FLOWS WITH VARIOUS TYPES OF SHADOWING

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Abstract. In the present paper we study the $C^1$-robustness of properties: (i) average shadowing; (ii) asymptotic average shadowing; (iii) limit shadowing. We obtain that the first two cases guarantee dominated splitting for flows on the whole manifold, and the third case implies that the flow is a transitive Anosov flow. We discuss the problems within three classes of flows: the dissipative, the incompressible and the Hamiltonian ones and in any dimension. In low dimension we recover sharper results by assuming the hypothesis of $C^1$-robustness of (asymptotic) average shadowing: dissipative/incompressible 3-flows are Anosov (cf. [4, 6]), and obtain a new one for 4-dimensional Hamiltonian flows by showing that they have Anosov energy levels.

1. Introduction: Basic Definitions and Statement of the Results

1.1. Dissipative and incompressible flows setting. Along this paper we consider vector fields $X : M \to TM$, where $M$ is a $d$-dimensional ($d \geq 3$) connected and closed $C^\infty$ Riemannian manifold $M$ and $TM$ its tangent bundle. Given a vector field $X$ we have an associated flow $X^t$ which is the infinitesimal generator of $X$ in a sense that $\frac{dX^t}{dt}|_{t=s}(p) = X(X^s(p))$. If the divergence of $X$, defined by $\nabla \cdot X = \sum_{i=1}^d \frac{\partial X_i}{\partial x_i}$, is zero we say that $X$ is divergence-free. The flow $X^t$ has a tangent flow $DX^t_p$ which is the solution of the non-autonomous linear variational equation $\dot{u}(t) = DX^t_p \cdot u(t)$. Moreover, due to Liouville’s formula, if $X$ is divergence-free, the associated flow $X^t$ preserves the volume-measure and for this reason we call it incompressible. If the vector field is not divergence-free its flow is dissipative. We denote by $\mathcal{X}^1(M)$ the set of all dissipative $C^1$ vector fields and by $\mathcal{X}^1_\mu(M) \subset \mathcal{X}^1(M)$ the set of all $C^1$ vector fields that preserve the volume, or equivalently the set of all incompressible flows. We assume that both $\mathcal{X}^1(M)$ and $\mathcal{X}^1_\mu(M)$ are endowed with the $C^1$ Whitney (or strong) vector field topology which turn these two vector spaces completed, thus a Baire space. We denote by $\mathcal{R}$ the set of regular points of $X$, that is, those points $x$ such that $X(x) \neq \vec{0}$ and by $\text{Sing}(X) = M \setminus \mathcal{R}$ the set of singularities of $X$. Let us denote by $\text{Crit}(X)$ the set of critical orbits of $X$, that is, the set formed by all periodic orbits and all singularities of $X$.

The Riemannian structure on $M$ induces a norm $\|\cdot\|$ on the fibers $T_pM$, $\forall p \in M$. We will use the standard norm of a bounded linear map $L$ given by

$$\|L\| = \sup_{\|u\|=1} \|L(u)\|.
$$

A metric on $M$ can be derived in the usual way by using the exponential map or through the Moser volume-charts (cf. [36]) in the case of volume manifolds, and it will be denoted by $d(\cdot, \cdot)$. Hence, we define the open balls $B(x, r)$ of the points $y \in M$ satisfying $d(x, y) < r$ by using those charts.

1991 Mathematics Subject Classification. Primary: 37D20, 37D30; Secondary: 37C10, 37C50.

Key words and phrases. Average shadowing, limit shadowing, hyperbolicity, robusty property.
Dissipative flows appear often in models given by differential equations in mathematical physics, economics, biology, engineering and many diverse areas. Incompressible flows arise naturally in the fluid mechanics formalism and has long been one of the most challenging research fields in mathematical physics. We suggest the reference [34] for more details on fluid/geophysical dynamics theories.

1.2. The Hamiltonian flow formalism. Let \((M, \omega)\) be a compact symplectic manifold, where \(M\) is a 2d-dimensional (\(d \geq 2\)), smooth and compact Riemannian manifold endowed with a symplectic structure \(\omega\), that is, a skew-symmetric and nondegenerate 2-form on the tangent bundle \(TM\). We notice that we use the same notation for manifolds supporting Hamiltonian flows and also flows as in Subsection 1.1 which we hope will not be ambiguous.

We will be interested in the Hamiltonian dynamics of real-valued \(C^2\) functions on \(M\), constant on each connected component of the boundary of \(M\), called Hamiltonians, whose set we denote by \(C^2(M, \mathbb{R})\). For any Hamiltonian function \(H : M \rightarrow \mathbb{R}\) there is a corresponding Hamiltonian vector field \(X_H : M \rightarrow TM\), tangent to the boundary of \(M\), and determined by the equality

\[ \nabla_p H(u) = \omega(X_H(p), u), \forall u \in T_p M, \]

where \(p \in M\).

Observe that \(H\) is \(C^2\) if and only if \(X_H\) is \(C^1\). Here we consider the space of the Hamiltonian vector fields endowed with the \(C^1\) topology, and for that we consider \(C^2(M, \mathbb{R})\) equipped with the \(C^2\) topology.

The Hamiltonian vector field \(X_H\) generates the Hamiltonian flow \(X_H^t\), a smooth 1-parameter group of symplectomorphisms on \(M\) satisfying \(\frac{d}{dt} X_H^t = X_H(X_H^t)\) and \(X_H^0 = id\). We also consider the tangent flow \(D_p X_H^t : T_p M \rightarrow T_{X_H^t(p)} M\), for \(p \in M\), that satisfies the linearized differential equality \(\frac{d}{dt} D_p X_H^t = (D_{X_H^t(p)} X_H) \cdot D_p X_H^t\), where \(D_p X_H : T_p M \rightarrow T_{X_H(p)} M\).

Since \(\omega\) is non-degenerate, given \(p \in M\), \(\nabla_p H = 0\) is equivalent to \(X_H(p) = 0\), and we say that \(p\) is a singularity of \(X_H\). A point is said to be regular if it is not a singularity. We denote by \(\mathcal{R}\) the set of regular points of \(H\), by \(\text{Sing}(X_H)\) the set of singularities of \(X_H\) and by \(\text{Crit}(H)\) the set of critical orbits of \(H\).

By the theorem of Liouville ([1 Proposition 3.3.4]), the symplectic manifold \((M, \omega)\) is also a volume manifold, that is, the 2d-form \(\omega^d = \omega \wedge \ldots \wedge \omega\) is a volume form and induces a measure \(\mu\) on \(M\), which is called the Lebesgue measure associated to \(\omega^d\). Notice that the measure \(\mu\) on \(M\) is invariant by the Hamiltonian flow.

Fixed a Hamiltonian \(H \in C^2(M, \mathbb{R})\) any scalar \(e \in H(M) \subset \mathbb{R}\) is called an energy of \(H\) and \(H^{-1}(\{e\}) = \{p \in M : H(p) = e\}\) is the corresponding energy level set which is \(X_H^1\)-invariant. An energy surface \(\mathcal{E}_{H,e}\) is a connected component of \(H^{-1}(\{e\})\); we say that it is regular if it does not contain singularity points and in this case \(\mathcal{E}_{H,e}\) is a regular compact (2d - 1)-manifold. Moreover, \(H\) is constant on each connected component \(\mathcal{E}_{H,e}\) of the boundary \(\partial M\).

A Hamiltonian system is a triple \((H, e, \mathcal{E}_{H,e})\), where \(H\) is a Hamiltonian, \(e\) is an energy and \(\mathcal{E}_{H,e}\) is a regular connected component of \(H^{-1}(\{e\})\).

Due to the compactness of \(M\), given a Hamiltonian function \(H\) and \(e \in H(M)\) the energy level \(H^{-1}(\{e\})\) is the union of a finite number of disjoint compact connected components, separated by a positive distance. Given \(e \in H(M)\), the pair \((H, e) \subset C^2(M, \mathbb{R}) \times \mathbb{R}\) is called a Hamiltonian level; if we fix \(\mathcal{E}_{H,e}\) and a small neighbourhood \(W\) of \(\mathcal{E}_{H,e}\) there
exist a small neighbourhood $\mathcal{U}$ of $H$ and $\delta > 0$ such that for all $\tilde{H} \in \mathcal{U}$ and $\varepsilon \in ]e-\delta, e+\delta[$ one has that $H^{-1}(\{\varepsilon\}) \cap \mathcal{W} = \mathcal{E}_{\tilde{H}, \varepsilon}$. We call $\mathcal{E}_{\tilde{H}, \varepsilon}$ the analytic continuation of $\mathcal{E}_{H, e}$.

