Inverse Optimal Adaptive $H_\infty$ Consensus Control of Multi-Agent Systems on Directed Network Graphs

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Abstract: Design methods of inverse optimal adaptive $H_\infty$ consensus control of multi-agent systems denoted by the first-order and the second-order regression models on directed network graphs are presented in this paper. The proposed control schemes are derived as solutions of certain $H_\infty$ control problems, where estimation errors of tuning parameters are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and that the desirable consensus tracking is achieved approximately via adaptation schemes and design parameters.

Key Words: adaptive control, consensus control, multi-agent system, $H_\infty$ control.

1. Introduction

In the studies of cooperative control problems of multi-agent systems, distributed consensus tracking of multi-agent systems with limited communication structures has been a basic and important issue, and various control methodologies have been reported for various processes and under various conditions such as [1]–[10]. In those research works, adaptive control or sliding mode control strategies were also proposed in order to deal with uncertainties of agents, and stability of control systems was assured via Lyapunov function analysis [4],[7]. Furthermore, basic mathematical formulas and robustness properties of the control schemes were also discussed [1],[4]. However, so much attention does not have been paid on control performance of the control schemes were also discussed [1], [4]. However, so much attention does not have been paid on control performance and time-varying) and that the desirable consensus tracking is achieved approximately via adaptation schemes and design parameters. Several simulation studies also confirm the effectiveness of the proposed methodologies.

It should be noted that the cost functional in the inverse optimal controller design is to be deduced at the final stage, as a possible choice, and is not specified in advance.

2. Multi-Agent Systems and Information Network Graphs

Mathematical preliminaries on directed network graph of multi-agent systems are summarized [6],[8],[9]. As a model of interaction among agents, a weighted directed graph $G = (\mathcal{V}, \mathcal{E})$ is considered, where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, which corresponds to a set of agents. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, and an edge $(i, j) \in \mathcal{E}$ indicates that agent $j$ can obtain information from $i$, but not necessarily vice versa. In the edge $(i, j)$, $i$ is called as a parent node and $j$ is called as a child node, and the in-degree of the node $i$ is the number of its parents, and the out-degree of $i$ is the number of its children. Especially, an agent having no parent (or with the in-degree 0), is called as a root. A directed path is a sequence of edges in the form $(i_1, i_2), \ldots, (i_l, i_{l+1})$, where $i_j \in \mathcal{V}$. A directed tree is a directed graph in which every node has exactly one parent except for a unique root, and the root has directed paths to all other nodes. An directed spanning tree $\mathcal{G}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subgraph of $\mathcal{G}$ such that $\mathcal{G}_S$ is a directed tree and $\mathcal{V}_S = \mathcal{V}$.

Associated with the edge set $\mathcal{E}$, a weighted adjacency matrix is introduced such that $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, and the entry $a_{ij}$ is defined by

$$a_{ij} = \begin{cases} > 0 & : (j, i) \in \mathcal{E}, \\ 0 & : \text{otherwise}. \end{cases}$$

For the adjacency matrix $A = [a_{ij}]$, the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined by

...
The Laplacian matrix has at least one zero eigenvalue and all nonzero eigenvalues have positive real parts. Especially, the Laplacian matrix has a simple 0 eigenvalue with the associated eigenvector \( \mathbf{1} = [1 \cdots 1]^T \), and all other eigenvalues have positive real parts if and only if the corresponding directed graph has a directed spanning tree.

In this manuscript, a consensus control problem of leader-follower type is considered, where each agent \( i \in \mathcal{V} \) (a follower) follows a leader. For the leader and the followers \( i, a_{i0} \) is defined such as

\[
a_{i0} = \begin{cases} 
> 0 & \text{leader’s information is available to follower } i, \\
0 & \text{otherwise},
\end{cases}
\] (1)

and from \( a_{i0} \) and \( L \), the matrix \( M \in \mathbb{R}^{n \times n} \) is defined by

\[
M = L + \text{diag} (a_{i0} \cdots a_{n0}).
\] (2)

For the above matrix \( M \), it follows that \( -M \) is a Hurwitz matrix if and only if the graph \( \mathcal{G} \) has a directed spanning tree with the root \( i = 0 \), which requires that at least one \( a_{i0} \) (\( 1 \leq i \leq N \)) is positive. Hereafter, it is assumed that the condition (\( \mathcal{G} \) has a directed spanning tree with the root \( i = 0 \)) is satisfied.

In the following, two adjacency matrices \( A = [a_{ij}], C = [c_{ij}] \in \mathbb{R}^{n \times n} \) are considered for a directed graph \( \mathcal{G} \) and the corresponding matrices are denoted as \( L_a, L_c \) (Laplacian matrices), and \( M_a, M_c \), respectively.

**Remark 1.** Hereafter, two independent adjacency matrices \( A \) and \( C \) are to be utilized in the leader state estimator and the feedback control scheme. It should be noted that those two adjacency matrices can be assigned independently for the separate two purposes, and are not required to be the same as each other; that is, we can choose \( A = C \) or \( A \neq C \).

