HORIZONTAL VECTOR FIELDS AND SEIFERT FIBERINGS

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ABSTRACT. This paper gives a classification of vector fields which are nowhere tangent to the fibers of a Seifert fibering.

1. INTRODUCTION

This paper gives a classification of horizontal vector fields on Seifert fiber spaces. Here, horizontal means that the vector field is nowhere tangent to the Seifert fibering. This work was inspired in part by the classification of horizontal foliations on Seifert fiber spaces [Nai94]. Depending on the geometry of the 3-manifold, the condition of having a horizontal vector field can be either more restrictive or less restrictive than having a horizontal foliation.

The key observation is that a horizontal vector field corresponds to a fiber-preserving map from the Seifert fiber space to the unit tangent bundle of the base orbifold. This map is explained in detail in the next section, but first we list out the possibilities based on the geometry of the base orbifold.

First and most interesting are Seifert fiber spaces where the base orbifold is a bad orbifold. These are lens spaces and each lens space $L(p, q)$ supports infinitely many distinct Seifert fiberings.

Theorem 1.1. Consider the lens space $L(p, q)$ with $p \geq 0$.

1. If $p = 0$, then no Seifert fibering on $L(p, q)$ has a horizontal vector field.
2. If $p = 1$ or $p = 2$, then every Seifert fibering on $L(p, q)$ has a horizontal vector field.
3. If $p \geq 3$ and $q \equiv \pm 1 \pmod{p}$, then $L(p, q)$ has infinitely many Seifert fiberings which support horizontal vector fields and infinitely many which do not.
4. If $q \not\equiv \pm 1 \pmod{p}$, then no Seifert fibering on $L(p, q)$ has a horizontal vector field.

We next consider Seifert fiberings where the base orbifold is elliptic; that is, the orbifold is finitely covered by the 2-sphere. Some of these spaces, such as the 3-sphere, are also lens spaces and are handled by the previous theorem.

Theorem 1.2. Suppose $M$ is a Seifert fiber space with elliptic base orbifold and $M$ is not a lens space. Then $M$ has a horizontal vector field if and only if $M$ is homeomorphic to the unit tangent bundle of the base orbifold.
Next consider Seifert fiber spaces over parabolic orbifolds. Such an orbifold is covered by the Euclidean plane and corresponds to a wallpaper group with only rotational symmetry.

**Theorem 1.3.** Suppose $M$ is a Seifert fiber space with parabolic base orbifold. Then $M$ has a horizontal vector field if and only if either

1. $M$ is homeomorphic to the unit tangent bundle of the base orbifold, or
2. the base orbifold is a surface (either the 2-torus or the Klein bottle).

Section 5 explains both of the above cases in detail. Note that the two cases overlap when the base orbifold is a surface.

Finally, the most general case is for hyperbolic orbifolds.

**Theorem 1.4.** Suppose $M$ is a Seifert fiber space with hyperbolic base orbifold. Then the following are equivalent:

1. $M$ has a horizontal vector field,
2. $M$ finitely covers the unit tangent bundle of the base orbifold,
3. $M$ supports an Anosov flow.

Theorem 1.4 was part of the original motivation for exploring the properties of these vector fields. In joint work with Mario Shannon and Rafael Potrie, we use this theorem to analyse partially hyperbolic dynamical systems on Seifert fiber spaces [HPS17]. The equivalence $(2) \iff (3)$ in theorem 1.4 was proved by Barbot, extending a result of Ghys on circle bundles [Ghy84, Bar96].

For an introduction to Seifert fiber spaces and 2-orbifolds, see [Sco83]. For lens spaces and the uniqueness or non-uniqueness of Seifert fiberings on a manifold, see [IN83, Hat07]. For a detailed treatment of the geometry of 2-orbifolds, see [Cho12]. In this paper, we consider orbifolds only in dimension two and only with cone points (also called elliptic points), and not with corner reflectors or silvered edges. In places, we use the orbifold notation of Thurston and Conley. For example, the 237 orbifold is a sphere with cone points of order 2, 3, and 7 added, and the 22× orbifold is the sphere with two cones points of order two and one cross-cap added.

2. General properties

We first consider vector fields on standard fibered tori. Let $\mathbb{D}^2$ be the open unit disk and, for a pair of coprime integers $(a, b)$ with $a > 0$, define $R_{ab}: \mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{D}^2 \times \mathbb{S}^1$ as rotation by an angle $2\pi b/a$ in the first coordinate and rotation by $-2\pi/a$ in the second coordinate. This defines a finite quotient

$$U_{ab} := \mathbb{D}^2 \times \mathbb{S}^1 / R_{ab}.$$  

Consider the projection $p: \mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{D}^2$. Its fibers are circles which quotient down to circles on $U_{ab}$. The solid torus $U_{ab}$ along with its fibering by circles is called a standard fibered torus. The fibering may also be viewed as the
fibers of a map from \( U_{ab} \) to \( \mathbb{D}^2/r_a \) where \( r_a \) is rotation by the angle \( 2\pi/a \) on the unit disc. If \( a > 1 \), then \( \mathbb{D}^2/r_a \) has the structure of an orbifold and not a smooth surface.

