Diagonalization of the elliptic Macdonald-Ruijsenaars difference system of type $C_2$

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Abstract

We study a pair of commuting difference operators arising from the elliptic solution of the dynamical Yang-Baxter equation of type $C_2$. The operators act on the space of meromorphic functions on the weight space of $\mathfrak{sp}(4, \mathbb{C})$. We show that these operators can be identified with the system by van Diejen and by Komori-Hikami with special parameters. It turns out that our case can be related to the difference Lamé operator (two-body Ruijsenaars operator) and thereby we diagonalize the system on the finite dimensional space spanned by the level one characters of the $C_2^{(1)}$-affine Lie algebra.

1 Introduction

The Ruijsenaars system of difference operators [R], are difference analogue of the Calogero-Moser integrable system of differential operators. The operators of the system is defined in terms of elliptic function, and in the trigonometric limit, they degenerate to the Macdonald $q$-difference operators [M]. The Ruijsenaars system has been studied extensively. Especially, Hasegawa shows that this system can be obtained as transfer matrices associated to the Sklyanin algebra [H3] and Felder-Varchenko reconstructed them as transfer matrices associated to the dynamical $R$-matrices [FV2]. These two approaches are related by the vertex-IRF correspondence [Bax] [JMO1].

Extending these works, in [HIK] we construct a pair of commuting difference operators acting on the space of functions on the $C_2$ type weight space. The method therein is based on the elliptic solution of the dynamical Yang-Baxter equation of type $C_2$ (or Boltzmann weights of the $C_2^{(1)}$ face model [JMO2]). We have also shown that the space spanned by the level one characters of the affine Lie algebra $\mathfrak{sp}(4, \mathbb{C})$ is invariant under the action of the difference operators.

On the other hand, a generalization of the Ruijsenaars system to $BC_n$ case is studied by van Diejen [vD1] and Komori-Hikami [KH1] [KH2]. First, van Diejen constructed two elliptic commuting operators, one is of the first order and the other is of the $n$-th order. Therefore he obtained an elliptic extension of difference Calogero-Moser system of type $BC_2$ [vD2]. Extending this work by van Diejen, Komori and Hikami obtained a general family of $n$ commuting difference operators with elliptic function coefficients. Besides the step parameter of difference operator and the modulus of elliptic functions, the family contains nine arbitrary parameters. Their construction uses Shibukawa-Ueno’s elliptic $R$-operator [SU] together with the elliptic $K$-operators [KH3] [KH4], the elliptic solution to the reflection equation. It can be regarded as an elliptic generalization of Dunkl type operator approach to Macdonald systems, which have been extensively used by Cherednik [Ch] (see [N] for $BC_n$ case).

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This paper has two goals. One is to establish the relationship between our system of difference operators and the van Diejen-Komori-Hikami system. The other is to diagonalize our difference operators on the finite dimensional space spanned by theta functions. The first goal is attained in section 2 and second is in section 3.

In section 2 we review the construction of the elliptic difference system of type $C_2$ and give a new form of our operators. After this, we will establish an identity consisting of theta functions (Lemma 1), and explain how our system can be identified with from van Diejen-Komori-Hikami system with special choice of parameters (Theorem 2). That is, our approach to the difference operators as transfer matrices, based on the knowledge of the Boltzmann weights, reproduces a special case among the family of commuting operators obtained by Dunkl-type approach. It should be also mentioned that those two approaches to the system are not yet related, although the resulting commuting operators have the relationship as above.

In section 3, we introduce the finite dimensional space of theta functions invariant under the action of Weyl group and its basis after Kac-Peterson [KP]. Our aim is to diagonalize our operators on this space (Theorem 4). This is an elliptic analogue of the eigenvalue problem of Macdonald operators on the space of symmetric polynomials. Their eigenfunctions, called Macdonald-Koornwinder polynomials, are much investigated in $q$-orthogonal polynomial theory [K] [N].

2 The difference operators of type $C_2$

2.1 Construction of the difference operators of type $C_2$

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}(4, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. We realize the root system $R$ for $(\mathfrak{g}, \mathfrak{h})$ as $R := \{\pm (\varepsilon_1 \pm \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\} \subset \mathfrak{h}^*$. A normalized Killing form $\langle , \rangle$ is given by

\[ \langle \varepsilon_j, \varepsilon_k \rangle = \frac{1}{2} \delta_{jk}, \]  

(2.1)

and the square length of the long roots $\pm 2\varepsilon_1$ is two. We will identify the space $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via the form $\langle , \rangle$. The fundamental weights are given by $\Lambda_1 = \varepsilon_1, \Lambda_2 = \varepsilon_1 + \varepsilon_2$. Let $P_d$ be the set of weights for the fundamental representation $L(A_d)$. We have

\[ P_1 = \{\pm \varepsilon_1, \pm \varepsilon_2\}, \quad P_2 = \{\pm (\varepsilon_1 \pm \varepsilon_2), 0\}. \]  

(2.2)

Let $d, d'$ be 1 or 2. The $C_2^{(1)}$ type Boltzmann weights of type $(d, d')$ are given as follows. Fix a complex parameter $\hbar \in \mathbb{C}$. For any pair $\lambda, \mu, \nu, \kappa \in \mathfrak{h}^*$ of weights, the Boltzmann weight

\[ W_{dd'} \left( \begin{array}{c} \lambda \\ \kappa \\ \mu \\ \nu \end{array} \middle| u \right) \]

is a function of the spectral parameter $u \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$. They satisfy the condition

\[ W_{dd'} \left( \begin{array}{c} \lambda \\ \kappa \\ \mu \\ \nu \end{array} \middle| u \right) = 0 \quad \text{unless} \quad \mu - \lambda, \nu - \kappa \in 2hP_d, \quad \kappa - \lambda, \nu - \mu \in 2hP_d', \]

and solve the Yang-Baxter equation of the face type,

\[
\sum_{\eta} W_{dd'} \left( \begin{array}{c} \rho \\ \sigma \\ \eta \\ \kappa \end{array} \middle| u - v \right) W_{dd''} \left( \begin{array}{c} \lambda \\ \rho \\ \mu \\ \eta \end{array} \middle| u - w \right) W_{d'd''} \left( \begin{array}{c} \mu \\ \kappa \end{array} \middle| v - w \right) \\
= \sum_{\eta} W_{d'd''} \left( \begin{array}{c} \lambda \\ \eta \\ \nu \\ \kappa \end{array} \middle| v - w \right) W_{dd''} \left( \begin{array}{c} \eta \\ \sigma \\ \nu \\ \kappa \end{array} \middle| u - w \right) W_{dd'} \left( \begin{array}{c} \lambda \\ \eta \\ \mu \\ \nu \end{array} \middle| u - v \right), \]  