Using the Darboux charts (cf. [37]) we define a metric on $M$ which we also denote by $d(\cdot, \cdot)$. Let $B(x, r)$ stand for the open balls centered in $x$ and with radius $r$ by using Darboux’s charts.

The Hamiltonian formalism appears in various branches of pure and applied mathematics and, due to its ubiquity, it is completely undeniable the importance and impact of this fundamental concept in science today. We refer the book [37] for a full detailed exposition about Hamiltonian formalism.

1.3. Properties of the Shadowing. The concept of shadowing in dynamical systems has both applications in numerical theoretical analysis and also to structural stability and hyperbolicity. In rough terms shadowing is supported in the idea of estimating differences between exact and approximate solutions along orbits and to understand the influence of the errors that we commit and allow on each iterate. We usually ask if it is possible to obtain shadowing of “almost” trajectories in a given dynamical system by exact ones.

It is interesting to take a more general context where the errors of the “almost” trajectories can be large, however, on average they remain small. This concept, much more relaxed than shadowing, and called average shadowing was introduced by Blank [16] and is one of the main subjects of this work.

A sequence $(x_i, t_i)_{i=0}^m$, with $m \in \mathbb{Z}$, is called a $\delta$-pseudo-orbit of a given vector field $X$ (dissipative, incompressible or Hamiltonian) if for every $0 \leq i \leq m-1$ we have $t_i \geq 1$ and

$$d(X^{t_i}(x_i), x_{i+1}) < \delta.$$ 

A sequence $(x_i, t_i)_{i \in \mathbb{Z}}$ is a $\delta$-average-pseudo-orbit of $X$, if $t_i \geq 1$ for every $i \in \mathbb{Z}$ and there is a number $N$ such that for any $n \geq N$ and $k \in \mathbb{Z}$ we have

$$\frac{1}{n} \sum_{k=1}^{n} d(X^{t_{i+k}}(x_{i+k}), x_{i+k+1}) < \delta.$$ 

A sequence $(x_i, t_i)_{i \in \mathbb{Z}}$ is positively $\epsilon$-shadowed in average by the orbit of $X$ through a point $x$, if there exists an orientation preserving homeomorphism $h: \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{s_i}^{s_{i+1}} d(X^{h(t)}(x), X^{t-s_i}(x))dt < \epsilon,$$

where $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i$ and $s_{-n} = \sum_{i=-n}^{-1} t_i$ for $n \in \mathbb{N}$. Analogously, we define negatively $\epsilon$-shadowed in average.

We say that $X$ has the average shadowing property if for any $\epsilon > 0$, there is a $\delta > 0$ such that any $\delta$-average-pseudo-orbit, $(x_i, t_i)_{i \in \mathbb{Z}}$, of $X$ can be positively and negatively $\epsilon$-shadowed in average by some orbit of $X$.

In the other hand, the authors in [21] posed the notion of the limit-shadowing property. From the numerical point of view this property on a dynamical system means that if we apply a numerical method of approximation to $X$ with “improving accuracy” so that one step errors tend to zero as time goes to infinity then the numerically obtained trajectories tend to real ones. Such situations arise, for example, when one is not so interested on the initial (transient) behavior of trajectories but wants to reach areas where “interesting things” happen (e.g. attractors) and then improve accuracy. To be more precise, we say
that a sequence \((x_i, t_i)_{i \in \mathbb{Z}}\) is a limit-pseudo orbit of \(X\) (dissipative, incompressible or Hamiltonian) if \(t_i \geq 1\) for every \(i \in \mathbb{Z}\) and
\[
\lim_{|i| \to \infty} d(X^{t_i}(x_i), x_{i+1}) = 0.
\]
A limit-pseudo orbit \((x_i, t_i)_{i \in \mathbb{Z}}\) of \(X\) is positively shadowed in limit by an orbit of \(X\) through a point \(x\), if there is an orientation preserving homeomorphism \(h : \mathbb{R} \to \mathbb{R}\) with \(h(0) = 0\) such that
\[
\lim_{i \to \infty} \int_{x_i}^{x_{i+1}} d(X^{h(t)}(x), X^{t-s_i}(x_i)) dt = 0.
\]
Analogously, as we did before, we define when a limit-pseudo orbit is said to be negatively shadowed in limit by an orbit.

We say that \(X\) has the limit shadowing property if any limit pseudo-orbit, \((x_i, t_i)_{i \in \mathbb{Z}}\), of \(X\) can be positively and negatively shadowed in limit by some orbit of \(X\).

Finally, Gu [26] introduced the notion of the asymptotic average shadowing property for flows which is particularly well adapted to random dynamical systems. A sequence \((x_i, t_i)_{i \in \mathbb{Z}}\) is an asymptotic average-pseudo orbit of \(X\) (dissipative, incompressible or Hamiltonian) if \(t_i \geq 1\) for every \(i \in \mathbb{Z}\) and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=-n}^{n} d(X^{t_i}(x_i), x_{i+1}) = 0.
\]
A sequence \((x_i, t_i)_{i \in \mathbb{Z}}\) is positively asymptotically shadowed in average by an orbit of \(X\) through \(x\), if there exists an orientation preserving homeomorphism \(h : \mathbb{R} \to \mathbb{R}\) such that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} d(X^{h(t)}(x), X^{t-s_i}(x_i)) dt = 0.
\]
Again, where \(s_0 = 0\) and \(s_n = \sum_{i=0}^{n-1} t_i\), \(n \in \mathbb{N}\). Similarly an asymptotic average-pseudo orbit is negatively asymptotically shadowed in average.

We say that \(X\) has the asymptotic average shadowing property if for any \(\epsilon > 0\), any asymptotic average-pseudo-orbit, \((x_i, t_i)_{i \in \mathbb{Z}}\), of \(X\) can be positively and negatively asymptotically shadowed in average by some orbit of \(X\).

Note that the definitions of shadowing above allows the presence of a reparametrization of the trajectory. In this case we have the difficulty of analyzing the existence (or not) of an orbit that shadows the pseudo orbit, because we cannot control how fast the orbit moves. In fact, the reparametrization allows, in a short time, an orbit possibly distant pseudo orbit, approach of it so that the limits (1), (2) and (3) are satisfied. This does not occur when the function \(h\) is equal to the identity, for example, as in [6].

We observe also that the above shadowing concepts are not equivalent. Recall that Morse-Smale vector fields, are a class within the dissipative flows, admitting sinks and sources. By [4] Lemma 6] and Proposition[5] the average shadowing, the asymptotic average shadowing, and the limit shadowing properties each imply that there are neither sinks nor sources in the system. Thus, a Morse-Smale vector field is an example of a vector field which has the shadowing property but do not have average shadowing property, or asymptotic average shadowing property or limit shadowing property. See [38] for more examples. Examples of systems which have the asymptotic average shadowing property or the limit shadowing property, but do not have the shadowing property can found in [27] and [40], respectively.

Finally, we define the set of \(C^1\)-stably average shadowing flows:
• If \( X \in \mathcal{X}^1(M) \) we say that \( X^t \) is a \( C^1 \)-stably average shadowable flow if any \( Y \in \mathcal{X}^1(M) \) sufficiently \( C^1 \)-close to \( X \) has the average shadowing property;

• If \( X \in \mathcal{X}^1_{\mu}(M) \) we say that \( X^t \) is a \( C^1 \)-stably average shadowable incompressible flow if any \( Y \in \mathcal{X}^1_{\mu}(M) \) sufficiently \( C^1 \)-close to \( X \) has the average shadowing property and

• The Hamiltonian system \((H, e, \mathcal{E}_{H, e})\) is stably average shadowable if there exists a neighbourhood \( \mathcal{V} \) of \((H, e, \mathcal{E}_{H, e})\) such that any \((\hat{H}, \hat{e}, \mathcal{E}_{\hat{H}, \hat{e}}) \in \mathcal{V}\) has the average shadowing property.

Analogously we define the sets of \( C^1 \)-stably asymptotic average shadowable flows and \( C^1 \)-stably limit shadowable flows.

Here we will include in our studies flows that in general can have singularities. Posteriori we will derive that the three properties of shadowing in fact imply the absence of singularities. We refer to \([3]\) and \([30]\) for a discussion of types of shadowing in the Lorenz flow containing one singularity.

