Hardy spaces associated with One-dimensional Dunkl transform for $\frac{2\lambda}{2\lambda+1} < p \leq 1$ *†

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Abstract

This paper mainly contains two parts. In the first part, we will characterize the Homogeneous Hardy spaces on the real line by a kernel with a compact support for $\frac{1}{1+\gamma} < p \leq 1$ where $0 < \gamma \leq 1$.

In the second part of this paper, we will study the Hardy spaces associated with One-Dimensional Dunkl transform. The usual analytic function is replaced by the $\lambda$-analytic function which is based upon the $\lambda$-Cauchy-Riemann equations:

$$Du - \partial_y v = 0, \partial_y u - D_x v = 0,$$

where $D_x$ is the Dunkl operator:

$$D_x f(x) = f'(x) + \frac{\lambda}{x} [f(x) - f(-x)].$$

The real characterization of the Complex-Hardy Spaces $H^p_\lambda(\mathbb{R}^2)$ will be obtained for $p > \frac{2\lambda}{2\lambda+1}$. We will also prove that the Real Hardy spaces $H^p_\lambda(\mathbb{R})$ is Homogeneous Hardy spaces for $\frac{1}{1+\gamma} < p \leq 1$ where $\gamma_\lambda = \frac{1}{2(2\lambda+1)}$ ($\lambda > 0$) from which we could obtain the real-variable method of $H^p_\lambda(\mathbb{R})$. These results extend the results about the Hankel transform of Muckenhoupt and Stein in [23] to a general case and contain a number of further results.

2000 MS Classification:

Key Words and Phrases: Hardy spaces, Dunkl transform, Dunkl setting, Kernel, Homogeneous Hardy spaces

0.1 Introduction

In 1965, Muckenhoupt and Stein studied the Hardy spaces associated with the Hankel transform in [23]. Their starting point is the generalized Cauchy-Riemann equations:

$$u_x - v_y = 0, \quad u_y + v_x + \frac{2\lambda}{x} v = 0$$

for functions $u(x,y), v(x,y)$ on the domain $\{(x,y) : x > 0, y > 0\}$. And they introduced a notion of conjugacy associated with the Bessel operators $\Delta_{\lambda}$, $\lambda > 0$, defined by

$$\Delta_{\lambda} f(x) = -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x} f(x), \quad x > 0.$$

They developed in this setting a theory parallel to the classical case associated to the Euclidean Laplacian. In [23], definitions of Poisson kernels, harmonic functions, conjugate functions and fractional integrals associated with $\Delta_{\lambda}$ are given. Results parallel to the classical case about $L^p((0,\infty), x^{2\lambda} dx)$-boundedness, $1 \leq p < \infty$, for these operators were obtained. In sight of the whole half-plane $\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y > 0\}$, the study in [23] is restricted to the case when $u$ is even in $x$ and $v$ is odd in $x$, and the nonsymmetry of $u$ and $v$ lead to some ambiguous treatments in any further study. And very little progress has been made on the real characterization and the real-variable method in [23] on the upper half plane for the case $p < 1$.

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The "Laplace Equation" associated with the Dunkl setting is given by:

\[ \lambda \text{translation of usual Fourier transform. We assume } \lambda > 0 \text{ is given by:} \]

\[ \lambda \text{Poisson integral, and the associated maximal functions are studied in [20]. Theory of the real characterization of } H^\lambda_p(\mathbb{R}^2_+) \text{ and the real-variable method of } H^\lambda_p(\mathbb{R}) \text{ are still unknown in [20]. By the theory of Uchiyama's result in [26], } H^\lambda_p(\mathbb{R}) \text{ is Homogeneous Hardy spaces for } p_1 < p \leq 1 \text{ (for some } p_1 \text{ close to 1) in [17]. In this paper, we will give a real characterization of the } H^\lambda_p(\mathbb{R}^2_+) \text{ for the range of } p > \frac{2\lambda}{2\lambda + 1}, \text{ and we also prove that } H^\lambda_p(\mathbb{R}) \text{ is Homogeneous Hardy spaces for the range of } 1 \geq p > \frac{1}{1 + \gamma_\lambda}, \text{ where } \gamma_\lambda = \frac{1}{2\lambda + 1}. \text{ Thus the real-variable method of } H^\lambda_p(\mathbb{R}) \text{ could be obtained by the properties of Homogeneous Hardy spaces. These results extend the results in [19] and [20].}

For } 0 < p < \infty, \text{ } L^p_\Lambda(\mathbb{R}) \text{ is the set of measurable functions satisfying } \|f\|_{L^p_\Lambda} = \left( c_\lambda \int_\mathbb{R} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} < \infty, c_\lambda^{-1} = 2^{\lambda+1/2} \Gamma(\lambda+1/2), \text{ and } p = \infty \text{ is the usual } L^\infty(\mathbb{R}) \text{ space. For } \lambda \geq 0, \text{ the Dunkl operator on the line is:}

\[ D_x f(x) = f'(x) + \frac{\lambda}{x} [f(x) - f(-x)] \]

involving a reflection part. The associated Fourier transform for the Dunkl setting for } f \in L^1_\Lambda(\mathbb{R}) \text{ is given by:}

\[ (\mathcal{F}_\lambda f)(\xi) = c_\lambda \int_\mathbb{R} f(x) E_\lambda(-ix\xi)|x|^{2\lambda} dx, \quad \xi \in \mathbb{R}, \ f \in L^1_\Lambda(\mathbb{R}). \] (2)

\[ E_\lambda(-ix\xi) \text{ is the Dunkl kernel} \]

\[ E_\lambda(iz) = j_{\lambda-1/2}(z) + \frac{iz}{2\lambda + 1} j_{\lambda+1/2}(z), \quad z \in \mathbb{C} \]

where } j_\alpha(z) \text{ is the normalized Bessel function:

\[ j_\alpha(z) = 2^{\alpha} \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} (-1)^n (z/2)^{2n} \]

\[ n! \Gamma(n + \alpha + 1) \]

Since } j_{\lambda-1/2}(z) = \cos z, j_{\lambda+1/2}(z) = z^{-1} \sin z \text{, it follows that } E_0(iz) = e^{iz}, \text{ and } \mathcal{F}_0 \text{ agrees with the usual Fourier transform. We assume } \lambda > 0 \text{ in what follows. And the associated } \lambda\text{-translation in Dunkl setting is}

\[ \tau_y f(x) = c_\lambda \int_\mathbb{R} (\mathcal{F}_\lambda f)(\xi) E(iy\xi) |\xi|^{2\lambda} d\xi, \quad x, y \in \mathbb{R}. \] (3)

The } \lambda\text{-convolution } (f \ast_{\lambda} g)(x) \text{ of two appropriate functions } f \text{ and } g \text{ on } \mathbb{R} \text{ associated to the } \lambda\text{-translation } \tau_t \text{ is defined by:

\[ (f \ast_{\lambda} g)(x) = c_\lambda \int_\mathbb{R} f(t) \tau_x g(-t)|t|^{2\lambda} dt. \]

The "Laplace Equation" associated with the Dunkl setting is given by:

\[ (\Delta_\lambda u)(x, y) = (D_x^2 + \partial_y^2) u(x, y) = (\partial_x^2 + \partial_y^2) u + \frac{\lambda}{x} \partial_x u - \frac{\lambda}{x^2} (u(x, y) - u(-x, y)). \]

A } C^2 \text{ function } u(x, y) \text{ satisfying } \Delta_\lambda u = 0 \text{ is } \lambda\text{-harmonic. When } u \text{ and } v \text{ are } \lambda\text{-harmonic functions satisfying } \lambda\text{-Cauchy-Riemann equations:

\[ \begin{cases} D_x u - \partial_y v = 0, \\ \partial_y u + D_x v = 0 \end{cases} \] (4)

the function } F(z) = F(x, y) - iu(x, y) + iv(x, y) \text{ (} z = x + iy) \text{ is a } \lambda\text{-analytic function. We define the Complex-Hardy spaces } H^\lambda_\Gamma(\mathbb{R}^2_+) \text{ to be the set of } \lambda\text{-analytic functions } F = u + iv \text{ on } \mathbb{R}^2_+ \text{ satisfying:

\[ \|F\|_{H^\lambda_\Gamma(\mathbb{R}^2_+)} = \sup_{y > 0} \left\{ c_\lambda \int_\mathbb{R} |F(x + iy)|^p |x|^{2\lambda} dx \right\}^{1/p} < \infty. \]
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We use the symbol $D^+$ and $C^+$ to denote the Disk $D^+ = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1, y > 0\}$ and half plane $C^+ = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. In [3], Hardy spaces associated with Bessel operator is introduced for the case $p = 1$. In [5] the characterization of $H^1_A(C^+(\mathbb{C}))$ of maximal functions and atomic decomposition could be obtained by the theory in [26]. In [19], the Complex-Hardy spaces associated with the Dunkl setting on the Disk $H^1_A(D^+)$ have been studied for the range of $\frac{2\lambda}{2\lambda+1} < p \leq 1$. In [22] the Homogeneous Hardy spaces could be characterized by atoms for $\frac{2\lambda}{2\lambda+1} < p \leq 1$. In [15] the real-variable theory of Homogeneous Hardy spaces is studied by the way of Littlewood–Paley function for $p \in (\omega/\omega + \eta), 1]$. In [1], the Real-Hardy spaces $H^1$ in high dimensions have been studied. In [2], the Complex-Hardy spaces in the rational Dunkl setting $H^1$ in high dimensions have been studied. The following is the main structure of this paper:

**b. Summary of Section 1.**

In Section 1, we will characterize the Homogeneous Hardy spaces by a kernel. The theory of $H^p_{\rho,\gamma}(\mathbb{R})$ is studied when $\frac{1}{1+\gamma} < p \leq 1$ with $0 < \gamma < 1$ by [22]. However, we will use a way different to [22] to characterize the Homogeneous Hardy spaces $H^p_{\rho,\gamma}(\mathbb{R})$ when $0 < p \leq 1$, with $\beta > p^{-1} - 1$ in Theorem 1.28. For any $f \in A^{p,\gamma}(\mathbb{R})$ and $n \geq [p^{-1} - 1]$, we could obtain

$$A^{p,\gamma}(\mathbb{R}) = H^p_{\rho,\gamma}(\mathbb{R}) = H^p_{\rho,\beta}(\mathbb{R}), \text{ for } \beta_1, \beta_2 > p^{-1} - 1$$

$$\|f\|_{A^{p,\gamma}(\mathbb{R})} \sim \|f\|_{H^p_{\rho,\beta}(\mathbb{R})} \sim \|f\|_{H^p_{\rho,\gamma}(\mathbb{R})}.$$

Kernel is introduced in [26] to characterize the Homogeneous Hardy spaces. Let $X$ be a topological space, $\rho$ a quasi-distance and $\mu$ a Borel doubling measure on $X$, then Hardy spaces $H^p(X)$ associated to this type $(X, \rho, \mu)$ is investigated in a series of studies. $H^p(X)$ becomes trivial when $p$ is near to 1. Let

$$F(r, x, f) = \int_X K(r, x, y)f(y)d\mu(y)/r, \quad f^\gamma(x) = \sup_{r > 0}|F(r, x, f)|$$

where $K(r, x, y)$ is a kind of nonnegative function on $X \times X$ enjoying several properties. Uchiyama showed that for $1 - p > 0$ small enough, the maximal function $f^\gamma(x)$ can be used to characterize the atomic Hardy spaces $H^p(X)$.

**Theorem 0.1.** [26] $\exists p_1$ with $1 \geq p_1$, such that the following inequality holds:

$$\|f^\gamma\|_{L^p(X, \mu)} \leq c_1 \|f^\gamma\|_{L^p(X, \mu)} \text{ for } p > p_1$$

$c_1$ is a constant depending only on $X$ and $p$, $1 \geq \gamma > 0$.

Notice that the topological space $X$ of Real-Hardy spaces $H^p(X)$ is $\mathbb{R}$. Thus we will extend Uchiyama’s result in [26] from $p_1 < p \leq 1$ (for some $p_1$ close to 1) to the range $\frac{1}{1+\gamma} < p \leq 1$ ($0 < \gamma < 1$) when the topological space $X$ is $\mathbb{R}$ with a quasi-distance $\rho$. Then we will obtain Theorem 1.35: the maximal function $f^\gamma(x)$ can be used to characterize the atomic Hardy spaces $H^p_{\rho}(X)$: for $f \in S^\gamma(\mathbb{R}, d, \mu)$, $\frac{1}{1+\gamma} < p \leq 1$, $0 < \gamma < 1$.

$$\|f^\gamma\|_{L^p(\mathbb{R}, \mu)} \sim \|f^\gamma\|_{L^p(\mathbb{R}, \mu)} \sim \|f^\gamma\|_{L^p(\mathbb{R}, \mu)} \sim \|f^\gamma\|_{L^p(\mathbb{R}, \mu)}.$$

where the kernels satisfy Definition 1.6. (We do not need the kernels $K_1(r, x, y)$ or $K_2(r, x, y)$ to be continuous on $r$ variable.)

**c. Summary of Section 2.** Section 2 mainly deals with the real characterization of $H^p_{\lambda}(\mathbb{R}^d)$ and the real-variable method of $H^p_{\lambda}(\mathbb{R})$. One of our results is that we will prove Theorem 2.8 in §2.1. We will use another way different from Burkholder–Gundy–Silverstein theorem in [4].

Then we will characterize the Real-Hardy spaces $H^p_{\lambda}(\mathbb{R})$ by Definition 2.10 and Theorem 2.22. The relation of Complex-Hardy spaces $H^p(\mathbb{R}^d)$, Real-Hardy spaces $H^p_{\lambda}(\mathbb{R})$ and Homogeneous Hardy spaces is characterized by Definition 2.10, Theorem 2.22 and Proposition 2.23.

In §2.2, the $\lambda$-Poisson kernel is introduced. We will prove that the Real-Hardy spaces $H^p_{\lambda}(\mathbb{R})$ is a kind of Homogeneous Hardy spaces for $\frac{2\lambda}{2\lambda+1} < p \leq 1$ in Theorem 2.22. Thus the $H^p_{\lambda}(\mathbb{R})$ can be characterized by the maximal functions in Homogeneous Hardy spaces, and the definition of $H^p_{\lambda}(\mathbb{R})$ can be evolved from the properties of $\lambda$-analytic functions.

**Main Result** The main result of this paper is Theorem 2.8 and Theorem 2.22. By Theorem 2.8, we could know that $H^p_{\lambda}(\mathbb{R}^d)$ can be characterized by $H^p_{\lambda}(\mathbb{R})$ for $\frac{2\lambda}{2\lambda+1} < p \leq 1$. By Theorem 2.22,
$H^p_\lambda(\mathbb{R})$ is Homogeneous Hardy spaces for $\frac{1}{1+\gamma}\lambda < p \leq 1$. The Homogeneous Hardy spaces have many good properties including atomic decomposition.

**e. Notation.** Let $S(\mathbb{R}, dx)$ the space of $C^\infty$ functions on $\mathbb{R}$ with the Euclidean distance rapidly decreasing together with their derivatives(Classic Schwartz Class), $L_{\lambda, \text{loc}}(\mathbb{R})$ the set of locally integrable functions on $\mathbb{R}$ associated with the measure $|x|^{2\lambda} dx$. $\mathcal{F}_\lambda$ is the Dunkl transform and $\mathcal{F}$ the Fourier transform.

We use $A \lesssim B$ to denote the estimate $|A| \leq CB$ for some absolute universal constant $C > 0$, which may vary from line to line, $A \gtrsim B$ to denote the estimate $|A| \geq CB$ for some absolute universal constant $C > 0$, $A \sim B$ to denote the estimate $|A| \leq C_1B$, $|A| \geq C_2B$ for some absolute universal constant $C_1, C_2$.

We use $B(x_0, r_0)$ or $B_\lambda(x_0, r_0)$ to denote the ball in the homogenous space in the Dunkl setting: $B(x_0, r_0) = B_\lambda(x_0, r_0) = \{y : d_\lambda(y, x_0) < r_0\}$, $d_\lambda(x, y)$ to denote the distance in the homogeneous space associated with Dunkl setting: $d_\lambda(x, y) = |(2\lambda + 1) \int_y^x |t|^{2\lambda} dt|$, $p_0$ to denote $p_0 = \frac{2\lambda}{2\lambda + 1}$, $\Omega$ to denote a domain and $\partial \Omega$ to denote the boundary of $\Omega$, $\gamma_\lambda$ to denote $\gamma_\lambda = \frac{2\lambda}{2\lambda + 1}$, $d_\mu(x, y)$ to denote the distance in the homogeneous space associated with a positive Radon measure $\mu$ on the real line satisfying $\mu(x, y) = \int_y^x d\mu(t)$ and $d_\mu(x, y) = |\mu(x, y)|$, $B_\mu(x_0, r_0)$ to denote the ball in the homogeneous space: $B_\mu(x_0, r_0) = \{y : d_\mu(y, x_0) < r_0\}$. For a measurable set $E \subseteq \mathbb{R}$, we use $E^c$ to denote the set $E^c = \{x \in \mathbb{R} : x \notin E\}$. For two sets $A$ and $B$, $A \setminus B$ means that $A \cap B^c$. Throughout this paper, we assume $\lambda > 0$ and $0 < \gamma \leq 1$. In section 2, $\psi_t(x)$ denotes $\psi_t(x) = \left(\frac{1}{t}\right)^{2\lambda+1} \psi\left(\frac{x}{t}\right)$.

## 1 Homogeneous Hardy spaces on $\mathbb{R}$ with a kernel

In this section 1, we will characterize the Homogeneous Hardy spaces on the real line by a kernel. We will extend the Uchiyama’s result in [26] when the topological space $X$ is $\mathbb{R}$ with a quasi-distance $\rho$.

**Definition 1.1 ($d_\mu(x, y)$).** $d_\mu(x, y)$ is a quasi-distance on the real line $\mathbb{R}$ endowed with a positive Radon measure $\mu$, $\mu(x, y) = \int_y^x d\mu(t)$, $d_\mu(x, y) = |\mu(x, y)|$, satisfying the following conditions (for some fixed constant $A > 0$):

(i) $d_\mu(x, y) = d_\mu(y, x)$, for any $x, y \in \mathbb{R}$;

(ii) $d_\mu(x, y) > 0$, if $x \neq y$;

(iii) $d_\mu(x, z) \leq A(d_\mu(x, y) + d_\mu(y, z))$, for any $x, y, z \in \mathbb{R}$

(iv) $A^{-1}r \leq \mu(B_\mu(x, r)) \leq r$, for any $r > 0$.

(v) $B_\mu(x, r) = \{y \in \mathbb{R} : d_\mu(x, y) < r\}$ form a basis of open neighbourhoods of the point $x$.

(vi) $f(u) = \mu(x, u)$ is a continuous bijection on $\mathbb{R}$ for any fixed $x \in \mathbb{R}$.

**Definition 1.2 ($S(\mathbb{R}, d_\mu x)$).** The derivative associated with the quasi-distance $d_\mu(x, y)$ is defined as follows:

$$\frac{d}{d_\mu x} \phi(x) = \lim_{\varepsilon \to 0, d_\mu(x, y) < \varepsilon} \frac{\phi(y) - \phi(x)}{\mu(y, x)}.$$

Then the Schwartz Class $S$ associated with the quasi-distance $d_\mu(x, y)$ could be defined as:

$$\|\phi\|_{(\alpha, \beta), \mu} = \sup_{x \in \mathbb{R}} \left| (d_\mu(x, 0))^{\alpha} \left( \frac{d}{d_\mu x} \right)^{\beta} \phi(x) \right| < \infty$$

for natural numbers $\alpha$ and $\beta$. This kind of Schwartz Class is denoted as $S(\mathbb{R}, d_\mu x)$, $\phi(u) \in C(\mathbb{R}, dx)$ means $\phi(u) \to \phi(u_0)$ as $u \to u_0$ in Euclid space, $\phi(u) \in C(\mathbb{R}, d_\mu x)$ means $\phi(u) \to \phi(u_0)$ as $d_\mu(u, u_0) \to 0$.

**Proposition 1.3.** For any $\phi \in S(\mathbb{R}, d_\mu x)$ with $\text{supp} \phi(u) \subseteq B_\mu(x_0, r_0)$, there exists $\psi(t) \in S(\mathbb{R}, dt)$ with $\text{supp} \psi(t) \subseteq [-1, 1]$ satisfying $\psi\left(\frac{d(x, u)}{r_0}\right) = \phi(u)$ for $u \in B_\mu(x_0, r_0)$ in $S(\mathbb{R}, d_\mu x)$ space.

**Proof.** Let $f(u) = \frac{d(x, u)}{r_0}$ for fixed $x_0 \in \mathbb{R}$ and $r_0 > 0$. Thus $f(u)$ is a bijection and has an inverse function. Let $g(x)$ to be the inverse function of $f(x)$: $g \circ f(u) = u$. Thus for any $\phi \in S(\mathbb{R}, d_\mu x)$,
we could write $\phi$ as:

$$\phi(u) = \phi(g \circ f) = \phi(g \left( \frac{\mu(x_0, u)}{r_0} \right)).$$

We use $\psi$ to denote $\psi = \phi \circ g$ and $\psi^{(n)}(t)$ to denote $\psi^{(n)}(t) = \frac{d^n}{dt^n} \psi(t)$. Then we could deduce that:

$$\frac{d}{d\mu x} \phi(x) = \lim_{\epsilon \to 0, d\mu(x,y) < \epsilon} \frac{\phi(y) - \phi(x)}{\mu(y, x)}.$$

$$= \lim_{\epsilon \to 0, d\mu(x,y) < \epsilon} \frac{1}{r_0} \psi \left( \frac{\mu(x_0, y)}{r_0} \right) - \psi \left( \frac{\mu(x_0, x)}{r_0} \right).$$

$$= -\frac{1}{r_0} \psi^{(1)} \left( \frac{\mu(x_0, x)}{r_0} \right).$$

Thus

$$\left( \frac{d}{d\mu x} \right)^n \phi(x) = \left( -\frac{1}{r_0} \right)^n \psi^{(n)} \left( \frac{\mu(x_0, x)}{r_0} \right).$$

Notice that $\mu$ is a bijection on $\mathbb{R}$, together with the fact $\psi \in S(\mathbb{R}, d\mu x)$, we could deduce that $\psi \in S(\mathbb{R}, dx)$. This proves the proposition.

In the same way as Proposition 1.3, we could obtain:

**Proposition 1.4.** For any $\phi \in C(\mathbb{R}, d\mu x)$, there exists $\psi \in C(\mathbb{R}, dx)$, satisfying $\psi \left( \frac{\mu(x_0, u)}{r_0} \right) = \phi(u)$ in $C(\mathbb{R}, d\mu x)$ space.

By Proposition 1.4, together with the fact that $S(\mathbb{R}, dx)$ is dense in $C_0(\mathbb{R}, dx)$, we could know that $S(\mathbb{R}, d\mu x)$ is dense in $C_0(\mathbb{R}, d\mu x)$.

**Definition 1.5 (S*(\mathbb{R}, d\mu x)).** A tempered distribution is a linear functional on $S(\mathbb{R}, d\mu x)$ that is continuous in the topology on $S(\mathbb{R}, d\mu x)$ induced by this family of seminorms. We shall refer to tempered distributions simply as distributions. Similar to the classical definition, we say a distribution $f$ is bounded if

$$\left| \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) \right| \in L^\infty(\mathbb{R}, \mu)$$

whenever $\phi \in S(\mathbb{R}, d\mu x)$. We use $S'(\mathbb{R}, d\mu x)$ to denote the bounded distributions.

Then we will define the kernels $K_1(r, x, y)$ and $K_2(r, x, y)$ as follows:

**Definition 1.6 (kernel $K_1(r, x, y)$).** For constant $A > 0$ and constant $1 \geq \gamma > 0$, let $K_1(r, x, y)$ be a nonnegative continuous function defined on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ satisfying the following conditions:

(i) $K_1(r, x, x) > 1/A$, for $r > 0, x \in \mathbb{R}$;

(ii) $0 \leq K_1(r, x, t) \leq 1$, for $r > 0, x, t \in \mathbb{R}$;

(iii) For $r > 0, x, t, z \in \mathbb{R}$

$$|K_1(r, x, t) - K_1(r, x, z)| \leq \left( \frac{d\mu(t, z)}{r} \right)^\gamma.$$

(iv) $K_1(r, x, y) = 0$, if $d\mu(x, y) > r$.

(v) $K_1(r, y, x) = K_1(r, y, x)$.

**Definition 1.7 (kernel $K_2(r, x, y)$).** For constants $C_i > 0$, $i = 1, 2, 3, 4$ and constant $1 \geq \gamma > 0$, let $K_2(r, x, y)$ be a nonnegative continuous function defined on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ satisfying the following conditions:

(i) $K_2(r, x, x) > C_1$, for $r > 0, x \in \mathbb{R}$;

(ii) $0 \leq K_2(r, x, t) \leq C_2 \left( 1 + \frac{d\mu(t, x)}{r} \right)^{-\gamma - 1}$, for $r > 0, x, t \in \mathbb{R}$;

(iii) For $r > 0, x, t, z \in \mathbb{R}$, if $\frac{d\mu(t, z)}{r} \leq C_3 \min \{1 + \frac{d\mu(t, x)}{r}, 1 + \frac{d\mu(x, z)}{r}\}$, then

$$|K_2(r, x, t) - K_2(r, x, z)| \leq C_4 \left( \frac{d\mu(t, z)}{r} \right)^\gamma \left( 1 + \frac{d\mu(x, t)}{r} \right)^{-2\gamma - 1}.$$

(iv) $K_2(r, y, x) = K_2(r, y, x)$.  