3. Adaptive \( H_{\infty} \) Consensus Control for First-Order Models

3.1 Problem Statement

A multi-agent system composed of the first-order regression models is considered as follows:

\[
\hat{x}_i(t) = \Omega_i(t)\theta_i + B_i u_i(t), \quad (i = 1, \ldots, N),
\] (3)

where \( x_i \in \mathbb{R}^n \) is a state, \( u_i \in \mathbb{R}^n \) is an input, \( \theta_i \in \mathbb{R}^n \) is an unknown parameter vector, and \( \Omega_i \in \mathbb{R}^{m \times n} \) is a regressor matrix composed of \( x_i \) and its structure is known. \( \Omega_i \) is assumed to be bounded for bounded \( x_i \). \( B_i \in \mathbb{R}^{m \times n} \) is an unknown matrix of the form

\[
B_i = \text{diag} (b_{i1}, \ldots, b_{im}),
\] (4)

and the sign of \( b_{ij} \) is known a priori. Hereafter, let \( b_{ij} > 0 \) without loss of generality, and furthermore, its lower-bound (\( > 0 \)) and upper-bound (\( < \infty \)) are also known a priori. The communication structure among agents and a leader is prescribed by the information network graph \( \mathcal{G} \) with the associated adjacency matrices \( A \) and \( C \), the Laplacian matrices \( L_a \) and \( L_c \), and the matrices \( M_a \) and \( M_c \). The control objective is to achieve consensus tracking of the leader-follower type such as

\[
x_i \rightarrow x_j, \quad (i, j = 1, \ldots, N),
\] (5)

\[
x_i \rightarrow x_0, \quad (i = 1, \ldots, N),
\] (6)

where \( x_0 \) is a leader to be followed by all agents \( i \in \mathcal{V} \).

**Remark 2.** Another objective from the viewpoint of adaptive control is to obtain an adaptive control structure whose stability is not seriously affected by the adaptation scheme, or whose performance is not significantly degraded by the estimation errors of the tuning parameters. For that purpose, in order to attenuate the effects of the estimation errors, the control scheme is to be deduced as a solution of certain \( H_{\infty} \) control problem, where estimation errors of tuning parameters are regarded as external disturbances to the process.

3.2 Control Law and Error Equation

As the first step of the design of the proposed control system, an estimation scheme of the leader \( x_0 \) is constructed via available information from the follower \( i \). A similar estimator was proposed in [5] for the second-order Euler-Lagrange systems:

\[
\dot{\hat{x}}_i(t) = -\beta \sum_{j=1}^{N} c_{ij} (\dot{\hat{x}}_j(t) - \dot{\hat{x}}_i(t)) - \beta c_{i0} (\dot{\hat{x}}_i(t) - x_0(t)) + n_{i0} \theta_i(t),
\] (7)

where \( \dot{\hat{x}}_i \) is a current estimate of \( x_0 \), and is synthesized from the data available to the follower \( i \). \( c_{ij}(1 \leq i \leq N, 0 \leq j \leq N) \) is defined as the entry of the adjacency matrix \( C \) and (1), and \( \beta > 0 \) is a design parameter. Concerned with \( c_{i0}, n_{i0} \) is defined as follows:

\[
n_{i0} = \begin{cases} 
1 & \text{if } c_{i0} > 0, \\
0 & \text{otherwise}.
\end{cases}
\] (8)

By utilizing the estimate \( \dot{\hat{x}}_i \), the following control law is employed.

\[
\dot{x}_i(t) = \dot{\hat{x}}_i(t) - \alpha \sum_{j=0}^{N} a_{ij} (x_j(t) - x_i(t)),
\] (9)

\[
s_i(t) = x_i(t) - x_0(t),
\] (10)

\[
u_i(t) = \tilde{P}_i(t) [\tilde{\Omega}_i(t)\tilde{\theta}_i(t) + \dot{s}_i(t)] + v_i(t) \equiv \tilde{P}_i(t) u_{i0}(t) + v_i(t),
\] (11)

where \( a_{ij}(1 \leq i \leq N, 0 \leq j \leq N) \) is defined similarly from the entry of the adjacency matrix \( A \) and (1), and \( \alpha > 0 \) is a design parameter. \( \dot{s}_i \) is denoted as a current estimate of \( s_i \), and \( P_i \) is defined by

\[
P_i = \text{diag} (p_{i1}, \ldots, p_{im}), \quad p_{ij} = 1/b_{ij}.
\] (12)

Furthermore, \( v_i \) is a stabilizing signal to be determined later based on \( H_{\infty} \) control criterion (see Remark 1). The signal \( s_{ij} \) is considered as an estimate of \( s_0 \) with additive feedback terms, and \( s_i \) is considered as a tracking error between the output of the agent and the estimated output of the leader.
An estimation error between the leader \( x_0 \) and the estimate \( \hat{x}_0 \) is defined by
\[
\tilde{z}_i(t) = \hat{z}_i(t) - x_0(t),
\]
and the following relations are deduced for \( s_i \) and \( \tilde{z}_i \).
\[
\dot{\tilde{z}}_i(t) = -\beta \sum_{j=1}^{N} c_{ij}[\tilde{z}_j(t) - \hat{z}_j(t)]
\]

\[
\dot{s}_i(t) = \Omega(t)\{\theta - \hat{\theta}(t)\} + U_0(t)B[\hat{p}(t) - p_1] + B\tilde{v}_i(t),
\]

where
\[
U_0 = \text{diag}(u_{01}, \ldots, u_{0N}),
\]
\[
u_0 = [v_{01}, \ldots, v_{0N}]^T,
\]
\[
p_1 = [p_{11}, \ldots, p_{1N}]^T.
\]

