On the validity of ADM formulation in 2d quantum gravity

K. Ghoroku\textsuperscript{1}, K. Kaneko\textsuperscript{2}

\textsuperscript{1} Fukuoka Institute of Technology
Wajiro, Higashi-ku, Fukuoka 811-02, Japan
\textsuperscript{2} Kyushu Sangyo University
Matsukadai, Fukuoka 813, Japan

\textbf{ABSTRACT}

We investigate 2d gravity quantized in the ADM formulation, where only the loop length $l(z)$ is retained as a dynamical variable of the gravitation, in order to get an intuitive physical insight of the theory. The effective action of $l(z)$ is calculated by adding scalar fields of conformal coupling, and the problems of the critical dimension and the time development of $l$ are addressed.

\textsuperscript{*} e-mail address: gouroku@dontaku.fit.ac.jp
\textsuperscript{†} e-mail address: kaneko@daisy.te.kyusan-u.ac.jp
1 Introduction

Two-dimensional (2d) quantum gravity has been extensively studied for the last few years, since it serves as a toy model of 4d quantum gravity as well as a prototype of string theories. In spite of great progress, there remains a difficulty in 2d quantum gravity. There appears a strong restriction on matters coupled to gravity [1] [2]. In fact, until now we have no way to couple the matters whose central charge is greater than one. In order to find a breakthrough in this direction, various kinds of trials would be necessary.

There have been mainly two continuum approaches so far. One is based on the conformal gauge [2], and the other on the light-cone gauge [1]. Recently an approach, which could provide a more intuitive understanding of the dynamics, based on the ADM (Arnowitt-Deser-Misner) formalism has been proposed [3] [4], and the time development of the loop variable was discussed.

In the present paper, we reconsider this formalism in terms of a slightly different but simple gauge condition, where only the loop variable is retained as the dynamical variable. In this formalism, the time direction is treated specially so that we can easily see the time development of the loop. But the lack of the covariance is fatal since it is difficult to get the quantum measure of the diffeomorphism with a definite form. In fact, a non-renormalizable divergent term appears if we proceed a straightforward calculation of the measure without any care of the properties of the operators. One way to get a meaningful and definite answer is to impose a consistent restriction on the operators. Such an example of a self-consistent calculation is given here. This example gives the same result with [3], where the induced action has a divergent pre-factor, so it leads to a functional delta function, i.e. a strong constraint on the loop variable. And this constraint is equivalent to the former restriction imposed on the operators.

On the other hand, the matter fields give a finite induced action. As a result, the resultant effective action is dominated by the quantum measure of the diffeomorphism. And the scalar fields could not give any effect on the induced action. This seems to be strange since the effective action is insensitive to the number of the scalar fields. So it could be said that the formalism based on the ADM decomposition is not suitable for seeing the dynamical effect of the matter fields on the surface. Therefore, the critical number of the scalar fields (the critical dimension in terms of the string theory terminology), which is obtained in the conformal gauge [5], can not be determined in this formalism.
2 Quantum measure and ADM decomposition

Here we firstly consider the case of pure gravity. ADM decomposition is the most popular and convenient one in general relativity, and it is given as follows in terms of a metric $g_{\mu\nu}$ (with Euclidian signature) on a two-dimensional manifold,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (N dx^0)^2 + h (\lambda dx^0 + dx^1)^2,$$

i.e.,

$$g_{\mu\nu} = \begin{bmatrix} N^2 + h\lambda^2 & h\lambda \\ h\lambda & h \end{bmatrix}.$$

Here $N$ and $\lambda$ are the lapse- and shift-functions, and $h$ is the metric on the time slice at $x^0$.

According to ref.\[3\], the amplitude of a cylinder with two loop boundaries $C$ and $C'$ of their length $l_0$ and $l_1$ respectively is defined as follows,

$$F(l_1, l_0) \equiv \int \frac{Dg_{\mu\nu}}{\text{Vol}_{\text{diff}}} \exp \left\{ -\mu_0 \int d^2 x \sqrt{g} \right\},$$

where $\mu_0$ denotes the bare cosmological constant, and the constraints

$$\delta \left( \int_C \sqrt{g_{\mu\nu} dx^\mu dx^\nu} - l_0 \right) \delta \left( \int_{C'} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} - l_1 \right)$$

