A NOTE ON QUASILINEAR SCHRÖDINGER EQUATIONS
WITH SINGULAR OR VANISHING RADIAL POTENTIALS

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Abstract. In this note we complete the study of [3], where we got existence results for the quasilinear elliptic equation
\[-\Delta w + V(|x|) w − w (\Delta w^2) = K(|x|)g(w) \quad \text{in} \ \mathbb{R}^N,\]
with singular or vanishing continuous radial potentials \(V(r), K(r)\). In [3] we assumed, for technical reasons, that \(K(r)\) was vanishing as \(r \to 0\), while in the present paper we remove this obstruction. To face the problem we apply a suitable change of variables \(w = f(u)\) and we find existence of non negative solutions by the application of variational methods. Our solutions satisfy a weak formulations of the above equation, but they are in fact classical solutions in \(\mathbb{R}^N \setminus \{0\}\). The nonlinearity \(g\) has a double-power behavior, whose standard example is
\[g(t) = \min\{t^{q_1-1}, t^{q_2-1}\} \quad (t > 0),\]
recovering the usual case of a single-power behavior when \(q_1 = q_2\).

1. Introduction

In this paper we complete the study of [3], where we got existence results for the quasilinear elliptic equation
\[-\Delta w + V(|x|) w − w (\Delta w^2) = K(|x|)g(w) \quad \text{in} \ \mathbb{R}^N.\] (1.1)

Such an equation arises in the search of standing waves for an evolution Schrödinger equation which has been used to study several physical phenomena (see [14, 17, 12] and the references therein), such as laser beams in

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matter [5] and quasi-solitons in superfluids films [11]. In recent times a great amount of work has been made on equation (1.1) to overcome the non-trivial technical difficulties associated with it (see [1, 6, 7, 8, 9, 10, 15, 16, 18, 19] and the references therein). In [3] and in the present paper we follow an idea introduced in [14] and then used by several authors, which exploits a suitable change of variable \( w = f(u) \): the problem in the new unknown \( u \) can be then faced with usual variational methods, by working in an Orlicz-Sobolev space.

In almost all the papers dealing with (1.1), the potential \( V \) (be it radial or nonradial) is supposed to be positive and not vanishing at infinity. At the best of our knowledge, the only papers dealing with a potential \( V \) allowed to vanish at infinity are [1, 7, 12, 13]. In [1] and [12] the authors assume that \( V \) is bounded, while in [13] existence of solutions is proved for possibly singular \( V \)'s but bounded \( K \)'s. In [7], which is the paper that inspired our work, both \( V \) and \( K \) can be singular or vanishing at zero or at infinity, but the authors assume that they are radial and essentially behave as powers of \( |x| \) as \( |x| \to 0 \) and \( |x| \to \infty \). In [3], instead, we have studied the case in which both \( V \) and \( K \) are radial potentials that can be singular or vanishing at zero or at infinity, and do not need to exhibit a powel-like behavior. The main novelties of our approach are the application of compact embeddings into the sum Lebesgue space \( L^{q_1}_K + L^{q_2}_K \) (see Section 2 below) and the use of assumptions on the potentials ratio \( K/V \), not on the two potentials separately. Nevertheless, for technical reasons, in almost all cases we had to assume that \( K \) was vanishing at the origin. In the present paper we remove this obstruction.

This note is organized as follows. In Section 2 we introduce our hypotheses on \( V \) and \( K \), the change of variables \( w = f(u) \) and the function space \( E \) in which we will work. Then we state a general result concerning the compactness of the embedding of \( E \) into \( L^{q_1}_K + L^{q_2}_K \) (Theorem 2.3), and we give some explicit conditions ensuring that the embedding is compact (Theorems 2.4 and 2.5). In Section 3 we introduce our hypotheses on the nonlinearity \( g \) and we study the main properties of the functional \( I \) associated to the dual problem, and in particular of its critical points, which give rise to the solutions of (1.1). In Section 4 we apply our embedding results to get existence of nonnegative solutions to (1.1), stating and proving the main existence result of the paper, which is Theorem 4.1. In section 5, we give concrete examples of potentials \( V, K \) satisfying our hypotheses and non included in the previous literature.
Notations. We end this introductory section by collecting some notations used in the paper.

