The Einstein-Podolsky-Rosen Paradox and Entanglement 2: Application to Proof of Security for Continuous Variable Quantum Cryptography

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In a previous paper certain measurable criteria have been derived, that are sufficient to demonstrate the existence of Einstein-Podolsky-Rosen (EPR) correlations for measurements with continuous variable outcomes. Here it is shown how such EPR criteria, which do not demand perfect EPR correlations, can be used to prove the extent of security for continuous variable quantum cryptographic schemes (in analogy to that proposed by Ekert) where Alice and Bob hope to construct a secure sequence of values from measurements performed on continuous-variable EPR-correlated fields sent from a distant source. It is proven that the demonstration of the EPR criterion on Alice’s and Bob’s joint statistics compels a necessary loss in the ability to infer the results shared by Alice and Bob, by measurements performed on any third channel potentially representing an eavesdropper (Eve). This result makes no assumption about the nature of the quantum source of the fields transmitted to Alice and Bob, except that the EPR correlations are observed at the final detector locations. In this way a means is provided to establish security in the presence of some loss and less than optimal correlation, and against any eavesdropping strategy employed by Eve prior to detection of the fields by Alice and Bob.

I. INTRODUCTION

Einstein, Podolsky and Rosen [1] (EPR) presented a now famous argument in 1935 in an attempt to show that quantum mechanics is an “incomplete” theory. Their argument was based on the premise of “no action-at-a-distance” and made assumptions about the nature of “reality”. In 1966 Bell [2] showed that the predictions of all theories (called local hidden variable theories) consistent with these EPR premises would satisfy certain constraints called Bell inequalities. He also showed that for some situations the predictions of quantum mechanics will violate these Bell inequalities, meaning an incompatibility of quantum mechanics with local hidden variables.

While Bell’s original work, and subsequent experiments relating to it, applied to situation of discrete spin measurements, the original EPR argument was presented for “position” and “momentum” measurements with continuous variable outcomes. The experimental observation of such continuous variable EPR correlations have been achieved using fields, where the conjugate “position” and “momentum” observables are replaced by the two orthogonal noncommuting quadrature phase amplitudes of the field [1]. The theoretical proposal relating to these experiments employed a two-mode squeezed state as the source of EPR fields. For such experiments it is not possible to demonstrate the perfect correlation as discussed originally by EPR. The experimental signatures are based on a criterion first presented in 1989, and expanded on in a recent paper [10]. The EPR fields generated through the two-mode squeezing interaction have enabled the experimental realization of a continuous variable quantum teleportation.

Quantum cryptography using squeezed or two-mode squeezed states predicting EPR correlations for quadrature phase amplitude measurements with continuous variable outcomes have been recently investigated [12,13]. Of particular interest here is the continuous variable quantum cryptographic scheme analogous to that discussed by Ekert [14] for spin-1/2 systems where Alice and Bob wish to construct a secure key from correlated data sent to each of their locations from an entangled continuous variable EPR source. The original proposal of Ekert proposed to use the correlated spin state shown by Bohm [21] to demonstrate a version of the EPR paradox relating to measurements with discrete outcomes. Bell showed in 1966 that this state (the Bell-state) violates a Bell inequality, and in Ekert’s proposal the violation of the Bell inequality is used to demonstrate security.

The direct continuous variable “position /momentum” measurements that demonstrate the EPR paradox for the two-mode squeezed state cannot by any simple rotation of measurement angle demonstrate a violation of a Bell inequality. The point of this paper is to emphasize that this does not however diminish the usefulness of such a state in for example providing secure mechanisms for quantum cryptography protocols, since one can replace the Bell-inequality used in Ekert’s protocol by an EPR-criterion to test for security.

In this paper we prove how the demonstration of EPR correlations, using the 1989 EPR criterion, by Alice and Bob on their two detected channels puts a limit on the accuracy of any inference made by Eve, about the results of the measurements performed by Alice and Bob. Importantly this is proved for any quantum source, meaning security against any strategies Eve is able to employ prior to Alice and Bob detecting the fields.

To summarize the conclusions of this paper, it is shown that the determination by Alice and Bob of a perfect, maximum EPR correlation in their detected fields implies security against any hypothetical Eve obtaining the
key sequence. If Eve has intercepted one or both of the EPR channels (from the source to Bob or from the source to Alice) in any manner, to obtain the key sequence with any degree of accuracy (to give a noninfinite variance in her estimate of the values), then it is proved that the EPR correlation detected by Alice and Bob could no longer be maximum. It has also been proved that there is no alternative set of quantum fields (source) available to Eve, that would enable her to obtain the key sequence with any degree of accuracy, and still retain the optimal EPR correlation measurable between Bob and Alice.

The situation where a reduced EPR correlation is observed between Alice and Bob is more subtle. Alice and Bob would expect a certain degree of EPR correlation based on measurements performed on their EPR source (and perhaps an expected degree of loss on transmission). If their measured EPR correlation is noticeably reduced (and perhaps an expected degree of loss on transmission). If Eve has intercepted one key sequence, based only on the reliable measurement of a certain amount of EPR correlation between Alice and Bob. This result is of current relevance in that fixed amounts of EPR correlation for continuous variable measurements Alice and Bob. This result is of current relevance in that fixed amounts of EPR correlation for continuous variable outcomes have been (irrefutably) confirmed experimentally (whereas Bell inequality violations have not). The proposal is to then use these limits to encode the message in such a way as to elude Eve.

II. EPR CRITERIA BASED ON CONDITIONAL MEASUREMENTS

We first need to define the EPR criteria, and here the results of a previous paper [7] are summarized. Consider two quantum fields propagating towards two spatially separated location at A and B respectively. The fields will be generated by a appropriate quantum source so that the results of certain measurements are correlated. Two observables \( \hat{x} \) (the “position”) and \( \hat{p} \) (“momentum”) are defined for the subsystem at location A. These observables satisfy an uncertainty relation

\[
\Delta \hat{x} \Delta \hat{p} \geq C
\]  

(1)

but where we will consider from this point on that with appropriate scaling the \( \hat{x} \) and \( \hat{p} \) are now dimensionless and \( C = 1 \). A measurement \( \hat{x}^B \) made at B gives a result \( x_i^B \). In this paper, \( i \) is used to label the possible results, discrete or otherwise, of the measurement \( \hat{x}^B \). The results of measurements \( \hat{x}^A \) and \( \hat{x}^B \) at A and B are correlated, so that the measurement at B enables a prediction to be made about the result of a measurement \( \hat{x}^A \) at A. We define a set of distributions \( P(x|x_i^B) \) giving the probability of a result \( x \) for the measurement at A, conditional on a result \( x_i^B \) for measurement at B. The variance and mean of the conditional distribution \( P(x|x_i^B) \) are denoted by \( \Delta^2 x \) and \( \mu \), respectively.

