A NOTE ON MALLIAVIN FRACTIONAL SMOOTHNESS FOR LÉVY PROCESSES AND APPROXIMATION

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ABSTRACT. Assume a Lévy process \((X_t)_{t \in [0,1]}\) that is an \(L_2\)-martingale and let \(Y\) be either its stochastic exponential or \(X\) itself. For certain integrands \(\varphi\) we investigate the behavior of
\[
\left\| \int_{(0,1]} \varphi_t dX_t - \sum_{k=1}^{N} v_{k-1} (Y_{t_k} - Y_{t_{k-1}}) \right\|_{L_2},
\]
where \(v_{k-1}\) is \(\mathcal{F}_{t_{k-1}}\)-measurable, in dependence on the fractional smoothness in the Malliavin sense of \(\int_{(0,1]} \varphi_t dX_t\). A typical situation where these techniques apply occurs if the stochastic integral is obtained by the Galtchouk-Kunita-Watanabe decomposition of some \(f(X_1)\). Moreover, using the example \(f(X_1) = \mathbf{1}_{(K,\infty)}(X_1)\) we show how fractional smoothness depends on the distribution of the Lévy process.

1. INTRODUCTION

We consider the quantitative Riemann approximation of stochastic integrals driven by Lévy processes and its relation to the fractional smoothness in the Malliavin sense. Besides the interest on its own, the problem is of interest for numerical algorithms and for Stochastic Finance. To explain the latter aspect, assume a price process \((S_t)_{t \in [0,1]}\) given under the martingale measure by a diffusion
\[
S_t = s_0 + \int_0^t \sigma(S_r) dW_r,
\]
where \(W\) is the Brownian motion and where usual conditions on \(\sigma\) are imposed. For a polynomially bounded Borel function \(f : \mathbb{R} \to \mathbb{R}\) we obtain a representation
\[
(1) \quad f(S_1) = V_0 + \int_0^1 \varphi_t dS_t
\]
where \((\varphi_t)_{t \in [0,1]}\) is a continuous adapted process which can be obtained via the gradient of a solution to a parabolic backward PDE related to

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σ with terminal condition f. The process \((\varphi_t)_{t \in [0,1]}\) is interpreted as a trading strategy. In practice one can trade only finitely many times which corresponds to a replacement of the stochastic integral in (1) by the sum \(\sum_{k=1}^{N} \varphi_{t_{k-1}}(S_{t_k} - S_{t_{k-1}})\) with \(0 = t_0 < t_1 < \cdots < t_N = 1\). The error

\[
\int_0^1 \varphi_t dS_t - \sum_{k=1}^{N} \varphi_{t_{k-1}}(S_{t_k} - S_{t_{k-1}})
\]

caused by this replacement is often measured in \(L_2\) and has been studied by various authors, for example by Zhang [21], Gobet and Temam [11], S. Geiss [8], S. Geiss and Hujo [9] and C. Geiss and S. Geiss [7]. For results concerning \(L_p\) with \(p \in (2, \infty)\) we refer to [20], the weak convergence is considered in [10] and [19] and by other authors. In particular, if \(S\) is the Brownian motion or the geometric Brownian motion, S. Geiss and Hujo investigated in [9] the relation between the Malliavin fractional smoothness of \(f(S_1)\) and the \(L_2\)-rate of the discretization error (2).

It is natural to extend these results to Lévy processes. A first step was done by M. Brodén and P. Tankov [5] (see Remark 4.11). The aim of this paper is to extend results of [9] into the following directions:

(a) The Brownian motion and the geometric Brownian motion are generalized to Lévy processes \((X_t)_{t \in [0,1]}\) that are \(L_2\)-martingales and their Doléans-Dade exponentials \(S = \mathcal{E}(X)\),

\[ S_t = 1 + \int_{(0,t]} S_u dX_u, \]

respectively. For certain stochastic integrals

\[ F = \int_{(0,1]} \varphi_s dX_s \]

and for \(Y \in \{X, \mathcal{E}(X)\}\) we study the connection of the Malliavin fractional smoothness of \(F\) (introduced by the real interpolation method) and the behavior of

\[
a_Y^{opt}(F; (t_k)_{k=0}^N) = \inf \left\| F - \sum_{k=1}^{N} v_{k-1}(Y_{t_k} - Y_{t_{k-1}}) \right\|_{L_2},
\]

where the infimum is taken over \(\mathcal{F}_{t_{k-1}}\)-measurable \(v_{k-1}\) such that \(\mathbb{E}v_{k-1}^2(Y_{t_k} - Y_{t_{k-1}})^2 < \infty\) and where \(0 = t_0 < \cdots < t_N = 1\) is a deterministic time-net.

(b) In contrast to [9], where the reduction of the stochastic approximation problem to a deterministic one is based on Itô’s formula and was
done in [8, 7], we prove an analogous reduction in Theorems 3.3 and 3.4 by techniques based on the Itô chaos decomposition.

(c) One more principal difference to [9] is the fact that Lévy processes do in general not satisfy the representation property and therefore there are \( F \in L_2 \) that cannot be approximated by sums of the form \( \sum_{k=1}^{N} v_{k-1}(Y_{t_k} - Y_{t_{k-1}}) \) in \( L_2 \). As a consequence we have to use the (orthogonal) Galtschouk-Kunita-Watanabe projection that projects \( L_2 \) onto the subspace \( I(X) \) of stochastic integrals

\[
\int_{(0,1]} \lambda_s dX_s
\]

with \( \mathbb{E} \int_{0}^{1} |\lambda_s|^2 ds < \infty \) that can be defined in our setting as the \( L_2 \)-closure of

\[
\left\{ \sum_{k=1}^{N} v_{a_{k-1}}(X_{a_k} - X_{a_{k-1}}) : v_{a_{k-1}} \in L_2(F_{a_{k-1}}), \ 0 = a_0 < \cdots < a_N = 1 \right\} \quad N = 1, 2, \ldots
\]

to deal with our approximation problem.

The paper is organized as follows. In Section 2 we recall some facts about real interpolation and Lévy processes. In Section 3 we investigate the discrete time approximation. The basic statement is Theorem 3.3 that reduces the stochastic approximation problem to a deterministic one in case of the Riemann-approximation (2) (which we call simple approximation in the sequel). The difference between the simple and optimal approximation (3) is shown in Theorem 3.4 to be sufficiently small. Theorem 3.5 provides a lower bound for the optimal \( L_2 \)-approximation. Finally, Theorems 3.6 and 3.8 give the connection to the Besov spaces defined by real interpolation. We conclude with Section 4 where we use the example \( f(x) = \mathbb{1}_{(K, \infty)}(x) \) to demonstrate how the fractional smoothness depends on the underlying Lévy process.

2. Preliminaries

2.1. Notation. Throughout this paper we will use for \( A, B, C \geq 0 \) and \( c \geq 1 \) the notation \( A \sim_c B \) for \( \frac{1}{c}B \leq A \leq cB \) and \( A = B \pm C \) for \( B - C \leq A \leq B + C \). The phrase \( \text{càdlàg} \) stands for a path which is right-continuous and has left limits. Given \( q \in [1, \infty] \), the sequence space \( \ell_q \) consists of all \( \alpha = (\alpha_N)_{N \geq 1} \subseteq \mathbb{R} \) such that \( \|\alpha\|_{\ell_q} := (\sum_{N=1}^{\infty} |\alpha_N|^q)^{1/q} < \infty \) for \( q < \infty \) and \( \|\alpha\|_{\ell_\infty} := \sup_{N \geq 1} |\alpha_N| < \infty \), respectively.

2.2. Real interpolation. First we recall some facts about the real interpolation method.
Definition 2.1. For Banach spaces $X_1 \subseteq X_0$, where $X_1$ is continuously embedded into $X_0$, we define for $u > 0$ the $K$-functional

$$K(u, x; X_0, X_1) := \inf_{x=x_0+x_1} \{\|x_0\|_{X_0} + u\|x_1\|_{X_1}\}.$$  

For $\theta \in (0, 1)$ and $q \in [1, \infty]$ the real interpolation space $(X_0, X_1)_{\theta, q}$ consists of all elements $x \in X_0$ such that $\|x\|_{(X_0, X_1)_{\theta, q}} < \infty$ where

$$\|x\|_{(X_0, X_1)_{\theta, q}} := \left\{ \begin{array}{ll}
\left[ \int_0^\infty [u^{-\theta}K(u, x; X_0, X_1)]^{\frac{q}{q-1}} du \right]^\frac{1}{\theta}, & q \in [1, \infty) \\
\sup_{u>0} u^{-\theta}K(u, x; X_0, X_1), & q = \infty.
\end{array} \right.$$  

The spaces $(X_0, X_1)_{\theta, q}$ equipped with $\| \cdot \|_{(X_0, X_1)_{\theta, q}}$ become Banach spaces and form a lexicographical scale, i.e. for any $0 < \theta_1 < \theta_2 < 1$ and $q_1, q_2 \in [1, \infty]$ it holds that

$$X_0 \supseteq (X_0, X_1)_{\theta_1, q_1} \supseteq (X_0, X_1)_{\theta_2, q_2} \supseteq (X_0, X_1)_{\theta, \min\{q_1, q_2\}} \supseteq X_1.$$  

For more information the reader is referred to [3, 4].

