The $L^p$ Cauchy sequence for one-dimensional BSDEs with linear growth generators

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Abstract
In this paper, the existence of $L^p$ ($p > 1$) solutions for one-dimensional backward stochastic differential equations will be shown directly by proving that an approximation sequence is a Cauchy one in the $L^p$ sense.

1 Introduction

In this paper, we consider the following one-dimensional backward stochastic differential equation (BSDE in short):

$\begin{cases}
-dY_t = f(t, Y_t, Z_t)dt - Z_t \cdot dW_t, & 0 \leq t \leq T, \\
Y_T = \xi,
\end{cases}$

where $T > 0$, $\xi$ is a random variable, $f$ is a real-valued random function, and $W$ is a $d$-dimensional Brownian motion with $W_0 = 0$. The function $f$ is called the generator. The equation above is also written in

\begin{equation}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \quad 0 \leq t \leq T. \tag{1.1}
\end{equation}

A pair $(Y, Z)$ of adapted processes satisfying the equation is called a solution.

As for $L^p$ ($p > 1$) solutions to the BSDE, El Karoui et al. [2] proved an existence and uniqueness result when $f$ is Lipschitz continuous and $\xi$ is in $L^p$ by using a fixed-point theorem. A natural question then arises whether the Lipschitz condition can be relaxed. On account of the standard forward SDEs, the linear growth condition of the generator seems to be a candidate for a weaker condition to guarantee the existence and the $L^p$-integrability of solutions. Hereinafter, we

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assume that \( f \) is continuous and of linear growth order and \( \xi \) is in \( L^p \). In this case, the existence results were shown by Lepeltier and San Martin [3] for \( p = 2 \), by Chen [5] for \( 1 < p \leq 2 \) and after them by Fan and Jiang [6] for general \( p > 1 \). In these papers, a key role is played by an approximation sequence. When \( 1 < p \leq 2 \), the existence was obtained by proving that the sequence is a Cauchy one. When \( p > 2 \), an \( L^p \) solution was constructed by taking advantage of a stopping time argument. And, it remains open to prove the sequence to be a Cauchy one when \( p > 2 \).

This paper is organized as follows. In Section 2, a priori estimates are obtained by using Itô’s formula. In Section 3, the approximation sequence is constructed. Then, it is proved that the sequence is a Cauchy one and converges to an \( L^p \) solution to the BSDE (1.1).

2 Preliminaries

2.1 Notations

Let \((W_t)_{0 \leq t \leq T}\) be a \(d\)-dimensional Brownian motion with \(W_0 = 0\) defined on a complete probability space \((\Omega, \mathcal{F}, P)\), and \((\mathcal{F}_t)_{0 \leq t \leq T}\) be the natural filtration of the Brownian motion \(W\) augmented by the \(P\)-null sets of \(\mathcal{F}\). Throughout the paper, we are working on with only one filtration \((\mathcal{F}_t)\) and for the sake of simplicity, we omit the prefix “\((\mathcal{F}_t)\)”; for example, we just say “adapted” instead of “\((\mathcal{F}_t)\)-adapted”. We denote by \(\mathcal{P}\) the predictable sub-\(\sigma\)-field of \(\mathcal{B}([0,T]) \otimes \mathcal{F}\), and let the generator \(f\), which is defined on \([0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d\), be \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})/\mathcal{B}(\mathbb{R})\)-measurable. For a given \(p > 1\), we denote by \(S^p\) the set of \(\mathbb{R}\)-valued, continuous and adapted processes \((\eta_t)_{0 \leq t \leq T}\) such that

\[
\|\eta\|_{S^p} := \left\{ E \left[ \sup_{0 \leq t \leq T} |\eta_t|^p \right] \right\}^{1/p} < \infty.
\]

\(\mathcal{H}^p\) stands for the set of \(\mathbb{R}^d\)-valued predictable processes \((\zeta_t)_{0 \leq t \leq T}\) such that

\[
\|\zeta\|_{\mathcal{H}^p} := \left\{ E \left[ \left( \int_0^T |\zeta_s|^2 ds \right)^{\frac{p}{2}} \right] \right\}^{1/p} < \infty.
\]

