Diploma Thesis

by Alexander Getmanenko

Geometry and Moduli Space of Certain Rank-4 Vector Bundles on $\mathbb{P}^4$

under the Supervision of
Prof. Dr. G. Trautmann

June, 2000
I thank Prof. Dr. G. Trautmann, my supervisor for this Diploma thesis, for the interesting problem, for many hours of discussions and for deep mathematical ideas I could learn from him.

I express my gratitude to Dr. B. Kreußler who took a lot of care of me and my education in the field of algebraic geometry, and to all members and students of the Group for Algebraic Geometry at the University Kaiserslautern for their sympathy and helpfulness, mathematical and personal.

My studies in Germany were made possible by the scholarship I was granted by the Faculty of Mathematics of the University Kaiserslautern as a participant of the program "Mathematics International".
Contents

1 Introduction 1
2 Basics 3
3 The Dual Bundle 16
4 The Construction of the Moduli Space 25
5 Smoothness of the Moduli Space 31
6 The Jumping Lines 37
7 Subsets of the Moduli Space 41
8 A Description of the Moduli Space as a Fibration 44
9 Restriction to a 3-Plane 46
10 Sections of the Kernel Bundle 49
References 50
1 Introduction

The topic of this diploma thesis are instanton bundles of rank 4 with the second Chern class (quantum number) 2 on $\mathbb{P}^4$ which are, as will be shown in the section 2, precisely the bundles appearing as cohomologies of short monads

$$2\Omega^4_{\mathbb{P}^4}(4) \xrightarrow{M} 2\Omega^1_{\mathbb{P}^4}(1) \xrightarrow{N} 2\mathcal{O}_{\mathbb{P}^4}$$

We use the following general definition of instanton bundles:

**Definition 2.1** A mathematical instanton bundle on $\mathbb{P}^s$ with quantum number $n$ or an $n$-instanton is an algebraic vector bundle $E$ on $\mathbb{P}^s$ which satisfies

(i) $E$ has Chern polynomial $(1 - h^2)^{-n}$ and $s - 1 \leq r = \text{rank} E \leq (s - 1)n$

(ii) for any $d$ in the range $-s \leq d \leq 0$ we have $h^i E(d) \neq 0$ for at most one $i$.

In [1] you can find a collection of general results and basic instructions that are useful to work with bundles with these properties.

The moduli problem for instanton bundles on $\mathbb{P}^3$ is being intensively studied for different Chern classes. Originally instanton bundles of rank 2 on $\mathbb{P}^3$ appeared as liftings of bundles on $S^4$ via the twistor construction, see [13]. They have been studied in a number of papers, e.g. [3], [14], [17], [18], [19], [20]. The most recent result in the direction is that in [23] where it is proved the regularity of the moduli space of mathematical instanton bundles on $\mathbb{P}^3$ with $c_2 = 5$. Together with the moduli problem, different questions about the geometry can be considered, e.g. geometry of zero sets of sections and sets of jumping lines, cf. [3], [15], [16].

Instanton bundles of rank $s - 1$ on $\mathbb{P}^s$ for odd $s$ have been studied in [11], [21], [22], [23], [24].

Another reason making mathematical instanton bundles interesting is, that they give nontrivial examples of simple vector bundles on projective spaces of different dimension.

In this diploma thesis I am discussing instanton bundles of rank 4 with quantum number 2 on $\mathbb{P}^4$. This case is similar to that considered in papers [3], [17] on $\mathbb{P}^3$ and to special ’t Hooft bundles on $\mathbb{P}^5$ in [11].

The main results of my diploma thesis are:

**Theorem 4.7** There exist a 29-dimensional irreducible quasi-projective variety $M$ that is a coarse moduli space of instanton bundles on $\mathbb{P}^4$ with quantum number 2 and of rank 4.

**Theorem 5.1** The moduli space $M$ is smooth.

**Theorem 3.1** Let an instanton bundle $E$ of rank 4 with quantum number 2 on $\mathbb{P}^4 = \mathbb{P}V$ ($V$ is a 5-dimensional vector space) be given by a monad

$$2\Omega^4_{\mathbb{P}^4}(4) \xrightarrow{M} 2\Omega^1_{\mathbb{P}^4}(1) \xrightarrow{N} 2\mathcal{O}_{\mathbb{P}^4}$$

and let the matrix $P$ be defined as the kernel in

$$0 \to k^p \xrightarrow{P} k^2 \otimes V \xrightarrow{\tilde{M}} k^2 \otimes \bigwedge^4 V,$$

where $\tilde{M}$ is the contraction with the matrix $M$ and can be proved to be of rank 7 or 8. Then the dual bundle $E^\vee$ is either the cohomology bundle of a monad

$$2\Omega^4(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 2\mathcal{O}$$
if \( \text{rank} \tilde{M} = 8 \), and is therefore an instanton bundle, too, or, if \( \text{rank} \tilde{M} = 7 \), it is an extension

\[
0 \to \mathcal{O} \to \mathcal{E}^\vee \to \mathcal{E}_1^\vee \to 0
\]

where \( \mathcal{E}_1^\vee \) is a reflexive sheaf appearing as the cohomology of the monad

\[
2\Omega^4(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 3\mathcal{O},
\]

**Theorem 6.1** In the notation of theorem 3.1, let \( \ell \) be a line in \( \mathbb{P}^4 \) identified via the Plücker embedding with an element of \( \mathbb{P}^{\wedge^2 V} \supset G(2,5) \ni \ell \). If \( (\ell) = \langle x \wedge y \rangle \) in \( \mathbb{P}^{\wedge^2 V} \) and \( M = (m_{i,j})_{i,j=1,2} \), denote by \( M \wedge \ell \) a matrix \((m_{i,j} \wedge x \wedge y)_{i,j=1,2} \in \text{Mat}_{2 \times 2}(\wedge^2 V) \cong \text{Mat}_{2 \times 2}(k) \) (well defined up to proportionality).

Then:
(i) if \( \text{rank} M \wedge \ell = 2 \), then \( \ell \) is not a jumping line;
(ii) if \( \text{rank} M \wedge \ell = 1 \), then \( \ell \) is a jumping line with splitting

\[
\mathcal{E}_\ell \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(1);
\]

(iii) if \( M \wedge \ell = 0 \), then \( \ell \) is a jumping line with

\[
\mathcal{E}_\ell \cong \mathcal{O}_\ell(-1)^{\oplus 2} \oplus \mathcal{F} \quad \text{or} \quad \mathcal{E}_\ell \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell \oplus \mathcal{F}
\]

where \( \mathcal{F} \cong \mathcal{O}_\ell(1)^{\oplus 2} \) or \( \mathcal{F} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2) \);
(iv) in the situation of (iii), the jumping lines \( \ell \) with the property \( \mathcal{E}_\ell = \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell \oplus \mathcal{F} \) form a smooth conic on the Grassmannian \( G(2,5) \);
(v) in the situation of (iii), the jumping lines \( \ell \) with the property \( \mathcal{F} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2) \) form either a smooth conic, if \( \text{rank} \tilde{M} = 8 \), or, if \( \text{rank} \tilde{M} = 7 \), a surface on the Grassmannian \( G(2,5) \).

**Theorem 9.2** In the notation of theorem 3.1, denote by \( W = \text{span} N \) the vector subspace of \( V \) generated by the entries of \( N \). If \( \dim W = 4 \) and \( H = \mathbb{P}W \cong \mathbb{P}^3 \) is a hyperplane in \( \mathbb{P}^4 \), then \( \mathcal{E}|_H \cong H \oplus 2\mathcal{O}_H \), where \( H \) is an instanton bundle on \( \mathbb{P}^3 \) of rank 2 with \( c_2 = 2 \).

In addition to these theorems, the following results are obtained:

- computation of dimensions and proof of irreducibility of certain subsets of the moduli space that naturally arise in the previous steps (section 7), namely:
  \( \mathcal{M}^3 := \) sheaves in whose monads \( \dim \text{span} N = 3 \)
  \( \mathcal{M}^7 := \) sheaves \( \mathcal{E} \) in whose monads \( \text{rank} \tilde{M} = 7 \) or, equivalently (cf. corollary 3.3), \( H^0\mathcal{E}^\vee = 1 \)
  \( \mathcal{M}^{sd} := \) self-dual bundles \( \mathcal{E} \), i.e. bundles with \( \mathcal{E} \cong \mathcal{E}^\vee \).

In particular, it is shown (theorem 7.2) that \( \mathcal{M}^7 \subset \mathcal{M}^3 \).

- construction of a fibration of the moduli space over an open dense subset of the set of singular hyperquadrics in \( \mathbb{P}^4 \) (section 8);

- some remarks on geometry of sections of the kernel bundle of monads defining instanton bundles on \( \mathbb{P}^4 \) with \( c_2 = 2 \) (section 10).
2 Basics

In this preparatory section we are giving some basic definitions and facts about mathematical instanton bundles in general we shall use in the sequel and also describe the objects we are going to work with in our situation.

The main reference here is [1]. We are working with algebraic schemes over an algebraically closed field \( k \) of characteristic 0 and therefore may do most of our proofs for closed points only.

Given a projectivization \( \mathbb{P}W \) of a \( k \)-vector space \( W \), we denote by \( \langle w \rangle \) the image of \( w \in W \) under the canonical projection \( W - \{0\} \rightarrow \mathbb{P}W \).

Given a \( k \)-vector space, a sheaf, a module etc. \( A \) and an integer \( n \geq 0 \), the symbols \( A^{\oplus n} \), \( nA \) and \( k^n \otimes A \) mean the same; in each particular situation we try to use the most convenient notation.

To fix the notation, we take the \( s \)-dimensional projective space \( \mathbb{P}^s \) to be the projectivization of an \( (s+1) \)-dimensional vector space \( V \) and write \( \mathbb{P}^s = \mathbb{P}V \).

**Definition 2.1** A mathematical instanton bundle on \( \mathbb{P}^s \) with quantum number \( n \) or an \( n \)-instanton is an algebraic vector bundle \( E \) on \( \mathbb{P}^s \) which satisfies

(i) \( E \) has Chern polynomial \((1 - h^2)^{-n}\) and \( s - 1 \leq r = \text{rank} E \leq (s - 1)n \)

(ii) for any \( d \) in the range \(-s \leq d \leq 0\) we have \( h^iE(d) \neq 0 \) for at most one \( i \).

**Lemma 2.2** [Lemma 1.3] Assuming condition (i), condition (ii) is equivalent to condition

(ii') \( a \) \( h^iE = 0 \) for \( i \neq 1 \) and \( h^1E = (s - 1)n - r \)

(\( b \)) \( h^iE(-1) = 0 \) for \( i \neq 1 \) and \( h^1E(-1) = n \)

(\( c \)) \( h^iE(d) = 0 \) for \(-s < d < -1 \) and all \( i \)

(\( d \)) \( h^iE(-s) = 0 \) for \( i \neq s - 1 \) and \( h^{s - 1}E(-s) = n \)

The objects of this study are rank-4 instanton bundles on \( \mathbb{P}^4 \) with quantum number 2.

The conditions can be written in this case as:

(i) The Chern polynomial is \( c(E) = (1 - h^2)^{-2} = 1 + 2h^2 + 3h^4 \)

(ii') \( a \) \( h^iE = 0 \) for \( i \neq 1 \) and \( h^1E = 2 \)

(\( b \)) \( h^iE(-1) = 0 \) for \( i \neq 1 \) and \( h^1E(-1) = 2 \)

(\( c \)) \( h^iE(d) = 0 \) for \(-4 < d < -1 \) and all \( i \)

(\( d \)) \( h^iE(-4) = 0 \) for \( i \neq 3 \) and \( h^3E(-4) = 2 \)

There is a convenient way to describe instanton bundles through Beilinson monads.

**Beilinson monads**

One of the usual methods to describe sheaves is by constructing some exact sequences made up of simpler sheaves such that the required sheaf is the kernel, cokernel, etc. of a certain morphism. For our problem the building blocks will be the sheaves \( \Omega^i(j) \) on \( \mathbb{P}^s \), where \( \Omega = \bigwedge T \) and \( T \) is a tangent bundle of \( \mathbb{P}^s \). Invariants of the sheaves \( \Omega^i(j) \) and their morphisms are classically known (the so called Bott formulae can be found, for example, in [5, Ch.I.1]). We shall represent instanton bundles as cohomologies of complexes of sheaves called monads and then use this representation to introduce parametrization of the bundles by matrices.

**Definition 2.3** If \( X \) is a scheme, a monad over \( X \) is a complex

\[ \cdots \mathcal{F}_{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \cdots \]

of coherent sheaves over \( X \) which is exact everywhere except at \( \mathcal{F}_0 \). The cohomology \( E \) at \( \mathcal{F}_0 \) is called the cohomology of this monad, and we speak of a monad for \( E \).
An intermediate step we shall use to construct a monad for instanton bundles will be the Beilinson spectral sequence. The theorem of Beilinson proves the existence of such a spectral sequence. Because of importance of its construction we quote this theorem together with the sketch of its proof done in details in [7, Ch.II.3].

**Theorem 2.4** Let \( E \) be a vector bundle over \( \mathbb{P}^s \). There is a Beilinson spectral sequence of type II \( E^{p,q}_* \) with \( E^1 \)-term
\[
E^q_1 = H^q(E(p)) \otimes \Omega^{-p}(-p)
\]
which converges to \( E^i \) with
\[
E^i = \begin{cases} E & \text{for } i = 0, \\ 0 & \text{otherwise}, \end{cases}
\]
i.e. \( E^{p,q}_\infty = 0 \) for \( p + q \neq 0 \) and \( \bigoplus_{p=0}^n E^{p,q}_\infty \) is the associated graded sheaf of a filtration of \( E \).

**Sketch of the proof.** Consider the product projections \( p_1, p_2 : \mathbb{P}^s \times \mathbb{P}^s \rightarrow \mathbb{P}^s \). We write \( \mathcal{A} \boxtimes \mathcal{B} = p_1^* \mathcal{A} \otimes p_2^* \mathcal{B} \) for any bundles \( \mathcal{A}, \mathcal{B} \).

There is a free (Koszul) resolution of \( \mathcal{O}_\Delta \) where \( \Delta \) is the diagonal in \( \mathbb{P}^s \times \mathbb{P}^s \):
\[
0 \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}_x \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}(1) \rightarrow \cdots \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}(s-1) \rightarrow \mathcal{O}(s) \rightarrow \mathcal{O}_\Delta \rightarrow 0
\]
or, written in another form,
\[
0 \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}_x \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}(1) \rightarrow \cdots \rightarrow \bigoplus_{x \in \mathbb{P}^s} \mathcal{O}(s-1) \rightarrow \mathcal{O}(s) \rightarrow \mathcal{O}_\Delta \rightarrow 0.
\]

Let \( k(\langle u \rangle, \langle v \rangle) \cong k \) be the residue field at a point \( \langle u \rangle, \langle v \rangle \in \mathbb{P}^s \times \mathbb{P}^s = \mathbb{P} V \times \mathbb{P} V \) where \( u, v \in V \).

Let \( k(\langle u \rangle, \langle v \rangle) = k \) as of a skyscraper sheaf at the point \( \langle u \rangle, \langle v \rangle \in \mathbb{P}^s \times \mathbb{P}^s \) and tensorizing with it means simply taking a fiber at this point.

The maps of this Koszul complex are defined up to a twist by the following maps on fibers of the vector bundles \( \langle u \rangle, \langle v \rangle \in \mathbb{P}^s \times \mathbb{P}^s \):
\[
\bigotimes_{x \in \mathbb{P}^s} V^\vee \otimes k(\langle u \rangle, \langle v \rangle) \rightarrow \bigotimes_{x \in \mathbb{P}^s} V^\vee \otimes k(\langle u \rangle, \langle v \rangle)
\]
for every point \( \langle u \rangle, \langle v \rangle \in \mathbb{P}^s \times \mathbb{P}^s \) and \( x, \theta \) standing for the result of contraction
\[
V \otimes \bigotimes_{x} V^\vee \otimes k(\langle u \rangle, \langle v \rangle) \rightarrow \bigotimes_{x} V^\vee
\]
Here \( \mathcal{O}_x \) denotes the fiber of the vector bundles \( \mathcal{O}(1) \) and \( \mathcal{O}(j) \) over a point \( \langle x \rangle \) as
\[
\mathcal{O}_x(1) = (k \cdot x) \subset V \\
\mathcal{O}_x(j) = \bigotimes_{x} V^\vee \subset \bigotimes_{x} V^\vee
\]
Here \( k \cdot x \) means the subspace of \( V \) generated by a vector \( x \).

Write \( \mathcal{C}^{-k} = \mathcal{E}(-k) \otimes \mathcal{O}^k \langle k \rangle \), where \( 0 \leq k \leq s \). The Koszul sequence above gives after tensorizing with \( p_1^* \mathcal{E} \) yields a complex
\[
0 \rightarrow \mathcal{C}^{-s} \rightarrow \mathcal{C}^{-s+1} \rightarrow \cdots \rightarrow \mathcal{C}^0 \rightarrow 0.
\]
By the theory of hyperdirect images of sheaves \([8]\), there are two spectral sequences
\[
\begin{align*}
' E^{p,q}_2 &= H^p(R^q p_2_*(\mathcal{C})), \\
'' E^{p,q}_2 &= R^p p_2_*(\mathcal{C})
\end{align*}
\]
converging to the \((p + q)\)-th hyperdirect image sheaf \(R^{p+q}p_{2*}(\mathcal{C})\). A reader may consult [8] for definitions of higher direct images and of hypercohomologies; they are not that important for this exposition. Important is that the two spectral sequences have the same limit.

Since \(\mathcal{C}\) is a locally free resolution of \(p_1^*\mathcal{E}|_\Delta\), it follows that

\[
H^q(\mathcal{C}) \cong \begin{cases} 
\mathcal{E} & \text{for } p = q = 0, \\
0 & \text{otherwise}
\end{cases}
\]

and therefore

\[
''E_2^{p,q} = R^p p_{2*}(H^q(\mathcal{C})) \cong \begin{cases} 
\mathcal{E} & \text{for } p = q = 0, \\
0 & \text{otherwise}
\end{cases}
\]

The spectral sequence \(''E_2^{p,q}\) converges to \(R^{p+q}p_{2*}(\mathcal{C})\) and thus

\[
R^i p_{2*}(\mathcal{C}) \cong \begin{cases} 
\mathcal{E} & \text{for } p = q = 0, \\
0 & \text{otherwise}
\end{cases}
\]

On the other hand, one proves that we can construct the \(E_1\)-level of the spectral sequence \(\prime E_1\) by setting

\[
\prime E_1^{p,q} = R^q p_{2*}(\mathcal{C}^p) = R^q p_{2*}(\mathcal{E}(p) \otimes \Omega^{-p}(-p)) = R^q p_{2*}(p_1^*\mathcal{E}(p) \otimes \Omega^{-p}(-p)) = H^q(\mathcal{P}_s, \mathcal{E}(p)) \otimes \Omega^{-p}(-p),
\]

where \(0 \leq q \leq s\) and \(-s \leq p \leq 0\), with maps \(\prime E_1^{p,q} \to \prime E_1^{p+1,q}\) induced by maps in the Koszul resolution above and described as

\[
\prime E_1^{p,q} = H^q(\mathcal{P}_s, \mathcal{E}(p)) \otimes \Omega^{-p}(-p) \to H^q(\mathcal{P}_s, \mathcal{E}(p)) \otimes \mathcal{V} \otimes \Omega^{-p-1}(-p-1) \cong H^q(\mathcal{P}_s, \mathcal{E}(p)) \otimes H^0(\mathcal{P}_s, \mathcal{O}(1)) \otimes \Omega^{-p-1}(-p-1) \to H^q(\mathcal{P}_s, \mathcal{E}(p+1)) \otimes \Omega^{-p-1}(-p-1) = \prime E_1^{p+1,q}.
\]

Here the first map is induced by the contraction homomorphism \(\Omega^{-p}(-p) \otimes \mathcal{V} \to \Omega^{-p-1}(-p-1)\), the isomorphism \(H^0(\mathcal{P}_s, \mathcal{O}(1)) \cong \mathcal{V}\) is canonical and the last map is the cup-product in cohomologies.

\[\square\]

There is also another variant of the Beilinson theorem:

**Theorem 2.5** Let \(\mathcal{E}\) be a vector bundle over \(\mathbb{P}^s\). There is a Beilinson spectral sequence of type I \(E^p,q\) with \(E^1\)-term

\[
E_1^{p,q} = H^q(\mathcal{E} \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(-p)
\]

which converges to

\[
E^i = \begin{cases} 
\mathcal{E} & \text{for } i = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

The next step is to pass from the Beilinson spectral sequence of type II to a monad for instanton bundles given by the invariants from the lemma 2.4.

