OPTIMAL Z-EIGENVALUE INCLUSION INTERVALS OF TENSORS AND THEIR APPLICATIONS

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ABSTRACT. Firstly, a weakness of Theorem 3.2 in [Journal of Industrial and Management Optimization, 17(2) (2021) 687-693] is pointed out. Secondly, a new Geršgorin-type $Z$-eigenvalue inclusion interval for tensors is given. Subsequently, another Geršgorin-type $Z$-eigenvalue inclusion interval with parameters for even order tensors is presented. Thirdly, by selecting appropriate parameters some optimal intervals are provided and proved to be tighter than some existing results. Finally, as an application, some sufficient conditions for the positive definiteness of homogeneous polynomial forms as well as the asymptotically stability of time-invariant polynomial systems are obtained. As another application, bounds of $Z$-spectral radius of weakly symmetric nonnegative tensors are presented, which are used to estimate the convergence rate of the greedy rank-one update algorithm and derive bounds of the geometric measure of entanglement of symmetric pure state with nonnegative amplitudes.

1. Introduction. Let $m$ and $n$ be two positive integers and $n \geq 2$, $\{n\}$ be the set \{1, 2, \ldots, $n$\}, $\mathbb{C}$ (or, respectively, $\mathbb{R}$) be the set of all complex (or, respectively, real) numbers, $\mathbb{R}^n$ be the set of all dimension $n$ real vectors, $\mathbb{R}_{+}^{[m,n]}$ be the set of all order $m$ dimension $n$ real tensors, and $\mathbb{R}_{+}^{[m,n]}$ be the set of all order $m$ dimension $n$ nonnegative tensors. Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, i.e.,

$$a_{i_1i_2\cdots i_m} \in \mathbb{R}, \quad i_j \in [n], \quad j \in [m].$$

$A$ is called symmetric [29] if $a_{i_1\cdots i_m} = a_{i_{\pi(1)}\cdots i_{\pi(m)}}$, $\forall \pi \in \Pi_m$, where $\Pi_m$ is the permutation group of $m$ indices. $A$ is called weakly symmetric [5] if the associated homogeneous polynomial

$$f(x) = Ax^m = \sum_{i_1,\ldots,i_m \in [n]} a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m} \quad (1)$$

satisfies $\nabla A x^m = m A x^{m-1}$, where $x = (x_1,x_2,\ldots,x_n)^\top \in \mathbb{R}^n$, and $Ax^{m-1}$ is an $n$ dimension vector whose $i$th component is

$$(Ax^{m-1})_i = \sum_{i_2,\ldots,i_m \in [n]} a_{i_2\cdots i_m}x_{i_2}\cdots x_{i_m}.$$
It is showed in [5] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Next, the $Z$-identity tensor is recalled; for details, see [16, 24, 29].

**Definition 1.1.** [16, 24, 29] A tensor $\mathcal{E} \in \mathbb{R}^{[m,n]}$ with $m$ being even is called a $Z$-identity tensor if

$$\mathcal{E} x^{m-1} = x,$$

for any vector $x \in \mathbb{R}^n$ with $x^\top x = 1$.

In general, the even-order $n$ dimension $Z$-identity tensor is not unique. For instance, each of the following tensors is a $Z$-identity tensor:

**Case I.** (see [24, Definition 2.1]): Let $\mathcal{E}_1 = (e_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 \cdots i_m} = \begin{cases} 1, & i_1 = i_2, i_3 = i_4, \ldots, i_{m-1} = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

**Case II.** (see [16, Property 2.4]): Let $\mathcal{E}_2 = (e_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$, where

$$e_{i_1 \cdots i_m} = \frac{1}{m!} \sum_{\pi \in \Pi_m} \delta_{\pi(1)\pi(2)} \delta_{\pi(3)\pi(4)} \cdots \delta_{\pi(m-1)\pi(m)}$$

for $i_1, \ldots, i_m \in [n]$, and $\delta$ is the standard Kronecker delta, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

For convenient applications, the $Z$-identity tensor $\mathcal{E}_2 = (e_{ijkl}) \in \mathbb{R}^{[4,n]}$ and the $Z$-identity tensor $\mathcal{E}_2 = (e_{i_1i_2\cdots i_6}) \in \mathbb{R}^{[6,n]}$ are listed as follows:

$$e_{ijkl} = \begin{cases} 1, & i = j = k = l, \\ 1/3, & i = j \neq k = l, \\ 1/3, & i = k \neq j = l, \\ 1/3, & i = l \neq j = k, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$e_{i_1i_2\cdots i_6} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_6, \\ 1/5, & (i_1, i_2, \ldots, i_6) \in \bigcup_{\pi \neq \pi', \pi', \pi' \in [n]} \{\pi(i, i, i, i, j, j)\}, \\ 1/15, & (i_1, i_2, \ldots, i_6) \in \bigcup_{\pi \neq \pi', \pi' \in [n]} \{\pi(i, i, j, j, k, k)\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{\pi(i, j, k, l, s, t)\}$ is the set of all combinations of $i, j, k, l, s, t$.

Now, let us recall the definitions of $Z$-eigenvalues of tensors. Given a tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, if there are $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \setminus \{0\}$ such that

$$\mathcal{A} x^{m-1} = \lambda x \quad \text{and} \quad x^\top x = 1,$$

then $\lambda$ is called an $E$-eigenvalue of $\mathcal{A}$ and $x$ an $E$-eigenvector of $\mathcal{A}$ associated with $\lambda$. If $\lambda$ and $x$ are all real, then $\lambda$ is called a $Z$-eigenvalue of $\mathcal{A}$ and $x$ a $Z$-eigenvector of $\mathcal{A}$ associated with $\lambda$; for details, see [26, 29, 33, 34]. Let $\sigma(\mathcal{A})$ be the set of all $Z$-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq 0$, then the $Z$-spectral radius [5] of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$
As shown in (1) that the tensor $A$ defines an $m$-th degree homogeneous polynomial $f(x)$. The positive definiteness of $f(x)$ is widely used in spectral hypergraph theory [33] and the stability study of nonlinear autonomous systems via Lyapunov’s direct method in automatic control [14, 3, 4, 6, 13, 28]. As pointed out in [29] that theory [33] and the stability study of nonlinear autonomous systems via Lyapunov’s nonnegative amplitudes in quantum mechanics [44]. The related results are showed to estimate the geometric measure of entanglement for symmetric pure states with spectral radius of weakly symmetric nonnegative tensors, which play a fundamental role in best rank-one approximation [15, 29]. Besides that the bounds can be used to estimate the geometric measure of entanglement for symmetric pure states with nonnegative amplitudes in quantum mechanics [44]. The related results are showed in [10, 11, 12, 25, 27, 35, 37, 38, 39, 40, 41, 43, 46, 47, 48].

In 2017, Wang et al. [41] established a Geršgorin-type $Z$-eigenvalue inclusion set for tensors as follows.

**Theorem 1.2.** [41, Theorem 3.1] Let $A = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$. Then

$$
\sigma(A) \subseteq K(A) := \bigcup_{i \in [n]} K_i(A),
$$

where

$$
K_i(A) = \{ z \in \mathbb{R} : |z| \leq R_i(A) \} \text{ and } R_i(A) = \sum_{i_2, \ldots, i_m \in [n]} |a_{i_1i_2\ldots i_m}|.
$$

Apparently, the inclusion interval $K(A)$ always includes zero, and consequently it cannot be used to judge the positive definiteness of an even-order real symmetric tensor (also an even-order homogeneous polynomial form). To overcome the drawback, Li et al. [24] presented another $Z$-eigenvalue inclusion interval with $n$ parameters for even-order real tensors as follows.

**Theorem 1.3.** [24, Theorem 2.2] Let $A = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ with $m$ being even. Then for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,

$$
\sigma(A) \subseteq G(A, \alpha) := \bigcup_{i \in [n]} \left( G_i(A, \alpha) := \{ z \in \mathbb{R} : |z - \alpha_i| \leq R_i(A, \alpha_i) \} \right),
$$

where

$$
R_i(A, \alpha_i) = \sum_{(i_2, \ldots, i_m) \in \Lambda_1} |a_{i_1i_2\ldots i_m} - \alpha_i e_{i_1i_2\ldots i_m}| + \sum_{(i_2, \ldots, i_m) \in \overline{\Lambda}_1} |a_{i_1i_2\ldots i_m}| \tag{5}
$$

and

\[
\Lambda_1 = \{(i_2, \ldots, i_m) : e_{i_1i_2\ldots i_m} \neq 0, \ i_2, \ldots, i_m \in [n]\},
\]

\[
\overline{\Lambda}_1 = \{(i_2, \ldots, i_m) : e_{i_1i_2\ldots i_m} = 0, \ i_2, \ldots, i_m \in [n]\}.
\]
The form of $R_i(\mathcal{A}, \alpha_i)$ in Theorem 1.3 is closely related to the $Z$-identify tensor $\mathcal{E}$. Li et al. in [24] obtained the following result by taking the $Z$-identify tensor $\mathcal{E}_1$.

**Corollary 1.** [24, Corollary 1] Let $\mathcal{A} = (a_{ij, \ldots, k}) \in \mathbb{R}^{[m,n]}$ with $m = 2k$ being even. Then (4) holds for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, where

$$R_i(\mathcal{A}, \alpha_i) = \sum_{i_2, \ldots, i_k \in [n]} |a_{ii_2i_3 \ldots i_k} - \alpha_i| + \bar{r}_i(\mathcal{A})$$

(6)

and

$$\bar{r}_i(\mathcal{A}) = R_i(\mathcal{A}) - \sum_{i_2, \ldots, i_k \in [n]} |a_{ii_2i_3 \ldots i_k}|.$$  

(7)

Based on the inclusion interval $G(\mathcal{A}, \alpha)$ in Theorem 1.3, Li et al. in [24] provided a sufficient condition for the positive definiteness of even-order real symmetric tensors.

**Theorem 1.4.** [24, Definition 3.1, Theorem 3.2] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ with $m$ being even, and $\lambda$ be a $Z$-eigenvalue of $\mathcal{A}$. If $\alpha_i > R_i(\mathcal{A}, \alpha_i)$ for each $i \in [n]$, then $\lambda > 0$. Furthermore, if $\mathcal{A}$ is also symmetric, then $\mathcal{A}$ is positive definite, consequently, $f(x)$ defined in (1) is positive definite.

However, Theorem 1.4 cannot be used to judge the positive definiteness of an even order $n$ dimension real symmetric tensor $\mathcal{A}$ if $R_i(\mathcal{A}, \alpha_i)$ is taken as in (6), that is, $\mathcal{E}$ is taken as $\mathcal{E}_1$ in Theorem 1.3. Now, an example for an order 4 dimension 2 symmetric tensor is given to verify this fact.

**Example 1.5.** Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor with elements defined as follows:

$$a_{1111} = a, \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = b,$$

$$a_{1122} = a_{1212} = a_{1221} = a_{2121} = a_{2211} = c,$$

$$a_{1222} = a_{2122} = a_{2212} = a_{2221} = d, \quad a_{2222} = e.$$

In Theorem 1.4, taking the $Z$-identity tensor $\mathcal{E} = (e_{ijkl})$ as $\mathcal{E}_1$, then $e_{1111} = e_{1122} = e_{2211} = e_{2222} = 1$ and $e_{ijkl} = 0$ for others. Assume that there exists a real number $\alpha_1$ such that $\alpha_1 > R_1(\mathcal{A}, \alpha_1)$. Then, by (6), we have

$$\alpha_1 > R_1(\mathcal{A}, \alpha_1) = |a_{1111} - \alpha_1| + |a_{1122} - \alpha_1|$$

$$+ |a_{1112}| + |a_{1121}| + |a_{1211}| + |a_{1212}| + |a_{2111}| + |a_{2121}| + |a_{2211}| + |a_{2212}|$$

$$= |a - \alpha_1| + |c - \alpha_1| + 3|b| + 2|c| + |d|,$$

that is,

$$|a - \alpha_1| - |c - \alpha_1| - 3|b| + 2|c| + |d|,$$

which implies that $\alpha_1$ needs to satisfy the condition

$$\max_{\alpha_1 \in \mathbb{R}} \{ |a - \alpha_1| - |c - \alpha_1| - 3|b| + 2|c| + |d| \} > 3|b| + 2|c| + |d|. \quad (8)$$

If $a \geq c$, then $\max_{\alpha_1 \in \mathbb{R}} \{ |a - \alpha_1| - |c - \alpha_1| \} = c$, and by $c \leq |c| \leq 3|b| + 2|c| + |d|$, it can be seen that (8) does not hold. If $a < c$, then $\max_{\alpha_1 \in \mathbb{R}} \{ |a - \alpha_1| - |c - \alpha_1| \} = a$, and by $a < c \leq |c| \leq 3|b| + 2|c| + |d|$, it can be seen that (8) still does not hold. Hence, there is no parameter $\alpha_1$ such that (8) holds, which implies that when $R_i(\mathcal{A}, \alpha_i)$ is taken as in (6), that is, $\mathcal{E}$ is taken as $\mathcal{E}_1$ in Theorem 1.3, we cannot use Theorem 1.4 to judge the positive definiteness of a fourth-order 2 dimension real symmetric tensor.
Next, similar to Example 1.5, a proposition can be presented.

**Proposition 1.** Let $A \in \mathbb{R}^{[m,n]}$ be a symmetric tensor with $m$ being even. Then Theorem 1.4 cannot be used to judge the positive definiteness of $A$ if $R_i(A, \alpha_i)$ is taken as in (6), that is, taking the $Z$-identify tensor $E$ as $E_1$ in $R_i(A, \alpha_i)$ in (5).

**Proof.** Let $m = 2k$, $\ell = n^{k-1}$, and $R_i(A, \alpha_i)$ be as in (6). For each $i \in [n], \alpha_i > R_i(A, \alpha_i) = \sum_{i_2, \ldots, i_k \in [n]} |a_{i_2i_3 \ldots i_k} - \alpha_i| + \bar{r}_i(A)$ is equivalent to

$$f(\alpha_i) := \alpha_i - \sum_{i_2, \ldots, i_k \in [n]} |a_{i_2i_3 \ldots i_k} - \alpha_i| > \bar{r}_i(A). \quad (9)$$

Let $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ be an arrangement in non-decreasing order of $a_{i_2i_3 \ldots i_k}$ for $i_2, \ldots, i_k \in [n]$. If $n$ is odd, then

$$\max_{\alpha_i \in \mathbb{R}} f(\alpha_i) = \sum_{t=1}^{\frac{\ell+1}{2}} b_{i,t} - \sum_{t=\frac{\ell+1}{2}}^{\ell} b_{i,t} = b_{i,1} + \sum_{t=2}^{\frac{\ell+1}{2}} b_{i,t} - \sum_{t=\frac{\ell+1}{2}}^{\ell} b_{i,t} \leq b_{i,1}. \quad (10)$$

And if $n$ is even, then

$$\max_{\alpha_i \in \mathbb{R}} f(\alpha_i) = \sum_{t=1}^{\frac{\ell+3}{2}} b_{i,t} - \sum_{t=\frac{\ell+3}{2}}^{\ell} b_{i,t} = b_{i,1} + \sum_{t=2}^{\frac{\ell+3}{2}} b_{i,t} - \sum_{t=\frac{\ell+3}{2}}^{\ell} b_{i,t} \leq b_{i,1}. \quad (11)$$

(Note here that the proofs of (10) and (11) are showed in Lemma 4.2.) If $b_{i,1} = a_{ii \ldots ii}$, then for any $j \in [n]$ and $j \neq i$, \n
$$\max_{\alpha_i \in \mathbb{R}} f(\alpha_i) \leq b_{i,1} = a_{ii \ldots ii} \leq a_{ii \ldots ijj} = a_{iij \ldots ij} \leq |a_{ii \ldots iij}| \leq \bar{r}_i(A),$$

which contradicts (9). If $b_{i,1} = a_{iij \ldots iik}$ for some $i_2, \ldots, i_k \in [n]$, where at least one of $i_2, \ldots, i_k$ is not $i$ (without loss of generality, assume that $i_2 \neq i$), then

$$\max_{\alpha_i \in \mathbb{R}} f(\alpha_i) \leq b_{i,1} = a_{iij \ldots iik} = a_{iiij \ldots iik} \leq |a_{iiij \ldots iik}| \leq \bar{r}_i(A),$$

which also contradicts (9). Hence, there is no $\alpha_i$ satisfying (9), which implies that Theorem 1.4 (also Corollary 1) is not used to judge the positive definiteness of even order real symmetric tensors.