are abbreviated since they are not necessary hereafter. The functional measure

$$Dg_{\mu\nu} = DNDhD\lambda$$

is defined by the following norm,

$$\| \delta g_{\mu\nu} \|_g^2 \equiv \int d^2 x \sqrt{g} g^{\alpha\beta} \delta g_{\mu\nu}|_g \delta g_{\alpha\beta}|_g = 4 \int d^2 x N\sqrt{h} \left[ \left( \frac{\delta h}{2h} \right)^2 + \left( \frac{\delta N}{N} \right)^2 + \frac{1}{2} \left( \frac{\sqrt{h}}{N} \delta \lambda \right)^2 \right].$$

The volume of diffeomorphism group $\text{Vol}_{\text{diff}}$ is identified with the functional integral over all diffeomorphisms,

$$\text{Vol}_{\text{diff}} \equiv \int D\nu \nu,$$

where $D\nu$ is determined by the norm of infinitesimal diffeomorphism ($x^\mu \mapsto x^\mu - \delta x^\mu$) with

$$\| \delta \nu \|_g^2 = \int d^2 x \sqrt{g} g^{\mu\nu} \delta \nu \delta \nu.$$
Next, we take the following gauge fixing condition, \( \bar{g}_{\mu\nu}, \)

\[
\begin{align*}
\bar{N} &\equiv 1, & \quad (2.7) \\
\bar{\lambda} &\equiv k = \text{constant}, & \quad (2.8)
\end{align*}
\]

or equivalently,

\[
\bar{g}_{\mu\nu}(x^0 = t, x^1 = x) \equiv \left[ \begin{array}{cc} 1 + l(t, x)^2k^2 & l(t, x)^2k \\ l(t, x)^2k & l(t, x)^2 \end{array} \right]. \quad (2.9)
\]

Here only \( l(t, x)^2 \) \((= \bar{h})\) is retained as a free variable since \( k \) is taken as a constant. In [3], \( k \) is retained as a variable and \( l \) is restricted as \( l = l(t) \). This point is essentially different from our gauge condition. Eq. (2.7) implies that the time coordinate \( x^0 \) is chosen directly as the geodesic distance from the incoming loop \( C \). In this gauge, the variations of the original three independent variables, \( \{N, \lambda, h\} \), are replaced by the two gauge parameters, \( \delta V_\mu \), and \( \delta l \).

In the following, we take the coordinate \((x^0, x^1) = (t, x)\) on \( M \), such that \( 0 \leq x \leq 1 \), and the times at \( C \) and \( C' \) are represented by \( t = 0 \) and \( t = D \), respectively. For this parametrization, it may be useful to introduce two vectors \( \bar{n} \) and \( \bar{s} \) which are normal and tangential to time slices, respectively;

\[
n_\mu \equiv (1, 0), \quad s_\mu \equiv l(k, 1). \quad (2.10)
\]

Then eq. (2.6) is rewritten at \( g_{\mu\nu} = \bar{g}_{\mu\nu} \) in the following form:

\[
\| \delta V_\mu \|_{\bar{g}}^2 = \int d^2x \, l \left[ (\delta v^n)^2 + (\delta v^s)^2 \right], \quad (2.11)
\]

where \( \delta v^n \) and \( \delta v^s \) are infinitesimal diffeomorphisms in the normal and tangential directions, respectively,

\[
\delta v^n \equiv \bar{g}_{\mu\nu} n^\mu \delta V^\nu, \quad \delta v^s \equiv \bar{g}_{\mu\nu} s^\mu \delta V^\nu. \quad (2.12)
\]

Eq. (2.11) implies that \( \text{Vol}_{\text{diff}} \) depends on \( l \), but its \( l \) dependent part is expected as,

\[
\exp \left\{ \alpha \delta(0) \int d^2x \ln (l) \right\},
\]

where \( \alpha = -1/2 \). And this divergent term can be neglected if it is regularized by the dimensional regularization since it can be set zero. As a result, \( \text{Vol}_{\text{diff}} \) can be considered as a \( l \)-independent infinite number, so it can be factored out and be devided out as in the usual gauge theory.

