Generalized Space-time Periodic Circuits for Arbitrary Structures

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Abstract—Time periodic ($T$-periodic) parameters and circuit relations in the complex frequency domain are derived from first principles and shown to be the vectorized extension of linear time invariant circuits. Using the $T$-periodic ABCD parameters, an expression for the dispersion relation of an arbitrary space-time modulated structure is obtained. The relation is valid for general structures even when the spatial granularity is comparable to the operating and modulation wavelengths. In the limit of infinitesimal unit cell, the dispersion relation reduces identically to its continuous counterpart. For homogeneous space-time modulated media, the $T$-periodic circuit approach allows the extension of the well-known telegraphist’s equations. The time harmonics are coupled together to form an infinite system of coupled differential equations. At the scattering centres, where the interaction is mediated via the modulation (pump wave), the telegraphist’s equations reduce to the interaction of two waves only: the signal and idler. The interaction can then be described using three wave mixing, satisfying the phase matching condition. It is demonstrated that the $T$-periodic $S$ parameters provide an alternative and appealing visualization of the modal conversion and the emergence of non-reciprocity inside the bandgaps.

Index Terms—Time Periodic, Space-time periodic, Nonreciprocity

I. Introduction

The asymmetric interaction of space time harmonics in spatio-temporal modulated media has been recently exploited to design multitude of novel non-reciprocal devices such as magnet-less circulators [1]–[3], nonreciprocal antenna [4]–[6], one-way beam splitters [7]. Fundamentally, the modulation of a medium constitutive parameter by a travelling wave biases the transmission in one direction over the other(s), resulting in a skewed dispersion relation that cannot be achieved using space only or time only modulation [8], [9]. The theory of space-time modulated media dates back to the mid of last century when there was an immense interest in the study of distributed parametric interactions [10]–[15]. During that period, the theoretical foundation was established. The framework is based upon Bloch-Floquet theory, where the variables (voltages and currents or electric and magnetic fields) are represented by the infinite sum of the space time harmonics. Upon the substitution of the variables into the governing differential equation (in this case the wave equation), the system behaviour can be expressed as the interaction of the infinite number of space time harmonics. Usually only few harmonics need to be considered, particularly in the vicinity of the scattering centres [9].

Practically speaking, lumped elements are used to synthesize space-time modulated structures. Nonlinear elements (eg. varactors are periodically inserted in a host microstrip transmission line [16]. The modulation is introduced either using a strong pump wave on the same line or via a special arrangement where the modulation comes from another coupled transmission line [16], [17]. The operating regime is generally assumed and/or designed such that the granularity of the structure is small compared to the operating and modulation wavelengths; hence the structure is considered homogeneous.

Recently, it was shown that the scattering magnitude and hence the corresponding non-reciprocity can be enhanced by reducing the modulation wavelength $\lambda_m$ [9]. The analysis assumed the homogeneity of the underlying structure. However, as $\lambda_m$ decreases so does the wavelength at the scattering centre. For instance, at the Anti-Stokes’ scattering centre the wavelength becomes $\lambda = \lambda_m/2(1-v^2)$ [9]. Additionally, the wavelength of the $+1$ harmonic becomes even shorter. Eventually, the wavelengths are reduced to the extent that they are just few unit cells, raising the question about the validity of the analysis and behaviour of the medium when it is becoming more like a photonic crystal. To have a satisfactory answer to this and similar questions, it is imperative to take the granularity of the unit cells into account, hence the necessity of using a circuit based formalism. A circuit based approach must reduce to the homogeneous case in the limit of an infinitesimal unit cell. It is the aim of the current manuscript to develop such framework and show that it can rigorously explain the system behaviour when the wavelengths become comparable with the dimension of the unit cell and eventually tends to the homogeneous case when the unit cell becomes infinitesimally small.

