TWO CODIMENSION-TWO BIFURCATIONS OF A SECOND-ORDER DIFFERENCE EQUATION FROM MACROECONOMICS

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ABSTRACT. In this paper we mainly investigate two codimension-two bifurcations of a second-order difference equation from macroeconomics. Applying the center manifold theorem and the normal form analysis, we firstly give the parameter conditions for the generalized flip bifurcation, and prove that the system does not produce a strong resonance. Then, we compute the normal forms to obtain the parameter conditions for the Neimark-Sacker bifurcation, from which we present the conditions for the Chenciner bifurcation. In order to verify the correctness of our results, we also numerically simulate a half stable invariant circle and two invariant circles, one stable and one unstable, arising from the Chenciner bifurcation.

1. Introduction. The business cycle has been and continues to be an important source of significant analytical problems in economic dynamics. The theory of the business cycle aims to explain the observed and well documented fluctuations in employment, consumption, investment, total output, etc. In order to address these issues, the mathematical model is widely recognized as a convenient theoretical tool as it often helps reveal generic tendencies in some situation. One of the earliest discrete time models for the business cycle is Samuelson’s business cycle model [12]

$$Y_n = cY_{n-1} + \alpha(Y_{n-1} - Y_{n-2}) + A_0.$$ 

Here the constant $A_0 = C_0 + I_0 + G_0$ represents the sum of the minimum consumption, the autonomous investment and the fixed government spending, $Y_n$ is the output-national income or GDP in period $n$, $\alpha(Y_{n-1} - Y_{n-2})$ is the net investment amount in the same period, the coefficient $\alpha > 0$ is the accelerator, and the constant $c \in [0, 1)$ denotes Keynes’ marginal propensity to consume. In 1997, Sedaghat [13]
generalized this model and considered a much more general class of the nonlinear difference equation

\[ x_{n+1} = px_n + f(x_n - x_{n-1}), \quad n \in \mathbb{N}, \tag{1} \]

where \( p \in [0, 1) \) is a constant and \( f : \mathbb{R} \to \mathbb{R} \) is a real function.

Over the last few decades, there have been a lot of papers devoted to equation (1) and its good properties have also been obtained. For example, Sedaghat [13] in 1997 discussed the boundedness, the permanence and the convergence results of the solutions for certain types of the function \( f \). Some years later, he [14] further proposed some open problems and conjectures about the boundedness, stability and oscillations. In 2004, Kent and Sedaghat [8] gave sufficient conditions on the existence and continuous dependence of invariant curves in the nonlinear case are discussed in [11]. Especially, if \( f \) is an analytic real function and has the Taylor expansion \( f(s) = f(0) + qs + \sum_{n=2}^{\infty} q_n s^n \), using the center manifold theorem and the implicit function theorem, Li and Zhang [10] analyzed the flip bifurcation for \( q_3 \neq 0 \), or \( q_3 = 0 \) and \( K(p, q_2, q_4, q_5) \neq 0 \) respectively, where

\[ K(p, q_2, q_4, q_5) = q_5 p^3 + (9q_5 + 4q_4 q_2)p^2 + (27q_5 + 24q_4 q_2)p + 27q_5 + 36q_4 q_2 - 80q_3^2. \tag{2} \]

They pointed out that the higher degeneracy of bifurcations will appear when \( q_3 = 0 \) and \( K(p, q_2, q_4, q_5) = 0 \). Moreover, assuming that

(H) \( q_k \neq 0 \) but \( q_i = 0 \) for all \( i = 2, \ldots, k-1 \) and either that there exists an integer \( n \) with \( k \leq 2n + 1 < 2k - 1 \) such that \( q_{2n+1} \neq 0 \) but \( q_{2i+1} = 0 \) for all integers \( i \) with \( k \leq 2i + 1 < 2n + 1 \) and \( p \neq 2 \cos(2l\pi/j) - 1 \) for all \( j = 1, 2, \ldots, 2n + 2 \) and all integers \( l \), or that \( q_{2n+1} = 0 \) for all integers \( i \) with \( k \leq 2i + 1 < 2k - 1 \) and \( p \neq 2 \cos(2l\pi/j) - 1 \) for all \( j = 1, 2, \ldots, 2k \) and all integers \( l \),

by calculating the normal form, they investigated the Neimark-Sacker bifurcation if the \( n \)-th Lyapunov coefficient is nonzero, and obtained the existence of \( j \) invariant circles for arbitrary \( 1 \leq j \leq n \). In particular, take \( n = 1 \) which corresponds to the first Lyapunov coefficient. Assumption (H) implies that \( k = 3 \) and then \( q_2 = 0 \) and \( q_3 \neq 0 \). In this case, there exists only one invariant circle.

However, to the best of our knowledge, the following two questions are still not discussed. One is whether or not the system can produce the generalized flip bifurcation, as called in [9], and the other is whether the system can undergo the Chenciner bifurcation (also called generalized Neimark-Sacker bifurcation) or not. Actually, these problems are related to higher degeneracy. In this paper, we will
rigorously investigate them. For the first question, we consider \( q_3 \) as a small parameter and require only \( q_5 \) nonzero \((K(p,q_2,q_4,q_5)\) in (2) might be zero). Using the center manifold theorem and the normal form analysis, we show that equation (1) undergoes a generalized flip bifurcation and give the parameter conditions under which the system possesses two, one and no 2-cycles respectively. See Theorems 2.1 and 2.2. In order to answer the second question, we also discuss the Neimark-Sacker bifurcation and give the first Lyapunov coefficient but without Assumption (H) and in particular \( q_2 = 0, q_3 \neq 0 \) in [10], from which we further present the conditions of the Chenciner bifurcation such that equation (1) has two invariant circles, one invariant circle and none separately.

This paper is organized as follows. In Section 2 we study the generalized flip bifurcation. In Section 3 we prove that equation (1) does not produce a strong resonance, and give the parameter conditions for the Neimark-Sacker bifurcation. Section 4 is devoted to the parameter conditions for the Chenciner bifurcation. In Section 5, the invariant circles arising from the Chenciner bifurcation are simulated numerically, which verifies the correctness of our results. Conclusions are drawn in Section 6. Some coefficients involved in the proofs are given in Appendix.

2. Generalized flip bifurcation. The dynamics of equation (1) can be described equivalently by a planar mapping \( F : \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
F(x,y) = (y,py + f(y-x)).
\]

As shown in [10], system (3) has a unique fixed point \( E_0 : (f(0)/(1-p), f(0)/(1-p)) \).

In this paper, we assume that the function \( f \) is analytic. Then it has the Taylor expansion

\[
f(s) = f(0) + qs + \sum_{n=2}^{\infty} q_n s^n,
\]

where \( q_n = f^{(n)}(0)/n! \), \( n = 2,3,\ldots \). Translating \( E_0 \) to the origin with the transformation \((w_1 + f(0)/(1-p), w_2 + f(0)/(1-p))\), we change the mapping \( F \) into the mapping \( G : \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
G(w_1, w_2) = \left( w_2, pw_2 + q(w_2 - w_1) + \sum_{n=2}^{\infty} q_n(w_2 - w_1)^n \right).
\]

Since the translation transformation does not change the topological properties of system (3), i.e., the mapping \( G \) is equivalent to the mapping \( F \), we focus on the dynamics of system (5) throughout this paper.

