The eigenvalue equation on the Eguchi-Hanson space

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Abstract

We consider the eigenvalue equation for the Laplace-Beltrami operator acting on scalar functions on the non-compact Eguchi-Hanson space. The corresponding differential equation is reducible to a confluent Heun equation with Ince symbol \([0, 2, 1/2]\). We construct approximations for the eigenfunctions and their asymptotic scattering phases with the help of the Liouville-Green approximation (WKB). Furthermore, for specific discrete eigenvalues obtained by a continued T-fraction we construct the solution by the Frobenius method and determine its scattering phase by a monodromy computation.

1 Introduction

The Eguchi-Hanson metric \([\text{EH79}]\) is a self-dual, asymptotically locally Euclidean (ALE) metric on the cotangent bundle of the 2-sphere \(T^*S^2\). Geometrically, this corresponds to a Ricci-flat metric on the smooth resolution space of an \(A_1\)-singularity – such a singularity looks like the origin of \(\mathbb{C}^2/\mathbb{Z}_2\) where the \(\mathbb{Z}_2\)-group acts by point reflection at the origin.

Apart from Euclidean quantum gravity this metric has physical applications in string compactification. In fact, it is well known that the only nontrivial two dimensional Calabi-Yau manifold – this is a K3-surface – can be obtained from the \(\mathbb{Z}_2\)-orbifold limit \(T^2_{\mathbb{C}}/\mathbb{Z}_2\) by blowing up its 16 \(A_1\)-singularities (where \(T^2_{\mathbb{C}}\) is the complex two-dimensional torus). By gluing the Eguchi-Hanson metric together with the flat torus metric one can explicitly construct an almost Ricci-flat metric on a K3-surface \([\text{Boz88}]\) which is related to a Ricci-flat metric on K3 by a gauge transformation \([\text{Tau82}]\).

In this paper we shall examine the eigenvalue equation for the Laplace-Beltrami operator on the Eguchi-Hanson space. The problem is interesting from a mathematical point of view since the differential equation is separable and reduces to an ordinary differential equation which due to its singularities can be identified as confluent Heun equation with corresponding Ince symbol \([0, 2, 1/2]\) (see \([\text{DMR78}]\) for definitions). The Heun equation is an ordinary differential equation with four regular singularities on the punctured Riemann sphere. By coalescing two of the regular singularities to one irregular singularity one obtains the confluent Heun equation (analogously to the procedure by which one obtains the confluent hyper-geometric differential equation from the hyper-geometric one).

The problem is also interesting from a physical point of view since the functions that are obtained by gluing the eigenfunctions on the Eguchi-Hanson space together with the well known eigenfunctions on the flat torus describe the quantum mechanical limit of string fields on K3.

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The plan of the paper is as follows: In Sect. 2 we introduce the ordinary differential equation that describes the radial part of the eigenvalue equation of the Laplace operator on the Eguchi-Hanson space. In Sect. 3 we construct its solutions and their corresponding scattering phases by the Liouville-Green approximation (WKB). This extends and corrects a result in [Mig91]. In [Mig91] the author used the ad-hoc version of the Liouville-Green approximation. This version did not give approximations for the wave functions which are valid over the whole range. Furthermore, we give error bounds for the constructed solutions and their scattering phases. The explicit calculations can be found in the Appendix (cf. App. A and B). In Sect. 4 we construct the exact solutions for specific discrete eigenvalues by the Frobenius method. These special values are given by the vanishing condition for a continued fraction. This approach is similar to the treatment of the generalized spheroidal wave equation in [Wil28], [Lea86]. In addition, we can determine the exact scattering phase by a monodromy computation. Finally, we will show how this information can be used to compute the asymptotic scattering phase for this discrete set of eigenvalues. In Sect. 5 we present the numerical results obtained by the method of Sect. 3 and Sect. 4 and show that they match up to a high accuracy. In Sect. 6 we give the conclusions of this article and a brief outlook.

2 The Eigenvalue Equation

In this section we introduce the eigenvalue equation for the Laplace-Beltrami operator of the Eguchi-Hanson space. Using the $SU(2) \times U(1)$ symmetry of the Eguchi-Hanson space this eigenvalue equation will reduce to an ordinary differential equation of second order. A complete derivation of these and similar results can be found in [GM86], [Per78].

The Riemannian metric on $T^*S^2$ that is known as Eguchi-Hanson metric is Hyperkähler. Moreover, a complex structure which is compatible with the Hyperkähler structure can be introduced by identifying $T^*S^2$ with the complex manifold $T^*\mathbb{C}P^1$. In particular, the latter can be covered by two coordinate charts $U \cong U' \cong \mathbb{C}^2$ with coordinates $(u, \xi), (u', \xi')$, respectively. Here, $u, u'$ denote Euclidean coordinates on the base $\mathbb{C}P^1$, and $\xi, \xi'$ parameterize the fiber of the bundle $T^*\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$. This means that

$$U \cap U' \cong \mathbb{C}^* \times \mathbb{C}, \quad (u', \xi') \in U \cap U' : (u', \xi') = (u^{-1}, u^2 \xi). \quad (1)$$

Note that $T^*\mathbb{C}P^1$ describes the minimal resolution $\mathbb{C}^2/\mathbb{Z}_2$ of $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $(z^1, z^2) \in \mathbb{C}^2$ by $(z^1, z^2) \mapsto (-z^1, -z^2)$. To see this, on $U_0 := \{(u, \xi) \in U \mid \xi \neq 0\}$ use $(u, \xi) = (z^1, \sqrt{i}(z^2)^2)$, and analogously for the other chart. Then (1) is given by $z^1 \leftrightarrow z^2$, and by the above we identify $\mathbb{C}^2/\mathbb{Z}_2$ with $T^*\mathbb{C}P^1$ with the zero section removed. The exceptional divisor of $\mathbb{C}^2/\mathbb{Z}_2$ therefore corresponds to the zero section of $T^*\mathbb{C}P^1$. In these coordinates, i.e. outside the exceptional divisor, the Eguchi-Hanson metric takes the form

$$g_{ij}^{EH} = \frac{\sqrt{c^4 + R^4}}{R^2} \left\{ \delta_{ij} - \frac{c^4 z_i z_j}{R^2 (c^4 + R^4)} \right\},$$

where $R^2 = |z^1|^2 + |z^2|^2$,

and $c > 0$ is the parameter of the Eguchi-Hanson metric. Introducing the Euler angles $(\theta, \phi, \psi)$ on $S^3$ by

$$(z^1, z^2) = R \left( \begin{array}{c} \cos \frac{\theta}{2} e^{i/2 (\psi + \phi)} \\
\sin \frac{\theta}{2} e^{i/2 (\psi - \phi)} \end{array} \right),$$
we can describe coordinates (in the coordinate patch with \(z_2 \neq 0\)) by a radial part \(R \in \mathbb{R}_{>0}\) and the angles \(\theta \in (0, \pi), \phi \in [0, 2\pi), \psi \in [0, 4\pi)\). In the following, it will be more convenient to use \(r := \sqrt{R^4 + c^4}\) instead of \(R\).

The singularity of \(\mathbb{C}^2/\mathbb{Z}_2\) then sits at \(r = c\). On the exceptional divisor, the Eguchi-Hanson metric takes the form of the Fubini-Study metric on \(\mathbb{CP}^1\), and therefore assigns volume \(\pi\) to the exceptional divisor [EH79].

The \(\mathbb{Z}_2\)-operation mentioned above takes the form

\[
\mathbb{Z}_2 : \begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} \mapsto -\begin{pmatrix} z_1^1 \\ z_2^2 \end{pmatrix}, \quad (\theta, \psi, \phi) \mapsto (\theta, \psi + 2\pi, \phi).
\]

In these new coordinates the metric takes the form originally found by Eguchi and Hanson. The Lie group \(SO(3)\) acts on itself by multiplication from the left and from the right. Let us call the vector fields that generate the right-multiplication \(\xi_1, \xi_2, \xi_3\), and the ones that generate the left-multiplication \(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\). Since \(SO(3) \cong SU(2)/\mathbb{Z}_2 \cong S^3/\mathbb{Z}_2\) we have an action of these vector fields on the Eguchi-Hanson space. In [GM86] the authors derive that \(\forall i, j : [\xi_i, \bar{\xi}_j] = 0\) and that \((\xi_1, \xi_2, \xi_3, \bar{\xi}_1)\) define Killing vector fields that generate an \(su(2)_L \oplus u(1)_R\) symmetry algebra of the Eguchi-Hanson space.

Let \(\Delta\) denote the Laplace-Beltrami operator on functions that is associated to the Eguchi-Hanson metric. Furthermore, let \(\Psi\) denote a smooth function on the Eguchi-Hanson space. We consider the eigenvalue equation for the real positive eigenvalue\(^1\), i.e. \((E + \Delta)\Psi = 0\) for \(E > 0\).

The operator \(E + \Delta\) can be expressed merely in terms of \(\frac{d}{dr}\), \(\xi^2 = \sum_{i=1}^{3} \xi^2_i, \xi^3_i, i.e.

\[
E + \Delta = E + \left(1 - \frac{c^4}{r^4}\right) \frac{d^2}{dr^2} + \left(\frac{3}{r} + \frac{c^4}{r^3}\right) \frac{d}{dr} + \frac{4c^2}{r^2} + \frac{4c^4 \xi^2_3}{r^2(r^4 - c^4)}.
\]

Due to their commutation relations we can diagonalize the operators \(\xi_3, \bar{\xi}_3, \xi^2\) simultaneously and expand the eigenfunctions in terms of Wigner functions \(D_{qm}^j(\theta, \psi, \phi)\) (see [AS55] for definitions), i.e.

\[
\Psi(r, \theta, \psi, \phi) = \sum_{j=0}^{\infty} \sum_{q,m=-j} \alpha_{qm}^j A(j, q, \beta | z) \ D_{qm}^j(\theta, \psi, \phi),
\]

where \(z := \frac{\xi^2}{c^2}, \beta := \frac{\xi^2}{c^4}\), and \(\alpha_{qm}^j\) are complex coefficients, and the \(A(j, q, \beta | z)\) are functions that depend only on the coordinate \(z\). The Wigner functions fulfill

\[
\begin{align*}
\bar{\xi}_3 D_{qm}^j &= -im D_{qm}^j, \\
\xi_3 D_{qm}^j &= iq D_{qm}^j, \\
\left(\xi^2_3 + \xi^2_2 + \xi^2_3\right) D_{qm}^j &= -j(j+1) D_{qm}^j,
\end{align*}
\]

where \(j \in \mathbb{N}\), \(q, m \in [-j,j]_\mathbb{N}\).