Let \( TU_{ab} \) denote the tangent bundle of \( U_{ab} \). A vector field \( u_0 \) on \( U_{ab} \) may be considered as a function \( u_0 : U_{ab} \rightarrow TU_{ab} \). This vector field may be lifted to a vector field \( u_1 \) on \( \mathbb{D}^2 \times S^1 \) and as \( R_{ab} \) is a deck transformation, the derivative \( DR_{ab} \) satisfies

\[
u_1 \circ R_{ab} = DR_{ab} \circ u_1.
\]

Define \( u_2 : \mathbb{D}^2 \times S^1 \rightarrow T\mathbb{D}^2 \) by \( u_2 = Dp \circ u_1 \). Then \( Dp \circ DR_{ab} = Dr_a^b \circ Dp \) implies \( u_2 \circ R_{ab} = Dr_a^b \circ u_2 \). Thus, \( u_2 \) quotients to a map \( u : U_{ab} \rightarrow TU/Dr_a \) where \( TU/Dr_a \) is the tangent bundle of the orbifold \( \mathbb{D}^2/r_a \). If \( u_0 \) is nowhere tangent to the fibers of \( U_{ab} \), then \( u \) is nowhere zero and may be rescaled to define a map from \( U_{ab} \) to the unit tangent bundle of \( \mathbb{D}^2/r_a \). Even though \( \mathbb{D}^2/r_a \) is not a surface, its unit tangent bundle is a smooth 3-manifold. All together, this shows the following result.

**Lemma 2.1.** Any horizontal vector field on a standard fibered torus \( U_{ab} \) induces a fiber-preserving map from \( U_{ab} \) to the unit tangent bundle of \( \mathbb{D}^2/r_a \).

A Seifert fibering of a closed 3-manifold \( M \) is given by the fibers of a map from \( M \) to a 2-orbifold \( \Sigma \). The unit tangent bundle \( UT\Sigma \) of the orbifold is a 3-manifold which naturally carries the structure of a Seifert fiber space. The local arguments in lemma 2.1 may then be adapted to prove the following global result.

**Lemma 2.2.** Suppose \( M \) is a Seifert fiber space and \( u_0 \) is a horizontal vector field. Let \( \Sigma \) be the 2-orbifold associated to the Seifert fibering and \( UT\Sigma \) its unit tangent bundle. Then \( u_0 \) induces a fiber-preserving map from \( M \) to \( UT\Sigma \).

We may also show that the converse holds.

**Proposition 2.3.** A Seifert fibering \( M \rightarrow \Sigma \) has a horizontal vector field if and only if there is a continuous map \( u : M \rightarrow UT\Sigma \) such that

\[
\begin{array}{ccc}
M & \xrightarrow{u} & UT\Sigma \\
\downarrow & & \downarrow \\
\Sigma & \xrightarrow{id} & \Sigma
\end{array}
\]

commutes.

**Proof.** One direction is given by lemma 2.2. For the converse direction, first note that for any Seifert fibering \( M \rightarrow \Sigma \), one may construct a plane field transverse to the fibers, say by using a partition of unity. Then for any fiber-preserving map \( u : M \rightarrow UT\Sigma \), one may define a horizontal vector field \( v : M \rightarrow TM \) by setting \( v(p) \) to be the unique vector in the plane at \( p \) which projects down to \( u(p) \).

Fiber-preserving maps must be of a specific form.
Proposition 2.4. Suppose $M_1$ and $M_2$ are Seifert fiber spaces over the same base orbifold $\Sigma$ and $u: M_1 \to M_2$ is a fiber-preserving map such that

\[
\begin{array}{ccc}
M_1 & \xrightarrow{u} & M_2 \\
\downarrow & & \downarrow \\
\Sigma & \xrightarrow{\text{id}} & \Sigma
\end{array}
\]

commutes. Then either

(1) $u$ is homotopic to a composition of the form $M_1 \to \Sigma \to M_2$, or
(2) $u$ is homotopic to a covering $M_1 \to M_2$.