In order to solve this question, new intervals are constructed in later sections.

2. **A new Geršgorin-type Z-eigenvalue inclusion interval for tensors.** In this section, a new Geršgorin-type Z-eigenvalue inclusion interval for tensors is presented and proved to be tighter than that in Theorem 1.2. Before that, some notations are listed. Let

$$\Delta = \{(i_2, \ldots, i_m) : i_2 \neq \cdots \neq i_m, \text{or only two of } i_2, \ldots, i_m \in [n] \text{ are the same}\},$$

$$\bar{\Delta} = \{(i_2, \ldots, i_m) : (i_2, \ldots, i_m) \notin \Delta, \ \text{or } i_2, \ldots, i_m \notin [n]\},$$

$$N = \{(i_2, \ldots, i_m) : i_2, \ldots, i_m \in [n]\}.$$

Apparently,

$$\Delta \cap \bar{\Delta} = \emptyset, \ \ N = \Delta \cup \bar{\Delta}, \ \ \text{and} \ \ \bar{\Delta} = N \ \text{when } \Delta = \emptyset.$$
For $i \in [n]$, let

$$ r_i^A(A) = \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \cdots i_m}|, \quad r_i^\pi(A) = \sum_{(i_2, \ldots, i_m) \in \Xi} |a_{i_2 \cdots i_m}|, $$

and

$$ r_i(A) = \frac{1}{(m-2)^{m-2}} r_i^A(A) + r_i^\pi(A). \quad (12) $$

Obviously, by $\frac{1}{(m-2)^{m-2}} \leq 1$ for $m \geq 3$, and

$$ R_i(A) = r_i^A(A) + r_i^\pi(A) = \sum_{i_2, \ldots, i_m \in [n]} |a_{i_2 \cdots i_m}|, \quad i \in [n], $$

it can be seen that

$$ r_i(A) \leq R_i(A), \quad i \in [n]. \quad (13) $$

**Lemma 2.1.** [17, pp. 38] Let $a_1, a_2, \ldots, a_k$ be nonnegative real numbers. Then

$$ \frac{a_1 + a_2 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \cdots a_k}. $$

**Lemma 2.2.** Let $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, where $x_i \in \mathbb{R}, i \in [n]$. If $y_1, y_2, \ldots, y_k$ are arbitrary $k$ entries of $x_1, x_2, \ldots, x_n$, then

$$ |y_1||y_2| \cdots |y_k| \leq \frac{1}{k^{\frac{1}{2}}} . $$

**Proof.** By Lemma 2.1, one can obtain

$$ 1 = x_1^2 + x_2^2 + \cdots + x_n^2 \geq y_1^2 + y_2^2 + \cdots + y_k^2 \geq k \sqrt[2k]{y_1^2 y_2^2 \cdots y_k^2} = k(|y_1||y_2| \cdots |y_k|)^{\frac{2}{k}}, $$

which leads to that $|y_1||y_2| \cdots |y_k| \leq \frac{1}{k^{\frac{1}{2}}}$. \hfill \Box

**Theorem 2.3.** Let $A = (a_{i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with $m \geq 3$. Then

$$ \sigma(A) \subseteq \Gamma(A) := \bigcup_{i \in [n]} \Gamma_i(A), $$

where

$$ \Gamma_i(A) = \{ z \in \mathbb{C} : |z| \leq r_i(A) \} . $$

Furthermore, $\Gamma(A) \subseteq \mathcal{K}(A)$.

**Proof.** Let $\lambda \in \sigma(A)$ with a corresponding $Z$-eigenvector $x \in \mathbb{R}^n \setminus \{0\}$. Then for any component $x_i$ of $x$, by $x^T x = 1$, we have $0 \leq |x_i|^{m-1} \leq |x_i| \leq 1, i \in [n]$. Let $|x_s| = \max_{i \in [n]} |x_i|$. Then $|x_s| > 0$. By the sth equation of (3), we have

$$ \lambda x_s = \sum_{i_2, \ldots, i_m \in [n]} a_{s i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} $$

$$ = \sum_{(i_2, \ldots, i_m) \in \Delta} a_{s i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2, \ldots, i_m) \in \Xi} a_{s i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}. $$
Taking modulus in the above equation and using the triangle inequality and Lemma 2.2 give
\[ |\lambda||x_s| \leq \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}||x_{i_2}| \cdots |x_{i_m}| \]
\[ \leq \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}||y_1| \cdots |y_{m-2}||x_s| + \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}||x_s|^{m-1} \]
\[ \leq \frac{1}{(m-2)^{\frac{n-2}{2}}} \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}|s \leq \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{s_1i_2 \cdots i_m}|s \]
\[ = \frac{1}{(m-2)^{\frac{n-2}{2}}} r_s^\Delta(\mathbf{A})|x_s| + r_s^\Gamma(\mathbf{A})|x_s| \]
\[ = r_s(\mathbf{A})|x_s|, \quad (14) \]
where \(|y_1|, \ldots, |y_{m-2}|\) are taken by the following methods:

**Case I.** If \(i_2 \neq \cdots \neq i_m\), then we can enlarge any one of \(|i_2|, \ldots, |i_m|\) to \(|x_s|\) and keep the others (can be taken as \(|y_1|, \ldots, |y_{m-2}|\) unchanged;

**Case II.** If only two of \(i_2, \ldots, i_m\) are the same, then we can enlarge one of the two same elements to \(|x_s|\) and keep the others (can be taken as \(|y_1|, \ldots, |y_{m-2}|\) unchanged.

Furthermore, (14) and \(|x_s| > 0\) yields \(|\lambda| \leq r_s(\mathbf{A})\), which implies that \(\lambda \in \Gamma_s(\mathbf{A}) \subseteq \Gamma(\mathbf{A})\), and consequently, \(\sigma(\mathbf{A}) \subseteq \Gamma(\mathbf{A})\).

Next, we will prove that \(\Gamma(\mathbf{A}) \subseteq \mathcal{K}(\mathbf{A})\). Let \(z \in \Gamma(\mathbf{A})\). Then there is some \(i \in [n]\) such that \(z \in \Gamma_i(\mathbf{A})\), i.e., \(|z| \leq r_i(\mathbf{A})\). By (13), we have \(|z| \leq R_i(\mathbf{A})\), which implies that \(z \in \mathcal{K}_i(\mathbf{A}) \subseteq \mathcal{K}(\mathbf{A})\). Hence, \(\Gamma(\mathbf{A}) \subseteq \mathcal{K}(\mathbf{A})\). \(\Box\)

### 3. Several Z-eigenvalue inclusion intervals for even-order real tensors

In this section, we first establish a new Z-eigenvalue inclusion interval for even-order tensors, which will be used to obtain a sufficient condition for the positive definiteness of even-order real symmetric tensors in Section 5. And then, we show the specific forms of this interval when the Z-identity tensor \(\mathcal{E}\) is taken as \(\mathcal{E}_1\) and \(\mathcal{E}_2\), respectively. Before that, some notations are listed. For any \(i \in [n]\), let
\[ r_i^{\Delta \cap \Lambda_i}(\mathbf{A}, \alpha_i) = \frac{1}{(m-2)^{\frac{n-2}{2}}} \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_i} |a_{i_2i_3 \cdots i_m} - \alpha_i e_{i_2i_3 \cdots i_m}|, \]
\[ r_i^{\Delta \cap \Lambda_i}(\mathbf{A}, \alpha_i) = \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_i} |a_{i_2i_3 \cdots i_m} - \alpha_i e_{i_2i_3 \cdots i_m}|, \]
\[ r_i^{\Delta \cap \Lambda_i}(\mathbf{A}) = \frac{1}{(m-2)^{\frac{n-2}{2}}} \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_i} |a_{i_2i_3 \cdots i_m}|, \]
\[ r_i^{\Delta \cap \Lambda_i}(\mathbf{A}) = \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_i} |a_{i_2i_3 \cdots i_m}|, \]
and
\[ r_i(\mathbf{A}, \alpha_i) = r_i^{\Delta \cap \Lambda_i}(\mathbf{A}, \alpha_i) + r_i^{\Delta \cap \Lambda_i}(\mathbf{A}, \alpha_i) + r_i^{\Delta \cap \Lambda_i}(\mathbf{A}) + r_i^{\Delta \cap \Lambda_i}(\mathbf{A}). \quad (15) \]

Then by \(\frac{1}{(m-2)^{\frac{n-2}{2}}} \leq 1\) for \(m \geq 3\), it can be seen that
\[ r_i(\mathbf{A}, \alpha_i) \leq R_i(\mathbf{A}, \alpha_i), \quad i \in [n]. \quad (16) \]
Theorem 3.1. Let $A = (a_{si_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ with $m$ being even, and $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$. Then

$$\sigma(A) \subseteq \mathcal{Y}(A, \alpha) := \bigcup_{i \in [n]} \mathcal{Y}_i(A, \alpha),$$

(17)

where

$$\mathcal{Y}_i(A, \alpha) := \{ z \in \mathbb{R} : |z - \alpha_i| \leq r_i(A, \alpha_i) \}.$$ 

Furthermore, $\mathcal{Y}(A, \alpha) \subseteq \mathcal{G}(A, \alpha)$.

Proof. Let $\lambda \in \sigma(A)$ with a corresponding $Z$-eigenvector $x \in \mathbb{R}^n \setminus \{0\}$. Then by (2) and (3), we have

$$Ax^{m-1} = \lambda x = E x^{m-1}$$

and

(18)

$$x^\top x = 1.$$ 

Let $|x_s| = \max_{i \in [n]} |x_i|$. Then $0 < |x_s|^{m-1} \leq |x_s| \leq 1$. From the $s$th equation of (18), we have

$$\sum_{i_2,\ldots,i_m \in [n]} a_{si_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_s = \lambda \sum_{i_2,\ldots,i_m \in [n]} e_{si_2\ldots i_m} x_{i_2} \cdots x_{i_m}.$$ 

Furthermore, for any real number $\alpha_s$, we have

$$(\lambda - \alpha_s)x_s = \sum_{i_2,\ldots,i_m \in [n]} (a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}) x_{i_2} \cdots x_{i_m}$$

$$= \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} (a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}) x_{i_2} \cdots x_{i_m}$$

$$+ \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} (a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}) x_{i_2} \cdots x_{i_m}$$

$$+ \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} a_{si_2\ldots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} a_{si_2\ldots i_m} x_{i_2} \cdots x_{i_m}.$$ 

Taking modulus in the above equation and using the triangle inequality and Lemma 2.2 give

$$|\lambda - \alpha_s||x_s| \leq \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} |a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}|$$

$$+ \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} |a_{si_2\ldots i_m} - \alpha_s e_{si_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}|$$

$$+ \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} |a_{si_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}|$$

$$+ \sum_{(i_2,\ldots,i_m) \in \Delta \cap \Lambda_s} |a_{si_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}|.$$
\[ \begin{align*}
& \leq \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m} - \alpha_s e_{si_2 \ldots i_m}| ||y_1| \cdots |y_{m-2}||x_s| \\
& + \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m} - \alpha_s e_{si_2 \ldots i_m}| |x_s|^{m-1} \\
& + \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m}| |z_1| \cdots |z_{m-2}| |x_s| + \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m}| |x_s|^{m-1} \\
& \leq \frac{1}{(m-2)\pi^2} \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m} - \alpha_s e_{si_2 \ldots i_m}| ||x_s| \\
& + \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m} - \alpha_s e_{si_2 \ldots i_m}| ||x_s| \\
& + \frac{1}{(m-2)\pi^2} \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m}| ||x_s| + \sum_{(i_2, \ldots, i_m) \in \Delta \cap \Lambda_s} |a_{si_2 \ldots i_m}| ||x_s| \\
& = r_s(\Delta \cap \Lambda_s)(A, \alpha_s)|x_s| + r_s(\Delta \cap \Lambda_s)(A, \alpha_s)|x_s| + r_s(\Delta \cap \Lambda_s)(A, \alpha_s)|x_s| + r_s(\Delta \cap \Lambda_s)(A, \alpha_s)|x_s| \\
& = r_s(A, \alpha_s)|x_s|,
\end{align*} \]

where \(|y_1|, \ldots, |y_{m-2}|\) and \(|z_1|, \ldots, |z_{m-2}|\) are taken by the methods in Theorem 2.3, which implies that
\[ |\lambda - \alpha_s| \leq r_s(A, \alpha_s), \quad \text{i.e.,} \quad \lambda \in \mathfrak{T}_s(A, \alpha) \subseteq \mathfrak{T}(A, \alpha), \]
and consequently, \(\sigma(A) \subseteq \mathfrak{T}(A, \alpha)\).

Moreover, by (16), it is easy to see that \(\mathfrak{T}(A, \alpha) \subseteq \mathcal{G}(A, \alpha)\). \(\square\)

The form of \(r_s(A, \alpha_i)\) for \(i \in [n]\) in Theorem 3.1 is not only closely related to the Z-identify tensor \(E\), but also closely related to the order \(m\) and dimension \(n\) of \(A\). For \(m = 4\), the specific form of Theorem 3.1 is listed as follows:

**Corollary 2.** Let \(A = (a_{ijkl}) \in \mathbb{R}^{[4,n]}\) and \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n\). Then (17) holds, where \(r_s(A, \alpha_i)\) is taken by the following methods:
(i) if the Z-identify tensor \(E\) is taken as \(E_1\), then
\[ r_s(A, \alpha_i) = \frac{1}{2} \sum_{j \neq i} |a_{iijj} - \alpha_i| + |a_{iiij} - \alpha_i| + \hat{r}_i(A), \]
where
\[ \hat{r}_i(A) = \frac{1}{2} \left( R_i(A) + \sum_{j \neq i} |a_{ijjj}| - \sum_{j \in [n]} |a_{ijjj}| \right). \] (19)
(ii) if the Z-identify tensor \(E\) is taken as \(E_2\), then
\[ r_s(A, \alpha_i) = \frac{1}{2} \sum_{j \neq i} \left( |a_{iijj} - \frac{1}{3} \alpha_i| + |a_{iiij} - \frac{1}{3} \alpha_i| + |a_{ijjj} - \frac{1}{3} \alpha_i| \right) + |a_{iiij} - \alpha_i| + \hat{r}_i(A), \]
where
\[ \hat{r}_i(A) = \frac{1}{2} \left( R_i(A) + \sum_{j \neq i} |a_{ijjj}| - \sum_{j \neq i} (|a_{iiij}| + |a_{ijij}| + |a_{ijji}|) - |a_{iiij}| \right). \] (20)
Moreover, for (i), it follows that \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha) \) and that if \( \sum_{j \neq i} |a_{ijjj} - \alpha_i| \neq 0 \) or \( R_i(\mathcal{A}) > \sum_{j \neq i} |a_{ijjj}| + \sum_{j \in [n]} |a_{ijjj}| \), then \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha) \).

