The Countable-armed Bandit with Vanishing Arms

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Abstract
We consider a bandit problem with countably many arms, partitioned into finitely many types, each characterized by a unique mean reward. A non-stationary distribution governs the relative abundance of each arm-type in the population of arms, aka the arm-reservoir. This non-stationarity is attributable to a probabilistic leakage of "optimal" arms from the reservoir over time, which we refer to as the vanishing arms phenomenon; this induces a time-varying (potentially "endogenous," policy-dependent) distribution over the reservoir. The objective is minimization of the expected cumulative regret. We characterize necessary and sufficient conditions for achievability of sub-linear regret in terms of a critical vanishing rate of optimal arms. We also discuss two reservoir distribution-oblivious algorithms that are long-run-average optimal whenever sub-linear regret is statistically achievable. Numerical experiments highlight a distinctive characteristic of this problem related to ex ante knowledge of the "gap" parameter (the difference between the top two mean rewards): in contrast to the stationary bandit formulation, regret in our setting may suffer substantial inflation under adaptive exploration-based (gap-oblivious) algorithms such as UCB vis-à-vis their non-adaptive forced exploration-based (gap-aware) counterparts like ETC.

1 Introduction

Background and motivation. The multi-armed bandit (MAB) problem is a widely studied machine learning paradigm encapsulating the tension between exploration and exploitation in online decision making. The earliest studies of this problem were conducted in the context of A/B testing for clinical trials (Thompson, 1933). Various formulations of the problem have since been studied, with applications ranging from scheduling, dynamic pricing, and online auctions to e-commerce and matching markets, among many others (see (Bubeck and Cesa-Bianchi, 2012) for a comprehensive survey). In the simplest MAB formulation, the decision maker (DM) must play at each time instant $t \in \{1, 2, \ldots\}$ one out of a set of $K$ possible arms, each characterized by an associated distribution of rewards. The DM, oblivious to the statistical properties of the arms, must balance exploring new arms and exploiting the best arm played thus far, in order to achieve her objective of maximizing the cumulative expected payoffs over the sequence of play. This objective is often converted to minimizing regret relative to an oracle with perfect ex ante knowledge of the best arm. The seminal work of (Lai and Robbins, 1985) was the first to show that the optimal order of this regret is asymptotically logarithmic in the number of plays in instances with “large” gaps, aka “well-separated” instances. This logarithmic instance-dependent regret should be contrasted with a polynomial “worst-case” instance-independent regret resulting from a minimax formulation of the problem (Theorem 15.2 in (Lattimore and Szepesvári, 2020)). Much of the focus in literature has been on the design and analysis of algorithms that can achieve near-optimal regret rates, both in the instance-dependent and the minimax sense, see, e.g., (Audibert et al., 2009; Russo and Van Roy, 2016; Agrawal and Goyal, 2017), etc.

Towards countably infinite arms. In applications of MAB, it is quite common for the number of arms to be "large" to the extent that it may potentially exceed even the horizon of play. Naturally, the canonical K-armed model is ill-suited for the study of such settings. Among problems of this nature, a simple but illuminating setting is one where an infinite population of arms is partitioned into $K$ different arm-types, each characterized uniquely by some reward statistic (e.g., the mean), and the relative abundance of each arm-type in the reservoir remains fixed over the horizon of play, with a fraction $\alpha$ of them being optimal. This is referred to as the countable-armed bandit (CAB) problem. The CAB setting has strong practical underpinnings in contemporary real-world applications, particularly, online matching markets. In the prototypical assignment problem, for example, the
platform must choose at each instant one out of a large set of agents that may potentially be endowed with a (latent) low-dimensional representation. Such finite-typed population models also appear frequently in the operations research literature in the analysis of systems such as online-retail and recommendation engines, among others, see, e.g., [Bui et al., 2012].

**The “vanishing arms” paradigm.** Beyond countable arms, an additional distinctive feature of many realistic MAB settings is the presence of a temporal variation in the low-dimensional sub-population structure. For example, the relative abundance of agents of different types may change over the horizon of play, even if the agent-types remain fixed. This is the case, for example, in online service-platforms, where arms may model agents capable of abandoning the platform if kept idle for protracted durations, or if the matching of agents to jobs results in an imbalance in the allocation of “similar” agent-types. These effects may catalyze an agent-departure process and a temporal non-stationarity that may seriously hinder the decision maker’s ability to discern “good” arms from “inferior” ones. There has been significant recent interest in non-stationary bandit models, however, most of the literature is largely limited to non-stationarities in rewards, and antecedents on non-stationarities in arm-reservoirs are markedly absent (see §1.1). In this paper, we provide the first systematic treatment of the latter setting by investigating the statistical limits of learning in the CAB problem with vanishing arms.

To distill the key insight pertaining to this setting, it will be convenient to focus on two arm-types in the reservoir; this serves to highlight the key statistical idiosyncrasies of the non-stationary problem and removes unnecessary mathematical detail. The statistical complexity of the “stationary” CAB problem with two arm-types is driven by three primitives: (i) the gap Δ between the mean rewards associated with the two arm-types; (ii) the fraction α of optimal arms in the reservoir; and (iii) the duration of play n. Temporal non-stationarity is incorporated into this model via the vanishing arms characteristic, under which the fraction of optimal arms, α, is no longer constant, but evolves as the (potentially “endogenous,” policy-dependent) stochastic process \( \{\alpha(t) : t = 1, 2, \ldots\} \). Thus, at any time \( t \), the probability that a query to the arm-reservoir returns an arm of the “optimal” type is given by (the sample path-dependent random variable) \( \alpha(t) \), and a “sub-optimal” type with probability \( 1 - \alpha(t) \). The vanishing arms regime corresponds to \( \alpha(t) \to 0 \) as \( t \to \infty \).

**Challenges due to “vanishing arms.”** It is non-trivial to design “good” policies that are reservoir-distribution-agnostic as well as gap-adaptive. In the simplest scenario where \( \alpha(t) \) is constant and ex ante known, an intuitive heuristic is to sample a large enough number of arms initially, and subsequently deploy a conventional bandit algorithm on said consideration set. In the absence of such information, it is unclear what the natural approach should be, and non-stationarity in \( \alpha(t) \) exacerbates the problem further. In classical “fixed-armed” bandit models, it is widely acknowledged that adaptive exploration-based policies such as UCB incur a smaller regret relative to non-adaptive policies such as ETC, that force all the exploration upfront. In contrast, exploration performed towards the end of the horizon (under UCB) seems to be harshly penalized under the vanishing arms paradigm vis-à-vis the initial rounds of exploration (under ETC); this plausibly results in the regret profile shown in Figure 1 (see §3 for details on the algorithms and Appendix A for the precise instance-specification). Figure 1 plots the (empirical) expected regret against the problem horizon under policies derived from the aforementioned families, and highlights the superiority of ETC-based policies over UCB in problems with fast depleting reservoirs (Figure 1(c,d)). The goal of this work is to provide a theoretical explanation for these observations and to investigate the performance limits of said algorithms vis-à-vis \( \alpha(t) \).

![Figure 1: Effects of vanishing rate on regret. Expected regret vs. horizon length under ETC and UCB-based policies when \( \alpha(t) = 0.5 t^{-\gamma} \).](image)

**Contributions.** We first establish a necessary condition for “complete learning.” Specifically, \( \sum_{t \in \mathbb{N}} \alpha(t) = \infty \)
is necessary for achieving sub-linear regret relative to an oracle that knows the identities of the optimal arms ex ante (Theorem 1). Conversely, if said condition is violated, i.e., if \( \{\alpha(t) : t = 1, 2, \ldots\} \) is summable, then linear regret is unavoidable. When sub-linear regret is achievable, we discuss gap-aware Explore-then-Commit (ETC) policies that achieve an expected regret which is poly-logarithmic in the horizon in “well-separated” instances (Theorem 2). In the same regime, we show that a gap-agnostic UCB-type policy achieves a substantially larger polynomial regret (Theorem 3), consistent with the behavior observed in Figure 1. We then consider the setting where \( \alpha(t) \) is “endogenous,” policy-dependent, and characterize sufficient conditions for asymptotic-optimality (sublinear regret) of the aforementioned policies (Theorem 4 and 5). The theoretical and empirical findings in this paper may have significant algorithm-design implications for more general non-stationary bandit problems.

Before proceeding with a formal description of our model, we provide a brief overview of related works below.

### 1.1 Related literature

**Bandits with state-dependent rewards.** The earliest work on MAB problems involving state and control-dependent arm-reward distributions dates back to the seminal paper (Gittins and Jones, 1974). Cited paper studied finite-state Markovian bandits where the state of an arm only changes upon execution of a pull and remains unchanged otherwise, prompting the term resting bandits for such problems. The celebrated Gittins index policy proposed in this paper maximizes the infinite horizon discounted cumulative expected reward. In the so-called restless bandits problem (Whittle, 1988), the states of all the arms may change simultaneously, irrespective of which arms are pulled by the decision maker. Furthermore, this formulation allows for any fixed number of arms to be pulled per period. This work and its follow-ups such as (Weber and Weiss, 1990) focus on heuristics that are optimal under an asymptotic scaling where the number of pulls per period scales linearly with the total number of arms. More recently, a finite-horizon variant of the restless bandits problem was studied in (Hu and Frazier, 2017) under a similar scaling. For a survey of variations of the restless bandits problem, see (Gittins et al., 2011; Zayas-Caban et al., 2019; Brown and Smith, 2020).

Our work is quite distinct from this strand of literature. (i) Asymptotic analyses in preceding papers are w.r.t. the number of arms, not the horizon of play. (ii) The number of pulls per period is fixed at 1 in our problem and does not scale with the number of arms. (iii) Most importantly, cited papers study a Markov Decision Process formulation of the problem where the transition kernels are typically assumed known. We consider a decision maker who is fully oblivious to the statistical properties of the arms as well as to the nature of non-stationarity in the problem (the vanishing arms feature).