On the \(E_1\)-level we have only 3 non-zero terms:

\[
\begin{align*}
E_1^{s,s-1} &= H^{s-1}(\mathcal{E}(-s)) \otimes \Omega^s(s) \\
E_1^{1,1} &= H^1(\mathcal{E}(-1)) \otimes \Omega^1(1) \\
E_1^{0,1} &= H^1(\mathcal{E}) \otimes \mathcal{O}
\end{align*}
\]
and when written out together with differentials looks like

\[
0 \rightarrow E_{1}^{-s,s-1} \rightarrow 0 \rightarrow \cdots \nRightarrow E_{1}^{-1,1} \nu \rightarrow E_{1}^{0,1} \rightarrow 0
\]

Computing the cohomologies of these differentials, we see \(E_{2}^{-s,s-1} = E_{1}^{-s,s-1}, \) \(E_{2}^{-1,1} = \text{Ker } \nu, \) \(E_{2}^{0,1} = \text{Coker } \nu.\) If \(s > 2,\) the differentials in \(E_{2}\) are zero:

\[
0 \rightarrow E_{k}^{-s,s-1} \rightarrow \cdots \nRightarrow \text{Ker } \nu \rightarrow \text{Coker } \nu \rightarrow 0 \rightarrow \cdots
\]

and analogous for all \(E_{k},\) \(1 < k < s.\) The last level with nonzero differentials is \(E_{s}:\)

\[
0 \rightarrow E_{s}^{-s,s-1} \rightarrow \cdots \nRightarrow \text{Ker } \nu \rightarrow \text{Coker } \nu \rightarrow 0 \rightarrow \cdots
\]

where \(\mu : E_{1}^{-s,s-1} = E_{1}^{-s,s-1} \rightarrow \text{Ker } \nu\) is some homomorphism.

We get the following information about \(E_{\infty}:\)

\[
E_{\infty}^{-s,s-1} = \text{Ker } \mu; \quad E_{\infty}^{-1,1} = \text{Coker } \mu; \quad E_{\infty}^{0,1} = \text{Coker } \nu,
\]

and all other terms vanish. In this circumstances the Beilinson theorem 2.4 implies:

\(E_{\infty}^{-s,s-1} = \text{Ker } \mu = 0\) hence \(\mu : E_{1}^{-s,s-1} \rightarrow \text{ker } \nu \subset E_{1}^{-1,1}\) is injective;

\(E_{\infty}^{0,1} = \text{Coker } \nu = 0\) hence \(\nu : E_{1}^{-1,1} \rightarrow E_{1}^{0,1}\) is surjective, and

\(\mathcal{E} \cong E_{\infty}^{-1,1} = \text{Coker } \mu,\) i.e. we have the

**Theorem 2.6** Every mathematical instanton bundle with invariants as in the definition 2.1 and lemma 2.2, is the cohomology of the monad

\[
0 \rightarrow H^{s-1}(\mathcal{E}(-s)) \otimes \Omega^{s}(s) \xrightarrow{\mu} H^{1}(\mathcal{E}(-1)) \otimes \Omega^{1}(1) \xrightarrow{\nu} H^{1}(\mathcal{E}) \otimes \mathcal{O} \rightarrow 0
\]

where \(\mu\) is a subbundle and \(\nu\) is surjective.
An analogous procedure can be carried out to get a monad from the spectral sequence of type I. There is also a stronger version of the Beilinson theorem, leading to the same monad in our case.

**Theorem 2.7 (Beilinson)** \[13\] Any coherent \( \mathcal{O} \)-module \( \mathcal{F} \) on \( \mathbb{P}^s \) is the cohomology at 0 of a complex

\[
0 \to C^{-s} \to \cdots \to C^0 \to C^1 \to \cdots \to C^s \to 0
\]

which is exact except at 0 and for which the sheaves are given by

\[
C^p = \bigoplus_i H^i(\mathcal{F} \otimes \Omega^{i-p}(i-p)) \otimes \mathcal{O}(p-i)
\]

or by

\[
C^{-p} = \bigoplus_i H^i(\mathcal{F}(-i-p)) \otimes \Omega^{p+i}(p+i)
\]

We call such complexes **Beilinson monads of types I and II** respectively.

Note that the homomorphisms of the monad are not a priori canonically given by \( \mathcal{F} \) in general; any two such monads for a sheaf \( \mathcal{F} \) are known only to be homotopically equivalent. The reason is basically that the higher differentials of a spectral sequence are induced in a non-unique way. In the case of mathematical instantons, however, one can express the homomorphisms in the monad via some natural maps in cohomologies, see later the supplement to the Beilinson theorem 2.9.

We shall be working with the monad of type II most of the time. Inserting the invariants for our rank-4 bundles on \( \mathbb{P}^4 \) with \( c_2 = 2 \), we can rewrite this monad as

\[
0 \to k^2 \otimes \Omega^4(4) \xrightarrow{\mu} k^2 \otimes \Omega^1(1) \xrightarrow{\nu} k^2 \otimes \mathcal{O} \to 0.
\]

**2.8 The morphisms of sheaves \( \mu, \nu \) from the monad**

\[
0 \to H^{s-1}(\mathcal{E}(-s)) \otimes \Omega^s(s) \xrightarrow{\mu} H^1(\mathcal{E}(-1)) \otimes \Omega^1(1) \xrightarrow{\nu} H^1(\mathcal{E}) \otimes \mathcal{O} \to 0
\]

can be identified with matrices \( M \in \text{Mat}_{n \times n}(\wedge^{s-1} V) \) and \( N \in \text{Mat}_{((s-1)n-r) \times n}(V) \) of suitable sizes using the canonical isomorphism \( \text{Hom}_{\mathbb{P}^s}(\Omega^k(k), \Omega^j(j)) = \wedge^{k-j} V \) for any \( 0 \leq j \leq k \leq n \).

There are at least two equivalent ways to describe this correspondence between sheaf homomorphism and matrices.

For the first construction note that \( \text{Hom}_{\mathbb{P}^s}(\Omega^k(k), \Omega^j(j)) = \text{Hom}_{\mathbb{P}^s}(\Omega^k(k+1), \Omega^j(j+1)) \). The sheaves \( \Omega^k(k+1), \Omega^j(j+1) \) are generated by their global sections, and

\[
H^0(\Omega^k(k+1)) = \bigwedge^{k+1} V^\vee, \quad H^0(\Omega^j(j+1)) = \bigwedge^{j+1} V^\vee.
\]

Take an exterior form \( \omega \in \wedge^{k-j} V \) and define a \( k \)-linear map by contraction with \( \omega \):

\[
\begin{align*}
H^0(\Omega^k(k+1)) &= \bigwedge^{k+1} V^\vee \xrightarrow{\omega} \bigwedge^{j+1} V^\vee \\
&\cong \bigwedge^{s-k} V \xrightarrow{\wedge \omega} \bigwedge^{s-j} V \\
&= H^0(\Omega^j(j+1))
\end{align*}
\]

One can prove that such linear map defines a sheaf homomorphism.
The second definition uses the Euler sequences

\[ 0 \to \Omega^k(k) \to \bigwedge^k V^\vee \otimes \mathcal{O} \to \Omega^k(k+1) \to 0, \]

\[ 0 \to \Omega^j(j) \to \bigwedge^j V^\vee \otimes \mathcal{O} \to \Omega^j(j+1) \to 0, \]

One can define a morphism

\[ \phi : \bigwedge^k V^\vee \otimes \mathcal{O} \xrightarrow{\cong} \bigwedge^j V^\vee \otimes \mathcal{O} \]

\[ \bigwedge^{s-k} V \otimes \mathcal{O} \xrightarrow{\cong} \bigwedge^{s-j} V \otimes \mathcal{O} \]

and show that \( \phi(\Omega^k(k)) \subseteq \Omega^j(j) \).

**2.9 Supplement / Special case of the Beilinson theorem.** (see [1]) Let \( \mu, \nu \) be the morphism from the type-II Beilinson monad for the instanton bundle \( \mathcal{E} \) as in theorem 2.8 and \( M \in \text{Mat}_{n \times n}(\bigwedge^{s-1} V) \), \( N \in \text{Mat}_{(s-1)n-r \times n}(V) \) \( M, N \) be matrices of \( \mu, \nu \) constructed as in 2.8, where \( n = h^{s-1}(\mathcal{E}(-s)) = h^1(\mathcal{E}(-1)) \) and \( (s-1)n-r = h^1(\mathcal{E}) \). Then via the identifications

\[ \text{Mat}_{n \times n}(\bigwedge V) = \text{Hom}_k(H^{s-1}(\mathcal{E}(-s)), H^1(\mathcal{E}(-1)) \otimes \bigwedge V) = \text{Hom}_k(H^{s-1}(\mathcal{E}(-s)) \otimes \bigwedge V^\vee, H^1(\mathcal{E}(-1))) \]

respectively

\[ \text{Mat}_{(s-1)n-r \times n}(V) = \text{Hom}_k(H^1(\mathcal{E}(-1)), H^1(\mathcal{E}) \otimes V) = \text{Hom}_k(H^1(\mathcal{E}(-1)) \otimes V^\vee, H^1(\mathcal{E})) \]

\( M \) and \( N \) can be canonically written as:

\[ N : H^1(\mathcal{E}(-1) \otimes V^\vee = H^1(\mathcal{E}(-1) \otimes H^0\mathcal{O}(1) \xrightarrow{\text{cup}} H^1(\mathcal{E}) \]

\[ M : H^{s-1}\mathcal{E}(-s) \otimes \bigwedge^s V^\vee = H^{s-1}(\mathcal{E}(-s) \otimes H^0(\mathcal{E} \otimes \mathcal{O}^{s-2}(s-1))) \xrightarrow{\text{cup}} H^{s-1}(\mathcal{E} \otimes \mathcal{O}^{s-2}(s-1)) \cong H^1(\mathcal{E}(-1)) \]

where the last isomorphism is the inverse to the composition

\[ H^1(\mathcal{E}(-1)) \xrightarrow{\cong} H^2(\mathcal{E} \otimes \mathcal{O}^1(-1)) \xrightarrow{\cong} H^3(\mathcal{E} \otimes \mathcal{O}^2(-1)) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H^{s-1}(\mathcal{E} \otimes \mathcal{O}^{s-2}(-1)) \]

of connecting homomorphisms of the long exact homology sequences associated to exact sequences of sheaves

\[ 0 \to \mathcal{E} \otimes \mathcal{O}^p(-1) \to \bigwedge^p V^\vee \otimes \mathcal{E}(-p-1) \to \mathcal{E} \otimes \mathcal{O}^{p-1}(-1) \to 0. \]

The connecting homomorphisms are isomorphisms because \( h^i(\mathcal{E}(p-1)) = 0 \) for all \( i \).

**Construction 2.10** There is a direct procedure to construct a monad (almost) of type I for an instanton bundle \( \mathcal{E} \) from a monad of type II for the same bundle \( \mathcal{E} \). In the additional assumption \( H^0\mathcal{E} = 0 \) we can also do the inverse procedure, i.e. construct a monad of type II from that of type I.

Given a monad of type II

\[ 0 \to A \otimes \mathcal{O}^s(s) \xrightarrow{\mu} B \otimes \mathcal{O}^1(1) \xrightarrow{\nu} C \otimes \mathcal{O} \to 0, \]
with some $k$-vector spaces $A, B, C$, we write the Euler sequence tensorized with $B$

$$0 \rightarrow B \otimes \Omega^1(1) \rightarrow B \otimes V^\vee \otimes \mathcal{O} \rightarrow B \otimes \mathcal{O}(1) \rightarrow 0$$

and embed the first map into the commutative diagram

$$B \otimes \Omega^1(1) \xrightarrow{\nu} C \otimes \mathcal{O} \rightarrow 0$$

$$0 \rightarrow H \otimes \mathcal{O} \rightarrow B \otimes V^\vee \otimes \mathcal{O} \xrightarrow{N} C \otimes \mathcal{O} \rightarrow 0$$

where $N$ is a matrix for $\nu$ as in 2.8 and we define $H := \text{Ker}(B \otimes V^\vee \xrightarrow{N} C)$. Now we can complete the diagram to the display

Diagram chase yields that the cohomology sheaf of the left column is again $\mathcal{E}$.

It can be shown that

**Lemma 2.11** [1, lemma 1.10] The left column when twisted by $-1$ gives the Beilinson monad of $\mathcal{E}(-1)$.

**Proof.** The proof of the lemma consists essentially in showing that the description of maps in the monad of type I as of cohomological operators coincides with the definition of $\alpha$ and $\beta$ by this display.

\[ \square \]

Conversely, if we start from a monad of the form

$$0 \rightarrow A \otimes \bigwedge^{s+1} V^\vee \otimes \mathcal{O}(-1) \rightarrow H \otimes \mathcal{O} \rightarrow B \otimes \mathcal{O}(1) \rightarrow 0$$

with the cohomology sheaf $\mathcal{E}$ such that $H^0\mathcal{E} = 0$, than we can factorize $\beta$ through $B \otimes \mathcal{O}$

$$
\begin{array}{c}
H \otimes \mathcal{O} \xrightarrow{b} B \otimes V^\vee \otimes \mathcal{O} \\
\beta \downarrow \quad \text{can} \\
B \otimes \mathcal{O}(1) \xrightarrow{\text{can}} B \otimes \mathcal{O}(1)
\end{array}
$$
and define $C := \text{Coker } b$. Further, $b$ is injective since

$$\text{Kerb} = \text{Ker } H^0(\beta) = H^0(N) = H^0(\mathcal{E}) = 0$$

where $N$ is the kernel sheaf of the monad, i.e. $N = \mathcal{Ker}(\beta)$ and the last equality comes from the exact sequence

$$0 \to A \otimes \bigwedge^{s+1} V^\vee \otimes \mathcal{O}(-1) \to N \to \mathcal{E} \to 0$$

as $H^1\mathcal{O}(-1) = 0$. All this said, we can reconstruct the display as above with the monad of type I as its upper row.

**Open conditions on $M$ and $N$**

2.12 Let $M \in \text{Mat}_{n \times n}(\bigwedge^{s-1} V), N \in \text{Mat}_{((s-1)n-r) \times n}(V)$ be two matrices defining maps $\mu$ and $\nu$ in

$$0 \to k^n \otimes \Omega^s(s) \xrightarrow{\mu} k^n \otimes \Omega^1(1) \xrightarrow{\nu} k^{((s-1)n-r)} \otimes \mathcal{O} \to 0.$$ 

In order to define a monad of an instanton bundle, $M$ and $N$, or, equivalently, $\mu$ and $\nu$ must subject to the following conditions:

(i) $\nu \circ \mu = 0$ or, equivalently, $N \wedge M = 0 \in \text{Mat}_{((s-1)n-r) \times n}(\bigwedge^s V)$; (ii) $\mu$ is a subbundle; (iii) $\nu$ is surjective.

In the condition (i), the operation $\wedge$ between matrices means their product as matrices over the exterior algebra $\wedge V$. The condition (i) here is a closed condition, and the conditions (ii),(iii) are open conditions with respect to the Zariski topology on the affine space $\text{Mat}_{n \times n}(\bigwedge^{s-1} V) \times \text{Mat}_{((s-1)n-r) \times n}(V)$.

Now it is time to describe these two open conditions in algebraic terms.

**Definition 2.13** (i) Nontrivial linear combinations of columns respectively rows of a matrix are also called its **generalized**, as in [2, Lecture 9], or **generated**, as in [1], columns respectively rows.

(ii) We say that a column or a row of a matrix with entries in some vector space $U$ has the form $\lambda \otimes z$, where $z \in U$, $\lambda \in k^p$, if it is equal to $(\lambda_1 \cdot z, \lambda_2 \cdot z, \ldots, \lambda_p \cdot z)$.

**Lemma 2.14** [1, 2.1] The following conditions are equivalent:

(i) The homomorphism $k^n \otimes \Omega^s(s) \xrightarrow{\mu} k^n \otimes \Omega^1(1) \xrightarrow{\nu} k^{((s-1)n-r)} \otimes \mathcal{O}$ is surjective;

(ii) There is no nontrivial linear combinations of the rows of $N$ of the form $\lambda \otimes x, x \in V, \lambda \in k^n - \{0\}$.

**Remark 2.15** In the case when $N$ is a $2 \times 2$-matrix we easily get one more condition equivalent to the previous two:

(ii') There is no nontrivial linear combinations of the **columns** of $N$ of the form $\lambda \otimes x, x \in V, \lambda \in k^n - \{0\}$.

**Lemma 2.16** [2, 2.2] The following conditions are equivalent:

(i) The homomorphism $k^n \otimes \Omega^s(s) \xrightarrow{\mu} k^n \otimes \Omega^1(1)$ is a subbundle;

(ii) For any $0 \neq x \in V$ the matrix $x \wedge M$ which induces the map

$$\mu(x) : \Omega^s(s)(x) = \bigwedge^s(V/x)^\vee \xrightarrow{\mu} (V/x)^\vee = \Omega^1(1)(x)$$

\[\begin{array}{c}
\bigwedge^{s+1} V^\vee \xrightarrow{\mu(x \wedge M)} V^\vee
\end{array}\]
is an injective linear operator;

(iii) There is no generalized column of the form \( x \wedge m \) where \( m \) is a column with entries in \( \Lambda^{s-1} V \) and \( x \in V \);

(iv) The matrix \( P \) representing the kernel of \( M \)

\[
0 \to k^p \xrightarrow{P} k^n \otimes V \xrightarrow{M} k^n \otimes \Lambda^s V
\]

has no generalized column of the form \( \lambda \otimes x \), \( x \in V \), \( \lambda \in k \).

\[\square\]

For the case \( \dim V = 5 \) and \( M \in \text{Mat}_{2 \times 2}(\Lambda^3 V) \) which is relevant to instantons with our invariants we can develop another criterion determining whether \( M \) defines a subbundle. This criterion will be used, e.g., in the proof of \( \mathbb{L} \).

Suppose we are given a generated column \( (\xi, \eta)^T \) of the matrix \( M \) and we want to know if this column is forbidden by the condition (iii) of the previous lemma.

**Lemma 2.17** Let \( V \) be a 5-dimensional \( k \)-vector space. Two exterior forms \( \eta, \xi \in \Lambda^3 V \) have a common linear factor if and only if the two equalities hold:

\[
(\xi^* \wedge \eta) = 0 ; \quad (\eta^* \wedge \xi) = 0 .
\]

where \( * \) denotes a dualization map \( \Lambda^s V \xrightarrow{\mathbf{*}} \Lambda^{5-s} V \).

**Proof.** If \( \xi \) or \( \eta \) is a zero form, we have nothing to prove.

Choose a basis of \( V \) such that \( \xi = e_{012} \) or \( \xi = e_0 \wedge (e_{12} + e_{34}) \) (we abbreviate \( e_i \wedge e_j \wedge \cdots \wedge e_k \) by \( e_{ij\ldots k} \)).

Case 1: \( \xi = e_{012} \) (we say that \( \xi \) is decomposable). Then up to a scalar multiple \( \xi^* = e_{45}^* \) and \( \xi^* \wedge \eta = 0 \). If \( \eta \) is decomposable, too, then, on one hand, conditions \( * \) are fulfilled and, on the other hand, \( k \cdot \xi = \Lambda^3 V_\xi \), \( k \cdot \eta = \Lambda^3 V_\eta \), where \( V_\xi, V_\eta \) are 3-dimensional subspaces in \( V \). Now a vector \( v \neq 0 \in V_\xi \cap V_\eta \) fulfills \( \xi = v \wedge \xi', \eta = v \wedge \eta' \).

Suppose one of \( \xi, \eta \), say, \( \xi \) is indecomposable and arrive at the

Case 2: \( \xi = e_{012} + e_{034} \). Then up to a scalar multiple \( \xi^* = e_{34}^* + e_{12}^* \), \( \xi^* \wedge \eta = e_{1234}^* \) and, finally, \( (\xi^* \wedge \eta) = 0 \). We see that the linear factor of \( \xi \) is precisely \( (\xi^* \wedge \eta) \). It is also a linear factor in \( \eta \) iff \( (\xi^* \wedge \eta) = 0 \).

If \( \eta \) is decomposable, then the second equality is trivially satisfied,

If both \( \xi \) and \( \eta \) are indecomposable, they both have linear factors \( (\xi^* \wedge \eta) \neq 0 \) and \( (\eta^* \wedge \xi) \neq 0 \) correspondently, and our equalities, equivalent in this situation, mean simply that they are collinear.