The Jacobian matrix of the mapping \( G \) at the origin \( O \),

\[
JG(O) = \begin{pmatrix} 0 & 1 \\ -q & p + q \end{pmatrix},
\]

has eigenvalues

\[
\lambda_1 = \frac{p + q - \sqrt{(p + q)^2 - 4q}}{2}, \quad \lambda_2 = \frac{p + q + \sqrt{(p + q)^2 - 4q}}{2}.
\]

If \( q = -(p + 1)/2 \), one can check that \( \lambda_1 = -1 \) and \( \lambda_2 = (p + 1)/2 \). As shown in [10, Theorem 1], if \( q_3 \neq 0 \), then system (3) undergoes the flip bifurcation at the fixed point \( E_0 \) when \( q \) crosses \(-(p + 1)/2\). In this section, we consider the more degenerate case, that is, \( q_3 \) is near 0. For the convenience of discussion, let \( \varepsilon_1 = q_3, \varepsilon_2 = q + (p + 1)/2 \) and \( \varepsilon = (\varepsilon_1, \varepsilon_2) \), where \( |\varepsilon| \) is sufficiently small. Then we have the following results.
Theorem 2.1. Suppose that $q_5 > 0$ and $|\varepsilon|$ is sufficiently small. Then system (5) undergoes a generalized flip bifurcation at the fixed point $E_0$. More specifically, as shown in Figure 1, if $(\varepsilon_1, \varepsilon_2) \in D_1$, where

$$D_1 := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 > \frac{\varepsilon_1^2}{4q_5} + O(\varepsilon_1^3), \varepsilon_1 < 0 \} \cup \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \},$$

system (5) has a unique stable equilibrium $E_0$ and no 2-cycle; As the parameter $(\varepsilon_1, \varepsilon_2)$ crosses the half line

$$\mathfrak{F}_+ = \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 = 0, \varepsilon_1 \geq 0 \}$$

from the region $D_1$ to the region $D_2$, where

$$D_2 := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 < 0 \} \cup \mathfrak{F}_-$$

and

$$\mathfrak{F}_- := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 = 0, \varepsilon_1 < 0 \},$$

system (5) undergoes a flip bifurcation at $E_0$, from which a stable 2-cycle is produced near $E_0$ and $E_0$ is unstable; As the parameter $(\varepsilon_1, \varepsilon_2)$ crosses the half line $\mathfrak{F}_-$ from the region $D_2$ to the region $D_3$, where

$$D_3 := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | 0 < \varepsilon_2 < \frac{\varepsilon_1^2}{4q_5} + O(\varepsilon_1^3), \varepsilon_1 < 0 \},$$

system (5) possesses two 2-cycles, one stable and one unstable, and $E_0$ becomes stable; The two 2-cycles coincide as the parameter $(\varepsilon_1, \varepsilon_2)$ lies on the curve

$$\mathfrak{C} := \{(\varepsilon_1, \varepsilon_2) | \varepsilon_2 = \frac{1}{4q_5} \varepsilon_1^2 + O(\varepsilon_1^3), \varepsilon_1 < 0 \},$$

and then disappear when $(\varepsilon_1, \varepsilon_2)$ enters the region $D_1$.

![Figure 1. Bifurcation diagram for $q_5 > 0$.](image-url)
Proof. Using the parameter \((\varepsilon_1, \varepsilon_2)\), the eigenvalues of the matrix \(JG(O)\) given in (6) are rewritten as

\[
\lambda_{1,2} = \frac{1}{2} \left( \varepsilon_2 + \frac{p - 1}{2} \pm \sqrt{\left( \varepsilon_2 + \frac{p - 1}{2} \right)^2 - 4 \left( \varepsilon_2 - \frac{p + 1}{2} \right)} \right).
\] (7)

The matrix has eigenvectors \((1, \lambda_1)^T\) and \((1, \lambda_2)^T\) corresponding to \(\lambda_1\) and \(\lambda_2\) respectively, where \(T\) denotes the transpose of matrices. For \(\varepsilon_2 = 0\), it follows from (7) that \(\lambda_1 \neq \lambda_2\) since \(p \in [0, 1]\). Hence, the matrix \(JG(O)\) can be diagonalized by the change of variables \((w_1, w_2)^T = H_1(u, v)^T\), where

\[
H_1 = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},
\]

and the mapping \(G\) can be changed into the mapping \(\Phi : \mathbb{R}^2 \to \mathbb{R}^2\),

\[
\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 u + \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n \\ \lambda_2 v - \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n \end{pmatrix}.
\] (8)

In order to involve the parameter \(\varepsilon_2\) explicitly in the discussion, rewrite (8) in the suspended form

\[
\begin{pmatrix} u \\ \varepsilon_2 \\ v \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 u + \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n \\ \varepsilon_2 \\ \lambda_2 v - \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n \end{pmatrix}.
\] (9)

Having two eigenvalues on the unit circle \(S^1\) for \(\varepsilon_2 = 0\), by the center manifold theorem [3], system (9) has a smooth two-dimensional center manifold with \(q_3 = \varepsilon_1\) small

\[
W^c(O) = \{(u, v, \varepsilon_2)|v = h(u, \varepsilon_2; \varepsilon_1) = \sum_{j=2}^{4} \sum_{i=0}^{j} c_{i,j-i}(\varepsilon_1) u^i \varepsilon_2^{j-i}
\]

\[
+ O((u, \varepsilon_2)^5((1, \varepsilon_1)))\}
\] (10)

in a neighborhood of \((u, \varepsilon_2)\)(0,0). By the invariance of the center manifold ([2, p. 34]), the coefficients \(c_{i,j-i}\) can be determined by

\[
h \left( \lambda_1 u + \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n, \varepsilon_2; \varepsilon_1 \right)
\]

\[
= \lambda_2 h(u, \varepsilon_2; \varepsilon_1) - \frac{1}{\lambda_1 - \lambda_2} \sum_{n=2}^{\infty} q_n \{ (\lambda_1 - 1)u + (\lambda_2 - 1)v \}^n.
\] (11)

Comparing coefficients of \(u^i \varepsilon_2^{j-i}\)(\(j = 2, 3, 4, i = 0, \ldots, j\)) in (11), we obtain

\[
c_{1,1}(\varepsilon_1) = c_{0,2}(\varepsilon_1) = c_{1,2}(\varepsilon_1) = c_{0,3}(\varepsilon_1) = c_{1,3}(\varepsilon_1) = c_{0,4}(\varepsilon_1) = 0,
\]

\[
c_{2,0}(\varepsilon_1) = -\frac{16q_2}{(p - 1)(3 + p)}, \quad c_{3,0}(\varepsilon_1) = \frac{32(p^2 - 4q_2^2 - \varepsilon_1)}{(3 + p)^2(p - 1)},
\]

\[
c_{2,1}(\varepsilon_1) = \frac{128(p^2 + 2p + 5)}{(p - 1)^2(3 + p)^3} q_2,
\]

\[
c_{4,0}(\varepsilon_1) = -\frac{64(q_4(p - 1)(3 + p)^2 - 2q_2\varepsilon_1(5p^2 - 2p + 29) + 20q_3(p - 1))}{(p - 1)^2(3 + p)^3},
\]

\[
c_{3,1}(\varepsilon_1) = \frac{64(q_4)}{(p - 1)^2(3 + p)^3}.
\]
where
\[ c_{3,1}(\varepsilon_1) = \frac{64 \left( \varepsilon_1 (5p - 9) (p - 1)^2 - 8q_2^2 \left( 3p^2 + 2p + 11 \right) \right)}{(p - 1)^3 (3 + p)^4}, \]
\[ c_{2,2}(\varepsilon_1) = -\frac{128 \left( 5p^4 + 18p^3 + 132p^2 + 174p + 183 \right) q_2}{(3 + p)^3 (p - 1)^3}. \]

