TRIANGULAR NUMBERS MULTIPLE OF TRIANGULAR NUMBERS AND SOLUTIONS OF PELL EQUATIONS

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Abstract. For all positive non-square integer multiplier \( k \), there is an infinity of multiples of triangular numbers which are also triangular numbers. With a simple change of variables, these triangular numbers can be found using solutions of Pell equations. With some conditions on parities of fundamental solutions of the simple and generalized Pell equations, only odd solutions of the generalized Pell equation are retained to provide many infinitely solutions found on branches corresponding to each of the generalized fundamental solutions. General algebraic expressions of fundamental solutions of the Pell equations are found for some values of the multiplier \( k \) in function of the closest natural square. Further, among the expressions of Pell equation solutions, a set of recurrent relations is identical to those found previously without the Pell equation solving method. It is found also that two constants of the problem of multiples of triangular numbers are directly related to the fundamental solutions of the simple Pell equation, which is an unexpected result as it means that simple Pell equation fundamental solutions in all generality, are related to constants in recurrent relations of the problem of finding triangular numbers multiple of other triangular numbers.

1. Introduction

Triangular numbers \( T_t = \frac{t(t+1)}{2} \) are figurate numbers with several interesting properties and formula (see, e.g., [1, 2]). In this paper, we investigate triangular numbers \( T_\xi \) that are multiples of other triangular numbers \( T_t \)

\[ T_\xi = kT_t \quad (1) \]

Several authors have investigated this Diophantine equation; see, e.g., [3, 4, 5, 6, 7, 8, 9]. Further historical accounts can be found in [6]. Recently, Pletser showed [10] that, for non-square integer values of \( k \), the four variables \( t, \xi, T_t \) and \( T_\xi \) can be represented by recurrent relations involving a rank \( r \) and parameters \( \kappa \) and \( \gamma \) which are respectively the sum and the product of the \((r - 1)\)th and the \(r\)th values of \( t \). The rank is being defined as the number of successive values of \( t \) solutions of (1) such that their successive ratios are slowly decreasing without jumps.

We only consider solutions of (1) for \( k > 1 \) as, for \( k = 0 \) and \( k = 1 \), solutions are trivial, respectively, \( \xi = 0 \) and \( \xi = t \) for any positive integer \( t \).

In this paper, we investigate how to find all solutions to (1) using the method of resolution of the simple and generalized Pell equations associated to (1). We show that the rank \( r \) and parameters \( \kappa \) and \( \gamma \) of recurrent relations can be

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deduced from fundamental solutions of Pell equations. Section 2 introduces the rank $r$ and recurrent relations. Section 3 gives a short reminder on how to find solutions of Pell equations. In Section 4, Pell equation methods are applied to find all multiples of triangular numbers that are triangular numbers. In certain cases, general expressions of fundamental solutions of the Pell equations associated to (1) are given for values of the multiplier $k$ in function of the closest natural square values $s^2$.

2. Rank and recurrent relations

The Online Encyclopedia of Integer Sequences (OEIS) lists sequences of solutions of (1) for $k = 2, 3, 5, 6, 7, 8$. Let us note first that, among all solutions, $(t_0, \xi_0) = (0, 0)$ is always a first solution of (1) for all non-square integer value of $k$.

Let’s consider the two cases of $k = 3$ and $k = 6$ yielding the successive solution pairs as shown in Table 1. We indicate also the ratios $t_n/t_{n-1}$ for both cases and $t_n/t_{n-2}$ for $k = 6$. It is seen that for $k = 3$, the ratio $t_n/t_{n-1}$ varies between close values, from 5 down to 3.737, while for $k = 6$, the ratio $t_n/t_{n-1}$ alternates between values 3 ... 2.385 and 4.667 ... 4.206, while the ratio $t_n/t_{n-2}$ decreases more regularly from 14 to 10.029 (corresponding approximately to the product of the alternating values of the ratio $t_n/t_{n-1}$).

We call rank $r$ the integer value such that $t_{2r}/t_r$ is approximately constant or, better, decreases regularly without jumps (a more precise definition is given further). So, here, the case $k = 3$ has rank $r = 1$ and the case $k = 6$ has rank $r = 2$.

Pletser showed that the rank $r$ is the index of $t_r$ and $\xi_r$ solutions of (1) such that

$$\kappa = t_r + t_{r-1} = \xi_r - \xi_{r-1} - 1$$

The rank $r$ is also such that the ratio $t_{2r}/t_r$, corrected by the ratio $t_{r-1}/t_r$, is equal to a constant $2\kappa + 3$

$$\frac{t_{2r} - t_{r-1}}{t_r} = 2\kappa + 3$$
For example, for $k = 6$ and $r = 2$, $\kappa = t_2 + t_1 = 3 + 1 = 4$, and $\kappa = \xi_2 - \xi_1 - 1 = 8 - 3 - 1 = 4$, yielding $2\kappa + 3 = 11$.

Pletser found [10] four recurrent equations for $t_n, \xi_n, T_t$ and $T_\xi$ for each non-square integer value of $k$

\[
\begin{align*}
t_n &= 2 (\kappa + 1) t_{n-r} - t_{n-2r} + \kappa \\
\xi_n &= 2 (\kappa + 1) \xi_{n-r} - \xi_{n-2r} + \kappa \\
T_t &= \left( 4 (\kappa + 1)^2 - 2 \right) T_{t_{n-r}} - T_{t_{n-2r}} + (T_\kappa - \gamma) \\
T_\xi &= \left( 4 (\kappa + 1)^2 - 2 \right) T_{\xi_{n-r}} - T_{\xi_{n-2r}} + k (T_\kappa - \gamma)
\end{align*}
\]

where coefficients are functions of two constants $\kappa$ and $\gamma$, respectively the sum (2) and the product $\gamma = t_r t_{r-1}$. Note that these four relations are independent from the value of $k$.