2.3. The spaces $\mathbb{B}_{2,q}^\theta(E)$.

Definition 2.2. For a sequence of Banach spaces $E = (E_n)_{n=0}^\infty$ with $E_n \neq \{0\}$ we let $\ell_2(E)$ and $d_{1,2}(E)$ be the Banach spaces of all $a = (a_n)_{n=0}^\infty \in E$ such that

$$\|a\|_{\ell_2(E)} := \left( \sum_{n=0}^\infty \|a_n\|_{E_n}^2 \right)^\frac{1}{2} \quad \text{and} \quad \|a\|_{d_{1,2}(E)} := \left( \sum_{n=0}^\infty (n + 1)\|a_n\|_{E_n}^2 \right)^\frac{1}{2},$$  

respectively, are finite. Moreover, for $\theta \in (0, 1)$ and $q \in [1, \infty]$ we let

$$\mathbb{B}_{2,q}^\theta(E) := \left\{ \begin{array}{ll}
(\ell_2(E), d_{1,2}(E))_{\theta, q} : & \theta \in (0, 1), q \in [1, \infty] \\
d_{1,2}(E) : & \theta = 1, q = 2
\end{array} \right..$$  

It can be shown that (cf. [3, Remark A.1])

$$\|a\|_{\mathbb{B}_{2,q}^\theta(E)}^2 \sim_{c_0^2} \sum_{n=0}^\infty (n + 1)^\theta \|a_n\|_{E_n}^2.$$  

To describe the interpolation spaces $\mathbb{B}_{2,q}^\theta(E)$ we use two types of functions. The first one is a generating function for $(\|a_n\|_{E_n}^2)_{n=0}^\infty$, i.e. for $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ we let

$$T_a(t) := \sum_{n=0}^\infty \|a_n\|_{E_n}^2 t^n.$$  

The second function will be used to describe our stochastic approximation in a deterministic way: For $a \in \ell_2(E)$ and a deterministic time-net $\tau = (t_k)_{k=0}^N$ with $0 = t_0 \leq \cdots \leq t_N = 1$ we let

$$A(a, \tau) := \left( \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t)(T_a''(t)) dt \right)^{\frac{1}{2}}.$$ 

For the formulation of the next two theorems which will connect approximation properties with fractional smoothness special time nets are needed. Given $\theta \in (0, 1]$ and $N \geq 1$, we let $\tau^{\theta}_N$ be the time-net

$$(t_{N,\theta}^k) = 1 - \left( 1 - \frac{k}{N} \right)^\theta \quad \text{for} \quad k = 0, 1, \ldots, N$$

for which one has (see [10, relation (4)])

$$(1 - t_{N,\theta}^k)^{1-\theta} \leq \frac{|t_{N,\theta}^k - t_{N,\theta}^{k-1}|}{(1 - t_{N,\theta}^{k-1})^{1-\theta}} \leq \frac{1}{\theta N} \quad \text{for} \quad k = 1, \ldots, N$$

and $t \in [t_{N,\theta}^{k-1}, t_{N,\theta}^k)$. For $\theta = 1$ we obtain equidistant time-nets. The following two theorems are taken from [9]. For the convenience of the reader we comment about the proofs in Remark 2.5 below.

Theorem 2.3 ([9]). For $\theta \in (0, 1)$, $q \in [1, \infty]$ and $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ one has

$$\|a\|_{B^{-\frac{1}{2}}_2(E)} \sim c \|a\|_{\ell_2(E)} + \left\| \left( N^{\frac{\theta-1}{2}} A(a, \tau_1^N) \right)_{N=1}^\infty \right\|_{\ell_q}$$

where $c \in [1, \infty)$ depends at most on $(\theta, q)$ and the expressions may be infinite.

Theorem 2.4 ([9]). For $\theta \in (0, 1)$ and $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ the following assertions are equivalent:

(i) $a \in B^{-\frac{1}{2}}_2(E)$.

(ii) $\int_0^1 (1 - t)^{1-\theta} T_F'(t) dt < \infty$.

(iii) There exists a constant $c > 0$ such that

$$A(a, \tau_1^N) \leq \frac{c}{\sqrt{N}} \quad \text{for} \quad N = 1, 2, \ldots$$

Remark 2.5. We fix $a = (a_n)_{n=0}^\infty \in \ell_2(E)$ and $(\theta, q)$ according to Theorems 2.3 and 2.4. Then we let $\beta_n := \|a_n\|_{E_n}$ and define $f = \sum_{n=0}^\infty \beta_n h_n \in L_2(\mathbb{R}, \gamma)$, where $\gamma$ is the standard Gaussian measure and $(h_n)_{n=0}^\infty$ the orthonormal basis of Hermite polynomials. As before, let

$$A(\beta, \tau) := \left( \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t)(T_{\beta''}(t)) dt \right)^{\frac{1}{2}}$$

with $T_{\beta}(t) := \sum_{n=0}^\infty \beta_n^2 t^n$. 
Omitting the notation \((E)\) in the case \(E = (\mathbb{R}, \mathbb{R}, ...),\) we have 
\[\|a\|_{\ell_2(E)} = \|\beta\|_{\ell_2} \text{ and } \|a\|_{d_1,2(E)} = \|\beta\|_{d_1,2}.\]
Moreover, [9, Theorem 2.2] gives that \(\|a\|_{\mathbb{E}_{\theta,q}(E)} \sim \|\beta\|_{\mathbb{E}_{\theta,q}}\) for \(\theta \in (0, 1)\) and \(q \in [1, \infty]\) because of \(T_0 = T_\beta.\)
Hence [9, Lemmas 3.9 and 3.10, Theorem 3.5 \((X=W)\)] imply Theorem 2.3 of this paper. The equivalence of (i) and (iii) of Theorem 2.4 follows in the same way by [9, Lemmas 3.9 and 3.10, Theorem 3.5 \((X=W)\)]. Finally, the equivalence of (i) and (ii) of Theorem 2.4 is a consequence of the proof of [9, Theorem 3.2 \((X=W)\)].

2.4. \textbf{Lévy processes}. We follow the setting and presentation of [17, Section 1.1] and assume a \textit{square integrable mean zero Lévy process} \(X = (X_t)_{t \in [0,1]}\) on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})\) satisfying the usual assumptions, i.e. \((\Omega, \mathcal{F}, \mathbb{P})\) is complete where the filtration \((\mathcal{F}_t)_{t \in [0,1]}\) is the augmented natural filtration of \(X\) and therefore right-continuous and \(\mathcal{F} := \mathcal{F}_1\) is assumed without loss of generality. The Lévy measure \(\nu\) with \(\nu(\{0\}) = 0\) satisfies
\[
\int_{\mathbb{R}} x^2 \nu(dx) < \infty
\]
by the square integrability of \(X\) (see [16, Theorem 25.3]). Let \(N\) be the associated Poisson random measure and \(d\tilde{N}(t, x) = d\tilde{N}(t, x) - dt d\nu(x)\) be the compensated Poisson random measure. The Lévy-Itô decomposition (see [16, Theorem 19.2]) can be written under our assumptions as
\[
X_t = \sigma W_t + \int_{(0,t) \times \mathbb{R} \setminus \{0\}} x \tilde{N}(ds, dx).
\]
We introduce the finite measures \(\mu\) on \(\mathcal{B}(\mathbb{R})\) and \(\mathfrak{m}\) on \(\mathcal{B}([0,1] \times \mathbb{R})\) by
\[
\mu(dx) := \sigma^2 \delta_0(dx) + x^2 \nu(dx),
\]
\[
\mathfrak{m}(dt, dx) := dt \mu(dx),
\]
where we agree about \(\mu(\mathbb{R}) > 0\) to avoid pathologies. For \(B \in \mathcal{B}((0,1] \times \mathbb{R})\) we define the random measure
\[
M(B) := \sigma \int_{\{t \in (0,1]: (t,0) \in B\}} dW_t + \int_{B \cap ([0,1] \times (\mathbb{R} \setminus \{0\}))} x \tilde{N}(dt, dx)
\]
and let
\[
L_n^2 := L_2(([0,1] \times \mathbb{R})^n, \mathcal{B}(([0,1] \times \mathbb{R})^n), \mathfrak{m}^\otimes n) \quad \text{for} \quad n \geq 1.
\]
By [12, Theorem 2] there is the chaos decomposition
\[
L_2 := L_2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} I_n(L_2^n),
\]
where \(I_0(L_2^n)\) is the space of the a.s. constant random variables and \(I_n(L_2^n) := \{I_n(f_n) : f_n \in L_2^n\}\) for \(n = 1, 2, \ldots\) and \(I_n(f_n)\) denotes
the multiple integral w.r.t. the random measure $M$. For properties of the multiple integral see \cite[Theorem 1]{12}. Especially, 
\[ \|I_n(f_n)\|_{L^2}^2 = n! \|\tilde{f}_n\|_{L^2}^2 \]
and
\[ \|F\|_{L^2}^2 = \sum_{n=0}^{\infty} n! \|\tilde{f}_n\|_{L^2}^2 \]
with $\tilde{f}_n$ being the symmetrization of $f_n$, i.e.
\[ \tilde{f}_n(z_1, \ldots, z_n) = \frac{1}{n!} \sum f_n(z_{\pi(1)}, \ldots, z_{\pi(n)}) \]
for all $z_i = (t_i, x_i) \in [0, 1] \times \mathbb{R}$, where the sum is taken over all permutations $\pi$ of $\{1, \ldots, n\}$. For $F \in L^2$ the $L^2$-representation
\[ F = \sum_{n=0}^{\infty} I_n(\tilde{f}_n), \]
with $I_0(f_0) = \mathbb{E}F$ a.s. is unique (note that $I_n(f_n) = I_n(\tilde{f}_n)$ a.s.).

