Invariant Measures and Long Time Behaviour for the Benjamin-Ono Equation III

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Abstract: We complete the program developed in our previous works aiming to construct an infinite sequence of invariant measures of Gaussian type associated with the conservation laws of the Benjamin-Ono equation.

1. Introduction

Our goal here is to complete the program developed in our previous works [10,25,27–29] aiming to construct an infinite sequence of invariant measures of Gaussian type associated with the conservation laws of the Benjamin-Ono equation.

The Benjamin-Ono equation reads

\[ \partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, \quad (1.1) \]

where \( \mathcal{H} \) denotes the Hilbert transform. We consider (1.1) with periodic boundary conditions, i.e., the spatial variable \( x \) is on the one dimensional torus. It is well known (see e.g. [12]) that, at least formally, the solutions of (1.1) satisfy an infinite number of conservation laws of the form

\[ E_{k/2}(u) = \| u \|^{2}_{H^{k/2}} + R_{k/2}(u), \quad k = 0, 1, 2, \ldots, \quad (1.2) \]

where \( R_{k/2} \) is a sum of terms homogeneous in \( u \) of order larger or equal to three (but contains “less derivatives”). Despite this remarkable algebraic property, which indicates that the Benjamin-Ono equation is “integrable”, there are some important analytical difficulties (related to the non local nature of (1.1)) to develop the inverse scattering method for (1.1). In particular we are aware of no reference implementing integrability methods in the context of the periodic Benjamin-Ono equation.

As already noticed in our previous works, the conservation laws (1.2) may be used to construct invariant measures supported by Sobolev spaces of increasing smoothness. This in turn implies some insights on the long time behaviour of the solution of (1.1).
Recall that the idea of using a conserved quantity to construct Gaussian type invariant measures goes back to [11]. It was then further developed by many authors, including [2–10,16,18,21,22,26–30].

Let us now briefly recall the construction of Gaussian type measures associated with (1.2). These measures are absolutely continuous with respect to the Gaussian measure \( \mu \) introduced by

\[
\varphi_{k/2}(x, \omega) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n(\omega)}{|n|^{k/2}} e^{inx},
\]

where, \( (g_n(\omega)) \) is a sequence of centered standard complex Gaussian variables such that \( g_n = \overline{g}_{-n} \) and \( (g_n(\omega))_{n>0} \) are independent. For any \( N \geq 1, k \geq 1 \) and \( R > 0 \) we introduce the function

\[
F_{k/2,N,R}(u) = \left( \prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N u)) \right) \chi_R(\pi_{(k-1)/2}(\pi_N u) - \alpha_N) e^{-R_{k/2}(|\pi_N u|)}
\]

where \( \alpha_N = \sum_{n=1}^{N} \frac{\bar{c}}{n} \) for a suitable constant \( c \), \( \pi_N \) denotes the projector on Fourier modes \( n \) such that \( |n| \leq N \), \( \chi_R \) is a cut-off function defined as \( \chi_R(x) = \chi(x/R) \) with \( \chi: \mathbb{R} \to \mathbb{R} \) a smooth, compactly supported function such that \( \chi(x) = 1 \) for every \( |x| < 1 \).

It is proved in [25,27] that for every \( k \in \mathbb{N} \) with \( k \geq 1 \) there exists a \( \mu_{k/2} \) measurable function \( F_{k/2,R}(u) \) such that \( (F_{k/2,N,R}(u))_{N \geq 1} \) converges to \( F_{k/2,R}(u) \) in \( L^q(d\mu_{k/2}) \) for every \( 1 \leq q < \infty \). This in particular implies that \( F_{k/2,R}(u) \in L^q(d\mu_{k/2}) \).

Set \( d\rho_{k/2,R} \equiv F_{k/2,R}(u)d\mu_{k/2} \). Then we have that

\[
\bigcup_{R>0} \text{supp}(\rho_{k/2,R}) = \text{supp}(\mu_{k/2})
\]

and one may conjecture that \( \rho_{k/2,R} \) is invariant under a well defined flow of (1.1).

This conjecture was proved for \( k = 1 \) in [10] and for \( k \geq 4 \) in [28,29]. Our goal in this paper is to treat the two remaining cases.

**Theorem 1.1.** The measures \( d\rho_{1,R} \) and \( d\rho_{3/2,R} \) are invariant under the flow associated with the Benjamin-Ono equation (1.1) established in [13].

There are two main sources of difficulties to prove the invariance of \( \rho_{k/2,R} \).

The first one, presented only for \( k \geq 2 \), is that even if \( E_{k/2} \) is conserved quantity for (1.1) it is no longer conserved by the approximated versions of (1.1). This difficulty was resolved for \( k \geq 4 \) in [28,29] by introducing an argument exploiting in an essential way the random oscillations of the initial data. This approach however does not quite see the time oscillations coming from the dispersive nature of (1.1). These time oscillations are quantitatively captured by the Bourgain spaces. We refer to [16] where the time oscillations are used in an essential way in a related context, i.e., to resolve the problem coming from the lack of conservation of the approximation versions of the original problem.

The second difficulty is the resolution of the Cauchy problem associated with (1.1) and its approximations on the support of \( \rho_{k/2,R} \). For \( k \geq 4 \) this can be done by standard methods for solving quasilinear hyperbolic PDE’s (the case \( k = 4 \) is slightly more delicate and already appeals to a dispersive effect). For \( k = 2, 3 \) the resolution of (1.1) on the support of \( \rho_{k/2,R} \) is a delicate issue, which is resolved in [13]. The case \( k = 1 \)
is even more delicate and was resolved in [10], which in turn led to the invariance of \( \rho_{1/2,k} \) because in the case \( k = 1 \) the first mentioned difficulty is absent.

In view of the above discussion, we see that the cases \( k = 2, 3 \) combine both difficulties and the aim of the present paper is to solve them simultaneously.

Let us now explain briefly the main novelty in this paper. Recall that in the construction of invariant measures, in the infinite dimensional situation, an important role is played by a suitable and careful choice of a “good” family of finite dimensional approximating problems. In many situations this approximation can be obtained by projection on the Fourier modes with at most \( N \) frequencies, via the sharp Dirichlet projectors \( \pi_N \), and letting \( N \to \infty \). It is, however, not quite clear whether in the case of the Benjamin-Ono equation at low level of regularity this family of finite dimensional problems approximate well the true solution of the Benjamin-Ono equation. To overcome this difficulty, we use the idea of [7,8,10] and we project the equation by using a family of smoothed projectors \( S^\epsilon_N \) with \( \epsilon > 0 \) and \( N \in \mathbb{N} \) (see the next paragraph). However, in contrast with the case treated in [10], where it is sufficient to work with a fixed parameter \( \epsilon \) and letting \( N \to \infty \), in the situation we face in this paper, it is important to consider both \( \epsilon \to 0 \) and \( N \to \infty \), which requires considerable care. In particular, it is of crucial importance that in Proposition 6.1 below, we have a bound proportional to \( t \) that enables us to glue local bounds on time intervals with very poor dependence on \( \epsilon \). In other words, the fact that we do not sacrifice any time integration and that we only exploit the random oscillations of the initial data in the estimates on the measure evolution is of importance for the analysis in this paper.

One may wish to ask whether it is possible get an invariant measure under a Benjamin-Ono flow in the case \( k = 0 \), i.e., starting from the \( L^2 \) conservation law. In this case the resulting measure is induced by the white noise (on the circle) and in view of the ill-posedness result of [14], the construction of a well-defined dynamics meets serious difficulties. On the other hand, in the case of the KdV equation, it is known that the white noise is invariant under a well-defined dynamics, see [17,19,20]. Proving the analogous result for the Benjamin-Ono equation is a challenging problem, which seems beyond our current understanding of the Benjamin-Ono equation.

We conclude this introduction by fixing some notations. We denote by \( \mathcal{B}(X) \) the Borel sets of the topological space \( X \) and by \( \mathcal{B}_M(Y) \) the ball of radius \( M \), centered at the origin of a Banach space \( Y \). For every fixed \( \epsilon \in (0,1) \) we denote by \( \psi_\epsilon \) a smooth function \( \psi_\epsilon: \mathbb{R} \to \mathbb{R} \) such that
\[
\psi_\epsilon(x) = 1 \quad \text{for } x \in [0,(1-\epsilon)], \quad \psi_\epsilon(x) = 0 \quad \text{for } x > 1, 
\]
\[
\|\psi_\epsilon\|_{L^\infty} = 1 \quad \text{and} \quad \psi_\epsilon(x) = \psi_\epsilon(|x|).
\]
We denote by \( S^\epsilon_N \) the Fourier multiplier:
\[
S^\epsilon_N(\sum_{j \in \mathbb{Z}} a_je^{ijx}) = \sum_{j \in \mathbb{Z}} a_j \psi_\epsilon\left(\frac{j}{N}\right)e^{ijx}. 
\]
We also denote by \( \Phi(t) \) the flow associated with (1.1) (well-defined on \( H^s, s \geq 0 \) thanks to [13], see also [15] for simpler proof) and by \( \Phi^\epsilon_N(t) \) the flow on \( H^s, s \geq 0 \) associated with
\[
\partial_t u + \mathcal{H} \partial_x^2 u + S^\epsilon_N(S^\epsilon_Nu \cdot S^\epsilon_Nu_x) = 0. 
\]
Since the \( x \) mean value is conserved by the flow of (1.1), we shall only consider solutions of (1.1) and its approximated version (1.7) with vanishing zero Fourier mode (this is the case in (1.3) as well).
2. Deterministic Theory

In this section, for a function \( u \) defined on \( \mathbb{T} \) or \( \mathbb{T} \times \mathbb{R} \), we denote
\[
\hat{u}(n), \quad u_{n, \tau} := (\mathcal{F}_{x,t}u)(n, \tau)
\]
for \( n \in \mathbb{Z} \) and \( \tau \in \mathbb{R} \). Moreover, we define the following notations: \( \mathcal{N}u \) is such that \( (\mathcal{N}u)_{n, \tau} = |u_{n, \tau}| \), and \( u^{(q,c)} \) is such that
\[
(u^{(q,c)})_{n, \tau} = \langle n \rangle^q \langle \tau - |n|n \rangle^c u_{n, \tau}.
\]
We also define the projectors \( \pi_0^> \) and \( \pi_0^< \) for functions on \( \mathbb{T} \) in the standard way. In this section we prove the following deterministic result.

**Proposition 2.1.** Let \( 0 < \epsilon < 1, \sigma > \sigma' > 0 \) and \( A > 0 \) be fixed, so that \( \sigma \) is small enough. We have, for some \( T = T(\epsilon, \sigma, \sigma', A) > 0 \) and \( C = C(\epsilon, \sigma, \sigma', A) > 0 \) that:
\[
\sup_{t \in \mathbb{R}} \sup_{|t| \leq T} \| \Phi_N(t)\phi - \Phi(t)\phi \|_{H^{1/2-\sigma}} \leq CN^{-\theta},
\]
where \( \theta = \theta(\sigma, \sigma') > 0 \).

**Definition 2.1.** Let the standard \( X_{s,b} \) space be defined by
\[
\| u \|_{X_{s,b}}^2 = \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \langle \tau - |n|n \rangle^{2b} |(\mathcal{F}_{x,t}u)(n, \tau)|^2 d\tau,
\]
and the \( Y_s \) space be
\[
\| u \|_{Y_s}^2 = \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left( \int_{\mathbb{R}} |(\mathcal{F}_{x,t}u)(n, \tau)| d\tau \right)^2.
\]
We then define the spaces \( U \) and \( V \) by
\[
\| u \|_U = \| u \|_{X^{-1/2-\sigma, 12/25}} + \| u \|_{Y^{1/2-\sigma}} \tag{2.1}
\]
and
\[
\| u \|_V = \| u \|_{X^{-1/2-\sigma', 12/25}} + \| u \|_{Y^{1/2-\sigma'}}. \tag{2.2}
\]
The space \( U_T \) is then defined by
\[
\| u \|_{U_T} = \sup \{ \| v \|_U : v|_{[-T,T]} = u|_{[-T,T]} \},
\]
while \( V_T \) and \( X_{s,b,T} \) are defined similarly.

In the proof below we will denote
\[
s = 1/2 - \sigma, \quad s' = 1/2 - \sigma', \quad r = 1/2 + \sigma = 1 - s.
\]
The norms we will control in the bootstrap estimate are \( X_{s',r,T} \) and \( X_{s,r,T} \) for the gauged function \( w \), and \( V_T \) and \( U_T \) for the original function \( u \).
2.1. Linear bounds. The content of this section is well-known. We just record it here for the reader’s convenience. In the sequel $\chi \in C_0^\infty(\mathbb{R})$ is a fixed function such that $\chi(t) = 1$ for $|t| < 1$, $\chi(t) = 0$ for $|t| > 2$.

**Proposition 2.2.** We have the following bounds for any $\rho$.

1. **the Strichartz estimates:**
   \[ \|u\|_{L_t^4 L_x^{3/8}} \lesssim \|u\|_{X^{0,3/8}}, \]
   and
   \[ \|u\|_{L_t^6 L_x^6} \lesssim \|u\|_{X^{\sigma,r}}. \]

2. **linear evolution bound:**
   \[ \|\chi(t) e^{-t\dd_x^2} \phi\|_{X^{\rho,b} \cap Y^\rho} \lesssim \|\phi\|_{H^\rho}, \]
   and
   \[ \|\mathcal{E}u\|_{X^{\rho,b}} \lesssim \|u\|_{X^{\rho,b-1}}, \quad 1/2 < b < 1, \]
   where the Duhamel operator is defined by
   \[ \mathcal{E}u(t) = \chi(t) \int_0^t \chi(t') e^{(t-t')\dd_x^2} u(t') \, dt'. \]

3. **the short-time bound:** for $T \leq 1$ and $-1/2 < b \leq b' < 1/2$ we have
   \[ \|\chi(T^{-1}t) u\|_{X^{\rho,b}} \lesssim T^{b'-b} \|u\|_{X^{\rho,b'}}. \]

4. **fixed-time estimate:**
   \[ \sup_t \|u(t)\|_{H^\rho} \lesssim \min(\|u\|_{X^{\rho,r}}, \|u\|_{Y^\rho}). \]

5. **if $u \in X^{\rho,b}$ with either $0 < b < 1/2$ or $1/2 < b < 1$ and $u(0) = 0$, then**
   \[ \lim_{T \to 0} \|\chi(T^{-1}t) u\|_{X^{\rho,b}} = 0. \]
   The same holds for $Y^b$.

**Proof.** Properties (1)–(4) are well-known properties of $X^{s,b}$ spaces, see [24]. The proof of (5) is basically contained in [9], but we include it here for the sake of completeness.

First, the limit result holds if $u$ is moreover assumed to be Schwartz (by checking the $X^{1,1}$ norm when $u(0) = 0$, say). Since $u$ can always be approximated by Schwartz functions (retaining the condition $u(0) = 0$ when $b > 1/2$), we only need to prove that multiplying by $\chi(T^{-1}t)$ is uniformly bounded. This is true for $Y^{s'}$ (by Hölder) and $X^{\rho,b}$ for $b < 1/2$ (by part (3) above), so we assume $1/2 < b < 1$. Let $v(t) = e^{t\dd_x^2} u(t)$, then
   \[ \|u\|_{X^{\rho,b}} = \|v\|_{H^\rho \cap H^b}, \]
   so we only need to prove that $f(t) \mapsto \chi(T^{-1}t) f(t)$ is uniformly bounded in $H^b$ under the assumption $f(0) = 0$. 
Now boundedness in $L^2$ is trivial, so we will consider boundedness in $\dot{H}^b$, and by scaling we can set $T = 1$. Decompose $f = f_1 + f_2$ where

$$\hat{f}_1(\tau) = 1_{|\tau| \geq 1} \cdot \hat{f}(\tau) - \frac{1}{2} \int_{|\lambda| \geq 1} \hat{f}(\lambda) \, d\lambda \cdot 1_{|\tau| \leq 2},$$

then $\hat{f}_1$ is supported in $|\tau| \geq 1$, $\hat{f}_2$ is supported in $|\tau| \leq 2$, and both $\hat{f}_i$ have integral zero. Moreover, since $\int_{|\lambda| \geq 1} |\hat{f}(\lambda)| \, d\lambda \lesssim \|f\|_{\dot{H}^b}$ by Hölder, we know that $\|f_i\|_{\dot{H}^b} \lesssim \|f\|_{\dot{H}^b}$.

Since multiplying by $\chi$ is bounded on $H^b$ (by interpolation between $L^2$ and $H^1$), we know that $\|\chi f_1\|_{\dot{H}^b} \lesssim \|f_1\|_{H^b} \sim \|f\|_{\dot{H}^b}$; for $f_2$ we have

$$\hat{\chi f_2}(\tau) = \int_{-2}^{2} (\hat{\chi}(\tau - \lambda) - \hat{\chi}(\tau)) \hat{f}_2(\lambda) \, d\lambda,$$

and by Hölder we have $|\hat{\chi f}_2(\tau)| \lesssim \langle \tau \rangle^{-5} \|f_2\|_{\dot{H}^b}$, so the desired bound follows.

2.2. The gauge transform. The gauge transform, which weakens the impact of the derivative loss, was introduced for the Benjamin-Ono equation by Tao [23] and refined, in the context of truncated equations, by the first author [10]. The computation here is a much simplified version of that in [10]. We will use the notation from [10], namely that

$$m_{ij} = m_i + \cdots + m_j$$

for $i \leq j$.

