We study approximate decimations in SU(N) LGT that connect the short to long distance regimes. Simple ‘bond-moving’ decimations turn out to provide both upper and lower bounds on the exact partition function. This leads to a representation of the exact partition function in terms of successive decimations whose effective couplings flows are related to those of the easily computable bond-moving decimations. The implications for a derivation of confinement from first principles are discussed.

1. Introduction

Over the last several years an enormous amount of work has been performed by lattice workers on the physics of the QCD vacuum. In particular, isolating the types of configurations in the functional measure that are responsible for maintaining confinement at (arbitrarily) weak coupling has been a central issue. A great deal of information concerning the confinement mechanism has been obtained from these investigations (for recent review, see\(^1\)). However, the goal of a direct derivation of confinement from first principles has remained elusive for the last thirty years.

The origin of the difficulty is clear. One is faced with a multi-scale problem involving the passage from the short-distance weakly coupled, ordered regime to the long distance strongly coupled, disordered, confining regime. Such variable multi-scale behavior can only be addressed by some nonperturbative block-spinning or decimation procedure capable of bridging these different regimes. Exact decimation schemes appear analytically hopeless, and numerically very difficult. It is not even clear what a good definition

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of block-spin variables would be. There is, however, a class of approximate simple decimation procedures which are known in many cases to give qualitatively correct results. They are generally known as ‘bond moving’ decimations. Here we will consider such decimations in a somewhat more general form and show that they can provide bounds on the exact theory. This leads to a representation of the partition function of the exact theory which allows a connection to be made to the behavior of the approximate, but easily computable, decimations at successive length scales. The implications for the question of an actual derivation of confinement in LGT will be discussed below.

The framework applies to general $SU(N)$, though explicit numerical or analytical calculations supporting the considerations below have for the most part been carried out only for $SU(2)$.

2. Bond moving decimations

Starting with some plaquette action, e.g the Wilson action $A_p(U) = \frac{\beta}{N} \text{Tr} U_p$, at lattice spacing $a$, we consider the character expansion of the exponential of the action:

$$F(U, a) = e^{A_p(U)} = \sum_j F_j(\beta, a) d_j \chi_j(U) \quad (1)$$

with Fourier coefficients:

$$F_j = \int dU F(U, a) \frac{1}{d_j} \chi_j(U). \quad (2)$$

Here $\chi_j$ denotes the character of the $j$-th representation of dimension $d_j$. $j = 0$ will always denote the trivial representation. E.g, for $SU(2)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and $d_j = (2j + 1)$. In terms of normalized coefficients:

$$c_j = \frac{F_j}{F_0}, \quad (3)$$

one then has

$$F(U, a) = F_0 \left[ 1 + \sum_{j \neq 0} c_j(\beta) d_j \chi_j(U) \right] \equiv F_0 f(U, a) \quad (4)$$

For a reflection positive action one necessarily has:

$$F_j \geq 0 \quad \text{hence} \quad 1 \geq c_j \geq 0 \quad \text{all} \quad j. \quad (5)$$
The partition function on lattice $\Lambda$ is then
\begin{equation}
Z^{\Lambda}(\beta) = F^{|\Lambda|}_0 \int dU_\Lambda \prod_p f_p(U, a).
\end{equation}

We now consider RG decimation transformations $a \to \lambda a$ in, say, the $x^1$-direction (Figure 1). Simple approximate transformations of the ‘bond moving’ type are implemented by ‘weakening’, i.e. decreasing the $c_j$’s of interior plaquettes (shaded), and ‘strengthening’, i.e. increasing $c_j$’s of boundary plaquettes (bold) in every decimation cell of side length $\lambda$. The simplest scheme\(^2\), which is adopted in the following, implements complete removal, $c_j = 0$, of interior plaquettes. This is performed simultaneously in all directions (Figure 2).

