SUBSONIC SOLUTIONS TO A SHOCK DIFFRACTION PROBLEM BY A CONVEX CORNERED WEDGE FOR THE PRESSURE GRADIENT SYSTEM

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Abstract. We establish the global existence of subsonic solutions to a two dimensional Riemann problem governed by a self-similar pressure gradient system for shock diffraction by a convex cornered wedge. Since the boundary of the subsonic region consists of a transonic shock and a part of a sonic circle, the governing equation becomes a free boundary problem for nonlinear degenerate elliptic equation of second order with a degenerate oblique derivative boundary condition. We also obtain the optimal $C^{0,1}$-regularity of the solutions across the degenerate sonic boundary.

1. Introduction. We consider the two-dimensional pressure gradient system

\[
\begin{align*}
    u_t + p_x &= 0, \\
    v_t + p_y &= 0, \\
    E_t + (pu)_x + (pv)_y &= 0,
\end{align*}
\]

where $t \geq 0$, $(x,y) \in \mathbb{R}^2$, $(u,v)$ is the velocity of fluid, $p$ is the pressure and $E = \frac{1}{2}(u^2 + v^2) + p$ is the energy. The three equations in (1.1) stand for the conservation of momentum along the $x$-direction and $y$-direction and the conservation of energy, respectively. System (1.1) as one of the approximated models for the Euler equations can be derived from the compressible gas dynamics for an ideal fluid through the flux-splitting method or asymptotic expansion. The readers can refer to [30, 31] for more background of this system.

Shock waves are fundamental in several physical nature processes. One of the most challenging researches is the mathematical analysis of the shock reflection-diffraction phenomena by concave or convex wedges. In 1878, E. Mach [25] observed two patterns of shock reflection configurations which are now named as the...
regular reflection and Mach reflection. In 1940s, von Neumann [28] and other mathematical and experimental scientists found various other patterns, which are more complicated than what E. Mach originally observed. The early studies of the shock reflection-diffraction problem can be found in [9, 12, 13, 15, 16, 17, 20, 23, 24, 26] and the references cited therein. Recently, Bae, Chen and Feldman [1, 7, 8] developed a rigorous mathematical analysis on the existence and optimal regularity of solutions to the regular shock reflection configurations for potential flow, allowing the wedge angle beyond the sonic angle up to the detachment angle. They [2] also established the stability theorem for the Prandtl-Meyer reflection configurations for unsteady potential flow, which has been a longstanding open problem. The shock reflection-diffraction phenomena have also been approached by many researchers through the unsteady transonic small disturbance equation (UTSD) [3], the nonlinear wave system [4] and the pressure gradient system [31], which are all reasonable simplified models derived from the compressible gas dynamics. Serre [27], Chen and Qu [10, 11], and Wang, Zhang and Yang [29] studied the shock reflection-diffraction problem for the Euler equations for a Chaplygin gas.

Let the convex cornered wedge be given by
\[
W := \{(x, y) | -\infty < x \leq y \cot \theta_w, \ y < 0\},
\]
where \(\theta_w\) is the angle of the cornered wedge. We consider two piecewise constant initial data, denoted by state (1) : \((p_1, u_1, 0)\) and state (0) : \((p_0, 0, 0)\), separately, where
\[
(p, u, v)|_{t=0} = \begin{cases} 
(p_0, 0, 0) & \text{in } \{-\pi + \theta_w \leq \arctan \frac{y}{x} \leq \frac{\pi}{2}\}, \\
(p_1, u_1, 0) & \text{in } \{x < 0, y > 0\},
\end{cases}
\]
(1.2)
satisfying
\[
u_1 = \frac{p_1 - p_0}{\sqrt{p_{01}}}, \quad p_{01} := \frac{p_0 + p_1}{2}, \quad p_1 > p_0.
\]
Suppose the two constant state are separated by a vertical shock \(S_0\). When the shock \(S_0\) passes through a convex cornered wedge \(W\), shock diffraction phenomenon occurs (see Figure 1 and Figure 2).

Since system (1.1) and initial data (1.2) are both invariant under the self-similar scaling
\[
(t, x, y) \rightarrow (\alpha t, \alpha x, \alpha y) \quad \text{for } \alpha > 0,
\]
we seek self-similar solutions with the form
\[
(p, u, v)(t, x, y) = (p, u, v)(\xi, \eta) \quad \text{for } (\xi, \eta) = (x/t, y/t).
\]

\textbf{Figure 1.} Shock \(S_0\) passes the wedge at \(t = 0\)
In the \((\xi, \eta)\)-coordinates, system (1.1) can be rewritten as
\[
\begin{align*}
-\xi u - \eta v + p \xi &= 0, \\
-\xi v - \eta u + p \eta &= 0, \\
-\xi E - \eta E + (pu)\xi + (pv)\eta &= 0.
\end{align*}
\] (1.3)

Eliminating \(u\) and \(v\), we can derive a second-order nonlinear equation for \(p\)
\[
(p - \xi^2)p\xi - 2\xi p\xi + (p - \eta^2)p\eta + \frac{1}{p}(\xi p\xi + \eta p\eta)^2 - 2(\xi p\xi + \eta p\eta) = 0. \quad (1.4)
\]

Equation (1.4) is of elliptic-hyperbolic mixed type. It is elliptic when \(\xi^2 + \eta^2 < p\),
and hyperbolic when \(\xi^2 + \eta^2 > p\). The sonic circle is given by \(\{(\xi, \eta) : \xi^2 + \eta^2 = p\}\).
The location of the incident shock \(S_0\) in the \((\xi, \eta)\)-coordinates is given by
\[
\xi_1 = \frac{p_1 - p_0}{\sqrt{P_{01}}} > 0.
\]

In the self-similar coordinates, the shock diffraction problem can be stated as a
boundary value problem at infinity \(\xi^2 + \eta^2 \to \infty\).

**Problem 1** (Boundary value problem). *Find a solution of system (1.3) with the*
boundary condition as \(\xi^2 + \eta^2 \to \infty\),
\[
(p, u, v) = \begin{cases} 
(p_0, 0, 0) & \text{in } \{\xi > \xi_1, \eta > 0\} \cup \{-\pi + \theta_w \leq \arctan \frac{\eta}{\xi} \leq 0\}, \\
(p_1, u_1, 0) & \text{in } \{\xi < \xi_1, \eta > 0\},
\end{cases}
\]
and the slip boundary condition
\[
(u, v) \cdot \nu|_{\partial W} = 0 \quad \text{for } t > 0
\]
along the wedge boundary \(\partial W\), where \(\nu\) is the outward unit normal to \(\partial W\).

Kim [18] studied the problem for a shock interaction with the 90-degree cornered
wedge by the nonlinear wave system. By utilizing the barrier methods and iterative
methods, Kim showed the well-posedness of the transonic shock in the entire sub-
sonic region and established the global existence of solutions. Later, Chen, Deng
and Xiang [5] obtained a rigorous global mathematical result on the shock diffraction
by any convex cornered wedge for nonlinear wave systems. Chen and Xiang [6]
also established the global theory of existence and regularity for this shock diffraction
problem for the potential flow equation. In the self-similar coordinates, as the
incident shock \(S_0\) passes through the wedge corner, \(S_0\) intersects with the sonic
circle \(C_1 = \{r = \sqrt{p_1}\}\) of state (1), and then becomes a transonic diffracted shock.
$\Gamma_{\text{shock}}$ (Figure 2). The position of $\Gamma_{\text{shock}}$ is a priori unknown. It may terminate on the wedge $\{\theta = \theta_w\}$ or on the sonic circle $C_0$ of state (0). In [18] Kim explained this fact from the physical point of view: the flow is a compression wave, that is, $\Gamma_{\text{shock}}$ must be a shock not a sonic circle. In [5] Chen, Deng and Xiang used the method of contradiction to prove that $\Gamma_{\text{shock}}$ cannot intersect with the sonic circle $C_0$. In this paper we provide a new physical explanation for this fact.

The organization of this paper is as follows. In Section 2, we reformulate the shock diffraction problem into a free boundary problem and state the main theorem. In Section 3-4, we establish the existence of global entropy solutions. The optimal $C^{0,1}$-regularity of the solution $p$ across the degenerate sonic boundary is obtained in Section 5.

2. Mathematical reformulation and main theorem. We first reformulate the shock diffraction problem as a free boundary problem. In the self-similar coordinates, the shock $S_0$ comes from infinity. When $S_0$ hits the sonic circle $C_1$ at point $P_1$, shock diffraction occurs. We denote the diffracted shock by $\Gamma_{\text{shock}}$ which becomes a transonic shock. Suppose $\Gamma_{\text{shock}}$ terminates on the wedge at point $P_2$. Let $P_3$ be the intersection point of $C_1$ and the boundary $\{\theta = \pi\}$ and $\Gamma_{\text{sonic}}$ be the portion $\hat{P}_1P_3$ of the sonic circle $C_1$ (Figure 2). Let $\Omega$ be an open domain enclosed by $\Gamma_{\text{sonic}}$, $\Gamma_{\text{shock}}$ and the portions of the wedges $OP_2$ and $OP_3$. Here we first assume that $\Gamma_{\text{shock}}$ will never intersect with the sonic circle $C_0$ of state (0), i.e. $p(P_2) \geq p_0 + \lambda$ for some constant $\lambda > 0$, which will be proved in Section 4.