In this section we are fixing an $\epsilon$ and an $N$, so we will denote $S_N^\epsilon$ simply by $S$ and $\psi^\epsilon$ by $\psi$. Let $u = \Phi_N^\epsilon(t)(\phi)$ be the global solution to (1.7) (with initial data $\phi$ of zero mean value), and consider the operators

$$P : g \mapsto (S\partial_x^{-1}u) \cdot g, \quad Q : g \mapsto (Su) \cdot g,$$

where $\partial_x^{-1}u$ is the unique mean-zero antiderivative of $u$. By abusing notation we will also call these operators $S\partial_x^{-1}u$ and $Su$. The exponential

$$M = \exp \left( \frac{i}{2} SP S \right) = \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} (i/2)^{\lambda} (SP S)^{\lambda}$$

is defined as a power series; we will then define

$$v = Mu; \quad w = \pi_{>0}(Mu).$$

The goal in this section is to prove the following.
Lemma 2.1. We have the evolution equation

\[(\partial_t - i\partial_{xx})w = N_2 + N_3,\]

where

\[(N_2)_{n_0} = \sum_{\lambda \geq 0} C_{\lambda} \sum_{n_0 = n_1 + n_2 + \lambda n_1} \Lambda_2 \cdot \nu_{n_1} \nu_{n_2} \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i},\]

and

\[(N_3)_{n_0} = \sum_{\lambda \geq 0} C_{\lambda} \sum_{n_0 = n_1 + n_2 + n_3 + \lambda n_1} \Lambda_3 \cdot \nu_{n_1} \nu_{n_2} \nu_{n_3} \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i}.\]

Here \(|C_{\lambda}| \leq C^\lambda / \lambda!\), \(\nu\) represents either \(w\) or \(\bar{w}\), and the weights are

\[\Lambda_j = \Lambda_j(n_0, \ldots, n_j, m_1, \ldots m_\lambda), \quad j \in \{2, 3\}\]

such that

\[|\Lambda_2| \lesssim \min(|n_0|, |n_1|, |n_2|), \quad |\Lambda_3| \lesssim 1.\]

All the implicit constants here are allowed to depend on \(\epsilon\).

Proof. Below, we will use \(C_{\lambda}\) to denote any quantity bounded by \(C^\lambda / \lambda!\). The evolution equation satisfied by \(w\) can be computed as follows:

\[\begin{aligned}
(\partial_t - i\partial_{xx})w &= \pi_{>0}(M(\partial_t - i\partial_{xx})u + \pi_{>0}[\partial_t, M]u - i\pi_{>0}[\partial_{xx}, M]u) \\
&= -2i\pi_{>0}(M\pi_{<0}u_{xx} + \pi_{>0}([\partial_t, M]u - i[\partial_t, [\partial_t, M]]u) \\
&\quad + \pi_{>0}(-M\mathcal{S}u \cdot Su_x - 2i[\partial_t, M]u_x) \\
&= -2i\pi_{>0}\partial_x(M\pi_{<0}u_x) \\
&\quad -2i\pi_{>0}([\partial_x, M] - \frac{i}{2}M(S\mathcal{Q}S))u_x \\
&\quad + 2i\pi_{>0}[\partial_x, M]\pi_{<0}u_x + \pi_{>0}([\partial_t, M] - i[\partial_t, [\partial_t, M]])u.
\end{aligned}\]

(2.4)

Expanding \(M\) as a power series and expressing it in Fourier space, we see that the term in (2.4) has the form \(\mathcal{M}_2\), where

\[\begin{aligned}
(\mathcal{M}_2)_{n_0} &= \sum_{\lambda \geq 0} C_{\lambda} \sum_{n_0, n_1, n_2 > n_1} \Phi \cdot (n_0 n_1 u_{n_1}) \cdot \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i},
\end{aligned}\]

and \(\Phi = \Phi(n_0, n_1, m_1, \ldots, m_\lambda)\) is a bounded factor. Notice that \(^1\)

\[|m_i| \gtrsim |n_0| + |n_1|\]

for at least one \(i\) (recall that \(m_i \neq 0\), so \(|m_i|\) is equivalent to \(|\langle m_i \rangle|\)), we can rearrange the indices and rewrite this term as

\(^1\) Actually this should be \(|m_i| \gtrsim (\lambda + 1)^{-1}(|n_0| + |n_1|)\), but this additional factor does not affect the proof; the same happens below.
\((M_2)_{n_0} = \sum_{\lambda \geq 0} C_{\lambda} \sum_{n_1+n_2+m_1\lambda=n_0} \Lambda_2 \cdot u_{n_1} u_{n_2} \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i},\)

where \(\Lambda_2\) verifies the bound

\[|\Lambda_2| \leq \min(\langle n_0 \rangle, \langle n_1 \rangle, \langle n_2 \rangle).\]

Next, to analyze the term in (2.5), notice that \([\partial_x, SPS] = SQS,\) we have that

\[\partial_x M = \frac{i}{2} M(SQS) = R,\]

where

\[R = \sum_{\lambda, \mu \geq 0} \frac{1}{(\lambda + \mu + 1)!} (i/2)^{\lambda+\mu+1} (SPS)^{\lambda} [SQS, (SPS)^{\mu}].\]

Note that

\([SPS, SQS] = SP[S^2, Q]S + SQ[P, S^2]S,\]

we have

\[R_{ux} = \sum_{\lambda, \mu \geq 0} \frac{\lambda + 1}{(\lambda + \mu + 2)!} (i/2)^{\lambda+\mu+2} (SPS)^{\lambda} (SP[S^2, Q]S + SQ[P, S^2]S)(SPS)^{\mu} \partial_x u.\]

We will only consider the term \(SP[S^2, Q]S\) in the parenthesis, since the other one is similar. First commute \(\partial_x\) and \((SPS)^{\mu}\) to move \(\partial_x\) left; since \([\partial_x, P] = Q,\) the term containing the commutator will be of form

\((\text{Remainder})_{n_0} = \sum_{\lambda \geq 0} C_{\lambda} \sum_{n_1+n_2+n_3+m_1\lambda=n_0} \Lambda_3 \cdot u_{n_1} u_{n_2} u_{n_3} \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i},\)

with \(|\Lambda_3| \lesssim 1,\) thus this remainder is in \(N_3.\)

After commuting, we may now omit the summation in \(\lambda\) and \(\mu\) since they only contribute to \(C_{\lambda};\) let \(v = S(SP)^{\mu} u,\) we only need to consider

\((SPS)^{\lambda} SP[S^2, Q] \partial_x v;\)

actually we only need to consider \([S^2, Q] \partial_x v,\) since applying \((SPS)^{\lambda} SP\) to any term with form \(M_2\) or \(N_3\) does not change its form. Noting that

\([S^2, Q] \partial_x v = i \sum_{n_0=n_1+m_1} n_1 (\psi^2(n_0/N) - \psi^2(n_1/N)) \psi(m_1/N) u_{m_1} v_{n_1},\]

and plugging in the expression of \(v_{n_1}\) in terms of \(u\) and renaming \(m_1\) by \(n_3,\) we obtain that

\([S^2, Q] \partial_x v = i \sum_{n_0=n_2+n_3+m_1\mu} \Phi(n_0, n_2, n_3, m_1, \ldots, m_\mu) u_{n_2} u_{n_3} \prod_{i=1}^{\mu} \frac{u_{m_i}}{m_i},\)

where \(\Phi\) is defined as in (2.3).
where

\[
\Phi(n_0, n_2, n_3, m_1, \ldots, m_\mu) = (n_0 - n_3) \left( \frac{n_0}{N} - \psi^2 \left( \frac{n_0 - n_3}{N} \right) \right) \\
\times \psi \left( \frac{n_2}{N} \right) \psi \left( \frac{n_3}{N} \right) \prod_{i=1}^{\mu} \psi^2 \left( \frac{n_2 + m_1 i}{N} \right) \prod_{i=1}^{\mu} \psi \left( \frac{m_i}{N} \right).
\]

In this sum, if \(|m_i| \gtrsim |n_2|\) for some \(i \geq 2\), then we may also assume

\[
|m_i| \gtrsim |n_2 + m_1\mu| = (\mu + 1)^{-1}|n_0 - n_3|,
\]

thus we have a term of form \(\mathcal{N}_3\); otherwise we have \(|n_0 - n_3| \sim |n_2|\). If \(|n_0| \gtrsim \min(|n_2|, |n_3|)\), notice that

\[
|n_0 - n_3| \cdot \left| \psi^2 \left( \frac{n_0}{N} \right) - \psi^2 \left( \frac{n_0 - n_3}{N} \right) \right| \lesssim \frac{|n_2 n_3|}{N} \lesssim \min(|n_2|, |n_3|)
\]

(with implicit constant depending on \(\epsilon\)), we obtain a term of form \(\mathcal{M}_2\). Similarly if \(|m_i| \gtrsim \min(|n_2|, |n_3|)\) for some \(i\), we will have a term in \(\mathcal{N}_3\). Finally, if \(|n_0| \ll \min(|n_2|, |n_3|)\) and \(|m_i| \ll \min(|n_2|, |n_3|)\), then we may symmetrize with respect to \(n_2\) and \(n_3\). Notice that in this situation we have

\[
\Phi(n_0, n_2, n_3, m_1, \ldots, m_\mu) = -n_3 \left( 1 - \psi^2 \left( \frac{n_3}{N} \right) \right) \psi^{2\mu+2} \left( \frac{n_2}{N} \right) + O \left( |n_0| + \sum_{i=1}^{\mu} |m_i| \right)
\]

by mean value theorem, we can symmetrize and obtain

\[
\Phi(n_0, n_2, n_3, m_1, \ldots, m_\mu) + \Phi(n_0, n_3, n_2, m_1, \ldots, m_\mu) = O \left( |n_0| + \sum_{i=1}^{\mu} |m_i| \right),
\]

where the constant again depends on \(\epsilon\). Thus we obtain a term of the form \(\mathcal{M}_2\) or \(\mathcal{N}_3\).

Next we consider the second term in (2.6). Recall from Leibniz rule that

\[
[\partial_x, M] = \sum_{\lambda, \mu \geq 0} \frac{1}{(\lambda + \mu + 1)!} (i/2)^{\lambda+\mu+1} (SPS)^\lambda (SQS) (SPS)^\mu;
\]

we will again omit the summation in \(\lambda\) and \(\mu\). If we then commute this with \(\partial_x\) again and the commutator hits one \(SPS\) factor, we will get a term of form \(\mathcal{N}_3\). Therefore, let

\[
y = \partial_x \partial_x^{-1} u - i \partial_x u,
\]

we only need to consider the contribution

\[
\pi_{>0}(SPS)^\lambda (S(Sy)S)(SPS)^\mu u.
\]

(2.7)

We have from (1.7) that

\[
y = -2i\pi_{<0} u_x - \frac{1}{2} \pi_{<0} S(Su)^2.
\]
Plugging the second component of $y$ into (2.7) yields a term of form $\mathcal{N}_3$; for the term obtained by plugging in the first component, we will combine it with the first term of line (2.6) to obtain

\[
\mathcal{N} := \pi > 0 (SPS)^{\lambda} S[(Su) S(PS)]^{\mu} (\pi < 0 u_x) - (S\pi < 0 u_x) S(PS)^{\mu} u].
\]

In Fourier space, this term has the expression

\[
(\mathcal{N})_{n_0} := \sum_{n_0 = n_1 + n_2 + m_1, n_0 > 0 > n_2} \Lambda \cdot u_{n_1} u_{n_2} \prod_{i=1}^{\tau} \frac{u_{m_i}}{m_i},
\]

where $\tau = \lambda + \mu$, and

\[
\Lambda = \Lambda_1 \cdot n_2 \left( \prod_{i=1}^{\mu} \psi^2 \left( \frac{n_2 + m_\mu}{N} \right) - \prod_{i=1}^{\mu} \psi^2 \left( \frac{n_1 + m_\mu}{N} \right) \right),
\]

$\Lambda_1$ being a product of $\psi$ factors. Clearly $\Lambda$ is nonzero only if all variables are $\lesssim N$, and using mean value theorem (and the fact that $\psi$ is even) we have

\[
|\Lambda| \lesssim \frac{|n_2|}{N} \left( |n_0| + \sum_{i=1}^{\tau} |m_i| \right) \lesssim \min(|n_0|, |n_2|) + \sum_{i=1}^{\tau} |m_i|
\]

with constants depending on $\epsilon$. Decomposing $\Lambda = \Lambda' + \Lambda''$ according to the two components above, then the term with $\Lambda''$ has form $\mathcal{N}_3$; for the term with $\Lambda'$, it would also have form $\mathcal{N}_3$ provided $|m_i| \gtrsim \min(|n_0|, |n_2|)$ for some $i$, and when $|m_i| \ll \min(|n_0|, |n_2|)$ for all $i$, we must have $|n_1| \gtrsim \min(|n_0|, |n_2|)$ since $n_0 n_2 < 0$, so this term has form $\mathcal{M}_2$.

Finally, we need to transform an $\mathcal{M}_2$ term, which has the form

\[
\sum_{n_1 + n_2 + m_{1\lambda} = n_0} \Lambda_2 \cdot u_{n_1} u_{n_2} \prod_{i=1}^{\lambda} \frac{u_{m_i}}{m_i},
\]

into an $\mathcal{N}_2$ term. Recall that

\[
u = M^{-1} v = \sum_{\lambda = 0}^{\infty} \frac{1}{\lambda!} (-i/2)^\lambda (SPS)^{\lambda} v,
\]

we have that for positive $n_1$,

\[
u_{n_1} = \sum_{\lambda \geq 0} C_\lambda \sum_{n_1 = n_3 + m_{1\lambda}} \Phi \cdot (\Phi)_{n_3} \prod_{i=1}^{\lambda} \frac{u_{m_i'}}{m_i'},
\]

where $m_{ij}'$ are defined similarly, and $|\Phi| \lesssim 1$. Since $u = \bar{u}$, for negative $n_1$ we have

\[
u_{n_1} = \sum_{\lambda \geq 0} C_\lambda \sum_{n_1 = n_3 + m_{1\lambda}} \Phi \cdot (\bar{\Phi})_{n_3} \prod_{i=1}^{\lambda} \frac{u_{m_i'}}{m_i'}.
\]

Clearly we may do the same for $n_2$. If $\langle m_{ij}' \rangle \ll \langle n_1 \rangle$ for all $i$, then we have $\langle n_1 \rangle \sim \langle n_3 \rangle$ and $n_1 n_3 > 0$, so that one can replace $v$ by $w$ and $\bar{v}$ by $\bar{w}$; if this is also true for $n_2$, then we already have an $\mathcal{N}_2$ term. Otherwise assume $\langle m_{ij}' \rangle \gtrsim \langle n_1 \rangle$, then the $(m_{ij}')^{-1}$ factor cancels the $\Lambda$ weight, thus by writing $v = M u$ and expanding again, we can reduce this to a term of form $\mathcal{N}_3$. This completes the proof. \[\square\]
2.3. The bootstrap estimate. In this section we prove the main a priori estimate, namely the following

**Proposition 2.3.** Let $\epsilon$ and $A$ be fixed. For each $N$, let $u^N$ be the solution to (1.7), with initial data $u^N(0) = \phi$, where $\|\phi\|_{H^s} \leq A$. If $N = \infty$ we assume $u^\infty$ solves (1.1). Moreover, let $w^N$ and $w^\infty$ be the corresponding gauge transforms. Then, when $T$ is small enough depending on $\epsilon$ and $A$, we can find some functions $\tilde{u}^N$, $\tilde{w}^N$ and $\tilde{u}^\infty$, $\tilde{w}^\infty$ on $[-T, T]$, such that

$$
\|\tilde{u}^N\|_V + \|\tilde{u}^\infty\|_V + N^\theta \|\tilde{u}^N - \tilde{u}^\infty\|_U \lesssim \epsilon, A
$$

and

$$
\|\tilde{w}^N\|_{X^s,r} + \|\tilde{w}^\infty\|_{X^s,r} + N^\theta \|\tilde{w}^N - \tilde{w}^\infty\|_{X^s,r} \lesssim \epsilon, A,$$

where $\theta > 0$ is some constant independent of $N$.

Noticing that

$$
\sup_t \|u(t)\|_{H^s} \lesssim \|u\|_U,
$$

which follows from Proposition 2.2, we can see that Proposition 2.1 is a consequence of Proposition 2.3.

**Proof.** We only prove the bound for $u^N$, since the bound for $u^\infty$ follows from a similar (and much easier) estimate. The bounds for the difference $u^N - u^\infty$ follows from a standard procedure of taking differences, which is briefly described at the end of the proof. Therefore, from now on we will fix one $N$ and drop all superscripts $N$.

Also the arguments in this part will involve only the $V$ and $X^s,r$ norms.

In order to initiate the bootstrap, the first step is to bound the norm $\|u\|_{V_T}$ and $\|w\|_{X^s,r,T}$ for very small $T$. In fact, let

$$
u^\circ(t) = \chi(t)e^{-iH_{d_x}}u(0) + \chi(T^{-1}t)(u(t) - \chi(t)e^{-iH_{d_x}}u(0))$$

be some function that coincides with $u$ on $[-T, T]$, then from parts (2) and (5) of Proposition 2.2, we have that

$$
\|u^\circ\|_V \lesssim \|u(0)\|_{H^s} \lesssim A
$$

for $T$ sufficiently small. In the same way, the bound for $\|w\|_{X^s,r,T}$ would follow if we have $\|w(0)\|_{H^s} \lesssim A$. Now from the expression of the gauge transform we know that

$$
|w(0)_{n_0}| \lesssim \sum_{\mu \geq 0} \frac{C}{u} \sum_{n_0 = n_1 + m_1\mu} |u(0)_{n_1}| \cdot \prod_{i=1}^\mu \frac{|u(0)_{m_i}|}{m_i}.
$$

Let the sum over $m_i$ be $y_{n_0 - n_1}$, then we have

$$
\sum_l \langle l \rangle^{3/4} |y_l| \lesssim \sum_{m_1, \ldots, m_\mu} \prod_{i=1}^\mu \langle m_i \rangle^{-1/4} |u(0)_{m_i}| \lesssim \|u(0)\|_{H^{1/4+\epsilon}} \lesssim (OA(1))\mu,
$$
and $s' < 1/2$, so by Minkowski we have

$$\|w(0)\|_{H^{s'}} \lesssim \sum_{l} |y_l| \cdot \left( \sum_{n} (n + l)^{2s'} |u(0)_n|^2 \right)^{1/2} \lesssim_A 1.$$  

Suppose that we have constructed some $\tilde{w}$ and $\tilde{u}$ for some time $T$ (which we may assume to be supported in $|t| \lesssim 1$), satisfying

$$\|\tilde{u}\|_V + \|\tilde{w}\|_{X^{s', r}} \leq A'$$

with some large $A'$ depending on $A$. We now need to improve these inequalities, with the same $T$, provided that $T \ll_{\epsilon', A'} 1$. We will first construct a new $\tilde{w}$, and this is done simply using the Eq. (2.3). We will define

$$w^* = \chi(t)e^{it\partial_x} w(0) + \mathcal{E}(\chi(T^{-1}t)) \cdot (N_2 + N_3),$$

where $\mathcal{E}$ is the Duhamel operator as in Proposition 2.2, and $N_2$ and $N_3$ are constructed using $\tilde{w}$ and $\tilde{u}$. Using Proposition 2.2, we now only need to bound

$$\| (\chi(T^{-1}t)N_2 \|_{X^{s', r-1}} + \| (\chi(T^{-1}t)N_3 \|_{X^{s', r-1}}.$$  

To bound $N_3$, using duality, let $h$ be a function such that $\|h\|_{X^{-s', 1-r}} \leq 1$, so for $\tilde{h} = \chi(T^{-1}t)h$ we have

$$\|\tilde{h}\|_{X^{-s', 1/2-2\sigma}} \lesssim T^\theta$$

for some positive $\theta$. Now we only need to bound

$$J = \sum_{\mu} C_\mu \sum_{n_0 = n_{13} + m_{1}\mu} \int_{\xi_0 = \xi_{13} + \eta_{1\mu}} A_3 \cdot \tilde{h}_{n_0, \xi_0} \bar{u}_{n_1, \xi_1} \bar{u}_{n_2, \xi_2} \bar{u}_{n_3, \xi_3} \prod_{i=1}^\mu \frac{\bar{u}_{m_i, \eta_i}}{m_i} d\Xi, \quad (2.8)$$

where $d\Xi$ denotes the surface measure on the hyperplane $\xi_0 = \xi_{13} + \eta_{1\mu}$. Below we will omit the summation in $\mu$, since the factor $C_\mu$ is always harmless; we also abbreviate the summation by $\sum_{(n,m)}$, and integration by $\int_{(\xi,\eta)}$.

