Decoherence and quantum steering of accelerated qubit–qutrit system

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Abstract
The bidirectional steerability between different-size subsystems is discussed for a single parameter accelerated qubit–qutrit system. The decoherence due to the mixing and acceleration parameters is investigated, where for the total system and the qutrit, it increases as the mixing parameter increases, while it decreases for the qubit. The non-classical correlations are quantified by using the local quantum uncertainty, where the uncertainty increases at large values of the acceleration parameter. The possibility that each subsystem steers each other is studied, where the behavior of the steering inequality predicts that the qubit has a large ability to steer the qutrit. The degree of steerability decays gradually when the qubit is accelerated. However, it decays suddenly when the qutrit or both subsystems are accelerated. The degree of steerability is shown for the qutrit vanishes at small values of the acceleration, while the qubit at large acceleration. The difference between the degrees of steerability depends on the initial state settings and the size of the accelerated subsystem.

Keywords Qubit–qutrit system · Decoherence · Steerability

1 Introduction

In 1935, Schrödinger made an effort to formalize the quantum phenomenon of Einstein–Podolsky–Rosen (EPR) steering, where there is the possibility that one observer steers or changes the quantum state of another party at a distance by perform-
ing appropriately chosen local measurements [1, 2]. The strict hierarchy of quantum non-separable states demonstrates that the EPR steering exists in the range between entanglement and Bell non-locality, where the three quantum correlations are inequivalent [3]. So, the Bell non-locality is contained in the steering states, and the steering states are subset in entanglement [4]. In the past few years, quantum steering is implemented in many branches of physics in experimental and information-theoretic tasks, such as self-testing of quantum states [5], secure teleportation [6], randomness generation [7], and quantum key distribution [8]. A nondegenerate optical parametric oscillator has accomplished the violation of continuous variable EPR steering inequality [9]. The phenomenon of EPR steering is detected in the optical system experimentally by using entangled two-photon states [10]. Theoretically, quantum steering and its steerability can be reported for different quantum systems via violating the steering inequalities. For example, it has been discussed for bipartite two-qubit X-state [11–13], optical cavity according to multipartite system [14], Heisenberg chain model [15, 16], two-level or three-level detectors [17, 18]. As quantum systems interact with several surrounding environments, it was essential to indicate the effect of these surroundings on the steering correlation [19, 20]. For instance, the efficacy of relativistic motion, the noisy channel, finite temperature, non-Markovian environment, and a cavity optomechanical system on the steering have been discussed [12, 21–24]. Mathematically, the degree of steerability and steering inequality have been diagnosed via different relations such as, Heisenberg uncertainty principle [25], steering witnesses [26], the standard geometric Bell inequalities [27, 28], and the maximal violation of the Clauser–Horne–Shimony–Holt inequality [29, 30].

The optimal steering inequality for a pair of arbitrary discrete observables was obtained by Walborn et al. [25] as

\[ H(R^B | R^A) + H(Q^B | Q^A) \geq \log_2(\Omega^B) \] with, \( \Omega^B \equiv \max_{i,j} \{ |\langle R^B | Q^B \rangle|^2 \} \),

(1)

with \( R^i, Q^i \) being observables, \( \Omega^B \equiv \max_{i,j} \{ |\langle R^B | Q^B \rangle|^2 \} \), where \( H(R^B | R^A) = H(\rho^{AB}) - H(\rho^A) \) is the conditional entropy with \( \rho^{AB} \) and \( \rho^A \) being the density operator of the two subsystems and reduced density operator of subsystem A. \( \{|R^B\}\) and \( \{|Q^B\}\) are the eigenstates of the observables \( R^B \) and \( Q^B \). By employing the Pauli spin matrices, one can find the optimal steering inequality as follows [31],

\[ H(S_x) + H(S_y) + H(S_z) \geq \gamma_s, \]  

(2)

where \( \gamma_s = 2, 3 \) for the qubit and the qutrit, respectively.

