The propagator of a relativistic particle via the path-dependent vector potential

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The proper time formalism for a particle propagator in an external electromagnetic field is combined with the path-dependent formulation of the gauge theory to simplify the quasiclassical propagator. The latter is achieved due to a specific choice of the gauge corresponding to the use of the classical path in the path-dependent formulation of the gauge theory, which leads to the cancellation of the interaction part of the action in the Feynman path integral. A simple expression for the quasiclassical propagator is obtained in all cases of the external field when the classical equation of motion in this field is integrable. As an example, new simple expressions for the propagators are derived for a spinless charged particle interacting with the following fields: an arbitrary constant and uniform electromagnetic field, an arbitrary plane wave and, finally, an arbitrary plane wave combined with an arbitrary constant and uniform electromagnetic field. In all these cases the quasiclassical propagator coincides with the exact result.

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I. INTRODUCTION

For those potentials which vary much more slowly than the corresponding wave function in position, the wave function given within the Wentzel-Kramers-Brilluin (WKB) approximation becomes legitimate [1]. In a sense, we are in the quasiclassical regime when the de Broglie wavelength of the particle varies slightly over the distance that characterizes the problem [2]. In fact, if the Lagrangian of a point particle is a quadratic function of the coordinate and the velocity, then the corresponding exact propagator coincides with the quasiclassical propagator, i.e., the classical path dominates the Feynman path integral [3–5]. It is interesting that the exact Volkov propagator, the propagator of a charged particle interacting with a plane electromagnetic wave, is also given by its quasiclassical limit [7–11], though the Lagrangian is not quadratic. Furthermore, there are many cases where the quasiclassical propagator is a good approximation. In this paper, we show that the expression of the quasiclassical propagator can be significantly simplified when employing the path dependent formulation of the propagator. This is achieved due to the gauge theory which is used, the interaction part of the action of the Feynman path integral can be expressed in terms of the electromagnetic flux through the area between the arbitrary Feynman path and the gauge path which generates the associated path-dependent vector potential [15]. The significant simplification of the quasiclassical propagator expression comes when one specifies the gauge path to coincide with the classical trajectory. In fact, as we will show here, the above mentioned flux vanishes in this case and, consequently, the interaction part of the action vanishes in the Feynman path integral.

The straightforward calculation of the relativistic propagator via the Feynman path integral is cumbersome due to the presence of the particle’s infinitesimal proper time [3–6]. However, one can overcome this difficulty via introducing a fifth parameter to the theory which is known as the proper time formalism [18–24]. In this method one defines an effective Lagrangian associated to the super-Hamiltonian $\mathcal{H}(P,X)$. The latter is defined by the quantum mechanical equation: any quantum mechanical equation can be written in the form of

$$\mathcal{H}(P,X)\partial \phi = 0,$$

with the four-momentum operator $P$ and four-position operator $X$. Then, the relativistic propagator is followed by the standard rules of the nonrelativistic Feynman path integral for an effective system governed by the effective Lagrangian.

In this paper, we unite two powerful methods for the calculation of the propagator of a relativistic charged particle interacting with an external electromagnetic field. In particular, the path-dependent formulation of the gauge theory is incorporated in the proper time formalism of propagators. This allows us to obtain simple expressions of the quasiclassical propagators for a constant and uniform electromagnetic field, for an arbitrary plane wave and for an arbitrary plane wave combined with an arbitrary constant and uniform electromagnetic field. In all these cases, the quasiclassical propagator coincides with the exact result.

The structure of the paper is the following. In Sec. I the path-dependent formulation of gauge theory is presented and its convenience for the Feynman path integral method for calculation of propagators is shown. In the next section, Sec. III...
the proper time formalism for calculation of propagators is introduced which incorporates the path-dependent gauge formalism. The explicit expressions of the propagators for a spinless charged particle interacting with a constant and uniform electromagnetic field, with a plane wave and with a plane wave combined with a constant and uniform electromagnetic field, respectively, are derived in Sec. [14] The conclusion is given in Sec. [15] The metric convention is \( g = (+, −, −, −) \) throughout the paper.