Definition 1.8 (maximal functions). For $f \in L^1(\mathbb{R}, \mu)$, $0 < \gamma \leq 1$, let

$$F_i(r, x, f) = \int_{\mathbb{R}} K_i(r, x, y) f(y) d\mu(y)/r, \quad f^*_i(x) = \sup_{r > 0} |F_i(r, x, f)|, \quad f^*_i(x) = \sup_{r > 0, d(x, s) < r} |F_i(r, s, f)|$$

for $i = 1, 2$. We use $L(f, 0)$ and $L(f, \alpha)$ to denote as following:

$$L(f, 0) = \sup_{x \in \mathbb{R}, r > 0} \inf_{c \in \mathbb{R}} \int_{B_n(x, r)} |f(y) - c| d\mu(y)/r,$$

$$L(f, \alpha) = \sup_{x, y \in \mathbb{R}, x \neq y} |f(x) - f(y)|/d_\mu(x, y)^\alpha, \quad \text{for } 1 \geq \alpha > 0.$$

We use $f^*_i(x)$ to denote as:

$$f^*_i(x) = \sup_{\phi, r} \left\{ \left( \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) \right)/r : r > 0, \sup \phi \subset B_\mu(x, r), L(\phi, \gamma) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\}.$$ (5)

The Hardy-Littlewood maximal operator $M_\mu$ is defined as:

$$M_\mu f(x) = \sup_{r > 0} \frac{1}{r} \int_{B_\mu(x, r)} |f(y)| d\mu(y).$$

Then $M_\mu$ is weak-(1, 1) bounded and $(p, p)$ bounded for $p > 1$.

Definition 1.9 (maximal functions). For $f \in S'(\mathbb{R}, d_\mu x)$, $0 < \gamma \leq 1$, we use $f^*_\gamma(x)$ to denote as:

$$f^*_\gamma(x) = \sup_{\phi, r} \left\{ \left( \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) \right)/r : r > 0, \sup \phi \subset B_\mu(x, r), L(\phi, \gamma) \leq r^{-\gamma}, \phi \in S(\mathbb{R}, d_\mu x), \|\phi\|_{L^\infty} \leq 1 \right\}.$$  

From the Definition 1.5, we could deduce that the above Definition 1.8 and Definition 1.9 associated with the maximal functions are meaningful.

Definition 1.10 ($\phi^{(n)}(x)$, $H^\alpha(\phi)$, $[\phi]_\beta$). For $\phi \in C(\mathbb{R}, dx)$, $n \in \mathbb{N}$, $1 \geq \alpha > 0$ and $\beta > 0$, we use $\{\beta\}$, $[\beta]$, $H^\alpha(\phi)$ and $\phi^{(n)}(x)$ to denote as:

$$\{\beta\} = \beta - \lfloor\beta\rfloor; \quad [\beta] = \max \{n : n \in \mathbb{Z}; n \leq \beta\};$$

$$H^\alpha(\phi) = \sup_{x, y \in \mathbb{R}, x \neq y} |\phi(x) - \phi(y)|/|x - y|^{\alpha};$$

$$\phi^{(n)}(x) = d^\alpha/dx^n \phi(x); \quad [\phi]_\beta = H^{\{\beta\}}(\phi^{([\beta])}).$$

Thus we could see that if $0 < \beta \leq 1$ then $[\phi]_\beta = H^\beta(\phi)$.

Thus it is clear that the following Propositions 1.11 and 1.12 hold:

Proposition 1.11. For $\phi \in C(\mathbb{R}, dx)$ satisfying $H^\alpha(\phi) \leq 1$, $|\phi| \leq 1$ ($1 \geq \alpha > 0$, $\beta > 0$), there exists $\phi_\tau(x) \in S(\mathbb{R}, dx)$ satisfying the following:

(i) $\lim_{\tau \to 0} \|\phi_\tau(x) - \phi(x)\|_{\infty} = 0$,

(ii) $\|\phi_\tau(x)\|_{\infty} \leq 1$,

(iii) $H^\alpha(\phi_\tau) \leq C$.

Proposition 1.12. $\beta \geq \beta_1 \geq 0$. $n \in \mathbb{Z}$, $n \leq \beta$. For any $\phi \in S(\mathbb{R}, dx)$, if $\|\phi(x)\|_{\infty} \leq 1$, $[\phi]_\beta \leq 1$, then the following holds:

$$\|\phi^{(n)}(x)\|_{\infty} \leq C, \quad [\phi]_{\beta_1} \leq C,$$

where $C$ is a constant independent on $\phi$.

Then we will prove the following Proposition 1.13.
Proposition 1.13. For $f \in L^1(\mathbb{R}, \mu)$, we could have
\[ f_{\gamma \nu}^\gamma(x) \lesssim_{\lambda} f_{\gamma}^\gamma(x) \quad i = 1, 2. \]
Then if $f_{\gamma}^\gamma(x) \in L^p(\mathbb{R}, \mu)$ for $p > 0$, we could have
\[ \| f_{\gamma \nu}^\gamma \|_{L^p(\mathbb{R}, \mu)} \lesssim_{\lambda} \| f_{\gamma}^\gamma \|_{L^p(\mathbb{R}, \mu)}. \]

Proof. When $i = 1$, it is clear to see that for fixed $r$ and $s$ the following hold:
\[ |K_1(r, s, y)| \lesssim 1 \]
\[ L K_1(r, s, y), \gamma \lesssim (r)^{-\gamma} \]
\[ \text{supp} K_1(r, s, y) \subseteq B_{\mu}(x, 2Ar) \]
then we could have
\[ f_{\gamma \nu}^\gamma(x) \lesssim f_{\gamma}^\gamma(x). \]

When $i = 2$, fix a positive $\phi(t) \in S(\mathbb{R}, dt)$ so that $\text{supp} \phi(t) \subseteq (-1, 1)$, and $\phi(t) = 1$ for $t \in (-1/2, 1/2)$. Let the functions $\psi_{k,x}(t)$ be defined as follows:
\[ \psi_{0,x}(t) = \phi\left(\frac{\mu(x,t)}{r}\right), \psi_{k,x}(t) = \phi\left(\frac{\mu(x,t)}{2^k r}\right) - \phi\left(\frac{\mu(x,t)}{2^k - 1 r}\right), \text{ for } k \geq 1. \]
Thus $\text{supp} \psi_{0,x}(t) \subseteq B_{\mu}(x, r)$ and $\text{supp} \psi_{k,x}(t) \subseteq B_{\mu}(x, 2^{k+1}r) \setminus B_{\mu}(x, 2^{k-2}r)$ for $k \geq 1$, $\psi_{k,x}(t) \in S(\mathbb{R}, d_{\mu} t)$ for $k \geq 0$. It is clear that
\[ \sum_{k=0}^{\infty} \psi_{k,x}(t) = 1. \]
Then we could conclude:
\[ f_{2\gamma \nu}^\gamma(x) = \sup_{r > 0, d_{\mu}(s,x) \leq r} \left| \int_{\mathbb{R}} K_2(r, s, y) \sum_{k=0}^{\infty} \psi_{k,x}(y) f(y) d\mu(y) \right|/r \]
\[ \leq \sum_{k=0}^{\infty} \sup_{r > 0, d_{\mu}(s,x) \leq r} \left| \int_{\mathbb{R}} K_2(r, s, y) \psi_{k,x}(y) f(y) d\mu(y) \right|/r. \]
It is clear that the kernel $K_2(r, s, y)$ satisfies:
\[ |(1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y)| \lesssim 1 \]
\[ L ( (1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y), \gamma \lesssim (2^{k} r)^{-\gamma} \]
\[ \text{supp}(1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y) \subseteq B_{\mu}(x, 2^{k+1}r) \setminus B_{\mu}(x, 2^{k-2}r) \text{ for } k \geq 1. \]
Then we could get
\[ f_{2\gamma \nu}^\gamma(x) = \sup_{r > 0, d_{\mu}(s,x) \leq r} \left| \int_{\mathbb{R}} K_2(r, s, y) f(y) d\mu(y) \right|/r \]
\[ \leq \sum_{k=0}^{\infty} \sup_{r > 0, d_{\mu}(s,x) \leq r} \left| \int_{\mathbb{R}} K_2(r, s, y) \psi_{k,x}(y) f(y) d\mu(y) \right|/r \]
\[ \lesssim \sum_{k=0}^{\infty} (2^{k})(1 + 2^{k})^{-1-\gamma} f_{\gamma}^\gamma(x) \]
\[ \lesssim_{\lambda} f_{\gamma}^\gamma(x). \]
This proves the proposition.\[\square\]
Proposition 1.14. For \( f \in L^1(\mathbb{R}, \mu), 1 \geq \gamma > 0, \infty > p > 0 \) we could obtain
\[
f_{\xi \gamma}(x) = f_\gamma^*(x) \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure.}
\]
Further more, if \( \int_{\mathbb{R}} |f_\gamma^*(x)|^p d\mu(x) \leq \infty \) or \( \int_{\mathbb{R}} |f_{\xi \gamma}(x)|^p d\mu(x) \leq \infty \), we could obtain
\[
\int_{\mathbb{R}} |f_\gamma^*(x)|^p d\mu(x) \sim \int_{\mathbb{R}} |f_{\xi \gamma}(x)|^p d\mu(x) < \infty.
\]

Proof. We will prove the following (6) first:
\[
f_{\xi \gamma}(x) = f_\gamma^*(x) \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure.} \tag{6}
\]

By the definition of \( f_{\xi \gamma}(x) \) and \( f_\gamma^*(x) \), it is clear that \( f_{\xi \gamma}(x) \leq f_\gamma^*(x) \). If \( \phi \) satisfies \( L(\phi, \gamma) \leq r^{-\gamma} \) and \( \text{supp} \phi \subset B_\mu(x, r) \), then \( \phi \) is a continuous function in \( \mu \) measure with compact support. Thus there exists sequence \( \{\psi_n\}_n \subset S(\mathbb{R}, d\mu x) \) with \( \lim_{n \to \infty} ||\phi_n(t) - \phi(t)||_\infty = 0, ||\psi_n(t) - \phi(t)||_\infty \neq 0 \). Denote \( \delta_n(x) \) as
\[
\delta_n(x) = \left| \int_{B_\mu(x, r)} f(y) (\phi(y) - \psi_n(y)) d\mu(y)/r \right|.
\]

Then we could conclude:
\[
\delta_n(x) \leq M_\mu f(x)||\psi_n(x) - \phi(x)||_\infty.
\]

We use \( i_n \) to denote as \( i_n = ||\psi_n(y) - \phi(y)||_\infty \), thus we could obtain that:
\[
\{x : \delta_n(x) > \alpha\} \subset \left\{ x : M_\mu f(x) > \frac{\alpha}{i_n} \right\}.
\]

Notice that \( M_\mu \) is weak-(1, 1) bounded, thus the following inequality holds for any \( \alpha > 0 \):
\[
||\{x : \delta_n(x) > \alpha\}||_\mu \leq \frac{1}{\alpha} ||f||_{L^1(\mathbb{R}, \mu)} ||\psi_n(y) - \phi(y)||_\infty.
\]

Thus
\[
\lim_{n \to +\infty} ||\{x : \delta_n(x) > \alpha\}||_\mu = 0.
\]

Then there exists a sequence \( \{n_j\} \subset \{n\} \) such that
\[
\int_{\mathbb{R}} f(y)\phi(y)d\mu(y)/r = \lim_{n_j \to \infty} \int_{\mathbb{R}} f(y)\psi_{n_j}(y)d\mu(y)/r, \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure}
\]
for \( f \in L^1(\mathbb{R}, \mu) \). Thus we could obtain:
\[
\int_{\mathbb{R}} f(y)\phi(y)d\mu(y)/r \leq f_{\xi \gamma}(x) \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure}
\]
for any \( \phi \) satisfies \( L(\phi, \gamma) \leq r^{-\gamma} \) and \( \text{supp} \phi \subset B_\mu(x, r) \). We could then deduce
\[
\sup_{\phi, r > 0} \left| \int_{\mathbb{R}} f(y)\phi(y)d\mu(y)/r \right| \leq f_{\xi \gamma}(x) \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure}.
\]

Thus
\[
f_{\xi \gamma}(x) = f_\gamma^*(x) \quad \text{a.e.} x \in \mathbb{R} \text{ in } \mu \text{ measure.}
\]

Let \( E \) denote a set defined as \( E = \{ x : f_{\xi \gamma}(x) = f_\gamma^*(x) \} \). Next we will prove that for any \( x_0 \in \mathbb{R} \), there is a point \( x_0 \in E \) such that
\[
f_{\xi \gamma}(x_0) \leq f_{\xi \gamma}(x_0).
\]

Notice that for \( x_0 \in \mathbb{R} \), there exist \( r_0 > 0 \) and \( \phi_0 \) satisfying: \( \text{supp} \phi_0 \subset B_\mu(x_0, r_0), \phi_0 \in S(\mathbb{R}, d\mu x), L(\phi_0, \gamma) \leq r_0^{-\gamma}, ||\phi_0||_{L^\infty} \leq 1 \). Then the following inequality could be obtained:
\[
\left| \frac{1}{r_0} \int_{r_0} f(y)\phi_0(y)d\mu(y) \right| \geq \frac{1}{2} f_\gamma^*(x_0).
\]
\[ |\mu(\mathbb{R}\backslash E)| = |\mu(E^c)| = 0 \text{ implies } E \text{ is dense in } \mathbb{R}. \] 
Then there exists a \( \overline{\tau}_0 \in E \) with \( d_\mu(x_0, \overline{\tau}_0) \leq \frac{r_0}{10} \).
Thus \( \text{supp } \phi_0 \subseteq B_\mu(\overline{\tau}_0, 4r_0) \) holds. Thus we could obtain
\[
\frac{1}{r_0} \int f(y) \phi_0(y) d\mu(y) \leq Cf_{\gamma, \chi}(\overline{\tau}_0),
\]
where \( C \) is a constant independent on \( f, \gamma \) and \( r_0 \). Then Formula (7) could be deduced. By Formula (7) we could deduce that:
\[
\int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty \Rightarrow \int_{\mathbb{R}} |f_{\gamma, \chi}^*(x)|^p d\mu(x) \sim \int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty. \quad (8)
\]
In the same way, we could conclude that
\[
\int_{\mathbb{R}} |f_{\gamma, \chi}^*(x)|^p d\mu(x) \sim \int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x). \quad (9)
\]
From Formula (6) we could deduce:
\[
\int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) = \int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty.
\]
The above Formula together with (8) (9) lead to
\[
\int_{\mathbb{R}} |f_{\gamma, \chi}^*(x)|^p d\mu(x) \sim \int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty
\]
holds if \( \int_E |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty \) or \( \int_{\mathbb{R}} |f_{\gamma, \chi}^*(x)|^p d\mu(x) < \infty \). This proves the proposition. \( \square \)

**Definition 1.15 (SS\( \beta \)).** We use \( SS\beta (\beta > 0) \) to denote as
\[
SS\beta = \left\{ \phi : \phi \in S(\mathbb{R}, d\mu), \text{supp } \phi \subseteq [-1, 1], ||\phi||_{L^\infty} \leq 1, ||\phi||_{\beta} \leq 1 \right\}.
\]

By Proposition 1.3 and Proposition 1.11, we could also define \( f_{\gamma, \chi}^*(1 \geq \gamma > 0) \) and \( f_{\gamma, \chi}^*(\beta > 0) \) for \( f \in S'(\mathbb{R}, d\mu) \) as following:
\[
f_{\gamma, \chi}^*(x) = \sup_{\psi, r > 0} \left\{ \int_{\mathbb{R}} f(y) \psi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \Bigg| r : r > 0, \psi(t) \in S(\mathbb{R}, d\mu), \text{supp } \psi(t) \subseteq [-1, 1], ||\psi||_{L^\infty} \leq 1, H^\gamma \psi \leq 1 \right\}. \tag{11}
\]
\[
f_{\chi, \beta}^*(x) = \sup_{\psi, r > 0} \left\{ \int_{\mathbb{R}} f(y) \psi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \Bigg| r : r > 0, \psi(t) \in SS\beta \right\}. \tag{12}
\]

**Definition 1.16 (\( M_{\phi, \beta} f(x) \)).** For \( f \in S'(\mathbb{R}, d\mu) \), \( M_{\phi, \beta} f(x) \) is defined as
\[
M_{\phi, \beta} f(x) = \sup_{r > 0} \left\{ \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \Bigg| r : r > 0, \phi(t) \in SS\beta \right\}.
\]
Thus it is easy to see that
\[
f_{\gamma, \chi}^*(x) \sim \sup_{\phi(t) \in SS\beta} M_{\phi, \beta} f(x). \tag{13}
\]

Let \( M_{\phi, \beta}^* f(x) \) be defined as
\[
M_{\phi, \beta}^* f(x) = \sup_{d_\mu(x, y) < r} \left\{ \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(y, u)}{r} \right) d\mu(u) \Bigg| r : r > 0, \phi(t) \in SS\beta \right\}. \tag{14}
\]

**Definition 1.17 (\( M_{\phi, \beta}^* f(x) \) and \( M_{\phi, \beta}^* f(x) \)).** Notice that \( \mu(y, u) = \mu(x, u) - \mu(x, y) \). For \( f \in S'(\mathbb{R}, d\mu) \), let \( s = \mu(x, y) \), \( M_{\phi, \beta}^* f(x) \) and \( M_{\phi, \beta}^* f(x) \) could be written as following:
\[
M_{\phi, \beta}^* f(x) = \sup_{|s| < r} \left\{ \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) d\mu(u) \Bigg| r : r > 0, \phi(t) \in SS\beta \right\}. \tag{15}
\]
The norm is defined as
\begin{equation}
M_{\mu \beta} f(x) = \sup_{s \in \mathbb{R}, r > 0} \left\{ \left| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) d\mu(u) \right| \right| f : r > 0, \phi(t) \in SS_{\beta} \right\}.
\end{equation}

**Definition 1.18** \((M_{\mu \beta} f(x))\). For \(f \in S'(\mathbb{R}, d_{\mu} x)\), \(M_{\mu \beta} f(x)\) is defined as:
\begin{equation}
M_{\mu \beta} f(x) = \sup_{s \in \mathbb{R}, r > 0} \left\{ \left| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(u) \right| \right| f : r > 0, \phi(t) \in SS_{\beta} \right\}
\end{equation}

Thus it is clear that
\begin{equation}
M_{\mu \beta} f(x) \lesssim M_{a \beta} f(x) \lesssim M_{\mu \beta} f(x).
\end{equation}

**Definition 1.19** \((H_{\mu \beta}^p)\) and \(\tilde{H}_{\mu \beta}^p\). \(\tilde{H}_{\mu \beta}^p(\mathbb{R})\) and \(H_{\mu \beta}^p(\mathbb{R})\) are defined as follows:
\begin{equation}
\tilde{H}_{\mu \beta}^p(\mathbb{R}) \triangleq \left\{ g \in L^1(\mathbb{R}, \mu) : g_{\mu \beta}^*(x) \in L^p(\mathbb{R}, \mu) \right\},
\end{equation}
\begin{equation}
H_{\mu \beta}^p(\mathbb{R}) \triangleq \left\{ g \in S'(\mathbb{R}, d_{\mu} x) : g_{\mu \beta}^*(x) \in L^p(\mathbb{R}, \mu) \right\}.
\end{equation}
The norm is defined as
\[
\|g\|_{\tilde{H}_{\mu \beta}^p(\mathbb{R})}^p = \int_{\mathbb{R}} |g_{\mu \beta}^*(x)|^p d\mu(x).
\]

When \(1 < p < \infty\), \(H_{\mu \beta}^p(\mathbb{R}) = L^p(\mathbb{R}, \mu)\), \(\tilde{H}_{\mu \beta}^p(\mathbb{R})\) is dense in \(L^p(\mathbb{R}, \mu)\).

**Proposition 1.20.** For fixed numbers \(a \geq b > 0\), \(F(x, r)\) is a function defined on \(\mathbb{R}^2_+\), its nontangential maximal function \(F^*_a(x)\) is defined as
\[
F^*_a(x) = \sup_{d_{\mu}(x, y) < ar} |F(y, r)|.\]

If \(F^*_a(x) \in L^1(\mathbb{R}, \mu)\) or \(F^*_b(x) \in L^1(\mathbb{R}, \mu)\), then we could have
\[
\int_{\mathbb{R}} \chi \{ x : F^*_a(x) > \alpha \} d\mu(x) \leq c \frac{a + b}{b} \int_{\mathbb{R}} \chi \{ x : F^*_b(x) > \alpha \} d\mu(x),
\]
c is a constant independent on \(F, a, b, \) and \(\alpha\).

**Proof.** First we could see that \(\{ x : F^*_a(x) > \alpha \}\) is an open set. It is clear that
\[
\{ x : F^*_a(x) > \alpha \} \subseteq \{ x : F^*_b(x) > \alpha \},
\]
when \(a \geq b > 0\). For any \(z\) with \(z \in \{ x : F^*_a(x) > \alpha \}\), there exists \(x_0, r_0\) such that \(|F(x_0, r_0)| > \alpha\) and \(d_{\mu}(z, x_0) < ar_0\) hold. It is clear that \(B_{\mu}(x_0, br_0) \subseteq \{ x : F^*_a(x) > \alpha \}\) and \(B_{\mu}(x_0, ar_0) \subseteq \{ x : F^*_a(x) > \alpha \}\) hold. Thus we could deduce that the following Formula hold:
\[
\frac{|B_{\mu}(z, (a + b)r_0)| \cap \{ x : F^*_a(x) > \alpha \}|_{\mu}}{|B_{\mu}(z, (a + b)r_0)|_{\mu}} \geq \frac{|B_{\mu}(x_0, br_0)|_{\mu}}{|B_{\mu}(x_0, (a + b)r_0)|_{\mu}} \geq \frac{b}{a + b}.
\]
Thus we could obtain
\[
\{ x : F^*_a(x) > \alpha \} \subseteq \left\{ x : M_{\mu} \chi \{ x : F^*_b(x) > \alpha \} > \frac{b}{a + b} \right\},
\]
where \(M_{\mu}\) is the Hardy-Littlewood maximal operator. With the fact that \(M_{\mu}\) is weak-(1, 1), we could deduce:
\[
\int_{\mathbb{R}} \chi \{ x : F^*_a(x) > \alpha \} d\mu(x) \leq c \frac{a + b}{b} \int_{\mathbb{R}} \chi \{ x : F^*_b(x) > \alpha \} d\mu(x).
\]
This proves the proposition.
When \( F^*_b(x) \in L^p(\mathbb{R}, \mu) \), by Proposition 1.20, we could obtain the following inequality for \( p > 0 \):
\[
\int_{\mathbb{R}} |F^*_b(x)|^p \, d\mu(x) \leq c \left( \frac{a + b}{b} \right) \int_{\mathbb{R}} |F^*_b(x)|^p \, d\mu(x).
\] (19)

**Proposition 1.21.** For \( f \in S'(\mathbb{R}, d\mu, x) \), if \( \|M_{\phi}^* f(x)\|_{L^p(\mathbb{R}, \mu)} < \infty \), then
\[
\|M_{\phi}^* f(x)\|_{L^p(\mathbb{R}, \mu)} \leq c_1 \|M_{\phi}^* f(x)\|_{L^p(\mathbb{R}, \mu)} \quad \text{for } p > 0, N > 1/p.
\]
c1 is independent on \( \phi \) and \( f \).

**Proof.** For \( \phi(t) \in SS_\beta \),
\[
M_{\phi, \beta}^* f(x) = \sup_{s \in \mathbb{R}, r > 0} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(y, u) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} \, d\mu(y) \right| / r
\]
\[
\lesssim \left( \sup_{0 < s \leq r} + \sum_{k=1}^{\infty} \sup_{2^{k-1} < r < 2^k r} \right) 2^{-kN} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(y, u) - s}{r} \right) \, d\mu(y) \right| / r
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{-kN} M_{\phi, \beta}^* f(x).
\]
Thus together with Formula (19), we could deduce the following inequality for \( N > 1/p \):
\[
\int_{\mathbb{R}} |M_{\phi}^* f(x)|^p \, d\mu(x) \leq c_1 \int_{\mathbb{R}} |M_{\phi}^* f(x)|^p \, d\mu(x).
\]
This proves our Proposition. \( \square \)

It is clear that the following Proposition holds from [25]:

**Proposition 1.22.** [25] Suppose \( \phi, \psi \in SS_\beta \), with \( \int \psi(x) \, dx = 1 \). Then there is a sequence \( \{\eta^k\} \), \( \eta^k \in S(\mathbb{R}, dx) \), so that
\[
\phi \left( \frac{\mu(y, u)}{r} \right) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \eta^k \left( \frac{s}{r} \right) \psi \left( \frac{\mu(y, u) - s}{2^{-k} r} \right) \, ds / 2^{-k} r.
\]
\( \eta^k \) satisfies
\[
\|\eta^k\|_{a, b} \leq C(2^{-kM}), \quad \text{as } k \to \infty.
\]

Now we need to prove that the nontangential maximal operator \( M_{\phi}^* f(x) \) allows the control of maximal function \( f_{\phi}^*(x) \).