**Proof.** Here, we only prove that \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha) \) when \( \mathcal{E} \) is taken as \( \mathcal{E}_1 \). By

\[
\hat{r}_i(\mathcal{A}) = \sum_{j \neq i} |a_{ijjj}| + \frac{1}{2} \left( R_i(\mathcal{A}) - \sum_{j \neq i} |a_{ijjj}| - \sum_{j \in [n]} |a_{ijjj}| \right) \\
\leq \sum_{j \neq i} |a_{ijjj}| + \left( R_i(\mathcal{A}) - \sum_{j \neq i} |a_{ijjj}| - \sum_{j \in [n]} |a_{ijjj}| \right) = \bar{r}_i(\mathcal{A}),
\]

(21)

it is not difficult to see that \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha) \), and that if \( \sum_{j \neq i} |a_{ijjj} - \alpha_i| \neq 0 \) or \( R_i(\mathcal{A}) > \sum_{j \neq i} |a_{ijjj}| + \sum_{j \in [n]} |a_{ijjj}| \), then \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha) \).

Taking \( \mathcal{E} \) as \( \mathcal{E}_1 \), the specific form of Theorem 3.1 for \( m \geq 6 \) is listed as follows:

**Corollary 3.** Let \( \mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m, n]} \) with \( m = 2k, \) \( k \geq 6, \) \( m \) be even, and \( \alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n \). Then

\[
\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}, \alpha) : \\
= \bigcup_{i \in [n]} \left\{ z \in \mathbb{R} : |z - \alpha_i| \leq \sum_{i_2, \ldots, i_k \in [n]} |a_{i_1 i_2 \ldots i_k} - \alpha_i| + \gamma_i(\mathcal{A}) \right\},
\]

(22)

where

\[
\gamma_i(\mathcal{A}) = \begin{cases} \bar{r}_i(\mathcal{A}), & 2 \leq n < m - 2; \\ \bar{r}_i(\mathcal{A}) - \left( 1 - \frac{1}{(m - 2)^{n-k+2}} \right) \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_1 i_2 \ldots i_m}|, & n \geq m - 2; \end{cases}
\]

(23)

(24)

and \( \bar{r}_i(\mathcal{A}) \) is as in (7).

Furthermore, if \( 2 \leq n < m - 2, \) or if \( n \geq m - 2 \) and \( \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_1 i_2 \ldots i_m}| = 0, \)
then \( \Upsilon(\mathcal{A}, \alpha) = \mathcal{G}(\mathcal{A}, \alpha); \) if \( n \geq m - 2 \) and \( \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_1 i_2 \ldots i_m}| > 0, \) then \( \Upsilon(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha). \)

**Proof.** Let \( \lambda \in \sigma(\mathcal{A}) \). By Theorem 3.1, there is \( i \in [n] \) such that \( |z - \alpha_i| \leq r_i(\mathcal{A}, \alpha_i) \). Let the Z-identify tensor \( \mathcal{E} \) in \( r_i(\mathcal{A}, \alpha_i) \) be \( \mathcal{E}_1 \). Next, two cases for \( r_i(\mathcal{A}, \alpha_i) \) are considered.

(i) If \( 2 \leq n < m - 2 \), then \( \Delta = \emptyset \) and \( \nabla = N \), and consequently, \( \Delta \cap \Lambda = \emptyset, \) \( \Lambda \cap \Lambda = \Lambda, \) and \( \nabla \cap \nabla = \nabla. \) Hence,

\[
r_{\Delta \cap \Lambda_i}(\mathcal{A}, \alpha_i) = r_{\Delta \cap \nabla_i}(\mathcal{A}) = 0,
\]

and consequently, \( r_i(\mathcal{A}, \alpha_i) = R_i(\mathcal{A}, \alpha_i) \), which implies \( \Upsilon(\mathcal{A}, \alpha) = \mathcal{G}(\mathcal{A}, \alpha) \). Apparently, Corollary 3 reduces to Corollary 1.
(ii) If \( n \geq m - 2 \), then \( \Delta \neq \emptyset \), but \( \Delta \cap \Lambda_i = \emptyset \), and consequently, \( \bar{\Delta} \cap \Lambda_i = \Lambda_i \) and \( \Delta \cap \bar{\Lambda}_i = \Delta \), which implies that (25) holds, \( r_i^{\Delta \cap \Lambda_i}(\mathcal{A}, \alpha_i) = 0 \) and
\[
r_i^{\Delta \cap \Lambda_i}(\mathcal{A}) = \left(\frac{m}{m-2}\right) \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \cdots i_m}|.
\]
By \( \bar{\Lambda}_i = N \cap \bar{\Lambda}_i = (\Delta \cup \bar{\Delta}) \cap \bar{\Lambda}_i = (\Delta \cap \bar{\Lambda}_i) \cup (\bar{\Delta} \cap \bar{\Lambda}_i) = \Delta \cup (\bar{\Delta} \cap \bar{\Lambda}_i) \) and \( N = \Lambda_i \cup \bar{\Lambda}_i \) for \( i \in [n] \), it follows that
\[
r_i^{\Delta \cup \bar{\Delta} \cap \bar{\Lambda}_i}(\mathcal{A}) = \sum_{(i_2, \ldots, i_m) \in \bar{\Lambda}_i} |a_{i_2 \cdots i_m}| - \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \cdots i_m}|
= \sum_{(i_2, \ldots, i_m) \in N} |a_{i_2 \cdots i_m}| - \sum_{(i_2, \ldots, i_m) \in \Lambda_i} |a_{i_2 \cdots i_m}|
= \sum_{i_2, \ldots, i_m \in [n]} |a_{i_2 \cdots i_m}| - \sum_{i_2, \ldots, i_m \in [n]} |a_{i_2 \cdots i_m}|
= \bar{r}_i(\mathcal{A}) - \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \cdots i_m}|.
\]
By (15), the conclusion (22) follows. Furthermore, if \( \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \cdots i_m}| > 0 \), then by \( 1 - \frac{1}{(m-2)} < 1 \) for \( m \geq 6 \), we have \( \gamma_i(\mathcal{A}) < \bar{r}_i(\mathcal{A}) \), which implies that \( r_i(\mathcal{A}, \alpha_i) < \bar{r}_i(\mathcal{A}, \alpha_i) \), and hence \( \Upsilon(\mathcal{A}, \alpha) \subset \mathcal{G}(\mathcal{A}, \alpha) \).

\[\square\]

4. Optimal \( Z \)-eigenvalue inclusion intervals for even order real tensors.

In this section, by choosing appropriate parameters \( \alpha_i \) in Theorems 1.3 and 3.1, several “optimal” \( Z \)-eigenvalue inclusion intervals are presented. Before that, two lemmas are given.

**Lemma 4.1.** Let
\[
f(x) = x + \frac{1}{a} \sum_{i \in [n]} |x - b_i| + c
\]
be a real valued function about \( x \), where \( a \) is a positive integer, \( b_i \in \mathbb{R} \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \), with \( n \geq a + 1 \), and \( c \in \mathbb{R} \).

(I) Assume that \( a \) is odd.
(I.i) If \( n \) is odd, then
\[
\min_{x \in \mathbb{R}} f(x) = \frac{1}{a} \left( \sum_{i=\frac{n-a}{a+1}}^{n} b_i - \sum_{i=1}^{\frac{a-1}{a+1}} b_i \right) + c \tag{26}
\]
and this takes place for every \( x \in \left[ b_{\frac{n-a}{a+1}}, b_{\frac{n-a}{a+1}+1} \right] \) if \( b_{\frac{n-a}{a}} \neq b_{\frac{n-a}{a+1}} \), and only for \( x = b_{\frac{n-a}{a}} \) if \( b_{\frac{n-a}{a}} = b_{\frac{n-a}{a+1}} \).

(I.ii) If \( n \) is even, then
\[
\min_{x \in \mathbb{R}} f(x) = \frac{1}{a} \left( \sum_{i=\frac{n-a}{a+3}}^{n} b_i - \sum_{i=1}^{\frac{a-1}{a+3}} b_i \right) + c \tag{27}
\]
and this minimum is reached when \( x = b_{\frac{n-a}{a+1}} \).

(II) Assume that \( a \) is even. If \( n \) is odd, then (27) holds. And if \( n \) is even, then (26) holds.
Proof. Writing \( f(x) \) as a piecewise function, we have
\[
\begin{cases}
(1 - \frac{n}{a})x + \frac{1}{a} \sum_{i=n}^{j} b_i + c, & x \in (-\infty, b_1]; \\
\frac{a+2j-n}{a}x + \frac{1}{a} \left( \sum_{i=n}^{j} b_i - \frac{j}{a} \right) + c, & x \in [b_j, b_{j+1}], \quad j \in [n-1]; \\
(1 + \frac{n}{a})x - \frac{1}{a} \sum_{i=n}^{j} b_i + c, & x \in [b_n, +\infty).
\end{cases}
\]

Now, let us consider the minimum of \( f(x) \) for \( x \in \mathbb{R} \). The proof is divided into two cases: whether \( a \) is odd or even. Here, only the proof for that \( a \) is odd is given (The case that \( a \) is even can be proved similarly).

(i) Assume that both \( a \) and \( n \) are odd.

When \( x \in (-\infty, b_1], f(x) \) is a decreasing function by \( 1 - \frac{n}{a} \leq 1 - \frac{a+1}{a} = -\frac{1}{a} < 0 \). When \( x \in [b_j, b_{j+1}], \) where \( 1 \leq j \leq \frac{n-a}{a} - 1, f(x) \) is a decreasing function by \( a+2j-n \leq -2 < 0 \) (Note here that the interval \( [b_j, b_{j+1}] \) does not exist if \( \frac{n-a}{2} \leq 1 \)).

When \( x \in [b_{\frac{n-a}{2}}, b_{\frac{n-a}{2}+1}], f(x) \equiv \frac{1}{a} \left( \sum_{i=\frac{n-a}{2}+1}^{n} b_i - \sum_{i=1}^{\frac{n-a}{2}} b_i \right) + c \) is a constant function by \( a + 2j - n = 0 \). When \( x \in [b_{\frac{n-a}{2}}, b_{\frac{n-a}{2}+1}], \) where \( \frac{n-a}{a} + 1 \leq j \leq n - 1, f(x) \) is an increasing function by \( a + 2j - n \geq 2 > 0 \). When \( x \in [b_n, +\infty), f(x) \) is also an increasing function by \( 1 + \frac{n}{a} > 0 \). By the continuity of \( f(x) \), it is not difficult to judge that

\[
\min_{x \in \mathbb{R}} f(x) = f(x_0) = \frac{1}{a} \left( \sum_{i=\frac{n-a}{2}+1}^{n} b_i - \sum_{i=1}^{\frac{n-a}{2}} b_i \right) + c, \quad x_0 \in [b_{\frac{n-a}{2}}, b_{\frac{n-a}{2}+1}].
\]

(ii) Assume that \( a \) is odd and \( n \) is even.

When \( x \in (-\infty, b_1], \) by \( 1 - \frac{n}{a} < 0, f(x) \) is a decreasing function. When \( x \in [b_j, b_{j+1}], \) where \( 1 \leq j \leq \frac{n-a-1}{a}, by a + 2j - n < 0, f(x) \) is also a decreasing function (Note here that the interval \( [b_j, b_{j+1}] \) does not exist if \( \frac{n-a-1}{2} \leq 0 \)). When \( x \in [b_{\frac{n-a+1}{2}}, b_{\frac{n-a+1}{2}}], \) where \( \frac{n-a+1}{a} \leq j \leq n - 1, by a + 2j - n > 0, f(x) \) is an increasing function. When \( x \in [b_n, +\infty), by 1 + \frac{n}{a} > 0, f(x) \) is also an increasing function. By the continuity of \( f(x) \), it is not difficult to judge that

\[
\min_{x \in \mathbb{R}} f(x) = f(b_{\frac{n-a+1}{2}}) = \frac{1}{a} \left( \sum_{i=\frac{n-a+1}{2}+1}^{n} b_i - \sum_{i=1}^{\frac{n-a-1}{a}} b_i \right) + c
\]

and this minimum is reached when \( x = b_{\frac{n-a+1}{2}} \).

Lemma 4.2. Let
\[
g(x) = x - \frac{1}{a} \sum_{i=n}^{j} |x - b_i| - c
\]
be a real valued function about \( x \), where \( a \) is a positive integer, \( b_i \in \mathbb{R} \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \) with \( n \geq a \), and \( c \in \mathbb{R} \).

(I) Assume that \( a \) is odd.

(I.i) If \( n \) is odd, then
\[
\max_{x \in \mathbb{R}} g(x) = \frac{1}{a} \left( \sum_{i=1}^{\frac{n-a}{2}+1} b_i - \sum_{i=\frac{n-a}{2}+1}^{n} b_i \right) - c,
\]
and this takes place for every \( x \in [b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}+1}] \) if \( b_{\frac{n+1}{2}} \neq b_{\frac{n+3}{2}+1} \), and only for \( x = b_{\frac{n+1}{2}} \) if \( b_{\frac{n+1}{2}} = b_{\frac{n+3}{2}+1} \). Note that let \( [b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}+1}] \) be \( [b_{\frac{n+1}{2}}, +\infty) \) if \( b_{\frac{n+3}{2}+1} \) does not exist.

(I.ii) If \( n \) is even, then
\[
\max_{x \in \mathbb{R}} g(x) = \frac{1}{a} \left( \sum_{i=1}^{\frac{n+1}{2}} b_i - \sum_{i=\frac{n+3}{2}+1}^{n} b_i \right) - c,
\]
and this maximum is reached when \( x = b_{\frac{n+3}{2}+1} \).

(II) Assume that \( a \) is even. If \( n \) is odd, then (29) holds. And if \( n \) is even, then (28) holds.

Proof. Writing \( g(x) \) as a piecewise function, we have
\[
g(x) = \begin{cases} 
(1 + \frac{a}{2})x - \frac{1}{a} \sum_{i=1}^{n} b_i - c, & x \in (-\infty, b_1]; \\
\frac{a-2j+n}{a}x - \frac{1}{a} \sum_{i=j+1}^{n} b_i - j, & x \in [b_j, b_{j+1}], \ j \in [n-1]; \\
(1 - \frac{n}{a})x + \frac{1}{a} \sum_{i=1}^{n} b_i - c, & x \in [b_n, +\infty). 
\end{cases}
\]
The proof is omitted as it is similar to the proof of Lemma 4.1.

4.1. The optimal Z-eigenvalue inclusion interval for Corollary 1. In this subsection, the optimal interval for Corollary 1 is considered.