Further, since the infinitesimal deformation of metric around \( \bar{g}_{\mu\nu} \) is generally expressed as\( ^1 \)

\[
\delta g_{\mu\nu} = \delta \bar{g}_{\mu\nu} + \nabla_\mu \delta V_\nu + \nabla_\nu \delta V_\mu, \quad (2.13)
\]

\( ^1 \) \( \nabla_\mu \) is the covariant derivative with respect to \( \bar{g}_{\mu\nu} \).
we obtain the following relations by using eqs. (2.2) and (2.9),

\[
\begin{align*}
\delta h &= 2l(\delta l - \Omega v^n + lD_s v^s), \\
\delta \lambda &= \frac{1}{l}[(D_n + \Omega) v^s + D_s v^n], \\
\delta N &= D_n v^n.
\end{align*}
\]

Here, \( \Omega = -\frac{1}{l} D_n l \), and \( D_n, D_s \) are the derivatives in the normal and tangential directions defined as,

\[
\begin{align*}
D_n &\equiv n^\mu \partial_\mu = \partial_0 - k \partial_1, \\
D_s &\equiv s^\mu \partial_\mu = l^{-1} \partial_1.
\end{align*}
\]

Their hermite conjugates, \( D^\dagger_n \) and \( D^\dagger_s \), under the diffeomorphism-invariant measure \( \int d^2x \sqrt{\bar{g}} = \int d^2x l \) are given as follows:

\[
\begin{align*}
D^\dagger_n &= -\partial_0 + k \partial_1 - \frac{i}{l}, \\
D^\dagger_s &= - l^{-1} \partial_1.
\end{align*}
\]

Then the original three components of \( \delta g_{\mu\nu} \) can be replaced by \( \delta l \) and two diffeomorphism generators. This change of variables is given by

\[
\mathcal{D}h \mathcal{D}\lambda \mathcal{D}N = \mathcal{D}l \mathcal{D}v^n \mathcal{D}v^s \left| \frac{\partial(h, \lambda, N)}{\partial(l, v^s, v^n)} \right|,
\]

where the Jacobian can be written as,

\[
\left| \frac{\partial(h, \lambda, N)}{\partial(l, v^s, v^n)} \right| = \text{Det}^{1/2} D^\dagger_n D_n \cdot \text{Det}^{1/2}(D^\dagger_n + \Omega)(D_n + \Omega).
\]

Here a numerical constant is abbreviated. We thus obtain

\[
\frac{\mathcal{D}g_{\mu\nu}}{\text{Vol}_{diff}} = \mathcal{D}l \cdot \text{Det}^{1/2} D^\dagger_n D_n \cdot \text{Det}^{1/2}(D^\dagger_n + \Omega)(D_n + \Omega).
\]

### 3 Evaluation of the measure and the amplitude

Next, our task is to evaluate the following functional of \( l \):

\[
F[l_1, l_0] = \int \mathcal{D}l \cdot \text{Det}^{1/2} D^\dagger_n D_n \cdot \text{Det}^{1/2}(D^\dagger_n + \Omega)(D_n + \Omega) e^{-\mu_0 \int d^2 x l}.
\]

\(^2\) We use the following abbreviation:

\[
\dot{f} \equiv \partial_0 f, \quad f' \equiv \partial_1 f, \nu
\]
The operators in the determinant are lacking the covariance, but we can estimate them from the invariant operators in terms of a limiting procedure adopted in [3] and the method in [7]. We can rewrite the Laplacian in the manifold defined by $\bar{g}_{\mu\nu}$ as,

$$\Delta[\bar{g}_{\mu\nu}] \equiv -\frac{1}{\sqrt{\bar{g}}} \partial_{\mu} \left( \sqrt{\bar{g}} \bar{g}^{\mu\nu} \partial_{\nu} \right) = D_{n}^\dagger D_{n} + D_{s}^\dagger D_{s}. \quad (3.2)$$

Noticing that $\Delta[\bar{g}_{\mu\nu}] = \Delta[l,k]$, and rescaling $l$ by a positive constant $\beta$, we find the following relation,

$$\Delta[\beta^{-1}l,k] = D_{n}^\dagger D_{n} + \beta^2 D_{s}^\dagger D_{s}. \quad (3.3)$$

Then $D_{n}^\dagger D_{n}$ can be obtained from the Laplacian as,

$$D_{n}^\dagger D_{n}[l,k] \equiv \lim_{\beta \to +0} \Delta[\beta^{-1}l,k]. \quad (3.3)$$