Then, the total representations of the multi-agent system are given as follows:
\[
\dot{\tilde{z}}_i(t) = -\beta (M_e \otimes I) [\tilde{z}_i(t) + [(N_0 - I) \otimes I] \hat{x}_0(t),
\]
\[
\dot{s}_i(t) = \Omega(t)[\theta - \hat{\theta}(t)] + U_0(t)B[p - \hat{p}(t)] + B\tilde{v}_i(t),
\]

where
\[
\tilde{z}_i = [\tilde{z}_{i1}^T, \ldots, \tilde{z}_{iN}^T]^T,
\]
\[
s_i = [s_{i1}^T, \ldots, s_{iN}^T]^T,
\]
\[
\Omega = \text{block diag}(\Omega_1, \ldots, \Omega_N),
\]
\[
\theta = [\theta_1^T, \ldots, \theta_N^T]^T,
\]
\[
U_0 = \text{block diag}(U_{10}, \ldots, U_{N0}),
\]
\[
B = \text{block diag}(B_1, \ldots, B_N),
\]
\[
p = [p_1^T, \ldots, p_N^T]^T,
\]
\[
N_0 = [n_{01}, \ldots, n_{0N}]^T,
\]
\[
\Gamma_1 = I_1 \otimes I_1 > 0, \quad \text{diagonal},
\]
\[
V = \text{block diag}(V_1, \ldots, V_N),
\]
\[
V_i = \text{diag}(v_{i1}, \ldots, v_{in}),
\]
\[
v_i = [v_{i1}, \ldots, v_{in}]^T.
\]

and Pr(\cdot) is projection operation in which tuning parameters are constrained to bounded regions deduced from upper-bounds (< \( \infty \)) and lower-bounds (> 0) of each element of \( b \) [15]. Then, the time derivative of \( W_0 \) along its trajectory is given as follows:
\[
\dot{W}_0(t) \leq s_i(t)\{[\Omega(t)\theta - \hat{\theta}(t)] + U_0(t)B[\hat{p}(t) - p] + B\tilde{v}_i(t)\}.\]

From the evaluation of \( W_0 \), the next virtual system is introduced.
\[
\dot{s} = f + g_{11}d_1 + g_{12}d_2 + g_{13}d_3 + g_2v,
\]
\[
f = 0,
\]
\[
g_{11}, g_{12} = U_0, \quad g_{13} = I, \quad g_2 = \hat{B},
\]
\[
d_1 = (\theta - \hat{\theta}), \quad d_2 = (\hat{B} - p),
\]
\[
d_3 = \text{virtual disturbance}.
\]

The virtual system is to be stabilized via a control input \( v \) by utilizing \( H_\infty \) criterion, where \( d_1, d_2, d_3 \) are regarded as external disturbances to the process [13,14]. (Note that \( d_1 \) and \( d_2 \) are estimation errors of tuning parameters \( \hat{\theta} \) and \( \hat{\theta} \), respectively. See also Remark 1.) For that purpose, the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution \( V_0 \) are considered.
\[
\mathcal{L}_fV_0 + \frac{1}{4} \sum_{i=1}^{N} \frac{\|L_{e_i}V_0\|^2}{\gamma_i^2} - (L_{e_i}V_0)R^{-1}(L_{e_i}^TV_0)^T
\]
\[
+ q = 0,
\]
\[
V_0 = \frac{1}{2} s^T s.
\]

where \( q \) and \( R \) are a positive function and a positive definite matrix respectively, and those are derived from the HJI equation based on inverse optimality for the given solution \( V_0 \) and the positive constants \( \gamma_1, \gamma_2 \). The substitution of the solution \( V_0 \) (44) into the HJI equation (43) yields
\[
\frac{1}{4} s^T \left\{ \frac{\Omega\Omega^T}{\gamma_1^2} + \frac{U_0U_0^T}{\gamma_2^2} + \frac{\hat{B}\hat{B}^T}{\gamma_2^2} - \hat{B}R^{-1}\hat{B} \right\} s + q = 0.
\]

Then, \( R \) and \( q \) are obtained such as
\[
R = \left( \frac{\hat{B}^{-1}\Omega\Omega^T\hat{B}^T}{\gamma_1^2} + \frac{\hat{B}^{-1}U_0U_0^T\hat{B}^T}{\gamma_2^2} + \frac{\hat{B}^{-1}\hat{B}}{\gamma_2^2} + K \right)^{-1},
\]
\[
q = \frac{1}{4} s^T \hat{B}K\hat{B}^Ts,
\]

where \( K \) is a diagonal positive definite matrix (a design parameter). From \( R \), \( v \) is derived as a solution of the corresponding \( H_\infty \) control problem as follows:
\[
v = -\frac{1}{2} R^{-1}(L_{e_i}^TV_0)^T = -\frac{1}{2} R^{-1} \hat{B}^Ts
\]
\[
= \frac{1}{2} \left( \frac{\hat{B}^{-1}\Omega\Omega^T\hat{B}^T}{\gamma_1^2} + \frac{\hat{B}^{-1}U_0U_0^T\hat{B}^T}{\gamma_2^2} + \frac{\hat{B}^{-1}\hat{B}}{\gamma_2^2} + K \right) \hat{B}^Ts,
\]
where the entries of $\hat{B}$ in $R$ (46) and $v$ (48) are constructed from the elements of $\hat{b}$ (34). Then, the time derivative of $W_0$ is evaluated by

$$W_0 \leq -q - v^T R v + \gamma_1^2 \|d_1\|^2 + \gamma_2^2 \|d_2\|^2.$$

(49)

and it follows that $s$ is bounded for bounded $\hat{b}$, $\hat{\rho}$ and for the stabilizing signal $v$ (48).

**Remark 3.** In the discussion hereafter, $q + v^T R v$ (46), (47) and $v$ (48) are shown to be a cost functional and an optimal control input for the corresponding $H_\infty$ control problem. The cost function in the inverse optimal controller design is deduced at the present stage, as a possible choice of the meaningful functional, and is not specified in advance.

