DESCRIPTION OF THE SCATTERING DATA FOR STURM–LIOUVILLE OPERATORS ON THE HALF-LINE

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Abstract. We describe the set of the scattering data for self-adjoint Sturm–Liouville operators on the half-line with potentials belonging to $L_1(\mathbb{R}_+, \rho(x) \, dx)$, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically nondecreasing function from some family $\mathcal{R}$. In particular, $\mathcal{R}$ includes the functions $\rho(x) = (1 + x)^\alpha$ with $\alpha \geq 1$.

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1. INTRODUCTION

In the Hilbert space $L_2(\mathbb{R}_+)$, we consider the Schrödinger operator generated by the differential expression

$$t_q(f) := -f'' + qf$$

and the boundary condition

$$f(0) = 0$$

with the potential $q$ belonging to the class

$$\mathcal{Q}_\rho := \{q \in L_1(\mathbb{R}_+, \rho(x) \, dx) \mid \text{Im } q = 0\}, \quad \rho \in \mathcal{R}_0.$$ 

Here $\mathcal{R}_0$ is the class of all monotonically nondecreasing weight functions $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $x \leq \rho(x)$ for all $x > 0$. In particular, the class $\mathcal{R}_0$ includes the weight function $\omega(x) := x$.

In the present paper, we study the problem of an efficient description of the scattering data for operators from the class $\mathcal{T}_\rho := \{T_q \mid q \in \mathcal{Q}_\rho\}$ (for more details on the operator $T_q$ see Appendix A). For the class $\mathcal{T}_\omega$, such description was given by V.A. Marchenko [3]. As shown in [4], the scattering data for operators from the class
can be efficiently described in terms of some functional Banach algebra introduced below. Our aim is to describe the class \( \mathcal{B} \) of weight functions \( \rho \in \mathcal{B}_0 \) for which a result analogous to that can be obtained.

To formulate the main result of the paper, let us recall some definitions. The scattering function \( S = S_q \) of the operator \( T_q \) is defined as

\[
S(\lambda) := \frac{e(-\lambda)}{e(\lambda)}, \quad \lambda \in \mathbb{R},
\]

where \( e(\lambda) := e(\lambda, 0) \) and \( e(\lambda, \cdot) \) is the Jost solution of the equation

\[
y'' + qy = \lambda^2 y, \quad \lambda \in \mathbb{C}_+ := \{ \lambda \in \mathbb{C} \mid \text{Im } \lambda \geq 0 \},
\]

i.e., a solution of (1.1) satisfying the asymptotics

\[
e(\lambda, x) = e^{i\lambda x}(1 + o(1)), \quad x \to +\infty.
\]

The spectrum of the operator \( T_q \) with \( q \in Q_\rho \) consists of the absolutely continuous part filling the whole positive half-axis and the point spectrum consisting of a finite number of negative simple eigenvalues (see, e.g., [3]). Let us enumerate these eigenvalues in the ascending order of their moduli and denote them by \(-\kappa_s^2, \ s = 1, \ldots, n\), where \( \kappa_s = \kappa_s(q) > 0 \). To each eigenvalue \( \lambda = -\kappa_s^2 \), there correspond the eigenfunction \( e(i\kappa_s, \cdot) \) and the norming constant \( m_s = m_s(q) \), which is defined as

\[
m_s = \left( \int_0^\infty |e(i\kappa_s, x)|^2 \, dx \right)^{-\frac{1}{2}}.
\]

The scattering data of the operator \( T_q \) are defined as the triple \( \mathbf{s}_q := (S_q, \vec{\kappa}_q, \vec{m}_q) \), where \( \vec{\kappa}_q := (\kappa_s(q))_{s=1}^n, \vec{m}_q := (m_s(q))_{s=1}^n \). If \( n = 0 \), then \( \mathbf{s}_q := (S_q, 0, 0) \). Let us put

\[
\Omega_n := \{ (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}_+^n \mid 0 < \kappa_1 < \cdots < \kappa_n \}, \quad n \in \mathbb{N}.
\]

For an arbitrary open set \( O \subset \mathbb{R} \), we denote by \( \text{AC}(O) \) the set of all functions \( f : O \to \mathbb{C} \) that are absolutely continuous on each compact interval \( \Delta \subset O \). For an arbitrary \( \rho \in \mathcal{B}_0 \), let us denote by \( X_\rho \) the Banach space consisting of functions \( u \in \text{AC}(\mathbb{R} \setminus \{0\}) \cap L_1(\mathbb{R}) \) with the norm

\[
\|u\|_{X_\rho} := \int_\mathbb{R} \rho(|x|)|u'(x)| \, dx < \infty.
\]

Similarly, we denote by \( X_\rho^+ \) and \( X_\rho^- \) the Banach spaces consisting of \( u_+ \in \text{AC}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+) \) and \( u_- \in \text{AC}(\mathbb{R}^-) \cap L_1(\mathbb{R}^-) \), respectively, with the norms

\[
\|u_\pm\|_{X_\rho^\pm} := \int_{\mathbb{R}^\pm} \rho(|x|)|u'_\pm(x)| \, dx < \infty.
\]
Let us agree to identify the spaces $X^\pm_\rho$ with the subspaces $\{f \in X_\rho \mid f|_{\mathbb{R}_-} = 0\}$ in the space $X_\rho$. Then $X_\rho = X^+_\rho + X^-_\rho$.

Recall that $\omega(x) = x$ and $\omega \leq \rho$. Therefore, $X_\rho \subset X_\omega$ and $X^\pm_\rho \subset X^\pm_\omega$. As will be shown in Section 2 of this paper, the space $X_\rho$ is continuously embedded in $L_1(\mathbb{R})$.

Consider the Banach space
$$B_\rho := \{\alpha 1 + \mathcal{F} \mid \alpha \in \mathbb{C}, \mathcal{F} \in X_\rho\}$$
with the norm
$$\|\alpha 1 + \mathcal{F}\|_{B_\rho} := |\alpha| + \|\mathcal{F}\|_{X_\rho}. \quad (1.2)$$
Here $1(x) \equiv 1$ and $\mathcal{F}$ is the Fourier transform of a function $\varphi$.

**Definition 1.1.** A weight function $\rho \in \mathcal{R}_0$ is called regular if
$$c(\rho) := \sup_{x>0} \rho(2x)/\rho(x) < \infty.$$  
Denote by $\mathcal{R}$ the set of all regular functions $\rho \in \mathcal{R}_0$.

**Theorem 1.2.** Let $\rho \in \mathcal{R}$. Then there is a norm on $B_\rho$ (see the formula (3.1) below) equivalent to the norm (1.2) which turns $B_\rho$ into a unital commutative Banach algebra in which the multiplication is the standard pointwise multiplication.

The main result of this paper is:

**Theorem 1.3.** Let $\rho \in \mathcal{R}$. Then the set $\{S_q \mid q \in \mathcal{Q}_\rho\}$ coincides with the set
$$S_\rho := \{S \in B_\rho \mid S(\infty) = 1 \text{ and } \forall \lambda \in \mathbb{R} \ S(\lambda)S(-\lambda) = |S(\lambda)| = 1\}.$$  

The following result follows from Theorem 1.3.

**Corollary 1.4.** Let $\rho \in \mathcal{R}$ and $n \in \mathbb{N}$ (resp. $n = 0$). A triple $(S, \vec{\kappa}, \vec{m})$ (resp. $(S, 0, 0)$), where $S : \mathbb{R} \to \mathbb{C}, \vec{\kappa} \in \mathcal{O}_n, \vec{m} \in \mathbb{R}_n^+$, is the scattering data of some $T \in \mathcal{T}_\rho$ if and only if $S \in S_\rho$ and $[\text{ind } S/2] = n$, where $\text{ind } S := ((\ln S)(\infty) - (\ln S)(-\infty))/2\pi i$ and $[x]$ is the integer part of $x$.

