The positive semi-definite cone and sum-of-squares cone of Hankel form

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Abstract

In this paper, the geometry properties of Hankel form are studied, including their positive semi-definite (PSD) cone and sum-of-squares (SOS) cone. We denote them by $HPSD(m, n)$ and $HSOS(m, n)$, respectively. We show that both $HPSD(m, n)$ and $HSOS(m, n)$ are closed convex cones. The dual cone of $HPSD(m, n)$ is the convex hull of all $m$-times convolutions of real vectors. Besides, we derive the dual cone of SOS tensors. By reformulation, it follows that the dual cone of $HSOS(m, n)$ can also be written explicitly. These results may lead further research on the Hilbert-Hankel problem.

Key words: Hankel form, positive semi-definite cone, SOS cone, dual cone

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1 Introduction

Hankel structures are frequently encountered in applications such as signal processing \[15\]. Besides Hankel matrices, tensors with different Hankel structures also find applications in high-order singular value decompositions (HOSVD) \[1\], exponential data fitting \[7, 16, 17\] and signal separation \[6\]. In this paper, we study the geometry propositions of Hankel form, including their positive semi-definite (PSD) cone and sum-of-squares (SOS) cone.

Let \( A = (a_{i_1 \cdots i_m}) \in T_{m,n} \). If there is a vector \( v = (v_0, v_1, \cdots, v_{(n-1)m})^\top \in \mathbb{R}^{(n-1)m+1} \) such that for \( i_1, \cdots, i_m \in [n] \), we have

\[
a_{i_1 \cdots i_m} \equiv v_{i_1+i_2+\cdots+i_m-m},
\]

then we say that \( A \) is an \( m \)th order Hankel tensor and the vector \( v \) is called the generating vector of \( A \) \[2, 4, 5, 7, 11, 13, 18, 20\]. Clearly, a Hankel tensor is symmetric. Let \( x = (x_0, x_1, \cdots, x_p)^\top \in \mathbb{R}^{p+1} \) and \( y = (y_0, y_1, \cdots, y_q)^\top \in \mathbb{R}^{q+1} \). The convolution of \( x \) and \( y \) is defined as \( z = x * y = (z_0, z_1, \cdots, z_{p+q}) \in \mathbb{R}^{p+q+1} \), with

\[
z_i = \sum_{0 \leq i_1 \leq p, 0 \leq i_2 \leq q} x_{i_1} y_{i_2}
\]

for \( i = 0, 1, \cdots, p + q \) \[8\]. If we regard the entries of \( x \) and \( y \) as the coefficients of two polynomials, then the entries of \( z \) is the coefficients of the product polynomial of these two polynomials. For any \( x \in \mathbb{R}^n \), we denote

\[
x^{*m} = \underbrace{x \ast \cdots \ast x}_{m}.
\]

Let \( v \in \mathbb{R}^{(n-1)m+1} \) and \( x \in \mathbb{R}^n \). Then \( H = v \cdot x^{*m} \) is called a Hankel form of order \( m \) and dimension \( n \), where ‘\( \cdot \)’ denotes the standard inner product. In fact, let \( A \in T_{m,n} \) be the Hankel tensor generated by \( v \), it follows that \( H = v \cdot x^{*m} = A x^{*m} \).

Let \( m \) be even. We say that a Hankel form \( H \) is PSD if for any \( x \in \mathbb{R}^n \), \( H \geq 0 \). We say that \( H \) is SOS if \( H \) can be written as a sum of squares of homogeneous polynomials of \( x \) with degree \( k \). Clearly, if \( H \) is SOS, then \( H \) is PSD. The main question is: if \( H \) is PSD, is it SOS? If the answer to this question is yes, then the problem for determining an even order Hankel tensor is positive semi-definite or not is solvable in polynomial-time \[13\]. In a certain sense, it is the 17th Hilbert problem with the Hankel constraint. And the problem raised by the above question is called the Hilbert-Hankel problem \[5\]. Recently, some work showed that there are no PSD non-SOS Hankel tensors under certain conditions \[3, 5, 11, 12, 13, 18\]. However, until now, it still remains an open problem.