**Remark 4.** The projection operations in the tuning of $\hat{b}$ are introduced in order to prevent the elements of $\hat{b}$ approaching to 0, and make $\hat{B}^{-1}$ bounded in the synthesis of $v$ (48). For detailed explanations of the projection operations together with the derivation of (38), see [12] (especially, (35) and Appendix A in [12]).

Next, for stability analysis of the estimation error $\hat{z}$, a positive function $V_1$ is introduced such as

$$V_1 = \hat{z}^T (\hat{P}_c \otimes \hat{I}) \hat{z},$$

(50)

$$\hat{P}_c \hat{M}_c + \hat{M}_c^T \hat{P}_c = I, \quad (\hat{P}_c = \hat{P}_c^T > 0),$$

(51)

where there exists a positive definite and symmetric matrix $\hat{P}_c$ satisfying (51), since $-\hat{M}_c$ is Hurwitz. Then, the time derivative of $V_1$ along its trajectory is evaluated as follows:

$$\dot{V}_1 = -\beta \|\hat{z}\|^2 - 2\gamma_1^2 (\hat{P}_c \otimes I)(\hat{N}_0 - 1) \otimes \hat{x}_0$$

$$\leq -\beta \|\hat{z}\|^2 - 2\gamma_1^2 \|\hat{P}_c \otimes I\| \|\hat{N}_0 - 1\| \|\hat{x}_0\|^2,$$

(52)

and it is shown that $\hat{z}$ is bounded for bounded $\hat{x}_0$.

Finally, for stability analysis of the control error $x_i - x_0$ and related terms, $\hat{x}_i$, $\tilde{y}_i$, $\hat{y}_i$ are defined such as

$$\hat{x}_i = x_i - x_0,$$

(53)

$$\tilde{y}_i = \hat{x}_i,$$

(54)

$$\hat{y}_i = [\tilde{y}_1^T, \ldots, \tilde{y}_N^T]^T.$$  

(55)

Then, the following relation holds

$$\hat{y}_i = \tilde{y} = s + \hat{z} - \alpha (\hat{M}_c \otimes I) \hat{y},$$

(56)

and $-\hat{M}_c$ is shown to be Hurwitz because of the assumption of the network graph $G$. From that property, a positive function is defined by

$$V_2 = \hat{y}^T (\hat{P}_e \otimes I) \hat{y},$$

(57)

$$P_c \hat{M}_c + \hat{M}_c^T P_c = I, \quad (P_c = P_c^T > 0).$$

(58)

Similarly to the previous case, there exists a positive definite and symmetric matrix $P_e$ satisfying (58), since $-\hat{M}_c$ is Hurwitz. Then, the time derivative of $V_2$ along its trajectory is evaluated as follows:

$$\dot{V}_2 = -\gamma_1^2 \|\hat{y}\|^2 + 4\gamma_2^2 (\hat{P}_e \otimes I) (s + \hat{z})$$

$$\leq -\frac{\alpha}{2} \|\hat{z}\|^2 + \frac{4}{\alpha} \|\hat{P}_e \otimes I\| \|s\|^2 + \|\hat{z}\|^2,$$

(59)

and it follows that $\hat{y}$ is bounded for bounded $s, \hat{z}$.

From the three stages of stability analysis (the evaluations of $W_0, V_1, V_2$), the next theorem is obtained.

**Theorem 1.** All signals in the partial adaptive control system (7), (9), (10), (11), (34), (48) are bounded for arbitrary bounded design parameters $\hat{\theta}, \hat{P}_e, \hat{P}_c$ and bounded $x_0, \hat{x}_0$, and $v$ is a sub-optimal control input which minimizes the upper bound on the following cost functional $J$.

$$J(t) = \sup_{d, d_1, d_2 \in L_2} \left[ \int_0^t (q + v^T R v) d\tau + W_0(t) \right] - \sum_{i=1}^3 \gamma_i^2 \int_0^t \|d_i\|^2 d\tau.$$

(60)

Also the next inequality holds.

$$\int_0^t (q + v^T R v) d\tau + W_0(t) \leq \frac{3}{\gamma_1^2} \int_0^t \|d_i\|^2 d\tau + W_0(0).$$

(61)

**Proof.** Boundedness of $s, \hat{z}$ and $\hat{y}$ is deduced from the evaluations of $W_0$ (49), $V_1$ (52), $V_2$ (59), respectively. Then, $x$ is shown to be bounded from (56), and boundedness of all signals is derived. From (49), it follows that

$$\int_0^t (q + v^T R v) d\tau + W_0(t) - \sum_{i=1}^3 \gamma_i^2 \int_0^t \|d_i\|^2 d\tau$$

$$\leq \int_0^t \left( v + \frac{1}{2} R^{-1} \hat{B}^T s \right)^T R \left( v + \frac{1}{2} R^{-1} \hat{B}^T s \right) d\tau$$

$$- \int_0^t \left( \gamma_1^2 \|d_1\|^2 + \gamma_2^2 \|d_2\|^2 + \frac{\alpha}{2} \|\hat{P}_e \otimes I\| \|s\|^2 + \|\hat{z}\|^2 \right) d\tau$$

$$+ \gamma^2 \int_0^t \|d_3\|^2 d\tau + W_0(0),$$

(62)

$$W_0(t) \leq -q - v^T R v + \sum_{i=1}^3 \gamma_i^2 \|d_i\|^2,$$

(63)

where the synthesis of $v$ (48) is considered to derive (63). From those two inequalities (62), (63), semi-optimality of $v$ for the cost functional $J(t)$ (60) and the relation (61) are deduced. □
can be applied to the properties 1, it follows that \( \dot{s}(t) \). Proof. Since all signals are assured to be bounded via Theorem 2, all signals are bounded for all followers (\( \hat{\theta} \)). Especially, if \( \dot{\theta} = \theta \). Theorem 2 states that the states of each element of \( p \) are generated from the elements of \( \hat{p} \). A positive function \( W_1 \) is defined by

\[
W_1(t) = \frac{1}{2} s(t)^T s(t) + \frac{1}{2} \left[ \dot{\theta} - \theta \right]^T \Gamma_1^{-1} \left[ \dot{\theta} - \theta \right] + \frac{1}{2} \left( \dot{p}(t) - p \right)^T B \Gamma_1^{-1} \left[ \dot{p}(t) - p \right].
\]

