On relativistic theory of spinning and deformable particles

A.N. Tarakanov *
Minsk State High Radiotechnical College
Independence Avenue 62, 220005, Minsk, Belarus

Abstract

A model of relativistic extended particle is considered with the help of generalization of space-time interval. Ten additional dimensions are connected with six rotational and four deformational degrees of freedom. An obtained 14-dimensional space is assumed to be an embedding one both for usual space-time and for 10-dimensional internal space of rotational and deformational variables. To describe such an internal space relativistic generalizations of inertia and deformation tensors are given. Independence of internal and external motions from each other gives rise to splitting the equation of motion and some conditions for 14-dimensional metric. Using the 14-dimensional ideology makes possible to assign a unique proper time for all points of extended object, if the metric will be degenerate. Properties of an internal space are discussed in details in the case of absence of spatial rotations.

1 Introduction

More and more attention is spared to relativistic description of extended objects, which could serve a basis for construction of dynamics of interacting particles. Necessity of introduction extended objects to elementary particle theory is out of doubt. Therefore, since H.A.Lorentz attempts to introduce particles of finite size were undertaken. However a relativization of extended body is found prove to be a difficult problem as at once there was a contradiction to Einstein’s relativity principle. Even for simplest model of absolutely rigid body [1] it is impossible for all points of a body to attribute the same proper time. Therefore considering of elementary particles with rotational degrees of freedom is often incorrect, when it is supposed that the particle possesses infinitesimal spatial sizes for this leads to formal treating of internal degrees of freedom [2]-[6]. Rotational degrees of freedom are degenerated in general, and as physical variables remain only internal angular momenta [7], [8]. This difficulty has not been overcame neither in extended electron theory

*E-mail: tarak-ph@mail.ru
by Markov [9] nor in bilocal and multilocal theories by Yukawa [10] and Takabayashi [11] nor in relativistic rotator by Nakano [12], causing large flow of works, nor in many other works devoted to extended objects [13] and relativistic continuous media [14]. It does not affect the further development of ideas, but one cannot say anything about the sizes of particles. At large distances from particle the account only internal momenta can appear sufficient, but considering of interactions must be correct only with taking into account particle sizes.

Usually the description of extended particle is carried out by means of consideration the moving 4-hedron, \( e^\mu = \{ e^\mu_{(\lambda)} \} \), connected with a point inside of object [12]. Thus, angular velocity \( \Omega^{[\mu\nu]} \) is defined according to the formula [15]

\[
\Omega^{[\mu\nu]} = \eta^{(\lambda)(\kappa)} e^\mu_{(\lambda)} \frac{D e^\nu_{(\kappa)}}{c d\tau} = \eta^{(\lambda)(\kappa)} e^\mu_{(\lambda)} \left[ \frac{d e^\nu_{(\kappa)}}{c d\tau} + \Gamma^\nu_{\rho\sigma} y^\rho e^\sigma_{(\kappa)} \right],
\]

where \( y^\mu \) are coordinates of a point with which the reference point is connected, \( \dot{y}^\mu = dy^\mu/cd\tau \),

\[
\Gamma^\nu_{\rho\sigma} = \frac{1}{2} g^{\nu\tau} \left[ \partial_\rho g_{\tau\sigma} + \partial_\sigma g_{\rho\tau} - \partial_\tau g_{\rho\sigma} \right],
\]

\( \partial_\rho = \partial/\partial y^\rho \), \( g_{\mu\nu} \) is the metrics describing complex movement of this point.

Another approach is the description of extended bodies in General Relativity by means of specifying energy-momentum tensor and its moments [15]-[18], going back to Mathisson [19] and Papapetrou [20]. However here even in the elementary case of a free rotating particle in flat space-time its momentum and velocity appear to be not parallel, what is not quite clear in case of the isolated particle. With the certain kind of Lagrangian first approach proves to be equivalent to the second one [15].

In this work another attempt is done to describe in details a relativistic extended particle by means of introducing internal rotational and deformational degrees of freedom. Our approach is based on dimensional extension of space-time interval connected with a world line of any point inside the particle. In this connection it is necessary to note the work [21] in which multidimensional space is used for the description of extended objects, and as an example the relativistic string is considered. In our work we connect additional measurements with both six rotation and hyper-rotation angles and four deformational degrees of freedom as it was partially made in [3].

In our opinion, many works, devoted to extended elementary particles (e.g., [3]-[6], [9]-[13], etc.), spare a little attention to classical description. Partly it is caused unobservability internal movements and connected with traditional attributing to microobjects of quantum properties. Passage to quantization is represented by natural way, but investigation of classical opportunities should be sufficiently complete and comprehensive. In this sense the description of extended bodies from the general relativistic viewpoint [8] is represented more consecutive. Here we also give only classical consideration. In Section 2 a generalization of space-time interval, connected with any point inside of an extended particle, is specified with the help of introducing internal rotational and deformational degrees of freedom. For such a particle we also define a notion of the ”center of inertia” as
a point representing the motion of the particle considered as a whole object. Relativistic inertia and deformation tensors, as well as general equations of motion are considered in Section 3. In Section 4 we split equations of motion and obtain some conditions for metrics with the help of embedding space formalism. In particular, certain condition imposed on angular and deformatinal velocities gives rise to possibility of introducing the unique proper time in the whole internal region of extended object. Section 5 shows that equations of motion obtained in Section 4 may be derived also under consideration of degenerated 14-dimensional metric, whose rank equals to the rank of the background 4-dimensional space-time. Properties of internal space are discussed and its metric is derived in Section 6 in the case of absence of spatial rotations \( (Ω_{[mn]} = 0) \). Here we also clarify the physical meaning of components \( Ω^{[0n]} \) of 4-dimensional angular velocity \( Ω^{[μν]} \), which are proportional to corresponding components of angular momentum of the internal point relative to the "center of inertia". This fact allows to connect \( Ω^{[0n]} \) with spin of the particle determined in Section 5 and redetermined in Section 6. Section 7 contains conclusive notes.

### 2 Rotating and deformable medium

Let there is some space-time (finite or infinite) domain, filled by some substance. We shall assume this domain to be moving in space-time and therefore it is possible to choose some point C, said to be the "center of inertia", which represents movement of the domain as a whole. Because the concept of the center of inertia cannot be uniquely defined in a relativistic case [22] we shall use it conditionally, implying as the "center of inertia" certain preferred point inside domain, whose movement in space-time defines movement of the domain as a whole. Domain, defined as a part of the Universe, may be both any physical system and fundamental particle. Here we consider such domain as a particle.

Let the observer be placed in the point O (origin of coordinates) of space-time. Then coordinates of the "center of inertia" C relative to O we shall denote through \( x^μ \), coordinates of a point M inside particle relative to O – through \( y^μ \), and coordinates of the point M relative to the "center of inertia" C – through \( r^μ \).

We wish to consider a rotating particle, inside of which internal substance undergoes some movement, so that a field of 4-velocities is specified inside of domain occupied by particle. Intuitively clearly, that world line of any point M should wind around a world line of the point C. If one postulate, that any movement always can be presented as geodesic movement (see, e.g., [23]-[26]), it means, that spaces, associated with movement of the "center of inertia" C and with movement of point M, are various and possess various metrics. The metric of the space, associated with point M, is complicated even when the background space and space of substance are Minkowski spaces. Should rotations and deformations be absent, these spaces would coincide, and it would possible to write \( y^μ = x^μ + r^μ \). In general case \( y^μ \) must be considered as some functions from \( x^μ \), \( r^μ \) and rotation angles \( ϕ^{μν} \). In the further we shall distinguish the indices connected with
various variables, and instead of \( x^\mu, r^\alpha, \varphi^{\mu\nu} \) we shall write \( x^{(\mu)}, r^{\dot{\alpha}}, \varphi^{[\mu\nu]} \). Thus, indices in parentheses, \((\mu), (\nu), (\lambda), \) ... are concerned to the “center of inertia” \( C \), moving in background space, indices with hats, \( \dot{\alpha}, \dot{\beta}, \dot{\gamma}, \) ..., are concerned to any point \( M \) of the substance, moving under internal deformations, while indices without any brackets, \( \mu, \nu, \lambda, \) ..., are concerned to any point \( M \) of the substance, moving relative to background space by trajectory being geodesic line in the internal space of the particle.

Infinitesimal variation of coordinates \( y^\mu \) of internal point \( M \), is obviously defined by expression

\[
dy^\mu = \partial_{(\nu)} y^\mu dx^{(\nu)} + \partial_\lambda y^\mu dr^{\dot{\alpha}} + \frac{1}{2} \frac{\partial y^\mu}{\partial \varphi^{[\lambda\kappa]}} d\varphi^{[\lambda\kappa]} .
\]

On the other hand, \( dy^\mu \) is caused by infinitesimal changes \( Dx^{(\mu)} \) and \( Dr^{\dot{\alpha}} \) of coordinates \( x^{(\mu)} \) and \( r^{\dot{\alpha}} \), taking into account rotations and deformation of substance. \( Dx^{(\mu)} \) is caused, firstly, by movement of the particle as a whole, and, secondly, by moving of substance within the vicinity of the point \( C \). \( Dr^{\dot{\alpha}} \) does not depend on external movement and is caused only by rotation of particle and moving of substance within the vicinity of the point \( M \). Therefore it is possible to write

\[
Dx^{(\mu)} = dx^{(\mu)} + V^{(\mu)}_C cdt ,
\]

\[
Dr^{\dot{\alpha}} = \frac{1}{2} (\delta^{\dot{\alpha}}_{(\lambda)} \delta^{(\kappa)}_\beta - \delta^{\dot{\alpha}}_{(\kappa)} \delta^{(\lambda)}_\beta) r^{\dot{\beta}} d\varphi^{[\kappa\lambda]} + V^{\dot{\alpha}}_M cdt ,
\]

where

\[
V^{(\mu)}_C = \delta^{(\mu)}_{\dot{\alpha}} \frac{dr^{\dot{\alpha}}}{cd\tau} \bigg|_{r^{\dot{\alpha}}=0, d\varphi^{[\kappa\lambda]}=0}
\]

is a velocity of substance within the vicinity of the ”center of inertia” \( C \) due to a deformation of substance,

\[
V^{\dot{\alpha}}_M = \frac{dr^{\dot{\alpha}}}{cd\tau} \bigg|_{d\varphi^{[\kappa\lambda]}=0}
\]

is a velocity of substance within the vicinity of the point \( M \) due to a deformation of substance, \( \tau \) is proper time of the point \( M \), \( \delta^{(\mu)}_{\dot{\alpha}} \) are usual Kronecker symbols (\( \delta^{(\mu)}_{\dot{\alpha}} = 1 \) when \( \alpha = \mu \) and \( \delta^{(\mu)}_{\dot{\alpha}} = 0 \) when \( \alpha \neq \mu \)). Thus, movement of the point \( M \) is characterized by infinitesimal variable

\[
dy^\mu = \delta^{\mu}_{(\lambda)} Dx^{(\lambda)} + \delta^{\mu}_{\dot{\alpha}} Dr^{\dot{\alpha}} = \delta^{\mu}_{(\lambda)} dx^{(\lambda)} +
\]

\[
+ \frac{1}{2} (\delta^{\mu}_{(\lambda)} \delta_{(\kappa)} - \delta^{\mu}_{(\kappa)} \delta_{(\lambda)}) r^{\dot{\beta}} d\varphi^{[\kappa\lambda]} + (\delta^{\mu}_{(\lambda)} V^{(\lambda)}_C + \delta^{\mu}_{\dot{\alpha}} V^{\dot{\alpha}}_M) cdt ,
\]

where \( \delta^{\mu}_{(\lambda)}, \delta^{\mu}_{\dot{\alpha}} \) are also Kronecker symbols. To eliminate unknown quantities \( V^{(\mu)}_C, V^{\dot{\alpha}}_M \), it is reasonable to assume, that they are connected with each other by transformation of a kind

\[
(\delta^{\mu}_{(\lambda)} V^{(\lambda)}_C + \delta^{\mu}_{\dot{\alpha}} V^{\dot{\alpha}}_M) cdt = f^{\mu}_{(\lambda)} dx^\lambda + b^{\mu}_{\dot{\alpha}} dr^{\dot{\alpha}} + \frac{1}{2} d^{[\mu}_{[\lambda\kappa]} d\varphi^{[\lambda\kappa]} ,
\]
where \( f_\mu^\nu \), \( b_\lambda^{\mu} \) and \( d^\nu_{\lambda \kappa} \) are some functions from coordinates \( x^{(\mu)} \), covering external space, internal coordinates \( r^\alpha \) and rotation angles \( \varphi^{[\mu \kappa]} \).

Comparing (2.1) and (2.6) and taking into account (2.7), we have

\[
\partial_{(\lambda)} y^\mu = \delta^\mu_{(\lambda)} + f_\nu^\mu_{(\lambda)} = e^\mu_{(\lambda)},
\]

\[
\partial_{\hat{\alpha}} y^\mu = b_\mu^{\hat{\alpha}},
\]

\[
\frac{\partial y^\mu}{\partial \varphi^{[\lambda \kappa]}} = d^\mu_{\lambda \kappa} - (\delta^\mu_{(\lambda)} \delta_{(\kappa)} - \delta^\mu_{(\kappa)} \delta_{(\lambda)}) r^\hat{\alpha}.
\]

For the description of movement of point \( M \) we shall write down a corresponding squared interval that will look like

\[
dS^2 = \Sigma c^2 d\tau^2 = g_{\mu \nu} dy^\mu dy^\nu = g_{\mu \nu} e_{(\lambda)}^\nu e_{(\kappa)}^\mu dx^{(\lambda)} dx^{(\kappa)} + g_{\mu \nu} b_\alpha^\mu b_\beta^\nu dr^\alpha dr^\beta +
\]

\[
+ \frac{1}{4} g_{\mu \nu} \left[ d^\mu_{\lambda \kappa} - (\delta^\mu_{(\lambda)} \eta_{(\kappa)\hat{\alpha}} - \delta^\mu_{(\kappa)} \eta_{(\lambda)\hat{\alpha}}) r^\hat{\alpha} \right] d\varphi^{[\lambda \kappa]} d\varphi^{[\mu \nu]} + 2 g_{\mu \nu} e_{(\lambda)}^\mu b_\alpha^\nu dx^{(\lambda)} dr^\alpha +
\]

\[
+ g_{\mu \nu} e_{(\lambda)}^\mu \left[ d^\nu_{[\rho \sigma]} - (\delta^\nu_{(\rho)} \eta_{(\sigma)\hat{\beta}} - \delta^\nu_{(\sigma)} \eta_{(\rho)\hat{\beta}}) r^\hat{\beta} \right] dx^{(\lambda)} d\varphi^{[\rho \sigma]} +
\]

\[
+ g_{\mu \nu} b_\alpha^\mu \left[ d^\nu_{[\rho \sigma]} - (\delta^\nu_{(\rho)} \eta_{(\sigma)\hat{\beta}} - \delta^\nu_{(\sigma)} \eta_{(\rho)\hat{\beta}}) r^\hat{\beta} \right] dr^\alpha d\varphi^{[\rho \sigma]},
\]

where \( \Sigma \) = \( \pm 1 \) or 0, and through \( g_{\mu \nu} \) the metric of space, associated with movement of point \( M \) is designated. Here it should be noted that using the metric \( g_{\mu \nu} \) is connected with unknown in advance character of movement of point \( M \). Should the trajectory of point \( M \) be known, it would be possible to use instead of \( g_{\mu \nu} \) the metric of background space \( \eta_{(\mu)\nu} = \delta_{(\mu)\lambda} \delta_{(\nu)\kappa} \eta_{\lambda \kappa} \) together with a restriction in the form of the equation of a trajectory of point \( M \), which in case of using the metric \( g_{\mu \nu} \) turns out to be an integral of the equations of motion

\[
\ddot{y}^\mu + \Gamma^\mu_{\nu \lambda} \dot{y}^\nu \dot{y}^\lambda = 0.
\]

Starting from expression (2.11) the ”center of inertia” may be defined as follows. Because movement of the ”center of inertia” should look as movement of a material point in background space it is reasonable to put

\[
g_{\mu \nu} e_{(\lambda)}^\nu e_{(\kappa)}^\mu = \eta_{(\lambda)\kappa},
\]

\[
\eta_{(\lambda)\kappa} e_{(\lambda)}^\mu e_{(\kappa)}^\nu = g^{\mu \nu},
\]

for all values of \( r^\hat{\alpha} \), which are not leaving for domain, occupied by substance. If \( x^{(\mu)} \) are not coordinates of the ”center of inertia” then relations (2.13), (2.14) should not be carried out.