Also, for certain correlated fields, we can infer the result of measurement \( \hat{p} \) at A, based on a measurement, \( \hat{p}^B \) say, at B. We denote the results of the measurement \( \hat{p}^B \) at B by \( p_j^B \). We also define the probability distribution, \( P(p_i|p_j^B) \) for obtaining the result \( p \) upon measurement of \( \hat{p} \) at A, conditional on the result \( p_j^B \) for the measurement \( \hat{p}^B \) at B. The variance of the conditional distribution \( P(p_i|p_j^B) \) is denoted by \( \Delta^2 p \).

The situation discussed by Einstein-Podolsky-Rosen demands a perfect correlation between the result of measurements \( \hat{x}^A \) and \( \hat{x}^B \), and also between \( \hat{p}^A \) and \( \hat{p}^B \) at B. For this case, the variances of the conditional distributions must satisfy

\[
\Delta_i x = \Delta_j p = 0
\]  

(2)

for all \( i, j \). This situation however is not achievable for continuous variable measurements.

It has been discussed in the previous paper [7] how EPR correlations would be demonstrated where one can establish that each of the conditional distributions \( P(x|x_i^B) \) is very narrow, so that

\[
P(x|x_i^B) = 0 \quad \text{where} \quad |x - \mu_i| > \delta
\]  

(3)

and \( \delta < 1 \). A similar result must be proved for each \( P(p_i|p_j^B) \). As discussed previously [7], this situation represents the spirit of the original EPR gedanken experiment in its truest form, but is difficult to achieve experimentally.

A more readily achievable criterion still sufficient to demonstrate EPR correlations was proposed in 1989 [10] and has been further explained in the previous paper [10]. We first define the weighted variance

\[
\Delta^2_{inj} \hat{x} = \sum_i P(x_i^B) \Delta^2 x
\]  

(4)

and similarly

\[
\Delta^2_{inj} \hat{p} = \sum_j P(p_j^B) \Delta^2 p
\]  

(5)

Here \( P(x_i^B) \) is the probability for a result \( x_i^B \) upon measurement of \( \hat{x}^B \), and \( P(p_j^B) \) is defined similarly. It has been shown that the observation of

\[
\Delta_{inj} \hat{x} \Delta_{inj} \hat{p} < 1
\]  

(6)
imply a demonstration of EPR correlations (the EPR paradox).

It is mentioned that the evaluation of the conditional distributions for each outcome of the continuous variable $x_i^B$ at $B$ is not always practical. It has been discussed previously how it is possible to perform other measurements, closely related to squeezing measurements, that are also sufficient to indicate the EPR/entangled nature of the system. This is the approach used experimentally to date to demonstrate EPR measurements, that are also sufficient to indicate the EPR/entangled nature of the system. The best linear estimate

$$\Delta_{\text{inf},L}^2 \hat{x} = \sum_{x_i^B} P(x_i^B) (\Delta x_i^B)^2 = \langle (\hat{x} - (g \hat{x}^B + d))^2 \rangle. \quad (7)$$

The best linear estimate $x_{\text{ext}}$ is the one that will minimize $\Delta_{\text{inf},L}^2 \hat{x}$. The best choice for $g$ is discussed in [11]. Where $x_{\text{ext}} = \mu_i$ it follows that the variance $\Delta_{\text{inf},L}^2 \hat{x} = \Delta_{\text{inf}}^2 x$. Generally however $\Delta_{\text{inf,L}}^2 \hat{x} \geq \Delta_{\text{inf}}^2 x$. The observation of $\Delta_{\text{inf,L}}^2 \hat{x} \Delta_{\text{inf,L}}^2 \hat{p} < 1 \quad (8)$

implies quantum inseparability, for any $g$ and $d$, and also the situation of the EPR paradox.

III. THE TWO-MODE SQUEEZED STATE AS THE QUANTUM EPR SOURCE

Suppose the two quantum fields are generated via the interaction Hamiltonian $H_I = i\hbar x (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger)$, where $\hat{a}$ and $\hat{b}$ symbolize the boson operators for the fields at $A$ and $B$ respectively. For vacuum initial states this interaction generates, after a finite time $t$, two-mode squeezed light [12]

$$|\psi> = \sum_{n=0}^{\infty} c_n |n>_a |n>_b \quad (9)$$

where $c_n = \tanh r/\cosh r$ and $r = \epsilon t$. This interaction provides a quantum model for the parametric amplifier.

This simple quantum state was shown to be EPR-correlated in reference [12], and EPR correlations using parametric interactions and the criteria (8) have been achieved experimentally. We define the quadrature phase amplitudes

$$\hat{x} = \hat{X}_a = (\hat{a} + \hat{a}^\dagger)$$
$$\hat{p} = \hat{P}_a = (\hat{a} - \hat{a}^\dagger)/i$$
$$\hat{x}^B = \hat{X}_b = (\hat{b} + \hat{b}^\dagger)$$
$$\hat{p}^B = \hat{P}_b = (\hat{b} - \hat{b}^\dagger)/i \quad (10)$$

that are measurable using local oscillator and homodyne detection techniques that were developed originally in efforts to generate and detect squeezed light. The Heisenberg uncertainty relation is $\Delta^2 X_a \Delta^2 P_a \geq 1$.

