TIME-CHANGES OF HOROCYCLE FLOWS

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ABSTRACT. We consider smooth time-changes of the classical horocycle flows on the unit tangent bundle of a compact hyperbolic surface and prove sharp bounds on the rate of equidistribution and the rate of mixing. We then derive results on the spectrum of smooth time-changes and show that the spectrum is absolutely continuous with respect to the Lebesgue measure on the real line and that the maximal spectral type is equivalent to Lebesgue.

1. INTRODUCTION

The classical horocycle flow is a fundamental example of a unipotent, parabolic (nonhyperbolic) flow. Its dynamical properties have been studied in great detail. For compact hyperbolic surfaces, it is known that the flow is minimal [10], uniquely ergodic [6], has Lebesgue spectrum and is therefore strongly mixing [18], in fact mixing of all orders [15], and has zero entropy [9]. Its finer ergodic and rigidity properties, as well as the rate of mixing, were investigated by M. Ratner in a series of papers [19, 20, 21, 22] (for results on the rate of mixing of the geodesic as well as horocycle flows see also the paper by C. Moore [16]). In joint work with L. Flaminio [5], the first author has proved precise bounds on ergodic integrals of smooth functions. In the case of finite-volume, noncompact surfaces, the horocycle flow is not uniquely ergodic and the classification of invariant measures is due to Dani [4]. The asymptotic behavior of averages along closed horocycles has been studied by D. Zagier [27], P. Sarnak [23], D. Hejhal [11] and more recently in [5] and by A. Strömbergsson [24]. Horocycle flows on general geometrically finite surfaces have been studied by M. Burger [3].

Not much is known for general smooth parabolic flows, not even for smooth perturbations of classical horocycle flows in the compact case. In fact, even the dynamics of nontrivial smooth time-changes is poorly understood. Our paper addresses the latter question. By the classification of horocycle invariant
distributions [5] and by the related results on the asymptotics of ergodic averages for classical horocycle flows (see [5, 2]), it is known that smooth time-changes which are measurably trivial form a subspace of countable codimension, so that the generic smooth time-change is not even measurably conjugate to the horocycle flow. It is therefore interesting to know, perhaps as a step towards a better understanding of parabolic dynamical systems, to what extent the dynamical properties of the horocycle flow persist after a smooth time-change. The most important result to date is the proof by B. Marcus more than thirty years ago that all time-changes satisfying a mild differentiability conditions are mixing [15]. Marcus’ results generalized earlier work by Kushnirenko who proved mixing for all time-changes with sufficiently small derivative in the geodesic direction [13].

A. Katok and J.-P. Thouvenot have conjectured that “Any flow obtained by a sufficiently smooth time change from a horocycle flow has countable Lebesgue spectrum” (see [12], Conjecture 6.8). In fact, the question on the spectral type of smooth time changes of horocycle flows was already asked in Kushnirenko’s paper [13]. There the author is able to prove the relative absolute continuity of the spectrum of (restricted) smooth perturbations of skew-shifts, but cannot extend his results to time-changes of horocycle flows. In our paper we prove sharp bounds on the rate of equidistribution and mixing of smooth time-changes of the classical horocycle flow on the unit tangent bundle of a compact hyperbolic surface (see Theorem 16 and Theorem 19 in Sections 4 and 5 respectively). We then derive results on the spectrum of smooth time-changes (in Section 6), most notably we prove that the spectrum is absolutely continuous with respect to the Lebesgue measure (Theorem 23 in Section 6.2) and that the maximal spectral type is indeed equivalent to Lebesgue (Theorem 25 in Section 6.3).

The guiding idea of our work is that Marcus’ mixing mechanism can be made quantitative by the more recent quantitative results on the rate of equidistribution for horocycle flows (see [3, 5, 2]). In fact, Marcus’ argument is based on the equidistribution of long horocycle-like arcs, that is, arcs which are long in the horocycle direction and bounded in the complementary directions. Sharp results on the rate of equidistribution of horocycle-like arcs were recently obtained in [2] as a refinement of earlier results for horocycle arcs [3, 5]. Finally, our estimates on the rate of mixing for time-changes would be far from optimal, and definitely too weak to derive any significant spectral result, without a key bootstrap trick. Thanks to this bootstrap trick we can prove that decay of correlations of the horocycle flows are indeed stable under any smooth time-change.

Spectral results are derived from sharp bounds on the decay of correlations of smooth functions, which are equivalent to bounds on the Fourier transform of their spectral measures. A well-known difficulty in this approach is that the decay of correlations of a general smooth function under the horocycle flow is
not square-integrable, so that it would seem hopeless to prove absolute continuity of the spectrum in this way. However, our results on decay of correlations of time-changes are precise enough, thanks to the bootstrap trick, to give optimal, and hence square-integrable, decay of correlations for smooth coboundaries. Once it is established that all smooth coboundaries have absolutely continuous spectral measures, it follows (for instance by a density argument) that the spectrum is purely absolutely continuous. Our estimates on decay of correlations of coboundaries are also crucial in the proof that the maximal spectral type is Lebesgue. After the completion of this paper, we found out that Tiedra de Aldecoa [25] had independently derived the absolute continuity of the spectrum, under the Kushnirenko condition on the time-change function, from a general ‘positive commutator’ method of spectral theory [17].

We stress that none of the spectral results of the paper (in Section 6) depends on the quantitative equidistribution results or on the estimates on the decay of correlations for general smooth observables. In particular, they do not depend in any way on the harmonic analysis, that is, on the theory of unitary representations for $SL(2,\mathbb{R})$, on which the results on invariant distributions, on solutions of the cohomological equation and on the speed of convergence of ergodic averages are based (see [5] and [2]). In fact, since our main spectral results follow from sharp estimates on the decay of correlations of coboundaries, they are based only on the ‘bootstrap trick’ (see Lemma 8), which in turn relies only on the unique ergodicity of the classical horocycle flow (established first in [6] and later in [14] by geometric arguments). The reader interested in the main spectral results may read Sections 2 and 3 and then go directly to Sections 6.2 and 6.3.

Finally, let us remark that while in this paper we only deal with horocycle flows for compact hyperbolic surfaces, most of the methods and results can presumably be extended to the noncompact, finite volume case with appropriate modifications.

