Spectrum of Hydrogen Atom in Space-Time Non-Commutativity

M. Mounni

Department of Matter Sciences, University of Biskra, Algeria and
m.mounni@univ-biskra.dz

A. BenSlama

Department of Physics, University of Constantine, Algeria and
a.benslama@yahoo.fr

S. Zaim

Department of Matter Sciences, University of Batna, Algeria and
zaim69slimane@yahoo.fr

(Date text June 13, 2011; Received text; Revised text; Accepted text; Published text)

Abstract

We study space-time noncommutativity applied to the hydrogen atom and its phenomenological effects. We find that it modifies the potential part of the Hamiltonian in such a way we get the Kratzer potential instead of the Coulomb one and this is similar to add a dipole potential or to consider the extended charged nature of the proton in the nucleus. By calculating the energies from the Schrödinger equation analytically and computing the fine structure corrections using perturbation theory, we study the modifications of the hydrogen spectrum. We find that it removes the degeneracy with respect to both the orbital quantum number \( l \) and the total angular momentum quantum number \( j \); it acts here like a Lamb shift. Comparing the results with the experimental values from spectroscopy, we get a new bound for the space-time non-commutative parameter. We do the same perturbative calculation for the relativistic case and compute the corrections of the Dirac energies; we find that in this case too, the corrections are similar to a Lamb shift and they remove the degeneracy with respect to \( j \); we get another bound for the parameter of non-commutativity.
I. INTRODUCTION:

The idea of taking non-commutative space-time coordinates is not new as it dates from the thirties. It had as objective to regulate the divergences of quantum field theory by introducing an effective cut-off coming from a non-commutative structure of space-time at small length scales. Then, it was abandoned because of problems caused by the violation of unitarity and causality, but the mathematical development of the theory continued and especially after the work of Connes in the eighties [1].

In 1999, during their work on string theory, Seiberg and Witten showed that the dynamics of the endpoints of an open string on a D-brane in the presence of a magnetic back-ground field is described by a theory of Yang-Mills on a non-commutative space-time [2]; this has renewed interest in the theory. Recently, there are many works on Lorentz invariant interpretation of the theory [3-7].

Today, we find non-commutativity in various fields of physics such as solid state physics, where it was shown that the Hall conductivity is quantized within this framework [8] and that non-commutativity is the right tool replacing Bloch’s theory whenever the translation invariance, that occurs in crystals, is broken in aperiodic solids [9]. Another example is fluid mechanics, where we define the non-commutative fluids by studying the quantum Hall effect [10] or bosonization of collective fermion states [11]. One can also mention the connection with quantum statistical physics [12], the equivalence between non-commutative quantum mechanics and the Landau problem in the lowest Landau level [13], the interpretation of
Ising-type models as a kind of field theory in the framework of non-commutative geometry [14] or the relation with Berry curvature in momentum space [15]. One can even find a manifestation of the non-commutativity in the physiology of the brain, where non-commutative computation in the vestibulo-ocular reflex was demonstrated in a way that is unattainable by any commutative system [16]. The article of Douglas and Nektasov [17] is an excellent reference for the different applications of noncommutative field theory.

The theory is a distortion of space-time where the coordinates \( x^\mu \) become Hermitian operators and thus do not commute:

\[
[x_{nc}^\mu, x_{nc}^\nu] = i\theta^{\mu\nu} = iC^{\mu\nu}/(\Lambda_{nc})^2; \mu, \nu = 0, 1, 2, 3
\] (1)

The \( nc \) indices denote noncommutative coordinates. \( \theta^{\mu\nu} \) is the parameter of the deformation, \( C^{\mu\nu} \) are dimensionless parameters and \( \Lambda_{nc} \) is the energy scale where the non-commutative effects of the space-time will be relevant. The non-commutative parameter is an anti-symmetric real matrix and ordinary space-time is obtained by making the limit \( \theta^{\mu\nu} \to 0 \). For a review, one can see reference [18].

In the literature, there are a lot of phenomenological studies giving bounds on the non-commutative parameter. For example, the OPAL collaboration founds \( \Lambda_{nc} \geq 140 \text{ GeV} \) [19], various non-commutative QED processes give the range \( \Lambda_{nc} \geq 500 \text{ GeV} - 1.7 \text{ TeV} \) [20], high precision atomic experiment on the Lamb shift in the hydrogen atom gives the limit \( \Lambda_{ss} \geq 6 \text{ GeV} \) [21] (This limit corrects the error made in calculating the bound in [22]); all these bounds deal with space-space non-commutativity. For the space-time case, the bound \( \theta \lesssim 9.51 \times 10^{-18} \text{ m.s} \) was found from quantum gravity considerations [23], the bound \( \theta_{st} \lesssim (0.6 \text{ GeV})^{-2} \) was determined in [21] from theoretical limit of the Lamb shift in H-atom. Some specific models gives the bound \( \Lambda_{nc} \geq 10 \text{ TeV} \) from CMB data [24] or the bound \( \Lambda_{nc} \gtrsim 10^{16} \text{ GeV} \) from particle phenomenology [25], but they are not direct constraints on the parameter because they use the loss of Lorentz invariance in the theory. A well documented review on non-commutative parameter bounds can be found in [26].

We are interested in the phenomenological consequences of space-time non-commutativity. We focus on the hydrogen atom because it is a simple and a well studied quantum system and so it can be taken as an excellent test for non-commutative signatures. The case of space-space non-commutativity was studied by Chaichian et al in [22] and [27]; here we work on the space-time case. We start by computing the corrections to
the Schrödinger energies then we study the contributions to the fine structure. Finally, we compute the corrections to the Dirac energies and we study the changes in the spectrum. This allows us to obtain a limit on the non-commutative parameter in each case.

The aim of this work is to study the effects of space-time noncommutativity on the spectrum of hydrogen atom and to find an upper limit for the non-commutative parameter by computing the corrections to the transition energies and comparing with the experimental results from hydrogen spectroscopy.