Then, the time derivative of \( W_1 \) along its trajectory is given by

\[
\dot{W}_1(t) \leq -\frac{1}{2} s(t)^T B R^{-1} B^T s(t) \leq 0.
\]

From the four stages of stability analysis (the evaluations of \( \hat{W}_0, W_1, V_1, V_2 \)), the next theorem is obtained.

Theorem 2. All signals in the total adaptive control system composed of the control laws (7), (9), (10), (11), (48), and the tuning laws of \( \hat{b}, \hat{b}, \hat{p}, \hat{p} \), (34), (64), (65), are bounded for bounded \( x_0, x_0 \), and it follows that

\[
\lim_{t \to \infty} s(t) = 0.
\]

Especially, if \( \dot{x}_0 = 0 \) or the information of the leader \( x_0 \) is available for all followers ((\( N_0 = 1 \)) \( I \), \( \dot{x}_0 = 0 \)), then it follows that

\[
\lim_{t \to \infty} \dot{x}(t) = 0,
\]

and the asymptotic consensus tracking is achieved. Otherwise, when \( \dot{x}_0 = 0 \) and the information of \( x_0 \) is not available for all followers ((\( N_0 = 1 \)) \( I \), \( \dot{x}_0 \neq 0 \)), then the next relation holds.

\[
\| \ddot{x} \| \sim \text{const} \times \frac{1}{\beta} \| (N_0 - 1) \otimes I \| \ddot{x}_0 \|
\]

Proof. Since all signals are assured to be bounded via Theorem 1, it follows that \( \dot{s} = \theta \), and thus Barbalat’s lemma can be applied to the properties \( s \in \mathcal{L}^2 \), \( \ddot{s} \in \mathcal{L}^\infty \), and \( (68) \) is deduced. Furthermore, it follows that

\[
\| \ddot{x} \| \sim \text{const} \times \frac{1}{\beta} \| (N_0 - 1) \otimes I \| \ddot{x}_0 \|
\]

(71)

\[
\| \ddot{s} \| \sim \text{const} \times \frac{1}{\alpha} \| \ddot{x} \|
\]

(72)

from (52) and (59), respectively. Then, (71) and (72) together with (56) yields (70). Especially, if ((\( N_0 - 1 \)) \( I \), \( \dot{x}_0 = 0 \)), \( (69) \) is easily derived.

Remark 5. The stability analysis via the positive functions \( W_0, W_1, V_1, V_2 \) only derives ultimate boundedness and convergence of \( \dot{x} \). Ultimate boundedness of the output \( x \) is assured by the assumption of bounded \( x_0 \).

Remark 6. Convergence property of \( \dot{x} \) is determined by \( \| (N_0 - 1) \otimes I \| \ddot{x}_0 \|. \) The term ((\( N_0 - 1 \)) \( I \), \( \ddot{x}_0 \)) shows how the information of the leader is directly available to each agents or not (that is, it means network density). If the information of the leader is directly available to certain agent \( i \), then \( \dot{x}_0 \) is still available to certain agent \( i \). When \( \| (N_0 - 1) \otimes I \| \) goes to 0 (\( n_0 \neq 1 \) for some \( i \)), the correspondent tracking error (\( \dot{x}_i \)) converges to 0.

4. Adaptive \( H_{\infty} \) Consensus Control for Second-Order Model

4.1 Problem Statement

Next, a multi-agent systems composed of the second-order regression models is considered as follows:

\[
\dot{x}_i(t) = \Omega(t) \theta_i + B_i u_i(t), \quad (i = 1, \ldots, N),
\]

where \( x_i, u_i, \theta_i, \Omega_i \), are defined similarly to the previous case, and the form of \( B_i \) is the same as the former one. \( \Omega_i \) is composed of \( x_i, \dot{x}_i \), and is assumed to be bounded for bounded \( x_i, \dot{x}_i \). The communication structure among agents is prescribed by the information network graph \( G \) with the associated adjacency matrix \( A, C \), the Laplacian matrix \( L_a \), \( L_c \), and the matrix \( M_a, M_c \). The control objective is to achieve consensus tracking of the leader-follower type together with velocity tracking such as

\[
x_i \rightarrow x_j, \quad \dot{x}_i \rightarrow \dot{x}_j, \quad (i, j = 1, \ldots, N),
\]

\[
x_i \rightarrow x_0, \quad \dot{x}_i \rightarrow \dot{x}_0, \quad (i = 1, \ldots, N),
\]

based on the notion of inverse optimality and \( H_{\infty} \) control criterion for estimation errors of tuning parameters (see also Remark 1).