**Theorem 4.3.** Let \( A = (a_{i_1, \ldots, i_m}) \in \mathbb{R}^{[m,n]} \) with \( m = 2k \) being even. Then
\[
\sigma(A) \subseteq G(A) := \bigcup_{i \in [n]} [g_i, h_i],
\]
where \( g_i \) and \( h_i \) are taken by the following methods:

(i) if \( n \) is odd, then
\[
g_i = \sum_{j=1}^{\frac{i+1}{2}} b_{i,j} - \sum_{j=\frac{i+1}{2}+1}^{\ell} b_{i,j} - \tilde{r}_i(A), \quad h_i = \sum_{j=\frac{i+1}{2}+1}^{\ell} b_{i,j} - \sum_{j=1}^{\frac{i-1}{2}} b_{i,j} + \tilde{r}_i(A);
\]

(ii) if \( n \) is even, then
\[
g_i = \sum_{j=1}^{\frac{i}{2}} b_{i,j} - \sum_{j=\frac{i}{2}+1}^{\ell} b_{i,j} - \tilde{r}_i(A), \quad h_i = \sum_{j=\frac{i}{2}+1}^{\ell} b_{i,j} - \sum_{j=1}^{\frac{i-2}{2}} b_{i,j} + \tilde{r}_i(A).
\]

Here, for each \( i \in [n] \), \( b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell} \) is an arrangement in non-decreasing order of \( a_{i_1 i_2 \cdots i_k} \) for \( i_2, \ldots, i_k \in [n] \), \( \ell = n^{k-1} \), and \( \tilde{r}_i(A) \) is defined in (7).

Proof. Let \( \lambda \in \sigma(A) \). By Corollary 1, there is some \( i \in [n] \) such that
\[
|\lambda - \alpha_i| \leq \sum_{i_2, \ldots, i_k \in [n]} |a_{i_1i_2\cdots i_k} - \alpha_i| + \tilde{r}_i(A)
\]
for any real number \( \alpha_i \), i.e.,
\[
\lambda \in [g(\alpha_i), f(\alpha_i)],
\]
(30)
and this minimum is reached for every \( \alpha \).

Let

\[
g(\alpha_i) = \alpha_i - \sum_{i_2, \ldots, i_k \in [n]} |a_{i_2i_1 \ldots i_k} - \alpha_i| - \bar{r}_i(\mathcal{A}) = \alpha_i - \sum_{j \in [\ell]} |h_{i,j} - \alpha_i| - \bar{r}_i(\mathcal{A}),
\]

\[
f(\alpha_i) = \alpha_i + \sum_{i_2, \ldots, i_k \in [n]} |a_{i_2i_1 \ldots i_k} - \alpha_i| + \bar{r}_i(\mathcal{A}) = \alpha_i + \sum_{j \in [\ell]} |h_{i,j} - \alpha_i| + \bar{r}_i(\mathcal{A}).
\]

Next, we consider a question: How to choose parameter \( \alpha \) to minimize the inclusion interval \([g(\alpha_i), f(\alpha_i)]\)?

(i) If \( n \) is odd, then \( \ell = n^{k-1} \) is odd. By Lemma 4.1 (taking \( a = 1 \)), we have

\[
\min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} + \bar{r}_i(\mathcal{A}),
\]

and this minimum is reached for every \( \alpha_i \in [b_{i,\ell-1}, b_{i,\ell+1}] \). By Lemma 4.2 (taking \( a = 1 \)), we have

\[
\max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} + \bar{r}_i(\mathcal{A}),
\]

and this maximum is reached for every \( \alpha_i \in [b_{i,\ell-1}, b_{i,\ell+1}] \). Taking \( \alpha_i = b_{i,\ell+1} \) in (30), by (31) and (32), we have

\[
\lambda \in \left[ \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\ell+2}^{\ell+2} b_{i,j} - \bar{r}_i(\mathcal{A}), \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell+1} b_{i,j} + \bar{r}_i(\mathcal{A}) \right],
\]

i.e., \( \lambda \in [g_i, h_i] \), and consequently, \( \lambda \in \bigcup_{i \in [n]} [g_i, h_i] \).

(ii) If \( n \) is even, then \( \ell = n^{k-1} \) is even. By Lemma 4.1 (taking \( a = 1 \)), we have

\[
\min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(b_{i,\ell}) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell-1} b_{i,j} + \bar{r}_i(\mathcal{A}) \leq f(b_{i,\ell+1}).
\]

By Lemma 4.2 (taking \( a = 1 \)), we have

\[
\max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(b_{i,\ell+1}) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\ell+2}^{\ell+2} b_{i,j} - \bar{r}_i(\mathcal{A}) \geq g(b_{i,\ell}).
\]

Taking \( \alpha_i = b_{i,\ell} \) and \( \alpha_i = b_{i,\ell+1} \) in (30), respectively, we have

\[
\lambda \in \left[ g(b_{i,\ell}), f(b_{i,\ell}) \right] \quad \text{and} \quad \lambda \in \left[ g(b_{i,\ell+1}), f(b_{i,\ell}) \right].
\]

By (33), (34) and the existence of \( \lambda \), we have \( \lambda \in \left[ g(b_{i,\ell+1}), f(b_{i,\ell}) \right] \), i.e., \( \lambda \in [g_i, h_i] \), and consequently \( \lambda \in \bigcup_{i \in [n]} [g_i, h_i] \).

**Remark 1.** Let \( \mathcal{A} \in \mathbb{R}^{m,n} \) with \( m \) being even, and \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n \) be any vector. By Corollary 1 and Theorem 4.3, it is easy to see that

\[
\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \alpha),
\]

which implies that \( \mathcal{G}(\mathcal{A}) \) is the “optimal” interval of \( \mathcal{G}(\mathcal{A}, \alpha) \).
4.2. The optimal Z-eigenvalue inclusion intervals for Corollaries 2 and 3. In this subsection, the optimal intervals for Corollaries 2 and 3 are considered.

**Theorem 4.4.** Let $A = (a_{ijkl}) \in \mathbb{R}^{[4,n]}$. Then

$$\sigma(A) \subseteq \Upsilon(A) := \bigcup_{i \in [n]} [l_i, u_i],$$

where $l_i$ and $u_i$ are taken by the following methods:

(i) Assume that the Z-identify tensor $E$ is taken as $E_1$. If $n$ is odd, then

$$l_i = \frac{1}{2} \left( \sum_{k=1}^{n+3} b_{i,k} - \sum_{k=\frac{n+5}{2}}^{n+1} b_{i,k} \right) - \hat{r}_i(A), \quad u_i = \frac{1}{2} \left( \sum_{k=\frac{n+1}{2}}^{n+1} b_{i,k} - \sum_{k=1}^{n-1} b_{i,k} \right) + \hat{r}_i(A).$$

If $n$ is even, then

$$l_i = \frac{1}{2} \left( \sum_{k=1}^{\frac{n+3}{2}} b_{i,k} - \sum_{k=\frac{n+5}{2}}^{n+1} b_{i,k} \right) - \hat{r}_i(A), \quad u_i = \frac{1}{2} \left( \sum_{k=\frac{n+1}{2}}^{n+1} b_{i,k} - \sum_{k=1}^{n-1} b_{i,k} \right) + \hat{r}_i(A).$$

Here, for each $i \in [n]$, $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,n+1}$ is an arrangement in non-decreasing order of $a_{iiii}$, $a_{iiij}$, and $a_{ijjj}$ for $j \in [n]$ and $j \neq i$, and $\hat{r}_i(A)$ is defined in (19). Note here that let $\sum_{k=\frac{n+1}{2}}^{n+1} b_{i,k} = 0$ if $n = 2$.

(ii) Assume that the Z-identify tensor $E$ is taken as $E_2$. If $n$ is odd, then

$$l_i = \frac{1}{6} \left( \sum_{k=1}^{\frac{3n+3}{2}} c_{i,k} - \sum_{k=\frac{3n+1}{2}}^{3n+3} c_{i,k} \right) - \hat{r}_i(A), \quad u_i = \frac{1}{6} \left( \sum_{k=\frac{3n-1}{2}}^{3n+3} c_{i,k} - \sum_{k=1}^{3n-3} c_{i,k} \right) + \hat{r}_i(A).$$

If $n$ is even, then

$$l_i = \frac{1}{6} \left( \sum_{k=1}^{\frac{3n+4}{2}} c_{i,k} - \sum_{k=\frac{3n+6}{2}}^{3n+3} c_{i,k} \right) - \hat{r}_i(A), \quad u_i = \frac{1}{6} \left( \sum_{k=\frac{3n+2}{2}}^{3n+3} c_{i,k} - \sum_{k=1}^{3n-3} c_{i,k} \right) + \hat{r}_i(A).$$

Here, for each $i \in [n]$, $c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,3n+3}$ is an arrangement in non-decreasing order of $a_{iiii}$ with its number $6$, $3aijj$, $3aijj$ and $3aijj$, for $j \in [n]$ and $j \neq i$, and $\hat{r}_i(A)$ is defined in (20).

**Proof.** Let $\lambda \in \sigma(A)$. By Corollary 2, there is some $i \in [n]$ such that

$$|\lambda - \alpha_i| \leq r_i(A, \alpha_i), \quad \text{i.e.,} \quad \lambda \in [\alpha_i - r_i(A, \alpha_i), \alpha_i + r_i(A, \alpha_i)] \quad (35)$$

for any real number $\alpha_i$. Because the form of $r_i(A, \alpha_i)$ is closely related to the Z-identify tensor $E$, the proof is divided into two cases: $E$ is taken as $E_1$ or $E_2$.

I. Assume that the Z-identify tensor $E$ is taken as $E_1$. Let

$$f(\alpha_i) = \alpha_i + r_i(A, \alpha_i) = \alpha_i + \frac{1}{2} \sum_{j \neq i} |a_{ijjj} - \alpha_i| + |a_{iiii} - \alpha_i| + \hat{r}_i(A)$$

$$= \alpha_i + \frac{1}{2} \left( \sum_{j \neq i} |a_{ijjj} - \alpha_i| + |a_{iiii} - \alpha_i| + |a_{ii} - \alpha_i| \right) + \hat{r}_i(A)$$
and
\[ g(\alpha_i) = \alpha_i - r_i(\mathcal{A}, \alpha_i) = \alpha_i - \frac{1}{2} \sum_{j \neq i} |a_{ijij} - \alpha_i| - |a_{iiii} - \alpha_i| - \hat{r}_i(\mathcal{A}) \]
= \alpha_i - \frac{1}{2} \left( \sum_{j \neq i} |a_{ijij} - \alpha_i| + |a_{iiii} - \alpha_i| + |a_{iiii} - \alpha_i| \right) - \hat{r}_i(\mathcal{A}),

where \( \hat{r}_i(\mathcal{A}) \) is defined in (19). For each \( i \in [n] \), let \( b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,n+1} \) be an arrangement in non-decreasing order of \( a_{iiii}, a_{iiii} \) and \( a_{ijij} \) for \( j \in [n] \) and \( j \neq i \). Then
\[ f(\alpha_i) = \alpha_i + \frac{1}{2} \sum_{k \in [n+1]} |b_{i,k} - \alpha_i| + \hat{r}_i(\mathcal{A}) \]
\[ = \alpha_i - \frac{1}{2} \sum_{k \in [n+1]} |b_{i,k} - \alpha_i| - \hat{r}_i(\mathcal{A}). \]

Next, two cases are considered depending on whether \( n \) is odd or even.

(i.i) Let \( n \) be odd. Then \( n + 1 \) is even. By Lemma 4.1 (taking \( a = 2 \)), we have
\[ \min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(b_{i,\frac{n+1}{2}}) = \frac{1}{2} \left( \sum_{k = \frac{n+1}{2}}^{n+1} b_{i,k} - \sum_{k = 1}^{n+1} b_{i,k} \right) + \hat{r}_i(\mathcal{A}) \leq f(b_{i,\frac{n+1}{2}}). \] (38)

By Lemma 4.2 (taking \( a = 2 \)), we have
\[ \max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(b_{i,\frac{n+1}{2}}) = \frac{1}{2} \left( \sum_{k = \frac{n+1}{2}}^{n+1} b_{i,k} - \sum_{k = 1}^{n+1} b_{i,k} \right) - \hat{r}_i(\mathcal{A}) \geq g(b_{i,\frac{n+1}{2}}). \] (39)

Taking \( \alpha_i = b_{i,\frac{n+1}{2}} \) and \( \alpha_i = b_{i,\frac{n+1}{2}} \) in (35), respectively, by (36) and (37), we have
\[ \lambda \in \left[ g(b_{i,\frac{n+1}{2}}), f(b_{i,\frac{n+1}{2}}) \right] \text{ and } \lambda \in \left[ g(b_{i,\frac{n+1}{2}}), f(b_{i,\frac{n+1}{2}}) \right]. \] (40)

Furthermore, by (38), (39), (40) and the existence of \( \lambda \), it is not difficult to judge that
\[ \lambda \in \left[ g(b_{i,\frac{n+1}{2}}), f(b_{i,\frac{n+1}{2}}) \right], \text{ i.e., } \lambda \in [l_i, u_i], \]
and consequently, \( \lambda \in \bigcup_{i \in [n]} [l_i, u_i] \).

(i.ii) Let \( n \) be even. Then \( n + 1 \) is odd. By Lemma 4.1 (taking \( a = 2 \)), we have
\[ \min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(b_{i,\frac{n}{2}}) = \frac{1}{2} \left( \sum_{k = \frac{n}{2} + 1}^{n+1} b_{i,k} - \sum_{k = 1}^{\frac{n}{2} + 1} b_{i,k} \right) + \hat{r}_i(\mathcal{A}) \leq f(b_{i,\frac{n}{2}} + 2). \] (41)

By Lemma 4.2 (taking \( a = 2 \)), we have
\[ \max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(b_{i,\frac{n}{2} + 2}) = \frac{1}{2} \left( \sum_{k = \frac{n}{2} + 1}^{\frac{n}{2} + 2} b_{i,k} - \sum_{k = 1}^{\frac{n}{2} + 2} b_{i,k} \right) - \hat{r}_i(\mathcal{A}) \geq g(b_{i,\frac{n}{2}}). \] (42)

Taking \( \alpha_i = b_{i,\frac{n}{2}} \) and \( \alpha_i = b_{i,\frac{n}{2} + 2} \) in (35), respectively, we by (36) and (37) have
\[ \lambda \in \left[ g(b_{i,\frac{n}{2}}), f(b_{i,\frac{n}{2}}) \right] \text{ and } \lambda \in \left[ g(b_{i,\frac{n}{2} + 2}), f(b_{i,\frac{n}{2} + 2}) \right], \]
which implies that

\[ \lambda \in [g(b_i, \frac{3n+1}{2}), f(b_i, \frac{3n+1}{2})], \quad \text{i.e.,} \quad \lambda \in [l_i, u_i], \]

by (41), (42) and the existence of \( \lambda \), and consequently, \( \lambda \in \bigcup_{i \in [n]} [l_i, u_i] \).

