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A quantitative vortex-fluid description of Nernst effect in Bi-based cuprate high-temperature superconductors

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Abstract
We present a completely analytical vortex-fluid model describing several vortex-fluid phases of hole-doped cuprate superconductors below and above \(T_c\). A key feature of the model is to synthesize several vortex damping mechanisms of scattering inside vortex core, core–core collisions and pinning. Accurate predictions for both magnetoresistance and Nernst effect are obtained, validated by measurements in six Bi-based cuprate samples over a wide range of temperature and magnetic fields, which strongly support the phase-fluctuation scenario in Bi-based cuprates. The model quantifies the unconventional vortex properties by a set of physical parameters, which adds quantitative contents to the discussions of pseudogap. A speculation is offered to use the vortex-fluid model as a first step towards achieving a comprehensive understanding of Nernst effect for all HTSC.

1. Introduction

It is well known that transport properties are powerful tools to address the pseudogap problem in high-temperature superconductors (HTSC) [1, 2]. A remarkable property is the so-called Nernst effect [3], in which a conductor exhibits a transverse detectable voltage when magnetic field is applied along \(z\) direction perpendicular to the direction of temperature gradient. In a conventional superconductor, large Nernst signal below \(T_c\) is a sign of vortex transport. But it was somewhat unexpected that strong Nernst signal were experimentally discovered above \(T_c\) in HTSC [4, 5], which has since stimulated extensive discussions on the nature of pseudogap state, e.g. superconducting fluctuations or competing order. Generally speaking, there are mainly three possible candidates, i.e., vortices [6, 7], fluctuating Cooper pairs [8], and quasiparticles (qp) generated from depair effect. A central issue in the study of hole-doped HTSC is thus: which component dominates the origin of Nernst signal in pseudogap state?

In underdoped cuprate, one typically observes positive qp Nernst signal above \(T_c\) and usual negative qp signal at very high temperature. A linear field dependence of positive Nernst signals is observed in a wide field regime at well above \(T_c\) in underdoped \(La_{2-x}Sr_xCuO_4\) (LSCO) and Eu-LSCO [5, 9], which is interpreted as a proof to support the qp scenario, which is thought [10–12] to originate from density wave order [13–15]. The theoretical prediction of peak signal variation with temperature reproduces qualitatively the observed behavior in experiments. However, the qp signal is not accurately quantified near or below \(T_c\), due to the fact that remarkable contributions of superconducting fluctuations to conductivity are neglected. Thus, the parametrization and validation of qp models call for a quantification of the superconducting fluctuations.

Gaussian fluctuations are conventional scenario due to fluctuating Cooper pairs above \(T_c\), which is analyzed by Ussishkin et al [8] in low fields of LSCO, and later, extended by a microscopic theory to describe temperature and field dependence for arbitrary temperatures and magnetic fields of NbSi, Eu-LSCO and PCCO [9, 16–18]. At low fields near \(T_c\), the signal of fluctuating Cooper pairs expressed by \(\alpha_{sy}/B\) (thermoelectric coefficient over field) is proportional to the square of coherence length, which is large in NbSi (conventional superconductors) and PCCO (electron-doped cuprate), and thus superconducting fluctuations are dominated by Gaussian fluctuations in these samples. On the other hand, \(\alpha_{sy}/B\) must be smaller in most hole-doped cuprates due to
their smaller coherence lengths. Indeed, experimentally observed signals are small in overdoped LSCO, as correctly predicted by the Gaussian-fluctuation theory. However, in underdoped LSCO and nearly optimally doped Bi-2212, Gaussian-fluctuation theory generally underestimates the signal (see section 3.1.3), which indicates an evidence that phase fluctuations are more important in these samples. We will show that the contribution of Gaussian fluctuations is minor in pseudogap state of hole-doped cuprates.

Due to two dimensionality (2D), low carrier density and short coherence length, phase fluctuations are believed to be especially strong in underdoped cuprates. According to the phase-fluctuation scenario [19], superconductivity is terminated at $T_c$ due to proliferation of thermal vortices. Nevertheless, local phase coherence is still present, driving a substantial vortex Nernst signal until upper critical temperature $T_c$, and field $H_{c2}$ [5, 6, 20] are reached. Experimentally, the widely observed tilted-hill profile of Nernst data and the continuity below and above $T_c$ are strong evidences for the existence of vortices. Theoretically, 2D XY model [7] and time-dependent Ginzburg–Landau (TDGL) equation [21–24] have been studied for transverse thermoelectric coefficient, but successful comparison is quantitatively achieved only in limited regimes, i.e., at at low field and high temperature in underdoped LSCO and at high fields and low temperature in overdoped LSCO. Furthermore, thermoelectric coefficient does not specify the whole transport; without the description of vortex damping, direct prediction of Nernst signal is not possible in these simulation studies.

In order to develop a whole description of the Nernst signal, Anderson proposed a phenomenological theory of vortex-tangles in pseudogap state [25]. However, an estimate indicates that his model presents an overestimation of two orders of magnitude in predicted resistivity compared to magnetoresistance data [26].

Thus, the validity of the vortex concept or phase-fluctuation scenario in pseudogap state remains to be established in a quantitative manner. In summary, it seems that there exists no quantitative theory capable of describing the Nernst signal for pseudogap state over the wide range of temperature and magnetic field. In our view, the situation originates from an oversimplification of the subject, i.e., insisting on finding a simple and single origin for describing the various samples and phase regimes. In fact, HTSC is a system composed of complicated physical environment where there are interplays between several interacting components, including several driven and damping mechanisms due to impurity and external fields. What is needed is a comprehensive theory synthesizing the three currents, in the light of quantitative measurements in a wide range of $T, B$ and with doping dependence. The theory should assert quantitatively at the end what are dominant components in different regimes, and what are the transitions between the multiple regimes in the $B$–$T$ phase diagram.

It is widely recognized that vortex fluid is the dominant component below $T_c$, thus a quantitatively accurate vortex-fluid model is the first step before achieving the comprehensive theory. In other words, without a proper quantification of vortex signal, the parametrization of HTSC physics is not reliable. We hereby attempt to build a sound vortex-fluid model which will serve as a basis for building a more complete theory, taking into account contributions of qp and amplitude fluctuations. In the present work, we focus on Bi-based cuprates in which vortex Nernst signal is the dominant contribution in most regimes, thus the vortex-fluid model can be quantitatively verified. Note that Bi-cuprates are extremely 2D superconductors in which phase fluctuations are strong, the qp signals are experimentally found to be small [5], and the Gaussian fluctuations are suppressed due to small coherence length. Experimentally, the onset temperature of pseudogap $T^*$ is much higher than $T_c$ in most doping of Bi-based cuprates [27, 28], indicating that amplitude fluctuations do not significantly affect the physics around $T_c$. Therefore, we expect that the vortex model alone can explain the Nernst signal over most regimes discussed below.

The main task in constructing a quantitative description of vortex Nernst signal is to model the complex damping process associated with vortex motions, which needs to go beyond Bardeen–Stephen model. Based on our recent work [26] which proposes a core–core collisions mechanism of vortex tangles, an unified vortex damping formula modeling several processes including scattering inside vortex core, core–core collisions and pinning is achieved, enabling accurate predictions of both magnetoresistance and Nernst signal over a wide range of $T$ and $B$. During the process of fitting a wide range of data, we have developed a systematic procedure, to determine several physical parameters which quantify the unconventional vortex properties in pseudogap state. The success yields a solid support to the underlying picture of vortex fluid in pseudogap state proposed by Anderson [6, 25, 29], as well as builds a basis for the comprehensive theory of transport integrating all three components in HTSC.

The paper is organized as follows. In section 2, an unified model is defined to quantify the damping viscosity, and to describe flux-flow resistivity and Nernst effect. In section 3, the model is first verified against magnetoresistance and Nernst data; for comparison, the validity of the Gaussian model is also assessed. Then, several physical parameters are discussed, displaying in particular the unconventional vortex properties in pseudogap state. Section 4 is devoted to the discussions on further applications and on the mechanism study of entropy.
In this work, HTSC refers to cuprate superconductors mostly and iron-based superconductors only in section 4. Some other acronyms are used to identify the cuprates, Bi-2201 for Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$, Bi-2212 for Bi$_2$Sr$_2$Ca$_2$Cu$_3$O$_{8+δ}$, UD, OP, and OV stand for underdoped, optimally doped, and overdoped, respectively.

2. Model description

2.1. Flux-flow resistivity

Consider a quasi-2D cuprate superconductor, and choose $T_c$ as the Berezinskii-Kosterlitz-Thouless (BKT) phase transition temperature [30–32]. When magnetic field is applied in HTSC, there are two types of vortices, namely magnetic and thermal vortices, whose density will be denoted as $n_B = B/\Phi_0$ and $n_T$. Since the thermal vortices appear in pairs, the density of vortices is $n_v = n_B + n_T/2$, and that of antivortices (with a reversed angular momentum) is $n_{-v} = n_T/2$.