Hence the lemma.

\[\square\]

The similar lemmata can be proved also for \( \Lambda^2 k^4 \) and for \( \Lambda^4 k^6 \), i.e. for spaces of exterior forms relevant to the cases of instantons on \( \mathbb{P}^3 \) and \( \mathbb{P}^5 \) respectively.

**Lemma.** Two exterior forms \( \eta, \xi \in \Lambda^4 k^6 \) have a common linear factor if and only if the four equalities hold:

\[
\xi^* = 0 ; \quad \eta^* = 0 ; \quad (\xi^* \wedge \eta) = 0 ; \quad (\eta^* \wedge \xi) = 0 .
\]

**Sketch of the proof.** The idea is the same as above. The main points are:

For every form \( 0 \neq \xi \in \Lambda^4 k^6 \) we can choose a basis of \( k^6 \) such that \( \xi^* = e_{12}^* + e_{34}^* \) or \( \xi^* = e_{12}^* + e_{34}^* + e_{56}^* \).

The condition \( \xi^* = 0 \) is equivalent to \( V_\xi \neq 0 \). If so, then either \( \xi^* = 0 \) and \( \dim V_\xi = 4 \) or \( \Lambda^2 V_\xi = k \cdot (\xi^*)^* \).
Lemma. Two exterior forms $\eta, \xi \in \bigwedge^2 k^4$ have a common linear factor if and only if the three equalities hold:

$$\xi^2 = 0; \; \eta^2 = 0; \; \xi \wedge \eta = 0.$$

Proof is obvious.

Proposition 2.18 Let a matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

define a sheaf homomorphism $2\Omega^4_{\mathbb{P}^4}(4) \to 2\Omega^1_{\mathbb{P}^4}(1)$, where $m_{ij} \in \bigwedge^3 V, \; V \cong k^5$. Then $M$ defines a subbundle if and only if for no $(s, t) \in \mathbb{P}^1$ the equalities

$$((s \cdot m_{11} + t \cdot m_{12})^2)^* \wedge (s \cdot m_{21} + t \cdot m_{22}) = 0; \; ((s \cdot m_{21} + t \cdot m_{22})^2)^* \wedge (s \cdot m_{11} + t \cdot m_{12}) = 0$$

are satisfied simultaneously.

Proof follows from lemmata 2.17 and 2.16.

The Group Action

From now on we consider only the case with our fixed invariants, i.e., rank-4 bundles on $\mathbb{P}^4$ with $c_2 = 2$. Our next aim is to construct a space parametrizing these instanton bundles.

2.19 There are two natural group operations on the set of all monads

$$0 \to A \otimes \Omega^4(4) \xrightarrow{M} B \otimes \Omega^1(1) \xrightarrow{N} C \otimes \mathcal{O} \to 0$$

where $A, B, C$ are 2-dimensional vector spaces, namely, those of $GL(A) \times GL(B) \times GL(C)$ and of $GL(V)$, where, as usual, $\mathbb{P}^4 = \mathbb{P}V$.

Let us first define the operation of the group $GL(A) \times GL(B) \times GL(C)$. If $(g_1, g_2, g_3) \in GL(A) \times GL(B) \times GL(C)$, we get a commutative diagram

$$
\begin{array}{ccc}
0 & \to & A \otimes \Omega^4(4) \\
g_1 \otimes 1 & \downarrow & B \otimes \Omega^1(1) \\
g_2 \otimes 1 & \downarrow & C \otimes \mathcal{O} \\
0 & \to & 0
\end{array}
$$

and the cohomologies of the both rows are clearly isomorphic. In particular, if we replace matrix $M$ by $\lambda M$ for some $\lambda \in k - \{0\}$, the cohomology sheaf will not change, and the same is true for $N$. Slightly reformulating, we see that the whole $G$-orbit of

$$((M), (N)) \in \mathbb{P}Mat_{2 \times 2}(\bigwedge^3 V) \times \mathbb{P}Mat_{2 \times 2}(V)$$

defines the same cohomology sheaf, where $G = SL(2) \times SL(2) \times SL(2)$ and the operation is given by

$$(g_1, g_2, g_3)((M), (N)) := ((g_2^{-1} M g_1), (g_3^{-1} N g_2)).$$
Later on we shall see, that there exist a 1:1 correspondence between $G$-orbits and the cohomology bundles.

We also have a $GL(V)$-operation on our monads that corresponds to changing coordinates in our $\mathbb{P}^4$ and pull-backs of the cohomology sheaves under these coordinate transformations.

On the set $\text{Mat}_{2 \times 2}(V)$ of matrices $N$ we have therefore the restricted group action of $GL(B) \times GL(C) \times GL(V)$ which should be thought of as of

$$\begin{pmatrix} \text{column transformations} \\ \text{row transformations} \end{pmatrix} \times \begin{pmatrix} \text{change of basis in } V \end{pmatrix}.$$

2.20 The questions listed below may be solved on representatives of $GL(B) \times GL(C) \times GL(V)$-orbits, i.e. if we know an answer/proof for one particular matrix $N_0$, we also have it for all matrices $N$ equivalent to $N_0$ modulo this group action:

(i) to describe the set of matrices $M \in \text{Mat}_{2 \times 2}(\wedge^3 V)$ such that $N \wedge M = 0$ (cf. 2.21 and, for different monads, proposition 3.7);

(ii) to study geometry of cohomology sheaves of monads

$$0 \to A \otimes \Omega^1(4) \xrightarrow{M} B \otimes \Omega^1(1) \xrightarrow{N} C \otimes \mathcal{O} \to 0$$

for all $M$’s with $N \wedge M = 0$;

(iii) to study properties of the parameter space $X_0$ (cf. 5.2) and moduli space (defined in the section 4) at points corresponding to the pairs of matrices $(M, N)$ with $N \wedge M = 0$,

etc.

After the next lemma we shall see that for our purposes we need to consider only two $GL(B) \times GL(C) \times GL(V)$-orbits and hence have to do a lot of things for two fixed matrices $N$ only.

Lemma 2.21 Given a matrix $A \in \text{Mat}_{2 \times 2}(U)$ with $U$ being a $k$-vector space. Suppose that

$$\text{span } A := \text{span } \{a_{11}, a_{12}, a_{21}, a_{22}\} \cong k^3$$

and that $A$ does not generate column of the form $\lambda \otimes v$ for $\lambda \in k^2$, $v \in U$. Then there exist an element $g \in GL(2)$ and three linearly independent vectors $e_1, e_2, e_3$ such that

$$A \cdot g = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Proof. Note first, that left multiplication of a matrix $A$ with a group element $g \in GL(2)$ corresponds to elementary column operations in $A$.

Without loss of generality, $A = \begin{pmatrix} a & b \\ c & \lambda a + \mu b + \nu c \end{pmatrix}$, where $a, b, c \in U; \lambda, \mu, \nu \in k$, where $a, b, c$ is a basis of $\text{span } N$ (otherwise permute rows and columns) Then we have the chain of transformations shown symbolically on the diagram and explained in details beneath:

$$\begin{pmatrix} a & b \\ c & \lambda a + \mu b + \nu c \end{pmatrix} \xrightarrow{\text{elem. oper.}} \begin{pmatrix} a & b - \nu a \\ c & \lambda a + \mu b \end{pmatrix} \xrightarrow{\text{new b, } \lambda} \begin{pmatrix} a & b \\ c & \lambda a + \mu b \end{pmatrix} \xrightarrow{\text{elem. oper.}} \begin{pmatrix} \lambda a & b \\ \lambda c & \lambda a + \mu b \end{pmatrix}$$

$$\lambda \neq 0 \text{ for } M \in X_0$$

$$\begin{pmatrix} a & b \\ c & a + \mu b \end{pmatrix} \xrightarrow{\text{elem. oper.}} \begin{pmatrix} a + \mu b \\ c + \mu a + \mu^2 b \end{pmatrix} \xrightarrow{\text{new } a, \mu} \begin{pmatrix} a & b \\ c & a \end{pmatrix}.$$
To perform these transformations, you have to do the following steps.

Start with \[
\begin{pmatrix}
  a & b \\
  c & \lambda a + \mu b + \nu c
\end{pmatrix}.
\]

On the first step subtract from the second column the first multiplied by \( \nu \), get \[
\begin{pmatrix}
  a & b - \nu a \\
  c & \lambda a + \mu b
\end{pmatrix}.
\]

Since \( a, b, c \) are linearly independent, so are \( a, b - \nu a, c \), and we may take \( b - \nu \) as a new vector \( b \); in order to have the same vector in the lower right corner, we need to choose a new value for \( \lambda \) as \( \lambda + \mu \nu \); the resulting matrix is then \[
\begin{pmatrix}
  \lambda a & \lambda b \\
  c & \lambda a + \mu b
\end{pmatrix}.
\]

If \( \lambda = 0 \), we have a column of forbidden form \((b, \mu b)\), hence \( \lambda \neq 0 \) and we may multiply the first column by \( \lambda \) and get \[
\begin{pmatrix}
  \lambda a & \lambda b \\
  c & \lambda a + \mu b
\end{pmatrix}.
\]

Scaling \( a := \lambda a, b := \lambda b \), we arrive at a matrix \[
\begin{pmatrix}
  a & b \\
  c & a + \mu b
\end{pmatrix};
\]

In the next step, add \( \mu \) times the second column to the first column and obtain \[
\begin{pmatrix}
  a + \mu b & b \\
  c + \mu a + \mu^2 b & a + \mu b
\end{pmatrix}.
\]

Since \( a + \mu b, b, c + \mu a + \mu^2 b \) are linearly independent because \( a, b, c \) are, we may call the three former vectors to be a new triple \( a, b, c \) and finally get \[
\begin{pmatrix}
  a & b \\
  c & a
\end{pmatrix}
\]
that completes the proof.

\[\square\]

**Corollary 2.22** A matrix \( N \) defines a surjection \( 2\Omega^1(1) \rightarrow 2\Omega \) if and only if \( N \) is equivalent modulo the \( GL(B) \times GL(C) \times GL(V) \)-action to one of the two matrices \[
\begin{pmatrix}
  e_1 & e_2 \\
  e_3 & e_4
\end{pmatrix}
\]

or \[
\begin{pmatrix}
  e_1 & e_2 \\
  e_3 & e_1
\end{pmatrix}.
\]

**Proof.** From lemma 2.14 it is clear that if \( N \) defines a surjection \( 2\Omega^1(1) \rightarrow 2\Omega \) than \( \dim \text{span} \ N \geq 3 \). If \( \dim \text{span} \ N = 4 \), the conclusion is clear, otherwise apply remark 2.13 and the lemma above.

\[\square\]

**Syzygies of \( N \)’s**

**2.23** For both possible types of \( N \), we can parametrize the set of all matrices \( M \in \text{Mat}_{2 \times 2}(\wedge^3 V) \) such that \( N \wedge M = 0 \) by means of some number (in fact, 20) of scalar parameters \( p_i, q_i \in k \) For both types of \( N \) there exist matrices \( M \) defining a subbundle \( 2\Omega^4(4) \rightarrow 2\Omega^1(1) \)

Let \( \Gamma \in \text{Mat}_{2 \times (r+1)}(\wedge^3 V) \) be a total matrix of syzygies of \( N \), i.e. \( N \wedge \Gamma = 0 \) and columns of \( \Gamma \) generate all the syzygies of degree 3 of the matrix \( N \). Then all possible matrices \( M \) with the condition \( N \wedge M = 0 \) are of the form

\[
M = \Gamma \cdot \left[ \begin{array}{c}
p_0, \ldots, p_r \\
q_0, \ldots, q_r \end{array} \right]^T.
\]

If \( N = \begin{pmatrix}
  e_1 & e_2 \\
  e_3 & e_4
\end{pmatrix} \), then

\[
\Gamma = \begin{pmatrix}
  0 & e_{024} & e_{023} + e_{014} & e_{013} & 0 & e_{124} & e_{123} & 0 & e_{234} & e_{134} & 0 \\
  e_{024} & e_{023} + e_{014} & e_{013} & 0 & e_{124} & e_{123} & 0 & e_{234} & e_{134} & 0
\end{pmatrix},
\]
and an example of $M$ defining a subbundle is
\[
\begin{pmatrix}
  e_{023} + e_{014} & e_{134} + e_{024} + e_{013} \\
  e_{124} + e_{013} & e_{023} + e_{014}
\end{pmatrix};
\]
and if $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$, we get
\[
\Gamma = \begin{pmatrix}
  0 & e_{012} & e_{023} & e_{013} & 0 & e_{123} & 0 & e_{124} & e_{234} & e_{134} \\
  e_{012} & -e_{023} & e_{013} & 0 & e_{123} & 0 & e_{124} & -e_{234} & e_{134} & 0
\end{pmatrix},
\]
and, for example, the following matrix $M$ defines a subbundle
\[
\begin{pmatrix}
  e_{234} + e_{012} & e_{134} + e_{124} + e_{013} \\
  e_{134} - e_{124} - e_{023} & -e_{234} + e_{123}
\end{pmatrix}.
\]

**Remark 2.24** We see that the case of instanton bundles given by a monad
\[
0 \to k^2 \otimes \Omega^s(s) \xrightarrow{M} k^2 \otimes \Omega^1(1) \xrightarrow{N} k^2 \otimes \mathcal{O} \to 0
\]
on $P^4$ differs from that in $P^3$ at the point that the $N$ with rank span $N = 3$ are allowed in our situation.

**The Parameter Space and its Irreducibility**

2.25 Denote:
\[
Y_0 := \{(N) \in P\text{Mat}_{2 \times 2}(V) : 2\Omega^1(1) \xrightarrow{N} 2\mathcal{O} \text{ is surjective}\}.
\]
\[
X := \{(\langle M \rangle, \langle N \rangle) : N \wedge M = 0 \} \subset P\text{Hom}(k^2, k^2 \otimes \bigwedge^3 V) \times P\text{Hom}(k^2, k^2 \otimes V),
\]
\[
X_0 := \{\langle (M), (N) \rangle \in X : M \text{ is subbundle}, N \text{ is surjective} \}.
\]
We refer to $X_0$ as to the **parameter space**.

**Theorem 2.26** $X_0$ is irreducible.

**Proof** We have an obvious projection map $X_0 \to Y_0$ and $X_0$ is open in $X \times_{P\text{Mat}_{2 \times 2}(V)} Y_0 =: X_{Y_0}$.

The projection $X_{Y_0} \to Y_0$ is clearly a projective morphism with equidimensional fibers that are 19-dimensional linear projective subspaces of $P\text{Mat}_{2 \times 2}(\bigwedge^3 V)$, hence $X_{Y_0}$ is irreducible by the proposition below, and so is $X_0$.

\[\square\]

**Proposition 2.27** Given a projective surjective morphism $p : S \to T$ where $T$ is irreducible and geometric fibers of $p$ are equidimensional and irreducible. Then $S$ is irreducible.

\[\square\]
3 The Dual Bundle

The aim of this section is to describe the duals to instanton bundles in our case of rank 4, quantum number 2 on $\mathbb{P}^4$. The main result is:

**Theorem 3.1** Given a monad

$$2\Omega^4_{\mathbb{P}^4}(4) \xrightarrow{M} 2\Omega^1_{\mathbb{P}^4}(1) \xrightarrow{N} 2\mathcal{O}_{\mathbb{P}^4}$$

with the cohomology bundle $\mathcal{E}$ and a matrix $P$ defined as the kernel in

$$0 \rightarrow k^p \xrightarrow{P} k^2 \otimes V \xrightarrow{\Lambda M} k^2 \otimes \bigwedge^4 V.$$ 

where $\tilde{M}$ is the contraction with the matrix $M$. Then the dual bundle $\mathcal{E}^\vee$ is either the cohomology bundle of the monad

$$2\Omega^4(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 2\mathcal{O}$$

if $\text{rank} \tilde{M} = 8$, and is therefore an instanton bundle, too, or, if $\text{rank} \tilde{M} = 7$, it is an extension

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee_1 \rightarrow 0$$

where $\mathcal{E}^\vee_1$ is a reflexive sheaf appearing as the cohomology of the monad

$$2\Omega^4(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 3\mathcal{O}.$$ 

The values of $\text{rank} \tilde{M}$ other than 8 or 7 do not appear.

In the proof of the theorem, we have to distinguish different cases depending on $h^0\mathcal{E}^\vee$. To be able to control this invariant, we use the following

**Proposition 3.2** [4, 1.15]. The complex of vector spaces

$$0 \rightarrow H^3(\mathcal{E} \otimes \Omega^4) \rightarrow H^3(\mathcal{E}(-4)) \otimes \bigwedge^4 V \xrightarrow{\tilde{M}} H^1(\mathcal{E}(-1)) \otimes V^\vee \xrightarrow{N} H^1\mathcal{E} \rightarrow 0,$$

where $\tilde{M}$ is contraction with $M$, is exact except at $H^1(\mathcal{E}(-1)) \otimes V^\vee$ where its cohomology is $H^1(\mathcal{E} \otimes \Omega^4)$.

**Corollary 3.3** $h^0\mathcal{E}^\vee = 8 - \text{rank} \tilde{M}$

**Proof.** Apply Serre duality and proposition above to get:

$$h^0\mathcal{E}^\vee = h^1(\mathcal{E} \otimes \Omega^4) = \dim h^1(\mathcal{E}(-1)) \otimes V^\vee - \text{rank} N - \text{rank} \tilde{M} = 10 - 2 - \text{rank} \tilde{M}. $$

**Lemma 3.4** The two equivalent inequalities $h^0\mathcal{E}^\vee \leq 1$ and $\text{rank} \tilde{M} \geq 7$ hold.
Proof. Firstly, we have to prove that cases $h^0E^\vee > 1$, i.e. rank $\tilde{M} \leq 6$ are impossible.

Recall that in 2.16 we have defined $P$ as the kernel of $\tilde{M}$, i.e. through the exact sequence

$$0 \to k^p \xrightarrow{P} k^2 \otimes V \xrightarrow{\tilde{M}} k^2 \otimes {}^sV$$

If rank $M = 6$, we get $p = 4$ and $P$ is a $4 \times 2$ matrix. By the theorem quoted below such a matrix can be assumed to be $P = \begin{pmatrix} e_0 & e_1 & e_2 & e_3 \\ e_1 & e_2 & e_3 & e_4 \end{pmatrix}$, because by lemma 2.16 $P$ does not generate a column of the form $(\lambda v, \mu v)$.

A computation shows that $2 \bigwedge^3 V \xrightarrow{P^\vee} 4 \bigwedge^4 V$ is a surjection of vector spaces and therefore we have an exact sequence

$$k^2 \xrightarrow{M^\vee} 2 \bigwedge^3 V \xrightarrow{P^\vee} 4 \bigwedge^4 V \to 0$$

which is impossible because $\dim_k(2 \bigwedge^3 V) = \dim_k(4 \bigwedge^4 V) = 20$ and $M \neq 0$.

If rank $M < 6$, then no matrix $P$ of the size $(\geq 5) \times 2$ without generalized columns $\lambda \otimes x$ exists (cf. [2, Lecture 9]) hence, the cases with rank $M < 7$ do not appear.

\[\square\]

Theorem 3.5 [2, Lecture 9] Consider a $k$-vector space $W$ and a matrix $A \in \text{Mat}_{2 \times r}(W)$ that has no generalized column of the form $(\lambda x, \mu x)$ and such that $\text{span} A = W$. Then modulo the group operation $GL(2) \times GL(r)$ on $\text{Mat}_{2 \times r}(W)$ the matrix $A$ is equivalent to a matrix of the form

$$\begin{pmatrix} e_0 & e_1 & e_2 & \ldots & e_{a_1-1} \\ e_1 & e_2 & e_3 & \ldots & e_{a_1} \end{pmatrix} \begin{pmatrix} e_{a_1+1} & e_{a_1+2} & \ldots & e_{a_2-1} \\ e_{a_1+2} & e_{a_1+3} & \ldots & e_{a_2} \end{pmatrix} \begin{pmatrix} \ldots & e_{a_1+2} & \ldots & e_n \end{pmatrix}$$

where $(e_0, \ldots, e_n)$ is a basis of $W$, $l = n - k$, and $a_1, \ldots, a_l$ is a sequence of integers.

\[\square\]

Having done this, we proceed with the easier case of $h^0E^\vee = 0$.