In Section II a circuit based approach is developed for time periodic ($T$-periodic) circuits, which is the direct extension of linear time invariant (LTI) circuits. Such circuits are widely used to describe nonlinear RF and microwave circuits such as mixers and oscillators, where the system is linearized around an oscillatory (limit cycle) steady state. In the context of space-time structures, such an approach was employed to calculate the dispersion characteristics of a capacitively loaded transmission line [3]. Next, the spatial periodicity is included to determine the dispersion relation of an arbitrary space-time periodic structure. Under the long wavelength approximation, the $T$-periodic circuit representation is shown to be the extension of the well-known telegraphist’s equations, representing space-time media by two coupled matrix equations. Such equations are valid for arbitrary series and shunt lumped elements.
In Section III we demonstrate the universality of the current approach by closely examining the behavior of a right-handed transmission line (RH-TL), which has been widely studied. We, however, determine the non-reciprocity when the modulation wavelength becomes just a few unit cells. It is shown that the scattering center shifts to a lower frequency due to the bending of the dispersion relation. To test the validity of the circuit method, the RH TL is solved in the time domain using a brute force Runge-Kutta method. The attenuation of the wave in the center of the bandgap is determined and compared to the theoretical prediction.

Additionally, in the limit of long wavelengths (or equivalently infinitesimal unit cells), it is demonstrated that the dispersion relation tends to the one rigorously derived for homogeneous media. We employ the coupled wave formalism to the long wavelength TL to develop a system of coupled wave equations. For backward propagation, it is shown that the system of equations reduces to the well-known three wave mixing approach and is exactly equivalent to the $2 \times 2$ dispersion relation. Nevertheless, the coupled wave approach provides a deeper insight into the system’s internal mechanisms.

II. Theory

A. Time Periodic Circuits

![Fig. 1. A generic T-periodic circuit, consisting of an arbitrary number of T-periodic inductors, capacitors and resistors.](image)

In this subsection, we will be dealing with circuits having elements that are T-periodic. In this case, the periodicity in the spatial domain is not taken into account. Hence, the analysis in the current subsection is valid for any T-periodic circuits, regardless of its behaviour in the spatial domain. Assume the arbitrary circuit shown in Fig. 1 where some of the circuit parameters (resistances, inductances or capacitances) are T-periodic with a period $T_m$, where the subscript $m$ stands for modulation. Although a T-periodic two port network is shown in Fig. 1, the framework is readily extendible to $n$ ports networks. Generally speaking, the circuit can described in the time domain using its states (voltages across capacitors and current through inductors) [18]. The number of states is denoted by $N_s$. The states $\mathbf{x}$ are coupled and governed by the well known state space matrix equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u(t),$$

where $\mathbf{A}$, $\mathbf{B}$ are real T-periodic matrices that are determined from the circuit topology via KVL/KCL and $\mathbf{u}$ is the input vector excitation. The dimensions of $\mathbf{A}$ and $\mathbf{B}$ are $N_s \times N_s$ and $N_s \times 1$ (assuming a single input excitation), respectively. Using the Floquet theorem, for sinusoidal excitations with frequency $\omega$, an arbitrary state $\mathbf{x}$ will attain the form [19], [20].

$$x_i(t) = p(t)e^{i\omega_0t} + c.c.,$$

where $p(t) = p(t + T_m)$ is a periodic function and $c.c$ stands for the complex conjugate. Since $p(t)$ is periodic, (2) can written as

$$x_i(t) = \sum_{k=-\infty}^{\infty} P_k e^{i\omega_k t} + c.c.,$$

where $P_k$ is the amplitude of the $k^{th}$ harmonic of $p(t)$ and $\omega_k \equiv \omega + k\omega_0$. According to this notation, input frequency $\omega \equiv \omega_0$ when $k = 0$, and hence $\omega$ and $\omega_0$ are used synonymously below.

