MORE ON GROUPS AND COUNTER AUTOMATA

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Abstract. Elder, Kambites, and Ostheimer showed that if the word problem of a finitely generated group $H$ is accepted by a $G$-automaton for an abelian group $G$, then $H$ is virtually abelian. We give a new, elementary, and purely combinatorial proof to the theorem. Furthermore, our method extracts an explicit connection between the two groups $G$ and $H$ from the automaton as a group homomorphism from a subgroup of $G$ onto a finite index subgroup of $H$.

1. Introduction

For a group $G$, a $G$-automaton is a finite automaton augmented with a register that stores an element of $G$. Such an automaton first initializes the register with the identity element $1_G$ of $G$ and may update the register content by multiplying by an element of $G$ during the computation. The automaton accepts an input word if the automaton can reach a terminal state and the register content is $1_G$ when the entire word is read. (For the precise definition, see Section 2.4.) For a positive integer $n$, $\mathbb{Z}^n$-automata are the same as blind $n$-counter automata, which were defined and studied by Greibach [14,15].

The notion of $G$-automata is discovered repeatedly by several different authors. The name “$G$-automaton” is due to Kambites [22]. (In fact, they defined the notion of $M$-automata for any monoid $M$.) Dassow–Mitrana [8] and Mitrana–Stiebe [26] use extended finite automata (EFA) over $G$ instead of $G$-automata.

For a finitely generated group $G$, the word problem of $G$, with respect to a fixed finite generating set of $G$, is the set of words over the generating set representing the identity element of $G$ (see Section 2.2 for the precise definitions). For several language classes, the class of finitely generated groups whose word problem is in the class is determined [1,2,10,17,19,27,28], and many attempts are made for other language classes [3,4,12,13,20,21,24,25]. One of the most remarkable theorems about word problems is the well-known result due to Muller and Schupp [27], which states that, with the theorem by Dunwoody [9], a group has a context-free word problem if and only if it is virtually free. These theorems suggest deep connections between group theory and formal language theory.

Involving both $G$-automata and word problems, the following broad question was posed implicitly by Elston and Ostheimer [11] and explicitly by Kambites [22].

Question 1. For a given group $G$, is there any connection between the structural property of $G$ and of the collection of groups whose word problems are accepted by non-deterministic $G$-automata?

Note that by $G$-automata, we always mean non-deterministic $G$-automata. As for deterministic $G$-automata, the following theorem is known.

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**Theorem 1** (Kambites [22, Theorem 1], 2006). Let $G$ and $H$ be groups with $H$ finitely generated. Then the word problem of $H$ is accepted by a deterministic $G$-automaton if and only if $H$ has a finite index subgroup which embeds in $G$.

For non-deterministic $G$-automata, several results are known for specific types of groups. For a free group $F$ of rank $\geq 2$, it is known that a language is accepted by an $F$-automaton if and only if it is context-free [5, Proposition 2; 7, Corollary 4.5; 23, Theorem 7]. Combining with the Muller–Schupp theorem, the class of groups whose word problems are accepted by $F$-automata is the class of virtually free groups. The class of groups whose word problems are accepted by $(F \times F)$-automata is exactly the class of recursively presentable groups [7, Corollary 3.5; 23, Theorem 8; 26, Theorem 10]. For the case where $G$ is (virtually) abelian, the following result was shown by Elder, Kambites, and Ostheimer.

**Theorem 2** (Elder, Kambites, and Ostheimer [10], 2008).

1. Let $H$ be a finitely generated group and $n$ be a positive integer. Then the word problem of $H$ is accepted by a $\mathbb{Z}^n$-automaton if and only if $H$ is virtually free abelian of rank at most $n$ [10, Theorem 1].
2. Let $G$ be a virtually abelian group and $H$ be a finitely generated group. Then the word problem of $H$ is accepted by a $G$-automaton if and only if $H$ has a finite index subgroup which embeds in $G$ [10, Theorem 4].