It is seen from the linear EPR criterion (8) that the following is also a criterion sufficient to demonstrate EPR correlations in the spirit of the original EPR paradox:

$$\Delta (X_a - g X_b) \Delta (P_a + h P_b) < 1 \quad (11)$$

where we have used the linear form

$$\Delta_{\text{inf}} x = \Delta_{\text{inf},L}^2 \hat{x} = \langle (X_a - g X_b)^2 \rangle \quad (12)$$

and $\Delta_{\text{inf}} p = \Delta_{\text{inf},L}^2 \hat{p} = \langle (P_a + h P_b)^2 \rangle$, and $g$ and $h$ are parameters chosen to minimise the variances (the choice of $d$ is best at 0 since the quadrature amplitude means for the vacuum squeezed state are zero).

The two-mode squeezed state predicts (g = h = tanh 2r) the correlations $X_a = X_b$, and $P_a = -P_b$ to give

$$\Delta_{\text{inf},L}^2 \hat{x} = \Delta_{\text{inf},L}^2 \hat{p} = 1/ \cosh 2r, \quad (13)$$

a clear demonstration of EPR correlations for all $r$.

IV. THE CRYPTOGRAPHIC SCHEME

We now consider the application of the EPR state to quantum cryptography with continuous variable outcomes. To summarize, an EPR source emits two fields, one which propagates towards Alice at location $A$, and the other to Bob at location $B$. As one possible strategy, Alice selects to measure randomly either quadrature phase amplitude $X_a = \hat{x}$, corresponding to angle $\theta = 0$, or quadrature phase amplitude $P_a = \hat{p}$, corresponding to angle choice $\theta = \pi/2$, say. Similarly Bob will measure randomly either quadrature phase amplitude $X_b = \hat{x}^B$, corresponding to angle $\phi = 0$, or quadrature phase amplitude $P_b = \hat{p}^B$, corresponding to angle choice $\phi = \pi/2$. As discussed in [11] and shown previously in [12], for the choices $\theta = \phi = 0$, as the two-mode squeeze parameter $r$ becomes large, the results $x$ of Alice’s measurement and $x_i^B$ for Bob’s measurement, will become identical. For the choices $\theta = \phi = \pi/2$ the results $p$ for Alice and $p_i^B$ are also correlated (anticorrelated in fact): for large $r$ we have $p = -p_i^B$.

In the style of the original quantum cryptographic proposals [20], we consider here the proposal where Alice communicates to Bob after the measurements (through a public channel) her choice of measurement angle $\theta$ and
the result for the measurement, for a subensemble, randomly selected after the detections. Bob is able to check his measurements and compare his value for the result of measurements that should be correlated with Alice’s.

Alice and Bob can then use their shared subensemble to calculate the conditional probabilities \( P(x|x_i^B) \) and \( P(p|p_i^B) \) and the associated respective variances \( \Delta_x^2 \) and \( \Delta_p^2 \) of these distributions defined in Section 2. Here \( x \) is Alice’s result for \( \hat{x} \), and \( x_i^B \) is the result for Bob’s measurement (which we have symbolized by \( \hat{x}^B \)) correlated with \( \hat{x} \). Similarly \( p \) is Alice’s result for \( \hat{p} \), and \( p_i^B \) is the result for Bob’s measurement (symbolized by \( \hat{p}^B \)) anti-correlated with \( \hat{p} \). For our particular two-mode squeezed state (5) the prediction is

\[
\Delta_i x = \Delta_i p = 1/\sqrt{\cosh 2r} \tag{14}
\]

for all \( i, j \).

The maintenance of the EPR correlation (between Alice and Bob’s fields) is determined through examination of the individual \( \Delta x \) or \( \Delta p \), and through a measured degree of violation of the 1989 EPR criteria (3). The proposal is that this is used to check, or establish a degree of security.

For example if perfect EPR correlations are established between Bob and Alice, it will be proved that there could have been no intervention on the channels from the source to Alice and Bob, or reconstruction of an alternative source, by an eavesdropper Eve. Where Alice and Bob are able to confirm narrow conditional distributions satisfying (5), it is possible to establish the necessity of a certain degree of fuzziness in Eve’s data. Once security is established, the measurement angle for the remaining subset is shared, and where the choice of angle is to predict correlation between results, the sequence of common shared values can be used in some manner to form a key.

It is the objective here to prove security against any strategy Eve could adopt prior to the detection of the transmitted fields by Alice and Bob. For example Eve might interfere with and retransmit one or both of the fields in some manner to forward to Alice and/ or Bob. Alternatively she might sabotage the EPR source to substitute an alternative three-channeled correlated source, where the three transmitted beams propagate to Alice, Bob and also Eve at a third location. Eve could then potentially perform a final measurement after public communication of Alice’s angle choice. Alternatively, where Alice and Bob use an EPR source with less than optimal correlations, as would be the realistic situation, Eve might replace the EPR source with one showing improved EPR correlations. This might enable her to tap some of the signal for the purpose of eavesdropping, while the decrease in EPR correlations that could be a consequence of her tapping would go unnoticed by Alice and Bob who expect a more weakly correlated signal anyway.

It is possible to consider cryptographic schemes where security is established on the basis of the assumption of a secure source, and also a secure channel from the source to Alice (see for example [17]). In such schemes, the tapping of the channel from the source to Bob shows as a loss of EPR correlations which can be detected by Alice and Bob, alerting them to Eve’s interference.

However for systems of perfect EPR correlation (such as generated from the Bohm-Bell state used in the original scheme of Ekert) a stronger proof of security is possible without these assumptions of secure source and second EPR channel. It then becomes relevant to determine the extent of such security possible for the continuous variable two-mode squeezed EPR state (5), whose EPR correlations have been experimentally confirmed, but which for finite squeeze parameter \( r \) is always less than optimally correlated.