2.1. Rankine-Hugoniot conditions on the diffracted shock. In the polar coordinates $(r, \theta) = (\sqrt{\xi^2 + \eta^2}, \arctan \frac{\eta}{\xi})$, system (1.3) can be rewritten as

\[
\begin{align*}
    r^2u_r - r \cos \theta p_r + \sin \theta p_\theta &= 0, \\
    r^2v_r - r \sin \theta p_r - \cos \theta p_\theta &= 0, \\
    r^2p_r - r p_r \cos \theta u_r + r p \sin \theta u_\theta - r p \sin \theta v_r - r \cos \theta v_\theta &= 0. 
\end{align*}
\]

The equation for $p$ takes the form

\[
(p - r^2)p_{rr} + \frac{p}{r^2} p_{r\theta} + \frac{p}{r} p_r + \frac{1}{p} (rp_r)^2 - 2rp_r = 0.
\]

It follows from (2.1) that the solution $(p,u,v)$ along $\Gamma_{\text{shock}}$ should satisfy the Rankine-Hugoniot conditions

\[
\begin{align*}
    -r[u] + [p] \cos \theta &= \frac{dr}{d\theta} \left( -\frac{\sin \theta}{r} \right)[p], \\
    -r[v] + [p] \sin \theta &= \frac{dr}{d\theta} \cos \theta \left[ \frac{r}{p} \right], \\
    -r[E] + [pu] \cos \theta + [pv] \sin \theta &= \frac{dr}{d\theta} \left( -\frac{\sin \theta}{r} [pu] + \cos \theta \left[ \frac{pv}{r} \right] \right).
\end{align*}
\]

It can be derived that

\[
\begin{align*}
    [u] &= (\cos \theta + \frac{r'}{r} \sin \theta) \left[ \frac{p}{r} \right], \\
    [v] &= (\sin \theta - \frac{r'}{r} \cos \theta) \left[ \frac{p}{r} \right].
\end{align*}
\]

Using

\[
E = p + \frac{1}{2}(u^2 + v^2) \quad \text{and} \quad [pu] = \overline{p}[u] + \overline{u}[p],
\]
where $\bar{p}$ is the average of two neighboring states of $p$, and eliminating $[u]$ and $[v]$ in the third equation in (2.3), we obtain

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2(r^2 - \bar{p})}{\bar{p}} \quad \text{with} \quad \bar{p} = \frac{p_0 + p}{2}. \quad (2.4)$$

Once shock diffraction occurs, the incident shock $S_0$ hits the sonic circle $C_1$ at $P_1$ and is created in a counter-clockwise direction. This implies $r'(\theta_1) > 0$, where $\theta_1$ is the $\theta$-coordinate of point $P_1$. Thus we choose $r'(\theta) > 0$ in our configuration:

$$\frac{dr}{d\theta} = r\sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} := g(r(\theta), \theta, p(r, \theta)). \quad (2.5)$$

Moreover, we have two useful formulas

$$[p] = r \cos \theta [u] + r \sin \theta [v] \quad (2.6)$$

and

$$[p]^2 = \bar{p}([u]^2 + [v]^2). \quad (2.7)$$

Let us assume the condition of $u_\eta = v_\xi$. From (2.7) by taking the derivative $r'\partial_r + \partial_\theta$ along the shock, we finally obtain the oblique derivative boundary condition

$$Mp := \beta_1 p_r + \beta_2 p_\theta = 0 \quad (2.8)$$

on $\Gamma_{\text{shock}} := \{(r(\theta), \theta) : r'(\theta) = g(r(\theta), \theta), \theta_w \leq \theta \leq \theta_1\}$, where $\beta_j$ ($j = 1, 2$) is a function of $(p_0, p, r(\theta), r'(\theta))$ with

$$\begin{align*}
\beta_1 &= r' \left(\frac{2(r^2 - \bar{p})}{r^2} - \frac{[p]}{2\bar{p}} + \frac{2\bar{p}(r^2 - p)}{r^2 p}\right), \\
\beta_2 &= \frac{4(r^2 - \bar{p})}{r^2} - \frac{[p]}{2\bar{p}}.
\end{align*} \quad (2.9)$$

Thus the obliqueness becomes

$$(\beta_1, \beta_2) \cdot (1, -r'(\theta)) = -\frac{2r'}{r^2} \left[\frac{\bar{p}(p - r^2)}{p} + r^2 - \bar{p}\right] = \kappa_0.$$ 

Note that $\kappa_0$ becomes zero when $r'(\theta) = 0$, that is $r^2 = \bar{p}$. When the obliqueness fails, we have

$$\beta_1 = 0, \quad \beta_2 = -\frac{[p]}{2\bar{p}} < 0,$$

since $[p] = p - p_0 > 0$. At point $P_1$, we let $r(\theta_1) := r_1$. At point $P_2$, the operator $M$ does not satisfy the oblique derivative boundary condition due to $r'(\theta_w) = 0$. We may express this as one-point Dirichlet condition $p = \tilde{p}$ by solving

$$p_0 + p(r(\theta_w), \theta_w) = 2r^2(\theta_w). \quad (2.10)$$

2.2. **Boundary condition on the wedge.** The condition on the wedge is the slip boundary condition, i.e.,

$$(u, v) \cdot \nu = 0,$$

which is equivalent to

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_{\text{wedge}} := \partial \Omega \cap \{\theta = \pi\} \cup \{\theta = \theta_w\}, \quad (2.11)$$

where $\nu$ is the outward unit normal to $\Gamma_{\text{wedge}}$. 
2.3. Boundary condition on sonic boundary. The Dirichlet boundary condition on \( \Gamma_{\text{sonic}} \) is

\[
p = p_1 \quad \text{on} \quad \Gamma_{\text{sonic}} := \{(\sqrt{p\rho}, \theta) : \theta_1 \leq \theta \leq \pi\}. \tag{2.12}
\]

Problem 1 can be reduced to the following free boundary problem.

**Problem 2** (Free boundary problem in the polar coordinates). *Seek a solution of the second-order nonlinear equation (2.2) for the pressure \( p \) in the domain \( \Omega \), satisfying the free boundary conditions (2.5), (2.8)-(2.10) on \( \Gamma_{\text{shock}} \), the Neumann boundary condition (2.11) on the wedge \( \Gamma_{\text{wedge}} \), and the Dirichlet boundary condition (2.12) on the sonic boundary \( \Gamma_{\text{sonic}} \).*

The main theorem of this paper is stated as follows.

**Theorem 2.1** (Main Theorem). *Let the wedge angle \( \theta_w \) be between \(-\pi\) and 0. There exists a global subsonic solution \( p(r, \theta) \) in the domain \( \Omega \) with the free boundary \( r = r(\theta), \theta \in [\theta_w, \theta_1] \) of Problem 2 satisfying

\[
p \in C^{2+\alpha}\left(\Omega\right) \cap C^{\alpha}\left(\Omega\right), \quad r \in C^{2+\alpha}\left([\theta_w, \theta_1]\right) \cap C^{1,1}\left([\theta_w, \theta_1]\right),
\]

for some \( \alpha \in (0,1) \) depending on the initial data. Moreover, the solution \((p(r,\theta), r(\theta))\) satisfies the following properties:

(i) \( p > p_0 \) on the shock \( \Gamma_{\text{shock}} \);

(ii) The shock \( \Gamma_{\text{shock}} \) is strictly convex up to the point \( P_1 \), except the point \( P_2 \), in the self-similar coordinates;

(iii) The solution \( p \) is \( C^{1,\alpha} \) up to \( \Gamma_{\text{sonic}} \) and Lipschitz continuous across \( \Gamma_{\text{sonic}} \) and at \( P_1 \) from the inside of \( \Omega \) is optimal.*

We will prove the main theorem in the following sections. In Section 3, we consider the regularized free boundary value problem to obtain a uniformly elliptic equation. In Section 4 we show that the limit of \( \varepsilon \to 0 \) gives actually a solution to Problem 2. Moreover, we prove that the diffracted shock \( \Gamma_{\text{shock}} \) will not intersect with the sonic circle \( C_0 \). The optimal regularity of solutions across \( \Gamma_{\text{sonic}} \) is obtained in Section 5.

3. The regularized free boundary value problem. For a fixed \( \varepsilon \in (0,1) \), we consider the regularized equation for \( p \) in the subsonic region

\[
Q^\varepsilon p := (p - r^2 + \varepsilon)p_{rr} + \frac{p + \varepsilon}{r^2} p_{\theta \theta} + \frac{p + \varepsilon}{r} p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \tag{3.1}
\]

For a given curve \( r(\theta) \), by solving the fixed boundary value problem, we obtain a new shock position \( \overline{r}(\theta) \) from (2.5):

\[
\overline{r}(\theta) = r_1 + \int_{\theta_1}^{\theta} g(r(s), s, p(s)) ds \quad \text{for} \quad \theta \in [\theta_w, \theta_1]\tag{3.2}
\]

in which we evaluate the right side along the old shock position \( r(\theta) \). Hence we can solve the free boundary problem by solving a fixed boundary problem and by obtaining a new shock.

According to (3.2), we define a mapping on \( K^\varepsilon \) such that

\[
J : r \mapsto \overline{r}, \tag{3.3}
\]

and the iteration set \( K^\varepsilon \) of \( r \), which is a closed, convex subset of the Hölder space \( C^{1+\alpha_1}\left([\theta_w, \theta_1]\right) \), where \( \alpha_1 \) depends on \( \varepsilon \) and \( \lambda \).
Definition 3.1. The functions $r(\theta)$ in $\mathcal{K}^\epsilon$ satisfy
\begin{align*}
(S1) \quad & r(\theta_1) = r_1, \\
(S2) \quad & r'(\theta_w) = 0, \\
(S3) \quad & \sqrt{p_0} \leq r(\theta) \leq \sqrt{p_1} \quad \text{for} \quad \theta \in [\theta_w, \theta_1], \\
(S4) \quad & 0 \leq r'(\theta) \leq \frac{r_1}{\sqrt{p_0}}.
\end{align*}

Let $V = \{P_1, P_2, O, P_3\}$ denote the corners of $\Omega$, and $V' = V \setminus \{P_2\}$. Set $\Omega' = \overline{\Omega \setminus (V \cup \Gamma_{\text{shock}})}$. For $P \in V$, we define the corner region
$$
\Omega_p(\sigma) := \{x \in \Omega : \text{dist}(x, P) \leq \sigma\}, \quad \Omega_V(\sigma) := \cup_{P \in V} \Omega_p(\sigma).
$$
Define
$$
\Gamma'(\sigma) := \{P \in \Gamma_{\text{shock}} : \text{dist}(P, P_1) > \sigma\}
$$
and
$$
\Gamma(\sigma) := \{x \in \Omega \cap (\cup_{P \in \Gamma'(\sigma)} B_{\sigma}(P))\},
$$
where $B_{\sigma}(P)$ is a ball of radius $\sigma$ centered at $P$. Hence $\Gamma(\sigma)$ is a region that is close to $\Gamma_{\text{shock}}$, but does not contain $P_1$. Let us introduce the weighted norm
$$
||u||^b_a := \sup_{\sigma > 0} (\sigma^{a+b} ||u||_{a, P(\sigma)}),
$$
(3.4)
for $a > 0$, $a + b \geq 0$. Let $C^\epsilon_0(\Omega)$ be the space of functions whose $|| \cdot ||^b_a$ norms are bounded. For a fixed $r(\theta) \in \mathcal{K}^\epsilon$, let
$$
\Gamma^\epsilon_{\text{shock}} := \{(r(\theta), \theta) : \theta_w \leq \theta \leq \theta_1\},
$$
and $\Omega^\epsilon$ be the domain bounded by $\Gamma_{\text{sonic}}, \Gamma_{\text{wedge}}$ and $\Gamma^\epsilon_{\text{shock}}$.