**Proposition 1.23.** There exists \( \beta > 0 \), such that for any \( \psi \in SS_\beta \), with \( \int \psi(x) \, dx = 1 \) and \( p > 0 \), the following holds for \( f \in S'(\mathbb{R}, d\mu, x) \) if \( \|M_{\phi}^* f(x)\|_{L^p(\mathbb{R}, \mu)} < \infty \):
\[
\|f_{\phi}^*(x)\|_{L^p(\mathbb{R}, \mu)} \leq c \|M_{\phi}^* f(x)\|_{L^p(\mathbb{R}, \mu)},
\]
\( C \) is independent on \( \beta \).

**Proof.** For any \( \phi, \psi \in SS_\beta \), with \( \int \psi(x) \, dx = 1 \) by Proposition 1.22, we have
\[
M_{\phi, \beta} f(x) = \sup_{r > 0} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(y, u)}{r} \right) \, d\mu(y) \right| / r
\]
\[
\lesssim \sup_{r > 0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \eta^k \left( \frac{s}{r} \right) \psi \left( \frac{\mu(y, u) - s}{2^{-k} r} \right) \, d\mu(y) \, ds / 2^{-k} r.
\]
Thus we could obtain:
\[
M_{\phi, \beta} f(x)
\]
\[
\lesssim \sup_{r > 0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \psi \left( \frac{\mu(y, u) - s}{2^{-k} r} \right) \left( 1 + \frac{|s|}{2^{-k} r} \right)^{-N} \, d\mu(y) \, ds / 2^{-k} r
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{-k} \int_{\mathbb{R}} \eta^k \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{-k} r} \right)^N \, ds / 2^{-k} r
\]
\[
\lesssim M_{\phi}^* f(x) \int_{\mathbb{R}} \eta^k \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{-k} r} \right)^N \, ds / 2^{-k} r
\]
\[
\lesssim M_{\phi}^* f(x) \int_{\mathbb{R}} \eta^k \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{-k} r} \right)^N \, ds / 2^{-k} r
\]
\[
\lesssim M_{\phi}^* f(x)
\]
\[
\lesssim M_{\phi}^* f(x),
\]
Hardy spaces associated with One-dimensional Dunkl transform for $\frac{2\lambda}{\lambda+\tau} < p \leq 1$

where $\|\eta^k\|_{a,b} = O(2^{-k(N+1)})$ for a suitable collection of seminorms. Thus

$$f_{\beta_\phi}(x) \sim \sup_{\phi \in SS_\beta} M_{\phi,\beta} f(x) \lesssim M_{\phi,\beta}^\ast f(x).$$

For all $x \in \mathbb{R}$, $N > 1/p$, from Proposition 1.21, we could get

$$\|f_{\beta_\phi}\|_{L^p(\mathbb{R},\mu)} \leq c\|M_{\phi,\beta}^\ast f\|_{L^p(\mathbb{R},\mu)}.$$

This proves our proposition. \(\square\)

**Proposition 1.24.** There exists $\beta > 0$, such that for $p > 0, \phi \in SS_\beta$, with $\int \phi(x)dx = 1$, the following holds for $f \in S'(\mathbb{R},d\mu)$ if $\|M_{\phi,\beta}^\ast f\|_{L^p(\mathbb{R},\mu)} < \infty$:

$$\|M_{\phi,\beta} f\|_{L^p(\mathbb{R},\mu)} \leq c\|M_{\phi,\beta}^\ast f\|_{L^p(\mathbb{R},\mu)}.$$

$C$ is dependent on $\beta$.

**Proof.** We assume $\|M_{\phi,\beta}^\ast f\|_{L^p(\mathbb{R},\mu)} < \infty$ first. Let $F$ be defined as $F = \{x : f_{\beta_\phi}(x) \leq \sigma M_{\phi,\beta} f(x)\}$. By Proposition 1.23, the following holds:

$$\int_F |M_{\phi,\beta} f(x)|^p d\mu(x) \lesssim \int_F |M_{\phi,\beta}^\ast f(x)|^p d\mu(x).$$

(20)

Choosing $\sigma^p \geq 2C$, we could have

$$\int_\mathbb{R} |M_{\phi,\beta}^\ast f(x)|^p d\mu(x) \lesssim \int_F |M_{\phi,\beta}^\ast f(x)|^p d\mu(x).$$

(21)

Next we will show that for any $q > 0$

$$\|M_{\phi,\beta}^\ast f(x)\|^q \leq cM_\mu(M_{\phi,\beta} f)^q(x).$$

Let $f(x, r)$ be defined as

$$f(x, r) = \int_\mathbb{R} f(u) \phi(\frac{|x-u|}{r}) d\mu(u)/r.$$

Then for any $x \in \mathbb{R}$, there exists $(y, r)$, satisfying $d_{\mu}(x,y) < r$ and $|f(y, r)| \geq M_{\phi,\beta}^\ast f(x)/2$. Choose $0 < \delta < 1$ and $x'$ satisfying $d_{\mu}(x', y) < \delta r$. Then there exists $\xi \in [x', y]$ such that:

$$|f(x', r) - f(y, r)| \leq \delta r \sup_{x \in B_{\mu}(y, \delta r)} \left| \frac{d}{d\mu} f(x, r) \right|$$

$$\leq \delta \sup_{\xi \in B_{\mu}(y, \delta r)} \left| \int_\mathbb{R} f(u) \phi^{(1)}(\frac{|x-u|}{r}) d\mu(u)/r \right|$$

$$\leq \delta \sup_{\xi \in B_{\mu}(y, \delta r)} \left| \int_\mathbb{R} f(u) \phi^{(1)}(\frac{|x-u| - \mu(x, \xi)}{r}) d\mu(u)/r \right|$$

$$\leq \delta \sup_{h \leq 1 + \delta} \int_\mathbb{R} f(u) \phi^{(1)}(\frac{|x-u| - h}{r}) d\mu(u)/r.$$  

Notice that $|h| \leq 1 + \delta < 2$ with $\|H_{\phi}^{(1)} f(x-h)\|_{\infty} \leq C_{\phi} \|\phi^{(1)}(x-h)\|_{\infty} \leq C$. By the definition of $f_{\beta_\phi}(x)$,

$$|f(x', r) - f(y, r)| \leq C_{\phi} d_{\mu} f_{\beta_\phi}(x) \leq C_{\phi} d_{\sigma} M_{\phi,\beta}^\ast f(x)$$

for $x \in F$.

Taking $\delta$ small enough such that $C_{\phi} d_{\sigma} \leq 1/4$, we obtain

$$|f(x', r)| \geq \frac{1}{4} M_{\phi,\beta}^\ast f(x).$$

Thus the following inequality holds:

$$\|M_{\phi,\beta}^\ast f(x)\|^q \leq \frac{1}{|B_{\mu}(y, \delta r)|} \int_{B_{\mu}(y, \delta r)} 4^q |f(x', r)|^q d\mu(x')$$

$$\leq \frac{|B_{\mu}(x, (1+\delta) r)|}{|B_{\mu}(y, \delta r)|} \frac{1}{|B_{\mu}(x, (1+\delta) r)|} \int_{B_{\mu}(x, (1+\delta) r)} 4^q |f(x', r)|^q d\mu(x')$$

$$\leq \frac{1+\delta}{\delta} \frac{1}{|B_{\mu}(x, (1+\delta) r)|} \int_{B_{\mu}(x, (1+\delta) r)} 4^q |f(x', r)|^q d\mu(x')$$

$$\leq CM_\mu([M_{\phi,\beta}^\ast f]^q)(x),$$
where $M_{\mu}$ is the Hardy-Littlewood Maximal Operator. Thus for $p$ satisfying $p > q$, using the
maximal theorem for $M_{\mu}$ leads to
\[ \int_F |M_{\alpha,\beta} f(x)|^p \, d\mu(x) \leq C \int_F (M_{\mu}([M_{\alpha,\beta} f])^q(x))^{p/q} \leq C \int F |M_{\alpha,\beta} f(x)|^p \, d\mu(x). \] (22)
Combining (21) and (22) together, we could prove the proposition. \hfill \Box

**Proposition 1.25.** [25/Classical Hardy spaces $H^p(R)$ in Euclid space]
Let $F = \{|\cdot|_{a,b}\}$ be any finite collection of seminorms on $S(R, dx)$. We use $S_F$ to denote the
subset of $S(R, dx)$ controlled by this collection of seminorms:
\[ S_F = \{ \phi \in S(R, dx) : \|\phi\|_{a,b} \leq 1 \text{ for any } \| \cdot \|_{a,b} \in F \}. \]
Let $M_F f(x)$ be defined as $M_F f(x) = \sup_{\phi \in S_F} \sup_{t > 0} \{f * \phi_t\}(x)$. If $f \in H^p(R)$, then $\|f\|_{H^p(R)}^p = \int R |M_F f(x)|^p \, dx$. Thus every $f \in H^p(R)$ can be written as a sum of $H^p(R)$ atoms: $f = \sum_k \lambda_k a_k$ in the sense of distribution. An $H^p(R)$ atom is a function $a(x)$ so that:
(i) $a(x)$ is supported in a ball $B$ in Euclid space;
(ii) $|a(x)| \leq |B|^{-1/p}$ almost everywhere;
(iii) $\int B x^r a(x) \, dx = 0$ for all $n \in Z$ with $|n| \leq p^{-1} - 1$. Further more
\[ \|f\|_{H^p(R)}^p = \int R |M_F f(x)|^p \, dx \sim \sum_k \lambda_k^p. \]

**Proposition 1.26.** For $\alpha$ and $\beta$ satisfying $\beta \geq \alpha > p^{-1} - 1 (0 < p \leq 1)$, we could deduce that
$\tilde{H}^p_{\mu,\beta}(R)$ is dense in $H^p_{\mu,\beta}(R)$ and we could also deduce that
\[ H^p_{\mu,\beta}(R) = H^p_{\mu,\alpha}(R). \]
For any $f \in H^p_{\mu,\beta}(R)$, we could also have
\[ C_2 \|f\|_{H^p_{\mu,\beta}(R)} \leq \|f\|_{H^p_{\mu,\alpha}(R)} \leq C_1 \|f\|_{H^p_{\mu,\beta}(R)}, \]
where $C_1$ and $C_2$ are independent on $f$.

**Proof.** First, with the fact $SS_\beta \subseteq SS_\alpha$, it is easy to see that
\[ H^p_{\mu,\beta}(R) \supseteq H^p_{\mu,\alpha}(R), \quad \|f\|_{H^p_{\mu,\beta}(R)} \leq C \|f\|_{H^p_{\mu,\alpha}(R)} \]
for $\beta \geq \alpha > p^{-1} - 1$. Thus we could deduce that $f \in H^p_{\mu,\beta}(R)$, if $f \in H^p_{\mu,\alpha}(R)$.

Next we will prove that $f \in H^p_{\mu,\alpha}(R)$, if $f \in H^p_{\mu,\beta}(R)$. Notice that $P(x) = \mu(x, 0)$ is a bijection on $R$. Let $P^{-1}(x)$ be the reverse map of $P(x)$. Let $g(t) = f \circ P^{-1}(t)$. From Definition 1.16, Definition 1.17, Definition 1.18, Definition 1.9 and Definition 1.19, Proposition 1.21, Proposition 1.23, Proposition 1.24 and Proposition 1.25, we could deduce that $g(t) \in H^p(R)$, if $f \in H^p_{\mu,\beta}(R)$. With the fact that $H^p(R) \cap L^1(R)$ is dense in $H^p(R)$, we could deduce that $\tilde{H}^p_{\mu,\beta}(R)$ is dense in $H^p_{\mu,\beta}(R)$. We could also deduce the that the following equation:
\[ \|f\|_{H^p_{\mu,\beta}(R)} = \|g\|_{H^p(R)}. \]
By Proposition 1.25, $g \in H^p(R)$ can be written as a sum of $H^p(R)$ atoms:
\[ g = \sum_k \lambda_k a_k \]
in the sense of distribution. Let $b_k(x) = a_k(P(x))$, then it is clear that the functions $\{b_k(x)\}_k$ satisfy the following:
(i) $b_k(x)$ is supported in a ball $B_{\mu}(x_k, r_k)$;
(ii) $|b_k(x)| \leq |B_{\mu}(x_k, r_k)|^{-1/p}$ almost everywhere in $\mu$ measure;
(iii) $\int \mu(x, 0)^n b_k(x) \, d\mu(x) = 0$ for all $n \in Z$ with $|n| \leq p^{-1} - 1$. Together with Proposition 1.25, we could deduce that
\[ \int_R f(x) \phi(x) \, d\mu(x) = \int_R \sum_k \lambda_k b_k(x) \phi(x) \, d\mu(x) = \sum_k \int_R \lambda_k b_k(x) \phi(x) \, d\mu(x) \]
holds for any $\phi(x) \in S(\mathbb{R}, d_\mu, x)$, and
\[ \|f\|_{H^p_{\mu_\alpha}(\mathbb{R})} = \|g\|_{H^p(\mathbb{R})} \sim \sum_k \lambda_k^p, \]
holds. For any $\psi(x) \in SS_\alpha$ satisfying $\int \psi(x) dx = 1$, we have:
\[ \int_{B_\mu(x_k, 4r_k)} |b_{k\alpha}(x)|^p d\mu(x) \leq C \int_{B_\mu(x_k, 4r_k)} |M_\mu b_k(x)|^p d\mu(x) \]
\[ \leq C \left( \int_{B_\mu(x_k, 4r_k)} |M_\mu b_k(x)|^2 d\mu(x) \right)^{p/2} \left( \int_{B_\mu(x_k, 4r_k)} 1 d\mu(x) \right)^{1-(p/2)} \leq C, \]
where $C$ is independent on $\psi$ and $b_k$. For $s \in \mathbb{Z}$, $s \leq \alpha$, by Taylor Expansion, there exists $\xi \in B_\mu(x_k, t)$ such that the following holds:
\[ \psi \left( \frac{\mu(t, x)}{r} \right) = \frac{[\alpha]!}{s!} \psi^{(s)} \left( \frac{\mu(x_k, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^s + \frac{1}{[\alpha]!} \psi^{(\alpha)} \left( \frac{\mu(t, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^{[\alpha]}. \]
Let $P(x, x_k)$ be defined as following:
\[ P(x, x_k) = \sum_{s=0}^{[\alpha]-1} \frac{1}{s!} \psi^{(s)} \left( \frac{\mu(x_k, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^s. \]
Thus we could obtain
\[ \left| P(x, x_k) - \psi \left( \frac{\mu(t, x)}{r} \right) \right| \leq \frac{1}{[\alpha]!} \left( \frac{\mu(t, x_k)}{r} \right)^{[\alpha]} . \]
(24)
Thus by Proposition 1.12 and the vanishing property of $b_k$ we could have:
\[ \int_{B_\mu(x_k, 4r_k)^c} \left| \int b_k(t) \psi \left( \frac{\mu(t, x)}{r} \right) \frac{d\mu(t)}{r} \right|^p d\mu(x) \]
\[ = \int_{B_\mu(x_k, 4r_k)^c} \left| \int b_k(t) \left( \psi \left( \frac{\mu(t, x)}{r} \right) - P(x, x_k) \right) \frac{d\mu(t)}{r} \right|^p d\mu(x) \]
\[ \leq C \int_{B_\mu(x_k, 4r_k)^c} \left[ \int_{r_k^{\alpha+1-p^{-1}}} r_k^{\beta} d\mu(x) \right] \]
(25)
\[ \leq C \int_{B_\mu(x_k, 4r_k)^c} \left[ \int_{r_k^{\alpha+1-p^{-1}}} r_k^{\beta} d\mu(x) \right] \]
(26)
Notice that $r > |\mu(x, x_k) - r_k|$, $\alpha > p^{-1} - 1$ and $0 < p \leq 1$, thus Formula (25) implies:
\[ \int_{B_\mu(x_k, 4r_k)^c} \left[ \int_{r_k^{\alpha+1-p^{-1}}} r_k^{\beta} d\mu(x) \right] \]
(26)
Formula (23) and Formula (26) imply:
\[ \int_{B_\mu(x_k, 4r_k)} |b_{k\alpha}(x)|^p d\mu(x) \leq C, \]
where $C$ is independent on $\psi$ and $b_k$. Thus
\[ \|f\|_{H^p_{\mu_\alpha}(\mathbb{R})} \leq C \sum_k \lambda_k^p \|b_k\|_{H^p_{\mu_\alpha}(\mathbb{R})} \leq C \sum_k \lambda_k^p \leq C \|f\|_{H^p_{\mu_\alpha}(\mathbb{R})}. \]
Thus $f \in H^p_{\mu_\alpha}(\mathbb{R})$, if $f \in H^p_{\mu_\beta}(\mathbb{R})$. Thus, we could deduce that
\[ H^p_{\mu_\alpha}(\mathbb{R}) = H^p_{\mu_\beta}(\mathbb{R}). \]

This proves the Proposition.
Definition 1.27. Let \( \{b_k^{n,p}(x)\} \) be functions as follows:
(i) \( b_k^{n,p}(x) \) is supported in a ball \( B_{r_k}(x_k, r_k) \);
(ii) \( |b_k^{n,p}(x)| \leq |B_{r_k}(x_k, r_k)|^{-1/p} \) almost everywhere in \( \mu \) measure;
(iii) \( \int \mu(x, 0)^m b_k^{n,p}(x) d\mu(x) = 0 \) for all \( m \in \mathbb{N} \) with \( m \leq n \).
For \( n \geq [p^{-1} - 1] \), \( A^{n,p}(\mathbb{R}) \) is defined as
\[
A^{n,p}(\mathbb{R}) = \left\{ f \in S'(\mathbb{R}, d_\mu) : \int_\mathbb{R} f(x)\phi(x)d\mu(x) = \sum_k \int_\mathbb{R} \lambda_k b_k^{n,p}(x)\phi(x)d\mu(x) \right. \\
\left. \text{for any } \phi(x) \in S(\mathbb{R}, d_\mu), \text{where } \sum_k |\lambda_k|^p < +\infty. \right\}
\]
The norm is defined by:
\[
\|f\|_{A^{n,p}(\mathbb{R})} = \inf \left( \sum_k |\lambda_k|^p \right)^{1/p}.
\]
Thus by Proposition 1.26, we could conclude that
\[
A^{n,p}(\mathbb{R}) = H^{p,\mu_0}(\mathbb{R}) = H^{p,\mu_0}(\mathbb{R})
\]
for \( \beta \geq \alpha > p^{-1} - 1 \) and \( n \geq [p^{-1} - 1](0 < p \leq 1) \).

Theorem 1.28. For \( \beta_1 \geq \beta_2 > p^{-1} - 1 \), \( n \geq [p^{-1} - 1] \), \( f \in A^{n,p}(\mathbb{R}) (0 < p \leq 1) \), we could obtain
\[
A^{n,p}(\mathbb{R}) = H^{p,\mu_{\beta_2}}(\mathbb{R}) = H^{p,\mu_{\beta_1}}(\mathbb{R}),
\]
and
\[
\|f\|_{A^{n,p}(\mathbb{R})} \sim \|f\|_{H^{p,\mu_{\beta_2}}(\mathbb{R})} \sim \|f\|_{H^{p,\mu_{\beta_1}}(\mathbb{R})}.
\]
We could also deduce that \( H^{p,\mu_{\beta_1}}(\mathbb{R}) \) is dense in \( H^{p,\mu_{\beta_1}}(\mathbb{R}) \) from Proposition 1.26.

Proposition 1.29. For the kernel \( K_1(r, x, y) \) as above, there exists sequence \( \{a^\tau_{x,r}(y) : a^\tau_{x,r}(y) \in C_c(\mathbb{R}, d_\mu_0) \cap S(\mathbb{R}, d_\mu_0)\} \) satisfying the following:
(i) \( a^\tau_{x,r}(y) = a^\tau_{x,r}(x) \),
(ii) \( \lim_{r \to \infty} \|K_1(r, x, y) - a^\tau_{x,r}(y)\|_\infty = 0 \),
(iii) \( 0 \leq a^\tau_{x,r}(y) \leq C \),
(iv) \( \text{For } r > 0, x, y, z \in \mathbb{R}, \)
\[
|a^\tau_{x,r}(y) - a^\tau_{x,r}(z)| \leq C \left( \frac{d_\mu_0(y, z)}{r} \right)^\gamma.
\]
\( C \) is constant independent on \( K_1(r, x, y) \) and \( a^\tau_{x,r}(y) \).
(v) \( \text{For } \tau \text{ small enough} \)
\[
|a^\tau_{x,r}(y) - K_1(r, x, y)| \leq C \left( \frac{r}{\tau} \right)^\gamma.
\]
(vi) \( a^\tau_{x,r}(x) > C, \text{ for } r > 0, x \in \mathbb{R} \).

Proof. Let \( \rho(x) \) to be a fixed function so that
\[
\rho(x) = \begin{cases} 
\vartheta \exp \left\{ \frac{1}{|x|^{\gamma-1}} \right\}, & \text{for } |x| < 1 \\
0, & \text{for } |x| \geq 1.
\end{cases}
\]
where \( \vartheta \) is a constant satisfying \( \int \rho(x)dx = 1 \). We use \( a^\tau_{x,r}(y) \) to denote as
\[
a^\tau_{x,r}(y) = \int_\mathbb{R} \int_\mathbb{R} K_1(r, t_1, t_2) \rho \left( \frac{\mu(x, t_1)}{\tau} \right) \rho \left( \frac{\mu(y, t_2)}{\tau} \right) d\mu(t_1) d\mu(t_2).
\]
It is clear that (i) (ii) and (iii) hold. We will prove (iv) next. Let \( \frac{\mu(y, t_2)}{\tau} = \frac{\mu(z, t_3)}{\tau} \). Notice that
\[
\rho \left( \frac{\mu(y, t_2)}{\tau} \right) = \rho \left( \frac{\mu(z, t_3)}{\tau} \right) \text{ and } \frac{d\mu(t_2)}{\tau} = \frac{d\mu(t_3)}{\tau}.
\]
Let \( \mu(y, t_2) = \mu(z, t_3) \). Thus (iv) holds. We will prove (v) next. Similar to Formula (29), we could obtain:

\[
|a^r_{x,r}(y) - K_1(r, x, y)| = \left| \int_{L^2} \int_{L^2} K_1(r, x, y) \rho \left( \frac{\mu(x, t_1)}{\tau} \right) \rho \left( \frac{\mu(y, t_2)}{\tau} \right) \frac{d\mu(t_1)}{\tau} \frac{d\mu(t_2)}{\tau} \right|
\]

Notice that \( \text{supp} \rho(x) \subseteq \{ x : |x| < 1 \} \). Thus we could deduce that \( d_\mu(y, t_2) < \tau \) and \( d_\mu(z, t_3) < \tau \). If we choose \( \tau \) small enough such that \( \frac{d_\mu(y, z) - d_\mu(t_2, t_3)}{\rho} \), then

\[
|K_1(r, t_1, t_2) - K_1(r, t_1, t_3)| \leq C \left( \frac{d_\mu(y, z)}{\tau} \right) \leq C \left( \frac{d_\mu(y, z)}{\tau} \right).
\]

Then together with Formula (29), we could conclude

\[
|a^r_{x,r}(y) - a^r_{x,r}(z)| \leq C \left( \frac{d_\mu(y, z)}{\tau} \right).
\]

Thus (iv) holds. We will prove (v) next. Similar to Formula (29), we could obtain:

\[
|a^r_{x,r}(y) - K_1(r, x, y)| = \left| \int_{L^2} \int_{L^2} K_1(r, x, y) \rho \left( \frac{\mu(x, t_1)}{\tau} \right) \rho \left( \frac{\mu(y, t_2)}{\tau} \right) \frac{d\mu(t_1)}{\tau} \frac{d\mu(t_2)}{\tau} \right|
\]

Notice that

\[
|K_1(r, t_1, t_2) - K_1(r, t_1, t_3)| \leq C |K_1(r, t_1, t_2) - K_1(r, t_1, t_3)| + C |K_1(r, t_1, t_3) - K_1(r, t_1, t_4)|
\]

\[
\leq C \left( \frac{d_\mu(t_2, t_3)}{\tau} \right)^\gamma + C \left( \frac{d_\mu(t_1, t_3)}{\tau} \right)^\gamma
\]

\[
\leq C \left( \frac{t\gamma}{\tau} \right)^\gamma.
\]

Together with Formula (30), we could conclude

\[
|a^r_{x,r}(y) - K_1(r, x, y)| \leq C \left( \frac{t\gamma}{\tau} \right)^\gamma,
\]

for \( \tau \) small enough. This proves our proposition.