However, their proof is somewhat indirect in the sense that it depends on a deep theorem by Gromov [16], which states that every finitely generated group with polynomial growth function is virtually nilpotent. In fact, their proof proceeds as follows. Let $H$ be a group whose word problem is accepted by a $\mathbb{Z}^n$-automaton. They first develop some techniques to compute several bounds for linear maps and semilinear sets. Then a map from $H$ to $\mathbb{Z}^n$ with certain geometric conditions is constructed to prove that $H$ has polynomial growth function. By Gromov’s theorem, $H$ is virtually nilpotent. Finally, they conclude that $H$ is virtually abelian, using some theorems about nilpotent groups and semilinear sets. Because of the indirectness of their proof, the embedding in Theorem 2 (2) is obtained only \textit{a posteriori} and hence has no relation with the combinatorial structure of the $G$-automaton.

To our knowledge, there are almost no attempts so far to obtain explicit algebraic connections between $G$ and $H$, where $H$ is a group that has a word problem accepted by a $G$-automaton. The only exception is the result due to Holt, Owens, and Thomas [19, Theorem 4.2], where they gave a combinatorial proof to a special case of Theorem 2 (1) for the case where $n = 1$. (In fact, their theorem is slightly stronger than Theorem 2 (1) for $n = 1$ because it is for \textit{non-blind} one-counter automata. See also [10, Section 7].)

In this paper, we give a new, elementary, and purely combinatorial proof to Theorem 2.

**Theorem 3.** Let $G$ be an abelian group and $H$ be a finitely generated group. Suppose that the word problem of $H$ is accepted by a $G$-automaton $A$. Then one can define a finite collection of monoids $(M(\mu,p))_{\mu,p}$, as in Definition 3, such that:

1. Each $M(\mu,p)$ consists of closed paths in $A$ with certain conditions,
2. Each $M(\mu,p)$ induces a group homomorphism $f_{\mu,p}$ from a subgroup $G(\mu,p)$ of $G$ onto a subgroup $H(\mu,p)$ of $H$, and
3. At least one of $H(\mu,p)$’s is a finite index subgroup of $H$.

For the implication from Theorem 3 to Theorem 2 see Section 2.4.
Note that the direction of the group homomorphisms $f_{\rho,\rho}$ in Theorem 3 is opposite to the embeddings in Theorem 1 and Theorem 2. This direction seems more natural for the non-deterministic case; this observation suggests the following question.

**Question 2.** Let $G$ and $H$ be groups with $H$ finitely generated. Suppose that the word problem of $H$ is accepted by a $G$-automaton. Does there exist a group homomorphism from a subgroup of $G$ onto a finite index subgroup of $H$? If so, is it obtained combinatorially?

Our Theorem 3 is the very first step for approaching Question 2. Note that an affirmative answer to Question 2 would generalize Theorem 1.

2. Preliminaries

2.1. Words, subwords, and scattered subwords. For a set $\Sigma$, we write $\Sigma^*$ for the free monoid generated by $\Sigma$, i.e., the set of words over $\Sigma$. For a word $u = a_1a_2\cdots a_n \in \Sigma^*$ ($n \geq 0, a_i \in \Sigma$), the number $n$ is called the length of $u$, which is denoted by $|u|$. For two words $u,v \in \Sigma^*$, the concatenation of $u$ and $v$ are denoted by $u \cdot v$, or simply $uv$. The identity element of $\Sigma^*$ is the empty word, denoted by $\varepsilon$, which is the unique word of length zero. For an integer $n \geq 0$, the $n$-fold concatenation of a word $u \in \Sigma^*$ is denoted by $u^n$. For an integer $n > 0$, we write $\Sigma^{<n}$ for the set of words of length less than $n$.

