplicative constants in the equivalences of \( \eta \) and \( \eta / \xi \) to nonincreasing functions.

**Proof.** First, note that the fact that \( v_2 u_1 \) satisfies the averaging condition (4.24) together with the monotonicity of \( u_1 \) implies that \( v_2 \), too, satisfies the averaging condition (4.24) (with the same multiplicative constant). Hence, we have that

\[
\| R_{u_1, v_1, \nu_1} ((R_{u_2, v_2, \nu_2}(f^*))^+) \|_{X(0,L)} \geq \| R_{u_2, v, \nu}(f^*) \|_{X(0,L)}
\]

thanks to Proposition 5.1. Therefore, we only need to prove the opposite inequality.

We may assume that \( u_2 \) is nondegenerate and \( \psi \in X(0,L) \), where \( \psi \) is defined as \( \psi(t) = u_2(\nu_2(t)) \chi_{(0,L)}(t) + u_2(t) \chi_{(L,\infty)}(t) \), \( t \in (0,L) \). Indeed, if it is not the case, then \( \| R_{u_2, v, \nu}(f^*) \|_{X(0,L)} = \infty \) for every \( f \in \mathcal{M}^+(0,L) \) that is not equivalent to 0 a.e. Proposition 3.1 with \( u = u_2 \) guarantees that there is a rearrangement-invariant function space \( Z(0,L) \) such that

\[
\| f \|_{Z(0,L)} = \| R_{u_2, v, \nu}(f^*) \|_{X(0,L)} \quad \text{for every } f \in \mathcal{M}^+(0,L).
\]

Furthermore, by (3.18) and the Hardy–Littlewood inequality (2.4), we have that

\[
\| H_{u_2, v, \nu}^{-1} g \|_{Z(0,L)} = 1.
\]

Note that, for every \( f \in \mathcal{M}^+(0,L) \), the function

\[
(0,L) \ni t \mapsto v_2(t) \int_0^{\nu_2(t)} s \xi(s) u_2(s) \sup_{\tau \in [s,L]} \frac{1}{\xi(\tau)} f^*(\tau) \, ds
\]

is nonincreasing. Indeed it is the integral mean of the nonincreasing function \((0,L) \ni s \mapsto u_2(s) \sup_{x \in [s,L]} \frac{1}{\xi(x)} f^*(x)\) over the interval \((0,v_2(t))\) with respect to the measure \( \xi(s) \, ds \). By (2.9) and (3.17), we have that

\[
\| R_{u_1, v_1, \nu_1} ((R_{u_2, v_2, \nu_2}(f^*))^+) \|_{X(0,L)} = \sup_{\| g \|_{X'(0,L)} \leq 1} \int_0^L (R_{u_2, v_2, \nu_2}(f^*))^+ (t) H_{u_1, v_1, \nu_1}^{-1} g(t) \, dt
\]

\[
= \sup_{\| g \|_{X'(0,L)} \leq 1} \int_0^L \left[ v_2(t) \int_0^{\nu_2(t)} u_2(\tau) f^*(\tau) \, d\tau \left]\right]^* (t) H_{u_1, v_1, \nu_1}^{-1} g(t) \, dt
\]

\[
\leq \sup_{\| g \|_{X'(0,L)} \leq 1} \int_0^L \left[ v_2(t) \int_0^{\nu_2(t)} \xi(\tau) u_2(\tau) \sup_{x \in [\tau,L]} \frac{1}{\xi(x)} f^*(x) \, d\tau \left]\right]^* (t) H_{u_1, v_1, \nu_1}^{-1} g(t) \, dt
\]

\[
= \sup_{\| g \|_{X'(0,L)} \leq 1} \int_0^L v_2(t) \int_0^{\nu_2(t)} \xi(s) u_2(s) \sup_{\tau \in [s,L]} \frac{1}{\xi(\tau)} f^*(\tau) ds H_{u_1, v_1, \nu_1}^{-1} g(t) \, dt
\]
\[
\begin{align*}
&= \sup_{\|g\|_{L^\infty(0,L)} \leq 1} \int_0^L \left( \xi(s) \sup_{\tau \in [s,L]} \frac{1}{\xi(\tau)} f^+(\tau) \right) \left( u_2(s) \int_{\nu_2^{-1}(s)}^L v_2(t)u_1(t) \int_{\nu_1^{-1}(t)}^L g(x)v_1(x)dx\,dt \right) ds \\
&\leq \left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)} \sup_{\|g\|_{L^\infty(0,L)} \leq 1} \left\| u_2(t) \int_{\nu_2^{-1}(t)}^L u_2(s)u_1(s) \int_{\nu_1^{-1}(s)}^L g(\tau)v_1(\tau)\,d\tau \right\|_{Z'(0,L)} \\
&= \left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)} \sup_{\|g\|_{L^\infty(0,L)} \leq 1} \left\| u_2(t) \int_{\nu_2^{-1}(t)}^L g(\tau)v_1(\tau) \int_{\nu_1^{-1}(s)}^L v_2(s)u_1(s)\,ds\,d\tau \right\|_{Z'(0,L)} \\
&\leq \left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)} \sup_{\|g\|_{L^\infty(0,L)} \leq 1} \left\| H u_2, v, \nu^{-1} g \right\|_{Z(0,L)} \\
&= \left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)},
\end{align*}
\]
for every \( f \in \mathfrak{M}^+(0,L) \). Here, we used Fubini’s theorem in the fourth and fifth equalities, the Hölder inequality (2.14) in the second inequality, the fact that \( u_1v_2 \) satisfies the averaging condition (4.24) in the last inequality, and (5.5) in the last equality. Therefore, the proof will be finished once we show that

\[
\left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)} \lesssim \|R u_2, v, f^+\|_{L^\infty(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L).
\]

