Multipliers and Unicentral Leibniz Algebras

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Abstract

This paper details the Leibniz generalization of Lie-theoretic results from Peggy Batten’s 1993 dissertation. We first show that the multiplier of a Leibniz algebra is characterized by its second cohomology group with coefficients in the field. We then establish criteria for when the center of a cover maps onto the center of the algebra. Along the way, we obtain a collection of exact sequences and a brief theory of unicentral Leibniz algebras.

1 Introduction

In [1], Peggy Batten established Lie analogues of results concerning the multipliers and covers of groups. In the first chapter of [1], she proves that covers of Lie algebras are unique, a result which deviates from the group case. Batten also characterizes the multiplier in terms of a free presentation. In [3], the author proves the Leibniz analogue of this first chapter.

The aim of the present paper is to generalize Chapters 3 and 4 in [1] to Leibniz algebras. Given a central ideal of a Leibniz algebra $L$ and a central $L$-module, we first construct a Hochschild-Serre type spectral sequence of low dimension. This sequence is used to characterize the multiplier $M(L)$ of $L$ in terms of the second cohomology group $H^2(L,F)$. We then establish conditions for when the center of any cover maps onto the center of the algebra, i.e., for when $\omega(Z(E)) = Z(L)$, where $E$ is a cover of $L$ and $\omega : E \rightarrow L$ is a surjective homomorphism. These conditions are the special case of a four-part equivalence theorem that highlights an extension-theoretic crossroads of unicentral algebras, multipliers and covers, free presentations, and the second cohomology group of Leibniz algebras.

2 Preliminaries

Let $F$ be a field. A Leibniz algebra $L$ is an $F$-vector space equipped with a bilinear multiplication which satisfies the Leibniz identity $x(yz) = (xy)z + y(xz)$ for all $x, y, z \in L$. Let $A$ and $B$ be Leibniz algebras. An extension of $A$ by $B$ is a short exact sequence $0 \rightarrow A \xrightarrow{\sigma} L \xrightarrow{\pi} B \rightarrow 0$ such that $\sigma$ and $\pi$ are homomorphisms and $L$ is a Leibniz algebra. One may assume that $\sigma$ is the identity map, and we make this assumption throughout. An extension is called central if $A \subseteq Z(L)$. A section of an extension is a linear map $\mu : B \rightarrow L$ such that $\pi \circ \mu = \text{id}_B$.

For a Leibniz algebra $L$, a pair of Leibniz algebras $(K, M)$ is called a defining pair for $L$ if $L \cong K/M$ and $M \subseteq Z(K) \cap K'$. We say such a pair is a maximal defining pair if the dimension of $K$ is maximal. In this case, $K$ is called a cover for $L$ and $M$ is called the multiplier for $L$, denoted $M(L)$. It immediately follows that $M$ is a unique Lie algebra since it is abelian, justifying $M(L)$ as being the multiplier of $L$. As in [1], $C(L)$ is used to denote the set of all pairs $(J, \lambda)$ such that $\lambda : J \rightarrow L$ is a surjective homomorphism and $\ker \lambda \subseteq J' \cap Z(J)$. An element $(T, \tau) \in C(L)$ is called a universal element in $C(L)$ if, for any $(J, \lambda) \in C(L)$, there exists a homomorphism $\beta : T \rightarrow J$ such that $\lambda \circ \beta = \tau$. 
As shown in [3], many results from Chapter 1 of [1] carry over to the Leibniz setting with few significant differences. One natural change is to consider algebras of the form $FR + RF$ as a replacement to the usual Lie product algebra $[F, R]$, which ensures a two-sided ideal for Leibniz algebras. A pair of dimension bounds are also notably different from the Lie case, but still ensure that both $K$ and $M$ have finite dimension. We now state the Leibniz version of the culminating result from the first chapter of [1], as proven in [3].

**Theorem 2.1.** Let $L$ be a finite-dimensional Leibniz algebra and let $0 \to R \to F \to L \to 0$ be a free presentation of $L$. Let

$$B = \frac{R}{FR + RF}, \quad C = \frac{F}{FR + RF}, \quad D = \frac{F' \cap R}{FR + RF}.$$

Then:

1. All covers of $L$ are isomorphic and have the form $C/E$ where $E$ is the complement to $D$ in $B$.
2. The multiplier $M(L)$ of $L$ is $D \cong B/E$.
3. The universal elements in $C(L)$ are the elements $(K, \lambda)$ where $K$ is a cover of $L$.

### 3 Cohomology

Consider a central extension $0 \to A \to L \to B \to 0$ and section $\mu : B \to L$. Define a bilinear form $f : B \times B \to A$ by $f(i, j) = \mu(i)\mu(j) - \mu(ij)$ for $i, j \in B$. By our work in [2], $f$ is a 2-cocycle of Leibniz algebras, meaning that $f(i, jk) = f(ij, k) + f(j, ik)$ for all $i, j, k \in B$. Moreover, the image of $f$ falls in $A$ by exactness. Since we are only working with central extensions for this section, we drop the trivial $\varphi$ and $\varphi'$ maps of central factor systems and let $Z^2(B, A)$ denote the set of all 2-cocycles. As usual, $B^2(B, A)$ is used to denote the set of all 2-coboundaries, i.e. 2-cocycles $f$ such that $f(i, j) = -\varepsilon(ij)$ for some linear transformation $\varepsilon : B \to A$. We next recall that any elements $f$ and $g$ in $Z^2(B, A)$ belong to equivalent extensions if and only if they differ by a linear map $\varepsilon : B \to A$, i.e. $f(i, j) - g(i, j) = -\varepsilon(ij)$ for all $i, j \in B$. In this case, we say $f$ and $g$ differ by a 2-coboundary. Therefore, extensions of $A$ by $B$ are equivalent if and only if they give rise to the same element of $H^2(B, A) = Z^2(B, A)/B^2(B, A)$, the second cohomology group of $B$ with coefficients in $A$. Finally, the work in [2] guarantees that each element $\overline{f} \in H^2(B, A)$ gives rise to a central extension $0 \to A \to L \to B \to 0$ with section $\mu$ such that $f(i, j) = \mu(i)\mu(j) - \mu(ij)$.

#### 3.1 Hochschild-Serre Spectral Sequence

Let $H$ be a central ideal of a Leibniz algebra $L$ and let $0 \to H \to L \xrightarrow{\beta} L/H \to 0$ be the natural central extension with section $\mu$ of $\beta$. Let $A$ be a central $L$-module. The following theorem concerns a five-term cohomological sequence that we refer to as the Hochschild-Serre spectral sequence of low dimensions.