So $\text{Det}^{1/2} D_{n}^\dagger D_{n}$ can be derived from the determinant of the Laplacian (3.2) through the the above relation. As for the operator $(D_{n}^\dagger + \Omega)(D_{n} + \Omega)$, it seems difficult to get a similar relation as above. However, the following relation can be easily found if we take $k = 0$;

$$(D_{n}^\dagger + \Omega)(D_{n} + \Omega)[l,k] \equiv \lim_{\beta \to +0} \bar{\Delta}[\beta^{-1}l,k], \quad (3.4)$$

where

$$\bar{\Delta}[g_{\mu\nu}] \equiv \Delta[g_{\mu\nu}] - \frac{1}{2} R + g^{\mu\nu} \partial_{\mu}( \ln \sqrt{g} ) \partial_{\nu}( \ln \sqrt{g} ). \quad (3.5)$$

So we evaluate (3.1) restricting to the case of $k = 0$, but the essential part of the theory will not be changed even if we take $k \neq 0$. Then we consider hereafter the case of $k = 0$ for the sake of the brevity.

First, we estimate the determinant of the Laplacian

$$\Delta[g_{\mu\nu}] \equiv -\frac{1}{\sqrt{g}} \partial_{\mu}(\sqrt{g}g^{\mu\nu} \partial_{\nu}).$$

It is defined by

$$\ln \text{Det} \Delta[g_{\mu\nu}] \equiv -\int_{\epsilon}^{\infty} \frac{ds}{s} \text{Tr} \ e^{-s\Delta[g_{\mu\nu}]}, \quad (3.6)$$

where $\epsilon$ is an ultraviolet cutoff. Its infinitesimal change under the Weyl transformation $g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu}$ can be derived as follows [6],

$$\delta \ln \text{Det} \Delta[g_{\mu\nu}] \equiv -2\text{Tr}(\delta \sigma e^{-\Delta}) = -2 \int d^{2}x \sqrt{g} \delta \sigma \left( \frac{1}{4\pi \epsilon} + \frac{1}{24\pi} R[g_{\mu\nu}] + O(\epsilon) \right). \quad (3.7)$$
Making use of this formula, we can obtain the result as a functional of $l$ if the infinitesimal deformation $\delta l$ is related to the infinitesimal Weyl transformation $\delta \sigma$.

For this purpose, consider the following reparametrization \footnote{Here we retain $k$ as non-zero as far as possible, but it is set to be zero at some stage as shown below.} \[ x^\mu \mapsto \tilde{x}^\mu(x), \]
with
\[
\begin{align*}
\dot{\tilde{l}} &= 1 + (1 - l^2 k^2) \delta \sigma, \\
\tilde{l} &= l [1 + (1 + l^2 k^2) \delta \sigma],
\end{align*}
\]
(3.9)

where the terms of $O(\delta \sigma^2)$ are neglected. Thus we obtain
\[
\delta l \equiv \tilde{l}(x, t) - l(x, t) = l(1 + l^2 k^2) \delta \sigma - \frac{\partial l}{\partial t} \delta \tau,
\]
(3.10)

where
\[
\delta \tau \equiv \tilde{t} - t = \int_0^t dt' (1 - l^2 k^2) \delta \sigma(t', x).
\]
(3.11)

Here we take $k = 0$, which is necessary for the calculation of $\tilde{\Delta}[g_{\mu\nu}]$. But the physical consequences would not depend on the value of $k$, so this is a shortcut to get a meaningful result. Thus we obtain from the above equations the following expression for $\delta \tau(t, x)$ and $\delta \sigma(t, x)$,
\[
\delta \tau(t, x) = l(t, x) \int_0^t dt' \frac{\delta l(t', x')}{l(t', x')^2},
\]
(3.12)

\[
\delta \sigma(t, x) = \frac{\partial l(t, x)}{\partial t} \int_0^t dt' \frac{\delta l(t', x')}{l(t', x')^2} + \frac{\delta l(t, x)}{l(t, x)}.
\]
(3.13)
Furthermore, the scalar curvature $R$ is given by

$$R[g_{\mu\nu}] = -2 \frac{\ddot{l}}{l}. \quad (3.14)$$

for $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $k = 0$. Finally, we arrive at the following result by using the eqs. (3.7), (3.13) and (3.14),

$$\delta \ln \text{Det} \Delta[\bar{g}_{\mu\nu}] = -\delta \int dz \left[ \mu_1 l(z) + \frac{1}{12\pi} \frac{\dot{l}(z)^2}{l(z)} \right]. \quad (3.15)$$

where $\mu_1 \equiv \lim_{\epsilon \to 0} 1/4\pi\epsilon$ is the divergent term.

