BROWNIAN MOTION IN RIEMANNIAN
ADMISSIBLE COMPLEXES.

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Abstract.
The purpose of this work is to construct a Brownian motion with values in simplicial complexes with piecewise differential structure. In order to state and prove the existence of such Brownian motion, we define a family of continuous Markov processes with values in an admissible complex; we call every process of this family, isotropic transport process. We show that the family of the isotropic processes contains a subsequence, which converges weakly to a measure; we name it the Wiener measure. Then, using the finite dimensional distributions of the obtained Wiener measure, we construct a new admissible complex valued continuous Markov process: the Brownian motion. We finished with a geometric analysis of this Brownian motion, to determine the recurrent or transient behavior of such process.

0. Introduction.

It has been proved in [19], [22] and [27] that, on a wide class of Riemannian manifolds, the Brownian motion can be approximated in law by Markov process, which generalizes the isotropic scattering transport process on euclidean space [31]. On the other hand, the Brownian motion was introduced as a tool to achieve important results in Riemannian geometry and potential theory, which is not surprising, since Brownian motion is intimately connected with harmonic functions [15], Laplacian, and other fundamental objects in mathematics. For instance, a complete Riemannian manifold is hyperbolic exactly when the Brownian motion is transient.

The purpose of this work is to consider the problem of defining the concept of continuous random walk in the admissible Riemannian complexes, in particular to construct a Brownian motion in singular spaces in spite of the absence of second order differential calculus.

The first section deals with some preliminaries on Riemannian admissible complexes [3] [7] and finishes with a brief survey on the theory of general Markov processes [14].

The second section is devoted firstly, to the construction of a Markov process with values in admissible complex that we name the isotropic transport process and secondly to show that this latter process is a strong Markov one.

In the third section, we construct a family of isotropic process and we show that this family contains a subsequence which converges weakly to a measure, we call it the Wiener measure.

1991 Mathematics Subject Classification. Primary: 60J65, 58J65, Secondary: 58A35.
Key words and phrases. Admissible complex, Brownian motion, Geodesic space, Hadamard space, Isotropic transport process, Martingale, Stochastic process, Wiener measure.

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The aim of the fourth section is to achieve the construction of a Brownian motion with values in an admissible complex by using the finite dimensional distributions of the obtained Wiener measure.

Finally, in the fifth section, we close the paper by studying the transience/recurrence properties of the Brownian motion. In particular, we show that, in the 2-dimensional case if the complex is complete simple connected of non positive curvature and the number of branching faces is always greater or equal to three then the Brownian motion is transient although (surprising) the Euclidean Brownian motion in dimension 2 is recurrent.

To our knowledge, there is an interesting study (I think it is the unique) of M. Brin and Y. Kifer [8] on the Brownian motion in singular spaces. In this study they consider the case of 2-dimensional simplicial complexes whose simplices are flat Euclidean where they describe the Brownian motion in such complex as the planer Brownian inside faces and, after hitting an edge, goes into each adjacent face "with equal probability". Thus actually, our work is the first one where it is shown the existence of Brownian motion, and not only in the case of 2-dimensional complexes with flat simplices but, in the general case of the admissible Riemannian complexes.

We notice that, the steps used for the construction of the admissible complex-valued Brownian motion, can be extended to the the general case of Hadamard spaces if we assume a given uniform probability (sub-probability) measure on the link of each point of the space.

Acknowledgement.

The author would like to express his thanks to Professor A. Verjovsky for encouraging him to investigate this subject and for his constant support over the years.

1. Preliminaries.

1.1 General theory [1] [2] [11] [17] [18].

Let \( X \) be a metric space with a metric \( d \). A curve \( c : I \to X \) is called a geodesic if there is \( v \geq 0 \), called the speed, such that every \( t \in I \) has neighborhood \( U \subset I \) with \( d(c(t_1), c(t_2)) = |t_1 - t_2|v \) for all \( t_1, t_2 \in U \). If the above equality holds for all \( t_1, t_2 \in I \), then \( c \) is called minimal geodesic.

The space \( X \) is called a geodesic space if every two points in \( X \) are connected by minimal geodesic. We assume from now on that \( X \) is complete geodesic space.

A triangle \( \Delta \) in \( X \) is a triple \( (\sigma_1, \sigma_2, \sigma_3) \) of geodesic segments whose end points match in the usual way. Denote by \( H_k \) the simply connected complete surface of constant Gauss curvature \( k \). A comparison triangle \( \bar{\Delta} \) for a triangle \( \Delta \subset X \) is a triangle in \( H_k \) with the same lengths of sides as \( \Delta \). A comparison triangle in \( H_k \) exists and is unique up to congruence if the lengths of sides of \( \Delta \) satisfy the triangle inequality and, in the case \( k > 0 \), if the perimeter of \( \Delta \) is \( < \frac{2\pi}{\sqrt{k}} \). Let \( \bar{\Delta} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3) \) be a comparison triangle for \( \Delta = (\sigma_1, \sigma_2, \sigma_3) \), then for every point \( x \in \sigma_i, i = 1, 2, 3 \), we denote by \( \bar{x} \) the unique point on \( \bar{\sigma}_i \) which lies at the same distances to the ends as \( x \).

Let \( d \) denote the distance functions in both \( X \) and \( H_k \). A triangle \( \Delta \) in \( X \) is \( CAT_k \) triangle if the sides satisfy the triangle inequality, the perimeter of \( \Delta \) is \( < \frac{2\pi}{\sqrt{k}} \) for \( k > 0 \), and if \( d(x, y) \leq d(\bar{x}, \bar{y}) \), for every two points \( x, y \in X \).
We say that $X$ has curvature at most $k$ and write $k_X \leq k$ if every point $x \in X$ has a neighborhood $U$ such that any triangle in $X$ with vertices in $U$ and minimizing sides is $CAT_k$. Note that we do not define $k_X$. If $X$ is Riemannian manifold, then $k_X \leq k$ iff $k$ is an upper bound for the sectional curvature of $X$.

A geodesic space $X$ is called geodesically complete iff every geodesic can be stretched in the two direction.

We say that a geodesic space $X$ is without conjugate points if every two points in $X$ are connected by unique geodesic.

1.2 Riemannian admissible complexes [26] [30].

Let $K$ be a locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric $g$; i.e. $g$ is a family of smooth Riemannian metrics $g_\Delta$ on simplices $\Delta$ of $K$ such that the restriction $g_\Delta|_{\Delta'} = g_{\Delta'}$ for any simplices $\Delta'$ and $\Delta$ with $\Delta' \subset \Delta$.

Let $K$ be a finite dimensional simplicial complex which is connected locally finite. A map $f$ from $[a, b]$ to $K$ is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < ... < t_{p+1} = b$ such that $f([t_i, t_{i+1}])$ is contained in some cell and the restriction of $f$ to $[t_i, t_{i+1}]$ is a geodesic inside that cell. Then define the length of the broken geodesic map $f$ to be:

$$L(f) = \sum_{i=0}^{i=p} d(f(t_i), f(t_{i+1})).$$

The length inside a cell is measured with respect to the metric of the cells.

Then define $\tilde{d}(x, y)$, for every two points $x, y$ in $K$, to be the lower bound of the lengths of broken geodesics from $x$ to $y$. $\tilde{d}$ is a pseudo-distance.

If $K$ is connected locally finite, then the space $(K, \tilde{d})$ is a length space and hence a geodesic space if is complete (see also [6]).

An $l$-simplex in $K$ is called a boundary simplex if it is adjacent to exactly one $l+1$ simplex. The complex $K$ is called boundaryless if there are no boundary simplices in $K$.