II. Assume that the Z-identify tensor \( E \) is taken as \( E_2 \). Let

\[ f(\alpha_i) = \alpha_i + r_i(A, \alpha_i) \]

\[ = \alpha_i + \frac{1}{2} \sum_{j \neq i} \left( |a_{ijj} - \frac{1}{3} \alpha_i| + |a_{ijj} - \frac{1}{3} \alpha_i| + |a_{ijj} - \frac{1}{3} \alpha_i| \right) + |a_{iii} - \alpha_i| + \tilde{r}_i(A) \]

\[ = \alpha_i + \frac{1}{6} \left( \sum_{j \neq i} (|3a_{ijj} - \alpha_i| + |3a_{ijj} - \alpha_i| + |3a_{ijj} - \alpha_i|) + |a_{iii} - \alpha_i| \right) + |a_{iii} - \alpha_i| + |a_{iii} - \alpha_i| + |a_{iii} - \alpha_i| + \tilde{r}_i(A) \]

and

\[ g(\alpha_i) = \alpha_i - r_i(A, \alpha_i) \]

\[ = \alpha_i - \frac{1}{2} \sum_{j \neq i} \left( |a_{ijj} - \frac{1}{3} \alpha_i| + |a_{ijj} - \frac{1}{3} \alpha_i| + |a_{ijj} - \frac{1}{3} \alpha_i| \right) - |a_{iii} - \alpha_i| - \tilde{r}_i(A) \]

\[ = \alpha_i - \frac{1}{6} \left( \sum_{j \neq i} (|3a_{ijj} - \alpha_i| + |3a_{ijj} - \alpha_i| + |3a_{ijj} - \alpha_i|) + |a_{iii} - \alpha_i| \right) - |a_{iii} - \alpha_i| + |a_{iii} - \alpha_i| + |a_{iii} - \alpha_i| - \tilde{r}_i(A), \]

where \( \tilde{r}_i(A) \) is defined in (20), \( i \in [n] \). For each \( i \in [n] \), let \( c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,3n+3} \) be an arrangement in non-decreasing order of \( a_{iii} \) with its number 6, 3\( a_{ijj} \), 3\( a_{ijj} \) and 3\( a_{ijj} \) for \( j \in [n] \) and \( j \neq i \). Then

\[ f(\alpha_i) = \alpha_i + \frac{1}{6} \sum_{k \in [3n+3]} |a_i - c_{i,k}| + \tilde{r}_i(A) \]  \hspace{1cm} (43)

and

\[ g(\alpha_i) = \alpha_i - \frac{1}{6} \sum_{k \in [3n+3]} |a_i - c_{i,k}| - \tilde{r}_i(A). \]  \hspace{1cm} (44)

Next, two cases are considered depending on that whether \( n \) is odd or even.

(II.i) Let \( n \) be odd. Then 3\( n + 3 \) is even. By Lemma 4.1 (taking \( a = 6 \)), we have

\[ \min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(c_i, \frac{3n+1}{2}) = \frac{1}{6} \left( \sum_{k=\frac{3n+1}{2}}^{\frac{3n+3}{2}} c_{i,k} - \sum_{k=\frac{3n+1}{2}}^{\frac{3n+3}{2}} c_{i,k} \right) + \tilde{r}_i(A) \leq f(c_i, \frac{3n+2}{2}). \]  \hspace{1cm} (45)

By Lemma 4.2 (taking \( a = 6 \)), we have

\[ \max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(c_i, \frac{3n+2}{2}) = \frac{1}{6} \left( \sum_{k=1}^{\frac{3n+3}{2}} c_{i,k} - \sum_{k=\frac{3n+1}{2}}^{\frac{3n+3}{2}} c_{i,k} \right) - \tilde{r}_i(A) \geq g(c_i, \frac{3n+1}{2}). \]  \hspace{1cm} (46)

Taking \( \alpha_i = c_i, \frac{3n+1}{2} \) and \( \alpha_i = c_i, \frac{3n+3}{2} \) in (35), respectively, by (43) and (44), we have

\[ \lambda \in \left[ g(c_i, \frac{3n+1}{2}), f(c_i, \frac{3n+1}{2}) \right] \text{ and } \lambda \in \left[ g(c_i, \frac{3n+3}{2}), f(c_i, \frac{3n+3}{2}) \right]. \]  \hspace{1cm} (47)
Furthermore, by (45), (46), (47) and the existence of $\lambda$, it can be seen that
\[ \lambda \in \left[ g(c_i, \frac{2n+9}{2}), f(c_i, \frac{2n-1}{2}) \right], \quad \text{i.e.}, \quad \lambda \in [l_i, u_i], \]
and consequently, $\lambda \in \bigcup_{i \in [n]} [l_i, u_i]$.

(II.ii) Let $n$ be even. Then $3n+3$ is odd. By Lemma 4.1 (taking $a = 6$), we have
\[ \min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(c_i, \frac{2n}{2} - 1) = \frac{1}{6} \left( \sum_{k=1}^{\frac{3n+3}{2}} c_{i,k} - \sum_{k=\frac{2n+2}{2}}^{\frac{3n+3}{2}} c_{i,k} \right) + \tilde{r}_i(A) \leq f(c_i, \frac{2n}{2} + 5). \quad (48) \]
By Lemma 4.2 (taking $a = 6$), we have
\[ \max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(c_i, \frac{2n}{2} + 5) = \frac{1}{6} \left( \sum_{k=1}^{\frac{3n+4}{2}} c_{i,k} - \sum_{k=\frac{2n+4+6}{2}}^{\frac{3n+4}{2}} c_{i,k} \right) - \tilde{r}_i(A) \geq g(c_i, \frac{2n}{2} - 1). \quad (49) \]
Taking $\alpha_i = c_i, \frac{2n}{2} - 1$ and $\alpha_i = c_i, \frac{2n}{2} + 5$ respectively in (35), we by (43) and (44) have
\[ \lambda \in \left[ g(c_i, \frac{2n}{2} - 1), f(c_i, \frac{2n}{2} - 1) \right] \quad \text{and} \quad \lambda \in \left[ g(c_i, \frac{2n}{2} + 5), f(c_i, \frac{2n}{2} + 5) \right], \quad (50) \]
and consequently,
\[ \lambda \in \left[ g(c_i, \frac{2n}{2} + 5), f(c_i, \frac{2n}{2} - 1) \right], \quad \text{i.e.}, \quad \lambda \in [l_i, u_i] \subseteq \bigcup_{i \in [n]} [l_i, u_i], \]
by (48), (49), (50) and the existence of $\lambda$.

Next, the optimal interval for Corollary 3 is considered.

**Theorem 4.5.** Let $A = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$, $m = 2k \geq 6$, and $m$ be even. Then
\[ \sigma(A) \subseteq \mathcal{Y}(A) := \bigcup_{i \in [n]} [l_i, u_i], \]
where $l_i$ and $u_i$ are taken by the following methods:

(i) if $n$ is odd, then
\[ l_i = \sum_{j=1}^{\ell+1} b_{i,j} - \sum_{j=\ell+2}^{\ell+1} b_{i,j} - \gamma_i(A), \quad u_i = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell-1} b_{i,j} + \gamma_i(A); \]

(ii) if $n$ is even, then
\[ l_i = \sum_{j=1}^{\frac{n}{2}+1} b_{i,j} - \sum_{j=\frac{n}{2}+2}^{\ell} b_{i,j} - \gamma_i(A), \quad u_i = \sum_{j=\frac{n}{2}+1}^{\ell} b_{i,j} - \sum_{j=1}^{\frac{n}{2}+1} b_{i,j} + \gamma_i(A). \]

Here, for each $i \in [n]$, $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ is an arrangement in non-decreasing order of $a_{ii_1i_2\ldots i_k}$ for $i_2, \ldots, i_k \in [n]$, where $\ell = n^{k-1}$; $\gamma_i(A)$ is taken as follows: if $2 \leq n < m - 2$, then $\gamma_i(A)$ is as in (23); and if $n \geq m - 2$, then $\gamma_i(A)$ is as in (24).

**Proof.** Let $\lambda \in \sigma(A)$. By Corollary 3, there is an $i \in [n]$ such that
\[ |\lambda - \alpha_i| \leq \sum_{i_2,\ldots,i_k \in [n]} |a_{ii_2i_3\ldots i_ki} - \alpha_i| + \gamma_i(A) \]

for any real number $\alpha_i$, which leads to (30), where
\[
\begin{align*}
g(\alpha_i) &= \alpha_i - \sum_{i_2, \ldots, i_k \in [n]} |a_{i_1i_2i_3} - \alpha_i| - \gamma_i(\mathbf{A}) = \alpha_i - \sum_{j \in [\ell]} |b_{i,j} - \alpha_i| - \gamma_i(\mathbf{A}), \\
f(\alpha_i) &= \alpha_i + \sum_{i_2, \ldots, i_k \in [n]} |a_{i_1i_2i_3} - \alpha_i| + \gamma_i(\mathbf{A}) = \alpha_i + \sum_{j \in [\ell]} |b_{i,j} - \alpha_i| + \gamma_i(\mathbf{A}),
\end{align*}
\]

$b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ is an arrangement in non-decreasing order of $a_{i_1i_2i_3} - \alpha_i$ for $i_2, \ldots, i_k \in [n]$, $\ell = n^{k-1}$, and $\gamma_i(\mathbf{A})$ is defined in Corollary 3.

(i) If $n$ is odd, then $\ell = n^{k-1}$ is even. By Lemma 4.1 (taking $a = 1$), we have
\[
\min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} + \gamma_i(\mathbf{A}),
\]
and this minimum is reached for every $\alpha_i \in [b_{i,\ell-1}, b_{i,\ell+1}]$. By Lemma 4.2 (taking $a = 1$), we have
\[
\max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} - \gamma_i(\mathbf{A}),
\]
and this maximum is reached for every $\alpha_i \in [b_{i,\ell-1}, b_{i,\ell+1}]$. Taking $\alpha_i = b_{i,\ell+1}$, by (30), (51) and (52), we have
\[
\lambda \in \left[ \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} - \gamma_i(\mathbf{A}), \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} + \gamma_i(\mathbf{A}) \right],
\]
i.e., $\lambda \in [l_i, u_i]$, and consequently, $\lambda \in \bigcup_{i \in [n]} [l_i, u_i]$.

(ii) If $n$ is even, then $\ell = n^{k-1}$ is even. By Lemma 4.1 (taking $a = 1$), we have
\[
\min_{\alpha_i \in \mathbb{R}} f(\alpha_i) = f(b_{i,\ell}) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=1}^{\ell} b_{i,j} + \gamma_i(\mathbf{A}) \leq f(b_{i,\ell+1}).
\]
By Lemma 4.2 (taking $a = 1$), we have
\[
\max_{\alpha_i \in \mathbb{R}} g(\alpha_i) = g(b_{i,\ell+1}) = \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\ell+2}^{\ell+1} b_{i,j} - \gamma_i(\mathbf{A}) \geq g(b_{i,\ell}).
\]
Taking $\alpha_i = b_{i,\ell}$ and $\alpha_i = b_{i,\ell+1}$, respectively, we by (30) have
\[
\lambda \in \left[ g(b_{i,\ell}), f(b_{i,\ell}) \right] \quad \text{and} \quad \lambda \in \left[ g(b_{i,\ell+1}), f(b_{i,\ell+1}) \right].
\]
Furthermore, by (53), (54) and the existence of $\lambda$, we have $\lambda \in \left[ g(b_{i,\ell+1}), f(b_{i,\ell}) \right]$, i.e., $\lambda \in [l_i, u_i]$, and consequently, $\lambda \in \bigcup_{i \in [n]} [l_i, u_i]$.

\[\square\]
4.3. Comparisons of three optimal $Z$-eigenvalue inclusion intervals. In this subsection, when the $Z$-identify tensor $E$ is taken as $E_1$, comparisons of the inclusion intervals in Theorems 4.3 and 4.4, and in Theorems 4.3 and 4.5 are considered.

**Theorem 4.6.** Let $A = (a_{i_1 \iota \ldots i_m}) \in \mathbb{R}^{[m,n]}$ with $m$ being even, and the $Z$-identify tensor $E$ be $E_1$.

(i) Let $m = 4$. Then $\Upsilon(A) \subseteq \mathcal{G}(A)$. Furthermore, if $R_i(A) > \sum_{j \neq i} |a_{ij} | + \sum_{j \in [n]} |a_{ij} |$ for each $i \in [n]$, then $\Upsilon(A) \subseteq \mathcal{G}(A)$.

(ii) Let $m \geq 6$. If $2 \leq n < m - 2$, or if $n \geq m - 2$ and $\sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \iota \ldots i_m}| = 0$, then $\Upsilon(A) = \mathcal{G}(A)$; If $n \geq m - 2$ and $\sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2 \iota \ldots i_m}| > 0$, then $\Upsilon(A) \subset \mathcal{G}(A)$.

**Proof.** I. Firstly, the case $m = 4$ is considered. For each $i \in [n]$, let $b_{i,1} \leq b_{i,2} \leq \ldots \leq b_{i,n}$ be an arrangement in non-decreasing order of $a_{ii}$ and $a_{ij}$ for $j \in [n]$ and $j \neq i$, and let $c_{i,1} \leq c_{i,2} \leq \ldots \leq c_{i,n+1}$ be an arrangement in non-decreasing order of $a_{ii}$, $a_{i}$ and $a_{ij}$ for $j \in [n]$ and $j \neq i$.