In this section we prove the following existence theorem.

Theorem 3.2. For any $\epsilon \in (0, 1)$, there exists a solution $(p^\epsilon, r^\epsilon) \in C^{2+\alpha}(\Omega^\epsilon) \times C^{1+\alpha_1}(\theta_w, \theta_1)$ to the regularized free boundary problem (3.1) and (2.8)-(2.12) such that
$$
p_0 < \bar{p}^\epsilon < p^\epsilon < p_1, \quad p^\epsilon > r^2 \quad \text{in} \quad \Omega^\epsilon,
$$
(3.5)
where $\alpha, \gamma$ both depend on $\epsilon$ and the initial data. The curve $r^\epsilon$ defining the position of the free boundary $\Gamma^\epsilon_{\text{shock}}$ is in $\mathcal{K}^\epsilon$. Here $\bar{p}^\epsilon$ is a constant satisfying $p_0 + \bar{p}^\epsilon (r^\epsilon(\theta_w), \theta_w) = 2(r^\epsilon(\theta_w))^2$.

Since the ellipticity of equation (3.1) is not known a priori, we impose a cut-off function in (3.1). Let $z(s) \in C^\infty(\mathbb{R})$ be a smooth increasing function such that
$$
z(s) = \begin{cases} 
s \quad & \text{if} \quad s \geq 0, \\
-\frac{\epsilon}{2} \quad & \text{if} \quad s < -\epsilon
\end{cases}
$$
with $|z'(s)| \leq 1$. With this function, we define a new equation
$$
Q^{\epsilon, +} p := (z(p - r^2) + \epsilon) p_{rr} + \frac{p + \epsilon}{r^2} p_{\theta\theta} + \frac{p + \epsilon}{r} p_r + \frac{1}{p} (r p_r)^2 - 2 r p_r = 0.
$$
(3.6)
First we will show the existence of a solution to the linear problem with the fixed boundary $\Gamma^\epsilon_{\text{shock}}$, defined by $r(\theta) \in \mathcal{K}^\epsilon$ and establish the Hölder and the Schauder estimates on $\Gamma^\epsilon_{\text{shock}}$. Next, using the Hölder gradient estimate to the linear problem, we will establish the existence results for the nonlinear fixed boundary problem, via the Schauder fixed point theorem. Finally we will conclude the existence of a solution to the free boundary problem with the oblique derivative boundary condition.
3.1. The regularized linear fixed boundary problem. Firstly, in order to linearize the equation, we define a function space $W$.

**Definition 3.3.** The elements $\omega$ of $W \subset C^2(-\gamma_1)$ satisfy

(W1) $p_0 < \bar{\rho} \leq \omega \leq p_1$, $\omega = p_1$ on $\Gamma_{\text{sonic}}$, $\frac{\partial \omega}{\partial n} = 0$ on $\Gamma_{\text{wedge}}$, and $p(P_2) = \bar{\rho}$,

(W2) $||\omega||_{\alpha_0} \leq K_0$, $||w||_{2 + \alpha_0, \Omega_{\text{isc}}} \leq K_0$, and $||w||_{1, \mu, \Gamma(d_0)} \leq K_0$,

(W3) $||\omega||_{2, \gamma_1} \leq K_1$.

The weighted Hölder space is defined by (3.4). The values of $\gamma_1, \alpha_0, \mu \in (0, 1)$ will be specified later, as well as the values of $K_0$ and $K_1$. The set $W$ is clearly closed, bounded and convex.

The quasilinear equations (3.1) and (2.8)-(2.9) are now replaced by the linearized equation

$L^{\epsilon, +} p = (z(\omega - r^2) + \epsilon)p_{rr} + \frac{\omega + \epsilon}{r^2} p_{r\theta} + \frac{\omega + \epsilon}{r} p_r + \frac{r^2 \omega}{\omega} p_r - 2r p_r = 0,$ (3.7)

and the linearized oblique derivative boundary condition on $\Gamma_{\text{shock}}$

$M p = \beta_1(\omega)p_r + \beta_2(\omega)p_\theta = 0$ (3.8)

with a given $\omega \in W$. Because of the bound of (W1), $L^{\epsilon, +}$ is uniformly elliptic in $\Omega$ with ellipticity ratio depending on the initial data and on $\epsilon$.

In this section we show the existence of the solution to the linear equation (3.7) with the derivative boundary condition (3.8) and the other remaining boundary conditions

$p = p_1$ on $\Gamma_{\text{sonic}}$, $\frac{\partial p}{\partial n} = 0$ on $\Gamma_{\text{wedge}}$, $p(P_2) = \bar{\rho}$. (3.9)

**Theorem 3.4** (Existence for the linearized fixed boundary problem). Let $\Gamma_{\text{shock}} := \{(r(\theta), \theta) : r(\theta) \in K^s\}$ for some $\alpha_1 \in (0, 1)$ and $\omega \in W$ for given $\alpha_0, \gamma_1, K_0$ and $K_1$. Then there exist $\gamma_V, \alpha_\Omega \in (0, 1)$ and $d_0 > 0$, which are independent of $\gamma_1$ and $\alpha_1$, such that there exists a solution

$p^\epsilon \in C^1(\bar{\Omega}) \cap C^{2, \alpha}(\Omega') \cap C^{\gamma}(\Omega V, (d))$ (3.10)

to the linear problem (3.7)-(3.9) for any $\alpha \leq \alpha_\Omega, \mu < \min\{\alpha_1, \gamma_1\}$, $\gamma \leq \gamma_V$ and $d \leq d_0$.

**Proof.** We can use the Perron method as framed in [19] to establish the existence of a global solution. The proof is based on a series of papers. More specifically, we refer to Lieberman [20] for handling mixed boundary value conditions and both points $P_3$ and $P_3$, and to [21] for handling the point $O$ where two oblique derivative boundary conditions are satisfied simultaneously. As for the interior and the Dirichlet boundary condition on the sonic arc $\Gamma_{\text{sonic}}$, we can use the classical theory for uniformly elliptic equations (see [14]).

We only need to provide a proof for the local existence at $P_2$. Let $B_2$ be a neighborhood of $P_2$ with a smooth boundary such that $O \notin B_2$, $\beta_1 \leq 0$, and $\beta_2 < 0$. To avoid considering the boundary condition on wedge, we reflect the region $B_2$ across $\theta = \theta_w$ to obtain a new domain, still denoted by $B_2$. Let $h$ be a continuous function on $\partial B_2 \cap \Omega$ satisfying $\bar{p}^\epsilon < h \leq p_1$. Consider the problem

$p = h$ on $\partial B_2 \cap \Omega$

for equation (3.7) in the domain $B_2 \cap \Omega$ with the oblique derivative boundary condition (3.8) restricted to $\partial B_2 \cap \Gamma_{\text{shock}}$. Following Lieberman [19], there exists a solution

$p \in C(\Omega \cap B_2) \cap C^{2, \alpha}(\Omega \cap B_2)$

Therefore, there exists a solution $p \in C(\Omega \cap B_2) \cap C^{2, \alpha}(\Omega \cap B_2)$.
for some small region $\hat{B}_2$. The local existence for such a region $\hat{B}_2$ implies the global existence of solution in $\Omega$. Moreover, one can construct a barrier function to obtain the continuity of $p$ at point $P_2$, see [5]. This completes the proof. □

Next we state some useful estimates. Since the linear problem is uniformly elliptic for $\varepsilon > 0$, we have $L^\infty$ a priori bounds for the solution $p$ by using the classical maximum principle.

**Lemma 3.5.** The solution $p$ to the linear problem (3.7)-(3.9) satisfies

$$p_0 < \hat{p}^2 < p < p_1 \quad \text{in} \quad \Omega \cup \Gamma_{\text{wedge}} \cup \Gamma_{\text{shock}},$$

and

$$p > p_0 > \xi^2 + \eta^2 \quad \text{in} \quad \Omega \cup \Gamma_{\text{wedge}} \cup \Gamma_{\text{shock}}.$$ 

The Schauder estimates up to the fixed boundary $\Gamma_{\text{sonic}}$ with Dirichlet boundary condition and to $\Gamma_{\text{wedge}}$ with the Neumann boundary condition, and the Hölder estimates at the corners in $V'$ are stated as follows:

**Lemma 3.6.** Let $\Gamma_{\text{shock}} := \{ (r(\theta), \theta) : r(\theta) \in K^\varepsilon \}$ for some $\alpha_1 \in (0, 1)$ and $\omega \in \mathcal{W}$ for given $\alpha_0, \gamma_1, K_0$ and $K_1$. Then there exist $\gamma_V, \alpha_\Omega \in (0, 1)$, with $\gamma_V$ depends on $p_0, p_1$ and $\theta_w$, and both $\gamma_V$ and $\alpha_\Omega$ depend on $\varepsilon$ but independent of $\alpha_1$ and $\gamma_1$, such that any solution $p \in C^{2+\alpha_\Omega}_\text{loc}(\Omega') \cap C^{\gamma_V}_V(\Omega_V(d_0))$ to the linear problem (3.7)-(3.9) satisfies

$$||p||_{1, \gamma_V, \Omega_V(d_0)} \leq C_1||p||_0 \quad \text{for any} \quad \gamma \leq \gamma_V,$$  

(3.11)

and

$$||p||_{2+\alpha, \Omega'} \leq C_2||p||_0 \quad \text{for any} \quad \alpha \leq \alpha_\Omega.$$  

(3.12)

Here, the constant $C_1$ is independent of $K_0$ and $K_1$, and the constant $C_2$ is independent of $K_1$ but depends on $K_0$.