1.4. Hyperbolicity and statement of the results. Let \( \Lambda \subseteq \mathcal{R} \) be an \( X^1 \)-invariant set. We say that \( \Lambda \) is hyperbolic with respect to the vector field \( X \in \mathcal{X}^1(M) \) if, there exists \( \lambda \in (0, 1) \) such that, for all \( x \in \Lambda \), the tangent vector bundle over \( x \) splits into three \( DX^1(x) \)-invariant subbundles \( T_x\Lambda = E_x^u \oplus E_x^s \oplus E_x^\sigma \), with \( \|DX^1(x)|_{E_x^s}\| \leq \lambda \) and \( \|DX^1(x)|_{E_x^\sigma}\| \leq \lambda \) and \( E_x^u \) stands for the one-dimensional flow direction. If \( \Lambda = M \) the vector field \( X \) is called Anosov. We observe that there are plenty Anosov flows which are not incompressible. Despite the fact that all incompressible Anosov flows are transitive, there exists dissipative ones which are not (see \([24]\)).

Given \( x \in \mathcal{R} \) we consider its normal bundle \( N_x = X(x)^\perp \subset T_xM \) and define the linear Poincaré flow by \( P_X^t(x) := \Pi_{X^t(x)} \circ DX^t_x \) where \( \Pi_{X^t(x)} : T_{X^t(x)}M \to N_{X^t(x)} \) is the projection along the flow direction \( E_x^0 \). Let \( \Lambda \subseteq \mathcal{R} \) be an \( X^1 \)-invariant set and \( N = N^1 \oplus N^2 \) be a \( P_X^0 \)-invariant splitting over \( \Lambda \) with \( N^1 \) and \( N^2 \) one-dimensional. Fixed \( \lambda \in (0, 1) \) we say that this splitting is an \( \lambda \)-dominated splitting for the linear Poincaré flow if for all \( x \in \Lambda \) we have:

\[
\|P_X^1(x)|_{N^2}\| \leq \lambda \text{ and } \|P_X^{-1}(X^t(x))|_{N^1_{X^t(x)}}\| \leq \lambda.
\]

This definition is weaker than hyperbolicity where it is required that

\[
\|P_X^1(x)|_{N^2}\| \leq \lambda \text{ and } \|P_X^{-1}(X^t(x))|_{N^1_{X^t(x)}}\| \leq \lambda.
\]

When \( \Lambda \) is compact this definition is equivalent to the usual definition of hyperbolic flow (\([20]\) Proposition 1.1)).

Let us recall that a periodic point \( p \) of period \( \pi \) is said to be hyperbolic if the linear Poincaré flow \( P_X^\pi(p) \) has no norm one eigenvalues. We say that \( p \) has trivial real spectrum if \( P_X^\pi(p) \) has only real eigenvalues of equal norm to one and there exists \( 0 \leq k \leq n - 1 \) such that \( 1 \) has multiplicity \( k \) and \(-1\) has multiplicity \( n - 1 - k \). Observe that, in the incompressible case, having trivial real spectrum is equivalent to the eigenvalues are equal to \( 1 \) or \(-1\).

Within Hamiltonian flows we define \( N_x := N_x \cap T_xH^{-1}(\{e\}) \), where \( T_xH^{-1}(\{e\}) = Ker\nabla H_x \) is the tangent space to the energy level set. As a consequence we get that \( N_x \) is a \((2d - 2)\)-dimensional subbundle. The transversal linear Poincaré flow associated to \( H \)
is given by

\[ \Phi_t \big( x \big) : N_x \to N_{X_t^H(x)} \]

\[ v \mapsto \Pi_{X_t^H(x)} \circ D_x X_t^H(v), \]

where \( \Pi_{X_t^H(x)} : T_{X_t^H(x)} M \to N_{X_t^H(x)} \) denotes the canonical orthogonal projection.

Analogously to was we did for the linear Poincaré flow in (4) we define hyperbolicity and also dominated splitting for the transversal linear Poincaré flow \( \Phi_t^H \). We say that a compact, \( X_t^H \)-invariant and regular set \( \Lambda \subset M \) is partially hyperbolic for the transversal linear Poincaré flow \( \Phi_t^H \) if there exists a \( \Phi_t^H \)-invariant splitting \( N = N^u \oplus N^c \oplus N^s \) over \( \Lambda \) such that all the subbundles have constant dimension and at least two of them are non-trivial and \( \lambda \in (0, 1) \) such that, \( N^u \) is \( \lambda \)-uniformly hyperbolic and expanding, \( N^s \) is \( \lambda \)-uniformly hyperbolic and contracting and \( N^c \) \( \lambda \)-dominates \( N^s \) and \( N^c \) \( \lambda \)-dominates \( N^u \).

Along this paper, we consider hyperbolicity, partial hyperbolicity and dominated splitting defined in a set \( \Lambda \) which is the whole energy level. It is quite interesting to observe that in Hamiltonians the existence of a dominated splitting implies partial hyperbolicity (see [17]).

We begin by presenting our main results. The first is a generalization of the result in [4, Theorem 4] for higher dimensional flows.

**Theorem A.** If \( X^t \) is \( C^1 \)-stably average shadowable, or \( C^1 \)-stably asymptotic average shadowable, then \( X^t \) admits a dominated splitting on \( M \).

**Theorem A’.** A \( d \)-dimensional flow (\( d \geq 3 \)) which is \( C^1 \)-stably limit shadowable is a transitive Anosov flow.

Now, our main result in the context of incompressible flows is a generalization of the main result in [6] also for higher dimensional flows.

**Theorem B.** If an incompressible flow \( X^t \) is \( C^1 \)-stably average shadowable, or \( C^1 \)-stably asymptotic average shadowable, then \( X^t \) admits a dominated splitting on \( M \).

**Theorem B’.** A \( d \)-dimensional incompressible flow (\( d \geq 3 \)) which is \( C^1 \)-stably limit shadowable is a topologically mixing Anosov flow.

Finally, we formulate our main results for Hamiltonian systems.

**Theorem C.** A Hamiltonian system \( (H, e, E_{H,e}) \) \( C^2 \)-stably average shadowable, or \( C^2 \)-robustly asymptotic average shadowable, is a partially hyperbolic Hamiltonian system.

**Theorem C’.** A Hamiltonian system \( (H, e, E_{H,e}) \) \( C^2 \)-stably limit shadowable is Anosov.

The analysis of shadowing for flows is certainly more complicated than for maps due to presence of reparametrizations of the trajectories and the (possible) presence of singularities. The result in [33] can now be obtained from the Theorem A by consideration of suspension flow. Complementing these result with the other types of shadowing which we consider here, we have the following.

**Corollary 1.** (1) A \( C^1 \)-diffeomorphism which is \( C^1 \)-stably (asymptotic) average shadowable admits a dominated splitting on \( M \), and a \( C^1 \)-stably limit shadowable is transitive Anosov.\(^1\)

\(^1\)The definitions of hyperbolicity and shadowing for the case of diffeomorphisms are analogous to the ones in the case of flows. See [43].
(2) A $C^1$-volume preserving diffeomorphism which is $C^1$-stably (asymptotic) average shadowable admits a dominated splitting on $M$, and a $C^1$-stably limit shadowable is transitive Anosov. 

(3) A symplectomorphism stably (asymptotic) average shadowable is partially hyperbolic, and a stably limit shadowable is Anosov.

It is noteworthy that, in [4, Theorem 4] it is proved that a $C^1$ stably (asymptotic) average shadowable flow is Anosov and transitive. In our incompressible 3-dimensional flow setting the stability of transitivity is obtained a priori since by [8] we know that away from Anosov incompressible flows we have elliptic orbits, thus invariant tori can be created by small $C^1$-perturbations using [2], and so robust transitivity is not feasible.

We end this introduction by recalling several results in the vein of ours proved in [15] - $C^1$-robust topologically stably incompressible flows are Anosov, in [23] - $C^1$-robust (Lipschitz) shadowing incompressible flows are Anosov and in [14] - $C^1$-robust weak shadowing incompressible flows are volume-hyperbolic. Another result which relates $C^1$-robust properties with hyperbolicity is the result in [9] which states that $C^1$-robustly transitive incompressible flows have dominated splitting. Some results that are related to the shadowing properties are studied in [5, 39]. See also the results in [11, 12, 41] for flows and in [32, 33, 47] for diffeomorphisms.

