Self-Consistent Conversion of a Viscous Fluid to Particles and Heavy-Ion Applications

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By Zachary J. Wolff

Entitled
Self-Consistent Conversion of a Viscous Fluid to Particles and Heavy-Ion Physics Applications

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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ABSTRACT

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The most widely used theoretical framework to model the early stages of a heavy-ion collision is viscous hydrodynamics. Comparing hydrodynamic simulations to heavy-ion data inevitably requires the conversion of the fluid to particles. This conversion, typically done in the Cooper-Frye formalism, is ambiguous for viscous fluids. In this thesis work, self-consistent phase space corrections are calculated by solving the linearized Boltzmann equation. These species-dependent solutions are contrasted with those obtained using the ad-hoc “democratic Grad” ansatz typically employed in the literature in which coefficients are independent of particle dynamics. Solutions are calculated analytically for a massless gas and numerically for the general case of a hadron resonance gas. For example, it is found that for a gas of massless particles interacting via isotropic, energy-independent $2 \rightarrow 2$ scatterings, the shear viscous corrections variationally prefer a momentum dependence close to $p^{3/2}$ rather than the quadratic dependence assumed in the Grad ansatz.

The self-consistent phase space distributions are then used to calculate transverse momentum spectra and differential flow coefficients, $v_n(p_T)$, to study the effects on heavy-ion identified particle observables. Using additive quark model cross sections, it is found that proton flow coefficients are higher than those for pions at moderately high $p_T$ in $Pb + Pb$ collisions at LHC, especially for the coefficients $v_4$ and $v_6$. 
1. Introduction

1.1 History and Motivation

From as far back as 400 BCE when Democritus and other Greek atomists were performing matter-cutting gedanken experiments, man has tried to find Nature’s most fundamental building blocks; those immutable particles whose properties and interactions govern the way the entire universe works. Throughout these same centuries and undoubtedly longer, man has also tried to understand the grand scale manifestation of Natural Laws; from the trees of their surroundings up to the stars in the heavens. These two questions have since moved from the thoughts of philosophers into the laboratories and chalkboards of physics. The same awe-inspiring stars have stoked modern day physicists into trying to answer these two questions through systematic trial and error: What are Nature’s most fundamental constituents and their interactions, and how is this microscopic interplay manifest in the macroscopic matter around us?

Much progress has been made since Democritus: The discovery of the electron as a constituent of the atom by J.J. Thomson in 1897 [1] and the discovery of the nucleus by Ernest Rutherford [2] based on experiments by Geiger and Marsden [3] a decade later lead to a newfound understanding of matter in atomic theory. After a few more decades, the behavior of a single atom could be understood in the theoretical framework of quantum mechanics. Since these paradigm-shifting discoveries, many other “fundamental” particles have been found. While the electron is still seen as a fundamental particle, the nucleus has since been deconstructed into protons and neutrons. Through “deep inelastic scattering” (DIS) experiments at the Stanford Linear Accelerator (SLAC) in 1968 [4] and a host of other experimental data and theoretical studies [5], these nucleons have since been shown to be made up of point-like parti-
cles called *quarks*. There have been six flavors of quarks postulated and subsequently found, commonly identified as up, down, charm, strange, top, and bottom. They participate in electromagnetic interactions like the electron, but have a fraction of its electric charge, either $2/3$ for $u$, $c$, and $t$ or $1/3$ for $d$, $s$, and $b$ \[^6\]. Quarks also carry another charge, commonly referred to as color charge, so just as electrons and protons form atoms by exchanging quanta of the electromagnetic field (*photons*), quarks bind together to form hadrons like protons and neutrons by exchanging quanta of this color field, the so-called *gluons*. Quarks and gluons are often collectively referred to as *partons* and their interaction is commonly known as the “strong force” described by the theory of Quantum Chromodynamics (QCD). QCD exhibits the distinctive properties of *asymptotic freedom* and *color confinement*. Asymptotic freedom \[^7, 8\] refers to the weak coupling of quarks and gluons at high energy and small length scales as a result of the gluons carrying color charge themselves, while confinement \[^9\] embodies the empirical fact that only colorless entities have been observed in Nature, i.e., isolated quarks and gluons have never been directly detected. Instead, experiments detect colorless two or three-quark bound states known respectively as *mesons* and *baryons*. Once many of these QCD bound states had been detected experimentally and understood theoretically, it was then time to turn to the second great question of how these QCD properties manifest themselves in Nature.

While a quark cannot be detected in isolation, one can still probe the diametric yet equally interesting limit of nuclear matter at high densities. This question, while being interesting in its own right, also has an important role in our history. Since the Big Bang, our universe has been expanding \[^10\] without creating more matter and energy, thus becoming less dense in the process. If we extrapolate this behavior back to the beginning of our universe, the properties of nuclear matter at high densities are then defining aspects of the early stages of space and time as we know it. It was noted back in the 1950s \[^11\] that the size of hadrons provides a natural density (or equivalently, temperature) where new behavior is likely to occur in nuclear matter.
Once the density reaches about one hadron per hadronic volume, the system should undergo a phase transition as the hadrons begin to overlap as illustrated in Fig. [1.1]

![Figure 1.1: Deconfinement emerges from dense hadron packing. Once the density reaches a critical value, borders separating overlapping hadrons blur as quarks (colored dots) in a single hadron become indistinguishable from quarks in adjacent hadrons.][12]

Using this extremely simple idea with a thermal gas of pions, the critical temperature was estimated to be around $T_C \simeq 190 \text{ MeV} \simeq 10^{12} \text{ K}$ by Pomeranchuk and reworked in [12]. It was also noted during the 1960s that the number of “elementary” particles and resonances produced in collider experiments grew with energy, seemingly without bound. “Atomists” at the time took this as evidence of the quark model as more fundamental entities were thought to comprise this zoo of resonances, but it also lead Hagedorn to the thermodynamics of strongly interacting matter [13]. Using the idea of a self-similar resonance mass spectrum allowed him to calculate an upper limit temperature for hadronic matter: $T_c \simeq 150 - 200 \text{ MeV}$. Both of these simple calculations seemed to hint at a transition from hadronic matter to that of deconfined quarks and gluons above 150 MeV.

The experimental search for this so-called *quark-gluon plasma (QGP)* began at CERN’s Super Proton Synchrotron (SPS) in the late twentieth century and continued into the 21st with the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National
Lab and the Large Hadron Collider (LHC) near Geneva, Switzerland. The goal of these colliders, now tens of miles in circumference, is to accelerate heavy nuclei (gold, lead, etc.) up to nearly the speed of light and collide them to produce an energy density high enough to create droplets of quark-gluon plasma. The QGP, expanding at a significant fraction of the speed of light, would be extremely short-lived ($\sim 10^{-23}$ s), but proof of its existence along with some of its properties are imprinted on the particles coming out of the collisions. The RHIC and LHC experiments have shown signatures \[14–20\] of a deconfined quark-gluon medium including collective flow patterns described in detail in the next section and anisotropic suppression of high $p_T$ particles (jets) as shown in Fig. 1.2.

![Figure 1.2.](image)

Figure 1.2.: Evidence of a deconfined color medium produced in a $\sqrt{s}_{NN} = 2.76$ TeV $Pb + Pb$ collision at the LHC based on high $p_T$ back to back jets. The subleading jet quark interacted with the bulk of the colored medium, losing a significant fraction of its initial energy (also see Figure 2.2 in the next Section). \[20\]
1.2 Heavy-Ion Standard Model

These results along with a host of theoretical insights have led to a “Standard Model” for modeling heavy-ion collisions as depicted in Fig. 1.3.

![Figure 1.3.](image)

Figure 1.3.: Stages in modeling a heavy-ion collision. The aim of this thesis work is to better model the transition from the hydrodynamic stage to the hadron gas stage. [21]

Modeling heavy-ion collisions includes many complex steps and a melding of several areas of physics. Less than a fermi/c after the initial collision of the Lorentz contracted nuclei, the medium is well described empirically by hydrodynamic expansion with a small shear viscosity compared to its entropy density. The initial conditions for the hydrodynamic simulation are typically modeled by sampling locations of nucleons in the nuclei and smearing the distributed energy (Monte Carlo Glauber approach [22]) or calculated using the QCD-motivated color glass condensate.
(CGC) model \[23\]. Once the medium expands to the point where it is too dilute to be
described accurately by hydrodynamics, one needs a description in terms of particles
(hadrons) and their interactions via kinetic theory. As this gas of hadrons expands
outward, interactions reshuffle hadron momenta and the chemical composition of
the gas. Eventually the gas becomes dilute enough to neglect further interactions
and the particles are said to “freeze-out” and free stream to the surrounding detec-
tors where observables such as identified particle momentum distributions (spectra)
and anisotropic flow coefficients \(v_n\) defined in (2.6)) can be measured. Whether a
simulation follows the particles through the hadronic stage with a transport model
(“afterburner” approach) or simply ignores these interactions, the transition from the
fluid description to one of distinct hadrons is inevitable. The prescription used to
calculate this so-called freeze-out is ambiguous in the case of viscous hydrodynamics,
and the effect different methods have on observables has not been calculated. In this
thesis work, these hadron distribution functions are calculated in a self-consistent way
using the Boltzmann equation to improve upon the widely employed method.

1.3 Modern Shortcomings: Original Thesis Contribution

A freeze-out prescription should correctly describe the distributions in momentum-
space (spectra) of particles coming from a specific chunk of the fluid at decoupling. For
non-dissipative (ideal) fluids, this instantaneous conversion is essentially straightfor-
ward in the ubiquitous Cooper-Frye formalism \[24\] outlined in Section 2.2.1. However,
dissipation (viscosity) distorts the phase space distributions of each particle species \(i\)
from local thermodynamic equilibrium by some amount \(\delta f_i\) (herein assumed small),
and an infinite number of choices for the form of \(\delta f_i\) are consistent with the hydrody-
namic fields. Most studies currently ignore this ambiguity and choose the ad-hoc (yet
obviously unphysical) “democratic Grad” \[25\] form for these corrections in which the
\(\delta f_i\)’s are proportional to the second power of momentum and have no dependence on
the interactions between the particles in the system.
In this thesis, I have calculated the shear viscous corrections to the phase space distributions at the transition surface self-consistently from the linearized Boltzmann equation, thus taking into account the different interactions of hadronic species through their respective cross sections. Under the assumption that the momentum dependence of the phase space corrections is quadratic ("Grad" ansatz), this uniquely and consistently constrains the viscous corrections. In Sections 4.1 through 4.2.2 I apply a variational approach to calculate the shear viscous corrections analytically for various systems, such as massless and nonrelativistic multiparticle systems interacting via $2 \rightarrow 2$ scatterings with energy independent, isotropic cross sections. I have also written a C++ code which uses the adaptive integration routines from the GNU Standard Library (GSL) [26] to do the nested 4-dimensional integrals that need to be calculated numerically to obtain the shear viscous corrections for fully relativistic multicomponent systems with arbitrary masses. In Sections 4.2.3 and 4.3 this code was applied to both a pion-nucleon system and a 49-species hadron gas system with $2 \rightarrow 2$ energy independent, isotropic cross sections. I also developed a parametrization for the fully energy dependent cross sections between pions and nucleons for different isospin channels in Section 4.2.4. These results were tabulated and used to calculate the shear viscous corrections for a fluid to particle transition to a pion-nucleon gas interacting via realistic $2 \rightarrow 2$ energy dependent, isotropic cross sections.

The variational method developed here also yields the shear viscosity of the gas after the transition, so I have also quoted results for the shear viscosity for the analytic and numerical systems discussed above. As a check of the variational method, in Section 4.2.1 I analytically calculated the viscosity of a single component relativistic massive gas following the approach in the kinetic theory “bible” by de Groot et al. [27], and found a typographical error in the widely circulated result. This new result for the viscosity was then verified numerically through the variational method and various transport simulations.

To test the effects my newly calculated distribution functions would have on heavy-ion observables, I wrote a viscous Cooper-Frye freezeout code in C++ to read in
freezeout hypersurfaces from hydrodynamic simulations of heavy-ion collisions and calculate the momentum distributions of particles coming from the fluid according to the Cooper-Frye prescription. The code involves integrating over all coordinate rapidity and gives momentum distributions as a function of the azimuthal angle. These particle distributions are then used to calculate the momentum spectra and harmonic flow coefficients, \( v_n(p_T) \), for identified particles. I have presented results for these observables for the pion-nucleon system in Section 4.2.3 and the full 49-species hadron gas assuming different isotropic, energy-independent cross section scenarios for interactions amongst the particles in Section 4.3. Short-lived particles in these calculations were decayed through the RESO code which is part of the AZHYDRO package. One of the projects done, but not discussed explicitly in the text, was the analytic calculation of 2-body and 3-body decays to verify the numerical accuracy of the RESO code.

Shear viscous corrections to the distribution functions outside the assumed quadratic momentum dependence of the Grad ansatz were also investigated in Chapter 5. I calculated the corrections analytically for a massless gas assuming a power series basis expansion for the momentum dependence. I used these results to find the variationally preferred single power of the momentum dependence analytically in a massless system by maximizing the functional. Finally I calculated the spectra and flow coefficients for a conversion to a 49-species hadron gas assuming the shear viscous corrections had a single power momentum dependence that was weaker than quadratic.
2. Theoretical Framework

2.1 Relativistic Hydrodynamics

In the field of heavy-ion physics and throughout this thesis, the term hydrodynamics, which would refer to water motion specifically, is used interchangeably with the more general term relativistic fluid dynamics. Relativistic fluid dynamics is a Lorentz covariant theoretical framework applicable as a long-wavelength effective theory of a system of particles whose mean-free-path is small compared to both the size of the system and the length scales involved in its spacetime gradients. Fluid dynamics describes a system near thermal equilibrium in terms of local macroscopic parameters. For example, in thermal equilibrium one uses the fields of temperature $T(x)$, pressure $P(x)$, energy density $\epsilon(x)$, and flow velocity $u^\mu(x) \equiv dx^\mu/d\tau$, along with any conserved charge densities in the system $N_c^\mu(x)$\(^1\) (For conventions concerning indices and units, see Appendix [A]). $\tau$ is the proper time whose increment is given by

\[
(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = (dt)^2 - (d\mathbf{r})^2 = (dt)^2 \left[ 1 - \left( \frac{d\mathbf{r}}{dt} \right)^2 \right] = (dt)^2 \left[ 1 - v^2 \right]. \tag{2.1}
\]

With these definitions,

\[
u^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1 - v^2}} (1, v) \equiv \gamma(|v|)(1, v), \tag{2.2}
\]

which is normalized such that

\[
u^2 \equiv \nu^\mu g_{\mu\nu} \nu^\nu = \gamma^2(|v|)(1 - v^2) = 1. \tag{2.3}
\]

In the local rest frame (LR) of the fluid, (2.2) reduces to $u^\mu_{LR} = (1, 0)$ by definition.

The pressure, energy density, and charge densities are related by an equation of state (EoS) that characterizes local equilibrium properties of the fluid and which

\(^1\)In heavy-ion physics, typically the baryon charge is the only conserved current considered, though strange \(^2\)\(^9\)\(^{31}\) and charm currents may have some applications for heavy flavor \(^3\)\(^2\)\(^3\) observables, while electric charge currents are of interest in chiral magnetic effect studies \(^34\).
hydrodynamics takes as input. The equations of motion governing fluid flow are con-
servation of energy and momentum, typically packaged into the energy-momentum
tensor, $T^{\mu\nu}(x)$, as well as conservation equations for any conserved charges in the system, $N^\mu_c(x)$. The equations that govern relativistic fluid flow with conserved charges $c$ are then

$$\partial_\nu T^{\mu\nu}(x) = 0 \quad , \quad \partial_\mu N^\mu_c(x) = 0 . \quad (2.4)$$

### 2.1.1 Ideal Hydrodynamics

A fluid that is in perfect local thermal equilibrium at all times, everywhere in
space is governed by ideal hydrodynamics. Requiring the energy-momentum tensor
to be symmetric and to transform as a Lorentz tensor, one can derive the ideal energy-
momentum tensor and charge current (see Appendix B.1):

$$T_{id}^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P \eta^{\mu\nu} , \quad N^\mu_{c,id} = n_c u^\mu , \quad (2.5)$$

where $N^{0,LR}_{c,id} = n_c$ is the charge density in the local rest frame of the fluid. In this
frame, the energy-momentum tensor takes the diagonal form $T_{id,LR}^{\mu\nu} = diag(\epsilon, P, P, P)$.

Early in the history of heavy-ion collisions, it was postulated that the hadronic sys-
tems created at the BEVALAC and later at the SPS energy of $\sqrt{s} = 17.3$ GeV/nucleon
could be modeled with ideal hydrodynamics [35,36]. Both hydrodynamic and hadron
kinetic theory based calculations [37] reproduced hadronic multiplicities and trans-
verse momentum spectra [38–40] at SPS energies as shown in Fig. 2.1, while the hydro-
dynamic calculation of the so-called elliptic flow observable was quantitatively below
the data [41].

Elliptic flow is the second coefficient in a Fourier expansion of the azimuthal
momentum distribution, at a fixed component of the momentum transverse to the
beam direction $p_T$, as defined by

$$E \frac{d^3N}{dp^3} = \frac{1}{2\pi} \frac{d^2N}{p_T dp_T dy} \left( 1 + 2 \sum_{n=1}^{\infty} v_n(p_T, y) \cos[n(\phi - \Psi_{RP})] \right) , \quad (2.6)$$
Figure 2.1: Charged pion and antiproton spectra for different centralities in $Au + Au$ collisions at $\sqrt{s_{NN}} = 130$ GeV at RHIC calculated from ideal hydrodynamics in qualitative agreement with experimental data. One does not expect hydrodynamics to reproduce data well for very peripheral collisions in which the overlap region is too small to fully thermalize, nor for high $p_T$ particles which also do not completely thermalize with the bulk \cite{42}.

where $\phi$ is the azimuthal angle around the beam $z$-axis, $y \equiv \frac{1}{2} \frac{E+p_z}{E-p_z}$ the rapidity, and $\Psi_{RP}$ the reaction plane angle corresponding to the angle the impact parameter makes with the $x$-axis as shown in Fig. 2.2. The coefficient $v_2$ is known as “elliptic flow” because it is the dominant term in off-center collisions between two spherical nuclei with smooth nucleon distributions, where the overlap region can be thought of as an elliptical almond shape as pictured in Fig. 2.2. It can be explicitly calculated from the spectrum as $v_2(p_T) = \langle \cos(2\phi) \rangle_{p_T}$.

Even in more realistic simulations with fluctuating initial nucleon positions, $v_2$ is typically larger than the next coefficient, triangular flow $v_3$, and all higher coefficients for all but the most central collision geometries \cite{44,45}. This makes elliptic flow one of the most important experimental probes of collective dynamics in heavy-ion collisions.

It was originally thought that the matter formed at the increased RHIC energy of $\sqrt{s} = 200$ GeV/nucleon could fall outside the small mean free path regime of hy-
Figure 2.2.: Illustration of the spatially anisotropic overlap region in a non-central head-on collision. The thermal pressure gradient is higher in the $x$ direction than the $y$ direction, causing higher momentum particles to be emitted preferentially in the $x$ direction as indicated by the arrows. [43]

dynamics due to the weak QCD coupling at high energy (asymptotic freedom). However, ideal hydrodynamics was shown to again reproduce most hadronic multiplicities and transverse momentum spectra [46–48], with the exception of proton and photon absolute yields (not shown), as well as a reasonable quantitative agreement for $v_2$ in the parameter space where a deconfined medium was thought to have formed as shown in Fig.2.3.

The agreement between ideal hydrodynamic models and experiment for elliptic flow strongly suggests that the medium created is at or near thermal equilibrium after successive final state parton rescatterings. Thus, the mean free path for partons must be significantly shorter than the size of the system and the scattering rate must be much larger than the local expansion rate of the plasma, allowing the QGP to be considered a strongly coupled thermal plasma of deconfined quarks and gluons.
Figure 2.3.: Charged pion and proton elliptic flow in $Au + Au$ at $\sqrt{s_{NN}} = 130$ GeV calculated from ideal hydrodynamics. As in Fig. 2.1, the disagreement for peripheral collisions in the left panel is not unexpected $^{42,49}$.

2.1.2 Viscous Hydrodynamics

As experimental data and theoretical models became more refined, it became clear that some of the early quantitative success of ideal hydrodynamics was fortuitous. The simple “bag model” equation of state $^{42}$ used for the plasma was inconsistent with thermal field theory calculations from lattice QCD $^{50}$. The former EoS was also no longer able to match data in more realistic simulations $^{51,52}$, in part due to the hydrodynamic models being run past the strongly coupled QGP phase and into the more dissipative hadron gas phase of the evolution. Not only did these realizations serve as an impetus for the development of hybrid models $^{53-55}$ which couple transport codes with hydrodynamics, but they also lead to a more conscious effort to explore the effects of dissipation on observables.

After a calculation applicable to a wide class of systems using the string theory-inspired AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence $^{56}$ obtained a conjectured lower bound of $1/4\pi$ for the shear viscosity to entropy density ratio $\eta/s$, many groups $^{57,59}$ extended their hydrodynamic models of heavy-ion col-
lisions to include shear viscosity. For recent reviews of hydrodynamics in heavy-ion collisions see \[60–63\].

In viscous hydrodynamics, one typically writes the charge currents and the energy-momentum tensor as a sum of the ideal, isotropic piece and a viscous correction:

\[ N^\mu_c \equiv N^\mu_{c, id} + \delta N^\mu_c , \quad T^{\mu\nu} \equiv T^{\mu\nu}_{id} + \delta T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \pi^{\mu\nu} - \Pi\Delta^{\mu\nu}, \quad (2.7) \]

with \( \Pi \equiv \frac{1}{3} \delta T^\mu_\mu \) defined as the trace of the viscous correction to the energy-momentum tensor, \( \pi^{\mu\nu} \equiv \delta T^{\mu\nu} - \frac{1}{3} \Pi^\alpha_\alpha g^{\mu\nu} \) as the traceless and symmetric part of the viscous correction often called the shear stress tensor, and \( \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu \) the projector orthogonal to the flow velocity. Note that the flow velocity \( u^\mu(x) \) is ambiguous in viscous hydrodynamics as the fluid rest frame can be defined as the frame in which there is no energy diffusion, no charge diffusion for some chosen charge \( c \), or some other less common choice.

Throughout this thesis work, the Landau \[64\] frame is used which defines the fluid rest frame as the frame in which there is no energy diffusion, i.e.

\[ u_\mu T^{\mu\nu} \Delta^\alpha_\nu \equiv 0 \Rightarrow u_\mu \delta T^{\mu\nu} \equiv 0 . \quad (2.8) \]

For further discussion of the flow velocity and choice of fluid rest frame in viscous hydrodynamics, see Appendix \[B.2\].

The assumption that viscous corrections are small, i.e. first order in flow gradients, and that dissipation gives rise to an increase in (equilibrium) entropy results in the Navier-Stokes form of \( \delta T^{\mu\nu} \) proportional to shear (\( \eta \)) and bulk (\( \zeta \)) viscosity, respectively (See Appendix \[B.2\]):

\[ \delta T^{\mu\nu}_{NS} = \eta(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \partial^\lambda u_\lambda) + \zeta \Delta^{\mu\nu} \partial^\lambda u_\lambda . \quad (2.9) \]

Energy-momentum conservation then yields the relativistic Navier-Stokes (NS) hydrodynamic equations of motion. The shear viscosity characterizes the fluid’s resistance to adjacent “layers” having different parallel speeds while the bulk viscosity quantifies its resistance to expansion or contraction. Thus, knowing the initial conditions and the possible temperature dependence of the transport coefficients specifies
the hydrodynamical evolution of the fluid in the Navier-Stokes regime. It has been noted \cite{66-69} that the NS equations are unstable to small wavelength perturbations. This can be thought of as a problem of principle as hydrodynamics is a theory to be used in the long wavelength regime, but it often causes problems for numerical NS equation solvers. This can be remedied by postulating an increasing entropy current that is out of equilibrium which results in the so-called Israel-Stewart \cite{70} (IS) dissipative hydrodynamic equations, which are second-order in flow gradients and lead to dissipative corrections such as $\pi^{\mu\nu}$ and $\Pi$ that relax dynamically toward their Navier-Stokes values in the late time regime.

2.2 Fluid to Particle Conversion

In order to test predictions from hydrodynamic models against experimental data from heavy-ion collisions, one must switch from a fluid dynamic description to one of particles. Whether these hadronic particles are then allowed to collide further in a hadronic transport code (so-called “hybrid” models) or just allowed to free stream to the “detectors”, this fluid to particle transition is unavoidable. The most common prescription for doing this particlization is the Cooper-Frye framework described below.

2.2.1 Cooper-Frye Particlization

The Cooper-Frye \cite{24} decoupling prescription gives the momentum distribution for particle species $i$ coming from a drop of fluid according to

$$E \frac{dN_i}{d^3p} = \int_{\sigma} p^\mu d\sigma(x) f_i(x, p),$$

\hspace{1cm} \text{(2.10)}

in terms of the phase space distribution function

$$f_i(x, p) \equiv \frac{dN_i(r, p, t)}{d^3r \ d^3p}.$$ \hspace{1cm} \text{(2.11)}

Here $\sigma(x)$ is a three-dimensional hypersurface in 4D spacetime defined using some critical decoupling criterion such as constant temperature or energy density. The four-
vector $d\sigma^\mu(x)$ is defined as in \cite{71} as being dual to the volume of the infinitesimal parallelepiped (i.e. the hypersurface “area”) denoted $d\sigma^{\alpha\beta\gamma}$ spanned by three four-vectors $dx^\alpha, dx'^\alpha, dx''^\alpha$ as

$$d\sigma^\mu = -\frac{1}{6} \epsilon^{\mu\alpha\beta\gamma} d\sigma_{\alpha\beta\gamma}, \quad d\sigma^{\alpha\beta\gamma} = \begin{vmatrix} dx^\alpha & dx'^\alpha & dx''^\alpha \\ dx^\beta & dx'^\beta & dx''^\beta \\ dx^\gamma & dx'^\gamma & dx''^\gamma \end{vmatrix},$$

where $\epsilon^{\mu\alpha\beta\gamma}$ is the fully antisymmetric symbol in 4-dimensions. The four-vector $d\sigma^\mu(x)$ thus has a magnitude equal to the 3-dimensional “area” of the hypersurface and is orthogonal to all vectors in this element at spacetime point $x$.

The instantaneous Cooper-Frye conversion is envisioned in spacetime regions where the hydrodynamic and particle descriptions are to good approximation equivalent, so we only switch “language” but the state of the system is unchanged\footnote{For an alternative approach that considers a rapid conversion process in a thin layer idealized as hypersurface in spacetime, with the process constrained by energy-momentum and current conservation across the hypersurface, and the nondecrease of entropy, see \cite{72,73}.}. For consistency, at the point of conversion the equation of state used in fluid dynamics must correspond to a gas of particles. In heavy-ion physics applications this typically means that if the hydrodynamic equation of state being used encodes a phase change from a deconfined color medium to a gas of hadrons, as is typically the case \cite{74}, the conversion should be done at a low enough temperature that corresponds to the hadron gas phase.

The principle challenge in converting a fluid to particles is that one needs to obtain phase space densities for each of the particle species solely from the hydrodynamic fields $T^{\mu\nu}(x)$ and $N^\mu_c(x)$. If the particles can be modeled as an ideal gas (as is usually assumed), one has to invert the kinetic theory definitions

$$T^{\mu\nu}(x) \equiv \sum_i \int \frac{d^3p}{E} p^\mu p^\nu f_i(x, p) \quad (2.12)$$

and

$$N^\mu_c(x) \equiv \sum_i q_{c,i} \int \frac{d^3p}{E} p^\mu f_i(x, p), \quad (2.13)$$
to find the distributions $f_i(x, p)$, where $q_{c,i}$ is the charge of type $c$ carried by a particle of species $i$.

### 2.2.2 Ideal Fluid Decoupling

For non-dissipative (ideal) fluids, which by definition are in local equilibrium everywhere in space at all times, the conversion is straightforward because in local thermal and chemical equilibrium particle distributions are\[^3\]

$$f_i(x, p) \equiv f_i^{eq}(x, p) = \frac{g_i}{(2\pi)^3} \exp \left[ \frac{\mu_i(x) - p_\alpha u^\alpha(x)}{T(x)} \right], \quad \mu_i \equiv \sum_c q_{c,i} \mu_c(x) , \quad (2.14)$$

where $g_i$ is the number of internal degrees of freedom for species $i$, often spin and/or isospin degeneracies, $p^\mu$ is the energy-momentum 4-vector, and the combination $p_\alpha u^\alpha$ is the energy of the particle in the local rest frame of the fluid. The local temperature $T(x)$, chemical potentials $\{\mu_c(x)\}$, and four-velocity $u^\mu(x)$ of the fluid are uniquely determined through the ideal hydrodynamic relations

$$T_{\mu\nu}^{id}(x) = \left[ \epsilon(x) + P(x) \right] u^\mu(x) u^\nu(x) - P(x) g^{\mu\nu} , \quad N_{c,\mu}^{id}(x) = n_c(x) u^\mu(x) , \quad (2.15)$$

with $\epsilon(T, \{\mu_c\})$, $P(T, \{\mu_c\})$, and $n_c(T, \{\mu_c\})$ given by the equation of state (these can be inverted for $T$ and $\{\mu_c\}$). Thus in local equilibrium there is a one-to-one mapping between the distribution function and hydrodynamic fields since knowledge of the temperature and chemical potentials gives the energy-momentum tensor in the local fluid rest frame:

$$T_{i,LR}^{\mu\nu} = diag(\epsilon_i, P_i, P_i, P_i) \iff f_i^{eq}(x, p) = \frac{g_i}{(2\pi)^3} \exp \left[ \frac{\mu_i(x) - p_\alpha u^\alpha(x)}{T(x)} \right] . \quad (2.16)$$

\[^3\]Throughout this work Boltzmann statistics is assumed but generalization to the Bose/Fermi case is straightforward.
2.2.3 Viscous Fluid Decoupling

If the fluid is dissipative, however, then it is not strictly in local thermal and chemical equilibrium and the viscosity of the fluid modifies the ideal forms of the energy-momentum tensor and charge currents in (2.15):

\[ T_{\mu\nu} = T_{\mu\nu}^{id} + \delta T_{\mu\nu}, \quad N_{\mu} = N_{\mu}^{id} + \delta N_{\mu}. \]  

(2.17)

This implies that the particles that make up the viscous fluid also have their distribution functions modified from the local thermal ones:

\[ f_i(x, p) \equiv \frac{dN_i(x, p)}{d^3r d^3p} = f_{eq}^i(x, p) + \delta f_i(x, p) \equiv f_{eq}^i(x, p)[1 + \phi_i(x, p)]. \]  

(2.18)

The general kinetic theory definitions (2.12) and (2.13), however, do remain valid and are the main constraints on the viscous corrections to the distribution functions

\[ \delta T_{\mu\nu}(x) = \sum_i \int \frac{d^3p}{E} p^\mu p^\nu \delta f_i(x, p), \quad \delta N_{\mu}(x) = \sum_i q_{c,i} \int \frac{d^3p}{E} p^\mu \delta f_i(x, p). \]  

(2.19)

Without additional information about the functional form of \( \delta f_i(x, p) \), this finite set of conditions can be satisfied with infinitely many different \( \delta f_i(x, p) \) (or equivalently, \( \phi_i(x, p) \) defined in 2.18), even if there is only a single particle species. Thus knowledge of the hydrodynamic fields does not uniquely determine the form of the \( \delta f_i \)'s, so they cannot be obtained from the output of a hydrodynamic simulation alone. One must postulate an ansatz for their functional form or calculate them using another theoretical framework.

2.3 Democratic Grad Ansatz

In heavy-ion physics, often the only dissipative correction considered is from shear stress. A commonly used prescription that satisfies the constraint (2.19) from shear alone is the so-called “democratic Grad” [25] ansatz, which assumes phase space corrections with quadratic momentum dependence

\[ \phi_i^{dem}(x, p) = \frac{T^{\mu\nu}(x)p_\mu p_\nu}{2[\epsilon(x) + P(x)]T^2(x)}. \]  

(2.20)
Note, the right hand side of (2.20) is independent of the particle species $i$ so the coefficient in this quadratic form is the same for all particle species. The reason this ansatz works is that for each species it gives a partial shear stress that is proportional to the partial enthalpy, 

$$h_i(x) \equiv \epsilon_i(x) + P_i(x),$$

as shown in Appendix D:

$$\pi_{\mu \nu}^{\text{dem}} \equiv \int \frac{d^3p}{E} p^\mu p^\nu \delta f_i^{\text{dem}} = \frac{\epsilon_i + P_i}{\epsilon + P} \pi^{\mu \nu} \Rightarrow \sum_i \pi_{i, \text{dem}}^{\mu \nu} = \pi^{\mu \nu}. \quad (2.21)$$

However, this simple choice ignores the very microscopic dynamics that keep the gas near local equilibrium. In particular, one expects species that interact more frequently to be better equilibrated than those that scatter less often, an effect totally absent from this ubiquitous ansatz. It is not hard to imagine that leaving out so much of the information contained in the particle dynamics could have a large effect on observables in a heavy-ion collision, especially identified particle observables. To quantify these effects one should calculate the distribution functions from another theoretical framework instead of using an ad-hoc ansatz for the form of the corrections. To this end, let us now turn to the fully nonthermal framework of kinetic theory.

### 2.4 Boltzmann Transport Equation

In contrast to proposing an ansatz for the distribution functions coming from a viscous fluid, a self-consistent set of dissipative corrections can be obtained from linearized covariant transport theory. The transport equation is derived assuming Boltzmann’s “molecular chaos” hypothesis. This hypothesis is a statistical assumption that there are no two or more body correlations before each individual collision such that the number of binary collisions is a product of the distribution functions of the colliding particles. Consider on-shell covariant transport theory for a multicomponent system with $2 \to 2$ interactions. For each particle species $i$, they are defined as particles whose momentum and energy are related by the relativistic expression $E = \sqrt{p^2 + m^2}$ where $m$ is the mass of the particle.
the evolution of the phase space density is given by the nonlinear Boltzmann transport equation:
\[ p^\mu \partial_\mu f_i(x, p) = S_i(x, p) + \sum_{jk\ell} C^{ij\rightarrow k\ell}[f_i, f_j, f_k, f_\ell](x, p), \quad (2.22) \]
where the source term \( S_i \) encodes the initial conditions, and the collision terms are
\[ C^{ij\rightarrow k\ell}[f_i, f_j, f_k, f_\ell](x, p_1) \equiv \iiint_{234} \left( \frac{g_i g_j}{g_k g_\ell} f_{3k} f_{4\ell} - f_{1i} f_{2j} \right) \bar{W}^{ij\rightarrow k\ell}_{12\rightarrow 34} \delta^4(12 - 34), \quad (2.23) \]
with shorthands
\[ \int_a \equiv \int d^3 p_a/(2E_a), \quad f_{ai} \equiv f_i(x, p_a), \quad \text{and} \quad \delta^4(ab - cd) \equiv \delta^4(p_a + p_b - p_c - p_d). \quad (2.24) \]
The transition probability \( \bar{W}^{ij\rightarrow k\ell}_{12\rightarrow 34} \) for the process \( i + j \rightarrow k + \ell \) with momenta \( p_1 + p_2 \rightarrow p_3 + p_4 \) is invariant under interchange of incoming or outgoing particles,
\[ \bar{W}^{ij\rightarrow k\ell}_{12\rightarrow 34} \equiv \bar{W}^{ji\rightarrow k\ell}_{21\rightarrow 34} \equiv \bar{W}^{ij\rightarrow \ell k}_{12\rightarrow 34} \equiv \bar{W}^{ji\rightarrow \ell k}_{21\rightarrow 34}, \quad (2.25) \]
satisfies detailed balance
\[ \bar{W}^{k\ell\rightarrow ij}_{34\rightarrow 12} \equiv \frac{g_i g_j}{g_k g_\ell} \bar{W}^{ij\rightarrow k\ell}_{12\rightarrow 34}, \quad (2.26) \]
and is given by the corresponding unpolarized scattering matrix element or differential cross section as
\[ \bar{W}^{ij\rightarrow k\ell}_{12\rightarrow 34} = \frac{1}{16\pi^2} |M^{ij\rightarrow k\ell}_{12\rightarrow 34}|^2 \equiv \frac{4}{\pi} s p_{cm}^2 \frac{d\sigma^{ij\rightarrow k\ell}_{12\rightarrow 34}}{dt} \equiv 4s p_{cm}' d\sigma^{ij\rightarrow k\ell}_{12\rightarrow 34}/d\Omega_{cm}. \quad (2.27) \]
Here \( s \equiv (p_1 + p_2)^2 \) and \( t \equiv (p_1 - p_3)^2 \) are standard Mandelstam variables, while
\[ p_{cm} \equiv \sqrt{(p_1 p_2)^2 - m_i^2 m_j^2 \over s}, \quad p_{cm}' \equiv \sqrt{(p_3 p_4)^2 - m_k^2 m_\ell^2 \over s} \quad (2.28) \]
are the magnitudes of incoming and outgoing particle momenta in the center of momentum frame of the microscopic two-body collision. The degeneracy factors \( g_i \) of
\[ \text{In (2.23) outgoing momenta } p_3 \text{ and } p_4 \text{ are understood to be integrated over full, unrestricted phasespace. This double-counts the rate for identical particles } (k = \ell) \text{ compared to nonidentical particles, however, that is compensated by double-counting in the sum for } k \neq \ell. \text{ See also } (H.3). \]
\[ \text{Note the 2 in the lorentz invariant integral measure. References without this factor of 2 may have different results for some of the intermediate quantities calculated in this work, though important quantities will, of course, be independent of this convention.} \]
the respective particle species appear explicitly in (2.26) because unpolarized matrix elements are summed over internal degrees of freedom (spin, polarization, color) of outgoing particles, but *averaged* over those of incoming particles. These factors also appear in (2.23) because distribution functions here are assumed to depend only on momentum and position but not on internal degrees of freedom, and thus the distribution of each species is summed over internal degrees of freedom (cf. the local equilibrium form (2.14)).
3. Self-Consistent Viscous Corrections from Linearized Transport

In this Chapter, a variational method is developed to obtain the particle distribution functions that solve the linearized Boltzmann transport equation. The shear viscosity of the system is also trivially obtained as a by-product of this variational method.