We say that the complex $K$ is admissible, if for every connected open subset $U$ of $K$, the open set $U \setminus \{U \cap \{the \ (k - 2) - skeleton\}\}$ is connected ($k$ is the dimension of $K$).

Let $x \in K$ be a vertex of $K$ so that $x$ is in the $l$-simplex $\Delta_l$. We view $\Delta_l$ as an affine simplex in $\mathbb{R}^l$, that is $\Delta_l = \bigcap_{i=0}^{l} H_i$, where $H_0, H_1, ..., H_l$ are closed half spaces in general position, and we suppose that $x$ is in the topological interior of $H_0$. The Riemannian metric $g_{\Delta_l}$ is the restriction to $\Delta_l$ of a smooth Riemannian metric defined in an open neighborhood $V$ of $\Delta_l$ in $\mathbb{R}^l$. The intersection $T_x \Delta_l = \bigcap_{i=1}^{l} H_i \cap T_x V$ is a cone with apex $0 \in T_x V$, and $g_{\Delta_l}(x)$ turns it into an euclidean cone. Let $\Delta_m \subset \Delta_l$ ($m < l$) be another simplex adjacent to $x$. Then, the face of $T_x \Delta_l$ corresponding to $\Delta_m$ is isomorphic to $T_x \Delta_m$ and we view $T_x \Delta_m$ as a subset of $T_x \Delta_l$.

Set $T_x K = \bigcup_{\Delta_l \ni x} T_x \Delta_l$, we call it the tangent cone of $K$ at $x$. Let $S_x \Delta_l$ denote the subset of all unit vectors in $T_x \Delta_l$ and set $S_x = S_x K = \bigcup_{\Delta_l \ni x} S_x \Delta_l$. The set $S_x$ is called the link of $x$ in $K$. If $\Delta_l$ is a simplex adjacent to $x$, then $g_{\Delta_l}(x)$ defines a Riemannian metric on the $(l - 1)$-simplex $S_x \Delta_l$. The family $g_x$ of
Riemannian metrics $g_{\Delta_{i}}(x)$ turns $S_{x}\Delta_{i}$ into a simplicial complex with a piecewise smooth Riemannian metric such that the simplices are spherical.

As it was characterized in [4], a two dimensional complete locally finite simplicial complexes $(K, g)$ is curvature bounded by $k$ ($k_{K} \leq k$) iff the following three conditions hold:

1. the Gauss curvature of the open faces is bounded from above by $k$;
2. for every edge $e$ of $K$, every two faces $f_{1}, f_{2}$ adjacent to $e$ and every interior point $x \in e$ the sum of the geodesic curvatures $k_{1}(x), k_{2}(x)$ of $e$ with respect to $f_{1}, f_{2}$ is nonpositive;
3. for every vertex $x$ of $K$, every simple loop in $S_{x}K$ has length at least $2\pi$ (i.e. $S_{x}K$ is CAT$_{1}$ space).

1.3 Liouville measure for the geodesic flow.

We assume that $K$ is an admissible $n$-dimensional Riemannian complex. We denote by $K^{(i)}$ the $i$-skeleton of $K$ and $K'$ the set of points $x \in K$ such that $x$ is contained in the interior of an $(n - 1)$-simplex.

Let $x \in K'$. Then $x$ is contained in the interior of an $(n - 1)$-simplex $\Delta'$. For any $n$-simplex $\Delta$ whose boundary $\partial\Delta$ contains $x$, let $S'_{x}\Delta$ denote the open hemisphere of unit tangent vectors at $x$ pointing inside $\Delta$. Let $\Delta_{1}, \ldots, \Delta_{m}, m \geq 2$, be the $n$-simplices containing $\Delta'$. We set $S'_{x} = \bigcup_{i=1}^{m} S'_{x} \Delta_{i}$, $S' = \bigcup_{x \in K'} S'_{x}$ and $S' = \bigcup_{x \in \partial\Delta \cap K'} S'_{x} \Delta$.

For $v \in S'_{x} \Delta$ denote by $\theta(v)$ the angle between $v$ and the interior normal $\nu_{\Delta_{i}}(x)$ of $\Delta'$ with respect to $\Delta$ at $x$. Let $d\lambda_{x}$ be the volume element on $K'$ and let $\lambda_{x}$ be the Lebesgue measure on $S'_{x}$. We define the Liouville measure on $S'$ by $d\mu'(x, v) = \cos\theta(v)d\lambda_{x}(x) \otimes dx$. Note that $d\mu'(x, v) \otimes dt$ is the ordinary Liouville measure invariant under the geodesic flow on each n-simplex $\Delta$ of $K$.

Therefore, for $\mu'$-a.e. $v \in S'\Delta$, the geodesic $\gamma_{v}$ in $\Delta$ determined by $\gamma_{v}(0) = v$ meets $\partial\Delta \cap K^{(n-1)} \setminus K^{(n-2)}$ after a finite time $t_{v} > 0$ so that $I(v) = -\gamma_{v}(t_{v}) \in S'\Delta$. Note that $\gamma_{v}(t_{v}) \in K'$ since $K$ is boundaryless. $\mu'$ is invariant under the involution $I$.

Let $I(v) = u + \cos\theta(I(v))\nu_{\Delta}(\gamma_{v}(t_{v}))$, where $u$ is tangent to $K'$ and set $F(v) = \bigcup\{-u + \cos\theta(I(v))\nu_{\Delta}(\gamma_{v}(t_{v}))\}$, where the union is taken over all $n$-simplices containing $\gamma_{v}(t_{v})$ except $\Delta$.

Thus there is a subset $S_{1} \subset S'$ of full $\mu'$-measure such that $F(v)$ is defined for any $v \in S_{1}$. We define recursively $S_{i+1} = \{(x, v) \in S_{1} \setminus F(v) \subset S_{i}\}$ and define $S_{\infty} = \bigcap_{i=0}^{\infty} S_{i}, V = S_{\infty} \cap I(S_{\infty})$. By construction, $V$ has full $\mu'$-measure.

We define the geodesic flow on the space SK (or TK) in the following way:

$$
\begin{align*}
g'(x, v) &= (X_{(x,v)}(t), \dot{X}_{(x,v)}(t)) \\
g^{0}(x, v) &= (x, v),
\end{align*}
$$

where $g'$ is the ordinary geodesic flow in the interior of every n-simplex and in the case where, for $t_{0} \in \mathbb{R}^{+}, X_{(x,v)}(t_{0}) \in K'$, we set $\dot{X}_{(x,v)}(t_{0}) = \dot{X}_{(x,v)}(t_{0}+)$ (therefore, $\dot{X}_{(x,v)}(t_{0}) \in F(\dot{X}_{(x,v)}(t_{0}-))$).

1.4 General Markov process.

Assume that $K$ is an admissible $n$-dimensional Riemannian complex, with the metric $g$ and corresponding distance function $d$. When $K$ is not compact, let $K_{D} = K \cup \{D\}$ be the one-point compactification of $K$. Then, we can define a
metric $\delta$ on $K_D$ such that the topology on $K$ generated by $\delta$ is the same as the topology generated by $d$. In case $K$ is already compact, we simply adjoin $D$ as an isolated point and define the metric $\delta$ on $K_D$ by letting $d = \delta$ on $K \times K$ and $\delta(p, D) = 1$ for $p \in K$. Therefore, the restriction of $\delta$ to $K \times K$ is uniformly continuous with respect to $d$.