On the other hand, investigating the effect of the Unruh framework is one of the significance studies in relativistic quantum processing, where the quantum regimes are basically non-inertial [32]. The Unruh framework effects on the quantum steering for the maximal entangled mixed state have been studied [33]. The dynamical behavior of quantum steering between two-modes of Dirac fields interact locally with thermal baths in the non-inertial frame has been investigated [34]. However, influence of the acceleration on the quantum correlation [35, 36], quantum coherence [37], estimation degree [38, 39], and the quantumness via Wigner distribution [40] have been discussed.
Our motivation in this study is to formalize a general form of entropic uncertainty steering inequality for the one-parameter family of qubit state (2D) interacting locally with qutrit state (3D), whether the qubit (small dimension) steers the qutrit (large dimension) or qutrit steers qubit. For a qubit system, the experimental estimation of one-parameter unitary gates for qubit systems in the presence of phase diffusion is proposed [41–43]. However, the experimental investigation of quantum correlation and quantum information scrambling of a qutrit is studied [44–46] state. Also, the Unruh framework effects are taken into account, where the qubit and the qutrit are accelerated individually, or simultaneously. Meanwhile, the degree of steerability of the accelerated system and the relation between the decoherence and steerability are investigated.

This article is organized as follows: In Sect. 2, we describe the one family parameter qubit–qutrit system and the acceleration process, where it is assumed that the qubit, qutrit, or both of them are accelerated. The decoherence due to the mixing parameter and the acceleration process is discussed in Sect. 3. The amounts of non-classical correlations are quantified by using the local quantum uncertainty in Sect. 4. Section 5 is devoted to investigate the bidirectional steerability process between the accelerated subsystems. Finally, we summarize our results in Sect. 6.

2 The model

Let us consider that a system of one parameter type that consists of a qubit system interacting locally with a qutrit system. In the computational basis, the system may be written as [47]

\[
\rho_q^{(p)} = \frac{p}{2} \left| 00 \right\rangle \left\langle 00 \right| + \left| 01 \right\rangle \left\langle 01 \right| + \left| 11 \right\rangle \left\langle 11 \right| + \left| 12 \right\rangle \left\langle 12 \right| + \left| 00 \right\rangle \left\langle 12 \right| + \left| 12 \right\rangle \left\langle 00 \right| + \frac{1 - 2p}{2} \left( \left| 02 \right\rangle \left\langle 02 \right| + \left| 02 \right\rangle \left\langle 10 \right| + \left| 10 \right\rangle \left\langle 02 \right| + \left| 10 \right\rangle \left\langle 10 \right| \right),
\]

(3)

where \( p \) is the initial setting state parameter, where \( 0 \leq p \leq 0.5 \) while the initial state \( \rho^0_t(p) \) is separable only when \( p = 1/3 \). The nature of the qubit system can be described by a two-level atom, which has a two-dimensional Hilbert space with the orthonormal basis \( \{ 0_q, 1_q \} \). However, the qutrit system can be described by a three-level atom with the orthonormal computational basis \( \{ 0_t, 1_t, 2_t \} \).

For a uniformly accelerated framework with a proper acceleration \( a \) in Minkowski coordinates \( (t, z) \), two different cases of Rindler coordinates \( (\tau, \xi) \) are expressed [48]:

\[
\begin{align*}
  a \ t &= e^{a \xi} \sinh a \tau & \quad a \ z &= e^{a \xi} \cosh a \tau & \quad & \text{region I}, \\
  a \ t &= -e^{a \xi} \sinh a \tau & \quad a \ z &= -e^{a \xi} \cosh a \tau & \quad & \text{region II}.
\end{align*}
\]

(4)

In terms energy solutions of Dirac equation \( \psi_{k,s}^\pm \), a fermionic field can be written in Minkowski coordinates as:

\[
\psi = \sum_s \int d^3k (a_{k,s} \psi_{k,s}^+ + b_{k,s}^\dagger \psi_{k,s}^-),
\]

(5)
where $a_i$ ($b_i^\dagger$) is the annihilation (creation) for the particle/antiparticle, $k$ is the momentum, and $s = \{ \uparrow, \downarrow \}$ is spin-up or spin-down along the quantization axis ($z$). The Minkowski states for 2 modes are $\{|0_k\rangle^\pm, \|\uparrow_k\rangle^\pm, |\downarrow_k\rangle^\pm, |\downarrow_k\rangle^\pm\}$.