(i) Let $n$ be odd. By (21), it follows that $\tilde{r}_i(A) \leq \bar{r}_i(A)$ for $i \in [n]$. Next, we prove that $g_i \leq l_i$ and $u_i \leq h_i$. Firstly, the conclusion that $g_i \leq l_i$ is proved. Let

$$
\zeta_i = \sum_{k \in \left[\frac{n+3}{2}\right]} c_{i,k} + \frac{1}{2} \sum_{k \in [n]} b_{i,k} - \frac{1}{2} a_{ii}, \quad i \in [n].
$$

If $a_{iii} \in \{ b_{i,1}, \ldots, b_{i, \frac{n+1}{2}} \}$, which implies that there is $b_{i,s}$ such that $b_{i,s} = a_{iii}$, where $1 \leq s \leq \frac{n+1}{2}$, then $\sum_{k \in [\frac{n+3}{2}]} c_{i,k} = a_{iii} + \sum_{k \in [\frac{n+3}{2}]} b_{i,k}$ and

$$
a_{iii} + \sum_{k = \frac{n+3}{2}}^{n} b_{i,k} \geq b_{i,s} + \sum_{k \in [\frac{n+3}{2}], k \neq s} b_{i,k} = \sum_{k \in [\frac{n+3}{2}]} b_{i,k},
$$

consequently,

$$
\zeta_i = \left( a_{iii} + \sum_{k \in \left[\frac{n+3}{2}\right]} b_{i,k} \right) + \frac{1}{2} \left( \sum_{k \in \left[\frac{n+3}{2}\right]} b_{i,k} + \sum_{k = \frac{n+3}{2}}^{n} b_{i,k} \right) - \frac{1}{2} a_{iii}
$$

$$
= \frac{3}{2} \sum_{k \in \left[\frac{n+3}{2}\right]} b_{i,k} + \frac{1}{2} \left( a_{iii} + \sum_{k = \frac{n+3}{2}}^{n} b_{i,k} \right) \geq 2 \sum_{k \in \left[\frac{n+3}{2}\right]} b_{i,k}.
$$

If $a_{iii} \in \{ b_{i, \frac{n+1}{2}}, \ldots, b_{i,n} \}$, that is to say, there is $b_{i,t}$ such that $b_{i,t} = a_{iii}$, where $\frac{n+3}{2} \leq t \leq n$, then $\sum_{k \in [\frac{n+3}{2}]} c_{i,k} = \sum_{k \in [\frac{n+3}{2}]} b_{i,k} + b_{i, \frac{n+3}{2}}$ and

$$
b_{i, \frac{n+3}{2}} + \frac{1}{2} \sum_{k = \frac{n+3}{2}}^{n} b_{i,k} - \frac{1}{2} a_{iii} = b_{i, \frac{n+3}{2}} + \frac{1}{2} \sum_{k = \frac{n+3}{2}, k \neq t}^{n} b_{i,k} + \frac{1}{2} b_{i,t} - \frac{1}{2} a_{iii}
$$

$$
= \frac{1}{2} b_{i, \frac{n+3}{2}} + \frac{1}{2} b_{i, \frac{n+3}{2}} + \frac{1}{2} \sum_{k \in [\frac{n+3}{2}], k \neq t}^{n} b_{i,k} \geq \frac{1}{2} \sum_{k \in [\frac{n+3}{2}]} b_{i,k}.
$$
consequently,
\[
\zeta_i = \left( \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + b_{i,\frac{n+3}{4}} \right) + \frac{1}{2} \left( \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + \sum_{k=\frac{n+3}{4}}^{n} b_{i,k} \right) - \frac{1}{2} a_{iii} = \frac{3}{2} \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + \left( b_{i,\frac{n+3}{4}} + \frac{1}{2} \sum_{k=\frac{n+3}{4}}^{n} b_{i,k} - \frac{1}{2} a_{iii} \right) \geq 2 \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k}.
\]

And then,
\[
l_i = \sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} - \frac{1}{2} \sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} - \hat{r}_i(A) = \sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} - \frac{1}{2} \left( \sum_{k \in \left[n\right]} b_{i,k} + a_{iii} \right) - \hat{r}_i(A) = \left( \sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} + \frac{1}{2} \sum_{k \in \left[n\right]} b_{i,k} - \frac{1}{2} a_{iii} \right) - \sum_{k \in \left[n\right]} b_{i,k} - \hat{r}_i(A) \geq 2 \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} \geq 2 \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} - \hat{r}_i(A) = g_i.
\]

Now, we prove that \( u_i \leq h_i \). Let
\[
\eta_i = \sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} + \frac{1}{2} \sum_{k \in \left[n\right]} b_{i,k} - \frac{1}{2} a_{iii}, \quad i \in \left[n\right].
\]

If \( a_{iii} \in \left\{ b_{i,1}, \ldots, b_{i,\frac{n-3}{2}} \right\} \), then \( a_{iii} + \sum_{k=\frac{n+5}{4}}^{n} b_{i,k} \geq \sum_{k=\frac{n+1}{2}}^{n} b_{i,k} \) and
\[
\sum_{k \in \left[\frac{n+1}{2}\right]} c_{i,k} = \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + a_{iii} = \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} - b_{i,\frac{n+1}{4}} + a_{iii},
\]

consequently,
\[
\eta_i = \left( \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} - b_{i,\frac{n+1}{4}} + a_{iii} \right) + \frac{1}{2} \left( \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + \sum_{k=\frac{n+5}{4}}^{n} b_{i,k} \right) - \frac{1}{2} a_{iii} = \frac{3}{2} \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + \frac{1}{2} \sum_{k=\frac{n+1}{2}}^{n} b_{i,k} - b_{i,\frac{n+1}{4}} + \frac{1}{2} a_{iii} = \frac{3}{2} \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k} + \frac{1}{2} \left( b_{i,\frac{n+3}{2}} + b_{i,\frac{n+3}{4}} - 2b_{i,\frac{n+1}{4}} + a_{iii} + \sum_{k=\frac{n+5}{4}}^{n} b_{i,k} \right) \geq 2 \sum_{k \in \left[\frac{n+1}{2}\right]} b_{i,k}.
\]
If \( a_{i_1\ldots i_n} \in \{ b_1, \ldots, b_n \} \), then
\[
\sum_{k \in \left[ \frac{n-1}{2} \right]} c_{i,k} = \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k} \quad \text{and} \quad \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k} \geq a_{i_1\ldots i_n} + \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k},
\]
consequently,
\[
\eta_i = \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k} + \frac{1}{2} \sum_{k \in \left[ \frac{n-1}{2} \right]} b_{i,k} + \frac{1}{2} \left( \sum_{k = \frac{n+1}{2}}^{n} b_{i,k} - a_{i_1\ldots i_n} \right) \geq 2 \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k}.
\]
Then, we have
\[
u_i = \frac{1}{2} \sum_{k \in \left[ n+1 \right]} c_{i,k} - \sum_{k \in \left[ \frac{n+1}{2} \right]} c_{i,k} + \hat{r}_i(A)
= \frac{1}{2} \left( a_{i_1\ldots i_n} + \sum_{k \in \left[ n \right]} b_{i,k} \right) - \sum_{k \in \left[ \frac{n+1}{2} \right]} c_{i,k} + \hat{r}_i(A)
= \sum_{k \in \left[ n \right]} b_{i,k} - \eta_i + \hat{r}_i(A) \leq \sum_{k \in \left[ n \right]} b_{i,k} - 2 \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k} + \hat{r}_i(A)
\leq \sum_{k \in \left[ n \right]} b_{i,k} - 2 \sum_{k \in \left[ \frac{n+1}{2} \right]} b_{i,k} + \hat{r}_i(A) = h_i.
\]

(I.ii) When \( n \) is even, the conclusion that \( g_i \leq l_i \) and \( u_i \leq h_i \) can be proved similarly. Here, it is omitted.

Combined with (I.i) and (I.ii), the conclusion \( \Upsilon(A) \subseteq \mathcal{G}(A) \) holds. By (21), it is not difficult to see that if \( R_i(A) > \sum_{j \neq i} |a_{ijjj}| + \sum_{j \in [n]} |a_{iijj}| \), then \( \hat{r}_i(A) < \hat{r}_i(A) \), which implies that \( g_i < l_i \) and \( u_i < h_i \), i.e., \([l_i, u_i] \subseteq [g_i, h_i] \), and consequently \( \Upsilon(A) \subseteq \mathcal{G}(A) \).

II. Next, when \( m \geq 6 \), the conclusion \( \Upsilon(A) \subseteq \mathcal{G}(A) \) is proved. By (23), (24) and Theorems 4.3 and 4.5, it is easy to see that if \( 2 \leq n < m - 2 \), or if \( n \geq m - 2 \) and \( \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2\ldots i_m}| = 0 \), then \( \gamma_i(A) = \hat{r}_i(A) \), and consequently, \( \Upsilon(A) = \mathcal{G}(A) \);
if \( n \geq m - 2 \) and \( \sum_{(i_2, \ldots, i_m) \in \Delta} |a_{i_2\ldots i_m}| \neq 0 \), then \( \gamma_i(A) < \hat{r}_i(A) \), and consequently, \( \Upsilon(A) \subseteq \mathcal{G}(A) \).

5. Applications.

5.1. **Positive definiteness of homogeneous polynomial forms.** Based on the inclusion interval in Theorem 4.3, a sufficient condition such that all \( Z \)-eigenvalues of an even-order tensor are positive is given.

**Corollary 4.** Let \( A = (a_{i_2\ldots i_m}) \in \mathbb{R}^{[m,n]} \) with \( m = 2k \) being even, and \( \lambda \) be any \( Z \)-eigenvalue of \( A \). If \( g_i > 0 \) for each \( i \in [n] \), then \( \lambda > 0 \), where
\[
g_i = \begin{cases} 
\frac{k-1}{2} \sum_{j=1}^{\frac{k+1}{2}} b_{ij} - \sum_{j=\frac{k+1}{2}}^{\frac{k+1}{2}} b_{ij} - \hat{r}_i(A), & n \text{ is odd}, \\
\frac{k}{2} \sum_{j=1}^{\frac{k}{2}} b_{ij} - \sum_{j=\frac{k}{2}}^{\frac{k}{2}} b_{ij} - \hat{r}_i(A), & n \text{ is even}; 
\end{cases}
\]
for each $i \in [n]$, $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ is an arrangement in non-decreasing order of $a_{i_{1}i_{2}\ldots i_{k}}$ for $i_{2}, \ldots, i_{k} \in [n]$, $\ell = nk^{-1}$; and $\hat{r}_{i}(A)$ is as in (7).

As shown in [19, 20, 21, 22, 23, 24, 29, 38, 48], an eigenvalue inclusion set can provide a sufficient condition for the positive definiteness of tensors. Based on the inclusion interval in Theorem 3.1, a sufficient condition for the positive definiteness of even-order tensors is given.

**Theorem 5.1.** Let $A = (a_{i_{1}i_{2}\ldots i_{m}}) \in \mathbb{R}^{[m,n]}$ with $m$ being even, and $\lambda$ be any Z-eigenvalue of $A$. If $\alpha_{i} > r_{i}(A, \alpha_{i})$ for each $i \in [n]$, then $\lambda > 0$. Furthermore, if $A$ is also symmetric, then $A$ is positive definite, consequently, $f(x)$ defined in (1) is positive definite.

**Proof.** Suppose on the contrary that $\lambda \leq 0$. For any vector $\alpha = (\alpha_{1}, \ldots, \alpha_{n})^{T} \in \mathbb{R}^{n}$, we by Theorem 3.1 have $\lambda \in \mathcal{Y}(A, \alpha)$, which implies that there is an index $i \in [n]$ such that $\lambda \in \mathcal{Y}_{i}(A, \alpha)$, i.e.,

$$|\lambda - \alpha_{i}| \leq r_{i}(A, \alpha_{i}).$$

(55)

On the other hand, we by $\alpha_{i} > 0$ have $|\lambda - \alpha_{i}| \geq \alpha_{i} > r_{i}(A, \alpha_{i})$. It contradicts (55). Hence, $\lambda > 0$. By the fact that an even-order real symmetric tensor with all positive Z-eigenvalues is positive definite, the conclusion follows. \qed

Based on the inclusion intervals in Theorems 4.4 and 4.5, two sufficient conditions for the positive definiteness of even-order tensors are given easily.

**Corollary 5.** Let $A = (a_{ij_{1}j_{2}}) \in \mathbb{R}^{[1,n]}$, and $\lambda$ be any Z-eigenvalue of $A$.

I. Assume that $\mathcal{E}$ is taken as $\mathcal{E}_{1}$.

(I.i) If $l_{i} > 0$ for each $i \in [n]$, then $\lambda > 0$, where

$$l_{i} = \begin{cases} 
\frac{1}{2} \left( n\frac{n+3}{2} \sum_{k=1}^{n+3} b_{i,k} - \sum_{k=\frac{n+3}{2}}^{n+1} b_{i,k} \right) - \hat{r}_{i}(A), & n \text{ is odd}, \\
\frac{1}{2} \left( n\frac{n+1}{2} \sum_{k=1}^{n+1} b_{i,k} - \sum_{k=\frac{n+1}{2}}^{n+3} b_{i,k} \right) - \hat{r}_{i}(A), & n \text{ is even},
\end{cases}$$

$b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,n+1}$ is an arrangement in non-decreasing order of $a_{i_{1}i_{2}\ldots i_{k}}$, $a_{i_{1}i_{2}}$ and $a_{i_{j_{1}}j_{2}}$ for $j \in [n]$ and $j \neq i$, and $\hat{r}_{i}(A)$ is defined in (19).

(I.ii) Furthermore, if $A$ is symmetric with $n = 2$, then $A$ is positive definite, consequently, $f(x)$ defined by (1) is positive definite.

II. Assume that $\mathcal{E}$ is taken as $\mathcal{E}_{2}$.

(II.i) If $l_{i} > 0$ for each $i \in [n]$, then $\lambda > 0$, where

$$l_{i} = \begin{cases} 
\frac{1}{6} \left( n\frac{3n+9}{2} \sum_{k=1}^{3n+9} c_{i,k} - \sum_{k=\frac{3n+9}{2}}^{3n+3} c_{i,k} \right) - \hat{r}_{i}(A), & n \text{ is odd}, \\
\frac{1}{6} \left( n\frac{3n+4}{2} \sum_{k=1}^{3n+4} c_{i,k} - \sum_{k=\frac{3n+4}{2}}^{3n+3} c_{i,k} \right) - \hat{r}_{i}(A), & n \text{ is even},
\end{cases}$$

$c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,3n+3}$ is an arrangement in non-decreasing order of $a_{i_{1}i_{2}\ldots i_{k}}$ with its number 6, $3a_{i_{1}j_{2}}$, $3a_{i_{1}j_{2}}$ and $3a_{i_{j_{1}}j_{2}}$ for $j \in [n]$ and $j \neq i$, and $\hat{r}_{i}(A)$ is defined in (20).
Corollary 6. Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$, $m = 2k \geq 6$, $\ell = n^{k-1}$, and $\lambda$ be any Z-eigenvalue of $\mathcal{A}$.

(i) If $l_i > 0$ for each $i \in [n]$, then $\lambda > 0$, where

$$ l_i = \begin{cases} \frac{\ell+1}{2} \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\frac{i+2}{2}}^{\ell} b_{i,j} - \gamma_i(\mathcal{A}), & n \text{ is odd}, \\ \frac{\ell}{2} \sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\frac{i+2}{2}}^{\ell} b_{i,j} - \gamma_i(\mathcal{A}), & n \text{ is even}, \end{cases} $$

and $b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ is an arrangement in non-decreasing order of $a_{i_1i_2\ldots i_k}$ for $i_2, \ldots, i_k \in [n]$. Here, $\gamma_i(\mathcal{A})$ is taken as follows: if $2 \leq n < m - 2$, then $\gamma_i(\mathcal{A})$ is as in (23); if $n \geq m - 2$, then $\gamma_i(\mathcal{A})$ is as in (24).

(ii) Furthermore, if $\mathcal{A}$ is symmetric with $n \geq m - 2$, then $\mathcal{A}$ is positive definite, consequently, $f(x)$ defined by (1) is positive definite.

Next, an example is given to show the efficiency of Theorem 5.1 and Corollary 5.

Example 5.2. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor with elements defined as follows:

$$ a_{1111} = a, \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = b, $$
$$ a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = c, $$
$$ a_{1222} = a_{2122} = a_{2212} = a_{2221} = d, \quad a_{2222} = e. $$

I. Firstly, we consider the tensor $\mathcal{A}_1$ in Example 1 of [24], where

$$ a = 10, \quad b = 0.5, \quad c = 3, \quad d = -0.2 \quad \text{and} \quad e = 9. $$

In Section 3.1 of [24], by taking $\mathcal{E}$ as $\mathcal{E}_2$ and $\alpha = (10,9)\top$, $\mathcal{A}_1$ is showed to be positive definite by Theorem 1.4. Proposition 1 shows the fact that if $R_t(\mathcal{A}, \alpha_t)$ in Theorem 1.4 is taken as in (6), which takes $\mathcal{E}$ as $\mathcal{E}_1$, then Theorem 1.4 cannot be used to judge the positive definiteness of $\mathcal{A}$.

