HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY \((\alpha, m)\)-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

MEHMET KUNT AND İMDAT İŞCAN

Abstract. In this paper, some Hermite-Hadamard type inequalities are established for harmonically \((\alpha, m)\)-convex functions via fractional integrals and some Hermite-Hadamard type inequalities are obtained for these classes of functions.

1. INTRODUCTION

Let \(f : I \subset \mathbb{R} \to \mathbb{R}\) be a convex function defined on the interval \(I\) of real numbers and \(a, b \in I\) with \(a < b\). The following inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.1}
\]

Both inequalities hold in the reversed direction if \(f\) is concave. Note that, some of the classical inequalities for means can be obtained from appropriate particular selections of the mapping \(f\). For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent paper [1]-[5] and references therein.

In [1], İşcan gave definition of harmonically convex functions and established some Hermite-Hadamard type inequalities for harmonically convex functions as follows:

**Definition 1.** Let \(I \subset \mathbb{R} \setminus \{0\}\) be a real interval. A function \(f : I \to \mathbb{R}\) is said to be harmonically convex, if

\[
f \left( \frac{xy}{tx + (1 - t)y} \right) \leq tf(y) + (1 - t)f(x) \tag{1.2}
\]

for all \(x, y \in I\) and \(t \in [0, 1]\). If the inequality in (1.2) is reversed, then \(f\) is said to be harmonically concave.

**Theorem 1.** [1]. Let \(f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}\) be a harmonically convex function and \(a, b \in I\) with \(a < b\). If \(f \in L[a, b]\) then the following inequalities hold:

\[
f \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_{a}^{b} f(x) \frac{x}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.3}
\]

**Theorem 2.** [1]. Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^0\), \(a, b \in I\) with \(a < b\), and \(f' \in L[a, b]\). If \(|f'|^q\) is harmonically convex on \([a, b]\) for \(q \geq 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_{a}^{b} f(x) \frac{x}{x^2} \, dx \right| \leq \frac{ab}{b - a} \int_{a}^{b} f(x) \frac{x}{x^2} \, dx.
\]

2000 Mathematics Subject Classification. Primary 26D15; Secondary 26A51.

Key words and phrases. Hermite-Hadamard type inequalities, harmonically \((\alpha, m)\)-convex functions, fractional integrals.
The function $f$ is said to be harmonically convex in the first sense, if we take $\alpha, m \in \mathbb{R}$ and $\alpha \neq 0, m \neq 0$. For recent results and generalizations concerning $(\alpha, m)$-convex functions we refer the reader to paper [8]-[12] and references therein.

In [8], Mihaşen gave definition of $(\alpha, m)$-convex functions as follows:

**Definition 2.** The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)$-convex where $(\alpha, m) \in [0,1]^2$, if we have

$$f(tx + (1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$.

It can be easily that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: increasing, $\alpha$-starshaped, starshaped, $m$-convex, convex, $\alpha$-convex.

For recent results and generalizations concerning $(\alpha, m)$-convex functions we refer the reader to paper [3]-[12] and references therein.

In [9], İşcan gave definition of harmonically $(\alpha, m)$-convex functions as follows:

**Definition 3.** The function $f : (0, b^*) \rightarrow \mathbb{R}$, $b^* > 0$, is said to be harmonically $(\alpha, m)$-convex, where $\alpha \in [0,1]$ and $m \in (0,1]$, if

$$f \left( \frac{mxy}{mty + (1-t)x} \right) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in (0, b^*)$ and $t \in [0,1]$. If the inequality in (1.5) is reversed, then $f$ is said to be harmonically $(\alpha, m)$-concave.

Note that $(\alpha, m) \in \{((1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: harmonically $m$-convex, harmonically convex, harmonically $\alpha$-convex (or harmonically $s$-convex in the first sense, if we take $s$ instead of $\alpha$).
We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively

\[ \beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \]

\[ 2F_1(a, b; c; z) = \frac{\beta(b, c-b)}{\Gamma(b)} \int_0^z t^{c-b-1} (1-t)^{b-1} \, dt, \]

\[ c > b > 0, |z| < 1 \text{ (see [13]).} \]

Lemma 1. [13], [15]. For \( 0 < \theta \leq 1 \) and \( 0 \leq a < b \) we have

\[ |a^\theta - b^\theta| \leq (b-a)^\theta. \]

Following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 4. Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J^\theta_{a+}f \) and \( J^\theta_{a-}f \) of order \( \theta > 0 \) with \( a \geq 0 \) are defined by

\[ J^\theta_{a+}f(x) = \frac{1}{\Gamma(\theta)} \int_a^x (x-t)^{\theta-1} f(t) \, dt, \quad x > a \]

and

\[ J^\theta_{a-}f(x) = \frac{1}{\Gamma(\theta)} \int_x^b (t-x)^{\theta-1} f(t) \, dt, \quad x < b \]

respectively, where \( \Gamma \) is the Euler Gamma function defined by \( \Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} \, dt \) and \( J^\theta_{a+}f(x) = J^\theta_{a-}f(x) = f(x) \).

Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \), throughout this paper we will take

\[ I_f(g; \theta, a, b) = \frac{f(a) + f(b)}{2} - \frac{\Gamma(\theta + 1)}{2} \left( \frac{ab}{b-a} \right)^\theta \times \left\{ J^\theta_{a-} (f \circ g) \left( \frac{1}{b} \right) + J^\theta_{b+} (f \circ g) \left( \frac{1}{a} \right) \right\}, \]

where \( a, b \in I \) with \( a < b, \theta > 0, g(x) = 1/x. \)

In [7], the authors represented Hermite-Hadamard’s inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 4. Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a function such that \( f \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is a harmonically convex function on \( [a, b] \), then the following inequalities for fractional integrals hold:

\[ f \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma(\theta + 1)}{2} \left( \frac{ab}{b-a} \right)^\theta \left\{ J^\theta_{a-} (f \circ g) \left( \frac{1}{b} \right) + J^\theta_{b+} (f \circ g) \left( \frac{1}{a} \right) \right\} \leq \frac{f(a) + f(b)}{2} \]

with \( \theta > 0. \)

In [7], the authors gave the following identity for differentiable functions.

Lemma 2. Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). Then the following equality for fractional integrals holds:

\[ I_f(g; \theta, a, b) = \frac{ab(b-a)}{2} \int_0^1 \frac{t^{\theta} - (1-t)^{\theta}}{(t(a+(1-t)b)} f'(\frac{ab}{t(a+(1-t)b)}) \, dt \quad (1.6) \]
Remark 1. The identity $I_f (g; θ, a, b)$ is equal to the following one

$$I_f (g; θ, a, b) = \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^θ - t^θ}{(tb + (1-t)a)^2} f'(tb + (1-t)a) \, dt. \quad (1.7)$$

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals. Recent results for this area, we refer the reader to paper [7], [15]-[18] and references therein.

In this paper, we aimed to establish Hermite-Hadamard’s inequalities for harmonically $(α, m)$-convex functions via fractional integrals. These results have some relations with [1].

2. Main Results

Theorem 5. Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^o$, $a, b/m \in I^o$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically $(α, m)$-convex on $[a, b/m]$ for some fixed $q \geq 1$, with $α \in [0, 1]$, then

$$|I_f (g; θ, a, b)| \leq \frac{ab(b-a)}{2} C_1^{1-1/q} (θ; a, b)$$

$$\times \left[ C_2 (θ; α; a, b) |f'(a)|^q + m C_3 (θ; α; a, b) |f'(b/m)|^q \right]^{1/q} \quad (2.1)$$

where

$$C_1 (θ; a, b) = \frac{b^{-2}}{θ + 1} \left[ \frac{2 F_1 (2, θ + 1; θ + 2; 1 - \frac{θ}{b})}{2 F_1 (2, θ + 1; θ + 2; 1 - \frac{θ}{b})} \right],$$

$$C_2 (θ; α; a, b) = \left[ \frac{β(θ+1,α+1)}{b^2} + \frac{b^{-2}}{θ + α + 1} \right] \frac{2 F_1 (2, θ + 1; θ + α + 2; 1 - \frac{θ}{b})}{2 F_1 (2, θ + 1; θ + α + 2; 1 - \frac{θ}{b})},$$

$$C_3 (θ; α; a, b) = C_1 (θ; a, b) - C_2 (θ; α; a, b).$$

Proof. Let $A_t = tb + (1-t)a$, $B_u = ua + (1-u)b$. Since $|f'|^q$ is harmonically $(α, m)$-convex, using (1.5)

$$|f' \left( \frac{ab}{A_t} \right)|^q = |f' \left( \frac{ab}{tb + (1-t)a} \right)|^q = \left| f' \left( \frac{ma(b/m)}{mt(b/m) + (1-t)a} \right) \right|^q \leq t^α |f'(a)|^q + m (1-t^α) |f'(b/m)|^q. \quad (2.2)$$
From (1.7), using the property of the modulus, the power mean inequality and (2.2), we find

\[ |I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\theta - t^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \]

\[ \leq \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \]

\[ \leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \right)^{1-1/q} \]

\[ \times \left( \int_0^1 \left( \frac{(1-t)^\theta + t^\theta}{A_t^2} \right)^q \left| f'(a) \right|^q dt \right)^{1/q} \]

\[ \leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \right)^{1-1/q} \]

\[ \times \left( \int_0^1 \left( \frac{(1-t)^\theta + t^\theta}{A_t^2} \right) (1-t^\alpha) dt \left| f'(b/m) \right|^q \right)^{1/q} \]

