THE MAXIMAL OPERATOR ASSOCIATED TO A NON-SYMMETRIC ORNSTEIN-UHLENBECK SEMIGROUP

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Abstract. Let $(H_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$ with covariance matrix $I$ and drift matrix $\lambda(R - I)$, where $\lambda > 0$ and $R$ is a skew-adjoint matrix and denote by $\gamma_\infty$ the invariant measure for $(H_t)_{t \geq 0}$. Semigroups of this form are the basic building blocks of Ornstein-Uhlenbeck semigroups which are normal on $L^2(\gamma_\infty)$. We prove that if the matrix $R$ generates a one-parameter group of periodic rotations then the maximal operator $H^*f(x) = \sup_{t \geq 0} |H_t f(x)|$ is of weak type 1 with respect to the invariant measure $\gamma_\infty$. We also prove that the maximal operator associated to an arbitrary normal Ornstein-Uhlenbeck semigroup is bounded on $L^p(\gamma_\infty)$ if and only if $1 < p \leq \infty$.

1. Introduction

Let $Q$ be a real, symmetric, positive definite $d \times d$-matrix and let $B$ be a nonzero real $d \times d$-matrix whose eigenvalues have negative real part. Then for every $t \in (0, \infty]$ we can define the family of Gaussian measures $\gamma_t$ on $\mathbb{R}^d$ with mean zero and covariance operators

\begin{equation}
Q_t = \int_0^t e^{sB}Qe^{sB^*} \, ds, \quad t \in (0, \infty],
\end{equation}

i.e. the measures

\begin{equation}
d\gamma_t(x) = (2\pi)^{-d/2}(\det Q_t)^{-1/2} e^{-\frac{1}{2}Q_t^{-1}x,x} \, d\lambda(x) \quad \forall t \in (0, \infty].
\end{equation}

The Ornstein-Uhlenbeck semigroup is the family of operators $(H^{Q,B}_t)_{t \geq 0}$ defined by

\begin{equation}
H^{Q,B}_t f(x) = \int_{\mathbb{R}^d} f(e^{tB}x - y) \, d\gamma_t(y)
\end{equation}

on the space $C_b(\mathbb{R}^d)$ of bounded continuous functions. The matrices $Q$ and $B$ are called the covariance and the drift matrix, respectively.

It is well known that $\gamma_\infty$ is the unique invariant measure for $H^{Q,B}_t$ and that $(H^{Q,B}_t)_{t \geq 0}$ is a diffusion semigroup on $(\mathbb{R}^d, \gamma_\infty)$ (see for instance [2]). Thus formula (1.2) defines a semigroup of positive contractions on $L^p(\gamma_\infty)$ for every $p \geq 1$, which we shall also denote by $(H^{Q,B}_t)_{t \geq 0}$.

2000 Mathematics Subject Classification. 42B25, 47D03.

Key words and phrases. Ornstein-Uhlenbeck semigroup, maximal operator, weak type.

The authors have received support by the Italian MIUR-PRIN 2005 project “Harmonic Analysis” and by the EU IHP 2002-2006 project “HARP”.
In this paper we are concerned with the boundedness of the maximal operator

$$\mathcal{H}^{Q,B}_t f(x) = \sup_{t \geq 0} |\mathcal{H}^{Q,B}_t f(x)|.$$ 

It is well known that by Banach’s principle (see [3]) this maximal operator is a key tool to investigate the almost everywhere convergence of \( \mathcal{H}^{Q,B}_t f \) to \( f \) as \( t \) tends to \( 0 \) for \( f \) in \( L^p(\gamma_\infty) \).

If the semigroup \( (\mathcal{H}^{Q,B}_t)^{t \geq 0} \) is symmetric, i.e. if \( \mathcal{H}^{Q,B}_t \) is self-adjoint on \( L^2(\gamma_\infty) \) for every \( t \geq 0 \), then \( \mathcal{H}^{Q,B}_t \) is bounded on \( L^p(\gamma_\infty) \) for every \( p \) in \( (1, \infty) \), by the Littlewood-Paley-Stein theory for symmetric semigroups of contractions on all \( L^p \) spaces [11]. Is the result still true if we drop the symmetry assumption? In the same monograph [11] Stein says that for general diffusion semigroups the condition of self-adjointness cannot be much modified. Indeed if one considers the semigroup of translations \( T_t f(x) = f(x + t) \) on the one-dimensional torus \( \mathbb{T} \), for every \( p \) in \( [1, \infty) \) it is easy to construct a function \( f \) in \( L^p(\mathbb{T}) \) such that \( \sup_{t \geq 0} |T_t f(x)| = \infty \) everywhere. Notice that \( (T_t)^{t \geq 0} \) is a semigroup of normal, actually unitary, operators.

However, in Theorem 4.2 below we show that Stein’s proof of the maximal theorem for semigroups of symmetric contractions on all \( L^p(\mu) \), \( 1 \leq p \leq \infty \), can be adapted to semigroups of normal contractions such that the generator of the semigroup on \( L^2(\mu) \) is a sectorial operator of angle \( \phi < \pi/2 \). Since the generator of the Ornstein-Uhlenbeck on \( L^2(\gamma_\infty) \) is sectorial of angle strictly less than \( \pi/2 \) this implies that if \( (\mathcal{H}^{Q,B}_t)^{t \geq 0} \) is normal on \( L^2(\gamma_\infty) \) then the maximal operator \( \mathcal{H}_* \) is bounded on \( L^p(\gamma_\infty) \) for every \( p \) in \( (1, \infty) \).

It remains to investigate the boundedness of the Ornstein-Uhlenbeck maximal operator \( \mathcal{H}^{Q,B}_* \) on \( L^1(\gamma_\infty) \). In Section 4 we show that \( \mathcal{H}^{Q,B}_* \) is always unbounded on \( L^1(\gamma_\infty) \). This still leaves open the question of the validity of the weak type 1 estimate

$$\gamma_\infty \{ x \in \mathbb{R}^d : |\mathcal{H}^{Q,B}_* f(x)| > \alpha \} \leq \frac{C \|f\|_1}{\alpha} \quad \forall f \in L^1(\gamma_\infty) \quad \forall \alpha > 0.$$ 

Even in the symmetric case very little is known about the weak type 1 boundedness of the Ornstein-Uhlenbeck maximal operator. The only result which is known for the semigroup with covariance matrix \( Q = I \) and drift matrix \( B = -I \) for which the weak type 1 boundedness of \( \mathcal{H}^{Q,B}_* \) is due to B. Muckenhoupt [9] in dimension one and to P. Sjögren [10] in arbitrary dimension. Sjögren’s proof was subsequently simplified in [6] and [4]. The arguments in these papers easily extend to the case where \( B = -\lambda I \) for some \( \lambda > 0 \). However, already the case where \( B \) is a diagonal matrix with at least two different eigenvalues seems to require new ideas.

In this paper we investigate the weak type 1 estimate for the maximal operator associated to the Ornstein-Uhlenbeck semigroup with covariance matrix \( Q = I \) and drift \( B = -\lambda (I - R) \), where \( \lambda > 0 \) and \( R \) is a nonzero real \( d \times d \) skew-adjoint matrix. The interest of these semigroups is motivated by the fact that they are the basic building blocks of normal Ornstein-Uhlenbeck semigroups. Indeed, in Section 2 we show that, after a change of variables, any normal Ornstein-Uhlenbeck semigroup can be written as the product of commuting semigroups of this form.
For these particular semigroups we shall prove two results. First we shall prove that the “truncated” maximal operator
\[ H_{Q,B}^{*,[0,T]} f(x) = \sup_{t \in [0,T]} |H_t^{Q,B} f(x)| \]
is of weak type 1. Second, we shall prove that if the one-parameter group of rotations \((e^{tR})_{t \in \mathbb{R}}\) generated by \(R\) is periodic then the full maximal operator \(H_{Q,B}^{*}\) is of weak type 1.

Finally we mention that, by using the results of the present paper, in [5] we have proved that first order Riesz transforms associated to the generator of these ‘periodic’ semigroups are of weak type 1.

We now briefly describe the content of the paper. In Section 2 we characterize the generators of normal Ornstein-Uhlenbeck semigroups and we show that, after a change of coordinates, normal semigroups are the product of commuting semigroups with covariance matrix \(Q = I\) and drift \(B = -\lambda(I - R)\), with \(\lambda > 0\) and \(R\) a real skew-adjoint matrix.

In Section 3 we give an explicit representation of the integral kernel of these semigroups with respect to the invariant measure. We show that, modulo an orthogonal change of coordinates, the semigroup kernel is the product of the kernel of a symmetric semigroup and some two-dimensional kernels. Ultimately, this will enable us to reduce the problem of the weak type 1 boundedness of the maximal operator to proving estimates of kernels defined on \(\mathbb{R}^2 \times \mathbb{R}^2\).