From (9) and (10), we get the reduced one-dimensional mapping \( \phi : \mathbb{R} \to \mathbb{R} \)
\[ \phi(u) = d_1(\varepsilon)u + d_2(\varepsilon)u^2 + d_3(\varepsilon)u^3 + d_4(\varepsilon)u^4 + d_5(\varepsilon)u^5 + O(|u|^6), \]
(12)
where
\[
\begin{align*}
  d_1(\varepsilon) &= -1 + \frac{4\varepsilon_2}{3 + p} + O(|\varepsilon|^2), \\
  d_2(\varepsilon) &= -\frac{8q_2}{3 + p} + 16 \frac{q_2 (3p + 1) \varepsilon_2}{(3 + p)^3} + O(|\varepsilon|^2), \\
  d_3(\varepsilon) &= -\frac{64q_2^2}{(3 + p)^2} + 16 \frac{1 + (512q_2^2 (p^2 + 2p + 5))}{(3 + p)^4 (p - 1)^2} + 128 \frac{(p - 5)q_2^2}{(3 + p)^4} \varepsilon_2 + O(|\varepsilon|^2), \\
  d_4(\varepsilon) &= -\frac{640q_2^3}{(3 + p)^3} - \frac{32q_4}{3 + p} + O(|\varepsilon|), \\
  d_5(\varepsilon) &= \frac{64q_5}{3 + p} - \frac{2048q_2^4}{(3 + p)^4} - \frac{256q_2 \left( 20 (p - 1)q_2^3 + (p - 1) (3 + p)^2 q_4 \right)}{(p - 1) (3 + p)^4} - \frac{512q_2q_2}{(3 + p)^4} + O(|\varepsilon|).
\end{align*}
\]

In order to find the normal form of (12), we need eliminate its quadratic and quartic terms. Applying a near identity transformation near \( u = 0 \),
\[ u = w + \delta(\varepsilon)w^2 + \theta(\varepsilon)w^4, \]
where
\[
\begin{align*}
  \delta(\varepsilon) &= \frac{d_2}{d_1(d_1 - 1)} = -\frac{4q_2}{3 + p} - \frac{64q_2}{(3 + p)^3} \varepsilon_2 + O(|\varepsilon|^2), \\
  \theta(\varepsilon) &= -\frac{(4d_1 - 1) d_2^2}{d_1^2 (d_1 - 1)^3 (d_1^2 + d_1 + 1)} + \frac{(2d_1 - 3) d_2 d_3}{d_1^2 (d_1 - 1)^2 (d_1^2 + d_1 + 1)} + \frac{4}{d_1 (d_1 - 1) (d_1^2 + d_1 + 1)} \\
  &\quad - \frac{16q_4}{3 + p} - \frac{160q_2}{(3 + p)^2} \varepsilon_1 - \frac{640 (7p^2 + 6p + 51) q_2^3}{(3 + p)^4 (p - 1)} + \frac{160q_4}{(3 + p)^2} \varepsilon_2 + O(|\varepsilon|^2),
\end{align*}
\]
we obtain a new mapping \( \psi : \mathbb{R} \to \mathbb{R} \)
\[ w \mapsto d_3(\varepsilon)w + B(\varepsilon)w^3 + D(\varepsilon)w^5 + O(w^6), \]
(14)
where
\[
B(\varepsilon) = d_3 + \frac{2d_2^2}{d_1(d_1 - 1)} = \frac{16}{3 + p} \varepsilon_1 + \frac{256q_2^2}{(p - 1) (3 + p)^2} \varepsilon_2 + O(|\varepsilon|^2),
\]
Thus, we obtain the bifurcation diagram showed in Figure 2. In a sufficiently small neighborhood of \( (\epsilon_1, \epsilon_2) = (0, 0) \) the system (17) is changed into the mapping 

\[
\eta \mapsto -(1 + \gamma_1)\eta + \frac{64q_5}{3 + p}w^5 + O(|\gamma| |w|^6).
\]

Setting 

\[
\beta_1 = \gamma_1, \quad \beta_2 = \gamma_2 \left( \frac{3 + p}{64q_5} \right)^{\frac{1}{2}}, \quad w = \left( \frac{3 + p}{64q_5} \right)^{\frac{1}{2}}\eta,
\]

system (17) is changed into the mapping 

\[
\eta \mapsto -(1 + \beta_1)\eta + \beta_2\eta^3 + s\eta^5 + O(|\beta||\eta|^5, |\eta|^6),
\]

where \( s = \text{sign}(q_5) = \pm 1 \). Now we consider the case \( s = 1 \), i.e., \( q_5 > 0 \). The case \( s = -1 \) will be discussed in Theorem 2.2. By Lemma 9.4 in [9, p. 403], system (19) is locally topologically equivalent near the origin to the mapping 

\[
\eta \mapsto -(1 + \beta_1)\eta + \beta_2\eta^3 + \eta^5.
\]

Analyzing the second iteration of system (20) gives 

\[
\eta \mapsto (1 + 2\beta_1 + \beta_2^2)\eta - (2\beta_2 + O(|\beta|^2))\eta^3 - (2 + 6\beta_1 + O(|\beta|^2))\eta^5 + O(|\eta|^7).
\]

Thus, we obtain the bifurcation diagram showed in Figure 2. In a sufficiently small neighborhood of \((\beta_1, \beta_2) = (0, 0)\), system (20) has a single stable fixed point \( \eta_0 = 0 \) in the region (1), i.e., the region 

\[
\left\{ (\beta_1, \beta_2) | \beta_1 < -\frac{1}{4}\beta_2^2 + O(\beta_2^3), \beta_2 < 0 \right\} \cup \left\{ (\beta_1, \beta_2) | \beta_1 \leq 0, \beta_2 \geq 0 \right\}.
\]

Its orbits approach the fixed point, “leap-frogging” around it. Crossing the half line 

\[
F_+ = \{ (\beta_1, \beta_2) | \beta_1 = 0, \beta_2 > 0 \}
\]

from the region (1) to the region (2), the right side of \( \beta_2 \)-axis and the negative axis of \( \beta_3 \)-axis, a flip bifurcation occurs. Specifically, the fixed point \( \eta_0 = 0 \) becomes unstable and a stable 2-cycle \{\( \eta_1, \eta_2 \)\} appears, where 

\[
\eta_{1,2} = \pm \sqrt{-\beta_2 + \sqrt{14\beta_1^2 + \beta_2^2 + 4\beta_1}}\sqrt{2(3\beta_1 + 1)} + O(|\beta|^\frac{3}{2}).
\]
Crossing the half line
\[ F_\pm = \{(\beta_1, \beta_2) | \beta_1 = 0, \beta_2 < 0\} \]
from the region 2 to the region 3, the left side of \( \beta_2 \)-axis and below \( L \), two different 2-cycles, a stable one \( \{\eta_1, \eta_2\} \) and an unstable one \( \{\eta_3, \eta_4\} \), coexist and \( \eta_0 \) regains the stability, where
\[ \eta_{3, 4} = \pm \sqrt{\frac{\beta_2 - \sqrt{14\beta_1^2 + \beta_2^2 + 4 \beta_1}}{2(3\beta_1 + 1)}} + O(|\beta|^\frac{3}{4}). \]
The two 2-cycles coincide via the fold bifurcation of the second iterate mapping as the parameter \( (\beta_1, \beta_2) \) lies on the curve \( L \).