We see that the following properties hold:

- If \(\|\eta^n - \eta^m\|_{S^p} \to 0\) as \(n, m \to \infty\), then there exists a unique \(\eta \in S^p\) such that \(\|\eta^n - \eta\|_{S^p} \to 0\) as \(n \to \infty\),

- if \(\|\zeta^n - \zeta^m\|_{\mathcal{H}^p} \to 0\) as \(n, m \to \infty\), then there exists a unique \(\zeta \in \mathcal{H}^p\) such that \(\|\zeta^n - \zeta\|_{\mathcal{H}^p} \to 0\) as \(n \to \infty\).
2.2 Assumptions

In this paper, we use the following assumptions (H1)-(H3):

(H1) There exists a positive constant $K$ and a non-negative predictable process $(g_t)_{0\leq t\leq T}$ such that

$$E\left[\left(\int_0^T g_s\,ds\right)^p\right] < \infty, \quad |f(t,\omega, y, z)| \leq g_t(\omega) + K(|y| + |z|)$$

for any $(t,\omega, y, z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$.

(H2) For each $(t,\omega) \in [0,T] \times \Omega$, $f(t,\omega, y, z)$ is continuous in $(y, z)$.

(H3) $\xi \in L^p$, i.e., $E[|\xi|^p] < \infty$.

**Definition 2.1.** A solution to the BSDE with the generator $f$ and the terminal value $\xi$ is a pair of continuous adapted processes $Y$ and predictable processes $Z$ such that

$$\int_0^T \{|f(s, Y_s, Z_s)| + |Z_s|^2\}\,ds < \infty \quad a.s.$$

and satisfies (1.1). In particular, we call a solution $(Y, Z) \in \mathcal{S}^p \times \mathcal{H}^p$ an $L^p$ solution to the BSDE.

In the case $p > 1$ and the generator is Lipschitz, the existence and uniqueness of $L^p$ solution is known ([2]).

**Theorem 2.2.** Assume that $f$ is uniformly Lipschitz in $(y, z)$, i.e., there exists a positive constant $C$ such that

$$|f(t,\omega, y_1, z_1) - f(t,\omega, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|)$$

for any $(t,\omega) \in [0,T] \times \Omega$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$.

And assume (H3) holds and

$$E\left[\left(\int_0^T |f(s,0,0)|\,ds\right)^p\right] < \infty.$$

Then, BSDE (1.1) has a unique $L^p$ solution.

It is also known ([2]) that
Theorem 2.3. For $i = 1, 2$, let $f^i$ be uniformly Lipschitz in $(y, z)$, $\xi^i$ satisfy (H3) and

$$E \left[ \left( \int_0^T |f^i(s, 0, 0)| ds \right)^p \right] < \infty.$$ 

In addition, assume that each $(Y^i, Z^i)$ is the $L^p$ solution to the BSDE with respect to $(f^i, \xi^i)$. Then, $\xi^1 \geq \xi^2$ a.s. and $f^1(t, Y^2_t, Z^2_t) \geq f^2(t, Y^1_t, Z^1_t) \, dt \times dP$-a.e. imply $Y^1 \geq Y^2$ a.s..

Remark 1. In [2], the assertion of Theorem 2.2 and 2.3 are stated under the assumptions like

$$E \left[ \left( \int_0^T |f(s, 0, 0)| ds \right)^2 \right] < \infty, \quad (2.1)$$

which is stronger than the ones in the theorems. Observing the proof in [2] carefully, we can weaken the assumption (2.1) to the one as we used.

2.3 A priori estimates

We prepare the following estimations which play a key role in the observation of this paper, by generalizing the ones in [5] used by Chen for specified solutions.