In Section 4 we study the boundedness of the maximal operator \(H_t^{Q,B}\) on \(L^p(\gamma_\infty)\), \(1 \leq p \leq \infty\), for arbitrary \(Q\) and \(B\). We prove that the truncated maximal operator is always unbounded on \(L^1(\gamma_\infty)\) and that, when the semigroup is normal, the full maximal operator is bounded on \(L^p(\gamma_\infty)\), \(1 < p \leq \infty\).

Finally, in Section 5 we prove the weak type estimate for the truncated and the full maximal operator when \(Q = I\) and \(B = -\lambda(I - R)\). By the results of Section 3 the kernel of the semigroup is a perturbation of the kernel of a symmetric semigroup. When \(t\) is close to zero the perturbation is small and the kernel of the nonsymmetric semigroup can be controlled by the kernel of the symmetric semigroup. The same thing happens in the periodic case when \(t\) is close to an integer multiple of a period. This enables us to apply the results of 4 to prove the weak type estimate for the truncated maximal operator and of the full maximal operator in the periodic case.

2. Preliminaries

The Schwartz space \(S(\mathbb{R}^d)\) is a core for the infinitesimal generator \(L_{Q,B}\) of the semigroup \((H_t^{Q,B})_{t \geq 0}\) on \(L^p(\gamma_\infty)\) for every \(p\), \(1 < p < \infty\), and
\[
L_{Q,B} f = \frac{1}{2} \text{tr}(Q \nabla^2) f + (B, \nabla) f \quad \forall f \in S(\mathbb{R}^d).
\]

By a result of G. Metafune, J. Prüss, A. Rhandi and R. Schnaubelt (see [8 Lemma 2.2]) there exists a linear change of coordinates in \(\mathbb{R}^d\) which allows us to reduce the analysis of the operator \(L_{Q,B}\) to the case where \(Q = I\) and \(Q_\infty\) is a diagonal matrix. Indeed, let \(M_1\) be an invertible real matrix such that \(M_1^T Q M_1 = I\) and \(M_2\) an orthogonal matrix such that \(M_2 M_1 Q_\infty M_1^T M_2 = \text{diag}(\lambda_1, \ldots, \lambda_d) := D\lambda\) for some \(\lambda_j > 0\). Then, if we take \(M = M_2 M_1\) and we denote by \(\Phi_M : S(\mathbb{R}^d) \to S(\mathbb{R}^d)\) the similarity transformation defined by \(\Phi_M f(x) = f(M^{-1} x)\) we
have that $\mathcal{L}_{Q,B} = \Phi_\lambda^{-1} \mathcal{L}_{1,B} \Phi_M$ where

$$
(2.1) \quad \hat{B} = -\frac{1}{2}D_{1/\lambda} + R
$$

and $R$ is a matrix such that

$$
(2.2) \quad RD_{\lambda} = -D_{\lambda} R^*.
$$

The invariant measure for the semigroup generated by $\mathcal{L}_{1,B}$ is

$$
d\bar{\gamma}_{\infty}(x) = (2\pi)^{-d/2}(\det D_{\lambda})^{-1/2}e^{-\frac{1}{2}g(D_{\lambda}^{-1}x,x)}d\lambda(x).
$$

Moreover $\bar{\gamma}_{\infty}(E) = \gamma_{\infty}(M^{-1}E)$ for every Borel subset $E$ of $\mathbb{R}^d$ and $\Phi_M$ extends to an isometry of $L^p(\gamma_{\infty})$ onto $L^p(\bar{\gamma}_{\infty})$.

By (2.1) we may write the operator $\mathcal{L}_{1,B}$ as the sum

$$
(2.3) \quad \mathcal{L}_{1,B} = \mathcal{L}^0 + \mathcal{R},
$$

where $\mathcal{L}^0 = \frac{1}{2}\Delta - \frac{1}{2}\langle D_{1/\lambda}x, \nabla \rangle$ and $\mathcal{R} = \langle Rx, \nabla \rangle$ are the symmetric and the antisymmetric part of $\mathcal{L}_{1,B}$ on $L^2(\gamma_{\infty})$, respectively. Thus, the operator $\mathcal{L}_{Q,B}$ is symmetric on $L^2(\gamma_{\infty})$ if and only if $R = 0$.

Let $(\mathcal{H}_{Q,B}^t)_{t \geq 0}$ be the semigroup generated by $\mathcal{L}_{1,B}$ and $\mathcal{H}_{Q,B}^t$ the corresponding maximal operator. Clearly, $\mathcal{H}_{Q,B}^t$ is bounded on $L^p(\gamma_{\infty})$ or of weak type 1 with respect to $\gamma_{\infty}$ if and only if $\mathcal{H}_{Q,B}^t$ is bounded on $L^p(\bar{\gamma}_{\infty})$ or of weak type 1 with respect to $\bar{\gamma}_{\infty}$. Thus, the analysis of the maximal operator $\mathcal{H}_{Q,B}^t$ may be reduced to the case where $Q = I$ and $Q_{\infty} = \text{diag}\{\lambda_1, \ldots, \lambda_d\}$ for some $\lambda_j > 0$.

**Proposition 2.1.** Let $\hat{B}$, $D_{\lambda}$ and $R$ be the matrices associated to $Q$ and $B$ as in (2.1). Denote by $\mathcal{L}^0$ and $\mathcal{R}$ the symmetric and the antisymmetric part of $\mathcal{L}_{1,B}$ as in (2.3). Then the following properties are equivalent

(i) the semigroup $(\mathcal{H}_{Q,B}^t)_{t \geq 0}$ is normal on $L^2(\gamma_{\infty})$;

(ii) the symmetric and the antisymmetric parts of $\mathcal{L}_{1,B}$ commute; i.e.

$$
[L^0, \mathcal{R}]\phi = 0 \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d);
$$

(iii) $R + R^* = 0$;

(iv) $D_{\lambda}$ and $R$ commute.

**Proof.** We claim that $\mathcal{L}_{1,B}^* = \mathcal{L}^0 - \mathcal{R}$. Indeed, on the one hand $(\mathcal{L}^0)^* = \mathcal{L}^0$ because $\mathcal{L}^0$ is symmetric. On the other hand, integrating by parts, we get that

$$
\mathcal{R}^* = -\mathcal{R} + \langle Rx, D_{\lambda}^{-1}x \rangle - \text{tr} R
$$

because $\text{tr} R = 0$ and $\langle Rx, D_{\lambda}^{-1}x \rangle = 0$ since $\langle Rx, D_{\lambda}^{-1}x \rangle = \langle x, R^* D_{\lambda}^{-1}x \rangle = -\langle x, D_{\lambda}^{-1}Rx \rangle = -\langle D_{\lambda}^{-1}x, Rx \rangle$ by (2.2).

The semigroup $(\mathcal{H}_{Q,B}^t)_{t \geq 0}$ is normal if and only if its generator $\mathcal{L}_{Q,B}$ on $L^2(\gamma_{\infty})$ is normal and this happens if and only if $\mathcal{L}_{1,B}$ is normal on $L^2(\gamma_{\infty})$, i.e.

$$
[L_{1,B}, \mathcal{L}_{1,B}^*] \phi = 0 \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).
$$

Now

$$
[L_{1,B}, \mathcal{L}_{1,B}^*] = [L^0 + \mathcal{R}, L^0 - \mathcal{R}] = 2[\mathcal{R}, L^0].
$$
This shows that (i) and (ii) are equivalent. Next observe that
\[
[R, L^0] = -\langle \nabla, R\nabla \rangle + \frac{1}{2}\langle (RD_{1/\lambda} - D_{1/\lambda}R)x, \nabla \rangle.
\] (2.4)
Hence \([R, L^0]\) vanishes if and only if \(\langle \nabla, R\nabla \rangle\) and \(\langle (RD_{1/\lambda} - D_{1/\lambda}R)x, \nabla \rangle\) both vanish, as can be easily seen by fixing any pair of indices \(j, k\) and an arbitrary point \(x_0\) and applying the commutator to a test function \(\phi\) which in a neighbourhood of \(x_0\) coincides with \((x - x_0)_j(x - x_0)_k\). Now, \(\langle \nabla, R\nabla \rangle\) vanishes if and only if \(R + R^* = 0\). Thus (ii) implies (iii). To prove the converse observe that by (2.2) the identity \(R + R^* = 0\) implies that \(R\) and \(D_{1/\lambda}\) commute. Thus also \(D_{1/\lambda}\) and \(R\) commute. Hence \([R, L^0]\) = 0 by (2.4). Finally, if (iv) holds then \(R + R^* = 0\) by (2.2). This concludes the proof of the proposition.