Structure of the paper. In Section 2 we introduce basic definitions, notation and properties of smooth time-changes of the classical horocycle flow. In Section 3 we refine Marcus’ mechanism based on averages along the push-forward of geodesic arcs and prove several related results. The key improvement is the bootstrap trick of Lemma 8. In Section 4 we recall the results on the invariant distributions for the classical horocycle flow from [5] and, from the results on the asymptotics of ergodic integrals in [5, 2], we derive analogous results for smooth-time changes (see Theorem 16). These quantitative equidistribution results are then applied in Section 5.2 to give a quantitative Marcus’ mixing argument (see Theorem 19) and in Section 5.3 to prove sharp bounds on the $L^2$ norm of twisted ergodic integrals of general smooth functions (see Theorem 20). In Section 6 we prove our main spectral results. We first prove a local estimate (Theorem 21 in Section 6.1), then absolute continuity of the spectrum (Theorem 23 in Section 6.2) and finally show that the maximal spectral type is Lebesgue (Theorem 25 in Section 6.3).
2. Time-changes of horocycle flows

Let \{U, V, X\} the basis of the Lie algebra \(\mathfrak{sl}(2, \mathbb{R})\) of PSL(2, \mathbb{R}) given by the generators \(U\) and \(V\) of the stable and unstable horocycle flows and by the generator \(X\) of the geodesic flow, respectively. The following commutation relations hold:

(1) \([U, V] = 2X, \quad [X, U] = U, \quad [X, V] = -V.\]

Let \(h_t^U\) and \(h_t^V\) denote respectively the stable and unstable horocycle flows and let \(\phi_t^X\) denote the geodesic flow on a compact homogeneous space \(M := \Gamma \backslash \text{PSL}(2, \mathbb{R}).\) They are defined respectively by the multiplicative action on the right of the 1-parameter subgroups of the group PSL(2, \mathbb{R}) listed below:

(2) \(\{\exp(tU)\}_{t \in \mathbb{R}}, \quad \{\exp(tV)\}_{t \in \mathbb{R}}, \quad \{\exp(tX)\}_{t \in \mathbb{R}}.\)

A smooth time-change of the (stable) horocycle flow is a flow \(\{h_t^U\}\) on \(M\) defined as follows. Let \(\tau: M \times \mathbb{R} \rightarrow \mathbb{R}\) be a smooth cocycle over the flow \(\{h_t^U\}\), that is, a function with the property that

\[\tau(x, t + t') = \tau(x, t) + \tau(h_t^U(x), t'), \quad \text{for all } (x, t, t') \in M \times \mathbb{R}^2.\]

We denote by \(\alpha: M \rightarrow \mathbb{R}^+\) the infinitesimal generator of the cocycle \(\tau: M \times \mathbb{R} \rightarrow \mathbb{R}\), that is, the function defined as follows:

\[\alpha(x) := \frac{\partial \tau}{\partial t}(x, t)|_{t=0}, \quad \text{for all } x \in M.\]

The time-change \(\{h_t^\alpha\}\) is the flow on \(M\) generated by the smooth vector field

(3) \(U_\alpha := U/\alpha.\)

One can check that \(h_t^\alpha\) is given by the formula

(4) \(h_t^\alpha(x) := h_t^U(x), \quad \text{for all } (x, t) \in M \times \mathbb{R}.\)

The flow \(\{h_t^\alpha\}\) preserves the (smooth) volume form \(\text{vol}_\alpha := \alpha \text{vol};\) in fact

\[\mathcal{L}_{U_\alpha} \text{vol}_\alpha = t_{U_\alpha} \text{vol}_\alpha = t_U \text{vol} = \mathcal{L}_U \text{vol} = 0.\]

We will assume below that the function \(\alpha: M \rightarrow \mathbb{R}\) is everywhere strictly positive and is normalized so that

(5) \(\int_M \text{vol}_\alpha = \int_M \alpha \text{vol} = 1.\)

We denote by \(L^2(M) := L^2(M, \text{vol})\) the Hilbert space of square-integrable function with respect to the standard volume form and by \(L^2(M, \text{vol}_\alpha)\) the Hilbert space of square-integrable function with respect to the \(\{h_t^\alpha\}\)-invariant volume form. For any \(r \geq 0,\) let \(W^r(M) \subset L^2(M)\) denote the standard Sobolev spaces on the compact manifold \(M\) and let \(W^{-r}(M)\) denote the dual Sobolev space. In this paper we will always assume that the time-change function \(\alpha \in W^r(M)\) for \(r > 5/2\) at least, hence by the Sobolev Embedding Theorem \(\alpha \in C^1(M).\)

The time-change \(\{h_t^\alpha\}\) is parabolic, in fact the infinitesimal divergence of trajectories is at most quadratic with respect to time (as is the case for the standard horocycle flow). The tangent flow \(\{Dh_t^\alpha\}\) on \(TM\) is described as follows.
**Lemma 1.** The tangent flow \( \{Dh^a_t\} \) on \( TM \) is given by the following formulas:

\[
Dh^a_t(V) = \left[ \int_0^t \left( \int_0^\tau \frac{1}{\alpha} \circ h^a_u \, du \right) \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t + \frac{V\alpha}{\alpha} \circ h^a_t \right] \, U_a \circ h^a_t + V \circ h^a_t + \left[ \int_0^t \frac{1}{\alpha} \circ h^a_t \, d\tau \right] X \circ h^a_t;
\]

\[
Dh^a_t(X) = \left[ \int_0^t \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t \, d\tau \right] U_a \circ h^a_t + X \circ h^a_t;
\]

**Proof.** For any vector field \( W \) on \( M \), let us write

\[
(h^a_t)_*(W) = a_t U_a + b_t V + c_t X.
\]

By the commutation relation

\[
[U_a, X] = \left( \frac{X\alpha}{\alpha} - 1 \right) U_a \quad \text{and} \quad [U_a, V] = X / \alpha + \frac{V\alpha}{\alpha} U_a,
\]

hence

\[
\frac{d}{dt} (h^a_t)_*(W) = \left[ c_t \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t + b_t \frac{V\alpha}{\alpha} \circ h^a_t \right] U_a + (b_t \circ h^a_t) X.
\]

It follows that the function \((a_t, b_t, c_t)\) satisfies the following system of O.D.E.’s:

\[
\begin{cases}
\frac{da_t}{dt} = c_t \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t + b_t \frac{V\alpha}{\alpha} \circ h^a_t; \\
\frac{db_t}{dt} = 0; \\
\frac{dc_t}{dt} = b_t \circ h^a_t.
\end{cases}
\]

If \( W = V \), the initial condition is \((a_0, b_0, c_0) = (0, 1, 0)\), hence the unique solution of the Cauchy problem is given by the functions:

\[
a_t = \int_0^t \left( \int_0^\tau \frac{1}{\alpha} \circ h^a_u \, du \right) \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t + \frac{V\alpha}{\alpha} \circ h^a_t \, d\tau,
\]

\[
b_t \equiv 1, \quad c_t = \int_0^t \frac{1}{\alpha} \circ h^a_t \, d\tau.
\]

If \( W = X \), the initial condition is \((a_0, b_0, c_0) = (0, 0, 1)\), hence the unique solution of the Cauchy problem is given by the functions:

\[
a_t = \int_0^t \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h^a_t \, d\tau, \quad b_t \equiv 0, \quad c_t \equiv 1.
\]

Formula (6) is therefore proved. \( \square \)
3. Push-forward of geodesic arcs

In this section we revisit the key idea of Marcus’ proof of mixing for smooth time-changes of horocycle flows [15], based on the analysis of the push-forward of a geodesic arc under the flow. Our main improvement is the bootstrap trick of Lemma 8, which in turn is based on the crucial bound derived in Lemma 7 from the formulas computed in Lemma 6.