**Theorem 3.1.** The sequence

$$0 \to \text{Hom}(L/H, A) \xrightarrow{\text{Inf}_1} \text{Hom}(L, A) \xrightarrow{\text{Res}} \text{Hom}(H, A) \xrightarrow{\text{Tra}} \mathcal{H}^2(L/H, A) \xrightarrow{\text{Inf}_2} \mathcal{H}^2(L, A)$$

is exact.
Before proving exactness, we need to define the maps of this sequence and check that they make sense. The first inflation map $\text{Inf}_1 : \text{Hom}(L/H, A) \to \text{Hom}(L, A)$ is defined by $\text{Inf}_1(\chi) = \chi \circ \theta$ for any homomorphism $\chi : L/H \to A$. Next, the restriction mapping $\text{Res} : \text{Hom}(L, A) \to \text{Hom}(H, A)$ is defined by $\text{Res}(\pi) = \pi \circ \iota$ where $\iota : H \to L$ is the inclusion map. It is readily verified that $\text{Inf}_1$ and $\text{Res}$ are well-defined and linear.

Third is the transgression map $\text{Tra} : \text{Hom}(H, A) \to \mathcal{H}^2(L/H, A)$. Let $f : L/H \times L/H \to H$ be defined by $f(x, y) = \mu(x)\mu(y) - \mu(xy)$ and consider $\chi \in \text{Hom}(H, A)$. Then $\chi \circ f \in \mathcal{Z}^2(L/H, A)$ since $\chi \circ f(x, y) = \chi \circ (\mu(x)\mu(y) - \mu(xy)) = \chi(0) = 0$ for all $x, y, z \in L$. If $\nu$ is another section of $\beta$, let $g(x, y) = \nu(x)\nu(y) - \nu(xy)$. Then $f$ and $g$ are cohomologous in $\mathcal{H}^2(L/H, H)$, which implies that there exists a linear transformation $\varepsilon : L/H \to H$ such that $f(x, y) - g(x, y) = -\varepsilon(xy)$. Clearly $\chi \circ \varepsilon : L/H \to A$ is also a linear transformation, and therefore $\chi \circ f$ and $\chi \circ g$ are cohomologous in $\mathcal{H}^2(L/H, A)$.

Letting $\chi = \chi \circ f$, we have shown that $\text{Tra}$ is well-defined. It is straightforward to verify that $\text{Tra}$ is linear.

Finally, let $\text{Inf}_2 : \mathcal{H}^2(L/H, A) \to \mathcal{H}^2(L, A)$ be defined by $\text{Inf}_2(f + \mathcal{B}^2(L/H, A)) = f' + \mathcal{B}^2(L, A)$, where $f'(x, y) = f(\beta(x), \beta(y))$ for $x, y \in L$ and $f \in \mathcal{Z}^2(L/H, A)$. It is straightforward to verify that $\text{Inf}_2$ is linear. To check that $\text{Inf}_2$ maps cocycles to cocycles, one computes

$$0 = f(\beta(x), \beta(y), \beta(z)) - f(\beta(x), \beta(y), \beta(z)) - f(\beta(y), \beta(x), \beta(z))$$

for all $x, y, z \in L$ since $f$ is a 2-cocycle. Hence $f' \in \mathcal{Z}^2(L, A)$. To check that $\text{Inf}_2$ maps coboundaries to coboundaries, suppose $f \in \mathcal{B}^2(L/H, A)$. Then there exists a linear transformation $\varepsilon : L/H \to A$ such that $f(x, y) = -\varepsilon(xy)$ for $x, y \in L$. Note that $\beta(x) = x + H = x$ for any $x \in L$. Therefore $f'(x, y) = f(\beta(x), \beta(y)) = -\varepsilon \circ \beta(xy)$, yielding $f' \in \mathcal{B}^2(L, A)$.

Proof. Once again, we are concerned with the central extension $0 \to H \to L \xrightarrow{\beta} L/H \to 0$, a section $\mu$ of $\beta$, and a central $L$-module $A$. One has $f \in \mathcal{Z}^2(L/H, H)$ for $f(x, y) = \mu(x)\mu(y) - \mu(xy)$. To show exactness at $\text{Hom}(L/H, A)$, it suffices to show that $\text{Inf}_1$ is injective. Suppose $\text{Inf}_1(\chi) = 0$ for $\chi \in \text{Hom}(L/H, A)$. Then $\chi \circ \beta(x) = 0$ for all $x \in L$, which means that $\chi = 0$ since $\beta$ is surjective.

To prove exactness at $\text{Hom}(L, A)$, first consider an element $\chi \in \text{Hom}(L/H, A)$. One computes $\text{Res}(\text{Inf}_1(\chi)) = \text{Res}(\chi \circ \beta) = \chi \circ \beta \circ \iota = 0$ since $\iota$ includes $H$ into $L$ and $\beta$ sends elements of $H$ to zero in $L/H$. Thus $\text{Im}(\text{Inf}_1) \subseteq \ker(\text{Res})$. Conversely, consider an element $\chi \in \ker(\text{Res})$. Then $\chi \circ \beta = 0$ implies that $H \subseteq \ker(\chi)$. By the fundamental theorem of homomorphisms, there exists $\hat{\chi} \in \text{Hom}(L/H, A)$ such that $\hat{\chi} \circ \bar{\beta} = \chi$. But $\text{Inf}_1(\hat{\chi}) = \bar{\chi} \circ \beta = \chi$. Hence $\ker(\text{Res}) \subseteq \text{Im}(\text{Inf}_1)$.

To show exactness at $\text{Hom}(H, A)$, first consider a map $\chi \in \text{Hom}(L, A)$. Then

$$\chi \circ f(x, y) = \chi \circ \mu(x)\chi \circ \mu(y) - \chi \circ \mu(xy)$$

by centrality, which implies that $\chi \circ f \in \mathcal{B}^2(L/H, A)$. Thus $\text{Tra}(\text{Res}(\chi)) = \text{Tra}(\chi \circ \theta) = \chi \circ \iota \circ f = 0$ and so $\text{Im}(\text{Res}) \subseteq \ker(\text{Tra})$. Conversely, let $\theta \in \text{Hom}(H, A)$ be such that $\text{Tra}(\theta) = \theta \circ f = 0$. Then $\theta \circ f \in \mathcal{B}^2(L/H, A)$ which implies that there exists a linear transformation $\varepsilon : L/H \to A$ such that $\theta \circ f(x, y) = -\varepsilon(xy)$. Let $x = \mu(x) + h_x$ and $y = \mu(y) + h_y$. Then $xy = \mu(xy) + h_{xy} = \mu(xy)\mu(y)$ implies that

$$\theta(h_{xy}) = \theta(\mu(xy))\mu(y) - \theta(\mu(y))h_x = \theta(\mu(xy)) + \varepsilon(xy) = -\varepsilon(xy). \quad (1)$$

Now let $\sigma(x) = \theta(h_x) + \varepsilon(x)$. Since $\text{Im} \sigma \subseteq A$, $\sigma(x)\sigma(y) = 0$ by centrality. By (1), $\sigma(xy) = \theta(h_{xy}) + \varepsilon(xy) = 0$. Hence $\sigma \in \text{Hom}(L, A)$ and $\sigma(h) = \theta(h) + \varepsilon(h + H) = \theta(h)$ for all $h \in H$, which means that $\text{Res}(\sigma) = \theta$ and thus $\ker(\text{Tra}) \subseteq \text{Im}(\text{Res})$.