In this case, the Dirac field in Rindler coordinates can be expanded as:

$$\psi = \sum_s \int d^3k (c_{k,s}^I \psi_{k,s}^I + d_{k,s}^{II} \psi_{k,s}^{II} + c_{k,s}^{I\dagger} \psi_{k,s}^{I\dagger} + d_{k,s}^{I\dagger} \psi_{k,s}^{I\dagger})$$

where $(c_{k,s}^I, d_{k,s}^{II})$ are annihilation for partial antiparticle operators corresponding region $i = I, II$. The relationship between Minkowski and Rindler particle/antiparticle operators is defined by Bogoliubov transformations as [49]:

$$a_{k,s} = \cos rc_{k,s}^I - e^{i\phi} \sin rd_{k,s}^{II\dagger}$$

$$b_{k,s}^I = \cos rd_{k,s}^{II\dagger} + e^{-i\phi} \sin rc_{k,s}^I$$

where $\phi$ a phase factor, which can be neglected, and $r = e^{-\pi k_0 c/\hbar}$ with $k_0$ is the single mode of Minkowski frame, which being transformed and $c$ is the speed of light.

Under the single mode approximation and applying Bogoliubov transformations, one can express the computational basis $\{|0, 1_t, 2_t\}$ in terms of Rindler space as [50, 51]:

$$|0_t\rangle = \cos^2 r_t |0\rangle |0\rangle_{II} + \cos r_t \sin r_t (|1\rangle |2\rangle_{II} + |2\rangle |1\rangle_{II}) + \sin^2 r_t |\uparrow\downarrow\rangle_{II},$$

$$|1_t\rangle = \cos r_t |1\rangle |0\rangle_{II} + \sin r_t |\uparrow\downarrow\rangle_{II},$$

$$|2_t\rangle = \cos r_t |2\rangle |0\rangle_{II} - \sin r_t |\uparrow\downarrow\rangle_{II},$$

where $|\uparrow\downarrow\rangle$ is pair state and $r_t \in [0, \pi/4]$ is the acceleration parameter of the qutrit. Likewise, the orthonormal basis $\{|0_q, 1_q\}$ of qubit in Minkowski is transformed in Rindler space as [49, 52]:

$$|0_q\rangle = \cos r_q |0_q\rangle |0_q\rangle_{II} + \sin r_q |1_q\rangle |1_q\rangle_{II}, \quad |1_q\rangle = |1_q\rangle |0_k\rangle_{II},$$

where $r_q \in [0, \pi/4]$ is the acceleration parameter of the qubit.

Subsequently, we consider that either the qubit and the qutrit are accelerated individually or the bipartite system is accelerated simultaneously. The output accelerated systems of the three cases in region $I$ may be written as

$$\rho_{acc}^{q, t, q t} = \rho_{11}^{q, t, q t} |00\rangle \langle 00| + \rho_{22}^{q, t, q t} |01\rangle \langle 01| + \rho_{33}^{q, t, q t} |02\rangle \langle 02|

+ \rho_{44}^{q, t, q t} |10\rangle \langle 10| + \rho_{55}^{q, t, q t} |11\rangle \langle 11| + \rho_{66}^{q, t, q t} |12\rangle \langle 12|

+ \rho_{77}^{q, t, q t} |\uparrow\downarrow\rangle \langle 0 | \uparrow\downarrow + \rho_{88}^{q, t, q t} |1 | \uparrow\downarrow \langle 1 | \uparrow\downarrow |

+ \rho_{34}^{q, t, q t} |02\rangle \langle 01| + \rho_{45}^{q, t, q t} |00\rangle \langle 12|

+ \rho_{56}^{q, t, q t} |11\rangle \langle 0 | \uparrow\downarrow + \rho_{67}^{q, t, q t} |01\rangle \langle 1 | \uparrow\downarrow + h.c.$$

where the superscripts $q, t, qt$ indicate to the three cases of accelerated qubit, qutrit, and qubit–qutrit system, respectively. The nonzero elements when only the qubit system is accelerated in Eq.(11) are given by
\[
\begin{align*}
\rho_{11}^q &= \frac{p}{2} c^2 = \rho_{22}^q, \quad \rho_{33}^q = \frac{1 - 2p}{2} c^2, \\
\rho_{44}^q &= \frac{p}{2} s^2 + \frac{1 - 2p}{2}, \quad \rho_{55}^q = \frac{p}{2} (s^2 + 1), \\
\rho_{66}^q &= \frac{1 - 2p}{2} s^2 + \frac{p}{2}, \quad \rho_{16}^q = \frac{p}{2} c = \rho_{61}^q, \\
\rho_{34}^q &= \frac{1 - 2p}{2} c = \rho_{43}^q, \\
\end{align*}
\]

