Linear-Quadratic Mixed Stackelberg–Nash Stochastic Differential Game with Major–Minor Agents

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Abstract

In this paper, we study a controlled linear-quadratic-Gaussian large population system combining three types of interactive agents mixed, which are respectively, major leader, minor leaders, and minor followers. In reality, they may represent three typical types of participants involved in market price formation: major supplier, minor suppliers and minor producers. The Stackelberg–Nash–Cournot (SNC) approximate equilibrium is derived from the combination of a major–minor mean-field game (MFG) and a leader–follower Stackelberg game. Although all agents are of forward states in that only initial conditions are specified in their dynamics, our SNC analysis provides an MFG framework that is naturally in a forward–backward state in that both initial and terminal conditions are specified. This result differs from those reported in the literature on standard MFG frameworks, mainly as a result of the adoption of a Stackelberg structure. Through variational analysis, the consistency condition system can be represented by some fully-coupled forward–backward-stochastic-differential-equations with a high-dimensional block structure in an open-loop case. To sufficiently address the related solvability, we also derive the feedback form of the SNC approximate equilibrium strategy via some coupled Riccati equations. Our study includes various mean-field game models as its special cases.

Keywords  Forward–backward-stochastic-differential-equations (FBSDE) · Leader–follower game · Major–minor (MM) game · Mean-field game (MFG) · Open-loop (OL) strategy · Stackelberg–Nash–Cournot (SNC) approximate equilibrium
1 Introduction

On a given finite decision horizon $[0, T]$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a $(1 + N_l + N_f)$-dimensional standard Brownian motion $(W_0(t), W_i(t), \tilde{W}_j(t))_{0 \leq t \leq T}$. In this paper, we consider a large-population system involving $(1 + N_l + N_f)$ individual agents (where $N_l$ and $N_f$ are sufficiently large), which are mixed with three types of agents: major leader $\mathcal{A}_0$; minor leaders $\mathcal{A}_i^f$, $1 \leq i \leq N_l$; and followers $\mathcal{A}_j^f$, $1 \leq j \leq N_f$. The dynamics of $\mathcal{A}_0$, $\{\mathcal{A}_i^f\}_{i=1}^{N_l}$, $\{\mathcal{A}_j^f\}_{j=1}^{N_f}$ are given sequentially by the following controlled linear stochastic differential equations:

$$\begin{align*}
\mathcal{A}_0 : & \quad \begin{cases}
\mathrm{d}X(t) = \{A_0X(t) + B_0u_0(t) + E_0^1X^{(N_l)}(t) + E_0^2X^{(N_f)}(t)\} \, \mathrm{d}t \\
\quad + \{C_0X(t) + D_0u_0(t) + E_0^2X^{(N_l)}(t) + E_0^2X^{(N_f)}(t)\} \, \mathrm{d}W(t), & (1) \\
X(0) = \xi_0,
\end{cases}
\end{align*}$$

$$\begin{align*}
\mathcal{A}_i^f : i = 1, 2, \ldots, N_l, & \quad \begin{cases}
\mathrm{d}X_i(t) = \{AX_i(t) + Bu_i(t) + E_1X^{(N_l)}(t)\} \, \mathrm{d}t \\
\quad + \{CX_i(t) + Du_i(t) + E_2X^{(N_l)}(t)\} \, \mathrm{d}W_i(t), & (2) \\
X_i(0) = \xi_i,
\end{cases}
\end{align*}$$

$$\begin{align*}
\mathcal{A}_j^f : j = 1, 2, \ldots, N_f, & \quad \begin{cases}
\mathrm{d}X_j(t) = \{\tilde{A}X_j(t) + \tilde{B}u_j(t) + F_1X^{(N_f)}(t)\} \, \mathrm{d}t \\
\quad + \{\tilde{C}X_j(t) + \tilde{D}u_j(t) + F_2X^{(N_f)}(t)\} \, \mathrm{d}\tilde{W}_j(t), & (3) \\
x_j(0) = \xi_j,
\end{cases}
\end{align*}$$

where $X^{(N_l)}(t) = \frac{1}{N_l} \sum_{i=1}^{N_l} X_i(t)$ and $X^{(N_f)}(t) = \frac{1}{N_f} \sum_{j=1}^{N_f} X_j(t)$ are called the state average or mean field term. In (1), (2) and (3), $X_0(\cdot), X_i(\cdot)$ and $x_j(\cdot)$ are called state processes; they take values in $\mathbb{R}^n$ with initial values $\xi_0, \xi_i, \xi_j$, which are random variables. $u_0(\cdot), u_i(\cdot)$ and $v_j(\cdot)$ are called admissible controls; they are applied by $(1 + N_l + N_f)$ agents in the game and take values in $\mathbb{R}^{m_1}, \mathbb{R}^{m_2}$ and $\mathbb{R}^{m_3}$, respectively.

Let $(u_0, u, v) = (u_0, u_1, \ldots, u_{N_l}, v_1, \ldots, v_{N_f})$ denote the strategy set of all $(1 + N_l + N_f)$ agents. Here, $u = (u_1, \ldots, u_{N_l})$ is the set of strategies of all $N_l$ minor-leader agents; $v = (v_1, \ldots, v_{N_f})$ is the set of strategies of all $N_f$ followers; $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{N_l})$ the control strategy set of minor-leader agents except $\mathcal{A}_i^f$; and $v_{-j} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{N_f})$ the control strategy set of followers except $\mathcal{A}_j^f$. The decision performance can be evaluated by the following cost functionals: for $\mathcal{A}_0$,

$$\begin{align*}
J_0(u_0(\cdot), u(\cdot), v(\cdot)) = & \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \|X_0(t) - (\lambda_0 X^{(N_l)}(t) + (1 - \lambda_0) X^{(N_f)}(t))\|_{Q_0}^2 \\
\quad + \|u_0(t)\|_{R_0}^2 \right) \, \mathrm{d}t + \|X_0(T)\|_{H_0}^2 \right\};
\end{align*}$$

(4)
for $A_i^f$, $1 \leq i \leq N_f$,

$$
\mathcal{J}_i^f(u_0(\cdot), u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left( \|X_i(t) - (\lambda X^{(N_f)})(t) + (1 - \lambda)X_0(t)\|_Q^2 + \|u_i(t)\|^2_R \right)dt + \|X_i(T)\|^2_{\tilde{H}} \right\};
$$

and for $A_j^f$, $1 \leq j \leq N_f$,

$$
\mathcal{J}_j^f(u_0(\cdot), u(\cdot), v_j(\cdot), v_{-j}(\cdot)) = \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left( \|x_j(t) - (\tilde{\lambda}_1X_0(t) + \tilde{\lambda}_2X^{(N_f)})(t) + \tilde{\lambda}_3x^{(N_f)})(t)\|_Q^2 + \|v_j(t)\|^2_R \right)dt + \|x_j(T)\|^2_{\tilde{H}} \right\},
$$

where for given vector $z$, $\|z\|_M^2 = \langle Mz, z \rangle$ for symmetric matrix $M$, and $Q_0$, $\tilde{Q}$, $R_0$, $R$, $\tilde{R}$, $H_0$, $H$, and $\tilde{H}$ are deterministic symmetric matrixes of suitable dimensions. Moreover, the convex factors $0 \leq \lambda_i, \lambda \leq 1$ and $\sum_{i=1}^3 \lambda_i = 1$ with $0 \leq \tilde{\lambda}_i \leq 1, i = 1, 2, 3$.

We notice that all agents are coupled not only in their state process but also in their cost functionals with convex combinations of state averages. This can be interpreted to track the centroid of population mass. In brief, the game to be studied is implemented as follows. First, the major leader $A_0$ and minor leaders $A_i^f$ announce their strategies $u_0(\cdot), u(\cdot)$ respectively and commit to implementing them. Next, the minor followers $A_j^f$ provide their best response to minimize their cost functional $\mathcal{J}_j^f(u_0(\cdot), u(\cdot), v_j(\cdot), v_{-j}(\cdot))$ accordingly. This reduces some best response functionals for minor followers, depending on the control law of leaders. With this functional in mind, before the announcement, agent $A_0$ designs the optimal strategy for minimizing his or her own cost functional $\mathcal{J}_0(u_0(\cdot), u(\cdot), v(\cdot))$. A similar design should simultaneously be employed by the minor leaders agents $A_i^f$ to minimize their own cost functional $\mathcal{J}_i^f(u_0(\cdot), u_i(\cdot), u_{-i}(\cdot))$ by selecting the optimal control $u_i(\cdot)$. As evidenced by the weakly-coupling among the agents in a large-population system, the aforementioned game problem is essentially a high-dimensional Stackelberg differential game. The influence of individual agents (leaders or followers) on the population should be evenly distributed when the population size tends to infinity; this is not the case for the $A_0$ because of its dominant effects on all minor agents through cost functionals.

We call the aforementioned formulated problem the **mixed Stackelberg major–minor differential game**, and hereafter for short, denoted by (S-MM). Here, “Mixed” refers to two subgame formulations involved in the aforementioned mechanism. The first is a game played by all leaders (minor and major) against all followers. This is formulated through the involvement of multiple leaders and multiple followers. The second is a major–minor (MM) game between major and minor leaders. These two subgames are further mixed in this large-population system when some mean-field
approximation argument is applied. Such a formulation is a non-trivial generalization of various well-studied game topics, and is further explained in the following.

(Single leader–follower game) In the case where \( N_l = 0 \) and \( N_f = 1 \), there are no minor leaders but rather only a single follower with one major leader, and our problem is reduced to the classical single-leader and single-follower game. The leader-follower (Stackelberg) game was proposed in 1934 by von Stackelberg [35], when he defined the concept of a hierarchical solution for markets in which some firms have more power than others and thus hold a dominant position. This solution concept is termed the Stackelberg equilibrium. An early study of stochastic Stackelberg differential games (SSDGs) was conducted by Basar [4]. Yong [38] conducted a relevant study introducing a linear-quadratic (LQ) leader–follower stochastic differential game and studied its open-loop (OL) information, concluding that its coefficients of system and cost functionals may be random, controls are used to enter the diffusion term of state dynamics, and the weight matrices for the controls of cost functionals are not necessarily positive definite. In a similar but nonlinear setting, Bensoussan et al. [8] obtained the global maximum principles for both OL and closed-loop (CL) SSDGs, but the diffusion term did not contain the controls. This simplifies the related analysis to a certain extent. The solvability of related Riccati equations in the special LQ setting is also discussed, and the state-feedback Stackelberg equilibrium is thus obtained.

(Multiple leaders–followers game) In case \( N_l, N_f \) are of medium or small size, then our problem is reduced to the Stackelberg game with multiple leaders and multiple followers. It is a natural extension of the single leader–follower game and examples include [7,9,31], etc. Such a setting has been adopted in various applications, such as [28,36], especially in markets for production planning, with leaders representing multiple suppliers and followers representing multiple producers.

(Mean-field-game with symmetric agents) In case \( N_l = 0 \) and no \( A_0 \) involved, then our problem is reduced to the standard mean-field game with a very large number of minor (symmetric) agents. Each single agent interacts with the mass-effect of other peers only through coupling in states/dynamics. For large-population systems with stochastic dynamics, the centralized strategy for achieving the exact Nash equilibrium is challenging to identify and implement. This may be attributable to the well-known “curse of dimensionality” intrinsic in such large-population systems. Alternatively, one effective scheme based on mean field game (MFG) theory is a search among its decentralized strategies. Consequently, an approximate Nash equilibrium can be designed with a less substantial computing burden. As a trade-off, some equilibrium performance loss amounting to \( o(1) \) occurs as \( N \rightarrow +\infty \).

Many studies have been conducted on MFGs. Since the independent research of Huang et al. [18,19] and Lasry and Lions [25–27], MFG theory and its applications have experienced rapid growth. Related developments in MFG theory may include those in the research of Bardi [3], Bensoussan et al. [6], Carmona and Delarue [10], Garnier et al. [14], Guéant et al. [15], and the references therein. Note that mean-field game differs from the mean-field type control such as [1,12].

(Major–minor mixed game) In case \( N_f = 0 \), then there are no followers but rather only major and minor leaders, and our problem becomes an MM-MFG. The MM-MFG was introduced in [17], and was thoroughly investigated by [30] as an MM mean-field LQG game. Our model generalizes [7,9,30,31] by adopting not only a leader–
follower structure but also an MM structure to thus elucidate the market mechanism of some monopoly suppliers, minor negligible suppliers, and numerous producers in the market.

(Convex combination) The convex combination in our functional is described as follows. Referencing Nourian, Caines, Malhamé and Huang (2012) [31], we consider a kind of general case of a cost functional and its likelihood ratio (i.e., convex combination). For example, the cost functional of the major leaders is based on a trade-off between maintaining cohesion among minor leaders and maintaining cohesion among followers (see (4)). However, we may be more interested in a special case such as \( \lambda_3 = 0 \) which means the cost functional of followers is not directly influenced by the major-leader.

In summary, the present work combines the leader–follower problem with the MM problem in the context of a large-scale population with a large number of agents. In the entire system, the major agent and some of the minor agents are regarded together as the leaders (referred to as the major leader and minor leaders, respectively), and the remaining minor agents are referred to as followers. Obviously, such a structure with advanced complexity engenders some technical difficulties. Thus, we adopt the current model by considering the balance between model’s generality and presentation’s tractability. Meanwhile, the current model, although somehow restrictive, can still cover most common-applied models in mean-field studies.

Let us now explain the basic analysis scheme of our problem, which can be implemented according to the following phrases.

(i): Fix the decision pair of the major leader and mass effect limit of the minor leaders, denoted as \((X_0, u_0)\) and \(\bar{m}_X\) respectively. Given these fixed quantities \((X_0, u_0, \bar{m}_X)\), introduce and solve the mean-field subgame encountered by all minor followers who also compete within their interaction cycle. For such a subgame, an auxiliary problem can be constructed and some decentralized response of minor followers can be derived; the related mass limit response of minor followers is denoted as \(m_x = \bar{m}_x(X_0, u_0, \bar{m}_X)\).

(ii): Given the fixed mass effect limit of minor leaders \(\bar{m}_X\) and the response functional of minor followers \(m_x\), solve the decentralized stochastic control problem of the major leader \(A_0\); the optimal solution pair is denoted as \((\bar{X}_0, \bar{u}_0) = (\bar{X}_0(m_x, \bar{m}_X), \bar{u}_0(m_x, \bar{m}_X))\).

(iii): Given \((\bar{m}_X, \bar{X}_0, \bar{u}_0)\), solve an auxiliary control problem for minor leaders with an optimal decentralized strategy \(\bar{u}(\cdot)\). Note that \(\bar{u}(\cdot)\) is influenced by the optimal control of the major leader \(\bar{u}_0(\cdot)\) and the response functional of the minor followers \(m_x\), which in turn depends on \(u(\cdot)\) selected by the minor leaders.

(iv): Derive a consistency condition (CC) system to determine \(\bar{m}_X\); subsequently, all decentralized strategies for the major leader, minor leaders and followers can sequentially be designed. An approximate SNC equilibrium can then be obtained.

The main contribution of this paper can be summarized as follows:

- The decentralized strategy profile of mixed leader–follower and MM games is studied for both OL and CL scenarios.
- The existence and uniqueness of the related CC system is investigated in a global solvability case.
• The CC system is represented by a fully coupled mean-field-type FBSDE in an OL case. Its equivalent representation by a nonstandard Riccati equation in a CL case is also addressed.
• The approximate SNC equilibrium is verified under a more general condition (unnecessary with the standard assumption of positive definitiveness).

The remainder of this paper is organized as follows. Section 2 provides the problem formulation and presents some preliminary details. In Sect. 3, we discuss the OL strategy of Stackelberg mixed MM games, and in Sect. 4, we derive a CC system based on OL and feedback strategies, which both provide fully coupled FBSDEs. Furthermore we introduce some Riccati equations that enable sufficient solvability of the relevant CC system. Section 5 is devoted to verifying the approximate equilibrium of OL strategies. Potential applications are further discussed in Sect. 6.

2 Preliminary and Formulation

The following notations are used throughout this paper. Let $\mathbb{R}^n$ denotes the $n$−dimensional Euclidean space, $\mathbb{R}^{n \times m}$ be the set of all $(n \times m)$ matrices, and let $\mathcal{S}^n$ be the set of all $(n \times n)$ symmetric matrices. We denote the transpose by subscript $\top$, the inner product by $\langle \cdot, \cdot \rangle$ and the norm by $| \cdot |$. For $t \in [0, T]$ and Euclidean space $\mathbb{H}$, we introduce the following function spaces:

\[
L^p(t, T; \mathbb{H}) = \left\{ \psi : [t, T] \to \mathbb{H} \left| \int_t^T |\psi(s)|^p ds < \infty \right. \right\}, \quad 1 \leq p < \infty,
\]
\[
L^\infty(t, T; \mathbb{H}) = \left\{ \psi : [t, T] \to \mathbb{H} \left| \text{ess} \sup_{s \in [t, T]} |\psi(s)| < \infty \right. \right\},
\]
\[
C([t, T]; \mathbb{H}) = \left\{ \psi : [t, T] \to \mathbb{H} \left| \psi(\cdot) \text{ is continuous} \right. \right\},
\]

and the spaces of process or random variables on given filtrated probability space with a general filtration $\mathcal{H} = \{ \mathcal{H}_t \}_{0 \leq t \leq T}$:

\[
L^2_{\mathcal{H}_t}(\Omega; \mathbb{H}) = \left\{ \xi : \Omega \to \mathbb{H} \left| \xi \text{ is } \mathcal{H}_t-\text{measurable, } \mathbb{E}[|\xi|^2] < \infty \right. \right\},
\]
\[
L^2_{\mathcal{H}}(t, T; \mathbb{H}) = \left\{ \psi : [t, T] \times \Omega \to \mathbb{H} \left| \psi(\cdot) \text{ is } \mathcal{H}_t-\text{progressively measurable, } \mathbb{E}\int_t^T |\psi(s)|^2 ds < \infty \right. \right\}.
\]

We set the following information structures that are important to admissible strategies defined soon: $\{ \mathcal{F}_t \}_{0 \leq t \leq T}$ is the natural filtration generated by all Brownian motion components $\{W_0(\cdot), \xi_0, W_i(\cdot), \xi_i, \widehat{W}_j(\cdot), \zeta_j\}$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$, $1 \leq i \leq N_i$, $1 \leq j \leq N_f$, it captures the full information of all states and noises; $\{ \mathcal{F}_i^0 \}_{0 \leq t \leq T}$ is the natural filtration generated by $\{W_0(\cdot), X_0(\cdot), \xi_0\}$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. It is the space on which the limiting state-average should be adapted; $\{ \mathcal{F}_i^1 \}_{0 \leq t \leq T}$ is the natural filtration generated by $\{W_i(\cdot), X_i(\cdot), \xi_i\}$ augmented
by all $\mathbb{P}$-null sets in $\mathcal{F}, 1 \leq i \leq N_l$; \{\mathcal{G}_i^j\}_{0 \leq j \leq T}$ is the natural filtration generated by \{\mathcal{W}_j(\cdot), x_j(\cdot), \zeta_j\} augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}, 1 \leq j \leq N_f$.

Given information structures, we can set the following Hilbert spaces to define *centralized* strategies for individual agents in an OL sense:

$$\mathcal{U}_i^c[0, T] \triangleq L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m_1}),$$

$$\mathcal{U}_i^d[0, T] \triangleq L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m_2}), \quad i = 1, 2, \ldots, N_l,$$

$$\mathcal{V}_j^c[0, T] \triangleq L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m_3}), \quad j = 1, 2, \ldots, N_f,$$

and *decentralized* open-loop strategies:

$$\mathcal{U}_i^d[0, T] \triangleq L^2_{\mathcal{F}_0}(0, T; \mathbb{R}^{m_1}),$$

$$\mathcal{U}_i^d[0, T] \triangleq L^2_{\mathcal{F}_0}(0, T; \mathbb{R}^{m_2}), \quad i = 1, 2, \ldots, N_l,$$

$$\mathcal{V}_j^d[0, T] \triangleq L^2_{\mathcal{F}_j}(0, T; \mathbb{R}^{m_3}), \quad j = 1, 2, \ldots, N_f.$$

When implementing the SNC strategy, it is sometimes helpful to establish the product spaces as follows.