V. PROOF OF SECURITY

We now need to give the proof of security for such an EPR scheme. It is assumed as usual that the choice of angles \( \theta \) and \( \phi \) (whether to measure \( \hat{x} \) or \( \hat{p} \)) for Bob’s and Alice’s measurements are randomly and independently chosen after the transmission of the fields to Alice and Bob at secure spatially separated locations. Alice and Bob make delayed-choice measurements. Therefore we assume that an eavesdropper (Eve) cannot anticipate the angle choice prior to Alice and Bob receiving the fields. This must also be true of the selection of the subensemble used by Alice and Bob to evaluate the statistics to test security. In this way it is assumed that the statistics evaluated by Bob and Alice on the subensemble accurately reflect the statistics of the entire fields received by Alice and Bob.
FIG. 1. EPR-correlated fields reach Alice and Bob who perform delayed-choice measurements ("position" or "momentum") as determined by the choice of \( \theta \) or \( \phi \) at spatially separated locations. The correlated results can then form the shared values for a key. If Alice and Bob can prove that their fields are EPR-correlated according to the 1989 criterion \( \delta \), then it is impossible for any eavesdropper Eve at a third location to have an accurate replica of the correlated results shared by Alice and Bob. Eve’s estimate of Alice’s values is necessarily less accurate than Bob’s, and the degree of inaccuracy can be evaluated from the details of the EPR-correlation. This is true regardless of the nature of the original quantum source.

The proof of security presented here then gives a method to determine the level of security based only on the nature of the measured statistics evaluated between Alice and Bob after the detection of their fields, and therefore involves no assumptions regarding the amount of loss occurring during propagation or the nature of the original quantum state. We present a proof of security by demonstrating the impossibility of Eve, an eavesdropper at a third spatial location, having (or being able to obtain) a perfect copy of the results \( x, p \) shared by Alice and Bob, if Alice and Bob measure through their subensemble an EPR correlation based on their measurements \( \Delta_i x \) and \( \Delta_i p \) (see Figure 1).

An eavesdropping process by Eve results in the generation of a final quantum state describable by a density operator symbolized by \( \rho \). For example if Eve attempts to extract information by intercepting Bob’s channel, quantum mechanically Eve’s measurement process is represented by a Hamiltonian that acts for some duration, there being an initial quantum state describing Bob’s and Alice’s fields. The final state \( \rho \) that is produced after the interaction (we may also consider a series of interactions that may involve destructive measurements and state generation) describes Alice, Bob and Eve’s final fields that are eventually detected and undergone measurements by Alice, Bob and Eve at their different final spatial locations.

Eve attempts to gain the results of Alice’s \( \hat{x} \) (or \( \hat{p} \)) through some measurement on her field symbolized by \( \hat{x}^E \) (and \( \hat{p}^E \)). The quantum state \( \rho \) predicts probability distributions for the outcomes of all possible measurements performed by Alice, Bob and Eve: for example a probability distribution \( P_{xx,p}(x, x_i^B, p_i^E) \) for the outcomes \( x, x_i^B, p_i^E \) of Alice’s, Bob’s and Eve’s results of measurement \( \hat{x}, \hat{x}_i^B, \hat{p}_i^E \); and a \( P_{p,p}(p, x_i^B, p_i^E) \) for the outcomes \( p, x_i^B, p_i^E \) of Alice’s, Bob’s and Eve’s results of measurement \( \hat{p}, \hat{x}_i^B, \hat{p}_i^E \).

We define the probability \( P(x\{x_i^B, p_i^E\}) \) of a result \( x \) for Alice’s measurement of \( \hat{x} \), conditional on the results \( x_i^B \) and \( p_i^E \) for Bob’s \( \hat{x}_i^B \) and Eve’s \( \hat{p}_i^E \) respectively. We also define the probability \( P(p\{x_i^B, p_i^E\}) \) of a result \( p \) for Alice’s measurement of \( \hat{p} \), given the results \( x_i^B \) and \( p_i^E \) for Bob’s \( \hat{x}_i^B \) and Eve’s \( \hat{p}_i^E \) respectively.

A constraint is placed on the variances \( \Delta_i^2 x \) and \( \Delta_i^2 p \) of the conditional distribution \( P(x\{x_i^B, p_i^E\}) \) and \( P(p\{x_i^B, p_i^E\}) \) respectively, for any possible quantum state \( \rho \). The predicted statistics of Alice’s measurements conditional on measurements performed by Bob and Eve are described by the reduced density operator \( \rho_A = \langle x_i^B | (\rho E)^p | x_i^B \rangle / N \) (where \( N \) is a normalization factor). The variance \( \Delta_i^2 x \) of the conditional distribution \( P(x\{x_i^B, p_i^E\}) \) gives the uncertainty in the estimate of Alice’s \( \hat{x} \) conditional on the results \( x_i^B \) and \( p_i^E \) for Bob’s \( \hat{x}_i^B \) and Eve’s \( \hat{p}_i^E \) measurements. Bob’s (and Eve’s) measurement constitutes a measurement of Alice’s \( \hat{x} \), with precision \( \Delta_i^2 x \). The uncertainty relation will imply the constraint (for the quadratures as defined by (10), \( C = 1 \))

\[
\Delta_i x \geq 1/\Delta_i p
\]

The marginal distribution \( P(x|x_i^B) \), the probability of Alice’s result \( x \) for measurement \( \hat{x}_i^B \) conditional on the result \( x_i^B \) for Bob’s measurement \( \hat{x}_i^B \), is given by

\[
P(x|x_i^B) = \frac{\sum_{p_i^E} P_{xx,p}(x, x_i^B, p_i^E)/P(x_i^B)}{\sum_{p_i^E} P(x\{x_i^B, p_i^E\})f_{p_i^E}}
\]

where \( f_{p_i^E} \) is the fraction \( f_{p_i^E} = P(x_i^B, p_i^E)/P(x_i^B) \); \( P(x_i^B, p_i^E) \) is the probability for result \( x_i^B \) and \( p_i^E \), respectively, upon joint measurement \( \hat{x}_i^B \) and \( \hat{p}_i^E \); and \( P(x_i^B) \) is the probability of \( x_i^B \) for Bob’s measurement of \( \hat{x}_i^B \). This implies the following relationship for the variance \( \Delta_i^2 x \) of the conditional distribution \( P(x|x_i^B) \).

\[
\Delta_i^2 x \geq \sum_{p_i^E} f_{p_i^E} \Delta_i^2 p_i^E
\]