A word $u \in \Sigma^*$ is a subword of a word $v \in \Sigma^*$, denoted by $u \subseteq v$, if there exist two words $u_1,u_2 \in \Sigma^*$ such that $u_1u_2 = v$. A word $u \in \Sigma^*$ is a scattered subword of a word $v \in \Sigma^*$, denoted by $u \subseteq_{sc} v$, if there exist two finite sequences of words $u_1,u_2,\ldots,u_n \in \Sigma^*$ $(n \geq 0)$ and $v_0,v_1,\ldots,v_n \in \Sigma^*$ such that $u = u_1u_2\cdots u_n$ and $v = v_0u_1v_1u_2v_2\cdots u_nv_n$. That is, $v$ is obtained from $u$ by inserting some words. Note that the two binary relations $\subseteq$ and $\subseteq_{sc}$ are both partial orders on $\Sigma^*$.

2.2. Word problem for groups. Let $H$ be a finitely generated group. A choice of generators for $H$ is a surjective monoid homomorphism $\rho$ from the free monoid $\Sigma^*$ on a finite alphabet $\Sigma$ onto $H$. The word problem of $H$ with respect to $\rho$, denoted by $WP_{\rho}(H)$, is the set of words in $\Sigma^*$ mapped to the identity element $1_H$ of $H$ via $\rho$, i.e., $WP_{\rho}(H) = \rho^{-1}(1_H)$.

Although the word problem $WP_{\rho}(H)$ depends on the choice of generators $\rho$, this does not cause problems:

**Lemma 1** (e.g., [20] Lemma 1). Let $\mathcal{C}$ be a class of languages closed under inverse homomorphisms and let $H$ be a finitely generated group. Then $WP_{\rho}(H) \in \mathcal{C}$ for some choice of generators $\rho$ if and only if $WP_{\rho}(H) \in \mathcal{C}$ for any choice of generators $\rho$. $\square$

This is the reason why we use “the word problem of $H$” rather than “a word problem of $H$.”

2.3. Graphs and paths. A graph is a 4-tuple $(V,E,s,t)$, where $V$ is the set of vertices, $E$ is the set of (directed) edges, $s$: $E \rightarrow V$ and $t$: $E \rightarrow V$ are functions assigning to every edge $e \in E$ the source $s(e) \in V$ and the target $t(e) \in V$, respectively. A graph is finite if it has only finitely many vertices and edges.

A path (of length $n$) in a graph $\Gamma = (V,E,s,t)$ is a word $e_1e_2\cdots e_n \in E^*$ $(n \geq 0)$ of edges $e_i \in E$ such that $t(e_i) = s(e_{i+1})$ for $i = 1,2,\ldots,n-1$. We usually use Greek letters to denote paths in a graph. For a non-empty path $\omega = e_1e_2\cdots e_n \in E^*$, the source and the target of $\omega$ are defined as $s(\omega) = s(e_1)$ and $t(\omega) = t(e_n)$, respectively. If $\omega = e_1e_2\cdots e_n$ and $\omega' = e'_1e'_2\cdots e'_k$ are non-empty paths such that $t(\omega) = s(\omega')$, or at least one of $\omega$ and $\omega'$ is empty, then the concatenation of $\omega$ and $\omega'$, denoted by $\omega \cdot \omega'$ or $\omega\omega'$, is the path $e_1e_2\cdots e_ne'_1e'_2\cdots e'_k$ of length $n+k$, i.e., the concatenation as words. A path $\omega$ in $\Gamma$ is closed if $t(\omega) = s(\omega)$.
if \( s(\omega) = t(\omega) \), or \( \omega = \epsilon \). For a closed path \( \sigma \) and an integer \( n \geq 0 \), we write \( \sigma^n \) for the \( n \)-fold concatenation of \( \sigma \).

For a graph \( \Gamma = (V, E, s, t) \), an edge-labeling function is a function \( \ell \) from \( E \) to a set \( M \). If \( M \) is a monoid and \( \omega = e_1 e_2 \cdots e_n \) is a path in \( \Gamma \), then the label of \( \omega \) is defined as \( \ell(\omega) = \ell(e_1) \ell(e_2) \cdots \ell(e_n) \) via the multiplication of \( M \).

