Generalized Forchheimer flows of isentropic gases

Emine Celik, Luan Hoang and Thinh Kieu

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Abstract
We consider generalized Forchheimer flows of either isentropic gases or slightly compressible fluids in porous media. By using Muskat’s and Ward’s general form of the Forchheimer equations, we describe the fluid dynamics by a doubly nonlinear parabolic equation for the appropriately defined pseudo-pressure. The volumetric flux boundary condition is converted to a time-dependent Robin-type boundary condition for this pseudo-pressure. We study the corresponding initial boundary value problem, and estimate the $L^\infty$ and $W^{1,2-a}$ (with $0 < a < 1$) norms for the solution on the entire domain in terms of the initial and boundary data. It is carried out by using a suitable trace theorem and an appropriate modification of Moser’s iteration.

1 Introduction

The most common equation to describe fluid flows in porous media is the Darcy law

$$-\nabla p = \frac{\mu}{k} v,$$

(1.1)

where $p$, $v$, $\mu$, $k$ are, respectively (resp.), the pressure, velocity, absolute viscosity and permeability.

However, this linear equation is not valid in many situations, particularly, when the Reynolds number increases, see [3,25]. Even in the early work, Darcy [4] already acknowledged the deviations from equation (1.1). There have been many investigations into what equations for hydrodynamics in porous media to replace Darcy’s law (1.1), see [3,25,26,28,33] and references therein. Forchheimer [5,9] established the following three nonlinear empirical models: two-term Forchheimer equation

$$-\nabla p = av + b|v|v,$$

(1.2)

three-term Forchheimer equation

$$-\nabla p = av + b|v|v + c|v|^2v,$$

(1.3)

and Forchheimer’s power law

$$-\nabla p = av + d|v|^{m-1}v, \text{ for some real number } m \in (1,2).$$

(1.4)
Above, the positive constants $a, b, c, d$ are obtained from experiments.

While mathematics of Darcy’s flows have been studied intensively for a long time with vast literature, see e.g. [31], there is a much smaller number of mathematical papers on Forchheimer flows and they appeared much later. Among those, there are even fewer papers dedicated to compressible fluids. (See [28] and references there in.)

In order to cover general nonlinear flows in porous media formulated from experiments, generalized Forchheimer equations were proposed. They extend the models (1.2)–(1.4) and are of the form

$$- \nabla p = \sum_{i=0}^{N} a_i |v|^{\alpha_i} v. \tag{1.5}$$

These equations are analyzed numerically in [7,19,27], theoretically in [2,10–13,16] for single-phase flows, and also in [14,15] for two-phase flows.

Our previous analysis [2,10–13,16] was focused on a simplified model for slightly compressible fluids. Though such a minor simplification is commonly used in reservoir engineering, the mathematical rigor is compromised. Furthermore, since the model does not specify the dependence on the density, its applications to gaseous flows would be inaccurate and might present artificial technical difficulties. The goals of this paper are: (a) Developing a more accurate model for generalized Forchheimer equations for gases, and (b) Analyzing it without making any simplifications.

For goal (a), we first have to modify (1.5) to reflect the dependence on the density. We return to an idea by Muskat and Ward. By using dimension analysis, Muskat [25] and then Ward [33] proposed the following equation for both laminar and turbulent flows in porous media:

$$- \nabla p = f(\nu^k \frac{\rho}{\alpha} k^2 \mu^{2-\alpha}), \text{ where } f \text{ is a function of one variable}. \tag{1.6}$$

In particular, when $\alpha = 1, 2$, Ward [33] established from experimental data that

$$- \nabla p = \frac{\mu}{k} v + c_F \frac{\rho}{\sqrt{k}} |v| v, \quad \text{where } c_F > 0. \tag{1.7}$$

Combining (1.5) with the suggestive form (1.6) for the dependence on $\rho$ and $v$, we propose the following equation

$$- \nabla p = \sum_{i=0}^{N} a_i \rho^{\alpha_i} |v|^{\alpha_i} v, \tag{1.8}$$

where $N \geq 1$, $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_N$ are real numbers, the coefficients $a_0, \ldots, a_N$ are positive.

Here, the viscosity and permeability are considered constant and we do not specify the dependence of $a_i$’s on them. Our mathematical exposition below will allow all $\alpha_i \geq 1$ in (1.8). In practice, we can simply take $\alpha_N \leq 2$ in (1.8) or use the popular model (1.7). Even in these cases, the results obtained in this paper are still new.

Multiplying both sides of (1.8) by $\rho$ gives

$$g(|\rho v|) \rho v = -\rho \nabla p, \tag{1.9}$$

where $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a generalized polynomial with positive coefficients defined by

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \cdots + a_N s^{\alpha_N} \quad \text{for } s \geq 0. \tag{1.10}$$

We will study the following two types of compressible fluids.

1. Isentropic gases. For isentropic gases, the constitutive law is

$$p = c \rho^\gamma \quad \text{for some } c, \gamma > 0. \tag{1.11}$$
Then from (1.9) and (1.11) follows

\[ g(|\rho v|) \rho v = -\rho \nabla p = -\nabla u \quad \text{with} \quad u = \frac{c\gamma \rho^{\gamma+1}}{\gamma + 1}. \]  

(1.12)

Solving for \( \rho v \) from this equation yields

\[ \rho v = -K(|\nabla u|) \nabla u, \]  

(1.13)

where the function \( K : \mathbb{R}^+ \to \mathbb{R}^+ \) is defined for \( \xi \geq 0 \) by

\[ K(\xi) = \frac{1}{g(s(\xi))} \]  

with \( s = s(\xi) \) being the unique non-negative solution of \( sg(s) = \xi \).

(1.14)

Recall the continuity equation

\[ \phi \rho_t + \text{div}(\rho v) = 0, \]  

(1.15)

where constant \( \phi \in (0, 1) \) is the porosity. Rewrite

\[ \rho = \left( \frac{\gamma + 1}{c\gamma} \right)^{\frac{1}{\gamma+1}} u^\lambda \]  

with \( \lambda = \frac{1}{\gamma + 1} \in (0, 1) \).

(1.16)

Combining (1.15) and (1.13) with relation (1.16), we have

\[ (u^\lambda)_t = \frac{1}{\phi} \left( \frac{c\gamma}{\gamma + 1} \right)^\lambda \nabla \cdot (K(|\nabla u|) \nabla u)). \]  

(1.17)

2. **Slightly compressible fluids.** The equation of state is

\[ \frac{1}{\rho} \frac{d \rho}{dp} = \frac{1}{\kappa} = \text{const.} > 0. \]  

(1.18)

Note from (1.18) that \( \rho \nabla p = \kappa \nabla \rho \). Then combining this with (1.9) gives

\[ g(|\rho v|) \rho v = -\nabla u \quad \text{with} \quad u = \kappa \rho. \]  

(1.19)

which is the same as (1.12). Thus, we obtain formula (1.13) for \( \rho v \). By combining (1.13) and (1.15) we have

\[ u_t = \frac{\kappa}{\phi} \nabla \cdot (K(|\nabla u|) \nabla u)). \]  

(1.20)

Observe that we can write \( u_t = (u^\lambda)_t \) with \( \lambda = 1 \) in (1.20). Therefore, the two equations (1.17) and (1.20) are the same except for the factors on the right-hand sides. Then by scaling the time variable in both (1.17) and (1.20), we obtain, for both isentropic gases and slightly compressible fluids, the unified equation

\[ (u^\lambda)_t = \nabla \cdot (K(|\nabla u|) \nabla u)) \]  

with \( \lambda \in (0, 1] \).

(1.21)

This will be the partial differential equation (PDE) of our interest. It is derived and will be analyzed without any simplifications (goal (b) above). Although the case of isentropic gases, i.e. \( \lambda < 1 \), is the main focus, the analysis will also cover the case of slightly compressible fluids, i.e. \( \lambda = 1 \), at no extra cost.

In case of ideal gases, i.e., \( \gamma = 1 \), we can derive from (1.21) a PDE for pseudo-pressure \( p^2 \). In general, we rewrite \( u \) in (1.12) as \( u = c' \rho^{\frac{\gamma}{\gamma+1}} \) for some \( c' > 0 \), hence it is approximately the
pseudo-pressure for isentropic gases \([11]\). For simplicity, we refer to \(u\) as the pseudo-pressure. Therefore, equation \([1.21]\) is a PDE describing the dynamics of the pseudo-pressure \(u\).

Regarding the boundary conditions, we focus on the volumetric flux condition \(v \cdot \vec{v} = \psi\), which results in \(p v \cdot \vec{v} = \psi \rho\), or,
\[
-K(\|\nabla u\|)\nabla u \cdot \vec{v} = \varphi u^\lambda,
\]
where \(\lambda \in (0,1), \varphi = (\frac{\gamma + 1}{\gamma}) \frac{1}{R^\gamma} \psi\) in case isentropic gases, and \(\lambda = 1, \varphi = \psi / \kappa\) in case of slightly compressible fluids. Here, \(\vec{v}\) is the outward normal vector on the boundary.

From mathematical point of view, equation \([1.21]\) for \(\lambda < 1\) is a doubly nonlinear parabolic equation, which is an interesting topic of its own. Research on doubly nonlinear parabolic equations follows the development of general parabolic equations \([21, 22]\) and degenerate/singular parabolic equations \([5, 6]\). However, it requires much more complicated techniques. See monograph \([17]\), review paper \([18]\) and references therein. For other developments, see e.g. \([1, 20, 23, 30, 32]\).

There are two issues that did not attract attention in most existing papers: (1) Robin boundary condition, and (2) Estimates for super-critical case. For instance, \([23, 30]\) give global \(L^\infty\)-estimates but for homogeneous Dirichlet boundary condition in the sub-critical case, see discussion in Remark \([3, 2]\) below. Regarding interior estimates, the common result is
\[
\|u\|_{L^\infty(Q(R/2,T/2))} \leq C_{R,T}(\|u\|_{L^{\alpha(q)}(Q(R,T))}^\alpha + 1),
\]
where \(q \in (1, \infty), C_{R,T} > 0\) depends explicitly on \(R\) and \(T\), while \(Q(R,T)\) denotes the cylinder \(B_R \times (-T,0)\). Surnachev \([29]\) improves it to
\[
\|u\|_{L^\infty(Q(R/2,T/2))} \leq C_{R,T}(\|u\|_{L^{\alpha(q)}(Q(R,T))}^\alpha + \|u\|_{L^{\beta(q)}(Q(R,T))}^\beta) \quad \text{with } \alpha(q), \beta(q) > 0.
\]
We call \([1.24]\) a quasi-homogeneous estimate (with respect to \(\|u\|_{L^\infty(Q(R,T))}\)). The global version of \([1.24]\) is not known for doubly nonlinear equations, though it was established for degenerate equations, see e.g. \([13]\).

In this paper, we focus on both topics (1) and (2) listed above. In this case, the boundary condition \([1.22]\) gives rise to high power boundary integral which cannot be treated by the standard trace theorem. Therefore, we derive and utilize a new, suitable trace inequality to obtain bounds for the solutions of \([1.21]\) in terms of initial and boundary data. For \(L^\infty\)-estimates, we make some technical improvements in order to overcome the non-homogeneity of function \(K(\cdot)\) and non-zero boundary data. We carefully modify Moser’s iteration \([24]\) and obtain quasi-homogeneous estimates. Our results are for both (spatially) interior and global estimates, hence, extend the previous interior improvement \([1.24]\).

Throughout this paper, \(U\) is an open, bounded subset of \(\mathbb{R}^n\), with \(n = 2, 3, \ldots\), and has \(C^1\)-boundary \(\Gamma = \partial U\). For physics problems \(n = 2, 3\), but we consider here any natural number \(n \geq 2\). Hereafter, we fix the functions \(g(s)\) in \([1.9]\) and \([1.10]\). Therefore, the exponents \(\alpha_i\) and coefficients \(a_i\) are all fixed, and so is the function \(K(\xi)\) in \([1.14]\). Also, our calculations frequently use the following exponent
\[
a = \frac{\alpha_N}{\alpha_N + 1} \in (0,1).
\]

We consider the initial boundary value problem associated with \([1.21]\) and \([1.22]\), specifically,
\[
\begin{cases}
\frac{\partial (u^\lambda)}{\partial t} = \nabla \cdot (K(\|\nabla u\|)\nabla u) & \text{in } U \times (0, \infty), \\
u(x,0) = u_0(x) & \text{in } U, \\
K(\|\nabla u\|) \frac{\partial u}{\partial \nu} + \varphi u^\lambda = 0 & \text{on } \Gamma \times (0, \infty),
\end{cases}
\]
where \( u_0(x) \) and \( \varphi(x,t) \) are given initial and boundary data, respectively. Again, \( \bar{\nu} \) denotes the outward normal vector on \( \Gamma \). Here, \( \lambda \) is a fixed number in \((0,1)\) for the remaining of the paper.

The current article is focused on studying non-negative solutions of problem (1.26). Section 2 contains new trace theorems and multiplicative Sobolev’s inequalities, which are suitable to the Robin-type boundary condition (1.22), as well as the nature of our equation’s double nonlinearity. In section 3 we estimate \( L^\alpha \)-norms of the solutions for all \( \alpha > 0 \), in terms of initial and boundary data. These will also be used for later gradient and \( L^\infty \) estimates. In section 4 we present estimates for the gradient’s \( L^{2-\alpha} \)-norm for time \( t > 0 \). In section 5 we estimate the \( L^\infty \)-norm of the solution in any compact subsets of the domain. Due to the basic Lebesgue norm relation in Proposition 5.2, the Moser’s iteration is of a non-homogeneous form (5.22). We deal with this by using Lemma A.2 and obtain in Theorem 5.3 the quasi-homogeneous estimate. Section 6 is focused on estimating the Moser’s iteration with non-homogeneous inequalities. In section 7 contains new trace theorems and multiplicative Sobolev’s inequalities, which are suitable to the Robin-type boundary condition (1.22), as well as the nature of our equation’s double nonlinearity. In section 3 we estimate \( L^{\alpha+\mu_1} \)-norms of the solutions for all \( \alpha > 0 \), \( \mu_1 > 0 \) depend on \( \alpha \), and the constant depends on the boundary data. To manage the powers during iterations, we construct in Lemma 6.3 two controlling sequences \( (\alpha_j)_{j=0}^\infty \) and \( (\beta_j)_{j=0}^\infty \). Using these sequences for iterations, we obtain in Theorem 6.6 the quasi-homogeneous estimates for the \( L^\infty \)-norm, and in Theorem 6.8 the ultimate estimates in terms of initial and boundary data. The Appendix contains key Lemma A.2 in implementing Moser’s iteration with non-homogeneous inequalities.

## 2 Auxiliaries

First, we recall elementary inequalities that will be used frequently. Let \( x, y \geq 0 \). Then

\[
(x + y)^p \leq 2^p(x^p + y^p) \quad \text{for all } p > 0,
\]

\[
(x + y)^p \leq x^p + y^p \quad \text{for all } 0 < p \leq 1,
\]

\[
(x + y)^p \leq 2^{p-1}(x^p + y^p) \quad \text{for all } p \geq 1,
\]

\[
x^\beta \leq x^\alpha + x^\gamma \quad \text{for all } 0 \leq \alpha \leq \beta \leq \gamma,
\]

particularly,

\[
x^\beta \leq 1 + x^\gamma \quad \text{for all } 0 \leq \beta \leq \gamma.
\]

Second, we establish particular Poincaré-Sobolev inequality and trace theorem for studying our doubly nonlinear equation with Robin-type boundary condition.

For any \( 1 \leq p < n \), we denote by \( p^* \) its Sobolev conjugate exponent, that is, \( p^* = \frac{np}{n-p} \).

**Lemma 2.1.** In the following statements, \( u(x) \) is a function defined on \( U \).

(i) If \( \alpha \geq s \geq 0, \alpha \geq 1, \) and \( p > 1 \), then for any \( |u|^\alpha \in W^{1,1}(U) \) and \( \varepsilon > 0 \) one has

\[
\int_U |u|^\alpha \, dx \leq \varepsilon \int_U |u|^{\alpha-s} |\nabla u|^p \, dx + c_1 \int_U |u|^\alpha \, dx + (c_2 \alpha)^{\frac{p}{p-1}} \varepsilon^{-\frac{1}{p-1}} \int_U |u|^{\alpha+\frac{np}{n-p}} \, dx,
\]

where \( c_1, c_2 > 0 \) are constants depending on \( U \), but not on \( u(x), \alpha, s, p, \).

(ii) If \( n > p > 1, r > 0, \alpha \geq s \geq 0, \alpha \geq \frac{n-s}{p-1}, \) and \( \alpha > \frac{n(r+s-p)}{p} \), then for any \( \varepsilon > 0 \) one has

\[
\int_U |u|^{\alpha+r} \, dx \leq \varepsilon \int_U |u|^{\alpha-s} |\nabla u|^p \, dx + \varepsilon^{-\frac{\theta}{1-s}} 2^{\frac{\theta(\alpha+s-p)}{1-s}} (c_3 m)^{\frac{\theta p}{1-s}} \|u\|_{L^{\alpha+\mu_1}}^{\alpha+\mu_1} + 2^{\theta(\alpha-s+p)} c_4^p |U| \frac{\|u\|_{L^{\alpha+r}}^{\alpha+\mu_1}}{\alpha},
\]
for all \( |u|^m \in W^{1,p}(U) \), where
\[
m = \frac{\alpha - s + p}{p}, \quad \theta = \frac{rn}{n(p-s) + \alpha p}, \quad \mu_1 = \frac{r + \theta(s-p)}{1 - \theta},
\]
and constants \( c_3, c_4 > 0 \) depend on \( U, p \), but not on \( u(x), \alpha, s \).

(iii) If \( n > p \geq 1, \alpha \geq s > p, \) and \( \alpha > \frac{n(s-p)}{p-1} \), then for any \( \varepsilon > 0 \), one has
\[
\int_{\Gamma} |u|^{\alpha} d\sigma \leq 2\varepsilon \int_U |u|^{\alpha-s} |\nabla u|^p dx + c_1 \|u\|^{\theta}_{L^\alpha}
+ \varepsilon^{-\frac{1}{p-1}} D_{1,\alpha} \|u\|^{\alpha+\mu_1}_{L^\alpha},
\]
for all functions \( u(x) \) satisfying \( |u|^\alpha \in W^{1,1}(U) \) and \( |u|^m \in W^{1,p}(U) \), where \( m \) is defined by (2.8),
\[
\theta = \frac{1}{(p-1)\left(\frac{\alpha p}{n(p-s)} - 1\right)},
\]
\[
D_{1,\alpha} = 2^{\theta(\alpha-s+p)}(c_2\alpha)^{\frac{3\theta p}{p-1}} c_4^\theta |U|^{\frac{\theta(p-1)(s-p)}{\alpha s-p}} - 1, \quad D_{2,\alpha} = 2^{\theta(\alpha-s+p)}(c_2\alpha)^{\frac{p}{p-1}}(c_3m)^{\frac{\theta p}{p-1}}.
\]

Proof. We make a couple of comments before starting the proof. First, the boundary integrals in (2.6) and (2.9) are in the sense of traces of \( |u|^\alpha \) on \( \Gamma \). Second, observe that the conditions on \( u(x) \) do not guarantee that the right-hand sides of (2.6), (2.7) and (2.9) are finite. In case they are not, these inequalities are understood to be trivially true. Therefore, the following proof only needs to cover the case when those right-hand sides are finite.

(i) We recall the trace theorem
\[
\int_{\Gamma} |\phi| d\sigma \leq c_1 \int_U |\phi| dx + c_2 \int_U |\nabla \phi| dx,
\]
for all \( \phi \in W^{1,1}(U) \), where \( c_1 \) and \( c_2 \) are positive constants depending on \( U \). Applying this trace theorem to \( \phi = |u|^\alpha \), we have
\[
\int_{\Gamma} |u|^\alpha d\sigma \leq c_1 \int_U |u|^\alpha dx + c_2 \alpha \int_U |u|^\alpha-1 |\nabla u| dx.
\]

Rewriting \( c_2 \alpha |u|^\alpha-1 |\nabla u| \) in the last integral as a product of \( \varepsilon^{1/p} u^\frac{\alpha-s}{p} |\nabla u| \) and \( c_2 \alpha \varepsilon^{-1/p} u^{\frac{(p-1)(s-p)}{p}} \), and applying Young’s inequality with exponent \( p \) and \( p/(p-1) \), we obtain inequality (2.6).