**Proof.** The corner estimates at $P_1$ and $P_3$ follow from Theorem 1 in [22]. Regarding the origin, we see that the interior angle $\angle P_3 OP_2$ is larger that $\pi$, thus Lerberman’s theory cannot be applied directly to yield $C^{1,\alpha}$ estimate. But we realize that the governing equation (3.7) is symmetric in the $\theta$-axis, so we introduce the following transformation

$$(r', \theta') := \left( r, \frac{\pi}{\pi-\theta_w}(\theta - \theta_w) \right), \quad (\xi', \eta') = (r' \cos \theta', r' \sin \theta').$$

Since

$$\det \left( \frac{D(\xi', \eta')}{D(\xi, \eta)} \right) = \frac{\pi}{\pi-\theta_w} > 0,$$

the $C^\alpha$ norms in $(\xi, \eta)$-coordinates and in $(\xi', \eta')$-coordinates are equivalent. Therefore, Lieberman’s theory can be applied to this case to yield the Hölder estimate of the solution at $O$. Finally using the standard interior and the boundary Schauder estimate we obtain the local estimate of (3.12). This completes the proof. □

Because the interior Schauder estimates can be further applied, a solution in $C^{2+\alpha}_\text{loc}(\Omega')$ is actually in $C^3_\text{loc}(\Omega)$. We next establish the Hölder gradient estimate on $\Gamma_{\text{shock}}$.

**Lemma 3.7.** Let $\Gamma_{\text{shock}} := \{ (r(\theta), \theta) : r(\theta) \in K^\varepsilon \}$ for some $\alpha_1 \in (0, 1)$ and $\omega \in \mathcal{W}$ for given $\alpha_0, \gamma_1, K_0$ and $K_1$. Then there exist a positive constant $d_0$ such that for every $d \leq d_0$, any solution $p \in C^1_\text{loc}(\Omega \cup \Gamma_{\text{shock}}) \cup C^3_\text{loc}(\Omega)$ to the linear problem (3.7)-(3.9) satisfies

$$||p||_{1, \gamma, \Gamma(d) \setminus B_{d_0}(P_1)} \leq C(\varepsilon, \alpha_1, \mu, \gamma_1, K_0, K_1)||p||_0$$  

(3.13)
for any $\mu < \min\{\alpha_1, \gamma_1\}$.

**Proof.** Note we can apply Theorem 6.30 in [14] to obtain (3.13) in $\Gamma(d)(B_{d_0}(P_1) \cup B_{d_0}(P_2))$ with a constant $C > 0$ depending on $\varepsilon$, $\alpha_1$, $\Omega$, $d_0$ and $K_0$ since the operator $M$ is oblique away from $P_2$. For the estimates in a region near $P_2$, we follow the idea in [4]. For a given solution $p$, we define

$$
\tilde{u} = \frac{p}{1 + \|Dp\|_0}, \quad \psi = M\tilde{u} = \sum_{i=1}^{2} \beta_i(p) D_i \tilde{u}.
$$

(3.14)

For sufficient small $d_0$, $O \notin B_{d_0}(P_2)$. We construct barrier function $g(x) = g_0 x^\mu$ for $\psi$ in $B = B_d(P_2) \cap \Omega$ such that $|\psi| \leq g$. The barrier function leads to

$$
|D(\psi + g)| \leq ||\psi + g||_{1+\gamma} d^{\mu-1} \leq C d^{\mu-1}
$$

for $d < d_0$.

which implies $\|\tilde{u}\|_{1+\mu} \leq C$. Finally, using the definition of $\tilde{u}$, we apply the interpolation inequality with small $\sigma$ to obtain

$$
\|p\|_{1+\mu} \leq C(1 + |Dp|_0) \leq C(1 + \sigma \|p\|_{1+\mu} + C_\sigma \|p\|_0),
$$

thus (3.13) holds.

3.2. The regularized nonlinear fixed boundary problem. This subsection is devoted to prove the existence of solutions to the nonlinear problem (3.1) with a fixed boundary $r^\varepsilon(\theta) \in K^\varepsilon$. We prove the following theorem.

**Theorem 3.38.** For each $\varepsilon \in (0,1)$, and for given $r^\varepsilon(\theta) \in K^\varepsilon$, there exists a solution $p^\varepsilon \in C^{2+\alpha}(\Omega^\varepsilon)$ to (3.1) and (2.8)-(2.12) such that

$$
p_0 < \hat{p} \leq p^\varepsilon < p_1
$$

(3.15)

in $\Omega^\varepsilon$. Moreover, for some $d_0 > 0$ the solution $p^\varepsilon$ satisfies

$$
\|p^\varepsilon\|_{1+\gamma, \Gamma(d_0) \cup B_{d_0}(P_1)} \leq K_1,
$$

(3.16)

where $\gamma$ and $K_1$ depend on $\varepsilon$, $\gamma_V$ and $K$ but both are independent of $\alpha_1$. Furthermore, the solutions satisfy the following properties:

(i) Ellipticity of equation (3.1): $p^\varepsilon - r^\varepsilon \geq 0$ in $\Omega^\varepsilon$,

(ii) (3.2) can always be integrated: $p \leq r$ on $\Gamma^\varepsilon_{\text{shock}}$,

(iii) $p^\varepsilon$ is monotone on $\Gamma^\varepsilon_{\text{shock}}$.

**Proof.** (1). For the notational simplicity, we write $p = p^\varepsilon$ throughout the proof. For any function $\omega \in \mathcal{W}$, we define a mapping

$$
T : \omega \in C^2_{(-\gamma_1)} \mapsto C^2_{(-\gamma_1)}
$$

by letting $p = T \omega$ be the solution to the linear regularized fixed boundary problem (3.1) and (2.8)-(2.12) solved in Theorem 3.34.

By Lemma 3.7, $T$ maps $\mathcal{W}$ into a bounded set in $C^{2+\alpha}_{(-\gamma_V)}$, where $\gamma_V$ is the value given in Theorem 3.34. Since $\gamma_V$ is independent of $\gamma_1$, we may take $\gamma_1 = \gamma_V/2$ so that $T(\mathcal{W})$ is precompact in $C^2_{(-\gamma_1)}$.

By the boundary conditions, the maximum principle and the standard interior and boundary Hölder estimates, we can prove that $T \omega$ satisfies (W1) and (W3). Next we verify (W2) to show $T$ maps $\mathcal{W}$ into itself. To achieve this, we need to find $K > 0$ such that

$$
\sup_{\delta > 0} \left( \delta^{2-\gamma_1} \|p\|_{2, \Gamma(d) \cup \Omega_V(\delta)}^\varepsilon \right) < K
$$

(3.17)
if \( ||\omega||_2^{-\gamma_1} \leq K \). To show (3.17), we consider domains \( \overline{\Omega} \setminus \{ \Gamma(\delta) \cup \Omega_{\nu}(\delta) \} \) with \( \delta > \tilde{d} \) and \( \delta \leq d \), separately, where \( \tilde{d} \leq d_0 \) is to be specified later.

In the domains \( \overline{\Omega} \setminus \{ \Gamma(\delta) \cup \Omega_{\nu}(\delta) \} \) with \( \delta > \tilde{d} \), the solution is smooth, and by uniform Hölder estimate, the bootstrap iteratively and the interpolation inequality, we can get

\[
\sup_{\delta > \tilde{d}} \left( \delta^{2-\gamma_1} ||p||_{2, \overline{\Omega} \setminus \{ \Gamma(\delta) \cup \Omega_{\nu}(\delta) \}} \right) \leq K',
\]

where \( K' \) depends on the size of the domain \( \Omega \) and \( C(K_0) \), but is independent of the distance to \( \Gamma_{\text{shock}} \).

Next we estimate \( \delta^{2-\gamma_1} ||p||_{2, \overline{\Omega} \setminus \{ \Gamma(\delta) \cup \Omega_{\nu}(\delta) \}} \) with \( \delta \leq \tilde{d} \). We use the estimates for the behavior of the solution near \( \Gamma_{\text{shock}} \). Let \( \gamma_1 = \gamma_\nu/2 \), we can obtain

\[
d^{2-\gamma_1} ||p||_2 \leq K_V \text{ for all } d \leq d_V,
\]

where \( K_V \) is independent of \( K \). We have a local bound for the weighted norm of \( p \) on \( \Gamma(d_0) \) of the form

\[
d^{2-\gamma} ||p||_2 \leq C d^{1-\gamma_1 + \nu},
\]

which holds for all \( d < d_0 \), where \( C \) depends on \( K \), \( \alpha_1 \) and \( \gamma_1 \). Therefore, we can choose

\[
\tilde{d} \leq \frac{\min\{d_0, d_V\}}{2}
\]

in (3.20) small enough so that \( C \tilde{d}^{1-\gamma_1 + \nu} < K \). Hence, (3.17) is satisfied, and \( T \) maps \( W \) into itself. By the Schauder fixed point theorem, there exists a fixed point \( p \) such that

\[Tp = p \in C^2(\Omega)^*.\]

Thus \( p \) is a solution of the nonlinear fixed boundary problem (3.1) and (2.8)-(2.12), and meets the estimates listed in Lemma 3.5-3.7.

(2). Next we show the properties briefly. First we prove property (i) by contradiction. If there exists a nonempty set

\[D' := \{ (\xi, \eta) \in \overline{\Omega} : p - r^2 < 0 \},\]

then it is easy to check that \( P_2 \notin D' \). Since \( (0, 0) \notin D' \), then

\[D' \cap \Omega_* := \{ (\xi, \eta) \in \overline{\Omega} \setminus \Omega : r^2 > \tilde{p}\}.
\]

First, since \( Q^* + r^2 = 4e + 2r^2 > 0 \), it means that the minimum point of \( p - r^2 \) can not obtain in \( D' \). Second, along \( \Gamma_{\text{shock}} \cap D' \),

\[Mr^2 = 2r \beta_1 = 2r \left( \frac{2(r^2 - \tilde{p})}{r^2} - \frac{[p]}{2\tilde{p}} + \frac{2[\tilde{p}(r^2 - \rho)]}{r^2 \rho} \right) > 0,
\]

where we have used the fact that \( r^2 > \tilde{p} \) in \( \Omega_* \). This means that the minimum point of \( p - r^2 \) can not obtain along \( \Gamma_{\text{shock}} \cap D' \). Third, on \( \Gamma_{\text{wedge}} \cap D' \), one has

\[\frac{\partial(p - r^2)}{\partial \nu} = 0.
\]

From the Hopf maximum principle, it follows that \( p - r^2 \) cannot obtain its minimum on \( \Gamma_{\text{wedge}} \cap D' \). Therefore, there is no minimum point, which implies that the set \( D' = \emptyset \). This completes the proof of property (i). We remark here that property (i) ensures the ellipticity of (3.1), so that we can remove the cut-off function \( z \) in (3.6).
We prove property (ii). Assume that there exists a non-empty set
\[ B = \{ (\xi, \eta) \in \Gamma_{\text{shock}}^\varepsilon, \, p > r^2 \} \]
and a point \( X \in B \) such that
\[ \max_{\overline{B}}(\overline{p} - r^2) = (\overline{p} - r^2)(X) = m > 0. \]
It is clear that \( P_1, P_2 \notin B \). Therefore, if such an \( X \) exists, then \( X \in \Gamma_{\text{shock}}^\varepsilon \setminus \{P_1, P_2\} \).