Note that we must have $\langle n^0 \rangle \lesssim \langle n^1 \rangle$ or $\langle n^0 \rangle \lesssim \langle m^1 \rangle$ for some $i$; by symmetry we may assume $i = 1$. If $\langle n^0 \rangle \lesssim \langle n^1 \rangle$, we have

$$|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle n^0 \rangle^{s'} \langle n^1 \rangle^{-s'} (\mathcal{N}\bar{u}(-s',0))_{n_0, \xi_0} (\mathcal{N}\bar{u}(s',0))_{n_1, \xi_1} (\mathcal{N}\bar{u})_{n_2, \xi_2} (\mathcal{N}\bar{u})_{n_3, \xi_3} \prod_{i=1}^\mu (\mathcal{N}\bar{u}(-1,0))_{m_i, \eta_i} d\Xi,$$

which is then bounded (using Hölder) by

$$\|\mathcal{N}\bar{u}(-s',0)\|_{L^4_{t,x}} \|\mathcal{N}\bar{u}(s',0)\|_{L^2_{t,x}} \|\mathcal{N}\bar{u}\|_{L^8_{t,x}}^2 \cdot \|\mathcal{N}\bar{u}(-1,0)\|_{L^\infty_{t,x}}^{\mu} \lesssim T^\theta (O_A'(1))^{\mu},$$
using Proposition 2.2, the bound for $\tilde{h}$, and bootstrap assumption on $\tilde{u}$. Now if $\langle n_0 \rangle \lesssim \langle m_1 \rangle$ then
\[
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle n_0 \rangle^{\xi} \langle m_1 \rangle^{-\xi} \langle \mathcal{N}\tilde{u}(s',0) \rangle_{n_0,\xi_0} \langle \mathcal{N}\tilde{u}(1+s',0) \rangle_{m_1,\eta_1}
\]
\[
\prod_{j=1}^{3} (\mathcal{N}\tilde{u})_{n_j,\xi_j} \prod_{j=2}^{\mu} (\mathcal{N}\tilde{u}(-1,0))_{m_j,\eta_j} \, d\Xi,
\]
which is then bounded (using Hölder) by
\[
\|\mathcal{N}\tilde{u}(s',0)\|_{L_{t,x}^4} \|\mathcal{N}\tilde{u}(1+s',0)\|_{L_{t,x}^\infty} \|\mathcal{N}\tilde{u}\|_{L_{t,x}^4}^{3} \cdot \|\mathcal{N}\tilde{u}(-1,0)\|_{L_{t,x}^\infty}^{\mu} \lesssim T^\theta (O_{A'}(1))^\mu.
\]

Now consider $N_2$. Again we only need to consider a sum of type
\[
J = \sum_{n_0=n_1+n_2+n_1,\mu} \int_{\xi_0=\xi_1+\xi_2+\eta_1,\mu} \Lambda_{2,\tilde{h}} \cdot \tilde{n}_{n_0,\xi_0} \tilde{v}_{n_1,\xi_1} \tilde{v}_{n_2,\xi_2} \prod_{i=1}^{\mu} \tilde{u}_{m_i,\eta_i} \, d\Xi,
\]
where $\|h\|_{X_{s',1-r}} \leq 1$. Moreover, we may assume in the summation that $\min \langle n_j \rangle \gg \max \langle m_i \rangle$, since otherwise we can bound this term in the same way as $N_3$ (recall that $\nu \in \{w, \bar{w}\}$ and enjoys a better estimate then $u$). We can now check algebraically that
\[
\max_i (\langle \xi_i - n_i | n_i \rangle, \langle \eta_i - m_i | m_i \rangle) \gtrsim |n_0|n_0| - n_1|n_1| - n_2|n_2| - \sum_{i=1}^{\mu} m_i|m_i| \gtrsim PQ,
\]
where $P = \max \langle n_j \rangle$ and $Q = \min \langle n_j \rangle$ satisfies
\[
\langle n_0 \rangle^{\xi} \langle n_1 \rangle^{-s' + \sigma} \langle n_2 \rangle^{-s' + \sigma} \min_{0 \leq j \leq 2} \langle n_j \rangle \lesssim P^\sigma Q^{1/2 + 2\sigma} \lesssim (PQ)^{13/50}.
\]
This means that, there must be some $i$, such that $\langle \xi_i - n_i | n_i \rangle \gtrsim PQ$, or $\langle \eta_i - m_i | m_i \rangle \gtrsim PQ$.

If $\langle \eta_i - m_i | m_i \rangle \gtrsim PQ$ for some $i$ (say $i = 1$), then we can bound
\[
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle n_0 \rangle^{\xi} \langle n_1 \rangle^{-s' + \sigma} \langle n_2 \rangle^{-s' + \sigma} \min_{0 \leq j \leq 2} \langle n_j \rangle \cdot \langle \eta_1 - m_1 | m_1 \rangle^{-13/50}
\]
\[
\times (\mathcal{N}\tilde{u}(s',0))_{n_0,\xi_0} (\mathcal{N}\tilde{u}(s',0))_{n_1,\xi_1} (\mathcal{N}\tilde{u}(s' - \sigma,0))_{n_2,\xi_2} (\mathcal{N}\tilde{u}(-1,13/50))_{m_1,\eta_1}
\]
\[
\prod_{j=2}^{\mu} (\mathcal{N}\tilde{u}(-1,0))_{m_j,\eta_j} \, d\Xi.
\]
Notice that $\mathcal{N}\tilde{u}(-1,0) \in L_{t,x}^\infty$, and that
\[
\mathcal{N}\tilde{u}(s',0) \in X^{0,1/2 - 2\sigma} \subset L_{t,x}^4, \quad \mathcal{N}\tilde{u}(s' - \sigma,0) \in X^{\sigma,r} \subset L_{t,x}^6, \quad \mathcal{N}\tilde{u}(-1,13/50)
\]
\[
eq X^{1/2 - \sigma,11/50} \subset L_{t,x}^{12/5}
\]
by Proposition 2.2 (and interpolation), we can use Hölder to bound $|J| \lesssim T^\theta (O_{A'}(1))^\mu + 2$ with some positive $\theta$. 

If $\langle \xi - n_i | n_i \rangle \gtrsim PQ$ with $i = 1$ (the case $i = 2$ being similar), then we will bound

$$|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} (n_0)^{s'} (n_1)^{-s' + \sigma} (n_2)^{-s' + \sigma} \min_{0 \leq j \leq 2} (n_j) \cdot (\xi_1 - n_1 | n_1 \rangle)^{-13/50} \times (\mathcal{M}_n(-s',0))_{n_0,\xi_0} (\mathcal{M}_n(s'-\sigma,13/50))_{n_1,\xi_1} (\mathcal{M}_n(s'-\sigma,0))_{n_2,\xi_2} \prod_{j=1}^\mu (\mathcal{M}_n(-1,0))_{m_j,\eta_j} d\Xi.$$

Here we use that

$$\mathcal{M}_n(-s',0) \in X^{0,1/2 - 2\sigma} \subset L^4_{t,x}, \quad \mathcal{M}_n(s'-\sigma,0) \in X^{\sigma,r} \subset L^4_{t,x}, \quad \mathcal{M}_n(s'-\sigma,13/50) \in X^{\sigma,11/50} \subset L^2_{t,x}$$

to bound $|J| \lesssim T^\theta (O_{A'}(1))^{\mu+2}$.

If $\langle \xi - n_i | n_i \rangle \gtrsim PQ$ with $i = 0$, then we will bound

$$|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} (n_0)^{s'} (n_1)^{-s' + \sigma} (n_2)^{-s' + \sigma} \min_{0 \leq j \leq 2} (n_j) \cdot (\xi_0 - n_0 | n_0 \rangle)^{-13/50} \times (\mathcal{M}_n(-s',13/50))_{n_0,\xi_0} (\mathcal{M}_n(s'-\sigma,0))_{n_1,\xi_1} (\mathcal{M}_n(s'-\sigma,0))_{n_2,\xi_2} \prod_{j=1}^\mu (\mathcal{M}_n(-1,0))_{m_j,\eta_j} d\Xi,$$

and use that

$$\mathcal{M}_n(s'-\sigma,13/50) \in X^{0,1/5} \subset L^2_{t,x}, \quad \mathcal{M}_n(s'-\sigma,0) \in X^{\sigma,r} \subset L^4_{t,x}$$

to bound $|J| \lesssim T^\theta (O_{A'}(1))^{\mu+2}$. This completes the estimate for $w$; in fact, we have constructed some $w^*$ such that $w^* = w$ on $[-T, T]$, and that

$$\|w^*\|_{X^{s',r}} \leq O_A(1) + T^\theta O_{A'}(1) \leq A'/4.$$

Now we turn to the bound for $u$; clearly we only need to consider $\pi_{>0} u$. First, let $u^*$ be defined by $\pi_{> K} u^* = \pi_{> K} \tilde{u}$, and

$$\pi_{[0,K]} u^* = \chi(t)e^{-\Delta_{\xi} t} \pi_{[0,K]} u(0) + \mathcal{E}(\chi(T^{-1} t) \cdot \pi_{[0,K]} S(S\tilde{u} \cdot S\tilde{u})),$$

where $K$ is a quantity which is large compared to $A'$, but small compared to $T^{-1}$. We claim that $\|\langle \partial_x \rangle^{-\sigma} u^*\|_V = O_A(1)$; in fact, first we have

$$\|\langle \partial_x \rangle^{-\sigma} u^*\|_V \lesssim K^{-\sigma} \|\tilde{u}\|_V \lesssim K^{-\sigma} O_{A'}(1) = O_A(1).$$

For $\pi_{[0,K]} u^*$, the initial data term is also bounded trivially. Using Proposition 2.2, we then only need to bound

$$\|\chi(T^{-1} t)\pi_{[0,K]} S(S\tilde{u} \cdot S\tilde{u})\|_{X^{1,r-1}} \lesssim T^\theta K^3$$

for some positive $\theta$. But using Proposition 2.2 again and noting the $\pi_{[0,K]}$ projection, we can bound

$$\|\chi(T^{-1} t)\pi_{[0,K]} S(S\tilde{u} \cdot S\tilde{u})\|_{X^{1,r-1}} \lesssim T^\theta K^2 \|\pi_{[0,K]} S(S\tilde{u})\|_{X^{0,2\sigma-1/2}} \lesssim T^\theta K^2 \|S(S\tilde{u})\|_{L^4_{t,x}} \lesssim T^\theta K^2 O_A(1) = O_A(1).$$
In the same way, we also have that \( \|u^*\|_V = O_A(1) \). Next, define \( u^{**} \) by \( \pi_{[0,K]}u^{**} = \pi_{[0,K]}u^* \), and

\[
\pi_{>K}u^{**} = \pi_{>K}(M^*)^{-1}(w^* + \pi_{\leq 0}M^*u^*).
\]

Here \( M^* \) is an operator defined in the same way as \( M \), but with the function \( u \) in the definition of \( M \) replaced by \( u^* \). Clearly \( u = u^{**} \) on \([−T, T] \), we now show that \( \|\pi_{>K}u^{**}\|_V \lesssim O_A(1) \). In fact, the \( Y^s' \) norm is easy to bound, since

\[
((M^*)^{-1}w^*)_{n_0,\xi_0} = \sum_{\mu \geq 0} C_{\mu} \sum_{n_0=n_1+m_{1\mu}} \int_{\xi_0=\xi_1+n_{1\mu}} \Phi \cdot (w^*)_{n_1,\xi_1} \frac{\prod_{i=1}^\mu (u^*)(m_i,\eta_i)}{m_i} \mathrm{d}\Xi
\]

(2.10)

with \( |\Phi| \lesssim 1 \), thus

\[
\| (M^*)^{-1}w^* \|_{Y^s'} \lesssim \sum_{\mu \geq 0} C_{\mu} \sum_{m_1,\ldots,m_{\mu}} \int_{|m_i|} \| u_{m_i,\eta_i} \| \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_\mu \cdot \| (n_1+m_{1\mu})^{\nu}(w^*)_{n_1,\xi_1} \|_{L^1_\xi}^2
\]

\[
\lesssim \| w^* \|_{Y^s'} \cdot \sum_{\mu \geq 0} C_{\mu} \sum_{m_1,\ldots,m_{\mu}} \langle m_{1\mu} \rangle^{\nu} \int_{|m_i|} \| u_{m_i,\eta_i} \| \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_\mu \| m_{1\mu} \|_{L^1_\eta}
\]

\[
\lesssim \| w^* \|_{X^s',r} \cdot \exp \left( C \| m \|^{-1+s'}(u^*)_{m,\eta} \|_{m_{1\mu}L^1_\eta} \right).
\]

which is \( O_A(1) \) since

\[
\| (m)^{-1+s'}(u^*)_{m,\eta} \|_{m_{1\mu}L^1_\eta} \lesssim \| \langle \partial_{\xi} \rangle^{-\sigma} u^* \|_{Y^s'} \lesssim O_A(1).
\]

As with \( \pi_{>K}(M^*)^{-1}\pi_{\leq 0}M^*u^* \), we have a similar expression as (2.10), but we must have at least one \( |m_i| \gtrsim K \) in the sum, due to the two projectors \( \pi_{>K} \) and \( \pi_{\leq 0} \); repeating the above argument gains us a small power \( K^{-\theta} \), which allows us to close using the bound \( \| u^* \|_{Y^s'} \leq O_A(1) \).

Next we prove that

\[
\| \pi_{>K}(M^*)^{-1}w^* \|_{X^{-1/2-s',12/25}} \leq O_A(1).
\]

Using duality (and omitting the summation in \( \mu \)), we only need to bound

\[
J = \sum_{n_0=n_1+m_{1\mu}} \int_{\xi_0=\xi_1+n_{1\mu}} \Phi \cdot h_{n_0,\xi_0}(w^*)_{n_1,\xi_1} \frac{\prod_{i=1}^\mu (u^*)(m_i,\eta_i)}{m_i} \mathrm{d}\Xi,
\]

(2.11)

where \( |\Phi| \lesssim 1 \) and \( \| h \|_{X^{1/2+s',-12/25}} \leq 1 \).

In (2.11), assume first that \( \langle m_i \rangle \ll \max(\langle n_0 \rangle, \langle n_1 \rangle) \), then

\[
|\Gamma| := \left| n_0|n_0| - n_1|n_1| - \sum_{i=1}^\mu m_i|m_i| \right| \lesssim (n_0) \cdot \max_i \langle m_i \rangle,
\]

and also that

\[
\langle \xi_0 - n_0|n_0| \rangle \lesssim \max_i \langle \Gamma \rangle, \langle \xi_1 - n_1|n_1| \rangle, \langle \eta_i - m_i|m_i| \rangle.
\]
If $\langle \xi_0 - n_0 | n_0 \rangle \lesssim \langle n_0 \rangle \cdot \langle m_1 \rangle$, then we can bound

$$
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle n_0 \rangle^{-1/2} \langle m_1 \rangle^{-1/2} \langle \xi_0 - n_0 | n_0 \rangle^{12/25} \\
\times (\mathcal{M}_{(1/2, -12/25)})_{n_0, \xi_0} (\mathcal{M} w^*)_n, \xi_1 (\mathcal{M} (u^*) (-1/2, 0))_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathcal{M} (u^*) (-1, 0))_{m_i, \eta_i} \, d\Xi
$$

(2.12)

which is bounded by $O_A(1)$ by estimating $h$ and $w^*$ factors in $L^2_{t,x}$ and other factors in $L^\infty_{t,x}$, using the bound $\| \{ \partial_x \}^{-\sigma} u^* \|_{Y^{\sigma'}} \leq O_A(1)$.

If $\langle \xi_0 - n_0 | n_0 \rangle \lesssim \langle \xi_1 - n_1 | n_1 \rangle$, then we will bound

$$
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle \xi_0 - n_0 | n_0 \rangle^{12/25} \langle \xi_1 - n_1 | n_1 \rangle^{-12/25} \\
\times (\mathcal{M}_{(0, -12/25)})_{n_0, \xi_0} (\mathcal{M} w^*)_n, \xi_1 (\mathcal{M} (u^*) (-1, 12/25))_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathcal{M} (u^*) (-1, 0))_{m_i, \eta_i} \, d\Xi,
$$

(2.13)

and estimate the $h$ and $w^*$ factors in $L^2_{t,x}$ and other factors in $L^\infty_{t,x}$. If $\langle \xi_0 - n_0 | n_0 \rangle \lesssim \langle \eta_1 - m_1 | m_1 \rangle$, then we will bound

$$
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle \xi_0 - n_0 | n_0 \rangle^{12/25} \langle \eta_1 - m_1 | m_1 \rangle^{-12/25} \\
\times (\mathcal{M}_{(0, -12/25)})_{n_0, \xi_0} (\mathcal{M} w^*)_n, \xi_1 (\mathcal{M} (u^*) (-1, 12/25))_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathcal{M} (u^*) (-1, 0))_{m_i, \eta_i} \, d\Xi.
$$

We then estimate in Fourier space and bound the $h$ factor in $l^{4/3}_{n} L^2_{\xi}$, the $w^*$ factor in $l^2_{\xi} L^1_{\xi}$, the separate $u^*$ factor in $l^{4/3}_{m} L^2_{\eta}$, and all the other $u^*$ factors in $l^1_{m} L^1_{\xi}$.

