Theoretical and experimental determination of the acoustic radiation force acting on an elastic cylinder in a plane progressive wave—far-field derivation approach

F G Mitri
Mayo Clinic, Department of Physiology and Biomedical Engineering, Ultrasound Research Laboratory, 200 First St. SW, Rochester, MN 55905, USA
E-mail: mitri@ieee.org and mitri.farid@mayo.edu

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Abstract. In this study, a new analytical expression of the radiation force function—which is the radiation force per unit energy density and unit cross sectional surface—for an elastic cylinder immersed in water and placed in a plane progressive acoustic wave is presented. The calculation is based on the far-field derivation approach. Furthermore, two solutions for the radiation force function are compared. The first is based on the far-field derivation and the second on the near-field. It is shown that for an ideal fluid both results are commensurate with the same result. Moreover, an experiment is conducted to verify the theoretical predictions for an elastic cylinder as well. The experimental results confirm the theory for elastic cylinders but show a quite different deviation from the rigid solution. These results are the first to confirm the theory for the radiation force on an elastic cylinder placed in a plane progressive wave-field.
1. Introduction

Physicists are familiar with the fact that acoustic waves [1] as well as electromagnetic waves [2] carry momentum. When an acoustic wave strikes an object, part of its momentum is transferred to the object, giving rise to the acoustic radiation force phenomenon, while the other part is transferred to the surrounding medium to form a dipole streaming [3], as long as real (viscous) fluids are concerned. Some publications refer to the force on an object as ‘radiation pressure’; however, this ‘pressure’ is a vector quantity and usually integrated over the surface area of the object, so it is more appropriate to use the term radiation force. Radiation force, is connected with energy densities, which are quadratic terms containing squares of velocities or displacements. Therefore, any theory dealing with acoustic radiation force must retain at least all second order terms to be valid even at small amplitudes.

In this paper, as is the common practice, we consider the case where the dimensions of the object are much greater than the acoustic boundary layer thickness. Here, the fluid is assumed to be nonviscous and the effects of streaming are neglected. King [4] was one of the first to analyse the radiation force on spheres. He published a landmark paper describing the radiation force on a sphere in a nonviscous fluid. He derived a formula for the second-order pressure and calculated the radiation force due to a standing wave and a progressive wave as well. Most studies presented have only considered spherical objects [5]–[14], but there have been a few investigations into the radiation force on a cylinder. These considered a nonviscous fluid where the cylinder is free to move in the acoustic field. Awatani [15] calculated the static radiation force on a rigid cylinder in both progressive and standing plane waves. In that study, the radiation force has been numerically evaluated in a small range of dimensionless frequency ($0 \leq ka \leq 5$; $k$ is the wave vector of the incident wave in the fluid medium, $a$ is the cylinder’s radius). Later on, Hasegawa et al [16] published theoretical calculations for elastic cylinders as well as for elastic cylindrical and spherical shells [17] in a plane progressive wave field, that have been extended to take into account the attenuation of sound within the solid material or when it is coated by a viscoelastic absorbing layer [18, 19]. Wu et al [20] also developed an analytical study for a rigid cylinder in a standing wave which was compared to experimental results. They found an agreement to within 20%. Ebenezer and Stepanishen [21, 22] presented two papers using numerical solutions to calculate the transient loading on a cylinder that is vibrating at an arbitrary number of natural frequencies. In a recent work, Wei et al [23] have studied theoretically the acoustic radiation force on compressible cylinders in plane standing waves in ideal nonviscous fluids based on the far-field
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acoustic scattering formulation. Haydock [24] formulated analytical equations for the radiation force on a rigid immovable and movable cylinder in a standing wave, based on the theory initially developed by King [4] for spheres. Recent theoretical work concerned the acoustic radiation force on elastic and viscoelastic cylinders [25] and shells [26] in stationary and quasi-stationary plane acoustic waves.

However, none of the abovementioned studies satisfactorily analyses experimentally the case of a cylinder in a progressive or travelling wave field. In this study, a new analytical calculation of the acoustic radiation force of a continuous plane wave field incident on an elastic cylinder in a nonviscous fluid (in this case water) is presented along with an experimental validation of the theory. The force is calculated from the far-field acoustic scattering field. Section 2 is devoted to the theoretical calculation of the radiation force on a cylinder in a progressive wave based on the far-field solution. In section 3, the experiments are described and the results are discussed, then in section 4 concluding remarks are presented.

