A MARKOV PARTITION FOR JEANDEL-RAO APERIODIC WANG TILINGS

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Abstract. We define a Markov partition for a $\mathbb{Z}^2$-rotation on the 2-dimensional torus whose associated symbolic dynamical system is a minimal and aperiodic Wang shift defined by 19 Wang tiles. We define another partition for another $\mathbb{Z}^2$-rotation on a 2-dimensional torus whose associated symbolic dynamical system is a minimal proper subshift of the Jeandel-Rao aperiodic Wang shift defined by 11 Wang tiles. As a result, we prove in both cases that the $\mathbb{Z}^2$-rotation on the torus is the maximal equicontinuous factor of the minimal subshifts. It provides a construction of these Wang tilings as model sets of 4-to-2 cut and project schemes. A do-it-yourself puzzle is available in the appendix to illustrate the results.

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Introduction

We build a biinfinite necklace by placing beads at integer positions on the real line:

\[ \cdots -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \cdots \]

Beads come in two colors: light red $\Diamond$ and dark blue $\bullet$. Given $\alpha > 0$, we would like to place the colored beads in such a way that the relative frequency

\[
\frac{\text{number of blue beads in } \{-n, -n+1, \ldots, n\}}{\text{number of red beads in } \{-n, -n+1, \ldots, n\}}
\]

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converges to $\alpha$ as $n$ goes to infinity.

A well-known approach is to use coding of rotations on a circle of circumference $\alpha + 1$ whose radius is $\frac{1}{2\pi}(\alpha + 1)$. The coding is given by the partition of the circle into one arc of length $\alpha$ associated to dark blue beads and another arc of length 1 associated to light red beads. The two end points of the arcs are associated to red and blue beads respectively in one way or the other. Then, we wrap the biinfinite necklace around the circle and each bead is given the color according to which of the two arcs it falls in. For example, when $\alpha = \frac{1+\sqrt{5}}{2}$ and if the zero position is assigned to one of the end points, we get the picture below:

Then, we unwrap the biinfinite necklace and we get an assignation of colored beads to each integer position such that the relative frequency between blue and red beads is $\alpha$. Here is what we get after zooming out a little:

We observe that this colored necklace has very few distinct patterns. The patterns of size 0, 1, 2 and 3 that we see in the necklace are shown in the table below:

| 0 | 1 | 2 | 3 |
|---|---|---|---|
|   |   |   |   |
|   |   |   |   |
|   |   |   |   |
|   |   |   |   |

We do not get other patterns of size 1, 2 or 3 in the whole biinfinite necklace since any pattern is uniquely determined by the position of its first bead on the circle. For each $n \in \mathbb{N}$ there exists a partition of the circle according to the pattern associated to the position of its first bead:

When $\alpha$ is irrational, we can prove that the partition of the circle for patterns of size $n$ is made of $n + 1$ parts. The proof follows from the fact that the distance between two consecutive beads on the necklace is equal to the length of one of the original arc (here, the red arc of length 1). So the partition at a given level is obtained from the previous one by adding exactly one separation which increases the number of patterns by 1. This shows that the colored necklace is a sturmian sequence, that is, a sequence whose pattern complexity is $n + 1$, see [Lot02]. When $\alpha = \frac{1+\sqrt{5}}{2}$, this is a construction of the biinfinite Fibonacci word [Ber80]. Note that it is known that sequences having strictly less than $n + 1$ patterns of length $n$, for some $n \in \mathbb{N}$, are eventually periodic [CN10, Thm 4.3.1]. Therefore, sturmian sequences are the most simple aperiodic sequences in terms of pattern (or factor) complexity.
What Coven and Hedlund proved in [CH73] based on the initial work of Morse and Hedlund [MH40] on sturmian sequences dating from 1940 is that a biinfinite sequence is sturmian if and only if it is the coding of an irrational rotation. Proving that the coding of an irrational rotation is a sturmian sequence is the easy part and corresponds to what we did above. The difficult part is to prove that a sturmian sequence can be obtained as the coding of an irrational rotation. The proof is explained nowadays in terms of $S$-adic development of sturmian sequences, Rauzy induction of rotations on the circle, the continued fraction expansion of real numbers and the Ostrowski numeration system, see [Fog02]. The fact that sturmian sequences are coding of rotations implies that they can be seen as model sets of cut and projects schemes, see [BMP05,BG13].

In this work, we want to consider the same question in a 2-dimensional setting. Suppose we have a 2-dimensional infinite necklace with colored beads to be placed at each integer coordinates:

Are there rules on the allowed coloring of 2-dimensional necklaces so that, like sturmian sequences, they are exactly obtained as the coding of rotations on a torus? While Berthé and Vuillon [BV00] considered the coding of two rotations on the 1-dimensional torus, we provide in this article an answer in terms of two rotations on the 2-dimensional torus.

Our contribution consists of two explicit examples, which naturally call for the search of others. We do as above for the 1-dimensional necklace and we use partitions of the 2-dimensional torus $\mathbb{T}^2$ and we construct colorings of $\mathbb{Z}^2$ obtained as the codings of orbits under $\mathbb{Z}^2$-rotations on the torus. We prove that such colorings of $\mathbb{Z}^2$ belongs to a specific class of colorings called subshift of finite type (SFT). More precisely we show that some codings of $\mathbb{Z}^2$-rotations on the torus are tilings of $\mathbb{Z}^2$ by tiles from a given set of Wang tiles. A Wang tile is a unit square tile which impose contraints on the neighboring tiles.

The main point of this article corresponds to the easy part of the proof illustrated above for sturmian sequences, namely that codings of irrational rotations have pattern complexity $n + 1$. Proving the converse, that is, that any tilings from a given Wang shift is obtained as the coding of a $\mathbb{Z}^2$-rotation needs more tools and space, for the same reason as proving that a sturmian sequence is a coding of an irrational rotation is more involved. This has lead to split the proof of the converse into more than one article.

In [Lab18a], a set $\mathcal{U}$ of 19 Wang tiles was introduced and its substitutive structure was described. It was proved that the associated Wang shift $\Omega_{\mathcal{U}}$ is self-similar, aperiodic and minimal. In the present article, we prove that a coloring of $\mathbb{Z}^2$ belongs to $\Omega_{\mathcal{U}}$ if and only if it is the symbolic representation of a irrational $\mathbb{Z}^2$-rotation on the 2-dimensional torus $\mathbb{T}^2$ with a partition onto polygons. Thus we provide a Markov partition for the Wang shift $\Omega_{\mathcal{U}}$. The fact that $\Omega_{\mathcal{U}}$ is minimal comes as a welcome shortcut in the proof. Note that Markov partitions remained abstract objects for a long time [Fog02, §7.1]. Explicit constructions of Markov partitions were originaly given for hyperbolic automorphisms of the torus, see [AW70,KV98,Ken99].

Our second example is related to the 11 Jeandel-Rao aperiodic Wang shift [JR15]. For some $\mathbb{Z}^2$-rotation on a 2-dimensional torus, we define another partition into polygons whose associated symbolic dynamical system is a minimal proper subshift of the Jeandel-Rao aperiodic Wang shift. Note that the substitutive structure of Jeandel-Rao aperiodic tilings was described in [Lab18b].
where it was proved, in particular, that the Jeandel-Rao Wang shift is not minimal. Thus, as opposed to $\Omega_\mathcal{U}$, we do not have a shortcut proof for the fact that we have a Markov partition for Jeandel-Rao tilings, but we believe it is one. This will be part of a forthcoming work.

We prove for both examples that the $\mathbb{Z}^2$-rotation on the torus is the maximal equicontinuous factor of the minimal subshifts. We prove that the two minimal subshifts are uniquely ergodic and are isomorphic as measure-preserving dynamical systems to the $\mathbb{Z}^2$-rotations on the torus. As a consequence, it provides a construction of Wang tilings as model sets of cut and project schemes. As opposed to Kari-Culik tilings for which a minimal subsystem is related to a dynamical system on $p$-adic numbers [Sie17], our window for the cut and project scheme is Euclidean. This can be compared with de Bruijn’s work describing Penrose tilings as model sets [dB81], the fact that Penrose tilings are coded by a Markov partition of the 4-dimensional torus [Rob96] and Lee and Moody’s description of Taylor-Socolar hexagonal tilings as model sets [LM13].

**Structure of the article.** This article is divided into two parts. In the first part, we construct Wang tile sets and Wang tilings as the coding of $\mathbb{Z}^2$-actions on the 2-dimensional torus. We illustrate the method on two examples including Jeandel-Rao aperiodic Wang tilings. In the second part, we express these Wang tilings in terms of model sets of cut and project schemes. In the appendix, we propose a do-it-yourself puzzle to explain the construction of Wang tilings as the coding of $\mathbb{Z}^2$-actions on the torus.

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**Part 1. Wang tilings as codings of $\mathbb{Z}^2$-actions**

In this part, we present a generic method for constructing Wang tile sets and Wang tilings as coding of $\mathbb{Z}^2$-action generated by two rotation on a torus. After introducing preliminary notions, we illustrate the method on two initial examples, and we show its application on a self-similar set of 19 Wang tiles and on Jeandel-Rao’s 11 tile set.

1. **Dynamical systems, subshifts and Wang tilings**

In this section, we introduce dynamical systems, subshifts, shifts of finite type, Wang tiles and Wang shifts. We denote by $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ the integers and by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the non-negative integers. If $d \geq 1$ is an integer and $1 \leq k \leq d$, we denote by $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ the vector of the canonical basis of $\mathbb{Z}^d$ with a 1 at position $k$ and 0 elsewhere.
1.1. Topological dynamical systems. Most of the notions introduced here can be found in [Wal82]. A dynamical system is a triple \((X, G, T)\), where \(X\) is a topological space, \(G\) is a topological group and \(T\) is a continuous function \(G \times X \to X\) defining a left action of \(G\) on \(X\): if \(x \in X\), \(e\) is the identity element of \(G\) and \(g, h \in G\), then using additive notation for the operation in \(G\) we have \(T(e, x) = x\) and \(T(g + h, x) = T(g, T(h, x))\). In other words, if one denotes the transformation \(x \mapsto T(g, x)\) by \(T^g\), then \(T^{g + h} = T^g T^h\). In this work, we consider the abelian group \(G = \mathbb{Z} \times \mathbb{Z}\).

If \(Y \subset X\), we denote the topological closure of \(Y\) as \(\overline{Y}\) and the \(T\)-closure of \(Y\) as \(\overline{Y}^T := \bigcup_{g \in G} T^g(Y)\). A subset \(Y \subset X\) is \(T\)-invariant if \(\overline{Y}^T = Y\). A dynamical system \((X, G, T)\) is called minimal if \(X\) does not contain any nonempty, proper, closed \(T\)-invariant subset. The left action of \(G\) on \(X\) is free if \(g = e\) whenever there exists \(x \in X\) such that \(T^g(x) = x\).

Let \((X, G, T)\) and \((Y, G, S)\) be two dynamical systems using the same topological group \(G\). A homomorphism \(\theta : (X, G, T) \to (Y, G, S)\) is a continuous function \(\theta : X \to Y\) satisfying the commuting property that \(T^g \circ \theta = \theta \circ S^g\) for every \(g \in G\). A homomorphism \(\theta : (X, G, T) \to (Y, G, S)\) is called an embedding if it is one-to-one, a factor map if it is onto, and a topological conjugacy if it is both one-to-one and onto and its inverse map is continuous. If \(\theta : (X, G, T) \to (Y, G, S)\) is a factor map, then \((Y, G, S)\) is called a factor of \((X, G, T)\) and \((X, G, T)\) is called an extension of \((Y, G, S)\). Two subshifts are topologically conjugate if there is a topological conjugacy between them. Also, a homomorphism \(\theta : (X, G, T) \to (Y, G, S)\) is topologically surjective if the range of \(\theta\) is dense in \(Y\), i.e., \(\overline{\theta(X)} = Y\), see [Rob07].

A factor map \(\theta : (X, G, T) \to (Y, G, S)\) is finite-to-one if there exists a constant \(K\) such that \(\text{card}(\theta^{-1}(y)) \leq K\) for every \(y \in Y\). Such a map is almost one-to-one if there is a point \(y \in Y\) such that \(\text{card}(\theta^{-1}(y)) = 1\). As in [Rob96], we define the thickness of the factor map by \(\eta = \sup\{\text{card}(\theta^{-1}(y)) : y \in Y\}\), and we call \(\{\text{card}(\theta^{-1}(y)) : y \in Y\}\) the thickness spectrum.

A metrizable dynamical system \((X, G, T)\) is called equicontinuous if the family of homeomorphisms \(\{T^g\}_{g \in G}\) is equicontinuous, i.e., if for all \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[d(T^g(x), T^g(y)) < \varepsilon\]

for all \(g \in G\) and all \(x, y \in X\) with \(d(x, y) < \delta\). According to a well-known theorem [ABKL15 Theorem 3.2], equicontinuous minimal systems are rotations on groups. We say that \((X_{\max}, G, T_{\max})\) is the maximal equicontinuous factor of \((X, G, T)\) if \((X_{\max}, G, T_{\max})\) is a equicontinuous factor of \((X, G, T)\) and whenever \((Y, G, S)\) is an equicontinuous factor of \((X, G, T)\) then \((Y, G, S)\) is a factor of \((X_{\max}, G, T_{\max})\). The maximal equicontinuous factor is unique [ABKL15 Theorem 3.8].