This paper is organized as follows. In Section 2, we study properties of the spaces $X_\rho$ and their subspaces $X^\pm_\rho$. In Section 3, we consider properties of the algebra $B_\rho$ and prove Theorem 1.2. In Section 4, we prove Theorem 1.3. Finally, in an Appendix, we give the explicit definition of the operator $T_q$.

**2. PROPERTIES OF THE SPACES $X_\rho$**

Denote by $\| \cdot \|_p$ the norm in the space $L_p(\mathbb{R})$, $p \in [1, \infty]$, and denote by $f * g$ the convolution of functions $f, g \in L_1(\mathbb{R})$, i.e.,
$$(f * g)(x) := \int_\mathbb{R} f(x-t)g(t)\,dt, \quad x \in \mathbb{R}.$$
It is well known that the convolution is a commutative operation in $L_1(\mathbb{R})$ and that
\[
\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in L_1(\mathbb{R}),
\]
and
\[
\hat{f} \ast \hat{g} = \hat{f} \hat{g},
\]
where $\hat{\varphi}$ is the Fourier transform of a function $\varphi$, i.e.,
\[
\hat{\varphi}(\lambda) := \int_{\mathbb{R}} e^{i\lambda t} \varphi(t) \, dt, \quad \lambda \in \mathbb{R}.
\]

Let us denote by $P_+$ and $P_-$ the projections in the space $L_1(\mathbb{R})$ acting by the formulas
\[
(P_+ f)(x) := \chi_+(x) f(x), \quad (P_- f)(x) := \chi_-(x) f(x), \quad x \in \mathbb{R},
\]
where $\chi_+$ (resp. $\chi_-$) is the indicator function of the half-line $\mathbb{R}_+$ (resp. of $\mathbb{R}_-$).

**Remark 2.1.** If $f, g \in L_1(\mathbb{R})$ and $P_- f = P_- g = 0$, then $P_- (f \ast g) = 0$ and
\[
(f \ast g)(x) = \int_0^x f(x-t) g(t) \, dt = \int_0^{x/2} f(x-t) g(t) \, dt + \int_0^{x/2} g(x-t) f(t) \, dt, \quad x > 0.
\]

Clearly, $P_+$ and $P_-$ are the projections in every space $X_\rho (\rho \in \mathbb{R}_0)$. Moreover, $P_\pm X_\rho = X_\rho ^\pm$ and
\[
\|f\|_{X_\rho} = \|P_+ f\|_{X_\rho} + \|P_- f\|_{X_\rho}, \quad f \in X_\rho. \tag{2.1}
\]

Note that the reflection operator $\Gamma$, given by the formula
\[
(\Gamma f)(x) = f(-x), \quad x \in \mathbb{R},
\]
is an isometry of $X_\rho$ onto itself and maps the space $X_\rho ^+ (X_\rho ^-)$ on $X_\rho ^- (X_\rho ^+)$. Moreover,
\[
(\Gamma f) \ast (\Gamma g) = \Gamma (f \ast g), \quad f, g \in L_1(\mathbb{R}). \tag{2.2}
\]

Next, denote by $\Lambda_\rho$ the operator acting on the space $L_{1,\text{loc}}(\mathbb{R})$ by the formula
\[
(\Lambda_\rho f)(x) := \rho(|x|) f(x), \quad x \in \mathbb{R}.
\]

**Lemma 2.2.** Let $\rho \in \mathbb{R}_0$. Then

(i) the space $X_\rho$ is continuously embedded in $L_1(\mathbb{R})$ and
\[
\|u\|_1 \leq \|u\|_{X_\rho}, \quad u \in X_\rho; \tag{2.3}
\]

(ii) the operator $\Lambda_\rho$ maps continuously the space $X_\rho$ into $L_\infty(\mathbb{R})$ and
\[
\|\Lambda_\rho u\|_\infty \leq \|u\|_{X_\rho}, \quad u \in X_\rho. \tag{2.4}
\]
Proof. Clearly, it suffices to prove the estimates (2.3), (2.4), and only for \( u \in X_\rho^+ \). Fix an arbitrary \( u \in X_\rho^+ \). Since \( u(x) \) vanishes at \(+\infty\) and thus

\[
|u(x)| \leq \int_x^{\infty} |u'(t)| \, dt, \quad x \in \mathbb{R}_+,
\]

we have

\[
\rho(x)|u(x)| \leq \rho(x) \int_x^{\infty} |u'(t)| \, dt \leq \int_x^{\infty} \rho(t)|u'(t)| \, dt, \quad x \in \mathbb{R}_+, \tag{2.5}
\]

and

\[
\int_0^\infty |u(x)| \, dx \leq \int_0^\infty \int_0^{\infty} |u'(t)| \, dt \, dx = \int_0^\infty |u'(t)| \, dx \leq \int_0^\infty \rho(t)|u'(t)| \, dt.
\]

Using these estimates, we obtain (2.3) and (2.4).

Consider the spaces

\[
Y_\pm := \{ f \in X_\rho^\pm \mid f \text{ has compact support and } f \in C^1(\mathbb{R}_\pm \cup \{0\}) \}.
\]

Lemma 2.3. Let \( \rho \in \mathcal{R}_0 \). Then the set \( Y^+ \) (resp. \( Y^− \)) is everywhere dense in the space \( X_\rho^+ \) (resp. in \( X_\rho^- \)).

Proof. Obviously, it suffices to prove the statement for the set \( Y^+ \) only. Take \( f \in X_\rho^+ \) and consider the sequence \( f_n := \theta_n f \) \((n \in \mathbb{N})\), where the functions \( \theta_n : \mathbb{R} \rightarrow [0,1] \) are defined as

\[
\theta_n(x) := \begin{cases} 
1, & \text{if } 0 \leq x \leq n, \\
2 - x/n, & \text{if } n < x \leq 2n, \\
0, & \text{if } x < 0 \text{ or } x > 2n.
\end{cases}
\]

It is easily seen that each function \( f_n \) belongs to \( X_\rho^+ \), has compact support and

\[
\|f - f_n\|_{X_\rho} = \int_0^\infty \rho(t)|f'(t) - f'_n(t)| \, dt \leq \int_0^\infty \rho(t)|f'(t)| \, dt + \frac{1}{n} \int_0^{2n} \rho(t)|f(t)| \, dt.
\]

It follows from (2.5) that

\[
\rho(x)|f(x)| \leq \int_0^{\infty} \rho(t)|f'(t)| \, dt, \quad x \geq n.
\]

