Appearance of branched motifs in the spectra of $BC_N$ type Polychronakos spin chains

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Abstract:

As is well known, energy levels appearing in the highly degenerate spectra of the $A_{N-1}$ type of Haldane-Shastry and Polychronakos spin chains can be classified through the motifs, which are characterized by some sequences of the binary digits like ‘0’ and ‘1’. In a similar way, at present we classify all energy levels appearing in the spectra of the $BC_N$ type of Polychronakos spin chains with Hamiltonians containing supersymmetric analogue of polarized spin reversal operators. To this end, we show that the $BC_N$ type of multivariate super Rogers-Szegö (SRS) polynomials, which at a certain limit reduce to the partition functions of the later type of Polychronakos spin chains, satisfy some recursion relation involving a $q$-deformation of the elementary supersymmetric polynomials. Subsequently, we use a Jacobi-Trudi like formula to define the corresponding $q$-deformed super Schur polynomials and derive a novel expression for the $BC_N$ type of multivariate SRS polynomials as suitable linear combinations of the $q$-deformed super Schur polynomials. Such an expression for SRS polynomials leads to a complete classification of all energy levels appearing in the spectra of the $BC_N$ type of Polychronakos spin chains through the ‘branched’ motifs, which are characterized by some sequences of integers of the form $(\delta_1, \delta_2, ..., \delta_{N-1}|l)$, where $\delta_i \in \{0, 1\}$ and $l \in \{0, 1, ..., N\}$. Finally, we derive an extended boson-fermion duality relation among the restricted super Schur polynomials and show that the partition functions of the $BC_N$ type of Polychronakos spin chains also exhibit similar type of duality relation.

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Dedicated to Artemio González-López on the occasion of his 60th birthday

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1. Introduction

The interplay between symmetry and exact solvability in one-dimensional quantum spin chains with long-range interactions \([1,27]\) has recently attracted a lot of attention due to its relevance in apparently diverse topics of physics and mathematics. Indeed, there are many important applications of this type of spin chains in various subjects like quantum electric transport phenomena \([28,29]\), condensed matter systems obeying generalized exclusion statistics \([5,27,30]\), random matrix theory \([31]\), ‘infinite matrix product states’ in conformal field theory \([32,37]\), planar \(\mathcal{N} = 4\) super Yang–Mills theory \([38–40]\) and Yangian quantum groups \([4,5,10,16,18]\). The study of such exactly solvable spin chains with long-range interactions was initiated through the pioneering works of Haldane and Shastry \([1,2]\). They derived the exact spectrum of a spin-\(\frac{1}{2}\) chain with lattice sites equally spaced on a circle and spins interacting with each other through a pairwise exchange interaction, inversely proportional to the square of their chord distances. An astonishing feature of this su(2) Haldane-Shastry (HS) spin chain and its su(m|n) supersymmetric generalizations is that they exhibit Yangian quantum group symmetry even for finite number of lattice sites. As a result, the degenerate multiplets (formed due to Yangian symmetry) in the energy spectra of these spin chains can be classified in an efficient way by using specific sequences of the binary digits ‘0’ and ‘1’, which are known as ‘motifs’ in the literature \([3,4]\). These motifs in fact characterize a class of irreducible representations of the Yangian algebra, which span the Fock space of the HS spin chains.

Interestingly, the Hamiltonian of su(m|n) HS spin chain may be obtained from that of su(m|n) spin Sutherland model where the particles are equipped with both the dynamical and the spin degrees of freedom \([6,7]\). In the limit of large coupling constant, the spin part of the Hamiltonian of this spin-dynamical model decouples from the dynamical part and reduces to the Hamiltonian of the HS spin chain. This technique of obtaining a spin chain from a spin-dynamical model in the limit of large coupling constant has a wide range of applicability for the case of integrable systems. Indeed, by using this technique, another spin chain with long-range interaction was derived by Polychronakos from the spin Calogero model with confining harmonic potential \([7]\). In this case, the lattice sites are inhomogeneously placed on a line and given by the roots of the \(N\)-th order Hermite Polynomial for \(N\) number of sites \([8]\). Such a spin chain is known as Polychronakos or Polychronakos-Frahm (PF) spin chain in the literature. The Hamiltonian of the su(m|n) supersymmetric version of ferromagnetic PF spin chain with \(N\) number of sites is given by \([12]\)

\[
H_N^{(m|n)} = \sum_{1 \leq i < j \leq N} \frac{1 - P_{ij}^{(m|n)}}{(\rho_i - \rho_j)^2},
\]

(1.1)

where \(P_{ij}^{(m|n)}\) is a supersymmetric spin exchange operator which interchanges the spins
of the $i$-th and $j$-th lattice sites (along with a phase factor), and $\rho_i$ is the $i$-th root of the $N$-th order Hermite polynomial. It may be noted that, for $n = 0$, (1.1) simply reduces to the Hamiltonian of the $su(m)$ ferromagnetic PF spin chain. Similar to the case of $su(m|n)$ supersymmetric HS spin chain, the PF spin chain (1.1) also exhibits $Y(gl_m|n)$ super Yangian quantum group symmetry for any value of $N$. As a result, degenerate multiplets appearing in the spectra of these spin chains can again be classified through the motifs.

Since the spin and coordinate degrees of freedom of the spin dynamical models related to the above mentioned PF and HS spin chains are decoupled from each other in the strong coupling limit, the partition functions of these spin chains can be computed by using the so called ‘freezing trick’. More precisely, the partition function of such a spin chain is obtained by taking the ratio of the partition function of the corresponding spin dynamical model to that of its spinless version in the strong coupling limit. However, it should be noted that, the partition functions of the spin chains obtained by using the freezing trick do not directly lead to the motif representation of the corresponding spectra. For this purpose, it is necessary to define the so-called generalized partition functions (which reproduce the standard partition functions at a certain limit) and to apply different techniques for expressing those generalized partition functions in terms of Schur polynomials associated with the motifs. By using such expressions of the generalized partition functions, one can immediately identify all degenerate multiplets within the spectra of the corresponding spin chains and write down the energy eigenvalues related to all motifs. Moreover, by applying a rather general framework, it is possible to show that all energy eigenvalues for a class of HS like Yangian invariant quantum spin chains can be reproduced from the energy functions of some one-dimensional classical vertex models having only local interactions. The above mentioned equivalence between the eigenvalues of Yangian invariant spin chains with long-range interactions and energy functions of one-dimensional vertex models with only local interactions can be extended even in the presence of chemical potentials. As a result, by using transfer matrices associated with those vertex models, one can calculate various thermodynamic quantities of this type of spin chains even in the presence of chemical potentials. Thus the expressions of the generalized partition functions in terms of Schur polynomials lead to a powerful method for classifying the degenerate multiplets of the corresponding spectra and for studying various thermodynamic properties of the related spin chains.

It may be noted that some homogeneous multivariate Rogers-Szegő (RS) polynomials, which can be expressed in terms of Schur polynomials, play the role of generalized partition functions for the case of non-supersymmetric PF spin chains. Similarly, for the case of $su(m|n)$ supersymmetric PF spin chain (1.1) with $N$ number of
lattice sites, one can define a multivariate super RS (SRS) polynomial of the form \[13\]

\[
H_{A,N}(x,y; q) = \sum_{a_i,b_j \in \mathbb{Z}_{\geq 0}} q^{\sum_{i=1}^{m} b_j} \prod_{i=1}^{m} x_{i}^{a_i} \prod_{j=1}^{n} y_{j}^{b_j}, \tag{1.2}
\]

where \(\mathbb{Z}_{\geq 0}\) represents the set of non-negative integers, \(x \equiv x_1, x_2, \ldots, x_m\) and \(y \equiv y_1, y_2, \ldots, y_n\) represent two different sets of variables, \(q\) is a free parameter and \((q)_n = (1-q)(1-q^2) \cdots (1-q^n)\). This SRS polynomial reduces to the partition function of the supersymmetric PF spin chain (1.1) at temperature \(T\) in the limit \(x_1 = x_2 = \cdots = x_m = 1, \ y_1 = y_2 = \cdots = y_n = 1\) and for \(q = e^{-1/(k_B T)}\). Moreover, such SRS polynomials corresponding to supersymmetric PF chains with different numbers of lattice sites \((N)\), but fixed values of internal degrees of freedom \((m\) and \(n))\), satisfy some recursion relations which lead to the desired expression of these polynomials through super Schur polynomials \[13\]. Hence these SRS polynomials can be treated as generalized partition functions for the supersymmetric PF spin chains.

In view of the above discussion, it is natural to ask whether multivariate RS or SRS polynomials can be used to analyse the spectra and partition functions of some other PF like spin chains. In this context it may be noted that, quantum integrable systems with long-range interactions can be classified according to their connections with different root systems related to the Lie algebra \[44, 45\]. In particular, the HS and PF spin chains which have been discussed so far are associated with the \(A_{N-1}\) type of root system. However, several exactly solvable variants of the PF spin chain, related to the \(BC_N\) and \(D_N\) root systems, have also been studied in the literature \[22, 25, 26, 46\]. The Hamiltonians of the PF spin chains related to the latter type of root systems contain reflection operators like \(S_i\) \((i = 1, \ldots, N)\), which satisfy the condition \(S_i^2 = 1\) and yield a representation of some elements belonging to the \(BC_N\) or \(D_N\) type of Weyl algebra. As a special case, \(S_i\) can be taken as the spin reversal operator \(P_i\) which flips the sign of the spin component on the \(i\)-th lattice site. Partition functions of non-supersymmetric \(BC_N\) and \(D_N\) types of PF spin chains, containing such spin reversal operators in their Hamiltonians, have been derived by using the freezing trick \[25, 46\]. Moreover, by taking reflection operators as supersymmetric analogue of spin reversal operators (SASRO), partition functions of corresponding PF spin chains related to the \(BC_N\) root system have also been computed by using the freezing trick \[26\].

However, it is possible to generate wider variants of \(BC_N\) or \(D_N\) type of PF spin chains by choosing the reflection operators in more general way than the above mentioned spin reversal operators and their supersymmetric analogues. For example, in the non-supersymmetric case, one can choose these reflection operators as arbitrarily polarized spin reversal operators (PSRO) denoted by \(P_i^{(m_1,m_2)}\), where \(m_1, m_2 \in \mathbb{Z}_{\geq 0}\). This \(P_i^{(m_1,m_2)}\) acts as an identity operator on the first \(m_1\) number of elements of the spin basis and as an identity operator with a negative sign on the remaining \(m_2\) number
of elements of the spin basis \([47]\). It can be shown that, in some particular cases like \(m_1 = m_2\) or \(m_1 = m_2 \pm 1\), \(P_{i}^{(m_1,m_2)}\) reduces to the spin reversal operator \(P_i\) through a similarity transformation. Choosing the reflection operators as such PSRO, new variants of \(BC_N\) and \(D_N\) type of PF spin chains have been obtained and the partition functions of these spin chains have also been calculated by using the freezing trick \([47, 48]\). Finally, by choosing the reflection operators as supersymmetric analogues of PSRO (SAPSRO), an even larger class of \(BC_N\) type of PF spin chains have been obtained \([49]\). These \(BC_N\) type of PF spin chains with SAPSRO can generate all of the previously obtained \(BC_N\) type of PF spin chains at different limits. The partition functions of these \(BC_N\) type of PF spin chains with SAPSRO have also been computed by using the freezing trick. Furthermore, it has been observed that such partition functions can be obtained by taking a certain limit of some \(BC_N\) type of multivariate SRS polynomials depending on four different sets of variables \([50]\).

In spite of the above mentioned works on \(BC_N\) type of PF spin chains, the important problem of classifying the degenerate multiplets of the corresponding spectra through some motif like representations have not been addressed so far. The main purpose of the present paper is to solve this problem by using a novel expression for \(BC_N\) type of multivariate SRS polynomials through some \(q\)-deformation of super Schur polynomials. The arrangement of this paper is as follows. In Section 2 of this paper, we define the Hamiltonian of the \(BC_N\) type of ferromagnetic PF spin chains with SAPSRO and briefly summarize some known properties of corresponding multivariate SRS polynomials. In Section 3, we show that these \(BC_N\) type of SRS polynomials satisfy some recursion relations involving a particular type of \(q\)-deformation of elementary supersymmetric polynomials. In Section 4, we use an analogue of the Jacobi-Trudi formula to define a \(q\)-deformed version of the super Schur polynomials in terms of the above mentioned \(q\)-deformed elementary supersymmetric polynomials. Subsequently, we expand those \(q\)-deformed super Schur polynomials as a power series of the parameter \(q\) to obtain the ‘restricted’ super Schur polynomials and also present some combinatorial form of such restricted super Schur polynomials. In Section 5, we derive novel expressions for the \(BC_N\) type of SRS polynomials through \(q\)-deformed super Schur polynomials and restricted super Schur polynomials. Such expressions for these SRS polynomials lead to a complete classification of the degenerate multiplets in the spectra of \(BC_N\) type of ferromagnetic PF spin chains through the ‘branched’ motifs. For a spin chain with \(N\) number of lattice sites, these branched motifs may be written as \((\delta_1, \delta_2, ..., \delta_{N-1} | l)\), where \(\delta_i \in \{0, 1\}\) and \(l \in \{0, 1, ..., N\}\). In Section 6, we briefly discuss similar classification of the degenerate multiplets in the spectra of \(BC_N\) type of anti-ferromagnetic PF spin chains. In section 7, we use an extended boson-fermion duality relation of the restricted super Schur polynomials to show that the partition functions of \(BC_N\) type of PF spin chains also exhibit similar duality relation. Section 8 is the concluding section.
2. $BC_N$ type of ferromagnetic PF spin chains and SRS polynomials

To describe a class of $BC_N$ type of PF spin chains on a superspace, let us consider a set of operators like $A_{j\alpha}^\dagger$ ($B_{j\alpha}$) which creates (annihilates) a particle of species $\alpha$ on the $j$-th lattice site. Let us assume that these creation (annihilation) operators are bosonic when $\alpha \in [1, 2, \ldots, m]$ and fermionic when $\alpha \in [m+1, m+2, \ldots, m+n]$. The parity of these operators are defined as

$$\pi(A_{j\alpha}) = \pi(A_{j\alpha}^\dagger) = 0 \text{ for } \alpha \in [1, 2, \ldots, m] ,$$

$$\pi(A_{j\alpha}) = \pi(A_{j\alpha}^\dagger) = 1 \text{ for } \alpha \in [m+1, m+2, \ldots, m+n] ,$$

and they satisfy the following commutation (anti-commutation) relations:

$$[A_{j\alpha}, A_{k\beta}]_\pm = 0 , \ [A_{j\alpha}^\dagger, A_{k\beta}^\dagger]_\pm = 0 , \ [A_{j\alpha}, A_{k\beta}^\dagger]_\pm = \delta_{jk}\delta_{\alpha\beta} , \quad (2.1)$$

where $[A, B]_\pm \equiv AB - (-1)^{\pi(A)\pi(B)}BA$. Now, let us consider a finite dimensional subspace of the corresponding Fock space, where each lattice site accommodates only one particle, i.e., $\sum_{\alpha=1}^{m+n} A_{ja}^\dagger A_{ja} = 1$ for all $j \in \{1, 2, \ldots, N\}$. The supersymmetric spin exchange operator $\hat{P}_{ij}^{(m|n)}$ can be defined on this subspace as

$$\hat{P}_{ij}^{(m|n)} = \sum_{\alpha,\beta=1}^{m+n} A_{i\alpha}^\dagger A_{j\beta}^\dagger A_{j\beta} A_{i\alpha} . \quad (2.2)$$