In this paper, we study the PSD cone and SOS cone of Hankel form respectively. This may lead further research on the Hilbert-Hankel problem. The rest of this chapter is organized as follows. In Section 2, we consider the PSD cone of Hankel form. And we show...
that its dual cone is the convex hull of all \( m \)-times convolutions of real vectors. The Hankel spectrahedra is introduced in Section 3. By reformulation, we give a new way to get the dual cone of the PSD cone and SOS cone of Hankel form. In Section 4, we study the SOS cone of Hankel form, as well as its dual cone.

## 2 The PSD cone of Hankel form

Suppose that \( m = 2k \), \( \mathbf{v} = (v_0, \cdots, v_{(n-1)m})^\top \) and \( \mathbf{x} = (x_0, \cdots, x_{n-1})^\top \). Define

\[
H_{PSD}(m, n) = \{ \mathbf{v} \in \mathbb{R}^{(n-1)m+1} : H = \mathbf{v} \bullet \mathbf{x}^{2k} \text{ is PSD} \}
\]

and

\[
H_{SOS}(m, n) = \{ \mathbf{v} \in \mathbb{R}^{(n-1)m+1} : H = \mathbf{v} \bullet \mathbf{x}^{2k} \text{ is SOS} \}.
\]

Obviously, \( H_{SOS}(m, n) \subseteq H_{PSD}(m, n) \).

### Proposition 1

Let \( m \) be even. Then \( H_{PSD}(m, n) \) and \( H_{SOS}(m, n) \) are closed convex cone.

**Proof.** Clearly, \( H_{PSD}(m, n) \) is a convex cone. Now we prove its closeness. Let \( \{ \mathbf{v}_k \} \subseteq H_{PSD}(m, n) \) with \( \mathbf{v}_k \to \mathbf{v} \). By definition, we have \( \mathbf{v}_k \bullet \mathbf{x}^{2m} \geq 0 \) for any \( \mathbf{x} \in \mathbb{R}^n \). It follows that \( \mathbf{v} \bullet \mathbf{x}^{2m} = \lim_{k \to \infty} \mathbf{v}_k \bullet \mathbf{x}^{2m} \geq 0 \) for any \( \mathbf{x} \in \mathbb{R}^n \). Hence, \( \mathbf{v} \in H_{PSD}(m, n) \). The convexity of the \( H_{SOS}(m, n) \) cone also directly follows from the definition. Let \( \{ \mathbf{v}_k \} \subseteq H_{SOS}(m, n) \) with \( \mathbf{v}_k \to \mathbf{v} \). Since \( H_{SOS}(m, n) \subseteq H_{PSD}(m, n) \), \( \mathbf{v} \in H_{PSD}(m, n) \). For \( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{v}_k \bullet \mathbf{x}^{2m} \) is an SOS polynomial and \( \mathbf{v}_k \bullet \mathbf{x}^{2m} \to \mathbf{v} \bullet \mathbf{x}^{2m} \). Note from [10] that the set of all SOS polynomials on \( \mathbb{R}^n \) with degree at most \( m \) is a closed cone. So, \( \mathbf{v} \bullet \mathbf{x}^{2m} \) is also an SOS polynomial. Therefore, \( \mathbf{v} \in H_{SOS}(m, n) \). □

Recall that for a given set \( S \) in the Euclidean space \( \mathbb{R}^n \), its dual cone \( S^* \) is defined as

\[
S^* = \{ \mathbf{y} \in \mathbb{R}^n : \mathbf{y} \bullet \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in S \}.
\]

Note that \( S^* \) is always a convex cone, even if \( S \) is neither convex nor a cone. When \( S \) is a closed convex cone, we have \( S^{**} = S \). To establish the dual cone of \( H_{PSD}(m, n) \), we introduce a set which is the convex hull of all \( m \)-times convolutions of real vectors.

