Collective excitation frequencies of Bosons in a parabolic potential with interparticle harmonic interactions

A. Minguzzi\textsuperscript{a}, M.P. Tosi\textsuperscript{a} and N.H. March\textsuperscript{b}

\textsuperscript{a}Istituto Nazionale di Fisica della Materia and Classe di Scienze, Scuola Normale Superiore, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
\textsuperscript{b}Oxford University, Oxford, England and Department of Physics, University of Antwerp (RUCA), Groenenborgerlaan, B-2020 Antwerpen, Belgium

Abstract

The fact that the ground-state first-order density matrix for Bosons in a parabolic potential with interparticle harmonic interactions is known in exact form is here exploited to study collective excitations in the weak-coupling regime. Oscillations about the ground-state density are treated analytically by a linearized equation of motion which includes a kinetic energy contribution. We show that the dipole mode has the frequency of the bare trap, in accord with the Kohn theorem, and derive explicit expressions for the frequencies of the higher-multipole modes in terms of a frequency renormalized by the interactions.

PACS numbers: 03.75.Fi, 05.30.Jp

1 Introduction

In a very recent investigation, Amoruso \textit{et al.} \cite{1} have given a comparative discussion of collective excitations in dilute Fermi and Bose gases subject to harmonic confinement at zero temperature. For this purpose these workers employed a linearized form of the equation of motion for the density profile $n(R, t)$ in the Hartree approximation \cite{2 - 4}, which is

$$\frac{\partial^2 n(R, t)}{\partial t^2} = \frac{1}{m} \nabla_R \cdot \left\{ n(R, t) \nabla_R \left[ V_{\text{ext}}(R) + \int d^3x v(R - x)n(x, t) \right] \right\}$$
$$+ \frac{1}{m} \sum_{\alpha, \beta} \nabla_\alpha \nabla_\beta \Pi_{\alpha, \beta}(R, t) \quad (1)$$
where \( V_{\text{ext}}(R) = m\omega_{ho}^2 R^2/2 \) is the confining potential, \( v(R - x) \) is the inter-particle interaction and \( \Pi_{\alpha\beta}(R, t) \) is the kinetic stress tensor.

Whereas for a Bose-condensed cloud Amoruso et al. [1] chose the interactions to be of contact form \( v(R - x) = g\delta(R - x) \) and adopted the Thomas-Fermi approximation (corresponding to \( \Pi_{\alpha\beta}(R, t) = 0 \)), the focus here is the N-Boson model discussed by earlier workers [5, 6]. In this model, all Boson properties are governed by harmonic forces, both confinement (which is usual) and, in contrast to the contact interactions in ref. [1],

\[
v(R - x) = \pm \frac{1}{2} \gamma^2 (R - x)^2 .
\]

Notice that the positive (negative) sign in Eq. (2) corresponds to attractions (repulsions).

One merit of this model is that the equilibrium density profile \( n_0(R) \) has the exact form

\[
n_0(R) = N(\kappa_N/\pi)^{3/2} \exp(-\kappa_N R^2)
\]

with

\[
\kappa_N = \frac{N\omega_{ho}\omega_N}{(N - 1)\omega_{ho} + \omega_N}
\]

and where

\[
\omega_N^2 = \omega_{ho}^2 \pm N \gamma^2
\]

\( N \) being the number of Bosons in the cloud. In the repulsive case one is assuming that the interactions are not so strong as to overcome the confinement.

In this Letter we show that within this model it is possible to determine in a self-consistent manner the equilibrium density profile and the collective excitation spectrum of the full Eq. (1) in the weak coupling limit. The advantage of having analytic results is that they can be used to study and test different ideas and numerical methods.

### 2 Equilibrium density profile at weak coupling

We have already pointed out that Eq. (1) involves the Hartree approximation, corresponding to the neglect of quantal fluctuations in treating the potential
energy terms. We introduce a parallel treatment of the kinetic stress tensor, by using a decoupling approximation on the one-body density matrix in terms of the product of two condensate wave functions. Namely, we write
\[ \langle \psi^\dagger(x',t)\psi(x'',t) \rangle \simeq \Phi^\ast(x',t)\Phi(x'',t) \] and correspondingly find
\[ \sum_{\alpha,\beta} \nabla_\alpha \nabla_\beta \Pi_{\alpha,\beta}(R,t) = -\frac{\hbar^2}{2m} \nabla_R \cdot \left\{ n(R,t)\nabla_R \left[ \frac{1}{\sqrt{n(R,t)}} \nabla^2 \sqrt{n(R,t)} \right] \right\} . \tag{6} \]

In the same approximation the Boson density is given by
\[ n(R,t) = |\Phi(R,t)|^2 , \] i.e. we neglect the depletion of the condensate due to the interactions. This approximate scheme has been shown to be equivalent to adopting the Gross-Pitaevskii equation for the condensate \[4\].