2. Static acoustic radiation force on a cylinder immersed in an ideal fluid and placed in a progressive plane wave—far-field derivation approach

The exact evaluation of the radiation force involves calculating the complete solution of the associated acoustic scattering problem. Hence, the acoustic scattering problem should be solved first based on the procedure described in [23].

Assume the cylinder is axisymmetric, and the outer fluid medium has density \( \rho_0 \) and sound speed \( c_0 \). A plane incident progressive wave is expressed by its velocity potential as [27]

\[
\Phi_i = \Phi_0 \sum_{n=0}^{N \to \infty} \epsilon_n(i)^n J_n(k_0 r) \cos(n\theta) e^{-i\omega t},
\]

(1)

where \( \Phi_0 \) is the amplitude, \( k_0 = \omega_0/c_0 \), \( J_n(\cdot) \) denotes the cylindrical Bessel function of order \( n \), \( \epsilon_n = 2 - \delta_{0n} \), and \( \delta_{0n} \) is the Kronecker delta.

The scattered wave due to the presence of the cylinder is expressed by [27]

\[
\Phi_{sc} = \Phi_0 \sum_{n=0}^{N \to \infty} \epsilon_n(i)^n C_n H_n^{(1)}(k_0 r) \cos(n\theta) e^{-i\omega t},
\]

(2)

where the dimensionless scattering coefficients \( C_n \) for each partial wave can be determined from the boundary condition at the surface of the cylinder, i.e. continuity of pressure and displacements (or velocities). The general solution for \( C_n \) is given by

\[
C_n = \begin{vmatrix}
v_1 & \lambda_{12} & \lambda_{13} \\
v_2 & \lambda_{22} & \lambda_{23} \\
v_3 & \lambda_{32} & \lambda_{33} \\
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{vmatrix},
\]

(3)
where \( v_i \) and \( \lambda_{ij} \) are the dimensionless elements of the determinants given by
\[
\lambda_{11} = \frac{\rho_0}{\rho} x_2^2 H_n^{(1)}(x), \quad \lambda_{12} = (2n^2 - x_2^2) J_n(x_1) - 2x_1 J'_n(x_1), \\
\lambda_{13} = 2n[x_2 J'_n(x_2) - J_n(x_2)], \\
\lambda_{21} = -x J'_n(x), \quad \lambda_{22} = x_1 J'_n(x_1), \quad \lambda_{23} = n J_n(x_2), \\
\lambda_{31} = 0, \quad \lambda_{32} = 2n[X_n(x_1) - x_1 J'_n(x_1)], \quad \lambda_{33} = 2x_2 J'_n(x_2) - (2n^2 - x_2^2) J_n(x_2), \\
v_1 = -\frac{\rho_0}{\rho} x_2^2 J_n(x), \quad v_2 = x J'_n(x), \quad v_3 = 0,
\]
where \( \rho \) is the mass density of the cylinder, \( x = k_0 a \), and is defined as the dimensionless size parameter, \( x_1 = x(c_0/c_1) \) and \( x_2 = x(c_0/c_2) \) where \( c_1 \) and \( c_2 \) are the compressional and shear wave velocities in the cylinder. Here, the primes denote derivatives with respect to the arguments.

In the far-field region, the following asymptotic approximation is used:
\[
(i)^n H_n^{(1)}(k_0 r) \rightarrow \sqrt{\frac{2}{\pi k_0 r}} e^{i(k_0 r - \pi/4)}, \quad (4)
\]
so that the total time-independent far-field is
\[
\lim_{k_0 r \rightarrow \infty} \bar{\Phi}_t = \bar{\Phi}_t + \lim_{k_0 r \rightarrow \infty} \bar{\Phi}_{sc} = \bar{\Phi}_t + \frac{f(\theta)}{\sqrt{k_0 r}} e^{i k_0 r}, \quad (5)
\]
where, \( \bar{\Phi}_t = \bar{\Phi}_t + \bar{\Phi}_{sc} = \Phi_t e^{i \omega t}, \bar{\Phi}_t = \Phi_t e^{i \omega t}, \Phi_t = \Phi_t + \Phi_{sc}, \Phi_{sc} = \Phi_{sc} e^{i \omega t}, \) and
\[
f(\theta) = \sqrt{\frac{2}{\pi}} e^{-i(\pi/4)} \sum_{n=0}^{N \rightarrow \infty} \epsilon_n C_n \cos(n \theta), \quad (6)
\]