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To show exactness at $\mathcal{H}^2(L/H, A)$, first consider a map $\chi \in \text{Hom}(H, A)$. Then $\text{Tra}(\chi) = \chi \circ f$ where, as before, $f(x, y) = \mu(x)\mu(y) - \mu(xy)$ and $\chi \circ f \in Z^2(L/H, A)$. By definition of $\text{Inf}_2$,

$$\text{Inf}_2(\chi \circ f) = (\chi \circ f)'$$

where $(\chi \circ f)'(x, y) = \chi \circ f(x, y)$. We want to show that $(\chi \circ f)'$ is a coboundary in $\mathcal{H}^2(L, A)$. To this end, we once again consider $x = \mu(x) + h_x$ and $y = \mu(y) + h_y$ with product $xy = \mu(x)\mu(y) - \mu(xy)$. Then $\chi \circ f(x, y) = \chi(\mu(x)\mu(y) - \mu(xy)) = \chi(hxy)$. Define $\epsilon(x) = -\chi(h_x)$. Then $\epsilon : L \to A$ and is linear. One computes $\epsilon(xy) = -\chi(hxy) = -\chi \circ f(x, y) = -(\chi \circ f')(x, y)$ which implies that $(\chi \circ f)' \in B^2(L, A)$. Therefore $(\chi \circ f)' = 0$ and we have $\text{Im}(\text{Tra}) \subseteq \ker(\text{Inf}_2)$. Conversely, suppose $g \in Z^2(L/H, A)$ such that $\overline{g} \in \ker(\text{Inf}_2)$. Then $g(x, y) = g'(x, y) = -\epsilon(xy)$ for some linear $\epsilon : L \to A$. Since $\epsilon$ is linear, $\epsilon \circ f \in Z^2(L/H, A)$. As before, let $x = \mu(x) + h_x \in L$ with $xy = \mu(x)\mu(y)$ the product of two such elements. Then

$$g'(x, y) = g(x, y)
= -\epsilon(\mu(x)\mu(y))
= -\epsilon \circ f(x, y) - \epsilon \circ \mu(xy)$$

where $\epsilon \circ \mu : L/H \to A$. Thus $\overline{g} = -\overline{\epsilon \circ f} = -\text{Tra}(\epsilon)$ which implies that $\ker(\text{Inf}_2) \subseteq \text{Im}(\text{Tra})$. \qed

### 3.2 Relation of Multipliers and Cohomology

The objective of this section is to prove that the multiplier $M(L)$ of a finite-dimensional Leibniz algebra $L$ is isomorphic to $\mathcal{H}^2(L, \mathbb{F})$, where $\mathbb{F}$ is considered as a central $L$-module. The following results rely on our ability to invoke the Hochschild-Serre spectral sequence, and so the labor of Theorem 3.1 begins to pay off.

**Theorem 3.2.** Let $Z$ be a central ideal in $L$. Then $L' \cap Z$ is isomorphic to the image of $\text{Hom}(Z, \mathbb{F})$ under the transgression map. In particular, if $\text{Tra}$ is surjective, then $L' \cap Z \cong \mathcal{H}^2(L/Z, \mathbb{F})$.

**Proof.** Let $0 \to Z \to L \to L/Z \to 0$ be the natural exact sequence for a central ideal $Z$ in $L$. Then the sequence $\text{Hom}(L, \mathbb{F}) \xrightarrow{\text{Res}} \text{Hom}(Z, \mathbb{F}) \xrightarrow{\text{Tra}} \mathcal{H}^2(L/Z, \mathbb{F})$ is exact by Theorem 3.1. Let $J$ denote the set of all homomorphisms $\chi : Z \to \mathbb{F}$ such that $\chi$ can be extended to an element of $\text{Hom}(L, \mathbb{F})$. Then $J$ is precisely the image of the restriction map in $\text{Hom}(Z, \mathbb{F})$, which is equal to the kernel of the transgression map by exactness. This means that $\text{Hom}(Z, \mathbb{F})/J \cong \text{Im}(\text{Tra})$ and thus it suffices to show that $\text{Hom}(Z, \mathbb{F})/J \cong L' \cap Z$.

Consider the natural restriction homomorphism $\text{Res}_2 : \text{Hom}(Z, \mathbb{F}) \to \text{Hom}(L' \cap Z, \mathbb{F})$. Since $Z$ and $L' \cap Z$ are both abelian, $\text{Res}_2$ is surjective and $\text{Hom}(L' \cap Z, \mathbb{F})$ is the dual space of $L' \cap Z$. Therefore

$$\frac{\text{Hom}(Z, \mathbb{F})}{\ker(\text{Res}_2)} \cong \text{Hom}(L' \cap Z, \mathbb{F}) \cong L' \cap Z$$

and it remains to show that $J \cong \ker(\text{Res}_2)$. For one direction, consider an element $\chi \in J$ with extension $\hat{\chi} \in \text{Hom}(L, \mathbb{F})$. Then $L' \subseteq \ker \hat{\chi}$ since $\mathbb{F}$ is abelian, which implies that $L' \cap Z \subseteq \ker \chi$. Thus $\chi \in \ker(\text{Res}_2)$ and we have $J \subseteq \ker(\text{Res}_2)$. Conversely, let $\chi \in \ker(\text{Res}_2)$. Then $\chi \in \text{Hom}(Z, \mathbb{F})$ is such that $L' \cap Z \subseteq \ker \chi$, which implies that $\chi$ induces a homomorphism $\hat{\chi}' : Z \to \mathbb{F}$.
defined by $\chi'(z + (L' \cap Z)) = \chi(z)$. Since

$$\frac{Z}{L' \cap Z} \cong \frac{Z + L'}{L'},$$

there exists a homomorphism

$$\chi^\prime : \frac{Z + L'}{L'} \to \mathbb{F}$$

defined by $\chi''(z + L') = \chi'(z + (L' \cap Z))$. But $\chi''$ can be extended to a homomorphism $\chi''' : L/L' \to \mathbb{F}$ which is defined by $\chi'''(x + L') = \chi''(x + L')$ for all $x \in Z$. Since $L/L'$ is abelian, $\chi'''$ can be extended to a homomorphism $\hat{\chi} : L \to \mathbb{F}$ which is defined by $\hat{\chi}(x) = \chi'''(x + L')$. Therefore $\chi \in J$ and the first statement holds. The second statement holds since $\text{Tra}$ maps $\text{Hom}(Z, \mathbb{F})$ to $\mathcal{H}^2(L/Z, \mathbb{F})$.