- $\mathbb{R}_+ = (0, +\infty) = \{ x \in \mathbb{R} : x > 0 \}$.
- For every $R > 0$, we set $B_R = \{ x \in \mathbb{R}^N : |x| < r \}$.
- $C^\infty_c(\mathbb{R}^N)$ is the space of the infinitely differentiable real functions with compact support. $C^\infty_c(\mathbb{R}^N)$ is the subspace of $C^\infty_c(\mathbb{R}^N)$ made of radial functions.
- For any measurable set $A \subseteq \mathbb{R}^N$, $L^q(A)$ and $L^q_{\text{loc}}(A)$ are the usual real Lebesgue spaces. If $\rho : A \rightarrow \mathbb{R}_+$ is a measurable function, then $L^p(A, \rho(z) \, dz)$ is the real Lebesgue space with respect to the measure $\rho(z) \, dz$ ($dz$ stands for the Lebesgue measure on $\mathbb{R}^N$). In particular, if $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is measurable, we denote $L^q_K(A) := L^q(A, K(|x|) \, dx)$.
- If $Y$ is a Banach space, $Y'$ is its dual.

2. Hypotheses and Preliminary Results

Throughout this paper we will assume $N \geq 3$ and the following hypotheses on $V, K$:

\begin{itemize}
  \item[(H)] $V : \mathbb{R}_+ \rightarrow [0, +\infty)$ and $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous, and there exists $C > 0$ such that for all $r \in (0, 1)$ one has $V(r) \leq \frac{C}{r^2}$.
\end{itemize}

As a first thing we introduce the function that we need to define the Orlicz-Sobolev space in which we will work. Such a function is defined as the solution $f$ of the following Cauchy problem:

\begin{equation}
\begin{aligned}
  f'(t) &= \frac{1}{\sqrt{1+2f^2(t)}} & \text{in } \mathbb{R} \\
  f(0) &= 0
\end{aligned}
\end{equation}

It is easy to check that this problem has a unique solution $f \in C^\infty(\mathbb{R}^N, \mathbb{R})$, which is odd, strictly increasing, and surjective (whence invertible). Other important properties of $f$ are listed in Lemma 2.1 of [3]. We use the function $f$ to define a suitable change of unknown, which is the following: we call $w$ the solution of (1.1) we are looking for and we set $w = f(u)$, where $u$ is the new unknown, living in a suitable space that we are going to define. To get solutions $w$ to (1.1) we will look for solutions $u$ to the following equation:

\begin{equation}
- \Delta u + V(|x|) f(u) f'(u) + K(|x|) g(f(u)) f'(u) = 0 \quad \text{in } \mathbb{R}^N,
\end{equation}
which will be obtained as critical points of the following functional:

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(|x|) f^2(u) dx - \int_{\mathbb{R}^N} K(|x|) G(f(u)) dx. \tag{2.3}
\]

The critical points of \(I\) and their relations with solutions of (1.1) will be studied in the next section. In the rest of the present section we introduce the function space \(E\) in which we will obtain the critical points of \(I\), and we study the relevant compactness results for \(E\). All the results of this section have been proven in [7] and [3].

To this aim, we introduce the space \(D^{1,2}_r(\mathbb{R}^N)\) as the closure of \(C^\infty_c(\mathbb{R}^N)\) with respect to the norm \(\|u\|_{1,2} = \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{1/2}\). It is well known that \(D^{1,2}_r(\mathbb{R}^N)\) is a Hilbert space. Then we define the space \(E\) as follows:

\[
E = \left\{ u \in D^{1,2}_r(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(|x|) f^2(u) dx < +\infty \right\}.
\]

In \(E\) we introduce the Orlicz norm

\[
\|u\|_o = \inf_{k>0} \frac{1}{k} \left[ 1 + \int_{\mathbb{R}^N} V(|x|) f^2(ku) dx \right],
\]

and then we set

\[
\|u\| = \|u\|_{1,2} + \|u\|_o.
\]

The space \(E\) endowed with the norm \(\|\cdot\|\) is an Orlicz-Sobolev space. In the following theorems we recall some of its properties. For the proofs, and other relevant properties of \(E\), see [3].

**Theorem 2.1.** \((E, \|\cdot\|)\) is a Banach space and the following continuous embedding holds true:

\[
E \hookrightarrow D^{1,2}_r(\mathbb{R}^N).
\]

**Lemma 2.2.** If \(u_n(x) \to u(x)\) a.e. in \(\mathbb{R}^N\) and

\[
\int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx \to \int_{\mathbb{R}^N} V(|x|) f^2(u) dx,
\]

then \(\|u_n - u\|_o \to 0\).