In the experiment of flux-flow resistivity as schematically described in figure 1, a transport current density $j$ is applied $|j|$ and magnetic field $H\hat{z}$. A vortex current perpendicular to $j$ is driven by a Lorentz force $fi = j \times \Phi_0/c$, where $c$ is the speed of light. And in the steady state, $fi$ must be balanced by a damping force, e.g., for a vortex,

$$j \times \frac{\Phi_0}{c} = \eta v_0, \quad (1)$$

where $\eta$ is the damping viscosity per unit length and $v_0$ is the drifting velocity of the vortex. Vortex flow generates a phase slippage [33, 34], which induces transverse electric fields $E = n_v v_0 \phi_0/c$, where $n_v = n_B + n_T$ is the total vortex density. Here, both vortex and antivortex contribute to the total electric field since they transport in opposite directions. Thus, the flux-flow resistivity $\rho$ can be derived as [35]

$$\rho = \frac{n_v \phi_0^2}{\eta c^2}. \quad (2)$$

In equation (2), two key parameters are involved, namely the damping viscosity $\eta$ and the thermal vortex density $n_T$.

Above $T_c$, $n_T$ can be determined by vortex correlation length $\xi_+$ as $n_T \approx \xi_+^{-2}$, where $\xi_+$ represents the characteristic scale beyond which thermal vortices begin to unbind [36]. At $T_c < T < T_a$, superconducting fluctuations dominate, thus we speculate that the superconducting fluctuations make thermal vortices undergo a critical behavior. This behavior is studied by Kosterlitz [37], and follow his idea, we can define the following effective fields for modeling effects of thermal vortices:

$$B_T = \frac{\Phi_0}{\xi_+^2} = B_0 e^{-2b(T/T_K-1)^{-1/2}}, \quad (3)$$

where $B_0$ represents the high-temperature limit of $B_T$. In a special situation when $b \ll 1$, $B_T$ saturates quickly to $B_0$ at $T > T_a$. Thus, at the high-temperature limit, the thermal vortices is as dense as the critical state described by $H_{c2}^0$, where $H_{c2}^0$ is the upper critical field at $T = T_c$. Thus, we can assume $B_0 \approx H_{c2}^0$ and use this approximation to achieve the comparison between theoretical predictions and experimental data.

2.2. Vortex Nernst effect

In figure 2, a temperature gradient $-\nabla T$ is applied $|\nabla T|$ and magnetic field $H\hat{z}$. As vortex contains higher entropy (and heat) than surrounding fluid, thus a vortex flow is driven by $-\nabla T$. This results in a heat transport with an energy $S_\phi = T S_\phi$, where $S_\phi$ is the transport entropy per unit length of a vortex [38, 39]. In the steady state, thermal force should be balanced by damping force,

$$S_\phi(-\nabla T) = \eta v_\phi. \quad (4)$$
Thermal diffusion and the transport velocity is the same for vortices and anti-vortices as they have the same thermodynamic properties, as illustrated in figure 2. Thus, their contributions of phase slippage cancel with each other. Then, only the magnetic vortices contributes to Nernst signal $e_{N} = E / (-\nabla T)$, then

$$h = f(\epsilon B c S).$$

The origin of $S_f$ is still under debate in the literature \[39\]. Nevertheless, inspired by Anderson’s simple idea based on BKT scenario near $T_c$, it is possible to develop an order-disorder balance argument to obtain an expression for $S_f$ in a dense vortex fluid, as we do now. In the superconducting state, the ordered motions (e.g., the supercurrents) obviously dominate over disordered motions (e.g., the qp currents), while it is the opposite in the normal state. In a 2D dense vortex fluid in Bi-based cuprates, strong phase fluctuations induce frequent vortex creation and annihilation, which yields an energy transfer and then a balance between ordered motions (i.e. the kinetic energy of supercurrents, $E_k$) and disordered motions (i.e. the heat, $\frac{f(U f TS)}{f}$) in a vortex. Thus, $S_f = E_k / T$. Neglecting the Kosterlitz–Thouless screening [32], the kinetic energy can be obtained by an integration from vortex core size $\xi$ to $l = \frac{l_0}{\sqrt{2\pi}}$,

$$S_0 = \frac{G_1 n_s}{c_0 T} \ln \frac{H_{cl}^2}{B},$$

where $n_s$ is the local superfluid density, $G_1 = \pi h^2 / 8m_e = 4.76 \times 10^{-28} \text{ cm}^4 \text{ gs}^{-2}$, $c_0$ is the $c$-axis lattice constant [40], and the effective mass of the hole is assumed as electron mass $m_e$. Note that the above balance argument is an extension of the BKT scenario, but we neglect the ordered motions inside the vortex core owing to cheap vortices scenario [41].

2.3. Damping model
The key theoretical contribution of the present work is the unified vortex damping model proposed in this subsection. Generally speaking, damping viscosity $\eta$ involves several mechanisms, which are difficult to calculate from the first principle, and will then be considered with phenomenological arguments. Three damping mechanisms are considered, namely impurity (or defect) scattering of qp inside the vortex core, core–core collisions, and an interaction between vortex and pinning center. We assume that these three mechanisms can be decomposed so that $\eta$ can be expressed as their linear superpositions,

$$\eta = \eta_0 + \eta_c + \eta_{pin},$$

where $\eta_0$, $\eta_c$ and $\eta_{pin}$ are the contribution of impurity scattering, vortex–vortex collision and pinning, respectively.

2.3.1. Impurity scattering of qp
Since the number of qp inside vortex core (about 10 holes in OP Bi-2212) and the phonon density at low temperature are both small, the scattering between phonon and qp inside vortex core is negligible in a vortex fluid below $T_c$. In this case, damping inside vortex core is dominated by impurity scattering, which is approximately temperature-independent. Thus, $\eta_0$ should be a constant for a given material, which can be expressed as $\eta_0 = H_{cl}^2 \rho_0 / l_0^2 c^2$ where $\rho_0$ is a characteristic resistivity. The steep rises above melting field in experiments [5] indicates that vortices move very fast in a dilute vortex fluid, thus $\rho_0$ should be much bigger than the normal state resistivity considered in Bardeen–Stephen model [42].
2.3.2 Core–core collisions

The crucial difference between HTSC and conventional SC is that for HTSC, core–core collisions are more important due to much higher vortex density under small coherence length. While the independent vortex assumption may be valid in low fields, it is certainly inappropriate in high fields in cuprates. When the sample is tested in high fields (say $B = 10^4$ T), the distance between vortex cores $l_q = \sqrt{\Phi_0 / B}$ becomes of the order of 100 Å, which is much smaller than the penetration depth $\lambda \sim 1000$ Å. Thus, vortices strongly interact with each other, leading to a remarkable increase of damping. In our recent work [26], we propose a new mechanism of core–core collision in dense vortex fluids: two vortex cores merge together to a bigger core, leading to an increase of qp DOS. This process induces a momentum transfer from circular supercurrents to qp, and then a damping.

Since the random motion of vortex core originates from quantum fluctuations of holes [26], the mean-free-path of the core–core collision is proportional to reciprocal of core density. Thus, a dimensional argument yields a linear $n_c$ dependence of $\eta_c$,

$$\eta_c = \frac{\phi_0^2 n_c}{c^2 \rho_0},$$

where $\rho_0$ is a characteristic normal state resistivity. This new mechanism offers an explanation [26] of a longstanding puzzle, i.e. the unconventional vortex dissipation in cuprate superconductors, as well as the nature of the discrepancy of Anderson’s damping model.

Express $\eta_c$ also in terms of $\rho_0$, we obtain

$$\eta_c = \frac{\phi_0 B_0}{c^2 \rho_0},$$

where $B_0 = H_c2 / \rho_0$ is the effective fields to describe the damping strength of impurity scattering relative to core–core collisions.

2.3.3 Pinning effect

Pinning effect introduces modification to the transport of a vortex fluid [43]. According to Blatter et al [43], pinning effect can be expressed in an effective damping viscosity. Since pinning and core–core collisions both depend on inter-vortex distance, their joint effect can then be written in terms of a multiplicative factor $\Gamma$ such that $\eta_c + \eta_{pin} = \Gamma \eta_c$. The value of $\Gamma$ must satisfy two limiting conditions. First, at high temperature and strong fields limit where the pinning effect vanishes, $\Gamma$ is equal to 1. At low temperature and weak field limit, it is the thermal assisted flux flow (TAFF) described by Arrhenius law $\rho \approx \rho_0 \exp[-U_{pl}(k_B T)]$, where $U_{pl}$ is the plastic deformation energy [43, 44]. A simple choice is thus $\Gamma = \exp(U_{pl}/k_B T)$, which yields the following expression:

$$\eta_{pin} = \frac{\phi_0 B_0 + B_T}{c^2 \rho_0}[\exp(U_{pl}/k_B T) - 1].$$

The $B$ and $T$ dependence of $U_{pl}$ may vary with samples [43, 45], which has been parameterized by Geshkenbein et al in terms of an energy involving deformations of the vortex lines on a scale of inter-vortices distance [46]. We introduce a parameter $B_p$ as an effective field strength quantifying pinning. Then, the model of Geshkenbein et al can be expressed as

$$U_{pl} = k_B(T_v - T) \sqrt{B_p/(B + B_T)}.$$  \(\text{(11)}\)

Note that this kind of $B$ and $T$ dependence was indeed discovered experimentally in Bi-2212 [47, 48]. Therefore, in the sequel, we use equation (11) to describe the pinning effect in Bi-based cuprates. $B_p$ is an intrinsic material parameter, which can be probed by the melting field $H_m$ as $B_p = H_m / (4c_1^2 t^2)$ according to Lindemann criterion [43, 49], where $c_1$ is the Lindemann number and $t = T_v / T - 1$.