The accuracy of the information obtainable by Eve is determined by the standard deviation \( \Delta_i^2 p \) of \( P^E(p\{p_i^E\}) \), the conditional distribution for result \( p \) for Alice’s measurement of \( \hat{p} \) given a result \( p_i^E \) for Eve’s \( \hat{p}_i^E \). This marginal distribution is given by

\[
P^E(p|p_i^E) = \sum_{x_i^B} P(p\{x_i^B, p_i^E\})f_{x_i^B}
\]

where the fraction \( f_{x_i^B} \) is defined as \( f_{x_i^B} = P(x_i^B, p_i^E)/P(p_i^E) \); \( P(p_i^E) \) is the probability for Eve’s result \( p_i^E \) upon measurement of \( \hat{p}_i^E \). The \( \Delta_i^2 p \) is related to an average of these variances as given by

\[
(\Delta_i^2 p)^2 \geq \sum_{x_i^B} f_{x_i^B} \Delta_i^2 p
\]
A. The case of perfect correlation

First we present a proof of security for the case where Alice and Bob measure perfect EPR correlations, meaning that all $\Delta_i x = \Delta_j p = 0$. This is the case if the quantum source is a simultaneous eigenstate of $X_a - X_b$ and $P_a + P_b$, which is closest to the situation originally defined by EPR. The variance $\Delta^2 x$ is zero in this case we consider initially of perfect EPR correlations. It must follow therefore from (17) that each $\Delta_i q x$ is also zero. Using the constraint (15) each $\Delta_i q p$ must then be infinite. Therefore each $\Delta^2 p$ must also be infinite. Each conditional variance, for all possible outcomes $p^E_q$, is infinite, meaning that any measurement performed by Eve will give an infinite uncertainty in the prediction of Alice’s $p$.

The same logic applied to joint measurements of Bob’s $p^B$ and Eve’s $x^E$ implies an infinite variance for Eve’s estimate of Alice’s $x$. In this way it is deduced that Eve’s estimates of each of Alice and Bob’s sequential $x, p$ values (these constitute the final key) will have an associated infinite uncertainty.

To summarize, the determination by Alice and Bob of an optimal EPR correlation

$$\Delta_i x = 0, \Delta_j p = 0$$

for all $i, j$, in their detected fields implies security that there can be no hypothetical Eve, at a third location as indicated in Figure 1, able to obtain the key sequence. This is proof that if Eve has intercepted one or both the EPR channels to obtain the key sequence with any degree of accuracy (so that there is a noninfinite variance in her estimate of the Alice and Bob’s key values), the EPR correlation would necessarily have been reduced to give a nonzero result for at least one of the $\Delta_i x, \Delta_j p$. It has also been proved that there is no alternative set of quantum fields (source) available to Eve, that would allow her to obtain the key sequence with any degree of accuracy and still retain the optimal EPR correlation between Bob and Alice.

B. The case of reduced correlation but where all conditional distributions are narrow

Considering that a practical experiment will not have perfect correlation (values of $\Delta_i x$ might typically be 0.7 for current situations), we need to argue more generally. It is still possible to derive limits on Eve’s knowledge of the key sequence, based only on the reliable measurement of a certain amount of EPR correlation between Alice and Bob (and the assumption that Eve does not have access to the random choice of Alice and Bob’s measurements and subensemble selection, both of which are selected after transmission and detection of the fields by Alice and Bob). The proposal, such as that discussed in Section 6, is to use these limits to encode the message in such a way as to defeat Eve.

Based on Alice and Bob’s measurements over the subensemble, the conditional probability distribution $P(x|x^B_i)$ (and $P(p|p^B_j)$) can be measured by Alice and Bob, and their associated variances $\Delta^2 x (p)$ can be evaluated. We examine in this subsection the case where all conditional distributions measured by Alice and Bob are shown to have a nonzero but small standard deviation, so that for example where these distributions are Gaussian Alice and Bob demonstrate $\Delta x < 1/3$ and $\Delta p < 1/3$.

Most generally the variances such as $\Delta^2 x$ are related to the individual variances $\Delta_i q x$ by the relation (17). Although the $\Delta_i x$ might be small, an individual $\Delta_i q x$ might not be. The possibility cannot be ruled out that Eve is able to obtain upon some (small) fraction of her measurements the result of Alice’s $p$ to good accuracy. The relationship given by (17) is certainly true however. Suppose all (that is for all $i$) the conditional distributions $P(x|x^B_i)$ measured by Alice and Bob are sufficiently narrow so that the probability of obtaining a result $x$ outside a range $\mu_i - \delta \leq x \leq \mu_i + \delta$ is zero i.e.

$$P(x|x^B_i) = 0, \text{ for } |x - \mu_i| > \delta$$

and we assume $\delta < 1$ ($C = 1$). A similar result must be proved for each $P(p|p^B_j)$. Recall here $\mu_i$ is the mean of the particular distribution $P(x|x^B_i)$. In this case based on (17) we can say for sure that, for a given fixed $x^B_i$, each of the $P(x|x^B_i, p^E_j)$ must also satisfy $P(x|x^B_i| p^E_j) = 0$ outside the range $\mu_i \pm \delta$. This implies that each variance $\Delta^2 q x$ (of $P(x|x^B_i, p^E_j)$) could not exceed the value of $\delta^2$, implying in turn by (17) that each $\Delta_i q x$ must satisfy

$$\Delta_i q x \geq 1/\delta.$$  

In this way, since this is true for all $i$, and using (17), it is proved that the uncertainty (standard deviation) in each of Eve’s estimates of Alice’s $p$ (this uncertainty is the standard deviation of the conditional distribution $P(p|p^B_j)$ as defined above) will exceed $1/\delta$.

$$\Delta^2 p \geq 1/\delta.$$  

For the two-mode EPR state (1), the conditional distributions are predicted to be Gaussian with variance given as $\Delta x = \Delta p = 1/\sqrt{\cosh 2r}$. Of course the actual distributions must be measured by Alice and Bob as part of the security procedure. For $r$ sufficiently large so that (recall $\delta \leq 1$)

$$\Delta x < \delta/3 \leq 1/3$$

the Gaussian distribution is predicted to be negligible at $x > \delta$, and there is then proof that Eve’s best possible estimates satisfy

$$\Delta^2 p \geq 1/\delta.$$  


\[ \Delta^E_T p \geq 1/\delta. \] (25)

This means that as Eve performs the measurement \( \hat{p}^E \) to obtain a result \( \hat{p}^E \), the standard deviation of the conditional distribution \( P^E(p|p^E_q) \) (for Alice’s result \( p \) conditional on Eve’s outcome) exceeds \( 1/\delta \) for every possible outcome \( p^E_q \), and for all possible measurements \( \hat{p}^E \). Similar logic applied to Alice and Bob’s conditional distributions \( P(p|p^B_j) \) gives a corresponding limit on the error in each of Eve’s estimates of Alice’s result for \( \hat{x} \).