(ii) Since \( \alpha \geq s \), the number \( m \) defined by (2.8) is greater or equal to 1. Then applying Sobolev-Poincaré inequality to \( |u|^m \) yields
\[
\| |u|^m \|_{L^\alpha} \leq c_3 \| \nabla (|u|^m) \|_{L^p} + c_4 \int_U |u|^m dx,
\]
where \( c_3 \) and \( c_4 \) are positive constants depending on \( U \) and \( p \). Note that by definition (2.8) of \( m \), we have \( (m-1)p = \alpha - s \). Hence (2.13) can be written as
\[
\left( \int_U |u|^p \right)^{m/p} \leq c_3 m \left( \int_U |u|^{\alpha-s} |\nabla u|^p dx \right)^{1/p} + c_4 \int_U |u|^m dx.
\]
Raising both sides to the power $1/m \leq 1$ and using inequality (2.22), we obtain
\[
\left( \int_U u^m dx \right)^{\frac{1}{m}} \leq (c_3 m)^{\frac{1}{m}} \left( \int_U u^{n - s} |\nabla u|^p dx \right)^{\frac{1}{n - s + p}} + c_4 \left( \int_U u^m dx \right)^{\frac{1}{m}}.
\]  
(2.14)

Note that
\[
q = p^* m = \frac{n(\alpha - s + p)}{n - p}.
\]  
(2.15)

Then (2.14) yields
\[
\|u\|_{L^q} \leq (c_3 m)^{\frac{1}{m}} \left( \int_U u^{n - s} |\nabla u|^p dx \right)^{\frac{1}{n - s + p}} + c_4 \|u\|_{L^m}.
\]  
(2.16)

Since $n > p$, $r > 0$, and $\alpha > \frac{n(r + s - p)}{p}$, one has $\alpha < \alpha + r < q$. Then
\[
\frac{1}{\alpha + r} = \frac{1}{\alpha} + \frac{1 - \theta_0}{q},
\]
where $\theta_0 \in (0, 1)$ is defined by
\[
\theta_0 = \frac{rq}{(\alpha + r)(q - \alpha)} \quad \text{with} \quad q = p^* m,
\]  
(2.17)

Then interpolation inequality and (2.16) give
\[
\|u\|_{L^{\alpha + r}} \leq \|u\|_{L^q}^{\theta_0} \|u\|_{L^m}^{1 - \theta_0} \leq \left\{ (c_3 m)^{\frac{1}{m}} \left( \int_U u^{n - s} |\nabla u|^p dx \right)^{\frac{1}{n - s + p}} + c_4 \|u\|_{L^m} \right\}^{\theta_0} \|u\|_{L^{\alpha + r}}^{1 - \theta_0}.
\]  
(2.18)

Raising both sides to the power $\alpha + r$ and applying inequality (2.11) for exponent $\theta_0(\alpha + r)$ yield
\[
\int_U u^{\alpha + r} dx \leq 2^{\theta_0(\alpha + r)} \left\{ (c_3 m)^{\frac{\theta_0(\alpha + r)}{m}} \left( \int_U u^{n - s} |\nabla u|^p dx \right)^{\frac{\theta_0(\alpha + r)}{n - s + p}} + c_4 \|u\|_{L^m} \right\} \|u\|_{L^{\alpha + r}}^{\theta_0(\alpha + r)}
\]
\[
+ 2^{\theta_0(\alpha + r)} c_4^{\frac{\theta_0(\alpha + r)}{m}} \|u\|_{L^m} \|u\|_{L^{\alpha + r}} \|u\|_{L^{\alpha + r}}.
\]
(2.19)

Since $\alpha > \frac{n(r + s - p)}{p}$, we have $\theta \in (0, 1)$. Then applying Young’s inequality to the first term on the right-hand side of (2.18) with powers $\frac{1}{\theta}$ and $\frac{1}{1 - \theta}$, we obtain
\[
\int_U u^{\alpha + r} dx \leq \varepsilon \int_U u^{n - s} |\nabla u|^p dx + \varepsilon^{-\frac{\theta}{1 - \theta}} 2^{\frac{\theta_0(\alpha + r)}{m}} (c_3 m)^{\frac{\theta_0(\alpha + r)}{m}} \|u\|_{L^m} \left[ \left(1 - \theta_0\right)\|u\|_{L^{\alpha + r}} \right]^{\frac{1}{1 - \theta}}
\]
\[
+ 2^{\theta_0(\alpha + r)} c_4^{\frac{\theta_0(\alpha + r)}{m}} \|u\|_{L^m} \|u\|_{L^{\alpha + r}} \|u\|_{L^{\alpha + r}}.
\]  
(2.20)

Since $\alpha \geq \frac{p^*}{p - 1}$, then $m \leq \alpha$. By applying Hölder’s inequality to bound the $L^m$-norm of $u$ on the right-hand side of (2.20) by $\|u\|_{L^m}^{\frac{1}{m} - \frac{1}{\alpha}}$, we obtain
\[
\int_U |u|^{\alpha + r} dx \leq \varepsilon \int_U |u|^{n - s} |\nabla u|^p dx + \varepsilon^{-\frac{\theta}{1 - \theta}} 2^{\frac{\theta_0(\alpha + r)}{m}} (c_3 m)^{\frac{\theta_0(\alpha + r)}{m}} \|u\|_{L^m} \left[ \left(1 - \theta_0\right)\|u\|_{L^{\alpha + r}} \right]^{\frac{1}{1 - \theta}}
\]
\[
+ 2^{\theta_0(\alpha + r)} c_4^{\frac{\theta_0(\alpha + r)}{m}} \left[ \|u\|_{L^m} \right]^{\frac{1}{m} - \frac{1}{\alpha}} \|u\|_{L^{\alpha + r}}.
\]  
(2.21)
Re-calculations of the powers:
\[ \theta_0(\alpha + r) = \theta(\alpha - s + p), \quad \theta_0(\alpha + r)/m = \theta p, \]
\[ (1 - \theta_0)(\alpha + r) \frac{1}{1 - \theta} = \left(1 - \theta \frac{\alpha - s + p}{\alpha + r} \right) \frac{\alpha + r}{1 - \theta} = \frac{(1 - \theta)\alpha + r + \theta(s - p)}{1 - \theta} = \alpha + \frac{r + \theta(s - p)}{1 - \theta} = \alpha + \mu_1, \]
\[ \theta_0(\alpha + r) \left( \frac{1}{m} - \frac{1}{\alpha} \right) = \theta(\alpha - s + p) \left( \frac{p}{\alpha - s + p} - \frac{1}{\alpha} \right) = \frac{\theta(\alpha(p - 1) + s - p)}{\alpha}. \]
Thus, inequality \( (2.7) \) follows \( (2.21) \).

(iii) Define
\[ r = \frac{s - p}{p - 1}. \]

Given \( \varepsilon > 0 \). First, we apply inequality \( (2.6) \), and then estimate the last integral \( \int_U |u|^{\alpha + r} \, dx \) in \( (2.6) \) by using \( (2.7) \) with the parameter \( \varepsilon \) in \( (2.7) \) being set as
\[ \varepsilon(c_2\alpha)^{-\frac{\mu}{p-1} \varepsilon^{-\frac{1}{p-1}}(c_2\alpha)^{\frac{\mu}{p-1}} = (\varepsilon^{-1}c_2\alpha)^{-\frac{\mu}{p-1}}}. \]
This results in
\[ \int_G u^\alpha \, d\sigma \leq 2\varepsilon \int_U u^{\alpha - s} |\nabla u|^p \, dx + c_1 \int_G u^\alpha \, dx + (c_2\alpha)^{-\frac{\mu}{p-1} \varepsilon^{-\frac{1}{p-1}}(c_2\alpha)^{\frac{\mu}{p-1}} \left( c_3m \right)^{\frac{\mu}{p-1}} \|u\|_{L^\alpha}^{\alpha + \mu_1} \right) + \varepsilon \left( c_2\alpha \right)^{-\frac{\mu}{p-1} \varepsilon^{-\frac{1}{p-1}}(c_2\alpha)^{\frac{\mu}{p-1}} 2^{\theta(p+1+s-p)} c_4^\frac{\mu}{p-1} U \left( \frac{\theta(\alpha(p - 1) + s - p)}{\alpha} \right) \|u\|_{L^\alpha}^{\alpha + r}, \]
where \( \theta \) is defined in \( (2.5) \), which is the same as in \( (2.10) \) since \( r \) is now specified by \( (2.22) \). Therefore \( (2.9) \) follows. Finally, we check the conditions on the exponents in (i) and (ii) to validate our calculations. Since \( \alpha > s > p > 1 \), we have \( r > 0, (p - s)/(p - 1) < 0, \) and only need to check \( \alpha > \frac{n(r + s - p)}{p} \). With \( r \) defined by \( (2.22) \), this, in fact, is \( \alpha > n(s - p)/(p - 1) \), which is already one of the assumptions on \( \alpha \). The proof is complete.

In our particular case, we have following lemma. Define
\[ \delta = 1 - \lambda \in [0,1), \quad \alpha_* = n(a - \delta)/(2 - a) \quad \text{and} \quad \mu_0 = \frac{a - \delta}{1 - \delta}. \]

**Lemma 2.2.** Assume \( a > \delta, \alpha > 2 - \delta \) and \( \alpha > n\mu_0 \). Let \( c_* = \max \{ c_1, c_2, c_3, c_4 \} \) with \( c_1, c_2, c_3, c_4 \) in Lemma 2.1 and
\[ \theta = \theta_\alpha \overset{\text{def}}{=} \frac{1}{(1 - a)(\alpha/\alpha_* - 1)} \in (0,1). \]

Then one has for any \( \varepsilon > 0 \) that
\[ \int_G |u|^\alpha \, d\sigma \leq 2\varepsilon \int_U |u|^\alpha \, d\sigma \leq 2\varepsilon \int_U \left| |u|^\alpha |\nabla u|^2 - a \, dx + c_* \right| u \|^\alpha \|_{U}^p + D_{4,\alpha} \varepsilon^{-\frac{1}{\alpha}} \|u\|_{L^\alpha}^{\alpha + \mu_1}, \]
\[ + D_{3,\alpha} \varepsilon^{-\frac{1}{\alpha}} \|u\|_{L^\alpha}^{\alpha + \mu_0} + D_{4,\alpha} \varepsilon^{-\frac{1}{\alpha}} \|u\|_{L^\alpha}^{\alpha + \mu_1}. \]
where
\begin{align*}
\mu_1 &= \mu_{1,\alpha} \overset{\text{def}}{=} \frac{\mu_0 (1 + \theta (1 - a))}{1 - \theta}, \quad \mu_2 = \mu_{2,\alpha} \overset{\text{def}}{=} \frac{1}{1 - \theta} + \frac{\theta (2 - a)}{(1 - \theta)(1 - a)}, \\
D_{3,\alpha} &= 2^\theta (\alpha + \delta - a) \frac{(2 - a)(1 + \theta (1 - a))}{\alpha^{1 - \theta}} \frac{\alpha^{1 - \theta}}{(1 - a)(1 - \theta)}, \\
D_{4,\alpha} &= 2 \frac{\theta (\alpha + \delta - a)}{1 - \theta} (c_\alpha \alpha) \frac{(2 - a)(1 + \theta (1 - a))}{(1 - a)(1 - \theta)} .
\end{align*}

Proof. We apply inequality (2.9) in Lemma 2.1 to \( p = 2 - a \) and \( s = 2 - \delta \). We recalculate exponents for these particular values. Note, \( \alpha - s + p = \alpha + \delta - a \). Then from (2.8) and (2.22),
\begin{align*}
m = \frac{\alpha + \delta - a}{2 - a}, \quad r = \frac{s - p}{p - 1} = \frac{a - \delta}{1 - a} = \mu_0 .
\end{align*}

Also, \( \theta \) in (2.11) becomes (2.24), the number \( \mu_1 \) in (2.8) is the same as in (2.26). The exponent of \( |U| \) is \( \theta (\alpha (1 - a) + a - \delta) / \alpha = \theta (\alpha (1 - a) + (1 - a) \mu_0) / \alpha = (1 - a) \theta (\alpha + \mu_0) / \alpha \). The exponent of \( \varepsilon^{-1} \) in the last term of (2.9) is \( \mu_2 \). Also using the fact \( c_1, c_2, c_3, c_4 \leq c_* \) and \( m \leq \alpha \), we have \( D_{1,\alpha} \leq D_{3,\alpha} \) and \( D_{2,\alpha} \leq D_{4,\alpha} \). Therefore, we obtain (2.25) from (2.9).

Next is a parabolic multiplicative Sobolev inequality.

**Lemma 2.3.** Assume
\begin{align*}
\alpha \geq 2 - \delta \quad \text{and} \quad \alpha > \alpha_* .
\end{align*}

If \( T > 0 \), then
\begin{align*}
\left( \int_0^T \int_U |u|^\kappa \alpha \, dx \, dt \right)^{\frac{1}{\kappa \alpha}} &\leq (c_5 \alpha^{2 - a}) \frac{1}{\kappa \alpha} \left( \int_0^T \int_U |u|^\alpha x^{\delta - a} \, dx \, dt + \int_0^T \int_U |u|^\alpha x^{\delta - 2} |\nabla u|^2 \, dx \, dt \right)^{\frac{\delta}{\alpha + \delta - a}} \\
& \cdot \sup_{t \in [0,T]} \left( \int_U |u(x, t)|^\alpha \, dx \right)^{\frac{1 - \delta}{\alpha}},
\end{align*}
where \( c_5 \geq 1 \) is independent of \( \alpha \) and \( T \), and
\begin{align*}
\bar{\theta} = \bar{\theta}_\alpha \overset{\text{def}}{=} \frac{1}{1 + \frac{\alpha (2 - a)}{n (\alpha + \delta - a)}}, \quad \kappa = \kappa (\alpha) \overset{\text{def}}{=} 1 + \frac{2 - a}{n} - \frac{a - \delta}{\alpha} = 1 + (a - \delta) \left( \frac{1}{\alpha_*} - \frac{1}{\alpha} \right) .
\end{align*}

In case \( U = B_R \) - a ball of radius \( R \) - one has
\begin{align*}
\left( \int_0^T \int_{B_R} |u|^\kappa \alpha \, dx \, dt \right)^{\frac{1}{\kappa \alpha}} &\leq [c_6 (1 + R^{-1})^2 a^{2 - a}] \frac{1}{\kappa \alpha} \\
& \cdot \left[ \int_0^T \int_{B_R} |u|^\alpha x^{\delta - a} \, dx \, dt + \int_0^T \int_{B_R} |u|^\alpha x^{\delta - 2} |\nabla u|^2 \, dx \, dt \right]^{\frac{\delta}{\alpha + \delta - a}} \\
& \cdot \sup_{t \in [0,T]} \left( \int_{B_R} |u(x, t)|^\alpha \, dx \right)^{\frac{1 - \delta}{\alpha}},
\end{align*}
where \( c_6 \geq 1 \) independent of \( \alpha, R \) and \( T \).
Proof. Note that by definition of \( \kappa \) and \( \tilde{\theta} \) we have \( \frac{1}{\kappa \alpha} = \frac{\tilde{\theta}}{\alpha} + \frac{1-\tilde{\theta}}{\alpha} \), where \( p_0 = \frac{n(\alpha+\delta-a)}{\alpha-2-\alpha} \). Then interpolation inequality gives

\[
\left( \int_U |u|^{\kappa \alpha} \, dx \right)^{\frac{1}{\kappa \alpha}} \leq \left( \int_U |u|^{p_0} \, dx \right)^{\frac{\tilde{\theta}}{p_0}} \cdot \left( \int_U |u|^{\alpha} \, dx \right)^{\frac{1-\tilde{\theta}}{\alpha}}. \tag{2.33}
\]

We estimate the first integral on the right-hand side of (2.33). We recall the following Sobolev’s inequality

\[
\|w\|_{L^{(2-a)^*}} \leq c_7 \left( \int_U |w|^{2-a} \, dx + \int_U |\nabla w|^{2-a} \, dx \right)^{\frac{1}{2-a}},
\]

where \((2-a)^*\) is the Sobolev conjugate of \(2-a\), and \(c_7 \geq 1\) is independent of \(\alpha\). Applying this inequality to \(w = |u|^m\) with \(m = \frac{\alpha+\delta-a}{2-\alpha} \geq 1\), we obtain

\[
\left( \int_U |u|^{\frac{n(\alpha+\delta-a)}{\alpha-2-\alpha}} \, dx \right)^{\frac{1}{2-a}} \leq c_7 \left( \int_U |u|^{\alpha+\delta-a} \, dx + m^2-a \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \, dx \right)^{\frac{1}{2-a}}. \tag{2.34}
\]

Note that \(\kappa \alpha \tilde{\theta} = \alpha + \delta - a\) and \((2-a)^*/p_0 = (2-a)/(\alpha + \delta - a)\). Then raising both sides of inequality (2.34) by \((2-a)^*/\kappa \alpha \tilde{\theta}/p_0 = 2-a\) results in

\[
\left( \int_U |u|^{\frac{n(\alpha+\delta-a)}{\alpha-2-\alpha}} \, dx \right)^{\frac{2-a}{\alpha}} \leq c_7^{2-a} \left( \int_U |u|^{\alpha+\delta-a} \, dx + m^2-a \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \, dx \right)^{\frac{1}{2-a}}. \tag{2.35}
\]

Raising both sides of (2.33) to the power \(\kappa \alpha\), and using inequality (2.35), we get

\[
\int_U |u|^{\kappa \alpha} \, dx \leq (c_7m)^{2-a} \left( \int_U |u|^{\alpha+\delta-a} \, dx + m^2-a \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \, dx \right) \left( \int_U |u|^{\alpha} \, dx \right)^{(1-\tilde{\theta})\kappa}.
\]

Integrating this inequality in \(t\) from 0 to \(T\) and taking the supremum of the last integral for \(t \in [0,T]\) yield

\[
\int_0^T \int_U |u|^{\kappa \alpha} \, dx \, dt \leq (c_7m)^{2-a} \left( \int_0^T \int_U |u|^{\alpha+\delta-a} \, dx \, dt + \int_0^T \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \, dx \, dt \right)
\]

\[
\cdot \sup_{t \in [0,T]} \left( \int_U |u(x,t)|^{\alpha} \, dx \right)^{(1-\tilde{\theta})\kappa}.
\]

Taking both sides to the power \(\frac{1}{\kappa \alpha} = \frac{\tilde{\theta}}{\alpha+\delta-a}\), we have

\[
\left( \int_0^T \int_U |u|^{\kappa \alpha} \, dx \, dt \right)^{\frac{1}{\kappa \alpha}} \leq (c_7m)^{\frac{2-a}{\kappa \alpha}} \left( \int_0^T \int_U |u|^{\alpha+\delta-a} \, dx \, dt + \int_0^T \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \, dx \, dt \right)^{\frac{\tilde{\theta}}{\alpha+\delta-a}}
\]

\[
\cdot \sup_{t \in [0,T]} \left( \int_U |u(x,t)|^{\alpha} \, dx \right)^{\frac{1-\tilde{\theta}}{\alpha}}.
\]

Note that \(m < \alpha\), then we obtain (2.30) with \(c_5 = c_7^{2-a}\).

Consider the case \(U = B_R\). The Sobolev inequality for \(B_R\) is

\[
\|w\|_{L^{(2-a)^*}(B_R)} \leq c_8 R^{-1} \left( \int_{B_R} |w|^{2-a} \, dx \right)^{\frac{1}{2-a}} + c_8 \left( \int_{B_R} |\nabla w|^{2-a} \, dx \right)^{\frac{1}{2-a}}, \tag{2.36}
\]

where \(c_8 \geq 1\) is independent of \(R\). Repeating the above proof with \(c_7\) replaced by \(c_8(1+R^{-1})\), we obtain (2.30) with \(c_5\) replaced by \([c_8(1+R^{-1})]^{2-a}\). Therefore, we obtain (2.32) with \(c_6 = c_8^{2-a}\). \(\square\)

It is noteworthy that the explicit exponents and constants in Lemmas 2.2 and 2.3 play an important role in Moser’s iterations in sections 5 and 6.
3 $L^\alpha$-estimates

We start studying problem (1.26). Hereafter, $u(x,t)$ denotes a non-negative solution of (1.26).