2.4. **G-automata.** For a group \( G \), a (non-deterministic) \( G \)-automaton over a finite alphabet \( \Sigma \) is defined as a 5-tuple \( (\Gamma, \ell_\Sigma, p_{init}, p_{ter}) \), where \( \Gamma = (V, E, s, t) \) is a finite graph, \( \ell_\Sigma : E \to G \) and \( \ell_\Sigma : E \to \Sigma^* \) are edge-labeling functions, \( p_{init} \in V \) is the initial vertex, and \( p_{ter} \in V \) is the terminal vertex. For simplicity, we assume that \( \ell_\Sigma(e) \in \Sigma \cup \{ \epsilon \} \) for each \( e \in E \).

Note that this assumption does not decrease the accepting power of \( G \)-automata. Indeed, if necessary, one can subdivide an edge \( e \) with labels \( \ell_\Sigma(e) = uv, \ell_\Sigma(e) = g \) into two new edges \( e_1, e_2 \) with labels \( \ell_\Sigma(e_1) = u, \ell_\Sigma(e_2) = g \) and \( \ell_\Sigma(e_2) = v, \ell_\Sigma(e_2) = 1_G \). An accepting path in a \( G \)-automaton \( A = (\Gamma, \ell_G, \ell_\Sigma, p_{init}, p_{ter}) \) is a path \( \alpha \) in \( \Gamma \) such that \( s(\alpha) = p_{init}, t(\alpha) = p_{ter} \), and \( \ell_G(\alpha) = 1_G \) (we consider that the empty path \( \epsilon \in E^* \) is accepting if and only if \( p_{init} = p_{ter} \).

We say that a path \( \omega \) in \( \Gamma \) is promising if \( \omega \) is a subword of some accepting path in \( A \), i.e., there exist two paths \( \omega_1, \omega_2 \in E^* \) such that the concatenation \( \omega_1 \omega_2 \in E^* \) is an accepting path in \( A \). The language accepted by a \( G \)-automaton \( A \), denoted by \( L(A) \), is the set of all words \( u \in \Sigma^* \) such that \( u \) is the label of some accepting path in \( A \), i.e., \( L(A) = \{ \ell_\Sigma(\alpha) \in \Sigma^* \mid \alpha \text{ is an accepting path in } A \} \).

**Proposition 1** (e.g., [23, Proposition 2]). For a group \( G \), the class of languages accepted by \( G \)-automata are closed under inverse homomorphisms. \( \square \)

Replacing the register group \( G \) by its finite index subgroup or finite index overgroup does not change the class of languages accepted by \( G \)-automata:

**Proposition 2** (e.g., [10, Proposition 8]). Let \( G \) be a group and \( H \) be a subgroup of \( G \). Then every language accepted by a \( H \)-automaton is accepted by a \( G \)-automaton. If \( H \) has finite index in \( G \), then the converse holds. \( \square \)

3. **Proof of the main theorem**

Throughout this section, we fix an abelian group \( G \), a finitely generated group \( H \), a choice of generators \( \rho : \Sigma^* \to H \), and a \( G \)-automaton \( A = (\Gamma, \ell_G, \ell_\Sigma, p_{init}, p_{ter}) \) such that \( WP_\rho(H) = L(A) \). We write the group operation of \( G \) additively and \( 0_G \) for the identity element of \( G \).

The following lemma is a starting point of our proof.

**Lemma 2.** Let \( \omega \) and \( \omega' \) be paths in \( \Gamma \) such that \( s(\omega) = s(\omega') \) and \( t(\omega) = t(\omega') \), and suppose that \( \omega \) is promising. Then \( \ell_G(\omega) = \ell_G(\omega') \) implies \( \rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega')) \).