In this way it is derived that a minimum degree of uncertainty or fuzziness (as given by \( \Delta^E_T p \geq 1/\delta, \Delta^E_T p \geq 1/\delta \)) exists for Eve’s estimate of every piece of the key sequence shared by Alice and Bob. If \( \eta_{Eve} \) is Eve’s estimate of the particular key value, and \( \eta \) is Alice’s actual key value, then we have \( (\langle \eta_{Eve} - \eta \rangle^2)^{1/2} \geq 1/\delta \).

VI. A POSSIBLE ENCRYPTION PROTOCOL

Using the prediction for the two-mode squeezed state \( \Delta_{mf} \hat{x} = \Delta_{mf} \hat{p} = \Delta \hat{x} = 1/\sqrt{\cosh 2r} \to 0 \) as \( r \to \infty \), we see that as the squeeze parameter \( r = \kappa t \) it increases it becomes possible for Bob to resolve Alice’s \( x \) or \( p \) value while Eve can only resolve with error \( \to \infty \). The measured quadrature phase amplitude values shared between Alice and Bob form a secure key sequence denoted by the sequence \( \eta_m, m = 1, 2, \ldots \). We will define the key \( \eta_m \) to consist of Alice’s relevant measured values \( x \) or \( p \), though in the limit of \( r \to \infty \) there is no deviation of Bob’s measured values \( x^B_i \) (or \( p^B_j \)) from Alice’s.

The data given by the variable \( z_m \) is encoded, suppose simplistic to give a transmitted classical amplitude or number \( y_{m,.sent} = z_m + A\eta_m \) (where \( A \) is a relative amplification factor). The key \( \eta_m \) known to both Alice and Bob enables Bob to decode the signal \( z_m \), whereas Eve will have an infinite uncertainty in her measurement of \( \eta_m \), and therefore \( z_m \).

The chief difficulty for Alice and Bob comes for finite \( r \) where the EPR correlation is reduced. Suppose initially that Alice and Bob’s measurements of the EPR correlation and associated conditional distributions, enable them to establish that the probability distribution \( P(x|x^B_i) \) of Alice’s result \( x \) conditional on Bob’s result \( x^B_i \) is a distribution with mean \( \mu_x \) and standard deviation \( \sigma \).

The two-mode squeezed state (9) predicts a Gaussian distribution \( \mu_x = \tanh r x^B_i / \theta = 0 \) and \( \sigma = 1/\sqrt{\cosh 2r} \). We suppose a similar result is achieved for \( P(p|p^B_j) \): (9) predicts the Gaussian with \( \mu_j = -\tanh r p^B_j / \theta = 0 \) and \( \sigma = 1/\sqrt{\cosh 2r} \).

Bob’s key sequence is the sequence \( \eta_{m,.Bob} \) that he builds up by selecting, for each of his relevant measurements, \( \eta_{m,.Bob} = \mu_j \) where he obtained an outcome \( x^B_i \) upon measurement of \( \hat{x}^B \), or \( \eta_{m,.Bob} = \mu_j \) where he obtained \( p^B_j \) upon measurement of \( \hat{p}^B \). The deviation of Bob’s key value from Alice’s key value is then

\[ (\langle \eta_m - \eta_{m,.Bob} \rangle^2) = \sigma \] (26)

(The choice \( \eta_{m,.Bob} = \mu_x, \mu_j \) made by Bob minimizes this rms error.) Bob’s estimate of the decoded data \( z_m \) is \( z_{m,.Bob} = y_{m,.sent} - A\eta_{m,.Bob} \). His rms error is

\[ (\langle z_m - z_{m,.Bob} \rangle^2) = A^2 (\langle \eta_m - \eta_{m,.Bob} \rangle^2) = A^2 \sigma^2. \] (27)

A satisfactory binning by Alice of her data \( z_m \) enable Alice and Bob to share precisely such a signal sent by Alice. This is determined by Alice and Bob, based on their knowledge of the conditional statistics measured over the subensemble. For example let us assume that Alice and Bob’s distributions are Gaussian. In this case there is a negligible chance (0.0027) of \( |\eta_m - \eta_{m,.Bob}| \) being greater than \( 3\sigma \), so that if Alice restricts the \( z_m \) to be one of a series of numbers separated by \( 6A\sigma \), then Bob will round off correctly to reconstruct the correct signal.

However Eve’s decoded data is \( z_{m,.E} = y_{m,.sent} - A\eta_{m,.Eve} \) where \( \eta_{m,.Eve} \) is Eve’s key. Consider the situation discussed above in equation (24) where every one of Bob’s measured conditionals are Gaussian and satisfy \( \Delta x = \Delta x^p = \sigma = \delta/3 (\delta < 1) \). With (23) we conclude that each standard deviation of Eve’s conditional distributions \( P^E(x|x^E_r) \) and \( P^E(p|p^E_r) \) satisfies \( \Delta^E_T p > 1/\delta, \Delta^E_T x > 1/\delta \). The rms error of Eve’s signal must satisfy \( (\sigma = \delta/3) \)

\[ (\langle z_m - z_{m,.Eve} \rangle^2) = A^2 (\langle \eta_m - \eta_{m,.Eve} \rangle^2) \geq A^2/\delta^2 = A^2/9\sigma^2 \] (28)

Suppose Bob and Alice’s correlation reveals \( \sigma = 1/3 \) (which is the largest value that is sensible to this particular approach). Then Eve’s best could not do better than \( (\langle z_m - z_{m,.Eve} \rangle^2)^{1/2} = A \).

