ON A CONJECTURE FOR TRIGONOMETRIC SUMS BY S. KOUMANDOS AND S. RUSCHEWEYH

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ABSTRACT. S. Koumandos and S. Ruscheweyh [5] posed the following conjecture: For $\rho \in (0, 1]$ and $0 < \mu \leq \mu^*(\rho)$, the partial sum $s_\mu^n(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} z^k$, $0 < \mu \leq 1$, $|z| < 1$, satisfies

$$(1 - z)^\rho s_\mu^n(z) \prec \left(1 + \frac{z}{1 - z}\right)^\rho, \quad n \in \mathbb{N},$$

where $\mu^*(\rho)$ is the unique solution of

$$\int_0^{(\rho+1)\pi} \sin(t - \rho \pi) t^{\mu-1} dt = 0.$$

This conjecture is already settled for $\rho = \frac{1}{3}, \frac{1}{4}, \frac{3}{4}$ and $\rho = 1$. In this work, we validate this conjecture for an open neighbourhood of $\rho = \frac{1}{3}$ and in a weaker form for $\rho = \frac{2}{3}$. The particular value of the conjecture leads to several consequences related to starlike functions.

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1. Preliminaries and Main results

Let $A$ be the class of analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$. Let $s_n(f, z)$ be the $n$th partial sum of the power series expansion of $f(z) = \sum_{k=0}^{\infty} a_k z^k$. For $\mu > 0$, let $s_\mu^n(z)$ be the $n$th partial sum of $(1 - z)^{-\mu} = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} z^k$ where $(\mu)_k$ is the Pochhammer symbol defined by $(\mu)_k = \mu(\mu + 1) \cdots (\mu + k - 1)$. An important concept namely subordination plays a vital role in this work. For two analytic functions $f, g \in A$, $f$ is subordinate to $g$, defined as $f \prec g$ by $f(z) = g(\omega(z))$ if $\omega(z)$ satisfies $|\omega(z)| \leq |z|$, $z \in \mathbb{D}$. If $g$ is univalent in $\mathbb{D}$, then $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$ are the sufficient conditions for $f \prec g$.

Let $S^*(\gamma)$ be the class of starlike function of order $0 \leq \gamma < 1$ defined as

$$S^*(\gamma) := \left\{ f \in A : f(0) = 0 = f'(0) - 1, \frac{zf'}{f} - \gamma \prec \frac{1 + z}{1 - z} \right\}$$

for $z \in \mathbb{D}$. Clearly, the function $zf_2(2\gamma)(z) := \frac{z}{(1-z)^2-\gamma}$ belongs to $S^*(\gamma)$. For $\gamma \in [1/2, 1)$, functions of $S^*(\gamma)$ are of particular interest. In [11] the following result was proved.

Theorem 1. [11] Let $f \in S^*(\gamma)$, $\gamma \in [1/2, 1)$. Then $s_n(f/z, z) \neq 0$, for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$.

Theorem [11] implies that the partial sums of starlike functions of order $\gamma$ where $1/2 \leq \gamma < 1$ are non-vanishing in $\mathbb{D}$ and takes value 1 at the origin. So it is interesting to find the information about the image domain of these polynomials. This, in particular, leads to find the best possible range of $\gamma$ such that $\text{Re}(s_n(f/z, z)) > 0$ where $f \in S^*(\gamma)$.
and \( z \in \mathbb{D} \). A more general statement was proved in \([13]\) which is given as follows. For \( zf \in S^*(\gamma) \),

\[
\frac{s_n(f, z)}{f(z)} < \frac{1}{f_{2-2\gamma}}, \quad \gamma \in [1/2, 1), \quad z \in \mathbb{D},
\]

(1)

which for \( zf \in S^*(1 - \rho/2) \) where \( \rho \in (0, 1] \) is equivalent to

\[
|\arg s_n(f/\gamma, z)| \leq \rho \pi, \quad \text{for} \quad z \in \mathbb{D}.
\]

The condition (1) can be replaced by a more general condition

\[
(1 - z)^{\mu} s_n^\mu(z) \prec \left(\frac{1 + z}{1 - z}\right)^{\rho}, \quad \text{for all} \quad n \in \mathbb{N}
\]

(2)

holds for \( 0 < \mu \leq \mu(\rho) \) where for \( \rho \in (0, 1] \), \( \mu(\rho) \) is defined as maximal number. To determine the value of \( \mu(\rho) \), Koumandos and Ruscheweyh \([5]\) proposed the following conjectures.

**Conjecture 1.** For \( \rho \in (0, 1] \), the number \( \mu(\rho) \) is equal to \( \mu^*(\rho) \) where \( \mu^*(\rho) \) is defined to be the unique solution in \( (0, 1] \) of the equation

\[
\int_0^{(\rho+1)\pi} \frac{\sin(t - \rho \pi)}{t^{1-\mu}} dt = 0.
\]

Conjecture 1 contains the following weaker form.

**Conjecture 2.** For \( \rho \in (0, 1] \), the following inequality

\[
\text{Re}((1 - z)^{2\rho-1}s_n^\mu(z)) > 0, \quad n \in \mathbb{N}
\]

(3)

holds for all \( z \in \mathbb{D} \) and \( 0 < \mu \leq \mu^*(\rho) \). Moreover, \( \mu^*(\rho) \) is the largest number with this property.

Generalizations of the above two conjectures are also given in \([9]\) and \([14]\) by considering Cesàro mean of order \( \delta \) and Cesàro mean of type \((b - 1; c)\) respectively. Several interesting properties of \( \mu^*(\rho) \), i.e. analytic and strictly increasing studied in \([8]\). Conjecture 1 is verified for \( \rho = 1/2, 1/4, \) and \( 1/5 \) respectively in \([5], [6]\) and \([8]\) while Conjecture 2 is verified for \( \rho = 1/2, \) and \( \rho = 3/4 \) in \([5]\). The objective of the manuscript is to establish Conjecture 1 for harmonic mean of \( 1/2 \) and \( 1/4 \) which is \( 1/3 \). In fact we establish Conjecture 1 in the open neighbourhood of \( \rho = 1/3 \). Further, we also prove Conjecture 2 for the harmonic mean of \( 1/2 \) and \( 1 \) which is \( 2/3 \).

