Bijective Proof of Kasteleyn’s Toroidal Perfect Matching Cancellation Theorem

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Abstract

We give a constructive bijective proof for an identity that generalizes an observation of Kasteleyn: Let $m, n$ be positive even numbers and let $T_{m,n}$ be the toroidal square grid which consists of $m$ horizontal cycles $C_0^H, \ldots, C_{m-1}^H$ and $n$ vertical cycles $C_0^V, \ldots, C_{n-1}^V$. Let $A$ be one layer of the horizontal edges of $T_{m,n}$ and let $B$ be one layer of the vertical edges of $T_{m,n}$. We say that a perfect matching is \textit{even} if it has an even number of elements of each of $A, B$. We call a perfect matching \textit{odd} if it is not even. If $h = (h_0, \ldots, h_{m-1})$ and $v = (v_0, \ldots, v_{n-1})$ are positive integer vectors then we let $E_{m,n}(h,v)$ denote the set of the \textit{even} perfect matchings of $T_{m,n}$ with exactly $h_i$ edges from cycle $C_i^H$ and exactly $v_j$ edges from cycle $C_j^V$, for $0 \leq i < m$ and $0 \leq j < n$. Analogously let $O_{m,n}(h,v)$ denote the set of the \textit{odd} perfect matchings of $T_{m,n}$ with exactly $h_i$ edges from cycle $C_i^H$ and exactly $v_j$ edges from cycle $C_j^V$, for $0 \leq i < m$ and $0 \leq j < n$. We show (combinatorially) that $|E_{m,n}(h,v)| = |O_{m,n}(h,v)|$, for all positive integer vectors $h$ and $v$.

1 Introduction

A perfect matching of a graph $G = (V, E)$ is a collection of edges with the property that each vertex is adjacent to exactly one of these edges. In statistical physics a Perfect matching is

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called dimer arrangement. Very often graph $G$ comes with an edge-weight function $w : E \to \mathbb{Q}$. The statistics of the weighted perfect matchings is then given by the generating function

$$P(G, x) = \sum_{M \text{ perfect matching of } G} x^{\sum_{e \in M} w(e)},$$

also known as the dimer partition function on $G$. We note that if $G$ is bipartite, with bipartition classes $V_1$ and $V_2$, then $P(G, x)$ equals the permanent of $A(G)$, where $A = A(G)$ is the $|V_1| \times |V_2|$ matrix given by $A_{uv} = x^{w(uv)}$.

Dimer partition functions are among the most studied in statistical mechanics. One of their remarkable properties is that, if $g$ denotes the minimum genus of an orientable surface in which $G$ embeds, then the partition function of a dimer model on a graph $G$ can be written as a linear combination of $2^{2g}$ Pfaffians of Kasteleyn matrices. These $2^{2g}$ Kasteleyn matrices are skew-symmetric $|V| \times |V|$ matrices determined by $2^{2g}$ orientations of $G$, called Kasteleyn (or Pfaffian) orientations. P. W. Kasteleyn himself proved this Pfaffian formula in the planar case and for the toroidal square grids where all the horizontal edge-weights and all the vertical edge-weights are the same [Kas61], and stated the general fact [Kas67]. A complete combinatorial proof of this statement was first obtained much later by Gallucio and Loebl [GL99], and independently by Tesler [Tes00]. In his seminal paper [Kas61], Kasteleyn found an analytic expression for the four Pfaffians in the case of the toroidal square grid where all the horizontal edge-weights and all the vertical edge-weights are the same, and noticed that:

*One of these four Pfaffians, which corresponds to a particular Kasteleyn orientation, vanishes.*

This observation can be restated as follows: Let $T$ be a toroidal square grid where we fix one layer of the horizontal edges (and denote it by $A$) and one layer of the vertical edges (and denote it by $B$). We will not do technical work with the Pfaffians and we thus only state the relevant facts regarding them (for details see [Kas61]). The Pfaffian is a signed generating function of the perfect matchings. In the particular Pfaffian which vanishes, a perfect matching of $T$ has sign $+1$ if and only if it contains an even number of edges from each of $A, B$; we call these perfect matchings *even* and the remaining ones *odd*. Hence, the above observation of Kasteleyn may be stated as follows:

**Theorem 1** [Kas61] Let $T$ be a toroidal square grid, where $A$ (respectively, $B$) is one layer of the horizontal (respectively, vertical) edges. For each $k, l$, the number of odd perfect matchings of $T$ with exactly $k$ vertical edges and $l$ horizontal edges is equal to the number of even perfect matchings of $T$ with exactly $k$ vertical edges and $l$ horizontal edges.