(2.3)
This equation is also known as the dynamical Yang-Baxter equation. Here we give the explicit formula for $W_{11}$ and see [HKK] for the other type $W_{dd'}$ ($(d, d') = (1, 2), (2, 1), (2, 2)$) which are obtained by fusion procedure. They are expressed by the Jacobi theta function $\theta_1(u) = \theta_1(u|\tau)$ with elliptic modulus $\tau$ in the upper half plane $\mathcal{H}_+$. (See Appendix B for the definition of $\theta_1(u)$). For $p, q, r, s \in \mathcal{P}$ such that $p + q = r + s$, we will write

$$s \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{p} \end{array} = W_{11} \left( \begin{array}{cc} 1 & \lambda + 2hp \\ \lambda + 2hs & \lambda + 2h(p + q) \end{array} \right) u.$$  

The explicit formula for $W_{11}$ is given as follows:

$$p \begin{array}{l} \text{p} \end{array} \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{p} \end{array} = \frac{\theta_1(c - u) \theta_1(u + h)}{\theta_1(c) \theta_1(h)},$$  

$$p \begin{array}{l} \text{p} \end{array} \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{q} \end{array} = \frac{\theta_1(c - u) \theta_1(\lambda_{p-q} - u)}{\theta_1(c) \theta_1(\lambda_{p-q})} (p \neq \pm q),$$  

$$p \begin{array}{l} \text{q} \end{array} \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{p} \end{array} = \frac{\theta_1(c - u) \theta_1(u) \theta_1(\lambda_{p-q} + h)}{\theta_1(c) \theta_1(h) \theta_1(\lambda_{p-q})} (p \neq \pm q),$$  

$$p \begin{array}{l} \text{q} \end{array} \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{q} \end{array} = -\frac{\theta_1(u) \theta_1(\lambda_{p+q} + h) \theta_1(\lambda_{p+q} + h) \prod_{r \neq \pm p} \theta_1(\lambda_{p+r} + h)}{u \theta_1(2\lambda_p)} \prod_{r \neq \pm q} \theta_1(\lambda_{q+r}) (p \neq q),$$  

$$p \begin{array}{l} \text{q} \end{array} \begin{array}{l} \text{s} \end{array} \begin{array}{l} \text{u} \end{array} \begin{array}{l} \text{r} \end{array} q \begin{array}{l} \text{p} \end{array} = \frac{\theta_1(c - u) \theta_1(2\lambda_p + h - u)}{\theta_1(c) \theta_1(2\lambda_p + h)}$$  

$$- \frac{\theta_1(u) \theta_1(2\lambda_p + h + c - u) \theta_1(2\lambda_p + h) \prod_{q \neq \pm p} \theta_1(\lambda_{p+q} + h)}{\theta_1(2\lambda_p)} \prod_{q \neq \pm q} \theta_1(\lambda_{q+r}).$$

Here the crossing parameter $c$ is fixed to be $c := -3h$.

We define the difference operators $M_d(u)$ $(u \in \mathbb{C}, d = 1, 2)$ acting on the functions on $\mathfrak{h}$ by means of the Boltzmann weights of type $(1, 2)$ and $(2, 2)$.

$$(M_d(u)f)(\lambda) := \sum_{p \in \mathcal{P}_d} W_{d2} \left( \begin{array}{cc} \lambda & \lambda + 2hp \\ \lambda + 2hp & \lambda + 2h(p + q) \end{array} \right) T^h_{2p} f(\lambda).$$

Here the shift operator $T^h_{2p}$ is defined as

$$T^h_{2p} f(\lambda) := f(\lambda + 2hp).$$

For $\lambda \in \mathfrak{h}^*$ and $p \in \mathcal{P}_d (d = 1, 2)$, we put

$$\lambda_p := (\lambda, p).$$

Note that if we denote $\lambda_i = (\lambda, \varepsilon_i) (i = 1, 2)$ and $f(\lambda) = f(\lambda_1, \lambda_2)$, then

$$T^h_{\pm 2\varepsilon_1} f(\lambda_1, \lambda_2) = f(\lambda_1 \pm h, \lambda_2), \quad T^h_{\pm 2\varepsilon_2} f(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2 \pm h).$$
Theorem 1 [HIK]

(i) For each \( u, v \in \mathbb{C} \), we have \( M_d(u)M_{d'}(v) = M_{d'}(v)M_d(u) \) \( (d, d' = 1, 2) \).

(ii) The explicit form of \( M_d(u) \) are as follows:

\[
M_1(u) = F(u) \sum_{p \in P_1} \prod_{q \neq p \in P_1} \frac{\theta_1(\lambda_{p+q} - h)T_2}{\theta_1(\lambda_{p+q})T_2}, \tag{2.9}
\]

\[
M_2(u) = G(u) \left( \sum_{p \neq \pm \epsilon_1} \frac{\theta_1(\lambda_{p+q} - h)T_2}{\theta_1(\lambda_{p+q} + h)}T_2 + U(\lambda_p, \lambda_q) - H(u) \right). \tag{2.10}
\]