Going back to the original parameters, by (13), (15), (16) and (18), we obtain
\[ \beta_1 = -\frac{4\varepsilon_2}{3 + p}, \quad \beta_2 = \left(\frac{3 + p}{64q_5}\right)^\frac{1}{2} \left(\frac{16\varepsilon_1}{3 + p} + \frac{256q_2^2\varepsilon_2}{(p - 1)(3 + p)^2}\right) + O(|\varepsilon|^2). \]
One can check that \( F_+, F_- \) and \( L \) correspond to the curves \( \mathcal{F}_+, \mathcal{F}_- \) and \( \mathcal{F} \) respectively. The proof is completed.

We now focus on the case \( s = -1 \) corresponding to \( q_5 < 0 \) in (19). From (13), (15), (16) and (18), it follows that
\[ \beta_1 = -\frac{4\varepsilon_2}{3 + p}, \quad \beta_2 = \left(\frac{3 + p}{-64q_5}\right)^\frac{1}{2} \left(\frac{16\varepsilon_1}{3 + p} + \frac{256q_2^2\varepsilon_2}{(p - 1)(3 + p)^2}\right) + O(|\varepsilon|^2). \]
Using a similar proof to one of Theorem 2.1, we have the following result.
Theorem 2.2. Suppose that $q_5 < 0$ and $|\varepsilon|$ is sufficiently small. Then system (3) undergoes a generalized flip bifurcation at the fixed point $E_0$. More specifically, as shown in Figure 3, if $(\varepsilon_1, \varepsilon_2) \in \tilde{D}_1$, where

$$\tilde{D}_1 := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 < \frac{\varepsilon_1^2}{4q_5} + O(\varepsilon_1^3), \varepsilon_1 > 0 \} \cup \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_1 \leq 0, \varepsilon_2 \leq 0 \},$$

system (3) has no 2-cycle; When $(\varepsilon_1, \varepsilon_2)$ crosses the half line $\tilde{\mathcal{F}}_-$ defined in Theorem 2.1 from the region $\tilde{D}_1$ to the region $\tilde{D}_2$, where

$$\tilde{D}_2 = \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 > 0 \} \cup \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_1 > 0, \varepsilon_2 = 0 \},$$

a flip bifurcation happens at the fixed point $E_0$ and an unstable 2-cycle appears. Crossing the half line $\tilde{\mathcal{F}}_+$ defined in Theorem 2.1 from the region $\tilde{D}_2$ to the region $\tilde{D}_3$, where

$$\tilde{D}_3 := \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | 0 > \varepsilon_2 > \frac{\varepsilon_1^2}{4q_5} + O(\varepsilon_1^3), \varepsilon_1 > 0 \},$$

system (3) possesses two 2-cycles, one stable and one unstable. The two 2-cycles coincide as the parameter $(\varepsilon_1, \varepsilon_2)$ lies on the curve

$$\tilde{\mathcal{F}} := \left\{ (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 | \varepsilon_2 = \frac{\varepsilon_1^2}{4q_5} + O(\varepsilon_1^3), \varepsilon_1 > 0 \right\},$$

and disappear when $(\varepsilon_1, \varepsilon_2)$ enters the region $\tilde{D}_1$.

Remark 2.1. If $q_3 \neq 0$ or $q_3 = 0$ (i.e. $\varepsilon_1 = 0$) but $K(p, q_2, q_4, q_5) \neq 0$ in (2), Li and Zhang [10] proved that system (3) undergoes the flip bifurcation at the fixed point $E_0$, which is a codimension 1 bifurcation. Here, we present the generalized flip bifurcation, which is a codimension 2 bifurcation.
In what follows, we will discuss the other codimension 2 bifurcation: the Chenciner bifurcation (or generalized Neimark-Sacker bifurcation). For this purpose, we need to discuss the resonant conditions and the Neimark-Sacker bifurcation of system (3).

3. Resonance and Neimark-Sacker bifurcation. In Theorem 2 in [10], if some coefficients \( q_a \) of (4) are zero or nonzero, Li and Zhang investigated the Neimark-Sacker bifurcation as \((p, q)\) crosses the segment \( \mathcal{L}_0 = \{(p, q) \in \mathbb{R}^2 | q = 1, 0 \leq p < 1\} \), and gave the expression of the \( n \)-th Lyapunov coefficient \( a_{2n+1} \). If \( a_{2n+1} \neq 0 \), they obtained that equation (1) possesses \( j \) invariant circles for any \( 1 \leq j \leq n \). In this section, we only focus on the existence of the first Lyapunov coefficient in order to study the Chenciner bifurcation.

Near the segment \( \mathcal{L}_0 \), \( JG(O) \), given above (6), has eigenvalues

\[
\lambda_\pm(q) = \mu \pm \omega i = \frac{p + q}{2} \pm \frac{i \sqrt{4q - (p + q)^2}}{2},
\]

where \( i \) denotes the imaginary unit, \( \mu = \frac{p+q}{2} \) and \( \omega = \sqrt{4q-(p+q)^2}/2 \).

3.1. Resonance. For the convenience of our discussion, let

\[
\lambda_0 = \lambda_+(1) = \frac{p + 1}{2} + i \frac{\sqrt{4 - (p + 1)^2}}{2},
\]

which lies on the unit circle. If \( \lambda_0 = \exp(i2\pi n/m) \), where \( m \) and \( n \) are coprime positive integers, then system (3) is resonant by the theory of the normal form for the mapping (see [18]). If \( 1 \leq m \leq 4 \), it is strong resonant, and if \( m \geq 5 \), it is weak resonant. In the resonant case, we have the following results.

**Lemma 3.1.** Suppose that the eigenvalue \( \lambda_0 = \exp(i2\pi n/m) \), where \( m \) and \( n \) are coprime positive integers. Then for \( p \in [0,1) \), \( 0 < n/m \leq 1/6 \). Moreover, the integer \( m \) is determined uniquely by \( p \).

**Proof.** Rewriting the complex number \( \lambda_0 \) in (22) in the exponential form \( \lambda_0 = \exp(i2\pi n/m) \), we get

\[
\cos\left(\frac{2\pi n}{m}\right) = \frac{p + 1}{2}.
\]

For \( p \in [0,1) \), we have \( 1/2 \leq (p + 1)/2 < 1 \). It follows from (23) that either \( 0 < n/m \leq 1/6 \) or \( 5/6 < n/m \leq 1 \). Since

\[
\sin\left(\frac{2\pi n}{m}\right) = \Im(\lambda_0) = \frac{\sqrt{4 - (p + 1)^2}}{2} > 0
\]

for \( p \in [0,1) \), we obtain \( 0 < n/m \leq 1/6 \). In this case the minimal option of \( m \) is 6 and then \( n = 1 \).

Furthermore, assume that there are two positive integers \( m_1 \) and \( m_2 \) such that equation (23) holds for the same \( p \). Then we can find two positive integers \( n_1 \) and \( n_2 \), which satisfy that \( n_1 < m_1 \), \( n_1 \) and \( m_i \) are coprime, \( i = 1, 2 \), and \( \cos(2\pi n_1/m_1) = \cos(2\pi n_2/m_2) \). It follows that either \( n_1/m_1 = n_2/m_2 \) or \( n_1/m_1 = 1 - n_2/m_2 \). In the case that \( n_1/m_1 = n_2/m_2 \), we get \( m_1 = m_2 \) since \( n_1 \) and \( m_i \) are coprime, \( i = 1, 2 \). In the other case, one can check that \( m_2 - n_2 \) and \( m_2 \) are also coprime. From the equality \( n_1/m_1 = (m_2 - n_2)/m_2 \), we also conclude that \( m_1 = m_2 \). Both cases indicate that \( m \) is determined uniquely by \( p \). This completes the proof.