The main result of this paper is a bijective combinatorial proof of a generalization of the preceding theorem. Let $m, n$ be positive even numbers and let $T_{m,n}$ be a toroidal square grid which consists of $m$ horizontal cycles $C^H_0, \ldots, C^H_{m-1}$ and $n$ vertical cycles $C^V_0, \ldots, C^V_{n-1}$. For $T = T_{m,n}$, let $A$ and $B$ be as in Theorem 1. If $h = (h_0, \ldots, h_{m-1})$ and $v = (v_0, \ldots, v_{n-1})$ are positive integer vectors then we let $E_{m,n}(h, v)$ denote the set of the even perfect matchings
of $T_{m,n}$ with exactly $h_i$ edges from cycle $C_i^H$ and exactly $v_j$ edges from cycle $C_j^V$, for $0 \leq i < m$ and $0 \leq j < n$. Analogously, let $O_{m,n}(h,v)$ be defined as $E_{m,n}(h,v)$ but with respect to the set of odd perfect matchings. We show:

**Theorem 2** For all positive integer vectors $h$ and $v$, there is a polynomial time computable bijection between the sets $E_{m,n}(h,v)$ and $O_{m,n}(h,v)$.

The partition function of free fermions on a closed Riemann surface of genus $g$ is also a linear combination of $2^{2g}$ determinants of Dirac operators [AGMV86]. It is widely assumed in theoretical physics that dimer models are discrete analogues of free fermions and thus a relation between Kasteleyn matrices and Dirac operators is expected. This is well understood in the planar case due to the work of Kenyon [Ken02], and in the genus one case by the work of Ferdinand [Fer67] that extend results of Kasteleyn [Kas61]. The higher genus $g > 1$ case remains mysterious. The only results available are numerical evidence for one graph embedded on the double torus [CSM02]. In particular, $2^{g-1}(2^g - 1)$ determinants of the Dirac operators vanish, and a natural open problem is whether a discrete analogue of this fact holds for the Pfaffians of Kasteleyn matrices. The case $g = 1$ is affirmative by Theorem 1. However, the algebraic techniques used to prove this fact are not suitable even for addressing the genus 2 case, where 6 out of 16 Pfaffians are required to vanish (or vanish 'in a limit').

The motivation for this paper has been to give a bijective combinatorial proof of Theorem 1. We hope that this may help understanding the genus 2 case.

2 Preliminaries

Throughout this work, let $m$ and $n$ be two positive integers. We solely consider the case where both $m$ and $n$ are even. We will introduce a couple of distinct graphs and/or digraphs with node set $V_{m,n} = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ all of them satisfying the following property: the endpoints $(i,j)$ and $(i',j')$ of any of their edges differ in at most one coordinate and exactly by 1, i.e.,

$$(i - i' = \pm 1 \wedge j = j') \vee (i = i' \wedge j - j' = \pm 1).$$

Edges between nodes $(i,j)$ and $(i',j')$ such that $i = i'$ (respectively, $j = j'$) will be referred to as horizontal (respectively, vertical) edges.

The collection of nodes $(i,j) \in V_{m,n}$ for which $0 \leq j < n$ (respectively, $0 \leq i < m$) will be called the $i$-th row (respectively, $j$-th column), and $i$ (respectively, $j$) will be called its index. Nodes that belong to a row (respectively, column) of even index will be called even row nodes (respectively, even column nodes). If a node is both an even (respectively, odd) row and column node, then it will be simply referred to as an even node (respectively, odd node).

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1Here and throughout this work, arithmetic involving node labels is always modulo $m$ and $n$ over the first and second coordinates, respectively
The collection of nodes \((i, j) \in V_{m,n}\) such that \(i + j\) is even (respectively, odd) will henceforth be referred to as **black** (respectively, **white**) nodes, denoted \(B_{m,n}\) (respectively, \(W_{m,n}\)). From our ongoing discussion it follows that the graphs we will work with are all bipartite. In fact, \(\{B_{m,n}, W_{m,n}\}\) will always be a feasible color class bipartition of \(V_{m,n}\).