II. HYDROGEN ATOM IN SPACE-TIME NON-COMMUTATIVITY:

We work here on the space-time version of the non-commutativity; thus instead of (1), we use:

\[
[x^j_{st} ; x^0_{st}] = i\theta^{j0}
\]  

(2)

the \(st\) subscripts are for non-commutative space-time coordinates. The 0 denotes time and \(j\) is used for space coordinates. As a solution to these relations, we choose the transformations:

\[
x^j_{st} = x^j - i\theta^{j0}\partial_0
\]  

(3)

The usual coordinates of space \(x^j\) satisfy the usual canonical permutation relations. For convenience we use the vectorial notation:

\[
\vec{r}_{st} = \vec{r} - i\vec{\theta}\partial_0
\]  

(4)

where we have used the notation:

\[
\vec{\theta} \equiv (\theta^{10}, \theta^{20}, \theta^{30}) = (\theta^1, \theta^2, \theta^3)
\]  

(5)

The relations (3) and (4) can be seen as a Bopp’s shift [28].

We are dealing with the stationary quantum equations, and this allows us to consider the energy as a constant parameter. In our computation, we follow the work done by Chaichian et al. in space-space non-commutativity whatsoever in the non relativistic case ([22] and [29]) or in the relativistic case [27]; we use the standard Schrödinger and Dirac equations. This is possible because our choice in the transformations (5) leaves the coordinate \(x^0\) and all the momentums \(p^a\) unchanged. One can cite other studies that have used the same procedure. In the case of non-commutativity being only between time and space coordinates
as is the case in our work, it was shown in [30] that one has to use the new non-commutative coordinates and momentums instead of the usual ones in the Schrödinger equation. If there is both space-space and space-time cases of non-commutativity, it has been shown in [23] and [31] by studying the neutron in the gravitational field, that only the potential part of the standard equation varies. In a more simple way, we say that the kinetic energy does not change since it depends on the momentum that remains unchanged, thus we take the Coulomb potential and construct its non-commutative image. To achieve this, we have to write the expression of \( r_{st}^{-1} \):

\[
\frac{1}{r_{st}} = \frac{1}{\| \vec{r} - i \vec{\Theta} \partial_0 \|}
\]  

(6)

We make the development in series of the expression and because of the smallness of the non-commutative parameter, as one can see from the bounds given in the introduction, we restrict ourselves to the 1st order in \( \theta \) and neglect the higher order terms:

\[
r_{st}^{-1} = \left( (\vec{r} - i \vec{\Theta} \partial_0)^2 \right)^{-1/2} = \left( 1 + \frac{i \partial_0 \vec{r} \cdot \vec{\Theta}}{r^3} + O(\theta^2) \right)
\]

(7)

and thus, one can write the non-commutative Coulomb potential in the space-time case (up to the 1st order \( \theta \)) as follows:

\[
V_{nc}(r) = -\frac{e^2}{r} \left( 1 + \frac{i \partial_0 \vec{\Theta} \cdot \vec{r}}{r^3} \right)
\]

(8a)

\[
= -\frac{e^2}{r} - \frac{e^2 E \vec{\Theta} \cdot \vec{r}}{\hbar r^3}
\]

(8b)

where we have used the fact that \( i \partial_0 \psi = H \psi = (E/\hbar) \psi \).

An adequate choice of the parameter is \( \vec{\Theta} = \theta^{\mu\nu} \vec{r} / r \); It is equivalent to write:

\[
\Theta^{\mu\nu} = \begin{pmatrix}
0 & -\theta & 0 & 0 \\
\theta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(9)

Here \( \mu, \nu = 0, 1, 2, 3 \) and \( (1, 2, 3) \) means the spherical coordinates \( (r, \vartheta, \varphi) \). This writing is similar to that in [32] for the case of non-commutative space-time and in [33] for the space-space case (Another possible choice is \( \vec{\Theta} = \theta^{0\nu} \vec{k} \) and we have examined it in [34] in the non-relativistic case, but it has not been studied in the relativistic case until now). The
choice made in this paper allows us to write the non-commutative Coulomb potential as (we note \( r^0 = \theta^{st} \)):

\[
V_{nc}(r) = -\frac{e^2}{r} - \frac{e^2 E\theta^{st}}{\hbar} \frac{1}{r^2} + O(\theta^{st^2})
\]  

(10)

The expression is similar to the Kratzer potential [35]:

\[
V(r) = -\frac{e^2}{r} - \frac{C e^2}{r^2}
\]  

(11)

(where \( C = E\theta^{st}/\hbar \)). This kind of potential is introduced to study the spectrum of the external electron in alkali metals where it is subject to the effect of the nucleus (Coulomb term) and to the influence of inner electrons represented by the additional term and which may be interpreted as the potential of a central dipole; a dipole with its axis directed towards the electron and thus retains a spherical symmetry [36].

In a same way, the non-commutative Coulomb potential (10) can be interpreted as the potential energy of a negative charge under the influence of the superposition of a field produced by a point charge and another field coming from an electric central dipole; both are placed at the origin.

In other words; the non-commutative Coulomb potential is equivalent to an electron in a field of a charge distribution whose characteristics are:

- it is not neutral (a positive net charge here) so it gives the usual Coulomb contribution,
- it is not spherically symmetric and this adds the dipole contribution.

Such a distribution exists in the hydrogen atom in the proton; it is an extended positively charged system composed of three quarks. So applying space-time non-commutativity to the electron in the hydrogen atom is equivalent to consider the extended charged nature of the nucleus or the proton.

The fact that the proton has a structure and is a composite particle implies that non-commutativity cannot be applied to it as for elementary particles like electron, and it behaves essentially as a commutative particle in the non-commutative hydrogen atom as mentioned in [28]. This is why we apply non-commutativity only to the electron in this work.

