The stress distribution heterogeneity influence on the limiting state of a bent curvelinear beam

S Yu Kalashnikov, E V Gurova, S A Kalinovsky, R Kh Kuramshin
Volgograd State Technical University, 28, Lenin Ave., Volgograd, 400005, Russia

E-mail: kalashnikovsu@mail.ru

Abstract. The limiting elastic state of a beam with large curvature under pure and transverse bending is considered. The gradient plasticity condition which determines the yield onset in an inhomogeneous stress state is applied as a criterion of the ultimate state. The analytical expressions for calculating the corresponding stresses and loads are obtained.

1. Introduction
Any theory of strength in the structural elements’ calculation is aimed at achieving two seemingly opposite goals: ensuring strength and operational safety while reducing the material consumption. The resolution of such a dilemma is provided by comparing the objectively obtained strength characteristics of the material with the calculated values of the stress state in dangerous sections. A similar approach is simple, but has two fundamental errors: the characteristics of the material are established when testing the material samples for uniaxial tension or compression, and the stress state of the structure is determined in the framework of the material’s calculation model, built on the simplest mathematical relations that describe the problem’s physical side. None of the noted aspects takes into account the nonlinearity of the material and the possible types of real stress state in the elements and structures.

The experimental data analysis [1-5] shows that the yield strength of a material, determined from the uniaxial tensile experiments, does not allow reliably solving the strength problems with an inhomogeneous stress distribution. Currently, among a large number of strength theories, there are a number of gradient strength criteria that take into account the stress distribution heterogeneity in the vicinity of the point under consideration for various materials [6-11].

In this work, we use the plasticity criterion for an elastically plastic isotropic material with a yield area. Its application leads to an analytical expression that makes it possible to calculate the corresponding stress and load. The authors are not aware of other works leading to an analytical solution.

2. Problem statement and solution method
Let us consider the pure bending of a beam of high curvature, the axis of which is outlined in a circle. In a polar coordinate system with a pole in the curvature center O (Figure 1), the normal stresses in the circumferential direction at any point in the cross section are determined by the well-known formula for the resistance of materials [12]:
where $A = b(R_2 - R_1)$ is the cross-sectional area; $e$ is the distance from the gravity center to the neutral axis; $R$ is the radius of the neutral layer.

For a rectangular cross section, the latter is calculated by the formula:

$$R = \frac{R_2 - R_1}{\ln k},$$

where $k = R_2/R_1$.

The classical formulas (1) - (2) were obtained using the hypotheses and assumptions of the resistance of materials. In further considerations, we leave the hypothesis of flat sections valid, and we transform the lateral pressures’ absence hypothesis into the concept of constraining deformations from the side of less stressed volumes of material. As a criterion of the ultimate state, we use the incremental plasticity condition, which establishes the moment of the yield onset in inhomogeneous stress [13-16].

![Figure 1. Pure bending of a high curvature beam](image-url)

The criterion proceeds from the above-mentioned concept of the shear deformations constraint along the sliding platforms from the side of less stressed material volumes and takes into account the effect of an increase in ultimate elastic stresses in inhomogeneous fields. The neighboring volumes influence degree is characterized by a gradient vector of the scalar stress field, and the vector field of gradients, in turn, characterizes the direction and rate of the stresses change. The more inhomogeneous the stress state, the greater the adjacent areas’ influence on each other in the material. In the case of the one-dimensional problem under consideration, the vector gradient is directed along the radii, therefore, deformation is constrained in the circumferential direction.

To write the incremental plasticity conditions, the form of the Tresca-Saint-Venant or Huber-Mises-Hencky criteria is used, taking into account that the yield strength of the material is in a certain way related to a certain function of the stress gradient and its derivatives - with respect to the coordinates. As shown in [17], this function can be linear fractional, exponential, irrational, or logarithmic.

