Easy Proofs of Löwenheim-Skolem Theorems by Means of Evaluation Games

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Abstract

We propose a proof of the downward Löwenheim-Skolem that relies on strategies deriving from evaluation games instead of the Skolem normal forms. This proof is simpler, and easily understood by the students, although it requires, when defining the semantics of first-order logic to introduce first a few notions inherited from game theory such as the one of an evaluation game.

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1 Introduction

Each mathematical logic course focuses on first-order logic. Once the basic definitions about syntax and semantics have been introduced and the notion of the cardinality of a model has been exposed, sooner or later at least a couple of hours are dedicated to the Löwenheim-Skolem theorem. This statement holds actually two different results: the downward Löwenheim-Skolem theorem (LS\textsuperscript{↓}) and the upward Löwenheim-Skolem theorem (LS\textsuperscript{↑}).

\begin{itemize}
\item \textbf{Theorem 1} (Downward Löwenheim-Skolem). Let $\mathcal{L}$ be a first-order language, $T$ some $\mathcal{L}$-theory, and $\kappa = \max\{\text{card}(\mathcal{L}), \aleph_0\}$. If $T$ has a model of cardinality $\lambda > \kappa$, then $T$ has a model of cardinality $\kappa$.
\item \textbf{Theorem 2} (Upward Löwenheim-Skolem). Let $\mathcal{L}$ be some first-order language with equality, $T$ some $\mathcal{L}$-theory, and $\kappa = \max\{\text{card}(\mathcal{L}), \aleph_0\}$. If $T$ has an infinite model, then $T$ has a model of cardinality $\lambda$, for any $\lambda > \kappa$.
\end{itemize}

The proof of the second theorem (LS\textsuperscript{↑}) is a simple exercise that relies on an easy application of the compactness theorem joined with a straightforward utilization of LS\textsuperscript{↓}. The proof of the first theorem is more involved and not that easy to understand for undergraduate students at EPFL. For some basic background and notations we refer the reader to \cite{1, 5}.

The usual approach to proving LS\textsuperscript{↓} goes through several steps which involve reducing the original theory $T$ to another theory $T'$ on an extended language $\mathcal{L}'$ where all statements are in Skolem normal form. Then obtaining some $\mathcal{L}'$-structure of the right cardinality that satisfies $T'$, from which one goes back to a model that satisfies $T$.

The burden of going through the Skolem normal forms is regarded as bothersome by the student.
We would like to advocate that the use of evaluation games greatly simplifies the proof of \( \text{LS}_1 \). Of course, this requires to talk about finite two-player perfect games and the related notions of player, strategies, winning strategies, etc. But it is worth the candle, especially if these games are introduced for explaining the semantics of first-order logic. Anyhow, determining whether a first-order formula holds true in a given structure is very similar to solving the underlying evaluation game as put forward by Jaako Hintikka [4]. We refer the readers unfamiliar with these notions to [6, 7], where the tight relations between logic and games are disclosed.

2 Evaluation games for first-order logic

We noticed that when presented with both the classical semantics of first-order logic and the semantics that makes use of evaluation games, the audience gets much more involved in the second approach. Indeed, most students are more eager to solving games than to checking whether a formula holds true.

Moreover, introducing first-order formulas not as (linear) sequences of symbols, but rather as trees (usually denoted as decomposition tree) makes it even easier to give evidence in support of the game-theoretical way of dealing with satisfaction. The reason is twofold:

1. the whole arena of the evaluation is very similar to the tree decomposition of the formula. It has the very same height and the same branching except when quantifiers are involved, where it depends on the cardinality of the domain of the model.

2. the task of pointing out the occurrences of variables that are bound by a given quantifier – in order to replace them by an element of the domain chosen by one of the players – is easily taken care of by taking the path along the unique branch that leads from a given leaf of the tree where the occurrence of the variable is situated, to the root of the tree. The first – if any – quantifier acting on this variable that is encountered is the one that bounds it.

We recall the definition of the evaluation game for first-order logic.

\( \textbf{Definition 3.} \) Let \( \mathcal{L} \) be a first-order language, \( \phi \) some closed formula whose logical connectors are among \( \{\neg, \lor, \land\} \), and \( \mathcal{M} \) some \( \mathcal{L} \)-structure.