Denote by $C(K)$ the space of bounded continuous real-valued functions on $K$, $C_0(K)$ the subspace of $C(K)$ such that the functions have a null limit at infinity and $C_c(K)$ the space of functions in $C(K)$ with compact support. Clearly, these tree spaces are the same if $K$ is compact. $C(K)$ endowed with supnorm is a (real) Banach space and $C_0(K)$, $C_c(K)$ are Banach subspaces of $C(K)$. The space $C_c(K)$ is dense in the space $C_0(K)$.

Finally, whenever the term measurable is used it will refer to the basic $\sigma$-algebra of Borrel sets in $K$ (or $K_D$).

The usual setup for the theory of temporarily homogeneous Markov process defined on measurable space $(\Omega \times [0, \infty[; \mathcal{M} \times \mathcal{R})$ ($\mathcal{R}$ is the Borrel $\sigma$-algebra in $[0, \infty]$) with values in topological measurable space $(E, \mathcal{B})$ is to consider the following objects:

1. We adjoin a point $D$ to the space $E$. We write $E_D = E \cup \{D\}$ and $\mathcal{B}_D$ the $\sigma$-algebra in $E_D$ generated by $\mathcal{B}$.
2. For each $x \in E_D$, a probability measure $P_x$ on $(\Omega, \mathcal{M})$.
3. An increasing family (a filtration) $(\mathcal{M}_t)_{t \geq 0}$ of sub-$\sigma$-algebras of $\mathcal{M}$ and distinguished point $\omega_D$ of $\Omega$.
4. For each $t \in [0, \infty[ a measurable map $Y_t : (\Omega, \mathcal{M}) \to (E_D, \mathcal{B}_D)$ such that if $Y_t(\omega) = D$ then $Y_s(\omega) = D$ for all $s \geq t$, $Y_\infty(\omega) = D$ for all $\omega$ and $Y_0(\omega_D) = D$.
5. For each $t \in [0, \infty[ a translation operator $\theta_t : \Omega \to \Omega$ such that $\theta_\infty(\omega) = \omega_D$ for all $\omega$.

We call the collection $Y = (\Omega, \mathcal{M}, \mathcal{M}_t, Y_t, \theta_t, P_x)$ a (temporally homogeneous) Markov process with state space $(E, \mathcal{B})$ if and only if the following axioms hold:

1. For each $t \geq 0$ and fixed $\Gamma \in \mathcal{B}$, the function $x \mapsto P(t, x, \Gamma) = P_x\{Y_t \in \Gamma\}$ is $\mathcal{B}$ measurable.
2. For all $x \in E$, $P(0, x, E \setminus \{x\}) = 0$ and $P_D\{X_0 = D\}$ = 1.
3. For all $t, h \geq 0$, $Y_t \circ \theta_h = Y_{t+h}$ (homogeneity).
4. For all $s, t \in \mathbb{R}^+$, $x \in E_D$ and $\Gamma \in \mathcal{B}_D$, $P_x\{X_{t+s} \in \Gamma | \mathcal{M}_t\} = P(s, X_t, \Gamma)$ (Markov property).

The point $D$ may be always thought of as a ”cemetery” when we regard $t \mapsto Y_t(\omega)$ as the trajectory of particle moving randomly in the space $E$. With this interpretation in mind, we name the random variable $\xi(\omega) = \inf\{t; X_t(\omega) = D\}$ the lifetime.

2. Isotropic transport process

In this section, $K$ will denote a complete admissible Riemannian complex with dimension $n$ and we will use all notations of the first section.

2.1 An intuitive approach.

Let $\Sigma K$ denote the space of links of the complex $K$. Choose a point $(x_0, v_0)$ from the space $\Sigma K$ and assume that the point $x_0$ is in the topological interior
of a maximal simplex $\Delta_0$. Intuitively, a particle starting from the point $x_0$ travels geodesically, in direction $v_0$ chosen randomly, during exponentially distributed waiting time $s_1$ to a new position $x_1$ supposed in the interior of $\Delta_0$. At $x_1$, the particle chooses a new direction $v_1$ in the link $S_{x_1}$ over $x_1$ with the uniform probability $\mathbb{P}[v_1 \in d\lambda] = \lambda_{x_1}(d\lambda)$, where $\lambda$ denotes the normalized Lebesgue measure on $S_{x_1}$. From the point $x_1$ and in the direction $v_1$, the particle travels geodesically during exponentially distributed waiting time $s_2$ to a position $x_2$ in the interior of the simplex $\Delta_0$. So the particle continues its motion in the interior of $\Delta_0$ until it hits transversally (because of the construction of the generalized geodesic flow on the admissible complexes) the border of the simplex $\Delta_0$ at an interior point of a $(n - 1)$-simplex adjacent to $\Delta_0$. Note this hit point $x_n$. Starting now from $x_n$ and choosing randomly a new direction in the link over $x_n$, the particle travels geodesically during exponentially waiting time $s_n$ to a new position in the interior of a maximal simplex (which could be $\Delta_0$) and so on.

### 2.2 Mathematical approach.

Right now, we will give a mathematical form to the random walk just described above.

Consider the product space $L = \Sigma K \times \mathbb{R}^+$ and the product $\sigma$-algebra $\mathfrak{F} = \mathcal{E} \times \mathfrak{B}$, where $\mathcal{E}$ and $\mathfrak{B}$ are respectively Borrel $\sigma$-algebra of $\Sigma K$ and of $\mathbb{R}^+$. Note $\Omega = L^\mathbb{N}$ and $\mathfrak{F} = \mathfrak{F}^\mathbb{N}$, where $\mathbb{N}$ is the set of positive entire numbers. Thus $(L, \mathfrak{F})$ and $(\Omega, \mathfrak{F})$ are measurable spaces and the points $\omega \in \Omega$ are sequences $\{(x_l, v_l, t_l) : l \in \mathbb{N}\}$.

Let $\{(x_l, v_l, t_l) : l \in \mathbb{N}\}$ be a point of $\Omega$ and set $\tilde{Y}_k(\omega) = ((x_k, v_k), t_k)$, $Z_k(\omega) = (x_k, v_k)$ and $\tau_k(\omega) = t_k$. The functions $\tilde{Y}_k : (\Omega, \mathfrak{F}) \to (L, \mathfrak{F})$, $Z_k : (\Omega, \mathfrak{F}) \to (\Sigma K, \mathcal{E})$ and $\tau_k : (\Omega, \mathfrak{F}) \to (\mathbb{R}^+, \mathfrak{B})$ are measurable.

Finally, we shall consider the following space of events:

$$\Omega' = \{\omega \in \Omega | \forall k \in \mathbb{N}, Z_{k+1}(\omega) \neq Z_k(\omega), \tau_0 = 0, \tau_{k+1}(\omega) > \tau_k(\omega)\}.$$  

Put $\tau(\omega) = \lim_{n \to \infty} \tau_n(\omega)$ (life time) and let $K_D = K \cup \{D\}$ denote the one point compactification. The space $K$ is assumed semi-compact so we shall endow $K_D$ with a metric $d'$ such that the space $(K_D, d')$ is compact and the restriction of $d'$ to $K$ coincides with the beginning metric of $K$.

Now, we will define the $K$-valued geodesically random walk which interests us. Let, for $t \geq 0$:

$$Y_t(\omega) = \begin{cases} X_{Z_i(\omega)}(t - \tau_i(\omega)) & \text{if } \tau_i(\omega) \leq t \leq \tau_{i+1}(\omega), \\ D & \text{if } \tau_i(\omega) \leq t, \end{cases}$$

where $X$ is the $K$-projection of the generalized geodesic flow on the complex $K$. According to the latest definition, we have for every $\omega \in \Omega, Y_\infty = D$.

### 2.3 Markov property.

In the following paragraph, we will complete the preceding construction to define the admissible complex-valued isotropic transport process and then we will show that the last process is a strong Markov one.