Thus

\[
\|f - f_n\|_{X_\rho} \leq 2 \int_0^{\infty} \rho(t)|f'(t)| \, dt
\]

and hence \( f_n \xrightarrow{X_\rho} f \) as \( n \rightarrow \infty \).
It remains to prove that every function $u \in X^+_{\rho}$ of compact support can be approximated by elements from $Y^+_{\rho}$. Let $u \in X^+_{\rho}$ be a function of compact support. Fix an arbitrary non-negative function $\phi \in C^\infty(\mathbb{R})$ for which

$$\text{supp} \phi \subset [0,1], \quad \int_{\mathbb{R}} \phi(t) \, dt = 1.$$ 

Obviously, for an arbitrary $\varepsilon > 0$, the function

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} \int_{\mathbb{R}} u(t) \phi \left( \frac{t-x}{\varepsilon} \right) \, dt, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

belongs to $Y^+$. Note that for $x > 0$,

$$u(x) - u_\varepsilon(x) = \int_{0}^{1} (u(x) - u(x + \varepsilon y)) \phi(y) \, dy,$$

and

$$\rho(x) \frac{d}{dx} (u(x) - u(x + \varepsilon y)) = v(x) - v(x + \varepsilon y) + v(x + \varepsilon y) m_\varepsilon(x, y),$$

where $v(x) := \rho(x) u'(x)$ and $m_\varepsilon(x, y) := 1 - \frac{\rho(x)}{\rho(x+\varepsilon y)}$. Thus

$$\|u - u_\varepsilon\|_{X^\rho} \leq \int_{0}^{\infty} \int_{0}^{1} |v(x) - v(x + \varepsilon y)| \phi(y) \, dy \, dx + \int_{0}^{\infty} \int_{0}^{1} |v(x + \varepsilon y)| m_\varepsilon(x, y) \phi(y) \, dy \, dx.$$

Since $v \in L_1(\mathbb{R})$, $0 \leq m_\varepsilon \leq 1$, and $m_\varepsilon(x, y) \to 0$ as $\varepsilon \to 0$ almost everywhere on $\mathbb{R}_+ \times [0,1]$, we conclude that $u_\varepsilon \overset{X^\rho}{\to} u$ as $\varepsilon \to +0$. 

**Proposition 2.4.** Let $\rho \in \mathcal{R}$ and $c = c(\rho)$. Then for an arbitrary $f, g \in X_{\rho}$, the convolution $f * g$ belongs to $X_{\rho}$ and

$$\|f * g\|_{X_{\rho}} \leq 4c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}}.$$  \hspace{1cm} (2.6)

**Proof.** Note that in view of Definition 1.1,

$$\rho(2x) \leq c\rho(x), \quad x > 0.$$ \hspace{1cm} (2.7)

1) Let $f, g \in Y^+$. Then (see Remark 2.1) $(f * g)(x) = 0$ for $x < 0$ and

$$(f * g)'(x) = f(x/2)g(x/2) + \int_{0}^{x/2} f'(x-t)g(t) \, dt + \int_{0}^{x/2} g'(x-t)f(t) \, dt, \quad x > 0.$$
Using this fact and the estimate (2.7), we obtain that for \( x > 0 \)
\[
\rho(x)(f * g)'(x) \leq c \rho(x/2)|f(x/2)| |g(x/2)|
\]
\[
+ c \int_{x/2}^{x} \rho(x-t)|f'(x-t)||g(t)| \, dt
\]
\[
+ c \int_{0}^{x/2} \rho(x-t)|g'(x-t)||f(t)| \, dt.
\]
Therefore, taking into account (2.3) and (2.4), we get that for all \( f, g \in Y^+ \), \( x > 0 \)
\[
\|f * g\|_{X_{\rho}} \leq 2c \|\Lambda_{\rho} f\|_{\infty} \|g\|_1 + c \|f\|_{X_{\rho}} \|g\|_1 + c \|g\|_{X_{\rho}} \|f\|_1 \leq 4c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}}.
\]  
(2.8)

2) Since the reflection operator \( \Gamma \) maps \( Y^+ \) onto \( Y^- \) and is an isometry of the spaces \( X_{\rho} \), taking into account (2.2) and (2.8), we obtain that
\[
\|f * g\|_{X_{\rho}} \leq 4c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}}, \quad f, g \in Y^-.
\]  
(2.9)

3) Let \( f \in Y^+ \) and \( g \in Y^- \). Then
\[
\rho(x)(f * g)'(x) \leq \rho(x) \int_{-\infty}^{0} |f'(x-t)||g(t)| \, dt \leq \int_{-\infty}^{0} \rho(x-t)|f'(x-t)||g(t)| \, dt
\]
for \( x > 0 \) and
\[
\rho(|x|)(f * g)'(x) \leq \rho(|x|) \int_{0}^{\infty} |g'(x-t)||f(t)| \, dt \leq \int_{0}^{\infty} \rho(|x-t|)g'(x-t)||f(t)| \, dt
\]
for \( x < 0 \). Since \( c \geq 1 \), using the estimate (2.3), we get
\[
\|f * g\|_{X_{\rho}} \leq \|f\|_{X_{\rho}} \|g\|_1 + \|g\|_{X_{\rho}} \|f\|_1 \leq 2c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}}, \quad f \in Y^+, \quad g \in Y^-.
\]  
(2.10)

4) Let \( f, g \in Y^+ \oplus Y^- \) and \( f_{\pm} := P_{\pm} f, \quad g_{\pm} := P_{\pm} g \). Then
\[
f * g = f_{+} * g_{+} + f_{-} * g_{-} + f_{+} * g_{-} + f_{-} * g_{+}.
\]

Taking into account (2.9), (2.10) and (2.1), we obtain
\[
\|f * g\|_{X_{\rho}} \leq 4c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}}, \quad f, g \in Y^+ \oplus Y^-.
\]  
(2.11)

Let \( f, g \in X_{\rho} \) and \( u = f * g \). In view of Lemma 2.3, there exist sequences \( (f_n)_{n \in \mathbb{N}} \) and \( (g_n)_{n \in \mathbb{N}} \) in \( Y^+ \oplus Y^- \) converging in \( X_{\rho} \) to \( f \) and \( g \), respectively. It follows from (2.11) that the sequence \( (f_n * g_n)_{n \in \mathbb{N}} \) is Cauchy in \( X_{\rho} \) and
\[
\|f_n * g_n\|_{X_{\rho}} \leq 4c \|f_n\|_{X_{\rho}} \|g_n\|_{X_{\rho}}, \quad n \in \mathbb{N}.
\]

Since the space \( X_{\rho} \) is complete and continuously embedded in \( L_1(\mathbb{R}) \), we conclude that the sequence \( (f_n * g_n)_{n \in \mathbb{N}} \) converges in \( X_{\rho} \) to some \( u \in X_{\rho} \). Thus, letting \( n \to \infty \), we get that \( \|f * g\|_{X_{\rho}} \leq 4c \|f\|_{X_{\rho}} \|g\|_{X_{\rho}} \), and the proof is complete. \( \square \)
3. PROPERTIES OF THE SPACES $\mathcal{B}_\rho$

Let us consider the classical Wiener algebra (see, e.g., [7, 8]), i.e., the commutative Banach algebra

\[ \mathcal{A} := \{ \alpha 1 + \varphi \mid \alpha \in \mathbb{C}, \varphi \in L_1(\mathbb{R}) \} \]

with the norm

\[ \| \alpha 1 + \varphi \|_{\mathcal{A}} := |\alpha| + \| \varphi \|_1. \]

The multiplication in $\mathcal{A}$ is the usual pointwise multiplication and

\[ \| fg \|_{\mathcal{A}} \leq \| f \|_{\mathcal{A}} \| g \|_{\mathcal{A}}, \quad f, g \in \mathcal{A}. \]

It is known that every function $f \in \mathcal{A}$ is continuous on $\mathbb{R} \cup \{ \infty \}$.