Here \( U(\lambda_p, \lambda_q) \) is given by:

\[
U(\lambda_p, \lambda_q) = \frac{\theta_1(2h) \theta_1(2\lambda_p + 2h) \theta_1(2\lambda_q + 2h) \theta_1(\lambda_{p+q} - 5h) \theta_1(\lambda_{p+q} + 2h)}{\theta_1(6h) \theta_1(2\lambda_p) \theta_1(2\lambda_q) \theta_1(\lambda_{p+q}) \theta_1(\lambda_{p+q} + h)},
\]

and \( F(u), G(u), H(u) \) are the following functions depend only on \( u \) and \( h \):

\[
F(u) := \frac{\theta_1(u) \theta_1(u + 2h)^2 \theta_1(u + 4h)}{\theta_1(-3h)^2 \theta_1(h)^2},
\]

\[
G(u) := \frac{\theta_1(u - h) \theta_1(u)^2 \theta_1(u + h) \theta_1(u + 2h) \theta_1(u + 3h)^2 \theta_1(u + 4h)}{\theta_1(-3h)^4 \theta_1(h)^4},
\]

and

\[
H(u) := \frac{\theta_1(u + 6h) \theta_1(u - 3h) \theta_1(2h)}{\theta_1(u) \theta_1(u + 3h) \theta_1(6h)}.
\]

The following Lemma is the key for the identification with van Diejen’s system as well as for the diagonalization of our difference operators. The author is grateful to van Diejen for the information.

Lemma 1 We have

\[
\sum_{p = \pm \epsilon_1, q = \pm \epsilon_2} U(\lambda_p, \lambda_q) - \sum_{p = \pm \epsilon_1, q = \pm \epsilon_2} \frac{\theta_1(\lambda_{p+q} - h) \theta_1(\lambda_{p+q} + 2h)}{\theta_1(\lambda_{p+q}) \theta_1(\lambda_{p+q} + h)} = K, \tag{2.11}
\]

where \( K \) is a constant given by

\[
K = \frac{\theta_1(8h) \theta_1(h)}{\theta_1(6h) \theta_1(5h)} + \frac{\theta_1(5h) \theta_1(2h)}{\theta_1(4h) \theta_1(3h)} + \frac{\theta_1(6h) \theta_1(3h)}{\theta_1(5h) \theta_1(4h)} + \frac{\theta_1(4h) \theta_1(h)}{\theta_1(3h) \theta_1(2h)}.
\]

Proof. Let \( f(\lambda_p) \) be the left-hand side of (2.11), regarded as a function of \( \lambda_p \) \( (p \in I) \). It is doubly periodic function of the periods 1, \( \tau \). Let us show that it is entire. The apparent poles of \( f(\lambda_p) \) are located at

\[
\lambda_p = \lambda_q, \lambda_p = \lambda_q - h \ (p, q \in I, p + q \neq 0), \lambda_p = 0 \ (p \in I).
\]

Note that \( f(\lambda_p) \) is \( W \)-invariant, then the points \( \lambda_p = \lambda_q \) and \( \lambda_p = 0 \) are regular. Also, the residue of \( f(\lambda_p) \) at \( \lambda_p = -\lambda_q - h \) is

\[
\frac{\theta_1(2h) \theta_1(-2\lambda_q + 2\lambda_p + 2h) \theta_1(-6h) \theta_1(h)}{\theta_1(6h) \theta_1(-2\lambda_q - 2h) \theta_1(2\lambda_q) \theta_1(-h)} - \frac{\theta_1(-2h) \theta_1(h)}{\theta_1(-h)} = 0.
\]
Now we have proved that \( f(\lambda_p) \) is independent of \( \lambda_p \), then we consider \( g(\lambda_q) = f(-\lambda_q - 2\hbar) \) as a function of \( \lambda_q \) \((q \neq p \in I)\):

\[
g(\lambda_q) = \frac{\theta_1(2\hbar) (\theta_1(2\lambda_q + 2\hbar) \theta_1(2\lambda_q - 2\hbar) \theta_1(2\lambda_q + 7\hbar)}{\theta_1(6\hbar) \theta_1(2\lambda_q + 4\hbar) \theta_1(2\lambda_q + 2\hbar) \theta_1(2\lambda_q + 3\hbar)} + \frac{\theta_1(2\lambda_q + 6\hbar) \theta_1(2\lambda_q + 2\hbar) \theta_1(2\lambda_q - 3\hbar)}{\theta_1(2\lambda_q) \theta_1(2\lambda_q + 2\hbar) \theta_1(2\lambda_q + 3\hbar)} + \frac{\theta_1(2\lambda_q + 6\hbar) \theta_1(2\lambda_q - 2\hbar) \theta_1(-3\hbar) \theta_1(4\hbar)}{\theta_1(2\lambda_q) \theta_1(2\lambda_q + 2\hbar) \theta_1(2\lambda_q + 3\hbar)} - \frac{\theta_1(2\lambda_q + h) \theta_1(2\lambda_q + 4\hbar)}{\theta_1(2\lambda_q + h) \theta_1(2\lambda_q + 3\hbar)} - \frac{\theta_1(h) \theta_1(4\hbar)}{\theta_1(2\lambda_q) \theta_1(2\lambda_q + 3\hbar)}.
\]

By the same argument we can show that \( g(\lambda_q) \) is independent of \( \lambda_q \). Therefore we get \( K \) by putting \( \lambda_q = h \) in \( g(\lambda_q) \) and the proof completes. \( \square \)

### 2.2 Identification with van Diejen’s system

We define the difference operators \( \widetilde{M}_d \) to be the components of \( M_d(u) \) independent of \( u \):

\[
\widetilde{M}_1 = \sum_{p \in \mathcal{P}_1 \, q \neq \pm p} \frac{\theta_1(\lambda_{p+q} - h) - \theta_1(\lambda_{p+q} + h)}{\theta_1(\lambda_{p+q})} T_{p,q}^h,
\]

\[
\widetilde{M}_2 = \sum_{p \in \mathcal{P}_1 \, q = \pm 2, \, r = \pm 1} \left( \frac{\theta_1(\lambda_{p+q} - h)}{\theta_1(\lambda_{p+q} + h)} T_{p,q}^h T_{r,p}^h + \frac{\theta_1(\lambda_{p+q} - h) \theta_1(\lambda_{p+q} + 2h)}{\theta_1(\lambda_{p+q})} \theta_1(\lambda_{p+q} + h) \right)
\]