Proof. Equip each of $M_1$ and $M_2$ with a metric so that every regular fiber has length exactly one. (The exceptional fibers will then have lengths $1/\alpha_i$ for integers $\alpha_i > 1$.) Consider a point $x \in M_1$ and let $L_1$ be the fiber through $x$ and $L_2$ the fiber through $u(x)$. Define length-preserving covering maps $\pi_1: \mathbb{R} \to L_1$ and $\pi_2: \mathbb{R} \to L_2$ and choose a lift $\tilde{u}: \mathbb{R} \to \mathbb{R}$ such that $\pi_2 \circ \tilde{u} = u \circ \pi_1$. Define $\hat{u}: \mathbb{R} \to \mathbb{R}$ by

\[
\hat{u}(t) = \int_{t-1/2}^{t+1/2} \tilde{u}(s) \, ds.
\]

so that

\[
\frac{d \hat{u}}{dt} = \tilde{u}\left(t + \frac{1}{2}\right) - \tilde{u}\left(t - \frac{1}{2}\right) = \deg(u).
\]

There is then a unique function $h: L_1 \to L_2$ such that $h \circ \pi_1 = \pi_2 \circ \hat{u}$ and $h$ is independent of the choice of lift $\hat{u}$. Moreover, when defined fiber by fiber on all of $M_1$, this gives a continuous map $h: M_1 \to M_2$ with the desired properties. □

When is the case $M \to \Sigma \to UT\Sigma$ possible? This requires a horizontal section $\Sigma \to UT\Sigma$. In a neighbourhood of a cone point, this is impossible, so $\Sigma$ has no cone points and is a surface admitting a non-zero vector field. By the Poincaré–Hopf Theorem, $\Sigma$ is either the torus, $\mathbb{T}^2$, or the Klein bottle, $K$. This situation is discussed in further detail in section 5.

The interval $[t - 1/2, t + 1/2]$ was used in the above proof in order to handle the case where the fiber direction is not orientable. If $M_2$ is orientable, then in case (2) of the theorem, the covering implies that $M_1$ is orientable as well. Note that the unit tangent bundle of an orbifold always has a well-defined orientation, even if the orbifold itself does not.

For a Seifert fibering of a closed oriented 3-manifold, we adopt the conventions given in [JN83] and write the Seifert invariant as

\[(2.5) \quad (g; (a_0, \beta_0), (a_1, \beta_1), \ldots, (a_k, \beta_k)).\]

We also write

\[M(g; (a_0, \beta_0), (a_1, \beta_1), \ldots, (a_k, \beta_k))\]

to denote the 3-manifold equipped with this fibering. The integer $g$ is the genus of the topological surface $\Sigma_0$ from which the orbifold was constructed (with $g < 0$.
if \( \Sigma_0 \) is not orientable). The condition \( \gcd(\alpha_i, \beta_i) = 1 \) holds for all pairs as otherwise the Seifert fibering is not well-defined. In general, \( \beta_i / \alpha_i \) is allowed to be any rational number. Two invariants define the same fibering if and only if one can be transformed into the other by changes of the following form:

1. adding an integer to one ratio \( \beta_i / \alpha_i \) and subtracting it from another,
2. re-ordering the pairs \((\alpha_i, \beta_i)\), and
3. inserting or removing pairs of the form \((1, 0)\).

The Euler number of a fibering is defined as

\[
e(M \to \Sigma) = - \sum_{i=0}^{k} \beta_i / \alpha_i.
\]

Now consider a covering \( M_1 \to M_2 \) of degree \( d \) as in case (2) of proposition 2.4 and where both manifolds are oriented. If \( M_1 \) has the invariant given by (2.5), then the \( d \)-fold covering of fibers implies that \( M_2 \) has invariant

\[(g; (\alpha_0, d \beta_0), (\alpha_1, d \beta_1), \ldots, (\alpha_k, d \beta_k)).\]

and consequently that \( d \cdot e(M_1 \to \Sigma) = e(M_2 \to \Sigma) \). See §3 of [JN83] for details. For such a covering to exist, each \( \alpha_i \) must be coprime to \( d \) in order for the \( d \)-fold covering on each regular fiber to extend continuously to a \( d \)-fold covering on each exceptional fiber.

Consider the specific case where \( M_2 \) is the unit tangent bundle \( UT \Sigma \). The orientation of \( UT \Sigma \) is determined by the condition that \( e(UT \Sigma \to \Sigma) \) equals the Euler characteristic \( \chi(\Sigma) \) of the orbifold. Further, the Seifert fibering \( UT \Sigma \to \Sigma \) has invariant

\[(g; (1, k - \chi(\Sigma_0)), (\alpha_1, -1), \ldots, (\alpha_k, -1))\]

where \( \Sigma_0 \) is the underlying topological surface of \( \Sigma \). See §5 of [EHN81] for details. The uniqueness conditions on Seifert invariants then give us a straightforward test for the existence of the covering.