Construct the type-I monad for $E$, which is the left column of the display:

$$0 \longrightarrow 2\mathcal{O}(-1) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{N} 2\mathcal{O} \longrightarrow 0$$

Dualizing the left column, we get

$$0 \longrightarrow 2\mathcal{O}(-1) \xrightarrow{B^\vee} 8\mathcal{O} \xrightarrow{A^\vee} 2\mathcal{O}(1) \longrightarrow 0$$
Since $H^0 \mathcal{E}^\vee = 0$, we use the similar diagram to pass back to the monad of type-II for $\mathcal{E}^\vee$, getting

$$0 \to 2\Omega^4(4) \to 2\Omega^1(1) \to 2\mathcal{O} \to 0.$$ 

**Claim.** $M_1 = M^T$.

By the construction of the Beilinson monad, $M$ is given as the cup-product

$$\bigwedge^3 V^\vee \otimes H^3 \mathcal{E}(-4) = H^0 \Omega^2(3) \otimes H^3 \mathcal{E}(-4) \to H^3(\mathcal{E}(-1) \otimes \Omega^2) \cong H^2(\mathcal{E}(-1) \otimes \Omega^1) \cong H^1(\mathcal{E}(-1))$$

Now by the naturality of the Serre duality

$$H^3(\mathcal{E}^\vee(-4)) \otimes \bigwedge^3 V^\vee \cong H^1 \mathcal{E}^\vee(-1) \otimes \bigwedge^3 V^\vee \cong H^1 \mathcal{E}^\vee(-1) \otimes \bigwedge^3 V^\vee \cong H^3(\mathcal{E}(-4)) \otimes \bigwedge^3 V^\vee$$

the claim follows.

From the sequence

$$0 \to k^2 \to k^2 \otimes V \to k^2 \otimes \bigwedge^4 V \to k^2 \otimes \bigwedge^5 V \to 0$$

and the nondegeneracy of $P$ we see that $k^2 \otimes \Omega^1(1) \to k^N \otimes \mathcal{O}$ is surjective and $N_1 = P^\vee$, $H^0 \mathcal{E}^\vee = 0$. This proves the theorem in the case rank $\tilde{M} = 8$.

If $h^0 \mathcal{E}^\vee = 1$ and rank $\tilde{M} = 7$, then $n = 3$ and $P$ is a $2 \times 3$ matrix and we have a cokernel of $\tilde{M}$ bigger than $N$. The following commutative diagram with the exact row appears:

$$k^2 \otimes V \xrightarrow{\tilde{M}} k^2 \otimes \bigwedge^4 V \xrightarrow{N_1} k^3 \otimes \bigwedge^5 V \to 0$$

We can write $N_1$ in the form $N_1 = \left(\frac{N}{x y}\right)$ for some $x, y \in V$.

The next important step of the proof is the

**Lemma 3.6** *In the notation introduced, $N_1 : 2\Omega^1(1) \to 3\mathcal{O}$ is never surjective.*
Proof Suppose the contrary and get the commutative diagrams

But then we can derive the relation of Chern polynomials $c(E_1) = c(E) = 1 + 2h^2 + 3h^4$ which is impossible because $c_4(E_1) = 0$ as rank $E_1 = 3$, that proves the lemma.

Hence $N_1$ is degenerate, i.e. without loss of generality $N_1 = \binom{N}{2 \pi}$ and the two diagrams look like

where $\mathcal{I}$ is an ideal sheaf and $\mathcal{C} = \mathcal{O}/\mathcal{I}$ is a sheaf of length 3 as we shall see later. Dualizing the right column of the right matrix, we get

$$0 \longrightarrow \mathcal{I}^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{E}_1^\vee \longrightarrow \text{Ext}^1(\mathcal{I}, \mathcal{O}) \longrightarrow 0$$

or, since $\mathcal{I}^\vee \cong \mathcal{O}$, $\text{Ext}^1(\mathcal{I}, \mathcal{O}) \cong \mathcal{O}$,

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{E}_1^\vee \longrightarrow 0$$

As $H^0(\mathcal{E}_1^\vee) = 1$, we obtain $H^0(\mathcal{E}_1^\vee) = 0$ and hence we may dualize the monad for $E_1$ by intermediate passing to the monad of type I, dualizing that one and passing back to a monad of type II for $E_1^\vee$. So,
we have a diagram

\[
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\begin{array}{c}
\xrightarrow{M} \\
\xrightarrow{N_1} \\
\xrightarrow{\Omega^1(1)} \\
\xrightarrow{C} \\
\xrightarrow{\Omega^1(1)} \\
\xrightarrow{P^\vee} \\
\xrightarrow{3\Omega} \\
\xrightarrow{2\Omega(1)} \\
\xrightarrow{2\Omega(1)}
\end{array}
\]

Now we have a monad for \( E_1 \):

\[
0 \rightarrow 2\Omega(-1) \xrightarrow{A} 7\Omega \xrightarrow{B} 2\Omega(1) \rightarrow C \rightarrow 0
\]

we dualize it (\( B^\vee \) is no subbundle!) to get a monad for \( E_1^\vee \)

\[
0 \rightarrow 2\Omega(-1) \xrightarrow{B^\vee} 7\Omega \xrightarrow{A^\vee} 2\Omega(1) \rightarrow 0
\]

and end up with a type-II monad

\[
2\Omega^4(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 3\Omega
\]

where \( M^\vee \) is no subbundle.

As in the previous case, the left matrix is \( M^\vee \) by the same argument using the naturality of the Serre duality and the right matrix is \( P^\vee = \text{Coker} \hat{M} \).

What remains is to prove the following

**Proposition 3.7** Let \( (\langle M \rangle, \langle N \rangle) \) define a monad with a bundle as its cohomology with rank \( \hat{M} = 7 \) and \( N_1 = (\frac{N}{x}) = \text{Coker} \hat{M} \). Then:

(i) \( \dim \text{span} N = 3 \) and \( x \in \text{span} N \);

(ii) if \( C = \text{Coker}(2\Omega^1(1) \xrightarrow{N_1} 3\Omega) \), then \( \dim \text{Supp} C = 0 \) and \( \text{length} C = 3 \).

**Proof of the proposition** We do the proof in two steps. We begin with classifying all possible matrices \( N_1 \) appearing as \( (\frac{N}{x}) \) up to the natural action of the group \( GL(2) \times GL(2) \times GL(5) \) under assumption, that \( N \) generates no column and no row of the form \( (\kappa_1 v, \kappa_2 v) \) for \( v \in V, \kappa_1, \kappa_2 \in k \) (cf. lemma 2.14 and remark 2.15). Compare 2.20 for the motivation.

Throughout the proof we denote \( N_1 = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix} \)

**Case 1.** \( \dim \text{span} N = 4 \)

**Case 1.1.** \( x \in \text{span} N \)
Obviously

\[
N_1 \sim \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \\ e_0 & 0 \end{pmatrix} =: T_1
\]

**Case 1.2.** \(x \notin \text{span } N\)

*Case 1.2.1* \(x, n_{1,2}, n_{2,1}, n_{2,2}\) form a basis of \(\text{span } N\).

Then we have the following sequence of transformations (explanations are given after the diagram):

\[
\begin{pmatrix} 0 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix} =: T_2
\]

\[
\begin{pmatrix} e_1 + \lambda e_2 + \mu e_3 + \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix} \xrightarrow{\nu = 0} \begin{pmatrix} e_1 + \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix} \xrightarrow{\nu \neq 0} \begin{pmatrix} \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix} =: T_3
\]

In this diagram, the first transformation from \(\begin{pmatrix} e_1 + \lambda e_2 + \mu e_3 + \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix}\) to \(\begin{pmatrix} e_1 + \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix}\) is done by subtracting \(\lambda\) times the second column from the first column, subtracting \(\mu\) times the second row from the first row and choosing new value for \(\nu\);

then we substract the third row from the first and get \(\begin{pmatrix} \nu e_4 & e_2 \\ e_3 & e_4 \\ e_1 & 0 \end{pmatrix}\).

If now \(\nu = 0\), we have already finished at the matrix \(T_2\) from the diagram; if \(\nu \neq 0\), choose a new \(e_1\) to be \(\nu e_1\) and a new \(e_3\) to be \(\nu e_3\) and arrive at the matrix \(T_3\).

*Case 1.2.2.** Neither \(x, n_{1,2}, n_{2,1}, n_{2,2}\) form a basis of \(\text{span } N\), nor do \(x, n_{1,1}, n_{1,2}, n_{2,2}\). Then, without loss of geneality, \(x, n_{1,1}, n_{2,1}, n_{2,2}\) are a basis of \(\text{span } N\).

\[
N_1 = \begin{pmatrix} e_2 & e_1 + \lambda e_2 + \mu e_3 + \nu e_4 \\ e_4 & e_3 \\ e_1 & 0 \end{pmatrix} = \begin{pmatrix} e_2 & e_1 + \lambda e_2 \\ e_4 & e_3 \\ e_1 & 0 \end{pmatrix}
\]

because \(\mu = \nu = 0\) due to our assumption. If \(\lambda = 0\), we get

\[
N_1 = \begin{pmatrix} e_2 & e_1 \\ e_4 & e_3 \\ e_1 & 0 \end{pmatrix} =: T_4
\]

otherwise:

add to the first row \(1/\lambda\) times the third;

take \(e_1 + \lambda e_2\) to be a new \(e_2\);

substract \(1/\lambda\) times the second column from the first one;

and put \(e_4 - \frac{1}{\lambda} e_3\) to be a new \(e_4\) finally arriving at the matrix \(\begin{pmatrix} 0 & e_2 \\ e_4 & e_3 \\ e_1 & 0 \end{pmatrix}\) \(\sim T_2\),

and so we get nothing new.
The transformations that have been described just can be illustrated by the diagram:

\[
\begin{pmatrix}
  e_2 & e_1 + \lambda e_2 \\
  e_4 & e_3 \\
  e_1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  \frac{1}{\lambda} e_1 + e_2 & e_1 + \lambda e_2 \\
  e_4 & e_3 \\
  e_1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  \frac{1}{\lambda} e_2 & e_2 \\
  e_4 & e_3 \\
  e_1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  0 & e_2 \\
  e_4 & e_3 \\
  e_1 & 0
\end{pmatrix}
\sim T_2
\]

**Case 2.** \( \dim \text{span} N = 3 \).

**Case 2.1.** \( x \in \text{span} N \). Since we can bring \( N_1 \) to the form with \( n_{1,1} = n_{2,2} \) by elementary transformations between first two rows, we get

\[
N_1 \sim \begin{pmatrix}
  e_1 & e_2 \\
  e_3 & e_1 \\
  e_0 & 0
\end{pmatrix} \sim T_3.
\]

**Case 2.2.** \( x \notin \text{span} N \).

First note that we cannot have \( N_1 \sim \begin{pmatrix} w & 0 \\ w & 0 \\ u & v \end{pmatrix} \) or \( N_1 \sim \begin{pmatrix} w & 0 \\ 0 & w \\ u & v \end{pmatrix} \) because such matrices do not generate a \( 2 \times 2 \)-submatrix that can play the role of \( N \).

Denote by \( S \) the set of points \( v \in PV \) such that \( N_1 \) has a generalized row of the form \((\alpha v, \beta v)\). By what we have remarked it is clear that \(|S| \leq 3\) for matrices \( N_1 \) of our type and the elements of \( S \) are linearly independent.

If \( S = \{e_1, e_2, e_3\} \) then \( N_1 \sim \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \\ e_3 & e_1 \end{pmatrix} =: U_1 \)

If \( S = \{e_1, e_2\} \), we get \( N_1 \sim \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \\ e_3 + \lambda e_2 & e_1 + \mu e_1 \end{pmatrix} \sim \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \\ e_3 & e_1 + \mu e_1 \end{pmatrix} \), where \( \varepsilon = 0, 1 \). We have \( \mu \neq 0 \) for otherwise \( e_3 \in S \), and we also may assume \( \varepsilon = 0 \) in order not to have \( e_3 + \mu e_1 \in S \).

We arrive at the matrix

\[
N_1 \sim \begin{pmatrix}
  e_1 & 0 \\
  0 & e_2 \\
  e_3 & e_1
\end{pmatrix} =: U_2
\]

Finally, assume \( S = \{e_1\} \) and suppose that \( n_{3,1} = e_1, n_{3,2} = 0 \). and that \( n_{1,1} = n_{2,2} \) (this can be attained by elementary transformations between the first two rows Since \(|S| = 1\) we have \( n_{2,1} \) is linearly independent of \( e_1 \). If now \( n_{1,1} = n_{2,2} = \lambda e_1 + \mu e_2 \) then \( N_1 = \begin{pmatrix} \lambda e_1 + \mu e_2 & e_3 \\ e_2 & \lambda e_1 + \mu e_2 \\ e_1 & 0 \end{pmatrix} \) and \( e_3 - \lambda (\lambda e_1 + \mu e_2) \in S \), contradiction.

Therefore we may suppose \( N_1 = \begin{pmatrix} e_3 & e_1 + \lambda e_2 + \mu e_3 \\ e_2 & e_3 \\ e_1 & 0 \end{pmatrix} \). The condition that \((e_1, 0)\) is the only degenerated generalized row of the matrix can be rewritten as an assumption that the equation

\[
(x e_3 + y e_2 + z e_1) \land (x(e_1 + \lambda e_2 + \mu e_3) + y e_3) = 0
\]

or, which is the same,

\[
\text{rank} \begin{pmatrix}
  z & y & x \\ x & \lambda x & \mu x + y
\end{pmatrix} = 1
\]
has the only solution \((x, y, z) = (0, 0, 1) \in \mathbb{P}^2\).

If in this equation \(x = 0\), then also \(y = 0\) and we arrive at the solution \((0, 0, 1)\). Otherwise put \(x = 1\) and get \(y = \lambda z\) and \(\lambda z^2 + \mu z - 1 = 0\). If \(\lambda \neq 0\) or \(\mu \neq 0\) then we have an additional solution. Hence \(\lambda = \mu = 0\) and

\[
N_1 \sim \begin{pmatrix}
e_3 & e_1 \\
e_2 & e_3 \\
e_1 & 0
\end{pmatrix} =: U_3,
\]

that completes the classification.

We are starting now with the second part of the proof where we shall see that for matrices \(N_1\) of types \(T_{1,2,3,4}\) there exist no matrices \(M\) defining a subbundle \(2\Omega^4(4) \xrightarrow{M} 2\Omega^1(1)\) and that for types \(U_{1,2,3,4}\) they do exist.

Let \(\Gamma \in \text{Mat}_{2 \times (r+1)}(\wedge^3 V)\) be a total matrix of syzygies of \(N_1\), i.e. \(N_1\Gamma = 0\) and columns of \(\Gamma\) generate all the syzygies of degree 3 of the matrix \(N_1\). Then all possible matrices \(M\) with the condition \(N_1M = 0\) are of the form

\[
M = G \cdot \begin{pmatrix}
p_0, & \ldots, & p_r \\
q_0, & \ldots, & q_r
\end{pmatrix}^T
\]

Here we shall several times use lemma 2.18.

If \(N_1 = T_1\), we get

\[
\Gamma = \begin{pmatrix}
0 & 0 & 0 & e_{024} & e_{013} & e_{023} + e_{014} \\
e_{024} & e_{124} & e_{234} & e_{023} + e_{014} & 0 & e_{013}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
p_3e_{024} + p_4e_{013} + p_5(e_{023} + e_{014}) & q_3e_{024} + q_4e_{013} + q_5(e_{023} + e_{014}) \\
p_0e_{024} + p_1e_{124} + p_2e_{234} + p_3(e_{023} + e_{014}) + p_5e_{013} & q_0e_{024} + q_1e_{124} + q_2e_{234} + q_3(e_{023} + e_{014}) + q_5e_{013}
\end{pmatrix}
\]

The exterior forms \(\xi, \eta\) being as in lemma 2.18, we get:

\[
(\eta^2)^* \wedge \xi = -2(q_1t + p_1s)f e_{0123} - 2(q_2t + p_2s)f e_{0134} = 0
\]

\[
(\xi^2)^* \wedge \eta = -2(q_1t + p_1s)f e_{0124} - 2(q_2t + p_2s)f e_{0234} = 0
\]

where

\[
f = (q_3q_4 - q_2^2)t^2 + (p_3q_3 + p_3q_4 - 2p_5q_5)st + (p_3p_4 - p_5^2)s^2
\]

and there always exists \((s, t)\) with \(f(s, t) = 0\).

If \(N_1 = T_2\), we get

\[
\Gamma = \begin{pmatrix}
0 & 0 & 0 & e_{013} & e_{123} & e_{134} & e_{124} \\
e_{024} & e_{124} & e_{234} & 0 & 0 & 0 & e_{123}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
p_3e_{013} + p_4e_{123} + p_5e_{134} + p_6e_{124} & q_3e_{013} + q_4e_{123} + q_5e_{134} + q_6e_{124} \\
p_0e_{024} + p_1e_{124} + p_2e_{234} + p_3e_{123} & q_0e_{024} + q_1e_{124} + q_2e_{234} + q_3e_{123}
\end{pmatrix}
\]

\[
(\eta^2)^* \wedge \xi = 2(q_0t + p_0s)(q_3t + p_3s)(q_0t + p_0s) 
\cdot e_{0123} - 2(q_0t + p_0s)(q_5t + p_5s)(q_0t + p_0s) 
\cdot e_{1234} = 0
\]

\[
(\xi^2)^* \wedge \eta = 2(q_0t + p_0s)(q_3t + p_3s)(q_0t + p_0s) 
\cdot e_{0124} - 2(q_2t + p_2s)(q_3t + p_3s)(q_0t + p_0s) 
\cdot e_{1234} = 0
\]

and the left hand sides of the equations have a common linear factor \((q_0t + p_0s)(q_0t + p_0s)\).

In case \(N_1 = T_3\)

\[
\Gamma = \begin{pmatrix}
0 & 0 & 0 & -e_{123} & e_{124} \\
e_{024} & e_{124} & e_{234} & 0 & e_{134}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
p_3e_{134} - p_4e_{123} + p_5e_{124} & q_3e_{134} - q_4e_{123} + q_5e_{124} \\
p_0e_{024} + p_1e_{124} + p_2e_{234} + p_3e_{134} + p_6e_{123} & q_0e_{024} + q_1e_{124} + q_2e_{234} + q_3e_{134} + q_6e_{123}
\end{pmatrix}
\]

\[
(\eta^2)^* \wedge \xi = f \cdot e_{1234} = 0 ; \quad (\xi^2)^* \wedge \eta = 0
\]
where \( f \) is a polynomial of degree 3 in \((s,t)\) and has a zero.

If \( N_1 = T_4 \) then

\[
\Gamma = \begin{pmatrix} 0 & 0 & 0 & e_{014} & e_{124} & e_{134} & e_{123} \\ e_{014} & e_{124} & e_{134} & e_{013} + e_{024} & e_{123} & e_{234} & 0 \end{pmatrix}
\]

\[
M = \begin{pmatrix} p_3 e_{014} + p_4 e_{124} + p_5 e_{134} + p_6 e_{123} \\ p_0 e_{014} + p_1 e_{124} + p_2 e_{134} + p_3 (e_{013} + e_{024}) + p_4 e_{123} + p_5 e_{234} \\ q_0 e_{014} + q_1 e_{124} + q_2 e_{134} + q_3 (e_{013} + e_{024}) + q_4 e_{123} + q_5 e_{234} \end{pmatrix}
\]

\[
(\eta^2)^\ast \wedge \xi = -2(q_6 t + p_6 s)(q_3 t + p_3 s)^2 \cdot e_{0123} + 2(q_6 t + p_6 s)F \cdot e_{1234} = 0
\]

and take, for example,

\[
M = \begin{pmatrix} e_{012} + e_{134} & e_{123} + e_{124} + e_{134} \\ -e_{012} + e_{234} & e_{023} - e_{124} \end{pmatrix}
\]

We have the following examples of matrices \( M \) defining subbundles \( 2\Omega^4(4) \rightarrow 2\Omega^1(1) \) with \( N_1 \wedge M = 0 \) for \( N_1 \) of the types \( U_{1,2,3} \).

If \( N_1 = U_1 \), we get

\[
\Gamma = \begin{pmatrix} 0 & 0 & 0 & e_{013} & e_{123} & e_{134} & e_{012} & e_{124} \\ e_{023} & e_{123} & e_{234} & 0 & 0 & 0 & -e_{012} & -e_{124} \end{pmatrix}
\]

where

\[
F = (q_2 t - p_2 s)(q_3 t + p_3 s) - (q_6 t + p_6 s)(q_5 t + p_5 s)
\]

and the left hand sides of the equations have a common linear factor \((q_6 t + p_6 s)\).