### 3. Pell equations: A Reminder

The Diophantine bivariate quadratic equation

\[
X^2 - DY^2 = N,
\]

with integers $X,Y,D,N$ and square free $D$, is called the Pell equation. Several mathematicians have investigated this equation (see historical accounts in [12, 13, 14, 15, 16]), treatments and solutions are described in several classical text books (see e.g. [17, 19, 18, 20] and references therein). We remind here some general formulas and how to calculate solutions. Details can be found in references.

For $N = 1$, (8) is called the simple Pell equation

\[
x^2 - Dy^2 = 1
\]

This equation admits the obvious trivial solution $(x_0, y_0) = (1, 0)$ and infinitely many solutions given by

\[
(x_n, y_n) = \left( \frac{(x_f + \sqrt{Dy_f})^n + (x_f - \sqrt{Dy_f})^n}{2} \right)
\]

\[
\left( \frac{(x_f + \sqrt{Dy_f})^n - (x_f - \sqrt{Dy_f})^n}{2\sqrt{D}} \right)
\]

where $n$ are positive integers and $(x_f, y_f)$ is the least solution to (9), i.e. the smallest integer solution different from the trivial solution, $x_f > 1, y_f > 0$. We call this least solution the fundamental solution. Obviously, having found the fundamental solution $(x_f, y_f)$ yields directly three other solutions, $(−x_f, y_f), (x_f, −y_f), (−x_f, −y_f)$.

Lagrange devised a method to find the fundamental solution, based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$, that can be
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summarized as follows. One computes the \( j^{th} \) convergent \((p_j/q_j)\) of the continued fraction \([\alpha_0; \alpha_1, \ldots, \alpha_j, \alpha_{j+1}, \ldots]\) of \(\sqrt{D}\), with \(\alpha_0 = \left\lfloor \sqrt{D} \right\rfloor\), i.e., the greatest integer \(\leq \sqrt{D}\). This continued fraction becomes periodic after the following term, \(\alpha_{j+1} = 2\alpha_0\) if \(\sqrt{D}\) is a quadratic irrational. The recurrence relations

\[
p_i = \alpha_ip_{i-1} + p_{i-2}, \quad q_i = \alpha_iq_{i-1} + q_{i-2}
\]

yield the terms \(p_i\) and \(q_i\) of the convergent, with \(p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0\). The fundamental solution is then \((x_f, y_f) = (p_j, q_j)\) if \(j\) is odd, or \((x_f, y_f) = (p_{2j+1}, q_{2j+1})\) if \(j\) is even.

For \(N \neq 1\), (8) is called the generalized Pell equation, which can have either no solution, one, or several fundamental solutions \((X_{f_i}, Y_{f_i})\), with positive integers \(i\) such that \(1 \leq i \leq \rho\), where \(\rho\) is the total number of fundamental solutions admitted by (8). All integer solutions, if they exist, are found on double infinite branches that can be expressed in terms of the fundamental solution(s) \((X_{f_i}, Y_{f_i})\) and \((-X_{f_i}, Y_{f_i})\). Methods to calculate the fundamental solution(s) of the generalized Pell equation (see e.g. [17, 19, 20, 21, 22, 23, 24, 25] and references therein) are all based on Lagrange’s method of continued fractions, sometime adapted (see e.g. [20]). The nearest integer continued fraction method and the Lagrange-Mollin-Matthews method [25] are used further to calculate the fundamental solutions of respecti vely, the simple and the generalized Pell equations.

Once fundamental solutions are known, the other solutions \((X_n, Y_n)\) of (8) are calculated by

\[
X_n + \sqrt{D}Y_n = \pm \left(X_{f_i} + \sqrt{D}Y_{f_i}\right) \left(x_f + \sqrt{D}y_f\right)^n
\]

for a proper choice of sign \(\pm\) [20], yielding respectively, for \(n = 0, 1, 2\) (assuming a + sign),

\[
(X_0, Y_0) = (X_{f_i}, Y_{f_i}) \quad (X_1, Y_1) = (X_{f_i}x_f + DY_{f_i}y_f, X_{f_i}y_f + Y_{f_i}x_f) \quad (X_2, Y_2) = (X_{f_i}(x_f^2 + Dy_f^2) + 2DY_{f_i}x_fy_f, Y_{f_i}(x_f^2 + Dy_f^2) + 2X_{f_i}x_fy_f)
\]

Note that, for each value of \(n\), one can have several (up to \(\rho\)) solutions depending on the different values of the generalized fundamental solutions \((X_{f_i}, Y_{f_i})\).

The other solutions \((X_n, Y_n)\) of (8) can also be represented by recurrence relations

\[
(X_n, Y_n) = (x_fX_{n-1} + Dy_fY_{n-1}, x_fY_{n-1} + y_fX_{n-1})
\]

that can also be written as

\[
(X_n, Y_n) = (2x_fX_{n-1} - X_{n-2}, 2x_fY_{n-1} - Y_{n-2})
\]
or by Chebyshev polynomials of the first kind $T_{n-1}(x_f)$ and of the second kind $U_{n-2}(x_f)$, evaluated at $x_f$ (see [27]),

$$(X_n, Y_n) = (X_f T_{n-1}(x_f) + D Y_f y_f U_{n-2}(x_f), Y_f T_{n-1}(x_f))$$

(17)

One notices that the second recurrent relations (16) is similar in form to recurrent relations (4) and (5) found in [10].

4. Pell Equations and Multiples of Triangular Numbers

4.1. Solutions of Pell equations. For non-square integers $k$ and with the change of variables

$$(X, Y) = (2 \xi + 1, 2 t + 1)$$

(18)

becomes a generalized Pell equation [7, 8], with $D = k$ and $N = 1 - k$ negative, as $k > 1$,

$$X^2 - kY^2 = 1 - k$$

(19)

and the associated simple Pell equation reads

$$x^2 - ky^2 = 1$$

(20)

Odd solutions $(X, Y)$ of (19) provide then pairs $(\xi, t)$, solutions of (1). Following the procedure of Section 2, the fundamental solutions of the simple and generalized Pell equations are calculated and shown in Tables 2 to 4 for non-square $k$ between 2 and 102. The second and third columns give the rank $r$ found in [10] and the total number $\rho$ of fundamental solutions of the generalized Pell equation. The fourth column shows the single fundamental solution of the simple Pell equations; the fifth and sixth columns give the fundamental solutions of the generalized Pell equations, the fifth column for those solutions with both $X_f$ and $Y_f$ odd or having different parities, while the sixth column give those solutions with both $X_f$ and $Y_f$ even (except for $k = 56$, see discussion further).