Introduction of relativistic generalizations of inertia and deformation tensors allows to write down the interval (2.11) in a more compact form. That will be a theme of the following Section.
3 Inertia and deformation tensors

Let us consider in more details the quantities entering into the interval (2.11) before differentials. First of all, for rotating particle we need to enter a relativistic inertia tensor. We shall consider specific quantities, i.e., quantities associated with a unit mass. It is natural to do so, as for as the concept of mass and mass distribution inside of considered substance still are not determined, the more so in the inertia tensor for a material point the mass of it is only constant multiplier with physical dimension of mass.

To give a relativistic generalization of inertia tensor, we shall start from nonrelativistic description of absolutely rigid body. Proper inertia tensor of a unit mass particle relative to the beginning of coordinate system in three-dimensional space looks like

\[ j_{ik} = r^2 \delta_{ik} - r_i r_k , \]  

(3.1)

where \( r_k \) are coordinates of the particle (see, e.g., [27]). Introduction of (specific) linear momentum of the particle relative to the beginning of coordinate system,

\[ i_{ik} = \varepsilon_{ikm} r_m , \]  

(3.2)

allows to write the inertia tensor (3.1) as product of the linear momenta

\[ j_{ik} = -j_{im} j_{mk} . \]  

(3.3)

In a relativistic case for the description of rotational movement except for three angles \( \varphi_1^{12} , \varphi_2^{23} , \varphi_3^{31} \), connected with rotations in three-dimensional space, one should require three more angular variables \( \varphi_0^{0i} \), connected with hyper-rotations in planes \( (0i) \). Therefore, there should be also components of generalized inertia tensor corresponding to these planes. As it is well known, \( \varepsilon_{ijk} \) is 0-component of four-dimensional tensor density by Levy-Civita: \( \varepsilon_{ijk} = \varepsilon_{0ijk} \). Hence, instead of linear momentum tensor of the second rank (3.2) in three-dimensional space there arises a tensor of the third rank

\[ i^0_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda\kappa} r^\kappa \]  

(3.4)

in four-dimensional space, and, obviously, \( i_{ik} = i^0_{0ik} \). Accordingly, instead of inertia tensor of the second rank (3.3) there arises a tensor of the fourth rank:

\[ j^0_{\mu\nu,\lambda\kappa} = j^0_{\lambda\kappa,\mu\nu} = \eta^{\alpha\beta} j^0_{\mu\nu\alpha\beta} \lambda\kappa = \eta^{\alpha\beta} \varepsilon_{\mu\nu\alpha\beta\gamma} \lambda\kappa r^\gamma r^\sigma = \eta_{\mu\nu\alpha\beta,\lambda\kappa} r^\alpha r^\beta r^\gamma r^\sigma = \]

\[ = [\eta_{\mu\lambda} \eta_{\nu\kappa} \eta_{\rho\sigma} + \eta_{\mu\kappa} \eta_{\nu\sigma} \eta_{\rho\lambda} + \eta_{\mu\sigma} \eta_{\nu\lambda} \eta_{\rho\kappa} - \eta_{\mu\kappa} \eta_{\nu\lambda} \eta_{\rho\sigma} - \eta_{\mu\lambda} \eta_{\nu\sigma} \eta_{\rho\kappa} - \eta_{\mu\sigma} \eta_{\nu\kappa} \eta_{\rho\lambda}] r^\rho r^\sigma = \]

\[ = \left[ (\eta_{\mu\lambda} \eta_{\nu\kappa} - \eta_{\mu\kappa} \eta_{\nu\lambda}) r_\eta^2 + \eta_{\mu\kappa} r^\nu r^\lambda + \eta_{\nu\lambda} r^\mu r^\kappa - \eta_{\mu\lambda} r^\nu r^\kappa - \eta_{\nu\kappa} r^\mu r^\lambda \right] . \]  

(3.5)

It should be noted, that this tensor can be written down in other form:

\[ j^0_{\mu\nu,\lambda\kappa} = -\frac{i}{2} C_{\mu\nu,\lambda\kappa,\rho\sigma} D^{\rho\sigma} r_\eta^2 , \]  

(3.6)

where

\[ r_\eta^2 = \eta_{\alpha\beta} r^\alpha r^\beta . \]  

(3.7)
Quantities

\[ C_{\mu\nu, \lambda\kappa, \rho\sigma} = i [\eta_{\mu\lambda} \eta_{\nu\rho} \eta_{\kappa\sigma} + \eta_{\mu\rho} \eta_{\nu\lambda} \eta_{\kappa\sigma} - \eta_{\mu\kappa} \eta_{\nu\rho} \eta_{\lambda\sigma} - \eta_{\mu\rho} \eta_{\nu\lambda} \eta_{\kappa\sigma}] \]  

(3.8)

and

\[ D^{\mu\nu} = \eta^{\mu\nu} - \frac{2 \tau^{\mu\nu}}{r_\eta^2} \]  

(3.9)

satisfy to relations

\[ \frac{1}{4} \eta^{\mu\nu, \lambda\kappa} C_{\mu\nu, \lambda\kappa, \rho\sigma} = \frac{1}{4} (\eta^{\mu\lambda} \eta_{\mu\kappa} - \eta^{\mu\kappa} \eta_{\mu\lambda}) C_{\mu\nu, \lambda\kappa, \rho\sigma} = i (d - 1) \eta_{\rho\sigma}; \]  

(3.10)

\[ \eta_{\rho\sigma} D^{\mu\rho} D^{\nu\sigma} = \eta_{\mu\nu}, \]  

(3.11)

\[ D^{\mu\rho} D_{\mu\nu} = \delta^\rho_\nu, \]  

(3.12)

where

\[ D_{\mu\nu} = \eta_{\mu\nu} - \frac{2 \eta_{\mu\rho} \eta_{\nu\sigma} \tau^{\rho\sigma}}{r_\eta^2}, \]  

(3.13)

d = \eta^{\mu\nu} \eta_{\mu\nu} = D^{\mu\nu} D_{\mu\nu} = 4 \] is the dimension of the background space.

Let us note also, that if four-dimensional background space is the Minkowski space \( \mathbb{E}^4_{1,3} \), then quantities (3.8) are structural constants of the Lorentz group. It is obvious, the relation that connects three-dimensional inertia tensor with a four-dimensional tensor is \( j_{\alpha \beta} = j^0_{00} \).

A relativistic generalization of the inertia tensor above is natural and differs from often used inertia tensor of the second rank (see, e.g., [3], [15], [16]). There exists also an attempt to express inertia tensor of the second rank through tensor of the fourth rank [22].

Taking into account various character of indices, one should written (3.4) and (3.5) as

\[ \tilde{t}^0_{[\mu\nu](\lambda)} = \varepsilon_{[\mu\nu][\lambda\kappa]} \delta^\kappa_\lambda \tilde{t}^\lambda_\alpha, \]  

(3.14)

\[ \tilde{J}^0_{[\mu\nu][\lambda\kappa]} = - \eta^{(\rho)(\sigma)} \tilde{t}^0_{[\mu\nu](\rho)} \tilde{t}^0_{[\lambda\kappa](\sigma)} = - \frac{i}{2} \eta_{\rho\sigma} D^{(\rho)(\sigma)} \tilde{t}^\lambda_\alpha \tilde{r}^\lambda_\beta, \]  

(3.15)

where

\[ D^{(\rho)(\sigma)} = \eta^{(\rho)(\sigma)} - \frac{2 \delta^{(\rho)}_\alpha \delta^{(\sigma)}_\beta \tilde{r}^\alpha_\alpha \tilde{r}^\beta_\beta}{r_\eta^2}, \]  

(3.16)

\[ r_\eta^2 = \eta^{(\rho)(\sigma)} \delta^{(\rho)}_\alpha \delta^{(\sigma)}_\beta \tilde{r}^\alpha_\alpha \tilde{r}^\beta_\beta = h^0_{\alpha\beta} \tilde{r}^\alpha_\alpha \tilde{r}^\beta_\beta. \]  

(3.17)

Let us define also dual quantities:

\[ \tilde{t}^0_{[\mu\nu](\lambda)} = \frac{1}{2} \varepsilon_{[\mu\nu][\rho\sigma]} \tilde{t}^0_{[\rho\sigma](\lambda)} = (\eta_{(\nu)(\lambda)} \eta_{(\mu)(\rho)} - \eta_{(\mu)(\lambda)} \eta_{(\nu)(\rho)}) \tilde{r}^\alpha_\alpha, \]  

(3.18)

\[ \tilde{J}^0_{[\mu\nu][\lambda\kappa]} = \tilde{J}^0_{[\lambda\kappa][\mu\nu]} = \frac{1}{2} \varepsilon_{[\mu\nu][\rho\sigma]} \tilde{J}^0_{[\rho\sigma][\lambda\kappa]} = - \eta^{(\rho)(\sigma)} \tilde{t}^0_{[\mu\nu](\rho)} \tilde{t}^0_{[\lambda\kappa](\sigma)} = \]  

\[ = (\tilde{t}^0_{[\lambda\kappa](\mu)} \eta_{(\nu)(\rho)} \tilde{t}^0_{[\lambda\kappa](\nu)} \eta_{(\mu)(\sigma)} \tilde{t}^0_{[\lambda\kappa](\sigma)} (\varepsilon_{[\lambda\kappa][\mu\sigma]} \eta_{(\nu)(\rho)} - \varepsilon_{[\lambda\kappa][\nu\sigma]} \eta_{(\mu)(\rho)}) \delta^{(\sigma)}_\beta \tilde{r}^\alpha_\alpha \tilde{r}^\beta_\beta, \]  

(3.19)
Then the quantity
\[ j_{[\mu\nu][\lambda\kappa]}^{0} \]
and quantity
\[ j_{[\rho\sigma][\tau\omega]}^{0} \]
additional terms. We shall introduce a denotation
\[ j_{[\mu\nu][\lambda\kappa]}^{0} = \frac{1}{4} \varepsilon_{[\mu\nu]} \varepsilon_{[\lambda\kappa]} j_{[\rho\sigma][\tau\omega]}^{0} = \]

\[ = -\eta^{(\rho)(\sigma)} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\sigma]}^{0} = -iC_{[\mu\nu][\lambda\kappa]}(\rho)(\sigma) \delta^{(\rho)}_{\alpha} \delta^{(\sigma)}_{\beta} r^{\alpha} r^{\beta} = \]

\[ = [\eta_{(\mu)}^{\alpha} (\eta_{(\nu)}^{(\kappa)} \eta_{(\lambda)}^{(\beta)} - \eta_{(\nu)}^{(\lambda)} \eta_{(\kappa)}^{(\beta)}) - \]

\[ -\eta_{(\nu)}^{(\alpha)} (\eta_{(\mu)}^{(\kappa)} \eta_{(\lambda)}^{(\beta)} - \eta_{(\mu)}^{(\lambda)} \eta_{(\kappa)}^{(\beta)})] r^{\alpha} r^{\beta} \]  

(3.20)

It is not difficult to show, that following relations take place:

\[ \frac{1}{4} \eta_{[\mu\nu][\lambda\kappa]}^{0} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\sigma]}^{0} = r_{\eta}^{2} N_{(\rho)(\sigma)}^{0} ; \]  

(3.21)

\[ \frac{1}{4} \eta_{[\rho\sigma][\tau\omega]}^{0} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\sigma]}^{0} = r_{\eta}^{2} N_{(\rho)(\sigma)}^{0} ; \]  

(3.22)

\[ \frac{1}{4} \eta_{[\mu\nu][\lambda\kappa]}^{0} j_{[\rho\sigma][\tau\omega]}^{0} j_{[\lambda\kappa][\tau\omega]}^{0} = 0 ; \]  

(3.23)

\[ \frac{1}{4} \eta_{[\rho\sigma][\tau\omega]}^{0} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\tau\omega]}^{0} = r_{\eta}^{2} j_{[\mu\nu][\lambda\kappa]}^{0} ; \]  

(3.24)

\[ \frac{1}{4} \eta_{[\rho\sigma][\tau\omega]}^{0} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\tau\omega]}^{0} = -r_{\eta}^{2} j_{[\mu\nu][\lambda\kappa]}^{0} ; \]  

(3.25)

\[ \frac{1}{4} \eta_{[\rho\sigma][\tau\omega]}^{0} j_{[\mu\nu][\rho]}^{0} j_{[\lambda\kappa][\tau\omega]}^{0} = 0 , \]  

(3.26)

where

\[ \eta^{[\rho\sigma][\tau\omega]} = \eta^{(\rho)(\tau)} \eta^{(\sigma)(\omega)} - \eta^{(\rho)(\omega)} \eta^{(\sigma)(\tau)} , \]  

(3.27)

and quantity

\[ N_{(\rho)(\sigma)}^{0} = \eta_{(\rho)}^{(\lambda)} - \frac{\eta_{(\rho)}^{[\alpha} \eta_{(\sigma)}^{[\beta} r^{\alpha} r^{\beta}}{r_{\eta}^{2}} = \frac{1}{2} (D_{(\rho)}^{(\sigma)} + \eta_{(\rho)}^{(\sigma)}) \]  

(3.28)

satisfies to relation

\[ \eta^{(\rho)(\sigma)} N_{(\mu)}^{0} N_{(\nu)}^{0} = N_{(\mu)}^{0} . \]  

(3.29)

It is easy to see now, that quantity standing before product of angular differentials in (2.11) is actually represents the sum of twice dual inertia tensor \( j_{[\mu\nu][\lambda\kappa]}^{0} \) and some additional terms. We shall introduce a denotation

\[ \tilde{j}_{[\mu\nu][\lambda\kappa]} = j_{[\mu\nu][\lambda\kappa]}^{0} - g_{\rho\sigma} d_{[\mu\nu]} \eta^{\sigma}_{[\lambda\kappa]} - g_{\rho\sigma} \eta^{\rho(\tau)} \left[ j_{[\mu\nu][\tau]}^{0} d_{[\lambda\kappa]}^{\sigma} + j_{[\lambda\kappa][\tau]}^{0} d_{[\mu\nu]}^{\sigma} \right] + \]

\[ + (g_{\rho\sigma} \eta^{\rho(\tau)} \eta^{\sigma(\omega)} - \eta^{\rho(\omega)} \eta^{\sigma(\tau)}) j_{[\mu\nu][\tau]}^{0} j_{[\lambda\kappa][\tau]}^{0} = \]

\[ = -g_{\rho\sigma} \left[ d_{[\mu\nu]}^{\sigma} + \eta^{\rho(\tau)} j_{[\mu\nu][\tau]}^{0} \right] \left[ d_{[\lambda\kappa]}^{\sigma} + \eta^{\sigma(\omega)} j_{[\lambda\kappa][\tau]}^{0} \right] . \]  

(3.30)

Then the quantity

\[ j_{[\mu\nu][\lambda\kappa]} = \frac{1}{4} \varepsilon_{[\mu\nu]} \varepsilon_{[\lambda\kappa]} j_{[\rho\sigma][\tau\omega]} \]  

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\[ j^0_{\mu\nu|\lambda\kappa} - g_{\rho\sigma} \tilde{d}^\rho_{\mu\nu} \tilde{d}^\sigma_{\lambda\kappa} + g_{\rho\sigma} \eta^{(\rho)} \left[ i^0_{\mu\nu|(\tau)} \tilde{d}^\sigma_{\lambda\kappa} + i^0_{\lambda\kappa|(\tau)} \tilde{d}^\rho_{\mu\nu} \right] + \\
+ \left( g_{\rho\sigma} \eta^{(\rho)} \eta^{(\sigma)} - \eta^{(\tau)(\omega)} \right) i^0_{\mu\nu|(\tau)} i^0_{\lambda\kappa|(\omega)} = \\
= -g_{\rho\sigma} \left[ \eta^{(\rho)} i^0_{\mu\nu|(\tau)} - \tilde{d}^\rho_{\mu\nu} \right] \left[ \eta^{(\sigma)} i^0_{\lambda\kappa|(\omega)} - \tilde{d}^\sigma_{\lambda\kappa} \right] \] (3.31)

will be called a dynamical inertia tensor of the point M relative to the “center of inertia” C.