2.5. **Besov spaces.** Here we recall the construction of Besov spaces (or spaces of random variables of fractional smoothness) based on the above chaos expansion.

**Definition 2.6.** Let $\mathcal{D}_{1,2}$ be the space of all $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2$ such that
\[ \|F\|_{\mathcal{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1) \|I_n(f_n)\|_{L^2}^2 < \infty. \]

Moreover,
\[ \mathbb{B}^\theta_{2,q} := \left\{ (L^2, \mathcal{D}_{1,2})_{\theta,q} : \theta \in (0,1), q \in [1,\infty) \right\} \cup \{ \mathcal{D}_{1,2} : \theta = 1, q = 2 \}. \]

2.6. **The space of the random variables to approximate.** We will approximate random variables from the following space $\mathbb{M}$:

**Definition 2.7.** The closed subspace $\mathbb{M} \subseteq L^2$ consists of all mean zero $F \in L^2$ such that there exists a representation
\[ F = \sum_{n=1}^{\infty} I_n(f_n) \]
with symmetric $f_n$ such that there are $h_0 \in \mathbb{R}$ and symmetric $h_n \in L^2(\mu^{\otimes n})$ for $n \geq 1$ with
\[ f_n((t_1, x_1), \ldots, (t_n, x_n)) = h_{n-1}(x_1, \ldots, x_{n-1}) \quad \text{for } 0 < t_1 < \cdots < t_n < 1. \]
The orthogonal projection onto $\mathbb{M}$ is denoted by $\Pi : L^2 \to \mathbb{M} \subseteq L^2$. 

Let us summarize some facts about the space $\mathcal{M}$:

(a) **Representation of $\Pi$.** For

$$G = \sum_{n=0}^{\infty} I_n(\alpha_n) \in L_2$$

with symmetric $\alpha_n \in L_2^n$ one computes the functions $h_n$ of the projection $F = \Pi(G)$ by

$$h_{n-1}(x_1, \ldots, x_{n-1}) = n! \int_0^1 \int_0^{t_{n-1}} \cdots \int_0^{t_1} \alpha_n((t_1, x_1), (t_{n-1}, x_{n-1}), (t_n, x_n))$$

$$\times \frac{\mu(dx_n)}{\mu(\mathbb{R})} dt_1 \cdots dt_n \quad \text{for } n \geq 1. \quad (7)$$

(b) **Integral representation of the elements of $\mathcal{M}$.** Given $F \in \mathcal{M}$ with a representation like in Definition 2.7 (the functions $h_n$ are unique as elements of $L_2(\mu^{\otimes n})$), we define the martingale $\varphi = (\varphi_t)_{t \in [0,1]}$ by the $L_2$-sum

$$\varphi_t := h_0 + \sum_{n=1}^{\infty} (n+1) I_n \left( h_n \mathbb{1}_{(0,t]} \right), \quad (8)$$

which we will assume to be path-wise càdlàg. It follows that

$$\|\varphi_t\|_{L_2}^2 = h_0^2 + \sum_{n=1}^{\infty} (n+1)^2 n! t^n \|h_n\|_{L_2(\mu^{\otimes n})}^2$$

$$= h_0^2 + \frac{1}{\mu(\mathbb{R})} \sum_{n=1}^{\infty} (n+1)^2 n! t^n \|f_{n+1}\|_{L_2^{n+1}}^2$$

$$= h_0^2 + \frac{1}{\mu(\mathbb{R})} \sum_{n=1}^{\infty} t^n (n+1) \|I_{n+1} f_{n+1}\|_{L_2}^2$$

so that

$$\mu(\mathbb{R}) \sup_{t \in [0,1]} \|\varphi_t\|_{L_2}^2 + \|F\|_{L_2}^2 = \sum_{n=0}^{\infty} (n+1) \|I_n f_n\|_{L_2}^2. \quad (9)$$

Moreover, for $t \in [0,1]$ we get that, a.s.,

$$F_t := \mathbb{E}(F | \mathcal{F}_t) = \int_{(0,t]} \varphi_{s-} dX_s.$$

This is analog to the Brownian motion case considered in [7] and [9], where the representation $F = \mathbb{E}F + \int_{(0,1]} \varphi_s dB_s$ was used together with
the regularity assumption that \((\varphi_s)_{s \in [0,1]}\) is a martingale or close to a martingale in some sense.

(c) **Basic examples for elements for** \(\mathbb{M}\) **are taken from Lemma 4.2** below: Let \(\Pi_X : L_2 \rightarrow I(X) \subseteq L_2\) be the orthogonal projection onto \(I(X)\) defined in (1) and let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a Borel function with \(f(X_1) \in L_2\), then

\[
\Pi_X(f(X_1)) = \Pi(f(X_1)).
\]

This means the elements of \(\mathbb{M}\) occur naturally when applying the Galtchouk-Kunita-Watanabe projection. It should be noted, that in the case that \(\sigma = 0\) and \(\nu = \alpha \delta_{x_0}\) with \(\alpha > 0\) and \(x_0 \in \mathbb{R} \setminus \{0\}\) we have a chaos decomposition of the form \(f(X_1) = \mathbb{E}f(X_1) + \sum_{n=1}^{\infty} \beta_n I_n(\mathbb{1}_{[0,1]})\) with \(\beta_n \in \mathbb{R}\), so that already \(f(X_1) \in \mathbb{M}\).

2.7. **Doléans-Dade stochastic exponential.**

**Definition 2.8.** For \(0 \leq a \leq t \leq 1\) we let

\[
S^a_t := 1 + \sum_{n=1}^{\infty} \frac{I_n(\mathbb{1}_{[a,t]})}{n!},
\]

where we can assume that all paths of \((S^a_t)_{t \in [a,1]}\) are càdlàg for any fixed \(a \in [0,1]\). In particular, we let \(S = (S_t)_{t \in [0,1]} := (S^0_t)_{t \in [0,1]}\).

The following lemma is standard and we omit its proof.

**Lemma 2.9.** For \(0 \leq a \leq t \leq 1\) one has that

(i) \(S^a_t = 1 + \int_{[a,t]} S^a_u \, dX_u\) a.s.,
(ii) \(S_t = S^a_t S^a_t\) a.s.,
(iii) \(S^a_t\) is independent from \(\mathcal{F}_a\) and \(\mathbb{E}(S^a_t)^2 = e^{\mu(\mathbb{R})(t-a)}\).

3. **Approximation of stochastic integrals**

In the sequel we will use

\[
\mathcal{T}_N := \{\tau = (t_k)_{k=0}^{N} : 0 = t_0 < \cdots < t_N = 1\} \quad \text{and} \quad \mathcal{T} := \bigcup_{N=1}^{\infty} \mathcal{T}_N
\]

as sets of deterministic time-nets and define \(|\tau| := \max_{1 \leq k \leq N} |t_k - t_{k-1}|\).

We will consider the following approximations of a random variable \(F \in \mathbb{M}\) with respect to the processes \(X\) and \(S\):
Definition 3.1. For $N \geq 1$, $Y \in \{X, S\}$, $F = \int_{[0,1]} \varphi_s dX_s \in \mathcal{M}$, $A = (A_k)_{k=1}^N \subseteq \mathcal{F}$ and $\tau \in \mathcal{T}_N$ we let

(i) $a_{S}^{\text{sim}}(F; \tau, A) := \left\| F - \sum_{k=1}^{N} \varphi_{t_{k-1}} 1_{A_k}(S_{t_k}^{t_{k-1}} - 1) \right\|_{L^2}$,

(ii) $a_{Y}^{\text{opt}}(F; \tau) := \inf \left\| F - \sum_{k=1}^{N} v_{t_{k-1}}(Y_{t_k} - Y_{t_{k-1}}) \right\|_{L^2}$, where the infimum is taken over all $\mathcal{F}_{t_{k-1}}$-measurable $v_{t_{k-1}}: \Omega \rightarrow \mathbb{R}$ such that $E|v_{t_{k-1}}(Y_{t_k} - Y_{t_{k-1}})|^2 < \infty$.

Remark 3.2. (i) The definition of $a_{S}^{\text{sim}}$ takes into account the additional sets $(A_k)_{k=1}^N$ to avoid problems with the case that $S$ vanishes. These extra sets $A$ in $a_{S}^{\text{sim}}(F; \tau, A)$ play different roles in Theorem 3.3, Theorem 3.4, and in Theorems 3.5, 3.6 and 3.8. To recover a more standard form of $a_{S}^{\text{sim}}$ assume that $(S_t)_{t \in [0,1]}$ and $(S_{t-})_{t \in [0,1]}$ are positive so that we can write

$$F = \int_{[0,1]} \psi_u (S_u - dX_u) \quad \text{with} \quad \psi_u := \frac{\varphi_u}{S_u},$$

and obtain that

$$F - \sum_{k=1}^{N} \varphi_{t_{k-1}} (S_{t_k}^{t_{k-1}} - 1) = F - \sum_{k=1}^{N} \psi_{t_{k-1}} S_{t_k}^{t_{k-1}} (S_{t_k}^{t_{k-1}} - 1)$$

$$= F - \sum_{k=1}^{N} \psi_{t_{k-1}} (S_{t_k} - S_{t_{k-1}})$$

which is what one expects.

(ii) In the sequel the crucial assumption will be

$$\Omega = \{S_t \neq 0\} \quad \text{for all} \quad t \in [0,1].$$

This can be achieved by the condition $\nu((-\infty, -1]) = 0$ which implies the almost sure positivity of $S$ and we can adjust $S$ on a set of measure zero; see [13, Theorem I.4.61] and [16, Theorem 19.2].