The yield onset is determined by such a deformation stage when the maximum stress at a hazardous point reaches a certain increased value $\sigma_{max} = \sigma_{gr}$, more that the yield strength $\sigma_0$ in uniaxial stress state. The greater this increase is, the more heterogeneous is the stress distribution in the vicinity of the point under consideration. At the same time, the increase cannot be unlimited, so it should be limited to some maximum $\sigma_m$. When using the form of the second of the indicated conditions, the asymptotic linear-fractional dependence for the shear stress intensities is taken as follows:

$$T_{gr} = T_0 + (T_m - T_0) \cdot \left[ \frac{\text{grad} T}{T} / \left( \lambda_{T,gr} \text{grad} T / T \right) \right],$$

(3)
where $T_m$ – is the largest of the stresses possible with an inhomogeneous stress state; we accept $T_m = 1.5T_0$; $\lambda_{T,g}$ – defines some elastic characteristic of the material, having a dimension inverse to the length. In [16,19] a numerical value was obtained as $\lambda_{T,g} = 20,1587$ m$^{-1}$ from the experimental data [2] for the building steel as an isotropic polycrystalline material with a random arrangement of component grains.

The proposed gradient approach is implemented in solving various problems of the elements and structures’ strength: in [15,18,19] - in one-dimensional and two-dimensional problems of the rods bending, in round plates [20,21], in the torsion of round shafts [18], in the axisymmetric deformation of pipes and ball reservoirs [22], in the eccentric compression of the rods [18], plate tension, weakened by a circular hole [23], as well as the continuum problems with a singularity [24]. A feature of this condition’s application, unlike others, is the receipt of an analytical solution in the form of either resolving an equation or an algebraic expression. At the same time, the obtained results have an unambiguous interpretation: depending on the design scheme and the specified parameters of the structural element, the plastic deformation onset in dangerous sections corresponds to the only possible value of the increased gradient stress $T_{gr} > T_0$.

We apply the proposed condition to the problem under consideration. In polar coordinates, the shear stress intensity is:

$$\sigma = \frac{1}{\sqrt{3}} \sqrt{\sigma_r^2 + \sigma_\theta^2 - 2\sigma_r\sigma_\theta + 3\tau_r^2}$$

the substitution of (1) gives us:

$$T = \sigma / \sqrt{3}.$$  \hfill (5)

The shear stress intensity modulus is:

$$gradT = \sqrt{\left(\frac{\partial T}{\partial \rho}\right)^2 + \frac{1}{\rho^2} \left(\frac{\partial T}{\partial \theta}\right)^2}$$

The view of (5) has the form

$$gradT = \frac{MR}{\sqrt{3}Aep^2}.$$ \hfill (7)

Then the function of the heterogeneity measure taking into account (2), has the form:

$$g = \frac{gradT}{T} = \frac{R_2 - R_1}{\rho(\rho h k - R_2 + R_3)}.$$ \hfill (8)

We assume that the beam material equally works in tension and compression. Then, regardless of the bending moment sign, the material’s transition into a plastic state will start in extreme fibers of small radius at $\rho = R_1$. Substituting (8) in (3), we obtain the expression:

$$T_{gr} = T_0 \cdot \frac{2\lambda_{T,g}(R_1(ink - R_2 + R_3) - 3(1-k))}{2\lambda_{T,g}(R_1(ink - R_2 + R_3) - 2(1-k))}.$$ \hfill (9)

A graphical representation (9) for the beams with different geometric parameters is shown in Figure 2. An analysis of the last expression and the results obtained shows that the value of the increased stress corresponding to the plastic flow beginning is most significantly manifested in beams of high curvature. A decrease in the cross-section height expectedly leads to an increase in the gradient effect (as in straight rods [15.18.19]), but to a slightly lesser extent than a decrease in the radius of curvature.
Equalization of the right-hand sides (1) and (9) allows to obtain the value of the increased load $M_{gr}$, causing the yield onset, shown graphically in Figure 3 in the half-reverse coordinates for different relative heights of the cross sections.

\begin{equation}
\sigma_r = (2A \rho - 2B \rho^3 + D \rho) \sin \theta; \\
\sigma_\theta = (6A \rho + 2B \rho^3 + D \rho) \sin \theta;
\end{equation}

\begin{equation}
\tau_{r\theta} = -(2A \rho - 2B \rho^3 + D \rho) \cos \theta,
\end{equation}

Where $A = F/2N$; $B = -FR_1^2 R_2^2/2N$; $D = -F(R_1^2 + R_2^2)/N$; $N = R_1^2 - R_2^2 + (R_1^2 + R_2^2) \ln k$.