4.2 Control Law and Error Equation

An estimation scheme of \( \dot{x}_0 \) (the leader’s information) is constructed via available data from the follower \( i \). A similar estimator was proposed in [5] for the second-order Euler-Lagrange systems:

\[
\hat{z}_i(t) = -\beta \sum_{j=1}^{N} c_{ij} (\dot{z}_j(t) - \dot{z}_i(t))
\]

\[
-\beta c_{i0} (\dot{z}_i(t) - \dot{x}_0(t)) + n_{i0} \dot{x}_0(t),
\]

where \( \dot{z}_i \) is a current estimate of \( \dot{x}_0 \), and is synthesized from the data available to the follower \( i \). The definitions of \( c_{ij} (1 \leq i \leq N, 0 \leq j \leq N), \beta > 0, n_{ij} \) are the same as the previous case. By utilizing the estimate \( \dot{z}_i \), the following control law is employed.

\[
\dot{x}_i(t) = \dot{z}_i(t) - \alpha \sum_{j=0}^{N} a_{ij} (x_j(t) - x_i(t)),
\]

\[
x_i(t) = \dot{x}_i(t) - \dot{x}_0(t),
\]

\[
u_i(t) = \hat{p}_i(t) \left[ -\Omega(t) \dot{\theta}_i(t) + \dot{x}_i(t) \right] + v_i(t)
\]

\[
\bullet \hat{p}_i(t) \nu_i(t) + v_i(t),
\]

(79)
\[ \hat{\varphi}(t) = \Phi(t) = \hat{b}(t) + U_0(t)B[\hat{p}(t) - p_t] + \hat{B}(t)v(t), \]  

where the definitions of \( U_0, u_0, p_t \) are the same as the previous case. Then, the total representations of the multi-agent system are given as follows:

\[ \begin{align*}
\dot{\hat{x}}(t) &= -\beta(M_e \otimes I)\hat{x}(t) + [(N_0 - I) \otimes I]x_0(t), \\
\dot{s}(t) &= \Omega(t)(\theta - \hat{b}(t)) + U_0(t)B[\hat{p}(t) - p_t] + Bv(t),
\end{align*} \]

where \( \hat{x}, s, x, \Omega, \theta, U_0, B, p, N_0, v, \otimes, \) and \( \oplus \) are defined similarly to the previous case. Also, \(-M_e\) is a Hurwitz matrix.

**Remark 7.** The design procedure for the second-order model is similar to the first-order case, but the major differences are that \( \hat{x}_0 \) is a current estimate of \( x_0 \), and that \( si \) is an estimation of the time derivative of the tracking error. Thus, related signal analysis utilizing \( W_0, V_1, V_2, W_1 \) is to be altered in Subsection 4.3.

### 4.3 Adaptive \( H_{\infty} \) Consensus Control for Second-Order Models

For stability analysis of \( s \) and the related terms, a positive function \( W_0(t) \) is defined by

\[ W_0(t) = \frac{1}{2}s(t)^T s(t) + \frac{1}{2}[(\hat{b}(t) - b(t))^T \Gamma_1^{-1} [(\hat{b}(t) - b(t)), \]

where the definition of \( b \) is the same as the previous case. The tuning law of \( \hat{b} \) is chosen as follows:

\[ \dot{\hat{b}}(t) = P \Gamma_1 V(t)^T s(t), \]

where \( V \) and \( P \) are defined similarly to the previous ones. Then, the time derivative of \( W_0 \) along its trajectory is evaluated as follows:

\[ \dot{W}_0(t) \leq s(t)^T [\Omega(t)(\theta - \hat{b}(t)) + U_0(t)B[\hat{p}(t) - p_t] + \dot{B}(t)v(t)]. \]

From the evaluation of \( W_0 \), the next virtual system is introduced.

\[ \begin{align*}
\dot{s} &= f + g_{11}d_1 + g_{12}d_2 + g_{13}d_3 + g_{21}v, \\
f &= 0, \\
g_{11} &= \Omega, \quad g_{12} = U_0, \quad g_{13} = I, \quad g_{21} = \hat{b}, \\
d_1 &= (\theta - \hat{b}), \quad d_2 = B(\hat{b} - p), \\
d_3 &= \text{virtual disturbance},
\end{align*} \]

The virtual system is to be stabilized via a control input \( v \) by utilizing \( H_{\infty} \) criterion, where \( d_1, d_2, d_3 \) are regarded as external disturbances to the process. (Note that \( d_1 \) and \( d_2 \) are estimation errors of the tuning parameters. See also Remark 1.) Then, by repeating the similar discussions to the first-order case, the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution \( V_0 \) are introduced such that

\[ \mathcal{L}_f V_0 + \frac{1}{4} \sum_{i=1}^{\infty} \left\{ \frac{1}{\gamma^2_1} \frac{[\mathcal{L}_g V_0]^2}{\gamma^2_1} - (\mathcal{L}_g V_0)^2 \mathcal{L}_g V_0 \right\} \frac{1}{\gamma^2_1} \]

\[ V_0 = \frac{1}{2} s^T s, \]

and the substitution of the solution \( V_0 \) (93) into the HJI equation (92) yields

\[ \frac{1}{4} s^T \left\{ \Omega \Omega^T \frac{y_1}{\gamma^2_1} + \frac{y_1}{\gamma^2_2} \frac{B B^T}{\gamma^2_2} + \frac{y_1}{\gamma^2_1} \frac{B B^T}{\gamma^2_2} + K \right\} s + q = 0. \]