Because of the martingale property of $(\varphi_t)_{t \in [0,1)}$ it is easy to check that

$$a_{X}^{\text{opt}}(F; \tau) = \left\| F - \sum_{k=1}^{N} \varphi_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}) \right\|_{L^2}$$

so that for $Y = X$ the simple and optimal approximation coincide. The theorem below gives a description of the simple approximation by a function $H_Y(t)$ that describes, in some sense, the curvature of $F \in \mathcal{M}$ with respect to $Y$. 
Theorem 3.3. Let $F \in \mathcal{M}$,
\[ H^2_Y(t) := \mu(\mathbb{R}) \sum_{n=1}^{\infty} n n! t^{n-1} \| A_n^Y \|_{L_2(\mu^{\otimes n})}^2 \]
with
\[ A_n^Y(x_1, \ldots, x_n) := \begin{cases} (n+1)h_n(x_1, \ldots, x_n) & : Y = X \\ (n+1)h_n(x_1, \ldots, x_n) - h_{n-1}(x_1, \ldots, x_{n-1}) & : Y = S \end{cases} \]
Then, for $\tau \in \mathcal{T}$, one has
\[ a_{\text{opt}}^X(F; \tau) = \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_X(t) dt \right)^{\frac{1}{2}} \]
\[ a_{\text{sim}}^S(F; \tau, \Omega_N) \sim_c \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_S(t) dt \right)^{\frac{1}{2}} \]
where in the last equivalence $|\tau| < 1/\mu(\mathbb{R})$ and $c := (1 - \sqrt{\mu(\mathbb{R}) |\tau|})^{-1}$ and $\Omega_N = (\Omega, \ldots, \Omega)$.

Proof. Case $Y = X$: We get that
\[ \mathbb{E} |\varphi_t - \varphi_{t_{k-1}}|^2 = \sum_{n=1}^{\infty} (t^n - t_{k-1}^n)(n + 1)^2 n! ||h_n||_{L_2(\mu^{\otimes n})}^2 \]
\[ = \sum_{n=1}^{\infty} (n + 1)^2 n! \int_{t_{k-1}}^{t} u^{n-1} du ||h_n||_{L_2(\mu^{\otimes n})}^2 \]
\[ = \frac{1}{\mu(\mathbb{R})} \int_{t_{k-1}}^{t} H^2_X(u) du \]
which implies for $a_{\text{sim}}^S(F; \tau) = a_{\text{opt}}^X(F; \tau) =: a_X(F; \tau)$ that
\[ |a_X(F; \tau)|^2 = \mu(\mathbb{R}) \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \mathbb{E} |\varphi_t - \varphi_{t_{k-1}}|^2 dt \]
\[ = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - u) H^2_X(u) du. \]
Case $Y = S$: Here we get that
\[ a_{\text{sim}}^S(F; \tau, \Omega_N) \]
where we used $S_{tk-1} = S_{tk-1}^{tk-1}$ a.s. for $t \in (tk-1, tk]$ and the martingale property of $\int_{tk-1, t} \varphi_u dX_u - \varphi_{tk-1} (S_{tk-1}^{tk-1} - 1)$. Finally,

$$
\left( \mu(\mathbb{R}) \sum_{k=1}^{N} \int_{tk-1}^{tk} \mathbb{E} \left[ \varphi_t - \varphi_{tk-1} - \int_{tk-1, t} \varphi_u dX_u \right]^2 dt \right)^{\frac{1}{2}}
$$

$$
= \left( \mu(\mathbb{R}) \sum_{k=1}^{N} \int_{tk-1}^{tk} \mathbb{E} \left[ (\varphi_t - \varphi_{tk-1}) - (F_t - F_{tk-1}) \right]^2 dt \right)^{\frac{1}{2}}
$$

$$
= \left( \sum_{k=1}^{N} \int_{tk-1}^{tk} \int_{tk-1}^{t} H_s^2(u) dudt \right)^{\frac{1}{2}}.
$$

The next theorem states that the simple and optimal approximation are equivalent whenever $A_k := \{S_{tk-1} \neq 0\}$ is taken.
Theorem 3.4. For $F \in \mathcal{M}$ and $\tau \in \mathcal{T}$ one has that
\[
|a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)| \leq c[|\tau|\|F\|_{L_2} + \sqrt{|\tau|}a_X^{\text{opt}}(F; \tau)]
\]
where $c > 0$ depends on $\mu$ only and $A_k := \{S_{t_{k-1}} \neq 0\}$.

Proof. (a) In the first step we determine an optimal sequence of $(v_k)_{k=1}^{N-1}$. For $0 \leq a < b \leq 1$ we get from Lemma 2.9 that
\[
\inf \left\{ \left\| v(S_b - S_a) - \int_{(a,b]} \varphi_u dX_u \right\|_{L_2} \colon v \text{ is } \mathcal{F}_a\text{-measurable} \right\}
\]
\[
= \inf \left\{ \left\| vS_a(S_b^a - 1) - \int_{(a,b]} \varphi_u dX_u \right\|_{L_2} \colon v \text{ is } \mathcal{F}_a\text{-measurable} \right\}
\]
\[
= \inf \left\{ \left\| \mathbb{P}_{\{S_a \neq 0\}}(S_b^a - 1) - \int_{(a,b]} \varphi_u dX_u \right\|_{L_2} \colon \mathbb{P} \text{ is } \mathcal{F}_a\text{-measurable} \right\}.
\]
The infimum is obtained with
\[
\bar{\nu} = \frac{\mathbb{E} \left( \int_a^b \varphi_{t-S_t^a} dt | \mathcal{F}_a \right)}{\mathbb{E} \left( \int_a^b (S_t^a)^2 dt | \mathcal{F}_a \right)} = \kappa(a, b)
\]
and
\[
v := \left\{ \begin{array}{ll}
\frac{1}{S_a(a,b)} \mathbb{E} \left( \int_a^b \varphi_t S_t^a dt | \mathcal{F}_a \right) & : S_a \neq 0 \\
0 & : S_a = 0
\end{array} \right.
\]
where we used that
\[
(10) \quad \varphi_{t-} = \varphi_t \text{ a.s. and } S_{t-}^a = S_t^a \text{ a.s. on } (a, b).
\]

(b) Now it holds that
\[
|a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)|
\]
\[
= \left\| F - \mathbb{E}F - \sum_{k=1}^N \varphi_{t_{k-1}} 1_{A_k}(S_{t_{k-1}}^a - 1) \right\|_{L_2}
\]
\[
- \left\| F - \mathbb{E}F - \sum_{k=1}^N v_{k-1}(S_{t_k} - S_{t_{k-1}}) \right\|_{L_2}
\]
\[
\leq \left\| \sum_{k=1}^N [\varphi_{t_{k-1}} - v_{k-1}S_{t_{k-1}}](S_{t_{k-1}}^a - 1) 1_{A_k} \right\|_{L_2}
\]
\[
= \left( \sum_{k=1}^{N} \left[ \| \varphi_{t_{k-1}} - v_{k-1}S_{t_{k-1}} \|_2 \right]^2 \left[ e^{\mu(\mathbb{R}))(t_k - t_{k-1})} - 1 \right] \right)^{\frac{1}{2}}.
\]

Moreover (using again (10)) we have
\[
\| \varphi_{t_{k-1}} - v_{k-1}S_{t_{k-1}} \|_2 \leq \| \varphi_{t_{k-1}}(1 - \frac{t_k - t_{k-1}}{\kappa(t_{k-1}, t_k)}) \|_2 + \left\| \frac{1}{\kappa(t_{k-1}, t_k)} \mathbb{E} \left( \int_{t_{k-1}}^{t_k} (\varphi_t - \varphi_{t_{k-1}})(S_t^{t_{k-1}} - 1) dt \bigg| \mathcal{F}_{t_{k-1}} \right) \right\|_2.
\]

The first term on the right-hand side can be bounded from above by
\[
\mu(\mathbb{R})(t_k - t_{k-1}) \| \varphi_{t_{k-1}} \|_2. \quad \text{For the second term we let } a = t_{k-1} < t_k = b \text{ and } \lambda_t = \frac{1}{\kappa(t_{k-1}, t_k)}(\varphi_t - \varphi_{t_{k-1}}) \text{ and obtain}
\]
\[
\mathbb{E} \left( \int_a^b \lambda_t(S_t^a - 1) dt \bigg| \mathcal{F}_a \right) \leq \left( \mathbb{E} \left( \int_a^b |\lambda_t|^2 dt \bigg| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \int_a^b (S_t^a - 1)^2 dt \bigg| \mathcal{F}_a \right) \right)^{\frac{1}{2}} = \left( \mathbb{E} \left( \int_a^b |\lambda_t|^2 dt \bigg| \mathcal{F}_a \right) \right)^{\frac{1}{2}} \sqrt{\frac{\mu(\mathbb{R})}{2} \kappa(a, b)}
\]

where the last inequality follows from
\[
\int_a^b \| S_t^a - 1 \|_2^2 dt = \int_a^b \mu(\mathbb{R}) \kappa(a, t) dt \leq \int_a^b \mu(\mathbb{R}) \kappa(a, t) \left( \frac{d}{dt} \kappa(a, t) \right) dt = \frac{\mu(\mathbb{R})}{2} \kappa(a, b)^2.
\]