where \( c = \cos r \), \( s = \sin r \). Likewise, the nonzero elements of the output accelerated system \( \rho_{\text{acc}}^t \) when only the qutrit is separately accelerated are obtained by

\[
\begin{align*}
\rho_{11}^t &= \frac{p}{2} c^4, \quad \rho_{22}^t = \frac{c^2}{2} (s^2 + 1), \\
\rho_{33}^t &= \frac{1}{2} (1 - 2p) c^4, \\
\rho_{44}^t &= \frac{c^2}{2} (ps^2 - 2p + 1), \\
\rho_{55}^t &= \frac{c^2}{2} ((1 - 2p)s^2 + p) = \rho_{66}^t, \quad \rho_{77}^t = \frac{s^2}{2} (ps^2 - p + 1), \\
\rho_{88}^t &= s^2 \left( \frac{1 - 2p}{2} s^2 + p \right), \\
\rho_{16}^t &= \frac{p}{2} c^3, \quad \rho_{28}^t = -\frac{p}{2} c s^2, \\
\rho_{34}^t &= \frac{1 - 2p}{2} c^3, \quad \rho_{57}^t = \frac{2p - 1}{2} c s^2. \\
\end{align*}
\]

Finally, the nonzero elements when the total system is accelerated are given by

\[
\begin{align*}
\rho_{11}^{qt} &= c^2 \rho_{11}^t, \quad \rho_{22}^{qt} = c^2 \rho_{22}^t, \quad \rho_{33}^{qt} = c^2 \rho_{33}^t, \quad \rho_{44}^{qt} = \rho_{55}^t + s^2 \rho_{11}^t, \\
\rho_{55}^{qt} &= \rho_{55}^t + s^2 \rho_{22}^t, \quad \rho_{66}^{qt} = \rho_{66}^t + s^2 \rho_{33}^t, \quad \rho_{77}^{qt} = c^2 \rho_{77}^t, \\
\rho_{88}^{qt} &= \rho_{88}^t + s^2 \rho_{77}^t, \quad \rho_{16}^{qt} = c \rho_{16}^t, \quad \rho_{28}^{qt} = c \rho_{28}^t, \\
\rho_{34}^{qt} &= c \rho_{34}^t, \quad \rho_{57}^{qt} = c \rho_{57}^t, \quad \rho_{34}^{qt} = c \rho_{34}^t. \\
\end{align*}
\]

Our motivation is to quantify the amount of the steerability from the qubit to the qutrit and vice versa.

### 3 Degree of decoherence

The degree of quantum decoherence has been constructed in terms of linear entropy and von Neumann entropy [53, 54]. Indeed, the quantum decoherence phenomenon means the off-diagonal elements which generate the coherence between quantum regimes in the density operator are terminated. The decoherence degree is dissipated automatically for the mixed and the pure states when they interact with their environment, where the pure states have a minimum value of decoherence (zero value). The degree
of quantum decoherence is based on linear entropy which is defined by

\[ D_{qt,q,t} = 1 - \text{Tr}[\rho_{qt,q,t}^2], \]

where \( qt, q, \) and \( t \) refer to the density operator of the qubit–qutrit, qubit, and qutrit states, respectively.

It is worth studying the amount of decoherence that arises from the mixture parameter \( p \) and the acceleration \( r \). Figure 1 displays the behavior of the decoherence for the non-accelerated/accelerated qubit–qutrit system. For the non-accelerated system, the behavior of the decoherence is depicted in Fig. 1a. It is clear that, at \( p = 0 \), the initial qubit–qutrit system \( \rho_{qt} \) is maximum entangled state and the system is a decoherence free. However, as \( p \) increases, the initial state \( \rho_{qt} \) loses its coherence due to the mixture of a non pure state. The decoherence increases gradually as the mixture parameter \( p \) increases. Also, the qubit state of this initial state, namely \( \rho^q = \text{Tr}_t(\rho_{qt}) = \frac{1}{2} I_{2 \times 2} \), is independent of the mixture parameter \( p \). Therefore, as it is displayed in Fig. 1a, there is no decoherence depicted on the initial qubit system. The initial qutrit system, namely \( \rho^t = \text{Tr}_q(\rho_{qt}) = p |1⟩⟨1| + \frac{1-p}{2} (|0⟩⟨0| + |2⟩⟨2|) \), depends on the mixture parameter \( p \). However, as \( p \) increases, the decoherence that displayed for the initial qutrit increases gradually to reach its maximum values at \( p = 0.5 \).