Proposition 2.4. (i) Let $p > 1$. If $(Y, Z)$ is an $L^p$ solution to the BSDE (1.1), then there exists a positive constant $C_p$ depending only on $p$ such that

$$\|Y\|_{S^p}^p \leq C_p E \left[ |\xi|^p + \int_0^T |Y_s|^{p-1} |f(s, Y_s, Z_s)| ds \right],$$

$$\|Z\|_{H^p}^p \leq C_p \left\{ E \left[ |\xi|^p + \left( \int_0^T |Y_s| |f(s, Y_s, Z_s)| ds \right)^{\frac{p}{2}} \right] + \|Y\|_{S^p}^p \right\}. $$

Moreover, if $f$ satisfies (H1), then

$$\|Z\|_{H^p}^p \leq C(1 + \|Y\|_{S^p}^\frac{p}{2} + \|Y\|_{S^p}^p),$$

where $C$ is a positive constant which depends only on $p, K, T, E[|\xi|^p]$ and $E[\int_0^T g_s ds]^p$.

(ii) Let $p > 1$. If $(Y^i, Z^i)$ is an $L^p$ solution to the BSDE with respect to $(f^i, \xi^i)$, $i = 1, 2$, respectively, then there exists a positive constant $C_p$ depending only on $p$ such that

$$\|\delta Y\|_{S^p}^p \leq C_p E \left[ |\delta Y_T|^p + \int_0^T |\delta Y_s|^{p-1} |\delta f_s| ds \right],$$

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\[ \|\delta Z\|_{\mathcal{H}^p} \leq C_p \left\{ E \left[ |\delta Y_T|^p + \left( \int_0^T |\delta Y_s| |\delta f_s| ds \right)^{\frac{p}{2}} \right] \right\}, \]

where \( \delta Y := Y^1 - Y^2, \delta Z := Z^1 - Z^2, \delta f_s := f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s). \)

**Proof.** The assertion (ii) follows from (i). Namely, put \( \tilde{f}(t, y, z) = f^1(t, Y^2_t + y, Z^2_t + z) - f^2(t, Y^2_t, Z^2_t) \). Then, \( \delta f_t = \tilde{f}(t, \delta Y_t, \delta Z_t) \) and the pair \((\delta Y, \delta Z) \in \mathcal{S}^p \times \mathcal{H}^p \) satisfies

\[
\delta Y_t = \delta Y_T + \int_t^T \tilde{f}(s, \delta Y_s, \delta Z_s) ds - \int_t^T \delta Z_s \cdot dW_s, \quad 0 \leq t \leq T.
\]

Thus, we only prove (i).

Let \( p > 1 \). We first estimate \( Y \). As an elementary application of Itô’s formula, we obtain

\[
|Y_t|^p + \frac{p(p-1)}{2} \int_t^T |Y_s|^{p-2} \tilde{1}(Y_s)|Z_s|^2 ds
\]

\[
= |\xi|^p + p \int_t^T \text{sgn}(Y_s)|Y_s|^{p-1} f(s, Y_s, Z_s) ds
\]

\[- p \int_t^T \text{sgn}(Y_s)|Y_s|^{p-1} Z_s \cdot dW_s, \quad 0 \leq t \leq T, \quad (2.2)\]

where

\[
\tilde{1}(y) := \begin{cases} 1, & 1 < p < 2 \\ 1, & 2 \leq p \end{cases}, \quad \text{sgn}(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}.
\]

See also [4, Lemma 2.2]. Hence, we get

\[
\sup_{0 \leq t \leq T} |Y_t|^p \leq |\xi|^p + p \int_0^T |Y_s|^{p-1} |f(s, Y_s, Z_s)| ds
\]

\[
+ 2p \sup_{0 \leq t \leq T} \left| \int_0^t \text{sgn}(Y_s)|Y_s|^{p-1} Z_s \cdot dW_s \right|. \quad (2.3)
\]