Here, \(j, q\) are the quantum numbers that label the \(SU(2)\) representation. The eigenvalue equation for \(\Psi\) then reduces to an ordinary differential equation for \(A(j, q, \beta | z)\).

\[
0 = \left[\frac{d^2}{dz^2} + \frac{2z}{z^2 - 1} \frac{d}{dz} + \frac{(\beta z - j(j+1))(z^2 - 1) - q^2}{(z^2 - 1)^2}\right] A(j, q, \beta | z). \tag{2}
\]

\(^1\text{We are considering only the scattering case since we want to glue together the eigenfunctions on the Eguchi-Hanson space and the flat torus. However, eigenfunctions for } E < 0 \text{ would correspond to bounded states.}\)
The differential equation (2) has three singularities which we have summarized in the following table

| $z$ | $r$ | singularity | roots |
|-----|-----|-------------|-------|
| 1   | $e$ | regular     | $\pm \frac{q}{2}$ |
| -1  | $i\,e$ | regular     | $\pm \frac{q}{2}$ |
| $\infty$ | $\infty$ | irregular | $-$ |

(for $q = 0$ the regular singularities are logarithmic).

One should notice that our differential equation has an irregular singularity at infinity, thus is not of Fuchsian type. More precisely, it is a confluent Heun equation with Ince symbol $[0,2,1_2]$. To construct a continuous solution valid on the whole Eguchi-Hanson sphere we have to choose the regular boundary condition at $z = 1$ (then we can extend this solution to $z = 1$, i.e. the blown up $A_1$-singularity). This means that we are looking for the recessive solutions that behave like $(z - 1)^{\frac{q}{2}}$ for $z \to 1$. This means that all the eigenfunctions obtain a constant value on the entire $\mathbb{CP}^1$ which gives the exceptional divisor of the blow-up. By approaching the singular point of $\mathbb{C}^2/\mathbb{Z}_2$ with different slopes, one reaches different points in the exceptional divisor. But for a well-defined solution of the differential equation, the limit does not depend on the chosen slope.

We remark that the differential equation (2) does not depend on $q$ but only on $q^2$. Therefore, the function $\Psi$ takes the following form

$$\Psi(r, \theta, \psi, \phi) = \sum_{j=0}^{\infty} \sum_{q=0}^{j} A(j, q, \beta | z) \sum_{m=-j}^{j} \left( \alpha_{qm} D_{qm}^{j}(\theta, \psi, \phi) + \alpha_{-qm} D_{-qm}^{j}(\theta, \psi, \phi) \right).$$

On these functions the differential operator of Eq. (2) is self-adjoint: Since the differential operator in Eq. (2) is already formally self-adjoint the statement follows from the application of the results in [CL55, Sect. 3.9] and a detailed study of the singularity at $z = 1$ for $q = 0$, where the differential equation is of a limit-circle type, and for $q \geq 1$, where it is of limit-point type – for more details see [CL55].

The specific problems that arise in the treatment of the differential equation (2) are due to the structure of singularities. In particular, the major issue is to treat three singularities – one being an irregular singularity – at the same time. The problem is to derive the connection between the different bases which provide expansions of the solutions in the neighborhood of the singular points. Another important question is how given a system of solutions the solutions transform into each other when passing through a cycle around the corresponding singularity.

For the thrice-punctured Riemann sphere with only regular singularities (this is a generalized hyper-geometric equation) this can be done in terms of the Meijer transcendental functions – as they were recently applied in [GL01]. Here, the key technique lies in a representation of the solutions in terms of Mellin-Barnes integrals.

For only two singularities – one being regular, one being singular – techniques can be applied which are familiar from the treatment of the Bessel differential equation. This corresponds to the confluent hyper-geometric equation, i.e. a hyper-geometric equation where two regular singularities are coalescing and forming one irregular singularity. Here, the key technique is a generalized Borel transformation [GM94b], [GM94a] which relates a cycle around the regular singularity at $z = 1$ to the irregular one at $z = \infty$.

In the case of two regular and one irregular singularities we cannot apply Mellin-Barnes integrals. Due to the irregular singularity solutions are oscillating at large real values. Thus, the
conditions for convergence for the Mellin-Barnes integrals when continuing into the complex plane to close the path of integration to a cycle are not satisfied any more.

On the other hand, due to the third singularity at \( z = -1 \) a cycle around infinity is also not homologous to a cycle around \( z = 1 \). Thus, an asymptotic expansion of the regular solution cannot be derived by a Borel transformation. However, as we will show in Sect. 4.2 in some particular cases one can still use a similar argument.

3 WKB type solutions

In this section we give approximations to the recessive solutions of differential equation (2) that can be obtained by the Liouville-Green approximation (WKB). The explicit derivation of the results can be found in App. A.1 to A.4.

From now on we suppress the labels \( \beta, j, q \) in \( A(\beta, j, q | z) \). Substituting \( A(z) = (z^2 - 1)^{-\frac{1}{2}} w(u, z) \) in Eq. (2) we find

\[
\frac{d^2}{dz^2} w(u, z) = \left[ -u^2 f(z) + g(z) \right] w(u, z),
\]

where \( u^2 = \beta, \ g(z) = -(z^2 - 1)^{-2}, \ a = \frac{b(j+1)}{\beta}, \ b = \frac{2^2}{\beta} \) and

\[
f(z) = \frac{(z - a)(z^2 - 1) - b}{(z^2 - 1)^{\frac{3}{2}}}.\]

We remark that due to the possible values for \( j, q, \beta \) we will always have \( 1 \leq b + 1 < a \) and \( 0 \leq a \).

Eq. (3) is the standard form of a differential equation of second order considered for Liouville-Green approximation (WKB). Following the discussion in [Olv74], the specific approximation depends on the order of the pole at \( z = 1 \) as well as the number of simple transition points (tps), i.e. simple zeros, in the region \( z > 1 \) which is of interest in connection with the geometry discussed in Sect. 2. A simple analysis of the function \( f \) shows that we have to deal with the four different cases (I to IV) which are summarized in the table below.

| case | condition | order of pole at \( z = 1 \) | tp for \( z > 1 \) |
|------|-----------|----------------|----------------|
| I    | \( 1 < b + 1 < a \) | 2 | yes |
| II   | \( b = 0, 1 < a \) | 1 | yes |
| III  | \( b = 0, a = 1 \) | 0 | no |
| IV   | \( b = 0, 0 \leq a < 1 \) | 1 | no |

For the different cases one can then apply the Liouville-Green approximation: First, one performs a Liouville-transformation of the variable \( z \) to the new variable \( \zeta \). The transformation is given by an integral equation of the form

\[
\forall z \geq z_0 : G(\zeta) = \int_{z_0}^{z} \sqrt{|f(t)|} \ dt
\]

such that \( \zeta \) and \( z \) are analytic functions of each other, and where the function \( G \) and the point \( z_0 \) depend on the case (I to IV) we are dealing with. Simultaneously, we replace the function \( w \) by a function \( W \) according to \( w(u, z) = \sqrt{|\frac{d\zeta}{dz}|} W(u, \zeta) \). The choices are as follows:
\[
\begin{array}{c|c|c}
\text{case} & G(\zeta) & z_0 \\
\hline
I & \frac{2}{3}(-\zeta)^{\frac{5}{2}} & > 1 \text{ such that } f(z_0) = 0 \\
II & \zeta^{\frac{1}{2}}(\zeta-\alpha)^{\frac{3}{2}} - \frac{\theta}{2} \ln \left(\frac{2\zeta-\alpha+2\zeta^{\frac{1}{2}}(\zeta-\alpha)^{\frac{1}{2}}}{\alpha}\right) & a \\
III & \zeta & 1 \\
IV & (-\zeta)^{\frac{5}{2}} & 1 \\
\end{array}
\]

and

\[
\alpha := \frac{2}{\pi} \int_1^a dt \sqrt{-f(t)}. 
\]

The aim of this transformation is to transform the differential equation (3) to a differential equation of the form

\[
\frac{d^2}{d\zeta^2} W(u, \zeta) = \left( T_1(u, \zeta) + T_2(\zeta) \right) W(u, \zeta),
\]

where \(T_1\) and \(T_2\) are real-valued functions such that the approximating differential equation obtained by omitting \(T_2\) has solutions which are functions of a single variable. However, this transformation has been done in a way that the approximate solution will still reflect the right behavior of the solution at the singularity at \(z = 1\) and for \(z \to \infty\). From now on we suppress the dependence of \(u\) in \(w\) and \(W\).

We have found the following approximations for the solutions of differential equation (2) that are regular at \(z = 1\)

\[
\begin{array}{c|c|c}
\text{case} & \text{approx. function} & \text{form} \\
\hline
I & \text{Airy} & w(z) = 4\sqrt{-\zeta f(z)} \text{Ai}(u^{\frac{1}{3}} \zeta) \\
II & \text{Whittaker} & w(z) = 4\sqrt{-\zeta f(z)} e^{-\frac{2u}{\pi} M_{\frac{3}{2},0} (2iu \zeta)} \\
III & \text{Bessel} & w(z) = \sqrt{\frac{2}{f(z)}} J_0(u \zeta) \\
IV & \text{Bessel} & w(z) = \sqrt{\frac{2}{f(z)}} J_0(u |\zeta|^{\frac{1}{2}}) \\
\end{array}
\]

Asymptotically, the Eguchi-Hanson metric becomes the flat metric. Therefore, the solutions of the differential equation (2) must have the following behavior

\[
A(z) \underset{z \to \infty}{\sim} \frac{1}{z^{\frac{3}{2}}} \sin \left(2\sqrt{\beta z} + \Delta_{j,q}\right),
\]

where \(\Delta_{j,q}\) is called scattering phase. Based on the results of Olver et al. (see App. A.1 to A.4 for details), for the scattering phase we obtain the results listed below, where in the last column we give the equation number for the error bounds that are determined in App. A.

\[
\begin{array}{c|c|c}
\text{case} & \Delta_{j,q}(\beta) & \text{error} \\
\hline
I & \sqrt{\beta} \lim_{z \to \infty} \left(\int_{\infty}^z \sqrt{f(t)} dt - 2\sqrt{z} + \frac{\pi}{4}\right) & (21) \\
II & -2\sqrt{\beta} \sqrt{1+a} \ E \left[\sqrt{\frac{2}{1+a}}\right] + \frac{\alpha \sqrt{\beta}}{2} - \frac{\alpha \sqrt{\beta}}{2} \ln \left(\frac{a \sqrt{\beta}}{2}\right) + \arg \Gamma \left(\frac{1}{2} + \frac{ia \sqrt{\beta}}{2}\right) + \frac{\pi}{4} & (28) \\
III & -\sqrt{8\beta + \frac{\pi}{4}} & (26) \\
IV & (1-a)2\sqrt{\beta} K \left[\sqrt{\frac{1+a}{2}}\right] - 2\sqrt{2\beta} \ E \left[\sqrt{\frac{1+a}{2}}\right] + \frac{\pi}{4} & (29) \\
\end{array}
\]

Here, \(F(\phi, m), E(\phi, m)\) denote elliptic integrals of the first and second kind, and \(K(m) = F\left(\frac{\pi}{2}, m\right)\) and \(E(m) = E\left(\frac{\pi}{2}, m\right)\) are the corresponding complete elliptic integrals (c.f. [AS55, Sect. 17]).