### Definition 1

Let \( m \) be even and \( n \in \mathbb{N} \). We denote by \( U(m, n) \) the convex hull of all \( m \)-times convolutions of real vectors in \( \mathbb{R}^n \), i.e.,

\[
U(m, n) := \text{conv}\{ \mathbf{x}^{2m} : \mathbf{x} \in \mathbb{R}^n \}.
\]

In fact, \( U(m, n) \) is a closed convex cone. Before that, a useful lemma is needed.
Lemma 1  Let $m$ be even and $n \in \mathbb{N}$. For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, there exists a constant $c > 0$ such that
\[ c \|x^m + y^m\| \geq \max\{\|x\|^m, \|y\|^m\}. \]

Proof. Let $z = x^m + y^m$. For any $t \in \mathbb{R}$, denote
\[ t = (1, t, \ldots, t^{n-1})^\top \quad \text{and} \quad \tilde{t} = (1, t, \ldots, t^{(n-1)m})^\top. \]

By definition, we have $z \bullet \tilde{t} = (x \bullet t)^m + (y \bullet t)^m \geq \max\{(x \bullet t)^m, (y \bullet t)^m\}$. According to Cauchy-Schwarz inequality, one can obtain that
\[ \|z\|\|\tilde{t}\| \geq \max\{\|x \bullet t\|^m, \|y \bullet t\|^m\}. \]

Now for $k = 0, \ldots, n - 1$, we choose $t = t_k \in \mathbb{R}$ such that $t_i \neq t_j$ when $i \neq j$. Denote
\[ t_k = (1, t_k, \ldots, t_k^{n-1})^\top \quad \text{and} \quad \tilde{t}_k = (1, t_k, \ldots, t_k^{(n-1)m})^\top. \]

Let $c_k = \|\tilde{t}_k\| > 0$. Define $w = (w_0, \ldots, w_{n-1})^\top \in \mathbb{R}^n$ with $w_k = x \bullet t_k$. Then we have $w = T x$, where $T$ is the transpose of a Vandermonde matrix with the $k$th column $t_k$, $k = 0, \ldots, n - 1$. It follows that
\[ \|z\| \sum_{k=1}^{n-1} c_k \geq \sum_{k=0}^{n-1} w_k^m = \|w\|^m. \]

On the other hand, there exists a constant $C > 0$ such that $\|w\| \leq C\|w\|^m$. Since $T$ is nonsingular, we have $\|x\| \leq \|T^{-1}\|\|w\|$, where $\|T^{-1}\| > 0$ is the largest singular of $T^{-1}$. So
\[ \|x\|^m \leq \|T^{-1}\|^m \sum_{k=0}^{n-1} c_k \|z\|. \]

Let $c = \|T^{-1}\|^m \sum_{k=0}^{n-1} c_k > 0$. Then $\|x\|^m \leq c \|z\|$. By a similar way, we have $\|y\|^m \leq c \|z\|$. So the proof is completed. \(\square\)

From the proof above, we have the following corollary.

Corollary 1  Let $m$ be even and $n \in \mathbb{N}$. Let $s \geq 1$ be a given integer. Then for any $x^j \in \mathbb{R}^n$, $j = 1, \ldots, s$, there exists a constant $c > 0$ such that
\[ c \sum_{j=1}^{s} (x^j)^m \geq \max_{1 \leq j \leq s} \|x^j\|^m. \]

We are now ready to prove that $U(m, n)$ is a closed convex cone.