However, if taking $\mathcal{E}$ as $\mathcal{E}_1$, then Theorem 5.1 and Corollary 5 can be used to judge the positive definiteness of $\mathcal{A}_1$. In fact, taking $\alpha = (10,9)\top$ and $\mathcal{E}$ as $\mathcal{E}_1$, we have

$$ \alpha_1 = 10 > 7.45 = r_1(\mathcal{A}_1, \alpha_1) \quad \text{and} \quad \alpha_2 = 9 > 6.80 = r_2(\mathcal{A}_1, \alpha_2), $$

which implies that $\mathcal{A}_1$ satisfies all conditions of Theorem 5.1, and hence $\mathcal{A}_1$ is positive definite. Moreover, by computations, we have

$$ b_{1,1} = 3, \quad b_{1,2} = b_{1,3} = 10, \quad \hat{r}_1(\mathcal{A}_1) = 3.95, \quad b_{2,1} = 3, \quad b_{2,2} = b_{2,3} = 9, \quad \hat{r}_1(\mathcal{A}_1) = 3.8, $$

and

$$ l_1 = \frac{1}{2} (b_{1,1} + b_{1,2}) - \hat{r}_1(\mathcal{A}_1) = 2.55 > 0, \quad l_2 = \frac{1}{2} (b_{2,1} + b_{2,2}) - \hat{r}_2(\mathcal{A}_1) = 2.2 > 0. $$

Then, by Corollary 5, one can also judge that $\mathcal{A}_1$ is positive definite.

II. Secondly, we consider another tensor $\mathcal{A}$, where

$$ a = 11, \quad b = -2, \quad c = 2, \quad d = -1 \quad \text{and} \quad e = 10. $$

By computations, $\sigma(\mathcal{A}) = \{5.2335, 12.3746\}$. 
sider the asymptotically stability of the time-invariant polynomial system.

Asymptotically stability of time-invariant polynomial systems.

Then, by Corollary 5, \( A \) is positive definite. Moreover, by computations, we have

\( \alpha_1 = 11 > 10.5 = r_1(A, \alpha_1) \quad \text{and} \quad \alpha_2 = 10 > 9.5 = r_2(A, \alpha_2), \)

that is to say, \( A \) satisfies all conditions of Theorem 5.1, and hence \( A \) is positive definite. Moreover, by computations, we have

\[
\begin{align*}
  b_{1,1} &= 2, \quad b_{1,2} = 11, \quad b_{1,3} = 11, \quad \hat{r}_1(A) = 6, \\
  b_{2,1} &= 2, \quad b_{2,2} = 10, \quad b_{2,3} = 10, \quad \hat{r}_2(A) = 5.5,
\end{align*}
\]

and

\[
\begin{align*}
  l_1 = \frac{1}{2}(b_{1,1} + b_{1,2}) - \hat{r}_1(A_1) = 0.5 > 0, \quad l_2 = \frac{1}{2}(b_{2,1} + b_{2,2}) - \hat{r}_2(A_1) = 0.5 > 0.
\end{align*}
\]

Then, by Corollary 5, \( A \) is positive definite.

5.2. Asymptotically stability of time-invariant polynomial systems. Consider the asymptotically stability of the time-invariant polynomial system

\[
\Sigma : \dot{x} = A^{(2)}x + A^{(4)}x^3 + \cdots + A^{(2k)}x^{2k-1},
\]

where \( A^{(t)} = (a_{i_1 \cdots i_t}) \in \mathbb{R}^{[t,n]}, \quad t = 2, 4, \ldots, 2k, \) and \( x = (x_1, \ldots, x_n)^\top \); see \([6, 8]\).

A sufficient condition such that the nonlinear system (56) above is asymptotically stable is gave by Deng et al. in \([6]\) as follows.

**Theorem 5.3.** \([6, \text{Theorem 3.3}]\) For the nonlinear system \( \Sigma \) in (56), if \( -A^{(t)} \) is positive definite, where \( t = 2, 4, \ldots, 2k \), then the equilibrium point of \( \Sigma \) is asymptotically stable.

By Theorems 5.1 and 5.3, a sufficient condition for the asymptotically stability can be given.

**Corollary 7.** For the nonlinear system \( \Sigma \) in (56), if \( -A^{(t)} \) satisfies all conditions of Theorem 5.1, where \( t = 2, 4, \ldots, 2k \), then the equilibrium point of \( \Sigma \) is asymptotically stable.

**Example 5.4.** Consider the following polynomial system

\[
\Sigma : \begin{align*}
  \dot{x}_1 &= -3x_1 + x_2 + x_3 - 2.6x_1^3 - 1.5x_1^2x_2 - 3.3x_1x_2^2 - 2.7x_1x_2^3 - 1.2x_2x_3^2, \\
  \dot{x}_2 &= x_1 - 3x_2 + x_3 - 0.5x_1^3 - 3.2x_2^3 - 1.2x_2x_3^2 - 3x_2x_3^2 + 3.3x_1x_3^2 - 2.7x_2^2x_3 - 1.2x_1x_3^2, \\
  \dot{x}_3 &= x_1 + x_2 - 3x_3 - 0.4x_1^3 - 2x_3^3 - 3.3x_1^2x_3 - 3x_2^2x_3 - 2.4x_1x_2x_3.
\end{align*}
\]

Then \( \Sigma \) can be written as \( \dot{x} = A^{(2)}x + A^{(4)}x^3 \), where \( x = (x_1, x_2, x_3)^\top \),

\[
A^{(2)} = \begin{pmatrix}
-3 & 1 & 1 \\
1 & -3 & 1 \\
1 & 1 & -3
\end{pmatrix}
\]
and $\mathcal{A}^{(4)} = (a_{ijkl}) \in \mathbb{R}^{[4]}$ with
\[
\begin{align*}
& a_{1111} = -2.6; \quad a_{2222} = -3.2; \quad a_{3333} = -2; \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = -0.5; \\
& a_{1122} = a_{1212} = a_{1221} = a_{2121} = a_{2211} = -0.9; \\
& a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = -1.1; \\
& a_{1233} = a_{1323} = a_{2133} = a_{2313} = a_{2331} = -0.4; \\
& a_{3123} = a_{3213} = a_{3231} = a_{3321} = -0.4; \\
& a_{2233} = a_{2323} = a_{2332} = a_{3232} = a_{3322} = -1; \\
& a_{2223} = a_{2232} = a_{2332} = a_{3232} = -0.4; \quad a_{ijkl} = 0, \text{otherwise}.
\end{align*}
\]

Clearly, $-\mathcal{A}^{(2)}$ is positive definite.

Next, the positive definiteness of the symmetric tensor $-\mathcal{A}^{(4)}$ is considered. In (5), taking the Z-identify tensor $\mathcal{E}$ as $\mathcal{E}_2$, then
\[
R_i(-\mathcal{A}^{(4)}, \alpha_i) = |a_{iiii} + \alpha_i| + \sum_{j \neq i} \left( |a_{iijj} + \frac{1}{3} \alpha_i| + |a_{ijij} + \frac{1}{3} \alpha_i| + |a_{ijji} + \frac{1}{3} \alpha_i| \right) + \omega_i,
\]
where
\[
\omega_i = R_i(-\mathcal{A}^{(4)}) - |a_{iiii}| - \sum_{j \neq i} \left( |a_{iijj}| + |a_{ijij}| + |a_{ijji}| \right), \quad i \in [3].
\]

Suppose that for each $i \in [3]$ there is $\alpha_i$ such that
\[
\alpha_i > R_i(-\mathcal{A}^{(4)}, \alpha_i),
\]
i.e.,
\[
g(\alpha_i) = \alpha_i - |a_{iiii} + \alpha_i| - \sum_{j \neq i} \left( |a_{iijj}| + \frac{1}{3} \alpha_i| + |a_{ijij}| + \frac{1}{3} \alpha_i| + |a_{ijji}| + \frac{1}{3} \alpha_i| \right) > \omega_i,
\]
which implies that $\max_{\alpha_i \in \mathbb{R}} g(\alpha_i) > \omega_i$ holds. By Lemma 4.2, we have
\[
\max_{\alpha_1 \in \mathbb{R}} g(\alpha_1) = 2 \leq 2.7 = \omega_1,
\]
\[
\max_{\alpha_2 \in \mathbb{R}} g(\alpha_2) = 2.5 \leq 2.9 = \omega_2,
\]
\[
\max_{\alpha_3 \in \mathbb{R}} g(\alpha_3) = 1.7 \leq 2.8 = \omega_3.
\]

Hence, for each $i \in [3]$, there is no $\alpha_i$ such that (57) holds, that is to say, the conditions of Theorem 1.4 do not satisfy, and consequently, we can not use Theorem 1.4 to judge the positive definiteness of $-\mathcal{A}^{(4)}$.

However, Corollary 5 (also Theorem 5.1) can be used to judge the positive definiteness of $-\mathcal{A}^{(4)}$. In fact, taking $\mathcal{E}$ as $\mathcal{E}_2$, by Corollary 5, we have $\tilde{r}_1(-\mathcal{A}^{(4)}) = 1.35, \tilde{r}_2(-\mathcal{A}^{(4)}) = 1.7, \tilde{r}_3(-\mathcal{A}^{(4)}) = 1.6; \text{the arrangement } c_{1.1} \leq c_{1.2} \leq \cdots \leq c_{1.12} \text{ is } 2.6, 2.6, 2.6, 2.6, 2.6, 2.7, 2.7, 2.7, 3.3, 3.3, 3.3; \text{the arrangement } c_{2.1} \leq c_{2.2} \leq \cdots \leq c_{2.12} \text{ is } 2.7, 2.7, 2.7, 3.3, 3.3, 3.3, 3.2, 3.2, 3.2, 3.2, 3.2; \text{the arrangement}
c_{3,1} \leq c_{3,2} \leq \cdots \leq c_{3,12} is 2, 2, 2, 2, 2, 3, 3, 3, 3.3, 3.3, 3.3; and consequently
\[l_1 = \frac{1}{6}(c_{1,1} + \cdots + c_{1,9} - c_{1,10} - c_{1,11} - c_{1,12}) - \tilde{r}_1(-\mathcal{A}^{(4)}) = 0.95 > 0,\]
\[l_2 = \frac{1}{6}(c_{2,1} + \cdots + c_{2,9} - c_{2,10} - c_{2,11} - c_{2,12}) - \tilde{r}_2(-\mathcal{A}^{(4)}) = 1.15 > 0,\]
\[l_3 = \frac{1}{6}(c_{3,1} + \cdots + c_{3,9} - c_{3,10} - c_{3,11} - c_{3,12}) - \tilde{r}_3(-\mathcal{A}^{(4)}) = 0.25 > 0.\]

Hence, $-\mathcal{A}^{(4)}$ is positive definite by Corollary 5. Furthermore, from Corollary 7, the equilibrium point of $\Sigma$ is asymptotically stable. In fact, all different Z-eigenvalues of $-\mathcal{A}^{(4)}$ are 1.9011, 1.9995, 2.0000, 2.1542, 3.3290, 3.5414 and 3.8187.

5.3. Bounds for the Z-spectral radius of weakly symmetric nonnegative tensors. Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and the Frobenius norm of $\mathcal{A}$ be
\[\|\mathcal{A}\|_F := \sqrt{\sum_{i_1,i_2,\ldots,i_m \in [n]} a_{i_1i_2\cdots i_m}^2}.\]

The symmetric best rank-one approximation of $\mathcal{A}$ is a rank-one tensor $\kappa x_m = (\kappa x_1, x_2, \ldots, x_m)$ such that $\|\mathcal{A} - \kappa x_m\|_F$ is minimized, where $\kappa \in \mathbb{R}$, $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ and $x^\top x = 1$. According to [30], $\kappa x_m$ is a symmetric best rank-one approximation of $\mathcal{A}$ if and only if $\kappa$ is a Z-eigenvalue of $\mathcal{A}$ with the largest absolute value and $x$ is a Z-eigenvector of $\mathcal{A}$ associated with $\kappa$.

Chang et al. in [5] proved that if $\mathcal{A}$ is a weakly symmetric nonnegative tensor, then $\varrho(\mathcal{A})$ is a positive Z-eigenvalue of $\mathcal{A}$. Hence, if $\mathcal{A}$ is nonnegative and weakly symmetric, then $\varrho(\mathcal{A}) x_0^m$ is a best rank-one approximation of $\mathcal{A}$, where $x_0$ is a Z-eigenvector associated with $\varrho(\mathcal{A})$, i.e.,
\[
\min_{\kappa \in \mathbb{R}, x \in \mathbb{R}^n, x^\top x = 1} \|\mathcal{A} - \kappa x_m\|_F = \|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \varrho(\mathcal{A})^2}. \tag{58}
\]

Furthermore, one can obtain
\[
\frac{\|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F}{\|\mathcal{A}\|_F} = \sqrt{1 - \frac{\varrho(\mathcal{A})^2}{\|\mathcal{A}\|_F^2}}, \tag{59}
\]
which gives a convergence rate for the greedy rank-one update algorithm; see [1, 7, 9, 15, 27, 32, 42] for details.

If the obtained upper bound of $\varrho(\mathcal{A})$ is small than or equal to $\|\mathcal{A}\|_F$, then the corresponding lower bounds of the equalities in (58) and (59) could be derived. Hence, the Z-spectral radius of weakly symmetric nonnegative tensors plays a fundamental role in the symmetric best rank-one approximation, which has numerous applications in engineering and higher order statistics, such as Statistical Data Analysis [15, 18, 2, 27, 31, 45].

Many researchers focus on bounding the Z-spectral radius of weakly symmetric nonnegative tensors [5, 12, 10, 11, 25, 27, 35, 39, 41, 43, 46, 47]. Recently, Song and Qi [39] presented an upper bound of $\varrho(\mathcal{A})$ for a nonnegative tensor $\mathcal{A}$ as follows.

Theorem 5.5. [39, Corollary 4.5] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_+$. Then
\[\varrho(\mathcal{A}) \leq \max_{i \in [n]} R_i(\mathcal{A}).\]

As an application of the sets in Theorems 2.3 and 3.1, we in this section give some new upper bounds for the Z-spectral radius $\varrho(\mathcal{A})$ of a weakly symmetric nonnegative
Let $A$ be a weakly symmetric tensor. Firstly, from the set $\Gamma(A)$ in Theorem 2.3, a new upper bound of $\varrho(A)$ can be obtained easily, which is sharper than that in Theorem 5.5 by (13).

**Theorem 5.6.** Let $A \in \mathbb{R}^{[m,n]}_+$ be a weakly symmetric tensor with $m \geq 3$. Then

$$\varrho(A) \leq \max_{i \in [n]} r_i(A),$$

where $r_i(A)$ is as in (12). Furthermore, $\max_{i \in [n]} r_i(A) \leq \max_{i \in [n]} R_i(A)$.

From the set in Theorem 3.1, another upper bound of $\varrho(A)$ for an even order tensor $A$ is given easily.

**Theorem 5.7.** Let $A \in \mathbb{R}^{[m,n]}_+$ be a weakly symmetric tensor with $m$ being even. Then for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$,

$$\varrho(A) \leq \max_{i \in [n]} \{\alpha_i + r_i(A, \alpha_i)\}.$$

Based on the intervals in Theorems 4.3, 4.4 and 4.5, several upper bounds of $\varrho(A)$ for an even order tensor $A$ are given as follows.