We now state the main compactness results concerning the space \(E\), which have been proven in [3]. They concern the embedding properties of \(E\) into the sum space

\[
L^{q_1}_K + L^{q_2}_K := \{ u_1 + u_2 : u_1 \in L^{q_1}_K(\mathbb{R}^N), u_2 \in L^{q_2}_K(\mathbb{R}^N) \}, \quad 1 < q_i < \infty.
\]
We recall from [4] that such a space can be characterized as the set of measurable mappings $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $A \subseteq \mathbb{R}^N$ such that $u \in L^{q_1}_K (A) \cap L^{q_2}_K (A^c)$. It is a Banach space with respect to the norm

$$\|u\|_{L^{q_1}_K + L^{q_2}_K} := \inf_{u_1 + u_2 = u} \max \left\{ \|u_1\|_{L^{q_1}_K (\mathbb{R}^N)} , \|u_2\|_{L^{q_2}_K (\mathbb{R}^N)} \right\}$$

and the continuous embedding $L^q_K \hookrightarrow L^{q_1}_K + L^{q_2}_K$ holds true for all $q \in \left[ \min \{ q_1 , q_2 \} , \max \{ q_1 , q_2 \} \right]$. The first compactness result is Theorem 2.3 below. Its assumptions are rather general but not so easy to check, so more handy conditions ensuring such assumptions will be provided in the next Theorems 2.4 and 2.5. To state the results, we need to preliminarily introduce the following functions of $R > 0$ and $q > 1$:

$$S_0 (q, R) := \sup_{u \in E , \|u\|=1} \int_{B_R} K (|x|) |u|^q \, dx, \quad (2.4)$$

$$S_\infty (q, R) := \sup_{u \in E , \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K (|x|) |u|^q \, dx. \quad (2.5)$$

Clearly $S_0 (q, \cdot)$ is nondecreasing, $S_\infty (q, \cdot)$ is nonincreasing and both of them can be infinite at some $R$.

**Theorem 2.3.** Let $N \geq 3$, let $V$ and $K$ be as in (H) and let $q_1 , q_2 > 1$. If

$$\lim_{R \to 0^+} S_0 (q_1 , R) = \lim_{R \to +\infty} S_\infty (q_2 , R) = 0, \quad (2.6)$$

then $E$ is compactly embedded into $L^{q_1}_K (\mathbb{R}^N) + L^{q_2}_K (\mathbb{R}^N)$.

We notice that assumption (2.6) can hold with $q_1 = q_2 = q$ and therefore Theorem 2.3 also concerns the compact embedding properties of $E$ into $L^q_K$, $1 < q < \infty$.

We now look for explicit conditions on $V$ and $K$ implying (2.6) for some $q_1$ and $q_2$. More precisely, in Theorem 2.4 we will find a range of exponents $q_1$ such that $\lim_{R \to 0^+} S_0 (q_1 , R) = 0$, while in Theorem 2.5 we will do the same for exponents $q_2$ such that $\lim_{R \to +\infty} S_\infty (q_2 , R) = 0$.

For $\alpha \in \mathbb{R}$, $\beta \in [0, 1]$, we define two functions $q_0^* (\alpha, \beta)$ and $q_\infty^* (\alpha, \beta)$ by setting

$$q_0^* (\alpha, \beta) := \frac{2\alpha + 2N - \beta(N + 2)}{N - 2} , \quad q_\infty^* (\alpha, \beta) := 2 \frac{\alpha + N - 2\beta}{N - 2}.$$
Theorem 2.4. Let $V$, $K$ be as in (H). Assume that there exists $R_1 > 0$ such that
\[
\sup_{r \in (0, R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty \quad \text{for some } 0 \leq \beta_0 \leq 1 \text{ and } \alpha_0 \in \mathbb{R}.
\] (2.7)
Assume also that
\[
\max \{1, 2 \beta_0\} < q_0^* (\alpha_0, \beta_0).
\] Then
\[
\lim_{R \to 0^+} S_0 (q_1, R) = 0 \quad \text{for every } q_1 \in \mathbb{R} \text{ such that }
\max \{1, 2 \beta_0\} < q_1 < q_0^* (\alpha_0, \beta_0).
\] (2.8)

Theorem 2.5. Let $V$, $K$ be as in (H). Assume that there exists $R_2 > 0$ such that
\[
\sup_{r > R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \leq \beta_\infty \leq 1 \text{ and } \alpha_\infty \in \mathbb{R}.
\] (2.9)
Then
\[
\lim_{R \to +\infty} S_\infty (q_2, R) = 0 \quad \text{for every } q_2 \in \mathbb{R} \text{ such that }
q_2 > \max \{1, 2 \beta_\infty, q_\infty^* (\alpha_\infty, \beta_\infty)\}.
\] (2.10)

Remark 2.6. We mean $V(r)^0 = 1$ for every $r$ (even if $V(r) = 0$). In particular, if $V(r) = 0$ for $r > R_2$, then Theorem 2.5 can be applied with $\beta_\infty = 0$ and assumption (2.9) means
\[
\lim_{r \to +\infty} \frac{K(r)}{r^{\alpha_\infty}} < +\infty \quad \text{for some } \alpha_\infty \in \mathbb{R}.
\] Similarly for Theorem 2.4 and assumption (2.7), if $V(r) = 0$ for $r \in (0, R_1)$.