Let $L$ be a Leibniz algebra with free presentation $0 \to R \xrightarrow{\omega} F \xrightarrow{\phi} L \to 0$. The induced sequence

$$0 \to \frac{R}{FR + RF} \xrightarrow{F} \frac{F}{FR + RF} \to L \to 0$$

is a central extension since $RF$ and $FR$ are both contained in $FR + RF$. It is not unique, but has the following property.

**Lemma 3.3.** Let $0 \to A \to B \xrightarrow{\alpha} C \to 0$ be a central extension and $\alpha : L \to C$ be a homomorphism. Then there exists a homomorphism $\beta : F/(FR + RF) \to B$ such that

$$\begin{array}{ccc}
0 & \to & \frac{R}{FR + RF} \\
\downarrow{\gamma} & & \downarrow{\beta} \\
0 & \to & A \\
\downarrow{\alpha} & & \downarrow{\phi} \\
0 & \to & B & \xrightarrow{\phi} & C & \to 0
\end{array}$$

is commutative, where $\gamma$ is the restriction of $\beta$ to $R/(FR + RF)$.

**Proof.** Since $F$ is free, there exists a homomorphism $\sigma : F \to B$ such that

$$\begin{array}{ccc}
F & \xrightarrow{\omega} & L \\
\downarrow{\sigma} & & \downarrow{\alpha} \\
B & \xrightarrow{\phi} & C
\end{array}$$

is commutative. Let $r \in R \subseteq F$. Then $\omega(r) = 0$ since $\ker \omega = R$. Therefore $0 = \alpha \circ \omega(r) = \phi \circ \sigma(r)$ and so $\sigma(R) \subseteq \ker \phi$. We want to show that $FR + RF \subseteq \ker \sigma$. If $x \in F$ and $r \in R$, then $\sigma(xr) = \sigma(x)\sigma(r) = 0$ and $\sigma(rx) = \sigma(r)\sigma(x) = 0$ since $\sigma(r) \in \ker \phi = A \subseteq Z(B)$. Now $\sigma$ induces a homomorphism $\beta : F/(FR + RF) \to B$. The left diagram commutes since we may take $A \to B$ to be the inclusion map.

**Lemma 3.4.** Let $0 \to R \to F \to L \to 0$ be a free presentation of $L$ and let $A$ be a central $L$-module. Then the transgression map $\text{Tra} : \text{Hom}(R/(FR + RF), A) \to \mathcal{H}^2(L, A)$ associated with

$$0 \to \frac{R}{FR + RF} \xrightarrow{F} \frac{F}{FR + RF} \xrightarrow{\phi} L \to 0$$

is surjective.
Proof. Consider \( \overline{f} \in \mathcal{H}^2(L, A) \) and let \( 0 \to A \to E \xrightarrow{\phi} L \to 0 \) be an associated central extension. By Lemma 3.3, there exists a homomorphism \( \theta \) such that

\[
\begin{array}{cccccc}
0 & \to & R_{FR+RF} & \to & F_{FR+RF} & \phi \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & A & \to & E & \varphi \\
& & & & \downarrow & \\
& & & & \leftarrow & \\
& & & & id & \\
\end{array}
\]

is commutative and \( \gamma = \theta|_{R/(FR+RF)} \). Let \( \mu \) be a section of \( \phi \). Then \( \varphi \circ \theta \circ \mu = \phi \circ \mu = id_L \) and so \( \theta \circ \mu \) is a section of \( \varphi \). Let \( \lambda = \theta \circ \mu \) and define \( \beta(x, y) = \lambda(x)\lambda(y) - \lambda(xy) \). Then \( \beta \in \mathcal{Z}^2(L, A) \) and \( \beta \) is cohomologous with \( g \) since they are associated with the same extension. One computes

\[
\begin{align*}
\beta(x, y) &= \theta(\mu(x))\theta(\mu(y)) - \theta(\mu(xy)) \\
&= \theta(\mu(x)\mu(y) - \mu(xy)) \\
&= \gamma(\mu(x)\mu(y) - \mu(xy)) \\
&= \gamma(f(x, y))
\end{align*}
\]

where \( f(x, y) = \mu(x)\mu(y) - \mu(xy) \) and \( \gamma = \theta|_{R/(FR+RF)} \). Thus \( \text{Tra}(\gamma) = \overline{\gamma} = \overline{f} = \overline{\beta} = \overline{g} \). \( \square \)

Lemma 3.5. If \( C \subseteq A \) and \( C \subseteq B \), then \( A/C \cap B/C = (A \cap B)/C \).

Proof. Clearly \((A\cap B)/C \subseteq A/C \cap B/C \). Let \( x \in A/C \cap B/C \). Then \( x = a + c_1 = b + c_2 \) for \( a \in A \), \( b \in B \), and \( c_1, c_2 \in C \). Since \( C \subseteq B \), \( a = b + c_2 - c_1 \in B \), which implies that \( a \in A \cap B \). Then \( x = a + c \in (A \cap B)/C \) and so \( A/C \cap B/C \subseteq (A \cap B)/C \). \( \square \)

Theorem 3.6. Let \( L \) be a Leibniz algebra over a field \( \mathbb{F} \) and \( 0 \to R \to F \to L \to 0 \) be a free presentation of \( L \). Then

\[
\mathcal{H}^2(L, \mathbb{F}) \cong F' \cap R_{FR+RF}.
\]

In particular, if \( L \) is finite-dimensional, then \( M(L) \cong \mathcal{H}^2(L, \mathbb{F}) \).

Proof. Let \( \overline{R} = R/(FR+RF) \) and \( \overline{F} = F/(FR+RF) \). Then \( 0 \to \overline{R} \to \overline{F} \to L \to 0 \) is a central extension. By Lemma 3.4,

\[
\text{Tra} : \text{Hom}(\overline{R}, \mathbb{F}) \to \mathcal{H}^2(L, \mathbb{F})
\]

is surjective. By Theorem 3.2,

\[
\overline{F}' \cap \overline{R} \cong \mathcal{H}^2(\overline{F}/\overline{R}, \mathbb{F}) \cong \mathcal{H}^2(L, \mathbb{F}).
\]

By Lemma 3.5,

\[
\overline{F}' \cap \overline{R} \cong \frac{F'}{FR+RF} \cap \frac{R}{FR+RF} = \frac{F' \cap R}{FR+RF}.
\]

Therefore,

\[
M(L) = \frac{F' \cap R}{FR+RF} \cong \mathcal{H}^2(L, \mathbb{F})
\]

by the characterization of \( M(L) \) from Theorem 2.1. \( \square \)

Thus is the multiplier \( M(L) \) characterized by \( \mathcal{H}^2(L, \mathbb{F}) \). We have now proven the main result of Batten’s Chapter 3 for the Leibniz case. We conclude this section with the Leibniz analogue of a corollary which appears at the end of said chapter.
Corollary 3.7. For any cover \( E \) of \( L \) and any subalgebra \( A \) of \( E \) satisfying

1. \( A \subseteq Z(E) \cap E' \),
2. \( A \cong M(L) \),
3. \( L \cong E/A \),

the associated transgression map \( \text{Tra} : \text{Hom}(A, F) \to M(L) \) is bijective.