In this approximation the equilibrium density profile \( n_0(R) \) is the solution of the following equation,
\[ V_{ext}(R) \pm \frac{\gamma^2}{2} \int d^3x (R - x)^2 n_0(x) - \frac{\hbar^2}{2m\sqrt{n_0(R)}} \nabla^2 \sqrt{n_0(R)} = \mu , \tag{7} \]
\( \mu \) being the chemical potential. We choose as an Ansatz for the solution of Eq. (7) a gaussian profile normalized to the total number of particles:
\[ n_0(R) = N(\kappa/\pi)^{3/2} \exp(-\kappa R^2) . \tag{8} \]

Upon substitution of Eq. (8) in Eq. (7) and equating to zero the coefficients of the \( R^0 \) and \( R^2 \) terms we find
\[ \mu = \frac{3}{2} \left( \kappa \pm \frac{N\gamma^2}{\kappa} \right) \] \( \kappa^2 = \omega_{ho}^2 \pm N\gamma^2 \).

Of course, the width of the equilibrium profile is narrowed (broadened) by attractive (repulsive) interactions.

We can now compare the approximate result in Eqs. (8) and (10) with the exact result in Eqs. (3) - (5). It is seen at once that the two results agree in the weak coupling limit \( (\omega_{ho}^2 \gg N\gamma^2) \), where Eqs. (4) and (5) yield \( \kappa_N^2 \simeq \omega_{ho}^2 \pm N\gamma^2 \) (we are neglecting unity relative to \( N \)). This comparison emphasizes the limits of validity of our approach, which involves the neglect of quantum fluctuations and of the depletion of the condensate.
3 Collective excitations at weak coupling

We proceed to evaluate the small-amplitude oscillations of the Bose cloud around the approximate density profile given in Eq. (8). We set \( n(R, t) = n_0(R) + n_1(R, t) \) in the equation of motion (1) with the approximate form of the kinetic stress tensor in Eq. (6) and linearize it in the fluctuation \( n_1(R, t) \). We emphasize that this procedure is treating the equilibrium profile and the dynamic fluctuations of the profile in a consistent manner. The linearized equation of motion is

\[
\frac{\partial^2 n_1(R, t)}{\partial t^2} = \frac{1}{m} \nabla_R \cdot \left\{ n_0(R) \nabla_R \left[ \pm \frac{\gamma^2}{2} \int d^3x (R - x)^2 n_1(x, t) \right] \right\}
- \frac{\hbar^2}{4m^2} \nabla_R \cdot \left\{ n_0(R) \nabla_R \left[ \frac{1}{\sqrt{n_0(R)}} \nabla^2 \frac{n_1(R, t)}{\sqrt{n_0(R)}} \right] \right\}
- \frac{n_1(R, t)}{(n_0(R))^{3/2}} \nabla^2 \sqrt{n_0(R)} \right\}. \tag{11}
\]

Carrying out a Fourier transform with respect to the time variable and using Eq. (8), the equation obeyed by \( n_1(R, \omega) \) becomes

\[
\omega^2 n_1(R, \omega) = \mp 2\gamma^2 \kappa n_0(R) R \cdot d(\omega)
+ \frac{1}{4} \left\{ \nabla^2_R A(R, \omega) + \frac{2\kappa}{R^2} \frac{\partial}{\partial R} \left[ R^3 A(R, \omega) \right] \right\} \tag{12}
\]

where

\[
A(R, \omega) = \nabla^2_R n_1(R, \omega) + 2\kappa R \frac{\partial}{\partial R} n_1(R, \omega) + 6\kappa n_1(R, \omega) \tag{13}
\]

and we have defined the dipole moment of the excitation as

\[
d(\omega) = \int d^3x x n_1(x, \omega) . \tag{14}
\]

Furthermore, we have used rescaled units such that \( m = 1 \) and \( \hbar = 1 \).