Let \( \sigma \in \mathbb{R}^+ \) and \((x, t) \in M \times \mathbb{R}\). Let \( \gamma^\sigma_{x,t} : [0, \sigma] \to M \) be the parametrized path defined as follows:

\[
\gamma^\sigma_{x,t}(s) := h_t^0 \circ \phi_s^X(x), \quad \text{for all } s \in [0, \sigma].
\]

We begin by computing the velocity of the path \( \gamma^\sigma_{x,t} \) and its length. Let

\[
v_t(x, s) := \int_0^t \left( \frac{X\alpha}{\alpha} - 1 \right) \circ h_t^0 \circ \phi_s^X(x) \, d\tau
\]

**Lemma 2.** The following identity holds for all \((x, t, s) \in M \times \mathbb{R} \times [0, \sigma] \):

\[
\frac{d\gamma^\sigma_{x,t}}{ds}(s) := v_t(x, s)U_\alpha(\gamma^\sigma_{x,t}(s)) + X(\gamma^\sigma_{x,t}(s)).
\]

**Proof.** The velocity of the geodesic path \([\phi_s^X(x) | s \in [0, \sigma]]\) is given at all points by the geodesic vector field \(X\) on \(M\), hence

\[
\frac{d\gamma^\sigma_{x,t}}{ds}(s) = Dh_t^0(X) \circ \phi_s^X(x).
\]

The formula for the velocity of the path \( \gamma^\sigma_{x,t} \) then follows from Lemma 1. \( \square \)

By Lemma 2 the path \( \gamma^\sigma_{x,t} \) is contained in a single leaf of the weak-stable foliation of the geodesic flow, tangent to the frame \([U, X]\). Its length is estimated below. Let \([\hat{U}, \hat{V}, \hat{X}]\) denote the frame of the cotangent bundle \(T^*M\) dual to the frame \([U, V, X]\) of the tangent bundle \(TM\).

**Lemma 3.** For all \(r > 5/2\) and for all \(\alpha \in W^r(M)\), there exists a constant \(C'_r(\alpha) > 0\) such that, for all \(\sigma > 0\) and for all \((x, t) \in M \times \mathbb{R}^+\),

\[
\int_{\gamma^\sigma_{x,t}} |\hat{X}| \leq C'_r(\alpha) \sigma \quad \text{and} \quad \int_{\gamma^\sigma_{x,t}} |\hat{U}| \leq C'_r(\alpha) \sigma t.
\]

**Proof.** By Lemma 2 we have

\[
\int_{\gamma^\sigma_{x,t}} |\hat{X}| = \int_0^\sigma ds \quad \text{and} \quad \int_{\gamma^\sigma_{x,t}} |\hat{U}| = \int_0^\sigma \frac{|v_t(x, s)|}{\alpha} \circ h_t^0 \circ \phi_s^X(x) \, ds,
\]

hence the first estimate in the statement is immediate, while the second estimate follows from a uniform linear bound on \(|v_t(x, s)|\). The latter is an easy consequence of the Sobolev Embedding Theorem, which implies that under our regularity assumptions the function \(X\alpha / \alpha - 1\) is uniformly bounded. \( \square \)

We then estimate the asymptotics (as \(t \to +\infty\)) of the integral

\[
\int_0^\sigma (f \circ h_t^0 \circ \phi_s^X)(x) \, ds.
\]
Let $U_\alpha := a\hat{U}$ be the 1-form on $M$ uniquely defined by the conditions

$$\iota_U a U_\alpha = 1 \quad \text{and} \quad \iota_X U_\alpha = \iota_V U_\alpha = 0.$$ 

**Lemma 4.** For any continuous function $f$ on $M$, for all $\sigma > 0$ and $(x, t) \in M \times \mathbb{R}$,

$$\int_0^\sigma (f \circ h_t^\alpha \circ \phi_s^X)(x) \, ds = -\frac{1}{t} \int_{y_{x,t}}^\sigma f \circ \hat{U}_\alpha + \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x)(\frac{\nu_t(x,s)}{t} + 1) \, ds.$$ 

**Proof.** By Lemma 2 and by the above definitions,

$$\int_{y_{x,t}}^\sigma f \circ \hat{U}_\alpha = \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) \nu_t(x,s) \, ds.$$ 

The formula follows immediately. $\square$

The above formula can be refined by integration by parts by:

**Lemma 5.** For any continuous function $f$ on $M$, for all $\sigma > 0$ and $(x, t) \in M \times \mathbb{R}$,

$$\int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) \, ds = -\frac{1}{t} \int_{y_{x,t}}^\sigma f \circ \hat{U}_\alpha + \left(\frac{\nu_t(x,s)}{t} + 1\right) \int_0^\sigma f \circ h_t^\alpha \circ \phi_s^X(x) \, ds$$

$$-\frac{1}{t} \int_0^\sigma \frac{\partial \nu_t}{\partial S}(x, s) \left[ \left( \int_0^S \frac{\partial \phi_s^X(x)}{\partial S} (s) \right) - \left( \int_0^S \frac{\partial \phi_s^X(x)}{\partial S} (s) \right) \right] \, ds.$$ 

**Lemma 6.** For all $s > 0$ and all $(x, t) \in M \times \mathbb{R}^+$, we have:

$$\frac{\partial \nu_t}{\partial S}(x, s) = \nu_t(x, s) \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x)$$

$$- \int_0^t \left( \left( \frac{X_\alpha}{\alpha} - 1 \right) \left( \frac{X_\alpha}{\alpha} \right) - X \left( \frac{X_\alpha}{\alpha} \right) \right) h_t^\alpha \circ \phi_s^X(x) \, d\tau.$$ 

**Proof.** By Lemma 1 and by formula (8) it follows that

$$\frac{\partial \nu_t}{\partial S}(x, s) = \int_0^t \left[ \nu_t(x, s) U_\alpha + X \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) \right] \, d\tau.$$ 

By integration by parts we also have

$$\int_0^t \nu_t(x, s) U_\alpha \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) \, d\tau = \int_0^t \nu_t(x, s) \frac{d}{d\tau} \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) \, d\tau$$

$$= \nu_t(x, s) \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) - \int_0^t \frac{d\nu_t}{d\tau}(x, s) \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) \, d\tau$$

$$= \nu_t(x, s) \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) - \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \left( \frac{X_\alpha}{\alpha} \right) h_t^\alpha \circ \phi_s^X(x) \, d\tau,$$

as claimed in the statement of the lemma. $\square$

By the Sobolev Embedding Theorem, we have the following estimate:

**Lemma 7.** For any $r > 7/2$, there exists a constant $C_r(\alpha) > 0$ such that the following holds. Let $\alpha \in W^r(M)$. For all $(x, s) \in M \times [0, \sigma]$ and for all $t > 0$,

$$\left| \frac{\partial \nu_t}{\partial S}(x, s) \right| \leq C_r(\alpha) t.$$
For any \( r > 7/2 \), let \( C_r(\alpha) > 0 \) be the constant of Lemma 7 and let

\[
(9) \quad \sigma_r(\alpha) := \frac{1}{C_r(\alpha)} > 0.
\]

By a bootstrap argument we derive the following bound.