II. THE PATH-DEPENDENT FORMALISM OF THE GAUGE THEORY

The Maxwell equations allow to express the electromagnetic field strength tensor \( F^{\mu \nu} \) in terms of a four-vector potential \( A^\mu = (\phi, A) \) as \( F_{\mu \nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \). The gauge transformation

\[
A^\mu(x) \rightarrow A^\mu(x) + \partial^\nu \chi(x)
\]

(2)

leaves the electromagnetic field strength tensor invariant, which is a consequence of the conservation of the electric charge under the local symmetry transformation via Noether’s theorem. The conservation of the electric charge follows from the Schrödinger equation is invariant under the transformations Eq. (2) as long as the wave function transforms as

\[
\psi(x) \rightarrow \exp \left( \frac{i e \chi(x)}{\hbar c} \right) \psi(x).
\]

(3)

In fact, the gauge function \( \chi \) can be identified via the path integral

\[
\chi(x) = -\int_p^x A_\mu(y) dy^\mu.
\]

(4)

This path-dependent phase (nonintegrable phase [14]), then, yields a gauge invariant but path-dependent vector potential \( A_\mu(x) \) as

\[
A_\mu(x) = \int_0^1 F_{\mu \nu}(y) \frac{\partial y^\nu}{\partial s} \frac{\partial y^\lambda}{\partial \xi^\mu} ds
\]

(5)

with the boundary conditions \( y(1, x) = x, \ y(0, x) = x' \), where the vector potential vanishes at \( x' \). In this equivalent formulation of the gauge theory the Schrödinger equation is invariant under path transformations

\[
\Psi[p', x] = \exp \left( \frac{i e}{\hbar c} \Phi_{EM}(\Sigma, x) \right) \Theta[p', x], \quad \Phi_{EM}(\Sigma, x) = \int_{\partial \Sigma} A^\mu dy_\mu
\]

(7)

with the electromagnetic flux

\[
\Phi_{EM}(\Sigma, x) = \int_{\partial \Sigma} A^\mu dy_\mu = \frac{1}{2} \int_{\Sigma} F^{\mu \nu} d\sigma_{\mu \nu}
\]

(8)

on the closed loop \( \partial \Sigma = \mathcal{P} - \mathcal{P}' \). The electromagnetic flux can further be identified as

\[
\Phi_{EM}(\Sigma, x) = \int_{\mathcal{P}} A_\mu(y) dy^\mu,
\]

(9)

which implies that the line integral of a path dependent vector potential along a path \( \mathcal{P} \) vanishes as long as the vector potential is evaluated via the same path \( \mathcal{P}' \), i.e.,

\[
\int_{\mathcal{P}} A_\mu(y) dy^\mu = 0.
\]

(10)

The full machinery of the path-dependent formulation of the gauge theory provides some fundamental simplifications for the path integral formulation of quantum mechanics. Namely, let us consider the propagator in terms of the Feynman path integral which is defined by

\[
K_F(x, x'; t) = \int \mathcal{D}[\mathcal{P}_F] \exp \left( \frac{i}{\hbar} S(\mathcal{P}_F) \right)
\]

(11)

where \( D[\mathcal{P}_F] \) and \( S(\mathcal{P}_F) \) represent the sum over all paths and the action evaluated along the path \( \mathcal{P}_F \), respectively. The action of interest of the present manuscript is the action of a spinless charged particle interacting with an electromagnetic field, which can be written as

\[
S(\mathcal{P}_F) = -mc^2 \int_{\mathcal{P}_F} d\tau - \frac{e}{c} \int_{\mathcal{P}_G} A_\mu P^\mu dy^\mu,
\]

(12)

with the particle’s infinitesimal proper time \( d\tau = \sqrt{d\tau^\mu d\tau_\mu}/c \). Here we should emphasize that paths appeared in the Feynman path integrals \( \mathcal{P}_F \) are real paths in the sense that the transition amplitude of a particle from spacetime point \( x' \) to \( x \) depends on these paths. Nonetheless, paths used for the vector potential \( \mathcal{P}_G \) are just gauge paths. Moreover, the action \( S(\mathcal{P}_F) \) can be defined via the electromagnetic flux \( \Phi_{EM} \) as

\[
S(\mathcal{P}_F) = -mc^2 \int_{\mathcal{P}_F} d\tau - \frac{e}{c} \Phi_{EM}(\Sigma, x),
\]

(13)

with \( \partial \Sigma = \mathcal{P}_F - \mathcal{P}_G \).