The above mentioned supersymmetric spin exchange operator can equivalently be expressed as an operator on the total internal space of $N$ number of spins, denoted by $\Sigma^{(m_1, m_2|n_1, n_2)}$, where $m_1$, $m_2$, $n_1$, $n_2 \in \mathbb{Z}_{\geq 0}$ satisfying the conditions $m_1 + m_2 = m$ and $n_1 + n_2 = n$ [51] [49]. This $\Sigma^{(m_1, m_2|n_1, n_2)}$ is spanned by some orthonormal state vectors of the form $|s_1, \ldots, s_i, \ldots, s_N\rangle$, where each local spin $s_i \in S \equiv \{1, 2, \ldots, m+n\}$ is endowed with two different types of parities. For the sake of defining these parities, it is convenient to write $S$ as the union of four sets given by

$$S_{+,+}^{(m_1)} = \{1, 2, \ldots, m_1\} ,$$

$$S_{+,=}^{(m_1)} = \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\} ,$$

$$S_{-,+}^{(n_1)} = \{m_1 + m_2 + 1, m_1 + m_2 + 2, \ldots, m_1 + m_2 + n_1\} ,$$

$$S_{-,-}^{(n_2)} = \{m_1 + m_2 + n_1 + 1, m_1 + m_2 + n_1 + 2, \ldots, m_1 + m_2 + n_1 + n_2\} . \quad (2.3)$$

The boson-fermion type parity of the local spin $s_i$ is defined as

$$\pi(s_i) = 0 \text{ if } s_i \in S_{+,+}^{(m_1)} \cup S_{+,=}^{(m_2)} ,$$

$$\pi(s_i) = 1 \text{ if } s_i \in S_{-,+}^{(n_1)} \cup S_{-,-}^{(n_2)} , \quad (2.4)$$

and another parity of $s_i$, related to the action of SAPSRO, is defined as

$$f(s_i) = 0, \text{ if } s_i \in S_{+,+}^{(m_1)} \cup S_{-,+}^{(n_1)} ,$$

$$f(s_i) = 1, \text{ if } s_i \in S_{+,=}^{(m_2)} \cup S_{-,-}^{(n_2)} . \quad (2.5)$$
Indeed the SAPSRO, denoted by \( P_i^{(m_1,m_2|n_1,n_2)} \), acts on the basis vectors of the space \( \Sigma^{(m_1,m_2|n_1,n_2)} \) as [49]

\[
P_i^{(m_1,m_2|n_1,n_2)} |s_1, \cdots, s_i, \cdots, s_N \rangle = (-1)^{f(s_i)} |s_1, \cdots, s_i, \cdots, s_N \rangle.
\]

(2.6)

Since each local spin vector \( s_i \) may be chosen in \( (m+n) \) number of different ways, \( \Sigma^{(m_1,m_2|n_1,n_2)} \) can be expressed as a direct product of the form

\[
\Sigma^{(m_1,m_2|n_1,n_2)} \equiv \mathcal{C}_{m+n} \otimes \mathcal{C}_{m+n} \otimes \cdots \otimes \mathcal{C}_{m+n},
\]

(2.7)

where \( \mathcal{C}_{m+n} \) is an \( (m+n) \)-dimensional complex vector space. Hence \( \Sigma^{(m_1,m_2|n_1,n_2)} \) is isomorphic to the subspace of the Fock space, on which \( \hat{P}^{(m|n)}_{ij} \) in (2.2) is defined. A supersymmetric spin exchange operator \( P^{(m|n)}_{ij} \) can be defined on the space \( \Sigma^{(m_1,m_2|n_1,n_2)} \) as [15, 51]

\[
P^{(m|n)}_{ij} |s_1, \cdots, s_i, \cdots, s_j, \cdots, s_N \rangle = (-1)^{\alpha_{ij}(s)} |s_1, \cdots, s_j, \cdots, s_i, \cdots, s_N \rangle,
\]

(2.8)

where \( \alpha_{ij}(s) = \pi(s_i)\pi(s_j) + (\pi(s_i) + \pi(s_j)) \rho_{ij}(s) \) and \( \rho_{ij}(s) = \sum_{k=i+1}^{j-1} \pi(s_k) \). The above equation implies that if two spins \( s_i \) and \( s_j \) with \( \pi(s_i) = \pi(s_j) = 0 \) or \( \pi(s_i) = \pi(s_j) = 1 \) are exchanged, then one gets a phase factor of 1 or \(-1\) respectively. Hence \( s_i \) can be considered as a ‘bosonic’ spin if \( \pi(s_i) = 0 \) and a ‘fermionic’ spin if \( \pi(s_i) = 1 \). However, it must be noted that the exchange of a bosonic spin with a fermionic spin (or, vice versa) produces a nontrivial phase factor of \((-1)^{\rho_{ij}(s)}\), where \( \rho_{ij}(s) \) represents the number of fermionic spins between the \( i \)-th and \( j \)-th lattice sites. Applying the commutation (anti-commutation) relations given in (2.1), one can easily show that \( \hat{P}^{(m|n)}_{ij} \) in (2.2) is completely equivalent to \( P^{(m|n)}_{ij} \) in (2.8).

It may be noted that, \( \hat{P}^{(m|n)}_{ij} \) in (2.8) and \( P^{(m_1,m_2|n_1,n_2)}_i \) in (2.6) yield a representation of the \( BC_N \) type of Weyl algebra [49]. Using this representation of Weyl algebra, the Hamiltonian of a class of exactly solvable \( BC_N \) type of ferromagnetic PF spin chains has been defined in the latter reference as

\[
\mathcal{H}^{(m_1,m_2|n_1,n_2)}_N = \sum_{i,j=1}^{N} \frac{1 - P^{(m|n)}_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - \hat{P}^{(m_1,m_2|n_1,n_2)}_{ij}}{(\xi_i + \xi_j)^2} + \beta \mathcal{C} \sum_{i=1}^{N} \frac{1 - P^{(m_1,m_2|n_1,n_2)}_i}{\xi_i^2},
\]

(2.9)

where \( \beta > 0 \) is a real parameter, \( \xi_i = \sqrt{2y_i} \) with \( y_i \) being the \( i \)-th root of the generalized Laguerre polynomial \( L^{\beta-1}_N \) and \( \hat{P}^{(m_1,m_2|n_1,n_2)}_{ij} = P^{(m_1,m_2|n_1,n_2)}_i P^{(m_1,m_2|n_1,n_2)}_j P^{(m|n)}_{ij} \). The above mentioned Hamiltonian is able to reproduce all of the previously studied \( BC_N \) type of PF spin chains for different values of the discrete parameters \( m_1, m_2, n_1 \) and \( n_2 \). For example, in the case when all the spins are either bosonic or fermionic, i.e., either \( n_1 = n_2 = 0 \) or \( m_1 = m_2 = 0 \), \( \mathcal{H}^{(m_1,m_2|n_1,n_2)}_N \) given by (2.9) reduces to the non-supersymmetric PF spin chain containing PSRO [47]. Moreover, in another case, where
the discrete parameters in (2.9) satisfy the following relations:

\[ m_1 = \frac{1}{2} (m + \epsilon \tilde{m}), \quad m_2 = \frac{1}{2} (m - \epsilon \tilde{m}), \quad n_1 = \frac{1}{2} (n + \epsilon' \tilde{n}), \quad n_2 = \frac{1}{2} (n - \epsilon' \tilde{n}), \quad (2.10) \]

with \( \epsilon, \epsilon' = \pm 1, \tilde{m} \equiv m \mod 2 \) and \( \tilde{n} \equiv n \mod 2 \), one can obtain the Hamiltonian (depending on the parameters \( m, n, \epsilon, \epsilon' \)) of the \( BC_N \) type of PF spin chains with SASRO [26] by using a unitary transformation [49].

Expanding the grand canonical partition function of the corresponding spin Calogero model as a power series of the fugacity parameter and applying the freezing trick, the canonical partition function of the \( BC_N \) type of ferromagnetic PF spin chain (2.9) has been derived in the form [50]

\[
Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{a_i+b_i+c_k+d_l \in \mathbb{Z}_{\geq 0}} \frac{(q^2)_{m_1} \cdot q^{\sum b_j + \sum_{k=1}^{n_1} c_k (c_k-1) + \sum_{l=1}^{n_2} d_l} \cdot \prod_{i=1}^{m_1} (q^2)^{a_i} \cdot \prod_{j=1}^{m_2} (q^2)^{b_j} \cdot \prod_{k=1}^{n_1} (q^2)^{c_k} \cdot \prod_{l=1}^{n_2} (q^2)^{d_l}}{\prod_{i=1}^{m_1} (q^2)^{a_i} \cdot \prod_{j=1}^{m_2} (q^2)^{b_j} \cdot \prod_{k=1}^{n_1} (q^2)^{c_k} \cdot \prod_{l=1}^{n_2} (q^2)^{d_l}}, \quad (2.11)
\]

where \( q = e^{-1/(k_BT)} \) and, for the sake of convenience, the above partition function is defined as a function of \( q \) instead of \( T \). It may be noted that this partition function does not depend on the parameter \( \beta \) which is present in the Hamiltonian (2.9). Motivated by the form of this partition function, a class of \( BC_N \) type of SRS polynomials has also been introduced in the later reference as

\[
H_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{a_i+b_i+c_k+d_l \in \mathbb{Z}_{\geq 0}} \frac{(q^2)_{m_1} \cdot q^{\sum b_j + \sum_{k=1}^{n_1} c_k (c_k-1) + \sum_{l=1}^{n_2} d_l} \cdot \prod_{i=1}^{m_1} (q^2)^{a_i} \cdot \prod_{j=1}^{m_2} (q^2)^{b_j} \cdot \prod_{k=1}^{n_1} (q^2)^{c_k} \cdot \prod_{l=1}^{n_2} (q^2)^{d_l}}{\prod_{i=1}^{m_1} (q^2)^{a_i} \cdot \prod_{j=1}^{m_2} (q^2)^{b_j} \cdot \prod_{k=1}^{n_1} (q^2)^{c_k} \cdot \prod_{l=1}^{n_2} (q^2)^{d_l}}, \quad (2.12)
\]

(with \( H_{B,0}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}, y, \bar{y}; q) = 1 \)), where \( x \equiv x_1, x_2, \ldots, x_{m_1}, \bar{x} \equiv \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{m_2}, y \equiv y_1, y_2, \ldots, y_{n_1} \) and \( \bar{y} \equiv \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n_2} \) denote four different sets of variables. It is evident that the partition function (2.11) can be obtained by taking a limit of the SRS polynomial (2.12) as

\[
Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = H_{B,N}^{(m_1,m_2|n_1,n_2)}(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1; q). \quad (2.13)
\]

The generating function corresponding to the \( BC_N \) type of SRS polynomials (2.12) has been defined as [50]

\[
G_{B}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q, t) = G_{1}^{(m_1)}(x; q, t) \cdot G_{2}^{(m_2)}(\bar{x}; q, t) \cdot G_{3}^{(n_1)}(y; q, t) \cdot G_{4}^{(n_2)}(\bar{y}; q, t), \quad (2.14)
\]
where

\[ \mathcal{G}^{(m_1)}_1(x; q, t) = \frac{1}{\prod_{i=1}^{m_1} (tx_i; q^2)^\infty}, \]

\[ \mathcal{G}^{(m_2)}_2(\bar{x}; q, t) = \frac{1}{\prod_{j=1}^{m_2} (tq\bar{x}_j; q^2)^\infty}, \]

\[ \mathcal{G}^{(n_1)}_3(y; q, t) = \frac{1}{\prod_{k=1}^{n_1} (-tq^{-2}y_k; q^{-2})^\infty}, \]

\[ \mathcal{G}^{(n_2)}_4(\bar{y}; q, t) = \frac{1}{\prod_{l=1}^{n_2} (-tq^{-1}\bar{y}_l; q^{-2})^\infty}, \]

and the notation \((t; q)^\infty \equiv \prod_{i=1}^{\infty} (1 - tq^{i-1})\) has been used. Expanding the generating function (2.14) as a power series of the parameter \(t\) by using the identity \[52\]

\[ \frac{1}{(t; q)^\infty} = \sum_{N=0}^{\infty} \frac{t^N}{(q)_N}, \]

one can show that

\[ \mathcal{G}^{(m_1, m_2| n_1, n_2)}_B(x, \bar{x}; y, \bar{y}; q, t) = \sum_{N=0}^{\infty} \frac{\mathcal{H}^{(m_1, m_2| n_1, n_2)}_{B,N}(x, \bar{x}; y, \bar{y}; q)}{(q^2)_N} t^N. \]

Using this generating function, some recursion relations for the \(BC_N\) type of SRS polynomials associated with different number of lattice sites and internal degrees of freedom have been found in Ref. \[50\]. However, one can not use such recursion relations for expressing the \(BC_N\) type of SRS polynomials through some super Schur like polynomials. So, in the next section, we shall derive a different type of recursion relations for the \(BC_N\) type of SRS polynomials involving different number of lattice sites (\(N\)), but with fixed values of the internal degrees of freedom \((m_1, m_2, n_1, n_2)\). This later type of recursion relation will enable us to express the \(BC_N\) type of SRS polynomials through some \(q\)-deformation of super Schur polynomials.