(2.3)

calculating following integrals, we have

\[ \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \right| dt = \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_2^2} du \]

\[ = \frac{b-2}{\theta+1} \left[ {}_2F_1 \left( 2, \theta + 1; \theta + 2; 1 - \frac{2}{b} \right) \right] \]

\[ = C_1 (\theta; a, b) \quad (2.4) \]

\[ \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} t^\alpha dt = \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_2^2} (1-u)^\alpha du \]

\[ = \left[ \frac{b(\theta+1,\alpha+1)}{b^\theta} \right] {}_2F_1 \left( 2, \theta + 1; \theta + \alpha + 2; 1 - \frac{2}{b} \right) \]

\[ + \frac{b^{-2}}{\theta+2} \left[ {}_2F_1 \left( 2, 1; \theta + \alpha + 2; 1 - \frac{2}{b} \right) \right] \]

\[ = C_2 (\theta; \alpha; a, b) \quad (2.5) \]

\[ \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} (1-t^\alpha) dt = \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} dt - \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} t^\alpha dt \]

\[ = C_1 (\theta; a, b) - C_2 (\theta; \alpha; a, b) \]

\[ = C_3 (\theta; \alpha; a, b) \quad (2.6) \]

Thus, if we use (2.4)-(2.6) in (2.3) we get the inequality of (2.1) and this completes the proof. \[ \square \]

**Corollary 1.** In Theorem 5,

(a) If we take \( \alpha = 1, m = 1 \) we have the following inequality for harmonically convex functions:

\[ |I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} C_1 (\theta; a, b) \]

\[ \times \left( C_2 (\theta; 1; a, b) |f'(a)|^q + C_3 (\theta; 1; a, b) |f'(b)|^q \right)^{1/q}, \]
(b) If we take $\alpha = 1$ we have the following inequality for harmonically $m$-convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2}C_1^{1-1/q} \left(\theta; a, b\right) \times \left[C_2(\theta; 1; a, b) |f'(a)|^q + mC_3(\theta; 1; a, b) |f'(b/m)|^q\right]^{1/q},$$

(c) If we take $m = 1$ we have the following inequality for harmonically $\alpha$-convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2}C_1^{1-1/q} \left(\theta; a, b\right) \times \left[C_2(\theta; \alpha; a, b) |f'(a)|^q + C_3(\theta; \alpha; a, b) |f'(b)|^q\right]^{1/q}.$$

When $0 < \theta \leq 1$, using Lemma 1 we shall give another result for harmonically $(\alpha, m)$-convex functions as follows:

**Theorem 6.** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b/m \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically $(\alpha, m)$-convex on $[a, b/m]$ for some fixed $q \geq 1$, with $\alpha \in [0, 1]$, then

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2}C_4^{1-1/q} \left(\theta; a, b\right) \times \left[C_5(\theta; \alpha; a, b) |f'(a)|^q + mC_6(\theta; \alpha; a, b) |f'(b/m)|^q\right]^{1/q} \quad (2.7)$$

where $0 < \theta \leq 1$ and

$$C_4(\theta; a, b) = \begin{bmatrix} \frac{b^2}{b+1} & \frac{b^2}{b+1} & 2F_1\left(2, 1; \theta + 2; 1 - \frac{\theta}{b}\right) \\ \frac{b^2}{b+1} & 2F_1\left(2, \theta + 1; \theta + 2; 1 - \frac{2}{b}\right) \\ \left(\frac{a+b}{2}\right)^{-2} & 2F_1\left(2, \theta + 1; \theta + 2; \frac{b-a}{a+b}\right) \end{bmatrix},$$