Then any $(u_0, u, v) \in \mathcal{U}_0^c[0, T] \times \mathcal{U}_0^c[0, T] \times \mathcal{V}_0^c[0, T]$ is called an *admissible centralized strategy*, and any $(u_0, u, v) \in \mathcal{U}_0^d[0, T] \times \mathcal{U}_0^d[0, T] \times \mathcal{V}_0^d[0, T]$ is called an *admissible decentralized strategy*.

Let us introduce the following hypotheses regarding the coefficients of state dynamics and cost functionals:

**\(\text{H1} \)** The coefficients of the state equations and cost functionals satisfy the following:

$$\begin{cases}
  A_0, A, \tilde{A}, C_0, C, \tilde{C}, E_0^1, E_0^2, E_1, E_2, F_0^1, F_0^2, F_1, F_2 \in \mathbb{R}^{n \times n}; \\
  B_0, D_0 \in \mathbb{R}^{n \times m_1}; \quad B, D \in \mathbb{R}^{n \times m_2}; \quad \tilde{B}, \tilde{D} \in \mathbb{R}^{n \times m_3}; \\
  Q_0, \tilde{Q}, H_0, H, \tilde{H} \in S^n; \\
  R_0 \in S^{m_1}; \quad R \in S^{m_2}; \quad \tilde{R} \in S^{m_3}.
\end{cases}$$
The initial states \( \xi_0, \xi_i, \xi_j \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \) are mutually independent of Brownian motions \( W_0, W_i, W_j ; \mathbb{E}[\xi_i] = \mathbb{E}[\xi_j] = 0 \), for each \( i = 1, \ldots, N_i \), \( j = 1, 2, \ldots, N_f \); and there exists \( c_0 < \infty \) independent of \( N_i \) and \( N_f \) such that \( \sup_{i \geq 0} \mathbb{E}[|\xi_i|^2] \leq c_0 \) and \( \sup_{j \geq 1} \mathbb{E}[|\xi_j|^2] \leq c_0 \).

Because no positive definitive conditions for the weighting matrix are imposed in (H1), the hypothesis generalizes the standard assumption for the LQ control setup. Moreover, the coefficients of the convex combination \( 0 \leq \lambda_0, \lambda, \lambda_1, \lambda_2, \lambda_3 \leq 1 \). Under (H1)–(H2), for any \( (\xi_0, u_0(\cdot)) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{U}^c_0[0, T] \) (resp., \( L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{U}^d_0[0, T] \)), \( (\xi_i, u_i(\cdot)) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{U}^c_i[0, T] \) (resp., \( L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{U}^d_i[0, T] \)), \( (\xi_j, v_j(\cdot)) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{V}_j[0, T] \) (resp., \( L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times \mathcal{V}_j^d[0, T] \)), (1), (2), and (3) admit unique (strong) solutions, and the cost functionals (4), (5), and (6) are also well-defined.

For simplicity, (H2) assumes that all minor leaders and followers have the same (zero) initial means. It is straightforward to generalize to the case with different (non-zero) initial means, provided the sets \( \{\mathbb{E}[\xi_i], i \geq 1\} \) and \( \{\mathbb{E}[\xi_j], j \geq 1\} \) admit a limiting empirical distribution. We can then introduce the Stackelberg–Nash–Cournot (SNC) equilibrium as follows.

**Definition 2.1** A \((1 + N_i + N_f)\)-tuple \( (\overline{\mathcal{U}}_0[\cdot], \bar{u}(\cdot), \overline{\mathcal{V}}[\cdot]) \), is called an open-loop Stackelberg–Nash–Cournot equilibrium for the initial states \( \xi_0, \xi_i, \xi_j \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \) if:

\[
\mathcal{J}_j^f(u_0(\cdot), u(\cdot), v_j(\cdot), v_{-j}(\cdot)) = \inf_{v_j(\cdot) \in \mathcal{V}_j[0, T]} \mathcal{J}_j^f(u_0(\cdot), u(\cdot), v_j(\cdot), v_{-j}(\cdot)),
\]

\( \forall u_0(\cdot) \in \mathcal{U}^c_0[0, T], \quad u(\cdot) \in \mathcal{U}^c[0, T], \quad \mathcal{J}_0(\overline{\mathcal{U}}_0[\cdot], \xi_0, \xi_i, \xi_j[\cdot], u(\cdot), \overline{\mathcal{V}}[\mathcal{U}_0[\cdot], \xi_0, \xi_i, \xi_j[\cdot])] \) (9)

\[
\mathcal{J}_i^f(\overline{\mathcal{U}}_i[\cdot], \mathcal{U}_{-i}(\cdot), \xi_0, \xi_i, \xi_j[\cdot], \overline{\mathcal{U}}_i[\cdot], \mathcal{U}_{-i}(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}^c_i[0, T]} \mathcal{J}_i^f(\overline{\mathcal{U}}_i[\cdot], \mathcal{U}_{-i}(\cdot), \xi_0, \xi_i, \xi_j[\cdot], u_i(\cdot), \mathcal{U}_{-i}(\cdot)),
\]

where \( \overline{\mathcal{V}}_j : \mathcal{U}^c_0[0, T] \times \mathcal{U}^c[0, T] \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \rightarrow \mathcal{V}_j^c[0, T], \) and \( \overline{\mathcal{U}}_0 : \mathcal{U}^c[0, T] \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \rightarrow \mathcal{U}^c_0[0, T]. \)

The aforementioned equilibrium definition is in OL and centralized case. Analog definition can be introduced for CL in decentralized case. Noticing this definition specifies some exact equilibrium. The following analysis aims to derive its counterpart in an approximate equilibrium when the population size tends to infinity. To this end, we should analyze a combined MM and leader–follower game, to be discussed below.
2.1 Mixed SNC Equilibrium Analysis

Analysis of a mixed leader–follower MM game using MFG theory should proceed with the following two schemes. The first scheme is a leader–follower subgame for which the key point is to specify the mean-field response functional of a large number of followers for any decision announced by major or minor leaders. The decentralized decisions of all leaders must then be analyzed in the second scheme, which entails an MM mean-field subgame. The two schemes are further coupled in the CC argument, which is a crucial component for determining the decentralized decision in a mean-field game through fixed-point analysis.

The described analysis path differs from that of a pure MM-MFG or leader–follower game; for these pure game types, only one analysis scheme must be adopted, and the relevant analysis is relatively tractable. Our analysis is more comprehensive and can be described in a sequential manner as follows.

Step 1: MFG analysis of followers
The MFG analysis of followers can be further decomposed into substeps.

Step 1.1 Let us introduce the auxiliary limiting LQG optimization problems. Firstly, in the Stackelberg game, for given strategy announced by major leader and minor leaders, the minor followers face minimization of the cost functionals given by

$$J_f^j(\xi_0, \zeta_j, \overline{m}_X(\cdot); u_0(\cdot), u(\cdot), v_j(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \|x_j(t) - (\tilde{\lambda}_1 X_0(t) + \tilde{\lambda}_2 \overline{m}_X(t) + \tilde{\lambda}_3 x^{(N_f)}(t))\|^2_{\tilde{Q}} + \|v_j(t)\|^2_R \right) dt + \|x_j(T)\|^2_H \right\},$$

where $\overline{m}_X(\cdot) = \lim_{N_l \to +\infty} X^{(N_l)}(\cdot)$. Furthermore, as $N_f \to +\infty$, we suppose that $x^{(N_f)}(\cdot)$ can be approximated by the $\mathcal{F}_t^0$-adapted function $\overline{m}_x(\cdot)$. Similarly, as $N_l \to +\infty$, we suppose that $X^{(N_l)}(\cdot)$ can be approximated by the $\mathcal{F}_t^0$-adapted function $\overline{m}_X(\cdot)$. The state process of the follower is then given by

$$\begin{cases} d\overline{x}_j(t) = [\tilde{A}\overline{x}_j(t) + \tilde{B}v_j(t) + F_1 \overline{m}_x(t)]dt \\ + [\tilde{C}\overline{x}_j(t) + \tilde{D}v_j(t) + F_2 \overline{m}_x(t)]d\tilde{W}_j(t), \end{cases} \quad \overline{x}_j(0) = \zeta_j,$$

with the auxiliary cost functionals

$$J_f^f(\xi_0, \zeta_j, \overline{m}_X(\cdot), \overline{m}_x(\cdot); u_0(\cdot), v_j(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \|x_j(t) - (\tilde{\lambda}_1 X_0(t) + \tilde{\lambda}_2 \overline{m}_X(t) + \tilde{\lambda}_3 \overline{m}_x(t))\|^2_{\tilde{Q}} + \|v_j(t)\|^2_R \right) dt + \|x_j(T)\|^2_H \right\}.$$
for $A^j_f$, $1 \leq j \leq N_f$. To distinguish this problem from the original problem, we use the new state variables $\overline{x}_j$ and we will denote $\overline{x}_0$ and $\overline{x}_i$ the new state variables later. However, we still use the same set of variables $u_0, u_i, v_j, W_0, W_i, \tilde{W}_j$ in this auxiliary limiting problem to avoid confusion. Next, we introduce the following auxiliary limiting problem for followers, using its OL solution, which is denoted by (OL1), as a reference.

**Problem (OL1).** For the given $\xi_0, \zeta_j \in L_2^2(\Omega; \mathbb{R}^n)$, $\mathcal{F}_t^0$-adapted functions $\overline{m}_X(\cdot), \overline{m}_x(\cdot)$, and the control $u_0(\cdot)$ of the major leader $\mathcal{A}_0$, find the optimal response functional $\overline{v}_j[\cdot] : \mathcal{U}_0^d[0, T] \times L_2^2(\Omega; \mathbb{R}^n) \times L_2^2(0, T; \mathbb{R}^n) \times L_2^2(\Omega; \mathbb{R}^n) \rightarrow \mathcal{V}_j^d[0, T]$ of the following differential games among followers:

$$J_f^j(\xi_0, \zeta_j, \overline{m}_X(\cdot), \overline{m}_x(\cdot); u_0(\cdot), \overline{v}_j[u_0(\cdot), \overline{m}_X(\cdot), \overline{m}_x(\cdot), \xi_0, \zeta_j])$$

(OL1) : \[ \inf_{v_j(\cdot) \in \mathcal{V}_j^d[0, T]} J_f^j(\xi_0, \zeta_j, \overline{m}_X(\cdot), \overline{m}_x(\cdot); u_0(\cdot), v_j(\cdot)). \]

**Step 1.2** Fix $\overline{m}_x$, and consider the optimal response functional of the above problem (OL1) for the representative follower agent denoted by $\overline{v}_j[\cdot]$. For given $\xi_0, \zeta_j \in L_2^2(\Omega; \mathbb{R}^n)$, $\mathcal{F}_t^0$-adapted functions $\overline{m}_X(\cdot), \overline{m}_x(\cdot)$, and the control $u_0(\cdot)$ of the major leader $\mathcal{A}_0$, find an OL strategy $\overline{v}_j(\cdot) = \overline{v}_j[u_0(\cdot), \overline{m}_X(\cdot), \overline{m}_x(\cdot), \xi_0, \zeta_j] \in \mathcal{V}_j^d[0, T]$, $1 \leq j \leq N_f$. In other words, find the optimal response functional $\overline{v}_j[\cdot] : \mathcal{U}_0^d[0, T] \times L_2^2(0, T; \mathbb{R}^n) \times L_2^2(\Omega; \mathbb{R}^n) \times L_2^2(\Omega; \mathbb{R}^n) \rightarrow \mathcal{V}_j^d[0, T]$ of the following differential games among followers:

$$J_f^j(\xi_0, \zeta_j, \overline{m}_X(\cdot), \overline{m}_x(\cdot); u_0(\cdot), \overline{v}_j[u_0(\cdot), \overline{m}_X(\cdot), \overline{m}_x(\cdot), \xi_0, \zeta_j])$$

= \[ \inf_{v_j(\cdot) \in \mathcal{V}_j^d[0, T]} J_f^j(\xi_0, \zeta_j, \overline{m}_X(\cdot), \overline{m}_x(\cdot); u_0(\cdot), v_j(\cdot)). \]

**Step 1.3** Apply the state-aggregation method to determine the state-average limit $\overline{m}_x$ according to the following CC qualification:

$$\mathbb{E}\left[\overline{x}_j(\overline{v}_j[u_0(\cdot), \overline{m}_X(\cdot), \overline{m}_x(\cdot), \xi_0, \zeta_j])|\mathcal{F}_t^0\right] = \overline{m}_x.$$ 

By following this step, the optimal response functional of the follower and $\overline{m}_x = \overline{m}_x(\xi_0, \zeta_j, u_0, \overline{m}_X)$ can be determined for any admissible strategy announced by leaders.

**Step 2:** **MFG analysis of the major leader** By anticipating the optimal response functional of followers $\overline{m}_x = \overline{m}_x(\xi_0, \zeta_j, u_0, \overline{m}_X)$, the major leader could solve his or her own problem. The state process of the major leader is then given by

$$\begin{align*}
\mathrm{d}\overline{X}_0(t) &= \{A_0\overline{X}_0(t) + B_0u_0(t) + E_0^1\overline{m}_X(t) + F_0^1\overline{m}_x(t)\}dt \\
&\quad + \{C_0\overline{X}_0(t) + D_0u_0(t) + E_0^2\overline{m}_X(t) + F_0^2\overline{m}_x(t)\}dW_0(t), \\
\overline{X}_0(0) &= \xi_0.
\end{align*}$$

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with the auxiliary cost functional
\[
J_0(\xi_0, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \| X_0(t) - (\lambda_0 \bar{m}_X(t) + (1 - \lambda_0)\bar{m}_x(t)) \|^2_{Q_0} \\
+ \| u_0(t) \|^2_{R_0} \right) dt + \| X_0(T) \|^2_{H_0} \right\},
\]
for \( \mathcal{A}_0 \). Subsequently, we can set the following auxiliary problem for the major-leader.

**Problem (OL2).** For the given \( \xi_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \) and \( \mathcal{F}_x^0 \)-adapted functions \( \bar{m}_X(\cdot) \), find an OL strategy \( \bar{u}_0(\cdot) \in \mathcal{U}_0^d[0, T] \) such that
\[
\text{(OL2): } J_0(\xi_0, \bar{m}_X(\cdot), \bar{m}_x(\cdot); \bar{u}_0(\cdot)) = \inf_{u_0(\cdot) \in \mathcal{U}_0^d[0, T]} J_0(\xi_0, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot)).
\]

**Step 3: MFG analysis of minor leaders** Predict the optimal response functional of followers \( \bar{m}_x = \bar{m}_x(\xi_0, \zeta_j, u_0, \bar{m}_X) \) and the optimal control \( \bar{u}_0 \) of the major leader. Use the state process
\[
\begin{aligned}
d\bar{X}_i(t) &= (A\bar{X}_i(t) + Bu_i(t) + E_1\bar{m}_X(t))dt \\
&\quad + [C\bar{X}_i(t) + Du_i(t) + E_2\bar{m}_X(t)]dW_i(t), \\
\bar{X}_i(0) &= \xi_i,
\end{aligned}
\]
with the auxiliary cost functionals
\[
J^i_1(\xi_0, \xi_i, \bar{m}_X(\cdot); u_i(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( \| X_i(t) - (\lambda \bar{m}_X(t) + (1 - \lambda)\bar{X}_0(t)) \|^2_{Q_0} \\
+ \| u_i(t) \|^2_{R} \right) dt + \| X_i(T) \|^2_{H_i} \right\},
\]
for \( \mathcal{A}^i_1 \), \( 1 \leq i \leq N_l \). We then address the following problem for minor leaders. For reference, this problem is denoted as (OL3) because we also address its OL solution.

**Problem (OL3).** For the given \( \xi_0, \xi_i \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \times L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \), and \( \mathcal{F}_x^0 \)-adapted functions \( \bar{m}_X(\cdot) \), find an OL strategy \( \bar{u}_i(\cdot) \in \mathcal{U}_i^d[0, T] \), \( 1 \leq i \leq N_l \), such that
\[
\text{(OL3): } J^i_1(\xi_0, \xi_i, \bar{m}_X(\cdot); \bar{u}_i(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_i^d[0, T]} J^i_1(\xi_0, \xi_i, \bar{m}_X(\cdot); u_i(\cdot)).
\]

**Step 4: CC of the SNC equilibrium** Apply the CC argument to determine the frozen \( \bar{m}_X \) by using
\[
\mathbb{E} \left[ \bar{X}_i(\bar{u}_i(\bar{m}_X)) \right] = \bar{m}_X,
\]
and demonstrate the global solvability of the related CC system.

For clearer illustration, the steps of this scheme are summarized in the following diagram.
3 Open-Loop Strategies of Stackelberg Mixed Major–Minor Games

From now on, we might suppress time variable $t$ in case no confusion occurs. In this section, we study the Mixed S-MM game strategy in its open-loop (OL) sense.

3.1 Open-Loop Strategies for Followers

In this subsection, we solve out Problem (OL1) firstly. The main result of this section can be stated as follows.

**Theorem 3.1** Under assumptions (H1), (H2), and let $\xi_j \in L^2_{\mathcal{F}}(\Omega; \mathbb{R}^n)$, $u_0(\cdot) \in U_0^d[0, T]$, $X_0(\cdot), m_X(\cdot), \bar{m}_x(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ be given. Then for initial value $\xi_j$, Problem (OL1) admits an open-loop decentralized optimal control $v_j(\cdot) \in V_d^j[0, T]$ if and only if the following two conditions hold:

(i) For $j = 1, 2, \ldots, N_f$, the adapted solution $(\bar{x}_j(\cdot), \bar{y}_j(\cdot), \bar{z}_j(\cdot), \bar{z}_j(0))$ to the FBSDE on $[0, T]$ satisfies the following stationarity condition:

\[
\bar{B}^\top \bar{y}_j + \bar{D}^\top \bar{z}_j = 0, \quad \text{a.e. } t \in [0, T], \text{ a.s.}
\]

(ii) For $j = 1, 2, \ldots, N_f$, the following convexity condition holds:

\[
\mathbb{E}\left\{ \int_0^T \left( \left\langle \bar{Q}x_j, x_j \right\rangle + \left\langle \bar{R}v_j, v_j \right\rangle \right) dt + \left\langle \bar{H}x_j(T), x_j(T) \right\rangle \right\} \geq 0,
\]

$\forall v_j(\cdot) \in V_d^j[0, T]$, (18)
where $x_j(\cdot)$ is the solution of

$$
\begin{align*}
\begin{cases}
    dx_j = \left\{ \tilde{A}x_j + \tilde{B}v_j \right\} dt + \left\{ \tilde{C}Y_j + \tilde{D}v_j \right\} d\tilde{W}_j(t), & t \in [0, T], \\
    x_j(0) = 0.
\end{cases}
\end{align*}
$$

Or, equivalently, the mapping $\nu_j(\cdot) \mapsto J^f_j(\xi_j, \zeta_j, m_X(\cdot), \bar{m}_X(\cdot); u_0(\cdot), \nu_j(\cdot))$, defined by (11), is convex (for $j = 1, 2, \ldots, N_f$).