Lemma 3.1 shows that equation (1) has not a strong resonance.
3.2. Neimark-Sacker bifurcation. Now we investigate the Neimark-Sacker bifurcation of system (3). The basic idea is to verify that all the conditions of Theorem 3.5.2 in [6, p. 162] hold.

**Lemma 3.2.** Suppose that $2q_2^2 - 3q_3 \neq 0$ and $q$ is sufficiently close to 1. Then for $p \in (0, 1)$ system (3) undergoes the Neimark-Sacker bifurcation at the fixed point $E_0$ as $q$ crosses 1. Specifically, system (3) possesses a unique attracting (resp. repelling) invariant circle $C$ near $E_0$ if $2q_2^2 - 3q_3 > 0$ (resp. $2q_2^2 - 3q_3 < 0$) and $q > 1$ (resp. $q < 1$).

**Proof.** Lemma 3.1 implies that the condition (SH1) of Theorem 3.5.2 in [6] for the Neimark-Sacker bifurcation is satisfied, i.e., $\lambda_0^k \neq 1, k = 1, 2, 3, 4$. Using (21), we obtain

$$\frac{d|\lambda_+(q)|}{dq} \bigg|_{q=1} = \frac{1}{2} > 0,$$

which shows that the transversality condition (SH2) of Theorem 3.5.2 in [6] holds.

The condition (SH3) of Theorem 3.5.2 in [6] involves the first Lyapunov coefficient. In order to get it, we compute the normal form of system (5). Applying an invertible linear transformation $(w_1, w_2)^T = H_2(\varphi, \psi)^T$, we change the mapping in (5) into the mapping

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + \begin{pmatrix} 0 \\ \Phi(\varphi, \psi) \end{pmatrix},$$

where

$$H_2 = \begin{pmatrix} 1 & 0 \\ \mu & -\omega \end{pmatrix}$$

and

$$\Phi(\varphi, \psi) = -\frac{q_2 (\mu - 1)^2}{\omega} \varphi^2 - 2q_2 (\mu - 1) \psi \varphi - q_2 \varphi \psi^2 - \frac{q_3 (\mu - 1)^3}{\omega} \varphi^3$$

$$+ 3q_3 (\mu - 1)^2 \psi \varphi^2 - 3 \omega q_3 (\mu - 1) \psi^2 \varphi + q_3 \omega^2 \psi^3 - \frac{q_4 (\mu - 1)^4}{\omega} \varphi^4$$

$$+ 4q_4 (\mu - 1)^3 \psi \varphi^3 - 6 \omega q_4 (\mu - 1)^2 \psi^2 \varphi^2 + 4 \omega^2 q_4 (\mu - 1) \psi^3 \varphi$$

$$- q_3 \omega^3 \psi^4 - \frac{q_5 (\mu - 1)^5}{\omega} \varphi^5 + 5 \omega q_5 (\mu - 1)^4 \psi \varphi^4 - 10 \omega q_5 (\mu - 1)^3 \psi^2 \varphi^3$$

$$+ 10 \omega^2 q_5 (\mu - 1)^2 \psi^3 \varphi^2 - 5 \omega^3 q_5 (\mu - 1) \psi^4 \varphi + q_5 \omega^4 \psi^5 + O((\varphi, \psi)^6).$$

Let $z = \varphi + i \psi$. Then near the origin $O$ the mapping in (25) is transformed to the following mapping

$$z \mapsto \lambda_+(q) z + \sum_{k=2}^{\infty} G_k(z),$$

where $G_k(z) = \sum_{j=0}^{k} g_{k-j,j} z^{k-j} \bar{z}^j$ is the sum of all terms with degree $k$ and

$$g_{k-j,j} = \frac{k! (\lambda_+(q) - 1)^{k-j} (\lambda_+(q) - 1)^j}{2^k (k-j)!} q_k.$$

By the theory of the normal form for the Neimark-Sacker bifurcation (see [9, p. 129-135] and [18, p. 285-288]), for sufficiently small $|q - 1|$, using the transformation

$$z = w + \sum_{2 \leq k+l \leq 3} h_{k,l} w^k \bar{w}^l,$$
where \( h_{k,l}(q,p), 2 \leq k + l \leq 3 \) are indeterminate, we can reduce the mapping in (26) into the mapping
\[
w \mapsto \lambda_+(q)w + \alpha_{2,1}w^2 \bar{w} + O(|w|^4),
\]
where \( \alpha_{2,1}(q,p) \) is indeterminate. Combining (26)-(28) and comparing the coefficients of \( w^k \bar{w}^l, 2 \leq k + l \leq 3 \), we get
\[
h_{2,0} = \frac{g_{2,0}}{\lambda_+(q) \left(1 - \lambda_+(q) \right)}, \quad h_{1,1} = \frac{g_{1,1}}{\lambda_+(q) \left(1 - \lambda_+(q) \right)}, \quad h_{0,2} = \frac{g_{0,2}}{\lambda_+(q) \left(\lambda_+(q) - \left(\lambda_+(q) \right)^2\right)},
\]
\[
h_{3,0} = \frac{g_{3,0} + 2\lambda_+(q)h_{2,0}g_{2,0} + \lambda_+(q)h_{1,1}g_{0,2}}{\lambda_+(q) \left(1 - \lambda_+(q)^2\right)}, \quad h_{2,1} = 0,
\]
\[
h_{1,2} = \frac{2\lambda_+(q)h_{2,0}g_{1,1} + g_{1,2} + \lambda_+(q)h_{1,1}g_{1,1} + 2\lambda_+(q)h_{2,0}g_{0,2} + \lambda_+(q)h_{1,1}g_{2,0}}{\lambda_+(q) \left(1 - \left(\lambda_+(q) \right)^2\right)},
\]
\[
h_{0,3} = \frac{\lambda_+(q)h_{1,1}g_{0,2} + g_{0,3} + 2\lambda_+(q)h_{0,2}g_{2,0}}{\lambda_+(q) - \lambda_+(q)}.
\]
Using the polar coordinate \( w = re^{i\theta} \) and writing \( \lambda_+(q) \) in the exponential form \( \lambda_+(q)e^{i\beta} \), we reduce system (28) to
\[
\begin{pmatrix}
\rho \\
\theta
\end{pmatrix} \mapsto \begin{pmatrix}
\left(\lambda_+(q)\right)\rho + c_3(q,p)\rho^3 + O(\rho^4) \\
\theta + \beta + d_2(q,p)\rho^2 + O(\rho^3)
\end{pmatrix},
\]
where
\[
c_3(q,p) = \text{Re}\left(\frac{\alpha_{2,1}(q)}{\lambda_+(q)}\right), \quad d_2(q,p) = \text{Im}\left(\frac{\alpha_{2,1}(q)}{\lambda_+(q)}\right).
\]
As called in [9], \( c_3(q,p) \) is the first Lyapunov coefficient. One can check from (26), (29) and (31) that
\[
c_3(1,p) = \frac{1}{8} (p - 1) \left(2q_2^2 - 3q_3\right).
\]
Under our assumption, we know that \( c_3(1,p) \neq 0 \), which yields that the condition (SH3) of Theorem 3.5.2 in [6] is satisfied. Moreover, as shown in [18, p. 378-385], from (24) and (32), it follows that a unique attracting (resp. repelling) invariant circle \( C \) appears near the fixed point \( E_0 \) if \( 2q_2^2 - 3q_3 > 0 \) (resp. \( 2q_2^2 - 3q_3 < 0 \)) and \( q > 1 \) (resp. \( q < 1 \)). This completes the proof.