3. The Functional I

In this section we study the functional $I$, defined in (2.3), whose critical points will give rise to solutions to (2.2), and thus to (1.1). To this aim, we need a set of hypotheses on the nonlinearity $g$.

First of all we make the following assumptions, to be maintained from now on:

(h1) we assume both hypotheses (2.7) and (2.9) of Theorems 2.4 and 2.5;
(h2) we let $q_1, q_2 \in \mathbb{R}$ be such that $4 < q_1 < 2 q_0^* (\alpha_0, \beta_0)$ and $q_2 > \max \{4, 2 q_\infty^* (\alpha_\infty, \beta_\infty)\}$.

Then we introduce the conditions we will use about the continuous $g$, meaning $G(t) = \int_0^t g(s) \, ds$:

(g1) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function;
We notice that these hypotheses imply that $q_1, q_2 \geq 2\theta$, and that there exists $C > 0$ such that the following estimate holds for all $t \in \mathbb{R}$:

\[
|G(t)| \leq C \min \{|t|^{q_1-1}, |t|^{q_2-1}\}.
\]

We also observe that, if $q_1 \neq q_2$, the double-power growth condition $(g_{q_1, q_2})$ is more stringent than the more usual single-power one, since it implies $|g(t)| \leq C|t|^{q-1}$ for $q = q_1, q = q_2$ and every $q$ in between. On the other hand, we will never require $q_1 \neq q_2$ in $(g_{q_1, q_2})$, so that our results will also concern single-power nonlinearities as long as we can take $q_1 = q_2$.

We begin with the following lemma.

**Lemma 3.1.** Assume $(h_1), (h_2)$. Then $E$ is compactly embedded into $L^{q_1/2}(\mathbb{R}^N) + L^{q_2/2}(\mathbb{R}^N)$.

**Proof.** The hypothesis $(h_2)$ easily gives that $q_1/2$ and $q_2/2$ satisfy (2.8) and (2.10). Together with $(h_1)$, this implies that the hypotheses of Theorems 2.4 and 2.5 are satisfied. Together with Theorem 2.3, this gives the result. □

We now state the main properties of the functional $I : E \to \mathbb{R}$ defined by (2.3).

**Theorem 3.2.** Let $N \geq 3$ and assume $(H), (h_1), (h_2)$. Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies $(g_1), (g_2), (g_3)$ and $(g_{q_1, q_2})$. Then

1. $I$ is well defined and continuous in $E$;
2. $I$ is a $C^1$ map on $E$ and for all $u \in E$ its differential $I'(u)$ is given by

\[
I'(u)h = \int_{\mathbb{R}^N} \nabla u \nabla h dx + \int_{\mathbb{R}^N} V(|x|) f(u) f'(u) h dx
\]

\[
- \int_{\mathbb{R}^N} K(|x|) g(f(u)) f'(u) h dx
\]

for all $h \in E$.

**Proof.** Define

\[
I_1(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad I_2(u) = \frac{1}{2} \int_{\mathbb{R}^N} V(|x|) f^2(u) dx,
\]

\[
I_3(u) = \int_{\mathbb{R}^N} K(|x|) G(f(u)) dx,
\]

and
where $G(t) = \int_0^t g(s) ds$. We study these three functionals separately.

As to $I_1$, it is a standard task to get that $I_1$ is $C^1$ on $E$ with differential given by $I_1'(u)h = \int_{\mathbb{R}^N} \nabla u \nabla h dx$.

As to $I_3$, we notice that, setting $h(x,t) = K(|x|)G(f(t))$, we have $h(x,t) = \int_0^t K(|x|)g(f(s))f'(s)ds$. Exploiting the properties of $f$ (see Lemma 2.1 in [3]), we have that

\[ |g(f(t))| |f'(t)| \leq C|f(t)|^{q_1-1}|t|^{-1} |t f'(t)| \leq C|t|^{q_1/2-1}. \]

and therefore

\[ |K(|x|)g(f(t))f'(t)| \leq CK(|x|) \min \left\{|t|^{q_1/2-1}, |t|^{q_2/2-1}\right\}. \]

Then we can apply the results of [4] (in particular Proposition 3.8) and the fact that $E \hookrightarrow L^{q_1}_{R} + L^{q_2}_{R}$ (see Theorem 2.3), to get that also $I_3$ is $C^1$ on $E$, with differential given by

\[ I_3'(u)h = \int_{\mathbb{R}^N} K(|x|)g(f(u))f'(u)h dx. \]

As to $I_2$, we can repeat here the arguments of [7] or [3], which also work under our hypotheses, to get that $I_2$ is well defined, continuous and Gateaux differentiable on $E$, with differential given by

\[ I_2'(u)h = \int_{\mathbb{R}^N} V(|x|)f(u)f'(u)h dx. \]

We now give the main properties of the critical points of $I$, under the hypotheses of Theorem 3.2. In particular, we will point out the relation between equation (2.2) and the original equation (1.1). Clearly a critical point $u$ of $I$ satisfies $I'(u)h = 0$, i.e.

\[ \int_{\mathbb{R}^N} \nabla u \nabla h dx + \int_{\mathbb{R}^N} V(|x|)f(u)f'(u)h dx - \int_{\mathbb{R}^N} K(|x|)g(f(u))f'(u)h dx = 0 \]

for all $h \in E$, which is, of course, a weak formulation of equation (2.2). The other relevant properties of the critical points of $I$ are summarized by the following theorem, which we proved in [3].