**Proof** For given $\xi_j \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$, $u_0(\cdot) \in \mathcal{U}_0^d[0, T]$, $X_0(\cdot), \bar{m}_X(\cdot), \bar{m}_x(\cdot) \in L^2_{\mathcal{F}_0}(0, T; \mathbb{R}^n)$ and $\nu_j(\cdot) \in \mathcal{V}_j^d[0, T]$, let $(\bar{x}_j(\cdot), \bar{y}_j(\cdot), \bar{z}_j(\cdot), \bar{z}_0(\cdot))$ be adapted solution to FBSDE (16). For any $\nu_j(\cdot) \in \mathcal{V}_j^d[0, T]$ and $\varepsilon \in \mathbb{R}$, let $x_j^\varepsilon(\cdot)$ be the solution to the following perturbed state equation on $[0, T]$:

$$
\begin{align*}
\begin{cases}
    dx_j^\varepsilon = \left\{ \bar{A}x_j^\varepsilon + \bar{B}(\bar{y}_j + \varepsilon v_j) + F_1 \bar{m}_x \right\} dt \\
    + \left\{ \bar{C}x_j^\varepsilon + \bar{D}(\bar{y}_j + \varepsilon v_j) + F_2 \bar{m}_x \right\} d\tilde{W}_j(t), \\
    x_j^\varepsilon(0) = \zeta_j.
\end{cases}
\end{align*}
$$

Then denoting $x_j(\cdot)$ the solution of (19), we have $x_j^\varepsilon(\cdot) = \bar{x}_j(\cdot) + \varepsilon x_j(\cdot)$ and

$$
J^f_j(\xi_0, \zeta_j, m_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot), \nu_j(\cdot) + \varepsilon v_j(\cdot))
\begin{align*}
&= \frac{\varepsilon}{2} \mathbb{E} \left\{ \int_0^T \left( \langle \bar{Q} \left( 2x_j - 2(\bar{\lambda}_1 X_0 + \bar{\lambda}_2 \bar{m}_X + \bar{\lambda}_3 \bar{m}_x) \right) + \varepsilon x_j \right), x_j \right\} dt \\
&\quad + \left( \langle \bar{R} (2\bar{y}_j + \varepsilon v_j), v_j \rangle \right) \right\} \\
&= \varepsilon \mathbb{E} \left\{ \int_0^T \left( \langle \bar{Q} \left( \bar{x}_j - (\bar{\lambda}_1 X_0 + \bar{\lambda}_2 \bar{m}_X + \bar{\lambda}_3 \bar{m}_x) \right), x_j \right) + \left( \bar{R} v_j, v_j \right) \right\} dt \\
&\quad + \langle \bar{H} x_j(T), x_j(T) \rangle \\
&\quad + \frac{\varepsilon^2}{2} \mathbb{E} \left\{ \int_0^T \left( \langle \bar{Q} x_j, x_j \rangle + \langle \bar{R} v_j, v_j \rangle \right) dt + \langle \bar{H} x_j(T), x_j(T) \rangle \right\}.
\end{align*}
$$

On the other hand, applying Itô’s formula to $\langle \bar{y}_j, x_j \rangle$, and taking expectation, we obtain

$$
\mathbb{E} \left[ \langle \bar{H} x_j(T), x_j(T) \rangle \right] = \mathbb{E} \left\{ \int_0^T \left( \langle \bar{B}^\top \bar{y}_j + \bar{D}^\top \bar{z}_j, v_j \right) - \left( \bar{Q} \left( \bar{x}_j - (\bar{\lambda}_1 X_0 + \bar{\lambda}_2 \bar{m}_X + \bar{\lambda}_3 \bar{m}_x) \right), x_j \right) \right\} dt \right\}.
$$
Hence,

\[
J_f^j(\xi_0, \zeta_j, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot), v_j(\cdot) + \varepsilon v_j(\cdot))
- J_f^j(\xi_0, \zeta_j, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot), \bar{v}_j(\cdot))
= \varepsilon \mathbb{E}\left\{ \int_0^T \left( \tilde{B}^\top \tilde{y}_j + \tilde{R} \tilde{v}_j + \tilde{D}^\top \tilde{z}_j, v_j \right) dt \right\}
+ \frac{\varepsilon^2}{2} \mathbb{E}\left\{ \int_0^T \left( \left( \tilde{Q} x_j, x_j \right) + \left( \tilde{R} v_j, v_j \right) \right) dt + \left( \tilde{H} x_j(T), x_j(T) \right) \right\}.
\]

It follows that

\[
J_f^j(\xi_0, \zeta_j, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot), v_j(\cdot))
\leq J_f^j(\xi_0, \zeta_j, \bar{m}_X(\cdot), \bar{m}_x(\cdot); u_0(\cdot), \bar{v}_j(\cdot) + \varepsilon v_j(\cdot)),
\forall v_j(\cdot) \in \mathcal{V}_d^j[0, T], \forall \varepsilon \in \mathbb{R},
\]

if and only if (17) and (18) hold. \hfill \Box

Furthermore, if we assume \( \tilde{R} \) is invertible, then the optimal control satisfies:

\[
v_j = -\tilde{R}^{-1}(\tilde{B}^\top \tilde{y}_j + \tilde{D}^\top \tilde{z}_j),
\]

so the related Hamiltonian system can be represented by

\[
\begin{aligned}
\text{d}x_j &= \left\{ \tilde{A} x_j - \tilde{B} \tilde{R}^{-1}(\tilde{B}^\top \tilde{y}_j + \tilde{D}^\top \tilde{z}_j) + F_1 \bar{m}_x \right\} dt \\
&\quad + \left\{ \tilde{C} x_j - \tilde{D} \tilde{R}^{-1}(\tilde{B}^\top \tilde{y}_j + \tilde{D}^\top \tilde{z}_j) + F_2 \bar{m}_x \right\} d\tilde{W}_j(t),
\end{aligned}
\]

\[
\begin{aligned}
\text{d}y_j &= -\left\{ \tilde{A}^\top \tilde{y}_j + \tilde{C}^\top \tilde{z}_j + \tilde{Q} \left( x_j - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \bar{m}_X + \tilde{\lambda}_3 \bar{m}_x) \right) \right\} dt \\
&\quad + \tilde{z}_j d\tilde{W}_j(t) + \tilde{z}_j_0 dW_0(t),
\end{aligned}
\]

\[
\begin{aligned}
x_j(0) &= \zeta_j, & y_j(T) &= \tilde{H} x_j(T), & j &= 1, 2, \ldots, N_f.
\end{aligned}
\]

Based on above analysis, we have

\[
\bar{m}_x(\cdot) = \lim_{N_f \to +\infty} \frac{1}{N_f} \sum_{j=1}^{N_f} \bar{x}_j(\cdot) = \mathbb{E}[\bar{x}_j(\cdot)|\mathcal{F}_t^0].
\]

Here, the first equality of (21) is due to the consistency condition which means that the frozen term \( \bar{m}_x(\cdot) \) should equal to the average limit of all realized states \( \bar{x}_j(\cdot) \); the second equality is due to the law of large numbers for conditionally independence (refer to [21]). Actually, we have the following arguments to above limit. First, when we do not consider their individual controls, all followers should be statistically symmetric because they are endowed with same system coefficients (see (1)–(3)). In addition, the input/controls applied by minor-followers, based on the auxiliary optimization
Hamiltonian systems should be exchangeable, and this is also the case for their states \( \{ \tilde{x}_j \}_{j=1}^{\mathcal{N}_f} \) embedded in Hamiltonian systems. Second, their Hamiltonian systems are driven by individual noises \( \{ \tilde{W}_j \}_{1 \leq j \leq \mathcal{N}_f} \) and common noise \( W_0 \), thus, \( \{ \tilde{x}_j \}_{j=1}^{\mathcal{N}_f} \) should be conditionally-independence on filtration generated by \( W_0 \). Then, applying the similar arguments in [21] or [10], it follows that the second equality of (21) holds. Here, we apply the exchangeability of \( \{ \tilde{x}_1(\cdot), \tilde{x}_2(\cdot), \ldots \} \) and law of large number for conditionally independence. Same result can be derived by using de Finetti’s law of large numbers as in [10]. Thus, by replacing \( \tilde{m}_x \) by \( \mathbb{E}_t[\tilde{x}_j] \triangleq \mathbb{E}[\tilde{x}_j(\cdot)|\mathcal{F}_t^0] \), we get the following system

\[
\begin{align*}
    \frac{d\tilde{x}_j}{dt} &= \{ \tilde{A}_{\tilde{x}} \tilde{x}_j - \tilde{B}_{\tilde{x}} \tilde{R}^{-1}(\tilde{B}_{\tilde{x}}^T \tilde{y}_j + \tilde{D}_{\tilde{x}}^T \tilde{z}_j) + F_1 \mathbb{E}_t[\tilde{x}_j] \} dt \\
    &\quad + \{ \tilde{C}_{\tilde{x}} \tilde{x}_j - \tilde{D}_{\tilde{x}} \tilde{R}^{-1}(\tilde{B}_{\tilde{x}}^T \tilde{y}_j + \tilde{D}_{\tilde{x}}^T \tilde{z}_j) + F_2 \mathbb{E}_t[\tilde{x}_j] \} d\tilde{W}_j(t), \\
    \frac{d\tilde{y}_j}{dt} &= - \left\{ \tilde{A}_{\tilde{y}}^T \tilde{y}_j + \tilde{C}_{\tilde{y}}^T \tilde{z}_j + \tilde{Q} \left( \tilde{x}_j - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \tilde{m}_X + \tilde{\lambda}_3 \mathbb{E}_t[\tilde{x}_j]) \right) \right\} dt \\
    &\quad + \tilde{z}_j d\tilde{W}_j(t) + \tilde{z}_0 dW_0(t), \\
    \tilde{x}_j(0) &= \zeta_j, \quad \tilde{y}_j(T) = \tilde{H}\tilde{x}_j(T), \quad j = 1, 2, \ldots, \mathcal{N}_f.
\end{align*}
\]

As all agents are statistically identical, thus we may suppress subscript “\( j \)” and the following consistency condition system arises for a “representative” agent:

\[
\begin{align*}
    \frac{d\tilde{x}}{dt} &= \{ \tilde{A}_{\tilde{x}} \tilde{x} - \tilde{B}_{\tilde{x}} \tilde{R}^{-1}(\tilde{B}_{\tilde{x}}^T \tilde{y} + \tilde{D}_{\tilde{x}}^T \tilde{z}) + F_1 \mathbb{E}_t[\tilde{x}] \} dt \\
    &\quad + \{ \tilde{C}_{\tilde{x}} \tilde{x} - \tilde{D}_{\tilde{x}} \tilde{R}^{-1}(\tilde{B}_{\tilde{x}}^T \tilde{y} + \tilde{D}_{\tilde{x}}^T \tilde{z}) + F_2 \mathbb{E}_t[\tilde{x}] \} d\tilde{W}(t), \\
    \frac{d\tilde{y}}{dt} &= - \left\{ \tilde{A}_{\tilde{y}}^T \tilde{y} + \tilde{C}_{\tilde{y}}^T \tilde{z} + \tilde{Q} \left( \tilde{x} - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \tilde{m}_X + \tilde{\lambda}_3 \mathbb{E}_t[\tilde{x}]) \right) \right\} dt \\
    &\quad + \tilde{z} d\tilde{W}(t) + \tilde{z}_0 dW_0(t), \\
    \tilde{x}(0) &= \zeta, \quad \tilde{y}(T) = \tilde{H}\tilde{x}(T),
\end{align*}
\]

where \( \tilde{W} \) stands for a generic Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}) \) that is independent of \( W_0 \). \( \zeta \) is a representative element of \( \{ \zeta_j \}_{1 \leq j \leq \mathcal{N}_f} \), and \( (X_0(\cdot), \tilde{m}_X(\cdot)) \) are two quantities need to be determined by further consistency condition analysis, to be given later.

### 3.2 Open-Loop Strategies for the Major Leader

Once Problem (OL1) is solved, we turn to solve Problem (OL2) for major leader (agent \( A_0 \)). Note that when the followers take their optimal response \( \tilde{v}_j(\cdot) \) given by
(20), the major leader ends up with the following state equation system:

\[
\begin{align*}
\frac{d\bar{X}_0}{dt} &= (A_0\bar{X}_0 + B_0u_0 + E_0^1m_X + F_0^1\mathbb{E}_t[\xi])dt \\
&\quad + [C_0\bar{X}_0 + D_0u_0 + E_0^2m_X + F_0^2\mathbb{E}_t[\xi]]dW_0(t), \\
\frac{d\bar{X}}{dt} &= (\bar{A}\bar{X} - \bar{B}\bar{R}^{-1}(\bar{B}^T\bar{Y} + \bar{D}^T\bar{Z}) + F_1\mathbb{E}_t[\xi])dt \\
&\quad + [\bar{C}\bar{X} - \bar{D}\bar{R}^{-1}(\bar{D}^T\bar{Y} + \bar{D}^T\bar{Z}) + F_2\mathbb{E}_t[\xi]]d\bar{W}(t), \\
\frac{d\bar{Y}}{dt} &= -\left\{\bar{A}^T\bar{Y} + \bar{C}^T\bar{Z} + \tilde{Q} \left(\bar{X} - (\tilde{x}_1\bar{X}_0 + \tilde{x}_2m_X + \tilde{x}_3\mathbb{E}_t[\xi])\right)\right\}dt \\
&\quad + \tilde{z}d\bar{W}(t) + \tilde{z}_0dW_0(t), \\
\bar{X}_0(0) &= \xi_0, \quad \bar{X}(0) = \xi, \quad \bar{Y}(T) = \tilde{H}\bar{X}(T).
\end{align*}
\]

(23)

In addition, its cost functional is given by (13). Notice that Eq. (23) is a two-point boundary value problem for SDEs, which we call a forward–backward stochastic differential equation (FBSDE; see [29,37–39]) and the cost functional is still of quadratic form. Hence, Problem (OL2) is actually a LQ problem for FBSDE system. Similar problems have been studied in, for example, [20] and [2]. In those works, the states are directly assumed to be mean-field FBSDEs, in contrast to our mean-field FBSDE (23) which is initially linked to the forward system (1). Let us keep in mind that the “state” for (23) is a five-tuple \((\bar{X}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{Z}_0(\cdot))\). The main result of this section can be stated as follows.

**Theorem 3.2** Under assumptions (H1), (H2), and let \(\xi_0, \xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n), \bar{m}_X(\cdot) \in L^2_{\mathcal{F}_0}(0, T; \mathbb{R}^n)\) be given. Then \(\bar{u}(\cdot) \in \mathcal{U}^d_{\mathcal{F}_0}[0, T]\) is an open-loop decentralized optimal control of Problem (OL2) for initial value \(\xi_0\) if and only if the following two conditions hold:

(i) The adapted solution \((\bar{X}(\cdot), \bar{X}(\cdot), (\bar{Y}(\cdot), \bar{Z}(\cdot), \bar{Z}_0(\cdot)), (Y_0(\cdot), Z_0(\cdot)), (p(\cdot), q(\cdot)), K(\cdot))\) to the FBSDE on \([0, T]\)

\[
\begin{align*}
\frac{d\bar{X}_0}{dt} &= (A_0\bar{X}_0 + B_0\bar{u}_0 + E_0^1\bar{m}_X + F_0^1\mathbb{E}_t[\xi])dt \\
&\quad + [C_0\bar{X}_0 + D_0\bar{u}_0 + E_0^2\bar{m}_X + F_0^2\mathbb{E}_t[\xi]]dW_0(t), \\
\frac{d\bar{X}}{dt} &= (\bar{A}\bar{X} - \bar{B}\bar{R}^{-1}(\bar{B}^T\bar{Y} + \bar{D}^T\bar{Z}) + F_1\mathbb{E}_t[\xi])dt \\
&\quad + [\bar{C}\bar{X} - \bar{D}\bar{R}^{-1}(\bar{D}^T\bar{Y} + \bar{D}^T\bar{Z}) + F_2\mathbb{E}_t[\xi]]d\bar{W}(t), \\
\frac{d\bar{Y}}{dt} &= -\left\{\bar{A}^T\bar{Y} + \bar{C}^T\bar{Z} + \tilde{Q} \left(\bar{X} - (\tilde{x}_1\bar{X}_0 + \tilde{x}_2\bar{m}_X + \tilde{x}_3\mathbb{E}_t[\xi])\right)\right\}dt \\
&\quad + \tilde{z}d\bar{W}(t) + \tilde{z}_0dW_0(t), \\
\frac{dY_0}{dt} &= -[A_0^TY_0 + C_0^TZ_0 + Q_0(\bar{X}_0 - (\lambda_0\bar{m}_X + (1 - \lambda_0)\mathbb{E}_t[\xi]) + \tilde{Q}\tilde{x}_1K)dt + Z_0dW_0(t), \\
P(0) &= \bar{P}(\bar{X}_0) - (\lambda_0\bar{m}_X + (1 - \lambda_0)\mathbb{E}_t[\xi]) - \tilde{Q}\tilde{K}d\bar{W}(t), \\
\frac{dK}{dt} &= [\tilde{A}\tilde{K} + \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{p} + \tilde{D}\tilde{R}^{-1}\tilde{D}^T\tilde{q})dt + [\tilde{C}\tilde{K} + \tilde{D}\tilde{R}^{-1}\tilde{D}^T\tilde{p} + \tilde{D}\tilde{R}^{-1}\tilde{D}^T\tilde{q}]d\tilde{W}(t), \\
\bar{X}_0(0) &= \xi_0, \quad \bar{X}(0) = \xi, \quad \bar{Y}(T) = \tilde{H}\bar{X}(T), \quad Y_0(T) = H_0\bar{X}_0(T), \quad p(T) = -\tilde{H}K(T), \quad K(0) = 0.
\end{align*}
\]

satisfies the following stationarity condition:

\[
B_0^TY_0 + D_0^TZ_0 + R_0\bar{u}_0 = 0, \quad \text{a.e. } t \in [0, T], \text{ a.s.}
\]
The following convexity condition holds:

\[ \mathbb{E} \left\{ \int_0^T \left( \begin{array}{c}
Q_0(\theta X_0 - (1 - \lambda_0)x) + (1 - \lambda_0)x
\end{array} \right) + \left( \begin{array}{c}
R_0 u_0, u_0
\end{array} \right) dt \right\} \geq 0, \quad \forall u_0(\cdot) \in \mathcal{U}_0^d[0, T], \]

(26)

where \((X_0(\cdot), x(\cdot))\) is the solution of FBSDE

\[
\begin{aligned}
\frac{dX_0}{dt} &= \{A_0 X_0 + B_0 u_0 + F_0^1 \mathbb{E}t[x] \} dt + \{C_0 X_0 + D_0 u_0 + F_0^2 \mathbb{E}t[x] \} dW_0(t), \\
\frac{dx}{dt} &= \{\tilde{A}_x - \tilde{B}_1 \tilde{R}^{-1} (\tilde{B}_1 y + \tilde{D}^T z) + F_1 \mathbb{E}t[x] \} dt \\
&\quad + \{\tilde{C}_x - \tilde{D} \tilde{R}^{-1} (\tilde{B}_1 y + \tilde{D}^T z) + F_2 \mathbb{E}t[x] \} d\tilde{W}(t), \\
\frac{dy}{dt} &= -\{\tilde{A}^T y + \tilde{C}^T z + \tilde{Q} \left( x - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_3 \mathbb{E}t[x]) \right) \} dt + z_0 dW_0(t), \\
X_0(0) &= \xi_0, \quad x(0) = 0, \quad y(T) = \tilde{H}x(0).
\end{aligned}
\]

Or, equivalently, the mapping \(u_0(\cdot) \mapsto J_0(\xi_0, \bar{m}X(\cdot), \bar{m}x(\cdot); u_0(\cdot))\), defined by (13), is convex.