In the last part of this section we show that operators of the form \(L_{Q,B}\) with \(Q = I\) and \(B = \frac{1}{2\alpha}(R - I)\) where \(\alpha > 0\) and \(R\) is a \(d \times d\) skew-symmetric real matrix, are the basic building blocks of normal Ornstein-Uhlenbeck operators. This motivates the interest in studying the maximal operator associated to semigroups generated by them.

To simplify notation we write
\[
L(\alpha, R) = L_{I, \frac{1}{2\alpha}(R - I)} = \frac{1}{2}\Delta - \frac{1}{2\alpha}\langle x, \nabla \rangle + \frac{1}{2\alpha}\langle Rx, \nabla \rangle,
\] (2.5)
Let \((H_t^{Q,B})_{t \geq 0}\) be a normal Ornstein-Uhlenbeck semigroup. By (2.1) after a change of variables we may assume that its generator is of the form
\[
L_{I, \tilde{B}} = \frac{1}{2}\Delta - \frac{1}{2}\langle D_{1/\lambda}x, \nabla \rangle + \langle Rx, \nabla \rangle,
\]
where \(R + R^* = 0\) and \(R\) commutes with \(D_{1/\lambda}\) by Proposition (2.1). Let \(\alpha_1, \ldots, \alpha_\ell\) be the distinct eigenvalues of \(D_{1/\lambda}\) and let
\[
D_{1/\lambda} = \alpha_1 P_1 + \cdots + \alpha_\ell P_\ell
\]
be the spectral resolution of \(D_{1/\lambda}\). The matrix \(R\) commutes with the projections \(P_j\) and if we set \(R_j = 2\alpha R P_j\) then \(R_j^* = -R_j^\top\) and \(R = \sum_{j=1}^\ell R_j\). Thus, denoting by \(\Delta_j = \text{tr}(P_j \nabla^2)\) and \(\nabla_j = P_j \nabla\) the Laplacian and the gradient with respect to the variables in \(P_j \mathbb{R}^d\), we have
\[
L_{I, \tilde{B}} = \sum_{j=1}^\ell L(\alpha_j, R_j),
\]
where
\[
L(\alpha_j, R_j) = \frac{1}{2}\Delta_j - \frac{1}{2\alpha_j}\langle x, \nabla_j \rangle + \frac{1}{2\alpha_j}\langle R_j x, \nabla_j \rangle.
\]
The semigroup generated by \(L_{I, \tilde{B}}\) is the product of the commuting semigroups \((e^{tL(\alpha_j, R_j)})_{t \geq 0}\) generated by the operators \(L(\alpha_j, R_j)\), \(j = 1, \ldots, \ell\), which are therefore the basic building blocks of normal Ornstein-Uhlenbeck semigroups.
3. The kernel of the semigroup with respect to the invariant measure

For our purposes it is convenient to write the Ornstein-Uhlenbeck semigroup as a semigroup of integral operators with respect to the invariant measure \( \gamma_{\infty} \). We recall that the Gauss measure with mean zero and covariance matrix \( Q_t \) on \( \mathbb{R}^d \) is the measure

\[
d\gamma_t(x) = (2\pi)^{-d/2}(\det Q_t)^{-1/2}e^{-\frac{1}{2}(Q_t^{-1}x,x)}d\lambda(x) \quad \forall t \in (0, \infty],
\]

where \( \lambda \) denotes the Lebesgue measure. In the following, with a slight abuse of notation, we shall denote by the same symbol \( \gamma_t \) also the density of the measure with respect to \( \lambda \). A simple change of variables in (1.2) yields

\[
\mathcal{H}^Q f(x) = \int h_t(x,y) f(y) d\gamma_\infty(y),
\]

where

\[
(3.1) \quad h_t(x,y) = \det(Q_\infty Q_t^{-1})^{1/2} e^{-\frac{1}{2}[(Q_t^{-1}(e^{Bt}x-y),(e^{Bt}x-y))-Q_\infty^{-1}y,y]]}
\]

The main result of this section is that, after an orthogonal change of coordinates, the kernel of the semigroup generated by an operator of the form

\[
(3.2) \quad \mathcal{L}(\alpha, R) = \frac{1}{2} \Delta - \frac{1}{2\alpha} \langle x, \nabla \rangle + \frac{1}{2\alpha} \langle Rx, \nabla \rangle,
\]

with \( \alpha > 0 \) and \( R + R^* = 0 \), can be written as the product of the kernel of the semigroup generated by its symmetric part \( \mathcal{L}(\alpha, 0) \) and some two-dimensional kernels (see Theorem 3.1 and formula (3.9)). To simplify notation, for the rest of this section we write \( \mathcal{L} = \mathcal{L}(\alpha, R) \) and \( \mathcal{L}^0 = \mathcal{L}(\alpha, 0) \). Thus

\[
\mathcal{L}^0 = \frac{1}{2} \Delta - \frac{1}{2\alpha} \langle x, \nabla \rangle, \quad \mathcal{L} = \mathcal{L}^0 + \frac{1}{2\alpha} \langle Rx, \nabla \rangle
\]

Henceforth we shall denote by \( (e^{t\mathcal{L}})_{t \geq 0} \) and by \( (e^{t\mathcal{L}})_{t \geq 0} \) the semigroups generated by \( \mathcal{L}^0 \) and by \( \mathcal{L} \), respectively, and by \( h^0_t(x,y) \) and \( h_t(x,y) \) their kernels with respect to the invariant measure

\[
d\gamma_\infty(x) = (2\pi\alpha)^{-d/2} e^{-\frac{\alpha \|x\|^2}{2}}.
\]

By the results of the previous section, the operator \( \mathcal{L}^0 \) is symmetric and \( \mathcal{L} \) is normal.

To avoid having many \( \alpha \)'s floating around and to be consistent with the notation in [4], we fix \( \alpha = 1/2 \). The formulas for arbitrary \( \alpha > 0 \) can be obtained from this special case by replacing \( t \) by \( t/2\alpha \) and \( (x,y) \) by \( (x/\sqrt{2\alpha}, y/\sqrt{2\alpha}) \) in formulas (3.2) and (3.3) below.

The kernel of the semigroup \( (e^{t\mathcal{L}})_{t \geq 0} \) is

\[
(3.2) \quad h^0_t(x,y) = (1 - e^{-2t})^{-d/2} \exp \left\{ \frac{1}{2} \left[ \frac{|x+y|^2}{e^t + 1} - \frac{|x-y|^2}{e^t - 1} \right] \right\}.
\]

The operator \( R = \langle Rx, \nabla \rangle \) generates the semigroup of isometries \( e^{tR} f(x) = f(e^{tR}x) \) of \( L^p(\gamma_\infty) \), \( 1 \leq p \leq \infty \). Since \( e^{tR} \) commutes with \( e^{t\mathcal{L}^0} \) for every \( t \geq 0 \), the kernel of \( (e^{t\mathcal{L}})_{t \geq 0} \) is

\[
(3.3) \quad h_t(x,y) = h^0_t(e^{tR}x,y).
\]
We shall exploit the facts that the matrix $R$ is skew-adjoint and that the symmetric semigroup $(e^{tL})_{t \geq 0}$ commutes with orthogonal transformations to prove that, after an orthogonal change of coordinates, the operator $L$ and the kernel $h_t(x, y)$ can be written in a more convenient form.

First we consider a special two-dimensional case. For every real number $\theta$ we denote by $R(\theta)$ the $2 \times 2$ matrix

\begin{equation}
R(\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.
\end{equation}

Let $x \wedge y$ denote the skew-symmetric bilinear form on $\mathbb{R}^2$ defined by

$x \wedge y = x_1 y_2 - x_2 y_1$.

Then

\begin{equation}
e^{tR(\theta)x \pm y} = |x|^2 + |y|^2 + 2 \cos(t\theta)\langle x, y \rangle \pm \sin(t\theta)x \wedge y \quad \forall x, y \in \mathbb{R}^2.
\end{equation}

Now, consider the Ornstein-Uhlenbeck operator $L(\frac{1}{2}, R(\theta))$ on $\mathbb{R}^2$. To simplify notation henceforth we write $L_\theta = L(\frac{1}{2}, R(\theta))$. Thus

$L_\theta = \frac{1}{2} \Delta - \langle x, \nabla \rangle + \langle R(\theta)x, \nabla \rangle$,

is the operator with covariance matrix $Q = I$ and drift $B = -I + R(\theta)$. By using \cite{3.2, 3.3, 3.5} it is straightforward to see that the kernel of the semigroup generated by $L_\theta$ is

\begin{equation}
h_t^0(x, y) = h_t^0(x, y) \ k_t^\theta(x, y),
\end{equation}

where $h_t^0(x, y)$ is as in \cite{3.2} with $d = 2$ and

\begin{equation}
k_t^\theta(x, y) = \exp \left\{ -\frac{e^{-t}}{1 - e^{-2t}} \left[ (1 - \cos(t\theta)) \langle x, y \rangle + \sin(t\theta)x \wedge y \right] \right\}.
\end{equation}

Next we consider the case when the matrix $R$ is a $d \times d$ matrix in block diagonal form, with $2 \times 2$ blocks of the form \cite{3.4}. Let $n = \lfloor d/2 \rfloor$ be the greatest integer less than or equal to $d/2$. If $\Theta = (\theta_1, \ldots, \theta_n)$ is in $\mathbb{R}^n$ we denote by $R(\Theta)$ the $d \times d$ block-diagonal matrix

\[
\begin{pmatrix}
R(\theta_1) & 0 & \cdots & 0 \\
0 & R(\theta_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R(\theta_n)
\end{pmatrix}
or
\begin{pmatrix}
R(\theta_1) & 0 & \cdots & 0 \\
0 & R(\theta_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

according to whether $d$ is even or odd, respectively.