**Lemma 8.** For any \( r > 7/2 \) and for any \( \sigma \in (0, \sigma_r(\alpha)) \), there exist a time \( t_{r,\sigma}(\alpha) > 0 \) and a constant \( C_{r,\sigma}(\alpha) > 0 \) such that, for any continuous function \( f \) on \( M \), for any \( x \in M \) and for all \( t > t_{r,\sigma}(\alpha) \),

\[
\sup_{S \in [0,\sigma]} \left| \int_0^S f \circ h_{t,\sigma} \circ \phi_s^X(x) \, ds \right| \leq C_{r,\sigma}(\alpha) \sup_{S \in [0,\sigma]} \left| \frac{1}{t} \int_{Y_{t,s}} f \hat{U}_\alpha \right|.
\]

**Proof.** By Lemmas 5 and 7, it follows that

\[
\sup_{S \in [0,\sigma]} \left| \int_0^S f \circ h_{t} \circ \phi_s^X(x) \, ds \right| \leq \sup_{S \in [0,\sigma]} \left| \frac{1}{t} \int_{Y_{t,s}} f \hat{U}_\alpha \right| + \left[ \max_{x \in M} \frac{\nu_t(x,\sigma)}{t} + 1 \right] + C_r(\alpha) \sigma \sup_{S \in [0,\sigma]} \left| \int_0^S f \circ h_{t,\sigma} \circ \phi_s^X(x) \, ds \right|.
\]

By the unique ergodicity of the classical horocycle flow (and its time-changes) on compact hyperbolic surfaces, proved in [6], [14] we have

\[
(10) \quad \max_{(x,\sigma) \in M \times \mathbb{R}} \left| \frac{\nu_t(x,\sigma)}{t} + 1 \right| \to 0 \quad \text{as} \quad t \to +\infty.
\]

In fact, the function \( \nu_t(x, s) / t \) (see formula (8)) is given by an ergodic average of the function \( Xa/\alpha - 1 \) along the trajectories of the time-change \( \{h_s^X\} \) (evaluated at the point \( \phi_s^X(x) \in M \)). The \( \{h_s^X\} \)-invariant volume \( \text{vol}_\alpha \) is given by the formula \( \text{vol}_\alpha = \alpha \text{vol} \), hence by the normalization condition (5) and the invariance of the volume \( \text{vol} \) under the geodesic flow, we have

\[
\int_M \left( \frac{Xa}{\alpha} - 1 \right) \text{vol}_\alpha = \int_M (Xa - \alpha) \text{vol} = -1.
\]

(We note that bounds on the speed of the convergence in formula (10) can be derived from Theorem 16 on the quantitative equidistribution of smooth time-changes of the horocycle flow, proved below in Section 4.)

By definition \( C_r(\alpha) \sigma < 1 \) for any \( \sigma \in (0, \sigma_r(\alpha)) \), hence by formula (10) there exists a real number \( t_{r,\sigma}(\alpha) > 0 \) such that, for all \( t > t_{r,\sigma}(\alpha) \),

\[
\max_{x \in M} \left| \frac{\nu_t(x,\sigma)}{t} + 1 \right| + C_r(\alpha) \sigma < 1.
\]

Thus we conclude that the statement holds if we set

\[
\sigma_{r,\sigma}(\alpha) := \left[ 1 - \max_{x \in M} \left| \frac{\nu_t(x,\sigma)}{t} + 1 \right| + C_r(\alpha) \sigma \right]^{-1}.
\]

It is essential to be able to derive bounds on correlations from bounds on integrals along the push-forward of geodesic arcs. By the invariance of the standard volume under the geodesic flow and by integration by parts we prove the following formula.
**Lemma 9.** Let $\sigma > 0$. For all $f \in L^2(M)$ and for all $g \in L^2(M)$ which belong to the maximal domain $\text{dom}(L_X) \subset L^2(M)$ of the densely defined Lie derivative operator $L_X : L^2(M) \to L^2(M)$, for all $t \in \mathbb{R}$,

$$\langle f \circ h_t^\sigma, g \rangle = \frac{1}{\sigma} \langle \int_0^\sigma f \circ h_t^\sigma \circ \phi_s^X ds, g \circ \phi_s^X \rangle - \frac{1}{\sigma} \int_0^\sigma \langle \int_0^S f \circ h_t^\sigma \circ \phi_s^X ds, L_X g \circ \phi_s^X \rangle dS.$$

Following Marcus’ method, integrals along the push-forward of geodesic arcs are approximately ergodic integrals for the horocycle flow. In Section 5.1 we will derive sharp bounds on integrals along the push-forward of geodesic arcs from the refined equidistribution results of [2].

For our local estimates on spectral measures (see Theorem 21 in Section 6.1), we prove an $L^2$ bound on twisted ergodic integrals.

**Lemma 10.** Let $r > 11/2$ and let $\alpha \in \mathcal{W}^r(M)$. There exists $\sigma_\alpha > 0$ such that for all $\sigma \in (0, \sigma_\alpha)$ the following holds. There exists a constant $C_{\sigma, \alpha} > 0$ such that for any bounded weight function $w \in L^\infty(\mathbb{R}^+, \mathbb{C})$, for any continuous function $f$ in $\text{dom}(L_X) \subset L^2(M)$ and for all $\mathcal{T} > 0$,

$$\| \int_0^\mathcal{T} w(t) f \circ h_t^\sigma dt \|_{L^2(M, \text{vol}_a)} \leq C_{\sigma, \alpha} \| w \|_\infty \| f \|_{X^\mathcal{T}}^{1/2} \times \left( \int_0^\mathcal{T} \int_0^\mathcal{T} \| \sup_{0 < \tau < \sigma} \int_{I_{\tau, \sigma}} f d\tau \|_0^2 d\tau \right)^{1/2}.$$

**Proof.** By the invariance of the volume form under the reparametrized horocycle flow and by change of variables

$$\| \int_0^\mathcal{T} w(t) f \circ h_t^\sigma dt \|_{L^2(M, \text{vol}_a)} = 2 \text{Re} \langle \int_0^\mathcal{T} w(t) w(t-\tau) f \circ h_t^\sigma \circ \phi_s^X ds d\tau, \alpha f \rangle.$$

For every fixed $t \in \mathbb{R}$, let $w_t \in C^0(\mathbb{R}, \mathbb{C})$ be the bounded weight function defined as

$$w_t(\tau) := \overline{w(t-\tau)}, \quad \text{for all } \tau \in \mathbb{R}.$$

By the formula of Lemma 9

$$\langle \int_0^\mathcal{T} \int_0^\mathcal{T} w(t) w_t(\tau) f \circ h_t^\sigma d\tau d\tau, \alpha f \rangle$$

$$= \frac{1}{\sigma} \int_0^\mathcal{T} \int_0^\mathcal{T} w(t) w_t(\tau) \langle \int_0^\sigma f \circ h_t^\sigma \circ \phi_s^X ds, (\alpha f) \circ \phi_s^X \rangle d\tau d\tau$$

$$- \frac{1}{\sigma} \int_0^\mathcal{T} \int_0^\mathcal{T} w(t) w_t(\tau) \int_0^\mathcal{T} \langle \int_0^S f \circ h_t^\sigma \circ \phi_s^X ds, L_X (\alpha f) \circ \phi_s^X \rangle dS d\tau d\tau.$$

The statement of the lemma then follows from Lemma 8 and from the estimate

$$\| \int_0^\mathcal{T} \int_0^\mathcal{T} w(t) w_t(\tau) f \circ h_t^\sigma \circ \phi_s^X ds d\tau d\tau \|_0 \leq \| w \|_\infty \langle f \rangle_\infty t_{\sigma, \alpha}(\alpha) S\mathcal{T}. \quad \blacksquare$$

We finally estimate integrals of smooth coboundaries along the arcs $\gamma_{x,t}$, for all $\sigma > 0$ and for all $(x, t) \in M \times \mathbb{R}^+$. This estimate is motivated by the well-known, elementary fact that, in order to prove the absolute continuity of the
spectrum of an ergodic dynamical system by a density argument, it is enough to establish the absolute continuity of the spectral measures of all functions in a dense subspace of the space of coboundaries.

