Gauge-fixing for the completion problem of reconstructed metric perturbations of a Kerr spacetime

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We provide a prescription to solve the metric completion problem in gravitational self-force calculations on a Kerr spacetime by fixing the remaining gauge freedom. We discuss the explicit example of eccentric equatorial orbits, recovering all limiting cases already studied in the literature of eccentric orbits in Schwarzschild as well as circular orbits in both Schwarzschild and Kerr spacetimes.

I. INTRODUCTION

The issue of metric completion in gravitational perturbation theory is a longstanding problem, since the seminal works of Regge, Wheeler and Zerilli (RWZ) [1,2], who provided the necessary formalism to study the first order perturbations of the Schwarzschild solution. This approach was soon applied to the analysis of gravitational radiation emitted by a small compact body [3], which captured the main interest of the gravity community. Since the computation of energy and angular momentum fluxes only involves the radiating part of the metric, the problem of low multipoles remained overlooked for many years. Zerilli himself showed that these multipoles simply correspond to “shifts” in the mass and angular momentum of the background spacetime. Their relevance for self-force calculations became clear after the work of Detweiler and Poisson [4], who emphasized that such shifts are as important as the radiating multipoles for describing the motion of a small body orbiting a black hole, since their contribution to the (conservative piece of the) self-force affects the dynamics even at the Newtonian level.

The nonradiative part of the perturbed metric thus plays a crucial role in gravitational self-force (GSF) calculations, for which the complete reconstruction of the metric perturbations is necessary to compute the various orbital invariants, like redshift, periastron advance, spin-precession angle and tidal invariants, which encode a gauge-invariant information on the dynamics, useful to compare results from different approximation methods, either analytic or numeric. We refer to Ref. [5] for a recent review on GSF computational techniques, and on the increasing importance of GSF calculations for an even more accurate modelling of the dynamics of extreme-mass-ratio inspirals (EMRIs), which will be a primary source of gravitational waves for the planned low-frequency space-based detector eLISA [6]. The RWZ formalism provides all necessary tools for fully determining the nonradiative piece of the perturbed metric in the Schwarzschild case, directly using the Zerilli’s results together with suitable gauge adjustments to ensure regularity as well as asymptotic flatness of the perturbation [4,7,8]. Solving the same problem in Kerr, instead, is still a challenge since many years.

The basic theory of gravitational perturbations of a Kerr spacetime was developed by Teukolsky using the Newman-Penrose formalism [9,11]. The Einstein field equations combined with the Bianchi identities lead to a single master wave equation, the Teukolsky equation, for the perturbed Weyl scalars $\psi_0$ ($s = 2$) or $\psi_4$ ($s = -2$), which can be solved by separation of variables (using spheroidal harmonics and Fourier decomposition). The radiative part of the metric perturbation can then be reconstructed from a scalar function, the Hertz potential, in a radiation gauge through the Chranowski-Cohen-Kegles (CCK) procedure [12–14]. The latter was originally developed for vacuum perturbations, and more recently extended to metric perturbations sourced by a particle moving along a bound geodesic orbit around a Kerr black hole [15]. Despite the appearance of irregular behaviors (string-like singularities) caused by the presence of matter (both within and outside the region where the source is located) [16], the use of radiation gauge metric perturbations and related CCK formalism to obtain self-force corrections to the particle’s motion is now well established in GSF theory [17,20].

The perturbed metric should then be completed by the nonradiative modes, which cannot be determined by the Teukolsky equation, since the spheroidal harmonics are not defined for $l < |s| = 2$, in contrast with the Schwarzschild case, where these lower multipoles associated with $l = 0, 1$ can be expressed appropriately in terms of spherical harmonics using the RWZ formalism. The remaining part must be stationary and axially symmetric, simply corresponding to mass and angular momentum perturbations of the Kerr background in the vacuum region away from the particle’s location, up to gauge modes [21,22]. This completion piece can be computed, in principle, in any gauge, and no general prescription has been found yet.

According to Ref. [16] a regular (“no-string”) solution can be formed by joining together the regular sides of two “half-string” solutions along a hypersurface containing the particle’s world line, supporting a gauge discontinuity (and possibly distributional singularities). The latter separates the spacetime region spatially inside the particle’s location (interior region, −) from that outside it (exterior region, +), so that the full metric perturbation can be split in three different contributions [23]

$$h_{\alpha\beta}^{\pm} = h_{\alpha\beta}^{\text{rec} \pm} + h_{\alpha\beta}^{\text{comp} \pm} + h_{\alpha\beta}^{\text{gauge} \pm}. \quad (1)$$

The reconstructed metric perturbation $h_{\alpha\beta}^{\text{rec} \pm}$ is obtained...
by the CCK procedure, and represents the radiative part also referred to as $h_{\alpha\beta}^{\text{rad}}$ below. The sum of the completion piece and the gauge piece instead gives the nonradiative part $h_{\alpha\beta}^{\text{nonrad}}$. A method to compute $h_{\alpha\beta}^{\text{comp}}$ has been recently proposed in Ref. [24] for eccentric equatorial motion (see also references therein for a review of previous attempts), based on the construction of certain gauge-invariant fields from the full perturbed metric. Imposing the continuity of these quantities across the hypersurface containing the particle world line fixes the completion piece of the metric perturbation in a way that in the spacetime region outside such a hypersurface the mass and angular momentum are given by the particle’s conserved energy and angular momentum, whereas both vanish in the region inside it ($h_{\alpha\beta}^{\text{comp}} = 0$). This result has been then generalized to any bound orbit around a Kerr black hole in Ref. [22].

A last problem still remains unsolved, how to determine the gauge part $h_{\alpha\beta}^{\text{gauge}}$. The “gauge-smoothing” of the perturbation across the particle’s world line is crucial for the GSF calculation of orbital invariants. In fact, if the gauge part is not calibrated properly, the computation will not give the correct result. The orbital invariants are indeed invariants under a class of gauge transformations which are sufficiently smooth functions of the coordinates and also preserve the symmetries of the perturbed spacetime, e.g., the helical symmetry for circular orbits [20]. In contrast, the perturbed metric is discontinuous at the location of the particle, so that a gauge transformation connecting the interior and exterior parts must be discontinuous too. This fact may affect or not the invariance of the quantity constructed with the full perturbation. A useful check is the comparison with the results obtained through different methods, e.g., within the PN theory. In any case, one should require the gauge part to share the same regularity as well as symmetry properties as the full perturbation. For instance, in the simplest case of circular motion one can reduce the gauge freedom by demanding that the metric perturbation preserves the helical symmetry, besides the usual conditions of regularity and asymptotic flatness. Further imposing the continuity of certain metric components has been shown in Ref. [25] to completely fix the gauge on a Schwarzschild background. Unfortunately, the same reasoning cannot be applied to more general situations, like eccentric orbits, and still for circular orbits in Kerr. A different strategy has been suggested in Refs. [21, 28, 29], but not fully implemented yet, consisting in requiring the continuity at the particle position of suitably chosen “quasi-invariant” fields built with the metric perturbation (see also the related discussion in Section 7.6 of Ref. [3]). So far, the only possibility to overcome this difficulty has been to choose a “reasonable” gauge, do the calculation of an orbital invariant (or any other gauge-invariant function), and check the agreement of the first few terms of its PN expansion with available PN results, eventually adjusting the gauge part if needed. This is the reason why the redshift invariant was the first orbital invariant to be computed in a Kerr spacetime [19, 20, 21, 22]. In fact, it is defined in terms of the double contraction of the perturbed metric with the particle’s four-velocity, which is a continuous function across the particle already at the level of individual radiative multipole modes, before summing over them and regularizing. Therefore, the knowledge of the nonradiative part in the exterior region (where the gauge part must vanish for regularity reason, as we will show below) suffices in the case of the redshift. As soon as one is considering more singular quantities at the particle’s location, like gyroscope precession (i.e., a connection term) and tidal invariants (i.e., curvature terms), involving first and second derivatives of the perturbed metric, respectively, one cannot expect such a simple feature, so that a more general procedure is needed.