Hence
\[
\| \varphi_{t_{k-1}} - v_{k-1}S_{t_{k-1}} \|_2 \leq \mu(\mathbb{R})(t_k - t_{k-1}) \| \varphi_{t_{k-1}} \|_2 + \sqrt{\frac{\mu(\mathbb{R})}{2}} \left( \int_{t_{k-1}}^{t_k} \| \varphi_t - \varphi_{t_{k-1}} \|_2^2 dt \right)^{\frac{1}{2}}.
\]
Using $e^{\mu(R)(t_k - t_{k-1})} - 1 \leq \mu(R)e^{\mu(R)(t_k - t_{k-1})}$ we conclude with

$$|a_S^{\text{sim}}(F; \tau, A) - a_S^{\text{opt}}(F; \tau)|$$

$$\leq \left( \sum_{k=1}^{N} \left[ \mu(R)(t_k - t_{k-1}) \| \varphi_{t_{k-1}} \|_{L_2} \right]^{2} \mu(R)e^{\mu(R)(t_k - t_{k-1})} \right)^{\frac{1}{2}}$$

$$+ \left( \sum_{k=1}^{N} \left[ \frac{\mu(R)}{2} \int_{t_{k-1}}^{t_k} \| \varphi_{t_{k-1}}(\varphi_t - \varphi_{t_{k-1}}) \|_{L_2} dt \right] \mu(R)e^{\mu(R)(t_k - t_{k-1})} \right)^{\frac{1}{2}}$$

$$\leq |\tau|\mu(R)e^{\mu(R)/2} \| F \|_{L_2} + \sqrt{|\tau|}
\sqrt{\frac{\mu(R)}{2}} e^{\mu(R)/2} a_X^{\text{opt}}(F; \tau).$$

Now we show that $1/\sqrt{N}$ is the lower bound for our approximation if time-nets of cardinality $N + 1$ are used.

**Theorem 3.5.** Let $F \in M$ and $Y \in \{X, S\}$, where in the case $X = S$ we assume that $\Omega = \{S_t \neq 0\}$ for all $t \in [0, 1]$. Unless there are $a, b \in \mathbb{R}$ such that $F = a + bY_1$ a.s., one has that

$$\liminf_{N \to \infty} \sqrt{N} \left[ \inf_{\tau_N \in T_N} a_Y^{\text{opt}}(F; \tau_N) \right] > 0.$$

**Proof.** Case $Y = X$: We have $H_X(t) = 0$ for some $t \in (0, 1)$ if and only if $h_n = 0 \mu^\otimes_n$ a.e. for all $n = 1, 2, \ldots$ which implies that $F = I_1(f_1) = I_1(h_0) = h_0X_1$. This means that our assumption on $F$ implies that $H_X(t) > 0$ for all $t \in (0, 1)$. Consequently, Theorem [3.3] gives for any fixed $s \in (0, 1)$ that

$$N |a_X^{\text{opt}}(F; \tau_N)|^2 = N \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t)H_X^2(t)dt$$

$$\geq N \int_s^1 \left[ \sum_{k=1}^{N} (t_k - t)1_{[t_{k-1}, t_k]}(t)H_X^2(s) \right] dt$$

$$= \frac{1}{2}H_X^2(s)N \sum_{k=1}^{N} (t_k \vee s - t_{k-1} \vee s)^2$$

$$\geq \frac{1}{2}H_X^2(s)(1 - s)^2$$

which proves the statement for $Y = X$. 
Case $Y = S$: Similarly as in the previous case our assumption on $F$ implies that $H_S(t) > 0$ for all $t \in (0, 1)$. In fact, assuming that $H_0(t) = 0$ for some $t \in (0, 1)$ implies

$$(n + 1)h_n(x_1, ..., x_n) = h_{n-1}(x_1, ..., x_{n-1}) \mu^{\otimes n}\text{-a.e.}$$

for all $n = 1, 2, ...$. By induction we derive that

$$h_n = \frac{h_0}{(n+1)!} \mu^{\otimes n}\text{-a.e. for } n \geq 0$$

so that $f_n = h_0/n! m^{\otimes n}\text{-a.e. for } n \geq 1$. This would give that $F = h_0(S_1 - 1) a.s.$

Hence applying Theorem 3.3 as in the case $Y = X$ implies that there is an $\varepsilon > 0$ such that

$$\sqrt{Na_{opt}^S(F; \tau_N, \Omega^N)} \geq \varepsilon > 0 \text{ for all } \tau_N \in \mathcal{T}_N \text{ with } |\tau_N| \leq \frac{1}{2\mu(\mathbb{R})}.$$

For an arbitrary $N \geq 1$ and $\tau_N \in \mathcal{T}_N$ Theorem 3.4 gives

$$a_{opt}^S(F; \tau_N) \geq a_{opt}^S(F; \tau_N, \Omega^N) - c_{3.4} \sqrt{N} \|F\|_{L_2} + \sqrt{|\tau_N|} a_{opt}^X(F; \tau_N).$$

Letting $\tilde{\tau}_N := \tau_N \cup \{k/N : k = 1, ..., N - 1\} \in \bigcup_{k=N}^{2N-1} \mathcal{T}_k$, $N \geq 2\mu(\mathbb{R}) \vee 2$ implies $|\tilde{\tau}_N| \leq 1/N \leq 1/(2\mu(\mathbb{R}))$ and

$$\sqrt{Na_{opt}^S(F; \tau_N)} \geq \sqrt{Na_{opt}^S(F; \tilde{\tau}_N)} \geq \sqrt{N} \frac{\varepsilon}{\sqrt{2N}} - c_{3.4} \sqrt{N} \left[|\tilde{\tau}_N| \|F\|_{L_2} + \sqrt{|\tilde{\tau}_N|} a_{opt}^X(F; \tilde{\tau}_N)\right] \geq \frac{\varepsilon}{\sqrt{2}} - c_{3.4} \left[\|F\|_{L_2} \sqrt{N} + a_{opt}^X(F; (k/N)_{k=0}^N)\right].$$

The convergence $a_{X}^{opt}(F; (k/N)_{k=0}^N) \to 0$ as $N \to \infty$ follows from Theorem 3.3 because of $\int_0^1 (1-t)H_X^2(t)dt < \infty$ which can be seen by considering the trivial time-net $\{0, 1\}$. Consequently,

$$\liminf_{N \to \infty} \sqrt{N} \left[\inf_{\tau_N \in \mathcal{T}_N} a_{opt}^S(F; \tau_N)\right] \geq \frac{\varepsilon}{\sqrt{2}}.$$

Now we relate the approximation properties to the Besov regularity. We recall that the nets $\tau_N^\theta$ were introduced in (5) and that for $\theta = 1$ we obtain the equidistant nets.
Theorem 3.6. For \( \theta \in (0, 1) \), \( q \in [1, \infty) \), \( Y \in \{X, S\} \) and \( F \in \mathbb{M} \) the following assertions are equivalent:

(i) \( F \in \mathbb{B}_{2,q}^\theta \).

(ii) \( \left\| (N^{\frac{\theta}{2}} - \frac{1}{\sigma} a^\text{opt}_X (F; \tau^1_N) ) \right\|_{\ell_q}^\infty < \infty \).

If \( \Omega = \{ S_t \neq 0 \} \) for all \( t \in [0, 1] \), then (i) and (ii) are equivalent to:

(iii) \( \left\| (N^{\frac{\theta}{2}} - \frac{1}{\sigma} a^\text{opt}_S (F; \tau^1_N) ) \right\|_{\ell_q}^\infty < \infty \).

(iv) \( \left\| (N^{\frac{\theta}{2}} - \frac{1}{\sigma} a^\text{sim}_S (F; \tau^1_N, \Omega^N) ) \right\|_{\ell_q}^\infty < \infty \).

For the proof the following lemma is needed.

Lemma 3.7. For \( F \in \mathbb{M} \) and \( t \in [0, 1) \) one has that

\[ |H_S(t) - H_X(t)| \leq \mu(\mathbb{R}) \| \varphi_t \|_{L_2}. \]

Moreover,

\[ \left| \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_S(t) dt \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_X(t) dt \right)^{\frac{1}{2}} \right| \leq \sqrt{\mu(\mathbb{R}) |\tau| \| F \|_{L_2}}. \]

Proof. From the definition we get that

\[ |H_S(t) - H_X(t)| \leq \left( \mu(\mathbb{R}) \sum_{n=1}^{\infty} nn! t^{n-1} \| h_{n-1} \|_{L_2(\mu^{\otimes n})} \right)^{\frac{1}{2}} = \left( \mu(\mathbb{R}) \sum_{n=1}^{\infty} (n-1)! t^{n-1} \| nh_{n-1} \|_{L_2(\mu^{\otimes (n-1)})} \right)^{\frac{1}{2}} = \mu(\mathbb{R}) \| \varphi_t \|_{L_2}. \]

Finally,

\[ \left| \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_S(t) dt \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) H^2_X(t) dt \right)^{\frac{1}{2}} \right| \leq \left( \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (t_k - t) |H_S(t) - H_X(t)|^2 dt \right)^{\frac{1}{2}} \leq \left\| \tau \right\|_{L_2(\mathbb{R})}^{\frac{1}{2}} \left( \int_{0}^{1} \| \varphi_t \|_{L_2}^2 dt \mu(\mathbb{R}) \right)^{\frac{1}{2}} = \left\| \tau \right\|_{L_2(\mathbb{R})}^{\frac{1}{2}} \| F \|_{L_2}.
Proof of Theorem 3.6. (i) $\iff$ (ii) follows from Theorem 2.3 and Theorem 3.3 because
\[ H_X^2(t) = \frac{d^2}{dt^2} \left( \sum_{n=1}^{\infty} \| I_n(f_n) \|_{L^2}^2 t^n \right) \quad \text{if} \quad F = \sum_{n=1}^{\infty} I_n(f_n). \]

(iii) $\iff$ (iv) follows from Theorem 3.4 and (ii) $\iff$ (iv) from Theorem 3.3 and Lemma 3.7.
\[ \square \]

Theorem 3.8. (a) For $F \in M$ and $\theta \in (0, 1]$ the following assertions are equivalent:

(i) $F \in \mathcal{B}_{2,2}^\theta$.