**Corollary 8.** Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}_+$ be weakly symmetric and $m = 2k$ be even. Then

$$\varrho(A) \leq \max_{i \in [n]} h_i,$$

where

$$h_i = \begin{cases} 
\sum_{j=\ell \frac{\ell-1}{2}}^{\ell} b_{i,j} - \sum_{j=1}^{\ell-1} b_{i,j} + \tilde{r}_i(A), & n \text{ is odd}, \\
\sum_{j=\ell \frac{\ell+1}{2}}^{\ell} b_{i,j} - \sum_{j=1}^{\ell-1} b_{i,j} + \tilde{r}_i(A), & n \text{ is even},
\end{cases}$$

$b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}$ is an arrangement in non-decreasing order of $a_{i_1i_2\cdots i_{\ell}i_{k}}$ for $i_2, \ldots, i_k \in [n]$, $\ell = n^{k-1}$, and $\tilde{r}_i(A)$ is as in (7).

**Corollary 9.** Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}_+$ be a weakly symmetric tensor with $m = 2k$ being even. Then

$$\min_{i \in [n]} \nu_i = \nu \leq \varrho(A) \leq \mu = \max_{i \in [n]} \mu_i,$$

where $\nu_i$ and $\mu_i$ are taken as follows:

(i) if $m = 4$ and $n$ is odd, then

$$\nu_i = \max \left\{ \frac{1}{2} \left( \sum_{t=1}^{n+1} b_{i,t} - \sum_{t=\frac{n+5}{2}}^{n+1} b_{i,t} \right) - \hat{r}_i(A), \frac{1}{6} \left( \sum_{t=1}^{3n+9} c_{i,t} - \sum_{t=\frac{3n+11}{2}}^{3n+9} c_{i,t} \right) - \hat{r}_i(A), 0 \right\}$$

and

$$\mu_i = \min \left\{ \frac{1}{2} \left( \sum_{t=2+1}^{n+1} b_{i,t} - \sum_{t=1}^{n+1} b_{i,t} \right) + \hat{r}_i(A), \frac{1}{6} \left( \sum_{t=1}^{3n+3} c_{i,t} - \sum_{t=3n+1}^{3n+3} c_{i,t} \right) + \hat{r}_i(A) \right\};$$

if $m = 4$ and $n$ is even, then

$$\nu_i = \max\{\hat{l}_i, \hat{l}_i, 0\} \quad \text{and} \quad \mu_i = \min\{\hat{u}_i, \hat{u}_i\},$$
where
\[
\hat{t}_i = \frac{1}{2} \left( \sum_{t=1}^{\frac{n+1}{2}} b_{i,t} - \sum_{t=\frac{n+3}{2}}^{n+1} b_{i,t} \right) - \hat{r}_i(A), \quad \tilde{t}_i = \frac{1}{6} \left( \sum_{t=1}^{\frac{3n+4}{2}} c_{i,t} - \sum_{t=\frac{3n+6}{2}}^{3n+3} c_{i,t} \right) - \tilde{r}_i(A),
\]
\[
\hat{u}_i = \frac{1}{2} \left( \sum_{t=\frac{n+1}{2}}^{\frac{n+1}{2}+m} b_{i,t} - \sum_{t=1}^{\frac{n-1}{2}} b_{i,t} \right) + \hat{r}_i(A), \quad \tilde{u}_i = \frac{1}{6} \left( \sum_{t=\frac{3n+3}{2}}^{3n+3} c_{i,t} - \sum_{t=\frac{3n+6}{2}}^{\frac{3n+6}{2}+m} c_{i,t} \right) + \tilde{r}_i(A).
\]
Here, for each \(i \in [n]\), \(b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,n+1}\) is an arrangement in non-decreasing order of \(a_{iiii}, a_{iiij}, a_{iiijj}\) for \(j \in [n]\) and \(j \neq i\); \(c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,3n+3}\) is an arrangement in non-decreasing order of \(a_{iiii}\) with its number \(6, 3a_{iiij}, 3a_{iiijj}\) and \(3a_{iiijj}\) for \(j \in [n]\) and \(j \neq i\); and \(\tilde{r}_i(A)\) and \(\hat{r}_i(A)\) are defined in (19) and (20), respectively. Note here that \(\sum_{t} b_{i,t} = 0\) if \(n = 2\).

(ii) if \(m = 2k \geq 6\), then
\[
\nu_i = \begin{cases} 
\frac{\ell+1}{2} \sum_{j=1}^{\frac{n}{2}} b_{i,j} - \sum_{j=\frac{\ell+3}{2}}^{\frac{n+1}{2}} b_{i,j} - \gamma_i(A), & \text{n is odd,} \\
\sum_{j=1}^{\ell} b_{i,j} - \sum_{j=\frac{\ell+3}{2}}^{\frac{n}{2}+2} b_{i,j} - \gamma_i(A), & \text{n is even,}
\end{cases}
\]
and
\[
\mu_i = \begin{cases} 
\sum_{j=1}^{\frac{n-2}{2}} b_{i,j} - \sum_{j=1}^{\frac{n}{2}} b_{i,j} + \gamma_i(A), & \text{n is odd,} \\
\sum_{j=1}^{\frac{\ell}{2}+1} b_{i,j} - \sum_{j=1}^{\frac{\ell}{2}+1} b_{i,j} + \gamma_i(A), & \text{n is even,}
\end{cases}
\]
where \(\ell = n^{k-1}\), and for each \(i \in [n]\), \(b_{i,1} \leq b_{i,2} \leq \cdots \leq b_{i,\ell}\) is an arrangement in non-decreasing order of \(a_{iiiz_1z_2\cdots i_k}\) for \(i_2, \ldots, i_k \in [n]\). Here, \(\gamma_i(A)\) is taken as follows: if \(2 \leq n < m - 2\), then \(\gamma_i(A)\) is as in (23); if \(n \geq m - 2\), then \(\gamma_i(A)\) is as in (24).

Finally, we show that the upper bounds in Theorem 5.6 and Corollary 9 are smaller than those in \([10, 11, 25, 27, 35, 36, 39, 41, 43, 44, 46, 47]\) by the following example.

**Example 5.8.** Let \(A = (a_{ijkl}) \in \mathbb{R}^{4,2}\) be a weakly symmetric tensor with entries defined as follows:
\[
a_{1111} = 7, \quad a_{2222} = 4, \quad a_{1211} = a_{1122} = 3, \quad a_{2111} = a_{2211} = a_{2121} = a_{2112} = 1, \\
a_{1121} = a_{1221} = a_{1112} = a_{1212} = 0, \quad a_{1222} = 6, \quad a_{2122} = 5, \quad a_{2221} = 13.
\]
By computations, we have \((\varrho(A), x) = (12.0995, (0.5426, 0.8400)^T)\) and
\[
\|A\|_F = 17.8045.
\]
Numerical results obtained by Theorem 5.6, Corollary 9 and those corresponding bounds in \([10, 11, 25, 27, 35, 36, 39, 41, 43, 44, 46, 47]\) are listed in Table 1. From
Table 1, we can see that the bounds in Theorem 5.6, Corollary 9 are smaller than the others.

| Method | ϱ(A) ≤ |
|--------|--------|
| Theorem 5.5, i.e., Corollary 4.5 of [39] | 26.0000 |
| Theorem 3.3 of [25] | 25.7771 |
| Theorem 3.4 of [47], where \( S = \{1\}, \bar{S} = \{2\} \) | 25.7382 |
| Theorem 4.5 of [41] | 25.7382 |
| Theorem 3.5 of [10] | 25.6437 |
| Theorem 6 of [11] | 25.6437 |
| Theorem 4 of [43], where \( S = \{1\}, \bar{S} = \{2\} \) | 25.6437 |
| Theorem 7 of [35] | 25.4807 |
| Theorem 7 of [44] | 25.4807 |
| Theorem 2.9 of [27] | 23.8617 |
| Theorem 5 of [46] | 22.5426 |
| Theorem 3.1 of [36] | 21.8172 |
| Theorem 5.6 | 16.0000 |
| Corollary 9 | 14.5000 |

As stated above, only when the upper bound of \( ϱ(A) \) is less than or equal to \( \|A\|_F \) can the formulas (58) and (59) be calculated. From Table 1, only the bounds obtained by Theorem 5.6 and Corollary 9 are smaller than \( \|A\|_F \). Then by Corollary 9 we have

\[
\min_{\kappa \in \mathbb{R}, x \in \mathbb{R}^n, x^\top x = 1} \|A - \kappa x^m\|_F = \sqrt{\|A\|_F^2 - ϱ(A)^2} \geq 10.3320,
\]

and

\[
\frac{\|A - ϱ(A)x^m\|_F}{\|A\|_F} = \sqrt{1 - \frac{ϱ(A)^2}{\|A\|_F^2}} \geq 0.5803,
\]

which gives an estimation for the convergence rate of the greedy rank-one update algorithm.

5.4. The geometric measure of entanglement of multipartite pure states.

In this section, upper and lower bounds for the geometric measures of entanglement of symmetric pure states with nonnegative amplitudes are given.

In a composite quantum \( m \)-partite system, a pure state \( |Ψ⟩ \) is a normalized element of a tensor product Hilbert space \( \mathcal{H} = \bigotimes_{k=1}^{m} \mathcal{H}_k \), where the dimension of \( \mathcal{H}_k \) is \( d_k \) and the orthonormal basis of \( \mathcal{H}_k \) is \( \{|e_{ik}^{(k)}⟩\}, k \in [m] \). A separable pure \( m \)-partite state \( |Ψ⟩ \in \mathcal{H} \) can be described by a product state \( |Φ⟩ = \bigotimes_{k=1}^{m} |ϕ^{(k)}⟩ \) with \( |ϕ^{(k)}⟩ = \sum_{i_k} u_{i_k}^{(k)} |e_{i_k}^{(k)}⟩ \in \mathcal{H}_k \), \( \|ϕ^{(k)}⟩\| = 1 \), \( k \in [m] \). The set of all separable pure states in \( \mathcal{H} \) is denoted by \( \text{Separ}(\mathcal{H}) \). If the state \( |Ψ⟩ \) is inseparable, it is called entangled state and its geometric measure of entanglement is defined as

\[
\text{GME}_Ψ = \min\{\|Ψ⟩ - |Φ⟩\| : |Φ⟩ = \bigotimes_{k=1}^{m} |ϕ^{(k)}⟩ \in \text{Separ}(\mathcal{H})\}. \tag{60}
\]

The authors of [44] pointed out that the geometric measures of entanglement \( \text{GME}_Ψ \) can be represented by tensors. Let \( \mathcal{A}_Ψ = (a_{i_1, \ldots, i_m}) \in \mathbb{C}^{d_1 \times \cdots \times d_m} \) be an
associated tensor of the pure state $|\Psi\rangle$, where $a_{i_1...i_m}$ is the amplitudes of the pure state $|\Psi\rangle$. Denote the spectral radius of the tensor $A_\Psi$ by

$$\rho(A_\Psi) = \max_{\|\phi^{(k)}\| = 1, k \in [m]} |A_\Psi u^{(1)} \cdots u^{(m)}|,$$

where

$$A_\Psi u^{(1)} \cdots u^{(m)} = \sum a_{i_1...i_m} u^{(1)}_{i_1} \cdots u^{(m)}_{i_m},$$

and each $|\phi^{(k)}\rangle$ is associated with a column vector $u^{(k)} \in \mathbb{C}^{d_k}$, $k \in [m]$. Then $GME_\Psi$ in (60) is equivalent to

$$GME_\Psi = \sqrt{2 - 2\rho(A_\Psi)}.$$  (61)

When $H_1 = \cdots = H_m$, $A_\Psi$ is symmetric if and only if $|\Psi\rangle$ is permutation symmetric. Given a symmetric pure state with nonnegative amplitudes $|\Psi\rangle \in H$, if a lower bound $\underline{\eta}$ and an upper bound $\overline{\eta}$ of $\rho(A_\Psi)$ are given, i.e., $\underline{\eta} \leq \rho(A_\Psi) \leq \overline{\eta}$, then the geometric measure of entanglement $GME_\Psi$ for $|\Psi\rangle$ has the following lower and upper bounds:

$$\sqrt{2 - 2\overline{\eta}} \leq GME_\Psi \leq \sqrt{2 - 2\underline{\eta}}.$$

For details, see [44].

**Example 5.9.** Consider the following general 4-qutrit symmetric state with nonnegative amplitudes

$$|\Psi\rangle = \frac{9}{4\sqrt{14}}(|0000\rangle + |1111\rangle) + \frac{1}{4\sqrt{14}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$$

$$+ \frac{1}{4\sqrt{14}}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)$$

$$+ \frac{3}{4\sqrt{14}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle).$$

It can be verified that $\| A_\Psi \|_F = 1$, $\rho(A_\Psi) = 0.7350$ and $GME_\Psi = 0.7280$. By Corollary 9, we have

$$0.4343 \leq \rho(A_\Psi) \leq 0.7684,$$

which implies that

$$0.6806 \leq GME_\Psi \leq 1.0637.$$

6. **Conclusion.** In this paper, we first in Proposition 1 pointed out a weakness of Theorem 3.2 of [24] (i.e., Theorem 1.4). That is, if $R_i(A, \alpha_i)$ is as in (6), that is to say, if taking the $Z$-identity tensor $E$ as $E_1$ in $R_i(A, \alpha_i)$, then Theorem 1.4 cannot be used to judge the positive definiteness of even order symmetric tensors. Subsequently, we in Theorem 2.3 presented a new Geršgorin-type $Z$-eigenvalue inclusion interval $\Gamma(A)$ for tensors.

Secondly, in order to overcome the weakness of Theorem 3.2 of [24], we in Theorem 3.1 presented the Geršgorin-type $Z$-eigenvalue inclusion interval $\Upsilon(A, \alpha)$ with parameters $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$ for even order tensors, and proved that it is tighter than that in Theorem 3.2 of [24]. Thirdly, by selecting appropriate parameters $\alpha$, the optimal interval $\Upsilon(A)$ in Theorems 4.4 and 4.5, respectively, for $m = 4$ and $m \geq 6$ are given and proved to be tighter than the interval $G(A)$ in Theorem 4.3, which is the optimal interval obtained by Corollary 1 (i.e., Corollary 1 of [24]).
Finally, as applications of these obtained intervals, we in Subsection 5.1 and Subsection 5.2 provided some sufficient conditions for the positive definiteness of homogeneous polynomial forms as well as the asymptotically stability of time-invariant polynomial systems. We in Subsection 5.3 presented lower and upper bounds for the \( Z \)-spectral radius \( \rho(A) \) of a weakly symmetric nonnegative tensor \( A \), which are used to estimate the convergence rate of the greedy rank-one update algorithm and derive bounds of entanglement of symmetric pure state with nonnegative amplitudes in Subsection 5.4. Moreover, we showed that the bounds in Theorem 5.6 and Corollary 9 are smaller than those in [10, 11, 25, 27, 35, 36, 39, 41, 43, 44, 46, 47] in some cases by numerical examples.

However, there are still many problems unsolved. For instance, what is the specific form of the \( Z \)-identity tensor \( E_2 \) for the order \( m \geq 8 \) and \( m \) is even? What is the specific form of Theorem 3.1 for \( m \geq 6 \) and \( m \) is even if the \( Z \)-identity tensor \( E_2 \)? These two problems have not been solved yet. We will continue to study these problems in the future.

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