$$C_5(\theta; \alpha; a, b) = \begin{bmatrix} \frac{b^2}{\theta+1} & \frac{b^2}{\theta+1} & 2F_1\left(2, 1; \theta + \alpha + 2; 1 - \frac{\theta}{b}\right) \\ -\frac{2(\theta+1, \alpha+1)}{\theta+1} & 2F_1\left(2, \theta + 1; \theta + \alpha + 2; 1 - \frac{\theta}{b}\right) \\ \frac{b^2}{\alpha+b} & 2F_1\left(2, \theta + 1; \theta + \alpha + 2; \frac{b-a}{\alpha+b}\right) \end{bmatrix},$$

$$C_6(\theta; \alpha; a, b) = C_4(\theta; a, b) - C_5(\theta; \alpha; a, b).$$
Proof. Let $A_t = tb + (1 - t)a$, $B_u = ua + (1 - u)b$. From (1.7), using the power mean inequality and (2.2), we find

$$|I_1 (g; \theta, a, b)| \leq \frac{ab (b - a)}{2} \int_0^1 \left| (1 - t)^\theta - t^\theta \right| \left| f' \left( \frac{ab}{A_t} \right) \right| dt$$

$$\leq \frac{ab (b - a)}{2} \left( \int_0^1 \left| (1 - t)^\theta - t^\theta \right| dt \right)^{1 - 1/q} \times \left( \int_0^1 \left| (1 - t)^\theta - t^\theta \right| \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}$$

$$\leq \frac{ab (b - a)}{2} \left( \int_0^1 \left| (1 - t)^\theta - t^\theta \right| dt \right)^{1 - 1/q} \times \left( \frac{1}{A_t} \int_0^1 \left| (1 - t)^\theta - t^\theta \right| \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}$$

$$\leq \frac{ab (b - a)}{2} \left( \int_0^1 \left| (1 - t)^\theta - t^\theta \right| dt \right)^{1 - 1/q} \times \left( \frac{1}{A_t} \int_0^1 \left| (1 - t)^\theta - t^\theta \right| \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}$$

Calculating following integrals by Lemma 1, we have

$$\int_0^1 \left| (1 - t)^\theta - t^\theta \right| dt = \int_0^{1/2} \left( (1 - t)^\theta - t^\theta \right) dt + \int_{1/2}^1 \left( (1 - t)^\theta - t^\theta \right) dt$$

$$= \int_0^{1/2} \left( t^\theta - (1 - t)^\theta \right) dt + 2 \int_0^{1/2} \left( (1 - t)^\theta - t^\theta \right) dt$$

$$\leq \int_0^{1/2} \left( t^\theta - (1 - t)^\theta \right) dt + 2 \int_0^{1/2} \left( (1 - t)^\theta - t^\theta \right) dt$$

$$= \int_0^{1/2} \left( t^\theta - (1 - t)^\theta \right) dt$$

$$+ \int_0^{1/2} \left( \frac{a + b}{2} \right)^{-2} \left( 1 - v \left( \frac{b - a}{b + a} \right) \right)^{-2} dv$$

$$= \left[ \begin{array}{c}
\frac{\Gamma^2 (2, \theta + 2; 1 - \frac{a}{b + a})}{\Gamma (2) \Gamma (2, \theta + 2; 1 - \frac{a}{b + a})} \\
\frac{\Gamma^2 (2, 1; \theta + 1; 1 - \frac{a}{b + a})}{\Gamma (2) \Gamma (2, 1; \theta + 1; 1 - \frac{a}{b + a})}
\end{array} \right]$$

$$= C_4 (\theta; a, b)$$

(2.9)
and similarly we get
\[
\int_0^1 \frac{(1-t)^{\alpha} - t^{\alpha}}{A_t^2} t^{\alpha} dt \leq \int_0^1 \frac{t^{\alpha} + \theta}{A_t^2} dt - \int_0^1 \frac{(1-t)^{\theta} t^{\alpha}}{A_t^2} dt + 2 \int_0^{1/2} \frac{(1-2t)^{\theta} t^{\alpha}}{A_t^2} dt
\]
\[
= \int_0^1 \frac{(1-u)^{\alpha+\theta}}{B_u^2} du \left( \int_0^u \frac{w^{\alpha}}{B_w^2} dw \right) + \int_0^1 \frac{(1-u)^{\theta} (u^2)}{B_u^2} du 
\]
\[
= \int_0^1 \frac{(1-u)^{\alpha+\theta}}{B_u^2} du - \int_0^1 \frac{u^{\theta} (1-u)^{\alpha}}{B_u^2} du 
\]
\[
+ \frac{(a+b)^{-2}}{2^{\alpha}} \int_0^1 u^\theta (1-u)^\alpha \left(1 - v \left( \frac{b-a}{b+a} \right) \right)^{-2} dv
\]
\[
= C_5 (\theta; \alpha; a, b)
\]
\[
(2.10)
\]
\[
\int_0^1 \frac{(1-t)^{\theta} - t^{\theta}}{A_t^2} (1-t^{\alpha}) dt = \int_0^1 \frac{(1-t)^{\theta} - t^{\theta}}{A_t^2} dt - \int_0^1 \frac{(1-t)^{\theta} - t^{\theta}}{A_t^2} t^{\alpha} dt
\]
\[
= C_4 (\theta; a, b) - C_5 (\theta; \alpha; a, b)
\]
\[
= C_6 (\theta; \alpha; a, b)
\]
\[
(2.11)
\]
Thus, if we use (2.9) - (2.11) in (2.8) we get the inequality of (2.7) and this completes the proof. \(\Box\)