(ii) $\sup_N N^{\gamma} a^\text{opt}_X(F; \tau_N^\theta) < \infty$.

If $\Omega = \{ S_t \neq 0 \}$ for all $t \in [0, 1]$, then (i) and (ii) are equivalent to:

(iii) $\sup_N N^{\gamma} a^\text{opt}_S(F; \tau_N^\theta) < \infty$.

(iv) $\sup_N N^{\gamma} a^\text{sim}_S(F; \tau_N^\theta, \Omega^N) < \infty$.

(b) If the assertions (i) - (ii) hold, then we have
\[ \lim_{N \to \infty} N \left| a^\text{opt}_X(F; \tau_N^\theta) \right|^2 = \frac{1}{2\theta} \int_0^1 (1 - t)^{1 - \theta} H_X^2(t) dt \]
and if in addition $\Omega = \{ S_t \neq 0 \}$ for all $t \in [0, 1]$, then
\[ \lim_{N \to \infty} N \left| a^\text{opt}_S(F; \tau_N^\theta) \right|^2 = \lim_{N \to \infty} N \left| a^\text{sim}_S(F; \tau_N^\theta, \Omega^N) \right|^2 = \frac{1}{2\theta} \int_0^1 (1 - t)^{1 - \theta} H_S^2(t) dt. \]

Proof. Part (a): (i) $\iff$ (ii) follows from Theorems 2.4 and 3.3 because of (11).

(ii) $\iff$ (iv) From [9, Lemma 3.8] and Theorem 3.3 it follows that the desired equivalence is equivalent to
\[ \int_0^1 (1 - t)^{1 - \theta} H_X^2(t) dt < \infty \quad \text{if and only if} \quad \int_0^1 (1 - t)^{1 - \theta} H_S^2(t) dt < \infty. \]

In view of Lemma 3.7 it is therefore sufficient to check that $\int_0^1 (1 - t)^{1 - \theta} \| \varphi_t \|_{L^2}^2 dt < \infty$ which follows from $\int_0^1 \| \varphi_t \|_{L^2}^2 \mu(\mathbb{R}) dt = \| F - \mathbb{E} F \|_{L^2}^2 < \infty$.

(iv) $\iff$ (iii) follows from Theorem 3.4 $a^\text{opt}_X(F, \tau) \leq \| F \|_{L^2}$ and $| \tau_N^\theta | \leq 1/(\theta N)$ by (6).
Part (b): Let \( \alpha(s) := 1 - (1 - s)^{\frac{1}{\theta}} \) and \( H : [0, 1) \to [0, \infty) \) be non-decreasing and continuous such that \( \int_0^1 (1-t)^{1-\theta} H^2(t) dt < \infty \). For any \( \delta \in (0, 1) \) and \( \eta := \alpha^{-1}(\delta) \) we observe that

\[
\frac{1}{2\theta} \int_0^\delta (1-t)^{1-\theta} H^2(t) dt = \frac{1}{2} \int_0^\delta \alpha'(\alpha^{-1}(t)) H^2(t) dt = \frac{1}{2} \int_0^\eta \alpha'(s) [H^2(\alpha(s)) \alpha'(s)] ds.
\]

Because

\[
\alpha'(s) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N N \left[ \alpha \left( \frac{k}{N} \wedge \eta \right) - \alpha \left( \frac{k - 1}{N} \wedge \eta \right) \right] [H^2(\alpha(s)) \alpha'(s)] ds
\]

for \( s \in [0, \eta) \) and all terms on the right-hand side are bounded by the Lipschitz constant of \( \alpha \) on \( [0, \eta] \), dominated convergence implies that

\[
\frac{1}{2\theta} \int_0^\delta (1-t)^{1-\theta} H^2(t) dt = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \int_{t_{k-1}^N \wedge \delta}^{t_k^N \wedge \delta} (t_k^N \wedge \delta - t_{k-1}^N \wedge \delta)^2 dt
\]

where we use that \( H \) is uniformly continuous on \( [0, \delta] \). From this we deduce that

\[
\liminf_{N \to \infty} N \sum_{k=1}^N \int_{t_{k-1}^N \wedge \delta}^{t_k^N \wedge \delta} (t_k^N - t) H^2(t) dt \geq \liminf_{N \to \infty} N \sum_{k=1}^N \int_{t_{k-1}^N \wedge \delta}^{t_k^N \wedge \delta} (t_k^N \wedge \delta - t) H^2(t_k^N \wedge \delta) dt = \frac{1}{2\theta} \int_0^\delta (1-t)^{1-\theta} H^2(t) dt
\]
for all $\delta \in (0, 1)$ and therefore
\[
\liminf_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta}}^{t_{k}^{N, \theta}} (t_{k}^{N, \theta} - t) H^2(t) dt \geq \frac{1}{2\theta} \int_{0}^{1} (1 - t)^{1-\theta} H^2(t) dt.
\]

On the other hand, (6) implies
\[
\int_{\delta}^{1} \sum_{k=1}^{N} \left( (t_{k}^{N, \theta} - t) H^2(t) dt \right) \leq \frac{1}{\theta} \int_{0}^{1} (1 - t)^{1-\theta} H^2(t) dt
\]
for $\delta \in (0, 1)$. Choose $\delta$ such that the right hand side is less than $\varepsilon > 0$.

We conclude (also using the previous computations of part (b) and the uniform continuity of $H$ on $[0, \delta]$)
\[
\limsup_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta}}^{t_{k}^{N, \theta}} (t_{k}^{N, \theta} - t) H^2(t) dt
\]
\[
\leq \limsup_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta} \land \delta}^{t_{k}^{N, \theta} \land \delta} (t_{k}^{N, \theta} - t) H^2(t) dt + \varepsilon
\]
\[
= \lim_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta} \land \delta}^{t_{k}^{N, \theta} \land \delta} (t_{k}^{N, \theta} \land \delta - t) H^2(t) dt + \varepsilon
\]
\[
= \frac{1}{2\theta} \int_{0}^{\delta} (1 - t)^{1-\theta} H^2(t) dt + \varepsilon
\]
\[
\leq \frac{1}{2\theta} \int_{0}^{1} (1 - t)^{1-\theta} H^2(t) dt + \varepsilon
\]
and
\[
\limsup_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta}}^{t_{k}^{N, \theta}} (t_{k}^{N, \theta} - t) H^2(t) dt \leq \frac{1}{2\theta} \int_{0}^{1} (1 - t)^{1-\theta} H^2(t) dt.
\]

Consequently,
\[
\lim_{N \to \infty} N \sum_{k=1}^{N} \int_{t_{k-1}^{N, \theta}}^{t_{k}^{N, \theta}} (t_{k}^{N, \theta} - t) H^2(t) dt = \frac{1}{2\theta} \int_{0}^{1} (1 - t)^{1-\theta} H^2(t) dt.
\]

It follows from (12) that for $H \in \{H_X, H_S\}$ our assumptions on $H$ are satisfied. Hence Theorem 3.3 implies the limit expressions for $a_{X}^{\text{opt}}$ and $a_{S}^{\text{sim}}(\cdot, \cdot, \Omega^{N})$ (note that $c \to 1$ for $|\tau| \to 0$ in Theorem 3.3). The relation for $a_{S}^{\text{opt}}$ follows from that one for $a_{S}^{\text{sim}}(\cdot, \cdot, \Omega^{N})$, Theorem 3.4 and the fact that
\[
\lim_{N \to \infty} \sqrt{N} \sqrt{\int_{\tau_{N}^{\theta}} H^2(F; \tau_{N}^{\theta}) dt} \leq \limsup_{N \to \infty} \sqrt{\frac{1}{N} a_{X}^{\text{opt}}(F; \tau_{N}^{\theta})} = 0
\]
where we have used (6) and, as in the proof of Theorem 3.5, the relation
\[ \int_0^1 (1 - t) H_x^2(t) dt < \infty \]
together with Theorem 3.3. □

Using the results from [15, Theorem 2.4] one can derive from Theorem 3.3 for example the following assertion.

**Corollary 3.9.** For \( F \in \mathbb{M} \) one has the following equivalences:

(i) There is a constant \( c > 0 \) such that
\[ \inf_{\tau_N \in \mathcal{T}_N} a^\text{opt}_X(F; \tau_N) \leq \frac{c}{\sqrt{N}} \text{ for } N = 1, 2, \ldots \text{ iff } \int_0^1 H_x(t) dt < \infty. \]

(ii) There is a constant \( c > 0 \) such that
\[ \inf_{\tau_N \in \mathcal{T}_N} a^\text{sim}_S(F; \tau_N, \Omega_N) \leq \frac{c}{\sqrt{N}} \text{ for } N = 1, 2, \ldots \text{ iff } \int_0^1 H_S(t) dt < \infty. \]

### 4. Examples

#### 4.1. Preparations.

The following two lemmas provide information about the orthogonal projection \( \Pi : L^2 \to \mathbb{M} \subseteq L^2 \).