3. Novel recursion relations for \(BC_N\) type of SRS polynomials

For the purpose of deriving recursion relations for the \(BC_N\) type of SRS polynomials, involving different number of lattice sites at fixed values of internal degrees of freedom,
we may proceed in the following way. By using (2.15a), we obtain
\[
\mathcal{G}^{(m_1)}_1(x; q, t) = \frac{1}{\prod_{i=1}^{m_1} \{ (1 - tx_i)(1 - q^2tx_i)(1 - q^4tx_i) \ldots \}}
= \frac{1}{\prod_{i=1}^{m_1} (1 - tx_i) \prod_{i=1}^{m_1} (q^2tx_i; q^2)_x},
\]
which implies that
\[
\mathcal{G}^{(m_1)}_1(x; q, q^2t) = \prod_{i=1}^{m_1} (1 - tx_i) \cdot \mathcal{G}^{(m_1)}_1(x; q, t). \tag{3.1}
\]
Similarly, by using (2.15b), one can show that
\[
\mathcal{G}^{(m_2)}_2(\bar{x}; q, q^2t) = \prod_{j=1}^{m_2} (1 - tq\bar{x}_j) \cdot \mathcal{G}^{(m_2)}_2(\bar{x}; q, t) \tag{3.2}
\]
Next, by using (2.15c), we obtain
\[
\mathcal{G}^{(n_1)}_3(y; q, t) = \frac{1}{\prod_{k=1}^{n_1} \{ (1 + ty_kq^{-2})(1 + ty_kq^{-4}) \ldots \}}
= \frac{1}{\prod_{k=1}^{n_1} (1 + ty_k) \prod_{k=1}^{n_1} (1 + ty_kq^{-2})(1 + ty_kq^{-4}) \ldots }}
\]
which implies that
\[
\mathcal{G}^{(n_1)}_3(y; q, q^2t) = \prod_{k=1}^{n_1} \frac{1}{(1 + ty_k)} \mathcal{G}^{(n_1)}_3(y; q, t). \tag{3.3}
\]
Similarly, by using (2.15d), we find that
\[
\mathcal{G}^{(n_2)}_4(\bar{y}; q, q^2t) = \frac{1}{\prod_{l=1}^{n_2} (1 + tq\bar{y}_l)} \mathcal{G}^{(n_2)}_4(\bar{y}; q, t). \tag{3.4}
\]
Combining Eqs. (3.1), (3.2), (3.3) and (3.4), and using the definition of the generating function (2.14), we obtain the following \(q^2\)-difference relation
\[
\mathcal{G}^{(m_1,m_2|n_1,n_2)}_B(x, \bar{x}; y, \bar{y}; q, q^2t) = \frac{\prod_{i=1}^{m_1} (1 - tx_i) \prod_{j=1}^{m_2} (1 - tq\bar{x}_j)}{\prod_{k=1}^{n_1} (1 + ty_k) \prod_{j=1}^{n_2} (1 + tq\bar{y}_l)} \mathcal{G}^{(m_1,m_2|n_1,n_2)}_B(x, \bar{x}; y, \bar{y}; q, t). \tag{3.5}
\]
In this context it may be noted that, the generating function $G_A^{(m|n)}(x, y; q, t)$ for the $A_{N-1}$ type of SRS polynomials satisfies a similar $q$-difference relation given by

$$G_A^{(m|n)}(x, y; q, t) = \frac{\prod_{i=1}^{m} (1 - tx_i)}{\prod_{j=1}^{n} (1 + ty_j)} G_A^{(m|n)}(x, y; q, t).$$  \hspace{1cm} (3.6)

The product $\prod_{i=1}^{m} (1 - tx_i)$ appearing in the above equation can be expanded as a polynomial in $t$ as

$$\prod_{i=1}^{m} (1 - tx_i) = \sum_{r=0}^{m} (-1)^r t^r e_r^{(m)}(x),$$  \hspace{1cm} (3.7)

where

$$e_r^{(m)}(x) \equiv \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq m} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_r},$$

and $e_0^{(m)}(x) = 1$. This $e_r^{(m)}(x)$ is known as the elementary symmetric polynomial of $m$-variables $(x_1, ..., x_m)$ with degree $r$, which vanishes for $r > m$. For the special case $x = 1$, this polynomial reduces to

$$e_r^{(m)}(x)|_{x=1} = C_r^m,$$  \hspace{1cm} (3.8)

with $C_r^m$ being the binomial coefficient. Similarly, the product $\prod_{j=1}^{n} \frac{1}{1 + ty_j}$ can be expanded as an infinite power series of $t$ as

$$\prod_{j=1}^{n} \frac{1}{1 + ty_j} = \sum_{r=0}^{\infty} (-1)^r t^r h_r^{(n)}(y),$$  \hspace{1cm} (3.9)

where

$$h_r^{(n)}(y) \equiv \sum_{l_1 + \cdots + l_n = r} \prod_{i=1}^{l_i} y_i,$$

(and $h_0^{(n)}(y) = 1$) is known as the completely symmetric polynomial of $n$-variables with degree $r$. It may be noted that

$$h_r^{(n)}(y)|_{y=1} = C_r^n r^{-1},$$  \hspace{1cm} (3.10)

which is non-zero for any non-negative value of $r$. Combining Eqs. (3.7) and (3.9), one obtains

$$\prod_{i=1}^{m} (1 - tx_i) = \sum_{k=0}^{\infty} (-1)^k t^k E_k^{(m|n)}(x; y),$$  \hspace{1cm} (3.11)
where the elementary supersymmetric polynomial $E_{k}^{(m|n)}(x; y)$ is defined as

$$E_{k}^{(m|n)}(x; y) = \sum_{l=0}^{k} e_{l}^{(m)}(x) h_{k-l}^{(n)}(y). \quad (3.12)$$

Substituting (3.11) into Eq. (3.6), and also using the expansion of $G_{A}^{(m|n)}(x, y; q, t)$ in terms of the corresponding SRS polynomials (1.2), it has been found that these polynomials satisfy a recursion relation of the form [13]

$$H_{A,N}^{(m|n)}(x, y; q) = \sum_{k=1}^{N} (-1)^{k+1} \frac{(q)_{N-1}}{(q)_{N-k}} E_{k}^{(m|n)}(x; y) \cdot H_{A,N-k}^{(m|n)}(x, y; q). \quad (3.13)$$

At present, our goal is to find out an analogue of the recursion relation (3.13) for the case of $BC_{N}$ type of SRS polynomials. To this end, we use the expansion (3.7) to obtain

$$\prod_{i=1}^{m_{1}}(1 - tx_{i}) \prod_{j=1}^{m_{2}}(1 - tq \bar{x}_{j}) = \sum_{k=0}^{m_{1}+m_{2}} (-1)^{k} t^{k} e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q), \quad (3.14)$$

where $e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q)$ is defined as

$$e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q) = \sum_{r=0}^{k} q^{k-r} e_{r}^{(m_{1})}(x) e_{k-r}^{(m_{2})(\bar{x})}. \quad (3.15)$$

Hence, $e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q)$ is a homogeneous polynomial of degree $k$ of the variables $x$ and $\bar{x}$, and this polynomial vanishes for $k > m_{1} + m_{2}$. It may be noted that, for the particular case $q = 1$, (3.14) reduces to

$$\prod_{i=1}^{m_{1}}(1 - tx_{i}) \prod_{j=1}^{m_{2}}(1 - t\bar{x}_{j}) = \sum_{k=0}^{m_{1}+m_{2}} (-1)^{k} t^{k} e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q = 1). \quad (3.16)$$

Comparing this equation with the direct expansion of $\prod_{i=1}^{m_{1}}(1 - tx_{i}) \prod_{j=1}^{m_{2}}(1 - t\bar{x}_{j})$ by using (3.7), we find that

$$e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q = 1) = e_{k}^{(m_{1}+m_{2})}(x, \bar{x}), \quad (3.17)$$

where $e_{k}^{(m_{1}+m_{2})}(x, \bar{x}) \equiv e_{k}^{(m)}(x)$, with $x = x, \bar{x}$. Hence $e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q)$ in (3.15) may be considered as a particular type of $q$-deformation of the elementary symmetric polynomial $e_{k}^{(m_{1}+m_{2})}(x, \bar{x})$. Moreover, since $e_{k-r}^{(m_{2})}(\bar{x})$ is a homogeneous polynomial of degree $k - r$, Eq. (3.15) can also be expressed as

$$e_{k}^{(m_{1}, m_{2})}(x, \bar{x}; q) = \sum_{r=0}^{k} e_{r}^{(m_{1})}(x) e_{k-r}^{(m_{2})(q \bar{x})} = e_{k}^{(m_{1}+m_{2})}(x, \bar{x})|_{\bar{x} \to q \bar{x}}. \quad (3.18)$$
Next, by using the expansion (3.9), we similarly find that

\[
\frac{1}{\prod_{k=1}^{n_1}(1 + t y_k) \prod_{j=1}^{n_2}(1 + t q \bar{y}_j)} = \sum_{k=0}^{\infty} (-1)^k t^k h_k^{(n_1, n_2)}(y, \bar{y}; q), \tag{3.19}
\]

where \( h_k^{(n_1, n_2)}(y, \bar{y}; q) \) is defined as

\[
h_k^{(n_1, n_2)}(y, \bar{y}; q) = \sum_{r=0}^{k} q^{k-r} h_r^{(n_1)}(y) h_{k-r}^{(n_2)}(\bar{y}). \tag{3.20}
\]

Moreover, for the case \( q = 1 \), the above equation yields

\[
h_k^{(n_1, n_2)}(y, \bar{y}; q = 1) = h_k^{(n_1 + n_2)}(y, \bar{y}), \tag{3.21}
\]

where \( h_k^{(n_1 + n_2)}(y, \bar{y}) \equiv h_k^{(n)}(y) \), with \( y \equiv y, \bar{y} \). Hence \( h_k^{(n_1, n_2)}(y, \bar{y}; q) \) in (3.20) is a homogeneous polynomial of degree \( k \) of the variables \( y \) and \( \bar{y} \), which may be considered as a \( q \)-deformation of the completely symmetric polynomial \( h_k^{(n_1 + n_2)}(y, \bar{y}) \). Also, in analogy with the case of \( q \)-deformed elementary symmetric polynomials, we can rewrite this \( h_k^{(n_1, n_2)}(y, \bar{y}; q) \) as

\[
h_k^{(n_1, n_2)}(y, \bar{y}; q) = \sum_{r=0}^{k} h_r^{(n_1)}(y) h_{k-r}^{(n_2)}(q \bar{y}) = h_k^{(n_1 + n_2)}(y, \bar{y})|_{\bar{y} \to q \bar{y}}. \tag{3.22}
\]

Combining Eqs. (3.14) and (3.19), we obtain

\[
\frac{m_1}{\prod_{i=1}^{m_1}(1 - t x_i)} \frac{m_2}{\prod_{j=1}^{m_2}(1 - t q \bar{x}_j)} = \sum_{k=0}^{\infty} (-1)^k t^k E_k^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q), \tag{3.23}
\]

where \( E_k^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) \) is defined as

\[
E_k^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{k_1=0}^{k} e_{k_1}^{(m_1, m_2)}(x, \bar{x}; q) h_{k-k_1}^{(n_1, n_2)}(y, \bar{y}; q). \tag{3.24}
\]

Putting \( q = 1 \) in the above equation and also using Eqs. (3.17), (3.21) and (3.12) consecutively, we find that

\[
E_k^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q = 1) = \sum_{k_1=0}^{k} e_{k_1}^{(m_1 + m_2)}(x, \bar{x}) h_{k-k_1}^{(n_1 + n_2)}(y, \bar{y})
= E_k^{(m_1 + m_2|n_1 + n_2)}(x, \bar{x}; y, \bar{y}), \tag{3.25}
\]
where \( E_k^{(m_1 + m_2 | n_1 + n_2)}(x, \bar{x}; y, \bar{y}) \equiv F_k^{(m | n)}(x; y) \), with \( x \equiv x, \bar{x} \) and \( y \equiv y, \bar{y} \). Thus \( E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) \) may be seen as a \( q \)-deformation of the elementary supersymmetric polynomial \( E_k^{(m_1 + m_2 | n_1 + n_2)}(x, \bar{x}; y, \bar{y}) \). Moreover, by using Eqs. (3.24), (3.18), (3.22) and (3.12) consecutively, we find that \( E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) \) can be obtained from \( E_k^{(m_1 + m_2 | n_1 + n_2)}(x, \bar{x}; y, \bar{y}) \) by scaling some of its variables as

\[
E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = E_k^{(m_1 + m_2 | n_1 + n_2)}(x, \bar{x}; y, \bar{y})|_{\bar{x} \rightarrow q\bar{x}, \bar{y} \rightarrow q\bar{y}}. \tag{3.26}
\]

Subsequently, expanding both sides of Eq. (3.26) in powers of \( t \) by using Eqs. (2.17) and (3.23), we get

\[
\sum_{N=0}^{\infty} (q^2 t)^N \frac{H_{B,N}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)}{(q^2)^N} = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} (-1)^k \frac{1}{(q^2)^s} E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) H_{B,s}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}, y, \bar{y}; q).
\]

Redefining the variable \( k \) as \( k = N - s \), one can rewrite the above equation as

\[
\sum_{N=0}^{\infty} (q^2 t)^N \frac{H_{B,N}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)}{(q^2)^N} = \sum_{k=0}^{\infty} t^N \sum_{k=0}^{N} (-1)^k \frac{1}{(q^2)^{N-k}} E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) H_{B,N-k}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}, y, \bar{y}; q). \tag{3.27}
\]

Comparing the powers of \( t^N \) from both sides of the above equation, we finally obtain the following recursion relation

\[
H_{B,N}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{k=1}^{N} (-1)^{k-1} \frac{(q^2)^{N-k}}{(q^2)^{N-k}} E_k^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) H_{B,N-k}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q), \tag{3.28}
\]

which involves \( BC_N \) type of SRS polynomials associated with different number of lattice sites and fixed values of the internal degrees of freedom. It may be noted that, the above relation can generate any \( H_{B,N}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) \) in a recursive way from the given initial condition \( H_{B,0}^{(m_1, m_2 | n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = 1 \). The first few SRS polynomials of \( BC_N \) type are found to be

\[
H_{B,1}(q) = E_1(q), \tag{3.29}
\]
\[
H_{B,2}(q) = (E_1(q))^2 - (1 - q^2)E_2(q), \tag{3.30}
\]
\[
H_{B,3}(q) = (E_1(q))^3 + (q^2 - 1)(q^2 + 2)E_1(q)E_2(q) + (1 - q^2)(1 - q^4)E_3(q), \tag{3.31}
\]
where we have used the following shorthand notation:
\[
H_{B,N}(q) \equiv H^{(m_1,m_2|n_1,n_2)}_{B,N}(x, \bar{x}; y, \bar{y}; q), \quad E_k(q) \equiv E^{(m_1,m_2|n_1,n_2)}_k(x, \bar{x}; y, \bar{y}; q). \tag{3.32a,b}
\]

By putting \(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1\) in the recursion relation (3.28) and then by using (2.13), we can also write a similar recursion relation in terms of the partition function of the \(BC_N\) type of PF spin chain as
\[
Z^{(m_1,m_2|n_1,n_2)}_{B,N}(q)
= \sum_{k=1}^{N} (-1)^{k-1} \frac{(q^2)^{N-k}}{(q^2)_{N-k}} E_k^{(m_1,m_2|n_1,n_2)}(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1; q) Z^{(m_1,m_2|n_1,n_2)}_{B,N-k}(q), \tag{3.33}
\]

where
\[
E_k^{(m_1,m_2|n_1,n_2)}(x = 1, \bar{x} = 1; y = 1, \bar{y} = 1; q)
= \sum_{r=0}^{k} \sum_{s=0}^{r} \sum_{t=0}^{k-r} q^{k-t-s} C_{s}^{m_1} C_{r-s}^{m_2} C_{t}^{n_1+t-1} C_{k-r-t}^{n_2+k-r-t-1}, \tag{3.34}
\]

obtained by substituting \(x = 1, \bar{x} = 1; y = 1, \bar{y} = 1\) in Eq. (3.24), and subsequently using Eqs. (3.8), (3.10), (3.15) and (3.20).