**Remark 2.** If we take \(\theta = 1, \alpha = 1, m = 1\) in Theorem 6, then inequality (2.7) becomes inequality (1.3) of Theorem 2.

**Corollary 2.** In Theorem 6,

(a) If we take \(\alpha = 1, m = 1\) we have the following inequality for harmonically convex functions:
\[
|I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} c_4^{-1/q} (\theta; a, b)
\]
\[
\times \left[ C_5 (\theta; 1; a, b) |f' (a)|^q + C_6 (\theta; 1; a, b) |f' (b)|^q \right]^{1/q},
\]

(b) If we take \(\alpha = 1\) we have the following inequality for harmonically \(m\)-convex functions:
\[
|I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} c_4^{-1/q} (\theta; a, b)
\]
\[
\times \left[ C_5 (\theta; 1; a, b) |f' (a)|^q + m C_6 (\theta; 1; a, b) |f' (b/m)|^q \right]^{1/q},
\]

(c) If we take \(m = 1\) we have the following inequality for harmonically \(\alpha\)-convex functions:
\[
|I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} c_4^{-1/q} (\theta; a, b)
\]
\[
\times \left[ C_5 (\theta; \alpha; a, b) |f' (a)|^q + C_6 (\theta; \alpha; a, b) |f' (b)|^q \right]^{1/q}.
\]
Theorem 7. Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b/m \in I^\circ$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically $(\alpha, m)$-convex on $[a, b/m]$ for some fixed $q > 1$, with $\alpha \in [0, 1]$, then

$$|I_f (g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q}$$

$$\times \left[ 2F_1^{1/p} (2p, \theta p + 1; \theta p + 2; 1 - \frac{\theta}{p}) \right]$$

(2.12)

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $A_t = tb + (1 - t)a$, $B_u = ua + (1 - u)b$. From (1.7), using the property of the modulus, the Hölder inequality and (2.2), we find

$$|I_f (g; \theta, a, b)| \leq \frac{ab(b-a)}{2} \left[ \int_0^1 \frac{(1-t)\theta}{A_t^q} |f' \left( \frac{ab}{A_t} \right)| dt + \int_0^1 \frac{t\theta}{A_t^q} |f' \left( \frac{ab}{A_t} \right)| dt \right]$$

$$\leq \frac{ab(b-a)}{2} \left[ \left( \int_0^1 \frac{1-t}{A_t^q} du \right)^{1/p} \left( \int_0^1 \frac{(1-t)^q}{A_t^q} dt \right)^{1/q} \right]$$

$$\times \left( \int_0^1 t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q dt \right)^{1/q}$$

$$= \frac{ab(b-a)}{2} \left( K_1^{1/p} + K_2^{1/p} \right) \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q}$$

(2.13)

calculating $K_1$ and $K_2$ we have

$$K_1 = \int_0^1 \frac{u^{\theta p}}{B_u^{\theta p}} du = \frac{b^{-2p}}{\theta p + 1} 2F_1 \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{\theta}{p} \right)$$

(2.14)

$$K_1 = \int_0^1 \frac{(1-u)^{\theta p}}{B_u^{\theta p}} du = \frac{b^{-2p}}{\theta p + 1} 2F_1 \left( 2p, 1; \theta p + 2; 1 - \frac{\theta}{p} \right)$$

(2.15)

Thus, if we use (2.14), (2.15) in (2.13) we get the inequality of (2.12) and this completes the proof. \qed