Assume first that $d$ is even. Given a vector $x$ in $\mathbb{R}^d \simeq (\mathbb{R}^2)^n$ we write $x = (\xi_1, \ldots, \xi_n)$, where $\xi_k = (x_{2k-1}, x_{2k}) \in \mathbb{R}^2$ for $k = 1, \ldots, n$. Let $L_\Theta = L(\frac{1}{2}, R(\Theta))$ be the Ornstein-Uhlenbeck operator on $\mathbb{R}^d$ of the form

\begin{equation}
L_\Theta = \frac{1}{2} \Delta - \langle x, \nabla \rangle + \langle R(\Theta)x, \nabla \rangle.
\end{equation}

Then $L_\Theta = L_{\theta_1} + \cdots + L_{\theta_n}$, where each $L_{\theta_k}$ for $k = 1, \ldots, n$ is a two-dimensional Ornstein-Uhlenbeck operator acting in the variables $\xi_k = (x_{2k-1}, x_{2k})$ of the form

$L_{\theta_k} = \frac{1}{2} \Delta_k - \langle \xi_k, \nabla_k \rangle + \langle R(\theta_k)\xi_k, \nabla_k \rangle$. 


Here $\Delta_k$ and $\nabla_k$ denote the two-dimensional Laplacian and gradient in the variables $(x_{2k-1}, x_{2k})$.

Thus the operators $L_{\Theta_k}$, $k = 1, \ldots, n$ commute as do the semigroups generated by them. This implies that the kernel $h_t^\Theta(x, y)$ of the semigroup $(e^{tL_\Theta})_{t \geq 0}$ is the product of the kernels of the semigroups $(e^{tL_{\Theta_k}})_{t \geq 0}$, $k = 1, \ldots, n$; i.e.

$$h_t^\Theta(x, y) = \prod_{k=1}^n h_t^{\Theta_k}(\xi_k, \eta_k)$$

with $\xi_k = (x_{2k-1}, x_{2k})$ and $\eta_k = (y_{2k-1}, y_{2k})$ in $\mathbb{R}^2$, where $h_t^{\Theta_k}(\xi_k, \eta_k)$ are as in (3.6).

If $d$ is odd then $L_\Theta = L_{\Theta_1} + \ldots + L_{\Theta_n} + L_{n+1}$ where $L_{\Theta_k}$, $k = 1, \ldots, n$, are as before and $L_{n+1}$ is the one-dimensional symmetric Ornstein-Uhlenbeck operator $\frac{1}{2}\partial^2_{x_{n+1}} - x_{n+1}\partial_{x_{n+1}}$, acting in the variable $x_{n+1}$. Thus the kernel $h_t(x, y)$ has an additional factor $h_t^{\Theta}(x_{n+1}, y_{n+1})$, which is the kernel of a one-dimensional symmetric Ornstein-Uhlenbeck semigroup.

In any case, regardless of the parity of $d$, by (3.6) we may write the kernel of $e^{tL_\Theta}$ in the following way

$$h_t^\Theta(x, y) = h_0^\Theta(x, y) \prod_{n=1}^\infty k_{\Theta_n}(\xi_j, \eta_j)$$

(3.9)

where $h_0^\Theta(x, y)$ is the kernel of the $d$-dimensional symmetric semigroup generated by $\frac{1}{2}\Delta - \langle x, \nabla \rangle$ and each $k_{\Theta_n}$ is a two-dimensional kernel as in (3.7).

Finally, we show that the analysis of any operator $L = \frac{1}{2}\Delta - \langle x, \nabla \rangle + \langle Rx, \nabla \rangle$, where $R$ is a skew adjoint matrix, may be reduced to that of an operator of the form $L_\Theta$. As in Section 2, given an invertible real $d \times d$-matrix $M$, we denote by $\Phi_M : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ the transformation defined by $\Phi_M u(y) = u(M^{-1}y)$.

**Theorem 3.1.** Let $n = [d/2]$ be the greatest integer less than or equal to $d/2$ and let $L$ be the operator $\frac{1}{2}\Delta - \langle x, \nabla \rangle + \langle Rx, \nabla \rangle$, where $R$ is a $d \times d$ real, skew-adjoint matrix. Then there exists a $d \times d$ orthogonal matrix $g$ and a vector $\Theta = (\theta_1, \ldots, \theta_n)$ with $\theta_j \geq 0$ such that $\Phi_g L \Phi_g^{-1} = L_\Theta$. Moreover the kernels $h_t(x, y)$ and $h_t^\Theta(x, y)$ of the semigroups generated by $L$ and $L_\Theta$, respectively, satisfy the identity

$$h_t(x, y) = h_t^\Theta(gx, gy) \quad \forall x, y \in \mathbb{R}^d, \ t > 0.$$

**Proof.** The set $a = \{R(\Theta) : \Theta \in \mathbb{R}^n\}$ is a maximal abelian subalgebra of the Lie algebra $\mathfrak{so}(d)$ of skew-symmetric $d \times d$ matrices. Since, by a well known result of Lie algebras (see [1]), every element of $\mathfrak{so}(d)$ is conjugated to an element of $a^+ = \{R(\Theta) : \Theta \in \mathbb{R}_{++}^n\}$, given a skew-symmetric matrix $R$ there exists an orthogonal matrix $g$ and a vector $\Theta = (\theta_1, \ldots, \theta_n)$, with $\theta_j \geq 0$, such that $R = gR(\Theta)g^{-1}$. The identity $\Phi_g L \Phi_g^{-1} = L_\Theta$ follows, because the symmetric part $\frac{1}{2}\Delta - \langle x, \nabla \rangle$ of the operator $L$ commutes with $\Phi_g$.

This implies that $\Phi_g e^{tL_\Theta} \Phi_g^{-1} = e^{tL_\Theta}$ for every $t \geq 0$. The identity between the kernels of the semigroups follows immediately from it.
4. Strong type estimates

In this section we return to consider a Ornstein-Uhlenbeck semigroup \((\mathcal{H}^Q_t)_{t \geq 0}\) with arbitrary covariance \(Q\) and drift \(B\). We prove that the truncated Ornstein-Uhlenbeck maximal operator \(\mathcal{H}^Q_t\) is always unbounded on \(L^1(\gamma_\infty)\) and when the semigroup is normal the full maximal operator \(\mathcal{H}^Q\) is bounded on \(L^p(\gamma_\infty)\), \(1 < p \leq \infty\).

**Theorem 4.1.** For all \(T > 0\) the operator \(\mathcal{H}^Q_{[0,T]}\) is unbounded on \(L^1(\gamma_\infty)\).

**Proof.** Suppose, by contradiction, that \(\mathcal{H}^Q_{[0,T]}\) is bounded on \(L^1(\gamma_\infty)\) for some \(T > 0\). Denote by \(\gamma_\infty\) the density of the invariant measure with respect to the Lebesgue measure. Let \((f_n)\) be a sequence of nonnegative functions of norm 1 in \(L^1(\gamma_\infty)\) which converges in sense of distributions to \(\gamma_\infty(0)^{-1}\delta_0\). Then there exists a constant \(C\) such that

\[
\|\mathcal{H}^Q_{[0,T]} f_n\|_1 \leq C \quad \text{for every } n.
\]

Moreover

\[
\lim_{n \to \infty} \mathcal{H}^Q_t f_n(x) = \lim_{n \to \infty} h_t(x,y) f_n(y) d\gamma_\infty(y) = h_t(x,0)
\]

uniformly on compact subsets of \(\mathbb{R}^d\). Thus, for \(n\) sufficiently large,

\[
\mathcal{H}^Q_{[0,T]} f_n(x) \geq \mathcal{H}^Q_t f_n(x) \geq h_t(x,0) - 1 \quad \forall x \in B(0,1) \quad \forall t \in [0,T].
\]

Hence

\[
(4.1) \quad \int_{|x| \leq 1} \sup_{t \in [0,T]} h_t(x,0) d\gamma_\infty(x) \leq C.
\]