Under successive decimations
\begin{align*}
a &\to \lambda a \\
\Lambda &\to \Lambda^{(1)} \to \Lambda^{(2)} \to \cdots \to \Lambda^{(n)}
\end{align*}
the RG transformation rule is then:
\begin{equation}
f(U, n-1) \to f(U, n) = \left[ 1 + \sum_{j \not= 0} c_j(n) d_j \chi_j(U) \right]
\end{equation}
with:
\begin{align}
c_j(n) &= F_j(n)/F_0(n) \, , \quad F_j(n) = \left[ \hat{F}_j(n) \right]^\lambda \, , \\
\hat{F}_j(n) &= \int dU \left[ f(U, n-1) \right]^{\nu} \frac{1}{d_j} \chi_j(U).
\end{align}

The parameter $\nu$ controls by how much the remaining plaquettes have been strengthened to compensate for the removed plaquettes. What has been

Figure 1. Basic bond (plaquette) moving operation.
considered in the literature before is \( \nu = \lambda^{(d-2)} \), where \( d \) is the spacetime dimension. This choice of \( \nu \) defines the MK recursions\(^2\). Here we generalize to consider \( \nu \) an arbitrary parameter.

The resulting partition function after \( n \) decimation steps is:

\[
Z_\Lambda(\beta, n) = \prod_{m=0}^{n} F_0(m)^{\Lambda/\lambda^{md}} \int dU_{\Lambda(n)} \prod_p f_p(U, n) .
\]  

(10)

It is important to note that after each decimation step the resulting action retains the original one-plaquette form but will, in general, contain all representations:

\[
A_p(U, n) = \sum_j \hat{\beta}_j(\beta) \chi_j(U) .
\]  

(11)

Furthermore, among the effective couplings \( \hat{\beta}_j \) some negative ones may in general occur. These features are present even after a single decimation step \( a \to \lambda a \) starting with the usual single representation (fundamental) Wilson action.

Preservation of the one-plaquette form of the action is of course what makes these decimations simple to explore. The rule specified by (7)- (9) is meaningful for any real (positive) \( \nu \). Here, however, a basic distinction can be made. For integer \( \nu \), the important property of positivity of the Fourier
coefficients in (1), (4):
\[ F_0(n) \geq 0, \quad c_j(n) \geq 0, \quad (12) \]
and hence reflection positivity are maintained at each decimation step.
This, in general, is not the case for non-integer \( \nu \). Thus non-integer \( \nu \) results in approximate RG transformations that violate the reflection positivity of the theory (assuming a reflection positive starting action).\(^a\)

There are various other interesting features of such decimations. The following property, in particular, is important. Define a normalized \( \hat{F}_j(n) \) (cf. (9)):
\[ \hat{c}_j(n) = \hat{F}_j(n) / \hat{F}_0(n) \leq 1, \quad \text{so that} \quad c_j(n) = \hat{c}_j(n)^{\lambda^2}. \quad (13) \]
Then it is possible to prove that
\[ \sum_j c_j(n) (\hat{c}_j(n + 1) - c_j(n)) \geq 0. \quad (14) \]
It follows from (14) that the norm (\( l_2 \) norm) of the vector formed from the \( \hat{c}_j(n + 1) \) coefficients is bigger than that of the vector of the \( c_j(n) \). In fact one finds in explicit numerical evaluations that (14) holds component-wise, i.e.
\[ \hat{c}_j(n + 1) \geq c_j(n). \]

As can be seen from the relation between \( \hat{c}_j(n + 1) \) and \( c_j(n + 1) \) in (13), (9), however, it can still be that the norm of the \( c_j(n + 1) \)'s is smaller than that of the \( c_j(n) \)'s. i.e. the norm of the normalized coefficients \( c_j(n) \) in (7) decreases under successive decimations. Note, in particular, that when \( \nu \) is taken to depend on \( \lambda, d \), the resulting highly nonlinear dependence can give very nontrivial behavior. This is in fact what happens in the case of the MK recursions where \( \nu = \lambda(d-2) \); the normalized coefficients \( c_j(n) \) do decrease under successive decimations in the approach to a fixed point in lower dimensions. But an upper critical dimension arises above which the \( \hat{c}_j(n + 1) \)'s become sufficiently large compared to the \( c_j(n) \)'s so that this is no longer the case, and triviality ensues (for the RG flow on the weak coupling side).