Now assume in (2.11) that

$$
R := \langle m_1 \rangle \gtrsim \max(\langle n_0 \rangle, \langle n_1 \rangle, \langle m_i \rangle)_{i \neq 1},
$$

then we have

$$
|\Gamma| := \left| n_0 | n_0 | - n_1 | n_1 | - \sum_{i=1}^{\mu} m_i | m_i | \right| \lesssim R^2.
$$

If we have $\langle \xi_0 - n_0 | n_0 \rangle \lesssim \langle \xi_1 - n_1 | n_1 \rangle$ or $\langle \xi_0 - n_0 | n_0 \rangle \lesssim \langle \eta_i - m_i | m_i \rangle$, then we can use exactly the same arguments as above (they do not require any information on the relative size of $n_j$ and $m_i$). Now let us assume that $\langle \xi_0 - n_0 | n_0 \rangle \lesssim R^2$, then we can bound

$$
|J| \lesssim \sum_{(n,m)} \int_{(\xi,\eta)} \langle \xi_0 - n_0 | n_0 \rangle^{12/25} \langle m_1 \rangle^{-1} (\mathcal{M}_{(0, -12/25)})_{n_0, \xi_0} (\mathcal{M} w^*)_n, \xi_1 (\mathcal{M} u^*)_{m_1, \eta_1} \prod_{i=2}^{\mu} (\mathcal{M} (u^*) (-1, 0))_{m_i, \eta_i} \, d\Xi,
$$

(2.14)

which is bounded by $O_A(1)$, by estimating the $h$ factor in $l^{4/3}_{n} L^2_{\xi}$, the $w^*$ factor in $l^2_{\xi} L^2_{\xi}$, the separate $u^*$ factor in $l^{4/3}_{m} L^1_{\xi}$, and all the other $u^*$ factors in $l^1_{m} L^1_{\xi}$. 

The estimate for
\[ \| \pi_{> \omega} (M^*)^{-1} \pi_{< \sigma} M^* u^* \|_{X_{-1/2,\sigma,12/25}} \]
can be done basically in the same way as above. Notice that in the above argument, except in the case when \( \langle \xi_0 - n_0 \mid n_0 \rangle \lesssim \langle \xi_1 - n_1 \mid n_1 \rangle \), we can close the estimate using only the \( V \) norm for \( u^* \); thus this proof will also work for \( u^* \). The bound we get seems to be only \( O_A'(1) \), but due to the projections \( \pi_{> \omega} \) and \( \pi_{< \sigma} \), we must be one \( m_i \) (say \( m_1 \)) such that \( \langle m_i \rangle \gtrsim \max(n_1, K) \), we can gain a power of \( K \) (from the fact that the bounds used involving \( m_1 \) are not sharp) and thus recover the \( O_A(1) \) bound.

If \( \langle \xi_0 - n_0 \mid n_0 \rangle \lesssim \langle \xi_1 - n_1 \mid n_1 \rangle \), then again there must be one \( m_i \) (say \( m_1 \)) such that \( \langle m_i \rangle \gtrsim \max(n_1, K) \). Now instead of (2.13) we have
\[
|J| \lesssim \sum_{n,m} \int_{(\xi, \eta)} \bigg( \xi_0 - n_0 |n_0| \bigg)^{12/25} \bigg( \xi_1 - n_1 |n_1| \bigg)^{-12/25} \bigg( n_1 \bigg)^{1/2+\sigma'} \bigg( m_1 \bigg)^{-1/2-\sigma'}
\times \left( \mathfrak{M}(u^*)^{(0,-12/25)} \right)_{n_0,0,0} \left( \mathfrak{M}(u^*)^{(-1/2-\sigma',12/25)} \right)_{n_1,\xi_1} \left( \mathfrak{M}(u^*)^{(-1/2+\sigma',0)} \right)_{m_1,n_1}
\prod_{i=2}^{\mu} \left( \mathfrak{M}(u^*)^{(-1,0)} \right)_{m_i,n_i}, \, d\Xi,
\]
so we can bound the \( h \) factor and the first \( u^* \) factor in \( L^2_{t,x} \), and other factors in \( L^\infty_{t,x} \). Again, here we gain a power of \( K \) and thus can improve the \( O_A'(1) \) bound to an \( O_A(1) \) bound.

Finally, we show how to estimate the differences. Suppose \( (\tilde{u}^N, \tilde{w}^N) \) and \( (\tilde{u}^\infty, \tilde{w}^\infty) \) are constructed like above, and that they are bounded in \( V \) and \( X^{s',r} \) respectively. Note that all the above arguments remain valid if \( V \) is replaced by \( U \) and \( X^{s',r} \) is replaced by \( X^{s,r} \); we will use this to analyze the difference
\[
\gamma := \| \tilde{u}^N - \tilde{u}^\infty \|_U + \| \tilde{w}^N - \tilde{w}^\infty \|_{X^{s,r}}.
\]
In fact, when doing this, the nonlinearity will involve two kinds of terms: one involves a factor of \( \gamma \), and the other involves terms where at least one of \( n_j \) or \( m_i \) is \( \gtrsim N \) (since the expression for (1.7) is the same as (1.1) for frequencies \( \ll N \)). When bounding the second kind of terms, we gain a small power of \( N \) since the corresponding factor is bounded in a stronger space, and we only estimate it in a weaker space. In this way we get that
\[
\gamma = T^\theta O_A'(1) \gamma + O(N^{-\theta})
\]
for some positive \( \theta \), thus \( \gamma = O(N^{-\theta}) \) as \( N \to \infty \). \( \square \)

3. Some Useful Orthogonality Relations

In this section we recall for the sake of completeness some useful results from [29] on the orthogonality of multilinear products of Gaussian variables \( g_k(\omega) \) that appear in (1.3). Introduce the sets:
\[
A(n) = \{ (j_1, \ldots, j_n) \in \mathbb{Z}^n | j_k \neq 0, k = 1, \ldots, n, \sum_{k=1}^n j_k = 0 \},
\]
\[
\tilde{A}(n) = \{ (j_1, \ldots, j_n) \in A(n) | j_k \neq -j_l, \forall k, l \},
\]
\[ \bar{\mathcal{A}}^c(n) = \mathcal{A}(n) \setminus \bar{\mathcal{A}}(n) \text{ and} \]
\[ \bar{\mathcal{A}}^{c,j}(n) = \{(j_1, \ldots, j_n) \mid (j_1, \ldots, j_n) \in \bar{\mathcal{A}}^c(n) \text{ for some } 1 \leq l \neq m \leq n\}. \]

**Proposition 3.1.** Assume that
\[
(j_1, \ldots, j_n), (i_1, \ldots, i_n) \in \bar{\mathcal{A}}(n), (j_1, \ldots, j_n) \neq (i_1, \ldots, i_n),
\]
then \( \int g_{j_1} \cdots g_{j_n} \overline{g_{i_1}} \cdots \overline{g_{i_n}} \, dp = 0. \)

**Proposition 3.2.** Let \( i, j > 0 \) be fixed and assume
\[
(j_1, j_2, j_3, j_4, j_5) \in \bar{\mathcal{A}}^{c,j}(5), (i_1, i_2, i_3, i_4, i_5) \in \bar{\mathcal{A}}^{c,j}(5),
\]
\[
\{(j_1, j_2, j_3, j_4, j_5) \setminus \{j_i, j_j\} \neq \{(i_1, i_2, i_3, i_4, i_5) \setminus \{i_i, i_j\}\},
\]
then \( \int g_{j_1}g_{j_2}g_{j_3}g_{j_4}g_{j_5} \overline{g_{i_1}} \overline{g_{i_2}} \overline{g_{i_3}} \overline{g_{i_4}} \overline{g_{i_5}} \, dp = 0. \)

The proof of the propositions above are based on the following lemma.

**Lemma 3.1.** Let
\[
(j_1, \ldots, j_n), (i_1, \ldots, i_n) \in \mathcal{A}(n), (j_1, \ldots, j_n) \neq (i_1, \ldots, i_n)
\]
be such that:
\[ \int g_{j_1} \cdots g_{j_n} \overline{g_{i_1}} \cdots \overline{g_{i_n}} \, dp \neq 0. \] (3.2)

Then there exist \( 1 \leq l, m \leq n, \) with \( l \neq m \) and such that at least one of the following occurs: either \( i_l = -i_m \) or \( j_l = -j_m. \)

**Proof.** By (3.1) we get the existence of \( l \in \{1, \ldots, n\} \) such that:
\[ |\{k = 1, \ldots, n \mid i_l = i_k\}| \neq |\{k = 1, \ldots, n \mid j_l = i_k\}| \] (3.3)
where \(|.|\) denotes the cardinality. Next we introduce
\[
\mathcal{N}_l = \{k = 1, \ldots, n \mid i_l = -i_k\}, \\
\mathcal{M}_l = \{k = 1, \ldots, n \mid i_l = i_k\}, \\
\mathcal{P}_l = \{k = 1, \ldots, n \mid j_l = i_k\}, \\
\mathcal{L}_l = \{k = 1, \ldots, n \mid j_l = -i_k\},
\]
Notice that \( \mathcal{M}_l \neq \emptyset \) since it contains at least the element \( l, \) and also by (3.3) \( |\mathcal{M}_l| \neq |\mathcal{P}_l|. \)
We can assume \( |\mathcal{M}_l| > |\mathcal{P}_l| \) (the case \( |\mathcal{M}_l| > |\mathcal{P}_l| \) is similar). Our aim is to prove that \( \mathcal{N}_l \neq \emptyset. \) Next assume by the absurd that \( \mathcal{N}_l = \emptyset, \) then by independence we get
\[ \int g_{j_1} \cdots g_{j_n} \overline{g_{i_1}} \cdots \overline{g_{i_n}} \, dp \]
\[ = \int |g_{i_l}|^{2|\mathcal{P}_l|} \overline{g_{i_l}}^{\mathcal{M}_l} \overline{g_{i_l}}^{\mathcal{L}_l} \, dp \int (\Pi_{k \notin \mathcal{M}_l} \overline{g_{i_k}}) (\Pi_{k \notin \mathcal{L}_l} g_{j_k}) \, dp = 0 \] (3.4)
where at the last step we used \( |\mathcal{M}_l| + |\mathcal{L}_l| - |\mathcal{P}_l| > 0. \) Hence we get an absurd by (3.2).
4. On the Approximation of the Measures $d\rho_{1,R}$ and $d\rho_{3/2,R}$

We first introduce the modified energies:

$$E_N^\epsilon(u) = \|u\|_{H^1}^2 - \|S_N^\epsilon u\|_{H^1}^2 + E_1(S_N^\epsilon u),$$

(4.1)

$$G_N^\epsilon(u) = \|u\|_{H^{3/2}}^2 - \|S_N^\epsilon u\|_{H^{3/2}}^2 + E_3/2(S_N^\epsilon u),$$

(4.2)

and the approximating modified densities:

$$F_{N,R}^\epsilon = \chi_R(\|\pi_N u\|_{L^2}) \times \chi_R(\|\pi_N u\|_{H^{1/2}}^2 - \alpha_N + 1/3 \int (S_{N}^\epsilon u)^3 \, dx)$$

$$\times \exp\left(\|S_N^\epsilon u\|_{H^1}^2 - E_1(S_N^\epsilon u)\right),$$

(4.3)

$$H_{N,R}^\epsilon = \chi_R(\|\pi_N u\|_{L^2}) \times \chi_R(\|\pi_N u\|_{H^{1/2}}^2 + 1/3 \int (S_N^\epsilon u)^3 \, dx)$$

$$\times \chi_R(E_3^\epsilon(\pi_N u) - \alpha_N) \times \exp\left(\|S_N^\epsilon u\|_{H^{3/2}}^2 - E_{3/2}(S_N^\epsilon u)\right).$$

(4.4)

We recall the explicit expressions of $E_1$ and $E_{3/2}$:

$$E_1(u) = \|u\|_{H^1}^2 + \frac{3}{4} \int u^2 \partial_x u + \frac{1}{8} \int u^4$$

and

$$E_{3/2}(u) = \|u\|_{H^{3/2}}^2 - \left(\int \frac{3}{2} uu_x^2 + \frac{1}{2} u(\partial_x u)^2\right) - \left(\frac{1}{3} u^3 \partial_x u + \frac{1}{4} u^2 \partial_x uu_x\right) - \frac{1}{20} \int u^5.$$

Next we prove that as $N \to \infty$ the measures $F_{N,R}^\epsilon d\mu_1$ (for $\epsilon > 0$ fixed) converge to $d\rho_{1,R}$ and $H_{N,R}^\epsilon d\mu_{3/2}$ converge to $d\rho_{3/2,R}$ (in a strong sense).

**Proposition 4.1.** Let $R, \sigma > 0$ and $\epsilon_0 > 0$ be fixed, then:

$$\lim_{N \to \infty} \sup_{A \in B(H^{1/2-\sigma})} \left| \int_A F_{N,R}^{\epsilon_0} d\mu_1 - \int_A d\rho_{1,R} \right| = 0,$$

(4.5)

$$\lim_{N \to \infty} \sup_{A \in B(H^{-\sigma})} \left| \int_A H_{N,R}^{\epsilon_0} d\mu_1 - \int_A d\rho_{3/2,R} \right| = 0.$$

(4.6)

The next lemma will be of importance in the sequel.

**Lemma 4.1.** For every fixed $R > 0, \epsilon_0 > 0$, $p \in [1, \infty)$ we have

$$\sup_{N} \| F_{N,R}^{\epsilon_0} \|_{L^p(d\mu_1)}, \| H_{N,R}^{\epsilon_0} \|_{L^p(d\mu_{3/2})} \leq \infty.$$

This lemma follows modulo minor changes by the same arguments presented along the proof of Propositions 6.5 and 7.10 in [27]. The only difference is that in this paper we use smoothed projectors $S_N^{\epsilon_0}$ in the definition of the approximating measures, while in [27] we use the sharp projectors $\pi_N$. This difference however does not affect the argument presented in [27].
Lemma 4.2. Let \( \epsilon_0, R > 0 \) be fixed and \( \sigma > 0 \) be small. For every sequence \( N_k \) in \( \mathbb{N} \), there exists a subsequence \( N_{kh} \) such that:

\[
\lim_{h \to \infty} F_{N_{kh}, R}^{\epsilon_0}(u) - F_{N_{kh}, R}(u) = 0, \; \text{a.e. (w.r.t.} \; d\mu_1) \; u \in H^{1/2-\sigma},
\]

\[
\lim_{h \to \infty} H_{N_{kh}, R}^{\epsilon_0}(u) - H_{N_{kh}, R}(u) = 0, \; \text{a.e. (w.r.t.} \; d\mu_{3/2}) \; u \in H^{1-\sigma},
\]

where

\[
F_{N,R} = \chi_R(\|\pi_N u\|_{L^2}^2) \chi_R \left( \|\pi_N u\|_{H^{1/2}}^2 - \alpha_N + 1/3 \int (\pi_N u)^3 dx \right) e^{-R_1(\pi_N u)}
\]

and

\[
H_{N,R} = \chi_R(\|\pi_N u\|_{L^2}^2) \chi_R \left( (E_{1/2}(\pi_N u))^{1/2} - \alpha_N \right) e^{-R_{3/2}(\pi_N u)}
\]

are two of the functions introduced in (1.4).

Proof. First we focus on the proof of (4.7). Notice that if we prove

\[
\left\| \int (S_{N}^{\epsilon_0} u)^2 \mathcal{H}\partial_x (S_{N}^{\epsilon_0} u) - \int (\pi_N u)^2 \mathcal{H}\partial_x (\pi_N u) \right\|_{L^2(d\mu_1)} \to 0 \; \text{as} \; N \to \infty,
\]

then up to a subsequence we get

\[
\left| \int (S_{N}^{\epsilon_0} u)^2 \mathcal{H}\partial_x (S_{N}^{\epsilon_0} u) - \int (\pi_N u)^2 \mathcal{H}\partial_x (\pi_N u) \right| \to 0, \; \text{a.e. (w.r.t.} \; d\mu_1) \; u \in H^{1/2-\sigma}.
\]

On the other hand

\[
\int (\pi_N u)^4 - \int (S_{N}^{\epsilon_0} u)^4 \to 0, \; \forall u \in H^{1/2-\sigma},
\]

provided that \( \sigma > 0 \) is small enough in such a way that \( H^{1/2-\sigma} \subset L^4 \). Hence summa-

izing we get

\[
|R_{N_{kh}}^{\epsilon_0}(u) - R_{N_{kh}}(u)| \to 0 \; \text{as} \; h \to \infty \; \text{a.e. (w.r.t.} \; d\mu_1) \; u \in H^{1/2-\sigma},
\]

where:

\[
R_{N}(u) = 3/4 \int (\pi_N u)^2 \mathcal{H}\partial_x (\pi_N u) + 1/8 \int (\pi_N u)^4
\]

and

\[
R_{N}^{\epsilon_0} = 3/4 \int (S_{N}^{\epsilon_0} u)^2 \mathcal{H}\partial_x (S_{N}^{\epsilon_0} u) + 1/8 \int (S_{N}^{\epsilon_0} u)^4.
\]

Recall also that following [27] one can show that there exists \( L \) such that

\[
\int (\pi_N u)^2 \mathcal{H}\partial_x (\pi_N u) \to L
\]
in $L^2(d\mu_1)$, in particular we have up to a subsequence convergence a.e. w.r.t. $d\mu_1$ and hence we can assume that up to a subsequence $R_N(u)$ is bounded a.e. w.r.t. $d\mu_1$. By combining this fact with (4.11) we deduce:

$$\lim_{h \to \infty} \exp(-R_{N_{kh}}^0(u)) - \exp(-R_{N_{kh}}(u)) = 0, \text{ a.e. (w.r.t. } d\mu_1) \text{ } u \in H^{1/2-\sigma}. \quad (4.12)$$

On the other hand we have

$$\int (\pi_N u)^3 - \int (S_{N}^{\epsilon_0} u)^3 \to 0, \forall u \in H^{1/2-\sigma},$$

and hence

$$\lim_{N \to \infty} \left[ \chi_R(\|\pi_N u\|_{H^{1/2}}^2 - \alpha_N + 1/3 \int (S_{N}^{\epsilon_0} u)^3) - \chi_R(\|\pi_N u\|_{H^{1/2}}^2 - \alpha_N + 1/3 \int (\pi_N u)^3) \right] = 0 \text{ a.e. (w.r.t. } d\mu_1) u \in H^{1/2-\sigma}. \quad (4.13)$$

We conclude by combining (4.12) and (4.13).