By the Burkholder-Davis-Gundy inequality (the BDG inequality in short), there exists a positive constant \( C_1 \) such that

\[
2p E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \text{sgn}(Y_s)|Y_s|^{p-1} Z_s \cdot dW_s \right| \right]
\]

\[
\leq 2p C_1 E \left[ \left( \int_0^T |Y_s|^{2p-2} \tilde{1}(Y_s)|Z_s|^2 ds \right)^{\frac{1}{2}} \right].
\]
\[ \leq 2pC_1E \left[ \sup_{0 \leq t \leq T} |Y_t|^{2p} \left( \int_0^T |Y_s|^{p-2} \mathbf{1}(Y_s)|Z_s|^2 ds \right) \right] \]
\[ \leq \frac{1}{2} E \left[ \int_0^T |Y_s|^p ds \right] + 2p^2 C_1^2 E \left[ \int_0^T |Y_s|^{p-2} \mathbf{1}(Y_s)|Z_s|^2 ds \right], \quad (2.4) \]

where, to see the third inequality above, we have used the inequality
\[ 2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2, \quad \varepsilon > 0, \quad a, b \geq 0 \]
with \( \varepsilon = 1/2 \).

By the Hölder inequality, we have
\[ E \left[ \left( \int_0^T |Y_s|^{2p-2} \mathbf{1}(Y_s)|Z_s|^2 ds \right)^{\frac{1}{2}} \right] \]
\[ \leq E \left[ \sup_{0 \leq t \leq T} |Y_t|^{p-1} \left( \int_0^T |Z_s|^2 ds \right)^{\frac{1}{2}} \right] \]
\[ \leq \left\{ E \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] \right\}^{\frac{1}{p-1}} \left\{ E \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \right\}^{\frac{1}{2}} < \infty. \]

Thus, \( \int_0^t \text{sgn}(Y_s)|Y_s|^{p-1}Z_s \cdot dW_s \) is a martingale. Then, taking the expectations of \( (2.2) \), we get
\[ \frac{p(p-1)}{2} E \left[ \int_0^T |Y_s|^{p-2} \mathbf{1}(Y_s)|Z_s|^2 ds \right] \]
\[ \leq E \left[ |\xi|^p + p \int_0^T |Y_s|^{p-1} |f(s, Y_s, Z_s)| ds \right]. \quad (2.5) \]

Then \( (2.3) \), \( (2.4) \) and \( (2.5) \) yield the estimation of \( Y \).

Next, we estimate \( Z \). By \( (2.2) \) with \( p = 2 \), we deduce that
\[ \int_0^T |Z_s|^2 ds \leq |\xi|^2 + 2 \int_0^T |Y_s||f(s, Y_s, Z_s)| ds + 2 \sup_{0 \leq t \leq T} \left| \int_0^t Y_sZ_s \cdot dW_s \right|. \]

Hence, it follows that
\[ \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \]
\[ \leq C_2 \left\{ |\xi|^p + \left( \int_0^T |Y_s||f(s, Y_s, Z_s)| ds \right)^{\frac{p}{2}} + \sup_{0 \leq t \leq T} \left| \int_0^t Y_sZ_s \cdot dW_s \right|^2 \right\}, \quad (2.6) \]
where $C_2$ is a positive constant depending only on $p$. By the BDG inequality, there exists a positive constant $C_3$ depending only on $p$ such that

$$C_2 E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t Y_s Z_s \cdot dW_s \right|^p \right]$$

$$\leq C_2 C_3 E \left[ \left( \int_0^T |Y_s|^2 |Z_s|^2 ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_2 C_3 \sup_{0 \leq t \leq T} |Y_t|^\frac{p}{2} \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{4}}$$

$$\leq 2C_2^2 C_3^2 E \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] + \frac{1}{2} E \left[ \left( \int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right], \quad (2.7)$$

where, to see the third inequality above, we have used (2.6) again with $\varepsilon = 1/2$. Then, we get the second estimation from (2.6) and (2.7).