For \( R \) and \( q \) derived from (94) such as

\[ \begin{align*}
R &= \left( \frac{B B^T}{\gamma_1^2} - \frac{y_1}{\gamma_2^2} \frac{B B^T}{\gamma_2^2} + \frac{y_1}{\gamma_1^2} \frac{B B^T}{\gamma_2^2} + K \right)^{-1}, \\
q &= \frac{1}{4} s^T B K \hat{B}^T s,
\end{align*} \]

where \( K \) is a diagonal positive definite matrix (a design parameter), and for \( v \) deduced from \( R \) such as

\[ \begin{align*}
v &= -\frac{1}{2} R^{-1} (\mathcal{L}_g V_0) s = -\frac{1}{2} R^{-1} \hat{B}^T s, \\
&= -\frac{1}{2} \left( \frac{B B^T}{\gamma_1^2} - \frac{y_1}{\gamma_2^2} \frac{B B^T}{\gamma_2^2} + \frac{y_1}{\gamma_1^2} \frac{B B^T}{\gamma_2^2} + K \right)^{-1} \hat{B}^T s, \end{align*} \]

the time derivative of \( W_0 \) is evaluated as follows:

\[ \dot{W}_0 \leq -q - v^T R v \]

\[ + \left( y_1 + \frac{1}{2} R^{-1} \hat{B}^T s \right) R \left( y_1 + \frac{1}{2} R^{-1} \hat{B}^T s \right) + \frac{y_1^2}{\gamma_2} \|d_1\|^2 - \frac{\gamma_1}{2\gamma_2^2} \|d_1\|^2 \]

\[ + \frac{y_1^2}{\gamma_2} \|d_2\|^2 - \gamma_2 \|d_2\|^2 \]

\[ + \frac{y_1^2}{\gamma_2} \|d_3\|^2 - \frac{\gamma_2}{2\gamma_1^2} \|d_3\|^2. \]

Thus, it follows that \( s \) is bounded for bounded \( \hat{b}, \hat{p} \), and for the stabilizing signal \( v \).

Next, for stability analysis of the estimation error \( \hat{z} \), a positive function \( V_1 \) is introduced such as

\[ V_1 = z^T (P_\varepsilon \otimes I) \hat{z}, \]

\[ P_\varepsilon, M_e + M_e^T P_\varepsilon = I, \ (P_\varepsilon = P_\varepsilon^T > 0), \]

where there exists a positive definite and symmetric matrix \( P_\varepsilon \), satisfying (100) since \(-M_e\) is Hurwitz. Then, the time derivative of \( V_1 \) along its trajectory is evaluated as follows:
\[ V_1 = -\beta\|\xi\|^2 - 2\beta^T(P_c \otimes I)((N_0 - I) \otimes I)\tilde{x}_0 \]
\[ \leq -\frac{\beta}{2}\|\xi\|^2 + \frac{2}{\beta}\|P_c \otimes I\|\|(N_0 - I) \otimes I\|\tilde{x}_0\|^2, \quad (101) \]
and it is shown that \( \tilde{z} \) is bounded for bounded \( \tilde{x}_0 \).

Finally, for stability analysis of the control error \( x_t - x_0 \) and the related terms, \( \tilde{x}_t \) are defined by
\[ \dot{\tilde{x}}_t = \tilde{x}_t - x_0, \quad (102) \]
\[ \hat{x}_t = [\hat{x}_1, \ldots, \hat{x}_n]^T. \quad (103) \]

Then, the following relation holds
\[ \dot{\tilde{x}}_t = s + \tilde{z} - \alpha(M_o \otimes I)\tilde{x}, \quad (104) \]
and \(-M_o\) is shown to be Hurwitz because of the assumption of the network graph \( G \). From that property, a positive function \( V_2 \) is defined by
\[ V_2 = \tilde{x}^T(P_o \otimes I)\tilde{x}, \quad (105) \]
\[ P_oM_o + M_o^T P_o = I, \quad (P_o = P_o^T > 0). \quad (106) \]

Similarly to the previous case, there exists a positive definite and symmetric matrix \( P_o \) satisfying \( (106) \) since \(-M_o\) is Hurwitz. Then, the time derivative of \( V_2 \) along its trajectory is evaluated as follows:
\[ \dot{V}_2 = -\alpha\|\hat{z}\|^2 + 2\tilde{x}^T(P_o \otimes I)(s + \tilde{z}) \]
\[ \leq \frac{\alpha}{2}\|\hat{z}\|^2 + \frac{4}{\alpha}\|P_o \otimes I\|\|(s)^T + \|\hat{z}\|^T\|\|^2. \quad (107) \]

From the three stages of stability analysis (the evaluations of \( V_0, V_1, V_2 \)), the next theorem is obtained.

**Theorem 3.** All signals in the partial adaptive control system (76), (77), (78), (79), (86), (97) are bounded for arbitrary bounded design parameters \( \beta, \hat{P}, \) and bounded \( x_0, \tilde{x}_0, \) and \( v \) is a sub-optimal control input which minimizes the upper bound on the following cost functional \( J \).

\[ J(t) = \sup_{d,t} \left[ \int_0^t (q + v^T R_v)dt + W_0(t) \right] \]
\[ - \sum_{i=1}^{3} \gamma_i^2 \int_0^t \|d_i\|^2 dt \]. \quad (108)

Also the next inequality holds.
\[ \int_0^t (q + v^T R_v)dt + W_0(t) \]
\[ \leq \sum_{i=1}^{3} \gamma_i^2 \int_0^t \|d_i\|^2 dt + W_0(0). \quad (109) \]

**Proof.** The proof is carried out almost similarly to the first-order case.

**Theorem 4.** All signals in the partial adaptive control system composed of the control laws (76), (77), (78), (79), (86), (97), and the tuning laws of \( \hat{b}, \hat{b}, \hat{P}, \) (88), (110), (111), are bounded for bounded \( x_0, \tilde{x}_0, \) and it follows that
\[ \lim_{t \to \infty} \tilde{x}_t = 0. \quad (114) \]