Lemma 2  Let $m$ be even and $n \in \mathbb{N}$. Then $U(m, n)$ is a closed convex cone with dimension at most $(n - 1)m + 1$. 

\[ \tag{4} \]
Proof. Since $U(m, n) \subseteq \mathbb{R}^{(n-1)m+1}$, the dimension of $U(m, n)$ is at most $(n-1)m + 1$. The convexity of the $U(m, n)$ cone also directly follows from the definition. To see the closeness, let $\{y_k\}_{k=1}^{\infty} \subseteq U(m, n)$ such that $y_k \to y$. Clearly, $y_k$ is bounded. For any $y_k$, by the Carathéodory theorem, there exist $x^j_k$, $j = 0, \cdots, (n-1)m$ such that

$$y_k = \sum_{j=0}^{(n-1)m} (x^j_k)^m.$$ 

By Corollary 1 we can see that for any $j = 0, \cdots, (n-1)m$, the sequence $\{x^j_k\}_{k=1}^{\infty}$ is also bounded. By passing to the subsequences, we can assume that $x^j_k \to x^j$, $j = 0, \cdots, (n-1)m$. It follows that

$$y = \lim_{k \to \infty} y_k = \sum_{j=0}^{(n-1)m} (x^j)^m \in U(m, n).$$

Thus, the proof is completed.

Theorem 1 Let $m$ be even and $n \in \mathbb{N}$. Then

$$U(m, n)^* = HPSD(m, n) \quad \text{and} \quad HPSD(m, n)^* = U(m, n).$$

Proof. First, we show that $U(m, n)^* = HPSD(m, n)$. Suppose $v \in U(m, n)^*$. Then, by definition, we have $v \cdot y \geq 0$ for all $y \in U(m, n)$. In particular, $v \cdot x^m \geq 0$ for all $x \in \mathbb{R}^n$. It follows that $v \in HPSD(m, n)$. Hence, $U(m, n)^* \subseteq HPSD(m, n)$. On the other hand, the dimension of $U(m, n)$ is at most $(n-1)m + 1$. For any $y \in U(m, n)$, by the Carathéodory theorem, there exist $x^j \in \mathbb{R}^n$, $j = 0, \cdots, (n-1)m$ such that

$$y = \sum_{j=0}^{(n-1)m} (x^j)^m.$$ 

Suppose $v \in HPSD(m, n)$. Then $v \cdot y \geq 0$ since $v \cdot (x^j)^m \geq 0$ for any $j = 0, \cdots, (n-1)m$. So $HPSD(m, n) \subseteq U(m, n)^*$. Therefore, we have $U(m, n)^* = HPSD(m, n)$.

For the second part, the equality $HPSD(m, n)^* = U(m, n)^{**} = U(m, n)$ holds since $U(m, n)$ is a closed convex cone by Lemma 2.

According to the fact $HSOS(m, n) \subseteq HPSD(m, n)$, we have the following corollary.

Corollary 2 Let $m$ be even and $n \in \mathbb{N}$. Then

$$HSOS(m, n) \subseteq U(m, n)^* \quad \text{and} \quad U(m, n) \subseteq HSOS(m, n)^*.$$
3 Hankel Spectrahedra

As we know, Semidefinite programming is a very important and valuable tool in polynomial optimization. And it can be solved in polynomial time. The feasible region of a Semidefinite Program forms a closed convex set which has the form

$$S = \{ x \in \mathbb{R}^m : Q_0 + \sum_{i=1}^{m} x_i Q_i \succeq 0 \},$$

where $Q_i \in S_{2,n}$, $i = 0, 1, \cdots, m$. The feasible region of a Semidefinite Program is called a Spectrahedra \[19\]. When $Q_0 = 0$, $S$ is a closed convex cone.

Given a set of Vandermonde vectors

$$u_k = (1, u_k, \cdots, u_k^{n-1})^\top \in \mathbb{R}^n, \quad k = 0, \cdots, (n-1)m,$$

where $u_i \neq u_j$ when $i \neq j$. It has been shown \[18\] that a symmetric tensor $A \in S_{m,n}$ is a Hankel tensor if and only if it has a Vandermonde decomposition, i.e., there exists a vector $\alpha = (\alpha_0, \cdots, \alpha_{(n-1)m})^\top \in \mathbb{R}^{(n-1)m+1}$ such that

$$A = \sum_{k=0}^{(n-1)m} \alpha_k u_k^{\otimes m},$$

where $u_k^{\otimes m} \in S_{m,n}$ is the rank-one tensor with entries $u_1 \cdots u_m$. In fact, this representation of a Hankel tensor is unique when the set of Vandermonde vectors is given.