Now we compute the corrections induced by this additional term in both the non-relativistic and the relativistic cases.
A. Corrections of the Bohr Energies:

We work on the standard Schrödinger equation and use the non-commutative Coulomb potential instead of the usual one, because as mentioned above, the kinetic energy depends on the momentum which remains unchanged and thus does not change. For this potential, the Schrödinger equation is:

\[ i\hbar \partial_\theta \psi = -\frac{\hbar^2}{2m} \Delta \psi - \frac{e^2}{r} \left( 1 - i\partial_\theta \frac{\vec{r} \cdot \vec{\theta}}{r^2} \right) \psi \] (12)

We are dealing with the stationary solutions, so we consider the energy as a constant parameter and write:

\[ H \psi = -\frac{\hbar^2}{2m} \Delta \psi - \frac{e^2}{r} \left( 1 + \frac{E}{\hbar} \frac{\vec{r} \cdot \vec{\theta}}{r^2} \right) \psi \] (13)

The spectrum of the Kratzer potential is obtained by defining a new orbital quantum number:

\[ l_K(l_K + 1) = l(l + 1) - 2me^2C/\hbar^2 \] (14)

and solving the Schrödinger equation with this new number for the Coulomb potential; by doing this, we obtain the Bohr energies with \( n = n_r + l_K + 1 \), and using this transformation, one can easily find the following expression of the energy (see [36] for example):

\[ E_{n,l} = -\frac{2me^4}{\hbar^2} \left( 2n - (2l + 1) + \sqrt{(2l + 1)^2 - 8mCe^2/\hbar^2} \right)^{-2} \] (15)

Making the replacement \( C = E_{\theta st}/\hbar \), we obtain the relation:

\[ E_{n,l} = -\frac{2me^4}{\hbar^2} \left( 2n - (2l + 1) + \sqrt{(2l + 1)^2 - 8mE_{\theta st}e^2/\hbar^3} \right)^{-2} \] (16)

The smallness of the non-commutative parameter allows us to expand the relation and we restrict ourselves to the 1st order in \( \theta \). Doing this, we solve the resulting expression for the energy (we take the energy as a constant parameter since we are dealing with stationary solutions of the Schrödinger equation) and find:

\[ E_{n,l} = -\frac{me^4}{2\hbar^2} \frac{1}{n^2} \left( 1 - \frac{m^2e^6}{\hbar^5} \frac{\theta_{\text{st}}}{(l + 1/2)n^3} \right) \] (17)

At the outset, we see that the accidental degeneracy with respect to \( l \) is removed. Noticing that the first term is the Bohr’s formula, we conclude that the space-time non-commutative corrections of the hydrogen energy levels in the framework of the Schrödinger equation are:

\[ \Delta E_{n,l} (nc) = \frac{m^3c^{10}}{\hbar^7} \frac{\theta_{\text{st}}}{(2l + 1)n^5} \] (18)
One can obtain a limit for $\theta$ by comparing the corrections to transition energies obtained using (17) with the experimental results from hydrogen spectroscopy. We take as test levels, $1S$ and $2S$ because we have the best experimental precision for the transition between them [37]:

$$f_{1S-2S} = (2446061102474851 \pm 34) \, Hz$$  \hspace{1cm} (19)

The non-commutative correction for this transition writes:

$$\Delta E_{nc} (1S - 2S) = \Delta E_{2,0} - \Delta E_{1,0} = 0.969 \left( m^3 e^{10}/h^7 \right) \theta_{st}$$  \hspace{1cm} (20)

Comparing with the precision of the experimental value in (19), we obtain:

$$\theta_{st} \lesssim 1.05 \cdot 10^{-19} \, eV^{-2} \approx (3 \, GeV)^{-2}$$  \hspace{1cm} (21)

This value is better than the limit obtained in both [21] and [23] and it justifies our expansion of (16) to obtain the energy. If one consider the limit for the accuracy of the resonance frequency measurements for this transition found to be $\sim 10^{-5} \, Hz$ in [38], then the new bound will be $\sim (10 \, TeV)^{-2}$.

**B. Corrections of the Fine Structure Expression :**

The fine structure Hamiltonian of the hydrogen can be written as:

$$H_{fs} = \frac{p^2}{2m} + V - \frac{p^4}{8m^3c^2} + \frac{\hbar}{4m^2c^2} \mathbf{\sigma} \cdot (\nabla V \times \mathbf{p})$$  \hspace{1cm} (22)

Where the first two terms are the Schrödinger Hamiltonian, the third one is the relativistic correction of the kinetic energy and the last term represents the spin-orbit contribution (we write his general expression from the non-relativistic limit of the Dirac equation for convenience as one can find in [39] and [40] for example). $\sigma$ are Pauli matrices.

The value of $\theta$ obtained in (21) allows us to consider non-commutative corrections with perturbation theory; to the 1st order in $\theta$, the corrections of the eigenvalues are:

$$\Delta E_{n,l}(nc) = \langle n, l, m_l | H_{fs} | n, l, m_l \rangle$$  \hspace{1cm} (23)

We follow [40] and [41] and neglect the changes of the eigenvectors; so from the Schrödinger equation, we can write the relativistic correction of the kinetic energy as:

$$\frac{p^4}{8m^3c^2} = \frac{1}{2mc^2} \left( \frac{p^2}{2m} \right)^2 = \frac{1}{2mc^2} (E - V)^2$$  \hspace{1cm} (24)
We use the relationship $\vec{\nabla} V = (\vec{r}/r)V(r)$ that applies for spherical potentials like our non-commutative Coulomb potential and write:

$$H_{fs} = \frac{p^2}{2m} + V - \frac{1}{2mc^2}(E - V)^2 + \frac{1}{4m^2c^2r} \frac{dV}{dr} \cdot \vec{l}$$  \hspace{1cm} (25)$$

where we have used the fact that $\vec{r} \times \vec{p} = \vec{l}$ and $\hbar \vec{\sigma} = \vec{s}$. Now, we develop (25) using the expression of the potential from (10) and we find:

$$H_{fs} = H_{fs}^{(Coulomb)} + \left[ -\frac{e^2}{r^2} - \frac{1}{mc^2} \left( E^2 + \frac{e^4}{r^2} \right) + \frac{\vec{s} \cdot \vec{l}}{2m^2c^2} \left( \frac{e^2}{r^4} \right) \right] C$$  \hspace{1cm} (26)$$

where we have considered the usual fine structure Hamiltonian for the habitual Coulomb potential:

$$H_{fs}^{(Coulomb)} = \frac{p^2}{2m} - \frac{e^2}{r} - \frac{1}{2mc^2} \left( E^2 + \frac{e^4}{r^2} + \frac{2Ee^2}{r^2} \right) + \frac{e^2}{4m^2c^2} \frac{\vec{s} \cdot \vec{l}}{r^3}$$  \hspace{1cm} (27)$$

and the new non-commutative correction to this Hamiltonian is ($C = E\theta_{st}/\hbar$):

$$H_{fs}^{(nc)} = \left[ -\frac{Ee^2}{\hbar r^2} - \frac{E}{\hbar mc^2} \left( \frac{Ee^2}{r^2} + \frac{e^4}{r^2} \right) + \frac{Ee^2}{2\hbar m^2c^2} \frac{\vec{s} \cdot \vec{l}}{r^4} \right] \theta_{st}$$  \hspace{1cm} (28)$$