More general commuting difference operators \( \mathcal{H}_1, \mathcal{H}_2 \) are obtained by van Diejen and later by Komori-Hikami in a different way. In this subsection we identify our operators \( \widetilde{M}_1, \widetilde{M}_2 \) as van Diejen’s system of difference operators with special values of parameters. The operators \( \mathcal{H}_1, \mathcal{H}_2 \) depend on nine complex parameters \( \mu, \mu_r, \mu_r' \) \((r = 0, 1, 2, 3)\) satisfying the condition

\[
\sum_r (\mu_r + \mu_r') = 0
\]

and are defined by

\[
\mathcal{H}_1 = \sum_{\varepsilon = \pm 1} w(\varepsilon x_1)v(\varepsilon x_1 + x_2)v(\varepsilon x_1 - x_2) T_{\varepsilon_1}^\gamma + \sum_{\varepsilon = \pm 1} w(\varepsilon x_2)v(\varepsilon x_2 + x_1)v(\varepsilon x_2 - x_1) T_{\varepsilon_2}^\gamma + U_{\{1,2\},1},
\]

\[
\mathcal{H}_2 = \sum_{\varepsilon, \varepsilon' = \pm 1} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1 + \varepsilon' x_2)v(\varepsilon x_1 + \varepsilon' x_2 + \gamma) T_{\varepsilon_1}^\gamma T_{\varepsilon_2}^\gamma + U_{\{2\},1} \sum_{\varepsilon = \pm 1} w(\varepsilon x_1)v(\varepsilon x_1 + x_2)v(\varepsilon x_1 - x_2) T_{\varepsilon_1}^\gamma + U_{\{1\},1} \sum_{\varepsilon = \pm 1} w(\varepsilon x_2)v(\varepsilon x_2 + x_1)v(\varepsilon x_2 - x_1) T_{\varepsilon_2}^\gamma + U_{\{1,2\},2}.
\]
Here $T_{\pm i}^\gamma$ ($i = 1, 2$) stand for the shift operators
\[ T_{\pm 1}^\gamma f(x_1, x_2) = f(x_1 \pm \gamma, x_2), \quad T_{\pm 2}^\gamma f(x_1, x_2) = f(x_1, x_2 \pm \gamma) \]
and
\[
v(z) := \frac{\sigma(z + \mu)}{\sigma(z)}, \quad w(z) := \prod_{0 \leq r \leq 3} \frac{\sigma_r(z + \mu_r) \sigma_r(z + \mu'_r + \gamma/2)}{\sigma_r(z) \sigma_r(z + \gamma/2)}, \tag{2.15}\]
where $\sigma(z) = \sigma_0(z)$ denotes the sigma function with two quasi periods $\omega_1$, $\omega_2$ and $\sigma_r(z)$ ($r = 1, 2, 3$) associated function obtained by shift of argument over the half periods (See Appendix B for more detail). The functions $U_{(j),1}, U_{(1,2),1}$ ($j = 1, 2$) are defined as follows:
\[
U_{(j),1} = -w(x_1) - w(-x_2) \quad (j = 1, 2),
\]

\[
U_{(1,2),1} = \sum_{0 \leq r \leq 3} c_r \prod_{j=1,2} \frac{\sigma_r(\mu - \gamma/2 + x_j) \sigma_r(\mu - \gamma/2 - x_j)}{\sigma_r(-\gamma/2 + x_j) \sigma_r(-\gamma/2 - x_j)},
\]
where
\[
c_r = \frac{2}{\sigma(\mu) \sigma(\mu - \gamma)} \prod_{0 \leq s \leq 3} \sigma_s(\mu_{\pi_r(s)} - \gamma/2) \sigma_s(\mu'_{\pi_r(s)}),
\]
with $\pi_r$ denoting the permutation $\pi_0 = id, \pi_1 = (01)(23), \pi_2 = (02)(13), \pi_3 = (03)(12)$.

\[
U_{(1,2),2} = \sum_{\varepsilon, \varepsilon' \in \{1,-1\}} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1 + \varepsilon' x_2)v(-\varepsilon x_1 - \varepsilon' x_2 - \gamma) \tag{2.16}
\]

We mention that the Komori-Hikami system in [KH2] is of more complicated form and has nine arbitrary parameters, that is, they removed the condition (2.14).

In $H_1, H_2$, we specialize parameters $\mu, \mu_r, \mu'_r$ ($r = 0, 1, 2, 3$) as $\mu = -\gamma$, $\mu_r = \mu'_r = 0$. Then $w(z) = 1$ and $U_{(1,2),1} = 0$. Let us denote these specialized operators by $\tilde{H}_1, \tilde{H}_2$. Because of these simplifications, we immediately obtain the following from Lemma 1, giving the identification of our system $\{\tilde{M}_1, \tilde{M}_2\}$ and van Diejen’s $\{\tilde{H}_1, \tilde{H}_2\}$.

**Theorem 2** For a function $f(\lambda_1, \lambda_2)$ on $\mathfrak{h}$, we set $\varphi(f)(x_1, x_2)$ by
\[
\varphi(f)(x_1, x_2) := \exp \frac{\eta_1(x_1^4 + x_2^4)}{\omega_1} f\left(\frac{x_1}{2\omega_1}, \frac{x_2}{2\omega_1}\right),
\]
and let $\gamma = 2\omega_1 h$, we have
\[
\varphi \tilde{M}_1 \varphi^{-1} = e^{2\pi \gamma^2 / \omega_1} \tilde{H}_1,
\]
\[
\varphi \tilde{M}_2 \varphi^{-1} = e^{2\pi \gamma^2 / \omega_1} (\tilde{H}_2 + 2\tilde{H}_1).
\]

*Proof.* Use the connection between the theta function and sigma function (B.8) in Appendix B and (2.11) to compare (2.10) and (2.16). \(\square\)
3 Diagonalization of the system