While the proof of (4.9) basically follows by Proposition 5.1 and Lemma 5.3 in [27], we present the argument for the completeness of the paper. More precisely (4.9) follows by the estimate:

$$\| \sum_{(j,k,l) \in \mathbb{Z}^3} \frac{1}{|j||k|} (1 - \psi_{\epsilon_0}(j/N) \psi_{\epsilon_0}(k/N) \psi_{\epsilon_0}(l/N)) g_j g_k g_l \|_{L^2}^2 \to 0 \text{ as } N \to \infty,$$

that in turn, by an orthogonality argument (as in [29]), is equivalent to:

$$\sum_{(j,k,l) \in \mathbb{Z}^3} \frac{1}{|j|^2|k|^2} |1 - \psi_{\epsilon_0}(j/N) \psi_{\epsilon_0}(k/N) \psi_{\epsilon_0}(l/N)|^2 \to 0 \text{ as } N \to \infty.$$

Notice that due to the cut-off $\psi_\epsilon$ we can restrict the sum on the set

$$\{(j, k, l) \in \mathbb{Z}^3 | j + k + l = 0, 0 < |j|, |k|, |l| \leq N, \max\{|j|/N, |k|/N, |l|/N\} \geq (1 - \epsilon_0)\}$$

and hence we can control the sum above by

$$\sum_{\max\{|j|, |k| \geq (1 - \epsilon_0)N/2} \frac{1}{|j|^2|k|^2} \to 0 \text{ as } N \to \infty.$$

The proof of (4.8) is similar to the proof of (4.7), provided that we show:

$$\| \int (S_{N}^{\epsilon_0} u)(S_{N}^{\epsilon_0} u_s)^2 - \int (\pi_N u)(\pi_N u_s)^2 \|_{L^2(d\mu_{3/2})} \to 0 \text{ as } N \to \infty, \quad (4.14)$$

$$\| \int (S_{N}^{\epsilon_0} u)(\mathcal{H} S_{N}^{\epsilon_0} u_s)^2 - \int (\pi_N u)(\mathcal{H} \pi_N u_s)^2 \|_{L^2(d\mu_{3/2})} \to 0 \text{ as } N \to \infty, \quad (4.15)$$
and

\[ | \int (S_N^0 u)^3 H(S_N^0 u)_x - \int (\pi_N u)^3 H(\pi_N u)_x | \to 0, \]

(4.16)

\[ | \int (S_N^0 u)^2 H(S_N^0 u S_N^0 u_x) - \int (\pi_N u)^2 H(\pi_N u \pi_N u_x) | \to 0, \]

(4.17)

\[ | \int (S_N^0 u)^5 - \int (\pi_N u)^5 | \to 0, \]

(4.18)

a.e. (w.r.t. \( d\mu_{3/2} \)) \( u \in H^{1-\sigma} \)

The proof of (4.18) follows by the Sobolev embedding \( H^{1-\sigma} \subseteq L^5 \). To prove (4.16) (and by a similar argument (4.17)) we use the following inequality (that follows by fractional integration by parts, see page 283 in [27]):

\[
| \int v_1 v_2 v_3 \partial_x v_4 d\mu_1 | \leq C( ||v_2||_{L^\infty} ||v_3||_{L^\infty} ||v_1||_{H^{1/2}} ||v_4||_{H^{1/2}} \\
+ ||v_1||_{L^\infty} ||v_3||_{L^\infty} ||v_2||_{H^{1/2}} ||v_4||_{H^{1/2}} + ||v_1||_{L^\infty} ||v_2||_{L^\infty} ||v_3||_{H^{1/2}} ||v_4||_{H^{1/2}} )
\]

and hence (4.16) follows provided that \( ||S_N^0 u - \pi_N u||_{L^\infty} \to 0, ||S_N^0 u - \pi_N u||_{H^{1/2}} \to 0 \) a.e. (w.r.t. \( d\mu_{3/2} \)) \( u \in H^{1-\sigma} \). The second estimate is trivial and the first one follows since we can select \( \rho > 0 \) small in such a way that \( H^{1/2+\rho} \subseteq L^\infty \) and also \( ||u||_{H^{1/2+\rho}} < \infty \) a.e. (w.r.t. \( d\mu_{3/2} \)) \( u \in H^{1-\sigma} \). The proof of (4.14) and (4.15) follows the same orthogonality argument as the proof of (4.9). More precisely we get

\[
\sum_{(j,k,l) \in \mathbb{Z}^3, j+k+l=0 \atop 0 < |j||k||l| \leq N} \frac{1}{|j||k||l|^3} < \sum_{0 < |j||k||l| > N^2} \frac{1}{|j||k||l|^3} \leq O(\frac{1}{N^\alpha})
\]

for some \( \alpha > 0 \). In the last estimate we used Lemma 2.1 in [25]. \( \square \)

**Proof of Proposition 4.1.** The proof of (4.5) and (4.6) since now on are the same, hence we focus on the first one. It is sufficient to prove that given any sequence \( N_k \) in \( \mathbb{N} \) there exists a subsequence \( N_{kh} \) such that (4.5) occurs. Recall from Theorem 1.1 in [27] that

\[
\lim_{N \to \infty} \sup_{A \in B(H^{1/2-\sigma})} | \int_A F_{N,R} d\mu_1 - \int_A d\rho_{1,R} | = 0.
\]

By combining Lemma 4.2 with the Egoroff theorem we get that, up to subsequence, for every \( \epsilon > 0 \) there exists \( \Omega_\epsilon \subset H^{1/2-\sigma} \), with \( \sigma > 0 \), such that \( \mu_1(\Omega_\epsilon) < 1 - \epsilon \) and \( F_{N,R}(u) - F_{N,R}(u) \to 0 \) in \( L^\infty(\Omega_\epsilon) \). As a consequence we get

\[
| \int_{A \cap \Omega_\epsilon} F_{N,R}^0(u) d\mu_1 - \int_{A \cap \Omega_\epsilon} F_{N,R}(u) d\mu_1 | < \epsilon \text{ for } N > N(\epsilon). 
\]

(4.19)
On the other hand by the Hölder inequality

$$\left| \int_{A \cap \Omega_1^c} F_{N,R}(u) d\mu_1 - \int_{A \cap \Omega_1^c} F_{N,R}(u) d\mu_1 \right| \lesssim \sup_N \left( \| F_{N,R} \|_{L^2(d\mu_1)} + \| F_{N,R} \|_{L^2(d\mu_1)} \right) \times |\mu_1(\Omega_1^c)|^{1/2} \lesssim \epsilon^{1/2} \quad (4.20)$$

where we used Lemma 4.1 and [27, Proposition 6.5]. The proof follows by combining (4.19) with (4.20). □

5. A-Priori Gaussian Bounds w.r.t. $d\mu_1$

Recall that for every $\epsilon > 0$ we denote by $\psi_\epsilon$ any function that satisfies (1.5) and by $S_N^\epsilon$ the associated multiplier defined by (1.6). The main aim of this section is the proof of the following result.

**Proposition 5.1.** Let us denote by $S$ the family of operators $S_N^\epsilon$, for every $N \in \mathbb{N}$, $\epsilon > 0$. Then we have:

$$\| \int S\varphi H(S\varphi_x) S\varphi S\varphi_x - S\varphi H(S\varphi_x) S^2(S\varphi S\varphi_x) \|_{L^2(d\mu_1(\varphi))} = O(\sqrt{\epsilon}); \quad (5.1)$$

$$\| \int (S\varphi)^3 (S\varphi S\varphi_x) - (S\varphi)^3 S^2(S\varphi S\varphi_x) \|_{L^2(d\mu_1(\varphi))} = O(\epsilon) + O\left( \frac{\ln N}{\sqrt{N}} \right). \quad (5.2)$$

The estimate (5.1) is equivalent to:

$$\| \sum_{(a,b,c,d) \in A_N(4)} \Lambda_N^\epsilon(a,b,c,d) \frac{\text{sign}(d)}{|a||b|} g_a g_b g_c g_d \|_{L^2_{\mu_\epsilon}} = O(\sqrt{\epsilon}) \quad (5.3)$$

where $g_\epsilon$ are the Gaussian independent variables in (1.3),

$$\Lambda_N^\epsilon(a,b,c,d) = \psi_\epsilon\left( \frac{a}{N} \right) \psi_\epsilon\left( \frac{b}{N} \right) \psi_\epsilon\left( \frac{c}{N} \right) \psi_\epsilon\left( \frac{d}{N} \right) [\psi_\epsilon^2\left( \frac{a+c}{N} \right) - 1] \quad (5.4)$$

and

$$A_N(4) = \{(a,b,c,d) \in \mathbb{Z}^4 | 0 < |a|, |b|, |c|, |d| \leq N, a + b + c + d = 0\}. \quad (5.5)$$

The proof of (5.3) (and hence (5.1)) is splitted in several lemmas.

**Lemma 5.1.** We have $\sum_{(a,b,c,d) \in B_N^\epsilon} \Lambda_N^\epsilon(a,b,c,d) \frac{\text{sign}(d)}{|a||b|} g_a g_b g_c g_d = 0$, where

$$B_N^\epsilon = \{(a, b, c, d) \in A_N(4) | 0 < |a|, |b|, |c|, |d| \leq N(1 - \epsilon)\}$$

for every $N \in \mathbb{N}$, $\epsilon > 0$. 
Proof. Let us fix \((a, b, c, d) \in B_N^e\). First we assume \(c, d > 0\) (the case \(c, d < 0\) is similar). By the condition \(a + b + c + d = 0\) we deduce that \(\min\{a, b\} < 0\). Since we are assuming \(0 < |a|, |b|, |c|, |d| \leq N(1 - \epsilon)\), we get \(|a + c| = |b + d| < N(1 - \epsilon)\). Hence we obtain by the cut–off properties of \(\psi_\epsilon\) that \(\Lambda_N^e(a, b, c, d) = 0\).

Hence we have to consider the case \(c \cdot d < 0\). Under the extra assumption \(a \neq b\) we deduce that the vectors \((a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c)\) are distinct and belong to \(B_N^e\). Moreover \(g_ag_bg_cgd = g_ag_bg_cgd = gbgagcda = gb_ag_cda\) and by simple algebra (recall \(a + b + c + d = 0\)) we get:

\[
\frac{1}{|a||b|} [\Lambda_N^e(a, b, c, d)sign(d) + \Lambda_N^e(a, b, d, c)sign(c)] + \Lambda_N^e(b, a, c, d)sign(d) + \Lambda_N^e(b, a, d, c)sign(c) = 0.
\]

The same argument works for \(a = b\) (in fact in this case \((a, a, c, d), (a, a, d, c) \in B_N^e\) are distinct since \(c\) and \(d\) have opposite sign, \(g_ag_ag_cgd = g_ag_ag_cgd\) and we have the identity \(\frac{1}{a^2} \Lambda_N^e(a, a, c, d)sign(d) + \frac{1}{a^2} \Lambda_N^e(a, a, d, c)sign(c) = 0\)). The proof is concluded. \(\square\)

Lemma 5.2. We have

\[
\sup_{N} \left\{ \sum_{(a, b, c, d) \in C_N} \Lambda_N^e(a, b, c, d) \frac{sign(d)}{|a||b|} gb_gcdg \right\}^2 L^2_{\infty} = O(\sqrt{\epsilon}),
\]

where \(C_N^e = \{(a, b, c, d) \in A_N(4)| \max\{|a|, |b| > N(1 - \epsilon)\}\) and \(N \in \mathbb{N}, \epsilon > 0\).

Proof. Assume for simplicity that \(|a| > N(1 - \epsilon)\) (the case \(|b| > N(1 - \epsilon)\) is similar). Next we split \(C_N^e = \tilde{C}_N^e \cup \tilde{C}_N^e\) where

\[
\tilde{C}_N^e = \{(a, b, c, d) \in C_N^e| a \neq -b, a \neq -c, a \neq -d, b \neq -c, b \neq -d, c \neq -d\}
\]

and \(\tilde{C}_N^e = C_N^e \setminus \tilde{C}_N^e\). Then by orthogonality and Proposition 3.1 we can estimate

\[
\left( \sum_{(a, b, c, d) \in \tilde{C}_N^e} \Lambda_N^e(a, b, c, d) \frac{(sign(d))}{|a||b|} gb_gcdg \right)^2 L^2_{\infty} \\ \lesssim \left( \sum_{N(1-\epsilon) < |a| < N} \frac{1}{|a|^2} \right) \cdot \left( \sum_{0 < |b| < N} \frac{1}{|b|^2} \right) \cdot \left( \sum_{0 < |c| < N} 1 \right) = O(\epsilon).
\]

Concerning the sum on the set \(\tilde{C}_N^{e,c}\) notice that:

\[
\tilde{C}_N^{e,c} = \left\{ \{(a, -a, -a, a)| |a| \neq 0\} \cup \{(a, -a, a, -a)| |a| \neq 0\} \cup \{(a, a, -a, -a)| |a| \neq |b|\} \cup \{(a, b, -a, -b)| |a| \neq |b|\} \right\} \cap C_N^e.
\]

As a consequence we get \((a, b, c, d) \in \tilde{C}_N^{e,c}\) implies \((-a, -b, -c, -d) \in \tilde{C}_N^{e,c}\) and moreover \(g_ag_bg_cgd = gbgagdca = gb_ag_cda\). Since we have the identity

\[
sign(d) \Lambda_N^e(a, b, c, d) + sign(-d) \Lambda_N^e(-a, -b, -c, -d) = 0,
\]

we deduce that the vectors \((a, b, c, d), (-a, -b, -c, -d)\) belong to \(\tilde{C}_N^{e,c}\) and by simple algebra (recall \(a + b + c + d = 0\)) we get:

\[
\frac{1}{|a||b|} [\Lambda_N^e(a, b, c, d)sign(d) + \Lambda_N^e(a, b, d, c)sign(c)] + \Lambda_N^e(b, a, c, d)sign(d) + \Lambda_N^e(b, a, d, c)sign(c) = 0.
\]

The same argument works for \(a = b\) (in fact in this case \((a, a, c, d), (a, a, d, c) \in \tilde{C}_N^{e,c}\) are distinct since \(c\) and \(d\) have opposite sign, \(g_ag_ag_cgd = g_ag_ag_cgd\) and we have the identity \(\frac{1}{a^2} \Lambda_N^e(a, a, c, d)sign(d) + \frac{1}{a^2} \Lambda_N^e(a, a, d, c)sign(c) = 0\)). The proof is concluded. \(\square\)
it is easy to deduce that
\[
\sum_{(a,b,c,d) \in \mathcal{E}_N^\infty} \Lambda_N^\infty(a, b, c, d) \frac{\text{sign}(d)}{|a||b|} g_a g_b g_c g_d = 0.
\]

\[
\square
\]

**Lemma 5.3.** We have \(\sum_{(a,b,c,d) \in \mathcal{D}_N^\infty} \Lambda_N^\infty(a, b, c, d) \frac{\text{sign}(d)}{|a||b|} g_a g_b g_c g_d = 0\), where
\[
\mathcal{D}_N^\infty = \{(a, b, c, d) \in \mathcal{A}_N(4)|0 < |a|, |b| \leq N(1 - \epsilon), |c|, |d| > N(1 - \epsilon)\}
\]
for every \(N \in \mathbb{N}, \epsilon > 0\).

**Proof.** We notice that if \((a, b, c, d) \in \mathcal{D}_N^\infty\) then \(c \cdot d < 0\). In fact assume by the absurd that \(c, d > 0\) or \(c, d < 0\) then we have \(|c+d| > 2N(1-\epsilon)\) and it implies \(|a+b| > 2N(1-\epsilon)\). This is in contradiction with \(|a+b| \leq |a| + |b| \leq 2N(1-\epsilon)\).

The proof can be concluded arguing as in Lemma 5.1 in the case \(c \cdot d < 0\). \(\square\)

**Lemma 5.4.** We have
\[
\sup_N || \sum_{(a,b,c,d) \in \mathcal{E}_N^\infty} \Lambda_N^\infty(a, b, c, d) \frac{\text{sign}(d)}{|a||b|} g_a g_b g_c g_d ||_{L_2^\infty} = O(\sqrt{\epsilon}),
\]
where
\[
\mathcal{E}_N^\infty = \{(a, b, c, d) \in \mathcal{A}_N(4)|0 < |a|, |b|, |c| \leq N(1 - \epsilon), |d| > N(1 - \epsilon)\}
\]
\[
\bigcup\{(a, b, c, d) \in \mathcal{A}_N(4)|0 < |a|, |b|, |d| \leq N(1 - \epsilon), |c| > N(1 - \epsilon)\},
\]
for every \(N \in \mathbb{N}\) and \(\epsilon > 0\).

**Proof.** Arguing as in Lemma 5.1 in the case \(c \cdot d < 0\), we can restrict to the case \((a, b, c, d) \in \mathcal{E}_N^\infty\) with \(c \cdot d > 0\). Next we split the sum on two constraints (see Sect. 3 for the definition of \(\tilde{\mathcal{A}}(4)\) and \(\tilde{\mathcal{A}}^c(4)\):
\[
\mathcal{E}_N^\infty \cap \tilde{\mathcal{A}}(4) \cap \{(a, b, c, d)|c \cdot d > 0\}
\]
and
\[
\mathcal{E}_N^\infty \cap \tilde{\mathcal{A}}^c(4) \cap \{(a, b, c, d)|c \cdot d > 0\}.
\]
By combining an orthogonality argument with Proposition 3.1 we can estimate the sum on the first constraint by
\[
(\sum_{|a+b|>N(1-\epsilon)} \frac{1}{a^2b^2}) \cdot (\sum_{N(1-\epsilon)<|d| \leq N} 1) = O(\epsilon),
\]
where we used that \(c \cdot d > 0\) implies \(|a+b| = |c+d| > N(1-\epsilon)\). Concerning the sum on the second constraint we have
\[
\mathcal{E}_N^\infty \cap \tilde{\mathcal{A}}^c(4) \cap \{(a, b, c, d)|c \cdot d > 0\}
\]
\[
\subset \mathcal{E}_N^\infty \cap \{(a, b, -a, -b)\} \cup \{(a, b, -b, -a)\} \cup \{(a, -a, b, -b)\} = \emptyset.
\]
\(\square\)
Proof of Proposition 5.1. The proof of (5.3) (and hence (5.1)) follows by combining Lemmas 5.1, 5.2, 5.3 and 5.4.