In the algebra $\mathcal{A}$, we consider the closed subalgebras

\[ \mathcal{A}^+ := \{ f = \alpha 1 + \hat{h} \mid \alpha \in \mathbb{C}, h \in L_1(\mathbb{R}), h\big|_{\mathbb{R}_-} = 0 \}, \]
\[ \mathcal{A}_0 := \{ f = \hat{h} \mid h \in L_1(\mathbb{R}) \}, \quad \mathcal{A}_0^+ := \mathcal{A}_0 \cap \mathcal{A}^+. \]

**Remark 3.1.** Each function $\varphi \in \mathcal{A}^+$ is the restriction onto $\mathbb{R}$ of a function $\Phi$ which is analytic in the upper half-plane $\mathbb{C}_+$ and continuous in $\mathbb{C}_+ \cup \{ \infty \}$. We will identify the functions $\varphi$ and $\Phi$.

The following statement follows from the well known results of Wiener (see, e.g., [2], Chapter VIII, 6) and is an analogue of classical Wiener’s lemma.

**Lemma 3.2** (Wiener). An element $f \in \mathcal{A}$ ($f \in \mathcal{A}^+$) is invertible in the algebra $\mathcal{A}$ (resp., in $\mathcal{A}^+$) if and only if $f$ does not vanish on $\mathbb{R} \cup \{ \infty \}$ (resp., in $\mathbb{C}_+ \cup \{ \infty \}$).

**Remark 3.3.** Since $\hat{X}_\rho$ and $X_\rho$ are isometric, then $\hat{X}_\rho$ and $\mathcal{B}_\rho$ are Banach spaces. It follows from (2.3) that the space $\hat{X}_\rho$ is continuously embedded in $\mathcal{A}_0$. Thus the algebra $\mathcal{B}_\rho$ is continuously embedded in $\mathcal{A}$.

**Proof of Theorem 1.2.** Let $\rho \in \mathcal{B}$ and $f, g \in X_\rho$. In view of Proposition 2.4, the convolution $f \ast g$ belongs to $X_\rho$. Since $\hat{f} \ast \hat{g} = \hat{f \ast g}$, the product $\hat{f} \hat{g}$ belongs to $\hat{X}_\rho$. Thus $\hat{X}_\rho$ is a complex algebra. By the definition of $\mathcal{B}_\rho$,

\[ \mathcal{B}_\rho = \hat{X}_\rho + \{ \alpha 1 \mid \alpha \in \mathbb{C} \}. \]

Hence $\mathcal{B}_\rho$ is a complex algebra with unit $1$.

Let $c$ be the constant from Definition 1.1. Obviously, the formula

\[ \| \alpha 1 + \varphi \|_{\mathcal{B}_\rho,c} := |\alpha| + 4c\| \varphi \|_{X_\rho}, \quad \alpha \in \mathbb{C}, \varphi \in X_\rho, \quad (3.1) \]

defines a norm on $\mathcal{B}_\rho$ which is equivalent to the norm (1.2). We now show that $\mathcal{B}_\rho$ with the norm $\| \cdot \|_{\mathcal{B}_\rho,c}$ is a Banach algebra with unit. Clearly, it suffices to prove that the norm $\| \cdot \|_{\mathcal{B}_\rho,c}$ satisfies the multiplicative inequality. Let $f = \alpha 1 + \varphi$ and $g = \beta 1 + \psi$, where $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in X_\rho$. Then

\[ \| fg \|_{\mathcal{B}_\rho,c} \leq |\alpha| |\beta| + |\beta| \| \varphi \|_{\mathcal{B}_\rho,c} + |\alpha| \| \psi \|_{\mathcal{B}_\rho,c} + \| \varphi \psi \|_{\mathcal{B}_\rho,c}. \]
It follows from the inequality (2.6) that
\[ \| \hat{\varphi} \|_{\rho,c} = 4c\| \varphi \|_{X_{\rho}} \leq 16c^2\| \varphi \|_{X_{\rho}}\| \psi \|_{X_{\rho}} \leq \| \hat{\varphi} \|_{\rho,c} \| \hat{\psi} \|_{\rho,c}. \]

Thus
\[ \| fg \|_{\rho,c} \leq (|\alpha| + \| \hat{\varphi} \|_{\rho,c})(|\beta| + \| \hat{\psi} \|_{\rho,c}) = \| f \|_{\rho,c} \| g \|_{\rho,c} \]
as claimed. \(\square\)

In the algebra \(B_{\rho}\), we consider the closed subalgebras \(B_{\rho}^+ := B_{\rho} \cap A^+\).

**Lemma 3.4.**

(i) Let \(\rho \in \mathcal{R}\) and \(b\) be a rational function that has only simple zeros and does not vanish on \(\mathbb{R} \cup \{ \infty \}\). Then \(1/b \in B_{\rho}\).

(ii) Let \(\rho \in \mathcal{R}\) and \(u \in Y^+\) and, moreover, assume that the function \(g = 1 + \hat{u}\) does not vanish in \(\mathbb{C}_+ \cup \{ \infty \}\). Then \(1/g \in B_{\rho}^+\).

**Proof.** Let the conditions of (i) be satisfied. Then
\[ \frac{1}{b(\lambda)} = c_0 + \sum_{j=1}^{n} \frac{c_j}{\lambda + \alpha_j}, \quad \lambda \in \mathbb{R}, \]
where \(\{c_j\}_{j=0}^{n} \subset \mathbb{C}\) and \(\{\alpha_j\}_{j=1}^{n} \subset \mathbb{C} \setminus \mathbb{R}\). Thus, it suffices to show that the functions \(f_\alpha(\lambda) = (\lambda + \alpha)^{-1}\) with \(\alpha \in \mathbb{C}_+\) belong to \(B_{\rho}^+\). Note that \(f_\alpha\) is the Fourier transform of the function \(u_\alpha(x) := -ie^{i\alpha x}\chi_+(x)\). Since \(\lim_{x \to +\infty} \rho(x)e^{-\gamma x} = 0\) for \(\gamma > 0\), then \(f_\alpha \in \hat{X}_{\rho}\).

Let the conditions of (ii) be satisfied. We consider the function \(v(x) := iu(x) + iu'(x) (x \neq 0)\). This function belongs to \(L_2(\mathbb{R})\), has compact support and
\[ \hat{v}(\lambda) = i\hat{u}(\lambda) + i \int_{\mathbb{R}} e^{i\lambda x}u'(x)\,dx = (\lambda + i)\hat{u}(\lambda) - i(u(+0) - u(-0)). \]

Thus
\[ \hat{u}(\lambda) = \frac{i(u(+0) - u(-0))}{\lambda + i} + \frac{\hat{v}(\lambda)}{\lambda + i}, \quad \lambda \in \mathbb{C}. \]

Using this fact, we conclude that
\[ \hat{u}(\lambda) = o(\lambda^{-1}), \quad \lambda \to \infty, \]
uniformly in each strip \(\{z \in \mathbb{C} \mid |\text{Im}z| < \gamma\} \) (\(\gamma > 0\)). Thus
\[ \frac{1}{g(\lambda)} = 1 - \hat{u}(\lambda) + \frac{\hat{u}(\lambda)^2}{1 + \hat{u}(\lambda)} = 1 - \hat{u}(\lambda) + h(\lambda), \]
where the function \(h\) is analytic in some half-plane \(\{z \in \mathbb{C} \mid \text{Im}z > -\delta\} \) (\(\delta > 0\)) and
\[ \sup_{|y| < \delta} \int_{\mathbb{R}} |(x + iy)h(x + iy)|^2\,dx < \infty. \]
Therefore, it suffices to show that \( h \in \hat{X}^+ \). It follows from (3.2) that \( h = w \), where \( w \) belongs to the Sobolev space \( W^2_2(\mathbb{R}) \). From known properties of the Fourier transform (see, e.g., [6, Chapter 5]), we obtain that

\[
2\pi \int_{\mathbb{R}} e^{-2\nu \xi} |w'(|xi)|^2 \, d\xi = \int_{\mathbb{R}} |(x + iy)h(x + iy)|^2 \, dx, \quad y \in (-\delta, \delta).
\]

Using this fact and (3.2), we get that

\[
J(y) := \int_{\mathbb{R}} e^{2\nu \xi} |w'(|xi)|^2 \, d\xi < \infty, \quad y \in (0, \delta).
\]

Using the Cauchy–Schwarz inequality, we derive that

\[
\left( \int_{\mathbb{R}} e^{y \xi} |w'(|xi)| \, d\xi \right)^2 \leq J(u) \int_{\mathbb{R}} e^{2(y-u)\xi} \, d\xi < \infty, \quad 0 < y < u < \delta.
\]

Since \( \lim_{x \to +\infty} \rho(x)e^{-yx} = 0 \) for \( y > 0 \), we conclude that \( w \in X^+_\rho \), and hence \( h \in \hat{X}^+ \).