Figure 1b displays the behavior of the decoherence on the total initial system and its composite systems, where it is assumed that only the qubit is accelerated and we set the mixing parameter \( p = 0.1 \). As it is displayed from this figure, at \( r = 0 \), the predicted decoherence is due to the mixing parameter. However, the decoherence of the total system \( \rho_{qt} \) increases gradually as \( r \) increases, where the maximum decoherence is
depicted at large acceleration, i.e., \( r \approx 0.8 \). Similarly, the decoherence of the qutrit system is only due to the mixing parameter, where its value does not change as the qubit is accelerated. The effect of the acceleration on the degree of decoherence is displayed on the accelerated qubit, where for \( r > 0.4 \), the decoherence decreases.

The behavior of the decoherence for the three states when only the qutrit is accelerated is shown in Fig. 1c. It is clear the decoherence on the qutrit system that due to the acceleration is depicted at \( r > 0.2 \). However, as \( r \) increases, the qubit system loses its coherence gradually. As it is expected, the accelerating process on the qutrit has no effect on the coherence degree of the qubit, where the displayed decoherence is only due to the mixing parameter. The coherent degree on the qubit–qutrit system is displayed clearly, where the decoherence of \( \rho^q_t \) is depicted at small accelerations and increases fast as \( r \) increases. Moreover, the upper bounds of this decoherence of the total state are smaller than that displayed for the qutrit for any \( r < 0.6 \). At further values of \( r > 0.6 \), the decoherence depicted for \( \rho^q_t \) is larger than that displayed for \( \rho^t \).

Figure 1d describes the effect of the accelerated qubit–qutrit on the decoherence of the initial total system and its composite states. It is clear that, at \( r = 0 \), the decoherence on the three states is due to the mixing parameter, where we set \( p = 0.1 \). However, the initial decoherence of the qutrit is the largest one, while it is the smallest one for the total state \( \rho^q_t \). As one increases the acceleration of the qubit and the qutrit, the decoherence of the total state and the qutrit increases gradually, while the degree of coherence of the qubit improves.

From Fig. 1, one may conclude that, for this suggested qubit–qutrit system, the accelerating process on the qubit improves its coherence. The decoherence due to the mixing parameter depicted for the qutrit is larger than that displayed for the total state. If the qutrit is accelerated, the total state and the reduced state of the qutrit lose their coherence as the acceleration increases.

### 4 Local quantum uncertainty

In this section, we investigate the local/nonlocal behavior of the accelerated initial qubit–qutrit state. For this aim, we consider the local quantum uncertainty, LQU as a predictor of the non-locality and quantifier of the amount of the non-classical correlations NCCs. If the operator \( O \) is an observable, we need to quantify the minimum amount of LQU in state \( \rho_{A,B} \). Thus, the LQU with respect to the qutrit \( A \) is defined as a minimum Wigner–Yanase skew information, which is given by [55]

\[
U_A(\rho_{A,B}) = \min_{O_A} \mathcal{I}(\rho_{A,B}, O_A),
\]

where \( \mathcal{I}(\rho_{A,B}, O_A) := \frac{1}{2} Tr[\rho_{A,B}, O_A]^2 \), and \( O_A := O_A \otimes I_b \). For \( 2 \otimes 3 \), the expression of LQU is written as

\[
U_A(\rho_{A,B}) = 1 - \max[\Gamma_1, \Gamma_2, \Gamma_3]
\]
where \( \Gamma_i \) are the eigenvalues of the \( 3 \times 3 \) symmetric matrix \( \Xi_{AB} \) with entries \( \Xi_{AB}^{ij} = Tr[\sqrt{\rho_{AB}}(S_i \otimes I_3)\sqrt{\rho_{AB}}(S_j \otimes I_3)] \), and \( S_i, S_j \) are the Pauli operators for the qubit system.