We provide here a general prescription to completely fix the remaining gauge freedom, with specific applications to eccentric equatorial orbits in the Kerr spacetime. The exterior part vanishes identically ($h_{\alpha\beta}^{\text{comp}} = 0$) due to the request of asymptotic flatness. The components of the most general gauge vector generating the interior part $h_{\alpha\beta}^{\text{gauge}}$, instead, are determined by requiring that the causality condition of the particle’s four velocity be preserved at every spacetime point, including the location of the particle, and by imposing on the full perturbed metric the Ricci identity across the hypersurface containing the particle’s world line.

The paper is organized as follows. We will start by reviewing the problem of low multipoles in the case of a Schwarzschild spacetime and its solution in the RWZ formalism. In the simplest case of perturbations due to a particle moving along an equatorial circular orbit we will show that the request of asymptotic flatness and continuity of certain metric components implies a gauge-adjustment of the nonradiative part of the metric by a necessarily discontinuous gauge vector, which generates an additional energy momentum tensor contributing to the Einstein’s equations. We will pass then to the Kerr case, discussing the various parts of the perturbations in the same line of reasoning: radiative and nonradiative, the latter further splitting into completion and gauge parts. Finally, we will provide a prescription to completely determine the gauge part, and apply it to the case of eccentric equatorial orbits. We will use geometrical units $G = 1 = c$.

II. PERTURBATIONS ON A SCHWARzsCHILD SPACETIME

Let us start by reviewing the gauge problem for gravitational perturbations on a Schwarzschild black hole background, with line element written in standard spherical-like coordinates $(t, r, \theta, \phi)$ given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  

(2)
The nonradiative part of the metric is obtained by solving the perturbation equations corresponding to the lowest multipoles $l = 0, 1$ in a spherical harmonic decomposition of the metric, following the original approach of Zerilli. We will consider below the simplest case of a particle moving along a circular equatorial orbit of radius $r = r_0$, with four velocity

$$U = U^t (\partial_t + \Omega \partial_\phi),$$

with

$$U^t = \left(1 - \frac{3M}{r_0}\right)^{-1/2}, \quad \Omega = \sqrt{\frac{M}{r_0}}.$$  \hspace{1cm} (4)

The particle’s energy-momentum tensor

$$T^{\mu\nu} = \rho \frac{U^\mu U^\nu}{r_0^2 U^t} \delta(r - r_0) \delta(\theta - \frac{\pi}{2}) \delta(\phi - \Omega t)$$

$$= \rho \sum_{lm} \frac{U^\mu U^\nu}{r_0^2 U^t} \delta(r - r_0) Y_{lm}(\theta, \phi) Y^*_{lm}(\frac{\pi}{2}, \Omega t)$$

(5)

can be decomposed into a radiative part

$$T^{rad}_{\mu\nu} = \sum_{l \geq 2, m} T^{lm}_{\mu\nu},$$

and a completion part

$$T^{comp}_{\mu\nu} = \sum_{l = 0, 1, m} T^{lm}_{\mu\nu},$$

(7)

associated with the radiative part $h^{rad}_{\mu\nu}$ ($l \geq 2$) and the completion part $h^{comp}_{\mu\nu}$ ($l = 0, 1$) of the perturbation, respectively. We will refer to the original work of Zerilli for notation and conventions.

After decomposing both the perturbed metric and the energy-momentum tensor in spherical harmonics, the Einstein’s field equations reduce to two sets of perturbation equations of different parity: the even sector consists of seven equations for the perturbation functions $H_0, H_1, H_2, K, h^e_{0,l}, h^e_{1,l}$, whereas the odd sector consists of three equations for the perturbation functions $h^o_{0,l}, h^o_{1,l}, h^o_{2,l}$. This general form of the perturbation can be simplified by performing a suitable gauge choice. Consider the infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu,$$

(8)

where the infinitesimal displacement $\xi^\mu$ is a function of coordinates $x^\mu$ and transforms like a vector. This transformation induces the following transformation of the metric perturbation

$$h'_{\mu\nu} = h_{\mu\nu} - 2\xi_{(\mu;\nu)},$$

(9)

where the multipole expansion of the second term can be easily obtained by expanding the gauge vector $\xi$ in vector harmonics of both types, i.e., $\xi = \xi^{even} + \xi^{odd}$, with

$$\xi^{even} = \sum_{lm} \left[ A_0 Y_{lm} dt + A_1 Y_{lm} dr \right.$$

$$\left. + A_2 \left( \frac{\partial Y_{lm}}{\partial \theta} d\theta + \frac{\partial Y_{lm}}{\partial \phi} d\phi \right) \right],$$

(10)

and

$$\xi^{odd} = \sum_{lm} A_3 \left[ \sin \theta \left( \frac{\partial Y_{lm}}{\partial \phi} d\theta - \sin^2 \theta \frac{\partial Y_{lm}}{\partial \phi} d\phi \right) \right],$$

(11)

where $A_\alpha = A_\alpha (l, r)$. Under such a transformation the even and odd perturbation functions change according to

$$H'_0 = H_0 - \frac{2}{N^2} \partial_t A_0 + \frac{1 - N^2}{r} A_1,$$

$$H'_1 = H_1 - \partial_t A_1 - \partial_r A_0 + \frac{1 - N^2}{r N^2} A_0,$$

$$H'_2 = H_2 - 2 N^2 \partial_t A_1 - \frac{1 - N^2}{r} A_1,$$

$$K' = K - \frac{2 N^2}{r} A_1 + \frac{2}{r^2} A_2,$$

$$G' = G - \frac{2}{r^2} A_2,$$

$$h^e_{0,l} = h^e_{0,l} - A_0 - \partial_t A_2,$$

$$h^e_{1,l} = h^e_{1,l} - A_1 + \frac{2}{r} A_2 - \partial_r A_2,$$

(12)

and

$$h^o_{0,l} = h^o_{0,l} + \partial_t A_3,$$

$$h^o_{1,l} = h^o_{1,l} - \frac{2}{r} A_3 + \partial_r A_3,$$

$$h^o_{2,l} = h^o_{2,l} - 2 A_3,$$

(13)

respectively. The Regge-Wheeler gauge sets to zero the functions $G, h^e_{0,l}, h^e_{1,l}$ and $h^o_{2,l}$.

If the gauge vector is a sufficiently smooth function of the coordinates, i.e., at least twice differentiable, then the gauge part $h^{gauge}_{\mu\nu} = 2\xi_{(\mu;\nu)}$ is a “pure gauge” metric perturbation, which is automatically a solution of the vacuum Einstein’s equations $\delta G_{\mu\nu}[h^{gauge}_{\mu\nu}] = 0$. In that case, the radiative part of the perturbation satisfies the perturbed equations

$$\delta G_{\mu\nu}[h^{rad}_{\mu\nu}] = 8\pi T^{rad}_{\mu\nu},$$

(14)

whereas the nonradiative one

$$\delta G_{\mu\nu}[h^{comp}_{\mu\nu}] = 8\pi T^{comp}_{\mu\nu},$$

(15)

so that

$$\delta G_{\mu\nu}[h^{rad}_{\mu\nu}] + \delta G_{\mu\nu}[h^{comp}_{\mu\nu}] = 8\pi (T^{rad}_{\mu\nu} + T^{comp}_{\mu\nu}) = 8\pi T_{\mu\nu}.$$  \hspace{1cm} (16)

In contrast, we will show below that in order to get a nonradiative metric perturbation with the desired properties the (nonradiative part of the) gauge vector must
be a discontinuous function across the particle’s world
line, thus generating an Einstein tensor or equivalently an
additional energy-momentum tensor with distributional
singularities there to be added to the right-hand-side of Eq. (15).