The compact form of the action \( S(\mathcal{P}_F) \) provides us a further simplification for the quasiclassical propagators. The quasiclassical propagator can be defined via the classical action \( S_c \), which is the action evaluated along the classical trajectory (world line) \( \mathcal{P}_c \). Furthermore, using the VanVleck-Pauli-Morrette formula \([25, 28]\), it reads

\[
K_F(x, x'; t) = \sqrt{\left( \frac{1}{2\pi \hbar} \right)^3 \det \left( \frac{\partial^2 S_c}{\partial x^\mu \partial x'^\nu} \right)} \times \exp \left( \frac{i mc^2}{\hbar} \int_{\mathcal{P}_c} d\tau - \frac{e}{c} \Phi_{EM}(\mathcal{P}_c - \mathcal{P}_G, x) \right).
\]

(14)

Now, if one specifies the classical path for the gauge path, the flux term in the above expression vanishes and then the quasiclassical propagator reduces to

\[
K_F(x, x'; t) = \left( \frac{1}{2\pi \hbar} \right)^3 \det \left( \frac{\partial^2 S_c}{\partial x^\mu \partial x'^\nu} \right) \exp \left( \frac{i mc^2}{\hbar} \int_{\mathcal{P}_c} d\tau \right),
\]

(15)
where the classical path $\mathcal{P}_c$ satisfies the Lorentz force law
\[ m \frac{\partial^2 y_c(t)}{\partial t^2} = e F^{\mu\nu}(y_c) \frac{\partial y_c}{\partial \tau}. \] (16)

The latter defines the path dependent vector potential $A_{\nu}$ as
\[ A_{\nu}(\mathcal{P}_c, x) = \frac{mc}{e} \int_{y_c}^{y} \frac{\partial^2 y_c}{\partial \tau^2} \frac{\partial y_c}{\partial x^{\nu}} \, d\tau, \] (17)

where the dependence on the electromagnetic fields contains only in the definition of the classical path via Eq. (16). Furthermore, in terms of the reparametrization invariant form of the equations of motion
\[ \frac{\partial^2 p^{\mu}}{\partial s^2} = e F^{\mu\nu}(y) \frac{\partial y^{\nu}}{\partial s}, \] (18)

with the particle’s four momentum $p^{\mu}$, the vector potential for the classical path reads
\[ A_{\nu}(\mathcal{P}_c, x) = \frac{c}{e} \int_0^{\tau} \frac{\partial p^{\nu}}{\partial \tau} \frac{\partial y^{\nu}}{\partial x^{\mu}} \, d\tau. \] (19)

For a nonrelativistic particle, on the other hand, the classical trajectory can be parametrized with the physical time $t$. Then, the path dependent vector potential for the non-relativistic classical path $y(t')$ with the boundary conditions $y(0) = x'$ and $y(t) = x$ can be written as
\[ A_{\nu}(\mathcal{P}_c, x) = \frac{mc}{e} \int_0^{\tau} \frac{\partial^2 y(t')}{\partial t'^2} \frac{\partial y(t')}{\partial x^{\nu}} \, dt' = \frac{c}{e} \int_0^{\tau} \frac{\partial p^{\nu}}{\partial t'} \frac{\partial y(t')}{\partial x^{\nu}} \, dt', \] (20)

where we have used $(\partial t'/\partial x' = 0$. This form of the vector potential provides great convenience for the quasiclassical propagator for a nonrelativistic particle. Since the integral of the corresponding potential terms in the action vanishes, i.e.,
\[ \int_0^{\tau} dt' \left( \frac{1}{c} \mathcal{A} \cdot \dot{y}(t') - e A^{0} \right) = 0, \] (21)

the quasiclassical propagator for a nonrelativistic particle in the classical path gauge yields
\[ K_{\mu}(x, x': t) = \sqrt{\left( \frac{1}{2\pi i\hbar} \right)^3 \det \left( \frac{-\partial^2 S_\mu}{\partial x^{\nu} \partial x^{\rho}} \right) \exp \left( \frac{i}{\hbar} \int_{\mathcal{P}_c}^{\tau} \frac{m \dot{y}^2}{2} \, dt' \right). \] (22)

In fact, this result can also be obtained directly via approximating the particle’s infinitesimal proper time with the usual non-relativistic kinetic term of the Lagrangian in Eq. (15).

In summary, since the equations of motion are gauge invariant, they are first found in any convenient gauge, then the propagator for a nonrelativistic particle as well as for a relativistic particle can be calculated in the corresponding classical path gauge via Eq. (22) and Eq. (15), respectively. The compact form of the quasiclassical propagator can be applied any type of potential, when the classical equations of motion are known.