Let us recall that a function \( f \) on \( M \) is called a coboundary for the flow \( \{ h^\alpha_t \} \) if there exists a function \( u \) on \( M \), called the transfer function, such that \( U_\alpha u = f \).

**Lemma 11.** There exists a constant \( C > 0 \) such that for all \( \sigma > 0 \), for all continuous coboundaries \( f = U_\alpha u \) with transfer function \( u \in L^\infty(M) \) such that \( Xu \in L^\infty(M) \), for all \( x \in M \) and all \( t > 0 \),

\[
\sup_{S \in [0, \sigma]} \left| \int_{Y^\alpha_{x,t}} f \hat{U}_\alpha \right| \leq C \max \{ 1, \sigma \} \max \{ \| u \|_{L^\infty(M)}, \| Xu \|_{L^\infty(M)} \}.
\]

**Proof.** By the definition of the path \( Y^\alpha_{x,t} \) in formula (7) and by Lemma 2 we have

\[
\int_{Y^\alpha_{x,t}} f \hat{U}_\alpha = \int_{Y^\alpha_{x,t}} du - \int_{Y^\alpha_{x,t}} Xu \hat{X} = u \circ h^\alpha_t \circ \phi^X_s(x) - u \circ h^\alpha_t(x) - \int_0^S Xu \circ h^\alpha_t \circ \phi^X_s(x) ds,
\]

hence the statement of the lemma follows. \( \Box \)

4. Cohomological equation and quantitative equidistribution

It is a general fact that all properties of the cohomological equation and of the asymptotics of ergodic integrals for all smooth time-changes of any smooth flow can be read from the corresponding properties of the flow itself.

Let \( \mathcal{D}'(M) \) be the space of distributions on \( M \). For any distribution \( D \in \mathcal{D}'(M) \), let \( D_\alpha \) be the distribution defined as follows:

\[
D_\alpha(f) := D(\alpha f), \quad \text{for all } f \in C^\infty(M).
\]

The distribution \( D_\alpha \) is well-defined and belongs to the Sobolev space \( W^{-r}(M) \) whenever \( \alpha \in W^r(M) \) and \( D \in W^{-r}(M) \) for any \( r > 3/2 \). In fact, the Sobolev space \( W^r(M) \), endowed with the standard structure of Hilbert space and with the standard product of functions, is a Banach algebra for \( r > \dim(M)/2 \), which is here the case since \( M \) is a 3-dimensional manifold. The subspace \( \mathcal{F}^{-r}_\alpha(M) \subset W^{-r}(M) \) of invariant distributions for the time-change \( \{ h^\alpha_t \} \) can be described in terms of the subspace \( \mathcal{F}^{-r}_U(M) \subset W^{-r}(M) \) of invariant distributions for the horocycle flow (described in [5]).

**Lemma 12.** Let \( r > 3/2 \) and let \( \alpha \in W^r(M) \). The following holds:

\[
\mathcal{F}^{-r}_\alpha(M) := \{ D_\alpha | D \in \mathcal{F}^{-r}_U(M) \}.
\]

**Proof.** By the algebra property of the Sobolev space \( W^r(M) \) for \( r > 3/2 \), for any nonvanishing function \( \alpha \in W^r(M) \), the map \( D \to D_\alpha \) is an automorphism of the Sobolev space \( W^{-r}(M) \) (it is continuous, invertible with continuous inverse). By definition, a distribution \( D_\alpha \in \mathcal{F}^{-r}_\alpha(M) \), that is, the distribution \( D_\alpha \in \mathcal{F}^{-r}_\alpha(M) \),
$W^{-r}(M)$ is invariant for the time-change $\{h_t^\alpha\}$ of generator $U_\alpha = U/\alpha$, if and only if

$$D_\alpha(U_\alpha f) = D_\alpha(U f/\alpha) = D(U f) = 0, \quad \text{for all } f \in W^{r+1}(M),$$

if and only if the distribution $D \in W^{-r}(M)$ is invariant under the horocycle flow $\{h_t^\alpha\}$, that is, if and only if $D \in \mathcal{I}_{U}^{-r}(M)$, as stated. \hfill $\square$

The subspace of coboundaries for the time-changes of the horocycle flow is described by the following dictionary.

**Lemma 13.** A function $f$ on $M$ is a coboundary for the time-change $\{h_t^\alpha\}$ with transfer function $u$ on $M$ if and only if the function $\alpha f$ is a coboundary for the flow $\{h_t^\beta\}$ with transfer function $u$ on $M$.

**Proof.** It follows immediately from the definition of coboundary recalling that by definition $U_\alpha = U/\alpha$. \hfill $\square$

The theory of the cohomological equation for time-changes is thus reduced, by the above lemmas, to that for the classical horocycle flow, developed in [5]. We state the main results below for the convenience of the reader.

By the theory of unitary representations for the group $SL(2, \mathbb{R})$ (see for instance [1, 7, 8]), the Sobolev spaces $W^{r}(M)$ split as direct sums of irreducible subrepresentations and each irreducible subrepresentations characterized up to unitary equivalence by the spectral value $\mu \in \sigma(\square)$ of the restriction of the Casimir operator $\square$ (a normalized generator of the center of the enveloping algebra), that is, for all $r \in \mathbb{R}$, the following splitting holds:

$$W^{r}(M) = \bigoplus_{\mu \in \sigma(\square)} W^{r}(H_\mu)$$

Non-trivial irreducible unitary representations belong to three different series: the principal series ($\mu \geq 1/4$), the complementary series ($0 < \mu < 1/4$) and discrete series ($\mu \leq 0$). We recall that the positive spectral values of the Casimir operator coincide with the eigenvalues of the Laplace–Beltrami operator on the (compact) hyperbolic surface, while the nonpositive spectral values are given by the set of nonpositive integers $\{-n^2 + n | n \in \mathbb{Z}^+\}$.

Let $\alpha \in W^r(M)$ for any $r > 3/2$ be any strictly positive function. For every Casimir parameter $\mu \in \mathbb{R}$, let

$$W_\alpha^{r}(H_\mu) := \{f \in L^2(M, \text{vol}_\alpha) | \alpha f \in W^{r}(H_\mu)\}.$$ 

By the above splitting (12), there is also a splitting

$$W^{r}(M) = \bigoplus_{\mu \in \sigma(\square)} W_\alpha^{r}(H_\mu)$$

For every $\mu \in \sigma(\square)$, let $\mathcal{S}_{U}^{-r}(H_\mu) := \mathcal{S}_{U}^{-r}(M) \cap W^{-r}(H_\mu)$ and let $\mathcal{S}_{\alpha}^{-r}(H_\mu)$ be the distributional space defined as

$$\mathcal{S}_{\alpha}^{-r}(H_\mu) := \{D_\alpha | D \in \mathcal{S}_{U}^{-r}(H_\mu)\}.$$
Theorem 14. Let $\alpha \in \mathcal{W}^r(M)$ for any $r > 3/2$. The space of $U_\alpha$-invariant distributions has a splitting

$$\mathcal{F}_a^{-r}(M) = \bigoplus_{\mu \in \sigma(\square)} \mathcal{F}_a^{-r}(H_\mu).$$

The subspace $\mathcal{F}_a^{-r}(H_\mu)$ has dimension 2 for all irreducible subrepresentations of the principal and complementary series ($\mu > 0$), it has dimension 1 for irreducible subrepresentations of the discrete series if $r > \frac{1+\sqrt{1-4\mu}}{2}$ and it is trivial otherwise.