Then \( X \) can be either a local maximum point or a saddle point in \( \Omega \cup \Gamma_{\text{shock}} \). We can show that both cases cannot occur, hence \( B = \emptyset \). The method is similar to [18] and we omit it here. The proof of property (iii) is also followed as in [18] as well as in [4]. \( \square \)

3.3. The regularized nonlinear free boundary problem. We now prove the existence of a solution to the regularized free boundary problem.

For convenience we suppress the \( \varepsilon \)-dependence. For each \( r(\theta) \in \mathcal{K}^\varepsilon \), using the solution \( p \) to the nonlinear fixed boundary problem given by, we define the map \( J \) on \( \mathcal{K}^\varepsilon \):
\[ \tilde{r} = Jr = r_1 + \int_{\theta_1}^\theta g(r(s), s, p(s, r(s)))ds. \]

First, we check that \( J \) maps \( \mathcal{K}^\varepsilon \) into itself. Property (S1) follows from (3.21). By the definition of \( g \) and \( r^2(\theta_{w}) = \bar{p} \), the property (S2) holds, while the upper and lower bounds in (S4) hold and imply (S3).

In order to use the Schauder fixed point theorem, we need to prove that the mapping \( J \) is compact and continuous on \( \mathcal{K}^\varepsilon \). Evaluating \( g(r, \theta, p) \), we get a bound \( |g|_{\gamma V/2} \leq C(K_1) \), and thus \( |\tilde{r}|_{1+\gamma V/2} \leq C(K_1) \).

Here \( \gamma_V \) is independent of \( \alpha_1 \) and the Hölder exponent of the space \( \mathcal{K}^\varepsilon \). Thus we conclude
\[ J(\mathcal{K}^\varepsilon) \subset C^{1+\gamma V/2} \quad \text{and} \quad J(\mathcal{K}^\varepsilon) \subset \mathcal{K}^\varepsilon \quad \text{if} \ \alpha_1 \leq \gamma V/2. \]

We can take \( \alpha_1 = \gamma V/3 \) to guarantee that \( J \) is compact. Furthermore, let \( r_m, r \in \mathcal{K}^\varepsilon \) and \( r_m \to r \) as \( m \to \infty \). Assume that \( p_m \) is the solution to the nonlinear fixed boundary problem with the shock \( \Gamma_{\text{shock}}^\varepsilon \) defined by \( r_m \). By the standard argument as in [4], we have \( p_m \to p \), which solves the problem for \( r \). Therefore,
\[ g(r_m(\theta), \theta, p(r_m(\theta), \theta)) \to g(r(\theta), \theta, p(r(\theta), \theta)) \quad \text{as} \ \ m \to \infty, \]
which implies \( Jr_m \to Jr \) as \( m \to \infty \). Therefore, the mapping \( J \) has a fixed point \( r^\varepsilon \in C^{1+\gamma V/3}([\theta_{w}, \theta_1]) \) by the Schauder fixed point theorem.

Together with the corresponding solution \( p^\varepsilon \), this establishes the existence of a solution \( (p^\varepsilon, r^\varepsilon) \in H^2_{\gamma V}(\Omega^\varepsilon) \times C^{1+\gamma V/3}([\theta_{w}, \theta_1]) \) to the regularized free boundary problem (3.1) and (2.8)-(2.12). The proof of Theorem 3.2 is completed.

4. Subsonic solutions to Problem 2.

4.1. The limiting solution. In this section we prove the limit of \( (p^\varepsilon, r^\varepsilon) \) as \( \varepsilon \to 0 \) is actually a solution to Problem 2.

Lemma 4.1. There exists a positive function \( \phi \), independent of \( \varepsilon \), such that
\[ p - (\xi^2 + \eta^2) \geq \phi \quad \text{in} \ \overline{\Omega} \setminus \Gamma_{\text{sonic}}, \quad (4.1) \]
and \( \phi \to 0 \) as \( \text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \to 0 \).
There exist functions \( \phi \) independent of \( \varepsilon \). For the free boundary Lemma 4.3, let us define
\[
\phi = \delta_0(\zeta(X))^\tau,
\]
where \( \delta_0 \) and \( \tau \) are two positive constants. We can obtain a local uniform lower barrier
\[
p - (\xi^2 + \eta^2) \geq \phi = \delta_0(\zeta(X))^\tau \text{ in } B_{\frac{\delta}{4}}(X_0) \cap \overline{\Omega},
\]
where \( \delta_0 \) and \( \tau \) are independent of \( \varepsilon \). Moreover, \( \delta_0 \to 0 \) as \( \text{dist}(X, \Gamma_{\text{sonic}}) \to 0 \), so does \( \phi \). See [4] for more details.

This lemma implies that the governing equation has local uniform ellipticity and is independent of \( \varepsilon \). Hence we can apply the standard local compactness arguments to obtain the local existence of the limit \( p \) in the interior of the domain.

**Lemma 4.2.** There exist functions \( r(\theta) \in C^1(\{\theta_w, \theta_1\}) \) and \( p \in C^{2+\alpha}_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \), such that
\[
r^x \to r \text{ in } C(\{\theta_w, \theta_1\}), \quad p^x \to p \text{ in } C^{2+\alpha}_{\text{loc}}(\Omega),
\]
and \( (p, r) \) is a solution of Problem 2.

**Proof.** We have obtained that
\[
\|r^x(\theta)\|_{C^1(\{\theta_w, \theta_1\})} \leq C,
\]
where \( C \) is independent of \( \varepsilon \). Then there exists a limit function \( r(\theta) \) in \( C^\alpha(\{\theta_w, \theta_1\}) \) as \( \varepsilon \to 0 \) for any \( \alpha \in (0, 1) \) by the Arzela-Ascoli theorem. By the local ellipticity and the standard interior Schauder estimate, there exists a function \( p \in C^{2+\alpha}_{\text{loc}}(\Omega) \) such that \( p^x \to p \) in any compact subset contained in \( \overline{\Omega \setminus (\Gamma_{\text{sonic}} \cup \Gamma_{\text{shock}})} \), satisfying \( Qp = 0 \) in \( \Omega \).

Since the shock will never meet the sonic circle \( C_0 \), we have \( p^x > p_0 + \lambda \) for some \( \lambda \). Thus, we have the uniform ellipticity, as well as the uniform negativity of \( (\beta_1, \beta_2) \cdot \nu \) locally. Hence, we can pass the limit to obtain \( p \in C^{1+\alpha} \) and
\[
Mp = 0 \text{ on } \Gamma_{\text{shock}}
\]
such that \( r'(\theta) = r \sqrt{\frac{r^2 + p}{p}} \).

**Lemma 4.3.** For the free boundary \( \Gamma_{\text{shock}} = \{(\xi, \eta(\xi)) : \xi_w < \xi < \xi_1\} \) determined, where \( \xi_w \) and \( \xi_1 \) are the \( \xi \)-coordinates of points \( P_2 \) and \( P_1 \), respectively, we have
\[
\eta(\xi) \in C^2([\xi_w, \xi_1]).
\]
Moreover, \( \eta(\xi) \) is strictly convex for \( \xi \in [\xi_w, \xi_1] \).

**Proof.** Defining
\[
F(\xi, \eta) = \xi^2 + \eta^2 - r^2(\theta(\xi, \eta)) = 0 \text{ on } \Gamma_{\text{shock}},
\]
we have
\[
F_{\eta|\xi=\xi_w} = (2\eta - 2r'\theta_\eta)|_{\xi=\xi_w} = 2\eta(\xi_w) \neq 0.
\]
By the implicit function theorem, there exists \( \eta = \eta(\xi) \) defined locally on \( \Gamma_{\text{shock}} \) near \( \xi = \xi_w \). That is, there exists \( \bar{\xi} > 0 \) such that \( (\xi, \eta(\xi)) \in \Gamma_{\text{shock}} \) for \( \xi_w < \xi < \bar{\xi} \). From the fact of
\[
\eta'(\xi) = f(\xi, \eta(\xi), \frac{p}{\xi^2 - \frac{p}{\xi^2}}) = \frac{\xi \eta + \sqrt{p(\xi^2 + \eta^2 - p)}}{\xi^2 - \frac{p}{\xi^2 - \frac{p}}},
\]
we have
\[ \eta''(\xi) = f_\xi + \eta' f_\eta + f_\eta' \]
Noticing that
\[ f_\xi = \frac{\eta}{\xi^2 - \rho} + \frac{\xi \rho}{(\xi^2 - \rho)\sqrt{\rho (\xi^2 + \eta^2 - \rho)}}, \]
\[ f_\eta = \frac{\xi}{\xi^2 - \rho} + \frac{\eta \rho}{(\xi^2 - \rho)\sqrt{\rho (\xi^2 + \eta^2 - \rho)}}, \]
we have \( f_\xi + \eta' f_\eta = 0 \). Therefore, the sign of \( \eta'' \) is determined by the signs of \( f_\rho \) and \( \rho' \). Note that \( \rho \) is increasing, \( \rho' > 0 \), and \( \frac{\partial \rho}{\partial \rho} > 0 \). Moreover, we have
\[ \frac{\partial f}{\partial \rho} = \frac{(\eta \sqrt{\rho} + \xi \sqrt{\xi^2 + \eta^2 - \rho})^2}{2(\xi^2 - \rho)^2\sqrt{\rho (\xi^2 + \eta^2 - \rho)}} > 0. \]
These imply that \( \eta(\xi) \) is strictly convex for \( \xi \in [\xi_w, \xi_1] \).