III. THE PROPER TIME FORMALISM AND THE FEYNMAN KERNEL VIA THE PATH DEPENDENT VECTOR POTENTIAL

The calculation of the relativistic propagator is, in general, tedious due to the presence of the particle’s infinitesimal proper time. Nevertheless, this can be overcome via the proper time (eigentime, fifth parameter or einbein) formalism $\mathcal{G}_\nu$.

The proper time formalism is based on the fact that any quantum mechanical equation can be written in the form of
\[ \mathcal{H}(P, X) |\psi \rangle = 0, \] (23)

with a certain operator $\mathcal{H}(P, X)$, so called super-Hamiltonian. For instance, in the case of Klein-Gordon equation the super-Hamiltonian is
\[ \mathcal{H} = \left( P - \frac{e}{c} \mathcal{A}(P, X) \right)^2 - m^2 c^2. \] (24)

The four-momentum operator $P^{\mu}$ and the four-position operator $X^{\mu}$ in Eq. (24) satisfy the eigenvalue equations
\[ P^{\mu}|p \rangle = p^{\mu}|p \rangle, \] (25)
\[ X^{\mu}|x \rangle = x^{\mu}|x \rangle, \] (26)

respectively, with $p^{\mu} = (E/c, p)$ and $x^{\mu} = (c t, x)$. Furthermore, the assumption of the canonical commutation relation $[X^{\mu}, P^{\nu}] = i \hbar g^{\mu\nu}$ implies the relations $\langle x|P_{\mu}|\psi \rangle = i \hbar \partial_{\mu}\psi(x)$, $\langle x|X_{\mu}|\psi \rangle = x_{\mu}\psi(x)$,

\[ \langle x|\mathcal{H}(P, X)|\psi \rangle = \mathcal{H}(i \hbar \partial_{\mu}, x) \psi(x) = 0. \] (29)

Then, the corresponding Green’s function satisfies
\[ \mathcal{H}(i \hbar \partial_{\mu}, x) G(x, x') = \delta^{4}(x - x'). \] (30)

Hence, the Green’s function can be identified as
\[ G(x, x') = \mathcal{H}^{-1}(i \hbar \partial_{\mu}, x) \delta^{4}(x - x'). \] (31)

which can further read
\[ G(x, x') = \langle x|\mathcal{H}^{-1}(P, X)|x' \rangle = \langle x| \frac{1}{\mathcal{H}(P, X) + i \epsilon} |x' \rangle \] (32)

with the Feynman $i \epsilon$ prescription. Furthermore, Eq. (32) may also be defined as
\[ G(x, x') = -\frac{i}{\hbar} \int_{0}^{\infty} d\tau \langle x| \exp \left[ \frac{i}{\hbar} \mathcal{H}(P, X) \alpha \tau \right] |x' \rangle e^{-\alpha \tau} \] (33)

with an auxiliary field $\alpha$ (einbein field). Here the parameter $\alpha$ is introduced in order to fix the right classical equations of motion.
motion of the corresponding super-Hamiltonian $\mathcal{H}(P, X)$ [23, 24].

Let us define the integrand in Eq. (33) as a Feynman Kernel

$$
K_F(x, x'; \tau) = \langle x | \exp \left[ \frac{i}{\hbar} \mathcal{L} (P, X) / \alpha \tau \right] | x' \rangle,
$$

(34)

with the effective Hamiltonian $\alpha \mathcal{H}$ in the space of $\tau$, which determines the propagator

$$
G(x, x') = - \frac{i}{\hbar} \alpha \int_0^\infty d\tau K_F(x, x' ; \tau) e^{-i\tau}.
$$

(35)

The Feynman Kernel in terms of the path integration reads

$$
K_F(x, x'; \tau) = \int D(\mathcal{P}_F) \exp \left( - \frac{i}{\hbar} \mathcal{S}(\mathcal{P}_F) \right),
$$

(36)

with the action $\mathcal{S}(\mathcal{P}_F)$, which is derived introducing an effective Lagrangian via common Legendre transformation

$$
\mathcal{L} = p \cdot \dot{x} - \alpha \mathcal{H}
$$

(37)

with $p^\mu = \partial \mathcal{L} / \partial \dot{x}_\mu$. Then,

$$
K_F(x, x'; \tau) = \int D[y] \exp \left( - \frac{i}{\hbar} \int_0^\tau d\sigma \mathcal{L}(y, \dot{y}) \right),
$$

(38)

where the parametrized path $y(\sigma)$ satisfies the boundary conditions $y^\mu(0) = x^\mu$, $y(\tau) = x'^\mu$.