\textbf{Proof} For given \(\xi_0, \zeta \in L^2(\Omega; \mathbb{R}^n), \bar{m}X(\cdot) \in L^2(\mathcal{F}_0^d(0, T; \mathbb{R}^n), \mathbb{P}), \) and \(\bar{m}u(\cdot) \in \mathcal{U}_0^d[0, T], \)

let \((\bar{X}_0(\cdot), \bar{x}(\cdot), (\bar{y}(\cdot), \bar{z}(\cdot), \bar{z}_0(\cdot)))\), \((Y_0(\cdot), Z_0(\cdot), (p(\cdot), q(\cdot), K(\cdot)))\) be adapted solution to FBSDE (24). For any \(u_0(\cdot) \in \mathcal{U}_0^d[0, T] \) and \(\varepsilon \in \mathbb{R}, \) let \(X^\varepsilon_0(\cdot), x^\varepsilon(\cdot), (y^\varepsilon(\cdot), z^\varepsilon(\cdot))\) be the solution to the following perturbed state equation on \([0, T]:\)

\[
\begin{aligned}
\frac{dX^\varepsilon_0}{dt} &= \{A_0 X^\varepsilon_0 + B_0 (u_0 + \varepsilon u_0) + E_0^1 \mathbb{E}t[x^\varepsilon] \} dt \\
&\quad + \{C_0 X^\varepsilon_0 + D_0 (u_0 + \varepsilon u_0) + E_0^2 \mathbb{E}t[x^\varepsilon] \} dW_0(t), \\
\frac{dx^\varepsilon}{dt} &= \{\tilde{A}_x - \tilde{B}_1 \tilde{R}^{-1} (\tilde{B}_1 y^\varepsilon + \tilde{D}^T z^\varepsilon) + F_1 \mathbb{E}t[x^\varepsilon] \} dt \\
&\quad + \{\tilde{C}_x - \tilde{D} \tilde{R}^{-1} (\tilde{B}_1 y^\varepsilon + \tilde{D}^T z^\varepsilon) + F_2 \mathbb{E}t[x^\varepsilon] \} d\tilde{W}(t), \\
\frac{dy^\varepsilon}{dt} &= -\{\tilde{A}^T y^\varepsilon + \tilde{C}^T z^\varepsilon + \tilde{Q} \left( x^\varepsilon - (\tilde{\lambda}_1 X^\varepsilon_0 + \tilde{\lambda}_3 \mathbb{E}t[x^\varepsilon]) \right) \} dt \\
&\quad + z^\varepsilon d\tilde{W}(t) + z_0^\varepsilon dW_0(t), \\
X^\varepsilon_0(0) &= \xi_0, \quad x^\varepsilon(0) = \zeta, \quad y^\varepsilon(T) = \tilde{H}x^\varepsilon(0).
\end{aligned}
\]

Then denoting \((X_0(\cdot), x(\cdot), (y(\cdot), z(\cdot), z_0(\cdot)))\) the solution to the FBSDE (27), we have

\[ X^\varepsilon_0(\cdot) = \bar{X}_0(\cdot) + \varepsilon X_0(\cdot), x^\varepsilon(\cdot) = \bar{x}(\cdot) + \varepsilon x(\cdot), y^\varepsilon(\cdot) = \bar{y}(\cdot) + \varepsilon y(\cdot), z^\varepsilon(\cdot) = \bar{z}(\cdot) + \varepsilon z(\cdot) \]
Similarly, if we assume \( R_0 \) is invertible, then we can represent the optimal control by

\[
\bar{u}_0 = -R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0). \tag{28}
\]
Then the following coupled system follows

\[
\begin{align*}
\dot{X}_0 &= \{A_0 X_0 - B_0 R_0^{-1}(B_0^T Y_0 + D_0^T Z_0) + E_0^1 \overline{m}_X + F_0^1 \mathbb{E}_t(X)\}dt \\
&\quad + \{C_0 \overline{X}_0 - D_0 R_0^{-1}(B_0^T Y_0 + D_0^T Z_0) + E_0^2 \overline{m}_X + F_0^2 \mathbb{E}_t(X)\}dW_0(t), \\
\dot{\overline{X}} &= \{\tilde{A} \overline{X} - \tilde{B} \tilde{R}^{-1}(\tilde{B}^T \overline{Y} + \tilde{D}^T \overline{Z}) + F_1 \mathbb{E}_t(X)\}dt \\
&\quad + \{\tilde{C} \overline{X} - \tilde{D} \tilde{R}^{-1}(\tilde{B}^T \overline{Y} + \tilde{D}^T \overline{Z}) + F_2 \mathbb{E}_t(X)\}d\tilde{W}(t), \\
\dot{K} &= \{\tilde{A} K + \tilde{B} \tilde{R}^{-1} \tilde{B}^T p + \tilde{D} \tilde{R}^{-1} \tilde{D}^T q\}dt \\
&\quad + \{\tilde{C} K + \tilde{D} \tilde{R}^{-1} \tilde{B}^T p + \tilde{D} \tilde{R}^{-1} \tilde{D}^T q\}d\tilde{W}(t), \\
\dot{Y}_0 &= -\{A_0^T Y_0 + C_0^T Z_0 + Q_0(\overline{X}_0 - (\lambda_0 \overline{m}_X + (1 - \lambda_0) \mathbb{E}_t[X])) + \tilde{Q}\lambda_1 K\}dt \\
&\quad + Z_0 dW_0(t), \\
\dot{\overline{Y}} &= -\left\{\tilde{A}^T \overline{Y} + \tilde{C}^T \overline{Z} + \tilde{Q}\left(\overline{X} - (\lambda_1 \overline{X}_0 + \lambda_2 \overline{m}_X + \lambda_3 \mathbb{E}_t[X])\right)\right\}dt + \tilde{z} d\tilde{W}(t), \\
\dot{p} &= -\{\tilde{A}^T p + \tilde{C}^T q + F_0^T \mathbb{E}_t[Y_0] + F_0^T_2 \mathbb{E}_t[Z_0] + F_1^T \mathbb{E}_t[p] + F_2^T \mathbb{E}_t[q] \\
&\quad + \tilde{Q}\lambda_3 \mathbb{E}_t[K] - (1 - \lambda_0) Q_0(\overline{X}_0 - (\lambda_0 \overline{m}_X + (1 - \lambda_0) \mathbb{E}_t[X])) - \tilde{Q} K\}dt + \tilde{q} d\tilde{W}(t), \\
\overline{X}_0(0) &= \xi_0, \quad \overline{Y}(0) = \zeta, \quad K(0) = 0, \quad Y_0(T) = H_0 \overline{X}_0(T), \quad \overline{Y}(T) = H \overline{X}(T), \quad p(T) = -\tilde{H} K(T), \\
\end{align*}
\]

(29)

where \(\overline{m}_X(\cdot)\) is to be determined.

### 3.3 Open-Loop Strategies for the Minor Leaders

Once **Problem (OL2)** is solved, we turn to solve **Problem (OL3)** for the minor leaders (agents \(A_i', 1 \leq i \leq N_i\)). Note that when the followers takes their optimal responses \(\overline{v}_j(\cdot)\) given by (20), and the major leader takes his optimal control \(\overline{u}_0(\cdot)\) given by (28), the minor leaders ends up with the following state equation system:

\[
\begin{align*}
\dot{\overline{X}}_i &= \{A \overline{X}_i + B u_i + E_1 \overline{m}_X\}dt + \{C \overline{X}_i + D u_i + E_2 \overline{m}_X\}dW_i(t), \\
\overline{X}_i(0) &= \xi_i, \quad i = 1, 2, \ldots, N_i.
\end{align*}
\]

And its cost functional is given by (15) with \(\overline{X}_0(\cdot)\) being from (29). So it is similar to solve **Problem (OL1)**, and the main result in this section can be stated as follows.

**Theorem 3.3** Under assumptions (**H1**), (**H2**), and let \(\xi_0, \xi_i \in L^2_{\mathbb{F}_0}(T; \mathbb{R}^n), \overline{u}_0(\cdot) \in U_0^d[0, T], \overline{m}_X(\cdot) \in L^2_{\mathbb{F}_0}(0, T; \mathbb{R}^n)\) be given. Then \(\overline{u}_i(\cdot) \in U_i^d[0, T] \) is a decentralized optimal control of **Problem (OL3)** for initial value \(\xi_i\) if and only if the following two conditions hold:

\[\Xi\] Springer
(i) For \( i = 1, 2, \ldots, N_l \), the adapted solution \((X_i(\cdot), Y_i(\cdot), Z_i(\cdot), Z_{i0}(\cdot))\) to the FBSDE on \([0, T]\)

\[
\begin{align*}
\frac{dX_i}{dt} &= \left\{ AX_i + Bu_i + E_1 mX \right\} dt + \left\{ CX_i + Du_i + E_2 mX \right\} dW_i(t), \\
\frac{dY_i}{dt} &= -\left\{ A^\top Y_i + C^\top Z_i + Q(X_i - (\lambda mX + (1 - \lambda)X_0)) \right\} dt \\
&\quad + \frac{dZ_i}{dt} W(t) + \frac{dZ_{i0}}{dt} W_0(t), \\
X_i(0) &= \xi_i, \quad Y_i(T) = H X_i(T),
\end{align*}
\]

satisfies the following stationarity condition:

\[
B^\top Y_i + Ru_i + D^\top Z_i = 0, \quad \text{a.e. } t \in [0, T], \text{ a.s.} \tag{31}
\]

(ii) For \( i = 1, 2, \ldots, N_l \), the following convexity condition holds:

\[
\mathbb{E}\left\{ \int_0^T \left( Q X_i, X_i \right) + \left( R u_i, u_i \right) \right\} dt + \left( H X_i(T), X_i(T) \right) \geq 0, \\
\forall u_i(\cdot) \in \mathcal{U}_i^d[0, T],
\]

where \( X_i(\cdot) \) is the solution of

\[
\begin{align*}
\frac{dX_i}{dt} &= \left\{ AX_i + Bu_i \right\} dt + \left\{ CX_i + Du_i \right\} dW_i(t), \\
X_i(0) &= x.
\end{align*}
\]

Or, equivalently, the mapping \( u_i(\cdot) \mapsto J_i^l(\xi_0, \xi_i, mX(\cdot); u_0(\cdot), u_i(\cdot)) \), defined by (15), is convex (for \( i = 1, 2, \ldots, N_l \)).

**Proof** For given \( \xi_0, \xi_i \in L^2_\mathcal{F}_0(\Omega; \mathbb{R}^n) \), \( u_0(\cdot) \in \mathcal{U}_0^d[0, T] \), \( mX(\cdot) \in L^2_\mathcal{F}_T(0, T; \mathbb{R}^n) \), and \( \bar{u}_i(\cdot) \in \mathcal{U}_i^d[0, T] \), let \((\tilde{X}_i(\cdot), \tilde{Y}_i(\cdot), \tilde{Z}_i(\cdot), \tilde{Z}_{i0}(\cdot))\) be adapted solution to FBSDE (30). For any \( u_i(\cdot) \in \mathcal{U}_i^d[0, T] \) and \( \varepsilon \in \mathbb{R} \), let \( X^\varepsilon_i(\cdot) \) be the solution of the following perturbed state equation on \([0, T]\):

\[
\begin{align*}
\frac{dX^\varepsilon_i}{dt} &= \left\{ AX^\varepsilon_i + B(u_i + \varepsilon u_i) + E_1 mX \right\} dt \\
&\quad + \left\{ CX^\varepsilon_i + D(u_i + \varepsilon u_i) + E_2 mX \right\} dW_i(t), \\
X^\varepsilon_i(0) &= x.
\end{align*}
\]
Then denoting $X_i(\cdot)$ the solution of (33), we have $X_i^\varepsilon(\cdot) = \overline{X}_i(\cdot) + \varepsilon X_i(\cdot)$ and

\[
J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot) + \varepsilon u_i(\cdot)) - J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot))
= \frac{\varepsilon}{2} \mathbb{E} \left\{ \int_0^T \left( \left\langle Q \left( 2 \overline{X}_i - 2(\lambda \overline{m}_X + (1 - \lambda)\overline{X}_0) + \varepsilon X_i \right), X_i \right\rangle + \left\langle R(2\overline{u}_i + \varepsilon u_i), u_i \right\rangle \right) dt + \left\langle H(2\overline{X}_i(T) + \varepsilon X_i(T)), X_i(T) \right\rangle \right\}
+ \varepsilon \mathbb{E} \left\{ \int_0^T \left( \left\langle Q X_i, X_i \right\rangle + \left\langle R u_i, u_i \right\rangle \right) dt + \left\langle H X_i(T), X_i(T) \right\rangle \right\}.
\]

On the other hand, applying Itô’s formula to $\left\langle Y_i, X_i \right\rangle$, and taking expectation, we obtain

\[
\mathbb{E} \left\langle H \overline{X}_i(T), X_i(T) \right\rangle
= \mathbb{E} \int_0^T \left( \left\langle B^\top Y_i + D^\top Z_i, u_i \right\rangle - \left\langle Q \left( \overline{X}_i - (\lambda \overline{m}_X + (1 - \lambda)\overline{X}_0) \right), X_i \right\rangle \right) dt.
\]

Hence,

\[
J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot) + \varepsilon u_i(\cdot)) - J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot))
= \frac{\varepsilon^2}{2} \mathbb{E} \left\{ \int_0^T \left( \left\langle Q X_i, X_i \right\rangle + \left\langle R u_i, u_i \right\rangle \right) dt + \left\langle H X_i(T), X_i(T) \right\rangle \right\}
+ \mathbb{E} \int_0^T \left( B^\top \overline{Y}_i + R \overline{u}_i + D^\top \overline{Z}_i, u_i \right) dt.
\]

It follows that

\[
J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot)) \leq J^I_i(\xi_0, \xi_i, \overline{m}_X(\cdot); \overline{u}_0(\cdot), \overline{u}_i(\cdot) + \varepsilon u_i(\cdot)),
\forall u_i(\cdot) \in U^d \left[0, T\right], \forall \varepsilon \in \mathbb{R},
\]

if and only if (31) and (32) hold. \hfill \Box

Furthermore, if we assume that $R$ is invertible, then we have

\[
\overline{u}_i = -R^{-1}(B^\top \overline{Y}_i + D^\top \overline{Z}_i),
\]

(34)
so the related Hamiltonian system can be represented by

\[
\begin{align*}
\frac{dX_i}{dt} &= \{A X_i - B R^{-1}(B^T Y_i + D^T Z_i) + E_1 \bar{m} X\} dt \\
&\quad + \{C X_i - D R^{-1}(B^T Y_i + D^T Z_i) + E_2 \bar{m} X\} dW_i(t), \\
\frac{dY_i}{dt} &= -\left\{A^T Y_i + C^T Z_i + Q\left(\bar{X} - (\lambda \bar{m} X + (1 - \lambda) \bar{X}_0)\right)\right\} dt \\
&\quad + Z_i dW_i(t) + Z_{i0} dW_0(t), \\
\bar{X}_i(0) &= \xi_i, \quad \bar{Y}_i(T) = H \bar{X}_i(T).
\end{align*}
\]

Based on the above analysis, it follows that

\[
\bar{m}(\cdot) = \lim_{N_l \to +\infty} \frac{1}{N_l} \sum_{i=1}^{N_l} \bar{X}_i(\cdot) = \mathbb{E}[\bar{X}_i(\cdot) | F_0^T].
\]

(35)

Here, the first equality of (35) is due to the consistency condition: the frozen term \(\bar{m}(\cdot)\) should equal to the average limit of all realized states \(\bar{X}_i(\cdot)\); the second equality is due to the law of large numbers for conditionally independence (apply for the same argument in Sect. 3.1). Thus, by replacing \(\bar{m}(\cdot)\) by \(\mathbb{E}[\bar{X}_i(\cdot) | F_0^T] \) we get the following system

\[
\begin{align*}
\frac{d\bar{X}}{dt} &= \{A \bar{X} - B R^{-1}(B^T \bar{Y} + D^T \bar{Z}) + E_1 \mathbb{E}[\bar{X}]\} dt \\
&\quad + \{C \bar{X} - D R^{-1}(B^T \bar{Y} + D^T \bar{Z}) + E_2 \mathbb{E}[\bar{X}]\} dW(t), \\
\frac{d\bar{Y}}{dt} &= -\left\{A^T \bar{Y} + C^T \bar{Z} + Q\left(\bar{X} - (\lambda \mathbb{E}[\bar{X}] + (1 - \lambda) \bar{X}_0)\right)\right\} dt \\
&\quad + \bar{Z} dW(t) + \bar{Z}_0 dW_0(t), \\
\bar{X}(0) &= \xi, \quad \bar{Y}(T) = H \bar{X}(T).
\end{align*}
\]

(36)

As all agents are statistically identical, thus we can suppress subscript “\(i\)” and the following consistency condition system arises for a representative agent:

\[
\begin{align*}
\frac{d\bar{X}}{dt} &= \{A \bar{X} - B R^{-1}(B^T \bar{Y} + D^T \bar{Z}) + E_1 \mathbb{E}[\bar{X}]\} dt \\
&\quad + \{C \bar{X} - D R^{-1}(B^T \bar{Y} + D^T \bar{Z}) + E_2 \mathbb{E}[\bar{X}]\} dW(t), \\
\frac{d\bar{Y}}{dt} &= -\left\{A^T \bar{Y} + C^T \bar{Z} + Q\left(\bar{X} - (\lambda \mathbb{E}[\bar{X}] + (1 - \lambda) \bar{X}_0)\right)\right\} dt \\
&\quad + \bar{Z} dW(t) + \bar{Z}_0 dW_0(t), \\
\bar{X}(0) &= \xi, \quad \bar{Y}(T) = H \bar{X}(T),
\end{align*}
\]

where \(W\) stands for a generic Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), and it is independent of \(W_0, \tilde{W}\). \(\xi\) is a representative element from the exchangeable family \(\{\xi_i\}_{1 \leq i \leq N_l}\).
Combine (29) and (36), replacing $\overline{m}_X$ by $E_t[\overline{X}]$, we get consistency condition system for open-loop strategy as follows, a high dimensional fully-coupled McKean–Vlasov type FBSDE:

$$\begin{align*}
\text{d}\overline{X}_0 &= \{ A_0\overline{X}_0 - B_0R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^1E_t[\overline{X}] + F_0^1E_t[\overline{Y}]\}\text{d}t \\
+ &\{ C_0\overline{X}_0 - D_0R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^2E_t[\overline{X}] + F_0^2E_t[\overline{Y}]\}\text{d}W(t), \\
\text{d}\overline{Y}_0 &= \{ \overline{A}\overline{X} - \overline{B}R^{-1}(\overline{B}^\top \overline{Y} + \overline{D}^\top \overline{Z}) + E_1E_t[\overline{X}]\}\text{d}t \\
+ &\{ \overline{C}\overline{X} - DR^{-1}(\overline{B}^\top \overline{Y} + \overline{D}^\top \overline{Z}) + E_2E_t[\overline{Y}]\}\text{d}W(t), \\
\text{d}\overline{K} &= \{ \overline{A}\overline{K} + \overline{B}R^{-1}\overline{B}^\top p + \overline{B}R^{-1}\overline{D}^\top q\}\text{d}t \\
+ &\{ \overline{C}\overline{K} + DR^{-1}\overline{B}^\top p + DR^{-1}\overline{D}^\top q\}\text{d}W(t), \\
Y_0(T) &= H_0\overline{X}_0(T), \quad \overline{Y}(T) = H\overline{X}(T), \quad \overline{Y}(T) = \overline{H}\overline{X}(T), \quad p(T) = -\overline{H}K(T). \tag{37}
\end{align*}$$

By observation, we can find that $\overline{z}_0$ and $\overline{Z}_0$ do not enter the equations which indicates $\overline{z}_0 = \overline{Z}_0 = 0$ is the (special) solution of (37). Therefore, we finally simplify the equation to

$$\begin{align*}
\text{d}\overline{X}_0 &= \{ A_0\overline{X}_0 - B_0R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^1E_t[\overline{X}] + F_0^1E_t[\overline{Y}]\}\text{d}t \\
+ &\{ C_0\overline{X}_0 - D_0R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^2E_t[\overline{X}] + F_0^2E_t[\overline{Y}]\}\text{d}W(t), \\
\text{d}\overline{Y}_0 &= \{ \overline{A}\overline{X} - \overline{B}R^{-1}(\overline{B}^\top \overline{Y} + \overline{D}^\top \overline{Z}) + E_1E_t[\overline{X}]\}\text{d}t \\
+ &\{ \overline{C}\overline{X} - DR^{-1}(\overline{B}^\top \overline{Y} + \overline{D}^\top \overline{Z}) + E_2E_t[\overline{Y}]\}\text{d}W(t), \\
\text{d}\overline{K} &= \{ \overline{A}\overline{K} + \overline{B}R^{-1}\overline{B}^\top p + \overline{B}R^{-1}\overline{D}^\top q\}\text{d}t \\
+ &\{ \overline{C}\overline{K} + DR^{-1}\overline{B}^\top p + DR^{-1}\overline{D}^\top q\}\text{d}W(t), \\
Y_0(T) &= H_0\overline{X}_0(T), \quad \overline{Y}(T) = H\overline{X}(T), \quad \overline{Y}(T) = \overline{H}\overline{X}(T), \quad p(T) = -\overline{H}K(T). \tag{38}
\end{align*}$$