The fine structure Coulomb energies can be found in the literature (For example [40] or [41]):

$$E_{n,j} = -\frac{me^4}{2\hbar^2 n^2} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{j + 1/2} - \frac{3}{4n} \right) \right]$$  \hspace{1cm} (29)$$

$\alpha = e^2/\hbar c$ is the fine structure constant and $j = l \pm 1/2$ is the quantum number associated to the total angular momentum $\vec{j} = \vec{l} + \vec{s}$. The non-commutative correction to this energies are:

$$\langle H_{fs}^{(nc)} \rangle = \left[ -\frac{Ee^2}{\hbar} \langle \frac{1}{r^2} \rangle - \frac{Ee^2}{\hbar mc^2} \left( E \langle \frac{1}{r^2} \rangle + \frac{e^2}{r^2} \langle \frac{1}{r^2} \rangle \right) - \frac{1}{2m} \langle \frac{\vec{s} \cdot \vec{l}}{r^4} \rangle \right] \theta_{st}$$  \hspace{1cm} (30)$$

To compute these terms, we use the Kramer’s recursive relations:

$$\langle \frac{1}{r^2} \rangle = \left( \frac{1}{a_0} \right)^2 \frac{1}{n^3(l + 1/2)}$$  \hspace{1cm} (31a)$$

$$\langle \frac{1}{r^3} \rangle = \left( \frac{1}{a_0} \right)^3 \frac{1}{n^3(l + 1/2)(l + 1)l}$$  \hspace{1cm} (31b)$$

$$\langle \frac{1}{r^4} \rangle = \left( \frac{1}{a_0} \right)^4 \frac{3n^2 - l(l + 1)}{2n^5(l + 3/2)(l + 1)(l + 1/2)(l - 1/2)}$$  \hspace{1cm} (31c)$$
where \( a_0 = \hbar^2 / me^2 \) is the 1st Bohr radius. Using the expression of the Bohr energies \( E = -me^4 / 2\hbar^2 n^2 \) and writing:

\[
\langle \vec{s} \cdot \vec{l} \rangle = \langle \frac{\hbar^2}{2r^4} \left( \vec{j}^2 - \vec{l}^2 - \vec{s}^2 \right) \rangle = \frac{\hbar^2 K}{2} \langle \frac{1}{r^4} \rangle
\]

with \( K = j(j+1) - l(l+1) - s(s+1) \), we find:

\[
\langle H_{fs}^{(nc)} \rangle = \langle H_0^{(nc)} \rangle + \langle H_{\alpha^2}^{(nc)} \rangle
\]

where:

\[
\langle H_0^{(nc)} \rangle = \left( \frac{m^3 e^{10}}{\hbar^7} \right) \frac{\theta_{st}}{2n^5(l+1/2)} \tag{33a}
\]

\[
\langle H_{\alpha^2}^{(nc)} \rangle = \langle H_0^{(nc)} \rangle \alpha^2 \left[ \frac{-1}{2n^2} \left( 1 - \frac{K}{4(l+3/2)(l-1/2)} \right) + \frac{1}{l+1/2} \left( 1 - \frac{3K}{8(l+3/2)(l-1/2)} \right) \right] \tag{33b}
\]

The first term is the non-commutative correction found in (17) and the last terms are the fine structure non-commutative corrections and are proportional to \( \alpha^2 \) as it should. We have two cases for the value of the orbital quantum number \( l \) since it can be zero or not, and because \( s = 1/2 \), this gives three possibilities for \( j \). We will study the two cases \( j = l \pm 1/2 \) corresponding for \( l \neq 0 \) first then we will treat the case \( j = 1/2 \) when \( l = 0 \) separately.

The case \( j = l + 1/2 \) and \( l \neq 0 \):

\[
\langle H_{fs}^{(nc)} \rangle_{n,j^+} = \left( \frac{m^3 e^{10}}{\hbar^7} \right) \frac{\theta_{st}}{2n^5 j} \left\{ 1 + \frac{\alpha^2/4}{(j^2 - 1)} \left[ \frac{-8j^2 + 2j + 7}{4n^2} + \frac{16j^2 - 6j - 13}{(4j^2 - 1)} \right] \right\}
\]

The case \( j = l - 1/2 \):

\[
\langle H_{fs}^{(nc)} \rangle_{n,j^-} = \left( \frac{m^3 e^{10}}{\hbar^7} \right) \frac{\theta_{st}}{2n^5(j + 1)} \left\{ 1 + \frac{\alpha^2/4}{j(j+2)} \left[ \frac{-8j^2 - 18j - 3}{2n^2} + \frac{16j^2 + 38j + 9}{(2j+1)(2j+3)} \right] \right\}
\]

We see that the expression depends on the way to obtain the number \( j \) from \( l \) unlike the usual fine structure correction in (28) which is the same for all the possible values of \( j \). It implies that the non-commutativity removes the degeneracy \( j = l + 1/2 = (l+1) - 1/2 \) in the hydrogen atom, in states like \( nP_{3/2} \) and \( nD_{3/2} \) (the same degeneracy exists in the Dirac energies for Coulomb potential).