The evaluation game \( \text{Ev} (\mathcal{M}, \phi) \) is defined as follows:

1. there are two players, called \textbf{Verifier} and \textbf{Falsifier}. \textbf{Verifier} (V) has incentive to show that the formula holds in the \( \mathcal{L} \)-structure \( (\mathcal{M} \models \phi) \), whereas the goal of \textbf{Falsifier} (F) is to show that it does not hold \( (\mathcal{M} \not\models \phi) \).

The moves of the players essentially consist of pushing a token down the tree decomposition of the formula \( \phi \) – as a way to choose sub-formulas – and must comply with the rules below:
if the current position is...  whose turn...  the game continues with...

|                              |                             |                             |
|------------------------------|-----------------------------|-----------------------------|
| $\varphi_0 \lor \varphi_1$  | $V$ chooses $j \in \{0,1\}$ | $\varphi_j$                |
| $\varphi_0 \land \varphi_1$ | $F$ chooses $j \in \{0,1\}$ | $\varphi_j$                |
| $\neg \varphi$              | $F$ and $V$ switch roles    | $\varphi$                  |
| $\exists x_i \varphi$       | $V$ chooses $a_i \in |M|$    | $\varphi[a_i/x_i]$         |
| $\forall x_i \varphi$       | $F$ chooses $a_i \in |M|$    | $\varphi[a_i/x_i]$         |
| $R(t_1,\ldots,t_k)\mid a_1/x_1,\ldots,a_n/x_n$ | the game stops | $V$ wins if $R(t_1,\ldots,t_k)$ is satisfied in the extended $L$-structure $M, a_1/x_1,\ldots,a_n/x_n \vDash R(t_1,\ldots,t_k)$; $F$ wins otherwise. |

2. The winning condition arises when the remaining formula becomes atomic, i.e. of the form $R(t_1,\ldots,t_k)\mid a_1/x_1,\ldots,a_n/x_n$. Notice that the rules guarantee that, since the initial formula is closed, one always ends up with an atomic formula that does not contain any more variable as each of them has been replaced by some element of the domain of the model.

Player $V$ wins if $R(t_1,\ldots,t_k)$ is satisfied in the extended $L$-structure $M, a_1/x_1,\ldots,a_n/x_n$ ($M, a_1/x_1,\ldots,a_n/x_n \vDash R(t_1,\ldots,t_k)$); $F$ wins otherwise.

Example 4. Suppose we have a language that contains a unary relation symbol $P$, a binary relation symbol $R$, and a unary function symbol $f$. We then consider the formula $\forall x \left( P(x) \lor \exists y R(f(x),y) \right)$ whose tree decomposition is

$$
\begin{array}{c}
\forall x \\
\lor \\
\exists y \\
R(f(b),a)
\end{array}
$$

\footnote{Formally we should not say that we replace each variable $x_i$ by some element $a_i$, but rather that we replace $x_i$ by some brand new constant symbol $c_{a_i}$, whose interpretation is precisely this element $a_i$ and the formula we reach at the end is of the form $R(t_1,\ldots,t_k)\mid c_{a_1}/x_1,\ldots,c_{a_n}/x_n)$.}
the model $\mathcal{M}$ is defined by:

\[
\begin{align*}
|\mathcal{M}| &= \{a, b\}, \\
\mathcal{M}(a) &= b, \\
\mathcal{M}(b) &= a, \\
\mathcal{M}^R &= \{(b, a)\}, \\
\mathcal{M}^P &= \{b\},
\end{align*}
\]

The game tree that represents the arena for the evaluation game $\mathcal{E}_v(\mathcal{M}, \forall x (P(x) \lor \exists y R(f(x), y)))$ is played on the arena represented by the following game tree:

The green leaves are the ones where the atomic formula holds true in the model, and the opposite for the red ones.

We then proceed by backward induction and assign either the colour green or the colour red to every node depending on whether the Verifier or the Falsifier has a winning strategy if the game were to start from that particular node.