The calculation for the acoustic radiation force in an ideal fluid is based on integrating the time-averaged momentum-flux tensor over the surface of the cylinder. Therefore the total force is expressed by [28]
\[
\langle F \rangle = \iint_{S_0} \langle \bar{\Pi} \rangle \, dS_0, \quad (7)
\]
where \( dS = \mathbf{n} \, dS \) (\( \mathbf{n} \) is the normal directed away from the object), \( S_0 \) is the cylinder’s surface at its equilibrium position, and \( \langle \bar{\Pi} \rangle \) is the time-averaged momentum-flux tensor [29] given by
\[
\langle \bar{\Pi} \rangle = \langle (P - P_0) + \rho_0 \mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)} \rangle, \quad (8)
\]
where $⟨P − P_0⟩$ is ‘a mean excess pressure’ [30], $\rho_0\mathbf{v}^{(i)} \cdot \mathbf{v}^{(1)}$ is the Reynolds stress resulting from changing from the Eulerian to the Lagrangian reference frame due to the motion of the fluid particle at the boundary, in which $\mathbf{v}^{(1)} = −\nabla(\text{Re}[\Phi_1])$ is the fluid particle velocity taken up to the first order, and the integration is over the cylinder’s surface. The time-average of the first order perturbation acoustic pressure field is [30]

$$⟨P − P_0⟩ = \frac{1}{2\rho_0 c_0^2}⟨P_1^2⟩ − \frac{1}{2} \rho_0 ⟨|\nabla(\text{Re}[\Phi_1])|^2⟩,$$

(9)

where $P_1 = \rho_0 \partial(\text{Re}[\Phi_1])/\partial t$.

Substituting equations (8) and (9) into equation (7), the total averaged force is then expressed by

$$⟨F⟩ = \iint_{S_0} \left[ \frac{1}{2} \rho_0 (|\mathbf{v}^{(1)}|^2) − \frac{1}{2} \rho_0 P_1^2 − \rho_0 \langle\mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)}⟩ \right] dS_0.$$

(10)

The surface integral in equation (7) is calculated over the cylinder’s surface $S_0$. Let us consider a circular cylindrical surface $S_{\text{Far-field}}$ at large radius that encloses $S_0$. The total force in the far-field is therefore expressed by

$$⟨F⟩_{\text{Far-field}} = −\iint_{S_{\text{Far-field}}} ⟨\mathbf{P}⟩ dS_{\text{Far-field}}.$$

(11)

Using the Gauss divergence theorem, and taking the difference of equations (7) and (11), we have

$$⟨F⟩ − ⟨F⟩_{\text{Far-field}} = \iint_{S_{\text{Far-field}}} ⟨\mathbf{P}⟩ dS_{\text{Far-field}} − \iint_{S_0} ⟨\mathbf{P}⟩ dS_0,$$

$$= \iiint_{V_{\text{Far-field}}} \text{div}⟨\mathbf{P}⟩ dV_{\text{Far-field}} − \iiint_{V_0} \text{div}⟨\mathbf{P}⟩ dV_0,$$

$$= \iiint_{δV} \text{div}⟨\mathbf{P}⟩ d(δV),$$

(12)

where $δV = V_{\text{Far-field}} − V_0$ is the volume between the cylinder’s surface $S_0$ and the circular cylindrical surface $S_{\text{Far-field}}$ at a large radius.

An ideal fluid, by definition, cannot absorb momentum. Therefore, $⟨\mathbf{P}⟩$ is a solenoidal tensor and the volume integral $\iiint_{δV} \text{div}⟨\mathbf{P}⟩ d(δV) = 0$. Physically, this means that the mean momentum change in a unit time which occurs inside $S_{\text{Far-field}}$ entirely goes into the force experienced by the particle.
Therefore,
\[
\langle F \rangle = \langle F \rangle_{\text{Far-field}}, \\
= - \int \int \langle \mathbf{P} \rangle \, dS_0,  \\
= - \int \int \langle \mathbf{P} \rangle \, dS_{\text{Far-field}},  \\
= \int \int \left[ \frac{1}{2} \rho_0 \langle |\mathbf{v}^{(1)}|^2 \rangle - \frac{1}{2} \rho_0 c_0^2 \langle P^2 \rangle - \rho_0 \langle \mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)} \rangle \right] dS_{\text{Far-field}}.
\]

It is important to note that the differential area \( dS_{\text{Far-field}} \) is directed away from the object. \( dS_{\text{Far-field}} = Lr \, d\theta \, u_r \), where \( L \) is the length of the circular cylindrical surface and \( u_r \) is a radial unit vector. Moreover, equation (13) gives the expression for the force on the object in terms of far-field properties.