The case \( l = 0 \) must be treated separately because \( l \) appears in the denominator of two terms in (33) and this gives a divergent result. This singularity is only apparent because
the terms come from the fact that we have neglected the potential energy of the electron with respect to $mc^2$, but this is not true at the limit $r \to 0$. The same thing appears in the calculation of the spin-orbit term in usual hydrogen fine structure. It has been showed in [39] that the spin orbit term is zero in this case because in the $\langle \vec{s} \cdot \vec{l} \rangle$ term, the numerator $\vec{s} \cdot \vec{l}$ vanishes exactly for $l = 0$ while the denominator $\langle r^{-4} \rangle$ approaches the limit zero. Things are not the same for the $\langle 1/r^3 \rangle$ term as there is no vanishing numerator here and one has to be replace $\langle 1/r^{-3} \rangle$ by $\langle 1/r^{-3} (1 - a_0 \alpha^2/2r) \rangle$ by using $(2mc^2 + e^2/r)$ instead of $(2mc^2)$ [39]; this gives us a finite result. We use the same argument to eliminate the spin-orbit term and the same transformation to compute $\langle 1/r^3 \rangle$; we get:

$$\langle H^{(nc)}_{fs} \rangle_{n,l=0} = \frac{n^3 e^{10}}{\hbar^7 n^2} \theta_{st} \left[ \frac{1}{n^3} - \frac{\alpha^2}{2} \left( \frac{1}{n^5} - \langle r^{-3} \left( 1 - \frac{a_0 \alpha^2}{2r} \right) \rangle \right) \right]$$

(36)

As an example, we compute the corrections of the states $1s$ and $2s$:

$$\langle H^{(nc)}_{\alpha^2} \rangle_{1s} = -\frac{m^3 e^{10}}{\hbar^7 \theta_{st}} \left[ \frac{\alpha^2}{2} (1 - 37.053) \right] = -\left( 0.96 \cdot 10^{-3} \right) \frac{m^3 e^{10}}{\hbar^7} \theta_{st}$$

(37)

$$\langle H^{(nc)}_{\alpha^2} \rangle_{2s} = -\frac{m^3 e^{10}}{\hbar^7 \theta_{st}} \left[ \frac{\alpha^2}{8} \left( \frac{1}{32} - 4.603 \right) \right] = -\left( 0.03 \cdot 10^{-3} \right) \frac{m^3 e^{10}}{\hbar^7} \theta_{st}$$

These terms are added to (18). We see that the additional contribution to the transition correction (20) is $10^{-3}$ smaller and this do not affect the limit (21).

We can also compute the correction to the Lamb shift $2P_{1/2} \rightarrow 2S_{1/2}$ as an example. From (17), (35) and (37), we have:

$$\langle H^{(nc)}_{fs} \rangle_{2P_{1/2}} = 0.010 \left( m^3 e^{10}/\hbar^7 \right) \theta_{st} \quad (38a)$$

$$\langle H^{(nc)}_{fs} \rangle_{2S_{1/2}} = 0.031 \left( m^3 e^{10}/\hbar^7 \right) \theta_{st} \quad (38b)$$

and the Lamb Shift correction follows:

$$\Delta E_{nc} \left( 2P_{1/2} \rightarrow 2S_{1/2} \right) = 0.021 \left( m^3 e^{10}/\hbar^7 \right) \theta_{st}$$

(39)

We compare this result to the current theoretical accuracy 0.08 kHz from [42] and find the bound $\theta_{st} \lesssim (0.4 \ GeV)^{-2}$ which is bigger than the precedent one in (21).

C. Corrections of the Dirac Energies:

For the relativistic case, we write the Dirac equation:

$$i\hbar \partial_0 = H \psi = (\vec{\alpha} \cdot \vec{p}) + m\gamma^0 + eA_0$$

(40)
where \( \alpha_i = \gamma_0 \gamma_i \) and \( \gamma_\mu \) are the Dirac matrices.

We use the same argument as in the non-relativistic case and employ the standard Dirac equation but with the non-commutative Coulomb potential:

\[
A^{(nc)}_0 = -\frac{e}{r_{st}} = -e \left( \left( \frac{\mathbf{r}^2 - i \mathbf{\theta} \partial_0}{r} \right)^2 \right)^{-1/2} = -\frac{e}{r} \left( 1 + i \partial_0 \frac{\mathbf{r} \cdot \mathbf{\theta}}{r^2} + O(\theta^2) \right)
\]

(41)

We restrict ourselves to the 1st order in \( \theta \) and neglect the higher order terms in the development in series of the expression:

\[
A^{(nc)}_0 = -\frac{e}{r} \left( 1 + i \partial_0 \frac{\theta_{st}}{r} \right) + O(\theta^2) = -\frac{e}{r} - \frac{eE_{st}}{\hbar} \frac{1}{r^2} + O(\theta^2)
\]

(42)

The Hamiltonian can now be expressed as:

\[
H = (\mathbf{\alpha} \cdot \mathbf{p}) + m\gamma^0 - e \left( \frac{e}{r} + e \left( \frac{E}{\hbar} \right) \frac{\theta_{st}}{r^2} \right) = H_0 + H_{nc}
\]

(43)

\( H_0 \) is the Dirac Hamiltonian in the usual relativistic theory and \( H_{nc} \) is the non-commutative correction to this Hamiltonian:

\[
H_{nc} = -e^2 \left( \frac{E}{\hbar} \right) \theta_{st} r^{-2}
\]

(44)

The smallness of the parameter \( \theta \) from the different bounds mentioned in the introduction allows us to consider noncommutative corrections with perturbation theory; to the 1st order in \( \theta \), the corrections of the eigenvalues are:

\[
\Delta E_{nc} = \langle H_{nc} \rangle = - \left( \frac{e^2 \theta_{st}}{\hbar} \right) \langle r^{-2} \rangle
\]

(45)

From [43], one has:

\[
\langle \frac{1}{r^2} \rangle = \frac{2\kappa (2\kappa \varepsilon - 1) (1 - \varepsilon^2)^{3/2}}{\alpha \sqrt{\kappa^2 - \alpha^2} [4 (\kappa^2 - \alpha^2) - 1]} \left( \frac{mc}{\hbar} \right)^2 \left( \frac{1}{a_0} \right)^2
\]

(46)

where \( a_0 = \hbar^2 / me^2 \) is the 1st Bohr radius and \( \varepsilon = E / mc^2 \); \( E \) is the Dirac energy:

\[
E_{n,j} = mc^2 \left\{ 1 + \alpha^2 \left[ (n - j - 1/2) + \sqrt{(j + 1/2)^2 - \alpha^2} \right]^2 \right\}^{-1/2}
\]

(47)

\( \alpha = e^2 / hc \) is the fine structure constant and \( j = l \pm 1/2 \) is the quantum number associated to the total angular momentum \( \mathbf{j} = \mathbf{l} + \mathbf{s} \). The number \( \kappa \) is giving by the two relations \( \kappa = -(j + 1/2) \) if \( j = (l + 1/2) \) and \( \kappa = (j + 1/2) \) if \( j = (l - 1/2) \). We see that through \( \kappa \),
the energy depends not only on the value of $j$ but also on the manner to get this value (or on $l$), unlike the usual Dirac energies in (48) which is the same for all the possible ways to obtain $j$. This implies that the non-commutativity removes the degeneracy $j = l+1/2 = (l+1)−1/2$ in hydrogen atom ($nP_{3/2}$ and $nD_{3/2}$ for example) and acts like the Lamb shift.