Let $m$ be even. A tensor $A \in S_{m,n}$ is called positive semi-definite if $A x^m \geq 0$ for all $x \in \mathbb{R}^n$. Let $PSD_{m,n}$ be the set of all positive semi-definite tensors. Denote

$$G = \{ \alpha \in \mathbb{R}^{(n-1)m+1} : \sum_{k=0}^{(n-1)m} \alpha_k u_k^{\otimes m} \in PSD_{m,n} \}.$$

This can be seen as the generalized spectrahedra for the tensor case. Clearly, $\mathbb{R}^{(n-1)m+1} \nsubseteq G$. It is easy to see that $G$ is also a closed convex cone. And there is a one-to-one mapping between $G$ and $HPSD(m,n)$. In particular, let $U \in \mathbb{R}^{[(n-1)m+1] \times [(n-1)m+1]}$ be the Vandermonde matrix with the $k$th column $(1, u_k, \cdots, u_k^{(n-1)m})^\top, \ k = 0, 1, \cdots, (n-1)m$. Since $u_i \neq u_j$ when $i \neq j$, the Vandermonde matrix is nonsingular. And the linear mapping $f : G \rightarrow HPSD(m,n)$ is a bijection defined by $f(\alpha) = U\alpha$.

By this reformulation, we give a new way to get the dual cone of $HPSD(m,n)$. Denote by $U_{m,n}$ the convex hull of all $m$th-order $n$-dimensional symmetric rank-one tensors, i.e.,

$$U_{m,n} = \text{conv}\{ x^{\otimes m} : x \in \mathbb{R}^n \}.$$

It has been shown \[9\] that $PSD_{m,n}$ and $U_{m,n}$ are dual cones, i.e.,

$$PSD^*_{m,n} = U_{m,n} \quad \text{and} \quad U_{m,n}^* = PSD_{m,n}.$$
Theorem 2 Let $G$ be defined as above. The dual cone of $G$ is the closure of the set

$$H := \{ (A u_0^m, A u_1^m, \cdots, A u_{(n-1)m})^\top \in \mathbb{R}^{(n-1)m+1} : A \in U_{m,n} \}.$$ 

Proof. Clearly, $H$ is convex cone. Suppose $\alpha \in H^*$. By definition, we have that

$$\sum_{k=0}^{(n-1)m} \alpha_k u_k^m = \left( A, \sum_{k=0}^{(n-1)m} \alpha_k u_k^m \right) \geq 0, \quad \forall A \in U_{m,n}.$$ 

It means that $\alpha \in H^*$ if and only if $\sum_{k=0}^{(n-1)m} \alpha_k u_k^m \in U_{m,n}^* = PSD_{m,n}$. So we have $H^* = G^* = H^{**} = \text{cl}H$. \hfill \Box

Corollary 3 Let $m$ be even and $n \in \mathbb{N}$. Let $U \in \mathbb{R}^{((n-1)m+1) \times ((n-1)m+1)}$ be the Vandermonde matrix generated by the set $\{u_k\}_{k=0}^{(n-1)m}$ with $u_i \neq u_j$ when $i \neq j$. Then

$$\{ U^\top y : y \in U(m,n) \} = \text{cl}H,$$

where $U(m,n)$ and $H$ are defined in Definition 1 and Theorem 2, respectively.