3.1 The space of theta functions

Let $Q$ and $Q^\vee$ be the root and coroot lattice, $P$ and $P^\vee$ the weight and coweight lattice respectively. Under the identification $\mathfrak{h} = \mathfrak{h}^*$ via the form $(\cdot, \cdot)$, they are given by

$$P = \sum_{j=1,2} \mathbb{Z}\varepsilon_j, \quad Q^\vee = \sum_{j=1,2} \mathbb{Z}2\varepsilon_j,$$

and

$$P^\vee = Q = \mathbb{Z}2\varepsilon_1 + \mathbb{Z}2\varepsilon_2 + \mathbb{Z}(\varepsilon_1 + \varepsilon_2).$$

For $\beta \in \mathfrak{h}^*$, we introduce the following operators $T_{\tau \beta}, T_{\beta}$ acting on the functions on $\mathfrak{h}^*$:

$$(T_{\tau \beta} f)(\lambda) := f(\lambda + \beta),$$

$$(T_{\beta} f)(\lambda) := \exp \left[ 2\pi i \left( (\lambda, \beta) + \frac{(\beta, \beta)}{2} \right) \right] f(\lambda + \tau \beta).$$

We define the space of theta functions (of level 1) by

$$Th_1 := \{ f \text{ is holomorphic on } \mathfrak{h}^* \mid T_{\tau \alpha} f = T_{\alpha} f = f \quad (\forall \alpha \in Q^\vee) \}.$$

For each $\mu \in P$ and fixed $\tau \in \mathfrak{h}_+$, we define the classical theta function $\Theta_\mu(\lambda)$ of $\lambda \in \mathfrak{h}^*$ by

$$\Theta_\mu(\lambda) := \sum_{\gamma \in \mu + Q^\vee} \exp \left[ 2\pi i \left( (\gamma, \lambda) + \frac{(\gamma, \gamma)}{2} \right) \right].$$

It is known that

$$\{ \Theta_\mu(\lambda) \mid \mu \equiv 0, \varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 \mod Q^\vee \}$$

gives a basis for $Th_1$ over $\mathbb{C}$ [KP].

Let $W \subset GL(\mathfrak{h}^*)$ denote the Weyl group for $(\mathfrak{g}, \mathfrak{h})$, and consider the $W$-invariants in $Th_1$:

$$Th_1^W := \{ f \in Th_1 \mid f(w\lambda) = f(\lambda) \quad (\forall w \in W) \}.$$

**Theorem 3** [HIK] The operators $\tilde{M}_1, \tilde{M}_2$ preserves $Th_1^W$.

For $\mu \in P$, we define $W_\mu := \{ w \in W \mid w\mu = \mu \}$ and introduce the following symmetric sum of theta functions,

$$S_\mu(\lambda) := \frac{1}{|W_\mu|} \sum_{w \in W} \Theta_{w(\mu)}(\lambda).$$

Then

$$\{ S_\mu(\lambda) \mid \mu \equiv 0, \Lambda_1 (= \varepsilon_1), \Lambda_2 (= \varepsilon_1 + \varepsilon_2) \mod Q^\vee \}$$

forms a basis for $Th_1^W$ over $\mathbb{C}$.

It is known that $Th_1^W$ is also spanned by the level 1 characters of the affine Lie algebra $\hat{\mathfrak{sp}}(4, \mathbb{C})$. Note that $\Theta_{-\mu}(\lambda) = \Theta_\mu(\lambda)$ and $\Theta_{\varepsilon_1+\varepsilon_2}(\lambda) = \Theta_{\varepsilon_1-\varepsilon_2}(\lambda)$. So that we have

$$S_0(\lambda) = \Theta_0(\lambda), \quad S_{\Lambda_1}(\lambda) = 2(\Theta_{\varepsilon_1}(\lambda) + \Theta_{\varepsilon_2}(\lambda)), \quad S_{\Lambda_2}(\lambda) = 4\Theta_{\varepsilon_1+\varepsilon_2}(\lambda).$$
3.2 Diagonalization of $\tilde{M}_d$

In this subsection, we diagonalize the operators $\tilde{M}_d$ on the space $Th^W$. We set
\[
f_1(\lambda) := \Theta_{\varepsilon_1}(\lambda) + \Theta_{\varepsilon_2}(\lambda), \quad f_2(\lambda) := \Theta_0(\lambda) + \Theta_{\varepsilon_1+\varepsilon_2}(\lambda) \quad \text{and} \quad f_3(\lambda) := \Theta_0(\lambda) - \Theta_{\varepsilon_1+\varepsilon_2}(\lambda).
\]

They are linearly independent in the space $Th^W$.

**Theorem 4** The functions $f_i(\lambda)$ $(i = 1, 2, 3)$ are common eigenfunctions of $\tilde{M}_d$:
\[
\tilde{M}_d f_i(\lambda) = E_{d,i} f_i(\lambda) \quad (d = 1, 2, \ i = 1, 2, 3).
\]

The eigenvalues are given by
\[
E_{1,i} = \left( \frac{\theta_i(2\hbar)\theta_{i+1}(0)}{\theta_i(\hbar)\theta_{i+1}(\hbar)} \right)^2
\]
and $E_{2,i} = 2E_{1,i}$, where the Jacobi theta functions $\theta_i(z) = \theta_i(z|\tau)$ $(i = 2, 3, 4)$ are defined as in Appendix B.

We will prove this theorem by using the following three lemmas. First, we show that the operators $\tilde{M}_d$ split into two $A_1$-type components.