Plugging (3) back in (1) and noting that $\mathbf{x}$ represents currents and voltages, the coefficients of the $p^{th}$ harmonic can be matched, resulting in a system of $N_s$ linear algebraic equations. The T-periodic elements embedded in $\mathbf{A}$ and $\mathbf{B}$ couple the $p^{th}$ harmonic with other time harmonics. In the limiting case of linear time invariant (LTI) systems, the time harmonics are decoupled, allowing the state space variables to be determined at any given frequency $\omega$, with no knowledge of the behaviour at other frequencies. For the general time periodic case, there is an infinite number of such equations (a set for each frequency $\omega_k$). Practically, only a few such harmonics are necessary to characterise performance. For instance if one is interested in the behaviour at the fundamental frequency $\omega_0$, the first finite time harmonics $2N+1$ ($|k| \leq N$ need only be considered, where $N$ is usually small ($\pm 1, \pm 2$ or $\pm 3$). Therefore, the problem is reduced to the solution of $(2N + 1) \times N_s$ algebraic equations.

To elucidate the general approach, a simple circuit that consists of one shunt time periodic capacitance is analysed. Not only does it provide insight into the process, but also demonstrates the concept of divide and conquer, where complex systems can be divided into simpler cascaded subsystems. This concept is very useful in describing space-time periodic circuits, as will be shown in the next subsection. The current through the T-periodic capacitance $\tilde{C}$ is given by

$$i(t) = \frac{d}{dt} \tilde{C}(t) v(t).$$

Since $\tilde{C}(t)$ is T-periodic, it can be expanded in a Fourier series

$$\tilde{C}(t) = \sum_{k=-\infty}^{\infty} C_k e^{i\omega_0 t}.$$ 

Since $\tilde{C}(t)$ is real, $C_k = \tilde{C}_{-k}$. Furthermore using (5), both $v(t)$ and $i(t)$ can be written as

$$v(t) = \sum_{f=\omega_0}^{\infty} V_f e^{i\omega_0 t} + c.c \text{ and } i(t) = \sum_{f=\omega_0}^{\infty} I_f e^{i\omega_0 t} + c.c \tag{6}$$

Substituting (5) and (6) in (4), and matching the exp ($i\omega_0 t$) give

$$I_p = \sum_{f=\omega_0}^{\infty} i\omega_0 p C_{p-1} V_f = \sum_{f=\omega_0}^{\infty} Y_{p-1} V_f, \tag{7}$$

where $Y_{p-1} \equiv i\omega_0 p C_{p-1}$ can be interpreted as the admittance connecting the $p^{th}$ harmonic voltage with the $p^{th}$ harmonic.
current. In another words, the \( p \)th harmonic current is the sum of all currents due to all harmonic voltages. Defining the \( T \)-periodic current and voltage as the infinite-dimensional vectors

\[
I = \begin{bmatrix} I_{p-1}, I_p, I_{p+1}, \cdots \end{bmatrix}^T \quad \text{and} \quad V = \begin{bmatrix} V_{p-1}, V_p, V_{p+1}, \cdots \end{bmatrix}^T,
\]

\(^7\) can be compactly written in a matrix form as

\[
I = YV = i\Omega CV,
\]

(8)

where \( Y \) is the \( T \)-periodic admittance, \( \Omega \) is a diagonal matrix storing all harmonic frequencies \( \omega_k \) and \( C \) is the \( T \)-periodic capacitance matrix. Similar expressions for \( T \)-periodic inductors and resistors can be obtained as shown in Table II. It is clear from the previous analysis and Table II that the \( T \)-periodic circuit elements relations are the direct extension of the corresponding relations in LTI networks. As time harmonics go from \(-\infty \) to \( \infty \), where \( 0 \) labels the fundamental frequency, it is convenient to number the rows and columns of the matrices with reference to the fundamental component. According to this notation, the \( 0 \)th row denotes the fundamental component. If a \( T \)-periodic capacitance is connected in shunt

![Fig. 2. Scattering from a generic \( T \)-periodic circuit represented by the circle. The left half of the circle represents port 1.](image)