A. The monopole mode \( l = 0 \)

The \( l = 0 \) mode represents the perturbation in the total
mass of the system due to the addition of the particle’s
conserved Killing energy \( \delta M \equiv E = -U_r \). For this mode
there are two gauge degrees of freedom and the choice
done by Zerilli is

\[
H^Z_1(t, r) = 0 = K^Z(t, r) .
\]

(17)

The solution for the remaining perturbation functions \( H_0 \)
and \( H_2 \) is

\[
H^Z_0 = H^Z_2 = \frac{a}{r - 2M} \theta(r - r_0) ,
\]

(18)

with

\[
a = 2\sqrt{4\pi} \delta M ,
\]

(19)

and \( \theta(x) \) denoting the Heaviside step function. The non-
vanishing metric components are then given by

\[
\begin{align*}
  h^Z_{\tau\tau} &= \frac{N^2 H_0}{4\pi} = \frac{2\delta M}{r} \theta(r - r_0) , \\
  h^Z_{rr} &= \frac{H_2}{\sqrt{4\pi} N^2} = \frac{2\delta M}{rn^4} \theta(r - r_0) .
\end{align*}
\]

(20)

One can use the gauge freedom to make the \( h_{\tau\tau} \) com-
ponent continuous across the particle’s position without
modifying the \( h_{rr} \) component. This is done by choosing

\[
A_0 = -\frac{at}{2r} \frac{r - 2M}{r_0 - 2M} \theta(r_0 - r) , \quad A_1 = 0 ,
\]

(21)

so that

\[
\begin{align*}
  H_0 &= \frac{a}{r_0 - 2M} \theta(r_0 - r) + \frac{a}{r - 2M} \theta(r - r_0) , \\
  H_1 &= \frac{-at}{2r_0} \delta(r - r_0) , \\
  H_2 &= \frac{a}{r - 2M} \theta(r - r_0) .
\end{align*}
\]

(22)

The new metric perturbation is then given by

\[
\begin{align*}
  h^\prime_{\tau\tau} &= \frac{2\delta M}{r} \left[ \frac{rN^2}{r_0 N_0^2} \theta(r_0 - r) + \theta(r - r_0) \right] , \\
  h^\prime_{\tau r} &= \frac{-2\delta M}{2r_0} \delta(r - r_0) , \\
  h^\prime_{rr} &= \frac{2\delta M}{rn^4} \theta(r - r_0) .
\end{align*}
\]

(23)

B. The dipole \( l = 1 \) odd mode

The \( l = 1 \) odd mode represents the angular momen-
tum perturbation \( \delta J = L = U_\phi \) added by the particle to
the system. Zerilli takes \( h^{(\text{odd})}_0 \) added by the particle to
the system. Zerilli takes \( h^{(\text{odd})}_0 \) and the remaining
perturbation equations imply

\[
h^{(\text{odd})}_1 = b \left( \frac{2}{r_0} \theta(r_0 - r) + \frac{r_0}{r} \theta(r - r_0) \right) ,
\]

(24)

with

\[
b = 2 \sqrt{4\pi} \frac{\delta J}{r_0} ,
\]

(25)

and only the \( m = 0 \) mode is nonzero. The only nonvan-
ishing metric component is then given by

\[
h^{(\text{odd})}_r = -\sqrt{\frac{3}{4\pi}} h^{(\text{odd})}_0 \sin^2 \theta
\]

(26)

Choosing

\[
A_3 = h^{(\text{odd})}_0 \theta(r_0 - r) ,
\]

(27)

allows one to make the \( h_{\tau\phi} \) component continuous across
the particle’s position. In fact, the new function \( h^{(\text{odd})}_0 \)
becomes

\[
h^{(\text{odd})}_0 = b \left[ \frac{2}{r_0^2} \theta(r_0 - r) + \frac{r_0}{r} \theta(r - r_0) \right] ,
\]

(28)

but a nonzero function \( h^{(\text{odd})}_1 \) is also generated

\[
h^{(\text{odd})}_1 = -b \delta(r - r_0) .
\]

(29)

The new metric perturbation is then given by

\[
\begin{align*}
  h^\prime_{\tau\phi} &= -\sqrt{\frac{3}{4\pi}} h^{(\text{odd})}_1 \sin^2 \theta \\
  h^\prime_{\tau\phi} &= \frac{2\delta J}{r_0} \delta(r - r_0) \sin^2 \theta .
\end{align*}
\]

(30)

C. The dipole \( l = 1 \) even mode

The \( l = 1 \) even mode is related to the shift of the center
of momentum of the system. Zerilli assumes \( K^Z = 0 \) in
addition to \( h^{(\text{even})}_0 = h^{(\text{even})}_1 = 0 \). The solution for
the remaining metric functions in the Zerilli gauge turns out
to be

\[
\begin{align*}
  H^Z_0 &= \frac{c(t) r_0 N_0^2}{3 \pi r_0^4 N^4} \left( 1 - \Omega^2 r^{-3} M^4 \right) \delta M \theta(r - r_0) , \\
  H^Z_1 &= i mc(t) r_0 N_0^2 \frac{r_0}{r^2 N^4} \delta M \theta(r - r_0) , \\
  H^Z_2 &= c(t) r_0 N_0^2 \frac{r_0}{r^2 N^4} \delta M \theta(r - r_0) .
\end{align*}
\]

(31)
with
\[ c(t) = -2\sqrt{6\pi}me^{-im\Omega t}. \] (32)

The nonvanishing metric components are then given by
\[
\begin{align*}
\bar{h}^Z_{tt} &= 2\frac{r^2N^2_0}{r^2N^4} \left(1 - \Omega^2 \frac{r^3}{M}\right) \delta M \theta(r - r_0) \sin \theta \cos \tilde{\phi}, \\
\bar{h}^Z_{tr} &= -6\frac{r^2N^2_0}{r^2N^4} \delta M \theta(r - r_0) \sin \theta \sin \tilde{\phi}, \\
\bar{h}^Z_{rr} &= 6\frac{r^2N^2_0}{r^2N^4} \delta M \theta(r - r_0) \sin \theta \cos \tilde{\phi},
\end{align*}
\] (33)

where \( \tilde{\phi} = \phi - \Omega t \). Therefore, the solution is not asymptotically flat. In vacuum, a dipole even-parity perturbation can be completely removed by a gauge transformation, as shown by Zerilli. This is no more true in the nonvacuum case, when a source term is present given by the particle's energy momentum tensor. However, one can always perform a “singular gauge” transformation \[2, 4\] which removes the perturbation in the vacuum region outside of the particle’s location, leaving a nonvanishing contribution just at \( r = r_0 \).