Finally, the total damping can be written as

$$\eta = \frac{\phi_0}{\rho_0 c^2} \{B_0 + (B + B_T)\exp[t(\sqrt{B_p/(B + B_T)})]\}. \ \text{(12)}$$

2.4 Final model

The integrated model equation (12) of $\eta$ enables us to obtain an analytic model of flux-flow resistivity:

$$\rho = \rho_0 \frac{B + B_T}{B_0 + (B + B_T) \exp[t(\sqrt{B_p/(B + B_T)})]}, \ \text{(13)}$$

where $B_T = B_0 \exp[-2b(T/T_v - 1)^{1/2}]$ is the effective field for modeling effects of thermal vortices.
On the other hand, combining equations (5) and (12), the prediction of $e_N$ is:

$$e_N = e_T - \frac{B \ln[H(\Delta/B)]}{B_0 + (B + B_T)\exp[\sqrt{B_T/(B + B_T)}]} ,$$

(14)

where

$$e_T = \frac{C_0 c_0 n_s}{\phi_0 c_0} T,$$

(15)

is a $T$-dependent characteristic signal.

3. Results

3.1. Validation and comparison

3.1.1. Magnetoresistance

In this section, in-plane magnetoresistance is used to validate our damping model. In OP ($\rho = 0.16$) sample of $\text{Bi}_2\text{Sr}_2\text{CaCu}_3\text{O}_{8+x}$, $\rho$ is measured in a wide range of temperature (20–120 K) and fields (0–17.5 T) [50]. As is shown in figure 3, $\rho$ first shows a slow increase regime at low temperature, which is a TAFF regime, and then the curves converge at a characteristic temperature. The transition between the two regimes varies with $H$.

Equation (13) is applied to describe the data, see solid lines in figure 3, showing very good agreement at all five magnetic field values. The agreement is specially good at $H = 0$, where all vortices are thermal. Thus, our model of $\rho$ is clearly validated.

Since the formula involves seven parameters, a systematic procedure must be developed to determine them. A two-step procedure is developed, with a rough estimate (RE) first and then a fine tuning (FT) process, involving a search for optimal parameters around the RE values, minimizing the errors between the set of predictions and data. At the end of the FT step, precise parameter values are obtained, and its uncertainties are assessed, see table 1.

Let us discuss the reliability of each parameter determination. First, the upper critical temperature $T_c$ is the onset temperature of short-range coherence and thermal vortices, when decreasing the temperature. In the
vortex–fluid model, it indicates the temperature, beyond which the vortex magnetoresistance collapses and vortex Nernst signal vanishes. Using the data of Usui et al, $T_v = 112$ K is obtained in RE and corrected to 104 K in FT, and the two values are very close, indicating a consistency. Besides, using the magnetoresistance data and Nernst signal measured by Ri et al [51], we find that $T_v$ is restricted in the range of [100, 110] K. In the next section, we will show that $T_v$ derived from the Nernst signal of Wang et al is also within the range (e.g. 107 ± 1.6 K). All these results indicate that $T_v$ is a well-defined, intrinsic parameter in HTSC samples.

$B_0$ is the parameter to determine pinning strength and melting field, and thus can be determined with data in TAff regime. The significant difference between RE and FT is attributed to measurement error due to low signal to noise ratio of the exponentially small signal, which amplifies the quantitative imperfection of Geshkenbein et al’s pinning model, i.e., equation (11). Note that more precise data in TAff regime can improve the quality of $B_0$, as we show below with the fitting of Nernst signal.

$B_0$ is the contribution from the qp scattering (inside vortex core) by defects. Comparing to pinning strength $B_p$ and $H_d$, $B_0$ is weak, which indicates a fast-vortices behavior of cuprates superconductors [52, 53], i.e., the steep rise of $\rho$ at low fields and near $T_c$ with weak damping characterized by $B_0$ [5]. For this sample, $B_0 \ll B_p$, thus the determination of $B_0$ is strongly influenced by fluctuations of $B_p$, which results in a substantial difference between the RE value (4 T) and FT value (0.4 ± 0.4 T).

$b$ is the unique parameter to determine the $T$ dependence of thermal vortex activation. And, the value $b = 0.42$ is obtained in the RE, and 0.56 ± 0.11 in the FT. Note that the classic procedure [34] yields an estimate of $b = 0.55$; thus, our determination is quite accurate.

In summary, the magnetoresistance measured by Usui [50] is well described by the vortex–fluid model, equation (13), with $T_v$ and $\rho_0$ well determined, as well as the Kosterlitz coefficient $b$. But, the quantitative deviation is still present for the description of pinning effect in the TAff regime below $T_v$, which results in an ill-determination of $B_p$. In addition, $B_0$ is also not very reliable, because it is too small in this sample.

### 3.1.2. Nernst effect

Wang et al have made systematic measurement of Nernst signal in cuprates superconductor, see figure 4 [5, 55, 58]. These observations reveal a characteristic tilted–hill profile of vortex–Nernst signal. At low $T$ and $H$, the exponentially small signal indicate a typical TAff behavior. On the other hand, a linear decay of Nernst signal in strong field regime is found, due to vortex–vortex interaction which dominates in $\eta$. Between these two regimes, a peak of maximum Nernst signal appears, and the peak vanishes at a characteristic temperature, called the upper critical temperature $T_v$, in the present work.

The Nernst signal encompasses a rich set of properties which enable us to determine a set of physical parameters. Comparing to the magnetoresistance $\rho$ in equation (13), the Nernst signal comprises extra complexity from $S_{\eta}$, which introduces two $T$-dependent parameters $c_\eta$ and $H_\eta$ in equation (14). However, our two-step procedure still applies, see appendix A.2 for a discussion of the sample OP Bi-2201.

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**Figure 4.** Comparison between predictions (solid lines) via equation (14) and experimental data (symbols) of six samples in Bi-2201 [5] and Bi-2212 [55, 56], using parameters determined by the FT procedure. (a)–(c) are UD, OP and OV samples in Bi-2201. (d)–(f) are UD, OP and OV samples in Bi-2212. In UD samples, the characteristic field scale increases at high temperature (e.g., above 20 K in UD Bi-2201), which is often identified as qp signal [37]; we thus only deal with the curves at low temperature in these samples.
are discussed in section 3.2. The italic number at the second line represents the roughly estimated value. $T_v, B_p, B_b, B_l$ and $b$ are determined from equation (14) with error bar estimated at root mean square error (RMSE) of $0.1 \mu \text{V K}^{-1}$ except for UD Bi-2201 (RMSE $=0.2 \mu \text{V K}^{-1}$). $B_l = H_{c2}^l$ where $H_{c2}^l$ is the upper critical field at $T_v$. The hole doping $p$ is estimated from the empirical formula $T_v(p) = T_{c,\text{max}}\left(1 - 82.6(p - 0.16)\right)$ with $T_{c,\text{max}}$ $=$ 28 K and 90 K in Bi-2201 and Bi-2212, respectively. $T_v$ are quoted from the literatures identical with Nernst signal.

| Sample      | $p$  | $T_v$ (K) | $T_p$ (K) | $B_p$ (T) | $B_b$ (T) | $B_l$ (T) | $b$   |
|-------------|------|-----------|-----------|-----------|-----------|-----------|------|
| OP Bi-2201  | 0.16 | 28        | 71.5      | 0         | 2.5       | 48.9      | 0.27 |
| UD Bi-2201  | 0.077| 12        | 64.9 ± 5.1| 0.5 ± 0.5 | 0.3 ± 0.1 | 63.5 ± 5.9| 0.672± 0.131|
| UD Bi-2201  | 0.16 | 28        | 65.4 ± 0.2| 0.1 ± 0.1 | 2.9 ± 0.1 | 50.4 ± 1.0 | 0.639 ± 0.032|
| UD Bi-2201  | 0.087| 90        | 107 ± 1.6 | 19.8 ± 1.9 | 9.1 ± 2 | 542 ± 4.9 | 0.186 ± 0.066|
| OV Bi-2212  | 0.16 | 50        | 105 ± 7.5 | 0.8 ± 0.8 | 9.1 ± 2 | 182 ± 17 | 0.490 ± 0.062|

The accurate agreements between the predictions and data are shown in figure 4, which yields a determination of the parameters shown in table 2. What is important is that the precise agreement extends to the whole range of $H$, $T$ and doping in both monolayer (Bi-2201) and bilayer (Bi-2212) samples. The minimum root mean square error (RMSE) between predictions and data is less than $0.1 \mu \text{V K}^{-1}$ in most figures, excepts in figure 4(a) where RMSE $=0.152 \mu \text{V K}^{-1}$. This is considered to be remarkable, supporting the validity of this vortex-fluid model and the reliability of the parameter values. In the following, we discuss $T_v$, $B_p$, $B_b$ and $b$, while $n_p$ and $H_{c2}$ are discussed in section 3.2.