**Theorem 2.** Conjecture 2 is true for \( \rho = 2/3 \).

**Theorem 3.** Conjecture 1 is true for all \( \rho \) in an open neighbourhood of \( 1/3 \).

The proofs of Theorem 2 and Theorem 3 are, respectively, given in Section 2 and Section 3. In establishing so, the method of sturm sequence and MAPLE 12 software will be used. For more details on this method, we refer to Kwong \([7]\). Theorem 3 leads to the following interesting results regarding starlike functions. Clearly, the function \( zf_{\rho} = \frac{z}{(1 - z)^{\rho}} \in S^*(1 - \mu/2) \). We define a function \( \phi_{\rho, \mu} := z_2 F_1(\rho, 1; \mu; z) \) which satisfies,

\[
\frac{z}{(1 - z)^{\mu}} \star \phi_{\rho, \mu}(z) = \frac{z}{(1 - z)^{\rho}}.
\]
Here \( _2F_1(a, b; c; z) \) is the Gaussian hypergeometric function defined by the infinite series \( \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k \), \( z \in \mathbb{D} \). Now for \( \mu > 0 \) we define a class \( \mathcal{F}_\mu \) by,

\[
\mathcal{F}_\mu := \left\{ g \in \mathcal{A}_0 : \Re \left( \frac{z g'}{g} \right) > \frac{-\mu}{2}, z \in \mathbb{D} \right\}.
\]

Clearly, \( g_\mu \in \mathcal{F}_\mu \) and \( g \in \mathcal{F}_\mu \iff zg \in \mathcal{S}^*(1-\mu/2) \). Moreover, \( g \in \mathcal{F}_\mu \) whenever \( g \prec g_\mu \). As \( \mathcal{F} \) corresponds to the class \( \mathcal{S}^* \), similarly, we define class analogous to prestarlike functions by \( \mathcal{PF}_\mu \) as

\[
\mathcal{PF}_\mu := \{ g \in \mathcal{A}_0 : g * g_\mu \in \mathcal{F}_\mu \}.
\]

Similar type of results for prestarlike class also hold for the class \( \mathcal{PF}_\mu \). A function \( \tilde{g}_\mu \in \mathcal{A}_0 \) can be defined such that \( g_\mu \ast \tilde{g}_\mu = \frac{1}{1-z} \).

**Theorem 4.** For \( f \in \mathcal{S}^*(1-\mu/2) \) and \( 0 < \mu \leq \mu^*(1/3) \), we have

\[
\frac{s_n(f, z)}{\phi_{1,3,\mu} * f} < \left( \frac{1+z}{1-z} \right)^{1/3}, \quad n \in \mathbb{N}.
\]

**Proof.** For proving the theorem we need to show that

\[
\frac{s_n(g, z)}{\psi_{1,3,\mu} * g} < \left( \frac{1+z}{1-z} \right)^{1/3}, \quad n \in \mathbb{N},
\]

where \( \psi_{\rho,\mu}(z) = \frac{1}{\rho} \phi_{\rho,\mu}(z) \) and \( g(z) = \frac{1}{\rho} f(z) \in \mathcal{F}_\mu \). By writing \( \frac{s_n(g, z)}{\psi_{1,3,\mu} * g} \) as follows and using the convolution theory for prestarlike functions \([12\text{ p.36}]\) we obtain

\[
\frac{s_n(g, z)}{\psi_{1,3,\mu} * g} = \frac{((1-z)^{1/3} s_{\mu}(z) g_{1/3}) * (g \ast \tilde{g}_\mu)}{g_{1/3} * (g \ast \tilde{g}_\mu)} < \left( \frac{1+z}{1-z} \right)^{1/3}, \quad z \in \mathbb{D}
\]

where the latter inequality holds for \( 0 < \mu \leq \mu^*(1/3) \). \( \square \)

Theorem 4 also settles the particular case \( \rho = 1/3 \) of \([5\text{ Conjecture 3}]\). Validity of Theorem 3 leads to the proof of Theorem 5.

**Theorem 5.** For \( f \in \mathcal{S}^*(1-\mu/2) \) and \( 0 < \mu \leq \mu^*(1/3) \), we have

\[
\frac{1}{z} s_n(f, z) < \left( \frac{1+z}{1-z} \right)^{2/3}.
\]

**Proof.** Since \( (1-z)^{1/3} < \left( \frac{1+z}{1-z} \right)^{1/3} \) holds. For \( f_\mu \in \mathcal{S}^*(1-\mu/2) \), this can be written as

\[
\phi_{1,3,\mu}(z) * f_\mu(z) < \left( \frac{1+z}{1-z} \right)^{1/3}, \quad z \in \mathbb{D}.
\]

Using convolution theory of prestarlike functions for \( f \in \mathcal{S}^*(1-\mu/2) \), we get

\[
\phi_{1,3,\mu}(z) * f(z) < \left( \frac{1+z}{1-z} \right)^{1/3}, \quad z \in \mathbb{D}.
\]

Hence Theorem 4 concludes that

\[
\frac{1}{z} s_n(f, z) = \frac{s_n(f, z)}{\phi_{1,3,\mu}(z) * f(z)} \cdot \phi_{1,3,\mu}(z) * f(z) < \left( \frac{1+z}{1-z} \right)^{2/3}.
\]

\( \square \)
For another interpretation of Theorem 5 the following definition of Kaplan class is of considerable interest.

**Definition 1** (Kaplan Class). \[15\] For \( \alpha \geq 0, \beta \geq 0, f \in K(\alpha, \beta) \) if \( f(z) \) can be written as

\[
 f(z) = H(z)k(z)
\]

where \( H \in A \) satisfies \( |\arg H(z)| \leq \frac{\pi}{2} \min(\alpha, \beta) \) and \( k(z) \in \prod_{\alpha-\beta} \) in \( z \in \mathbb{D} \).

The class \( \prod_{\alpha} \) is defined as follows. For real \( \alpha \), a function \( k(z) \in A \) is in \( \prod_{\alpha} \) where

\[
 Re \frac{zk'(z)}{k(z)} \begin{cases} < \frac{\alpha}{2}, & \text{if } \alpha > 0; \\ > \frac{\alpha}{2}, & \text{if } \alpha < 0; \\ \equiv 0, & \alpha = 0. \end{cases}
\]

In terms of Kaplan classes \( K(\alpha, \beta) \) \[12, p.32\], (4) can be replaced by a stronger statement

\[
 \frac{1}{z}s_n(f, z) \in K(1/3, 2/3), \quad n \in \mathbb{N}.
\]

**Remark 1.** Note that (4) is equivalent to \( \text{Re}(s_n(f, z)/z^{3/2}) > 0 \) for \( z \in \mathbb{D} \). It is obvious that \( zG_\lambda(z, x) \in S^*(1 - \lambda) \) where \( G_\lambda(z, x) \) is the generating function for the Gegenbauer polynomials defined by,

\[
 G_\lambda(z, x) := \frac{1}{(1 - 2xz + z^2)^\lambda} = \sum_{k=0}^{\infty} C^\lambda_k(x)z^k, \quad x \in [-1, 1].
\]

Therefore (4) implies the following inequality holds for all \( x \in (-1, 1) \) and \( 0 < \lambda \leq \frac{1}{2} \mu^*(1/3) = 0.2483 \ldots \)

\[
 |\arg \sum_{k=0}^{n} C^\lambda_k(x)z^k| < \frac{\pi}{3}, \quad z \in \mathbb{D}.
\]