In Lemma 3.2, we need the condition \( 2q_2^2 - 3q_3 \neq 0 \), which implies that the first Lyapunov coefficient \( c_3(1,p) \neq 0 \) in (32). If \( c_3(1,p) = 0 \) or \( q_3 = 2q_2^2/3 \), then system (5) may have the Chenciner bifurcation (or the generalized Neimark-Sacker bifurcation), as called in [9].

4. Chenciner bifurcation. In this section, we deal with the case: \( 2q_2^2 - 3q_3 \) is close to 0. Let \( \epsilon_1 = q - 1, \epsilon_2 = q_3 - 2q_2^2/3 \) and \( \epsilon = (\epsilon_1, \epsilon_2) \). Then the eigenvalue \( \lambda_+(q) \) in (21) is rewritten as
\[
\lambda(\epsilon_1) := \lambda_+(1 + \epsilon_1) = \frac{p + 1 + \epsilon_1}{2} + i\sqrt{3 - 2(p - \epsilon_1) - (p + \epsilon_1)^2}.
\]
For $\epsilon = 0$, two conjugate complex eigenvalues lie on the unit circle and the first Lyapunov coefficient $c_3(1,p)$ in (32) is also equal to zero. Let

$$\mathcal{L} = 2(p - 1)(p + 3)(p + 2)(4p + 9)q_2^3 + 12(3p + 5)(p + 1)q_2^3 - 6(5p + 11)(p - 1)(p + 3)(p + 2)q_2q_4 + 12(p + 3)(p + 2)^2q_2 + 15(p + 3)(p - 1)(p + 2)^2q_5.$$ (34)

Then we have the following result.

**Theorem 4.1.** Suppose that $p \in (0,1)$, $q_3$ is close to $2q_2^2/3$ and $\mathcal{L} > 0$. Then system (3) undergoes the Chenciner bifurcation. Specifically, in a small neighborhood of $(\epsilon_1, \epsilon_2) = (0,0)$, if $(\epsilon_1, \epsilon_2) \in \Omega_1$ (see Figure 4) where

$$\Omega_1 := \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 < \frac{27(p-1)(p+3)(p+2)^2}{8\mathcal{L}}\epsilon_2^2 + O(\epsilon_2), \epsilon_2 > 0\}$$

$$\cup \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 \leq 0, \epsilon_2 \leq 0\},$$

the fixed point $E_0$ is stable. As the parameter $(\epsilon_1, \epsilon_2)$ crosses the curve

$$\mathcal{N}^- = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = 0, \epsilon_2 \leq 0\}$$

from the region $\Omega_1$ to the region

$$\Omega_2 = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 > 0\} \cup \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = 0, \epsilon_2 > 0\},$$

system (3) undergoes the Neimark-Sacker bifurcation and produces a stable invariant cycle. Simultaneously, $E_0$ becomes unstable. As $(\epsilon_1, \epsilon_2)$ crosses the curve

$$\mathcal{N}^+ = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = 0, \epsilon_2 > 0\}$$

from the region $\Omega_2$ to the region $\Omega_3$, where

$$\Omega_3 = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \frac{27(p-1)(p+3)(p+2)^2}{8\mathcal{L}}\epsilon_2^2 + O(\epsilon_2) < \epsilon_1 < 0, \epsilon_2 > 0\},$$

system (3) possesses two invariant circles, a stable “outer” one $\Gamma_s$ and an unstable “inner” one $\Gamma_u$. When the parameter $(\epsilon_1, \epsilon_2)$ lies on the curve

$$\mathcal{T}_c = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = \frac{27(p-1)(p+3)(p+2)^2}{8\mathcal{L}}\epsilon_2^2 + O(\epsilon_2), \epsilon_2 > 0\},$$

system (3) has a unique invariant circle $\Gamma_{su}$, stable from the outside and unstable from the inside.

**Proof.** In order to apply Lemma 9.5 in [9, p. 404] for the Chenciner bifurcation, we have to check all its conditions.

By Lemma 3.1, the non-resonant condition (CH0) of Lemma 9.5 in [9] is satisfied for $p \in (0,1)$, that is, $\lambda(0)^k \neq 1, k = 1, 2, 3, 4, 5, 6$, where $\lambda(\epsilon_1)$ is given in (33). Note that we here assume $p \neq 0$. If $p = 0$, then $\lambda(0)^6 = 1$. This becomes much more difficult and is left for further investigation.

From (32) the first Lyapunov coefficient $c_3(1,p) = 0$ under our assumptions. To obtain the second Lyapunov coefficient, we continue computing the normal form of system (26) for $p \in (0,1)$. By the theory of the normal form for the Chenciner bifurcation (see [9, p. 404-407] and [7, p. 44]), for sufficiently small $|\epsilon|$, there exists a near identity transformation

$$z = w + \sum_{2 \leq k+l \leq 5} h_{k,l}w^kw^l,$$ (35)
where $h_{k,l}, 2 \leq k + l \leq 3$, are given in (29) with $\lambda_+(q)$ and $q_3$ replaced by $\lambda(\epsilon_1)$ and $2/3q_2^2 + \epsilon_2$ respectively, and $h_{i,j}, 4 \leq i + j \leq 5$ are indeterminate. Thus, the mapping in (26) can be transformed into the system

$$w \mapsto \lambda(\epsilon_1) w + \alpha_{2,1} w^2 \bar{w} + \alpha_{3,2} w^3 \bar{w}^2 + O(|w|^6),$$

where $\alpha_{2,1}$ is given in (29) with $\lambda_+(q)$ and $q_3$ replaced by $\lambda(\epsilon_1)$ and $2/3q_2^2 + \epsilon_2$ respectively, and $\alpha_{3,2}$ is indeterminate. Combining (26), (35) and (36) and comparing the coefficients of $w^k \bar{w}^l, 4 \leq k + l \leq 5$, we get $h_{k,l}, 4 \leq k + l \leq 5$ and $\alpha_{3,2}$ given in Appendix since their expressions are very complicated. Applying the polar coordinate transformation, we change system (36) into the following form

$$
\begin{pmatrix}
\rho \\
\theta
\end{pmatrix} \mapsto 
\begin{pmatrix}
|\lambda(\epsilon_1)|\rho + c_3(\epsilon)\rho^3 + c_5(\epsilon)\rho^5 + O(\rho^6) \\
\theta + \beta + d_2(\epsilon)\rho^2 + d_4(\epsilon)\rho^4 + O(\rho^5)
\end{pmatrix},
$$

where

$$c_3(\epsilon) = \text{Re} \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right), \quad c_5(\epsilon) = \text{Re} \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right) + \frac{1}{2} |\lambda(\epsilon_1)| \text{Im}^2 \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right),$$

$$d_2(\epsilon) = \text{Im} \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right), \quad d_4(\epsilon) = \text{Im} \left( \frac{\alpha_{3,2}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right) - \text{Im} \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right) \text{Re} \left( \frac{\alpha_{2,1}(\lambda(\epsilon_1))}{\lambda(\epsilon_1)} \right).$$

One can check that by $p \in (0, 1)$ and $\mathcal{L} \neq 0$

$$c_5(0) = \frac{(p - 1)(2(p - 1)(p + 3)(p + 2)(4p + 9)q_2^3 + 12(3p + 5)(p + 1)q_2^3 - 6(5p + 11)(p - 1)(p + 2)q_2q_4 + 12(p + 3)(p + 2)^2q_2 + 15(p + 3)(p + 2)^2q_5) / (48(p + 3)(p + 2)^2)}{(p - 1)\mathcal{L}} \neq 0,$$

or

$$|\lambda_+(q)| \approx \frac{c_5}{\mathcal{L}} = \frac{1}{\mathcal{L}}.$$
which is the second Lyapunov coefficient at $\epsilon = 0$. Thus, the condition (CH2) of the Chenciner bifurcation in [9] is satisfied.