This corrects a result in [Mig91].
4 Solutions related to continued fractions

In this section we determine for which values of $\beta, j, q$ the exact solution of the differential equation (2) can be obtained by a formal power series expansion (Frobenius method) around the singularity at $z = 1$.

For an expansion around the regular singularity at $z = 1$ a transformation according to $\zeta = \frac{1}{2}(z - 1)$ is suitable. Substitution of

\[ A(z) = (z^2 - 1)^{\frac{q}{2}} u(\zeta) \]  

in Eq. (2) yields

\[ 0 = \zeta(\zeta + 1) u''(\zeta) + (q + 1)(2\zeta + 1) u'(\zeta) + \left( \beta(2\zeta + 1) + \mu \right) u(\zeta) , \]  

where $\mu = q(q + 1) - j(j + 1)$. In a neighborhood of $z = 1$ the two linearly independent solutions can be represented by the series

\[ u_{\text{reg}}(\zeta) = \sum_{k=0}^{\infty} a_k(\beta, j, q) \zeta^k , \]  

\[ u_{\text{sing}}(\zeta) = u_{\text{reg}}(\zeta) \left( \ln \zeta + B_u \right) + \frac{1}{\zeta_q} \sum_{k=0}^{\infty} b_k(\beta, j, q) \zeta^k , \]

where $B_u$ is a real number. Since the singular solution is not unique (one can always add a multiple of the regular solution), the parameter $B_u$ is not uniquely determined. However, the parameter can be fixed by fixing the asymptotic scattering phase of the singular solution.

By the Frobenius method (cf. [Rab72, Chapt. 3.6]) we obtain the coefficients $(a_k)_{k \geq -1}$ as solutions of the following three-term recurrence relation

\[ a_{k+1} = -\frac{k(k + 2q + 1) + \mu + \beta}{(k + 1)(k + q + 1)} a_k - \frac{2\beta}{(k + 1)(k + q + 1)} a_{k-1} . \]  

Notice that a rescaling of the parameter $a_0$ results in a general rescaling of all the coefficients $(a_k)_{k \geq -1}$ since Eq. (8) is linear. However, the crucial information, i.e. the ratio of $a_1$ and $a_0$ is fixed by the recurrence relation (8). By standard methods one can show that there are only two types of solutions for $(a_k)_{k \in \mathbb{N}}$ in (8) depending on the radius of convergence $r_c$ of the series $\sum_{k=0}^{\infty} a_k \zeta^k$

\[ \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \begin{cases} \frac{-1}{r_c} & r_c = 1, \\ 0 & r_c = \infty. \end{cases} \]

A solution with $r_c = 1$ corresponds to a solution of the recurrence relation for generic values of $(\beta, j, q)$ whereas the solution with $r_c = \infty$ is the minimal solution.\(^2\) The question for which values of the parameters $(\beta, j, q)$ this minimal solution exists will be considered in Chap. 4.\(^1\)

One should mention that it is possible to derive an explicit representation of the coefficients $(a_k)_{k \in \mathbb{N}}$ by the use of Babister’s inhomogeneous hyper-geometric functions [Ext91a]. However,\(^2\)

\(^2\)A minimal solution $(g_n)_{n \in \mathbb{N}}$ is defined by the universal property that for any other solution $(h_n)_{n \in \mathbb{N}}$ one obtains $\lim_{n \to \infty} \frac{2\pi}{\ln n} = 0$.\(^1\)
this approach did not enable us to derive the asymptotic behavior in the case of a minimal solution. To solve the differential equation (6) by a Laplace transformation as in [Ext91b] for similar differential equations or an Euler transformation as in [Kaz98] is not possible due to the structure of the coefficients in Eq. (6).

To gain a better understanding of solutions to (6), let us take a look at the solution $u_{\text{reg}}$ which is regular at $z = 1$ ($\zeta = 0$) in terms of the singularity at $z = -1$ ($\zeta = -1$). Let $v_{\text{reg}}(\zeta + 1)$ be the solution regular at $z = -1$ ($\zeta = -1$). Its continuation to $\zeta \to \infty$ shall have an asymptotic scattering phase of $\gamma$, which by (4) and (5) gives an asymptotic expansion

$$v_{\text{reg}}(\zeta + 1) \xrightarrow{\zeta \to \infty} \frac{\sin (2\sqrt{2\beta \zeta} + \gamma)}{\zeta^{q + \frac{3}{4}}} \left(1 + O \left(\frac{1}{\zeta}\right)\right).$$

We fix a solution singular at $z = -1$ ($\zeta = -1$) by the requirement that its asymptotic scattering phase differs from the regular solution by a phase of $\frac{\pi}{2}$, giving an asymptotic expansion

$$v_{\text{sing}}(\zeta + 1) \xrightarrow{\zeta \to \infty} \frac{\cos (2\sqrt{2\beta \zeta} + \gamma)}{\zeta^{q + \frac{3}{4}}} \left(1 + O \left(\frac{1}{\zeta}\right)\right).$$

To resume, looking for the nontrivial solutions of the differential equation (2) we have already found two contributions: for a generic set of parameters $(\beta, j, q)$ the solution regular at $z = 1$ ($\zeta = 0$) has a regular singularity at $z = -1$ ($\zeta = -1$). This means that it is a linear combination of $v_{\text{reg}}$ and $v_{\text{sing}}$, without loss of generality:

$$u_{\text{reg}}(\zeta) = \cos (\alpha) v_{\text{reg}}(\zeta + 1) + \sin (\alpha) v_{\text{sing}}(\zeta + 1) \xrightarrow{\zeta \to \infty} \frac{1}{\zeta^{q + \frac{3}{4}}} \sin (2\sqrt{2\beta \zeta} + \gamma + \alpha) \left(1 + O \left(\frac{1}{\zeta}\right)\right) = \Delta_{j,q}(\beta).$$

Because of the singularity at $z = -1$ ($\zeta = -1$) the representation of $u_{\text{reg}}$ by a power series expansion must break down for all values $|\zeta| > 1$. Therefore, the asymptotic scattering phase of $u_{\text{reg}}$ can in general not be obtained from (7).

Only for those values $(\beta_n)_{n \in \mathbb{N}}$ that lead to a minimal solution of the recurrence relation (8) the solution regular at $z = 1$ is regular at $z = -1$ as well, and therefore of class $C^\infty(\mathbb{R})$. This means $\sin \alpha = 0$ and therefore

$$\Delta_{j,q}(\beta_n) = \gamma \mod \pi.$$

Since we can always change the sign of the solution by an overall factor the scattering phase is only determined up to an integer multiple of $\pi$. In Chapt. 1.2 we will eventually determine $\gamma$.

If we had made an expansion around the regular singularity at $z = -1$ a transformation according to $\chi = \frac{1}{2}(z + 1)$ would have been suitable. Substitution of

$$A(z) = (z^2 - 1)^{\frac{3}{2}} v(\chi)$$

in Eq. (2) then yields

$$0 = v''(\chi) + \left(\frac{q + 1}{\chi} + \frac{q + 1}{\chi - 1}\right) v'(\chi) + \left(\frac{\mu - \beta}{\chi(\chi - 1)} + \frac{2\beta}{\chi - 1}\right) v(\chi).$$

Eq. (10) can also be understood as the isomonodromic deformation of the confluent Heun equation with an additional apparent regular singularity moving towards $\chi = 1$ [Sla99]. By a
theorem: Eq. (10) is then related to a Painlevé equation, in our case $P^V$: The Painlevé equation is the Newtonian equation of motion corresponding to the quantum Hamiltonian given by means of the Heun equation. This is clear since the $SU(2)$-invariant self-dual metrics are specified exactly by a solution of the Painlevé equation $P^V$. 

4.1 Minimal solutions and continued fractions

According to Pincherle’s Theorem (see [JT80, Sect. 5.3]) a three-term recurrence relation

$$\forall n \geq 1 \quad y_{n+1} = -\delta_n y_n + \gamma_n y_{n-1}$$

has a minimal solution if and only if the following continued fraction converges:

$$\sum_{k=1}^{\infty} \frac{|\gamma_k|}{|\delta_k|} = \frac{\gamma_1}{\delta_1 + \frac{\gamma_2}{\delta_2 + \frac{\gamma_3}{\delta_3 + \ldots}}}. $$

In particular, if $(a_n)_{n \in \mathbb{N}}$ is the minimal solution it follows that $a_0 \neq 0$ and

$$-\frac{a_1}{a_0} + \sum_{k=1}^{\infty} \frac{|\gamma_k|}{|\delta_k|} = 0 .$$

For any set of parameters $(j, q)$, where as before $q \geq 0$, we now define the function $\tilde{M}(j, q \mid x)$ by the continued fraction of Thron type (or T-fraction)

$$\tilde{M}(j, q \mid x) := \delta_0 + \sum_{k=1}^{\infty} \frac{|\gamma_k|}{|\delta_k|},$$

where $\forall \; k \geq 0 \quad \delta_k := \frac{k(k+2q+1) + \mu - x}{(k+1)(k+q+1)}$, $\forall \; k \geq 1 \quad \gamma_k := \frac{2x}{(k+1)(k+q+1)}$.