In the present manuscript, we restrict ourselves to spinless charged particles, although the results can be easily generalized to the Fermionic particles. Then using Eq. (37) for Eq. (24), the corresponding effective Lagrangian is found

$$
\mathcal{L} = \frac{1}{4\alpha} \dot{y}^2 + \frac{e}{c} \dot{y} \cdot \mathbf{A}(\mathcal{P}_G, y) + \alpha m^2 c^2.
$$

(39)

The Euler-Lagrange equation of the effective Lagrangian provides the effective equation of motion

$$
\frac{1}{2\alpha} \dot{y}^\mu(\sigma), y^\mu = \frac{e}{c} F^{\mu\nu}(y_c) y(\sigma), y^\nu
$$

(40)

which coincides with the classical equation of motion in the given external electromagnetic field for $\alpha = 1/(2m)$ as long as the path is parametrized with the particle’s proper time $\tau$, see [29]. In the quasiclassical approximation and using the gauge determined by the classical path, see Eq. (15), the Feynman Kernel yields

$$
K_F(x, x'; \tau) = \sqrt{\frac{1}{2\pi i\hbar}} \det \left( \frac{\partial^2 S_\epsilon}{\partial x_\mu \partial x'_\nu} \right) \times \exp \left( - \frac{i}{\hbar} m^2 c^2 \alpha \tau - \frac{i}{\hbar} \int_0^\tau d\sigma \frac{\sigma^2}{4\alpha} \right).
$$

(41)

The Green function is, then, obtained via Eq. (35).

The equation (41) is the main result of the paper, which shows how to derive the fundamental Feynman Kernel for the quasiclassical Green’s function in a simple way when the classical equation of motion Eq. (40) is integrable in the given field.

**IV. EXAMPLES**

In this section we will apply the developed formalism to some important cases where the quasiclassical propagator coincides with the exact propagator.

**A. Constant and uniform electromagnetic field case**

Let us first consider a spinless charged particle interacting with a constant and uniform electromagnetic field $F_{\mu\nu}$. Since the Lagrangian of a constant and uniform electromagnetic field (39) is a quadratic function of $y$ and $\dot{y}$, the quasiclassical formula gives the exact result.

The effective equation of motion Eq. (40) in the constant and uniform electromagnetic field is solved providing the classical path

$$
y(\sigma)^\nu = \left( \frac{e \mathbf{F}_c - 1}{e i \mathbf{F}_c - 1} \right)^\nu \mu (x - x')^\nu + x'^\nu
$$

(42)

with $\lambda = 2\alpha e / c$, and corresponding to the boundary conditions $y(0)^\mu = x^\mu$, $y(\tau)^\mu = x'^\mu$. Here it should be understood that

$$
\left( \frac{e i \mathbf{F}_c - 1}{e i \mathbf{F}_c - 1} \right)^\nu \mu = \frac{\lambda}{\tau} \delta^\nu \mu + \frac{\lambda (\sigma^2 - \sigma^\nu)}{2\tau} F^\nu_{\nu} + \cdots
$$

(43)

As a result, in the classical path gauge the Feynman Kernel of Eq. (41) reads

$$
K_F(x, x'; \tau) = \exp \left( - \frac{i}{\hbar} \Phi_{EM}(x) \right) K_F(x, x'; \tau) \exp \left( \frac{i}{\hbar} \Phi_{EM}(x') \right)
$$

(45)

where the electromagnetic flux $\Phi_{EM}$ is calculated for the area bounded by the loop $\partial \Sigma = \mathcal{P}_c - \mathcal{P}_G$ with desired gauge $\mathcal{P}_G$.

Equivalently, the Kernel in an arbitrary gauge can also be found using the corresponding gauge function $\chi$, taking into
account that the vector potential in the classical path gauge is

$$\mathcal{A}_\mu(\mathcal{P}_c, x) = -\frac{1}{\alpha} \int_0^\tau d\sigma \frac{\partial^2 y_{\nu}}{\partial x^\nu} \frac{\partial y_{\nu}}{\partial x^\nu}$$  \hspace{1cm} (46)$$

where we have used Eq. (40) in Eq. (5).