To verify that the mapping
\[
(\epsilon_1, \epsilon_2) \mapsto (|\lambda(\epsilon_1)| - 1, c_3(\epsilon))
\] is regular, by (26), (29), (33) and (38), we obtain that
\[
|\lambda(\epsilon_1)| = \sqrt{1 + \epsilon_1}, \\
c_3(\epsilon) = -\frac{(p^3 + 4p^2 + 3p - 1)q_2^2}{4(p + 2)^2}\epsilon_1 + \frac{3}{8}(1 - p)\epsilon_2 + O(|\epsilon|^2).
\]
Hence,
\[
\det \left( \frac{\partial (|\lambda(\epsilon_1)| - 1, c_3(\epsilon))}{\partial (\epsilon_1, \epsilon_2)} \right)_{(0,0)} = \frac{3}{16}(1 - p) \neq 0
\]
for $p \in (0, 1)$. The mapping in (40) is regular so that the condition (CH1) of Chenciner bifurcation given in [9, p. 405] is true.

Therefore, system (5) undergoes the Chenciner bifurcation. In what follows, we discuss specifically the phenomena of the bifurcation. Applying a parameter transformation
\[
\beta_1 = |\lambda(\epsilon_1)| - 1, \quad \beta_2 = c_3(\epsilon),
\]
we change system (37) into the mapping
\[
\left( \begin{array}{l}
\rho \\
\theta
\end{array} \right) \mapsto \left( \begin{array}{l}
\rho + \beta_1\rho + \beta_2\rho^2 + (c_5(0) + O(|\beta|))\rho^3 + O(\rho^4) \\
\theta + \beta + d_2(\epsilon(\beta))\rho^2 + d_4(\epsilon(\beta))\rho^4 + O(\rho^6)
\end{array} \right),
\]
where $\epsilon$ is regarded as a function of $(\beta_1, \beta_2)$. We firstly consider that $\mathcal{L} > 0$ so that $c_5(0) < 0$. The case $\mathcal{L} < 0$ is given later in Theorem 4.2. Then the complete bifurcation diagram of system (44) can be seen in Figure 5, which is given in [9, p. 407], where
\[
N_+ = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 > 0 \}, \quad N_- = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 < 0 \}, \\
T_0 = \{(\beta_1, \beta_2) : \beta_1 = \frac{1}{4c_5(0)}\beta_2^2 + o(\beta_2^2), \beta_2 > 0 \}.
\]

The bifurcations of system (44), given in [1] and [9], are as follows:

(1) As the parameter $(\beta_1, \beta_2)$ is in the region (1), i.e., the region
\[
\left\{ (\beta_1, \beta_2) | \beta_1 < \frac{1}{4c_5(0)}\beta_2^2 + o(\beta_2^2), \beta_2 > 0 \right\} \cup \left\{ (\beta_1, \beta_2) | \beta_1 \leq 0, \beta_2 \leq 0 \right\},
\]
system (44) has a stable fixed point $O$. As the parameter $(\beta_1, \beta_2)$ crosses the half line $N_-$ from the region (1) to the region (2) (the right side of $\beta_2$-axis) and $N_+$, system (44) undergoes the Neimark-Sacker bifurcation and a stable invariant circle appears.

(2) The Neimark-Sacker bifurcation happens again and an unstable invariant circle appears as the parameter $(\beta_1, \beta_2)$ crosses the half line $N_+$ from the region (2) to the region (3), above the curve $T_c$ and the left side of $\beta_2$-axis. Hence there are two invariant circles, stable “outer” one and unstable “inner” one, in the region (3).

(3) When the parameter $(\beta_1, \beta_2)$ lies on the curve $T_c$, system (44) has a unique invariant circle, stable from the outside and unstable from the inside.
Figure 5. Chenciner bifurcation of system (44) in the case $c_5(0) < 0$.

Going back to the original parameter $(\epsilon_1, \epsilon_2)$ and computing the second order derivative of the implicit function, from (41) and (43), we see that the curves $N_+, N_-$ and $T_c$ correspond to $N^+, N^-$ and $T_c$ given in the theorem respectively, and the regions $\Omega_3$ and $\Omega_4$ in $(\beta_1, \beta_2)$-plane correspond to the regions $\Omega_1, \Omega_2$ and $\Omega_3$ in $(\epsilon_1, \epsilon_2)$-plane. The proof is completed.

For the case $L < 0$, we similarly obtain the following results.

**Theorem 4.2.** Assume that $p \in (0, 1)$, $q_3$ is close to $2q_3^2/3$ and $L < 0$, then system (3) undergoes the Chenciner bifurcation at $E_0$. Specifically, in a small neighborhood of $(\epsilon_1, \epsilon_2) = (0, 0)$, if $(\epsilon_1, \epsilon_2) \in \tilde{\Omega}_1$ (see Figure 6), where

$$\tilde{\Omega}_1 := \left\{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 > \frac{27(p-1)(p+3)(p+2)^2}{8L} \epsilon_2^2 + O(\epsilon_3^2), \epsilon_2 < 0 \right\} \cup \left\{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 \geq 0, \epsilon_2 \geq 0 \right\},$$

the fixed point $E_0$ is unstable; As $(\epsilon_1, \epsilon_2)$ crosses the half line $N^+$ given in Theorem 4.1 from the region $\tilde{\Omega}_1$ to the region

$$\tilde{\Omega}_2 := \left\{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 < 0 \right\} \cup \left\{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = 0, \epsilon_2 < 0 \right\},$$

system (3) undergoes the Neimark-Sacker bifurcation and an unstable invariant circle appears. Simultaneously, the fixed point $E_0$ becomes stable; As $(\epsilon_1, \epsilon_2)$ crosses the half line $N^-$ given in Theorem 4.1 from the region $\tilde{\Omega}_2$ to the region $\tilde{\Omega}_3$, where

$$\tilde{\Omega}_3 := \left\{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | 0 < \epsilon_1 < \frac{27(p-1)(p+3)(p+2)^2}{8L} \epsilon_2^2 + O(\epsilon_3^2), \epsilon_2 < 0 \right\},$$
system (3) possesses two invariant circles, an unstable “outer” one and a stable “inner” one; When the parameter \((\epsilon_1, \epsilon_2)\) lies on the curve
\[
\tilde{T}_c := \{ (\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = \frac{9(p-1)^2}{256c_5(0)} \epsilon_2^2 + O(\epsilon_2^3), \epsilon_2 < 0 \},
\]

system (3) has a unique invariant circle, unstable from the outside and stable from the inside.

Figure 6. Chenciner bifurcation of system (3) in the case \(L < 0\).

Remark 4.1. In [10], Li and Zhang required Assumption (H) in Section 1 in order to compute the \(n\)-th Lyapunov coefficient for any positive integer \(n\). In particular, if \(n = 1\), Assumption (H) implies that \(k = 3, q_2 = 0\) and \(q_3 \neq 0\). Our main purpose here is to analyze the Chenciner bifurcation of system (3) when the first Lyapunov coefficient is zero. Hence we do not need the above conditions.

As called in [9], the Chenciner bifurcation is also called the generalized Neimark-Sacker bifurcation. This bifurcation is similar to the Bautin bifurcation of a vector field. However, the Chenciner bifurcation possesses more complicated dynamical properties than the Bautin one. For instance, there are Arnold tongues near the \(\epsilon_2\)-axis such that system (3) has periodic orbits on the invariant circle as the parameter \((\epsilon_1, \epsilon_2)\) lies in the tongues. When the parameter is near \(T_c\) (resp. \(\tilde{T}_c\)), the system possesses more complicated dynamical properties such as homoclinic structure (see [1]).