**Theorem 3.3.** Assume the hypotheses of Theorem 3.2. Let $u \in E$ be a critical point of $I$ and set $w = f(u)$. Then

1. $u \in C^2(\mathbb{R}^N \setminus \{0\})$ and $u$ is a classical solution of equation (2.2) in

\[ \mathbb{R}^N \setminus \{0\}, \]
(2) \( w \in C^2(\mathbb{R}^N \setminus \{0\}) \) and \( w \) is a classical solution of equation (1.1) in \( \mathbb{R}^N \setminus \{0\} \);
(3) \( w \in X \) and \( w \) satisfies the following weak formulation of (1.1):
\[
\int_{\mathbb{R}^N} (1 + 2w^2) \nabla w \cdot \nabla h \, dx + \int_{\mathbb{R}^N} 2w|\nabla w|^2 h \, dx + \int_{\mathbb{R}^N} V(|x|) wh \, dx = \int_{\mathbb{R}^N} K(|x|)g(w)h \, dx
\]
(3.4)
for all \( h \in C^\infty_c(\mathbb{R}^N) \).

4. Existence of solutions

Our main existence result is the following.

Theorem 4.1. Assume the hypotheses of Theorem 3.2. Then the functional \( I : E \to \mathbb{R} \) has a nonnegative critical point \( u \neq 0 \).

Remark 4.2. In Theorem 4.1, as we look for nonnegative solutions, we may assume \( g(t) = 0 \) for all \( t \leq 0 \). Indeed, if we have a nonlinearity \( g \) satisfying the hypotheses of the theorem, we can replace \( g(t) \) with \( \chi_{\mathbb{R}_+}(t)g(t) \) (\( \chi_{\mathbb{R}_+} \) is the characteristic function of \( \mathbb{R}_+ \)) and the new nonlinearity still satisfies the hypotheses.

Remark 4.3. As concerns examples of nonlinearities satisfying the hypotheses of Theorem 4.1, the simplest \( g \in C(\mathbb{R}; \mathbb{R}) \) such that \( (g_{q_1,q_2}) \) holds is
\[
g(t) = \min \left\{ |t|^{q_1-2}t, |t|^{q_2-2}t \right\},
\]
which also ensures \( (g_2) \) if \( q_1, q_2 > 4 \) (with \( \theta = \min \left\{ \frac{q_1}{2}, \frac{q_2}{2} \right\} \)). Another model example is
\[
g(t) = \frac{|t|^{q_2-2}t}{1 + |t|^{q_2-q_1}} \quad \text{with} \quad 1 < q_1 \leq q_2,
\]
which ensures \( (g_2) \) if \( q_1 > 4 \) (with \( \theta = \frac{q_1}{2} \)). Note that, in both these cases, also \( (g_3) \) holds true. Moreover, both of these functions \( g \) become \( g(t) = |t|^{q_2-2}t \) if \( q_1 = q_2 = q \).

In order to prove Theorem 4.1, we first recall the Palais-Smale condition and a version of the well-known Mountain-Pass Lemma (see [2, Chapter 2]).

Definition 4.4. Let \( Y \) be a Banach space and \( \Phi : Y \to \mathbb{R} \) a \( C^1 \) functional. We say that \( \Phi \) satisfies the Palais-Smale condition if for any sequence \( \{x_n\}_n \) such that \( \Phi(x_n) \) is bounded in \( \mathbb{R} \) and \( \Phi'(x_n) \to 0 \) in \( Y' \), there exists a subsequence \( \{x_{n_k}\}_k \) converging in \( Y \).
Theorem 4.5. (Mountain Pass Lemma)

Let $Y$ be a Banach space and $\Phi : Y \to \mathbb{R}$ a $C^1$ functional such that $\Phi(0) = 0$.
Assume that $\Phi$ satisfies the Palais-Smale condition and that there exist a subset $S \subseteq Y$ and $\alpha > 0$ such that:

(i) $Y \setminus S$ is not arcwise connected;
(ii) $\Phi(x) \geq \alpha$ for all $x \in S$;
(iii) there exists $y \in Y \setminus (C_0 \cup S)$ such that $\Phi(y) < 0$, where $C_0$ is the
connected component of $Y \setminus S$ such that $0 \in C_0$.