Next we focus on the proof of (5.2), that can be written as follows:

\[ \| \sum_{(a,b,c,d,e) \in \mathcal{A}_N(5)} \Gamma_E(a,b,c,d,e) \frac{\text{sign}(e)}{|a||b||c||d|} g_{a,b,c,d,e} \|_{L^2_w} = O(\epsilon) + O\left( \frac{\ln N}{\sqrt{N}} \right), \]  

(5.6)

where:

\[ \Gamma_E(a,b,c,d,e) = \psi_e(a) \psi_e(b) \psi_e(c) \psi_e(d) \psi_e(e) \left[ 1 - \psi_e^2 \left( \frac{d+e}{N} \right) \right], \]  

(5.7)

and

\[ \mathcal{A}_N(5) = \{(a,b,c,d,e) \in \mathbb{Z}^5 | 0 < |a|, |b|, |c|, |d|, |e| \leq N, a + b + c + d + e = 0\}. \]

(5.8)

Due to the cut-off properties of \( \psi_e \) the sum in (5.6) can be replaced by the sum on the set

\[ \mathcal{A}_N(5) \cap \{(a,b,c,d,e)||d+e| > N(1-\epsilon)\} = \mathcal{A}_N(5) \cap \{(a,b,c,d,e)||a+b+c| > N(1-\epsilon)\}. \]

Next we split

\[ \mathcal{A}_N(5) = \tilde{\mathcal{A}}_N(5) \cup \tilde{\mathcal{A}}^c_N(5), \]  

(5.9)

where:

\[ \tilde{\mathcal{A}}_N(5) = \{(a,b,c,d,e) \in \mathcal{A}_N(5) | a \notin \{-b,-c,-d,-e\}, b \notin \{-c,-d,-e\}, c \notin \{-d,-e\}, d \neq -e\} \]

and \( \tilde{\mathcal{A}}^c_N(5) = \mathcal{A}_N(5) \setminus \tilde{\mathcal{A}}_N(5) \). By orthogonality and Proposition 3.1 we get:

\[ \| \sum_{(a,b,c,d,e) \in \tilde{\mathcal{A}}^c_N(5)} \Gamma_E(a,b,c,d,e) \frac{\text{sign}(e)}{|a||b||c||d|} g_{a,b,c,d,e} \|_{L^2_w}^2 \lesssim \frac{1}{|a|^2 |b|^2 |c|^2 |d|^2} = O\left( \frac{1}{N^3} \right). \]

Concerning the sum on the set \( \tilde{\mathcal{A}}^c_N(5) \cap \{(a,b,c,d,e)||a+b+c| > N(1-\epsilon)\} \) we first consider the splitting:

\[ \tilde{\mathcal{A}}^c_N(5) = \tilde{\mathcal{A}}^c_{N,a=-b} \cup \tilde{\mathcal{A}}^c_{N,a=-c} \cup \tilde{\mathcal{A}}^c_{N,a=-d} \cup \tilde{\mathcal{A}}^c_{N,a=-e} \cup \tilde{\mathcal{A}}^c_{N,b=-c} \]

\[ \cup \tilde{\mathcal{A}}^c_{N,b=-d} \cup \tilde{\mathcal{A}}^c_{N,b=-e} \cup \tilde{\mathcal{A}}^c_{N,c=-d} \cup \tilde{\mathcal{A}}^c_{N,c=-e} \cup \tilde{\mathcal{A}}^c_{N,d=-e}, \]

where

\[ \tilde{\mathcal{A}}^c_{N,a=-b} = \{(a,b,c,d,e) \in \mathcal{A}_N(5) | a = -b\} \]  

(5.10)
and analogous definition for the other sets. First notice that
\[ \tilde{A}_{N,d=-e}^c \cap \{(a, b, c, d, e) \mid |d + e| > N(1 - \epsilon)\} = \emptyset. \]
Moreover by orthogonality and Proposition 3.2 we can estimate the sum on
\[ \tilde{A}_{N,a=-b}^c \cap \{(a, b, c, d, e) \mid |a + b + c| > N(1 - \epsilon)\} \]
(notice that since \( a = -b \) we get \(|c| > N(1 - \epsilon)\)) by:
\[
\sum_{0 < |a| < N} \frac{1}{|a|^2} \left( \sum_{|c| > N(1 - \epsilon), 0 < |c|, |d| < N} \frac{1}{c^2d^2} \right)^{1/2} = O\left(\frac{1}{\sqrt{N}}\right).
\]
In a similar way we can treat the sum on \( \tilde{A}_{N,a=-e}^c, \tilde{A}_{N,b=-c}^c \). Next we treat the sum on
\( \tilde{A}_{N,a=-d}^c \) (which by symmetry is equivalent to \( \tilde{A}_{N,b=-d}^c, \tilde{A}_{N,c=-d}^c \)). In this case we can control the sum by
\[
\sum_{0 < |a| \leq N} \frac{1}{|a|^2} \left( \sum_{|c| > N(1 - \epsilon), 0 < |c|, |d| \leq N} \frac{1}{b^2c^2} \right)^{1/2} \leq \sum_{0 < |a| < N/2(1 - \epsilon)} \frac{1}{|a|^2} \left( \sum_{|b+c| > N(1 - \epsilon)/2, 0 < |b|, |c| \leq N} \frac{1}{b^2c^2} \right)^{1/2}
+ \sum_{N/2(1 - \epsilon)}^{N} \frac{1}{|a|^2} \left( \sum_{0 < |b|, |c| \leq N} \frac{1}{b^2c^2} \right)^{1/2} = O\left(\frac{1}{\sqrt{N}}\right).
\]
Next we treat the sum on the set \( \tilde{A}_{N,a=-e}^c \) (in the same way we can treat the remaining cases). We reduce by orthogonality to the estimate:
\[
\sum_{0 < |e| \leq N} \frac{1}{|e|} \left( \sum_{|e+d| > N(1 - \epsilon), 0 < |d| \leq N} \frac{1}{b^2c^2d^2} \right)^{1/2} \leq \sum_{0 < |e| \leq N} \frac{1}{|e|} \left( \sum_{|e+d| > N(1 - \epsilon), 0 < |d| \leq N} \frac{1}{d^2} \right)^{1/2}
= \sum_{N(1 - \epsilon)}^{N} \frac{1}{|e|} \left( \sum_{|e+d| > N(1 - \epsilon), 0 < |d| \leq N} \frac{1}{d^2} \right)^{1/2} + \sum_{0 < |e| \leq N(1 - \epsilon)} \frac{1}{|e|} \left( \sum_{|e+d| > N(1 - \epsilon), 0 < |d| \leq N} \frac{1}{d^2} \right)^{1/2}
= O\left(\frac{\ln N}{\sqrt{N}}\right) + \sum_{0 < |e| \leq N(1 - \epsilon)} \frac{1}{|e|} \left( \sum_{|e+d| > N(1 - \epsilon), 0 < |d| \leq N} \frac{1}{d^2} \right)^{1/2}.
\]
Notice that if \(|e| \leq N(1 - \epsilon)\) and \(|e+d| > N(1 - \epsilon)\) then either \(|d| \geq N(1 - \epsilon)\) or \(e \cdot d > 0\). In the first case the sum on the r.h.s. of (5.11) can be estimated by \(O\left(\frac{\ln N}{\sqrt{N}}\right)\) and in the second case it can be controlled by:
\[
\sum_{0 < e \leq N(1 - \epsilon)} \frac{1}{e} \left( \sum_{N(1 - \epsilon) - e < d \leq N} \frac{1}{d^2} \right)^{1/2}
\]
We can suppose that the $e$ summations ranges up to $N(1 - \epsilon) - 2$ since the contribution of the remaining terms is $O\left(\frac{1}{N}\right)$. Finally, we can estimate
\[
\sum_{0 < e < N(1 - \epsilon) - 2} \frac{1}{e} \left( \sum_{N(1 - \epsilon) - e < d \leq N} \frac{1}{d^2} \right)^{1/2}
\]
by
\[
\approx \sum_{0 < e < N(1 - \epsilon) - 2} \frac{1}{e} \left( \frac{1}{N(1 - \epsilon) - e} - \frac{1}{N} \right)^{1/2}
\]
\[
\approx \sum_{0 < e < N(1 - \epsilon)} \frac{1}{e} \frac{\sqrt{\epsilon N + e}}{\sqrt{N(1 - \epsilon) - e}}
\]
\[
\approx \sum_{0 < e < N(1 - \epsilon)} \frac{1}{e} \frac{1}{\sqrt{N(1 - \epsilon) - e}}.
\]
By using the variable $\epsilon N + e$ we can rewrite the last sum as follows:
\[
\ldots = \sum_{\epsilon N < a < N} \frac{1}{(a - \epsilon N)\sqrt{N - a}}
\]
\[
= \sum_{\epsilon N < a < N/2} \frac{1}{(a - \epsilon N)\sqrt{N - a}} + \sum_{N/2 \leq a < N} \frac{1}{(a - \epsilon N)\sqrt{N - a}}
\]
\[
\approx \frac{1}{\sqrt{N}} \sum_{\epsilon N < a < N/2} \frac{1}{a - \epsilon N} + \frac{1}{N} \sum_{N/2 \leq a < N} \frac{1}{\sqrt{N - a}} = O\left(\frac{\ln N}{\sqrt{N}}\right).
\]
It concludes the proof of (5.2). □

6. Almost Invariance of $F_{N, R}^{\epsilon} d\mu_1$

The main result of this section is the following proposition, where $F_{N, R}^{\epsilon}$ is defined in (4.3).

**Proposition 6.1.** Let $\sigma, R > 0$ be fixed. Then for every $\delta > 0$ there exists $N = N(\delta) > 0$ and $\epsilon = \epsilon(\delta) > 0$ such that
\[
| \int_{A} F_{N, R}^{\epsilon} d\mu_1 - \int_{\Phi_{N}(t)A} F_{N, R}^{\epsilon} d\mu_1 | \leq \delta t
\]
for every $A \in B(H^{1/2-\sigma})$ and for every $t$.

**Remark 6.1.** Notice that the proposition follows provided that we show
\[
\sup_{t, A} \left| \frac{d}{dt} \int_{\Phi_{N}(t)A} F_{N, R}^{\epsilon} d\mu_1 \right| \to 0 \text{ as } \epsilon \to 0, N \to \infty.
\]

Using that the modified energies associated with $E_0$ and $E_{1/2}$ are true conservation laws for the approximated flows, we obtain that the proof of Proposition 6.1 can be completed by combining the proof of Proposition 5.3 in [28] with the following statement.
**Proposition 6.2.** We have the following estimate:

$$\lim_{\epsilon \to 0} \left( \limsup_{N \to \infty} \| \frac{d}{dt} E_N^\epsilon (\pi_N \Phi_N^\epsilon (t) \varphi)_{t=0} \|_{L^2(d\mu_1(\varphi))} \right) = 0,$$

where the energies $E_N^\epsilon (u)$ are defined by (4.1).

**Proof.** In the sequel we make computations with $N \in \mathbb{N}, \epsilon > 0$ fixed. Hence for simplicity we use the following notations: $S_N^\epsilon = S$, $E_N^\epsilon = \tilde{E}$. Notice that if we denote by $u(t) = \pi_N \Phi_N^\epsilon (t) \varphi$ the solution to (1.7), then $Su(t)$ solves

$$Su_t + \mathcal{H} Su_{xx} + (Su Su_x) + (S^2 - Id)(Su Su_x) = 0. \quad (6.1)$$

Next we recall that

$$\tilde{E}(u) = E_1(Su) + (\| u \|_{H^1}^2 - \| Su \|_{H^1}^2), \quad (6.2)$$

where

$$E_1(u) = \| u \|_{H^1}^2 + \frac{3}{4} \int u^2 \mathcal{H} \partial_x u + \frac{1}{8} \int u^4.$$  

Arguing as in [27] and recalling that $Su$ solves (6.1), then we get:

$$\frac{d}{dt} E_1(Su)_{t=0}$$

$$= 2 \int S \varphi_x (Id - S^2)(S \varphi S \varphi_x)_x + \frac{3}{2} \int (S \varphi) \mathcal{H}(S \varphi_x)(Id - S^2)(S \varphi S \varphi_x)$$

$$+ \frac{3}{4} \int (S \varphi)^2 \mathcal{H}((Id - S^2)(S \varphi S \varphi_x)_x) + \frac{1}{2} \int (S \varphi)^3(Id - S^2)(S \varphi S \varphi_x). \quad (6.3)$$

Moreover we have (use that $u(t)$ solves (1.7)):

$$\frac{d}{dt} (\| u \|_{H^1}^2 - \| Su \|_{H^1}^2)_{t=0}$$

$$= 2 \left( -\mathcal{H} \varphi_{xx} - S(S \varphi S \varphi_x)_x \varphi_x \right) - 2 \left( -\mathcal{H} S \varphi_{xx} - S^2(S \varphi S \varphi_x)_x \varphi_x \right)$$

$$= 2 \int S \varphi S \varphi_x S \varphi_{xx} - 2 \int S \varphi S \varphi_x S^3 \varphi_{xx}. \quad (6.4)$$

(here we used integration by parts and $\int v \mathcal{H} v = 0$). By combining (6.2), (6.3), (6.4) we get:

$$\frac{d}{dt} \tilde{E}(\pi_N \Phi_N^\epsilon (t) \varphi)_{t=0} = 3/2 \int (S \varphi) \mathcal{H}(S \varphi_x)(Id - S^2)(S \varphi S \varphi_x)$$

$$+ \frac{3}{4} \int (S \varphi)^2 \mathcal{H}((Id - S^2)(S \varphi S \varphi_x)_x) + \frac{1}{2} \int (S \varphi)^3(Id - S^2)(S \varphi S \varphi_x).$$
and by integration by parts
\[ ... = 3/2 \int S\varphi H(S\varphi_x)(S\varphi S\varphi_x) - 3/2 \int S\varphi H(S\varphi_x)S^2(S\varphi S\varphi_x) - 3/2 \int S\varphi S\varphi_x H(S\varphi S\varphi_x) + 3/2 \int S\varphi S\varphi_x H S^2(S\varphi S\varphi_x) + 1/2 \int (S\varphi)^3(S\varphi S\varphi_x) - 1/2 \int (S\varphi)^3 S^2(S\varphi S\varphi_x). \]

By using the property \( \int vHv = 0 \) we deduce
\[ ... = 3/2 \int S\varphi H(S\varphi_x)(Id - S^2)(S\varphi S\varphi_x) + 1/2 \int (S\varphi)^3(Id - S^2)(S\varphi S\varphi_x). \]

We conclude by Proposition 5.1.  \( \square \)

7. Invariance of \( d\rho_{1,R} \)

The proof of the invariance of \( d\rho_{1,R} \) follows via standard arguments (see e.g. [28]) by the following proposition.

**Proposition 7.1.** Let \( \sigma > 0 \) small enough and \( \bar{t} > 0 \) be fixed. Then for every compact set \( A \subset H^{1/2-\sigma} \) we have:
\[ \int_A d\rho_{1,R} \leq \int_{\Phi(\bar{t})A} d\rho_{1,R}. \] (7.1)

**Proof.** We fix \( M > 0 \) such that \( A \subset B_M(H^{1/2-\sigma}) \) and we choose \( L > 0 \) such that
\[ \Phi(t)(B_M(H^{1/2-\sigma})) \subset B_L(H^{1/2-\sigma}) \] (7.2)
for every \( t \in [0, \bar{t}] \) (the existence of \( L \) follows by [13]).

Next we fix \( k > 0 \) and by Proposition 6.1 we get \( N_k \in \mathbb{N} \) and \( \epsilon_k > 0 \) such that:
\[ |\int_A F_{N,k}^\epsilon d\mu_1 - \int_{\Phi(\bar{t})A} F_{N,k}^\epsilon d\mu_1| \leq t/k, \quad \forall N > N_k, \quad \forall t. \] (7.3)

On the other hand we have by Proposition 2.1 the existence of \( t_1 = t_1(L, k) > 0 \) and \( C = C(L, k) > 0 \) such that
\[ \sup_{u \in B_k(H^{1/2-\sigma})} \| \Phi_{N,k}^\epsilon(t)u - \Phi(t)u \|_{H^{1/2-\sigma-\delta}} \leq CN^{-\theta} \]
for some \( \delta > 0 \), and hence
\[ \int_{\Phi(\bar{t})A} d\rho_{1,R} \leq \int_{\Phi(\bar{t})A+B_{C_{N^{-\theta}}(H^{1/2-\sigma-\delta})}} d\rho_{1,R}, \quad \forall t \in [0, t_1]. \] (7.4)

In turn by combining (7.3) with Proposition 4.1 we get the existence of \( \tilde{N}_k \in \mathbb{N} \) such that
\[ |\int_A d\rho_{1,R} - \int_{\Phi(\bar{t})A} d\rho_{1,R}| \leq 3t_1/k, \quad \forall N > \tilde{N}_k, \quad \forall t \in [0, t_1]. \] (7.5)
By combining (7.4) with (7.5) we get
\[
\int_A d\rho_{1,R} \leq \int_{\Phi(t)A+B_{C,N^-0}(H^{1/2-\delta})} d\rho_{1,R} + 3t_1/k, \forall t \in [0,t_1]
\] (7.6)
that by taking the limit as \(N \to \infty\) gives:
\[
\int_A d\rho_{1,R} \leq \int_{\Phi(t)A} d\rho_{1,R} + 3t_1/k, \forall t \in [0,t_1].
\]
It is sufficient to iterate the bound at most \(\lceil \bar{t}/t_1 \rceil + 1\) times and to take the limit as \(k \to \infty\) in order to get (7.1) (notice that we can iterate thanks to (7.2)). \(\square\)

8. The Measures \(H_{N,R}^\epsilon d\mu_{3/2}\) and the Invariance of \(d\rho_{3/2,R}\)

The proof of the invariance of \(d\rho_{3/2,R}\) is similar to the proof of the invariance of \(d\rho_{1,R}\), once we establish the following analogue of Proposition 6.1.

**Proposition 8.1.** Let \(\sigma, R > 0\) be fixed. Then for every \(\delta > 0\) there exists \(N = N(\delta) > 0\) and \(\epsilon = \epsilon(\delta) > 0\) such that
\[
|\int_A H_{N,R}^\epsilon d\mu_{3/2} - \int_{\Phi_N(t)A} H_{N,R}^\epsilon d\mu_{3/2}| \leq \delta t
\]
for every \(A \in B(H^{1-\sigma})\) and for every \(t\).