Paths and cycles will always be considered as subgraphs, say \(S\), of whatever the graph or digraphs, we are dealing with. We thus rely on the standard notation \(V(S)\) and \(E(S)\) to denote \(S\)’s nodes and edges, respectively.

For a simple path or cycle \(S\) in whatever graph we consider, we say that node \(v \in V(S)\) is a **corner** of \(S\) if one of the edges of \(S\) incident on \(v\) is a horizontal edge and the other one is a vertical edge.

We now establish a simple, yet key property, of certain type of cycles of \(T_{m,n}\). The property concerns the number of nodes encircled by such cycles.

**Lemma 3** If \(C\) is a contractible cycle of \(T_{m,n}\) all of whose corners have the same parity, then the disk encircled by \(C\) contains a single connected component of \(T_{m,n} \setminus C\).

**Proof** For the sake of contradiction, assume the disk encircled by \(C\) contains two connected components of \(T_{m,n} \setminus C\). The only way in which this could happen is if the intersection of \(C\) with a face of \(T_{m,n}\) is not connected. Since the corners of \(C\) all have the same parity, one of the following two cases holds for some \((i, j) \in V_{m,n}\): (1) there is a horizontal edge of \(C\) between \((i, j)\) and \((i, j + 1)\), and also between \((i + 1, j)\) and \((i + 1, j + 1)\), or (2) there is a vertical edge of \(C\) between \((i, j)\) and \((i + 1, j)\), and also between \((i, j + 1)\) and \((i + 1, j + 1)\).

Because of the grid-like structure of \(T_{m,n}\), for case (1) to occur, one of the corners of \(C\) must belong to the row of index \(i\) and another corner to the row of index \(i + 1\), contradicting the parity assumption. The last case can be dealt with similarly as the first case. □

**Lemma 4** If \(C\) is a contractible cycle of \(T_{m,n}\) all of whose corners have the same parity, then the disk encircled by \(C\) contains an odd number of nodes of \(T_{m,n} \setminus C\).

**Proof** By Lemma 3, we know that \(C\) encircles one connected component of \(T_{m,n} \setminus C\). Let \(P\) be the shortest segment of \(C\) (ties broken arbitrarily), whose endpoints, say \(c_1\) and \(c_2\) are corners of \(C\). Clearly, all nodes of \(P\) must belong to either the same column or the same row. Without loss of generality, assume that the former case holds, and that the nodes of \(P\) are \((i_1, j), (i_1 + 1, j), \ldots, (i_2, j)\). Let \(R_1\) (respectively, \(R_2\)) be the rectangle (cycle with four corners) of \(T_{m,n}\) that contains \(P\) as a segment and whose opposite corners are \((i_1, j)\) and \((i_2, j + 2)\) (respectively, \((i_2, j - 2)\)). Because the parity hypothesis concerning \(C\)’s corners, the minimality of \(P\), and the structure of \(T_{m,n}\), either \(R_1\) or \(R_2\) must encircle an area of the torus also encircled by \(C\). Without loss of generality, assume \(R_1\) is such a rectangle.

Thus, either \(C = R_1\) or the situation is one of the three depicted in Figure 1. To conclude, observe that any path fully contained in a single row or column whose endpoints have the same parity has an odd number of nodes. The desired conclusion follows immediately by induction and the fact that any rectangle whose corners have the same parity encircles a set of nodes of size a product of two odd numbers, i.e., an odd number of nodes. □
Consider now a perfect matching $M$ of $T_{m,n}$ (in contrast to our view of paths and cycles as subgraphs, we consider matchings as subsets of edges). We will associate to $M$ a di-graph $D_{m,n} = D_{m,n}(M) = (U, D)$ over the same set of nodes of $T_{m,n}$, that is $U = V_{m,n}$, and such that (see Figure 3):

- If $v$ is black, then $vw \in D$ if and only if $vw$ is an edge of $M$.
- If $v$ is white, then $vw \in D$ provided: (1) $vw$ is not an edge of $M$, and (2) if $uv$ is the edge of the matching $M$ incident to $v$, then $u$, $v$, and $w$ all lie in the same row or the same column of $T_{m,n}$.