3. The exact partition function
Since our decimations are not exact decimation transformations, the partition function does not in general remain invariant under them. The subse-\(^a\)It is worth noting in this context that numerical investigations of the standard MK recursions, at least for gauge theories, appear to have been carried out for the most part for fractional \( \lambda \), (1 < \( \lambda < 2 \), which corresponds to non-integer \( \nu \); e.g. see\(^4\).
quent development hinges on the following two basic statements that can now be proved:

(I) With \( \nu = \lambda^{d-2} \):

\[
Z_{\Lambda}(\beta, n) \leq Z_{\Lambda}(\beta, n + 1). \tag{15}
\]

(II) With \( \nu = 1 \):

\[
Z_{\Lambda}(\beta, n + 1) \leq Z_{\Lambda}(\beta, n). \tag{16}
\]

Note that for \( d = 2 \) (15) - (16) express the well-known fact that the decimations become exact. For \( d > 2 \), in both (I), (II) one in fact has strict inequality.

(I) says that modifying the couplings of the remaining plaquettes after decimation by taking \( \nu = \lambda^{d-2} \) (standard MK choice\(^2\)) results into overcompensation (upper bound on the partition function). Translation invariance and convexity of the free energy as a function of the couplings in the action underlie (15).

(II) says that decimating plaquettes while leaving the couplings of the remaining plaquettes unaffected results in a lower bound on the partition function. Reflection positivity (positivity of Fourier coefficients) is crucial for this to hold.

Consider now the, say, \((n-1)\)-th decimation step with Fourier coefficients \( c_j(n-1) \), which we relabel \( \tilde{c}_j(n-1) = c_j(n-1) \). Given these \( \tilde{c}_j(n-1) \), we proceed to compute the coefficients \( F_0(n) \), \( c_j(n) \) of the next decimation step according to (7)-(9) above with \( \nu = \lambda^{d-2} \).

Then introducing a parameter \( \alpha, (0 \leq \alpha) \), define the interpolating coefficients:

\[
\tilde{c}_j(n, \alpha) = \tilde{c}_j(n-1) \lambda^2(1-\alpha) c_j(n)^\alpha. \tag{17}
\]

Then,

\[
\tilde{c}_j(n, \alpha) = \begin{cases} c_j(n) & : \alpha = 1 \\ \tilde{c}_j(n-1) \lambda^2 & : \alpha = 0 \end{cases} \tag{18}
\]

The \( \alpha = 0 \) value is that of the \( n \)-th step coefficients resulting from (7)-(9) above with \( \nu = \lambda^{d-2} \).

Thus defining the corresponding partition function

\[
Z_{\Lambda}(\beta, \alpha, n) = \left( \prod_{m=0}^{n-1} F_0(m)^{|A|/\lambda^m d} \right) F_0(n)^\alpha \cdot \int dU_{\Lambda(n)} \prod_p f_p(U, n, \alpha) \tag{19}
\]
where
\[ f_p(U, n, \alpha) = \left[ 1 + \sum_{j \neq 0} \tilde{c}_j(n, \alpha) d_j \chi_j(U) \right], \]
we have from (15), (16), and (18) above:
\[ Z_\Lambda(\beta, 0, n) \leq Z_\Lambda(\beta, n - 1) \leq Z_\Lambda(\beta, 1, n). \] (21)

Now the partition function (19) is a continuous, in fact analytic, in \( \alpha \). So (21) implies that, by continuity, there exist a value of \( \alpha \):
\[ \alpha = \alpha^{(n)}(\beta, \lambda, \Lambda), \quad 0 < \alpha^{(n)}(\beta, \lambda, \Lambda) < 1 \]
such that
\[ Z_\Lambda(\beta, \alpha^{(n)}, n) = Z_\Lambda(\beta, n - 1). \] (22)

In other words there is an \( \alpha \) at which the \( n \)-th decimation step partition function equals that obtained at the previous decimation step; the partition function does not change its value under the decimation step \( \lambda^{n-1} a \rightarrow \lambda^n a \).