3.1 Linearized Transport Equation

For small departures from local equilibrium one can split each phase space density into a local equilibrium part and a dissipative correction as in (2.18), and linearize the transport equation (2.22) in $\delta f$:

$$p^\mu \partial_\mu f_i^{eq} + p^\mu \partial_\mu \delta f_i = \sum_{jkl} \left( C^{ij\rightarrow kl}[\delta f_i, f_j^{eq}, \delta f_k, f_\ell^{eq}] + C^{ij\rightarrow kl}[f_i^{eq}, \delta f_j, f_k^{eq}, \delta f_\ell] \right).$$

(3.1)

Here the source term has been dropped as described below and spacetime and momentum arguments have been suppressed for legibility. The collision term for equilibrium distributions, $C[f^{eq}, f^{eq}]$ vanishes due to energy-momentum conservation in the microscopic scattering amplitudes.

In general, the solutions $\delta f_i$ to this coupled set of equations depend on both the matrix elements in the collision terms and the initial conditions. However, typical systems quickly relax on microscopic scattering timescales to a solution dictated by gradients of the equilibrium distribution on the left hand side of (3.1). The asymptotic solution, for given gradients, is then uniquely determined by the interactions in the system and independent of the initial conditions and past history of the system. (To see this transient relaxation worked out explicitly, see [79].) In this so-called Navier-Stokes regime, one can neglect the time derivative of $\delta f_i$, and if gradients of $f_i^{eq}$ are
small, one can also ignore the spatial derivatives of \( \delta f_i \). The viscous corrections to the distribution function now appear only on the right side of (3.1). At each spacetime point \( x \) one now has a linear integral equation to solve. This is also the starting point of the standard calculation of transport coefficients in kinetic theory. For example, the shear viscosity \( \eta \) and bulk viscosity \( \zeta \) are defined in this Navier-Stokes limit through

\[
\delta T_{\mu \nu}^{NS} \equiv \eta \sigma_{\mu \nu} + \zeta \Delta_{\mu \nu} \partial_\alpha u^\alpha ,
\]

(3.2)

with the symmetric and traceless part of the flow derivative tensor

\[
\sigma_{\mu \nu} \equiv \nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \Delta_{\mu \nu} \partial_\alpha u^\alpha .
\]

(3.3)

The remaining derivative on the left side of the linearized transport equation (3.1) can be expanded as

\[
p_\alpha \partial_\alpha f_{eq}^i = p_\alpha (\nabla_\alpha + u^\alpha D) f_{eq}^i = f_{eq}^i p_\alpha (\nabla_\alpha + u^\alpha D) \left[ \frac{\mu_i(x) - p_\beta u^\beta(x)}{T(x)} \right]
\]

\[
= f_{eq}^i p_\alpha \left\{ \left[ \nabla_\alpha \left( \frac{\mu_i}{T} \right) - p_\beta \nabla_\alpha \left( \frac{u^\beta}{T} \right) \right] + \left[ u^\alpha D \left( \frac{\mu_i}{T} \right) - u^\alpha p_\beta D \left( \frac{u^\beta}{T} \right) \right] \right\} .
\]

(3.4)

Separating the flow derivatives from the temperature ones using the product rule and again isolating the symmetric and traceless part of the flow derivative tensor yields the form

\[
p_\alpha \partial_\alpha f_{eq}^i = f_{eq}^i \left\{ p_\alpha \left[ \nabla_\alpha \left( \frac{\mu_i}{T} \right) - (p \cdot u) \nabla_\alpha \left( \frac{1}{T} \right) \right] + (p \cdot u) D \left( \frac{\mu_i}{T} \right) - (p \cdot u)^2 D \left( \frac{1}{T} \right)
\]

\[
- \frac{p_\alpha p_\beta}{2T} \left[ \left( \nabla_\alpha u^\beta + \nabla_\beta u^\alpha - \frac{2}{3} \Delta_{\alpha \beta} (\partial \cdot u) \right) + \frac{2}{3} \Delta_{\alpha \beta} (\partial \cdot u) \right] - \frac{(p \cdot u)}{T} p_\alpha D u^\alpha \right\} .
\]

(3.5)

To isolate the response to shear, consider a system with uniform temperature and chemical potentials \( T = \text{const}, \mu_i = \text{const} \), with \( \sigma_{\mu \nu} \neq 0 \) but \( (\partial \cdot u) = 0 \). Only terms on the second line remain; the ones in the square bracket contribute to \( \delta T_{\mu \nu} \), whereas the last term with the convective time derivative \( Du^\alpha \) can be dropped as long as

\(^1\delta f_i \) is to leading order proportional to the gradients of \( f_{eq}^i \), and if those are small due to a large length scale \( L \) in the problem \( \nabla_{\mu} f_{eq}^i \sim 1/L \), then \( \nabla_{\mu} \delta f_i \sim 1/L^2 \) is suppressed compared to \( \nabla_{\mu} f_i \).

\(^2\)See, e.g., Chapter VI of Ref. [27], or for a more recent presentation Section 3 of Ref. [80].
gradients are weak, as time derivatives of hydrodynamic quantities can be replaced with spatial ones using conservation laws\textsuperscript{3} e.g., (B.31):

$$\Delta^\mu_\beta \partial_\alpha T^\alpha_\beta (x) = 0 \Rightarrow D u^\mu = \frac{1}{\epsilon + P} \left( \nabla^\mu P - \Delta^\mu_\beta \nabla_\alpha \delta T^\alpha_\beta \right).$$

(3.6)

In shear viscosity calculations, pressure, energy density, and charge densities are uniform by assumption, so derivatives of those vanish as well as derivatives of $T$ and $\mu_i$. What remains after application of the conservation laws are first derivatives of dissipative corrections, and dissipative corrections times first derivatives of ideal hydrodynamic fields like the second term in (3.6). In the Navier-Stokes regime these are both second order in gradients of the ideal hydrodynamic fields and thus can be discarded at this order in small gradients.

Introduction of the irreducible, dimensionless, symmetric, and traceless tensors:

$$P^\mu_\nu \equiv \frac{1}{T^2} \left[ \Delta^\mu_\alpha \Delta^\nu_\beta p^\alpha p^\beta - \frac{1}{3} \Delta^\mu_\nu (\Delta^\alpha_\alpha p^\alpha p^\beta) \right], \quad X^\mu_\nu \equiv \frac{\sigma^\mu_\nu}{T} = \frac{\pi^\mu_\nu_{NS}}{\eta T},$$

(3.7)

which are purely spatial (in LR) allows the left side of the linearized transport equation (3.1) to be written in the compact form

$$p^\alpha \partial_\alpha f^\text{eq}_i = -\frac{T^2}{2} f^\text{eq}_i (p) X^\mu_\nu (x).$$

(3.8)

Expanding the viscous corrections in irreducible tensors and noting that the $\phi_i(x, p)$ function must carry all the angular dependence of the RHS of (3.1) since the equilibrium distribution depends only on the magnitude of the rest frame momentum, one can derive the tensor form of the viscous corrections in response to shear to be\textsuperscript{5}

$$\phi_i(x, p) = \chi_i(|\mathbf{p}|) P^\mu_\nu X^\mu_\nu.$$  

(3.9)

\textsuperscript{3}This procedure of gradient expansion is known as the Chapman-Enskog\textsuperscript{81,83} procedure and the Navier-Stokes limit used here is the first approximation in this expansion. For expansion details see Chapter V of\textsuperscript{27} or Chapter V of\textsuperscript{84}.

\textsuperscript{4}Note that in the derivation of (3.6) the term $-D(u_\nu T^\nu_\mu)$ was dropped assuming the Landau frame (See Appendix\textsuperscript{B.2} for discussion of Landau flow velocity). In general this term will involve the heat flow $q^\mu$, which vanishes for a system with uniform temperature.

\textsuperscript{5}For details on how the form (3.9) comes about from irreducible tensor expansion, see Appendix\textsuperscript{E} and Refs.\textsuperscript{27,80}.
Here $\tilde{p}$ is the LR frame three-momentum normalized by temperature, i.e.,

$$\frac{1}{T} \Delta^{\mu\nu} p_{\nu} \bigg|_{LR} \equiv (0, \tilde{p})$$

such that $|\tilde{p}| \equiv \frac{\sqrt{(p \cdot u)^2 - m^2}}{T}$. (3.10)

This means that in the Navier-Stokes limit, $\delta f_i$ are solely determined by real, dimensionless scalar functions $\chi_i(|\tilde{p}|)$ of the rescaled momentum for each particle species $i$. An expansion in irreducible tensors is unique, so the terms multiplying the "$\ell = 2$" tensor $X_{\mu\nu}$ must be the same on both sides of the transport equation. Substituting (3.9) and (3.8) into (3.1) and using

$$g_i g_j g_k g_\ell f_{eq}^3 f_{eq}^4 \delta^4(12 - 34) \equiv f_{eq}^1 f_{eq}^2 \delta^4(12 - 34),$$

(3.11)

to combine the two collision terms one obtains a tensor equation from the linearized Boltzmann equation:

$$-\frac{1}{2} P_{1i}^\nu f_{1i}^{eq} = \frac{1}{T^2} \sum_{jk\ell} \int \int \int \int f_{1i}^{eq} f_{2j}^{eq} W_{12 \rightarrow 34}^{ij \rightarrow k\ell} \delta^4(12 - 34) (\chi_{3k} P_3 + \chi_{4\ell} P_4 - \chi_{1i} P_1 - \chi_{2j} P_2).$$

(3.12)

Contracting both sides of (3.12) with $P_{1,\mu\nu}$ yields a scalar integral equation for $\chi_i(|\tilde{p}|)$ written succinctly as $S_i(\tilde{p}_1) = (C\chi)_i(\tilde{p}_1)$, or explicitly

$$-\frac{1}{2} P_1 \cdot P_1 f_{1i}^{eq} = \frac{1}{T^2} \sum_{jk\ell} \int \int \int \int f_{1i}^{eq} f_{2j}^{eq} W_{12 \rightarrow 34}^{ij \rightarrow k\ell} \delta^4(12 - 34) \cdot$$

$$(\chi_{3k} P_3 + \chi_{4\ell} P_4 - \chi_{1i} P_1 - \chi_{2j} P_2) \chi_{ai}$$

(3.13)

with the introduction of the shorthands

$$\chi_{ai} \equiv \chi_i(|\tilde{p}_a|), \quad P_a \cdot P_b \equiv P_a^{\mu\nu} P_{b,\mu\nu} = (\tilde{p}_a \cdot \tilde{p}_b)^2 - \frac{1}{3} |\tilde{p}_a|^2 |\tilde{p}_b|^2.$$ (3.14)

### 3.2 Variational Solution of Linearized Transport

Using the symmetry properties of the transition probability and the $\delta$-function, (2.25), (2.26), and (3.11), one can show that the dimensionless functional

$$Q[\chi] \equiv (\chi, S) - \frac{1}{2} (\chi, (C\chi))$$

with $(f, g) \equiv \frac{1}{T^2} \sum_i \int \frac{d^3 p}{E_i} f^i g^i.$ (3.15)
is the sum of squares and thus positive definite. This allows one to solve the linearized transport equation variationally, as (3.13) is reproduced by varying $Q[\chi]$ with respect to $\chi$, and demanding the result vanishes to first order in variations of $\chi$, i.e.,

$$\delta Q[\chi] = 0 + O(\delta \chi^2) \Rightarrow S^i(\vec{p}) = (C\chi_{\text{max}})^i(\vec{p}) .$$  

(3.16)

The functional defined in (3.15) can be written out explicitly as

$$Q[\chi] \equiv \frac{1}{2T^2} \sum_{i} \int P_1 \cdot P_{1i} f_{eq}^{1i} \chi_{1i}$$

$$+ \frac{1}{2T^4} \sum_{ijkl} \int \int \int \int f_{eq}^{ij} f_{eq}^{jk} W_{12\rightarrow34}^{ij\rightarrow k\ell} \delta^4(12-34) (\chi_{3k} P_3 \cdot P_1 + \chi_{4l} P_4 \cdot P_1 - \chi_{1i} P_1 \cdot P_1 - \chi_{2j} P_2 \cdot P_1) \chi_{1i}$$

$$\equiv \sum_{i} B_i + \sum_{ijkl} \left( Q_{31}^{ij\rightarrow k\ell} + Q_{41}^{ij\rightarrow k\ell} - Q_{11}^{ij\rightarrow k\ell} - Q_{21}^{ij\rightarrow k\ell} \right)$$  

(3.17)

This variational procedure, which closely follows the approach in [80], allows one to estimate $\chi_i$ using a finite basis expansion:

$$\chi_i(|\vec{p}|) = \sum_{n=1}^{N} c_{i,n} \Psi_{i,n}(|\vec{p}|) ,$$  

(3.18)

since imposing (3.16) allows one to calculate optimal coefficients $\{c_{i,n}\}$ that maximize $Q$. If the basis $\{\Psi_{i,n}\}$ is complete, the limit $n \rightarrow \infty$ reproduces the exact solution of (3.13). Numerical evaluation of $Q$ is discussed in Appendix H.

The variational method not only allows one to calculate the coefficients $\{c_{i,n}\}$ given a finite basis $\{\Psi_{i,n}\}$, but it also allows for a natural quantitative comparison of different basis choices. Given a basis, the set of coefficients that is a solution to (3.16) defines a set of $\{\chi_i^{\text{max}}\}$ such that the dimensionless number $Q[\{\chi_i^{\text{max}}\}] \equiv Q_{\text{max}}$ is maximal. Extremizing $Q$ in (3.15), one sees that the maximum value of the functional can be written in two equivalent ways

$$Q_{\text{max}} = \frac{1}{2} \langle \chi, (C\chi) \rangle|_{\chi=\chi_{\text{max}}} = \frac{1}{2} \langle \chi, S \rangle|_{\chi=\chi_{\text{max}}} \equiv \frac{1}{2} \sum_{i} B_i ,$$  

(3.19)

One can in principle use different sets of basis functions $\{\Psi_n\}$ for different particle species. For example, the basis that works best for each particle may change depending on the momentum scale when resonance formation for that particle becomes important in scattering amplitudes or a host of other factors.
where the second form, rewritten in the $B_i$ notation of (3.17), is easily evaluated once the function $\chi_{\text{max}}$ has been calculated. The value of the scalar $Q_{\text{max}}$ can be easily compared for different basis choices and the largest of these can be thought of as the “best” solution in this variational sense. While this comparison concept is a useful one, it will turn out that drastically different functional forms for $\{\Psi_{i,n}\}$ can, and often do, give very similar results for $Q_{\text{max}}$. Therefore, small improvements in the value of $Q_{\text{max}}$ can mask much more rapid changes in the functions $\chi_{i,\text{max}}$ as one converges to the exact solution, especially at very small and very large momenta. In other words, computation of $\chi_{i,\text{max}}$ accurately is orders of magnitude more challenging numerically than that of $Q_{\text{max}}$.

In addition to the two attractive features already discussed, the variational solution can also be used to calculate the shear viscosity of the system almost trivially.

### 3.3 Shear Viscosity from Variational Method

The extremal value of the functional $Q$ defined in (3.19) is directly proportional to the shear viscosity in the Navier-Stokes regime of the system. Throughout this work it is assumed that $\partial \cdot u = 0$, i.e. there are no bulk viscous effects. In this limit, the shear viscosity $\eta$ is defined through (3.2) as $\delta T^{\mu\nu} \equiv \eta \sigma^{\mu\nu} \equiv \eta T X^{\mu\nu}$. Plugging the previously derived form of the viscous corrections (3.9) into the general kinetic theory form of $\delta T^{\mu\nu}$ in (2.19) gives

$$\eta T X^{\mu\nu} = \sum_i \int \frac{d^3 p}{E_i} \frac{p^\mu p^\nu}{f_i} f_{\text{eq}}^i \chi_i P_{\alpha\beta} X^{\alpha\beta}.$$  (3.20)

To explicitly calculate $\eta$, one can take the $x-z$ component of the tensor equation (3.20) and evaluate in the local fluid rest frame. Just as in Appendix D, Eq. (D.5), if $\mu, \nu$ are spatial then $\alpha, \beta$ must be the same components as $\mu, \nu$ to give a nonvanishing result for the integral. The symmetric $X^{xz}$ will then be the only nonvanishing terms in $X$ on both sides and thus cancel, giving

$$\eta = \frac{1}{T} \sum_i \int \frac{d^3 p}{E_i} \frac{p^\mu p^\nu}{f_i} f_{\text{eq}}^i \chi_i \frac{p^x p^z + p^z p^x}{T^2} = \frac{2 \pi}{T^3} \frac{4 \pi}{15} \sum_i \int \frac{dp}{E_i} p^2 f_{\text{eq}}^i \chi_i,$$  (3.21)
where the factor $4\pi/15$ came from performing the angular integrals as in (D.9) and $p$ here is the magnitude of the 3-momentum in the fluid rest frame.

The integrals for $Q_{\text{max}}$ are of the same form since

$$Q_{\text{max}} = \frac{1}{2} \sum_i B_i \equiv \frac{1}{2} \frac{1}{2T^2} \sum_i \int \frac{d^3p}{2E_i} P_1 \cdot P_1 f_{i\text{eq}}^i X_i \ . \quad (3.22)$$

Evaluating the tensor dot product in (3.22) in the local fluid rest frame, one can see from (3.14) that

$$P_1 \cdot P_1 = (\tilde{p}_1 \cdot \tilde{p}_1)^2 - \frac{1}{3} |\tilde{p}_1|^2 |\tilde{p}_1|^2 = |\tilde{p}_1|^4 - \frac{1}{3} |\tilde{p}_1|^4 \equiv \frac{1}{T^4} \frac{2}{3} |P_{1,LR}|^4 , \quad (3.23)$$

so in terms of local rest frame variables, we have

$$Q_{\text{max}} = \frac{1}{4T^2} \frac{2}{3} \sum_i \int \frac{d^3p}{2E_i} \frac{p^4}{T^4} f_{i\text{eq}}^i X_i = \frac{1}{T^6} \frac{4\pi}{4} \sum_i \int \frac{dp}{E_i} \frac{p^2}{p^4} f_{i\text{eq}}^i X_i \ . \quad (3.24)$$

Comparing (3.24) with (3.21) gives the shear viscosity in terms of the functional maximum as

$$\eta = \frac{8}{5} Q_{\text{max}} T^3 \ . \quad (3.25)$$

One can also find what basis functions and coefficients the widely used democratic Grad ansatz corresponds to by comparing the democratic form of $\phi$,

$$\phi_i^{\text{dem}}(x, p) = \phi_i^{\text{dem}}(x, p) = \frac{\pi^{\mu\nu}(x)p_\mu p_\nu}{2(\epsilon(x) + P(x)T^2(x))} \ , \quad (3.26)$$

to the form (3.9) using the expansion, (3.18), for $\chi$:

$$\phi_i(x, p) = \chi_i(|\tilde{p}|) P^{\mu\nu} X_{\mu\nu} = \left( \sum_n c_{i,n} \Psi_{i,n}(|\tilde{p}|) \right) P^{\mu\nu} X_{\mu\nu} \ . \quad (3.27)$$

In the Navier-Stokes limit (3.2),

$$\pi^{\mu\nu} = \delta T^{\mu\nu}_{NS} = \eta T X^{\mu\nu} \ , \quad (3.28)$$

so comparing (3.26) with (3.27) gives

$$\left( \sum_n c_{i,n} \Psi_{i,n}(|\tilde{p}|) \right) P_{\alpha\beta} X^{\alpha\beta} = \frac{1}{2(\epsilon + P)} \frac{p_\alpha p_\beta}{T^2} (\eta T X^{\alpha\beta}) \ . \quad (3.29)$$
It is now apparent that the democratic Grad ansatz corresponds to a single basis function that is independent of momentum, $\Psi \equiv 1$, with coefficient given by

$$\chi^\text{dem}_i = \frac{\eta T}{2(\epsilon + P)} \rightarrow \frac{\eta}{2s}.$$  \ \ (3.30)

In the last step the thermodynamic identity $Ts = \epsilon + P - \sum \mu_c n_c$ was employed in the limit of vanishing chemical potentials appropriate for the midrapidity region in heavy-ion collisions at RHIC and LHC energies. Not only is the democratic Grad approach limited to only quadratic momentum dependence, but it also ignores all particle species dependence in $\phi_i(x, p)$ as the coefficients (3.30) are the same for all species in the system: $\chi^\text{dem}_i$ is simply one-half the dimensionless shear viscosity to entropy density ratio (assuming no chemical potentials at the transition). The rest of this thesis work will be devoted to calculating the species dependence of $\chi_i$ self-consistently from microscopic dynamics.
4. Dynamic Grad Ansatz

In the field of heavy-ion physics, most collision simulations thus far have employed the so-called democratic Grad ansatz \[ (2.20) \] to obtain the distribution functions needed to calculate observables through the Cooper-Frye prescription \[ (2.10) \]. This simple choice corresponds to corrections from local thermal equilibrium distributions of the form

\[
\delta f_i^{\text{dem}}(x, p) \equiv f_i^{\text{eq}}(x, p) \phi_i^{\text{dem}}(x, p) = f_i^{\text{eq}}(x, p) \frac{\pi^{\mu\nu}(x) p_\mu p_\nu}{2\left[ \epsilon(x) + P(x) \right] T^2(x)}
\]

\[
\equiv f_i^{\text{eq}}(x, p) \chi_i \chi_i^{\text{dem}} P^{\mu\nu}(p) X_{\mu\nu}(x), \quad (4.1)
\]

where the only dependence on particle species \( i \) is contained in the equilibrium distribution and there is no dependence on microscopic scattering rates in the system. The moniker “democratic Grad” was coined in [25] with “democratic” referring to the lack of dynamical or species-specific properties in \( \phi_i(x, p) \), and therefore \( \chi_i(\tilde{p}) \), and “Grad”\(^1\) referring to the quadratic momentum dependence of \( \phi_i(x, p) \) in (4.1), yielding a \( \chi_i \) independent of momentum.

The choice to use the same \( \chi_i(\tilde{p}) \) for each particle species is obviously unphysical for the transition of a viscous fluid to a gas of particles whose properties are governed by scattering rates between the particles. Even for a system with only a single particle species, deriving the properties of the gas, e.g. the shear viscosity, and the form of the distribution function, i.e. \( \chi_i(\tilde{p}) \), in terms of the scattering cross-section is useful. To this end, now consider the “dynamic Grad”\(^2\) ansatz where the “democratic” assumption is relaxed and particles are distinguished by their individual properties and scattering rates, but the “Grad” assumption of phase space corrections \( \phi_i(x, p) \) being quadratic in momentum still holds.

\(^1\)Harold Grad is the eponym of this phrase due to his seminal works in the kinetic theory of gases [85].

\(^2\)For brevity, the dynamic Grad ansatz will often be refered to as just the Grad ansatz as dynamics play a pivotal role for the rest of the calculations in this work.
4.1 Massless Systems in Grad Ansatz

The simplest system to first consider is a fluid to particle transition for a gas of massless particles interacting via energy independent, isotropic, $2 \rightarrow 2$ cross sections for which analytic results can be calculated to validate the variational approach, build intuition, and later verify general results in the ultrarelativistic limit. For simplicity, the case of a single massless particle species is considered first.

4.1.1 Massless Single-Component System

In this section the constant value of $\chi$ and the shear viscosity are calculated analytically in the Navier-Stokes limit for a gas containing a single massless particle species with a constant, isotropic scattering cross-section, $\sigma$, using the aforementioned variational method. This involves explicitly calculating the functional integrals defined in (3.17). The $B$ integral was already partially calculated in Section 3.3 as it is the only term independent of the microscopic dynamics of the system:

$$B \equiv \frac{1}{2T^2} \int \frac{d^3p}{2E} P_1 \cdot P_1 f^\text{eq} \chi = \frac{1}{T^6} \frac{4\pi}{6} \int \frac{dp}{E} \frac{p^2}{E} \frac{1}{p^4} f^\text{eq} \chi , \quad (4.2)$$

where for massless particles we can now substitute

$$E = p \equiv p_{LR} \Rightarrow f^\text{eq} = \frac{g}{(2\pi)^3} e^{\frac{\mu}{T}} \cdot (4.3)$$

As noted previously, $p$ will refer to the magnitude of the 3-momentum in the local rest frame of the fluid. Substituting the massless relationships (4.3) into (4.2) gives

$$B = \frac{1}{T^6} \frac{4\pi}{6} \chi e^{\frac{\mu}{T}} \int \frac{dp}{p} \frac{p^2}{p} \frac{g}{(2\pi)^3} e^{-\frac{\mu}{T}} = \chi \frac{g}{T^6} \frac{2\pi}{3(2\pi)^3} e^{\frac{\mu}{T}} \int \frac{dp}{p} p^5 e^{-\frac{\mu}{T}}$$

$$= \chi \frac{g}{T^6} \frac{1}{12\pi^2} e^{\frac{\mu}{T}} T^6 5! = 10\chi \frac{g}{\pi^2} e^{\frac{\mu}{T}} = 10\chi \frac{n}{T^3} , \quad (4.4)$$

where $n$ is the number density of particles in thermal equilibrium in the rest frame as derived in Appendix C.
The explicit calculation of the other terms in the functional (3.17) for a massless single component gas with isotropic $2 \rightarrow 2$ interactions can be found in Appendix F. The results for the five terms are:

$$B = 10 \chi \frac{n}{T^3}, \quad Q_{11} = 60 \chi^2 \sigma \frac{n^2}{T^4}, \quad Q_{21} = 0, \quad Q_{31} = 40 \frac{\sigma n^2}{T^4} \chi^2.$$ (4.5)

Note that for identical or massless particles, the expressions for $Q_{31}$ and $Q_{41}$ in (3.17) are equivalent. So for a one-component massless system, the functional evaluates to

$$Q[\chi] \equiv B + 2Q_{31} - Q_{11} - Q_{21} = 10 \chi \frac{n}{T^3} + \left( \frac{80}{3} - \frac{180}{3} \right) \sigma \frac{n^2}{T^4} \chi^2 = 10 \chi \frac{n}{T^3} - 100 \frac{\sigma n^2}{T^4} \chi^2.$$ (4.6)

Finding $\chi_{\text{max}}$ by requiring $\delta Q[\chi_{\text{max}}] = 0$ gives

$$10 \frac{n}{T^3} - \frac{200}{3} \frac{\sigma n^2}{T^4} \chi_{\text{max}} = 0.$$ (4.7)

Thus for a single component, massless system in the Grad ansatz

$$\chi_{\text{Grad}, m=0} = \frac{3}{20} \frac{T}{n \sigma} = \frac{3}{20} \lambda_{\text{MFP}} T.$$ (4.8)

The viscous correction $\chi$ is simply a dimensionless measure of the mean free path for this system.

The shear viscosity can also be calculated using the functional maximum from (3.25):

$$\eta = \frac{8}{5} Q_{\text{max}} T^3 = \frac{8}{5} T^3 \frac{1}{2} 10 \left( \frac{3}{20} \frac{T}{n \sigma} \right) \frac{n}{T^3}.$$ (4.9)

So the functional method reproduces the well-known Grad result [27]

$$\eta_{\text{Grad}, m=0} = \frac{6}{5} \frac{T}{\sigma}.$$ (4.10)

Note that the shear viscosity is inversely proportional to the cross-section and independent of the density. Both of these properties of shear viscosity are general and often found to be counter-intuitive. After a little more thought, one’s intuition about the cross section is fixed by noticing if the scattering rate is infinite, one is back to ideal hydrodynamics with zero viscosity as gradients are immediately communicated.
to all layers of the fluid. In the opposite limit of vanishing cross section, particles
never talk to each other and thus can maintain arbitrarily large gradients between
adjacent layers, in which case the viscosity would be infinite. The lack of dependence
on density is also surprising to some as a naive intuition often leads one to assume the
force transmitted between layers of the gas should scale with the number of particles.
While this intuition is indeed correct, its effect on the viscosity is perfectly balanced by
a reduction of the mean free path of the particles; there are more particles transfering
momentum, but they do it less effectively.

The functional method has returned results for the shear viscosity consistent with
known values from other methods for a single massless system, so it is now brought
to bear on more complicated multi-component systems.

4.1.2 Massless Two-Component System

Now consider extension to a minimalist multicomponent system with two massless
particle species which interact via elastic two-body scatterings with three interaction
channels: \( A + A \rightarrow A + A \), \( B + B \rightarrow B + B \), and \( A + B \rightarrow A + B \). Crossing symmetry
would also imply the inelastic channels \( A + A \rightarrow B + B \) and \( B + B \rightarrow A + A \), but
these are ignored here in order to isolate shear only\(^3\). Considering again isotropic,
energy-independent cross sections \( \sigma_{AA}, \sigma_{BB}, \text{ and } \sigma_{AB} \), and phase space corrections \( \phi_i \)
still assumed quadratic in momentum (Grad ansatz), the functional integrals in \( Q[\chi] \)
evaluate to (see Appendix H):

\[
B_i = 10 \frac{n_i}{T^3} \chi_i, \quad Q_{11}^{ij \rightarrow k\ell} = 30(1 + \delta_{k\ell}) \frac{\sigma_{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i^2, \quad Q_{21}^{ij \rightarrow k\ell} = 0
\]

\[
Q_{21}^{ij \rightarrow k\ell} = \frac{20}{3} (1 + \delta_{k\ell}) \frac{\sigma_{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i \chi_k, \quad Q_{41}^{ij \rightarrow k\ell} = \frac{20}{3} (1 + \delta_{k\ell}) \frac{\sigma_{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i \chi_\ell, \quad (4.11)
\]

where \((H.4), (H.5), \text{ and } (H.11)\) were used with \( E_a = p_a \), and \( \gamma_3 = \beta_3 = 1/2 \).
Inspection of these more general results shows an agreement with those for a single
massless species shown in the last Section and derived in Appendix E.

\(^3\)If particle densities are allowed to change, there will be dissipative effects due to a particle diffusion
current in addition to those from shear as discussed in Section 2.1.2.
The value of the functional (3.17) is calculated by summing the terms in (4.11) over all particle species, giving
\[
Q = \frac{10}{T}(n_A\chi_A + n_B\chi_B) - \frac{100}{3T^4}(\sigma_{AA}n_A^2\chi_A^2 + \sigma_{BB}n_B^2\chi_B^2) + \frac{20\sigma_{AB}n_A n_B}{3T^4}(4\chi_A\chi_B - 7\chi_A^2 - 7\chi_B^2),
\]
which is maximized when
\[
\chi_{A,Grad} = \frac{3LT}{20K_A(A)[5K_B(B) + 7K_B(A)] + K_A(A)[9K_B(B) + 7K_B(A)]},
\]
\[
\chi_{B,Grad} = \frac{3LT}{20K_B(B)[5K_A(A) + 7K_A(B)] + K_B(A)[9K_A(B) + 7K_A(A)]}.
\]
Here \(K_{i(j)} \equiv L/\lambda_{i(j)} \equiv Ln_j\sigma_{ij}\) denote partial inverse Knudsen numbers characterizing the scattering of species \(i\) off of species \(j\) and \(L\) is the characteristic macroscopic length scale for gradients in the system. All four \(K_{i(j)}\) appear because the solution to (3.13) is influenced by any particle in the microscopic scattering process that is out of equilibrium, whether that particle is incoming or outgoing. The partial inverse Knudsen numbers also come with different weights, therefore, in contrast to the single-component system, the result cannot in general be written with just the simple mean free path as \(\chi_i \sim T\lambda_i \equiv LT/K_i = LT/\sum_j K_{i(j)}\). The shear viscosity for this 2-component massless system in the Grad ansatz, calculated from (3.25) is
\[
\eta_{Grad|m_{i} = 0} = \frac{6T}{5} \frac{\sigma_{AB}(7r + 7r^{-1} + 4) + 5(\sigma_{AA} + \sigma_{BB})}{7\sigma_{AB}(\sigma_{AA}r + \sigma_{BB}r^{-1}) + 9\sigma_{AB}^2 + 5\sigma_{AA}\sigma_{BB}}, \quad r \equiv \frac{n_A}{n_B}.
\]
4.1.3 Comparison to Nonlinear Transport

The linearized transport results calculated in the previous two Sections correspond to the Navier-Stokes limit where the system has relaxed to a solution dictated only by the gradients of hydrodynamic variables and independent of the initial conditions and history of the system. For expanding systems, such as in heavy-ion collisions, relaxation to local equilibrium has to compete with dilution and cooling of the fluid. It is therefore important to check how well this late-time limit applies when local equilibrium is no longer a static fixed point in time.

A convenient test scenario for heavy-ion applications is a massless system undergoing boost-invariant 0+1D Bjorken expansion\footnote{Longitudinal boost invariance means that the state of the system at each point in spacetime with $t > 0$ and coordinate rapidity $\eta \neq 0$ can be obtained from the state on the $\eta = 0$ midrapidity sheet trivially via Lorentz boost along the $z$ direction. This ubiquitous assumption in the field of heavy-ion physics was first used by J.D. Bjorken in 1982 \cite{86} based on empirical observations.} with homogeneous and isotropic transverse directions $(x, y)$, as done in Ref. \cite{87}, but here with a two-component $A+B$ mixture.