For \( N_1 = U_2 \) we have the total matrix of syzygies

\[
\Gamma = \begin{pmatrix} e_{013} & e_{123} & e_{134} & e_{012} & e_{124} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{023} & -e_{234} & e_{012} & e_{123} & e_{124} \end{pmatrix}
\]

and an example of \( M \) is

\[
M = \begin{pmatrix} e_{013} + e_{124} & e_{012} + e_{124} + e_{134} \\ e_{124} - e_{234} & -e_{023} - e_{234} \end{pmatrix}
\]

Finally, if \( N_1 = U_3 \), then

\[
\Gamma = \begin{pmatrix} e_{013} & e_{134} & e_{023} & e_{123} & 0 & e_{024} & e_{124} & e_{123} & e_{124} \end{pmatrix}
\]

We see that \( N_1 \) allows a subbundle \( M \) iff \( r + 1 = \text{rank} \Gamma = 8 \). But we always have an exact sequence of sheaves

\[
0 \rightarrow (r + 1)\Omega^4(5) \rightarrow \Omega^1(2) \overset{N_1}{\rightarrow} 3\mathcal{O}(1) \rightarrow \mathcal{C} \rightarrow 0
\]

where \( \Gamma \) defines an injection of the two \( \mathcal{O} \)-modules generated by their global sections because it defines an injection of these global sections. We obtain

\[
\text{length} \mathcal{C} = \dim \mathcal{H}^0(\mathcal{C}) = \dim \mathcal{H}^0((r + 1)\Omega^4(5)) - \dim \mathcal{H}^0(\Omega^1(2)) + \dim \mathcal{H}^0(3\mathcal{O}(1)) = (r + 1) - 2 \cdot \dim V + 3 \cdot \dim V = 8 - 20 + 15 = 3,
\]

Q.E.D.
4 The Construction of the Moduli Space

Here we shall define the moduli problem we are working with (4.9) and show that this problem admits a coarse moduli space (theorem 4.7).

In this chapter we are making use of the following general theory.

Definition 4.1 Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, where $\mathbb{P}^n \cong \mathbb{P}V$, where $V$ is an $(n+1)$-dimensional vector space. If $G$ is an affine algebraic group, acting on $X$, define a linearization of this action to be a group homomorphism $\varphi : G \to \text{GL}(V)$ such that

$$g \circ \langle x \rangle = \langle \varphi(g) \circ x \rangle,$$

for any $g \in G$, $\langle x \rangle \in X$.

Definition 4.2 If $\lambda \in X_*(G)$ is a 1-parameter subgroup of $G$, it defines a 1-parameter subgroup of $\text{GL}(V)$ by $\varphi \circ \lambda : k^\ast \to \text{GL}(V)$. As the image of 1-parameter subgroup is a torus and, in particular, diagonalizable, after a suitable basis change we have $\varphi \circ \lambda(t) = \text{Diag}(t^{a_0}, \ldots, t^{a_n})$. For a vector $x \in V$ define

$$\mu(x, \lambda) := \min\{a_k : x_k \neq 0\} = \min\{ m : \exists \lim_{t \to 0} t^{-m} \cdot \varphi \circ \lambda(t)(x) \}$$

where the second expression is independent on a diagonalization.

Definition 4.3 We call a point $x \in V$ stable if $\mu(x, \lambda) < 0$ for all nontrivial 1-parameter subgroups $\lambda \in X_*(G) - \{1\}$.

Definition 4.4 We say that the (categorical) quotient $\pi : X \to Y$ (denoted: $Y =: X/G$) is the geometric quotient, if the following conditions are satisfied:

(i) $\pi$ is $G$-equivariant, surjective and affine;
(ii) for any open subset $U \subseteq Y$ the ring homomorphism $\pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}U)^G$ is an isomorphism;
(iii) for any closed $G$-equivariant subset $W \subseteq Y$ the image $\pi(W) \subseteq Y$ is closed;
(iv) for every (closed) point $y \in Y$ the preimage $\pi^{-1}(y)$ is precisely a $G$-orbit.

Theorem 4.5 [3, theorem 3.14] In the above situation, if $X$ is contained in the set $\mathbb{P}^s$ of stable points with respect to some linearization $\varphi$ of the $G$-action, then there exists a geometric quotient $X/G$ which is again quasi-projective.

And now we start applying this theory to our situation.

Consider $\mathbb{P}^3 \text{Hom}(k^2, k^2 \otimes \bigwedge^3 V) \times \mathbb{P}^3 \text{Hom}(k^2, k^2 \otimes V)$ with the action of the group $G = SL(2) \times SL(2) \times SL(2)$ given by

$$(g_1, g_2, g_3)(\langle M \rangle, \langle N \rangle) := (\langle g_2^{-1} M g_1 \rangle, \langle g_3^{-1} M g_2 \rangle).$$

Recall the notation:

$$X = \{\langle M \rangle, \langle N \rangle : N \wedge M = 0\} \subset \mathbb{P}^3 \text{Hom}(k^2, k^2 \otimes \bigwedge^3 V) \times \mathbb{P}^3 \text{Hom}(k^2, k^2 \otimes V),$$

$$X_0 = \{\langle (M), (N) \rangle \in X : M \text{ is subbundle, } N \text{ is surjective} \}$$

25
We have the $G$-equivariant inclusions

$$X_0 \subset X \subset \mathbb{P} \text{Hom}(k^2, k^2 \otimes V^3) \times \mathbb{P} \text{Hom}(k^2, k^2 \otimes V) \xrightarrow{\text{Segre}} \mathbb{P} \text{Hom}(k^2, k^2 \otimes V^3) \otimes \text{Hom}(k^2, k^2 \otimes V),$$

where we define the $G$-action on $\mathbb{P} \text{Hom}(k^2, k^2 \otimes V^3) \otimes \text{Hom}(k^2, k^2 \otimes V)$ by

$$(g_1, g_2, g_3)((M \otimes N)) := (g_2^{-1} Mg_1 \otimes g_3^{-1} Mg_2)).$$

**Proposition 4.6** $X_0$ consists of stable points with respect to the last group action, i.e.,

$$X_0 \subset \mathbb{P} \left( \text{Hom}(k^2, k^2 \otimes V) \right)^s \times \mathbb{P} \text{Hom}(k^2, k^2 \otimes V^3).$$

**Proof.** In order to prove this, we are showing that $\mu(M \otimes N, \lambda) < 0$ for any nontrivial 1-parameter subgroup $\lambda \in X_*(G)$. Any 1-parameter subgroup of $G$ is of the form

$$\lambda(t) = \left( U_1 \begin{pmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{-r_1} \end{pmatrix} V_1, U_2 \begin{pmatrix} t^{r_2} & 0 & 0 \\ 0 & t^{-r_2} \end{pmatrix} V_2, U_3 \begin{pmatrix} t^{r_3} & 0 & 0 \\ 0 & t^{-r_3} \end{pmatrix} V_3 \right).$$

Correspondently, the $k^*$-orbit of $(M, N)$ is then given by

$$(M, N) \xrightarrow{\lambda(t)} U_2^{-1} \begin{pmatrix} t^{-r_2} & 0 & 0 \\ 0 & t^{r_2} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{-r_1} \end{pmatrix} U_1, U_3^{-1} \begin{pmatrix} t^{-r_3} & 0 & 0 \\ 0 & t^{r_3} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} t^{r_2} & 0 & 0 \\ 0 & t^{-r_2} \end{pmatrix} U_2$$

where $V_2^{-1} MV_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $V_3^{-1} NV_2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Since $U_1, U_2, U_3$ are irrelevant for the computation of $\mu(M, N, \lambda)$, we shall omit them in the sequel. The coordinates of $\lambda(t)(M, N)$ in $\text{Hom}(k^2, k^2 \otimes V^3) \otimes \text{Hom}(k^2, k^2 \otimes V)$ will be the collection of 4·4 vectors of the form

$$(\text{entry of } M') \otimes (\text{entry of } N') \in \bigwedge^3 V \otimes V \cong k^{50},$$

where we abbreviate

$$M' = \begin{pmatrix} t^{-r_2} & 0 & 0 \\ 0 & t^{r_2} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{-r_1} \end{pmatrix}; \quad N' = \begin{pmatrix} t^{-r_3} & 0 & 0 \\ 0 & t^{r_3} \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} t^{r_2} & 0 & 0 \\ 0 & t^{-r_2} \end{pmatrix},$$

i.e. there will be $50 \cdot 4 \cdot 4 = 800$ coordinates. To keep notation easier, we shall think of these vectors in $\bigwedge^3 V \otimes V \cong k^{50}$ as of coordinates of the pair $(M', N')$. Note that $\mu((M, N), \lambda) = \mu(M, \lambda) + \mu(N, \lambda)$. Suppose $\mu((M, N), \lambda) \geq 0$. But

$$N' = \begin{pmatrix} x \cdot t^{r_2-r_3} & y \cdot t^{-r_2+r_3} \\ z \cdot t^{r_2+r_3} & w \cdot t^{-r_2-r_3} \end{pmatrix}.$$  

Since $N$ generates no row of the form $(\kappa_1 v, \kappa_2 v)$, all of the vectors $x, y, z, w$ are nonzero. If $\mu(N, \lambda) \geq 0$, then all the four expressions $(r_2 - r_3), -(r_2 - r_3), (r_2 + r_3), -(r_2 + r_3) \geq 0$, but this happens only if $r_2 = r_3 = 0$. The same argument applied to $M$ shows that $r_1 = r_2 = 0$, i.e. $\lambda$ is a trivial 1-parameter subgroup. So the claim follows.

By the theorem above we have a geometric quotient $X_0/G =: \mathcal{M}$, which is again a quasiprojective variety.

**Theorem 4.7** The 29-dimensional irreducible quasiprojective variety $\mathcal{M} := X_0/G$ is a coarse moduli space (cf. definition 4.3 and 4.9) of instanton bundles on $\mathbb{P}^4$ with quantum number 2 and of rank 4.
**Proof.** Dimension and irreducibility follow immediately from the definition of $\mathcal{M}$, because $X_0$ is irreducible by theorem 2.26.

The rest of this sections consists of showing that $\mathcal{M}$ fulfills the axioms of a coarse moduli space.

**Definition 4.8** Given a moduli problem defined by a contravariant functor $\mathcal{F} : (\text{Sch}/k) \to (\text{Sets})$, we call a scheme $\mathcal{M}$ a coarse moduli space for our functor $\mathcal{F}$, if it satisfies the following three axioms:

(i) There exists a natural transformations of functors $\ldots$

\[ \beta : \mathcal{F} \to \text{Hom}(-, \mathcal{M}) \]

(ii) $\ldots$ which is a bijection on a $k$-point:

\[ \beta : \mathcal{F}(\text{Spec}(k)) \xrightarrow{1:1} (\text{closed points of } \mathcal{M}) = \text{Hom}(\text{Spec}(k), \mathcal{M}) \]

(iii) and universal in the sense that for any natural transformation of functors $\mathcal{F} \to \text{Mor}(-, N)$, there exists a unique morphism $\psi : \mathcal{M} \to N$ such that the diagram commutes:

\[ \begin{array}{ccc} \mathcal{F} & \xrightarrow{\beta} & \text{Mor}(-, \mathcal{M}) \\ \downarrow & & \downarrow \text{Mor}(-, \psi) \\ \text{Mor}(-, N). \end{array} \]

**4.9** We are representing the functor

\[ 2\mathcal{M}(S) := \left\{ \mathcal{F} : \begin{array}{c} \text{\mathcal{F} is a rank-4 vector bundle on } S \times \mathbb{P}^4 \\ \text{flat over } S \\ \text{and } \forall s \in S \text{ closed } \mathcal{F}_s \text{ is an instanton on } \mathbb{P}_s^4 \end{array} \right\} / \sim \]

where two such families are defined to be equivalent

\[ \mathcal{F} \sim \mathcal{F}' : \iff \mathcal{F}' = \mathcal{F} \otimes \text{pr}_S^* \mathcal{L} \]

for a suitable line bundle $\mathcal{L}$ on $S$.

First we construct a natural transformation of functors

\[ 2\mathcal{M} \to \text{Hom}(-, \mathcal{M}) \]

For given $S$ we take a flat family $\mathcal{F}$ of locally free sheaves on $S$, i.e. a locally free sheaf on $S \times \mathbb{P}^4$ satisfying our conditions. $\mathcal{F}$ is then the cohomology of the relative Beilinson monad

\[ K \otimes \Omega^4(4) \to L \otimes \Omega^1(1) \to P \otimes \mathcal{O}, \]

where $K = R^3p_*(\mathcal{F} \otimes \mathcal{O}(-3)), L = R^1p_*(\mathcal{F} \otimes \mathcal{O}(-1)), P = R^1p_*(\mathcal{F} \otimes \mathcal{O}(-1))$ and $p$ is the product projection $S \times \mathbb{P}^4 \to S$. These three sheaves are locally free of rank 2 because for every $s \in S$

\[ H^4p_*(\mathcal{F}_s \otimes \mathcal{O}(-3)) = H^2p_*(\mathcal{F}_s \otimes \mathcal{O}(-1)) = H^2p_*(\mathcal{F}_s \otimes \mathcal{O}(-1)) = 0 \]

and we can apply the following theorem.

**Theorem 4.10** [4, III.12.11] Let $P$ be a projective space, $p : S \times P \to S$ the product projection, and $\mathcal{F}$ a coherent sheaf on $S \times P$ flat over $S$. If the natural maps

\[ (R^i p_* \mathcal{F}) \otimes k(s) \to H^j(P_s, \mathcal{F} \otimes \mathcal{O}_P) \]

are surjective in all points $s \in S$, then there are natural isomorphisms

\[ (R^{i-1} p_* \mathcal{F}) \otimes k(s) \cong H^{j-1}(P_s, \mathcal{F} \otimes \mathcal{O}_P) \]

for any point $s \in S$. 

\[ 27 \]
Over an open subset $U \subset S$ trivializing all of $K, L, P$ we get

$$2\mathcal{O}_U \boxtimes \Omega^4(4) \xrightarrow{\beta} 2\mathcal{O}_U \boxtimes \Omega^1(1) \xrightarrow{N} 2\mathcal{O}_U \boxtimes \mathcal{O},$$

where the matrices $M, N$ depend on a point of $U$. We get a composite map

$$U \ni u \mapsto (M(u), N(u)) \xrightarrow{\text{mod } G} \mathcal{M}.$$  

This composite map does not depend on the choice of $U$ and on a particular trivialization because the class $(\langle M(u) \rangle, \langle N(u) \rangle) \mod G$ is uniquely determined by the isomorphism class of $F_u$. Therefore we get a global morphism $S \to \mathcal{M}$ (cf. below).

The next axiom:

$$\mathfrak{M}(pt) = \{\text{isom classes of } \mathcal{F} : \mathcal{F} \text{ is an instanton on } \mathbb{P}^4\} \overset{1:1}{\longleftrightarrow} (\text{closed points of } \mathcal{M})$$

To check this, we apply the

**Lemma 4.11** [7, II.4.1.3] Let $\mathcal{E}, \mathcal{E}'$ be the cohomology sheaves of two monads

$$M : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

$$M' : 0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$$

over a smooth variety $X$. The mapping

$$\text{Hom}(M', M) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}')$$

which associates to each homomorphism of monads the induced homomorphism of cohomology bundles is bijective if the following hypotheses are satisfied:

$$\text{Hom}(B, A') = \text{Hom}(C, B') = 0,$$

$$H^1(C^\ast \otimes A') = H^1(B^\ast \otimes A') = H^1(C^\ast \otimes B') = H^2(C^\ast \otimes A') = 0$$

We put in the lemma

$$M = M' : 0 \rightarrow 2\Omega^4(4) \rightarrow \Omega^1(1) \rightarrow 2\mathcal{O} \rightarrow 0$$

the conditions become then

(i) $\text{Hom}(\Omega^1(1), \Omega^4(4)) = 0, \text{Hom}(\mathcal{O}, \Omega^1(1)) = 0$, that is obvious

(ii) $H^1(\Omega^4(4)) = 0, H^1(\Omega^1(1)) = 0, H^2(\Omega^4(4)) = 0$, that is no less obvious

(iii) $H^1\text{Hom}(\Omega^1(1), \Omega^4(4)) = 0$, that can be proven by writing the isomorphisms

$$H^1\text{Hom}(\Omega^1(1), \Omega^4(4)) \cong H^1(\Omega^3(3) \otimes \mathcal{O}(-1)) = H^1(\Omega^3(2)) = 0$$

These conditions fulfilled, the isomorphism classes of monads of our type correspond 1:1 to the isomorphism classes of the cohomology sheaves.

The 3rd axiom. Given a natural transformation of functors $\mathfrak{M} \rightarrow \text{Mor}(-, N)$, we have to construct a morphism $\psi : \mathcal{M} \to N$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{M} & \xrightarrow{\beta} & \text{Mor}(-, \mathcal{M}) \\
\downarrow & & \downarrow \\
\text{Mor}(-, \psi) & \rightarrow & \text{Mor}(-, N)
\end{array}$$
By the properties of a quotient, construction of $\psi$ is equivalent to construction of a $G$-equivariant morphism $\Psi : X_0 \to N$. We have the universal family $\mathcal{F}$ on $X_0$; take $\Psi$ to be its image under the map

$$\beta(X_0) : \mathcal{M}(X_0) \to Mor(X_0, N)$$

$$\mathcal{F} \mapsto \Psi \circ g$$

Since the diagram

$$[g^*\mathcal{F}] \in \mathcal{M}(X_0) \to Mor(X_0, N) \ni \Psi \circ g$$

$$\mathcal{F} \in \mathcal{M}(X_0) \to Mor(X_0, N) \ni \Psi$$

commutes, it suffices to show that $[g^*\mathcal{F}] = [\mathcal{F}]$, i.e. $g^*\mathcal{F} \cong \mathcal{F} \otimes pr^*_X L$ for a suitable line bundle $L$ on $X_0$.

It is clear that $(g^*\mathcal{F})_x \cong \mathcal{F}_x$ for all $x \in X_0$. Further, $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}_x, \mathcal{F}_x) \cong k$ for we know that automorphisms of $\mathcal{F}_x$ correspond to automorphisms of the monad, an the latters can be computed as follows.

Suppose the diagram below is commutative and the vertical arrows are isomorphisms:

$$2\Omega^4(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{(e_1, e_2)} 2\mathcal{O}$$

$$2\Omega^4(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{(e_3, e_4)} 2\mathcal{O}$$

We have to show that $g = h = \lambda \cdot Id$.

Write

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}; \quad h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}.$$  

We get:

$$\begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \cdot \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \cdot \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix};$$

$$\begin{pmatrix} h_1e_1 + h_2e_3 & h_1e_2 + h_2e_1 \\ h_3e_1 + h_4e_3 & h_3e_2 + h_4e_1 \end{pmatrix} = \begin{pmatrix} f_1e_1 + f_3e_3 & f_2e_2 + f_4e_2 \\ f_1e_3 + f_3e_1 & f_2e_3 + f_4e_1 \end{pmatrix}.$$  

Comparison of the entries yields

- upper left : $f_1 = h_1, h_2 = f_3 = 0$
- lower left : $h_3 = f_3 = 0, h_4 = f_1 = h_1 =: \lambda$
- upper right : $f_4 = h_1 = \lambda, f_2 = h_2 = 0$

i.e.

$$f = h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

QED.
Define $\mathcal{L} := \pi_* \mathcal{Hom}(\mathcal{F}, g^* \mathcal{F})$ a line bundle on $X_0$, where $\pi = pr_{X_0}$. Now we can check fiberwise that

$$
\pi^* \mathcal{L} \otimes \mathcal{F} = [\pi^* \pi_* \mathcal{Hom}(\mathcal{F}, g^* \mathcal{F})] \otimes \mathcal{F} \xrightarrow{\text{can} \otimes id} \mathcal{Hom}(\mathcal{F}, g^* \mathcal{F}) \otimes \mathcal{F} \xrightarrow{\text{ev}} g^* \mathcal{F}
$$

is an isomorphism.

So, we have checked the three axioms and therefore realized the coarse moduli space as the geometric quotient quasi-projective variety.

\[\square\]
5 Smoothness of the Moduli Space

The fact we are proving in this section is:

**Theorem 5.1** The moduli space \( \mathcal{M} \) is smooth.

We start the proof by showing that \( X_0 \) is smooth.

**Theorem 5.2** \( X_0 \) is smooth.

**Proof.** We know that \( X_0 \) is 38-dimensional, i.e. has codimension 20 in \( \mathbb{P}Mat_{2 \times 2}(\wedge^3 V) \times \mathbb{P}Mat_{2 \times 2}(V) \).

Define the affine cone of \( X_0 \) to be

\[
CX_0 = \{(M, N) : M \neq 0, N \neq 0, ((M, \langle N \rangle)) \in X_0\} \subset Mat_{2 \times 2}(\wedge^3 V) \times Mat_{2 \times 2}(V)
\]

and analogously the affine cone \( CX \) over \( X \).