The proof is complete. \( \square \)

**Lemma 3.5.** Let \( \rho \in \mathcal{R} \), \( c = c(\rho) \), \( u \in Y^+ \) and \( \|u\|_1 \leq 1/4c \). Then the function \( g = 1 + \hat{u} \) is invertible in the algebra \( \mathcal{B}^+_\rho \) and, moreover, (see (3.1))

\[
\|1/g\|_{\rho,c} \leq 4\|g\|_{\rho,c}.
\]

**Proof.** Since \( c \geq 1 \), we conclude that the element \( g = 1 + \hat{u} \) is invertible in the algebra \( \mathcal{A}^+ \) and, moreover, \( 1/g = 1 + \hat{v} \), where \( v \in L_1(\mathbb{R}) \) and

\[
\|v\|_1 = \|1/g - 1\|_A \leq \sum_{n=1}^{\infty} \|\hat{u}\|_A^n = \frac{\|\hat{u}\|_A}{1 - \|\hat{u}\|_A} = \frac{\|u\|_1}{1 - \|u\|_1} \leq \frac{1}{2c}.
\]  

(3.3)

In view of the Wiener Lemma and Lemma 3.4, we obtain that \( v \in X^+_\rho \). Since \( (1 + \hat{u})(1 + \hat{v}) = 1 \), we have that \( u + v + u \ast v = 0 \). Taking into account that \( u \in Y^+ \) and \( v \in X^+_\rho \), we get the equality

\[
u(x) + v(x) + \int_0^x u(x-t)v(t) \, dt = 0, \quad x > 0,
\]

from which we can easily see that \( v \in C^1[0, \infty) \). We represent the convolution \( u \ast v \) in the form \( u \ast v = w_1 + w_2 \), where (see Remark 2.1)

\[
w_1(x) := \int_0^{x/2} u(x-t)v(t) \, dt, \quad w_2(x) := \int_0^{x/2} v(x-t)u(t) \, dt, \quad x \geq 0,
\]
and \( w_1(x) = w_2(x) = 0 \) for \( x < 0 \). It is clear that \( w_1, w_2 \in C^1[0, \infty) \) and

\[
\begin{align*}
w_1'(x) &= \frac{1}{2} u(x/2) v(x/2) + \int_0^{x/2} u'(x-t)v(t) \, dt, \quad x > 0, \\
w_2'(x) &= \frac{1}{2} u(x/2) v(x/2) + \int_0^{x/2} v'(x-t)u(t) \, dt, \quad x > 0.
\end{align*}
\]

Let us estimate the norm \( \|w_1\|_{X_\rho} \). Taking into account the inequality (2.7), we have that for an arbitrary \( x > 0 \),

\[
\rho(x)|w_1'(x)| \leq \frac{c}{2} \rho(x/2)u(x/2)||v(x/2)|| + c \int_0^{x/2} \rho(x-t)|u'(x-t)||v(t)|| \, dt.
\]

Thus, using (2.4) and (3.3), we get

\[
\|w_1\|_{X_\rho} \leq c\|u\|_{X_\rho}\|v\|_1 + c\|u\|_{X_\rho}\|v\|_1 \leq 2c\|u\|_{X_\rho}\|v\|_1 \leq \|u\|_{X_\rho}.
\]  

(3.4)

Similarly, we obtain that

\[
\|w_2\|_{X_\rho} \leq 2c\|v\|_{X_\rho}\|u\|_1 \leq \frac{1}{2}\|v\|_{X_\rho}.
\]  

(3.5)

It is easily seen that \( \|v\|_{X_\rho} \leq \|u\|_{X_\rho} + \|w_1\|_{X_\rho} + \|w_2\|_{X_\rho} \). Taking into account (3.4) and (3.5), we obtain that \( \|v\|_{X_\rho} \leq 4\|u\|_{X_\rho}, \) so that

\[
\|1/g\|_{\rho,c} = 1 + 4c\|v\|_{X_\rho} \leq 4(1 + 4c\|u\|_{X_\rho}) = 4\|g\|_{\rho,c}
\]

as claimed. \( \square \)

The main result of this section is following analogue of the Wiener Lemma.

**Theorem 3.6.** Let \( \rho \in \mathcal{R} \). Then \( g \in B^+_\rho \) is invertible in the Banach algebra \( B^+_\rho \) if and only if \( g \) does not vanish in \( \overline{C_+} \cup \{\infty\} \).

**Proof.** Let \( g \) be invertible in the algebra \( B^+_\rho \). Since \( B^+_\rho \subset A^+ \), the element \( g \) is invertible in the algebra \( A^+ \). Thus, in view of Wiener Lemma, \( g \) does not vanish in \( \overline{C_+} \cup \{\infty\} \).

Conversely, let \( g \in B^+_\rho \) not vanish in \( \overline{C_+} \cup \{\infty\} \). From Wiener Lemma, we can conclude that \( 1/g \in A^+ \). Let us show that \( 1/g \in B^+_\rho \). Without loss of generality, we can assume that \( g = 1 + \hat{u} \), where \( u \in X^+_\rho \).

First, we consider the case \( \|u\|_1 \leq 1/4c \). By Lemma 2.3, there exists a sequence \((u_n)_{n \in \mathbb{N}}\) in \( Y^+ \) converging to \( u \) in \( X^+_\rho \). Since the space \( X_\rho \) is continuously embedded in \( L_1(\mathbb{R}) \), we can assume that \( \|u_n\|_1 \leq 1/4c \) for all \( n \in \mathbb{N} \). Let \( g_n := 1 + \hat{u}_n \), \( n \in \mathbb{N} \). Then the sequence \((g_n)_{n \in \mathbb{N}}\) converges to \( g \) in \( B^+_\rho \) and, in view of Lemma 3.5,

\[
1/g_n \in B^+_\rho, \quad \|1/g_n\|_{\rho,c} \leq 4\|g_n\|_{\rho,c}, \quad n \in \mathbb{N}.
\]
Since the sequence \((1/g_n)_{n \in \mathbb{N}}\) is bounded in \(B^+_\rho\), we conclude (see, e.g., [5, Chapter 10]) that \(1/g \in B^+_\rho\).