A measure-preserving dynamical system is defined as a system \((X, G, T, \mu, \mathcal{B})\), where \(\mu\) is a probability measure defined on the \(\sigma\)-algebra \(\mathcal{B}\) of subsets of \(X\), and \(T^g : X \to X\) is a measurable map which preserves the measure \(\mu\) for all \(g \in G\), that is, \(\mu(T^g(B)) = \mu(B)\) for all \(B \in \mathcal{B}\). The measure \(\mu\) is said to be \(T\)-invariant.

The set of all \(T\)-invariant probability measures of a dynamical system \((X, G, T)\) is denoted \(\mathcal{M}^T(X)\). An invariant probability measure on \(X\) is said ergodic if for every set \(B \in \mathcal{B}\), \(T^g(B) = B\) with \(g \in G \setminus \{e\}\) implies that \(B\) has either zero of full measure. A dynamical system \((X, G, T)\) is uniquely ergodic if it has only invariant probability measure, i.e., \(|\mathcal{M}^T(X)| = 1\). A dynamical system \((X, G, T)\) is said strictly ergodic if it is uniquely ergodic and minimal.

Let \((X, T, \mu, \mathcal{B})\) and \((X', T', \mu', \mathcal{B}')\) be two measure-preserving dynamical systems. We say that the two systems are isomorphic if there exists measurable sets \(X_0 \subset X\) and \(X'_0 \subset X'\) of full measure (i.e., \(\mu(X \setminus X_0) = 0\) and \(\mu'(X' \setminus X'_0) = 0\)) with \(T(X_0) \subset X_0\), \(T'(X'_0) \subset X'_0\), and there exists a map \(\phi : X_0 \to X'_0\), called an isomorphism, that is one-to-one and onto and such that for all \(A \in \mathcal{B}'(X'_0)\),

\[\phi^{-1}(A) \in \mathcal{B}(X_0),\]
• \( \mu(\phi^{-1}(A)) = \mu'(A) \), and
• \( \phi \circ T(x) = T' \circ \phi(x) \) for all \( x \in X_0 \).

The role of the set \( X_0 \) is to make precise the fact that the properties of the isomorphism need to hold only on a set of full measure.

1.2. Subshifts and shifts of finite type. We follow the notations of [Sch01]. Let \( \mathcal{A} \) be a finite set, \( d \geq 1 \), and let \( \mathcal{A}^{\mathbb{Z}^d} \) be the set of all maps \( x : \mathbb{Z}^d \to \mathcal{A} \), furnished with the compact product topology. We write a typical point \( x \in \mathcal{A}^{\mathbb{Z}^d} \) as \( x = (x_m) = (x_m : m \in \mathbb{Z}^d) \), where \( x_m \in \mathcal{A} \) denotes the value of \( x \) at \( m \). The topology \( \mathcal{A}^{\mathbb{Z}^d} \) is compatible with the metric \( \delta \) defined for all colorings \( x, x' \in \mathcal{A}^{\mathbb{Z}^d} \) by \( \delta(x, x') = 2^{-\min\{|m| : x_m \neq x'_m\}} \). The shift action \( \sigma : n \mapsto \sigma^n \) of \( \mathbb{Z}^d \) on \( \mathcal{A}^{\mathbb{Z}^d} \) is defined by

\[
(\sigma^n(x))_m = x_{m+n}
\]

for every \( x = (x_m) \in \mathcal{A}^{\mathbb{Z}^d} \) and \( n \in \mathbb{Z}^d \). If \( X \subset \mathcal{A}^{\mathbb{Z}^d} \), we denote the topological closure of \( X \) as \( \overline{X} \) and the shift-closure of \( X \) as \( \sigma(X) = \{\sigma^n(x) \mid x \in X, n \in \mathbb{Z}^d\} \). A subset \( X \subset \mathcal{A}^{\mathbb{Z}^d} \) is shift-invariant if \( \sigma(X) = X \) and a closed, shift-invariant subset \( X \subset \mathcal{A}^{\mathbb{Z}^d} \) is a subshift. If \( X \subset \mathcal{A}^{\mathbb{Z}^d} \) is a subshift we write \( \sigma = \sigma^X \) for the restriction of the shift-action (1) to \( X \). If \( X \subset \mathcal{A}^{\mathbb{Z}^d} \) is a subshift it will sometimes be helpful to specify the shift-action of \( \mathbb{Z}^d \) explicitly and to write \( (X, \sigma) \) instead of \( X \). When \( X \) is a subshift, the triple \( (X, \mathbb{Z}^d, \sigma) \) is a dynamical system and the notions presented in the previous section hold. We say that a nonempty subshift \( X \) is aperiodic if the shift-action \( \sigma \) on \( X \) is free.

For any subset \( S \subset \mathbb{Z}^d \) we denote by \( \pi_S : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^S \) the projection map which restricts every \( x \in \mathcal{A}^{\mathbb{Z}^d} \) to \( S \). A pattern is a function \( p : S \to \mathcal{A} \) for some finite subset \( S \subset \mathbb{Z}^d \). To every pattern \( p : S \to \mathcal{A} \) correspond a subset \( \pi_S^{-1}(p) \subset \mathcal{A}^{\mathbb{Z}^d} \) called cylinder. A subshift \( X \subset \mathcal{A}^{\mathbb{Z}^d} \) is a shift of finite type (SFT) if there exists a finite set \( \mathcal{F} \) of forbidden patterns such that

\[
X = \{ x \in \mathcal{A}^{\mathbb{Z}^d} \mid \pi_S \cdot \sigma^n(x) \notin \mathcal{F} \text{ for every } n \in \mathbb{Z}^d \text{ and } S \subset \mathbb{Z}^d \}.
\]

In this case, we write \( X = \text{SFT}(\mathcal{F}) \). In this contribution, we consider tilings of \( \mathbb{Z} \times \mathbb{Z} \), that is, the case \( d = 2 \).

1.3. Wang shifts. A Wang tile \( \tau = \begin{array}{c|c}
\text{a} & \text{b} \\
\hline
\text{c} & \text{d}
\end{array} \) is a unit square with colored edges formally represented as a tuple of four colors \( (a, b, c, d) \in I \times J \times I \times J \) where \( I, J \) are two finite sets (the vertical and horizontal colors respectively). For each Wang tile \( \tau = (a, b, c, d) \), we denote by \( \text{RIGHT}(\tau) = a, \text{TOP}(\tau) = b, \text{LEFT}(\tau) = c, \text{BOTTOM}(\tau) = d \) the colors of the right, top, left and bottom edges of \( \tau \) [Wan61,Rob71].

Let \( \mathcal{T} \) be a set of Wang tiles. A tiling of \( \mathbb{Z}^2 \) by \( \mathcal{T} \) is a function \( x : \mathbb{Z}^2 \to \mathcal{T} \) which assigns tiles to each position of \( \mathbb{Z}^2 \) so that contiguous edges have the same color, that is,

\[
\text{RIGHT}(x_n) = \text{LEFT}(x_{n+e_1})
\]
\[
\text{TOP}(x_n) = \text{BOTTOM}(x_{n+e_2})
\]

for every \( n \in \mathbb{Z}^2 \). We denote by \( \Omega_\mathcal{T} \subset \mathcal{T}^{\mathbb{Z}^2} \) the set of all Wang tilings of \( \mathbb{Z}^2 \) by \( \mathcal{T} \) and we call it the Wang shift of \( \mathcal{T} \). Together with the shift action \( \sigma \) of \( \mathbb{Z}^2 \) on \( \mathcal{T}^{\mathbb{Z}^2} \), it is a SFT of the form (2).

A set of Wang tiles \( \mathcal{T} \) tiles the plane if \( \Omega_\mathcal{T} \neq \emptyset \) and does not tile the plane if \( \Omega_\mathcal{T} = \emptyset \). A tiling \( x \in \Omega_\mathcal{T} \) is periodic if there is a nonzero period \( n \in \mathbb{Z}^2 \setminus \{(0,0)\} \) such that \( x = \sigma^n(x) \) and otherwise it is said nonperiodic. A set of Wang tiles \( \mathcal{T} \) is periodic if there is a tiling \( x \in \Omega_\mathcal{T} \) which is periodic. A Wang tile set \( \mathcal{T} \) is aperiodic if the subshift \( \Omega_\mathcal{T} \neq \emptyset \) is aperiodic.
2. Symbolic representation of $\mathbb{Z}^2$-actions

This section follows the section of the same name [LM95 §6.5] where we adapt it to the case of invertible $\mathbb{Z}^2$-actions. A \textbf{topological partition} of a metric space $M$ is a finite collection \{\(P_0, P_1, \ldots, P_{r-1}\)\} of disjoint open sets whose closures \(\overline{P}_j\) together cover $M$ in the sense that $M = \overline{P}_0 \cup \overline{P}_1 \cup \cdots \cup \overline{P}_{r-1}$.

Suppose that $M$ is a compact metric space, $(M, \mathbb{Z}^2, R)$ is a invertible dynamical system and that $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$ is a topological partition of $M$. Let $\mathcal{A} = \{0, 1, \ldots, r-1\}$ and $S \subset \mathbb{Z}^2$ be a finite set. We say that a pattern $w : S \to \mathcal{A}$ is \textbf{allowed} for $\mathcal{P}, R$ if

\[
\bigcap_{k \in S} R^{-k}(P_{w_k}) \neq \emptyset.
\]

Let $\mathcal{L}_{\mathcal{P}, R}$ be the collection of all allowed patterns for $\mathcal{P}, R$. It can be checked that $\mathcal{L}_{\mathcal{P}, R}$ is the language of a subshift. Hence, using standard arguments [LM95 Prop. 1.3.4], there is a unique subshift $\mathcal{X}_{\mathcal{P}, R} \subset \mathcal{A}^{\mathbb{Z}^2}$ whose language is $\mathcal{L}_{\mathcal{P}, R}$. We call $\mathcal{X}_{\mathcal{P}, R}$ the \textbf{symbolic representation of $\mathcal{P}, R$} corresponding to $\mathcal{P}, R$. For each $w \in \mathcal{X}_{\mathcal{P}, R} \subset \mathcal{A}^{\mathbb{Z}^2}$ and $n \geq 0$ there is a corresponding nonempty open set

\[
D_n(w) = \bigcap_{\|k\| \leq n} R^{-k}(P_{w_k}) \subseteq M.
\]

The closures $\overline{D}_n(w)$ of these sets are compact and decrease with $n$, so that $\overline{D}_0(w) \supseteq \overline{D}_1(w) \supseteq \overline{D}_2(w) \supseteq \ldots$. It follows that $\bigcap_{n=0}^{\infty} \overline{D}_n(w) \neq \emptyset$. In order for points in $\mathcal{X}_{\mathcal{P}, R}$ to correspond to points in $M$, this intersection should contain only one point. This leads to the following definition.

A topological partition $\mathcal{P}$ of $M$ gives a \textbf{symbolic representation} of $(M, \mathbb{Z}^2, R)$ if for every $w \in \mathcal{X}_{\mathcal{P}, R}$ the intersection $\bigcap_{n=0}^{\infty} \overline{D}_n(w)$ consists of exactly one point $m \in M$. We call $w$ a \textbf{symbolic representation of $m$}. We call $\mathcal{P}$ a \textbf{Markov partition} for $(M, \mathbb{Z}^2, R)$ if $\mathcal{P}$ gives a symbolic representation of $(M, \mathbb{Z}^2, R)$ and furthermore $\mathcal{X}_{\mathcal{P}, R}$ is a shift of finite type.