**Bandits with non-stationary rewards.** The focus in this literature is on policies that minimize the expected cumulative regret relative to a dynamic oracle that plays at each time \( t \) an arm with the highest mean reward at \( t \). Some of the early work in this area is premised on a formulation where the identity of the best arm may change a finite number of times adversarially during the horizon of play, see, e.g., (Auer et al., 2002b). While other works, e.g., (Slivkins and Upfal, 2008) study specific models of temporal variation, for example, where rewards evolve according to a Brownian motion, much of the traditional work focuses on a finite number of changes in the mean rewards, see, e.g., (Garivier and Moulines, 2011) and references therein. (Besbes et al., 2014) subsequently provided a general framework for studying the aforementioned problem classes by introducing a variation budget that controls and bounds the evolution of mean rewards over the horizon of play. Several other forms of non-stationarity such as rotting rewards (Levine et al., 2017), recharging rewards (Kleinberg and Immorlica, 2018) and more recently, delay-dependent rewards (Cella and Cesa-Bianchi, 2020), have also been studied (see, e.g., (Besbes et al., 2019) for an overview of literature). The fundamental distinction from our work lies in that cited papers study finite-armed models where non-stationarity can be ascribed to changes in arm-means, whereas our work is premised on an infinite-armed formulation (the CAB setting) with non-stationarity attributable to the vanishing arms characteristic. In a nutshell, preceding work focuses on distributional shifts in rewards associated with a fixed set of arms, whereas we propose a new paradigm where the arm-reservoir itself undergoes distributional shifts (the reservoir may “leak” arms), which is functionally a very different concept.

**Bandits with infinitely many arms.** These problems involve an infinite population of arms and a fixed reservoir distribution over a (typically uncountable) set of arm-types; a common reward statistic (usually the mean) uniquely characterizes each arm-type. The infinite-armed bandit problem traces its roots to (Berry et al., 1997) where the problem was studied under Bernoulli rewards and a reservoir distribution of Bernoulli parameters that is Uniform on \([0, 1]\). Subsequent works have considered more general reward and reservoir distributions on
In terms of the statistical complexity of regret minimization, an uncountably rich set of arm-types is tantamount to the minimal achievable regret being polynomial in the number of plays (see cited works). In contrast, the recently introduced countable-armed bandit (CAB) problem (Kalvit and Zeevi, 2020; de Heide et al., 2021) that our work is most closely related to, is fundamentally simpler owing to a finite set of arm-types; this is central to the achievability of logarithmic regret in the CAB problem. Kalvit and Zeevi (2020) study the CAB problem with $K = 2$ types, and propose an online adaptive algorithm that achieves an expected cumulative regret of $O(\log n)$ after any number of plays $n$. Notably, their algorithm does not require ex ante knowledge of (a lower bound on) the fraction $\alpha$ of “optimal” arms in the arm-reservoir, and its regret analysis relies purely on certain novel properties of the UCB algorithm (Auer et al., 2002a); these properties are elucidated in (Kalvit and Zeevi, 2021). The general CAB problem with multiple arm-types was subsequently studied in (de Heide et al., 2021), where it was shown that achieving logarithmic regret absent ex ante knowledge of $\alpha$ is impossible if $K > 2$. Cited paper also proposed a UCB-styled algorithm to achieve $O(\log n)$ expected cumulative regret, assuming ex ante knowledge of a lower bound on $\alpha$.

Our work in this paper investigates the best achievable statistical rates in a non-stationary formulation of the CAB problem characterized by the vanishing arms feature, modeling the natural variations in $\alpha$ over the duration of play – a commonly observed phenomenon in practice.

In addition to the aforementioned works, there are other problem classes also based on infinite population models, e.g., continuum-armed bandits and online stochastic optimization. However, these problems are predicated on an entirely different set of assumptions involving the topological embedding of the arms and functional regularities of the mean-reward, and share little similarity with our stochastic model; see, e.g., (Kleinberg et al., 2008; Hazan, 2019) for a survey.

1.2 Outline of the paper

A formal description of the model is provided in §2 thereafter we discuss two reservoir distribution-oblivious algorithms (non-adaptive and adaptive) in §3. Their performance guarantees and fundamental achievability results under the two reservoir models, exogenous and endogenous (see §4), are stated in §5 and §6 respectively. All proofs are relegated to the appendix.

2 Problem formulation

The set of arm-types is given by $\mathcal{T} := \{1, 2\}$. The decision maker (DM) only knows the cardinality of $\mathcal{T}$, i.e., $K = |\mathcal{T}| = 2$. Each type $i \in \mathcal{T}$ is characterized by a unique mean reward $\mu_i$. Without loss of generality, we assume $\mu_1 > \mu_2$ and call type 1, the optimal type. $\Delta := \mu_1 - \mu_2$ denotes the gap (or separability) between the types (This is an important driver of the statistical complexity of the regret minimization problem.). The horizon of play is $n$, and the DM must play one arm at each time $t \in \{1, ..., n\}$.

The set of arms that have been played up to and including time $t \in \{1, 2, ..., \}$ is denoted by $\mathcal{I}_t$ (and $\mathcal{I}_0 := \emptyset$). The set of actions available to the DM at time $t$ is given by $\mathcal{A}_t = \mathcal{I}_{t-1} \cup \{\text{new}_t\}$: at time $t$, the DM must either play an arm from $\mathcal{I}_{t-1}$ or select the action “new$_t$” which corresponds to playing a new arm queried from the arm-reservoir. This new arm is optimal-typed with probability $\alpha(t)$ and sub-optimal otherwise. Regardless, the DM remains oblivious to the type of a newly queried arm, as well as to the nature and functional form of $\alpha(t)$. A policy $\pi := (\pi_1, \pi_2, ...) \in \Pi$ is an adaptive allocation rule that prescribes at time $t$ an action $\pi_t$ (possibly randomized) based on information contained in $\mathcal{A}_t$. The sequence of rewards realized in the first $k$ pulls of an arm labeled $i$ (henceforth called arm $i$) is denoted by $(X_{i,j})_{1 \leq j \leq k}$. The realized rewards are independent across arms, i.i.d. for each arm, and bounded in $[0, 1]$. The natural filtration at time $t$, denoted by $\mathcal{F}_t$ and defined w.r.t. the sequence of rewards realized up to and including time $t$, is given by $\mathcal{F}_t := \sigma \left\{ (\pi_s)_{1 \leq s \leq t}, (X_{i,j})_{1 \leq j \leq N_i(t)} : i \in \mathcal{I}_t \right\}$ (with $\mathcal{F}_0 := \emptyset$), where $N_i(t)$ denotes the number of pulls of arm $i$ up to and including time $t$. The cumulative pseudo-regret of policy $\pi$ after $n$ plays is defined as $R_n^\pi := \sum_{t=1}^n (\mu_1 - \mu_{\tau(t)})$, where $\tau(t) \in \mathcal{T}$ denotes the type of the arm played by $\pi$ at time $t$; note that $R_n^\pi$ is a sample path-dependent quantity. The DM is interested in the
problem of minimizing the expected cumulative regret\(^2\), given by

\[
\inf_{\pi \in \Pi} \mathbb{E} R_n^\pi = \inf_{\pi \in \Pi} \mathbb{E} \left[ \sum_{t=1}^n (\mu_1 - \mu_{\pi(t)}) \right],
\]

where the infimum is over the set of policies \( \Pi \) satisfying the non-anticipation property \( \pi_t : \mathcal{F}_{t-1} \to \mathcal{P}(\mathcal{A}_t); t \in \{1, 2, \ldots\} \) (\( \mathcal{P}(\mathcal{A}_t) \) denotes the probability simplex on \( \mathcal{A}_t \)), and the expectation in \( \Pi \) is w.r.t. all the possible sources of randomness in the problem (rewards, policy, and the reservoir distribution). We next introduce some asymptotic conventions that will be used in this paper.

**Notation.** We say \( f(n) = o(g(n)) \) or \( g(n) = \omega(f(n)) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \). Similarly, \( f(n) = O(g(n)) \) or \( g(n) = \Omega(f(n)) \) if \( \limsup_{n \to \infty} \frac{f(n)}{g(n)} \leq C \) for some constant \( C \). If \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \) hold simultaneously, we say \( f(n) = \Theta(g(n)) \), or \( f(n) \asymp g(n) \), and we write \( f(n) \sim g(n) \) in the special case where \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \). If either sequence \( f(n) \) or \( g(n) \) is stochastic, and one of the aforementioned ratio conditions holds in probability, we use the subscript “\( p \)” with the corresponding Landau symbol. For example, \( f(n) = o_p(g(n)) \) if \( f(n)/g(n) \xrightarrow{p} 0 \) as \( n \to \infty \). Lastly, the “tilde” notation (e.g., \( \tilde{O}(\cdot) \)) is used to hide logarithmic factors.

**Asymptotically optimal policies.** We will use the standard notion of long-run-average optimality for performance evaluation of candidate policies. Though weaker than rate-optimality, this benchmark serves well to highlight the key statistical features of this new problem class, as well as algorithm-design considerations that separate it from conventional “stationary” bandit models. Formally, a policy \( \pi \) is deemed long-run-average or first-order optimal if it incurs sub-linear regret, i.e., \( \mathbb{E} R_n^\pi = o(n) \), where the \( o(\cdot) \) may hide instance-specific constants. We note that unlike traditional bandit literature which focuses, by and large, on finite-sample guarantees, our results resonate instead with large-sample limits. This is because the limits of statistical learning in the vanishing arms paradigm are best elucidated by way of this asymptotic lens. Furthermore, despite the asymptotic lens, characterization of optimal rates for a given model of \( \alpha(t) \) is reasonably challenging. Instead, the primary objective of this work is to establish necessary conditions on \( \alpha(t) \) for the existence of a first-order optimal policy, and sufficient conditions for first-order optimality of the CAB policies discussed in \(^3\) below.