Regarding the nonlinearity of the PDE in (1.26), we recall that the function $K(\xi)$ has
the following properties: it is decreasing in $\xi$, maps $[0,\infty)$ onto $(0,\frac{1}{\mu_0}]$, and

\[
\frac{d_1}{(1+\xi)\alpha} \leq K(\xi) \leq \frac{d_2}{(1+\xi)\alpha},
\]

\[
d_3(\xi^{2-\alpha} - 1) \leq K(\xi)\xi^{2-\alpha} \leq d_2\xi^{2-\alpha},
\]

where $d_1, d_2, d_3$ are positive constants depending on $g(s)$. (See [2] for the proof.)

**Notation.** The symbol $C$ denotes a generic positive constant with varying values in different places, while $C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ have their values fixed. The constants $C, C', C_i, C_j, \text{ for } i, j = 1, 2, 3, \ldots$, used in calculations can, otherwise stated, implicitly depend on number $\lambda$, function $g$, the space dimension $n$, and $U$. We also use the positive and negative parts notation $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, and brief notation $\|\varphi(t)\|_{L^\infty(\Gamma)}$ for $\|\varphi(\cdot, t)\|_{L^\infty(\Gamma)}$.

Recall $\delta, \alpha$, and $\mu_0$ are defined in (2.23). We assume hereafter that

\[
a > \delta, \text{ i.e., the number } \mu_0 \text{ is positive}.\]

The case $a \leq \delta$ is much simpler, see Remark [3,2] below. We begin with a differential inequality for $\int_U u^\alpha dx$.

**Lemma 3.1.** Assume $\alpha \geq 2 - \delta$ and $\alpha > n\mu_0$. Then

\[
\frac{d}{dt}\int_U u^\alpha(x,t)dx + C_1\int_U |\nabla u(x,t)|^{2-\alpha}u^{\alpha+\delta-2}(x,t)dx \leq C\|u(t)\|_{L^\alpha(U)}^{\alpha+\delta-2} + C\|\varphi^-(t)\|_{L^\infty(\Gamma)}\|u(t)\|_{L^\alpha(U)}^\alpha + C\|\varphi^-(t)\|_{L^\infty(\Gamma)}^{2-\alpha}\|u(t)\|_{L^\alpha(U)}^{\alpha+\mu_0},
\]

(3.4)

with $\mu_1$ and $\theta$ in Lemma [2,2], where the positive constants $C_1 = C_{1,\alpha}$ and $C > 0$ depend on $\alpha$. Consequently, there is $C_2 = C_{2,\alpha} > 0$ such that

\[
\frac{d}{dt}\int_U u^\alpha dx + C_1\int_U |\nabla u|^{2-\alpha}u^{\alpha+\delta-2}dx \leq C_2\left(1 + \|\varphi^-(t)\|_{L^\infty(\Gamma)}^{2-\alpha}\right)\left(1 + \int_U u^\alpha dx\right)^{1+\mu_1/\alpha}.
\]

(3.5)

**Proof.** Multiplying both sides of the first equation in (1.26) by $u^{\alpha+\delta-1}$, integrating over domain $U$, and using integration by parts, we have

\[
\frac{\lambda}{\alpha} d\int_U u^\alpha dx = \int_U \nabla \cdot (K(|\nabla u|)|\nabla u|)u^{\alpha+\delta-1} dx
\]

\[
= - (\alpha - \lambda) \int_U K(|\nabla u|)|\nabla u|^2u^{\alpha+\delta-2} dx + \int_\Gamma K(|\nabla u|)\frac{\partial u}{\partial n}u^{\alpha+\delta-1} d\sigma
\]

\[
= - (\alpha - \lambda) \int_U K(|\nabla u|)|\nabla u|^2u^{\alpha+\delta-2} dx - \int_\Gamma u^\alpha \varphi d\sigma.
\]

(3.6)

Using relation $K(\xi)\xi^2 \geq d_3(\xi^{2-\alpha} - 1)$ in (3.2), one has

\[
- \int_U K(|\nabla u|)|\nabla u|^2u^{\alpha+\delta-2} dx \leq -d_3\int_U |\nabla u|^{2-\alpha}u^{\alpha+\delta-2} dx + d_3\int_U u^{\alpha+\delta-2} dx
\]

\[
\leq -d_3\int_U |\nabla u|^{2-\alpha}u^{\alpha+\delta-2} dx + C\|u\|_{L^\alpha(U)}^{\alpha+\delta-2}.
\]

(3.7)
Estimate the last integral of (3.6) by using the trace theorem in Lemma 2.2 we have

\[-\int_{\Gamma} u^\alpha \varphi d\sigma \leq \int_{\Gamma} u^\alpha \varphi^- d\sigma \leq \|\varphi^-\|_{L^\infty(\Gamma)} \int_{\Gamma} u^\alpha d\sigma\]

\[\leq \|\varphi^-\|_{L^\infty(\Gamma)} \left\{ 2\varepsilon \int_{U} |u|^{\alpha+\delta-2} |\nabla u|^{2-a} dx + C\|u\|_{L^\infty(U)}^\alpha + C\varepsilon^{-\frac{1}{1-a}} \|u\|_{L^\infty(U)}^{\alpha+\mu_0} + C\varepsilon^{-\mu_2} \|u\|_{L^\infty(U)}^{\alpha+\mu_1} \right\}. \tag{3.8}\]

Combining (3.6), (3.7) and (3.8), we have

\[\frac{\lambda}{\alpha} \frac{d}{dt} \int_{U} u^\alpha dx \leq -(\lambda - \frac{1}{2})d_3 \int_{U} |\nabla u|^{2-a} u^{\alpha+\delta-2} dx + C\|u\|_{L^\infty(U)}^{\alpha+\delta-2} + C\|\varphi^-\|_{L^\infty(\Gamma)}
\[\cdot \left\{ 2\varepsilon \int_{U} |u|^{\alpha+\delta-2} |\nabla u|^{2-a} dx + C\|u\|_{L^\infty(U)}^{\alpha} + C\varepsilon^{-\frac{1}{1-a}} \|u\|_{L^\infty(U)}^{\alpha+\mu_0} + C\varepsilon^{-\mu_2} \|u\|_{L^\infty(U)}^{\alpha+\mu_1} \right\}. \tag{3.9}\]

The case \(\|\varphi^-\|_{L^\infty(\Gamma)} = 0\), inequality (3.4) immediately follows (3.9). Consider \(\|\varphi^-\|_{L^\infty(\Gamma)} \neq 0\). Select \(\varepsilon = \frac{d_3}{4\|\varphi^-\|_{L^\infty(\Gamma)}} \) in (3.9). Then we obtain

\[\frac{\lambda}{\alpha} \frac{d}{dt} \int_{U} u^\alpha dx \leq -(\lambda - \frac{1}{2})d_3 \int_{U} |\nabla u|^{2-a} u^{\alpha+\delta-2} dx + C\|u\|_{L^\infty(U)}^{\alpha+\delta-2} + C\|\varphi^-\|_{L^\infty(\Gamma)}\|u\|_{L^\infty(U)}^{\alpha}
\[+ C\|\varphi^-\|_{L^\infty(\Gamma)}^{2-a} \|u\|_{L^\infty(U)}^{\alpha+\mu_0} + C\|\varphi^-\|_{L^\infty(\Gamma)}^{\mu_2+1} \|u\|_{L^\infty(U)}^{\alpha+\mu_1}. \tag{3.10}\]

Note that \(\mu_2 + 1 = \frac{2-a}{(1-a)(1-\beta)}\). Multiplying both sides of (3.10) by \(\alpha/\lambda\), we get inequality (3.4).

Now, we prove (3.5). Denote \(\beta = 1 + \mu_1/\alpha\). Note that, on the right-hand side of (3.4), the maximum power of \(\|u\|_{L^\alpha}\) is \(\alpha + \mu_1 = \alpha\beta\), and the maximum power of \(\|\varphi^-\|_{L^\infty(\Gamma)}\) is \(\frac{2-a}{(1-a)(1-\beta)}\). By applying inequality (2.5),

\[\|u\|_{L^\infty(U)}^{\alpha+\delta-2}, \|u\|_{L^\infty(U)}^{\alpha}, \|u\|_{L^\infty(U)}^{\alpha+\mu_0} \leq 1 + \|u\|_{L^\alpha}^{\alpha\beta}, \]

\[\|\varphi^-\|_{L^\infty(\Gamma)}, \|\varphi^-\|_{L^\infty(\Gamma)}^{2-a} \leq 1 + \|\varphi^-\|_{L^\infty(\Gamma)}^{\frac{2-a}{(1-a)(1-\beta)}}. \]

Therefore, it follows from (3.10) that

\[\frac{d}{dt} \int_{U} u^\alpha dx + C_1 \int_{U} |\nabla u|^{2-a} u^{\alpha+\delta-2} dx \leq C(1 + \|\varphi^-\|_{L^\infty(\Gamma)}^{\frac{2-a}{(1-a)(1-\beta)}})(1 + \|u\|_{L^\alpha}^{\alpha\beta}) \]

\[\leq C(1 + \|\varphi^-\|_{L^\infty(\Gamma)}^{\frac{2-a}{(1-a)(1-\beta)}})(1 + \|u\|_{L^\alpha})^{\beta}. \]

Thus, inequality (3.5) follows.

**Remark 3.2.** In case \(a \leq \delta\), by combining (3.6) and the simple trace inequality (2.6), we can find

\[\frac{\lambda}{\alpha} \frac{d}{dt} \int_{U} u^\alpha dx = \int_{U} \nabla \cdot (K(|\nabla u|)\nabla u) u^{\alpha+\delta-1} dx \]

\[= -C_1 \int_{U} K(|\nabla u|)|\nabla u|^2 u^{\alpha+\delta-2} dx + C\varphi\left( \int_{U} |u|^\alpha dx + \int_{U} |u|^{\alpha+\mu_0} dx \right), \tag{3.11}\]

where \(C\varphi > 0\) depends on the function \(\varphi\). Since \(\mu_0 \leq 0\) in this case, we can apply Hölder’s inequality and easily derive a differential inequality for \(\int_{U} u^\alpha dx\), and consequently obtain its estimates. Hence, there is no need to use more involved trace inequality for \(\int_{U} u^\alpha dx\), which is essential in the proof of Lemma 2.7 for the case \(a > \delta\). Therefore, we refer to \(a > \delta\) as the super-critical case, and \(a < \delta\) as the sub-critical case. Note that papers \([23, 30]\) fall into the latter case.
We now have local (in time) estimates for solutions.

**Theorem 3.3.** Assume \( \alpha \geq 2 - \delta \) and \( \alpha > n\mu_0 \).

(i) If \( T > 0 \) satisfies
\[
\frac{1}{C_3} \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\frac{\mu_1}{\alpha}} \leq \int_0^T (1 + \| \varphi^- (t) \|_{L^{\infty}(\Gamma)}) dt < \frac{1}{C_3} \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\frac{\mu_1}{\alpha}},
\]  
where \( C_3 = C_{3,\alpha} \) defined by \( \int_0^T (1 + \| \varphi^- (t) \|_{L^{\infty}(\Gamma)}) dt > 0 \), then for all \( t \in (0, T] \):
\[
\int_U u^\alpha(x, t)dx \leq \left\{ \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\frac{\mu_1}{\alpha}} - C_3 \int_0^t (1 + \| \varphi^- (\tau) \|_{L^{\infty}(\Gamma)}) d\tau \right\}^{-\frac{\alpha}{\mu_1}}.
\]  

(ii) Consequently, if \( T > 0 \) satisfies
\[
\frac{1}{C_3} \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\frac{\mu_1}{\alpha}} \leq \int_0^T (1 + \| \varphi^- (t) \|_{L^{\infty}(\Gamma)}) dt \leq \frac{1}{C_3} \left( 1 + \int_U u_0^\alpha(x)dx \right)^{-\frac{\mu_1}{\alpha}},
\]  
then
\[
\int_U u^\alpha(x, t)dx \leq 2 \left( 1 + \int_U u_0^\alpha(x)dx \right) \text{ for all } t \in (0, T],
\]  
\[
\int_0^T \int_U u^{\alpha+\delta-2}(x, t) |\nabla u(x, t)|^{2-a} dx dt \leq C \left( 1 + \int_U u_0^\alpha(x)dx \right),
\]  
where \( C > 0 \) depends on \( \alpha \).

**Proof.** (i) Let \( V(t) = 1 + \int_U u^\alpha(x, t)dx \). Then inequality (3.5) can be written as
\[
\frac{d}{dt} V(t) \leq C_2(1 + \| \varphi^- \|_{L^{\infty}(\Gamma)}) V(t)^{1+\mu_1/\alpha}.
\]  
Solving this differential inequality gives
\[
V(t) \leq \left\{ V(0)^{-\mu_1/\alpha} - \frac{\mu_1 C_2}{\alpha} \int_0^t (1 + \| \varphi^- (\tau) \|_{L^{\infty}(\Gamma)})^{\frac{2-a}{\delta}(1-\gamma)} d\tau \right\}^{-\alpha/\mu_1},
\]  
for all \( t \in (0, T] \), with \( T > 0 \) satisfying (3.12). Hence we have (3.13).

(ii) When \( T > 0 \) satisfies (3.14), inequality (3.15) easily follows (3.13).

Integrating inequality (3.5) in time and using (3.15), we have
\[
\int_0^T \int_U u^{\alpha+\delta-2}(x, t) |\nabla u(x, t)|^{2-a} dx dt
\leq C \int_U u_0^\alpha(x)dx + C \int_0^T \left( 1 + \int_U u^\alpha(x, t)dx \right)^{1+\mu_1/\alpha} \left( 1 + \| \varphi^- (t) \|_{L^{\infty}(\Gamma)}) dt.
\]  
\[
\leq C \int_U u_0^\alpha(x)dx + C \left( 1 + \int_U u_0^\alpha(x)dx \right)^{1+\mu_1/\alpha} \int_0^T \left( 1 + \| \varphi^- (t) \|_{L^{\infty}(\Gamma)}) dt.
\]  
Combining this with the bound (3.14) for the last integral, we obtain (3.16). \( \square \)
4 Gradient estimates

In this section, we estimate \( \int_U |\nabla u|^{2-a}(x,t)\,dx \) for \( t > 0 \). Same as in \([2]\), we will use the following function

\[
H(\xi) = \int_0^{\xi^2} K(\sqrt{s})\,ds \quad \text{for} \quad \xi \geq 0.
\]  

(4.1)

The function \( H(\xi) \) can be compared with \( \xi \) and \( K(\xi) \) by

\[
K(\xi)\xi^2 \leq H(\xi) \leq 2K(\xi)\xi^2,
\]  

(4.2)

and hence, as a consequence of (3.2) and (4.2), we have

\[
d_3(\xi^{2-a} - 1) \leq H(\xi) \leq 2d_2\xi^{2-a}.
\]  

(4.3)

\[E.\text{ Celik}, L.\text{ Hoang}, \text{ and T. Kieu}\]

\[\text{Proposition 4.1. Assume}\]

\[
\alpha > \max\{n\mu_0, \lambda + 1 + \mu_0\}.
\]  

(4.4)

If \( t > 0 \) then

\[
\int_0^t \int_U u^{1-\lambda}[(u^\lambda)_t]^2\,dx\,d\tau + \int_U |\nabla u(x,t)|^{2-a}\,dx
\leq C\mathcal{E}_0 + CK(t) + C \int_0^t \left(1 + \|\varphi^{-\lambda}(\tau)\|_{L^\infty(\Gamma)}^{2-a}\right)(1 + \|u(\tau)\|_{L^{\alpha}_\lambda(U)})\,d\tau,
\]  

(4.5)

where \( C > 0 \) depends on \( \alpha \),

\[
\mathcal{E}_0 = \int_U u_0^\alpha(x)\,dx + \int_U |\nabla u_0(x)|^{2-a}\,dx + \int_\Gamma u^{\lambda+1}_0(x)\varphi^+(x,0)\,d\sigma,
\]  

(4.6)

\[
K(t) = 1 + \|\varphi^{-\lambda}(t)\|_{L^\infty(\Gamma)}^{\mu_3} + \int_0^t \int_\Gamma |\varphi^{-\lambda}(x,\tau)|^\alpha\,d\sigma\,d\tau,
\]  

(4.7)

with

\[
\mu_3 = \mu_{3,\alpha} = \frac{(2-a)\alpha}{(1-a)(\alpha - (\lambda + 1 + \mu_0))}.
\]

Proof. Multiplying both sides of the PDE in (1.26) by \( u = \frac{1}{\lambda}(u^\lambda)_t u^{1-\lambda} \), integrating over \( U \) and using the boundary condition, we obtain

\[
\frac{1}{\lambda} \int_U u_t(u^\lambda)_t\,dx + \frac{d}{dt} \int_U H(|\nabla u(x,t)|)\,dx = -\int_\Gamma \varphi u^\lambda u_t\,d\sigma
\]

\[
= -\frac{1}{\lambda + 1} \frac{d}{dt} \int_\Gamma u^{\lambda+1}\varphi\,d\sigma + \frac{1}{\lambda + 1} \int_\Gamma u^{\lambda+1}\varphi_t\,d\sigma.
\]  

(4.8)

Multiplying the equation by \( \lambda + 1 \), and applying Young’s inequality to the last integral yield

\[
\frac{\lambda + 1}{\lambda} \int_U u^{1-\lambda}[(u^\lambda)_t]^2\,dx + \frac{d}{dt} \left(\frac{\lambda + 1}{2} \int_U H(|\nabla u(x,t)|)\,dx + \int_\Gamma u^{\lambda+1}\varphi\,d\sigma\right)
\leq \int_\Gamma u^{\lambda+1}\varphi_t\,d\sigma + \int_\Gamma |\varphi_t|^{\alpha-\lambda-1}\,d\sigma.
\]  