Its proof follows by the following analogue of Proposition 6.2.

**Proposition 8.2.** We have the following estimate:
\[
\lim_{\epsilon \to 0} \left( \limsup_{N \to \infty} \left\| \frac{d}{dt} G_N^\epsilon(\pi_N \Phi_N^\epsilon(t) \varphi)_{t=0} \right\|_{L^2(d\mu_{3/2}(\varphi))} \right) = 0,
\]
where \(G_N^\epsilon(u)\) are defined in (4.2).

In turn we split the proof of this proposition in several steps.

**Proposition 8.3.** We have the following estimates:
\[
\| \int S \varphi(\mathcal{H} \varphi_x) \mathcal{H}(I - S^2)(S \varphi S \varphi_x) \|_{L^2(d\mu_{3/2}(\varphi))} = O(\sqrt{\frac{\ln N}{N}} + O(\epsilon)), \quad (8.1)
\]
\[
\| \int (S \varphi_x)^2 (I - S^2)(S \varphi S \varphi_x) \|_{L^2(d\mu_{3/2}(\varphi))} = O(\sqrt{\frac{\ln^3 N}{N}}) + O(\sqrt{\epsilon}), \quad (8.2)
\]
\[
\| \int (\mathcal{H} \varphi_x)^2 (I - S^2)(S \varphi S \varphi_x) \|_{L^2(d\mu_{3/2}(\varphi))} = O(\sqrt{\frac{\ln^3 N}{N}}) + O(\sqrt{\epsilon}), \quad (8.3)
\]
where \(S = S_N^\epsilon\).
We split the proof of (8.1) in several lemmas. Notice that we have
\[
\| \int S\varphi(HS\varphi_x)\mathcal{H}(Id - S^2)(S\varphi S\varphi_x)_x \|_{L^2(d\mu^{3/2})}
\]
\[
= \| \sum_{(a,b,c,d) \in \mathcal{A}_N(4)} \Delta_N^\varepsilon(a, b, c, d) \frac{|c + d| \text{sign}(d)}{|a|^{3/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} g_a g_b g_c g_d \|_{L^2},
\]
where \( g_a \) are the independent Gaussian random variables in (1.3),
\[
\Delta_N^\varepsilon(a, b, c, d) = \psi_e \left( \frac{a}{N} \right) \psi_e \left( \frac{b}{N} \right) \psi_e \left( \frac{c}{N} \right) \psi_e \left( \frac{d}{N} \right) \left[ 1 - \psi_e^2 \left( \frac{c + d}{N} \right) \right]
\]
(8.4) and
\[
\mathcal{A}_N(4) = \{(a, b, c, d) \in \mathbb{Z}^4 | 0 < |a|, |b|, |c|, |d| \leq N, a + b + c + d = 0 \}.
\]
(8.5)

**Lemma 8.1.** We have
\[
\sum_{(a,b,c,d) \in \mathcal{B}_N^\varepsilon} \frac{|c + d| \text{sign}(d)}{|a|^{3/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} \Delta_N^\varepsilon(a, b, c, d) g_a g_b g_c g_d = 0
\]
where:
\[
\mathcal{B}_N^\varepsilon = \{(a, b, c, d) \in \mathcal{A}_N(4) | 0 < |a|, |b|, |c|, |d| \leq N(1 - \varepsilon) \}
\]
for any \( N \in \mathbb{N}, \varepsilon > 0 \).

**Proof.** Notice that in the case \( \text{sign}(b) \cdot \text{sign}(d) < 0 \) we have:
\[
\frac{|c + d| \text{sign}(d)}{|a|^{3/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} \Delta_N^\varepsilon(a, b, c, d) + \frac{|a + b| \text{sign}(b)}{|a|^{3/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} \Delta_N^\varepsilon(c, a, b, d) = 0.
\]
Hence we can assume that \( \text{sign}(b) = \text{sign}(d) \). By the condition \( a + b + c + d = 0 \) we get that at least one of the following occurs: either \( \text{sign}(b) \neq \text{sign}(a) \) or \( \text{sign}(c) \neq \text{sign}(d) \).
In any case we have \( |a + b| = |c + d| < N(1 - \varepsilon) \) and hence we conclude by the cut–off properties of \( \psi_e \) that \( \Delta_N^\varepsilon(a, b, c, d) = 0 \). \( \Box \)

**Lemma 8.2.** We have
\[
\| \sum_{(a,b,c,d) \in \mathcal{C}_N^\varepsilon} \frac{|c + d| \text{sign}(d)}{|a|^{3/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} \Delta_N^\varepsilon(a, b, c, d) g_a g_b g_c g_d \|_{L^2} = O(\sqrt{\frac{\ln N}{N}}) + O(\varepsilon),
\]
where:
\[
\mathcal{C}_N^\varepsilon = \{(a, b, c, d) \in \mathcal{A}_N(4) | \max(|a|, |c|) > N(1 - \varepsilon) \}
\]
for every \( N \in \mathbb{N}, \varepsilon > 0 \).

**Proof.** We will only treat the case \( |a| > N(1 - \varepsilon) \) (for \( |c| > N(1 - \varepsilon) \) the same argument works). Notice that by the cut–off property of \( \psi_e \) we can work on the set \( \mathcal{C}_N^\varepsilon \cap \{|a| > N(1 - \varepsilon)\} \cap \{|a + b| = |c + d| > N(1 - \varepsilon)\} \).
We argue as in Lemma 5.2 and we split $\mathcal{C}_N^\varepsilon = \tilde{\mathcal{C}}_N^\varepsilon \cup \tilde{\mathcal{C}}_N^{\varepsilon,c}$. When we sum on the set $\tilde{\mathcal{C}}_N^\varepsilon$ then we can combine an orthogonality argument with Proposition 3.1 and with the identity $d = -a - b - c$, and we are reduced to:

$$\sum_{|a| > N(1-\varepsilon)} \frac{1}{|a|^3 |b| |c|} + \sum_{|a| > N(1-\varepsilon)} \frac{|a + b + c|}{|a|^3 |b| |c|^3} \leq \sum_{|a| > N(1-\varepsilon)} \frac{1}{|a|^3 |b| |c|}$$

$$+ \sum_{|a| > N(1-\varepsilon)} \frac{1}{|a|^2 |b| |c|^2} + \sum_{|a| > N(1-\varepsilon)} \frac{1}{|a|^3 |c|^3} + \sum_{|a| > N(1-\varepsilon)} \frac{1}{|a|^3 |b| |c|^2} = O\left(\frac{\ln N}{N}\right).$$

Concerning the sum on $\mathcal{C}_N^{\varepsilon,c}$ we don’t consider the contribution on the set

$$\{(a, -a, -a, a)|a| \neq 0\} \cup \{(a, -a, a, -a)|a| \neq 0\}$$

since in this case $|c + d| = 0$, and we work on the set:

$$\{(a, a, -a, -a)||a| \neq 0\} \cup \{(a, b, -a, -b), |a| \neq |b|\} \cup \{(a, b, -b, -a), |a| \neq |b|\}.$$ Notice that

$$\frac{|a + b| \text{sign}(-b)}{|a|^{3/2} |b|^{1/2} |c|^{1/2} |d|^{1/2}} \Delta_N^\varepsilon(a, b, -a, -b) + \frac{|a + b| \text{sign}(b)}{|a|^{3/2} |b|^{1/2} |c|^{1/2} |d|^{1/2}} \Delta_N^\varepsilon(-a, -b, a, b) = 0$$

and hence we have to consider the sum on the set:

$$\{(a, a, -a, -a)||a| \neq 0\} \cup \{(a, b, -b, -a), |a| \neq |b|\}.$$ We can conclude by combining the Minkowski inequality with the following estimates:

$$\sum_{2|a| > N(1-\varepsilon)} \frac{1}{|a|^3} + \sum_{0 < |a|, |b| \leq N} \frac{|a + b|}{|a|^2 |b|^2}$$

$$\leq O\left(\frac{1}{N^2}\right) + \sum_{0 < |b| \leq N} \frac{1}{N(1-\varepsilon) < |a| \leq N} \frac{|a||b|^2}{N} + \sum_{0 < |b| \leq N} \frac{1}{N(1-\varepsilon) < |a| \leq N} \frac{1}{|a|^2 |b|} = O\left(\frac{\ln N}{N}\right) + O(\varepsilon).$$

□

Lemma 8.3. We have

$$\sum_{(a, b, c, d) \in \mathcal{D}_N^\varepsilon} \frac{|c + d| \text{sign}(d)}{|a|^{3/2} |b|^{1/2} |c|^{1/2} |d|^{1/2}} \Delta_N^\varepsilon(a, b, c, d)g_ag_bg_cg_d = 0,$$

where:

$$\mathcal{D}_N^\varepsilon = \{(a, b, c, d) \in \mathcal{A}_N(4)|0 < |a|, |c| \leq N(1-\varepsilon), \max(|b|, |d|) > N(1-\varepsilon)\}$$

for every $N \in \mathbb{N}$, $\varepsilon > 0$. 
Proof. Arguing as in Lemma 8.1 we can assume that $\text{sign}(b) = \text{sign}(d)$, and also by the cut-off property of $\psi_\epsilon$ we can restrict to the set $|a + b| = |c + d| > N(1 - \epsilon)$. We claim that $\mathcal{D}_N^\epsilon \cap \{\text{sign}(b) = \text{sign}(d)\} \cap \{|a + b| > N(1 - \epsilon)\} = \emptyset$, and it will conclude the proof. We can assume $b, d > 0$ (the case $b, d < 0$ can be treated in a similar way). This implies that $0 < \min\{b, d\} \leq N(1 - \epsilon)$. In fact in case it is not true then we get $b + d > 2N(1 - \epsilon)$ which implies $|a + c| = b + d > 2N(1 - \epsilon)$, and it is in contradiction with $0 < |a|, |c| \leq N(1 - \epsilon)$. We assume for simplicity $0 < b \leq N(1 - \epsilon).$ Moreover by the condition $|a + b| > N(1 - \epsilon)$ we get (since $|a| \leq N(1 - \epsilon)$ and $0 < b \leq N(1 - \epsilon)$) $a > 0$. By combining $a + b + c + d = 0$ and $a, b, d > 0$, we get $c < 0$. Hence we have $|c| = a + b + d$ and it is absurd since $0 < |c| \leq N(1 - \epsilon)$ and $a + b + d > d \geq N(1 - \epsilon)$. \hfill $\Box$

Proof of Proposition 8.3. The proof of (8.1) follows by combining Lemmas 8.1, 8.2 and 8.3. Concerning the proof of (8.2) first notice that

$$\| \int (S\varphi_\alpha)^2 (Id - S^2)(S\varphi S\varphi_\alpha) \|_{L^2(d\mu_{3/2})}^2$$

$$= \sum_{(a, b, c, d) \in \mathcal{A}_N(4)} \Delta_N^\epsilon(a, b, c, d) \text{sign}(a) \text{sign}(b) \text{sign}(d) \frac{1}{|a|^{1/2}|b|^{1/2}|c|^{3/2}|d|^{1/2}} \|a b c d\|_{L^2}^2,$$  \hspace{1cm} (8.6)

where $\Delta_N^\epsilon(a, b, c, d)$ and $\mathcal{A}_N(4)$ are defined in (8.4) and (8.5). Next (following Sect. 3) we split $\mathcal{A}_N(4) = \tilde{\mathcal{A}}_N(4) \cup \check{\mathcal{A}}_N^\epsilon(4)$, where:

$$\tilde{\mathcal{A}}_N(4) = \{(a, b, c, d) \in \mathcal{A}_N(4) | a \neq -b, a \neq -c, a \neq -d, b \neq -c, b \neq -d, c \neq -d\}$$

and

$$\check{\mathcal{A}}_N^\epsilon(4) = \{(a, a, -a, -a) | |a| \neq 0\} \cup \{(a, b, -a, -b), |a| \neq |b|\} \cup \{(a, b, -a, -a), |a| \neq |b|\}.$$

(Notice that we have excluded the vectors $(a, -a, b, -b)$ that in principle belong to $\mathcal{A}_N(4)$, but due to the cut-off property of $\psi_\epsilon$, they give a trivial contribution in the sum in (8.6)). When we consider the sum on $\tilde{\mathcal{A}}_N(4)$, and we recall that $\Delta_N^\epsilon(a, b, c, d) = 0$ when $|c + d| \leq N(1 - \epsilon)$, then by orthogonality and Proposition 3.1 we are reduced to:

$$\sum_{0 < |a|, |b|, |c|, |d| \leq N} \frac{1}{|a||b||c|^3|d|} \leq O(\ln^2 N) \sum_{0 < |c|, |d| \leq N(1 - \epsilon)} \frac{1}{|c|^3|d|}$$

$$+ O(\ln^2 N) \sum_{0 < |c|, |d| \leq N} \frac{1}{|c|^3|d|}.$$  

The first and second terms on the r.h.s. are $O(\frac{\ln N}{N})$ (in particular the first term is estimated by Lemma 10.1 in [27]). Concerning the third sum we can estimate it by

$$\sum_{n > N(1 - \epsilon)} \left( \sum_{0 < |a|, |b| < N} \frac{1}{|a||b|} \right) \left( \sum_{0 < |c| \leq N} \frac{1}{|c|^3|d|} \right).$$
and since by elementary considerations
\[ \sum_{0<|a|,|b| < N \atop |a+b| = n} \frac{1}{|a||b|} \lesssim \frac{N}{n}, \]
we can continue the estimate above:
\[ \ldots \lesssim \sum_{n>N(1-\epsilon)} \sum_{0<|c| \leq N \atop N(1-\epsilon)<|d| \leq N} \frac{1}{|c|^3|d|} \lesssim \sum_{0<|c| \leq N \atop N(1-\epsilon)<|d| \leq N} \frac{1}{|c|^3|d|} = O(\epsilon). \]
Concerning the sum on \( \tilde{A}_N^-(4) \) we can apply the Minkowski inequality and we get:
\[ \sum_{2|a|>N(1-\epsilon)} \frac{1}{|a|^3} + \sum_{0<|a|,|b| \leq N \atop |a+b| > N(1-\epsilon)} \frac{1}{|a||b|^2} = O\left(\frac{\ln N}{N}\right) + O(\epsilon), \]
where the estimate of the second term is obtained as follows:
\[ \sum_{0<|a|,|b| \leq N \atop |a+b| > N(1-\epsilon)} \frac{1}{|a||b|^2} \leq \sum_{0<|a|,|b| \leq N(1-\epsilon) \atop |a+b| > N(1-\epsilon)} \frac{1}{|a||b|^2} + \sum_{0<|a|,|b| \leq N \atop |b|>N(1-\epsilon) \atop |a+b| > N(1-\epsilon)} \frac{1}{|a||b|^2} + \sum_{0<|a|,|b| \leq N \atop N(1-\epsilon)<|a| \leq N \atop |a+b| > N(1-\epsilon)} \frac{1}{|a||b|^2} = O\left(\frac{\ln N}{N}\right) + O(\epsilon). \quad (8.7) \]
Here we have used Lemma 10.1 in [27] to estimate the first term on the r.h.s. The proof of (8.3) is similar. \( \square \)

We shall also need the following estimates.

**Proposition 8.4.** We have the following estimates:

\[
\begin{align*}
\| \int (S\varphi)^2 (HS\varphi_x)(I d - S^2)(S\varphi S\varphi_x) \|_{L^2(d\mu_{3/2})} & = O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon}), \\
\| \int (S\varphi)^3 H[(I d - S^2)(S\varphi S\varphi_x)_x] \|_{L^2(d\mu_{3/2})} & = O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon}), \\
\| \int (S\varphi)^2 \mathcal{H}(S\varphi S\varphi_x)(I d - S^2)(S\varphi S\varphi_x) \|_{L^2(d\mu_{3/2})} & = O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon}), \\
\| \int (S\varphi)^2 \mathcal{H}(S\varphi(I d - S^2)(S\varphi S\varphi_x)_x) \|_{L^2(d\mu_{3/2})} & = O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon}), \\
\| \int (S\varphi)^2 \mathcal{H}[S\varphi(I d - S^2)(S\varphi S\varphi_x)_x] \|_{L^2(d\mu_{3/2})} & = O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon}).
\end{align*}
\]
Proof. We focus on the first estimate (the others can be treated by the same arguments as below). We have to prove

\[
\| \sum_{(a,b,c,d,e) \in \mathcal{A}_N(5)} \frac{\text{sign}(\epsilon) \Gamma_N^e(a,b,c,d,e)}{|a|^{3/2}|b|^{3/2}|c|^{1/2}|d|^{3/2}|e|^{1/2} \overline{g}_a b \overline{g}_b c \overline{g}_d e} \|_{L^2_{\omega}}^2
\]

\[
= O\left(\frac{\ln N}{N}\right) + O(\sqrt{\epsilon}),
\]

(8.8)

where \(\Gamma_N^e(a, b, c, d, e)\) and \(\mathcal{A}_N(5)\) are defined in (5.7) and (5.8). Next we split \(\mathcal{A}_N(5) = \bar{\mathcal{A}}_N(5) \cup \bar{\mathcal{A}}_N^c(5)\) (see (5.9)) and we get by Proposition 3.1

\[
\| \sum_{(a,b,c,d,e) \in \bar{\mathcal{A}}_N(5)} \frac{\text{sign}(\epsilon) \Gamma_N^e(a,b,c,d,e)}{|a|^{3/2}|b|^{3/2}|c|^{1/2}|d|^{3/2}|e|^{1/2} \overline{g}_a b \overline{g}_b c \overline{g}_d e} \|_{L^2_{\omega}}^2
\]

\[
\leq \sum_{0 < |a|, |b|, |c|, |d|, |e| \leq N} \frac{1}{|a|^3|b|^3|c||d|^3|e|}
\]

\[
\leq \left( \sum_{|a|, |b|, |c| \leq N} \frac{1}{|a|^3|b|^3|c|} \right) \cdot \left( \sum_{|d|, |e| \leq N} \frac{1}{|d|^3|e|} \right),
\]

(8.9)

(here the constraint \(|d + e| > N(1 - \epsilon)\) comes from the cut–off \(\psi_\epsilon\)). Next notice that

\[
\sum_{|d|, |e| \leq N} \frac{1}{|d|^3|e|}
\]

\[
\leq \sum_{0 < |d|, |e| \leq N(1 - \epsilon)} \frac{1}{|d|^3|e|} + \sum_{|d| \geq N(1 - \epsilon)} \frac{1}{|d|^3|e|} + \sum_{N(1 - \epsilon) \leq |e| \leq N} \frac{1}{|d|^3|e|}
\]

\[
= O\left(\frac{\ln N}{N}\right) + O(\epsilon)
\]

(8.10)

where we have used Lemma 10.1 of [27] to estimate the first term on the r.h.s. By a similar argument

\[
\sum_{|a|, |b|, |c| \leq N} \frac{1}{|a|^3|b|^3|c|} = O\left(\frac{\ln N}{N}\right) + O(\epsilon).
\]

Hence the l.h.s. in (8.9) can be estimated by \(O\left(\frac{\ln^2 N}{N^2}\right) + O(\epsilon^2)\). Next we consider the set

\[
\bar{\mathcal{A}}_{N,a=-b}^c = \{(a, b, c, d, e) \in \mathcal{A}_N(5)|a = -b\}.
\]

(8.11)

By Proposition 3.2 we can estimate the sum on the set \(\bar{\mathcal{A}}_{N,a=-b}^c \cap \{(a, b, c, d, e)|a + b + c| > N(1 - \epsilon)\}\) by
\[ \sum_{0 < |a| \leq N} \frac{1}{|a|^3} \left( \sum_{|d+e| > N(1-\epsilon), 0 < |d|,|e| \leq N} \frac{1}{|c||d|^3|e|} \right)^{\frac{1}{2}} \leq \sqrt{\epsilon} \left( \sum_{|d+e| > N(1-\epsilon), 0 < |d|,|e| \leq N} \frac{1}{|d|^3|e|} \right)^{\frac{1}{2}} \]

(the condition \(|c| > N(1-\epsilon)\) comes from \(a + b + c = c\) on the set \(a = -b\), and by (8.10) we can continue the estimate

\[ \ldots \leq O(\sqrt{\epsilon})[O\left(\sqrt{\frac{\ln N}{N}}\right) + O(\sqrt{\epsilon})]. \]

Next we treat the sum on the set \(\tilde{A}_{N,b=-c}^c\). First notice that the constraint \(|a + b + c| > N(1-\epsilon)\) becomes \(|a| > N(1-\epsilon)\) and hence the estimate follows by

\[ \sum_{0 < |b| \leq N} \frac{1}{|b|^2} \left( \sum_{|a| > N(1-\epsilon)} \frac{1}{|a|^3|d|^3|e|} \right)^{\frac{1}{2}} = O\left(\frac{\sqrt{\ln N}}{N}\right). \]

By symmetry we can treat in the same way the sum on \(\tilde{A}_{N,a=-d}^c\).