**Proposition 1.30.** For \( p > \frac{1}{1 + \gamma} \), \( f \in L^1(\mathbb{R}, \mu) \), \( 1 \geq \gamma > 0 \), there exists some \( \beta \) with \( \beta > \gamma \) such that the following inequality holds:

\[
\|f^r_{x,\gamma}\|_{L^p(\mathbb{R}, \mu)} \leq c\|f_{x,\gamma}\|_{L^p(\mathbb{R}, \mu)}.
\]

**Proof.** Let \( \phi \in SS_\beta \) first. Notice that \( C_\mu(\mathbb{R}, dx) \) is dense in \( C_0(\mathbb{R}, dx) \), by Proposition 1.3 and Proposition 1.4, \( C_\mu(\mathbb{R}, dx) \) is dense in \( C_0(\mathbb{R}, d\mu, dx) \). By the fact that \( K_1(r, x, y) = K_1(r, y, x) \) and \( |\int_{\mathbb{R}} K_1(r, x, y) d\mu(y)/r| \geq m > 0 \), together with Proposition 1.29, there exists sequence \( \{ \phi_{x,r}^\gamma(y) \in S(\mathbb{R}, d\mu(y)) \} \) satisfying the following conditions:

\[
\phi_{x, r}^\gamma(y) = \phi_{x, r}^\gamma(x), \quad \phi_{x, r}^\gamma(y) \in S(\mathbb{R}, d\mu,y), \quad \text{supp} \phi_{x, r}^\gamma(y) \subseteq B_\mu(x, r), \quad \left| \int_{\mathbb{R}} \phi_{x, r}^\gamma(y) d\mu(y)/r \right| \geq m/2 > 0
\]

\[
L(\phi_{x, r}^\gamma(y), \gamma) \leq r^{-\gamma}, 0 \leq \phi_{x, r}^\gamma(y) \leq C, \text{ for } r > 0
\]

\[
\lim_{\tau \to 0} \phi_{x, r}^\gamma(y) = K_1(r, x, y).
\]
Thus by Proposition 1.3, Proposition 1.4 and Proposition 1.29, there exists sequence \( \{ \phi^\tau(x) : \phi^\tau(y) \in S(\mathbb{R},dy) \} \) satisfying:

\[
\begin{cases}
\phi^\tau_z(x) = \phi^\tau_z \left( \frac{\mu(x,y)}{r} \right) = \phi^\tau_y \left( \frac{\mu(x,y)}{r} \right), \quad \| \phi^\tau_z(t) \|_{L^\infty} \leq 1, H^\gamma \phi^\tau_z(t) \lesssim 1 \\
\phi^\tau_z(y) \in S(\mathbb{R},dy), \quad \lim_{\tau \to 0} \phi^\tau_z \left( \frac{\mu(x,y)}{r} \right) = K_1(r,x,y) \\
\left| \int I \phi^\tau_z(t)dt \right| \geq \frac{m}{2} > 0, \quad \text{supp} \phi^\tau_z(t) \subseteq [-1,1].
\end{cases}
\]

Notice that \( \int t^2 \phi^\tau_y(t)dt \) \( \lesssim C_\beta \), thus we could deduce the following inequality:

\[
\left| \int \phi^\tau_z(t)dt \right| \lesssim C_\beta,
\]

where \( C_\beta \) is a constant independent on \( \tau \). Notice that \( (\mathcal{F} \phi^\tau_y)(0) = 1 \), thus by Formula (32), we could also deduce that there exists a \( k_\alpha \) independent on \( \tau \), such that

\[
|(\mathcal{F} \phi^\tau_y)(2^{-k_\alpha}x)| \geq 1/2 \quad \text{for } |x| \leq 2.
\]

Fix a function \( \varphi \in S(\mathbb{R},dx) \) so that

\[
\begin{cases}
\varphi(\xi) = 0 \quad \text{for } |\xi| \geq 1 \\
\varphi(\xi) = 1 \quad \text{for } |\xi| \leq 1/2.
\end{cases}
\]

The function \( \varphi^k \in S(\mathbb{R},dx) \) is defined as:

\[
\varphi^k(\xi) = \varphi(\xi), \quad \text{for } k = 0,
\]

\[
\varphi^k(\xi) = \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi) \quad \text{for } k \geq 1.
\]

We use \( \eta^\tau \) to denote as

\[
(\mathcal{F} \eta^\tau)(\xi) = \frac{\varphi^k(\xi)(\mathcal{F} \varphi)(\xi)}{(\mathcal{F} \phi^\tau_y)(2^{-k}2^{-k_\alpha}x)},
\]

where \( \mathcal{F} \) is the Fourier transform. By the fact that \( \sup_{\xi \in \mathbb{R}} \left| \frac{d^2}{d\xi^2} (\mathcal{F} \phi_y^\tau)(2^{-k}2^{-k_\alpha}x) \right| \leq C_{\beta,k_\alpha} \) and

\[
\left| \mathcal{F} \eta^\tau \right| \leq C_{\alpha,\beta,1},
\]

where \( C_{\alpha,\beta,1} \) is a constant independent on \( \tau \), we could deduce that for any \( M > 0 \), the following inequality holds:

\[
\sup_{\xi \in \mathbb{R}} \left| \xi^\alpha \frac{d^3}{d\xi^3} (\mathcal{F} \eta^\tau)(\xi) \right| \leq C_{\alpha,\beta,1} 2^{-k_\alpha}.
\]

Then by Formula (35) with the fact that \( f \in L^1(\mathbb{R},\mu) \) we have

\[
M_{\phi^\tau} f(x) = \sup_{r > 0} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x,y)}{r} \right) d\mu(y) \right| / r \leq \sup_{r > 0} \left| \int_0^\infty \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x,y)}{r} \right) d\mu(y) \right| / r
\]

Then by Proposition 1.22, for any \( \phi(t) \in \mathbb{S}(\mathbb{R}) \) with \( \int_{\mathbb{R}} \phi(t)dt = 1 \), we could deduce:

\[
\phi \left( \frac{\mu(x,y)}{r} \right) = \sum_{k=0}^{\infty} \int_\mathbb{R} \eta^\tau_k \left( \frac{x}{r} \right) \phi^\tau_y \left( \frac{\mu(x,y) - s}{2^{-k}2^{-k_\alpha}r} \right) ds / 2^{-k_\alpha}.
\]
By Formula (34), we could deduce that

\[ \sum_{k=0}^{\infty} \int_{\mathbb{R}} \eta_s^k \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{k-\delta} r} \right)^N \frac{ds}{r} \leq C_{N,k_0} \sum_{k=0}^{\infty} 2^{-k}, \]

where \( C_{N,k_0} \) is a constant independent on \( \tau \). Together with Formula (36), we could obtain:

\[
M_{\phi,\beta} f(x) \lesssim \sup_{r > 0, s \in \mathbb{R}} \left| \int_{\mathbb{R}} f(y) \phi_y^\tau \left( \frac{\mu(x,y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| \tag{37}
\]

\[
\lesssim \left( \sup_{0 < s \leq r} \sup_{k=1}^{\infty} \sum_{0 \leq s \leq 2^k r} \sup_{0 \leq s \leq 2^k r} \left| \int_{\mathbb{R}} f(y) \phi_y^\tau \left( \frac{\mu(x,y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| \right. \\
\left. \lesssim \sum_{k=0}^{\infty} 2^{-(k-1)N} \sup_{0 \leq s \leq 2^k r} \left| \int_{\mathbb{R}} f(y) \phi_y^\tau \left( \frac{\mu(x,y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| . \right.
\]

Thus by Formula (37) the following holds:

\[
f_{S_{\beta}}^*(x) \overset{\phi \in SS_{\beta}}{=} M_{\phi,\beta} f(x) \tag{38}
\]

\[
\lesssim \sum_{k=0}^{\infty} 2^{-(k-1)N} \sup_{0 \leq s \leq 2^k r} \left| \int_{\mathbb{R}} f(y) \phi_y^\tau \left( \frac{\mu(x,y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| .
\]

For a positive measure \( \mu \) where \( \mu(x,u) \) is a bijection on \( \mathbb{R} \), let \( s = \mu(x,u) \) with \( d_\mu(x,u) < 2^k r \). We use \( T(x,k,\tau) \), \( (F^\tau f)(u,r) \) and \( (K_1 f)(u,r) \) to denote as:

\[
T(x,k,\tau) = \sup_{0 \leq s \leq 2^k r} \left| \int_{\mathbb{R}} f(y) \phi_y^\tau \left( \frac{\mu(x,y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| ,
\]

\[
(F^\tau f)(u,r) = \int_{\mathbb{R}} f(y) \phi_{u,r}^\tau(y) \frac{d\mu(y)}{r}, \quad (K_1 f)(u,r) = \int_{\mathbb{R}} f(y) K_1(r,u,y) \frac{d\mu(y)}{r}.
\]

\[
\int_{\mathbb{R}} |T(x,k,\tau)|^p d\mu(x) < \infty \quad \text{and Formula (19) lead to}
\]

\[
\int_{\mathbb{R}} |T(x,k,\tau)|^p d\mu(x) \leq c \left( 1 + 2^k \right) \int_{\mathbb{R}} |T(x,0,\tau)|^p d\mu(x). \tag{39}
\]

For \( N > 1/p \), we could obtain

\[
\int_{\mathbb{R}} |f_{S_{\beta}}^*(x)|^p d\mu(x) \leq C_{p,n,\beta} \int_{\mathbb{R}} |T(x,0,\tau)|^p d\mu(x), \tag{40}
\]

where \( C_{p,n,\beta} \) is a constant independent on \( \tau \). By Formula (31) it is clear that (taking \( \tau = \frac{\alpha}{r} \))

\[
| (F^\tau f)(u,r) - (K_1 f)(u,r) | \leq \int_{\mathbb{R}} |f(y)| |\phi_{u,r}^\tau(y) - K_1(r,u,y)| \frac{d\mu(y)}{r} \leq C_\gamma |M_\mu f(u)| \left( \frac{1}{r} \right)^\gamma,
\]

where \( C_\gamma \) is dependent on \( \gamma \), and \( M_\mu \) is the Hardy-Littlewood Maximal Operator. Let us set

\[
\delta_n(u) = |(F^\tau f)(u,r) - (K_1 f)(u,r)|.
\]

Thus we could deduce the following:

\[
\{ x : \delta_n(x) > \alpha \} \subseteq \left\{ x : M_\mu f(x) > \frac{1}{C_\gamma} \right\}.
\]

Notice that \( M_\mu \) is weak-(1, 1) bounded. Then the following holds for any \( \alpha > 0 \):

\[
|\{ x : \delta_n(x) > \alpha \}| \leq \frac{C_\gamma}{\alpha} \| f \|_{L^1(\mathbb{R},\mu)} \left( \frac{1}{r} \right)^\gamma.
\]
Thus
\[ \lim_{n \to +\infty} \| \{ x : \delta_n(x) > \alpha \} \|_\mu = 0. \]

Thus there exists a sequence \( \{ \tau_j \} \subseteq \{ \tau \} \) such that the following holds:
\[ \lim_{\tau_j \to 0} (F^{\tau_j} f)(u, r) = (K_1 f)(u, r), \quad \text{a.e.} u \in \mathbb{R} \text{ in } \mu \text{ measure} \]
for \( f \in L^1(\mathbb{R}, \mu) \). Denote
\[ E = \{ u \in \mathbb{R} : \lim_{\tau_j \to 0} (F^{\tau_j} f)(u, r) = (K_1 f)(u, r) \}. \]

That \( E \) is dense in \( \mathbb{R} \) could be deduced from the fact \( |E| = 0 \). Notice that for any \( x_0 \in \mathbb{R} \) and any \( \tau_j \in \{ \tau \} \), there exists a \( (u_0, r_0) \) with \( r_0 > 0, u_0 \in \mathbb{R}, d_\mu(u_0, x_0) < r_0 \) such that the following holds:
\[ |(F^{\tau_j} f)(u_0, r_0)| \geq \frac{1}{2} |T(x_0, 0, \tau_j)|. \]

Because \( (F^{\tau_j} f)(u, r_0) \) is a continuous function in \( u \) variable and \( E \) is dense in \( \mathbb{R} \). There exists a \( \tilde{u}_0 \in E \) with \( d_\mu(\tilde{u}_0, x_0) < r_0 \) such that
\[ |(F^{\tau_j} f)(\tilde{u}_0, r_0)| \geq \frac{1}{4} |T(x_0, 0, \tau_j)|. \]

Thus we could deduce that
\[ \sup_{\{ u \in E : d_\mu(u, x) < r \}} |(F^{\tau_j} f)(u, r)| \sim \sup_{\{ u \in \mathbb{R} : d_\mu(u, x) < r \}} |(F^{\tau_j} f)(u, r)|. \]  \tag{42}  

Formula \( 42 \) together with the dominated convergence theorem (Proposition 1.29(iii)), we could conclude:
\[ \lim_{\tau_j \to 0} \int_{\mathbb{R}} |T(x, 0, \tau_j)|^p d\mu(x) \sim \sup_{\{ u \in \mathbb{R} : d_\mu(u, x) < r \}} |(F^{\tau_j} f)(u, r)|^p d\mu(x) \]
\[ \leq C \int_{\mathbb{R}} \lim_{\tau_j \to 0} \sup_{\{ u \in E : d_\mu(u, x) < r \}} |(F^{\tau_j} f)(u, r)|^p d\mu(x) \]
\[ \leq C \int_{\mathbb{R}} \sup_{\{ u \in E : d_\mu(u, x) < r \}} |(K_1 f)(u, r)|^p d\mu(x) \]
\[ \leq C \int_{\mathbb{R}} \sup_{\{ u \in \mathbb{R} : d_\mu(u, x) < r \}} |(K_1 f)(u, r)|^p d\mu(x). \]  \tag{43}  

That is
\[ \| f^\times_\beta \|_{L^p(\mathbb{R}, \mu)} \leq C \| f^\times_\gamma \|_{L^p(\mathbb{R}, \mu)}. \]

This proves our proposition. \( \square \)

**Proposition 1.31.** \( K_2(r, x, y) \) is the kernel in Definition 1.7. Then for any fixed \( \alpha \) with \( 0 < \alpha < \gamma \leq 1 \), the following holds:
\[ 0 \leq |K_2(r, a, y) - K_2(r, b, y)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( 1 + \frac{d_\mu(x, y)}{r} \right)^{-(\gamma-\alpha)-1}, \]
and
\[ |(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \]
\[ \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( \frac{d_\mu(y, z)}{r} \right)^{\gamma-\alpha} \left( 1 + \frac{d_\mu(x, y)}{r} \right)^{-(\gamma-\alpha)-1}, \]
for \( d_\mu(a, b) \leq r, \frac{d_\mu(y, z)}{r} \leq C_3 \min \{ 1 + \frac{d_\mu(a, u)}{r}, 1 + \frac{d_\mu(a, z)}{r} \}, x \in B_\mu(a, 2r) \cap B_\mu(b, 2r). \)

**Proof.** First, we consider the case when \( d_\mu(a, b) \leq d_\mu(y, z). \)
From the fact that \( d_\mu(a, b) \lesssim r, \frac{d_\mu(y, z)}{r} \leq C_3 \min\{1 + \frac{d_\mu(a, y)}{r}, 1 + \frac{d_\mu(a, z)}{r}\}, \) the following relations could be obtained:

\[
1 + \frac{d_\mu(a, y)}{r} \sim 1 + \frac{d_\mu(b, y)}{r}, 1 + \frac{d_\mu(a, z)}{r} \sim 1 + \frac{d_\mu(b, z)}{r}, \text{ and } 1 + \frac{d_\mu(a, z)}{r} \sim 1 + \frac{d_\mu(a, y)}{r}.
\]  

(44)

Notice that

\[
K_2(r, x, y) = K_2(r, y, x).
\]

Then we could get

\[
|K_2(r, a, y) - K_2(r, b, y)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1},
\]  

(45)

and

\[
|K_2(r, a, z) - K_2(r, b, z)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, z)}{r} \right)^{-2\gamma - 1}.
\]

Together with Formula (44), we could conclude

\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]

By the fact \( d_\mu(a, b) \leq d_\mu(y, z) \) and \( 1 \leq 1 + \frac{d_\mu(a, y)}{r} \), we could obtain:

\[
\left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1} \leq \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]

Then for \( d_\mu(a, b) \leq d_\mu(y, z) \), the Formula

\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]

(46)

holds. In a similar way, we will obtain the Formula (46) for the case when \( d_\mu(a, b) \geq d_\mu(y, z) \). Notice that by Formula (44),

\[
|K_2(r, a, y) - K_2(r, a, z)| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1},
\]

and

\[
|K_2(r, b, y) - K_2(r, b, z)| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(b, y)}{r} \right)^{-2\gamma - 1}
\]

hold. Then we could obtain

\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]

By the fact \( d_\mu(a, b) \geq d_\mu(y, z) \) and \( 1 \leq 1 + \frac{d_\mu(a, y)}{r} \), the following holds:

\[
\left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1} \leq \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]
Then for \( d_\mu(a,b) \geq d_\mu(y,z) \), we could get
\[
\left| (K_2(r,a,y) - K_2(r,b,y) - (K_2(r,a,z) - K_2(r,b,z)) \right|
\leq C \left( \frac{d_\mu(a,b)}{r} \right)^\gamma \left( \frac{d_\mu(y,z)}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{d_\mu(a,y)}{r} \right)^{-2(\gamma - \alpha) - 1}.
\] (47)

By the fact that \( x \in B_\mu(a,2r) \cap B_\mu(b,2r) \), we could deduce that:
\[
1 + \frac{d_\mu(a,y)}{r} \sim 1 + \frac{d_\mu(x,y)}{r}.
\] (48)

Formulas (45) (46) (47) (48) yeald the Proposition.

\[
\square
\]

Proposition 1.32. For any \( 0 < \gamma \leq 1 \), \( f \in L^1(\mathbb{R},\mu) \), if the following inequality holds
\[
\| f^{x_1}_{x_2} \|_{L^p(\mathbb{R},\mu)} \sim \| f^{x_2}_{x_1} \|_{L^p(\mathbb{R},\mu)}
\] then for \( 1 \geq p > \frac{1}{1 + \gamma} \), we could deduce that:
\[
\| f^{x_1}_{x_2} \|_{L^p(\mathbb{R},\mu)} \leq C \| f^{x_2}_{x_1} \|_{L^p(\mathbb{R},\mu)}
\]
where \( C \) is dependent on \( p \) and \( \gamma \), and \( i = 1, 2 \).

**Proof.** We will only prove the proposition when \( i = 2 \). For any fixed \( \alpha \) satisfying \( 0 < \alpha < \gamma \) and \( p > \frac{1}{1 + \gamma - \alpha} \), let \( F \) denote as:
\[
F = \left\{ x \in \mathbb{R} : f^{x_1}_{x_2}(x) \leq \sigma f^{x_2}_{x_1}(x) \right\}.
\]

By Proposition 1.14 and Proposition 1.26, we could deduce that the following holds for \( f \in L^1(\mathbb{R},\mu) \):
\[
\| f^{x_1}_{x_2} \|_{L^p(\mathbb{R},\mu)} \sim_{\gamma,\alpha} \| f^{x_2}_{x_1} \|_{L^p(\mathbb{R},\mu)} \sim_{\gamma,\alpha} \| f^{x_2}_{x_1} \|_{L^p(\mathbb{R},\mu)} \sim_{\gamma,\alpha} \| f^{x_1}_{x_2} \|_{L^p(\mathbb{R},\mu)}.
\]

Then it is clear that
\[
\int_{F^c} |f^{x_1}_{x_2}(x)|^p d\mu(x) \leq C \int_{F^c} |f^{x_2}_{x_1}(x)|^p d\mu(x) \leq \frac{C_{\alpha}}{\sigma^p} \int_{\mathbb{R}} |f^{x_2}_{x_1}(x)|^p d\mu(x) \leq \frac{C_{\alpha}}{\sigma^p} \int_{\mathbb{R}} |f^{x_1}_{x_2}(x)|^p d\mu(x). \] (49)

Choosing \( \sigma^p \geq 2C_{\alpha} \), we could have
\[
\int_{\mathbb{R}} |f^{x_1}_{x_2}(x)|^p d\mu(x) \leq \int_{F} |f^{x_1}_{x_2}(x)|^p d\mu(x). \] (50)

We use \( Df(x) \) and \( F(x,r) \) to denote as:
\[
Df(x) = \sup_{r > 0} \left| \int_{\mathbb{R}} f(t) K_2(r,x,t) \frac{d\mu(t)}{r} \right|, \quad F(x,r) = \int_{\mathbb{R}} f(t) K_2(r,x,t) \frac{d\mu(t)}{r}.
\]

Next, we will show that for any \( q > 0 \),
\[
|f^{x_1}_{x_2}(x)| \leq C [M_\mu(Df)^q(x)]^{1/q} \quad \text{for} \ x \in F, \] (51)

where \( M_\mu \) is the Hardy-Littlewood maximal operator. For any fixed \( x_0 \in F \), there exists \( (u_0,r_0) \) satisfying \( d_\mu(u_0,x_0) < r_0 \) such that the following inequality holds:
\[
|F(u_0,r_0)| > \frac{1}{2} f^{x_1}_{x_2}(x_0). \] (52)

Choosing \( \delta < 1 \) small enough and \( u \) satisfying \( d_\mu(u,u_0) < \delta r_0 \), we could deduce that
\[
|F(u,r_0) - F(u_0,r_0)| = \left| \int_{\mathbb{R}} f(y) K_2(r,u,y) d\mu(y)/r_0 - \int_{\mathbb{R}} f(y) K_2(r,u_0,y) d\mu(y)/r_0 \right|
\leq \left| \int_{\mathbb{R}} f(y) (K_2(r,u,y) - K_2(r,u_0,y)) d\mu(y)/r_0 \right|.
\]
We could consider \( K_2(r_0, u, y) - K_2(r_0, u_0, y) \) as a new kernel. By Proposition 1.31 and Proposition 1.13, we could obtain:
\[
|F(u, r_0) - F(u_0, r_0)| \leq C \delta^\alpha f_{2\gamma}^\alpha(x_0) \leq C \delta^\alpha \sigma f_{2\gamma}^\alpha(x_0) \quad \text{for } x_0 \in F.
\]

Taking \( \delta \) small enough such that \( C \delta^\alpha \sigma \leq 1/4 \), we obtain
\[
|F(u, r_0)| \geq \frac{1}{4} f_{2\gamma}^\alpha(x_0) \quad \text{for } u \in B_\mu(u_0, \delta r_0).
\]

Thus the following inequality holds: for any \( x_0 \in F \),
\[
\left| f_{2\gamma}^\alpha(x_0) \right|^q \leq \frac{1}{C \mu(u_0, \delta r_0)} \int_{B_\mu(u_0, \delta r_0)} 4^q |F(u, r_0)|^q d\mu(u)
\]
\[
\leq \frac{B_\mu(x_0, (1 + \delta)r_0)}{B_\mu(u_0, \delta r_0)} \left| \int_{B_\mu(x_0, (1 + \delta)r_0)} \int_{B_\mu(u_0, \delta r_0)} 4^q |F(u, r_0)|^q d\mu(u)
\]
\[
\leq \frac{1 + \delta}{\delta} \left| \int_{B_\mu(x_0, (1 + \delta)r_0)} \int_{B_\mu(u_0, \delta r_0)} 4^q |F(u, r_0)|^q d\mu(u)
\]
\[
\leq CM_\mu[(DF)^q](x_0)
\]

\( C \) is independent on \( x_0 \). Finally, using the maximal theorem for \( M_\mu \) when \( q < p \) leads to
\[
\int_F \left| f_{2\gamma}^\alpha(x) d\mu(x) \right|^p dx \leq C \int \left\{ M_\mu[(DF)^q](x) \right\}^{p/q} d\mu(x) \leq C \int \left| f_{2\gamma}^\alpha(x) \right|^p d\mu(x). \quad (53)
\]

Thus for any fixed \( \alpha \) satisfying \( 0 < \alpha < \gamma \) and \( p > \frac{1}{1 + \gamma - \alpha} \), the above Formula (53) combined with Formula (50) leads to
\[
\|f_{2\gamma}^\alpha\|_{L^p(\mathbb{R}, \mu)} \leq C\|f_{2\gamma}^\alpha\|_{L^p(\mathbb{R}, \mu)} , \quad (54)
\]

where \( C \) is dependent on \( p \) and \( \alpha \). Next we will remove the number \( \alpha \). For any \( p > \frac{1}{1 + \gamma} \), let
\[
p_0 = \frac{1}{2} \left( p + \frac{1}{1 + \gamma} \right) \quad \text{with} \quad p > p_0 > \frac{1}{1 + \gamma} \quad \text{and let} \quad \alpha = 1 + \gamma - \frac{1}{p_0}.
\]

Thus it is clear that
\[
p_0 = \frac{1}{1 + \gamma - \alpha}, \quad p > p_0.
\]

Thus by Formula (54), we could obtain the following inequality holds for \( 1 \geq p > \frac{1}{1 + \gamma} \)
\[
\|f_{2\gamma}^\alpha(x)\|_{L^p(\mathbb{R}, \mu)} \leq C\|f_{2\gamma}^\alpha(x)\|_{L^p(\mathbb{R}, \mu)}
\]

\( C \) is dependent on \( p \) and \( \gamma \). This proves the Proposition. \( \square \)

At last we will prove the following Proposition:

**Proposition 1.33.** For \( \frac{1}{1 + \gamma} < p \leq 1 \), \( 0 < \gamma \leq 1 \), \( f \in L^1(\mathbb{R}, \mu) \), there exists \( \beta > 0 \), such that the following conditions are equivalent:

(i) \( f_\beta^\gamma \in L^p(\mathbb{R}, \mu) \).