Let $K$ denote an admissible Riemannian complex and define the next transition density on the measurable space $(L, \mathfrak{F})$ as:

$$N(z, t; dz, ds) = \begin{cases} 0 & \text{if } t < s, \\ \lambda_z(ds) e^{-(s-t)} ds & \text{if } s \geq t, \end{cases}$$
Lemma 2.3.

Definition 2.2.

Theorem 2.3.

Lemma 2.4.

Proof.

Proposition 2.5.

Proposition 2.6.

Proof.

Definition 2.1.

Proposition 2.7.

Proof.

Definition 2.2.

Lemma 2.3.

Proof.
We can easily check that, for all \( A \in \mathcal{E}_D \), the sets \( \{ Y_s \in A \}_{s \leq t} \) belong to the \( \sigma \)-algebra \( \mathcal{G}_t \). Thus, we end the proof; indeed the sets \( \{ Y_s \in A \}_{s \leq t} \) generate the \( \sigma \)-algebra \( \mathcal{G}_t \).

\[ \square \]

We set, for real functions \( g \in C_0(\Sigma K) \) and \( f \in C_0(K) \) (or simply measurable functions):

1. \( P g(x) := \int_{\Sigma \times K} g(x, \eta) d\lambda_x(\eta) \).
2. \( \forall t > 0, T^0_t f(x) := \int_{\Sigma \times K} f(X(x, \eta)(t)) d\lambda_x(\eta) \) and \( T_t f(x) := E_x[f(Y_t)] \), the latest is the expectation with respect to \( Y_t \).
3. \( \forall \lambda > 0, R^0_\lambda f(x) := \int_{R^+} e^{-\lambda t} T^0_t f(x) dt \) and \( R_\lambda f(x) := \int_{R^+} e^{-\lambda t} T_t f(x) dt \), respectively the resolvent operator of \( T^0_t \) and of \( T_t \).

**Proposition 2.4.**

Let \( f \in C^0(K) \), then, for all \( \lambda > 0 \) we have:

\[ R_\lambda f = \sum_{n=0}^{\infty} (R^0_{\lambda + 1})^{n+1} f, \]

where \( (R^0_{n+1})^0 := Id \) the identity map.

**Proof.**

First we write:

\[ R_\lambda f(x) = \int_0^{\tau_1} + \sum_{i=1}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-\lambda t} f(Y_t) dt. \]

Taking into account the distribution of \( \tau_1 \) and the initial distribution of the process \( Y \), the first integral becomes:

\[ \int_0^{\infty} e^{-(1+\lambda)s} T^0_s f(x) ds = R^0_{1+\lambda} f(x). \]

For the second part of the decomposition, we will prove by induction argument that for all \( i \geq 1 \) the following equality:

\[ \mathbb{R} \int_{\tau_i}^{\tau_{i+1}} e^{-\lambda t} f(Y_t) dt = (R^0_{\lambda + 1})^{i+1} f(x). \]

Let see the case \( i = 1 \):

\[ \int_{\tau_1}^{\tau_2} e^{-\lambda t} f(Y_t) dt = \int_0^{\tau_2 - \tau_1} e^{-\lambda t} f(Y_{t+\tau_1}) dt, \]

which is equal to:

\[ [e^{-\lambda \tau_1} (R^0_{\lambda + 1}) f(X_{Z_1}(0))] = [e^{-\lambda \tau_1} (P R^0_{\lambda + 1}) f(X_{Z_0}(\tau_1))]. \]

Using the distribution of \( \tau_1 \), we obtain:

\[ (R^0_{\lambda + 1})(R^0_{\lambda + 1}) f(x). \]

Assume the property \( \mathbb{R} \) until the order \( l \), and see what happens at the order \( l + 1 \):
\[
\int_{\tau_l}^{\tau_{l+1}} e^{-\lambda t} f(Y_t) dt = [e^{-\lambda \tau_{l+1}} \int_{0}^{\tau_{l+2} - \tau_{l+1}} e^{-\lambda t} f(Y_{t+\tau_{l+1}}) dt],
\]
which is equal to:
\[
[e^{-\lambda \tau_{l+1}} (R^0_{\lambda+1}) f(X_{\tau_{l+1}}(0))] = [e^{-\lambda \tau_l} e^{-\lambda (\tau_{l+1} - \tau_l)} (PR^0_{\lambda+1}) f(X_{\tau_l}(\tau_{l+1} - \tau_l))].
\]
Using the distribution of \((\tau_{l+1} - \tau_l)\):
\[
= [e^{-\lambda \tau_l} (R^0_{\lambda+1}) (R^0_{\lambda+1}) f(X_{\tau_l}(0))],
\]
this latest expectation is equal to:
\[
\int_{\tau_l}^{\tau_{l+1}} e^{-\lambda t} R^0_{\lambda+1} f(Y_t) dt.
\]
Hence, using the recurrence hypothesis applied to the function \(R^0_{\lambda+1} f\), we obtain the equality \(\Box\) at the order \(l+1\).

Lemma 2.5.
Let \(f\) be a measurable real (positive) function on \((K, \mathcal{B})\). Then, for all \(t \geq 0\) and \(\lambda > 0\), we have:
\[
E\{ \int_{t}^{\infty} e^{-\lambda u} f(Y_u) du | \mathcal{F}_t \} = e^{-\lambda t} R_{\lambda} f(Y_t).
\]

Remark 2.6.
By the lemma 2.3 of this paragraph, to establish the lemma 2.5 it suffices to show the same equality(s) on the sets \(\Lambda_n \in \mathcal{F}_t\) with
\[
\Lambda_n \cap \{\tau_n \leq t < \tau_{n+1}\} = \Lambda_n \cap \{t < \tau_{n+1}\}.
\]
i.e:
\[
\Box \quad E\{ \int_{t}^{\infty} e^{-\lambda u} f(Y_u) du | \Lambda_n \} = E\{e^{-\lambda t} R_{\lambda} f(Y_t) | \Lambda_n \}.
\]

Proof of Lemma 2.5.
Consider the left side of the equality \(\Box\) and set it in the following way:
\[
E\{ \int_{t}^{\infty} e^{-\lambda u} f(Y_u) du | \Lambda_n \} = [\int_{t}^{\tau_{n+1}} + \sum_{i=n+1}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-\lambda u} f(Y_u) du | \Lambda_n].
\]
Using, the Markov property of the process \(\{\tilde{Y}_n; n \geq 0\}\), the fact that \(\Lambda_n \subset \{\tau_n \leq t \leq \tau_{n+1}\}\) and the exponential distribution of the random variable \(\tau_{n+1} - t \land \tau_{n+1} - \tau_n\), the first integral of the decomposition becomes:
\[
[ e^{-\lambda \tau_n} e^{-(t-\tau_n)} \int_{0}^{\infty} e^{-(\lambda+1) u} P f(X_{\tau_n}(u + (t-\tau_n))) du | \Lambda_n]
\]
which is equal to:

\[ e^{-\lambda t} e^{-(t-\tau_n)} R_{\lambda+1}^0 f(X_{Z_n}(t - \tau_n)) | \Lambda_n ] \]

For the second half of the decomposition, we will show by induction argument that for all \( i \geq 1 \), we have the equality:

\[ \mathbb{R}^t \int_{\tau_{n+1}}^{\tau_{n+i+1}} e^{-\lambda u} f(Y_u) du | \Lambda_n ] = [e^{-\lambda t} e^{-(t-\tau_n)} (R_{\lambda+1}^0)^i f(X_{Z_n}(t - \tau_n)) | \Lambda_n]. \]