**Lemma 2** Let us denote $\lambda_{\pm} := (\lambda, \varepsilon_1 \pm \varepsilon_2)$ and define
\[
H_{\pm} := \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} T^\hbar_{\varepsilon_1 \pm \varepsilon_2} + \frac{\theta_1(-\lambda_{\pm} - \hbar)}{\theta_1(-\lambda_{\pm})} T^\hbar_{\varepsilon_1\varepsilon_2}.
\]

Then we have
\[
\tilde{M}_1 = H_+ H_-, \quad \tilde{M}_2 = H_+^2 + H_-^2. \tag{3.2}
\]

**Proof.** To prove the first identity, we note that
\[
\frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} T^\hbar_{\varepsilon_1 + \varepsilon_2} T^\hbar_{\varepsilon_1 - \varepsilon_2} = \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} \frac{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))) - \hbar}{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))) - \hbar} T^\hbar_{\varepsilon_1 + \varepsilon_2} T^\hbar_{\varepsilon_1 - \varepsilon_2} = \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} T^\hbar_{2\varepsilon_1}.
\]

Here we used the identity $(\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2) = 0$. The second identity follows from, for instance,
\[
\frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} T^\hbar_{\varepsilon_1 + \varepsilon_2} T^\hbar_{\varepsilon_1 + \varepsilon_2} = \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} \frac{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))) + \hbar}{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))) + \hbar} T^\hbar_{\varepsilon_1 + \varepsilon_2} T^\hbar_{\varepsilon_1 + \varepsilon_2} = \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm})} \frac{\theta_1(\lambda_{\pm} + h)}{\theta_1(\lambda_{\pm} + h)} T^\hbar_{2\varepsilon_1} T^\hbar_{2\varepsilon_2} = \frac{\theta_1(\lambda_{\pm} - \hbar)}{\theta_1(\lambda_{\pm} + h)} T^\hbar_{2\varepsilon_1} T^\hbar_{2\varepsilon_2}.
\]
Here we used the identity \((\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2) = 1\). □

Second, we consider the eigenvalue problem for the \(A_1\)-type difference operator (difference Lamé or two-body Ruijsenaars operator)

\[
\frac{\theta_1(z - \ell \hbar)}{\theta_1(z)} f(z + \hbar) + \frac{\theta_1(z + \ell \hbar)}{\theta_1(z)} f(z - \hbar) = E f(z).
\] (3.3)

**Lemma 3** For the special coupling constant \(\ell = 1\), the functions

\[
\theta_i(z) \quad (i = 2, 3, 4)
\]

are solutions of the equation (3.3) with eigenvalues

\[
E = E_i = \frac{\theta_1(2\hbar) \theta_1(0)}{\theta_1(h) \theta_1(h)} \quad (i = 2, 3, 4).
\]

**Proof.** We note that the functions \(\theta_2(z), \theta_3(z), \) and \(\theta_4(z)\) can be rewritten in \(e^{cz}\theta_1(z + t)\) up to a constant, where

\[
(t, c) = \left(\frac{1}{2}, 0\right), \left(\frac{1 + \tau}{2}, \pi \imath\right) \text{ and } \left(\frac{\tau}{2}, \pi \imath\right)
\] (3.4)

respectively (See the formula (B.3) in Appendix B). Let \((t, c)\) be one of these, and we denote by \(g(z)\) the function which obtained by the action of the \(A_1\)-type difference operator (3.3) with \(\ell = 1\) to \(e^{cz}\theta_1(z + t)\): \[
g(z) := \frac{\theta_1(z - h) \theta_1(z + t + h)}{\theta_1(z)} e^{c(z + h)} + \frac{\theta_1(z + h) \theta_1(z + t - h)}{\theta_1(z)} e^{c(z - h)}.
\]

This is holomorphic and doubly quasi-periodic function:

\[
g(z + 1) = -e^{c}g(z), \quad g(z + \tau) = -e^{\pi i c - 2\pi i z + ct}g(z).
\]

Moreover, \(g(z) = 0\) at \(z = -t\). Therefore, \(g(z)\) is equal to \(e^{cz}\theta_1(z + t)\) up to a constant, which is the value of

\[
\frac{\theta_1(z - h) \theta_1(z + t + h)}{\theta_1(z)} e^{c h} + \frac{\theta_1(z + h) \theta_1(z + t - h)}{\theta_1(z)} e^{-c h}
\] (3.5)

at any chosen point. If we choose \(z = h\), then the first term in (3.5) vanishes and we have

\[
\frac{\theta_1(2h) \theta_1(t)}{\theta_1(h) \theta_1(h + t)} e^{-c h} = \frac{\theta_1(2h) \theta_1(0)}{\theta_1(h) \theta_1(h)},
\]

where \(i = 2, 3\) and \(4\) corresponding to the values of \((t, c)\) in (3.4), as an eigenvalue. □

**Remark.** This can be regarded as a special case of Felder-Varchenko’s study [FV1]. They expressed the solutions of (3.3) in terms of the algebraic Bethe Ansatz method, which is originally developed and applied to the spin chain model. In fact, the operator in the left hand side of (3.3) can be regarded as the transfer matrix of the simplest spin chain, that is, it consists of only one site of freedom with spin \(\ell = 1\). In this case, the Bethe Ansatz equation

\[
\frac{\theta_1(t - h)}{\theta_1(t + h)} = e^{2hc}
\] (3.6)
is exactly the same as the condition that the function \((3.5)\) dose not have pole at \(z = -t\).

Because of Lemma 3, the product of the theta functions
\[
\theta_i(\lambda_+) \theta_j(\lambda_-) \quad (i, j = 2, 3, 4)
\]
are simultaneous eigenfunctions of the operators \(H^2_+, H^2_+\) and \(H_+ H_-\) with eigenvalues
\[
\frac{\theta_i(2h)^2 \theta_i(0)^2}{\theta_i(h)^2 \theta_i(h)^2}, \quad \frac{\theta_j(2h)^2 \theta_j(0)^2}{\theta_j(h)^2 \theta_j(h)^2}, \quad \frac{\theta_i(2h)^2 \theta_i(0) \theta_j(0)}{\theta_i(h) \theta_j(h)}
\]
respectively. Finally, we shall establish the relationship of these Bethe Ansatz solutions and the bases of \(Th^W_1\).