Using the Umordnungssatz [Per77, Kap. 6.42, Satz 2] and [JT80, Sect. 7.3, Th. 7.23], we can conclude that the function $\tilde{M}(j, q \mid x)$ is a meromorphic function on $\mathbb{C}$. Moreover, by the above its real zeros determine the values of $x = -\beta$ for which the recurrence relation \((8)\) has a minimal solution, i.e. the radius of convergence of the power series expansion \((7)\) becomes infinite.

The corresponding solution $u_{\text{reg}}$ is then of class $C^\infty(\mathbb{R})$. Since the relation $A(\beta, j, q \mid z) = A(-\beta, j, q \mid -z)$ holds for the differential equation \((2)\), it follows that any such smooth solution for $\beta, j, q$ in $z$ is simultaneously a smooth solution for $-\beta, j, q$ in the variable $-z$. Therefore, $x = \beta$ must be another zero of $\tilde{M}(j, q \mid x)$. This is

$$\tilde{M}(j, q \mid \beta) = 0 \iff \tilde{M}(j, q \mid -\beta) = 0 .$$

Using the Umordnungssatz [Per77, Kap. 6.42, Satz 2], one can see that one can cancel the factor $(k+1)(k+q+1)$ in $\gamma_k, \delta_k$. Namely, it is equivalent to calculate the zeros of the meromorphic function $M(j, q \mid x)$ instead of $\tilde{M}(j, q \mid x)$ where $M(j, q \mid x)$ is defined by the continued fraction

$$M(j, q \mid x) := d_0 + \sum_{k=1}^{\infty} \frac{|c_k|}{|d_k|},$$

where $\forall \; k \geq 0 \quad d_k := k(k+2q+1) + \mu - x$, $\forall \; k \geq 1 \quad c_k := 2k(k+q)x$.
For this continued fraction one can even prove separate convergence (see [Thr91] for definitions). To see this, recall that $\mu = (q - j)(j + q + 1)$, and rewrite the continued fraction in the following way:

\[ \forall j_{-q} \geq 1 \quad M(j, q \mid x) = d_0 + \sum_{k=1}^{j-q-1} \frac{c_k}{d_k} + \frac{2j(q-j)}{x-1-m(j,q|x)}, \]

\[ \forall j \geq 0 \quad M(j, j \mid x) = x \left(-1 + \frac{m(j,j|x)}{x}\right). \]

Here, $m(j, q \mid x) := \sum_{k=j-q+1}^{\infty} \frac{c_k}{d_k}$ can be written as a T-fraction using again the Umordnungssatz [Per77, Kap. 6.42, Satz 2], i.e.

\[ m(j, q \mid x) = \sum_{k=j-q+1}^{\infty} \frac{F_k x}{1 + G_k x}, \quad (12) \]

where $\forall k_{j-q+1} \geq 1 \quad G_k := \frac{1}{(k-j+q)(k+j+q+1)}$, $\forall k_{j-q+2} \geq 1 \quad F_k := \frac{2k(k+q)}{(k-j+q+1)(k-j+q)(k+j+q)(k+j+q+1)}$, $F_{j-q+1} := j-q+1$.

Since we have both $\sum_{k=j-q+1}^{\infty} |F_k| < \infty$ and $\sum_{k=j-q+1}^{\infty} |G_k| < \infty$ we can apply [Thr91] Sect. 3, Th. 3.1] to the T-fraction (12): let $A_k(x)$ and $B_k(x)$ be the numerators and denominators, respectively, of the $n$th approximant of the T-fraction (12). Then the sequences $(A_k(x))_{k \geq j-q+1}$ and $(B_k(x))_{k \geq j-q+1}$ converge, uniformly on compact subsets of $\mathbb{C}$, to entire functions $A(x)$ and $B(x)$ of order at most one. Further $B(0) = 1$, $A(0) = 0$, $A'(0) = F_{j-q+1}$ so that neither function is identically zero, and $\frac{1}{x} m(j, q \mid x)$ is well defined at $x = 0$.

### 4.2 The determination of the scattering phase and the monodromy

In this section we calculate the scattering phase of the regular solution $u_{\text{reg}}$ in the case that it is also regular at $\zeta = -1$.

Let us first look at the asymptotic expansion of the solutions of Eq. (6). For an asymptotic expansion a Fabry transformation [Olv74] is suitable, i.e. a change of the variable according to $x^2 = \zeta$. With $u(\zeta) = U(x)$ Eq. (6) becomes

\[ (x^2 + 1)U''(x) + \left(-\frac{x^2+1}{x} + \frac{2(q+1)(2x^2+1)}{x}\right) U'(x) + \left(4\beta(2x^2+1) + 4\beta \right) U(x) = 0. \]

By standard methods (c.f. [CL55]) one finds that this equation has two linearly independent solutions $H_1, H_2$, such that for some $\delta > 0$ the following asymptotic expansions hold:

\[ \forall x \quad -2\pi + \delta \leq \arg x \leq \pi - \delta : \quad H_1(x) \sim \frac{1}{x^{2^\beta+\frac{1}{2}}} e^{2i\sqrt{2\beta x}} \sum_{k=0}^{\infty} \frac{(-1)^k p_k}{(4i\sqrt{2\beta x})^k}, \]

\[ \forall x \quad -\pi + \delta \leq \arg x \leq 2\pi - \delta : \quad H_2(x) \sim \frac{1}{x^{2^\beta+\frac{1}{2}}} e^{-2i\sqrt{2\beta x}} \sum_{k=0}^{\infty} \frac{p_k}{(4i\sqrt{2\beta x})^k}, \quad (13) \]
where the coefficients \((p_k)_{k \in \mathbb{N}}\) fulfill the following four-term recurrence relation

\[
\forall k \geq 0 : \quad p_{k+1} = \frac{k(k+1) - 4\beta - 4j(j+1) - \frac{3}{k+1}}{k+1} p_k + 32\beta \frac{k+q}{k+1} p_{k-1} - 32\beta \frac{k^2 + (2q-1)k - q + \frac{1}{4}}{k+1} p_{k-2}.
\] (14)

It follows from this recurrence relation that for \(p_0 \in \mathbb{R}\) the coefficients \((p_k)_{k \in \mathbb{N}}\) are real. In particular, we have \(H_1(x) = \overline{H_2(x)}\). This property remains true not only asymptotically, but also for the actual solutions: Since all the coefficients in the differential equation are real, the complex conjugate of any solution is again a solution.

The Stokes phenomenon will lead to a non-trivial monodromy of the solutions if we pass through a cycle around infinity \(\zeta \mapsto \zeta e^{2\pi i}\). This is because for \(x \mapsto xe^{\pi i}\) we will leave the sector in which the asymptotic expansion \([14]\) for \(H_2\) holds \([DO95]\). However, since the solutions \(H_1, H_2\) are a complete system of solutions the new solutions can be expressed as linear combinations of them. These are the connection formulae. In the case of a second order differential equation they had been explicitly calculated, namely in \([DO91]\). We obtain for \(|x| \gg 0\)

\[
\lim_{\theta \to 1} H_1(xe^{\pi i \theta}) = -2\pi P H_1(x) + i H_2(x),
\]

\[
\lim_{\theta \to 1} H_2(xe^{\pi i \theta}) = i H_1(x).
\] (15)

The parameter \(P\) can be determined by a generalized Borel transformation of the asymptotic solution as pointed out in \([GM94a]\), \([GM94b]\). For ordinary differential equations of second order this has been done explicitly by Daalhuis and Olver \([DO94]\). Applying these results we can determine the parameter \(P\) from the coefficients in the asymptotic expansion \([14]\), i.e.

\[
P = \lim_{k \to \infty} \frac{1}{(k-1)!} \frac{p_k}{p_0}.
\] (16)

In particular, the parameter \(P\) is real. We subsume the solutions \(H_1, H_2\) in the vector \(\overrightarrow{H} := t(H_1, H_2)\).

We need to calculate the linear combination of the regular solution \(u_{\text{reg}}(\zeta)\) in terms of the solutions \(\overrightarrow{H}(x)\) derived earlier. This amounts to determining a complex parameter \(\lambda \in U(1)\) such that

\[
u_{\text{reg}}(\zeta) = (\lambda \overline{\lambda}) \cdot \overrightarrow{H}(x).
\]

The occurrence of \(\lambda\) and \(\overline{\lambda}\) is due to the fact that \(u_{\text{reg}}(\zeta)\) is real, i.e. \(\overline{u_{\text{reg}}(\zeta)} = u_{\text{reg}}(\zeta)\), and \(\overline{H_1(x)} = H_2(x)\).

Now, the representation of \(u_{\text{reg}}(\zeta)\) by the power series \([7]\) is valid for \(|\zeta| > 1\) iff the coefficients \((a_k)_{k \geq 1}\) constitute a minimal solution of the recurrence relation \([8]\). Therefore, \(u_{\text{reg}}(\zeta)\) has a trivial monodromy around both of the regular singularities iff the \((a_k)_{k \geq 1}\) constitute a minimal solution of Eq. \([8]\). Hence, iff for any set of parameters \((\beta, j, q)\) the equation \(M(j, q|\beta) = 0\) holds, then the following equation must hold:

\[
u_{\text{reg}}(\zeta) = \lim_{\theta \to 1} u_{\text{reg}}(\zeta e^{2\pi i \theta}) = (\lambda \overline{\lambda}) \cdot \lim_{\theta \to 1} \overrightarrow{H}(x e^{\pi i \theta})
\]

\[
= (\lambda \overline{\lambda}) \cdot \left( \begin{array}{cc} -2\pi P & i \\ i & 0 \end{array} \right) \cdot \overrightarrow{H}(x).
\]
Thus, the equation $\mathcal{M}(j, q|\beta) = 0$ holds iff

\[
\begin{pmatrix}
\lambda & \bar{\lambda}
\end{pmatrix} = \begin{pmatrix}
\lambda & \bar{\lambda}
\end{pmatrix} \cdot \begin{pmatrix}
-2\pi P & i \\
0 & 0
\end{pmatrix}.
\]

If we set $\lambda = i e^{-i\gamma}$ with $\gamma \in \mathbb{R}$ we obtain (since $P$ is real)

\[
\gamma = \frac{3\pi}{4} \mod \pi, \quad P = -\frac{1}{\pi}.
\]

Thus, $\lambda$ encodes the crucial information for the asymptotic expansion of the regular solution, i.e.

\[
\begin{aligned}
\nu_{\text{reg}}(\zeta) &= \frac{1}{i} \left( e^{i\gamma}H_2(x) - e^{-i\gamma}H_1(x) \right) \\
&\sim \frac{2p_0}{\zeta^{q+\frac{1}{2}}} \left( \sin\left(2\sqrt{2\beta\zeta + \gamma}\right) + O\left(\frac{1}{\sqrt{\zeta}}\right) \right).
\end{aligned}
\]

This is the desired formula for the scattering phase in the case that the set of parameters $(\beta, j, q)$ induces a minimal solution of the recurrence relation (8).