We end up this way with the root being coloured green which shows that the Verifier has a winning strategy. We indicate below by blue arrows such a winning strategy for the Verifier.
The classical proof of LS↓ with Skolemization

Given any theory $T$ as in Theorem 4 without loss of generality, one first assumes that every formula $\phi \in T$ is in prenex normal form i.e. $\phi = Q_1 x_1 \ldots Q_k x_k \psi$, where, for all $1 \leq i < j \leq k$, $Q_i, Q_j \in \{\forall, \exists\}$, $x_i \neq x_j$ and $\psi$ is quantifier free. Then, the usual proof of (LS↓) goes through the following steps

1. The skolemization of each such $\phi$, which consists of
   a. first extending the language by adding, for each existential quantifier that $\phi$ contains, a new function symbol $f^{\phi(q_k)}_k$ of arity $q_k$.

   $$L' = L \cup \left\{ f^{\phi(q_k)}_k \mid Q_k = \exists, q_k = card \left( \{ m \mid 1 \leq m < k \land Q_m = \forall \} \right) \right\}.$$

   With this definition $q_k$ coincides with the number of universal quantifiers which precede the existential $Q_k$.

   b. For each existentially quantified variable $x_k$, replacing inside $\psi$, each occurrence of $x_k$ by the term $f^{\phi(q_k)}_k(x_{p_1}, \ldots, x_{p_{q_k}})$, where $x_{p_1}, \ldots, x_{p_{q_k}}$ are the universally quantified variables that precede $Q_k$. More formally:

   $$t_k = f^{\phi(q_k)}_k(x_{p_1}, \ldots, x_{p_{q_k}})$$

   where $\{ x_{p_i} \mid 1 \leq i \leq m \}$ and the sequence of subscripts $(p_i)_{1 \leq i \leq q_k}$ is strictly increasing. We then obtain

   $$\tilde{\psi} = \psi\left[t_{k_1}/x_{k_1}, \ldots, t_{k_m}/x_{k_m}\right]$$

   where $\{ x_{k_i} \mid 1 \leq i \leq m \}$ is the set of all existentially quantified variables of $\phi$.

   c. Removing all universal quantifiers from $\phi$. The Skolem normal form of $\phi$ – denoted $\sigma_\phi$ – becomes:

   $$\sigma_\phi = Q_1 x_{i_1} \ldots Q_t x_{i_t} \tilde{\psi}$$

   where the sequence of subscripts $(x_{i_j})_{1 \leq j \leq t}$ runs through all universally quantified variables of $\phi$.

   This way, any $L$-theory $T$ is turned into its skolemized version: some $L'$-theory $\sigma_T = \{ \sigma_\phi \mid \phi \in T \}$.

2. One shows that the cardinality of the set of new function and constant symbols that have been added to the language is either countable if the language is finite, and it is the same as the one of the original language $L$ if it is infinite. Therefore the extended language $L'$ has cardinality $\max\{ card(L), \aleph_0 \}$.

3. One takes any $L$-structure $M$ of cardinality $\lambda > \kappa$ such that $M \models T$ and construct some $L'$-structure $M'$ by extending $M$ from $L$ to $L'$. This is done by providing for every new symbol $f^{\phi(q_k)}_k$ an interpretation

   $$f^{\phi(q_k)}_k^M : |M|^{q_k} \rightarrow |M|$$

   such that it satisfies $M' \models \sigma_T$.

With the classical approach, the description of the extension $M'$ is usually messy, whereas it simply does not exist with the game-theoretical approach.

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2 A function symbol whose arity is zero is simply a constant symbol.
One constructs some $\mathcal{L}'$-structure $\mathcal{N}'$ of cardinality $\kappa$ such that $\mathcal{N}' \models \sigma_T$ holds. So, one selects some sub-domain of $|\mathcal{M}'|$ which is closed under all interpretations of functions $\mathcal{F}$ of $\mathcal{L}'$. For this purpose, one starts with any subset $N_0 \subseteq |\mathcal{M}'|$ of cardinality $\kappa$ which contains all interpretations of constant symbols from $\mathcal{L}'$. We let $\mathcal{F}_k(\mathcal{L}')$ denote the set of function symbols of $\mathcal{L}'$ of arity $k$. By induction, one defines

$$N_{n+1} = N_n \cup \{ f^{\mathcal{M}'}(a_1, \ldots, a_k) \mid k \in \mathbb{N}, f \in \mathcal{F}_k(\mathcal{L}'), a_1, \ldots, a_k \in N_n \},$$

and one sets $N_\omega = \bigcup_{n \in \mathbb{N}} N_n$. One observes that:

a. $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_\omega$.