**Lemma 4** The functions \(f_i(\lambda) \in Th^W_1\) are expressed in terms of the Jacobi theta functions as follows:
\[
f_1(\lambda) = \theta_2(\lambda_+) \theta_2(\lambda_-), \quad f_2(\lambda) = \theta_3(\lambda_+) \theta_3(\lambda_-), \quad f_3(\lambda) = \theta_4(\lambda_+) \theta_4(\lambda_-).
\]

**Proof.** Because of the definitions of coroot lattice \(Q^\vee (3.1)\) and Killing form \((2.1)\), each basis of \(Th_1\) is expressed as
\[
\Theta_0(\lambda) = \theta_3(2\lambda_1|2\tau) \theta_3(2\lambda_2|2\tau),
\Theta_{\varepsilon_1}(\lambda) = \theta_3(2\lambda_1|2\tau) \theta_2(2\lambda_2|2\tau),
\Theta_{\varepsilon_2}(\lambda) = \theta_2(2\lambda_1|2\tau) \theta_3(2\lambda_2|2\tau),
\Theta_{\varepsilon_1+\varepsilon_2}(\lambda) = \theta_2(2\lambda_1|2\tau) \theta_2(2\lambda_2|2\tau).
\]
Here \(\lambda_i = \lambda_{\varepsilon_i} \quad (i = 1, 2)\). Therefore we can prove this lemma by using the identities of theta functions (addition theorems) \((B.4), (B.5), (B.6), (B.7)\) in appendix. \(\square\)

We note that the anti-symmetric function \(\Theta_{\varepsilon_1}(\lambda) - \Theta_{\varepsilon_2}(\lambda) = \theta_1(\lambda_+) \theta_1(\lambda_-)\) is also the eigenfunction with eigenvalue zero.

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**Appendix A: Differential limit**

Let us clarify the connection between our system of difference operators and a quantization of the Inozemtsev Hamiltonian \([IM]\) \([I]\). By expanding in \(h\) one infers that
\[
\tilde{M}_1 = 4 + M_{1,2}h^2 + M_{1,4}h^4 + O(h^5),
\tilde{M}_2 = 8 + M_{2,2}h^2 + M_{2,4}h^4 + O(h^5).
\]
If we abbreviate a function \( f(\lambda_{1,2}) \) as \( f(\pm) \), \( \partial_1 = \frac{\partial}{\partial \lambda_1} \) (\( i = 1, 2 \)), and \( \theta'_1(z) = \frac{d}{dz} \theta_1(z) \) etc. We have

\[
M_{1,2} = \partial_1^2 + \partial_2^2 \quad -2 \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right) \partial_1 \partial_2 - 2 \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right) \partial_1 \partial_2 \\
+ 2 \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)
\]

and

\[
M_{2,4} - 2M_{1,4} = \partial_1^2 \partial_2^2 \quad -2 \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right) \partial_1 \partial_2 - 2 \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right) \partial_1 \partial_2 \\
+ 2 \left( \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right)^2 \right) - \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + 2 \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right) \right) \partial_1^2 \\
+ 2 \left( \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta'_1(\pm)}{\theta_1(\pm)} \right)^2 \right) - \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + 2 \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right) \right) \partial_2^2 \\
+ 4 \left( \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 - \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right) \partial_1 \partial_2 \\
+ \left\{ \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right\} \partial_1 \\
+ \left\{ \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right\} \partial_2 \\
+ \frac{1}{2} \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right) - 4 \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} + \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right) \\
+ 2 \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right) - \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)
\]

We set \( \Delta = \theta_1(\pm) \theta_1(-) \), then

\[
\Delta^{-1} \cdot M_{2,2} \cdot \Delta = \partial_1^2 + \partial_2^2 + 4 \left( \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right) \]

\[
= \partial_1^2 + \partial_2^2 + 4 \left( \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 + \left( \frac{\theta''_1(\pm)}{\theta_1(\pm)} \right)^2 \right) \quad \text{(A.1)}
\]
\[
\begin{align*}
\Delta^{-1} \cdot (M_{2A} - 2M_{1A}) \cdot \Delta &= \partial^2_t \partial^2_2 \\
& + 4 \left( \left( \frac{\theta'''}{\theta_1} + \frac{\theta''^2}{\theta_1^2} \right) (+) - \left( \frac{\theta'''}{\theta_1} + \frac{\theta''^2}{\theta_1^2} \right) (-) \right) \partial_1 \partial_2 \\
& + 2 \left( \frac{\theta'''}{\theta_1} - 3 \frac{\theta''^2}{\theta_1^2} + 2 \frac{\theta'''_1}{\theta_1^3} \right) (+) + \left( \frac{\theta'''}{\theta_1} - 3 \frac{\theta''^2}{\theta_1^2} + 2 \frac{\theta'''_1}{\theta_1^3} \right) (-) \partial_1 \\
& + 2 \left( \frac{\theta'''}{\theta_1} - 3 \frac{\theta''^2}{\theta_1^2} + 2 \frac{\theta'''_1}{\theta_1^3} \right) (+) - \left( \frac{\theta'''}{\theta_1} - 3 \frac{\theta''^2}{\theta_1^2} + 2 \frac{\theta'''_1}{\theta_1^3} \right) (-) \partial_2 \\
& + 2 \left( \frac{\theta'''}{\theta_1} (+) + \frac{\theta''^2}{\theta_1^2} (-) \right) - 8 \left( \frac{\theta'''}{\theta_1^2} (+) + \frac{\theta''^2}{\theta_1^2} (-) \right) \\
& - 8 \left( \frac{\theta''^2}{\theta_1} (+) + \frac{\theta''^4}{\theta_1^2} (-) \right) + 8 \left( \frac{\theta''^2}{\theta_1} (+) + \frac{\theta''^4}{\theta_1^2} (-) \right) \\
& - 8 \left( \frac{\theta''^4}{\theta_1} (+) + \frac{\theta''^4}{\theta_1^2} (-) \right) \\
& = \partial^2_t \partial^2_2 \\
& + 4 \{ \log \theta_1'' (+) - (\log \theta_1)''' (-) \} \partial_1 \partial_2 \\
& + 2 \{ \log \theta_1''' (+) + (\log \theta_1)'''' (-) \} \partial_1 + 2 \{ \log \theta_1''' (+) - (\log \theta_1)'''' (-) \} \partial_2 \\
& + 2 \{ (\log \theta_1)'''' (+) + (\log \theta_1)'''' (-) \} \\
& + 4 \{ (\log \theta_1)'''' (+) - (\log \theta_1)'''' (-) \}^2 \\
& = \{ \partial_1 \partial_2 + 2 \{ (\log \theta_1)'''' (+) - (\log \theta_1)'''' (-) \} \}^2
\end{align*}
\]