(4.9)
To estimate the second to last boundary integral, we use the trace inequality (2.25) in Lemma 2.2:
\[
\int_\Gamma |u|^\alpha d\sigma \leq 2\varepsilon \int_U |u|^{\alpha + \delta - 2} |\nabla u|^{2-a} dx + C \|u\|_{L^\alpha(U)}^\alpha + C \varepsilon^{-\frac{1}{1-a}} \|u\|_{L^\alpha(U)}^{\alpha + \mu_0} + C \varepsilon^{-\mu_2} \|u\|_{L^\alpha(U)}^{\alpha + \mu_1},
\]
for any \(\varepsilon > 0\). Using this inequality in (4.10), we have
\[
\int_\Gamma |u|^\alpha d\sigma \leq 2\varepsilon \int_U |u|^{\alpha + \delta - 2} |\nabla u|^{2-a} dx + C \|u\|_{L^\alpha(U)}^\alpha + C \varepsilon^{-\frac{1}{1-a}} \|u\|_{L^\alpha(U)}^{\alpha + \mu_0} + C \varepsilon^{-\mu_2} \|u\|_{L^\alpha(U)}^{\alpha + \mu_1} + \int_\Gamma |\varphi_t|^{\alpha - \frac{\alpha}{\lambda - 1}} d\sigma.
\]
Define
\[
\mathcal{E}(t) = \frac{\lambda + 1}{2} \int_U H(|\nabla u(x,t)|) dx + \int_\Gamma u^{\lambda+1} \varphi d\sigma + \int_U u^\alpha dx.
\]
Adding (4.10) to (4.11) yields
\[
\frac{\lambda + 1}{\lambda} \int_U u^{1-\lambda}(u^\lambda)^2 dx + \frac{d}{dt} \mathcal{E}(t) + (C_1 - 2\varepsilon) \int_U |u|^{\alpha + \delta - 2} |\nabla u|^{2-a} dx
\]
\[
\leq C(1 + \|\varphi^+\|_{L^\infty(\Gamma)}) \|u\|_{L^\alpha(U)}^\alpha + C \|u\|_{L^\alpha(U)}^{\alpha + \mu_1} + C \varepsilon^{-\frac{2-a}{1-a}} \|u\|_{L^\alpha(U)}^{\alpha + \mu_0} + C \varepsilon^{-\mu_2} \|u\|_{L^\alpha(U)}^{\alpha + \mu_1} + \int_\Gamma |\varphi_t|^{\alpha - \frac{\alpha}{\lambda - 1}} d\sigma.
\]
Choosing \(\varepsilon \) sufficiently small such that \(C_1 - 2\varepsilon > 0\), we derive
\[
\frac{\lambda + 1}{\lambda} \int_U u^{1-\lambda}(u^\lambda)^2 dx + \frac{d}{dt} \mathcal{E}(t) \leq C(1 + \|\varphi^+\|_{L^\infty(\Gamma)}) \|u\|_{L^\alpha(U)}^\alpha + C \|u\|_{L^\alpha(U)}^{\alpha + \mu_1}
\]
\[
+ C(1 + \|\varphi^-\|_{L^\infty(\Gamma)}) \|u\|_{L^\alpha(U)}^{\alpha + \mu_0} + C \varepsilon^{-\frac{2-a}{1-a}} \|u\|_{L^\alpha(U)}^{\alpha + \mu_1} + C \varepsilon^{-\mu_2} \|u\|_{L^\alpha(U)}^{\alpha + \mu_1} + \int_\Gamma |\varphi_t|^{\alpha - \frac{\alpha}{\lambda - 1}} d\sigma.
\]
Choosing \(\alpha + \delta - 2 < \alpha < \alpha + \mu_0 < \alpha + \mu_1\) and \(1 < \frac{2-a}{1-a} < \frac{2-a}{(1-a)(1-\theta)}\). We apply Young’s inequality for each norm on the right-hand side and obtain
\[
\frac{\lambda + 1}{\lambda} \int_U u^{1-\lambda}(u^\lambda)^2 dx + \frac{d}{dt} \mathcal{E}(t) \leq C \int_\Gamma |\varphi_t|^{\alpha - \frac{\alpha}{\lambda - 1}} d\sigma + C(1 + \|\varphi^-\|_{L^\infty(\Gamma)^{1-\theta}}^{\frac{2-a}{1-\theta}})(1 + \|u\|_{L^\alpha(U)}^{\alpha + \mu_1}).
\]
Let \(t \in (0, T)\). Integrating both sides of previous inequality in \(t\), we obtain
\[
\frac{\lambda + 1}{\lambda} \int_0^t \int_U u^{1-\lambda}(u^\lambda)^2 dx + \frac{\lambda + 1}{\lambda} \int_U H(|\nabla u(x,t)|) dx + \int_U u^\alpha dx
\]
\[
\leq \mathcal{E}(0) - \int_\Gamma u^{\lambda+1} \varphi d\sigma + C \int_0^t \int_\Gamma |\varphi_t|^{\alpha - \frac{\alpha}{\lambda - 1}} d\sigma d\tau + C \int_0^t (1 + \|\varphi^-\|_{L^\infty(\Gamma)^{1-\theta}})(1 + \|u\|_{L^\alpha(U)}^{\alpha + \mu_1}) d\tau.
\]
For the first integral on the right-hand side, applying inequality (2.6) in Lemma 2.1 with \(\alpha = s = \lambda + 1 = 2 - \delta\) and \(p = 2 - a\), we have
\[
- \int_\Gamma u^{\lambda+1} \varphi d\sigma \leq \|\varphi^-\|_{L^\infty} \int_\Gamma u^{\lambda+1} d\sigma
\]
\[
\leq \|\varphi^-\|_{L^\infty} \left\{ \varepsilon \int_U |\nabla u|^{2-a} dx + C \int_U |u|^{\lambda+1} dx + C \varepsilon^{-\frac{1}{1-a}} \int_U |u|^{\lambda+1+\mu_0} dx \right\},
\]
for any $\varepsilon > 0$. Now, using $H(|\nabla u|) \geq C(|\nabla u|^{2-a} - 1)$ from (4.3) and applying Young’s inequality to the last two integrals with $\alpha > \lambda + 1 + \mu_0$ we obtain

$$- \int_{\Gamma} u^{\lambda+1} \varphi d\sigma \leq C_8 \varepsilon \|\varphi\|_{L^{\infty}(\Gamma)} \int_{U} (H(|\nabla u(x,t)|) + 1) dx$$

$$+ \left(\frac{1}{4}\right) \int_{U} u^\alpha dx + C \|\varphi\|_{L^{\infty}(\Gamma)} + \left(\frac{1}{4}\right) \int_{U} u^\alpha dx + C \{ \varepsilon \|\varphi\|_{L^{\infty}(\Gamma)} \} \frac{\alpha - \lambda - 1 - \mu_0}{\alpha} \right).$$

Selecting $\varepsilon = \frac{\lambda + 1}{4C_8 (1+\|\varphi\|_{L^{\infty}(\Gamma)})}$, we obtain

$$\frac{\lambda + 1}{\lambda} \int_0^t \int_{U} u^{1-\lambda} (u^\lambda)^2 dx d\tau + \frac{\lambda + 1}{\lambda} \int_0^t H(|\nabla u(x,t)|) dx + \frac{1}{2} \int_{U} u^\alpha dx$$

$$\leq \mathcal{E}(0) + C + C \|\varphi\|_{L^{\infty}(\Gamma)} \frac{\alpha}{\lambda - 1} + C \{ (1 + \|\varphi\|_{L^{\infty}(\Gamma)}) \frac{\alpha - \lambda - 1 - \mu_0}{\alpha} \} \int_0^t \int_{\Gamma} |\varphi(t)| \frac{\alpha}{\lambda - 1} d\sigma d\tau + C \int_0^t (1 + \|\varphi\|_{L^{\infty}(\Gamma)}) (1 + \|u\|_{L^\alpha(U)}) d\tau$$

which gives

$$\int_0^t \int_{U} u^{1-\lambda} (u^\lambda)^2 dx d\tau + \int_{U} H(|\nabla u(x,t)|) dx$$

$$\leq C \left\{ \mathcal{E}(0) + \mathcal{K}(t) + \int_0^t (1 + \|\varphi\|_{L^{\infty}(\Gamma)} \frac{2-a}{1-a} (1+\|u\|_{L^\alpha(U)}) d\tau \right\}. \quad (4.13)$$

where

$$\mathcal{K}(t) = 1 + \|\varphi^-(t)\|_1 \frac{2-a}{1-a} \alpha + \|\varphi^-(t)\|_1 \frac{\alpha}{\lambda - 1} + \int_0^t \int_{\Gamma} |\varphi(t)| \frac{\alpha}{\lambda - 1} d\sigma d\tau.$$
By using (3.14) to bound the last integral, we obtain
\[ \int_0^t \int_U u^{-\lambda}(u^\lambda)^2 \, dx \, dt + \int_U |\nabla u|^{2-a} \, dx \leq C \varepsilon_0 + CK(t) + C \left( 1 + \int_U u_0^a(x) \, dx \right). \]
Then (4.15) follows.

5 Interior $L^\infty$-estimates

In this section, we estimate the $L^\infty$-norm of the solution in the interior of the domain by using Moser’s iteration.

**Lemma 5.1.** Assume $\alpha \geq 2 - \delta$. Suppose $B_R$ and $B_\rho$, with $R > \rho > 0$, are two concentric balls in a compact subset of $U$. If $T > T_2 > T_1 \geq 0$ then
\[ \sup_{t \in [T_2, T]} \int_{B_\rho} u^\alpha(x, t) \, dx + \int_{T_2}^T \int_{B_R} |\nabla u|^{2-a} u^{\alpha+\delta-2} \, dx \, dt \leq C_\alpha S, \]
where
\[ S = \int_{T_1}^T \int_{B_R} u^\alpha \, dx \, dt + \left( \int_{T_1}^T \int_{B_R} u^\alpha \, dx \, dt \right)^{\frac{\alpha+\delta-2}{\alpha}}, \]
\[ C_\alpha = C_\alpha(\rho, R, T_1, T_2, T) \overset{\text{def}}{=} c_9 \alpha^2 (1 + |B_R|) \left( 1 + \frac{1}{T_2 - T_1} + \frac{1}{(R - \rho)^{2-a}} \right), \]
with $c_9 \geq 1$ independent of $U$, $\alpha$, $\rho$, $R$, $T_1$, $T_2$, $T$.

**Proof.** Let $\xi(x, t) = \xi_1(|x|) \xi_2(t)$ be the the cut-off function which is 1 on $B_\rho \times (T_2, T)$ and has compact support in $B_R \times (T_1, T)$. More specifically, $\xi_1(|x|), \xi_2(t) \in [0, 1]$ and satisfy
\[ \xi_1(|x|) = \begin{cases} 1 & \text{if } |x| < \rho, \\ 0 & \text{if } |x| > R, \end{cases} \quad \text{and} \quad \xi_2(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq T_1, \\ 1 & \text{if } T_2 < t < T. \end{cases} \]

Also, there is $C > 0$ such that
\[ |\xi_x| \leq \frac{C}{T_2 - T_1} \quad \text{and} \quad |\nabla \xi| \leq \frac{C}{R - \rho}. \]

In the calculations within this proof, notation $C$ denotes a generic constant independent of $\alpha$, $\rho$, $R$, $T_1$, $T_2$, $T$.

Recall that $\delta = 1 - \lambda$. Multiplying the PDE in (1.26) by $u^{\alpha+\delta-1} \xi^2$, integrating over $U$, and using integration by parts, we obtain
\[ \lambda \int_U u^{-1} \xi^2 \frac{\partial u}{\partial t} \, dx = \int_U \nabla \cdot (K(|\nabla u|) \nabla u) u^{\alpha+\delta-1} \xi^2 \, dx \]
\[ = -(\alpha - \lambda) \int_U K(|\nabla u|)|\nabla u|^2 u^{\alpha+\delta-2} \xi^2 \, dx - 2 \int_U K(|\nabla u|) \nabla u \cdot \nabla u^{\alpha+\delta-1} \xi \, dx. \]

Using properties (3.2), resp., (3.1) of function $K(\cdot)$ in the first, resp., second integral on the right-hand side of the last identity, we find
\[ \lambda \int_U u^{-1} \xi^2 \frac{\partial u}{\partial t} \, dx \leq -d_2 (\alpha - \lambda) \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 \, dx + d_1 (\alpha - \lambda) \int_U u^{\alpha+\delta-2} \xi^2 \, dx \]
\[ + 2d_2 \int_U |\nabla u|^{1-a} u^{\alpha+\delta-1} \xi \, dx. \]
Let $\varepsilon > 0$. Applying Young’s inequality to the last integral of (5.5), for conjugate exponents $\frac{2}{2-a}$ and $2-a$, we have

$$
\lambda \int_U \frac{\partial u^\alpha}{\partial t} \xi^2 dx \leq -d_3(\alpha - \lambda) \int_U |\nabla u|^{2-a} u^{\alpha + \delta - 2} \xi^2 dx + d_3(\alpha - \lambda) \int_U u^{\alpha + \delta - 2} \xi^2 dx
$$

$$
+ \varepsilon \int_U |\nabla u|^{2-a} u^{\alpha + \delta - 2} \xi^2 dx + C \varepsilon \int_U |u|^{\alpha - a} \xi \nabla |\xi|^{2-a} dx.
$$

Choosing $\varepsilon = \frac{d_3(\alpha - \lambda)}{2}$, we then have

$$
\frac{\lambda}{\alpha} \int_U u^\alpha \xi^2 dx - \frac{2\lambda}{\alpha} \int_U u^\alpha \xi_t dx \leq -\frac{d_3(\alpha - \lambda)}{2} \int_U |\nabla u|^{2-a} u^{\alpha + \delta - 2} \xi^2 dx
$$

$$
+ C(\alpha - \lambda) \int_U u^{\alpha + \delta - 2} \xi^2 dx + \frac{C}{(\alpha - \lambda)(1-a)} \int_U |u|^{\alpha - a} \xi \nabla |\xi|^{2-a} dx.
$$

Then integrating the inequality in time from 0 to $t$ gives

$$
\frac{T}{\alpha} \int_T T u^\alpha(x,t) \xi^2(x,t) dx + \frac{d_3(\alpha - \lambda)}{2} \int_T T |\nabla u|^{2-a} u^{\alpha + \delta - 2} \xi^2 dx d\tau
$$

$$
\leq \frac{2\lambda}{\alpha} \int_T T u^\alpha \xi_t dx + C(\alpha - \lambda) \int_T T u^{\alpha + \delta - 2} \xi^2 dx d\tau
$$

$$
+ \frac{C}{(\alpha - \lambda)(1-a)} \int_T T |u|^{\alpha - a} \xi \nabla |\xi|^{2-a} dx d\tau.
$$

Using (5.4) to estimate $\xi_t$ and $\nabla \xi$ on the right-hand side of (5.6), the bound and support of $\xi(x,t)$, we have

$$
J \overset{\text{def}}{=} \frac{\lambda}{\alpha} \sup_{t \in [T_2,T]} \int_{B_{\rho}} u^\alpha(x,t) dx + \frac{d_3(\alpha - \lambda)}{2} \int_{T_2} T \int_{B_{\rho}} |\nabla u|^{2-a} u^{\alpha + \delta - 2} dx d\tau
$$

$$
\leq \frac{\lambda C}{\alpha(T_2 - T_1)} \int_{T_1} T \int_{B_R} u^\alpha dx d\tau + C(\alpha - \lambda) \int_{T_1} T \int_{B_R} u^{\alpha + \delta - 2} dx d\tau
$$

$$
+ \frac{C}{(\alpha - \lambda)(1-a)(R - \rho)^{2-a}} \int_{T_1} T \int_{B_R} |u|^{\alpha - a} dx d\tau.
$$

Note that $\lambda \leq 1$ and $\alpha - \lambda \geq 1$. We then have

$$
J \leq \frac{C}{T_2 - T_1} \int_{T_1} T \int_{B_R} u^\alpha dx d\tau + C(\alpha - \lambda) \int_{T_1} T \int_{B_R} u^{\alpha + \delta - 2} dx d\tau + \frac{C}{(R - \rho)^{2-a}} \int_{T_1} T \int_{B_R} |u|^{\alpha - a} dx d\tau.
$$

Inequality (2.4) gives $u^{\alpha + \delta - a} \leq u^{\alpha + \delta - 2} + u^\alpha$, hence

$$
J \leq C(\alpha - \lambda) \left[ 1 + \frac{1}{T_2 - T_1} + \frac{1}{(R - \rho)^{2-a}} \right] \cdot J_1,
$$

where $J_1 = \int_{T_1} T \int_{B_R} u^\alpha dx d\tau + \int_{T_1} T \int_{B_R} u^{\alpha + \delta - 2} dx d\tau$. Therefore,

$$
\sup_{t \in [T_2,T]} \int_{B_{\rho}} u^\alpha(x,t) dx \leq C \alpha^2 \left[ 1 + \frac{1}{T_2 - T_1} + \frac{1}{(R - \rho)^{2-a}} \right] \cdot J_1,
$$

(5.7)
where \( \kappa \) gives

\[
\int_{T_2} \int_{B_\rho} |\nabla u|^{2-a} u^{\alpha+\delta-2} dxd\tau \leq C \left[ 1 + \frac{1}{T_2 - T_1} + \frac{1}{(R - \rho)^{2-a}} \right] \cdot J_1.
\] (5.8)

Applying Hölder’s inequality to the second integral of \( J_1 \) using exponents \( \frac{\alpha}{\alpha + \delta - 2} \) and \( \frac{\alpha}{2 - \delta} \) we have

\[
J_1 \leq (1 + (|B_R|T)^{\frac{\alpha}{\alpha + \delta - 2}}) S \leq 2(1 + |B_R|T)S.
\] (5.9)

Hence estimate (5.1) follows from (5.7), (5.8) and (5.9).

**Proposition 5.2.** Assume (2.29). Let \( B_R, B_\rho, \) and \( T, T_2, T_1 \) and \( C_\alpha \) be as in Lemma 5.1. Then

\[
\|u\|_{L^\alpha(B_\rho \times (T_2, T))} \leq A_\alpha^{\frac{1}{\alpha}} \left( \|u\|_{L^\alpha(B_R \times (T_1, T))}^\theta \right)^{\frac{1}{\theta}},
\] (5.10)

where \( \kappa \) is defined in (2.31),

\[
r = r(\alpha) \equiv \alpha + \delta - 2, \quad s = s(\alpha) \equiv \alpha^2 \frac{\alpha}{\alpha + \delta - 2},
\] (5.11)

\[
A_\alpha = c_{10}(1 + \rho^{-1})^{2-a} \alpha^6 - \alpha(1 + |B_R|T)^{2} \left( 1 + \frac{1}{T_2 - T_1} + \frac{1}{(R - \rho)^{2-a}} \right)^{2},
\] (5.12)

with \( c_{10} \geq 1 \) independent of \( U, \alpha, \rho, R, T_1, T_2, T \).

**Proof.** Applying Sobolev inequality (2.32) to the ball \( B_\rho \) in place of \( B_R \), we have

\[
J \equiv \left( \int_{T_2} \int_{B_\rho} |u|^\alpha \kappa \alpha dxdt \right)^{\frac{1}{\alpha}} \leq \hat{c}_\alpha \left\{ \hat{\theta} \cdot \sup_{t \in [T_2, T]} \left( \int_{B_\rho} |u|^\alpha \right)^{1-\theta} \right\}^{\frac{1}{\theta}},
\] (5.13)

where \( \hat{c} = c_6(1 + \rho^{-1})^{2-a} \alpha^2-a \), exponent \( \hat{\theta} \) is defined in (2.31), and

\[
I = \left( \int_{T_2} \int_{B_\rho} |u|^{\alpha+\delta-a} dxdt \right) \int_{T_2} \int_{B_\rho} |u|^{\alpha+\delta-2} |\nabla u|^{2-a} dxdt \right)^{\frac{\alpha}{\alpha + \delta - a}}.
\]

Using (2.1), we have

\[
I \leq C_4 \left[ \int_{T_2} \int_{B_\rho} |u|^{\alpha+\delta-a} dxdt \right]^{\frac{\alpha}{\alpha + \delta - a}} + C_4 \left[ \int_{T_2} \int_{B_\rho} |u|^{\alpha+\delta-2} |\nabla u|^{2-a} dxdt \right]^{\frac{\alpha}{\alpha + \delta - a}},
\]

where \( C_4 = 2^{\frac{\alpha}{\alpha + \delta - a}} \). Then applying Hölder’s inequality to the first integral on the right-hand side yields

\[
I \leq C_4(T|B_\rho|) \frac{\alpha-\delta}{\alpha + \delta - a} \int_{T_2} \int_{B_\rho} |u|^{\alpha} dxdt + C_4 \left( \int_{T_2} \int_{B_\rho} |u|^{\alpha+\delta-2} |\nabla u|^{2-a} dxdt \right)^{\frac{\alpha}{\alpha + \delta - a}}.
\] (5.14)

Estimating the second integral on the right-hand side of (5.14) by (5.1), and combining with (5.13) give

\[
J \leq \hat{c}_\alpha^{\frac{1}{\alpha}} \left\{ \left( C_4(T|B_\rho|) \frac{\alpha-\delta}{\alpha + \delta - a} S + C_4(C_\alpha S) \frac{\alpha}{\alpha + \delta - a} \right)^{\hat{\theta}} \right\} \left( C_\alpha S \right)^{\hat{\theta} + 1 - \hat{\theta}} \frac{1}{\alpha},
\]

\[
\leq \hat{c}_\alpha^{\frac{1}{\alpha}} \left\{ \left( C_4(T|B_\rho|) \frac{\alpha-\delta}{\alpha + \delta - a} + C_\alpha S \right) \frac{\alpha}{\alpha + \delta - a} \right\} \left( C_\alpha S \right)^{\hat{\theta} + 1 - \hat{\theta}} \frac{1}{\alpha}.
\]
Note by definition $[5.2]$ that $S = y_0^\alpha + y_0^{\alpha+\delta-2}$, where $y_\alpha = \left( \int_{T_1}^T \int_{B_R} |u|^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}}$. Thus, we find that

$$
J \leq \hat{c} \frac{1}{\alpha} \left\{ \left( C_4(T|B_\rho) \right)^{\frac{\alpha-\delta}{\alpha+\delta-a}} + C_\alpha (y_0^\alpha + y_0^{\alpha+\delta-2}) + C_4 C_\alpha^{\frac{\alpha}{\alpha+\delta-a}} (y_0^\alpha + y_0^{\alpha+\delta-2})^{\frac{\alpha}{\alpha+\delta-a}} \right\}^{\frac{1}{\alpha}}
$$

$$
\leq \frac{1}{\alpha} \left\{ \left( M_1 y_0^\alpha + M_1 y_0^{\alpha+\delta-2} + M_2 y_0^{\alpha+\delta-2} + M_2 y_0^{\alpha+\delta-2} \right)^\alpha \right\}^{\frac{1}{\alpha}}
$$

where

$$
M_1 = C_4(T|B_\rho)^{\frac{\alpha-\delta}{\alpha+\delta-a}} + C_\alpha, \quad M_2 = C_4 C_\alpha^{\frac{\alpha}{\alpha+\delta-a}} 2^{\frac{\alpha}{\alpha+\delta-a}} = C_4 C_\alpha^{\frac{\alpha}{\alpha+\delta-a}}.
$$

(5.15)

Note that $\alpha + \delta - 2 < \frac{\alpha}{\alpha+\delta-a} < \alpha < \frac{\alpha^2}{\alpha+\delta-a}$, then by (2.4) we have

$$
y_0^\alpha, \frac{(\alpha+\delta-2)\alpha}{\alpha+\delta-a} \leq y_0^{\alpha+\delta-2} + \frac{\alpha^2}{\alpha+\delta-a}.
$$

Therefore, we obtain

$$
J \leq \left[ 3\hat{c}^{\frac{1}{\alpha}} (M_1 + M_2) \right]^{\frac{1}{\alpha}} (y_0^\alpha + y_0^{\alpha+\delta-2})^{\frac{1}{\alpha}}.
$$

(5.16)

Because $\frac{\alpha-\delta}{\alpha+\delta-a} \leq 1$, $\frac{\alpha}{\alpha+\delta-a} \leq 2$, $C_\alpha \geq 1 + |B_R| T > 1$, and $1 < C_4 < 4$, we have

$$
M_1 + M_2 \leq C_4 (1 + |B_R| T) + C_\alpha + C_4 C_\alpha^2 \leq C_4 C_\alpha + C_\alpha + C_4 C_\alpha^2 \leq 3C_4 C_\alpha^2 \leq 3(4C_\alpha)^2.
$$

(5.17)

Also, $\hat{c}^{\frac{1}{\alpha}} \leq \hat{c}$. Combining (5.16) and (5.17), we obtain

$$
J \leq \left[ 12^{\frac{1}{\alpha}} (y_0^\alpha + y_0^{\alpha+\delta-2}) \right]^{\frac{1}{\alpha}}.
$$

Then (5.10) follows.