We finally show the last assertion of (i). To do this, it is sufficient to estimate the second term of the estimation with respect to $Z$. By (H1) and the Hölder inequality, there exists positive constants $C_{p,K}, C_{p,K,T}$ and $C'_{p,K,T}$ which depend only on the subscripts such that

$$E \left[ \left( \int_0^T |Y_s||f(s,Y_s,Z_s)| ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_{p,K} \left\{ E \left[ \left( \int_0^T |Y_s|g_s ds \right)^{\frac{p}{2}} \right] \right\} + E \left[ \left( \int_0^T |Y_s||Z_s| ds \right)^{\frac{p}{2}} \right]$$

$$\leq C_{p,K,T} \left\{ \|Y\|_{S^p}^{\frac{p}{2}} \left\{ E \left[ \left( \int_0^T g_s ds \right)^{p} \right] \right\}^{\frac{1}{2}} \right\} + \|Y\|_{S^p}^{p} + E \left[ \left( \int_0^T (\varepsilon^{-1}|Y_s|^2 + \varepsilon|Z_s|^2) ds \right)^{\frac{p}{2}} \right]$$

$$\leq C'_{p,K,T} \left\{ \|Y\|_{S^p}^{\frac{p}{2}} \left\{ E \left[ \left( \int_0^T g_s ds \right)^{p} \right] \right\}^{\frac{1}{2}} + \varepsilon^{-\frac{p}{2}} \|Y\|_{S^p}^{p} + \varepsilon^{\frac{p}{2}} \|Z\|_{H^p}^{p} \right\},$$

where, to see the second inequality above, we have used (3) with $C_p C'_{p,K,T} \varepsilon^{\frac{p}{2}} = 1/2$. Then, we obtain the desired estimation.
3 Existence of an $L^p$ solution

3.1 Approximation of linear growth functions

According to [3], linear growth functions can be approximated by Lipschitz functions. Precisely speaking, when a generator $f$ satisfies (H1) and (H2),

$$f_n(t, y, z) := \inf_{(u, v) \in \mathbb{R}^{d+1}} \{f(t, u, v) + n(|y - u| + |z - v|), \quad n \geq K$$ (3.1)

is a Lipschitz function and approximates the linear growth function $f$, where $K$ is a constant appeared in (H1).

**Lemma 3.1.** Assume (H1) and (H2) hold. Then, (3.1) is well-defined for $n \geq K$ and the following properties i)-iv) hold:

i) $|f_n(t, \omega, y, z)| \leq g_t(\omega) + K(|y| + |z|)$ for any $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$,

ii) $f_n \leq f_{n+1} \leq f$, $n \geq K$,

iii) $|f_n(t, \omega, y_1, z_1) - f_n(t, \omega, y_2, z_2)| \leq n(|y_1 - y_2| + |z_1 - z_2|)$ for any $(t, \omega) \in [0, T] \times \Omega$,

iv) if $(y_n, z_n) \rightarrow (y, z)$, then $f_n(t, \omega, y_n, z_n) \rightarrow f(t, \omega, y, z)$ for any $(t, \omega) \in [0, T] \times \Omega$.

3.2 Approximation of a solution

Let $p > 1$ and assumptions (H1)-(H3) hold. We consider the following one-dimensional BSDEs:

$$Y^n_t = \xi + \int_t^T f_n(s, Y^n_s, Z^n_s)ds - \int_t^T Z^n_s \cdot dW_s, \quad n \geq K,$$ (3.2)

$$U_t = \xi + \int_t^T \{g_s + K(|U_s| + |V_s|)\}ds - \int_t^T V_s \cdot dW_s.$$

Theorem 2.2 assures the existence and uniqueness of $L^p$ solution to these BSDEs. Thus, $(Y^n, Z^n)$ and $(U, V)$ are well-defined for $n \geq K$. Moreover, by Theorem 2.3 and Lemma 3.1 ii), we have

$$Y^n \leq Y^{n+1} \leq U, \quad n \geq K.$$ (3.3)

**Theorem 3.2.** $(Y^n, Z^n)$ is a Cauchy sequence in $S^p \times H^p$. 

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Proof. The assertion for $1 < p \leq 2$ can be proved in the same manner as [5, Lemma 4]. Thus, we give the proof only for the case $p > 2$.