Let \( h^0_{\text{even}} = 0 = h^e_{\text{even}} \), in order that the RW gauge be still holding. This implies
\[
A_0 = -\partial_t A_2, \quad A_1 = \frac{2}{r} A_2 - \partial_r A_2. \quad (34)
\]

Choosing then
\[
A_2 = f_2(t) \frac{r^2}{r^2 - 2Mr^2} \theta(r - r_0), \quad f_2(t) = -\frac{c(t)}{6} \frac{r_0}{M} N^2_0 \delta M,
\] (35)
leads to the new metric components of the type
\[
h'_\alpha = h^0_{\alpha} \delta(r - r_0) + h^e_{\alpha} \delta'(r - r_0), \quad (36)
\]
that is, explicitly
\[
\begin{align*}
h^0_{tt} &= 2N^2_0 \delta M \theta(r - r_0) \sin \theta \cos \tilde{\phi}, \\
h^0_{tr} &= -\frac{2}{M} N^2_0 \delta M \theta(r - r_0) \sin \theta \sin \tilde{\phi}, \\
h^0_{rr} &= \frac{2\delta M}{M} \left[ (r_0 - M) \delta'(r - r_0) \right] \sin \theta \cos \tilde{\phi}, \\
h^0_{\theta\theta} &= -\frac{2}{M} N^2_0 \delta M \delta(r - r_0) \sin \theta \cos \tilde{\phi}, \\
h^0_{\phi\phi} &= \sin^2 \theta h^0_{\theta\theta}
\end{align*}
\]
which are all delta-singular (meaning that they generically include \( \delta(r - r_0) \), \( \delta'(r - r_0) \), etc. contributions) at the particle’s position.

D. Final form of the nonradiative part of the perturbation

In the Zerilli gauge the completion part of the metric perturbation can be written as
\[
h^\text{comp}_{\alpha\beta} \equiv \bar{h}^Z_{\alpha\beta} = \tilde{h}^Z_{\alpha\beta} \theta(r - r_0), \quad (38)
\]
with nonvanishing components
\[
\begin{align*}
\tilde{h}^Z_{tt} &= \frac{2\delta M}{r}, \\
\tilde{h}^Z_{rr} &= \frac{2\delta M}{r r_0}, \\
\tilde{h}^Z_{\theta\theta} &= -\frac{2\delta J}{r} \sin^2 \theta.
\end{align*}
\] (39)

Therefore, the interior metric is identically vanishing, and the monopole and dipole (exterior) perturbations describe the sudden shift in mass and angular momentum parameters induced by the particle occurring at \( r = r_0 \), respectively, modulo the presence of gauge terms associated with a change in the system’s center of mass, which are delta-singular there \[2, 4]\. The latter will be included in the gauge piece.

The gauge freedom can be used to modify the final form of the perturbation by requiring, e.g., asymptotic flatness and the continuity of certain metric components. The gauge part of the perturbation
\[
h^\text{gauge}_{\alpha\beta} = 2\xi_{(\alpha, \beta)}, \quad (40)
\]
is generated by the gauge field
\[
\begin{align*}
\xi &= -\frac{A_0}{N^2_0 r^2} \delta M \theta(r - r_0) + \frac{A_3}{r^2} \delta \theta \delta \phi,
\end{align*}
\]
\[= t \left[ \frac{\delta M}{r_0 N^2_0} \delta \theta + \frac{2\delta J}{r_0^3} \delta \phi \right] \theta(r_0 - r), \quad (41)
\]
and can be written as
\[
h^\text{gauge}_{\alpha\beta} = \tilde{h}^\text{gauge}_{\alpha\beta} \theta(r_0 - r) + \text{delta-singular terms}, \quad (42)
\]
with nonvanishing components
\[
\begin{align*}
\tilde{h}^\text{gauge}_{tt} &= \frac{2\delta M}{r} \frac{r N^2_0}{r_0 N^4_0}, \\
\tilde{h}^\text{gauge}_{r\phi} &= -\frac{2\delta J}{r r_0} \sin^2 \theta. \quad (43)
\end{align*}
\]

Taking into account these gauge modes in the computation of orbital invariants is crucial for obtaining the correct result.

The final form of the perturbation is then
\[
h^\text{nonrad}_{\alpha\beta} = \tilde{h}^\text{gauge}_{\alpha\beta} \theta(r_0 - r) + \tilde{h}^Z_{\alpha\beta} \theta(r - r_0) + \text{delta-singular terms}. \quad (44)
\]

The completion piece \( h^\text{comp}_{\alpha\beta} \) satisfies the perturbed Einstein’s equations \[15\] sourced by the completion part \( (l = 0, 1) \) of the particle’s energy momentum tensor. The gauge part \( h^\text{gauge}_{\alpha\beta} \) introduces additional (higher) delta-singular terms (containing also derivatives of the Dirac-delta function at \( r = r_0 \))
\[
T^\text{gauge}_{\mu\nu} \equiv \frac{1}{8\pi} \delta G_{\mu\nu}[h^\text{gauge}_{\alpha\beta}], \quad (45)
\]
so that the nonradiative metric perturbation \[44\] satisfies the field equations
\[
\delta G_{\mu\nu}[h^\text{nonrad}] = 8\pi (T^\text{comp}_{\mu\nu} + T^\text{gauge}_{\mu\nu}). \quad (46)
\]
Finally, the Einstein’s equations associated with the full perturbation read

$$\delta g_{\mu\nu}[h_{\alpha\beta}] = 8\pi(T_{\mu\nu} + \mathcal{T}^{\text{gauss}}_{\mu\nu}).$$

We have recalled how the completion problem of metric perturbations is handled in the case of Schwarzschild spacetime and circular orbits. The original approach of Zerilli suffices in determining the completion part of the nonradiative metric for arbitrary motion, i.e., the exterior metric describing the sudden shift in the spacetime’s mass and angular momentum induced by the particle. Performing a discontinuous gauge transformation then leads to a nonvanishing interior metric, which can be adjusted in order that the whole nonradiative perturbation shares some additional regularity properties, e.g., asymptotic flatness and continuity of certain metric components. Passing then to the Kerr case this problem is more difficult to be addressed, due to the lack of field equations governing the nonradiative part of the perturbations. However, the above considerations apply also to Kerr perturbations, so that we can follow the same line of reasoning as before.

III. PERTURBATIONS ON A KERR SPACETIME

Let us consider the Kerr spacetime (with mass $M$ and angular momentum $J = Ma$), whose line element written in Boyer-Lindquist coordinates $x^\alpha = (t, r, \theta, \phi)$, with $\alpha = 0, 1, 2, 3$, is given by

$$ds^2 = g^K_{\alpha\beta}dx^\alpha dx^\beta$$

$$= \left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{4aMr \sin^2 \theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(\mathcal{T}^2 + 2\frac{Mr^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2,$$

where

$$\Delta = r^2 + a^2 - 2Mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta.$$  \hspace{1cm} (49)

The outer horizons $r_+$ is located at $r_+ = M + \sqrt{M^2 - a^2}$. Note that the signature has been switched from $+2$ to $-2$ with respect to the Schwarzschild case, as customary within the Teukolsky formalism.