Since the function \( \frac{\eta}{\xi} \) is equivalent to a nonincreasing function, we have that

\[
\left\| \xi(t) \sup_{s \in [t,L]} \frac{1}{\xi(s)} f^+(s) \right\|_{Z(0,L)} \lesssim \left\| \eta(t) \sup_{s \in [t,L]} \frac{1}{\eta(s)} f^+(s) \right\|_{Z(0,L)},
\]
for every \( f \in \mathfrak{M}^+(0,L) \). Hence, it is sufficient to show that

\[
\left\| \eta(t) \sup_{s \in [t,L]} \frac{1}{\eta(s)} f^+(s) \right\|_{Z(0,L)} \lesssim \|R u_2, v, f^+\|_{L^\infty(0,L)} \quad \text{for every } f \in \mathfrak{M}^+(0,L). \tag{5.6}
\]

Note that, for every \( f \in \mathfrak{M}^+(0,L) \),

\[
\left\| \eta(t) \sup_{s \in [t,L]} \frac{1}{\eta(s)} f^+(s) \right\|_{Z(0,L)} \approx \left\| v(t) \int_0^{\nu(t)} u_2(s)\eta(s) \sup_{r \in [s,\nu(t)]} \frac{1}{\eta(r)} f^+(r)\,ds \right\|_{L^\infty(0,L)} \\
\lesssim \left\| v(t) \int_0^{\nu(t)} u_2(s)\eta(s) \sup_{r \in [s,\nu(t)]} \frac{1}{\eta(r)} f^+(r)\,ds \right\|_{L^\infty(0,L)} \\
+ \left\| v(t) \left( \sup_{r \in [s,\nu(t)]} \frac{1}{\eta(r)} f^+(r) \right) \int_0^{\nu(t)} u_2(s)\eta(s)\,ds \right\|_{L^\infty(0,L)}, \tag{5.7}
\]
inasmuch as \( \eta \) is equivalent to a nonincreasing function. Furthermore, since \( \eta \) is equivalent to a nonincreasing function and satisfies (5.3), [34, Theorem 3.2] guarantees that

\[
\int_0^{\nu(t)} u_2(s)\eta(s) \sup_{r \in [s,\nu(t)]} \frac{1}{\eta(r)} f^+(r)\,ds \lesssim \int_0^{\nu(t)} f^+(s)u_2(s)\,ds
\]
for every \( t \in (0, L) \) and every \( f \in \mathcal{M}^+(0, L) \). Here, the multiplicative constant depends only on \( C_2 \). Hence,

\[
\left\| v(t) \int_0^{\nu(t)} u_2(s) \eta(s) \sup_{\tau \in [s, \nu(t)]} \frac{1}{\eta(\tau)} f^+(\tau) \, ds \right\|_{X(0,L)} \lesssim \left\| v(t) \int_0^{\nu(t)} f^+(s) u_2(s) \, ds \right\|_{X(0,L)} = \| R_{u_2,v} (f^+) \|_{X(0,L)}
\]

(5.8)

for every \( f \in \mathcal{M}^+(0, L) \). Furthermore, thanks to the fact that \( \eta \) satisfies (5.3) again, we have that

\[
\left\| v(t) \left( \sup_{\tau \in [\nu(t), L]} \frac{1}{\eta(\tau)} f^+(\tau) \right) \int_0^{\nu(t)} u_2(s) \eta(s) \, ds \right\|_{X(0,L)} \lesssim \left\| v(t) U_2(\nu(t)) \eta(\nu(t)) \sup_{\tau \in [\nu(t), L]} \frac{1}{\eta(\tau)} f^+(\tau) \right\|_{X(0,L)} = \left\| \sup_{\tau \in [\nu(t), L]} \frac{1}{\eta(\tau)} f^+(\tau) \right\|_{X(0,L)}
\]

(5.9)

for every \( f \in \mathcal{M}^+(0, L) \). We claim that

\[
\left\| \sup_{\tau \in [t, L]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \right\|_{X(0,L)} \lesssim \| R_{u_2,v} (f^+) \|_{X(0,L)}.
\]

(5.10)

Thanks to the Hardy–Littlewood–Pólya principle (2.7), it is sufficient to show that

\[
\int_0^t \sup_{\tau \in [s, t]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \, ds \lesssim \int_0^t \left( R_{u_2,v} (f^+) \right)^{\ast}(s) \, ds \quad \text{for every } t \in (0, L).
\]

(5.11)

To this end, we have that

\[
\int_0^t \sup_{\tau \in [s, t]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \, ds \lesssim \int_0^t \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \, ds = \int_0^t U_2(\nu(s)) \nu(s) f^+(\nu(s)) \, ds \\
\leq \int_0^t R_{u_2,v} (f^+) (s) \, ds \lesssim \int_0^t \left( R_{u_2,v} (f^+) \right)^{\ast}(s) \, ds
\]

(5.12)

for every \( t \in (0, L) \). Here, the first inequality follows from [34, Theorem 3.2] (the fact that the function \( (0, L) \ni s \mapsto \frac{1}{\eta(\nu(s))} = U_2(\nu(s)) \nu(s) \) is equivalent to a nondecreasing function and satisfies (5.4) was used here), the second inequality follows from the monotonicity of \( f^+ \), and the last one follows from the Hardy–Littlewood inequality (2.4). Furthermore, owing to (5.4) again, we have that

\[
\sup_{\tau \in [t, L]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) = \sup_{\tau \in [t, L]} U_2(\nu(\tau)) \nu(\tau) f^+(\nu(\tau)) \lesssim \sup_{\tau \in [t, L]} \left( \frac{1}{\tau} \int_0^\tau U_2(\nu(s)) \nu(s) \, ds \right) f^+(\nu(\tau)) \\
\leq \sup_{\tau \in [t, L]} \frac{1}{\tau} \int_0^\tau U_2(\nu(s)) \nu(s) f^+(\nu(s)) \, ds \leq \sup_{\tau \in [t, L]} \frac{1}{\tau} \int_0^\tau R_{u_2,v} (f^+) (s) \, ds \\
\leq \sup_{\tau \in [t, L]} \frac{1}{\tau} \int_0^\tau \left( R_{u_2,v} (f^+) \right)^{\ast}(s) \, ds = \frac{1}{t} \int_0^t \left( R_{u_2,v} (f^+) \right)^{\ast}(s) \, ds.
\]

(5.13)

Inequality (5.11) now follows from (5.12) and (5.13) inasmuch as

\[
\int_0^t \sup_{\tau \in [s, L]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \, ds \lesssim \int_0^t \sup_{\tau \in [s, t]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau)) \, ds + t \sup_{\tau \in [t, L]} \frac{1}{\eta(\nu(\tau))} f^+(\nu(\tau))
\]

for every \( t \in (0, L) \).