Since $T_v$ is the temperature where vortex Nernst signal vanish, it can be determined from the Nernst signal from the vanishing point of the peak (i.e. maximum Nernst signal). For instance, in OP Bi-2212, we have determined $T_v = 107 \pm 1.6$ K from $c_n$, which is consistent with both the result determined from $\rho$ (see section 2.1) and the data from Bi et al. [51]. This convergence of the multiple estimations strongly support the validity of the concept of $T_v$. Our final determined $T_v$ for Bi-2212 are shown in figure 5, which increases slowly in the UD regime, but decreases quickly in the OV regime. Wang et al. defined $T_v$ as the onset temperature where the signal deviates from a background linear $T$ dependence at a given magnetic field [5], see black open hexagons in figure 5. Note that Wang et al.’s definition may involve contributions of $qp$ and contributions of amplitude fluctuations; in contrast, our definition is more precise for vortex since it involves only the onset effect of vortex signal.

In OP Bi-2201, $B_p$ is found to be more precise than what is determined from $\rho$, varying only 16% from RE and FT. The resulted uncertainty of $B_p$ is around ±10% for all samples, as shown in table 2. This reliability of $B_p$ is due to the clear existence of the TAFF regime in OV Bi-2212, OP and OV Bi-2201, as shown in figure 4. On the other hand, the data at low temperature in UD Bi-2201, UD and OP Bi-2212 are lacking, $B_p$ in these samples requires further investigation.

$B_b$ in Bi-2212 is found to be bigger than in Bi-2201, which is an interesting feature to be explored. It implies that the damping force of impurity scattering is stronger in Bi-2212, for which a qualitative explanation is proposed as follows. As explained in section 2.3.1, damping contributed by impurity scattering can be expressed by resistivity of impurity scattering of the holes inside vortex core, i.e., $\rho_s$. Based on Drude model of $\rho_s$, one
finds: $B_0 \propto H_{c2} n_0 \tau$, where $\tau$ is the relaxation time. Since $n_0$ and $H_{c2}$ are significantly bigger in Bi-2212 than in Bi-2201, the higher $B_0$ or damping in Bi-2212 is expected, although a quantitative understanding requires a more careful estimate of $\tau$ which is also sample-dependent.

The determination of $b$ in various samples under different doping is made for the first time. In Bi-2201 and Bi-2212, $b$ decreases monotonically from 0.672 to 0.310, and 0.490 to 0.181 respectively, with increasing doping. The uncertainties of $b$ in UD Bi-2212, UD and OP Bi-2201 are around $\pm 11\%$, for which the determination of $b$ is reliable. On the other hand, for OV Bi-2201, OP and OV Bi-2212, the uncertainties of $b$ are bigger (around 30%), due to the fact that the signals above $T_c$ are too small. However, since the result $b = 0.186 \pm 0.066$ in OP Bi-2212 is very close to the results determined from magnetoresistance: 0.183 in Bi-2212 ($T_c = 84.7$ K) [54], we consider that the estimates are reliable.

### 3.1.3. Comparison to Gaussian fluctuations

The identification of the dominant component of Nernst effect slightly above $T_c$ is important to clarify the physics in pseudogap phase. In this section, we carry out a comparison between Gaussian theory and the present vortex-fluid model, and conclude that the Gaussian fluctuations cannot be, but the vortex fluid is the dominant contribution to Nernst signal in Bi-based cuprates.

First, let us carry out an estimate of the thermoelectric coefficient $\alpha_{xy}^{\text{VF}} = \frac{\varepsilon n}{\rho} / \rho$ in 2D system using our vortex Nernst model equation (14). Taking the same $\eta$ as in magnetoresistance equation (13), we obtain

$$\alpha_{xy}^{\text{VF}} = \frac{C_{\text{c}} n_0}{\phi_0} \frac{B}{T} + \frac{B_T}{B} \ln \frac{H_{c2}}{B}.$$  \hspace{1cm} (16)

In addition, we make an assumption that $n_s$ decays linearly from $n_{s0}$ (at zero temperature) to zero (at $T_s$), i.e., $n_s = n_{s0}(1 - T/T_s)$.

The comparison is made with the data (red open circles) at low field ($B = 1$ T) and above $T_c = 85$ K of (nearly) OP Bi-2212 measured by Ri et al [51]; figure 6 shows a precise agreement between the prediction (red solid line) of the present vortex-fluid model and data at $\ln(T/T_s) \leq 0.2$. Note that every parameter is determined independently: $T_c = 106$ K is determined from the onset temperature of $B$ dependence of magnetoresistance, and $H_{c2} = 68.1$ T is estimated from $\phi_0/2\pi \xi^2$ with the coherence length determined as in [5], $n_{s0} = n_s/2$ is determined with data of $\alpha_{xy}$ at $T_c$, and $b = 0.5$ is determined from the $T$ dependence above $T_c$. Note that an extra confirmation of the vortex-fluid model in Bi-2212 was obtained in Ginzburg–Landau–Lawrence–Doniach study of diamagnetism, where a mean field of order parameters was introduced to revise the discrepancy between Gaussian theory and data [59].

On the other hand, in the limit of $B/H_{c2}(0) \ll \ln(T/T_s) \ll 1$, for 2D superconductors, Gaussian-fluctuation theory predicts $\alpha_{xy}$ as [16]

$$\alpha_{xy}^{\text{GF}} = \frac{2eB}{3h} \frac{1}{H_{c2}(0)} \frac{1}{\ln(T/T_s)}.$$  \hspace{1cm} (17)

Previously, equation (17) was claimed to agree with data in NbSi (conventional superconductor) [16, 17], OV LSCO (hole doped) [8], UD Eu-LSCO (hole doped) [9] and PCCO (electron doped) [18]. We have made a quantitative comparison between the fitting parameter $H_{c2}(0)$ from Gaussian-fluctuation theory and low
temperature $H_{c2}$ measured in those samples. It turns out that the ratio $H_{c2}(0)/H_{c2}$ is 1.8 for nearly OP ($p = 0.15$) PCCO, 1.9 for OV ($p = 0.17$) LSCO, 2.8 for NbSi, 0.25 for UD ($p = 0.12$) LSCO and 14 for UD ($p = 0.11$) Eu-LSCO. While the first three measured upper critical fields are slightly below the predicted one, the last two (UD LSCO and Eu-U-LSCO) shows a considerable departure.

This result can be interpreted as following. Since coherence length in conventional superconductors (e.g., NbSi) and electron-doped cuprate (e.g., PCCO) are large, signal above $T_c$ in these samples is dominated by Gaussian fluctuations. However, signals due to Gaussian fluctuations must be smaller in hole-doped cuprates due to their smaller coherence lengths, and thus the contribution of phase fluctuations may not be neglected. Indeed, in OV LSCO, experimental signals are smaller, as consistent with Gaussian-fluctuation theory [8]. On the other hand, in UD samples, strong phase fluctuations result in a larger $\alpha_{3d}/B$. So, Gaussian-fluctuation theory generally underestimates the signal in UD samples (e.g., UD LSCO) [8]. Surprisingly, Gaussian-fluctuation theory overestimates the signal in Eu-LSCO, indicating that the theory predicting only the dependence on coherence length is too simplified, with some physics left over in strong stripe order [15]. In summary, in hole-doped cuprates, Gaussian-fluctuation theory is incapable to describe the UD samples, despite its successful application in OV samples.

Furthermore, we consider the application of equation (17) in Bi-cuprates, e.g. taking Bi-2212 specifically. In figure 6, the black dash line is predicted by Gaussian model with $H_{c2}(0) = 10.5$ T, which only describe the data in 0.03 $\ll \ln(T/T_c) \ll 0.085$, but significantly higher outside that range. Unfortunately, the fitting value ($H_{c2}(0) = 10.5$ T) is far from being reasonable, since it is much smaller than the estimate from $H_{c2} \approx \phi_0/2\pi\xi^2 = 68.1$ T with $\xi$ measured with STM [60]. If the physical parameter $H_{c2}(0) = 68.1$ T is used, the Gaussian theory’s prediction (see, the blue dash dot line in figure 6) is much lower than the data. Note that previously, the Gaussian-fluctuation theory of diamagnetism for (nearly) OP Bi-2212 also yields unreasonable fitting parameter (e.g., $H_{c2} = 330$ T) and irrational $B$ dependence [61, 62]. We thus conclude that Gaussian fluctuation alone is unable to describe the physics in pseudogap state, at lest for Bi-based cuprates, for which the phase fluctuations are dominant.