**Proof.** Since \( \omega \) is promising, there exist two paths \( \omega_1, \omega_2 \) in \( \Gamma \) such that \( \omega_1 \omega_2 \) is an accepting path in \( A \). It follows from the assumption that \( \ell_G(\omega_1 \omega_2) = \ell_G(\omega_1) + \ell_G(\omega') + \ell_G(\omega_2) = \ell_G(\omega) + \ell_G(\omega_2) = 0_G, \) and \( \omega_1 \omega_2 \) is also an accepting path in \( A \). That is, \( \ell_\Sigma(\omega_1 \omega_2), \ell_\Sigma(\omega_1 \omega_2) \in WP_\rho(H) \), and \( \rho(\ell_\Sigma(\omega_1))\rho(\ell_\Sigma(\omega))\rho(\ell_\Sigma(\omega_2)) = 1_H = \rho(\ell_\Sigma(\omega_1))\rho(\ell_\Sigma(\omega'))\rho(\ell_\Sigma(\omega_2)) \) in \( H \). Thus we have \( \rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega')) \). \( \square \)
Each Lemma 3. Definition 3. For a minimal accepting path are accepting paths in $A$. Definition 2. A $A$ accepting paths in relation $\sqsubseteq$.

Proof. Since both Remarks 1. Note that, by Higman’s lemma [18, Theorem 4.4], the scattered subword relation $\sqsubseteq_{sc}$ on $\Sigma^*$ is a well-quasi-order. In particular, there are only finitely many minimal accepting paths in $A$, and every accepting path on $A$ dominates some minimal accepting path in $A$.

Definition 2. Let $\mu = e_1e_2\cdots e_n \in E^*$ ($e_i \in E$) be a minimal accepting path in $A$. A closed path $\sigma \in E^*$ in $\Gamma$ is pumpable in $\mu$ if there exists an accepting path $\alpha$ in $A$ dominating $\mu$ such that $\alpha = \alpha_0 e_1 \alpha_1 e_2 \alpha_2 \cdots e_n \alpha_n$ for some paths $\alpha_0, \alpha_1, \ldots, \alpha_n \in E^*$ in $\Gamma$ and $\sigma \sqsubseteq \alpha_j$ for some $j \in \{0, 1, \ldots, n\}$.

Remarks 1.

(1) In Definition 2 each $\alpha_i$ is a closed path in $\Gamma$ and satisfies $\ell_G(\alpha_0) + \ell_G(\alpha_1) + \cdots + \ell_G(\alpha_n) = \ell_G(\alpha)$ since $\ell_G(\mu) = 0_G$ and $G$ is abelian.

(2) Every closed path pumpable in a minimal accepting path $\mu$ is promising.

Definition 3. For a minimal accepting path $\mu$ in $A$ and a vertex $p \in V$, define $M(\mu, p) = \{ \sigma \mid \sigma$ is a closed path in $\Gamma$ pumpable in $\mu$ such that $s(\sigma) = p, \text{ or } \sigma = \varepsilon \}$. Lemma 3. Each $M(\mu, p)$ is a monoid with respect to the concatenation operation, i.e., $\sigma_1, \sigma_2 \in M(\mu, p)$ implies $\sigma_1 \sigma_2 \in M(\mu, p)$.

Proof. Since both $\sigma_1$ and $\sigma_2$ are pumpable in $\mu = e_1 e_2 \cdots e_n \in E^*$ ($e_i \in E$), there exist two accepting paths $\alpha = \alpha_0 e_1 \alpha_1 e_2 \alpha_2 \cdots e_n \alpha_n$ ($\alpha_i \in E^*$) and $\beta = \beta_0 e_1 \beta_1 e_2 \beta_2 \cdots e_n \beta_n$ ($\beta_i \in E^*$) such that $\sigma_1 \sqsubseteq \alpha_i$ and $\sigma_2 \sqsubseteq \beta_j$ for some $i, j \in \{0, 1, \ldots, n\}$. Then we have $\alpha_i = \alpha'_i \alpha''_i$ for some $\alpha'_i, \alpha''_i \in E^*$ and $\beta_j = \beta'_j \beta''_j$ for some $\beta'_j, \beta''_j \in E^*$. We may assume that $i \leq j$. Since $G$ is abelian, the merged path $\gamma = (\alpha_0 \beta_0)e_1(\alpha_1 \beta_1)e_2(\alpha_2 \beta_2)\cdots e_n(\alpha_n \beta_n)$ and its permutation (1) $\gamma' = (\alpha_0 \beta_0)e_1(\alpha_1 \beta_1)e_2(\alpha_2 \beta_2)\cdots e_i(\alpha'_i \sigma_1 \alpha''_i)\cdots e_j(\beta'_j \beta''_j)e_{j+1} \cdots e_n(\alpha_n \beta_n)$ are accepting paths in $A$ (Figure 1).