The system is initialized in local thermal equilibrium at longitudinal proper time $\tau \equiv \sqrt{t^2 - z^2} = \tau_0$, but due to expansion, dissipative corrections quickly develop which can be conveniently quantified by the partial shear stresses of the two species. Due to scaling of the transport solutions \footnote{In a scale invariant system all cross sections are set by the temperature, i.e., $\sigma \propto 1/T^2$. However, as shown in Ref. \cite{87}, $\sigma \propto 1/T^2$ is well approximated by $\sigma \propto \tau^{2/3}$ because for 0+1D Bjorken expansion $T \propto \tau^{-1/3}$ as long as the system is near local equilibrium.}, the evolution only depends on the dimensionless ratio $\tilde{\tau} \equiv \tau/\tau_0$ along with the partial inverse Knudsen numbers defined previously, $K_{i(j)} \equiv \tau/\lambda_{i(j)} = \tau n_j \sigma_{ij}$, where the characteristic scale for gradients, $L$, is the proper time $\tau$. The initial temperature $T_0$ plays no role beyond setting the momentum scale (all momenta are proportional to $T_0$). As in Section 4.1.2, only elastic two-body interactions $A + A \rightarrow A + A$, $B + B \rightarrow B + B$, and $A + B \rightarrow A + B$ are included. All three cross sections are set to grow with time as $\sigma_{ij} \propto \tau^{2/3}$, which ensures approximately scale invariant dynamics with $\eta/s \approx \text{const}$. In such a scenario,
longitudinal expansion first drives the system out of local equilibrium but at late times
the system returns asymptotically to local equilibrium.

By symmetry, the phase space densities \( f_i(\tau, p_T, \xi) \) only depend on proper time \( \tau \),
transverse momentum magnitude \( p_T \), and the difference \( \xi \equiv \eta - y \) between coordinate
rapidity \( \eta \) and momentum rapidity \( y \) (see Appendix K for definitions). The flow
velocity is constrained to \( u^\mu = (\text{ch} \eta, 0, 0, \text{sh} \eta) \), and for both species shear stress is
diagonal in the LR \((\eta = 0)\) frame, i.e., \( \pi^\mu_\nu_{i,LR} = \text{diag}(0, -\pi_{L,i}/2, -\pi_{L,i}/2, \pi_{L,i}) \), where \( \pi_{L,i} \) is the longitudinal shear stress for species \( i \).
Assuming dissipative corrections are quadratic in momentum, we have
\[
\phi_i = c_i \frac{\pi^\mu_\nu p_\mu p_\nu}{2(\epsilon + P)T^2} = c_i(\tau) \frac{\pi_L(\tau)}{8P(\tau)T^2(\tau)} \left( \text{sh}^2 \xi - \frac{1}{2} \right), \quad c_i(\tau) = \frac{\pi_{L,i}(\tau)}{P_i(\tau)} \frac{P(\tau)}{\pi_L(\tau)}
\]
where \( \epsilon_i = 3P_i \) was substituted for massless particles. Up to the factor \( P/\pi_L \) that
is common to all species, \( c_i \) describes how far species \( i \) is from local equilibrium. In
the late-time Navier-Stokes regime, linearized kinetic theory was used previously in
(4.14) to predict
\[
\frac{c_B}{c_A} = \frac{5K_A + 2(K_{A(B)} + K_{B(A)})}{5K_B + 2(K_{A(B)} + K_{B(A)})} \quad (K_i \equiv \sum_j K_{i(j)})
\]
(4.17)
In contrast, the “democratic Grad” approach postulates \( c_i = 1 \) for all species so that \( c_B/c_A = 1 \).

Figure 4.1 compares these two extremes to fully nonlinear transport solutions
obtained numerically using Molnar’s Parton Cascade code (MPC) version 1.8.13 [88].
The simulations are initialized with uniform coordinate rapidity distributions \( dN/d\eta \)
in a wide window \( |\eta| < 5 \). To avoid the \( |\eta| \gtrsim 4 \) edges of the system where boost
invariance is strongly violated, shear stress evolution is extracted only using particles
with \( |\eta| < 2 \) (properly boosted to the \( \eta = 0 \) frame). A variety of relative cross sections
and densities between the two species were explored in five different scenarios shown
in the table in Fig. 4.1, all of which keep species \( A \) closer to equilibrium than species
\( B \). In all five cases, the ratio of viscous corrections \( c_B/c_A \) starts from unity but then
relaxes to a constant value at late times that depends on the partial inverse Knudsen
numbers of the system. While the commonly used “democratic Grad” ansatz fails to account for the species dependence of viscous corrections, the variational method using linearized transport (Eq. (4.17)) captures the corrections with better than 10% accuracy in all five scenarios. Thus it seems that at late times, \( \tau \gtrsim 5\tau_0 \approx 2.5 \text{ fm} \), and for inverse Knudsen numbers \( 2 - 3 \) corresponding to \( \eta/s \approx 1/4\pi \) [87], the Navier-Stokes limit is still a reasonable approximation for heavy-ion applications despite rapid longitudinal expansion of the system.

![Graph showing dissipative corrections as a function of normalized proper time for a massless two-component system in a 0+1D Bjorken scenario.](image)

Figure 4.1.: *Left plot:* Ratio of dissipative corrections as a function of normalized proper time for a massless two-component system in a 0+1D Bjorken scenario, calculated from nonlinear \( 2 \rightarrow 2 \) covariant transport using MPC. Five different scenarios a) - e) with various cross sections and densities are shown, labeled with the ratio of inverse Knudsen numbers \( K_A/K_B \). The ratios of the densities and cross section in each scenario are shown in the table on the right. Thin, horizontal dotted lines and arrows on the right side of the plot correspond to the expectation from the self-consistent variational calculation based on linearized transport in the quadratic Grad approximation (“dynamical Grad” approach). Only four such lines and arrows are visible because scenarios b) and c) are identical except for the timescale of relaxation to Navier-Stokes regime; scenario b) relaxes 5/3 times quicker than c).
4.2 Simple Massive Systems in Grad Ansatz

In the last Section, phase space corrections and the shear viscosity were calculated analytically using the functional method based on the linear transport equation for a transition from a viscous fluid to a gas of massless particles. In this Section, the shear viscosity of a single-component system with arbitrary mass is calculated analytically in the Grad ansatz and will serve as a check for future numeric work. The distribution functions and shear viscosity of a two-component system in the non-relativistic limit are also calculated using the functional method. These analytic calculations will be used to aid intuition and add understanding to fully relativistic results for more realistic systems used in heavy-ion applications.

In the last part of this section, a transition to a two-component pion-nucleon gas is considered. This will involve fully relativistic numerical calculations and will be the first system used to test the effects on heavy-ion observables.

4.2.1 Viscosity of a Relativistic, Massive One-Component Gas

For a one-component system, the shear viscosity is known analytically in the Grad approximation for arbitrary $m/T \equiv z$ with fully relativistic kinematics. For calculational details, see Appendix G based on the method used in Chapter XI of Ref. [27]. The result turns out to be

$$\eta^{\text{Grad}} = \frac{T}{\sigma} \frac{15z^2 K_2^2(z)h^2(z)}{16[(15z^2 + 2)K_2(2z) + (3z^3 + 49z)K_3(2z)]} \quad , \quad h(z) = \frac{zK_3(z)}{K_2(2z)}$$

(4.18)

where $K_n(z)$ is a modified Bessel function of the second kind. The complete derivation of the formula in Ref. [27] was rechecked as shown in Appendix G. This calculation revealed a typographic error in the book; the correct coefficient in the denominator is 15, not 5. The numerical integration method in Appendix H using the functional approach also reproduces this result for the viscosity with the newly derived coefficient of 15.
4.2.2 Nonrelativistic Two-Component System

In the limit of nonrelativistic particles in the Grad approximation, the functional integrals evaluate to (see Appendix J),

\[
B_i = \frac{5 z_i n_i}{2} T^3 \chi_i
\]

\[
Q_{11}^{ij \rightarrow k\ell} = \frac{1}{3 \sqrt{2\pi} z_j^{1/2}} \frac{z_i^{3/2}}{(z_i + z_j)^{3/2}} \frac{15 z_i^2 + 40 z_i z_j + 24 z_j^2}{T^3} (1 + \delta_{k\ell}) \frac{\sigma^{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i^2
\]

\[
Q_{21}^{ij \rightarrow k\ell} = -\frac{1}{3 \sqrt{2\pi} z_j^{1/2}} \frac{z_i z_j}{(z_i + z_j)^{3/2}} (1 + \delta_{k\ell}) \frac{\sigma^{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i \chi_j
\]

\[
Q_{31}^{ij \rightarrow k\ell} = \frac{5}{\sqrt{2\pi} z_j^{1/2}} \frac{z_i^{3/2}}{(z_i + z_j)^{3/2}} \frac{z_k^2}{T^3} (1 + \delta_{k\ell}) \frac{\sigma^{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i \chi_k
\]

\[
Q_{41}^{ij \rightarrow k\ell} = \frac{5}{\sqrt{2\pi} z_j^{1/2}} \frac{z_i^{3/2}}{(z_i + z_j)^{3/2}} \frac{z_\ell^2}{T^3} (1 + \delta_{k\ell}) \frac{\sigma^{ij \rightarrow k\ell} n_i n_j}{T^4} \chi_i \chi_\ell
\]

where equilibrium densities were substituted,

\[
n_i^{NR} = \frac{g_i}{(2\pi)^{3/2}} (m_i T)^{3/2} e^{(\mu_i - m_i)/T}.
\]

For the special case of a one-component nonrelativistic system, the above imply

\[
\chi^{\text{Grad}} = \frac{5 \sqrt{\pi}}{32} \sqrt{\frac{T}{z n \sigma}} \Rightarrow \eta^{\text{Grad}} = \frac{5 \sqrt{\pi}}{16} \sqrt{\frac{T}{\sigma}}.
\]

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\]

reproducing the familiar nonrelativistic expression for shear viscosity at the end of Chapter 14 in [89].\(^6\)

The dependence on the dimensionful quantities \(T, n,\) and \(\sigma\) is the same as in the massless case in (4.8) and (4.10): \(T/(n \sigma)\) and \(T/\sigma,\) respectively.

It is important to note for future reference that for fixed density and cross section, the relative viscous correction \((\delta f/f^\text{eq} \equiv \phi)\) decreases when mass increases, while the shear viscosity increases with mass.

\(^6\)For an alternate derivation from kinetic theory, see Chapter 3 of [90].
More generally, for a two-component nonrelativistic \( A + B \) system with isotropic, energy-independent, elastic scattering in the Grad approximation, substituting (4.19) into (3.17) gives

\[
Q[\chi_A, \chi_B] = \frac{5z_A n_A \chi_A}{2T^3} - \frac{8\sigma_{AA} n_A^2 z_A^{3/2} \chi_A^2}{\sqrt{\pi} T^4} \\
+ \frac{8\sqrt{2} \sigma_{AB} n_A n_B z_A^{3/2} z_B^{1/2} (5z_A + 3z_B) \chi_A - 2z_B \chi_B] \chi_A}{3\sqrt{\pi} T^4(z_A + z_B)^{3/2}} + A \leftrightarrow B . \tag{4.22}
\]

The general structure of the solution is very similar to the massless case with all partial inverse Knudsen numbers contributing with different weights that now depend on the masses. In the limit when species \( B \) is much more dilute than species \( A \) (for example, because it is near thermal equilibrium and very heavy, e.g., protons compared to pions), one can approximate \( n_B \to 0 \) to obtain

\[
\chi_A^{\text{Grad}} \bigg|_{n_B \to 0} = \frac{5\sqrt{\pi}}{32} \sqrt{\frac{T}{m_A \sigma_{AA} n_A}} , \\
\chi_B^{\text{Grad}} \bigg|_{n_B \to 0} = \chi_A^{\text{Grad}} \frac{3(\mu + 1)^2 \sigma_{AA} + 2\sqrt{2\mu(1 + \mu)} \sigma_{AB}}{\sqrt{2\mu(1 + \mu)}(3 + 5\mu) \sigma_{AB}} \quad (\mu = \frac{m_B}{m_A}) . \tag{4.23}
\]

In this special case species \( A \) is unaffected by species \( B \) and \( \sigma_{BB} \) is irrelevant as scatterings off of \( B \) particles are vanishingly rare. Comparing the corrections for the two species in two special cases,

\[
\frac{\chi_B}{\chi_A} = \frac{3\sigma_{AA}}{4\sigma_{AB}} + \frac{1}{4} \quad \text{if} \quad m_A = m_B , \\
\frac{\chi_B}{\chi_A} \approx \frac{3\sigma_{AA}}{5\sqrt{2} \sigma_{AB}} \quad \text{if} \quad m_B \gg m_A . \tag{4.24}
\]

Thus, at least in the nonrelativistic limit, the much heavier species tends to have a smaller viscous correction even when its interaction cross section is the same as that of the lighter species. These results for one heavy species and one light species will be useful in trying to understand the results for the transition to a system of pions and nucleons in the next section, especially at lower transition temperatures.
4.2.3 Pion-Nucleon Gas: Constant Cross Section

Now consider a more realistic transition from a fluid to a gas of pions and nucleons with fully relativistic kinematics. Grouping spin and isospin states into a single species and choosing representative masses, this is equivalent to a two-component pion-nucleon system with $m_\pi = 140$ MeV, $g_\pi = 3$ for isospin-1 pions, and $m_N = 940$ MeV, $g_N = 4$ for spin $1/2$ protons and neutrons. For transition temperatures of interest, $120$ MeV $\lesssim T \lesssim 165$ MeV, one can approximate the two-body cross sections which, in general depend on the 4-momenta of all particles involved, with the isotropic, energy-independent, effective values $\sigma_{\pi\pi}^{\text{eff}} = 30$ mb, $\sigma_{\pi N}^{\text{eff}} = 50$ mb, and $\sigma_{NN}^{\text{eff}} = 20$ mb. These values were chosen so that for a static system ($u^\mu = (1, 0)$) in thermal and chemical equilibrium, the mean times $\bar{\tau}_{ij}$ between scatterings for particles of species $i$ with those of species $j$, defined as

$$\frac{1}{\bar{\tau}_{ij}} \equiv \langle n_j \sigma_{ij} v_{\text{rel}} \rangle \equiv \frac{1}{n_i} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} f_{eq}^i(p_1) f_{eq}^j(p_2) \sigma_{ij} F(s),$$

(4.25)

are comparable to the values shown in Figs. 2b and 5a of Ref. [91] for temperatures $120$ MeV $\lesssim T \lesssim 165$ MeV. The mean scattering times calculated with these constant effective cross section values are shown as a function of temperature in Table 4.1 which also includes values for $T = 100$ and 200 MeV outside the matching range. The term $F(s)$ in (4.25) is defined as

$$F(s) \equiv p_{cm} \sqrt{s} \equiv E_1 E_2 v_{\text{rel}} = \frac{1}{2} \sqrt{(s - m_i^2 - m_j^2)^2 - 4m_i^2 m_j^2},$$

(4.26)

and is often called the flux factor.

Note that at these temperatures pions are much more abundant than nucleons. (From (C.12) with vanishing chemical potentials, there are 84 times as many pions as nucleons at 120 MeV and 17 times as many at 165 MeV). If there are many more pions than nucleons in the system, then nucleon-nucleon scattering affects viscous corrections negligibly (i.e. one could put $\sigma_{NN} = 0$ to good approximation) as discussed in the previous section.

Calculating the viscous corrections using the functional method involves 12 integrals, 3 for each of the momenta of the 4 particles in the $2 \rightarrow 2$ scattering in (3.17).
| $T$ [MeV] | $\bar{\tau}_{\pi\pi}$ [fm] | $\bar{\tau}_{N(\pi)}$ [fm] | $\bar{\tau}_{NN}$ [fm] |
|----------|-----------------|-----------------|-----------------|
| 100      | 12.7            | 8.2             | 8300            |
| 120      | 6.6             | 4.2             | 1200            |
| 140      | 3.9             | 2.4             | 280             |
| 165      | 2.2             | 1.4             | 73              |
| 200      | 1.2             | 0.73            | 18              |

Table 4.1: Mean scattering times in a pion-nucleon gas with effective cross sections $\sigma^{\text{eff}}_{\pi\pi} = 30$ mb, $\sigma^{\text{eff}}_{\pi N} = 50$ mb, and $\sigma^{\text{eff}}_{NN} = 20$ mb. Values are rounded to the two most significant digits.
Four of the integrals can be eliminated using the energy-momentum $\delta$-function and for the case of isotropic, energy-independent scattering, one can do 4 more of the integrals analytically (see Appendix H). This leaves a nested 4-dimensional integral to be done numerically for this pion-nucleon system. Calculating 3-dimensional nested integrals with general math software such as Mathematica is time consuming; 4-dimensional ones are practically an impossible task. To numerically calculate these 4-dimensional nested integrals, moreover, in a timely fashion, one must write a computer code for the task. The code used here was written as part of this thesis work. It was coded in C++ and uses adaptive routines from the GNU Standard Library (GSL) [26] to do the calculation. The numerical solutions for the relative viscous corrections obtained by maximizing the functional are shown in Table 4.2.

| $T$ (MeV) | 100  | 120  | 140  | 165  |
|-----------|------|------|------|------|
| $c_\pi$   | 1.00 | 1.01 | 1.03 | 1.05 |
| $c_N$     | 0.48 | 0.51 | 0.55 | 0.60 |

Table 4.2: Self-consistent corrections $\chi_i$ relative to the democratic Grad value ($\eta/2s$), or equivalently, $c_i$ as defined in (4.16) for shear stress as a function of temperature for a chemically equilibrated pion-nucleon gas, in the Grad approximation, with effective cross sections $\sigma_{\pi\pi}^{\text{eff}} = 30$ mb, $\sigma_{\pi N}^{\text{eff}} = 50$ mb, and $\sigma_{NN}^{\text{eff}} = 20$ mb.

For the $\pi - N$ system, the ratio of viscous coefficients is $c_\pi / c_N \sim 2$ in the window of typical switching temperatures, $100 < T < 165$ MeV. This means that nucleons are about twice as close as pions to equilibrium (at the same momentum), in qualitative agreement with the analytic results in Section 4.2.2. For example, the nonrelativistic formula (4.23) would predict $c_\pi / c_N \approx 2.9$, which is not far off considering that pions are relativistic at these temperatures. The primary origin of the pion-nucleon difference is the larger $\pi N$ cross section; a nucleon scatters more frequently off pions than a pion scatters off another pion. However, based on the earlier discussion, one would still expect $c_\pi > c_N$ even if $\sigma_{\pi\pi} = \sigma_{\pi N}$. 
The above pion-nucleon difference is reflected in pion vs. proton identified particle observables if these self-consistent, species-dependent viscous corrections to the distribution functions are included in Cooper-Frye freezeout. To estimate the effects, one can perform a hydrodynamic simulation of $Au + Au$ at top RHIC energy $\sqrt{s_{NN}} = 200$ GeV with impact parameter $b = 7$ fm, and look at the difference between pion and proton elliptic flow. The calculations were done with AZHYDRO version 0.2p2 [28], which is a 2+1D relativistic ideal hydrodynamics code with longitudinal boost invariance. This version includes the fairly recent state-of-the-art s95-p1 equation of state parameterization [74] by Huovinen and Petreczky that matches lattice QCD results to a hadron resonance gas.

Because there is no dissipation in AZHYDRO, one must estimate the shear stress on the conversion hypersurface from the gradients of the ideal flow fields using the Navier-Stokes formula (3.2), i.e., $\pi^{\mu\nu}(x) = \eta \sigma^{\mu\nu}(x)$. This is in the same spirit as an early exploration of shear stress corrections by Teaney [94], except here real hydrodynamic solutions are used from the AZHYDRO simulation instead of a parameterization. The shear viscosity to entropy density is taken to be $\eta/s = 0.1$, near its AdS/CFT conjectured minimum, and the shear viscosity needed to calculate the viscous corrections is determined using the thermodynamic identity

$$\eta = \frac{\eta \epsilon(x) + P(x)}{s T(x)} \quad (\mu_B = 0), \quad (4.27)$$

evaluated on spacetime points $x$ on the isothermal conversion surface.

For initial conditions at Bjorken proper time $\tau_0 = 0.5$ fm, the transverse entropy density distribution $ds/d^2 x_T d\eta$ was set to a 25%+75% weighted sum of binary collision and wounded nucleon profiles ($\sigma_{NN}^{inel} = 40$ mb), with diffuse Woods-Saxon nuclear densities for gold nuclei (Woods-Saxon parameters $R = 6.37$ fm, $\delta = 0.54$ fm), a peak entropy density value $s_0 = \frac{1}{\tau_0} \frac{ds(x_T=0)}{dx_T d\eta} = 110/\text{fm}^3$, and vanishing baryon density $n_B = 0$ everywhere. The parameters used in the Glauber model of high energy nuclear collisions are reviewed in [22]. With ordinary, ideal ($\delta f = 0$) Cooper-Frye freezeout at temperature $T_{conv} = 140$ MeV, these initial conditions roughly reproduce the experimentally measured pion spectrum. In the following, the initial conditions
are kept fixed but the conversion temperature, $T_{\text{conv}}$, is varied to study pion and proton elliptic flow from fluid-to-particle conversion with self-consistent viscous $\delta f_i$ corrections. The viscous Cooper-Frye code was written as part of this thesis work and discussed in Appendix K (The AZHYDRO code package can only handle ideal freezeout with $\delta f = 0$).

![Figure 4.2](image)

**Figure 4.2.** Differential elliptic flow $v_2(p_T)$ of pions and protons in $Au + Au$ at $\sqrt{s_{NN}} = 200$ GeV at RHIC with impact parameter $b = 7$ fm, using 2+1D boost invariant hydrodynamic solutions from AZHYDRO [28, 48, 92, 93], and Cooper-Frye fluid-to-particle conversion at $T_{\text{conv}} = 165$ MeV. Dashed lines are for pions, while solid curves are for protons. The standard “democratic Grad” approach (open boxes) is compared to self-consistent shear corrections (crosses) computed for a pion-nucleon gas from linearized kinetic theory (see text). In both cases, $\eta/s = 0.1$ at conversion. Results with uncorrected, local equilibrium phase space distributions ($\delta f = 0$) are also shown (filled circles).

Figure 4.2 shows differential elliptic flow results $v_2(p_T)$ for pions and protons for freezeout at $T_{\text{conv}} = 165$ MeV. The pion and proton elliptic flow curves separate already in the ideal conversion case (filled circles), following the characteristic mass
ordering\footnote{This mass ordering is well understood in hydrodynamic simulations as coming from the fact that all particles coming from the plasma move with the same fluid velocity $u^\mu(x)$ as the expanding fluid (apart from thermal motion). This means the behavior of higher mass particles (protons) corresponds to lower mass particles (pions) at lower transverse momentum.} of $v_2$ in hydro, though this effect diminishes at high $p_T$. Viscous freezeout with the commonly used democratic ansatz (open boxes) preserves the mass ordering but with $v_2$ strongly suppressed by dissipation, even for the modest value $\eta/s = 0.1$ used here. In this calculation, dissipative effects are only present in the shear viscous phase space corrections $\delta f_i$ at fluid-to-particle conversion (ideal hydrodynamic evolution), but viscous corrections to the evolution of hydrodynamic flow and temperature fields are known\textsuperscript{95-97} to have smaller influence on $v_2$ then $\delta f$ itself. In contrast, self-consistent species-dependent freezeout (crosses) leads to a clear pion-proton elliptic flow splitting at moderately high transverse momentum, with the proton $v_2$ exceeding that of the pion by 30%. Both species exhibit a strong viscous suppression in $v_2$, however, the suppression is smaller for protons because they are more equilibrated than pions ($c_\pi/c_N \sim 2$). At low $p_T$ the mass effect is still present, which means that the pion and proton elliptic flow curves necessarily cross each other (at around $p_T \sim 1$ GeV in this calculation). The reason why the pion results are almost identical to “democratic” freezeout is that at $T = 165$ MeV the pion density is much higher than the proton density, i.e., the dynamics of pions is largely unaffected by the protons, and both the shear viscosity and the entropy density are then dominated by pions. In other words, almost all the particles present at high temperature are pions, so the democratic approach essentially treats all particles as pions already. The temperature $T_{\text{conv}} = 165$ MeV used here is the same as the typical switching temperature used in hybrid hydro+transport models\textsuperscript{98}. It would be very useful to initialize the transport stage of hybrid calculations with the self-consistent viscous distributions for each species calculated here, and check the effect on identified particle elliptic flow and other observables at the end of the hadron transport evolution. This application is the ultimate goal of calculations throughout this work where the higher conversion temperatures are used.
Figure 4.3.: Differential elliptic flow $v_2(p_T)$ of pions and protons in $Au + Au$ at $\sqrt{s_{NN}} = 200$ GeV at RHIC with impact parameter $b = 7$ fm, using 2+1D boost invariant hydrodynamic solutions from AZHYDRO \cite{28, 48, 92, 93}, and Cooper-Frye fluid-to-particle conversion at $T_{\text{conv}} = 140$ MeV. Dashed lines are for pions, while solid curves are for protons. The standard “democratic Grad” approach (open boxes) is compared to self-consistent shear corrections (crosses) computed for a pion-nucleon gas from linearized kinetic theory (see text). In both cases, $\eta/s = 0.1$ at conversion. Results with uncorrected, local equilibrium phase space distributions ($\delta f = 0$) are also shown (filled circles).

Figure 4.3 shows the same $v_2(p_T)$ calculation but with a lower $T_{\text{conv}} = 140$ MeV. The qualitative picture is the same, but for $T_{\text{conv}} = 140$ MeV the viscous suppression of $v_2$ is smaller in magnitude (closer to ideal) because, for the Navier-Stokes stresses \cite{3.2} used here, flow gradients $\partial \mu u^\nu \sim 1/\tau$ are smaller if one waits longer for the fluid to cool to a lower conversion temperature. The mass ordering is also stronger, which is expected as it is driven by $m/T$. At the higher values of $p_T \sim 2 - 2.5$ GeV, the relative difference between proton $v_2$ curves from the “democratic” and the self-consistent approaches is smaller than for $T_{\text{conv}} = 165$ MeV. However, the relative
change in viscous suppression of $v_2$ from ideal is actually larger; the difference for protons between ideal freezeout and the viscous result shrinks by a factor of two at $T_{\text{conv}} = 140$ MeV when the fluid is converted to particles with the self-consistent (species-dependent) procedure.

Figure 4.4.: Differential elliptic flow $v_2(p_T)$ of pions and protons in Au + Au at $\sqrt{s_{NN}} = 200$ GeV at RHIC with impact parameter $b = 7$ fm, using 2+1D boost invariant hydrodynamic solutions from AZHYDRO [28, 48, 92, 93], and Cooper-Frye fluid-to-particle conversion at $T_{\text{conv}} = 120$ MeV. Dashed lines are for pions, while solid curves are for protons. The standard “democratic Grad” approach (open boxes) is compared to self-consistent shear corrections (crosses) computed for a pion-nucleon gas from linearized kinetic theory (see text). In both cases, $\eta/s = 0.1$ at conversion. Results with uncorrected, local equilibrium phase space distributions ($\delta f = 0$) are also shown (filled circles).

As show in Fig. 4.4 for the even lower temperature $T_{\text{conv}} = 120$ MeV, dissipative corrections for $\eta/s = 0.1$ are basically negligible for protons for $p_T < 2.5$ GeV, at least with the Navier-Stokes shear stress used here. For pions there is a less than 10%
suppression in $v_2$ at high $p_T$; The flow gradients have all essentially been smeared out by this time in the ideal hydrodynamic evolution.

### 4.2.4 Pion-Nucleon Gas: Energy-Dependent Cross Section

Now that an understanding of the viscous corrections and their effects on elliptic flow has been developed, one can turn to a system governed by more realistic cross sections to see how different cross sections affect observables. In this section, consider the same pion-nucleon gas as before only now instead of constant cross sections, include the full energy-dependent elastic cross sections between the two different species while maintaining the isotropic scattering approximation. The nucleon-nucleon cross section is kept at a constant 20 mb since, as noted earlier in Section 4.2.2, this parameter is mostly unimportant due to the small number of protons in the system relative to pions. The other two cross sections, however, will be realistic cross sections for each isospin pair parametrized similarly to how it is done in hadron transport codes [99] to match the channels in the Review of Particle Properties handbook [6]. Isospin-weighted averages of these channels were then calculated using the appropriate Clebsch-Gordon coefficients so that one ends up with two different energy-dependent cross sections; one for pion-nucleon scattering and one for pion-pion scattering. The cross-sections were tabulated and interpolated for the 4-dimensional numerical integration. The resulting cross-sections are plotted in Fig. 4.5.

The numerical solutions for the relative viscous corrections obtained using the functional method with the energy-dependent cross sections plotted in Fig. 4.5 are shown in Table 4.3. The general trend for the viscous coefficients is a reduction in the ratio of $c_\pi/c_N$ from 2 for constant effective cross sections to roughly 1.5 using the energy-dependent ones; The protons are roughly 25% closer to equilibrium after using the energy-dependent cross sections.

To see what difference this complicated energy dependence in the cross sections can have on observables, the self-consistent corrections were again used in the viscous
Figure 4.5.: Binary hadron-hadron scattering cross sections as a function of center of mass energy $\sqrt{s}$ calculated from isospin-weighted channels using the Clebsch-Gordon coefficients compared to scattering data from the Particle Data Group [6] for pion-pion scattering (left) and pion-nucleon scattering (right).

| $T$ = 100 | 120 | 140 | 165 MeV |
|-----------|-----|-----|---------|
| $c_\pi$   | 1.00| 1.01| 1.02   | 1.03    |
| $c_N$     | 0.63| 0.67| 0.70   | 0.72    |

Table 4.3: Self-consistent corrections $\chi_i$ relative to the democratic Grad value ($\eta/2s$) as function of temperature for a chemically equilibrated pion-nucleon gas in the Grad approximation with realistic energy-dependent cross sections from isospin-weighted sums plotted in Fig. 4.5.

Cooper-Frye decoupling with freezeout hypersurfaces obtained from the same AZHYDRO simulation used previously. The result for differential elliptic flow, $v_2(p_T)$, for identified pions and protons is shown in Fig. 4.6.

The qualitative picture of elliptic flow after including realistic energy-dependent cross sections is the same as the previous case with the constant effective cross sections. The proton $v_2$ is suppressed a bit more with realistic cross sections, yielding a value exceeding that of the pion by 20% at higher $p_T$. Thus, the results for elliptic
Figure 4.6.: Differential elliptic flow $v_2(p_T)$ of pions and protons in $Au + Au$ at $\sqrt{s_{NN}} = 200$ GeV at RHIC with impact parameter $b = 7$ fm, using 2+1D boost invariant hydrodynamic solutions from AZHYDRO and Cooper-Frye fluid-to-particle conversion at $T_{\text{conv}} = 165$ MeV. Dashed lines are for pions, while solid curves are for protons. The “Dynamic” (triangles) and ideal (filled circles) curves are the same as Fig. [4.2] while the “Real $\sigma$” curve (upside-down triangles) was calculated using realistic energy-dependent cross-sections from isospin-weighted sums plotted in Fig. [4.5].

flow are closer to the democratic case after using realistic cross sections. This does not mean that the specific dynamics of the particles do not matter. On the contrary, it was shown that the elliptic flow of protons is quite different for the dynamic Grad assumption and the democratic Grad assumption. The effect from using the specific energy dependent elastic cross sections considered here just happens to bring the elliptic flow closer to the democratic case when the corrections are constrained to be quadratic in momentum and the simple model of a transition to a pion-nucleon gas is assumed.
4.3 Multi-Component Hadron Gas

In Section 4.2.3 self-consistent corrections to particle distribution functions were calculated for a Cooper-Frye transition from a hydrodynamic fluid to a pion-nucleon gas. This pion-nucleon gas is clearly only a crude approximation of reality as it ignores interactions of pions and nucleons with all other particle species in the system. It is natural to extend the previous investigation to mixtures with many hadronic species, in which case each species will accrue its own dissipative corrections based on its microscopic dynamics in the system. The problem is complicated, however, because it requires knowledge of hadronic scattering rates between all species. In principle, these are encoded in hadronic transport codes, such as UrQMD [100], AMPT [101], or JAM [102], but calculations using the energy-dependent cross sections with resonance formation for all hadronic species is left as a future direction of study beyond this thesis work. In this work, only two simple models are considered: 1) a hadron gas in which each species has the same, fixed scattering cross section akin to the model used in Ref. [103], and 2) a gas with more realistic cross sections that follow additive quark model [100,104,105] (AQM) scaling, i.e., constant meson-meson, meson-baryon, and baryon-baryon cross sections which scale with the number of quarks in the colliding hadrons such that the ratios are $\sigma_{MM} : \sigma_{MB} : \sigma_{BB} = 4 : 6 : 9$. In both models, only elastic $ij \rightarrow ij$ channels, including the $i = j$ channel, with energy-independent, isotropic cross sections are considered.

For the fixed cross section scenario 1, the value $\sigma_{ij} = 30$ mb is used, the same value as the effective $\sigma_{\pi\pi}$ for the pion-nucleon gas studied earlier (cf. Fig. 4.2). For the AQM model scenario 2, the same $\sigma_{MM} = 30$ mb is used for consistency, which implies $\sigma_{MB} = 45$ mb, and $\sigma_{BB} = 67.5$ mb. As done in Section 4.2.3 members of the same isospin multiplet, as well as their antiparticles, are grouped into one species with appropriately scaled degeneracy factors so that the number of degrees of freedom along with the particle densities remains the same. The system considered
Figure 4.7.: Self-consistent corrections $\chi_i$ relative to the democratic Grad value ($\eta/2\varsigma$), or equivalently, $c_i$ as defined in (4.16) for shear stress as a function of temperature for the chemically equilibrated hadron gas described in the text, in the Grad approximation, with effective cross sections $\sigma_{ij} = 30$ mb (left) and $\sigma_{MM} = 30$ mb, $\sigma_{MB} = 45$ mb, and $\sigma_{BB} = 67.5$ mb from the AQM (right).

| $\sigma = const$ | $T = 100$ | 120 | 140 | 165 MeV | AQM | $T = 100$ | 120 | 140 | 165 MeV |
|------------------|----------|-----|-----|--------|-----|----------|-----|-----|--------|
| Pion             | 1.07     | 1.12| 1.16| 1.19   | Pion| 1.08     | 1.13| 1.18| 1.22   |
| Kaon             | 0.89     | 0.95| 1.01| 1.06   | Kaon| 0.89     | 0.96| 1.03| 1.09   |
| eta              | 0.87     | 0.93| 0.99| 1.04   | eta | 0.87     | 0.94| 1.01| 1.07   |
| f0               | 0.85     | 0.91| 0.97| 1.03   | f0  | 0.85     | 0.92| 0.99| 1.06   |
| rho              | 0.80     | 0.86| 0.92| 0.97   | rho | 0.80     | 0.87| 0.93| 1.00   |
| omega            | 0.79     | 0.86| 0.91| 0.97   | omega| 0.80 | 0.86| 0.93| 1.00   |
| K*892            | 0.77     | 0.83| 0.88| 0.94   | K*892| 0.77 | 0.83| 0.9 | 0.97   |
| N                | 0.76     | 0.82| 0.87| 0.93   | N   | 0.56     | 0.61| 0.66| 0.71   |
| eta'(958)        | 0.75     | 0.81| 0.87| 0.92   | eta '(958)| 0.75 | 0.82| 0.88| 0.95   |
| f0(980)          | 0.75     | 0.81| 0.86| 0.92   | f0(980)| 0.75 | 0.81| 0.88| 0.95   |
| a0(980)          | 0.75     | 0.81| 0.86| 0.92   | a0(980)| 0.75 | 0.81| 0.88| 0.95   |
| phi(1020)        | 0.74     | 0.80| 0.85| 0.91   | phi(1020)| 0.74 | 0.81| 0.87| 0.94   |
| Lambda           | 0.72     | 0.78| 0.83| 0.89   | Lambda| 0.53 | 0.58| 0.63| 0.68   |

The results for the relative viscous correction coefficients $c_i \equiv \chi_i/\chi_{dem}$ for each of the 49 species obtained from numerically solving the functional integrals are tabulated in Appendix L. Table 4.7 shows a few of the values calculated for the lower mass hadrons in the system, including the pions and nucleons.