Corollary 3. In Theorem 7,

(a) If we take $\alpha = 1$, $m = 1$ we have the following inequality for harmonically convex functions:

$$|I_f (g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + |f'(b/m)|^q}{2} \right)^{1/q}$$

$$\times \left[ 2F_1^{1/p} (2p, \theta p + 1; \theta p + 2; 1 - \frac{\theta}{p}) \right]$$

(2.16)
(b) If we take $\alpha = 1$ we have the following inequality for harmonically $m$-convex functions:

$$
|I_f (g; \alpha, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta_p+1} \right)^{1/p} \left( \frac{|f'(a)|^q + m|f'(b/m)|^q}{2} \right)^{1/q} \times \left[ 2F_1^{1/p} (2p, \theta_p + 1; \theta_p + 2; 1 - \frac{2}{q}) + 2F_1^{1/p} (2p, 1; \theta_p + 2; 1 - \frac{2}{q}) \right],
$$

(c) If we take $m = 1$ we have the following inequality for harmonically $\alpha$-convex functions:

$$
|I_f (g; \alpha, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta_p+1} \right)^{1/p} \left( \frac{|f'(a)|^q + \alpha |f'(b)|^q}{\alpha + 1} \right)^{1/q} \times \left[ 2F_1^{1/p} (2p, \theta_p + 1; \theta_p + 2; 1 - \frac{2}{q}) + 2F_1^{1/p} (2p, 1; \theta_p + 2; 1 - \frac{2}{q}) \right].
$$

**Theorem 8.** Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^o$, $a, b/m \in I^o$ with $a < b$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically $(\alpha, m)$-convex on $[a, b/m]$ for some fixed $q > 1$, with $\alpha \in [0, 1]$, then

$$
|I_f (g; \alpha, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta_p+1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \times \left[ 1 + m \left[ 2F_1^{1/p} (2q, 1; \alpha + 2; 1 - \frac{2}{q}) \right] \left| f'(a) \right|^q - \left[ 2F_1^{1/p} (2q, 1; \alpha + 2; 1 - \frac{2}{q}) \right] \left| f'(b/m) \right|^q \right]^{1/q}
$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

**Proof.** Let $A_t = tb + (1-t)a$, $B_u = ua + (1-u)b$. From (1.1), using the Hölder inequality, Lemma 1, and (2.2), we find

$$
|I_f (g; \alpha, a, b)| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{1}{A_t^q} \left| f' \left( \frac{ab}{A_t} \right) \right| dt
$$

$$
\leq \frac{ab(b-a)}{2} \left( \int_0^1 \left| (1-t)^\theta - t^\theta \right|^p dt \right)^{1/p} \times \left( \int_0^1 \frac{1}{A_t^q} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}
$$

$$
\leq \frac{ab(b-a)}{2} \left( \int_0^1 \left| 1 - 2t \right|^{\theta p} dt \right)^{1/p} \times \left( \int_0^1 \frac{1}{A_t^q} \left[ \left| f'(a) \right|^q + m \left| (1 - t^\alpha) |f'(b/m)|^q \right| dt \right]^{1/q}
$$

calculating following integrals, we have

$$
\int_0^1 \left| 1 - 2t \right|^{\theta p} dt = \frac{1}{\theta_p + 1}
$$

$$
\int_0^1 \frac{\alpha}{A_t^q} dt = \int_0^1 \frac{(1 - u)\alpha}{B_u^q} dt = \frac{b^{-2q}}{\alpha + 1} 2F_1 (2q, 1; \alpha + 2; 1 - \frac{2}{q})
$$
where \(1 < b\) with \(a < b\) and \(m \geq 1\), then inequality \((2.20)\) becomes inequality \((1.4)\) of Theorem 3.

**Remark 3.** If we take \(\theta = 1\), \(\alpha = 1\), \(m = 1\) in Theorem 8, then inequality \((2.16)\) becomes inequality \((1.3)\) of Theorem 3.

**Corollary 4.** In Theorem 8,

(a) If we take \(\alpha = 1\), \(m = 1\) we have the following inequality for harmonically convex functions:

\[
|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2^{1+1/q}b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \\
\times \left[ 2F_1(2q; 1; \alpha + 2; 1 - \frac{\theta}{\alpha}) |f'(a)|^q + m \left( \frac{2}{2F_1(2q; 1; 2; 1 - \frac{\theta}{\alpha})} \right) |f'(b/m)|^q \right].
\]