### 4 The Consistency Condition System

Under assumptions (H1), (H2), when $\overline{R}(-)$, $R_0(-)$ and $R(-)$ are invertible, we get the CC system for OL strategy as (38). In this section, we turn to verify its well-posedness since it is important to the decentralized strategy design. For sake of
notation simplicity, denote $X^T = (\tilde{X}_0^T, \tilde{X}^T, \tilde{K}^T), \ Y^T = (Y_0^T, \tilde{Y}^T, \tilde{y}^T, \ p^T), \ Z^T = (Z_0^T, \tilde{Z}^T, \tilde{z}^T, \ q^T), \ W^T = (W_0^T, \tilde{W}^T, \tilde{\tilde{W}}^T, \tilde{W}^T)$, and then consistency condition system (38) can be rewritten as the following compact form:

$$
\begin{align*}
\text{d}X &= \{A \text{X} + \tilde{A} \bar{E}_t[X] + B \text{Y} + E \tilde{Z}\} \text{d}t + \{C \text{X} + \tilde{C} \bar{E}_t[X] + D \text{Y} + F \tilde{Z}\} \circ \text{d}W(t), \\
\text{d}Y &= -[A^T \text{Y} + A_0^T \bar{E}_t[Y] + C^T \text{Z} + C_0^T \bar{E}_t[Z] + Q \text{X} + \tilde{Q} \bar{E}_t[X]] \text{d}t \\
&\quad + \tilde{Z} \circ \text{d}W(t), \\
X(0) &= X_0, \quad Y(T) = H_0 X(T),
\end{align*}
$$

(39)

with the block structures defined as

$$
A = \begin{pmatrix}
A_0 & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & \tilde{A} & 0 \\
0 & 0 & 0 & \tilde{A}
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
0 & E_0^1 & F_0^1 & 0 \\
0 & E_1 & F_1 & 0 \\
0 & 0 & F_2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
C_0 & 0 & 0 & 0 \\
0 & C & 0 & 0 \\
0 & 0 & \tilde{C} & 0 \\
0 & 0 & 0 & \tilde{C}
\end{pmatrix}, \quad \tilde{C} = \begin{pmatrix}
0 & E_0^2 & F_0^2 & 0 \\
0 & E_2 & F_2 & 0 \\
0 & 0 & F_3 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
-B_0 R_0^{-1} B_0^T & 0 & 0 & 0 \\
0 & -B R_0^{-1} B^T & 0 & 0 \\
0 & 0 & -\tilde{B} R_0^{-1} \tilde{B}^T & 0 \\
0 & 0 & 0 & \tilde{B} R_0^{-1} \tilde{B}^T
\end{pmatrix},
$$

$$
D = \begin{pmatrix}
-D_0 R_0^{-1} D_0^T & 0 & 0 & 0 \\
0 & -D R_0^{-1} D^T & 0 & 0 \\
0 & 0 & -\tilde{D} R_0^{-1} \tilde{D}^T & 0 \\
0 & 0 & 0 & \tilde{D} R_0^{-1} \tilde{D}^T
\end{pmatrix},
$$

$$
E = \begin{pmatrix}
-D_0 R_0^{-1} D_0^T & 0 & 0 & 0 \\
0 & -D R_0^{-1} D^T & 0 & 0 \\
0 & 0 & -\tilde{D} R_0^{-1} \tilde{D}^T & 0 \\
0 & 0 & 0 & \tilde{D} R_0^{-1} \tilde{D}^T
\end{pmatrix},
$$

$$
F = \begin{pmatrix}
0 & -D R_0^{-1} D^T & 0 & 0 \\
0 & 0 & -\tilde{D} R_0^{-1} \tilde{D}^T & 0 \\
0 & 0 & 0 & \tilde{D} R_0^{-1} \tilde{D}^T \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
A_0 = \begin{pmatrix}
0 & 0 & 0 & F_0^1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_1 \\
0 & 0 & 0 & F_2
\end{pmatrix}, \quad C_0 = \begin{pmatrix}
0 & 0 & 0 & F_0^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & F_2
\end{pmatrix},
$$

$$
H_0 = \begin{pmatrix}
H_0 & 0 & 0 & 0 \\
0 & H & 0 & 0 \\
0 & 0 & \tilde{H} & 0 \\
0 & 0 & 0 & -\tilde{H}
\end{pmatrix}, \quad X_0 = \begin{pmatrix}
\xi_0 \\
\xi \\
\xi \\
0
\end{pmatrix},
$$

$$
Q = \begin{pmatrix}
-Q_0 & 0 & 0 & -\tilde{Q}_0 \\
Q(1-\lambda_1) & -Q & 0 & 0 \\
0 & -\tilde{Q} & 0 & 0 \\
Q_0(1-\lambda_0) & 0 & 0 & -\tilde{Q}_0 \\
\end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix}
0 & Q_0(1-\lambda_0) & 0 & 0 \\
0 & Q_0(1-\lambda_0) & 0 & 0 \\
0 & 0 & Q_0(1-\lambda_0) & 0 \\
0 & -Q_0(1-\lambda_0)^2 & -Q_0(1-\lambda_0) & 0 \\
\end{pmatrix}.
$$
where \( \circ \) denotes Hadamard product.

### 4.1 Decoupling for Open-Loop Strategy

Then, we turn to discuss the wellposedness of FBSDE (39) by decoupling method via some Riccati equations. Notice that

\[
\begin{align*}
\mathbf{d}E_t[\mathbf{X}] &= \left( (A + \bar{A})E_t[\mathbf{X}] + \mathbf{B}E_t[\mathbf{Y}] + \mathbf{E}E_t[\mathbf{Z}] \right)dt \\
&\quad + E_t\left( (\mathbf{C}X + \bar{C}E_t[\mathbf{X}] + \mathbf{D}Y + 
\end{align*}
\]

Next, we assume that

\[
Y(t) = P(t)X(t) + \Pi(t)E_t[X(t)], \quad t \in [0, T],
\]

for some deterministic and absolutely continuous functions \( P(\cdot) \) and for some continuous \( \mathcal{F}_t^0 \)-adapted process \( (\Pi(\cdot), \Theta(\cdot)) \), taking values in \( S^{kn} \), such that

\[
\begin{align*}
\mathbf{d}P(t) &= \dot{P}(t)dt, \\
\mathbf{d}\Pi(t) &= \dot{\Pi}(t)dt + \Theta(t)dW_0, \\
\Pi(T) &= 0.
\end{align*}
\]

Then

\[
\mathbf{E}_t[Y(t)] = \left( P(t) + \Pi(t) \right)\mathbf{E}_t[X(t)].
\]

Therefore,

\[
\begin{align*}
\mathbf{d}Y &= \left( \dot{P}X + \dot{\Pi}E_t[\mathbf{X}] \right)dt + P\cdot d\mathbf{X} + \Pi\cdot dE_t[\mathbf{X}] + \Theta \circ dW(t) \\
&= \left( \dot{P}X + \dot{\Pi}E_t[\mathbf{X}] + P\left( AX + \bar{A}E_t[\mathbf{X}] + BY + E\mathbf{Z}\right) \\
&\quad + \Pi\left( (A + \bar{A})E_t[\mathbf{X}] + \mathbf{B}E_t[\mathbf{Y}] + \mathbf{E}E_t[\mathbf{Z}] \right) \right)dt \quad (41) \\
&\quad + P\left( \mathbf{C}X + \bar{C}E_t[\mathbf{X}] + \mathbf{D}Y + \mathbf{F}\mathbf{Z} \right) \circ dW(t) + \Pi E_t[\left( \mathbf{C}X + \bar{C}E_t[\mathbf{X}] + \mathbf{D}Y + \mathbf{F}\mathbf{Z} \right) \circ dW(t)] + \Theta \circ dW_0(t).
\end{align*}
\]

Comparing the diffusion terms, we have

\[
Z = (I - P\mathbf{F})^{-1} P\left[ \mathbf{C} + \mathbf{D}P \right]X + (\bar{\mathbf{C}} + \mathbf{D}\Pi)E_t[\mathbf{X}] \quad (42)
\]

Then

\[
\mathbf{E}_t[Z] = (I - P\mathbf{F})^{-1} P\left[ \mathbf{C} + \bar{\mathbf{C}} + \mathbf{D}P + \mathbf{D}\Pi \right]E_t[\mathbf{X}].
\]

---

1 Hadamard product (also known as the Schur product or the entrywise product) is a binary operation that takes two matrices of the same dimensions, and produces another matrix where each element \((i, j)\) is the product of elements \((i, j)\) of the original two matrices.
Comparing the drift terms, we have

\[
0 = \dot{P}X + \Pi E_t[X] + P(AX + \overrightarrow{A} E_t[Y] + BY + EZ) + \Pi(\overrightarrow{A} + \overrightarrow{\overrightarrow{A}})E_t[X] + B E_t[Y] + E E_t[Z] \\
+ (A^T Y + A_0^T E_t[Y] + C^TZ + C_0^T E_t[Z] + QX + QE_t[Z]) \\
= \left\{ \dot{P} + PA + A^TP + (C + DP)^T (I - PF)^{-1}(C + DP) + Q + PB \right\} X
\]

Therefore, we denote \( P(\cdot) \) be the solution of the following Riccati equation and \((\Pi(\cdot), \Theta(\cdot))\) be the solution of the following BSDE, respectively:

\[
\left\{ \begin{align*}
\dot{P} + PA + A^TP + (C + DP)^T (I - PF)^{-1}(C + DP) + Q + PB &= 0, \\
P(T) &= H_0,
\end{align*} \right.
\tag{44}
\]

and

\[
\left\{ \begin{align*}
d\Pi &= -\left\{ \Pi(A + \overrightarrow{A}) + (A^T + A_0^T)\Pi + A_0^TP + P\overrightarrow{A} + PB\Pi + \Theta + \Pi \Pi \Pi + (C + DP)^T \\
&\quad (I - PF)^{-1}(C + DP) + (C_0 + D\Pi)^T (I - PF)^{-1}(C + \overrightarrow{C} + DP + D\Pi) \right\} dt \\
&\quad + \Theta dW_0(t), \\
\Pi(T) &= 0.
\end{align*} \right.
\tag{45}
\]

We remark that (44), (45) are asymmetric Riccati equation with high dimensional block structures, and they are also non-classical due to the term \((I - PF)^{-1}\). Notice that references [13,23] also deal with asymmetric Riccati equations, but in a more standard and simplified form (actually, the Riccati equations therein correspond to the case when the control variables do not enter the diffusion term of state equation. It differs fairly to our general formulation here where control enters diffusion terms. Also, our Riccati equation is non-standard since its variable enters the denominator of equation such as \((I - PF)^{-1}\). On the other hand, the Riccati equation discussed by Reference [24] is symmetric because it is not rooted from MFG (especially it is not involved with CC structure thus symmetric). Keep them in mind, we borrow one result/ideas from the non-degeneracy flow-condition of [16] to establish the wellposedness of our asymmetric and nonstandard Riccati equation here.

**Remark 4.1** We point out that there are a lot of existing papers to deal with such kind of Riccati equations. For example, we can introduce the following conditions to guarantee the solvability for such Riccati equations.

\( \text{(C)} \)

\[ 4 \text{eig}(A) < -2\|\overrightarrow{A}\|_{sp} - \|A_0\|_{sp}^2 - 2\|C\|_{sp}^2 - \|\overrightarrow{C}\|_{sp} - \|C_0\|_{sp}^2, \]

where \( \text{eig}(\cdot) \) denotes the maximal eigenvalue of the matrix and \( \|\cdot\|_{sp} \) denotes the spectral norm of the matrix.

As long as the coefficients of the Riccati equations satisfy the condition (C), then (44), (45) will admit unique (global) solution.
4.2 Decoupling for the Feedback Strategy

Besides the open-loop strategy, we can also introduce the following Riccati equations to represent some feedback strategy. At first, the Hamiltonian system of follower is (16), i.e.,

\[
\begin{align*}
\mathrm{d}\bar{x}_j &= \left[ \tilde{A}\bar{x}_j + \tilde{B}\bar{v}_j + F_1\bar{m}_x \right]\mathrm{d}t + \left[ \tilde{C}\bar{x}_j + \tilde{D}\bar{v}_j + F_2\bar{m}_x \right]\mathrm{d}\tilde{W}_j, \\
\mathrm{d}\bar{y}_j &= -\left[ \tilde{A}^\top\bar{y}_j + \tilde{C}^\top\bar{z}_j + \tilde{Q}(\bar{x}_j - (\tilde{\lambda}_1X_0 + \tilde{\lambda}_2\bar{m}_X + \tilde{\lambda}_3\bar{m}_x)) \right]\mathrm{d}t + \bar{z}_j\mathrm{d}\tilde{W}_j, \\
\bar{x}_j(0) &= \zeta_j, \quad \bar{y}_j(T) = \tilde{H}\bar{x}_j,
\end{align*}
\]

with the stationary condition

\[
\bar{v}_j = -\tilde{R}^{-1}(\tilde{B}^\top\bar{y}_j + \tilde{D}^\top\bar{z}_j).
\]

Assume that \( \bar{y}_j = P_1\bar{x}_j + \Phi_1 \), and we can get the Riccati equations

\[
\begin{align*}
\dot{P}_1 + \tilde{A}^\top P_1 + P_1\tilde{A} + \tilde{Q} - P_1\tilde{B}\tilde{R}^{-1}\tilde{B}^\top P_1 + \tilde{S}^\top\tilde{R}^{-1}P_1\tilde{S} &= 0, \\
P_1(T) &= \tilde{H},
\end{align*}
\]

(46)

and the related BSDE

\[
\begin{align*}
\mathrm{d}\Phi_1 &= \left( -\tilde{A}^\top\Phi_1 + P_1\tilde{B}\tilde{R}^{-1}\tilde{B}^\top\Phi_1 + \tilde{Q}(\tilde{\lambda}_1X_0 + \tilde{\lambda}_2\bar{m}_X + \tilde{\lambda}_3\bar{m}_x) \\
&\quad - P_1F_1\bar{m}_x - \tilde{S}^\top\tilde{R}^{-1}\tilde{f} \right)\mathrm{d}t \\
&\quad + \eta_1 W_0(t), \\
\Phi_1(T) &= 0,
\end{align*}
\]

(47)

where

\[
\begin{align*}
\tilde{R} &:= I + P_1\tilde{D}\tilde{R}^{-1}\tilde{D}^\top, \\
\tilde{S} &:= \tilde{C} - \tilde{D}\tilde{R}^{-1}\tilde{B}^\top P_1, \\
\tilde{f} &:= P_1F_2\bar{m}_x - P_1\tilde{D}\tilde{R}^{-1}\tilde{B}^\top\Phi_1.
\end{align*}
\]

Note that

\[
\text{follower: } \begin{cases} 
\bar{y}_j = P_1\bar{x}_j + \Phi_1, \\
\bar{z}_j = \tilde{R}^{-1}P_1\tilde{S}\bar{x}_j + \tilde{R}^{-1}\tilde{f},
\end{cases}
\]

so the feedback is

\[
\bar{v}_j = -\tilde{R}^{-1}\left( \tilde{B}^\top P_1 + \tilde{D}^\top\tilde{R}^{-1}P_1\tilde{S} \right)\bar{x}_j - \tilde{R}^{-1}\tilde{B}^\top\Phi_1 - \tilde{R}^{-1}\tilde{D}^\top\tilde{R}^{-1}\tilde{f}.
\]
Then under the above feedback strategy of follower, the major leader ends up with the following Hamiltonian system

\[
\begin{align*}
    d\tilde{X}_0 &= (A_0 \tilde{X}_0 + B_0 \bar{u}_0 + E_0^1 \bar{m}_x + F_0^1 \bar{m}_x)\,dt + (C_0 \tilde{X}_0 + D_0 \bar{u}_0 + E_0^2 \bar{m}_x + F_0^2 \bar{m}_x)\,dW_0, \\
    d\bar{m}_x &= (\tilde{A}\bar{m}_x + \tilde{B}\Phi_1)\,dt + (\tilde{C}\bar{m}_x + \tilde{D}\Phi_1)\,dW_0, \\
    d\Phi_1 &= (\tilde{Q}\tilde{\lambda}_1 \tilde{X}_0 + \tilde{Q}\bar{m}_x + \tilde{A}\Phi_1 + \tilde{Q}\tilde{\lambda}_2 \bar{m}_x)\,dt, \\
    dY_0 &= - \left\{ A_1^T Y_0 + C_0^T Z_0 + Q_0 \left( \tilde{X}_0 - (\tilde{\lambda}_0 \tilde{m}_x + (1 - \tilde{\lambda}_0) \bar{m}_x) \right) + \tilde{Q}\tilde{\lambda}_1 y_2 \right\} \,dt + Z_0 \,dW_0, \\
    dy_1 &= - \left\{ \tilde{\Lambda}^T y_1 + \tilde{C}^T z_1 + \tilde{Q}^T y_2 + F_1^T Y_0 + F_2^T Z_0 - (1 - \tilde{\lambda}_0) Q_0 \left( \tilde{X}_0 - (\tilde{\lambda}_0 \tilde{m}_x + (1 - \tilde{\lambda}_0) \bar{m}_x) \right) \right\} \,dt \\
    + z_1 \,dW_0, \\
    dy_2 &= - (\tilde{B}^T y_1 + \tilde{D}^T z_1 + \tilde{A}^T y_2) \,dt, \\
    \tilde{X}_0(0) &= \xi_0, \bar{m}_x(0) = 0, \Phi_1(T) = 0, Y_0(T) = H_0 \tilde{X}_0(T), y_1(T) = 0, y_2(0) = 0,
\end{align*}
\]

with the stationary condition

\[
\bar{u}_0 = - R_0^{-1} (B_0^T Y_0 + D_0^T Z_0),
\]

where

\[
\begin{align*}
    \tilde{\Lambda} &:= \tilde{A} + F_1 - \tilde{B} \tilde{R}^{-1} (\tilde{B}^T P_1 + \tilde{D}^T \tilde{R}^{-1} (P_1 S + P_1 F_2)), \\
    \tilde{C} &:= \tilde{C} + F_2 - \tilde{D} \tilde{R}^{-1} (\tilde{B}^T P_1 + \tilde{D}^T \tilde{R}^{-1} (P_1 S + P_1 F_2)), \\
    \tilde{A} &:= (\tilde{S}^T \tilde{R}^{-1} P_1 \tilde{D} + P_1 \tilde{B}) \tilde{R}^{-1} \tilde{B}^T - \tilde{A}^T, \\
    \tilde{B} &:= (\tilde{B} \tilde{R}^{-1} \tilde{D}^T \tilde{R}^{-1} P_1 \tilde{D} - \tilde{B}) \tilde{R}^{-1} \tilde{B}^T, \\
    \tilde{D} &:= (\tilde{D} \tilde{R}^{-1} \tilde{D}^T \tilde{R}^{-1} P_1 \tilde{D} - \tilde{D}) \tilde{R}^{-1} \tilde{B}^T, \\
    \tilde{Q} &:= \tilde{Q}\tilde{\lambda}_3 - P_1 F_1 - \tilde{S}^T \tilde{R}^{-1} P_1 F_2.
\end{align*}
\]