Now recall the expression of the kernel \(h_t(x,y)\) given in (4.1). Since \(Q_t \sim tQ\) for \(t \to 0^+\), if \(t \in (0,\epsilon)\) for some \(\epsilon > 0\) sufficiently small then there exist positive constants \(c_0, c_1\) and \(c_2\) such that

\[
h_t(x,0) = \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\left\{ -\frac{1}{4} (Q^{-1} e^{tB}_x, e^{tB}_x) \right\}
\]

\[
\geq c_0 t^{-d/2} \exp\left\{ -c_1 \frac{\|e^{tB}_x\|^2}{t} \right\}
\]

\[
\geq c_0 t^{-d/2} \exp\left\{ -c_2 \frac{|x|^2}{t} \right\}.
\]

Thus if \(|x| \leq 1\)

\[
\sup_{0 < t < \epsilon} h_t(x,0) \geq c_0 \sup_{0 < t < \epsilon} t^{-d/2} e^{-c_2 \frac{|x|^2}{t}} \geq c_t |x|^{-d},
\]

which contradicts (4.1). \(\square\)

The positive result for \(L^p(\gamma_\infty)\), \(1 < p \leq \infty\), for normal Ornstein-Uhlenbeck semigroups follows from a more general result for normal semigroups of contractions on all \(L^p\)-spaces, whose generator on \(L^2\) is sectorial. Indeed we have the following theorem.

**Theorem 4.2.** Let \((X, \mu)\) be a \(\sigma\)-finite measure space. Let \((T_t)_{t \geq 0}\) be a semigroup of contractions on \(L^p(\mu)\) for every \(p\) in \([1, \infty]\), which is strongly continuous for \(p < \infty\). Suppose that each \(T_t\) is normal on \(L^2(\mu)\) and that the spectrum of the
Corollary 4.3. Let \((\mathcal{H}_t^{Q,B})_{t \geq 0}\) be a normal Ornstein-Uhlenbeck semigroup. Then the maximal operator \(\mathcal{H}_*^{Q,B}\) is bounded on \(L^p(\gamma_\infty)\) for every \(p \in (1, \infty)\).

Proof. By \([11]\) the spectrum of the generator of \((\mathcal{H}_t^{Q,B})_{t \geq 0}\) is contained in a sector of angle less than \(\pi/2\). Hence the conclusion follows from Theorem 4.2. \(\square\)
5. THE WEAK TYPE ESTIMATE.

In this section we shall prove the weak type 1 estimate for the maximal operators associated to the normal Ornstein-Uhlenbeck semigroup \((H_t^{Q,R})_{t \geq 0}\) with covariance \(Q = I\) and drift \(B = \frac{1}{2\alpha}(R - I)\), where \(\alpha > 0\) and \(R\) is a skew-symmetric real matrix, i.e. for the semigroup generated by the operator

\[
\mathcal{L}(\alpha, R) = \frac{1}{2}\Delta - \frac{1}{2\alpha} \langle x, \nabla \rangle + \frac{1}{2\alpha}\langle Rx, \nabla \rangle.
\]

Namely, we shall prove the following theorem.

**Theorem 5.1.** For every \(T > 0\) the truncated maximal operator

\[
\mathcal{H}_{*,[0,T]}f(x) = \sup_{t \in [0,T]} |e^{t\mathcal{L}(\alpha, R)}f(x)|
\]

is of weak type 1. If the one-parameter group \((e^{tR})_{t \in \mathbb{R}}\) is periodic then the full maximal operator \(\mathcal{H}_*f(x) = \sup_{t \geq 0} |e^{t\mathcal{L}(\alpha, R)}f(x)|\) is of weak type 1.

As we have already remarked in Section 3 we may assume that \(2\alpha = 1\), by a scaling argument.

First we reduce the problem to proving that two smaller maximal operators are of weak type 1. For every subset \(A\) of \(\mathbb{R}_+\) denote by \(\mathcal{H}_{*,A}\) the maximal operator defined by

\[
\mathcal{H}_{*,A}f(x) = \sup_{t \in A} |e^{t\mathcal{L}(1/2, R)}f(x)|, \quad f \in L^1(\gamma_\infty).
\]

If \(I\) is a closed interval in \(\mathbb{R}_+\) and \(P\) is a positive number, we denote by \(I^d_P\) the union of \(PN\)-translates of \(I\), i.e. \(I^d_P = \bigcup_{n \in \mathbb{N}} (I + Pn)\).

**Lemma 5.2.** Suppose that for some \(t_0 > 0\) the maximal operator \(\mathcal{H}_{*,[0,t_0]}\) is of weak type 1. Then the truncated maximal operator \(\mathcal{H}_{*,[0,T]}\) is of weak type 1 for every \(T > 0\). If, furthermore, there exists an interval \(I\) in \(\mathbb{R}_+\) such that the operator \(\mathcal{H}_{*,I^d_P}\) is of weak type 1 then the full maximal operator \(\mathcal{H}_*\) is of weak type 1.

**Proof.** First we show that if \(A\) is a subset of \(\mathbb{R}_+\) such that the operator \(\mathcal{H}_{*,A}\) is of weak type 1 and \(B = \bigcup_{i=1}^N (A + t_i)\) is a finite union of translates of \(A\) then \(\mathcal{H}_{*,B}\) is of weak type 1. Indeed

\[
\mathcal{H}_{*,B}f(x) = \sup_{t \in B} |e^{t\mathcal{L}(1/2, R)}f(x)| = \max_{i=1,\ldots,N} \sup_{t \in A} |e^{(t+t_i)\mathcal{L}(1/2, R)}f(x)|
\]

\[
= \max_{i=1,\ldots,N} \sup_{t \in A} |e^{t\mathcal{L}(1/2, R)}e^{t_i\mathcal{L}(1/2, R)}f(x)|
\]

\[
= \max_{i=1,\ldots,N} \mathcal{H}_{*,A} e^{t_i\mathcal{L}(1/2, R)}f(x).
\]

Hence, for \(\lambda > 0\) fixed,

\[
\gamma_\infty(\{x \in \mathbb{R}^d : \mathcal{H}_{*,B}f(x) > \lambda\}) \leq \sum_{i=1}^N \gamma_\infty(\{x \in \mathbb{R}^d : \mathcal{H}_{*,A} e^{t_i\mathcal{L}(1/2, R)}f(x) > \lambda\})
\]

\[
\leq C \lambda \sum_{i=1}^N \|e^{t_i\mathcal{L}(1/2, R)}f\|_{L^1(\gamma_\infty)}
\]

\[
\leq C \frac{N}{\lambda} \|f\|_{L^1(\gamma_\infty)}.
\]
because \( e^{tL(1/2,R)} \) is a contraction on \( L^1(\gamma_\infty) \) for every \( i = 1, \ldots, N \).

The conclusion follows because the set \([0,T]\) is a finite union of translates of \((0,t_0)\) and \(\mathbb{R}_+\) is a finite union of translates of \([0,T]\) and \(I^*_P\).

Thus we only need to prove the weak type 1 estimate for the operator \( H_{s,A} \) when \( A = (0,t_0) \) and \( A = I^*_P \) for some \( t_0 > 0 \) and some closed interval \( I \) in \( \mathbb{R}_+ \). As in the analysis of the maximal operator for the symmetric Ornstein-Uhlenbeck semigroup \( \{e^{tL(1/2,0)}\}_{t \geq 0} \) (see [3]), we shall decompose each of these two maximal operators in a “local” part, given by a kernel living close to the diagonal, and the remaining or “global” part. To this end consider the set

\[
L = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq \min(1, |x + y|^{-1}) \right\}
\]

and denote by \( G \) its complement. We shall call \( L \) and \( G \) the ‘local’ and the ‘global’ region, respectively. The local and the global parts of the operator \( H_{s,A} \) are defined by

\[
H_{s,A}^{\text{loc}} f(x) = \sup_{t \in A} \left| \int h_t(x,y) 1_L(x,y) f(y) \, d\gamma(y) \right|,
\]

\[
H_{s,A}^{\text{glob}} f(x) = \sup_{t \in A} \left| \int h_t(x,y) 1_G(x,y) f(y) \, d\gamma(y) \right|,
\]

(5.1)

where \( 1_L \) and \( 1_G \) are the characteristic functions of the sets \( L \) and \( G \) respectively. Clearly

\[
H_{s,A} f(x) \leq H_{s,A}^{\text{loc}} f(x) + H_{s,A}^{\text{glob}} f(x).
\]

We shall prove separately the weak type 1 estimate for \( H_{s,A}^{\text{loc}} \) and \( H_{s,A}^{\text{glob}} \).

First we deal with the local part. We shall actually prove that for all Ornstein-Uhlenbeck semigroups \( \{H_t\}_{t \geq 0} \), without restrictions on covariance and drift, the local maximal operator \( H_{s}^{\text{loc}} = H_{s,A}^{\text{loc}} \) is of weak type 1.