4. \(q\)-Deformed and restricted super Schur polynomials

It has been found earlier that, super Schur polynomials associated with border strips play a key role in classifying the degenerate multiplets appearing in the spectra of \(A_{N-1}\) type of \(su(m|n)\) supersymmetric PF spin chains [13]. For the purpose of doing a similar classification in the case of \(BC_N\) type of PF spin chains, in this section we shall prescribe a Jacobi-Trudi like formula to define a \(q\)-deformed version of the super Schur polynomials in terms of the \(q\)-deformed elementary supersymmetric polynomials (3.24). Subsequently, by expanding such \(q\)-deformed super Schur polynomials as a power series of the parameter \(q\), we shall obtain the so called ‘restricted’ super Schur polynomials which are independent of \(q\). Moreover, we shall present some combinatorial forms for computing both of the \(q\)-deformed and the restricted super Schur polynomials.

It may be noted that, border strips represent a class of irreducible representations of the \(Y(gl(m))\) Yangian algebra (\(Y(gl(m|n))\) super Yangian algebra), which span the Fock spaces of \(su(m)\) (\(su(m|n)\)) HS and PF spin chains [11, 13]. There exists a one-to-one correspondence between these border strips and the motifs which we have mentioned earlier. For a spin chain with \(N\) lattice sites, a border strip is denoted as \(\langle k_1, \ldots, k_r \rangle\), where \(k_i\)’s are some positive integers satisfying the relation \(\sum_{i=1}^{r} k_r = N\) (see Fig. 1), and a
The inverse mapping from a motif to a border strip can be obtained by reading a motif \( p_{\delta_1}, \delta_2, ..., \delta_{N-1} \) from the left and adding a box under (resp. left) the box when \( \delta_j = 0 \) (resp. \( \delta_j = 1 \)) is encountered.

For the case of \( A_{N-1} \) type of su(\( m|n \)) PF spin chains, the super Schur polynomial associated with the border strip \( \langle k_1, k_2, ..., k_r \rangle \) is defined by using the following Jacobi-Trudi like determinant formula [13]:

\[
S_{\langle k_1, k_2, ..., k_r \rangle}^{(m|n)}(x, \bar{x}; y, \bar{y}) = \begin{vmatrix}
E_{k_1} & E_{k_1+k_r-1} & \cdots & \cdots & E_{k_r+...+k_1} \\
1 & E_{k_r-1} & E_{k_r-1+k_r-2} & \cdots & E_{k_r-1+...+k_1} \\
0 & 1 & E_{k_r-2} & \cdots & E_{k_r-2+...+k_1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & E_{k_1}
\end{vmatrix},
\]

(4.2)

where \( x \equiv x_1, x_2, \cdots, x_{m_1} \) and \( \bar{x} \equiv \bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{m_2} \) are two sets of bosonic variables (with total number \( m = m_1 + m_2 \)), \( y \equiv y_1, y_2, \cdots, y_{n_1} \) and \( \bar{y} \equiv \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{n_2} \) are two sets of fermionic variables (with total number \( n = n_1 + n_2 \)), and the shorthand notation \( E_k \) is used for the elementary supersymmetric polynomial \( E_k^{(m|n)}(x; y) \) (with \( x \equiv x, \bar{x} \)).
and $y \equiv y, \bar{y}$ defined in (3.12). In the limit $x = 1, \bar{x} = 1, y = 1, \bar{y} = 1$, this super Schur polynomial reduces to

$$S^{(m|n)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y})|_{x=1, \bar{x}=1, y=1, \bar{y}=1} = \mathcal{N}^{(m|n)}_{(k_1, k_2, ..., k_r)},$$

(4.3)

where $\mathcal{N}^{(m|n)}_{(k_1, k_2, ..., k_r)} \in \mathbb{Z}_{>0}$ gives the dimensionality of the irreducible representation of $Y(gl(m|n))$ super Yangian algebra associated with the border strip $\langle k_1, ..., k_r \rangle$ or the corresponding motif. Hence, $\mathcal{N}^{(m|n)}_{(k_1, k_2, ..., k_r)}$’s also determine the dimensions of the degenerate eigenspaces associated with the spectra of the Yangian invariant $A_{N-1}$ type of $su(m|n)$ PF spin chains.

In analogy with the Jacobi-Trudi like determinant formula (4.2), let us define a novel $q$-dependent super Schur polynomial associated with the border strip $\langle k_1, ..., k_r \rangle$ by using the $q$-deformed elementary supersymmetric polynomials in (3.24) as

$$S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q) = \begin{vmatrix}
E_{k_1}(q) & E_{k_1+k_2-1}(q) & \cdots & \cdots & E_{k_r+k_r-1}(q) \\
1 & E_{k_1-1}(q) & \cdots & \cdots & E_{k_r-1+k_r}(q) \\
0 & 1 & \cdots & \cdots & E_{k_r-k_r}(q) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & E_{k_1}(q)
\end{vmatrix},$$

(4.4)

where the notation $E_k(q) \equiv E^{(m_1, m_2|n_1, n_2)}_{k}(x, \bar{x}; y, \bar{y}; q)$ is used. By putting $q = 1$ in the above determinant and using Eqs. (3.25) and (4.2) consecutively, it can be readily seen that

$$S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q = 1) = S^{(m|n)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}).$$

(4.5)

Thus $S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q)$ may be interpreted as a $q$-deformation of the super Schur polynomial $S^{(m|n)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y})$. Moreover, by using Eq. (3.26) along with (4.2) and (4.4), we obtain a relation like

$$S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q) = S^{(m|n)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y})|_{x \to q\bar{x}, \bar{y} \to q\bar{y}}.$$  

(4.6)

Since $S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y})$ in (4.2) is a homogeneous polynomial of order $N$, it follows from the above relation that $S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q)$ is also a homogeneous polynomial of order $N$ in the variables $x, \bar{x}; y, \bar{y}$. Hence, Eq. (4.6) may be written in an alternative form like

$$S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q) = q^N \cdot S^{(m|n)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y})|_{x \to q^{-1}x, y \to q^{-1}y}.$$  

(4.7)

From Eq. (4.6) it is also evident that, $S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q)$ is a polynomial of order less than or equal to $N$ in the variable $q$. Consequently, one can formally expand $S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q)$ in a power series of $q$ as

$$S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r)}(x, \bar{x}; y, \bar{y}; q) = \sum_{l=0}^{N} q^l \cdot S^{(m_1, m_2|n_1, n_2)}_{(k_1, k_2, ..., k_r|l)}(x, \bar{x}; y, \bar{y}),$$  

(4.8)
where, according to the terminology used by us, the symbol \( \langle k_1, k_2, \ldots, k_r | l \rangle \) denotes a ‘branched’ border strip (more precisely, the \( l \)-th branch of the border strip \( \langle k_1, k_2, \ldots, k_r \rangle \)) and \( S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}) \) denotes a ‘restricted’ super Schur polynomial corresponding to the branched border strip \( \langle k_1, k_2, \ldots, k_r | l \rangle \). It will be shown shortly that the branched border strip \( \langle k_1, k_2, \ldots, k_r | l \rangle \) can be described through a set of skew Young tableaux. By putting \( q = 1 \) in (4.8) and using (4.5), we find that the super Schur polynomial is related to these restricted super Schur polynomials as

\[
S_{(k_1,k_2,\ldots,k_r)}^{(m|n)}(x, \bar{x}; y, \bar{y}) = \sum_{l=0}^{N} S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}).
\]

(4.9)

In the limit \( x = 1, \bar{x} = 1, y = 1, \bar{y} = 1 \), the restricted super Schur polynomial yields

\[
S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}) |_{x=1,\bar{x}=1,y=1,\bar{y}=1} = \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)},
\]

(4.10)

where \( \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)} \)'s are some non-negative integers which will be determined later. Inserting \( x = \bar{x} = y = \bar{y} = 1 \) in (4.9), and also using (4.3) and (4.10), we obtain

\[
\mathcal{N}_{(k_1,k_2,\ldots,k_r)}^{(m|n)} = \sum_{l=0}^{N} \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}.
\]

(4.11)

In analogy with the role played by \( \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m|n)} \)'s for the case of \( A_{N-1} \) type of spin chains, we shall show in the next section that \( \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)} \)'s determine the dimensions of the degenerate eigenspaces associated with the spectra of the \( BC_N \) type of PF spin chains. However, it seems to be difficult to explicitly compute \( S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}) \) and its limiting value \( \mathcal{N}_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)} \) by directly expanding \( S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) in (4.4) as a power series of \( q \).

To bypass the above mentioned problem it may be noted that, apart from the determinant relation (4.2), the super Schur polynomial \( S_{(k_1,k_2,\ldots,k_r)}^{(m|n)}(x, \bar{x}; y, \bar{y}) \) can also be expressed in the following combinatorial form \[16, 18\]. To begin with, let us recall that a skew Young tableau associated with the border strip \( \langle k_1, k_2, \ldots, k_r \rangle \) is constructed by filling up this border strip with the numbers \( 1, 2, \ldots, m+n \), such that their arrangement obey two rules given by:

(i) Entries in each row are increasing, allowing the repetition of elements in \{ \( i \mid i \in S_{+,+}^{(m_1)} \cup S_{+,+}^{(m_2)} \} \), but not permitting the repetition of elements in \{ \( i \mid i \in S_{-,+}^{(m_1)} \cup S_{-,+}^{(m_2)} \} \),

(ii) Entries in each column are increasing, allowing the repetition of elements in \{ \( i \mid i \in S_{-,+}^{(m_1)} \cup S_{-,+}^{(m_2)} \} \) but not permitting the repetition of elements in \{ \( i \mid i \in S_{-,+}^{(m_1)} \cup S_{-,+}^{(m_2)} \} \),

where the sets \( S_{+,+}^{(m_1)}, S_{+,+}^{(m_2)}, S_{-,+}^{(m_1)} \) and \( S_{-,+}^{(m_2)} \) are defined in \[2.3\]. Let \( \mathcal{G} \) be the set of all skew Young tableaux which are obtained by filling up the border strip \( \langle k_1, k_2, \ldots, k_r \rangle \)
through the above mentioned rules. The combinatorial form of $S^{(m|n)}_{(k_1,k_2,\ldots,k_r)}(x,\bar{x};y,\bar{y})$ may now be written as

$$S^{(m|n)}_{(k_1,k_2,\ldots,k_r)}(x,\bar{x};y,\bar{y}) = \sum_{\mathcal{T} \in \mathcal{G}} e^{\text{wt}(\mathcal{T})},$$  

(4.12)

where $\text{wt}(\mathcal{T})$ is the weight of the tableau $\mathcal{T}$ given by

$$\text{wt}(\mathcal{T}) = \sum_{\alpha=1}^{m+n} \alpha(\mathcal{T}) \cdot \epsilon_\alpha,$$  

(4.13)

$\alpha(\mathcal{T})$ denotes the multiplicity of the number $\alpha$ in the tableau $\mathcal{T}$, and the four sets of variables $x$, $\bar{x}$, $y$, $\bar{y}$ are defined as

$$
x_\alpha \equiv e^{\alpha}, \quad \text{if} \quad \alpha \in S^{(m_1)}_{+,+},
\bar{x}_{\alpha-m_1} \equiv e^{\alpha}, \quad \text{if} \quad \alpha \in S^{(m_2)}_{+,-},

y_{\alpha-(m_1+m_2)} \equiv e^{\alpha}, \quad \text{if} \quad \alpha \in S^{(n_1)}_{-,+},
\bar{y}_{\alpha-(m_1+m_2+n_1)} \equiv e^{\alpha}, \quad \text{if} \quad \alpha \in S^{(n_2)}_{-,+}.  
\ (4.14)

Inserting $x = 1, \bar{x} = 1, y = 1, \bar{y} = 1$ in (4.12) and comparing it with (4.13), it is easy to see that $\mathcal{N}^{(m|n)}_{(k_1,k_2,\ldots,k_r)}$ coincides with the number of distinct tableaux within the set $\mathcal{G}$.

At present, our aim is to suitably modify Eq. (4.12) to obtain the combinatorial form for both of the $q$-deformed and restricted super Schur polynomials. Since, due to Eq. (4.6), $\bar{x}$ and $\bar{y}$ in $S^{(m|n)}_{(k_1,k_2,\ldots,k_r)}(x,\bar{x};y,\bar{y})$ should be replaced by $q\bar{x}$ and $q\bar{y}$ respectively to obtain $S^{(m_1,m_2|n_1,n_2)}_{(k_1,k_2,\ldots,k_r)}(x,\bar{x};y,\bar{y};q)$, the combinatorial form of the $q$-deformed super Schur polynomial can easily be obtained by modifying Eq. (4.12) as

$$S^{(m_1,m_2|n_1,n_2)}_{(k_1,k_2,\ldots,k_r)}(x,\bar{x};y,\bar{y};q) = \sum_{\mathcal{T} \in \mathcal{G}} q^{\mathcal{F}(\mathcal{T})} e^{\text{wt}(\mathcal{T})},$$  

(4.15)

where $\mathcal{F}(\mathcal{T})$ for the tableau $\mathcal{T}$ is given by

$$\mathcal{F}(\mathcal{T}) = \sum_{\alpha=1}^{m_1+m_2+n_1+n_2} \alpha(\mathcal{T}) f^{(m_1,m_2|n_1,n_2)}(\alpha),$$  

(4.16)

and $f^{(m_1,m_2|n_1,n_2)}(\alpha)$ is defined as

$$f^{(m_1,m_2|n_1,n_2)}(\alpha) = 0, \quad \text{if} \quad \alpha \in S^{(m_1)}_{+,+} \cup S^{(n_1)}_{-,+},$$

$$f^{(m_1,m_2|n_1,n_2)}(\alpha) = 1, \quad \text{if} \quad \alpha \in S^{(m_2)}_{+,-} \cup S^{(n_2)}_{-,+}.  
\ (4.17)

From Eq. (4.16), it is evident that $\mathcal{F}(\mathcal{T})$ is a non-negative integer satisfying the relation $\mathcal{F}(\mathcal{T}) \leq N$. Let $\mathcal{G}_l$ be the set of all tableaux which are obtained by filling up the border strip $\langle k_1,k_2,\ldots,k_r \rangle$ through the previously mentioned rules (i) and (ii) and for which $\mathcal{F}(\mathcal{T}) = l$, i.e.,

$$\mathcal{G}_l = \{ \mathcal{T} \in \mathcal{G} | \mathcal{F}(\mathcal{T}) = l \}.  \ (4.18)$$
Hence, the set $G$ may be written as

$$G = \bigcup_{0 \leq t \leq N} G_t,$$  \hspace{1cm} (4.19)

and (4.15) can be expressed in the form

$$S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{l=0}^{N} q^l \sum_{T \in \mathcal{G}_l} e^{wt(T)}. \hspace{1cm} (4.20)$$

Comparing the above equation with (4.8), we finally obtain the combinatorial form of the restricted super Schur polynomials as

$$S_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}) = \sum_{T \in \mathcal{G}_l} e^{wt(T)}. \hspace{1cm} (4.21)$$

The above equation clearly shows that all information about the branched border strip $\langle k_1, k_2, \ldots, k_r|l \rangle$ is essentially encoded within the set $\mathcal{G}_l$. Substituting $x = 1, \bar{x} = 1, y = 1, \bar{y} = 1$ in (4.21) and using (4.11), we find that

$$N_{(k_1,k_2,\ldots,k_r|l)}^{(m_1,m_2|n_1,n_2)} = |\mathcal{G}_l|, \hspace{1cm} (4.22)$$

where $|\mathcal{G}_l|$ denotes the number of elements in the set $\mathcal{G}_l$.