2.1. Symbolic representation of $\mathbb{Z}^2$-rotations on the torus. From now on, we assume that the compact metric space $M$ is a 2-torus. Let $\Gamma$ be a \textbf{lattice} in $\mathbb{R}^2$, i.e., a discrete subgroup of the additive group $\mathbb{R}^2$ such that the quotient space has finite invariant measure. This defines a 2-torus $T = \mathbb{R}^2/\Gamma$. For some $\alpha, \beta \in T$, we consider the dynamical system $(T, \mathbb{Z}^2, R)$ where $R : \mathbb{Z}^2 \times T \to T$ is the action defined by

\[
R^n(x) := R(n, x) = x + n_1 \alpha + n_2 \beta
\]

for every $n = (n_1, n_2) \in \mathbb{Z}^2$. The map $R$ is a continuous $\mathbb{Z}^2$-action on $T$.

\[\textbf{Lemma 1.} \text{ Let } (T, \mathbb{Z}^2, R) \text{ be a minimal dynamical system and } \mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\} \text{ be a topological partition of } T. \text{ If there exists an atom } P_i \text{ which is invariant only under the trivial translation in } T, \text{ then } \mathcal{P} \text{ gives a symbolic representation of } (T, \mathbb{Z}^2, R). \]

\[\textbf{Proof.} \text{ Let } \mathcal{A} = \{0, 1, \ldots, r-1\}. \text{ Let } w \in \mathcal{X}_{\mathcal{P}, R} \subset \mathcal{A}^{\mathbb{Z}^2}. \text{ As already noticed, the closures } \overline{D}_n(w) \text{ are compact and decrease with } n, \text{ so that } \overline{D}_0(w) \supseteq \overline{D}_1(w) \supseteq \overline{D}_2(w) \supseteq \ldots. \text{ It follows that } \bigcap_{n=0}^{\infty} \overline{D}_n(w) \neq \emptyset.
\]

We show that $\bigcap_{n=0}^{\infty} \overline{D}_n(w)$ contains at most one element. Let $x, y \in T$. We assume $x \in \bigcap_{n=0}^{\infty} \overline{D}_n(w)$ and we want to show that $y \notin \bigcap_{n=0}^{\infty} \overline{D}_n(w)$ if $x \neq y$. Let $P = P_i \subset T$ for some $i \in \mathcal{A}$ be the atom which is invariant only under the trivial translation. Since $x \neq y$, $P \setminus (P - (y - x))$ contains an open set $O$. Since $(T, \mathbb{Z}^2, R)$ is minimal, any orbit $\{R^k x \mid k \in \mathbb{Z}^2\}$ is dense in $T^2$. Therefore, there exists $k \in \mathbb{Z}^2$ such that $R^k x \in O$. Then $R^k x \in \hat{P}$ and $x \in R^{-k}(\hat{P})$. Also $x \in \bigcap_{n=0}^{\infty} \overline{D}_n(w) \subset R^{-k}P_{w_k}$ which implies that $w_k = i$. Thus

\[
\bigcap_{n=0}^{\infty} \overline{D}_n(w) \subset R^{-k}P_{w_k} = R^{-k}P_{i} = R^{-k}P.
\]
The fact that $R^k x \in O$ also means that $R^k x \notin \mathcal{P} - (y - x)$ which can be rewritten as $R^k y \notin \mathcal{P}$ or $y \notin R^{-k} \mathcal{P}$ and we conclude that $y \notin \cap_{n=0}^{\infty} \mathcal{D}_n(w)$. Thus $\mathcal{P}$ gives a symbolic representation of $(T, \mathbb{Z}^2, R)$.

\[ \square \]

Remark 2. Note that minimality hypothesis in Lemma 1 is not a necessity. For example, the partition $\mathcal{Q}$ of the torus $\mathbb{T}^2$ gives a symbolic representation of the $\mathbb{Z}^2$-action defined by $R(n, x) = x + n_1(\sqrt{2}, 0) + n_2(\sqrt{3}, 0)$ even if $(T, \mathbb{Z}^2, R)$ is not minimal.

The set
\[ \Delta_{\mathcal{P}, R} := \bigcup_{n \in \mathbb{Z}^2} R^n \left( \bigcup_{a \in \mathcal{A}} \partial P_a \right) \subset T \]
is the set of points whose orbit under the $\mathbb{Z}^2$-action of $R$ intersect the boundary of the topological partition $\mathcal{P} = \{ P_a \}_{a \in \mathcal{A}}$. From Baire Category Theorem [LM95, Theorem 6.1.24], the set $T \setminus \Delta_{\mathcal{P}, R}$ is dense in $T$.

For every starting point $x \in T \setminus \Delta_{\mathcal{P}, R}$, the coding of its orbit under the $\mathbb{Z}^2$-action of $R$ is a well-defined 2-dimensional word:
\[ \text{ORBIT}_{x}^{\mathcal{P}, R} : \mathbb{Z} \times \mathbb{Z} \to A \]
\[ n \mapsto a \text{ if and only if } R^n(x) \in P_a. \]

Thus it defines a map
\[ \text{TILING} : T \setminus \Delta_{\mathcal{P}, R} \to A^{\mathbb{Z}^2} \]
\[ x \mapsto \text{ORBIT}_{x}^{\mathcal{P}, R}. \]

The map TILING can not be extended continously on $T$. Up to some choice to be made, it can still be extended to the whole domain $T$. Recall that for interval exchange transformations, one way to deal with this issue is to consider two copies $x^-$ and $x^+$ for each discontinuity point [Kea75].

Here we use this idea in order to extend TILING on the whole domain $T$ by approching any point from a chosen direction. Not all directions work, so we need some care to formalize this properly. Let $\Theta^p$ with $\{ 0 \} \subseteq \Theta^p \subset \mathbb{R}^2$ be the set of vectors parallel to a segment included in the boundary of some atom $P_a \in \mathcal{P}$. If all atoms have curved boundaries, then $\Theta^p = \{ 0 \}$. If the atoms are polygons like in this contribution, then the set $\Theta^p$ contains nonzero directions. In any case, we assume that $\mathbb{R} \Theta^p = \Theta^p$. For every $v \in \mathbb{R}^2 \setminus \Theta^p$ we define
\[ \text{TILING}^v : T \to A^{\mathbb{Z}^2} \]
\[ x \mapsto \lim_{\epsilon \to 0} \text{TILING}(x + \epsilon v). \]

We say that $\text{TILING}(x) = \text{ORBIT}_{x}^{\mathcal{P}, R}$ is a generic tiling if $x \in T \setminus \Delta_{\mathcal{P}, R}$ and that $\text{TILING}^v(x)$ is a singular tiling if $x \in \Delta_{\mathcal{P}, R}$ for some $v \in \mathbb{R}^2 \setminus \Theta^p$. The choice of direction $v$ is not so important since the topological closure of the range of $\text{TILING}^v$ does not depend on $v$ as shown in the next lemma. In other words, singular tilings are limits of generic tilings.

Lemma 3. For every $v \in \mathbb{R}^2 \setminus \Theta^p$, the following equality holds
\[ \text{TILING}^v(T) = \text{TILING}(T \setminus \Delta_{\mathcal{P}, R}) = \mathcal{X}_{\mathcal{P}, R}, \]
where $\mathcal{X}_{\mathcal{P}, R}$ is the symbolic dynamical system corresponding to $\mathcal{P}, R$.

Proof. If $x \in T \setminus \Delta_{\mathcal{P}, R}$, then $\text{TILING}(x) = \text{TILING}^v(x)$. Thus $\text{TILING}(T \setminus \Delta_{\mathcal{P}, R}) = \text{TILING}^v(T \setminus \Delta_{\mathcal{P}, R})$. ($\subseteq$) Then $\text{TILING}(T \setminus \Delta_{\mathcal{P}, R}) = \text{TILING}^v(T \setminus \Delta_{\mathcal{P}, R}) \subseteq \text{TILING}^v(T)$.

($\supseteq$) Let $w \in \text{TILING}^v(\Delta_{\mathcal{P}, R})$. Then $w = \lim_{\epsilon \to 0} \text{TILING}(x + \epsilon v)$ for some $x \in \Delta_{\mathcal{P}, R}$. We may extract a subsequence $(\text{TILING}(x + \epsilon_n v))_{n \in \mathbb{N}}$ with $\epsilon_n \in \mathbb{R}$ such that $x + \epsilon_n v \in T \setminus \Delta_{\mathcal{P}, R}$ for all
\[ n \in \mathbb{N} \]

This implies that \( w \in \overline{\text{TILING}(T \setminus \Delta_{\mathcal{P},R})} \). Therefore \( \text{TILING}^v(\Delta_{\mathcal{P},R}) \subseteq \overline{\text{TILING}(T \setminus \Delta_{\mathcal{P},R})} \).

We obtain

\[
\text{TILING}^v(T) = \text{TILING}^v(T \setminus \Delta_{\mathcal{P},R}) \cup \text{TILING}^v(\Delta_{\mathcal{P},R}) \subseteq \text{TILING}(T \setminus \Delta_{\mathcal{P},R})
\]

which proves the first equality.

Recall that the collection of all allowed patterns for \( \mathcal{P}, R \) is the language \( \mathcal{L}_{\mathcal{P},R} \). The set \( \text{TILING}(T \setminus \Delta_{\mathcal{P},R}) \) is a subshift and contains \( \mathcal{L}_{\mathcal{P},R} \). Moreover the language of \( \overline{\text{TILING}(T \setminus \Delta_{\mathcal{P},R})} \) is contained in \( \mathcal{L}_{\mathcal{P},R} \). The conclusion follows since the symbolic dynamical system \( \mathcal{X}_{\mathcal{P},R} \) is the unique subshift whose language if \( \mathcal{L}_{\mathcal{P},R} \).

**Lemma 4.** Let \( \mathcal{P} \) gives a symbolic representation of the dynamical system \( (T, \mathbb{Z}^2, R) \) and let \( v \in \mathbb{R}^2 \setminus \Theta^\mathcal{P} \). Then \( \text{TILING} : T \setminus \Delta_{\mathcal{P},R} \to \mathcal{X}_{\mathcal{P},R} \) and \( \text{TILING}^v : T \to \mathcal{X}_{\mathcal{P},R} \) are one-to-one. Moreover, the following diagrams commute:

\[
\begin{array}{ccc}
T \setminus \Delta_{\mathcal{P},R} & \xrightarrow{R^k} & T \setminus \Delta_{\mathcal{P},R} \\
\text{TILING} & & \text{TILING} \\
\mathcal{X}_{\mathcal{P},R} & \xrightarrow{\sigma^k} & \mathcal{X}_{\mathcal{P},R}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
T & \xrightarrow{R^k} & T \\
\text{TILING} & & \text{TILING}^v \\
\mathcal{X}_{\mathcal{P},R} & \xrightarrow{\sigma^k} & \mathcal{X}_{\mathcal{P},R}
\end{array}
\]

for every \( k \in \mathbb{Z}^2 \).

**Proof.** The fact that \( w = \text{TILING}^v(x) \) implies that \( x \in \cap_{n=0}^{\infty} \overline{D_n}(w) \). Therefore, if \( \text{TILING}^v(x) = \text{TILING}^v(y) = w \) then \( x, y \in \cap_{n=0}^{\infty} \overline{D_n}(w) \). Since \( \mathcal{P} \) gives a symbolic representation of the dynamical system \( (T, \mathbb{Z}^2, R) \), the set \( \cap_{n=0}^{\infty} \overline{D_n}(w) \) contains at most one element, and it implies that \( x = y \). Thus, \( \text{TILING}^v \) is one-to-one. As \( \text{TILING}^v \) and \( \text{TILING} \) agree on \( T \setminus \Delta_{\mathcal{P},R} \), we also have that \( \text{TILING} \) is one-to-one.

We now show conjugacy of \( \mathbb{Z}^2 \)-actions. Let \( k \in \mathbb{Z}^2 \), \( x \in T \setminus \Delta_{\mathcal{P},R} \) and \( n \in \mathbb{Z}^2 \). We have

\[
(\sigma^k \circ \text{TILING}(x))(n) = (\sigma^k \circ \text{ORBIT}_{x}^{\mathcal{P},R})(n) = \text{ORBIT}_{x}^{\mathcal{P},R}(n + k) = \text{ORBIT}_{R^k x}^{\mathcal{P},R}(n) = (\text{TILING}(R^k x))(n) = (\text{TILING} \circ R^k(x))(n).
\]

Therefore \( \sigma^k \circ \text{TILING} = \text{TILING} \circ R^k \). The conjugacy of \( \mathbb{Z}^2 \)-actions extends to \( \text{TILING}^v \). \( \square \)

The fact that \( \text{TILING}^v \) is one-to-one means that it admits a left-inverse map \( f : \text{TILING}^v(T^2) \to T^2 \) such that \( f \circ \text{TILING}^v = \text{Id}_{T^2} \). But we can say more and define the map \( f \) on the closure \( \overline{\text{TILING}^v(T^2)} = \mathcal{X}_{\mathcal{P},R} \). Indeed, if \( \mathcal{P} \) gives a symbolic representation of the invertible dynamical system \( (T, \mathbb{Z}^2, R) \), then there is a well-defined mapping \( f \) from \( \mathcal{X}_{\mathcal{P},R} \) to \( T \) which maps a point \( w \in \mathcal{X}_{\mathcal{P},R} \subseteq \mathcal{A}^{\mathbb{Z}^2} \) to the unique point \( f(w) \in T \) in the intersection \( \cap_{n=0}^{\infty} \overline{D_n}(w) \). In the spirit of [LM95, Prop. 6.5.8], the following result shows that \( f \) is a continuous and onto homomorphism and therefore a factor map from \( (\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma) \) to \( (T, \mathbb{Z}^2, R) \).