**Lemma 5.3.** Let \( \{H_t\}_{t \geq 0} \) be an Ornstein-Uhlenbeck semigroup with arbitrary covariance and drift. Then there exist positive constants \( c \) and \( C \) such that for all \( (x,y) \) in the local region \( L \)

\[
h_t(x,y) \leq C (1 - e^{-t})^{-d/2} \gamma_\infty(y)^{-1} \exp \left( -c \frac{|x-y|^2}{1 - e^{-t}} \right) \quad \forall t > 0.
\]

(5.2)

**Proof.** Since the real part of the eigenvalues of \( B \) is negative, there exist positive constants \( \alpha \leq \beta \) and \( C_0 \) such that \( C_0^{-1} e^{2\alpha s} |x|^2 \leq C_0 |e^{sB^*}| \leq e^{2\beta s} |x|^2 \) for all \( x \in \mathbb{R}^d \) and all \( s \in \mathbb{R} \). Thus, by (1.1) there exists a positive constant \( C \) such that

\[
C^{-1}(1 - e^{-t})I \leq Q_t \leq C(1 - e^{-t})I \quad \forall t \in (0, \infty).
\]

and, by (3.1), there exist two positive constants \( c \) and \( C \) such that

\[
h_t(x,y) \leq C (1 - e^{-t})^{-d/2} \gamma_\infty(y)^{-1} \exp \left( -c \frac{|e^{tB}x - y|^2}{1 - e^{-t}} \right).
\]

(5.3)
Now, for all \((x, y)\) in the local region \(L\)
\[
|e^{tB}x - y|^2 = |x - y + (e^{tB} - I)x|^2
\]
\[
= |x - y|^2 + |(e^{tB} - I)x|^2 + 2\langle x - y, (e^{tB} - I)x \rangle
\]
\[
\geq |x - y|^2 - 2\|e^{tB} - I\||x - y||x|
\]
\[
\geq |x - y|^2 - C(1 - e^{-t}),
\]
because \(\|e^{tB} - I\| \leq C(1 - e^{-t})\) and \(|x - y| |x| \leq C\) in the local region \(L\). □

**Proposition 5.4.** Let \((\mathcal{H}_t)_{t \geq 0}\) be a Ornstein-Uhlenbeck semigroup with arbitrary covariance and drift. Then the maximal operator \(\mathcal{H}_*^{loc}\) is of weak type \(1\).

**Proof.** By Lemma 5.3 one has that for each \(f \geq 0\)
\[
\mathcal{H}_*^{loc}f(x) \leq C \sup_{0 < s \leq 1} s^{-d/2} \int e^{-\frac{|x-y|^2}{s}} 1_L(x, y) f(y) \, d\lambda(y)
\]
\[
= \mathcal{W}f(x),
\]
say. Since the operator \(\mathcal{W}\) is of weak type 1 with respect to the Lebesgue measure and its kernel is supported in the local region \(L\), the conclusion follows by well-known arguments (see for instance [4, Section 3]). □

Now we turn to the proof of the weak type estimate for the global part of the maximal operator associated to the semigroup generated by the special Ornstein-Uhlenbeck operator
\[
\mathcal{L}(1/2, R) = \frac{1}{2}\Delta - \langle x, \nabla \rangle + (Rx, \nabla),
\]
where \(R\) is a skew-symmetric real matrix.

As in Section 3 we denote by \(h_t(x, y)\) and by \(h^0_t(x, y)\) the kernels with respect to the invariant measure of the semigroups generated by \(\mathcal{L}(1/2, R)\) and by its symmetric part
\[
\mathcal{L}^0 = \frac{1}{2}\Delta - \langle x, \nabla \rangle,
\]
respectively (see 3.2 and 3.3).

To estimate the semigroup kernel in the global region, it is convenient to simplify the expression of \(h^0_t(x, y)\) by means of the change of variables in the parameter \(t\) introduced in [4]. We denote by \(\tau\) the function defined by
\[
\tau(s) = \log \frac{1 + s}{1 - s}, \quad s \in (0, 1).
\]
Notice that \(\tau\) maps \((0, 1)\) onto \(\mathbb{R}_+\). It is straightforward to check (see [4]) that for all \(s\) in \((0, 1)\)
\[
h^0_{\tau(s)}(x, y) = (4s)^{-d/2}(1 + s)^{d/2}e^{\frac{|x|^2 + |y|^2}{4s} - \frac{1}{4}(s|x+y|^2 + \frac{1}{2}|x-y|^2)}.
\]
Next, as in [4], we introduce the quadratic form
\[
Q_s(x, y) = |(1 + s)x - (1 - s)y|^2, \quad x, y \in \mathbb{R}^d.
\]
Thus
\[
s|x + y|^2 + \frac{1}{s}|x - y|^2 = \frac{1}{s}Q_s(x, y) - 2|x|^2 + 2|y|^2.
\]
and
\[ h_{\tau(s)}^0(x, y) = s^{-d/2} \exp \left\{ \left| x \right|^2 - \frac{1}{4s} Q_s(x, y) \right\} \quad \forall s \in (0, 1). \]

**Lemma 5.5.** If \( t_0 > 0 \) is sufficiently small, there exists a positive constant \( C \) such that for all \( s \) in \((0, \tau^{-1}(t_0))\) and all \((x, y)\) in \( \mathbb{R}^d \times \mathbb{R}^d \)
\[ h_{\tau(s)}(x, y) \leq C s^{-d/2} e^{\left| x \right|^2 - \frac{1}{4s} Q_s(x, y)}. \]

**Proof.** Let \( n = \lfloor d/2 \rfloor \). The right hand side of the inequality to prove is invariant under orthogonal transformations. Hence, by Theorem 3.1, it is enough to prove the inequality for the kernel \( h_\Theta^\tau(x, y) \), with \( \Theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \), \( \theta_j \geq 0 \).

By (5.9) and (5.7)
\[ h_\Theta^\tau(x, y) \leq s^{-d/2} \exp \left\{ \left| x \right|^2 - \frac{1}{4s} Q_s(x, y) \right\} \prod_{\theta_j > 0} k_{\tau(s) \theta_j}(\xi_j, \eta_j), \quad \forall s \in (0, 1), \]
where \( \xi_j = (x_{2j}, x_{2j+1}) \) and \( \eta_j = (y_{2j}, y_{2j+1}) \) are in \( \mathbb{R}^2 \) and each \( k_{\theta_j} \) is a two-dimensional kernel as in (5.7).

Define
\[ M_s(x, y) = \exp \left\{ -\frac{9}{40 s} Q_s(x, y) \right\} \prod_{\theta_j > 0} k_{\tau(s) \theta_j}(\xi_j, \eta_j). \]

Then
\[ h_\Theta^\tau(x, y) \leq s^{-d/2} \exp \left\{ \left| x \right|^2 - \frac{1}{40 s} Q_s(x, y) \right\} M_s(x, y), \]
and to conclude the proof of the lemma all we need to show is that there exist a \( s_0 > 0 \) sufficiently small and a constant \( C \) such that
\[ M_s(x, y) \leq C \quad \forall s \in (0, s_0) \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \]

Let us denote by \( Q_s^{(m)} \) the quadratic form defined in (5.6) when considered as a function on \( \mathbb{R}^m \times \mathbb{R}^m \). Then
\[ Q_s^{(d)}(x, y) = \begin{cases} 
\sum_{j=1}^n Q_s^{(2)}(\xi_j, \eta_j) & \text{if } d \text{ is even} \\
\sum_{j=1}^n Q_s^{(2)}(\xi_j, \eta_j) + Q_s^{(1)}(x_{n+1}, y_{n+1}) & \text{if } d \text{ is odd}.
\end{cases} \]

Thus, since \( Q_s^{(m)} \) is nonnegative,
\[ M_s(x, y) \leq \prod_{\theta_j > 0} \exp \left\{ -\frac{9}{40 s} Q_s^{(2)}(\xi_j, \eta_j) \right\} k_{\tau(s) \theta_j}(\xi_j, \eta_j) \]
regardless of the parity of \( d \). Hence we only need to show that each factor is bounded, i.e. that for every \( \theta > 0 \) there exist \( s_0 \in (0, 1) \) and a constant \( C \) such that for all \((x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \)
\[ \exp \left\{ -\frac{9}{40 s} Q_s(x, y) \right\} k_{\tau(s) \theta}(x, y) \leq C \quad \forall s \in (0, s_0), \]
where now \( Q_s = Q_s^{(2)} \), for the sake of brevity.
To this end we fix $\beta$ in $(0,1)$, we let $\delta$ be a constant in $(0,1)$ to be chosen later and we denote by $\vartheta = \vartheta(x, y)$ the angle between the two vectors $x$ and $y$. The set $\mathbb{R}^2 \times \mathbb{R}^2$ is the disjoint union of the five sets

\begin{align*}
R_1 &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, y \rangle < 0\} \\
R_2 &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, y \rangle \geq 0, x \wedge y \geq 0\}, \\
R_3 &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, y \rangle \geq 0, x \wedge y < 0, |x - y| \geq \beta|y|\}, \\
R_4 &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, y \rangle \geq 0, x \wedge y < 0, |x - y| < \beta|y|, |\sin \vartheta| \geq \delta\}, \\
R_5 &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x, y \rangle \geq 0, x \wedge y < 0, |x - y| < \beta|y|, |\sin \vartheta| < \delta\}.
\end{align*}