3.2. Unconventional properties of vortex fluid

Unconventional properties of vortex fluid in pseudogap state raise an important issue in HTSC study. Our vortex-fluid model of Nernst effect enables us to determine physical parameters, including local superfluid density, upper critical field and thermal vortex density. These parameter values deliver important information about the physics of pseudogap state, e.g., whether Cooper pairs and vortex survive above $T_c$ (i.e., finite local superfluid density), and how large is the core size (i.e., finite upper critical field), and how thermal vortices density varies with temperature and doping. The properties illustrated through the present phenomenological consideration provide important guide for the further study by microscopic theory to account for quantitative effect of strong phase fluctuations and short range coherence in pseudogap state.

3.2.1. Local superfluid density

Anderson pointed out that, in a vortex fluid, ‘$n_s$ is not the conventional macroscopic quantity giving the penetration depth, which is its average over phase fluctuations, but is defined microscopically...’ [23]. Indeed, in the BKT scenario of 2D superconductor, thermal vortices emerge above $T_c$ and results in a dramatic increase of the penetration depth [63] and the macroscopic quantity of $n_s = m_v c^2/4\pi \hbar^2 v^2$ (see, open hexagon in figure 7) quickly decreases to zero. However, the local superfluid density which flows within interstitial puddles between vortex cores remains finite, which can induce a vortex Nernst effect. An advantage of the present vortex-fluid model is to allow one to determine the local superfluid density from Nernst signal, and then discuss its unconventional temperature dependence in pseudogap state.

Equation (15) yields an expression for the local superfluid density $n_s$:

$$n_s = \frac{\phi_0 c_0}{G_4 j_p} T_0 T_c,$$

where $c_0 = 24.6$ Å for Bi-2201, 30.9 Å for Bi-2212. In order to extract $n_s$, $\rho_0$ should be estimated. Since $\rho$ for the samples corresponding to the present Nernst data is not available in literature, $\rho_0$ needs to be estimated from $\rho$ of other (similar) samples with the same $T_c$. According to equation (13), $\rho_0$ is also the resistivity of holes in normal fluid. Based on the Drude model of normal metal, the conductivity is proportional to hole concentration $p$, thus

$$\rho_0 \propto 1/p.$$

Hence, $\rho_0$ can be determined for each sample, once a value of $\rho_0$ at some doping is obtained. The results are shown in table 3, using the data $\rho_0 \approx 0.11$ m$\Omega$ cm in OP ($p = 0.16$) Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ [50] and $\rho_0 \approx 0.13$ m$\Omega$ cm in the OV ($p = 0.24$) Bi-2201 [64].
In this work, we find that local superfluid density indeed exhibits interesting behaviors. As shown in figure 7, \( n_s \) of OP Bi-2201 and OP Bi-2212 determined from equation (18) nearly collapse on a straight line \( n_s = n_0 (1 - T/T_c) \), where \( n_0 \) is a fitting parameter. Two distinct features differ from the temperature dependence of macroscopic superfluid density (see, open hexagon in figure 7) are noteworthy: a finite \( n_s \) above \( T_c \) and a universal linear temperature dependence below \( T_c \). The finite \( n_s \) above \( T_c \) and vanishing of \( n_s \) at \( T_c \), but not \( T_f \), is very indicative of the nature of pseudogap state: full of thermal vortices. This indicates that the thermal effects above \( T_c \) and below \( T_f \) are restricted within the vortex core, perhaps in the form of qp holes, while a local superfluid around the (thermal) vortex core still survives. Note that a linear \( T \) dependence of \( n_s \) at low temperature has been widely observed in HTSC [65] and can be described by the standard BCS theory of d-wave superconductor; but the presently observed linear behavior of local superfluid density above \( T_c \) is exotic, and requires further theoretical explanation. For more discussion, see section 3.2.4.

Strictly speaking, the concentration of oxygen deficiency in the CuO2 plane are different in each sample so there exists a certain deviation of this estimated \( \rho_{0c} \) and hence of \( n_s \) [64]. However, the relative values of \( n_s \) with respect to temperature are considered to be reliable.

### Table 3. Characteristic resistivity \( \rho_{0c} \) estimated with equation (19).

| Sample      | UD Bi-2201 | OP Bi-2201 | OV Bi-2201 | UD Bi-2212 | OP Bi-2212 | OV Bi-2212 |
|-------------|------------|------------|------------|------------|------------|------------|
| \( \rho_{0c} \) (mΩ cm) | 0.41 | 0.20 | 0.15 | 0.19 | 0.11 | 0.076 |

3.2.2. Upper critical field

Upper critical field \( H_{c2} \) represents a crossover from a vortex fluid to the normal state due to overlap of magnetic vortex cores, and it is also a probe of the coherence length \( \xi \) because \( H_{c2} = \phi_0 / (2\pi \xi^2) \). Since core–core collisions are enhanced with increasing magnetic field (see, equation (13)) in cuprate HTSC, magnetoresistance presents a ‘knee’ behavior, instead of following a Bardeen–Stephen law [5, 42, 53]. Therefore, one can no longer make an exact definition for \( H_{c2} \) from magnetoresistance as in conventional superconductor. Indeed, empirically determined ‘\( H_{c2} \)’ from the ‘knee’ feature (see crosses in figure 8(a)) displays a sharply different behavior from that described by the conventional Werthamer–Helfand–Hohenberg (WHH) theory for BCS superconductors [66]. Recently, distinct progress has been achieved using Nernst signal [5, 55] and diamagnetism [67, 68], in which \( H_{c2} \) is defined as the field where \( e_N \) and \( M \) versus \( B \) extrapolates to zero. Our model enables one to make this determination in a wider range of \( T \) and \( B \), since it may be considered as a more sophisticated version of the simple method by Wang et al, especially when the measured profile is imperfect.

The \( T \) dependence of \( H_{c2} \) (see, section A.2) as shown in figure 8 displays a slow variation near \( T_c \), consistent with the results of linear extrapolation of Nernst and diamagnetism versus \( B \) [5, 68]. For instance, at \( T = 5–40 \) K in OP Bi-2201 (or \( T = 70–100 \) K in OP Bi-2212), \( H_{c2} \) is presently in \([43.3, 54.2]\) T (or \([40.3, 81.8]\) T), as compared to \([42, 52]\) T (or \([43, 90]\) T) according to extrapolations of Nernst or diamagnetism [5, 68]. This behavior of finite \( H_{c2} \) above \( T_c \) is different from the canonical BCS behavior described by WHH theory, in which \( H_{c2} = 0 \) at \( T_c \) due to gap-closing. If \( H_{c2} = 0 \) above \( T_c \) in Bi-based cuprates, the strong and positive \( e_N \) above \( T_c \) in OP Bi-2212...
must be interpreted by Gaussian fluctuations which is shown above (in section 3.1.3) to be unsuccessful, and the linear decaying of $\epsilon_n$ in high-field in OP Bi-2201 cannot be explained by the logarithmic decay of Gaussian fluctuations. Furthermore, the finite $H_{c2} = \epsilon_n/(2\pi\xi^2)$ also implies finite core size or coherence length $\xi$, which has a weak $T$ dependence near $T_c$. This is highly consistent with the Pippard coherence length $\xi_p \propto 1/\Delta_0$ determined from gap amplitude $\Delta_0$ which is also observed to be finite and varies slowly above $T_c$ in ARPES measurement [69]. In conclusion, the finite $H_{c2}$ above $T_c$ is physically sound in Bi-based cuprates, indicating that vortices with finite core size survive above $T_c$.

On the other hand, in figure 8(b), anomalous increase of $H_{c2}$ with increasing temperature occurs above $T_c$ in OV Bi-2201, which might be the evidence of the contribution of amplitude fluctuations. The signal above $T_c$ is one order of magnitude lower than the maximum signal at low temperature, thus can be influenced by weak amplitude fluctuations tail [5], leading to an overestimate of $H_{c2}$. In addition, for extremely UD samples, anomalous increase of $H_{c2}$ is also present at $T$ near and above $T_c$. A possible interpretation is exotic qp excitation due to density-wave order in this regime [12]. While the regular behaviors of $H_{c2}$ in most regimes ensure the validity of the vortex-fluid model, the irregular behaviors of $H_{c2}$ deduced with vortex fluid model provide a sensitive probe for additional contributions of qp or amplitude fluctuations. The present study thus serves a basis for developing a more comprehensive theory in the future.