Since $(Y^n)$ is non-decreasing, it admits the limit process $Y$. By (3.3), it follows that

$$Y^{[K]} \leq Y^n, \quad Y^k \leq U, \quad n \geq K,$$

where $\lceil \cdot \rceil$ represents the ceiling function. Thus, we have

$$|Y^n| \leq M, \quad |Y| \leq M, \quad n \geq K,$$

(3.4)

where $\sup_{0 \leq t \leq T} |Y^{[K]}_t| \lor \sup_{0 \leq t \leq T} |U_t| =: M \in L^p$. Then, by the dominated convergence theorem, it follows that

$$E \left[ \int_0^T |Y^n_s - Y^k_s|^{p-1} g_s ds \right] \to 0, \quad E \left[ \int_0^T |Y^n_s - Y^k_s|^{p} ds \right] \to 0,$$

and thus, we get

$$E \left[ \int_0^T |Y^n_s - Y^m_s|^{p-1} g_s ds \right] \to 0, \quad E \left[ \int_0^T |Y^n_s - Y^m_s|^{p} ds \right] \to 0,$$

as $n, m \to \infty$. (3.5)

By Proposition 2.4-(ii), we have

$$\|Y^n - Y^m\|_{S^p}^{i} \leq C_p \left( E \left[ \left( \int_0^T |Y^n_s - Y^m_s|^{p-1} |f_s(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s)| ds \right)^{\frac{p}{2}} \right] + \|Y^n - Y^m\|_{S^p}^{i} \right).$$

We first estimate the right hand side of (3.6). By Lemma 3.1-i), we get

$$E \left[ \left( \int_0^T |Y^n_s - Y^m_s|^{p-1} g_s ds \right)^{\frac{p}{2}} \right] \leq 2E \left[ \int_0^T |Y^n_s - Y^m_s|^{p-1} g_s ds \right] + KE \left[ \int_0^T |Y^n_s - Y^m_s|^{p-1} F_{n,m}(s) ds \right],$$

(3.8)
where $F_{n,m}(s) := |Y^n_s| + |Z^n_s| + |Y^m_s| + |Z^m_s|$. By (3.5), we know the first term of (3.8) converges to zero. Thus, we estimate the second term of this. By the Hölder inequality and (3.5), we have

\[
KE \left[ \int_0^T |Y^n_s - Y^m_s|^p F_{n,m}(s) ds \right] \\
\leq KE \left[ \left( \int_0^T |Y^n_s - Y^m_s|^{2p-2} ds \right)^{\frac{1}{2}} \left( \int_0^T \{F_{n,m}(s)\}^2 ds \right)^{\frac{1}{2}} \right] \\
\leq KE \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \left( \int_0^T |Y^n_s - Y^m_s|^{2p-2} ds \right)^{\frac{1}{2}} \left( \int_0^T \{F_{n,m}(s)\}^2 ds \right)^{\frac{1}{2}} \right] \\
\leq \varepsilon E \left[ \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^p \right] + \varepsilon^{-1} K^2 E \left[ \int_0^T |Y^n_s - Y^m_s|^{p-2} ds \int_0^T \{F_{n,m}(s)\}^2 ds \right] \\
\leq \varepsilon \|Y^n - Y^m\|_{S^p}^p + \varepsilon^{-1} K^2 T^\frac{1}{2} \left\{ E \left[ \left( \int_0^T |Y^n_s - Y^m_s|^{p-2} ds \right)^{\frac{p}{p-2}} \right] \right\}^{1-\frac{2}{p}} \\
\|Y^n - Y^m\|_{S^p}^p + \varepsilon^{-1} K^2 T^\frac{1}{2} \left\{ E \left[ \left( \int_0^T \{F_{n,m}(s)\}^2 ds \right)^{\frac{p}{2}} \right] \right\}^{\frac{2}{p}}. \tag{3.9}
\]

By (3.4), we have

\[
\sup_{n \geq K} \|Y^n\|_{S^p} < \infty.
\]

Thus, by Proposition 2.4 (i), we see that

\[
\sup_{n,m \geq K} E \left[ \left( \int_0^T \{F_{n,m}(s)\}^2 ds \right)^{\frac{p}{2}} \right] < \infty.
\]