TABLE I
**Circuit Relations of \( T \)-periodic elements in time and frequency domains.**

| Element | Time | Freq. | Circuit Parameter |
|---------|------|-------|-------------------|
| \( i = \frac{d}{dt}C(t)v \) | \( I = YV \) | \( Y = \begin{bmatrix} Y_{p-1}, Y_p, Y_{p+1}, \cdots \end{bmatrix} \) |
| \( v = \frac{d}{dt}L(t)i \) | \( V = ZI \) | \( Z = \begin{bmatrix} Z_{p-1}, Z_p, Z_{p+1}, \cdots \end{bmatrix} \) |
| \( v = \dot{R}(t)i \) | \( V = RI \) | \( R = \begin{bmatrix} R_{p-1}, R_p, R_{p+1}, \cdots \end{bmatrix} \) |

between input and output terminals, the structure forms a two port network, where at \( \omega_k \) the input voltage and current (\( V_1 \) and \( I_1 \)) are related to the output values (\( V_2 \) and \( I_2 \)) by the ABCD parameters

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},
\]

(9)

where \([A], [D]\) are the identity matrices, \([B]\) is the zero matrix and \([C]\) (not to be confused with the capacitance matrix) is the matrix where its \((m,n)\) element is given by \( C_{m,n} = Y_{m-n} \). The \( T \)-periodic ABCD matrix is a generalization of the well-known ABCD matrix of LTI systems. Depending on the circuit configuration, some circuit parameters can be more convenient than others. For instance, shunt elements are naturally represented by the \( Y \) parameters, while series elements are readily represented by the \( Z \) parameters. The conversion process between different sets of parameters parallel that of LTI systems \(^{21}\).

Generally, each parameter is an infinite dimensional square matrix (hence the inner brackets), but practically only few harmonic need to be considered. In the subsequent analysis, for simpler notation, the inner brackets will be dropped. For a \( T \)-periodic system, the conventional \( S \)-parameters (\( S_{11}, S_{21}, S_{12} \) and \( S_{22} \)) are extended to four matrices. The \( S_{11} \) represents the reflection coefficients at port number 1, when port 2 is terminated in the reference impedance. For instance, with reference to Fig. 2 consider an incident wave with frequency \( \omega \) and complex amplitude \( a_0 \). \( S_{11} \) represents the wave bouncing back at port 1 in the \( r \)th harmonic. Similarly, \( S_{21} \) is the wave transmitted to port 2 in the \( r \)th harmonic. The vertical dotted line conceptually separates between ports one and two.

The total current and voltage at any instant satisfy KCL and KVL, respectively. Matching the \( r \)th harmonic, both KCL and KVL are satisfied for each component, i.e.

\[
\sum_{i=1}^{Q} I_i = 0
\]

(10)

for all current branches \( Q \) entering a given node and

\[
\sum_{i=1}^{Q} V_i = 0
\]

(11)

for all voltages through a closed loop. Here \( 0 \) stands for the null vector, emphasizing the fact that each KCL and KVL are satisfied for each time harmonic \( \omega_k \).

**B. Space-time Periodic Structures**

The analysis in subsection II-A is general for time periodic systems. However sometimes, a system is also periodic in the spatial domain, forming a travelling wave modulation. This can result, for instance, from the linearization of nonlinear distributed structures \(^{22}\) or the Distributedly Modulated Capacitance technique \(^{16}\). In this subsection, we will show how the general circuit approach discussed in the previous section can be extended to the case of space-time periodic structures.

![Fig. 6. Shows a hypothetical space-time periodic structure, where the modulation, regardless of its source, is represented by a travelling wave \( G(t-x/v_m) \), where \( v_m \) is the wave front velocity. The wave front travels a unit cell length \( p \) in \( p/v_m \)](image)
units of time. The wave $G$ can be the modulated capacitance, inductance or resistance. Generally, the spatially modulated elements are placed $p$ units apart or at $x = np$, where $n$ is an integer. There can be any number of modulated elements per unit cell, for instance a shunt capacitance or both capacitance and inductance of a right handed transmission line [17]. In this case there will be a travelling wave $G(t - x/v_m)$ for each modulated element and all travel with the same speed $v_m$. Any travelling wave $G$ can then be written as