(b) If we take \(\alpha = 1\) we have the following inequality for harmonically \(m\)-convex functions:

\[
|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2^{1+1/q}b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \\
\times \left[ 2F_1(2q; 1; 3; 1 - \frac{\theta}{\alpha}) |f'(a)|^q + m \left( \frac{2}{2F_1(2q; 1; 2; 1 - \frac{\theta}{\alpha})} \right) |f'(b)|^q \right].
\]

(c) If we take \(m = 1\) we have the following inequality for harmonically \(\alpha\)-convex functions:

\[
|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \\
\times \left[ 2F_1(2q, 1; \alpha + 2; 1 - \frac{\theta}{\alpha}) |f'(a)|^q + m \left( \frac{\alpha + 1}{2F_1(2q, 1; \alpha + 2; 1 - \frac{\theta}{\alpha})} \right) |f'(b)|^q \right].
\]

**Theorem 9.** Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^\circ\), \(a, b/m \in I^\circ\) with \(a < b\), \(m \in (0, 1]\) and \(f' \in L[a, b]\). If \(|f'|^q\) is harmonically \((\alpha, m)\)-convex on \([a, b/m]\) for some fixed \(q > 1\), with \(\alpha \in (0, 1]\), then

\[
|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2^{1/p-1}(a+b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( |f'(a)|^q + m \alpha \max \left( |f'(b/m)|^q \right) \right)^{1/q} \\
\times \left[ 2F_1(2p, \theta p + 1; \theta p + 2; \frac{\theta m}{\theta p + \alpha}) + 2F_1(2p, \theta p + 1; \theta p + 2; \frac{\theta m}{\theta p + \alpha}) \right]^{1/p}
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\).
Proof. Let \( A_t = tb + (1 - t)a \). From (1.7), using the Hölder inequality, Lemma 1, and (2.23), we find

\[
|I_f (g; \theta, a, b)| \leq \frac{ab(b - a)}{2} \int_0^1 \left| \frac{(1-t)^\theta - t^\theta}{A_t^2} \right| f'(\frac{ab}{A_t}) \, dt
\]

\[
\leq \frac{ab(b - a)}{2} \int_0^1 \left| \frac{1 - 2t}{A_t^2} \right| f'(\frac{ab}{A_t}) \, dt
\]

\[
\leq \frac{ab(b - a)}{2} \left( \int_0^1 \left| \frac{1 - 2t}{A_t^2} \right| \, dt \right)^{1/p} \left( \int_0^1 \left| f'(\frac{ab}{A_t}) \right|^q \, dt \right)^{1/q}
\]

\[
\leq \frac{ab(b - a)}{2} \left( \int_0^1 \left| \frac{1 - 2t}{A_t^2} \right| \, dt \right)^{1/p}
\]

\[
\times \left( \int_0^1 t^\alpha \left| f'(a) \right|^q + m (1 - t^\alpha) \left| f'(b/m) \right|^q \, dt \right)^{1/q}
\]

\[
= \frac{ab(b - a)}{2} \left[ \int_0^{1/2} \left( \frac{1 - 2t}{A_t^2} \right)^{\theta p} \, dt + \int_{1/2}^1 \left( \frac{2t - 1}{A_t^2} \right)^{\theta p} \, dt \right]^{1/p}
\]

\[
\times \left( \frac{\left| f'(a) \right|^q + m\alpha \left| f'(b/m) \right|^q}{\alpha + 1} \right)^{1/q}
\]

(2.22)

Calculating following integrals, we have

\[
\int_0^{1/2} \left( \frac{1 - 2t}{A_t^2} \right)^{\theta p} \, dt = \frac{1}{2} \int_0^{1/2} \left( \frac{1 - u}{\frac{u}{2} + (1 - \frac{u}{2}) a} \right)^{\theta p} \, du
\]

\[
= \frac{(a + b)^{-2p}}{2^{1-2p}} \int_0^1 v^{\theta p} \left( 1 - v \left( \frac{b - a}{b + a} \right) \right)^{-2p} \, dv
\]

\[
= \frac{(a + b)^{-2p}}{2^{1-2p} (\theta p + 1)} \, \, _2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b - a}{b + a} \right)
\]

(2.23)

\[
\int_{1/2}^1 \left( \frac{2t - 1}{A_t^2} \right)^{\theta p} \, dt = \frac{1}{2} \int_{1/2}^1 \left( \frac{1 - u}{\frac{u}{2} + (1 - \frac{u}{2}) a} \right)^{\theta p} \, du
\]

\[
= \frac{(a + b)^{-2p}}{2^{1-2p}} \int_0^1 v^{\theta p} \left( 1 - v \left( \frac{a - b}{b + a} \right) \right)^{-2p} \, dv
\]

\[
= \frac{(a + b)^{-2p}}{2^{1-2p} (\theta p + 1)} \, \, _2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{a - b}{b + a} \right)
\]