**Remark 2.** Remark [7] implies that

\[
 \sum_{k=0}^{n} C^\lambda_k(x)z^k \neq 0 \quad \text{on} \quad z \in \mathbb{D} \quad \text{and} \quad 0 < \lambda \leq \frac{1}{2} \mu^*(1/3), \quad x \in [-1, 1].
\]

whereas from [2, Theorem 1] we have

\[
 \sum_{k=0}^{n} C^\lambda_k(x)z^k = 0, \quad |z| \leq 1, \quad 0 < \lambda \leq 1/2.
\]

So we can conclude that

\[
 \sum_{k=0}^{n} C^\lambda_k(x)z^k = 0 \quad \text{on} \quad z \in \partial \mathbb{D} \quad \text{for} \quad x \in [-1, 1]. \quad (5)
\]

Further by substituting \( x = \cos \theta \) and using the relation between Gegenbauer polynomials and Jacobi polynomials,

\[
 C^\lambda_k(x) = \frac{(2\lambda + 1)_k P_{k}^{(\lambda,\lambda)}(x)}{k! P_{k}^{(\lambda,\lambda)}(1)},
\]
we obtain
\[
\sum_{k=0}^{n} \frac{(2\lambda + 1)_k P_k^{(\lambda, \lambda)}(\cos \theta)}{k! P_k^{(\lambda, \lambda)}(1)} \neq 0, \quad \text{on } z \in \mathbb{D}, \quad -\pi < \theta < \pi,
\]
for \(-1/2 < \lambda \leq \frac{1}{2}(\mu^* (1/3) - 1)\).

For any convex function \(f \in C\), Wilf Theorem [11 Theorem 8.9] (known as Wilf Conjecture) leads us
\[
f \ast (1 - z)^{1/3}s_n^\mu(z) < f \ast \left(\frac{1 + z}{1 - z}\right)^{1/3}, \quad 0 < \mu \leq \mu^* (1/3).
\]

**Lemma 1.** [10] Lemma 2.7 Let \(\varphi(z)\) be convex and \(g(z)\) is starlike in \(\mathbb{D}\). Then for each function \(F(z)\) analytic in \(\mathbb{D}\) and satisfying \(\text{Re}F(z) > 0\), we have
\[
\text{Re}\left(\frac{\varphi \ast F g(z)}{\varphi \ast g(z)}\right) > 0, \quad z \in \mathbb{D}.
\]

**Remark 3.** For any convex function \(\varphi(z)\) the following holds.
\[
\left| \arg \left(\frac{\varphi \ast z s_n^\mu(z)}{\varphi \ast z(1 - z)^{-1/3}}\right) \right| < \frac{\pi}{6}, \quad 0 < \mu \leq \mu^* (1/3).
\]

Since \(\frac{1}{(1 - z)^{1/3}}\) is the derivative of a convex function say \(\varphi'\). By writing
\[
\frac{\varphi \ast z s_n^\mu(z)}{\varphi \ast z(1 - z)^{-1/3}} = \frac{\varphi \ast z(1 - z)^{1/3}s_n^\mu(z)}{\varphi \ast z(1 - z)^{-1/3}},
\]
which using Lemma 1 and Theorem 3 justifies the remark.

Since both the conjectures hold good for the harmonic mean of 1/2 & 1/4 and 1/2 & 1, affirmative solution to the following conjecture will settle both these conjectures.

**Conjecture 3.** If Conjecture 1 and Conjecture 2 hold for \(0 < \rho_1, \rho_2 \leq 1\). Then both conjectures also hold for harmonic mean of \(\rho_1\) and \(\rho_2\) which is \(\frac{2\rho_1\rho_2}{\rho_1 + \rho_2}\).

2. **Proof of Theorem 2**

Conjecture 1 and Conjecture 2 are consequences of the positivity of trigonometric sums having coefficients of type \(\frac{(\mu)_{k}}{k!}\). To prove the positivity of such trigonometric sums, we follow the procedure given by Koumandos [4]. In this method, we estimate the sequence \(\left\{\frac{(\mu)_{k}}{k!}\right\}\) in terms of the limiting sequence \(k^{\mu - 1}\). For \(\mu \in (0, 1)\), we consider \(\Delta_k\) as
\[
\Delta_k := \frac{k^{\mu - 1}}{\Gamma(\mu)} - \frac{(\mu)_{k}}{k!}, \quad \text{for } k \in \mathbb{N}.
\]

Now from [5, eq.3.8], \(\sum_{k=0}^{n} \frac{(\mu)_{k}}{k!} e^{2ik\theta}\) can be written as
\[
\sum_{k=0}^{n} \frac{(\mu)_{k}}{k!} e^{2ik\theta} = \sum_{k=0}^{\infty} \frac{(\mu)_{k}}{k!} e^{2ik\theta} - \frac{\theta}{\sin \theta} e^{i\mu \pi/2} + \frac{1}{\Gamma(\mu)} \frac{\theta}{\sin \theta} (2\theta)^{\mu} \int_{0}^{(2n+1)\theta} e^{it} t^{1-\mu} dt - \frac{1}{\Gamma(\mu)} \frac{\theta}{\sin \theta} \sum_{k=n+1}^{\infty} \{A_k(\theta) + B_k(\theta)\} + \sum_{k=n+1}^{\infty} \Delta_k e^{2ik\theta}, \quad (7)
\]
where
\[ A_k(\theta) := \int_0^{1/2} \int_0^t \left\{ \frac{1 - \mu}{(k + s)^{2-\mu}} - \frac{1 - \mu}{(k - t + s)^{2-\mu}} \right\} ds e^{2it(k-t)} dt \quad \text{and} \]
\[ B_k(\theta) := \int_0^{1/2} 2i \sin 2\theta t \int_0^t \frac{1 - \mu}{(k + s)^{2-\mu}} ds e^{2ik\theta} dt. \]

We have to estimate the terms in the right hand side of (7). We use the following estimates of \( A_k(\theta) \) and \( B_k(\theta) \) given in [3, Lemma 1]:

\[ \left| \sum_{k=n+1}^{\infty} A_k(\theta) \right| < \frac{1 - \mu}{8} n^{\mu - 2} \quad \text{and} \quad \left| \sum_{k=n+1}^{\infty} B_k(\theta) \right| < \frac{\theta}{\sin \theta} \frac{1 - \mu}{6} n^{\mu - 2}, \quad (8) \]

for \( \theta \in \mathbb{R} \). Further, using the completely monotonicity of \( x - \frac{\Gamma(x+\mu)}{\Gamma(x+1)} x^{2-\mu} \), the following estimate was obtained in [6, eqn.11].

\[ \left| \sum_{k=n+1}^{\infty} \Delta_k e^{2ik\theta} \right| \leq \frac{\mu(1 - \mu)}{2 \sin \alpha} \frac{1}{\Gamma(\mu)} \frac{1}{(n + 1)^{2-\mu}}, \quad (9) \]

for \( 0 < \alpha < \theta < \pi/2, n \in \mathbb{N} \) and \( 1/3 \leq \mu < 1 \). Now we are ready to give the proof of Theorem 2.