Finally, by combining (5.7) with (5.8), (5.9), and (5.10), we obtain (5.6).
Remark 5.3. Since Theorem 5.2 has several assumptions, it is instructive to provide a concrete, important example, which is also quite general. Let \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in (0, \infty) \). Set \( v_j(t) = t^{\alpha_j}, u_j(t) = t^{\beta_j-1}b_j(t), \) and \( v_j(t) = t^{\gamma_j-1}c_j(t), t \in (0, L), j = 1, 2, \) where \( b_j, c_j \) are continuous slowly varying functions. Set \( \nu_j(t) = t^{\alpha_j}, u_j(t) = t^{\beta_j-1}b_j(t), \) and \( v_j(t) = t^{\gamma_j-1}c_j(t), t \in (0, L), j = 1, 2, \) where \( b_j, c_j \) are continuous slowly varying functions. Set \( d = (b_1 \circ \nu_1) \cdot c_1 \cdot (c_2 \circ \nu_1) \) and \( \tilde{d} = (b_1 \circ \nu_1) \cdot c_1 \). Assume that \( \gamma_2 < 1, \beta_1 + \gamma_2 > 1, \) and

\[
\alpha_1(\beta_1 + \alpha_2 \beta_2 + \gamma_2 - 1) + \gamma_1 \geq 1, \quad \alpha_1(\beta_1 + \alpha_2 \beta_2 - \alpha_2) + \gamma_1 \geq 1, \quad \alpha_1(\beta_1 + \gamma_2 - 1) + \gamma_1 < 1.
\]

If \( \alpha_1(\beta_1 + \alpha_2 \beta_2 + \gamma_2 - 1) + \gamma_1 = 1 \) or \( \alpha_1(\beta_1 + \alpha_2 \beta_2 - \alpha_2) + \gamma_1 = 1 \), also assume that \( d \) or \( \tilde{d} \), respectively, is equivalent to a nondecreasing function. Under these assumptions, we can use Theorem 5.2 to obtain that

\[
\left| X(0, L) \right| \approx \left| \int_0^L \left( \int_0^\tau v_2(\sigma) \int_0^\tau f^* \left( \int_0^\tau u_2(\sigma) \, d\sigma \right) \, d\tau \right) \, \nu_1(s) \, ds \, d\tau \right| \]

for every \( f \in \mathcal{M}^+(0, L) \), where \( \delta = \alpha_1(\beta_1 + \gamma_2 - 1) + \gamma_1 - 1. \)

When \( \beta_j = 1 \) and \( b_j = c_j \equiv 1 \), \( j = 1, 2, \) the assumptions are satisfied provided that

\[
\alpha_1(\alpha_2 + \gamma_2) + \gamma_1 \geq 1, \quad \alpha_1 + \gamma_1 \geq 1, \quad \alpha_1 \gamma_2 + \gamma_1 < 1.
\]

(5.14)

In particular, (5.14) is satisfied if (cf. [21, Theorem 3.4])

\[
\alpha_2 + \gamma_2 \geq 1, \quad \alpha_1 + \gamma_1 \geq 1, \quad \alpha_1 \gamma_2 + \gamma_1 < 1.
\]

5.2 | Iteration principle for \( H_{u,v,v} \)

We conclude this section with a \( H_{u,v,v} \) counterpart to Theorem 5.2, whose proof is substantially simpler than that of the theorem.

Proposition 5.4. Let \( \| \cdot \|_{X(0, L)} \) be a rearrangement-invariant function norm.

(1) Let \( \nu_1, \nu_2 : (0, L) \to (0, \infty) \) be increasing bijections. Assume that \( \nu_1 \in D_0 \). If \( L = \infty \), assume that \( \nu_1 \in D_\infty \).

(2) Let \( u_1, u_2, v_1, v_2 : (0, L) \to (0, \infty) \) be measurable. Assume that the function \( v_1 u_2 \) is equivalent to a nonincreasing function and that it satisfies the averaging condition (4.24).

Set

\[
v(t) = v_2^{-1}(t)u_1(v_2^{-1}(t))u_2(v_2^{-1}(t))v_2(t), \quad t \in (0, L),
\]

and \( \nu = v_2 \circ \nu_1 \). We have that

\[
\| H_{u_1,v_1,v_1}(H_{u_2,v_2,v_2} f) \|_{X(0, L)} \approx \| H_{u_1,v,v} f \|_{X(0, L)} \quad \text{for every} \quad f \in \mathcal{M}^+(0, L). 
\]

(5.15)

Here, the multiplicative constants depend only on \( \nu_1 \), the averaging constant of \( v_1 u_2 \), and the multiplicative constants in the equivalence of \( v_1 u_2 \) to a nonincreasing function.

Finally, assume, in addition, that

(1) \( u_1 \) and \( u_2 \) are nonincreasing,

(2) \( v_1 \) is defined by

\[
\frac{1}{v_1(t)} = \int_0^{v_1^{-1}(t)} \xi(s) \, ds \quad \text{for every} \quad t \in (0, L),
\]

where \( \xi : (0, L) \to (0, \infty) \) is a measurable function,

(3) the operator \( T_\varphi \) defined by (4.2) with \( \varphi = u_1 / \xi \) is bounded on \( X'(0, L) \).
Then,
\[
\sup_{g \sim f} \sup_{h \sim H_{u_2,v_2,\nu_2} g} \|H_{u_1,v_1,\nu_1} h\|_{X(0,L)} \approx \sup_{g \sim f} \|H_{u_1,v,\nu} g\|_{X(0,L)} \text{ for every } f \in \mathcal{M}^+(0,L).
\]

Here, the multiplicative constants depend only on the norm of $T_\varphi$ on $X'(0,L)$ and the multiplicative constant in (5.15).