Then $\Phi$ has a critical point $u \in Y$ such that $\Phi(u) \geq \alpha$.

We will prove Theorem 4.1 by showing that the functional $I : E \to \mathbb{R}$ satisfies the hypotheses of the Mountain Pass Lemma. Obviously $I(0) = 0$. The proof of the other hypotheses is the subject of Lemmas 4.6 and 4.7. Recall the three functionals $I_1, I_2, I_3$ introduced in the proof of Lemma 3.2, and for $u \in E$ define $J(u) = I_1(u) + I_2(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(|x|) f^2(u) \, dx$.

Moreover, for any $\rho > 0$ set

$$S_\rho = \{ u \in E \mid J(u) = \rho \}.$$

Lemma 4.6. Assume the hypotheses of Theorem 4.1. Then:

(1) for every $\rho > 0$ the set $E \setminus S_\rho$ is not arcwise connected;
(2) there exist $\rho, \alpha > 0$ and $v \in E$ such that $J(v) > \rho$, $I(v) < 0$ and
$J(u) \geq \alpha$ for all $u \in S_\rho$;
(3) if $\{u_n\}_n \subseteq E$ is a Palais-Smale sequence for $I$, i.e., a sequence such that $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ in $E'$, then $\{u_n\}_n$ in bounded in $E$.

Proof. See [3].

The following lemma shows that $I$ satisfies the Palais-Smale condition. This relies on the compact embedding of $E$ into $L_{p/2}^{q/2}(\mathbb{R}^N) + L_{p/2}^{q/2}(\mathbb{R}^N)$, which is one of the main devices in our proof of Theorem 4.1 and holds true by Lemma 3.1.

Lemma 4.7. Under the assumptions of Theorem 4.1, the functional $I : E \to \mathbb{R}$ satisfies the Palais-Smale condition.

Proof. Let $\{u_n\}_n$ be a sequence in $E$ such that $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ in $E'$. From the previous lemma we get that $\{u_n\}_n$ is bounded in $E$, so that there is a subsequence, which we still call $\{u_n\}_n$, such that $u_n \rightharpoonup u$ in $E$ and in $D_1^{1,2}(\mathbb{R}^N)$, and $u_n(x) \to u(x)$ a.e.x. Recall that we have defined $J = I_1 + I_2$, so that $I = J - I_3$. We know that $I_3$ is $C^1$ as
a functional from $L^{q_1/2}_K + L^{q_2/2}_K$ to $\mathbb{R}$. By compactness of the embedding of $E$ into $L^{q_1/2}_K + L^{q_2/2}_K$, we have that $u_n \to u$ in $L^{q_1/2}_K + L^{q_2/2}_K$. Hence $I'_3(u_n) \to I'_3(u)$ in the dual space of $L^{q_1/2}_K + L^{q_2/2}_K$ and $I'_3(u_n)(u - u_n) \to 0$ in $\mathbb{R}$. We now notice that, as $f^2$ is a convex function (see Lemma 2.1 in [3]), $J$ is a convex functional on $E$. Hence

$$J(u) - J(u_n) \geq J'(u_n)(u - u_n) + I'_3(u_n)(u - u_n).$$

Since $I'(u_n) \to 0$ in $E'$ and $\{u - u_n\}$ is bounded in $E$, we have that $I'(u_n)(u - u_n) \to 0$. So

$$J(u) \geq J(u_n) + o(1).$$

Taking the $\liminf_n$, this gives

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(|x|) f^2(u) dx$$

$$\geq \liminf_n \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx \right)$$

$$\geq \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \liminf_n \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx.$$

By semicontinuity of the norm, one has

$$\liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

and hence (4.1) gives

$$\int_{\mathbb{R}^N} V(|x|) f^2(u) dx \geq \liminf_n \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx.$$

Fatou’s Lemma obviously implies

$$\int_{\mathbb{R}^N} V(|x|) f^2(u) dx \leq \liminf_n \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx,$$

whence

$$\int_{\mathbb{R}^N} V(|x|) f^2(u) dx = \liminf_n \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx.$$  (4.2)

Passing to a subsequence, which we still label $\{u_n\}_n$, we can assume

$$\int_{\mathbb{R}^N} V(|x|) f^2(u) dx = \liminf_n \int_{\mathbb{R}^N} V(|x|) f^2(u_n) dx.$$  (4.3)
By Lemma 2.2, we infer that $\|u - u_n\|_o \rightarrow 0$. Repeating the previous arguments for this subsequence, we get again (4.1), which now gives

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \lim \inf \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

and hence

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \lim \inf \int_{\mathbb{R}^N} |\nabla u_n|^2 dx.$$