### 4.3 Computation

The numerical determination of the scattering phase consists of two steps: First, we have to determine the successive zeros of the function $\mathcal{M}(j, q|x)$. However, one should mention that for the necessary evaluation of the continued fraction one has to use a backward algorithm since any forward algorithm must be numerically instable as shown in [Gau67].

We have determined the positive zeros $(\beta_n)_{n \in \mathbb{N}}$ of $\mathcal{M}(j, q|x)$ by a simple bisection algorithm where the evaluation of the continued fraction was accomplished by the Gau\ss chi algorithm. It is essential to use the function $\mathcal{M}(j, q|x)$ instead of the earlier defined $\tilde{\mathcal{M}}(j, q|x)$: Because of the structure of the coefficients in the continued fraction the typical values of $\mathcal{M}(j, q|x)$ become very small, and the determination of its zeros unstable. With $(\beta_n, j, q, \gamma)$ we then have the data needed for a numerical interpolation of the scattering phase over the whole range of $\beta$.

### 5 Numerical Results

#### 5.1 Numerical Results for the WKB approximation

The figures below show the scattering phases for different quantum numbers $j$ and $q$ with varying parameter $\beta$ obtained by the WKB approximation from Sect. 3.

In Fig. 1 to Fig. 4 we have applied the WKB approximation of case I.

- Fig. 1 shows (from the top to the bottom) the graphs for $\Delta_{j,q}$ for $j = 1, \ldots, 10$ and $q = 1$.
- Fig. 2 shows (from the top to the bottom) the graphs for $\Delta_{j,q}$ for $j = 3, \ldots, 10$ and $q = 3$.
- Fig. 3 shows (from the top to the bottom) the graphs for $\Delta_{j,q}$ for $j = 4, \ldots, 10$ and $q = 4$.
- Fig. 4 shows (from the top to the bottom) the graphs for $\Delta_{j,q}$ for $j = 5$ and $q = 1, 3, 5$.

In Fig. 5 and Fig. 6 we have compared the WKB approximation of case II to IV (left) with the WKB approximation of case I (right) for the values $j = 9, q = 0$. The vertical line in Fig. 5 is indicating the transition from case II to IV.

- Fig. 7 shows (from the top to the bottom) the graphs for $\Delta_{j,0}$ for $j = 1, 3, 5$.
- Fig. 8 shows the graph for $\Delta_{12,0}$. 

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5.2 Comparison between WKB and Frobenius method

In this section we compare the numerical results for the asymptotic scattering phase obtained by the WKB approximation (already shown in Sect. 5.1) with the numerical data obtained by the Frobenius/continued-fraction method as described in Sect. 4.3.

The figures below show the scattering phases for different quantum numbers $j$ and $q$ with varying parameter $\beta$. Each figure will show the graph $(\beta, \Delta^{WKB}_{j,q}(\beta))$ of the corresponding WKB approximation, and crosses will mark the points $\left(\beta_n, \Delta^{Fr}_{j,q}(\beta_n)\right)_{n \geq 1}$ obtained by the Frobenius/continued-fraction method where $\beta_n$ is the $n^{th}$ zero of $\mathcal{M}(j,q|\beta)$ and

$$\Delta^{Fr}_{j,q}(\beta_n) = -\frac{\pi}{4} - l_{j,q}(n)\pi - \max(0, j-1)\pi \quad (18)$$

with $\forall_{n \geq 1} : l_{j,q}(n) \in \mathbb{N}_0$. Notice that the scattering phase is only determined up to a multiple of $\pi$ since we can always change the sign of a solution by an overall factor. Therefore, Eq. (18) is equivalent to Eq. (17). We will also present an additional and more significant diagram of $\left(\Delta_{j,q}(\beta) \mod \pi\right)$ and varying parameter $\beta$.

Figs. 1 to 9, 11, 13, 15 show $\Delta_{j,0}(\beta)$ with varying $\beta$. For $j \leq 6$ it is $l(n) := n$ and all $\beta_n > j(j+1)$, i.e. $1 > a \geq 0$ and we have to compare with case IV of the WKB approximation. $j = 7$ is the lowest value where for $q = 0$ we have $\beta_1 < j(j+1)$, i.e. $1 < a$, and we have to compare with case II of the WKB approximation: we have to relabel the $\beta_n$. This can be done by setting $\forall_{j \geq 7} : l(n) := |n - \frac{3}{2}| - \frac{1}{2}$. However, since the phase can be obtained only up to multiples of $\pi$ the function $l(n)$ is irrelevant for the comparison of $\Delta^{WKB}_{j,0}(\beta)$ and $\Delta^{Fr}_{j,0}(\beta)$. For this purpose one can look at Figs. 10, 12, 14, 16 which are independent of $l(n)$. These figures show an exact match of the data sets obtained by the WKB approximation and the Frobenius/continued-fraction method.

Figs. 17 to 23 show $\Delta_{j,1}(\beta)$ with varying $\beta$. For $j \leq 8$ it is $l(n) := n$. $j = 9$ is the lowest value where for $q = 1$ we have $\beta_1 < \beta_{\text{max}}$, where $\beta_{\text{max}}$ denotes the local maximum of the WKB approximation: We have to relabel the $\beta_n$. This can be done by setting $\forall_{j \geq 9} : l(n) := |n - \frac{3}{2}| - \frac{1}{2}$. Figs. 17 to 23 show an exact match of the data sets obtained by the WKB approximation and the Frobenius/continued-fraction method.

Figs. 25 to 32 show $\Delta_{j,2}(\beta)$ with varying $\beta$. For $j \leq 11$ it is $l(n) := n$. Figs. 25 to 32 show an exact match of the data sets obtained by the WKB approximation and the Frobenius/continued-fraction method.

The accuracy of the WKB approximation can be explained as follows: from the error bound in Eq. (29) we can deduce the following numerical result for the error $\delta_{j,q}(\beta)$ of $\Delta_{j,q}(\beta)$

$$|\delta_{j,q=0}(\beta)| \leq \frac{\pi}{2} \min \left(1, 1.1 e^{\frac{\pi}{\sqrt{3}}} - 1\right).$$

This means that for $\beta > 4$ we already have

$$|\delta_{j,q=0}(\beta)| \leq \frac{\pi}{2} \left(e^{\frac{\pi}{\sqrt{3}}} - 1\right),$$

and for $\beta > 100$ we find $|\delta_{j,q=0}(\beta)| \leq \frac{\pi}{100}$.

The same computations can be made for $q \geq 1$ with similar results since we also have a similar error bound for the general $\delta_{j,q}(\beta)$ in Eq. (21).
6 Conclusions and outlook

The eigenvalue equation for the Laplace-Beltrami operator acting on scalar functions on the non-compact Eguchi-Hanson space reduces to a confluent Heun equation (2) or after a suitable substitution (6) with Ince symbol [0, 2, 1, 2].

With the help of the Liouville-Green approximation (WKB) we have constructed approximations for the eigenfunctions by special functions in Sect. 3. Depending on the quantum numbers that label the SU(2)-representation the approximating functions are either Airy, Whittaker, or Bessel functions. Furthermore, we have derived the scattering phases and error bounds in these cases.

Moreover, for specific discrete values of the eigenvalue in Sect. 4 we have constructed the exact solutions by the Frobenius methods. These eigenvalues are given by the zeros of a meromorphic function defined by the infinite continuous fraction (11). Together with a monodromy computation this has provided us with the data needed for a numerical interpolation of the scattering phases.

Finally, in Sect. 5 we have shown that these two sets of data (obtained by the WKB approximation and the Frobenius/continued-fraction method) agree to a high accuracy. This shows that one can find a discrete set of exact values for the spectral density of the Laplace-Beltrami operator by the method described in Sect. 4.3. Conversely, it shows that the expressions for the eigenfunctions and scattering phases which were derived in Sect. 3 and which have the advantage of being given in terms of explicit functions are very accurate approximations and can be used for all numerical purposes.

It is now interesting to ask whether the meromorphic function defined by the continued fraction (11) can be expressed explicitly as a ratio of special functions. If so we could obtain the discrete eigenvalues – for which we have already calculated the exact scattering phase and which we have also computed numerically by the continued fraction – as zeros of this ratio of special functions. A first step towards an explicit representation of the continued fraction might arise from the method of Pincherle. If we apply [Per77, Kap. 21.84, Satz 8] to the continued fraction (11) we obtain the following result: a representation of the meromorphic function defined by the T-fraction (11) is given by

\[ M(j, q|x) = q(q + 1) - j(j + 1) + x + \frac{\delta_{q0} 16\pi x^3 \phi \left( \frac{1}{2x} \right)}{\delta_{q0} 8\pi x^2 \phi \left( \frac{1}{2x} \right)} + \int_{\frac{1}{2x}}^{\infty} dz \phi(z) z^{-3}. \]

Here, \( \phi \) is the solution of the differential equation

\[ \left\{ z^3 \left( z - \frac{1}{2x} \right) \frac{d^2}{dz^2} + z^2 \left( (q + 1)z - \frac{q}{x} \right) \frac{d}{dz} + \left( \frac{x - q(q + 1) + j(j + 1)}{2x} \right) z - \frac{1}{2x} \right\} \phi(z) = 0 \]

that behaves as \((z - \frac{1}{2x})^q\) at the regular singularity \( z = \frac{1}{2x} \). An analysis of the differential equation shows that such a solution always exists. A further analysis of the behavior at the irregular singularity \( z = \infty \) then shows that all the appearing integrals also exist.

A closer investigation of the meromorphic function \( M(j, q|x) \) is subject of our ongoing research.

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A The different cases in WKB approximation

In this appendix we show the explicit construction of the approximate solutions of the differential equation (3) by the Liouville-Green approximation (WKB) in the cases I to IV (cf. Sect. 3). The construction of an error bound in cases I to III can be found in the references. The construction of the bound in case IV is given in App. B and might explain the construction and philosophy behind these error bounds.