Next, we evaluate the determinant of $\bar{\Delta}(g_{\mu\nu})$ in a similar way to the case of $\Delta(g_{\mu\nu})$. However we must notice that this operator is not positive definite for arbitrary values of $g_{\mu\nu}$ due to the second term of (3.5) even if the negative modes of $\Delta(g_{\mu\nu})$ are excluded. So we must assume the positive definiteness of the operator $\bar{\Delta}(g_{\mu\nu})$ and examine the validity of this assumption after the calculation.

If we assume that the eigenvalue of $\bar{\Delta}$ is positive, then the following formula is obtained,

$$\delta \ln \text{Det} \bar{\Delta}[g_{\mu\nu}] = -2\text{Tr}(\delta \sigma e^{-\epsilon \bar{\Delta}}) + \text{Tr}(\bar{\Delta}^{-1} e^{-\epsilon \bar{\Delta}} \nabla^2 + 4g^{\mu\nu} \partial_\mu \ln \sqrt{g} \partial_\nu) \delta \sigma. \quad (3.16)$$

The first term of (3.16) is written as,

$$-2\text{Tr}(\delta \sigma e^{-\epsilon \bar{\Delta}}) = \int d^2z \sqrt{g(z)} \delta \sigma(z) G(z, z'; \epsilon), \quad (3.17)$$

where

$$G(z, z'; \epsilon) \equiv \langle z | e^{-\epsilon \bar{\Delta}} | z' \rangle. \quad (3.18)$$

We estimate this diffusion evolution operator $G(z, z'; \epsilon)$ according to the method of [3] by taking the conformal flat metric, $g_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu}$, since the results are rewritten in a general form. In conformal coordinates,

$$\bar{\Delta} = -\partial^2 - \bar{V} \quad \bar{V} = (e^{-2\sigma} - 1)\partial^2 - e^{-2\sigma} [\partial^2 \sigma + 4(\partial \sigma)^2], \quad (3.19)$$

where the second term of $\bar{V}$ is the newly appeared one compared to the case of the operator $\Delta$. Further, choose a local conformal coordinate with the origin being the interesting point,

$$\sigma(0) = 0, \quad \partial_\mu \sigma(0) = 0. \quad (3.20)$$

7
Then we obtain the following result after a straightforward calculation,

$$G(0,0;\epsilon) = \frac{1}{4\pi\epsilon} + \frac{1}{6\pi} R + O(\epsilon).$$  \hspace{1cm} (3.21)

Comparing (3.21) and (3.7), we can see the difference of the coefficient of $R$.

As for the second term of (3.16), we write it as follows,

$$\int d^2z \sqrt{g(z)} \delta\sigma(z) [\nabla^2 - 4g^{\mu\nu} \partial_\mu \ln \sqrt{g}\partial_\nu] < z|\bar{\Delta}^{-1}e^{-\epsilon\bar{\Delta}}|z >,$$  \hspace{1cm} (3.22)

by making use of partial integrations. In the calculation, we expand the inverse operator $\bar{\Delta}$ as,

$$\bar{\Delta} = \frac{1}{-\partial^2} + \frac{1}{-\partial^2} \bar{V} \frac{1}{-\partial^2} + \cdots.$$

After a straightforward calculation, we get

$$\text{Tr}(\bar{\Delta}^{-1}e^{-\epsilon\bar{\Delta}}[\nabla^2 + 4g^{\mu\nu} \partial_\mu \ln \sqrt{g}\partial_\nu]\delta\sigma)$$

$$= \int d^2z \sqrt{g(z)} \delta\sigma(z) \left[\frac{-1}{16\pi} (2 + \Gamma(0) \ln (0^+))R + O(\epsilon)\right],$$  \hspace{1cm} (3.23)

where the divergent coefficient $\Gamma(0)$ appeared because of the infrared divergence of the loop of "massless propagator" $1/\partial^2$ in $\bar{\Delta}^{-1}$ expansion. So this could be regularized by introducing some small cutoff parameter. However the divergence due to $\ln (0^+)$ is ultraviolet, and this divergence can not be absorbed by the cosmological constant since this term is corresponding to the anomaly term. This result seems to indicate the non-renormalizability of the theory. But this statement would not be correct, and we should consider that our assumption of the positive definiteness of (3.5) is broken at some point. Then (3.16) would not be correct.