After some arithmetic manipulation, the force per unit length of the cylinder along the direction of wave propagation becomes
\[
\langle F \parallel \rangle = - \rho_0 r \int_0^{2\pi} \frac{1}{2} \operatorname{Re} \left[ \frac{\partial \tilde{\Phi}_i}{\partial r} \frac{\partial \tilde{\Phi}_i^*}{\partial \theta} \right] \, d\theta + \rho_0 r \int_0^{2\pi} \frac{1}{2} \operatorname{Re}[|\nabla \tilde{\Phi}_i||\nabla \tilde{\Phi}_i^*|] \cos \theta \, d\theta \\
- \rho_0 r \frac{1}{2 c_0^2} \int_0^{2\pi} \frac{1}{2} \operatorname{Re}[(-i\omega \tilde{\Phi}_i)(-i\omega \tilde{\Phi}_i^*)] \cos \theta \, d\theta.
\]

Substituting equation (5) into equation (14) gives
\[
\langle F \parallel \rangle = - \rho_0 k_0^2 \frac{1}{2} \int_0^{2\pi} f(\theta) f^*(\theta) \cos \theta \, d\theta - \rho_0 r \frac{1}{2} \int_0^{2\pi} \operatorname{Re} \left[ \frac{\partial \tilde{\Phi}_i}{\partial r} \frac{\partial \tilde{\Phi}_i^*}{\partial \theta} \right] \, d\theta \\
- \rho_0 r k_0^2 \frac{1}{2} \int_0^{2\pi} \frac{1}{2} \operatorname{Re} \left[ \frac{\tilde{\Phi}_i^* f(\theta) e^{ik_0 r}}{\sqrt{k_0 r}} \right] \cos \theta \, d\theta - \rho_0 r \frac{1}{4} \int_0^{2\pi} (k_0^2 |\tilde{\Phi}_i|^2 - |\nabla \tilde{\Phi}_i|^2) \cos \theta \, d\theta \\
+ \rho_0 r k_0 \frac{1}{2} \int_0^{2\pi} \operatorname{Im} \left( \frac{\partial \tilde{\Phi}_i^* f(\theta) e^{ik_0 r}}{\sqrt{k_0 r}} \right) \, d\theta.
\]

By definition, the acoustic radiation force is experienced by the object when subjected to incident acoustic waves. Hence, if there is no object such that \( f(\theta) = 0 \), the force should vanish. Therefore the sum of the second and third terms in equation (15) vanishes, and the force per unit length of the cylinder is expressed as
\[
\langle F \parallel \rangle = - \rho_0 k_0^2 \frac{1}{2} \int_0^{2\pi} f(\theta) f^*(\theta) \cos \theta \, d\theta - \rho_0 r k_0 \frac{1}{2} \int_0^{2\pi} \frac{1}{2} \operatorname{Re} \left( \frac{\tilde{\Phi}_i^* f(\theta) e^{ik_0 r}}{\sqrt{k_0 r}} \right) \cos \theta \, d\theta  \\
+ \rho_0 r k_0 \frac{1}{2} \int_0^{2\pi} \operatorname{Im} \left( \frac{\partial \tilde{\Phi}_i^* f(\theta) e^{ik_0 r}}{\sqrt{k_0 r}} \right) \, d\theta.
\]
Substituting the time-independent part of equations (1) and (6) into equation (16) using the following relations

\[
\int_0^{2\pi} \cos(n\theta) \cos(m\theta) \cos \theta \, d\theta = \begin{cases} 
\pi & (n + m = 1), \\
\pi/2 & (n - m = \pm 1, n \neq 0, m \neq 0), \\
0 & \text{(otherwise)},
\end{cases}
\]

(17)

\[
\int_0^{2\pi} \cos(n\theta) \cos \theta \, d\theta = \begin{cases} 
\pi & (n = 1), \\
0 & \text{(otherwise)},
\end{cases}
\]

(18)

the radiation force per unit length of the cylinder in a progressive wave field becomes

\[
\langle F \rangle = Y_p^{\text{Far-field}} S_c \langle E_p \rangle,
\]

(19)

where \( \langle E_p \rangle = \frac{1}{4} \rho_0 k_0^2 |\Phi_0|^2 \) is the time-average energy density, \( S_c = 2a \) is the cross-sectional surface for a unit-length cylinder, and \( Y_p^{\text{Far-field}} \) is the dimensionless radiation force function given in terms of the scattering coefficients \( C_n = (\alpha_n + i\beta_n) \), given by equation (3) as

\[
Y_p^{\text{Far-field}} = -\frac{2}{ka} \sum_{n=0}^{N \to \infty} [\epsilon_n \alpha_n + 2(\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1})].
\]

(20)

Equation (20) gives the radiation force function using the far-field solution.