Passing again to a subsequence if necessary, we may assume

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \lim \int_{\mathbb{R}^N} |\nabla u_n|^2 dx.$$

As $u_n \rightharpoonup u$ in $D^{1,2}\left(\mathbb{R}^N\right)$, we conclude that $u_n \rightarrow u$ in $D^{1,2}\left(\mathbb{R}^N\right)$, that is $\|u - u_n\|_{1,2} \rightarrow 0$, and therefore $\|u - u_n\| = \|u - u_n\|_o + \|u - u_n\|_{1,2} \rightarrow 0$. □

We can now easily conclude the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Taking $\rho$ and $v$ as in (2) of Lemma 4.6, we have $0 = J(0) < \rho < J(v)$, so that $v$ and 0 are in two distinct connected components of $E_1 \setminus S_\rho$. Together with Lemma 4.6, this shows that all the hypotheses of the Mountain Pass Lemma 4.5 are satisfied, so that we get a critical point $u \neq 0$ of $I$. As $g(t) = 0$ for $t < 0$ (cf. Remark 4.2), it is a standard task to get that the negative part $u^-$ of $u$ is zero (see for example the proof of Theorem 7.1 in [3]), that is, $u$ is nonnegative. □

5. Examples

In this section we give some examples of applications of our existence results. In all the analogous applications (except one) contained in our previous paper [3], we had to assume that $K$ vanishes as $r \rightarrow 0$. Therefore, here we limit ourselves to give examples in which this does not happen. We also point out that all our examples do not satisfy the hypotheses of the results of [7], which is the other main reference about the problem under consideration.

As usual, we let $N \geq 3$ and assume hypothesis (H). Just for simplicity, in all the examples we shall consider the model nonlinearity $g(t) = \min\{t^{q-1}, t^{q-1}\}$.

**Example 5.1.** This first example is a very simple one: set $K(r) = 1$ and $V(r) = 1/r^2$ for all $r > 0$. Computing the coefficients of [7], we find $a_0 = -2$ and $b_0 = 0$, so that the results in [7] cannot be applied, because they need
\(a_0 \geq b_0\) (see [7, Theorem 1.1]). Instead, if we set \(\alpha_0 = \alpha_\infty = \beta_0 = \beta_\infty = 0\), we get

\[
q^*_0(\alpha_0, \beta_0) = q^*_\infty(\alpha_\infty, \beta_\infty) = \frac{2N}{N-2}
\]

and thus we get existence results for nonlinearities \(g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}\) with \(4 < q_1 < \frac{2N}{N-2} < q_2\).

**Example 5.2.** Assume that

\[
K(r) = \frac{1}{r^\gamma} \quad \text{for } r \in (0, 1), \quad K(r) = \frac{1}{r^\delta} \quad \text{for } r > 1,
\]

where \(0 < \delta < \gamma < 2\), and suppose that \(V\) just satisfies (H), without any further assumption on its asymptotic behavior as \(r \to +\infty\).

Computing the coefficients \(b, b_0\) in [7], one gets \(b \geq -\delta\) and \(b_0 = -\gamma\), whence \(b > b_0\). As a consequence, the results of [7] cannot be applied, because they need \(b_0 \geq b\). If we set \(\beta_0 = \beta_\infty = 0, \alpha_0 = -\gamma, \alpha_\infty = -\delta\), we get

\[
q^*_0(\alpha_0, \beta_0) = \frac{2N - \gamma}{N-2}, \quad q^*_\infty(\alpha_\infty, \beta_\infty) = \frac{2N - \delta}{N-2},
\]

so that, by the hypotheses on \(\delta\) and \(\gamma\),

\[
2 < q^*_0(\alpha_0, \beta_0) < q^*_\infty(\alpha_\infty, \beta_\infty).
\]

Hence we can apply our existence result to nonlinearities of the form \(g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}\) with

\[
4 < q_1 < \frac{2N}{N-2} < 4 \frac{N - \gamma}{N-2} < q_2.
\]

**Example 5.3.** Assume that there exists \(\gamma > 1\) such that

\[
K(r) = \frac{1}{r^\gamma} \quad \text{as } r \to 0^+, \quad K(r) = \frac{1}{r^\gamma} \quad \text{as } r \to +\infty
\]

and, as in the previous example, suppose that \(V\) just satisfies (H), without further assumptions on its behavior at \(+\infty\).