### 3 Distribution-agnostic policies

As discussed previously, our goal is to investigate the achievable regret when algorithms are agnostic to the reservoir distribution. To this end, we discuss two algorithms (non-adaptive and adaptive) that are known to achieve logarithmic regret when \( \alpha(t) \) remains bounded away from 0 throughout the horizon of play, irrespective of the policy.

#### 3.1 A forced exploration-based policy

The non-adaptive ETC (Explore-then-Commit) approach outlined below is predicated on ex ante knowledge of the problem horizon\(^3\) \( n \) and a gap parameter \( \delta \in (0, \Delta] \). In what follows, a new arm refers to a newly queried arm from the reservoir.

**Policy dynamics.** The horizon is divided into epochs of length \( 2m = \mathcal{O} (\log n) \) each. In each epoch, the algorithm reinitializes by querying the arm-reservoir for a pair of new arms, and playing them \( m \) times each. Subsequently, the pair is classified as either “distinct” or “identical”-typed via a hypothesis test (step 9 of Algorithm \( \text{[1]} \)). If classified as “distinct,” the algorithm commits the residual budget of play to the empirically superior arm among the two (with ties broken arbitrarily). On the other hand, if the pair is classified as “identical,” the algorithm discards it permanently and ushers in a new epoch. This entire process repeats until a “distinct”-typed pair is identified.

\(^2\)the cumulative regret \( \sum_{t=1}^n (\mu_1 - X_{\pi(t)} N_{\pi(t)}(t)) \), has the same expectation as the cumulative pseudo-regret in the stochastic bandits setting.

\(^3\)The exponential doubling trick (Besson and Kaufmann, 2018) can be used to make it horizon-free.
3.2 Towards optimism in exploration

The assumption that the decision maker has ex ante knowledge of (a lower bound on) the gap $\Delta$ is not very practical, and in many applications, this information can only be learnt online. This necessitates designing algorithms that can adapt to the problem complexity online, without compromising on regret performance, relative to their non-adaptive counterparts which rely on ex ante information on $\Delta$. There exists a rich literature on adaptive policies for classical formulations of the MAB problem, however, theoretical development on non-stationarity variations such as the one studied in this paper is in a nascent stage at the moment. The closest related works (de Heide et al., 2021; Kalvit and Zeevi, 2020) propose UCB-styled approaches, but the former’s algorithm cannot be tailored to our setting as it requires knowledge of the minimal and maximal elements in the support of reward distributions; see (Kalvit and Zeevi, 2020) for details. The aforementioned assumption fixes these as $\{0, 1\}$. The implications of Assumption 1 for the performance guarantee of Algorithm 2 are discussed later.

Algorithm 1 ETC-$\infty$(2)

1: Input: Horizon $n$, gap parameter $\delta \in (0, \Delta]$.
2: Set exploration duration $L = \lceil 2\delta^{-2} \log n \rceil$. Set budget $T = n$.
3: Start a new epoch: Play a new arm (call arm 1); observe reward $X_{1,1}$.
4: Play a new arm (call arm 2); observe reward $X_{2,1}$.
5: Consideration set $A = \{1, 2\}$.
6: $m \leftarrow \min (L, T/2)$.
7: Play each arm in $A$ $m - 1$ times; observe rewards $(X_{1,j}, X_{2,j})_{2 \leq j \leq m}$.
8: Update budget: $T \leftarrow T - 2m$.
9: if $\left| \sum_{j=1}^{m} (X_{1,j} - X_{2,j}) \right| < \delta m$ then
10:   Permanently discard $A$ and go to step (3).
11: else
12:   Commit to arm $i^* \in \arg\max_{i \in A} \left\{ \sum_{j=1}^{m} X_{i,j} \right\}$.

Algorithm 2 ALG(Ξ, Φ, 2)

1: Input: A $\Delta$-agnostic policy $\Xi$, a deterministic sequence $\Phi \equiv \{\theta_m \in \mathbb{R}_+ : m = 1, 2, \ldots\}$.
2: Reset clock ($s \leftarrow 0$): At $s = 1$, play a new arm (call arm 1); observe reward $X_{1,1}$.
3: At $s = 2$, play a new arm (call arm 2); observe reward $X_{2,1}$.
4: Consideration set $A = \{1, 2\}$.
5: $m \leftarrow 1$.
6: for $s \in \{3, 4, \ldots\}$ do
7:    if $\left| \sum_{j=1}^{m} (X_{1,j} - X_{2,j}) \right| < \theta_m$ then
8:       Permanently discard $A$ and go to step (2).
9:    else
10:       Play arm $i \in A$ using $\Xi$; observe $X_{i, N_i(s)}$.
11:       $m \leftarrow \min_{i \in A} N_i(s)$.

\footnote{Assumption 1 can be dropped if the reward distributions are sub-Gaussian with an infinite support, e.g., Normally-distributed rewards.}
Discussion of Algorithm 2. ALG(Ξ, Φ, 2) takes as an input, any adaptive policy Ξ for finite-armed bandits such as UCB1 (Auer et al., 2002a), Thompson Sampling (Agrawal and Goyal, 2012), etc., and a sequence of separation thresholds Φ. For example, if Ξ = UCB1, then step (10) translates to playing at clock-tick $t$ an arm $i \in \mathcal{A}$ that maximizes the UCB statistic, i.e.,

$$
i \in \arg \max_{i \in \mathcal{A}} \left( \frac{\sum_{j=1}^{N_i(s-1)} X_{i,j}}{N_i(s-1)} + \sqrt{\frac{2\log(s-1)}{N_i(s-1)}} \right).$$

Like Algorithm 1, ALG(Ξ, Φ, 2) also runs in epochs with exactly one pair of arms played in each epoch. The distinction between the two, however, is three-pronged: firstly, arms are played according to Ξ in any epoch, unlike Algorithm 1 which plays arms equally often in each epoch. Secondly, the separation threshold $\theta_m$ used in step (7) of Algorithm 2, apart from being independent of any information on $\Delta$, also varies over the run of play in any epoch, and is a dynamic proxy for the static $\delta m$ threshold used in step (9) of Algorithm 1. Lastly, albeit Algorithm 2 has an episodic nature just like Algorithm 1, the duration of its epochs is not fixed apriori but determined online in an adaptive fashion.

Specification of Φ. It is intuitive that the $\theta_m$’s in step (7) of Algorithm 2 must satisfy certain properties in order for the algorithm to have a desirable regret performance. There are two possibilities for the $X_{1,j}$’s and $X_{2,j}$’s appearing in this step: (i) $\mathbb{E}[X_{1,j} - X_{2,j}] = 0$, or (ii) $\mathbb{E}[X_{1,j} - X_{2,j}] = \Delta$. In the former case, it is statistically impossible to infer whether $\mathbb{E}X_{1,j} = \mu_1$ or $\mu_2$ in the absence of any information on $\Delta$. Therefore, it is imperative in this case that we discard the consideration set $\mathcal{A}$ at the earliest possible to avoid incurring a large potential regret due to the possibility that $\mathbb{E}X_{1,j} = \mu_2$. Note that in case (i), the Law of the Iterated Logarithm (see Durrett, 2019, Theorem 8.5.2) dictates that $\left| \sum_{j=1}^{m} (X_{1,j} - X_{2,j}) \right| \sim \propto \sqrt{m \log \log m}$. On the other hand, in case (ii), it follows from the Strong Law of Large Numbers that $\left| \sum_{j=1}^{m} (X_{1,j} - X_{2,j}) \right| \sim \Delta m$, almost surely. Thus, a separation of the two cases can be achieved by setting $\theta_m$ such that it satisfies $\theta_m = o(\sqrt{m \log \log m})$ as well as $\theta_m = o(\Delta m)$. The precise specification is provided in (3) (see the appendix), but on a high level our choice of $\theta_m$ satisfies $\theta_m \propto \sqrt{m \log m}$.

Significance of Assumption 1 in specifying Φ. Consider step (7) of Algorithm 2 for $m = 1$. Notice that if $\theta_1$ is strictly larger than the maximum possible value of $|X_{1,1} - X_{2,1}|$, then the algorithm will end up discarding the consideration set $\mathcal{A}$ after just one round of play in every epoch. In such a scenario, linear regret will be unavoidable. Assumption 1 fixes the maximum possible value of $|X_{1,1} - X_{2,1}|$ at 1, irrespective of whether the arms are identical-typed or distinct. Under this assumption, it is possible to ensure that the aforementioned (linear regret) scenario is avoided by initializing $\theta_1$ to a value that is strictly smaller than 1. The exact specification of $\{\theta_m : m = 1, 2, \ldots\}$ is provided in (3) (see the appendix).

4 Modeling the arm-reservoir

The probability $\alpha(t)$ that a newly queried arm is optimal-typed, is expected to have a natural policy-dependence in realistic settings. In the context of matching platforms, the availability of “good” agents in the population will likely depend on their historical experience at the population-level. At any time $t$, a natural statistic to measure the platform’s level of engagement with the population is the number of new “hires,” i.e., reservoir queries until time $t$ (call this $J_t$). We consider two plausible ways in which $\alpha(t)$ may react to $J_t$ and $t$, resulting in an exogenous reservoir, or an endogenous one. In the former setting, $\alpha(t)$ depends only on $t$, i.e., the total time elapsed since the beginning of play; thus, the policy and reservoir are decoupled. This offers an instructive first order understanding of the more complicated model in (ii), where $\alpha(t)$ has dependencies on both $t$ as well as $J_t$.