Proof. First note that \( 0 \to A \to E \to L \to 0 \) is a central extension of \( L \). Invoking the Hochschild-Serre spectral sequence yields

\[
0 \to \text{Hom}(L, F) \xrightarrow{\text{Inf}_1} \text{Hom}(E, F) \xrightarrow{\text{Res}} \text{Hom}(A, F) \xrightarrow{\text{Tra}} \mathcal{H}^2(L, F) \xrightarrow{\text{Inf}_2} \mathcal{H}^2(E, F)
\]

with \( \text{Im}(\text{Res}) = \ker(\text{Tra}) \). Furthermore, any \( \theta \in \text{Hom}(E, F) \) yields \( \text{Res}(\theta) \in \text{Hom}(A, F) \). Now let \( a \in A \subseteq E' \). Then \( a = e_1e_2 \) for some \( e_1, e_2 \in E \) which implies that \( \text{Res}(\theta(a)) = \text{Res}(\theta(e_1)\theta(e_2)) = \text{Res}(0) = 0 \). Thus \( \text{Im}(\text{Res}) = 0 \), making \( \ker(\text{Tra}) = 0 \), and so \( \text{Tra} \) injective. Since \( \text{Hom}(A, F) \cong A \cong M(L) \), \( \text{Tra} \) is bijective. \( \square \)

4 Unicentral Leibniz Algebras

Let \( L \) be a Leibniz algebra. The objective of this section is to develop criteria for when the center of any cover of \( L \) maps onto the center of \( L \). One of these criteria will take the form of \( Z(L) \subseteq Z^*(L) \), where \( Z^*(L) \) denotes the intersection of all images \( \omega(Z(E)) \) such that \( 0 \to \ker \omega \to E \xrightarrow{\omega} L \to 0 \) is a central extension of \( L \). It is easy to see that \( Z^*(L) \subseteq Z(L) \). We say a Leibniz algebra \( L \) is **unicentral** if \( Z(L) = Z^*(L) \). To prove our result, we will establish conditions for a more general central ideal \( Z \) in \( L \) before specializing to \( Z(L) \).

4.1 Sequences

We begin by extending our Hochschild-Serre sequence. Let \( Z \) be a central ideal in \( L \) and consider the natural central extension \( 0 \to Z \to L \to L/Z \to 0 \). To define our \( \delta \) map, consider a cocycle \( f' \in Z^2(L, \mathbb{F}) \) and define two bilinear forms \( f_1' : L/L' \times Z \to \mathbb{F} \) and \( f_2' : Z \times L/L' \to \mathbb{F} \) by

\[
f_1''(x + L', z) = f'(x, z)
f_2''(z, x + L') = f'(z, x)
\]

for \( x \in L \) and \( z \in Z \). To check that they are well-defined, one computes

\[
f_1''(xy + L', z) = f'(xy, z)
= f'(x, yz) - f'(y, xz)
= 0
\]

and

\[
f_2''(z, xy + L') = f'(z, xy)
= f'(x, zy) - f(xz, y)
= 0
\]

since \( z \in Z(L) \). Hence \((f_1'', f_2'') \in \text{Bil}(L/L' \times Z, \mathbb{F}) \oplus \text{Bil}(Z \times L/L', \mathbb{F}) \cong L/L' \otimes Z \oplus Z \otimes L/L' \). Now consider a coboundary \( f' \in \mathcal{B}^2(L, \mathbb{F}) \). By definition, there exists a linear map \( \varepsilon : L \to \mathbb{F} \) such that \( f'(x, y) = -\varepsilon(xy) \). One computes \( f_1'(x + L', z) = f'(x, z) = -\varepsilon(xz) = 0 \) and \( f_2'(z, x + L') = f'(z, x) = -\varepsilon(zx) = 0 \) since \( z \in Z(L) \). Hence, a map \( \delta : f' + \mathcal{B}^2(L, \mathbb{F}) \to (f_1'', f_2'') \) is induced which is clearly linear since \( f' \), \( f_1'' \), and \( f_2'' \) are all in vector spaces of bilinear forms and the latter two are defined by \( f' \).

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Theorem 4.1. Let $Z$ be a central ideal of Leibniz algebra $L$. The sequence
\[ \mathcal{H}^2(L/Z, \mathbb{F}) \xrightarrow{\text{Inf}} \mathcal{H}^2(L, \mathbb{F}) \xrightarrow{\delta} L/L' \otimes Z \oplus Z \otimes L/L' \]
is exact.

Proof. Let $f \in \mathcal{Z}^2(L/Z, \mathbb{F})$. Then $\text{Inf}(f + \mathcal{B}^2(L/Z, \mathbb{F})) = f' + \mathcal{B}^2(L, \mathbb{F})$ where $f'$ is the cocycle defined by $f'(x, y) = f(x + Z, y + Z)$. We also have $\delta(f' + \mathcal{B}^2(L, \mathbb{F})) = (f''_1, f''_2)$ where, for all $x \in L$ and $z \in Z$,
\[ f''_1(x + L', z) = f'(x, z) = f(x + Z, z + Z) = 0, \]
\[ f''_2(z, x + L') = f'(z, x) = f(z + Z, x + Z) = 0, \]
which implies that $\delta(\text{Inf}(f + \mathcal{B}^2(L/Z, \mathbb{F}))) = (f''_1, f''_2) = (0, 0)$. Therefore $\text{Im}(\text{Inf}) \subseteq \text{ker} \delta$.