For each $M(\mu, p)$, Lemma 3 allows us to define a surjective monoid homomorphism $f_{\mu, p}: M(\mu, p) \rightarrow \rho(\ell_G(M(\mu, p)))$ as the composition function $\rho \circ \ell_G$. By Lemma 2 $f_{\mu, p}$ induces
a well-defined surjective monoid homomorphism \( \bar{f}_{\mu,p} : \ell_G(M(\mu,p)) \to \rho(\ell_G(M(\mu,p))) \). Let \( G(\mu,p) \) (resp. \( H(\mu,p) \)) denotes the subgroup of \( G \) generated by \( \ell_G(M(\mu,p)) \) (resp. the subgroup of \( H \) generated by \( \rho(\ell_G(M(\mu,p))) \). One can easily extend \( \bar{f}_{\mu,p} \) to a unique surjective group homomorphism \( \bar{f}_{\mu,p} : G(\mu,p) \to H(\mu,p) \). The remaining task is to prove that at least one of the \( (H(\mu,p))'s \) is a finite index subgroup of \( H \).

**Lemma 4.** Each \( M(\mu,p) \) is downward closed with respect to \( \sqsubseteq_{sc} \), i.e., if \( \sigma \) is an element of \( M(\mu,p) \) and \( \tau \) is a closed path in \( \Gamma \) with \( \sigma(\tau) = \mu \) such that \( \tau \sqsubseteq_{sc} \sigma \), then \( \tau \in M(\mu,p) \).

**Proof.** Suppose that \( \tau = e_{1}'e_{2}' \cdots e_{k}' \) \((k \geq 0, e_{i}' \in E)\) and \( \sigma = e_{0}'e_{1}'e_{2}' \cdots e_{k}' \sigma_{k}' \) \((\sigma_{i} \in E^*)\). Each \( \sigma_{i} \) is a closed path in \( \Gamma \). Since, by Lemma 3, \( \sigma_{2}' \) is pumpable in \( \mu = e_{1}e_{2} \cdots e_{n} \) \((n \geq 0, e_{i} \in E)\), there exists an accepting path \( \alpha = e_{0}e_{1}e_{2} \cdots e_{n} \alpha_{n} \) dominating \( \mu \) such that \( \sigma_{2}' \sqsubseteq_{sc} \alpha_{i} \) for some \( i \in \{0,1, \ldots, n\} \). If \( \alpha_{i} = \alpha_{i}' \sigma_{i}^{2} \alpha_{i}'' \), then the path

\[
\gamma = \alpha_{0} e_{1} \alpha_{1} e_{2} \alpha_{2} \cdots e_{i}(\alpha_{i}')^{\tau} \cdot (\sigma_{0}'e_{1}'e_{2}' \cdots e_{k}' \sigma_{k}') \cdot \alpha_{i}'' e_{i+1} \cdots e_{n} \alpha_{n}
\]

is an accepting path in \( A \) (Figure 2).

**Lemma 5.** Let \( \sigma \in M(\mu,p) \) and \( \omega \sqsubseteq \sigma \) be a path. Then there exist two paths \( \omega_{1}, \omega_{2} \in E^{<|V|} \) such that \( \omega_{1}\omega_{2} \in M(\mu,p) \).