From these Tables, we deduce the following.

First, the rank of solutions of (11) is equal to, or less than, the total number of fundamental solutions of the generalized Pell equations, $r \leq \rho$, as was expected.

Second, for all the single fundamental solutions $(x_f, y_f)$ of the simple Pell equation, both $x_f$ and $y_f$ are of different parities, i.e., one is odd, the other even (except for some cases of $k \equiv 0 \pmod{8}$, where both $x_f$ and $y_f$ are odd; see further). It is easy to see why: for (20) to hold, the following three conditions must hold:

(C1) $x_f$ and $y_f$ can not be simultaneously even, whatever the value of $k$ is;
(C2) if $k$ is even, $x_f$ must necessarily be odd and $y_f$ can be either even or odd;
(C3) if $k$ is odd, $x_f$ and $y_f$ must have different parities, one odd and the other even.

Third, the sets of fundamental solutions of the generalized Pell equation always include the two fundamental solutions $(X_{f_1}, Y_{f_1}) = (1, 1)$ and $(X_{f_2}, Y_{f_2}) =$
Table 2. Fundamental solutions of simple \((19)\) and generalized \((20)\) Pell equations

| \(k\) | \(r\) | \(\rho\) | \((x_f, y_f)\) | \((X_f, Y_f)\) |
|-------|-------|--------|----------------|----------------|
| 2     | 1     | 1      | (3, 2)         | (1, 1)         |
| 3     | 1     | 1      | (2, 1)         | (1, 1)         |
| 5     | 2     | 3      | (9, 4)         | (\(\pm 1, 1\)) | (4, 2) |
| 6     | 2     | 2      | (5, 2)         | (\(\pm 1, 1\)) |
| 7     | 2     | 2      | (8, 3)         | (\(\pm 1, 1\)) |
| 8     | 2     | 2      | (3, 1)         | (\(\pm 1, 1\)) |
| 10    | 3     | 3      | (19, 6)        | (\(\pm 1, 1\), (9, 3)) |
| 11    | 2     | 2      | (10, 3)        | (\(\pm 1, 1\)) |
| 12    | 2     | 2      | (7, 2)         | (\(\pm 1, 1\)) |
| 13    | 4     | 6      | (649, 180)     | (\(\pm 1, 1\), (\(\pm 25, 7\)) (\(\pm 14, 4\)) |
| 14    | 2     | 2      | (15, 4)        | (\(\pm 1, 1\)) |
| 15    | 2     | 2      | (4, 1)         | (\(\pm 1, 1\)) |
| 17    | 2     | 3      | (33, 8)        | (\(\pm 1, 1\)) (16, 4) |
| 18    | 2     | 2      | (17, 4)        | (\(\pm 1, 1\)) |
| 19    | 3     | 3      | (170, 39)      | (\(\pm 1, 1\), (39, 9)) |
| 20    | 2     | 2      | (9, 2)         | (\(\pm 1, 1\)) |
| 21    | 4     | 6      | (55, 12)       | (\(\pm 1, 1\), (\(\pm 13, 3\)) (\(\pm 8, 2\)) |
| 22    | 4     | 4      | (197, 42)      | (\(\pm 1, 1\), (\(\pm 23, 5\)) |
| 23    | 2     | 2      | (24, 5)        | (\(\pm 1, 1\)) |
| 24    | 2     | 2      | (5, 1)         | (\(\pm 1, 1\)) |
| 26    | 3     | 3      | (51, 10)       | (\(\pm 1, 1\), (25, 5)) |
| 27    | 2     | 2      | (26, 5)        | (\(\pm 1, 1\)) |
| 28    | 4     | 4      | (127, 24)      | (\(\pm 1, 1\), (\(\pm 15, 3\)) |
| 29    | 4     | 6      | (9801, 1820)   | (\(\pm 1, 1\), (\(\pm 59, 11\)) (\(\pm 86, 16\)) |
| 30    | 2     | 2      | (11, 2)        | (\(\pm 1, 1\)) |
| 31    | 4     | 4      | (1520, 273)    | (\(\pm 1, 1\), (\(\pm 61, 11\)) |
| 32    | 2     | 2      | (17, 3)        | (\(\pm 1, 1\)) |
| 33    | 2     | 4      | (23, 4)        | (\(\pm 1, 1\)) (\(\pm 10, 2\)) |
| 34    | 2     | 2      | (35, 6)        | (\(\pm 1, 1\)) |
| 35    | 2     | 2      | (6, 1)         | (\(\pm 1, 1\)) |
| 37    | 2     | 3      | (73, 12)       | (\(\pm 1, 1\)) (36, 6) |
| 38    | 2     | 2      | (37, 6)        | (\(\pm 1, 1\)) |
| 39    | 2     | 2      | (25, 4)        | (\(\pm 1, 1\)) |
| 40    | 4     | 4      | (19, 3)        | (\(\pm 1, 1\), (\(\pm 11, 2\)) |
| 41    | 4     | 4      | (2049, 320)    | (\(\pm 1, 1\), (\(\pm 83, 13\)) |
| 42    | 2     | 2      | (13, 2)        | (\(\pm 1, 1\)) |
| 43    | 4     | 4      | (3482, 531)    | (\(\pm 1, 1\), (\(\pm 85, 13\)) |
| 44    | 2     | 2      | (199, 30)      | (\(\pm 1, 1\)) |
| 45    | 4     | 6      | (161, 24)      | (\(\pm 1, 1\), (\(\pm 19, 3\)) (\(\pm 26, 4\)) |
Table 3. Fundamental solutions of simple \(^{19}\) and generalized \(^{20}\) Pell equations