2.1. **Proof of Theorem 2.** For \( \rho = 2/3 \), (3) reduces to \( \text{Re}((1 - z)^{1/3} s_n^\mu(z)) > 0 \) which further equivalent to

\[ \U_n(\phi) := \sum_{k=0}^{n} d_k \cos \left( 2k + 1 \right) \phi - \frac{\pi}{6} > 0 \quad \text{for} \quad 0 < \phi < \pi. \quad (10) \]

where \( d_k = \frac{(\mu)_k}{k!} \), \( k = 0, 1, \ldots, n \). To prove the theorem it is equivalent to establish the existence of (10). A numerical evaluation yields that

\[ \mu^*(2/3) = 0.8468555683 \ldots. \]

Using summation by parts, we observe that this particular case need to be established only for \( \mu = \mu^*(2/3) = 0.8468555683 \ldots. \) Since \( \U_n(\pi - \phi) = \U_n(\phi) \), it is sufficient to prove \( \U_n(\phi) > 0 \) for \( 0 < \phi \leq \pi/2 \).

For \( n = 1 \):

Simple calculation gives

\[ \U_1(\phi) = (1 - d_1) \sin \left( \frac{\phi}{3} + \frac{\pi}{3} \right) + 2d_1 \sin \left( \frac{4\phi}{3} + \frac{\pi}{3} \right) \cos \phi, \]

which is clearly positive for all \( 0 < \phi \leq \pi/2 \). For \( n \geq 2 \), the proof is divided into several cases.

**The case** \( 0 < \phi \leq \frac{\pi}{9} \) and \( \frac{\pi}{5} \leq \phi \leq \frac{\pi}{2} \) for \( n \geq 2 \):

For \( n \geq 2 \),

\[ 2 \sin \phi \U_n(\phi) = \sum_{k=0}^{n} d_k \left[ \sin \left( 2k + \frac{4}{3} \phi - \frac{\pi}{6} \right) - \sin \left( 2k + \frac{2}{3} \phi - \frac{\pi}{6} \right) \right], \]
which using summation by parts and simple trigonometric identities yield that
\[
\begin{align*}
&= \sum_{k=0}^{n-1} (d_k - d_{k+1}) \left[ \sin \left( \frac{2\phi}{3} + \frac{\pi}{3} \right) + \sin \left( \left( 2k + \frac{4}{3} \right) \phi - \frac{\pi}{6} \right) \right] \\
&\quad + d_n \left[ \sin \left( \frac{2\phi}{3} + \frac{\pi}{3} \right) + \sin \left( \left( 2n + \frac{4}{3} \right) \phi - \frac{\pi}{6} \right) \right] \\
&\geq (1 - d_1) \left[ \sin \left( \frac{2\phi}{3} + \frac{\pi}{6} \right) + \sin \left( \frac{4\phi}{3} - \frac{\pi}{6} \right) \right] + (d_1 - d_2) \\
&\quad + \left[ \sin \left( \frac{2\phi}{3} + \frac{\pi}{6} \right) + \sin \left( \frac{10\phi}{3} - \frac{\pi}{6} \right) \right] + d_2 \left[ -1 + \sin \left( \frac{2\phi}{3} + \frac{\pi}{6} \right) \right] \\
&= -d_2 + \cos \left( \frac{2\phi}{3} - \frac{\pi}{3} \right) + (1 - d_1) \cos \left( \frac{4\phi}{3} - \frac{2\pi}{3} \right) + (d_2 - d_1) \cos \left( \frac{10\phi}{3} - \frac{5\pi}{3} \right).
\end{align*}
\]

Substituting \( \frac{2\phi - \pi}{3} = t \), then
\[
2 \sin \phi \mathcal{U}_n(\phi) \geq -d_2 + \cos t + (1 - d_1) \cos 2t + (d_2 - d_1) \cos 5t
\]
\[
= -\mu \frac{(\mu + 1)}{2} + \cos t + (1 - \mu) \cos 2t + \mu \frac{(\mu + 1)}{2} \cos 5t =: P(t).
\]

Using the method of sturm sequence, we get that \( P(t) \) has no roots in \((-\frac{\pi}{3}, -\frac{7\pi}{27})\) and \([-\frac{\pi}{3}, 0]\). It can be directly verified that \( P(0) > 0 \) and \( P(-\pi/3) > 0 \). Hence for \( n \geq 2 \),
\[
\mathcal{U}_n(\phi) > 0 \quad \text{for} \quad 0 < \phi \leq \pi/9 \quad \text{and} \quad \pi/5 \leq \phi \leq \pi/2.
\]

**The case \( \frac{\pi}{9} < \phi < \frac{\pi}{5} \):**

We prove this case for \( n = 2, 3 \) and \( n \geq 4 \). For \( n = 2, 3 \) instead of \( \frac{\pi}{9} < \phi < \frac{\pi}{5} \), we establish for the larger range \( \phi \in (0, \pi/2] \).

**The case \( n = 2, 3 \):**

This can be verified again with sturm sequence. Using simple calculations,
\[
\mathcal{U}_2(\phi) = \sin t + \mu \sin 7t + \frac{\mu(\mu + 1)}{2} \sin 13t =: Q(t) \quad \text{and}
\]
\[
\mathcal{U}_3(\phi) = \sin t + \mu \sin 7t + \frac{\mu(\mu + 1)}{2} \sin 13t + \frac{\mu(\mu + 1)(\mu + 2)}{6} \sin 19t =: R(t),
\]

where \( t = \frac{\phi + \pi}{3} \). It can be verified that \( Q(t) \) and \( R(t) \) have no zeros in \((\frac{\pi}{3}, \frac{\pi}{2})\) and \( Q(\pi/2) > 0, R(\pi/2) > 0 \). Hence it implies that \( \mathcal{U}_2(\phi) > 0 \) and \( \mathcal{U}_3(\phi) > 0 \) in \( \phi \in (0, \pi/2] \).

**The case \( \frac{\pi}{9} < \phi < \frac{\pi}{5} \) for \( n \geq 4 \):**

For this case, we use the representation of \( \mathcal{U}_n(\phi) \) as
\[
\mathcal{U}_n(\phi) = \text{Re} \left( e^{i\left( \frac{\phi - \pi}{6} \right)} \sum_{k=0}^{n} d_k e^{2ik\phi} \right)
\]

and the following propositions are used.

**Proposition 2.1.** Let \( F(\phi) := \sum_{k=0}^{\infty} d_k e^{2ik\phi} - \frac{\phi}{\sin \phi} \frac{e^{i(\frac{\pi}{3} - \frac{\phi}{2})}}{(2\phi)^\mu}. \) Then for \( 0 < \phi \leq \frac{\pi}{5} \), we have
\[
\frac{2\mu}{\phi^{1-\mu}} \text{Re} \left\{ F(\phi) e^{i(\frac{\pi}{3} - \frac{\phi}{2})} \right\} \geq \mu \cos \left( \frac{2\pi}{3} - \frac{\mu\pi}{2} \right) - \lambda \left( \frac{\pi}{5} \right),
\]
where \( \Lambda(\phi) := \frac{1}{\sin \phi} \left( 1 - \left( \frac{\sin \phi}{\phi} \right)^{1-\mu} \right) \).