Let see the case \( i = 1 \):

\[ \int_{\tau_{n+1}}^{\tau_{n+2}} e^{-\lambda u} f(Y_u) du | \Lambda_n ] = [e^{-\lambda \tau_{n+1}} \int_{0}^{\tau_{n+2} - \tau_{n+1}} e^{-\lambda u} f(Y_{u+\tau_{n+1}}) du | \Lambda_n], \]

which is equal to:

\[ [e^{-\lambda \tau_{n+1}} (R_{\lambda+1}^0) f(X_{Z_{n+1}}(0)) | \Lambda_n] = [e^{-\lambda t} e^{-\lambda (\tau_{n+1} - t)} (R_{\lambda+1}^0) f(X_{Z_n}(\tau_{n+1} - \tau_n)) | \Lambda_n], \]

which is the same as:

\[ [e^{-\lambda t} e^{-\lambda (\tau_{n+1} - t)} (R_{\lambda+1}^0) f(X_{Z_n}((\tau_{n+1} - t) + (t - \tau_n)))) | \Lambda_n]. \]

Using the Markov property of \( \{ \tilde{Y}_n; n \geq 0 \} \) and the distribution of \((\tau_{n+1} - t) \wedge (\tau_n)\), we obtain:

\[ [e^{-\lambda t} e^{-(t-\tau_n)} \int_{0}^{\infty} e^{-(\lambda+1)u} P(R_{\lambda+1}^0) f(X_{Z_n}(u + (t - \tau_n))) du | \Lambda_n], \]

what is equal to:

\[ [e^{-\lambda t} e^{-(t-\tau_n)} R_{\lambda+1}^0 (R_{\lambda+1}^0) f(X_{Z_n}(t - \tau_n)) | \Lambda_n]. \]

Now assume that the property \( \mathbb{R}^t \) comes true until the order \( l \), and let’s see what will happen at order \( l + 1 \).

\[ \int_{\tau_{n+(l+1)}}^{\tau_{n+2}} e^{-\lambda u} f(Y_u) du | \Lambda_n ] = [e^{-\lambda \tau_{n+(l+1)}} \int_{0}^{\tau_{n+(l+2)} - \tau_{n+(l+1)}} e^{-\lambda u} f(Y_{u+\tau_{n+(l+1)}}) du | \Lambda_n], \]

what is equal to:

\[ [e^{-\lambda \tau_{n+(l+1)}} (R_{\lambda+1}^0) f(X_{Z_{n+(l+1)}}(0)) | \Lambda_n], \]

which is equal to:

\[ [e^{-\lambda \tau_{n+l}} e^{-\lambda (\tau_{n+(l+1)} - \tau_{n+l})} (P R_{\lambda+1}^0) f(X_{Z_{n+l}}((\tau_{n+1}) - \tau_{n+l})) | \Lambda_n]. \]

Then, using the distribution of \((\tau_{n+(l+1)} - \tau_{n+l})\) we get:

\[ = [e^{-\lambda \tau_{n+l}} (R_{\lambda+1}^0) (R_{\lambda+1}^0) f(X_{Z_{n+l}}(0)) | \Lambda_n], \]
this latest expectation is equal to:

\[ \left[ \int_{\tau_n+t}^{\tau_n+(l+1)} e^{-\lambda u} R_{\lambda+1}^0 f(Y_u) du \right] |\Lambda_n]. \]

Thus, if we apply the recurrence hypothesis to the function \( R_{\lambda+1}^0 \), we obtain the equality \( \mathbb{R}^l_\lambda f \) at the order \( l+1 \).

Up to now we have shown that the left side of the equality \( \mathbb{R}^l_\lambda f \) is equal to:

\[ [e^{-\lambda t} e^{-(t-\tau_n)} \{ R_{\lambda+1}^0 f(X_{\tau_n}(t-\tau_n)) + \sum_{i=1}^{\infty} (R_{\lambda+1}^0)^{i+1} f(X_{\tau_n}(t-\tau_n)) \}] |\Lambda_n]. \]

Thus by proposition 2.4 of this section, this sum is equal to:

\[ [e^{-\lambda t} e^{-(t-\tau_n)} R_\lambda f(X_{\tau_n}(t-\tau_n)) |\Lambda_n]. \]

Using once again the Markov property of \( \{ \tilde{Y}_n; n \geq 0 \} \), we get:

\[ [e^{-\lambda t} e^{-(t-\tau_n)} R_\lambda f(X_{\tau_n}(t-\tau_n)) |\Lambda_n] = [e^{-\lambda t} R_\lambda f(Y_t) |\Lambda_n] \]

which was to be proved. \( \square \)

Right now, we have collected all the ingredients to prove the following theorem:

**Theorem 2.7.**

Let \( Y = (\Omega, \mathfrak{F}_t, Y_t, \theta_t, P^x) \) be the isotropic transport process with values in the admissible Riemannian complex \( K \). Then \( Y \) is a strong Markov process.

**Remark 2.8 ([14] I pp 97-100).**

It suffices to show that the process \( Y \) is a Markov process because the right continuous (with right continuous trajectories) Markov process is always strongly Markov for the filtration \( \{ \mathfrak{F}_t \} \). But we know that, in case of continuous stochastic process, the filtration \( \{ \mathfrak{F}_t \} \) is equal to the filtration \( \{ \mathfrak{F}_t \} \), which includes the case of the isotropic transport process (it is trajectories continuous).

**Proof of Theorem 2.7.**

By Lemma 2.5 of this section we have:

\[ E \left\{ \int_t^{\infty} e^{-\lambda u} f(Y_u) du \mathfrak{F}_0^t \right\} = E^Y \{ \int_0^{\infty} e^{-\lambda(t+u)} f(Y_u) du \}. \]

Then, if the function \( f \) is bounded we have the next equality:

\[ E \left\{ \int_t^{\infty} \varphi(u) f(Y_u) du \mathfrak{F}_0^t \right\} = E^Y \{ \int_0^{\infty} \varphi(t+u) f(Y_u) du \}, \]

whenever the function \( \varphi \) is a linear combination of exponentials and hence, by uniform approximation, whenever \( \varphi \) is continuous and vanishes at infinity. Then, consider the following sequence of functions:

\[ \varphi_n(s + t + u) = \begin{cases} 0 & \text{if } \frac{1}{n} \leq u, \\ \frac{1}{n} - x & \text{if } 0 \leq u < \frac{1}{n}. \end{cases} \]
3.1 Construction.

Let \( Y = (\Omega, \mathcal{F}_t^0, Y_t, \theta_t, P^x) \) be the isotropic transport process in the complete admissible Riemannian complex \( K \) constructed in the last section. Set for a real \( \eta > 0 \) and \( z = (x, v) \in \Sigma K, \eta z := (x, \eta v) \).

Define now a process \( Y^\eta \) from \( Y = (\Omega, \mathcal{F}_t^0, Y_t, \theta_t, P^x) \) in the following way:

\[
Y_t^\eta(\omega) = \begin{cases} 
X_{\eta Z_i(\omega)}(\frac{s}{\eta} - \tau_i(\omega)) & \text{if } \tau_i(\omega) \leq \frac{s}{\eta} \leq \tau_{i+1}(\omega), \\
D & \text{if } \xi(\omega) \leq \frac{s}{\eta}.
\end{cases}
\]

Thus the process \( Y^\eta = (\Omega, \mathcal{F}_t^0, Y^\eta_t, \theta_t, P^x) \) is (trajectories) continuous and it is, as the process \( Y \), strongly Markov.

Proposition 3.1.

Let \( K \) be an admissible Riemannian and \( C(\mathbb{R}^+,K) \) be the space of continuous paths in \( K \). Then for each \( \eta > 0 \), the process \( Y^\eta \) generates a measure \( \mu^\eta \) on the space \( C(\mathbb{R}^+,K) \).