One can see from the results in Table 4.7 that the pion still receives a slightly larger viscous correction than in the democratic case ($c_\pi > 1$) at all temperatures considered, while most of the other hadrons receive a lower correction than the democratic value, especially at lower conversion temperatures. The pions have corrections that now
increase by a noticeable fraction however, up to 20% compared to only 5% for the pion-nucleon gas. This is easily understood as the pions now make up a smaller fraction of the gas particles once all the hadrons are included, especially at higher conversion temperatures when heavier hadron states become more populated. If the pion density does not totally dominate the gas, a sizeable change in the pion distribution can still satisfy the constraints (2.19) on the gas as a whole. The relative viscous coefficients in the additive quark model cross section scenario differ from those in the constant scenario most notably in the smaller values of the coefficients for baryons like the nucleons and the Lambda in the AQM. This is due simply to the fact that in the AQM the cross sections for baryon scattering are larger than that of mesons, so the baryons will be closer to equilibrium. For example, the pion to nucleon ratio in the AQM is roughly $c_\pi/c_N \sim 1.8$ while it is only $\sim 1.3$ in the constant cross section scenario. With the viscous corrections for the hadron gas now known, one can use them to calculate heavy-ion observables as well as the shear viscosity of the hadron gas model considered here.

4.3.1 Hadron Gas Shear Viscosity

In Section 3.3 the shear viscosity of a gas of particles governed by the linearized transport equation in the Navier-Stokes limit was shown to be proportional to the variational solution of the functional $Q$ in (3.17) as

$$\eta = \frac{8}{5} Q_{\text{max}} T^3,$$

(4.28)

with $Q_{\text{max}}$ being defined as a function of the viscous correction coefficients $\chi_{\text{max},i}$ in (3.19). Substituting the values of $\chi_{i,\text{max}}$ into (3.19) to find $Q_{\text{max}}$ gives the values$^8$ of the shear viscosity in units of GeV/fm$^2$ displayed in Table 5.2.

The property more commonly discussed in heavy-ion physics is the dimensionless ratio of shear viscosity to entropy density, $\eta/s$, as it is the property that governs

$^8$Note that everyday fluids have very small shear viscosities in units of GeV/fm$^2$, e.g. $\sim 10^{-15}$ for water and $\sim 10^{-8}$ for molasses.
Table 4.4: Shear viscosity $\eta$ in units of GeV/fm$^2$ calculated using the variational result (4.28) in the Grad approximation. Both the pion gas and the pion-nucleon gas results were calculated using the energy-dependent cross sections described in Sec. 4.2.4, while the full hadron gas results are for constant 30 mb and additive quark model (AQM) cross sections described in Sec. 4.3.

|               | $T = 50$ | 80   | 100  | 120  | 140  | 165 MeV |
|---------------|---------|------|------|------|------|---------|
| $\pi$        | 0.040   | 0.035| 0.045| 0.062| 0.085| 0.12    |
| $\pi$-N      | 0.040   | 0.035| 0.045| 0.063| 0.087| 0.13    |
| HG-30mb      | 0.024   | 0.036| 0.046| 0.057| 0.069| 0.084   |
| HG-AQM       | 0.024   | 0.036| 0.045| 0.055| 0.065| 0.072   |

viscous hydrodynamic evolution. The entropy density of the hadron gas was computed by summing the chemical equilibrium result derived in (C.6),

$$s_{\mu=0} = n \left[ 4 + \frac{zK_1(z)}{K_2(z)} \right] = \frac{gT^3}{\pi^2} \frac{z^2}{2} \left[ 4K_2(z) + zK_1(z) \right] , \quad (4.29)$$

over all particle species. The numerical results for $\eta/s$ are plotted in Fig. 4.8 and were discussed previously in [106].

As one can see in Fig. 4.8, the value of $\eta/s$ for the hadron gas falls close to the AdS/CFT conjectured minimum at 165 MeV due to the rapid rising of the entropy density at high temperatures. The values here are 0.13 and 0.11 for the constant and AQM cross section scenarios, respectively. The value of $s$ is of course independent of the cross section, but $\eta \sim 1/\sigma$ so the AQM gas will have a lower viscosity due to the higher cross sections in collisions with baryons. This difference in $\eta$ is small, however, due to the relatively low number of baryons compared to mesons at these temperatures. For recent alternate calculations of the shear viscosity of a hadron gas, see [107–111].
Figure 4.8.: Dimensionless shear viscosity to entropy density ratio $\eta/s$ in the Grad approximation for a realistic pion gas (squares), realistic pion-nucleon gas (triangles), and the 49-species hadron gas in the constant (circles) and AQM (inverted triangles) cross section scenarios described in Sec. 4.3. All values are calculated at the same temperatures as in Table 5.2, but are plotted with slight horizontal offsets for clarity.

4.3.2 Hadron Gas Elliptic Flow: $v_2(p_T)$

Figure 4.9 shows pion and proton elliptic flow $v_2(p_T)$ in $Au+Au$ at RHIC at impact parameter $b = 7$ fm from a calculation analogous to the $\pi - N$ system in Section 4.2.3. Cooper-Frye particle conversion is applied at the same $T_{conv} = 165$ MeV, except now with self-consistent phase space corrections $\delta f_i$ calculated for the multicomponent hadron gas dynamics outlined in the last section. The left plot is for the $\sigma_{ij} = const$ scenario, in which case pion and proton elliptic flow are close to results from the “democratic” approach. The lack of species dependence is very similar to the findings of Ref. [103]. The similarity is not surprising, as giving all species the same scattering cross section is very similar to the democratic approach of treating all particles the same. If one looks closely, however, at high $p_T$, proton flow is actually slightly higher than pion flow, reflecting the decrease in shear stress corrections with mass at fixed
cross section (cf. Section 4.2.2). The curves cross at roughly $p_T \sim 1.25$ GeV and there remains a relative difference of $\sim 15\%$ at $p_T \sim 2.5$ GeV.

Figure 4.9.: Same as Fig. 4.2, except the self-consistent viscous corrections are computed for a gas of all hadron species up to mass $m = 1.672$ GeV, $\Omega(1672)$, with members of each isospin multiplet (and antiparticles) combined together into a single effective species, effectively 49 species. **Left plot:** all hadron species interacting with the same constant isotropic cross section $\sigma_{ij} = 30$ mb. **Right plot:** Constant isotropic cross sections with additive quark model scaling for meson-meson, meson-baryon, and baryon-baryon interactions $\sigma_{MM} : \sigma_{MB} : \sigma_{BB} = 4 : 6 : 9$ and $\sigma_{MM} = 30$ mb. Calculations with the “democratic Grad” ansatz for $\eta/s = 0.1$ (open boxes) and with local equilibrium distribution (filled circles) are also shown. In all cases, and for both plots, the Cooper-Frye prescription is applied at $T_{conv} = 165$ MeV.

The right plot of Fig. 4.9 shows, on the other hand, that more realistic additive quark model cross sections do generate a pion-proton difference in elliptic flow, of similar magnitude to the difference seen for the pion-nucleon gas earlier. Crossing between pion and proton $v_2$ again happens at about the same $p_T \sim 1$ GeV and a relative difference of $\sim 30\%$ at $p_T \sim 2.5$ GeV is seen. The likely explanation for this similarity is that even though interactions with all species are now considered, interactions with pions dominate because at $T_{conv} = 165$ MeV pions have a much
higher density compared to all other species, including kaons, the second lightest species.

The results for differential elliptic flow obtained using $T_{\text{conv}} = 140$ MeV are plotted in Fig. 4.10. One finds qualitatively the same as for $T_{\text{conv}} = 165$ MeV: the fixed cross section scenario closely matches the “democratic” Grad results, whereas pion-proton splitting in the AQM scenario is very similar in magnitude to the $T_{\text{conv}} = 140$ MeV result of Fig. 4.3.

Figure 4.10.: Same as Fig. 4.9, except the Cooper-Frye prescription is applied at $T_{\text{conv}} = 140$ MeV.

The Cooper-Frye prescription gives the momentum distribution of particles emitted directly from the fluid (“primary” particles). In a pure hydrodynamic approach, i.e., without a hadronic afterburner, many of these particles later decay en route to the detectors. Figure 4.11 shows the $p_T$ dependence of pion and proton elliptic flow from the same calculation shown in Fig. 4.9 except the unstable resonances are decayed using the RESO code in the AZHYDRO package. This means all particles are decayed to the long lived final state hadrons in RESO: pions, kaons (not shown), and nucleons.

For ideal freezeout ($\delta f = 0$), the “democratic Grad” ansatz, and also the constant cross section scenario, the main effect of resonance decays on elliptic flow is a reduction of the pion-proton splitting at low $p_T$; the characteristic mass-ordering is no longer
Figure 4.11.: Same as Fig. 4.9, except after feeddown from resonance decays using the RESO code in the AZHYDRO package.

present. At high $p_T$, there is barely any effect on elliptic flow besides a further shrinking of the pion-proton difference. At $T_{\text{conv}} = 165$ MeV the difference between pions and protons for all three scenarios gets washed out almost completely (this is not universal at all temperatures, for lower $T_{\text{conv}} = 140$ or 120 MeV, a portion of the difference survives, see Fig. 4.12). In contrast, in the more realistic AQM scenario, with self-consistent viscous fluid-to-particle conversion, proton elliptic flow stays 30% higher at $p_T \sim 2$ GeV than pion elliptic flow even after resonance decays are taken into account. The same insensitivity to resonance decays is present at $T_{\text{conv}} = 140$ MeV plotted in Fig. 4.12 and 120 MeV (not shown).

4.3.3 Higher Flow Harmonics: $v_n(p_T)$

In the previous section the second flow harmonic, $v_2(p_T)$, as defined in (2.6) as

$$E \frac{d^3N}{d^3p} = \frac{1}{2\pi} \frac{d^2N}{p_T dp_T dy} \left( 1 + 2 \sum_{n=1}^{\infty} v_n(p_T, y) \cos[n(\phi - \Psi_{RP})] \right),$$

was calculated for two different particle distributions at freezeout. Viscous corrections to the distribution functions were calculated self-consistently from the Boltzmann equation as well as in the democratic Grad ansatz, and there was a 30% difference around $p_T \sim 2$ GeV for proton $v_2(p_T)$ when using additive quark model inspired cross
sections for the transition to a hadron gas. Here the effects of the different freezeout distributions on the higher flow harmonics $v_4(p_T)$ and $v_6(p_T)$ are investigated.

Figure 4.13.: Differential fourth flow harmonic $v_4(p_T)$ for the 49-species hadron gas in the Grad approximation with constant 30mb cross sections for all species (left plot) and additive quark model (AQM) cross sections (right plot). In both cases the Cooper-Frye prescription is applied at $T_{conv} = 165$ MeV with $\eta/s = 0.1$ and all resonances have decayed to long lived particles.

Figure 4.13 shows pion and proton $v_4(p_T)$ in the same RHIC $Au + Au$ collision at impact parameter $b = 7$ fm after resonances have been decayed for $T_{conv} = 165$ MeV.
The effects from using self consistent freeze-out distributions are similar to the ones seen in elliptic flow in the previous section. There is a slight difference between the two particles if constant cross sections are used, while in the AQM, the 4th harmonic separates at higher $p_T$. For $T_{\text{conv}} = 165$ MeV, the 4th harmonic turns out to be negative for all viscous conversion scenarios besides the null result for protons in the AQM, while in experiment this observable is positive and of a slightly higher magnitude [112] in min-bias events. This feature is not universal for all AZHYDRO simulations done here as shown for the lower conversion temperatures in Fig. 4.14.

![Graphs showing $v_4(p_T)$ for different freeze-out distributions and conversion temperatures]

Figure 4.14.: Same as Fig. 4.13 but with $T_{\text{conv}} = 140$ MeV (top plots) and $T_{\text{conv}} = 120$ MeV (bottom plots).

Figure 4.14 shows the effects the different freeze-out distributions have on pion and proton $v_4(p_T)$ for the lower particlization temperatures of $T_{\text{conv}} = 140$ MeV (top plots) and $T_{\text{conv}} = 120$ MeV (bottom plots). For $T_{\text{conv}} = 140$ MeV, $v_4(p_T)$ is
now positive over most of the $p_T$ range until about 2 GeV. The splitting between pions and protons in the constant cross section scenario is similar to the elliptic flow results, but here the difference is between negative and positive result, respectively, at higher $p_T$. In the additive quark model, however the curves separate almost twice as much, again similar to the elliptic flow results. This gives a noticeable difference between a measurable positive $v_4$ for protons and a null or negative value for pions above 1.5 GeV. The results for the $4^{th}$ harmonic at the transition temperature of $T_{\text{conv}} = 120$ MeV are also similar to the results for elliptic flow. The flow gradients have diminished so much that effects from viscosity are small at these late times.

Figure 4.15.: Differential sixth flow harmonic $v_6(p_T)$ for the 49-species hadron gas in the Grad approximation with constant 30mb cross sections for all species (left plot) and additive quark model (AQM) cross sections (right plot). In both cases the Cooper-Frye prescription is applied at $T_{\text{conv}} = 165$ MeV with $\eta/s = 0.1$ and all resonances have decayed to long lived particles.

Figure 4.15 shows the results for the $6^{th}$ flow harmonic at $T_{\text{conv}} = 165$ MeV. There is no difference in the pion and proton $v_6$ in the constant cross section scenario, and the difference in the AQM values is $\lesssim 10\%$. If the transition temperature is taken to be 140 MeV, however, there is again a measurable difference in the proton $v_6$ in the AQM above 1.5 GeV as shown in Figure 4.16.
4.3.4 Observables from Viscous Hydrodynamics

The results for differential flow harmonics $v_n(p_T)$ shown so far have been obtained from converting a fluid to a gas of particles at a constant temperature decoupling hypersurface provided by hydrodynamic simulations using AZHYDRO. The AZHYDRO simulation used previously comes with many approximations one would like to improve upon; namely the approximations of chemical equilibrium and vanishing dissipation in the hydrodynamic evolution. These simplifications will be improved upon and the effects on heavy-ion observables explored in this section.

The equation of state used in the ideal hydrodynamics code AZHYDRO assumes the fluid is in perfect thermal and chemical equilibrium throughout the entire evolution. This latter assumption is suspect in the spirit of a smooth transition to a hadron gas, as the cross sections for inelastic, particle number-changing processes are known to be smaller than those of elastic (and quasi-elastic resonance forming) processes that conserve particle number, especially at the lower temperatures considered \cite{113,114}. Final hadron abundance ratios seem to favor a chemical freeze-out at $T_{ch} \approx 160$-175 MeV \cite{115,116}, while the slopes of the spectra prefer a much lower temperature for thermal (kinetic) freeze-out \cite{117}. Thus one is lead to try a framework with separate chemical and kinetic freezeout temperatures. This chemical freeze-out approach, de-
scribed first in [118], was tried previously in [46, 93, 119, 120] but failed to reproduce data as the initial conditions were kept the same as in chemical equilibrium. When the initial conditions were adjusted, it was found in [121] that ideal hydrodynamics with chemical freezeout could describe the hadron ratios and transverse momentum spectra, but not the elliptic flow. This was in turn remedied by including a small but nonzero shear viscosity in the hydrodynamic evolution.

Once the system reaches the point where the inelastic collisions that maintain chemical equilibrium cannot keep up with the rapid expansion of the system, chemical potentials will build up. If a system is said to “chemically freeze-out” at a temperature \( T_{ch} \), then the relative abundances of particles are locked in at this temperature, i.e. for \( T < T_{ch} \) one has

\[
\frac{n_1(T, \mu_1)}{n_2(T, \mu_2)} = \frac{n_1(T_{ch}, 0)}{n_2(T_{ch}, 0)}.
\]  

(4.31)

The 49 − 1 = 48 ratio equations of the form (4.31) allow one to write the chemical potentials of all particle species in the system in terms of one unknown chemical potential, say \( \mu_\pi \). This last chemical potential can be fixed by requiring the ratio of particle density to entropy density remain unchanged along flow streamlines as done in [121],

\[
\frac{n_\pi(T, \mu_\pi)}{s(T, \{\mu_i\})} = \frac{n_\pi(T_{ch}, 0)}{s(T_{ch}, \{\mu_i = 0\})}.
\]  

(4.32)

Note that in [121], constraint (4.32) is said to hold as the entropy density, \( s(T, \{\mu_i\}) \), is conserved in ideal hydrodynamic evolution, similar to the AZHYDRO simulations discussed so far in this work.

Once the chemical potentials for all particles have been calculated from (4.31) and (4.32), one can solve for the viscous corrections to the distribution functions under the assumption of chemical freezeout using the functional method. Note that the democratic Grad coefficient \( \chi_{dem} \) in (3.30) is no longer \( \eta/2s \) if the system is not in

\[9\]The model of a certain energy density or temperature being the point when chemical potentials build up is obviously only an idealization of the real process. The transition in kinetic theory happens gradually and can depend on a host of other local factors in general.
chemical equilibrium. One must use the more general result including the chemical potentials,

\[ \chi_{i}^{\text{dem}} = \frac{\eta T}{2(\epsilon + P)} = \frac{\eta T}{2(Ts + \sum c \mu c n c)} . \]  

(4.33)

The results for the elliptic flow once these distributions are used in the Cooper-Frye decoupling prescription are shown in Fig. 4.17 using a chemical freezeout temperature \( T_{ch} = 175 \) MeV. One can see that the effects on elliptic flow from using chemical freezeout distributions (light grey curves) are essentially negligible, with the only noticeable effect being a very slight decrease in \( v_2(p_T) \) at higher \( p_T \) for \( T_{conv} = 140 \) MeV.

Figure 4.17.: Differential \( v_2(p_T) \) for the 49-species hadron gas in the Grad approximation with AQM cross sections with \( \eta/s = 0.1 \) after resonance decays with chemical equilibrium at \( T_{ch} = 175 \) MeV and Cooper-Frye particlization at \( T_{conv} = 140 \) MeV (left) and \( T_{conv} = 120 \) MeV (right). The previous results without chemical freezeout are shown without symbols in grey.

The effects using chemical freezeout distributions have on the higher harmonics \( v_4(p_T) \) and \( v_6(p_T) \) are similar to those on elliptic flow as depicted in Fig. 4.18. At \( T_{conv} = 140 \) MeV, the \( v_4(p_T) \) curves for pions and protons are slightly lower after accounting for chemical freezeout while the \( v_6(p_T) \) curves are slightly higher at the higher \( p_T \) values.
Figure 4.18.: Same as Fig. 4.17 but for $v_4(p_T)$ (top) and $v_6(p_T)$ (bottom).

Overall, the effect on harmonic flow of using chemical freezeout distributions in the Cooper-Frye decoupling is a small one in the AZHYDRO simulations considered thus far. However, AZHYDRO is an ideal hydrodynamic code without any dissipation and the approximation of estimating an effective shear stress on the transition hypersurface from the flow gradients, $\pi_{NS}^{\mu\nu} = \eta \sigma^{\mu\nu}$, with $\eta/s$ chosen to be near its conjectured lower limit is, at best, a large theoretical uncertainty.

In this section, results for heavy-ion observables are now calculated using viscous hydrodynamic simulations of the early stages of the collision with the actual value of the shear stress tensor, $\pi^{\mu\nu}(x)$, output on the constant temperature hypersurface rather than estimated using the Navier-Stokes ansatz (3.2). The viscous hypersurfaces were obtained from H. Niemi and recently used in [122] and previously in [123–125]. For details on the numerical algorithm used in the 2+1D viscous hydrodynamic
simulations, see [125, 126]. The simulations used the EoS s95p-PCE-v1 [74] with an empirically driven, somewhat higher chemical freeze-out temperature of $T_{ch} = 175$ MeV.

Results for the differential elliptic flow $v_2(p_T)$ from a RHIC $Au + Au$ collision simulation from viscous hydrodynamics at $\sqrt{s_{NN}} = 200$ GeV in the 20-30% centrality class are shown in Fig. 4.19. The previously used impact parameter of $b = 7$ fm falls roughly into this centrality class. The Cooper-Frye conversion was done assuming a transition to the 49-species hadron gas in the Grad approximation as before, but the kinetic freezeout conversion temperature is now taken to be $T_{conv} = 100$ MeV with chemical potentials corresponding to chemical freezeout at $T_{ch} = 175$ MeV. One can see that although the conversion temperature is below the lowest 120 MeV used previously, the viscous effects from the real dissipative hydrodynamic simulation are not negligible. The elliptic flow is now suppressed by 30% relative to ideal freezeout distributions for the democratic and constant cross section scenarios. However, the difference between the democratic and AQM dynamic freezeout scenarios for protons has been reduced to 15%, only about 1/2 of its relative difference with ideal evolution from AZHYDRO.

The same viscous hydrodynamics code was used to find the constant temperature hypersurface in an LHC $Pb + Pb$ collision simulation at $\sqrt{s_{NN}} = 2.76$ TeV in the same 20-30% centrality class. The results for differential elliptic flow from this simulation using the same Cooper-Frye conversion procedure are shown in Fig. 4.22. Other than an overall rise in the $v_2(p_T)$ for all scenarios considered, the results at LHC collision energy, one order of magnitude higher than RHIC energies, are similar to the results at RHIC. There remains a difference between the ideal and democratic scenarios even at this low temperature, $T_{conv} = 100$ MeV, due to real viscous hydrodynamic evolution, but the difference between democratic and self-consistent dynamic freezeout distributions is small.

To test the dependence of elliptic flow on the Cooper-Frye conversion temperature after viscous hydrodynamic evolution, hypersurfaces were generated at $T_{conv} =$
Figure 4.19.: Differential elliptic flow $v_2(p_T)$ of pions and protons after resonance decays in $Au + Au$ at $\sqrt{s_{NN}} = 200$ GeV at RHIC in the 20-30% centrality class using 2+1D boost invariant viscous hydrodynamic solutions from H. Niemi [125,126]. The Cooper-Frye fluid-to-particle transition to the 49-species hadron gas was done in the Grad approximation with chemical equilibrium at $T_{ch} = 175$ MeV and conversion temperature $T_{conv} = 100$ MeV for the constant cross section $\sigma_{ij} = 30$ mb scenario (left) and AQM cross sections (right).

120, 140, and 160 MeV for the same $Pb + Pb$ collision setup. From Figures 4.21 and 4.22 one can see that there is very little dependence on the conversion temperature for elliptic flow. In the AQM scenario after viscous hydrodynamic evolution, all conversion temperatures considered here show the same mass ordering at low $p_T$, crossing of the pion and proton curves around 2 GeV, and a difference of roughly 15% between democratic and dynamic freezeout for protons.

Differential $v_4(p_T)$ and $v_6(p_T)$ are plotted in Fig. 4.23 for the same RHIC $Au + Au$ collision simulation at $\sqrt{s_{NN}} = 200$ GeV in the 20-30% centrality class from viscous hydrodynamics. As was the case for elliptic flow, $v_2$, there remains a large viscous suppression relative to ideal freezeout of the higher harmonics after decoupling from viscous hydrodynamics at $T_{conv} = 100$ MeV. Unlike in elliptic flow, however, there is a noticeable difference in the democratic and dynamic freezeout cases. Proton $v_4$ is around 45% higher in the dynamic freezeout calculation while the relative effect on
Figure 4.20.: Differential elliptic flow $v_2(p_T)$ of pions and protons after resonance decays in $Pb + Pb$ at $\sqrt{s_{NN}} = 2.76$ TeV at LHC in the 20-30% centrality class using 2+1D boost invariant viscous hydrodynamic solutions from H. Niemi [125,126]. The Cooper-Frye fluid-to-particle transition to the 49-species hadron gas was done in the Grad approximation with chemical equilibrium at $T_{ch} = 175$ MeV and conversion temperature $T_{conv} = 100$ MeV for the constant cross section $\sigma_{ij} = 30$ mb scenario (left) and AQM cross sections (right).

Figure 4.21.: Same as the right plot of Fig. 4.22 with AQM cross sections, but for $T_{conv} = 120$ MeV (left) and $T_{conv} = 140$ MeV (right).

$v_6$ is even larger. In the RHIC simulations it appears that the effect on the higher
harmonics could be significant enough to change the qualitative understanding of heavy-ion data.

Figure 4.23.: Differential $v_4(p_T)$ (left) and $v_6(p_T)$ (right) of pions and protons after resonance decays in $Au + Au$ at $\sqrt{s_{NN}} = 200$ GeV at RHIC in the 20-30% centrality class using 2+1D boost invariant viscous hydrodynamic solutions from H. Niemi [125, 126]. The Cooper-Frye fluid-to-particle transition to the 49-species hadron gas was done in the Grad approximation with chemical equilibrium at $T_{ch} = 175$ MeV and conversion temperature $T_{conv} = 100$ MeV with AQM cross sections.

To see the temperature dependence of this effect on higher harmonics as well as test the significance at higher collisional energies, the higher harmonics were also
calculated in the LHC $Pb+Pb$ collision at $\sqrt{s_{NN}} = 2.76$ TeV in the 20-30% centrality class after viscous hydrodynamic evolution. One can see from Figure 4.24 that again the values of the flow coefficients $v_4$ and $v_6$ are a bit larger at the higher LHC energy, but the shape is also qualitatively different than the RHIC results. The viscous suppression of the $4^{th}$ harmonic is smaller at the LHC, as is the difference between the democratic and dynamic freezeout calculations. The difference between LHC and RHIC $v_6$ is even more pronounced as the viscous suppression is reduced to the point where results for $v_6$ stay positive even at high $p_T$ at the LHC. There does, however, remain a large 50% difference in the proton $v_6$ between the democratic and dynamic freezeout at higher $p_T$.

Figure 4.24.: Differential $v_4(p_T)$ (left) and $v_6(p_T)$ (right) of pions and protons after resonance decays in $Pb+Pb$ at $\sqrt{s_{NN}} = 2.76$ TeV at LHC in the 20-30% centrality class using 2+1D boost invariant viscous hydrodynamic solutions from H. Niemi [125, 126]. The Cooper-Frye fluid-to-particle transition to the 49-species hadron gas was done in the Grad approximation with chemical equilibrium at $T_{ch} = 175$ MeV and conversion temperature $T_{conv} = 100$ MeV with AQM cross sections.

Figure 4.25 shows the temperature dependence of the higher harmonics at LHC energies. The qualitative picture of $v_4(p_T)$ is the same at all temperatures, though there is a slight increase of the difference between democratic and dynamic freezeout.
with temperature; Around 2 GeV, the relative difference changes from only about 15% at $T_{\text{conv}} = 100$ MeV to a splitting of 30% at $T_{\text{conv}} = 160$ MeV.

Figure 4.25.: Same as Fig. 4.24 only for $T_{\text{conv}} = 120$ MeV (top), $T_{\text{conv}} = 140$ MeV (middle), and $T_{\text{conv}} = 160$ MeV (bottom).

The story for the 6th harmonic is similar. The shape of the curves is qualitatively the same at all conversion temperatures considered here, though the curves tend to
“turn over” above $p_T \sim 2$ GeV for conversion temperatures above 100 MeV. The relative difference between pion and proton $v_6$ at the higher $p_T$ values shown is quite noticeable. The self consistent dynamic freezeout leads to an identified proton $v_6$ nearly 3 times as large as that of the pions.
5. Beyond the Grad Ansatz

Thus far, all results for particle distributions and heavy-ion observables in this work have been calculated using the so-called “Grad ansatz”, in which the viscous corrections to the thermal distribution functions used in the Cooper-Frye prescription of fluid-to-particle conversion are assumed to be quadratic in momentum. The previous Chapter was dedicated to calculating these particle distributions under this assumption. Particle dynamics were included via the linearized transport equation such that the viscous corrections, $\phi_i(x, p)$, defined in (2.18) were different for each particle species $i$ and governed by the scattering rates between particles in the system. This is in stark contrast to the typically employed “democratic Grad” ansatz encoded in (2.20) in which all particle species receive the same relative correction $\phi_{dem}(x, p)$. The constraint of quadratic momentum dependence on the viscous corrections will now be lifted and the functional method will be used to calculate the power of momentum that maximizes the functional extremum in a single power basis ansatz. The effects on particle distributions and heavy-ion observables in different systems are also investigated for different single power basis ansatze.

The goal remains to describe systems that are near, but not in, thermodynamic equilibrium due to the presence of shear stress whose late time behavior in the Navier-Stokes regime is governed by the Boltzmann transport equation. The form of the viscous corrections in this limit was derived in Chapter 3 to be

$$\phi_i(x, p) = \chi_i(|\tilde{p}|)P^{\mu\nu}(p)X_{\mu\nu}(x).$$

The viscous correction for particle species $i$ is given by the dimensionless function $\chi_i(|\tilde{p}|)$, where $\tilde{p}$ is the local rest frame three-momentum normalized by temperature, i.e.,

$$\frac{1}{T}\Delta^{\mu\nu}p_{\nu}\bigg|_{LR} \equiv (0, \tilde{p}) \text{ such that } |\tilde{p}| \equiv \tilde{p} \equiv \frac{\sqrt{(p \cdot u)^2 - m^2}}{T}. $$
Since the tensor $P^{\mu\nu}$ defined in (3.7) already carries two powers of the momentum 4-vector, the Grad ansatz used thus far corresponds to a momentum independent $\chi_i$ for each particles species. This constraint on $\chi_i(\vec{p})$ is now relaxed. Given a finite basis expansion (3.18) for $\chi_i(\vec{p})$, one can solve the transport equation by equivalently maximizing the functional $Q[\{\chi_i(\vec{p})\}]$ in (3.17) with respect to the set of functions $\{\chi_i\}$ as was done in the last chapter. For simplicity, this method is again first brought to bear on massless systems in the next Section.

5.1 Massless Systems

In the massless limit, one can do the functional integrals analytically for certain classes of basis functions $\chi_i(\vec{p})$. In this section, the $\chi(\vec{p})$ function, and thus the viscous correction to the equilibrium distribution function, is calculated for a single massless particle species with energy-independent, isotropic scattering cross section, i.e. the simplest non-trivial case. Dividing Eq. (3.13) by $f_{eq}^1$ and performing the integrals over $y \equiv \tilde{p}_2$ and $t \equiv \cos \theta_{12}$ yields the integral equation for the rescaled $\tilde{\chi}(x) \equiv (n\sigma/T)\chi(x)$:

$$B(x) - D(x) \tilde{\chi}(x) + \int_0^\infty dw \tilde{\chi}(w) A(x, w) = 0 ,$$  \hspace{1cm} (5.3)

with $x \equiv \tilde{p}_1$ and $w \equiv \tilde{p}_3$. Here, $B(x) = \frac{1}{3}x^4$ comes from the left side of the transport equation and $D(x) = \frac{2}{3}x^5$ from the $P_1 \cdot P_1$ collision term. The collision term that includes particles with momentum labels 1 and 2 vanishes for massless particles and
\[ A(x, w) = \Theta(x - w)A_1(x, w) + \Theta(w - x)A_2(x, w) - A_3(x, w) \]
comes from the collision term symmetric in momenta 3 and 4 with

\[
A_1(x, w) = \frac{4}{3x} \left[ 36(60+36x+9x^2+x^3) - 12w(72+42x+10x^2+x^3) \right. \\
+ w^2(108+60x+13x^2+x^3) \bigg] \\
A_2(x, w) = \frac{4}{3x} e^{x-w} \left[ 108(20-8x+x^2)+12w(108-42x+5x^2) \right. \\
+ w^2(324-120x+13) + w^3(x-6)^2 \bigg] \\
A_3(x, w) = \frac{4}{3x} e^{-w} \left[ 36(60+36x+9x^2+x^3) + 12w(108+66x+17x^2+2x^3) \right. \\
+ w^2(324+204x+55x^2+7x^3) \bigg] \quad (5.4)
\]

Heaviside theta functions arise in (5.3) from the constrained limits of the integrals being interchanged to leave the integral over \( w \) to perform last. The variational functional is now

\[
Q[\tilde{\chi}] = \frac{1}{4} \int_0^\infty dx \, e^{-x} \tilde{\chi}(x) \left[ B(x) - \frac{1}{2} D(x) \tilde{\chi}(x) + \frac{1}{2} \int_0^\infty dw \, \tilde{\chi}(w) A(x, w) \right]. \quad (5.5)
\]

In principle, one can now solve for \( \chi(\tilde{p}) \) variationally given any finite set of basis functions, but only some basis functions lend themselves to an analytic treatment. For example, the functional integrals can be calculated analytically if the class of basis functions for the single massless species in the system is a power series,

\[
\chi(\tilde{p}) = \sum_{n=0}^N k_n \tilde{p}^{n a} \quad . \quad (5.6)
\]

The sum over basis functions will often be omitted and repeated basis indices are henceforth assumed summed over. Substituting the power series form (5.6) into the functional yields

\[
Q[\{k_n\}] = k_n S_n - \frac{1}{2} k_n C_{nm} k_m \quad , \quad (5.7)
\]

with shorthands for the inner product integrals

\[
S_n \equiv (\tilde{p}^a, S), \quad C_{nm} \equiv (\tilde{p}^a, C(\tilde{p}^m)) \quad , \quad (f,g) \equiv \frac{1}{T^2} \int \frac{d^3 p}{E} fg \quad (5.8)
\]
defined using the same notation as in (3.15) for the special case of a system with only one particle species. Extremizing (5.7) with respect to the vector $k$ gives a vector equation whose solution is found by inverting the collision matrix:

$$S_n = C_{nm} k_{m}^{\text{max}} \Rightarrow k_{m}^{\text{max}} = C_{nm}^{-1} S_n$$

(5.9)

Thus, the general solution for the viscous correction to the distribution function in a transition to a single component massless gas from a power series expansion is

$$\delta f(x, p) = f_i^{eq}(x, p)[1 + \phi_i(x, p)] , \quad \phi_i(x, p) = \sum_{n=0}^{N} k_n \tilde{p}^{\alpha_n} P^\mu\nu(\tilde{p}) X_{\mu\nu}(x)$$

(5.10)

with the vector $k^{\text{max}}$ found analytically through the linear algebra in (5.9). The first thing one can do is find the variationally preferred power solution if only one term is used, i.e. take the special case of (5.6) with one term, $\chi(\tilde{p}) = k \tilde{p}^\alpha$ and calculate the value of the functional in terms of the single power $\alpha$. One can then find what value of $\alpha$ is preferred in a variational sense by maximizing the functional and solving for $\alpha$ by imposing $dQ_{\text{max}}[\alpha]/d\alpha = 0$. The solution for the coefficient $k_{\text{max}}$ in (5.6) as a function of the single power $\alpha$ chosen is

$$k_{\text{max}}(\alpha) = \frac{T}{n\sigma}(1+\alpha)^2(2+\alpha)^2(3+\alpha)^2\Gamma(6+\alpha) \left\{ 4[22 + \alpha(8+\alpha)]\Gamma^2(6+\alpha) \\
+ 2[-5 + \alpha(2+\alpha)][172 + \alpha(60 + \alpha(13 + \alpha(6+\alpha)))]\Gamma(7+2\alpha) \right\}^{-1}.$$  

(5.11)

Putting (5.11) into (3.19) gives the maximum value of the functional as a function of $\alpha$:

$$Q_{\text{max}}(\alpha) = \frac{5}{8} \frac{1}{T^2\sigma}(1+\alpha)^2(2+\alpha)^2(3+\alpha)^2\Gamma^2(6+\alpha) \left\{ 60[22 + \alpha(8+\alpha)]\Gamma^2(6+\alpha) \\
+ 30[-5 + \alpha(2+\alpha)][172 + \alpha(60 + \alpha(13 + \alpha(6+\alpha)))]\Gamma(7+2\alpha) \right\}^{-1}.$$  

(5.12)

To find the $\alpha_{\text{max}}$ that gives the maximum value of the functional, one can impose $dQ_{\text{max}}[\alpha]/d\alpha = 0$. The value obtained from this second maximization for a single component, massless system with isotropic, energy-independent scattering is $\alpha_{\text{max}} = -0.532$ to three significant digits. Recall that (5.1) implies that the widely used
Grad ansatz corresponds to $\alpha = 0$ giving shear viscous corrections to the distribution function quadratic in momentum. The variational calculation done analytically here for a massless gas suggests viscous corrections are proportional to the $\sim 3/2$ power of the momentum.

This is not the first study to find a preference for this $3/2$ power, however. In [127], it was found that $\phi(x, p) \propto p^{3/2}$ for a mixture of massless quarks and gluons to leading order in perturbative QCD (pQCD) with forward-peaked $1 \leftrightarrow 2$ interactions, as shown in the left plot of Fig. 5.1. Single and two-component systems of massless particles with energy-independent, isotropic $2 \rightarrow 2$ cross sections were also found to have relative corrections close to $p^{3/2}$ in [25]. Different powers of momentum for the viscous corrections were also empirically explored using viscous hydrodynamics in [128]. There it was found that for the cases investigated, $\phi(x, p) \propto p^{3/2}$ was the only value in agreement with experimental data points on elliptic flow as shown in the right plot of Fig. 5.1.