**Proof.** The infinite trigonometric sum can be written as \( \sum_{k=0}^{\infty} d_k e^{2ik\phi} = \frac{e^{i\mu(\frac{\pi}{5}-\phi)}}{(2 \sin \phi)^\mu} \). Thus \( F(\phi) \) becomes,

\[
F(\phi) = \frac{\phi^{1-\mu} e^{i\mu\pi/2}}{2\mu} \sin \phi \left[ (e^{-i\mu\phi} - 1) - \left( 1 - \left( \frac{\sin \phi}{\phi} \right)^{1-\mu} \right) e^{-i\mu\phi} \right].
\]

Therefore

\[
\frac{2^\mu}{\phi^{1-\mu}} \text{Re} \left[ F(\phi) e^{i(\frac{\pi}{5} - \phi)} \right] = \sin \left( \frac{\mu(\pi - \phi)}{2} + \frac{\phi}{3} - \frac{\pi}{6} \right) \frac{1}{\cos \frac{\phi}{2}} \sin \frac{\phi}{2} - \frac{1}{\sin \phi} \left[ 1 - \left( \frac{\sin \phi}{\phi} \right)^{1-\mu} \right] \sin \left[ (\mu - \frac{1}{3}) \phi - \frac{\mu\pi}{2} + \frac{2\pi}{3} \right] \geq \mu \cos \left( \frac{2\pi}{3} - \frac{\mu\pi}{2} \right) - \frac{1}{\sin \phi} \left( 1 - \left( \frac{\sin \phi}{\phi} \right)^{1-\mu} \right).
\]

Since the function \( \Lambda(\phi) := \frac{1}{\sin \phi} \left( 1 - \left( \frac{\sin \phi}{\phi} \right)^{1-\mu} \right) \) is positive and strictly increasing on \((0, \frac{\pi}{5})\), we obtain

\[
\frac{2^\mu}{\phi^{1-\mu}} \text{Re} \left\{ F(\phi) e^{i(\frac{\pi}{5} - \phi)} \right\} \geq \mu \cos \left( \frac{2\pi}{3} - \frac{\mu\pi}{2} \right) - \Lambda \left( \frac{\pi}{5} \right),
\]

which completes the proof of Proposition 2.1. \( \square \)

**Proposition 2.2.** Let

\[
\chi_n(\phi) := \frac{1}{\sin(\phi)} \text{Re} \left( e^{i(\frac{\pi}{5} - \phi)} \int_0^{(2n+1)\phi} e^{it} t^{1-\mu} dt \right).
\]

For \( \frac{\pi}{2n+1} \leq \phi \leq \frac{\pi}{5} \) and \( n \geq 4 \) we have

\[
\chi_n(\phi) > \frac{1}{\sin \frac{\pi}{5}} \int_0^{\frac{8\pi}{5}} \frac{\cos \left( t - \frac{\pi}{10} \right)}{t^{1-\mu}} dt = -0.3212698190821 \ldots.
\]

**Proof.** Using simple trigonometric identities we obtain,

\[
\chi_n(\phi) = \frac{2 \sin \left( \frac{\phi}{5} + \frac{\pi}{5} \right)}{\sin \phi} \int_0^{(2n+1)\phi} \frac{\cos(t - \pi/6)}{t^{1-\mu}} dt - \frac{2 \sin \phi}{\sin \phi} \int_0^{(2n+1)\phi} \frac{\cos(t - \pi/3)}{t^{1-\mu}} dt.
\]

From the definition of \( \mu \), we observe that

\[
\int_0^x \frac{\cos(t - \pi/6)}{t^{1-\mu}} dt \geq 0 \quad \text{and} \quad \int_0^x \frac{\cos(t - \pi/3)}{t^{1-\mu}} dt \geq 0 \quad \text{for all} \quad x > 0.
\]

Further, \( p(\phi) := \frac{\sin(\frac{\phi}{5} + \frac{\pi}{5})}{\sin(\frac{\phi}{5})} \) and \( q(\phi) := \frac{\sin(\phi)}{\sin(\phi/3)} \) are positive and decreasing in \((0, \frac{\pi}{2})\). Since \( \phi \leq \frac{\pi}{5} \), clearly \( p(\phi) \geq p(\frac{\pi}{5}) \) and \( q(\phi) \geq q(\pi/5) \). Therefore,

\[
\chi_n(\phi) \geq \frac{1}{\sin \pi/5} \int_0^{(2n+1)\phi} \frac{\cos(t - \pi/10)}{t^{1-\mu}} dt \geq \frac{1}{\sin \pi/5} \int_0^{\frac{8\pi}{5}} \frac{\cos(t - \pi/10)}{t^{1-\mu}} dt,
\]
Lemma 2. Let \( \eta(\theta) \) be a real integrable function of \( \theta \in \mathbb{R} \), \( 1/3 \leq \mu < 1 \) and \( 0 < a < b \leq \pi/2 \). Then for \( g(\theta) = \sin \theta \) or \( \cos \theta \), and \( \theta \in [a,b] \) we have,

\[
\frac{2^\mu}{\theta^{1-\mu}} \Gamma(\mu) \sum_{k=0}^{n} \frac{(\mu)_k}{k!} g(2k\theta + \eta(\theta)) > \kappa_n(\theta) - X_n - Y_n - Z_n + \Gamma(\mu) \left( 2q(\theta) \frac{\sin \mu \theta}{\sin \theta} - r(\theta) \wedge (\theta) \right),
\]

where the latter inequality follows by minimizing the expression on the right-hand side over \( (2n+1)\phi \geq \pi \). This completes the proof of Proposition 2.2. \( \square \)

Now returning back to the proof of the theorem for the case \( \frac{\pi}{9} < \phi < \frac{\pi}{5} \) and \( n \geq 4 \), let

\[
\sigma_n(\phi) := 2^\mu \phi^{\mu-1} \frac{\phi}{\sin \phi} \sum_{k=n+1}^{\infty} \mathcal{A}_k(\phi) \quad \text{and} \quad \tau_n(\phi) := 2^\mu \phi^{\mu-1} \frac{\phi}{\sin \phi} \sum_{k=n+1}^{\infty} \mathcal{B}_k(\phi).
\]