Now we consider the general case when \(g = 1 + \hat{u}, u \in X^+_\rho\) and \(g\) does not vanish in \(\overline{\mathbb{C}^+} \cup \{\infty\}\). By Lemma 2.3, there exists a sequence \((u_n)_{n \in \mathbb{N}}\) in \(Y^+\) converging to \(u\) in \(X^+_\rho\). Since \(X^\rho\) is continuously embedded in \(L_1(\mathbb{R})\), we can assume that all functions \(g_n := 1 + \hat{u}_n\) (\(n \in \mathbb{N}\)) do not vanish in \(\overline{\mathbb{C}^+} \cup \{\infty\}\), so that (see Lemma 3.4) \(1/g_n \in B^+_\rho\) for all \(n\). Hence (see Theorem 1.2) the sequence \(f_n := g/g_n\) \((n \in \mathbb{N}\) belongs to the space \(B^+_\rho\) and, obviously, converges to \(1\) in the space \(A^+\). Using this fact, we conclude that \(f_n = 1 + \hat{v}_n\), where the sequence \((v_n)_{n \in \mathbb{N}}\) belongs to \(X^+_\rho\) and converges to zero in \(L_1(\mathbb{R})\). Thus (see Lemma 3.5) \(1/f_n \in B^+_\rho\) for sufficiently large \(n\). Let \(1/f_m \in B^+_\rho\) for some \(m \in \mathbb{N}\). Since \(1/g = 1/g_m \cdot 1/f_m\), in view of Theorem 1.2, we arrive at the conclusion that \(1/g \in B^+_\rho\) and the proof is complete. \(\square\)

4. PROOF OF THEOREM 1.3.

First, we prove two auxiliary Lemmas that are generalizations of the similar Lemmas in [3, Chapter 3].

**Lemma 4.1.** Let \(\rho \in \mathcal{R}_0\) and \(\varphi \in L_r(\mathbb{R}^+)(r \in [1, \infty])\). If a function \(\psi \in X^+_\rho\) is such that the function \(g\) is given by

\[
g(x) := \varphi(x) + \int_0^\infty \varphi(t)\psi(x + t)\,dt, \quad x \in \mathbb{R}^+, \tag{4.1}
\]

belongs to the space \(X^+_\rho\), then \(\varphi \in X^+_\rho\).

**Proof.** Let \(g, \psi \in X^+_\rho\). Since \(X^+_\rho \subset L_1(\mathbb{R}^+)\), then (see [4], Lemma 3.1) \(\varphi \in L_1(\mathbb{R}^+)\). Taking into account the equalities

\[
g(x) = -\int_x^\infty g'(\xi)\,d\xi, \quad \psi(x) = -\int_x^\infty \psi'(\xi)\,d\xi, \quad x \in \mathbb{R}^+,
\]

(4.1) can be represented as

\[
\varphi(x) = -\int_x^\infty g'(\xi)\,d\xi + \int_0^\infty \varphi(t)\int_x^\infty \psi'(\xi + t)\,d\xi\,dt. \tag{4.2}
\]

Since

\[
\int_0^\infty \int_0^\infty |\psi'(\xi + t)|\,d\xi\,dt = \int_0^\infty t|\psi'(t)|\,dt \leq \|\psi\|_{X^+_\rho},
\]

applying Fubini’s theorem to the iterated integral in (4.2), we get

\[
\varphi(x) = -\int_x^\infty \left(g'(\xi) - \int_0^\infty \varphi(t)\psi'(\xi + t)\,dt\right)\,d\xi, \quad x \in \mathbb{R}^+.
\]
Consequently, the function $\varphi$ belongs to $\text{AC}(\mathbb{R}_+)$ and

$$
\varphi'(x) = g'(x) - \int_0^\infty \varphi(t)\psi'(x + t) \, dt, \quad x \in \mathbb{R}_+.
$$

Thus

$$
\int_0^\infty \rho(x)|\varphi'(x)| \, dx \leq \int_0^\infty \rho(x)|g'(x)| \, dx + \int_0^\infty \int_0^\infty |\varphi(t)||\rho(x + t)\psi'(x + t)| \, dt \, dx,
$$

and, therefore, $\|\varphi\|_{X_\rho^+} \leq \|g\|_{X_\rho^+} + \|\varphi\|_1\|\psi\|_{X_\rho^+} < \infty$. \hfill \Box

**Lemma 4.2.** Let $\rho \in \mathcal{R}_0$ and $\varphi \in L_1(\mathbb{R}_+)$ and $\psi \in X_\rho^+$ be related via

$$
\varphi(x) + \psi(x) + \int_0^\infty \varphi(t)\psi(x + t) \, dt = 0, \quad x \in \mathbb{R}_+.
$$

(4.3)

If the function $f$ is given by the formula

$$
f(\lambda) = 1 + \int_0^\infty \varphi(t)e^{i\lambda t} \, dt, \quad \lambda \in \mathbb{R},
$$

and $f(0) = 0$, then there exists $g \in B_\rho^+$ such that $f(\lambda) = \frac{\lambda}{\lambda + 1} g(\lambda)$.

**Proof.** Let the conditions of the lemma be satisfied. From Lemma 4.1, it follows that $\varphi \in X_\rho^+$ and thus $f \in B_\rho^+$. Let us show that the function

$$
h(x) := \int_x^\infty \varphi(t) \, dt, \quad x \in \mathbb{R}_+,
$$

belongs to $X_\rho^+$. Note that it follows from the condition $f(0) = 0$ that $h(0) = -1$. Consider the auxiliary function

$$
\Phi(x) := \int_0^x h'(t) \int_{x+t}^\infty \psi(\xi) \, d\xi \, dt, \quad x \geq 0.
$$

(4.4)

Integrating by parts, we obtain that

$$
\Phi(x) = \int_x^\infty \psi(\xi) \, d\xi + \int_0^\infty h(t)\psi(x + t) \, dt.
$$

(4.5)
On the other hand, it follows from (4.4) that
\[ \Phi(x) = -\int_0^\infty \varphi(t) \int_x^\infty \psi(y+t) \, dy \, dt = -\int_x^\infty \varphi(t) \psi(y+t) \, dt \, dy. \tag{4.6} \]

Taking into account (4.3), (4.5) and (4.6), we get
\[ \int_x^\infty \psi(\xi) \, d\xi + \int_0^\infty h(t) \psi(x+t) \, dt = \int_x^\infty (\varphi(y) + \psi(y)) \, dy \]
and, therefore,
\[ h(x) + \int_0^\infty h(t)(-\psi(x+t)) \, dt = 0, \quad x \in \mathbb{R}_+. \]

Since \( h \in L_\infty(\mathbb{R}_+) \) and \( -\psi \in X_\rho^+ \), in view of Lemma 4.1, we conclude that \( h \in X_\rho^+ \). Consequently, the function
\[ g_1(\lambda) := i \int_0^\infty h(t) e^{i\lambda t} \, dt, \quad \lambda \in \mathbb{R}, \]
belongs to \( B_\rho^+ \). Integrating by parts, we get
\[ \lambda g_1(\lambda) = \int_0^\infty h(t) \left( \frac{d}{dt} e^{i\lambda t} \right) = -h(0) + \int_0^\infty \varphi(t) e^{i\lambda t} \, dt = f(\lambda). \]

Let \( g(\lambda) := f(\lambda) + ig_1(\lambda) \). Since \( g_1, f \in B_\rho^+ \), we deduce that \( g \in B_\rho^+ \). Moreover, \( \lambda(\lambda + i)^{-1} g(\lambda) = \lambda g_1(\lambda) = f(\lambda) \).