| \(k\) | \(r\) | \(\rho\) | \((x_r, y_r)\) | \((X_r, Y_r)\) |
|-------|-------|-------|----------------|----------------|
| 46    | 6     | 6     | (24335, 3598)  | (±1, 1), (±47, 7), (±183, 27) |
| 47    | 2     | 2     | (48, 7)        | (±1, 1)        |
| 48    | 2     | 2     | (7, 1)         | (±1, 1)        |
| 50    | 3     | 3     | (99, 14)       | (±1, 1), (49, 7) |
| 51    | 3     | 3     | (50, 7)        | (±1, 1), (35, 5) |
| 52    | 4     | 4     | (649, 90)      | (±1, 1), (±79, 11) |
| 53    | 4     | 6     | (66249, 9100)  | (±1, 1), (±211, 29), (±160, 22) |
| 54    | 2     | 2     | (485, 66)      | (±1, 1)        |
| 55    | 4     | 4     | (89, 12)       | (±1, 1), (±21, 3) |
| 56    | 2     | 4     | (15, 2)        | (±1, 1)        |
| 57    | 4     | 4     | (151, 20)      | (±1, 1), (±37, 5) |
| 58    | 4     | 4     | (19603, 2574)  | (±1, 1), (±175, 23) |
| 59    | 2     | 2     | (530, 69)      | (±1, 1)        |
| 60    | 2     | 2     | (31, 4)        | (±1, 1)        |
| 61    | 8     | 12    | (1766319049, 226153980) | (±1, 1), (±367, 47), (±6709, 859), (±94793, 12137), (±62, 8), (±5186, 664) |
| 62    | 2     | 2     | (63, 8)        | (±1, 1)        |
| 63    | 2     | 2     | (8, 1)         | (±1, 1)        |
| 65    | 2     | 4     | (129, 16)      | (±1, 1)        |
| 66    | 4     | 4     | (65, 8)        | (±1, 1), (±23, 3) |
| 67    | 4     | 4     | (48842, 5967)  | (±1, 1), (±401, 49) |
| 68    | 2     | 2     | (33, 4)        | (±1, 1)        |
| 69    | 4     | 6     | (7775, 936)    | (±1, 1), (±91, 11), (±116, 14) |
| 70    | 4     | 4     | (251, 30)      | (±1, 1), (±41, 5) |
| 71    | 4     | 4     | (3480, 413)    | (±1, 1), (±143, 17) |
| 72    | 2     | 2     | (17, 2)        | (±1, 1)        |
| 73    | 6     | 6     | (2281249, 267000) | (±1, 1), (±145, 17), (±1461, 171) |
| 74    | 2     | 2     | (3699, 430)    | (±1, 1)        |
| 75    | 2     | 2     | (26, 3)        | (±1, 1)        |
| 76    | 6     | 6     | (57799, 6630)  | (±1, 1), (±113, 13), (±305, 35) |
| 77    | 4     | 6     | (351, 40)      | (±1, 1), (±43, 5), (±34, 4) |
| 78    | 4     | 4     | (53, 6)        | (±1, 1), (±25, 3) |
| 79    | 2     | 2     | (80, 9)        | (±1, 1)        |
| 80    | 2     | 2     | (9, 1)         | (±1, 1)        |
| 82    | 3     | 3     | (163, 18)      | (±1, 1), (±81, 9) |
| 83    | 2     | 2     | (82, 9)        | (±1, 1)        |
| 84    | 2     | 2     | (55, 6)        | (±1, 1)        |
| 85    | 8     | 12    | (285769, 30996) | (±1, 1), (±101, 11), (±341, 37), (±1429, 155), (±16, 2), (±424, 46) |
(−1, 1), which is quite obvious from (19). The only two exceptions are for the cases $k = 2$ and 3. Although (−1, 1) is also a solution to (19) for these two cases, it does not bring a new branch of solutions calculated by (11) to (13) different from the one obtained with (1, 1). Therefore, there is only one fundamental solution, i.e., $\rho = 1$ for these two cases. Furthermore, the two pairs (1, −1) and (−1, −1) are also solutions of (19), but they do not yield new branches of solutions different from those obtained with (−1, 1) and (1, 1).

Fourth, all generalized fundamental solutions $(X_{f_i}, Y_{f_i})$ with $i > 2$, i.e., other than $(±1, 1)$, have both $X_{f_i}$ and $Y_{f_i}$ odd, except for $k = 40, 96, 208, \ldots$ where $Y_{f_i}$ is even.

Fifth, the generalized fundamental solutions with both $X_{f_i}$ and $Y_{f_i}$ even are shown separately as they do not bring any solutions to (11), and there are $\rho - r$ such solutions.

With the two generalized fundamental solutions $(X_{f_1,2}, Y_{f_1,2}) = (±1, 1)$, one has from (12) $(X_{01,2}, Y_{01,2}) = (±1, 1)$ and it yields the two trivial solutions $(s_{01,2}, t_{01,2}) = (\frac{±1}{2}, \frac{±1}{2}) = (0, 0)$ and (−1, 0) of (11). The next generalized solution (13) reads

$$(X_{11,2}, Y_{11,2}) = ((±x_f + k y_f), (±y_f + x_f))$$
yielding, from (18),

$$(\xi_{1,2}, t_{1,2}) = \left( \frac{\pm x_f + ky_f - 1}{2}, \frac{\pm y_f + x_f - 1}{2} \right)$$

(21)

with both terms integers under the three conditions C1 to C3 above.