**Proposition 2.6.** A Seifert fiber space \( M \) with invariant

\[(g; (\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))\]

covers the unit tangent bundle of the base orbifold \( \Sigma \), taking fibers to fibers, if and only if there is a non-zero integer \( d \) such that \( d \cdot e(M \to \Sigma) = \chi(\Sigma) \) and

\[d \beta_i / \alpha_i \equiv -1 / \alpha_i \mod \mathbb{Z}\]

for all \( i = 0, \ldots, k \).

3. Lens spaces

We now consider Seifert fiberings on lens spaces. We state several properties here and refer the reader to [JN83] for further details. For coprime integers \( p \) and \( q \), the lens space \( L(p, q) \) is defined by quotienting

\[S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \]
by the $\mathbb{Z}/p$-action generated by
\[(z_1, z_2) \mapsto (e^{2\pi i/p} \cdot z_1, e^{2\pi i q/p} \cdot z_2).\]

As in [JN83], we consider $L(1, 1) = \mathbb{S}^3$ and $L(0, 1) = \mathbb{S}^2 \times \mathbb{S}^1$ to be lens spaces. We write $M_1 \cong M_2$ if there is an orientation-preserving diffeomorphism between oriented manifolds $M_1$ and $M_2$. Then

1. $L(p, q) \cong L(-p, -q)$,
2. $L(p, q) \cong -L(p, -q)$, and
3. $L(p, q) \cong L(p, q')$ if and only if $q' \equiv q^{\pm 1} \mod p$.

When viewed as the unit tangent bundle of the real projective plane, the lens space $L(4, 1)$ has a natural fibering by circles. Apart from this single special case, all other Seifert fiberings on lens spaces may be produced by gluing together two standard fibered tori and their Seifert invariants may be written in the form $(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$. The lens space is determined from the fibering by the following.

**Theorem 3.1** (Theorem 4.4 of [JN83]). $L(p, q) \cong M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$ if
\[p = \det\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\beta_1 & \beta_2 \end{pmatrix} \quad \text{and} \quad q = \det\begin{pmatrix} \alpha_1 & \alpha'_2 \\ -\beta_1 & \beta'_2 \end{pmatrix} \quad \text{where} \quad \det\begin{pmatrix} \alpha_2 & \alpha'_2 \\ \beta_2 & \beta'_2 \end{pmatrix} = 1.\]

From this, we prove a number of results which together imply theorem 3.1.

**Lemma 3.2.** If $\Sigma$ is a 2-sphere with cone points of order $\alpha_1$ and $\alpha_2$ added, then $UT\Sigma \cong L(p, 1)$ where $p = \alpha_1 + \alpha_2$.

**Proof.** The Seifert invariant of $UT\Sigma$ is $(0; (\alpha_1, -1), (\alpha_2, -1))$. Define $p = \alpha_1 + \alpha_2$. Then theorem 3.1 with $\alpha'_2 = 1 - \alpha_2$ and $\beta'_2 = 1$ implies that $UT\Sigma \cong L(-p, -1 + p) \cong L(-p, -1) \cong L(p, 1)$. \[\Box\]

As is evident from the proof, lemma 3.2 also applies to the 2-sphere $(\alpha_1 = 1 = \alpha_2)$ and to teardrop orbifolds $(\alpha_1 = 1 < \alpha_2)$.

**Corollary 3.3.** No Seifert fibering of $L(0, 1)$ has a horizontal vector field.

**Proof.** $L(0, 1)$ cannot finitely cover $UT\Sigma \cong L(p, 1)$ as they have different universal covers. \[\Box\]

For a Seifert invariant of the form $(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$, define values $d = -\frac{1}{p}(\alpha_1 + \alpha_2)$ and $b = \frac{1}{p}(\beta_2 - \beta_1)$.

**Lemma 3.4.** A Seifert fibering has a horizontal vector field if and only if the associated values $d$ and $b$ are integers.

**Proof.** Note that $d$ and $b$ satisfy
\[d \cdot e(M \to \Sigma) = \chi(\Sigma), \quad \frac{d \beta_1}{\alpha_1} = -\frac{1}{\alpha_1} + b, \quad \text{and} \quad \frac{d \beta_2}{\alpha_2} = -\frac{1}{\alpha_2} - b.\]
This lemma then follows from propositions 2.3 and 2.6.

**Lemma 3.5.** A Seifert fibering has a horizontal vector field if and only if \( q \equiv -1 \) mod \( p \) where \( p \) and \( q \) are given by theorem 3.1.