**Lemma 4.1.** Given \( G \in L^2 \), \( \theta \in (0,1) \) and \( q \in [1,\infty] \), one has that

(i) \( G \in D_{1,2} \) implies \( \Pi(G) \in D_{1,2} \),

(ii) \( G \in B^\theta_{2,q} \) implies \( \Pi(G) \in B^\theta_{2,q} \).

**Proof.** The lemma follows from the fact that for
\[ G = \sum_{n=0}^{\infty} I_n(\alpha_n) \]
with symmetric \( \alpha_n \in L^2_\mu \) the function \( h_n \) from Definition 2.7 computes as in (7) so that \( \|f_n\|_{L^2} \leq \|\alpha_n\|_{L^2} \) where \( f_n \) is defined as in Definition 2.7. Hence, the statement can be derived (for example) from Theorem 2.3 using the monotonicity of \( A \) with respect to \( \|a_n\|_{E_n} \) and the definition of \( D_{1,2} \). □

**Lemma 4.2.** For a Borel function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(X_1) \in L^2_\mu \) there are symmetric \( g_n \in L^2_\mu \) such that
\[ f(X_1) = \mathbb{E}f(X_1) + \sum_{n=1}^{\infty} I_n(g_n 1_{(0,1)}^{\otimes n}). \]

Moreover, it holds that \( \Pi(f(X_1)) = \sum_{n=1}^{\infty} I_n(f_n) \) with symmetric \( f_n \) satisfying
\[ f_n((t_1, x_1), \ldots, (t_n, x_n)) = h_{n-1}(x_1, \ldots, x_{n-1}) \]
\begin{equation}
\int_{\mathbb{R}} g_n(x_1, ..., x_{n-1}, x) \frac{\mu(dx)}{\mu(\mathbb{R})}
\end{equation}
onumber

on $0 < t_1 < \cdots < t_n < 1$ and $\Pi(f(X_1))$ is the orthogonal projection of $f(X_1)$ onto $I(X)$ defined in \cite{4}.

The representation (13) is proved in \cite{1} and \cite{2} and is based on invariance properties of $f(X_1)$ that transfer to the chaos representation. One could also use \cite[Section 6]{6}.

\textbf{Lemma 4.3.} Let $f \in C^b_\infty(\mathbb{R})$ and $f(X_1) = \sum_{n=1}^{\infty} I_n(g_n \mathbb{1}_{(0,1]}) \in D_{1,2}$ with symmetric $g_n \in L_2(\mu^{\otimes n})$. Then the martingale $(\varphi_t)_{t \in [0,1)}$ given by (8) and (14) has a closure $\varphi_1$, i.e. $\mathbb{E}(\varphi_1|\mathcal{F}_t) = \varphi_t$ a.s., with

\begin{align*}
\varphi_1 &= \int_{\mathbb{R}} \left[ \mathbb{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbb{1}_{\{x = 0\}} f'(X_1) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} \text{ a.s.}
\end{align*}

\textbf{Proof.} From \cite[Proposition 5.1 and its proof]{6} it is known that

\begin{align*}
\mathbb{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbb{1}_{\{x = 0\}} f'(X_1) &= \sum_{n=1}^{\infty} nI_{n-1}(g_n(\cdot, x) \mathbb{1}_{(0,1]}^{\otimes(n-1)}) \mu \otimes \mathbb{P} \text{ a.e.}
\end{align*}

Consequently, (14) implies that, a.s.,

\begin{align*}
\int_{\mathbb{R}} \left[ \mathbb{1}_{\{x \neq 0\}} \frac{f(X_1 + x) - f(X_1)}{x} + \mathbb{1}_{\{x = 0\}} f'(X_1) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} &= \int_{\mathbb{R}} \left[ \sum_{n=1}^{\infty} nI_{n-1} \left( g_n(\cdot, x) \mathbb{1}_{(0,1]}^{\otimes(n-1)} \right) \right] \frac{\mu(dx)}{\mu(\mathbb{R})} \\
&= \sum_{n=1}^{\infty} nI_{n-1} \left( \int_{\mathbb{R}} g_n(\cdot, x) \frac{\mu(dx)}{\mu(\mathbb{R})} \mathbb{1}_{(0,1]}^{\otimes(n-1)} \right) \\
&= \sum_{n=1}^{\infty} nI_{n-1} \left( h_{n-1} \mathbb{1}_{(0,1]}^{\otimes(n-1)} \right) \\
&= \varphi_1
\end{align*}

where the second equality follows by a standard Fubini argument. \qed

\textbf{Definition 4.4.} For $\delta > 0$ we let

$$
\psi(\delta) := \sup_{\lambda \in \mathbb{R}} \mathbb{P}(|X_1 - \lambda| \leq \delta).
$$
Example 4.5. The small ball estimate
\begin{equation}
\psi(\delta) \leq c\delta
\end{equation}

can be deduced if $X_1$ has a bounded density. As an example we use tempered $\alpha$-stable processes with $\alpha \in (0, 2)$, given by the Lévy measure
\[
\nu_{\alpha}(dx) := \frac{d}{|x|^{1+\alpha}(1+|x|)^{-m}} \mathbf{1}_{ \{x \neq 0\} } dx
\]
with $d > 0$ and $m \in (2 - \alpha, \infty)$ being fixed parameters. Then [18, Theorem 5] implies that $X_1$ has a bounded density.

For $K \in \mathbb{R}$ and $\varepsilon \in (0, 1]$ we let $f_{K,\varepsilon} \in C_b^\infty(\mathbb{R})$ with $f_{K,\varepsilon}(x) = 0$ if $x \leq K$, $f_{K,\varepsilon}(x) = 1$ if $x \geq K + \varepsilon$, $0 \leq f_{K,\varepsilon}(x) \leq 1$ and $0 \leq f'_{K,\varepsilon}(x) \leq 2/\varepsilon$ for all $x \in \mathbb{R}$.

Lemma 4.6. For $K \in \mathbb{R}$ and $\varepsilon > 0$ we have that
\[
\int_{\mathbb{R}\setminus\{0\}} E \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx).
\]

Proof. We get that
\[
\int_{\mathbb{R}\setminus\{0\}} E \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) = E \int_{0 < |x| \leq \varepsilon} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) + E \int_{\varepsilon < |x| < \infty} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \\
\leq \frac{4}{\varepsilon^2} \mathbb{P}(X_1 \in [K - \varepsilon, K + 2\varepsilon]) \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \mathbb{P}(X_1 \leq K + \varepsilon, X_1 + x \geq K) \nu(dx) + \int_{|x| < \varepsilon} \mathbb{P}(|X_1 - K| \leq x) \nu(dx) \\
\leq \frac{4\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \mathbb{P}(|X_1 - K| \leq x) \nu(dx).
\[ + \int_{-\infty < x < -\varepsilon} \mathbb{P}(K \leq X_1 \leq K - 2x) \nu(dx) \leq 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 \nu(dx) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx). \]

**Lemma 4.7.** For \( K \in \mathbb{R} \) and \( \varepsilon > 0 \) the following assertions are true:

(i) \( \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq \nu(\mathbb{R}) \)

(ii) If \( \psi(\delta) \leq c\delta \), then

\[ \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq 9c \min \left\{ \frac{1}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(dx), \int_{\mathbb{R}} |x| \nu(dx) \right\}. \]

**Proof.** (i) Using \( \mu(dx) = x^2 \nu(dx) \) on \( \mathbb{R} \setminus \{0\} \) one has that

\[ \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left| \frac{f_{K,\varepsilon}(X_1 + x) - f_{K,\varepsilon}(X_1)}{x} \right|^2 \mu(dx) \leq 9c \min \left\{ \frac{1}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(dx), \int_{\mathbb{R}} |x| \nu(dx) \right\}. \]

(ii) If \( \psi(\delta) \leq c\delta \), then we can bound the right-hand side in Lemma 4.6 by

\[ 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 d\nu(x) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx) \leq 8c \int_{\mathbb{R}} x^2 d\nu(x) + c \int_{\varepsilon < |x| < \infty} |x| \nu(dx) \leq 8c \int_{\mathbb{R}} x^2 d\nu(x). \]

Moreover,

\[ 4 \frac{\psi(2\varepsilon)}{\varepsilon^2} \int_{0 < |x| \leq \varepsilon} x^2 d\nu(x) + \int_{\varepsilon < |x| < \infty} \psi(|x|) \nu(dx) \leq 8c \int_{0 < |x| \leq \varepsilon} |x| \nu(dx) + c \int_{\varepsilon < |x| < \infty} |x| \nu(dx) \leq 8c \int_{\mathbb{R}} |x| \nu(dx). \]
Lemma 4.8. Let \( f(x) = \chi_{(K,\infty)}(x) \) for some \( K \in \mathbb{R} \). Assume \( \sigma = 0 \), \( \int_{\mathbb{R}} |x|^{\frac{3}{2}} \nu(dx) < \infty \) and assume that there is a \( c > 0 \) such that \( \psi(\delta) \leq c\delta \) for all \( \delta > 0 \). Then one has that
\[
\mathbb{E} \left| \int_{\mathbb{R}\backslash\{0\}} \frac{f(X_1 + x) - f(X_1)}{x} \mu(dx) \right|^2 \leq \frac{c}{2} \left( \int_{\mathbb{R}} |x|^{\frac{3}{2}} \nu(dx) \right)^2.
\]