So starting at original spacing \( a \), at every decimation step \( m, (m = 0, 1, \cdots, n) \), there exist a value \( 0 < \alpha^{(m)} < 1 \) such that
\[ Z_\Lambda(\beta, \alpha^{(m+1)}, m + 1) = Z_\Lambda(\beta, \alpha^{(m)}, m). \] (23)

This then gives, after \( n \) successive decimations, an exact representation of the original partition function in the form:
\[
Z_\Lambda(\beta) = \int dU_\Lambda \prod_p f_p(U, a) = \prod_{m=0}^n F_0(m)^{\alpha^{(m)}|\Lambda|/\lambda^{md}} \cdot \int dU_\Lambda^{(n)} \prod_p f_p(U, n, \alpha^{(n)}), \] (24)
i.e. in terms of the successive bulk free energy contributions from the \( a \rightarrow \lambda \rightarrow \cdots \rightarrow \lambda^n a \) decimations and a one-plaquette effective action on the resulting lattice \( \Lambda^{(n)} \).

The \( \alpha^{(m)} \)'s in (24) may be viewed as effective couplings which, in addition to the \( \{\tilde{\beta}_j^{(m)}\} \), enter in the specification of the effective action and bulk free energy at each decimation step \( m \). Thus the flow from scale \( a \) to scale \( \lambda^n a \) is now specified by \( \{\tilde{\beta}(n, \beta, \lambda), \ \alpha(n, \beta, \lambda, \Lambda)\} \equiv \{ \{\tilde{\beta}_j^{(m)}(\beta, \lambda)\}, \alpha^{(m)}(\beta, \lambda, \Lambda) | m = 0, \ldots, n \} \). This dependence on the additional couplings \( \alpha \) may be considered as compensating for the absence in
(24) of additional terms, beyond the one-plaquette interaction, that would normally be expected in an effective action.

At weak and strong coupling $\alpha(m)$ may be estimated analytically. At large $\beta$, where the decimations approximate the free energy rather accurately, the appropriate $\alpha$ values are very close to unity. At strong coupling they may be estimated by comparison with the strong coupling expansion. On any finite lattice there is also a weak volume dependence as a correction which goes away as an inverse power of the lattice size.

For most purposes the exact values of the $\alpha(m)$'s, beyond the fact that are fixed between 0 and 1, are not immediately relevant. The main point of the representation (24) is that it can in principle relate the behavior of the exact theory to that of (modifications of) the easily computable approximate decimations.

Indeed, starting from the $\tilde{c}_j(n-1,\alpha^{(n-1)})$'s at the $(n-1)$-th step, consider the coefficients at the next step, and compare those evaluated at $\alpha = \alpha^{(n)}$, i.e. $\tilde{c}_j(n,\alpha = \alpha^{(n)})$, to those evaluated at $\alpha = 1$, i.e. $\tilde{c}_j(n,\alpha = 1) \equiv c_j(n)$. The latter will be referred to as the MK coefficients. (Recall that $\alpha = 1 \iff \nu = \lambda^{d-2}$, the standard MK choice. The absence of a tilde on a coefficient in the following always means that it is computed at $\alpha = 1$.) According to (I), the MK coefficients give an upper bound. To facilitate the comparison let us rewrite (17) in the form

$$\tilde{c}_j(n,\alpha) = \left( \frac{\tilde{c}_j(n-1)}{c_j(n)} \right)^{\lambda^2(1-\alpha)} c_j(n). \quad (25)$$

Now property (14) and the remark following it imply that the ratio in the brackets in (25) is less or equal to unity. It follows that

$$\tilde{c}_j(n,\alpha) \leq c_j(n) \quad \text{for any} \quad 0 \leq \alpha \leq 1. \quad (26)$$

This has the following important consequence.

Assume we are in a dimension $d$ such that under successive decimations the MK coefficients ($\alpha = 1$) are non-increasing. Then (26) implies:

$$\tilde{c}_j(n,\alpha^{(n)}) \geq c_j(n+1) \geq \tilde{c}_j(n+1,\alpha^{(n+1)}) \geq c_j(n+2) \geq \tilde{c}_j(n+2,\alpha^{(n+2)}) \geq \cdots$$