A particle with mass $\mu \ll M$ moving along a geodesic orbit with parametric equations $x^\alpha_p = x^\alpha_p(\tau)$, $\tau$ denoting the proper time parameter, and four velocity $U = \frac{dx^\alpha_p}{d\tau} \partial_\alpha$, with $U \cdot U = 1$, induces a perturbation $h_{\alpha\beta}$ on the background due to its energy-momentum tensor, which is Dirac-delta singular along its world line

$$T_{\mu\nu}(x^\alpha) = \mu \int_{-\infty}^{\infty} \frac{d^4 \delta(x^\alpha - x^\alpha_p(\tau))U_\mu U_\nu d\tau}{\sqrt{-g^K}},$$

where $g^{K}$ is the metric determinant for the background, and $\delta^{(4)}$ is the four dimensional delta function

$$\delta^{(4)}(x^\alpha - x^\alpha_p(\tau)) = \delta(t - t(\tau))\delta^{(3)}(x^\alpha - x^\alpha_p(\tau))$$

$$= \frac{\delta(t - t(\tau))}{U^t}\delta^{(3)}(x^\alpha - x^\alpha_p(t))$$  \hspace{1cm} (51)

with $(\alpha = 1, 2, 3)$

$$\delta^{(3)}(x^\alpha - x^\alpha_p(t)) = \delta(r - r_p(t))\delta(\theta - \theta_p(t))\delta(\phi - \phi_p(t)).$$  \hspace{1cm} (52)

A general geodesic motion is governed by the equations (see, e.g., Ref. [34])

$$\frac{dt_p}{d\tau} = \frac{1}{\Sigma} \left[aB + \frac{r_p^2 + a^2}{\Delta}P\right], \quad \frac{dr_p}{d\tau} = \epsilon_r \frac{1}{\Sigma} \sqrt{\mathcal{R}}, \quad \frac{d\theta_p}{d\tau} = \epsilon_\theta \frac{1}{\Sigma} \sqrt{\Theta}, \quad \frac{d\phi_p}{d\tau} = \frac{1}{\Sigma} \left[\frac{B}{\sin^2 \theta_p} + \frac{a}{\Delta}P\right],$$  \hspace{1cm} (53)

where $\epsilon_r$ and $\epsilon_\theta$ are sign indicators, and

$$P = E(r^2 + a^2) - aL, \quad B = L - aE \sin^2 \theta_p, \quad R = P^2 - \Delta(r^2 + \mathcal{K}), \quad \Theta = \mathcal{K} - a^2 \cos^2 \theta_p - \frac{B^2}{\sin^2 \theta_p}.$$  \hspace{1cm} (54)

Here $E = U_t$ and $L = -U_\phi$ denote the conserved Killing energy and angular momentum per unit mass, and $\mathcal{K}$ is a separation constant, usually called the Carter constant, which for equatorial plane orbits ($\theta = \frac{\pi}{2}$, $U^\theta = 0$) reduces to $K = (L - aE)^2$.

The perturbed metric is given by

$$g_{\alpha\beta}(x^\lambda) = g^K_{\alpha\beta}(x^\lambda) + h_{\alpha\beta}(x^\lambda),$$  \hspace{1cm} (55)

where the first order perturbation $h_{\alpha\beta}$ should be suitably regularized, being the retarded field divergent at the particle position. This is usually done by subtracting the Detweiler-Whiting singular field mode-by-mode through an extension of the particle’s 4-velocity $U$ to a field in the neighborhood of the world line and the use of suitable regularization parameters. These issues for radiation gauges and generic orbits in a Kerr spacetime are discussed, e.g., in Ref. [16]. The regularized field $h^R_{\alpha\beta}$ is thus a smooth vacuum perturbation, with the particle moving along a geodesic in the effective metric $g^R_{\alpha\beta} = g^K_{\alpha\beta} + h^R_{\alpha\beta}$.

We will conveniently use a parametrization of the world line in terms of the coordinate time $t$ instead of the proper time $\tau$. On each $t = \text{constant}$ spacetime slice the instantaneous position of the particle is represented by a single spatial point, associated with coordinates $x^\alpha = x^\alpha(t)$. Consider then a spacelike 3-volume $V$ defined by $t = \text{constant}$ and $r_* < r_1 < r < r_2$, for some values $r_1$ and $r_2$ of the radial variable. Such a hypersurface is pierced by the particle’s world line, which is thus instantaneously bounded by the two 2-surfaces $S_1 = V_{r=r_1}$ and $S_2 = V_{r=r_2}$, i.e., the inner and outer boundaries of $V$. Let $r_1 = r_p(t) - \epsilon \equiv r^-_p(t)$ and $r_2 = r_p(t) + \epsilon \equiv r^+_p(t)$, with $\epsilon \ll 1$. In the limit $\epsilon \to 0$
the two spherical-like 2-surfaces \( S_1 = V_{r=r^-} \equiv S_- \) and \( S_2 = V_{r=r^+} \equiv S_+ \) join smoothly and are identified with the interface \( S = V = r_0 \) between the interior (−) and exterior (+) vacuum regions. The perturbation can then be written in the form
\[
h_{\alpha\beta} = h_{\alpha\beta}^0 \theta(r_0(t) - r) + h_{\alpha\beta}^0 \theta(r - r_0(t)),
\]
where \( h_{\alpha\beta}^\pm \) is given by Eq. (1) and \( \theta(x) \) denotes the Heaviside step function. The regularization procedure basically involves the radiative part \( h_{\alpha\beta}^{\text{rec+}} \) of the perturbed metric, whose large-\( l \) behavior actually determines the regularization parameters and is enough to obtain a convergent series, once the average of the interior and exterior solutions is taken.

The radiative part of the perturbation is constructed by using the full energy-momentum tensor (50) as a source. In fact, according to the CCK procedure in a radiation gauge, the components of the metric perturbation tensor are obtained by applying a suitable differential operator on a scalar quantity, the Hertz potential. The latter is determined in two steps. It must satisfy the initial operator on a scalar quantity, the Hertz potential. The conservation of the full energy-momentum tensor, whose large-\( l \) behavior actually determines the regularization parameters and is enough to obtain a convergent series, once the average of the interior and exterior solutions is taken.

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where the change in mass $\delta M = \partial_M g_{\alpha\beta}^K$ and angular momentum $h^{\alpha\beta}_{\delta M} = \partial_M g_{\alpha\beta}^K$ are obtained by replacing $\delta M \rightarrow \delta M + \delta J$ and $J \rightarrow J + \delta J$ in the background metric [13] and retaining only terms which are linear in $\delta M$ and $\delta J$ [24]. The latter are then identified with the conserved energy and angular momentum of the perturbing particle, i.e., $\delta M = \mu E$ and $\delta J = \mu L$. We list below the nonvanishing components of $h^{\alpha\beta}_{\delta M}$, for completeness (see Eqs. (88)–(89) of Ref. [24])

$$h^{\alpha\beta}_{\delta M} = \frac{2}{M} \cos^2 \theta \delta M + \frac{2}{M} \sin \theta \sin \theta \delta J, \quad h^{\alpha\beta}_{\delta \phi} = \frac{2}{M} \sin \theta \cos \theta \delta M - \frac{2}{M} \cos \theta \sin \theta \delta J.$$ (62)

The interior metric $h^{\alpha\beta}_{\delta M}$ is instead identically vanishing, as already stated, so that

$$h^{\alpha\beta}_{\delta M} = h^{\alpha\beta}_{\delta M} \theta(r - r_p(t)).$$ (63)

Its double contraction with the particle’s four velocity is then given by

$$h^{\alpha\beta}_{U\theta} = h^{\alpha\beta}_{U\theta} U^{\alpha}U^{\beta} = h^{\alpha\beta}_{U\theta} \theta(r - r_p(t)).$$ (64)