Assume that \( \widehat{Y} = P_2 \widetilde{X} + \Phi_2 \), where

\[
\begin{align*}
    \widetilde{X} &= \begin{pmatrix} X_0 \\ \bar{m}_x \\ \Phi_1 \end{pmatrix}, \\
    \widehat{Y} &= \begin{pmatrix} Y_0 \\ y_1 \\ y_2 \end{pmatrix}, \\
    \widehat{Z} &= \begin{pmatrix} Z_0 \\ z_1 \\ 0 \end{pmatrix}, \\
    \widehat{W}_0 &= \begin{pmatrix} W_0 \\ 0 \\ 0 \end{pmatrix},
\end{align*}
\]

and for simplicity, we rewrite the Hamiltonian system by

\[
\begin{align*}
    d\widetilde{X} &= (L_{11} \widetilde{X} + L_{12} \widehat{Y} + L_{13} \widehat{Z} + f_1)\,dt + (L_{21} \widetilde{X} + L_{22} \widehat{Y} + L_{23} \widehat{Z} + f_2) \circ d\widehat{W}_0, \\
    d\widehat{Y} &= (L_{31} \widetilde{X} + L_{32} \widehat{Y} + L_{33} \widehat{Z} + f_3)\,dt + \widehat{Z} \circ d\widehat{W}_0,
\end{align*}
\]
where

\[
L_{11} = \begin{pmatrix}
A_0 & F_1^1 \\
0 & \overline{A} & \overline{B} \\
Q\xi & 0 & \overline{Q} & \overline{A}
\end{pmatrix}, \quad L_{12} = \begin{pmatrix}
-B_0 R_0^{-1} R_0^T & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
L_{13} = \begin{pmatrix}
-B_0 R_0^{-1} D_0^T & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
L_{21} = \begin{pmatrix}
0 & F_2^2 \\
0 & \overline{C} & \overline{D} \\
0 & 0 & 0
\end{pmatrix}, \quad L_{22} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
L_{23} = \begin{pmatrix}
-D_0 R_0^{-1} D_0^T & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
L_{31} = \begin{pmatrix}
0 & Q_0 (1-\lambda_0) \\
0 & 0
\end{pmatrix}, \quad L_{32} = \begin{pmatrix}
-A_0^T & 0 & -\tilde{Q}_x \\
0 & -\overline{A}^T & -\overline{Q}^T \\
0 & 0 & 0
\end{pmatrix},
\]

\[
L_{33} = \begin{pmatrix}
-C_0^T & 0 & 0 \\
-D_0^T & 0 & 0
\end{pmatrix},
\]

\[
f_1 = \begin{pmatrix}
E_0^1 \overline{m}_X \\
0 \\
Q_0 \overline{\lambda}_0 \overline{m}_X
\end{pmatrix}, \quad f_2 = \begin{pmatrix}
E_0 \overline{m}_X \\
0 \\
0
\end{pmatrix},
\]

\[
f_3 = \begin{pmatrix}
Q_0 \lambda_0 \overline{m}_X \\
0
\end{pmatrix},
\]

then we can get the following Riccati equations

\[
\begin{align*}
\dot{P}_2 + P_2 L_{11} - L_{32} P_2 - L_{31} + P_2 L_{12} P_2 \\
+ (P_2 L_{13} - L_{33})(I - P_2 L_{23})^{-1} P_2 (L_{21} + L_{22} P_2) &= 0, \\
P_2(T) &= H_0,
\end{align*}
\]

and the related BSDE

\[
\begin{align*}
d\Phi_2 &= - \left( (P_2 L_{12} - L_{32}) + (P_2 L_{13} - L_{33})(I - P_2 L_{23})^{-1} P_2 L_{22} \right) \Phi_2 \\
&\quad - P_2 f_1 - (P_2 L_{13} - L_{33})(I - P_2 L_{23})^{-1} f_2 + f_3 + \eta_2 W_0(t), \\
\Phi_2(T) &= 0.
\end{align*}
\]

So the feedback is

\[
\begin{pmatrix}
\pi_0 \\
0
\end{pmatrix} = - \begin{pmatrix}
R_0^{-1} \\
0
\end{pmatrix} \begin{pmatrix}
\pi_0 \\
0
\end{pmatrix} \\
\left[ \begin{pmatrix}
R_0^{-1} \\
0
\end{pmatrix} \begin{pmatrix}
P_2 \tilde{X} + \Phi_2 \\
0
\end{pmatrix} + \begin{pmatrix}
D_0^T \\
0
\end{pmatrix} \begin{pmatrix}
I - P_2 L_{23} \\
0
\end{pmatrix}^{-1} P_2 (L_{21} + L_{22} P_2) \tilde{X} + L_{22} \Phi_2
\right] \\
= - \begin{pmatrix}
R_0^{-1} \\
0
\end{pmatrix} \begin{pmatrix}
P_2 + \begin{pmatrix}
D_0^T \\
0
\end{pmatrix} \begin{pmatrix}
I - P_2 L_{23} \\
0
\end{pmatrix}^{-1} P_2 (L_{21} + L_{22} P_2) \\
0
\end{pmatrix} \tilde{X} \\
+ \begin{pmatrix}
D_0^T \\
0
\end{pmatrix} \begin{pmatrix}
I - P_2 L_{23} \\
0
\end{pmatrix}^{-1} P_2 L_{22} \Phi_2.
\]
Last, the Hamiltonian system of minor leader is (30), i.e.,

\[
\begin{aligned}
\frac{d}{dt} \bar{X}_i &= [A \bar{X}_i + B \bar{u}_i + E_1 \bar{m}_X] dt + [C \bar{X}_i + D \bar{u}_i + E_2 \bar{m}_X] dW_i, \\
\frac{d}{dt} \bar{Y}_i &= -[A^\top \bar{Y}_i + C^\top \bar{Z}_i + Q(\bar{X}_i - (\bar{\lambda}_i \bar{m}_X + (1 - \bar{\lambda}) \bar{X}_0))] dt + \bar{Z}_i dW_i, \\
\bar{X}_i(0) &= \xi_i, \quad \bar{Y}_i(T) = H \bar{X}_i,
\end{aligned}
\]

with the stationary condition

\[
\bar{u}_i = -R^{-1}(B^\top \bar{Y}_i + D^\top \bar{Z}_i).
\]

Assume that \( \bar{Y}_i = P_3 \bar{X}_i + \Phi_3 \), and we can get the Riccati equations

\[
\begin{aligned}
\dot{P}_3 + A^\top P_3 + P_3 A + Q - P_3 B R^{-1} B^\top P_3 + S^\top R^{-1} P_3 S &= 0, \\
P_3(T) &= H,
\end{aligned}
\]

and the related BSDE

\[
\begin{aligned}
\frac{d}{dt} \Phi_3 &= \left(-A^\top \Phi_3 + P_3 B R^{-1} B^\top \Phi_3 + Q(\lambda \bar{m}_X + (1 - \lambda) \bar{X}_0) - P_3 E_1 \bar{m}_X - S^\top R^{-1} f \right) dt \\
&\quad + \eta_3 W_0(t), \\
\Phi_3(T) &= 0,
\end{aligned}
\]

where

\[
\begin{aligned}
\mathcal{R} &:= I + P_3 D R^{-1} D^\top, & S &:= C - D R^{-1} B^\top P_3, \\
f &:= P_3 E_2 \bar{m}_X - P_3 D R^{-1} B^\top \Phi_3.
\end{aligned}
\]

Note that

\[
\text{minor leader:} \quad \begin{cases} 
\bar{Y}_i = P_3 \bar{X}_i + \Phi_3, \\
\bar{Z}_i = \mathcal{R}^{-1} P_3 S \bar{X}_i + \mathcal{R}^{-1} f,
\end{cases}
\]

so the feedback is

\[
\bar{u}_i = -R^{-1} \left( B^\top P_3 + D^\top \mathcal{R}^{-1} P_3 S \right) \bar{X}_i - R^{-1} B^\top \Phi_3 - R^{-1} D^\top \mathcal{R}^{-1} f.
\]
For such feedback strategy, we can get another CC equation as follows.

\[
\begin{aligned}
\frac{d\bar{X}}{dt} &= (A_0 \bar{X}_0 - B_0 R_0^{-1}(B_0^T Y_0 + D_0^T Z_0) + E_0^1 \bar{m}_X + F_0^1 \bar{m}_x) dt \\
&\quad + (C_0 \bar{X}_0 - D_0 R_0^{-1}(B_0^T Y_0 + D_0^T Z_0) + E_0^2 \bar{m}_X + F_0^2 \bar{m}_x) dW_0, \\
\frac{d\bar{m}}{dt} &= \left(\bar{A} \bar{m} + \tilde{B} \Phi_1\right) dt + \left(\tilde{C} \bar{m} + \tilde{D} \Phi_1\right) dW_0, \\
\frac{d\Phi_1}{dt} &= \left(\tilde{Q} \lambda_1 \bar{X}_0 + \tilde{Q} \bar{m}_x + \tilde{A} \Phi_1 + \tilde{Q} \lambda_2 \bar{m}_X\right) dt + \eta_1 W_0(t), \\
\frac{d\bar{y}_1}{dt} &= -\left(\bar{A}^T \bar{y}_1 + \tilde{C}^T \bar{z}_1 + \tilde{Q}^T \bar{y}_2 + F_0^1 \bar{y}_0 + F_0^2 \bar{z}_0 \\
&\quad - (1 - \lambda_0) Q_0 \left(\bar{X}_0 - (\lambda_0 \bar{m}_X + (1 - \lambda_0) \bar{m}_x)\right)\right) dt + \bar{z}_1 dW_0, \\
\frac{d\bar{y}_2}{dt} &= -(\tilde{B}^T \bar{y}_1 + \tilde{D}^T \bar{z}_1 + \tilde{A}^T \bar{y}_2) dt, \\
\frac{d\bar{m}_X}{dt} &= (\bar{A} \bar{m}_X + B \Phi_3) dt + (\tilde{C} \bar{m}_X + \tilde{D} \Phi_3) dW_0, \\
\frac{d\Phi_3}{dt} &= (Q(1 - \lambda) \bar{X}_0 + \bar{A} \Phi_3 + \tilde{Q} \bar{m}_X) dt + \eta_3 W_0(t), \\
\bar{X}_0(0) &= \bar{x}_0, \bar{m}_x(0) = 0, \Phi_1(T) = 0, Y_0(T) = H_0 \bar{X}_0(T), y_1(T) = 0, \\
y_2(0) = 0, \bar{m}_X(0) = 0, \Phi_3(T) = 0,
\end{aligned}
\]

where

\[
\begin{align*}
A &:= A + E_1 - B R^{-1}(B^T P_3 + D^T R^{-1}(P_3 S + P_3 E_2)), \\
B &:= (B R^{-1} D^T R^{-1} P_3 D - B) R^{-1} B^T, \\
C &:= C + E_2 - D R^{-1}(B^T P_3 + D^T R^{-1}(P_3 S + P_3 E_2)), \\
D &:= (D R^{-1} D^T R^{-1} P_3 D - D) R^{-1} B^T, \\
\bar{A} &:= (S^T R^{-1} P_3 D + P_3 B) R^{-1} B^T - A^T, \\
\bar{Q} &:= Q\lambda - P_3 E_1 - S^T R^{-1} P_3 E_2.
\end{align*}
\]

5 Stackelberg–Nash–Cournot (SNC) Approximate Equilibrium

In above sections, we obtained the decentralized open-loop strategy and feedback strategy of the mixed S-MM game through the consistency condition system. Now we turn to verify such decentralized strategy satisfying the SNC approximate equilibrium (i.e., $\varepsilon$-Stackelberg–Nash–Cournot equilibrium) property. To this end, we need first ensure the open-loop strategy and feedback strategy are well-defined by assuming all Riccati equations (44)–(51) admit a unique solution. Thus, we may impose additional assumptions as follows.

(H3) $\tilde{R}(\cdot), R_0(\cdot)$ and $R(\cdot)$ are invertible.
The Riccati equations (44)–(51) admit a unique solution.
We first present the definition of \(\varepsilon\)-SNC equilibrium.

**Definition 5.1** A set of controls \((\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_{N_l}, \tilde{v}_1, \ldots, \tilde{v}_{N_f}) \in \mathcal{U}_0^d \times \mathcal{U}_1^d \times \mathcal{V}_d\), for \((1 + N_l + N_f)\) agents is called to satisfy an \(\varepsilon\)-SNC equilibrium with respect to the costs \((J_0, J_1^f, \ldots, J_{N_l}^f, \mathcal{J}_f^{f_f}, \mathcal{J}_N^f)\), if there exists \(\varepsilon = \varepsilon(N) \geq 0 (N = \min\{N_l, N_f\})\), \(\lim_{N \to \infty} \varepsilon(N) = 0\) such that for any fixed \(i = 1, 2, \ldots, N_l, j = 1, 2, \ldots, N_f\), we have

\[
\begin{align*}
J_0(\tilde{u}_0, \tilde{u}, \tilde{v}) &\leq J_0(u_0, u, \tilde{v}) + \varepsilon, \\
J_i^f(\tilde{u}_0, \tilde{u}_i, \tilde{u}_{-i}) &\leq J_i^f(u_0, u_i, \tilde{u}_{-i}) + \varepsilon, \\
J_j^f(\tilde{u}_0, \tilde{u}, \tilde{v}_j, \tilde{v}_{-j}) &\leq J_j^f(u_0, \tilde{u}, v_j, \tilde{v}_{-j}) + \varepsilon,
\end{align*}
\]

(53)

when any alternative control \((u_0, u_i, v_j) \in \mathcal{U}_0^d \times \mathcal{U}_1^d \times \mathcal{V}_d\) is applied by \((A_0, A_i, B_j)\).

At first, we present the main result of this section and defer its proof in later part.

**Theorem 5.1** Under assumptions (H1)–(H4), and if the conditions of Theorems 3.1, 3.2, 3.3 hold, then \((\tilde{u}_0, \tilde{u}_i, \tilde{v}_j)\) is an \(\varepsilon\)-SNC equilibrium of mixed S-MM-MFG for major leader agent \(A_0\), each minor leader agent \(A_i^l\), \(i = 1, 2, \ldots, N_l\), and each follower agent \(A_j^f\), \(j = 1, 2, \ldots, N_f\). And \((\tilde{u}_0, \tilde{u}_i, \tilde{v}_j)\) is given by

\[
\begin{align*}
\tilde{u}_0(t) &= - R_0^{-1}(t)(B_0(t)^\top Y_0(t) + D_0(t)^\top Z_0(t)), \\
\tilde{u}_i(t) &= - R^{-1}(t)(B(t)^\top Y_i(t) + D(t)^\top Z_i(t)), \\
\tilde{v}_j(t) &= - \tilde{R}^{-1}(t)(\tilde{B}(t)^\top \tilde{Y}_j(t) + \tilde{D}(t)^\top \tilde{Z}_j(t)),
\end{align*}
\]

(54)

for \((Y_0, Z_0), (\tilde{Y}_i, \tilde{Z}_i), (\tilde{Y}_j, \tilde{Z}_j)\) solved by (38).

For major leader \(A_0\), minor leaders \(A_i^l\) and followers \(A_j^f\), the decentralized states \(\overline{X}_0, \overline{X}_i\) and \(\overline{Y}_j\) are given respectively by

\[
\begin{align*}
d\overline{X}_0 &= (A_0 \overline{X}_0 - B_0 R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^1 \overline{X}^{(N_l)} + F_0^1 \overline{X}^{(N_f)})dt \\
&\quad + \{C_0 \overline{X}_0 - D_0 R_0^{-1}(B_0^\top Y_0 + D_0^\top Z_0) + E_0^2 \overline{X}^{(N_l)} + F_0^2 \overline{X}^{(N_f)}\}dW_0, \\
d\overline{X}_i &= (A \overline{X}_i - BR^{-1}(B^\top \overline{Y}_i + D^\top \overline{Z}_i) + E_i \overline{X}^{(N_l)})dt \\
&\quad + \{C \overline{X}_i - DR^{-1}(B^\top \overline{Y}_i + D^\top \overline{Z}_i) + E_i \overline{X}^{(N_l)}\}dW_i, \\
d\overline{Y}_j &= (\tilde{A} \overline{X}_j - \tilde{B} \tilde{R}^{-1}(\tilde{B}^\top \overline{Y}_j + \tilde{D}^\top \overline{Z}_j) + \tilde{E}_j \overline{X}^{(N_f)})dt \\
&\quad + \{\tilde{C} \overline{X}_j - \tilde{D} \tilde{R}^{-1}(\tilde{B}^\top \overline{Y}_j + \tilde{D}^\top \overline{Z}_j) + \tilde{F}_j \overline{X}^{(N_f)}\}d\tilde{W}_j,
\end{align*}
\]

(55)

where the processes \((Y_0, Z_0), (\overline{Y}_i, \overline{Z}_i), (\overline{Y}_j, \overline{Z}_j)\) are solved by (38). Let us first present several lemmas to be used later. Here, we may abuse the inner product notation \(\langle \cdot, \cdot \rangle\) with \(| \cdot |^2\).
Lemma 5.1 Under assumptions (H1)–(H4), and if the conditions in the Theorems 3.1, 3.2, 3.3 hold, then there exists a constant $M$ independent of $N_l$ and $N_f$, such that

$$
\sup_{0 \leq i \leq N_l} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\overline{X}_i(t)|^2 \right] < M,
$$

$$
\sup_{1 \leq j \leq N_f} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\overline{X}_j(t)|^2 \right] < M.
$$

**Proof** From Theorems 3.1, 3.2, 3.3, the FBSDEs (16), (24) and (30) have a unique solution $(\overline{X}_0, \overline{Y}_0, \overline{Z}_0) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{3n})$, $(\overline{X}_i, \overline{Y}_i, \overline{Z}_i) \in L^2_{\mathcal{F}_i}(0, T; \mathbb{R}^{3n})$ and $(\overline{x}_j, \overline{y}_j, \overline{z}_j) \in L^2_{\mathcal{G}_j}(0, T; \mathbb{R}^{3n})$, $1 \leq i \leq N_l, 1 \leq j \leq N_f$. Thus, SDEs system (55) has also a unique solution

$$(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_{N_l}, \overline{x}_1, \ldots, \overline{x}_{N_f}) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}_i}(0, T; \mathbb{R}^n) \times \ldots \times L^2_{\mathcal{F}_l}(0, T; \mathbb{R}^n).$$

From (55), by using Burkholder–Davis–Gundy (BDG) inequality, there exists a constant $M$, independent of $N_l$ and $N_f$, such that for any $t \in [0, T]$,

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_0(s)|^2 \right] 
\leq M + M \mathbb{E} \left[ \int_0^t |\overline{X}_0(s)|^2 + |\overline{X}^{(N_l)}(s)|^2 + |\overline{x}^{(N_f)}(s)|^2 ds \right] 
\leq M + M \mathbb{E} \left[ \int_0^t |\overline{X}_0(s)|^2 + \frac{1}{N_l} \sum_{i=1}^{N_l} |\overline{X}_i(s)|^2 + \frac{1}{N_f} \sum_{j=1}^{N_f} |\overline{x}_j(s)|^2 ds \right]
$$

and by Gronwall’s inequality, we obtain

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_0(s)|^2 \right] \leq M + M \mathbb{E} \left[ \int_0^t \frac{1}{N_l} \sum_{i=1}^{N_l} |\overline{X}_i(s)|^2 + \frac{1}{N_f} \sum_{j=1}^{N_f} |\overline{x}_j(s)|^2 ds \right].
$$

(56)

Similarly, we have

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_i(s)|^2 \right] \leq M + M \mathbb{E} \left[ \int_0^t |\overline{X}_i(s)|^2 + \frac{1}{N_l} \sum_{i=1}^{N_l} |\overline{X}_i(s)|^2 ds \right],
$$

$1 \leq i \leq N_l,$

(57)

and

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |\overline{X}_j(s)|^2 \right] \leq M + M \mathbb{E} \left[ \int_0^t |\overline{X}_j(s)|^2 + \frac{1}{N_f} \sum_{j=1}^{N_f} |\overline{x}_j(s)|^2 ds \right],
$$

$1 \leq j \leq N_f.$

(58)
Thus
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} \sum_{i=1}^{N_l} |X_i(s)|^2 \right] \leq \mathbb{E}\left[ \sum_{i=1}^{N_l} \sup_{0 \leq s \leq t} |X_i(s)|^2 \right] \leq MN_l + 2M\mathbb{E}\left[ \int_0^t \sum_{i=1}^{N_l} |\bar{X}_i(s)|^2 \right],
\]
and
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} \sum_{j=1}^{N_f} |\bar{x}_j(s)|^2 \right] \leq \mathbb{E}\left[ \sum_{j=1}^{N_f} \sup_{0 \leq s \leq t} |\bar{x}_j(s)|^2 \right] \leq MN_f + 2M\mathbb{E}\left[ \int_0^t \sum_{j=1}^{N_f} |\bar{x}_j(s)|^2 \right].
\]

By Gronwall’s inequality, it follows that
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} \sum_{i=1}^{N_l} |X_i(s)|^2 \right] = O(N_l) \quad \text{and} \quad \mathbb{E}\left[ \sup_{0 \leq s \leq t} \sum_{j=1}^{N_f} |\bar{x}_j(s)|^2 \right] = O(N_f).
\]

Then, substituting this estimate to (57) and (58) and Gronwall’s inequality yields
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |\bar{X}_i(s)|^2 \right] \leq M \quad \text{and} \quad \mathbb{E}\left[ \sup_{0 \leq s \leq t} |\bar{x}_j(s)|^2 \right] \leq M.
\]

By applying this estimate to (56), we get
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |\bar{X}_0(s)|^2 \right] \leq M. \quad \square
\]

**Remark 5.1** In fact, $M$ is determined by not only the size of the population $N_l, N_f$ but also the terminal time $T$ and the Lipschitz constants of state dynamics.