We recall that the energy level without considering the rest mass energy ($mc^2$) is written as a function of the total energy by the relation $E_{n,j} = E - mc^2$ and so the corrections to these energies are: $\Delta E_{n,j}^{(nc)} = \Delta E^{(nc)}$. From now on, we note these corrections $E_{n,j}^{(nc)}$ or $E^{(nc)}(nL_j)$ where $L$ is the spectroscopic letter corresponding to a specific value of the angular quantum number $l$. We take as an example the levels $n = 1, 2$ and we compute the corrections to their energies:

$$E^{(nc)} (1S_{1/2}) = 1.065084 \times 10^{-4} \left( m^3 e^2 c^4 / \hbar^3 \right) \theta_{st}$$

$$E^{(nc)} (2S_{1/2}) = 1.331426 \times 10^{-5} \left( m^3 e^2 c^4 / \hbar^3 \right) \theta_{st}$$

$$E^{(nc)} (2P_{1/2}) = 0.443805 \times 10^{-5} \left( m^3 e^2 c^4 / \hbar^3 \right) \theta_{st}$$

$$E^{(nc)} (2P_{3/2}) = 0.443765 \times 10^{-5} \left( m^3 e^2 c^4 / \hbar^3 \right) \theta_{st}$$

We can get a limit for $\theta$ by comparing these shifts to experimental results from hydrogen spectroscopy. We take as test levels, $1S - 2S$ transition (19) and looking at (49), the non-commutative correction for this transition is:

$$\Delta E^{(nc)} (1S - 2S) = E^{(nc)} (1S_{1/2}) - E^{(nc)} (2S_{1/2})$$

$$= 0.931941 \times 10^{-4} \left( m^3 e^2 c^4 / \hbar^3 \right) \theta_{st}$$

Comparing with the precision of the experimental value in (15), we obtain:

$$\theta_{st} \lesssim 3.099 \times 10^{-24} \text{ eV}^{-2} \approx (0.57 \text{ TeV})^{-2}$$

It is a significant improvement of the previous bounds obtained in [22] [23] and [34] and it justifies the use of perturbation method to obtain the energy. If one consider the limit for the accuracy of the resonance frequency measurements for this transition found to be $\sim 10^{-5} \text{ Hz}$ in [38], then the new bound will be $\approx (1.5 \text{ PeV})^{-2}$.

To make the differences between the corrections of the levels more visible, we can write their expressions in a more elegant and appropriate way by using the development in series with respect to $\alpha$ (up to the 2nd order in $\alpha^2$ to do the comparison with the fine structure
corrections). For the Dirac energies, we have:

\[
E = mc^2 \left\{ 1 - \frac{\alpha^2}{2n^2} \left[ 1 + \left( \frac{2}{(2j + 1)n - \frac{3}{4n^2}} \right) \alpha^2 \right] + O(\alpha^6) \right\}
\]

(51)

We use this formula and the general expressions from (46) and (47) to compute the non-commutative corrections and find:

\[
E_{n,j=\frac{l+\frac{1}{2}}{2}}^{(nc)} = \frac{m^3e^2c^4\alpha^2}{j(n)\hbar^3} \left[ 1 + \left( \frac{6j^2 + 6j + 1}{j(j+1)(2j+1)^2} \right) + \frac{3}{(2j+1)n} - \frac{2j+1}{4j^3n^2} \alpha^2 \right] \theta_{st}
\]

(52a)

\[
E_{n,j=\frac{l-\frac{1}{2}}{2}}^{(nc)} = \frac{m^3e^2c^4\alpha^2}{(j+1)(n+1)\hbar^3} \left[ 1 + \left( \frac{6j^2 + 6j + 1}{j(j+1)(2j+1)^2} \right) + \frac{3}{(2j+1)n} - \frac{2j+1}{4j^3n^2} \alpha^2 \right] \theta_{st}
\]

(52b)

We see that the non-commutativity acts like a Lamb shift and remove the degeneracy \( j = l + 1/2 = (l + 1) - 1/2 \) in the hydrogen as we have mentioned before. We note that the relativistic corrections (53a,53b) are not the same as those found for the fine structure (35,36,37), whereas they coincide for the quantum theory of hydrogen atom in the usual case. This coincidence is accidental (as it is the case of Gauss theorem) and is due to the Coulomb potential which is a special case. So the additional term in \( r^{-2} \) breaks the equivalence and this induces the difference found.

The non-commutative correction to the Lamb shift follows from the previous expressions:

\[
\Delta E_{n,j}^{(nc)} (Lamb shift) = E_{n,j=\frac{l+1}{2}}^{(nc)} - E_{n,j=(l+1)-\frac{1}{2}}^{(nc)}
\]

\[
= \frac{m^3e^2c^4\alpha^2}{j(j+1)n^3\hbar^3} \left[ 1 + \left( \frac{6j^2 + 6j + 1}{j(j+1)(2j+1)^2} \right) + \frac{3}{(2j+1)n} - \frac{2j+1}{4j^3n^2} \alpha^2 \right] \theta_{st}
\]

(53)

To compare with experience, we apply he result to the \( n = 2 \) and \( j = 1/2 \) case or the \( 2P_{1/2} \rightarrow 2S_{1/2} \) Lamb shift (the 28cm line). From (53) (or from (49)), we have:

\[
\Delta E_{2,1/2}^{(nc)} (2P_{1/2} \rightarrow 2S_{1/2}) = 0.887621 \cdot 10^{-5} \left( m^3e^2c^4/h^3 \right) \theta_{st}
\]

(54)

We compare this result to the current theoretical accuracy 0.08 kHz from [42] and find the bound \( \theta_{st} \lesssim 3.254 \cdot 10^{-23} \text{ eV}^{-2} \approx (0.18 \text{ TeV})^{-2} \). It is larger than the previous one in (51) but it is still better than the ones from [22] [23] and [34].