Proof. On the one hand, we have that
\[
\|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2} f)\|_{X(0,L)} = \left\| u_1(t) \int_{v_1(t)}^L u_2(s) \int_{\nu_2^{-1}(t)}^{\nu_2^{-1}(s)} f(\tau) v_2(\tau) d\tau u_2(s) v_1(s) d\tau \right\|_{X(0,L)}
\]
\[
\leq \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v_2(\tau) \int_{\nu_2^{-1}(t)}^{\nu_2^{-1}(s)} u_2(s) v_1(s) d\tau d\tau \right\|_{X(0,L)}
\]
\[
\leq \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v_2(\tau) v_2^{-1}(t) u_2(\nu_2^{-1}(\tau)) v_1(\nu_2^{-1}(\tau)) d\tau \right\|_{X(0,L)}
\]
\[
= \|H_{u_1,v,\nu} f\|_{X(0,L)}
\]
for every $f \in \mathcal{M}^+(0,L)$, thanks to the fact that $v_1 u_2$ satisfies the averaging condition (4.24).

As for the opposite inequality, observe that $M = \inf_{t \in (0,L)} \frac{\nu_1(t)}{\nu_1(\theta t)} > 1$, where $\theta > 1$ is such that $\nu_1 \in D_0^\theta$ and, if $L = \infty$, also $\nu_1 \in D_0^\infty$. Set $K = \min\{\frac{1}{\theta}, \nu_1^{-1}(\frac{1}{M})\}$. We have that
\[
\|H_{u_1,v_1,\nu_1}(H_{u_2,v_2,\nu_2} f)\|_{X(0,L)}
\]
\[
= \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v_2(\tau) \int_{\nu_2^{-1}(t)}^{\nu_2^{-1}(s)} u_2(s) v_1(s) d\tau d\tau \right\|_{X(0,L)}
\]
\[
\geq \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v_2(\tau) \int_{\nu_2^{-1}(t)}^{\nu_2^{-1}(s)} u_2(s) v_1(s) d\tau d\tau \right\|_{X(0,L)}
\]
\[
\geq \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v_2(\tau) v_2^{-1}(t) u_2(\nu_2^{-1}(\tau)) v_1(\nu_2^{-1}(\tau)) (\nu_2^{-1}(\tau) - \nu_1(\tau)) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
\[
\geq \frac{M - 1}{M} \left\| u_1(t) \int_{v_1(t)}^L f(\tau) v(\tau) d\tau \right\|_{X(0,L)}
\]
for every \( f \in \mathfrak{M}^+(0, L) \). Here, we used the fact that \( v_1 u_2 \) is equivalent to a nonincreasing function and the boundedness of the dilation operator \( D^\frac{1}{L} \) (see (2.19)).

Finally, under the additional assumptions, we have that

\[
\sup_{g \sim f} \sup_{h \in \mathfrak{M}^+(0, L)} \| H_{u_1, v_1, \nu_1} h \|_{X(0, L)} \approx \sup_{g \sim f} \sup_{H_{u_2, v_2, \nu_2} g} \| H_{u_1, v_1, \nu_1} (H_{u_2, v_2, \nu_2} g) \|_{X(0, L)}
\]

\[
\approx \sup_{g \sim f} \| H_{u_1, v, \nu} g \|_{X(0, L)}
\]

for every \( f \in \mathfrak{M}^+(0, L) \), thanks to (4.4) combined with (5.15).

\section*{Remark 5.5.}

If \( T_\varphi \) is not bounded on \( X'(0, \infty) \), then, while we still have that

\[
\sup_{g \sim f} \sup_{h \in \mathfrak{M}^+(0, L)} \| H_{u_1, v_1, \nu_1} (H_{u_2, v_2, \nu_2} g) \|_{X(0, L)} \approx \sup_{g \sim f} \| H_{u_1, v, \nu} g \|_{X(0, L)}
\]

for every \( f \in \mathfrak{M}^+(0, L) \), it remains an open question whether the opposite inequality is valid.

\section*{6 \quad CONCRETE EXAMPLES OF OPTIMAL FUNCTION SPACES}

We conclude this paper with a few concrete examples of optimal function spaces. Let \( \gamma \in [0, 1) \) and \( \delta > 0 \). Set \( v(t) = t^{-1+\gamma} \) and \( \nu(t) = c t^\delta \), \( t \in (0, L) \), where \( c = 1 \) if \( L = \infty \) and \( c = L^{1-\delta} \) if \( L < \infty \). Throughout this section, \( R_{\gamma, \delta} \) and \( H_{\gamma, \delta} \) denote \( R_{1, v, \nu} \) and \( H_{1, v, \nu} \), respectively. In all the examples, the fixed function space is a Lorentz–Zygmund space. The class of Lorentz–Zygmund spaces contains several customary function spaces, for example, the Lebesgue spaces \( L^p \), the Lorentz spaces \( L^{p, q} \), and some Orlicz spaces—namely, logarithmic and exponential Orlicz spaces.