Below, we list some facts from [3, Chapter 3]. Let \( q \in \mathcal{Q}_\omega \) and
\[ \sigma(x) := \int_x^\infty |q(\xi)| \, d\xi, \quad \sigma_1(x) := \int_x^\infty \xi|q(\xi)| \, d\xi. \]

1°. The solution of the Jost equation (1.1) can be represented in the form
\[ e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x, t) e^{i\lambda t} \, dt, \quad \lambda \in \mathbb{C}_+, \quad x \in \mathbb{R}_+, \]
where the kernel \( K \) is continuous on the set \( \Omega := \{(x, t) \in \mathbb{R}_+^2 \mid x \leq t\} \) and
\[ |K(x, t)| \leq \sigma \left( \frac{x + t}{2} \right) \exp\{\sigma_1(x)\}, \quad (x, t) \in \Omega. \]
2°. For $\lambda \in \mathbb{R} \setminus \{0\}$, the estimate for the derivative of the Jost solution
\[ |e'(\lambda, x) - i\lambda e^{i\lambda x}| \leq \sigma(x) \exp\{\sigma_1(x)\}, \quad x \in \mathbb{R}_+, \tag{4.7} \]
holds, and the formula
\[ \omega(\lambda, x) := \frac{e(-\lambda, 0)e(\lambda, x) - e(\lambda, 0)e(-\lambda, x)}{2i\lambda}, \quad x \in \mathbb{R}_+, \tag{4.8} \]
defines a solution of the equation (1.1) satisfying
\[ \omega(\lambda, x) = x(1 + o(1)), \quad \omega'(\lambda, x) = 1 + o(1), \quad x \to +0. \tag{4.9} \]

3°. The function $C_+ \setminus \{0\} \ni \lambda \mapsto e(\lambda)$ has a finite number of zeros which are simple and lie on the imaginary line.

4°. The kernel $K$ is a solution of the Marchenko equation
\[ F(x + t) + K(x, t) + \int_x^\infty K(x, \xi)F(\xi + t)\,d\xi = 0, \quad (x, t) \in \Omega, \tag{4.10} \]
with $F$ given by
\[ F(x) := \sum_{s=1}^n m_se^{-\kappa_s x} + F_S(x), \quad x \geq 0, \tag{4.11} \]
where
\[ F_S(x) := \frac{1}{2\pi} \int_\mathbb{R} (1 - S(\lambda))e^{i\lambda x}\,d\lambda, \quad x \in \mathbb{R}. \tag{4.12} \]

5°. The function $F$ belongs to the class $AC(\mathbb{R}_+)$ and there exists a constant $C_1 > 0$ such that
\[ |F'(2x) - q(x)/4| \leq C_1 \sigma^2(x), \quad x > 0. \tag{4.13} \]

**Lemma 4.3.** Let $q \in Q_\omega$ and the function $F$ be given by formula (4.11). Then for each $\rho \in \mathcal{R}$ the function $q$ belongs to the class $Q_\rho$ if and only if $F \in X^+_\rho$.

**Proof.** 1° Let $\rho \in \mathcal{R}$ and $q \in Q_\rho$. Then for an arbitrary $\gamma \geq 0$,
\[ \rho(x)\sigma(x) \leq \int_x^\infty \rho(t)|q(t)|\,dt \leq \int_\gamma^\infty \rho(t)|q(t)|\,dt, \quad x \geq \gamma, \]
and
\[ \int_\gamma^\infty \sigma(x)\,dx = \int_\gamma^\infty \int_x^\infty |q(t)|\,dt\,dx \leq \int_\gamma^\infty t|q(t)|\,dt < \infty. \]
Thus
\[ \int_{\gamma}^{\infty} \rho(x) \sigma^2(x) \, dx \leq \left( \int_{\gamma}^{\infty} \rho(t) |q(t)| \, dt \right) \left( \int_{\gamma}^{\infty} \sigma(x) \, dx \right) \]
\[ \leq \left( \int_{\gamma}^{\infty} \rho(t) |q(t)| \, dt \right) \left( \int_{\gamma}^{\infty} t |q(t)| \, dt \right) < \infty. \quad (4.14) \]

It follows from (4.13) that
\[ |F'(2x)| \leq |q(x)| + C_1 \sigma^2(x), \quad x > 0. \]

Using this estimate and (2.7), we get
\[ \int_{0}^{\infty} \rho(2x) |F'(2x)| \, dx \leq c \int_{0}^{\infty} \rho(x) |F'(2x)| \, dx \]
\[ \leq c \int_{0}^{\infty} \rho(x) |q(x)| \, dx + cC_1 \int_{0}^{\infty} \rho(x) \sigma^2(x) \, dx < \infty, \]
and hence \( F \in X^+_{\rho} \) as claimed.

2) Let \( q \in Q_{\omega} \) and \( F \in X^+_{\rho} \). It follows from (4.13) that
\[ |q(x)| \leq 4|F'(2x)| + 4C_1 \sigma^2(x), \quad x > 0. \quad (4.15) \]

Let us fix \( \gamma > 0 \) for which
\[ \int_{\gamma}^{\infty} t |q(t)| \, dt \leq \frac{1}{8C_1}, \quad (4.16) \]
and put
\[ \rho_n(x) := \min\{\rho(x), n + x\}, \quad x \geq 0, \quad n \in \mathbb{N}. \]

Obviously, that \( \rho_n \in \mathcal{R} \). Using the estimate (4.15), we obtain that for an arbitrary \( n \in \mathbb{N} \),
\[ \int_{\gamma}^{\infty} \rho_n(x) |q(x)| \, dx \leq 4 \int_{\gamma}^{\infty} \rho_n(2x) |F'(2x)| \, dx + 4C_1 \int_{\gamma}^{\infty} \rho_n(x) \sigma^2(x) \, dx. \quad (4.17) \]

From (4.14) and (4.16), we deduce that
\[ 4C_1 \int_{\gamma}^{\infty} \rho_n(x) \sigma^2(x) \, dx \leq 4C_1 \int_{\gamma}^{\infty} \xi |q(\xi)| \, d\xi \int_{\gamma}^{\infty} \rho_n(t) |q(t)| \, dt \leq \frac{1}{2} \int_{\gamma}^{\infty} \rho_n(t) |q(t)| \, dt. \]
Thus, in view of (4.17), we get
\[ \int_\gamma \rho_n(x)|q(x)| \, dx \leq 8 \int_\gamma \rho_n(2x)|F'(2x)| \, dx \leq 4 \int_0^\infty \rho(x)|F'(x)| \, dx. \]

Using the monotone convergence theorem, we have
\[ \int_\gamma \rho(x)|q(x)| \, dx \leq 4 \int_0^\infty \rho(x)|F'(x)| \, dx < \infty, \]
and hence \( q \in Q_\rho \).

**Proof of Theorem 1.3.** First, we prove sufficiency. Let \( \rho \in \mathcal{R}, S \in \mathcal{S}_\rho \) and \( n := [-\text{ind} \, S/2] \). Since \( \mathcal{S}_\rho \subset \mathcal{S}_\omega \), in view of the results of [4], we conclude that \( S \) is the scattering function for some operator \( T_q \) with \( q \in \mathcal{Q}_\omega \). Since \( S \in \mathcal{S}_\rho \), the function \( F_S \) (see (4.12)) belongs to the space \( X_\rho \). Therefore, the function \( F \), given by the formula (4.11), belongs to the space \( X_\rho^+ \). In view of Lemma 4.3, we have that \( q \in \mathcal{Q}_\rho \) so that every function \( S \in \mathcal{S}_\rho \) is the scattering function of some operator \( T_q \) with \( q \in \mathcal{Q}_\rho \) as claimed.