Iterating relation (5.10), we obtain the following local estimate for $u$.

**Theorem 5.3.** Assume $\alpha_0 > 0$ such that $\alpha = \alpha_0$ satisfies (2.24). Let $B_R$, with $R > 0$, be a ball in a compact subset of $U$, and $T > 0$, $\sigma \in (0,1)$. Then

$$
\|u\|_{L^\infty(B_{R/2} \times \sigma T; T)} \leq \tilde{C}_{R,T,\sigma} \max \left\{ \|u\|_{L^{\mu_0}(B_R \times (0,T))}^{\mu}, \|u\|_{L^{\nu_0}(B_R \times (0,T))}^{\nu} \right\},
$$

where

$$
\mu = \prod_{j=0}^{\infty} \frac{\alpha_0 \kappa_j - 2 + \delta}{\alpha_0 \kappa_j}, \quad \nu = \prod_{j=0}^{\infty} \frac{\alpha_0 \kappa_j^2 + \delta - a}{\alpha_0 \kappa_j},
$$

(5.19)

$$
\tilde{C}_{R,T,\sigma} = \left[ 2^{11} c_{10} \alpha_0^{\omega-a} (1 + R^{-1})^2 (1 + |B_R| T)^3 \left( 1 + \frac{1}{\sigma T} + \frac{1}{R^2-a} \right)^2 \right]^{\omega}
$$

(5.20)

with $\kappa_\bullet = \kappa(\alpha_0)$ defined in (2.37), constant $c_{10}$ as in Proposition 5.2, and some positive number $\omega$ depending on $\alpha_0$.

**Proof.** For $j = 0, 1, 2, \ldots$, let

$$
t_j = \sigma T (1 - \frac{1}{2^j}), \quad \rho_j = \frac{R}{2} \left( 1 + \frac{1}{2^j} \right), \quad Q_j = B_{\rho_j} \times (t_j, T),
$$

where $B_{\rho_j}$ is the ball of radius $\rho_j$ having the same center as $B_R$. Note

$$
t_j - t_{j+1} = \frac{\sigma T}{2^{j+1}}, \quad \rho_j - \rho_{j+1} = \frac{R}{2^{j+2}}, \quad \lim_{j \to \infty} t_j = \sigma T, \quad \lim_{j \to \infty} \rho_j = R/2.
$$
Also, \( \kappa_* > 1 \). Let \( \alpha_j = \alpha_0 \kappa_*^j \). Then \( \alpha_j \geq \alpha_0 \) gives \( \kappa(\alpha_j) \geq \kappa(\alpha_0) = \kappa_* \).

Define \( Y_j = \|u\|_{L^{\kappa_0}(Q_j)} \). Note that \( (Q_j)_{j=0}^\infty \) is a sequence of nested cylinders. By Hölder’s inequality we obtain

\[
\|u\|_{L^{\kappa_0}(Q_{j+1})} \leq \|u\|_{L^{\kappa_0}(Q_{j})} \leq (1 + |Q_{0}|)^{\frac{1}{\kappa_0}} \|u\|_{L^{\kappa_0}(Q_{j+1})}.
\]

Hence applying (5.10) to \( \alpha = \alpha_j, \rho = \rho_j + 1, R = \rho_j, T_2 = t_{j+1} \) and \( T_1 = t_j \) gives

\[
Y_{j+1} \leq (1 + |Q_{0}|)^{\frac{1}{\kappa_0}} A_{\alpha_j} \left[ Y_j^{r(\alpha_j)} + Y_j^{s(\alpha_j)} \right]^{\frac{1}{\alpha_j}}.
\]

Using definitions in (5.11) and (5.12), we denote

\[
Y_j \leq (1 + |Q_{0}|)^{\frac{1}{\kappa_0}} A_{\alpha_j} \left[ Y_j^{r(\alpha_j)} + Y_j^{s(\alpha_j)} \right]^{\frac{1}{\alpha_j}}.
\]

Then

\[
Y_{j+1} \leq \hat{A}_{j} \left[ Y_j^{r(\alpha_j)} + Y_j^{s(\alpha_j)} \right]^{\frac{1}{\alpha_j}}.
\]

We estimate \( \hat{A}_{j} \). We have from (5.12) that

\[
\hat{A}_{j} \leq c_{10}(1 + |Q_{0}|) \cdot (1 + 2/R)^{2} \alpha_j^{6-a}(1 + |Q_{0}|)^{2} \left\{ 1 + \frac{2^{j+1}}{\sigma T} + \left( \frac{2^{j+2}}{R} \right)^{2-a} \right\}^{2}
\leq 4c_{10}(1 + R^{-1})^{2}(1 + |Q_{0}|)^{3}(\alpha_0 \kappa_*^{j})^{6-a} 16^{j+2}(1 + \frac{1}{\sigma T} + \frac{1}{R^{2-a}})^{2} \leq A_{R,T,\sigma}^{j+1},
\]

where

\[
A_{R,T,\sigma} = \max \left\{ 16 \kappa_*^{6-a}, 45 \alpha_0^{6-a}(1 + R^{-1})^{2}(1 + |Q_{0}|)^{3}(1 + \frac{1}{\sigma T} + \frac{1}{R^{2-a}})^{2} \right\}.
\]

Since \( \kappa_* \in (1, 2) \), we actually have

\[
A_{R,T,\sigma} = 45 \alpha_0^{6-a}(1 + R^{-1})^{2}(1 + |Q_{0}|)^{3}(1 + \frac{1}{\sigma T} + \frac{1}{R^{2-a}})^{2} \geq 1.
\]

Therefore,

\[
Y_{j+1} \leq A_{R,T,\sigma}^{j+1} \left[ Y_j^{r(\alpha_j)} + Y_j^{s(\alpha_j)} \right]^{\frac{1}{\alpha_j}} \forall j \geq 0.
\]

Since \( \kappa_* > 1 \), we clearly have \( \sum_{j=0}^{\infty} (j+1)/\alpha_j \) converges to a positive number. Note also that

\[
0 < \sum_{j=0}^{\infty} \ln \frac{s_j}{\alpha_j} = \sum_{j=0}^{\infty} \ln \left( 1 + \frac{a - \delta}{\alpha_0 \kappa_*^{j} + \delta - a} \right) \leq \sum_{j=0}^{\infty} \frac{a - \delta}{\alpha_0 \kappa_*^{j} + \delta - a} < \infty,
\]

and

\[
0 < -\sum_{j=0}^{\infty} \ln \frac{r_j}{\alpha_j} = \sum_{j=0}^{\infty} \ln \frac{\alpha_j}{r_j} = \sum_{j=0}^{\infty} \ln \left( 1 + \frac{2 + \delta}{\alpha_0 \kappa_*^{j} - 2 - \delta} \right) \leq \sum_{j=0}^{\infty} \frac{2 + \delta}{\alpha_0 \kappa_*^{j} - 2 - \delta} < \infty.
\]

Therefore, \( \Pi_{j=0}^{\infty}(r_j/\alpha_j) \) and \( \Pi_{j=0}^{\infty}(s_j/\alpha_j) \) converge to positive numbers \( \mu \) and \( \nu \), resp., given by (5.19).

By (5.24), and applying Lemma A.2 to sequence \( (Y_j)_{j=0}^{\infty} \), we obtain

\[
\|u\|_{L^{\infty}(B_{R/2} \times (\sigma T, T))} = \lim_{j \to \infty} Y_j \leq (2A_{R,T,\sigma})^{\omega} \max \{ Y_0^\mu, Y_0^\nu \},
\]

for some positive number \( \omega \). Then the desired estimate (5.18) follows. \( \square \)
Remark 5.4. Inequality (5.18) obviously leads to the quasi-homogeneous estimate (1.24), which was proved in [29] for equation

$$u_t = \nabla \cdot A(x, t, u, \nabla u)$$

(5.27)

with the homogeneous structure

$$A(x, t, u, \nabla u) \cdot \nabla u \geq c|u|^{m-1}|\nabla u|^p, \quad |A(x, t, u, \nabla u)| \leq c'|u|^{m-1}|\nabla u|^{p-1}. \quad (5.28)$$

Due to the non-homogeneity of function $K(\cdot)$, see (3.1), our equation (1.27) cannot be converted to (5.27), (5.28). Therefore, above inequality (5.18) is an extension of (1.24) to the class of equations (1.27) with non-homogeneous structure (3.1).

Now, we bound the $L^\infty$-norm of $u$, in any compact subsets of $U$, in terms of the initial and boundary data.

**Theorem 5.5.** Let $U'$ be an open, relatively compact subset of $U$, and $\alpha = \alpha_0$ satisfy (2.29). Then for $T > 0$, and $0 < \varepsilon < \min\{1, T\}$, one has

$$\|u\|_{L^\infty(U' \times (\varepsilon, T))} \leq C\varepsilon^{-2\omega} (1 + T)^{3\omega} \max \left\{ \|u\|_{L^{\alpha_0}(\bar{U} \times (0, T))}^{\mu}, \|u\|_{L^{\alpha_0}(\bar{U} \times (0, T))}^{\nu} \right\}, \quad (5.29)$$

where $\omega, \mu, \nu$ are the same as in Theorem 5.3.

In particular, if $T > 0$ satisfies (3.13) for $\alpha = \alpha_0$, then

$$\|u\|_{L^\infty(U' \times (\varepsilon, T))} \leq C\varepsilon^{-2\omega} (1 + T)^{3\omega} \max \left\{ \left( \int_0^T \int_U |u_0^{\alpha_0}(x)dx \right)^{\frac{\mu}{\alpha_0}}, \left( \int_0^T \int_U |u_\varphi,\alpha_0(t)dt \right)^{\frac{\nu}{\alpha_0}} \right\}, \quad (5.30)$$

where

$$U_{\alpha_0}(t) = \left\{ \left(1 + \int_U u_0^{\alpha_0}(x)dx \right)^{-\frac{\mu}{\alpha_0}} - C_{3, \alpha_0} \int_0^t (1 + \|\varphi^{-}(\tau)\|_{L^\infty(T)})^{2\omega - a}(1 - a)(1 - a_0) d\tau \right\}^{-\frac{\alpha_0}{\mu}}. \quad (5.31)$$

Furthermore, if $T > 0$ satisfies (3.14) for $\alpha = \alpha_0$ then

$$\|u\|_{L^\infty(U' \times (\varepsilon, T))} \leq C\varepsilon^{-2\omega} (1 + T)^{3\omega + \nu/\alpha_0} (1 + \|u_0\|_{L^{\alpha_0}(U)})^{\nu}. \quad (5.32)$$

Above, constant $C > 0$ depends on $U$, $U'$ and $\alpha_0$.

**Proof.** Denote $R = \frac{1}{2} dist(\bar{U}', \partial U') > 0$. Because the set $\bar{U}'$ is compact, there exists finitely many $x_i \in \bar{U}'$, $i = 1, \ldots, m$ for some $m \in \mathbb{N}$ such that $\{B_{R/2}(x_i)\}_{i=1}^m$ is an open covering of $\bar{U}'$. For each $i$, we have from (5.13) with $\varepsilon = \sigma T$ that

$$\|u\|_{L^\infty(B_{R/2}(x_i) \times (\varepsilon, T))} \leq C_{R, T, \varepsilon} \max \left\{ \|u\|_{L^{\alpha_0}(\bar{U} \times (0, T))}^{\mu}, \|u\|_{L^{\alpha_0}(\bar{U} \times (0, T))}^{\nu} \right\}, \quad (5.33)$$

where

$$C_{R, T, \varepsilon} = \left\{2^{11} C_{10} \alpha_0^{6-a} (1 + |B_R|T)^3(1 + R^{-1})^2(1 + \varepsilon^{-1} + (R^a - 2)^2)^{\omega} \right\} \leq C_R \varepsilon^{-2\omega} (1 + T)^{3\omega}$$

for some positive number $C_R$ depending on $R$ and $\alpha_0$. Summing up the estimates (5.33) in $i$, we obtain (5.29).

In case $T > 0$ satisfies (3.12) for $\alpha = \alpha_0$, we use (3.13) to estimate the $L^{\alpha_0}$-norm in (5.29), and obtain

$$\|u\|_{L^\infty(U' \times (\varepsilon, T))} \leq C\varepsilon^{-2\omega} (1 + T^{3\omega}) \max \left\{ \left( \int_0^T \int_{B_R} |u|^{\alpha_0} dx dt \right)^{\frac{\mu}{\alpha_0}}, \left( \int_0^T \int_{B_R} |u|^{\alpha_0} dx dt \right)^{\frac{\nu}{\alpha_0}} \right\}$$

$$\leq C\varepsilon^{-2\omega} (1 + T^{3\omega}) \max \left\{ \left( \int_0^T \int_{U_{\alpha_0}} |u|^{\alpha_0} dx dt \right)^{\frac{\mu}{\alpha_0}}, \left( \int_0^T \int_{U_{\alpha_0}} |u|^{\alpha_0} dx dt \right)^{\frac{\nu}{\alpha_0}} \right\},$$

This proves (5.30).

In case $T$ satisfies (3.14) for $\alpha = \alpha_0$, we use estimate (3.15) in (5.29) and note that $\nu \geq \mu$. Then we obtain (5.32). \qed
Global $L^\infty$-estimates

In this section, we estimate the $L^\infty$-norm on $U$ of the solution $u(x, t)$. We perform Moser’s iteration on the entire domain, and take into account the effect of the Robin boundary condition. Thanks to the contribution of the boundary terms, calculations need to be much more meticulous.

**Lemma 6.1.** Assume $\alpha > \max\{2 - \delta, n\mu_0\}$. Let $\theta, \mu_1, \mu_2, D_{3,\alpha}, D_{4,\alpha}$ be defined as in Lemma 2.2. If $T > T_2 > T_1 \geq 0$ then

$$
\sup_{t \in [T_1, T]} \int_U u^\alpha dx + \int_{T_1}^T \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} dx dt \leq \mathcal{M} \tilde{S}
$$

with

$$
\tilde{S} = \int_{T_1}^T \int_U u^{\alpha+\mu_1} dx dt + \left( \int_{T_1}^T \int_U u^{\alpha+\mu_1} dx dt \right) \frac{\alpha+\delta-2}{\alpha+\mu_1},
$$

$$
\mathcal{M} = \mathcal{M}(\alpha, \varphi, T_1, T_2, T), \quad \text{def} \quad c_{11} \alpha^2 (E_1 + E_2 + E_3 + E_4 + E_5),
$$

where constant $c_{11} > 0$ is independent of $\alpha, T_1, T_2, T$, while

$$
E_1 = (T_2 - T_1)^{-1} (|U|T)^{\frac{\mu_1}{\alpha+\mu_1}}, \quad E_2 = (|U|T)^{\frac{\mu_1-\delta+2}{\alpha+\mu_1}}, \quad E_3 = (|U|T)^{\frac{\mu_1}{\alpha+\mu_1}} \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))},
$$

$$
E_4 = D_{3,\alpha} |U|^{\frac{\mu_1(\alpha+\mu_1)}{\alpha(\alpha+\mu_1)}}, \quad E_5 = D_{4,\alpha} (d_3/4)^{-\mu_2} |U|^\frac{\mu_1}{\alpha} \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))}.
$$

with $d_3 > 0$ defined in (3.2), and

$$
\mu_4 = \mu_{4,\alpha} \overset{\text{def}}{=} \mu_2 + 1 = \frac{2 - a}{(1-a)(1-\theta)}.
$$

**Proof.** Let $\xi(t)$ be a smooth cut-off function with $\xi(t) \in [0, 1], \xi(t) = 0$ for $0 \leq t \leq T_1, \xi(t) = 1$ for $T_2 < t < T$, and

$$
|\xi_t| \leq C/(T_2 - T_1),
$$

for some $C > 0$ independent of $T_1, T_2$.