**Proof.** Let \((\omega_{1}, \omega_{2}) \in E^{*} \times E^{*}\) be a pair of two paths such that \( \omega_{1}\omega_{2} \in M(\mu,p) \) and \( \max\{|\omega_{1}|,|\omega_{2}|\} \) is minimum. Such a pair exists since \( \omega \sqsubseteq \sigma \in M(\mu,p) \). Suppose the contrary that \( \max\{|\omega_{1}|,|\omega_{2}|\} \geq |V| \), say \( |\omega_{1}| \geq |V| \). By the pigeonhole principle, \( \omega_{1} \) must visit some vertex \( p \in V \) at least twice. That is, there exist three paths \( \alpha, \beta, \gamma \) such that \( \omega_{1} = \alpha \beta \gamma \) and \( \beta \) is a non-empty closed path. Now we have \( \alpha \beta \gamma \omega_{2} \subseteq_{sc} \omega_{1}\omega_{2} \in M(\mu,p) \), and Lemma 4 implies \( \alpha \beta \gamma \omega_{2} \in M(\mu,p) \), which contradicts the minimality of \((\omega_{1}, \omega_{2})\).

**Proof of Theorem 3** Let \( h \in H \) and fix a word \( v \in \Sigma^{*} \) such that \( \rho(v) = h \). There exists a word \( \bar{v} \in \Sigma^{*} \) such that \( v\bar{v} \in WP_{\rho}(H) \). Define

\[
N = 1 + \max\{|\mu| \mid \mu \in E^{*} \text{ is a minimal accepting path in } A\},
\]

and then \( N < \infty \) by Remark 1. Since \((v\bar{v})^{N} \in WP_{\rho}(H)\), there exists an accepting path

\[
\alpha = \omega_{1}\omega_{1}\omega_{2}\cdots \omega_{N}\bar{\omega}_{N}
\]

in \( A \) such that \( \ell_{Ω}(\omega_{i}) = v \) and \( \ell_{Ω}(\bar{\omega}_{i}) = \bar{v} \) for \( i = 1, 2, \ldots, N \). Let \( \mu = e_{1}e_{2} \cdots e_{n} \) \((e_{i} \in E)\) be a minimal accepting path such that \( \alpha \) dominates \( \mu \). Then we have another decomposition

\[
\alpha = \alpha_{0}e_{1}e_{2} \cdots e_{n} \alpha_{n}
\]

for some closed paths \( \alpha_{0}, \alpha_{1}, \ldots, \alpha_{n} \in E^{*} \). Since \( N > n \) and each \( e_{i} \) in the decomposition \( \gamma \) is contained in at most one \( \omega_{i} \) in the decomposition \( \alpha \), at least one of the \( \omega_{i} 's \) is disjoint.

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Figure 2. Construction of the path \( \tau \cdot (\sigma_{0}'e_{1}'e_{2}' \cdots e_{k}' \sigma_{k}') \) in (2)
from all $e_i$’s, i.e., there exist $i \in \{1, 2, \ldots, N\}$ and $j \in \{0, 1, \ldots, n\}$ such that $\omega_i \subseteq \alpha_j$. Since $\alpha_j$ is a pumpable closed path in $\mu$, $\alpha_j$ is an element of $M(\mu, s(\alpha_j))$. By Lemma 3 there exist $\alpha_j', \alpha_j'' \in E^{|V|}$ such that $\alpha_j' \omega_i \alpha_j'' \in M(\mu, s(\alpha_j))$. Then we have $|\ell_\mu(\alpha_j')|, |\ell_\mu(\alpha_j)| < |V|$ and $\rho(\ell_\mu(\alpha_j'))\rho(\ell_\mu(\omega_i))\rho(\ell_\mu(\alpha_j'')) \in H(\mu, s(\alpha))$, hence

$$h = \rho(v) = \rho(\ell_\mu(\omega_i)) \in \rho(\ell_\mu(\alpha_j'))^{-1}H(\mu, s(\alpha))\rho(\ell_\mu(\alpha_j''))^{-1}.$$

From the above argument, we obtain

$$H = \bigcup \left\{ h_1^{-1}H(\mu, p)h_2^{-1} \bigg| \begin{array}{l} \mu \text{ is a minimal accepting path in } A, \\
p \in V, \text{ and } h_1, h_2 \in \rho(\Sigma^{<|V|}) \end{array} \right\},$$

where the right-hand side is a finite union of cosets of $H$ by Remark 1. Thus, by B. H. Neumann’s lemma (1978), no. 3, 311–324. MR513714 ↑

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