Now, we recall that
\[
\bar{X}^{(N_l)}(t) = \frac{1}{N_l} \sum_{i=1}^{N_l} \bar{X}_i(t) \quad \text{and} \quad \bar{x}^{(N_f)}(t) = \frac{1}{N_f} \sum_{j=1}^{N_f} \bar{x}_j(t),
\]
then we have

**Lemma 5.2** Under assumptions (H1)–(H4), and if the conditions of Theorems 3.1, 3.2, 3.3 hold, then there exists a constant $M$ independent of $N_l$ and $N_f$, such that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\bar{X}^{(N_l)}(t) - \bar{m}_X(t)|^2 \right] \leq \frac{M}{N_l},
\]
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\bar{x}^{(N_f)}(t) - \bar{m}_x(t)|^2 \right] \leq \frac{M}{N_f}.
\]
Proof} For the first one, we have
\begin{align}
\begin{cases}
d\left(\mathbf{X}^{(N_l)} - \bar{m}_X\right) = (A + E_1)\left(\mathbf{X}^{(N_l)} - \bar{m}_X\right)dt + \cdots dW_i, \\
\left(\mathbf{X}^{(N_l)} - \bar{m}_X\right)(0) = 0.
\end{cases}
\end{align}
(59)

From (59), by using Burkholder-Davis-Gundy (BDG) inequality and Lemma 5.1, there exists a constant \(M\), independent of \(N_l\) and \(N_f\), such that for any \(t \in [0, T]\),
\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} \left|\mathbf{X}^{(N_l)} - \bar{m}_X\right|^2(s)\right] \leq \frac{M}{N_l} + M \mathbb{E}\left[\int_0^t \left|\mathbf{X}^{(N_l)} - \bar{m}_X\right|^2(s)ds\right].
\]
and by Gronwall’s inequality, we obtain
\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} \left|\mathbf{X}^{(N_l)} - \bar{m}_X\right|^2(s)\right] \leq \frac{M}{N_l}.
\]

In the same way, we can prove the second formula. □

Remark 5.2 The above estimates are well-known results and common-employed in MFG analysis. Interested readers may refer [8–10,22] for similar results in detail.

Lemma 5.3 Under assumptions (H1)–(H4), and if the conditions of Theorems 3.1, 3.2, 3.3 hold, then there exists a constant \(M\) independent of \(N_l\) and \(N_f\), such that
\[
\begin{align*}
\left|J_0(\hat{u}_0, \hat{u}, \hat{v}) - J_0(\bar{u}_0)\right| &= O\left(\frac{1}{\sqrt{N}}\right), \\
\left|J^l_i(\hat{u}_0, \hat{u}_i, \bar{u}_{-i}) - J^l_i(\bar{u}_0, \bar{u}_i)\right| &= O\left(\frac{1}{\sqrt{N}}\right), \\
\left|J^f_j(\hat{u}_0, \hat{u}, \bar{v}_j, \bar{v}_{-j}) - J^f_j(\bar{u}_0, \bar{v}_j)\right| &= O\left(\frac{1}{\sqrt{N}}\right),
\end{align*}
\]
where \(N := \min\{N_l, N_f\}\).
Proof} Let us first consider the major leader agent. Recall (4) and (13), we have
\[ J_0(u_0, u, v) - J_0(u_0) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[ \| \bar{X}_0 - (\lambda_0 \bar{X}^{(N_l)} + (1 - \lambda_0)\bar{X}^{(N_f)}) \|_{Q_0}^2 \\
- \| \bar{X}_0 - (\lambda_0 \bar{m}_X + (1 - \lambda_0)\bar{m}_x) \|_{Q_0}^2 \right] dt \right\}. \]

(60)

By Hölder inequality and Lemma 5.1, there exists a constant \( M \) independent of \( N_l \) and \( N_f \) such that
\[ \mathbb{E} \left\{ \int_0^T \left| \mathcal{Q}_0(\lambda_0(\bar{m}_X - \bar{X}^{(N_l)}) + (1 - \lambda_0)(\bar{m}_x - \bar{X}^{(N_f)})) \right|^2 dt \right\}^{\frac{1}{2}} \leq \mathbb{E} \left\{ \int_0^T \left[ \lambda_0(\bar{m}_X - \bar{X}^{(N_l)}) + (1 - \lambda_0)(\bar{m}_x - \bar{X}^{(N_f)}) \right]^2 dt \right\}^{\frac{1}{2}} \]

(61)

Noting (60), (61) and Lemma 5.2, there exists a constant \( M \) independent of \( N_l \) and \( N_f \) such that
\[ \mathbb{E} \left[ \int_0^T \left| \mathcal{Q}_0(\lambda_0(\bar{m}_X - \bar{X}^{(N_l)}) + (1 - \lambda_0)(\bar{m}_x - \bar{X}^{(N_f)})) \right|^2 dt \right]^{\frac{1}{2}} \leq M \left[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\bar{X}^{(N_l)}|_2^2(s) \right] \int_0^T \left| \mathcal{Q}_0(\lambda_0) \right|^2 dt \right]^{\frac{1}{2}} \]

(62)

The rest two claims can be proved in the same way. □
Remark 5.3 We denote $M$ the common constant of different bounds. In above lemmas, the constant $M$ may vary line by line but it is always independent of the number of minor-leader agents $N_l$ and the number of follower agents $N_f$.

5.1 Major Leader Agent’s Perturbation

In this subsection, we will prove that the control strategies set $(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{N_l}, \overline{v}_1, \ldots, \overline{v}_{N_f})$ given by Theorem 5.1 is an $\varepsilon$-SNC equilibrium of mixed S-MM for major leader agent, i.e., there exists an $\varepsilon = \varepsilon(N) \geq 0$, $\lim_{N \to \infty} \varepsilon(N) = 0$ such that

$$J_0(\overline{u}_0, \overline{u}, \overline{v}) \leq J_0(u_0, \overline{u}, \overline{v}) + \varepsilon, \quad \forall u_0 \in U_0^c[0, T].$$

Let us consider that the major leader agent $A_0$ apply an alternative strategy $u_0$, each minor leader agent $A_l^j$ uses the control $\overline{u}_j = -R^{-1}(t)(B(t)^T Y_i(t) + D(t)^T \tilde{Z}_i(t))$ and each follower agent $A_l^j$ uses the control $\overline{v}_j = -\tilde{R}^{-1}(t)(\tilde{B}(t)^T \overline{v}_j(t) + \tilde{D}(t)^T \tilde{Z}_j(t))$. To prove $(\overline{u}_0, \overline{u}, \ldots, \overline{u}_{N_l}, \overline{v}_1, \ldots, \overline{v}_{N_f})$ is an $\varepsilon$-SNC equilibrium for major leader agent, we need to show that for possible alternative control $u_0$, $\inf_{u_0 \in U_0^c[0, T]} J_0(u_0, \overline{u}, \overline{v}[u_0]) \geq J_0(\overline{u}_0, \overline{u}, \overline{v}) - \varepsilon$. Then we only need to consider the perturbation $u_0 \in U_0^c[0, T]$ such that $J_0(u_0, \overline{u}, \overline{v}[u_0]) \leq J_0(\overline{u}_0, \overline{u}, \overline{v})$. By the representation of cost functional in [34,40], we can give the representation of cost functional as follows.

Proposition 5.1 Let (H1)-(H2) hold. There exists a bounded self-adjoint linear operator $N_0 : U_0^c[0, T] \to U_0^c[0, T]$, a bounded linear operator $H_0 : \mathbb{R}^n \to U_0^c[0, T]$, a bounded real-valued function $M_0 : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned}
J_0(x_0, x, y; u_0, v[u_0]) &= \frac{1}{2} \left\{ <N_0 u_0(\cdot), u_0(\cdot)> + 2 \left( H_0(x_0), u_0(\cdot) \right) + M_0(x_0) \right\}, \\
& \forall (x_0, u_0) \in \mathbb{R}^n \times U_0^c[0, T].
\end{aligned}$$

Proof Refer to Proposition 3.1 in [34]. □

So if we assume that $N_0 \gg 0$, from Lemma 5.3, then there exists a constant $c > 0$, such that

$$\mathbb{E} \int_0^T \left| N_0^{-\frac{1}{2}} u_0(t) + N_0^{-\frac{1}{2}} H_0(x_0) \right|^2 dt \leq J_0(u_0, \overline{u}, \overline{v}) + c \leq J_0(\overline{u}_0, \overline{u}, \overline{v}) + c \leq J_0(\overline{u}_0) + c + O\left( \frac{1}{\sqrt{N}} \right),$$

which implies that $\mathbb{E} \int_0^T \left| u_0(t) \right|^2 dt \leq M$, where $M$ is a constant independent of $N$. In fact, by bounded inverse theorem, $N_0^{-1}$ is bounded, so there exists a constant... Springer
\[ 0 < \gamma \leq \| N_0^{\frac{1}{2}} \|, \text{ such that} \]
\[ \gamma E \int_0^T |u_0(t)|^2 \, dt \leq \| N_0^{\frac{1}{2}} \| E \int_0^T \left| u_0(t) + N_0^{-1} H_0(x_0) \right|^2 \, dt \leq J_0(\bar{u}_0) + c + O \left( \frac{1}{\sqrt{N}} \right). \]

Then we have \( E \int_0^T |u_0(t)|^2 \, dt \leq M \). Similar to Lemma 5.1, we can show that
\[ E \left[ \sup_{0 \leq t \leq T} |X_0(t)|^2 \right] \leq M. \] (63)

**Lemma 5.4** Under assumptions (H1)–(H4), and if conditions of Theorems 3.1, 3.2, 3.3 hold, for the major leader agent’s perturbation control \( u_0 \), we have
\[ |J_0(u_0, \bar{u}, \bar{v}) - J_0(u_0)| = O \left( \frac{1}{\sqrt{N}} \right). \]

**Proof** Recall (4) and (13), we have
\[ J_0(u_0, \bar{u}, \bar{v}) - J_0(u_0) \]
\[ = \frac{1}{2} E \left\{ \int_0^T \left[ \| X_0 - (\lambda_0 \bar{x}^{(N_l)} + (1 - \lambda_0) \bar{x}^{(N_f)}) \|^2_{Q_0} \right. \right. \]
\[ - \| X_0 - (\lambda_0 \bar{m}X + (1 - \lambda_0) \bar{m}x) \|^2_{Q_0} \] \[ \left. \left. \right] \, dt \right\}, \]
\[ = E \left\{ \int_0^T \left\{ Q_0 \left( X_0 - (\lambda_0 \bar{m}X + (1 - \lambda_0) \bar{m}x) \right), \lambda_0 (\bar{m}X - \bar{x}^{(N_l)}) \right. \right. \]
\[ + (1 - \lambda_0)(\bar{m}x - \bar{x}^{(N_f)}) \left. \right\} \, dt \}
\[ + \frac{1}{2} E \left\{ \int_0^T \left\| \lambda_0 (\bar{x}^{(N_l)} - \bar{m}X) + (1 - \lambda_0)(\bar{x}^{(N_f)} - \bar{m}x) \right\|^2_{Q_0} \, dt \right\}. \] (64)

By Hölder inequality and (63), there exists a constant \( M \) independent of \( N_l \) and \( N_f \) such that
\[ E \left\{ \int_0^T \left\| Q_0 \left( X_0 - (\lambda_0 \bar{m}X + (1 - \lambda_0) \bar{m}x) \right), \lambda_0 (\bar{m}X - \bar{x}^{(N_l)}) + (1 - \lambda_0)(\bar{m}x - \bar{x}^{(N_f)}) \right\} \, dt \right\} \]
\[ \leq E \left\{ \int_0^T \left\| X_0 - (\lambda_0 \bar{m}X + (1 - \lambda_0) \bar{m}x) \right\|^2_{Q_0} \, dt \right\}^{\frac{1}{2}} \]
\[ \leq E \left\{ \int_0^T \left\| Q_0 \left( \lambda_0 (\bar{m}X - \bar{x}^{(N_l)}) + (1 - \lambda_0)(\bar{m}x - \bar{x}^{(N_f)}) \right) \right\|^2 \, dt \right\}^{\frac{1}{2}} \]
\[ \leq M E \left\{ \int_0^T \left\| Q_0 \left( \lambda_0 (\bar{m}X - \bar{x}^{(N_l)}) + (1 - \lambda_0)(\bar{m}x - \bar{x}^{(N_f)}) \right) \right\|^2 \, dt \right\}^{\frac{1}{2}}. \] (65)
At last, same as the Lemma 5.3, noting (64), (65), and Lemma 5.2, there exists a constant $M$ independent of $N_l$ and $N_f$ such that

$$
\mathbb{E}\left\{ \int_0^T \left| Q_0\left( \lambda_0(\bar{m}_X - \bar{X}^{(N_l)}) + (1 - \lambda_0)(\bar{m}_x - \bar{X}^{(N_f)}) \right) \right|^2 dt \right\}^{\frac{1}{2}}
\leq \left\{ \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| \bar{X}^{(N_l)} - \bar{m}_X \right|^2(s) \right] \int_0^T |Q_0\lambda_0|^2 dt \right\}^{\frac{1}{2}}
\leq M \left( \frac{1}{\sqrt{N_l}} + \frac{1}{\sqrt{N_f}} \right) = O\left(\frac{1}{\sqrt{N}}\right).
$$

(66)

Then, applying Lemmas 5.3 and 5.4, we can give the first part of proof of Theorem 5.1, i.e., the control strategies set $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_l}, \bar{v}_1, \ldots, \bar{v}_{N_f})$ given by Theorem 5.1 is an $\varepsilon$-SNC equilibrium of the mixed S-MM for major leader agent.

**Part A of the Proof to Theorem 5.1**

Combining Lemmas 5.3 and 5.4, we have

$$
J_0(\bar{u}_0, \bar{u}, \bar{v}) \leq J_0(\bar{u}_0) + O\left(\frac{1}{\sqrt{N}}\right) \leq J_0(u_0) + O\left(\frac{1}{\sqrt{N}}\right) \leq J_0(\bar{u}_0, \bar{u}, \bar{v}) + O\left(\frac{1}{\sqrt{N}}\right),
$$

where the second inequality comes from the fact that $J_0(\bar{u}_0) = \inf_{u_0 \in \mathcal{U}_0[0,T]} J_0(u_0)$. Consequently, the Theorem 5.1 holds for the major leader agent with $\varepsilon = O\left(\frac{1}{\sqrt{N}}\right)$.

5.2 Minor Leader Agent’s Perturbation

Now, let us consider the following perturbation: a given minor leader agent $\mathcal{A}_l^j$ uses an alternative strategy $u_i \in \mathcal{U}_i[0, T]$, the major leader agent uses $\bar{u}_0$, each follower agent $\mathcal{A}_f^j$ uses $\bar{v}_j$ while other minor leader agents use the control $\bar{u}_{-i}$. In fact, by the representation of cost functional (which is similar to the argument of major leader agent), to prove $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_l}, \bar{v}_1, \ldots, \bar{v}_{N_f})$ is an $\varepsilon$-SNC equilibrium for minor leader agent, we only need to consider the perturbation $u_i \in \mathcal{U}_i[0, T]$ satisfying

$$
\mathbb{E}\int_0^T |u_i(t)|^2 dt \leq M,
$$

where $M$ is a constant independent of $N_l$. Then similar to Lemma 5.1, we can show that

$$
\sup_{1 \leq i \leq N_l} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |X_i(t)|^2 \right] \leq M.
$$

(67)
Lemma 5.5 Under assumptions (H1)–(H4), and if conditions of Theorems 3.1, 3.2, 3.3 hold, then there exists a constant $M$ independent of $N_l$ and $N_f$, such that

$$
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| X^{(i,N_l)}(t) - \bar{m}X(t) \right|^2 \right] \leq \frac{M}{N_l},
$$

where $X^{(i,N_l)}(t) = \frac{1}{N_l} (X_i(t) + \sum_{k \neq i} X_k(t))$.

**Proof** In fact, we have

$$X^{(i,N_l)}(t) - \bar{X}^{(N_l)}(t) = \frac{1}{N_l} X_i(t),$$

by (67), it yields

$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| X^{(i,N_l)}(t) - \bar{X}^{(N_l)}(t) \right|^2 \right] \leq \frac{M}{N_l}.$$ 

Combined with Lemma 5.2, we can directly get

$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| X^{(i,N_l)}(t) - \bar{m}X(t) \right|^2 \right] \leq \frac{M}{N_l}.
\square
$$

Lemma 5.6 Under assumptions (H1)–(H4), and if the conditions of Theorems 3.1, 3.2, 3.3 hold, for the minor leader agent’s perturbation control $u_i$, we have

$$\left| J_i^l(\bar{u}_0, u_i, \bar{u}_{-i}) - J_i^l(\bar{u}_0, u_i) \right| = O\left( \frac{1}{\sqrt{N}} \right).$$

**Proof** Recall (5) and (15), we have

$$J_i^l(\bar{u}_0, u_i, \bar{u}_{-i}) - J_i^l(\bar{u}_0, u_i)$$

$$= \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left[ \left\| X_i - (\lambda \bar{X}^{(i,N_l)} + (1 - \lambda)X_0) \right\|_Q^2 
- \left\| X_i - (\lambda \bar{m}X + (1 - \lambda)X_0) \right\|_Q^2 \right] dt \right\},$$

$$= \mathbb{E}\left\{ \int_0^T \left( Q(X_i - (\lambda \bar{m}X + (1 - \lambda)X_0)), \lambda(\bar{m}X - \bar{X}^{(i,N_l)}) \right) dt \right\}$$

$$+ \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left\| \lambda(\bar{X}^{(i,N_l)} - \bar{m}X) \right\|_Q^2 dt \right\}. (68)$$

\ Springer
By same technique, applying Hölder inequality, Lemmas 5.5, and (67), there exists a constant $M$ independent of $N_l$ and $N_f$ such that

\[
\mathbb{E}\left\{ \int_0^T \left( \left| Q \left( X_i - (\lambda \bar{m} X + (1 - \lambda) X_0) \right) \right|, \left| \lambda (\bar{m} X - X^{(i,N_l)}) \right| \right) dt \right\} \\
\leq \mathbb{E}\left\{ \int_0^T \left| X_i - (\lambda \bar{m} X + (1 - \lambda) X_0) \right|^2 dt \right\}^{1/2} \\
\mathbb{E}\left\{ \int_0^T \left| Q \lambda (\bar{m} X - X^{(i,N_l)}) \right|^2 dt \right\}^{1/2} \\
\leq M \mathbb{E}\left\{ \int_0^T \left| Q \lambda (\bar{m} X - X^{(i,N_l)}) \right|^2 dt \right\}^{1/2} \\
\leq M \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| X^{(i,N_l)}(s) - \bar{m} X(s) \right|^2 \right] \int_0^T |Q \lambda|^2 dt \right\}^{1/2} \\
\leq \frac{M}{\sqrt{N_l}} = O\left( \frac{1}{\sqrt{N}} \right),
\]

Taking the advantage of Lemmas 5.3 and 5.6, we can give the second part of proof to Theorem 5.1, i.e., the control strategies set \((\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_l}, \bar{v}_1, \ldots, \bar{v}_{N_f})\) given by Theorem 5.1 is an \(\varepsilon\)-SNC equilibrium of the mixed S-MM for minor leader agent.