2. DISSIPATIVE FLOWS - PROOFS OF THEOREMS A AND A’

This section is divided into two subsections: At first we study the (asymptotic) average shadowing properties and prove Theorem A. The second part is dedicated the limit shadowing property and the results necessary to prove Theorem A’.

2.1. (Asymptotic) Average shadowing property - Proof of Theorem A. Firstly, we recall that $X^t$ is chain transitive, if for any points $x, y \in M$ and any $\delta > 0$, there exists a finite $\delta$-pseudo orbit $(x_i, t_i)_{0 \leq i \leq K}$ of $X$ such that $x_0 = x$ and $x_K = y$. Observe that transitivity implies chain transitivity.

We recall also that a vector field $X$ has a property $\Psi$ robustly if there exists a $C^1$-neighborhood $U$ of $X$ such that any $Y \in U$ has the property $\Psi$. Theorem A is a direct consequence of the more general result:

**Theorem 2.** If $X \in \mathcal{X}_1(M)$ is robustly chain transitive, then $X^t$ admits a dominated splitting on $M$.

A compact invariant set $\Lambda$ is attracting if $\Lambda = \bigcap_{t \geq 0} X^t(U)$ for some neighborhood $U$ of $\Lambda$ satisfying, $X^t(U) \subseteq U$ for all $t > 0$. An attractor of $X$ is a transitive attracting set of $X$ and a repeller is an attractor for $-X$. We say that $\Lambda$ is a proper attractor or repeller if $\emptyset \neq \Lambda \neq M$. A sink (source) of $X$ is a attracting (repelling) critical orbit of $X$.

We recall that the chain-transitivity rules out the presence of sinks and sources [4, Lemma 6] and that robustly chain transitive vector fields have no singularities [4, Theorem 15]. We also recall that the Hausdorff distance between two compact subsets $A$ and $B$ of $M$ is defined by:

$$d_H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}.$$  

We also make use of the following result.

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2The definitions of hyperbolicity and shadowing for the case of diffeomorphisms are analogous to the ones in the case of flows. See [15].
Theorem 3. [19] Theorem 4] There exists a residual set $\mathcal{R}$ of $\mathfrak{X}^1(M)$ such that for any vector field $X \in \mathcal{R}$, a compact invariant set $\Lambda$ is the limit, with respect to the Hausdorff distance, of a sequence of periodic orbits if and only if $X$ is chain transitive in $\Lambda$.

Finally, the next dichotomy will play an important role along our proof.

Theorem 4. [18] Corollary 2.22] Let $X \in \mathfrak{X}^1(M)$ be a vector field and $\mathcal{U}$ be a small $C^1$-neighborhood of $X$. For any $\varepsilon > 0$ there exist two integers $\ell$ and $\tau$ such that, for any $Y \in \mathcal{U}$ and any periodic point $p$ of $Y$ of period $\tau(p) \geq \tau$,

- either $P^{\ell}_{Y^p}$ admits an $\ell$-dominated splitting over the $Y$-orbit of $p$, or else
- for any tubular neighborhood $U$ of $p$, there exists an $\varepsilon$-$C^1$-perturbation $Z$ of $Y$ coinciding with $Y$ outside $U$ and along the orbit of $p$, and such that the linear Poincaré flow $P^t_Z(p)$ has trivial real spectrum; furthermore the eigenvalues can be chosen different from $1$ or $-1$, thus the orbit of $p$ is a source or a sink for $Y$.

To prove Theorem A is sufficient to prove Theorem 2 since vector fields with (asymptotic) average shadowing property are chain transitive (cf. [27, 28]). The proof of Theorem 2 follows in a manner analogous to [4, Theorem 15].

Proof. (Proof of Theorem 2)

Let $X$ be a robustly chain transitive vector field, $\mathcal{U}$ the neighborhood of $X$ as in the definition, and $\mathcal{R}$ the residual in $\mathcal{U}$ of the Theorem 5.

Hence, there exist a sequence $Y_n \in \mathcal{R}$, converging to $X$, and periodic orbits $O_{Y_n}(p_n)$ of $Y_n$ such that $M = \limsup O_{Y_n}(p_n)$. As $Y_n$ neither admits sinks nor sources, the Theorem assure that the linear Poincaré flow $P^t_{X^p}$ admits an $\ell$-dominated splitting over $O_{Y_n}(p_n)$, with $\ell$ independent of $n$. Therefore, $P^t_{X^p}$ admits an $\ell$-dominated splitting over $M$.  

2.2. Limit shadowing property - Proof of Theorem A'. We begin by recall that flows with the limit shadowing property have no proper attractor (repellers).

Proposition 5. [41] Proposition 18] If $X$ has the limit shadowing property, then $X$ has no proper attractor.

This proposition implies that all the critical orbits of a vector field with limit shadowing property are of saddle-type. We recall that by the stable manifold theory (cf. [29]), if $O$ is a hyperbolic closed orbit of $X$ with splitting $T_O M = E^s_O \oplus E^u_O \oplus E^0_O$ then its unstable set

$$W^u(O) = \{ y \in M; \alpha(y) = O \},$$

is an immersed submanifold tangent at $O$ to the subbundle $E^u_O \oplus E^0_O$, and its stable set

$$W^s(O) = \{ y \in M; \omega(y) = O \},$$

is an immersed submanifold tangent at $O$ to the subbundle $E^s_O \oplus E^0_O$. In this case $W^s(O)$ and $W^u(O)$ are called the stable and the unstable manifolds of $O$, respectively.

Lemma 6. [41] Lemma 23] If $X$ has the limit shadowing property then $W^s(O) \cap W^u(O') \neq \emptyset$ for any pair of orbits $O, O' \in \text{Crit}(X)$.

A vector field $X \in \mathfrak{X}^1(M)$ is Kupka-Smale if its critical orbits are all hyperbolic and moreover their stable and unstable manifolds intersect transversally. The Kupka-Smale vectors fields form a residual subset in $\mathfrak{X}^1(M)$, [31] (see [44] for diffeomorphisms).

Lemma 7. If $X$ is a $C^1$-stably limit shadowable flow, then all periodic orbits of $X$ are hyperbolic.
Proof. Suppose that the dimension of $M$ is 3. Let $U$ be a neighborhood of $X$ as in the definition of robustness of limit shadowing property. If $X$ has a non-hyperbolic periodic orbit $O(p)$, then $p$ is a non-hyperbolic point for the Poincaré map associated to $p$ which is defined on a surface. Thus we can create a sink or source for the Poincaré map by a perturbation ([35] Lemma 1.3]). This implies that there is a vector field $Y \in U$ with the limit shadowing property which has a sink or source, which contradicts the Proposition 5.

If the dimension on $M$ is greater than 3, we proceed as follows. Let $V$ be a neighborhood of $X$ whose limit shadowing property is verified. Suppose that $X$ has a non-hyperbolic periodic orbit $O(p)$, associated to periodic point $p$, then $p$ is a non-hyperbolic periodic point for the Poincaré map associated to $p$. By Franks’ Lemma ([35] Lemma 1.3] there exists $Y \in V$ with hyperbolic periodic orbits, $O_1$ and $O_2$, with different indices. Since we can consider $Y$ a Kupka Smale vector field we have that $W^s(O_1) \cap W^u(O_2) = \emptyset$. This contradicts Lemma 8 and prove the desired.

Now, we proceed as in [41] Lemma 7] and we obtain the following result.

Lemma 8. If $X^t$ is a $C^1$-stably limit shadowable flow then $X$ has no singularities.

Proof. Let $\gamma_0$ be a periodic orbit of $X$. Suppose that $X$ has a singularity $\gamma_0$, then by [35] Lemma 1.1] there exists $Y$ $C^1$-close to $X$ with a hyperbolic singularity $\sigma$ and a hyperbolic periodic orbit $\gamma$ of different indices $i$ and $j$, respectively.

If $j < i$, then
\[
\dim W^u(\sigma) + \dim W^s(\gamma) \leq \dim M.
\]
Since we can consider $Y$ a Kupka-Smale vector field we have $\dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. By Lemma 8, we can consider $x \in W^u(\sigma) \cap W^s(\gamma)$. Then $O(x) \subset W^u(\sigma) \cap W^s(\gamma)$ and we can split
\[
T_x(W^u(\sigma)) = T_x(O(x)) \oplus E^1 \quad \text{and} \quad T_x(W^s(\gamma)) = T_x(O(x)) \oplus E^2.
\]
So, $\dim(T_x(W^u(\sigma)) + T_x(W^s(\gamma))) < \dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. This is a contradiction, because $X$ is a Kupka-Smale vector field.