For the purpose of explaining the above mentioned procedure of computing the $q$-deformed and the restricted super Schur polynomials through a particular example, let us assume that $m_1 = 1, m_2 = 1, n_1 = 0, n_2 = 2$ and consider the border strip $\langle 2, 1 \rangle$ for $N = 3$. In this case, the sets in Eq. (2.3) are given by: $S^{(1)}_{+,-} = \{1\}, S^{(1)}_{+,+} = \{2\}, S^{(0)}_{-,-} = \{3, 4\}$. Moreover, by using Eq. (4.17), we obtain $f^{(1,1|0,2)}(1) = 0, f^{(1,1|0,2)}(2) = f^{(1,1|0,2)}(3) = f^{(1,1|0,2)}(4) = 1$. Now, for each value of $l \in \{0, 1, 2, 3\}$, we can construct all possible tableaux $T$ following the rules (ii) and (iii) for which $\mathcal{F}(T) = l$.

For $l = 0$, there will be no possible tableau in this case, i.e., $\mathcal{G}_0 = \{ \}$. For $l = 1$, the set of tableaux are given by:

$$\mathcal{G}_1 = \left\{ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 1 \\
\end{array} \right\},$$

for $l = 2$,

$$\mathcal{G}_2 = \left\{ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 3 \\
2 & 4 & 3 & 4 \\
1 & 3 & 1 & 4 \\
2 & 2 & 1 & 3 \\
3 & 1 & 4 & 1 \\
4 & 1 & 4 & 1 \\
\end{array} \right\},$$

and for $l = 3$,

$$\mathcal{G}_3 = \left\{ \begin{array}{cccc}
2 & 2 & 2 & 2 \\
2 & 3 & 2 & 4 \\
3 & 4 & 2 & 3 \\
2 & 3 & 3 & 4 \\
2 & 4 & 2 & 4 \\
3 & 4 & 3 & 4 \\
4 & 3 & 4 & 3 \\
\end{array} \right\}.$$
Using the combinatorial expression (4.21), the corresponding restricted super Schur polynomials are obtained

\[
S_{(2,1|0)}^{(1,1|0,2)}(x_1, \bar{x}_1; \bar{y}_1, \bar{y}_2) = 0,
\]
\[
S_{(2,1|1)}^{(1,1|0,2)}(x_1, \bar{x}_1; \bar{y}_1, \bar{y}_2) = x_1^2(\bar{x}_1 + \bar{y}_1 + \bar{y}_2),
\]
\[
S_{(2,1|2)}^{(1,1|0,2)}(x_1, \bar{x}_1; \bar{y}_1, \bar{y}_2) = x_1(\bar{x}_1 + \bar{y}_1 + \bar{y}_2)^2,
\]
\[
S_{(2,1|3)}^{(1,1|0,2)}(x_1, \bar{x}_1; \bar{y}_1, \bar{y}_2) = (\bar{y}_1 + \bar{y}_2)(\bar{x}_1 + \bar{x}_1\bar{y}_1 + \bar{x}_1\bar{y}_2 + \bar{y}_1\bar{y}_2). \tag{4.23}
\]

Inserting \( x_1 = \bar{x}_1 = \bar{y}_1 = \bar{y}_2 = 1 \) in the above equation or directly using Eq. (4.22), we obtain

\[
\mathcal{N}_{(2,1|0)}^{(1,1|0,2)} = 0, \quad \mathcal{N}_{(2,1|1)}^{(1,1|0,2)} = 3, \quad \mathcal{N}_{(2,1|2)}^{(1,1|0,2)} = 9, \quad \mathcal{N}_{(2,1|3)}^{(1,1|0,2)} = 8, \tag{4.24}
\]

Moreover, using Eqs. (4.23) and (4.8), we can write the corresponding \( q \)-deformed super Schur polynomial as

\[
S_{(2,1)}^{(1,1|0,2)}(x_1, \bar{x}_1; \bar{y}_1, \bar{y}_2; q) = q \cdot x_1^2(\bar{x}_1 + \bar{y}_1 + \bar{y}_2) + q^2 \cdot x_1(\bar{x}_1 + \bar{y}_1 + \bar{y}_2)^2
+ q^3 \cdot (\bar{y}_1 + \bar{y}_2)(\bar{x}_1 + \bar{x}_1\bar{y}_1 + \bar{x}_1\bar{y}_2 + \bar{y}_1\bar{y}_2). \tag{4.25}
\]

In the next section, we shall explain how this type of \( q \)-deformed and restricted super Schur polynomials play an important role in classifying the degenerate multiplets within the spectra of the \( BC_N \) type of PF spin chains.

5. Spectra of the \( BC_N \) type of ferromagnetic PF spin chains

It has been found earlier that the \( A_{N-1} \) type of homogeneous RS polynomials can be expressed through some suitable linear combinations of the Schur polynomials associated with the border strips \([11]\). Similarly, the \( A_{N-1} \) type of homogeneous SRS polynomials \([12]\) can be expressed through super Schur polynomials \([4.2]\) as \([13]\)

\[
\mathbb{H}_{A,N}^{(m|n)}(x, y; q) = \sum_{k \in \mathcal{P}_N} q^{\frac{N(N-1)}{2} - \sum_{i=1}^{r-1} K_i} \cdot S_{(k_1, k_2, \ldots, k_r)}^{(m|n)}(x, \bar{x}; y, \bar{y}), \tag{5.1}
\]

where \( x = x, \bar{x}, y = y, \bar{y} \), \( \mathcal{P}_N \) denotes the set of all ordered partitions of \( N \), \( \bar{k} \equiv \{k_1, k_2, \ldots, k_r\} \) (where \( r \) is an integer taking value from 1 to \( N \)) is an element of \( \mathcal{P}_N \) and \( K_i \)'s are partial sums given by \( K_i = \sum_{j=1}^{i} k_j \). Substituting \( x = \bar{x} = y = \bar{y} = 1 \) in \((5.1)\) and using \((4.3)\), one obtains

\[
\mathcal{Z}_{A,N}^{(m|n)}(q) = \sum_{k \in \mathcal{P}_N} q^{\frac{N(N-1)}{2} - \sum_{i=1}^{r-1} K_i} \cdot \mathcal{N}_{(k_1, \ldots, k_r)}^{(m|n)} \cdot \mathcal{N}_{(k_1, \ldots, k_r)}^{(m|n)}, \tag{5.2}
\]

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where $Z_{A,N}^{(m|n)}(q)$ denotes the partition function of the $A_{N-1}$ type of supersymmetric PF spin chain (1.1). From the structure of $Z_{A,N}^{(m|n)}(q)$ in (5.2) it is clear that, the exponent of $q$ yields the energy level for the spin chain (1.1) associated with the border strip $\langle k_1, \ldots, k_r \rangle$ as
\[
E_{\langle k_1, \ldots, k_r \rangle}^{(m|n)} = \frac{N(N-1)}{2} - \sum_{i=1}^{r-1} K_i ,
\]
and the degeneracy of this energy level is given by $N_{\langle k_1, \ldots, k_r \rangle}^{(m|n)}$. Using the mapping (4.1) between the border strips and the motifs, it is easy to show that
\[
\sum_{i=1}^{r-1} K_i = \sum_{j=1}^{N-1} j\delta_j .
\]
Hence, by substituting (5.4) into (5.3), one can write down all energy levels of the supersymmetric PF spin chain (1.1) in terms of motifs as
\[
E_{\langle \delta_1, \delta_2, \ldots \delta_{N-1} \rangle}^{(m|n)} = E_{\langle k_1, \ldots, k_r \rangle}^{(m|n)} = \sum_{j=1}^{N-1} j(1 - \delta_j)
\]
with degeneracy factor $N_{\langle \delta_1, \delta_2, \ldots \delta_{N-1} \rangle}^{(m|n)} = N_{\langle k_1, \ldots, k_r \rangle}^{(m|n)}$.

At present, our aim is to express the $BC_N$ type of SRS polynomials (2.12) through the $q$-deformed and the restricted super Schur polynomials which have been defined in Sec. 4 and subsequently use such expression for classifying the degenerate multiplets within the spectra of the $BC_N$ type of ferromagnetic PF spin chains (2.9) with SAPSRO. For this purpose, in analogy with $H_{A,N}^{(m|n)}(x, y; q)$ appearing in Eq. (5.1), let us define a polynomial $F_N^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)$ for $N > 0$ as
\[
F_N^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{k \in \mathbb{P}_N} q^{N(N-1)-2} \sum_{i=1}^{r-1} K_i \cdot S_{\langle k_1, \ldots, k_r \rangle}^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) ,
\]
and also assume that $F_0^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q) = 1$. Let us now evaluate the polynomial $F_N^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)$ for the cases $N = 1, 2$ and 3, by using shorthand notations like $F_N(q) = F_N^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)$, $S_{\langle k_1, \ldots, k_r \rangle}(q) = S_{\langle k_1, \ldots, k_r \rangle}^{(m_1, m_2|n_1, n_2)}(x, \bar{x}; y, \bar{y}; q)$, for sake of brevity. For the case $N = 1$, Eqs. (5.6) and (4.4) imply that
\[
F_1(q) = S_{\langle 1 \rangle}(q) = E_1(q) .
\]
Similarly, for the cases $N = 2$ and $N = 3$, we obtain
\[
F_2(q) = q^2 \cdot S_{\langle 2 \rangle}(q) + S_{\langle 1, 1 \rangle}(q) = q^2 \cdot E_2(q) + \begin{vmatrix} E_1(q) & E_2(q) \\ 1 & E_1(q) \end{vmatrix} = E_2^2(q) + (q^2 - 1)E_2(q) ,
\]

and
\[ \mathcal{F}_3(q) = q^6 \cdot S_{(3)}(q) + q^4 \cdot S_{(1,2)}(q) + q^2 \cdot S_{(2,1)}(q) + S_{(1,1,1)}(q) \]
\[ = q^6 \cdot E_3(q) + q^4 \cdot \left| \begin{array}{cc} E_2(q) & E_4(q) \\ 1 & E_1(q) \end{array} \right| + q^2 \cdot \left| \begin{array}{cc} E_4(q) & E_3(q) \\ 1 & E_2(q) \end{array} \right| \]
\[ + \left| \begin{array}{ccc} E_4(q) & E_3(q) & E_3(q) \\ 1 & E_2(q) & E_1(q) \\ 0 & 1 & E_1(q) \end{array} \right| \]
\[ = E_1^3(q) + (q^2 - 1)(q^2 + 2)E_1(q)E_2(q) + (1 - q^2)(1 - q^4)E_3(q). \]  
(5.9)

It can be readily seen that the right hand sides of Eqs. (5.7), (5.8) and (5.9) exactly match with those of Eqs. (5.29), (5.30) and (5.31) respectively. This result strongly suggests that the equality
\[ \mathcal{H}^{(m_1,m_2)}_{N}(x, \tilde{x}; y, \tilde{y}; q) = \mathcal{F}^{(m_1,m_2)}_{N}(x, \tilde{x}; y, \tilde{y}; q), \]  
(5.10)
would hold for arbitrary values of \( N \). In the following, we shall give a proof of this statement by using a procedure similar to what has been described in Appendix A of Ref. [11] for the case of \( A_{N-1} \) type of PF spin chain.

Expanding the determinant formula (4.4) along its first row, we obtain a recursion relation for the \( q \)-deformed super Schur polynomials as
\[ S_{(m_1,m_2)}^{(m_1,m_2)}(x, \tilde{x}; y, \tilde{y}; q) = \sum_{r=1}^{s} (-1)^{s+1} E_{k_r} E_{r-1} + \cdots + E_{r-s+1} (q) S_{(m_1,m_2)}^{(m_1,m_2)}(x, \tilde{x}; y, \tilde{y}; q). \]  
(5.11)

Substituting the above recursion relation into Eq. (5.6), we can rewrite the polynomial \( \mathcal{F}^{(m_1,m_2)}_{N}(x, \tilde{x}; y, \tilde{y}; q) \) in the following form for any \( N \geq 1 \):
\[ \mathcal{F}^{(m_1,m_2)}_{N}(x, \tilde{x}; y, \tilde{y}; q) = \sum_{k=1}^{N} A_{N,k}(q^2) E_k(q) \mathcal{F}^{(m_1,m_2)}_{N-k}(x, \tilde{x}; y, \tilde{y}; q), \]  
(5.12)
where
\[ A_{N,k}(q^2) = \sum_{j=1}^{k} \sum_{l_1 + \cdots + l_j = k} (-1)^{j+1}(q^2)^C_{N,k,l_1,\ldots,l_j}, \]  
(5.13)
with
\[ C_{N,k,l_1,\ldots,l_j} = \frac{1}{2}N(N+1) - \frac{1}{2}(N-k)(N-k+1) - \sum_{i=1}^{j} (l_1 + \cdots + l_i + N - k) \]
\[ = N(k - j) - \frac{1}{2}k(k + 1) + \sum_{i=1}^{j} il_i. \]
It may be noted that exactly this form of $C_{N,k}(l_1,\ldots,l_j)$ has appeared earlier in Eq. (A.6) of Ref. [11]. Hence, by using the method of latter reference, we compute the function $A_{N,k}(q^2)$ in (5.13) as

$$A_{N,k}(q^2) = (-1)^{k+1}(q^2)^{N-1}.$$

Substituting the above expression of $A_{N,k}(q^2)$ into Eq. (5.12), we find that the polynomial $F_{N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q)$ satisfies the recursion relation

$$F_{N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q) = \sum_{k=1}^{N} (-1)^{k+1}(q^2)^{N-1} E_k(q) F_{N-k}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q).$$  

(5.14)

It may be noted that the form of above recursion relation coincides with that of (3.28) satisfied by the $BC_N$ type of SRS polynomials. Since the $BC_N$ type of SRS polynomials are uniquely determined by the recursion relation (3.28) and the initial condition $H_{B,0}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q) = 1$, the equality (5.10) is proved for arbitrary values of $N$.