If \( L = \infty \) and \( A = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2 \), we define the broken logarithmic function \( \ell^A \) as \( \ell^A_j = \ell_1 \psi^A_j (0, 1] + \ell_\infty \psi^A_j (1, \infty) \), where \( \ell_j \) is defined by (4.23). Let \( p, q \in [1, \infty) \), \( A = (\alpha_0, \alpha_\infty) \), \( B = (\beta_0, \beta_\infty) \in \mathbb{R}^2 \), and \( \alpha, \beta \in \mathbb{R} \). If \( L = \infty \), the Lorentz–Zygmund space \( L^{p, q; A, B}(0, \infty) \) is defined as the collection of all the functions \( f \in \mathfrak{M}(0, \infty) \) for which the functional \( \| \cdot \|_{L^{p, q; A, B}(0, \infty)} \) defined as

\[
\| f \|_{L^{p, q; A, B}(0, \infty)} = \left\| \frac{1}{t^\frac{1}{p} q^{-1}} \ell^A(t) \ell^B(t) f^*(t) \right\|_{L^q(0, \infty)}
\]

is finite. If \( L < \infty \), the Lorentz–Zygmund space \( L^{p, q; \alpha, \beta}(0, L) \) is defined as the collection of all the functions \( f \in \mathfrak{M}(0, L) \) for which the functional \( \| \cdot \|_{L^{p, q; \alpha, \beta}(0, L)} \) defined as

\[
\| f \|_{L^{p, q; \alpha, \beta}(0, L)} = \left\| \frac{1}{t^\frac{1}{p} q^{-1}} \ell_1 \ell^\beta(t) f^*(t) \right\|_{L^q(0, L)}
\]

is finite. When \( B = (0, 0) \) and \( \beta = 0 \), we write \( L^{p, q; A, B}(0, \infty) \) and \( L^{p, q; \alpha, \beta}(0, L) \) for short, respectively. Similarly, when \( A = B = (0, 0) \) and \( \alpha = \beta = 0 \), we write \( L^p(0, \infty) \) and \( L^p(0, L) \) for short, respectively. Note that these are the usual Lorentz spaces. We shall also encounter Lorentz–Zygmund spaces \( L^{(p, q; A, B)}(0, \infty) \) and \( L^{(p, q; \alpha, \beta)}(0, L) \). In the definitions of these spaces, the nonincreasing rearrangement \( f^* \) is replaced by the maximal nonincreasing rearrangement \( f^{**} \). At one point, we will need a Lorentz–Zygmund space with three tiers of logarithm, which is defined in the obvious way. For more information on Lorentz–Zygmund spaces, see [50]. In particular, the functional \( \| \cdot \|_{L^{p, q; A, B}(0, \infty)} \) is equivalent to a rearrangement-invariant function norm if and only if \( p = q = 1 \), \( \alpha_0 \geq 0 \), and \( \alpha_\infty \leq 0 \), or if \( p \in (1, \infty) \) and \( q \in [1, \infty) \), or if \( p = \infty \), \( q \in [1, \infty) \), and \( \alpha_0 + 1/q < 0 \), or if \( p = q = \infty \) and \( \alpha_0 \leq 0 \). The functional \( \| \cdot \|_{L^{p, q; A, B}(0, L)} \) is equivalent to a rearrangement-invariant function norm if and only if \( p = q = 1 \), \( \alpha \geq 0 \), or if \( p \in (1, \infty) \) and \( q \in [1, \infty) \), or if \( p = \infty \), \( q \in [1, \infty) \), and \( \alpha + 1/q < 0 \), or if \( p = q = \infty \) and \( \alpha \leq 0 \). Throughout the rest of this section, we implicitly assume that the parameters \( p, q, A \), and \( \alpha \) satisfy one of these conditions.
We omit proofs in this section, which are to some extent straightforward but lengthy and technical. However, the interested reader can find detailed proofs in the author’s PhD thesis [45, section 2.2.3]. Some particular examples with detailed proofs can also be found in [21, 30, 46].

6.1 Optimal function spaces for $R_{\gamma, \delta}$

We start with optimal domain spaces for the operator $R_{\gamma, \delta}$.

**Proposition 6.1.** Let $\gamma \in [0, 1)$ and $\delta > 0$.
If $L = \infty$, the optimal domain space $X(0, \infty)$ for the operator $R_{\gamma, \delta}$ and the space $L^{p,q;\alpha}(0, \infty)$ satisfies

$$
X(0, \infty) = \begin{cases}
L^{1,1}(0, \alpha_0+1)(0, \infty) & \text{if } p = 1 - \gamma, q = 1, \alpha_0 + 1 < 0, \alpha_\infty + 1 < 0, \gamma \in (0, 1); \\
L^{1,1}(\alpha_0+1, \alpha_\infty+1)(0, \infty) & \text{if } p = 1 - \gamma, q = 1, \alpha_0 + 1 > 0, \alpha_\infty + 1 < 0, \gamma \in (0, 1) \text{ or } \\
& p = q = 1, \alpha_0 \geq 0, \alpha_\infty + 1 < 0, \gamma = 0; \\
L^{1,1}(0, \alpha_\infty+1)(1,0)(0, \infty) & \text{if } p = 1 - \gamma, q = 1, \alpha_0 + 1 = 0, \alpha_\infty + 1 < 0, \gamma \in (0, 1); \\
L^{1,q;\alpha}(0, \infty) & \text{if } p = 1 - \gamma, q \in (1, \infty), \alpha_\infty + \frac{1}{q} < 0, \gamma \in (0, 1) \text{ or } \\
& p = \frac{1}{1 - \gamma}, q = \infty, \alpha_\infty \leq 0, \gamma \in (0, 1); \\
L^{\frac{\delta}{p},p(r+\delta-1)q;\alpha}(0, \infty) & \text{if } p \in \left(1 - \gamma, \frac{1}{1 - \gamma - \delta}\right), \gamma + \delta < 1 \text{ or } \\
& p \in \left(\frac{1}{1 - \gamma}, \infty\right), \gamma + \delta \geq 1; \\
L^{\infty,q;\alpha}(0, \infty) & \text{if } p = \frac{1}{1 - \gamma - \delta}, \alpha_0 + \frac{1}{q} < 0, \gamma + \delta \leq 1 \text{ or } \\
& p = \frac{1}{1 - \gamma - \delta}, q = \infty, \alpha_0 \leq 0, \gamma + \delta \leq 1; \\
L^{1,1}\delta_{+\infty}(0, \infty) & \text{if } p = q = \infty, \alpha_0 \leq 0, \alpha_\infty \geq 0, \gamma + \delta > 1.
\end{cases}
$$