(2.24)

Thus, if we use (2.23) in (2.22) we get the inequality of (2.21) and this completes the proof.

\[\square\]

Corollary 5. In Theorem 9,

(a) If we take \( \alpha = 1, m = 1 \) we have the following inequality for harmonically convex functions:

\[
|I_f (g; \theta, a, b)| \leq \frac{ab(b - a)}{2^{(1/p) - 1} (a + b)} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{\left| f'(a) \right|^q + \left| f'(b) \right|^q}{2} \right)^{1/q}
\]

\[
\times \left[ \, \, _2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b - a}{b + a} \right) \right]^{1/p}
\]

(2.24)
(b) If we take $\alpha = 1$ we have the following inequality for harmonically $m$-convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b - a)}{2^{(1/p) - 1}(a + b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m|f'(b/m)|^q}{\theta p + 1} \right)^{1/q} \times \left[ 2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b - a}{b + a} \right) + 2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{a - b}{b + a} \right) \right]^{1/p}.$$

(c) If we take $m = 1$ we have the following inequality for harmonically $\alpha$-convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b - a)}{2^{(1/p) - 1}(a + b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + \alpha |f'(b)|^q}{\alpha + 1} \right)^{1/q} \times \left[ 2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b - a}{b + a} \right) + 2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{a - b}{b + a} \right) \right]^{1/p}.$$

REFERENCES

[1] İ. İcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacet. J. Math. Stat., 43 (6) (2014), 935-942.
[2] İ. İcan, New estimates on generalization of some integral inequalities for $(\alpha, m)$-convex functions, Contemp. Anal. Appl. Math., 1 (2) (2013) 253-264.
[3] İ. İcan, New estimates on generalization of some integral inequalities for $s$-convex functions and their applications, Int. J. Pure Appl. Math., 86 (4) (2013) 727-746.
[4] H. Kavurmacı, M. E. Özdemir, M. Avci, New Ostrowski type inequalities for fractional integrals, Appl. Anal., 93 (9) (2014) 1846-1862.
[5] E. Set, M. E. Özdemir, S. S. Dragomir, On Hadamard-type inequalities involving several kinds of convexity, J. Inequal. Appl. 2011 (2011) 1, http://dx.doi.org/10.1155/2011/286245 (Article ID 286245).
[6] İ. İcan, Hermite-Hadamard type inequalities for harmonically $(\alpha, m)$-convex functions (in press).
[7] İ. İcan, S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, Appl. Math. Comput., 238 (2014) 237-244.
[8] V. G. Milovan, A generalization of the convexity, Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, Romania, 1993.
[9] M. K. Bakula, M. E. Özdemir, J. Pecaric, Hadamard type inequalities for $m$-convex and $(\alpha, m)$-convex functions, J. Inequal. Pure Appl. Math., 9 (4), Article 96, p. 12, 2008.
[10] İ. İcan, A new generalization of some integral inequalities for $(\alpha, m)$-convex functions, Mathematical Sciences, 7 (22) (2013) 1-8.
[11] İ. İcan, Hermite-Hadamard type inequalities for functions whose derivatives are $(\alpha, m)$-convex, Int. J. Eng. Appl. Sci., 2 (3) (2013) 69-78.
[12] M. E. Özdemir, H. Kavurmacı, E. Set, Ostrowski’s type inequalities for $(\alpha, m)$-convex functions, Kyungpook Math. J., 50 (2010) 371-378.
[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[14] A. P. Prudnikov, Y. A. Brychkov, O. J. Marichev, Integral and series, Elementary Functions, vol. 1, Nauka, Moscow, 1981.
[15] J. Wang, C. Zho, Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, J. Inequal. Appl., 2013 (325) (2013) 15pp.
[16] J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal. 92 (11) (2012) 2241-2263.
[17] İ. İcan, Generalization of different type integral inequalities for $s$-convex functions via fractional integrals, Appl. Anal., 93 (9) (2014) 1846-1862.
[18] İ. İcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., 2013 (491) (2013) 15pp.
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, KARADENIZ TECHNICAL UNIVERSITY, TRABZON, TURKEY
E-mail address: mkunt@ktu.edu.tr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND ARTS, GİRESUN UNIVERSITY, GİRESUN, TURKEY
E-mail address: imdat.iscan@giresun.edu.tr