To give bounds for the different cases I to IV we need the notion of auxiliary weight, modulus, and phase function $E, M, \theta$: If $X, Y$ are solutions of the respective differential equation of the first and second kind, then $X = ME^{-1} \sin \theta$ and $Y = EM \cos \theta$.

In case I we will need $E, M, \theta$ for the Airy function (see [Olv74, Sect. 11.2] for more details):

$$ Ai(x) = \frac{M(x)}{E(x)} \sin \theta(x), \quad Bi(x) = E(x) M(x) \cos \theta(x). $$

In the cases III and IV, we will use the functions $E_0, M_0, \theta_0$ for the Bessel function (see [Olv74, Sect. 12.1, 12.3] for more details):

$$ J_0(x) = \frac{M_0(x)}{E_0(x)} \cos \theta_0(x), \quad Y_0(x) = E_0(x) M_0(x) \sin \theta_0(x). $$

For case II, the definition of modulus and weight function is the most complicated. Therefore, we refer to [Dun94, Chap. 2, Th. 1] for the quite extensive definitions in this case.

A.1 Case I

In case I it is easy to prove that the function $f$ has a transition point (i.e. simple zero) at $z > 1$ which we denote by $z_0$. The idea is now to perform the transformation of the variable $z$ and the function $w(z)$ to $\zeta, W(\zeta)$ according to

$$ \forall z \geq z_0 : \quad \frac{2}{3} (-\zeta)^{\frac{3}{2}} = \int_{z_0}^{z} \sqrt{f(t)} \, dt, $$

$$ \forall z \leq z_0 : \quad \frac{2}{3} \zeta^{\frac{3}{2}} = \int_{z}^{z_0} \sqrt{-f(t)} \, dt, $$

and $w(z) = \sqrt{-\frac{dx}{R}} W(\zeta).$ Note that $\zeta \to -\infty$ corresponds to $z \to \infty$, $\zeta \to 0-$ to $z \to z_0+$, and $\zeta \to \infty$ to $z \to 1$. Eq. (3) becomes

$$ \frac{d^2}{d\zeta^2} W(\zeta) = \left[ a^2 \zeta + \psi(\zeta) \right] W(\zeta), \quad (19) $$

$$ \psi(\zeta) = \frac{5}{16 \zeta^2} - \frac{[4f(z)J''(z) - 5f'(z)^2]}{16f(z)^3} - \frac{\zeta g(z)}{f(z)}. $$
Approximate solutions of (3), i.e. solutions of (19) with $\psi = 0$, that are regular at $z = 1$, are given by [Olv74, Sect. 11.3, Th. 3.1]

$$w(z) = 4\sqrt{-\zeta} \left[ \text{Ai}(u^2 \zeta) + \varepsilon(u, \zeta) \right],$$

with $|\varepsilon(u, \zeta)| \leq \frac{1}{\lambda} \frac{M(u^2 \zeta)}{E(u^2 \zeta)} \left[ e^{\frac{2}{\lambda} \nu_{\zeta, \infty}(|u^2 \zeta| B_0(\zeta))} - 1 \right].$ (20)

The constant $\lambda$ and the function $B_0$ are defined as follows:

$$\lambda := \sup_x \left\{ \pi |x| \frac{1}{2} M^2(x) \right\},$$

$$B_0(\zeta) := \frac{1}{2\sqrt{|\zeta|}} \int_\zeta^{\infty} \frac{dv}{|v|^{\frac{3}{2}}} \psi(u, v),$$

and $\mathcal{V}$ is the variational operator, i.e.

$$\nu_{\zeta, \infty}(|u|^{\frac{1}{2}} B_0(\zeta)) = \frac{1}{2} \int_\zeta^{\infty} \frac{dv}{|v|^{\frac{3}{2}}} |\psi(u, v)|.$$

Following the discussion in [Olv74, Sect. 13.7.2] we can determine the behavior for $z \to 1$ and $z \to \infty$:

$$A(z) \quad z \to 1 \sim \frac{\sqrt{\beta}}{2\sqrt{\pi q}} (z - 1)^{\frac{3}{2}},$$

$$A(z) \quad z \to \infty \sim \frac{1}{\sqrt{\pi z^{\frac{3}{2}}}} \sin \left( u \int_{z_0}^{z} \sqrt{f(t)} \, dt + \frac{\pi}{4} + \delta \right),$$

where we have used $\lim_{\zeta \to \infty} \varepsilon(u, \zeta) = 0$, and the phase $\delta$ is determined by $\lim_{\zeta \to -\infty} \varepsilon(u, \zeta)$. We know that the argument of the sin-function equals $2\sqrt{\beta z} + \Delta_{j,q}$ for large $z$, i.e.

$$A(z) \quad z \to \infty \sim \frac{1}{\sqrt{\pi z^{\frac{3}{2}}}} \sin \left( 2\sqrt{\beta z} + \Delta_{j,q} \right).$$

Therefore, for the scattering phase we obtain

$$\Delta_{j,q} = u \lim_{z \to \infty} \left( \int_{z_0}^{z} \sqrt{f(t)} \, dt - 2\sqrt{z} \right) + \frac{\pi}{4} + \delta,$$

and with the help of [Olv74, Sect. 11.2] it follows that

$$\frac{2|\delta|}{\pi} \leq \min \left\{ 1, \frac{1}{\lambda} \left[ e^{\frac{2}{\lambda} \nu_{\zeta, \infty}(|u^2 \zeta| B_0(\zeta))} - 1 \right] \right\}. \quad (21)$$

To resume, the solution (20) can be considered as relation between the recessive solution and its asymptotic expansion. In this sense, the solution is a connection formula and it is known as Gans-Jeffreys formula.
A.2 Case II

In case II, \( f(z) = \frac{z-a}{1-z} \), we can apply a result of Olver and Nestor that has been generalized in [Dun94]. The Liouville transformation takes the form

\[
\int_{\alpha}^{\zeta} \left( \frac{\tau - \alpha}{\tau} \right)^{\frac{1}{2}} d\tau = \int_{a}^{z} dt \sqrt{f(t)}
\]

with \( \alpha := \frac{2}{\pi} \int_{1}^{a} \sqrt{-f(t)} \, dt \).

Therefore, we perform the transformation of the variable \( z \) and the function \( w(z) \) to \( \zeta, W(\zeta) \) according to

\[
\zeta^{\frac{1}{2}}(\zeta - \alpha)^{\frac{1}{2}} - \frac{\alpha}{2} \ln \left( \frac{2\zeta - \alpha + 2\zeta^{\frac{1}{2}}(\zeta - \alpha)^{\frac{1}{2}}}{\zeta} \right) = \int_{a}^{z} \sqrt{f(t)} \, dt
\]

and \( w(z) = \sqrt{\frac{dz}{d\zeta}} W(\zeta) \). The integral can be expressed through an elliptic integral of the second kind:

\[
\int_{a}^{z} \sqrt{f(t)} \, dt = 2\sqrt{(z-a)(z+1)} - 2\sqrt{1+a} \, \text{E} \left[ \arcsin \left( \frac{z-a}{\sqrt{z-1}} \right), \sqrt{\frac{2}{1+a}} \right].
\]

Note that \( \zeta \to \alpha \) corresponds to \( z \to a, \zeta \to 0 \) to \( z \to 1 \), where branches of \( \zeta \) must be chosen such that it is an analytic function at both values. Moreover, \( \zeta \to \infty \) corresponds to \( z \to \infty \). Eq. (3) becomes

\[
\frac{d^2}{d\zeta^2} W(\zeta) = \left[ u^2 \alpha - \frac{1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta} \right] W(\zeta),
\]

\[
\psi(\zeta) = \frac{4\zeta^2 + \alpha^2}{16\zeta(\alpha - \zeta)^2} + \left[ 4f(z)f''(z) - 5f'(z)^2 \right] \frac{\zeta - \alpha}{16f(z)^3}
\]

\[
+ \frac{g(z)}{f(z)}.
\]

Approximate solutions of (3), i.e. solutions of (19) with \( \psi = 0 \), that are regular at \( z = 1 \), are given in [Dun94] Chap. 2, case 1, \( m = 0 \) in terms of the Whittaker function \( M \), i.e.

\[
w(z) = \sqrt{\frac{\zeta - \alpha}{\zeta f(z)}} e^{\frac{i\pi}{4}} \sqrt{\frac{2\pi}{1 + e^{2\pi i\alpha}}} M_{\frac{\alpha}{2},0} (2iu\zeta) + \varepsilon(u, \zeta)
\]

with \( |\varepsilon(u, \zeta)| \leq \frac{M_{\frac{\alpha}{2},0}(2u\zeta)}{E_{\frac{\alpha}{2},0}(2u\zeta)} \left[ e^{\frac{\chi_{1}^{(1)}(2u\zeta)}{2u} + 1} V_{p(0)}(F) - 1 \right] \).