To prove that \( X_0 \) is smooth is the same as to prove that \( CX \) is smooth at all points of \( X_0 \).

We shall write elements of \( Mat_{2 \times 2}(\wedge^3 V) \times Mat_{2 \times 2}(V) \) in the form

\[
\left( \left( \sum_{i<j<k} A_{ijk} e_{ijk} \right) \sum_{i<j<k} C_{ijk} e_{ijk} \right) \cdot \left( \sum_{i<j<k} D_{ijk} e_{ijk} \right)
\]

and thus introduce coordinates \( \{a_i, b_i, c_i, d_i, A_{ijk}, B_{ijk}, C_{ijk}, D_{ijk}\} \) in this 60-dimensional affine space.

The equations of \( CX \) can be written in the matrix form as

\[
\begin{pmatrix}
  f_1 \cdot e_{0123} + \cdots + f_{5} \cdot e_{1234} \\
  f_{11} \cdot e_{0123} + \cdots + f_{15} \cdot e_{1234} \\
  f_{16} \cdot e_{0123} + \cdots + f_{20} \cdot e_{1234}
\end{pmatrix}
= \left( \sum_{i=0}^{4} a_i e_i \right) \cdot \left( \sum_{i=0}^{4} b_i e_i \right) \cdot \left( \sum_{i=0}^{4} c_i e_i \right) \cdot \left( \sum_{i=0}^{4} d_i e_i \right)
\]

This corresponds to 20 scalar equations \( f_i(a_1, \ldots, D_{234}) = 0 \) for \( i = 1, \ldots, 20 \).

The smoothness of \( CX_0 \) and hence of \( X_0 \) will be proven, if we show that the rank of the Jacobi matrix

\[
J(M, N) = \left( \frac{\partial f_i}{\partial x} \right) (M, N)
\]

is 20 at any point \( (M, N) \in CX_0 \). Here \( J(M, N) \) is a \( 20 \times 60 \)-matrix, where \( i \in \{1, \ldots, 20\} \) and \( x \) ranges through the 60 variables \( \{a_1, \ldots, D_{234}\} \).

The Leibnitz rule for matrices implies

\[
\left( \frac{\partial f_i}{\partial x} \cdot e_{0123} + \cdots + \frac{\partial f_{20}}{\partial x} \cdot e_{1234} \right) = \frac{\partial}{\partial x} \left[ \left( \sum_{i=0}^{4} a_i e_i \right) \cdot \left( \sum_{i=0}^{4} b_i e_i \right) \right] \wedge M + \left( \sum_{i=0}^{4} c_i e_i \right) \cdot \left( \sum_{i=0}^{4} d_i e_i \right) \wedge M
\]

\[
= \frac{\partial}{\partial x} \left( \sum_{i=0}^{4} a_i e_i \right) \cdot \left( \sum_{i=0}^{4} b_i e_i \right) \wedge M + \left( \sum_{i=0}^{4} c_i e_i \right) \cdot \left( \sum_{i=0}^{4} d_i e_i \right) \wedge M + \frac{\partial}{\partial x} \left( \sum_{i=0}^{4} a_i e_i \right) \cdot \left( \sum_{i=0}^{4} b_i e_i \right) \wedge M
\]

31
To prove that \( \text{rank} \, J(M, N) = 20 \) it suffices to check that \( \text{rank} \, J_1(M, N) = 20 \), where \( J_1 \) is a submatrix of \( J \)

\[
J_1(M, N) = \left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq 20, \, x \in \{A_{012}, \ldots, D_{234}\}}
\]

If \( x \in \{A_{012}, \ldots, D_{234}\} \), the previous formula simplifies to

\[
\left( \frac{\partial f_1}{\partial x} \cdot e_{0123} + \cdots + \frac{\partial f_9}{\partial x} \cdot e_{1234} \right) = N \wedge \frac{\partial}{\partial x} \left( \sum_{i<j<k} A_{ijk} e_{ijk} \frac{\partial}{\partial x} \left( \sum_{i<j<k} B_{ijk} e_{ijk} \right) \right).
\]

By \( 2.20 \) and \( 2.22 \) we may assume \( N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \) or \( N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \). In the former case \( J_1 \) is:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]
Definition 5.5 Let \( m : X \rightarrow G \) be a morphism. If \( m \) is an isomorphism, then we call \( m \) an isomorphism, and we are done.

The rest of the work has to be postponed until the end of this section where all its ingredients will have been prepared.

Definition 5.3 We call a morphism of schemes \( f : S \rightarrow T \) locally isotrivial with fiber \( F \), if there exists an open covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( T \) such that for each \( i \in I \) there exists a diagram

\[
\begin{array}{ccc}
\hat{U}_i \times F & \longrightarrow & U_i \times_T S \\
\downarrow \quad \circ & \quad \downarrow \quad pr_{U_i} & \quad \downarrow \quad f \\
\hat{U}_i & \longrightarrow & U_i \\
\end{array}
\]

The following results are presented in this form in \([\mathbb{F}, 2.5]\)

Definition 5.4 An action of an algebraic group \( G \) on a variety \( X \) is called principal, if the induced morphism is a closed immersion:

\[
\omega : \quad G \times_k X \longrightarrow X_0 \times_k X_0 \\
(g, x) \quad \mapsto \quad (y, g \circ x)
\]

Definition 5.5 Let \( X \) and \( Y \) be separated algebraic varieties and let an affine algebraic group \( G \) operate on \( X \) with the geometric quotient morphism \( \pi : X \rightarrow Y \). If \( \pi \) is flat and the induced morphism

\[
\omega : \quad G \times_k X \xrightarrow{\sim} X \times_Y X \subset X \times_k X \\
(g, x) \quad \mapsto \quad (y, g \circ x)
\]

is an isomorphism, then we call \((X, f, Y; F)\) a Mumford principal fibration.
Theorem 5.6  Let $X$ and $Y$ be separated algebraic varieties and let an affine algebraic group $G$ operate principally on $X$ such that there exists a morphism $\pi : X \to Y$ making $Y$ into the geometrical quotient $X/G$. Then $(X, \pi, Y; G)$ is a Mumford principal fibration.

Theorem 5.7  Every Mumford principal fibration is locally isotrivial.

Proposition 5.8  The quotient map $X_0 \to X_0/G$ is a Mumford principal fibration and, in particular, locally isotrivial.

Proof  We have to show that:

(i) $X_0, X_0/G$ are separated, what is true because they are both quasiprojective;

(ii) $PGL(2) \times PGL(2) \times PGL(2)$ is an affine algebraic group. This is true because $PGL(2) \cong SL(2)/\{\pm 1\}$ and there is the following theorem:

Theorem 5.9  [3, 5.2.5] Let $H$ be a closed normal subgroup of the linear algebraic group $G$. Then $G/H$ with the induced group structure is a linear algebraic group.

(iii) the action of $G$ on $X_0$ is principal, i.e.

$$\omega : \quad PGL(2) \times X_0 \longrightarrow X_0 \times X_0$$

$$\quad (g, x) \mapsto (g \circ x)$$

$$\quad (f, g, h)((M), (N)) \mapsto (((M), (N)), ((gMf^{-1}), (hNg^{-1}))))$$

is an immersion, where $PG = PGL(2) \times PGL(2) \times PGL(2)$.

Step 1  We shall show first, that

$$\omega_Y : \quad PGL(2) \times X_0 \longrightarrow X_0 \times X_0$$

$$\quad (g, y) \mapsto (g \circ y)$$

$$\quad (f, g, h)((M), (N)) \mapsto (((M), (N)), ((gMf^{-1}), (hNg^{-1}))))$$

where $PGL(2) \times PGL(2)$ is quotient of $G$ modulo the third factor, and

$$Y_0 = \{(N) \in \mathbb{P}Mat_{2 \times 2}(V) : 2\Omega^1(1) \to 2\mathcal{O} \to 0\}.$$

Set-theoretical injectivity. Suppose

$$\left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right), \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right) \in PGL(2)$$

and

$$\left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right) \left( \begin{array}{cc} e_1 & e_2 \\ e_3 & e_1 \end{array} \right) \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right) = \lambda \cdot \left( \begin{array}{cc} e_1 & e_2 \\ e_3 & e_1 \end{array} \right), \quad \lambda \in k^*.$$

The comparison of the coefficients standing by $e_2$ and $e_3$ gives the matrix equalities:

$$e_2 : \left( \begin{array}{cc} g_1h_3 & g_1h_4 \\ g_3h_3 & g_3h_4 \end{array} \right) = \left( \begin{array}{cc} 0 & \lambda \\ 0 & 0 \end{array} \right) ; \quad e_3 : \left( \begin{array}{cc} g_2h_1 & g_2h_2 \\ g_4h_1 & g_4h_2 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ \lambda & 0 \end{array} \right),$$

hence $g_1h_4 = \lambda, g_3 = h_3 = 0$ and $g_4h_1 = \lambda, g_2 = h_2 = 0$. And now the coefficients of $e_1$ give

$$\left( \begin{array}{cc} g_1 & 0 \\ 0 & g_4 \end{array} \right) \left( \begin{array}{cc} \lambda/g_1 & 0 \\ 0 & \lambda/g_1 \end{array} \right) = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right)$$

and we deduce $g_1/g_4 = 1$, that means

$$\left( \begin{array}{cc} g_1 & g_2 \\ g_3 & g_4 \end{array} \right) = \left( \begin{array}{cc} h_1 & h_2 \\ h_3 & h_4 \end{array} \right),$$

and there is the following theorem:

Theorem 5.9  [3, 5.2.5] Let $H$ be a closed normal subgroup of the linear algebraic group $G$. Then $G/H$ with the induced group structure is a linear algebraic group.
The case of \( N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \) can be done similarly.

**Smoothness.** It suffices to check that for each \( N \) the map of the tangent spaces has the full rank equal to \( \dim Y_0 + \dim G = \dim Y_0 + 6 \):

\[
\tau : T_{\langle(N),\lambda,1\rangle}(Y_0 \times G') \rightarrow T_{\langle(N),\langle(N)\rangle\rangle}(Y_0 \times Y_0)
\]

\[
(N_1, g_1, h_1) \mapsto (N_1, N_1 + g_1 N - N h_1),
\]

where \( N_1 \in Mat_{2\times2}(V); g_1, h_1 \in \mathfrak{sl}(2) \). Again, we do the proof for \( N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \) and the other case is considered even easier.

Clearly it suffices to check that

\[
\text{rank}( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) ) \rightarrow Mat_{2\times2}(V) ) = 6
\]

\[
(g_1, h_1) \mapsto g_1 N - N h_1
\]

Studying the images of the basis vectors of \( \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \) under this map, we get:

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ -e_3 & -e_1 \end{pmatrix} ; \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e_1 & -e_2 \\ e_3 & -e_1 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} = \begin{pmatrix} e_3 & e_1 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e_1 \\ 0 & e_3 \end{pmatrix}
\]

\[
\begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_1 & e_2 \end{pmatrix} ; \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_2 & 0 \\ e_1 & 0 \end{pmatrix}
\]

Now choose as a basis in \( Mat_{2\times2}(V) \) the matrices

\[
\begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_i \end{pmatrix}, i = 0, 4
\]

and write the submatrix of the matrix of \( \tau \) corresponding to the values \( i = 1, 2, 3 \).

\[
\begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\end{pmatrix}
\]

\[
\begin{pmatrix} e_1 & e_2 & e_3 & e_1 & e_2 & e_3 & e_1 & e_2 & e_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\]

\[
=: A
\]

whose rank is obviously 6.

**Step 2.** Now we return to our \( \omega \).

By lemma 2.22, if \( (\langle M \rangle, \langle N \rangle) \in X_0 \) and \( \text{rank span } M = 3 \), then up to elementary transformations of columns \( M \) is of the form \( \begin{pmatrix} a & b \\ c & a \end{pmatrix} \), where \( a, b, c \) are linearly independent in \( \Lambda^3 V \).

**Set-theoretical injectivity** follows by step 1 applied to the equalities \( \langle fMg^{-1} \rangle = \langle M \rangle \) and \( \langle gNh^{-1} \rangle = \langle N \rangle \).

**Smoothness.** We have to check that the map of tangent spaces

\[
T_{\langle\langle(M),\langle(N)\rangle\rangle\rangle}(X_0 \times G') \rightarrow T_{\langle\langle(M),\langle(N)\rangle\rangle\rangle}(X_0 \times X_0)
\]

\[
((M_1, N_1), (f_1, g_1, h_1)) \mapsto ((M_1, N_1), (M_1 + f_1 M - M g_1, N_1 + g_1 N - N h_1)),
\]

35
has the full rank, where $f_1, g_1, h_1 \in \mathfrak{sl}(2)$, $M_1 \in Mat_{2 \times 2}(\Lambda^3 V)$, $N_1 \in Mat_{2 \times 2}(V)$.

Clearly it suffices to check that the linear map

$$(f_1, g_1, h_1) \mapsto (f_1 M - M g_1, g_1 N - N h_1)$$

has rank 9. But this map has a (generalized sub)matrix of the form

$$
\begin{bmatrix}
A & 0_{3 \times 12} \\
0_{3 \times 12} & A
\end{bmatrix}
$$

which is of rank 9 as $A$ is of rank 6.

Hence the proposition.

\[\square\]

**Proof of the theorem 5.1.** By the previous proposition the quotient projection $X_0 \to X_0/G$ is locally isotrivial, that means by definition, that there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $X_0/G$ such that for each $i \in I$ there exists a diagram

$$
\begin{align*}
\tilde{U}_i \times G & \longrightarrow U_i \times X_0/G \xrightarrow{\rho U_i} X_0 \\
\tilde{U}_i & \xrightarrow{\text{étale}} U_i \xrightarrow{\text{open}} X_0/G
\end{align*}
$$

Now we have the following chain of equivalences:

- $X_0/G$ is smooth $\iff$ all $U_i$ are smooth $\iff$ (by [loc. cit., VI, 4.9, p.117]) $\tilde{U}_i$ are smooth $\iff$ $\tilde{U}_i \times G$ are smooth $\iff$ (by loc cit, because $\tilde{U}_i \times G \to U_i \times_{X_0/G} X_0$ is étale, too) $U_i \times G$ are smooth $\iff$ $X_0$ is smooth,

and the last statement was the contents of theorem 5.2.

\[\square\]
6 The Jumping Lines

As the first Chern class of our bundles is 0, we may expect that restricting an instanton bundle \( \mathcal{E} \) on a generic line \( \ell \subset \mathbb{P}^4 \) gives \( \mathcal{E}|_{\ell} \cong 4\mathcal{O}_\ell \), and we shall see that this is indeed true. Then we define a **jumping line** as a line with \( \mathcal{E}|_{\ell} \neq 4\mathcal{O}_\ell \). The aim of this section is to understand to some extent the geometry of the subsets of the Grassmannian of lines on \( \mathbb{P}^4 \) corresponding to different nontrivial splittings of \( \mathcal{E}|_{\ell} \).

The theorems 6.2, 6.3, 6.5 and corollary 6.4 can be summarized in the

**Theorem 6.1** Let an instanton bundle \( \mathcal{E} \) of rank 4 with quantum number 2 on \( \mathbb{P}^4 = \mathbb{P}V \) (\( V \) is a 5-dimensional vector space) be given by a monad

\[
2\Omega^4_{\mathbb{P}^4}(4) \xrightarrow{M} 2\Omega^1_{\mathbb{P}^4}(1) \xrightarrow{N} 2\mathcal{O}_{\mathbb{P}^4}
\]

and let the linear map

\[
k^2 \otimes V \xrightarrow{\tilde{M}} k^2 \otimes \bigwedge^4 V.
\]

be the contraction with the matrix \( M \). Let \( \ell \) be a line in \( \mathbb{P}^4 \) identified via the Plücker embedding with an element of \( \mathbb{P}\Lambda^2 V \cong \mathbb{G}(2,5) \ni \ell \). If \( (\ell) = (x \wedge y) \) in \( \mathbb{P}\Lambda^2 V \) and \( M = (m_{i,j})_{i,j=1,2} \), denote by \( M \wedge \ell \) a matrix \( (m_{i,j} \wedge x \wedge y)_{i,j=1,2} \in \text{Mat}_{2 \times 2}(\Lambda^5 V) \cong \text{Mat}_{2 \times 2}(k) \) (well defined up to proportionality). Then:

(i) if \( \text{rank } M \wedge \ell = 2 \), then \( \ell \) is not a jumping line;
(ii) if \( \text{rank } M \wedge \ell = 1 \), then \( \ell \) is a jumping line with splitting

\[
\mathcal{E}|_{\ell} \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell^{\oplus 2} \oplus \mathcal{O}_\ell(1);
\]

(iii) if \( M \wedge \ell = 0 \), then \( \ell \) is a jumping line with

\[
\mathcal{E}|_{\ell} \cong \mathcal{O}_\ell(-1)^{\oplus 2} \oplus \mathcal{F} \quad \text{or} \quad \mathcal{E}|_{\ell} \cong \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell + \mathcal{F}
\]

where \( \mathcal{F} \cong \mathcal{O}_\ell(1)^{\oplus 2} \) or \( \mathcal{F} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2) \);
(iv) in the situation of (iii), the jumping lines \( \ell \) with the property \( \mathcal{E}|_{\ell} = \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell + \mathcal{F} \) form a smooth conic on the Grassmannian \( \mathbb{G}(2,5) \);
(v) in the situation of (iii), the jumping lines \( \ell \) with the property \( \mathcal{F} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2) \) form either a smooth conic, if \( \text{rank } \tilde{M} = 8 \), or, if \( \text{rank } \tilde{M} = 7 \), a surface on the Grassmannian \( \mathbb{G}(2,5) \).

Restricting a monad

\[
2\Omega^4(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{N} 2\mathcal{O}
\]

on \( \mathbb{P}^4 \) with the cohomology sheaf \( \mathcal{E} \) to a line \( \ell = \mathbb{P}W \subset \mathbb{P}^4 = \mathbb{P}V \), we get a monad of the restriction \( \mathcal{E}|_{\ell} \):

\[
2 \left[ \Omega^4_{\ell}(1) \otimes \Lambda^3(V/W)^\vee \right] \xrightarrow{M} 2 \left[ \Omega^1_{\ell}(1) \otimes (V/W)^\vee \otimes \mathcal{O}(1)|_{\ell} \right] \xrightarrow{N} 2\mathcal{O}
\]

Clearly \( \ell \) is a jumping line iff \( H^1(\mathcal{E}|_{\ell}(-1)) \neq 0 \).

The monad on \( \ell \) twisted down once decomposes into two exact sequences:

\[
0 \rightarrow 2\Omega^4(3)|_{\ell} \rightarrow \mathcal{K}(-1)|_{\ell} \rightarrow \mathcal{E}(-1)|_{\ell} \rightarrow 0
\]

\[
0 \rightarrow \mathcal{K}(-1)|_{\ell} \rightarrow 2\Omega^1|_{\ell} \rightarrow 2\mathcal{O}|_{\ell}(-1) \rightarrow 0
\]
whose long exact cohomology sequences give rise to the diagram:

\[
\begin{array}{ccc}
2H^1(\Omega^1_\ell \otimes \wedge^3(V/W)^\vee) & \rightarrow & H^1(K|_{\ell}(-1)) \\
\downarrow & & \downarrow \\
2k & \rightarrow & H^1(\mathcal{E}|_{\ell}(-1))
\end{array}
\]

\[
\begin{array}{ccc}
2\wedge^3(V/W)^\vee & \rightarrow & 2H^1(\Omega^1_\ell \oplus (V/W)^\vee \otimes \mathcal{O}|_{\ell}(-1)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

Therefore

\[
\dim_k H^1\mathcal{E}_\ell(1) = 2 - \text{rank}(2\wedge^3(V/W)^\vee \rightarrow 2k) = 2 - \text{rank}(M \wedge \ell).
\]

To justify the last equality sign, we use the diagram

\[
\begin{array}{ccc}
2\wedge^3(V/W)^\vee & \rightarrow & 2k \\
\downarrow & & \downarrow \\
2k & \rightarrow & 2\wedge^3(V/W)^\vee \otimes \wedge^2W^\vee \rightarrow 2k
\end{array}
\]

which can be designed to be commutative by choosing the isomorphism in such a way that the diagram

\[
\begin{array}{ccc}
\wedge^3(V/W)^\vee & \rightarrow & k \\
\downarrow & & \downarrow \\
k & \rightarrow & \wedge^3(V/W)^\vee \otimes \wedge^2W^\vee \rightarrow 2k
\end{array}
\]

commutes for contractions with any \( w \in \wedge^3V \)

So, we have proved

**Theorem 6.2** Depending on \( M \wedge \ell \), the following statements hold:

\[
\begin{align*}
\text{rank } M \wedge \ell = 2 & \Rightarrow \mathcal{E}_\ell \cong \mathcal{O}_\ell^{\oplus 4} \\
\text{rank } M \wedge \ell = 1 & \Rightarrow \mathcal{E}_\ell \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell^{\oplus 2} \oplus \mathcal{O}_\ell(1) \\
\text{rank } M \wedge \ell = 0 & \Rightarrow \mathcal{E}_\ell \cong \mathcal{O}_\ell(-1)^{\oplus 2} \oplus \mathcal{F} \text{ or } \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell \oplus \mathcal{F}
\end{align*}
\]

where \( \mathcal{F} \cong \mathcal{O}_\ell(1)^{\oplus 2} \) or \( \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2) \).