Letting $\varepsilon$ such that $C_p \varepsilon = 1/2$, by (3.5), (3.6), (3.8) and (3.9), it follows that

\[
\|Y^n - Y^m\|_{S^p} \to 0, \quad \text{as} \quad n, m \to \infty.
\]
By Lemma 3.1-i) and the Schwartz inequality, we get the following estimation for the first term of the right hand side of (3.7):

$$E \left[ \left( \int_0^T |Y^n_s - Y^m_s| f_n(s, Y^n_s, Z^n_s) - f_m(s, Y^m_s, Z^m_s) \, ds \right)^{\frac{p}{2}} \right]$$

$$\leq C \left\{ E \left[ \left( \int_0^T |Y^n_s - Y^m_s| g_s \, ds \right)^{\frac{p}{2}} \right] + E \left[ \left( \int_0^T |Y^n_s - Y^m_s| F_{n,m}(s) \, ds \right)^{\frac{p}{2}} \right] \right\}$$

$$\leq C \left[ \|Y^n - Y^m\|_{\mathcal{S}^p}^{\frac{p}{2}} \left\{ E \left[ \left( \int_0^T g_s \, ds \right)^p \right]^{\frac{1}{2}} \right\}^{\frac{1}{p}}$$

$$+ T^{\frac{p}{2}} \|Y^n - Y^m\|_{\mathcal{S}^p}^{\frac{p}{2}} \left\{ E \left[ \left( \int_0^T F_{n,m}(s)^2 \right)^p \right]^{\frac{1}{p}} \right\}^{\frac{1}{p}} \right],$$

where $C$ is a positive constant depending only on $p$. Since $\|Y^n - Y^m\|_{\mathcal{S}^p} \to 0$, we obtain $\|Z^n - Z^m\|_{\mathcal{H}^p} \to 0$. □

By Proposition 3.2, we denote by $(Y, Z)$ the limit of $(Y^n, Z^n)$ in $\mathcal{S}^p \times \mathcal{H}^p$.

**Theorem 3.3.** $(Y, Z)$ is an $L^p$ solution to the BSDE (1.1).

**Proof.** It is already seen that $\|Y^n - Y\|_{\mathcal{S}^p} \to 0$, as $n \to \infty$.

By the BDG inequality, we have

$$\sup_{0 \leq t \leq T} \left| \int_0^t (Z^n_s - Z_s) \cdot dW_s \right| \to 0 \text{ in } L^p, \text{ as } n \to \infty.$$ 

Since $\|Y^n - Y\|_{\mathcal{S}^p} \to 0$, $\|Z^n - Z\|_{\mathcal{H}^p} \to 0$ as $n \to \infty$, we may assume

$$Y^n_t \to Y_t, \quad 0 \leq t \leq T \quad a.s.,$$

$$Z^n_t \to Z_t, \quad dt \times dP\text{-a.e.}$$

by choosing a subsequence if necessary. Thus, by Lemma 3.1-iv), we get

$$f_n(t, Y^n_t, Z^n_t) \to f(t, Y_t, Z_t), \quad dt \times dP\text{-a.e.}.$$ 

Now, by Lemma 3.1-i), we have

$$|f_n(t, Y^n_t, Z^n_t)| \leq g_t + K(|Y^n_t| + |Z^n_t|).$$

By the Hölder inequality, $Y^n \to Y$, $Z^n \to Z$ in $L^1$ with respect to $dt \times dP$, and then, we see that $(Y^n)_{n \geq K}$ and $(Z^n)_{n \geq K}$ are uniformly integrable with respect to
\[ \frac{dt}{T} \times dP. \] Hence, \((f_n(\cdot, Y^n, Z^n))_{n \geq K}\) is uniformly integrable with respect to \(\frac{dt}{T} \times dP\).

Thus, we get

\[ \int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)|ds \to 0 \quad \text{in } L^1. \]

Therefore, letting \(n \to \infty\) in (3.2), we obtain (1.1). \(\square\)

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