![Figure 5.1: Left plot: Differential $v_2(p_T)$ for a perturbative gluon gas at leading order with linear, and quadratic ansatze shown for comparison taken from [127]. Right plot: Charged hadron $v_2(p_T)$ at $b = 8$ fm with color glass initial conditions for three choices of $\phi(x, p)$: Linear in momentum, momentum to the $3/2$, and the Grad case quadratic in momentum taken from [128]. The solid red ideal freezeout curve is also shown for reference in both plots.](image-url)
Consider now a power series ansatz for the basis functions in [5.6] that include more than a single term. The first case considered will be the integer power series,

\[ \chi(\tilde{p}) = \sum_{n=0}^{N} k_{n-2} \tilde{p}^{n-2}, \quad n \in \mathbb{N}. \]  

(5.13)

Recall that in this notation, the tensor \( P^{\mu\nu}(\tilde{p}) \) carries two powers of momentum, so the series starts at \( n = 0 \) as this would correspond to relative viscous corrections \( \phi(x, p) \) independent of momentum, while the \( n = 2 \) term corresponds to the typical Grad case where \( \phi(x, p) \propto \tilde{p}^2 \). Figure 5.2 shows the results for the momentum dependence of the relative viscous correction to the thermal distribution function, \( \chi(\tilde{p}) \cdot \tilde{p}^2 \), for different choices for the number of terms, \( N \), in (5.13). One can see from Figure 5.2 that for integer power basis functions, the solutions for \( \chi(\tilde{p}) \) are well-behaved out to \( p/T \sim 35 - 40 \), though quite a few terms are necessary to achieve this convergence with the integer basis. If one uses the finer-grained quarter-integer power basis shown in Figure 5.3 however, convergence is achieved past \( p/T = 60 \) using only \( N_4 = 16 \) terms. For another comparison, a half-integer basis solution is plotted in Fig. 5.4 out to higher dimensionless momentum to see just how high the convergence lasts. For the quarter-integer basis, convergence is achieved past \( p/T = 100 \) with 16 terms.

Apparentely one is better suited concentrating on the lower powers of momentum by using more intervals than including higher powers in a power series expansion. This is unsurprising as the best single power fit found using the functional method was \( n = -0.532 \), corresponding to \( \phi(x, p) \propto \tilde{p}^{1.468} \), for this massless system. It should also be noted that the variationally best general power basis solutions grow roughly like \( \tilde{p}^{1.5} \), but become closer to linear at very high \( p/T \). Thus it would seem, at least in this simple massless case, the relative viscous corrections seem much closer to \( \phi(x, p) \propto \tilde{p}^{3/2} \) than the naive quadratic Grad dependence.

It was noted earlier that the functional method provides a convenient measure, \( Q_{\text{max}} \), or equivalently the shear viscosity, \( \eta \), from [3.25], for how close a set of basis functions is to the “real” answer. The comparison of \( \eta \) values for different choices of basis functions is much easier than plotting and comparing the values of \( \chi(\tilde{p}) \cdot \tilde{p}^2 \) as
Figure 5.2.: $\tilde{p}$ dependence of relative viscous corrections, $\phi(x, p)$, using a power series basis of integers from 0 to $N$ for a single massless particle species with energy-independent, isotropic scattering cross section. The red line is the best single power calculated analytically by extremization of the integral equation 5.3 done in the previous three figures. The values of the shear viscosity for different basis functions in Table 5.1 shows why comparing only values of $\eta$ masks many important properties of the system. Even though the behavior of the $N = 10$ and $N = 60$ integer basis functions is drastically different at the modest $p/T$ of 30 in Fig. 5.2, the values of the shear viscosities in Table 5.1 hardly change at all. The shear viscosity of a system has been calculated in other frameworks mentioned previously, and is often not the troublesome parameter of the system to calculate. However, the shear viscosity seems to be largely insensitive to the shape of the distribution function at very low or very high momenta. The distribution functions themselves are harder to calculate.
Figure 5.3.: $\tilde{p}$ dependence of relative viscous corrections, $\phi(x, p)$, using a power series basis of $N_4$ quarter-integers ($n = 0$, $1/4$, $1/2$, $3/4$ etc.) for a single massless particle species with energy-independent, isotropic scattering cross section. The green and red curves are the same as in Fig. 5.2.

and constrain these new results for the shear viscous corrections to the thermal distribution functions derived in this thesis work are necessary to calculate results for heavy-ion observables in the hydrodynamic paradigm. For example, if one wants to calculate heavy-ion observables in the 100 MeV temperature range, one had better be careful how they handle particles with transverse momentum 3 GeV and higher. Note that this does not mean that the $N = 10$ basis function choice that is accurate out to $p/T \approx 30$ would be well suited to calculating the elliptic flow up to $p_T = 3$ GeV at a transition temperature of 100 MeV. Particles that reach the detectors at midrapidity come from fluid cells moving at different coordinate rapidities, $\eta_x$. The momentum
Figure 5.4.: $\tilde{p}$ dependence of relative viscous corrections, $\phi(x, p)$, using a power series basis of $N_2$ half-integers for a single massless particle species with energy-independent, isotropic scattering cross section. The brown and red curves are the same as in Fig. 5.3.

in the fluid rest frame that appears in $\chi(p_T^2)$ can be much larger than the transverse momentum $p_T$ in the lab frame from the longitudinal boost $p \cdot u = \gamma (m_T c h \xi - p_T v_T)$ where $\xi$ is the different between coordinate rapidity $\eta_x$ and momentum rapidity $y$ (see Appendix K). Thus particles with rest frame energy quite a bit larger than 3 GeV still make sizable contributions to the Cooper-Frye rapidity integrals, so one must be careful pushing too far out in $p_T$ for these and similar calculations.

The high momentum asymptotic solutions of Eq. (5.3) are thus of interest for this study and heavy-ion physics in general. So far some limits have been put on
Table 5.1: Shear viscosity in units of $T/\sigma$ of a single-component massless gas with isotropic, energy-independent cross sections as a function of the number of terms, $N$, used in the power series expansion of the correction to the thermal distribution functions.

| $N$ | $\eta$     |
|-----|------------|
|     | 0.7284     |
| Grad | 1.2        |
| 2    | 1.26707    |
| 10   | 1.26757155 |
| 30   | 1.26757157373 |
| 60   | 1.26757157374 |

the asymptotic solutions mathematically, but this problem mostly remains an open question for future research.

5.2 Hadron Gas Observables

Finally to investigate the theoretical uncertainty due to the assumed quadratic momentum dependence of dissipative phase space corrections (Grad ansatz) on heavy-ion observables, two more power law momentum dependence ansatze ($\phi(x, p) \propto p^\alpha$) are tested with $\phi(x, p) \propto p$ and $p^{3/2}$. These correspond to single term basis functions in (5.1) with

$$\chi_i^{(1)}(|\tilde{p}|) = k_i |\tilde{p}|^{-1}, \quad \chi_i^{(3/2)}(|\tilde{p}|) = k_i |\tilde{p}|^{-1/2}$$

$$(\chi_i^{\text{Grad}} = k_i |\tilde{p}|^0), \quad (5.14)$$

where the coefficients, $k_i$, vary among different particle species and are determined variationally via maximizing $Q[\chi]$. The values of the shear viscosity obtained variationally from (3.25) are shown in Table 5.2. One can see that the value $\alpha = 3/2$ is preferred variationally to the typically used Grad value of 2 in the hadron gas, in addition to the previously derived massless single-component system.
Table 5.2: Shear viscosity in units of fm$^{-3}$ of the 49-species hadron gas with AQM-inspired elastic and isotropic $2 \rightarrow 2$ cross sections assuming shear corrections of the form $\phi(x, p) \propto p^\alpha$ calculated variationally from (3.25).

| $\alpha$ | $\eta$ [fm$^{-3}$] |
|----------|-------------------|
| 1        | 0.361             |
| 1.5      | 0.379             |
| 2 (Grad) | 0.366             |

The two new forms considered here have a weaker momentum dependence than the quadratic Grad correction, as motivated by the previous analytic calculation for the massless gas. They will both generally exhibit smaller viscous effects than the dynamical Grad results thus far considered for the same hydrodynamics solutions. For example, one would expect smaller suppression of elliptic flow at high $p_T$ with these two lower power ansatze.

Figure 5.5 shows the ratio of the spectra of long-lived particles (pions, protons, and kaons) for $Pb+Pb$ collisions at the LHC emitted from the Cooper-Frye decoupling surface when the transition is done with self-consistent dynamical freezeout compared to the democratic Grad ansatz. The transition in all cases is for the 49-species hadron gas with AQM cross sections discussed previously after resonance decays. One can see there is only about a percent difference in the spectra ratio for the bulk low $p_T$ particles at both $T_{conv} = 160$ and 100 MeV shown. There is a reduction in the number of low $p_T$ protons if $T_{conv} = 100$ MeV, while there is a slight increase if $T_{conv} = 160$ MeV for all three powers of momentum in the viscous corrections considered. The larger effect comes at high $p_T$, especially for the distribution function corrections with weaker momentum dependence ansatze, as expected. There is a reduction of the number of all three long-lived particles by $\sim 3\%$ between the dynamic Grad and $p^{3/2}$ ansatze and $\sim 8\%$ between the Grad and $p^1$ at $p_T$ over 2 GeV.
The differential elliptic flows of pions and protons for the three different momentum basis functions for the viscous corrections to the distribution functions are shown in Fig. 5.6 for the same hadron gas calculation as in Fig. 4.22 again at $T_{\text{conv}} = 160$ and 100 MeV. The dynamic Grad plots from Section 4.3.4 are included in the top panel for reference. The results for the elliptic flow for the different power ansatze are qualitatively similar. There is a reduction of the viscous effects at high $p_T$ from the weaker momentum dependence of $p^{3/2}$ and $p^1$. The effect is similar in magnitude for all temperatures considered; Proton $v_2$ is about 5% and 10% higher in the $p^{3/2}$ and $p^1$ ansatze, respectively, than in the typically used democratic Grad calculation at high $p_T$.

Figure 5.7 shows the effects of the assumed momentum dependence of the shear corrections to the distribution functions on $v_4(p_T)$. The main effect of changing the momentum dependence of the corrections is again the reduction of viscous suppression of the flow at high $p_T$. There are a few residual effects of the conversion temperature on $v_4$; In the $p^{3/2}$ case at $T_{\text{conv}} = 160$ MeV, there remains a large $\sim 25\%$ difference in the pion and proton $v_4$ at high $p_T$, while these curves cross around 2 GeV in the $T_{\text{conv}} = 100$ MeV case leaving very little difference at high $p_T$. If one uses the $p^1$ ansatz, even the pion $v_4$ is 30% higher than the democratic Grad calculation at high $p_T$ for $T_{\text{conv}} = 160$ MeV.

Finally the effect of the different momentum ansatze proposed on $v_6(p_T)$ is shown in Fig. 5.8. The picture is qualitatively the same as the effects on $v_4$; namely there is less viscous suppression in $v_6$ if one assumes a weaker momentum dependence. There remains a sizeable difference between pion and proton $v_6$ at high $p_T$ if the conversion temperature is high appropriate for an early fluid conversion to be fed into a hadronic transport afterburner. The proton $v_6$ is only suppressed roughly half as much as in the democratic Grad calculation at high $p_T$ and this difference in pion $v_6$ becomes sizable if one assumes viscous corrections proportional to $p^1$. 
Figure 5.5.: Ratios of long-lived particle spectra from self consistent dynamic Cooper-Frye freezeout to freezeout in the democratic Grad ansatz in an LHC $Pb+Pb$ collision at $\sqrt{s_{NN}} = 2.76$ TeV in the 20-30% centrality class. The left plots are for $T_{\text{conv}} = 100$ MeV and the right for $T_{\text{conv}} = 160$ MeV. The momentum dependence of relative shear viscous corrections is taken to be proportional to $p^2$ (top), $p^{3/2}$ (middle), and $p^1$ (bottom). In all cases the transition is calculated for the 49-species hadron gas with AQM cross sections and in chemical equilibrium at 165 MeV after resonances have decayed as discussed in the text.
Figure 5.6.: Differential $v_2(p_T)$ for pions and protons in an LHC $Pb + Pb$ collision at $\sqrt{s_{NN}} = 2.76$ TeV in the 20-30% centrality class. The left plots are for $T_{\text{conv}} = 100$ MeV and the right for $T_{\text{conv}} = 160$ MeV. The momentum dependence of relative shear viscous corrections is taken to be proportional to $p^2$ (top), $p^{3/2}$ (middle), and $p^1$ (bottom). In all cases the transition is calculated for the 49-species hadron gas with AQM cross sections and in chemical equilibrium at 165 MeV after resonances have decayed as discussed in the text.
Figure 5.7.: Same as Fig. 5.6 except for the differential $4^{th}$ flow harmonic, $v_4(p_T)$. 
Figure 5.8.: Same as Fig. 5.6 except for the differential $6^{th}$ flow harmonic, $v_6(p_T)$. 
6. Extracting QGP Shear Viscosity

One of the main goals of heavy-ion physics is to study the quark-gluon plasma and extract quantitative values for its properties such as its shear viscosity to entropy density ratio, $\eta/s$. As discussed in Section 2.3, heavy-ion simulations using hydrodynamics have typically used the “democratic Grad” ansatz in Cooper-Frye particlization where shear viscous coefficients relative to the equilibrium distribution, $\phi_i(x, vp)$, are independent of particle species and assumed to be quadratic in momentum. To estimate the theoretical uncertainty in extracting $\eta/s$ from experimental elliptic flow data using such an assumption, one can scale $\eta/s$ in different dynamic ansatz calculations to match the democratic elliptic flow curves. This is what has been done in Figure 6.1 for the case preferred variationally in this work; namely, the dynamic ansatz assuming a momentum dependence of $\phi_i(x, vp) \propto p^{1.5}$ with AQM cross sections in a hadron gas.

The blue “Scaled $p^{1.5}$” curve in Fig. 6.1 was calculated using the same distribution functions as the green “Dyn $p^{1.5}$” curve, only $\eta/s$ was scaled by $3/2$ to make it fall roughly on the democratic curve. Thus, if one adjusts the simulation parameters to match proton elliptic flow assuming democratic Grad particlization distributions, the extracted value of $\eta/s$ can be off by more than 50% from the uncertainty in the freezeout distributions alone.
Figure 6.1.: Differential $v_2(p_T)$ for protons from the 49-species hadron gas calculation using AQM cross sections and $T_{conv} = 160$ MeV with chemical freezeout at $T_{ch} = 175$ MeV. The Ideal, Dem Grad, and Dyn $p^{1.5}$ curves are the same as the proton curves in the right middle plot of Fig. 5.6, while the Scaled $p^{1.5}$ curve is the same as Dyn $p^{1.5}$ only with shear viscous corrections scaled by a factor of $3/2$. 
7. Summary and Future Work

In the hydrodynamics-based standard model of simulating heavy-ion collisions, there is an ambiguity in transitioning from a viscous hydrodynamic fluid to a gas governed by hadron transport. Thus far in the field of heavy-ion physics, the viscous corrections to the distribution functions have typically been assumed to be quadratic in momentum and independent of particle species-specific dynamics, i.e., the simplest form suggested by the tensor structure. This thesis work has presented a self-consistent way to calculate these corrections based on solving the linearized Boltzmann equation using functional extremization. It was shown that the value of the extremized functional provides a natural comparison between different functional forms for these corrections to the thermal distribution functions, as well as being related trivially to the shear viscosity of the system.

This formalism was applied to a massless gas of particles interacting via isotropic, energy-independent $2 \rightarrow 2$ scatterings. The collision integrals were calculated analytically in the special case of shear viscous corrections expanded in a power series in momentum. It was shown that the variational solution was maximized for a momentum dependence close to $p^{3/2}$ rather than the commonly used Grad ansatz momentum dependence of $p^2$. In fact, in all cases studied in this work, from ultrarelativistic massless gases, to non-relativistic multi-particle systems, to the fully relativistic 49-species hadron gas with additive quark model cross sections, the self-consistent approach was shown to variationally prefer a power of 3/2 rather than 2. Such powers of 3/2 have emerged in other studies that applied the Boltzmann equation to systems with different interactions from those studied here. At this point, it is not clear if there is anything deep or universal about this 3/2 in the momentum dependence of shear viscous corrections to thermal distributions; more work is needed to uncover its origin and trace its implications.
The self-consistent phase space distributions obtained from the linearized Boltzmann equation were then used in conjunction with viscous hydrodynamic simulations to calculate heavy-ion observables for identified particles such as transverse momentum distributions and differential flow coefficients, $v_n(p_T)$. Using additive quark model cross sections in the dynamic Grad ansatz, it was found that proton flow coefficients are systematically higher than those for pions at moderately high $p_T$ in $Pb+Pb$ collisions at LHC, especially for the coefficients $v_4$ and $v_6$. It was also found that viscous effects on all flow coefficients investigated are reduced if the momentum dependence is taken to be weaker than quadratic, namely $p^{3/2}$. This highlights a theoretical uncertainty in the flow observables in heavy-ion collisions. The assumed form of the viscous corrections to the distribution functions in Cooper-Frye freezeout can have a noticeable effect ($\sim 50\%$) on the value of properties like the shear viscosity to entropy density ratio, $\eta/s$, extracted by comparing hydrodynamic simulations to heavy-ion data.

There are a few areas where the calculation can be improved in future studies. Heavy-ion observables were calculated assuming a transition to a hadron gas with 49 effective species interacting via energy-independent, isotropic $2 \to 2$ scatterings. One would like to tabulate the known energy-dependent cross sections for all particle pairs as is done in state-of-the-art hadron transport codes. The theoretical uncertainty of this effect was studied in a simplified pion-nucleon system in this work, but the effect of including energy-dependent cross sections, let alone ones with realistic angular dependence, remains a topic for future study.

The calculation also should be improved with proper Bose/Fermi quantum statistics for the particles in the final state hadron gas. The deviations from Boltzmann statistics are thought to be small here, but this has not been investigated in detail yet. Switching to quantum statistics may affect how much one can derive analytically and will lead to more difficult integrals to do numerically in the general case of a multi-component hadron gas, but the size of the effects should nevertheless be estimated in a future study.
Finally, and probably most importantly, throughout this work the effects of nonzero bulk viscosity have been ignored. Corrections from bulk viscosity were calculated early in the history of the field and thought to be small under some assumptions. More recently, there have been reasons for questioning the validity of these assumptions, and only now are groups beginning to look into bulk viscous effects [97][129][130].

As was shown in this work, the value of the shear viscosity is not sensitive to the form of the distribution functions at low or high momentum, making the calculation of $\eta/s$ fairly simple even without detailed knowledge of the form of the shear viscous corrections. This work is one of few that has concentrated on calculating the distribution functions themselves, which are vital in the calculation of heavy-ion observables like elliptic flow. The method developed here could be extended in a straightforward manner to calculate heavy-ion observables for systems with nonzero bulk viscosity.
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APPENDICES
A. Units and Conventions

Here the units and conventions used throughout this thesis are summarized.

As is common in relativistic physics, the argument $x$ will refer to the 4-vector $x^{\mu}$ with Greek indices denoting Minkowski 4-space ($\mu = 0,1,2,3$) with metric tensor

$$g^{\mu\nu}(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix} \equiv \text{diag}(1,-1,-1,-1) . \quad (A.1)$$

For brevity, the spacetime argument $x$ is often left out of equations. Latin indices will denote the spatial directions, $i = 1,2,3$. Repeated Lorentz indices are assumed to be summed over (Einstein convention), while sums over particle species are always written explicitly.

Through the relationships between time, length, energy, and mass there is enough freedom to conveniently set 3 physical constants to 1 in some system of units. In this thesis work, as in most particle physics literature, these are taken to be the so-called “natural units” with $c = k_B = \hbar = 1$ where $c$ is the speed of light in vacuum, $k_B$ is Boltzmann’s constant, and $\hbar \equiv \hbar/(2\pi)$ is Planck’s constant divided by $2\pi$. Therefore, the units typically used for energy, momentum, mass, and temperature are MeV or GeV (10⁶ or 10⁹ electron volts, respectively), while distances and times are usually expressed in fm (10⁻¹⁵ meters).
B. Hydrodynamics

Here the basic hydrodynamic relations and equations of motion are derived in both the ideal and viscous fluid cases. The hydrodynamic derivations here closely follow the opening section of [131].

B.1 Ideal Hydrodynamics Derivations

In this Appendix section, the energy-momentum tensor, equations of motion, and entropy are derived for an ideal fluid in thermal equilibrium.

B.1.1 Ideal Energy-Momentum Tensor \( T_{id}^{\mu\nu} \)

Here the form of the ideal energy-momentum tensor is derived by demanding it transform as a symmetric, rank-2 tensor under Lorentz transformations and be made up of the hydrodynamic variables: flow velocity vector \( u^\mu(x) \), the scalar energy density \( \epsilon(x) \) and pressure \( P(x) \), and the Minkowski metric tensor \( g^{\mu\nu}(x) \). The most general form using these building blocks is\(^\text{1}\)

\[
T_{id}^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + P(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu) .
\] (B.1)

In the local rest frame, then demand the \( T_{id,LR}^{00} \) component be the energy density of the fluid, the momentum density \( T_{id,LR}^{0i} \) vanish, and the space-like components to be the isotropic pressure \( T_{id,LR}^{ij} = P\delta^{ij} \). Applying the first condition and using the local rest frame flow velocity \( u_{LR}^\mu = (1, \vec{0}) \) gives

\[
\epsilon(c_0 + c_1) + P(c_2 + c_3) = \epsilon \Rightarrow c_0 + c_1 = 1, \quad c_2 + c_3 = 0
\] (B.2)

\(^{1}\)Note that \( u^\mu u_\mu = 1 \) and \( g^{\mu\nu}g_{\mu\nu} = 4 \) so no new invariant scalars can be formed from them.
while applying the second condition yields

$$\epsilon(c_0 \delta_{ij} + c_1 0) + P(c_2 \delta_{ij} + c_3 0) = P \delta_{ij} \Rightarrow c_0 = 0, \ c_2 = 1 \quad (B.3)$$

thus giving \(c_1 = 1\) and \(c_3 = -1\) and the ideal energy-momentum tensor is therefore

$$T_{id}^{\mu \nu} = (\epsilon + P) u^\mu u^\nu - P g^{\mu \nu} \quad (B.4)$$

The energy-momentum tensor is often rewritten in terms of the projector orthogonal to the fluid velocity

$$\Delta^{\mu \nu} \equiv g^{\mu \nu} - u^\mu u^\nu \quad (B.5)$$

which satisfies \(\Delta^{\mu \nu} u_\mu = \Delta^{\mu \nu} u_\nu = 0\) and \(\Delta^{\mu \nu} \Delta^\nu_\alpha = \Delta^{\mu \alpha}\).

Using this projector for future convenience the energy-momentum tensor can be written

$$T_{id}^{\mu \nu} = \epsilon u^\mu u^\nu - P \Delta^{\mu \nu} \quad (B.6)$$

### B.1.2 Ideal (Euler) Equations of Motion

In this section I derive the Euler equations of motion from conservation of the ideal energy-momentum tensor derived in the previous section. Without sources, the conservation equations are written in the compact form

$$\partial_\nu T_{id}^{\mu \nu} = 0 \quad (B.7)$$

It will be convenient to project these equations parallel and perpendicular to the fluid velocity (i.e., \(u_\mu \partial_\nu T_{id}^{\mu \nu}\) and \(\Delta^\alpha_\mu \partial_\nu T_{id}^{\mu \nu}\)). For the parallel projection one gets

$$u_\mu \partial_\nu T_{id}^{\mu \nu} = u_\mu \partial_\nu [\epsilon u^\mu u^\nu - P \Delta^{\mu \nu}]$$

$$= u_\mu u^\nu \partial_\nu \epsilon + \epsilon [u_\mu u^\nu (\partial_\nu u^\mu) + u_\mu u^\mu \partial_\nu u^\nu] - u_\mu (\partial_\nu P) \Delta^{\mu \nu} - Pu_\mu \partial_\nu \Delta^{\mu \nu} \quad (B.8)$$
The second and fourth term vanish by using the respective properties \( u_\mu(\partial_\nu u^\mu) = \partial_\nu(u^2 = 1) - u^\mu \partial_\nu u_\mu = \frac{1}{2} \partial_\nu(1) = 0 \) and \( u_\mu \Delta^{\mu\nu} = 0 \). The normalization of the flow velocity, \( u^\mu u_\mu = 1 \), also gives

\[
u(\partial_\nu u^\nu - P u_\mu \partial_\nu(-u^\mu u^\nu))
= \nu \partial_\nu \epsilon + \epsilon \partial_\nu \nu + P \partial_\nu \nu = (\epsilon + P) \partial_\nu \nu + u^\nu \partial_\nu \epsilon = 0 \quad (B.9)
\]

For the perpendicular projection one has

\[
\Delta_\mu^\alpha \partial_\nu T^\mu_{id} = \Delta_\mu^\alpha \partial_\nu \left[ \epsilon u^\mu u^\nu - P \Delta^{\mu\nu} \right]
\]

\[
= \Delta_\mu^\alpha \epsilon u^\nu \partial_\nu \epsilon + \Delta_\mu^\alpha \partial_\nu u^\nu + \Delta_\mu^\alpha \Delta^{\mu\nu} \partial_\nu P - \Delta_\mu^\alpha P \partial_\nu(-u^\mu u^\nu) \quad (B.10)
\]

The first two terms vanish by the property \( \Delta^\mu_\nu u^\nu = \Delta^{\mu\nu} u_\nu = 0 \) leaving

\[
\Delta_\mu^\alpha \epsilon u^\nu \partial_\nu \epsilon - \Delta_\mu^\alpha \partial_\nu P + \Delta_\mu^\alpha P \partial_\nu u^\mu = (\epsilon + P) \Delta_\mu^\alpha \partial_\nu u^\mu - \Delta_\mu^\alpha \partial_\nu P \quad (B.11)
\]

The term that multiplies \( \epsilon + P \) can be simplified as

\[
\Delta_\mu^\alpha \epsilon u^\nu \partial_\nu \epsilon = \epsilon \partial_\nu \nu + P \partial_\nu \nu = (\epsilon + P) \partial_\nu \nu + u^\nu \partial_\nu \epsilon = 0 \quad (B.12)
\]

Introducing shorthands for the derivatives projected parallel and perpendicular to the flow

\[
D \equiv u^\mu \partial_\mu, \quad \nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu
\]

the equations of motion for a relativistic ideal fluid are

\[
(\epsilon + P) \partial_\nu \nu + D \epsilon = 0, \quad (\epsilon + P) D u^\alpha - \nabla^\alpha P = 0 \quad (B.14)
\]

where the “expansion scalar”, \( \partial_\nu \nu \), is often denoted by \( \theta \).

In the non-relativistic limit, where the velocity \( \nu(u^\mu) \) defined in (2.2) is much less than 1 (the speed of light)

\[
D \equiv u^\mu \partial_\mu = \gamma(v)(1, \vec{v}) \cdot (\partial_t, \vec{\partial}) \simeq (1 + \frac{v^2}{2})(1, \vec{v}) \cdot (\partial_t, \vec{\partial}) \simeq \partial_t + \vec{v} \cdot \vec{\partial} + \mathcal{O}(v^2),
\]

\[
\nabla^\mu = \partial^\mu - u^\mu D \simeq \partial^\mu - (1 + \frac{v^2}{2})(1, \vec{v})(\partial_t + \vec{v} \cdot \vec{\partial}) \simeq (0 + \mathcal{O}(v), \vec{\partial} + \mathcal{O}(v)) \simeq (0, -\partial_t).
\]

\[
(B.15)
\]
Thus \( D \) and \( \nabla^\mu \) are essentially temporal and spatial derivatives, respectively, in the non-relativistic limit. For a non-relativistic fluid, \( P \ll \epsilon \) and the energy density is dominated by the mass \( \epsilon \simeq \rho \) so the non-relativistic equations of motion are the continuity equation

\[
D\epsilon + (\epsilon + P)\partial_\mu u^\mu \simeq (\partial_t - \vec{v} \cdot \vec{D})\rho + \rho(\partial_t u^0 + \partial_i u^i) = \partial_t \rho + \nabla^i(\rho v^i) = 0 , \tag{B.16}
\]

and the Euler [64] equation

\[
(\epsilon + P)Du^\alpha - \nabla^\alpha \simeq \rho(\partial_t + \vec{v} \cdot \vec{D})u^\alpha + \partial_\alpha P = \partial_t \vec{v} + (\vec{v} \cdot \vec{D})\vec{v} - \frac{1}{\rho} \vec{D}P = 0 . \tag{B.17}
\]

For reference, the quantity \( \partial_\nu T^\mu_\nu \) itself is calculated here and setting this quantity to 0 is equivalent to the projected equations (B.14).

\[
\partial_\nu T^\mu_\nu = \partial_\nu [(\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu}] = u^\mu u^\nu \partial_\nu(\epsilon + P) + (\epsilon + P)(u^\nu \partial_\nu u^\mu + u^\mu \partial_\nu u^\nu) - \partial^\mu P \\
= u^\mu D(\epsilon + P) + (\epsilon + P)(Du^\mu + \mu^\alpha \partial \cdot u) - \partial^\mu P = u^\mu D\epsilon - \nabla^\mu P + (\epsilon + P)Du^\mu + (\epsilon + P)u^\mu \partial \cdot u , \tag{B.18}
\]

where in the last line \( \partial^\mu = \nabla^\mu + u^\mu D \) was used. Note that \( \partial \cdot u \equiv \partial_\mu u^\mu \) is equivalent to \( \nabla \cdot u \) since \( u_\mu Du^\mu = 0 \).

\[
\boxed{\partial_\nu T^\mu_\nu = u^\mu D\epsilon - \nabla^\mu P + (\epsilon + P)Du^\mu + (\epsilon + P)u^\mu \partial \cdot u} \tag{B.19}
\]

There will also be a conservation equation for each conserved charge, \( c \), in the system:

\[
\partial_\mu N^\mu_{id,c} = 0 , \quad N^\mu_{id,c} = n_c u^\mu \tag{B.20}
\]

with \( n_c \) the charge density in the local rest frame. Thus the charge conservation equation for ideal fluids can be written as

\[
\partial_\mu (nu^\mu) = u^\mu \partial_\mu n + n \partial_\mu u^\mu = 0 \Rightarrow Dn = -n \partial \cdot u \tag{B.21}
\]
B.1.3 Ideal Entropy

Here the entropy density 4-vector \( s^\mu \equiv su^\mu \) is defined, where \( s \) is the equilibrium entropy density in the local rest frame that obeys the thermodynamic relation

\[
Ts = \epsilon + P - \sum_c \mu_c n_c ,
\]  
(B.22)

where \( \mu_c \) is the chemical potential for the conserved charge \( c \). The differential of \( s \) satisfies the first law of thermodynamics

\[
Ds = \frac{D\epsilon}{T} - \sum_c \frac{\mu_c}{T} Dn_c .
\]  
(B.23)

The relativistic second law of thermodynamics states that

\[
\partial_\mu s^\mu \geq 0 ,
\]  
(B.24)

where the equality holds in thermal equilibrium\(^2\). One can also write

\[
\partial_\mu s^\mu = \partial_\mu (su^\mu) = s \partial_\mu u^\mu + Ds = \frac{1}{T} (\epsilon + P - \sum_c \mu_c n_c) \partial_\mu u^\mu + \frac{D\epsilon}{T} - \sum_c \frac{\mu_c}{T} Dn_c
\]

\[
= \frac{1}{T} (\epsilon + P - \sum_c \mu_c n_c) \partial_\mu u^\mu - \frac{1}{T} (\epsilon + P) \partial_\mu u^\mu + \sum_c \frac{\mu_c}{T} n_c \partial_\mu u^\mu = 0 \]  
(B.25)

with the help of the equations of motion\(^1\) and \( \text{(B.21)} \) to rewrite \( D\epsilon \) and \( Dn_c \).

B.2 Viscous Hydrodynamics Derivations

In this section the equations of motion and energy-momentum tensor are derived for a viscous fluid near thermal equilibrium.

B.2.1 Viscous Equations of Motion

If the system has dissipation (entropy production), then the energy-momentum tensor and charge current vector will contain additional terms:

\[
T^{\mu\nu} \equiv T_{id}^{\mu\nu} + \delta T^{\mu\nu} , \quad N_c^\mu \equiv n_c u^\mu + \delta N^\mu . \]  
(B.26)

---

\(^2\)Entropy can still be produced for discontinuous solutions to the ideal equations, such as shocks as noted in \[60\].
There is an ambiguity to address in the definition of the flow velocity for viscous hydrodynamics. In a fluid cell there is a direction of net energy flow into or out of the cell as well as a direction of net charge flow for each conserved charge. These are, in general, not the same vector, so one needs to make a choice whether the local rest frame, i.e. the frame where $u^\mu = (1, \vec{0})$, is the frame where there is no energy flow (Landau frame) or the frame with no charge density flow (Eckart frame).

In the Landau frame one defines the flow velocity by

$$u_\mu T^{\mu \nu} \Delta^\alpha_\nu \equiv 0 \Rightarrow u_\mu \delta T^{\mu \nu} = 0 \ , \quad (B.27)$$

while in the Eckart frame one defines the flow velocity as satisfying

$$u^\mu N^\mu_c = n_c \Rightarrow u_\mu \delta N^\mu = 0 \ . \quad (B.28)$$

Throughout this work I will use the Landau definition. In ideal hydrodynamics, $\delta N^\mu = 0$ and $\delta T^{\mu \nu} = 0$ so both choices correspond to the same frame and therefore, no distinction is necessary in the previous sections.

As in the ideal case, the equations of motion for a viscous fluid can be derived from conservation of the energy-momentum tensor and the charge currents. Projecting the former onto the flow velocity gives

$$u_\mu \partial_\nu T^{\mu \nu} = u_\mu \partial_\nu T_{id}^{\mu \nu} + u_\mu \partial_\nu \delta T^{\mu \nu} = (\epsilon + P) \partial_\nu u^\mu + D\epsilon + \partial_\nu (u_\mu \delta T^{\mu \nu}) - \delta T^{\mu \nu} \partial_\nu u_\mu = (\epsilon + P) \partial_\mu u^\mu + D\epsilon + \delta T^{\mu \nu} (u_\nu D + \nabla_\nu) u_\mu = (\epsilon + P) \partial_\mu u^\mu + D\epsilon - \delta T^{\mu \nu} \nabla_\nu (u_\mu) = 0 \ , \quad (B.29)$$

where the symmetric $\delta T^{\mu \nu}$ projects out the symmetric piece of $\nabla_\nu u_\mu$ defined as

$$\nabla_{(\nu u_\mu)} \equiv \frac{1}{2} (\nabla_\nu u_\mu + \nabla_\mu u_\nu) \quad (B.30)$$

Projecting the energy-momentum conservation equation orthogonal to the flow velocity yields

$$\Delta^\alpha_\mu \partial_\nu T^{\mu \nu} = \Delta^\alpha_\mu \partial_\nu T_{id}^{\mu \nu} + \Delta^\alpha_\mu \partial_\nu \delta T^{\mu \nu} = (\epsilon + P) Du^\alpha - \nabla^\alpha P + \Delta^\alpha_\mu \partial_\nu \delta T^{\mu \nu} = 0 \ . \quad (B.31)$$
The projected equations of motion for a relativistic viscous fluid from energy-momentum conservation are then

\[(\epsilon + P)\partial_\mu u^\mu + D\epsilon - \delta T^{\mu\nu} \nabla_{(\nu} u_{\mu)} = 0, \quad (\epsilon + P)D u^\alpha - \nabla^\alpha P + \Delta_\mu^\alpha \partial_\nu \delta T^{\mu\nu} = 0.\] (B.32)

Conservation of charged currents in the system yields the equations

\[\partial_\mu N_c^\mu = \partial_\mu (n_c u^\mu) + \partial_\mu \delta N_c^\mu = Dn_c + n_c \partial_\mu u^\mu + \partial_\mu \delta N_c^\mu = 0.\] (B.33)

Equations (B.32) and (B.33) form part of the equations of motion of a relativistic viscous fluid, though the forms of the corrections $\delta T^{\mu\nu}$ and $\delta N_c^\mu$ are, as of yet, unspec-ified. Additional equations of motion are needed for $\delta T^{\mu\nu}$ and $\delta N_c^\mu$ (e.g. Israel-Stewart theory), or $\delta T^{\mu\nu}$ and $\delta N_c^\mu$ need to be related to gradients of the ideal hydrodynamic variables (e.g. the Navier-Stokes or Burnett equations).