3.2.3. Thermal vortex density
Equation (3) is able to predict the temperature dependence of thermal vortex density $n_T$, as shown in figure 9(a). Note that $n_T$ in UD sample is much larger than in OP and OV samples, in both Bi-2201 and Bi-2212. We argue that the energy cost to excite a thermal vortex-antivortex pair is proportional to $n_c$. Thus, thermal vortex is easier to be excited in UD sample. Furthermore, it is interesting to observe a similar temperature dependence for each sample. In figure 9(b), normalized densities $n_T/n_T(T_c) = \exp[-2b(T_c/T_c - 1)^{-1/2}(t^{-1/2} - 1)]$ collapse into three groups, where $n_T(T_c)$ is the maximum density at $T_c$ and $t' = (T - T_c)/(T_c - T_c)$. The difference is controlled by one material parameters $S_c = b(T_c/T_c - 1)^{-1/2}$ which may be treated as a normalized coherence strength to resist thermal vortex activation. For Bi-2212, $n_T/n_T(T_c)$ is nearly doping independent with $S_c$ ranging from 0.42 to 0.47, indicating a universal normalized resistance of thermal vortex activation for different doping of Bi-2212. For Bi-2201, however, two behaviors of $n_T/n_T(T_c)$ appear, one in UD and OV sample with $S_c \approx 0.31$, indicating that the normalized coherence strength in bilayer system is stronger than in monolayer bilayer system. But, an exception is found for OP Bi-2201 that $S_c$ reaches a maximum (i.e., 0.55) with a strong decline of $n_T/n_T(T_c)$; this last observation requires further theoretical explanation.

3.2.4. Vanishing of short-range phase coherence
In the BKT scenario, long range phase coherence is destroyed by the proliferation of thermal vortices. Here, the quantification of the Nernst signal allows us to examine the dynamical properties of thermal vortices, and to propose a mechanism for the vanishing of short-range phase coherence, namely the lifetime of thermal vortices versus the cycle period of Cooper pairs around the core, as we explain now.

For a vortex, short-range phase coherence can be quantified by the cycle period of Cooper pairs around the core, i.e., $\tau_c \sim 4\pi m_e \xi^2 / \hbar$. This coherence may be destroyed by the activation and annihilation of thermal vortices, whose characteristic time defines the average lifetime of thermal vortices (or coherence time) during which the local wavefunction becomes totally dephased. This lifetime can be estimated by $\tau_c = s_c / v_R$, where $s_c$ is

![Figure 8](image-url)
In the present work, we have developed a systematic procedure to analyzing the effect of vortex dynamics.  

4.1. Application to other HTSC  

Now, we argue that vanishing of short-range phase coherence requires that \( s_v = \pi \xi_v \), the perimeter of a circle with diameter \( \xi_v \), and \( v_R \) is the random velocity of the vortex. The latter can be obtained by an energy estimate due to quantum fluctuations of holes inside vortex core, \( v_R = \hbar/(m_v \xi_v) \). It follows that  
\[
\tau_v = \frac{\pi m_v \xi_v}{\hbar}, \quad \text{and the ratio of the two time scales is}
\]
\[
\frac{\tau_v}{\tau_c} = \frac{\pi H_c}{8BT}.
\]  

Now, we argue that vanishing of short-range phase coherence requires that \( \tau_v / \tau_c \) equal to 1 at the upper critical temperature \( T_c \). Using the parameters near \( T_c \), we can obtain \( \tau_v / \tau_c = 1.13 \pm 0.07, 0.91 \pm 0.10 \) and \( 0.98 \pm 0.30 \) for three samples, OP Bi-2201, OP Bi-2212 and OV Bi-2212, respectively. For other samples, \( H_{c2} \) at \( T_c \) is either unavailable or is unreliable due to too big errors. Thus, we consider that available data confirm our proposal. Physically, the near unity is originated from the key fact that motions of thermal vortices and Cooper pairs near the core both depend on the same core size \( \xi \).

Furthermore, we speculate that this understanding may also help in interpreting the temperature dependence of \( n_s \). This is because that the lifetime for a vortex (magnetic or thermal) can be defined as \( \tau_v \). When temperature increases, both \( \tau_v \) and \( n_s \) decays. If we assumed that the decay of \( n_s \) is induced by the dephasing effect of vortex activation and annihilation, a simple relation, i.e., \( n_s/\pi \tau_v(0) = 1 - \tau_v / \tau_c \), can be obtained, where \( n_s(0) \) is the superfluid density at zero temperature. At zero temperature, \( \tau_v \) is infinite thus \( n_s = n_s(0) \). While at \( T_c \), \( n_s \) vanishes. This interesting idea will be explored elsewhere in the future.

4. Discussion and conclusion

4.1. Application to other HTSC

In the present work, we have developed a systematic procedure to analyzing the effect of vortex dynamics embedded in magnetoresistance \( \rho \) and Nernst signal \( e_{N,0} \), and this procedure is applicable to other extremely type-II layered superconductor. Once \( \rho \) and \( e_{N,0} \) data of a new sample are available, the RE can be conducted first using asymptotic analysis at large and small limits of \( T \) and \( B \). Note that for electron-doped cuprates, some UD hole-doped cuprates [9], and even iron-based superconductor [70], the contributions of qf should be subtracted with a linear or nearly linear model [5, 57]. The procedure of RE can be summarized as following. Step one, intuitively estimate the upper critical temperature \( T_{c1} \) and characteristic resistivity \( \rho_0 \) from the convergent (or saturated) regime of \( \rho \). Step two, carry out a magnitude analysis of the maximum \( e_{N,0} \) signal with equation (15), and predict the characteristic superfluid density \( n_{s0} \) if the carrier effective mass is available (e.g., for Fe1+\( y \) Te0.6 Se0.4, \( m_{v} = 29 m_e \) [70]). The comparison between \( n_{s0} \) and carrier density in the sample allows one to judge the validity of the vortex-fluid scenario. Step three, determine \( H_{c2} \) (thus \( B_{0} \)) from the linear extrapolation of \( e_{N,0} \) in strong fields according to equation (A7). Sometimes, linear regime of \( e_{N,0} \) in strong fields is not measured (e.g., for Fe1+\( y \) Te0.6 Se0.4 [70]), then use the values from other measurements of similar sample [71] such that the determination of \( n_s \) in the final step can be achieved. Step four, use the data at low \( T \) of \( \rho \) (or \( e_{N,0} \)) (with significant pinning effect) to determine \( B_{0} \) with equation (A2) (or equation (A9)). Step five, determine \( B_{0} \) from the linear field dependence of \( \rho \) or \( e_{N,0} \) in low fields at \( T_{c1} \). Step six, determine the Kosterlitz coefficient \( b \) from the sharp increase near \( T_{c} \) of \( \rho \) in zero field with equation (3). Finally, estimate the \( T \)-dependent local superfluid density \( n_s \).  

Figure 9. Thermal vortex density \( n_v \) versus Tin Bi-2201 and Bi-2212. (a) \( n_v = (R/\phi_0) \exp[-2b(T/T_c - 1)^{1/2}] \) are calculated with parameters in table 2, and plotted in various lines and colors. (b) Normalized density \( n_v/n_{v0}(T_c) = \exp[-2b(T/T_c - 1)^{1/2}(T_c/T - 1)] \) of each sample in (a), where \( n_{v0}(T_c) \) is the maximum density at \( T_c \) and \( t' = (T - T_c)/(T_c - T_0) \).
by the calculation of the peak of $e_N$ versus $B$ with equations (A12) and (18). By the way, although it is better to obtain measurements of magnetoresistance and Nernst signal simultaneously and in wide $T$ and $B$ range, there are other ways to complement the data from the literatures using our vortex-fluid model. Details of the above procedure are presented in the appendix.

4.2. Validity of entropy model

It is important to discuss the validity of the entropy model, i.e., equation (6), which is similar to Anderson’s original proposal [29], but is more general. Although it appears that equation (6) is an extension of the BKT scenario, but we argue that in a dense vortex fluid of 2D superconductor, the fluctuations are strong, so that the probability of proliferation of (magnetic or thermal) vortices is substantial, and the balance equation (6) is approximately valid. Quantitatively, the current description of Bi-2212 [31], equation (6) (with a relative mean square root error equals 0.37) is more precise than the Maki’s model deduced form TDGL near $H_{c2}$ (with a relative mean square root error equals 0.58) [72]. Thus, equation (6) is a reasonable approximation (which is in fact the most precise one for Bi-2212 so far) of transport entropy in Bi-cuprates, despite its simplicity.

Strictly speaking, equation (6) is not valid at low temperature where ordered motions dominate so that heat in vortex is much smaller than the kinetic energy of supercurrents. In this case, an energy transformation mechanism between ordered and disordered motions may take place, and the ratio $TS_E/EB$ needs to be multiplied by a factor which increases from zero (at zero temperature) to a finite value at high temperature. This extension is beyond the scope of the present work, and will be considered in the future.