Figure 2 shows the behavior of LQU (\( \sqcap \)), where it is assumed that the qubit, qutrit, or both of them are accelerated. Figure 2a displays the behavior of \( U_A \) when only the qubit is accelerated and different values of the mixing parameter. It is clear that at \( p = 0 \), namely the initial state is maximally entangled state, the QLU decreases gradually as the acceleration increases. However, as one increases the value of the mixing parameter, the decreasing rate of the LQU increases. At further values of \( p \), the decoherence due to the acceleration is very small, where LQU is almost invariant and the amount of quantum correlations is slightly affected as \( r \) increases. In Fig. 2b, we investigate the effect of the accelerated qutrit on the behavior of the local quantum uncertainty. The behavior is similar to that displayed in Fig. 2a, but the decreasing rate of QLU is slightly smaller than that displayed in Fig. 2a and it faintly decreases as the acceleration increases. Moreover, as it is displayed in Fig. 2c, the local quantum uncertainty increases as the acceleration increases at large values of the mixing parameter.

5 Degree of steerability

In this section, we investigate the steerability from the local qubit to the local qutrit in the absence or presence of the acceleration. In terms of the steering inequality (1), the amount of steering from the local qubit system (A) to the local qutrit system (B) can be identified via violating of the following inequality

\[
S_{AB} = H(S_x^B | S_x^A) + H(S_y^B | S_y^A) + H(S_z^B | S_z^A) \geq 3. \tag{18}
\]

Hence, the steerability by the qubit’s measurements to render \( S_{AB} \in [0, 1] \) is expressed by

\[
S^{A\rightarrow B} = \max \left\{ 0, \frac{S_{AB} - 3}{S_{\text{max}} - 3} \right\}, \tag{19}
\]
where the $S_{\text{max}} = 4$. In this case, the denominator is a normalization factor. Also the inequality of steering from B to A is given by

$$S_{BA} = H(S_A^x | S_x^B) + H(S_A^y | S_y^B) + H(S_A^z | S_z^B) \geq 2.$$  

(20)

Likewise, the steerability by the qutrit’s measurements to make $S_{BA} \in [0, 1]$ is defined by

$$S^{B \rightarrow A} = \max \left\{ 0, \frac{S_{BA} - 2}{S_{\text{max}} - 2} \right\},$$

(21)

with $S_{\text{max}} = 3$. In the computational basis $\{|0\rangle, |1\rangle\}$, and $\{|1\rangle, |0\rangle, |2\rangle\}$, the operators $\hat{S}_i^{AB}, i = x, y, z$ that describe the local qubit and qutrit for the users A and B, respectively, are defined by [56],

$$\hat{S}_x^A = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \hat{S}_x^B = -i(|1\rangle\langle 1| - |1\rangle\langle 0|),$$

$$\hat{S}_y^A = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad \hat{S}_y^B = -i(|1\rangle\langle 2| - |2\rangle\langle 1|),$$

$$\hat{S}_z^A = |0\rangle\langle 0| + |1\rangle\langle 1|, \quad \hat{S}_z^B = -i(|0\rangle\langle 1| - |1\rangle\langle 0|).$$

(22)

According to Eqs. (18) and (11), the inequality of the steering from the qubit to the qutrit is given by

$$\mathcal{I}_{AB} = (1 - a) \log_2(1 - a) + a \log_2 a + b \log_2 b + \frac{c^+}{2} \log_2(c^+) + \frac{c^-}{2} \log_2(c^-) + \log_2(\rho_{11}^{q,t,qt} + \rho_{22}^{q,t,qt}) \log_2 2^5(\rho_{11}^{q,t,qt} + \rho_{22}^{q,t,qt}) + \rho_{33}^{q,t,qt} \log_2 2^5 \rho_{33}^{q,t,qt} + \rho_{66}^{q,t,qt} \log_2 2^5 \rho_{66}^{q,t,qt} + \log_2 \rho_{44}^{q,t,qt} + \rho_{55}^{q,t,qt} \log_2 2^5(\rho_{44}^{q,t,qt} + \rho_{55}^{q,t,qt}) - \frac{d^-}{2} \log_2(d^-) - \frac{d^+}{2} \log_2(d^+) \leq 3$$

(23)

where

$$a = \rho_{11}^{q,t,qt} + \rho_{44}^{q,t,qt}, \quad b = \rho_{22}^{q,t,qt} + \rho_{55}^{q,t,qt}, \quad c^\pm = 1 - b \pm 2(\rho_{16}^{q,t,qt} + \rho_{34}^{q,t,qt}),$$

and $d^\pm = 1 \pm (\rho_{11}^{q,t,qt} + \rho_{22}^{q,t,qt} + \rho_{33}^{q,t,qt} - \rho_{44}^{q,t,qt} - \rho_{55}^{q,t,qt} - \rho_{66}^{q,t,qt})$.