This definition differs from usual one (a geometrical inertia tensor) (3.1), (3.5) because the quantity (3.23) is defined by not only geometrical characteristics (coordinates \( r^\hat{\alpha} \)) of point M, but also physical processes in a vicinity of this point. This is due to presence of quantities \( d^\mu_{\lambda\kappa} \) in (3.23). If some closed domain bound the substance under consideration, then inertia tensor of the whole such an object can be obtained with multiplication of (3.23) by mass density and subsequent integration by volume of the domain.

Due to relations (2.13), (2.14) dynamical inertia tensor \( j_{\mu\nu|\lambda\kappa} \) and dual tensor \( \tilde{j}_{\mu\nu|\lambda\kappa} \) can be written down in the form

\[ j_{\mu\nu|\lambda\kappa} = -\eta^{(\rho)} i^0_{\mu\nu|(\rho)} i^0_{\lambda\kappa|(\sigma)} , \] (3.32)

\[ \tilde{j}_{\mu\nu|\lambda\kappa} = -\eta^{(\rho)} i^0_{\mu\nu|(\rho)} \tilde{i}^0_{\lambda\kappa|(\sigma)} , \] (3.33)

Quantity

\[ i^0_{\mu\nu|(\lambda)} = \eta^{(\lambda)}(\kappa) e^{(\kappa)}_\rho \left[ \eta^{(\rho)} i^0_{\mu\nu|(\tau)} - \tilde{d}^\rho_{\mu\nu} \right] \] (3.34)

will be called a dynamical momentum tensor of the point M, while quantity \( i^0_{\mu\nu|(\lambda)} \), specified in (3.11), will be called a geometrical momentum tensor of the point M relative to the “center of inertia” C.

Quantity

\[ h_{\alpha\beta} = g_{\mu\nu} b^\mu_{\alpha\beta} b^\nu_{\alpha\beta} \] (3.35)

may be called a deformation tensor, forasmuch as it is stipulated by moving of substance inside of the domain under consideration. Definition (3.27) is somewhat differs from the definition accepted in the mechanic of continuous media, where deformation tensor is defined as semi-difference between external background metrics and the metrics of deformed body [28]-[31] either in the Lagrange-Green form or in the Euler-Cauchy-Almansi form. However, the definition similar to (3.27) is given, for example, in [14], while semi-difference above is called a strain tensor. As it will be shown below, \( h_{\alpha\beta} \) actually represents the internal metric of the domain.

At last, if we introduce also following notations

\[ \ell_{\mu\nu|\alpha} = g_{\rho\sigma} b^\rho_{\alpha\beta} \left[ \eta^{(\sigma)} i^0_{\mu\nu|(\tau)} - \tilde{d}^\sigma_{\mu\nu} \right] = e^{(\tau)} b^\rho_{\alpha\beta} i^0_{\mu\nu|(\tau)} , \] (3.36)

\[ c_{\lambda\alpha} = g_{\rho\sigma} e^{(\lambda)}(\sigma) b^\sigma_{\alpha\beta} , \] (3.37)
the interval (2.11) will have the form

\[ dS^2 = \Sigma c^2 \, d\tau^2 = \eta(\mu) dx^{(\mu)} dx^{(\nu)} - \frac{1}{4} \tilde{\eta}_{[\rho] \sigma} \eta^{[\rho]} \eta_{[\sigma]} d\varphi^{[\rho\sigma]} d\varphi_{[\rho\sigma]} + h_{\alpha\beta} \, dr^{\alpha} dr^{\beta} + \]

\[ + \tilde{L}_{[\rho\sigma]} dx^{(\mu)} d\varphi_{[\rho\sigma]} + 2 \tilde{c}_{[\mu]} \, d\varphi_{[\rho]} d\varphi^{(\mu)} d\varphi_{[\rho]} + \tilde{L}_{[\sigma]} \, dr^{\alpha} d\varphi_{[\rho\sigma]} \], \quad (3.38) \]

where

\[ \tilde{L}_{[\rho\sigma]} = \eta(\lambda) \eta^{(\kappa)} \left[ \eta^{(\rho)} \tilde{a}_{[\mu\nu]}(\sigma) + d_{[\mu\nu]} \right], \]

\[ \tilde{L}_{[\sigma]} = g_{\rho\sigma} \beta_{\alpha} \left[ \eta^{(\sigma)} \tilde{a}_{[\mu\nu]}(\tau) + d_{[\mu\nu]} \right] = \eta^{(\sigma)} \beta_{\alpha} \tilde{a}_{[\mu\nu]}(\tau), \quad (3.39) \]

\[ \tau \text{ is proper time of the point } M. \]

Proper time of the "center of inertia" \( \tau_C \) may be defined from the relation

\[ \sigma c^2 \, d\tau_C^2 = \eta(\mu) dx^{(\mu)} dx^{(\nu)}, \]

where \( \sigma = \pm 1 \) (obviously, a concept of proper time of the "center of inertia" is undefinable for \( \sigma = 0 \)). Then it follows from (3.30) and (3.33) that

\[ \frac{d\tau_C}{d\tau} = \Gamma = \left[ \Sigma \left( \sigma - \frac{1}{4} \tilde{\eta}_{[\rho] \sigma} \eta^{[\rho]} \eta_{[\sigma]} \right) + h_{\alpha\beta} \hat{V}^\alpha \hat{V}^\beta + \right. \]

\[ + \tilde{L}_{[\rho\sigma]} C^{(\mu)} \eta^{[\rho]} \eta_{[\sigma]} + 2 \tilde{c}_{[\mu]} A^{(\mu)} \hat{V}^\alpha + \tilde{L}_{[\sigma]} \hat{V}^\alpha \Omega^{[\rho\sigma]} \right]^{-1/2}, \]  \quad (3.42)

where \( C^{(\mu)} = dx^{(\mu)}/d\tau_C \) is four-velocity of translational movement of the medium in the back-ground space, \( \Omega^{[\rho\sigma]} = d\varphi^{[\rho\sigma]}/d\tau_C \) is angular velocity of rotational movement of the medium relative to the "center of inertia", \( \hat{V}^\alpha = dr^\alpha/c d\tau \) is four-velocity of translations of substance of the medium in the vicinity of the point M relative to the "center of inertia".

The proper time of the point M will be coincided with the proper time of the "center of inertia" when \( \Gamma = 1 \). It is possible only in two cases: i) if internal substance of the domain is absolutely rigid and performs only translational movement, i.e. when \( \Omega^{[\rho\sigma]} = 0 \), \( \hat{V}^\alpha = dr^\alpha/c d\tau = 0 \); ii) there takes place a relation

\[ -\frac{1}{4} \tilde{\eta}_{[\rho] \sigma} \eta^{[\rho]} \eta_{[\sigma]} \hat{V}^\alpha \hat{V}^\beta + h_{\alpha\beta} \hat{V}^\alpha \hat{V}^\beta + \tilde{L}_{[\rho\sigma]} A^{(\mu)} \eta^{[\rho]} \eta_{[\sigma]} + \]

\[ + 2 \tilde{c}_{[\mu]} A^{(\mu)} \hat{V}^\alpha + \tilde{L}_{[\rho\sigma]} \hat{V}^\alpha \eta^{[\rho] \sigma} \Omega^{[\rho\sigma]} = 0, \quad \Sigma = \sigma. \quad (3.43) \]

If this relation is fulfilled for all points of the domain occupied by the extended object one may say about proper time of the object.

Expression (3.30) is turn out to be an interval in 14-dimensional space, covered by coordinates \( X^A \) \( (A = (\mu), [\rho\sigma], \hat{\alpha}) \) with

\[ X^{(\mu)} = x^{(\mu)} \text{, } X^{[\rho\sigma]} = \varphi^{[\rho\sigma]} \text{, } X^{\hat{\alpha}} = r^{\hat{\alpha}}. \]  \quad (3.44)

Then (3.30) may be written as

\[ dS^2 = G_{AB} dX^A dX^B. \]  \quad (3.45)
The interval (3.36) obtained in such a way, generally speaking, describes not necessarily extended particle, but any deformable medium performing translational and rotational movement. Moreover it is supposed, that the equations of motion of any point M of this medium look like geodesic equation in 14-dimensional space

\[ \dot{U}^A + \Gamma^A_{BC} U^B U^C = 0 , \] (3.46)

where

\[ \Gamma^A_{BC} = \frac{1}{2} G^{AD} (\partial_B G_{DC} + \partial_C G_{BD} - \partial_D G_{BC}) , \] (3.47)

\[ U^{(\mu)} = \frac{dx^{(\mu)}}{cd\tau} = \Gamma U^{(\mu)}_C , \] (3.48)

\[ U^{[\rho\sigma]} = \Omega^{[\rho\sigma]} = \frac{d\varphi^{[\rho\sigma]}}{cd\tau} = \Gamma \Omega^{[\rho\sigma]}_C , \] (3.49)

\[ U^{\dot{\alpha}} = V^{\dot{\alpha}} = \frac{dr^{\dot{\alpha}}}{cd\tau} = \Gamma V^{\dot{\alpha}}_C , \] (3.50)

To describe a rotating and deformable particle, it is necessary to impose the certain conditions on metrics \( G_{AB} \), which are considered in the following paragraph.

### 4 Splitting of the equations of motion

Let \( R_{14} \) denotes 14-dimensional space, and \( R_4, H_6 \) and \( \hat{R}_4 \) denote the background space, covered by coordinates \( x^{(\mu)} \), a space of rotation and hyper-rotation angles \( \varphi^{[\rho\sigma]} \), and internal space, covered by coordinates \( r^{\dot{\alpha}} \), respectively. Similarly, \( R_{10}, \hat{R}_{10} \) and \( \hat{R}_4 \) be denotations of spaces of coordinates \( \{x^{(\mu)}, \varphi^{[\rho\sigma]}\} \), \( \{r^{\dot{\alpha}}, \varphi^{[\rho\sigma]}\} \) and \( \{x^{(\mu)}, r^{\dot{\alpha}}\} \), respectively. Thus, \( R_4 = R_{10} \cap \hat{R}_8, H_6 = R_{10} \cap \hat{R}_{10}, \hat{R}_4 = R_8 \cap \hat{R}_{10} \). Obviously, it is possible to present \( R_{14} \) in the form of the direct sum of spaces: \( R_{14} = R_4 \oplus H_6 \oplus \hat{R}_4 \) with \( R_{10} = R_4 \oplus H_6, \hat{R}_{10} = R_4 \oplus H_6, \hat{R}_8 = \hat{R}_4 \oplus \hat{R}_4 \).

By definition, rotations and deformations inside of a particle are independent movements, which are not depending on movement in background space \( R_4 \) and from each other. Therefore, all three spaces, \( R_4, H_6 \) and \( \hat{R}_4 \), should be in certain respects independent and orthogonal to each other. This situation may be described from a viewpoint of the embedded space theory. Actually, the space \( R_{14} \) is trivial embedding space, so that \( R_4 \subset R_{14}, H_6 \subset R_{14}, \hat{R}_4 \subset R_{14} \). Because for a free particle we do not observe any dependence of external movement (in \( R_4 \)) on internal variables, the space \( R_4 \) appears to be stationary hypersurface [32] in \( R_{14} \) defined by conditions \( \varphi^{[\rho\sigma]} = 0, \Omega^{[\rho\sigma]} = 0, r^{\dot{\alpha}} = 0, V^{\dot{\alpha}} = 0 \). This leads to orthogonality of geodesics of the space \( R_4 \) to geodesics of space \( \hat{R}_{10} \), written as a condition

\[
\left[ \frac{1}{2} \tilde{i}^{[\rho\sigma](\mu)} \Omega^{[\rho\sigma]} + c_{(\mu)\dot{\alpha}} V^{\dot{\alpha}} \right] U^{(\mu)} = 0 .
\] (4.1)
Then \((\mu)\)-components of geodesic equations (3.35) in \(\mathbf{R}_{14}\) give geodesic equations in \(\mathbf{R}_{4}\):

\[
\dot{U}^{(\mu)} + \Gamma^{(\mu)}_{\nu\lambda}(U^{(\nu)}U^{(\lambda)}) = 0 ,
\]

(4.2)

\[
\nabla_{(\mu)}\tilde{\eta}^{[\rho\sigma]}_{(\nu)} + \nabla_{(\nu)}\tilde{\eta}^{[\rho\sigma]}_{(\mu)} = 2\frac{\partial\tilde{\eta}^{(\mu)(\nu)}}{\partial\phi^{[\rho\sigma]}} = 0 ,
\]

(4.3)

\[
\nabla_{(\mu)}c(\nu)\dot{\alpha} + \nabla_{(\nu)}c(\mu)\dot{\alpha} = \frac{\partial\tilde{\eta}^{(\mu)(\nu)}}{\partial\phi^{[\rho\sigma]}} = 0 ,
\]

(4.4)

where \(\nabla_{(\mu)}\) denotes covariant derivative in \(\mathbf{R}_{4}\),

\[
\Gamma^{(\mu)}_{\nu\lambda}(\eta^{(\nu)(\lambda)}) = \frac{1}{2}\eta^{(\mu)(\nu)}[\partial_{\nu}\eta^{(\nu)(\lambda)} + \partial_{\lambda}\eta^{(\nu)(\nu)} - \partial_{\nu}\eta^{(\nu)(\lambda)}]
\]

(4.5)

are coefficients of connection in \(\mathbf{R}_{4}\), which ought to be called an "external connection", where \(\eta^{(\mu)(\nu)}\) satisfy relations \(\eta^{(\mu)(\nu)}\eta^{(\nu)(\nu)} = \delta^{(\mu)}_{(\nu)}\).