Computing the coefficients \(a_0, b_0\) of [7], one gets \(a_0 = -2\) and \(b_0 = -1\), so that [7, Theorem 1.1] cannot be applied, because it needs \(a_0 \geq b_0\). If we set \(\beta_0 = \beta_\infty = 0, \alpha_0 = -1\) and \(\alpha_\infty = -\gamma\), we get

\[
q^*_0(\alpha_0, \beta_0) = \frac{-2 + 2N}{N-2} > 2, \quad q^*_\infty(\alpha_\infty, \beta_\infty) = \frac{2N - \gamma}{N-2} < q^*_0(\alpha_0, \beta_0).
\]
So we can apply our existence result to nonlinearities $g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ with
\[4 < q_1 < 4 \frac{N - 1}{N - 2}, \quad 4 \frac{N - \gamma}{N - 2} < q_2, \quad q_1 \leq q_2.\]
In particular, setting $\bar{q} = \max\left\{4, 4 \frac{N - \gamma}{N - 2}\right\}$, we can choose $q_1 = q_2 = q \in \left(\bar{q}, 4 \frac{N - \gamma}{N - 2}\right)$ and we get an existence result for a standard power-type nonlinearity $g(t) = t^{q-1}$. If $\gamma \geq 2$, we have $\bar{q} = 4$ and such a result holds for $q \in \left(4, 4 \frac{N - \gamma}{N - 2}\right)$.

**Example 5.4.** Assume that
\[K(r) = \frac{1}{r^{\gamma}} \quad \text{for } r \in (0, 1), \quad K(r) = \frac{1}{r^{\delta}} \quad \text{for } r > 1,\]
with $0 < \delta$ and $0 < \gamma < 2$, and that
\[V(r) = \frac{1}{r^2} \quad \text{for } r \in (0, 1), \quad V(r) = e^{-r} \quad \text{as } r \to +\infty.\]
The results of [7] cannot be applied, because they need a power-type decay of $V(r)$ at infinity. Instead, we can apply our results exactly as in Example 5.2, setting $\beta_0 = \beta_\infty = 0$, $\alpha_0 = -\gamma$ and $\alpha_\infty = -\delta$. Then we get existence results for nonlinearities $g(t) = \min\{t^{q_1-1}, t^{q_2-1}\}$ with
\[4 < q_1 < 4 \frac{N - \gamma}{N - 2}, \quad 4 \frac{N - \delta}{N - 2} < q_2, \quad q_1 \leq q_2.\]
If $\delta > 2$, we can pick $q_1 = q_2 = q \in \left(4, 4 \frac{N - \gamma}{N - 2}\right)$ and such results thus concern nonlinearities $g(t) = t^{q-1}$.

**Example 5.5.** Assume that
\[K(r) = \frac{e^r}{r} \quad \text{for all } r > 0,\]
and
\[V(r) = \frac{1}{r^2} \quad \text{for } r \in (0, 1), \quad V(r) = e^r \quad \text{as } r \to +\infty.\]
The results of [7] cannot be applied, because they need a power-type growth of $K(r)$ at infinity. We can apply our results setting $\alpha_0 = \alpha_\infty = -1$, $\beta_0 = 0$ and $\beta_\infty = 1$. Then we get
\[q_0^\ast(\alpha_0, \beta_0) = \frac{-2 + 2N}{N - 2} > 2, \quad q_\infty^\ast(\alpha_\infty, \beta_\infty) = 2 \frac{N - 3}{N - 2}.\]
which gives existence results for nonlinearities \( g(t) = \min\{t^{q_1-1}, t^{q_2-1}\} \) with
\[
4 < q_1 < 4 \frac{N-1}{N-2}, \quad q_2 > 4 \frac{N-3}{N-2}, \quad q_1 \leq q_2.
\]
In particular we can choose \( g(t) = t^{q-1} \) with \( q_1 = q_2 = q \in \left(4, 4 \frac{N-1}{N-2}\right)\).

**Example 5.6.** As a last example, assume
\[
K(r) = |\log r| \quad \text{for } r \to 0^+, \quad K(r) = \frac{1}{r^2} \quad \text{for } r > 1
\]
and suppose that \( V \) satisfies nothing other than (H). The results of [7] cannot be applied, because they need a power-type behavior of \( K(r) \) at zero. On the one hand, we can assume \( \beta_\infty = 0 \) and \( \alpha_\infty = -2 \), which gives \( q^*_\infty(\alpha_\infty, \beta_\infty) = 2 \). On the other hand, fix \( q \in \left(4, 4 \frac{N}{N-2}\right) \) and pick \( \alpha_0 < 0 \) small enough that \( q < 4 \frac{N+\alpha_0}{N-2} \). Together with \( \beta_0 = 0 \), this gives
\[
q^*_0(\alpha_0, \beta_0) = 2 \frac{N + \alpha_0}{N - 2}.
\]
So we obtain existence results for every \( 4 < q_1 < 4 \frac{N+\alpha_0}{N-2} \) and \( q_2 > 4 \). As we can take \( q_1 = q_2 = q \) (where \( q \) is the same we have chosen before), we get an existence result for every nonlinearity \( g(t) = t^{q-1} \) with \( q \in \left(4, 4 \frac{N}{N-2}\right)\).

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