4 The SOS cone of Hankel form

Let $m$ be even. A tensor $A \in S_{m,n}$ is called a SOS tensor if $A x^m \geq 0$ is a SOS polynomial of $x \in \mathbb{R}^n$ [14]. Let $SOS_{m,n}$ be the set of all SOS tensors. Denote

$$SG = \{ \alpha \in \mathbb{R}^{(n-1)m+1} : \sum_{k=0}^{(n-1)m} \alpha_k u_k^m \in SOS_{m,n} \},$$

where $u_k \in \mathbb{R}^n$ are Vandermonde vectors given in (1). Clearly, $\mathbb{R}^{(n-1)m+1} \subseteq SG$. From the analysis above, the linear mapping $\tilde{f} : SG \rightarrow HSOS(m,n)$ defined by $\tilde{f}(\alpha) = U \alpha$ is a one-to-one mapping between $SG$ and $HSOS(m,n)$, where $U \in \mathbb{R}^{((n-1)m+1) \times ((n-1)m+1)}$ is the Vandermonde matrix with the $k$th column $(1, u_k, \cdots, u_k^{(n-1)m})^\top, k = 0, 1, \cdots, (n-1)m$. Since $u_i \neq u_j$ when $i \neq j$, the Vandermonde matrix is nonsingular.

In fact, if we know the dual cone of $SOS_{m,n}$, the dual cone of $HSOS(m,n)$ can also be derived similarly. First, we consider the SOS cone of symmetric tensors with $(m,n) = (6,3)$, i.e., $SOS_{6,3}$. For any $A \in S_{6,3}$, there are 28 independent elements. We index these elements by the 3-tuples of degree 6, i.e.

$$A = (a_\alpha), \quad \alpha \in \left\{ (\alpha_0, \alpha_1, \alpha_2) : \sum_{i=0}^{2} \alpha_i = 6, \alpha_i \geq 0, \forall i = 0, 1, 2 \right\}.$$
For simplicity, we write $A = (a_\alpha)|_{|\alpha|=6} \in S_{6,3}$ to emphasis on these 28 independent elements. For any $w = (x, y, z)^T \in \mathbb{R}^3$, $Aw^6$ is a homogeneous polynomial with degree 6, i.e.,

$$Aw^6 = \sum_{|\alpha|=6} c_\alpha a_\alpha w^\alpha,$$

where $w^\alpha$ denotes the monomial $x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}$, and $c_\alpha$ is the number of the elements $a_\alpha$ in $A$. For example, $c_{600} = 1$, $a_{510} = 6$ and $c_{321} = 60$. By simple computation, there are exactly 28 monomials of $w$ of degree 6.

Let $[w]_3 := (x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^2z, yz^2, z^3)^T \in \mathbb{R}^{10}$ be the vector of all the monomials of $w$ of degree 3. Then $Aw^6$ is a sum of square with SOS rank $r$ if and only if there exist $c_i \in \mathbb{R}^{10} \setminus \{0\}$, $i = 1, \cdots, r$, such that

$$Aw^6 = \sum_{i=1}^r (c_i^T [w]_3)^2 = [w]_3^T \left( \sum_{i=1}^r c_i c_i^T \right) [w]_3,$$

i.e., there exists a positive semi-definite matrix $Q \in \mathbb{R}^{10 \times 10}$ with rank $r$ such that

$$Aw^6 = [w]_3^T Q [w]_3.$$

Moreover, by indexing the matrix $Q$ by the 10 monomials of $w$ of degree 3 (or, more precisely, the associated exponent tuples), we obtain the following conditions:

$$c_\alpha a_\alpha = \sum_{\beta+\gamma=\alpha} Q_{\beta\gamma}, \quad Q \succeq 0.$$

This is a system of 28 linear equations, one for each coefficient of $Aw^6$. We denote these linear equations by $\langle A_\alpha, Q \rangle = c_\alpha a_\alpha$ for any $\alpha$ satisfying $|\alpha| = 6$. In particular, the matrix $Q$ is expressed as follows:

$$Q = \begin{bmatrix}
q_{(300,300)} & q_{(300,210)} & q_{(300,201)} & q_{(300,120)} & q_{(300,111)} & q_{(300,030)} & q_{(300,021)} & q_{(300,012)} & q_{(300,003)} \\
q_{(210,300)} & q_{(210,210)} & q_{(210,201)} & q_{(210,120)} & q_{(210,111)} & q_{(210,030)} & q_{(210,021)} & q_{(210,012)} & q_{(210,003)} \\
q_{(201,300)} & q_{(201,210)} & q_{(201,201)} & q_{(201,120)} & q_{(201,111)} & q_{(201,030)} & q_{(201,021)} & q_{(201,012)} & q_{(201,003)} \\
q_{(120,300)} & q_{(120,210)} & q_{(120,201)} & q_{(120,120)} & q_{(120,111)} & q_{(120,030)} & q_{(120,021)} & q_{(120,012)} & q_{(120,003)} \\
q_{(111,300)} & q_{(111,210)} & q_{(111,201)} & q_{(111,120)} & q_{(111,111)} & q_{(111,030)} & q_{(111,021)} & q_{(111,012)} & q_{(111,003)} \\
q_{(103,300)} & q_{(103,210)} & q_{(103,201)} & q_{(103,120)} & q_{(103,111)} & q_{(103,030)} & q_{(103,021)} & q_{(103,012)} & q_{(103,003)} \\
q_{(102,300)} & q_{(102,210)} & q_{(102,201)} & q_{(102,120)} & q_{(102,111)} & q_{(102,030)} & q_{(102,021)} & q_{(102,012)} & q_{(102,003)} \\
q_{(021,300)} & q_{(021,210)} & q_{(021,201)} & q_{(021,120)} & q_{(021,111)} & q_{(021,030)} & q_{(021,021)} & q_{(021,012)} & q_{(021,003)} \\
q_{(012,300)} & q_{(012,210)} & q_{(012,201)} & q_{(012,120)} & q_{(012,111)} & q_{(012,030)} & q_{(012,021)} & q_{(012,012)} & q_{(012,003)} \\
q_{(003,300)} & q_{(003,210)} & q_{(003,201)} & q_{(003,120)} & q_{(003,111)} & q_{(003,030)} & q_{(003,021)} & q_{(003,012)} & q_{(003,003)}
\end{bmatrix}.$$

Let $E_{ij}$ be the matrix in $\mathbb{R}^{10 \times 10}$ with $(i, j)$-th entry 1 and 0 otherwise. It follows that $A_{600} = E_{11}$, $A_{510} = E_{12} + E_{21}$, $A_{501} = E_{13} + E_{31}$, $A_{420} = E_{14} + E_{22} + E_{41}$ and so on. We can see that $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$. By this reformulation, we have

$$SOS_{6,3} = \{(a_\alpha)|_{|\alpha|=6} \in S_{6,3} : \langle A_\alpha, Q \rangle = c_\alpha a_\alpha \text{ for all } |\alpha| = 6, \quad Q \in \mathbb{R}_+^{10 \times 10} \}. \quad (2)$$

In fact, we can generalize this result to $SOS_{m,n}$ where $m$ is even. Let $m = 2k$. For any $x = (x_0, \cdots, x_{n-1})^T \in \mathbb{R}^n$, there are $\binom{n+m-1}{m}$ monomials of $x$ of degree $m$, and there are $\binom{n+k-1}{k}$
monomials of \( x \) of degree \( k \). It means that for \( \text{SOS}_{m,n} \), there are \( \binom{n+m-1}{m} \) linear equations on the right hand side of (2), and \( Q \in \mathbb{R}_+^{\binom{n+k-1}{k} \times \binom{n+k-1}{k}} \). Moreover, we can write the constant \( c_\alpha \) precisely. Denote \( S = \{ (\alpha_0, \cdots, \alpha_{n-1}) : \sum_{i=0}^{n-1} \alpha_i = m, \alpha_i \geq 0, \forall i = 0, \cdots, n-1 \} \). For any tuple of exponents \( \alpha \in S \), we have

\[
c_\alpha = \left( \frac{m}{\alpha_0} \right) \left( \frac{m-\alpha_0}{\alpha_1} \right) \cdots \left( \frac{\alpha_{n-1}}{\alpha_{n-1}} \right).
\]

For instance, \( c_{m0\cdots0} = 1 \), \( c_{(m-1)1\cdots0} = m \) and \( c_{00\cdots m} = 1 \).