The completion perturbation is thus associated with the energy-momentum tensor

$$T^{\mu\nu}_{\delta M} = \frac{1}{8\pi} G_{\mu\nu}[h^{\alpha\beta}_{\delta M}],$$ (65)

the components of which are proportional to the Dirac-delta function and its first derivatives, i.e.,

$$T^{\mu\nu}_{\delta M} = T^{\mu\nu}_{\delta M} \delta(r - r_p(t)) + \tilde{T}^{\mu\nu}_{\delta M} \delta'(r - r_p(t)).$$ (66)

**B. Gauge piece**

The gauge part $h^{\alpha\beta}_{\delta \gamma} = 2\xi_{(\alpha;\beta)}$, with a discontinuous gauge field

$$\xi = \xi_-(r_p(t) - r) + \xi_+(r - r_p(t)), \quad \text{which generates delta-singular terms in the metric and consequently higher and higher singular terms in the connection, the curvature, etc. As in the Schwarzschild case, one can equivalently write} \quad h^{\alpha\beta}_{\delta \gamma} = h^{\alpha\beta}_{\delta \gamma} \theta(r_p(t) - r) + h^{\alpha\beta}_{\delta \gamma} \theta(r - r_p(t)) + \text{delta-singular terms},$$ (69)

where

$$h^{\alpha\beta}_{\delta \gamma} = 2\xi_{(\alpha;\beta)}.$$ (70)

The interior and exterior parts $\xi_{\pm}$ of the gauge field have the general form [23, 24]

$$\xi_{\pm} = \frac{\mu}{M} [\alpha_{\pm}(t) \partial_t + \beta_{\pm}(t) \partial_\phi].$$ (71)

The only nonvanishing components of $h^{\alpha\beta}_{\delta \gamma}$ then turn out to be

$$h^{\alpha\beta}_{\delta \gamma} = \frac{2\mu}{M} \left[ \frac{\partial \alpha_{\pm}}{\partial \phi} \left( 1 - \frac{2Mr}{\Sigma} \right) + \frac{\partial \beta_{\pm}}{\partial \phi} \sin^2 \theta \right],$$ (72)

$$h^{\alpha\beta}_{\delta \phi} = \frac{\mu}{M} \left\{ \frac{\partial \alpha_{\pm}}{\partial t} \left( 1 - \frac{2Mr}{\Sigma} \right) - \frac{\partial \beta_{\pm}}{\partial t} \left( \Delta + \frac{2Mr}{\Sigma} (r^2 + a^2) \right) \right\} \sin^2 \theta,$$ (73)

which we require bounded in time and asymptotically flat (i.e., they must vanish for large $r$). The latter request implies that $h^{\alpha\beta}_{\delta \gamma} \equiv 0$, so that $\alpha_{\pm}(t) = 0 = \beta_{\pm}(t)$, leading to

$$h^{\alpha\beta}_{\delta \gamma} = h^{\alpha\beta}_{\delta \gamma} \theta(r_p(t) - r) + h^{\alpha\beta}_{\delta \gamma} \delta(r - r_p(t)).$$ (74)

Only the functions $\alpha_{\pm}(t)$ and $\beta_{\pm}(t)$ remain to be specified. Therefore, we need two further conditions.

The singular part of the gauge perturbation (according to the notation of Eq. (73)) has nonvanishing components

$$h^{\alpha\beta}_{\delta \gamma} = \frac{2}{M} \frac{\partial \alpha_{\pm}}{\partial \phi} \left( 1 - \frac{2Mr}{\Sigma} \right) + \frac{\partial \beta_{\pm}}{\partial \phi} \sin^2 \theta,$$ (75)

$$h^{\alpha\beta}_{\delta \phi} = \frac{\mu}{M} \left\{ \frac{\partial \alpha_{\pm}}{\partial t} \left( 1 - \frac{2Mr}{\Sigma} \right) - \frac{\partial \beta_{\pm}}{\partial t} \left( \Delta + \frac{2Mr}{\Sigma} (r^2 + a^2) \right) \right\} \sin^2 \theta,$$ (76)

so that the double contraction with the particle’s four velocity is identically vanishing (i.e., $h^{\alpha\beta}_{\delta \gamma} = 0$), whence

$$h^{\alpha\beta}_{\delta \gamma} = h^{\alpha\beta}_{\delta \gamma} \theta(r_p(t) - r).$$ (77)
Adding gauge modes to the completion piece thus generates a further high singular source term

\[ T_{\mu\nu}^{\text{gauge}} \equiv \frac{1}{8\pi} \delta G_{\mu\nu}[h_{\alpha\beta}^{\text{gauge}}], \]  

(76)

with

\[ T_{\mu\nu}^{\text{gauge}} = \tilde{T}_{\mu\nu}^{\text{gauge},\delta}(r - r_p(t)) + \tilde{T}_{\mu\nu}^{\text{gauge},\delta'}(r - r_p(t)) + \tilde{T}_{\mu\nu}^{\text{gauge},\delta''}(r - r_p(t)). \]  

(77)

C. Gauge-fixing

The resulting metric perturbation associated with the nonradiative multipole is then

\[ h_{\alpha\beta}^{\text{nonrad}} = h_{\alpha\beta}^{\text{gauge},-}\theta(r_p(t) - r) + h_{\alpha\beta}^{\text{comp},+}\theta(r - r_p(t)) + h_{\alpha\beta}^{\text{gauge}}\delta(r - r_p(t)). \]  

(78)

Each metric component is a smooth function of the coordinates in either side, and is well behaved in the limit \( r \to r_p(t) \). However, the regular part of the metric is not continuous at the particle position, and derivatives generate distributional singularities there, which add to those arising from the singular part of the metric. Therefore, all relevant tensors associated with the metric (78) are meant in the sense of distributions.

The first condition comes from demanding that the causality property of the particle’s four velocity be preserved with respect to the full (regularized) perturbed metric \( g_{\alpha\beta}^{\text{R}} \) at every spacetime point, including the location of the particle. This is equivalent to impose the continuity of \( h_{UU}^{R} = h_{UU}^{\text{R}}U^\alpha U^\beta \) across the instantaneous position of the particle \( r = r_p(t) \), which for the metric (78) implies

\[ h_{UU}^{\text{comp},+}|_{\alpha=\beta(t)} = h_{UU}^{\text{gauge},-}|_{\alpha=\beta(t)}. \]  

(79)

The second condition can be derived as follows. Consider the full spacetime metric

\[ g_{\alpha\beta} = g_{\alpha\beta}^{K} + g_{\alpha\beta}^{\text{rec}} + h_{\alpha\beta}^{\text{nonrad}}, \]  

(80)

satisfying the Einstein’s equations

\[ G_{\mu\nu}[g_{\alpha\beta}] = 8\pi T_{\mu\nu}. \]  

(81)

For an arbitrary vector field \( v = v^\alpha \partial_\alpha \) the exterior derivative writes

\[ dv = \frac{1}{2}(dv)_{\alpha\beta} dx^\alpha \wedge dx^\beta, \]  

(82)

with components

\[ (dv)_{\alpha\beta} = 2v^\gamma_{,\alpha\beta}. \]  

(83)

The Ricci identity and the Einstein equations imply

\[ v^\alpha_{,\beta\alpha} - v^\alpha_{,\alpha\beta} = R^\alpha_{\gamma\alpha\beta}v^\gamma = R^\alpha_{\beta\gamma\alpha}v^\gamma - 8\pi T^{(\text{TR})}_{\alpha\beta}v^\gamma, \]  

(84)

where we have introduced the trace-reversed (TR) notation

\[ T_{\alpha\beta}^{(\text{TR})} = T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}, \quad T = T^\alpha_{\alpha}. \]  

(85)