4.2. The Position of Diffracted Shock. Next we want to make clear the position of the diffracted shock. There are three cases:
(1) \( r_0 < r(\theta) \leq r_1 \) for all \( \theta \in [\theta_a, \theta_1] \),
(2) \( r(\theta_a) = r_0 \) and \( r_0 < r(\theta) \leq r_1 \) for all \( \theta \in (\theta_a, \theta_1) \),
(3) there exists \( \theta^* \in (\theta_a, \theta_1) \) such that \( r(\theta) = r_0 \) for all \( \theta \in [\theta_a, \theta^*] \), and \( r_0 < r(\theta) \) for all \( \theta \in (\theta^*, \theta_1) \).

We will show that Case (2) and Case (3) do not actually occur. The method will be very similar to that of [5], but for the sake of completeness, we give the proof.

Lemma 4.4. Let \((p, r)\) be a solution to Problem 2. Then the diffracted shock \( \Gamma_{\text{shock}} \) does not meet the sonic circle \( C_0 = \{ r = \sqrt{p_0} \} \) at the wedge.

Proof. Suppose that there exists \( \theta^* \in (\theta_a, \theta_1) \) such that \( r(\theta^*) = r_0 \). Due to the monotonicity of \( r(\theta) \), one has
\[ r(\theta) = r_0 \quad \text{for all } \theta \in [\theta_a, \theta^*]. \]
For any \( \theta_a \in [\theta_a, \theta^*] \), we denote the small interior neighborhood \( \mathcal{N} \) of \((r(\theta_a), \theta_a)\) \( \in \Gamma_{\text{shock}} \). We first prove that the optimal regularity of \( p \) in \( \mathcal{N} \) near \( \Gamma_{\text{shock}} \) is \( C^{\frac{1}{2}} \). We introduce
\[ w(r, \theta) = p_0 + A(r_0 - r)^{\frac{1}{2}} - B(r_0 - r)^{\beta} + D(\theta - \theta_a)^2, \]
where \( A, B, D > 0 \) and \( \frac{1}{2} < \beta < 1 \) will be specified later. It can be computed that
\[ \dot{w} := \left( \frac{1}{4} - \beta^2 \right) AB(r_0 - r)^{\beta - \frac{1}{2}} + O_1 - \frac{A^2}{4p}(p - r^2)(r_0 - r)^{-1} \]
\[ + (B^2 \beta(2\beta - 1)(r_0 - r)^{2\beta - 2} + O_2) + \left( - \frac{AD}{4}(r_0 - r)^{-\frac{3}{2}}(\theta - \theta_a)^2 + O_3 \right), \]
where
\[ O_1 = \frac{p - r^2}{p} AB\beta(r_0 - r)^{\beta - \frac{1}{2}} + \frac{A(2r^2 - p)}{2r}(r_0 - r)^{-\frac{1}{2}} + \frac{p - 2r^2}{r} B\beta(r_0 - r)^{\beta - 1} - AB\beta(\theta - \theta_a)(r_0 - r)^{-\frac{1}{2}}. \]
For sufficiently small $r$ in $K$ deduce that $N$ at some points in $K$.

On the other hand, if $r$ is large $\theta$ which implies to $\rho > w$. Moreover, there exists $0 < \alpha < \frac{1}{2}$. Thus,

$$
\left| - \frac{A^2}{4p} (p - r^2)(r_0 - r)^{-1} \right| \leq M(p_0, p_1) A^2 (r_0 - r)^{\alpha - 1}
$$

for some constant $M > 0$ depending on $p_0$ and $p_1$.

Next we choose $\beta$ such that $\beta - \frac{3}{2} < \alpha - 1$, i.e. $\alpha > \beta - \frac{1}{2}$. Then we have

$$(\beta^2 - \frac{1}{4}) AB (r_0 - r)^{\beta - \frac{3}{2}} > 3M(p_0, p_1) A^2 (r_0 - r)^{\alpha - 1},$$

which implies that

$$B > AM(p_0, p_1, \beta) (r_0 - r)^{\alpha - \beta + \frac{1}{2}}. \quad (4.2)$$

On the other hand, if $r_0 - r$ is small enough,

$$3B^2 \beta (2\beta - 1) (r_0 - r)^{2\beta - 2} < (\beta^2 - \frac{1}{4}) AB (r_0 - r)^{\beta - \frac{3}{2}},$$

which implies

$$A > \frac{3\beta(2\beta - 1) B}{\beta^2 - \frac{3}{4}} (r_0 - r)^{\beta - \frac{3}{2}}. \quad (4.3)$$

Consider a relative neighborhood $N_1$ of $(r_0, \theta_a)$ to $\Omega$. We can choose sufficiently large $A$ and $D$ such that

$$w > p_0 + \frac{1}{2} A (r_0 - r)^{\frac{1}{2}} + D(\theta - \theta_a)^2 > p \quad (4.4)$$

at some points in $N_1$. Next we choose $B$ sufficiently small satisfying (4.2)-(4.3), we deduce that

$\hat{Q}w < 0$ in $N_1$.

For sufficiently small $r_0 - r$, it holds

$$w_{rr}(r, \theta) = -\frac{1}{4} A (r_0 - r)^{-\frac{3}{2}} + B \beta (1 - \beta) (r_0 - r)^{\beta - 2} < 0 \quad \text{in } N_1.$$

Suppose $K_1 = \{(r, \theta) \in N_1, p > w\} \neq \emptyset$. It is noted that on $\partial N_1 \cap \Gamma_{\text{shock}} \setminus (r_0, \theta_a)$, one has

$$w = p_0 + D(\theta - \theta_a)^2 > p_0 = p,$$

which implies that $\overline{K_1} \subseteq N_1 \cup (r_0, \theta_a)$.

Moreover,

$$Qp - Qw = Qp - \hat{Q}w + (w - p)w_{rr} > Qp - \hat{Q}w > 0$$

in $K_1$. Due to the maximum principle we obtain

$$\max_{K_1} \{p - w\} = \max_{\partial K_1} \{p - w\} = 0,$$

which contradicts to $p > w$. Thus $p \leq w$ in $N_1$. Especially, $p \leq w$ also holds when $\theta = \theta_a$.

Next we take

$$v(r, \theta) = p_0 + a(r_0 - r)^{\frac{1}{2}} + b(r_0 - r)^{\beta} - d(\theta - \theta_a)^2,$$
and in the same way we can prove that \( p \geq v \) in a neighborhood \( \mathcal{N}_2 \). Thus we derive
\[
a(r_0 - r)^{\frac{1}{2}} < p - p_0 < A(r_0 - r)^{\frac{1}{2}} \quad \text{in } \mathcal{N}_1 \cap \mathcal{N}_2.
\]
for some constants \( a, A > 0 \). So the optimal regularity of \( p \) near the sonic circle \( C_0 \) is \( C^1 \).

Next we introduce the coordinates \((x, y) = (r_0 - r, \theta - \theta_w)\), and let \( \varphi = p - p_0 \). We scale \( \varphi \) in \( \mathcal{N} \) such that
\[
\varphi = \frac{1}{s^{1/5}} \varphi(s^{-\frac{12}{5}}, y_0 + s^{-\frac{14}{5}} t) \quad \text{for } (s^{-\frac{12}{5}}, y_0 + s^{-\frac{14}{5}} t) \in \mathcal{N}.
\]
From the optimal regularity, we have
\[
0 < a \leq s^{\frac{2}{5}} u \leq A, \quad |s^{\frac{4}{5}} u_s| \leq A \quad \text{and} \quad |s^{\frac{8}{5}} u_t| \leq A.
\]
From (2.2) we obtain the governing equation for \( u \):
\[
Q(u) = a_{11} u_{ss} + a_{12} u_{ss} + a_{22} u_{tt} + b_1 u_s + b_2 u_t + c_1 u + \text{H.O.T.} = 0, \quad (4.5)
\]
where
\[
a_{11} = s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}, \quad a_{12} = \frac{28t}{5s} \left(s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}\right),
\]
\[
a_{22} = \frac{144}{25} + \frac{166Gt^2}{s^2} \left(s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}\right),
\]
\[
b_1 = \frac{19}{5s} \left(s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}\right) + \frac{12G}{5s^{\frac{14}{5}}} \left(s^{\frac{2}{5}} u + p_0 - 2\right) =: \hat{b}_1 s^{-1},
\]
\[
b_2 = \frac{294t}{25s^2} \left(s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}\right) + \frac{168Gt}{25s^{\frac{14}{5}}} \left(s^{\frac{2}{5}} u + p_0 - 2\right) =: \hat{b}_2 s^{-2} t,
\]
\[
c_1 = \frac{13}{25s^2} \left(s^{\frac{2}{5}} u + 2r_0 s^{-\frac{6}{5}} - s^{-\frac{13}{5}}\right) + \frac{12G}{25s^{\frac{14}{5}}} \left(s^{\frac{2}{5}} u + p_0 - 2\right) =: \hat{c}_1 s^{-2},
\]
\[
\text{H.O.T.} = \frac{G^2}{s^{\frac{2}{5}} u + p_0} \left(s^{\frac{2}{5}} u_s^2 + \frac{196}{25} s^{\frac{2}{5}} t^2 (u_s)^2 + \frac{28}{5} s^{\frac{2}{5}} u u_t + \frac{28}{25} s^{\frac{2}{5}} u u_{tt}\right).
\]
with \( G := r_0 - s^{-\frac{12}{5}} \). Here “H.O.T.” represents the higher order term. According to the optimal regularity we have
\[
0 < C^{-1} \leq a_{11}, a_{12}, \hat{b}_1, \hat{b}_2, \hat{c}_1 \leq C, \quad \text{H.O.T.} \to 0
\]
if \( s^{-1} \) and \( t \) are sufficiently small. Moreover, the eigenvalues of \((4.5)\) are positive and bounded, then \((4.5)\) is uniformly elliptic for \( u \) in \((s, t)\)-coordinates.

Let \( \tau_0^{-1} < s < \tau_0^{-2} \) with \( \tau_0 \) small enough. Then using Theorem 8.20 in [14], we have
\[
a \tau_0^{-\frac{2}{5}} \leq u(\tau_0^{-1}, 0) \leq \sup_{\tau_0^{-1} \leq s \leq \tau_0^{-2}} u(s, t)
\]
\[
\leq C \inf_{\tau_0^{-1} \leq s \leq \tau_0^{-2}} u(s, t) \leq C u(\tau_0^{-2}, 0) \leq C A \tau_0^{-\frac{14}{5}},
\]
where \( C \) is independent of \( \tau_0 \). This implies that \( \tau_0^{-\frac{2}{5}} \leq AC/a \), which is a contradiction if \( \tau_0 \) is sufficiently small. Hence Case (3) does not occur. In the similar way
we can prove that Case (2) is impossible. This shows that the diffracted shock can not intersect with the sonic circle $C_0$.