Combining Eqs. (5.10) and (5.6), we obtain the desired expression for $BC_N$ type of SRS polynomials in terms of linear combinations of $q$-deformed super Schur polynomials as

$$H_{N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q) = \sum_{k\in\mathbb{P}_N} q^{N(N-1)/2 + \sum_{i=1}^{r} K_i} \cdot S_{(k_1,\ldots,k_r)}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q).$$  

(5.15)

Substituting the expansion of $S_{(k_1,\ldots,k_r)}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q)$ in (4.8) to (5.15), we further express the $BC_N$ type of SRS polynomials through restricted super Schur polynomials as

$$H_{B,N}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y};q) = \sum_{k\in\mathbb{P}_N} q^{N(N-1)/2 + \sum_{i=1}^{l} K_i + l} \cdot S_{(k_1,\ldots,k_r,l)}^{(m_1,m_2|n_1,n_2)}(x,\bar{x};y,\bar{y}).$$  

(5.16)

Inserting $x = \bar{x} = y = \bar{y} = 1$ in Eq. (5.16) and subsequently using Eqs. (2.13) as well as (4.10), we find that the partition function of the $BC_N$ type of PF spin chain (2.9) can be written as

$$Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{k\in\mathbb{P}_N} q^{N(N-1)/2 + \sum_{i=1}^{l} K_i + l} \cdot N_{(k_1,\ldots,k_r,l)}^{(m_1,m_2|n_1,n_2)}.$$  

(5.17)

From the structure of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ in (5.17), it is evident that, the exponent of $q$ yields the energy level of the spin chain (2.9) corresponding to the branched border strip $\langle k_1,\ldots,k_r|l \rangle$ as

$$E_{(k_1,\ldots,k_r,l)}^{(m_1,m_2|n_1,n_2)} = N(N-1) - 2 \sum_{i=1}^{r} K_i + l,$$

(5.18)
and the degeneracy of this energy level is given by $\mathcal{N}_{\langle k_1, \ldots, k_r | l \rangle}^{(m_1, m_2 | n_1, n_2)}$. In analogy with Eq. (4.11), we may now construct a one-to-one mapping between border strips like $\langle k_1, \ldots, k_r | l \rangle$ and branched motifs like $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle$ as

$$\langle k_1, \ldots, k_r | l \rangle \Rightarrow \left( \begin{array}{c} 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0 \end{array} \right)_{k_1-1, k_2-1, \ldots, k_r-1}.$$  

(5.19)

Hence, by inserting (5.4) into (5.18), we finally obtain the energy level of the $BC_N$ type of PF spin chain with SAPSRO (2.9) associated with the branched motif $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle$ as

$$\mathcal{E}_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle}^{(m_1, m_2 | n_1, n_2)} \equiv \mathcal{E}_{\langle k_1, \ldots, k_r | l \rangle}^{(m_1, m_2 | n_1, n_2)} = 2 \sum_{j=1}^{N-1} j (1 - \delta_j) + l,$$

(5.20)

with degeneracy factor given by $\mathcal{N}_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle}^{(m_1, m_2 | n_1, n_2)} \equiv \mathcal{N}_{\langle k_1, \ldots, k_r | l \rangle}^{(m_1, m_2 | n_1, n_2)}$, which can be computed by using Eq. (4.22). One may observe that, the dimension of the Hilbert space associated with the spin chain (2.9) is given by $d_1 = (m + n)^N$ and the highest possible number of distinct energy levels of the form (5.20) is given by $d_2 = 2^{N-1}(N + 1)$. Since $d_1$ and $d_2$ satisfy the relation

$$\ln \left( \frac{d_2}{d_1} \right) = \ln (N + 1) - N \ln \left( \frac{m + n}{2} \right) - \ln 2,$$

assuming $m + n > 2$ and taking $N \to \infty$ limit we find that $\ln \left( \frac{d_2}{d_1} \right) = O(\ln N - O(N) \to -\infty$, i.e., $\frac{d_2}{d_1} \to 0$. This result clearly indicates that, in general, the energy levels of $BC_N$ type of PF spin chain are highly degenerate for large values of $N$.

It should be noted that, if the degeneracy factor $\mathcal{N}_{\langle k_1, \ldots, k_r | l \rangle}^{(m_1, m_2 | n_1, n_2)}$ becomes zero for some choice of the discrete parameters $m_1, m_2, n_1, n_2$, and branched motif $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle$, then the energy level $\mathcal{E}_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle}^{(m_1, m_2 | n_1, n_2)}$ in (5.20) would be absent from the spectrum of the corresponding spin chain. For example, in the case of $BC_N$ type of bosonic PF spin chain with $m_1 + m_2 = m \geq 2$ and $n_1 = n_2 = 0$, rule (i) of Sec. 4 implies that at most $m$ boxes can placed in any column of a border strip. As a result, branched border strips like $\langle k_1, \ldots, k_r | l \rangle$ with $k_i \leq m$ can only give nontrivial values of $\mathcal{N}_{\langle k_1, \ldots, k_r | l \rangle}^{(m_1, m_2 | n_1, n_2)}$. Hence, by using the mapping (5.19), we find that branched motifs like $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle$ with more than $(m - 1)$ consecutive $\delta_i = 0$ do not appear in the spectrum of this spin chain. Similarly, by using the rule (ii) of Sec. 4, one can show that branched motifs like $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} | l \rangle$ with more than $(n - 1)$ consecutive $\delta_i = 1$ do not appear in the spectrum of the $BC_N$ type of fermionic PF spin chain with $m_1 = m_2 = 0$ and $n_1 + n_2 = n \geq 2$. Similar type of selection rules, giving restrictions on possible values of $\delta_i$ within the motif $\langle \delta_1, \delta_2, \ldots, \delta_{N-1} \rangle$, have been found earlier in the context of $A_{N-1}$ type of non-supersymmetric HS and PF spin chains [4, 43].

However it is worth noting that, apart from the above mentioned restrictions on possible values of $\delta_i$, some restriction on allowed values of $l$ may also be present for the
case of branched motif \((\delta_1, \delta_2, \ldots, \delta_{N-1}|l)\). Such restriction on possible values of \(l\) leads to a new type of selection rule for the case of some \(BC_N\) type of PF spin chains. To demonstrate the existence of the latter type of selection rule through a simple example, let us consider a \(BC_N\) type of PF spin chain (2.9) with discrete parameters given by \(m_1 = m, m_2 = 0, n_1 = n, n_2 = 0\), where \(m + n \geq 2\). In this case, Eq. (2.5) implies that \(f(s_i) = 0\) for all possible value of the local spin \(s_i\). As a result, \(P_{ij}^{(m,0|n,0)}\) defined in (2.6) becomes an identity operator and \(P_{ij}^{(m,0|n,0)}\) reduces to the spin permutation operator \(P_{ij}^{(m,n)}\). Therefore, in this special case, the Hamiltonian (2.9) of the \(BC_N\) type of PF spin chain can be expressed in a simple form like

\[
H_N^{(m,0|n,0)} = \sum_{i \neq j} \frac{y_i + y_j}{(y_i - y_j)^2} (1 - P_{ij}^{(m,n)}),
\]

(5.21)

where \(y_j\)'s are the roots of the the generalized Laguerre polynomial \(L_N^{\beta-1}\) as mentioned earlier. Similarly it can be shown that, in another special case with discrete parameters given by \(m_1 = 0, m_2 = m, n_1 = 0, n_2 = n\), the corresponding Hamiltonian \(H_N^{(0,m|0,n)}\) would coincide with \(H_N^{(m,0|n,0)}\) in (5.21) up to an insignificant additive constant.

For the purpose of finding out the allowed branched motifs associated with Hamiltonian \(H_N^{(m,0|n,0)}\) in (5.21), we use Eq. (4.17) to find that \(f^{(m,0|n,0)}(\alpha) = 0\) for all possible value of \(\alpha\). Hence, Eqs. (4.16), (4.18) and (4.22) imply that \(F(\mathcal{T}) = 0\) for any \(\mathcal{T} \in \mathcal{G}\), \(\mathcal{G}_l = \{\}\) for \(l > 0\), and \(N_{\langle k_1, \ldots, k_l | l \rangle}^{(m,0|n,0)} = 0\) for \(l > 0\). As a consequence, the spectrum of \(H_N^{(m,0|n,0)}\) in (5.21) obeys a selection rule which forbids the occurrence of branched motifs like \((\delta_1, \delta_2, \ldots, \delta_{N-1}|l)\) for \(l > 0\). Therefore, by putting \(l = 0\) in (5.20), we find that the allowed energy levels of the Hamiltonian \(H_N^{(m,0|n,0)}\) can in general be written in the form

\[
E_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} \rangle}^{(m,0|n,0)} = 2 \sum_{j=1}^{N-1} j(1 - \delta_j).
\]

(5.22)

Using the rules (i) and (ii) of Sec. 4, it is easy to see that \(\delta_j\)'s in the above equation can be chosen without any restriction when both \(m\) and \(n\) take nonzero values. On the other hand, as we have already mentioned earlier, more than \((m-1)\) number of \((n-1)\) consecutive \(\delta_i = 0\) \((\delta_i = 1)\) would not appear in the above equation in the special case \(n = 0 \ (m = 0)\). Comparing Eq. (5.22) with Eq. (5.5), we interestingly find that

\[
E_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} \rangle}^{(m,0|n,0)} = 2 E_{\langle \delta_1, \delta_2, \ldots, \delta_{N-1} \rangle}^{(m,n)}
\]

(5.23)

and, by inserting \(N_{\langle k_1, \ldots, k_l | l \rangle}^{(m,0|n,0)} = 0\) for \(l > 0\) in Eq. (4.11), we also get

\[
N_{\langle k_1, k_2, \ldots, k_l | l \rangle}^{(m,0|n,0)} = N_{\langle k_1, k_2, \ldots, k_l \rangle}^{(m,n)}.
\]

(5.24)

Eqs. (5.23) and (5.24) clearly show that, except for an overall scale factor of two, the spectrum of the \(BC_N\) type of PF chain with Hamiltonian \(H_N^{(m,0|n,0)}\) in (5.21) coincides
with that of the $A_{N-1}$ type of supersymmetric PF chain with Hamiltonian $\mathcal{H}_N^{(m|n)}$ in (1.1). Thus the Hamiltonians $\mathcal{H}_N^{(m,0|n,0)}$ and $2\mathcal{H}_N^{(m|m)}$ must be connected through a unitary transformation. It is well known that the latter Hamiltonian exhibits $Y(gl(m))$ Yangian symmetry in the special case $n = 0$, $Y(gl(n))$ Yangian symmetry in the special case $m = 0$, and $Y(gl(m|n))$ super Yangian symmetry when both $m$ and $n$ take nonzero values. Hence, Eqs. (5.23) and (5.24) imply that the Hamiltonian $\mathcal{H}_N^{(m,0|0,0)}$ (and the related Hamiltonian $\mathcal{H}_N^{(0,m,0,0)}$) would also exhibit $Y(gl(m))$ Yangian symmetry in the special case $n = 0$, $Y(gl(n))$ Yangian symmetry in the special case $m = 0$, and $Y(gl(m|n))$ super Yangian symmetry when both $m$ and $n$ take nonzero values.

Let us now consider the class of Hamiltonians of the $BC_N$ type of PF spin chains (2.9), which can not be expressed in the above mentioned simple forms $\mathcal{H}_N^{(m,0|n,0)}$ and $\mathcal{H}_N^{(0,m|0,0)}$. It is interesting to observe that, Eq. (1.11) establishes a connection between the spectrum of a $BC_N$ type of PF chain with Hamiltonian $\mathcal{H}_N^{(m_1,m_2|n_1,n_2)}$ and that of the $A_{N-1}$ type of supersymmetric PF chain with Hamiltonian $\mathcal{H}_N^{(m|m)}$ in (1.1), where $m = m_1 + m_2$ and $n = n_1 + n_2$. More precisely, Eq. (1.11) shows that the degeneracy factor of the energy level (of the later spin chain) associated with the motif $(\delta_1, \delta_2, ..., \delta_{N-1})$ is exactly same as the sum of degeneracy factors of all possible energy levels (of the former spin chain) associated with branched motifs of the form $(\delta_1, \delta_2, ..., \delta_{N-1}|l)$ over the variable $l$. This type of splitting of energy levels, between two quantum systems occupying the same Hilbert space, can only appear when an irreducible representation associated with the bigger symmetry algebra of one quantum system is expressed as a direct sum of several irreducible representations associated with the smaller symmetry subalgebra of the other quantum system. Hence we conclude that, the symmetry algebra of the presently considered class of $BC_N$ type of PF spin chains (2.9) would be a proper subalgebra of the i) $Y(gl(m))$ Yangian algebra when $n = 0$ and $m_1, m_2$ are positive integers, ii) $Y(gl(n))$ Yangian algebra when $m = 0$ and $n_1, n_2$ are positive integers, iii) $Y(gl(m|n))$ super Yangian algebra when $m, n$ are positive integers and neither the relation $m_1 = n_1 = 0$ nor $m_2 = n_2 = 0$ is satisfied.

From the above discussion it is clear that, $BC_N$ type of PF spin chains with Hamiltonians of the form $\mathcal{H}_N^{(m,0|n,0)}$ and $\mathcal{H}_N^{(0,m|0,0)}$ possess the maximal Yangian or super Yangian symmetry. Indeed, by combining Eqs. (1.11) and (5.24) we find the relation

$$\mathcal{N}^{(m,0|n,0)}_{\{k_1,k_2,...,k_r\}} = \sum_{l=0}^{N} \mathcal{N}^{(m_1,m_2|n_1,n_2)}_{\{k_1,k_2,...,k_r,l\}},$$

which, along with Eqs. (5.22) and (5.20), implies that each highly degenerate energy level of the Hamiltonian $\mathcal{H}_N^{(m,0|n,0)}$ splits into many parts (depending on admissible values of $l$) and transforms into less degenerate energy levels of the Hamiltonian $\mathcal{H}_N^{(m_1,m_2|n_1,n_2)}$.

In Fig. 2 we have shown how the spectra of some $BC_N$ type of ferromagnetic PF spin chains with $N = 3$ are expressed through the branched motifs and compared their spectra in two diagrams. In the left diagram, we have compared the spectra of two
bosonic spin chains with Hamiltonians $\mathcal{H}_3^{(3,0)(0,0)}$ and $\mathcal{H}_3^{(2,1)(0,0)}$. In particular, we have shown that the energy levels of the former Hamiltonian split due to the possible non-zero values of $l$ and transform into the energy levels of the later Hamiltonian. It may also be noted that the branched motifs $(01|3)$, $(10|3)$, $(00|0)$, $(00|2)$ and $(00|3)$ are absent in the spectrum of the later spin chain, since the corresponding degeneracy factors are found to be zero by using (4.22). Similarly, in the right diagram, we have compared the spectra of two spin chains with Hamiltonians $\mathcal{H}_3^{(2,0)(2,0)}$ and $\mathcal{H}_3^{(2,0)(0,2)}$ which have both bosonic and fermionic spin degrees of freedom, and shown how the energy levels of the former Hamiltonian split into the energy levels of the later Hamiltonian. The branched motifs $(01|0)$, $(10|0)$, $(00|0)$ and $(00|1)$ are absent in the spectrum of the later spin chain since the corresponding degeneracy factors are found to be zero.