Proof.

For \( \eta > 0 \), set \( P^n_{s,t}(p, A) \) with \( p \in K \) and \( A \in \mathcal{B}(K_D) \), the transition probability of the process \( Y^\eta \) (i.e. \( P^n_{s,t}(p, A) := \text{Prob}\{Y^n_{t+s} \in A; Y^\eta_s = p\} \)).

Consider the finite sets of reals \( J = \{t_1 < t_2 < \ldots < t_n\} \subset (\mathbb{R}^+)^n \). Then, for each finite set \( J = \{t_1 < t_2 < \ldots < t_n\} \), we define probability measure in the following way:

\[
\text{for } B \subset K^n_D, P^n_J(B) = \int_B P^x(dx_0) \int P^n_{0,t_1}(x_0, dx_1) \int \ldots \int P^n_{t_{n-1},t_n}(x_{n-1}, dx_n).
\]

Let \( \Phi(\mathbb{R}^+) \) denote the set of the finite subset of \( \mathbb{R}^+ \). Then, the system \( \{P^n_J; J \in \Phi(\mathbb{R}^+)\} \), and thanks to the Markov property of \( Y^\eta \), is a projective system on \( (K_D, \mathcal{B}(K_D)) \) (i.e : if \( \pi^J_1 \) (respectively \( \pi^J_2 \)) is the natural projection of \( K^I \) (respectively \( \Omega \)) to \( K^J \) then \( P^n_J(\pi^J_2)^{-1} = P^n_I \)).

On the other hand, the trajectories of \( Y^\eta \) are continuous and the space \( K \) is Hausdorff and \( \sigma \)-compact. Consequently, and using the Kolmogorov theorem \([5]\), we get a probability measure \( \mu^\eta \) on the space \( C(\mathbb{R}^+,K) \).

3.2. Wiener measure.

Right now, we will announce the main theorem of this section.

Theorem 3.2.

Let \( K \) denote an admissible Riemannian complex, and consider the family \( \{Y^\eta\}_{\eta > 0} \) of the isotropic transport processes constructed in the paragraph above and let \( \{\mu^\eta\}_{\eta > 0} \) be the family of the generated probability measures on \( C(\mathbb{R}^+,K) \). The space \( C(\mathbb{R}^+,K) \)
is provided with the compact-open topology. Then the family \((\mu_\eta)_{\eta>0}\) has a convergent subsequence.

To prove the last theorem we need the following lemma:

**Lemma 3.3.**

Under the hypothesis of Theorem 3.2, the family of the probability measures \((\mu_\eta)_{\eta>0}\) is Tight, i.e.:

\[
\lim_{\eta \to 0} \sup_{c \to 0} \min_{t-c<t_1<t_2<t+c} \frac{d(Y^n_\eta(t_1),Y^n_\eta(t_2))}{\gamma} > \epsilon \}
\]

**Remark 3.4.**

Before proceeding to look at the proof of the lemma, recall first the two following facts:

1. When the space \(C(\mathbb{R}^+,K)\) is provided with the compact-open topology, the Tightness property is equivalent, following Stone [29], to the equality of the Lemma 3.3.

2. According to Jørgensen’s result [19, Lemma 1.4], if the following property:

\[
\text{for all } \epsilon > 0, \text{ there exists } \alpha > 0 \text{ such that, } \sup_{p \in K_D} \frac{1}{\lambda} P_{0,s}(p, B_D^\epsilon(p, \epsilon)) \leq \alpha ,
\]

comes true then the equality of Lemma 3.3 is also true.

**Proof of Lemma 3.3.**

By Remark 3.4, if we show the following:

\[
\forall \epsilon > 0, \exists \alpha > 0, \lim_{\eta \to 0} \sup_{p \in K_D} P_{t-\epsilon \leq t \leq t+\epsilon}^{0,s}(p, B_D^\epsilon(p, \epsilon)) \leq \alpha ,
\]

then the sequence \((\mu_\eta)_{\eta>0}\) is Tight.

We will assume that \(\epsilon < \eta\) (otherwise, the probability needed should be null) and \(\frac{t}{\eta^2} < \tau_1\) (see the recurrence in proof of Lemma 2.5) which doesn’t affect the result. On the other hand, \(\epsilon\) is necessarily lower or equal than \(\frac{t}{\eta^2}\) unless the sought after probability should vanish and then, there is nothing to prove.

Thus, we have:

\[
Prob\{Y^n_t \in B_D^\epsilon(p, \epsilon)\} = E\{I_{B_D^\epsilon(p, \epsilon)}(Y^n_t)|\epsilon \leq \frac{t}{\eta^2} < \tau_1\}.
\]

Using the Markov property, we obtain:

\[
Prob\{Y^n_t \in B_D^\epsilon(p, \epsilon)\} = E\{e^{-\frac{t}{\eta^2}} \int_0^{\infty} PI_{B_D^\epsilon(p, \epsilon)}(X(p,\eta\zeta)\frac{t}{\eta^2} + s) e^{-s} ds | \epsilon \leq \frac{t}{\eta^2} \},
\]

that is equal to:

\[
E\{e^{-\frac{t}{\eta^2}} R_1^{\eta} I_{B_D^\epsilon(p, \epsilon)}(X(p,\eta\zeta)\frac{t}{\eta^2})|\epsilon \leq \frac{t}{\eta^2} \}.
\]
Using the fact that $\|R_\eta^0\| \leq 1$ we obtain the following estimation:

$$E\left\{ e^{-\frac{t^2}{\eta^2}} R_\eta^0 I_{B_D(p,\epsilon)}(X(p,\eta\zeta)(\frac{t}{\eta^2})) \left| \epsilon \leq \frac{t}{\eta^2} \right\} \leq e^{-\frac{t^2}{\eta^2}}.$$ 

So for all $t > 0$ we get:

$$\frac{\text{Prob}\{Y^\eta_t \in B_D(p,\epsilon)\}}{\frac{1}{\eta^2}} \leq e^{-\frac{t^2}{\eta^2}}.$$ 

Thus, for all $t > 0$, if $\eta$ goes to zero, $\frac{e^{-\frac{t^2}{\eta^2}}}{\frac{1}{\eta^2}}$ goes also to zero, which was to be proved.

Right now, we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.**

Consider the space $C(\mathbb{R}^+, K)$ provided with the compact-open topology, where $K$ is an admissible Riemannian complex. Let $(\mu_\eta)_{\eta > 0}$ be the sequence of probability measures generated by the family of isotropic transport processes $\{Y^\eta\}_{\eta > 0}$.

By Lemma 3.3, the sequence $(\mu_\eta)_{\eta > 0}$ is Tight; moreover, the space $C(\mathbb{R}^+, K)$ endowed with the compact-open topology is separable. Thus, using Prohorov’s theorem (see [5]), the sequence $(\mu_\eta)_{\eta > 0}$ is relatively compact. The proof is now complete.

We showed above that the sequence $(\mu_\eta)_{\eta > 0}$ has a subsequence which converges to a probability measure. Let $W$ denote this limit; then we set the following definition:

**Definition 3.5.** The measure $W$ on the space $C(\mathbb{R}^+, K)$ is called a Wiener measure.

**Example 3.6 : The smooth case.**

Assume $K$ is a smooth Riemannian manifold of dimension $n$ and let $\triangle$ denote the operator of Laplace-Beltrami on $K$, then $\triangle$ is the infinitesimal generator of a Markov process, named the Brownian motion [16], and note it $\{B^x_t\}_{t<\zeta'}$. Let $(U_\epsilon)_{\epsilon > 0}$ denote the semigroup associated to the Brownian motion. Suppose that, for all $f \in C_0(K)$, $U_\epsilon f \in C_0(K)$. Then we have the following theorem:

**Theorem.**

The sequence of processes $\{Y^\eta\}_{\eta > 0}$ converge weakly to the process $\{B^x_t\}_{t<\zeta'}$.