If $j \geq i$, then $\dim W^u(\sigma) + \dim W^u(\gamma) \leq \dim M$ and by the same arguments we have a contradiction. Thus $X$ has no singularities.

We recall that a vector field $X \in \mathcal{X}^1(M)$ is a star flow if there exists a $C^1$-neighborhood $U$ of $X$ in $\mathcal{X}^1(M)$ such that any critical orbit of an $Y \in U$ is hyperbolic. As consequence of Lemmas 7 and 8 we obtain the following result.

Corollary 9. A $C^1$-stably limit shadowable flow is a star flow without any singularity.

Hence we can apply the following theorem by Gan and Wen [25].

Theorem 10. A star flow without singularities is Axiom A without cycle.

Proof. (Proof of Theorem A’)

Let $X^t$ be a $C^1$-stably limit shadowable flow. By Corollary 9 and Theorem 10, $X^t$ is Axiom A without cycle. Therefore, by Proposition 5, $X^t$ is a transitive Anosov flow. □

3. INCOMPRESSIBLE FLOWS - PROOFS OF THEOREMS B AND B’

In this section we study the (asymptotic) average shadowing and limit shadowing properties in the context of incompressible flows. This section is divided in two subsection: the first part is dedicated the some results about (asymptotic) average shadowing property which are necessary to prove the Theorem B. The second subsection is devoted to study the limit shadowing property and the proof of Theorem B’.
3.1. (Asymptotic) Average shadowing property - Proof of Theorem B. We begin by recalling the following result which was proved in [9, Proposition 2.4] and is the volume-preserving version of Theorem 4.

Lemma 11. Let $X \in X^1_{\mu}(M)$ and fix a small $\epsilon_0 > 0$. There exist $\pi_0, \ell \in \mathbb{N}$ such that for any closed orbit $x$ with period $\pi(x) > \pi_0$ we have either

(i) that $P^t_X$ has an $\ell$-dominated splitting along the orbit of $x$ or else

(ii) for any neighborhood $U$ of $\bigcup \ell \mathbb{X}^\ell(x)$, there exists an $\epsilon$-$C^1$-perturbation $Y$ of $X$, coinciding with $X$ outside $U$ and on $\bigcup \ell \mathbb{X}^\ell(x)$, and such that $P^t_X(x)$ has all eigenvalues equal to 1 and $-1$.

To study perturbations it is convenient to work with linear systems. Before enunciate the next result let us recall some of these notions. Given $X \in X^1_{\mu}(M)$ and a regular point $p$ we consider a linear differential system (see [9, 10] for full details on definitions) over the orbit of $p$ in the following way:

$$S^t: \mathbb{R}^d_p \to \mathbb{R}^d_{X^t(p)}$$

is such that

- $S^t \in \mathrm{SL}(d, \mathbb{R})$, for every $t$;
- $S^0 = \mathrm{id}$ and $S^{t+s} = S^t \circ S^s$, for every $s, t$ and
- $S^t$ is differentiable in $t$.

In [10, Subsection 2.2] it is developed a way to obtain “good coordinates” adequate to the continuous-time setting. Actually, we translate, in a conservative fashion, our flow from the manifold $M$ to the Euclidian space $\mathbb{R}^d$ with coordinates $(x, y_1, ..., y_{d-1})$, and, considering $v = \frac{\partial}{\partial x}$ we get the following local linear representation of the flow $Y$, say valid for $\|y, z\|$ very small:

$$\hat{X}^t((0, y_1, ..., y_{d-1})) = tv + S^t((0, y_1, ..., y_{d-1}),$$

(5)

where, in rough terms, $S^t$ represents the action of the linear Poincaré flow $P^t_Y$. We observe that, since the flow is linear, the linear Poincaré flow equals the Poincaré map itself (see [10] for full details). Finally, and after performing the perturbations we want, we use the Pasting Lemma (see [2]) to spread (in a volume-preserving way) the linear vector field into a divergence-free vector field that coincides with the original vector field outside a small neighborhood of the periodic orbit.

Lemma 12. If $X \in X^1_{\mu}(M)$ and $X^t$ is $C^1$-stably average shadowable, then any $Y \in X^1_{\mu}(M)$ sufficiently $C^1$-close to $X$ does not contains closed orbits with trivial real spectrum.

Proof. The proof is by contradiction. Taking into account Theorem [17] let us assume that there exists a $C^1$-stably average shadowable incompressible flow $X^t$ having a non-hyperbolic closed orbit $q$ of period $\pi$ and with trivial real spectrum.

Now, we consider a representation of $X^t$, say $\hat{X}^t$, in the linear coordinates given in (5). Thus, there exists an eigenvalue, $\lambda$, with modulus equal to one for $S^\pi(\hat{0})$ (where $\hat{0}$ is the image of $q$ in this change of coordinates). Thus,

$$\hat{X}^{2\pi}(0, y_1, ..., y_{d-1}) = 2\pi v + S^{2\pi}(0, y_1, ..., y_{d-1}) = \lambda id$$

holds, say in a $\xi$-neighborhood of $\hat{0}$. Recall that, since $\hat{X}^t$ has the average shadowing property $\hat{X}^{2\pi t}$ also has.
Take two points \( w_1 = (0, y_1, \ldots, y_{d-1}) \) and \( w_2 = -w_1 \), with \( d(w_1, w_2) = \xi/2 \) and take \( \epsilon := d(w_1, w_2)/3 \). Given \( \delta > 0 \), as in the definition of average shadowing, pick \( n_0 = n_0(\delta, \epsilon) \) a sufficiently large positive integer such that \( \xi/n_0 < \delta \).

Then, we define a sequence \( (x_i, t_i)_{i \in \mathbb{Z}} \) which \( x_i \) takes the values \( w_1 \) and \( w_2 \) alternately by steps of length \( 2^j \), in the following way:

\[
(7) \quad x_i = w_1, \quad t_i = 1 \quad \text{if } 2^{2j} \leq i < 2^{2j+1}, \\
x_i = w_2, \quad t_i = 1 \quad \text{if } 2^{2j+1} \leq i < 2^{2(j+1)}
\]

for \( j \in \mathbb{Z} \).

The sequence \( (x_i, t_i)_{i \in \mathbb{Z}} \) is a \( \delta \)-average-pseudo-orbit of \( \hat{X}^{2\pi} \). Indeed, take \( m \in \mathbb{N} \) large enough, such that \( m > 2^n_0 \). Then \( m = 2^n + r \), with \( n \geq n_0, n \in \mathbb{N} \), and \( r \in \{0, \ldots, 2^n - 1\} \). So,

\[
(8) \quad \frac{1}{m} \sum_{k=1}^{m} d(\hat{X}^{2\pi(t_{k+i})}(x_{k+i}), x_{k+i+1}) = \frac{1}{2^{n} + r} \sum_{k=1}^{2^n + r} d(\hat{X}^{2\pi(t_{k+i})}(x_{k+i}), x_{k+i+1}) \\
\leq \frac{1}{2^n} \sum_{k=1}^{2^n} d(\hat{X}^{2\pi(t_{k+i})}(x_{k+i}), x_{k+i+1}) \\
< \frac{n}{2^n} \cdot \frac{\xi}{2} < \frac{1}{n} \cdot \frac{\xi}{2} < \delta.
\]

Denote \( \hat{X}^{2\pi} \) by \( X \). So the \( \delta \)-pseudo orbit \( (x_i, t_i)_{i \in \mathbb{Z}} \) can be \( \epsilon/2 \) positively shadowed in average by the orbit of \( X \) through some point \( z \in M \), that is, there is an orientation preserving homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) with \( h(0) = 0 \) such that

\[
(9) \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{i}^{i+1} d(X^{h(t)}(x), X^{t-i}(x_i))dt < \epsilon/2.
\]

We can assume without loss of generality that \( d(z, w_1) < \epsilon/2 \). Thus,
\[
d(X^{h(t)}(z), X^{t-i}(w_2)) > 2\epsilon \quad \text{for all } t \in \mathbb{R}.
\]

This implies that for every \( n > n_0 \) enough large,

\[
\frac{1}{n} \sum_{i=1}^{n} \int_{i}^{i+1} d(X^{h(t)}(x), X^{t-i}(x_i))dt > \epsilon,
\]

and therefore,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{i}^{i+1} d(X^{h(t)}(x), X^{t-i}(x_i))dt > \frac{\epsilon}{2}.
\]

This contradicts \( (9) \).

\[\]

**Lemma 13.** If \( X \in \mathcal{X}^1(M) \) and \( X^t \) is \( C^1 \)-stably asymptotic average shadowable, then any \( Y \in \mathcal{X}^1(M) \) sufficiently \( C^1 \)-close to \( X \) does not contain closed orbits with trivial real spectrum.