For other generalized fundamental solutions $$(X_{f_i}, Y_{f_i})$$ (with $$i > 2$$) different from $$(\pm 1, 1)$$, one has from (12) $$(X_0, Y_0) = (X_{f_i}, Y_{f_i})$$, yielding

$$(\xi_{0,1,2}, t_{0,1,2}) = \left( \frac{X_{f_i} - 1}{2}, \frac{Y_{f_i} - 1}{2} \right)$$

(22)

integer solutions of (1) if $$X_{f_i}$$ and $$Y_{f_i}$$ are both odd. The next generalized solution (15) reads $$(X_1, Y_1) = (X_{f_i}, x_f + kY_{f_i}, y_f + X_{f_i}y_f, Y_{f_i}x_f)$$, yielding

$$(\xi_{1,1,2}, t_{1,1,2}) = \left( \frac{X_{f_i}x_f + kY_{f_i}y_f - 1}{2}, \frac{X_{f_i}y_f + Y_{f_i}x_f - 1}{2} \right)$$

(23)

One sees clearly that $$X_{f_i}$$ and $$Y_{f_i}$$ can not be simultaneously even for $$\xi_{1,1}$$ and $$t_{1,1}$$ to be integers. For $$X_{f_i}$$ and $$Y_{f_i}$$ both odd, the three conditions C1 to C3 above ensure that $$\xi_{1,1}$$ and $$t_{1,1}$$ are integers.

For the cases of $$X_{f_i}$$ odd and $$Y_{f_i}$$ even, like for $$k = 40$$ and 96 in Tables 2 to 4 one has that $$x_f$$ and $$y_f$$ must be simultaneously odd and, by condition C2 above, $$k$$ must be even for (23) to provide integer solutions.

Finally, for all single fundamental solutions $$(x_f, y_f)$$ of the simple Pell equation with both $$x_f$$ and $$y_f$$ odd, they appear for most of the values of $$k$$ such that $$k \equiv 0 \pmod{8}$$. The exceptions to this are for $$k = 56, 72, 112, 184, 240, 248, 264, 272, 376, \ldots$$, i.e., for some values of $$k$$ such that $$k \equiv \pm 8, \pm 16 \pmod{64}$$ (but not all), where $$y_f$$ is even. In these cases, one has that $$k$$ and $$y_f$$ are both even, then $$Y_{f_i}$$ can not be even for (23) to provide integer solutions. If this is not the case, i.e., if $$Y_{f_i}$$ is even, then the generalized fundamental solutions $$(X_{f_i}, Y_{f_i})$$ must be discarded as it does not provide integer solutions for $$t$$ in (23).

For the general case of $$k \equiv 0 \pmod{8}$$, the fact that $$y_f$$ is not odd can be explained as follows. As $$k \equiv 0 \pmod{8}$$ is not square free, the simple Pell equation (20) can be simplified posing $$k = c^2k'$$, with $$k'$$ square free, yielding

$$x^2 - k'y'^2 = 1$$

(24)

with $$y' = cy$$. The fundamental solution $$(x_f, y'_f)$$ of (21) yields then the fundamental solution $$(x_f, y_f) = \left( x_f, \frac{y'_f}{c} \right)$$ of (20). For example, for $$k = 8$$, let $$k' = 2$$ and $$c = 2$$, (21) yields $$(x_f, y'_f) = (3, 2)$$ and $$(x_f, y'_f) = (x_f, y_f) = (3, 1)$$. For most of the cases of $$k$$ such that $$k \equiv 0 \pmod{8}$$, $$y'_f$$ is divisible by $$c$$ such that $$\frac{y'_f}{c}$$ is odd yielding then $$y_f$$ odd.

For the exceptions of some values of $$k$$ such that $$k \equiv \pm 8, \pm 16 \pmod{64}$$, this procedure does not lead to an odd value of $$\frac{y'_f}{c}$$. For example, for $$k = 56,
let \( k' = 14 \) and \( c = 2 \), yielding \((x_f, y'_f) = (15, 4)\) and \( y_f = \frac{y'_f}{c} = 2 \). For \( k = 72 \), let \( k' = 2 \) and \( c = 6 \), yielding \((x_f, y'_f) = (3, 2)\). However, \( y'_f \) is not divisible by \( c = 6 \) and one must consider not the first fundamental solution of the simple Pell equation for \( k' = 2 \), but the second solution given by (10) for \( n = 2 \), yielding \((x_2, y'_2) = (17, 12)\) that gives \( y_f = \frac{y'_2}{c} = 2 \) and finally \((x_f, y_f) = (17, 2)\).

Furthermore, for some expressions of \( k \) in function of the closest natural square \( s^2 \), one can find general expressions of \((x_f, y_f)\) and \((X_{f_i}, Y_{f_i})\) in addition to \((\pm 1, 1)\) (i.e., for \( i > 2 \)) as shown in Table 5. All these expressions can easily be demonstrated by replacing the appropriate variables in (19) and (20).

Note that these general expressions for the fundamental solutions \((x_f, y_f)\) are valid in all generality for the simple Pell equation (20).

### 4.2. First \( r \) solutions of (11) for multiple of triangular numbers.

Before calculating all solutions of (11) yielding triangular numbers that are multiple of other triangular numbers, we have to find the first \( r \) solutions \((\xi_i, t_i)\) (with \( 0 \leq i \leq r \)) of (11), arranged in increasing value order, i.e., \( \xi_0 = 0 < \xi_1 < \ldots < \xi_i < \ldots < \xi_r \) and \( t_0 = 0 < t_1 < \ldots < t_i < \ldots < t_r \), and that correspond to the \( r \) fundamental solutions \((X_{f_i}, Y_{f_i})\) of the generalized Pell equation (19), with both \( X_{f_i} \) and \( Y_{f_i} \) odd or of different parities. The generalized fundamental solutions \((X_{f_1}, Y_{f_1}) = (1, 1)\) and \((X_{f_2}, Y_{f_2}) = (-1, 1)\) provide respectively, the solutions \((\xi_r, t_r)\) and \((\xi_{r-1}, t_{r-1})\) of (11) from (13), yielding successively

\[
(X_1, Y_1) = (X_{f_1} x_f + kY_{f_1} y_f, X_{f_1} y_f + Y_{f_1} x_f) \\
= (x_f + ky_f, y_f + x_f)
\]

\[
(X_2, Y_2) = (X_{f_2} x_f + kY_{f_2} y_f, X_{f_2} y_f + Y_{f_2} x_f) \\
= (-x_f + ky_f, -y_f + x_f)
\]

and

\[
(\xi_r, t_r) = \left( \frac{X_{1_r} - 1}{2}, \frac{Y_{1_r} - 1}{2} \right) \\
= \left( \frac{x_f + ky_f - 1}{2}, \frac{y_f + x_f - 1}{2} \right)
\]

\[
(\xi_{r-1}, t_{r-1}) = \left( \frac{X_{1_{r-1}} - 1}{2}, \frac{Y_{1_{r-1}} - 1}{2} \right) \\
= \left( \frac{-x_f + ky_f - 1}{2}, \frac{-y_f + x_f - 1}{2} \right)
\]