**Proof.**

Set $T^\eta_t f(x) = E^x[f(Y^\eta_t)]$ ; following a result of Pinsky (see [27]), we have:

$$\forall f \in C_0(K), \lim_{\eta \to 0} T^\eta_t f = U_t f,$$

where $n$ is the dimension of $K$.

By Theorem 3.2, there exists a subsequence $(\mu_{\eta'})_{\eta' > 0}$ of the sequence of probability measures $(\mu_\eta)_{\eta > 0}$, such that $(\mu_{\eta'})_{\eta' > 0}$ converges to a probability measure $W$ on the space $C(\mathbb{R}^+, K)$.

Thus, by Stone’s theorem [29], $W$ is then the classical Wiener measure generated by the Brownian motion $\{B^x_t\}_{t<\zeta'}$. □
4. BROWNIAN MOTION.

By $K$ we always denote a complete admissible Riemannian complex, consider $\{Y^0_n\}_{n>0}$ the family of the isotropic transport processes and $(\mu_\eta)_{\eta>0}$ the corresponding sequence of probability measures.

Let $(\mu_{\eta_k})_k$ be a subsequence of the sequence $(\mu_\eta)_{\eta>0}$ which converges to the Wiener measure $W$.

Note, for $\eta_k > 0$ and for each finite set $J = \{t_1 < t_2 < \ldots < t_n\}$, $P^n_{\eta_k}$ the probability measure defined on the product space $K^n$, as follows:

$$ P^n_{\eta_k}(B) = \int_B P^x(dx_0) \int P^n_{0,t_1}(x_0,dx_1) \int \ldots \int P^n_{t_{n-1},t_n}(x_{n-1},dx_n). $$

**Proposition 4.1.**

By $\Phi(\mathbb{R}^+)$ we note the set of all finite subsets of $\mathbb{R}^+$. Then, for all $J$ in the set $\Phi(\mathbb{R}^+)$, the sequence of probability measures $(P^n_{\eta_k})_k$ has a subsequence converging to a probability measure $\mu_J$ on the space $K^n_J$ ($|J|$ is the cardinal of $J$). Moreover, the system $\{\mu_J; J \in \Phi(\mathbb{R}^+)\}$ is projective on the space $(K_D, \mathcal{B}(K_D))$.

**Proof.**

Recall that for all $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$ and all $p \in K$, the sequence of transition functions $(P^n_{\eta_k}(p_0,\cdot))(P^n_{\eta_k}(p,A) := \text{Prob}\{Y^n_{t+s} \in A; Y^n_{t} = p\}$ where $A \in \mathcal{B}(K_D)$ defines a sequence of probability measures on the space $(K_D, \mathcal{B}(K_D))$.

Moreover, the space $K_D$ is $\sigma$-compact; Thus, following Prohorov’s theorem [5], there exists a probability measure $\mu^p_{s,t}$ and a subsequence $(P^n_{\eta_k}(p,\cdot))_k$ converging weakly to $\mu^p_{s,t}$.

By a diagonal argument, we obtain, for all $J = \{t_1 < t_2 < \ldots < t_n\} \in \Phi(\mathbb{R}^+)$, a probability measure $\mu_J$ on the product space $K^n_J$ in the following way:

$$ \mu_J(B) = \int_B P^x(dx_0) \int \mu_{0,t_1}^{s_0}(dx_1) \int \ldots \mu_{t_{n-1},t_n}^{s_{n-1}}(dx_n), $$

consequently the proof is now complete. \hfill $\square$

**Remark 4.2.**

The sequence $(\mu_{\eta_k})_k$ is weakly convergent to the Wiener measure $W$. Thus, for every set $J$ belonging to $\Phi(\mathbb{R}^+)$, the finite dimensional distribution $W(\pi_J)^{-1}$ coincides with $\mu_J$. In particular, for all $s > 0$, $t > 0$ and $p \in K_D$, we have $W(\pi_{s,t})^{-1} = \mu^p_{s,t}$.

**Corollary 4.3.**

The function which maps a point $(t,p,\Gamma) \in \mathbb{R}^+ \times K_D \times \mathcal{B}(K_D) \to W(t,p,\Gamma) := W(\pi_{\{t,0\}})^{-1}(\Gamma)$ is a transition function on the measurable space $(K_D, \mathcal{B}(K_D))$.

**Proof.**

The corollary is an immediate consequence of proposition 4.1 and remark 4.2. \hfill $\square$

Just now, we are ready to give the main theorem of this section.

**Theorem 4.4.**

Let $(t,p,\Gamma) \mapsto W(t,p,\Gamma)$ denote the transition function on the measurable space $(K_D, \mathcal{B}(K_D))$, corresponding to the Wiener measure on the space $C(\mathbb{R}^+, K)$ (see corollary 4.3). Then there exists a continuous $K_D$-valued Markov process $\{B^p_t\}_{t \geq 0}$ with $W(t,p,\Gamma)$ as transition function.

Before proceeding to look at the proof, we first give the following definition:
Definition 4.5.

The continuous $K_D$-valued Markov process $\{B_t^p\}_{t \geq 0}$, is called a Brownian motion.

Proof of theorem 4.4.

Using a corollary of the Kolmogorov’s theorem (see [14] I page 91 theorem 3.5), if we show that the transition functions $(t, p, \Gamma) \mapsto W(t, p, \Gamma)$ satisfy the following two conditions, for each compact $\Gamma \subset K_D$:

1. for all $N > 0$, $\lim_{y \to \infty} \sup_{t \leq N} W(t, y, \Gamma) = 0$,
2. for all $\epsilon > 0$, $\lim_{t \downarrow 0} \sup_{p \in \Gamma} \frac{1}{t} W(t, p, B_D(p, \epsilon)) = 0$,

then the conclusion of theorem 4.4 comes true.

For the first condition, consider a compact $\Gamma \subsetneq K_D$ (otherwise if $\Gamma = K_D$ then the first condition is trivially satisfied). Let $(\mu_{\eta_k})_k$ be a sequence of measures associated to the sequence of isotropic processes which converges weakly to the Wiener measure $W$ and let $P_{\eta_k}^t(p, A) := \text{Prob}\{Y_{\eta_k}^t \in A; Y_{\eta_k}^0 = p\}$ denote the associated transition functions.

Recall that, for each $\eta > 0$, all trajectories of the random walk $Y^n$ are concatenations of geodesic segments, with every geodesic segment’s length lower or equal to $\eta$. Consequently, we have for each $\eta_k > 0$, $d(p, Y_{\eta_k}^t) \leq \eta_k t$ if $Y_{\eta_k}^0 = p$.

Let $N > 0$ some (fixed) real, then for all $t \leq N$, if $Y_{\eta_k}^0 = y$, $d(y, Y_{\eta_k}^t) \leq \eta_k N$. Thus, if we consider the points $y \in K_D$ with the distance $d(y, \Gamma)$ strictly greater than $(\eta_k + \epsilon)N$, for some $\epsilon > 0$, then the probability $P_{\eta_k}^t(y, \Gamma)$ should vanish.

In a nutshell, we proved that, for all $\eta_k > 0$, $N > 0$ and $y \in K_D$:

For all $\epsilon > 0$, there exists $\alpha = (\eta_k + \epsilon)N$ such that if $d(y, \Gamma) \geq \alpha$ then $\sup_{t \leq N} P_{\eta_k}^t(y, \Gamma) < \epsilon$.