Especially, if \( \hat{x}_0(t) = 0 \) or the information of the leader \( \tilde{x}_0 \) is available for all followers \(((N_0 - I) \otimes I) \tilde{x}_0 = 0 \), then it follows that
\[ \lim_{t \to \infty} \tilde{x}_t = \lim_{t \to \infty} \hat{x}_t(0), \quad (115) \]
and the asymptotic consensus tracking is achieved. Otherwise, when \( \hat{x}_0(t) \neq 0 \) and the information of \( \tilde{x}_0 \) is not available for all followers \(((N_0 - 1) \otimes I) \hat{x}_0 \neq 0 \), then the next relation holds.
\[ \|\tilde{x}\| \sim \text{const.} \left[ \frac{1}{\alpha\beta} \|((N_0 - 1) \otimes I)\tilde{x}_0\| \right]. \quad (116) \]
\[ \|\hat{x}\| \sim \text{const.} \left[ \frac{1}{\beta} \|((N_0 - 1) \otimes I)\hat{x}_0\| \right]. \quad (117) \]

**Proof.** The proof is carried out almost similarly to the first-order case.

Theorem 4 states that the asymptotic consensus tracking is achieved under several conditions, and also shows that the approximate consensus tracking with the ratios of \( 1/(\alpha\beta), 1/\beta > 0 \) is still assured, even if those conditions are not satisfied.

5. Numerical Example

In order to show the effectiveness of the proposed methodology, numerical experiments for the second-order regression models are performed.

A multi-agent system composed of the second-order regression models is considered as follows:
\[ \dot{x}(t) = \theta x(t) + u(t), \quad (i = 1, 2, 3), \]
\[ (x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = -1), \]
where \( x_i \in \mathbb{R}, \) \( u_i \in \mathbb{R}, \) and \( \theta_i \in \mathbb{R} \) is an unknown parameter. Associated with the information network structure (Fig. 1), the adjacency matrix \( A = [a_{ij}] (= C) \) and \( a_{00} (= c_{00}) \) are chosen such that
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
\[ a_{10} = 1, \quad a_{20} = a_{30} = 0. \]
The control objective is to achieve consensus tracking

\[ x_i \rightarrow x_j, \quad \dot{x}_i \rightarrow \dot{x}_j, \]
\[ x_i \rightarrow x_0, \quad \dot{x}_i \rightarrow \dot{x}_0, \]
\[ (i, j = 1, 2, 3), \]

where the virtual leader \( x_0 \) is determined such as

\[ \ddot{x}_0 + 2\dot{x}_0(t) + x_0(t) = \sin t. \]

The design parameters are chosen as follows:

\[ \Gamma = 10I, \quad K = I, \quad \alpha = \beta = 10, \]
\[ \gamma_i = 0.01. \]

As the system parameters, we consider both time-invariant and time-varying cases such that

\[ \theta_1 = 1, \quad \theta_2 = 2, \quad \theta_3 = 3, \quad \text{(time - invariant case)}, \]
\[ \theta_1 = f_\theta(t), \quad \theta_2 = 2f_\theta(t), \quad \theta_3 = 3f_\theta(t), \quad \text{(time - varying case)}, \]

where (Fig. 2)

\[ f_\theta(t) = \begin{cases} 1 & 0 \leq t < 2.5, \quad 5 \leq t < 7.5, \quad \ldots, \\ 0 & 2.5 < t \leq 5, \quad 7.5 < t \leq 10, \quad \ldots. \end{cases} \]

The simulation results of the proposed design scheme (Theorem 3 and Theorem 4) are shown in Fig. 3 (time-invariant case) and Fig. 4 (time-varying case). For comparison, the adaptive control systems which do not contain \( H_\infty \) control scheme (that is, \( \gamma_i = \infty \)), are also shown for both cases in Fig. 5 (time-invariant case) and Fig. 6 (time-varying case).

From those results, it is seen that the proposed \( H_\infty \) adaptive control strategies achieve better tracking convergence properties together with robustness to abrupt changes of the system parameters, and those are owing disturbance attenuation properties of the \( H_\infty \) controllers.

Remark 8. The purpose of the present simulation study is to verify the proposed scheme under directed graphs with the spanning property. In order to compact the situation, the simple network structure with three followers is employed. The case of huge number of agents would be considered in the future application-oriented study.

Furthermore, the selection of \( \gamma_i \) seems similar to the choice of adaptation gains \( \Gamma_i \), that is, it depends on the condition of the numerical simulation tools. In the present case, the numerical computation sometimes became unstable for \( \gamma_i < 0.01 \). Until \( \gamma_i = 0.01 \), the response properties got better for smaller \( \gamma_i \).
6. Concluding Remarks

Design methods of adaptive $H_{\infty}$ consensus control for multi-agent systems denoted by the first-order and the second-order regression forms on directed information network graphs have been presented in this paper. The proposed control schemes are derived as solutions of certain $H_{\infty}$ control problems, where estimation errors of tuning parameters are regarded as external disturbances to the process. It was shown that the resulting control systems are robust to uncertain system parameters and that the desirable consensus is achieved approximately via adaptation schemes and design parameters. Effectiveness of the proposed design schemes was also confirmed by the simulation studies.

The present paper is an extended version of the conference paper [16], and includes detailed proofs of theorems and the explanation of the motive of the research work, and provides several numerical examples. Furthermore, passivity-based approaches were applied to handling the second-order models and the first-order ones in the present methodologies, and those approaches can be extended to the case of general order processes by utilizing iterative design procedures such as backstepping, although there remain several technical issues to be clarified.

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