The energy-momentum tensor is proportional to the mass ratio \( \mu/M \) like the components of the perturbed metric, so that indices can be raised/lowered with the background metric to first order. From Eq. (81) we then have for the particle’s energy-momentum tensor

\[ T_{\alpha\beta}^{(\text{TR})} = T(U_{\alpha} U_{\beta} - \frac{1}{2}K g_{\alpha\beta}) \quad T = \frac{\mu}{\Sigma_p \sin \theta_p U^\gamma (x - x_p(t))}. \]  

(86)

Taking the covariant derivative of both sides of Eq. (81) then gives

\[ (dv)_{\alpha\beta} = -8\pi T^{(\text{TR})}_{\alpha\beta}v^\beta + v^\beta_{,\beta\alpha} - v_{\alpha\beta}^\beta. \]  

(87)

The last term can be conveniently rewritten by separating its symmetric and antisymmetric parts as follows

\[ v_{\alpha\beta}^\beta = [v_{\alpha\beta}]^\beta + v_{(\alpha)}^\beta = -\frac{1}{2}(dv)_{\alpha\beta}^\beta + v_{(\alpha)}^\beta, \]  

(88)

whence Eq. (87) becomes

\[ \frac{1}{2}(dv)_{\alpha\beta}^\beta = -8\pi T^{(\text{TR})}_{\alpha\beta}v^\beta + v^\beta_{,\beta\alpha} - v_{\alpha\beta}^\beta. \]  

(89)

Note that if \( v \) is a Killing vector of the perturbed spacetime, i.e., \( v = K \), the last two terms vanish identically (because of the relations \( K^\beta_{,\beta} = 0 = K_{(\alpha\beta)} \)) and Eq. (89) simplifies to

\[ (dv)_{\alpha\beta}^\beta = -16\pi T^{(\text{TR})}_{\alpha\beta}K^\beta. \]  

(90)

Let us now integrate both sides of Eq. (81) over the spacelike 3-volume \( V \) with boundary \( S_- \cup S_+ \) introduced above, surrounding the instantaneous location of the particle \( r = r_p(t) \). This leads to the relation

\[ \frac{1}{2}I_1 = I_2 + I_3, \]  

(91)

where

\[ I_1 = \int_V (dv)_{\alpha\beta} v^\beta dV^\alpha, \]  

(92)

\[ I_2 = -8\pi \int_{S_+} T^{(\text{TR})}_{\alpha\beta} v^\beta dS^\alpha, \]  

(93)

\[ I_3 = \int_V [v^\beta_{,\beta\alpha} - v_{(\alpha)}^\beta] dV^\alpha, \]  

(94)

which allows one to connect interior and exterior metrics at the boundary. The volume element \( dV^\alpha \) and the surface element \( dS^\alpha \) are defined by

\[ dV^\alpha = \eta^\alpha_{r\theta\phi} dr d\theta d\phi, \quad dS^\alpha = \eta^{\alpha\beta}_{\theta\phi} d\theta d\phi, \]  

(95)
respectively, where \( \eta_{\alpha \beta \gamma \delta} = \sqrt{-g} \epsilon_{\alpha \beta \gamma \delta} \) is the unit volume 4-form, with \( \epsilon_{\alpha \beta \gamma \delta} \) \((\epsilon_{0123} = 1)\) being the Levi-Civita alternating symbol. Higher singular terms containing derivatives of the Dirac-delta function do not contribute to the integrals \((92)\).

The relations derived above are completely general, and are valid for any choice of the vector field \( v \) (with corresponding 1-form \( v^\mu \)). The latter can be naturally chosen as aligned with either the temporal or azimuthal Killing vector of the background spacetime, or even a combination of them, like the four velocity of the zero-angular-momentum observers (ZAMOs). For the application we are going to discuss below we will take

\[
\begin{align*}
  v &= \partial_t, \\
  v^\pm &= g_{\alpha \mu} dx^\alpha = (g^K_{\alpha \mu} + h^K_{\alpha \mu}) dx^\alpha,
\end{align*}
\]

which implies different expressions for the exterior derivatives \( dv^\pm \) in either region. In contrast, when considered as a vector and not as a 1-form, its components are continuous across the particle’s location. Notice that choosing \( v = \partial_\phi \) would not give any useful information. In fact, it is a Killing vector for both interior and exterior spacetime regions, implying that \( I_3 \equiv 0 \) and the relation \((91)\) reduces to a trivial identity not involving the functions \( \alpha_-\) and \( \beta_-\).

We have recalled above how the full particle’s energy momentum tensor \( T^{\mu \nu} \) fixes the completion amplitude by constructing with it Komar-like integrals over a closed surface enveloping the region containing the matter distribution, thus completely determining the completion piece. The radiative part of the perturbation does not carry any mass and angular momentum, as it follows from computing the conserved quantities associated with the currents \( j_\mu = T_{\mu \nu} \xi^\nu \) built with the background Killing vectors \( \xi(t) = \partial_t \) and \( \xi(\phi) = \partial_\phi \) (see Eq. \((59)\)). Applying then this result to the integral form \((91)\)–\((92)\) of the Ricci identity \((51)\)–\((52)\) of the particle’s location, we have that the radiative part of the metric perturbation does not contribute to the integrals \( I_1 \) and \( I_3 \). Hence, the latter can be computed simply by using the nonradiative part of the metric.

Let us consider the case of eccentric equatorial orbits, as an example. The particle’s four velocity is given by

\[
U = \frac{1}{r_p^2} \left( ax + \frac{r_p^2 + 2a^2}{\Delta} \right) \partial_t + \frac{e}{r_p^2} \left[ (p^2 - \Delta(r^2 + x^2))^{1/2} \partial_r + \frac{1}{r_p^2} \left( x + \frac{a}{\Delta} \right) \partial_\phi \right],
\]

as from Eq. \((52)\), with \( x = L - aE \). The first two integrals \((92)\) are easily computed

\[
I_1 = -8\pi \left[ \delta M - \mu \left( \frac{d\alpha_-}{dt} - 2a \frac{d\beta_-}{dt} \right) \right],
\]

\[
I_2 = -8\pi \mu \frac{E - 1}{U^t}.
\]

The evaluation of the third integral \( I_3 \) instead is much more involved, since differentiating twice the Heaviside function generates terms proportional to the Dirac-delta function and its first derivatives. We find

\[
I_3 = I_3^{M} \delta M + I_3^{L} \delta J,
\]

with

\[
\begin{align*}
I_3^{M} &= -aI_3^{L} + 4\pi r_p \left( \frac{r_p^2 + a^2}{\Delta} \right)^2 \left[ \frac{d^2 r_p}{dt^2} + \frac{1}{r_p} \left( \frac{dr_p}{dt} \right)^2 \left( 1 - \frac{4Mr_p r_p^2 - 2a^2}{\Delta (r_p^2 + a^2)} \right) \right], \\
I_3^{L} &= -8\pi r_p \frac{r_p^2 + a^2}{\Delta} \left\{ \frac{d^2 r_p}{dt^2} \left[ \frac{r_p}{a} \arctan \left( \frac{a}{r_p} \right) - 1 + \frac{a^2}{2} - \frac{a^2}{3Mr_p} \left( 1 + \frac{M r_p r_p^2}{r_p^2 + a^2} \right) \right] \right. \\
& \quad + \frac{2}{r_p} \left( \frac{dr_p}{dt} \right)^2 \left[ 1 - \frac{M r_p r_p^2}{r_p^2 + a^2} \right] \left[ \frac{r_p}{a} \arctan \left( \frac{a}{r_p} \right) - 1 + \frac{2a^2}{\Delta} - \frac{1}{2} \left( \frac{a^2}{r_p^2 + a^2} \right)^2 - \frac{a^2}{M r_p} \right] \left( \frac{2(r_p^2 + a^2)}{r_p^2 + a^2} \right) \right\}.
\end{align*}
\]