Clearly \( \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell \) appears in the splitting iff \( H^1\mathcal{E}_\ell \neq 0 \) The long exact cohomology sequences on \( \ell \)
yield

\[
\begin{array}{ccccc}
0 & \rightarrow & 2H^0(\Omega^1_\ell(1) \oplus (V/W)^\vee \otimes \mathcal{O}_\ell) & \rightarrow & 2(V/W)^\vee \\
& \downarrow & & \downarrow & N \\
& & 2H^0(\mathcal{O}_\ell) & \rightarrow & 2k \\
& \downarrow & & \downarrow & \cong \\
0 & \rightarrow & H^1K_\ell & \rightarrow & H^1E_\ell & \rightarrow & 0 \\
& & & & & & 0
\end{array}
\]

hence \(H^1E_\ell \cong \text{Coker}(2(V/W)^\vee \xrightarrow{N} 2k)\). Therefore \(H^1E_\ell \neq 0\) iff \(N : 2(V/W)^\vee \rightarrow 2k\) is not surjective iff \(N^\vee : 2k \rightarrow 2(V/W)\) is not injective.

In the case \(N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}\) it means that the intersection of linear subspaces of \(2V\)

\[
2W \cap \left( k \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + k \cdot \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} \right) \neq 0.
\]
i.e. there exists \((s : t) \in \mathbb{P}^1\) such that \(t e_1 + s e_3 \in W, t e_2 + s e_4 \in W\).

So, to describe all jumping lines with \(\mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell\) take the ruling lines of the scroll between the lines \(\overline{e_1e_3}\) and \(\overline{e_2e_4}\). This system of lines corresponds to a smooth conic in \(G(2, 5) \subset \mathbb{P}^5\).

If \(N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix}\), we get the conditions \(t e_1 + s e_3 \in W, t e_2 + s e_1 \in W\)
and then the jumping lines fill the plane \(\overline{e_1e_2e_3}\) and form again a smooth conic

\[
(te_1 + se_3) \wedge (te_2 + se_1) = t^2 e_{12} - tse_{23} - s^2 e_{13}
\]
in the Grassmannian \(G(2, 5)\).

We formulate the result as

**Theorem 6.3** The jumping lines \(\ell\) with the property \(\mathcal{E}_\ell = \mathcal{O}_\ell(-2) \oplus \mathcal{O}_\ell \oplus \mathcal{F}\) form a smooth conic on the Grassmannian \(G(2, 5)\).

Since in the case \(H^0\mathcal{E}^\vee = 0\) the dual bundle \(\mathcal{E}^\vee\) is an instanton bundle again, we get

**Corollary 6.4** If \(H^0\mathcal{E}^\vee = 0\), then the locus of jumping lines \(\ell\) with \(\mathcal{E}_\ell = \mathcal{G} \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2)\) is also a smooth conic on \(G(2, 5)\).

If \(h^0\mathcal{E}^\vee = 1\), then we have an extension

\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee_1 \rightarrow 0,
\]

where \(\mathcal{E}^\vee_1\) is a cohomology sheaf (not a bundle) of the monad

\[
2\Omega^1(4) \xrightarrow{M^\vee} 2\Omega^1(1) \xrightarrow{P^\vee} 3\mathcal{O}.
\]
If we restrict the above extension to a line $\ell$, we get

$$\mathcal{O}_\ell \to \mathcal{E}_\ell^\vee \to \mathcal{E}_{1,\ell}^\vee \to 0$$

where $\mathcal{E}_{1,\ell}^\vee$ is reflexive hence locally free. One can choose an isomorphism of $\mathcal{E}_\ell^\vee \cong \mathcal{L} \oplus \mathcal{O}(p)$ such that this sequence becomes

$$0 \to \mathcal{L} \to \mathcal{F} \to 0 \oplus \oplus \mathcal{O}_\ell \to \mathcal{O}_\ell(a) \to \mathcal{T} \to 0$$

where $a \geq 0$, $\mathcal{E}_{1,\ell}^\vee \cong \mathcal{F} \oplus \mathcal{T}$, and $\mathcal{T}$ is a torsion sheaf. Since $\mathcal{T}$ has to be 0, we get $a = 0$ and again $\mathcal{E}_\ell^\vee \cong \mathcal{O}_\ell \oplus \mathcal{E}_{1,\ell}^\vee$

We are interested in lines $\ell$ with the property that $\mathcal{O}_\ell(-2)$ is contained in the splitting of $\mathcal{E}_{1,\ell}^\vee$, i.e. $H^1(\mathcal{E}_{1,\ell}^\vee) \neq 0$. Such $\ell$’s will be precisely those giving $\mathcal{O}_\ell \oplus \mathcal{O}_\ell(2)$ in the splitting of $\mathcal{E}_\ell$.

But $H^1(\mathcal{E}_{1,\ell}^\vee) \cong \text{Coker}(2(V/W)^\vee \to 3k)$. Therefore $H^1 \mathcal{E}_\ell \neq 0$ iff $N : 2(V/W)^\vee \to 2k$ is not surjective iff $P : 3k \to 2(V/W)$ is not injective.

If $P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$, it will mean that the intersection of linear subspaces of $2V$:

$$2W \cap \left( k \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + k \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + k \cdot \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \right) \neq 0.$$  

i.e. there exists $(s : t : u) \in \mathbb{P}^2$ such that $sa_1 + ta_2 + ua_3 \in W$ and $sb_1 + tb_2 + ub_3 \in W$.

This system of lines corresponds to a surface in $G(2,5) \subset \mathbb{P}^5$ given by a parametrization $(sa_1 + ta_2 + ua_3) \wedge (sb_1 + tb_2 + ub_3)$. Note that this expression is nowhere zero because of the nondegeneracy of $P$. Also note that by theorem 3.5 after a suitable basis choice in $V$ $P$ can be written in one of the (possibly) two forms $P \sim \begin{pmatrix} e_0 & e_1 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$ or $P \sim \begin{pmatrix} e_0 & e_1 & e_3 \\ e_1 & e_2 & e_4 \end{pmatrix}$.

The argument presented proves the

**Theorem 6.5** If $h^0 \mathcal{E}^\vee = 1$ then the locus of jumping lines with $\mathcal{E}_\ell = \mathcal{G} \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(2)$ is a surface in $G(2,5)$.
7 Subsets of the Moduli Space

In this section we are studying the following three special subsets of $\mathcal{M}$:
\[ \mathcal{M}^3 := \text{sheaves in whose monads dim span } N = 3 \]
\[ \mathcal{M}^7 := \text{sheaves } \mathcal{E} \text{ in whose monads rank } \hat{\mathcal{M}} = 7 \text{ or, equivalently (cf. corollary 3.3), } H^0 \mathcal{E}^\vee = 1 \]
\[ \mathcal{M}^{sd} := \text{self-dual bundles } \mathcal{E}, \text{i.e. bundles with } \mathcal{E} \cong \mathcal{E}^\vee. \]

**Theorem 7.1** $\mathcal{M}^3$ is irreducible of codimension 2.

**Proof** Denote :
\[ Y^3 := \{ \langle N \rangle : \text{dim span } N = 3 \} \subset Y ; \quad Y^3_0 := Y^3 \cap Y_0, \]
\[ X^3 := \{ (\langle M \rangle, \langle N \rangle) \in X : \langle N \rangle \in Y^3 \} \subset X ; \quad X^3_0 := X^3 \cap X_0. \]

Since $Y^3_0$ is dense in $Y^3$ and $X_0$ is a fibration over $Y_0$ with irreducible fibers of constant dimension 19, we get
\[ \text{codim}_X X^3 = \text{codim}_Y Y^3. \]

If $\langle N \rangle \in Y^3$, it means that its entries are linearly dependent. But the entries of $N$ are a priori arbitrary vectors from the 5-dimensional vector space $V$, and it is classically known that the condition of the linear dependence of 4 vectors in $k^5$ has codimension 2.

Irreducibility can be verified by constructing the surjection from an irreducible set onto $Y^3_0/G'$ as $\mathcal{M}$ is a fibration over it with irreducible fibers. The composite map
\[ V^0 \subset 3V \rightarrow Y^3_0 \rightarrow Y^3_0/G' \]
\[ (v_1, v_2, v_3) \mapsto \begin{pmatrix} v_1 & v_2 \\ v_3 & v_1 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 & v_2 \\ v_3 & v_1 \end{pmatrix} \]

where $V^0$ is the set of triples of linear independent vectors in $V$ and, clearly, an open subset of $3V$. does the job.

$\square$

**Theorem 7.2** $\mathcal{M}^7$ is irreducible of codimension 3 and contained in $\mathcal{M}^3$.

**Proof.** Denote by $X^7_0 \subset X_0$ a set of all pairs $(\langle M \rangle, \langle N \rangle)$ with $H^0 \mathcal{E}^\vee = 1$. By the classification done above $X^7_0$ is an open nonempty subset in
\[ X^7 = \{ (\langle M \rangle, \langle N \rangle) : \left( \frac{N}{\lambda x \mu x} \right) \wedge M = 0, \exists x \in \mathbb{P}V, (\lambda, \mu) \in \mathbb{P}^1 \} \subset \mathbb{P}Mat_{2 \times 2}(\bigwedge^3 V) \times Y^3 \]

which, in turn, is an image of the set
\[ \tilde{X}^7 = \{ (\langle M \rangle, \langle N \rangle, x, (\lambda, \mu)) : \left( \frac{N}{\lambda x \mu x} \right) \wedge M = 0, \text{dim span } N = 3, \exists x \in \mathbb{P}\text{span } N, (\lambda, \mu) \in \mathbb{P}^1 \} \subset \mathbb{P}Mat_{2 \times 2}(\bigwedge^3 V) \times Y^3 \times \mathbb{P}V \times \mathbb{P}^1. \]

Now consider the morphism $\pi$ from the following diagram:

\[ \tilde{X}^7 \xrightarrow{\text{closed}} \mathbb{P}Mat_{2 \times 2}(\bigwedge^3 V) \times Y^3 \times \mathbb{P}V \times \mathbb{P}^1 \]
\[ \pi \]
\[ Y^3 \times \mathbb{P}V \times \mathbb{P}^1 \]
Clearly $\pi$ is a projective morphism onto an irreducible subvariety

$$\{\dim \text{span } N = 3; \ x \in \text{span } N\} \subset Y^3 \times \mathbb{P}V \times \mathbb{P}^1$$

with equidimensional irreducible fibers (which are, in fact, $(8 \cdot 2 - 1)$-dimensional linear subspaces of $\mathbb{P}\text{Mat}_{2 \times 2}(\mathbb{A}^3 V)$), hence $\tilde{X}^7$ and so are $X^7, X^7_0$ and $\mathcal{M}^7$.

Further, $\mathcal{M}^7 \subset \mathcal{M}^3$, the latter has codimension 2 and both are irreducible. Therefore, to prove the statement about the dimension it suffices to show that codim$_{X^7_0} X^7_0 \leq 3$.

The general form of the matrix $M$ as given in the introduction allows us to write the general form of $\tilde{M}$ as well. If $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$, a computation shows that

$$\tilde{M} = \begin{pmatrix}
- p_2 & - p_1 & 0 & 0 & p_8 & - q_2 & - q_1 & 0 & 0 & q_8 \\
p_3 & p_2 & 0 & 0 & - p_9 & q_3 & q_2 & 0 & 0 & - q_9 \\
0 & 0 & p_2 & p_1 & p_5 & 0 & 0 & q_2 & q_1 & q_5 \\
0 & 0 & - p_3 & - p_2 & - p_6 & 0 & 0 & - q_3 & - q_2 & - q_6 \\
p_5 & p_4 & p_8 & p_7 & 0 & q_5 & q_4 & q_8 & q_7 & 0 \\
- p_1 & - p_0 & 0 & 0 & p_7 & - q_1 & - q_0 & 0 & 0 & q_7 \\
p_2 & p_1 & 0 & 0 & - p_8 & q_2 & q_1 & 0 & 0 & - q_8 \\
0 & 0 & p_1 & p_0 & p_4 & 0 & 0 & q_1 & q_0 & q_4 \\
0 & 0 & - p_2 & - p_1 & - p_5 & 0 & 0 & - q_2 & - q_1 & - q_5
\end{pmatrix}$$

and in the case $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix}$

$$\tilde{M} = \begin{pmatrix}
p_5 & p_7 & p_9 & p_8 & 0 & q_5 & q_7 & q_9 & q_8 & 0 \\
- p_2 & 0 & 0 & 0 & p_8 & - q_2 & 0 & 0 & 0 & q_8 \\
p_3 & 0 & 0 & 0 & - p_9 & q_3 & 0 & 0 & 0 & - q_9 \\
- p_1 & 0 & 0 & 0 & p_7 & - q_1 & 0 & 0 & 0 & q_7 \\
0 & - p_1 & - p_3 & - p_2 & - p_5 & 0 & - q_1 & - q_3 & - q_2 & - q_5 \\
p_4 & p_6 & p_8 & - p_7 & 0 & q_4 & q_6 & q_8 & - q_7 & 0 \\
p_1 & 0 & 0 & 0 & - p_7 & q_1 & 0 & 0 & 0 & q_7 \\
p_2 & 0 & 0 & 0 & - p_8 & q_2 & 0 & 0 & 0 & - q_8 \\
- p_0 & 0 & 0 & 0 & p_6 & - q_0 & 0 & 0 & 0 & q_6 \\
0 & - p_0 & - p_2 & p_1 & - p_4 & 0 & - q_0 & - q_2 & q_1 & - q_4
\end{pmatrix}$$

We see that in the $10 \times 10$-matrices $\tilde{M}$ there are two pairs of equal rows. If we delete the redundant rows, we end up with $8 \times 10$ matrices. The condition that rank $\tilde{M} = 7$ can be then locally written by three equations as for all $M$’s we consider rank $\tilde{M} \geq 7$. This implies that codim$_{X^7_0} X^7_0 \leq 3$, Q.E.D.

\[ \square \]

**Theorem 7.3** The subset $\mathcal{M}^{sd}$ of the self-dual bundles is an irreducible closed subvariety of codimension 4

**Proof.** Denote by $X_0^{sd}$ the set of the parameter space corresponding to the self-dual bundles. We shall prove that $X_0^{sd}$ is a smooth irreducible closed $G$-invariant subvariety of codimension 4, and by the axioms of a geometric quotient the same is then true for $\mathcal{M}^{sd}$. First we give a more handier criterion of self-duality.
Lemma 7.4 A monad

\[ 2\Omega^1(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{N} 2\mathcal{O} \]

defines a self-dual bundle \( \mathcal{E} \) if and only if \( \dim \operatorname{span} M = 3 \). In this case, there exist a matrix \( M_0 = M_0^\vee \) and a group element \( g \in \operatorname{GL}(2) \) such that \( M = M_0 \cdot g \).

**Proof of the lemma.** We know from the chapter on the dual bundles that \( \mathcal{E} \cong \mathcal{E}^\vee \) iff \( M \sim M^\vee \). Obviously, if \( M \) is a symmetric matrix, then \( \dim \operatorname{span} M = 3 \). Conversely, if \( \dim \operatorname{span} M = 3 \) and there is no generalized column of the form \( \lambda \otimes x \), then by lemma 2.21 there exist \( M_0 = M_0^\vee \) and \( g \in \operatorname{GL}(2) \) such that \( M = M_0 \cdot g \), in particular, \( \dim \operatorname{span} M = \dim \operatorname{span} M_0 = 3 \), hence the lemma.

We can prove that \( \dim X_0 X_{sd} \leq 4 \) simply by looking at the equations. For any \( \langle N \rangle \in Y_0 \) consider the fiber \( X_{0,N} \) of \( X \) over \( \langle N \rangle \). Then \( X_{0,N} \) is isomorphic to an open subset of \( P^{19} \) with homogeneous coordinates \( p_0, \ldots, q_9 \). The condition \( \dim \operatorname{span} M = 3 \) can be written as

\[
\begin{pmatrix}
p_1 & p_2 & p_3 & p_5 & p_6 & p_8 & p_9 \\
p_0 & p_1 & p_2 & p_4 & p_5 & p_7 & p_8 \\
q_1 & q_2 & q_3 & q_5 & q_6 & q_8 & q_9 \\
q_0 & q_1 & q_2 & q_4 & q_5 & q_7 & q_8 \\
\end{pmatrix}
\begin{pmatrix}
p_1 & p_2 & p_3 & p_5 & p_6 & p_8 & p_9 \\
p_0 & p_1 & p_2 & p_4 & p_5 & p_7 & p_8 \\
q_1 & q_2 & q_3 & q_5 & q_6 & q_8 & q_9 \\
q_0 & q_1 & q_2 & q_4 & q_5 & q_7 & q_8 \\
\end{pmatrix}
\]

for the case \( \dim \operatorname{span} N = 4 \) and

\[
\begin{pmatrix}
p_1 & p_2 & p_3 & p_5 & p_7 & p_8 & p_9 \\
p_0 & p_1 & p_2 & p_4 & p_6 & p_7 & p_8 \\
q_1 & q_2 & q_3 & q_5 & q_7 & q_8 & q_9 \\
q_0 & q_1 & q_2 & q_4 & q_6 & q_7 & q_8 \\
\end{pmatrix}
\begin{pmatrix}
p_1 & p_2 & p_3 & p_5 & p_7 & p_8 & p_9 \\
p_0 & p_1 & p_2 & p_4 & p_6 & p_7 & p_8 \\
q_1 & q_2 & q_3 & q_5 & q_7 & q_8 & q_9 \\
q_0 & q_1 & q_2 & q_4 & q_6 & q_7 & q_8 \\
\end{pmatrix}
\]

if \( \dim \operatorname{span} N = 3 \). As these ranks are always \( \geq 3 \), the codimension of the subvariety \( X_{sd}^{sd} \subset X_{0,N} \) defined by these equations is \( \leq 4 \).

We shall construct a surjection of an irreducible \( 38 - 4 = 34 \)-dimensional variety onto \( X_{0}^{sd} \) and thus prove the inverse inequality \( \dim X_{0} X_{sd} \geq 4 \) and the irreducibility of \( X_{0}^{sd} \).

Consider the variety \( Z \in X_0 \) given by the condition \( \langle M \rangle = \langle M^\vee \rangle \). In each \( X_{0,N} \) it is given by 7 linear equations hence each \( Z \cap X_{0,N} \) a \( 19 - 7 = 12 \)-dimensional linear projective space and by proposition 2.27 \( Z \) is an irreducible \( 38 - 7 = 31 \)-dimensional variety. We define a map

\[ Z \times \text{SL}(2) \to X_{0}^{sd} : (\langle \langle M \rangle, \langle N \rangle \rangle, g) \mapsto (\langle M \cdot g \rangle, \langle N \rangle) \]

which is surjective by the lemma above and where \( Z \times \text{SL}(2) \) is \( 31 + 3 = 34 \)-dimensional and irreducible. This completes the proof.

\[ \square \]

The same argument shows that

**Corollary 7.5** The intersection \( \mathcal{M}^{sd} \cap \mathcal{M}^3 \) is an irreducible subvariety of codimension 4 in \( \mathcal{M}^3 \).

**Remark 7.6** From the section on the dual bundle we see, that \( \mathcal{M}^7 \cap \mathcal{M}^{sd} = \emptyset \).
8 A Description of the Moduli Space as a Fibration

From the proofs done in the sections 4 and 5 we can derive also the following

**Theorem 8.1** Consider the group $G' = SL(2) \times SL(2)$ operating on $Y_0$. Then:

(i) There exists a geometric quotient $Y_0/G' =: Q$ which is a smooth quasi-projective variety of dimension 13;

(ii) The quotient map $Y_0 \to Q$ is a Mumford principal fibration;

(iii) $Y_3^0$ is a $G'$-invariant closed subset in $Y_0$ and therefore defines a closed subset $Q^3 \subset Q$ of codimension 2;

(iv) The projection $X_0 \to Y_0$ induces the surjective map $\mathcal{M} \to Q$ whose geometric fibres are isomorphic to open subsets in the Grassmannian $G(2,10)$.