It should be noted that, in addition to the intrinsic degeneracy of the energy level
\[ \mathcal{E}^{(m_1,m_2|n_1,n_2)}_{(\delta_1,\delta_2,\ldots,\delta_{N-1}|l)} \] given by \( \mathcal{N}^{(m_1,m_2|n_1,n_2)}_{(k_1,k_2,\ldots,k_r|l)} \), accidental degeneracies may also occur in the spectrum of the \( BC_N \) type of PF spin chain. For example, let us assume that there exist two different branched motifs \((\delta_1,\delta_2,\ldots,\delta_{N-1}|l)\) and \((\delta'_1,\delta'_2,\ldots,\delta'_{N-1}|l')\) such that
\[
2 \sum_{j=1}^{N-1} j\delta_j - l = 2 \sum_{j=1}^{N-1} j\delta'_j - l'.
\]
Since by using (5.20) one obtains \( \mathcal{E}^{(m_1,m_2|n_1,n_2)}_{(\delta_1,\delta_2,\ldots,\delta_{N-1}|l)} = \mathcal{E}^{(m_1,m_2|n_1,n_2)}_{(\delta'_1,\delta'_2,\ldots,\delta'_{N-1}|l')} \), the energy levels associated with branched motifs \((\delta_1,\delta_2,\ldots,\delta_{N-1}|l)\) and \((\delta'_1,\delta'_2,\ldots,\delta'_{N-1}|l')\) coincide with each other. In this case, the total degeneracy of the corresponding energy level becomes the sum of the degeneracy factors associated with these two branched motifs. In fact, for some choices of the parameters \( m_1, m_2, n_1, n_2 \) and values of \( N \), there may exist multiple branched motifs with same energy level and in that case, the corresponding degeneracy will be the sum of degeneracies associated with all of these branched motifs. As shown in Fig. 2 due to the accidental degeneracy in the spectrum of Hamiltonian \( \mathcal{H}_3^{(1,1,0,0)} \), the energy levels corresponding to the branched motifs \((11|2), (11|3)\) and \((01|2)\) coincide with those of the branched motifs \((01|0), (01|1)\) and \((10|0)\) respectively. Similarly, for the case of Hamiltonian \( \mathcal{H}_3^{(1,1,0,2)} \), the energy levels corresponding to the branched motifs \((11|3)\) and \((01|3)\) coincide with those of the branched motifs \((01|1)\) and \((10|1)\) respectively.

6. Spectra of the \( BC_N \) type of anti-ferromagnetic PF spin chain

In analogy with the ferromagnetic case, the Hamiltonian for a class of exactly solvable \( BC_N \) type of anti-ferromagnetic PF spin chains with SAPSRO may be defined as
\[
\tilde{\mathcal{H}}^{(m_1,m_2|n_1,n_2)}_N = \sum_{i,j=1}^{N} \left[ \frac{1 + \tilde{P}_{ij}^{(m|n)}}{(\xi_i - \xi_j)^2} + \frac{1 + \tilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_{i=1}^{N} \frac{1 + \tilde{P}_{i}^{(m_1,m_2|n_1,n_2)}}{\xi_i}. \tag{6.1}
\]
It has been found that \( BC_N \) type of SRS polynomials of the second kind play the role of generalized partition functions for these anti-ferromagnetic PF spin chains [50]. In this section, at first we shall derive a recursion relation similar to (3.28) involving \( BC_N \) type of SRS polynomials of the second kind. Subsequently, by using the above mentioned recursion relation, we shall express the later type of SRS polynomials through restricted super Schur polynomials and find out the energy levels of anti-ferromagnetic PF spin chains [(6.1)] in terms of branched motifs.

By using the freezing trick, partition functions of the anti-ferromagnetic PF spin chains [(6.1)] have been derived in the form [50]
\[
\tilde{Z}^{(m_1,m_2|n_1,n_2)}_{B,N}(q) = \sum_{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} \frac{(q^2)^{m_1} \cdot \sum_{j=1}^{m_2} b_j (b_j - 1) + \sum_{k=1}^{n_1} c_k}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}. \tag{6.2}
\]
It may be noted that these partition functions can be reproduced by taking \( x = \bar{x} = y = \bar{y} = 1 \) limit of the \( BC_N \) type of SRS polynomials of the second kind given by

\[
\tilde{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} (q^2)^N \cdot q^{a_1 + \sum_{j=1}^{m_2} b_j} \prod_{k=1}^{n_1} \frac{x^{a_i}}{(q^2)^{a_i} (q^2)^{b_j} \prod_{k=1}^{n_1} \frac{y^{c_k}}{(q^2)^{c_k} \prod_{l=1}^{n_2} \frac{d_l}{(q^2)^{d_l}}}.
\]

(6.3)

Comparing \((2.12)\) with \((6.3)\), it is easy to see that the \( BC_N \) type of SRS polynomials of the second and first kind are related as

\[
\tilde{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \tilde{H}_{B,N}^{(n_2,n_1|m_2,m_1)}(\bar{y}, y; \bar{x}, x; q).
\]

(6.4)

In analogy with \((2.17)\), the generating function \( \tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q; t) \) corresponding to \( \tilde{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) may be defined as

\[
\tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q; t) = \sum_{N=0}^{\infty} \frac{\tilde{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q)}{(q^2)^N} t^N.
\]

(6.5)

With the help of Eqs. \((2.17)\), \((6.4)\) and \((6.3)\), this generating function can be related to the generating function associated with the ferromagnetic case as

\[
\tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q, t) = \tilde{G}_B^{(n_2,n_1|m_2,m_1)}(\bar{y}, y; \bar{x}, x; q, t).
\]

(6.6)

By using the above equation along with \((2.14)\), \( \tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q; t) \) can be expressed as

\[
\tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q, t) = \tilde{G}_4^{(m_1)}(x, q, t) \cdot \tilde{G}_3^{(n_1)}(\bar{x}, q, t) \cdot \tilde{G}_2^{(m_2)}(y, q, t) \cdot \tilde{G}_1^{(n_2)}(\bar{y}, q, t).
\]

(6.7)

Now, for the purpose of deriving a recursion relation for the \( BC_N \) type of SRS polynomials of the second kind, we use Eqs. \((3.1)\), \((3.2)\), \((3.3)\), \((3.4)\) and \((6.7)\), to obtain a \( q^2 \)-difference relation like

\[
\tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q, t) = \prod_{i=1}^{m_1} (1 + t q x_i) \prod_{j=1}^{m_2} (1 + t \bar{x}_j) \prod_{k=1}^{n_1} (1 - t q y_k) \prod_{l=1}^{n_2} (1 - t \bar{y}_l) \tilde{G}_B^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q, q^2 t).
\]

(6.8)

In analogy with \((3.23)\), we expand the prefactor in the right hand side of the above equation in a power series of \( t \) as

\[
\prod_{i=1}^{m_1} (1 + t q x_i) \prod_{j=1}^{m_2} (1 + t \bar{x}_j) \prod_{k=1}^{n_1} (1 - t q y_k) \prod_{l=1}^{n_2} (1 - t \bar{y}_l) = \sum_{k=0}^{\infty} t^k \tilde{E}_k^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q),
\]

(6.9)
where
\[ E_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = E_{k}^{(m_1+m_2|n_1+n_2)}(x, \bar{x}; y, \bar{y}) \big|_{x \rightarrow qx, y \rightarrow qy} \]  
(6.10)

Thus \( \tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) may be considered as a \( q \)-deformation of the second kind of the elementary supersymmetric polynomial \( E_{k}^{(m_1+m_2|n_1+n_2)}(x, \bar{x}; y, \bar{y}) \). Expanding both sides of Eq. (6.8) in powers of \( t \) by using Eqs. (6.3) and (6.9), and subsequently comparing the powers of \( t^N \), we obtain a recursion relation for the \( BC_N \) type of SRS polynomials of the second kind as
\[
\tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{k=1}^{N} \frac{q^{2(N-k)} \cdot (q^2)^{N-1}}{(q^2)^{N-k}} \tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \tilde{E}_{k+N-k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q). 
\]  
(6.11)

The above relation can generate any \( \tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) in a recursive way from the given initial condition \( \tilde{E}_{0}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = 1 \). In analogy with the ferromagnetic case, we find that \( \tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) satisfying the recursion relation (6.11) can equivalently be expressed as
\[
\tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{\vec{k} \in \mathcal{P}_N} q^{\sum_{i=1}^{r} 2k_{i}} \tilde{S}_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q), 
\]  
(6.12)

where \( \vec{k} = \{k_1, k_2, \ldots, k_r\} \) and \( \tilde{S}_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) is defined as
\[
\tilde{S}_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \left| \begin{array}{cccc}
\tilde{E}_{k_1}(q) & \tilde{E}_{k_2+k_{r-1}}(q) & \cdots & \tilde{E}_{k_r+\ldots+k_1}(q) \\
1 & \tilde{E}_{k_{r-1}}(q) & \tilde{E}_{k_{r-2}}(q) & \cdots \\
0 & 1 & \tilde{E}_{k_{r-2}}(q) & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1
\end{array} \right|, 
\]  
(6.13)

where \( \tilde{E}_{k}(q) \equiv \tilde{E}_{k}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \). Using Eqs. (6.13), (6.10) and (4.2), we obtain the relation
\[
\tilde{S}_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = S_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m|n)}(x, \bar{x}; y, \bar{y}) \big|_{x \rightarrow qx, y \rightarrow qy}, 
\]  
(6.14)

which shows that \( \tilde{S}_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) \) may be interpreted as a \( q \)-deformed super Schur polynomial of the second kind. Comparing Eq. (4.7) with (6.14), we find that the \( q \)-deformed super Schur polynomials of the first kind and the second kind are related as
\[
\tilde{S}_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = q^{N} S_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q^{-1}). 
\]  
(6.15)

Using the above relation along with (4.8), one can express the \( q \)-deformed super Schur polynomials of the second kind in terms of restricted super Schur polynomials as
\[
\tilde{S}_{\langle k_1, k_2, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = \sum_{l=0}^{N} q^{N-l} \cdot S_{\langle k_1, k_2, \ldots, k_{r+l} \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}). 
\]  
(6.16)
Combining Eqs. (6.12) and (6.16), we finally express the $BC_N$ type of SRS polynomials of the second kind through restricted super Schur polynomials as

$$
\tilde{P}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}, y, \bar{y}; q) = \sum_{k \in \mathcal{P}_N} \sum_{l=0}^{N} q^{\sum_{i=1}^{r-1} K_i + N - l} \cdot S_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}, y, \bar{y}).
$$

(6.17)

Now, putting $x = \bar{x} = y = \bar{y} = 1$ into (6.17) and also using (4.10), we find that the partition functions of the $BC_N$ type of anti-ferromagnetic PF spin chains (6.1) can be written as

$$
\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \tilde{P}_{B,N}^{(m_1,m_2|n_1,n_2)}(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1; q)
= \sum_{k \in \mathcal{P}_N} \sum_{l=0}^{N} q^{\sum_{i=1}^{r-1} K_i + N - l} \cdot \mathcal{N}_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}.
$$

(6.18)

It is evident that, the exponent of $q$ appearing in the r.h.s. of the above equation yields the energy level for the spin chains (6.1) associated with the branched border strip $\langle k_1, \ldots, k_r \rangle$ as

$$
\mathcal{E}_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)} = 2 \sum_{i=1}^{r-1} K_i + N - l,
$$

(6.19)

and the intrinsic degeneracy of this energy level is given by $\mathcal{N}_{\langle k_1, \ldots, k_r \rangle}^{(m_1,m_2|n_1,n_2)}$. Moreover, using the mapping defined in (5.19) along with the relation (5.4), the above energy level can also be written in terms of the corresponding branched motif as

$$
\mathcal{E}_{\langle \delta_1, \ldots, \delta_{N-1} \rangle}^{(m_1,m_2|n_1,n_2)} = 2 \sum_{j=1}^{N-1} j \delta_j + N - l.
$$

(6.20)

It may be noted that the intrinsic degeneracy factors corresponding to the energy levels $\mathcal{E}_{\langle \delta_1, \ldots, \delta_{N-1} \rangle}^{(m_1,m_2|n_1,n_2)}$ (in the ferromagnetic case) and $\mathcal{E}_{\langle \delta_1, \ldots, \delta_{N-1} \rangle}^{(m_1,m_2|n_1,n_2)}$ (in the anti-ferromagnetic case) are exactly same. In addition, by summing Eqs. (5.20) and (6.20), we obtain the relation

$$
\mathcal{E}_{\langle \delta_1, \ldots, \delta_{N-1} \rangle}^{(m_1,m_2|n_1,n_2)} + \mathcal{E}_{\langle \delta_1, \ldots, \delta_{N-1} \rangle}^{(m_1,m_2|n_1,n_2)} = 2 \sum_{j=1}^{N-1} j + N = N^2.
$$

(6.21)

Hence, if the spectrum of the ferromagnetic case is completely known, then one can easily obtain that of the anti-ferromagnetic case and vice-versa. Indeed, Eq. (6.21) is simply a manifestation of the fact that the sum of two Hamiltonians, corresponding to the ferromagnetic and anti-ferromagnetic cases given by (2.9) and (6.1) respectively, yields essentially same relation in operator form:

$$
\mathcal{H}_N^{(m_1,m_2|n_1,n_2)} + \mathcal{H}_N^{(m_1,m_2|n_1,n_2)} = 2 \sum_{i \neq j} [(\xi_i - \xi_j)^{-2} + (\xi_i + \xi_j)^{-2}] + 2\beta \sum \xi_i^{-2} = N^2,
$$

where the last sum involving zero points of the generalized Laguerre polynomials has been computed earlier [25].

32
7. Extended boson-fermion duality through microscopic approach

The partition functions of $A_{N-1}$ type of supersymmetric HS and PF spin chains are known to obey boson-fermion duality relations \[5, 12, 13, 15, 16\]. In the case of $A_{N-1}$ type of supersymmetric HS spin chains, the boson-fermion duality has been derived earlier by using two different techniques: the first one uses a unitary transformation which relates the Hamiltonians of the $\text{su}(m|n)$ and $\text{su}(n|m)$ spin chains and corresponding partition functions, while the second one is a microscopic approach which utilizes a more fundamental boson-fermion duality relation among the super Schur polynomials associated with the partition functions of these spin chains \[16\]. In the case of $BC_N$ type of PF spin chains (2.9) with SAPSRO, a similar type of duality relation has been obtained by using the first technique as \[49\]

\[
Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = q^{N^2} \cdot Z_{B,N}^{(n_2,n_1|m_2,m_1)}(q^{-1}), \tag{7.1}
\]

which not only involves the exchange of bosonic and fermionic degrees of freedom, but also involves the exchange of positive and negative parity degrees of freedom associated with SAPSRO (i.e., $m_1 \leftrightarrow m_2$ and $n_1 \leftrightarrow n_2$). In this section, our goal is to derive such ‘extended’ boson-fermion duality relation by using the microscopic approach. To this end, at first we shall derive some duality relations among the $q$-deformed and the restricted super Schur polynomials which have been introduced in Sec. 4. Subsequently, we shall show how the duality relation (7.1) emerges as a consequence of more fundamental duality relations among the $q$-deformed and the restricted super Schur polynomials.