**Proposition 5.** Let \( \mathcal{P} \) gives a symbolic representation of the dynamical system \( (T, \mathbb{Z}^2, R) \). The map \( f : \mathcal{X}_{\mathcal{P},R} \to T \) defined above is an almost one-to-one factor map from \( (\mathcal{X}_{\mathcal{P},R}, \mathbb{Z}^2, \sigma) \) to \( (T, \mathbb{Z}^2, R) \) which makes the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{X}_{\mathcal{P},R} & \xrightarrow{\sigma^k} & \mathcal{X}_{\mathcal{P},R} \\
\downarrow f & & \downarrow f \\
T & \xrightarrow{R^k} & T
\end{array}
\]

for every \( k \in \mathbb{Z}^2 \).
Proof. Let \( \mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\} \) and \( X = \mathcal{X}_{\mathcal{P}, R} \). We show that the map \( f \) is continuous. Let \( \varepsilon > 0 \). Let \( w \in X \). Since the partition is a symbolic representation, there exists \( n \in \mathbb{N} \) such that the diameter of

\[
\bar{D}_n(w) = \bigcap_{|k| \leq n} R^{-k} \left( \overline{P_{w_k}} \right)
\]

is smaller or equal to \( \varepsilon \). That set contains \( \left( \bigcap_{n=0}^{\infty} \overline{D}_n(w) \right) \cup \left( \bigcap_{n=0}^{\infty} \overline{D}_n(w') \right) \) if \( w' \in X \) is such that \( d_X(w, w') < \frac{1}{2\varepsilon} \). We conclude that if \( d_X(w, w') < \frac{1}{2\varepsilon} \), then \( d_T(f(w), f(w')) < \varepsilon \) which means that \( f \) is continuous.

We show that the map \( f \) is onto. Let \( x \in T \setminus \Delta_{\mathcal{P}, R} \) and \( w = \text{TLING}(x) \). Then \( x \in \cap_{n=0}^{\infty} \overline{D}_n(w) \). Since \( \mathcal{P} \) gives a symbolic representation of \((T, \mathbb{Z}^2, R)\), we have that

\[
\{x\} = \bigcap_{n=0}^{\infty} \overline{D}_n(w) = \bigcap_{n=0}^{\infty} D_n(w),
\]

so that \( f(w) = x \). Thus the image of \( f \) contains the dense set \( T \setminus \Delta_{\mathcal{P}, R} \). Since the image of a compact set via a continuous map is compact and therefore closed, it follows that the image of \( f \) is all of \( T \).

An alternate proof that \( f \) is onto uses \( \text{TLING}^v \). Let \( x \in T \) and \( w = \text{TLING}^v(x) \) for some \( v \in \mathbb{R}^2 \setminus \Theta^P \). We have that \( \cap_{n=0}^{\infty} \overline{D}_n(w) = \{x\} \). Therefore, \( f(w) = x \) and \( f \) is onto.

We show that the map \( f \) is a homomorphism:

\[
R^k\{f(w)\} = R^k \left( \bigcap_{n=0}^{\infty} \overline{D}_n(w) \right) = R^k \bigcap_{n \in \mathbb{Z}^2} R^{-n} \overline{P_{w_n}} = \bigcap_{n \in \mathbb{Z}^2} \overline{R^{-n}P_{\sigma^k w_{n-k}}} = \bigcap_{m \in \mathbb{Z}^2} \overline{R^{-m}P_{\sigma^k w_m}} = \bigcap_{n=0}^{\infty} \overline{D}_n(\sigma^k w) = \{f(\sigma^k w)\}
\]

where \( m = n - k \). Therefore \( R^k \circ f = f \circ \sigma^k \) for every \( k \in \mathbb{Z}^2 \) and \( f : \mathcal{X}_{\mathcal{P}, R} \to T \) is a factor map.

We show that \( f \) is one-to-one on \( f^{-1}(T \setminus \Delta_{\mathcal{P}, R}) \). Let \( x \in T \setminus \Delta_{\mathcal{P}, R} \) and suppose that \( w, w' \in f^{-1}(x) \). This means that \( \cap_{n=0}^{\infty} \overline{D}_n(w) = \cap_{n=0}^{\infty} \overline{D}_n(w') = \{x\} \). Therefore for every \( n \in \mathbb{Z}^2 \) we have

\[
x \in \left( R^{-n} \overline{P_{w_n}} \right)^\circ \cap \left( R^{-n} \overline{P_{w'_n}} \right)^\circ.
\]

Then \( w_n = w'_n \) for every \( n \in \mathbb{Z}^2 \) and \( w = w' \). Therefore for every \( x \in T \setminus \Delta_{\mathcal{P}, R} \), \( f^{-1}(x) \) contains exactly one element. This shows that the map \( f \) is almost one-to-one. \( \Box \)

**Corollary 6.** If \( \mathcal{P} \) gives a symbolic representation of the dynamical system \((T, \mathbb{Z}^2, R)\), then \((T, \mathbb{Z}^2, R)\) is the maximal equicontinuous factor of \((\mathcal{X}_{\mathcal{P}, R}, \mathbb{Z}^2, \sigma)\).

**Proof.** The dynamical system \((T, \mathbb{Z}^2, R)\) is equicontinuous. We proved in Proposition 5 that the factor map \( f \) is almost one-to-one. Therefore, \((T, \mathbb{Z}^2, R)\) is the maximal equicontinuous factor of \((\mathcal{X}_{\mathcal{P}, R}, \mathbb{Z}^2, \sigma)\) \cite{ABKL15} Lemma 3.11]. \( \Box \)

As mentioned in Remark 2, it is possible that \( \mathcal{X}_{\mathcal{P}, R} \) is not minimal. But as shown in the next Lemma it is minimal if \( R \) is minimal.

**Lemma 7.** Let \( \mathcal{P} \) gives a symbolic representation of the dynamical system \((T, \mathbb{Z}^2, R)\). Then

(i) if \((T, \mathbb{Z}^2, R)\) is minimal, then \((\mathcal{X}_{\mathcal{P}, R}, \mathbb{Z}^2, \sigma)\) is minimal,

(ii) if \( R \) is a free \( \mathbb{Z}^2 \)-action on \( T \), then \( \mathcal{X}_{\mathcal{P}, R} \) aperiodic.

**Proof.** Let \( f : \mathcal{X}_{\mathcal{P}, R} \to T \) be the almost one-to-one factor map from Proposition 5.

(i) Let \( Y \subseteq \mathcal{X}_{\mathcal{P}, R} \) be a nonempty subshift. Thus \( Y \) is compact. Continuous map preserve compact sets, thus \( f(Y) \) is compact. The set \( f(Y) \) is also \( R \)-invariant since \( R^k f(Y) = f(\sigma^k Y) = f(Y) \) for
every \( k \in \mathbb{Z}^2 \). Since \((T,\mathbb{Z}^2,R)\) is minimal, the only nonempty compact subset of \( T \) which is invariant under \( R \) is \( T \). Thus \( f(Y) = T \).

For every \( x \in T \), \( f^{-1}(x) \cap Y \neq \emptyset \). Then \( Y \) contains \( \text{TILING}(x) \) for every \( x \in T \) such that \( f^{-1}(x) \) is a singleton. Then \( Y \) contains \( \text{TILING}(T \setminus \Delta_{P,R}) \). Since \( Y \) is closed, it must contain \( \text{TILING}(T \setminus \Delta_{P,R}) \) so that \( Y = \mathcal{X}_{P,R} \). Thus \( \mathcal{X}_{P,R} \) is minimal.

(ii) Suppose that there exists \( w \in \mathcal{X}_{P,R} \) such that \( w \) is periodic, i.e. there exists \( k \in \mathbb{Z}^2 \) such that \( \sigma^k w = w \). Since \( f \) commutes the \( \mathbb{Z}^2 \)-actions, we obtain

\[
R^k f(w) = f(\sigma^k w) = f(w).
\]

Since we assume that \( R \) is a free \( \mathbb{Z}^2 \)-action, it implies that \( k = 0 \). Thus \( \mathcal{X}_{P,R} \) is aperiodic. \( \square \)

Let \( X \subset \mathcal{A}^{\mathbb{Z}^2} \) be a subshift. Recall that for any subset \( S \subset \mathbb{Z}^2 \) we denote by \( \pi_S : X \to \mathcal{A}^S \) the projection map which restricts every \( w \in X \) to \( S \). To every finite pattern \( p : S \to \mathcal{A} \) correspond a cylinder \( [p] = \pi_S^{-1}(p) \subset X \). The set of all cylinders

\[
\{ [p] \mid p \in \mathcal{A}^S \text{ with } S \subset \mathbb{Z}^2 \text{ finite} \}
\]

generates the Borel \( \sigma \)-algebra on \( X \).

Let \( \mathcal{P} = \{ P_a \}_{a \in \mathcal{A}} \) gives a symbolic representation of the dynamical system \((T,\mathbb{Z}^2,R)\) and let \( f : \mathcal{X}_{P,R} \to T \) be the almost one-to-one factor map from Proposition 5. For each \( a \in \mathcal{A} \), we have that

\[
f([a]) = \overline{P_a} \subset T
\]
is a closed set. Thus the image of a cylinder \([p]\) under \( f \) for some finite pattern \( p \in \mathcal{A}^S \) is a closed set called \textit{coding region} for the pattern \( p \) being the finite intersection of closed sets:

\[
f([p]) = \bigcap_{n \in \mathbb{Z}} R^{-n} \overline{P_{p_n}} \subset T.
\]

**Lemma 8.** Let \( \mathcal{P} \) gives a symbolic representation of a minimal dynamical system \((T,\mathbb{Z}^2,R)\). Suppose that \( \lambda(\partial P) = 0 \) for each atom \( P \in \mathcal{P} \) where \( \lambda \) is the Haar measure on \( T \). Then the dynamical system \((\mathcal{X}_{P,R},\mathbb{Z}^2,\sigma)\) is strictly ergodic and the measure-preserving dynamical system \((\mathcal{X}_{P,R},\mathbb{Z}^2,\sigma,\nu)\) is isomorphic to the toral \( \mathbb{Z}^2 \)-rotation \((T,\mathbb{Z}^2,R,\lambda)\) where \( \nu \) is the unique shift-invariant probability measure on \( \mathcal{X}_{P,R} \).

**Proof.** We prove that the factor map \( f : \mathcal{X}_{P,R} \to T \) from Proposition 5 provides the isomorphism. The map \( f \) is measurable as \( f \) is continuous and \( f^{-1}(K) \) is compact for any compact subset \( K \subset T \). Let \( \lambda \) be the Haar measure on \( T \). By hypothesis, \((T,\mathbb{Z}^2,R)\) is minimal. It is also strictly ergodic [Wal82] with \( \lambda \) being the only \( R \)-invariant probability measure on \( T \).

Since \( \sigma \) is continuous and \( \mathcal{X}_{P,R} \) is a compact metric space, the set \( \mathcal{M}^\sigma(\mathcal{X}_{P,R}) \) of \( \sigma \)-invariant probability measure on \( \mathcal{X}_{P,R} \) is nonempty following a result of Krylov and Bogolioubov [Wal82 Cor. 6.9.1]. Thus let \( \nu \in \mathcal{M}^\sigma(\mathcal{X}_{P,R}) \). Let \( Z = [p] \subset \mathcal{X}_{P,R} \) be the cylinder corresponding to some pattern \( p \in \mathcal{A}^S \) for some finite subset \( S \subset \mathbb{Z}^2 \). From Equation (6) we know that \( f(Z) \) is a closed set being the intersection of a finite number of closed sets. Closed sets as well as their interior are both measurable for the Haar measure \( \lambda \). Continuity of \( f \) implies that \( f^{-1}(f(Z)) \) and \( f^{-1}((f(Z))^c) \) are both measurable for \( \nu \).

For each letter \( a \in \mathcal{A} \), we have \( f^{-1}((f([a]))^c) \subset [a] \). Thus we have

\[
f^{-1}((f(Z))^c) \subset Z \subset f^{-1}(f(Z))
\]
so that

\[
\nu(f^{-1}((f(Z))^c)) \leq \nu(Z) \leq \nu(f^{-1}(f(Z))).
\]
We denote by \( f_* \) the pushforward map
\[
f_* : \mathcal{M}^\nu(\mathcal{X}_{\mathcal{P}, R}) \to \mathcal{M}^\nu(T) \quad \nu \mapsto \nu \circ f^{-1}
\]
which maps shift-invariant measure on \( \mathcal{X}_{\mathcal{P}, R} \) to \( R \)-invariant measure on \( T \). But there is only one such measure, so that \( f_* \nu = \lambda \) for every \( \nu \in \mathcal{M}^\nu(\mathcal{X}_{\mathcal{P}, R}) \). For every \( \nu \in \mathcal{M}^\nu(\mathcal{X}_{\mathcal{P}, R}) \), we have for the left-hand side
\[
\nu(f^{-1}((f(Z))^\circ)) = f_* \nu((f(Z))^\circ) = \lambda((f(Z))^\circ)
\]
and for the right-hand side
\[
\nu(f^{-1}(f(Z))) = f_* \nu(f(Z)) = \lambda(f(Z)).
\]
As the boundary of \( f(Z) \) is a \( \lambda \)-null set, we obtain
\[
\lambda(f(Z)) = \lambda((f(Z))^\circ) \leq \nu(Z) \leq \lambda(f(Z))
\]
and we conclude that
\[
\nu(Z) = \lambda(f(Z)).
\]
Since measures are defined from the measure of cylinders which generate the Borel \( \sigma \)-algebra, we conclude that there is a unique shift-invariant probability measure on \( \mathcal{X}_{\mathcal{P}, R} \). Thus \( \mathcal{X}_{\mathcal{P}, R} \) is uniquely ergodic and therefore strictly ergodic since minimality of \( \mathcal{X}_{\mathcal{P}, R} \) was proved in Lemma 7. \( \square \)

Lemma 8 implies uniform patch frequencies for tilings in \( \mathcal{X}_{\mathcal{P}, R} \).