The continuity \((53)\) of \( h^\text{monrad} \) at the particle’s position \( r = r_p(t) \) reads

\[
I^{\text{comp+}}_{UU}(r_p) = -2 \frac{\delta M}{r_p} (U^t)^2 + 4 \frac{\delta J}{r_p} U^t U^\phi - 2 \frac{r_p^2}{M \Delta^2} [(M r_p + a^2) \delta M - a \delta J] (U^r)^2 \\
+ 2 \frac{a}{M} \left[ a \left( 1 + \frac{M}{r_p} \right) \delta M - \left( 1 + \frac{2M}{r_p} \right) \delta J \right] (U^\phi)^2 \\
= 2 \frac{\mu}{M} U^t \left( E \frac{d\alpha_-}{dt} - L \frac{d\beta_-}{dt} \right) = h^{\text{gauge-}}_{UU}(r_p).
\]


The two conditions (91) and (99) can be finally solved for the time derivatives of \( \alpha_\cdot \) and \( \beta_\cdot \), leading to

\[
(L - 2aE) \frac{d\alpha_\cdot}{dt} = EL + \frac{L}{4\pi\mu} (I_2 + I_3) - \frac{Ma}{\mu U} h^{\text{comp+}}_{UU}(r_p),
\]

\[
(L - 2aE) \frac{d\beta_\cdot}{dt} = E^2 + \frac{E}{4\pi\mu} (I_2 + I_3) - \frac{M}{2\mu U} h^{\text{comp+}}_{UU}(r_p),
\]

where the quantities \( I_2, I_3 \) and \( h^{\text{comp+}}_{UU}(r_p) \) are all functions of \( r_p \) only, with \( \delta M = \mu E \) and \( \delta J = \mu L \) as already stated.

This result has been successfully applied in Ref. [35] to compute the first-order GSF correction to the gyroscope precession along slightly eccentric orbits and the related periastron advance in the circular orbit limit [36]. In the circular case \((r = r_0, U^r = 0)\) the previous relations reduce to

\[
\frac{d\alpha_\cdot}{dt} = -\frac{u_0(1 + 2\hat{a}u_0^{3/2} - \hat{a}^2u_0^2)}{(1 - 3u_0 + 2\hat{a}u_0^{3/2})^{1/2}(1 + \hat{a}u_0^{3/2})},
\]

\[
M\frac{d\beta_\cdot}{dt} = -\frac{u_0^{5/2}(2 - \hat{a}u_0^{1/2})}{(1 - 3u_0 + 2\hat{a}u_0^{3/2})^{1/2}(1 + \hat{a}u_0^{3/2})},
\]

(101)

where \( u_0 = M/r_0 \) and \( \hat{a} = a/M \), so that the corresponding gauge vector coincides with that used in Ref. [29] (see Eqs. (3.14)–(3.16) there).

In the Schwarzschild limit \((a = 0)\) Eq. (100) becomes

\[
\frac{d\alpha_\cdot}{dt} = -\frac{ML^2}{r_p^3E^2} + \frac{M}{r_p E} - \frac{2ME}{r_p} - 2M,
\]

\[
M\frac{d\beta_\cdot}{dt} = -\frac{2M^2L}{r_p^2},
\]

(102)

which for circular orbits reduces to

\[
\frac{d\alpha_\cdot}{dt} = -\frac{u_0}{\sqrt{1 - 3u_0}}, \quad M\frac{d\beta_\cdot}{dt} = -\frac{2u_0^{5/2}}{\sqrt{1 - 3u_0}},
\]

(103)

in agreement with Eq. (11). The resulting metric components thus reproduce previous results for both circular and eccentric orbits [3, 27, 9].

IV. DISCUSSION

GSF calculations of orbital invariants require the complete knowledge of the perturbed metric, including a radiative part and a completion piece, made of nonradiative modes and gauge modes. The gauge-smoothing of the perturbation on a Kerr spacetime across the particle position has been a challenge for many years. We have finally solved this problem, providing a prescription for fully determining the gauge part of the metric perturbation, resulting in two conditions to be imposed on the completion piece of the metric. These conditions are necessary to preserve the causality property of the particle’s four velocity at its instantaneous position (see Eq. (100)), and to fulfill the Ricci identity across the hypersurface containing the particle’s world line (see Eqs. (91)–(92)).

We have applied this prescription to the case of eccentric equatorial orbits, as an example. The resulting nonradiative metric perturbation has been used in Refs. [35, 36] to compute the first-order GSF correction to the gyroscope precession along slightly eccentric orbits and the related periastron advance in the circular orbit limit. We have also recovered all limiting cases already studied in the literature of eccentric orbits in Schwarzschild as well as circular orbits in both Schwarzschild and Kerr spacetimes. This result will allow for completing ongoing and future GSF calculations, including the case of inclined orbits, i.e., bound orbits not confined to the equatorial plane, which are expected to play a key role in the study of orbital resonances in EMRIs, strongly affecting the phasing of the inspiral [37].

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[1] T. Regge and J. A. Wheeler, “Stability of a Schwarzschild singularity,” Phys. Rev. 108, 1063 (1957). doi:10.1103/PhysRev.108.1063
[2] F. J. Zerilli, “Gravitational field of a particle falling in a Schwarzschild geometry analyzed in tensor harmonics,” Phys. Rev. D 2, 2141 (1970). doi:10.1103/PhysRevD.2.2141
[3] M. Davis, R. Ruffini, W. H. Press and R. H. Price, “Gravitational radiation from a particle falling radially into a Schwarzschild black hole,” Phys. Rev. Lett. 27, 1466
Newtonian order gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole,” Phys. Rev. D 93, no. 12, 124058 (2016) doi:10.1103/PhysRevD.93.124058 [arXiv:1602.08282 [gr-qc]].

[33] D. Bini and A. Geralico, “New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: redshift invariant,” Phys. Rev. D 100, no. 10, 104002 (2019) doi:10.1103/PhysRevD.100.104002 [arXiv:1907.11080 [gr-qc]].

[34] S. Chandrasekhar, “The mathematical theory of black holes,” (Clarendon, Oxford, UK, 1985).

[35] D. Bini and A. Geralico, “New gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole: gyroscope precession,” Phys. Rev. D 100, no. 10, 104003 (2019) doi:10.1103/PhysRevD.100.104003 [arXiv:1907.11082 [gr-qc]].

[36] D. Bini and A. Geralico, “Analytical determination of the periastron advance in spinning binaries from self-force computations,” Phys. Rev. D 100, no. 12, 121502 (2019) doi:10.1103/PhysRevD.100.121502 [arXiv:1907.11083 [gr-qc]].

[37] E. E. Flanagan and T. Hinderer, “Transient resonances in the inspirals of point particles into black holes,” Phys. Rev. Lett. 109, 071102 (2012) doi:10.1103/PhysRevLett.109.071102 [arXiv:1009.4923 [gr-qc]].

[38] S. R. Dolan and L. Barack, “Self-force via $m$-mode regularization and 2+1D evolution: III. Gravitational field on Schwarzschild spacetime,” Phys. Rev. D 87, 084066 (2013) doi:10.1103/PhysRevD.87.084066 [arXiv:1211.4586 [gr-qc]].