**Proof.** We may rewrite theorem 3.1 as

\[
\begin{pmatrix}
\alpha_1 & -\beta_1 \\
\alpha_2 & \beta_2
\end{pmatrix}
\begin{pmatrix}
\beta'_1 \\
\alpha'_1
\end{pmatrix}
= \begin{pmatrix} 1 \\ q \end{pmatrix}
\]

and \( p \) is the determinant of the leftmost of these matrices. Inverting this matrix yields

\[
\begin{pmatrix}
\beta'_1 \\
\alpha'_1
\end{pmatrix}
= \frac{1}{p}
\begin{pmatrix}
\beta_2 & \beta_1 \\
-\alpha_2 & \alpha_1
\end{pmatrix}
\begin{pmatrix} 1 \\ q \end{pmatrix}.
\]

Note for an integer \( j \) that \( q = -1 + jp \) holds if and only if

\[
\alpha'_1 = d + j \alpha_1 \quad \text{and} \quad \beta'_1 = b + j \beta_1.
\]

Hence, if \( q \equiv -1 \) mod \( p \), then \( d \) and \( b \) are integers. Conversely, if \( d \) and \( b \) are integers, then any integer solution to \( \alpha_1 \beta'_1 - \beta_1 \alpha'_1 = 1 \) must have \( \alpha'_1 \) and \( \beta'_1 \) as above for some integer \( j \).

In the proof of theorem 3.1 given in [JN83], the Seifert invariant is used to define a matrix

\[
\begin{pmatrix}
-q & p \\
r & s
\end{pmatrix}
\]

having integer entries and determinant \(-1\) and which represents the linear map that identifies the boundaries of the two solid tori that make up the lens space. This fact, taken with lemma 3.5, shows that the existence of a horizontal vector field depends not on the specific fiberings of the two standard fibered tori, but only on the gluing map between them.

For instance, consider the matrices

\[
\begin{pmatrix} 1 & -3 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}.
\]

The gluing map associated to either matrix yields the lens space \( L(3, 1) \). For the gluing map associated to the matrix on the left, any choice of a standard fibering on one of the solid tori induces a standard fibering on the other solid torus (so long as the slope is not zero). Lemma 3.5 then implies that the fibering over the whole lens space has a horizontal vector field. Hence, there are infinitely many Seifert fiberings on \( L(3, 1) \) with a horizontal vector field.

For the matrix on the right, lemma 3.5 implies that no fibering has a horizontal vector field. The obstruction is basically that such a vector field would have to rotate clockwise around fibers in one solid torus and counter-clockwise around fibers in the other. Hence, there are infinitely many Seifert fiberings on \( L(3, 1) \) without a horizontal vector field.

Matrices like the above can be constructed for any lens space \( L(p, \pm 1) \) with \( p \geq 3 \). If \( |p| = 1 \) or \( |p| = 2 \), then any valid matrix for the gluing map satisfies \( q \equiv -1 \) mod \( p \).
−1 mod $p$. Therefore, every Seifert fibering on $L(1, 1)$ or $L(2, 1)$ has a horizontal vector field. All together, the results of this section prove theorem 1.1.

4. Elliptic orbifolds

We now consider a Seifert fiber space $M$ over an elliptic orbifold $\Sigma$. The orbifold is finitely covered by $S^2$ and pulling back the fibering by the orbifold covering yields a Seifert fiber space $\hat{M}$ over $S^2$. If $M$ has a horizontal vector field, then the map $M \to UT\Sigma$ given by proposition 2.3 leads to the commutative diagram

\[
\begin{array}{ccc}
\hat{M} & \longrightarrow & M \\
\downarrow & & \downarrow \\
UTS^2 & \longrightarrow & UT\Sigma \\
\downarrow & & \downarrow \\
S^2 & \longrightarrow & \Sigma \\
\end{array}
\]

where the top two vertical arrows represent maps having same degree $d$. Up to a change of orientation, we may assume that $d$ is positive.

To prove theorem 1.2 by contradiction, we assume that $M$ is not a lens space and $d \neq 1$. The 3-sphere double covers $UTS^2$ and so $d = 2$. The results in section 2 then imply that $\Sigma$ has no cone points of even order. This rules out the $22p$ orbifold for any $p \geq 2$ and the $23q$ orbifold for $3 \leq q \leq 5$. The assumption that $M$ is not a lens space rules out the case that $\Sigma$ is the sphere with at most two cone points added.

For an elliptic orbifold, the only remaining possibility is that $\Sigma$ is the real projective plane with at most one cone point added. The Seifert invariant is then of the form $(-1; (\alpha, \beta))$ and proposition 2.3 implies that

\[-\frac{d\beta}{\alpha} = d \cdot e(M \to \Sigma) = \chi(\Sigma) - \frac{1}{\alpha},\]

which is impossible if $d = 2$. This completes the proof of theorem 1.2.

Note that it is possible for a lens space to double cover the unit tangent bundle of an elliptic orbifold. The examples of this are all of the form $M(0; (\alpha, \beta), (\alpha, \beta))$ where $\alpha \geq 1$ and $\beta = -\frac{1}{2}(\alpha + 1)$.