3. Comparison with the solution of Hasegawa et al

Hasegawa et al [16] have developed a solution for the radiation force function \( Y_p \) for cylinders in a progressive wave using a near-field solution. According to their theory,

\[
Y_p^{\text{Near-field}} = -\frac{2}{ka} \sum_{n=0}^{N \to \infty} [\alpha_n + \alpha_{n+1} + 2(\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1})],
\]

(21)

where \( \alpha_n \) and \( \beta_n \) are real and imaginary parts of the scattering coefficients \( C_n \) defined by equation (3).

Equations (20) and (21) are very alike in appearance, but apparently not equal. Following the procedure developed by Hasegawa [9], the \( n \)th terms on the right-hand sides of equations (20) and (21) are evaluated numerically:

\[
Y_N^{ff} = -\frac{2}{ka} [\epsilon_n \alpha_n + 2(\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1})],
\]

(22)

and

\[
Y_N^{nf} = -\frac{2}{ka} [\alpha_n + \alpha_{n+1} + 2(\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1})].
\]

(23)
Table 1. Experimental values of $Y_p^{\text{Far-field}}$.

| $ka$     | $Y_p^{\text{Far-field}}$ |
|----------|-------------------------|
| 25.1327  | 1.24                    |
| 25.76    | 1.173                   |
| 26.38    | 1.2086                  |
| 27.017   | 1.216                   |
| 27.646   | 1.28                    |
| 28.27    | 1.203                   |
| 28.9     | 1.201                   |
| 29.53    | 1.2                     |
| 30.15    | 1.19                    |
| 30.78    | 1.25                    |
| 31.416   | 1.2252                  |
| 33.30    | 1.26                    |
| 33.92    | 1.2524                  |

As an example of this comparison, a stainless steel cylinder ($\rho(=7800 \text{ kg m}^{-3})$, $c_1 = 5700 \text{ m s}^{-1}$ and $c_2 = 3000 \text{ m s}^{-1}$), suspended in a plane progressive wave in water is considered. The mechanical properties of this material used in the calculations are listed in table 1. It is also essential to extend the maximum index $N(=40$ for this example) to exceed $ka$ to ensure proper convergence. According to the theory of Hasegawa, $Y_p$ is 1.0468 at $ka = 27$.

Equations (22) and (23) are plotted versus $n$ in figure 1. As seen in this figure, one can clearly observe the great difference between the two plots. This makes serious doubt about the agreement between the two theories. However, in point of fact, it is the sum of $Y_{ff}^N$ or $Y_{nf}^N$ for all $n$, but not the $n$th term itself that should be evaluated. Therefore, summing up $Y_{ff}^N$ or $Y_{nf}^N$ for all values of $n$, we obtain the same value of 1.0468 for $Y_p$ values of both theories for $ka = 27$. Therefore, it is suggested that the two solutions are numerically equivalent in spite of the apparent discrepancy. This is explained as follows: if we denote by $\Delta_N$ the difference between the partial sums up to $n = N$ in equations (22) and (23), we obtain

$$
\Delta_N = \sum_{n=0}^{N} (Y_{ff}^N - Y_{nf}^N),
$$

$$
= \frac{2}{ka} \sum_{n=0}^{N} [\alpha_n (1 - \varepsilon_n) + \alpha_{n+1}],
$$

$$
= \frac{4}{ka} \alpha_{N+1}. \tag{24}
$$

It can be numerically proved that $\alpha_{N+1}$ is at most a small quantity so long as equations (20) or (21) are convergent. Hence, $\lim_{N \to \infty} \Delta_N = 0$, and the two solutions, i.e. equations (20) and (21) are numerically equivalent.
Figure 1. The values of $Y_{N}^{f}$ and $Y_{N}^{nf}$ given by equations (22) and (23) respectively versus $n$, for a stainless steel cylinder immersed in water at $ka = 27$.