(ii) There is a \( \phi(x) \in SS_\beta \) satisfying \( \int \phi(x) dx \neq 0 \) so that \( M_{\phi\beta} f(x) \in L^p(\mathbb{R}, \mu) \).

(iii) \( f_{1\gamma}^\alpha(x) = \sup_{d_\mu(x,y)<r} |F_1(r,y,f)| \in L^p(\mathbb{R}, \mu) \).

(iv) \( f_\gamma^\beta(x) = \sup_{r>0} |F_1(r,x,f)| \in L^p(\mathbb{R}, \mu) \).

(v) \( f_\gamma^\beta \in L^p(\mathbb{R}, \gamma) \).

**Proof.** (i) \( \Rightarrow \) (ii) is obvious. (ii) \( \Rightarrow \) (i) is deduced from Proposition 1.23 and Proposition 1.24. (i) \( \Rightarrow \) (v) is deduced from Proposition 1.26. (iii) \( \Rightarrow \) (i) is deduced from Proposition 1.30. (iv) \( \Rightarrow \) (iii) is deduced from Proposition 1.32, Proposition 1.26, Proposition 1.13 and Proposition 1.30. (iii) \( \Rightarrow \) (iv) is obvious. (v) \( \Rightarrow \) (iii) is deduced from Proposition 1.13. This proves the proposition. \( \square \)
We define $H^p_\mu(\mathbb{R})$ and $\tilde{H}^p_\mu(\mathbb{R})$ as:

**Definition 1.34 (\(\tilde{H}^p_\mu(\mathbb{R})\) and \(H^p_\mu(\mathbb{R})\)).** \(H^p_\mu(\mathbb{R})\) is defined as:

\[ H^p_\mu(\mathbb{R}) \triangleq \{ g \in S'(\mathbb{R},d\mu) : g*_{\mu}\beta(x) \in L^p(\mathbb{R},\mu), \text{ for any } \beta > p^{-1} - 1 \}. \]

And its norm is given by

\[ \|g\|_{H^p_\mu(\mathbb{R})} = \int_{\mathbb{R}} |g*_{\mu}\beta(x)|^p d\mu(x). \]

\(\tilde{H}^p_\mu(\mathbb{R})\) is defined as:

\[ \tilde{H}^p_\mu(\mathbb{R}) \triangleq \{ g \in L^1(\mathbb{R},\mu) : g*_{\mu}\beta(x) \in L^p(\mathbb{R},\mu), \text{ for any } \beta > p^{-1} - 1 \}. \]

From Theorem 1.28, we could know that \(H^p_\mu(\mathbb{R})\) space is the completion of \(\tilde{H}^p_\mu(\mathbb{R})\) with \(\| \cdot \|_{\tilde{H}^p_\mu(\mathbb{R})}\) norm. Thus by Proposition 1.33 and Hahn-Banach Theorem, we could deduce the following:

**Theorem 1.35.** For \(\frac{1}{1+\gamma} < p \leq 1, 0 < \gamma \leq 1, f \in S'(\mathbb{R},d\mu,x), \) there exists \(\beta > 0, \) such that the following conditions are equivalent:

(i) \(\int_{\mu} g*_{\mu}\beta \in L^p(\mathbb{R},\mu);\)

(ii) There is a \(\phi(x) \in SS_\beta\) satisfying \(\int \phi(x)dx \neq 0\) so that \(M_{\phi}\beta f(x) \in L^p(\mathbb{R},\mu);\)

(iii) \(\int_{\mu} f*_{\mu}\beta \in L^p(\mathbb{R},\mu);\)

(iv) \(\int_{\mu} f*_{\mu}\gamma \in L^p(\mathbb{R},\mu);\)

(v) \(\int_{\mu} f*_{\mu}\gamma \in L^p(\mathbb{R},\mu);\)

(vi) \(H^p_\mu(\mathbb{R})\) space is the completion of \(\tilde{H}^p_\mu(\mathbb{R})\) with \(\| \cdot \|_{\tilde{H}^p_\mu(\mathbb{R})}\) norm.

## 2 Hardy spaces associated with the Dunkl setting

In this Section we will discuss the Hardy spaces associated with the one dimensional Dunkl setting. In section§2.1, we will give a real characterization of \(H^p_\mu(\mathbb{R}_+^2)\). We will use another way different from Burkholder-Gundy-Silverstein in [4], in a very simple way. In section§2.2, we will prove that \(H^p_\mu(\mathbb{R}_+^2)\) is a kind of Homogeneous Hardy spaces for \(\frac{1}{1+\gamma} < p \leq 1,\) then we could obtain the real-variable method of \(H^p_\mu(\mathbb{R}_+^2)\) by the theory of Homogeneous Hardy spaces.

### 2.1 Real Parts of function in \(H^p_\mu(\mathbb{R}_+^2)\) and maximal function

**Definition 2.1.** [17][20] For \(f \in L_1^1(\mathbb{R}) \cap L_\infty^\alpha(\mathbb{R}, x \in \mathbb{R}, y \in (0, \infty), \) we can define \(\lambda\)-Possion integral and conjugate \(\lambda\)-Possion integral by

\[ (P f)(x,y) = (f \ast\lambda P_y)(x) = c_\lambda \int_\mathbb{R} f(t)(\tau_x P_y)(-t)|t|^{2\lambda} dt, \]

\[ (Q f)(x,y) = (f \ast\lambda Q_y)(x) = c_\lambda \int_\mathbb{R} f(t)(\tau_x Q_y)(-t)|t|^{2\lambda} dt, \]

where \(\lambda\)-Possion kernel \((\tau_x P_y)(-t)\) has the representation

\[ (\tau_x P_y)(-t) = \frac{\lambda x^{\lambda+1}}{2^{\lambda-1/2}\pi} \int_0^\pi \frac{g(1+\text{sgn}(xt)\cos \theta)}{(y^2+x^2+t^2-2|xt|\cos \theta)^{\lambda+\frac{1}{2}}} \sin^{\lambda-1} \theta d\theta, \]

and \((\tau_x Q_y)(-t)\) is the conjugate \(\lambda\)-Possion kernel, with the following representation:

\[ (\tau_x Q_y)(-t) = \frac{\lambda x^{\lambda+1}}{2^{\lambda-1/2}\pi} \int_0^\pi \frac{(x-t)(1+\text{sgn}(xt)\cos \theta)}{(y^2+x^2+t^2-2|xt|\cos \theta)^{\lambda+\frac{1}{2}}} \sin^{\lambda-1} \theta d\theta. \]

The maximal functions are:

\[
Qf^*(x) = \sup_{\|x<-y\|}(Qf)(s,y), \quad Pf^*(x) = \sup_{\|x<-y\|}(Pf)(s,y), \quad \text{and} \quad F^*(x) = \sup_{\|x<-y\|}|F(s,y)|.
\]
Proposition 2.2. [20] Let $F \in H^p_\lambda(\mathbb{R}^2_+)$ and $f(x) \in L^p_\lambda(\mathbb{R})$, then the following hold:

(i) For $1 < p < \infty$, $\|Q\|_{L^p_\lambda} \leq c_p \|f\|_{L^p_\lambda}$, where $\|Q\|_{L^p_\lambda} \leq c_p \|f\|_{L^p_\lambda}$.

(ii) For $\frac{2}{1+p} < p < 1$, $F \in H^p_\lambda(\mathbb{R}^2_+)$ if and only if $F \in L^p_\lambda(\mathbb{R})$, and moreover $\|F\|_{H^p_\lambda} \geq \|F\|_{L^p_\lambda} \geq c \|F\|_{H^p_\lambda}$.

(iii) For $1 \leq p < \infty$, $F(x, y)$ has boundary values, and let $f(x)$ be the real part of the boundary values of $F(x, y)$ satisfying $F(x, y) = P f(x, y) + i Q f(x, y)$.

(iv) For $1 \leq p < \infty$, $P f(x, y)$ and $Q f(x, y)$ satisfy the generalized Cauchy-Riemann system (4) on $\mathbb{R}^2_+$.

Proposition 2.3. [17]/[20] Let $F(x, y) \in H^p_\lambda(\mathbb{R}^2_+)$, $f(x)$ to be the boundary value of $F(x, y)$ for $p > p_0 = \frac{2}{1+p}$, then the following hold:

(i) For almost every $x \in \mathbb{R}$, $\lim F(t, y) = f(x)$ exists as $(x, 0)$ approaches the point $(t, 0)$ non-tangentially.

(ii) $\lim_{y \to 0} F(t, y) = f(x)$, for $\frac{2}{1+p} < p < 1$. $\|F\|_{H^p_\lambda} = \|f\|_{L^p_\lambda}$ for $1 \leq p$. $\|F\|_{H^p_\lambda} \geq \|f\|_{L^p_\lambda} \geq \frac{2^{1-2/p}}{p} \|F\|_{H^p_\lambda}$ for $\frac{2}{1+p} < p < 1$, where $\|f\|_{L^p_\lambda} = (c_p \int_\mathbb{R} \frac{|f(x)|^p}{x^2} dx)^{1/p}$.

(iii) Let $p > \frac{2}{1+p}$, $p_1 > \frac{2}{1+p}$, $F(x, y) \in H^p_\lambda(\mathbb{R}^2_+)$, and $f \in L_{\lambda}^{\infty}(\mathbb{R})$, then $F(x, y) \in H^{p_1}_\lambda(\mathbb{R}^2_+)$.

Proposition 2.4. [17]/[20] For simplicity, we write $\tau_t u(x, y) = [\tau_t (u(.)), y)](x)$.

(1) If $u$ is twice continuously differentiable on $\mathbb{R}^2_+$ and satisfies $\triangle u = 0$, then for $(x, 0, y_0) \in \mathbb{R}^2_+$, $0 < r < y_0$, we have

$$u(x, 0, y_0) = \sigma_\lambda \int_{-\pi}^{\pi} (\tau_{\cos \theta} u)(x, 0, y_0 + r \sin \theta) |\cos \theta|^{2\lambda} d\theta,$$

where $\sigma_\lambda^{-1} = \int_{-\pi}^{\pi} |\cos \theta|^{2\lambda} d\theta = 2 \sqrt{\pi} \Gamma(\lambda + \frac{1}{2}) / \Gamma(\lambda + 1)$.

(2) For $f \in S(\mathbb{R}, dx)$, for fixed $t \in \mathbb{R}$, the function $x \to \tau_t f(x) \in S(\mathbb{R})$, and the following holds:

$$D_x(\tau_t f(x)) = D_x(\tau_t f(x)) = (\tau_t(Df))(x).$$

(3) For $f \in L^\infty_{\lambda}(\mathbb{R})$, the following holds for $t \in \mathbb{R}$ (We could use $\|\cdot\|_{\infty}$ instead of $\|\cdot\|_{L^\infty_{\lambda}(\mathbb{R})}$ for convenience):

$$\|\tau_t f\|_{L^\infty_{\lambda}(\mathbb{R})} \leq 4 \|f\|_{L^\infty_{\lambda}(\mathbb{R})}.$$

(4) For $1 < p < \infty$, $u(x, y)$ is a $\lambda$-harmonic function on $\mathbb{R}^2_+$. $u(x, y)$ is the $\lambda$-Poisson integral of some function $F(x) \in L^p_{\lambda}(\mathbb{R})$ if and only if $u(x, y)$ satisfies the following:

$$\sup_{t > 0} c_\lambda \int_{\mathbb{R}} |u(t, y)|^p |x|^{2\lambda} dx < \infty.$$

In [18], the dual of intertwining operator are introduced as follows.

Definition 2.5 (Dual of intertwining operator). [18] We use $\mathcal{V}_\lambda^* f$ to denote as the dual of intertwining operator:

$$\mathcal{V}_\lambda^*(f) = \mathcal{F}^{-1} \mathcal{F}_\lambda(f),$$

$(\mathcal{V}_\lambda^*)^{-1}$ to denote as:

$$(\mathcal{V}_\lambda^*)^{-1}(f) = \mathcal{F}^{-1} \mathcal{F}(f).$$

The properties of the dual of intertwining operator are as follows:

Proposition 2.6. [18] (i) $\mathcal{V}_\lambda^*$ is a topological automorphism on $S(\mathbb{R}, dx)$;

(ii) If $\text{supp} f \subseteq B(0, a)$, then $\text{supp} \mathcal{V}_\lambda^*(f) \subseteq B(0, a)$ and $\text{supp} (\mathcal{V}_\lambda^*)^{-1}(f) \subseteq B(0, a)$;

(iii) $\mathcal{V}_\lambda^*(Df)(x) = \frac{d}{dx} \mathcal{V}_\lambda^*(f)(x)$ for any $f \in S(\mathbb{R}, dx)$, where $D$ is the Dunkl operator.

By Proposition 2.6, we could deduce the following Proposition 2.7:

Proposition 2.7. For any $\phi \in S(\mathbb{R}, dx)$,

$$\sup_{x \in \mathbb{R}} |x|^\alpha D^\beta \phi(x) | < \infty.$$
Theorem 2.8. Let \( u(x, y) \) be a \( \lambda \)-harmonic function satisfying \( u^2 \in L^1_\alpha(\mathbb{R}) \). For \( \frac{2\lambda}{2\lambda + 1} < p < \infty \), there exists a \( \lambda \)-analytic function \( F(z) \in H^1_\lambda(\mathbb{R}^2_+) \) satisfying \( u(x, y) = \Re F(z) \) and
\[
\|F\|_{H^1_\lambda(\mathbb{R}^2_+)} \sim \|u^2\|_{L^1_\alpha(\mathbb{R})}.
\]

Proof. Case 1 \( 1 < p < \infty \): It is clear that part (2) of this Theorem holds for \( 1 < p < \infty \) by Proposition 2.4(4) and Proposition 2.2(1)(ii)(iv).

Case 2 \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \): Notice that the following inequality holds for any \( h \in \{ h : |x - h| < t \} \):
\[
|u(x, t)| \leq \sup_{|h-s|<t} |u(s, t)|.
\]

We could also deduce that
\[
\int_{\{|x-h|<t\}} |h|^{2\lambda}dh \sim |x|^{2\lambda}|t| \gtrsim |t|^{2\lambda+1} \text{ for } 0 < t \leq |x|/2, \text{ and } \int_{\{|x-h|<t\}} |h|^{2\lambda}dh \sim |t|^{2\lambda+1} \text{ for } t \geq |x|/2.
\]

Then for \( 0 < t \), we could have:
\[
|u(x, t)|^p \lesssim \frac{1}{t^{2\lambda+1}} \int_{\{|x-h|<t\}} |h|^{2\lambda}dh \sup_{|h-s|<t} |u(s, l)|^p|h|^{2\lambda}dh.
\]

Thus we could deduce the following Formula (57) holds:
\[
|u(x, y)| \lesssim \|u^2\|_{L^1_\alpha(\mathbb{R})} y^{-(2\lambda+1)/p}.
\] (57)

We define \( v(x, y) \) as the conjugate \( \lambda \)-harmonic function of \( u(x, y) \) as following:
\[
v(x, y) = -\int_y^{+\infty} D_x u(x, r)dr.
\] (58)

Next we will show that \( v(x, y) \) is a well defined function. We use \( \psi_{(\rho)}(\zeta, \xi) \) \((0 < \rho < \infty)\) to denote a radial positive function on \( \mathbb{R}^2 \) satisfying
\[
\text{supp} \psi_{(\rho)}(\zeta, \xi) \subseteq \{ (\zeta, \xi) : \sqrt{\zeta^2 + \xi^2} < \frac{\rho}{100} \}, \psi_{(\rho)}(\zeta, \xi) \in S(\mathbb{R}^2, dx),
\]
and
\[
\int_{\mathbb{R}^2} \psi_{(\rho)}(\zeta, \xi)|\zeta|^{2\lambda}d\zeta d\xi = 1, \|\psi_{(\rho)}\|_\infty \sim \frac{1}{\rho^{2\lambda+2}}.
\]

Thus it is clear that
\[
\|D_\zeta \psi_{(\rho)}(\zeta, \xi)\|_\infty \lesssim \frac{1}{\rho^{2\lambda+2}}, \|D_\zeta^2 \psi_{(\rho)}(\zeta, \xi)\|_\infty \lesssim \frac{1}{\rho^{2\lambda+4}}, \|D_\zeta D_\xi \psi_{(\rho)}(\zeta, \xi)\|_\infty \lesssim \frac{1}{\rho^{2\lambda+4}}.
\] (59)

By Proposition 2.4(1), we could write \( u(x, r) \) as following:
\[
\begin{align*}
u(x, r) & = \sigma_\lambda \int_{0}^{+\infty} \int_\mathbb{R} (\tau_x u)(-\zeta, r - \xi)\psi_{(\tau)}(\zeta, \xi)|\zeta|^{2\lambda}d\zeta d\xi \\
& = \sigma_\lambda \int_{0}^{+\infty} \int_\mathbb{R} u(s, t)\rho_{-s}\psi_{(r)}(x, r - t)|s|^{2\lambda}ds dt,
\end{align*}
\] (60)

where \( \sigma_\lambda^{-1} = \int_\pi^\pi |\cos \theta|^{2\lambda}d\theta = 2\sqrt{\pi}\Gamma(\lambda + \frac{1}{2})/\Gamma(\lambda + 1) \). Thus we could deduce that
\[
(s, t) \in \left\{ (s, t) : \sqrt{(x - s)^2 + (r - t)^2} < \frac{r}{10} \right\} \cup \left\{ (s, t) : \sqrt{(x + s)^2 + (r - t)^2} < \frac{r}{10} \right\}.
\]

We use \( A_{\mu, \nu} \) to denote as the set:
\[
A_{\mu, \nu} = \left\{ (s, t) : \sqrt{(\mu - s)^2 + (\nu - t)^2} < \frac{\nu}{10} \right\} \cup \left\{ (s, t) : \sqrt{(\mu + s)^2 + (\nu - t)^2} < \frac{\nu}{10} \right\}.
\]
Thus by Proposition 2.4(2)(3), Formula (57), Formula (60), Formula (59) we could deduce the following inequality:

\[ |D_x u(x, r)| = \left| \sigma \int_0^{+\infty} \int_R u(s, t) r^{-s} (D\psi_{(r)})(x, r - t)|s|^{2\lambda} ds dt \right| \tag{61} \]

\[ \lesssim \sup_{(s, t) \in A_{x, r}} |u(s, t)||D\psi_{(r)}(\zeta, \xi)||_{\infty} r^{2\lambda + 2} \]

\[ \lesssim r^{-(2\lambda + 1)/p} \frac{1}{r^{2\lambda + 3}} r^{2\lambda + 2} \]

\[ \lesssim r^{-(2\lambda + 1)/p} r^{-1}. \]

In a similar way, we could obtain the following inequality

\[ |(D_x)^2 u(x, r)| = \left| \sigma \int_0^{+\infty} \int_R u(s, t) r^{-s} (D^2\psi_{(r)})(x, r - t)|s|^{2\lambda} ds dt \right| \tag{62} \]

\[ \lesssim r^{-(2\lambda + 1)/p} r^{-2}. \]

Thus from Formula (58), Formula (61) and Formula (62), we could know that the integral of \( D_x u(x, r) \) and \((D_x)^2 u(x, r)\) are meaningful. Thus \( v(x, y) \), \( D_x v(x, y) \) and \( \partial_y v(x, y) \) are well defined functions. Thus it is not difficult to check that \( v(x, y) \) and \( u(x, y) \) satisfy the \( \lambda \)-Cauchy-Riemann equations:

\[ \begin{cases} D_x u(x, y) - \partial_y v(x, y) = 0, \\
\partial_y u(x, y) + D_x v(x, y) = 0. \end{cases} \]

Thus the function \( F(z) = u(x, y) + iv(x, y) \) is a \( \lambda \)-harmonic function and \( u(x, y) = ReF(z) \). By Formula (58), it is clear that the following inequality holds:

\[ |v(x, y)| = \left| -\int_y^{+\infty} D_x u(x, r) dr \right| \tag{63} \]

\[ = \sigma \left| \int_y^{+\infty} \int_0^{+\infty} \int_R u(s, t) (r^{-s}(D\psi_{(r)}))(x, r - t)|s|^{2\lambda} ds dt dr \right| \]

\[ \lesssim \left| \int_y^{+\infty} \left( \sup_{(s, t) \in A_{x, r}} |ru(s, t)| \right) \frac{1}{r^2} dr \right| \]

\[ \lesssim \left( \sup_{r \geq y > 0} \sup_{(s, t) \in A_{x, r}} |tu(s, t)| \right) \int_y^{+\infty} \frac{1}{r^2} dr. \]

By Formula (57), we could know that

\[ \sup_{r \geq y > 0} \sup_{(s, t) \in A_{x, r}} |tu(s, t)| < \infty. \]

Notice that the balls \( \{(s, t) : (s, t) \in A_{x, r}\} \) are in the cone \( \{(s, t) : |s - x| < |t - y|, t > \frac{y}{2}\} \), that is:

\( \{(s, t) : (s, t) \in A_{x, r}\} \subset \{(s, t) : |s - x| < |t - y|, t > \frac{y}{2}\}. \)

Thus we could deduce that

\[ \sup_{r \geq y > 0, (s, t) \in A_{x, r}} |tu(s, t)| \lesssim |yu_{\psi_{(r)}}(x, \frac{y}{2})| + |yu_{\psi_{(r)}}(-x, \frac{y}{2})|, \tag{64} \]

where \( u_{\psi_{(r)}}(x, \frac{y}{2}) \) denotes \( \sup_{|s| < y} |u(x + s, \frac{y}{2} + t)| \).

Thus by Formula (63) and Formula (64), we could obtain that:

\[ |v(x, y)| \lesssim |yu_{\psi_{(r)}}(x, \frac{y}{2})| \frac{1}{y} + |yu_{\psi_{(r)}}(-x, \frac{y}{2})| \frac{1}{y} \]

\[ \lesssim u_{\psi_{(r)}}(x) + u_{\psi_{(r)}}(-x). \tag{65} \]

Thus by Formula (65), we could deduce the following inequality for any \( y > 0 \):

\[ \int_{-\infty}^{+\infty} |v(x, y)|^p |x|^{2\lambda} dx \lesssim \int_{-\infty}^{+\infty} |u_{\psi_{(r)}}(x)|^p |x|^{2\lambda} dx \quad \text{for} \quad \frac{2\lambda}{2\lambda + 1} < p \leq 1. \]
Then for \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \), we could deduce that:

\[
\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \leq c\|u_F\|_{L^p_{\lambda}}. \tag{66}
\]

By Formula (66) and Proposition 2.2, we deduce the following inequality for \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \):

\[
\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \sim \|u_F\|_{L^p_{\lambda}}.
\]

This proves the Theorem. \(\square\)

**Proposition 2.9.** \(H^p_{\lambda}(\mathbb{R}^2_+) \cap H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+)\) is dense in \(H^p_{\lambda}(\mathbb{R}^2_+)\), for \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \).

**Proof.** From [20], we could know that for \( F(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+) \) and \( s > 0 \)

\[
\left( \int_{\mathbb{R}} |F(x,y+s)|^2 |x|^{2\lambda}dx \right)^{\frac{1}{2}} \leq c s^{(1/2-1/p)(1+2\lambda)} \|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}
\]

and

\[
\left( \int_{\mathbb{R}} |F(x,y+s)||x|^{2\lambda}dx \right)^{\frac{1}{2}} \leq c s^{-(1/p-1/1)(1+2\lambda)} \|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}
\]

hold for \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \). Thus we could deduce that \( F(x,y+s) \in H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \). By Proposition 2.3(ii), we could see that \( \lim_{s \to 0} ||F(\cdot, y+s) - F(\cdot, y)||_{L^p_{\lambda}} = 0 \). Then we could see that \( H^p_{\lambda}(\mathbb{R}^2_+) \cap H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \) is dense in \( H^p_{\lambda}(\mathbb{R}^2_+) \). This proves the proposition. \(\square\)

**Definition 2.10.** By Proposition 2.3 and Theorem 2.8, \( \tilde{H}^p_{\lambda}(\mathbb{R}) \) \( (\frac{2\lambda}{2\lambda+1} < p < \infty) \) could be defined as

\[
\tilde{H}^p_{\lambda}(\mathbb{R}) \triangleq \left\{ g(x) : g(x) = \lim_{y \to 0} \text{Re} F(t, y), F \in H^p_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \right\},
\]

\((t, y)\) approaches the point \((x, 0)\) nontangentially\}

with the norm:

\[
\|g\|_{\tilde{H}^p_{\lambda}(\mathbb{R})} \triangleq \|P_{\lambda}^p g\|_{L^p_{\lambda}(\mathbb{R})}.
\]

Thus

\[
\tilde{H}^p_{\lambda}(\mathbb{R}) \triangleq \left\{ g(x) \in L^1_{\lambda}(\mathbb{R}) \cap L^2_{\lambda}(\mathbb{R}) : \|P_{\lambda}^p g\|_{L^p_{\lambda}(\mathbb{R})} < \infty \right\}.
\]

Thus \( \tilde{H}^p_{\lambda}(\mathbb{R}) \) is a linear space equipped with the norm: \( \| \cdot \|_{\tilde{H}^p_{\lambda}(\mathbb{R})} \), which is not complete. The completion of \( \tilde{H}^p_{\lambda}(\mathbb{R}) \) with the norm \( \| \cdot \|_{\tilde{H}^p_{\lambda}(\mathbb{R})} \) is denoted as \( H^p_{\lambda}(\mathbb{R}) \). (We will also define \( H^p_{\lambda}(\mathbb{R}) \) as Theorem 2.22.)