For every function $f \in W_\mu^r(H_\mu)$, the cohomological equation $U_\alpha u = f$ has a unique solution $u \in H_\mu \subset L^2(M)$ if and only if $f \in \text{Ann}[\Gamma_a^{-r}(H_\mu)]$. In case a solution $u \in H_\mu$ exists then $u \in W^r(H_\mu)$ for all $s < r - 1$ and the following a priori bound holds: there exists a constant $C_{5,r} > 0$, independent on $\mu \in \sigma(\square)$, such that

$$\|u\|_2 \leq C_{5,r}\|af\|_r.$$

The quantitative equidistribution for time-changes, and in fact the complete asymptotics of ergodic averages, can also be derived from the corresponding results for the classical horocycle flow, derived in [5] and [2]. By change of variable there exists a function $T': M \times \mathbb{R} \to \mathbb{R}$ such that, for any function $f$ on $M$,

$$\int_0^T f \circ h^a_t(x) \, dt = \int_0^{T(x,T')} f \circ h^a_t(x,t) \frac{\partial}{\partial t}(x,t) \, dt$$

$$= \int_0^{T(x,T')} (af) \circ h^U_t(x) \, dt. \tag{14}$$

By the above formula for the constant function $f = 1$, it follows that the function $T': M \times \mathbb{R} \to \mathbb{R}$ is given by the following identity: for all $(x,T) \in M \times \mathbb{R}$,

$$T = \int_0^{T(x,T')} a \circ h^U_t(x) \, dt. \tag{15}$$

The asymptotics of the function $T: M \times \mathbb{R} \to \mathbb{R}$ for large $T \in \mathbb{R}$, uniformly with respect to $x \in M$, can be derived by the quantitative equidistribution result of M. Burger [3] (see also [5]).

Let $\mu_0 > 0$ be the smallest strictly positive eigenvalue of the Casimir operator (that, as we remarked, coincide with the smallest eigenvalue for the hyperbolic Laplacian on the compact surface $\Gamma \setminus \{z \in \mathbb{C} | \text{Im}(z) > 0\}$). Let $\nu_0 \in [0,1)$ and $\epsilon_0 \in [0,1]$ be the parameters defined as follows:

$$\nu_0 := \begin{cases} \sqrt{1-4\mu_0}, & \text{if } \mu_0 < 1/4, \\ 0, & \text{if } \mu_0 \geq 1/4; \end{cases} \quad \epsilon_0 := \begin{cases} 0, & \text{if } \mu_0 \neq 1/4, \\ 1, & \text{if } \mu_0 = 1/4. \end{cases} \tag{16}$$

Lemma 15. For any $r > 3$, there exists a constant $C_r > 0$ such that the following estimate holds. Let $\alpha \in \mathcal{W}^r(M)$. For all $(x,T) \in M \times \mathbb{R}^+$,

$$|T(x,T) - T| \leq C_r \|a\|_r \mathcal{M}^{1+\nu_0}_{-r} (1 + \log^{+} T)^{\epsilon_0}.$$
Proof. Let us assume $\mu_0 \neq 1/4$. The argument in the case $\mu_0 = 1/4$ is similar. By the normalization condition (5), it follows from the identity (15) and from [5], Theorem 1.5, that there exists a constant $C_r' > 0$ such that, for all $(x, \mathcal{T}) \in M \times \mathbb{R},$

(17) \[ |\mathcal{T} - T(x, \mathcal{T})| \leq C_r'\|f\|_rT(x, \mathcal{T})^{1+\nu_0}. \]

Thus for any $\nu > 0$ there exists $\mathcal{T}_\nu > 0$ such that, for all $x \in M$ and $\mathcal{T} \geq \mathcal{T}_\nu,$

\[ \left| \frac{\mathcal{T}}{T(x, \mathcal{T})} - 1 \right| \leq \nu, \]

hence $T(x, \mathcal{T}) \leq \mathcal{T}/(1 - \sigma).$ Thus by formula (17), for all $x \in M$ and for all $\mathcal{T} \geq \mathcal{T}_\sigma,$

\[ |\mathcal{T} - T(x, \mathcal{T})| \leq \frac{C_r'}{(1 - \nu)^{1+\nu_0}} \|f\|_r\mathcal{T}^{1+\nu_0}. \]

The argument is thus completed. \qed

By Lemma 15, the asymptotics of ergodic integrals for smooth time-changes of horocycle flows is entirely analogous and can be derived from the corresponding results for horocycle flows, proved in [5] (see also [2]).

For any $\nu > 0$, let $\nu_\mu \in \mathbb{R}$ and $\epsilon_\mu \in [0, 1)$ be defined as follows:

\[ \nu_\mu := \begin{cases} \sqrt{1 - 4\mu}, & \text{if } \mu < 1/4, \\ 0, & \text{if } \mu \geq 1/4; \end{cases} \quad \epsilon_\mu := \begin{cases} 0, & \text{if } \mu \neq 1/4, \\ 1, & \text{if } \mu = 1/4. \end{cases} \]

**Theorem 16.** For any $r > 3$, there exists a constant $C_r > 0$ such that the following holds. Let $\alpha \in W^r(M)$. Let $H_\mu$ be an irreducible subrepresentation of the principal or complementary series ($\mu > 0$). There exists a basis $\{D_{a, \mu}, D_{a, \mu}\} \subset \mathcal{F}_a^{-1}(H_\mu)$ such that for any function $f \in W^r_a(H_\mu)$ and for all $(x, \mathcal{T}) \in M \times \mathbb{R}^+$,

\[ \left| \int_0^{\mathcal{T}} f \circ h^\tau_t(x) \, d\tau \right| \leq C_r |D_{a, \mu}(f)||\mathcal{T}|^{1+\nu_\mu} \]

\[ + C_r \left( |D_{a, \mu}(f)||\mathcal{T}|^{1+\nu_\mu} (1 + \log^+ \mathcal{T})^{\nu_\mu} + \|f\|_r \right). \]

Let $H_\mu$ be an irreducible subrepresentation of the discrete series ($\mu \leq 0$). For any function $f \in W^r_a(H_\mu)$ and for all $(x, \mathcal{T}) \in M \times \mathbb{R}^+$,

(18) \[ \left| \int_0^{\mathcal{T}} f \circ h^\tau_t(x) \, d\tau \right| \leq C_r \|f\|_r (1 + \log^+ \mathcal{T}). \]

5. **Quantitative mixing and bounds on twisted ergodic integrals**

In this section we derive sharp bounds on the decay of correlations (in Section 5.2) and on the $L^2$ norm of twisted ergodic integrals (in Section 5.3) for smooth time-changes of the horocycle flow. The results are based on the refined Marcus’ mechanism of Section 3 and on sharp bounds on integrals of smooth functions along the push-forward of geodesic arcs, which we derive below in Section 5.1 from the quantitative equidistribution results of Section 4.
5.1. **Equidistribution estimates.** From the results of [2] on the asymptotics of ergodic averages for the horocycle flow (see in particular [2], Theorem 1.3) we derive the following bound. Let

\[ L^2_0(\mathcal{M}, \text{vol}_\alpha) := \{ f \in L^2(\mathcal{M}, \text{vol}_\alpha) | \int_M f \text{vol}_\alpha = 0 \}. \]