$$G(t - x/v_m) = G(t'),$$

where $t' = t - x/v_m$. $G(t')$ is periodic with a minimum period $T_m$, therefore it can be expanded in its Fourier components

$$G(t') = \sum_{r=-\infty}^{\infty} G^r \exp(i r \omega_m t - \beta_m x),$$

where $\beta_m \equiv \omega_m/v_m$ is the modulation wave number. In the long wavelength approximation, the exact positions of the modulated elements as well as the distance between them are irrelevant as long as $p$ is much smaller than the operating and modulation wavelengths. We seek a general solution $\theta(t, x)$ that satisfies the Bloch-Floquet condition, i.e.,

$$\theta(t, x) = \theta_0 \exp(i(\omega t - \beta x))P(t'), \quad P(t' + T) = P(t').$$

The periodicity of $P(t')$ allows $\theta(t, x)$ to be written as

$$\theta(t, x) = \theta_0 \exp(i(\omega t - \beta x))P(t - x/v_m) = \sum_{r=-\infty}^{\infty} \Theta^r_0 \exp \left[ i \left( \hat{\omega}_r t - \hat{\beta}_r x \right) \right],$$

where $\Theta^r_0 = \Theta_0 P^r$. $\hat{\omega}_r = \omega + r\omega_m$, and $\hat{\beta}_r \equiv \beta + r\beta_m$. Using the above equation, the harmonics components of $v_n (i_n)$ are related to those of $v_{n+1} (i_{n+1})$ as

$$V_{n+1} = P_x V_n,$$

where $P_x$ is diagonal with $P_x^r = \exp(-i\hat{\beta}_r p)$. $P_x$ can be thought of as the spatial (hence the s subscript) propagator; it relates the voltage and current at $x = n+1$ to those at $x = np$.

$$dV = -Z'(x)I \quad (20)$$

and

$$dI = -Y'(x)V, \quad (21)$$

Equations (20) and (21) are general for any $T$-periodic structures, where $Z'(x)$ and $Y'(x)$ can be an arbitrary functions of $x$. It is worth noting that the state variables $V$ and $I$ are the time harmonics at a given position $x$. Eqs. (20) and (21) represent an infinite system of coupled differential equations. The time harmonics are generally coupled due to the off-diagonal terms in $Z'$ and/or $Y'$, arising from the $T$-periodic modulation.

III. Results and Discussion

The previous section proposes an expression that can be used to calculate the dispersion relation of a generic space-time modulated structure. It is the extension of the framework widely used to describe periodic structures [25], hence is

![Fig. 3. An arbitrary space-time periodic structure with a unit cell of $p$ units of length.](image-url)
readily applicable to a wide range of modulation wavelengths and frequencies. In this section, we will use the $T$-periodic circuit approach to derive the characteristic equation of a synthesized right handed transmission line (RH TL), where the length of the unit cell can be comparable to the modulation wavelength. Based on the derived expressions, it will be shown that when $\beta_m p \ll 1$ and for monotone modulation, the well-known dispersion relation of a homogeneous TL is automatically produced.

### A. RH TL Dispersion Relation

Similar to LTI systems, a single unit cell, shown in Fig. 4(b), can be considered as the cascade of two networks: the series LTI inductance and the shunt LTP capacitance. Hence

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} = \begin{bmatrix} I + ZY & Z \\ Y & I \end{bmatrix}$$ (22)

The matrices $Z$ and $Y$ are $2N + 1 \times 2N + 1$, where the $-N$ to $N$ harmonics are only considered. The situation where the temporal periodicity is applied to one element only (usually the shunt capacitance) is very well understood and hence will be used as our test-case in the following discussion. In this case, the impedance matrix is diagonal,

$$Z_{kk} = i\omega_k L.$$ (23)

Time periodicity appears in the $Y$ matrix, where

$$Y_{km} = i\omega_k C_{k-m}.$$ (24)