On the basis of the assumption of a particular form for Eve’s conditional distributions \( P(x|x^E_r) \) and \( P(p|p^E_r) \) (eg Gaussian), a minimum error rate for Eve’s information could be calculated. The probability of Eve evaluating Alice’s \( z_m \) outside of the range \( z_m \pm 3A\sigma \), and to therefore establish the incorrect value for \( z \), is significant if Eve’s conditional distributions are Gaussian (the probability of an incorrect \( z_{m,.Eve} \) being 0.3173). The Gaussian calculation for Eve’s error rate is relevant, in that the EPR channels generated from the source (4), and subsequently interfered with by Eve through mechanisms able to be modeled by linear interaction Hamiltonians/couplings such as \( \hat{a}_{Eve} \) and \( \hat{b} \) symbolize the boson operators for Eve’s and Bob’s fields respectively)

\[ H_1 = \kappa (\hat{a}_{Eve}^\dagger \hat{b}^\dagger + \hat{a}_{Eve} \hat{b}) \] (29)

with vacuum or squeezed state inputs, would predict such Gaussian conditional distributions. Examples of such linear eavesdropping strategies, have been discussed previously [12, 13].
VII. PROOF OF SECURITY FOR WEAKER CORRELATION

The above protocol requires narrow conditional distributions, \( \Delta_{x} \), \( \Delta_{p} < 1/3 \) (ie \( \sigma < 1/3 \) for Gaussian distributions). With reported measured values of \( \Delta_{x}^{2} = \sum P(x) \Delta_{x}^{2} \approx 0.7 \) such a value is probably not currently achievable. Here we present a more general strategy which can apply where variances satisfy \( \Delta_{x} < .57, \Delta_{p} < .57 \).

We define the set of probabilities \( \{ P(x_{E}^{i}), P(p_{E}^{i}) \} \) and uncertainties \( \{ \Delta_{x}^{E}, \Delta_{p}^{E} \} \) on the key sequence (this being Alice’s sequence of relevant \( \hat{x}, \hat{p} \) results). Here \( P(x_{E}^{i}) \) is the probability of Eve obtaining a result \( x_{E}^{i} \) upon a measurement \( \hat{x} \), and \( \Delta_{x}^{E} \) is the standard deviation of the probability distribution \( P^{E}(x|x_{E}^{i}) \) for Alice’s result \( x \) for \( \hat{x} \), conditional on Eve’s result \( x_{E}^{i} \). This set must be compared with the set of probabilities \( \{ P(x_{r}^{i}), P(p_{r}^{i}) \} \) and uncertainties \( \{ \Delta_{x}, \Delta_{p} \} \) that determine Bob’s accuracy of information of Alice’s \( x, p \) data.

First, where the correlation between Alice’s and Bob’s data is sufficient to satisfy the 1989 criteria \[ (1) \] for EPR, it can be shown that the sets of statistics are necessarily different: that determining Eve’s information involving greater uncertainties than that determining Bob’s information. We show this as follows.

If we assume that Eve’s set is identical to Bob’s, we obtain a contradiction. We could define the joint probability \( P_{i,q} = P(x_{r}^{i}, p_{r}^{i}) \) of the result \( x_{r}^{i} \) for Bob’s measurement \( \hat{x} \) and of the result \( p_{r}^{i} \) for Eve’s measurement \( \hat{p} \). As before we define the variances \( \Delta_{i,q}^{x} + \Delta_{i,q}^{p} \) of the probability distributions for Alice’s result of measurement \( \hat{x} \) and \( \hat{p} \) respectively, conditional on Bob’s and Eve’s results \( x_{E}^{i}, p_{E}^{i} \). The prediction for the average conditional variance (as measured by Alice and Bob) is given by \( \Delta_{inf}^{2} = \sum P(x_{E}^{i}) \Delta_{i,q}^{x} \). Also if Eve’s inferred statistics are to be the same as Bob’s, the quantity \( \Delta_{inf}^{2} \) measured on Eve’s statistics must equal Alice and Bob’s measure of the average \( \Delta_{inf}^{2} \). Applying \[ (13) \] and \[ (19) \] and the Cauchy-Schwarz inequality we would always predict

\[
\Delta_{inf}^{2} = \sum_{q} P(p_{q}^{E})(\Delta_{q}^{E})^{2} \geq \left( \sum_{i,q} P(x_{i}^{E} p_{q}^{E}) \right)^{2} \]  
\[
\geq \sum_{i,q} P(x_{i}^{E} p_{q}^{E}) \Delta_{i,q}^{2} \]  
\[
(\Delta_{inf}^{2})^{2} \geq 1 \]  
\[
(\Delta_{inf}^{2})^{2} \geq 1 \]  

This is not the case given that Bob’s and Alice’s statistics show the 1989 EPR criterion. In other words, the demonstration of the EPR criteria for Bob’s and Alice’s statistics ensures that there is a loss of information, as compared to Bob, on Eve’s channel.

An increase in Eve’s error of inference on the data shared by Alice and Bob follows necessarily from Alice and Bob’s measurements of the general EPR correlations using the 1989 criterion. It is required however to employ this fact in a satisfactory way to enable Bob full information on a signal transmitted by Alice, while leaving Eve unable to decode. Above we have considered strategies where all conditionals have narrow variances \( \Delta_{x} < .3, \Delta_{p} < .3 \) for all \( i, j \) in the fashion of a strong EPR paradox.

Now we consider particular strategies for the situation of inference variances \( \Delta_{inf}^{2} > .3 \). First it is possible to prove that

\[
\Delta_{inf}^{2} \geq 1/\Delta_{inf}^{x} \]  

(31)

From result \[ (19) \] we have

\[
(\Delta_{inf}^{2})^{2} = \sum_{q} P(p_{q}^{E})(\Delta_{q}^{E})^{2} \]  
\[
\geq \sum_{i,q} P(x_{i}^{E} p_{q}^{E}) \Delta_{i,q}^{2} \]  
\[
(\Delta_{inf}^{2})^{2} \geq 1 \]  

(32)

Using the Cauchy-Schwarz inequality and \[ (15) \] we obtain \[ (33) \].