So if we take the limit (of the adequate subsequence) then the fact required is obtained.

For the second condition, we recall that throughout the proof of Lemma 3.3 we obtained the following inequality:

$$\forall t > 0, \forall \eta_k > 0, \sup_{p \in K_D} \frac{P_{\eta_k}^t\{Y_{\eta_k}^t \in B_D^c(p, \epsilon)\}}{\frac{t}{\eta_k}} \leq e^{-\frac{\epsilon^2}{\eta_k^2}}.$$

Then it is enough to let $\eta_k$ and at the same time $t$ go to zero, to obtain the second condition, which ends the proof. \qed

5. Recurrent and transient behavior of the Brownian motion

Usually, in the literature about the Brownian motion in the smooth case, the authors question the recurrent or transient behavior of this stochastic process.

It is known, for example, that the euclidian Brownian motion is recurrent when it is two dimensional and it is transient if its dimension is greater or equal to three. Moreover, we know that the noncompact hyperbolic surface valued Brownian motion is transient.

For more results and details, we recommend to the reader the papers of H.P Mckean, D. Sullivan [23] and T.J. Lyons, H.P Mckean [21].
5.1 The geometric behavior of the admissible Riemannian complex valued Brownian motion.

Let $K$ denotes a complete admissible Riemannian complex of dimension $n$ and $p \in K$. We recall that the $K$-valued Brownian motion $\{B_t^p\}_{t \geq 0}$, was obtained as a weak limit of sequence of isotropic transport processes.

On the other hand, we have seen that the trajectories of the isotropic processes are concatenations of geodesic segments. When a trajectory joins (a.e.) transversally the $(n-1)$-skeleton\((n-2)\)-skeleton, it goes on choosing isotropically a new maximal face (i.e. all adjacent maximal faces have the same probability to be chosen).

Consequently, the $K$-valued Brownian motion $\{B_t^p\}_{t \geq 0}$ behaves, inside every $n$-simplex $\Delta_n$, as the standard Brownian motion with values in Riemannian $n$-dimensional manifold endowed with the metric $g_{\Delta_n}$.

Moreover, the process hits (a.e.) "transversally" the $(n-1)$-skeleton\((n-2)\)-skeleton, then it goes on choosing isotropically a maximal face. Thus, it results from this geometric description a new discreet random walk corresponding to the isotropic choices of the maximal faces.

To give a rigorous mathematical construction of such discreet process, let us consider some notions.

The dual graph $X$ of a complex $K$ is 1-dimensional simplicial complex defined as follows:

Consider one point inside (topological interior) each $n$-simplex of $K$ and, for every $(n-1)$-simplex a point in its topological interior, then, we connect the considered points with geodesic segments and let $E(X)$ denote the set of such segments. Thus, this dual graph consists of set $V_n(X)$ of vertices of degree $n+1$ (interior $n$-simplexes points) and a set $V_{n-1}(X)$ of vertices corresponding to the interior points of the $(n-1)$-simplexes, where every vertex has degree equal to the number of the $n$-simplexes adjacent to this vertex.

Consider now, the Markov chain (discreet Markov process) $\{C_n\}_{n \in \mathbb{N}}$ which has as a transition probability the function :

$$p(x, y) = \begin{cases} \frac{1}{\deg x} & \text{if } x, y \in V_{n-1}(X) \text{ and there exists } z \in V_n(X) \text{ such that } xz, zy \in E(X), \\ 0 & \text{unless,} \end{cases}$$

where $\deg x$ is the degree of $x$ and $xz$ is an edge (geodesic segment) connecting $x$ to $z$.

Thus the latest random walk is a discreet "jump" process on the set $V_{n-1}(X)$.

5.2 Brownian motion in an admissible complex with nonpositive curvature and with dimension at the most 2.

This subsection is devoted to the study of the transient or recurrent behavior of the Brownian motion in an admissible complex with nonpositive curvature (in the sense of Alexandrov) and with dimension at the most 2.

Now recall the definition of recurrent/transient process:

**Definition 5.2.**

Let $\{X_t^p\}_t$ denote a stochastic process in a metric space $K$. Then $\{X_t^p\}_t$ is said to be recurrent if, for every ball $B_p$ containing the point $p$, the process $\{X_t^p\}_t$ returns to the ball $B_p$ (and so infinitely) with probability equal to one ; in other words, the process is transient.
Remark 5.3.
When the space $K$ is a discreet space, we consider the point $p$ instead of the ball $B_p$ in the above definition.

**Theorem 5.4.**
Let $K$ denote a 2-dimensional (respectively 1-dimensional) non-compact simply connected admissible Riemannian complex with nonpositive curvature. Then, if for every 1-simplex (respectively a vertex) there is at least three 2-simplices (respectively 1-simplices) adjacent to it, the Brownian motion is transient.

Before proceeding to look at the proof of Theorem 5.4, let us first give a short treatise on simple random walk on a graph.

Let $X = (V(X), E(X))$ denote a connected locally finite graph (a 1-dimensional admissible Riemannian complex), where $V(X)$ is the set of vertexes and $E(X)$ is the set of edges. By simple random walk on the graph $X$, we mean the Markov chain for which the transition probability $p(x, y)$ from vertex $x$ to vertex $y$ is given by the function:

$$p(x, y) = \begin{cases} \frac{1}{\deg x}, & \text{if } xy \in E(X), \\ 0, & \text{unless,} \end{cases}$$

where $xy$ is an edge connecting $x$ to $y$.

We say that $X$ is recurrent (respectively transient) if the simple random walk is recurrent (respectively transient).

The word metric on the graph $X$ is an intrinsic metric in which each edge has unit length.

**Remark 5.5 [13, Ch 6].**
Let $X$ denote a connected locally finite graph with uncountably many ends. Assuming that every vertex has degree greater or equal to three, then $X$ is transient.

**Proof of Theorem 5.4.**
Let $K$ be an admissible complex and let $X$ denote the dual graph of $K$. Now in the following, we will construct a graph $Y$ from the graph $X$.

Let $x_1$ be a vertex belonging to the set $V_1(X)$ and $z_1 \in V_2(X)$ such that $x_1z_1 \in E(X)$. Recall that the degree of $z_1$ is equal to three. We delete an edge adjacent to $z_1$, different than $x_1z_1$. We do the same thing with the other faces adjacent to $x_1$.

Now go back to $z_1$, it is connected to another vertex $x_2 \in V_1(X)$ ($x_1$, $z_1$ and $x_2$ are all in the same 2-simplex). We do the same thing with $x_2$ as we have done with $x_1$. At the end of this construction, forgetting the vertexes of degree equal to two, and as a consequence of the hypothesis on the complex $K$, we get a graph $Y$ isometrically equivalent to connected locally finite graph with uncountably many ends and whose each vertex degree is greater or equal to three. Moreover, the random walk coming from the isotropic choice of maximal faces by the Brownian motion induces a simple random walk on the graph $Y$.

Just now, suppose that the $K$-valued Brownian motion $\{B_t^p\}_{t \geq 0}$ is recurrent. We can suppose that the point $p$ is in the interior of an edge. Take as compact neighborhood of the point $p$ the union of all its adjacent 2-simplices and note this neighborhood $B_p$.

Thus, if $\{B_t^p\}_{t \geq 0}$ returns to the ball $B_p$ with probability equal to one, then inevitably, the simple random walk on $Y$ returns to the point $p$ with probability 1.
In other words, the graph \( Y \) is recurrent which contradicts Remark 5.5, and so the theorem is now proven. \( \square \)

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