Concerning the sum on \(\tilde{A}_{N,a=-d}^c\), we are reduced to the estimate:

\[ \sum_{0 < |a| \leq N} \frac{1}{|a|^3} \left( \sum_{|a+b+c| > N(1-\epsilon), 0 < |d| \leq N} \frac{1}{|b|^3|c||e|} \right)^{\frac{1}{2}} \leq \sum_{0 < |a| \leq N} \frac{1}{|a|} \left[ \sum_{|a+b+c| > N(1-\epsilon), 0 < |d| \leq N} \frac{1}{|b|^3|c||d|^4|e|} \right]^{\frac{1}{2}} + \left( \sum_{|a+b+c| > N(1-\epsilon), 0 < |d| \leq N} \frac{1}{|b|^3|c||d|^4|e|} \right)^{1/2} \]

\[ + \sum_{0 < |a| \leq N} \frac{1}{|a|} \left( \sum_{|a+b+c| > N(1-\epsilon), |e+d| > N(1-\epsilon), N(1-\epsilon) \leq |e| \leq N} \frac{1}{|b|^3|c||d|^4|e|} \right)^{1/2}. \] (8.12)

Notice that the first two terms on the r.h.s. give a contribution \(O\left(\frac{\ln^2 N}{\sqrt{N}}\right)\) (in particular to estimate the first term we used Lemma 10.1 in [27]). Concerning the last term, due to the fact \(a = -d\) and \(|-d + b + c| > N(1-\epsilon)\), it can be controlled by
equivalent to

\[ \sum_{0 < |a| \leq N} \frac{1}{|a|^2} \left( \sum_{|a+b+c| > N(1-\epsilon)} \frac{1}{|b|^3 |c||d|^2 |e|} \right)^{1/2} \]

\[ \leq \sum_{0 < |a| \leq N} \frac{1}{|a|^2} \sqrt{\epsilon} \left( \sum_{|d+b+c| > N(1-\epsilon)} \frac{1}{|b|^3 |c||d|^2} \right)^{1/2} \]

\[ = O(\sqrt{\epsilon})[O(\sqrt{\frac{\ln N}{N}}) + O(\sqrt{\epsilon})]. \]

At the last step we have used the estimate

\[ \sum_{|d+b+c| > N(1-\epsilon)} \frac{1}{|b|^3 |c||d|^2} = O(\epsilon) + O(\frac{\ln N}{N}), \]

that in turn follows by splitting the constraint in four subdomains given by:

\[ \{| - d + b + c | > N(1 - \epsilon), |d|, |b|, |c| \leq N (1 - \epsilon)\}, \]

\[ \{| - d + b + c | > N(1 - \epsilon), |d| > N (1 - \epsilon)\}, \]

\[ \{| - d + b + c | > N(1 - \epsilon), |b| > N (1 - \epsilon)\}, \]

\[ \{| - d + b + c | > N(1 - \epsilon), |c| > N (1 - \epsilon)\}. \]

The sum on the first constraint can be estimated by Lemma 3.2 in [29], the estimate on the second and third one are trivial, and the last one gives a contribution \( O(\epsilon) \).

Concerning the sum on \( \tilde{A}_{N,a=e}^c \), notice that the constraint \( |d+e| > N(1-\epsilon) \) is equivalent to \( |d-a| > N(1-\epsilon) \) and hence we are reduced to treat:

\[ \sum_{0 < |a| \leq N} \frac{1}{|a|^2} \left( \sum_{|d-a| > N(1-\epsilon)} \frac{1}{|b|^3 |c||d|^3} \right)^{1/2} \]

\[ \lesssim \sqrt{\ln N} \left( \sum_{|a| \geq N(1-\epsilon)/2} \frac{1}{|a|^2} + \sum_{|a| \leq N(1-\epsilon)/2} \frac{1}{|a|^2} \left( \sum_{|d| \geq N(1-\epsilon)/2} \frac{1}{|d|^3} \right)^{1/2} \right) \]

\[ = O \left( \frac{\sqrt{\ln N}}{N} \right). \quad (8.13) \]

Notice that the sum on the sets \( \tilde{A}_{N,b=-d}^c \) and \( \tilde{A}_{N,b=-e}^c \) are similar to the sum on \( \tilde{A}_{N,a=-d}^c \) and \( \tilde{A}_{N,a=-e}^c \), that have been already treated. Next we focus on the sum on the set \( \tilde{A}_{N,c=-d}^c \). In this case the constraint \( |a+b+c| > N(1-\epsilon) \) becomes \( |a+b-d| > N(1-\epsilon) \) and we are reduced to estimate.
\[
\sum_{0 < |d| \leq N} \frac{1}{|d|^2} \left( \sum_{|a + b - d| > N(1 - \epsilon)} \frac{1}{|a|^3 |b|^3 |e|} \right)^{\frac{1}{2}} \leq \sqrt{\ln N} \left[ \sum_{|d| \geq N(1 - \epsilon)/2} \frac{1}{|d|^2} + \left( \sum_{|a + b| \geq N(1 - \epsilon)/2} \frac{1}{|a|^3 |b|^3} \right)^{\frac{1}{2}} \right] = O \left( \frac{\sqrt{\ln N}}{N} \right).
\]

The last sum to be considered is on the set \( \hat{\mathcal{A}}_{N, \epsilon} \), where the constraint \(|d + e| > N(1 - \epsilon)|\) can be rewritten as \(|d - c| > N(1 - \epsilon)|\), hence we conclude by the following estimates:

\[
\sum_{0 < |c| \leq N} \frac{1}{|c|} \left( \sum_{0 < |a|, |b|, |c|, |d| \leq N} \frac{1}{|a|^3 |b|^3 |d|^3} \right)^{\frac{1}{2}} \leq \sum_{0 < |c| \leq N(1 - \epsilon)} \left( \sum_{0 < |d| \leq N(1 - \epsilon)} \frac{1}{|c|^2 |d|^3} \right)^{\frac{1}{2}} + O(\epsilon) = O \left( \frac{1}{\sqrt{N}} \right) + O(\epsilon),
\]

where we used Lemma 3.3 in [29] to estimate the first sum on the r.h.s. \( \square \)

**Proposition 8.5.** We have the following estimates:

\[
\| \int (S\varphi)^4 (Id - S^2) (S\varphi S\varphi_x) \|_{L^2(\mu_{3/2})} = O \left( \frac{1}{\sqrt{N}} \right).
\]

**Proof.** We have to show

\[
\| \sum_{(a,b,c,d,e,f) \in \mathcal{A}_N(6)} \frac{\text{sign}(f) \Lambda_N^\epsilon(a, b, c, d, e, f) g_a g_b g_c g_d g_e g_f}{|a|^{3/2} |b|^{3/2} |c|^{3/2} |d|^{3/2} |e|^{3/2} |f|^{1/2}} \|_{L^2_\omega} = O \left( \frac{1}{\sqrt{N}} \right),
\]

where:

\[
\Lambda_N^\epsilon(a, b, c, d, e, f) = \psi_e \left( \frac{a}{N} \right) \psi_e \left( \frac{b}{N} \right) \psi_e \left( \frac{c}{N} \right) \psi_e \left( \frac{d}{N} \right) \psi_e \left( \frac{e}{N} \right) \psi_e \left( \frac{f}{N} \right) [1 - \psi_e^2 \left( \frac{e + f}{N} \right)]
\]

and

\[
\mathcal{A}_N(6) = \{(a, b, c, d, e, f) \in \mathbb{Z}^6 | 0 < |a|, |b|, |c|, |d|, |e|, |f| \leq N, a + b + c + d + e + f = 0 \}.
\]
By the Minkowski inequality and by the cut–off property of $\psi_\epsilon$ we can estimate the l.h.s. in (8.14) by

$$
\sum_{(a,b,c,d,e,f) \in \mathcal{A}_N(6)} \frac{1}{|a|^{3/2}|b|^{3/2}|c|^{3/2}|d|^{3/2}|e|^{3/2}|f|^{1/2}} \leq \sum_{0 < |a|, |b|, |c|, |d|, |e| \leq N} \frac{1}{|a|^{3/2}|b|^{3/2}|c|^{3/2}|d|^{3/2}|e|^{3/2}} = O\left(\frac{1}{\sqrt{N}}\right) (8.15)
$$

where we used $|a + b + c + d| > N(1 - \epsilon)$ implies $\max\{|a|, |b|, |c|, |d|\} \geq N/4(1 - \epsilon)$.

**Proof of Proposition 8.2.** For simplicity we denote $\tilde{G} = G_N^\epsilon$ and $S = S_N^\epsilon$, then we get for a general function $u$

$$
\tilde{G}(u) = E_{3/2}(Su) + (\|u\|^2_{H^{3/2}} - \|Su\|^2_{H^{3/2}}),
$$

where

$$
E_{3/2}(u) = \|u\|^2_{H^{3/2}} - \left( \int \frac{3}{2} uu_x^2 + \frac{1}{2} u(Hu_x)^2 \right) - \int \left( \frac{1}{3} H u_{xx} + \frac{1}{4} u^2 H(uu_x) \right) - \frac{1}{20} \int u^5. (8.17)
$$

Arguing as in Proposition 6.2 and by using the notation $u(t) = \pi_N \Phi^\epsilon_N(t) \varphi$ we get:

$$
\frac{d}{dt} E_{3/2}(Su)_{t=0} = \int Su_t \mathcal{H}\mathcal{S}\varphi_{xxx} + \int S\varphi \mathcal{H}Su_{txxx}
- (\frac{3}{2} Su_t(S\varphi_x)^2 + \frac{1}{2} Su_t(\mathcal{H}S\varphi_x)^2) - (\int 3S\varphi S\varphi_x Su_{tx} + S\varphi(\mathcal{H}S\varphi_x)\mathcal{H}Su_{tx})
- \int (S\varphi)^2 Su_t \mathcal{H}S\varphi_x - \int \frac{1}{3}(S\varphi)^3 \mathcal{H}Su_{tx}
- \frac{1}{2} \int (S\varphi) Su_t \mathcal{H}(S\varphi S\varphi_x) - \frac{1}{4} \int (S\varphi)^2 \mathcal{H}(Su, S\varphi_x)
- \frac{1}{4} \int (S\varphi)^2 \mathcal{H}(S\varphi Su_{tx}) - \frac{1}{4} \int (S\varphi)^4 Su_{t=\pi_N^1(t)}(Id - S^2) \varphi S\varphi_x. (8.18)
$$

Moreover by using the equation solved by $u(t) = \pi_N \Phi^\epsilon_N(t) \varphi$ we get:

$$
\frac{d}{dt} (\|u\|^2_{H^{3/2}} - \|Su\|^2_{H^{3/2}})_{t=0}
= \int (-\mathcal{H}\varphi_{xx} - S(S\varphi S\varphi_x)) \mathcal{H}\varphi_{xxx} + \int \varphi \mathcal{H}(-\mathcal{H}\varphi_{xx} - S(S\varphi S\varphi_x))_{xxx}
+ \int (\mathcal{H}S\varphi_{xx} + S^2(S\varphi S\varphi_x))\mathcal{H}S\varphi_{xxx} + \int S\varphi \mathcal{H}(\mathcal{H}S\varphi_{xx} + S^2(S\varphi S\varphi_x))_{xxx}.
$$
By properties of $\mathcal{H}$ and integration by parts we get:

$$
\frac{d}{dt} \left( \| u \|_{\mathcal{H}^{3/2}}^2 - \| Su \|_{\mathcal{H}^{3/2}}^2 \right)_{t=0} = \int ( -S(S\varphi S\varphi_x)) \mathcal{H} \varphi_{xxx} + \int \varphi \mathcal{H}(-S(S\varphi S\varphi_x))_{xxx} + \int (S^2(S\varphi S\varphi_x)) \mathcal{H} S \varphi_{xxx} + \int S \varphi \mathcal{H}(S^2(S\varphi S\varphi_x))_{xxx} = 2 \int \varphi \mathcal{H}(-S(S\varphi S\varphi_x))_{xxx} + 2 \int (S^2(S\varphi S\varphi_x)) \mathcal{H} S \varphi_{xxx}. 
$$

(8.19)

By combining (8.18), (8.19) with (8.16) we deduce that the terms of order three in $\frac{d}{dt}(\tilde{G}(\pi_N \Phi^\epsilon_N(t)\varphi))_{t=0}$ give a trivial contribution, as the following computation shows:

**cubic terms in (8.18) + (8.19)**

$$
= 2 \int \varphi \mathcal{H}(-S(S\varphi S\varphi_x))_{xxx} + 2 \int (S^2(S\varphi S\varphi_x)) \mathcal{H} S \varphi_{xxx} + \int (Id - S^2)(S\varphi S\varphi_x) \mathcal{H} S \varphi_{xxx} + \int S \varphi \mathcal{H}(Id - S^2)(S\varphi S\varphi_x)_{xxx} = -2 \int \varphi \mathcal{H}(S(S\varphi S\varphi_x))_{xxx} + 2 \int (S^2(S\varphi S\varphi_x)) \mathcal{H} S \varphi_{xxx} + \int (S\varphi S\varphi_x) \mathcal{H} S \varphi_{xxx} + \int S \varphi \mathcal{H}(S\varphi S\varphi_x)_{xxx} + \int S^2(S\varphi S\varphi_x) \mathcal{H} S \varphi_{xxx} - \int S \varphi \mathcal{H} S^2(S\varphi S\varphi_x)_{xxx} = 0.
$$

Next by combining again (8.18), (8.19) with (8.16) we compute the contribution to $\frac{d}{dt}(\tilde{G}(\pi_N \Phi^\epsilon_N(t)\varphi))_{t=0}$ given by terms of order four:

**quartic terms in (8.18) + (8.19)**

$$
= - \int \frac{3}{2} (Id - S^2)(S\varphi S\varphi_x)(S\varphi_x)^2 - 3 \int S\varphi S\varphi_x(Id - S^2)(S\varphi S\varphi_x)_x - \int \frac{1}{2} (Id - S^2)(S\varphi S\varphi_x)(\mathcal{H} S \varphi_x)^2 - \int S \varphi (\mathcal{H} S \varphi_x) \mathcal{H}(Id - S^2)(S\varphi S\varphi_x)_x = - \int \frac{3}{2} (Id - S^2)(S\varphi S\varphi_x)(S\varphi_x)^2 - \int \frac{1}{2} (Id - S^2)(S\varphi S\varphi_x)(\mathcal{H} S \varphi_x)^2 - \int S \varphi (\mathcal{H} S \varphi_x) \mathcal{H}(Id - S^2)(S\varphi S\varphi_x)_x,
$$

where we used

$$
2 \int S\varphi S\varphi_x(Id - S^2)(S\varphi S\varphi_x)_x = \int \partial_x((Id - S^2)^{1/2}(S\varphi S\varphi_x))^2 = 0.
$$
By Proposition 8.3 we can estimate the terms above. Next we focus on the quintic terms:

quintic terms in (8.18) + (8.19)

\[-\int (S\varphi)^2(HS\varphi_x)((Id - S^2)(S\varphi S\varphi_x)) - \frac{1}{3} \int (S\varphi)^3H((Id - S^2)(S\varphi S\varphi_x)_x)\]
\[-\frac{1}{2} \int (S\varphi)H(S\varphi S\varphi_x)((Id - S^2)(S\varphi S\varphi_x))\]
\[-\frac{1}{4} \int (S\varphi)^2H(S\varphi_x(Id - S^2)(S\varphi S\varphi_x)) - \frac{1}{4} \int (S\varphi)^2H(S\varphi(Id - S^2)(S\varphi S\varphi_x)_x)\]

Notice that those terms can be controlled by Proposition 8.4. Next we notice that

sixth order terms in (8.18) + (8.19)

\[= -\frac{1}{4} \int (S\varphi)^4(Id - S^2)(S\varphi S\varphi_x)\]

and they can be controlled thanks to Proposition 8.5.

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