The space \(\hat{\mathbf{R}}_{10} = \hat{\mathbf{R}}_{4} \oplus \mathbf{H}_{6}\) in turn is an embedding space for \(\hat{\mathbf{R}}_{4}\) and \(\mathbf{H}_{6}\). \(\hat{\mathbf{R}}_{4}\) is stationary hypersurface in \(\hat{\mathbf{R}}_{10}\) defined by conditions \(\varphi^{[\rho\sigma]} = 0, \Phi^{[\rho\sigma]} = 0\). It leads both to orthogonality of geodesics in \(\hat{\mathbf{R}}_{4}\) and \(\mathbf{H}_{6}\).

\[
\frac{1}{2}\tilde{\ell}^{[\rho\sigma]}_{\dot{\alpha}}\Phi^{[\rho\sigma]}V^{\dot{\alpha}} = 0 ,
\]

(4.6)

and to equations of motion

\[
\dot{V}^{\dot{\alpha}} + H^{\dot{\alpha}}_{\beta\dot{\gamma}}V^{\dot{\beta}}U^{\dot{\gamma}} = 0 ,
\]

(4.7)

\[
\nabla_{\dot{\alpha}}c(\nu)\dot{\beta} + \nabla_{\dot{\beta}}c(\mu)\dot{\alpha} = \partial_{(\mu)}h^{\dot{\alpha}\dot{\beta}} ,
\]

(4.8)

\[
\nabla_{\dot{\alpha}}\tilde{\ell}^{[\rho\sigma]}_{\dot{\beta}} + \nabla_{\dot{\beta}}\tilde{\ell}^{[\rho\sigma]}_{\dot{\alpha}} = 2\frac{\partial h^{\dot{\alpha}\dot{\beta}}}{\partial\phi^{[\rho\sigma]}} = 0 ,
\]

(4.9)

where \(\nabla_{\dot{\alpha}}\) denotes covariant derivative in \(\hat{\mathbf{R}}_{4}\),

\[
H^{\dot{\alpha}}_{\dot{\beta}\dot{\gamma}} = \frac{1}{2}h^{\dot{\alpha}\dot{\beta}}[\partial_{\dot{\beta}}\eta^{\dot{\delta}\dot{\gamma}} + \partial_{\dot{\gamma}}\eta^{\dot{\beta}\dot{\delta}} - \partial_{\dot{\delta}}\eta^{\dot{\beta}\dot{\gamma}}]
\]

(4.10)

are coefficients of connection in \(\hat{\mathbf{R}}_{4}\), which ought to be called an "internal connection" of extended particle; \(h^{\dot{\alpha}\dot{\beta}}\) satisfy relations \(h^{\dot{\alpha}\dot{\beta}}h_{\dot{\gamma}\dot{\delta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}\).

The space \(\mathbf{R}_{14}\) may be represented also as \(\mathbf{R}_{14} = \hat{\mathbf{R}}_{8} \oplus \mathbf{H}_{6}\) and considered as embedding space for \(\hat{\mathbf{R}}_{8} \subset \mathbf{R}_{14}\). Then \(\hat{\mathbf{R}}_{8}\) is stationary hypersurface in \(\mathbf{R}_{14}\) defined by conditions \(\varphi^{[\rho\sigma]} = 0, \Phi^{[\rho\sigma]} = 0\). It leads both to orthogonality of geodesics in \(\hat{\mathbf{R}}_{8}\) and \(\mathbf{H}_{6}\) and to equations of motion, which in view of (4.2)-(4.4), (4.7)-(4.9) are led to the equations

\[
\partial_{(\mu)}c(\nu)\dot{\alpha} - \partial_{(\nu)}c(\mu)\dot{\alpha} = \frac{\partial\tilde{\eta}^{(\mu)(\nu)}}{\partial\phi^{[\rho\sigma]}} = 0 ,
\]

(4.11)

\[
\partial_{(\mu)}\tilde{\ell}^{[\rho\sigma]}_{\dot{\alpha}} + \partial_{\dot{\alpha}}\tilde{\ell}^{[\rho\sigma]}_{(\mu)} = 2\frac{\partial c(\mu)\dot{\alpha}}{\partial\phi^{[\rho\sigma]}} = 0 ,
\]

(4.12)
Orthogonality condition (4.6) is complemented with two more conditions

$$\partial_\beta c(\mu)\hat{\alpha} - \partial_\alpha c(\mu)\hat{\beta} = \partial(\mu) h_{\hat{\alpha}\hat{\beta}} = 0 .$$

(4.13)

In deriving equations (4.2)-(4.4), (4.7)-(4.9), (4.11)-(4.13) we have been using an independence of the internal metric on coordinates $\varphi[\rho\sigma]$ and $r\hat{\alpha}$, as well as independence of the internal metric of the space $\hat{g}$ metric, defined by a character of internal movements. On the other hand, in this case simple form may write the equation (2.7) does not depend on angular variations, i.e.

$$\partial R \text{space}$$

(4.14)

$$c(\mu)\hat{\alpha} U(\mu) V^{\hat{\beta}} = 0 .$$

(4.15)

Let us clarify this relation in simplest case when i) background space is the Minkowski space $\mathbf{R}_4 \equiv \mathbf{E}^4_{\mu\nu}$ with the metric $\eta = \{\eta(\mu)(\nu)\} = \text{diag}(+1, -1, -1, -1)$, and ii) l.h.s. of the equation (2.7) does not depend on angular variations, i.e. $d^{\mu}_{\ [\lambda\kappa]} = 0$. In this case we have $i_{\mu\nu}[\lambda] = i^{0}_{\ [\mu\nu][\lambda]}$, $j_{\mu\nu}[\lambda\kappa] = j^{0}_{\ [\mu\nu][\lambda\kappa]}$. In the reference frame of the "center of inertia" the motion of the point M is describing by the deformation tensor $h_{\hat{\alpha}\hat{\beta}}$, being an internal metric, defined by a character of internal movements. On the other hand, in this case $h_{\hat{\alpha}\hat{\beta}}$ should coincide with an external metric $g_{\mu\nu}$, so that $b^{\mu}_{\ [\hat{\alpha}]} = \delta^{\mu}_{\hat{\alpha}}$, and relation (3.35) takes a form $h_{\hat{\alpha}\hat{\beta}} = g_{\mu\nu} \delta^{\mu}_{\hat{\alpha}} \delta^{\nu}_{\hat{\beta}}$, or $g_{\mu\nu} = h_{\hat{\alpha}\hat{\beta}} \delta_{\hat{\alpha}}^\mu \delta_{\hat{\beta}}^\nu$.

Let us introduce following notations

$$\Omega_{(k)} = \frac{1}{2} \varepsilon_{(k)(l)(m)} \Omega^{[lm]}_C , \quad B^{(k)} = \Omega^{[0k]}_C .$$

(4.16)

Let us introduce following notations

$$\Omega_{(k)} = \frac{1}{2} \varepsilon_{(k)(l)(m)} \Omega^{[lm]}_C , \quad B^{(k)} = \Omega^{[0k]}_C .$$

(4.17)

Obviously, $\Omega_{(k)}$ is a component of pseudovector $\Omega = \{\Omega_{(k)}\}$ of angular velocity of rotation of the object in the plane ($l$, $m$), perpendicular to the axis ($k$); $B = \{B^{(k)}\}$ is some vector, whose physical meaning will have to clarify yet. Then the equation (4.17) may be write down in the form

$$[\Omega \times r]^2 - 2(0)(B \cdot [\Omega \times r]) + (r\hat{0})^2 B^2 - (r \cdot B)^2 = 0 ,$$

(4.18)

where scalar and vector products are determined relative to background metric:

$$[\Omega \times r]^{(k)} = \varepsilon^{(k)(l)(m)} \eta_{(m)\hat{\alpha}} \Omega^{(l)} r\hat{\alpha} , \quad [\Omega \times r]^2 = r^2 \Omega^2 - (r \cdot \Omega)^2 ,$$

(4.19)

$$B = B^{(m)} B^{(m)}, \quad \Omega^2 = -\eta^{(m)(n)} \Omega^{(m)} \Omega^{(n)} ,$$

(4.20)
\[(\mathbf{r} \cdot \mathbf{\Omega}) = \delta_{,\alpha}^{(m)} r^{\hat{\alpha}} \Omega^{(m)} , \quad (\mathbf{r} \cdot \mathbf{B}) = -\eta_{(m)\hat{\alpha}} B^{(m)} r^{\hat{\alpha}} . \quad (4.23)\]

In the reference frame of the "center of inertia" the orthogonality condition (4.6) looks as follows
\[
\frac{1}{2} \epsilon_{[\mu\nu\lambda]} \Omega^{[\mu\nu]} V_C^{\hat{\alpha}} = [e_{(0)}^{(m)} (\mathbf{r} \cdot \mathbf{B}) - \hat{r}^{\hat{\alpha}} (\mathbf{e}_\alpha \cdot \mathbf{B}) - (\mathbf{e}_\alpha \cdot [\mathbf{\Omega} \times \mathbf{r}])] V_C^{\hat{\alpha}} = 0 , \quad (4.24)\]

where vectors \(\mathbf{e}_\alpha = \{e_{(m)}^{(n)}\}\) are spatial components of moving 4-hedron and
\[
(\mathbf{e}_\alpha \cdot \mathbf{B}) = -\eta_{(m)(n)} e_{(m)}^{(n)} B^{(n)} , \quad (\mathbf{e}_\alpha \cdot [\mathbf{\Omega} \times \mathbf{r}]) = -\eta_{(m)(n)} e_{(m)}^{(n)} [\mathbf{\Omega} \times \mathbf{r}]^{(n)} . \quad (4.25)\]

Conditions (4.14) and (4.15) in arbitrary reference frame look as
\[
\frac{1}{2} \epsilon_{[\mu\nu\lambda]} U^{(\lambda)} V_C^{\hat{\alpha}} = V_C^{\hat{\alpha}} (\mathbf{r} \cdot \mathbf{B}) - \hat{r}^{\hat{\alpha}} (\mathbf{V}_C \cdot \mathbf{B}) - (\mathbf{V}_C \cdot [\mathbf{\Omega} \times \mathbf{r}]) = 0 , \quad (4.26)\]
\[
c_{(\mu)\hat{\alpha}} U^{(\mu)} V_C^{\hat{\alpha}} = c_{(0)\hat{\alpha}} U^{(0)} V_C^{\hat{\alpha}} + c_{(m)\hat{\alpha}} U^{(m)} V_C^{\hat{\alpha}} = 0 . \quad (4.27)\]

In the reference frame of the "center of inertia", where \(U^{(0)} = 1, U^{(m)} = 0\), conditions (4.24) (4.26), (4.27) are reduced to
\[
(\mathbf{V}_C \cdot \{ -\hat{r}^{\hat{\alpha}} \mathbf{B} + [\mathbf{\Omega} \times \mathbf{r}] \}) = 0 , \quad (4.28)\]
\[
(\mathbf{r} \cdot \mathbf{B}) = 0 , \quad (4.29)\]
\[
h_{\hat{\alpha}\hat{\beta}} e_{\hat{\alpha}}^{(0)} V_C^{\hat{\beta}} = 0 . \quad (4.30)\]

where
\[
\mathbf{V}_C = V_C^{\hat{\alpha}} \mathbf{e}_\alpha = \{V_C^{\hat{\alpha}} e_{\hat{\alpha}}^{(m)}\} = \{V_C^{\hat{\alpha}} e_{\hat{\alpha}}^{(m)}\} . \quad (4.31)\]

It follows from (4.28), that
\[
\hat{r}^{\hat{\alpha}} \mathbf{B} = [\mathbf{\Omega} \times \mathbf{r}] + [\mathbf{V}_C \times \mathbf{K}] . \quad (4.32)\]

As well it follows from (4.29) and (4.32), that \((\mathbf{r} \cdot [\mathbf{V}_C \times \mathbf{K}]) = (\mathbf{K} \cdot [\mathbf{r} \times \mathbf{V}_C]) = 0\). Hence, pseudovector \(\mathbf{K}\), perpendicular to the vector \(\mathbf{V}_C\) and lying in the plane formed by the vectors \(\mathbf{r}\) and \(\mathbf{V}_C\), equals to \(\mathbf{K} = a [\mathbf{V}_C \times [\mathbf{r} \times \mathbf{V}_C]]\), whence
\[
\hat{r}^{\hat{\alpha}} \mathbf{B} = [(\mathbf{\Omega} + a \mathbf{V}_C^2) \mathbf{V}_C] \times \mathbf{r} . \quad (4.33)\]

If pseudoscalar \(a = 0\), then the vector \(\mathbf{B}\) is parallel to the vector \([\mathbf{\Omega} \times \mathbf{r}]\) of linear velocity of the motion of the point \(M\) around the axis \(\mathbf{\Omega}\) due to only rotation of extended object, and condition (4.29) is fulfilled automatically. In view of (4.33) the condition (4.19) in the case above takes the form
\[
a^2 \mathbf{V}_C^4 [\mathbf{r} \times \mathbf{V}_C]^2 - 2[\mathbf{\Omega} \times \mathbf{r}]^2 + h_{\hat{\alpha}\hat{\beta}} V_C^{\hat{\alpha}} V_C^{\hat{\beta}} = 0 . \quad (4.34)\]

Orthogonality conditions (4.6), (4.14) and (4.15) imply, that equations of motion (3.46) in the space \(\mathbf{R}_{14}\) are splitted in equations (4.2), (4.3), (4.7), (4.9), (4.12) and
(4.16). These equations may be written down in the other form by the introduction of the generalized momenta, conjugated with coordinates $x^{(\mu)}$, $p^{(\alpha)}$, $\varphi^{[\rho\sigma]}$.

As it is well known, equations (3.46), determining a motion of point M, one can obtain from the action, representable similar to the relativistic mass point action in the form

$$J = \int Lcd\tau = -c \int [\Sigma dS^2]^{1/2},$$

(4.35)

where $dS^2$ is defined in (3.30), and the mass of the point is supposed to be unit.

We define the specific generalized momenta

$$p_{(\mu)} = -\frac{\partial L}{c\partial U^{(\mu)}} = \Sigma c \left[ \eta_{(\mu)(\nu)}U^{(\nu)} + \frac{1}{2}t_{[\rho\sigma](\mu)}\Omega^{[\rho\sigma]} + c_{(\mu)\alpha}V^{\alpha} \right] =$$

$$= p^{C}_{(\mu)} + p^{rot}_{(\mu)} + p^{M}_{(\mu)};$$

$$s_{[\lambda\kappa]} = -\frac{\partial L}{c\partial \Omega^{[\lambda\kappa]}} = \Sigma c \left[ -\frac{1}{4}t_{[\lambda\kappa][\rho\sigma]}\Omega^{[\rho\sigma]} + \frac{1}{2}t_{[\lambda\kappa](\nu)}U^{(\nu)} + \frac{1}{2}\tilde{\ell}_{[\lambda\kappa]\alpha}V^{\alpha} \right] =$$

$$= \frac{1}{2}t_{[\lambda\kappa](\mu)}p^{(\nu)} = s^{rot}_{[\lambda\kappa]} + c^{C}_{[\lambda\kappa]} + s^{M}_{[\lambda\kappa]},$$

(4.37)

$$\pi_{\alpha} = -\frac{\partial L}{c\partial V^{\alpha}} = \Sigma c \left[ h^{(\alpha\beta)}V^{\beta} + c_{(\nu)\alpha}U^{(\nu)} + \frac{1}{2}\tilde{\ell}_{[\rho\sigma]\alpha}\Omega^{[\rho\sigma]} \right] =$$

$$= c_{(\mu)\alpha}\eta^{(\mu)(\nu)}p^{(\nu)} = \pi^{M}_{\alpha} + \pi^{C}_{\alpha} + \pi^{M}_{\alpha},$$

(4.38)

and generalized forces $f_{A} = -\partial L/\partial X^{A}$. Then in view of (4.2)-(4.18) equations (3.35) are splitted in system of the following equations

$$\frac{D_{h}p^{C}_{(\mu)}}{cd\tau} = \frac{dp^{C}_{(\mu)}}{cd\tau} - \Gamma^{(\mu)(\nu)}_{(\rho)}U^{(\nu)}p^{C}_{(\rho)} = 0,$$

(4.39)

$$\frac{D_{h}p^{rot}_{(\mu)}}{cd\tau} = \frac{\Sigma c}{16} \left[ \frac{\partial t_{[\lambda\kappa](\mu)}}{\partial \varphi^{[\rho\sigma]}} - \frac{1}{2}\partial_{(\mu)}\tilde{\ell}_{[\lambda\kappa][\rho\sigma]} \right] \Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]},$$