5. Parabolic orbifolds

We now consider fiberings $M \to \Sigma$ where the base orbifold is parabolic; that is, $\chi(\Sigma) = 0$. First consider case (2) of proposition 2.3 where there is a $d$-fold covering $M \to UT\Sigma$ and $d \cdot e(M \to \Sigma) = \chi(\Sigma)$ implies $e(M \to \Sigma) = 0$. There are exactly seven Seifert fiberings of this form. (See §2.1 of [Hat07] or §8.2 of [Orl72].) These are all unit tangent bundles of orbifolds and we list out the seven cases below. A self-covering map of degree $d > 1$ is possible in all of the cases.
Every parabolic orbifold is associated to a wallpaper group consisting of affine isometries acting on $\mathbb{R}^2$. Consider the unit tangent bundle $UT\mathbb{R}^2$ of the Euclidean plane. An element of $UT\mathbb{R}^2 \cong \mathbb{R}^2 \times S^1$ may be represented by a pair $(x, \theta)$ where $x \in \mathbb{R}^2$ and $\theta$ is an angle. Then $UT\mathbb{R}^2$ has self-covering maps of the form $(x, \theta) \mapsto (x, d\theta)$ for $d \geq 1$.

If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving isometry, then its derivative is of the form $(x, \theta) \mapsto (A(x), \theta + \theta_0)$ for some constant $\theta_0$. In some cases, this will commute with the $d$-fold self-covering; for instance, when $A$ is a rotation by an angle $\theta_0 = \frac{2\pi}{k}$ and $d \equiv 1 \mod k$. Using this, one may determine the values of $d$ for which the self-covering of $UT\mathbb{R}^2$ descends to a self-covering of $UT\Sigma$. The possible coverings may also be established directly from the Seifert invariants and the results in section 2.

The possible coverings are as follows:

1. If $\Sigma = \mathbb{T}^2$, then $UT\Sigma \cong -UT\Sigma$ and has a $d$-fold self-covering for all $d \neq 0$.
2. If $\Sigma = K$ is the Klein bottle, then $UT\Sigma \cong -UT\Sigma$ and has a $d$-fold self-covering for all $d \neq 0$.
3. If $\Sigma$ is the 236 orbifold, then $UT\Sigma$ $d$-fold covers $UT\Sigma$ if $d \equiv +1 \mod 6$, and $UT\Sigma$ $d$-fold covers $-UT\Sigma$ if $d \equiv -1 \mod 6$.
4. If $\Sigma$ is the 244 orbifold, then $UT\Sigma$ $d$-fold covers $UT\Sigma$ if $d \equiv +1 \mod 4$, and $UT\Sigma$ $d$-fold covers $-UT\Sigma$ if $d \equiv -1 \mod 4$.
5. If $\Sigma$ is the 333 orbifold, then $UT\Sigma$ $d$-fold covers $UT\Sigma$ if $d \equiv +1 \mod 3$, and $UT\Sigma$ $d$-fold covers $-UT\Sigma$ if $d \equiv -1 \mod 3$.
6. If $\Sigma$ is the 2222 orbifold, then $UT\Sigma \cong -UT\Sigma$ and $d$-fold covers itself if $d \equiv 1 \mod 2$.
7. If $\Sigma$ is the $2 \times$ orbifold, then $UT\Sigma \cong -UT\Sigma$ and $d$-fold covers itself if $d \equiv 1 \mod 2$.

We now consider case (1) of proposition 2.4. Here, $d = 0$ and the map factors as a composition $M \to \Sigma \to UT\Sigma$. As noted in section 2, this is only possible if $\Sigma = \mathbb{T}^2$ or $K$. Conversely, for any fibering over $\mathbb{T}^2$ or $K$, we may choose a non-zero vector field $\Sigma \to UT\Sigma$ and lift it to a horizontal vector field on the Seifert fiber space.

For each of $\mathbb{T}^2$ and $K$, there are infinitely many oriented Seifert fiberings, one for each integer $e(M \to \Sigma)$. These manifolds have Euclidean geometry if $e(M \to \Sigma) = 0$ and Nil geometry otherwise.

There are exactly four non-orientable 3-manifolds with Euclidean geometry. (See §8.2 of [Orl72].) Each of these has a Seifert fibering over $K$ and therefore has a horizontal vector field. Two of the manifolds also have Seifert fiberings over $\mathbb{T}^2$ which therefore also have horizontal vector fields.
Together, the results in this section prove theorem 1.3.