**Lemma 17.** Let \( r > 11/2. \) For any \( \alpha \in W^r(M) \) and for any \( \sigma > 0, \) there exists a constant \( C_{r,\sigma}(\alpha) > 0 \) such that the following holds. For any zero-average function \( f \in W^r(M) \cap L^2_0(M, \text{vol}_\alpha), \) for all \( x \in M, \) for all \( S \in (0, \sigma], \) and for all \( t > 0, \)

\[ \left| \int_{\gamma_{x,t}} f \hat{U}_\alpha \right| \leq C_{r,\sigma}(\alpha) \| f \|_r (S t)^{1+\nu_0} \left[ 1 + \log^+ (S t) \right]^{\nu_0}. \]

**Proof.** Precise bounds on the integrals

\[ \int_{\gamma_{x,t}} f \hat{U}_\alpha = \int_{\gamma_{x,t}} (\alpha f) \hat{U}. \]

can be derived from [2], Theorem 1.3, if the function \( \alpha f \in W^r(M) \) is supported on irreducible unitary components of the principal and complementary series. By Lemma 3, it follows from [2], Theorem 1.3, that there exists a constant \( C_{r,\alpha}(\alpha) > 0 \) such that, for all \( S \in (0, \sigma] \) and for all \( t > 0, \)

\[ \left| \int_{\gamma_{x,t}} f \hat{U}_\alpha \right| \leq C_{r,\sigma}(\alpha) \| f \|_r (S t)^{1+\nu_0} \left[ 1 + \log^+ (S t) \right]^{\nu_0}. \]

In case \( \alpha f \in W^r(M) \) is supported on irreducible unitary components of the discrete series, since the path \( \gamma_{x,t}^\alpha \) is contained in a leaf of the weak stable foliation of the geodesic flow (tangent to the integrable distribution \( \{ X, U \} \) it follows from the methods of [2] that

\[ \left| \int_{\gamma_{x,t}} f \hat{U}_\alpha \right| \leq C_{r,\sigma}(\alpha) \| f \|_r \left( 1 + \int_{\gamma_{x,t}} |\hat{X}| \right) \log \left( 1 + \int_{\gamma_{x,t}} |\hat{U}| \right). \]

By the above formula and by Lemma 3 the argument is completed. \( \Box \)

**Lemma 18.** Let \( r > 11/2. \) For any \( \alpha \in W^r(M) \) and \( \sigma \in (0, \sigma_r(\alpha) \), there exists a constant \( C_{r,\sigma}(\alpha) > 0 \) such that the following holds. For any zero-average function \( f \in W^r(M) \cap L^2_0(M, \text{vol}_\alpha), \) for all \( x \in M, \) for all \( S \in (0, \sigma], \) and for all \( t > t_{r,\sigma}(\alpha), \)

\[ \left| \int_0^S (f \circ h_t^\alpha \circ \phi_t^X)(x) ds \right| \leq C_{r,\sigma}(\alpha) \| f \|_{r,\alpha} (S t)^{1+\nu_0} \left[ 1 + \log^+ (S t) \right]^{\nu_0} / t. \]

**Proof.** By the bootstrap bound proved in Lemma 8, the statement follows immediately from the above Lemma 17. \( \Box \)

5.2. **Quantitative Mixing.** Let us denote \( \| \cdot \|_X \) the graph norm of the densely defined Lie derivative operator \( \mathcal{L}_X : L^2(M) \to L^2(M), \) that is, for all functions \( g \in L^2(M) \) which belong to the maximal domain \( \text{dom}(\mathcal{L}_X) \subset L^2(M) \) of \( \mathcal{L}_X, \)

\[ \| g \|_X := (\| g \|_0^2 + \| Xg \|_0^2)^{1/2}. \]
The statement of the theorem then follows from Lemma 10 by integration.

Proof. By Lemma 9 and Lemma 18, there exists a constant \( C_1(a) > 0 \) such that
\[
|\langle f \circ h^t, g \rangle| \leq C_1(a) \| f \| R \| g \|_X t^{-1/2} (1 + \log t)^\epsilon_0.
\]
Finally, by taking into account that, for all \( f, g \in \mathcal{W}_r(M) \) and all \( t \geq 0 \),
\[
\langle f \circ h^t, g \rangle_{L^2(M, \text{vol}_a)} = \langle f \circ h^t, \alpha g \rangle,
\]
the theorem follows from the estimate in formula (19). In fact, by the Sobolev Embedding Theorem, for all \( g \in \text{dom}(\mathcal{L}_X) \),
\[
\| \alpha g \|_X \leq 2 \| \alpha \|_R \| g \|_X.
\]

5.3. Twisted ergodic integrals. We prove below \( L^2 \) bounds on twisted ergodic integrals of smooth functions, that is, on integrals of the form
\[
\int_0^T w(t) f \circ h^t(x) \, dt
\]
for any weight function \( w \in L^\infty(\mathbb{R}, \mathbb{C}) \) and for any sufficiently smooth function \( f \) on \( M \). The main class of weight functions is given by the exponential functions (the characters of the additive group of real numbers) which are related to the Fourier transforms of the spectral measures. We note that the question on optimal uniform bounds for twisted ergodic integrals (even with exponential weight functions) is open even for the classical horocycle flow. Non-optimal, polynomial bounds can be derived from estimates on the rate of equidistribution and on the rate of mixing by an argument due to A. Venkatesh [26]. Such uniform bounds are closely related to estimates on the rate of equidistribution of time-\( \mathcal{F} \) maps of horocycle flows.

Theorem 20. Let \( r > 11/2 \). For any \( a \in \mathcal{W}_r(M) \), there exists a constant \( C_r(a) > 0 \) such that the following holds. For any bounded weight function \( w \in L^\infty(\mathbb{R}^+, \mathbb{C}) \), for any zero-average function \( f \in \mathcal{W}_r(M) \cap L^2_0(M, \text{vol}_a) \) and for all \( \mathcal{F} > 0 \),
\[
\left\| \int_0^T w(t) f \circ h^t \, dt \right\|_{L^2(M, \text{vol}_a)} \leq C_r(a) |w|_\infty \| f \|_R \mathcal{F}^{1/2} \left[ 1 + \log^+ \mathcal{F} \right]^{\epsilon_0}.
\]

Proof. By the equidistribution estimates proved in Lemma 17, for any \( \sigma > 0 \) there exists a constant \( C_{r,\sigma}(a) > 0 \) such that for all \( f \in \mathcal{W}_r(M) \cap L^2_0(M, \text{vol}_a) \) and for all \( x \in M \) and all \( \tau > 0 \),
\[
\sup_{S \in [0, \sigma]} \left| \int_{Y^{u,\tau}_x} f \hat{U}_a \right| \leq C_{r,\sigma}(a) \| f \|_R \| \tau \|^\epsilon_0 \left[ 1 + \log^+ \tau \right]^{\epsilon_0}.
\]
The statement of the theorem then follows from Lemma 10 by integration. \( \square \)
We remark that more refined estimates can be proved for functions supported on finite codimensional subspaces orthogonal to irreducible components of the complementary series and for coboundaries. In particular, from the estimate on twisted ergodic integrals of coboundaries, based on Lemma 10 and Lemma 11, we will derive in Section 6.1 local bounds on spectral measures.