(25)

(26)

Then for \( r > 2 \), the next two generalized fundamental solutions \((X_{f_3}, Y_{f_3})\) and \((X_{f_4}, Y_{f_4}) = (-X_{f_3}, Y_{f_3})\) yield respectively \((\xi_1, t_1)\) and \((\xi_2, t_2)\). If both
| $k$ | $s$  | $r$ | $(x_f, y_f)$ | $(X_f, Y_f)$ |
|-----|------|-----|-------------|-------------|
| $s^2 + 1$ | even | 2   | $(\pm (2s^2 + 1), 2s)$ | $(-, -)$ |
|       | odd  | 3   | $(\pm (2s^2 + 1), 2s)$ | $(s^2, s)$ |
| $s^2 + 2$ | any  | $2^{(a)}$ | $(\pm (s^2 + 1), s)$ | $(-, -)$ |
| $s^2 + 4$ | even | $2^{(b)}$ | $(\pm (s^2 + 2), . \frac{s}{2})$ | $(-, -)$ |
| 1 (mod 4) | $4^{(c)}$ | $(\pm (s^2 + 1)(s^2 + 4) - 2, \frac{s^2 + 3}{2})$ | $(-, -)$ |
|           |       | $(\pm (s (\frac{s^2 + 4}{2} - 1)) \cdot \frac{s(s-1)}{2} + 1)$ | $(-, -)$ |
| 3 (mod 4) | 4    | $(\pm (s^2 + 1)(s^2 + 4) - 2, \frac{s^2 + 3}{4})$ | $(-, -)$ |
|           |       | $(\pm (s (\frac{s^2 + 4}{2} + 1)) \cdot \frac{s(s+1)}{2} + 1)$ | $(-, -)$ |
| $s^2 + 8$ | 0 (mod 4) | 2   | $(\pm (s^2 + 1), . \frac{s}{4})$ | $(-, -)$ |
| $s^2 + s$ | any  | 2   | $(\pm (2s + 1), 2)$ | $(-, -)$ |
| $s^2 + s - 1$ | any  | $\geq 2$ | $(\pm (s^2 + s - 1), 2s + 1)$ | $(s, *)$ |
| $s^2 + s - 2$ | 0 (mod 3) | $\geq 4$ | $(s, *)$ | $(\pm (s^2 + 3s - 3), 2s + 1)$ |
|           | 1 (mod 3) | 2   | $(s, *)$ | $(-, -)$ |
|           | 2 (mod 3) | 4   | $(s, *)$ | $(\pm \frac{2s^2 - 5}{3} 2s - 2, 3)$ |
| $s^2 + s + 1$ | 1 (mod 3) | 4   | $(\pm \frac{2(s + 1)^2}{3} + 1, 4 \frac{2(s - 1)}{3} + 1)$ | $(-, -)$ |
|           |       | $\geq 4$ | $(\pm \frac{2s^2 + 2s - 1}{3} 2s - 1, 3)$ |
| $s^2 + 2s$ | any  | 2   | $(\pm (s + 1), 1)$ | $(-, -)$ |
| $s^2 + 2s - 1$ | any  | 2   | $(\pm (s^2 + 2s), s + 1)$ | $(-, -)$ |
| $s^2 + 2s - 2$ | 2 (mod 3) | $2^{(g)}$ | $(\pm \frac{2s^2 + 4s - 1}{3}, 2(s + 1))$ | $(-, -)$ |
| $s^2 + 2s - 3$ | 0 (mod 4) | 4   | $(\pm (s + 1)(s^2 + 2s - 2), 2s + 2)$ | $(-, -)$ |
|           | 2 (mod 4) | $4^{(h)}$ | $(\pm (s + 1)(s^2 + 2s - 2), 2s + 2)$ | $(\pm \frac{2s^2 + 4s - 4}{4}, 4)$ |
| odd    | $2^{(i)}$ | $(\pm \frac{s^2 + 2s - 1}{2}, 2s + 1)$ | $(-, -)$ |
| $s^2 + 2s - 7$ | 3 (mod 4) | $2^{(j)}$ | $(\pm \frac{s^2 + 2s - 3}{4}, 1 + \frac{s + 1}{4})$ | $(-, -)$ |
|           | 1 (mod 4) | 4   | $(\pm \frac{s^2(s^2 + 2s - 3) + 1}{8}, \frac{s^2(s + 3) + 1}{8})$ | $(-, -)$ |
|           |       | $\frac{2s^2 + 3s - 5}{2} 1 + 2s + 1$ | $(-, -)$ |

$(-, -)$: no solutions exist as $r = 2(s, *)$: no apparent pattern; $\pm$: plus/minus sign independent from other $\pm$ sign; (a) except for $k = 51, 66 (r = 3, 4)$; (b) except for $k = 40 (r = 4)$; (c) except for $k = 85 (r = 8)$; (d) except for $k = 197 - 1$, with $\sigma$ even; (e) except for $k = 5, 11, 55, \ldots$; (f) except for $k = 40; (g)$ except for $k = 78 (r = 4)$; (h) except for $k = 5 (r = 2); (i)$ except for $k = 96 (r = 4); (j)$ except for $k = 136 (r = 4)$
\(X_{f_1}\) and \(Y_{f_1}\) are odd, then (12) \((n = 0)\) can be used for \((\xi_1, t_1)\), yielding
\[
(\xi_1, t_1) = \left( \frac{X_{f_1} - 1}{2}, \frac{Y_{f_1} - 1}{2} \right)
\]
(27)

Equation (12) could also be used for \((\xi_2, t_2)\) with \((-X_{f_1}, Y_{f_1})\), but it would provide a negative value for \(\xi_2\). Instead, we use (13) \((n = 1)\), giving
\[
(\xi_2, t_2) = \left( \frac{-X_{f_1}x_f + kY_{f_1}y_f - 1}{2}, \frac{-X_{f_1}y_f + Y_{f_1}x_f - 1}{2} \right)
\]
(28)

The next two generalized fundamental solutions \((X_{f_5}, Y_{f_5})\) and \((X_{f_6}, Y_{f_6})\) yield similarly the next two solutions \((\xi_i, t_i)\) that are put in the right increasing order.