Proof. For \( d\nu_0(x) := |x|^{\frac{3}{2}} \nu(dx) \) we get that
\[
\mathbb{E} \left| \int_{\mathbb{R}\backslash\{0\}} \frac{f(X_1 + x) - f(X_1)}{x} \mu(dx) \right|^2 \\
\leq \mathbb{E} \int_{\mathbb{R}} |f(X_1 + x) - f(X_1)||x|^{-\frac{1}{2}} \nu_0(dx)^2 \\
\leq \nu_0(\mathbb{R}) \mathbb{E} \int_{\mathbb{R}} |f(X_1 + x) - f(X_1)|^2 |x|^{-1} \nu_0(dx) \\
\leq \nu_0(\mathbb{R}) \left( \int_{\mathbb{R}} \psi \left( \frac{|x|}{2} \right) |x|^{-1} \nu_0(dx) \right) \\
\leq \frac{c}{2} \nu_0(\mathbb{R})^2.
\]

4.2. Examples. Throughout the whole subsection we fix a real number \( K \) and let
\[
f(x) := \mathbb{1}_{(K,\infty)}(x).
\]
(a) Without projection on \( \mathbb{M} \): We will obtain the (fractional) smoothness of \( \mathbb{1}_{(K,\infty)}(X_1) \) in dependence of distributional properties of \( X \). Note that Lemma 4.1 ensures that \( \Pi(\mathbb{1}_{(K,\infty)}(X_1)) \) has at least the (fractional) smoothness of \( \mathbb{1}_{(K,\infty)}(X_1) \). Our standing assumption, as mentioned in the beginning, is \( \int_{\mathbb{R}} x^2 \nu(dx) < \infty \). The case \( C_1 \) below confirms that for a compound Poisson process \( X \) we have \( \mathbb{1}_{(K,\infty)}(X_1) \in \mathbb{D}_{1,2} \).

| \( \sigma \) | \( \psi \) | additional assumption on \( \nu \) | Smoothness |
|---|---|---|---|
| \( C_1 \) | \( \sigma = 0 \) | arbitrary | \( \frac{\nu}{1} \nu(dx) < \infty \) | \( \mathbb{D}_{1,2} \) |
| \( C_2 \) | \( \sigma = 0 \) | \( \psi(\delta) \leq c\delta \) | \( \int_{|x| \leq 1} |x| \nu(dx) < \infty \) | \( \mathbb{D}_{1,2} \) |
| \( C_3 \) | arbitrary | \( \psi(\delta) \leq c\delta \) | | \( \mathbb{B}_{2,\infty} \) |

To check this table assume that the chaos-decomposition of \( f_{K,\varepsilon}(X_1) \) is described by symmetric \( g_{n}^{K,\varepsilon} \in L_2(\mu^{\otimes n}) \). From (15) we derive in the
case $\sigma = 0$ that
\[
\sum_{n=1}^{\infty} nn! \| g_n^{K, \varepsilon} \|^2_{L_2(\mu^{\otimes n})} = \sum_{n=1}^{\infty} n^2 \int_{\mathbb{R}} (n - 1)! \| g_n^{K, \varepsilon} (\cdot, x) \|^2_{L_2(\mu^{\otimes(n-1)})} \mu(dx)
\]
\[
= \sum_{n=1}^{\infty} n^2 \mathbb{E} \int_{\mathbb{R}} I_{n-1}(g_n^{K, \varepsilon} (\cdot, x) \mathbb{1}^{\otimes(n-1)})^2 \mu(dx)
\]
\[
= \int_{\mathbb{R}} \mathbb{E} \left| \sum_{n=1}^{\infty} n I_{n-1}(g_n^{K, \varepsilon} (\cdot, x) \mathbb{1}^{\otimes(n-1)}) \right|^2 \mu(dx)
\]
\[
= \int_{\mathbb{R}\setminus\{0\}} \mathbb{E} \left| f_{K, \varepsilon} (X_1 + x) - f_{K, \varepsilon} (X_1) \right|^2 \mu(dx)
\]
so that
\[
\| f_{K, \varepsilon} (X_1) \|^2_{\mathcal{D}_{1,2}} \leq 1 + \int_{\mathbb{R}\setminus\{0\}} \mathbb{E} \left| f_{K, \varepsilon} (X_1 + x) - f_{K, \varepsilon} (X_1) \right|^2 \mu(dx).
\]

Cases $C_1$ and $C_2$: Exploiting Lemma 4.7 gives that
\[
\sup_{m=1,2,\ldots} \| f_{K,1/m} (X_1) \|_{\mathcal{D}_{1,2}} < \infty.
\]
Moreover $\| f_{K,1/m} (X_1) - \chi_{(K,\infty)} (X_1) \|_{L_2} \to_m 0$ by dominated convergence so that $C_1$ and $C_2$ follow by a standard argument.

Case $C_3$: As before we get from (15) that
\[
\| f_{K, \varepsilon} (X_1) \|^2_{\mathcal{D}_{1,2}} \leq 1 + \int_{\mathbb{R}} \mathbb{E} \left| f_{K, \varepsilon} (X_1 + x) - f_{K, \varepsilon} (X_1) \right|^2 \mu(dx).
\]
Exploiting Lemma 4.7 and the property $0 \leq f'_{K, \varepsilon} (x) \leq \frac{2}{\varepsilon}$ we continue with
\[
\| f_{K, \varepsilon} (X_1) \|^2_{\mathcal{D}_{1,2}} \leq 1 + \frac{9c}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(x) + \frac{\sigma^2}{\varepsilon} \psi \left( \frac{\varepsilon}{2} \right)
\]
\[
\leq 1 + \frac{9c}{\varepsilon} \int_{\mathbb{R}} x^2 \nu(x) + \frac{\sigma^2}{\varepsilon} \frac{2c}{\varepsilon}.
\]
On the other hand,
\[
\| \chi_{(K,\infty)} (X_1) - f_{K, \varepsilon} (X_1) \|_{L_2} \leq \sqrt{\psi \left( \frac{\varepsilon}{2} \right)} \leq \sqrt{\frac{c \varepsilon}{2}}.
\]
Estimating the $K$-functional $K(u, \chi_{(K,\infty)} (X_1); L_2, \mathcal{D}_{1,2})$ by the help of the decomposition $\chi_{(K,\infty)} (X_1) = \left[ \chi_{(K,\infty)} (X_1) - f_{K, \varepsilon} (X_1) \right] + f_{K, \varepsilon} (X_1)$ and optimizing over $\varepsilon > 0$ gives $\chi_{(K,\infty)} (X_1) \in \mathcal{D}_{2,\infty}^{1/2}$. 

After projection on $\mathcal{M}$: Here we have the following

**Proposition 4.9.** Assume that $\sigma = 0$, $0 < \int_\mathbb{R} |x|^\frac{3}{2} \nu(dx) < \infty$ and that $\psi(\delta) \leq c\delta$. Then one has for all $K \in \mathbb{R}$ that

$$\Pi(\mathbb{I}_{(K,\infty)}(X_1)) \in \mathbb{D}_{1,2}.$$

**Proof.** By the same reasoning as in the cases $C_1$ and $C_2$ it is sufficient to show that

$$\sup_{m=1,2,...} \|\Pi(f_{K,1/m}(X_1))\|_{\mathbb{D}_{1,2}} < \infty.$$

By (9) and Lemma 4.3 it suffices to check that

$$\sup_{m=1,2,...} \mathbb{E} \left[ \int_{\mathbb{R} \setminus \{0\}} \left| \frac{f_{K,\frac{1}{m}}(X_1 + x) - f_{K,\frac{1}{m}}(X_1)}{x} \right|^2 d\mu(x) \right] < \infty.$$

But this estimate follows from Lemma 4.8 and the representation

$$f_{K,\varepsilon}(x) = \int_{-\infty}^{x} f'_{K,\varepsilon}(y)dy = \int_{\mathbb{R}} \mathbb{I}_{[y,\infty)}(x)f'_{K,\varepsilon}(y)dy$$

and $\int_{\mathbb{R}} f'_{K,\varepsilon}(y)dy = 1$. 

**Example 4.10.** An example for Proposition 4.9 is obtained from Example 4.5. Considering

$$\nu_\alpha(dx) = \frac{d}{|x|^{1+\alpha}}(1 + |x|)^{-m}\mathbb{I}_{\{x \neq 0\}}dx$$

for $d > 0$, $\alpha \in (0, \frac{3}{2})$ and $m \in (2 - \alpha, \infty)$ gives $\psi(\delta) \leq c\delta$ and $0 < \int_\mathbb{R} |x|^\frac{3}{2} d\nu_\alpha(x) < \infty$, where $\alpha$ turns out to be the Blumenthal-Getoor index. Using the results of [14] one can also show that $\mathbb{I}_{(K,\infty)}(X_1) \notin \mathbb{D}_{1,2}$ for $\alpha \geq 1$ so that the projection $\Pi$ improves the smoothness of $\mathbb{I}_{(K,\infty)}(X_1)$ for $\alpha \in \left[1, \frac{3}{2}\right]$.

**Remark 4.11.** Using a Fourier transform approach Brodén and Tankov [5] compute the discretization error under the historical measure for the delta hedging as well as for a strategy which is optimal under a given equivalent martingale measure. Using the equivalences of Theorem 3.6 (i) $\iff$ (iv) and Theorem 3.8 (i) $\iff$ (iv) one can also conclude about the fractional smoothness of the projection of the considered digital option from the computed convergence rate for equidistant time nets.
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