4.3. Effects of fluctuating vortices

In previous studies, vortex motions are believed to originate only from phase fluctuations while amplitude is frozen, implying that amplitude fluctuations play no role in vortex fluids [7]. Here, we argue, based on an estimate below of the lifetime $\tau_{CL}$ of fluctuating Cooper pairs [73], that magnetic field can also drive fluctuating vortex pairs. In fact, a dimensional estimate of $\tau_{CL}$ yields that $\tau_{CL} \sim h/k_B(T - T_c)$, while the cycle period of the fluctuating pair near the core is $\tau_f \sim 4\pi m_c \xi_c^2/h \ln(T/T_c)$. In overdoped samples (due to small $\xi$), one finds that $\tau_f \sim \tau_{CL}$, e.g., $\tau_f/\tau_{CL} = 2.2$ in OV Bi-2201 ($T_c = 22$ K), which implies that a group of fluctuating Cooper pairs can form a fluctuating vortex in about a half cycle period. The total heat and charge transport, thus the Nernst signal, must be a superposition of the contribution of conventional vortices and fluctuating vortices. Therefore, we can transform the Gaussian fluctuations model $\epsilon_N^{GF} = \rho \alpha_{xy}^{GF}$ to a fluctuating vortex-fluid model $\epsilon_N^{NV} = \rho S_0^{NV}/\phi_0$, where $S_0^{NV}$ is the transport entropy of fluctuating vortices,

$$S_0^{NV} = \alpha_{xy}^{GF} \phi_0$$  (21)

Following equation (21), we introduce an integrated entropy model of conventional vortices and fluctuating vortices $S_0 = S_0^{NV} + S_0^{PV}$, where $S_0^{NV}$ is the contribution of conventional vortices. In UD samples where phase fluctuations dominate, the fluid is composed of conventional vortices; while in extremely OV samples, the fluctuating vortices play an important role. Thus, this integrated model may sets a basis to describe the intermediate regime of phase and amplitude fluctuations.

4.4. Conclusion

In order to describe the Nernst effect in pseudogap state, many insightful ideas have been proposed, including vortex tangles [25] and $q_p$ associated with a density wave order [10]. However, the signal’s dependence on $T$ and $B$ is complicated, due to multiple components and mechanisms in play, and hence has not yet been addressed quantitatively. The present phenomenological model takes upon this challenge, and provides a completely analytical description for the dominant vortex component, which covers several phase regimes and involves several relevant processes such as thermal and magnetic vortex activation, impurity scattering, core–core collisions and pinning.

The main conclusion of the work is that both magnetoresistance and Nernst signal in Bi-based samples are accurately described by vortex-fluid model at low and high fields and below and above $T_c$, which quantitatively confirms the indispensability of vortex fluid and phase fluctuations in pseudogap state. The model quantifies the unconventional vortex properties by local superfluid density, upper critical field and thermal vortices density, which adds quantitative contents to discuss the nature of pseudogap. The fully analytic nature of the model enables the signal separation of vortex and non-vortex components, thus represents the first step towards achieving a comprehensive theory of Nernst effect for all HTSC, by integrating multiple components. Furthermore, the model provides an opportunity to comparing the physical properties of Fe-based to cuprates HTSC according to the temperature, doping and layer dependence of parameters.
Acknowledgments

We thank L Yin for many constructive discussion during the early stage of this work. We thank Y Wang for a very helpful discussion. This work is partially supported by National Nature Science (China) Fund 11452002.

Appendix. Parameters determination

A key result of the present work is to demonstrate that it is possible to derive reliably the values of the model parameters, as we show below.

The determination of model parameters is conducted in two steps. The first step is a rough estimation (RE), using asymptotic analysis at high and low limits of \( T \) and \( B \), where naturally one or two leading mechanisms are dominant. In this case, only one or two parameters appear, so their approximate values can be obtained. This determination is approximate because empirical data do not go very far in the asymptotic limits; more often the effects of variation of the multiple parameters are entangled together, so that a global optimization is necessary to find more accurate description of data. This is then accomplished by the second step of fine tuning, which define the best set of parameter values by minimizing the error between the predictions and empirical data. In other words, the parameters are tuned around the roughly estimated values to achieve a better fitting of a set of data (multi-curves).

The fine tuning (FT) is realized by a least square fitting process. For example, in the determination with Nernst signal, we divide the entire set of parameters into groups so that the FT can be conducted one group after another, with the former results substituted into the later calculation. This iterative process runs until a local minimum of the error is reached. If there are more than one local minimums, a comparison between them decides the local minimum, and then the FT ends with a set of precisely determined parameters. Sometimes, the physically optimal point is not the global minimum but the local minimum. Therefore, to determine the local minimum, we need to define the range of parameters to be explored, to be specified in the main text.

The reliability of the parameters depends on the precision of the comparison between predictions and the data, and the uncertainties of the parameters. The later can be expressed by error bar. Once the optimal set of parameters are obtained, the error bar is determined once the upper boundary of the RMSE between the predictions and empirical data reaches a threshold equals to the measurement uncertainty.

A.1. Determine parameters with flux-flow resistivity

In the resistivity model equation (13), there are seven parameters, namely \( T_c, \rho_0, T_v, B_0, B_p, B_l \) and \( \xi \), to be determined by experimental data. \( T_c \) can be easily estimated from the transition temperature of zero resistivity. In this section, we explain how other parameters are determined, using magnetoresistance data of optimally doped \((p = 0.16)\) Bi2S2Sr2CaCu2O8+\( \delta \) [50].

A.1.1. Rough estimate. First, let us discuss the RE. In section 2.1, we assume that the high \( T \) limit of thermal vortex density is as dense as magnetic vortices at \( H_\text{c2} \), thus \( B_l \approx H_\text{c2} \). \( H_\text{c2} \) can be estimated as \( \phi_0/(2\pi\xi^2) \). Thus, \( B_l \approx \phi_0/(2\pi\xi^2) = 68.1\text{ T} \) as \( \xi = 22\text{ Å} \) is measured by STM [60].

Near \( T_v \), core–core collisions are dominant in \( \eta \) since the density of thermal vortices is dense, thus \( B_0 \) can be neglected. Then, \( \rho \) can be expressed as

\[
\rho \approx \rho_0 \exp \left[ -\frac{T_v}{T} - 1 \right] \sqrt{\frac{B_p}{B + B_T}}
\]

which predicts that \( \rho \) increases with \( B \) for \( T < T_v \) and converge to \( \rho_0 \) for \( T \) approaching \( T_v \). We use the convergent regime of the experimental curves (see figure 3) to carry out a RE of \( T_v \) and \( \rho_0 \). The convergent regime corresponds to the range where the curves at the highest (17.5 T) and lowest (0) become close (or overlapped). In practice, we consider the overlap starts where the difference of the two curves equals the scatters of each curve, which is about 5 \( \mu \Omega \text{ cm} \). Applying this criteria, we obtain \( T_v \in [104, 120] \text{ K} \) and \( \rho_0 \in [0.098, 0.133] \text{ m}\Omega \text{ cm} \). The middle values, e.g., \( T_v = 112\text{ K} \) and \( \rho_0 = 0.116\text{ m}\Omega \text{ cm} \), are taken to be the roughly estimated values.

\( B_p \) is determined from the low \( T \) limit (i.e., the TAF regime) of the resistivity profiles. At \( T \ll T_v \), pinning effect dominates and \( B_p \) is neglected. We define \( y = \ln(\rho_0/\rho), x = T_v/T - 1 \), then a linear formula is obtained,

\[
y = a_0 x + b_0
\]

where \( a_0 \approx 1.85 \) so that \( B_p \approx 30.8\text{ T} \), as shown in figure 10.
$B_0$ is supposed to dominate in low fields where vortices flow individually in materials with weak pinning. However, for $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ [50], the melting field $H_m \geq 17.5 \text{T}$ indicates a significant pinning effect, thus $B_0$ is nearly invisible at low temperature. The contribution of core–core collisions and pinning effect is presented by $f$,

$$f = B \exp\left[\left(\frac{T_c}{T} - 1\right)\sqrt{\frac{B_p}{B}}\right].$$

To suppress this contribution and avoid the effect of thermal vortices, we propose to determine $B_0$ at $T_c$ and near $H'$, where $H'$ is the minimum of $f$ at $T_c$. At $T_c$, the equation (13) simplifies to

$$\rho = \rho_0 \frac{B}{B_0 + f}.$$  

Since $T_c$ and $B_p$ are determined, the minimum of $f$ is found to be $f_{\text{min}} = (e^2/4)(T_c/T_c - 1)^2B_p = 4.7 \text{T}$ at $H' = 0.64 \text{T}$. According to calculation, $f$ is approximately a constant near $H'$, thus the data at $H = 0.5 \text{T}$ ($\rho = 10.9 \, \mu\Omega \text{cm}$) and $1 \text{T}$ ($\rho = 16.6 \, \mu\Omega \text{cm}$) are used in the linear fitting. Finally, $B_0 \approx 4 \text{T}$ is obtained.

$b$ is determined near and above $T_c$ with Kosterlitz model [37]. At $T > T_c$, we have the following relation:

$$B_0 = (B_i + B_T) \left\{ \frac{\rho_0}{\rho_i} - \exp\left[\frac{i \frac{B_p}{\sqrt{B_i + B_T}}}{}\right]\right\},$$

where $i = a, b$. Equaling the case $a$ and $b$, we obtain $B_T$, $B_T$ calculated from $B_0 = 0.5 \text{T}$ and $B_0 = 9 \text{T}$ in the range $T \in [90, 100] \text{K}$ is shown in figure 11. According to equation (3), the Kosterlitz coefficient $b = 0.42$ is fitted from the data at $T \in [90, 94] \text{K}$.