We shall prove that (5.11) holds in each region $R_j$, $j = 1, \ldots, 5$. Note that by (3.7) and (5.6)

\begin{equation}
(5.12) \quad k_{\tau(s)}(x, y) = e^{-\frac{\tau(s)}{2}[1 - \cos(\tau(s))\langle x, y \rangle + \sin(\tau(s))x \wedge y]}
\end{equation}

and that the function $s \mapsto \tau(s)$ is positive and increasing in $(0,1)$ and $\tau(s) \sim 2s$ as $s \to 0^+$. To prove the estimate in $R_1$, we observe that there exists a constant $C_1$ such that

\begin{equation}
(5.13) \quad k_{\tau(s)}(x, y) \leq \exp\{C_1|x||y|\} \quad \forall x, y \in \mathbb{R}^2, \forall s \in (0,1).
\end{equation}

Since $Q_s(x, y) \geq (1 - s^2)(|x|^2 + |y|^2)$, because $\langle x, y \rangle < 0$ in $R_1$, we have that if $s_0$ is sufficiently small

\begin{equation}
(5.14) \quad -\frac{9}{40}s Q_s(x, y) + C_1|x||y| < 0 \quad \forall (x, y) \in R_1, \forall t \in (0, s_0).
\end{equation}

Together (5.13) and (5.14) imply (5.11) in $R_1$.

The proof of (5.11) in $R_2$ is straightforward, because in this region $Q_s(x, y) \geq 0$ and $k_{\tau(s)}(x, y) \leq 1$.

Next suppose that $(x, y)$ is in $R_3$. Since $\langle x, y \rangle \geq 0$, there exists a constant $C_2$ such that

\begin{equation}
(5.15) \quad k_{\tau(s)}(x, y) \leq \exp\{C_2|x| \wedge y\} = \exp\{C_2|x| |y| |\sin \vartheta|\} \quad \forall s \in (0,1).
\end{equation}

We claim that there exists $s_0 \in (0,1)$ such that

\begin{equation}
(5.16) \quad -\frac{9}{40}s Q_s(x, y) + C_2|x| |y| |\sin \vartheta| \leq 0 \quad \forall s \in (0, s_0),
\end{equation}

To prove the claim first consider the case where $|x| \geq |y|$. Then $Q_s(x, y) \geq |x - y|^2$ and hence, since $|x - y| \geq |\sin \vartheta||x|$ and $|x - y| \geq \beta|y|$, $-\frac{9}{40}s Q_s(x, y) + C_2|x| |y| |\sin \vartheta| \leq \left(-\frac{9}{40}s \beta + C_2\right) |x| |y| |\sin \vartheta| \leq 0$, provided that $s < \frac{9\beta}{40C_2}$.

Next consider the case where $|x| < |y|$. In this case we have that $Q_s(x, y) \geq |x - y|^2 - 2s|y|^2$. Thus, since $|x| < |y|$ and $|x - y| \geq \beta|y|$, $-\frac{9}{40}s Q_s(x, y) + C_2|x| |y| |\sin \vartheta| \leq \left(-\frac{9}{40}s \beta^2 + \frac{9}{20} + C_2\right) |y|^2 \leq 0$ provided that $s < \frac{9\beta^2}{40C_2+18}$. 
Thus (5.16) holds for all \((x, y)\) in \(R^3\) with \(s_0 \leq \min \left\{ \frac{9\beta}{40C_2}, \frac{9\beta^2}{40C_2+18} \right\}\). Together (5.15) and (5.16) imply (5.11) in \(R_3\).

The proof of estimate (5.11) in \(R_4\) is similar. Indeed, first of all (5.16) holds in \(R_4\) because here too \((x, y) > 0\). Moreover, arguing much as before, one can show that (5.16) holds also for all \((x, y)\) in \(R_4\) with \(s_0 \leq \min \left\{ \frac{9\beta^2}{40C_2}, \frac{9\beta^2}{40C_2+18} \right\}\). The only difference is that one uses the estimates

\[
Q_s(x, y) \geq |x - y|^2 \geq (\sin \vartheta)^2 |x| \geq \delta^2 |x| |y|
\]

when \(|x| \geq |y|\) and

\[
Q_s(x, y) \geq |x - y|^2 - 2s|x|^2 \geq (\sin \vartheta)^2 |y|^2 - 2s |y|^2 \geq (\delta^2 - 2s) |y|^2
\]

when \(|x| < |y|\). We omit the details. Notice that, so far, we did not need to impose any restriction on \(\delta\), which therefore could be any number in \((0, 1)\).

It remains to estimate \(h_4(x, y)\) in \(R_5\). We observe that since \(\tau(s) \sim 2s\) as \(s \to 0^+\) and \((x, y) \geq 0\) and \(x \wedge y < 0\) in \(R_5\), by (5.12) there exist \(s_0 > 0\) and two positive constants \(c_0 < 2 < c_1\) such that

\[
k_{\tau(s)\vartheta}(x, y) \leq \exp \left\{ -c_0 \frac{\theta^2}{4} s \langle x, y \rangle - c_1 \frac{\theta}{4} x \wedge y \right\} \quad \forall s \in (0, s_0).
\]

Moreover, we can choose \(c_0\) and \(c_1\) as close to 2 as we want, provided that we choose \(s_0\) sufficiently small; in particular, we may take

\[
c_1^2/c_0 < 18/5.
\]

Now we are ready to prove estimate (5.11) in \(R_5\). Define

\[
E_s(x, y) = -\frac{9}{10} Q_s(x, y) - c_0 \theta^2 s^2 \langle x, y \rangle - c_1 \theta s x \wedge y.
\]

By (5.17)

\[
\exp \left\{ -\frac{9}{40s} Q_s(x, y) \right\} k_{\tau(s)\vartheta}(x, y) \leq \exp \left\{ \frac{1}{4s} E_s(x, y) \right\}.
\]

Thus, to prove (5.11) in \(R_5\) it is enough to show that

\[
E_s(x, y) \leq 0 \quad \forall s \in (0, s_0) \quad \forall (x, y) \in R_5.
\]

provided that \(s_0, \beta\) and \(\delta\) are sufficiently small.

Observe that

\[
E_s(x, y) = \lambda(x, y) s^2 + \mu(x, y) s + \nu(x, y),
\]

where

\[
\lambda(x, y) = -\frac{9}{10} |x + y|^2 - c_0 \theta^2 \langle x, y \rangle,
\]

\[
\mu(x, y) = \frac{18}{10} (|y|^2 - |x|^2) - c_1 \theta x \wedge y,
\]

\[
\nu(x, y) = -\frac{9}{10} |x - y|^2.
\]

It turns out that, instead of \(E_s(x, y)\), it is more convenient to consider the function \(|x|^{-1} |y|^{-1} E_s(x, y)\) because the latter function depends only on the variables \(s\),
Indeed, if we denote by $\Psi$ the function defined by $\Psi(s, x, y) = (s, X, \vartheta)$,
\begin{equation}
|y|^{-1} |y|^{-1} E_s(x, y) = F(\Psi(s, x, y)),
\end{equation}
where
\begin{equation}
F(s, X, \vartheta) = \tilde{\lambda}(X, \vartheta)s^2 + \tilde{\mu}(X, \vartheta)s + \tilde{\nu}(X, \vartheta),
\end{equation}
and
\begin{align*}
\tilde{\lambda}(x, y) &= -\frac{9}{10} (X + X^{-1} + 2 \cos \vartheta) - c_0 \theta^2 \cos \vartheta \\
\tilde{\mu}(x, y) &= \frac{18}{10} (X^{-1} - X) - c_1 \theta \sin \vartheta \\
\tilde{\nu}(x, y) &= -\frac{9}{10} (X + X^{-1} - 2 \cos \vartheta).
\end{align*}

It is easy to see that $(0, 1, 0)$ is a critical point of $F$ and the Hessian $\nabla^2 F(0, 1, 0)$ is
 definite negative because $c_2^2 - \frac{18}{12} c_0 < 0$ by (3.13). Thus $(0, 1, 0)$ is a local maximum
of $F$ and, since $F(0, 1, 0) = 0$ there exists a neighbourhood $U$ of $(0, 1, 0)$ in which $F$ is
$\leq 0$. Now, since
\[
\Psi((0, s_0) \times R) \subset \{(s, X, \vartheta) : s \in (0, s_0), |X - 1| < \beta, \delta < \sin(\vartheta) \leq 0\},
\]
we can choose $s_0$, $\beta$ and $\delta$ so small that $\Psi((0, s_0) \times R) \subset U$. Hence $F \circ \Psi \leq 0$ in
$(0, s_0) \times R$. Thus (5.19) is satisfied and the proof of the lemma is complete. \qed

To prove the boundedness of the non-truncated maximal operator we need to assume
that the one-parameter group $(e^{tR})_{t \in \mathbb{R}}$ generated by the skew-adjoint ma-
natrix $R$ is periodic. We recall that if $I$ is an interval contained in $\mathbb{R}_+$ and $P > 0$ we
denote by $I^*_P$ the set $\cup_{n \in \mathbb{N}}(I + nP)$.