(4.40)

$$\frac{D_{h}p^{M}_{(\mu)}}{cd\tau} = \frac{\Sigma c}{2} \left[ \tilde{t}_{[\lambda\kappa](\mu)}\tilde{\Omega}^{[\lambda\kappa]} - \frac{1}{8}\partial_{(\mu)}\tilde{\ell}_{[\lambda\kappa][\rho\sigma]} \Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]} \right] =$$

$$= \frac{\Sigma c}{2}\tilde{t}_{[\lambda\kappa](\mu)} \left[ \delta^{(\nu)}_{(\mu)}\tilde{\Omega}^{[\lambda\kappa]} + \frac{1}{4}\eta^{(\nu)(\tau)}\nabla_{(\mu)}\tilde{t}_{[\rho\sigma](\tau)} \Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]} \right];$$

(4.41)

$$\frac{D_{h}\pi^{M}_{\alpha}}{cd\tau} = \frac{d\pi^{M}_{\alpha}}{cd\tau} - H^{(\gamma)}_{\cdot\alpha\beta}V^{\beta}\pi^{M}_{\gamma} = 0,$$

(4.42)

$$\frac{D_{h}\pi^{rot}_{\alpha}}{cd\tau} = \frac{\Sigma c}{8} \left[ \frac{\partial \tilde{\ell}_{[\lambda\kappa]\alpha}}{\partial \varphi^{[\rho\sigma]}} - \frac{1}{4}\partial_{\alpha}\tilde{\ell}_{[\lambda\kappa][\rho\sigma]} \right] \Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]},$$

(4.43)

$$\frac{D_{h}\pi^{M}_{\alpha}}{cd\tau} = \frac{\Sigma c}{2} \left[ \tilde{\ell}_{[\lambda\kappa]\alpha}\tilde{\Omega}^{[\lambda\kappa]} - \frac{1}{8}\partial_{\alpha}\tilde{\ell}_{[\lambda\kappa][\rho\sigma]} \Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]} \right] =$$

15
\[
\frac{ds_{[\mu\nu]}^{rot}}{cd\tau} = \frac{2}{c} \frac{d(s_C^{[\mu\nu]} + s_M^{[\mu\nu]})}{cd\tau} + \frac{\Sigma c}{2} \tilde{z}^{[\mu\nu][\rho\sigma]} \partial_{[\rho\sigma]} \tilde{z}^{[\mu\nu]} - \eta^{(\lambda)(\kappa)} (p_C^{(\lambda)} + p_M^{(\lambda)}) \frac{\partial_{[\rho\sigma]} \tilde{j}^{[\mu\nu](\kappa)}}{\partial_{[\rho\sigma]} \tilde{j}^{[\mu\nu](\kappa)}} \Omega^{[\rho\sigma]}.
\]

Orthogonality conditions (4.6), (4.14), (4.15) reduce the interval (3.38) to the form

\[
dS^2 = \Sigma c^2 dt^2 = \eta_{(\mu)(\nu)} dx^{(\mu)} dx^{(\nu)} - \frac{1}{4} \tilde{z}^{[\lambda\kappa][\rho\sigma]} \partial_{[\rho\sigma]} \tilde{z}^{[\lambda\kappa]} - h^{\hat{\alpha}\hat{\beta}} df^{\hat{\alpha}} df^{\hat{\beta}}.
\]

with additional condition that \(z^\hat{\alpha}\) does not depend on \(x^{(\mu)}\), but, generally speaking, may depend on \(\varphi^{[\lambda\kappa]}\). Starting from (4.3), (4.9), (4.12) and (4.16) one may show, that

\[
\partial_{\tilde{\alpha}} \tilde{j}^{[\mu\nu](\mu)} = \eta^{(\nu)(\tau)} c_{(\nu)} \tilde{\alpha} \nabla_{(\tau)} \tilde{j}^{[\mu\nu](\mu)},
\]

\[
\partial_{\tilde{\alpha}} \tilde{j}^{[\lambda\kappa][\rho\sigma]} = \eta^{(\nu)(\tau)} c_{(\nu)} \tilde{\alpha} \partial_{(\tau)} \tilde{j}^{[\lambda\kappa][\rho\sigma]}.
\]

Now, independent conditions (4.3), (4.12) and (4.48) are remained instead of inter-dependent conditions (4.3), (4.9), (4.12) and (4.16). All the other relations (4.4), (4.8), (4.9), (4.11)-(4.13), (4.16) and (4.49) are their consequences. Interdependency of these relations is result of degeneracy of the metric \(G_{AB}\). It is manifested also in connection (4.37), (4.38) between \(p_{(\mu)}\), \(s_{[\lambda\kappa]}\) and \(\pi_{\hat{\alpha}}\). Owing to this fact equations of motion (4.39)-(4.46) reduce to only three ones: (4.39), (4.42) and

\[
\frac{ds_{[\mu\nu]}^{rot}}{cd\tau} = -\frac{\Sigma c}{8} \partial_{\tilde{\alpha}} \tilde{j}^{[\lambda\kappa][\rho\sigma]} \partial_{[\rho\sigma]} \tilde{j}^{[\lambda\kappa]} \Omega^{[\rho\sigma]}.
\]

Quantity \(p_C^{(\mu)}\) is a specific momentum of the "center of inertia", \(\pi_{\hat{\alpha}}^M\) is a momentum of point M relative to the "center of inertia", caused by movement of substance inside extended object, \(s_{[\mu\nu]}^{rot}\) is angular momentum of point M relative to the "center of inertia", caused by rotation of this object. As to dependence of the inertia tensor on \(\varphi^{[\mu\nu]}\) it is reasonable to require it to be invariant under translations on angular variables \(\varphi^{[mn]}\), \(m, n = 1, 2, 3\). Because of an obscurity of variables \(\varphi^{[0n]}\) it would be necessary, generally speaking, to conserve some freedom in dependence of the inertia tensor on these variables. However, linking \(s_{[\mu\nu]}^{rot}\) with spin of the particle (which should be determined individually) and taking into account the observable fact, that spin of free particles conserves its value, we shall require a translational invariance on \(\varphi^{[0n]}\) as well. It leads to the equation \(ds_{[\mu\nu]}^{rot}/cd\tau = 0\), which means \(\Omega^{[\mu\nu]} = const\).

Thus, we have gave one of possible variants of the description of free relativistic extended particle, where it is necessary to start from the equations of motion (4.39),
(4.42) and (4.50). If the background space is the Minkowski space, then conservation of spin (or angular velocity of rotation) as well as conservation of momentum of extended particle take place as consequences of these equations. Even in this case the internal space is curved.

5 Degeneracy of the metric

Let us summarize the results obtained above. Firstly, representation of the space-time interval (3.38) of the world line of a point M inside of a particle in 14-dimensional space, covered by the center-of-inertia, angular and deformational coordinates, in the form (4.47) is possible provided imposing the orthogonality conditions (4.6), (4.14), (4.15). This is equivalent to splitting of $R_{14}$ in three subspaces $R_{4}$, $H_{6}$ and $R_{4}$. In this case the general equation of motion (3.46) also splits in equation (4.2) (or (4.39)) of center-of-inertia motion, equation (4.7) (or (4.42)) of motion of the point M in $R_{4}$ relative to the center of inertia, which does not depend on external motions, and equation (4.50) of rotational motion of the point M around the center of inertia. Secondly, the interval (4.47) along world line of a point M in $R_{14}$ turns out to be equivalent to the interval (3.41) along center-of-inertia world line in $R_{4}$ provided fulfillment of condition (4.17). This fact makes possible to introduce time coinciding with proper time of the center of inertia in all points of the region in $R_{4}$, occupied by an extended object. Thirdly, equivalence of intervals (4.47) and (3.41) means the metric $G_{AB}$ of the space $R_{14}$ be degenerate and have rank which is equal to the rank of background metric.

Obtained results are compatible with the theory of degenerate Riemannian spaces [33], according to which the space $R_{14}$ is reducible and there take place relations:

\[ \partial_{\alpha} \eta_{(\mu)(\nu)} = 0, \quad \partial_{[\lambda\kappa]} \eta_{(\mu)(\nu)} = 0, \quad \partial_{[\lambda\kappa]} h_{\dot{\alpha}\dot{\beta}} = 0; \quad (5.1) \]

\[ \nabla_{(\mu)} c_{(\nu)} = 0, \quad \nabla_{\dot{\alpha}} c_{(\mu)\dot{\beta}} = 0, \quad \partial_{[\lambda\kappa]} c_{(\mu)\dot{\alpha}} = 0; \quad (5.2) \]

\[ \nabla_{(\mu)} \tilde{t}_{(\nu)[\lambda\kappa]} = 0, \quad \partial_{\dot{\alpha}} \tilde{t}_{(\mu)[\lambda\kappa]} = 0, \quad \partial_{[\lambda\kappa]} \tilde{t}_{(\mu)[\rho\sigma]} = 0; \quad (5.3) \]

\[ \partial_{(\mu)} \tilde{\ell}_{\dot{\alpha}[\lambda\kappa]} = \partial_{(\mu)} \tilde{\ell}_{[\lambda\kappa]\dot{\alpha}} = 0, \quad \nabla_{\dot{\beta}} \tilde{\ell}_{\dot{\alpha}[\lambda\kappa]} = 0, \quad \partial_{[\lambda\kappa]} \tilde{\ell}_{\dot{\alpha}[\rho\sigma]} = 0; \quad (5.4) \]

\[ \partial_{(\mu)} h_{\dot{\alpha}\dot{\beta}} = 0, \quad \partial_{(\mu)} j_{[\lambda\kappa][\rho\sigma]} = 0, \quad \partial_{\dot{\alpha}} j_{[\lambda\kappa][\rho\sigma]} = 0. \quad (5.5) \]

Relations (5.1)-(5.3) are independent and compatible with both relations (4.3), (4.16) and equations of motion (4.39), (4.42) and (4.50). However, it follows from the last equation in (5.5) that even for absolutely rigid body, when inertia tensor has the form (3.1) (or (3.15)), $j^0_{[\lambda\kappa][\rho\sigma]} = 0$ due to identity

\[ r^{\dot{\alpha}} \partial_{\dot{\alpha}} (r^2 \dot{D}_{(\mu)(\nu)}) = 2r^2 \dot{D}_{(\mu)(\nu)}, \quad (5.6) \]

whence

\[ r^{\dot{\alpha}} \partial_{\dot{\alpha}} j^0_{[\lambda\kappa][\rho\sigma]} = 2j^0_{[\lambda\kappa][\rho\sigma]}, \quad r^{\dot{\alpha}} \partial_{\dot{\alpha}} \tilde{\dot{\alpha}} e^0_{[\lambda\kappa][\rho\sigma]} = 2\tilde{\dot{\alpha}} e^0_{[\lambda\kappa][\rho\sigma]}. \quad (5.7) \]
In the general case condition \( \partial_\dot{\alpha} \tilde{\lambda}_\kappa|_{[\rho|\sigma]} = 0 \) (more exactly, two conditions, \( \nabla_\dot{\alpha} c_{(\mu)\dot{\beta}} = 0 \) and \( \partial_\dot{\alpha} \hat{\kappa}(\mu)[\lambda\kappa] = 0 \)) expresses restriction to movement of substance inside of an extended object, and it is not necessarily this condition leads to \( j|_{[\lambda\kappa]|_{[\rho|\sigma]} = 0 \).

It follows from the last condition in (5.3) that \( \partial_{[\mu\nu]} j|_{[\lambda\kappa]|_{[\rho|\sigma]} = 0 \). Then equation of motion (4.50) means, that all internal points are rotating with the same constant angular velocity. In the general case it is not quite necessarily. Therefore condition \( \partial_{[\mu\nu]} \tilde{\lambda}(\mu)|_{[\rho|\sigma]} = 0 \), as well as some other conditions (5.1)-(5.3), is not necessary too.

It should be noted that the main result of splitting of equation of motion have to be in geodesic motion of the center of inertia in the background space \( \mathbf{R}_4 \) and geodesic motion of the point M in the internal space \( \hat{\mathbf{R}}_4 \) provided the rotation is absent. Moreover, total proper angular momentum, which may be defined as

\[
S_{[\mu\nu]} = \int_V \rho s_{[\mu\nu]}^{\text{rot}} dV ,
\]

where \( \rho \) is mass density, \( V \) is volume of the region, occupied by extended object, should be conserved. Obviously, it follows from \( j|_{[\lambda\kappa]|_{[\rho|\sigma]} = 0 \) that \( S_{[\mu\nu]} = 0 \). Such an object may be associated with scalar particle.

We shall consider more adequate model, in which \( \mathbf{R}_4 \) and \( \hat{\mathbf{R}}_4 \) is trivial embedding space for \( \mathbf{R}_4 \) and \( \hat{\mathbf{R}}_{10} \), but \( \hat{\mathbf{R}}_{10} \) is not an embedding space for \( \hat{\mathbf{R}}_4 \) and \( \mathbf{H}_6 \). In this case coefficients of connection \( \Gamma^\alpha_{\cdot \beta[\lambda\kappa]}, \Gamma_{\cdot \beta[\lambda\kappa]}, \Gamma_{\cdot \beta[\lambda\kappa]}, \Gamma_{\cdot \beta[\lambda\kappa]} \) are, in general, not equal to zero and instead of conditions (5.1)-(5.5) there will be fulfilled conditions \([33]\)

\[
\partial_\dot{\alpha} \eta(\mu)(\nu) = 0 , \quad \partial_{[\lambda\kappa]} \eta(\mu)(\nu) = 0 ;
\]

\[
\nabla_{(\mu)} c(\nu)_{\dot{\alpha}} = \partial_{(\mu)} c(\nu)_{\dot{\alpha}} - \Gamma^{(\lambda)}_{\cdot (\mu)(\nu)} c(\lambda)_{\dot{\alpha}} = 0 ;
\]

\[
\nabla_\dot{\alpha} c_{(\mu)\dot{\beta}} = \partial_\dot{\alpha} c_{(\mu)\dot{\beta}} - H^{\gamma}_\dot{\alpha} c(\mu)^\gamma - \frac{1}{2} \Gamma^{[\ldots]}_{\cdot [\ldots][\rho|\sigma]} \cdot \dot{\gamma} = 0 ;
\]

\[
\nabla_{[\lambda\kappa]} c(\mu)_{\dot{\alpha}} = \partial_{[\lambda\kappa]} c(\mu)_{\dot{\alpha}} - \Gamma^{\dot{\gamma}}_{\cdot [\lambda\kappa] \alpha} c(\mu)^\gamma - \frac{1}{2} \Gamma_{\cdot [\ldots]}^{[\ldots]} \cdot \dot{\gamma} = 0 ;
\]

\[
\nabla_{(\mu)} \dot{\kappa}(\nu)_{[\lambda\kappa]} = \partial_{(\mu)} \dot{\kappa}(\nu)_{[\lambda\kappa]} - \Gamma_{\cdot [\mu\nu]} \cdot \dot{\gamma} = 0 ,
\]

\[
\nabla_\dot{\alpha} \dot{\kappa}(\mu)_{[\lambda\kappa]} = \partial_\dot{\alpha} \dot{\kappa}(\mu)_{[\lambda\kappa]} - 2 \Gamma^{\gamma}_{\cdot \alpha[\lambda\kappa]} c(\mu)^\gamma - \Gamma_{\cdot [\ldots]}^{[\ldots]} \cdot \dot{\gamma} = 0 ,
\]

\[
\nabla_{[\lambda\kappa]} \dot{\kappa}(\mu)_{[\rho|\sigma]} = \partial_{[\lambda\kappa]} \dot{\kappa}(\mu)_{[\rho|\sigma]} - 2 \Gamma^{\gamma}_{\cdot [\lambda\kappa] \rho|\sigma} c(\mu)^\gamma - \Gamma_{\cdot [\ldots]}^{[\ldots]} \cdot \dot{\gamma} = 0 ,
\]