The constant and uniform crossed field with equal amplitude

$$E = E_0 \mathbf{x}, \quad B = E_0 \mathbf{y},$$  \hspace{1cm} (47)

is an important particular case of the considered field, corresponding to the asymptotic of the laser field at large field parameters $a_0 \equiv E_0 / mc^2 \gg 1$. Since the third power of the field tensor vanishes for the above field, the Feynman Kernel in the classical path gauge yields

$$K_F(x, x'; \tau)_{FS} = \frac{1}{(4\pi \hbar \alpha \tau)^2} \times \exp \left( -\frac{i}{\hbar} \frac{(x-x')^2}{4\alpha \tau} + \frac{i\lambda^2 \tau}{48 \hbar \alpha} \right) \left( F(x-x') F_\mu(x-x') - \frac{im^2 c^2 \alpha \tau}{\hbar} \right)$$  \hspace{1cm} (48)

which corresponds to the vector potential in the classical path gauge

$$\mathcal{A}_\mu(\mathcal{P}_c, x) = -\frac{1}{2} \left( F + \frac{\lambda \tau}{3} F^2 \right) (x-x')^\nu. \hspace{1cm} (49)$$

For comparison, in the Fock-Schwinger gauge, the vector potential

$$\mathcal{A}_\mu(\mathcal{P}_{FS}, x) = -\frac{1}{2} F_{\mu\nu}(x-x')^\nu \hspace{1cm} (50)$$

is obtained via the gauge function

$$\chi = \frac{\lambda \tau}{12} (x-x')^\mu F_{\mu\nu}(x-x')^\nu. \hspace{1cm} (51)$$

In terms of the gauge paths, the above gauge function corresponds to the flux through the area bounded by the classical path \( \mathcal{P}_{FS} \) and the Fock-Schwinger path which is a straight line

$$\mathcal{P}_{FS} : y^\mu(\sigma) = \sigma x^\mu + (1 - \sigma) x'^\mu \hspace{1cm} (52)$$

with the boundary conditions $y^\mu(1) = x^\mu$, $y^\mu(0) = x'^\mu$.\[48\]

### B. Plane wave case: Volkov Propagator

Another case where the quasiclassical approximation yields the exact result is the interaction of a charged particle with a plane electromagnetic wave \[7–11\]. The corresponding propagator for this case is called Volkov propagator. The field strength tensor $F_{\mu\nu}$ of a plane wave can be written as

$$F_{\mu\nu}(\phi) = \epsilon_{\mu\nu} f'(\phi) \hspace{1cm} (53)$$

where the phase of the wave is defined as $\phi = kx$ and the antisymmetric tensor is $\epsilon_{\mu\nu} = k_\mu \epsilon_{\nu} - k_\nu \epsilon_{\mu}$ with the propagation and polarization directions $k_\mu$ and $\epsilon_{\nu}$, respectively.

The classical trajectory $\mathcal{P}_c$ via Eq. (40) for a plane wave reads:

$$k y_c(\sigma) = \sigma \frac{\tau}{k} (x-x') + kx',$$  \hspace{1cm} (54)

$$e y_c(\sigma) = \frac{\sigma}{\tau} e (x-x') + e x' + \frac{\lambda}{\tau} \left( \sigma g_1(\sigma) - \sigma g_1(\tau) \right),$$  \hspace{1cm} (55)

$$\overline{e} y_c(\sigma) = \frac{\sigma}{\tau} \overline{e} (x-x') + \overline{e} x',$$  \hspace{1cm} (56)

$$\overline{k} y_c(\sigma) = \frac{\lambda}{\tau} \left( \sigma g_1(\sigma) - \sigma g_1(\tau) \right) \frac{k(x-x') \tau}{k(x-x')^\nu} (e(x-x') - \lambda g_1(\tau))$$

$$+ \frac{\lambda^2}{2 k(x-x')^\nu} \left( \sigma g_2(\sigma) - \sigma g_2(\tau) \right) + \frac{\sigma^2}{\tau} k(x-x') + \overline{k} x', \hspace{1cm} (57)$$

where $g_1(\sigma) = \int_0^\sigma f(k y_c) \, d\sigma'$ and the basis $k$, $e$, $\overline{e}$, $\overline{k}$ is introduced such that they satisfy $k^2 = \overline{k}^2 = k \epsilon k = \epsilon k = \overline{e} \overline{k} = \epsilon \overline{e} = 0$, $\overline{e}^2 = \overline{k}^2 = -1$, and $k \overline{k} = 1$.