Then,

\[
\text{Re} \left( \sigma_n(\phi) e^{i(\frac{\pi}{9} - \phi)} \right) \leq 2^\mu \phi^{\mu-1} \frac{1 - \mu}{\sin \phi} \frac{1}{8} \frac{1}{n^{2-\mu}} \leq 1 - \mu 4 \frac{\pi}{5} \frac{9}{2n\pi} \left( \frac{\pi}{n\pi} \right)^{1-\mu} \frac{1}{n} < 1 - \mu \frac{\pi}{80} \frac{\sin \pi/5}{\sin \pi/5}
\]

and

\[
\text{Re} \left( \tau_n(\phi) e^{i(\frac{\pi}{9} - \phi)} \right) \leq 2^\mu \phi^{\mu-1} \left( \frac{\phi}{\sin \phi} \right)^2 \frac{1 - \mu}{6} \frac{1}{n^{2-\mu}} \leq 1 - \mu 3 \left( \frac{\pi}{\sin \pi/5} \right)^2 \left( \frac{9}{2n\pi} \right)^{1-\mu} \frac{1}{n} < 1 - \mu \frac{300}{\sin \pi/5} \left( \frac{\pi}{\sin \pi/5} \right)^2
\]

Using (9) we find that for \( \frac{\pi}{9} \leq \phi \leq \frac{\pi}{5} \),

\[
2^\mu \phi^{\mu-1} \Gamma(\mu) \text{Re} \left( e^{i(\frac{\pi}{9} - \phi)} \sum_{k=n+1}^{\infty} \Delta_k e^{2ik\phi} \right) > - \frac{\mu(1 - \mu)}{(2n + 2)\phi} \theta^{1-\mu} > - \frac{\mu(1 - \mu)}{\pi^{1-\mu}}.
\]

Finally using Propositions 2.1 - 2.2 and equations (11) - (13) we conclude that

\[
2^\mu \phi^{\mu-1} \Gamma(\mu) \mathcal{U}_n(\phi) > 0 \quad \text{which establishes the case} \quad n \geq 4 \quad \text{and} \quad \frac{\pi}{9} < \phi < \frac{\pi}{5}.
\]

Combining all these cases completes the proof of the theorem.

3. Proof of Theorem

To prove Conjecture 1 for all \( \rho \) in neighbourhood of 1/3 the following modification of (7) obtained in [6] Lemma 1 is required. Note that this modification plays an important role in proving Conjecture 1 for \( \rho = 1/4 \) [6] and we follow the same as well.

**Lemma 2.** [6] Lemma 1 | Let \( \eta(\theta) \) be a real integrable function of \( \theta \in \mathbb{R} \), \( 1/3 \leq \mu < 1 \) and \( 0 < a < b \leq \pi/2 \). Then for \( g(\theta) = \sin \theta \) or \( \cos \theta \), and \( \theta \in [a,b] \) we have,

\[
\frac{2^\mu}{\theta^{1-\mu}} \Gamma(\mu) \sum_{k=0}^{n} \frac{(\mu)_k}{k!} g(2k\theta + \eta(\theta)) > \kappa_n(\theta) - X_n - Y_n - Z_n + \Gamma(\mu) \left( 2q(\theta) \frac{\sin \mu \theta}{\sin \theta} - r(\theta) \wedge (\theta) \right),
\]
where
\[\kappa_n(\theta) := \frac{1}{\sin \theta} \int_0^{(2n+1)\theta} g(t + \eta(\theta))t^{\mu-1} dt,\]
\[X_n := \frac{b}{\sin b} \frac{1 - \mu (2an)^{\mu-1}}{4n},\]
\[Y_n := \frac{b^2}{\sin^2 b} \frac{1 - \mu (2an)^{\mu-1}}{3n},\]
\[q(\theta) := g\left(\frac{\mu(\pi - \theta) - \pi}{2} + \eta(\theta)\right),\]
\[r(\theta) := g\left(\frac{\mu(\pi - \theta)}{2} + \eta(\theta)\right),\]
\[\wedge(\theta) := \frac{1}{\sin \theta} \left(1 - \left(\frac{\sin \theta}{\theta}\right)^{1-\mu}\right).\]

The function \(\wedge(\theta)\) is positive and increasing on \((0, \pi)\).

It follows from [8] Lemma 2.9 that proving Conjecture [1] is equivalent to proving non-negativity of the trigonometric sum \(s(\rho, \mu, \theta)\) for \(0 < \mu \leq \mu^*(\rho)\) where,
\[s_n(\rho, \mu, \theta) := \sum_{k=0}^n \frac{\mu^k}{k!} \sin[(2k + \rho)\theta], \quad \theta \in [0, \pi], \quad n \in \mathbb{N}. \quad (14)\]

3.1. **Proof of Theorem [8]**: We write \(\nu_0 := \mu^*(\frac{1}{2}) = 0.4966913651 \ldots < \frac{1}{2}\), \(\nu := \mu^*(\rho)\), \(s_n(\rho, \rho) := s_n(\rho, \mu^*(\rho), \theta)\) and \(s_n(\theta) := s_n(\frac{1}{2}, \theta)\) for \(\rho \in (0, 1)\) and \(\theta \in (0, \pi]\).

At first we prove Theorem [8] for \(n = 1, 2\). Since \(\nu_0 = 0.49 \ldots < \frac{1}{2}\), the sequence \(\left\{\frac{\nu_0}{(1/2)^k}\right\}_{k \in \mathbb{N}_0}\) is the decreasing sequence. Therefore using summation by parts, it is enough to verify that
\[\omega_n(\theta) := \sum_{k=0}^n \frac{(1/2)^k}{k!} \sin \left(\left(\frac{2k + 1}{3}\right)\theta\right) > 0 \quad \text{in} \quad \theta \in (0, \pi].\]

It is easy to write \(\omega_n(\theta) = \sin \theta/3 q_n(\cos^2 \theta/3)\) for \(n = 1, 2\), where
\[q_1(x) := \frac{1}{2} + 12x - 40x^2 + 32x^3 \quad \text{and}\]
\[q_2(x) := \frac{7}{8} - \frac{39}{2} x + 380x^2 - 1984x^3 + 4320x^4 - 4224x^5 + 1536x^6.\]

Using the method of sturm sequence, we observe that \(q_n(x)\) does not have any zero in \((0, 1)\) and \(q_n(0) > 0\) for \(n = 1, 2\). Therefore \((14)\) is positive for \(\mu = \nu_0\) and \(n = 1, 2\). For \(n \geq 3\), first we observe that,
\[s_n(\rho, \pi - \theta) = \sum_{k=0}^n \frac{\rho^k}{k!} \cos \left[(2k + \rho)\theta - \left(\rho - \frac{1}{2}\right)\pi\right] := \ell_n(\rho, \theta)\]

It is obvious that \(s_n(\rho, \theta) > 0\) for \(\theta \in (0, \pi]\) if and only if \(\ell_n(\rho, \pi - \theta) > 0\). For \(n \geq 3\), we split the interval \([0, \pi]\) into several subintervals.