Conditions (5.5), which also should be fulfilled, are consequence of (5.9)-(5.15). Connections \( \Gamma_{\cdot BC}^A \) are determined from vanishing of covariant derivatives \([33]\):

\[
G_{AB;C} = 0 , \quad g^{A}_{\epsilon;B} = 0 , \quad (A, B = (\mu), \dot{\alpha}, [\lambda\kappa]) ,
\]

(5.16)
where \( G_{AB} \) is degenerate metric of original 14-dimensional space (\( \text{rank}(G_{AB}) = 4 \)), and \( g^A_e \) \((a = 1, 2, ..., 10)\) are ten linearly independent solutions of the equation \( G_{AB} g^B_e = 0 \), which gives
\[
 g^{(\mu)}_e = -\eta^{(\mu)(\nu)} \left[ c(\nu) g^\alpha_e + \frac{1}{2} \hat{\Gamma}_e^{[\mu\nu]} g^{[\lambda\kappa]}_e \right], \tag{5.17}
\]
with the requirement \( \det(g^a_e) \neq 0 \) \((a = \hat{\alpha}, [\lambda\kappa])\). For \( \Gamma^A_{BC} = \Gamma^A_{CB} \) we receive following expressions:
\[
 \Gamma^{(\lambda)}_{(\mu)(\nu)} = \frac{1}{2} \eta^{(\lambda)(\kappa)} \partial^{(\mu)}(\eta^{(\nu)(\kappa)}) + \partial^{(\nu)}(\eta^{(\mu)(\kappa)}) - \partial^{(\kappa)}(\eta^{(\mu)(\nu)}) \right] , \tag{5.18}
\]
\[
 \Gamma^{(\lambda)}_{ab} = 0 . \tag{5.19}
\]
As it was shown above, imposing of orthogonality conditions (4.6), (4.14), (4.15) leads to equivalency of metrics\(^1\) (3.38) (or (3.45)) and (4.47) and gives following expressions for the rest coefficients of connection:
\[
 \Gamma^a_{(\mu)(\nu)} = 0 , \quad \Gamma^a_{(\mu)b} = 0 , \quad \Gamma^{[\mu\nu]}_{.. \hat{a}\hat{b}} = 0 , \tag{5.20}
\]
\[
 \Gamma_{\hat{a}\hat{b}\gamma} = \frac{1}{2} h^{\hat{a}\hat{b}} \left[ \partial_{\hat{a}} h_{\hat{b}\gamma} + \partial_{\hat{b}} h_{\hat{a}\gamma} - \partial_{\gamma} h_{\hat{a}\hat{b}} \right] = H^{\hat{a}\hat{b}\gamma} , \tag{5.21}
\]
\[
 \Gamma^\alpha_{\hat{b}[\lambda\kappa]} = \frac{1}{4} h^{\hat{a}\hat{b}} \partial^{\hat{a}\hat{b}} h_{\lambda\kappa} , \tag{5.22}
\]
\[
 \Gamma^\hat{a}_{[\mu\nu][\lambda\kappa]} = \frac{1}{8} h^{\hat{a}\hat{b}} \partial^{\hat{a}\hat{b}} h_{[\mu\nu][\lambda\kappa]} , \tag{5.23}
\]
Coefficients of connection \( \Gamma^{[\mu\nu]}_{.. \hat{a}[\lambda\kappa]} \) and \( \Gamma^{[\mu\nu]}_{.. [\lambda\kappa][\rho\sigma]} \) may be found from the metric (4.47). Then equivalency of two metrics gives a system of differential equations for \( g^a_e \):
\[
 \partial_{\hat{b}} g^{\hat{a}}_e + H^{\hat{a}\hat{b}} g^{\hat{b}}_e + \frac{1}{2} \Gamma^{[\lambda\kappa]}_{\hat{a}\hat{b}} g^{[\lambda\kappa]}_e = 0 , \tag{5.24}
\]
\[
 \partial_{\lambda\kappa} g^{\hat{a}}_e + \Gamma^\alpha_{\hat{b}[\lambda\kappa]} g^{\hat{b}}_e + \frac{1}{2} \Gamma^{[\lambda\kappa]}_{\hat{a}[\rho\sigma]} g^{[\rho\sigma]}_e = 0 , \tag{5.25}
\]
\[
 \partial_{\hat{a}} g^{[\lambda\kappa]}_e + \frac{1}{2} \Gamma^{[\lambda\kappa]}_{\hat{a}[\rho\sigma]} g^{[\rho\sigma]}_e = 0 , \tag{5.26}
\]
\[
 \partial_{[\mu\nu]} g^{[\lambda\kappa]}_e + \Gamma^{[\lambda\kappa]}_{[\mu\nu]} g^{\hat{b}}_e + \frac{1}{2} \Gamma^{[\lambda\kappa]}_{[\rho\sigma]} g^{[\rho\sigma]}_e = 0 . \tag{5.27}
\]
Equations of motion for the case above may be written in the form
\[
 \frac{D_{\eta P^{(\mu)}}}{cdT} = \frac{dP^{(\mu)}}{cdT} - \Gamma^{(\mu)}_{(\rho)(\nu)} U^{(\nu)} p^{(\rho)} = 0 , \tag{5.28}
\]
\(^1\)Metrics, which give the same equations of motion, are equivalent. If they are not degenerate, then equivalence reduces to proportionality of the metrics with constant coefficient of proportionality.
\[
\frac{ds_{[\mu\nu]}}{cd\tau} = -\frac{\Sigma c}{8}\partial_{[\mu\nu]}\Omega^{[\lambda\kappa]\rho\sigma}\Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]} + \frac{\Sigma c}{2}\partial_{[\mu\nu]}h_{\alpha\beta}\dot{V}^\alpha\dot{V}^\beta, \tag{5.29}
\]

\[
\frac{D_h\pi_\alpha}{cd\tau} = \frac{dp_\alpha}{cd\tau} - H_{\dot{\alpha}\dot{\beta}}\dot{V}^\beta\pi_\dot{\alpha} = -\frac{\Sigma c}{8}\partial_{\alpha\rho}\Omega^{[\lambda\kappa]\rho\sigma}\Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]}, \tag{5.30}
\]

where

\[
p_{(\mu)} = -\frac{\partial L}{cdU(\mu)} = \Sigma c\eta_{(\mu)(\nu)}U^{(\nu)} = p^C_{(\mu)}, \tag{5.31}
\]

\[
s_{[\lambda\kappa]} = -\frac{\partial L}{cd\Omega^{[\lambda\kappa]}} = -\frac{\Sigma c\dot{\xi}}{4}j_{[\lambda\kappa][\rho\sigma]}\Omega^{[\rho\sigma]} = s^{rot}_{[\lambda\kappa]}, \tag{5.32}
\]

\[
\pi_\alpha = -\frac{\partial L}{cdV^\alpha} = \Sigma ch_{\alpha\beta}\dot{V}^\beta = \pi_M^\alpha. \tag{5.33}
\]

The requirement of conservation of proper angular momentum of an extended object gives conditions

\[
\partial_{[mn]}\dot{z} = 0, \quad \partial_{[mn]}h_{\alpha\beta} = 0, \tag{5.34}
\]

\[
\partial_{[0n]}\left[ h_{\alpha\beta}\dot{V}^\alpha\dot{V}^\beta - \frac{1}{4}\zeta_{[\lambda\kappa][\rho\sigma]}\Omega^{[\lambda\kappa]}\Omega^{[\rho\sigma]} \right] = 0. \tag{5.35}
\]

In the variant considered above, as before, there takes place geodesic motion of the center of inertia in \(R_4\). The motion of the point \(M\) in \(\hat{R}_4\), being not geodesic in general, becomes geodesic one when \(\Omega^{[\mu\nu]} = 0\).

## 6 Internal metric

Up to this place a structure and size of extended objects were bounded by nothing. Therefore in this Section we shall discuss properties of internal space and try to build its metric in the case when spatial rotations are absent, i.e. when \(\Omega^{[\mu\nu]} = 0\).

Internal space \(\hat{R}_4\) is covered with vectors \(r^\alpha\) of arbitrary points \(M\) relative to the center of inertia \(C\). In the interval (4.47), describing a motion of the point \(M\), quantity \(\tau\) is proper time of the center of inertia. Therefore this interval should have the same value for another point \(M'\) from the manifold \(\hat{R}_4\), i.e. for all vectors \(r^\alpha\) labelling all points of an extended object. In other words, this interval must be invariant relative to any admissible transformations in \(\hat{R}_4\). Thus, all points of extended object have the same proper time coinciding with the proper time of the center of inertia. In Section 4 there were shown that it is possible provided condition (4.17) be implied. In the case above this condition, having the form (equation (4.34))

\[
a^2V_C^\alpha[r \times V_C]^2 + h_{\alpha\beta}\dot{V}^\alpha\dot{V}^\beta = 0, \tag{6.1}
\]

when \(\Omega^{[mn]} = 0\), should be considered along with orthogonality condition (4.15), reducing to condition (4.30).

Because a particle is bounded in space-time \(R_4\), one should choose as \(\hat{R}_4\) manifolds with finite geodesics (not necessarily closed ones), which are bounded by some domain in
when the center of inertia is in rest. Such a domain may be considered as interiority of an extended particle. Obviously, this domain will seem as world four-dimensional tube in $\mathbb{R}_4$ in the case of moving center of inertia. For simplest approximations one may choose the Minkowski space-time as $\mathbb{R}_4$, corresponding to infinitely extended particle, internal substance of which is moving with constant velocity, for $b^{\alpha^*}_{\hat{a}}$ in this case form a matrix of Lorentz transformations (see formulae (2.1), (2.9) and (3.35)). Although such an approximation, certainly, does not both correspond to reality and satisfy to condition of finiteness of geodesics, it may lead in quantum case to sufficiently simple relativistic wave equations of harmonic oscillator type.

Usually physical bodies and elementary particles are considered to be extended in three-dimensional space, and this fact is observable. Theoretically it is attained by imposing of specific limitations (such as Yukawa condition $p_\mu x^\mu = 0$, which looks in our case as $\eta^{(\mu)(\nu)} c_{(\nu)\hat{a}} p_\mu r^{\hat{a}} = 0$) to relativistic description for elimination of temporal excitations. In the model under consideration conditions (4.30) and (6.1) may be treated as limitations of mentioned type.

In so far as we give here a relativistic description, it is reasonably to suppose, that particles (and maybe also macroscopic bodies) have an extension in temporal axis too, so that visible three-dimensional objects are cross sections of four-dimensional ones by the hyperplane $r^{\hat{a}} = 0$. This idea logically follows from successive interpretation of the space and time as four-dimensional continuum. It should be noted in this connection, that components $\varphi^{[0a]}$ are not specified by the Lorentz transformations. It is naturally to require certain principle of correspondence to be fulfilled between relativistic and non-relativistic descriptions. Obviously, passage from relativity to nonrelativistic descriptions means not only $U \ll c$, where $U$ is a velocity of the center of inertia, but also $r^{\hat{a}} = 0$, which is equivalent to Yukawa condition.

Let us define square of the vector $r^{\hat{a}}$ with the help of components $c_{(\mu)\hat{a}}$ of fixed 4-hedron in $\mathbb{R}_4$ (see (3.37)) by means of the formula

$$ r^2 = \eta^{(\mu)(\nu)} c_{(\mu)\hat{a}} c_{(\nu)\hat{b}} r^{\hat{a}} r^{\hat{b}} = h_{\hat{a}\hat{b}}^0 r^{\hat{a}} r^{\hat{b}}, $$

(6.2)

where $h_{\hat{a}\hat{b}}^0 = h_{\hat{a}\hat{b}}(r^{\hat{0}} = 0)$ is the metric of the space, which is associated with deformational geodesic motion of substance in the vicinity of the center of inertia. (6.2) represents a squared pseudo-distance between points in $\mathbb{R}_4$ relative to background metric. It reduces to $r^2 = (r^\hat{0})^2 - r^2$ in the Minkowski space. In the non-relativistic limit the quantity $r^2$ should reduce to $r^2_{\text{NR}} = h_{\hat{a}\hat{b}}^0 r^{\hat{a}} r^{\hat{b}} < 0$, $\hat{a}, \hat{b} = \hat{1}, \hat{2}, \hat{3}$ ($r^2_{\text{NR}} = -r^2$ in the Minkowski space). Therefore one should require the vector $r^{\hat{a}}$ to be space-like one, $r^2 < 0$ (see also [31]). If $r^{\hat{a}}$ will be time-like vector, then $r^\hat{0} = 0$ cannot be assumed in this case, i.e. one cannot pass to non-relativistic limit. However, one may put $r^{\hat{a}} = 0$, which corresponds to collapsing of the object to a point in three-dimensional space $\hat{\mathbb{R}}_3 \subset \hat{\mathbb{R}}_4$. But it means not passage to mass point, but a disappearance of the object from $\hat{\mathbb{R}}_3$. Therefore, for confining of the

\[2\]For the first time an idea of four-dimensionally extended particles was promoted by M.A.Markov [9].
for any vector \( r^\hat{\alpha} \) from the manifold \( \hat{\mathbf{R}}_4 \). Here \( r^2 = 0 \) implies \( r^\hat{\alpha} = 0 \), i.e. motion of the point \( M \) coincides in this case with the motion of the center of inertia.

The interval (4.47) must be invariant under any transformation of the vector \( r^\hat{\alpha} \), satisfying to condition (6.3). Relevant transformation may be written similar to formula of velocity composition in Special Relativity [35], [36]. To do so we shall consider five-dimensional space with the interval

\[
ds^2 = \eta_{\alpha \beta} d\xi^\alpha d\xi^\beta = h_{\hat{\alpha} \hat{\beta}} d\xi^\hat{\alpha} d\xi^\hat{\beta} + \eta_{\hat{5} \hat{5}} (d\xi^\hat{5})^2 =
\]

\[
= (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 - (d\xi^\hat{5})^2,
\]

where \( \eta_{\hat{5} \hat{5}} = \pm 1 \). In this space one may consider analogs of inertial reference frame of standard Minkowski space. Let \( \rho^\hat{\alpha} \) denotes a 3+1-dimensional analog of 3-vector of relative velocity of inertial frames. Like as coordinates in two reference frames are coupled by the Lorentz transformation, where 3-vector of relative velocity is a parameter, in our case an analog of the Lorentz transformation will have the form [35]

\[
L = 1 + \frac{\gamma}{R_0} \hat{\rho} + \frac{\gamma - 1}{R_0^2 \beta^2} \rho^2,
\]

where \( \gamma = (1 - \beta^2)^{-1/2} \), \( \hat{\rho} = h^0_{\hat{\alpha} \hat{\beta}} \rho^\hat{\alpha} (e^{\hat{\beta} \hat{\delta}} - e^{5 \hat{\delta}}) \), \( e^{ab} \) are elements of complete matrix algebra, satisfying to relations [37], [38]

\[
e^{ab} e^{cd} = \eta^{bc} e^{ad}, \quad (e^{ab})_{c d} = \eta^{ab} \delta_{c d}, \quad (a, ..., d = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5}).
\]

Quantity

\[
\beta^2 = \frac{1}{2R_0^2} \text{Sp}(\rho^2) = -\frac{\eta_{\hat{5} \hat{5}}}{R_0^2} \eta_{\hat{\alpha} \hat{\beta}} \rho^\hat{\alpha} \rho^\hat{\beta} = -\frac{\eta^0_{\hat{5}}}{R_0^2} h^0_{\hat{\alpha} \hat{\beta}} \rho^\hat{\alpha} \rho^\hat{\beta} = -\frac{\eta^0_{\hat{5}}}{R_0^2} ((\rho^\hat{0})^2 - \rho^2) = -\frac{\eta^0_{\hat{5}}}{R_0^2} \rho^2
\]

varies in the interval \( 0 \leq \beta^2 < 1 \), i.e. \( -R_0^2 < \rho^2 \leq 0 \) for \( \eta_{\hat{5} \hat{5}} = +1 \) and \( 0 \leq \rho^2 < R_0^2 \) for \( \eta_{\hat{5} \hat{5}} = -1 \). Obviously, in the case under consideration one should choose \( \eta_{\hat{5} \hat{5}} = +1 \). Defining in systems \( K \) and \( K' \) quantities, similar to velocities

\[
r^\hat{\alpha} = R_0 \frac{d\xi^\hat{\alpha}}{d\xi^\hat{5}}, \quad 'r'^\hat{\alpha} = R_0 \frac{d'\xi^\hat{\alpha}}{d'\xi^\hat{5}},
\]

we find a transformation of the velocity composition law type

\[
'r'^\hat{\alpha} = R_0 L^\hat{\alpha}_\hat{\beta} r^\hat{\beta} + R_0 L^\hat{\alpha}_5 R_0 L^5_\hat{\beta} + R_0 L^5_\hat{5}
\]

\[
= \frac{r^\hat{\alpha} + \rho^\hat{\alpha} + (\gamma - 1) \left[ 1 - \frac{h^0_{\hat{5}}}{\beta^2 R_0^2} \right] \rho^\hat{0}}{\gamma \left[ 1 - \frac{h^0_{\hat{5}} \rho^\hat{0}}{R_0} \right]}.\]

(6.9)
This is a desired transformation, which should be completed with three-dimensional rotations of the vector $r^\alpha$. It is easily to see, that vectors $r^\alpha$ and $'r^\alpha$ satisfy condition (6.3). Thus, variations of parameters $\rho^\alpha$ and parameters, corresponding to three-dimensional rotations of the vectors $r^\alpha$, give all points of the space $\mathbb{R}_4$.