**Proof** of (i) and (ii) was in fact done in the step 1 of the proof of proposition 5.8 together with the proof of theorem 5.1. Part (iii) follows from the proof of 7.1. Part (iv) is obvious from definitions of $X_0$, $Y_0$ and of group actions.

Proposition 8.2 There exists a generically 2:1 map $\delta : Q \to Z$ where $Z = \mathbb{P}S^2 V$ is the space of quadratic hypersurfaces in $\mathbb{P}V^\vee$ and $\delta$ has the properties: (i) the ramification locus of delta is precisely $Q^3$ (ii) the image $\delta(Q)$ is contained in the set of singular quadric hypersurfaces, i.e. in the set of quadratic cones.

**Sketch of the proof** (cf. [6] for more details). Define the morphism

$$
\det : \quad Y_0 \to Z = \mathbb{P}S^2 V \\
\langle N \rangle \mapsto \langle \det(N) \rangle
$$

This morphism is $G'$-equivariant (in the situation of reduced quasi-projective varieties it suffices to check this set-theoretically on closed points) hence factors through a map $\delta : Q \to Z$.

Calculations of the primages show that

$$
\delta^{-1}(e_1 e_4 - e_2 e_3) = \left\{ \left[ \begin{array}{cc} e_1 & e_2 \\ e_3 & e_4 \end{array} \right], \left[ \begin{array}{cc} e_1 & e_3 \\ e_2 & e_4 \end{array} \right] \right\} \\
\delta^{-1}(e_1^2 e_2 - e_2 e_3) = \left\{ \left[ \begin{array}{cc} e_1 & e_2 \\ e_3 & e_1 \end{array} \right] \right\}
$$

where $[N]$ denotes the class of $\langle N \rangle$ modulo the action of $G'$. This implies the property (i), while the property (ii) is obvious.

There is also another observation that helps to understand the structure of $Q$ and hence that of $\mathcal{M}$. The construction we are presenting now will be implicitly used in the section 9.

**Construction 8.3** of a rational map $\sigma : Q \to \mathbb{P}V^\vee$.  

44
Similarly to the last proposition, induce $\sigma$ by defining a $G'$-equivariant morphism

$$\text{span} : Y_0^4 \rightarrow \mathbb{P}V^\vee$$

$$\langle N \rangle \mapsto \text{span} N$$

where $Y_0^4 = Y_0 - Y_0^3$. Then $\sigma$ is singular along $Q^3$.

However, we can construct a birational modification $\pi : \tilde{Q} \rightarrow Q$ such that $\sigma$ lifts to $Q$ as a regular map. Define $\tilde{Q}$ as the graph of an incidence correspondence

$$\tilde{Q} := \{ ([N], H) : \text{span} N \subseteq H \} \subset Q \times \mathbb{P}V^\vee$$

and $\pi$ as the natural projection.

Clearly, $\pi$ is birational and an isomorphism outside $\pi^{-1}Q^3$. The morphism $\tilde{\sigma}$ appears from the commutative diagram:

$$\tilde{Q} \subset Q \times \mathbb{P}V^\vee \xrightarrow{pr_2} \mathbb{P}V^\vee$$

It is also easy to see that fibers of $\pi$ over $Q^3$ are isomorphic to $\mathbb{P}^1$. 
9 Restriction to a 3-Plane

**Proposition 9.1** Let $E$ be the instanton bundle defined by a monad

$$0 \to 2\Omega^1(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{N} 2\mathcal{O} \to 0$$

and $H = \mathbb{P}W \subset \mathbb{P}V$ a projective 3-plane. Then $h^0(E|_H) = 2$ iff $W \supset \text{span } N$.

**Proof.** Restricting the monad to $H$ and denoting by $N$ its kernel bundle, we get:

$$0 \to 2\Omega^3_H(3) \otimes (V/W)^\vee \xrightarrow{M(H)} 2[\Omega^1_H(1) \oplus (V/W)^\vee \otimes \mathcal{O}_H] \xrightarrow{N(H)} 2\mathcal{O}_H \to 0$$

$$H^0(E|_H) = H^0(N|_H) = \text{Ker } (2(V/W)^\vee \xrightarrow{N} 2k)$$

But $\dim H^0(E|_H) = 2$ means precisely that $2(V/W)^\vee \xrightarrow{N} 2k$ is the zero map, or, equivalently, that the dual map $2k \xrightarrow{N^\vee} 2(V/W)$ is zero, i.e. span $N \subset W$. Hence the proposition., \[\square\]

**Theorem 9.2** If $\dim \text{span } N = 4$, denote $W = \text{span } N$ and $H = \mathbb{P}W \subset \mathbb{P}V$ will be a projective 3-plane. Then $E|_H \cong \mathcal{F} \oplus 2\mathcal{O}_H$.

**Proof.** Put $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ and write the matrix $M$ as $M = M' \wedge e_0 + M''$ where $M' \in \text{Mat}_{2 \times 2}(\wedge^2 W)$ and $M'' \in \text{Mat}_{2 \times 2}(\wedge^3 W)$.

The restricted monad gives us

\[
\begin{array}{ccc}
0 & \xrightarrow{} & 2\Omega^3_H(3) \\
\downarrow & & \downarrow \\
2\mathcal{O}_H(-1) & \xrightarrow{\nu} & 2\Omega^1_H(1) \oplus 2\mathcal{O}_H \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & \mathcal{N}' \oplus 2\mathcal{O}_H \\
\downarrow & & \downarrow \\
\mathcal{E}_H & \xrightarrow{} & 0 \\
\downarrow & & \\
2\mathcal{O}_H & \xrightarrow{} & 2\mathcal{O}_H \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & 2\mathcal{O}_H(-1) \\
\downarrow & & \downarrow \\
\mathcal{N}' & \xrightarrow{} & \mathcal{F} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & 0.
\end{array}
\]

in particular, $\mathcal{N}' = \text{Ker } (N : 2\Omega^1_H(1) \to 2\mathcal{O})$, and hence we have a diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{} & 2\mathcal{O}_H \\
\downarrow & & \downarrow \\
2\mathcal{O}_H & \xrightarrow{} & 2\mathcal{O}_H \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & 2\mathcal{O}_H(-1) \\
\downarrow & & \downarrow \\
\mathcal{N}' & \xrightarrow{} & \mathcal{F} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & 0.
\end{array}
\]
Claim. $M'$ defines an injection of sheaves $2\Omega^3_H(3) \to 2\Omega^1_H(1)$.

Indeed, if $M'$ is not injective as a sheaf homomorphism, we may assume $M' = \left( \begin{array}{cc} 0 & b' \\ 0 & d' \end{array} \right)$. Then the first column of $M$ has two elements $a, c \in \mathcal{O}^3_H$. But if so, $a^*, c^*$ are divisible by $e_0$ and hence $a^{*2} = c^{*2} = 0$ and by lemma 2.17 $M$ doesn’t define a subbundle, contradiction.

We are looking for a splitting $\rho$ as shown in the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & 2\mathcal{O}_H & \rightarrow & 2\mathcal{O}_H & \rightarrow & 2\mathcal{O}_H & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 2\mathcal{O}_H(-1) & \rightarrow & \mathcal{N}' & \rightarrow & \mathcal{E}_H & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 2\mathcal{O}_H(-1) & \rightarrow & \mathcal{N}' & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

If $\rho$ exists, then we also have a splitting $\sigma = \rho \pi$ as $\sigma j = \rho \pi j = \rho \varepsilon = 1_{2\mathcal{O}_H}$.

Denote $\sigma = (S' S'')$. Since $\sigma$ is a splitting, we get

\[
1 = (S' S'') \cdot \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = S''
\]

and $\sigma = (S' 1)$.

Given any such splitting $\sigma$, it induces a splitting $\rho$ if and only if $(S' 1) \cdot \left( \begin{array}{c} M' \\ M'' \end{array} \right) = 0$. Therefore the question is reduced to the following:

When does a sheaf homomorphism $S' : \mathcal{N}' \rightarrow 2\mathcal{O}_H$ exist such that $S'M' + M'' = 0$?

Since we have an exact sequence

\[
0 \rightarrow \mathcal{N}'_H \rightarrow 2\Omega^1_H(1) \xrightarrow{N} 2\mathcal{O}_H \rightarrow 0
\]

and $\text{Ext}^1(2\mathcal{O}_H, 2\mathcal{O}_H) = 0$, every $S'$ lifts to some $S_1 : 2\Omega^1_H(1) \rightarrow 2\mathcal{O}_H$.

If we take two different liftings $S_{1,2} : 2\Omega^1_H(1) \rightarrow 2\mathcal{O}_H$ of $S'$ then $S_1 - S_2 = R \cdot N$ for some $R : 2\mathcal{O}_H \rightarrow 2\mathcal{O}_H$, then $S_2 \wedge M' = S_1 \wedge M' + R \cdot N \wedge M'' = S_1 \wedge M'$.

We see now that the existence of $\rho$ is equivalent to existence of a matrix $S_1 \in \text{Mat}_{2 \times 2}(W)$ such that $S \wedge M' + M''$.

Let

\[
S_1 = \left( \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right), \quad M' = \left( \begin{array}{cc} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{array} \right), \quad M'' = \left( \begin{array}{cc} m''_{11} & m''_{12} \\ m''_{21} & m''_{22} \end{array} \right);
\]

then the conditions $S_1 \wedge M' + M'' = 0$ can be rewritten as

\[
\begin{align*}
s_{11} \wedge m'_{11} + s_{12} \wedge m'_{21} &= m''_{11} \\
s_{11} \wedge m'_{12} + s_{12} \wedge m'_{22} &= m''_{12} \\
s_{21} \wedge m'_{11} + s_{22} \wedge m'_{21} &= m''_{21} \\
s_{21} \wedge m'_{12} + s_{22} \wedge m'_{22} &= m''_{22}
\end{align*}
\]
Put \( s_{11} = \sum_{i=1}^{4} x_i e_i \) and \( s_{12} = \sum_{i=1}^{4} y_i e_i \),

\[
\begin{align*}
m''_{11} &= \mu_1 e_{123} + \mu_2 e_{124} + \mu_3 e_{134} + \mu_4 e_{234} \\
m''_{12} &= \nu_1 e_{123} + \nu_2 e_{124} + \nu_3 e_{134} + \nu_4 e_{234}
\end{align*}
\]

A computation shows that the first two conditions give in fact a non-homogeneous system of linear equations

\[
\begin{pmatrix}
-p_2 & p_3 & 0 & 0 \\
-p_1 & p_2 & 0 & 0 \\
0 & 0 & -p_2 & p_3 \\
0 & 0 & -p_1 & p_2 \\
-q_2 & q_3 & 0 & 0 \\
-q_1 & q_2 & 0 & 0 \\
0 & 0 & -q_2 & q_3 \\
0 & 0 & -q_1 & q_2 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\end{pmatrix}
\]

with the matrix nondegenerated on an open dense subset of \( X_0 - X^3_0 \). Hence the set of monads with the property that \( \mathcal{E}|_{\text{span } N} \) splits form an open dense subset of \( X_0 - X^3_0 \).

Now take \( (X_0 - X^3_0) \times \mathbb{P}^3 \) and consider the diagram of flat families over \( X_0 - X^3_0 \), where \( p \) is the product projection on \( \mathbb{P}^3 \).

\[
\begin{array}{cccccccccc}
0 & \rightarrow & 2p^*\Omega^3_H(3) & \rightarrow & 2p^*\Omega^1(1) \oplus 2\mathcal{O} & \rightarrow & 2\mathcal{O} & \rightarrow & 0 \\
2p^*\mathcal{O}(-1) & \rightarrow & \mathcal{N} \oplus 2\mathcal{O} & \rightarrow & \mathcal{E} & \rightarrow & 0 \\
\end{array}
\]

which gives the diagram from the very beginning of the proof over each \( (\langle M \rangle, \langle N \rangle) \) with \( W = \text{span } N \). By what we have proven over each closed point, the composition \( 2\mathcal{O} \rightarrow \mathcal{N} \oplus 2\mathcal{O} \rightarrow \mathcal{E} \) is an injection and we may consider a short exact sequence

\[0 \rightarrow 2\mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0\]

where \( \mathcal{F} \) is a cokernel sheaf. Since the families are flat over \( X_0 - X^3_0 \), this sequence splits over its closed subset.

But \( X_0 - X^3_0 \) is irreducible, hence, putting things together, we see, that a splitting exists for each pair \( (\langle M \rangle, \langle N \rangle) \in (X_0 - X^3_0) \), Q.E.D.

\[\Box\]

**Remark 9.3** As you can see from the proof, \( \mathcal{F} \) is a family of instanton bundles on \( \mathbb{P}^3 \) with \( c_2 = 2 \). Such bundles have been studied, e.g., in [4].
10 Sections of the Kernel Bundle

In this section we describe the sections of the twisted kernel bundle $\mathcal{N}(1)$ of the monad

$$0 \to 2\Omega^4(4) \xrightarrow{M} 2\Omega^1(1) \xrightarrow{N} \mathcal{O} \to 0$$

which vanish on some 2-plane in $\mathbb{P}^4$.

We know that $h^0\mathcal{N}(1) = 10$ and $H^0(\mathcal{N}(1)) \subset 2 \wedge^2 V^\vee \cong 2 \wedge^3 V$

It is easy to see that if $\xi \in H^0\Omega^1(2) = 2 \wedge^3 V$ then the zero scheme of this section is precisely $\mathbb{P}V_\xi$ with the reduced structure, where $V_\xi = \{v \in V : v \wedge \xi = 0\}$ is the space of linear factors of the 3-form $\xi$. Therefore a section $(\xi, \eta) \in H^0(\mathcal{N}(1)) \cong 2 \wedge^3 V$ vanishes if and only if

$$\xi = \lambda \eta, \exists \lambda \in k - \{0\}; \quad \xi^* = \eta^* = 0$$

because, as we have seen in the proof of lemma 2.17, $\xi^* = 0$ is equivalent to $\dim V_\xi = 3$.

If $\Gamma$ is the total matrix of syzygies of $\mathcal{N}$, then

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Gamma \cdot (p_0, \ldots, p_9)^T.$$ 

If $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ then the condition $\xi = \lambda \eta$ can be written as

$$\text{rank} \begin{pmatrix} p_1 & p_2 & p_3 & p_5 & p_6 & p_8 & p_9 \\ p_0 & p_1 & p_2 & p_4 & p_5 & p_7 & p_8 \end{pmatrix} = 1$$

and

$$\xi^* = 2 \cdot [(p_2 p_8 - p_1 p_9)e_{0123} + (p_3 p_8 - p_2 p_9)e_{0124} + (p_1 p_6 - p_2 p_5)e_{0134} + (p_2 p_6 - p_3 p_5)e_{0234} + (p_2^2 - p_1 p_3)e_{1234}].$$

If $N = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_1 \end{pmatrix}$, the condition $\xi = \lambda \eta$ becomes

$$\text{rank} \begin{pmatrix} p_1 & p_2 & p_3 & p_5 & p_7 & p_8 & p_9 \\ p_0 & p_1 & p_2 & p_4 & p_6 & p_7 & p_8 \end{pmatrix} = 1$$

and

$$\xi^* = 2 \cdot [(p_3 p_8 - p_2 p_9)e_{0124} + (p_1 p_8 - p_2 p_7)e_{0134} + (p_1 p_9 - p_3 p_7)e_{0234}].$$

In both cases we see that, if the condition on the rank is fulfilled, $\xi^* = \eta^* = 0$ automatically. The conditions on the rank define a variety which is classically known as a 3-dimensional rational normal scrolls in $\mathbb{P}^9 = \mathbb{P}H^0(\mathcal{N}(1))$. See [3, 8.26] for other equivalent definitions and geometry of rational normal scrolls.

We have proved

**Theorem 10.1** The subset of $\mathbb{P}H^0(\mathcal{N}(1))$ corresponding to sections vanishing on a 2-plane, is a 3-dimensional rational normal scroll.
References

[1] G. Trautmann. Generalities about Instanton Bundles.

[2] J. Harris. Algebraic Geometry. A first Course. - Springer, GTM 133

[3] Newstead, P.E. Lectures on introduction to moduli problems and orbit spaces. Published for the Tata Institute of Fundamental Research, Bombay. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Mathematics. 51. Berlin - Heidelberg - New York: Springer-Verlag.

[4] Hartshorne, R. Algebraic geometry. Graduate Texts in Mathematics. 52. New York - Heidelberg - Berlin: Springer-Verlag.

[5] H. Weigand, Faserbündel Techniken in der Schnitttheorie geometrischer Quotienten. - Aachen: Shaker, 2000

[6] Narasimhan, M.S.; Trautmann, G. Compactification of $\mathbb{M}_P^3(0,2)$ and Poncelet pairs of conics. Pac. J. Math. 145, No.2, 255-365 (1990).

[7] Okonek, C.; Schneider, M.; Spindler, H. Vector bundles on complex projective spaces. Progress in Mathematics. 3. Boston - Basel - Stuttgart: Birkhaeuser.

[8] Grothendieck, A. Elements de geometrie algebrique. III: Etude cohomologique des faisceaux coherents. Publ. Math., Inst. Hautes Etud. Sci. 17, 137-223 (1963).

[9] Springer, T.A. Linear algebraic groups. Progress in Mathematics, Vol. 9. Boston - Basel - Stuttgart: Birkhaeuser

[10] Altman, A.; Kleiman, S. Introduction to Grothendieck duality theory - Berlin-Heidelberg-New York: Springer-Verlag

[11] Spindler, H.; Trautmann, G. Rational normal curves and the geometry of special instanton bundles on $\mathbb{P}^{2n+1}$. Math. Gottingensis, Schriftenr. Sonderforschungsbereichs Geom. Anal. 18, 47 p. (1987).

[12] Ancona, V.; Ottaviani, G. Canonical resolutions of sheaves on Schubert and Brieskorn varieties. In Proceedings Complex Analysis, Wuppertal 1990, Vieweg 1991.

[13] Atiyah, M.F.; Hitchin, N. J.; Drinfeld, V.G.; Manin, Yu.I. Construction of instantons. Phys. Lett. 65A, 185-187 (1978)

[14] Hartshorne, R.; Hirschowitz, A. Cohomology of a general instanton bundle. Ann. Sci. Ec. Norm. Super., IV. Ser. 15, 365-390 (1982).

[15] Brun, J.; Hirschowitz, A. Restrictions planes du fibre instanton general. J. Reine Angew. Math. 399, 27-37 (1989).

[16] Brun, J.; Hirschowitz, A. Variete des droites sauteuses du fibre instanton general. Compos. Math. 53, 325-336 (1984).

[17] Hirschowitz, A.; Narasimhan, M.S. Fibres De ’t Hooft speciaux et applications. Enumerative geometry and classical algebraic geometry, Prog. Math. 24, 143-164 (1982)

[18] Nüssler, Th.; Trautmann, G. Multiple Koszul structures on lines and instanton bundles. Int. J. Math. 5, No.3, 373-388 (1994)
[19] Maruyama, M.; Trautmann, G. Limits of instantons. Int. J. Math. 3, No.2, 213-276 (1992).

[20] Barth, W. Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$. Math. Ann. 258, 81-106 (1981).

[21] Okonek, Ch.; Spindler, H. Mathematical instanton bundles on $\mathbb{P}^{2n+1}$. J. Reine Angew. Math. 364, 35-50 (1986).

[22] Ancona, V.; Ottaviani, G. On moduli of instanton bundles on $\mathbb{P}^{2n+1}$. Pac. J. Math. 171, No.2, 343-351 (1995).

[23] Ottaviani, G.; Trautmann, G. The tangent space at a special symplectic instanton bundle on $\mathbb{P}^{2n+1}$. Manuscr. Math. 85, No.1, 97-107 (1994).

[24] Ancona, V.; Ottaviani, G. Stability of special instanton bundles on $\mathbb{P}^{2n+1}$. Trans. Am. Math. Soc. 341, No.2, 677-693 (1994).

[25] Katsylo, P.; Ottaviani, G. Regularity of the Moduli Space of Instanton Bundles $MI_{\mathbb{P}^3}(5)$.

Hiermit ekläre ich, dass ich die vorliegende Arbeit selbstständig erstellt und keine anderen als die angegebenen Hilfsmittel verwendet habe.