For monotone modulation

$$\tilde{C} = C_0 \left(1 + M \cos(\omega t - \beta_m np)\right).$$ (25)

Therefore, $C_{\ell k} = C_0$, $C_{k,k+1} = MC_0/2$ and $C_{k,k} = 0$ for $|k - s| > 1$. Substituting (23) and (24) in (22), and using (17), one arrives at the system of homogeneous equations

$$e^{\beta_m p} V_{k+1} + e^{-\beta_m p} V_{k-1} + \frac{2}{M} \left[1 - \left(\frac{2 \sin \beta_m p/2}{\beta_m c}\right)^2\right] V_k = 0, \quad (26)$$

where $k = 0, \pm 1, \pm 2, \ldots$. The system of equations (26) is valid for an arbitrary modulation wavelength. The dispersion relation is determined from the non-trivial solution of (26). When the modulation wavelength is considerably larger than the unit cell, i.e., $\beta_m p \ll 1$ and operating in the range where the structure is considered homogeneous (i.e., $\beta_k p \ll 1$), $2 \sin \beta_k p/2 = \beta_k p$ and (26) reduces to

$$V_{k-1} + V_{k+1} + \frac{2}{M} \left[1 - \left(\frac{\beta + k\beta_m}{\omega + k\omega_m}\right)^2\right] V_k = 0, \quad (27)$$

where

$$c = \lim_{p \to 0} \omega/\sqrt{LC_0}$$

is the speed of the homogeneous TL. It is convenient to normalize the wave-numbers and frequencies to be multiples of the modulation wavelength $\lambda_m \equiv 2\pi/\beta_m$. Therefore, the above equation reduces to

$$V_{k-1} + V_{k+1} + \frac{2}{M} \left[1 - \left(\frac{\beta \lambda_m + 2\pi k}{K\lambda_m + 2\pi v k}\right)^2\right] V_k = 0, \quad (28)$$

where $K \equiv \omega/c$, the wave number of the unmodulated TL, $\nu \equiv v_m/c$. Equation (28) is identical to the one derived for a modulated homogeneous medium. The circuit approach shows that such relation is only valid under the long wavelength approximations:

$$\beta_m p \ll 1, \quad \beta_k p \ll 1. \quad (29)$$

When the above two conditions are not satisfied, one should resort to (26). This may be necessary in practical scenarios, where the TL is synthesized from finite unit cells. Fig. 5 shows the dispersion relations calculated using (26) and compared to (27) in the limit $M \to 0$. In this case, the dispersion relation of a general TL, determined by (26), reduces to

$$2 \sin \beta_k p/2 = \pm \frac{\omega_k}{c} p.$$ (30)

Additionally, the dispersion relation for a homogeneous TL is calculated from (27) as

$$\beta_k = \frac{\omega_k}{c}.$$ (31)

As illustrated in Fig. 5 the scattering centers are shifted down in frequency due to the bending of the dispersion curves, where $2 \sin \beta_k p/2$ deviates from $\beta_k p$. This is particularly true when $\beta_m p$ becomes large and is exhibited more in higher harmonics $\beta_k = \beta + k\beta_m$, $k \geq 1$. The scattering frequencies represent the intersection of two curves given by the above equation. For instance, the first Anti-Stokes’ center is determined from the intersection of the negative branch of the $0^\text{th}$ curve with the positive branch of the $+1$ curve, which are given by

$$\sin(\beta + 2\pi) p/2 + \sin(\beta p)/2 = \pi \nu p,$$ and $K p = -2 \sin \beta p/2$. (32)

To examine the behaviour when the unit cell is macroscopic, the dispersion relations (26) and (27) are solved for different $\lambda_m$ values when $\nu = 0.3$ and $M = 0.5$. The results are reported in Fig. 4 where the Anti-Stokes’ (Backward) center is only considered. Inside the bandgap, the incident wave number become complex: $\beta - i\alpha$, where $\alpha$ is the attenuation constant. As $\lambda_m$ decreases the scattering center is shifted to a lower frequency value, as expected from the bending of the dispersion curves (Fig. 5). Since the scattering center is
the intersection of the 0th and 1st harmonics, the system of equations (26) can be reduced to the $2 \times 2$ secular equation

$$
\begin{pmatrix}
D_0 & e^{\beta m p}
\end{pmatrix}
\begin{pmatrix}
\phi_{m p}
\end{pmatrix}
= 0,
$$

or $D_0 D_{+1} = 1$. (33)