The association of $M$ to $D_{m,n}(M)$ is inspired on one due to Thiant [Thi06, Ch. 5] which maps domino tilings of rectangles (equivalently, perfect matching of grids) to rooted trees. However, neither our context, nor the properties of the derived objects is related to [Thi06]. In particular, we consider toroidal grids in contrast to square grids, and as a consequence $D_{m,n}(M)$ is not necessarily even a forest.

The digraph $D_{m,n}(M)$ has a fair amount of interesting structure. Our next result states the most obvious one.

**Lemma 5** If $M$ is a perfect matching of $T_{m,n}$ and $D_{m,n} = D_{m,n}(M)$, then every node of $D_{m,n}$ has out-degree one.

**Proof** Simply observe that the edges outgoing from a node are completely determined by the edges of $M$ incident to that node, and observing that there is exactly one such matching edge per node. □
Figure 3: Perfect matching $M$ of $T_{6,8}$ and associated $D_{6,8}(M)$. Even (respectively, odd) edges shown as solid (respectively, dashed) lines.

3 Bijection

Henceforth, let $A$ (respectively, $B$), be the set of vertical (respectively, horizontal) edges whose endpoints are in the rows (respectively, columns) of index 0 and $m - 1$ (respectively, $n - 1$). Whether or not we consider the collection of edges in $A$ and $B$ are undirected or directed will be clear from context.

Let $\mathcal{M}_{m,n}$ be the collection of perfect matchings of $T_{m,n}$. We say that $M \in \mathcal{M}_{m,n}$ is of:

- **Type $EE$**: If $|M \cap A|$ and $|M \cap B|$ are both even.
- **Type $EO$**: If $|M \cap A|$ is even and $|M \cap B|$ is odd.
- **Type $OE$**: If $|M \cap A|$ is odd and $|M \cap B|$ is even.
- **Type $OO$**: If $|M \cap A|$ and $|M \cap B|$ are both odd.

We henceforth let $\mathcal{M}^{(EE)}_{m,n}, \mathcal{M}^{(EO)}_{m,n}, \mathcal{M}^{(OE)}_{m,n}$, and $\mathcal{M}^{(OO)}_{m,n}$ be the collection of matchings of $T_{m,n}$ of type $EE, EO, OE$, and $OO$, respectively.

We now explicitly construct a mapping $\Phi_{m,n} : \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$. In order to define it, we fix a total ordering of the dicycles of digraphs over the node set $V_{m,n}$ which satisfies the orientation independence property, i.e. the ordering only depends on the node set of the dicycles (in particular, it ignores the orientation of its edges). Also, for a subgraph $H$ of a digraph $D$, we denote by $U(H)$ the collection of edges of $H$ ignoring orientations. Now, consider a perfect matching $M$ of $T_{m,n}$. Let $C_M$ be the first (according to the aforementioned fixed total ordering) dicycle of $D_{m,n}(M)$ and let

$$\Phi_{m,n}(M) = M \triangle U(C_M).$$
Since $C_M$ alternates between edges and non-edges of $M$, it follows that $\Phi_{m,n}(M) \in M_{m,n}$.

(For the matching $M$ of Figure 3(a) the matching $M' = \Phi_{m,n}(M)$ is illustrated in Figure 4(a)).

**Proposition 6** If a perfect matching $M \in M_{m,n}$ has exactly $h_i$ edges in row $i$ and exactly $v_j$ edges in column $j$, $0 \leq i < m$, $0 \leq j < n$, then so does $\Phi_{m,n}(M)$.

**Proof** Let $e_0, e_1, \ldots, e_\ell = e_0$ be the sequence of (undirected) edges of $T_{m,n}$ traversed by $C_M$. Since $D_{m,n}$ is bipartite, we have that $\ell$ must be even. Without loss of generality, we can assume $e_0 \in M$. The way in which $D_{m,n}$ was constructed immediately guarantees that the following two claims hold for all $0 \leq i < \ell$: (1) $e_i \in M$ if and only if $i$ is even, and (2) if $i$ is even, then $e_i$ and $e_{i+1}$ are either both horizontal or both vertical edges. By definition of $\Phi_{m,n}(),$ the desired conclusion follows. \(\square\)

Now, consider a perfect matching $M$ of $T_{m,n}$ and let $D_{m,n} = D_{m,n}(M)$. We say that a dicycle $C$ of $D_{m,n}$ is of:

- Type ee: If $|U(C) \cap A|$ and $|U(C) \cap B|$ are both even.
- Type eo: If $|U(C) \cap A|$ is even and $|U(C) \cap B|$ is odd.
- Type oe: If $|U(C) \cap A|$ is odd and $|U(C) \cap B|$ is even.
- Type oo: If $|U(C) \cap A|$ and $|U(C) \cap B|$ are both odd.