III. CONCLUSION:

In this work, we look for space-time non-commutative hydrogen atom and induced phenomenological effects. We found that applying space-time non-commutativity to the electron
in the H-atom modifies the Coulomb potential to give us the potential of Kratzer. The additional term is proportional to $r^{-2}$ and we assimilate it to the field of a central dipole. In other words, the action of space-time non-commutativity is equivalent to consider the extended charged nature of the proton in the nucleus.

We started by solving the Schrödinger equation for this potential; we have calculated the corrections induced to energy levels by this non-commutative effect and we find that the non-commutative corrections remove the degeneracy of the Bohr energies with respect to the orbital quantum number $l$ and the energies are labelled $E_{n,l}$. By comparing to experimental results from high precision hydrogen spectroscopy, we get a new bound for the parameter of non-commutativity (around $(3 \text{ GeV})^{-2}$).

In a second step, we study the contributions to the fine structure and find that they have no significant effect on the energies. But we found that the non-commutative Coulomb potential remove the degeneracy $j = l + 1/2 = (l + 1) - 1/2$ of the hydrogen fine structure and the energies write $E_{n,j,l}$; the non-commutativity acts here like a Lamb shift. This is explained by the fact that Lamb correction can be interpreted as a shift of $r$ in the Coulomb potential due to interactions of the bound electron with the fluctuating vacuum electric field [44], and non-commutativity is also a shift of $r$ as we can see from the Bopp’s shift.

The same thing was done for the relativistic case where by solving the Dirac equation, we have calculated the corrections induced to energy levels by this non-commutative effect. By comparison with experimental results, we get a new bound for the non-commutative parameter (about $(0.57 \text{ TeV})^{-2}$).

The non-commutative corrections to the Dirac theory of hydrogen atom remove the degeneracy of the Bohr energies with respect to the orbital quantum number $l$ and also the degeneracy of the Dirac energies with respect to the total angular momentum quantum number $j$ ($j = l + 1/2 = (l + 1) - 1/2$), and the energies are labelled $E_{n,j,l}$. As in the case of the fine structure, the non-commutativity has an effect similar to that of the Lamb Shift.

Recently, there has been a certain amount of activity around the theme of cosmological and astrophysical applications of non-commutative geometry models of particle physics, for example [45-47]. One can study such applications of non-commutativity via the Lamb shift line and the $2S - 1S$ transition and also via the Lyman-$\alpha$ ray. We draw attention to the fact that $2S - 1S$ transition is used in high precision spectroscopy because of the implication of these measurements on the values of fundamental physical constants like the fine structure
constant $\alpha$ and the Rydberg constant $R$ [37] and in tests of Lorentz invariance [47]. The possible variation of the fine structure constant has relation with primordial light nuclei abundance in the early universe [48], with f(R) theories in Einstein frame and quintessence models [49] or with the inhomogeneity of the mass distribution in the early universe and the cosmological constant [50] (One can find a good review in this last reference). If we take the bound obtained from the Lamb shift theoretical accuracy 0.08 kHz, we find that it corresponds to a shift in the $2S - 1S$ transition frequency equal to $\approx 0.8$ kHz. This value is greater than the experimental accuracy in (19) and thus the space-time non-commutativity can be tested here (If we consider the bound from the 28cm line).

We have to mention that there is another challenge in the study of the hydrogen atom in the context of space-time non-commutativity, which is to determine the spectrum for the choice $\vec{\theta} = \theta^{30} \vec{k}$ in the relativistic case. As has been demonstrated in our article for the non-relativistic case [34], the additional term to the Coulomb potential is proportional to $\cos \vartheta$ and therefore non-commutative contributions are no longer diagonals and the perturbation theory of order one is no longer valid. We need, in this case, to write the Hamiltonian matrix for corrections of first and second order in $\theta$ and then compute the eigenvalues for this system; this is in preparation.

[1] A. Connes, *Noncommutative Geometry*, Academic Press, CA, (1994)
[2] N. Seiberg and E. Witten, *String theory and Noncommutative Geometry*, JHEP 09, 032 (1999)
[3] M. Chaichian, P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT*, Phys.Lett. B604, 98-102 (2004)
[4] M. Ihl and C. Saemann, *Drinfeld-Twisted Supersymmetry And Non-Anticommutative Super-space*, JHEP 0601, 065 (2006)
[5] C.D. Carone and H.J. Kwee, *Unusual High-Energy Phenomenology of Lorentz-Invariant Noncommutative Field Theories*, Phys.Rev. D73, 096005 (2006)
[6] P. Aschieri, L. Castellani, *Noncommutative Supergravity in D = 3 and D = 4*, JHEP 0906, 086 (2009)
[7] N. Mebarki, S. Zaim, L. Khodja and H. Aissaoui, *Gauge Gravity in Noncommutative De Sitter
Space and Pair Creation, Phys.Scr. 78, 045101 (2008)

[8] J. Bellissard, H. Schulz-Baldes and A. Van Elst, The Non Commutative Geometry of the Quantum Hall Effect, J. Math. Phys. 35 5373-5471 (1994)

[9] J. Bellissard, Noncommutative Geometry of Aperiodic Solids, in ”Geometric and Topological Methods for Quantum Field Theory”, (Villa de Leyva, 2001), pp. 86-156, World Sci. Publishing, River Edge, NJ, (2003).