Conversely, suppose $f' \in \mathcal{Z}^2(L, \mathbb{F})$ is such that $\delta(f' + \mathcal{B}^2(L, \mathbb{F})) = (f''_1, f''_2) = (0, 0)$. Then, for all $x \in L$ and $z \in Z$, one has $0 = f''_1(x + L', z) = f'(x, z)$ and $0 = f''_2(z, x + L') = f'(z, x)$. Hence, for all $z, z' \in Z$ and $x, y \in L$, one computes
\[ f'(x + z, y + z') = f'(x, y) + f''_1(x + L', z') + f''_2(z, y + L') + f''_1(z + L', z') \]
which yields a bilinear form $g : L/Z \times L/Z \to \mathbb{F}$, defined by $g(x + Z, y + Z) = f'(x, y)$, that is well-defined. Furthermore, $g \in \mathcal{Z}^2(L/Z, \mathbb{F})$ since $f'$ is a cocycle. Thus $\text{Inf}(g + \mathcal{B}^2(L/Z, \mathbb{F})) = f' + \mathcal{B}^2(L, \mathbb{F})$ and so $\text{ker} \delta \subseteq \text{Im}(\text{Inf})$. \hfill \Box

Theorem 4.2. (Ganea Sequence) Let $Z$ be a central ideal in a finite-dimensional Leibniz algebra $L$. Then the sequence
\[ L/L' \otimes Z \oplus Z \otimes L/L' \to M(L) \to M(L/Z) \to L' \cap Z \to 0 \]
is exact.

Proof. Let $F$ be a free Leibniz algebra such that $L = F/R$ and $Z = T/R$ for some ideals $T$ and $R$ of $F$. Since $Z \subseteq Z(L)$, one has $T/R \subseteq Z(F/R)$ and $FT + TF \subseteq R$. Inclusion maps $\beta : R \cap F' \to T \cap F'$ and $\gamma : T \cap F' \to T \cap (F' + R)$ induce homomorphisms
\[ \frac{R \cap F'}{FR + RF} \xrightarrow{\beta} \frac{T \cap F'}{FT + TF} \xrightarrow{\gamma} \frac{T \cap (F' + R)}{R} \to 0. \]
Since $R \subseteq T$, one has
\[ \frac{T \cap (F' + R)}{R} = \frac{(T + R) \cap (F' + R)}{R} \cong \frac{(T \cap F') + R}{R} \]
which implies that $\gamma$ is surjective. By Theorem 211
\[ M(L) \cong \frac{R \cap F'}{FR + RF} \quad \text{and} \quad M(L/Z) \cong \frac{T \cap F'}{FT + TF}. \]
Also
\[ L' \cap Z \cong (F/R)' \cap (T/R) \cong \frac{F' + R}{R} \cap \frac{T}{R} \cong \frac{(F' + R) \cap T}{R}. \]
Therefore, the sequence $M(L/Z) \xrightarrow{\gamma} L' \cap Z \rightarrow 0$ is exact. Since 
\[
\ker \gamma = \frac{(T \cap F') \cap R}{FT + TF} = \frac{R \cap F'}{FT + TF} = \text{Im } \beta,
\]
the sequence $M(L) \xrightarrow{\beta} M(L/Z) \xrightarrow{\gamma} L' \cap Z$ is exact.

It remains to show that $L/L' \otimes Z \oplus Z \otimes L/L' \rightarrow M(L) \xrightarrow{\beta} M(L/Z)$ is exact. Define a pair of maps 
\[
\begin{align*}
\theta_1 : & \frac{F}{R + F'} \times \frac{T}{R} \rightarrow \frac{R \cap F'}{FR + RF'}, \\
\theta_2 : & \frac{T}{R} \times \frac{F}{R + F'} \rightarrow \frac{R \cap F'}{FR + RF'}
\end{align*}
\]
by $\theta_1(f + (R + F'), t + R) = ft + (FR + RF)$ and $\theta_2(t + R, f + (R + F')) = tf + (FR + RF)$. Both are bilinear because multiplication is bilinear. To check that $\theta_1$ and $\theta_2$ are well-defined, suppose $(f + (R + F'), t + R) = (f' + (R + F'), t' + R)$ for $t, t' \in T$ and $f, f' \in F$. Then $t - t' \in R$ and $f - f' \in R + F'$ which implies that $t = t' + r$ for $r \in R$ and $f = f' + x$ for $x \in R + F'$. One computes 
\[
\begin{align*}
t f - t' f' &= (t' + r)(f' + x) - t' f' \\
&= t' x + r f' + r x
\end{align*}
\]
and 
\[
\begin{align*}
f t - f' t' &= (f' + x)(t' + r) - f' t' \\
&= x t' + f' r + x r
\end{align*}
\]
which both fall in $FR + RF$ by the Leibniz identity and the fact that $FT + TF \subseteq R$. Thus $\theta_1$ and $\theta_2$ are well-defined, and so induce linear maps 
\[
\begin{align*}
\overline{\theta}_1 : & \frac{F}{R + F'} \otimes \frac{T}{R} \rightarrow \frac{R \cap F'}{FR + RF'}, \\
\overline{\theta}_2 : & \frac{T}{R} \otimes \frac{F}{R + F'} \rightarrow \frac{R \cap F'}{FR + RF'}
\end{align*}
\]
These, in turn, yield a linear transformation 
\[
\overline{\theta} : \frac{F}{R + F'} \otimes \frac{T}{R} \otimes \frac{T}{R} \otimes \frac{F}{R + F'} \rightarrow \frac{R \cap F'}{FR + RF'}
\]
defined by $\overline{\theta}(a, b) = \overline{\theta}_1(a) + \overline{\theta}_2(b)$. The image of $\overline{\theta}$ is 
\[
\frac{FT + TF}{FR + RF}
\]
which is precisely equal to $\{x + (FR + RF) \mid x \in R \cap F', x \in FT + TF\} = \ker \beta$. Thus the sequence 
\[
\frac{F}{R + F'} \otimes \frac{T}{R} \oplus \frac{T}{R} \otimes \frac{F}{R + F'} \cong L/L' \otimes Z \oplus Z \otimes L/L' \rightarrow \frac{R \cap F'}{FR + RF} \cong M(L)
\]
\[
\rightarrow \frac{F' \cap T}{FT + TF} \cong M(L/Z)
\]
is exact. \qed
Corollary 4.3. (Stallings Sequence) Let $Z$ be a central ideal of a Leibniz algebra $L$. Then the following map is exact:

$$M(L) \to M(L/Z) \to Z \to L/L' \to \frac{L}{Z+L'} \to 0.$$ 

Proof. Let $F$ be a free Leibniz algebra such that $L = F/R$ and $Z = T/R$ for ideals $T$ and $R$ of $F$. Then $FT + TF \subseteq R$ since $Z \subseteq Z(L)$. The inclusion maps $R \cap F' \to T \cap F' \to T \to F \to F'$ induce the following sequence of homomorphisms:

$$\frac{R \cap F'}{FR + RF} \xrightarrow{\beta} \frac{T \cap F'}{FT + TF} \xrightarrow{\theta} \frac{T}{R} \xrightarrow{\alpha} \frac{F}{R + F'} \xrightarrow{\omega} \frac{F}{T + F'} \xrightarrow{\gamma} 0$$

To prove exactness for our desired sequence, we make use of the following facts:

1. $M(L) \cong \frac{R \cap F'}{FR + RF}$,
2. $M(L/Z) \cong \frac{T \cap F'}{FT + TF}$,
3. $Z \cong T/R$,
4. $\frac{F}{R + F'} \cong L/L'$,
5. $\frac{F}{T + F'} = \frac{F}{T + F'} \cong \frac{(F/R)/(T + F'/R)}{R} \cong \frac{F}{T + (F'+R)/R} \cong \frac{L}{Z+L'}$.