Thus we could have the following conclusions:

**Proposition 2.11.** \( H^p_{\lambda}(\mathbb{R}) \cap H^2_{\lambda}(\mathbb{R}) \cap H^1_{\lambda}(\mathbb{R}) \) is dense in \( H^p_{\lambda}(\mathbb{R}) \) for \( \frac{2\lambda}{2\lambda+1} < p < \infty \). \( H^p_{\lambda}(\mathbb{R}) = L^p_{\lambda}(\mathbb{R}) \), for \( 1 < p < \infty \). \( H^1_{\lambda}(\mathbb{R}) \subset L^1_{\lambda}(\mathbb{R}). \)

### 2.2 Homogeneous type Hardy Spaces on Dunkl setting

In Definition 2.10, we have introduced the real-variable Hardy spaces: \( H^p_{\lambda}(\mathbb{R}) \) which is associated with the Complex-Hardy spaces \( H^p_{\lambda}(\mathbb{R}^2_+) \). In this section, we will prove that the \( H^p_{\lambda}(\mathbb{R}) \) is Homogeneous Hardy spaces.

We use \( d\mu_{\lambda}(x, y) \) to denote as: \( d\mu_{\lambda}(x, y) = (2\lambda + 1) \int_{\mathbb{R}} \frac{|t|^{2\lambda}dt}{\lambda} \). And the ball \( B(x, r) \) is denoted as: \( B(x, r) = \{ y : d(x, y) < r \} \).

We will introduce a new kernel \( K(r, x, t) \) as following:

\[
K(r, x, t) = \begin{cases} 
(\tau_x P_{\lambda})_{r^{-2\lambda+1}}(t) & \text{for } r < |x|^{2\lambda+1}, \\
(r \tau_x P_{\lambda+1, 2\lambda+1})_{r^{-2\lambda+1}}(t) & \text{for } r \geq |x|^{2\lambda+1}. 
\end{cases} \tag{67}
\]
Thus $K(r, x, t) = r(\pi x P_{y})(-t)$, where $y$ has the representation

$$y = \begin{cases} r|x|^{-2\lambda} & \text{for } r < |x|^{2\lambda+1}, \\ r^{1/(2\lambda+1)} & \text{for } r \geq |x|^{2\lambda+1}. \end{cases} \quad (68)$$

Then for any $f(x) \in L^{2}_{\lambda}(\mathbb{R}) \cap L^{1}_{\lambda}(\mathbb{R}) \cap H^{2}_{\lambda}(\mathbb{R})$, $\frac{2\lambda}{2\lambda+1} < p \leq 1$, the following holds:

$$\sup_{r > 0} \int_{\mathbb{R}} K(r, x, t) f(t) \frac{d^2u}{r} = \sup_{y > 0} (P_{y} * f)(x). \quad (69)$$

From[20], the following inequality holds:

$$(\pi x P_{y})(-t) \sim \frac{y^2 + (|x| + |t|)^2}{y^2 + (x-t)^2} \ln \left( \frac{y^2 + (x-t)^2 + 2}{y^2 + (x+t)^2 + 2} \right). \quad (70)$$

Then we will prove the following Theorem 2.12.

**Theorem 2.12.** $K(r, x, t) = r(\pi x P_{y})(-t)$ is a kernel satisfying the following:

(i) $K(r, x, t) \geq 1$, for $r > 0, x \in \mathbb{R}$;

(ii) $0 \leq K(r, x, t) \lesssim \left( 1 + \frac{2(\pi x t)}{r} \right)^{-1-\gamma_{\lambda}}$, for $r > 0, x, t \in \mathbb{R}$;

(iii) For $r > 0, x, t, z \in \mathbb{R}$, if $d_{\lambda}(zt) \leq C \min \{ 1 + \frac{d_{\lambda}(xz)}{r}, 1 + \frac{d_{\lambda}(xz)}{r} \}$

$$|K(r, x, t) - K(r, x, z)| \lesssim \left( \frac{d_{\lambda}(t, z)}{r} \right)^{\gamma_{\lambda}} \left( 1 + \frac{d_{\lambda}(x, t)}{r} \right)^{-1-2\gamma_{\lambda}};$$

(iv)

$$K(r, x, y) = K(r, y, x),$$

where $\gamma_{\lambda} = \frac{1}{2(2\lambda+1)}$.

**Proof.** $K(r, x, y) = K(r, y, x)$ can be deduced from the fact that $(\pi x P_{y})(-t) = (\pi y P_{y})(-x)$. Notice that for any $s \neq 0$, we have

$$K(|s|^{2\lambda+1} r, sx, st) = K(r, x, t), \quad d_{\lambda}(sx, st) = |s|^{-2\lambda-1} d_{\lambda}(x, t).$$

Thus we need to only prove the theorem for the case when $x = 0$ and $x = 1$. First, we will prove $K(r, x, y) \geq c > 0$ for some constant $c$.

**Case 1** $x = 0$. By Formula (68), we could deduce that $y = r^{1/(2\lambda+1)}$. Thus from Formula (70), we could deduce that

$$K(r, 0, 0) = \frac{r * r^{1/(2\lambda+1)}}{(r^{2/(2\lambda+1)})^{\lambda+1}} \geq 1.$$  

**Case 2** $x \neq 0$, we need only to consider the case when $x = 1$.

When $r < 1$, by Formula (67) and Formula (68), we have $y = r < 1$. Thus from Formula (55), we could deduce that:

$$K(r, 1, 1) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{-\lambda-1/2} \pi} \int_{0}^{\pi} \frac{rg(1 + \cos \theta)}{(y^2 + 2 - 2 \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta.$$

$$\geq c \int_{0}^{\pi/4} \frac{rg(1 + \cos \theta)}{(y^2 + 2 - 2 \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta$$

$$\geq c.$$

When $r \geq 1$, from Formula (67) and Formula (68), we could deduce that $y = r^{1/(2\lambda+1)} \geq 1$. Thus we could obtain the following from Formula (70):

$$K(r, 1, 1) \geq \frac{r^{1/(2\lambda+1)}/r}{(r^{2/(2\lambda+1)} + 2)^{\lambda+1}} \geq c.$$
Second, we will prove that $0 \leq K(r, x, t) \leq A(1 + \frac{d_{\lambda}(x, t)}{r})^{-1-\gamma_{\lambda}}$, for $r > 0, x, t \in \mathbb{R}$.

Case 1 When $x=0$, by Formula (68), we could deduce that $y = r^{1-\lambda}$. Thus from Formula (70) the following holds:

$$K(r, 0, t) \sim C\left(1 + \frac{t^2}{r^{2/(2\lambda+1)}}\right)^{\lambda-1} \sim A\left(1 + \frac{|t|^{2\lambda+1}}{(2\lambda+1)r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}} = A\left(1 + \frac{d_{\lambda}(0, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

Case 2 When $x \neq 0$, we need only to consider the case for $x=1$. Notice that $y = r^{1-\lambda} \geq 1$ for $r \geq 1$, and $y = r$ for $r < 1$. By Formula (70), we could have

\begin{equation}
\begin{aligned}
\text{when } r \geq 1 & \quad K(r, 1, t) \sim \begin{cases}
\frac{2\lambda+2}{r^{2/(2\lambda+1)} + t^2 + 1} \lambda \ln \left(\frac{r^2 + t^2 + 1}{r^2 + (t+1)^2} + 1\right) & \text{for } t < 0, \\
\frac{2\lambda+2}{r^{2/(2\lambda+1)} + t^2 + 1} \lambda \left(r^{2/(2\lambda+1)} + (1-t)^2\right) & \text{for } t \geq 0.
\end{cases} \\
\text{when } r < 1 & \quad K(r, 1, t) \sim \begin{cases}
\frac{r^2}{(r^2 + t^2 + 1)^{\lambda+1}} & \text{for } t < 0, \\
\frac{r^2}{(r^2 + t^2 + 1)^{\lambda+1}} \left(r^2 + (1-t)^2\right) & \text{for } t \geq 0.
\end{cases}
\end{aligned}
\end{equation}

If $r < 1$, $1/2 \leq t \leq 3/2$, we have $d_{\lambda}(1, t) \sim |1-t|$. Then

$$K(r, 1, t) \lesssim \left(1 + \frac{|1-t|}{r}\right)^{-2} \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-2}.$$  

If $r \geq 1$, $1/2 \leq t \leq 3/2$, we have $d_{\lambda}(1, t) \sim |1-t|$. Then

$$K(r, 1, t) \lesssim r^{\frac{2\lambda+1}{2\lambda+1}} (r + |1-t|)^{-\frac{2\lambda+1}{2\lambda+1}} \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

If $r < 1$, $t \geq 3/2$, we have $d_{\lambda}(1, t) \sim |1-t|^{2\lambda+1}$. Then

$$K(r, 1, t) \lesssim r^{2} (|1-t|)^{-2(\lambda+1)} \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

If $r \geq 1$, $t \geq 3/2$, we have $d_{\lambda}(1, t) \sim |1-t|^{2\lambda+1}$. Then

$$K(r, 1, t) \lesssim r^{\frac{2\lambda+1}{2\lambda+1}} \left(r^{2/(2\lambda+1)} + |1-t|^2\right)^{-(\lambda+1)} \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

If $r < 1$, $-2 \leq t \leq 1/2$, we have $d_{\lambda}(1, t) \sim 1$. Then

$$K(r, 1, t) \lesssim r^{2} \ln(r^{-1} + 1) \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

If $r \geq 1$, $-2 \leq t \leq 1/2$, we have $d_{\lambda}(1, t) \sim 1$. Then

$$K(r, 1, t) \lesssim C \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$  

If $r < 1$, $t \leq -2$, we have $d_{\lambda}(1, t) \sim t^{2\lambda+1}$. Then

$$K(r, 1, t) \lesssim C \frac{r^2}{|t|^{2(\lambda+1)}} \lesssim \left(1 + \frac{d_{\lambda}(1, t)}{r}\right)^{-\frac{2(\lambda+1)}{2\lambda+1}}.$$
If \( r \geq 1, t \leq -2 \), we have \( d_\lambda(1, t) \sim t^{2\lambda+1} \). Then
\[
K(r, 1, t) \lesssim C \left( \frac{r^{2\lambda+2}}{(r^{2\lambda+1} + t^2)^{\lambda+1}} \right) \lesssim \left( 1 + \left( \frac{d_\lambda(1, t)}{r} \right) \right)^{-2(\lambda+1)\left(\frac{2\lambda+1}{2\lambda+2}\right)}.
\]
Thus we have established
\[
0 \leq K(r, x, t) \lesssim \left( 1 + \frac{d_\lambda(x, t)}{r} \right)^{-2(\lambda+1)\left(\frac{2\lambda+1}{2\lambda+2}\right)}, \text{ for } r > 0, x, t \in \mathbb{R}.
\] (73)

From the above Formula (73), we could deduce that
\[
0 \leq K(r, x, t) \lesssim \left( 1 + \frac{d_\lambda(x, t)}{r} \right)^{-1-\gamma}, \text{ for } r > 0, x, t \in \mathbb{R}.
\]

At last, if \( \frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(x, t)}{r}, 1 + \frac{d_\lambda(x, z)}{r} \} \), we will prove the following inequality
\[
|K(r, x, t) - K(r, x, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{-1-\gamma} \left( 1 + \frac{d_\lambda(x, t)}{r} \right)^{-1-\gamma}
\]
for \( r > 0, x, t, z \in \mathbb{R} \). If \( \frac{d_\lambda(t, z)}{r} \lesssim 1 + \frac{d_\lambda(x, t)}{r} \), then we could deduce the following inequality:
\[
\frac{d_\lambda(x, z)}{r} \lesssim \left( \frac{d_\lambda(x, t)}{r} + \frac{d_\lambda(t, z)}{r} \right) \lesssim \left( \frac{d_\lambda(x, t)}{r} + 1 + \frac{d_\lambda(x, t)}{r} \right) \lesssim 1 + \frac{d_\lambda(x, t)}{r}.
\]

Then
\[
1 + \frac{d_\lambda(x, z)}{r} \lesssim 1 + \frac{d_\lambda(x, t)}{r}.
\]

Thus we could deduce:
\[
1 + \frac{d_\lambda(x, z)}{r} \sim 1 + \frac{d_\lambda(x, t)}{r}.
\] (74)

For \( u \in \mathbb{R} \) satisfying \((u-t)(u-z) \leq 0\), we could obtain
\[
\frac{d_\lambda(u, t)}{r} \lesssim \frac{d_\lambda(t, z)}{r} \lesssim C \min\{1 + \frac{d_\lambda(x, t)}{r}, 1 + \frac{d_\lambda(x, z)}{r} \}.
\]
Thus:
\[
1 + \frac{d_\lambda(x, u)}{r} \sim 1 + \frac{d_\lambda(x, t)}{r}, \text{ when } (u-t)(u-z) \leq 0.
\] (75)

It is enough to prove that if \( \frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(x, t)}{r}, 1 + \frac{d_\lambda(x, z)}{r} \} \), then
\[
\left( 1 + \frac{d_\lambda(x, t)}{r} \right)^{-1+2\gamma} |K(r, x, t) - K(r, x, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma}
\]
(76)

Let \( t, z \) to be fixed first. We could see that
\[
|t - z| \lesssim_\lambda \left( \frac{d_\lambda(t, z)}{r} \right)^{\frac{2\lambda+1}{2\lambda+2}}.
\] (77)

**Case 1** When \( x=0 \) \((y = \frac{r^{2\lambda+1}}{2\lambda+2})\), we suppose that \( z > 0 \) first. By Formula (75), we could obtain the following inequality for \((u-t)(u-z) \leq 0\):
\[
1 + \frac{d_\lambda(0, u)}{r} \sim 1 + \frac{d_\lambda(0, z)}{r} \sim 1 + \frac{d_\lambda(0, t)}{r} \sim 1 + \frac{u^{2\lambda+1}}{r}.
\]

By the Mean value theorems for definite integrals, we could have:
\[
\left( 1 + \frac{d_\lambda(0, z)}{r} \right)^{\frac{2\lambda+3}{2\lambda+2}} |K(r, 0, t) - K(r, 0, z)|
\]
\[
= \ c_\lambda (1 + \frac{d_\lambda(0, z)}{r} \right)^{\frac{2\lambda+3}{2\lambda+2}} \int_0^\pi r \left( \frac{y}{(y^2 + t^2)^{\lambda+1}} - \frac{y}{(y^2 + z^2)^{\lambda+1}} \right) \sin^{2\lambda-1} \theta \, d\theta
\]
\[
\lesssim \left( 1 + \frac{u^{2\lambda+1}}{r} \right)^{\frac{2\lambda+3}{2\lambda+2}} \left( \frac{ur^{\frac{2\lambda+3}{2\lambda+2}}}{(r^{\frac{2\lambda+1}{2\lambda+2}} + u^2)^{\lambda+2}} \right) |t - z|.
\]
Thus when \( \frac{d_{\lambda}(0,t)}{r} \leq C \min\{1 + \frac{d_{\lambda}(0,t)}{r}, 1 + \frac{(0,t)}{r}\} \), the following inequality holds:

\[
\left( 1 + \frac{d_{\lambda}(0,t)}{r} \right)^{\frac{2\lambda+3}{\lambda+1}} |K(r,0,t) - K(r,0,z)| \lesssim \frac{|t-z|}{r^{\frac{2\lambda+3}{\lambda+1}}} \lesssim \left( \frac{d_{\lambda}(t,z)}{r} \right)^{\frac{1}{\lambda+1}}. \tag{78}
\]

**Case 2** When \( x \neq 0 \), it will be enough to prove Formula (76) for the case when \( x = 1 \). From Formula (55), we could write \( K(r,1,t) = r(\tau_{1}P_{0})(-t) \) as following:

\[
r(\tau_{1}P_{0})(-t) = \frac{\lambda!}{2^{-\lambda-1/2}\pi} \int_{-1}^{1} \frac{ry}{(y^{2} + 1 + t^{2} - 2ts)^{\lambda+1}} (1 + s)(1-s^{2})^{-1}ds. \tag{79}
\]

By Formula (79) and Mean value theorem for definite integrals, we could obtain:

\[
|K(r,1,t) - K(r,1,z)| \sim \left| \int_{-1}^{1} \left( \frac{ry(1-s^{2})^{\lambda-1}(1+s)}{(y^{2} + 1 + t^{2} - 2ts)^{\lambda+1}} - \frac{ry(1-s^{2})^{\lambda-1}(1+s)}{(y^{2} + 1 + z^{2} - 2zs)^{\lambda+1}} \right) ds \right| 
\lesssim \left| \int_{-1}^{1} \frac{ry|u-s|}{(y^{2} + 1 + u^{2} - 2us)^{\lambda+2}} (1 - s^{2})^{\lambda-1}(1+s)ds \right| |t-z|, \tag{80}
\]

where \( u \) satisfies \((u-t)(u-z) \leq 0 \). Then we will discuss the Formula (80) for three conditions:

**Condition A** \( u \geq 0 \), **Condition B** \( u \leq -3/2 \) or \(-1/2 \leq u \leq 0 \), and **Condition C** \(-3/2 \leq u \leq -1/2 \).

**Condition A** \( u \geq 0 \).

When \( u \geq 0 \), for \( \frac{d_{\lambda}(0,t)}{r} \leq C \min\{1 + \frac{d_{\lambda}(0,t)}{r}, 1 + \frac{(0,t)}{r}\} \), we will prove the following inequality:

\[
\left( 1 + \frac{d_{\lambda}(0,t)}{r} \right)^{\frac{2\lambda+3}{\lambda+1}} |K(r,1,t) - K(r,1,z)| \lesssim \left( \frac{d_{\lambda}(t,z)}{r} \right)^{\frac{1}{\lambda+1}}. \tag{81}
\]

By Formula (79), Formula (75) and Mean value theorems for definite integrals, we could obtain:

\[
\left( 1 + \frac{d_{\lambda}(1,t)}{r} \right)^{\frac{2\lambda+3}{\lambda+1}} |K(r,1,t) - K(r,1,z)| 
\lesssim \left| \left( 1 + \frac{d_{\lambda}(1,t)}{r} \right)^{\frac{2\lambda+3}{\lambda+1}} \int_{-1}^{1} \frac{ry|u-s|}{(y^{2} + 1 + u^{2} - 2us)^{\lambda+2}} (1 - s^{2})^{\lambda-1}(1+s)ds \right| |t-z|, \tag{81}
\]

where \( u \) satisfies \((u-t)(u-z) \leq 0 \).

Notice that the following Formulas (82), (83), (84) hold for \(-1 \leq s \leq 1 \) and \( u \geq 0 \):

\[
\left| \frac{u - 1}{(y^{2} + 1 + u^{2} - 2us)} \right| < \left| \frac{u - 1}{(y^{2} + 1 + u^{2} - 2u)} \right|. \tag{82}
\]

For \( 0 \leq s \leq 1 \), we have:

\[
\left| \frac{1 - s}{(y^{2} + 1 + u^{2} - 2us)} \right| \lesssim \frac{1}{(y^{2} + 1 + u^{2})}. \tag{83}
\]

For \(-1 \leq s \leq 0 \), we have:

\[
\left| \frac{1}{(y^{2} + 1 + u^{2} - 2us)} \right| \lesssim \frac{1}{(y^{2} + 1 + u^{2})}. \tag{84}
\]
From Formula (81), Formula (82), Formula (83), Formula (84), and Formula (70), we could obtain the following Formula (85):

\[
\begin{aligned}
&\left\{ r y|u - s| \left( \frac{1}{(y^2 + 1 + u^2 - 2us)^{\lambda/2}} \right) \left( 1 - s^2 \right)^{\lambda-1} (1 + s) ds \right\} (t - z) \\
\leq &\left\{ r y|u - 1| \left( \frac{1}{(y^2 + 1 + u^2 - 2us)^{\lambda/2}} \right) \left( 1 - s^2 \right)^{\lambda-1} (1 + s) ds \right\} (t - z) \\
\leq &\left\{ C \left| \frac{|u - 1|}{(y^2 + 1 + u^2 - 2us)^{\lambda/2}} \right| r y \left| \tau_0 P_{y} \right| (-u) (t - z) \right\} + C \left| \frac{1}{(y^2 + 1 + u^2)^{\lambda/2}} \right| r y \left| \tau_0 P_{y} \right| (-u) (t - z) \\
\leq &\left\{ C \left| \frac{(t - z) y r \left( 1 - |u| \right)^2 + y^2 + (1 + u^2 + y^2) (1 - |u|)}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \right| \\
\right. \\
\end{aligned}
\]

i: If \( r < 1 \), then \( y = r \).

**ConditionA.** For \( r < 1 \), \( |1 - |u|| \geq \frac{1}{10C} \) (for some constant \( C > 1 \)), we could deduce that \( d_{\lambda}(1,|u|) \gtrsim \frac{1}{10C} \). Thus the following could be obtained by Formula (77):

\[
\begin{aligned}
&\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
= &\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\lesssim &\left\{ \frac{1}{r} \frac{d_{\lambda}(t,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\end{aligned}
\]

**ConditionA.** For \( r < \frac{1}{20C} \leq |1 - |u|| \leq \frac{1}{10C} \), it is clear that \( d_{\lambda}(1,|u|) \sim |1 - |u|| \), \( d_{\lambda}(t,|u|) \lesssim r + d_{\lambda}(1,|u|) \leq C \left| d_{\lambda}(1,|u|) \right| \leq \frac{1}{10C} \). Let \( C \) to be a constant satisfying \( \frac{C}{2} \leq 1 \), thus we could deduce that \( d_{\lambda}(t,|u|) \leq \frac{1}{10C} \). Then we could obtain that \( d_{\lambda}(t,|u|) \sim |t - z| \). Thus

\[
\begin{aligned}
&\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
= &\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\lesssim &\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\end{aligned}
\]

**ConditionA.** For \( r < 1 \), \( |1 - |u|| \leq \frac{1}{20C} \), we have \( d_{\lambda}(1,|u|) \sim |1 - |u|| \), \( |t - z| \sim d_{\lambda}(t,|u|) \lesssim r + d_{\lambda}(1,|u|) \lesssim r \), then

\[
\begin{aligned}
&\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
= &\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\lesssim &\left\{ \frac{1}{r} \frac{d_{\lambda}(t,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\end{aligned}
\]

**ConditionA.** ii: If \( r \geq 1 \), then \( y = r \). Thus

\[
\begin{aligned}
&\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
= &\left\{ \frac{1}{r} \frac{d_{\lambda}(1,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\lesssim &\left\{ \frac{1}{r} \frac{d_{\lambda}(t,|u|)}{d_{\lambda}(t,|u|)} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\end{aligned}
\]

\[
\begin{aligned}
&\left\{ \frac{|u|^{2\lambda+1}}{r^{2\lambda+1}} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \lesssim \frac{d_{\lambda}(t,|u|)}{r^{2\lambda+1}} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \right\} \left( t - z \right) \frac{2}{\lambda + 1} y r \frac{1 - |u|}{((1 - |u|)^2 + y^2)^2 (1 + u^2 + y^2)^{\lambda+1}} \\
\end{aligned}
\]
Thus we have proved the following inequality when \( u \geq 0 \):

\[
\left( 1 + \frac{d_{x}(1,t)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \left| K(r, 1, t) - K(r, 1, z) \right| \lesssim \left( \frac{d_{x}(t, z)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \tag{86}
\]

for \( \frac{d_{x}(t, z)}{r} \leq C \min\{1 + \frac{d_{x}(1,t)}{r}, 1 + \frac{d_{x}(1,z)}{r}\} \).