Model 1 (Exogenous reservoir) $\{\alpha(t) : t = 1, 2, \ldots\}$ is a deterministic process evolving independent of the decision maker’s actions, satisfying $\lim_{t \to \infty} \alpha(t) = 0$.

Model 2 (Endogenous reservoir) $\{\alpha(t) : t = 1, 2, \ldots\}$ is a stochastic process evolving as $\alpha(t) = g(J_{t-1}/(t-1))$ for $t \geq 2$ with $\alpha(1) := c \in (0, 1)$, where $g : [0, 1] \to [0, c]$ is a deterministic, monotone increasing function with $g(0) = 0$, $g(1) = c$, and $J_t$ denotes the (policy and sample path-dependent) number of reservoir queries up to (and including) time $t$. 
Model 1 offers a good first-order understanding of the endogeneity issue in Model 2 by highlighting fundamental characteristics of the learning problem under the *vanishing arms* paradigm, sans any policy-dependencies. Model 2 builds upon this feature to capture the realistic phenomenon that a reservoir (agent-population) queried sufficiently often will retain healthy levels of “good” arms (agents) as they are incentivized to stay on and be available for assignments, owing to the high new-arm selection-rate \( J_i/t \). In Model 2 the condition \( c < 1 \) is clearly necessary for non-triviality of the learning problem; otherwise, the naive policy that simply pulls a new arm at each time step will be (asymptotically) optimal.

## 5 Exogenous reservoirs

In this section, we will focus on settings specified by Model 1. Our first result stated below provides a necessary condition on \( \alpha(t) \) for achievability of sub-linear instance-dependent regret in the problem.

**Theorem 1 (Achieving “complete learning”)** Under Model 1, if \( \sum_{t \in \mathbb{N}} \alpha(t) < \infty \), then the expected cumulative regret of any policy \( \pi \) grows linearly in the number of samples, i.e., \( E R_n^\pi = \Omega(\Delta n) \). Note that the \( \Omega(\cdot) \) only hides multiplicative constants independent of \( \Delta \) and \( n \). Equivalently, a necessary condition for achieving sub-linear regret is \( \sum_{t \in \mathbb{N}} \alpha(t) = \infty \).

In what follows, we show a slightly more refined characteristic: \( \tilde{\Theta}(t^{-1}) \) is, in fact, a critical rate for “complete learning,” in that policies achieving sub-linear regret exist if \( \alpha(t) = \omega((\log t)/t) \).

To elucidate the regret performance of Algorithm 1 and the role of the critical \( \tilde{\Theta}(t^{-1}) \) rate, it will be convenient to consider a parameterization of \( \alpha(t) \) given by \( \alpha(t) \sim ct^{-\gamma} \), where \( c, \gamma \in (0, 1) \) are fixed parameters. This parameterization offers meaningful insights as to the statistical complexity of the problem w.r.t. the control parameters, and facilitates an easier comparison of the regret guarantees of Algorithm 1 and 2. We begin with an upper bound on the theoretical performance of Algorithm 1.

**Theorem 2 (Upper bound for \( ETC^{-\infty}(2) \))** Under Model 1 with \( \alpha(t) \sim ct^{-\gamma} \), the expected cumulative regret of the policy \( \pi \) given by Algorithm 1 satisfies

\[
\limsup_{n \to \infty} \frac{E R_n^\pi}{(\log n)^{1/\gamma}} \leq 24\Delta \left( \frac{8}{\delta^2c} \right)^{1/\gamma} \tilde{\Theta} \left( \left\lceil \frac{\gamma}{1-\gamma} \right\rceil \right),
\]

where \( \tilde{\Theta}(\cdot) \) denotes the factorial function.

The proof of Theorem 2 involves technical details beyond the scope of a succinct discussion here (see the appendix). The above result establishes that the inflation in regret due to *vanishing arms* is at most polynomial in the horizon for “slowly decaying” \( \alpha(t) \), until a sharp phase transition to linear regret occurs around the \( \alpha(t) = \tilde{\Theta}(t^{-1}) \) “critical rate” (Theorem 1).

**Remarks.** (i) **Logarithmic regret.** Theorem 2 suggests that the regret approaches \( \mathcal{O}(\Delta / (\delta^2 c) \log n) \) if the reservoir has no “leakage.” This is consistent with known results for the stationary CAB problem (Kalvit and Zeevi, 2020; de Heide et al., 2021). (ii) **Improving sample-efficiency.** Instead of discarding both arms after step 6 of Algorithm 1 one can, in principle, discard only one arm, and query only one new arm from the reservoir as replacement. The regret of this procedure will differ only in absolute constants. The given design only intends to unify it structurally with Algorithm 2 so as to highlight differences. (iii) **Extension to \( K \) types.** For \( K \)-typed reservoirs, the regret of an appropriate modification of the aforementioned procedure can be shown to scale as \( \mathcal{O}(K \log K) \) (the \( \log K \) factor is non-superfluous and the scaling, in fact, has connections to the “coupon-collector” problem with a universe of size \( K \)). This should be contrasted with the classical \( \mathcal{O}(K) \) scaling achievable in the standard \( K \)-armed formulation.

We next discuss results for Algorithm 2.

**Theorem 3 (Upper bound for \( ALG(UCB1, \Phi, 2) \))** Under Model 2 and Assumption 2, the expected cumulative regret of the policy \( \pi \) given by Algorithm 2 with \( \Xi = UCB1 \) as defined in 2 and \( \Phi = \{\theta_m : m = 1, 2, \ldots\} \) (given explicitly in 3 in Appendix B), satisfies

\[
\limsup_{n \to \infty} \frac{\alpha(n)ER_n^\pi}{\log n} \leq \frac{4}{\Delta \beta \Delta},
\]

\( \beta \) denotes the factorial function.
where \( \beta_\Delta \) is an instance-specific constant that depends only on the reward distributions associated with the arm-types, and \( \beta_\Delta \to 0 \) as \( \Delta \to 0 \). The full specification of \( \beta_\Delta \) is provided in [8] (see the appendix).

**Discussion.** It follows directly from the above result that \( \alpha(n) = \omega(\log n/n) \) is sufficient for ALG(UCB1, \( \Phi \), 2) to be first-order optimal. On the other hand, we have already identified \( \alpha(n) = \omega(1/n) \) as a necessary condition for the existence of a first-order optimal policy (Theorem 1). Thus, the characterization of \( t^{-1} \) as a critical rate in Theorem 1 is sharp up to a logarithmic scaling term. The proof is technical and relegated to the appendix.

**Remarks.** (i) Choice of \( \Xi \). The choice of \( \Xi \) may have significant impact on the performance of Algorithm 2. For example, we observe empirically that the algorithm incurs linear regret if initialized with Thompson Sampling instead of UCB1. (ii) Extension to \( K \) types. Unlike Algorithm 1, Algorithm 2 cannot be directly adapted to the \( K \)-typed problem. In fact, designing a \((\Delta, \alpha)\)-agnostic algorithm with \( \mathcal{O}(\log n) \) regret is impossible for \( K > 2 \) even when \( \alpha(t) = \alpha \) is constant [de Heide et al. 2021]. However, whether a first-order optimal algorithm exists for \( K > 2 \) remains an open problem at present.

**Comparison of theoretical performance.** To facilitate a direct comparison between the upper bounds for the two algorithms, consider \( \alpha(n) = cn^{-\gamma} \) with \( \gamma < 1 \) in Theorem 3. Evidently, ALG(UCB1, \( \Phi \), 2) pays a substantially heavier price for its independence of ex ante information on \( \Delta \), which is reflected in a polynomial \( \tilde{O}(n^\gamma) \) upper bound via-à-vis the poly-logarithmic \( \mathcal{O}\left((\log n)^{1+\epsilon}\right) \) one for ETC-\( \infty \)(2). The fundamental reason underlying this disparity is tied to the exploratory nature of ALG(UCB1, \( \Phi \), 2) owing to UCB being the basal motif. Under non-stationarity, new consideration sets queried in “far future” are almost certainly going to contain arms exclusively of the inferior type (since \( \lim_{t \to \infty} \alpha(t) = 0 \)). In contrast, ETC-\( \infty \)(2) commits to an empirically superior arm earlier, with the error probability vanishing fast enough in the number of pulls to guarantee poly-log regret. This is also evident numerically for large \( \gamma \) (see Figure 1(c,d)).

### 6 Endogenous reservoirs

Unlike the exogenous setting, when \( \alpha(t) \) is no longer guaranteed to be asymptotically vanishing in \( t \) (due to the sampling rate \( J_t/t \) being policy-dependent, possibly even approaching 1 as \( t \to \infty \)), it is clearly difficult to characterize conditions similar to Theorem 1 which would be necessary for achieving sub-linear regret. However, one could read off algorithm-specific sufficient conditions on \( g(\cdot) \) from the corresponding regret bounds stated below.

**Theorem 4 (Upper bound for ETC-\( \infty \)(2))** Under Model 2, the expected cumulative regret of the policy \( \pi \) given by Algorithm 2 satisfies

\[
\limsup_{n \to \infty} g\left(\frac{\delta^2}{3 \log n}\right) \frac{E R_n^\pi}{\log n} \leq \frac{48\Delta}{\delta^2}.
\]

The performance of an appropriately specified Algorithm 2 under this model is stated below.

**Theorem 5 (Upper bound for ALG(UCB1, \( \Phi \), 2))** Under Model 2 and Assumption 1, the expected cumulative regret of the policy \( \pi \) given by Algorithm 2 with \( \Xi = UCB1 \) as defined in 2 and \( \Phi = \{\theta_m : m = 1, 2, \ldots\} \) (given explicitly in 3 in Appendix B), satisfies

\[
\limsup_{n \to \infty} g\left(\frac{1}{n}\right) \frac{E R_n^\pi}{\log n} \leq \frac{4}{\Delta \beta_\Delta},
\]

where \( \beta_\Delta \) is the instance-specific constant appearing in Theorem 3.