The position and strength of the Anti-Stokes' center are calculated using (32) and (33) as shown in Fig. 6 (c). The $2 \times 2$ system accurately predicts the frequency of the scattering center. However, it slightly overestimates the value of the attenuation constant.

Additionally, Fig. 6 (d) shows that even when $\lambda_m$ becomes just a few unit cells long, $\alpha$ is still inversely proportional to $\lambda_m$. This implies that, given a fixed modulation speed $\nu_m$, the insertion loss is directly proportional to the modulation frequency and such a relation is valid even when $\lambda_m$ is just few times larger than $p$. It is worth noting too that the behaviour when the unit cell is macroscopic is solely dependent on the bend of the dispersion curves. The coefficients $\exp(-\beta m p)$ and $\exp(\beta m p)$ appearing in the off-diagonal terms in the $2 \times 2$ determinant (33) have no influence. They only affect the phase difference between the 0th and +1st harmonics.

To verify the behaviour for small $\lambda_m$ values, a time domain analysis of the transmission line is performed using a state space model (SSM) [26]. Fig. 7 (a) shows $\alpha$ calculated using SSM and secular equation (17) at the Anti-Stokes’ center ($Ka = \pi(1 - \nu)$) for different values of $\lambda_m/p$. As $\lambda_m/p$ decreases, $\alpha$ increases. Fig. 7 (b) shows the amplitude of the incident and scattered waves superimposed on the modulation wave. The modulating wavelength is four unit cells. The incident wave attenuates as it propagates through the structure due to the scattering in the +1 mode that bounces back toward the source. Furthermore, when $\lambda_m/p = 4$, the normalized frequency at which the maximum attenuation occurs is shifted from 0.7$\pi$ to 0.66$\pi$ due to the bend of the dispersion curves (Fig. 5). At this frequency the SSM predicts $\alpha_{max}$ to be 99.57 m$^{-1}$, very close to the value predicted by the $2 \times 2$ secular equation ($\alpha_{max} = 98.12$ m$^{-1}$).

B. T-periodic S parameters

Instead of the dispersion relation, the circuit model presents another complementary view of the interaction process that may lead to non-reciprocity. Given the number of stages $N$, the total ABCD matrix can be calculated using (19), from which the scattering parameters are obtained. It is worth noting that the spatial modulation is only used to modify the appropriate terms in the ABCD matrices as given by (15). The $T$-periodic S parameters are shown in Fig. 8. Noting that $S_{21}^{(0)}$ and $S_{12}^{(0)}$ represent the scattering in the forward (backward) direction, it is clear that the isolation occurs in the bandgaps, where the transmission coefficients reduce significantly. The $T$-periodic $S_{11}^{(2,0)}$ parameters show that, inside the forward...
bandgap, scattering occurs at the $-1$ harmonic as expected (1st Stokes' centers) \([9], [14]\). Additionally, scattering in the +1 harmonic occurs for the backward interaction (1st Anti-Stokes' center). Moreover, there is another interaction at the 2nd backward scattering center due to the coupling between the fundamental signal and its second harmonic.