For the error bound we have adopted the notation of [Dun94] Chap. 2, Th. 1. Using formula (2.19) in [Dun94] we determine the behavior of the solution for \( z \to 1 \)

\[
A(z) \xrightarrow{z \to 1} \sqrt{\frac{2\pi}{1 + e^{2\pi i\alpha}}} \sqrt{\frac{2\alpha u^2}{a - 1}} \left( \frac{\zeta}{z - 1} \right)^{\frac{1}{2}},
\]

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where we have used \( \lim_{\zeta \to 0} \varepsilon(u, \zeta) = 0 \). A close examination of the Liouville transformation then shows that \( \zeta/(z-1) \to (a-1)/(2\alpha) \) for \( \zeta \to 0 \) and \( z \to 1 \). Using [Dun94 (2.16)-(2.18)] we obtain for \( z \to \infty \)

\[
A(z) \xrightarrow{z \to \infty} \frac{2}{z^2} \sin \left( \frac{u\zeta - u\alpha}{2} \ln(2u\zeta) + \frac{\pi}{4} + \gamma + \delta \right),
\]

where

\[
\gamma = \arg \Gamma \left( \frac{1}{2} + \frac{iu\alpha}{2} \right)
\]

and the phase \( \delta \) is determined by \( \lim_{\zeta \to \infty} \varepsilon(u, \zeta) \). We know that the argument in the \( \sin \)-function equals \( 2\sqrt{\beta z + \Delta_{j,0} |_{a>1}} \), i.e.

\[
A(z) \xrightarrow{z \to \infty} \frac{1}{z^2} \sin \left( 2\sqrt{\beta z + \Delta_{j,0} |_{a>1}} \right).
\]

A careful examination of the Liouville transformation reveals the following relation between large \( \zeta \) and \( z \)

\[
\zeta - \frac{\alpha}{2} - \frac{\alpha}{2} \ln(\zeta) + \frac{\alpha}{2} \ln \left( \frac{\alpha}{4} \right) + O(\zeta^{-1}) = 2\sqrt{\zeta - 2\sqrt{1 + a} \left( \sqrt{\frac{2}{1 + a}} \right) + O(\zeta^{-\frac{1}{2}})}.
\]

Therefore, for the scattering phase we obtain

\[
\Delta_{j,0} |_{a>1} = u \lim_{z \to \infty} \left( \zeta - \frac{\alpha}{2} \ln(2u\zeta) - 2\sqrt{\zeta} \right) + \frac{\pi}{4} + \gamma + \delta
\]

\[
= -2u\sqrt{1 + a} \left( \sqrt{\frac{2}{1 + a}} \right) + \frac{au}{2} - \frac{au}{2} \ln \left( \frac{au}{2} \right) + \frac{\pi}{4} + \gamma
\]

\[
\gamma = \arg \Gamma \left( \frac{1}{2} + \frac{iu\alpha}{2} \right)
\]

\[
\frac{2|\delta|}{\pi} \leq \min \left\{ 1,\left[ e^{\frac{e^{2\gamma}}{2u} + 1} \right]^\frac{1}{\nu_{(0)}(F)} - 1 \right\}.
\]

The phase \( \Delta_{j,0} |_{a>1} \) differs from the one obtained in [Mig91].

### A.3 Case III

In case III, the function \( f \) has neither a transition point nor a pole for \( z \geq 1 \). This case can be understood as the limit \( a \to 1 \) of case II, and applying further results of Olver’s [Olv77], in this limit the Whittaker function will become the Bessel function \( J_0 \).

The idea is to perform the transformation of the variable \( z \) and the function \( w(z) \) to \( \zeta \), \( W(\zeta) \) according to

\[
\zeta = \int_{1}^{z} \sqrt{f(t)} \, dt = 2\sqrt{z + 1} - 2\sqrt{2}
\]

and \( w(z) = \sqrt{\frac{dz}{d\zeta}} W(\zeta) \). Note that \( \zeta \to \infty \) corresponds to \( z \to \infty \) and \( \zeta \to 0 \) to \( z \to 1 \). Eq. (3) becomes

\[
\frac{d^2}{d\zeta^2} W(\zeta) = \left[ -u^2 - \frac{1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta} \right] W(\zeta),
\]

\[
\psi(\zeta) = \frac{1}{4\zeta} + \left[ 4f(z)f''(z) - 5f'(z)^2 \right] \frac{\zeta}{16f(z)^3} + \frac{\zeta g(z)}{f(z)}.
\]
Approximate solutions of (3), i.e. solutions of (24) with \( \psi = 0 \), that are regular at \( z = 1 \), are given by [Olv77, Chap. 5, Th. 2, case 2, \( \nu = 0, \mu = 0 \)], i.e.

\[
w(z) = \frac{1}{\sqrt{f(z)}} \left[ \zeta \frac{1}{2} J_0(u\zeta) + \varepsilon(u, \zeta) \right],
\]

with \( |\varepsilon(u, \zeta)| \leq \frac{l_{0,1}}{l_{0,0}} \frac{M_0(u\zeta)}{E_0(u\zeta)} \left[ e^{\rho_0 \nu_0 \zeta \rho(H)} - 1 \right]. \)

The constants \( l_{0,0}, l_{0,1} \), and the function \( H \) are defined as follows:

\[
l_{0,0} := \sup_x \{ \pi \Omega_0(x) M_0^2(x) \},
\]

\[
l_{0,1} := \sup_x \{ \pi \Omega_0(x) |J_0(x)| E_0(x) M_0(x) \},
\]

\[
H(u, \zeta) := \frac{1}{2} \int \frac{\psi(u, \zeta)}{\Omega_0(\zeta)} d\zeta,
\]

\[
\Omega_0(x) := \frac{1 + x}{\ln(e + \frac{1}{x})},
\]

and \( \mathcal{V} \) is the variational operator, i.e.

\[
\mathcal{V}_{0,\zeta}(H) = \frac{1}{2} \int_0^\zeta \frac{dv}{\Omega_0(uv)} |\psi(u, v)|.
\]

We determine the behavior of the solutions for \( z \to 1 \) and \( z \to \infty \):

\[
A(z) \xrightarrow{z \to 1} \frac{1}{\sqrt{2}} J_0 \left( u \frac{z - 1}{\sqrt{2}} \right) \to \frac{1}{\sqrt{2}},
\]

\[
A(z) \xrightarrow{z \to \infty} \frac{\sqrt{2}}{\sqrt{\pi uz^2}} \sin \left( u \int_1^z \sqrt{f(t)} dt + \frac{\pi}{4} + \delta \right),
\]

where we have used \( \lim_{\zeta \to 0^+} \varepsilon(u, \zeta) = 0 \), and the phase \( \delta \) is determined by \( \lim_{\zeta \to \infty} \varepsilon(u, \zeta) \). We know that the argument of the sin-function equals \( 2\sqrt{\beta z} + \Delta_{j,0} |_{a=1} \) for large \( z \), i.e.

\[
A(z) \xrightarrow{z \to \infty} \frac{\sqrt{2}}{\sqrt{\pi uz^2}} \sin \left( 2\sqrt{\beta z} + \Delta_{j,0} |_{a=1} \right).
\]

Therefore, for the scattering phase we obtain

\[
\Delta_{j,0} |_{a=1} = u \lim_{z \to \infty} \left( \int_1^z \sqrt{f(t)} dt - 2\sqrt{z} \right) + \frac{\pi}{4} + \delta
\]

\[
= -\sqrt{8\beta} + \frac{\pi}{4} + \delta
\]

and with the help of [Olv77, Chapt. 5] and analogous to App. [3] the bound for \( \varepsilon \) in (25) gives a bound for the phase \( \delta \). It follows that

\[
\frac{2|\delta|}{\pi} \leq \min \left\{ 1, \frac{l_{0,1}}{l_{0,0}} \left[ e^{\rho_0 \nu_0 \zeta \rho(H)} - 1 \right] \right\}.
\]

(26)

The phase \( \Delta_{j,0} \) differs from the one obtained in [Mig91] by \( \frac{\pi}{4} \).
A.4 Case IV

In case IV the function $f$ has no transition point for $z \geq 1$ and takes the form $f(z) = \frac{z-a}{z}$. The idea is now to perform the transformation of the variable $z$ and the function $w(z)$ to $\zeta$, $W(\zeta)$ according to

$$(-\zeta)\frac{1}{2} = \int_{1}^{z} \sqrt{f(t)} \, dt$$

and $w(z) = \sqrt{-\frac{dz}{d\zeta}} W(\zeta)$. The integral can be expressed through elliptic integrals of the first and second kind:

$$\int_{1}^{z} \sqrt{f(t)} \, dt = 2\sqrt{\frac{z^2-1}{z-a}} + (1-a) \sqrt{2} F \left[ \arcsin \left( \frac{z-1}{z-a} \right), \sqrt{\frac{1+a}{2}} \right] - 2\sqrt{2} E \left[ \arcsin \left( \frac{z-1}{z-a} \right), \sqrt{\frac{1+a}{2}} \right].$$

Note that $\zeta \to -\infty$ corresponds to $z \to \infty$ and $\zeta \to 0^-$ to $z \to 1^+$. Eq. (3) becomes

$$\frac{d^2}{d\zeta^2} W(\zeta) = \left[ \frac{u^2}{4\zeta^2} - \frac{1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta} \right] W(\zeta), \quad (27)$$

$$\psi(\zeta) = \frac{1}{16\zeta} + g(z) \frac{f(z) + f''(z) - 5f'(z)^2}{64f(z)^3}.$$ 

Approximate solutions of (3), i.e. solutions of (27) with $\psi = 0$, that are regular at $z = 1$, are then given by [Olv74] Sect. 12.4, Th. 4.1, $\nu = 0$, $n = 0$, i.e.

$$w(z) = \frac{1}{\sqrt{4\zeta J(f(z)) \zeta}} \left[ |\zeta|^{3/2} J_0(u|\zeta|^{1/2}) + \varepsilon(u, \zeta) \right] \quad (28)$$

with $|\varepsilon(u, \zeta)| \leq \frac{\lambda_{0,1}}{\lambda_{0,0}} \left| \frac{M_0(u\zeta)}{E_0(u\zeta)} \right| \left[ \frac{\lambda_{0,0}}{e^{\frac{\lambda_{0,0}}{\alpha}} V_{\zeta,0}(|\zeta|^{1/2} B_0(\zeta)) - 1} \right]$.

The bound for the error $\varepsilon$ is derived in App. [B] The constants $\lambda_{0,0}, \lambda_{0,1}$, as well as the function $B_0$ are defined as follows:

$$\lambda_{0,0} := \sup_{x \geq 0} \left\{ \pi x M_0^2(x) \right\},$$

$$\lambda_{0,1} := \sup_{x \geq 0} \left\{ \pi x M_0^2(x), \cos \theta_0(x) \right\},$$

$$B_0(\zeta) := \frac{1}{|\zeta|^{3/2}} \int_{\zeta}^{0} \frac{dv}{|v|^{1/2}} \psi(v),$$

and $V$ is the variational operator, i.e.

$$V_{\zeta,0} \left( |\zeta|^{1/2} B_0(\zeta) \right) = \int_{\zeta}^{0} \frac{dv}{|v|^{1/2}} |\psi(v)|.$$ 

We determine the behavior of the solutions for $z \to 1$ and $z \to \infty$:

$$A(z) \underset{z \to 1}{\sim} \frac{1}{\sqrt{2}} J_0 \left( u \sqrt{2(1-a) \sqrt{z-1}} \right) \to \frac{1}{\sqrt{2}},$$

$$A(z) \underset{z \to \infty}{\sim} \frac{1}{\sqrt{\pi uz^2}} \sin \left( u \int_{1}^{z} \sqrt{f(t)} \, dt + \frac{\pi}{4} + \delta \right),$$

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where we have used that \( \lim_{\zeta \rightarrow 0-} \varepsilon(u, \zeta) = 0 \) and the phase \( \delta \) is determined by \( \lim_{\zeta \rightarrow -\infty} \varepsilon(u, \zeta) \).