### B.2.2 Derivation of $\delta T^{\mu\nu}$ from Equilibrium Entropy

If dissipative terms are included then the system considered is by definition, out of equilibrium. Here the form of $\delta T^{\mu\nu}$ is derived to first order in flow gradients from the second law of thermodynamics using the equilibrium form of the entropy current, which remains valid at the first Chapman-Enskog approximation used here, alongside the viscous equations of motion.

\[\partial_\mu s^\mu = s\partial_\mu u^\mu + Ds = \frac{1}{T}(\epsilon + P - \sum_c \mu_c n_c)\partial_\mu u^\mu + \frac{D\epsilon}{T} - \sum_c \frac{\mu_c}{T} Dn_c =\]

\[= \frac{1}{T}(\epsilon + P - \sum_c \mu_c n_c)\partial_\mu u^\mu - \frac{1}{T}(\epsilon + P)\partial_\mu u^\mu + \frac{1}{T} \delta T^{\mu\nu} \nabla_{(\nu} u_{\mu)} + \sum_c \frac{\mu_c}{T} n_c \partial_\mu u^\mu\]

\[= \frac{1}{T} \delta T^{\mu\nu} \nabla_{(\nu} u_{\mu)} \geq 0 \quad (B.34)\]

It is customary to rewrite $\delta T^{\mu\nu}$ in terms of irreducible tensors

\(^3\)For details on the form of the entropy current in the Chapman-Enskog procedure, see Chapter V of [27].
\[ \delta T^{\mu\nu} = \pi^{\mu\nu} + \Pi \Delta^{\mu\nu} \]  

(B.35)

where \( \pi^{\mu\nu} \) is the traceless symmetric part of \( \delta T^{\mu\nu} \), i.e. \( \pi^\mu_\mu = 0 \), and \( \Pi \) is \( \frac{1}{3} Tr[\delta T^{\mu\nu}] \) since \( Tr[\Delta^{\mu\nu}] \equiv \Delta^\mu_\mu = 4 - 1 = 3 \).

Similarly I define the traceless symmetric part of \( \nabla_(u\mu) \) projected out by \( \pi^{\mu\nu} \)

\[ \sigma_{\mu\nu} \equiv 2\nabla_(u\mu) - \frac{2}{3} \Delta_{\mu\nu} \nabla^\alpha u^\alpha \]  

(B.36)

With these decompositions the second law gives

\[ \frac{1}{2T} \pi^{\mu\nu} \sigma_{\mu\nu} + \frac{1}{T} \Pi \nabla^\alpha u^\alpha \geq 0 \]  

(B.37)

The inequality is guaranteed if the left hand side is the sum of squares, i.e. if

\[ \pi^{\mu\nu} = \eta \sigma_{\mu\nu}, \quad \Pi = \zeta \nabla^\alpha u^\alpha, \quad \eta \geq 0, \quad \zeta \geq 0 \]  

(B.38)

Comparing these definitions with the nonrelativistic limit, one sees these are the relativistic generalizations of the Navier-Stokes equations of motion if one identifies \( \eta \) as the shear viscosity and \( \zeta \) as the bulk viscosity.
C. Thermodynamic Calculations

In local thermal equilibrium the distribution function \( f(x, p) \) can be written as

\[
f_{\text{eq}}(x) = \frac{g}{(2\pi)^3} \left\{ \exp \left[ \frac{p \cdot u(x) - \mu(x)}{T(x)} \right] + a \right\}^{-1},
\]

where \( g \) is the number of internal degrees of freedom, \( p \cdot u \) is the energy in the fluid rest frame, and \( \mu \) is the chemical potential. The parameter \( a = 0 \) for Boltzmann statistics, \(+1\) for Fermi-Dirac, and \(-1\) for Bose-Einstein.

For particles obeying Boltzmann statistics, (C.1) takes the form

\[
f^\text{eq} = \frac{g}{(2\pi)^3} e^{\frac{\mu - E}{T}}
\]

where \( E \) is the on-shell energy of the particle in the rest frame of the fluid; i.e. \( E = \sqrt{p^2 + m^2} \), where \( m \) is the mass of the particle and \( E = |p| \) for massless particles in natural units.

C.1 Number Density: Massless Limit

In this section of the Appendix, the number density for particles in thermal equilibrium is calculated for massless particles. Throughout this section \( p \equiv |p| \) will refer to the magnitude of the 3-momentum in the fluid rest frame and \( \tilde{p} \equiv \frac{p}{T} \) is the dimensionless magnitude of this 3-momentum. In the massless limit, (C.2) takes the form

\[
n(r, t) \equiv \frac{dN(r, t)}{d^3 r} = \int d^3 p \ f^\text{eq}(x, p) = \frac{g}{(2\pi)^3} \frac{e^\frac{\mu}{T}}{4\pi T^3} \frac{2}{\pi^2} \int_0^\infty d\tilde{p} \tilde{p}^2 e^{-\tilde{p}}.
\]

The \( d\tilde{p} \) integral is from zero to infinity, so integration by parts gives

\[
n = \frac{gT^3}{2\pi^2} \frac{e^\frac{\mu}{T}}{(2!)} = \frac{gT^3}{\pi^2} \frac{e^\frac{\mu}{T}}{}.
\]
Defining the dimensionless chemical potential as \( \bar{\mu} \equiv \mu / T \) allows one to write the thermal density of massless particles as

\[
n_{m=0} = \frac{g T^3}{\pi^2} e^{\bar{\mu}}. \tag{C.5}
\]

### C.2 Number Density: Arbitrary Mass

For particles of arbitrary mass, the steps leading to (C.3) now give

\[
n = \int d^3 p \; f = \frac{g}{(2\pi)^3} e^{\bar{\mu}} \int d^3 p \; e^{-E} = \frac{g}{(2\pi)^3} e^{\bar{\mu}} 4\pi T^3 \int_0^\infty dp \; p^2 e^{-\tilde{E}}, \tag{C.6}
\]

where \( \tilde{E} \equiv E / T \). Using

\[
dE dp = \frac{1}{2} \frac{1}{\sqrt{p^2 + m^2}} \tag{C.7}
\]

yields

\[
n = \frac{g T^3}{2\pi^2} e^{\bar{\mu}} \int d\tilde{E} \; \frac{\tilde{E}}{\tilde{p}} \tilde{p}^2 e^{-\tilde{E}} = \frac{g T^3}{2\pi^2} e^{\bar{\mu}} \int_{\tilde{z}}^\infty d\tilde{E} \; \tilde{E} \sqrt{\tilde{E}^2 - \tilde{z}^2} e^{-\tilde{E}}, \tag{C.8}
\]

where I have defined the dimensionless mass, \( \tilde{z} \equiv m / T \). The integral in (C.8) can be found in [132](page 351, Eq. (3.389.4)):

\[
\int_{\tilde{z}}^\infty d\tilde{E} \; \tilde{E} \sqrt{\tilde{E}^2 - \tilde{z}^2} e^{-\tilde{E}} = 2^1 (\sqrt{\pi})^{-1} \tilde{z}^2 \Gamma\left(\frac{3}{2}\right) K_2(z) = \tilde{z}^2 K_2(z), \tag{C.9}
\]

where the gamma function is defined as [132]

\[
\Gamma(z) = \int_0^\infty dte^{-t} t^{z-1}, \quad \Re[z] > 0 \tag{C.10}
\]

and obeys \( \Gamma(\frac{3}{2}) = \frac{1}{2} \Gamma(\frac{1}{2}) = \frac{1}{2} \sqrt{\pi} \). \( K_{\nu}(z) \) is the modified Bessel function of the second kind, which has an integral representation ([132], page 917)

\[
K_{\nu}(z) = \left(\frac{\tilde{z}}{2}\right)^{\nu} \left(\frac{1}{2}\right) \frac{\Gamma(\frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty dt e^{-\tilde{z}t} (t^2 - 1)^{\nu - \frac{1}{2}}. \tag{C.11}
\]

Thus one obtains the density of massive particles in thermal equilibrium to be

\[
n = \frac{g T^3}{\pi^2} e^{\bar{\mu}} \frac{z^2}{2} K_2(z). \tag{C.12}
\]
For \( z \ll 1 \) the modified Bessel function of the second kind has the asymptotic form

\[
K_{\nu}(z \ll 1) \to \frac{1}{2} \Gamma(\nu) \left(\frac{2}{z}\right)^{\nu},
\]

(C.13)

so in the massless limit

\[
n_{m=0} \to \frac{g T^3}{\pi^2} \hat{e}^\mu \frac{z^2}{2} \left(\frac{2}{z}\right)^2 \frac{g T^3}{\pi^2} \hat{e}^\nu (1) = n_{m=0},
\]

(C.14)

and one recovers the massless result (C.5).

### C.3 Energy Density: Massless Limit

In this section of the Appendix, the energy density of massless particles in thermal equilibrium is calculated using classical Boltzmann statistics. Substituting (C.2) into the definition of the rest frame energy density gives

\[
\epsilon(x) = \int d^3 p \, f^{eq}(x, p)E = \frac{g}{(2\pi)^3} e^{\hat{p} \cdot \hat{\mu}} \int d^3 p \, e^{-\frac{\hat{p} \cdot \hat{T}}{T}} p = \frac{g}{(2\pi)^3} e^{\hat{\mu}} 4\pi T^4 \int_0^\infty d\tilde{p} \, \tilde{p}^3 e^{-\tilde{p}}. \quad (C.15)
\]

Integration by parts gives

\[
\epsilon = \frac{g T^4}{2\pi^2} e^{\hat{\mu}} (3!) = \frac{3g T^4}{\pi^2} e^{\hat{\mu}} = 3nT. \quad (C.16)
\]

So for massless particles, the energy density in thermal equilibrium is

\[
\epsilon_{m=0} = \frac{3g T^4}{\pi^2} e^{\hat{\mu}} = 3n_{m=0} T. \quad (C.17)
\]

### C.4 Energy Density: Arbitrary Mass

For particles of nonzero mass, the steps leading to (C.15) give

\[
\epsilon = \int d^3 p \, f \cdot E = \frac{g}{(2\pi)^3} e^{\hat{p} \cdot \hat{\mu}} \int d^3 p \, e^{-\frac{\hat{p} \cdot \hat{T}}{T}} E = \frac{g}{(2\pi)^3} e^{\hat{\mu}} 4\pi \int_0^\infty dp \, p^2 \, E e^{-\frac{\hat{p} \cdot \hat{T}}{T}}
\]

\[
\quad = \frac{g}{2\pi^2} e^{\hat{\mu}} \int_m^\infty dE \, E^2 \sqrt{E^2 - m^2} e^{-\frac{E}{T}}. \quad (C.18)
\]

A different method of integration from the one used to calculate the number density is needed as the integrand in (C.18) does not fall into the same class as the
integrand in (C.8). To get (C.18) into a similar form as (C.8) one can use the fact that
\[ \frac{\partial}{\partial T} e^{-\frac{E}{T}} = \left( \frac{E}{T^2} \right) e^{-\frac{E}{T}} \]  
(C.19)
to rewrite
\[
\int_{m}^{\infty} dE \ E^2 \sqrt{E^2 - m^2} e^{-\frac{E}{T}} = \int_{m}^{\infty} dE \ T^2 \frac{\partial}{\partial T} (E \sqrt{E^2 - m^2} e^{-\frac{E}{T}}) = T^2 \frac{\partial}{\partial T} (T^3 \int_{z}^{\infty} d\tilde{E} \ \tilde{E} \sqrt{\tilde{E}^2 - z^2} e^{-\tilde{E}}) = T^2 \frac{\partial}{\partial T} \left[ T^3 z^2 K_2(z) \right] = T^2 m^2 \frac{\partial}{\partial T} \left[ TK_2 \left( \frac{m}{T} \right) \right], \quad (C.20)
\]
where
\[
\frac{\partial}{\partial T} \left[ TK_2 \left( \frac{m}{T} \right) \right] = K_2 \left( \frac{m}{T} \right) + T \frac{\partial}{\partial T} K_2 \left( \frac{m}{T} \right) = K_2(z) + T \frac{\partial z}{\partial T} \frac{\partial}{\partial z} K_2(z)
= K_2(z) - T \frac{m}{T^2} \frac{\partial}{\partial z} K_2(z). \quad (C.21)
\]
Taking the derivative of (C.11) with respect to \( z \) one can derive a recurrence relation for the modified Bessel functions of the second kind (Ref. [132], page 929, Eq. (8.486.12)):
\[
\frac{\partial}{\partial z} K_\nu(z) = -\frac{\nu}{z} K_\nu(z) - K_{\nu-1}(z). \quad (C.22)
\]
The energy density can now be expressed as \[ \epsilon = g \frac{e^\mu}{2\pi^2} m^2 T^2 \left\{ K_2(z) - z \left[ -2 K_2(z) - K_1(z) \right] \right\} = g \frac{e^\mu T^4 z^2}{2\pi^2} [3K_2(z) + zK_1(z)]. \quad (C.23) \]
Rewriting (C.23) in terms of the number density (C.12) gives the energy density of massive particles in thermal equilibrium:
\[
\epsilon = \frac{g T^4}{\pi^2} e^\mu z^2 K_2(z) \left[ 3 + z \frac{K_1(z)}{K_2(z)} \right] = nT \left[ 3 + z \frac{K_1(z)}{K_2(z)} \right]. \quad (C.24)
\]
Again using the asymptotic form for the Bessel functions (C.13)
\[
z \frac{K_1(z)}{K_2(z)} \rightarrow z(1 - \frac{2}{z} \left( \frac{z}{2} \right)^2 = \frac{z^2}{2} \rightarrow 0 \quad (C.25)
\]
one recovers the massless limit result for the energy density, \( 3nT \) in (C.17).

\[ ^3 \text{There are many equivalent ways to write these expressions because of Bessel function identities, such as } K_{n-1}(x) + 2nK_n(x)/x = K_{n+1}(x). \]
C.5 Pressure

In this section, the pressure of an ideal gas in thermal equilibrium is calculated. The procedure closely follows the derivation in [89], and the result is valid both for ultrarelativistic massless particles as well as nonrelativistic systems.

Consider a gas of massless particles hitting the wall of a box. The pressure of the gas is defined as the force per unit area on the wall. The force of a particle on the wall is equal in magnitude to the force of the wall on the particle, $F$, and the particles that will hit the wall in a time increment $dt$ are within a distance of $L = v_x dt$. If there are $N$ particles a distance $L$ from a wall in the $y-z$ plane of area $A$, then the pressure is

$$P = N F/A = N \frac{dp_x}{Adt} = \frac{N}{2} \frac{dp_x}{A \frac{L}{v_x}} = \frac{N}{LA} \frac{dp_x v_x}{2}.$$  \hspace{1cm} (C.26)

If one assumes isotropic velocities of the particles in thermal equilibrium, then on average half of the $N$ particles will be moving toward the wall and half away, thus the factor of 2 in (C.26). If the particles have an isotropic velocity distribution then on average $v_x \simeq v_y \simeq v_z$ and thus $v \simeq \sqrt{3} v_x$ and $p \simeq \sqrt{3} p_x$. If the collisions with the wall are elastic, then the particles rebound with the same magnitude of momentum with which they hit, only with the opposite x-component so that $dp_x = 2p_x$. Defining $LA$ as the volume of the particles, $V$, Eqn (C.26) then becomes

$$P = \frac{N}{V} \frac{1}{2} \frac{dp}{\sqrt{3} \sqrt{3}} \frac{v}{\sqrt{3}} = \frac{1}{3} npv = \frac{1}{3} nE(1) = \frac{\epsilon}{3},$$  \hspace{1cm} (C.27)

where the density of particles $n \equiv N/V$ and the energy density $\epsilon \equiv nE$ were defined. For massless particles, one can put $p = E$ and $v = c \equiv 1$ in (C.27). This gives the equation of state for a massless ideal gas in thermal equilibrium:

$$P = \frac{\epsilon}{3} = nT.$$  \hspace{1cm} (C.28)

Though this derivation was for massless particles, the relation $P = nT$ is also valid for the general massive ideal gas.
C.6 Entropy Density

The entropy density \( s \) of an ideal massless gas in thermodynamic equilibrium can be calculated using the thermodynamic relationship

\[
\epsilon = Ts - P + \mu n \Rightarrow s = \frac{\epsilon + P - \mu n}{T} .
\] (C.29)

Substituting (C.17) and (C.28) into (C.29) gives

\[
s_{m=0} = \frac{3nT + nT - \mu n}{T} = n(4 - \bar{\mu}) .
\] (C.30)

The entropy density of an ideal gas of massive particles is given similarly by substituting (C.24) and (C.28) into (C.29) yielding

\[
s = n \left[ 4 + \frac{zK_1(z)}{K_2(z)} - \bar{\mu} \right] .
\] (C.31)
D. Democratic Grad Ansatz: Partial Enthalpy

This section involves calculations for the democratic Grad ansatz defined originally in (2.20) as

$$\phi^\text{dem}_i(x, p) = \frac{\pi_{\mu\nu}(x)p_\mu p_\nu}{2[\epsilon(x) + P(x)]T^2(x)}, \quad (D.1)$$

with $\phi^\text{dem}_i(x, p)$ defined by (2.18) as

$$f_i(x, p) = f^\text{eq}_i(x, p) + \delta f_i(x, p) \equiv f^\text{eq}_i(x, p)[1 + \phi_i(x, p)]. \quad (D.2)$$

Here it is shown explicitly how the democratic Grad ansatz (D.1) defined above satisfies the constraint 2.19 for a system with shear stress only ($\Pi = 0$); i.e. Eq. (2.21),

$$\pi_{\mu\nu}^\text{dem}_i \equiv \int d^3 p E p^\mu p^\nu \delta f^\text{dem}_i = \epsilon_i + P_i \epsilon + P \pi_{\alpha\beta} p^\alpha p^\beta 2(\epsilon + P)T^2 = \pi_{\alpha\beta}^{\text{dem}} \frac{\epsilon_i - P_i}{2(\epsilon + P)T^2}. \quad (D.3)$$

is derived explicitly here. Substituting the democratic form of the correction (D.1) into the shear stress tensor defined in the constraint on the energy-momentum tensor (2.19) gives

$$\pi_{\mu\nu}^\text{dem} = \int \frac{d^3 p}{E_i} p^\mu p^\nu f^\text{eq}_i \frac{\epsilon_i + P_i}{2(\epsilon + P)T^2} \int \frac{d^3 p}{E_i} p^\mu p^\nu p^\alpha p^\beta g_i \frac{\epsilon_i - P_i}{2(\epsilon + P)T^2}. \quad (D.4)$$

For simplicity, here only one component of the shear tensor is calculated in the local rest frame and all Lorentz invariant combinations are evaluated in this frame as well. The $x, z = 1, 3$ tensor component is chosen since the components with 0 index in the LR frame vanish for the Landau flow velocity due to $u_\nu \pi_{\mu\nu} = 0$.

$$\pi_{i, \text{dem}}^{xz} = \frac{g_i}{(2\pi)^3} \frac{\pi^{\alpha\beta} p^\alpha p^\beta}{2(\epsilon + P)T^2} \int \frac{d^3 p}{E_i} p^x p^z p^\alpha p^\beta e^{\frac{\mu_i - E_i}{T}}. \quad (D.5)$$
If the integrand is odd in any component of the 3-momentum, the integral will vanish as each component goes from $-\infty$ to $\infty$, thus the only contributions come from $(\alpha, \beta) = (x, z)$ or $(z, x)$

$$\pi_{i, \text{dem}}^{xx} = \frac{g_i}{(2\pi)^32(\epsilon + P)T^2} \left[ \pi^{xx} \int \frac{d^3p}{E_i} p^\mu p^\nu p_x p_z e^{\mu_i - E_i} + \pi^{xx} \int \frac{d^3p}{E_i} p^\mu p^\nu p_z p_x e^{\mu_i - E_i} \right]$$

(D.6)

The two terms are, in fact, equivalent since $\pi^{\mu\nu}$ and the integrand are both symmetric. Rewriting the covariant components as contravariant ones then gives

$$\pi_{i, \text{dem}}^{xx} = \frac{g_i}{(2\pi)^3} \frac{\pi^{xx}}{2(\epsilon + P)T^2} e^{\mu_i} \int \frac{d^3p}{E_i} p^\mu p^\nu (-p^x)(-p^z)e^{-E_i}.$$  

(D.7)

Using the conventional spherical coordinate system with $\theta$ the angle from the $z$-axis and $\phi$ the angle from the $x$-axis, one can evaluate the integral

$$\int \frac{d^3p}{E_i} p^2 p_x^2 e^{-E_i} = \int \frac{p^2 dp}{E_i} d (\cos \theta) \frac{d\phi}{E_i} e^{-E_i} p^2 \sin^2 \theta \cos^2 \phi p^2 \cos^2 \theta.$$  

(D.8)

First compute the angular integrals:

$$\int_0^{2\pi} d\phi \cos^2 \phi = \frac{1}{2} \cdot 2\pi = \pi$$

(D.9)

$$\int_{-1}^{1} d\cos \theta \sin^2 \theta \cos^2 \theta = \int_{-1}^{1} d\cos \theta (1 - \cos^2 \theta) \cos^2 \theta$$

$$= \int_{-1}^{1} d\cos \theta \left(\cos^2 \theta - \cos^4 \theta\right) = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

(D.10)

so

$$\pi_{i, \text{dem}}^{xx} = \frac{g_i}{(2\pi)^3} \frac{\pi^{xx}}{(\epsilon + P)T^2} e^{E_i} = \frac{\pi^{xx}}{\pi^{xx}} \frac{g_i T^4}{30\pi^2} \epsilon^{\mu} \int_0^{\infty} \frac{d\bar{E}_i}{E_i} (\epsilon + P) e^{-\bar{E}_i}$$

(D.11)

The remaining energy integral can be evaluated using [132] (page 350, Eq. (3.387.6)):

$$\int_z^{\infty} dE (E^2 - z^2)^{\frac{5}{2}} e^{-E} = \frac{1}{\sqrt{\pi}} (2z)^3 \Gamma \left( \frac{7}{2} \right) K_3(z) = 8z^3 \frac{5}{2} \frac{3}{2} \frac{1}{2} K_3(z) = 15z^3 K_3(z),$$

(D.12)

giving

$$\pi_{i, \text{dem}}^{xx} = \frac{g_i T^4}{(\epsilon + P) 30\pi^2} e^{E_i} 15z^3 K_3(z) = \frac{\pi^{xx}}{\pi^{xx}} \frac{g_i T^4}{\pi^2} e^{\mu} \frac{z^2}{2} z K_3(z).$$

(D.13)
$K_3(z)$ can also be written in terms of $K_2(z)$ and $K_1(z)$ using [132] (page 929, Eq. (8.486.10)):

$$zK_1(z) - zK_3(z) = -4K_2(z) \Rightarrow K_3(z) = K_1(z) + \frac{4}{z}K_2(z), \quad (D.14)$$

so the shear stress tensor can be written as

$$\pi_{i, dem}^{xz} = \frac{\pi^{xz}}{(\epsilon + P)} \frac{g_i T^4}{\pi^2} e^\mu z^2 \left[ zK_1(z) + 4K_2(z) \right] = \frac{\pi^{xz}}{(\epsilon + P)} \frac{g_i T^4}{\pi^2} e^\mu z^2 \left[ 4 + \frac{z K_1(z)}{K_2(z)} \right] \quad (D.15)$$

Comparison with the thermal results for energy density (C.24) and pressure (C.28) shows that

$$\pi_{i, dem}^{xz} = \frac{\epsilon_i + P_i \pi^{xz}}{\epsilon + P \pi^{xz}}, \quad (D.16)$$

and thus (2.21) follows from (2.20).
E. General form of $\phi_i(x, p)$

In this Appendix, the functions $\phi_i(x, p)$ that satisfy (3.8) are shown to be of the form (3.9) using the uniqueness of an expansion in irreducible tensors. Recall the distribution function correction $\phi_i(x, p)$ is defined through the relationship (2.18):

$$f_i(x, p) = f_i^{eq}(x, p) + \delta f_i(x, p) \equiv f_i^{eq}(x, p)[1 + \phi_i(x, p)].$$  \hfill (E.1)

The form (3.9):

$$\phi_i(x, p) = \chi_i(|\tilde{p}|) P_{\mu\nu} X_{\mu\nu}$$  \hfill (E.2)

comes from expanding $\phi_i(x, p)$ in terms of irreducible tensors\footnote{For properties of irreducible tensors, see Section VI.2a of Ref. [27] or Appendix F of Ref. [79].}

$$\phi_i(x, p) = \sum_{r=0}^{\infty} a_r(|\tilde{p}|) P^{(r)}(p) \cdot X^{(r)}(x),$$ \hfill (E.3)

which is just a Lorentz covariant way to write an expansion over spherical harmonics in the LR frame (the $\cdot$ denotes full contraction of tensors $P^{(r)}$ and $X^{(r)}$). $P^{(r)}$ is a rank-$r$ irreducible tensor projected out from the fully symmetric, rank-$r$ Lorentz tensor $p^{\mu_1}p^{\mu_2} \cdots p^{\mu_r}$ such that $P^{(r)}$ is purely spatial in the LR frame (orthogonal to the flow velocity $u$ in any index) and vanishes under contraction of any two of its indices, so it is the irreducible representation with maximal angular momentum $r$ from the tensor product of $r$ three-dimensional (spin-1) vectors in the LR frame, $\tilde{p} \otimes \tilde{p} \otimes \cdots \otimes \tilde{p}$. For example, with suitable normalization, $P^{(2)}_{\mu\nu}(p) = P_{\mu\nu}$ defined in (3.7). Because $\phi_i$ is a Lorentz scalar, $X^{(r)}$ is also a rank-$r$ irreducible tensor, while the coefficients $a_r$ are invariant under rotations in the LR frame, so their momentum dependence is only through the LR-frame particle energy, or equivalently, the normalized momentum magnitude $|\tilde{p}|$. The expansion (E.3) can be inverted for $X^{(r)}$ through integration using the orthogonality of invariant tensors:

$$X^{(r)}(x) \propto \int \frac{d^3p}{E} P^{(r)}(p) \phi_i(x, p),$$ \hfill (E.4)
where the omitted proportionality constant depends on $|\tilde{p}|$. Inverting both sides of (3.5), the shear source term (3.8) only contributes for $r = 2$, and the result is proportional to $X^{\mu\nu}$, so the RHS must give a similar contribution only for $r = 2$. Because the linearized collision operator commutes with Lorentz transformations, contains scalar functions of momentum, and the functions $f_{i}^{eq}$ only depend on $|\tilde{p}|$, the collision operator preserves the expansion (E.3) except for the coefficients $a_{r}$. Thus, (3.9) indeed follows.
F. Massless Single-Component Grad $Q_{ab}$ Terms

In this Appendix the terms $Q_{11}$, $Q_{21}$, and $Q_{31} = Q_{41}$ in (3.17) are calculated for a gas containing a single massless particle species with a constant, isotropic total cross-section $\sigma$.

F.1 $Q_{11}$ Term

From (3.17) we have

$$Q_{11} \equiv \frac{1}{2T} \int_1^{4} \int_1^{4} f_1^eq f_2^eq \bar{W}_{12\rightarrow34} \delta^4(p_1 + p_2 - p_3 - p_4) \chi_2^2 P_1 \cdot P_1.$$  \hfill (F.1)

The transition probability (2.27) simplifies for massless particles to

$$\bar{W}_{12\rightarrow34} \equiv \frac{4}{\pi s p_{cm}^2} \frac{d\sigma_{12\rightarrow34}}{dt} = \frac{s^2}{\pi} \frac{d\sigma}{dt},$$  \hfill (F.2)

with Mandelstam $t$ for massless particles given by

$$t \equiv -(p_1^\mu + p_2^\mu)^2 = 2p_1^\mu p_{3,\mu} = 2p_1^{cm} p_3^{cm} (1 - \cos \theta_{cm}) = 2 \frac{s^2}{4} (1 - \cos \theta_{cm}) = \frac{s}{2} (1 - \cos \theta_{cm}),$$  \hfill (F.3)

and $dt = \frac{d}{d \cos \theta_{cm}}$. Calculating $\sigma$ for isotropic (angle independent in cm) scattering gives

$$\sigma = \int \frac{d\sigma}{dt} dt = \frac{d\sigma}{dt} \frac{s}{2} \int_0^1 d \cos \theta_{cm} = \frac{d\sigma}{dt} \frac{s}{2} \int_0^1 d \cos \theta_{cm} = \frac{d\sigma}{dt} \frac{s}{2} \Rightarrow d\sigma = \frac{\sigma}{s} dt,$$

$$\sigma = \int d\cos \theta_{cm} d\sigma = \int \frac{s}{2} \frac{d\sigma}{dt} dt = \int \frac{d\sigma}{dt}$$

where the $\cos \theta_{cm}$ integration limits go from 0 to 1 due to the reduced phase space for identical particles. This gives the result for $\bar{W}_{12\rightarrow34}$ for massless isotropic scattering

$$\bar{W}_{12\rightarrow34} = \frac{s^2}{\pi} \frac{2}{s} = \frac{2s}{\pi}.$$

$$\bar{W}_{12\rightarrow34} = \frac{s^2}{\pi} \frac{2}{s} = \frac{2s}{\pi}.$$  \hfill (F.5)
Now evaluate the integrals involving particles 3 and 4 with $E = p$ for massless particles

\[
\int_3^4 \bar{W}_{12 \rightarrow 34} \delta^4(12 - 34) = \int \frac{d^3p_3}{2p_3} \int \frac{d^3p_4}{2p_4} \left( \frac{2s}{\pi} \right) \delta^3(p_1 + p_2 - p_3 - p_4) \delta(p_1 + p_2 - p_3 - p_4),
\]

\[\text{(F.6)}\]

where here I evaluate the invariant measure for the integrals and 4-dimensional delta function in the center-of-momentum (cm) frame of the collision such that $p_1 + p_2 = 0$. Integrating the $\delta$-function over $p_4$ sets $p_4 = p_1 + p_2 - p_3 = -p_3$ in the cm frame.

\[
\frac{1}{4} \int \frac{d^3p_3}{p_3} \frac{1}{p_3} \left( \frac{2s}{\pi} \right) \delta(p_1 + p_2 - 2p_3) = \frac{1}{4} \int d\Omega_3 \frac{d\Omega_3 p_3^2}{p_3^2} \left( \frac{2s}{\pi} \right) \delta(\sqrt{s} - 2p_3)
\]

\[\text{(F.7)}\]

with Mandelstam $s = \sqrt{E_{cm}^2} = \sqrt{p_1 + p_2}$ in the cm frame. Using the $\delta$-function property

\[
\delta(\sqrt{s} - 2p_3) = \left| \frac{\partial}{\partial p_3} (\sqrt{s} - 2p_3) \right|^{-1} \delta(p_3 - \frac{\sqrt{s}}{2}) = \frac{1}{2} \delta \left( p_3 - \frac{\sqrt{s}}{2} \right)
\]

\[\text{(F.8)}\]

allows the $p_3$ integral to set $p_3 = \frac{\sqrt{s}}{2}$, though all the $p_3$ terms here cancel. This leaves

\[
\int_3^4 \bar{W}_{12 \rightarrow 34} \delta^4(12 - 34) = \frac{1}{4} \int d\Omega_3 \left( \frac{2s}{\pi} \right) \frac{1}{2} = \frac{1}{8} \sigma \frac{2s}{\pi} \int d\Omega_3 = \frac{s}{4\pi} \sigma 4\pi.
\]

\[\text{(F.9)}\]

Thus for identical massless particles

\[
\int_3^4 \bar{W}_{12 \rightarrow 34} \delta^4(12 - 34) = s(p_1, p_2)\sigma.
\]

\[\text{(F.10)}\]

Putting this result into the $Q_{11}$ term and evaluating $p_1$ and $p_2$ in the LR frame of the fluid where the equilibrium distribution is isotropic gives

\[
Q_{11} = \frac{\chi^2}{2T^4} \left( \frac{ge\tilde{\mu}}{(2\pi)^3} \right)^2 \int_{12} e^{-\frac{p_1^2}{8}} e^{-\frac{p_2^2}{8}} \left( \frac{2}{3} \frac{p_1^4}{T^4} \right) (s\sigma).
\]

\[\text{(F.11)}\]

Evaluating the invariant $s$ in the LR frame gives

\[
s \equiv (p_1^\mu + p_2^\mu)^2 = m_1^2 + m_2^2 + 2(p_1^\mu p_2,\mu) = 0 + 0 + 2(E_1^{cm} E_2^{cm} - p_1 \cdot p_2) = 2p_1 p_2 (1 - \cos \theta_{12})
\]

\[\text{(F.12)}\]
\[ Q_{11} = \frac{\chi^2}{2T^4} \left( \frac{g e^\mu}{(2\pi)^3} \right)^2 \left( \frac{2}{3T^4} \sigma \right) \int d^3 p_1 d^3 p_2 e^{-\frac{p_1}{2p_2}} e^{-\frac{p_2}{2p_1}} p_1^4 2p_1p_2(1 - \cos \theta_{12}) \]
\[ = \frac{\chi^2}{3T^8} \left( \frac{g e^\mu}{(2\pi)^3} \right)^2 \left( \frac{1}{4} \pi \int \frac{dp_1 d\Omega_1 p_1^4 e^{-\frac{p_1}{2p_2}} (4\pi) \int \frac{dp_2 p_2^2 e^{-\frac{p_2}{2p_1}} 2p_1p_2(1 - \cos \theta_{12})}{p_1} \right) \]
\[ = \frac{\chi^2}{3T^8} \left( \frac{g e^\mu}{(2\pi)^3} \right)^2 \left( \frac{1}{4} \pi \int \frac{dp_1 d\Omega_1 p_1^6 e^{-\frac{p_1}{2p_2}} (1 - \cos \theta_{12}) \int dp_2 p_2^2 e^{-\frac{p_2}{2p_1}}}{p_1} \right) \]
\[ = \frac{\chi^2}{3T^8} \left( \frac{g e^\mu}{(2\pi)^3} \right)^2 \left( \frac{2}{\pi^2} \int d\tilde{p}_1 p_1^6 e^{-\tilde{p}_1} [4\pi] T^3 \int d\tilde{p}_2 p_2^2 e^{-\tilde{p}_2} \right), \quad (F.13) \]

where the factor of \([4\pi]\) comes from the 1 in the integration over the direction of \(p_1\) while the \(\cos \theta_{12}\) term gives zero. Doing the momentum integrals by parts gives the result
\[ Q_{11} = \frac{\chi^2}{3T^8} \left( \frac{g e^\mu}{(2\pi)^3} \right)^2 \left( \frac{8\pi^2 T^10 6!}{3} \right) = \frac{\chi^2 T^2 16\pi^2}{3} \left( \frac{1}{T^6} \frac{1}{2\pi^2} \right) = 60\chi^2 \sigma \frac{n^2}{T^4}. \quad (F.14) \]

The final result is thus
\[ Q_{11} = 60\chi^2 \sigma \frac{n^2}{T^4}. \quad (F.15) \]

**F.2 \(Q_{21}\) Term**

From (3.17) we have
\[ Q_{21} \equiv \frac{1}{2T^4} \int_{1234} f_1^{eq} f_2^{eq} W_{1234} \delta^4(p_1 + p_2 - p_3 - p_4) \chi^2 P_2 \cdot P_1. \quad (F.16) \]

The \(p_3\) and \(p_4\) are evaluated the same way as in \(Q_{11}\) while
\[ P_1 \cdot P_2 = (p_1 \cdot p_2)^2 - \frac{1}{3} p_1^2 p_2^2 = p_1^2 p_2^2 \left( \cos^2 \theta - \frac{1}{3} \right) \quad (F.17) \]

in the fluid rest frame. The angular integral now gives
\[ \int_{-1}^{1} d\cos \theta \left( \cos^2 \theta - \frac{1}{3} \right)(1 - \cos \theta) = \int_{-1}^{1} d\cos \theta \left( \cos^2 \theta - \frac{1}{3} \right) = \frac{1^3 - (-1)^3}{3} - \frac{2}{3} = 0, \quad (F.18) \]

where the first equality holds since terms odd in \(\cos \theta\) vanish upon integration. The terms left will integrate to 0 by inspection since 1 and \(\left( \cos^2 \theta - \frac{1}{3} \right)\) are proportional
to orthogonal Legendre polynomials $P_0$ and $P_2$. So for massless particles this term vanishes.

\[
Q_{21} = 0
\]  

**F.3 $Q_{31} = Q_{41}$ Term**

From (3.17),

\[
Q_{31} \equiv \frac{1}{2T^4} \int \int \int f_1 f_2 \tilde{W}_{12 \rightarrow 34} \delta^4(p_1 + p_2 - p_3 - p_4) \chi^2 P_3 \cdot P_1.
\]  

(F.20)

One cannot directly apply (F.10) since there is now another angle in the calculation besides $\theta_{21}$ from $P_3 \cdot P_1 = p_1^2 p_3^2 \left( \cos^2 \theta_{31} - \frac{1}{3} \right)$. The calculation is complicated by the fact that this dot product is simplest in the fluid rest frame, which is not the best frame to evaluate the $\delta$-function in. Care must be taken to keep these two frames separate for $p_3$. The bar notation will be used to indicate the momentum in the CM frame with $\bar{p}_3 \equiv p_{3m}$ and $\bar{p}_3 \equiv p_{3m} n_{3m}$.