For example, for \(k = 13, r = 4, (x_f, y_f) = (649, 180), (X_{f_1}, Y_{f_1}) = (\pm 1, 1), (\pm 25, 7), (25)\) and (26) yield respectively, \((\xi_r, t_r) = (\xi_4, t_4) = (1494, 414)\) and \((\xi_{r-1}, t_{r-1}) = (\xi_3, t_3) = (845, 234)\); (27) and (28) yield respectively, \((\xi_1, t_1) = (12, 3)\) and \((\xi_2, t_2) = (77, 21)\).

Another example, for \(k = 46, r = 6, (x_f, y_f) = (24335, 3588), (X_{f_1}, Y_{f_1}) = (\pm 1, 1), (\pm 47, 7), (\pm 183, 27)\). With \((X_{f_1, 2}, Y_{f_1, 2}) = (\pm 1, 1), (25)\) and (26) yield respectively, \((\xi_5, t_5) = (94691, 13961)\), \((\xi_5, t_5) = (70356, 10373)\). With \((X_{f_3, 4}, Y_{f_3, 4}) = (\pm 47, 7), (12)\) yields \((\xi_1, t_1) = \left( \frac{X_{f_1} - 1}{2}, \frac{Y_{f_1} - 1}{2} \right) = (23, 3)\) and (13) yields
\[
(\xi_4, t_4) = \left( \frac{-X_{f_1}x_f + kY_{f_1}y_f - 1}{2}, \frac{-X_{f_1}y_f + Y_{f_1}x_f - 1}{2} \right) = (5795, 854)
\]

Finally, with \((X_{f_5, 6}, Y_{f_5, 6}) = (\pm 183, 27), (12)\) yields \((\xi_2, t_2) = \left( \frac{X_{f_1} - 1}{2}, \frac{Y_{f_1} - 1}{2} \right) = (91, 13)\) and (13) yields \((\xi_3, t_3) = (1495, 220)\).

For the case where \(Y_{f_1}\) is even, i.e., \(k = 40, 96, 120, \ldots\), (24), (26) and (27) cannot be used with \((X_{f_1, 2}, Y_{f_1, 2}) = (\pm 1, 1)\) as both \(k\) and \(Y_{f_1}\) are even, yielding non-integer solutions for \(\xi\) and \(t\). Instead, the other generalized fundamental solution have to be used with (13) \((n = 1)\) and (14) \((n = 2)\). For example, for \(k = 40, r = 4, (x_f, y_f) = (19, 3), (X_{f_1}, Y_{f_1}) = (\pm 1, 1), (\pm 11, 2), (13)\) yields, first, with \((X_{f_1}, Y_{f_1}) = (11, 2), (X_{f_3, 1}, Y_{f_3, 1}) = (X_{f_1}x_f + kY_{f_1}y_f, X_{f_1}y_f + Y_{f_1}x_f) = (449, 71)\), yielding \((\xi_2, t_2) = (224, 35)\), and second, with \((X_{f_1}, Y_{f_1}) = (-11, 2), (X_{f_1}, Y_{f_1}) = (31, 5)\), giving \((\xi_1, t_1) = (15, 2)\). Next, (14) yields, first, with \((X_{f_1}, Y_{f_1}) = (1, 1)\), \((X_{f_2, 1}, Y_{f_2, 1}) = (x_f^2 + ky_f^2 + 2kx_fy_f, x_f^2 + ky_f^2 + 2y_fy_f) = (5281, 835)\), yielding \((\xi_4, t_4) = (2640, 417)\), and second, with \((X_{f_2}, Y_{f_2}) = (-1, 1), (X_{f_2}, Y_{f_2}) = (-x_f^2 + ky_f^2 + 2kx_fy_f, x_f^2 + ky_f^2 - 2y_fy_f) = (3839, 697)\), yielding \((\xi_3, t_3) = (1919, 303)\).

4.3. All solutions of (11) for multiple of triangular numbers. Once that the first \(r\) values of \((\xi_i, t_i)\) have been found, each corresponding to
one of the $r$ generalized fundamental solutions $(X_{f_i}, Y_{f_i})$, the $r$ branches of infinitely many other solutions can be found using either:

1) the $r$ general solutions (11) (assuming a + sign), yielding

$$\xi_n + \sqrt{kt_n} = \left(\xi_i + \sqrt{kt_i} + \left(\frac{1 + \sqrt{k}}{2}\right)\right)\left(x_f + \sqrt{ky_f}\right)^n - \left(\frac{1 + \sqrt{k}}{2}\right)$$

(29)

where $(\xi_i, t_i)$ must be replaced successively by the $r$ values of $(\xi, t)$; or,

2) the first recurrence relation (15), yielding

$$(\xi_n, t_n) = \left((x_f\xi_{n-r} + kyt_{n-r}) + \left(\frac{x_f + ky_f - 1}{2}\right)\right),$$

$$t_2 + y_f\xi_{n-r} + \left(\frac{x_f + y_f - 1}{2}\right)$$

(30)

where indices of $\xi_{n-r}$ and $t_{n-r}$ (instead of $\xi_{n-1}$ and $t_{n-1}$) in the right part of (30) refer to the preceding values of $\xi$ and $t$ in the same branch of solutions; or,