The polar angle is counted counterclockwise from the vertical. Substituting (10) into (4), we obtain:

\[ T = \frac{1}{\sqrt{3}} \left\{ (2A \rho - 2B \rho^3 + D \rho)^2 + (6A \rho + 2B \rho^3 + D \rho)^2 - (2A \rho - 2B \rho^3 + D \rho)(6A \rho + 2B \rho^3 + D \rho) \sin^2 \Theta + 3(2A \rho - 2B \rho^3 + D \rho)^2 \cos^2 \Theta \right\}^{1/2}. \tag{11} \]

Further, from (6) and (11) we obtain an expression for the shear stress intensity modulus, which is omitted due to bulkiness. The dangerous section is in the seal, and the dangerous point is the fiber on the inner edge near the seal, where $T = T_{max}$. 

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{Graph of the gradient stress variation in various curved beams with a pure bending}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{The graph of the increase in the bearing capacity of a curved beam with a pure bending}
\end{figure}
Figure 4. Transverse bending of a high curvature beam

In this way, $\theta = \pi/2$, and $\rho = R_1$. Substituting these values in the expressions (11) for $T$ and $\text{grad}T$, after the routine transformations we get the heterogeneity measure function

$$g = \frac{\text{grad}T}{T} = \frac{[(1 - k^2)(3 + 5k^2)^2]^{1/2}}{2R_1(1 - k^2)^2}.$$ 

The last expression and (3) imply the dependence:

$$T_{gr} = T_0 \frac{4kT_0}{4kT_0 R_1(1-k^2)^2 + 3[(1-k)^2(3+5k^2)^2]^{1/2}}.$$ \hspace{1cm} \text{(12)}$$

the graph of which is shown in Figure 5 for the real $R$ and $k$ value region.

Figure 5. Graph of the gradient stress variation in various curved beams with lateral bending

As with pure bending, the gradient stress in the beams of high curvature and at a low cross-sectional height changes most significantly.

Figure 6. The schedule for increasing the bearing capacity of a curved beam with lateral bending
Equating the right-hand sides of (11) and (12), we obtain an increased value of the force $F_{gr}$, causing the yield onset. The result is shown illustratively in Fig. 6 in semi-inverse coordinates for some values of the relative section height.

As with pure bending, an increase in the bearing capacity takes place, but in the beams with different parameters and sizes, it is estimated from 8% to 40% due to the multicomponent stress state.

3. Summary

Solving the theory problems of the strength and subsequent material consumption in relation to the building structures’ calculation, taking into account the specifics of their work, requires a deeper development. The correctly selected criterion makes it possible to more accurately assess the work of the structure in the elastic stage of deformation and to clarify the bearing capacity resource, which will lead to a more rational engineering solution.

The use of the gradient yield condition in the present work leads to analytical formulas that give the satisfactory results. The curvature of the beam determines more significant manifestations of gradient effects compared to the pure and lateral bending of the straight rods.

References

[1] Baldin V A, 1977 On regard to plastic deformations in case of non-uniform distribution of stresses over a cross-section Structural Mechanics and Analysis of Constructions 1 29-31.

[2] Baldin V A, Potapov V N,Fadeev A A 1982 On the resistance of steel to deformation at non-uniform stress distribution Structural Mechanics and Analysis of Constructions 5 23-26.

[3] Fadeev A A 1982 About the transformation of extra-soft steel into elasto-plastic state under non-uniform distribution of tension (under pure bending) in: Institute Works. Research on Strength of Elements for Metal Building Structures 85-91.

[4] Campus F 1963 Plastification de l'acier doux en flexion plane simple Bull. de la Classe des Sciences de l'Academie R. de Belgique Serie 5 (49) 303-314.

[5] Dehouss N M 1962 Note relative a un phenomen de superelasticite en flexion constate lors d'essais d'un beamereau en acier doux Bull. de la Classe des Sciences de l'Academie R. de Belgique Serie 5 (48) 329-334.

[6] König J A, Olszak W 1974 The YIELD Criterion in the General Case of Nonhomogeneous Stress and Deformation Fields (in: Zeman J L., Ziegler F. (eds) Topics in Applied Continuum Mechanics. Springer, Vienna). DOI: 10.1007/978-3-7091-4188-5_4.