5. Numerical simulations. In this section, we make numerical simulations to illustrate two invariant circles arising from the Chenciner bifurcation given in Section 4. For this purpose, we need to simulate system (3) near the fixed point \(E_0\)
when parameter \((\epsilon_1, \epsilon_2)\) lies in \(T_c\) and \(\Omega_3\) given in Theorem 4.1 for small \(|\epsilon|\). For convenience, choose 
\[f(y - x) = 0.5y + q(y - x) + 3(y - x)^2 + q_3(y - x)^3 + 10(y - x)^4 + 4(y - x)^5 + (y - x)^6.\]
Then \(L = 6125.25\) given in (34), from which we obtain 
\[T_c = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | \epsilon_1 = -0.006026539732\epsilon_2^2 + O(\epsilon_2^3), \epsilon_2 > 0\}\]
and 
\[\Omega_3 = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 | -0.006026539732\epsilon_2^2 + O(\epsilon_2^3) < \epsilon_1 < 0, \epsilon_2 > 0\}.\]
They both require that \(\epsilon_2 = \frac{q_3 - 2q_2^2}{3} = q_3 - 6 > 0\). Thus, we choose \(q_3 = 6.1\), that is, \(\epsilon_2 = 0.1\). In order to make \((\epsilon_1, \epsilon_2) \in T_c\), we choose \(\epsilon_1 = q - 1 \approx -0.000175\), which implies \(q \approx 0.999825\). Near the invariant circle \(\Gamma_{su}\), take two initial values \((x_{01}, y_{01}) = (0.15, 0)\) and \((x_{02}, y_{02}) = (0.071, 0)\), which are outside and inside the \(\Gamma_{su}\) respectively. Using the mathematical software MATLAB Version 7.11, after 200000 steps, we respectively simulate two orbits, the blue one approaching \(\Gamma_{su}\) and the red one leaving \(\Gamma_{su}\) and approaching \(E_0\) (see Figure 7 (a)). In order to make \((\epsilon_1, \epsilon_2) \in \Omega_3\), we choose \(q = 0.99986\), which implies \(\epsilon_1 = q - 1 = -0.00014\). Near the unstable invariant circle \(\Gamma_u\), take two initial values \((x_{01}, y_{01}) = (0.052553, 0)\) and \((x_{02}, y_{02}) = (0.052554, 0)\) which are outside the unstable invariant circle \(\Gamma_u\), sufficiently close to \(\Gamma_u\) and inside \(\Gamma_s\) respectively. Again using the MATLAB, after 200000 steps, we respectively obtain two orbits, the red one leaving \(\Gamma_u\) and approaching the stable fixed point \(E_0\) and the green one leaving \(\Gamma_u\) and approaching the stable invariant circle \(\Gamma_s\) (see Figure 7 (b)). Take an initial value \((0.15, 0)\) outside \(\Gamma_s\), using the same mathematical software and the same steps, we get a blue orbit, which approaches \(\Gamma_s\) (see Figure 7 (b)).

![Invariant circles arising from Chenciner bifurcation.](image)

**Figure 7.** Invariant circles arising from Chenciner bifurcation.

6. **Conclusions.** In this paper, applying the center manifold theorem and the normal form theory, we prove that system (3) undergoes the generalized flip bifurcation, and give the parameter conditions such that the system possesses two, one and no 2-cycles. In our discussion, we require the coefficient \(q_5 \neq 0\). However, if the coefficient \(q_5 = 0\), the system may produce a codimension 3 flip bifurcation. Similarly, in the Chenciner bifurcation, if \(L = 0\) in (34), the codimension 3 Neimark-Sacker
The coefficients and comments and suggestions. Acknowledgments.

Appendix. The coefficients $h_{i,j}$, $4 \leq i + j \leq 5$ in (35) and the coefficient $\alpha_{3,2}$ in (36) are listed in the following:

$$h_{4,0} = \left\{-h_{1,92,0} \hat{\gamma} + h_{0,2} \hat{\gamma}_{2,0} - h_{0,2} \hat{\gamma}_{2,0}^2 - \lambda(\epsilon) h_{1,92,0} - 2 h_{2,0} \hat{\gamma}_{2,0}^2 - h_{2,0} \hat{\gamma}_{2,0}^3 - 94.0 - 3 h_{3,0} \hat{\gamma}_{2,0}^2 \right\}.$$ $$h_{3,1} = \left\{2 h_{0,2} \hat{\gamma}_{2,0} + h_{0,2} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^2 + \lambda(\epsilon) h_{1,92,0} + 93.1 + h_{1,92,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0} - 94.0 - 3 h_{3,0} \hat{\gamma}_{2,0}^2 \right\}.$$ $$h_{2,2} = \left\{h_{1,92,0} \hat{\gamma}_{2,0} + 3 h_{3,0} \hat{\gamma}_{2,0}^2 + 2 h_{0,2} \hat{\gamma}_{2,0} + 2 h_{0,2} \hat{\gamma}_{2,0}^2 + h_{0,2} \hat{\gamma}_{2,0}^3 + h_{1,92,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^3 + 91.3 + 2 h_{2,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^3 \right\}.$$ $$h_{1,3} = \left\{h_{2,0} \hat{\gamma}_{2,0} + 2 h_{0,2} \hat{\gamma}_{2,0} + 2 h_{0,2} \hat{\gamma}_{2,0}^2 + h_{0,2} \hat{\gamma}_{2,0}^3 + h_{1,92,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^3 \right\}.$$ $$h_{0,4} = \left\{g_{90,0} + h_{1,92,0} \hat{\gamma}_{2,0} + 2 h_{0,2} \hat{\gamma}_{2,0} + 2 h_{0,2} \hat{\gamma}_{2,0}^2 + h_{1,92,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0} + 2 h_{2,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^3 \right\}.$$ $$h_{3,2} = \left\{2 h_{2,0} \hat{\gamma}_{2,0} + 3 h_{3,0} \hat{\gamma}_{2,0}^2 \right\}.$$ $\alpha_{3,2} = \left\{6 h_{0,2} \hat{\gamma}_{2,0} + 6 h_{0,2} \hat{\gamma}_{2,0}^2 + 6 h_{0,2} \hat{\gamma}_{2,0}^3 + h_{1,92,0} \hat{\gamma}_{2,0} + \lambda(\epsilon) h_{1,92,0} + h_{1,92,0} \hat{\gamma}_{2,0} + h_{1,92,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0} + h_{2,0} \hat{\gamma}_{2,0}^2 + h_{2,0} \hat{\gamma}_{2,0}^3 \right\}.$$ $g_{k-j,j} = -\frac{k! \left(\lambda(\epsilon) - 1\right)^k \left(\lambda(\epsilon) - 1\right)^j}{2^k \left(k - j\right)!} q_k,$ $k = 1, \ldots, 6, j = 0, \ldots, k.$

$q_3 = 2 / 3 q_3^2 + \epsilon_2,$ $h_{i,j} = h_{i,j}(\epsilon) \quad (1 \leq i + j \leq 3), \alpha_{3,1}$ are given in (29) with $\lambda_1(g)$ replaced by $\lambda(\epsilon),$ $h_{i,j} = h_{i,j}(\epsilon) \quad (4 \leq i + j \leq 5),$ and $\alpha_{3,2} = \alpha_{3,2}(\epsilon).$
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