If $L < \infty$, the optimal domain space $X(0, L)$ for the operator $R_{\gamma, \delta}$ and the space $L^{p,q;\alpha}(0, L)$ satisfies

$$
X(0, L) = \begin{cases}
L^{1}(0, L) & \text{if } p \in [1, \frac{1}{1 - \gamma}), \gamma \in (0, 1) \text{ or } \\
& p = \frac{1}{1 - \gamma}, \alpha + \frac{1}{q} < 0, \gamma \in (0, 1) \text{ or } \\
& p = \frac{1}{1 - \gamma}, q = \infty, \alpha \leq 0, \gamma \in (0, 1); \\
L^{1,1}\alpha_{+1}(0, L) & \text{if } p = \frac{1}{1 - \gamma}, q = 1, \gamma \in (0, 1) \text{ or } \\
& p = q = 1, \gamma \in (0, 1); \\
L^{1,0,1}(0, L) & \text{if } p = \frac{1}{1 - \gamma}, q = 1, \gamma \in (0, 1); \\
L^{1,\alpha,1}(0, L) & \text{if } p = \frac{1}{1 - \gamma}, q \in (1, \infty), \alpha + \frac{1}{q} \geq 0, \gamma \in (0, 1) \text{ or } \\
& p = \frac{1}{1 - \gamma}, q = \infty, \alpha \geq 0, \gamma \in (0, 1); \\
L^{\frac{\delta}{p},p(r+\delta-1)q;\alpha}(0, L) & \text{if } p \in \left(1 - \gamma, \frac{1}{1 - \gamma - \delta}\right), \gamma + \delta < 1 \text{ or } \\
& p \in \left(\frac{1}{1 - \gamma}, \infty\right), \gamma + \delta \geq 1; \\
L^{\infty,q;\alpha}(0, L) & \text{if } p = \frac{1}{1 - \gamma - \delta}, \alpha + \frac{1}{q} < 0, \gamma + \delta \leq 1 \text{ or } \\
& p = \frac{1}{1 - \gamma - \delta}, q = \infty, \alpha \leq 0, \gamma + \delta \leq 1; \\
L^{1,1}\delta_{+\infty}(0, L) & \text{if } p = q = \infty, \alpha \leq 0, \gamma + \delta > 1.
\end{cases}
$$

The following proposition describes optimal target spaces for the operator $R_{\gamma, \delta}$.
Proposition 6.2. Let $\gamma \in [0, 1)$ and $\delta > 0$.
If $L = \infty$, the optimal target space $X(0, \infty)$ for the operator $R_{\gamma, \delta}$ and the space $L^{p,q;A}(0, \infty)$ satisfies

$$X(0, \infty) = \begin{cases} 
L^{1,\infty}(0, \infty) & \text{if } p = q = 1, \alpha_0 = \alpha_\infty = 0, \gamma \in (0, 1); \\
L^{1,1;\alpha_0-1,\alpha_\infty-1}(0, \infty) & \text{if } p = q = 1, \alpha_0 \geq 1, \alpha_\infty < 0, \gamma = 0; \\
L^{p,\frac{\gamma}{\gamma+\delta};q;A}(0, \infty) & \text{if } p \in (1, \infty), \gamma + \delta \leq 1, \gamma \in (0, 1) \text{ or } \\
p \in (1, \infty), \delta < 1, \gamma = 0 \text{ or } \\
p \in \left(1, \frac{\delta}{\gamma+\delta-1}\right), \gamma + \delta > 1; \\
L^{p,q;A}(0, \infty) & \text{if } p \in (1, \infty), \delta = 1, \gamma = 0 \text{ or } \\
p = \infty, q \in [1, \infty), \alpha_0 + \frac{1}{q} < 0, \delta = 1, \gamma = 0 \text{ or } \\
p = q = \infty, \alpha_0 \leq 0, \delta = 1, \gamma = 0; \\
L^{1,\infty;A}(0, \infty) & \text{if } p = q = \infty, \alpha_0 \leq 0, \alpha_\infty \geq 0, \gamma + \delta \leq 1, \gamma \in (0, 1) \text{ or } \\
p = q = \infty, \alpha_0 \leq 0, \alpha_\infty \geq 0, \delta < 1, \gamma = 0; \\
L^{\infty,\infty;A}(0, \infty) & \text{if } p = q = \infty, \alpha_0 \leq 0, \alpha_\infty \geq 0, \gamma + \delta > 1; \\
L^{\infty}(0, \infty) & \text{if } p = q = \infty, \alpha_0 = \alpha_\infty = 0, \gamma + \delta > 1, \gamma \in (0, 1). 
\end{cases}$$

If $L < \infty$, the optimal target space $X(0, L)$ for the operator $R_{\gamma, \delta}$ and the space $L^{p,q;A}(0, L)$ satisfies

$$X(0, L) = \begin{cases} 
L^{1,\infty}(0, L) & \text{if } p = q = 1, \alpha = 0, \gamma \in (0, 1); \\
L^{1,1;\alpha_0-1,\alpha_\infty-1}(0, L) & \text{if } p = q = 1, \alpha \geq 1, \gamma = 0; \\
L^{p,\frac{\gamma}{\gamma+\delta};q;A}(0, L) & \text{if } p \in (1, \infty), \gamma + \delta \leq 1, \gamma \in (0, 1) \text{ or } \\
p \in (1, \infty), \delta < 1, \gamma = 0 \text{ or } \\
p \in \left(1, \frac{\delta}{\gamma+\delta-1}\right), \gamma + \delta > 1; \\
L^{p,q;A}(0, L) & \text{if } p \in (1, \infty), \delta = 1, \gamma = 0 \text{ or } \\
p = \infty, q \in [1, \infty), \alpha + \frac{1}{q} < 0, \delta = 1, \gamma = 0 \text{ or } \\
p = q = \infty, \alpha \leq 0, \delta = 1, \gamma = 0; \\
L^{1,\infty;A}(0, L) & \text{if } p = q = \infty, \alpha \leq 0, \gamma + \delta \leq 1, \gamma \in (0, 1) \text{ or } \\
p = q = \infty, \alpha \leq 0, \delta < 1, \gamma = 0; \\
L^{\infty,\infty;A}(0, L) & \text{if } p = q = \infty, \alpha \leq 0, \gamma + \delta > 1; \\
L^{\infty}(0, L) & \text{if } p = q = \infty, \alpha = 0, \gamma + \delta > 1, \gamma \in (0, 1) \text{ or } \\
p \in \left(1, \frac{\delta}{\gamma+\delta-1}\right), \gamma + \delta > 1. 
\end{cases}$$