Now we require the interval (4.47) to be invariant under these transformations. Thereby a domain, occupied by four-dimensionally extended object, is bounded with size $R_0$. Because temporal sizes are not observable $R_0$ be maximal radius of observable three-dimensional cross section of this object. As it follows from (6.9), if points $M$ and $M'$ are at a short distance, infinitesimal transformation, connecting their radius vectors $r^\alpha$ and $'r^\alpha$, has the form

$$
't^\alpha = r^\alpha \left[ 1 + \frac{h^0_{\beta\gamma} r^\beta}{R_0^2} \right] + \rho^\alpha. \tag{6.10}
$$

Variation $d't^\alpha$ in the point $M'$ may be found from differential of (6.10) when $\rho^\alpha = \text{const}$:

$$
d't^\alpha = \left[ \left( 1 + \frac{h^0_{\gamma\delta} r^\gamma}{R_0^2} \right) \delta^\alpha_{\beta} + \frac{r^\alpha h^0_{\beta\gamma} \rho^\gamma}{R_0^2} \right] d\rho^\beta. \tag{6.11}
$$

Value of a metric $h_{\hat{\alpha}\hat{\beta}}$ in the point $M'$ is defined by

$$
h_{\hat{\alpha}\hat{\beta}}(r^\sigma) = h_{\hat{\alpha}\hat{\beta}}(r^\sigma) + \frac{\partial h_{\hat{\alpha}\hat{\beta}}}{\partial r^\gamma} \left[ \rho^\gamma + r^\gamma h^0_{\delta\sigma} \frac{r^\delta \rho^\sigma}{R_0^2} \right] =
$$

$$
= \left[ \delta^\mu_{\hat{\alpha}} \delta^\nu_{\hat{\beta}} + (H^\mu_{\hat{\alpha}\gamma \delta} \delta^\nu_{\hat{\beta}} + H^\nu_{\hat{\beta}\gamma \delta} \delta^\mu_{\hat{\alpha}}) \left( \rho^\gamma + r^\gamma h^0_{\delta\sigma} \frac{r^\delta \rho^\sigma}{R_0^2} \right) \right] h_{\mu\nu}(\hat{r}^\sigma). \tag{6.12}
$$

When rotations are absent an invariance of the interval (4.47) under transformations (6.9) reduces to condition $h_{\hat{\alpha}\hat{\beta}}(r^\sigma)dr^\alpha dr^\beta = h_{\hat{\alpha}\hat{\beta}}(r^\sigma)d't^\alpha d't^\beta$. From here we get a relation

$$
(h_{\hat{\alpha}\hat{\beta}} h^0_{\gamma\delta} + h_{\hat{\alpha}\hat{\sigma}} h^0_{\gamma\delta}) r^\sigma + (R_0^2 \delta^\nu_{\hat{\beta} \gamma} + h^0_{\hat{\beta}\gamma \delta} r^\gamma) H^\delta_{\hat{\alpha}\hat{\sigma}} h_{\delta\beta} = 0, \tag{6.13}
$$

being actually an equation for metric. Assuming, that the most general form of the metric is

$$
h_{\hat{\alpha}\hat{\beta}} = \chi_{\hat{\alpha}\hat{\beta}} + (a_{\hat{\alpha}} h^0_{\hat{\beta}\gamma} + a_{\hat{\beta}} h^0_{\hat{\alpha}\gamma}) r^\gamma + b h^0_{\hat{\gamma}\delta} h^0_{\hat{\beta}\delta} r^\gamma r^\delta, \tag{6.14}
$$

where $\chi_{\hat{\alpha}\hat{\beta}}$, $a_{\hat{\alpha}}$, $b$ are functions only on variables $r^2 = h^0_{\hat{\alpha}\hat{\beta}} r^\alpha r^\beta$ and $\eta = k^\alpha r^\alpha$ ($k^\alpha$ is constant vector), we find $\partial h_{\hat{\alpha}\hat{\beta}} / \partial \eta = 0$, $a_{\hat{\alpha}} = 0$ and

$$
h_{\hat{\alpha}\hat{\beta}} = \left( 1 + \frac{r^2}{R_0^2} \right)^{-1} h^0_{\hat{\alpha}\hat{\beta}} - \left( 1 + \frac{r^2}{R_0^2} \right)^{-2} h^0_{\hat{\alpha}\hat{\gamma}} h^0_{\hat{\beta}\delta} r^\gamma r^\delta, \tag{6.15}
$$

$$
h_{\hat{\alpha}\hat{\beta}} = \left( 1 + \frac{r^2}{R_0^2} \right) \left[ h_{\hat{\alpha}\hat{\beta}} + \frac{r^\alpha r^\beta}{R_0^2} \right], \quad h_{\hat{\alpha}\hat{\gamma}} h_{\hat{\gamma}\delta} = \delta^\delta_{\hat{\alpha}}. \tag{6.16}
$$
Let us find field equation for this metric. Connections, curvature tensor, Ricci tensor, scalar curvature and Einstein tensor are

\[
H^\alpha{}_{\beta\gamma} = -\frac{1}{R_0} \left( 1 + \frac{r^2}{R_0^2} \right)^{-1} \left( \delta^\alpha{}_{\beta} h^\gamma_0 + \delta^\alpha{}_{\gamma} h^\beta_0 \right) \rho^\delta. 
\]

\[
R^\mu{}_{\nu\alpha\beta} = \partial_\alpha H^\mu{}_{\nu|\beta|} + H^\mu{}_{\nu|\alpha| H^\gamma{}_{\beta|\gamma|}} = \frac{\eta}{R_0^2} \left( h_{\beta\alpha} \delta^\mu{}_{\lambda} - h_{\alpha\lambda} \delta^\mu{}_{\beta} \right),
\]

\[
R_{\alpha\beta} = R^\mu{}_{\beta\mu\alpha} = \frac{3}{R_0^2} h_{\alpha\beta} = \lambda h_{\alpha\beta},
\]

\[
R = h^\alpha{}_{\alpha} R_{\alpha\beta} = \frac{12}{R_0},
\]

\[
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} = -\frac{3}{R_0^2} h_{\alpha\beta} = -\kappa_0 T_{\alpha\beta},
\]

respectively, where \( T_{\alpha\beta} = 3\kappa_0^{-1} R^2_0 h_{\alpha\beta} = \lambda \kappa_0^{-1} h_{\alpha\beta} \) is the energy-momentum tensor, \( \lambda = 3R_0^2 \) is cosmological constant, \( \kappa_0 = 8\pi G/c^4 = 2,0759 \cdot 10^{-43} \text{ N}^{-1} \) is Einstein gravitational constant. Equation (6.19) shows, that the Ricci tensor is of \( \Lambda \)-term type, while the energy-momentum tensor describes a perfect fluid with pressure \( p = \lambda \kappa_0^{-1} \) and energy density \( \mu \), satisfying to relation \( p + \mu = 0 \) (6.2), which gives \( \mu = -\lambda \kappa_0^{-1} < 0 \). To satisfy a condition of positiveness of the energy density, we shall make a substitution \( \mu' = \mu + \Lambda \kappa_0^{-1} \), \( p' = p - \Lambda \kappa_0^{-1} \), so that condition \( \mu' > 0 \) be fulfilled. Then, obviously, \( \mu' + \mu' = p + \mu = 0 \) and equation (6.21) will be written down in the next form

\[
R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} + \Lambda h_{\alpha\beta} = -\left( \frac{3}{R_0^2} - \Lambda \right) h_{\alpha\beta} = -\kappa_0 T'_{\alpha\beta} = -\kappa_0 T_{\alpha\beta} + \Lambda h_{\alpha\beta}. 
\]

Thus, introduction of cosmological constant is rather artificial step here.

We shall say also some words about the requirement \( \mu > 0 \). It is stipulated by the fact, that Einstein equations (6.21) describe, as it is generally adopted, an external field of gravitating masses, obeying to the Newton’s world gravity law \( F_\gamma = -G m_1 m_2 r^{-3} \) in the non-relativistic approximation. However, Einstein equations have wider interpretation, for they describe arbitrary curved space (without torsion). Particularly, one may consider they to describe also a field of electric charge, which in the non-relativistic limit of point charge reduces to the Coulomb law \( F_e = (4\pi \varepsilon_0 r^3)^{-1} e_1 e_2 \). Taking into account that Newton’s and Coulomb’s laws have the same form and differ, besides of the sense of quantities, with only sign, it is reasonably to assume, that in the case, when Einstein equations describe a field of electron, the constant \( G \) should be substituted with \( -e^2(4\pi \varepsilon_0 m_e^2)^{-1} = -\alpha hc/m_e^2 \approx -2.78 \cdot 10^{32} \text{ m}^3/(\text{kg} \cdot \text{s}^2) \), where \( m_e \) is the electron rest mass. Then the constant \( \kappa_0 \) will have the next value

\[
\kappa_0 = -\frac{8\pi e^2}{4\pi \varepsilon_0 m_e^2 c^4} = -\frac{4\alpha h}{m_e^2 c^4} \approx -0,865 \text{ N}^{-1} .
\]
Since energy density is the $\hat{0}\hat{0}$-component of the energy-momentum tensor

$$\mu = T^{\hat{0}\hat{0}} = h^{\hat{0}\hat{0}}h^{\hat{0}\hat{0}}T_{\hat{0}\hat{0}} = -\frac{3}{\kappa_0 R_0^2} \left( 1 + \frac{(r^\hat{0})^2 - r^2}{R_0^2} \right) \left( 1 + \frac{(r^\hat{0})^2}{R_0^2} \right),$$

(6.24) then mass of extended object will be equal to

$$m_0 = \frac{1}{c^2} \int_V \mu dV = -\frac{3}{\kappa_0 c^2 R_0^2} \int_V \left( 1 + \frac{r^2}{R_0^2} \right) \left( 1 + \frac{(r^\hat{0})^2}{R_0^2} \right) dV.$$  

(6.25)

If $\mu$ does not depend on $r^\hat{0}$ (what is rather strong condition), then (6.24) leads to

$$\frac{\partial}{\partial r^\hat{0}} \left[ \left( 1 + \frac{(r^\hat{0})^2 - r^2}{R_0^2} \right) \left( 1 + \frac{(r^\hat{0})^2}{R_0^2} \right) \right] = \frac{2r^\hat{0}}{R_0^2} \left( 2 + \frac{2(r^\hat{0})^2 - r^2}{R_0^2} \right) = 0,$$

(6.26)

whence there follow two possibilities

$$a) \ r^\hat{0} = 0,$$

(6.27)

and

$$b) \ 1 + \frac{(r^\hat{0})^2}{R_0^2} = \frac{r^2}{2R_0^2}.$$  

(6.28)

Condition (6.28) contradicts to condition (6.3), for combination of them gives

$$0 \leq r^2 - (r^\hat{0})^2 = \frac{r^2}{2} + R_0^2 < R_0^2,$$

(6.29)

whence it follows, that (6.29) is inconsistent when $r^2 \neq 0$, while $r^2 = 0$ is inconsistent with (6.28). Condition (6.27) is consistent with (6.3), which gives $0 \leq r^2 < R_0^2$.

Thus, substitution (6.27) in (6.25) and taking into account spherical symmetry of the solution leads to

$$m_0 = -\frac{3}{\kappa_0 c^2 R_0^2} \int_V \left( 1 - \frac{r^2}{R_0^2} \right) dV =$$

$$= -\frac{12\pi R_0}{\kappa_0 c^2 R_0^2} \int_0^R \left( 1 - \frac{r^2}{R_0^2} \right) r^2 dr = -\frac{8\pi R_0}{5\kappa_0 c^2} = \frac{R_0 m_e c}{5\alpha \hbar} = \frac{R_0 m_e}{5a_0},$$

(6.30)

where $a_0 = \alpha \hbar/m_e c = 2.81751 \cdot 10^{-15}$ m is the classical radius of the electron. Under assumption $m_0 = m_e$ we get a following estimation for radius of an extended object $R_0 = 5a_0 = 1.41 \cdot 10^{-14}$ m.

Calculations above should not attach absolute importance to, for there were made too many assumptions of intuitive character such, for example, as independence of the
energy density $\mu$ on $r^\delta$, description of internal space of the electron by the metric (6.15), interpolation of the Coulomb law into electron interiority, and so on. It is sufficient to point out, that modern indirect measurings of the electron radius give the value $r_e = 0.452 \cdot 10^{-12}$ m [40]-[41].