**Condition** if \( u \leq -3/2 \) or \(-1/2 \leq u \leq 0 \).

When \( u \leq -3/2 \) or \(-1/2 \leq u \leq 0 \), we could obtain:

\[
\left( 1 + \frac{d_{x}(1,t)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \left| K(r, 1, t) - K(r, 1, z) \right| \lesssim \left( \frac{d_{x}(t, z)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \tag{87}
\]

Notice that \( \frac{d_{x}(1,t)}{r} \sim \left( 1 + \frac{d_{x}(1,u)}{r} \right) \) when \( u \leq -3/2 \) or \(-1/2 \leq u \leq 0 \). Thus by Formula (79), Formula (75) and Mean value theorems for definite integrals, we could obtain:

\[
\left( 1 + \frac{d_{x}(1,u)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \left| K(r, 1, t) - K(r, 1, z) \right| \lesssim \left( \frac{d_{x}(t, z)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \tag{88}
\]

Notice that the following inequality hold for \(-1 \leq s \leq 1 \):

\[
\left| \frac{u + 1}{(y^2 + 1 + u^2 - 2us)} \right| \leq \frac{1}{(y^2 + 1 + u^2)} \tag{89}
\]

For \(-1 \leq s \leq 0 \), we have:

\[
\left| \frac{1}{(y^2 + 1 + u^2 - 2us)} \right| \lesssim \frac{1}{(y^2 + 1 + u^2)} \tag{90}
\]

From Formula (87) Formula (88) Formula (89) Formula (90) and Formula (70), we could obtain

\[
\left| \int_{-1}^{1} \frac{ry|u - s|}{(y^2 + 1 + u^2 - 2us)^{\lambda+2}} (1 - s^2)^{\lambda-1}(1 + s)ds (t - z) \right|
\]

\[
\leq \left( \int_{-1}^{1} \frac{ry|u + 1|(1 - s^2)^{\lambda-1}(1 + s)}{(y^2 + 1 + u^2 - 2us)^{\lambda+2}}ds + \int_{-1}^{1} \frac{ry(1 - s^2)^{\lambda-1}(1 + s)^2}{(y^2 + 1 + u^2 - 2us)^{\lambda+2}}ds \right) |(t - z)|
\]

\[
\lesssim \frac{|u + 1|}{(y^2 + 1 + u^2 - 2|u|)} |r(\tau_{1}P_{u})(-u)| (t - z) + \frac{1}{(y^2 + 1 + u^2)} \frac{|r(\tau_{1}P_{u})(-u)| (t - z)}{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^2(1 - |u|)^2 + y^2} \tag{91}
\]

\[
\leq C (t - z) yr \frac{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^2(1 - |u|)}{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^2} \tag{92}
\]

From Formula (92), similar to the case **Condition** if \( u \geq 0 \), we could deduce the following inequality:

\[
\left( 1 + \frac{d_{x}(1,t)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \left| K(r, 1, t) - K(r, 1, z) \right| \lesssim \left( \frac{d_{x}(t, z)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \tag{93}
\]

**Condition** if \( u \geq 0 \), we could deduce the following inequality:

\[
\left( 1 + \frac{d_{x}(1,t)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \left| K(r, 1, t) - K(r, 1, z) \right| \lesssim \left( \frac{d_{x}(t, z)}{r} \right)^{\frac{2\lambda}{2\lambda+1}} \tag{93}
\]
Thus we have proved the following inequality when \( u \leq -3/2 \) or \(-3/2 \leq u \leq 0\):
\[
\left(1 + \frac{d_\lambda(1,t)}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} |K(r,1,t) - K(r,1,z)| \lesssim \left(\frac{d_\lambda(t,z)}{r}\right)^{\frac{1}{2\lambda + 1}}
\]
(94)
for \( \frac{d_\lambda(t,z)}{r} \leq C \min\{1 + \frac{d_\lambda(1,t)}{r}, 1 + \frac{d_\lambda(1,z)}{r}\} \).

**Condition C:** \( -3/2 \leq u \leq -1/2 \)

Notice that \( d_\lambda(1,u) \sim 1 \) and \( 1 + \frac{d_\lambda(1,u)}{r} \sim 1 + \frac{d_\lambda(1,t)}{r} \sim 1 + \frac{d_\lambda(1,z)}{r} \) for \( \frac{d_\lambda(t,z)}{r} \leq C \min\{1 + \frac{d_\lambda(1,t)}{r}, 1 + \frac{d_\lambda(1,z)}{r}\} \). Thus by Formula (92), we could deduce that:
\[
\left(1 + \frac{d_\lambda(1,t)}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} |K(r,1,t) - K(r,1,z)|
\lesssim \left(1 + \frac{d_\lambda(1,u)}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} |K(r,1,t) - K(r,1,z)|
\lesssim \left(1 + \frac{1}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} y r (1 - |u|)^2 + y^2 + (1 + u^2 + y^2)(1 - |u|) \frac{|(t - z)|}{(1 - |u|)^2 + y^2) (1 + u^2 + y^2)^{\lambda+2}}
\]
(95)

**Condition C1:** When \( r > 1 \), we could deduce that \( y = r^{\frac{1}{2\lambda + 1}} \). By Formula (77) and Formula (95), we could deduce that
\[
\left(1 + \frac{d_\lambda(1,t)}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} |K(r,1,t) - K(r,1,z)|
\lesssim \left(1 + \frac{1}{r}\right)^{\frac{2\lambda + 3}{2\lambda + 1}} y r (1 - |u|)^2 + y^2 + (1 + u^2 + y^2)(1 - |u|) \frac{|(t - z)|}{(1 - |u|)^2 + y^2) (1 + u^2 + y^2)^{\lambda+2}}
\]
\[
\lesssim \left(\frac{d_\lambda(t,z)}{r}\right)^{\frac{2\lambda + 1}{2\lambda + 1}} \frac{|(t - z)|}{r^{\frac{1}{2\lambda + 1}}}
\]
(96)

**Condition C2:** When \( 0 < r \leq 1 \) and \( |t - z| \geq 1/4 \), we could deduce that \( y = r \) and \( d_\lambda(t,z) \geq C \) for some constant. Also it is clear that
\[
\frac{(\frac{1}{2})^{\frac{2\lambda + 1}{2\lambda + 1}}}{(1 + \frac{1}{2})^{\frac{2\lambda + 1}{2\lambda + 1}}} \sim 1.
\]

Thus from the above Formula (73), we could deduce that
\[
|K(r,1,t) - K(r,1,z)| \lesssim \left(1 + \frac{d_\lambda(1,t)}{r}\right)^{-\frac{2\lambda + 1}{2\lambda + 1}} (1 + \frac{d_\lambda(1,t)}{r})^{-\frac{2\lambda + 1}{2\lambda + 1}} d_\lambda(t,z)^{\frac{1}{2\lambda + 1}}
\]
(97)

**Condition C3:** When \( 0 < r \leq 1 \) and \( r/4 \leq |t - z| \leq 1/4 \), with the fact that \(-3/2 \leq u \leq -1/2 \) we could deduce that \( y = r \) and \( d_\lambda(t,z) \sim |t - z| \). Thus it is clear that
\[
1 \lesssim \left(\frac{d_\lambda(t,z)}{r}\right)^{\frac{1}{\gamma}}
\]
And we could also deduce that \( d_\lambda(1,t) \sim d_\lambda(1,u) \sim d_\lambda(1,z) \sim 1 \). Thus from the above Formula (70), we could obtain:
\[
|K(r,1,t) - K(r,1,z)| \lesssim \left(1 + \frac{d_\lambda(1,t)}{r}\right)^{-\frac{2\lambda + 1}{2\lambda + 1}} (1 + \frac{d_\lambda(1,t)}{r})^{-\frac{2\lambda + 1}{2\lambda + 1}} d_\lambda(t,z)^{\frac{1}{2\lambda + 1}}
\]
(98)
Thus by Formula (91) and Formula (70), we could deduce that:

\[
(1 + \frac{d_\lambda(1, t)}{r})^{\frac{2}{\lambda + 1}} |K(r, 1, t) - K(r, 1, z)| \\
\lesssim |(t - z)| \frac{r \ln r}{r^{\frac{2}{\lambda + 1}}} \\
\lesssim \frac{d_\lambda(t, z)}{r} \leq \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda}.
\]

Thus we could obtain that

\[
|K(r, 1, t) - K(r, 1, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda} \left(1 + \frac{d_\lambda(1, t)}{r}\right)^{-1 - 2\gamma_\lambda}.
\] (99)

Notice that

\[
\left(1 + \frac{d_\lambda(1, t)}{r}\right)^{-1} \frac{d_\lambda(t, z)}{r} \leq \left(1 + \frac{d_\lambda(1, t)}{r}\right)^{-1} \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda}.
\]

Thus from Formula (78), Formula (86), Formula (96), Formula (97), Formula (98), Formula (99) and Formula (94), we could deduce that for \(\frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(1, t)}{r}, 1 + \frac{d_\lambda(t, z)}{r}\}\), the following inequality holds:

\[
|K(r, 1, t) - K(r, 1, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda} \left(1 + \frac{d_\lambda(1, t)}{r}\right)^{-1 - 2\gamma_\lambda}.
\]

This proves the Theorem.

**Proposition 2.13.** For any \(\phi \in S(\mathbb{R}, dx)\), where \(\phi\) is an even function,

(i) \(|r_t \phi_y(-t)| \lesssim \left(1 + \frac{d_\lambda(x, t)}{r}\right)^{-1 - \gamma_\lambda}\), for \(r > 0, x, t \in \mathbb{R}\);

(ii) For \(r > 0, x, t, z \in \mathbb{R}\), if \(\frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(1, t)}{r}, 1 + \frac{d_\lambda(t, z)}{r}\}\)

\[
|r_t \phi_y(-t) - r_t \phi_y(-z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda} \left(1 + \frac{d_\lambda(x, t)}{r}\right)^{-1 - 2\gamma_\lambda}.
\]

(iii) \(r_t \phi_y(-z) = r_t \phi_y(-x)\).

\(y\) has the representation

\[
y = \left\{ \begin{array}{ll}
|x|^{-2\lambda} & \text{for } 0 < r < |x|^{2\lambda + 1}, \\
\lambda^{1/(2\lambda + 1)} & \text{for } r \geq |x|^{2\lambda + 1}.
\end{array} \right.
\]

**Proof.** When \(\phi\) is even, we could write \(r_t \phi_y(-t)\) as:

\[
r_t \phi_y(-t) = c'_\lambda \int_{\lambda} \frac{r}{y^{2\lambda + 1}} \left( \frac{x^2 + t^2 - 2|xt| \cos \theta}{y} \right) (1 + \text{sgn}(xt) \cos \theta) \sin^{2\lambda - 1} \theta d\theta
\]

\[
= c'_\lambda \int_{-\lambda}^{\lambda} \frac{r}{y^{2\lambda + 1}} \left( \frac{x^2 + t^2 - 2xts}{y} \right) (1 + s)^{\lambda}(1 - s)^{\lambda - 1} ds,
\]

where \(c'_\lambda = \frac{\Gamma(\lambda + (1/2))}{\Gamma(\lambda)\Gamma(1/2)}\).

Thus it is clear that the following holds

\[
|r_t \phi_y(-t)| \lesssim |r_t \phi_y(-t)|, \quad r_t \phi_y(-z) = r_t \phi_y(-x),
\]

then we could deduce (ii) and (iii) of the Proposition. Next we will prove (i) of the Proposition. Similar to Theorem 2.12, we will only consider the cases for \(x = 0\) and \(x = 1\).
Case 1 When \( x = 0 \), we suppose the theorem we could deduce that:

\[
|r\phi_y(-t) - r\phi_y(-z)| = \frac{r}{y^{2\lambda+1}} \left| \phi \left( \frac{t}{y} \right) - \phi \left( \frac{z}{y} \right) \right| = \frac{r}{y^{2\lambda+2}} \left| \frac{t}{y} \phi' \left( \frac{y_\lambda}{y} \right) \right| |t - z| \lesssim r \frac{y|\xi|}{(y^2 + |\xi|^2)^{\lambda+2}} |t - z|.
\]

Then by Theorem 2.12, we could obtain:

\[
|r\phi_y(-t) - r\phi_y(-z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma_\lambda} (1 + \frac{d_\lambda(0, t)}{r})^{-1 - 2\gamma_\lambda}.
\]

Case 2 When \( x = 1 \), by the mean value theorem we could deduce that:

\[
|r\tau_1\phi_y(-t) - r\tau_1\phi_y(-z)| = \left| c_\lambda \frac{r}{y^{2\lambda+1}} \int_{-1}^1 \left( \phi \left( \frac{\sqrt{1 + t^2 - 2ts}}{y} \right) - \phi \left( \frac{-s \sqrt{1 + t^2 - 2ts}}{y} \right) \right) \left( 1 - s^2 \right)^{\lambda-1} (1 + s) ds \right| |t - z|.
\]

Notice that \( \phi^{(1)} \) is an odd function and \( \phi^{(1)} \in S(\mathbb{R}, dx) \), thus we could deduce the following:

\[
\left| \frac{\phi^{(1)} \left( \frac{\sqrt{t^2 + 2t - 2\xi s}}{y} \right)}{\sqrt{t^2 + 2t - 2\xi s}} \left( \frac{y^2 + 1 + \xi^2 - 2\xi s}{y^2} \right)^{\lambda+2} \right| \lesssim 1.
\]

Thus Formula (100) and Formula (101) lead to:

\[
|r\tau_1\phi_y(-t) - r\tau_1\phi_y(-z)| \lesssim \left| \frac{r y |\xi - s|}{\left( y^2 + 1 + \xi^2 - 2\xi s \right)^{\lambda+2}} (1 - s^2)^{\lambda-1}(1 + s) ds \right| |t - z|.
\]

Thus we could obtain (ii) of this Proposition by Formula (102) and Theorem 2.12 for the case \( x = 1 \). This proves the Proposition. \( \square \)

**Proposition 2.14.** Let \( B(x_0, r_0) \) satisfying \( x_0 > 0 \) and \( r_0 \frac{1}{\lambda+1} < |x_0/2| \) be the ball in the homogeneous type space: \( B(x_0, r_0) = \{ y : d_\lambda(y, x_0) < r_0 \} \), \( I_0 \) the Euclidean interval: \( I_0 = (x_0 - \delta_2, x_0 + \delta_1) = B(x_0, r_0) \). For any \( t \in B(x_0, r_0) \), the following inequalities hold:

\[
\delta_1 < r_0 \frac{1}{\lambda+1} < |x_0/2|, \quad \delta_2 < r_0 \frac{1}{\lambda+1} < |x_0/2|,
\]

\[
|x_0| \sim |s| \quad \text{for any } s \in B(x_0, r_0), \quad \delta_1 \sim \delta_2 \sim \frac{r_0}{x_0}.\]

**Proof.** When \( r_0 \frac{1}{\lambda+1} < |x_0/2| \), it is easy to see that:

\[
|x_0| \sim |s| \quad \text{for any } s \in B(x_0, r_0).
\]

We could see that in fact \( \delta_1 \) and \( \delta_2 \) have the representation:

\[
\delta_2 = \left| (x_0^{2\lambda+1} - r_0) \frac{1}{\lambda+1} - x_0 \right|, \quad \delta_1 = \left| (x_0^{2\lambda+1} + r_0) \frac{1}{\lambda+1} - x_0 \right|.
\]

With the fact that

\[
|y - x|^{2\lambda+1} < |y^{2\lambda+1} - x^{2\lambda+1}|
\]
holds for \( x, y > 0 \), it is easy to see that \( \delta_1 \leq r_0^{2\lambda+1} \) and \( \delta_2 \leq r_0^{2\lambda+1} \). By Taylor expansion near the origin, for \( r_0^{2\lambda+1} < |x_0/2| \), we could obtain that

\[
\left| \left(x_0^{2\lambda+1} \pm r_0 \right)^{\frac{1}{2\lambda+1}} - x_0 \right| \sim x_0 \left| \left(1 \pm \frac{r_0}{x_0^{2\lambda+1}} \right)^{\frac{1}{2\lambda+1}} - 1 \right| \sim \frac{r_0}{x_0^{2\lambda}}.
\]

Therefore:

\[
\delta_1 \sim \delta_2 \sim \frac{r_0}{x_0^{2\lambda}}.
\]

This proves the proposition.

**Proposition 2.15.** Let \( B(x_0, r_0) \) satisfying \( x_0 > 0 \) and \( r_0^{\frac{1}{2\lambda+1}} < |x_0/2| \) be the ball in the homogeneous type space: \( B(x_0, r_0) = \{ y : d_\lambda(y, x_0) < r_0 \} \), \( I(x_0, t) \) be the Euclid interval: \( I(x_0, t) = (x_0 - t, x_0 + t) \). There exists constants \( c_1 > 0 \) and \( c_2 > 0 \) independent on \( x_0 \) and \( r_0 \), such that the following holds:

\[
I(x_0, c_2 \frac{r_0}{x_0^{2\lambda}}) \subseteq B(x_0, r_0) \subseteq I(x_0, c_1 \frac{r_0}{x_0^{2\lambda}}).
\]

And the following holds:

\[
B(x_0, r_0) \subseteq I(x_0, r_0^{\frac{1}{2\lambda+1}}).
\]

**Proof.** Notice that the following inequality holds when \( x > 0 \) and \( y > 0 \):

\[
|y - x| < |y^{2\lambda+1} - x^{2\lambda+1}|^{\frac{1}{2\lambda+1}}.
\]

Then we could obtain \( B(x_0, r_0) \subseteq I(x_0, r_0^{\frac{1}{2\lambda+1}}) \). By Proposition 2.14, we could obtain that

\[
\max_{y, x \in B(x_0, r_0)} |y - x| \sim \frac{r_0}{x_0^{2\lambda}}.
\]

Therefore there are constants \( c_1 > 0 \) and \( c_2 > 0 \) independent on \( x_0 \) and \( r_0 \), such that

\[
I(x_0, c_2 \frac{r_0}{x_0^{2\lambda}}) \subseteq B(x_0, r_0) \subseteq I(x_0, c_1 \frac{r_0}{x_0^{2\lambda}}).
\]

Hence the Proposition holds.

**Proposition 2.16.** For any fixed \( \phi \in S(\mathbb{R}, dx) \), where \( \phi \) is an even function with \( \text{supp}\phi \subseteq [-1, 1] \), \( 0 \leq \phi \leq 1 \), \( \phi(0) = 1 \), then we could obtain the following:

(i) \( 0 < r r_\tau \phi_y (-t) \leq \left(1 + \frac{d_\lambda(t, x)}{r}\right)^{1-\gamma} \), for \( r > 0, x, t \in \mathbb{R} \);

(ii) For \( r > 0, x, t, z \in \mathbb{R} \), if \( \frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(t, x)}{r}, 1 + \frac{d_\lambda(t, x)}{r}\} \),

\[
|r r_\tau \phi_y (-t) - r r_\tau \phi_y (-z)| \leq \left(\frac{d_\lambda(t, z)}{r}\right)^{\gamma}\left(1 + \frac{d_\lambda(t, x)}{r}\right)^{1-2\gamma};
\]

(iii) \( r r_\tau \phi_y (-z) = r r_\tau \phi_y (-z) \);

(iv) \( |r r_\tau \phi_y (-x)| \sim 1 \);

(v) \( \text{supp} r r_\tau \phi_y (-t) \subseteq B(x, cr) \cup B(-x, cr) \), where \( c \) is constant independent on \( r, x, y, t \). There exists a constant \( C_0 < \frac{1}{2\lambda+1} \), such that \( B(x, cr) \cap B(-x, cr) = \emptyset \) for \( 0 < y < C_0 |x| \);

\( y \) has the representation

\[
y = \begin{cases} 
  r|x|^{-2\lambda} & \text{for } 0 < r < |x|^{2\lambda+1}, \\
  r^{1/(2\lambda+1)} & \text{for } r \geq |x|^{2\lambda+1}.
\end{cases}
\]

**Proof.** (i), (ii), (iii) and (iv) of the Proposition could be deduced from Proposition 2.13. We will prove (iv) next, then we need to consider the cases for \( x = 0 \) and \( x = 1 \). It is clear that
\[ |r_{t,z} \phi_y(-x)|_{x=0} = \phi(0) \sim 1 \text{ for the case } x = 0. \] When \( 0 < y < 1 = x = t, r = y \), we could deduce that for some fixed \( 0 < \delta < 1 \), the following holds:

\[
|r_{t,z} \phi_y(-1)| = \left| \int_{-1}^{1} c'_\lambda \frac{r}{y^{2\lambda+1}} \phi \left( \frac{\sqrt{2 - 2s}}{y} \right) (1 - s^2)^{\lambda - 1}(1 + s) \, ds \right| \\
\geq \left| \int_{-1}^{1} c'_\lambda \frac{r}{y^{2\lambda+1}} \phi \left( \frac{\sqrt{2 - 2s}}{y} \right) (1 - s^2)^{\lambda - 1}(1 + s) \, ds \right| \\
\geq C_\delta.
\]

When \( y > 1 = x = t, r = y^{2\lambda+1} \), we could deduce the following inequality:

\[
|r_{t,z} \phi_y(-1)| = \left| \int_{-1}^{1} c'_\lambda \frac{r}{y^{2\lambda+1}} \phi \left( \frac{\sqrt{2 - 2s}}{y} \right) (1 - s^2)^{\lambda - 1}(1 + s) \, ds \right| \\
\geq \left| \int_{-1/4}^{1} c'_\lambda \frac{r}{y^{2\lambda+1}} \phi \left( \frac{\sqrt{2 - 2s}}{y} \right) (1 - s^2)^{\lambda - 1}(1 + s) \, ds \right| \\
\geq C.
\]

Thus (iv) of this Proposition holds. We will prove (v) of this Proposition at last.

For \( x, t, z \in \mathbb{R} \), we use \( W_\lambda(x, t, z) \) to denote as: \( W_\lambda(x, t, z) = W_\lambda^0(x, t, z)(1 - \sigma_{x,t,z} + \sigma_{x,0,t} + \sigma_{z,t,x}) \), where

\[
W_\lambda^0(x, t, z) = c'_\lambda \frac{|xtz|^{1-2\lambda} \chi(|x||t|, |x|+|t|, |z|)}{|(|x|+|t|)|^{2} - z^2} (|x| + |t|)^{2 - 2\lambda} (|x| + |t|)^{2} (|x| + |t|)^{2} |z|^{2}.
\]

\[ c'_\lambda = 2^{3/2-\lambda} \left( \Gamma(\lambda + 1/2) \right)^2/|\sqrt{\pi} \Gamma(\lambda)|. \] And \( \sigma_{x,t,z} = \frac{x^2 + y^2 - 2}{2xt} \), for \( x \neq 0 \) and \( t \neq 0 \). \( \sigma_{x,0,t} = 0 \), for \( x = 0 \) or \( t = 0 \). For \( t \neq 0 \), we could write \((\tau_{x}\phi)(-t)\)

\[
(\tau_{x}\phi)(-t) = c_\lambda \int_{\mathbb{R}} \phi(z) W_\lambda(-t, x, z) |z|^2 \, dz.
\]

(103)

It is clear that \( \tau_{x} \phi_y(-t) = 0 \) when \( \frac{|x|-|t|}{y} \geq 1 \). Thus the function \( t \rightarrow \tau_{x} \phi_y(-t) \) satisfies \( \text{supp}\tau_{x} \phi_y(-t) \subseteq \{|x| - |y|, |x| + |y|\} \cup \{|x| - |y|, |x| + |y|\} \cup \{|x| - |y|, |x| + |y|\} \cup \{|x| - |y|, |x| + |y|\} \subseteq B(0, c_\lambda) \) and \( \text{supp}\tau_{x} \phi_y(-t) \subseteq B(0, c_\lambda) \) for \( x \neq 0 \). This proves (v) of this Proposition.

**Proposition 2.17.** For any fixed \( \phi \in S(\mathbb{R}, dx) \), where \( \phi \) is an even function with \( \text{supp}\phi \subseteq [-1, 1] \), \( 0 \leq \phi \leq 1, \phi(0) = 1 \), we use \( K_3(x, r, t) \) to denote as:

\[
K_3(x, r, t) = r_{x} \phi_y(-t) - r_{x} \phi_y(t), \quad x \neq 0
\]

where \( y \) has the representation

\[
y = \begin{cases} 
\frac{r|x|^{-2\lambda}}{0 < y < C_0|x|} & \quad \text{for } 0 < r < |x|^{2\lambda+1}, \\
0 & \quad \text{for } x = 0.
\end{cases}
\]

(2.16)

Then we could obtain the following:

(i) \( |K_3(r, x, t)| \leq \left( 1 + d_\lambda(x, t) \right)^{-1-\gamma} \), for \( r > 0, x, t \in \mathbb{R} \);

(ii) For \( r > 0, x, t, z \in \mathbb{R} \), if \( \frac{d_\lambda(t, z)}{r} \leq C \min \left\{ 1 + \frac{d_\lambda(x, t)}{r}, 1 + \frac{d_\lambda(x, t)}{r} \right\} \)

\[
|K_3(r, x, t) - K_3(r, x, z)| \leq \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma} \left( 1 + \frac{d_\lambda(x, t)}{r} \right)^{-1-2\gamma}.
\]
K_3(r, x, t) = K_3(r, t, x);

(iv) K_3(r, x, x) \sim 1 and K_3(r, x, t) = -K_3(r, x, -t);

(v) supp K_3(r, x, t) \subseteq B(x, cr) \cup B(-x, cr) with B(x, cr) \cap \{x = 0\} = \emptyset, where c is a constant independent on r, x, y, t;

(vi) 0 < K_3(r, x, t) \leq C when x > 0, and -C \leq K_3(r, x, t) < 0 when x < 0 for some constant C independent on r, x, t.