The following result establishes that the structure of $D_{m,n}()$ is such that it forbids the appearance of cycles of type ee.
Lemma 7 For every perfect matching $M$ of $T_{m,n}$, the digraph $D_{m,n} = D_{m,n}(M)$ does not contain cycles of type $ee$.

Proof For the sake of contradiction, assume $C$ is a cycle in $D_{m,n}$ of type $ee$. Note that $C$ must be the symmetric difference of boundaries of faces $F_1, \ldots, F_k$ of $T_{m,n}$. Since $C$ is connected, it follows that $C$ is contractible. By Lemma 4 the interior of the disk encircled by $C$ must contain an odd number of vertices of $T_{m,n}$, contradicting the fact that $C$ is the symmetric difference of two perfect matchings of $T_{m,n}$, and hence must encircle an even number of vertices of $T_{m,n}$. □

Next, we show that for each perfect matching $M$, only vertex disjoint cycles of equal type can arise in $D_{m,n}(M)$.

Lemma 8 Let $M$ be a perfect matching of $T_{m,n}$ and let $D_{m,n} = D_{m,n}(M)$. If $C$ and $C'$ are two distinct dicycles of $D_{m,n}$, then they are vertex disjoint.

Proof Direct consequence of the fact that every node of $D_{m,n}$ has out-degree 1 (see claim (??) of Lemma 5). □

We are now ready to establish that $\Phi_{m,n}(\cdot)$ is one-to-one. In fact, we can prove something stronger.

Corollary 9 The mapping $\Phi_{m,n}$ is an involution of $\mathcal{M}_{m,n}$.

Proof Consider $M \in \mathcal{M}_{m,n}$ and let $M' = \Phi_{m,n}(M)$. Note that the only difference between $D_{m,n}(M)$ and $D_{m,n}(M')$ is that $C_M$ and $C_{M'}$ have opposite orientations. Hence, because of the orientation independence property, $U(C_{M'}) = U(C_M)$, which implies that

$$\Phi_{m,n}(M') = M' \triangle U(C_{M'}) = \left( M \cap U(C_M) \right) \triangle U(C_{M'}) = M.$$

□

We still need to show that $\Phi_{m,n}(\cdot)$ maps matchings of type $EE$ onto matchings of type distinct than $EE$, and vice versa.

Proposition 10 The image of $\mathcal{M}_{m,n}^{(EE)}$ via $\Phi_{m,n}(\cdot)$ is $\mathcal{M}_{m,n}^{(EO)} \cup \mathcal{M}_{m,n}^{(OE)} \cup \mathcal{M}_{m,n}^{(OO)}$.

Proof Let $M$ be a perfect matching of $\mathcal{M}_{m,n}^{(EE)}$. Observe that,

$$|\Phi_{m,n}(M) \cap A| = |(M \triangle U(C_M)) \cap A| = |(M \cap A) \triangle (U(C_M) \cap A)|$$

$$= |M \cap A| + |U(C_M) \cap A| - 2|M \cap A \cap U(C_M)|.$$

Since $M$ is of type $EE$, we have that $|M \cap A|$ is even. Hence, the parity of $|\Phi_{m,n}(M) \cap A|$ equals the parity of $|U(C_M) \cap A|$. Similarly, the parity of $|\Phi_{m,n}(M) \cap B|$ equals the parity of $|U(C_M) \cap B|$. Because of Lemma 7 dicycle $C_M$ must be of type $ee$, $oe$, or $oo$, in which case, by the ongoing discussion, $\Phi_{m,n}(M)$ would be of type $EO$, $OE$, or $OO$, respectively. □

Theorem 2 is an immediate consequence of Proposition 6, Corollary 9 and Proposition 10.
4 Final comments

The proof argument for establishing Theorem 2 seems specially fit to handle the case where both $m$ and $n$ are even. If both were odd, the result is trivially true, since then $T_{m,n}$ does not have perfect matchings. It is somewhat puzzling that we could not find a simple adaptation of our proof argument for the case when exactly one of the integers $m$ or $n$ is even.

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