Furthermore, the classical action in the classical path gauge can be written in terms of the new basis as

$$S_c = m^2 c^2 \alpha \tau + \int_0^\tau d\sigma \frac{1}{4\alpha} \left( 2 k y_c \overline{k} y_c - \epsilon^2 y_c^2 - \overline{e}^2 y_c^2 \right). \hspace{1cm} (58)$$

Consequently, the Feynman Kernel in the classical path gauge yields

$$K_F(x, x'; \tau) = \frac{1}{(4\pi \hbar \alpha \tau)^2} \exp \left( -\frac{i}{\hbar} \frac{(x-x')^2}{4\alpha \tau} + \frac{im^2 c^2 \alpha \tau}{\hbar} \right) - \frac{i \lambda^2}{4 \hbar \alpha} \left( \int_0^\tau d\sigma f(k y_c) \right)^2 + \frac{i \lambda^2}{4 \hbar \alpha} \int_0^\tau d\sigma f(k y_c)^2. \hspace{1cm} (59)$$

The result is more compact due to the absence of the interaction term, see for instance Eq. (31) of \[10\]. For a constant and uniform plane wave one naturally covers Eq. (48) and in the absence of the field one obtains the relativistic propagator for a free particle, see Eq. (19.28) of \[8\]. Furthermore, the Volkov propagator can be written in an arbitrary gauge via the
C. Plane wave combined with a constant and uniform electromagnetic field case

In the last example we will obtain the relativistic propagator for a charged particle interacting with an arbitrary plane wave combined with a constant and uniform electromagnetic field.

The associated field strength tensor can be written as

\[ F^{\mu\nu}(\phi) = F_0^{\mu\nu} + f^{\mu\nu}(\phi) \]  

(60)

where \( F_0^{\mu\nu} \) and \( f^{\mu\nu}(\phi) \) are the field strength tensors of the constant and uniform electromagnetic field and the plane wave, respectively.

Before calculating the classical action, note that since the action is a Lorentz scalar, one can calculate it in an arbitrary frame of reference. In fact, in an arbitrary reference frame, there are two fundamental Lorentz invariants which have to be satisfied by any field strength tensor

\[ F_{\mu\nu} F^{\mu\nu} = 2 (B^2 - E^2), \]

(61)

\[ G_{\mu\nu} F^{\mu\nu} = -4 E \cdot B \]

(62)

with the dual of the field strength tensor \( G_{\mu\nu} = e^{\mu\nu\rho} F_{\rho\sigma}/2 \).

Hence, for the constant and uniform electromagnetic field there exists such a reference frame that the magnetic field and the electric field can be parallel to each other. Furthermore, the direction of the parallel magnetic and electric fields can be chosen along the propagation direction of the plane wave [30]. As a consequence, the field strength tensor of the constant and uniform electromagnetic field \( F_0^{\mu\nu} \) can be written as

\[ F_0^{\mu\nu} = E_0 (k^\mu k^\nu - \kappa^\mu \kappa^\nu) - i B_0 (e_\nu^\mu e_\rho^\sigma - e_\rho^\mu e_\nu^\sigma), \]

(63)

where \( E_0 \) and \( B_0 \) are the electric field and the magnetic field in the aforementioned frame, respectively, and the new basis \( \epsilon_\mu = 1/\sqrt{2} (\epsilon \pm \bar{\epsilon})^\mu \) satisfy \( \epsilon_+ \epsilon_- = -1 \), \( \epsilon_+^2 = 0 \). Moreover, one can recover the field strength tensor as

\[ E_0 = \frac{1}{2} \sqrt{I_1^2 + I_2^2 - I_1}, \]

(64)

\[ B_0 = \frac{1}{2} \sqrt{I_1^2 + I_2^2 + I_1} \]

(65)

with \( I_1 = F_{0\mu\nu} F_0^{\mu\nu} \) and \( I_2 = G_{0\mu\nu} F_0^{\mu\nu} \).

Then, in the frame of reference where the electric field and the magnetic field of \( F_0^{\mu\nu} \) and the propagation direction of \( f^{\mu\nu}(\phi) \) are all parallel to each other, the equations of motion are governed by

\[ k \dot{y}_\epsilon(\sigma) = -\lambda E_0 k y_\epsilon(\sigma), \]

(66)

\[ \bar{k} \dot{y}_\epsilon(\sigma) = \lambda E_0 \bar{k} y_\epsilon(\sigma) + \frac{\lambda f}{k y_\epsilon}, \]

(67)

\[ \epsilon_+ \dot{y}_\epsilon(\sigma) = -i \lambda B_0 \epsilon_+ \dot{y}_\epsilon(\sigma) + \frac{\lambda f}{\sqrt{2}}, \]