**The case** \(\theta \in \left(0, \frac{\pi}{n+1}\right) \cup \left[\pi - \frac{\pi}{n+\rho}, \pi\right]\) **and** \(n \geq 3\):

Using a simple trigonometric identity, we write
\[e^{ik\theta} \sum_{k=0}^n e^{ik\theta} = e^{i(m+n\theta/2)} \frac{\sin \frac{n+1}{2} \theta}{\sin \frac{\theta}{2}}, \quad n \in \mathbb{N}, \quad \theta \in (0, \pi],\]
where \( m \in \mathbb{R} \) can be a function of \( \theta \). Since the sequence \( \{ \frac{(\nu)_{n}}{k!} \}_{k \in \mathbb{N}_0} \) is decreasing, again using summation by parts and the above trigonometric identity yields that for all \( \rho \) in the neighbourhood of 1/3, \( s_n(\rho, \theta) > 0 \) for \( \theta \in (0, \frac{\pi}{n+1}] \) and \( \ell_n(\rho, \theta) > 0 \) for \( \theta \in [0, \frac{\rho\pi}{n+\rho}] \).

The case \( \theta \in \left[ \frac{\pi}{n+1}, \frac{\pi}{3} \right] \) and \( n \geq 3 \):

Here we apply Lemma 2 with \( n \) using summation by parts and the above trigonometric identity yields that for all \( \theta \) in \( \left[ \frac{\pi}{n+1}, \frac{\pi}{3} \right] \) and \( \ell_n(\rho, \theta) > 0 \) for \( \theta \in [0, \frac{\rho\pi}{n+\rho}] \).

\[ \kappa_n(\theta) \geq \frac{\cos \rho b}{\sin b} S((2n+1)\theta) + \rho C((2n+1)\theta) \geq \frac{\cos \rho b}{\sin b} S(2\pi) + \rho C(\pi/4) =: L_1^{(1)}(\rho). \]

where the latter inequality is obtained by minimizing the integrals \( S(x) \) and \( C(x) \) over \( x \geq (2n+1)\pi/(n+1) \) for \( n \geq 3 \). It follows from [8, Lemma 2.13], the functions \( q(\theta) = -\sin \left[ (\nu - 1)\frac{\pi}{3} + \theta(\rho - \frac{\pi}{3}) \right] \) and \( r(\theta) = \sin \left[ \frac{\pi}{3} (\pi - 2\theta) + \rho \theta \right] \) are positive and decreasing on \([a_n, b] \). Thus, we obtain

\[ \Gamma(\nu) \left[ 2q(0) \frac{\sin \frac{\nu\theta}{2}}{\sin \theta} - r(0) \wedge (\theta) \right] \geq \Gamma(\nu) \left[ 2q(0) \frac{\sin \frac{\nu\theta}{2}}{\sin b} - r(0) \wedge (b) \right] =: L_2^{(1)}(\rho). \]

Further for \( n \geq 3 \) and \( \theta \in I \), it is clear that the expression \( X_n + Y_n + Z_n \) is smaller than

\[ \frac{b}{\sin b} \frac{1 - \nu}{12} \frac{1}{(\frac{2\pi}{3})^{1-\nu}} + \frac{b^2}{\sin^2 b} \frac{1 - \nu}{9} \frac{1}{(\frac{2\pi}{3})^{1-\nu}} + \frac{\nu(1 - \nu)\pi}{(2\pi)^{2-\nu}} =: L_3^{(1)}(\rho). \]

Thus for \( \rho \) in an open neighbourhood of 1/3,

\[ 2^\nu \theta^{\nu-1} \Gamma(\nu) s_n(\rho, \theta) \geq L^{(1)}(\rho) := L_1^{(1)}(\rho) + L_2^{(1)}(\rho) - L_3^{(1)}(\rho). \]

Since \( \nu = \mu^*(\rho) \) continuously depend on \( \rho \), the function \( L^{(1)}(\rho) \) is continuous in \((0, 1)\) and \( L^{(1)}(\frac{1}{3}) = 1.00046 \ldots \). This leads to \( s_n(\rho, \theta) > 0 \) for all \( \theta \in I, n \geq 3 \) and \( \rho \) in an open neighbourhood of \( \frac{1}{3} \).

The case \( \theta \in \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \) for \( n \geq 3 \):

It can be verified that in the limiting case \( n \to \infty \), \( s_n(\rho, \theta) \to \frac{\sin[(\rho - \nu)\theta + \frac{\pi\nu}{2}]}{(2\sin \theta)^\nu}. \) Since the sequence \( \left\{ \frac{(\nu)_{n}}{n!} \right\} \) is monotonically decreasing, we obtain

\[ s_n(\rho, \theta) \geq \frac{\sin[(\rho - \nu)\theta + \frac{\pi\nu}{2}]}{(2\sin \theta)^\nu} \frac{(\nu)_{n+1}}{(n+1)! \sin \theta} \]

\[ \geq (2\sin 2\pi/3)^{-\nu} \sin \left( \frac{\pi}{6}(4\rho - \nu) \right) - \frac{(\nu)_{4}}{24 \sin \pi/3} =: L^{(2)}(\rho). \]

Again \( L^{(2)}(\rho) \) is a continuous function in \( \rho \) and \( L(1/3) = 0.0106517 \ldots \). Therefore, we obtain \( s_n(\rho, \theta) > 0 \) for all \( \rho \) in an open neighbourhood of 1/3 and \( \theta \in \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \) and \( n \geq 3 \).

The case \( \theta \in \left[ \frac{2\pi}{3}, \pi - \frac{\rho\pi}{n+\rho} \right] \):

For this case, instead of proving \( s_n(\rho, \theta) > 0 \) in \( \theta \in \left[ \frac{2\pi}{3}, \pi - \frac{\rho\pi}{n+\rho} \right] \), we will show the equivalent form that \( \ell(\rho, \theta) > 0 \) in \( \left[ \frac{\rho\pi}{n+\rho}, \frac{\pi}{3} \right] \). Further, we split the interval \( \left[ \frac{\rho\pi}{n+\rho}, \frac{\pi}{3} \right] \) into
three subintervals $I_k = [a_k, b_k]$ where,

\[
I_1 = [a_1, b_1] := \left[\frac{\rho \pi}{n + \rho}, \frac{\pi}{2n + 2}\right], \quad n \geq 3,
\]

\[
I_2 = [a_2, b_2] := \left[\frac{\pi}{2n + 2}, \frac{\pi}{n + 2}\right], \quad n \geq 4 \quad \text{and}
\]

\[
I_3 = [a_3, b_3] := \left[\frac{\pi}{n + 2}, \frac{\pi}{3}\right], \quad n \geq 4.
\]

For each interval $I_k$, we estimate $\kappa_n(\theta), r(\theta), \wedge(\theta), q(\theta), X_n, Y_n$ and $Z_n$. It follows from [8 Lemma 2.13] that for $0 < \theta \leq \pi/2$ and $\rho \in (0, \pi/2)$, we have $\kappa_n(\theta) \geq \mathcal{R}(b, (2n+1)\theta)$, where

\[
\mathcal{R}(b, x) := \frac{1}{\sin b} \int_x^\pi \cos(t + \rho b - (\rho - 1/2)\pi) t^{1-\mu} dt.
\]