6. Hyperbolic Orbifolds

The final class of orbifolds to consider are those with hyperbolic geometry. In contrast to the other cases, here it is relatively easy to find coverings $M \rightarrow UT\Sigma$ of degree $d > 1$. If $\Sigma$ is a surface of genus $g \geq 2$, then there is a cover of degree $d$ for every factor $d$ of $\chi(g) = 2 - 2g$. Even for orbifolds without handles, non-trivial covers are possible. For instance,

$$M(0; (1, -1), (5, 2), (5, 2), (5, 2))$$

double covers the unit tangent bundle of the 555 orbifold. For some choices of cone points, there may be no non-trivial covers. For instance, if $\Sigma$ is the 237 orbifold, one may show that only $d = \pm 1$ is possible.

If $\Sigma$ is a hyperbolic orbifold, the geodesic flow on $UT\Sigma$ is an Anosov flow and the flow is generated by a vector field which is horizontal. This flow lifts to any finite cover and is still Anosov on the cover. Ghys and Barbot showed that (up to orbit equivalency) every Anosov flow on a Seifert fiber space is of this form \[\text{Ghy84, Bar96}\]. This establishes the equivalence (2) $\iff$ (3) in theorem 1.4. The equivalence (1) $\iff$ (2) is a re-statement of the results in section 2.

7. Homotopies of Horizontal Vector Fields

For a given Seifert fibering $M \rightarrow \Sigma$, consider the space of all horizontal vector fields. What are the connected components of this space? That is, when can one vector field be deformed into another along a path of horizontal vector fields? For simplicity, we only consider this in the case where the base orbifold $\Sigma$ is oriented. By proposition 2.3, the question reduces to studying homotopy classes of maps of the form

$$M \xrightarrow{u} UT\Sigma \xrightarrow{\text{id}} \Sigma.$$

Consider two such maps $u, v : M \rightarrow UT\Sigma$ and suppose they have the same degree $d$. (Otherwise, they are clearly not homotopic.) By the averaging method used in the proof of proposition 2.4, we may assume there are metrics on the fibers of $M$ and $UT\Sigma$ and that $u$ and $v$ have the same constant speed on all fibers. As $\Sigma$ is oriented, the fibers of $UT\Sigma$ are oriented. Using the metric, there is then a well-defined difference map $u - v : M \rightarrow S^1$ which is constant on fibers and therefore quotients to a map $g : \Sigma \rightarrow S^1$. The maps $u$ and $v$ are homotopic if and only if $g$ is homotopic to a constant map. Since we are only concerned with the homotopy class of $g$, the smooth orbifold structure of $\Sigma$ is unimportant and we may consider $g$ as a map from $\Sigma_0$ to $S^1$ where $\Sigma_0$ is the underlying topological surface.
There are canonical isomorphisms identifying

1. the homotopy classes of maps from $\Sigma_0$ to $\mathbb{S}^1$,
2. homomorphisms from $\pi_1(\Sigma_0)$ to $\mathbb{Z} = \pi_1(\mathbb{S}^1)$,
3. homomorphisms from the first homology group $H_1(\Sigma_0)$ to $\mathbb{Z}$, and
4. elements of the first cohomology group $H^1(\Sigma_0, \mathbb{Z})$.

For $\Sigma_0 = \mathbb{S}^2$, this follows because all of the above are trivial. For other oriented surfaces, it follows because both $\Sigma_0$ and $\mathbb{S}^1$ are $K(\pi, 1)$. (See also Theorem 4.57 and the discussion on page 198 of [Hat02].)

Each connected component of the space of horizontal vector fields is therefore uniquely determined by the degree $d$ (which is also the number of turns that the vector field makes around any fiber) and an element of $H^1(\Sigma_0, \mathbb{Z})$. The above reasoning then implies the following:

**Theorem 7.1.** For a Seifert fibering $M \to \Sigma$, there is a canonical bijection between the connected components of the space of horizontal vector fields and pairs of the form $(d, \varphi) \in \mathbb{Z} \times H^1(\Sigma_0, \mathbb{Z})$ where $d \cdot e(M \to \Sigma) = \chi(\Sigma)$.

Except for parabolic orbifolds, the degree $d$ is uniquely determined by the fibering. In particular, for a bad, elliptic, or hyperbolic orbifold with $\Sigma_0 = \mathbb{S}^2$, theorem 7.1 implies that the Seifert fibering uniquely determines the horizontal vector field up to homotopy. Conversely if $H^1(\Sigma_0, \mathbb{Z})$ is non-trivial, then many homotopy classes are possible. For instance, consider the 3-torus $\mathbb{T}^3$ with fibers tangent to the vertical $z$-direction. Then $UT \mathbb{T}^2$ may be identified with $\mathbb{T}^3$ and the classes of horizontal vector fields correspond to maps of the form

$$\mathbb{T}^3 \to \mathbb{T}^3, \quad (x, y, z) \mapsto (x, y, ax + by + dz)$$

for all $(a, b, d) \in \mathbb{Z}^3$.