**Discussion.** The above result establishes \( g(1/n) = \omega(\log n/n) \) as a sufficient condition for achieving sub-linear regret under Algorithm 2. Contrast this with Theorem 4 which guarantees sub-linear regret under Algorithm 1 even when \( g(\cdot) \) vanishes much faster near 0; to wit, if \( g(x) = cx \) for \( x \in [0, 1] \), then the upper bound in Theorem 4 is linear and therefore vacuous, whereas that in Theorem 4 is \( \mathcal{O}\left((\log n)^2\right) \). Similarly, for \( g(x) = c\sqrt{x} \), the bound in Theorem 5 is \( \mathcal{O}\left(\sqrt{n}\right) \), whereas that in Theorem 4 is \( \mathcal{O}\left((\log n)^2\right) \). These bounds highlight the robustness of Algorithm 1 to non-stationarity as well as endogeneity issues vis-à-vis Algorithm 2. However, unlike said policy-dependent sufficient conditions, a policy-independent necessary condition for sub-linear regret is much
more challenging to derive in this setting owing to the non-monotonicity of the sampling rate. In addition, investigation into the limits of learning under more “reasonable” models of endogeneity also remains an open problem.

A Numerical illustration in Figure 1

The reservoir contains arms with Bernoulli(0.6) and Bernoulli(0.5) rewards. ETC-$\infty(2)$ is initialized with $\delta = \Delta/2 = 0.05$. ALG(UCB1, $\Phi, 2$) is initialized with $\Phi$ specified as per (3) below. Maximum duration of play is $n = 10^6$ pulls. All plots are averaged over 20,000 experiments.

B Specification of $\Phi$ in Algorithm 2

We consider the sequence $\Phi \equiv \{\theta_m \in \mathbb{R}_+ : m = 1, 2, \ldots\}$, where

$$\theta_m := \sqrt{\frac{m^2 (4 \log (m + m_0) + \lambda \log (m + m_0))}{m + m_0}}.$$  \hfill (3)

Here, $m_0 \geq 0$ and $\lambda > 2$ are user-defined parameters to ensure that the sequence \{\theta_m/m : m = 1, 2, \ldots\} is monotone decreasing in $m$ with $\theta_1 < 1$ (the monotonicity condition is not necessary, but it helps keep the proofs simple). For example, $(m_0, \lambda) = (11, 2.1)$ is a valid specification. Ensuring $\theta < 1$ is necessary to avoid linear regret; this is related to Assumption 1 and discussed in §3.2 (main text). Note that the $\theta_m$’s follow the asymptotic $\theta_m \sim 2\sqrt{m \log m}$.

C Proof of Theorem 1

In order to prove this result, we consider an oracle that can perfectly observe whether an arm is “optimal” or “inferior”-typed immediately upon pulling it. If such an oracle incurs linear regret, then every policy that only gets to observe a noisy realization of the mean rewards associated with the types, must necessarily incur linear regret as well.

Clearly, the optimal oracle policy $\pi^*$ is one that keeps pulling new arms until it finds one of the optimal type (type 1), which it then persists with for the remaining duration of play. Let $Y$ denote the time at which an arm of the optimal type is pulled for the first time under $\pi^*$. Then,

$$\mathbb{P}(Y \geq k) = \prod_{t=1}^{k-1} (1 - \alpha(t)) \quad \text{for } k \geq 2, \quad \mathbb{P}(Y \geq 1) = 1.$$  \hfill (4)

The expected cumulative regret of the aforementioned policy at time $n$ is

$$\mathbb{E} R_n^{\pi^*} = \sum_{k=1}^{n} \mathbb{P}(Y = k) \Delta(k - 1) + \mathbb{P}(Y > n) \Delta n > \mathbb{P}(Y > n) \Delta n > \mathbb{P}(Y = \infty) \Delta n.$$  

Thus, if $\mathbb{P}(Y = \infty)$ is bounded away from 0, linear regret is unavoidable. Since $\lim_{t \to \infty} \alpha(t) = 0$, we know that $\exists \ t_0 \in \mathbb{N}$ s.t. $\alpha(t) < 1/2$ for all $t > t_0$. Then,

$$\mathbb{P}(Y = \infty) = \prod_{t=1}^{\infty} (1 - \alpha(t)) = \exp \left( \sum_{t=1}^{\infty} \log (1 - \alpha(t)) \right) = \prod_{t=t_0+1}^{\infty} (1 - \alpha(t)) \exp \left( \sum_{t=t_0+1}^{\infty} \log (1 - \alpha(t)) \right) > \prod_{t=1}^{t_0} (1 - \alpha(t)) \exp \left( -2 \sum_{t=t_0+1}^{\infty} \alpha(t) \right),$$  \hfill (5)

where the final inequality follows since $\alpha(t) < 1/2$ for $t > t_0$ and $\log(1 - x) > -2x$ for $x \in (0, 1/2]$. Since $t_0$ is finite, it is clear from (5) that a sufficient condition for $\mathbb{P}(Y = \infty)$ to be bounded away from 0 is the summability of $\alpha(t)$, i.e., $\sum_{t \in \mathbb{N}} \alpha(t) < \infty.$ \hfill $\square$
D Analysis of Algorithm 1

The horizon of play is divided into epochs of length $2m = 2 \left\lceil (2/\delta^2) \log n \right\rceil$ each (exactly one pair of arms is played $m$ times each per epoch), e.g., epoch 1 starts at $t = 1$, epoch 2 at $t = 2m + 1$, and so on. The decision to commit forever to an empirically superior arm or to discard the consideration set of arms and reinitialize the policy, is taken after an epoch ends. For each $k \geq 1$, let $S_k$ denote the cumulative pseudo-regret incurred by Algorithm 1 when it is initialized at the beginning of epoch $k$ and run until the end of the horizon, i.e., from $t = (2k-2)m + 1$ to $t = n$. Let $S_k'$ denote an i.i.d. copy of $S_k$. We are interested in an upper bound on $\mathbb{E}R_n^\pi = \mathbb{E}S_1$, where $\pi = \text{Algorithm 1}$. To this end, suppose that $\pi$ is initialized at time $(2k-2)m + 1$ (beginning of epoch $k$). Label the arms played in this epoch as $\{1,2\}$ (arm 1 is played first). Let $X_t$ denote the empirical mean reward from the $m$ plays of arm $i \in \{1,2\}$ in this epoch. Let $\tau(i) \in \mathcal{T} = \{1,2\}$ denote the type of arm $i$. Recall that type 1 is optimal and that, the probability of a new arm (queried from the reservoir) at time $t$ being optimal-type is $\alpha(t)$. Let $\mathbb{1}\{E\}$ denote the indicator corresponding to an event $E$. Consider the following events:

\begin{align*}
A := & \{\tau(1) = 1, \tau(2) = 2\}, \\
B := & \{\tau(1) = 2, \tau(2) = 1\}, \\
C := & \{\tau(1) = 2, \tau(2) = 2\}, \\
D := & \{\tau(1) = 1, \tau(2) = 1\},
\end{align*}

where $\tau(1)$ and $\tau(2)$ are independent random variables with distributions given by $\mathbb{P}(\tau(1) = 1) = \alpha((2k-2)m + 1) =: \alpha_k$ and $\mathbb{P}(\tau(2) = 1) = \alpha((2k-2)m + 2) =: \tilde{\alpha}_k$ respectively. Now, observe that $S_k$ evolves according to the following stochastic recursion:

\begin{align*}
S_k = & \mathbb{1}\{A\} \left[ \Delta m + \mathbb{1}\{|\bar{X}_2 - \bar{X}_1| > \delta\} \Delta (n - 2km) + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}' \right] \\
+ & \mathbb{1}\{B\} \left[ \Delta m + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| > \delta\} \Delta (n - 2km) + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}' \right] \\
+ & \mathbb{1}\{C\} \left[ 2\Delta m + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| > \delta\} \Delta (n - 2km) + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}' \right] \\
+ & \mathbb{1}\{D\} \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}'.
\end{align*}

Collecting like terms in (10) together,

\begin{align*}
S_k = & \mathbb{1}\{A\} \mathbb{1}\{|\bar{X}_2 - \bar{X}_1| > \delta\} \Delta (n - 2km) + \mathbb{1}\{B\} \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| > \delta\} \Delta (n - 2km) \\
+ & \mathbb{1}\{C\} \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| > \delta\} \Delta (n - 2km) + \mathbb{1}\{A \cup B\} + 2\mathbb{1}\{C\} \Delta m + \mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}'.
\end{align*}

Define the following conditional events:

\begin{align*}
E_1 := & \{\bar{X}_2 - \bar{X}_1 > \delta \mid A\} , \\
E_2 := & \{\bar{X}_1 - \bar{X}_2 > \delta \mid B\} , \\
E_3 := & \{|\bar{X}_1 - \bar{X}_2| > \delta \mid C\} , \\
E_4 := & \{\bar{X}_1 - \bar{X}_2 < \delta \mid C \cup D\} , \\
E_5 := & \{|\bar{X}_1 - \bar{X}_2| < \delta \mid A \cup B\} .
\end{align*}

Taking expectations on both sides in (11) and rearranging, one obtains the following using (12), (13), (14), (15), (16):

\begin{align*}
\mathbb{E}S_k = & \left[ \alpha_k (1 - \tilde{\alpha}_k) \mathbb{P}(E_1) + \tilde{\alpha}_k (1 - \alpha_k) \mathbb{P}(E_2) \right] \Delta (n - 2km) + \left[ (1 - \alpha_k) (1 - \tilde{\alpha}_k) \mathbb{P}(E_3) \right] \Delta (n - 2km) \\
+ & \left[ \alpha_k (1 - \tilde{\alpha}_k) + \tilde{\alpha}_k (1 - \alpha_k) + 2 (1 - \alpha_k) (1 - \tilde{\alpha}_k) \right] \Delta m + \mathbb{P}\left(\bar{X}_1 - \bar{X}_2 < \delta\right) \mathbb{E}S_{k+1}.
\end{align*}