3. Wang tilings as codings of a \( \mathbb{Z}^2 \)-action

We consider the 2-torus \( T = \mathbb{R}^2/\Gamma \) where \( \Gamma \) is a lattice in \( \mathbb{R}^2 \). We suppose that \((T, \mathbb{Z}^2, R)\) is a dynamical system where \( R \) is a \( \mathbb{Z}^2 \)-rotation. Let \( \mathcal{Y} = \{Y_i\}_{i \in I}, Z = \{Z_j\}_{j \in J} \) be two finite topological partitions of \( T \). For each \((i, j, k, \ell) \in I \times J \times I \times J\) we define the intersection of 4 atoms in the following way
\[
P_{(i,j,k,\ell)} = Y_i \cap Z_j \cap R^{e_1}(Y_k) \cap R^{e_2}(Z_\ell).
\]
The quadruples \( \tau \) for which the intersection \( P_\tau \) is nonempty defines a set
\[
\mathcal{T} = \{\tau \in I \times J \times I \times J \mid P_\tau \neq \emptyset\}
\]
that we see as a set of Wang tiles. Naturally, this comes with a topological partition
\[
\mathcal{P} = \{P_\tau\}_{\tau \in \mathcal{T}}
\]
of \( T \) which is the refinement of the four partitions \( \mathcal{Y} \) (the right color), \( Z \) (the top color), \( R^{e_1}(\mathcal{Y}) \) (the left color) and \( R^{e_2}(Z) \) (the bottom color). Thus to each \( x \in T \setminus \Delta_{\mathcal{P}, R} \) corresponds a unique Wang tile, that is, a right, a top, a left and a bottom color according to which atom it belongs in each of the four partitions.

**Proposition 9.** Let \((T, \mathbb{Z}^2, R)\) be a dynamical system where \( R \) is a \( \mathbb{Z}^2 \)-rotation and let \( \mathcal{Y} \) and \( Z \) be two finite topological partitions of \( T \). Let \( \mathcal{P} = \mathcal{Y} \cap Z \cap R^{e_1}(\mathcal{Y}) \cap R^{e_2}(Z) \) be the refinement of four partitions. Let \( \mathcal{T} \) be the Wang tile set defined above as the set of quadruples \( \tau \) such that \( P_\tau \) is a nonempty atom of the partition \( \mathcal{P} \). Then \( \mathcal{X}_{\mathcal{P}, R} \) is a subshift of the Wang shift \( \Omega_\mathcal{T} \).

**Proof.** Let \( x \in T \setminus \Delta_{\mathcal{P}, R} \) and \( w = \text{TILING}(x) \). Let \( n \in \mathbb{Z}^2 \). First we check that Equation (3) is satisfied. There exists \( i \in I \) such that \( R^n(x) \in Y_i \). Equivalently, \( R^{n+e_1}(x) \in R^{e_1}(Y_i) \). Thus we have
\[
\text{RIGHT}(w_n) = \text{RIGHT}(\text{ORBIT}_{x}^{P, R}(n)) = i = \text{LEFT}(\text{ORBIT}_{x}^{P, R}(n + e_1)) = \text{LEFT}(w_{n + e_1}).
\]
Similarly we check that Equation \((4)\) is satisfied. There exists \(j \in J\) such that \(R^n(x) \in Z_j\). Equivalently, \(R^{n+e_2}(x) \in R^{e_2}(Z_j)\). Thus we have

\[
\text{top}(w_n) = \text{top}(\text{orbit}_{x,R}^n(n)) = j = \text{bottom}(\text{orbit}_{x,R}^n(n + e_2)) = \text{bottom}(w_{n+e_2}).
\]

Then \(w \in \Omega_T\) is a valid Wang tiling of the plane. Thus \(Tiling(T \setminus \Delta_{P,R}) \subseteq \Omega_T\).

Remark that \(\Omega_T\) is closed since it is a subshift. Therefore the topological closure of the image of \(Tiling\) is in the Wang shift \(\Omega_T\) using Lemma \(3\) \(Tiling(T \setminus \Delta_{P,R}) = \mathcal{X}_{P,R} \subseteq \Omega_T\). \(\square\)

**Lemma 10.** With the same hypothesis as in Proposition \(9\), for every \(v \in \mathbb{R}^2 \setminus \Theta^P\), \(Tiling^v\) is a map \(T \to \Omega_T\).

**Proof.** Follows from Lemma \(3\) and Proposition \(9\). \(\square\)

The strategy that we use in the present contribution is described in the following remark.

**Remark 11.** If for some independent reason \(\mathcal{X}_{P,R}\) is exactly equal to \(\Omega_T\) and \(P\) gives a symbolic representation of \((T, \mathbb{Z}^2, R)\), then we have that \(P\) is a Markov partition for \((T, \mathbb{Z}^2, R)\).

We end the section with two non-examples. We use the indices \(Z\) (zero) and \(I\) (one) to identify them. The motivation for presenting those two non-examples is to illustrate that properties of partitions associated with \(\mathbb{Z}^2\)-actions that are presented thereafter are not shared by randomly chosen toral partition and \(\mathbb{Z}^2\)-action.

### 3.1. Example Zero (Z).

Let \(\alpha_1, \alpha_2 \in \mathbb{R}\). On the torus \(\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2\), we consider the \(\mathbb{Z}^2\)-action \(R_Z : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2\) defined by

\[
R_Z^n(x) := R_Z(n, x) = x + (n_1 \alpha_1, n_2 \alpha_2).
\]

for every \(n = (n_1, n_2) \in \mathbb{Z}^2\). Let \(Y = \{Y_A\}\) and \(Z = \{Z_B\}\) be trivial partitions with \(Y_A = Z_B = \mathbb{T}^2\). The refined partition is \(P_Z = Y \wedge Z \wedge R_{Z}^{e_1}(Y) \wedge R_{Z}^{e_2}(Z) = \{P_{(A,B,A,B)}\}\) where \(P_{(A,B,A,B)} = \mathbb{T}^2\). The Wang tile set \(T_Z = \{\tau\}\) is a singleton set with \(\tau = (A, B, A, B)\). The associated color partitions and the tile coding partition are shown in Figure \(1\).

\[
\begin{array}{cccc}
Y & Z & R_{e_1}(Y) & R_{e_2}(Z) \\
\begin{array}{|c|}
\hline
A \\
\hline
\end{array} & \begin{array}{|c|}
\hline
B \\
\hline
\end{array} & \begin{array}{|c|}
\hline
A \\
\hline
\end{array} & \begin{array}{|c|}
\hline
B \\
\hline
\end{array} \\
\end{array}
\]

\[
\tau = \begin{array}{|c|}
\hline
B \\
\hline
A \\
\hline
A \\
\hline
B \\
\hline
\end{array}
\]

**Figure 1.** Partitions for the Example Zero (Z). From left to right, the partition \(Y\) for the right color, \(Z\) for the top color, \(R_{e_1}(Y)\) for the left color and \(R_{e_2}(Z)\) for the bottom color. Their refinement is the partition \(P_Z\) where each part is associated to a Wang tile.

The map \(Tiling_Z : \mathbb{T}^2 \to \Omega_{T_Z}\) is clearly not one-to-one, but it is onto.

**Lemma 12.** We have \(\mathcal{X}_{P_Z,R_Z} = \Omega_{T_Z}\), but the partition \(P_Z\) does not give a symbolic representation of \((\mathbb{T}^2, \mathbb{Z}^2, R_Z)\).
Proof. The Wang tile set $\mathcal{T}_Z = \{\tau\}$ is a singleton set with $\tau = (A, B, A, B)$. Therefore $\Omega_{\mathcal{T}_Z}$ contains a unique tiling corresponding to the constant map $(m, n) \mapsto \tau$ for all $m, n \in \mathbb{Z}$. The fact that $\mathcal{X}_{\mathcal{P}_Z, R_Z} \subseteq \Omega_{\mathcal{T}_Z}$ follows from Proposition 9. The unique constant tiling in $\Omega_{\mathcal{T}_Z}$ can be obtained as $\text{tiling}_Z(x) = \text{orbit}_{x}^{\mathcal{P}_Z, R_Z}$ for any $x \in \mathbb{T}$. Therefore $\text{tiling}_Z$ is onto.

The partition $\mathcal{P}_Z$ does not give a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R_Z)$ as every point of $\mathbb{T}^2$ are associated to the same tiling. \hfill \square

3.2. Example One (I). Let $\varphi = \frac{1+\sqrt{5}}{2}$. On the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, we consider the $\mathbb{Z}^2$-action $R_I : \mathbb{Z}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined by

$$R^n_I(x) := R_I(n, x) = x + (n_1 \varphi, n_2 \varphi).$$

for every $n = (n_1, n_2) \in \mathbb{Z}^2$. Let $I = \{A, B\}$ and $J = \{C, D\}$ be sets of colors. We consider the partitions $\mathcal{Y} = \{Y_A, Y_B\}$ and $\mathcal{Z} = \{Z_C, Z_D\}$ shown in Figure 2 involving slopes 1 and $-1$ in the partition of $\mathbb{T}^2$ into polygons. The refined partition is $\mathcal{P}_I = \mathcal{Y} \wedge \mathcal{Z} \wedge R^{e_1}_I(\mathcal{Y}) \wedge R^{e_2}_I(\mathcal{Z}) = \{P_i\}_{i \in I}$ where $\mathcal{T}_I$ is the Wang tile set made of 20 tiles shown in Figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example-one.png}
\caption{Partitions for the Example One (I). From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R^{e_1}(\mathcal{Y})$ for the left color and $R^{e_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}_I$ where each part is associated to a Wang tile.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example-one-tiles.png}
\caption{The set of 20 tiles $\mathcal{T}_I = \{\tau_0, \ldots, \tau_{19}\}$. Each index $i \in \{0, \ldots, 19\}$ written in the middle of a tile corresponds to a tile $\tau_i$. The Wang shift $\Omega_{\mathcal{T}_I}$ contains periodic points.}
\end{figure}

Lemma 13. The partition $\mathcal{P}_I$ gives a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$ and $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_I, R_I}, \mathbb{Z}^2, \sigma)$. We have that $\mathcal{X}_{\mathcal{P}_I, R_I}$ is a strictly ergodic and aperiodic subshift of $\Omega_{\mathcal{T}_I}$. But the Wang shift $\Omega_{\mathcal{T}_I}$ contains a periodic point so $\mathcal{X}_{\mathcal{P}_I, R_I} \subsetneq \Omega_{\mathcal{T}_I}$.

Proof. The dynamical system $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$ is minimal. The atom $P_{\tau_0}$ is invariant only under the trivial translation. Therefore, from Lemma 1 $\mathcal{P}_I$ gives a symbolic representation of $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$. From Proposition 9, there exists an almost one-to-one factor map from $(\mathcal{X}_{\mathcal{P}_I, R_I}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$ and from Corollary 8, $(\mathbb{T}^2, \mathbb{Z}^2, R_I)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_I, R_I}, \mathbb{Z}^2, \sigma)$.

Since $R^{e_1}_I$ and $R^{e_2}_I$ are linearly independent irrational rotations, we have that $R_I$ is a free $\mathbb{Z}^2$-action. Thus from Lemma 7 $\mathcal{X}_{\mathcal{P}_I, R_I}$ is minimal and aperiodic. From Lemma 8 $\mathcal{X}_{\mathcal{P}_I, R_I}$ is uniquely
ergodic thus strictly ergodic. From Proposition 9 we have $X_{\mathcal{P}_1, R_1} \subseteq \Omega_{T_1}$. The Wang tile set $T_1$ contains the tile $\tau_0 = (A, C, A, C)$. Let $w$ be the constant map $(m, n) \mapsto \tau_0$ for all $m, n \in \mathbb{Z}$. The tiling $w$ is a valid periodic Wang tiling, thus $w \in \Omega_{T_1 \setminus \mathcal{X}_{\mathcal{P}_1, R_1}}$.  

4. A Markov partition for Wang tilings in $\Omega_{U}$

On the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$, we consider the $\mathbb{Z}^2$-action $R_U : \mathbb{Z}^2 \times T^2 \to T^2$ defined by

$$R_U^n(x) := R_U(n, x) = x + \varphi^{-1}n$$

for every $n \in \mathbb{Z}^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$. Let $I = \{A, B, C, D, E, F, G, H, I, J\}$ and $J = \{K, L, M, N, O, P\}$ be sets of colors and consider the partitions $\mathcal{Y} = \{Y_j\}_{j \in I}$ and $\mathcal{Z} = \{Z_j\}_{j \in J}$ shown in Figure 4.