**Remark 1.** Since the two states (0) and (1) are connected by a shock initially, we see that the pressure $p$ is discontinuous along any streamline from state (0) to state (1). But when either Case (2) or Case (3) occurs, we can find two streamlines such that along one of them the pressure $p$ is continuous, while along the other $p$ is discontinuous, which is physically impossible.

5. **Optimal regularity across the sonic boundary.** In this section we first establish the Lipschitz continuity for the solution near the degenerate sonic boundary $\Gamma_{\text{sonic}}$.

**Lemma 5.1.** The solution $p$ to Problem 2 is Lipschitz continuous up to the sonic boundary $\Gamma_{\text{sonic}}$.

**Proof.** Since $p \leq p_1$ in $\Omega$, we have

$$p - \xi^2 - \eta^2 < p_1 - \xi^2 - \eta^2.$$  

On the other hand, $p - \xi^2 - \eta^2 > \xi^2 + \eta^2 - p_1$ in $\Omega$ by Lemma 4.1. Then we obtain

$$|p - p_1| \leq |p - \xi^2 - \eta^2| + |p_1 - \xi^2 - \eta^2|$$

$$\leq 2|p_1 - \xi^2 - \eta^2| \leq 4\sqrt{p_1} \sqrt{\xi^2 + \eta^2},$$

which implies that $p$ is Lipschitz continuous up to the boundary $\Gamma_{\text{sonic}}$. □

Next, we want to show that the Lipschitz continuity is the optimal regularity for $p$ across the sonic boundary $\Gamma_{\text{sonic}}$, as well as the intersection point $P_1$.

For $0 < \varepsilon < r_1/4$, we denote the $\varepsilon$-neighborhood of the sonic boundary $\Gamma_{\text{sonic}}$ within $\Omega$ by

$$\Omega_\varepsilon := \Omega \cap \{(r, \theta) : 0 < r_1 - r < \varepsilon\},$$

and we introduce new coordinates

$$(x, y) := (r_1 - r, \theta - \theta_1)$$

in $\Omega_\varepsilon$. Then $\Gamma_{\text{sonic}} = \{(0, y) : 0 < y < \pi - \theta_1\}$ and $P_1 = (0, 0)$. We can take $P_1$ as an interior point of

$$\Gamma_{\text{ext}}^{\text{sonic}} := \{(0, y) : 0 \leq y < |\pi - \theta_1|\},$$

which is obtained by reflecting $\Gamma_{\text{sonic}}$ with respect to $y = 0$. Let

$$Q_{r,R}^+ := \{(x, y) : x \in (0, r), |y| < R\} \text{ with } R \leq \pi - \theta_1.$$  

Letting $\varphi = p_1 - p$,

$$\varphi > 0 \text{ in } Q_{r,R}^+, \text{ and } \varphi = 0 \text{ on } \partial Q_{r,R}^+ \cap \{x = 0\}. \quad (5.2)$$

It follows from (2.2) that $\varphi$ satisfies

$$L\varphi := (2r_1 x - \varphi + \Lambda_1)\varphi_{xx} + (1 + \Lambda_2)\varphi_{yy} + (r_1 + \Lambda_3)\varphi_x - (1 + \Lambda_4)(\varphi_x)^2 = 0 \quad (5.3)$$

in $Q_{r,R}^+$, where

$$\begin{align*}
\Lambda_1(x, \varphi) &= -x^2, \\
\Lambda_2(x, \varphi) &= \frac{x(2r_1 - x) - \varphi}{(r_1 - x)^2}, \\
\Lambda_3(x, \varphi) &= \frac{2x^2 - 3r_1 x + \varphi}{r_1 - x}, \\
\Lambda_4(x, \varphi) &= \frac{x^2 - 2r_1 x + \varphi}{r_1^2 - \varphi}. 
\end{align*}$$  

(5.4)
It follows from Lemmas 4.1 and 4.4 that
\[ 0 \leq \varphi \leq (2r_1 - \mu_0)x \] (5.5)
in \( \Omega_\varepsilon \) for some constant \( \mu_0 > 0 \). Then it is easy to see that \( \Lambda_i(x, \varphi) (i = 1, \ldots, 4) \) are continuously differentiable and satisfy
\[
\frac{|\Lambda_1(x, y)|}{x^2} + \frac{|\Lambda_k(x, y)|}{x^2} + \frac{|D\Lambda_1(x, y)|}{x} + |D\Lambda_k(x, y)| \leq N \quad \text{for } k = 2, 3, 4, \quad (5.6)
\]
which imply that \( \Lambda_i (k = 1, 2, 3, 4) \) are infinitesimals of higher order in the related coefficients. Hence the main terms of (5.3) form the following equation
\[
(2r_1x - \varphi)\varphi_{xx} + \varphi_{yy} + r_1\varphi_x - \varphi^2_x = 0, \quad (5.7)
\]
which is uniformly elliptic in every subdomain \( \{ x > \delta \} \) with \( \delta > 0 \).

Furthermore, for sufficiently small \( \hat{r} > 0 \), (5.5) and (5.6) imply that (5.3) is uniformly elliptic with respect to \( \varphi \) in \( Q^+_{r,R} \cap \{ x > \delta \} \) for any \( \delta \in (0, \hat{r}) \).

**Lemma 5.2.** Let \( \varphi \in C^2(\Omega^+_{r,R}) \cap C(\partial \Omega^+_{r,R}) \) be the solution of the equation (5.3) with the condition (5.2). Then for any \( \alpha \in (0, 1) \), \( |y| < \hat{r}/r \), we have
\[
\varphi \in C^{1+\alpha}(Q^+_{r/2,R/2}), \quad \text{and} \quad \varphi_x(0, y) = r_1, \quad \varphi_y(0, y) = 0. \quad (5.8)
\]

*Proof.* The proof is similar to that in [1, 5], and we only list the major procedure and the difference here. First by constructing barrier functions and maximum principle for strictly elliptic equations, we can prove that \( \varphi = p_1 - p \) has a positive lower bound, i.e., there exist \( \hat{r} > 0 \) and \( \mu > 0 \), depending on the initial data and \( \inf_{Q^+_{r,R}} \varphi \), such that for all \( r \in (0, \hat{r}) \),
\[
\varphi \geq \mu r_1 x \quad \text{in } Q^+_{r,15r/8}.
\]
Moreover, we can obtain more precise estimates for \( \varphi \) near the sonic boundary:
\[
|\varphi(x, y) - r_1x| \leq C x^{1+\alpha} \quad \text{for all } Q^+_{r,R/8}
\]
for any \( \alpha \in (0, 1) \) and constant \( C \) depending on \( \hat{r}, R, \alpha \) and initial data. Denote \( W(x, y) = r_1x - \varphi(x, y) \), and introduce a cut-off function \( \zeta(x) \in C^\infty(\mathbb{R}) \) such that
\[
\zeta(s) = \begin{cases} 
  s & \text{if } s \in (-r_1, r_1), \\
  0 & \text{if } s \in \mathbb{R}\setminus(-r_1 - 1, r_1 + 1).
\end{cases}
\]
Then \( W(x, y) \) satisfies the following equation
\[
x \left( r_1 + \zeta \left( \frac{W}{x} \right) + \frac{\Lambda_1}{x} \right) W_{xx} + (1 + \Lambda_2)W_{yy} + (1 + \Lambda_4)W^2_x
- (r_1 - \Lambda_3 + 2r_1\Lambda_4)W_x = r_1\Lambda_3 - r_1^2\Lambda_4, \quad (5.9)
\]
where \( \Lambda_i (i = 1, \ldots, 4) \) are given in (5.4).

Next for fixed \( q_0 = (x_0, y_0) \in Q^+_{r/2,R/2} \), we define
\[
W^{(q_0)}(S, T) = \frac{1}{x_0^{1+\alpha}} W \left( x_0 + \frac{x_0}{8} S, y_0 + \frac{\sqrt{x_0}}{8} T \right), \quad (S, T) \in Q_1,
\]
where \( Q_1 = (-1,1)^2 \). By estimating the coefficients carefully we can show that (5.9) is uniformly elliptic with constants independent of \( q_0 \). Then by Theorem A.1 in [7], we derive
\[
\|W^{(q_0)}\|_{C^{2,\alpha}(Q_{1/2})} \leq C \left( \frac{r_1}{r_0^{1+\alpha}} + r_1^{-\alpha} \right)
\]
where $C$ depends only on the data and $\alpha$. Thus,

$$|D_x^i D_y^j W(x_0, y_0)| \leq C x_0^{2+\alpha-i-j/2}$$

for all $(x_0, y_0) \in Q^+_{r/2,R/2}$, $0 \leq i + j \leq 1$, which implies $DW(0, y) = 0$. This completes the proof of Lemma 5.2.

Furthermore, we have

**Lemma 5.3.** Let $p \in C^{2+\alpha}(\Omega) \cap C(\partial)\overline{\Omega}$ be the solution of Problem 2 satisfying $p_0 < \hat{p} \leq p < p_1$, $p > r^2$ in $\Omega$. Then $p$ cannot be $C^1$ across the degenerate sonic boundary $\Gamma_{\text{sonic}}$.

**Proof.** Suppose that $p$ is $C^1$ across $\Gamma_{\text{sonic}}$, so is $\varphi = p_1 - p$. Since (5.3) is invariant under the transformation $(x, y) \rightarrow (x, y - y_0)$, we have $\hat{\varphi}(0, y) = 0$ for any $(0, y) \in \Gamma_{\text{sonic}}$.