**Lemma 5.6.** Suppose that the skew-adjoint matrix $R$ generates a one-parameter
group $(e^{tR})_{t \in \mathbb{R}}$ which is periodic of period $P$. Then there exist an interval $I$ and
a constant $C$ such that for all $s$ in $\tau^{-1}(I^*_P)$ and all $(x, y)$ in $\mathbb{R}^d \times \mathbb{R}^d$
\[
h_{\tau(s)}(x, y) \leq C e^{\left\| x \right\|^2 - \frac{4}{3P} \mathcal{Q}_s(x, y)},
\]

**Proof.** As in the proof of Lemma 5.3 it is enough to prove the inequality for the kernel
$h^{\Theta}_s(x, y)$, with $\Theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\theta_j \geq 0$. Let $\{\theta_1, \ldots, \theta_m\}$ be
the nonzero components of $\Theta$, i.e. the absolute values of the nonzero eigenvalues of $R$.
Denote by $\theta_{\max}$ the maximum of $\{\theta_1, \ldots, \theta_m\}$.

Fix $\delta = \min \left\{ \theta_{\max}^{-1}, 1/10 \right\}$ and let $\epsilon$ be a small positive constant ($\epsilon \leq 1/10$ will
do). Define $I = [\delta, (1 + \epsilon)\delta]$. For all $\theta \in (\theta_1, \ldots, \theta_m)$ the functions $t \mapsto \cos(\theta t)$ and
$t \mapsto \sin(\theta t)$ are periodic of period $P$ and by considering their Taylor expansions at
zero it is easy to see that for all $\theta \in (\theta_1, \ldots, \theta_m)$
\begin{equation}
c_0 \leq 1 - \cos(\theta t) \leq c_2, \quad \sin(\theta t) \leq c_1 \quad \forall t \in I^*_P,
\end{equation}
where
\begin{equation}
c_0 = \frac{5}{12} \epsilon \theta^2 \delta^2 \quad c_1 = (1 + \epsilon) \theta \delta \quad \text{and} \quad c_2 = \frac{(1 + \epsilon) \theta^2 \delta^2 \theta^2}{2}.
\end{equation}

Arguing as in the proof of Lemma 5.5 we may reduce matters to showing that if
$\vartheta \in (\theta_1, \ldots, \theta_m)$ then there exists a constant $C$ such that
\begin{equation}
e^{\frac{-4}{3P} \mathcal{Q}_s(x, y)} k_{\tau(s)}(x, y) \leq C \quad \forall s \in \tau^{-1}(I^*_P) \quad \forall (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2.
\end{equation}
Thus, to prove (5.24) in $\mathbb{R}^2 \times \mathbb{R}^2$ we see that \((5.25)\) and the constant term \(c\) it is enough to observe that here \(k_{t^{\theta}}(x, y) \leq 1\) for all \(t\) in \(\mathbb{R}_+\).

Now suppose that \((x, y)\) is in \(R_2\). Then, by \((5.22)\) and \((5.25)\) we have that
\[
\exp \left\{ -\frac{9}{40} s Q_s(x, y) \right\} k_{t^{\theta}}(x, y) \leq \exp \left\{ \frac{1}{4} s F_s(x, y) \right\},
\]
where
\[
F_s(x, y) = p(x, y) s^2 + q(x, y) s + r(x, y)
\]
\[
p(x, y) = -\frac{9}{10} |x + y|^2 + c_0(x, y) + c_1 x \wedge y,
\]
\[
q(x, y) = \frac{18}{10} (|y|^2 - |x|^2),
\]
\[
r(x, y) = -\frac{9}{10} |x - y|^2 - c_0(x, y) - c_1 x \wedge y.
\]

Thus, to prove \((5.24)\) in \(R_2\), we only need to show that \(F_s(x, y) \leq 0\) for all \((x, y)\) in \(R_2\).

It is an easy matter to see that, with \(c_0\) and \(c_1\) as in \((5.23)\), the leading coefficient \(p(x, y)\) and the constant term \(r(x, y)\) are negative for all \((x, y)\) in \(R_2\). Thus it suffices to show that the discriminant \(q^2 - 4pr\) is nonpositive in \(R_2\). If \(|y| = |x|\) this is obvious, because then \(q(x, y) = 0\). If \(|y| \neq |x|\), after some simple algebra using the identity
\[
|x + y|^2 |x - y|^2 = (|y|^2 - |x|^2)^2 + 4 \sin^2(\theta) |x|^2 |y|^2,
\]
we see that \((q^2 - 4pr) |x|^{-2} |y|^{-2}\) is only a function of the angle \(\theta\) between \(x\) and \(y\). Thus its sign does not change if we rescale in \(x\). In particular we may reduce matters to the case \(|y| = |x|\), where \(q = 0\). This proves that \(F_s(x, y) \leq 0\) for all \((x, y)\) in \(R_2\) and \(s\) in \(\mathbb{R}\). By \((5.27)\) this implies that \((5.24)\) holds in \(R_2\).

Finally suppose that \((x, y)\) is in \(R_3\). We have that
\[
\exp \left\{ -\frac{9}{40} s Q_s(x, y) \right\} k_{t^{\theta}}(x, y) \leq \exp \left\{ \frac{1}{4} s G_s(x, y) \right\},
\]
where
\[
G_s(x, y) = \tilde{p}(x, y) s^2 + q(x, y) s + \tilde{r}(x, y),
\]
\[
\tilde{p}(x, y) = -\frac{9}{10} |x + y|^2 - c_2 |\langle x, y \rangle| + c_1 |x \wedge y|,
\]
\[
q(x, y) = \frac{18}{10} (|y|^2 - |x|^2),
\]
\[
\tilde{r}(x, y) = -\frac{9}{10} |x - y|^2 + c_2 |\langle x, y \rangle| - c_1 |x \wedge y|,
\]
and \(c_1, c_2\) are as in (5.23). Thus to prove the desired inequality (5.24), we only need to show that \(G_s(x, y) \leq 0\) in \(R^3\). Since it is easy to see that both \(\tilde{p}\) and \(\tilde{r}\) are negative in \(R^3\), as before we only need to prove that \(q^2 - 4\tilde{p}\tilde{r} \leq 0\) in \(R^3\). This can be proved by an argument similar to that used in \(R^2\). We omit the details.

Hence (5.24) holds for all \((x, y)\) in \(R^2 \times R^2\). This concludes the proof of the lemma.

We recall two lemmas from [4].

**Lemma 5.7.** Let \(\vartheta = \vartheta(x, y)\) denote the angle between the non-zero vectors \(x\) and \(y\). There exists a constant \(C\) such that for all \((x, y)\) in the global region \(G\)
\[
\sup_{0 < s \leq 1} s^{-d/2} e^{-\frac{d}{4}Q_s(x, y)} \leq C \min \left\{ \left(1 + |x|\right)^d, \left(|x| \sin \vartheta\right)^{-d} \right\}
\]

**Lemma 5.8.** The operator
\[
T f(x) = e^{\left|x\right|^2} \int \min \left\{ \left(1 + \left|x\right|\right)^d, \left(|x| \sin \vartheta\right)^{-d} \right\} f(y) d\gamma_\infty(y)
\]
is of weak type 1.

We are now ready to conclude the proof of Theorem 5.1

**Proof.** Let \(A\) denote either the set \([0, \tau_0]\) or \(I^p_\tau\). By Proposition 5.4 the local part of the operator \(H_{s, A}\) is of weak type 1. Thus it remains only to prove that the global part is of weak type 1. By (5.1), Lemma 5.5, Lemma 5.6 and Lemma 5.7, the global part of the operator \(H_{s, A}\) is controlled by the operator \(T\), which is of weak type 1 by Lemma 5.8. The conclusion follows by Lemma 5.2.

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