In order to proceed the estimation of the measure, we restrict the last two terms of (3.5) such that they can be written as

$$\bar{\Delta}[g_{\mu\nu}] = \Delta[g_{\mu\nu}] + m^2.$$  \hspace{1cm} (3.24)

This replacement of the operator $\bar{\Delta}[g_{\mu\nu}]$ is corresponding to giving a restriction on the loop variable as,

$$l(x,t) = l_0(x) \exp(\pm \frac{m}{\sqrt{2}}t).$$  \hspace{1cm} (3.25)

It can be seen that this restriction is consistent with the final result of the calculation of the determinants as seen below. And it should be noted that the restriction (3.25) is implied by the calculation of $\text{Det}^{1/2}D_n^\dagger D_n$. If we could perform the calculation of $\text{Det}^{1/2}(D_n^\dagger + \Omega)(D_n + \Omega)$ under a more loose restriction, it might lead to an interesting result. But it is beyond our present work.
In this case, the eq.(3.16) is changed as follows,
\[ \delta \ln \text{Det} \bar{\Delta} [g_{\mu\nu}] = -2 \text{Tr}(\delta \sigma e^{-\bar{\Delta}}) \]
\[ + 2m^2 \text{Tr}(\bar{\Delta}^{-1} e^{-\bar{\Delta}} \delta \sigma). \]  
(3.26)

Then we obtain after a straightforward calculation the following form,
\[ \delta \ln \text{Det} \bar{\Delta} [g_{\mu\nu}] = -2 \int d^2 z \sqrt{g(z)} \delta \sigma(z) \left( \frac{1}{4\pi \epsilon} + \frac{m^2}{4\pi} [1 + e^{m^2 \epsilon} E_i(-m^2 \epsilon)] \right) \]
\[ + \frac{1}{24\pi} R + O(\epsilon) \]
(3.27)

where \( E_i \) is the error function,
\[ E_i(-x) = - \int_x^\infty \frac{dt}{t} e^{-t}. \]

And (3.27) is easily rewritten in terms of the variable \( l \),
\[ \delta \ln \text{Det} \bar{\Delta} [g_{\mu\nu}] = - \delta \int d^2 z \left[ \mu_2 l(z) + \frac{1}{12\pi} \frac{\dot{l}(z)^2}{l(z)} \right], \]
(3.28)

where
\[ \mu_2 = \frac{1}{4\pi \epsilon} + \frac{m^2}{4\pi} [1 + e^{m^2 \epsilon} E_i(-m^2 \epsilon)]. \]
(3.29)

From (3.15) and (3.28), we get the following final result,
\[ F[l] = \int \mathcal{D}l e^{-S_{\text{eff}}(l)} \]
\[ S_{\text{eff}}(l) = \frac{c}{24\pi} \int d^2 z \left[ \frac{\dot{l}(z)^2}{l(z)} - \frac{m^2}{2} l(z) \right] + \mu^2 \int d^2 z l(z), \]
(3.30)

where the constant \( c \) is defined as,
\[ c \equiv \lim_{\beta,\gamma \to +0} \left( \frac{1}{\beta} + \frac{1}{\gamma} \right), \]
(3.31)

and \( \mu^2 \) is the renormalized cosmological constant. The parameter \( m \) has been taken into the resultant formula because of the consistency of the previous assumption (3.25) which was adopted in the calculation of \( \text{Det} \bar{\Delta} \). It can be easily seen that we can obtain the relation (3.25) from (3.30) since \( c \) is divergent. Then we arrive at
\[ S_{\text{eff}}(l) = \mu^2 \int d^2 z l(z), \]
(3.32)

where the constraint (3.25) was used. This implies the area law of the time development amplitude of the loop, and the loop develops according to (3.25). It indicates an inflation or a deflation of one dimensional universe depending on the sign in (3.25).
4 Matter fields and the critical dimension