We know that the argument of the \( \sin \)-function equals \( 2\sqrt{\beta z} + \Delta_{j,0} \) for large \( z \), i.e.

\[
A(z) \sim \frac{1}{\sqrt{\pi uz^2}} \sin \left(2\sqrt{\beta z} + \Delta_{j,0}\right).
\]

Therefore, we obtain for the scattering phase

\[
\Delta_{j,0} |_{a<1} = u \lim_{z \rightarrow \infty} \left( \int_1^z \sqrt{f(t)} dt - 2\sqrt{z} \right) + \frac{\pi}{4} + \delta
\]

\[
= (1 - a) \sqrt{2\beta} \ K \left[ \sqrt{\frac{1 + a}{2}} \right] - 2\sqrt{2\beta} \ E \left[ \sqrt{\frac{1 + a}{2}} \right] + \frac{\pi}{4} + \delta.
\]

In addition, we will derive in App. B

\[
\frac{2|\delta|}{\pi} \leq \min \left\{ 1, \frac{\lambda_{0,1}}{\lambda_{0,0}} \exp \left( \frac{\lambda_{0,0}}{\lambda_{0,0}} \nu_{-\infty,0} \left( \frac{\varepsilon(u, \zeta)}{\varepsilon(u, \zeta)} \right) - 1 \right) \right\}.
\]

The phase \( \Delta_{j,0} |_{a<1} \) differs from the one obtained in Mig91 by \( \frac{\pi}{4} \) and the factor in front of the elliptic integral of the first kind.

**B Error Bounds**

In this chapter we apply the construction of error bounds presented in [Olv74] Sect. 11.3 to case IV (cf. [Olv74] Sect. 12.4, Ex. 4.4). In case IV the Bessel functions \( |\zeta|^2 J_0(u|\zeta|^2) \) and \( |\zeta|^2 Y_0(u|\zeta|^2) \) of first and second type, are first approximations for \( W \). They are exact solutions if \( \psi \equiv 0 \). Substituting \( W(\zeta) = |\zeta|^2 J_0(u|\zeta|^2) + \varepsilon(u, \zeta) \), the differential equation becomes

\[
\frac{d^2}{d\zeta^2} \varepsilon(u, \zeta) - \left[ \frac{u^2}{4\zeta} - \frac{1}{4\zeta^2} \right] \varepsilon(u, \zeta) = \frac{\psi(\zeta)}{\zeta} \left[ \varepsilon(u, \zeta) + |\zeta|^2 J_0(u|\zeta|^2) \right].
\]

If we rewrite it as an integral equation we obtain a Volterra integral equation

\[
\varepsilon(u, \zeta) = \int_{\zeta}^{0} K(\zeta, v) \frac{\psi(v)}{|v|^2} \left[ \varepsilon(v) + |v|^2 J_0(u|v|^2) \right] dv,
\]

where

\[
K(\zeta, v) = \pi |v|^2 \left[ J_0(u|\zeta|^2) Y_0(u|v|^2) - Y_0(u|\zeta|^2) J_0(u|v|^2) \right].
\]

Using the fact that \( E_0 \) is a non-increasing function we obtain

\[
\forall \zeta \leq v \leq 0 : \ |K(\zeta, v)| \leq \frac{M_0(u|\zeta|^2)}{E_0(u|\zeta|^2)} \exp \left( \frac{\pi |v|^2 M_0(u|v|^2)}{E_0(u|v|^2)} \right) .
\]

We introduce the parameters \( \kappa_0, \kappa \) by

\[
\kappa_0 := \sup_{v \in [\zeta, 0]} \left\{ P_0(v) Q(v) \right\} \leq \frac{\lambda_{0,0}}{u},
\]

\[
\kappa := \sup_{v \in [\zeta, 0]} \left\{ Q(v) \left| |v|^2 J_0(u|v|^2) \right| \right\} \leq \frac{|\zeta|^2 \lambda_{0,1}}{u}.
\]
With the help of [Olv74, Sect. 6.10, Th. 10.2] which establishes a bound for the solution of a Volterra integral equation we end up with a bound for $\varepsilon$,

$$\frac{|\varepsilon(u, \zeta)|}{P_0(\zeta)} \leq \frac{k}{k_0} \left[ \exp \left\{ k_0 V_{C,0} \left( |\zeta|^{\frac{1}{2}} B_0(\zeta) \right) \right\} - 1 \right]$$

$$\leq |\zeta|^{\frac{1}{2}} \frac{\lambda_{0,1}}{\lambda_{0,0}} \left[ \exp \left\{ \frac{\lambda_{0,0}}{u} V_{C,0} \left( |\zeta|^{\frac{1}{2}} B_0(\zeta) \right) \right\} - 1 \right] .$$

We are interested in an error bound for the phase in the asymptotic expansion of the solution. Assume that the solution $W$ takes the following form for large arguments $\zeta$:

$$W(u, \zeta) \sim \left( \frac{2|\zeta|^{\frac{1}{2}}}{\pi u} \right)^{\frac{1}{2}} \left( (1 + \rho) \sin \left( u|\zeta|^{\frac{1}{2}} + \frac{\pi}{4} + \delta \right) + o(1) \right) .$$

As in [Olv74] Sect. 6.7 we rewrite the difference of the exact and the approximate solution as a trigonometric function

$$\varepsilon(u, \zeta) = W(u, \zeta) - |\zeta|^{\frac{1}{2}} J_0(u|\zeta|^{\frac{1}{2}})$$

$$\sim \left( \frac{2|\zeta|^{\frac{1}{2}}}{\pi u} \right)^{\frac{1}{2}} \left\{ [1 + \rho] \sin \left( u|\zeta|^{\frac{1}{2}} + \frac{\pi}{4} + \delta \right) - \sin \left( u|\zeta|^{\frac{1}{2}} + \frac{\pi}{4} \right) \right\}$$

$$= \left( \frac{2|\zeta|^{\frac{1}{2}}}{\pi u} \right)^{\frac{1}{2}} \sigma \sin \left( u|\zeta|^{\frac{1}{2}} + \frac{\pi}{4} + \eta \right) ,$$

where the new parameters $\sigma$ (always considered to be positive) and $\eta$ are related to $\rho$ and $\delta$ by

$$(1 + \rho) e^{i\delta} = 1 + \sigma e^{i\eta} .$$

By elementary geometry it follows that $\frac{2|\delta|}{\pi} \leq \sigma$. Then, the bound for $\varepsilon$ in (30) gives a bound for $\sigma$. Now, we choose a sequence $(\zeta_n)$ with $\lim_{n \to \infty} \zeta_n = -\infty$ for which $u|\zeta|^{\frac{1}{2}} + \frac{\pi}{4} + \eta$ is an odd integer multiple of $\frac{\pi}{2}$. This shows that

$$\sigma_{\zeta \to -\infty} = \lim_{\zeta \to -\infty} \left( \frac{\pi u}{2|\zeta|^{\frac{1}{2}}} \right)^{\frac{1}{2}} |\varepsilon(u, \zeta)| .$$

Finally, since

$$M_0(x) \xrightarrow{x \to \infty} \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} ,$$

$$E_0(x) \equiv 1 ,$$

we obtain a bound for $\delta$:

$$\frac{2|\delta|}{\pi} \leq \min \left\{ 1, \frac{\lambda_{0,1}}{\lambda_{0,0}} \left[ e^{-\infty} V_{-\infty,0} \left( |\zeta|^{\frac{1}{2}} B_0(\zeta) \right) - 1 \right] \right\} .$$

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Figure 1: $\Delta_{j,q}$ for $j = 1, \ldots, 10, \ q = 1$

Figure 2: $\Delta_{j,q}$ for $j = 3, \ldots, 10, \ q = 3$

Figure 3: $\Delta_{j,q}$ for $j = 4, \ldots, 10, \ q = 4$

Figure 4: $\Delta_{j,q}$ for $j = 10, \ q = 1, \ldots, 10$

Figure 5: $\Delta_{9,0}$ for case II, III, IV

Figure 6: $\Delta_{9,0}$ for case I

Figure 7: $\Delta_{j,0}$ for $j = 1, 3, 5$

Figure 8: $\Delta_{12,0}$
Figure 9: $\Delta_{0,0}^{\text{WKB}}$ and $\Delta_{0,0}^{\text{Fr}}$

Figure 10: $\Delta_{0,0}^{\text{WKB}}$ and $\Delta_{0,0}^{\text{Fr}}$ modulo $\pi$

Figure 11: $\Delta_{1,0}^{\text{WKB}}$ and $\Delta_{1,0}^{\text{Fr}}$

Figure 12: $\Delta_{1,0}^{\text{WKB}}$ and $\Delta_{1,0}^{\text{Fr}}$ modulo $\pi$

Figure 13: $\Delta_{7,0}^{\text{WKB}}$ and $\Delta_{7,0}^{\text{Fr}}$

Figure 14: $\Delta_{7,0}^{\text{WKB}}$ and $\Delta_{7,0}^{\text{Fr}}$ modulo $\pi$

Figure 15: $\Delta_{9,0}^{\text{WKB}}$ and $\Delta_{9,0}^{\text{Fr}}$

Figure 16: $\Delta_{9,0}^{\text{WKB}}$ and $\Delta_{9,0}^{\text{Fr}}$ modulo $\pi$
Figure 17: $\Delta_{2,1}^{WKB}$ and $\Delta_{2,1}^{Fr}$

Figure 18: $\Delta_{3,1}^{WKB}$ and $\Delta_{3,1}^{Fr}$

Figure 19: $\Delta_{4,1}^{WKB}$ and $\Delta_{4,1}^{Fr}$

Figure 20: $\Delta_{5,1}^{WKB}$ and $\Delta_{5,1}^{Fr}$

Figure 21: $\Delta_{6,1}^{WKB}$ and $\Delta_{6,1}^{Fr}$

Figure 22: $\Delta_{7,1}^{WKB}$ and $\Delta_{7,1}^{Fr}$

Figure 23: $\Delta_{8,1}^{WKB}$ and $\Delta_{8,1}^{Fr}$

Figure 24: $\Delta_{9,1}^{WKB}$ and $\Delta_{9,1}^{Fr}$