On the other hand, for $(0, y_0) \in \Gamma_{\text{sonic}}$ and small $r, R > 0$, we have

$$\hat{\Omega}^+_{r,R} := \{(x, y) : x \in (0, r), |y - y_0| \leq R\} \subset \Omega_{\varepsilon}.$$

Since (5.3) is invariant under the transformation $(x, y) \rightarrow (x, y - y_0)$, we can let $(0, y_0) = (0, 0)$ so that $\hat{\Omega}^+_{r,R} = \hat{\Omega}_{r,R}^+ \subset \Omega_{\varepsilon}$. From the proof of Lemma 5.2, we know that there exist $r, \mu > 0$ such that

$$\varphi \geq \mu r_1 x \quad \text{in} \quad \hat{\Omega}_{r,15R/16}^+,$$

which contradicts $\hat{D} \varphi(0, y) = 0$.

Finally we show the regularities of solutions to Problem 2.

**Theorem 5.4.** Let $p$ be the solution of Problem 2 and satisfy the properties: There exists a neighborhood $\mathcal{N}(\Gamma_{\text{sonic}})$ such that, for $\varphi = p_1 - p$,

(a) $\varphi$ is $C^{0,1}$ across the degenerate boundary $\Gamma_{\text{sonic}}$.

(b) there exists $\mu_0 > 0$ such that in $(x, y)$-coordinates

$$0 \leq \varphi \leq (2r_1 - \mu_0)x \quad \text{in} \quad \Omega \cap \mathcal{N}(\Gamma_{\text{sonic}}),$$

(c) there exist $\varepsilon_0 > 0$, $\omega > 0$ and $y = f(x) \in C^{1,1}([0, \varepsilon_0])$, such that

$$\Gamma_{\text{shock}} \cap \partial \Omega_{\varepsilon_0} = \{(x, y) : x \in (0, \varepsilon_0), y = f(x)\},$$

$$\Omega_{\varepsilon_0} = \{(x, y) : x \in (0, \varepsilon_0), f(x) < y < \omega x,\}$$

and $\frac{\partial f}{\partial x} > \omega > 0$ for $x \in (0, \varepsilon_0)$.

Then we have that

(i) there exists $\varepsilon_0 > 0$ such that $\varphi$ is $C^{1,\alpha}$ in $\Omega$ up to $\Gamma_{\text{sonic}}$ away from point $P_1$ for any $\alpha \in (0, 1)$. That is, for any $\alpha \in (0, 1)$ and $(\xi, \eta_0) \in \Gamma_{\text{sonic}} \setminus P_1$, there exists $K < \infty$ so that

$$\|\varphi\|_{1,\alpha, B_{2/(1+\varepsilon)}(\xi_0, \eta_0)} \leq K,$$

(ii) for any $(\xi, \eta_0) \in \Gamma_{\text{sonic}} \setminus \{P_1\}$, $\lim_{(\xi, \eta) \to (\xi, \eta_0)} D_r \varphi = -r_1$,

(iii) $D_r \varphi$ has a jump across $\Gamma_{\text{sonic}}$; for any $(\xi, \eta) \in \Gamma_{\text{sonic}} \setminus \{P_1\}$,

$$\lim_{(\xi, \eta) \to (\xi, \eta_0)} \varphi_r(\xi, \eta) = \lim_{(\xi, \eta) \to (\xi, \eta_0)} \varphi_r(\xi, \eta) = -r_1,$$

(iv) the limit $\lim_{(\xi, \eta) \to P_1} D_r \varphi$ does not exist.
Proof. The proof consists of four steps.

Step 1. Let \( \varphi = p_1 - p \). We have
\[
\varphi(0, y) = \varphi_y(0, y) = 0, \quad \varphi_x(0, y) = r_1 \quad \text{on } \Gamma_{\text{sonic}}.
\]
Using condition (b) and reducing \( \varepsilon_0 \) if necessary, we conclude that (5.3) is uniformly elliptic on \( \Omega_{\varepsilon_0} \cap \{ x > \delta \} \) for any \( \delta \in (0, \varepsilon_0) \). Due to the slip boundary, we find
\[
\varphi_y(x, \pi - \theta_1) = 0 \quad \text{for all } x \in (0, \varepsilon_0).
\]
By the standard regularity theory for the oblique derivative problem for uniformly elliptic equations, we obtain
\[
\varphi \in C^{0,1}(\overline{\Omega_{\varepsilon_0}}) \cap C^2(\Omega_{\varepsilon_0} \cup \Gamma_{\text{wedge}}),
\]
where \( \Gamma_{\text{wedge}} = \{ (x, \pi - \theta_1) : 0 < x < \varepsilon_0 \} \). Reflecting \( \Omega_{\varepsilon_0} \) with respect to \( y = \pi - \theta_1 \), we define
\[
\tilde{\Omega}_{\varepsilon_0} := \{ (x, y) : x \in (0, \varepsilon), f(x) < y < 2(\pi - \theta_1) + f(x) \}.
\]
Extend \( \varphi \) from \( \Omega_{\varepsilon_0} \) to \( \Omega_{\varepsilon_0} \) by the even reflection
\[
\varphi(x, y) = \varphi(x, 2(\pi - \theta_1) - y)
\]
for \( (x, y) \in \Omega_{\varepsilon_0} \). Moreover, \( \varphi \) satisfies (5.3) with (5.2). Also, \( \varphi(0, y) = 0 \) for all \( y \in (0, 2(\pi - \theta_1)) \), and \( \varphi(x, y) \geq 0 \) in \( \tilde{\Omega}_{\varepsilon_0} \). Then we conclude that the extended function \( \varphi(x, y) \) satisfies
\[
\varphi \in C^{0,1}(\overline{\Omega_{\varepsilon_0}}) \cap C^2(\tilde{\Omega}_{\varepsilon_0}).
\]
Step 2. Let \( (\xi^*, \eta^*) \in \Gamma_{\text{shock}} \setminus \{ P_1 \} \). In the \((x, y)\)-coordinates, \( P = (0, y^*) \) with \( y \in [0, \pi - \theta_1] \). There exists \( r, R \) depending on \( \varepsilon_0, p_1 \) and \( \text{dist}((\xi^*, \eta^*), \Gamma_{\text{shock}}) \) such that
\[
\tilde{Q}_{r, R}^+ := \{ (x, y) : x \in (0, r), |y - y^*| < R \} \subset \tilde{\Omega}_{\varepsilon_0}.
\]
Then in \( \tilde{Q}_{r, R}^+ \), the function \( \tilde{\varphi}(x, y) = \varphi(x, y - y^*) \) satisfies all the conditions of Lemma 5.2. Thus we obtain that, for all \( y_0 \in [0, \pi - \theta_1] \),
\[
\lim_{(x, y) \rightarrow (0, y_0)} \varphi_x(x, y) = r_1, \quad \lim_{(x, y) \rightarrow (0, y_0)} \varphi_y(x, y) = 0. \tag{5.10}
\]
Since \( D_r \varphi = -\varphi_x \), this implies assertions (i) and (ii).

Step 3. Since \( p = p_1 \) in \( B^\varepsilon(\xi^*, \eta^*) \setminus \Omega \) for small \( \varepsilon > 0 \), and \( p_1 \) is a smooth function in \( \mathbb{R}^2 \), we have
\[
\varphi_r(\xi, \eta) \rightarrow 0 \quad \text{as } (\xi, \eta) \rightarrow (\xi^*, \eta^*), \quad (\xi, \eta) \in B^\varepsilon(\xi^*, \eta^*) \setminus \Omega.
\]
Then assertion (iii) follows from (ii).

Step 4. We prove (iv) by contradiction. We choose two different sequence of points converging to \( P_1 \), and show that the limit \( \varphi_x \) along the two sequences are different, which reaches to a contradiction.

We first choose a sequence close to \( \Gamma_{\text{sonic}} \). Let \( \{ y_m \}_{m=1}^\infty \) be a sequence such that \( y_m \in (0, \pi - \theta_1) \) and \( \lim_{m \rightarrow \infty} y_m = 0 \). By (5.10) there exists \( x_m \in (0, 1/m) \) such that
\[
|\varphi_x(x_m, y_m) - r_1| < \frac{1}{m}.
\]
Thus we have \( (x_m, y_m) \in \Omega \), \( \lim_{m \rightarrow \infty}(x_m, y_m) = (0, 0) \), and
\[
\lim_{m \rightarrow \infty} \varphi_x(x_m, y_m) = r_1, \quad \lim_{m \rightarrow \infty} \varphi_y(x_m, y_m) = 0.
\]
We construct the second sequence close to $\Gamma_{\text{shock}}$. Suppose that the limit

$$\lim_{(\xi, \eta) \to P_1((\xi, \eta) \in \Omega)} D_r \varphi$$

exists. We have

$$\lim_{x \to 0} \varphi(x, f(x)) = \varphi(0, 0) = 0.$$  

From Lemma 5.2, we get

$$\lim_{x \to 0} \varphi_y(x, f(x)) = 0.$$  

When we rewrite the Rankine-Hugoniot conditions (2.8)-(2.9) for $\varphi$ in $(x, y)$-coordinates as

$$\hat{\beta}_1 \varphi_x + \hat{\beta}_2 \varphi_y = 0,$$

it is easy to see that there exists $\lambda_0 > 0$ such that $\hat{\beta}_1 > \lambda_0$, $|\hat{\beta}_2| \leq \frac{1}{\lambda_0}$ on $\Gamma_{\text{shock}} \cap \partial \Omega_x$. Thus, we find

$$|\varphi_x(x, f(x))| \leq K|\varphi_y(x, f(x))|$$

for some $K > 0$, which implies that $\lim_{x \to 0} \varphi_x(x, f(x)) = 0$. Let $H(x) = \varphi(x, f(x) + \frac{\varphi}{2} x)$. Then there exists $\{x_k\}_{k=1}^{\infty}$, $x_k \in (0, \varepsilon_0)$ such that

$$\lim_{k \to \infty} x_k = \lim_{k \to \infty} H'(x_k) = 0.$$  

Moreover, since

$$H'(x) = \varphi_x(x, h(x)) + \varphi_h(x, h(x)) h'(x)$$

with $h(x) = f(x) + \frac{\varphi}{2} x$, $|h'| \leq K$, we conclude that

$$\lim_{k \to \infty} \varphi_x(x_k, h(x_k)) = 0, \quad (5.11)$$

where $(x_k, h(x_k)) \in \Omega_x$. It means that we find an sequence $(x_k, h(x_k))$, along which the limit of $\varphi_x$ is 0. Combining with (5), we prove that $\varphi_x$ does not have a limit at $P_1$ from $\Omega$, which implies assertion (iv). This completes the proof of Theorem 5.4. \qed

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