Here we consider the cilinder amplitude $F[l]$ by including the matter fields contribution. The action is written as follows in this case,

$$S = \mu_0^2 \int d^2z \sqrt{g} + \int d^2z \sqrt{g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i, \quad (4.1)$$

where $i = 1 \sim d$. The measure of the scalars is defined by the norm,

$$|| \delta \phi_i ||_g^2 = \int d^2x \sqrt{g} (\delta \phi_i)^2. \quad (4.2)$$

Then the term, $-\frac{d}{2} \ln \det \Delta [g_{\mu\nu}]$, is added to the effective action after integration over the scalar fields. So, $S_{\text{eff}}$ in (3.30) is modified here as follows,

$$S_{\text{eff}}(l) = \frac{c - d}{24\pi} \int d^2z \left[ \frac{l(z)^2}{l(z)} - \frac{m^2}{2} l(z) \right] + \mu^2 \int d^2z l(z), \quad (4.3)$$

where the same notation is used for the renormalized cosmological constant $\mu^2$, and $c$ is given in (3.31).

As shown in the previous section, the value of $c$ is infinite. So we arrive at the same result (3.25) and (3.32) for any value of $d$. Then the scalar field could not give any influence on the effective action, but this conclusion seems to be contradicted with the analyses performed in terms of the conformal gauge where the coefficient of the induced Liouville term is proportional to $26 - d$. The reason why we could not get this critical dimension would be reduced to the incomplete calculation of $\det \Delta$. It has been obtained under a consistent constraint on $l(z)$ and the result has the same form with the one of $\det \Delta$. However there is a possibility that we obtain a finite $c$ if it has the form,

$$c = \lim_{\beta, \gamma \to +0} \left( \frac{1}{\beta} - \frac{c_1}{\gamma} \right), \quad (4.4)$$

where $c_1$ is some positive number. In this case, both the infinities might cancel out leaving a finite number. And the critical dimension could be determined if $c$ is obtained as a definite number, which is expected to be 26.

From the result obtained so far, the classical path of $l$ is constrained by (3.25). In order to investigate the quantum mechanical problem of the loop dynamics, the quantum measure of $l$ must be determined. However it seems to be difficult since the norm of $\delta l$ is defined as

$$|| \delta l ||_g^2 = \int d^2x l(\delta l)^2, \quad (4.5)$$
then the measure has $l$-dependence. In the case of the conformal gauge, this problem was solved by exploiting conformal invariance with respect to the fiducial metric. But there is no such a principle to determine the measure here. So we could not proceed the work to find the so-called dressed factor, which has been given in the case of the conformal gauge formulation, of the perturbation operator like the cosmological term. As shown in the case of the conformal gauge [5], it is also expected that a new instability of the surface would be seen if we could calculate the string susceptibility. These analyses seem to be impossible here.

5 Conclusion and Discussions

We have examined the formulation of 2d gravity in terms of the ADM decomposition in order to see the time development of the cylinder, the closed one dimensional space. This gauge is unfortunately not covariant, so it is difficult to see the critical dimension, the critical number of the scalar fields which couple to the gravity with the conformal coupling. In fact, we can not determine the coefficient of the induced kinetic term of the loop variable, $l$, coming from the measure of the diffeomorphism. The main reason of this difficulty is in the lack of the covariance of the quadratic operators. In our self-consistent calculation, the coefficient of the kinetic term of $l$ in the induced action is divergent. This situation means that this coefficient is not affected by the quantum effect of scalar fields even if we add any number of scalar fields. This result implies that the formalism based on ADM decomposition is not valid to see the dynamics of the surface which should be sensible to the fields on the surface.
References

[1] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[2] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509; F. David, Mod. Phys. Lett. A3 (1988) 1651.

[3] M. Fukuma, N. Ishibashi, H. Kawai and M. Ninomiya, Nucl. Phys. B427 (1994) 139.

[4] T. Kawano, Prog. Theor. Phys. 93 (1995) 455.

[5] A. Polyakov, Polyakov, Phys. Lett. 103B (1981)207.

[6] O. Alvarez, Nucl. Phys. B216 (1983) 125.

[7] A. Polyakov, Gauge Fields and Strings, Harwood Academic Publishers (1987).