Note that (17) follows from (11) since $\mathbb{E}\left(\mathbb{1}\{|\bar{X}_1 - \bar{X}_2| < \delta\} S_{k+1}'\right) = \mathbb{P}\left(\bar{X}_1 - \bar{X}_2 < \delta\right) \mathbb{E}S_{k+1}$ due to the independence of $S_{k+1}'$. Further note that

\begin{align*}
\mathbb{P}\left(\bar{X}_1 - \bar{X}_2 < \delta\right) = & \left[ \alpha_k \tilde{\alpha}_k + (1 - \alpha_k) (1 - \tilde{\alpha}_k) \right] \mathbb{P}(E_4) + \left[ \alpha_k (1 - \tilde{\alpha}_k) + \tilde{\alpha}_k (1 - \alpha_k) \right] \mathbb{P}(E_5).
\end{align*}

From (17) and (18), it follows after a little rearrangement that

\begin{align*}
\mathbb{E}S_k = & \xi_1(k) - k\xi_2(k) + \xi_3(k) \mathbb{E}S_{k+1},
\end{align*}

where \( \xi_1(k) \), \( \xi_2(k) \), and \( \xi_3(k) \) are constants.
where the $\xi_i(k)$'s are given by

\begin{align}
\xi_1(k) &:= \Delta [\alpha_k (1 - \tilde{\alpha}_k) \mathbb{P}(E_1) + \tilde{\alpha}_k (1 - \alpha_k) \mathbb{P}(E_2)] n + \Delta \left[(1 - \alpha_k) (1 - \tilde{\alpha}_k) \mathbb{P}(E_3)\right] n + \Delta (2 - \alpha_k - \tilde{\alpha}_k)m, \\
\xi_2(k) &:= 2\Delta [\alpha_k (1 - \tilde{\alpha}_k) \mathbb{P}(E_1) + \tilde{\alpha}_k (1 - \alpha_k) \mathbb{P}(E_2)] m + 2\Delta \left[(1 - \alpha_k) (1 - \tilde{\alpha}_k) \mathbb{P}(E_3)\right] m, \\
\xi_3(k) &:= [\alpha_k \tilde{\alpha}_k + (1 - \alpha_k) (1 - \tilde{\alpha}_k)] \mathbb{P}(E_4) + [\alpha_k (1 - \tilde{\alpha}_k) + \tilde{\alpha}_k (1 - \alpha_k)] \mathbb{P}(E_5).
\end{align}

Observe that the recursion in (14) is solvable in closed-form and admits the following solution:

\begin{equation}
\mathbb{E}S_t = \sum_{k=1}^{l} \left( \xi_1(k) \prod_{j=1}^{k-1} \xi_3(j) \right) - \sum_{k=1}^{l} \left( k\xi_2(k) \prod_{j=1}^{k-1} \xi_3(j) \right) + \mathbb{E}S_{t+1} \left( \prod_{k=1}^{l} \xi_3(k) \right),
\end{equation}

where $l := \lfloor n/(2m) \rfloor$, $\lfloor \cdot \rfloor$ being the “floor” operator. Since the $\xi_i(k)$’s are non-negative for all $i \in \{1, 2, 3\}$, $k \in \mathbb{N}$ and $\mathbb{E}S_{t+1} \leq 2\Delta m$, it follows that

\begin{equation}
\mathbb{E}R_n^* = \mathbb{ES}_1 \leq \sum_{k=1}^{l} \left( \xi_1(k) \prod_{j=1}^{k-1} \xi_3(j) \right) + 2\Delta m,
\end{equation}

where the inequality follows since $\xi_3(k)$ is a convex combination of $\mathbb{P}(E_4)$ and $\mathbb{P}(E_5)$ (see (22)); hence $\xi_3(k) \leq 1 \ \forall \ k \in \mathbb{N}$. Now using (12), (13), (14), (15), (16) and Hoeffding’s inequality (Hoeffding, 1963) together with the fact that the rewards are bounded in $[0, 1]$, we conclude

\begin{align}
\{ \mathbb{P}(E_1), \mathbb{P}(E_2) \} &\leq \exp \left(- (\Delta + \delta)^2 m/2 \right), \\
\{ \mathbb{P}(E_3), \mathbb{P}(E_3^c) \} &\leq 2 \exp \left(- \delta^2 m/2 \right), \\
\mathbb{P}(E_5) &\leq \exp \left(- (\Delta - \delta)^2 m/2 \right). \quad \text{(27)}
\end{align}

From (24), (25) and (26), we then have

\begin{equation}
\xi_1(k) \leq 2\Delta \exp \left(- \delta^2 m/2 \right) n + 2\Delta m \leq 2 \Delta + 2\Delta m,
\end{equation}

where the last inequality follows since $m = \lceil (2/\delta^2) \log n \rceil$, $\lceil \cdot \rceil$ being the “ceiling” operator. Using (24) and (28), we now have

\begin{equation}
\mathbb{E}R_n^* \leq 2\Delta \left[ 1 + \sum_{k=1}^{l} \prod_{j=1}^{k-1} \xi_3(j) \right] (m + 1).
\end{equation}

From (22), observe that

\begin{equation}
\xi_3(k) \leq 1 - (\alpha_k + \tilde{\alpha}_k - 2\alpha_k \tilde{\alpha}_k) \mathbb{P}(E_5^c) \leq \exp \left[-(\alpha_k + \tilde{\alpha}_k - 2\alpha_k \tilde{\alpha}_k) \mathbb{P}(E_5^c) \right] \ \forall \ k \in \mathbb{N},
\end{equation}

where the last inequality follows since $1 - x \leq \exp(-x)$. From (29) and (30), we obtain

\begin{equation}
\mathbb{E}R_n^* \leq 2\Delta \left[ 1 + \sum_{k=1}^{l} \exp \left(- \mathbb{P}(E_5^c) \sum_{j=1}^{k-1} \left( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \right) \right) \right] (m + 1).
\end{equation}

Recall from (24) that $\mathbb{P}(E_5^c) \geq 1 - \exp \left(- (\Delta - \delta)^2 m/2 \right)$. Since $m = \lceil (2/\delta^2) \log n \rceil$, it follows that $\exp \left(- (\Delta - \delta)^2 m/2 \right) < 1/2$ for $n > 2(\frac{1}{\Delta - \delta})^2$. Therefore, for $n$ large enough, we have

\begin{align}
\mathbb{E}R_n^* &\leq 2\Delta \left[ 1 + \sum_{k=1}^{l} \exp \left(- \frac{1}{2} \sum_{j=1}^{k-1} \left( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \right) \right) \right] (m + 1) \\
&\leq 2\Delta \left[ 3 + \sum_{k=3}^{l} \exp \left(- \frac{1}{2} \sum_{j=1}^{k-1} \left( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \right) \right) \right] (m + 1) \\
&\leq 2\Delta \left[ 3 + \sum_{k=3}^{l} \exp \left(- \frac{1}{2} \sum_{j=2}^{k-1} \left( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \right) \right) \right] (m + 1). \quad \text{(31)}
\end{align}

This concludes the basic analysis of Algorithm 1. We will use these results in subsequent sub-sections to provide the proofs for specific functional forms of $\alpha(t)$. 
D.1 Proof of Theorem 2

Recall that \( \alpha_j := \alpha_e((2j - 2)m + 1) \). Since \( \alpha(t) \sim ct^{-\gamma} \), it follows that for \( n \) large enough (equivalently, \( m \) large enough since \( m = \lfloor (2/\delta^2) \log n \rfloor \)), we have \( \alpha_j \leq 1/2 \) for all \( j \geq 2 \), which implies \( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \geq \alpha_j \) for all \( j \geq 2 \). Therefore, it follows from (31) that for \( n \) large enough,

\[
ER_n^\gamma \leq 2\Delta \left[ 3 + \sum_{k=3}^l \exp \left( -\frac{c}{m^{\gamma+1}} \sum_{j=2}^{k-1} j^{-\gamma} \right) \right] (m + 1). \tag{32}
\]

Now, since \( \alpha(t) \sim ct^{-\gamma} \), it follows that for \( n \) large enough (equivalently, \( m \) large enough),

\[
\alpha_j \geq \frac{c}{(2jm)\gamma} \tag{33}
\]

Combining (32) and (33), we get that for \( n \) large enough, \( \alpha_j \geq 2 \), which implies \( \alpha_j + \tilde{\alpha}_j - 2\alpha_j \tilde{\alpha}_j \geq \alpha_j \) for all \( j \geq 2 \).