### 8. Manifolds with Boundary

In this final section, we consider Seifert fiberings on manifolds with boundary. First, consider the case where there are boundary conditions for the vector field. We could require that the vector field be either tangent or transverse to the boundary and the resulting restrictions on $M$ will be the same.

**Theorem 8.1.** For a Seifert fibering $M \to \Sigma$ on a manifold with boundary, the following are equivalent:

1. $M$ has a horizontal vector field everywhere tangent to the boundary,
2. $M$ has a horizontal vector field everywhere transverse to the boundary,
3. the base orbifold $\Sigma$ is either the annulus or the Möbius band.

**Proof:** We show (2) ⇔ (3). The proof of (1) ⇔ (3) is similar and left to the reader. Note that proposition 2.4 holds for manifolds with boundary using the same proof. Suppose $M$ supports a horizontal vector field which is everywhere transverse to $\partial M$ and consider the associated map $u : M \to UT \Sigma$. For a point $x \in \partial \Sigma$, ...
the restriction of $u$ to the fiber over $x$ cannot be surjective as its range omits the two unit vectors at $x$ which are tangent to $\partial \Sigma$. This implies that $u$ has degree zero and Proposition 2.4 shows that $u$ is homotopic to a composition $M \to \Sigma \to U T \Sigma$. In the proof of the proposition, the vector field $\Sigma \to U T \Sigma$ is defined by averaging and one can verify that is it transverse to $\partial \Sigma$. Further, the same argument as given in Section 2 shows that $\Sigma$ has no cone points and is therefore a surface with boundary. The Poincaré–Hopf theorem then implies that $\Sigma$ is either the annulus or Möbius band.

Conversely, for any circle bundle over an annulus or Möbius band, we may compose the projection $M \to \Sigma$ with a vector field $\Sigma \to U T \Sigma$ transverse to $\partial \Sigma$ and produce a horizontal vector field on $M$ transverse to $\partial M$.

For the remainder of the section, we consider horizontal vector fields with no boundary conditions.

**Theorem 8.2.** Let $M \to \Sigma$ be a Seifert fibering on a manifold with boundary. Then $M$ has a horizontal vector field if and only if one or both of the following hold:

1. the base orbifold $\Sigma$ is a surface with boundary, or
2. $M$ finitely covers the unit tangent bundle of $\Sigma$.

To prove this, we use the following.

**Lemma 8.3.** An orbifold with boundary supports a non-zero vector field if and only if it has no cone points (i.e., it is a surface with boundary).

**Proof.** As noted earlier, it is impossible to define a section $u : \Sigma \to U T \Sigma$ in the neighborhood of a cone point and so the existence of such a $u$ implies that $\Sigma$ is a surface with boundary. Conversely, if we have no boundary conditions, we may construct a non-zero vector field on any surface with boundary. For instance, we may take a generic vector field on a closed surface. This is zero at finitely many points, and we can excise one or more topological disks to remove all of these points.

Using this lemma and adapting the results of Section 2, it is straightforward to prove Theorem 8.2.

For a manifold with boundary, write the Seifert invariant as

$$(g, n; (\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$$

where $n > 0$ is the number of boundary components. We can reorder the pairs, add or remove pairs of the form $(1, 0)$, and (specifically for $\partial M \neq \emptyset$) add an integer to any of the ratios $\beta_i / \alpha_i$ without changing the fibering. Because of this, the Euler number $e(M \to \Sigma)$ is not defined. See §2 of [Hat07] for more details.

By adapting arguments in Section 2, one can prove the following analogue of Proposition 2.6. Note that the condition $d \cdot e(M \to \Sigma) = \chi(\Sigma)$ has been removed.

**Proposition 8.4.** On a manifold with boundary, a Seifert fibering with invariant

$$(g, n; (\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$$
covers the unit tangent bundle of the base orbifold, taking fibers to fibers, if and only if there is a non-zero integer $d$ such that

$$d \beta_i / \alpha_i \equiv -1 / \alpha_i \mod Z$$

for all $i = 0, \ldots, k$.

Even with the Euler number condition removed, it may not be possible to find a horizontal vector field. Consider, for instance $(g, n; (3, 1), (3, 2))$. No integer $d$ satisfies $\frac{1}{3}d \equiv \frac{2}{3}d \equiv -1 \mod Z$ and so no horizontal vector field exists.

If a Seifert fibering satisfies proposition [8.4] for some integer $d$, then it also satisfies the proposition when $d$ is replaced by $d + m\ell$ where $m$ is any integer and $\ell$ is the least common multiple of $\{\alpha_0, \ldots, \alpha_k\}$. Using this and adapting the results in section[7] one may show that if a Seifert fibering on a manifold with boundary has one horizontal vector field, then it has infinitely many homotopy classes of such vector fields.

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