The refined partition is $\mathcal{P}_U = \mathcal{Y} \cap \mathcal{Z} \cap R_U^{e_1}(\mathcal{Y}) \cap R_U^{e_2}(\mathcal{Z}) = \{P_u\}_{u \in \mathcal{U}}$ where $\mathcal{U} = \{u_0, u_1, \ldots, u_{18}\}$ is the Wang tile set made of 19 tiles shown in Figure 5 which was introduced in [Lab18a].

![Figure 4. Partitions for the 19 self-similar Wang tile example. From left to right, the partition $\mathcal{Y}$ for the right color, $\mathcal{Z}$ for the top color, $R^{e_1}(\mathcal{Y})$ for the left color and $R^{e_2}(\mathcal{Z})$ for the bottom color. Their refinement is the partition $\mathcal{P}_U$ where each part is associated to a Wang tile.](image)

![Figure 5. The set $\mathcal{U} = \{u_0, \ldots, u_{18}\}$ of 19 Wang tiles. Each index $i \in \{0, \ldots, 18\}$ written in the middle of a tile corresponds to a tile $u_i$.](image)

We now prove the first main result of the contribution. Note that the fact that $\Omega_{U}$ is minimal (proved in [Lab18a]) allows to conclude that $\mathcal{P}_U$ is a Markov partition without having to compute the substitutive structure of $X_{\mathcal{P}_1, R_1}$.

**Theorem 14.** The partition $\mathcal{P}_U$ gives a symbolic representation of $(T^2, \mathbb{Z}^2, R_U)$ which is the maximal equicontinuous factor of $(X_{\mathcal{P}_1, R_1}, \mathbb{Z}^2, \sigma)$. Also $X_{\mathcal{P}_U, R_U} = \Omega_U$ is minimal and aperiodic.

**Proof.** The dynamical system $(T^2, \mathbb{Z}^2, R_U)$ is minimal. The atom $P_{u_0}$ is invariant only under the trivial translation. Therefore, from Lemma 1 $\mathcal{P}_U$ gives a symbolic representation of $(T^2, \mathbb{Z}^2, R_U)$. From Proposition 5 there exists an almost one-to-one factor map $f_U$ from $(X_{\mathcal{P}_1, R_1}, \mathbb{Z}^2, \sigma)$ to $(T^2, \mathbb{Z}^2, R_U)$ and from Corollary 6 $(T^2, \mathbb{Z}^2, R_U)$ is the maximal equicontinuous factor of $(X_{\mathcal{P}_1, R_1}, \mathbb{Z}^2, \sigma)$.
Since $R^e_{\mathcal{U}}$ and $R^e_{\mathcal{W}}$ are linearly independent rotations, we have that $R_{\mathcal{U}}$ is a free $\mathbb{Z}^2$-action. Thus from Lemma 7, $X_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}}$ is minimal and aperiodic. From Proposition 9 we have $X_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}} \subseteq \Omega_{\mathcal{U}}$. It was proved in Lab18a that $\Omega_{\mathcal{U}}$ is minimal. Thus $X_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}} = \Omega_{\mathcal{U}}$.

**Corollary 15.** The dynamical system $(\Omega_{\mathcal{U}}, \mathbb{Z}^2, \sigma)$ is strictly ergodic and the measure-preserving dynamical system $(\Omega_{\mathcal{U}}, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to the toral $\mathbb{Z}^2$-rotation $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}}, \lambda)$ where $\nu$ is the unique shift-invariant probability measure on $\Omega_{\mathcal{U}}$ and $\lambda$ is the Haar measure on $\mathbb{T}^2$.

**Proof.** It follows from Theorem 14 and Lemma 8.

**Corollary 16.** $\mathcal{P}_{\mathcal{U}}$ is a Markov partition for $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$.

**Proof.** $X_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}} = \Omega_{\mathcal{U}}$ is a shift of finite type. Therefore, $\mathcal{P}_{\mathcal{U}}$ is a Markov partition for $(\mathbb{T}^2, \mathbb{Z}^2, R_{\mathcal{U}})$.

Recall that the thickness spectrum for Penrose tilings is $\{1, 2, 10\}$ [Rob96] and $\{1, 2, 6, 12\}$ for Taylor-Socolar hexagonal tilings [LM13].

**Remark 17.** The thickness spectrum of the factor map $f_{\mathcal{U}} : \Omega_{\mathcal{U}} \to \mathbb{T}^2$ is $\{1, 2, 8\}$.

**Proof sketch.** In Proposition 5 we proved that $f_{\mathcal{U}}$ is one-to-one on $\mathbb{T}^2 \setminus \Delta_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}}$. Suppose that $x \in \Delta_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}}$. We have $\text{card}(f_{\mathcal{U}}^{-1}(x)) \geq 2$. If $\text{card}(f_{\mathcal{U}}^{-1}(x)) > 2$, then we may show that there exists $n \in \mathbb{Z}^2$ such that $x = R_{\mathcal{U}}^n(0)$. If $x = R_{\mathcal{U}}^n(0)$ for some $n \in \mathbb{Z}^2$, then the set $f_{\mathcal{U}}^{-1}(x)$ contains 8 different tilings of the form $\text{TILING}_{\mathcal{U}}^v(0)$ for some $v \in \mathbb{R}^2 \setminus \Theta^{P_{\mathcal{U}}}$ where $\Theta^{P_{\mathcal{U}}} = \mathbb{R} \cdot \{(0, 0), (0, 1), (1, -1), (1, -\varphi)\}$. If $x \in \Delta_{\mathcal{P}_{\mathcal{U}, R_{\mathcal{U}}}}$ but not in the orbit of 0 under $R_{\mathcal{U}}$, then $\text{card}(f_{\mathcal{U}}^{-1}(x)) = 2$.

5. **Jeandel-Rao tilings as the coding of a $\mathbb{Z}^2$-action on a 2-torus**

Consider the lattice $\Gamma_0 = \langle (\varphi, 0), (1, \varphi + 3) \rangle_{\mathbb{Z}}$ where $\varphi = \frac{1 + \sqrt{5}}{2}$. On the torus $\mathbb{R}^2 / \Gamma_0$, we consider the $\mathbb{Z}^2$-action $R_0 : \mathbb{Z}^2 \times \mathbb{R}^2 / \Gamma_0 \to \mathbb{R}^2 / \Gamma_0$ defined by

$$R^0_0(x) := R_0(n, x) = x + n$$

for every $n \in \mathbb{Z}^2$. We consider the fundamental domain $D = [0, \varphi] \times [0, \varphi + 3]$ of $\mathbb{R}^2$ for the group of translations $\Gamma_0$. Let $I = \{0, 1, 2, 3\}$ and $J = \{0, 1, 2, 3, 4\}$ be sets of colors and consider the partitions $\mathcal{Y} = \{Y_i\}_{i \in I}$ and $\mathcal{Z} = \{Z_j\}_{j \in J}$ shown in Figure 6.

The refined partition is $\mathcal{P}_0 = \mathcal{Y} \wedge \mathcal{Z} \wedge R^\mathcal{Y}_0(\mathcal{Y}) \wedge R^\mathcal{Z}_0(\mathcal{Z}) = \{P_t\}_{t \in \mathcal{T}_0}$ where $\mathcal{T}_0 = \{t_0, t_1, \ldots, t_{10}\}$ is the Wang tile set

$$\mathcal{T}_0 = \begin{cases}
  t_0 = \begin{array}{c}
  \begin{array}{c}
  2 \ \ 4 \\
  3 \ \ 2
  \end{array}
  \end{array}, & t_1 = \begin{array}{c}
  \begin{array}{c}
  2 \ \ 2 \\
  0 \ \ 2
  \end{array}
  \end{array}, & t_2 = \begin{array}{c}
  \begin{array}{c}
  1 \ \ 3 \\
  2 \ \ 1
  \end{array}
  \end{array}, & t_3 = \begin{array}{c}
  \begin{array}{c}
  2 \ \ 1 \\
  3 \ \ 2
  \end{array}
  \end{array}, & t_4 = \begin{array}{c}
  \begin{array}{c}
  3 \ \ 3 \\
  4 \ \ 1
  \end{array}
  \end{array}, & t_5 = \begin{array}{c}
  \begin{array}{c}
  4 \ \ 0 \\
  3 \ \ 1
  \end{array}
  \end{array},
  \\
  t_6 = \begin{array}{c}
  \begin{array}{c}
  0 \ \ 1 \\
  2 \ \ 1
  \end{array}
  \end{array}, & t_7 = \begin{array}{c}
  \begin{array}{c}
  3 \ \ 1 \\
  2 \ \ 0
  \end{array}
  \end{array}, & t_8 = \begin{array}{c}
  \begin{array}{c}
  2 \ \ 0 \\
  3 \ \ 2
  \end{array}
  \end{array}, & t_9 = \begin{array}{c}
  \begin{array}{c}
  1 \ \ 4 \\
  2 \ \ 4
  \end{array}
  \end{array}, & t_{10} = \begin{array}{c}
  \begin{array}{c}
  4 \ \ 3 \\
  3 \ \ 0
  \end{array}
  \end{array}.
\end{cases}$$

We observe that $\mathcal{T}_0$ is the Jeandel-Rao’s 11 tiles set [JR15] and let $\Omega_0 = \Omega^{\mathcal{T}_0}$ denote the Jeandel-Rao Wang shift.

**Theorem 18.** The partition $\mathcal{P}_0$ gives a symbolic representation of $(\mathbb{R}^2 / \Gamma_0, \mathbb{Z}^2, R_0)$ which is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_0, R_0}, \mathbb{Z}^2, \sigma)$. Also $\mathcal{X}_{\mathcal{P}_0, R_0}$ is a proper minimal and aperiodic subshift of the Jeandel-Rao Wang shift, i.e., $\mathcal{X}_{\mathcal{P}_0, R_0} \subseteq \Omega_0$.

**Proof.** The dynamical system $(\mathbb{R}^2 / \Gamma_0, \mathbb{Z}^2, R_0)$ is minimal. The atom $P_{t_{10}}$ is invariant only under the trivial translation. Therefore, from Lemma 1, $\mathcal{P}_0$ gives a symbolic representation of $(\mathbb{R}^2 / \Gamma_0, \mathbb{Z}^2, R_0)$. From Proposition 5, there exists an almost one-to-one factor map $f_0$ from $(\mathcal{X}_{\mathcal{P}_0, R_0}, \mathbb{Z}^2, \sigma)$ to $(\mathbb{R}^2 / \Gamma_0, \mathbb{Z}^2, R_0)$ and from Corollary 6, $(\mathbb{R}^2 / \Gamma_0, \mathbb{Z}^2, R_0)$ is the maximal equicontinuous factor of $(\mathcal{X}_{\mathcal{P}_0, R_0}, \mathbb{Z}^2, \sigma)$.
Since $R^e_1$ and $R^e_2$ are linearly independent irrational rotations on $\mathbb{R}^2/\Gamma_0$, we have that $R_0$ is a free $\mathbb{Z}^2$-action. Thus from Lemma 7, $\mathcal{X}_{P_0, R_0}$ is minimal and aperiodic. From Proposition 9, we have $\mathcal{X}_{P_0, R_0} \subseteq \Omega_U$. It was proved in [Lab18b] that the Jeandel-Rao Wang shift $\Omega_0$ is not minimal. Thus $\mathcal{X}_{P_0, R_0} \not\subseteq \Omega_0$.

Corollary 19. The dynamical system $(\mathcal{X}_{P_0, R_0}, \mathbb{Z}^2, \sigma)$ is strictly ergodic and the measure-preserving dynamical system $(\mathcal{X}_{P_0, R_0}, \mathbb{Z}^2, \sigma, \nu)$ is isomorphic to the toral $\mathbb{Z}^2$-rotation $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0, \lambda)$ where $\nu$ is the unique shift-invariant probability measure on $\mathcal{X}_{P_0, R_0}$ and $\lambda$ is the Haar measure on $\mathbb{R}^2/\Gamma_0$.

Proof. It follows from Theorem 18 and Lemma 8.

Conjecture 20. $\mathcal{P}_0$ is a Markov partition for $(\mathbb{R}^2/\Gamma_0, \mathbb{Z}^2, R_0)$.

As done in Remark 17, we may compute the thickness spectrum of the factor map $f_0$.

Remark 21. The thickness spectrum of the factor map $f_0 : \mathcal{X}_{P_0, R_0} \to \mathbb{R}^2/\Gamma_0$ is $\{1, 2, 8\}$.

Part 2. Wang tilings as model sets of cut and project schemes

This part is divided into three sections. Its goal is to show that Wang tilings in $\Omega_U$ and a minimal subshift of Jeandel-Rao tilings are obtained as 4-to-2 cut and project schemes.
6. Cut and project schemes and model sets

We recall from [BG13 §7.2] the definition of cut and project scheme and we reuse their notations.