b. For all $n \in \mathbb{N}$, $\text{card}(N_{n+1}) = \text{card}(N_n) = \kappa$, since the induction yields

$$\kappa \leq \text{card}(N_n \cup \{ f^{\mathcal{M}'}(a_1, \ldots, a_k) \mid k \in \mathbb{N}, f \in \mathcal{F}(\mathcal{L}'), a_1, \ldots, a_k \in N_n \})$$

$$\leq \text{card}(\mathcal{L}' \times N_{\omega})$$

$$\leq \text{card}(\kappa \times \kappa)$$

$$\leq \text{card}(N_\omega)$$

$$\leq \kappa.$$  

c. Hence $\text{card}(N_\omega) = \kappa$ holds, since

$$\kappa \leq \text{card}(N_\omega) \leq \text{card}(\prod_{n \in \omega} N_n) \leq \text{card}(\kappa \times \kappa) \leq \kappa.$$  

d. $N_\omega$ is closed under all interpretations of function symbols from $\mathcal{L}'$. For if $k \in \mathbb{N}$, $f \in \mathcal{F}_k(\mathcal{L}')$, and $a_1, \ldots, a_k \in N_\omega$, there would then exist some $n \in \mathbb{N}$ such that $a_1, \ldots, a_k \in N_n$, henceforth $f^{\mathcal{M}'}(a_1, \ldots, a_k) \in N_{n+1} \subseteq N_\omega$.

Therefore $\mathcal{N}'$ is defined by:

- $|\mathcal{N}'| = N_\omega$;
- for all constant symbols $c$, $c^{\mathcal{N}'} = c^{\mathcal{M}'}$;
- for all function symbols $f$ of arity $k$, $f^{\mathcal{N}'} = f^{\mathcal{M}'}|_{N_\omega^k}$;
- for all relation symbols $R$ of arity $k$, $R^{\mathcal{N}'} = R^{\mathcal{M}'} \cap (N_\omega)^k$.

5. Finally, one shows that the model $\mathcal{N}$ defined as the restriction of $\mathcal{N}'$ to the language $\mathcal{L}$ satisfies $\mathcal{N} \models T$. This requires once again to review the construction of $\mathcal{M}'$, by backward this time: another wearisome moment for the students.

### 4 An alternative game-theoretical proof of LS↓

Compared to the previous proof, the game-theoretical proof that we advocate is simpler for it only focuses on the original model and fixed winning strategies that witness that the theory holds in this model. Indeed, this proof contains

- no Skolem normal form
- no extended language $\mathcal{L}'$
- no extension $\mathcal{M}'$ of $\mathcal{M}$
- no restriction $\mathcal{N}'$ of $\mathcal{N}'$

In this proof, we start with any model $\mathcal{M}$ such that both $\text{card}(|\mathcal{M}|) \geq \kappa$ and $\mathcal{M} \models T$ hold. We also assume that every formula $\phi \in T$ is in prenex normal form$^3$.

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$^3$ Including the constants which are the ones of arity 0.

$^4$ We suppose that $T \neq \emptyset$ holds; otherwise, the result is simply straightforward.
1. For each \( \phi \in T \) we pick any winning strategy \( \sigma_\phi \) for the \textbf{Verifier} in \( \text{Ev} ( \mathcal{M}, \phi ) \). By looking at the very rules of the game, every student realises immediately that for each existential \( Q_i x_i \) in \( \phi \), the given strategy secures one element from \( | \mathcal{M} | \) that only depends on the previous choices made by her opponent (\textbf{Falsifier}). Since \( \phi \) is in prenex normal form, these choices made by \textbf{Falsifier} correspond precisely to the universal quantifiers preceding \( Q_i x_i \). For instance, if \( \phi \) is of the form

\[
\forall x_1 \exists x_2 \forall x_3 \forall x_4 \forall x_5 \psi
\]

then the choices that \textbf{Verifier} makes – following \( \sigma_\phi \) – of an element \( a_2 \in | \mathcal{M} | \) for \( x_2 \) and an element \( a_5 \in | \mathcal{M} | \) for \( x_5 \) depend respectively on the choices made by \textbf{Falsifier} of an element \( a_1 \in | \mathcal{M} | \) for \( x_1 \), and of elements \( a_3 \in | \mathcal{M} | \) for \( x_3 \), \( a_4 \in | \mathcal{M} | \) for \( x_4 \). In other words, the winning strategy picks for \( x_i \) an element that is function of – meaning that it only depends on – the choices made for the universally quantified variables that come before \( x_i \). Assuming there is \( k \)-many such universally quantified variables, this induces a unique function \( f_\psi^\sigma : | \mathcal{M} |^k \mapsto | \mathcal{M} | \). So we come up with a set \( \mathcal{F} = \{ f_\psi^\sigma \mid \psi \in T \} \cup \{ f^\sigma \mid f \in \mathcal{L} \} \) of functions of different arities\(^5\) whose cardinality is at most \( k = \max \{ \text{card} ( \mathcal{L} ), \aleph_0 \} \).