Similar to LTI systems, the scattering parameters can be used to estimate the dispersion relation. In particular, at any given frequency, $\alpha$ and $\beta$ are estimated as

$$ \beta^F = \text{Im} \frac{\ln(S_{11}^{(0,0)})}{Np}, \quad \alpha^F = \text{Re} \frac{\ln(S_{11}^{(0,0)})}{Np} \tag{34} $$

for forward propagation and

$$ \beta^B = \text{Im} \frac{\ln(S_{12}^{(0,0)})}{Np}, \quad \alpha^B = \text{Re} \frac{\ln(S_{12}^{(0,0)})}{Np} \tag{35} $$

for backward propagation. Fig. 7 shows the calculated $\alpha$ and $\beta$ using (34) and (35) for 100 unit cells. Qualitatively, the S parameters can be used to estimate the position and width of the bandgaps. It is, however, not as accurate as the dispersion relations \([17]\) mainly due to errors accumulating due to the multiplication of the cascaded unit cells \([19]\).

\[ C. \text{Coupled Wave Equations of spacetime periodic RH TL} \]

The $T$-periodic telegraphist’s equations \([20]\) and \([21]\) are readily applicable to the RH TL, where the shunt capacitance is spacetime modulated. The telegraphist equations can be combined to produce the second order matrix equation

$$ \frac{d^2 \mathbf{V}}{dx^2} = \mathbf{Z}'(x) \mathbf{Y}'(x) \mathbf{V}, \tag{36} $$

which represents the interaction of infinite number of waves (a fundamental with its time harmonics). To illustrate the usefulness of the coupled wave representation, the interaction of the fundamental with its +1 harmonic is analyzed using (36). Such interaction is significant at the Anti-Stokes’ scattering center \([9], [14]\).

For monotone modulation, using \([23]\) and \([24]\),

$$ \frac{d^2 V_0}{dx^2} = -\left(\frac{\omega}{c}\right)^2 V_0 - \left(\frac{\omega}{c}\right)^2 \frac{M}{2} e^{i\varphi \omega x} V_1 \tag{37} $$

and

$$ \frac{d^2 V_1}{dx^2} = -\left(\frac{\omega + \omega m}{c}\right)^2 \frac{M}{2} e^{i\varphi \omega x} V_0 - \left(\frac{\omega + \omega m}{c}\right)^2 V_1 \tag{38} $$

Fig. 7. (a) Attenuation Constant $\alpha$ calculated at the normalized frequency $K_a = \nu (1 - \nu)$ using the dispersion relation and state space model. (b) Amplitude of incident and scattered signals near the Anti-Stoke’s scattering center calculated using SSM.

Fig. 8. Generalized S parameters for $\nu = 0.3, M = 0.5$, using the $T$-periodic circuit when $\lambda m / p = 8$. Forward direction: (a) Scattering. (b) Transmission. (c) Dispersion. Backward: (d) Scattering. (e) Transmission. (f) Dispersion.

Fig. 9. (Left) Dispersion Relation calculated using the secular equation \([17]\) and the $T$-periodic S parameters. (Right) Attenation constant calculated \([17]\) and S parameters.
If \( V_0 = A_0 \exp(-\beta x) \) it follows that \( V_1 = A_1 \exp(-i(\beta + \omega_0) x) \); implying that the harmonics satisfy the phase matching condition. Substitution these expressions back in (37) and (38) results in a secular equation in \( \omega \) and \( \beta \)

\[
\left( \frac{2}{M} \right)^2 \left[ 1 - \frac{\beta \lambda_m}{K_m} \right]^2 \left[ 1 - \frac{\beta \lambda_m + 2\pi}{K_m + 2\pi r} \right]^2 - 1 = 0, \tag{39}
\]

identical to (33) under the long wavelength approximation.

IV. CONCLUSION

A circuit formalism for \( T \)-periodic circuits is used to determine the dispersion relation of an arbitrary space-time periodic structure. The relation is valid even when the length of the spatial periodicity is comparable to the modulating and operating wavelengths. In this case, the scattering centers are shifted to lower frequencies due to the bending of the dispersion relation. For infinitesimal unit cells, the relation retains the formula that was perviously derived for homogeneous media. Additionally, the system can be described by a generalized telegraphist’s equations. Generally, the time harmonics are coupled by the \( T \)-periodic circuit elements. The circuit based approach permits the use of S parameters to describe scattering and modal conversion in different time harmonics.

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