**Definition 22.** A cut and project scheme (CPS) is a triple \((\mathbb{R}^d, H, \mathcal{L})\) with a (compactly generated) locally compact Abelian group (LCAG) \(H\), a lattice \(\mathcal{L}\) in \(\mathbb{R}^d \times H\) and the two natural projections \(\pi : \mathbb{R}^d \times H \to \mathbb{R}^d\) and \(\pi_{\text{int}} : \mathbb{R}^d \times H \to H\), subject to the conditions that \(\pi_{\mathcal{L}}\) is injective and that \(\pi_{\text{int}}(\mathcal{L})\) is dense in \(H\).

A CPS is called **Euclidean** when \(H = \mathbb{R}^m\) for some \(m \in \mathbb{Z}\). A general CPS is summarised in the following diagram.

\[
\begin{array}{ccc}
\mathbb{R}^d & \leftarrow & \mathbb{R}^d \times H \\
\pi \downarrow & & \pi_{\text{int}} \downarrow \\
\pi(\mathcal{L}) & \leftarrow & \mathcal{L} \quad \mathcal{L} \rightarrow \pi_{\text{int}}(\mathcal{L}) \\
\pi(\mathcal{L}) & \mapsto & \pi(\mathcal{L}) \\
L & \rightarrow & L^* \\
\end{array}
\]

The image \(L = \pi(\mathcal{L})\). Since for a given CPS, \(\pi\) is a bijection between \(\mathcal{L}\) and \(L\), there is a well-defined mapping \(\star : L \to H\) given by

\[
x \mapsto x^* := \pi_{\text{int}} \left( (\pi_{\mathcal{L}})^{-1}(x) \right)
\]

where \((\pi_{\mathcal{L}})^{-1}(x)\) is the unique point in the set \(\mathcal{L} \cap \pi^{-1}(x)\). This mapping is called the **star map** of the CPS. The \(\star\)-image of \(L\) is denoted \(L^*\). The set \(\mathcal{L}\) can be viewed as a diagonal embedding of \(L\) as

\[
\mathcal{L} = \{(x, x^*) \mid x \in L\}.
\]

For a given CPS \((\mathbb{R}^d, H, \mathcal{L})\) and a (general) set \(A \subset H\),

\[
\lambda(A) := \{x \in L \mid x^* \in A\}
\]

denotes the projection set within the CPS. The set \(A\) is called its acceptance set, coding set or window.

**Definition 23.** If \(A \subset H\) is a relatively compact set with non-empty interior, the projection set \(\lambda(A)\), or any translate \(t + \lambda(A)\) with \(t \in \mathbb{R}^d\), is called a model set.

A model set is termed **regular** when \(\mu_H(\partial A) = 0\), where \(\mu_H\) is the Haar measure of \(H\). If \(L^* \cap \partial A = \emptyset\), the model set is called **generic**. If the window is not in a generic position (meaning that \(L^* \cap \partial A \neq \emptyset\)), the corresponding model set is called **singular**.

The shape of the accepting set \(A\) is important and implies structure on the model set \(\Lambda = t + \lambda(A)\). For example, if \(A\) is relatively compact, \(\Lambda\) has finite local complexity and thus also is uniformly discrete; if \(A^c \neq \emptyset\), \(\Lambda\) is relatively dense. If \(\Lambda\) is a model set, it is also a Meyer set, [BG13 Prop. 7.5]. For regular model set \(\Lambda = \lambda(A)\) with a compact window \(A = \overline{A}\), it is known [BG13 Thm. 7.2] that the points \(\{x^* \mid x \in \Lambda\}\) are uniformly distributed in \(A\).

Linear repetitivity of model sets is an important notion. Recall that a Delone set \(Y \subset \mathbb{R}^d\) is called **linearly repetitive** if there exists a constant \(C > 0\) such that, for any \(r \geq 1\), every patch of size \(r\) in \(Y\) occurs in every ball of diameter \(Cr\) in \(\mathbb{R}^d\). It was shown by Lagarias and Pleasants in [LP03 Thm 6.1] that linear repetitivity of a Delone set implies the existence of strict uniform patch frequencies, equivalently the associated dynamical system on the hull of the point set is strictly ergodic (minimal and uniquely ergodic). As a consequence [LP03 Cor 6.1], a linearly repetitive Delone set \(X \subset \mathbb{R}^d\) has a unique autocorrelation measure \(\gamma_X\). This measure \(\gamma_X\) is a pure discrete measure supported on \(X - X\). In particular \(X\) is diffractive. A characterization of linearly
repetitive model sets \( \lambda(A) \) for cubical accepting set \( A \) was recently proved by Haynes, Koivusalo and Walton [HKW18].

**Polygon exchange transformations.** We end this section with a concept that will be useful for the next two sections. Suppose that \((T, Z^2, R)\) is a dynamical system where \( R \) is a \( \mathbb{Z}^2 \)-action defined by rotations. The rotations \( R^{e_1} \) and \( R^{e_2} \) can be seen as polygon exchange transformations [Sch14] on a fundamental domain of \( T \).

**Definition 24.** [AKY18] Let \( X \) be a polygon together with two topological partitions of \( X \) into polygons

\[
X = \bigcup_{k=0}^{N} P_k = \bigcup_{k=0}^{N} Q_k
\]

such that for each \( k \), \( P_k \) and \( Q_k \) are translation equivalent, i.e., there exists \( v_k \in \mathbb{R}^2 \) such that \( P_k = Q_k + v_k \). A **polygon exchange transformation (PET)** is the piecewise translation on \( X \) defined for \( x \in P_k \) by \( T(x) = x + v_k \). The map is not defined for points \( x \in \cup_{k=0}^{N} \partial P_k \).

7. **Wang tilings in \( \Omega_{\mathcal{U}} \) as model sets**

We want to describe the positions \( S \subseteq \mathbb{Z}^2 \) of tiles in Wang tilings in \( \Omega_{\mathcal{U}} \). Because of that, in the construction of a proper cut and project scheme, we need to be careful in the choice of the locally compact Abelian group \( H \) so that \( \pi|_{\mathcal{L}} \) is an injective map. This is why we introduce the submodule \( \Lambda = \langle (1,1,0,0), (0,0,1,-1) \rangle_{\mathbb{Z}} \) and define the projections \( \pi \) and \( \pi_{\text{int}} \) on \( \mathbb{R}^4/\Lambda \) as:

\[
\pi : \mathbb{R}^4/\Lambda \rightarrow \mathbb{R}^2 \\
(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_3 + x_4)
\]

and

\[
\pi_{\text{int}} : \mathbb{R}^4/\Lambda \rightarrow \mathbb{T}^2 \\
(x_1, x_2, x_3, x_4) \mapsto \left( \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_2, \frac{1}{\sqrt{2}} x_3 - \frac{1}{\sqrt{2}} x_4 \right)
\]

where \( \varphi = \frac{1+\sqrt{5}}{2} \). The product \( \pi \times \pi_{\text{int}} : \mathbb{R}^4/\Lambda \rightarrow \mathbb{R}^2 \times \mathbb{T}^2 \) of the projections is one-to-one and onto so we may identify the domain of the projections as \( \mathbb{R}^4/\Lambda \cong \mathbb{R}^2 \times \mathbb{T}^2 \), in agreement with the definition of cut and project schemes.

Note that if \( \mathcal{L} = \mathbb{Z}^4 \subseteq \mathbb{R}^4/\Lambda \), then \( \pi|_{\mathcal{L}} \) is injective and \( L = \pi(\mathcal{L}) = \mathbb{Z}^2 \). If the accepting set is the whole cubical window \( A = \mathbb{T}^2 \), we obtain a description of the positions of Wang tiles in a Wang tilings as a model set, that is, \( \mathbb{Z}^2 = \lambda(A) \). In the result below, noncubical accepting set \( A \subseteq \mathbb{T}^2 \) are used to describe the positions of patterns occuring in Wang tilings.

Recall that we proved among others things in Theorem [14] that \( X_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} = \Omega_{\mathcal{U}} \) and that there exists an almost one-to-one factor map \( f_\mathcal{U} \) from \( (X_{\mathcal{P}_\mathcal{U},R_\mathcal{U}}, \mathbb{Z}^2, \sigma) \) to \( (\mathbb{T}^2, \mathbb{Z}^2, R_\mathcal{U}) \). Therefore any tiling \( w \in \Omega_{\mathcal{U}} \) can be qualified as a singular or generic tiling according to whether \( f_\mathcal{U}(w) \) is in the set \( \Delta_{\mathcal{P}_\mathcal{U},R_\mathcal{U}} \subseteq \mathbb{T}^2 \) or not.

**Theorem 25.** Let \( \mathcal{U} \) be the self-similar Wang tile set shown in Figure [3]. For every Wang tiling \( w \in \Omega_{\mathcal{U}} \), the set \( Q \subseteq \mathbb{Z}^2 \) of occurrences of a pattern in \( w \) is a regular model set. If \( w \) is a generic (resp. singular) tiling, then \( Q \) is a generic (resp. singular) model set.

**Proof.** Let \( w \in \Omega_{\mathcal{U}} \) be a Wang tiling and let \( x = (r, s) = f_\mathcal{U}(w) \in \mathbb{T}^2 \). We consider \( \mathcal{L} = \mathbb{Z}^4 + (r, -r, s, -s) \subseteq \mathbb{R}^4/\Lambda \). We have that \( \pi|_{\mathcal{L}} \) is injective and \( L = \pi(\mathcal{L}) = \mathbb{Z}^2 \). We also have that \( \pi_{\text{int}}(\mathcal{L}) \) is dense in \( H = \mathbb{T}^2 \). Also \( \pi_{\text{int}}(r, -r, s, -s) = (r, s) \).

Since \( \pi \) is a bijection between \( \mathcal{L} \) and \( L \), there is a well-defined mapping \( * : L \rightarrow H \) given by

\[
x \mapsto x^* := \pi_{\text{int}} \left( (\pi|_{\mathcal{L}})^{-1}(x) \right)
\]

where \((\pi|_{\mathcal{L}})^{-1}(x)\) is the unique point in the set \( \mathcal{L} \cap \pi^{-1}(x) \).
Recall that the $\mathbb{Z}^2$-action $R_\mathcal{U}$ is defined on the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ as:

$$R_\mathcal{U}^n(x) = x + \varphi^{-n}n$$

for every $n \in \mathbb{Z}^2$. The maps $(R_\mathcal{U})^{e_1}$ and $(R_\mathcal{U})^{e_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,1)^2$ of $\mathbb{T}^2$:

$$(R_\mathcal{U})^{e_1}(x) = \begin{cases} x + v_a & \text{if } x \in P_a, \\ x + v_b & \text{if } x \in P_b, \end{cases} \quad \text{and} \quad (R_\mathcal{U})^{e_2}(x) = \begin{cases} x + v_c & \text{if } x \in P_c, \\ x + v_d & \text{if } x \in P_d, \end{cases}$$

with $v_a = (\varphi^{-2}, 0)$, $v_b = (-\varphi^{-1}, 0)$, $v_c = (0, \varphi^{-2})$ and $v_d = (0, -\varphi^{-1})$, see Figure 7. Notice that the base of $\mathbb{Z}^2$ can be written in terms of the translations as

$$e_1 = v_a - v_b \quad \text{and} \quad e_2 = v_c - v_d.$$ 

Since $W$ is a fundamental domain for $\mathbb{Z}^2 = \langle e_1, e_2 \rangle_\mathbb{Z}$, by definition for every $x \in \mathbb{R}^2$, there exist unique $k, \ell \in \mathbb{Z}$ such that $x + ke_1 + \ell e_2 \in W$. Therefore, for every $(m, n) \in \mathbb{Z}^2$ there exist unique $k, \ell \in \mathbb{Z}$ such that the following holds

$$R_\mathcal{U}^{(m,n)}(r, s) = (r, s) + \frac{1}{\varphi^2}(m, n) \mod \mathbb{Z}^2$$

$$= r e_1 + s e_2 + m v_a + n v_c \mod \mathbb{Z}^2$$

$$= r e_1 + s e_2 + m v_a + n v_c + k e_1 + \ell e_2 \in W$$

$$= m v_a + n v_c + (r + k)(v_a - v_b) + (s + \ell)(v_c - v_d)$$

$$= (m + r + k)v_a - (r + k)v_b + (n + s + \ell)v_c - (s + \ell)v_d$$

$$= \pi_{\text{int}}((m + r + k, -r - k, n + s + \ell, -s - \ell) + \Lambda) \in \pi_{\text{int}}(\mathcal{L}).$$

Notice that the projection into the physical space is

$$\pi((m + r + k, -r - k, n + s + \ell, -s - \ell) + \Lambda) = (m, n).$$

Thus

$$(m + r + k, -r - k, n + s + \ell, -s - \ell) + \Lambda = (\pi|_{\mathcal{L}})^{-1}(m, n)$$

so that

$$(m, n)^* = \pi_{\text{int}}\left((\pi|_{\mathcal{L}})^{-1}(m, n)\right)$$

$$= \pi_{\text{int}}\left(((m + r + k, -r - k, n + s + \ell, -s - \ell) + \Lambda\right)$$

$$= R_\mathcal{U}^{(m,n)}(r, s) = \{r + \varphi m, s + \varphi n\}$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of $x$. 