A.1.2. Fine tuning. In this step, we vary the five parameters ($\rho_0, T_c, B_0, B_p$ and $b$) around their roughly estimated values to get a precise fitting of the experimental curves, while keeping $B_i = 68.1 \text{T}$ unchanged, following the
Figure 12. Linear extrapolation to determine $T_c$ and $H_{cl}$ in OP Bi-2201. (a) Linear extrapolations (blue solid line) of $e_N^0$ at 50 and 60 K to determine $T_c$. The blue dashed line represents the error bar of extrapolation. (b) Linear extrapolations (blue solid line) of $e_N$ in high fields to determine $H_{cl}(T = 20 K)$.

A.2. Determine parameters with Nernst effect

The systematically measured Nernst signal allows us to do reliable parameter determinations. In this part, equation (14) is applied to the six samples in Bi-2201 and Bi-2212 [50, 52, 56]. The process of parameters determination with $e_N$ is similar to the one with $\rho$ but there are some more steps due to the two more $T$ dependent parameters $e_T$ and $H_{cl}$. In the following, OP Bi-2201 is taken as an example to show the specific procedures.

A.2.1. Rough estimate. In RE, the asymptotic analysis at high and low limits of $T$ and $B$ is also the key strategy. $T_c$ can be determined from the zero point of the linear extrapolation of the peak at very high temperature. Using similar linear extrapolation in high fields, $H_{cl}$ is obtained. $e_T$ and $B_0$ can be determined from field dependence of the signal in the TAFF regime. The nearly linear field dependence of $e_N$ near the minimum of $f$ below and near $T_c$ can be used to determine $B_0$. Meanwhile, $b$ can be determined from the peaks of the profiles above $T_c$.

On the profile of $e_N$ versus $B$, when $T \rightarrow T_c$,

$$ e_N \rightarrow e_T \frac{B}{B_0 + B + B_T} \ln \left( \frac{H_{cl}}{B} \right). $$

(A6)

Since $n_s$ approaches 0 at $T_c$, the peak $e_N^0$ of the profile approaches zero. Since $e_N^0$ versus $T$ is a nonlinear curve at large $T$ range, only the data (50 and 65 K) nearest to zero are used in the linear extrapolation, shown in figure 12(a). Finally, $T_c = 71.5$ K is obtained. The uncertainty of the data is estimated to be $\pm 100$ nV K$^{-1}$ (discussed in the end of this section), and the error bar of $T_c$ is found to be $\pm 6.2$, thus $T_c = 71.5 \pm 6.2$ K.

When $B \rightarrow H_{cl}$, a linear dependence of $B$ can be obtained from equation (14),

$$ e_N \approx e_T \frac{H_{cl} - B}{B_0 + (H_{cl} + B_T) \exp \left[ I \sqrt{B_T / (H_{cl} + B_T)} \right]}. $$

(A7)

$H_{cl}$ represents the zero point of the signal. In figure 12(b), a linear extrapolation of the high fields data, applied to the curve at 20 K, yields $H_{cl} = 45 \pm 0.7$ T. Results of $H_{cl}$ at other temperatures are shown in table 4. Since $B_f \approx H_{cl}^\prime (T_c = 28$ K), $B_f$ can be estimated as $H_{cl} (30$ K) = 48 T.

At low temperature and fields (i.e., the TAFF regime), pinning effect is dominant in damping viscosity. Thus, $B_0$ can be neglected in $e_N$.
Taking the logarithm of the two sides of $e_N$ in equation (A8), we can obtain

$$y = a_1 x + b_1,$$  \hspace{1cm} (A9)

where $y = \ln\left[\ln\left(\frac{H_{\alpha}}{B}\right)\right] - \ln e_N$, $a_1 = \left(\frac{T_c}{T} - 1\right)\sqrt{\frac{B_p}{B}}$, $b_1 = -\ln e_T$, $x = B^{-1/2}$. The linear fitting of the data allows us to determine the parameters,

$$e_T = \exp(-b_1),$$

$$B_p = \frac{a_1^2}{\left(\frac{T_c}{T} - 1\right)^2}.$$  \hspace{1cm} (A10)

The fitting at $T = 20$ K is shown in figure 13, and $B_p = 2.5$ T, $e_T = 11.0 \mu$V K$^{-1}$ are obtained. Data in very low fields are not used because the signals are very weak thus the relative error are big.

A nearly linear equation can be obtained from the model equation (14) of the Nernst effect at the minimum of $f$ and below $T_c$,

$$g = \frac{B}{B_0 + f},$$  \hspace{1cm} (A11)

where $g = e_N / \left(\frac{B}{B_T} \ln\left(\frac{H_{\alpha}}{B}\right)\right)$. Similar to equation (A4), the data in low fields and high temperature should be used to determine $B_0$. In figure 14(a), $g$ versus $B$ at $T = 20$ K shows a linear regime in low fields (from 1 to 8 T), which supports a linear fit ($g = 0.032 \times H$). But the minimum of $f$ is so big (31.1 T) that $B_0$ is found to be very small (0.1 T). In addition, the error of the slope yields that $B_0$ varies in $\pm 1$ T. Thus, $B_0$ can be reasonably chosen as 0.

Since $B_1 \approx 48$ T is estimated, $b$ can be determined from the peak value $e_N^m$ above $T_c$, where

$$e_N^m = e_T \left(\frac{B^m \ln(H_{\alpha}/B^m)}{B_0 + (B^m + B_T) \exp\left(\frac{B_p}{B^m + B_T}\right)}\right).$$  \hspace{1cm} (A12)

$B_T$ can be calculated from this formula directly. According to Korsterlitz model equation (3), a linear model can be obtained as

$$y = a_2 x + b_2,$$  \hspace{1cm} (A13)

where $y = \ln\left(\frac{B_0}{B_T}\right)$, $a_2 = 2b$, $x = \left(\frac{T_c}{T} - 1\right)^{-1/2}$ and $b_2$ contains the errors. $b$ can be determined from the linear fit of equation (A13). In figure 14(b), the data near $T_c$ (30 and 35 K) are used to conduct the linear fitting and $b = 0.27$ is obtained. The results of RE are shown in table 2.

| $T$ (K) | 5   | 11  | 20  | 30  | 40  |
|---|---|---|---|---|---|
| $H_{\alpha}$ (T) | 50.0 ± 3.3 | 45.0 ± 1 | 45.0 ± 0.7 | 48.0 ± 1.3 | 50.0 ± 3.6 |

Figure 13. Linear fitting of $\ln\left[\ln\left(\frac{H_{\alpha}}{B}\right)\right] - \ln e_N$ versus $H^{-1/2}$ to determine $B_p$ and $e_T$ at $T = 20$ K.
A.2.2. Fine tuning. Since both \(e_f\) and \(H_{\text{c2}}\) in equation (14) are \(T\)-dependent, the FT procedure of the Nernst effect is a little more complicated, which requires to run an iteration program. First, the parameters are divided into two groups. One group is made up of intrinsic parameters \(B_0, B_p, B_b\) and \(T_c\). The other contains the \(T\)-dependent parameters \(e_f\) and \(H_{\text{c2}}\). The RE helps to define the ranges of the fitting parameters, e.g., \(T_c \in [65.3, 77.7] \text{K}, H_{\text{c2}} \in [44, 53.6] \text{K}, B_0 \in [46.7, 49.3] \text{T}\) and \(B_p = 0\). Besides, \(B_p > 0, b > 0\) and \(e_f > 0\) are set to be constraints. Second, we treat the empirical data of a group, a new group and another, with the former results substituted into the later calculation. This iterative process runs until a local minimum (RMSE \(\approx 0.0993 \mu \text{VK}^{-1}\)) in the parameter space specified as above is reached.

In the Nernst experiment [3], the error of the voltage measured by nanovoltmeter is \(\pm 5 \text{nV}\), thus the uncertainty of the Nernst signal \(\delta E = \delta E / \partial T\) is estimated to be \(\pm 25 \text{nV K}^{-1}\). By private communication, Y. Wang indicates that the noise in strong fields should be larger, thus \(\delta E\) is larger than \(\pm 50 \text{nVK}^{-1}\) (the maximum value may be \(\pm 0.1 \mu \text{VK}^{-1}\) at \(45 \text{T}\)), consistent with the typical fluctuations of the data. In the present work, only data below 30 \text{T} are used. We choose a rational fitting error of \(\pm 100 \text{nVK}^{-1}\) as the upper boundary of the error function for the samples except for UD Bi-2201. For UD Bi-2201, the local minimum error is 0.152 \(\mu \text{VK}^{-1}\), thus we set 0.2 \(\mu \text{VK}^{-1}\) as the upper boundary, to reveal the sensitivity of the model. The procedure to determine the error bar is similar to that above for \(\rho_\text{fl}\) except \(B_0\) is assumed as equals to \(H_{\text{c2}}\) at \(T_c\) (with error bar). The final results of FT are shown in table 2 and the multi-curves fitting figure is shown in figure 4(b).

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