Thus do the following equalities suffice for this proof:

i. $\ker \theta = \{x + (FT + TF) \mid x \in T \cap F', \ x \in R\} = \frac{R \cap (F')}{FT + TF} = \frac{FR + RF'}{FR + RF} = \text{Im } \beta$,

ii. $\ker \alpha = \{x + R \mid x \in T, \ x \in (R + F')\} = \frac{T \cap (R + F')}{R} = \frac{R + (T \cap F')}{R} = \text{Im } \theta$,

iii. $\ker \omega = \{x + (R + F') \mid x \in F, \ x \in (T + F')\} = \frac{F \cap (T + F')}{R + F'} = \frac{T + F'}{R + F'} = \text{Im } \alpha$,

iv. $\ker \gamma = \frac{F}{T + F'} = \text{Im } \omega$.

4.2 The Main Result

In the previous subsection, we defined maps $\delta$ and $\beta$ that appeared in the extended Hochschild-Serre and Ganea sequences respectively. The latter of these is called the natural map. The following statements, two of which involve these maps, make up the conditions of our four-part theorem.

1. $\delta$ is the trivial map,
2. $\beta$ is injective,
3. $M(L) \cong \frac{M(L/Z)}{L \cap Z}$,
4. $Z \subseteq Z^*(L)$.

The following pair of lemmas shows that the first three are equivalent.
Lemma 4.4. Let $Z$ be a central ideal of finite-dimensional Leibniz algebra $L$ and let $\delta : M(L) \to L/L' \otimes Z \oplus Z \otimes L/L'$ be as in Theorem 4.1. Then

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if $\delta$ is the trivial map. Here we have identified $L' \cap Z$ with its image in $M(L/Z)$.

Proof. We invoke Theorems 3.1 and 3.1 yielding an exact sequence

$$0 \to \text{Hom}(L/Z, \mathbb{F})^{\text{Inf}_1} \to \text{Hom}(L, \mathbb{F})^{\text{Res}} \to \text{Hom}(Z, \mathbb{F})^{\text{Tra}} \to M(L/Z)^{\text{Inf}_2} \to M(L) \delta \to L/L' \otimes Z \oplus Z \otimes L/L'.$$

In one direction, suppose $\delta$ is the zero map. Then $M(L) \cong \ker \delta \cong \text{Im}(\text{Inf}_2)$. Since

$$\text{Im}(\text{Inf}_2) \cong \frac{M(L/Z)}{\ker(\text{Inf}_2)},$$

and $\ker(\text{Inf}_2) = \text{Im}(\text{Tra}) \cong L' \cap Z$ by Theorem 3.2 we have

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}.$$

Conversely, the isomorphism

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z} \cong \frac{M(L/Z)}{\ker(\text{Inf}_2)} \cong \text{Im}(\text{Inf}_2) \cong \ker \delta$$

implies that $\delta$ is trivial.

Lemma 4.5. Let $Z$ be a central ideal of a finite-dimensional Leibniz algebra $L$ and let $\beta : M(L) \to M(L/Z)$ be as in Theorem 4.2. Then

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z}$$

if and only if $\beta$ is injective.

Proof. By Theorem 4.2 the sequence $M(L) \xrightarrow{\beta} M(L/Z) \xrightarrow{\alpha} L' \cap Z \xrightarrow{\omega} 0$ is exact. Suppose $\beta$ is injective. Then $\ker \beta = 0$, which implies that

$$M(L) \cong \text{Im} \beta \cong \ker \alpha \cong \frac{M(L/Z)}{\text{Im} \alpha} = \frac{M(L/Z)}{\ker \omega} \cong \frac{M(L/Z)}{L' \cap Z}.$$

Conversely, the isomorphism

$$M(L) \cong \frac{M(L/Z)}{L' \cap Z} \cong \text{Im} \beta$$

implies that $\beta$ is injective.

It remains to show that these conditions are equivalent to $Z \subseteq Z^*(L)$. This will lead to criteria for when $\omega(Z(E)) = Z(L)$, where $E$ is any cover of $L$ and $0 \to \ker \omega \to E \xrightarrow{\omega} L \to 0$ is a central extension. Such an extension is called a stem extension, i.e., a central extension $0 \to A \to B \to C \to 0$ for which $A \subseteq B'$.
Consider a free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$ of $L$ and let $\overline{X}$ denote the quotient algebra $\frac{X}{FR+RF}$ for any $X$ such that $FR+RF \subseteq X \subseteq F$. Since $R = \ker \pi$ and $FR+RF \subseteq R$, $\pi$ induces a homomorphism $\overline{\pi} : \overline{F} \to L$ such that the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\pi} & L \\
\downarrow & & \downarrow \\
\overline{F} & \xrightarrow{\overline{\pi}} & \overline{L}
\end{array}
$$

commutes. Since $R \subseteq Z(\overline{L})$, there exists a complement $\frac{S}{FR+RF}$ to $\frac{R}{FR+RF}$ such that $S \subseteq R \subseteq \ker \pi$ and $S \subseteq R \subseteq \ker \overline{\pi}$. Thus $\overline{\pi}$ induces a homomorphism $\pi_S : F/S \to L$ and a central extension $0 \to R/S \to F/S \xrightarrow{\pi_S} L \to 0$. This extension is stem since $R/S \cong \frac{R \cap E'}{FR+RF} = \ker \pi_S$ implies that $F/S$ is a cover of $L$.

**Lemma 4.6.** For every free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$ of $L$ and every central extension $0 \to \ker \omega \to E \xrightarrow{\omega} L \to 0$, one has $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$.

**Proof.** Since the identity map $\text{id} : L \to L$ is a homomorphism, we can invoke Lemma 3.13 yielding a homomorphism $\beta : \overline{F} \to E$ such that the diagram

$$
\begin{array}{ccc}
0 & \to & \frac{F}{FR+RF} \\
\gamma & \downarrow & \downarrow \beta \\
0 & \to & \frac{E}{FR+RF}
\end{array}
$$

is commutative (where $\gamma$ is the restriction of $\beta$ to $\overline{F}$).

Let $A = \ker \omega$. Our first claim is that $E = A + \beta(\overline{F})$. Indeed, let $e \in E$. Then $\omega(e) = \overline{\pi}(f)$ for some $f \in \overline{F}$, and so $\omega(e) = \omega \circ \beta(f)$ by diagram commutativity. This implies that $e - \beta(f) \in \ker \omega = A$, meaning $e - \beta(f) = a$ for some $a \in A$. Thus $e = a + \beta(f)$.