(68)

\[ \epsilon_- \dot{y}_\epsilon(\sigma) = i \lambda B_0 \epsilon_- \dot{y}_\epsilon(\sigma) + \frac{\lambda f}{\sqrt{2}}. \]

(69)

As a consequence, the Feynman Kernel becomes

\[ K_F(x, x') = \frac{1}{(2\pi \hbar)^4} \det \left( \frac{\partial^2 S_c}{\partial x_\epsilon, \partial x'_\epsilon} \right) \exp \left( \frac{i}{\hbar} S_c \right), \]

(70a)

where the classical action is

\[ S_c = m^2 c^2 \alpha \tau + \int_0^\tau d\tau \frac{1}{2a} \left( k y_\epsilon \bar{k} y_\epsilon - \epsilon_+ y_\epsilon \epsilon_- y_\epsilon \right). \]

(70b)

with the following solutions of the equations of motion

\[ k y_\epsilon(\sigma) = \frac{e^{-\lambda E_0 \sigma}}{e^{\lambda E_0 \tau} - 1} k(x - x') + k x', \]

(70c)

\[ \epsilon_+ y_\epsilon(\sigma) = \frac{e^{-\lambda i B_0 \sigma}}{e^{\lambda i B_0 \tau} - 1} \left[ \epsilon_+(x - x') - \frac{i}{\sqrt{2} B_0} \left( e^{\lambda i B_0(\sigma - \tau)} - 1 \right) \int_0^\tau (1 - e^{-\lambda E_0 \rho}) f \, d\rho + \int_\tau^\sigma (1 - e^{-\lambda E_0(\sigma - \tau)}) f \, d\rho \right] + \epsilon_+ x', \]

(70d)

\[ \epsilon_- y_\epsilon(\sigma) = \frac{e^{\lambda i B_0 \sigma}}{e^{\lambda i B_0 \tau} - 1} \left[ \epsilon_-(x - x') + \frac{i}{\sqrt{2} B_0} \left( e^{-\lambda i B_0(\sigma - \tau)} - 1 \right) \int_0^\tau (1 - e^{-\lambda E_0 \rho}) f \, d\rho + \int_\tau^\sigma (1 - e^{-\lambda E_0(\sigma - \tau)}) f \, d\rho \right] + \epsilon_- x', \]

(70e)

\[ \bar{k} y_\epsilon(\sigma) = \frac{e^{\lambda E_0 \sigma}}{e^{\lambda E_0 \tau} - 1} \left[ k(x - x') - \frac{1}{E_0} \left( e^{\lambda E_0 \tau} - e^{-\lambda E_0 \sigma} \right) \int_0^\tau (1 - e^{-\lambda E_0 \rho}) \frac{\epsilon_+ y_\epsilon}{k y_\epsilon} f \, d\rho + \int_\tau^\sigma (1 - e^{-\lambda E_0(\sigma - \tau)}) \frac{\epsilon_+ y_\epsilon}{k y_\epsilon} f \, d\rho \right] + \bar{k} x'. \]

(70f)

Although the closed expression for the Feynman Kernel is very cumbersome and is not shown here, it is derived by straightforward calculation when the plane wave function \( f(\phi) \) and the components of the field strength tensor of the constant and uniform electromagnetic field \( F_0^{\mu\nu} \) are known. Moreover, the form of the Feynman Kernel given by Eq. (70) provides considerable convenience to the numerical calculations. Simpler expressions for the propagator can be obtained in a limit \( E_0 \to 0 \) (\( B_0 \to 0 \)), corresponding to a plane wave combined with a constant and uniform magnetic (electric) field along the propagation direction of the plane wave.
V. CONCLUSION

We have applied the path-dependent formulation of the gauge theory within the Feynman path integral formalism of quantum mechanics for a Klein-Gordon particle in an external electromagnetic field. The applied formalism points to a specific gauge when a significant simplification of the expression of quasiclassical propagators is obtained. The simplification is due to the fact that the interaction part of the classical action vanishes in this gauge. In the path-dependent formulation the optimal gauge corresponds to choice of the classical path in the definition of the vector potential.

Specifically, we have calculated the quasiclassical propagators of a scalar charged particle interacting with an arbitrary constant and uniform electromagnetic field, an arbitrary plane wave and, finally, an arbitrary plane wave combined with an arbitrary constant and uniform electromagnetic field. It is shown that in the classical path gauge the expressions for the quasiclassical propagators, which yield the exact result for above configurations, are more compact.

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