**Part B of the Proof to Theorem 5.1**

Combining Lemmas 5.3 and 5.6, we have

\[
\mathcal{J}^f_i(\bar{u}_0, \bar{u}_i, \bar{u}_{-i}) \leq \mathcal{J}^f_i(\bar{u}_0, \bar{u}_i) + O\left( \frac{1}{\sqrt{N}} \right) \leq \mathcal{J}^f_i(\bar{u}_0, u_i) + O\left( \frac{1}{\sqrt{N}} \right) \\
\leq \mathcal{J}^f_i(\bar{u}_0, u_i, \bar{u}_{-i}) + O\left( \frac{1}{\sqrt{N}} \right),
\]

where the second inequality comes from the fact that \(\mathcal{J}^f_i(\bar{u}_0, \bar{u}_i) = \inf_{u_i \in \mathcal{U}^c_i[0,T]} \mathcal{J}^f_i(\bar{u}_0, u_i)\). Consequently, the Theorem 5.1 holds for the minor leader agent with \(\varepsilon = O\left( \frac{1}{\sqrt{N}} \right)\). □

### 5.3 Follower Agent’s Perturbation

As last step, we consider the following perturbation case: a given follower agent \(A^f_j\) applies an alternative strategy \(v_j \in \mathcal{V}^c_j[0,T]\), the major leader agent applies \(\bar{u}_0\), each minor leader agent \(A^l_i\) uses \(\bar{u}_i\) while other follower agents use control \(\bar{v}_{-j}\). Actually, by the cost functional representation (which is similar to the argument of major leader agent), to prove \((\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_l}, \bar{v}_1, \ldots, \bar{v}_{N_f})\) is an \(\varepsilon\)-SNC equilibrium for the follower agent, we only need to consider the perturbation \(v_j \in \mathcal{V}^c_j[0,T]\) satisfying

\[
\mathbb{E} \int_0^T |v_j(t)|^2 dt \leq M,
\]

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where $M$ is some constant independent of $N$. Then similar to Lemma 5.1, we can show that

$$\sup_{1 \leq j \leq N_f} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x_j(t)|^2 \right] \leq M. \quad (70)$$

**Lemma 5.7** Under assumptions (H1)–(H4), and if the conditions of Theorems 3.1, 3.2, 3.3 hold, then there exists a constant $M$ independent of $N_l$ and $N_f$, such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^{(j,N_f)}(t) - \bar{m}_x(t)|^2 \right] \leq \frac{M}{N_f},$$

where $x^{(j,N_f)}(t) = \frac{1}{N_f} (x_j(t) + \sum_{k \neq j} x_k(t))$.

**Proof** In fact, we have

$$x^{(j,N_f)}(t) - \bar{x}^{(N_f)}(t) = \frac{1}{N_f} x_j(t),$$

by (70), it yields

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^{(j,N_f)}(t) - \bar{x}^{(N_f)}(t)|^2 \right] \leq \frac{M}{N_f}.$$

Combined with Lemma 5.2, we can directly get

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^{(j,N_f)}(t) - \bar{m}_x(t)|^2 \right] \leq \frac{M}{N_f}.$$

\[\square\]

**Lemma 5.8** Under assumptions (H1)–(H4), and if the conditions in Theorems 3.1, 3.2, 3.3 hold, for the follower agent’s perturbation control $v_j$, we have

$$\left| J_j^f (\bar{u}_0, \bar{u}, v_j, \bar{v}_{-j}) - J_j^f (\bar{u}_0, v_j) \right| = O \left( \frac{1}{\sqrt{N}} \right).$$
**Proof** Recall (6) and (11), we have

\[
J_f(u_0, u, v_j, v_{-j}) - J_f(u_0, v_j) \\
= \frac{1}{2}\mathbb{E}\left\{ \int_0^T \left[ \left\| x_j - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \tilde{m}X + \tilde{\lambda}_3 \tilde{m}x) \right\|^2 \right] \right\},
\]

\[
= \mathbb{E}\left\{ \int_0^T \left[ \left( x_j - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \tilde{m}X + \tilde{\lambda}_3 \tilde{m}x) \right) \right. \\
\left. \left( \tilde{\lambda}_2 (\tilde{m}X - \tilde{X}^{(N_j)}) + \tilde{\lambda}_3 (\tilde{m}x - x^{(j,N_f)}) \right) \right] \right\} \\
+ \frac{1}{2}\mathbb{E}\left\{ \int_0^T \left\| \tilde{\lambda}_2 (\tilde{m}X - \tilde{X}^{(N_j)}) + \tilde{\lambda}_3 (\tilde{m}x - x^{(j,N_f)}) \right\|^2 \right\}.
\]

(71)

By Hölder inequality and (70) there exists a constant \( M \) independent of \( N_l \) and \( N_f \) such that

\[
\mathbb{E}\left\{ \int_0^T \left| \tilde{Q} \left( x_j - (\tilde{\lambda}_1 X_0 + \tilde{\lambda}_2 \tilde{m}X + \tilde{\lambda}_3 \tilde{m}x) \right) \right| \right\} \\
\leq \mathbb{E}\left\{ \int_0^T \left( \tilde{\lambda}_2 (\tilde{m}X - \tilde{X}^{(N_j)}) + \tilde{\lambda}_3 (\tilde{m}x - x^{(j,N_f)}) \right) \right\}^{\half} \\
\leq M \mathbb{E}\left\{ \int_0^T \left| \tilde{Q} \left( \tilde{\lambda}_2 (\tilde{m}X - \tilde{X}^{(N_j)}) + \tilde{\lambda}_3 (\tilde{m}x - x^{(j,N_f)}) \right) \right| \right\}^{\half}.
\]

(72)

Last, same as the Lemma 5.3, noting (71), (72), and Lemma 5.7, there exists a constant \( M \) independent of \( N_l \) and \( N_f \) such that

\[
\mathbb{E}\left\{ \int_0^T \left| \tilde{Q} \left( \tilde{\lambda}_2 (\tilde{m}X - \tilde{X}^{(N_j)}) + \tilde{\lambda}_3 (\tilde{m}x - x^{(j,N_f)}) \right) \right| \right\}^{\half} \\
\leq M \left\{ \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| \tilde{X}^{(N_j)} - \tilde{m}X \right|^2(s) \right] \right\}^{\half} \left\{ \mathbb{E}\left[ \sup_{0 \leq s \leq t} \left| x^{(j,N_f)} - \tilde{m}x \right|^2(s) \right] \right\}^{\half} \\
\leq M \left( \frac{1}{\sqrt{N_l}} + \frac{1}{\sqrt{N_f}} \right) = O\left( \frac{1}{\sqrt{N}} \right).
\]

(73)
With Lemmas 5.3 and 5.8, we can give the last part of the proof to Theorem 5.1, i.e., the control strategies set \((\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{N_l}, \overline{v}_1, \ldots, \overline{v}_{N_f})\) given by Theorem 5.1 is an \(\varepsilon\)-SNC equilibrium of the mixed S-MM for follower agent.

**Part C of the Proof to Theorem 5.1**
Combining Lemmas 5.3 and 5.8, we have

\[
J^f_j(\overline{u}_0, \overline{u}, \overline{v}_j, \overline{v}_{-j}) \leq J^f_j(\overline{u}_0, \overline{v}_j) + O\left(\frac{1}{\sqrt{N}}\right) \leq J^f_j(\overline{u}_0, v_j) + O\left(\frac{1}{\sqrt{N}}\right),
\]

where the second inequality comes from the fact that \(J^f_j(\overline{u}_0, v_j) = \inf_{v_j \in \mathcal{V}_j[0,T]} J^f_j(\overline{u}_0, v_j)\). Consequently, the Theorem 5.1 holds for the follower agent with \(\varepsilon = O\left(\frac{1}{\sqrt{N}}\right)\). Finally, combined with the Part A, Part B, we complete the proof to Theorem 5.1. 

\[\square\]

**6 Application: Keynes’ Beauty Contest Games**

This section addresses an application based on our theoretical results. Our study can be applied to various practical problems in different fields. A typical application is related to the Keynes’ beauty contest games (see [31] for a recent study using mean-field theory). A similar setup focused on opinion dynamics was used in [5,11,32,33]. Here, we consider a model that can be viewed as a dynamic version of the Keynes’ beauty contest but framed in a leader–follower major–minor setting. Keynes proposed a beauty contest in which a newspaper prints photographs of people, and people vote for the prettiest faces. People who selected the most attractive face are automatically entered into a lottery to win a prize. Keynes remarked that the stock market is similar to such a beauty contest because each investor estimates the values of shares as estimated by other investors. On this basis, we can formulate a leader–follower major–minor LQG dynamic version of the Keynes’ beauty contest games. To this end, we begin with a large population of agents divided into three groups:

(i) The major leader as the well-informed player (e.g., the regulatory authority in a stock market);
(ii) Minor leaders as the well-informed players (e.g., institutional investors in a stock market);
(iii) And followers (e.g., retail or individual investors in a stock market).

The state of each player is its publicly announced prediction of the prettiest face where \(X_i(\cdot)\) denotes the state of the \(i\)-th leader \((0 \leq i \leq N_l)\) and \(x_j(\cdot)\) denotes the state of the \(j\)-th follower \((1 \leq j \leq N_f)\). The leaders and followers have linear stochastic dynamics defined in (1)–(3). The average prediction of minor leaders and followers is defined by their centroids \(X^{(N_l)} := \sum_{i=1}^{N_l} X_i(\cdot)\) and \(x^{(N_f)} := \sum_{j=1}^{N_f} x_j(\cdot)\), respectively. Based on the quadratic payoff functions considered in [31], we formulate the cost functionals of the agents as follows. The major leader intends to minimize
its cost functional (4) by making guesses close to some convex combination of the average prediction of minor leaders \( X^{(N)}(\cdot) \) and the average prediction of followers \( x^{(N_f)}(\cdot) \), i.e., \( \lambda_0 X^{(N)}(\cdot) + (1 - \lambda_0) x^{(N_f)}(\cdot) \) for some \( \lambda_0 \). In addition, the minor leaders intend to minimize their cost functionals (5) by making guesses close to some convex combination of the major leader’s prediction and their average prediction \( X^{(N)}(\cdot) \), i.e., \( \lambda X^{(N)}(\cdot) + (1 - \lambda) X_0(\cdot) \) for some \( \lambda \). Followers intend to make guesses close to some convex combination of their own average prediction \( x^{(N_f)}(\cdot) \), the average prediction of minor leaders \( X^{(N)}(\cdot) \) and the major leader’s guess (see (6)), i.e., \( \tilde{\lambda}_1 X_0(\cdot) + \tilde{\lambda}_2 X^{(N)}(\cdot) + \tilde{\lambda}_3 x^{(N_f)}(\cdot) \) for some \( \tilde{\lambda}_i \) such that \( \sum_{i=1}^{3} \tilde{\lambda}_i = 1 \). On the basis of our results and Theorem 5.1, \( (u_0, \bar{u}_i, \bar{v}_j) \) given by (54) is an \( \varepsilon \)-SNC equilibrium of mixed S-MM-MFG for major leader agent \( A_0 \), minor leaders \( A_i^j, i = 1, 2, \ldots, N_l \), and minor followers \( A_j^i, j = 1, 2, \ldots, N_f \). That is (54), where \( (Y_0, Z_0), (\bar{Y}_i, \bar{Z}_i), (\bar{v}_j, \bar{z}_j) \) can be solved by the related CC equation (38). By observation, we can find the major leader influences the strategies of minor leaders through coefficients \( E_0^1, E_0^2, \lambda_0, \lambda \) and those of followers through coefficients \( F_0^1, F_0^2, \lambda_0, \lambda_1 \). Moreover, we can investigate specifically how the strategies of minor leaders and followers will change by related Riccati equations (44)–(51). As mentioned, explicit solutions to such high-dimensional Riccati equations are generally intractable. Therefore, we present two simplified examples to illustrate how the major leader influences the associated equilibrium. We first consider Example 6.1 as follows.

**Example 6.1** Let \( n = p = q = 1 \), and set the coefficients as

\[
\begin{align*}
A_0 = A &= \tilde{A} = -1/2, & B_0 = B &= \tilde{B} = 0, & E_0^1 = E_1 = E_2 = 0, \\
C_0 = C &= \tilde{C} = 0, & D_0 = D &= \tilde{D} = 1, & F_0^1 = F_1 = F_2 = 0, \\
Q_0 = Q &= \tilde{Q} = 1, & R_0 = R &= \tilde{R} = 1, & H_0 = H = \tilde{H} = 1, \\
\lambda_0 = 1, & \lambda = 1, & \tilde{\lambda}_1 = 0, & \tilde{\lambda}_2 = \tilde{\lambda}, & \lambda_3 = 1 - \tilde{\lambda}.
\end{align*}
\]

We have the state equation

\[
\begin{align*}
\frac{dX_0(t)}{dt} &= -\frac{1}{2} X_0(t) dt + u_0(t) dW_0(t), \\
\frac{dX_i(t)}{dt} &= -\frac{1}{2} X_i(t) dt + u_i(t) dW_i(t), & 1 \leq i \leq N_l, \\
\frac{dx_j(t)}{dt} &= -\frac{1}{2} x_j(t) dt + v_j(t) d\tilde{W}_j(t), & 1 \leq j \leq N_f, \\
X_0(0) &= \xi_0, & X_i(0) &= \xi_i, & x_j(0) &= \xi_j.
\end{align*}
\]

The cost functional reads as

\[
\begin{align*}
\mathcal{J}_0(u_0(\cdot), u(\cdot)) &= \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left( \left| X_0(t) - X^{(N)}(t) \right|^2 + |u_0(t)|^2 \right) dt + |X_0(T)|^2 \right\}, \\
\mathcal{J}_i^l(u_i(\cdot), u_{-i}(\cdot)) &= \frac{1}{2} \mathbb{E}\left\{ \int_0^T \left( \left| X_i(t) - X^{(N)}(t) \right|^2 + |u_i(t)|^2 \right) dt + |X_i(T)|^2 \right\}.
\end{align*}
\]
and
\[ J_f^{\mathcal{F}}(u(\cdot), v_j(\cdot), v_{-j}(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( |x_j(t) - (\tilde{\lambda}x^{(N_l)}(t) + (1 - \tilde{\lambda})x^{(N_f)}(t))|^2 + |v_j(t)|^2 \right) dt + |x_j(T)|^2 \right\}, \]

and following (54) and (38), we get the solutions of the Riccati equation (46)–(51)
\[
\begin{align*}
P_1(t) &= 1, \\
P_2(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
P_3(t) &= 1.
\end{align*}
\] (74)

Hence, we get the SNC equilibrium strategies \( \overline{u}_0 = -Z_0 = 0, \overline{u}_i = -\overline{Z} = 0, \overline{v}_j = -\tilde{\xi} = 0. \)

**Remark 6.1** In Example 6.1, the derived SNC equilibrium strategies are the same as those when no major leader is involved. The coefficients we selected lead to such results that the major leader can not influence the strategies of minor leaders and followers. However, if we modify the levels of system coefficients and parameters, then the major leader can influence the minor leaders and followers. To illustrate this point, we present another example with changed drift coefficients of the control variables.

**Example 6.2** Let \( n = p = q = 1, \) and set the coefficients as
\[
\begin{align*}
A_0 &= A = \tilde{A} = -\frac{1}{2}, & B_0 &= B = \tilde{B} = 1, & E_0^1 &= E_1 = E_2 = 0, \\
C_0 &= C = \tilde{C} = 0, & D_0 &= D = \tilde{D} = 1, & F_0^1 &= F_1 = F_2 = 0, \\
Q_0 &= Q = \tilde{Q} = 1, & R_0 &= R = \tilde{R} = 1, & H_0 &= H = \tilde{H} = 1, \\
\lambda_0 &= 1, & \lambda &= 1, & \tilde{\lambda}_1 &= 0, & \tilde{\lambda}_2 &= \tilde{\lambda}, & \lambda_3 &= 1 - \tilde{\lambda}.
\end{align*}
\]

We have the state equation
\[
\begin{align*}
dX_0(t) &= \left( -\frac{1}{2} X_0(t) + u_0(t) \right) dt + u_0(t) dW_0(t), \\
dX_i(t) &= \left( -\frac{1}{2} X_i(t) + u_i(t) \right) dt + u_i(t) dW_i(t), \quad 1 \leq i \leq N_l, \\
dx_j(t) &= \left( -\frac{1}{2} x_j(t) + v_j(t) \right) dt + v_j(t) d\tilde{W}_j(t), \quad 1 \leq j \leq N_f, \\
X_0(0) &= \xi_0, & X_i(0) &= \xi_i, & x_j(0) &= \xi_j.
\end{align*}
\]
The cost functional reads as

\[ J_0(u_0(\cdot), u(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( |X_0(t) - X^{(N_l)}(t)|^2 + |u_0(t)|^2 \right) dt + |X_0(T)|^2 \right\}, \]

\[ J_l(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( |X_i(t) - X^{(N_l)}(t)|^2 + |u_i(t)|^2 \right) dt + |X_i(T)|^2 \right\}, \]

and

\[ J_f(u(\cdot), v_j(\cdot), v_{-j}(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left( |x_j(t) - (\tilde{\lambda} X^{(N_l)}(t) + (1 - \tilde{\lambda}) x^{(N_f)}(t))|^2 + |v_j(t)|^2 \right) dt + |x_j(T)|^2 \right\}. \]

We only focus on the decisions of followers and compare the strategy difference. Following (54) and (38), the Riccati equation of followers turns out to be:

\[
\begin{cases} 
\dot{P}_1 - P_1 + 1 - P_1^2 + \frac{P_1^3}{1 + P_1} = 0, \\
P_1(T) = 1.
\end{cases}
\]  

Equation (75) should not admit the explicit solution, but with the help of Matlab, we can still simulate its solution based on classical numerical algorithm for ordinary differential equation (ODE). For comparison purpose, we fix other coefficients in system and only vary the parameter \( \tilde{B} \). In accordance with Riccati equation (46), the corresponding equation should read as

\[
\begin{cases} 
\dot{P}_1 - P_1 + 1 - \tilde{B}^2 P_1^2 + \frac{\tilde{B}^2 P_1^3}{1 + P_1} = 0, \\
P_1(T) = 1.
\end{cases}
\]  

Accordingly, the related numerical results are reported by Fig. 1. We remark that without loss of generality, only the influence of major leaders upon followers is highlighted here. From Fig. 1, it is easy to observe that when the coefficient \( \tilde{B} \) increases, the initial value of related solution \( P_1 \) should decrease. That implies that the cost functional (i.e., \( \langle P_1(0) \xi_i, \xi_i \rangle \)) of followers should reduce in the presence of larger values of \( \tilde{B} \). As indicated in Example 6.1, \( \tilde{B} = 0 \) corresponds to the case when there has no influence from major leader and the cost functional of followers should attain its maximal value among all simulated cases. The other simulation cases also show that the more influence coming from major leader, the less cost will be generated. We thus conclude that the introduction of major leader do influence the strategy of the followers as well as the associated functional to be reached.

**Remark 6.2** Let’s further present another remark based on above simulation. Notice that in above examples, the major leader directly influence the large-population system.
via the mean-field coupling term $X^{(N_l)}$. Consequently, it will influence the minor leader and followers in principle. However, if we set the coefficients as in Example 6.1, the influence from major leader is only presented in diffusion term and in our deduction of CC equation, the control of major leader $u_0(\cdot)$ will not influence the mean field average $X^{(N_l)}$. Therefore, the major leader does not influence the minor leaders and followers in Example 6.1. In Example 6.2, the major leader directly influences the system via $X^{(N_l)}$ in context such as $B_0 = B = \tilde{B} \neq 0$. Then it will influence the minor leaders and followers when varying the relevant values of $B_0, B, \tilde{B}$. When we mainly focus on the influence on followers, we need only vary the levels of parameter $\tilde{B}$. We also notice that by (38), if we increase the value of $\tilde{B}$, then the influence from major leader should become more significant. This is mainly because the weight of $\tilde{y}$, $\tilde{z}$ become larger accordingly.

Finally, the aforementioned examples suggest that our setup can be degenerated to the classical leader–follower problem. As such, our current work can be viewed as an extension of classical work that introduces an agent with a high-priority decision-making position (i.e., the major-leader agent). By comparing Examples 6.1 and 6.2, we demonstrate that the major-leader does influence the decisions of minor leaders and followers. As mentioned in the introduction, our model can be applied to various practical problems. Here, we provide only one application consistent with the Keynes’ beauty contest problem and demonstrate the influence of the major leader on the associated equilibrium. We intend to address more applications in the future.
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