The proof follows analogously to Lemma 12.

**Proof.** Suppose that there exists a \( C^1 \)-stably asymptotic average shadowable incompressible flow \( X^t \) having a non-hyperbolic closed orbit \( q \) of period \( \pi \) and with trivial real spectrum, since, by Theorem 17, the vector field \( X \) has no singularities.

Consider \( \hat{X}^t \) a representation of \( X^t \). Then, by \( (6) \), \( \hat{X}^{2\pi t} = id \) in a \( \xi \)-neighborhood of \( 0 \). Recall that, since \( \hat{X}^t \) has the asymptotic average shadowing property, \( \hat{X}^{2\pi t} \) also has.
Take two points \( w_1 = (0, y_1, \ldots, y_{d-1}) \) and \( w_2 = -w_1 \), with \( d(w_1, w_2) = \xi/2 \) and take \( \epsilon := d(w_1, w_2)/3 \). Consider \( (x_i, t_i)_{i \in \mathbb{Z}} \) the sequence defined in (7). Observe that the sequence is a asymptotic average pseudo orbit. In fact, for \( m \in \mathbb{N} \) large enough, the inequality (8) implies that,

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} d(X^{t_k}(x_k), x_{k+1}) = 0.
\]

Denote \( \hat{X}^{2\pi} \) by \( X^1 \). So the asymptotic average pseudo orbit \( (x_i, t_i)_{i \in \mathbb{Z}} \) can be \( \epsilon/2 \) positively asymptotic shadowed in average by the orbit of \( X \) through some point \( z \in M \), that is, there is an orientation preserving homeomorphism \( h: \mathbb{R} \to \mathbb{R} \) with \( h(0) = 0 \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} d(X^{h(t)}(x), X^{t-i}(x)) dt = 0.
\]

We can assume without loss of generality that \( d(z, w_1) < \epsilon/2 \). This implies that for \( n \) sufficiently big,

\[
\frac{1}{n} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} d(X^{h(t)}(x), X^{t-i}(x)) dt > \epsilon.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} d(X^{h(t)}(x), X^{t-i}(x)) dt = 0,
\]

which contradicts (10). \( \square \)

3.2. **Rule out singularities.** In order to prove that there are no coexistence of singularities and the \( C^1 \)-stably (asymptotic) average shadowable incompressible flows we will recall some useful results.

**Lemma 14.** [9, Lemma 3.3] Let \( \sigma \) be a singularity of \( X \in \mathcal{X}_\mu^1(M) \). For any \( \epsilon > 0 \) there exists \( Y \in \mathcal{X}_\mu^\infty(M) \), such that \( Y \) is \( \epsilon \)-\( C^1 \)-close to \( X \) and \( \sigma \) is a linear hyperbolic singularity of \( Y \).

The second one, was proved in [46, Proposition 4.1] generalizing the Doering theorem in [20]. Observe that, in our volume-preserving context, the singularities of hyperbolic type are all saddles.

**Proposition 15.** If \( Y \in \mathcal{X}_\mu^1(M) \) admits a linear hyperbolic singularity of saddle-type, then the linear Poincaré flow of \( Y \) does not admit any dominated splitting over \( M \setminus \text{Sing}(Y) \).

Finally, since by Poincaré recurrence, any \( X \in \mathcal{X}_\mu^1(M) \) is chain transitive, the following result is a direct consequence of [7].

**Proposition 16.** In \( \mathcal{X}_\mu^1(M) \) chain transitive flows are equal topologically mixing flows in a \( C^1 \)-residual subset.

The following theorem is proved using an analog reasoning as the one in [4, Theorem 15].

**Theorem 17.** If \( X^t \) is a \( C^1 \)-stably (asymptotic) average shadowable incompressible flow, then \( X^t \) has no singularities.
Proof. Let $X \in \mathcal{X}^1_\mu(M)$ be with $X^t$ $C^1$-stably (asymptotic) average shadowable. Fix a small $C^1$ neighborhood $\mathcal{U} \subset \mathcal{X}^1_\mu(M)$ of $X$. The proof is by contradiction. Assume that $\text{Sing}(X) \neq \emptyset$. Using Lemma 14 there exists $Y \in \mathcal{U}$ with a linear saddle-type singularity $\sigma \in \text{Sing}(Y)$. By Proposition 16 there exist $Z_n \in \mathcal{X}^1_\mu(M)$ $C^1$-close to $Y$ which is topologically mixing. We can find $W_n \in \mathcal{X}^1_\mu(M)$ $C^1$-close to $Z_n$ having a $W_n$-closed orbit $\mathcal{O}(p_n)$ such that the Hausdorff distance between $M$ and $\cup_t W_n^t(\mathcal{O}(p_n))$ is less than $1/n$.

Now we consider jointly Lemma 11 and Lemma 12 (Lemma 13) and obtain that $P^t_{W_n}$ is $\ell$-dominated over the $W_n$-orbit of $\mathcal{O}(p_n)$ where $\ell$ is uniform on $n$. Since $W_n$ converges in the $C^1$-sense to $Y$ and $\limsup_n \cup_t W_n^t(\mathcal{O}(p_n)) = M$ we obtain that $M \setminus \text{Sing}(Y)$ has an $\ell$-dominated splitting which contradicts Proposition 15.

The proof of Theorem B similarly follows the proof of the Theorem 17.

Proof. (of Theorem B)

Let $X \in \mathcal{X}^1_\mu(M)$ be a vector field with $X^t$ $C^1$-stably average shadowable ($C^1$-stably asymptotic average shadowable) flow and fix a small $C^1$ neighborhood $\mathcal{U} \subset \mathcal{X}^1_\mu(M)$ of $X$. By Proposition 16 there exists $Z_n \in \mathcal{X}^1_\mu(M)$ $C^1$-close to $X$ which is topologically mixing. We can find $W_n \in \mathcal{X}^1_\mu(M)$ $C^1$-close to $Z_n$ having a $W_n$-closed orbit $\mathcal{O}(p_n)$ such that the Hausdorff distance between $M$ and $\cup_t W_n^t(\mathcal{O}(p_n))$ is less than $1/n$. Now, we consider together Lemmas 11, 12 (Lemma 13) obtaining that $P^t_{W_n}$ is $\ell$-dominated over the $W_n$-orbit of $\mathcal{O}(p_n)$ where $\ell$ is uniform on $n$. Since $W_n$ converges in the $C^1$-sense to $X$ and

$$\limsup_n \cup_t W_n^t(\mathcal{O}(p_n)) = M$$

we obtain that $M \setminus \text{Sing}(X)$ has an $\ell$-dominated splitting. Now, if $X$ has a singularity $\sigma$, then using Lemma 14 there exists $Y \in \mathcal{X}^1_\mu(M)$ $C^1$-close to $X$ with a linear saddle-type singularity, and proceeding as above $P^t_Y$ admits a $\ell$-dominated splitting over $M \setminus \text{Sing}(Y)$, which contradicts Proposition 15. Therefore, $M$ admits an $\ell$-dominated splitting.

3.3. Limit shadowing property - Proof of Theorem B’. The proof of Theorem B’ is analogous to the proof of Theorem A’ and so we will not give the proof in details. Instead, we will point out the necessary steps to adapt the proof of Theorem A’ in this context.

We observe that Lemma 6 stated in Section 2.2 holds in this setting (incompressible flows). Now, we need to check Lemmas 7 and 8 for incompressible flows.

We start recalling that for conservative vector fields $\mathcal{X}^1_\mu(M)$, whose dimension $d$ of $M$ is greater than or equal to 3, it is proved in [12] the existence of a residual subset of vector fields such that every singularity and periodic orbit is hyperbolic (or elliptic, if $d = 3$), and the corresponding invariant manifolds intersect transversely. We have also a kind of Franks’ lemma for conservative flows [9, Lemma 3.2]. Using these notions, we can give a version of Lemma 7 for conservative vector fields.