2. We take any subset \( N_0 \subseteq | \mathcal{M} | \) of cardinality \( \kappa \) and proceed as in (4)(a-d) to obtain the least (for inclusion) subset \( N \subseteq | \mathcal{M} | \) of cardinality \( \kappa \) that satisfies both \( N_0 \subseteq N \) and \( N \) is closed under all functions in \( \mathcal{F} \).

3. We form \( \mathcal{N} \) as the restriction of \( \mathcal{M} \) from \( | \mathcal{M} | \) to \( N \), and show that \( \mathcal{N} \models T \) in a straightforward manner this time, since for every formula \( \phi \in T \) the very same strategy\(^6\) \( \sigma_\phi \) which is winning for the \textbf{Verifier} in \( \text{Ev} ( \mathcal{M}, \phi ) \) is also winning for the \textbf{Verifier} in \( \text{Ev} ( \mathcal{N}, \phi ) \).

5 An even simpler proof of \( \text{LS}_\downarrow \)

This proof does not even require to go through the formulas of \( T \) to be in prenex normal form.

1. for each \( \phi \in T \), pick any winning strategy \( \sigma_\phi \) for the \textbf{Verifier} in \( \text{Ev} ( \mathcal{M}, \phi ) \) and for any \( A \subseteq | \mathcal{M} | \) consider all possible plays in \( \text{Ev} ( \mathcal{M}, \phi ) \) such that

\begin{itemize}
  \item[a.] \textbf{Falsifier} restricts his \( \forall \)-moves to choosing elements of \( A \), and
  \item[b.] \textbf{Verifier} applies her winning strategy \( \sigma_\phi \).
\end{itemize}

set \( (A)^{\sigma_\phi} \) as the subset of \( | \mathcal{M} | \) formed of all the \( \exists \)-moves made by \textbf{Verifier}. Set also

\[
(A)^F = \{ f^\sigma (\bar{a}) \mid f \in \mathcal{L} \text{ a function symbol with arity } k, \bar{a} \in N_n^{k^0} \}.
\]

2. inductively define \( N \) by:

\begin{itemize}
  \item[a.] \( N_0^F \subseteq | \mathcal{M} | \) any set s.t. \( \text{card} ( N_0^F ) = \kappa \), \( \{ c^m \mid c \text{ constant } \in \mathcal{L} \} \subseteq N_0^F \),
  \item[b.] \( N_{n+1}^F = N_n^F \cup ( N_n^F )^F \cup \bigcup_{\phi \in T} ( N_n^F )^{\sigma_\phi} \), and \( N = \bigcup_{n \in \mathbb{N}} N_n^F \). Then,

\[
\text{card} (N) = \kappa = (N)^{\sigma_\phi} \subseteq N = (N)^F \subseteq N.
\]
\end{itemize}

3. Form \( \mathcal{N} \) as the restriction of \( \mathcal{M} \) to \( N \), and easily verify that \( \mathcal{N} \models T \), since the very same \( \sigma_\phi \) is winning for \textbf{Verifier} in \( \text{Ev} ( \mathcal{N}, \phi ) \).

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\(^5\) Functions of arity 0 being identified with elements of the domain \( | \mathcal{M} | \).

\(^6\) Strictly speaking this is the restriction of this strategy to \( N \).
Conclusion

We tried these proofs on our own students at EPFL. It turns out that they understand much better the proof of LS↓ that we recommend than they buy the classic one. It also requires much less time to teach, and above all we deeply believe that this proof highlights what is essential in this result now devoid of all the technical details of the skolemization. On the other hand, there is a price to pay in doing so: one has to present the semantics of first-order logic through evaluation games. But here also we have noticed that the students learn more easily this other part of the course. Another advantage of the game-theoretical approach is also that it paves the way for the back-and-forth method \[3\] or the Ehrenfeucht-Fraïssé games \[2\] that are intensively used in model theory \[5\].

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