[10] S. Hellerman and M. Van Raamsdonk, Quantum Hall Physics = Noncommutative Field Theory, JHEP 10 039 (2001)

[11] A.P. Polychronakos, Non-Commutative Fluids, Prog. Math. Phys. 53 109-159 (2007) (and the references therein)

[12] A. Connes and M. Marcolli, From Physics to Number Theory via Noncommutative Geometry, Frontiers in Numbers Theory, Physics, and Geometry I, Part II, 269-349 (2006)

[13] J. Gamboa, M. Loewe, F. Mendez and J.C. Rojas, The Landau Problem and Noncommutative Quantum Mechanics, Mod. Phys. Lett. A16 2075-2078 (2001)

[14] A. Sitarz, Non-Commutative Geometry and the Ising Model, J. Phys. A: Math.Gen. 26 5305-5312 (1993)

[15] A. Bérard and H. Mohrbach, Spin Hall Effect and Berry Phase of Spinning Particles, Phys. Lett. A352 190-195(2006)

[16] D.B. Tweed, T.P. HalsWanter, V. Happe and M. Fetter, Non-Commutativity in the Brain, Nature 399 261-263 (1999)

[17] M.R. Douglas and N.A. Nekrasov, Noncommutative Field Theory, Rev. Mod. Phys. 73 977-1029 (2001)

[18] R.J. Szabo, Quantum Field Theory on Noncommutative Spaces, Phys.Rept. 378, 207-299 (2003)

[19] G. Abbiedi et al, Test of Non-Commutative QED Effect in the Process \( e^+ e^- \rightarrow \gamma \gamma \), Phys.Lett. B568, 181-190 (2003)

[20] J.L. Hewett, F.J. Petrielloand T.G. Rizzo, Signals for Non-Commutative Interactions at Linear Colliders, Phys.Rev. D64, 075012 (2001)

[21] A. Stern, Particlelike Solutions to Classical Noncommutative Gauge Theory, Phys.Rev. D78, 065006 (2008)

[22] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Hydrogen Atom Spectrum and the Lamb
Shift in Noncommutative QED, Phys.Rev.Lett. 86, 2716 (2001)

[23] A. Saha, Time-Space Noncommutativity in Gravitational Quantum Well scenario, Eur.Phys.J. C51, 199-205 (2007) (and the references therein)

[24] E. Akofor, A.P. Balachandran, A. Joseph, L. Pekowsky, B.A. Qureshi, Constraints from CMB on Spacetime Noncommutativity and Causality Violation, Phys.Rev. D79, 063004 (2009)

[25] A. Joseph, Particle Phenomenology on Noncommutative Spacetime, Phys.Rev. D79, 096004 (2009)

[26] R.J. Szabo, Quantum Gravity, Field Theory and Signatures of Noncommutative Spacetime, Gen.Relativ.Gravit. 42, 1-29 (2010)

[27] T. C. Adorno, M. C. Baldiotti, M. Chaichian, D. M. Gitman and A. Tureanu, Dirac Equation in Noncommutative Space for Hydrogen Atom, Phys.Lett. B682, 235-239 (2009)

[28] S. Dulat and K. Li, The Aharonov-Casher Effect for Spin-1 Particles in Non-Commutative Quantum Mechanics, Eur.Phys.J. C54, 333-337 (2008)

[29] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, Comments on the Hydrogen Atom Spectrum in the Noncommutative Space, Eur.Phys.J. C36, 251-252 (2004)

[30] A.P. Balachandran, T.R. Govindarajan, C. Molina and P. Teotonio-Sobrinho, Unitary Quantum Physics with Time-Space Noncommutativity, JHEP. 0410 (2004) 072.

[31] A. Saha, Galilean Symmetry in a Noncommutative Gravitational Quantum Well, Phys. Rev. D 81, 125002 (2010)

[32] S. Fabi, B. Harms and A. Stern, Noncommutative Corrections to the Robertson-Walker Metric, Phys.Rev. D78, 065037 (2008)

[33] M. Chaichian, A. Tureanu, M. R. Setare and G. Zet, On Black Holes and Cosmological Constant in Noncommutative Gauge Theory of Gravity, JHEP 0804, 064 (2008)

[34] M. Moumni, A. BenSlama and S. Zaim, A New Limit for the Noncommutative Space-Time Parameter, J. Geom. Phys. 61, 151-156 (2011)

[35] A. Kratzer, Die Ultraroten RotationsSpektren der HalogenWasserstoffe, Z. Phys. 3, 289 (1920)

[36] D. Sivoukhine, Cours de Physique Générale Tome V Physique Atomique et Nucléaire, Traduction Française Editions Mir, Moscou, (1986)

[37] T.W. Hänisch et al, Precision Spectroscopy of Hydrogen and Femtosecond Laser Frequency Combs, Phil.Trans.R.Soc. A363, 2155–2163 (2005)

[38] L. Labzowsky, G. Schedrin, D. Solovyev and G. Plunien, Theoretical Study of the Accuracy
Limits for the Optical Frequency Measurements, Phys. Rev. Lett. 98, 203003 (2007)

[39] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, Academic Press, NY, (1957)

[40] E. Chpolski, *Physique Atomique*, Traduction Française Editions Mir, Moscou, (1978)

[41] V. Berestetski, E. Lifchitz and L. Pitayevski, *Théorie Quantique Relativiste 1ère partie*, Traduction Française Editions Mir, Moscou, (1972)

[42] M. I. Eides, H. Grotch and V. A. Shelyuto, *Theory of Light Hydrogenlike Atoms*, Phys.Rept. 342, 63-261 (2001)

[43] S.K. Suslov, B. Trey, *The Hahn Polynomials in the Nonrelativistic and Relativistic Coulomb Problems*, J. Math. Phys. 49, 012104 (2008)

[44] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw-Hill Inc (1980)

[45] B. Malekolkalami and M. Farhoudi, *Noncommutativity Effects in FRW Scalar Field Cosmology*, Phys.Lett. B678, 174-180 (2009)

[46] O. Bertolami and C. A. D. Zarro, *Towards a Noncommutative Astrophysics*, Phys.Rev. D81, 025005 (2010)

[47] B. Altschul, *Testing Electron Boost Invariance with 2S-1S Hydrogen Spectroscopy*, Phys.Rev. D81, 041701(R) (2010)

[48] S. J. Landau, M. E. Mosquera, C. G. Scóccola and H. Vucetich, *Early Universe Constraints on Time Variation of Fundamental Constants*, Phys.Rev. D78, 083527 (2008)

[49] Y. Bisabr, *Constraining f(R) Theories with Temporal Variation of Fine Structure Constant*, Phys.Lett. B688, 4-8 (2010)

[50] A. Bhattacharya, B. Chakrabarti and S. Mani, *On Some Properties of the Fine Structure Constant*, Acta.Phys.Polon. B39, 235-240 (2008)