Lemma 18. If $X^t$ is an incompressible flow which is $C^1$-stably limit shadowable, then all periodic orbits of $X$ are hyperbolic.

Proof. Suppose that the dimension of $M$ is 3. Let $\mathcal{U}$ be a neighborhood of $X$ as definition of robustness of the limit shadowing property. Let $p$ and $q$ periodic points of $X$, and suppose $\mathcal{O}(p)$ is a non-hyperbolic periodic orbit of $X$. Using the [8, Theorem 1.3] there exists $Y$ in $\mathcal{U}$ with an elliptic periodic orbit $\mathcal{O}(p_1)$ and a hyperbolic periodic orbit $\mathcal{O}(p_2)$. By stability of elliptic points ([8]), we can consider $Y$ a Kupka-Smale. Then, dim
$W^s(O(p_1)) + \dim W^u(O(p_2)) \leq \dim M$. As $Y$ is Kupka-Smale vector field, we have that $W^s(O(p_1)) \cap W^u(O(p_2)) = \emptyset$, which contradicts the Lemma 8 and prove the desired.

Now, we prove the case that dimension of $M$ is greater than $3$. Let $\mathcal{V}$ be a neighborhood of $X$ such that the limit shadowing property is verified. Let $p$ and $q$ periodic points of $X$ and suppose that $O(p)$ is a non-hyperbolic periodic orbit of $X$. By [9] Lemma 3.2, there exists $Y$ $C^1$-close to $X$ with $O(p_1)$ and $O(q_1)$ hyperbolic periodic orbit of $Y$. Since we can consider $Y$ a Kupka Smale vector field ([12]), we have that $W^s(O(p_1)) \cap W^u(O(q_1)) = \emptyset$. This contradicts Lemma 6 for incompressible flows, and prove the desired. □

Now, proceeding as Lemma 8 we obtain the following result.

Lemma 19. A incompressible flow $X^t$ which is $C^1$-stably limit shadowable has no singularities.

We recall that an incompreensible flow is said to be an incompressible star flow if there exists a $C^1$-neighborhood $\mathcal{U}$ of $X$ in $X^1_\mu(M)$ such that any critical orbit of any $Y \in \mathcal{U}$ is hyperbolic. A consequence of Lemmas 18 and 19 is the following result.

Corollary 20. If $X \in X^1_\mu(M)$ is $C^1$-stably limit shadowing shadowable then $X^t$ is a star flow without singularities.

Theorem 21. ([12] Theorem 1) If $X \in X^1_\mu(M)$ is a star flow without singularities, then $X^t$ is a transitive Anosov flow.

The proof of Theorem B’ follows directly by Corollary 20 and Theorem 21.

4. Hamiltonian flows-proof of Theorems C and C'

Observe that, in the Hamiltonian context, we only consider regular energy surfaces, thus we do not have to deal with singularities.

In [13] it is proved the following result which is the Hamiltonian version of the main result in [7].

Theorem 22. ([13] Theorem 1) There is a residual $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H_0 \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$, the Hamiltonian level $(H, e)$ is topologically mixing.

Now, we state the following Hamiltonian version of Theorem 4 and Lemma 11 and which was proved in [11] Theorem 3.4:

Theorem 23. Let $H \in C^2(M, \mathbb{R})$ and $U$ be a neighborhood of $H$ in the $C^2$-topology. Then for any $\varepsilon > 0$ there are $\ell, \pi_0 \in \mathbb{N}$ such that, for any $H_0 \in U$ and for any periodic point $p$ of period $\pi(p) \geq \pi_0$:

1. either $\Phi^t_{H_0}(p)$ admits an $\ell$-dominated splitting along the orbit of $p$;

2. or, for any tubular flowbox neighborhood $\mathcal{T}$ of the orbit of $p$, there exists an $\varepsilon$-$C^2$-perturbation $H_1$ coinciding with $H_0$ outside $\mathcal{T}$ and whose transversal linear Poincaré flow $\Phi^t_{H_1}(p)$ has all eigenvalues with modulus equal to 1.

Proof. (of Theorem C)

By Theorem 22 exist $H_n \in C^2(M, \mathbb{R})$ $C^2$-close to $H$ and $\varepsilon$ arbitrarily close to $e$, such that $\mathcal{E}_{H_n, \varepsilon}$ is topologically mixing. We can find $H_{n} \in C^2(M, \mathbb{R})$ $C^2$-close to $H_n$ and $\varepsilon$ close to $e$ having a $X^t_{H_n}$-closed orbit $p_n$ such that the Hausdorff distance between $\mathcal{E}_{H_n, \varepsilon}$ and $\bigcup_{\ell} X^t_{H_n}(p_n)$ is less than $1/n$. 

□
Then, we follow the steps of Lemma 12 (Lemma 13) but using the formalism developed in Lemma 6.1 obtaining that $P_{H,n}^t$ is $\ell$-dominated over the $X_{H,n}^t$-orbit of $p_n$ where $\ell$ is uniform on $n$. Since $\hat{H}_n$ converges in the $C^2$-sense to $H$ and

$$\lim_{n} \sup_{t} \bigcup_{p_n} X_{H,n}^t(p_n) = E_{\hat{H}_n, \hat{e}}$$

we obtain that $E_{\hat{H}_n, \hat{e}}$ has an $\ell$-dominated splitting. Therefore, $E_{\hat{H}_n, \hat{e}}$ admits an $\ell$-dominated splitting. By Remark 2.1 $E_{\hat{H}_n, \hat{e}}$ is partial hyperbolic. Finally, we observe that partial hyperbolicity spreads to the closure and we are over.

4.1. Limit shadowing property - Proof of Theorem C'. Now, we prove Theorem C'.

The proof of this theorem follows as in Theorem A'. So, we only mention the version for Hamiltonian systems of results in Subsection 2.2 and the modifications necessary to obtain our result.

Firstly, note that Lemma 6 is true for Hamiltonian systems. Now, we need to check the Lemmas 7 and 8 in this context. For this we recall the notions of Kupka-Smale Hamiltonian systems. A Hamiltonian system $(H, e, E_{H,e})$ is a Kupka-Smale Hamiltonian system if the union of the hyperbolic and $k$-elliptic closed orbits $(1 \leq k \leq n-1)$ in $E_{H,e}$ is dense in $E_{H,e}$ and the intersection of invariant of the closed orbits intersect transversally. Furthermore, the Kupka Smale Hamiltonian systems form a residual in $C^2(M, \mathbb{R})$. See Theorem 1 and 2.

Now, we state Lemma 7 for Hamiltonian systems.

Lemma 24. If a Hamiltonian system $(H, e, E_{H,e})$ is $C^2$-stably limit shadowable, then all its periodic orbits are hyperbolic.

Proof. Let $U$ be a neighborhood of $(H, e, E_{H,e})$ as in the definition of robustness of limit shadowing property. Let $O(p)$ and $O(q)$ periodic orbits of $H$. Suppose that $O(p)$ is a non-hyperbolic periodic orbit associated to point $p \in E_{H,e}$. By Lemma 6.6 there exists a Kupka Smale Hamiltonian system $(H_0, e_0, E_{H_0,e_0}) \in U$ such that $H_0$ has a non-hyperbolic periodic orbit $O(p_1)$ and a hyperbolic periodic orbit $O(q_1)$.

Then $\dim W^u(O(p_1)) + \dim W^s(O(q_1)) \leq 3 \dim M$. Since the Hamiltonian system is Kupka Smale, we have that $W^u(O(p_1)) \cap W^s(O(q_1)) = \emptyset$. This contradicts the Lemma 6 and prove the desired. \hfill \Box

Finally, using the Franks Lemma for Hamiltonians Theorem 1) we obtain the version for Hamiltonian systems of Lemma 8. To complete the proof we recall the notion of star Hamiltonian system. A Hamiltonian systems $(H, e, E_{H,e})$ is a star Hamiltonian system if there exists a neighborhood $V$ of $(H, e, E_{H,e})$ such that, for any $(\tilde{H}, \tilde{e}, \tilde{E}_{H,e}) \in V$, the correspondent regular energy hypersurface $\tilde{E}_{H,e}$ has all the critical orbits hyperbolic.

As consequence of this results for Hamiltonian systems, we obtain the following result.