The complete integrable Hamiltonian of type $BC_n$ is introduced by Olshanetsky-Perelomov [OP], and later generated by Inozemtsev-Meshcheryakov [IM] [I]. In the rank two case, the Hamiltonian is

\[
H = -\frac{1}{2} (\partial^2_1 + \partial^2_2) + g(g - 1) (\varphi(x_1 + x_2) + \varphi(x_1 - x_2)) \\
+ \sum_{0 \leq r \leq 3} g_r (g_r - 1) (\varphi(\omega_r + x_1) + \varphi(\omega_r + x_2)),
\]

where $\varphi(x)$ denotes the Weierstrass $\wp$-function with two periods $2\omega_1$ and $2\omega_2$, and $\omega_0 = 0$, $\omega_3 = -\omega_1 - \omega_2$. By the connection between theta function and $\wp$ function (B.9) in Appendix B, our differential limit (A.1) is identified with this Hamiltonian for the special coupling constants $g(g - 1) = 2$, and $g_r (g_r - 1) = 0$ ($0 \leq r \leq 3$).

### Appendix B: Theta function

We establish notations and identities on the theta functions [WW]. The Jacobi theta functions are defined for $\tau \in \mathfrak{H}_+$ as follows:

\[
\theta_1(z|\tau) = \sum_{k \in \mathbb{Z}} \exp \left[ 2\pi i \left( z + \frac{1}{2} \right) \left( k + \frac{1}{2} \right) + \frac{1}{2} \left( k + \frac{1}{2} \right)^2 \right]
\]
\[
\theta_2(z|\tau) = \sum_{k \in \mathbb{Z}} \exp \left[ 2\pi i \left( z(k + \frac{1}{2}) + \frac{1}{2}(k + \frac{1}{2})^2 \tau \right) \right]
\]

\[
\theta_3(z|\tau) = \sum_{k \in \mathbb{Z}} \exp \left[ 2\pi i \left( zk + \frac{k^2}{2} \tau \right) \right]
\]

\[
\theta_4(z|\tau) = \sum_{k \in \mathbb{Z}} \exp \left[ 2\pi i \left( (z + \frac{1}{2})k + \frac{k^2}{2} \tau \right) \right]
\]

Note that \(\theta_1(z)\) is odd and the other three are even. These functions have quasi-periodicity:

\[
\theta_1(z + m|\tau) = (-1)^m \theta_1(z|\tau), \quad (m \in \mathbb{Z}),
\]

while other three can be expressed by \(\theta_1(z)\)

\[
\theta_1(z + \frac{1}{2}|\tau) = \theta_2(z|\tau), \quad \theta_1(z + \frac{3}{2}|\tau) = e^{-\pi i(z + \frac{2}{3})} \theta_3(z|\tau),
\]

We use these identities in the computations in Lemma 4.

\[
\theta_4(x|\tau)\theta_4(y|\tau) = \theta_3(x + y|2\tau)\theta_3(x - y|2\tau) - \theta_2(x + y|2\tau)\theta_2(x - y|2\tau), \quad (B.4)
\]

\[
\theta_3(x|\tau)\theta_3(y|\tau) = \theta_3(x + y|2\tau)\theta_3(x - y|2\tau) + \theta_2(x + y|2\tau)\theta_2(x - y|2\tau), \quad (B.5)
\]

\[
\theta_2(x|\tau)\theta_2(y|\tau) = \theta_3(x + y|2\tau)\theta_3(x - y|2\tau) + \theta_2(x + y|2\tau)\theta_3(x - y|2\tau) + \theta_2(x + y|2\tau)\theta_3(x - y|2\tau), \quad (B.6)
\]

\[
\theta_1(x|\tau)\theta_1(y|\tau) = \theta_3(x + y|2\tau)\theta_3(x - y|2\tau) - \theta_2(x + y|2\tau)\theta_3(x - y|2\tau). \quad (B.7)
\]

The sigma function \(\sigma(z)\) is an entire, odd, and quasi-periodic function with two primitive quasi-periods \(2\omega_1, 2\omega_2\).

\[
\sigma(z + 2n\omega_1 + 2m\omega_2) = (-1)^{n+m+nm}e^{(2n\eta_1 + 2m\eta_2)(z+n\omega_1+m\omega_2)}\sigma(z)
\]

with \(\eta_i = \zeta(\omega_i)\) \((i = 1, 2)\), where \(\zeta(z) = \sigma'(z)/\sigma(z)\) denotes the Weierstrass \(\zeta\)-function. The connection between the Jacobi theta functions and the sigma functions are

\[
\sigma(z) = \left( \exp \frac{\eta_1 z^2}{2\omega_1} \right) \frac{\theta_1(z/2\omega_1)}{\theta_1(0)},
\]

\[
\sigma_r(z) = \left( \exp \frac{\eta_1 z^2}{2\omega_1} \right) \frac{\theta_{r+1}(z/2\omega_1)}{\theta_{r+1}(0)} \quad (r = 1, 2, 3).
\]

Then, for the function \(v(z)\) in van Diejen’s system (2.15), we have

\[
v(z) := \frac{\sigma(z + \mu)}{\sigma(z)} = \left( \exp \frac{\eta_1 (2z\mu + \mu^2)}{2\omega_1} \right) \frac{\theta_1((z + \mu)/2\omega_1)}{\theta_1(z/2\omega_1)}.
\]

The connection with \(\varphi\) function is

\[
\varphi(z) = -\frac{d^2}{dz^2} \log \sigma(z) = -\frac{1}{4\omega_1^2} \left( \frac{d^2}{dz^2} \log \theta_1(z/2\omega_1) \right) - \frac{\eta_1}{\omega_1}.
\]
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