Another way to analyse this apparent discrepancy is to rewrite the series $\sum_{n=0}^{N} \varepsilon_n \alpha_n$ appearing in equation (20) as:

$$\sum_{n=0}^{N} \varepsilon_n \alpha_n = \alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots = (\alpha_0 + \alpha_1) + (\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) + (\alpha_3 + \cdots)$$

$$= \sum_{n=0}^{N} (\alpha_n + \alpha_{n+1}).$$

It follows from equation (25) that equations (20) and (21) are equal and numerically equivalent.

4. Experiments

The corresponding experiment was performed in a water tank (figure 2). A 12.5 mm diameter flat piston transducer (Panametrics, V309) was used with CW (continuous wave) ultrasound. This transducer operated at a central frequency around 4.7 MHz. The transducer was connected to a custom power amplifier and the radio frequency (RF) signal was obtained from a function generator (Agilent, 33250A). The transducer was mounted on a 3D positioning system immersed within the water tank.

To calibrate this transducer, a PVDF membrane hydrophone (GEC—Marconi, Y-33-7611) with a sensitivity of $0.168 \mu$V Pa$^{-1}$, was placed at a distance of 15 cm from the flat surface of the piston transducer (far-field region) where the sound intensity is relatively constant. The frequency was swept in the range between 4 and 5.4 MHz in increments of 100 kHz, and the rms
Figure 2. Schematic diagram of the experimental apparatus. The ultrasound radiation force deflects the cylinder from its position. The deflection $\delta$ is measured by the laser positioned on a micro-station.

Pressure was measured using a digital oscilloscope (Tektronix, TDS 3014). Hence, the energy density was evaluated from the measured rms pressure using the formula:

$$\langle E \rangle = \frac{P^2}{\rho c^2},$$

(26)

where $P_{(\text{in}P_0)} = 1.2 \times P_{\text{rms}}/\text{sensitivity}$. The factor 1.2 accounted for the electrical loading of the PVDF membrane hydrophone by the oscilloscope.

After the transducer calibration, the membrane hydrophone was replaced by the cylinder placed at the same position (i.e. 15 cm) from the flat surface of the piston transducer. The stainless steel cylinder (3 mm in diameter, 10 mm in length) was suspended with a human hair (0.07 mm in diameter, 100 mm in length). The deflection of the cylinder was measured with an alignment laser (Polytec, OFV 303) that was perpendicular to the ultrasound beam (figure 2). The laser was first focused on the cylinder at its equilibrium position when the ultrasound was initially turned off. Then, the ultrasound was turned on and the laser was moved laterally on a linear micrometer stage to align with the cylinder at its deflection position. The deflection distance $\delta$ (or displacement) of the cylinder can be measured with 10 $\mu$m resolution. The frequency was swept in the range between 4 and 5.4 MHz in increments of 100 kHz and the deflection of the cylinder was measured for each 100 kHz frequency sweep in a step-by-step manner.
5. Method for measuring the radiation force

As shown in figure 2, the static radiation force along the direction of wave propagation is balanced by the gravitational force on the cylinder and the tension of the string. Knowing that the buoyancy on the cylinder is equal to the weight of water with identical volume as that of the cylinder, and neglecting the mass of the string (in this case human hair), the deflection distance $\delta$ of the cylinder is related to the radiation force as:

$$F_{\parallel} \simeq m_a g \frac{\delta}{\sqrt{(L + h/2)^2 - \delta^2}},$$

$$\simeq \pi a^2 h (\rho - \rho_0) g \frac{\delta}{\sqrt{(L + h/2)^2 - \delta^2}},$$

where $m_a$ is the mass of the cylinder corrected for buoyancy (known also by apparent mass), $L(=100 \text{ mm})$ is the suspension length, $\rho(=8064 \text{ kg m}^{-3})$, $h(=10 \text{ mm})$ and $a(=1.5 \text{ mm})$ are the cylinder’s mass density, length and radius respectively, $g(=9.8 \text{ m s}^{-2})$ is the acceleration of gravity, and $\rho_0(=1000 \text{ kg m}^{-3})$ is the density of the surrounding medium (in this case water).

Therefore, using equation (19), the radiation force function $Y_{\text{Exp}}^{\text{Far-field}}$ can be rewritten as

$$Y_{\text{Exp}}^{\text{Far-field}} \simeq \frac{\pi a h (\rho - \rho_0) g \delta}{2\langle E \rangle \sqrt{(L + h/2)^2 - \delta^2}},$$

(28)

where $\langle E \rangle$ is the energy density determined previously in section 4. Hence, $Y_{\text{Exp}}^{\text{Far-field}}$ can be evaluated accurately by measuring the deflection $\delta$ using equation (28).