Thus, if the $c_j(n)$'s are non-increasing, so are the $\tilde{c}_j(n,\alpha)$. The $c_j(n)$'s must then approach a fixed point, and hence so must the $\tilde{c}_j(n,\alpha)$'s, since $c_j(n), \tilde{c}_j(n,\alpha) \geq 0$. Note the fact that this conclusion is independent of the specific value of the $\alpha$'s at every decimation step.
In particular, if the $c_j(n)$’s approach the strong coupling fixed point, i.e. $F_0 \rightarrow 1$, $c_j(n) \rightarrow 0$ as $n \rightarrow \infty$, so must the $\tilde{c}_j(n, \alpha)$’s of the exact representation. If the MK decimations confine, so do those in the exact representation (24). As it is well-known by explicit numerical evaluation, the MK decimations for $SU(2)$ and $SU(3)$ indeed confine for all $\beta < \infty$ and $d \leq 4$. Above the critical dimension $d = 4$, the decimations result in free spin wave behavior.

4. Discussion and outlook

What do the above results say about the question of confinement in the exact theory? They are clearly strongly suggestive of confinement for all $\beta$ in the exact theory. They cannot, however, as yet be taken to constitute an actual proof. The statement at the end of the previous section concerns the behavior of the long distance action part in the representation (24). Now (24) also includes the large free energy bulk contributions from integration over all scales from $a$ to $\lambda^na$. It is the complete representation involving both contributions that provides an equality to the value of the exact partition function. This, just by itself, does not suffice to rigorously isolate, at least in any direct way, the actual behavior of the corresponding long distance part in the exact theory. To do this one needs to consider order parameters which can directly couple to the corresponding long distance parts of the effective action in the exact theory and any representation of it like that given by (24). In other words, one would need to carry through the above derivation given for the partition function also for the case of appropriate order parameters.

The derivation of the basic two statements (I) and (II) above (eqs. (15), (16)) assumes translation invariance and reflection positivity. In the presence of observables such as a Wilson loop, translation invariance is broken and reflection positivity is reduced to hold only in the plane bisecting the loop. This does not allow the above derivations to be carried through in any obvious way. Fortunately, there are other order parameters that can characterize the possible phases of the theory while maintaining translational invariance. They are the well-known vortex free energy, and its $Z(N)$ Fourier transform (electric flux free energy). They are in fact the natural order parameters in the present context since they are constructed out of partition functions. Recall that the vortex free energy is defined by

$$e^{-F_v(\tau)} = \frac{Z(\Lambda(\tau))}{Z(\Lambda)}.$$  

Here $Z(\Lambda(\tau))$ denotes the partition function with action modified by the
‘twist’ $\tau \in Z(N)$ for every plaquette on a coclosed set of plaquettes $V$ winding through the periodic lattice in $(d - 2)$ directions; e.g. for the Wilson action one has the replacement $\frac{\beta}{N} \text{Re} \text{tr} U_p \rightarrow \frac{\beta}{N} \text{Re} \text{tr} U_p \tau$ for every $p \in V$. The twist represents a discontinuous gauge transformation on the set $V$ which introduces $\pi_1(SU(N)/Z(N))$ vortex flux rendered topologically stable by being wrapped around the lattice torus. As indicated by the notation, $Z_\Lambda(\tau)$ depends only on the presence of the flux, and is invariant under changes in the exact location of $V$. The vortex free energy is then the ratio of the partition function in the presence of this external flux to the partition function in the absence of the flux (the latter is what was considered above). The above development, in particular the derivation of (24), should then be repeated also for $Z_\Lambda(\tau)$. Bulk (local) free energy contributions resulting from integrating over successive scales are insensitive to the presence of the flux. Thus in the analog to (24) for $Z_\Lambda(\tau)$ only the long distance effective action part would be affected by its presence, and the bulk contributions would cancel in (27). Statements, as the ones obtained in the previous section, concerning the behavior of the long distance parts in such representations of the two factors in (27) would then directly constrain the corresponding behavior in the exact theory.

There is, however, an immediate technical complication in obtaining the analog to (24) for $Z_\Lambda(\tau)$. The presence of the flux reduces reflection positivity to hold only in planes perpendicular to the directions in which the flux winds through the lattice. The simple nature of the decimations, however, makes it plausible that this still suffices to allow a generalization of the previous derivation for $Z_\Lambda$ to go through also in the case of $Z_\Lambda(\tau)$. Further investigation of this and related questions will be reported elsewhere.

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