Let us suppose then that Alice and Bob establish a uniform set of Gaussian conditional distributions with variances

\[
\Delta_{i} x = \Delta_{i} p = \Delta_{inf} x = \sigma < 1. \]  

(34)
Alice can choose to bin her signal values to the nearest number of a sequence separated by 6.4\(\sigma\), as described in Section 6. For the case where the two-mode squeezed state is used as a source the conditional distributions are Gaussian (this must be verified by Alice and Bob upon measurements). The probability of Bob decoding the wrong signal value (this is the probability that his value for the key deviates from Alice’s by more than three standard deviations) is therefore negligible (0.0027), meaning that Bob can use his slightly fuzzy key to decode correct signal values.

The average deviation of Eve’s estimate (the mean of her conditional distribution) from Alice’s measured key value \(x\) (or \(p\)) is given by the average variances of her conditional distributions. These must satisfy

\[
\Delta_{inf} E p \geq 1/\sigma
\]

\[
\Delta_{inf} E x \geq 1/\sigma.
\] (35)

The probability of an error (that Eve will decode Alice’s signal \(z_m\) incorrectly) is the probability that Eve’s conditional distributions (\(P_E(x/x_E^p), P_E(p/p_E^x)\)) deviate from the mean by an amount greater than 3\(\sigma\).

First, provided 1/\(\sigma\) > 3\(\sigma\) (\(\sigma < .57\)), it is necessary that Eve, in order to achieve an average variance \(\Delta_{inf} E p\) or \(\Delta_{inf} E x\) satisfying (35), will have key values deviating from Alice’s key value by an amount greater than 3\(\sigma\): |\(\eta_m - \eta_{m,Eve}\) > 3\(\sigma\) for some value of \(\eta_{m,Eve}\). This is true for any hypothetical eavesdropping scheme Eve might have employed. In other words it is proven that Eve will decode at some point to obtain wrong signal value \(z_m\) sent by Alice. Of course the signal values \(z_m\) that are now shared accurately by Alice and Bob, but not by Eve, need not form the final message, but can be used as a discrete key to encode a further signal.

A calculation of the Eve’s error rate based on the assumption that her conditionals are Gaussian distributions with equal \(\Delta_q^E\) is however immediately possible, for any \(\delta < 1\). For example, where \(\sigma = \Delta_q x = \Delta_q p = \Delta_{inf} x = \Delta_{inf} p = .57\), the probability of Eve’s error is .32.

Of course as discussed in Section 6, for absolutely secure cryptography, the exact nature of Eve’s conditional distributions cannot be measured by Alice and Bob and therefore cannot be assumed. Since in (35) we only restrict the average inference error, we have not ruled out that Eve is able to achieve very narrow conditional distributions for most \(q\), to obtain the correct result for Alice’s signal for most of the signal sequence. This situation however could only be achieved if Eve has a very significant \(\Delta_q x\), for some \(q\), and therefore a high deviation between Alice’s and Eve’s measured key values for some of the key sequence, which would cause a large deviation \(z_m, Eve - z_m\) of Eve’s decoded signal from Alice’s, for some \(m\). As discussed in Section 6, the encoding protocol would then need to make use of, not only an Eve’s error rate, but of possibly large individual errors, to reduce her ability to decipher any final message.

**VIII. CONCLUSION**

It is proved in this paper that fields demonstrated by Alice and Bob (at two spatially separated locations) to have certain EPR correlations, enable Alice and Bob to share the results of measurements to a great accuracy. This accurate knowledge of a sequence of results of certain measurements cannot be shared by a third experimenter or eavesdropper Eve at a different location. In the case of EPR correlations that are less than ideal, certain limits on Eve’s accuracy of inference of the sequence of values that form the key have been derived. This conclusion makes no assumptions about loss or the nature of the quantum source, except that EPR correlations are measured by Alice and Bob, and therefore provides a security against all strategies Eve may take to eavesdrop prior to Alice and Bob receiving the fields.

For situations where EPR correlations between Bob and Alice are not perfect, it is still possible for Alice and Bob to reconstruct a shared key or signal sequence where the values are shared with perfect accuracy. The fuzziness placed on Eve’s key values means that Eve will necessarily at some point decipher incorrectly, and some specific strategies are presented for the case where the averages of the variances of Bob’s conditional distributions (determining Alice’s result based on his measurements) satisfy \(\Delta_{inf} x < .57\) and \(\Delta_{inf} p < .57\). Specific error rates for Eve’s key or decoded signal can be established where a particular form (for example a Gaussian as would be the case for various linear eavesdropping strategies) for her conditional distributions are assumed. Generally, the given encoding scheme must use Eve’s proven nonzero error rate for a key sequence to establish that it is not practical for Eve to decipher a final message.

Lastly a comment must be made on what could be concluded on the basis of measurements that would appear to be currently achievable (\(\Delta_{inf} x < .57\) probably is not). Reported measurements are close to \(\Delta_{inf} x = \Delta_{inf} p = .7\). The inference variances are measured experimentally in this case through the linear estimate described by (12). Our results (11) then prove that Eve’s conditionals satisfy \(\Delta_{inf} E x > 1.4, \Delta_{inf} E p > 1.4\). Assuming that the conditionals could be measured to be Gaussian and uniform with a standard deviation \(\sigma = \Delta_q x = \Delta_q p = .7\), we could apply the strategy discussed in Section 6 to allow Bob to share Alice’s discretized data without significant error. The probability of Bob making an error is the probability that the conditional distribution gives greater than 3\(\sigma = 2.1\) (a negligible error rate of .0027).

If we assume Eve’s conditional distributions are also Gaussian, then her standard deviation is at least 1.4 and her error rate is at least 0.136, fifty times greater.
than Bob’s. This gives a proven measure of a level of security against all eavesdropping strategies employed by Eve that would result in her conditionals being Gaussian. Such strategies include the use of any two-mode squeezed state-EPR source of the type (9) generated by the parametric amplification discussed in Section 3, in conjunction with any lossy mechanisms or eavesdropping strategies involving linear beamsplitters such as given by the coupling (29). We cannot prove this generally however for any strategy taken by Eve, since she may have a conditional distribution with standard deviation 1 but where none of her results deviate from the mean by more than 2.1 (to give the same error rate on Alice’s discretized data as Bob’s). It is noted in conclusion however that it has been proved generally (see equation (30)) that Eve’s estimates of the continuous values that form Alice’s original key sequence are more fuzzy than Bob’s.

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