\[
ER_n^\gamma \leq 2\Delta \left[ 3 + \sum_{k=3}^l \exp \left( -\frac{c}{m^{\gamma+1}} \int_2^k x^{-\gamma} dx \right) \right] (m + 1)
\]

where the last inequality holds since \( m = \lfloor (2/\delta^2) \log n \rfloor \) and \( n \) is large enough. Now observe that

\[
ER_n^\gamma \leq 6\Delta \left[ 1 + \sum_{k=3}^l \exp \left( -\frac{ck^{1-\gamma}}{4(1-\gamma)m^\gamma} \right) \right] (m + 1), \tag{35}
\]

where the last inequality follows since \( l = \lfloor n/(2m) \rfloor \). We now focus on solving the integral. Define

\[
I := \int_2^{n/(2m)} \exp \left( -\frac{cx^{1-\gamma}}{4(1-\gamma)m^\gamma} \right) dx
\]

\[
\leq \int_0^\infty \exp \left( -\frac{cx^{1-\gamma}}{4(1-\gamma)m^\gamma} \right) dx
\]

\[
= (1-\gamma)^{1-\gamma} \left( \frac{4m^\gamma}{c} \right)^{1-\gamma} \int_0^\infty z^{1-\gamma} \exp(-z)dz
\]

\[
\leq \left( \frac{4}{c} \right)^{1-\gamma} \left( \int_0^\infty z^{1-\gamma} \exp(-z)dz \right) m^{1-\gamma}
\]

\[
\leq \left( \frac{4}{c} \right)^{1-\gamma} \left( \int_0^\infty z^{1-\gamma} \exp(-z)dz \right) m^{1-\gamma}
\]
where (4) follows after the variable substitution $z = \frac{e^{x+\gamma}}{a(1-\gamma)m}$. Now, the $\frac{\gamma}{1-\gamma}$th moment of a unit rate exponential random variable is given by the factorial of $\frac{\gamma}{1-\gamma}$, denoted by $\mathfrak{F}\left(\frac{\gamma}{1-\gamma}\right)$. Thus, we have

$$I \leq \left(\frac{4}{c}\right)^{\frac{1}{\gamma}} \mathfrak{F}\left(\left[\frac{\gamma}{1-\gamma}\right]\right) m^{\frac{1}{\gamma}}.$$

Combining (36) and (37), we conclude that for large enough $m$,

$$\mathbb{E}R_n^\pi \leq 24\Delta \left(\frac{4}{c}\right)^{\frac{1}{\gamma}} \mathfrak{F}\left(\left[\frac{\gamma}{1-\gamma}\right]\right) m^{\frac{1}{\gamma}}.$$

Finally, since $m = \lceil (2/\delta^2) \log n \rceil$, the stated assertion follows, i.e.,

$$\limsup_{n \to \infty} \frac{\mathbb{E}R_n^\pi}{(\log n)^{\frac{1}{\gamma}}} \leq 24\Delta \left(\frac{8}{\delta^2c}\right)^{\frac{1}{\gamma}} \mathfrak{F}\left(\left[\frac{\gamma}{1-\gamma}\right]\right).$$

\[ \square \]

D.2 Proof of Theorem 4

Again, we pick things up from (31). We know that

$$\mathbb{E}R_n^\pi \leq 2\Delta \left[3 + \sum_{k=3}^{t} \exp\left(-\frac{1}{2} \sum_{j=2}^{k-1} (\alpha_j + \bar{\alpha}_j - 2\alpha_j\bar{\alpha}_j)\right)\right] (m+1),$$

where $m = \lceil (2/\delta^2) \log n \rceil$, $\alpha_j = \alpha ((2j-2)/m) + 1$ and $\bar{\alpha}_j = \alpha ((2j-2)/m + 2)$. Since $\alpha(t) = g(Jt^{-1}/(t-1))$, we have $\alpha_j = g((2j-2)/(2j-2)m) = g(1/m)$ and $\bar{\alpha}_j = g((2j-1)/[(2j-2)m + 1])$. Since $g(1/m)$ is asymptotically vanishing in $m$ (equivalently in $n$), it follows that for $n$ large enough, $\alpha_j \leq 1/2$ for all $j \geq 2$, which implies $\alpha_j + \bar{\alpha}_j - 2\alpha_j\bar{\alpha}_j \geq \alpha_j = g(1/m)$ for all $j \geq 2$. Therefore, it follows that for $n$ large enough,

$$\mathbb{E}R_n^\pi \leq 2\Delta \left[3 + \sum_{k=3}^{t} \exp\left(-\frac{1}{2} \sum_{j=2}^{k-1} g(1/m)\right)\right] (m+1)$$

$$= 2\Delta \left[3 + \sum_{k=3}^{t} \exp\left(-\frac{g(1/m)(k-2)}{2}\right)\right] (m+1)$$

$$= 2\Delta \left[3 + \sum_{k=1}^{t-2} \exp\left(-\frac{g(1/m)k}{2}\right)\right] (m+1)$$

$$\leq 2\Delta \left[3 + \frac{\exp(-g(1/m)/2)}{1 - \exp(-g(1/m)/2)}\right] (m+1)$$

$$= 2\Delta \left[3 + \frac{1}{\exp(g(1/m)/2) - 1}\right] (m+1).$$

Using $e^x - 1 \geq x$ in the above inequality, we obtain

$$\mathbb{E}R_n^\pi \leq 2\Delta \left[3 + \frac{1}{g(1/m)/2}\right] (m+1) \leq \frac{16\Delta m}{g(1/m)},$$

where the last inequality holds for $m$ large enough (equivalently, $n$ large enough). Finally, using $m = \lceil (2/\delta^2) \log n \rceil$ and the fact that $g(\cdot)$ is monotone increasing, we conclude that for $n$ large enough,

$$g\left(\frac{\delta^2}{3 \log n}\right) \mathbb{E}R_n^\pi \leq \frac{48\log n}{\delta^2}.$$
E Auxiliary results used in the analysis of Algorithm 2

Below, we state two results from (Kalvit and Zeevi, 2020) that will be useful towards the analysis of Algorithm 2 that follows in §C.

Fact 1 (Simplified version of Proposition 1 (Kalvit and Zeevi, 2020)) Consider two distributions \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) satisfying Assumption 1 (see main text). Suppose that \( \mathcal{Y}_i \) denotes the distribution of rewards associated with type \( i \) arms, with mean \( \mu_i \). Define \( \Delta := \mu_1 - \mu_2 > 0 \). Let \( (Y_{1,j})_{j \geq 1} \) denote an i.i.d. sequence of samples from distribution \( \mathcal{Y}_1 \). Consider the sequence \( \{\theta_m : m = 1, 2, \ldots\} \) specified by \( \mathcal{Y}_1 \). Then, the following probability is bounded away from 0 by a constant that depends exclusively on \( \Delta \):

\[
\beta_{\Delta} := \mathbb{P} \left( \bigcap_{m=1}^{\infty} \sum_{j=1}^{m} (Y_{1,j} - Y_{2,j}) \geq \theta_m \right) > 0. \tag{38}
\]

In particular, \( \beta_{\Delta} \) satisfies \( \lim_{\Delta \to 0} \beta_{\Delta} = 0 \).

Fact 2 (Lemma 2 of Kalvit and Zeevi (2020)) Consider a stochastic two-armed bandit with rewards satisfying Assumption 1 (see main text). Suppose that \( \Delta \geq 0 \) denotes the difference between the mean rewards of the two arms. Let the reward sequence associated with arm \( i \in \{1, 2\} \) be denoted by \( (X_{i,j})_{j \geq 1} \). Let \( N_i(n) \) denote the number of pulls of arm \( i \) until time \( n \) under UCB1 (Auer et al., 2002a), and define \( M_n := \min(N_1(n), N_2(n)) \).

Consider the following stopping time:

\[
T_1 := \inf \left\{ n \geq 2 : \sum_{k=1}^{M_n} (X_{1,k} - X_{2,k}) < \theta_{M_n} \right\},
\]

where \( \{\theta_m : m = 1, 2, \ldots\} \) is specified as per (38). Then, the following results hold:

1. If \( \Delta > 0 \), then \( \mathbb{P}(T_1 = \infty) \geq \beta_{\Delta} > 0 \), where \( \beta_{\Delta} \) is as given in (38).

2. If \( \Delta = 0 \), then \( \mathbb{E}T_1 \leq C_0 < \infty \), where \( C_0 \) is a constant that only depends on the user-defined parameters \((m_0, \gamma)\) in the specification of \( \{\theta_m : m = 1, 2, \ldots\} \) in (38).

We will not provide the proofs for these results (Fact 1 and 2); the reader is referred to (Kalvit and Zeevi, 2020) for the same.

F Analysis of Algorithm 2

Algorithm 2 runs in epochs of stochastic durations determined online adaptively. Let the sequence of epoch durations be \( \{T_k ; k = 1, 2, \ldots\} \). Define \( W_n := \inf \left\{ l \in \mathbb{N} : \sum_{k=1}^{l} T_k \geq n \right\} \). Let \( I_k(i) \) denote the event that each of the two arms queried at the beginning of epoch \( k \) has type \( i \) (\( i \) is an element of \( \mathcal{T} = \{1, 2\} \)). Similarly, let \( D_k \) denote the event that the same two arms have “distinct” types. Suppose that \( S_n \) denotes the cumulative pseudo-regret of UCB1 (Auer et al., 2002a) after \( n \) plays in an independent instance of a two-armed bandit with gap \( \Delta \). Then, the cumulative pseudo-regret \( R_n^x \) of \( x = \text{Algorithm 2} \) after any number of plays \( n \), satisfies

\[
R_n^x = \sum_{k=1}^{W_n-1} \left[ \mathbb{1}\{D_k\} S_{T_k} + \mathbb{1}\{I_k(2)\} \Delta T_k \right] + \mathbb{1}\{D_{W_n}\} S_{(n-\sum_{k=1}^{W_n-1} T_k)} + \mathbb{1}\{I_{W_n}(2)\} \Delta \left( n - \sum_{k=1}^{W_n-1} T_k \right)
\]

\[
\leq \sum_{k=1}^{W_n} \left[ \mathbb{1}\{D_k\} S_n + \mathbb{1}\{I_k(2)\} \Delta T_k \right] + \mathbb{1}\{D_{W_n}\} S_n + \mathbb{1}\{I_{W_n}(2)\} \Delta T_{W_n}
\]

\[
= \sum_{k=1}^{W_n} \left[ \mathbb{1}\{D_k\} S_n + \mathbb{1}\{I_k(2)\} \Delta T_k \right]
\]

\[
= \sum_{k=1}^{\infty} \mathbb{1}\{W_n \geq k\} \left[ \mathbb{1}\{D_k\} S_n + \sum_{k=1}^{\infty} \mathbb{1}\{W_n \geq k\} \mathbb{1}\{I_k(2)\} \Delta T_k \right]
\]
where (†) follows since the pseudo-regret of UCB1 is weakly increasing in the horizon, and \( n - \sum_{k=1}^{W_n-1} T_k \leq T_{W_n} \). Taking expectations and invoking Tonelli’s theorem to interchange expectation and infinite-sum (since all summands are non-negative), we obtain

\[
\mathbb{E}R_n^* \leq \mathbb{E}S_n \sum_{k=1}^{\infty} \mathbb{P}(D_k, W_n \geq k) + \Delta \sum_{k=1}^{\infty} \mathbb{P}(I_k(2), W_n \geq k) \mathbb{E}[T_k \mid I_k(2), W_n \geq k]. \tag{39}
\]

(Note that \( \mathbb{E}[S_n \mid W_n \geq k, D_k] = \mathbb{E}S_n \) since \( S_n \), by definition, is independent of \( W_n \) and \( D_k \).)