**Proof.** (i) (ii) and (v) of this Proposition can be deduced from Proposition 2.16 directly. Notice that we could write $K_3(r, x, t)$ as following:

$$K_3(r, x, t) = \int_{-1}^{1} c'_{\lambda} r \gamma y^{2\lambda + 1} \left( \frac{\sqrt{x^2 + t^2 - 2|x|s}}{y} \right) 2\text{sgn}(xt)(1 - s^2)^{\lambda-1} \, ds.$$

Thus we could deduce (iii) of this Proposition. We will prove (iv) of this Proposition at last. From (v) we could deduce that $\text{sgn}(xt) > 0$, thus we could write $K_3(r, x, t)$ as:

$$K_3(r, x, t) = \int_{0}^{1} c'_{\lambda} r \gamma y^{2\lambda + 1} \left( \phi \left( \frac{\sqrt{x^2 + t^2 - 2|x|s}}{y} \right) - \phi \left( \frac{\sqrt{x^2 + t^2 + 2|x|s}}{y} \right) \right) 2(1 - s^2)^{\lambda-1} \, ds.$$

We will prove (iv) of this Proposition next, then we need to consider the cases for $x = 1$. When $0 < y < C_0 < 1 = x = t, r = y$, we could deduce that for some fixed $0 < \delta < 1$, the following holds:

$$K_3(1, 1) \geq \int_{1 - \frac{1}{2s}}^{1} c'_{\lambda} r \gamma y^{2\lambda + 1} \left( \phi \left( \frac{2 - 2s}{y} \right) - \phi \left( \frac{2 + 2s}{y} \right) \right) 2(1 - s^2)^{\lambda-1} \, ds \geq C \delta.$$

Also it is clear that $K_3(r, x, t)$ is an odd function in $t$, thus $K_3(r, -1, -1) \sim -1$. Thus we obtain (iv) of this Proposition. Thus we could also deduce (vi) of this Proposition. This proves the Proposition. \qed

In a similar way, we could obtain the following Proposition:

**Proposition 2.18.** For any fixed $\phi \in S(\mathbb{R}, dx)$, where $\phi$ is an even function with $\text{supp} \phi \subseteq [-1, 1]$, $0 \leq \phi \leq 1, \phi(0) = 1$, we use $K_4(r, x, t)$ to denote as:

$$K_4(r, x, t) = r\tau_x \phi_y(-t) + r\tau_x \phi_y(t), \text{ for } x \neq 0,$$

where $y$ has the representation

$$y = \begin{cases} r|x|^{-2\lambda} & \text{for } 0 < r < |x|^{2\lambda + 1}, \\ 0 < y < C_0|x| & (C_0 \text{ is the constant in Proposition 2.16}) \text{and } x \neq 0. \end{cases}$$

Then the following holds:

(i) $|K_4(r, x, t)| \leq \left(1 + \frac{d_\lambda(x,t)}{r}\right)^{-1-\gamma}$, for $r > 0, x, t \in \mathbb{R};$

(ii) For $r > 0, x, t, z \in \mathbb{R},$ if $\frac{d_\lambda(x,z)}{r} \leq C \min\{1 + \frac{d_\lambda(x,t)}{r}, 1 + \frac{d_\lambda(x,z)}{r}\}$

$$|K_4(r, x, t) - K_4(r, x, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{\gamma} \left(1 + \frac{d_\lambda(x, t)}{r}\right)^{-1-2\gamma};$$

(iii) $K_4(r, x, t) = K_4(r, t, x);$ 

(iv) $K_4(r, x, x) \sim 1$ and $K_4(r, x, t) = K_4(r, x, -t);$ 

(v) $\text{supp} K_4(r, x, t) \subseteq B(x, cr) \cup B(-x, cr)$ with $B(x, cr) \cap \{x = 0\} = \emptyset$, where $c$ is a constant independent on $r, x, y, t;$

(vi) $0 < K_4(r, x, t) \leq C.$
Proposition 2.19. We use $F_{\nabla}(x)$ to denote as $F_{\nabla}(x) = \sup_{|x-u|<y}|F(u,y)|$, $F_{\nabla \lambda}(x)$ to denote as $F_{\nabla \lambda}(x) = \sup_{d_{x}(x,u)<r}|F(u,y)|$, where $y$ has the representation

$$y = \begin{cases} r|x|^{-2}\lambda & \text{for } 0 < r < |x|^{2\lambda+1}, \\ r^{1/(2\lambda+1)} & \text{for } r \geq |x|^{2\lambda+1}. \end{cases} \quad (104)$$

Then we could have:

$$\|F_{\nabla \lambda}\|_{L_{\lambda}^{p}(\mathbb{R})} \sim_{\lambda,p} \|F_{\nabla}\|_{L_{\lambda}^{p}(\mathbb{R})}. \quad (105)$$

We also use $F_{+}(x)$ to denote as $F_{+}(x) = \sup_{y>0}|F(x,y)|$, $F_{+ \lambda}(x)$ to denote as $F_{+ \lambda}(x) = \sup_{y>0}|F(x,y)|$. Thus it is clear that $F_{+}(x) = F_{+ \lambda}(x)$.

Proof. Case 1: When $0 < y < \frac{|x|}{2\lambda}$, by Proposition 2.15 we could deduce that for some constants $c_{1}$ and $c_{2}$

$$I(x, c_{2} \frac{r}{x^{2\lambda}}) \subseteq B(x, r) \subseteq I(x, c_{1} \frac{r}{x^{2\lambda}}).$$

Thus we could deduce that

$$I(x, c_{2}y) \subseteq B(x, r) \subseteq I(x, c_{1}y). \quad (106)$$

Case 2: When $y \geq \frac{|x|}{2\lambda}$, it is clear that $r \sim y^{2\lambda+1}$. Then we could see that there exists $c_{1}$ and $c_{2}$ independent on $x, r, y$, such that

$$I(x, c_{2}y) \subseteq B(x, r) \subseteq I(x, c_{1}y). \quad (107)$$

Then by Formulas (106) and (107), together with Proposition 1.20, we could deduce that Formula (105) holds. This proves the Proposition.

We use $(f *_{\lambda} \phi)_{\nabla \lambda}(x)$, $(f *_{\lambda} \phi)_{\nabla}(x)$ and $(f *_{\lambda} \phi)_{+}(x)$ to denote as following:

$$(f *_{\lambda} \phi)_{\nabla \lambda}(x) = \sup_{d_{x}(u,x)<r} |f *_{\lambda} \phi_{y}(u)|, \quad (f *_{\lambda} \phi)_{\nabla}(x) = \sup_{|x-u|<y} |f *_{\lambda} \phi_{y}(u)|,$$

$$(f *_{\lambda} \phi)_{+}(x) = \sup_{y>0} |f *_{\lambda} \phi_{y}(x)|,$$

where $y$ has the representation as Formula (104) and $\phi_{y}(x) = \frac{1}{y^{2\lambda+1}} \phi \left( \frac{x}{y} \right)$.

Theorem 2.20. For any fixed $\phi \in S(\mathbb{R}, dx)$, where $\phi$ is an even function with $\text{supp} \phi \subseteq [-1, 1]$, $0 \leq \phi \leq 1$, $\phi(0) = 1$, we could deduce that for $f \in L_{\lambda}^{1}(\mathbb{R})$:

$$\|f *_{\lambda} \phi\|_{L_{\lambda}^{p}(\mathbb{R})} \sim_{\lambda, p, \beta, \phi} \|f *_{\lambda} \phi_{y}\|_{L_{\lambda}^{p}(\mathbb{R})} \sim_{\lambda, p, \beta, \phi} \|f *_{\lambda} \phi_{y}\|_{L_{\lambda}^{p}(\mathbb{R})}, \quad (108)$$

for $p > \frac{\lambda}{1+y^{2}\lambda}$, for some $\beta > 0$.

Proof. We use $f_{o}$ and $f_{e}$ to denote as:

$$f_{o}(x) = \frac{f(x) - f(-x)}{2}, \quad f_{e}(x) = \frac{f(x) + f(-x)}{2}.$$

We use $\tilde{K}(r, x, t)$, $\tilde{K}_{o}(r, x, t)$, $\tilde{K}_{e}(r, x, t)$ to denote as:

$$\tilde{K}(r, x, t) = r \tau_{x} \phi_{y}(-t),$$

$$2\tilde{K}_{o}(r, x, t) = r \tau_{x} \phi_{y}(-t) - r \tau_{x} \phi_{y}(t),$$

$$2\tilde{K}_{e}(r, x, t) = r \tau_{x} \phi_{y}(-t) + r \tau_{x} \phi_{y}(t),$$

where $y$ has the representation

$$y = \begin{cases} r|x|^{-2\lambda} & \text{for } 0 < r < |x|^{2\lambda+1}, \\ r^{1/(2\lambda+1)} & \text{for } r \geq |x|^{2\lambda+1}. \end{cases} \quad (109)$$
One obvious fact is that the following two formulas hold:

\[
\|f \ast \phi\|_{L^p_\lambda}\lesssim \|(f_o) \ast \phi\|_{L^p_\lambda} + \|(f_e) \ast \phi\|_{L^p_\lambda} \lesssim \|f \ast \phi\|_{L^p_\lambda},
\]

(109)

\[
\|f \ast \lambda \phi\|_{L^p_\lambda}\lesssim \|(f_o) \ast \lambda \phi\|_{L^p_\lambda} + \|(f_e) \ast \lambda \phi\|_{L^p_\lambda} \lesssim \|f \ast \lambda \phi\|_{L^p_\lambda}.
\]

(110)

Next, we will define new kernels as follows (\(C_0\) is the constant in Proposition 2.16):

Case 1: \(x > 0\)

\[
K_0(r, x, t) = \begin{cases} 
\tilde{K}(r, x, t) & \text{for } y \geq C_0|x|,
\end{cases}
\]

\[
= \begin{cases} 
\bar{K}(r, x, t) & \text{for } 0 < y < C_0|x|,
\end{cases}
\]

Case 2: \(x < 0\)

\[
K_0(r, x, t) = \begin{cases} 
\tilde{K}(r, x, t) & \text{for } y \geq C_0|x|,
\end{cases}
\]

\[
= \begin{cases} 
\bar{K}(r, x, t) & \text{for } 0 < y < C_0|x|,
\end{cases}
\]

Case 3: \(x = 0\)

\[
K_0(r, x, t) = K_e(r, x, t) = \tilde{K}(r, x, t).
\]

Thus we could see that the following two formulas hold:

\[
((f_o) \ast \lambda \phi \nabla \lambda(x) \sim \sup_{d_\lambda(x, r)} \left| \int \bar{K}_0(r, u, t) f_o(t) |t|^{2\lambda} dt/r \right|,
\]

(111)

\[
((f_e) \ast \lambda \phi \nabla \lambda(x) \sim \sup_{d_\lambda(x, r)} \left| \int \bar{K}_e(r, u, t) f_e(t) |t|^{2\lambda} dt/r \right|.
\]

(112)

By Proposition 2.16, Proposition 2.17, Proposition 2.18, we could deduce that \(K_0(r, x, t)\) and \(K_e(r, x, t)\) are just the kind of kernel \(K_1(r, x, t)\) with compact support in Section 1: Theorem 1.35. Thus by Formula (111), Formula (112), and Theorem 1.35, we could deduce the following:

\[
\|f_o\|_{L^p_\lambda} \sim \|(f_e) \ast \lambda \phi \nabla \lambda\|_{L^p_\lambda},
\]

(113)

Thus from Formula (109) Formula (110) Formula (113) Formula (114) and Proposition 2.19, we could prove the theorem. \(\Box\)

**Proposition 2.21.** For \(p > \frac{1}{1+\lambda}, \phi\) is an even function with \(\text{supp} \phi \subseteq [-1, 1], 0 \leq \phi \leq 1, \phi(0) = 1, \psi\) is an even function, \(\int \psi(t) |t|^{2\lambda} dt \sim 1\) with \(\phi, \psi \in \mathcal{S}(\mathbb{R}, dx)\), then we could deduce the following for \(f \in L^1_\lambda(\mathbb{R})\):

\[
\|f \ast \phi\|_{L^p_\lambda} \sim \lambda, p, \phi, \psi \|f \ast \psi\|_{L^p_\lambda} \sim \lambda, p, \phi, \psi \|f \ast \psi\|_{L^p_\lambda},
\]

(115)

**Proof.** Fix a function \(\phi \in \mathcal{S}(\mathbb{R}, dx)\) so that:

\[
\begin{cases} 
\phi(\xi) = 0 & \text{for } |\xi| \geq 1 \\
\phi(\xi) = 1 & \text{for } |\xi| \leq 1/2,
\end{cases}
\]
where $\varphi$ is an even function. Then $\varphi^k \in S(\mathbb{R}, dx)$ can be defined as:

$$
\left\{
\begin{array}{l}
\varphi^k(\xi) = \varphi(\xi) \text{ for } k = 0, \\
\varphi^k(\xi) = \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi) \text{ for } k \geq 1.
\end{array}
\right.
$$

By Proposition 2.7 and 2.6, we could deduce that $\sup_{\xi \in \mathbb{R}} |\xi|^{\alpha} D^\beta (\mathcal{F}_\lambda \varphi)(\xi) | \lesssim C_{\beta, \alpha}$, when $\psi(t) \in S(\mathbb{R}, dx)$. Thus together with the fact that $(\mathcal{F}_\lambda \psi)(0) \sim 1$, we could deduce that there exists a $k_0$, such that

$$
| (\mathcal{F}_\lambda \psi)(2^{-k_0}\xi) | \gtrsim 1/2 \text{ for } |\xi| \leq 2.
$$

We use $\eta^{k,\lambda}$ to denote as

$$(\mathcal{F}_\lambda \eta^{k,\lambda})(\xi) = \frac{\varphi^k(\xi)(\mathcal{F}_\lambda \varphi)(\xi)}{(\mathcal{F}_\lambda \psi)(2^{-k}2^{-k-1}\xi)},$$

where $\mathcal{F}_\lambda$ denotes the Dunkl transform.

Then

$$\phi(x) = \sum_{k=0}^{+\infty} \eta^{k,\lambda} \star \lambda \psi_{2^{-k-1}u}(x).$$

By the fact that $\sup_{\xi \in \mathbb{R}} |D^\beta (\mathcal{F}_\lambda \psi)(\xi) | \lesssim 1$ and $\sup_{\xi \in \mathbb{R}} |\xi|^{\alpha} D^\beta (\mathcal{F}_\lambda \varphi)(\xi) | \lesssim 1$, where $D$ is the Dunkl operator, we could deduce that for any $M > 0$

$$
\sup_{\xi \in \mathbb{R}} |\xi|^{\alpha} D^\beta (\mathcal{F}_\lambda \eta^{k,\lambda})(\xi) | \lesssim \alpha, \beta, M, k_0 \ 2^{-M}.
$$

Thus we could deduce that

$$
\left| \int_\mathbb{R} \eta^{k,\lambda}(x) \left( 1 + 2^{k+k_0}|x| \right)^N |x|^{2\lambda} dx \right| \leq C2^{-k}.
$$

By Formula (118), we could deduce that

$$
\sum_{k=0}^{+\infty} \int_\mathbb{R} \eta^{k,\lambda} \left( \frac{x}{t} \right) \left( 1 + \frac{|x|}{2^{k+k_0}t} \right)^N |x|^{2\lambda} dx \leq C_{k_0,N} \sum_{k=0}^{+\infty} 2^{-k}.
$$

Then by Formula (116) and Formula (118), we could deduce the following:

$$
\sup_{t > 0} |f \star \lambda \psi_t(x)| = \sup_{t > 0} \left| \int_\mathbb{R} \eta^{k,\lambda} \left( \frac{u}{t} \right) \left( 1 + \frac{|u|}{2^{k+k_0}t} \right)^N |u|^{2\lambda} du \right|
$$

$$
\leq \sup_{t > 0} \sum_{k=0}^{+\infty} \int \tau_{-u} (f \star \lambda \psi_{2^{-k-1}u})(x) \eta^{k,\lambda}(u) |u|^{2\lambda} du
$$

$$
\leq \sup_{t > 0, u \in \mathbb{R}} \left| \tau_{-u} (f \star \lambda \psi_{l})(x) \right| \left( 1 + \frac{|u|}{t} \right)^{-N} \sum_{k=0}^{+\infty} \left| \int \eta^{k,\lambda}(u) \left( 1 + \frac{|u|}{2^{k+k_0}t} \right)^N |u|^{2\lambda} du \right|
$$

$$
\leq \sum_{m=0}^{+\infty} \sup_{t > 0, 2^{m-1}t < |u| \leq 2^m} 2^{-mN} \left| \tau_{-u} (f \star \lambda \psi_{l})(x) \right| + \sup_{t > 0, |u| \leq t} \left| \tau_{-u} (f \star \lambda \psi_{l})(x) \right|
$$

$$
\leq \sum_{m=0}^{+\infty} \sup_{t > 0, |u| \leq 2^m} 2^{-mN} \left| \tau_{-u} (f \star \lambda \psi_{l})(x) \right|.
$$

For $x \neq 0$, we could write $\tau_{-u} (f \star \lambda \psi_{l})(x)$ as

$$
\tau_{-u} (f \star \lambda \psi_{l})(x) = c\lambda \int_\mathbb{R} (f \star \lambda \psi_{l})(z) W_\lambda(x, -u, z) |z|^{2\lambda} dz.
$$

For $x = 0$, we could write $\tau_u (f \star \lambda \psi_{l})(0)$ as

$$
\tau_u (f \star \lambda \psi_{l})(0) = (f \star \lambda \psi_{l})(u).
$$
Notice that \(|x| - |u| \leq |z| \leq |x| + |u|\), thus by Formula (120) Formula (121) and Formula (122) with the fact that \(\int_\mathbb{R} |W_\lambda(x, -u, z)| |z|^{2\lambda}dz \leq 4\), we could deduce that:

\[
\sup_{t>0} |f * \phi_t(x)| \lesssim \sum_{m=0}^{\infty} \left| \sup_{|z-x| \leq 2^mt} 2^{-mN} (f * \psi_t)(z) \right| + \sum_{m=0}^{\infty} \left| \sup_{|z+x| \leq 2^mt} 2^{-mN} (f * \psi_t)(z) \right| . \tag{123}
\]

Thus Proposition 1.20 and Formula (123) lead to the following inequality for \(N > \frac{1}{p}\):

\[
\| (f * \phi)_+ \|_{L^p_\mathbf{R}(N)} \lesssim \| (f * \psi)_\nabla \|_{L^p_\mathbf{R}(N)}. \tag{124}
\]

Proposition 2.19 Proposition 2.13 and Proposition 1.13 lead to

\[
\| (f * \psi)_\nabla \|_{L^p_\mathbf{R}(N)} \lesssim \| f^*_{\gamma} \|_{L^p_\mathbf{R}(N)}. \tag{125}
\]

Formula (124) Proposition 2.19 Proposition 1.33 and Theorem 2.20 lead to the following:

\[
\| f^*_{\gamma} \|_{L^p_\mathbf{R}(N)} \lesssim \| (f * \phi)_+ \|_{L^p_\mathbf{R}(N)}. \tag{126}
\]

Formula (124) Formula (125) Formula (126) Proposition 1.32 and Theorem 2.20 lead to Formula (115). This proves the Proposition.

**Theorem 2.22** (\(H^p_\mathbf{R}(\lambda), \tilde{H}^p_\mathbf{R}(\lambda)\) for \(p > \frac{1}{1+\gamma}\)). For \(p > \frac{1}{1+\gamma}\), let \(f(x) \in H^p_\mu_\lambda(\mathbf{R})\). Let \(\gamma_\lambda = \frac{1}{2(2\lambda+1)}\), then we could obtain:

\[
\| f^*_{\gamma_\lambda} \|_{L^p_\mathbf{R}(\lambda)} \sim \| P_{\chi}^\nabla f \|_{L^p_\mathbf{R}(\lambda)}. \tag{127}
\]

Thus \(\tilde{H}^p_\lambda(\mathbf{R})\) and \(H^p_\lambda(\mathbf{R})\) can be defined as follows:

\[
\tilde{H}^p_\lambda(\mathbf{R}) = \tilde{H}^p_{\mu_\lambda}(\mathbf{R}) = \left\{ g \in L^1_\mathbf{R}(\mathbf{R}) \cap L^p_\lambda(\mathbf{R}) : g^*_{\gamma_\lambda}(x) \in L^p_\lambda(\mathbf{R}) \right\}
\]

\[
H^p_\lambda(\mathbf{R}) = H^p_{\mu_\lambda}(\mathbf{R}) = \left\{ g \in S'((\mathbf{R},|x|^{2\lambda}dx) : g^*_{\gamma_\lambda}(x) \in L^p_\lambda(\mathbf{R}) \right\}.
\]

(remark: \(H^p_{\mu_\lambda}(\mathbf{R})\) with the \(\mu_\lambda\) measure is not \(H^p_\mu(\mathbf{R})\), as in Definition 1.34.)

**Proof.** Let \(f \in L^1_\mathbf{R}(\lambda)\) first. By Proposition 2.19 Theorem 2.12 and Proposition 1.13, we could deduce that:

\[
\| P_{\chi}^\nabla f \|_{L^p_\mathbf{R}(\lambda)} \lesssim \| f^*_{\gamma_\lambda} \|_{L^p_\mathbf{R}(\lambda)}. \tag{128}
\]

Next we will prove

\[
\| f^*_{\gamma_\lambda} \|_{L^p_\mathbf{R}(\lambda)} \lesssim \| P_{\chi}^\nabla f \|_{L^p_\mathbf{R}(\lambda)}. \tag{129}
\]

Notice the \(\lambda\)-Poisson kernel is \(\tau_xP_x(-t)\) with \(P_x(x) = a_\lambda y (y^2 + x^2)^{-\lambda-1}\), where \(a_\lambda = 2^{\lambda+1}\Gamma(\lambda + 1)/\sqrt{\pi}\). We use similar idea in [25]. There exists a function \(\eta\) defined on \([1, \infty)\) that is rapidly decreasing at \(\infty\) and satisfies the moment conditions:

\[
\int_1^\infty \eta(s)ds = 1, \text{ and } \int_1^\infty s^k\eta(s)ds = 0, \text{ for } k = 1, 2, \ldots . \tag{130}
\]

Then we could check that the function \(\Phi(x)\)

\[
\Phi(x) = \int_1^\infty \eta(s)P_x(x)ds,
\]

is rapidly decreasing and is an even function: \(\Phi(x) \in S(\mathbf{R}, dx)\) is even. Also it is clear that

\[
\int \Phi(x)|x|^{2\lambda}dx = C \int_1^\infty \eta(s)ds \sim 1.
\]
Thus we could deduce that:

\[
(f * \Phi_y)_+(x) = \sup_{y > 0} \left| \int f(t) \tau_{-t} \Phi_y(x) |t|^{2\lambda} dt \right| \leq \sup_{y > 0} \left| \int \tau_{-t} f(x) \Phi_y(t) |t|^{2\lambda} dt \right|
\]

Thus the above Formula (131), Proposition 1.33, Theorem 2.20, and Proposition 2.21, we could deduce Formula (129). Thus Formula (127) holds for \( f \in L^1_{\lambda}(\mathbb{R}) \). Notice that \( \tilde{H}^p_{\mu_\lambda}(\mathbb{R}) \) is dense in \( H^p_{\mu_\lambda}(\mathbb{R}) \). Thus by the Hahn-Banach theorem, we could deduce that Formula (127) holds for \( f \in H^p_{\mu_\lambda}(\mathbb{R}) \). Thus together with Theorem 1.35, \( \tilde{H}^p_{\mu_\lambda}(\mathbb{R}) \) and \( H^p_{\mu_\lambda}(\mathbb{R}) \) can be defined as follows:

\[-\begin{align*}
\tilde{H}^p_{\mu_\lambda}(\mathbb{R}) &= \tilde{H}^{p*}_{\mu_\lambda}(\mathbb{R}) = \{ g \in L^2_{\mu_\lambda}(\mathbb{R}) \cap L^1_{\mu_\lambda}(\mathbb{R}) : g^{\ast}_S(x) \in L^p_{\lambda}(\mathbb{R}) \} \\
H^p_{\mu_\lambda}(\mathbb{R}) &= H^p_{\mu_\lambda}(\mathbb{R}) = \{ g \in S'(\mathbb{R}), |x|^{2\lambda} dx : g^{\ast}_S(x) \in L^p_{\lambda}(\mathbb{R}) \},
\end{align*} \]

where \( \gamma_\lambda = \frac{1}{2(2\lambda + 1)} \). This proves the Theorem.

Thus we could obtain the following Proposition:

**Proposition 2.23.** \( u(x, y) \) is a \( \lambda \)-harmonic function, for \( 1 \geq p > \frac{1}{4 + \gamma_\lambda} \)

*case 1,* \( u^{\ast}_\lambda(x) \in L^p_{\lambda}(\mathbb{R}) \cap L^2_{\lambda}(\mathbb{R}) \cap L^1_{\lambda}(\mathbb{R}) \), then there exists \( f \in \tilde{H}^p_{\mu_\lambda}(\mathbb{R}) \), such that

\[
u(x, y) = f \ast \lambda P_y(x).
\]

*case 2,* \( u^{\ast}_\lambda(x) \in L^p_{\lambda}(\mathbb{R}) \), then there exists \( f \in H^p_{\mu_\lambda}(\mathbb{R}) \), such that

\[
\int |x - s| < y \sup |u(s) - f \ast \lambda P_y(s)|^p |x|^{2\lambda} dx = 0,
\]

Moreover,

\[
\|u^{\ast}_\lambda\|_{L^p_{\mu_\lambda}(\mathbb{R})} \sim \|f\|_{H^p_{\mu_\lambda}(\mathbb{R})}.
\]

**Proof.** By Proposition 2.4(4), we could deduce Formula (132). By Theorem 2.8(2), Proposition 2.9, Formula (132), together with the fact that \( \tilde{H}^p_{\mu_\lambda}(\mathbb{R}) \) is dense in \( H^p_{\mu_\lambda}(\mathbb{R}) \), we could deduce that Formula (133) holds. This proves the Proposition.

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