The $p_4$ integral, however can be done the same way in the cm frame giving

\[
Q_{31} = \frac{\chi^2}{2T^4} \int f_1 f_2 \left( \frac{2s}{\pi} \right) \int d\bar{p}_3 \bar{p}_3^2 d\bar{\Omega}_3 \frac{1}{T^4} \left[ (p_1 \cdot p_3)^2 - \frac{1}{3} p_1^2 p_3^2 \right] \frac{1}{2\bar{p}_3} \delta \left( \sqrt{s} - 2\bar{p}_3 \right)
\]

\[
= \frac{\chi^2}{2T^8} \frac{2\sigma}{\pi} \frac{1}{4} \int f_1 f_2 \int d\bar{p} \bar{p}_3 \frac{1}{3} \left[ 3(p_1 \cdot p_3)^2 - p_1^2 p_3^2 \right] \frac{1}{2} \delta \left( \bar{p}_3 - \frac{\sqrt{s}}{2} \right)
\]

\[
= \frac{\chi^2}{8T^8} \frac{2\sigma}{\pi} \frac{1}{6} \int f_1 f_2 \left\{ \int d\bar{\Omega}_3 \left[ 3(p_1 \cdot p_3)^2 - p_1^2 p_3^2 \right] \right\} \bigg| \bar{p}_3 = \frac{\sqrt{s}}{2}.
\]  

(F.21)

Now $p_3$ in the fluid rest frame must be written in terms of $\bar{p}_3$ in the collision CM frame using the boost that connects these frames: $p_3 = \Lambda_{cm \rightarrow fluid} \bar{p}_3$.

\[
\begin{bmatrix}
 p_3 \\
 \bar{p}_3
\end{bmatrix} = \begin{bmatrix}
 \gamma & \gamma \beta \\
 \gamma \beta & \delta_{ij} + \frac{\gamma - 1}{\gamma} \beta^i \beta^j
\end{bmatrix} \begin{bmatrix}
 \bar{p}_3 \\
 p_3
\end{bmatrix}
\]  

(F.22)

with $\beta = \frac{p_1 + p_2}{E_1 + E_2} = \frac{p_1 + p_2}{p_1 + p_2}$, the fluid rest frame velocity only since the velocity of the collision CM frame is 0.

\[\text{For details on the orthogonal Legendre polynomials, see Ch. 3 of [133] or Ch. 12 of [134].}\]
Calculating the RHS of (F.22) gives

\[
\begin{bmatrix}
  p_3 \\
  p_3
\end{bmatrix}
= \begin{bmatrix}
  \gamma (\bar{p}_3 + \beta \cdot \bar{p}_3) \\
  \bar{p}_3 + \gamma \bar{p}_3 \beta + \gamma^{-1} (\beta \cdot \bar{p}_3) \beta
\end{bmatrix}.
\]  

\tag{F.23}

Replacing the LR frame values with these CM values in (F.21) gives for the term in curly brackets

\[
\int d\tilde{\Omega}_3 \left[ 3 \left( p_1 \cdot \bar{p}_3 + \gamma \bar{p}_3 \beta + \gamma^{-1} (\beta \cdot \bar{p}_3) p_1 \cdot \beta \right)^2 - p_1^2 (\gamma \bar{p}_3 + \gamma \beta \cdot \bar{p}_3)^2 \right] 
\]  

\tag{F.24}

Expanding the brackets and performing the integrals over \(d\tilde{\Omega}_3\) in conveniently rotated frames and then integrating over \(p_1\) and \(p_2\) as done in the \(Q_{11}\) term, one arrives at the result

\[
Q_{31} = Q_{41} = \frac{40 \sigma n^2}{3 T^4 \chi^2}.
\]  

\tag{F.25}
G. Viscosity for One-Component System

In this Appendix, the shear viscosity of an arbitrary mass,single-component system in the Grad ansatz is calculated following the approach of Chapter XI in Ref. [27] and the typographical error is corrected in the final result shown in Eq.(24) therein.

In Chapter VI of Ref. [27], it was shown (and verified) that the first term in the Chapman-Enskog expansion of the shear viscosity is (Section 3, Eq.(61))

\[ \eta_1 = \frac{1}{10} \frac{T}{\sigma} \left( \frac{\gamma_0}{C_{00}} \right)^2 \equiv \frac{10T\tilde{h}}{\sigma} [C_{00}]^{-1} , \quad (G.1) \]

where \( \tilde{h} \) is the reduced enthalpy in natural units

\[ \tilde{h} \equiv h/T \equiv \frac{\epsilon + P}{T} = \frac{zK_3(z)}{K_2(z)} . \quad (G.2) \]

\( C_{00} \) is a collision bracket involving twelve-fold integrals written in bracket notation as

\[ C_{00} = [P_{\mu\nu}, P_{\mu\nu}] , \quad (G.3) \]

if one uses the notation for \( P_{\mu\nu} \) in (3.7). The bracket notation is defined in Ref. [27] (page 381, Eq.(4)) for

\[ [F,G] \equiv \frac{1}{2T^0\sigma[\pi z^2 K_2(z)]^2} \int_{1\ 2\ 3\ 4} \int_{1\ 2\ 3\ 4} \int_{1\ 2\ 3\ 4} e^{-\tilde{p}_1-\tilde{p}_2} (F_1 + F_2 - F_3 - F_4) GW(p_3, p_4 | p_1, p_2) \ , \quad (G.4) \]

where \( F_a \) is a function of \( p_a \) and the previously used integral notation in (2.24) with 2 in the denominator was employed. The energy-momentum conserving \( \delta \)-function is included in \( W \). It is convenient to rewrite the integral expressions with the total and relative four-momenta defined according to

\[ P^\mu \equiv p_1^\mu + p_2^\mu \ , \quad P'^\mu \equiv p_3^\mu + p_4^\mu \ , \quad Q^\mu \equiv p_1^\mu - p_2^\mu \ , \quad Q'^\mu \equiv p_3^\mu - p_4^\mu . \quad (G.5) \]
Substituting (G.5) into (G.4), one is left with integrals of the generic form

\[ J(a,b,d,e,f) \equiv \frac{1}{2T^6 \sigma [\pi z^2 K_2(z)]^2} \int_1^2 \int_3^4 e^{-P \cdot u T - 2a - b - d - e - 2f} P^{2a} \]

\[ (P \cdot u)^b (Q \cdot u)^d (Q' \cdot u)^e (Q \cdot Q')^f W(p_3, p_4 | p_1, p_2) . \]

(G.6)

The integrals needed to calculate the transport coefficients involve the combinations

\[ J'(a,b,d,e,0) \equiv J(a,b,d+e,0,0) - J(a,b,d,e,0) . \]

(G.7)

Evaluating the \( P', Q, \) and \( Q' \) integrals in (G.6) in the CM frame and expanding in Legendre polynomials, one can make use of the spherical harmonics addition theorem (see Chapter 12 of [134])

\[ \int_0^{2\pi} d\phi \int_0^{\pi} d\phi' P_g(\cos \Theta) = 4\pi^2 P_g(\cos \theta) P_g(\cos \theta') , \]

(G.8)

where \( \Theta \) is the angle between \( Q \) and \( Q' \), the flow velocity is oriented along the \( z \)-axis, and \( \theta \) and \( \theta' \) are the angles of \( Q \) and \( Q' \) with respect to this axis. This leaves only the integral over \( P \), or equivalently \( v \equiv \tilde{P} \), to be evaluated in the rest frame of the fluid as Eq. (40) on page 380 of [27]. Considering isotropic scattering only, this integral can be performed with a change of variables \( v \rightarrow 2zu \). Expanding in Legendre polynomials and applying their orthogonality relation gives \( J' \) in (G.7) in terms of integrals of Bessel functions over the dummy variable \( u \) written in Eq. (18) of [27]:

\[ J'(a,b,d,e,f) = \frac{\pi (d+e-1)!! \sigma^{(d,e,f)} [b/2]}{z^4 K_2^2(z)} \sum_{h=0}^{[b/2]} (-1)^h (2h-1)!! \left( \begin{array}{c} b \\ 2h \end{array} \right) \]

\[ I \left( 2a+2f+3, \frac{1}{2}(d+e)+f+1, b+\frac{1}{2}(d+e)-h+1 \right| 2z \) , \]

(G.9)

with the “transfer cross section”

\[ \sigma^{(d,e,f)} = \frac{1}{2} \left[ 1-(d+e+1)de!f! \sum_{g=0}^{\min(d,e)} \right] \frac{2g+1}{(d-g)!!(e-g)!!(f-g)!!(d+g+1)!!(e+g+1)!!(f+g+1)!!} \] , \( (G.10) \)

\( u \) without the Greek index stands for a dummy variable in [27].
and with the function

\[ I(r,s,n|x) \equiv x^{r+n+1} \int_1^\infty du \; u^{r-2s+n}(u^2 - 1)^s K_n(xu) \quad \text{.} \tag{G.11} \]

The prime on the sum over \( g \) in (G.10) indicates that only the values of \( g \) that make \( d-g, e-g, \) and \( f-g \) even contribute to the sum, while \( g \leq f \).

Using the binomial theorem and Eqs. (12-17) on page 381 of [27], one can show that the general form of \( C^{rs} \) is given by Eq. (22) including \( A^{rs} \) and \( B^{rs} \) terms given by Eq. (21). Specifically the \( C^{00} \) term is

\[ C^{00} = -\frac{4}{3} A^{22} + \frac{1}{3} z^2 (A^{20} + A^{02}) - \frac{1}{3} z^4 A^{00} + 2B^{11} + \left( \frac{1}{2} \right)^3 \left[ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(2,0,0,0,0)} + J'^{(0,0,0,0,2)} - 2(0) \right] \quad \text{.} \tag{G.12} \]

The \( A^{rs} \) terms in (G.12) evaluate to

\[ A^{22} = \left( \frac{1}{2} \right)^3 \left[ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,4,0,0,0)} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,2,2,0,0)} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,0,2,2,0)} \right] = \frac{1}{8} J'^{(0,0,2,2,0)} \quad , \tag{G.13} \]

\[ A^{20} + A^{02} = \frac{1}{2} \left[ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,2,0,0,0)} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,0,2,0,0)} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,0,0,2,0)} \right] = 0 \quad , \tag{G.14} \]

and

\[ A^{00} = J'^{(0,0,0,0,0)} = 0 \quad . \tag{G.15} \]

The \( B \) term in (G.12) is

\[ B^{11} = A^{22} + \frac{1}{8} \left[ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} J'^{(0,0,1,1,1)} - 0 \right] = \frac{1}{8} \left[ J'^{(0,0,2,2,0)} + J'^{(0,0,1,1,1)} \right] \quad . \tag{G.16} \]

Putting Eqs. (G.13), (G.14), (G.15), and (G.16) into (G.12) gives \( C^{00} \) in terms of the \( J' \) integrals

\[ C^{00} = -\frac{4}{3} \frac{1}{8} J'^{(0,0,2,2,0)} + 0 - 0 + \frac{1}{8} \left( J'^{(0,0,2,2,0)} + J'^{(0,0,1,1,1)} \right) + \frac{1}{8} J'^{(0,0,0,0,2)} \]

\[ = \frac{1}{8} \left[ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} J'^{(0,0,2,2,0)} + 2J'^{(0,0,1,1,1)} + J'^{(0,0,0,0,2)} \right] \quad . \tag{G.17} \]
Using (G.9), one can find the three $J'$ integrals in (G.17) terms of the $I$ integrals defined in (G.11):

\[ J^{(0,0,2,2,0)} = \frac{2}{3} \frac{\pi}{z^4 K_2^2(z)} I(3,3,3|2z) , \]
\[ J^{(0,0,1,1,1)} = \frac{1}{3} \frac{\pi}{z^4 K_2^2(z)} I(5,3,2|2z) , \]
\[ J^{(0,0,0,2)} = \frac{1}{3} \frac{\pi}{z^4 K_2^2(z)} I(7,3,1|2z) . \]  

G.18

Substituting (G.18) into (G.1) gives the shear viscosity in terms of the $I$ integrals,

\[ \eta = \frac{80T}{\pi \sigma} \frac{z^6 K_2^2(z)}{\frac{2}{3} I(3,3,3|2z) + \frac{1}{3} I(5,3,2|2z) + I(7,3,1|2z)} \]  

G.19

Using the binomial expansion of $(u^2 - 1)^n$ allows one to expand $I(r, s, n|x)$ as a series of $I(r',0,n|x)$ and Eq. (11.3.5) in Ref. 135 gives a recurrence relation for these:

\[ I(r,0,n|x) = [(r+n-1)^2 - n^2] I(r-2,0,n|x) + (r-1)x^{r+n-1}K_n(x) + x^{r+n}K_{n+1}(x) . \]  

G.20

One only needs to be able to solve for one specific $I(r,0,n|x)$ and the rest can be found using (G.20). Using Eq. (6.592.12) of Ref. 132 one can derive

\[ I(-3,0,2|x) = \frac{K_1(x)}{x} . \]  

G.21

The three $I(r,0,n|x)$ in (G.19) all have different $n$ values, so one must also find a recurrence relation between them. Using recurrence relations for the Bessel functions (Eq. (8.486.13) in 132) one can show that

\[ I(r,0,n|x) = \frac{1}{r + 2n + 1} I(r,0,n+1|x) - \frac{x^{r+n+1}}{r + 2n + 1} K_n(x) , \]
\[ I(r,0,n|x) = (r + 2n - 1)I(r,0,n-1|x) + x^{r+n}K_{n-1}(x) . \]  

G.22

To calculate $I(3,3,3|2z)$ in (G.19), one can now use the binomial expansion of $(u^2-1)^3$ in (G.11) to show that

\[ I(3,3,3|x) = I(3,0,3|x) - 3x^2 I(1,0,3|x) + 3x^4 I(-1,0,3|x) - x^6 I(-3,0,3|x) , \]  

G.23
and using the recurrence relation (G.22) one has

\[ I(3, 3, 3|x) = 192x^2K_2(x) + 48x^3K_3(x) . \]  (G.24)

Similarly, one can expand \( I(5, 3, 2|x) \) and \( I(7, 3, 1|x) \). The results for these remaining two integrals can be expressed as

\[ I(5, 3, 2|x) = 384x^3K_3(x) + 48x^4K_2(x) , \]
\[ I(7, 3, 1|x) = 2304x^3K_3(x) + 384x^4K_2(x) + 48x^5K_3(x) . \]  (G.25)

Substituting (G.24) and (G.25) into (G.19) gives the shear viscosity of a single component gas in the Grad approximation

\[ \frac{15}{32\pi} \frac{T}{\sigma^4} \frac{z^4K_3^2(z)}{(15z^2 + 2)K_2(2z) + (3z^3 + 49z)K_3(2z)} , \]  (G.26)

exposing the typographic error of 5 instead of 15 in Eq. (24) of Chapter XI, Section 1 of \[27\].
H. Calculation of Momentum Integrals in $Q[\chi]$ 

All required integrals are scalars, so it is convenient to integrate momenta 3 and 4 in the center-of-mass (CM) of the scattering process (momentum conservation is simpler), while momenta 1 and 2 in the LR frame of the fluid (so that $f_{eq} \propto e^{-E/T}$ is isotropic). For brevity, in this entire Section LR subscripts are omitted, while CM variables are distinguished with an overbar wherever confusion might arise. Spherical coordinates are also helpful.

$B_1$ can be reduced to one dimensional integration, $Q_{11}$ and $Q_{22}$ to three dimensions, while $Q_{31}$ and $Q_{41}$ to five dimensions in general, or four in the case of isotropic cross sections. All remaining integrals were performed numerically using adaptive integration routines from the GNU Scientific Library (GSL) [26].

H.1 Reduction of terms $B$, $Q_{11}$, and $Q_{21}$

The source term $B_i$ in (3.17), which is linear in $\chi_i$, immediately reduces this way to

$$B_i = \frac{2\pi}{3T^6} \int_{m_i}^{\infty} dE_1 \ p_i^5 f_{eq1i} \chi_1i . \quad (H.1)$$

In the terms quadratic in $\chi$, $\vec{p}_4$ can be eliminated using the $\delta$-function in three-momentum, and the magnitude of $|\vec{p}_3|$ is set by the $\delta$-function in energy:

$$\int \delta^4(12-34) \ldots = \frac{1}{4} \int d\Omega_3 d\vec{p}_3 \frac{p_3^2}{E_3E_4} \delta(E_3 + E_4 - \sqrt{s}) \ldots = \frac{p_{cm}'}{4\sqrt{s}} \int d\Omega_3 \ldots \bigg|_{\vec{p}_3 = p_{cm}'} . \quad (H.2)$$

For the $\chi_1^2_i$ and $\chi_{1i} \chi_{2j}$ terms one can substitute (2.27) to obtain

$$\int \delta^4(12 - 34)W_{12\rightarrow 34}^{ij \rightarrow k\ell} = p_{cm} \sqrt{s} (1 + \delta_{k\ell}) \sigma_{TOT}^{ij \rightarrow k\ell} (s) , \quad (H.3)$$
and the calculation is then analogous to the scattering rate in Appendix I. Keeping $t_{12} \equiv \cos \theta_{12}$, one has

\[ Q_{11}^{ij \rightarrow k\ell} = \frac{2\pi^2}{3T^8} (1 + \delta_{k\ell}) \int_{m_i} \int_{m_j} dE_1 p_1^5 f_{1i}^{eq} \int_{m_j} dE_2 p_2^2 f_{2j}^{eq} \int_{-1}^1 dt_{12} F(s) \sigma_{ij \rightarrow k\ell}^{\text{TOT}}(s) \]  

(H.4)

and

\[ Q_{21}^{ij \rightarrow k\ell} = \frac{\pi^2}{3T^8} (1 + \delta_{k\ell}) \int_{m_i} dE_1 p_1^3 f_{1i}^{eq} \int_{m_j} dE_2 p_2^3 f_{2j}^{eq} \int_{-1}^1 dt_{12} (3t_{12}^2 - 1) F(s) \sigma_{ij \rightarrow k\ell}^{\text{TOT}}(s) \] 

(H.5)

where $F$ is given by (4.26).

**H.2 Reduction of terms $Q_{31}$ and $Q_{41}$**

The last two $\chi_1 \chi_3$ and $\chi_1 \chi_4$ terms in general involve numerical integration in 9-4=5 dimensions (three momentum integrals with a 4D $\delta$-function constraint) because $\chi_3$ and $\chi_4$ depend on outgoing three-momenta in the LR frame. Interchange symmetry (2.25) with $3 \leftrightarrow 4$, $k \leftrightarrow \ell$ implies $Q_{41}^{ij \rightarrow k\ell} = Q_{31}^{ij \rightarrow k\ell}$, so it is enough to discuss $Q_{31}$. For isotropic cross section, it is possible to do one more integral analytically, if the LR frame momentum $p_3$ is expressed using the CM frame momentum $\bar{p}_3 \equiv p'_{cm} \bar{n}_3$ (here $|\bar{n}_3| = 1$). Lorentz boost from CM to LR gives

\[ E_3 = \gamma_3 E_T + \beta_3 p_T \bar{n}_3 , \quad p_3 = p'_{cm} \bar{n}_3 + p_T \left( \gamma_3 + \beta_3 \frac{p_T \bar{n}_3}{E_T + \sqrt{s}} \right) , \]  

(H.6)

where

\[ \beta_3 \equiv \frac{p'_{cm}}{\sqrt{s}} , \quad \gamma_3 \equiv \frac{E_3}{\sqrt{s}} = \sqrt{\beta_3^2 + \frac{m_k^2}{s}} , \quad E_T \equiv E_1 + E_2 , \quad p_T \equiv p_1 + p_2 \]  

(H.7)

only depend on $p_1$ and $p_2$ but not on $\bar{p}_3$. With convenient angles $\bar{n}_3(\phi_3, \theta_3)$ for the $d\Omega_3$ integration such that the zenith direction is parallel to $p_T$,

\[ \bar{n}_3 p_T = p_T \cos \theta_3 , \quad \bar{n}_3 p_1 = p_1(\sin \theta_1 \sin \theta_3 \cos \phi_3 + \cos \theta_1 \cos \theta_3) , \]  

(H.8)

where

\[ \cos \theta_1 \equiv \frac{p_T p_1}{p_T p_1} = \frac{p_1 + p_2 t_{12}}{p_T} . \]  

(H.9)
Because $|\mathbf{p}_3|$ does not depend on $\phi_3$, the only $\phi_3$ dependence is in the $(\mathbf{p}_3 \mathbf{p}_1)^2$ term from $\mathbf{P}_3 \cdot \mathbf{P}_1$, which can be integrated. So even if the total cross section depends on energy, we have only four integrals remaining:

$$\int H.10 \int \frac{d\Omega_3}{4} = \int H.10 \int dE_1 p_1 \int dE_2 p_2 \int dt_{12} \int d\phi_3 \langle \cdots \rangle_{\phi_3}$$

i.e.,

$$Q_{ij}^{\ell} = \frac{\pi^2}{2T^8} (1 + \delta_{ik}) \int H.11 \int dE_1 p_1 f_{i1}^e \chi_{i1} \int H.11 \int dE_2 p_2 f_{2j}^e \int \int H.11 \int dt_{12} F(s) \sigma_{TOT}^{ij} \int \int H.11 \int dt_3 \chi_{3k} \langle \mathbf{P}_3 \cdot \mathbf{P}_1 \rangle_{\phi_3},$$

where $t_3 \equiv \cos \theta_3$, $p_3 = |\mathbf{p}_3| = \sqrt{(\gamma_3 E_T + \beta_3 p_T t_3)^2 - m_k^2}$, and

$$\langle \cdots \rangle_{\phi_3} \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi_3 \langle \cdots \rangle$$

denotes averaging over $\phi_3$. The following $\phi_3$ averages appear:

$$\langle \mathbf{n}_3 \mathbf{p}_1 \rangle_{\phi_3} = p_1 \cos \theta_1 t_3 \ , \ \langle (\mathbf{n}_3 \mathbf{p}_1)^2 \rangle_{\phi_3} = \frac{p_1^2}{2} [(3t_3^2 - 1) \cos^2 \theta_1 + 1 - t_3^2] ,$$

in terms of which

$$\langle \mathbf{P}_3 \cdot \mathbf{P}_1 \rangle_{\phi_3} = \frac{1}{T} \left[ (p_{cm}^e)^2 \langle (\mathbf{n}_3 \mathbf{p}_1)^2 \rangle_{\phi} + p_1^2 (p_1 + p_2 t_{12}) \left( \gamma_3 + \beta_3 \frac{p_{T T 3}}{E_1 + E_2 + \sqrt{s}} \right)^2 \right. \right.$$

$$+ 2p_{cm}^e p_1 (p_1 + p_2 t_{12}) \left( \gamma_3 + \beta_3 \frac{p_{T T 3}}{E_1 + E_2 + \sqrt{s}} \right) \langle \mathbf{n}_3 \mathbf{p}_1 \rangle_{\phi} \left] - \frac{p_1^2 p_3^2}{3T^4} \right)$$

\[ H.14 \]

**H.3 Integration using auxiliary variable $\omega$**

The method outlined above is practical but limited to isotropic cross section. For general $d\sigma(s, t)/dt$, one can evaluate $Q_{31}$ and $Q_{41}$ via extending the technique used in Ref. [80] to massive particles. The key elements of that technique are splitting the energy conservation integral with the help of the energy transfer $\omega$ as

$$\delta(E_1 + E_2 - E_3 - E_4) \equiv \int_{-\infty}^{\infty} \delta(\omega + E_1 - E_3) \delta(\omega - E_2 + E_4) ,$$

\[ H.15 \]
eliminating $p_4$ through momentum conservation, and swapping $p_3$ for the momentum transfer $q \equiv p_3 - p_1$. Exploiting rotation invariance, introduce angles such that

$$q = q(0, 0, 1), \quad p_1 = p_1(\sin \theta_{1q}, 0, \cos \theta_{1q}), \quad p_2 = p_2(\cos \phi \sin \theta_{2q}, \sin \phi \sin \theta_{2q}, \cos \theta_{2q}).$$

(H.16)

Then the Mandelstam variables for the scattering process are

$$s = m_i^2 + m_j^2 + 2(E_1 E_2 - p_1 p_2), \quad t = \omega^2 - q^2,$$

(H.17)

the magnitudes of outgoing momenta are

$$p_3 = \sqrt{(E_1 + \omega)^2 - m_k^2}, \quad p_4 = \sqrt{(E_2 - \omega)^2 - m_\ell^2},$$

(H.18)

and the scalar products that appear in $s$ and $P \cdot P$ are

$$p_1 p_2 = p_1 p_2(\cos \theta_{1q} \cos \theta_{2q} + \cos \phi \sin \theta_{1q} \sin \theta_{2q}), \quad p_1 p_3 = p_1^2 + \frac{m_i^2 - m_k^2 + 2E_1 \omega + t}{2},$$

$$p_1 p_4 = p_1^2 + p_1 p_2 - p_1 p_3,$$

(H.19)

where the $\theta$ angles are fixed by the $\delta$-functions:

$$\cos \theta_{1q} = \frac{m_i^2 - m_k^2 + 2E_1 \omega + t}{2p_1 q}, \quad \cos \theta_{2q} = \frac{m_\ell^2 - m_j^2 + 2E_2 \omega - t}{2p_2 q}.$$  

(H.20)

Five integrals remain:

$$\prod_{1234} \delta^4(12 - 34) (\ldots) = \frac{\pi^2}{2} \int_{m_1}^{\infty} dE_1 \int_{m_2}^{\infty} dE_2 \int_{0}^{2\pi} d\phi \int_{q}^{q} dq \int_{-\infty}^{\infty} d\omega \Theta(1 - \cos^2 \theta_{1q}) \Theta(1 - \cos^2 \theta_{2q}) (\ldots),$$

(H.21)

where the Heaviside functions set the integration limits.

For equal masses $m_i = m_j = m_k = m_\ell \equiv m$,

$$\prod_{1234} \delta^4(12 - 34) (\ldots) = \frac{\pi^2}{2} \int_{0}^{q} dq \int_{-q}^{q} d\omega \int_{q \Lambda(q, -\omega)}^{q \Lambda(q, \omega)} dE_1 \int_{0}^{2\pi} dE_2 \int_{\Lambda(q, -\omega)}^{\Lambda(q, \omega)} d\phi (\ldots),$$

(H.22)

where

$$\Lambda(q, \omega) = \sqrt{m^2 + \Lambda^2(q, \omega)}, \quad \Lambda(q, \omega) = \left| q + \omega \sqrt{1 - \frac{4m^2}{t}} \right|,$$

(H.23)
and we verified that both methods give numerically identical results with isotropic cross sections. The main disadvantage compared to the method in the previous Subsection is speed - for isotropic cross section one still has five numerical integrals to do compared to four in (H.11).
I. Evaluation of Scattering Rates

The scattering rate integral (4.25) right away reduces from six dimensions to only three because in the static case the phase space density $f^{eq} \propto e^{-E/T}$ and Mandelstam

$$s \equiv m_i^2 + m_j^2 + 2(E_1E_2 - \mathbf{p}_1\cdot\mathbf{p}_2)$$

only depend on the magnitudes of momenta and the angle $\theta_{12}$ between them. Replacing $\cos \theta_{12}$ with $s$, in spherical coordinates we then have

$$\int\frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} (\ldots) = 4\pi \cdot 2\pi \int_{m_i}^{\infty} dE_1 \int_{m_j}^{\infty} dE_2 \int_{s^-}^{s^+} ds (\ldots)$$

with limits $s_{\pm} = m_i^2 + m_j^2 + 2 \left[ E_1E_2 \pm \sqrt{(E_1^2 - m_i^2)(E_2^2 - m_j^2)} \right]$.

Though not pursued here, further simplification of the $2 \to 2$ scattering rate is possible. If speed of evaluation is a concern, consult Appendix A of Ref. [136] (integrated rate for equal mass particles), Appendix B of Ref. [137] (rate for fixed particle momentum), or [138] (integrated rate for arbitrary masses).
J. Grad results in the Nonrelativistic Limit

In this Appendix, the terms in the functional are evaluated for a multicomponent system in the Grad ansatz in the nonrelativistic limit. In the nonrelativistic limit one can replace terms in (H.4), (H.5), and (H.11) with their nonrelativistic counterparts

\[ \frac{d^3p}{E} \rightarrow \frac{d^3p}{m}, \quad dE \rightarrow \frac{dp}{m}, \quad \exp\left(-\frac{E}{T}\right) \rightarrow \exp\left(-\frac{m}{T} - \frac{p^2}{2mT}\right), \quad F(s) \rightarrow m_im_j|v_1-v_2|. \]

Similarly, in (H.14)

\[ \gamma_3 \rightarrow \frac{m_3}{m_1+m_2} = \frac{m_3}{m_3+m_4}, \quad \beta_3 \rightarrow 0, \quad p_3^2 \rightarrow (p'_c)^2 + \gamma_3p_T^2 + 2p'_cm_3p_Tt_3. \]

Note that it is simpler to get the above result for \( p_3 \) from \( p_3 \approx \bar{p}_3 + \gamma_3p_T \) than from \( \sqrt{E_3^2-m_k^2} \) because there is an almost perfect cancellation in the latter.

It is further convenient to switch variables from \( p_1 \) and \( p_2 \) to total momentum and relative velocity

\[ p_T = p_1 + p_2, \quad v_{rel} \equiv v_1-v_2 \Leftrightarrow p_1 = \frac{m_1}{m_1+m_2}(p_T+m_2v_{rel}), \quad p_2 = \frac{m_2}{m_1+m_2}(p_T-m_1v_{rel}) \]

for which

\[ d^3p_1d^3p_2 = \left(\frac{m_1m_2}{m_1+m_2}\right)^3 d^3p_T d^3v_{rel} \]

so

\[ \int_{m_i}^{\infty} \int_{m_j}^{\infty} \int_{-1}^{1} F(s) \ldots \rightarrow \left(\frac{m_im_j}{m_i+m_j}\right)^3 \int_{0}^{\infty} d\bar{p}_T \int_{0}^{\infty} dv_{rel} \int_{-1}^{1} d\cos \tilde{\theta} \frac{p_T^2v_{rel}^3}{p_1p_2} \ldots, \]

where \( \tilde{\theta} \) is the angle between \( p_T \) and \( v_{rel} \), while in the exponents

\[ \frac{p_1^2}{2m_1T} + \frac{p_2^2}{2m_2T} = \frac{p_T^2 + m_1m_2v_{rel}^2}{2(m_1+m_2)T}. \]

Straightforward integration leads then to (4.19).
K. Boost Invariance and Cooper-Frye Integrals

Both the hydrodynamic simulations used in this work are 2+1D simulations which assume longitudinal boost invariance. This means that the state of the system at each point in spacetime with \( t > 0 \) and nonzero coordinate rapidity, in this section denoted by \( \eta \), can be obtained from the state on the \( \eta = 0 \) midrapidity sheet via trivial Lorentz boost along the longitudinal beam direction, taken to be the \( z- \)axis.

For longitudinally boost invariant systems, hyperbolic coordinates

\[
\eta \equiv \frac{1}{2} \ln \frac{t + z}{t - z} , \quad \tau \equiv \sqrt{t^2 - z^2}
\]  

(K.1)

are most convenient for spacetime, while rapidity and transverse mass are used for momenta

\[
y \equiv \frac{1}{2} \ln \frac{E + p_z}{E - p_z} , \quad m_T \equiv \sqrt{p_T^2 + m^2} .
\]

(K.2)

In these hyperbolic rapidity coordinates, the position and momentum four-vectors are written as

\[
x^\mu = (\tau \text{ ch } \eta, x_T, \tau \text{ sh } \eta) , \quad p^\mu = (m_T \text{ ch } y, p_T, m_T \text{ sh } y) .
\]

(K.3)

The Cooper-Frye formula for the distribution of particles emitted from a surface element \( d\sigma^\mu \) of a 3D spacetime hypersurface is

\[
E \frac{dN_i(x, p)}{d^3 p} \equiv \frac{dN_i(x, p_T, y)}{d^2 p_T dy} = p^\mu d\sigma_\mu(x) f_i(x, p) .
\]

(K.4)

Often a \( \Theta(p^\mu d\sigma_\mu) \) factor is also included to cut out potential negative contributions from spacelike surface elements, but it is not used in this work. With boost invariance,

\[
d\sigma^\mu = n^\mu \tau d\eta d^2 x_T , \quad n^\mu = (n^0 \text{ ch } \eta, n_T, n^0 \text{ sh } \eta) ,
\]

(K.5)

i.e.,

\[
p^\mu d\sigma_\mu = \tau [m_T n^0 \text{ ch } \xi - p_T n_T] d\eta d^2 x_T ,
\]

(K.6)
where \( n^\mu(x) \) is a unit vector normal to the hypersurface at spacetime point \( x \) and \( \xi \equiv \eta - y \). In the thermal equilibrium distribution \( \text{(2.14)} \)

\[
\gamma(x, \eta, \mathbf{v}_T, \mathbf{sh} \eta), \quad \gamma \equiv \frac{1}{\sqrt{1 - v_T^2}} \Rightarrow p \cdot u = \gamma (m_T \mathbf{c} - \mathbf{p}_T \mathbf{v}_T),
\]

and in the shear correction \( \chi_i(\bar{p}) \) defined in \( \text{(3.9)} \), \( \bar{p} \equiv |\bar{p}| = \sqrt{(p \cdot u)^2 - m^2/T} \)

\[
\pi^{\mu\nu} p_\mu p_\nu = m_T^2 (\pi^{00} \mathbf{c}^2 + \pi^{zz} \mathbf{sh}^2 \xi) - 2m_T \mathbf{c} \cdot \mathbf{v} (p_x \pi^{0x} + p_y \pi^{0y}) + p_x^2 \pi^{xx} + p_y^2 \pi^{yy} + 2p_x p_y \pi^{xy},
\]

with shear stress components all taken at \( \eta = 0 \). For several equivalent forms of this expression, see Ref. \( \text{[139]} \).