**Figure 7.** The maps $(R_\mathcal{U})^{e_1}$ and $(R_\mathcal{U})^{e_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,1)^2$ of $\mathbb{T}^2$. 



Let \( p = \pi_S(w) \in \mathcal{U}^g \) be a pattern occurring in the tiling \( w \) for some subset \( S \subset \mathbb{Z}^2 \). Let \([p]\) be the polygon associated to the pattern \( p \) and \( A = f_{\mathcal{U}}([p]) \subset \mathcal{W} \) be the accepting set. The set \( A \) is a polygon by construction (see Equation 6). Therefore the Lebesgue measure of \( \partial A \) is zero. Assume for now that \( w \) is a generic tiling. Since \( R^{(m,n)}_{\mathcal{U}}(r,s) = (m,n) \notin \partial A \) for every \( m, n \in \mathbb{Z} \), the set \( Q \subset \mathbb{Z}^2 \) of occurrences of \( p \) in \( w \) is

\[
Q = \{(m, n) \in \mathbb{Z}^2 \mid R^{(m,n)}_{\mathcal{U}}(r,s) \in A \} = \{(m, n) \in L \mid (m,n) \notin A \} = \lambda(A)
\]

which is a regular and generic model set. If \( w \) is a singular tiling, then \( w = \text{TILING}^{(m,n)}_{\mathcal{U}}(r,s) \) for some \( v \in \mathbb{R}^2 \setminus \Theta^{\mathcal{U}} \). If \( A = f_{\mathcal{U}}([p]) \subset \mathcal{W} \), then we take \( A' = \lim_{\epsilon \to 0} A \cap (A - \epsilon v) \) as accepting set and we have

\[
Q = \{(m, n) \in \mathbb{Z}^2 \mid R^{(m,n)}_{\mathcal{U}}(r,s) \in A' \} = \{(m, n) \in L \mid (m,n) \notin A' \} = \lambda(A')
\]

which is a regular and singular model set.

In [Lab18], \( \Omega_{\mathcal{U}} \) was proved to be self-similar being invariant under the application of an expansive and primitive substitution. It follows that \( \Omega_{\mathcal{U}} \) is linearly repetitive. Based on [HKW18], an alternate proof of linear repetitivity of \( \Omega_{\mathcal{U}} \) could be obtained now that \( \Omega_{\mathcal{U}} \) is described as a model set. Some more work has to be done as the characterization of linearly repetitive model sets provided in [HKW18] is stated for cubical windows only.

In the present work, we made the choice of uniform \( 1 \times 1 \) size for Wang tiles but we can make the following remark on the use of other rectangular shapes and stone inflations.

**Remark 26.** To use the natural size for Wang tiles in \( \mathcal{U} \) as stone inflation deduced from its self-similarity, see [Lab18] §7, one must use

\[
\pi' : \mathbb{R}^4 \to \mathbb{R}^2 \quad (x_1, x_2, x_3, x_4) \mapsto (x_1 + \frac{1}{\varphi} x_2, x_3 + \frac{1}{\varphi} x_4).
\]

as projection into the physical space. In that case, \( \pi'|_\mathcal{L} \) is injective making it a proper cut and project scheme. Another way to construct the cut and project scheme is to use the Minkowski embedding of \( \mathbb{Z}[\sqrt{5}] \times \mathbb{Z}[\sqrt{5}] \)

\[
\mathcal{L} = \{(x, y, x^*, y^*) \mid x, y \in \mathbb{Z}[\sqrt{5}]\}
\]

where the star map \( \star \) corresponds to the algebraic conjugation \((\sqrt{5})^* = -\sqrt{5}\) in the quadratic field \( \mathbb{Q}(\sqrt{5}) \), see [BG13] §7. In this setup, the natural window to be used should be \( \mathcal{W} = [-1, \varphi - 1] \times [-1, \varphi - 1] \) instead of \([0, 1] \times [0, 1]\) following known construction in the Fibonacci case. We do not provide this construction here.

### 8. Jeandel-Rao tilings as model sets

As in previous section we use the submodule \( \Lambda = \langle (1, -1, 0, 0), (0, 0, 1, -1) \rangle_{\mathbb{Z}} \) and define the projections on \( \mathbb{R}^4/\Lambda \) as:

\[
\pi : \mathbb{R}^4/\Lambda \to \mathbb{R}^2 \quad (x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_3 + x_4)
\]

and

\[
\pi_{\text{int}} : \mathbb{R}^4/\Lambda \to \mathbb{R}^2/\Gamma_0 \quad (x_1, x_2, x_3, x_4) \mapsto \left(x_1 - \frac{1}{\varphi} x_2 + \frac{1}{\varphi} x_4, x_3 - (\varphi + 2)x_4\right)
\]

where \( \varphi = \frac{1 + \sqrt{5}}{2} \). The product \( \pi \times \pi_{\text{int}} : \mathbb{R}^4/\Lambda \to \mathbb{R}^2 \times \mathbb{R}^2/\Gamma_0 \) of the projections is one-to-one and onto. Therefore, the projections define a Euclidean cut and project scheme with \( d = 2 \) and \( H = \mathbb{R}^2/\Gamma_0 \) on \( \mathbb{R}^4/\Lambda \simeq \mathbb{R}^2 \times H \).
Recall that we proved in Theorem 18 that $\mathcal{X}_{P_0,R_0} \subset \Omega_0$ and that there exists an almost one-to-one factor map $f_0$ from $(\mathcal{X}_{P_0,R_0},\mathbb{Z}^2,\sigma)$ to $(\mathbb{R}^2/\Gamma_0,\mathbb{Z}^2,R_0)$. Therefore any Jeandel-Rao tiling $w \in \mathcal{X}_{P_0,R_0} \subset \Omega_0$ can be qualified as a singular or generic tiling according to whether $f_0(w)$ is in the set $\Delta_{P_0,R_0} \subset \mathbb{R}^2/\Gamma_0$ or not.

**Theorem 27.** Let $\mathcal{X}_{P_0,R_0} \subset \Omega_0$ be the proper minimal subshift of Jeandel-Rao Wang shift. For every tiling $w \in \mathcal{X}_{P_0,R_0}$, the set $Q \subset \mathbb{Z}^2$ of occurrences of a pattern in $w$ is a regular model set. If $w$ is a generic (resp. singular) tiling, then $Q$ is a generic (resp. singular) model set.

**Proof.** Let $w \in \mathcal{X}_{P_0,R_0}$. Let $x = (r,s) = r'(\varphi,0) + s'(1,\varphi + 3) = f_0(w) \in \mathbb{R}^2/\Gamma_0$. We consider the lattice $L = \mathbb{Z}^4 + (r' + s',-r' - s',s',-s') \subset \mathbb{R}^4/\Lambda$. We have that $\pi|_L$ is injective. Also $L = \pi(L) = \mathbb{Z}^2$ since $\pi(r' + s',-r' - s',s',-s') = 0$. We also have that $\pi_{\text{int}}(L)$ is dense in $H = \mathbb{R}^2/\Gamma_0$. Also $\pi_{\text{int}}(r' + s',-r' - s',s',-s') = (r,s)$.

Recall that the $\mathbb{Z}^2$-action $R_0$ is defined on the torus $\mathbb{R}^2/\Gamma_0$ as:

$$R_0^n(x) = x + n$$

for every $n \in \mathbb{Z}^2$. The maps $(R_0)^{e_1}$ and $(R_0)^{e_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,\varphi] \times [0,\varphi + 3]$ of $\mathbb{R}^2/\Gamma_0$ (see Figure 8):

$$R_0^{e_1}(x) = \begin{cases} x + v_a & \text{if } x \in P_a, \\ x + v_b & \text{if } x \in P_b, \end{cases}$$

and

$$R_0^{e_2}(x) = \begin{cases} x + v_c & \text{if } x \in P_c, \\ x + v_d & \text{if } x \in P_d, \\ x + v_e & \text{if } x \in P_e. \end{cases}$$

The translations written in terms of the base of $\mathbb{Z}^2$ and $\Gamma_0$ and vice versa:

![Figure 8](image-url)

**Figure 8.** The maps $(R_0)^{e_1}$ and $(R_0)^{e_2}$ can be seen as polygon exchange transformations on the fundamental domain $W = [0,\varphi] \times [0,\varphi + 3]$ of $\mathbb{R}^2/\Gamma_0$. 
Since $W$ is a fundamental domain for $\Gamma_0 = \langle (\varphi,0),(1,\varphi + 3) \rangle_{\mathbb{Z}}$, by definition for every $x \in \mathbb{R}^2$, there exist unique $k, \ell \in \mathbb{Z}$ such that $x + k(\varphi,0) + \ell(1,\varphi + 3) \in W$. Therefore, for every $(m,n) \in \mathbb{Z}^2$ there exist unique $k, \ell \in \mathbb{Z}$ such that the following holds

$$R_0^{(m,n)}(r,s) = (r,s) + (m,n) \text{ mod } \Gamma_0$$

$$= r'(\varphi,0) + s'(1,\varphi + 3) + m e_1 + n e_2 + k(\varphi,0) + \ell(1,\varphi + 3) \in W$$

$$= m e_1 + n e_2 + (r' + k)(\varphi,0) + (s' + \ell)(1,\varphi + 3)$$

$$= m v_a + n v_c + (r' + k)(v_a - v_b) + (s' + \ell)(v_a - v_b + v_c - v_d)$$

$$= (m + r' + k + s' + \ell)v_a - (r' + k + s' + \ell)v_b + (n + s' + \ell)v_c - (s' + \ell)v_d$$

$$= \pi_{\text{int}}( (m + r' + k + s' + \ell, -r' - k - s' - \ell, n + s' + \ell, -s' - \ell) + \Lambda ) \in \pi_{\text{int}}(\mathcal{L}).$$

Notice that the projection into the physical space is

$$\pi((m + r' + k + s' + \ell, -r' - k - s' - \ell, n + s' + \ell, -s' - \ell) + \Lambda) = (m,n).$$

Thus

$$(m + r' + k + s' + \ell, -r' - k - s' - \ell, n + s' + \ell, -s' - \ell) + \Lambda = (\pi|_{\mathcal{L}})^{-1}(m,n)$$

so that

$$(m,n)^* = \pi_{\text{int}}\left((\pi|_{\mathcal{L}})^{-1}(m,n)\right)$$

$$= \pi_{\text{int}}((m + r' + k + s' + \ell, -r' - k - s' - \ell, n + s' + \ell, -s' - \ell) + \Lambda)$$

$$= R_0^{(m,n)}(r,s).$$

Let $p = \pi_S(w) \in T_0^S$ be a pattern occurring in the tiling $w$ for some subset $S \subset \mathbb{Z}^2$. Let $[p]$ be the cylinder associated to the pattern $p$ and $A = f_0([p]) \subset W$ be the accepting set. The set $A$ is a polygon by construction (see Equation 6). Therefore the Lebesgue measure of $\partial A$ is zero. Assume for now that $w$ is a generic tiling. Since $R_0^{(m,n)}(r,s) = (m,n)^* \notin \partial A$ for every $m,n \in \mathbb{Z}$, the set $Q \subseteq \mathbb{Z}^2$ of occurrences of $p$ in $w$ is

$$Q = \{(m,n) \in \mathbb{Z}^2 \mid R_0^{(m,n)}(r,s) \in A\} = \{(m,n) \in L \mid (m,n)^* \in A\} = \lambda(A)$$

which is a regular and generic model set. If $w$ is a singular tiling, then $w = \text{TILING}_0^{(r,s)}$ for some $v \in \mathbb{R}^2 \setminus \Theta^p_0$. If $A = f_0([p]) \subset W$, then we take $A' = \lim_{\epsilon \to 0} A \cap (A - \epsilon v)$ as accepting set and we have

$$Q = \{(m,n) \in \mathbb{Z}^2 \mid R_0^{(m,n)}(r,s) \in A'\} = \{(m,n) \in L \mid (m,n)^* \in A'\} = \lambda(A')$$

which is a regular and singular model set. \qed

References and appendix

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Appendix – A DIY Puzzle to illustrate the results

We encode the 11 Jeandel-Rao tiles as follows where each integer color in \{0, 1, 2, 3, 4\} is replaced by an equal number of triangular or circular bumps:

Print one or more copies of this page and cut each of the 25 tiles shown in Figure 9 with scissors. Use the tiles and the Universal solver for Jeandel-Rao tilings shown in Figure 10 to construct any pattern seen in the proper minimal subshift of the Jeandel-Rao Wang shift.

Figure 9. A $5 \times 5$ pattern with Jeandel-Rao tiles ready for laser cut (red means cut and blue means engrave. Tiles should have 3cm size when printed in A4 format.

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Figure 10. The Universal solver for Jeandel-Rao tilings. Any pattern in the minimal subshift of Jeandel-Rao tilings is the coding of the orbit of some starting point by the action of horizontal and vertical translations by 1 unit (3cm when printed in A4 format).