6.2 | Optimal function spaces for $H_{\gamma, \delta}$

The following proposition describes optimal domain spaces for the operator $H_{\gamma, \delta}$.

Proposition 6.3. Let $\gamma \in [0, 1)$ and $\delta > 0$.
If $L = \infty$, the optimal domain space $X(0, \infty)$ for the operator $H_{\gamma, \delta}$ and the space $L^{p,q;A}(0, \infty)$ satisfies
\[ X(0, \infty) = \begin{cases} 
\frac{1}{L^{\frac{1}{\gamma+\delta}}} (0, \infty) & \text{if } p = q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta \leq 1; \\
L^{1,\frac{1}{\gamma+\delta}} (0, \infty) & \text{if } p = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta > 1; \\
L^{1} (0, \infty) & \text{if } p = \frac{\delta}{1-\gamma}, \alpha_0 = \alpha_\infty = 0, \gamma + \delta > 1, \gamma \in (0, 1); \\
L^{q,\frac{1}{\gamma+\delta}} (0, \infty) & \text{if } p \in (1, \infty), \gamma + \delta \leq 1 \text{ or } p \in \left( \frac{\delta}{1-\gamma}, \infty \right), \gamma + \delta > 1; \\
L^{\frac{1}{\gamma+\delta}} (0, \infty) & \text{if } p = q = \infty, \alpha_0 = \alpha_\infty = 0, \gamma \in (0, 1); \\
L^{\infty,\infty} (0, \infty) & \text{if } p = q = \infty, \alpha_0 + 1 \leq 0, \alpha_\infty > 0, \gamma = 0. 
\end{cases} \]

If \( L < \infty \), the optimal domain space \( X(0, L) \) for the operator \( H_{\gamma, \delta} \) and the space \( L^{p,q,:}(0, L) \) satisfies

\[
X(0, L) = \begin{cases} 
\frac{1}{L^{\frac{1}{\gamma+\delta}}} (0, L) & \text{if } p = q = 1, \alpha_0 \geq 0, \gamma + \delta \leq 1; \\
L^{1} (0, L) & \text{if } p \in [1, \frac{\delta}{1-\gamma}), \gamma + \delta > 1 \text{ or } p = \frac{\delta}{1-\gamma}, \alpha_0 = 0, \gamma + \delta > 1, \gamma \in (0, 1); \\
L^{1,1,\frac{1}{\gamma+\delta}} (0, L) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \gamma + \delta > 1, \gamma \in (0, 1); \\
L^{1,\infty} (0, L) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \gamma + \delta > 1; \\
L^{1,1,\infty} (0, L) & \text{if } p \in (1, \infty), \gamma + \delta \leq 1 \text{ or } p \in \left( \frac{\delta}{1-\gamma}, \infty \right), \gamma + \delta > 1; \\
\frac{1}{L^{\infty,\infty}} (0, L) & \text{if } p = q = \infty, \alpha_0 = 0, \gamma \in (0, 1); \\
\frac{1}{L^{\infty,\infty}} (0, L) & \text{if } p = q = \infty, \alpha_0 + 1 \leq 0, \alpha_\infty > 0, \gamma = 0. 
\end{cases} \]

Finally, we end with optimal target spaces for the operator \( H_{\gamma, \delta} \).

**Proposition 6.4.** Let \( \gamma \in (0, 1) \) and \( \delta > 0 \).

If \( L = \infty \), the optimal target space \( X(0, \infty) \) for the operator \( H_{\gamma, \delta} \) and the space \( L^{p,q,:}(0, \infty) \) satisfies

\[
X(0, \infty) = \begin{cases} 
\frac{p}{L^{\frac{1}{\gamma+\delta}}} (0, \infty) & \text{if } p = q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta \geq 1 \text{ or } p \in (1, \frac{1}{\gamma}), \gamma + \delta \geq 1 \text{ or } p \in \left( \frac{1}{\gamma+\delta}, \frac{1}{\gamma} \right), \gamma + \delta < 1; \\
L^{1,\infty} (0, \infty) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta < 1; \\
L^{1,1,\infty} (0, \infty) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \alpha_\infty > 0, \gamma + \delta < 1; \\
L^{1,1,\frac{1}{\gamma+\delta}} (0, \infty) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta > 1; \\
L^{1,1,\frac{1}{\gamma+\delta}} (0, \infty) & \text{if } p = \frac{\delta}{1-\gamma}, q = 1, \alpha_0 \geq 0, \alpha_\infty > 0, \gamma + \delta < 1; \\
\frac{1}{L^{\infty,\infty}} (0, \infty) & \text{if } p = q = \infty, \alpha_0 \geq 0, \alpha_\infty \leq 0, \gamma + \delta < 1; \\
\frac{1}{L^{\infty,\infty}} (0, \infty) & \text{if } p = q = \infty, \alpha_0 + 1 \leq 0, \alpha_\infty > 0, \gamma = 0. 
\end{cases} \]
Here, $X_1(0, \infty)$, $X_2(0, \infty)$, and $X_3(0, \infty)$ are rearrangement-invariant function spaces such that

$$\|f\|_{X_1(0, \infty)} \approx \|t^{-\frac{1}{q} \ell^{\alpha_0-1}}(t)^{f^{**}(t)}\|_{L^1(0,1)} + \|f\|_{L^1(0, \infty)},$$

$$\|f\|_{X_2(0, \infty)} \approx \|t^{-\frac{1}{q} \ell^{\alpha_\infty-1}}(t)^{f^{**}(t)}\|_{L^q(0,\infty)} + \|f\|_{L^\infty(0, \infty)},$$

$$\|f\|_{X_3(0, \infty)} \approx \|t^{-1/\ell^{\alpha_0-1}}(t)^{f^{**}(t)}\|_{L^1(0,1)},$$

for every $f \in \mathcal{M}(0, \infty)$.