Corollary 25. If $(H, e, E_{H,e})$ is a Hamiltonian system $C^2$-robustly limit shadowable, then $(H, e, E_{H,e})$ is a Hamiltonian star systems.

In Theorem 1) is was proved that a Hamiltonian star system, defined on a 2d-dimensional $(d \geq 2)$ is Anosov. This concluded the proof of Theorem C'.

\(^3\)Notice that, in the symplectic context, hyperbolic closed orbits have constant index, however, non hyperbolic closed orbits may not display this property.
ACKNOWLEDGEMENTS

The authors would like to thank Alexander Arbieto for suggestions given during the preparation of this work. Mário Bessa was partially supported by National Funds through FCT - “Fundação para a Ciência e a Tecnologia”, project PEst-OE/MAT/UI0212/2011.

REFERENCES

[1] R. Abraham and J.E. Marsden, Foundations of Mechanics. The Benjamin/Cummings Publishing Company. Advanced Book Program, 2nd edition (1980).

[2] A. Arbieto and C. Matheus, A pasting lemma and some applications for conservative systems. With an appendix by David Dílica and Yakov Simpson-Weller, Ergod. Th. & Dynam. Sys. 27 (2007), 1399–1477.

[3] A. Arbieto, J. Reis and R. Ribeiro, On various types of shadowing for geometric Lorenz flows. Preprint ArXiv: 1306.2061, 2013.

[4] A. Arbieto and R. Ribeiro, Flows with the (asymptotic) average shadowing property on three-dimensional closed manifolds. Dynamical Systems, 26, 4 (2011), 425–432.

[5] M. U. Akhmet, Shadowing and dynamical synthesis. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19 (2009), no. 10, 3339–3346.

[6] A. Z. Bahabadi, Divergence-free vector fields with average and asymptotic average shadowing property. Chaos Solitons Fractals, 45 (2012), 1358–1360.

[7] M. Bessa, A Generic incompressible flows are topological mixing. Comptes Rendus Mathematique, 346 (2008), 1169–1174.

[8] M. Bessa and P. Duarte, Abundance of elliptic dynamics on conservative 3-flows. Dynamical Systems – An international Journal, 23, 4 (2008), 409–424.

[9] M. Bessa and J. Rocha, On C\(^1\)-robust transitivity of volume-preserving flows. Jr. Diff. Eq., 245, 11 (2008), 3127–3143.

[10] M. Bessa and J. Rocha, Homoclinic tangencies versus uniform hyperbolicity for conservative 3-flows. Jr. Diff. Eq., 247, 11 (2009), 2913–2923.

[11] M. Bessa, J. Rocha and M. J. Torres, Hyperbolicity and Stability for Hamiltonian flows. Jr. Diff. Eq., vol 254, 1 (2013), 309–322.

[12] M. Bessa, J. Rocha and M. J. Torres, Shades of Hyperbolicity for Hamiltonians. Preprint ArXiv: 1212.4874, 2012.

[13] M. Bessa, C. Ferreira and J. Rocha, Generic Hamiltonian Dynamics. Preprint [arXiv:1203.3839], 2012.

[14] M. Bessa, M. Lee and S. Vaz, Stably weakly shadowable volume-preserving systems are volume-hyperbolic. Preprint ArXiv: 1207.5546, 2012.

[15] M. Bessa and J. Rocha, Topological stability for conservative systems. Jr. Diff. Eq. 250, 10 (2011), 3960–3966.

[16] M. L. Blank, Metric properties of minimal solutions of discrete periodical variational problems. Nonlinearity, 2 (1989), no. 1, 1–22.

[17] J. Bochi and M. Viana, Lyapunov exponents: How frequently are dynamical systems hyperbolic? in Modern Dynamical Systems and Applications, 271–297, Cambridge Univ. Press, Cambridge, 2004.

[18] C. Bonatti, N. Gourmelon and T. Vivier, Perturbations of the derivative along periodic orbits. Ergod. Th. & Dynam. Sys. 26, 5 (2006), 1307–1337.

[19] S. Crovisier, Periodic orbits and chain-transitive sets of C\(^1\)-diffeomorphisms. Publ. Math. Inst. Hautes Etudes Sci. no. 104 (2006), 87–141.

[20] C. Doering, Persistently transitive vector fields on three-dimensional manifolds. Proceedings on Dynamical Systems and Bifurcation Theory, Vol. 160 (1987), 59–89, Pitman.

[21] T. Eirola, O. Nevanlinna and S. Y. Pilyugin, Limit shadowing property. Numer. Funct. Anal. Optim. 18 (1997), no. 1-2, 7592.

[22] C. Ferreira, Stability properties of divergence-free vector fields. Dyn. Syst. 27 (2012), no. 2, 223238.

[23] C. Ferreira, Shadowing, expansiveness and stability of divergence-free vector fields. Preprint ArXiv: 1011.3546, 2010.

[24] J. Franks and B. Williams, Anomalous Anosov flows. Global Theory of Dynamical Systems Lecture Notes in Mathematics, Volume 819 (1980), 158–174.

[25] S. Gan and L. Wen, Nonsingular star flows satisfy Axiom A and the no-cycle condition. Invent. Math. 164 (2006), no. 2, 279-315.

[26] R. Gu, The asymptotic average shadowing property and transitivity. Nonlinear Anal. 67 (2007), no. 6, 1680-1689.
[27] R. Gu, *The asymptotic average-shadowing property and transitivity for flows*. Chaos Solitons Fractals 41 (2009), no. 5, 2234-2240.
[28] R. Gu, Y. Sheng and Z. Xia, *The average-shadowing property and transitivity for continuous flows*. Chaos Solitons Fractals, 23 (2005), no. 3, 989-995.
[29] M. Hirsh, C. Pugh and M. Shub, *Invariant Manifolds*. Lecture Notes in Math. 583 1997, Springer-Verlag.
[30] M. Komuro, *Lorenz attractors do not have the pseudo-orbit tracing property*. J. Math. Soc. Japan 37 (1985), no. 3, 489-514.
[31] I. Kupka, *Contribution à la théorie des champs génériques*. Contributions to Differential Equations, 2 (1963), 457-484.
[32] M. Lee, *Average shadowing property for volume preserving diffeomorphisms*. Far East J. Math. Sci. 64 (2012), 2, 261–267.
[33] M. Lee and X. Wen, *Diffeomorphisms with C^1-stably average shadowing*. Acta Math. Sin. 29, 1 (2013), 85–92.
[34] T. Ma and S. Wang, *Geometric Theory of Incompressible Flows with Applications to Fluid Dynamics*. Math. Surveys and Monographs, 119, AMS 2005.
[35] K. Moriyasu, K. Sakai and S. Naoya, *Vector Fields with Topological Stability*. Trans. Amer. Math. Soc. 353, no. 8, 3391-3408, 2001
[36] J. Moser, *On the volume elements on a manifold*. Trans. Amer. Math. Soc. 120 (1965), 286–294.
[37] J. Moser and E. Zehnder, *Notes on dynamical systems*. Courant Lecture Notes in Mathematics, 12. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2005.
[38] Y. Niu, *The average-shadowing property and strong ergodicity*. J. Math. Anal. Appl. 376 (2011), no. 2, 528-534.
[39] D. Orrell, *Estimating error growth and shadow behavior in nonlinear dynamical systems*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 15 (2005), no. 10, 32633280.
[40] S. Pilyugin, *Shadowing in Dynamical Systems*. Lecture Notes in Math., 1706, Springer-Verlag, Berlin, 1999.
[41] R. Ribeiro, *Hyperbolicity and types of shadowing for C^1-generic vector fields*. Preprint [arXiv:1305.2977], 2013.
[42] C. Robinson, *Generic properties of conservative systems*. Amer. J. Math. 92 (1970), 562-603.
[43] M. Shub, *Global Stability of Dynamical Systems*. Springer-Verlag, New York, 1987.
[44] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*. Ann. Sc. Norm. Super. Pisa, III. Ser. 17, (1963) 97-116.
[45] P. Walters, *On the pseudo-orbit tracing property and its relationship to stability*. The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D. 1977, 231–244, Lecture Notes in Math. 668, Springer, Berlin, 1978.
[46] T. Vivier, *Projective hyperbolicity and fixed points*. Ergod. Th. & Dynam. Sys. 26 (2006), 923–936.
[47] L. Xiliang, L. Xuemei and Z. Zuohuan, *Homoclinic shadowing and its application to chaotic systems*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 18 (2008), no. 5, 13631375.

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