6. Results and concluding remarks

Table 1 lists the experimental values of $Y_{\text{Exp}}^{\text{Far-field}}$. Figure 3 shows the comparison between experimental and theoretical values of $Y_{\text{Exp}}^{\text{Far-field}}$. In this figure, the solid line is the theoretical and the circles are the corresponding experimental values $Y_{\text{Exp}}^{\text{Exp}}$ listed in table 1. It should be pointed out that the experimental results are in very good agreement with the theory within the experimental accuracy. Moreover, the radiation force function for a rigid immovable cylinder is also computed and shown as a dashed line in figure 3. This additional calculation is added for comparison. The rigid immovable cylinder solution is obtained by letting $\rho, c_1$ and $c_2 \to \infty$ in equation (3). It is obvious that the experimental results show much deviation from the rigid cylinder solution.

The analysis of the present paper shows that both near and far-field derivation approaches are equivalent as long as ideal fluids are concerned. However, it is anticipated that these two approaches will not be equivalent for a viscous fluid. In a recent work, Haydock [31] computed with a lattice-Boltzmann simulation the time-average force on rigid moveable and immovable small cylinders in standing wave in a viscous fluid. His computations show that viscous corrections for cylinders will be small when the oscillating viscous boundary layer thickness is much less that the cylinder radius. On the other hand, Doinikov [32]–[36] showed analytically that the viscous effect for a sphere immersed in a viscous fluid and placed in progressive and standing waves can be neglected when the oscillating viscous boundary layer thickness is much less that the sphere radius. In this experiment, the acoustic viscous boundary layer thickness,
Figure 3. Comparison between experimental (circles) and theoretical (solid line) values of the radiation force function $Y_{\text{Far-field}}$. Dashed curve: theoretical $Y_{p}^{\text{Far-field}}$ for a rigid immovable cylinder immersed in water.

defined as $\varepsilon = \sqrt{2\eta/\rho\omega}$, where $\eta(=0.89 \times 10^{-3} \text{ Pa s})$ is the dynamic viscosity, $\rho(=1000 \text{ kg m}^{-3})$ is the fluid mass density, and $\omega(=2\pi \times 4.7 \times 10^6)$ is the angular frequency of the wave. These parameters give an acoustic viscous boundary layer of $0.24 \mu \text{m} \ll a$ (cylinder radius $=1.5 \text{ mm}$). Therefore, according to Haydock [31] and Doinikov [32]–[36], the viscous boundary layer effects on the radiation force measurements can be safely neglected.

It should be also mentioned that a real (viscous) fluid is capable of absorbing the momentum and converting it into acoustic streaming in the near-field. Therefore in order to calculate the mean force acting on the cylinder in a viscous fluid (not to be confused with the total radiation force [3], [32]–[36]), one should use the near-field derivation approach. It is worth mentioning that one must solve the time-averaged equations of the fluid motion with accuracy up to the second-order terms to calculate the mean force. However, to calculate the total acoustic radiation force in a viscous fluid, i.e. the force that acts on the whole medium (fluid + obstacle) when ultrasound is switched on, one should use the far-field derivation approach. The reason is that the generation of incident sound field is accompanied by production of acoustic streaming. This external streaming is dependent on the geometry of the domain as well as on the boundary conditions. The drag force coming from this streaming is not connected with the momentum flux carried by the sound wave to the obstacle. For this experiment, an approximate estimation of the streaming has been performed; the pressure of the incident ultrasound used in the experiments was about $7.8 \times 10^4 \text{ Pa}$ at 4.7 MHz. Using the method of Nyborg [37] the acoustic streaming velocity was estimated to be around $v_s = 1.59 \text{ mm s}^{-1}$ in water. The drag force caused by streaming can be calculated by Stokes’ formula (for a sphere) [28]. To have an approximate estimation about the magnitude of

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the force, in the following calculation, the cylinder was replaced by a sphere of the same radius. The drag force is determined by [28]

$$F_{\text{drag}} = 6\pi \eta v_s,$$

(29)

where $v_s$ is the streaming velocity ($v_s = 1.59 \text{ mm s}^{-1}$). According to Stokes’ formula, the force due to the drag is at the order of $4 \times 10^{-8} \text{ N}$. In the experiment, the radiation force measured on the cylinder at 4.7 MHz (i.e. $ka = 29.53$) was found to be equal to $9.69 \times 10^{-5} \text{ N}$, which is about $\approx 2500$ higher than the drag force. This result can be expected since Danilov and Mironov [3] have previously showed that when the acoustic viscous boundary layer thickness $\varepsilon \ll a$, the streaming is unimportant and its effects on the radiation force measurements can also be safely neglected.