Boost invariant 2+1D viscous fluid dynamics provides hydrodynamic fields \( (T, \{\mu_c\}, \mathbf{v}_T, \pi^{\mu\nu}) \) and hypersurface elements \( (n^0, \mathbf{n}_T) \) in the \( \eta = 0 \) frame, as a function of \( \tau \) and \( \mathbf{x}_T \). If one is only interested in the momentum distribution, one integrates \( \text{(K.4)} \) over the hypersurface, which includes at each \( \tau \) and \( \mathbf{x}_T \) integration over \( \eta \):

\[
\tau \int_{-\infty}^{\infty} d\eta \left[ m_T n^0 \mathbf{c} \cdot \mathbf{n}_T f_i(\tau, \mathbf{x}_T, \mathbf{p}_T, \xi) \right] = 2\tau \int_0^\infty d\xi \left[ m_T n^0 \mathbf{c} \cdot \mathbf{n}_T f_i(\tau, \mathbf{x}_T, \mathbf{p}_T, \xi) \right]
\]

\( \text{(K.9)} \)

with reflection symmetry along the beam axis assumed. For the ideal piece involving the thermal equilibrium distribution function, \( \text{(K.9)} \) yields

\[
2\tau \frac{g_i}{(2\pi)^3} e^{\alpha_T} [m_T n^0 K_1(z_T) - \mathbf{p}_T \cdot \mathbf{n}_T K_0(z_T)] , \quad z_T \equiv \frac{\gamma m_T}{T}, \quad \alpha_T \equiv \frac{\mu_i + \gamma \mathbf{p}_T \cdot \mathbf{v}_T}{T} \quad \text{(K.10)}
\]

For the viscous correction \( \text{(3.9)} \), the integral can only be evaluated analytically in special cases. For example, for viscous corrections quadratic in momentum (Grad ansatz), \( \chi_i(\bar{p}) = \text{const} \) and one has

\[
\frac{\chi^{\text{Grad}}_i}{\eta_{\text{shear}}} \frac{g_i}{2\tau} e^{\alpha_T} \left\{ m_T n^0 \left[ m_T^2 \left( K_1(z_T) + \frac{K_2(z_T)}{z_T} \right) \right] \pi^{00} + m_T^2 \frac{K_2(z_T)}{z_T} \pi^{zz} - 2m_T \left( K_0(z_T) + \frac{K_1(z_T)}{z_T} \right) \left( p_x \pi^{0x} + p_y \pi^{0y} \right) + p_x^2 \pi^{xx} + p_y^2 \pi^{yy} + 2p_x p_y \pi^{xy} \right\} - \mathbf{p}_T \cdot \mathbf{n}_T \left[ m_T^2 \left( K_0(z_T) + \frac{K_1(z_T)}{z_T} \right) \pi^{00} + m_T^2 \frac{K_1(z_T)}{z_T} \pi^{zz} - 2m_T K_0(z_T) \left( p_x \pi^{0x} + p_y \pi^{0y} \right) \right.

\[ + \left. K_0(z_T) \left( p_x^2 \pi^{xx} + p_y^2 \pi^{yy} + 2p_x p_y \pi^{xy} \right) \right\}. \quad \text{(K.11)}
\]
(Again, there are many equivalent ways one can write these Bessel function expressions using the recurrence relations such as $K_{n+1}(x) = K_{n-1}(x) + 2nK_n(x)/x$.)
L. Self-consistent Grad Coefficient Tables and Fits

Tables L.1 L.2 and L.3 tabulate self-consistent viscous phase space corrections for the 49-species gas of hadrons in Section 4.3 using $\delta f/f_{eq} \propto p^2, p^{3/2}$ and $p$, respectively, while Tables L.7 L.8 and L.9 are for the hadron gas chemically frozen at $T_{ch} = 175$ MeV in Section 4.3.4. In all tables, correction factors relative to the “democratic Grad” form (2.20) are printed. To apply the dynamical correction for species $i$, read the coefficient $c_i$ from the table for the species and multiply democratic viscous corrections by the expression in (5.14) that corresponds to the desired momentum dependence.

The corrections depend rather smoothly on hadron (pole) mass, despite varying degeneracy factors, and therefore can also be well represented by fits of the form

$$c(x) = \delta + \alpha \left[ 1 + \left( \frac{x}{\gamma} \right)^\beta \right]^{-1} \quad \text{or} \quad c(x) = \alpha + \beta |x - \gamma|^\delta, \quad x \equiv \frac{m}{\text{1 GeV}}, \quad \text{(L.1)}$$

where $x$ is the hadron (pole) mass $m$ in GeV units. Tables L.4-L.6 list the best fit values for the parameters $\alpha, \beta, \gamma, \text{and } \delta$ as a function of temperature for the various scenarios in Tables L.1-L.3, while Tables L.10-L.12 list the best fit parameter values for scenarios in Tables L.7-L.9.

The fits are done to the original unrounded $c_i$ values. Note that there are separate fits for mesons and baryons in the case of additive quark model (AQM) cross sections. There is no specific physics motivation behind the forms (L.1); the functions are chosen solely for accuracy (the relative accuracy is better than $8.5 \times 10^{-4}$ in all cases).
constant cross section is used for all species, and the fit parameters are \( \delta f / f \). Despite the fluctuating particle degeneracy values, the tables can be well fit

| Species | \( T = 100 \) | 120 | 140 | 165 MeV |
|---------|----------------|-----|-----|--------|
| \( \pi \) | 1.08 | 1.13 | 1.17 | 1.21 |
| \( K \) | 0.89 | 0.96 | 1.02 | 1.08 |
| \( \eta \) | 0.87 | 0.94 | 1.00 | 1.06 |
| \( f_0 \) | 0.85 | 0.92 | 0.98 | 1.04 |
| \( \rho \) | 0.80 | 0.87 | 0.93 | 0.99 |
| \( \omega \) | 0.80 | 0.86 | 0.93 | 0.99 |
| \( K^\ast 892 \) | 0.77 | 0.83 | 0.90 | 0.96 |
| \( N \) | 0.76 | 0.82 | 0.88 | 0.94 |
| \( \eta'(958) \) | 0.75 | 0.82 | 0.88 | 0.94 |
| \( f_0(980) \) | 0.75 | 0.81 | 0.87 | 0.93 |
| \( a_0(980) \) | 0.75 | 0.81 | 0.87 | 0.93 |
| \( \phi(1020) \) | 0.74 | 0.81 | 0.86 | 0.92 |
| \( \Lambda \) | 0.72 | 0.79 | 0.84 | 0.90 |
| \( h_1(1170) \) | 0.72 | 0.78 | 0.83 | 0.89 |
| \( \Sigma \) | 0.71 | 0.77 | 0.83 | 0.89 |
| \( b_1(1235) \) | 0.71 | 0.76 | 0.82 | 0.88 |
| \( \Delta(1232) \) | 0.71 | 0.76 | 0.82 | 0.88 |
| \( a_1(1260) \) | 0.71 | 0.77 | 0.82 | 0.88 |
| \( \Lambda(1270) \) | 0.70 | 0.76 | 0.81 | 0.87 |
| \( f_2(1270) \) | 0.70 | 0.76 | 0.81 | 0.87 |
| \( f_1(1285) \) | 0.70 | 0.76 | 0.81 | 0.87 |
| \( \eta(1295) \) | 0.70 | 0.75 | 0.81 | 0.87 |
| \( \pi(1300) \) | 0.70 | 0.75 | 0.81 | 0.87 |
| \( \Xi \) | 0.69 | 0.75 | 0.81 | 0.86 |
| \( a_2(1320) \) | 0.69 | 0.75 | 0.81 | 0.86 |
| \( \Sigma(1385) \) | 0.68 | 0.74 | 0.80 | 0.85 |
| \( f_0(1370) \) | 0.69 | 0.74 | 0.80 | 0.85 |
| \( K_1(1400) \) | 0.68 | 0.74 | 0.79 | 0.85 |
| \( \Lambda(1405) \) | 0.68 | 0.74 | 0.79 | 0.85 |
| \( K^\ast(1410) \) | 0.68 | 0.74 | 0.79 | 0.85 |
| \( \eta(1405) \) | 0.68 | 0.74 | 0.79 | 0.85 |
| \( \omega(1420) \) | 0.68 | 0.74 | 0.79 | 0.84 |
| \( f_1(1420) \) | 0.68 | 0.73 | 0.79 | 0.84 |
| \( K_2^0(1430) \) | 0.68 | 0.73 | 0.79 | 0.84 |
| \( K_2^+ (1430) \) | 0.68 | 0.73 | 0.79 | 0.84 |
| \( N(1440) \) | 0.68 | 0.73 | 0.79 | 0.84 |
| \( \eta(1450) \) | 0.68 | 0.73 | 0.79 | 0.84 |
| \( \rho(1450) \) | 0.67 | 0.73 | 0.78 | 0.83 |
| \( f_0(1500) \) | 0.67 | 0.72 | 0.78 | 0.83 |
| \( \Lambda(1520) \) | 0.67 | 0.72 | 0.77 | 0.83 |
| \( N(1520) \) | 0.67 | 0.72 | 0.77 | 0.83 |
| \( \Xi(1530) \) | 0.67 | 0.72 | 0.77 | 0.83 |
| \( N(1535) \) | 0.67 | 0.72 | 0.77 | 0.83 |
| \( \Delta(1600) \) | 0.66 | 0.71 | 0.76 | 0.82 |
| \( \Lambda(1600) \) | 0.66 | 0.71 | 0.76 | 0.81 |
| \( \Delta(1620) \) | 0.66 | 0.71 | 0.76 | 0.81 |
| \( \omega(1650) \) | 0.65 | 0.71 | 0.76 | 0.81 |
| \( \Omega \) | 0.65 | 0.70 | 0.75 | 0.81 |

Table L.1: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass \( m = 1.672 \) GeV, assuming quadratic momentum dependence \( \delta f / f \propto p^2 \) (dynamical Grad approximation). Despite the fluctuating particle degeneracy values, the tables can be well fit by a function of the form \( a - bm^c \) with \( m \) the mass of the particle in GeV. Left table: The same constant cross section is used for all species, and the fit parameters are \( a = 1.39, b = 0.461, c = 0.489 \). Right table: The additive quark model (AQM) cross sections are used so that mesons and baryons are fit by separate functions with \( a_M = 1.44, b_M = 0.453, c_M = 0.535 \) and \( a_B = 4.72, b_B = 3.99, c_B = 0.0492 \).
A constant cross section is used for all species, and the fit parameters can be well fit by a function of the form $m^{a}$ up to mass $B = 8$.

| Species $\times 10^3$ | $m$ (GeV) | $f$ (GeV) |
|------------------------|---------|----------|
| $\pi$ | 2.56 | 2.68 |
| $m$ | 2.46 | 2.46 |
| $f_0$ | 2.26 | 2.39 |
| $p$ | 2.19 | 2.32 |
| $\omega$ | 2.19 | 2.32 |
| $K^*892$ | 2.15 | 2.28 |
| $N$ | 2.14 | 2.27 |
| $\eta'(958)$ | 2.14 | 2.26 |
| $f_0(980)$ | 2.13 | 2.26 |
| $a_0(980)$ | 2.13 | 2.25 |
| $\phi(1020)$ | 2.12 | 2.24 |
| $\Lambda(1170)$ | 2.09 | 2.21 |
| $\Sigma^+_0(1385)$ | 2.09 | 2.20 |
| $b_1(1235)$ | 2.08 | 2.20 |
| $\Delta(1232)$ | 2.08 | 2.20 |
| $a_1(1260)$ | 2.08 | 2.20 |
| $K_1(1270)$ | 2.07 | 2.19 |
| $f_2(1270)$ | 2.07 | 2.19 |
| $f_2(1285)$ | 2.07 | 2.19 |
| $\eta(1295)$ | 2.07 | 2.18 |
| $\eta'(1300)$ | 2.07 | 2.18 |
| $\Xi(1385)$ | 2.07 | 2.18 |
| $a_2(1320)$ | 2.07 | 2.18 |
| $\Sigma(1385)$ | 2.06 | 2.17 |
| $f_0(1370)$ | 2.06 | 2.17 |
| $K_0(1400)$ | 2.06 | 2.17 |
| $\Lambda(1405)$ | 2.06 | 2.16 |
| $K^{+(1410)}$ | 2.06 | 2.16 |
| $\eta(1405)$ | 2.06 | 2.16 |
| $\omega(1420)$ | 2.05 | 2.16 |
| $f_0(1450)$ | 2.05 | 2.16 |
| $K_0^+(1430)$ | 2.05 | 2.16 |
| $K_2^+(1430)$ | 2.05 | 2.16 |
| $\eta(1450)$ | 2.05 | 2.16 |
| $f_2(1520)$ | 2.04 | 2.15 |
| $\Lambda(1520)$ | 2.04 | 2.15 |
| $N(1520)$ | 2.04 | 2.15 |
| $f_0(1525)$ | 2.04 | 2.15 |
| $\Xi(1530)$ | 2.04 | 2.15 |
| $N(1535)$ | 2.04 | 2.15 |
| $\Delta(1600)$ | 2.03 | 2.14 |
| $\Lambda(1600)$ | 2.03 | 2.14 |
| $\omega(1650)$ | 2.03 | 2.13 |
| $\Omega$ | 2.03 | 2.13 |

Table L.2: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming power-law momentum dependence $\delta f / f \propto p^{b/c}$. The tables can be well fit by a function of the form $a - bm^c$ with $m$ the mass in GeV. Left table: The same constant cross section is used for all species, and the fit parameters are $a = 3.18$, $b = 0.0856$, and $c = 0.438$. Right table: The additive quark model (AQIM) cross sections are used so that mesons and baryons are fit by separate functions with $a_M = 3.30$, $b_M = 0.673$, $c_M = 0.480$ and $a_B = 8.01$, $b_B = 6.07$, and $c_B = 0.0445$. 

![104](105)
Table L.3: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming linear momentum dependence $\delta f/f^{eq} \propto \rho$. The tables can be well fit by a shallow quadratic function of the form $a + b(m - c)^2$ with $m$ the mass in GeV. Left table: The same constant cross section is used for all species, and the fit parameters are $a = 6.46$, $b = 0.0738$, and $c = 0.75$. Right table: The additive quark model [104, 105] (AQM) cross sections are used so that mesons and baryons are fit by separate functions with $a_M = 6.81$, $b_M = 0.0813$, $c_M = 0.74$, and $a_B = 4.96$, $b_B = 0.0475$, and $c_B = 1.29$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Species & $T = 100$ & 120 & 140 & 165 MeV \\
\hline
$\pi$ & 5.81 & 6.06 & 6.29 & 6.49 \\
K & 5.78 & 6.03 & 6.27 & 6.46 \\
$\eta$ & 5.79 & 6.03 & 6.27 & 6.46 \\
$f_0$ & 5.79 & 6.03 & 6.27 & 6.46 \\
$\rho$ & 5.82 & 6.04 & 6.27 & 6.46 \\
$\omega$ & 5.82 & 6.04 & 6.27 & 6.46 \\
$K^{*}\!892$ & 5.85 & 6.06 & 6.28 & 6.46 \\
N & 5.86 & 6.07 & 6.28 & 6.46 \\
$\eta'(958)$ & 5.87 & 6.07 & 6.28 & 6.46 \\
f$_0(980)$ & 5.87 & 6.07 & 6.28 & 6.46 \\
a$_0(980)$ & 5.87 & 6.07 & 6.28 & 6.46 \\
$\phi(1020)$ & 5.88 & 6.08 & 6.29 & 6.46 \\
$\Lambda$ & 5.91 & 6.10 & 6.30 & 6.47 \\
h$_0(1170)$ & 5.93 & 6.11 & 6.31 & 6.47 \\
$\Sigma$ & 5.94 & 6.11 & 6.31 & 6.47 \\
b$_1(1235)$ & 5.95 & 6.12 & 6.32 & 6.48 \\
$\Delta(1232)$ & 5.95 & 6.12 & 6.32 & 6.48 \\
a$_1(1260)$ & 5.95 & 6.12 & 6.32 & 6.48 \\
K$_1(1270)$ & 5.96 & 6.13 & 6.32 & 6.48 \\
f$_2(1270)$ & 5.96 & 6.13 & 6.32 & 6.48 \\
f$_1(1285)$ & 5.97 & 6.14 & 6.32 & 6.48 \\
$\eta(1295)$ & 5.97 & 6.14 & 6.33 & 6.48 \\
$\pi(1300)$ & 5.97 & 6.14 & 6.33 & 6.48 \\
$\Xi$ & 5.98 & 6.14 & 6.33 & 6.48 \\
a$_2(1320)$ & 5.98 & 6.14 & 6.33 & 6.48 \\
$\Sigma(1385)$ & 6.00 & 6.16 & 6.34 & 6.49 \\
f$_0(1370)$ & 6.00 & 6.16 & 6.34 & 6.49 \\
K$_1(1400)$ & 6.01 & 6.16 & 6.34 & 6.49 \\
$\Lambda(1405)$ & 6.01 & 6.17 & 6.34 & 6.49 \\
$K^{*}(1410)$ & 6.01 & 6.17 & 6.34 & 6.49 \\
$\eta(1445)$ & 6.01 & 6.17 & 6.34 & 6.49 \\
$\omega(1420)$ & 6.02 & 6.17 & 6.35 & 6.49 \\
f$_1(1420)$ & 6.02 & 6.17 & 6.35 & 6.49 \\
K$_0^*(1430)$ & 6.02 & 6.17 & 6.35 & 6.49 \\
K$_2^*(1430)$ & 6.02 & 6.17 & 6.35 & 6.49 \\
N(1440) & 6.02 & 6.17 & 6.35 & 6.49 \\
p(1450) & 6.03 & 6.18 & 6.35 & 6.50 \\
f$_0(1500)$ & 6.05 & 6.19 & 6.36 & 6.50 \\
f$_0(1500)$ & 6.07 & 6.28 & 6.56 & 6.86 \\
$\Lambda(1520)$ & 6.05 & 6.19 & 6.36 & 6.50 \\
N(1520) & 6.05 & 6.20 & 6.36 & 6.50 \\
f$_2(1525)$ & 6.05 & 6.20 & 6.36 & 6.50 \\
$\Xi(1530)$ & 6.06 & 6.20 & 6.37 & 6.50 \\
N(1535) & 6.06 & 6.20 & 6.37 & 6.50 \\
$\Delta(1600)$ & 6.08 & 6.22 & 6.38 & 6.51 \\
$\Lambda(1600)$ & 6.08 & 6.22 & 6.38 & 6.51 \\
$\Delta(1620)$ & 6.09 & 6.22 & 6.38 & 6.51 \\
$\omega(1650)$ & 6.10 & 6.23 & 6.39 & 6.52 \\
N(1650) & 6.10 & 6.23 & 6.39 & 6.52 \\
$\Omega$ & 6.11 & 6.24 & 6.39 & 6.52 \\
\hline
\end{tabular}
\end{table}
fits for $\delta f / f_{eq} \propto p^2 \text{ (Grad)}$

using $c(x) = \delta + \alpha \left[ 1 + \left( \frac{x}{\gamma} \right)^\beta \right]^{-1}$

$\sigma = \text{ const }$ scenario

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|--------|----------|---------|
| 100       | 0.698    | 1.204  | 0.715    | 0.467   |
| 120       | 0.700    | 1.266  | 0.862    | 0.493   |
| 140       | 0.702    | 1.326  | 0.996    | 0.519   |
| 165       | 0.693    | 1.397  | 1.140    | 0.551   |

AQM scenario, mesons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|--------|----------|---------|
| 100       | 0.696    | 1.214  | 0.712    | 0.472   |
| 120       | 0.704    | 1.278  | 0.856    | 0.505   |
| 140       | 0.715    | 1.342  | 0.985    | 0.543   |
| 165       | 0.717    | 1.414  | 1.124    | 0.591   |

AQM scenario, baryons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|--------|----------|---------|
| 100       | 0.687    | 1.009  | 0.623    | 0.286   |
| 120       | 0.698    | 1.037  | 0.801    | 0.297   |
| 140       | 0.710    | 1.075  | 0.987    | 0.311   |
| 165       | 0.711    | 1.129  | 1.204    | 0.334   |

Table L.4: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.1.
fits for $\delta f / f_{eq} \propto p^{3/2}$

using $c(x) = \delta + \alpha \left[ 1 + \left( \frac{x}{\gamma} \right)^{\beta} \right]^{-1}$

$\sigma = \text{const} \text{ scenario}$

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 0.748    | 1.446    | 0.559    | 1.900    |
| 120       | 0.823    | 1.375    | 0.712    | 1.933    |
| 140       | 0.883    | 1.363    | 0.876    | 1.969    |
| 165       | 0.921    | 1.386    | 1.061    | 2.006    |

AQM scenario, mesons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 0.759    | 1.427    | 0.561    | 1.904    |
| 120       | 0.839    | 1.370    | 0.714    | 1.959    |
| 140       | 0.908    | 1.367    | 0.874    | 2.032    |
| 165       | 0.957    | 1.397    | 1.047    | 2.126    |

AQM scenario, baryons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 0.540    | 1.627    | 0.760    | 1.344    |
| 120       | 0.691    | 1.396    | 0.836    | 1.369    |
| 140       | 0.806    | 1.288    | 0.974    | 1.400    |
| 165       | 0.890    | 1.243    | 1.177    | 1.439    |

Table L.5: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.2.
fits for $\delta f / f_{eq} \propto p$

using $c(x) = \alpha + \beta |x - \gamma|^\delta$

| $\sigma = const$ scenario | $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|--------------------------|----------|---------|---------|---------|---------|
|                          | 100      | 5.775   | 0.240   | 0.419   | 1.521   |
|                          | 120      | 6.025   | 0.166   | 0.502   | 1.633   |
|                          | 140      | 6.265   | 0.114   | 0.599   | 1.734   |
|                          | 165      | 6.458   | 0.073   | 0.747   | 1.882   |

| AQM scenario, mesons    | $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-------------------------|----------|---------|---------|---------|---------|
|                         | 100      | 5.802   | 0.239   | 0.419   | 1.546   |
|                         | 120      | 6.114   | 0.167   | 0.504   | 1.655   |
|                         | 140      | 6.459   | 0.118   | 0.603   | 1.743   |
|                         | 165      | 6.808   | 0.080   | 0.742   | 1.865   |

| AQM scenario, baryons   | $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-------------------------|----------|---------|---------|---------|---------|
|                         | 100      | 4.245   | 0.156   | 0.848   | 1.404   |
|                         | 120      | 4.466   | 0.097   | 0.897   | 1.603   |
|                         | 140      | 4.715   | 0.062   | 1.015   | 1.842   |
|                         | 165      | 4.963   | 0.045   | 1.293   | 1.931   |

Table L.6: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.3.
Table L.7: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming quadratic momentum dependence $\delta f / f_{eq} \propto p^2$ (dynamical Grad approximation). Despite the fluctuating particle degeneracy values, the tables can be well fit by a 4 parameter function as shown in Table L.10. Left table: The same constant cross section is used for all species Right table: The additive quark model [104, 105] (AQM) cross sections are used so that mesons and baryons are fit by separate functions.
Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming quadratic momentum dependence $\delta f / f^{eq} \propto p^{3/2}$. Despite the fluctuating particle degeneracy values, the tables can be well fit by a 4 parameter function as shown in Table L.10. *Left table:* The same constant cross section is used for all species. *Right table:* The additive quark model (AQM) cross sections are used so that mesons and baryons are fit by separate functions.

| Species  | $T = 100$ | 120 | 140 | 165 MeV |
|----------|-----------|-----|-----|---------|
| $\pi$    | 3.33      | 3.17| 3.06| 2.95    |
| K        | 3.00      | 2.90| 2.82| 2.76    |
| $\eta$   | 2.96      | 2.87| 2.79| 2.73    |
| $f_0$    | 2.93      | 2.84| 2.76| 2.71    |
| $\omega$ | 2.83      | 2.74| 2.68| 2.63    |
| $K^*$(892)| 2.77      | 2.69| 2.63| 2.58    |
| $\eta'(958)$ | 2.74  | 2.66| 2.60| 2.56    |
| $f_0(980)$ | 2.73  | 2.66| 2.60| 2.55    |
| $a_0(980)$ | 2.73  | 2.65| 2.60| 2.55    |
| $\phi(1020)$ | 2.72  | 2.64| 2.58| 2.54    |
| $\Lambda$ | 2.68      | 2.61| 2.55| 2.51    |
| $h_1(1170)$ | 2.67  | 2.59| 2.53| 2.49    |
| $\Sigma$  | 2.66      | 2.58| 2.53| 2.49    |
| $b_1(1235)$ | 2.65  | 2.57| 2.52| 2.48    |
| $\Delta(1232)$ | 2.65  | 2.57| 2.52| 2.48    |
| $a_1(1260)$ | 2.65  | 2.57| 2.52| 2.48    |
| $K^*_1(1270)$ | 2.63  | 2.56| 2.51| 2.46    |
| $f_2(1270)$ | 2.63  | 2.56| 2.51| 2.46    |
| $f_1(1285)$ | 2.63  | 2.56| 2.50| 2.46    |
| $\eta(1295)$ | 2.63  | 2.55| 2.50| 2.46    |
| $\pi(1300)$ | 2.63  | 2.55| 2.50| 2.46    |
| $\Xi$     | 2.62      | 2.55| 2.49| 2.45    |
| $a_0(1320)$ | 2.62  | 2.55| 2.49| 2.45    |
| $\Sigma(1385)$ | 2.60  | 2.53| 2.48| 2.44    |
| $f_0(1370)$ | 2.60  | 2.53| 2.48| 2.44    |
| $K^*_1(1400)$ | 2.60  | 2.53| 2.47| 2.43    |
| $\Lambda(1405)$ | 2.60  | 2.53| 2.47| 2.43    |
| $K^*(1410)$ | 2.60  | 2.52| 2.47| 2.43    |
| $\eta(1405)$ | 2.60  | 2.52| 2.47| 2.43    |
| $\omega(1420)$ | 2.59  | 2.52| 2.47| 2.43    |
| $f_1(1420)$ | 2.59  | 2.52| 2.47| 2.43    |
| $K^*_1(1430)$ | 2.59  | 2.52| 2.47| 2.43    |
| $\omega(1430)$ | 2.59  | 2.52| 2.47| 2.43    |
| $f_0(1500)$ | 2.57  | 2.50| 2.45| 2.41    |
| $\Lambda(1520)$ | 2.57  | 2.50| 2.45| 2.41    |
| $\omega(1520)$ | 2.57  | 2.50| 2.45| 2.41    |
| $f_2^*(1525)$ | 2.57  | 2.50| 2.45| 2.41    |
| $\Xi(1530)$ | 2.57  | 2.50| 2.44| 2.40    |
| $\eta(1535)$ | 2.57  | 2.50| 2.44| 2.40    |
| $\Delta(1600)$ | 2.55  | 2.48| 2.43| 2.39    |
| $\Lambda(1600)$ | 2.55  | 2.48| 2.43| 2.39    |
| $\omega(1650)$ | 2.54  | 2.47| 2.42| 2.38    |
| $\Omega(1650)$ | 2.54  | 2.47| 2.42| 2.38    |
| $\rho(1540)$ | 2.58  | 2.51| 2.46| 2.42    |
| $\rho(1540)$ | 2.58  | 2.51| 2.46| 2.42    |
| $\omega(1580)$ | 2.54  | 2.47| 2.42| 2.38    |
| $\Omega(1600)$ | 2.54  | 2.47| 2.42| 2.38    |
| $N(1650)$ | 2.54      | 2.47| 2.42| 2.38    |
| $\Omega(1600)$ | 2.54  | 2.47| 2.42| 2.38    |

Table L.8: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming quadratic momentum dependence $\delta f / f^{eq} \propto p^{3/2}$. Despite the fluctuating particle degeneracy values, the tables can be well fit by a 4 parameter function as shown in Table L.10. *Left table:* The same constant cross section is used for all species. *Right table:* The additive quark model (AQM) cross sections are used so that mesons and baryons are fit by separate functions.
| Species | $T = 100$ | $120$ | $140$ | $165$ MeV |
|---------|-----------|-------|-------|-----------|
| $\pi$   | 7.71      | 7.27  | 6.94  | 6.68      |
| $K$     | 7.72      | 7.26  | 6.93  | 6.66      |
| $\eta$  | 7.72      | 7.26  | 6.92  | 6.66      |
| $f_0$   | 7.72      | 7.26  | 6.92  | 6.66      |
| $\rho$  | 7.73      | 7.27  | 6.92  | 6.66      |
| $\omega$| 7.73      | 7.27  | 6.92  | 6.66      |
| $K^*$(892) | 7.74 | 7.27  | 6.93  | 6.66      |
| $\eta'(958)$ | 7.75 | 7.28  | 6.93  | 6.66      |
| $f_0$(980) | 7.75 | 7.28  | 6.93  | 6.66      |
| $\rho_0$(980) | 7.75 | 7.28  | 6.93  | 6.66      |
| $\phi$(1020) | 7.76 | 7.29  | 6.93  | 6.66      |
| $\Lambda$ | 7.77     | 7.29  | 6.94  | 6.67      |
| $h_1$(1170) | 7.78 | 7.30  | 6.95  | 6.67      |
| $\Sigma$ | 7.78     | 7.30  | 6.95  | 6.67      |
| $b_1$(1235) | 7.79 | 7.31  | 6.95  | 6.68      |
| $\Delta$(1232) | 7.79 | 7.31  | 6.95  | 6.68      |
| $a_1$(1260) | 7.79 | 7.31  | 6.95  | 6.68      |
| $K_1$(1270) | 7.80   | 7.31  | 6.95  | 6.67      |
| $f_2$(1270) | 7.80   | 7.31  | 6.96  | 6.68      |
| $f_1$(1285) | 7.80   | 7.31  | 6.96  | 6.68      |
| $\eta$(1295) | 7.79   | 7.32  | 6.96  | 6.68      |
| $\pi$(1300) | 7.78    | 7.32  | 6.96  | 6.68      |
| $\Xi$ | 7.80     | 7.32  | 6.96  | 6.68      |
| $a_2$(1320) | 7.81    | 7.32  | 6.96  | 6.68      |
| $\Sigma$(1385) | 7.82   | 7.33  | 6.97  | 6.69      |
| $f_0$(1370) | 7.82   | 7.33  | 6.97  | 6.69      |
| $K_1$(1400) | 7.82   | 7.33  | 6.97  | 6.69      |
| $\Lambda$(1405) | 7.83   | 7.33  | 6.97  | 6.69      |
| $K^*(1410)$ | 7.82    | 7.33  | 6.97  | 6.69      |
| $\eta$(1405) | 7.82    | 7.33  | 6.97  | 6.69      |
| $\omega$(1420) | 7.82   | 7.34  | 6.97  | 6.69      |
| $f_1$(1420) | 7.83    | 7.34  | 6.97  | 6.69      |
| $K^*_0$(1430) | 7.83    | 7.34  | 6.97  | 6.69      |
| $K^*_1$(1430) | 7.83    | 7.34  | 6.97  | 6.69      |
| $N$(1440) | 7.83    | 7.34  | 6.97  | 6.69      |
| $\rho$(1450) | 7.83    | 7.34  | 6.98  | 6.70      |
| $f_0$(1500) | 7.84    | 7.35  | 6.98  | 6.70      |
| $\Lambda$(1520) | 7.85   | 7.35  | 6.98  | 6.70      |
| $N$(1520) | 7.85    | 7.35  | 6.98  | 6.70      |
| $f_2$(1525) | 7.85    | 7.35  | 6.98  | 6.70      |
| $\Xi$(1530) | 7.85    | 7.35  | 6.98  | 6.70      |
| $N$(1535) | 7.85    | 7.35  | 6.99  | 6.70      |
| $\Delta$(1600) | 7.86   | 7.36  | 6.99  | 6.71      |
| $\Lambda$(1600) | 7.86   | 7.36  | 6.99  | 6.71      |
| $\omega$(1650) | 7.87    | 7.37  | 7.00  | 6.72      |
| $\Omega$ | 7.88     | 7.38  | 7.00  | 6.72      |

Table L.9: Species-dependent shear viscous phase space corrections, calculated for a gas of hadrons up to mass $m = 1.672$ GeV, assuming quadratic momentum dependence $\delta f/f^{eq} \propto p$. Despite the fluctuating particle degeneracy values, the tables can be well fit by a 4 parameter function as shown in Table L.10. **Left table:** The same constant cross section is used for all species **Right table:** The additive quark model [104, 105] (AQM) cross sections are used so that mesons and baryons are fit by separate functions.
fits for $\delta f / f_{eq} \propto p^2 \langle \text{Grad} \rangle$ with $T_{ch} = 175 \text{ MeV}$

using $c(x) = \delta + \alpha \left[ 1 + \left( \frac{x}{\gamma} \right)^\beta \right]^{-1}$

| $\sigma = \text{const}$ scenario |
|-------------------------------|
| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
| 100  | 0.950 | 1.223 | 0.833 | 0.511 |
| 120  | 0.852 | 1.287 | 0.943 | 0.527 |
| 140  | 0.779 | 1.342 | 1.041 | 0.541 |
| 160  | 0.721 | 1.390 | 1.130 | 0.553 |

| AQM scenario, mesons |
|----------------------|
| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
| 100  | 1.001 | 1.237 | 0.824 | 0.555 |
| 120  | 0.894 | 1.302 | 0.931 | 0.572 |
| 140  | 0.815 | 1.359 | 1.026 | 0.587 |
| 160  | 0.752 | 1.407 | 1.112 | 0.600 |

| AQM scenario, baryons |
|-----------------------|
| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
| 100  | 0.955 | 1.014 | 0.784 | 0.317 |
| 120  | 0.867 | 1.052 | 0.925 | 0.323 |
| 140  | 0.798 | 1.089 | 1.061 | 0.330 |
| 160  | 0.742 | 1.124 | 1.190 | 0.337 |

Table L.10: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.7.
fits for \( \delta f/f_{eq} \propto p^{3/2} \) with \( T_{ch} = 175 \) MeV using \( c(x) = \delta + \alpha \left[ 1 + \left( \frac{x}{\gamma} \right)^\beta \right]^{-1} \)

\[ \sigma = \text{const} \] scenario

| \( T \) [MeV] | \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) |
|-------------|---------|---------|--------|--------|
| 100         | 1.281   | 1.255   | 0.791  | 2.179  |
| 120         | 1.150   | 1.299   | 0.890  | 2.116  |
| 140         | 1.051   | 1.340   | 0.981  | 2.071  |
| 160         | 0.973   | 1.376   | 1.065  | 2.037  |

AQM scenario, mesons

| \( T \) [MeV] | \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) |
|-------------|---------|---------|--------|--------|
| 100         | 1.361   | 1.261   | 0.783  | 2.331  |
| 120         | 1.215   | 1.308   | 0.879  | 2.265  |
| 140         | 1.104   | 1.350   | 0.967  | 2.217  |
| 160         | 1.018   | 1.388   | 1.048  | 2.180  |

AQM scenario, baryons

| \( T \) [MeV] | \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) |
|-------------|---------|---------|--------|--------|
| 100         | 1.127   | 1.229   | 0.897  | 1.611  |
| 120         | 1.055   | 1.225   | 0.987  | 1.552  |
| 140         | 0.994   | 1.227   | 1.083  | 1.507  |
| 160         | 0.942   | 1.233   | 1.182  | 1.473  |

Table L.11: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.8
fits for $\delta f / f_{eq} \propto p$ with $T_{ch} = 175$ MeV
using $c(x) = \alpha + \beta |x - \gamma|^\delta$

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 7.711    | 0.091    | 0.298    | 1.950    |
| 120       | 7.261    | 0.083    | 0.484    | 1.953    |
| 140       | 6.923    | 0.075    | 0.625    | 1.951    |
| 160       | 6.659    | 0.068    | 0.739    | 1.946    |

AQM scenario, mesons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 8.232    | 0.108    | 0.365    | 1.922    |
| 120       | 7.741    | 0.096    | 0.517    | 1.927    |
| 140       | 7.371    | 0.086    | 0.636    | 1.920    |
| 160       | 7.084    | 0.076    | 0.734    | 1.912    |

AQM scenario, baryons

| $T$ [MeV] | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|----------|----------|----------|----------|
| 100       | 6.030    | 0.065    | 0.957    | 1.887    |
| 120       | 5.658    | 0.056    | 1.083    | 1.942    |
| 140       | 5.380    | 0.049    | 1.208    | 1.942    |
| 160       | 5.165    | 0.046    | 1.322    | 1.981    |

Table L.12: Fit functions for the species-dependent shear viscous phase space corrections listed in Table L.9
VITA

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