[7] Kolodezev V E 1996 To the issue of interrelation of the gradient criteria of strength with linear fracture mechanics in: Collec. of Scient. Art., NGASU Publ., N. Novgorod 3 (5) 37-42.

[8] Legan M A 1990 The gradient approach to estimating the strength of brittle materials in the maximum-stress region in: The Dynamics of a Continuous Medium. Collected Papers, Science Academy USSR, Hydrodynamics 98 179-182.

[9] Fadeev A A 1983 Specific features of steel work in elements of metal structures under non-uniform stress distribution (Abstract of Cand. of Tech. Sci. Thesis, Central Research Institute of Building Structures Publ., Moscow).

[10] Harlab V D 1993 Gradient criteria of brittle failure, in: Research on the Mechanics of Building Structures and Materials: Inter-University Thematic Collection of Works (St. Petersburg Institute of Civil Engineering Publ., St. Petersburg).

[11] Harlab V D, Minin V A 1989 Strength Criterion Allowing for the Effect of Stress State Gradient in: Research on Mechanics of Building Structures and Materials (Inter-University Thematic Collection of Works), Leningrad.

[12] Timoshenko S P 1960 Strength of materials. In two volumes (Vol.1. Elementary theory and problems.-M.: Gos. Publishing House Phys.-Math. Literature).

[13] Timoshenko S P 1960 Strength of materials. In two volumes (Vol. 1. Elementary theory and problems, Moscow).
[14] Geniev G A, Kalashnikov S Yu 1985 Influence of stress gradients, geometry and scales of sections on the transition of bending elements to plastic behavior in: Institute Works. Investigations in Structural Mechanics, Central Res. Institute of Building Structures Publ., Moscow.

[15] Geniev G A, Kalashnikov S Yu 1988 On regarding the influence of stress state non-uniformity on the transition of material to plastic behavior Struct. Mech. and Analysis of Constr 6 12-15.

[16] Geniev G A, Kalashnikov S Yu 1984 On the Plotting of Incremental Plasticity Conditions (Deposited at VNIIS, Moscow).

[17] Kalashnikov S Yu 2003 Criteria of plasticity allowing for the influence of stress state non-uniformity Trudy NGASU 6 142-148.

[18] Kalashnikov S Yu Experimental Testing of Strain Model of Material under the Conditions of Non-Uniform Stress State (IUNL VSTU Publ., Volgograd).

[19] Kalashnikov S Yu 1984 On the solution of some two-dimensional problems applying incremental yield condition: depos. man., Dep. at Deposited Manuscript 1984-01-01. 5015 16.

[20] Kalashnikov S Yu, Levin A V 1988 On the solution of problems of in-plane transverse bending applying incremental yield condition: deposited manuscript, Dep. at Deposited Manuscript 1988-01-01. 8382 10.

[21] Kalashnikov S Yu 2003 Implementation of incremental yield condition at axisymmetric deformation of circular plate in: Numerical Methods to Solve the Problems of Elasticity and Plasticity Theory: Proceed. of XVIII Inter-Republic Conf. Kemerovo, June 1-3 2003, Nonparel Publ., Novosibirsk 82-84.

[22] Kalashnikov S Yu 2003 Application of incremental yield condition in problems of axisymmetric bending of circular plate University News North-Caucasian Region - Technical Sciences ser. 5 169-174.

[23] Kalashnikov S Yu 2003 Implementation of incremental yield conditions in spherical and cylindrical coordinates University News North-Caucasian Region - Technical Sciences series 3 44-48.

[24] Kalashnikov S Yu, Gurova E V, Kuramshin R Kh, Starov A V 2018 Influence of stress gradients on the limit state of the plate weakened by a circular hole IOP Conf. Series: Materials Science and Engineering 456 012111. DOI: 10.1088/1757-899x/456/1/012111

[25] Kalashnikov Sergey Y, Gurova Elena, Kuramshin Renat, and Kharlanov Vladimir 2019 Implementation of Incremental Yield Condition for Continual Problems with Singularity Materials Science Forum 974 (December 2019) 620–26. doi.org/10.4028/www.scientific.net/msf.974.620

[26] Timoshenko S P, Goodyear J 1979 Theory of Elasticity (Nauka, Moscow).