Let us consider now restrictions (4.30) and (6.1) imposed to velocity $V_C^\delta$ for obtained internal metric (6.15). In the center-of-inertia system the metric $g_{\mu\nu}$, describing a motion of the point M in the background space, must coincide with internal metric, so that $g_{\mu\nu} = \delta_\mu^\alpha \delta_\nu^\beta h_{\delta\beta}$, where $\delta_\mu^\alpha = 1$, if $\mu = \alpha$ and $\delta_\mu^\alpha = 0$, if $\mu \neq \alpha$. Then equations (2.13)-(2.14) shall have the form

$$h_{\delta\beta} e^{\dot{\delta}}_{\cdot (\lambda)} e^{\dot{\beta}}_{\cdot (\kappa)} = \eta(\lambda)(\kappa) ,$$

$$\eta(\lambda)(\kappa) e^{\dot{\alpha}}_{\cdot (\lambda)} e^{\dot{\beta}}_{\cdot (\kappa)} = h^{\dot{\alpha}\dot{\beta}} .$$

(6.31)

(6.32)

It is not difficult to show these equations to give following expressions for components of reciprocal 4-hedron $e^\mu = \{e^{\mu}_{\cdot (\lambda)}\} = \{\delta^{\mu}_{\cdot \alpha} e^{\dot{\alpha}}_{\cdot (\lambda)}\}:

$$e^{\dot{\alpha}}_{\cdot (\mu)} = \delta^{\dot{\alpha}}_{\cdot \lambda} e^{\lambda}_{\cdot (\mu)} = \left(1 + \frac{r^2}{R_0^2}\right)^{1/2} \left[\delta^{\dot{\alpha}}_{\cdot (\mu)} - \left[1 \pm \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2}\right] \frac{\eta_{(\mu)(\nu)} e^{\cdot (\nu)}_{\cdot \alpha}}{r^2}\right] .$$

(6.33)

For components of moving 4-hedron $\bar{e}_{\cdot \lambda} = \{e^{\cdot (\mu)}_{\cdot \lambda}\} = \{\delta^{\cdot (\mu)}_{\cdot \alpha} \delta^{\dot{\alpha}}_{\cdot \lambda}\}$, satisfying to relations

$$e_{\cdot (\mu)} e^{\cdot (\nu)}_{\cdot \alpha} = \delta_{\cdot (\mu)}^{\cdot (\nu)} , \quad e^{\cdot (\mu)}_{\cdot \alpha} e^{\cdot (\nu)}_{\cdot \beta} = \delta^{\cdot (\mu)}_{\cdot \beta} ,$$

$$\eta_{(\mu)(\nu)} e^{\cdot (\mu)}_{\cdot \alpha} e^{\cdot (\nu)}_{\cdot \beta} = h_{\delta\beta} , \quad h_{\delta\beta} e^{\cdot (\mu)}_{\cdot \alpha} e^{\cdot (\nu)}_{\cdot \beta} = \eta_{(\mu)(\nu)} ,$$

we get

$$e^{\cdot (\mu)}_{\cdot \alpha} = \delta^{\beta}_{\cdot \alpha} e^{\cdot (\mu)}_{\cdot \beta} = \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2} \left[\delta^{(\mu)}_{\cdot \alpha} - \left[1 \pm \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2}\right] \frac{\delta^{(\mu)}_{\cdot \beta} h_{\beta\gamma} e^{\gamma}_{\cdot (\nu)}}{r^2}\right] .$$

(6.34)

(6.35)

Substituting (6.15) and (6.33) into (4.30) we come to the next relation between $V_C^\delta$ and $r^\delta$

$$h_{\delta\beta} e^{\alpha}_{\cdot (0)} V_C^\beta = \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2} \left[V_C^{\bar{\delta}} - \left[1 \pm \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2}\right] \frac{r^\delta h_{\alpha\beta} e^{\dot{\alpha}}_{\cdot (0)} V_C^{\dot{\beta}}}{r^2}\right] = 0 ,$$

(6.37)

which may be written down in the form

$$\left[1 - \frac{r^2}{R_0^2} + \left(1 + \frac{r^2}{R_0^2}\right)^{1/2}\right] V_C^{\bar{\delta}} = \frac{r^\delta h_{\alpha\beta} e^{\dot{\alpha}}_{\cdot (0)} V_C^{\dot{\beta}}}{R_0^2} ,$$

(6.38)

where $r^2 = -h_{\delta\beta} e^{\dot{\alpha}}_{\cdot (0)} e^{\dot{\beta}}_{\cdot (0)}$ is determined in (4.22).
In view of (6.37) squared vector (4.31) is

\[ \mathbf{V}_C^2 = -\eta^{(m)(n)} e^{(m)} e^{(n)} V^\alpha C V^\beta C = -h_{\dot{a}\dot{b}} V^\dot{a}_C V^\dot{b}_C = \]

\[ = \left(1 + \frac{r^2}{R_0^2}\right)^{-1} \left\{ -h^{0}_{\dot{a}\dot{b}} V^\dot{a}_C V^\dot{b}_C - \frac{r^2(V_0^0)^2}{(r^0)^2} + 2 \left[ 1 - \left(1 + \frac{r^2}{R_0^2}\right)^{1/2} \right] \frac{R_0^2(V_0^0)^2}{(r^0)^2} \right\} , \quad (6.39) \]

where only upper sign in square brackets should be left, for lower sign is inconsistent with obvious inequality \( V^2_C = -h_{\dot{a}\dot{b}} V^\dot{a}_C V^\dot{b}_C \geq 0 \), where equality sign takes place only when \( V^\dot{a}_C = 0 \). Then restriction (6.1) will be written down as

\[ (a^2 V^2_C | r \times V_C|^2 - 1) h_{\dot{a}\dot{b}} V^\dot{a}_C V^\dot{b}_C = 0 , \quad (6.40) \]

whence it follows

\[ a V^2_C = \pm \sqrt{ \frac{V^2_C}{| r \times V_C|^2} } = \pm \frac{1}{r \sin \vartheta} , \quad (6.41) \]

where \( \vartheta \) is an angle between vectors \( r \) and \( V_C \),

\[ (r \cdot V_C) = -\eta^{(m)} e^{(m)} r^\alpha V^\beta_C = - \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2} \left[ 1 - \left(1 + \frac{r^2}{R_0^2}\right)^{-1/2} \right] \frac{R_0^2 V_0^0}{r^0} , \quad (6.42) \]

\[ | r \times V_C|^2 = r^2 V^2_C - (r \cdot V_C)^2 = \]

\[ = \left(1 + \frac{r^2}{R_0^2}\right)^{-1} \left\{ -r^2 h^{0}_{\dot{a}\dot{b}} V^\dot{a}_C V^\dot{b}_C - R_0^2(V_0^0)^2 + \frac{r^2(R_0^2 - r^2)(V_0^0)^2}{r^0} \right\} - 2 \left[ 1 - \left(1 + \frac{r^2}{R_0^2}\right)^{1/2} \right] \frac{R_0^2(R_0^2 - r^2)(V_0^0)^2}{r^0} . \quad (6.43) \]

Due to \( V^\dot{a}_C = dr^\dot{a}/cd\tau_C \) equation (6.38) represents in fact an equation of hypersurface, along which a point M is moving. Setting \( u = r^\dot{0}/R_0 \), \( A = 1 - r^2/R_0^2 \) we get from (6.38) the next equation

\[ 2(A - \sqrt{A + u^2}) du = udA , \quad (6.44) \]

solution of which is

\[ \sqrt{A + u^2} - 1 = gu , \quad (6.45) \]

or

\[ (1 - g^2) \left[r^\dot{0} - \frac{g R_0}{1 - g^2}\right]^2 - r^2 = \frac{g^2 R_0^2}{1 - g^2} , \quad (6.46) \]

where \( g \) is constant parameterizing the hypersurface. According to (6.3) \(-1 < \sqrt{A + u^2} - 1 \leq 0 \). Hence \( -1 < gu \leq 0 \), or \( -R_0 < gr^\dot{0} \leq 0 \). Therefore (6.46) gives

\[ 0 \leq \frac{r^\dot{0}}{R_0} = -\frac{g}{g^2 - 1} - \sqrt{\frac{g^2}{(g^2 - 1)^2} - \frac{r^2}{R_0^2(g^2 - 1)}} \leq -\frac{1}{g} , \quad g < 0 , \quad (6.47) \]
\[-\frac{1}{g} \leq \frac{r^0}{R_0} = -\frac{g}{g^2 - 1} + \sqrt{g^2 - 1} - \frac{r^2}{R_0^2 g^2 - 1} \leq 0, \quad g > 0. \quad (6.48)\]

Here \( g = \pm \infty \) corresponds to \( r^0 = 0 \), and \( g = 0 \) corresponds to light cone
\[ (r^0)^2 - r^2 = 0. \quad (6.49) \]

At \( g = \pm 1 \) equation (6.46) is degenerated into the equation of paraboloid
\[ r^0 = \pm r^2 / 2R_0. \quad (6.50) \]

In all cases internal coordinates vary in the next boundaries
\[ -\frac{R_0 \sqrt{g^2 + 1}}{g} \leq r^\hat{a} \leq \frac{R_0 \sqrt{g^2 + 1}}{g}, \quad g > 0; \]
\[ \frac{R_0 \sqrt{g^2 + 1}}{g} \leq r^\hat{a} \leq -\frac{R_0 \sqrt{g^2 + 1}}{g}, \quad g < 0. \quad (6.51) \]

These hypersurfaces are represented in Fig.1.

Figure 1.

Calculations above give a possibility to set a sense of components \( B^{(k)} = \Omega_C^{[k]} \) forming the vector \( \mathbf{B} \). The sign “±” in (6.41) corresponds both to the sign of \( r^0 \) and to direction of the pseudo-vector \( [\mathbf{r} \times \mathbf{V}_C] \), which is a moment of the vector \( \mathbf{V}_C \) relative
to the center of inertia. Consequently, substituting (6.41) in (4.33) one may put (when \( \Omega = 0 \))

\[
B^{(k)} = \Omega^{[0]}_C = |r^0|^{-1} \sqrt{\frac{V_C^2}{|r \times V_C|^2}} \eta^{(k)(l)}[r \times V_C](l) =
\]

\[
= |r^0|^{-1} \sqrt{\frac{V_C^2}{|r \times V_C|^2}} \eta^{(k)(l)} \varepsilon(l)(m)(n) \delta^{(m)}_a r^\alpha V^{(n)}_C ,
\]

(6.52)

where \( r^0 \) is determined in (6.47)-(6.50).

At last, in the case under consideration, when \( j_{[\mu\nu][\lambda\kappa]} = j_{[\mu\nu][\lambda\kappa]}^0 \) and \( \Omega = 0 \), tensor (5.32) has the form

\[
s_{[\mu\nu]}^{\text{rot}} = -\frac{\Sigma_C}{4} j_{[\mu\nu][\lambda\kappa]} \Omega^{[\lambda\kappa]}_C = -\frac{\Sigma_C}{2} \left[ \eta(\mu)\dot{a} \eta(\nu)(\kappa) - \eta(\nu)\dot{a} \eta(\mu)(\kappa) \right] \eta(\lambda)\dot{\beta} r^\beta \Omega^{[\lambda\kappa]}_C =
\]

\[
= \frac{\Sigma_C}{2} \left[ \eta(\mu)\dot{a} \left( \eta(\nu)(0) \eta(\kappa) - \eta(\nu)(k) \right) r^\beta \right. - \left. \eta(\nu)\dot{a} \left( \eta(\mu)(0) \eta(\kappa) - \eta(\mu)(k) \right) r^\beta \right] r^\alpha \Omega^{[0k]}_C ,
\]

(6.53)

whence it follows

\[
s_{[m\eta]}^{\text{rot}} = \frac{\Sigma_C}{2} \left[ \eta(m)\dot{a} \eta(\eta)(k) - \eta(m)\dot{a} \eta(\eta)(k) \right] r^\alpha r^\beta \Omega^{[0k]}_C = \varepsilon(m)(n)(k) s^{(k)} ,
\]

(6.54)

\[
s^{(k)} = \frac{1}{2} \varepsilon^{(m)(n)(k)} s_{[m\eta]}^{\text{rot}} = \frac{\Sigma_C}{2} \left[ r^\beta B \times r \right]^{(k)} =
\]

\[
= -\frac{\Sigma_C}{2} \text{sign}(r^0) \sqrt{\frac{V_C^2}{|r \times V_C|^2}} [r \times [r \times V_C]]^{(k)} ,
\]

(6.55)

\[
s_{[m\eta]}^{\text{rot}} = -\frac{\Sigma_C}{2} \left( r^\beta \right)^2 \eta(m)(k) \Omega^{[0k]}_C = -\frac{\Sigma_C}{2} |r^0| \sqrt{\frac{V_C^2}{|r \times V_C|^2}} [r \times V_C]^{(n)} .
\]

(6.56)

The latter relation shows, that \( s_{[m\eta]}^{\text{rot}} \) represents a specific angular momentum of the point \( M \) relative to the center of inertia.

In so far as extended particle model in question implies, that a particle has infinite extension in temporal axis, expression (5.8) for proper angular momentum (spin) of the particle should be redetermined in the form

\[
S_{[\mu\nu]} = \int_V \rho^{(4)}(r^\alpha) s_{[\mu\nu]}^{\text{rot}} d^4r = \int_D \rho^{(4)}(r^\alpha) s_{[\mu\nu]}^{\text{rot}} d^4r = \int_V d^4V \int_{-\infty}^{+\infty} \rho^{(4)}(r^\alpha) s_{[\mu\nu]}^{\text{rot}} d\tilde{r}^0 ,
\]

(6.57)

where \( \rho^{(4)}(r^\alpha) \) is four-dimensional mass density of the particle, \( d^4r = dr^0 dV = dr^0 dr^1 dr^2 dr^3 \) is an elementary volume of four-dimensional domain \( D \), occupied with particle substance,
which is bounded in this model by hypersurfaces of light cone (6.49) and one-sheet hyperboloid $r^2 = h^0_{\alpha\beta} r^\alpha r^\beta = 0$. One can see from (6.56), that pseudovector $S = \{S_{[\alpha]}\}$ should be implied as spin of the particle, interpreted as proper mechanical angular momentum.

In our case transformations (6.9) do not exhaust all transformations satisfying to condition (6.3). As an example we can give a class of transformations, deriving from five-dimensional interval (6.4). Analog of transformation (6.5) may be written as \[35, 42\]

$$L = 1 + \frac{1}{R_0} \varphi(\alpha_\omega) \sinh \sqrt{\alpha_\omega} \hat{\rho} + \frac{1}{R_0^2} (\cosh \sqrt{\alpha_\omega} - 1) \varphi^2(\alpha_\omega) \rho^2 ,$$

(6.58)

where $\varphi(\alpha_\omega)$ is arbitrary function on $\alpha_\omega$, satisfying to condition $0 \leq \varphi^{-2}(\alpha_\omega) < 1$, and

$$\varphi^{-2}(\alpha_\omega) = \frac{1}{2R_0^2} \left( g^5_{\alpha_\beta} g_{\alpha_\beta} \rho^\alpha \rho^\beta = -1 \right) \frac{1}{R_0^2} h^0_{\alpha_\beta} \rho^\alpha \rho^\beta = -\frac{\rho^2}{R_0^2} = \beta^2 .$$

(6.59)

It should be express $\alpha_\omega$ through $\beta^2$ from (6.59) and substitute it to (6.58). Then transformation law of vectors $r^\alpha$ would be derived similar to formula (6.9).

Moreover, using of transformation (6.58) admits a generalization of condition (6.3) in the form

$$-R_{k+1}^2 < r^2 = h^0_{\alpha_\beta} r^\alpha r^\beta < -R_k^2 , \quad k = 0, 1, 2, ..., K ,$$

(6.60)

which leads to layered structure of internal space \[43\]. It is quite obvious also, that to construct a real metric one can use nonlinear transformations, based on principles, which differ from the principle, used here.

## 7 Conclusive notes

Essential moment in our attempt to describe an extended elementary particle is using a postulate of geodesic lines \[23]-\[25\]. Applying it to an extended particle we specified a concept of the center of inertia by condition that its motion should pass along geodesic line of external space. Geodesic lines of internal space should represent a motion of substance inside of a particle relative to the center of inertia. Physically it is expressed in independency of internal motions on external motion. Geometrically it turns out to be expressed in terms of degenerated embedding spaces. If it is hardly valid for macroscopic extended bodies, for internal and external motions are very strongly coupled. But unobservability of internal motions and constancy of some characteristics of particles at least at not very large energies gives possibility to postulate this assumption just for elementary particles. This is a basic distinction of suggested method from the description, adopted in General Relativity, equations of which are used only for interpretation of the metric.

Here we considered only a one-particle problem. Therefore, it was not necessary to introduce such characteristic of the particle as mass and charge. Apparently, they must appear as parameters of interaction between two extended objects. But, in return, there appears a quantity $R_0$, characterizing a size of a particle. Instead of quantum-mechanical
spin we consider mechanical proper angular momentum, whose presence or absence of association with spin should be determined later about. We think such association to be existing, and this method may be applied to stable particles, such as electron and proton, but when their interactions are considering it should be modified.

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