**Lemma (I).** The following is true for all \( k, n \in \mathbb{N} \):

\[
\mathbb{E}[T_k \mid I_k(2), W_n \geq k] = \mathbb{E}[T_k \mid I_k(2)] \leq C_0 < \infty,
\]

where \( C_0 \) is as given in Fact 2.

**Proof.** Note that

\[
\mathbb{E}[T_k \mid I_k(2)] = \mathbb{E}[T_k \mid I_k(2), W_n \geq k] \mathbb{P}(W_n \geq k \mid I_k(2)) + \mathbb{E}[T_k \mid I_k(2), W_n < k] \mathbb{P}(W_n < k \mid I_k(2))
\]

(\( \dagger \))

\[
\Rightarrow \mathbb{E}[T_k \mid I_k(2), W_n \geq k] = \mathbb{E}[T_k \mid I_k(2)] \leq C_0,
\]

where (†) follows since \( T_k \) is independent of \( W_n \), given \( I_k(2) \) and \( k > W_n \); the last inequality follows using Fact 2. \( \square \)

Now, from (39) and Lemma I, it follows that

\[
\mathbb{E}R_n^* \leq \mathbb{E}S_n \sum_{k=1}^{\infty} \mathbb{P}(D_k, W_n \geq k) + C_0 \Delta \sum_{k=1}^{\infty} \mathbb{P}(I_k(2), W_n \geq k). \tag{40}
\]

Note that

\[
\mathbb{P}(W_n \geq k) = \mathbb{P}\left(\sum_{m=1}^{k-1} T_m < n\right)
\]

\[
= \mathbb{P}\left(\bigcap_{j=1}^{k-1} \left\{ \sum_{m=1}^{j} T_m < n \right\} \right)
\]

\[
= \mathbb{P}(T_1 < n) \prod_{j=2}^{k-1} \mathbb{P}\left(\sum_{m=1}^{j} T_m < n \mid T_1 < n, \ldots, \sum_{m=1}^{j-1} T_m < n \right)
\]

\[
= \mathbb{P}(T_1 < n) \prod_{j=2}^{k-1} \mathbb{P}\left(\sum_{m=1}^{j} T_m < n \mid \sum_{m=1}^{j-1} T_m < n \right)
\]

\[
\leq \mathbb{P}(T_1 < \infty) \prod_{j=2}^{k-1} \mathbb{P}\left(T_j < \infty \mid \sum_{m=1}^{j-1} T_m < n \right). \tag{41}
\]

This concludes the basic analysis of Algorithm 2. We will use these results in subsequent sub-sections to provide the proofs for specific functional forms of \( \alpha(t) \).

**F.1 Proof of Theorem 3**

In this setting, \( \{\alpha(t) : t = 1, 2, \ldots\} \) is a deterministic process with \( \lim_{t \to \infty} \alpha(t) = 0 \).
In this setting, \(F.2\) Proof of Theorem 5

\[
P(T_j < \infty | E_i) = P(T_j < \infty | E_i, I_j(1) \cup I_j(2)) P(I_j(1) \cup I_j(2) | E_i) + P(T_j < \infty | E_i, D_j) P(D_j | E_i)
\]

\[
\leq P(I_j(1) \cup I_j(2) | E_i) + P(T_j < \infty | D_j) P(D_j | E_i)
\]

\[
= 1 - \P(D_j | E_i) \beta\Delta,
\]

\[
= 1 - [\alpha(l + 1) (1 - \alpha(l + 2)) + \alpha(l + 2) (1 - \alpha(l + 1))] \beta\Delta,
\]

where (\(t\)) follows because \(T_j\) is independent of \(E_i\), given \(D_j\), and (\(\dagger\)) follows using Fact 2.1. For \(n\) large enough, both \(\alpha(l + 1) (1 - \alpha(l + 2))\) as well as \(\alpha(l + 2) (1 - \alpha(l + 1))\) are lower bounded by \(\alpha(n) (1 - \alpha(n))\) since \(\alpha(\cdot)\) is asymptotically vanishing. Therefore, for \(n\) large enough, we have that \(P(T_j < \infty | E_i) \leq 1 - 2\alpha(n) (1 - \alpha(n)) \beta\Delta\)

\[
\text{when } j \geq 2.
\]

The same bound trivially holds for \(j = 1\) as well: For \(j = 1\), we have \(P(T_1 < \infty) \leq 1 - [\alpha(1) (1 - \alpha(2)) + \alpha(2) (1 - \alpha(1))] \beta\Delta \leq 1 - 2\alpha(n) (1 - \alpha(n)) \beta\Delta\) for \(n\) large enough. Using this result in (41), it follows that for \(n\) large enough,

\[
P(W_n \geq k) \leq (1 - 2\alpha(n) (1 - \alpha(n)) \beta\Delta)^{k-1}.
\]

Now, we know from (40) that

\[
\mathbb{E} R_n^T \leq \mathbb{E} S_n \sum_{k=1}^{\infty} P(W_n \geq k) + C_0 \Delta \sum_{k=1}^{\infty} P(W_n \geq k)
\]

\[
\leq \frac{\mathbb{E} S_n + C_0 \Delta}{2\alpha(n) (1 - \alpha(n)) \beta\Delta},
\]

where the last step holds for \(n\) large enough (follows using (42)). Finally, using \(\mathbb{E} S_n \leq (8/\Delta) \log n + (1 + \pi^2/3) \Delta\) (Auer et al. 2002a) and taking appropriate limits, the stated assertion follows. \(\square\)

### F.2 Proof of Theorem 5

In this setting, \(\{\alpha(t) : t = 1, 2, \ldots\}\) is a stochastic process that evolves as \(\alpha(t) = g(\mathcal{J}_{t-1} / (t-1))\) for \(t \geq 2\) with \(\alpha(1) := c \in (0,1)\), where \(g : [0,1] \mapsto [0,c]\) is a deterministic monotone increasing function satisfying \(g(0) = 0\) and \(g(1) = c\). Consider the following for \(j \geq 2\):

\[
P(T_j < \infty | \sum_{m=1}^{j-1} T_m < n)
\]

\[
= E \left[ P(T_j < \infty | T_1, \ldots, T_{j-1}) \left| \sum_{m=1}^{j-1} T_m < n \right. \right]
\]

\[
\leq E \left[ 1 - \left\{ \alpha \left( 1 + \sum_{m=1}^{j-1} T_m \right) \left( 1 - \alpha \left( 2 + \sum_{m=1}^{j-1} T_m \right) \right) + \alpha \left( 2 + \sum_{m=1}^{j-1} T_m \right) \left( 1 - \alpha \left( 1 + \sum_{m=1}^{j-1} T_m \right) \right) \right\} \beta\Delta \left| \sum_{m=1}^{j-1} T_m < n \right. \right]
\]

\[
\leq 1 - 2g \left( \frac{1}{n} \right) \left( 1 - g \left( \frac{1}{n} \right) \right) \beta\Delta,
\]

where (\(t\)) follows from Fact 2.1 after conditioning on \(I_j(1) \cup I_j(2)\) and \(D_j\), and arguing in a manner similar to step (\(\dagger\)) in the proof of Theorem 3; (\(\dagger\)) holds for \(n\) large enough since \(g(\cdot)\) is monotone increasing with \(g(0) = 0\).
From (41) and (43), we therefore have that for $n$ large enough,

$$
P(W_n \geq k) \leq P(T_1 < \infty) \left(1 - 2g \left(\frac{1}{n}\right) \left(1 - g \left(\frac{1}{n}\right)\right) \beta\Delta\right)^{k-2}
\leq (1 - 2c \left(1 - c\right) \beta\Delta) \left(1 - 2g \left(\frac{1}{n}\right) \left(1 - g \left(\frac{1}{n}\right)\right) \beta\Delta\right)^{k-2}
\leq \left(1 - 2g \left(\frac{1}{n}\right) \left(1 - g \left(\frac{1}{n}\right)\right) \beta\Delta\right)^{k-1},
$$

(44)

where the last step again follows for $n$ large enough. Combining (40) and (44), we obtain for $n$ large enough that

$$
ER_n^x \leq ES_n \sum_{k=1}^{\infty} P(W_n \geq k) + C_0 \Delta \sum_{k=1}^{\infty} P(W_n \geq k) \leq \frac{ES_n + C_0 \Delta}{2g \left(\frac{1}{n}\right) \left(1 - g \left(\frac{1}{n}\right)\right) \beta\Delta}.
$$

Finally, using $ES_n \leq (8/\Delta) \log n + (1 + \pi^2/3) \Delta$ (Auer et al., 2002a) and taking appropriate limits, the stated assertion follows.

□

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