We now focus first of all on the equation of motion for a dipolar density fluctuation, which is obtained from Eq. (12) by taking its dipole moment. Setting

\[
n_1(R, \omega) \propto R Y_{1m}(\theta, \phi) \exp(-\kappa R^2) \tag{15}
\]
in this case, it is easily seen that \( A(R, \omega) \propto n_1(R, \omega) \). The average of the Hartree term is

\[
\int d^3 R R_\alpha R_\beta n_0(R) = (N/2\kappa)\delta_{\alpha\beta}
\]  

while the average of the contribution from the kinetic stress tensor yields

\[
\frac{1}{4} \int d^3 R R_\alpha \left\{ \nabla^2_R A(R, \omega) + \frac{2\kappa}{R^2} \frac{\partial}{\partial R} \left[ R^3 A(R, \omega) \right] \right\} = \kappa^2 d_\alpha(\omega) .
\]  

The equation of motion of the dipole therefore takes the form

\[
\omega^2 d_\alpha(\omega) = (\kappa^2 \mp N\gamma^2) d_\alpha(\omega) .
\]  

From Eqs. (18) and (10) we see that the dipole mode oscillates at the bare trap frequency, i.e. \( \omega = \omega_{ho} \), in agreement with the Kohn theorem [7 - 9]. The theorem would be violated if we used the exact equilibrium profile (3) - (5) in the approximate equation of motion (11).

We next turn to consider the dispersion relation for all the other modes. What is shown below is that this dispersion behaviour for the higher-multipole modes is given by multiples of the renormalized harmonic oscillator frequency \( \kappa \).

We begin by taking as an Ansatz for the density fluctuations about the spherical equilibrium profile the form

\[
n_1(R, \omega) = \Phi_0(R)\Phi_{nlm}(R)
\]  

where \( \Phi_0(R) \propto \exp(-\kappa R^2/2) \) and \( \Phi_{nlm}(R) \propto R^l L_n^{l+1/2}(\kappa R^2) Y_{lm}(\theta, \phi) \exp(-\kappa R^2/2) \) are the ground-state and excited-state wave functions of a non-interacting gas trapped by an harmonic (3D spherical) oscillator of frequency \( \kappa \), \( L_n^{l+1/2}(\kappa R^2) \) being the Laguerre polynomials. Consistent with this Ansatz, the Hartree term vanishes for all higher multipoles, from orthogonality of \( \Phi_{nlm}(R) \) to the wave function \( R_\alpha \Phi_0(R) \) of the dipolar excitation. We are therefore left with the following equation of motion for the density fluctuation:

\[
\omega^2 n_1(R, \omega) = \frac{1}{4} \nabla_R \cdot \left\{ n_0(R) \nabla_R \left[ \frac{1}{\sqrt{n_0(R)}} \nabla^2 n_1(R, \omega) \sqrt{n_0(R)} \right] - \frac{n_1(R, \omega)}{(n_0(R))^{3/2}} \nabla^2 \sqrt{n_0(R)} \right\} .
\]  

Since the equilibrium density is already of the form appropriate to the ground state of an harmonic oscillator, Eq. (20) is identically satisfied by the Ansatz
made in Eq. (19). Therefore, the frequencies of the modes with \( n \geq 2 \) are \( \omega = n\kappa \), as anticipated.

4 Summary

In summary, we have shown that for the \( T = 0 \) limit of the N-Boson model in a parabolic potential and with harmonic interparticle interactions, the linearized equation of motion involves kinetic contributions of the kind exhibited in Eq. (11). The corresponding Gross-Pitaevskii equation (7) for the equilibrium density profile has a solution which agrees with the weak-coupling limit of the exact profile given in ref. [5]. Inputting this equilibrium profile in Eq. (11) enables the dispersion relation of the collective excitations to be obtained analytically. The frequency of the dipole mode is, in accord with the Kohn theorem [7 - 9], that of the bare trap. The frequencies of the higher-multipole modes are directly related to the trap frequency renormalized by the interactions.

Acknowledgements

One of us (NHM) wishes to acknowledge generous hospitality from the Scuola Normale Superiore di Pisa during the period in which his contribution to this study was completed.

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