Multiplying the PDE in (1.26) by $u^{\alpha+\delta-1} \xi^2$, integrating over $U$, and using integration by parts yield

$$
\lambda \int_U u^{\alpha-1} \xi^2 \frac{\partial u}{\partial t} dx = \int_U u^{\alpha+\delta-1} \xi^2 \frac{\partial (u^\lambda)}{\partial t} = \int_U \nabla \cdot (K(|\nabla u|) \nabla u) u^{\alpha+\delta-1} \xi^2
$$

$$
= -(\alpha - \lambda) \int_U K(|\nabla u|) |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx - \int_{\Gamma} \varphi u^\alpha \xi^2 d\sigma.
$$

Using relation (3.2) we obtain

$$
\lambda \int_U u^{\alpha-1} \xi^2 \frac{\partial u}{\partial t} dx \leq -d_3 (\alpha - \lambda) \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx + d_3 (\alpha - \lambda) \int_U u^{\alpha+\delta-2} \xi^2 dx + J(t),
$$

where

$$
J(t) = \|\varphi^-(t)\|_{L^\infty(\Gamma)} \int_{\Gamma} u^{\alpha}(x, t) \xi^2(t) d\sigma.
$$

Since $\alpha - \lambda \geq 1$, we have

$$
\lambda \int_U u^{\alpha-1} \xi^2 \frac{\partial u}{\partial t} dx \leq -d_3 \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx + d_3 \alpha \int_U u^{\alpha+\delta-2} \xi^2 dx + J(t),
$$
Using the product rule on the left-hand side of the inequality we have
\[
\frac{\lambda}{\alpha} \frac{d}{dt} \int_U u^\alpha \xi^2 dx + d_3 \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx \leq \frac{2\lambda}{\alpha} \int_U u^\alpha \xi_t dx + d_3 \alpha \int_U u^{\alpha+\delta-2} \xi^2 dx + J(t). \quad (6.6)
\]

Integrating from 0 to \(t\), and taking supremum for \(t \in [0, T]\) give
\[
\frac{\lambda}{\alpha} \sup_{[0, T]} \int_U u^\alpha(x, t) \xi^2(t) dx + d_3 \int_0^T \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx dt \leq \frac{4\lambda}{\alpha} \int_0^T \int_U u^\alpha \xi |\xi| dt dt + 2d_3 \alpha \int_U u^{\alpha+\delta-2} \xi^2 dx dt + 2 \int_0^T J(t) dt. \quad (6.7)
\]

Using the trace inequality [22] and noting that \(\xi(t)\) is independent of \(x\), we can estimate \(J\) by
\[
J(t) \leq \|\varphi^-(t)\|_{L^\infty(\Gamma)} \left\{ 2\varepsilon \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \xi^2 dx + c_\ast \int_U u^\alpha \xi^2 dx + D_{3, \alpha} \varepsilon^{-\frac{1}{1-a}} \left( \int_U u^\alpha \xi^2 dx \right)^{\frac{\alpha+\mu_0}{\alpha}} + D_{4, \alpha} \varepsilon^{-\mu_2} \left( \int_U u^\alpha \xi^2 dx \right)^{\frac{\alpha+\mu_1}{\alpha}} \right\},
\]

where \(\mu_2\) is defined by (2.26). Hence,
\[
2 \int_0^T J(t) dt \leq \|\varphi^-(t)\|_{L^\infty(\Gamma \times (0, T))} \left\{ 4\varepsilon \int_0^T \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \xi^2 dx dt + C \int_0^T \int_U u^\alpha \xi^2 dx dt + 2D_{3, \alpha} \varepsilon^{-\frac{1}{1-a}} \int_0^T \left( \int_U u^\alpha \xi^2 dx \right)^{\frac{\alpha+\mu_0}{\alpha}} dt + 2D_{4, \alpha} \varepsilon^{-\mu_2} \int_0^T \left( \int_U u^\alpha \xi^2 dx \right)^{\frac{\alpha+\mu_1}{\alpha}} dt \right\}.
\]

Next, applying Hölder’s inequality to the last three integrals yields
\[
2 \int_0^T J(t) dt \leq 4\varepsilon \|\varphi^-(t)\|_{L^\infty(\Gamma \times (0, T))} \int_0^T \int_U |u|^{\alpha+\delta-2} |\nabla u|^{2-a} \xi^2 dx dt + C \|\varphi^-. \|_{L^\infty(\Gamma \times (0, T))} J_0, \quad (6.8)
\]

where
\[
J_0 = (|U(T)|^{\frac{\mu_1}{\alpha+\mu_1}} Y^{\frac{\alpha}{\alpha+\mu_1}} + D_{3, \alpha} \varepsilon^{-\frac{1}{1-a}} |U|^{\frac{\mu_1(\alpha+\mu_0)}{\alpha(\alpha+\mu_1)}} T^{\frac{\mu_1-\mu_0}{\alpha+\mu_1}} Y^{\frac{\alpha+\mu_0}{\alpha+\mu_1}} + D_{4, \alpha} \varepsilon^{-\mu_2} |U|^{\frac{\mu_1}{\alpha+\mu_1}} Y)
\]

with \(Y = \int_0^T \int_U u^{\alpha+\mu_1} \xi^2 dx dt\). Combining (6.7) and (6.8) with properties of \(\xi(t)\) gives
\[
\frac{\lambda}{\alpha} \sup_{[0, T]} \int_U u^\alpha(x, t) \xi^2(t) dx + \left( d_3 - 4\varepsilon \|\varphi^-\|_{L^\infty(U \times (0, T))} \right) \int_0^T \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} \xi^2 dx dt \\
\leq \frac{C}{\alpha(T_2 - T_1)} \int_0^T \int_U u^\alpha \xi dx dt + 2d_3 \alpha \int_0^T \int_U u^{\alpha+\delta-2} \xi^2 dx dt + C \|\varphi^-\|_{L^\infty(\Gamma \times (0, T))} J_0 \\
\leq \frac{C}{\alpha(T_2 - T_1)} (|U(T)|^{\frac{\mu_1}{\alpha+\mu_1}} Y^{\frac{\alpha}{\alpha+\mu_1}} + 2d_3 \alpha (|U(T)|^{\frac{\mu_1-\mu_2}{\alpha+\mu_1}} Y^{\frac{\alpha+\mu_2}{\alpha+\mu_1}} + C \|\varphi^-\|_{L^\infty(\Gamma \times (0, T))} J_0. \quad (6.9)
\]

Choosing \(\varepsilon = \frac{d_3}{\beta \|\varphi^-\|_{L^\infty(U \times (0, T))}}\), and using properties of \(\xi(t)\), we have
\[
\frac{\lambda}{\alpha} \sup_{t \in [T_2, T]} \int_U u^\alpha(x, t) dx + \frac{d_3}{2} \int_{T_2}^T \int_U |\nabla u|^{2-a} u^{\alpha+\delta-2} dx dt \\
\leq C(E_1 + E_3) Y^{\frac{\alpha}{\alpha+\mu_1}} + C a E Y^{\frac{\alpha+\delta-2}{\alpha+\mu_1}} + C E_4 Y^{\frac{\alpha+\mu_0}{\alpha+\mu_1}} + C E_5 Y.
\]
Note from the choice of the cut-off function $\xi(t)$ that $Y \leq \tilde{Y}$ and we obtain (6.1).

Proof. Assume $\alpha > \max\{2 - \delta, n\mu_0\}$. If $T > T_2 > T_1 > 0$ then

$$\|u\|_{L^\infty(U \times (T_2, T_1))] \leq \tilde{A}_\alpha \left( \|u\|_{L^\infty(U \times (T_1, T))]}^\tilde{r} + \|u\|_{L^\infty(U \times (T_1, T))]}^\tilde{s} \right)^{\tilde{\delta}},$$

(6.11)

where $\kappa$ is defined by (2.31), $\mu_1$ is defined by (2.26),

$$\tilde{r} = \tilde{r}(\alpha) = \frac{\alpha + \tilde{\delta} - 2}{\alpha + \mu_1}, \quad \tilde{s} = \tilde{s}(\alpha) = \frac{\alpha + \mu_1}{\alpha + \tilde{\delta} - \alpha},$$

(6.12)

$$\tilde{A}_\alpha = c_{12} 2^{-a - \alpha} \left[ (T|U|)^{(\mu_1 + \alpha + \tilde{\delta})\alpha} + M^{\alpha \alpha + \mu_1} + M^{\alpha \alpha + \tilde{\delta} - \alpha} \right],$$

(6.13)

with $M$ defined by (6.3), and $c_{12} \geq 1$ independent of $\alpha, T_1, T_2, T$.

Proposition 6.2. Assume $\alpha > \max\{2 - \delta, n\mu_0\}$. If $T > T_2 > T_1 > 0$ then

$$\|u\|_{L^\infty(U \times (T_2, T_1))] \leq \tilde{A}_\alpha \left( \|u\|_{L^\infty(U \times (T_1, T))]}^\tilde{r} + \|u\|_{L^\infty(U \times (T_1, T))]}^\tilde{s} \right)^{\tilde{\delta}},$$

(6.11)

where $\kappa$ is defined by (2.31), $\mu_1$ is defined by (2.26),

$$\tilde{r} = \tilde{r}(\alpha) = \frac{\alpha + \tilde{\delta} - 2}{\alpha + \mu_1}, \quad \tilde{s} = \tilde{s}(\alpha) = \frac{\alpha + \mu_1}{\alpha + \tilde{\delta} - \alpha},$$

(6.12)

$$\tilde{A}_\alpha = c_{12} 2^{-a - \alpha} \left[ (T|U|)^{(\mu_1 + \alpha + \tilde{\delta})\alpha} + M^{\alpha \alpha + \mu_1} + M^{\alpha \alpha + \tilde{\delta} - \alpha} \right],$$

(6.13)

with $M$ defined by (6.3), and $c_{12} \geq 1$ independent of $\alpha, T_1, T_2, T$.

Proof. We use Sobolev inequality (2.30) in Lemma 2.3

$$J \overset{\text{def}}{=} \left( \int_{T_2}^T \int_U |u|^{\alpha \alpha \alpha} dx dt \right)^{\frac{\alpha}{\alpha + \alpha \alpha \alpha}} \leq \tilde{c} \\sup_{t \in [T_2, T]} \left( \int_U |u|^{\alpha \alpha \alpha} dx \right)^{1 - \tilde{\delta}} \left( \int_U |u|^{\alpha \alpha \alpha} dx \right)^{\tilde{\delta}},$$

(6.14)

where $\tilde{c} = c_5 \alpha^{a - \alpha}$, the numbers $\tilde{\delta}$ and $\kappa$ are defined in (2.31) and

$$I = \left[ \int_{T_2}^T \int_U |u|^{\alpha + \alpha \alpha \alpha} dx dt + \int_{T_2}^T \int_U |u|^{\alpha + \alpha \alpha \alpha} dx dt \right]^{\frac{\alpha}{\alpha + \alpha \alpha \alpha}} \left( \int_{T_2}^T \int_U |u|^{\alpha + \alpha \alpha \alpha} dx dt \right)^{\frac{\alpha}{\alpha + \alpha \alpha \alpha}},$$

(6.15)

Applying inequality (2.3), we find that

$$I \leq C_5 \left( \int_{T_2}^T \int_U |u|^{\alpha + \alpha \alpha \alpha} dx dt \right)^{\frac{\alpha}{\alpha + \alpha \alpha \alpha}} + C_5 \left( \int_{T_2}^T \int_U |u|^{\alpha + \alpha \alpha \alpha} dx dt \right)^{\frac{\alpha}{\alpha + \alpha \alpha \alpha}},$$

(6.15)

where $C_5 = 2^\mu_1 \frac{\alpha}{\mu_1 - a}$. Applying Hölder’s inequality to the first integral on the right-hand side of (6.15) with conjugate exponents $\frac{\alpha + \mu_1}{\alpha + \mu_1}$ and $\frac{\alpha + \mu_1}{\mu_1 - a}$, we get

$$I \leq C_5 (T|U|)^{(\mu_1 + \alpha + \tilde{\delta})\alpha \alpha \alpha} \left( \int_{T_2}^T \int_U |u|^{\alpha + \mu_1} dx dt \right)^{\frac{\alpha}{\alpha + \mu_1}} + C_5 \left( \int_{T_2}^T \int_U |u|^{\alpha + \tilde{\delta} - \alpha \alpha \alpha} dx dt \right)^{\frac{\alpha}{\alpha + \tilde{\delta} - \alpha \alpha \alpha}}. \quad (6.16)$$
Next, we use (6.1) to estimate right-hand side of (6.10). Hence combining (6.14) and (6.16) yields

\[ J \leq \hat{c}_{\kappa} \left\{ C_5(T|U|)^{\frac{\mu_1+\delta\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + C_5(\mathcal{M}\tilde{S})^{\frac{\alpha}{\alpha+\beta}} (\mathcal{M}\tilde{S})^{1-\hat{\beta}} \right\} \hat{\beta}^\frac{1}{2} \]

\[ \leq \hat{c}_{\kappa} \left\{ C_5(T|U|)^{\frac{\mu_1+\delta\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + C_5(\mathcal{M}\tilde{S})^{\frac{\alpha}{\alpha+\beta}} + C_\alpha \tilde{S} \right\} \hat{\beta}^\frac{1}{2} \]

\[ = \hat{c}_{\kappa} \left\{ C_5(T|U|)^{\frac{\mu_1+\delta\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + \mathcal{M}\tilde{S} + C_5\mathcal{M}^{\frac{\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\beta}} \right\} \hat{\beta}^\frac{1}{2}. \]

Since \( \frac{\alpha}{\alpha+\mu_1} < 1 < \frac{\alpha}{\alpha+\delta-a} \), we use (2.4) to estimate \( \tilde{S} \leq \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + \tilde{S}^{\frac{\alpha}{\alpha+\delta-a}} \). Thus, we have

\[ J \leq \hat{c}_{\kappa} \left\{ C_5(T|U|)^{\frac{\mu_1+\delta\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + \mathcal{M}\tilde{S} + [1 + C_5]\mathcal{M}^{\frac{\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\beta}} \right\} \hat{\beta}^\frac{1}{2}. \]

Denote \( J_1 = \int_T T \int_U |u|^{\alpha+\mu_1} dx \), then by definition (6.2) of \( \tilde{S} \), and applying (2.2), resp. (2.3), we find

\[ \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} = (J_1^{\alpha+\mu_1} + J_1^\alpha) \leq J_1^{\alpha+\mu_1} \]

resp.,

\[ \tilde{S}^{\frac{\alpha}{\alpha+\delta-a}} = (J_1^{\alpha+\mu_1} + J_1^\alpha)^2 \leq C_5(J_1^{\alpha+\mu_1} + J_1^\alpha). \]

Therefore,

\[ J \leq \hat{c}_{\kappa} \left\{ \tilde{M}_1^{\alpha} + \tilde{M}_2^{\alpha} \right\} \hat{\beta}^\frac{1}{2}, \]

(6.17)

where

\[ \tilde{M}_1 = C_5(T|U|)^{\frac{\mu_1+\delta\alpha}{\alpha+\beta}} \tilde{S}^{\frac{\alpha}{\alpha+\mu_1}} + \mathcal{M}\tilde{S}, \quad \tilde{M}_2 = C_5(1 + C_5)\mathcal{M}^{\frac{\alpha}{\alpha+\beta}}. \]

Since \( \tilde{r} < \alpha, \frac{\alpha+\delta-2\alpha}{\alpha+\delta-a} < \tilde{s} \), we apply (2.4) to estimate the first and last summands on the right-hand side of (6.17) by

\[ J_1^{\alpha+\mu_1}, J_1^\alpha \leq J_1^{\alpha+\mu_1} + J_1^\alpha. \]

Then it follows

\[ J \leq \hat{c}_{\kappa} \left\{ 3(\tilde{M}_1 + \tilde{M}_2)(J_1^\alpha + J_1^{\alpha+\mu_1}) \right\} \hat{\beta}^\frac{1}{2}. \]

(6.18)

Since \( \alpha > 2 - \delta > 2(a - \delta) \), we have \( \tilde{M}_1 + \tilde{M}_2 \leq 9\tilde{M} \). Note also that \( \hat{c}_{\kappa} \leq \tilde{c} \), then we obtain (6.11) from (6.18).

Now to perform the iteration we need to start with an initial exponent \( \kappa(\alpha_0)\alpha_0 \) such that \( \kappa(\alpha_0)\alpha_0 > \alpha_0 + \mu_1, \alpha_0 \), that is, \( \kappa(\alpha_0)\alpha_0/(\alpha_0 + \mu_1, \alpha_0) > 1 \). We define

\[ \tilde{\kappa}(\alpha) = \frac{\kappa(\alpha)\alpha}{\alpha + \mu_1, \alpha}. \]

(6.19)

The following properties are useful in later iterations.

**Lemma 6.3.** For \( \alpha \in (2 - a)\alpha_*/(1 - a), \infty \), the functions \( \alpha \rightarrow \mu_1, \alpha \) in (2.20) and \( \alpha \rightarrow \mu_4, \alpha \) in (6.4) are decreasing, while the function \( \alpha \rightarrow \tilde{\kappa}(\alpha) \) in (6.19) is increasing.
Proof. First of all, we note that $\kappa(\alpha)$ defined in (2.31) is increasing, while $\theta_\alpha$ defined in (2.24) is decreasing. Since $\mu_1$ defined in terms of $\theta$ in (2.26) is increasing in $\theta$, and $\theta = \theta_\alpha$ is decreasing in $\alpha$, then, as a composition, $\mu_{1,\alpha}$ is decreasing in $\alpha$. Similar argument applies to $\mu_{4,\alpha}$.

Next, if $\alpha' > \alpha$ we have $\kappa(\alpha') \geq \kappa(\alpha)$ and $\mu_{1,\alpha'} \leq \mu_{1,\alpha}$, hence

$$
\bar{\kappa}(\alpha') = \frac{\kappa(\alpha')\alpha'}{\alpha' + \mu_{1,\alpha'}} \geq \frac{\kappa(\alpha)\alpha'}{\alpha' + \mu_{1,\alpha}} \geq \frac{\kappa(\alpha)\alpha}{\alpha + \mu_{1,\alpha}} = \check{\kappa}(\alpha).
$$

Therefore, $\bar{\kappa}(\alpha)$ is increasing in $\alpha$. \qed

We construct two sequences of exponents in order to implement Moser’s iteration. (Regarding the notation, the numbers $\alpha_j$’s below are newly constructed and are not the exponents in (1.10).)

**Lemma 6.4 (Construction of $\alpha_j$’s and $\beta_j$’s).** Let

$$
x_* = \frac{2 + \sqrt{(2 - a)(2 + \frac{1}{n}) - 1}}{1 - a}.
$$

(6.20)

Assume $\alpha_0 > (1 + x_*)\alpha_s$, let $\theta_* = \theta_{\alpha_0}$, $\mu_* = \mu_{1,\alpha_0}$, $\kappa_* = \kappa(\alpha_0)$, and $\bar{\kappa}_* = \check{\kappa}(\alpha_0)$. Define the sequence $(\beta_j)_{j=0}^\infty$ by

$$
\beta_0 = \alpha_0 + \mu_* , \quad \beta_j = \bar{\kappa}_* \beta_0 \quad \text{for } j \geq 1.
$$

(6.21)

Then:

(i) $\bar{\kappa}_* > 1$.

(ii) There exists a strictly increasing sequence $(\alpha_j)_{j=0}^\infty$ such that

$$
\beta_j = \alpha_j + \mu_{1,\alpha_j} \quad \forall j \geq 0.
$$

(6.22)

(iii) For all $j \geq 0$,

$$
\theta_{\alpha_j} \leq \theta_*, \quad \mu_{1,\alpha_j} \leq \mu_*, \quad \kappa(\alpha_j) \geq \kappa_*, \quad \bar{\kappa}(\alpha_j) \geq \bar{\kappa}_*.
$$

(6.23)

(iv) For all $j \geq 0$,

$$
\alpha_j < \bar{\kappa}_*^j \beta_0 ,
$$

and there exists a number $\hat{\kappa}_* > 1$ such that

$$
\alpha_j \geq \hat{\kappa}_*^j \alpha_0 \quad \forall j \geq 0.
$$

(6.25)

Proof. (i) Using definitions of $\kappa$, $\mu_1$ and $\mu_0$ in (2.31), (2.26) and (2.23), the inequality $\bar{\kappa}_* > 1$ is rewritten explicitly as

$$
\alpha_0 \left(1 + (a - \delta)\frac{1}{\alpha_*} - \frac{1}{\alpha_0}\right) > \alpha_0 + \frac{\frac{a - \delta}{1 - a} + \theta_{\alpha_0}(a - \delta)}{1 - \theta_{\alpha_0}},
$$

which is equivalent to

$$
\frac{\alpha_0}{\alpha_*} > \frac{2 - a}{(1 - a)(1 - \theta_{\alpha_0})}.
$$

Using formula (2.24) for $\theta_{\alpha_0}$, we convert this inequality to a quadratic inequality in $\alpha_0$ as

$$
(1 - a)\alpha_0^2 - 2(2 - a)\alpha_* \alpha_0 + (2 - a)\alpha_*^2 > 0.
$$
Its positive solutions are
\[
\alpha_0 > (1 + \frac{1 + \sqrt{2-a}}{1-a})\alpha_*.
\] (6.26)
This is satisfied by our choice of \(\alpha_0\). Therefore \(\bar{\kappa}_s > 1\).