On the other hand, the steering inequality from the qutrit to qubit is given by

$$\mathcal{I}_{BA} = \frac{c^+}{2} \log_2(c^+) + \frac{c^-}{2} \log_2(c^-) + (\rho_{11}^{q,t,qt} + \rho_{22}^{q,t,qt}) \log_2 4(\rho_{11}^{q,t,qt} + \rho_{22}^{q,t,qt}) + \rho_{33}^{q,t,qt} \log_2 4 \rho_{33}^{q,t,qt} + \log_2 4(\rho_{44}^{q,t,qt} + \rho_{55}^{q,t,qt}) + \rho_{66}^{q,t,qt} \log_2 4 \rho_{66}^{q,t,qt} + \log_2 4 \rho_{44}^{q,t,qt} \log_2 4(\rho_{44}^{q,t,qt} + \rho_{55}^{q,t,qt}) + \rho_{55}^{q,t,qt} - (1 - b) \log_2(1 - b) - (1 - g) \log_2(1 - g) - g \log_2 g \leq 2,$$

(24)
Fig. 3 steering inequality if the qubit is accelerated. a steering from qubit to the qutrit $I_{AB}$, b steering from the qutrit to the qubit $I_{BA}$. c, d the same as a, b, respectively, but the qubit is accelerated. e, f the same as a, b, respectively, but the qubit and the qutrit are accelerated.

with $g = \rho_{33}^{t.q.l.} + \rho_{66}^{q.t.q.l}$. Hereinafter, we introduce a comparative study for the steerability between the two subsystems.

The possibility that either the qubit or the qubit steers each other is displayed in Fig. 3, for different initial states. In general, the steering inequality decreases gradually as the acceleration parameter increases. The decreasing rate depends on the initial settings of the accelerated state. The behavior of the steering inequalities shows the smallest decay depicted for the maximum entangled state, i.e., the mixing parameter $p = 0$. As one increases the mixing parameter, namely increasing the degree of decoherence, the steering inequality decreases fast as the acceleration increases. As it is displayed in Fig. 3a–d, the size of the accelerated particle (qubit/qutrit) has a remarkable effect on the behavior of the steering inequalities. It is shown that if the qutrit is accelerated, the depicted decreasing rate is larger than that displayed if the qubit is accelerated. Moreover, these inequalities decrease faster when both subsystems (qubit and the qutrit) are accelerated. On the other hand, the steering of qutrit via the qubit and vice versa depends on which object is accelerated.

The degree of steerability that the qubit steers the qutrit and vice versa is displayed in Fig. 4, where different values of the mixing parameter are considered, and only the qubit is accelerated. This figure shows that the degree of steerability decreases as the degree of decoherence increases; namely, either $p$ or $r$ increases. The possibility that
the qubit steers the qutrit, $S_{AB}$ is larger than that displayed for $S_{BA}$. The steerability that the qubit steers the qutrit vanishes at large values of $r$ compared with that for $S_{BA}$. The difference between the degree of steerabilities $|S_{AB} - S_{BA}|$ increases as the mixing and the accelerating parameters increase.

Figure 5 displays the behavior of the steerability degrees $S_{AB}$ and $S_{BA}$, when only the qutrit is accelerated. The behavior is similar to that displayed in Fig. 4, where the degree of steerabilities decreases as the decoherence parameters increase. However, the possibility that the qubit steers the qutrit is larger than that shown when the qutrit steers the qubit, namely $S_{AB} > S_{BA}$. Moreover, the sudden decreasing phenomenon of the steerabilities is depicted, while the gradual decay is displayed in Fig. 4. The steerability from the qutrit to the qubit vanishes at smaller values of the acceleration parameter. The difference between the degree of steerability $|S_{AB} - S_{BA}|$ is larger than that displayed in Fig. 4, where only the qutrit is accelerated.