**Theorem 3** Let \( m = 2k \) be even and \( n \in \mathbb{N} \). Then

\[
\text{SOS}_{m,n}^* = \left\{ (b_\alpha)_{|\alpha|=m} \in S_{m,n} : \langle A_\alpha, Q \rangle = c_\alpha a_\alpha \text{ for all } |\alpha| = m, \quad Q \in \mathbb{R}_+^{\binom{n+k-1}{k} \times \binom{n+k-1}{k}} \right\},
\]

where \( c_\alpha \) are given by (3) and the \( \binom{n+m-1}{m} \) matrices \( \{A_\alpha\}_{|\alpha|=m} \) are only depend on \( (m,n) \), satisfying \( \langle A_\alpha, A_\beta \rangle = 0 \) if \( \alpha \neq \beta \).

Based on this result, the dual cone of \( \text{SOS}_{m,n}^* \) can be derived immediately.

**Theorem 4** Let \( m = 2k \) be even and \( n \in \mathbb{N} \). Then,

\[
\text{SOS}_{m,n}^* = \left\{ (b_\alpha)_{|\alpha|=m} \in S_{m,n} : \sum_{|\alpha|=m} b_\alpha A_\alpha \succeq 0 \right\},
\]

where \( \{A_\alpha\}_{|\alpha|=m} \) are symmetric \( \binom{n+k-1}{k} \times \binom{n+k-1}{k} \) matrices given in Theorem 3.

**Proof.** Let \( M = \left\{ (b_\alpha)_{|\alpha|=m} \in S_{m,n} : \sum_{|\alpha|=m} b_\alpha A_\alpha \succeq 0 \right\} \). Let \( (b_\alpha)_{|\alpha|=m} \in S_{m,n} \). It follows from Theorem 3 that \( (b_\alpha) \in \text{SOS}_{m,n}^* \) if and only if \( \langle (b_\alpha), (a_\alpha) \rangle = \sum_{|\alpha|=m} c_\alpha b_\alpha a_\alpha \geq 0 \) for any \( (a_\alpha) \in \text{SOS}_{m,n} \), i.e.,

\[
\sum_{|\alpha|=m} b_\alpha \langle A_\alpha, Q \rangle = \left( \sum_{|\alpha|=m} b_\alpha A_\alpha, Q \right) \succeq 0, \quad \forall Q \in \mathbb{R}_+^{\binom{n+k-1}{k} \times \binom{n+k-1}{k}}.
\]

Since all the positive semi-definite matrices are self-dual, this is equivalent to say that \( (b_\alpha)_{|\alpha|=m} \in M \). The proof is completed. \( \square \)

It can be seen that the dual cone of \( \text{SOS}_{m,n} \) is nothing but a spectrahedra.

In the following, we establish the dual cone of \( \text{HSOS}(m,n) \). Let \( SG \) be the set defined in the beginning of this section. Similar with Theorem 2 we have

\[
SG^* = \text{cl} \left\{ (A u_0^m, A u_1^m, \cdots, A u_{(n-1)m})^\top \in \mathbb{R}^{(n-1)m+1} : A \in \text{SOS}_{m,n}^* \right\},
\]

where \( u_k \in \mathbb{R}^n \) are Vandermonde vectors given in (1). Since there is a one-to-one mapping between \( SG \) and \( \text{HSOS}(m,n) \), we have the following conclusion.
Theorem 5. Let $m$ be even and $n \in \mathbb{N}$. Let $U \in \mathbb{R}^{\binom{(n-1)m+1}{2}}$ be the Vandermonde matrix generated by the set $\{u_k\}_{k=0}^{\binom{n-1}{m}}$ with $u_i \neq u_j$ when $i \neq j$. Then

$$HSOS(m, n)^* = \{(U^\top)^{-1}\alpha : \alpha \in SG^*\},$$

where $SG^*$ is given by [4].

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