Penrose Inequality for Asymptotically AdS Spaces

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Abstract

In general relativity, the Penrose inequality relates the mass and the entropy associated with a gravitational background. If the inequality is violated by an initial Cauchy data, it suggests a creation of a naked singularity, thus providing means to consider the cosmic censorship hypothesis. We propose a general form of Penrose inequality for asymptotically locally $AdS$ spaces.

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I. INTRODUCTION

The Penrose inequality [1] (for reviews see e.g. [2, 3]) relates the mass and the horizon area of Cauchy initial data that if violated suggests a creation of a naked singularity. Consider, for instance, the evolution in time of an asymptotically flat initial Cauchy data with a mass $M_i$ and horizon area $A_i$. The solution to Einstein equations with this initial data is expected to settle down at a late time to a Kerr black hole with mass $M$ and horizon area $A$. The Kerr black hole satisfies the inequality (we use $c = G_N = 1$) $M \geq \sqrt{A/6\pi}$, which is saturated by the Schwarzschild black hole. The event horizon area does not decrease with time $A \geq A_i$ [4], while the mass cannot increase and may only decrease due to radiation loss $M \leq M_i$. Thus,

$$M_i \geq M \geq \sqrt{A/16\pi} \geq \sqrt{A_i/16\pi},$$

and the initial Cauchy data should also satisfy the Penrose inequality $M_i \geq \sqrt{A_i/16\pi}$. The argument relies on the Hawking area theorem and the relaxation at late times to a Kerr solution, both assume the weak censorship hypothesis (for a review see [5]). Therefore, a Cauchy data that violates the Penrose inequality is likely to generate a naked singularity.

In view of the AdS/CFT correspondence (for a review see [6]) and the interest in higher-dimensional black holes, it is of interest to have a general form of Penrose inequality for asymptotically $AdS$ spaces. A special form of such inequality has been considered recently in the context of the fluid/gravity correspondence in [7]. The purpose of this note is to construct a general form of the Penrose inequality for asymptotically locally $AdS_{d+1}$ spaces, with electrical charge and a general boundary topology.

Consider the Einstein equations

$$G_{AB} + \Lambda g_{AB} + 8\pi T_{AB} = 0, \quad A, B = 0, \ldots, d,$$

where $G_{AB}$ is the Einstein tensor, $\Lambda = -\frac{d(d-1)}{2l^2}$ is a negative cosmological constant and $T_{AB}$ is the stress-energy tensor for the electromagnetic field $F^{AB}$, whose field equation is $\nabla_A F^{AB} = 0$. The equilibrium background that saturates the inequality is the electrically charged black brane solution with charge $q$. Its background takes the form

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{k, d-1}^2, \quad F^{tr} = \frac{q}{r^{d-1}},$$

where

$$f = k + \frac{r^2}{l^2} - \frac{m}{r^{d-2}} + \left(\frac{q}{r^{d-2}}\right)^2,$$
and \(d \Omega_{k,d-1}^2\) is the \((d - 1)\)-dimensional metric on a flat space, a sphere or hyperboloid for \(k = 0\), \(k = 1\) and \(k = -1\) respectively. We denote the corresponding \((d - 1)\)-dimensional volume by \(\Omega_{k,d-1}\).

In order to derive the Penrose inequality we have to construct the entropy \(S\) and mass \(M\) associated with the background. The entropy density \(s\) reads

\[
s = \frac{R^{d-1}}{4},
\]

where \(R\) is the horizon location, \(f(r = R) = 0\).

The energy density \(\varepsilon\) can be obtained from the \(T_{00}\) component of the \(d\)-dimensional boundary stress-energy tensor. One writes the \((d + 1)\)-dimensional metric in an ADM-like decomposition

\[
ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr),
\]

where \(\gamma_{\mu\nu}\) is the boundary \((r = const.)\) metric

\[
\gamma_{\mu\nu} dx^\mu dx^\nu = -f dt^2 + r^2 d \Omega_{k,d-1}^2,
\]

and \(N\) and \(N^\mu\) are the lapse and shift functions. The boundary stress-energy tensor takes the form

\[
T_{\mu\nu} = \frac{1}{8\pi} \lim_{r \to \infty} r^{d-2} (\theta_{\mu\nu} - \theta \gamma_{\mu\nu}) + \text{counter terms},
\]

where \(\theta_{\mu\nu}\) is the extrinsic curvature tensor of the boundary

\[
\theta_{\mu\nu} = -\frac{1}{2} \sqrt{f} \partial_r \gamma_{\mu\nu},
\]

and \(\theta = \gamma^{\mu\nu} \theta_{\mu\nu}\).

One gets

\[
\varepsilon = \frac{(d - 1)}{16\pi} \left[ \frac{q^2}{R^{d-2}} + k R^{d-2} + \frac{R^d}{l^2} \right] + \varepsilon_0,
\]

where \(\varepsilon_0\) is the Casimir energy

\[
\varepsilon_0 = \frac{1}{8\pi} (-k)^{d/2} (d-1)!2^{d-2} d^{d-2}
\]

for \(d\) even and zero for \(d\) odd.

Using (5) and (10) we get the equilibrium relation between the energy density, the entropy density and electric charge:

\[
\varepsilon = \varepsilon_0 + \frac{d - 1}{16\pi} \left[ q^2 \left( \frac{1}{4s} \right)^{d-2} + k \left( \frac{d-1}{4s} \right) \left( \frac{d-1}{4s} + 1 \right) \right].
\]
Going away from equilibrium we obtain the inequality relation among the mass $M$, area $A$ and charge $q$

$$M - M_0 \geq \frac{(d-1)\Omega_{d-1,k}}{16\pi} \left[ q^2 \left( \frac{\Omega_{d-1,k}}{A} \right)^{\frac{d-2}{d}} + k \left( \frac{A}{\Omega_{d-1,k}} \right)^{\frac{d-2}{d}} + \frac{l^2}{l^2} \left( \frac{A}{\Omega_{d-1,k}} \right)^{\frac{d-2}{d}} \right]$$  \hspace{1cm} (13)

where

$$M_0 = \frac{1}{8\pi} (-k)^{d/2} \frac{(d-1)!}{d!} l^{d-2} \Omega_{d-1,k}$$  \hspace{1cm} (14)

for $d$ even and zero for $d$ odd. The form (13) together with (14) is our proposal for a general form of the Penrose inequality for asymptotically locally $AdS_{d+1}$ spaces, with electric charge $q$ and a boundary topology characterized by $k$. The inequality (13) has been formally established when $d = 3$ by Gibbons, using the inverse mean curvature flow [10]. Partial forms of (13) were written in [11, 12], where evidence for its correctness was given.

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