From Figs. 4 and 5, it is clear that if the qubit or the qutrit is accelerated, the $S_{AB} > S_{BA}$. The possibility that the qubit steers the qutrit survives at large values of the acceleration, while that depicted for the qutrit vanishes at small accelerations. The degree of steerability decreases as the mixing parameter increases.

In this context, it is important to investigate the degree of steerability as a function of the initial state settings, namely $S(p)$. Figure 6 displays the behavior of the steerability for different cases, where the phenomena of the sudden death is depicted for all cases. However, if the particles are initially in a rest, the steerability vanishes at large values of $p$, while in the presence of acceleration the steerability vanishes at smaller values of $p$. Moreover, the steerability $S^{A\rightarrow B}$ vanishes at large value compared with $S^{B\rightarrow A}$. 

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**Fig. 4** Steerability if the qubit is accelerated, $S_{AB}$ (green curve), $S_{BA}$ (red curve), and $|S_{AB} - S_{BA}|$ (blue curve) a $p = 0$, b $p = 0.01$, and c $p = 0.05$ (Color figure online)

**Fig. 5** Same as Fig. 4, but only the qutrit is accelerated
Fig. 6 Steerability $S^{A\to B}$ (green curve), $S^{B\to A}$ (red curve), and $|S^{B\to A} - S^{A\to B}|$ (blue curve) a the qubit and the qutrit are rested $r = 0$, b the qubit is accelerated with $r = 0.2$, c the qutrit is accelerated with $r = 0.2$, d the qubit and the qutrit are accelerated with $r = 0.2$ (Color figure online)

6 Conclusion

In this article, we investigate the steering process for an accelerated qubit–qutrit system. The initial system depends on a mixing parameter, which is considered as a decoherence parameter. Due to the acceleration process, there is an additional decoherence depending on which subsystem is accelerated. Therefore, we quantify the decoherence rate that rises from the initial state settings and from the acceleration process. Moreover, the non-locality, as well as the amount of quantum correlations, is quantified by using the local quantum uncertainty. Finally, we discuss the possibility that the qubit/qutrit steers each other besides we quantify the degree of steerability.

Our results show that, at the absences of the mixing parameter, namely the initial state is maximally entangled state, the purity of the qubit–qutrit system is maximum. However, as one increases the mixing parameter, the decoherence of the total qubit–qutrit system and the qutrit subsystems increase gradually. On the other hand, the decoherence could be improved if the qubit is accelerated with large accelerations. Moreover, the decoherence arises from the acceleration process, where it increases as the acceleration increases. The decoherence rate depends on the accelerated subsystem, where the decay rate that is displayed when the qutrit is accelerated is larger than that displayed if the qubit is accelerated.

Due to the decoherence, the accelerated qubit–qutrit system loses its coherence, and consequently, the amount of the non-classical correlation decreases. The local quantum uncertainty is used to quantify the quantum correlations, where at small values of the mixing parameter, it decreases as the acceleration increases. However, the amount of quantum correlations increases at large values of the mixing and acceleration parameters. This behavior of these correlations is exhibited clearly when both subsystems are accelerated.

The possibility that each subsystem steers the other is investigated at different initial state settings. It is clear that the steerability decreases as the mixing and acceleration.
parameters increase. The predicted steerability that the small size subsystem steers the large size subsystem is larger than that displayed for the large subsystem steers the small subsystem. The degree of steerability decreases gradually if only the qubit is accelerated, while it vanishes suddenly, when the qutrit or both subsystems are accelerated. Moreover, the qubit has the ability to steer the qutrit at large accelerations. Furthermore, when the qutrit is accelerated, the degree of steerability vanishes at small accelerations. The difference between the degrees of steerability increases when the qutrit or both subsystems are accelerated.

Finally, for this family of the initial state, the acceleration process improves the purity of the initial state and its subsystems. The steerability between different sizes of accelerated subsystems is possible. The degree of steerability depends on the initial state settings and the values of the acceleration parameter. Moreover, the phenomena of steerability’s sudden death is displayed at small values of the initial state settings and the speed of vanishing depends on whether the particles are accelerated or at rest.

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