3) the second recurrence relation (16), yielding

$$(\xi_n, t_n) = \left(2x_f\xi_{n-r} - \xi_{n-2r} + (x_f - 1), 2x_ft_{n-r} - t_{n-2r} + (x_f - 1)\right)$$

(31)

where indices of $\xi_{n-r}, \xi_{n-2r}$ and $t_{n-r}, t_{n-2r}$ (instead of $\xi_{n-1}, \xi_{n-2}$ and $t_{n-1}, t_{n-2}$) in the right part of (31) refer to the preceding and the one before values of $\xi$ and $t$ in the same branch of solutions; or,

4) the Chebyshev polynomial solution (17), yielding

$$(\xi_n, t_n) = \left(\left(\xi_i + \frac{1}{2}\right)T_{n-1}(x_f) + k\left(t_i + \frac{1}{2}\right)y_fU_{n-2}(x_f) - \frac{1}{2},
\right.$$

$$\left.\left(\xi_i + \frac{1}{2}\right)y_fU_{n-2}(x_f) + \left(t_i + \frac{1}{2}\right)T_{n-1}(x_f) - \frac{1}{2}\right)$$

(32)

where $(\xi_i, t_i)$ must be replaced successively by the $r$ values of $(\xi, t)$.

4.4. Relation between Pell equation solutions and recurrent relations. We can give now a new definition of the rank $r$ introduced in Section 2. The rank $r$ is the number of fundamental solutions $(X_{f_i}, Y_{f_i})$ of the generalized Pell equation (19), with $X_{f_i}$ odd and $Y_{f_i}$ odd or even (if $y_f$ is not even), with $r \leq \rho$, the total number of generalized solution of (19).

Furthermore, we see that the second recurrent relations (31) for both $\xi$ and $t$ have $x_f$ as the only parameter, and that the two relations are independent from the value of $k$ and $y_f$. This fundamental solution $x_f$ of the simple Pell equation (20) acts like a constant of the problem for each value of $k$. Note further that summing the expressions of $t_r$ and $t_{r-1}$ in (25) and (26) yields $t_r + t_{r-1} = x_f - 1$. As this sum $t_r + t_{r-1}$ was already defined in (2), the constant $\kappa$ is related to $x_f$

$$\kappa = x_f - 1$$

(33)
Furthermore, (33) yields also that $y_f$ is related to the difference $\delta = t_r - t_{r-1}$ through the simple Pell equation $20 \left( \kappa + 1 \right)^2 - ky_f^2 = 1$, which is verified if $y_f^2 = \kappa^2 - 4t_r t_{r-1} = \left( t_r - t_{r-1} \right)^2 = \delta^2$, giving

$$\delta = y_f$$  \hspace{1cm} (34)

for all non-square values of $k$ (except for some values of $k$ such that $k \equiv 0 \pmod{8}$, see further). Replacing in the simple Pell equation $20 \left( \kappa + 1 \right)^2 - k\delta^2 = 1$ yields the condition between the sum and the difference of $t_r$ and $t_{r-1}$

$$\delta = \sqrt{\frac{\kappa^2 + \kappa}{k}}$$  \hspace{1cm} (35)

With the exception of $k = 56, 72, 112, 184, 240, 248, 264, 272, 376, ..., \text{ i.e., }$ for some values of $k$ such that $k \equiv \pm 8, \pm 16 \pmod{64}$ (for which (34) is valid), the relation (33) is not valid for the other cases of $k \equiv 0 \pmod{8}$. In these cases, $\delta > y_f$ and one must find the next pair of solutions to the simple Pell equation by (10) for $n = 2$, i.e., $(x_2, y_2) = \left( x_f^2 + ky_f^2, 2x_f y_f \right)$. Then for these cases,

$$\kappa = x_f^2 + ky_f^2 - 1$$  \hspace{1cm} (36)

$$\delta = 2x_f y_f$$  \hspace{1cm} (37)

Finally, replacing $x_f$ in (31) from (33) yields

$$\left( \xi_n, t_n \right) = \left( 2 \left( \kappa + 1 \right) \xi_{n-r} - \xi_{n-2r} + \kappa, 2 \left( \kappa + 1 \right) t_{n-r} - t_{n-2r} + \kappa \right)$$  \hspace{1cm} (38)

which are the same recurrent relations given in (4) and (5).

5. Conclusions

We have shown that the problem of finding all triangular numbers that are multiples of other triangular numbers with non-square integer multiplier $k$ can be solved using solutions of Pell equations with a simple change of variables. Only those $r$ fundamental solutions $(X_{f_i}, Y_{f_i})$ of the generalized Pell equation with $X_{f_i}$ odd and $Y_{f_i}$ odd or even (if $y_f$ is not even) provide solutions to the problem of finding triangular numbers that are multiple of other triangular numbers. General expressions of fundamental solutions of the Pell equations are given for some values of the multiplier $k$ in function of the closest natural square values $s^2$. Many infinitely solutions are then found on $r$ branches corresponding to each of the $r$ generalized fundamental solutions $(X_{f_i}, Y_{f_i})$ and these solutions can be found either by a general relation involving $\sqrt{k}$, or by a first set of recurrent relations, or by a second set of recurrent relations, or by Chebyshev polynomial solutions. Among these, the second set of recurrent relations are found to be the same as those found previously without using the Pell equation solving method.

Furthermore, the number $r$ of generalized fundamental solutions $(X_{f_i}, Y_{f_i})$ with $X_{f_i}$ odd and $Y_{f_i}$ odd or even (if $y_f$ is not even) corresponds to the rank of these second set recurrent relations. Finally, the two constants $\kappa = t_r + t_{r-1}$...
and $\delta = t_r - t_{r-1}$ are also related to respectively the fundamental solutions $x_f$ and $y_f$ of the simple Pell equation, as $\kappa = x_f - 1$ and $\delta = y_f$ or $\delta = 2x_fy_f$.

These are an unexpected result as this means that the fundamental solutions of the simple Pell equation, in all its generality, are related to constants in recurrent relations of the problem of finding triangular numbers multiple of other triangular numbers.

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