If $L < \infty$, the optimal target space $X(0,L)$ for the operator $H_{\gamma, \delta}$ and the space $L^{p,q;\alpha}(0,L)$ satisfies

$$X(0,L) = \begin{cases}
L^\frac{\rho d}{q} - 1 \frac{p d}{q} (0,L) & \text{if } p = q = 1, \alpha \geq 0, \gamma + \delta \geq 1 \text{ or } p \in (1, \frac{1}{\gamma + \delta}, \gamma + \delta \geq 1 \text{ or } p \in \left(\frac{1}{\gamma + \delta}, \frac{1}{\gamma}\right), \gamma + \delta < 1;\\
L^{1,1;\alpha} (0,L) & \text{if } p = \frac{1}{\gamma + \delta}, q = 1, \alpha \geq 0, \gamma + \delta < 1;\\
L^{1,\alpha - 1,1} (0,L) & \text{if } p = \frac{1}{\gamma + \delta}, q \in (1, \infty], \alpha > 1 - \frac{1}{q}, \gamma + \delta < 1;\\
L^{\infty, \frac{1}{q} - 1 \frac{1}{q} - 1} (0,L) & \text{if } p = \frac{1}{\gamma}, q \in (1, \infty], \alpha = 1 - \frac{1}{q}, \gamma \in (0,1);\\
L^{\infty}(0,L) & \text{if } p = \frac{1}{\gamma}, \alpha > 1 - \frac{1}{q}, \gamma \in (0,1) \text{ or } p = \frac{1}{\gamma}, q = 1, \alpha \geq 0, \gamma \in (0,1) \text{ or } p > \frac{1}{\gamma}, \gamma \in (0,1);\\
L^{\infty, \frac{1}{q} - 1 \frac{1}{q} - 1} (0,L) & \text{if } p = q = \infty, \alpha \leq 0, \gamma = 0.
\end{cases}$$

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The author declares no potential conflict of interests.

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REFERENCES
[1] E. Agora, J. Antezana, S. Baena-Miret, and M. Carro, From weak-type weighted inequality to pointwise estimate for the decreasing rearrangement, J. Geom. Anal. 32 (2022), no. 2, 56.
[2] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková, Sharp Sobolev type embeddings on the entire Euclidean space, Commun. Pure Appl. Anal. 17 (2018), no. 5, 2011–2037.
[3] C. Bennett and K. Rudnick, On Lorentz-Zygmund spaces, Dissertationes Math. (Rozprawy Mat.) 175 (1980), 67.
[4] C. Bennett and R. Sharpley, Weak-type inequalities for $H^p$ and BMO, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, MA, 1978), Part 1, Proc. Sympos. Pure Math., Part XXXV, Amer. Math. Soc., Providence, RI, pp. 201–229.
[5] C. Bennett and R. Sharpley, Interpolation of operators, Pure Appl. Math. 129 (1988), xiv+469.
[6] N. Bingham, C. Goldie, and J. Teugels, Regular variation, Encyclopedia of mathematics and its applications, vol. 27, Cambridge University Press, Cambridge, MA, 1989, pp. xx+494.
[7] D. Breit and A. Cianchi, Symmetric gradient Sobolev spaces endowed with rearrangement-invariant norms, Adv. Math. 391 (2021), 107954.
[8] Y. Brudnyi and N. Krugljak, Interpolation functors and interpolation spaces. Vol. I, North-Holland Mathematical Library, vol. 47, North-Holland Publishing Co., Amsterdam, 1991, pp. xvi+718.
[9] A.-P. Calderón, Spaces between $L^1$ and $L^\infty$ and the theorem of Marcinkiewicz, Studia Math. 26 (1966), 273–299.
[52] L. Pick, A. Kufner, O. John, and S. Fučík, *Function spaces. Volume I*, De Gruyter series in nonlinear analysis and applications, vol. 14, extended edn., Walter de Gruyter & Co., Berlin, 2013, pp. xvi+479.

[53] M. Rao and Z. Ren, *Theory of Orlicz spaces*, Monographs and textbooks in pure and applied mathematics, vol. 146, Marcel Dekker, Inc., New York, 1991, pp. xii+449.

[54] F. Riesz, *Sur un Theoreme de Maximum de Mm. Hardy et Littlewood*, J. London Math. Soc. 7 (1932), no. 1, 10–13.

[55] J. Ryff, *Measure preserving transformations and rearrangements*, J. Math. Anal. Appl. 31 (1970), 449–458.

[56] R. Sharply, *Spaces Λ𝛼(𝑋) and interpolation*, J. Functional Analysis 11 (1972), 479–513.

[57] R. Sharply, *Counterexamples for classical operators on Lorentz-Zygmund spaces*, Studia Math. 68 (1980), no. 2, 141–158.

[58] J. Soria and P. Tradacete, *Optimal rearrangement invariant range for Hardy-type operators*, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016), no. 4, 865–893.

[59] V. Stepanov, *The weighted Hardy’s inequality for nonincreasing functions*, Trans. Amer. Math. Soc. 338 (1993), no. 1, 173–186.

[60] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), no. 3, 511–544.

[61] N. Wiener, *The ergodic theorem*, Duke Math. J. 5 (1939), no. 1, 1–18.

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