Let us prove necessity. Let \( q \in \mathcal{Q}_\rho \). We need to prove that \( S_q \in \mathcal{S}_\rho \). Since \( q \in \mathcal{Q}_\rho \), in view of Lemma 4.3, we conclude that \( F \in X_\rho^+ \). It follows from the Marchenko equation (4.10) that
\[ F(t) + K(0,t) + \int_0^\infty K(0,\xi)F(\xi + t) \, d\xi = 0, \quad t > 0. \]

Thus in view of Lemma 4.1 the function \( \mathbb{R}_+ \ni t \mapsto K(0,t) \) belongs to the space \( X_\rho^+ \) and, therefore, the Jost function
\[ e(\lambda) = 1 + \int_0^\infty K(0,t)e^{\lambda t} \, dt, \quad \lambda \in \mathbb{C}_+ \]
belongs to the space \( \mathcal{B}_\rho^+ \).

1) Suppose that \( e(0) \neq 0 \). Then, in view of 3\(^2\), the function \( e \) has a finite number of zeros in \( \overline{\mathbb{C}_+} \cup \{\infty\} \). All these zeros are simple and can be represented as \( z = i\kappa_j \), where \( \{\kappa_j\}_{j=1}^n \subset \mathbb{R}_+ \). Let us consider the Blaschke product
\[ b(\lambda) = \prod_{j=1}^n \frac{\lambda - i\kappa_j}{\lambda + i\kappa_j} \quad \text{(4.18)} \]
and the functions
\[ f(\lambda) := \frac{e(-\lambda)}{b(\lambda)}, \quad g(\lambda) := \frac{e(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{R}. \]
It follows from Lemma 3.4 and Theorem 1.2 that \( f, g \in B_\rho \). Obviously, \( g \in A^+ \), and thus \( g \in B_\rho^+ \). Moreover, the function \( g \) does not vanish in \( \overline{C_+} \cup \{\infty\} \). Therefore, in view of Theorem 3.6, we obtain that \( 1/g \in B_\rho^+ \). Since \( S = f/g \) and \( B_\rho \) is an algebra, we deduce that \( S \in B_\rho \).

2) Suppose that \( e(0) = 0 \). Taking into account (4.10) and Lemma 4.2, we get that
\[
e(\lambda) = \frac{\lambda}{\lambda^2 + b} h(\lambda),
\]
where \( h \in B_\rho^+ \). Let us show that \( h(0) \neq 0 \). It follows from (4.7) that there exists \( C > 0 \) such that \( |e'(\lambda, x)| \leq C \) for \( x \in \mathbb{R}_+ \) and \( \lambda \in [-1, 1] \setminus \{0\} \). Thus (see (4.8))
\[
|\omega'(\lambda, x)| \leq C(|h(-\lambda)| + |h(\lambda)|), \quad x \in \mathbb{R}_+, \quad \lambda \in [-1, 1] \setminus \{0\}.
\]
Therefore, taking into account (4.9), we have
\[
1 = \lim_{x \to +0} |\omega'(\lambda, x)| \leq C(|h(-\lambda)| + |h(\lambda)|), \quad \lambda \in [-1, 1] \setminus \{0\}.
\]
Since the function \( h \) is continuous, we obtain that \( h(0) \neq 0 \). In view of 3°, the function \( h \) has a finite number of zeros in \( \overline{C_+} \cup \{\infty\} \). All these zeros are simple and can be represented as \( z = i\kappa_j \), where \( \{\kappa_j\}_{j=1}^n \subset \mathbb{R}_+ \). Let us consider the functions
\[
f(\lambda) := \frac{\lambda + i}{\lambda - i} \frac{h(-\lambda)}{b(\lambda)}, \quad g(\lambda) := \frac{h(\lambda)}{b(\lambda)}, \quad \lambda \in \mathbb{R},
\]
where \( b \) is the Blaschke product given by the formula (4.18). It follows from Lemma 3.4 and Theorem 1.2 that \( f, g \in B_\rho \). Obviously, \( g \in B_\rho^+ \) and the function \( g \) does not vanish in \( \overline{C_+} \cup \{\infty\} \). It follows from Theorem 3.6 that \( 1/g \in B_\rho \). Since \( S = f/g \) and \( B_\rho \) is an algebra, we arrive at the conclusion that \( S \in B_\rho \). Therefore, the proof is complete.

\[\square\]

APPENDIX

A. OPERATOR \( T_q \)

In this appendix, we will give the explicit definition of the operator \( T_q \).

We denote by \( C_0^\infty \) the linear space of all functions on the half-line with compact support that are infinitely often differentiable. Also we denote by \( W_2^1 \) the Sobolev space of functions \( f \in AC[0, \infty) \) for which
\[
||f||_{W_2^1}^2 := \int_0^\infty (|f(x)|^2 + |f'(x)|^2) \, dx < \infty.
\]

Let \( q \) be a locally integrable real-valued function on \( \mathbb{R}_+ \) and
\[
\int_0^\infty |xq(x)| \, dx < \infty. \tag{A.1}
\]
We consider the symmetric sesquilinear forms $t_0$ and $q$ that are defined on the common domain $W_{2,0}^1 := \{ f \in W_2^1 \mid f(0) = 0 \}$ by the formulas

$$t_0[f, g] := \int_0^\infty f'(x) \overline{g'(x)} \, dx, \quad q[f, g] := \int_0^\infty q(x) f(x) \overline{g(x)} \, dx.$$ 

Note that the form $t_0$ is nonnegative and closed (see [1], Ch.VI.§1.3). We will show that the form $q$ is $t_0$-bounded (see [1], Ch.VI.§1.6). We represent the function $q$ (see (A.1)) as the sum $q_1 + q_2$, where $q_1 \in C_0^\infty$ and $q_2$ satisfies the following condition:

$$\int_0^\infty x|q_2(x)| \, dx \leq b < 1.$$

Using the Cauchy–Schwarz inequality, we get that for $f \in W_{2,0}^1$

$$|f(x)|^2 = \left| \int_0^x f'(t) \, dt \right|^2 \leq x \int_0^x |f'(t)|^2 \, dt \leq x t_0[f], \quad x \in \mathbb{R}_+,$$

where $t_0[f] := t_0[f, f]$. Thus for all $f \in W_{2,0}^1$

$$|q[f]| \leq \int_0^\infty |q_1(x)||f(x)|^2 \, dx + \int_0^\infty |q_2(x)||f(x)|^2 \, dx \leq a \|f\|^2 + b t_0[f],$$

where $a := \max |q_1(x)|$. Consequently, the form $q$ is $t_0$-bounded with $b < 1$. Therefore (see [1, Chapter VI, §1.6]), the symmetric form $t = t_0 + s$ is bounded from below and closed. By the first representation theorem (see [1, Chapter VI, §2.1]), there exists the unique self-adjoint operator $T_q$ that is associated with $t$. Its domain consists of functions $f \in W_{2,0}^1$ for which there exists $h \in L_2(\mathbb{R}_+)$ such that

$$t[f, g] = (h \mid g), \quad g \in W_{2,0}^1. \quad (A.2)$$

If (A.2) holds, then $T_q f = h$. Let $f \in \text{dom} \, T_q$. Then for some $h \in L_2(\mathbb{R}_+)$

$$(f' \mid g') = (h - qf \mid g), \quad g \in C_0^\infty.$$ 

Thus we have that $-f'' = h - qf$ in the sense of distribution theory. It means that $f' \in \text{AC}(0, \infty)$ and $(-f'' + qf) = h \in L_2(0, \infty)$. Therefore,

$$\text{dom} \, T_q := \{ f \in W_{2,0}^1 \mid f' \in \text{AC}(0, \infty), \ ( -f'' + qf) \in L_2(\mathbb{R}_+) \}$$

and

$$T_q f := -f'' + qf, \quad f \in \text{dom} \, T_q.$$
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