For $\theta \in I_1$, $\kappa_n(\theta) \geq \mathcal{R}_1((2n+1)\theta) := \mathcal{R}(\frac{\pi}{12}, (2n+1)\theta)$. For $x \in [0, \pi]$, there is only one $x_{\mathcal{R}_1}$ such that $\mathcal{R}_1(x) > 0$ for $x \in [0, x_{\mathcal{R}_1})$ and $\mathcal{R}_1(x) < 0$ for $x \in (x_{\mathcal{R}_1}, \pi)$. Therefore,

\[
\mathcal{R}_1(x) > L_1^{(31)}(\rho) := \mathcal{R}_1(\pi).
\]

Further, for $\rho \in (0, 1/2)$ and $\theta \in I_1$ the functions $q(\theta), \wedge(\theta)$ and $-r(\theta)$ are increasing. Hence, we obtain

\[
\Gamma(\nu) \left(2q(\theta) \frac{\sin \nu \theta / 2}{\sin \theta} - r(\theta) \wedge (\theta)\right) \geq \Gamma(\nu) (\nu q(0) - r(\pi/8) \wedge (\pi/8)) =: L_2^{(31)}(\rho).
\]

For $n \geq 3$ and $\theta \in I_1$, the expression $X_n + Y_n + Z_n$ is smaller than $L_3^{(31)}(\rho)$, where

\[
L_3^{(31)}(\rho) := \frac{\pi/8}{\sin \pi/8} \frac{1 - \nu}{12} + \frac{1}{(\frac{\pi}{8})^{1-\nu}} + \frac{(\pi/8)^2}{\sin^2 \pi/8} \frac{1 - \nu}{9} + \frac{1}{(\frac{2\pi}{5})^{1-\nu}} + \frac{\nu(1 - \nu) \pi}{(2\pi/3)^{2-\nu}}.
\]

For $\theta \in I_1$, we have

\[
2^\nu (\nu - 1) \Gamma(\nu) \ell_n(\rho, \theta) \geq L_1^{(31)}(\rho) + L_2^{(31)}(\rho) - L_3^{(31)}(\rho) =: L^{(31)}(\rho).
\]

Since $L^{(31)}(\rho)$ is a continuous function in $\rho$ and $L^{(31)}(1/3) = 0.435939 \ldots$. This mean $L^{(31)}(\rho) > 0$ in open neighbourhood of $1/3$. For $\theta \in I_2, I_3$ and $n \geq 4$ can be dealt in similar way. Now for $\theta \in I_2$ and $\theta \in I_3$, $\kappa_n(\theta) \geq \mathcal{R}_2((2n+1)\theta) := \mathcal{R}(\frac{\pi}{6}, (2n+1)\theta)$ and $\kappa_n(\theta) \geq \mathcal{R}_3((2n+1)\theta) := \mathcal{R}(\frac{\pi}{7}, (2n+1)\theta)$ respectively. Following the reasoning similar to previous subcase, we get $\mathcal{R}_2(x) \geq \mathcal{R}_2((1 + 5\rho/6)\pi) \mathcal{R}_3(x) \geq \mathcal{R}_3(3\pi/2)$ respectively for $x \geq 0$ and $x \geq 3\pi/2$. Thus for all $\rho$ in neighbourhood of $1/3$ and $n \geq 4$, we have

\[
\kappa_n(\theta) \geq L_1^{(32)}(\rho) := \mathcal{R}_2((1 + 5\rho/6)\pi), \quad \theta \in I_2,
\]

\[
\kappa_n(\theta) \geq L_1^{(33)}(\rho) := \mathcal{R}_3(3\pi/2), \quad \theta \in I_3.
\]

Moreover, for $n \geq 4$ and $\theta \in I_2$ and $I_3$, $\Gamma(\nu) \left(2q(\theta) \frac{\sin \nu \theta / 2}{\sin \theta} - r(\theta) \wedge (\theta)\right)$ is larger than $\Gamma(\nu) (\nu q(0) - r(\pi/6) \wedge (\pi/6)) =: L_2^{(32)}(\rho)$ and $\Gamma(\nu) (\nu q(0) - r(\pi/3) \wedge (\pi/3)) =: L_2^{(33)}(\rho)$ respectively. Clearly $X_n + Y_n + Z_n$ is smaller than $L_3^{(32)}(\rho)$ and $L_3^{(33)}(\rho)$ respectively for
$\theta \in I_2$ and $I_3$, where $L_3^{(32)}(\rho)$ and $L_3^{(33)}(\rho)$ are given by,

$$L_3^{(32)}(\rho) := \frac{\pi/6}{\sin \pi/6} \frac{1 - \nu}{16(4\pi/5)^{1-\nu}} + \frac{\left(\frac{\pi/6}{\sin \pi/6}\right)^2}{12(4\pi/5)^{1-\nu}} \frac{1 - \nu}{\pi^{1-\nu}} + \frac{\nu(1 - \nu)}{\pi^{1-\nu}}$$

and

$$L_3^{(33)}(\rho) := \frac{\pi/3}{\sin \pi/3} \frac{1 - \nu}{16(4\pi/3)^{1-\nu}} + \frac{\left(\frac{\pi/3}{\sin \pi/3}\right)^2}{12(4\pi/3)^{1-\nu}} \frac{1 - \nu}{\pi^{1-\nu}} + \frac{\nu(1 - \nu)\pi}{(5\pi/3)^{2-\nu}}.$$

Hence for $\theta \in I_2$ and $I_3$ and for all $\rho$ in an open neighbourhood of $1/3$, we have

$$2^\nu \theta^{-1} \Gamma(\nu) \ell_n(\rho, \theta) \geq L_1^{(k)}(\rho) + L_2^{(k)}(\rho) + L_3^{(k)}(\rho) =: L^{(k)}(\rho), \quad k = 32, 33.$$  

It is obvious that the functions $L^{(k)}(\rho)$ are continuous and $L^{(32)}(1/3) = 0.00620342\ldots$ and $L^{(33)}(1/3) = 0.123105\ldots$ This gives $L^{(k)}(\rho) > 0$ for $k = 32, 33$. The only remaining case of $\ell_3(\rho, \theta) > 0$ is for $\theta \in \left[\frac{2\pi}{3n+2}, \frac{\pi}{3}\right]$ and it can be verified equivalently for $\omega_3(\rho, \theta) > 0$ with $\theta \in \left[\frac{2\pi}{3}, \frac{7\pi}{8}\right]$ using method of sturm sequence. Now $\omega_3(\theta)$ can be written as $\omega_3(\theta) = \sin \theta/3q_3(\cos^2 \theta/3)$ for $\theta \in \left[\frac{2\pi}{3}, \frac{7\pi}{8}\right]$, where

$$q_3(x) := \frac{9}{16} + \frac{147}{4} x - 1270x^2 + 16496x^3 - 98640x^4 + 316096x^5 - 580864x^6 + 614400x^7 - 348160x^8 + 81920x^9.$$  

Now $q_3(x)$ does not have any zero in $[0.37059, 1]$ and $q(1) > 0$ yields that $\omega_3(\theta) > 0$ and this completes the proof of the theorem.

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