Our second claim is that $\beta(Z(\overline{F}))$ centralizes both $A$ and $\beta(\overline{F})$. To see this, one first computes $\beta(Z(\overline{F}))\beta(\overline{F}) = \beta(Z(F)F) = \beta(0) = 0$ and $\beta(\overline{F})\beta(Z(\overline{F})) = \beta(FZ(\overline{F})) = \beta(0) = 0$. Next, we know that $AE$ and $EA$ are both zero, and so $A\beta(Z(\overline{F}))$ and $\beta(Z(\overline{F}))A$ are zero as well. But this implies that $\beta(Z(\overline{F}))$ centralizes $E$ by the first claim. Hence $\beta(Z(\overline{F})) \subseteq Z(E)$ and $\omega \circ \beta(Z(\overline{F})) \subseteq \omega(Z(E))$, which yields $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$.

**Theorem 4.7.** For every free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$ of $L$ and every stem extension $0 \to \ker \omega \to E \xrightarrow{\omega} L \to 0$, one has $Z^*(L) = \overline{\pi}(Z(\overline{F})) = \omega(Z(E))$.

**Proof.** By Lemma 4.6, $\overline{\pi}(Z(\overline{F}))$ is contained in $\omega'(Z(E'))$ for every central extension $0 \to \ker \omega' \to E' \xrightarrow{\omega'} L \to 0$. Thus $\overline{\pi}(Z(\overline{F})) \subseteq \omega(Z(E))$ for our stem extension. We also know that $Z^*(L)$ is the intersection of all images $\omega'(Z(E'))$, and that $\overline{\pi}(Z(\overline{F}))$ is one of these images since $0 \to \overline{R} \to \overline{F} \xrightarrow{\overline{\pi}} L \to 0$ is central. Therefore $\overline{\pi}(Z(\overline{F})) = Z^*(L)$. Since this equality holds for all $F$, we can assume that $0 \to R/S \to F/S \xrightarrow{\pi_S} L \to 0$ is a stem extension where $S$ is defined as above. Since the cover $F/S$ is unique up to isomorphism, it now suffices to show that $\pi_S(Z(F/S)) = \overline{\pi}(Z(\overline{F}))$.

Let $T$ be the inverse image of $Z(F/S)$ in $F$ and consider the commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\pi_S} & F/S \\
\downarrow & & \downarrow \\
\overline{F} & \xrightarrow{\overline{\pi}} & \overline{F/S}
\end{array}
$$

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where all mappings are the natural ones. Then $\overline{T} = \pi_1(T)$ by definition and $\pi_2(\overline{T}) = \pi_2 \circ \pi_1(T) = \pi_3(T) = Z(F/S)$, yielding the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\pi_3} & Z(F/S) \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
\overline{T} & \xrightarrow{\pi_3} & \pi_3(T) = Z(F/S)
\end{array}
$$

where all maps denote their restrictions. Now let $x \in Z(F)$. Then $\pi_2(x) \in Z(F/S)$, which implies that there exists $y \in T$ such that $\pi_3(y) = \pi_2(x)$. The resulting equality $\pi_2 \circ \pi_1(y) = \pi_2(x)$ yields an element $\pi_1(y) - x \in \ker\pi_2 = S \subseteq \overline{T}$, where $S \subseteq \overline{T}$ since $S \subseteq T$. Therefore $x \in \overline{T}$ and $Z(\overline{T}) \subseteq \overline{T}$. For the reverse inclusion, we first note that $T/S = Z(F/S)$, and so $FT + TF \subseteq S$. Thus $\overline{FT} + \overline{TF} \subseteq \overline{S}$. Also $\overline{FT} + \overline{TF} \subseteq \overline{R}$ since $\overline{S} \subseteq \overline{R}$ and $\overline{FT} + \overline{TF} \subseteq \overline{F}$ by definition. Hence $\overline{FT} + \overline{TF} \subseteq \overline{S} \cap (\overline{R} \cap \overline{F}) = 0$ which implies that $\overline{T} \subseteq Z(\overline{F})$ and thus $\overline{T} = Z(\overline{F})$. Thus, the commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\pi} & L \\
\downarrow{\pi} & & \downarrow{\pi} \\
F/S & \xrightarrow{\pi_2} & L/S
\end{array}
$$

yields the equality $\overline{\pi}(Z(F)) = \overline{\pi}(T) = \pi_3(T/S) = \pi_3(Z(F/S))$ by the definition of $T$. \hfill \Box

**Lemma 4.8.** Let $Z$ be a central ideal of finite-dimensional Leibniz algebra $L$ and let $\beta : M(L) \to M(L/Z)$ be as in Theorem 4.2. Then $Z \subseteq Z^*(L)$ if and only if $\beta$ is injective.

**Proof.** In the proof of the Ganea sequence, we saw that $\ker\beta$ can be interpreted as $\overline{FT} + \overline{TF}$. If $\beta$ is injective, then $\overline{FT} + \overline{TF} = 0$, which implies that $\overline{F} \subseteq Z(\overline{F})$. By the proof of Theorem 4.7, $Z \subseteq Z^*(L)$. Conversely, if $Z \subseteq Z^*(L)$, then $\overline{F} \subseteq Z(\overline{F})$, which implies that $\overline{FT} + \overline{TF} = 0$. Thus $\ker\beta = 0$ and $\beta$ is injective. \hfill \Box

**Theorem 4.9.** Let $Z$ be a central ideal of a finite-dimensional Leibniz algebra $L$ and

$$
\delta : M(L) \to L/L' \otimes Z \oplus Z \otimes L/L'
$$

be as in Theorem 4.1. Then the following are equivalent:

1. $\delta$ is the trivial map,
2. the natural map $\beta$ is injective,
3. $M(L) \cong M(L/Z)_{L/Z}$,
4. $Z \subseteq Z^*(L)$.

We conclude this section by narrowing our focus to when the conditions of Theorem 4.9 hold for $Z = Z(L)$.

**Theorem 4.10.** Let $L$ be a Leibniz algebra and $Z(L)$ be the center of $L$. If $Z(L) \subseteq Z^*(L)$, then $\omega(Z(E)) = Z(L)$ for every stem extension $0 \to \ker\omega \to E \xrightarrow{\omega} L \to 0$.

**Proof.** By definition, $Z^*(L) \subseteq \omega(Z(E)) \subseteq Z(L)$ for any stem extension $0 \to \ker\omega \to E \xrightarrow{\omega} L \to 0$. By hypothesis, $Z(L) \subseteq Z^*(L)$. Therefore $Z^*(L) = \omega(Z(E)) = Z(L)$. \hfill \Box
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