Moreover, Danilov and Mironov [3] pointed out that even if this drag was absent, the total radiation force is not the mean force exerted on the obstacle because another acoustic dipole streaming is produced locally near the obstacle and carries away part of the momentum flux brought by sound waves [3]. Therefore, to calculate the total radiation force in a viscous fluid, the surface over which the integration is performed should be moved to infinity in order to capture the momentum change of the whole fluid. Once again, in the limit $\varepsilon \ll a$, the effect of the acoustic dipole streaming can also be safely neglected.

References

[1] Borgnis F E 1953 Rev. Mod. Phys. 25 653
[2] Ashkin A 1970 Phys. Rev. Lett. 24 156
[3] Danilov S D and Mironov M A 2000 J. Acoust. Soc. Am. 107 143
[4] King L V 1934 Proc. R. Soc. Lond. A 147 212
[5] Klein E 1938 J. Acoust. Soc. Am. 9 312
[6] Fox F E 1940 J. Acoust. Soc. Am. 12 147
[7] Hasegawa T and Yosioka K 1969 J. Acoust. Soc. Am. 46 1139
[8] Crum L A 1971 J. Acoust. Soc. Am. 50 157
[9] Hasegawa T 1977 J. Acoust. Soc. Am. 61 1445
[10] Hasegawa T 1979 J. Acoust. Soc. Am. 65 32
[11] Hasegawa T 1979 J. Acoust. Soc. Am. 65 41
[12] Hertz H M 1995 J. Appl. Phys. 78 4845
[13] Yasuda K and Kamakura T 1997 Appl. Phys. Lett. 71 1771
[14] Brandt D T 2001 Nature 413 474
[15] Mitri F G 2005 Ultrasonics 43 681
[16] Mitri F G 2005 Wave Motion 43 271
[17] Mitri F G 2006 Wave Motion 43 445
[18] Mitri F G 2006 Ultrasonics 44 244
[19] Mitri F G 2006 Wave Motion 43 12
[20] Wu J, Du G, Work S S and Warshaw D M 1990 J. Acoust. Soc. Am. 87 581
[21] Ebenezer D D and Stepanshen P R 1991 J. Acoust. Soc. Am. 89 39
[22] Ebenezer D D and Stepanshen P R 1991 J. Acoust. Soc. Am. 89 2532
[23] Wei W, Thiessen D B and Marston P L 2004 J. Acoust. Soc. Am. 116 201

Marston P L and Thiessen D B 2004 Am. NY Acad. Sci. 1027 414

New Journal of Physics 8 (2006) 138 (http://www.njp.org/)
[24] Haydock D 2005 J. Phys. A: Math. Gen. 38 3279
[25] Mitri F G 2005 Eur. Phys. J. B 44 71
[26] Mitri F G 2005 J. Phys. A: Math. Gen. 38 9395
  Mitri F G and Fellah Z E A 2006 J. Phys. A: Math. Gen. 39 6085
[27] Morse P M 1981 Vibration and Sound (New York: Acoustical Society of America)
[28] Landau L D and Lifshitz E M 1987 Fluid Mechanics 2nd edn (Oxford: Butterworth-Heinemann)
[29] Gor’kov L P 1962 Sov. Phys.—Dokl. 6 773
[30] Lee C P and Wang T G 1993 J. Acoust. Soc. Am. 94 1099
[31] Haydock D 2005 J. Phys. A: Math. Gen. 38 3265
[32] Doinikov A A 1994 Proc. R. Soc. Lond. A 447 447
[33] Doinikov A A 1996 Phys. Rev. E 54 6297
[34] Doinikov A A 1997 J. Acoust. Soc. Am. 101 713
[35] Doinikov A A 1997 J. Acoust. Soc. Am. 101 722
[36] Doinikov A A 1997 J. Acoust. Soc. Am. 101 731
[37] Hamilton M F and Blackstock D T (ed) 1998 Nonlinear Acoustics (New York: Academic) chapter 7, pp 214–7