To begin with, let us note that the $A_{N-1}$ type of super Schur polynomials satisfy the boson-fermion duality relation given by \[16, 53, 54\]

\[
S_{\langle \vec{k} \rangle}^{(m|n)}(x, \bar{x}; y, \bar{y}) = S_{\langle \vec{k}' \rangle}^{(n|m)}(y, \bar{y}; x, \bar{x}), \tag{7.2}
\]

where $\langle \vec{k} \rangle \equiv \langle k_1, k_2, ..., k_r \rangle$ is the border strip defined in Fig. 1 and $\langle \vec{k}' \rangle \equiv \langle k_1', ..., k_{N-r+1}' \rangle$ is the corresponding conjugate border strip which is obtained by flipping $\langle \vec{k} \rangle$ over its main diagonal. The structure of this conjugate border strip is shown in Fig. 3. Using the symmetry of $S_{\langle \vec{k} \rangle}^{(n|m)}(y, \bar{y}; x, \bar{x})$ under the exchange of $x \leftrightarrow \bar{x}$ and $y \leftrightarrow \bar{y}$, Eq. (7.2) can be also written as

\[
S_{\langle \vec{k} \rangle}^{(m|n)}(x, \bar{x}; y, \bar{y}) = S_{\langle \vec{k}' \rangle}^{(n|m)}(\bar{y}, y; \bar{x}, x). \tag{7.3}
\]

Multiplying both sides of the above equation with $q^N$, scaling the sets of variables $x$ and $y$ as $x \rightarrow q^{-1}x$ and $y \rightarrow q^{-1}y$ respectively, and finally using Eqs. (4.6) as well as (4.7), we obtain an extended boson-fermion duality relation between two deformed super Schur polynomials as

\[
S_{\langle \vec{k} \rangle}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = q^{N} \cdot S_{\langle \vec{k}' \rangle}^{(n_2,n_1|m_2,m_1)}(\bar{y}, y; \bar{x}, x; q^{-1}). \tag{7.4}
\]
Expanding both sides of the above equation by using (4.8) and redefining the variable $N - l$ as $l$, we obtain
\[
\sum_{l=0}^{N} q^l \cdot S_{\langle \vec{k} | \ell \rangle}^{(m_1, m_2 | n_1, n_2)} (x, \bar{x}; y, \bar{y}) = \sum_{l=0}^{N} q^l \cdot S_{\langle \vec{k}' | N-l \rangle}^{(n_2, n_1 | m_2, m_1)} (y, y; \bar{x}, x).
\]

Comparing the powers of $q$ in both sides of the above equation, we find that restricted super Schur polynomials satisfy an extended boson-fermion duality relation of the form
\[
S_{\langle \vec{k} | \ell \rangle}^{(m_1, m_2 | n_1, n_2)} (x, \bar{x}; y, \bar{y}) = S_{\langle \vec{k}' | N-l \rangle}^{(n_2, n_1 | m_2, m_1)} (y, y; \bar{x}, x).
\] (7.5)

Next, we consider the reversed border strip $\langle \vec{k}'_{\text{rev}} \rangle \equiv \langle k_r, k_{r-1}, \ldots, k_1 \rangle$ corresponding to the border strip $\langle \vec{k} \rangle \equiv \langle k_1, k_2, \ldots, k_r \rangle$, and the reverse conjugate border strip $\langle \vec{k}'_{\text{rev}} \rangle \equiv \langle k_{N-r+1}', k_{N-r}', \ldots, k_1' \rangle$ corresponding to $\langle \vec{k}' \rangle \equiv \langle k_1', k_2', \ldots, k_{N-r+1}' \rangle$. Following the procedure outlined in the Appendix of Ref. [41], it can be easily shown that any $q$-deformed super Schur Polynomial, defined through a determinant relation of the form (4.4), would remain invariant under the reversal of the corresponding border strip:
\[
S_{\langle \vec{k}' | \ell \rangle}^{(n_2, n_1 | m_2, m_1)} (x, \bar{x}; y, \bar{y}; q) = S_{\langle \vec{k}'_{\text{rev}} | \ell \rangle}^{(n_2, n_1 | m_2, m_1)} (x, \bar{x}; y, \bar{y}; q).
\] (7.6)

Expanding both sides of the above equation by using (4.8), we find that the restricted super Schur polynomials also remain invariant under the above mentioned reversal, i.e.,
\[
S_{\langle \vec{k}' | \ell \rangle}^{(n_2, n_1 | m_2, m_1)} (x, \bar{x}; y, \bar{y}) = S_{\langle \vec{k}'_{\text{rev}} | \ell \rangle}^{(n_2, n_1 | m_2, m_1)} (x, \bar{x}; y, \bar{y}).
\] (7.7)

Combining Eqs. (7.5) and (7.7), we obtain an extended boson-fermion duality relation of the form
\[
S_{\langle \vec{k} | \ell \rangle}^{(m_1, m_2 | n_1, n_2)} (x, \bar{x}; y, \bar{y}) = S_{\langle \vec{k}'_{\text{rev}} | N-l \rangle}^{(n_2, n_1 | m_2, m_1)} (y, y; \bar{x}, x).
\] (7.8)

For our purpose of proving Eq. (7.1), it is also needed to find out a relation between the partial sums associated with the border strip $\langle \vec{k} \rangle$ and those of the corresponding
reversed conjugate border strip \(\langle \vec{k}'_{\text{rev}} \rangle\). To this end it may be noted that, the set of partial sums \(\{K'_1, K'_2, ..., K'_{N-r+1}\}\) corresponding to the conjugate border strip \(\langle \vec{k}' \rangle\) are given by [16]

\[
\{K'_1, K'_2, ..., K'_{N-r} \} = \{N - K_{r+1}, N - K_{r+2}, ..., N - K_N\}, \tag{7.9}
\]

where \(\{K_{r+1}, K_{r+2}, ..., K_N\}\) represents the set of complementary partial sums corresponding to the border strip \(\langle \vec{k} \rangle\), i.e.,

\[
\{K_{r+1}, K_{r+2}, ..., K_N\} = \{1, 2, ..., N\} - \{K_1, K_2, ..., K_r\}. \tag{7.10}
\]

Using Eqs. (7.9) and (7.10), and inserting \(K_r = N\), we get

\[
\sum_{i=1}^{N-r} K'_i = N(N - r) - \sum_{i=r+1}^{N} K_i = N(N - r) - \frac{N(N - 1)}{2} + \sum_{i=1}^{r-1} K_i. \tag{7.11}
\]

Next, we note that the sums of the partial sums for the border strip \(\langle \vec{k} \rangle\) and the corresponding reversed border strip \(\langle \vec{k}_{\text{rev}} \rangle\) can be written as

\[
\sum_{i=1}^{r-1} K_i = k_1 + (k_1 + k_2) + \cdots + (k_1 + \cdots + k_{r-1}) = \sum_{j=1}^{r} (r - j) k_j, \tag{7.12}
\]

and

\[
\sum_{i=1}^{r-1} \tilde{K}_i = k_r + (k_r + k_{r-1}) + \cdots + (k_r + \cdots + k_2) = \sum_{j=1}^{r} (j - 1) k_j, \tag{7.13}
\]

respectively, where \(\tilde{K}_i\) denotes the \(i\)-th partial sum associated with the reversed border strip \(\langle \vec{k}_{\text{rev}} \rangle\). Adding Eqs. (7.12) and (7.13), we obtain the relation

\[
\sum_{i=1}^{r-1} K_i + \sum_{i=1}^{r-1} \tilde{K}_i = N(r - 1). \tag{7.14}
\]

It is evident that, for the case of conjugate border strip \(\langle \vec{k} \rangle\) and the reversed conjugate border strip \(\langle \vec{k}'_{\text{rev}} \rangle\), the above relation can be written as

\[
\sum_{i=1}^{N-r} K'_i + \sum_{i=1}^{N-r} \tilde{K}'_i = N(N - r), \tag{7.15}
\]

where \(\tilde{K}'_i\) denotes the \(i\)-th partial sum associated with the reversed conjugate border strip \(\langle \vec{k}'_{\text{rev}} \rangle\). Combining (7.15) and (7.11), we finally obtain a relation between the partial sums associated with border strip \(\langle \vec{k} \rangle\) and those of \(\langle \vec{k}'_{\text{rev}} \rangle\) as

\[
\sum_{i=1}^{r-1} K_i = \frac{N(N - 1)}{2} - \sum_{i=1}^{N-r} \tilde{K}'_i. \tag{7.16}
\]
Substituting the relations (7.8) and (7.16) into Eq. (5.16), and making a change of the summation variables, we obtain

\[
\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = q^{N^2} \sum_{\vec{k}_{\text{rev}} \in \mathcal{P}_N} \sum_{t=0}^{N} (q^{-1})^{N(N-1)-2} \sum_{i=1}^{N-r} \vec{k}_{i}^{t+1} \cdot S_{(i,|l)}^{(n_2,n_1|m_2,m_1)}(\bar{y}, y; \bar{x}, x). \tag{7.17}
\]

It is interesting to observe that the power of \( q \) in Eq. (5.16) and the power of \( q^{-1} \) in Eq. (7.17) can be represented through essentially same function of the variables \( \vec{k} \) and \( \vec{k}_{\text{rev}} \) respectively. Consequently, Eq. (7.17) can be expressed in the form

\[
\mathbb{H}_{B,N}^{(m_1,m_2|n_1,n_2)}(x, \bar{x}; y, \bar{y}; q) = q^{N^2} \mathbb{H}_{B,N}^{(n_2,n_1|m_2,m_1)}(\bar{y}, y; \bar{x}, x; q^{-1}). \tag{7.18}
\]

Inserting \( x = \bar{x} = y = \bar{y} = 1 \) to the above equation and using (2.13), we finally obtain the extended boson-fermion duality relation (7.1). Using the results of section 6, it can be shown that the partition functions associated with \( BC_N \) type of anti-ferromagnetic PF spin chains obey similar type of duality relation.

8. Concluding remarks

Here we establish that the spectra of the \( BC_N \) type of PF spin chains with SAP-SRO, including the degeneracy factors associated with all energy levels, can be described completely through the branched motifs. Recently introduced \( BC_N \) type of multivariate SRS polynomials, which are closely connected with the partition functions of the above mentioned spin chains, play the central role in our approach. At first, we show that these SRS polynomials satisfy the recursion relation (3.28) involving a particular type of \( q \)-deformation of the elementary supersymmetric polynomials. With the help of such \( q \)-deformed elementary supersymmetric polynomials, subsequently we define a few mathematical entities which would be helpful for further analysis. For example, we use a Jacobi-Trudi like formula (4.4) to define the corresponding \( q \)-deformed super Schur polynomials. Moreover, by expanding these \( q \)-deformed super Schur polynomials in powers of \( q \), we obtain the so called restricted super Schur polynomials. With the help of skew Young tableaux associated with border strips, we also present some combinatorial expressions (4.15) and (4.21) for the \( q \)-deformed and the restricted super Schur polynomials respectively.

Applying the recursion relation (3.28), subsequently we derive novel expressions (5.15) and (5.16) for the \( BC_N \) type of SRS polynomials through suitable linear combinations of the \( q \)-deformed and the restricted super Schur polynomials respectively. As shown in Eq. (5.20), the later expressions for the \( BC_N \) type of SRS polynomials enable us to describe the spectra of the corresponding ferromagnetic PF spin chains with \( N \) number of lattice sites in terms of the branched motifs like \( (\delta_1, \delta_2, ..., \delta_{N-1}|l) \), where

\[36\]
\( \delta_i \in \{0, 1\} \) and \( l \in \{0, 1, \ldots, N\} \). By taking a particular limit of the restricted super Schur polynomials, we obtain the combinatorial expression (4.22) which determines the intrinsic degeneracy of the energy level associated with the branched motif \((\delta_1, \delta_2, \ldots, \delta_{N-1})|l\). However, as discussed in Section 5, there may exist more than one branched motifs with coincident energy levels. In that case, the total degeneracy of the coincident energy level is determined by the sum of the intrinsic degeneracy factors corresponding to those branched motifs. In analogy with the ferromagnetic case, we also obtain a complete classification for the spectra of the \(BC_N\) type of anti-ferromagnetic PF spin chains through the branched motifs. Moreover, we derive an extended boson-fermion duality relation (7.5) among the restricted super Schur polynomials and show that the partition functions of the \(BC_N\) type of PF spin chains exhibit a similar duality relation.

From Eq. (4.22), one can easily see that the intrinsic degeneracy of the energy level associated with the branched motif \((\delta_1, \delta_2, \ldots, \delta_{N-1})|l\) depends on the discrete parameters \(m_1, m_2, n_1, n_2\) of the Hamiltonian \(H_{N}^{(m_1,m_2|n_1,n_2)}\) in (2.9). It may also be noted that, Eq. (4.11) establishes an interesting connection between the intrinsic degeneracy factors corresponding to the energy levels of the \(BC_N\) type Hamiltonian \(H_{N}^{(m_1,m_2|n_1,n_2)}\) and those of the \(A_{N-1}\) type of Hamiltonian \(H_{N}^{(m|n)}\) in (1.1), where \(m = m_1 + m_2\) and \(n = n_1 + n_2\). By using the above mentioned equations, we find that the spectrum of \(H_{N}^{(m,0|n,0)}\) in (5.21) obeys a selection rule which forbids the occurrence of branched motifs like \((\delta_1, \delta_2, \ldots, \delta_{N-1})|l\) for \(l > 0\). Moreover, the nontrivial intrinsic degeneracy of the branched motif \((\delta_1, \delta_2, \ldots, \delta_{N-1})|0\) coincides with that of the usual motif \((\delta_1, \delta_2, \ldots, \delta_{N-1})\) associated with the Hamiltonian \(H_{N}^{(m|n)}\). Since the latter Hamiltonian exhibits the \(Y(gl(m|n))\) Yangian symmetry, our analysis indirectly proves that the Hamiltonian \(H_{N}^{(m,0|n,0)}\) (and also the related Hamiltonian \(H_{N}^{(0,m|0,n)}\)) has the same symmetry. Our analysis using Eqs. (4.22) and (4.11) also suggests that, except for the above mentioned special cases where either \(m_1 = n_1 = 0\) or \(m_2 = n_2 = 0\), the symmetry algebra of the Hamiltonian \(H_{N}^{(m_1,m_2|n_1,n_2)}\) would be a proper subalgebra of the \(Y(gl(m|n))\) Yangian algebra. Even though the symmetry algebras of some spin Calogero models associated with the \(BC_N\) root system have been studied earlier [55, 56], as far as we know the symmetry algebras of the \(BC_N\) type of PF spin chains like (2.9) have not received much attention. So it might be interesting as a future study to find out the symmetry algebras of these \(BC_N\) type of spin chains and explore how the representations of such symmetry algebras are connected with the \(q\)-deformed and the restricted super Schur polynomials.

Finally, we would like to make a few comments about some further developments which we are making at present by using the results of this paper. First of all, it is possible to extend the connection between \(BC_N\) type of multivariate SRS polynomials and the partition functions of the \(BC_N\) type of PF spin chains even in the presence of the chemical potentials. By using such connection, one can express the complete spectra and partition functions of the \(BC_N\) type of PF spin chains with chemical potentials in terms of some one-dimensional classical vertex models. Moreover, by employing the
transfer matrices associated with those classical vertex models, one can study various thermodynamic properties and criticality of the $BC_N$ type of PF spin chains with chemical potentials. We plan to describe these interesting developments about vertex models and thermodynamic properties of these spin chains in some forthcoming publications.

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