(ii) It follows from definition of \(\beta_j\) that \((\beta_j)_{j=0}^{\infty}\) is unique and strictly increasing. Consider the equation
\[
\beta_j = x + \mu_1(x) \overset{\text{def}}{=} f(x),
\] (6.27)
where, for \(x > 0\),
\[
f(x) = x + \mu_0 + \frac{(a - \delta)\Theta(x)}{1 - \Theta(x)} = x - (a - \delta) + \frac{(2-a)(a-\delta)}{(1-a)(1-\Theta(x))},
\]
with \(\Theta(x) = \frac{\alpha_*}{(1-a)(x-\alpha_*)}\). We have \(f'(x) = 1 - \frac{(2-a)^2\Theta^2(x)}{n(1-\Theta(x))}\). Then \(f'(x) > 0\) if
\[
1/\Theta(x) > 1 + \frac{2-a}{\sqrt{n}}, \text{ that is, } x > \alpha_* \left\{ \frac{1}{1-a} \left(1 + \frac{2-a}{\sqrt{n}}\right) + 1 \right\}.
\]
Note that \(\frac{2-a}{\sqrt{n}} \leq \sqrt{2-a}\), we already have from (6.26) that
\[
\alpha_0 > \left[ \frac{1}{1-a} \left(\frac{2-a}{\sqrt{n}} + 1\right) + 1 \right] \alpha_*.
\]
Hence, \(f\) is strictly increasing on \([\alpha_0, \infty)\), \(f(\alpha_0) = \beta_0\) and \(f(\infty) = \infty\). Since the sequence \((\beta_j)\) is strictly increasing, we have for any \(j \geq 1\) that \(\beta_j > \beta_0\), and hence the number \(\alpha_j = f^{-1}(\beta_j)\) solves (6.22). Clearly, the sequence \((\alpha_j)_{j=0}^{\infty}\) is also strictly increasing.

(iii) By the monotonicity of \(\theta_\alpha\), \(\mu_1,\alpha\), \(\kappa(\alpha)\), \(\bar{\kappa}(\alpha)\) (Lemma 6.3 and its proof), and the fact \(\alpha_j \geq \alpha_0\), we have
\[
\theta_{\alpha_j} \leq \theta_{\alpha_0} = \theta_*, \quad \mu_{1,\alpha_j} \leq \mu_{1,\alpha_0} = \mu_*, \quad \kappa(\alpha_j) \geq \kappa(\alpha_0) = \kappa_*, \quad \bar{\kappa}(\alpha_j) \geq \bar{\kappa}(\alpha_0) = \bar{\kappa}_s.
\]

(iv) From (6.22), we have \(\alpha_j < \beta_j\) and hence inequality (6.24) follows.

By (6.22) and the fact \(\mu_{1,\alpha}\) is decreasing in \(\alpha\), we have for \(j \geq 1\) that
\[
\bar{\kappa}_s = \frac{\beta_j}{\beta_{j-1}} < \frac{\alpha_j + \mu_{1,\alpha_0}}{\alpha_{j-1}} < \frac{\alpha_j}{\alpha_{j-1}} + \frac{\mu_0}{\alpha_0} \frac{1}{1 - \frac{1}{(1-a)(\alpha_0/\alpha_*)-1}}.
\]
Set \(\alpha_0 = (1 + \hat{x})\alpha_*\), then \(\hat{x} > x_*\). Thus,
\[
\bar{\kappa}_s < \frac{\alpha_j}{\alpha_{j-1}} + \frac{\mu_0(1-a)}{\alpha_0(1-a)x_* - 1} = \frac{\alpha_j}{\alpha_{j-1}} + \frac{2-a}{n[(1-a)x_* - 1]}.
\]
Hence,
\[
\frac{\alpha_j}{\alpha_{j-1}} - 1 = \hat{\varepsilon} \overset{\text{def}}{=} \bar{\kappa}_s - 1 - \frac{2-a}{n[(1-a)x_* - 1]}.
\] (6.28)
We have
\[
\bar{\kappa}_s - 1 = \frac{\kappa_\alpha \alpha_0}{\alpha_0 + \mu_*} - 1 = \frac{(1 + (a-\delta)(1/\alpha_* - 1/\alpha_0))\alpha_0}{\alpha_0 + \mu_*} - 1 = \frac{(a-\delta)\hat{x} - \mu_*}{(1 + \hat{x})(\frac{a-\delta}{2-a})n + \mu_*}.
\]
Note that \( \theta_{\alpha_0} = \frac{1}{1-a} \), hence,

\[
\mu_* = \frac{\mu_0(1 + \theta_{\alpha_0}(1-a))}{1 - \theta_{\alpha_0}} = \frac{(a - \delta)(1 + \frac{1}{n})}{(1-a)(1 - \frac{1}{(1-a)x})} = \frac{(a - \delta)(\hat{x} + 1)}{(1-a)\hat{x} - 1}.
\]

Thus,

\[
\kappa_* - 1 = \frac{\hat{x} + \frac{1}{n} - \frac{1}{(1-a)x - 1}}{\frac{1}{2-a} + \frac{1}{(1-a)x - 1}}.
\]

We aim at finding \( \hat{x} \) such that

\[
\hat{x} = \frac{\frac{\hat{x} + \frac{1}{n}}{\frac{1}{2-a} + \frac{1}{(1-a)x - 1}} - \frac{2-a}{n(1-a)x - 1}} > 0.
\] (6.29)

If inequality (6.29) holds true then we choose \( \kappa_* = 1 + \hat{\varepsilon} > 1 \), and by (6.28), \( \alpha_j/\alpha_{j-1} \geq 1 + \hat{\varepsilon} = \kappa_* \). Thus, \( \alpha_j \geq \kappa_* \alpha_{j-1} \), and by induction, \( \alpha_j \geq \kappa_*^j \alpha_0 \) for all \( j \geq 0 \).

It remains to verify (6.29). For \( \hat{x} > \frac{1}{1-a} \), inequality (6.29) is equivalent to

\[
(1-a)^2\hat{x}^3 - 4(1-a)x^2 + (1 + 2a - \frac{2-a}{n})\hat{x} + 2 - \frac{2-a}{n} > 0.
\] (6.30)

Since \( 2 - \frac{2-a}{n} > 0 \), a sufficient condition for (6.30) is

\[
(1-a)^2\hat{x}^2 - 4(1-a)x + (1 + 2a - \frac{2-a}{n}) > 0.
\]

Solving this inequality gives

\[
\hat{x} > \frac{2 + \sqrt{3 - 2a + \frac{2-a}{n}}}{1-a} = x_*,
\]

which is satisfied by the choice of \( \hat{x} \). The proof is complete.

The final preparation for Moser’s iteration is to estimate \( \hat{A}_\alpha \) in (6.11).

**Lemma 6.5.** Let \( \alpha_0 \) be a positive number such that \( \alpha = \alpha_0 \) satisfies (2.22). Then one has for any \( \alpha \geq \alpha_0 \) that

\[
\hat{A}_\alpha \leq \hat{C}\alpha^{\mu_5}(1 + T)^{\mu_6}(1 + \frac{1}{T_2 - T_1})^2(1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\mu_7},
\] (6.31)

where \( \mu_5, \mu_6, \mu_7 > 0 \) and \( \hat{C} > 0 \) depend on \( \alpha_0 \) but not on \( \alpha \).

**Proof.** In this proof, \( \hat{C} \) denotes a generic positive constant depending on \( \alpha_0 \), but not on \( \alpha \).

In order to estimate \( \hat{A}_\alpha \), we estimate \( \mathcal{M} \) from (6.3) first. Note that \( 0 \leq \mu_2 \leq \mu_{2,0} \), hence \( (d_3/4)^{-\mu_2} \leq \hat{C} \). Then from (6.3), we find

\[
\mathcal{M} \leq \hat{C}\alpha^2(1 + \frac{1}{T_2 - T_1})^2(1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\mu_7} \left\{ (|U|T)^{\frac{\mu_1}{\alpha + \mu_1}} + (|U|T)^{\frac{\mu_1 - \delta + 2}{\alpha + \mu_2}} + D_{3,\alpha}|U|^{\frac{\mu_1}{\alpha + 1}} T^{\frac{\mu_1 - \mu_0}{\alpha + \mu_1} + D_{4,\alpha}|U|^{\frac{\mu_1}{\alpha}} \right\}.
\]
Above, we used the fact that $\mu_4 > 1$ is the maximum among the exponents of $\|\varphi^-\|_{L^\infty(\Gamma \times (0,T))}$.

Next, using definitions of $D_{3,\alpha}$ and $D_{4,\alpha}$ in (2.27) and (2.28), we have

$$
\mathcal{M} \leq \hat{C} \alpha^2 (1 + \frac{1}{T_2 - T_1}) (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\mu_4} \left\{ (|U|^T)^{\frac{\mu_1}{\alpha + \mu_1}} + (|U|^T)^{\frac{\mu_1 - \delta + 2}{\alpha + \mu_1}} \right. \\
+ 2^{\theta(\alpha + \delta - a)} c_* \frac{1}{1 - \alpha} \left| \left( \begin{array}{c} (2 - a)(1 + \theta(1 - a)) \\ \frac{1 - a}{\alpha} \end{array} \right) \right| |U|^{\frac{1 - a(\alpha + \mu_0)}{\alpha}} \cdot |U|^{\frac{\mu_1(\alpha + \mu_0)}{\alpha(\alpha + \mu_1)} T^{-\frac{\mu_1 - \mu_0}{\alpha + \mu_1}}} \\
+ 2^{\frac{\theta(\alpha + \delta - a)}{1 - \theta}} (c_* \alpha) \frac{1}{1 - \alpha} \left( \begin{array}{c} (2 - a)(1 + \theta(1 - a)) \\ \frac{1 - a}{\alpha} \end{array} \right) |U|^{\frac{1 - a(\alpha + \mu_0)}{\alpha}} \cdot |U|^{\frac{\mu_1(\alpha + \mu_0)}{\alpha(\alpha + \mu_1)} T^{-\frac{\mu_1 - \mu_0}{\alpha + \mu_1}}} \},
$$

where constants are as in Lemma 2.22. For convenience in calculations below, we keep using $c_*$ to denote max\{1, $c_*$\}. Also, using $\theta < 1$ and $m \leq \alpha$, we have

$$
\mathcal{M} \leq \hat{C} \alpha^2 (1 + \frac{1}{T_2 - T_1}) \left\{ (|U|^T)^{\frac{\mu_1}{\alpha + \mu_1}} + (|U|^T)^{\frac{\mu_1 - \delta + 2}{\alpha + \mu_1}} \right. \\
+ 2^{\theta(\alpha + \delta - a)} c_* \frac{1}{1 - \alpha} \left( \begin{array}{c} (2 - a)(1 + \theta(1 - a)) \\ \frac{1 - a}{\alpha} \end{array} \right) |U|^{\frac{1 - a(\alpha + \mu_0)}{\alpha}} \cdot |U|^{\frac{\mu_1(\alpha + \mu_0)}{\alpha(\alpha + \mu_1)} T^{-\frac{\mu_1 - \mu_0}{\alpha + \mu_1}}} \},
$$

Set $\theta_0 = \theta_{a_0}$ and $\mu_0 = \mu_{1,a_0}$. Since $\alpha \geq \alpha_0$, we have $\theta_0 \leq \theta_\ast$. Hence

$$(c_* \alpha)^{\frac{(2 - a)(1 + \theta(1 - a))}{(1 - a)(1 - \theta)}} \leq (c_* \alpha)^{\frac{(2 - a)^2}{(1 - a)(1 - \theta)}} \leq \hat{C} \alpha^{z_1}, \text{ where } z_1 = \frac{(2 - a)^2}{(1 - a)(1 - \theta_\ast)}.
$$

Next, we want to bound $2^{\frac{\theta(\alpha + \delta - a)}{1 - \theta}}$ by some number independent of $\alpha$. From (2.24),

$$
\theta(\alpha + \delta - a) = \frac{\alpha + \delta - a}{(1 - a)(\alpha/\alpha_\ast - 1)},
$$

which, due to the fact that $\alpha_\ast > a - \delta$, is decreasing in $\alpha$. Hence,

$$
\theta(\alpha + \delta - a) \leq z_2 \underset{\text{def}}{=} \theta_\ast (\alpha_0 + \delta - a) \text{ and } 2^{\frac{\theta(\alpha + \delta - a)}{1 - \theta}} \leq 2^{\frac{z_2}{1 - \theta_\ast}}.
$$

For exponents of $|U|^T$, we note that

$$
\frac{\mu_1}{\alpha + \mu_1}, \frac{\mu_1 - \delta + 2}{\alpha + \mu_1} \leq 1.
$$

For the remaining power of $T$,

$$
\frac{\mu_1 - \mu_0}{\alpha + \mu_1} \leq 1.
$$

For the remaining powers of $|U|$,

$$(1 - a) \theta \frac{\alpha + \mu_0}{\alpha} \leq \frac{\alpha + \mu_0}{\alpha} \leq 2, \text{ and } \frac{\mu_1(\alpha + \mu_0)}{\alpha(\alpha + \mu_1)} \leq \frac{\mu_1}{\alpha} \leq \mu_\ast,$$

Also, we have for the power of $\|\varphi^-\|_{L^\infty(\Gamma \times (0,T))}$:

$$
\mu_4 \leq z_3 \underset{\text{def}}{=} \mu_{4,a_0} = \frac{2 - a}{(1 - a)(1 - \theta_\ast)},
$$

due to the decrease of $\mu_4$ in $\alpha$, see Lemma 6.3.

So we find

$$
\mathcal{M} \leq \hat{C} \alpha^{2 + z_1} (1 + \frac{1}{T_2 - T_1}) 2^{\frac{z_2}{1 - \theta_\ast}} (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{z_3} (1 + |U|)^{2 + \mu_\ast} (1 + T).
$$
Summing up, we obtain
\[ \tilde{M} \leq \tilde{M}^\text{def} = \hat{C}\alpha^{2+z_1}(1 + \frac{1}{T_2 - T_1})(1 + T)(1 + \|\varphi^-\|_{L^\infty(T \times (0,T))}^{2z_3}). \]  
(6.32)

Since \(\tilde{M} > 1\) and \(\frac{\alpha}{\alpha + \mu_1} < \frac{\alpha}{\alpha + \delta - a} \leq 2\), using (6.32) in (6.13) we have
\[ \tilde{A}_\alpha \leq c_{12}\alpha^{2-a}[\max\{2, \mu_0\}] \left(1 + \|\varphi^-\|_{L^\infty(T \times (0,T))}^{2z_3}\right)^2. \]  
(6.33)

Note that \(a - \delta = (1 - a)\mu_0\) and \(\alpha + \delta - a \geq 2 - a\), hence
\[ \frac{(\mu_1 + a - \delta)\alpha}{(\alpha + \mu_1)(\alpha + \delta - a)} = \frac{\mu_0(2 - a)\alpha}{(1 - \theta)(\alpha + \mu_1)(\alpha + \delta - a)} \leq \frac{\mu_0(2 - a)}{1 - \theta}. \]  
(6.34)

Hence by (6.32), (6.33) and (6.34),
\[ \tilde{A}_\alpha \leq \hat{C}\alpha^{2-a+2(2+z_1)}(1 + \frac{1}{T_2 - T_1})^2(1 + T)^{\max\{2, \mu_0\}} \left(1 + \|\varphi^-\|_{L^\infty(T \times (0,T))}^{2z_3}\right)^2. \]  
(6.35)

Hence we obtain (6.31) from (6.35) with \(\mu_5 = 6 - a + 2z_1\), \(\mu_6 = \max\{2, \mu_0\}\), and \(\mu_7 = 2z_3\). \(\square\)

Applying iteration process, we obtain:

**Theorem 6.6.** Assume
\[ \alpha_0 > \max\{2 - \delta, (1 + x_*)\alpha_*\} \]  
(6.36)

with \(x_*\) defined by (6.20). There are \(C, \hat{\mu}, \hat{\nu}, \omega_1, \omega_2, \omega_3 > 0\) such that if \(T > 0\) and \(\sigma \in (0,1)\) then
\[ \|u\|_{L^\infty(U \times (\sigma T, T))} \leq C \left(1 + \frac{1}{\sigma T}\right)^{\omega_1}(1 + T)^{\omega_2}(1 + \|\varphi^-\|_{L^\infty(T \times (0,T))}^{\omega_3}} \cdot \max\{\|u\|^{\mu_0}_{L^0(U \times (0,T))}, \|\tilde{u}\|^{\mu_0}_{L^0(U \times (0,T))}\}, \]  
(6.37)

where \(\beta_0 = \alpha_0 + \mu_1, \alpha_0\).

**Proof.** Note that \(\alpha_* \geq a - \delta\) and \(n\mu_0 = \alpha_*(2 - a)/(1 - a)\), then it is easy to check that \(\alpha_0 \geq 2(a - \delta)\) and \(\alpha_0 > n\mu_0\).

Let \(\mu_5, \mu_6\) and \(\mu_7\) be defined as in Lemmas 6.3 and 6.4, \(\alpha_j, \beta_j, \kappa_\epsilon, \tilde{\kappa}_\epsilon, \hat{\kappa}_\epsilon\) be as in Lemma 6.3.

For \(j \geq 0\), let \(t_j = \sigma T(1 - \frac{1}{2^j})\), \(Q_j = U \times (t_j, T)\), and define \(Y_j = \|u\|^{\beta_j}_{L^0(Q_j)}\).

Applying (6.11) of Theorem 6.2 with \(\alpha = \alpha_j, T_2 = t_{j+1}\) and \(T_1 = t_j\), we have
\[ \|u\|^{\beta_j}_{L^\infty(Q_j)} \leq \tilde{A}^{\alpha_j}_{\beta_j} \left(\|u\|_{L^\infty(U \times (t_j+1, T))}^{\hat{r}(\alpha_j)} + \|\tilde{u}\|_{L^\infty(U \times (t_j+1, T))}^{\hat{s}(\alpha_j)}\right)^{\frac{1}{\alpha_j}}, \]  
(6.38)
Now we estimate $\tilde{A}_j$. From (6.31), (6.38), the fact that $\alpha_j < \beta_j + 1$, and (6.24), we have
\begin{align*}
\tilde{A}_j &\leq C(1 + |Q_{j+1}|)^{\frac{\alpha_j}{\beta_j}} \alpha_j^{\mu_5} (1 + \frac{2j+1}{\sigma T})^2 (1 + |T|)^{\mu_6} (1 + ||\varphi||_{L^\infty(\Gamma \times (0,T))})^{\mu_7} \\
&\leq C(1 + |Q_0|)(\kappa_s^j \beta_0)^{\mu_5} A_0^j (1 + \frac{1}{\sigma T})^2 (1 + |U|^{(1 + |T|) \mu_6} (1 + ||\varphi||_{L^\infty(\Gamma \times (0,T))})^{\mu_7} \leq A_j^{j+1},
\end{align*}
where

$$A_{T,\sigma,\varphi} = \max \{ 4\kappa_s^{\mu_5} |C\kappa_0^{\mu_5} (1 + \frac{1}{\sigma T})^2 (1 + |U|^{(1 + |T|) \mu_6} (1 + ||\varphi||_{L^\infty(\Gamma \times (0,T))})^{\mu_7} \} > 1.$$ 
Hence
\begin{equation}
Y_{j+1} \leq A_{T,\sigma,\varphi}^{j+1} (Y_j^{\tilde{r}_j} + Y_j^{\tilde{r}_j}) \frac{1}{\alpha_j}. 
\end{equation}

From (6.25) we have
\begin{equation}
\sum_{j=1}^{\infty} \frac{j+1}{\alpha_j} \leq \frac{1}{\sigma_0} \sum_{j=1}^{\infty} \frac{j+1}{\kappa_s^j} < \infty.
\end{equation}
Note that
\begin{align}
1 &\geq \frac{\tilde{r}_j}{\alpha_j} = \frac{\alpha_j + \delta - 2}{\alpha_j + \mu_1,\alpha_j} \geq \frac{\alpha_j + \delta - 2}{\alpha_j + \mu_1} \geq \frac{\kappa_s^j \alpha_0 + \delta - 2}{\kappa_s^j \alpha_0 + \mu_1} \geq 1 - \frac{\mu_1 + 2 - \delta}{\kappa_s^j \alpha_0}, \\
\frac{1}{\alpha_j} &\leq \frac{\tilde{r}_j}{\alpha_j} = \frac{\alpha_j + \mu_1,\alpha_j}{\alpha_j + \delta - a} \leq \frac{\kappa_s^j \alpha_0 + \mu_1}{\kappa_s^j \alpha_0 + \delta - a} = 1 + \frac{\mu_1 + a - \delta}{\kappa_s^j \alpha_0 + \delta - a}.
\end{align}
Then it is elementary, see (5.25) and (5.26), to show that the products
\begin{equation}
\bar{\mu} = \prod_{j=0}^{\infty} \frac{\alpha_j + \delta - 2}{\alpha_j + \mu_1,\alpha_j} \quad \text{and} \quad \bar{\nu} = \prod_{j=0}^{\infty} \frac{\alpha_j + \mu_1,\alpha_j}{\alpha_j + \delta - a},
\end{equation}
converge to positive numbers. By (6.39) and Lemma A.2 we obtain
\begin{equation}
\limsup_{j \to \infty} Y_j \leq (2A_{T,\sigma,\varphi})^{\omega} \max \{ Y_0^{\bar{\mu}}, Y_0^{\bar{\nu}} \},
\end{equation}
where $\omega = \mathcal{G} \sum_{j=1}^{\infty} \frac{j+1}{\alpha_j}$ with $\mathcal{G} = \prod_{k=1}^{\infty} (\tilde{s}_k/\alpha_k) \in (0, \infty)$.

Note that
\begin{equation}
(2A_{T,\sigma,\varphi})^{\omega} \leq C(1 + \frac{1}{\sigma T})^{\omega_1} (1 + T)^{\omega_2} (1 + ||\varphi||_{L^\infty(\Gamma \times (0,T))})^{\omega_3},
\end{equation}
where $\omega_1 = 2\omega$, $\omega_2 = (1 + \mu_0)\omega$ and $\omega_3 = \mu_\tau \omega$. Then estimate (6.37) follows (6.43).

**Remark 6.7.** (i) The exponents $\bar{\mu}$ and $\bar{\nu}$ in (6.37) are given by (6.42) but can, in fact, be replaced by simpler and more explicit ones such as
\begin{align*}
\bar{\mu} &= \prod_{j=0}^{\infty} \frac{\alpha_0 \kappa_s^j - 2 + \delta}{\alpha_0 \kappa_s^j + \mu_1}, \quad \bar{\nu} = \prod_{j=0}^{\infty} \frac{\alpha_0 \kappa_s^j + \mu_1}{\alpha_0 \kappa_s^j + \delta - a}.
\end{align*}
Indeed, it follows from estimates (6.40) and (6.41) that $\mu \leq \bar{\mu} \leq \bar{\nu} \leq \nu$, and then applying (2.4) gives
\begin{equation}
\max \{ Y_0^{\bar{\mu}}, Y_0^{\bar{\nu}} \} \leq Y_0^{\bar{\mu}} + Y_0^{\bar{\nu}} \leq 2 \max \{ Y_0^{\bar{\mu}}, Y_0^{\bar{\nu}} \}.
\end{equation}
(ii) Estimate (6.37) is a global version of the improvement (1.24) on interior estimates. See also (13) for a similar global result for degenerate equations with the use of De Giorgi’s iteration instead.
We now have global $L^\infty$-estimates in terms of the initial and boundary data.

**Theorem 6.8.** Assume $\alpha_0$ satisfies (6.30). Let $\beta_0 = \alpha_0 + \mu_{1,0}$. Then there are positive numbers $C, \omega_1, \omega_2, \omega_3, \tilde{\mu}, \tilde{\nu}$ such that:

(i) If $T > 0$ satisfies (3.12) with $\alpha = \beta_0$, and $0 < \varepsilon < \min\{1, T\}$, then

$$
\|u\|_{L^\infty(U \times (\varepsilon,T))} \leq C \varepsilon^{-\omega_1} (1 + T)^{\omega_2} (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\omega_3}
\cdot \max \left\{ \left( \int_0^T U_{\beta_0}(t) dt \right)^{\frac{\beta_0}{\beta_0}}, \left( \int_0^T U_{\beta_0}(t) dt \right)^{\frac{\beta_0}{\rho_0}} \right\},
$$

where $U_{\beta_0}(t)$ is defined by (5.31) with $\beta_0$ replacing $\alpha_0$.

(ii) If $T > 0$ satisfies (3.14) with $\alpha = \beta_0$, then

$$
\|u\|_{L^\infty(U \times (\varepsilon,T))} \leq C \varepsilon^{-\omega_1} (1 + T)^{\omega_2 + \tilde{\beta}/\beta_0} (1 + \|u_0\|_{L^{\beta_0}(U)})^{\tilde{\nu}} (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\omega_3}.
$$

**Proof.** (i) Applying (6.37) to $\sigma T = \varepsilon > 0$, we have

$$
\|u\|_{L^\infty(U \times (\varepsilon,T))} \leq C \varepsilon^{-\omega_1} (1 + T)^{\omega_2} (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\omega_3}
\cdot \max \left\{ \left( \int_0^T \int_U |u|^\beta_0 dxdt \right)^{\frac{\beta_0}{\beta_0}}, \left( \int_0^T \int_U |u|^\beta_0 dxdt \right)^{\frac{\beta_0}{\rho_0}} \right\}.
$$

Using (3.13) we have

$$
\int_U |u(x,t)|^{\beta_0} dx \leq U_{\beta_0}(t).
$$

Then (6.44) follows (6.46) and (6.47).

(ii) If $T > 0$ satisfies (3.14) with $\alpha = \beta_0$, then (3.15) gives

$$
\int_U u^{\beta_0}(x,t) dx \leq 2 \left( 1 + \int_U u_0^{\beta_0}(x) dx \right).
$$

Combining this with (6.46) yields

$$
\|u\|_{L^\infty(U \times (\varepsilon,T))} \leq C \varepsilon^{-\omega_1} (1 + T)^{\omega_2} (1 + \|\varphi^-\|_{L^\infty(\Gamma \times (0,T))})^{\omega_3}
\cdot \max \left\{ \left( 2 \int_0^T \left( 1 + \int_U u_0^{\beta_0}(x) dx \right) dt \right)^{\frac{\beta_0}{\beta_0}}, \left( 2 \int_0^T \left( 1 + \int_U u_0^{\beta_0}(x) dx \right) dt \right)^{\frac{\beta_0}{\rho_0}} \right\}.
$$

Since $\tilde{\nu} > \tilde{\mu}$, then we obtain (6.45). \qed

**A Appendix**

Let $(y_j)_{j=0}^\infty$ be a sequence of non-negative numbers.

**Lemma A.1.** Let $(\alpha_j)_{j=0}^\infty$ and $(\beta_j)_{j=0}^\infty$ be sequences of positive numbers with

$$
\alpha \overset{\text{def}}{=} \sum_{j=0}^\infty \alpha_j < \infty \text{ and } \beta \overset{\text{def}}{=} \prod_{j=0}^\infty \beta_j \text{ exists and belongs to } \in (0, \infty).
$$
Suppose there is \( A \geq 1 \) such that
\[
y_{j+1} \leq A^{\alpha_j} y_j^{\beta_j} \quad \forall j \geq 0.
\] (A.1)
Then \((y_j)_{j=0}^\infty\) is a bounded sequence. More specifically, for all \( j \geq 1 \),
\[
y_j \leq A B_j \sum_{i=0}^{j-1} \alpha_i \beta_0 \beta_1 \cdots \beta_{j-1},
\] (A.2)
and consequently,
\[
\limsup_{j \to \infty} y_j \leq A^{B \bar{\alpha}} y_0^{\bar{\beta}},
\] (A.3)
where \( B_j = \max \{1, \beta_m \beta_{m+1} \beta_{m+2} \cdots \beta_n : 1 \leq m \leq n < j\} \), and \( B = \limsup_{j \to \infty} B_j \).

Proof. Applying (A.1) recursively, we have
\[
y_{j+1} \leq A^{\alpha_j} (A^{\alpha_{j-1}} y_{j-1}^{\beta_{j-1}})^{\beta_j} = A^{\alpha_j + \alpha_{j-1} \beta_j} y_{j-1}^{\beta_j} \times \cdots \times A^{\alpha_1 + \alpha_{j-2} \beta_{j-1} + \alpha_{j-3} \beta_{j-2} \beta_{j-1} + \cdots + \alpha_0 \beta_1 \beta_2 \cdots \beta_j} y_0^{\beta_0 \cdots \beta_j}.
\]
It follows that \( y_{j+1} \leq A^{B_{j+1}} y_0^{\beta_0 \cdots \beta_j} \). Hence, we obtain (A.2). Taking the limit superior of (A.2) as \( j \to \infty \), we obtain (A.3). Note that \( B < \infty \) by Cauchy’s criterion. \( \square \)

The next lemma is the main adaptation used in this paper.

Lemma A.2. Let \( \kappa_j > 0, s_j \geq r_j > 0 \) and \( \omega_j \geq 1 \) for all \( j \geq 0 \). Suppose there is \( A \geq 1 \) such that
\[
y_{j+1} \leq A^{\bar{\alpha}_j} (y_j^{s_j} + y_j^{r_j})^{\bar{\beta}_j} \quad \forall j \geq 0.
\]
Denote \( \beta_j = r_j / \kappa_j \) and \( \gamma_j = s_j / \kappa_j \). Assume
\[
\bar{\alpha} \overset{\text{def}}{=} \sum_{j=0}^\infty \frac{\omega_j}{\kappa_j} < \infty \quad \text{and the products } \prod_{j=0}^\infty \beta_j, \prod_{j=0}^\infty \gamma_j \text{ converge to positive numbers } \bar{\beta}, \bar{\gamma}, \text{ resp.}
\]
Then
\[
y_j \leq (2A)^{G_j} \max \{y_0^{\gamma_0 \cdots \gamma_{j-1}}, y_0^{\beta_0 \cdots \beta_{j-1}}\} \quad \forall j \geq 1,
\] (A.4)
where \( G_j = \max \{1, \gamma_m \gamma_{m+1} \cdots \gamma_n : 1 \leq m \leq n < j\} \). Consequently,
\[
\limsup_{j \to \infty} y_j \leq (2A)^{\bar{G} \bar{\alpha}} \max \{y_0^{\bar{\gamma}}, y_0^{\bar{\beta}}\}, \quad \text{where } G = \limsup_{j \to \infty} G_j.
\] (A.5)

Proof. We prove (A.4) first. Define a sequence \((z_j)_{j=0}^\infty\) by \( z_0 = y_0 \) and \( z_{j+1} = A^{\bar{\alpha}_j} (z_j^{r_j} + z_j^{s_j})^{\bar{\beta}_j} \) for \( j \geq 0 \). Then
\[
y_j \leq z_j \quad \text{for all } j \geq 0.
\] (A.6)
Therefore, it suffices to bound \( z_j \). We consider three cases.

Case 1: \( z_0 \geq 1 \). Clearly, \( z_j \geq 1 \) for all \( j \). Hence,
\[
z_{j+1} \leq A^{\bar{\alpha}_j} (2 z_j^{s_j})^{\bar{\beta}_j} \leq (2A)^{\bar{\alpha}_j} z_j^{\bar{\gamma}_j}.
\]
Then using Lemma A.1, we have

\[ z_j \leq (2A)^G_j \sum_{i=0}^{j-1} \omega_i/\kappa_i z_0^{\alpha_0 \gamma_1 \ldots \gamma_j - 1}. \] (A.7)

Together with (A.6), we obtain (A.4).

**Case 2:** \( z_j < 1 \) for all \( j \geq 0 \). Then

\[ z_{j+1} \leq A^{\bar{\gamma}_j} (2z_j^r \frac{1}{\bar{r}_j}) \leq (2A)^{\bar{\gamma}_j} z_j^{\bar{\gamma}_j}. \]

Applying Lemma A.1 gives

\[ z_j \leq (2A)^{B_j} \sum_{i=0}^{j-1} \omega_i/\kappa_i z_0^{\beta_0 \beta_1 \ldots \beta_{j-1}}. \] (A.8)

where \( B_j = \max\{1, \beta_m \beta_{m+1} \ldots \beta_n : 1 \leq m \leq n < j\} < \infty \). Note that \( B_j \leq G_j \). Then, again, (A.8) and (A.6) yield (A.4).

**Case 3:** There is \( j_0 \geq 1 \) such that \( z_j < 1 \) for \( 0 \leq j < j_0 \) and \( z_{j_0} \geq 1 \). Applying (A.8) to \( 1 \leq j \leq j_0 \),

\[ z_j \leq (2A)^{B_j} \sum_{i=0}^{j-1} \omega_i/\kappa_i z_0^{\beta_0 \beta_1 \ldots \beta_{j-1}} \leq (2A)^{G_j} \sum_{i=0}^{j-1} \omega_i/\kappa_i z_0^{\gamma_0 \gamma_1 \ldots \gamma_j - 1}. \] (A.9)

Same as Case 1, \( z_j \geq 1 \) for all \( j \geq j_0 \). Then applying (A.7) for \( j > j_0 \) gives

\[ z_j \leq (2A)^{G_{j_0,j}} \sum_{i=0}^{j-1} \omega_i/\kappa_i z_0^{\gamma_0 \gamma_1 \ldots \gamma_{j_0+1} \gamma_{j_0+2} - 1}. \] (A.10)

where \( G_{j_0,j} = \sup\{1, \gamma_{j_0+m} \gamma_{j_0+m+1} \ldots \gamma_{j_0+n} : 1 \leq m \leq n < j - j_0\} < \infty \). Using inequality (A.9) with \( j = j_0 \) to estimate \( z_{j_0} \) in (A.10), we have for \( j > j_0 \) that

\[ z_j \leq (2A)^{G_{j_0,j}} \sum_{i=0}^{j-1} \omega_i/\kappa_i \left\{ (2A)^{G_{j_0,j}} \sum_{i=0}^{j_0-1} \omega_i/\kappa_i z_0^{\beta_0 \beta_1 \ldots \beta_{j_0-1}} \right\} \gamma_0 \gamma_0 + 1 \gamma_{j_0} \gamma_{j_0+1} \ldots \gamma_{j_0+1} - 1 \]

\[ = (2A)^{G_{j_0,j}} (\sum_{i=0}^{j_0-1} \omega_i/\kappa_i + G_{j_0,j} \gamma_{j_0} \gamma_{j_0+1} \ldots \gamma_{j_0-1} (\sum_{i=0}^{j_0-1} \omega_i/\kappa_i) z_0^{\beta_0 \beta_1 \ldots \beta_{j_0-1} - \gamma_0 \gamma_0 + 1 \gamma_{j_0} \gamma_{j_0+1} \ldots \gamma_{j_0+1} - 1}. \]

Since \( z_0 < 1, \beta_i \leq \gamma_i \) for all \( i \), and \( G_{j_0,j}, G_{j_0} \gamma_{j_0} \gamma_{j_0+1} \ldots \gamma_{j_0+1} - 1 \leq G_j \), we obtain

\[ z_j \leq (2A)^{\max\{G_{j_0,j}, G_{j_0} \gamma_{j_0} \gamma_{j_0+1} \ldots \gamma_{j_0+1} - 1}\} \sum_{i=0}^{j_0-1} \omega_i/\kappa_i z_0^{\beta_0 \beta_1 \ldots \beta_{j_0-1}} \leq (2A)^{G_j} \sum_{i=0}^{j_0-1} \omega_i/\kappa_i z_0^{\beta_0 \beta_1 \ldots \beta_{j_0-1}} \] (A.11)

for all \( j > j_0 \). Then, (A.9), (A.11) and (A.6) imply (A.4). This completes the proof of (A.4) for all cases.

Now, taking the limit superior of (A.4) as \( j \to \infty \) yields (A.5).

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**References**

[1] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[2] E. Aulisa, L. Bloshanskaya, L. Hoang, and A. Ibragimov. Analysis of generalized Forchheimer flows of compressible fluids in porous media. *J. Math. Phys.*, 50(10):103102, 44, 2009.

[3] J. Bear. *Dynamics of Fluids in Porous Media*. Dover, New York, 1972.
E. Celik, L. Hoang, and T. Kieu

[4] H. Darcy. *Les Fontaines Publiques de la Ville de Dijon*. Dalmond, Paris, 1856.

[5] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[6] E. DiBenedetto, U. Gianazza, and V. Vesprì. *Harnack’s inequality for degenerate and singular parabolic equations*. Springer Monographs in Mathematics. Springer, New York, 2012.

[7] J. J. Douglas, P. J. Paes-Leme, and T. Giorgi. Generalized Forchheimer flow in porous media. In *Boundary value problems for partial differential equations and applications*, volume 29 of *RMA Res. Notes Appl. Math.*, pages 99–111. Masson, Paris, 1993.

[8] P. Forchheimer. Wasserbewegung durch Boden. *Zeit. Ver. Deut. Ing.*, 45:1781–1788, 1901.

[9] P. Forchheimer. *Hydraulik*. Number Leipzig, Berlin, B. G. Teubner. 1930. 3rd edition.

[10] L. Hoang and A. Ibragimov. Qualitative Study of Generalized Forchheimer Flows with the Flux Boundary Condition. *Adv. Diff. Eq.*, 17(5–6):511–556, 2012.

[11] L. Hoang, A. Ibragimov, T. Kieu, and Z. Sobol. Stability of solutions to generalized Forchheimer equations of any degree. *IMA Preprint Series #2391*, pages 1–63, April 2012. submitted.

[12] L. Hoang and T. Kieu. Interior estimates for generalized Forchheimer flows of slightly compressible fluids. 2014. submitted, preprint http://arxiv.org/abs/1404.6517.

[13] L. Hoang and T. Kieu. Global estimates for generalized Forchheimer flows of slightly compressible fluids. 2015. submitted, preprint http://arxiv.org/abs/1502.04732.

[14] L. T. Hoang, A. Ibragimov, and T. T. Kieu. One-dimensional two-phase generalized Forchheimer flows of incompressible fluids. *J. Math. Anal. Appl.*, 401(2):921–938, 2013.

[15] L. T. Hoang, A. Ibragimov, and T. T. Kieu. A family of steady two-phase generalized Forchheimer flows and their linear stability analysis. *J. Math. Phys.*, 55:123101, 2014.

[16] L. T. Hoang, T. T. Kieu, and T. V. Phan. Properties of generalized Forchheimer flows in porous media. *J. Math. Sci.*, 202(2):259–332, 2014.

[17] A. V. Ivanov. Second-order quasilinear degenerate and nonuniformly elliptic and parabolic equations. *Trudy Mat. Inst. Steklov.*, 160:285, 1982.

[18] A. V. Ivanov. The regularity theory for \((M,L)\)-Laplacian parabolic equation. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 243(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 28):87–110, 339, 1997.

[19] T. Kieu. Analysis of expanded mixed finite element methods for the generalized Forchheimer flows of slightly compressible fluids. *Numer. Methods Partial Differential Equations*, 2015. accepted, preprint http://arxiv.org/abs/1409.7821.

[20] J. Kinnunen and T. Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. *Mathematische Annalen*, 337(3):705–728, 2007.

[21] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
[22] G. M. Lieberman. *Second order parabolic differential equations.* World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[23] J. J. Manfredi and V. Vespri. Large time behavior of solutions to a class of doubly nonlinear parabolic equations. *Electron. J. Differential Equations*, pages No. 02, approx. 17 pp. (electronic only), 1994.

[24] J. Moser. On a pointwise estimate for parabolic differential equations. *Communications on Pure and Applied Mathematics*, 24(5):727–740, 1971.

[25] M. Muskat. *The flow of homogeneous fluids through porous media.* McGraw-Hill Book Company, inc., 1937.

[26] D. A. Nield and A. Bejan. *Convection in porous media.* Springer-Verlag, New York, fourth edition, 2013.

[27] E.-J. Park. Mixed finite element methods for generalized Forchheimer flow in porous media. *Numer. Methods Partial Differential Equations*, 21(2):213–228, 2005.

[28] B. Straughan. *Stability and wave motion in porous media*, volume 165 of *Applied Mathematical Sciences*. Springer, New York, 2008.

[29] M. D. Surnachev. On improved estimates for parabolic equations with double degeneration. *Tr. Mat. Inst. Steklova*, 278(Differentsialnye Uravneniya i Dinamicheskie Sistemy):250–259, 2012.

[30] M. Tsutsumi. On solutions of some doubly nonlinear degenerate parabolic equations with absorption. *J. Math. Anal. Appl.*, 132(1):187–212, 1988.

[31] J. L. Vázquez. *The porous medium equation.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.

[32] V. Vespri. On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations. *Manuscripta Math.*, 75(1):65–80, 1992.

[33] J. C. Ward. Turbulent flow in porous media. *Journal of the Hydraulics Division, Proc. Am. Soc. Civ. Eng.*, 90(HY5):1–12, 1964.