6. SPECTRAL THEORY

In this final section we prove spectral results for smooth time-changes of the horocycle flow. In Section 6.1 we prove a local estimate on spectral measures of smooth functions (see Theorem 21). In Section 6.2 we prove the absolute continuity of the spectrum (with countable multiplicity) for all smooth time-changes of the classical horocycle flow (see Theorem 23). Finally in Section 6.3 we show that the maximal spectral type is always equivalent to Lebesgue (see Theorem 25). All our spectral results are based only on the results of Section 3 and thus do not depend on our results on quantitative unique ergodicity, on quantitative mixing and on our bounds on twisted ergodic integrals for general smooth functions (proved in Sections 4 and 5). In fact, as we have already indicated, the crucial step in the derivation of our spectral results following Marcus’ approach is the ‘bootstrap trick’ of Lemma 8 (which depends only on the unique ergodicity of the horocycle flow and of its time-changes).

6.1. Local estimates. For any function \( f \in L^2(M, \text{vol}_\alpha) \), let \( \mu_f \) denote its spectral measure with respect to a time-change of the horocycle flow. We recall that \( \mu_f \) is a complex measure on the real line. We prove the following result.

**Theorem 21.** Let \( r > 11/2 \) and let \( \alpha \in W^r(M) \). There exists a constant \( C_r(\alpha) > 0 \) such that, for any function \( u \in W^{r+1}(M) \) and for any \( \xi \in \mathbb{R} \sim \{0\} \),

\[
|\mu_u(\xi - \delta, \xi + \delta)| \leq C_r(\alpha) \| u \|_{r+1} \frac{\delta |\log \delta|}{\xi^2}, \quad \text{for all } \delta \in (0, |\xi|/2),
\]

hence the measure \( \mu_u \) has local dimension 1 at all points \( \xi \in \mathbb{R} \sim \{0\} \), that is,

\[
\lim_{\delta \to 0^+} \frac{\log |\mu_u(\xi - \delta, \xi + \delta)|}{\log \delta} = 1.
\]

The above result implies that the local dimension of the spectral measures of smooth functions is everywhere equal to 1, but it is off by a logarithmic term from the sharpest possible bound, which would imply that spectral measures of sufficiently smooth functions are absolutely continuous with bounded densities. In the following section (§6.2) we nevertheless show how one can derive absolutely continuity from bounds on the correlations of coboundaries.

Let us estimate the twisted ergodic integrals of smooth coboundaries.

**Lemma 22.** Let \( r > 11/2 \) and let \( \alpha \in W^r(M) \). There exists a constant \( C_r(\alpha) > 0 \) such that for any bounded weight function \( w \in L^\infty(\mathbb{R}^+, \mathbb{C}) \), for all coboundaries \( f = U_\alpha u \) with a transfer function \( u \in W^{r+1}(M) \) and for all \( \mathcal{F} > 0 \),

\[
\left\| \int_0^\mathcal{F} w(t) f \circ h_t^\alpha \, dt \right\|_{L^2(M, \text{vol}_\alpha)} \leq C_r(\alpha) \| w \|_{\infty} \| u \|_{r+1} \mathcal{F}^{1/2} \left( 1 + \log^+ \mathcal{F} \right)^{1/2}.
\]
Proof. By Sobolev Embedding Theorem if \( u \in W^{r+1}(M) \) then \( u, Xu \in L^\infty(M) \) and the following estimate holds: there exists a constant \( C_r > 0 \) such that
\[
\max \{ \| u \|_{L^\infty(M)}, \| Xu \|_{L^\infty(M)} \} \leq C_r \| u \|_{r+1}.
\]
The statement then follows by integration from Lemma 10 and Lemma 11.

Proof of Theorem 21. By the spectral theorem, for any function \( f \in L^2(M, \text{vol}_\alpha) \) and any \( \xi \in \mathbb{R} \),
\[
\left\| \frac{e^{i(\xi+\eta)t}}{t} - 1 \right\|_{L^2_\alpha(\mathbb{R}, d\mu_f)}^2 = \left\| \int_{-\infty}^{\infty} \frac{e^{i(\xi+\eta)t}}{t} d\mu_f(\eta) \right\|_{L^2_\alpha(\mathbb{R}, d\mu_f)}^2.
\]
By a simple computation, there exists a constant \( C > 0 \) such that
\[
\mu_f\left(\xi - \frac{1}{T}, \xi + \frac{1}{T}\right) \leq C \int_{-\frac{1}{T}}^{\frac{1}{T}} \left| \frac{e^{i(\xi+\eta)t} - 1}{i(\xi+\eta)t} \right|^2 d\mu_f(\eta)
\]
(21)
\[
\leq \frac{C}{T^2} \left\| \frac{e^{i(\xi+\eta)t}}{i(\xi+\eta)} - 1 \right\|_{L^2_\alpha(\mathbb{R}, d\mu_f)}^2.
\]
Let \( u \in \text{dom}(U_\alpha) \subset L^2(M, \text{vol}_\alpha) \) and let \( f := U_\alpha u \in L^2(M, \text{vol}_\alpha) \). By the spectral theorem
\[
d\mu_f(\xi) = \xi^2 d\mu_u(\xi), \quad \text{for all } \xi \in \mathbb{R},
\]
hence there exists a constant \( C' > 0 \) such that, for all \( \xi \in \mathbb{R} \setminus \{0\} \),
\[
\mu_u(\xi - \delta, \xi + \delta) \leq C' \frac{\mu_f(\xi - \delta, \xi + \delta)}{\xi^2}, \quad \text{for all } \delta \in (0, |\xi|/2).
\]
(23)
By formulas (20) and (21) and by Lemma 22, it follows that there exists a constant \( C_r(\alpha) > 0 \) such that, for all \( \xi \in \mathbb{R} \) and for all \( \mathcal{F} > 0 \),
\[
\mu_f\left(\xi - \frac{1}{\mathcal{F}}, \xi + \frac{1}{\mathcal{F}}\right) \leq C_r(\alpha) \| u \|_{r+1} \left(1 + \log^- \mathcal{F}\right).
\]
(24)
The statement of the theorem can readily be derived from formulas (23) and (24).

6.2. Absolute continuity. We prove in this section that any smooth time-change of the horocycle flow has absolutely continuous spectrum.

Theorem 23. Let \( r > 11/2 \) and let \( \alpha \in W^r(M) \). The time-change \( h^\alpha \) of the (stable) horocycle flow \( h_U \) with infinitesimal generator \( U_\alpha := U/\alpha \) has purely absolutely continuous spectrum.

The Theorem is derived below from the following estimate on decay of correlations of coboundaries.
Lemma 24. Let \( r > 11/2 \) and let \( \alpha \in W^r(M) \). There exist constants \( C_r(\alpha) > 0 \) and \( t_r(\alpha) > 0 \) such that the following holds. For any continuous coboundary \( f = U_\alpha u \) with a transfer function \( u \in L^\infty(M) \) such that \( Xu \in L^\infty(M) \) and for any \( g \in \text{dom}(X) \), for all \( t > t_r(\alpha) \),

\[
|\langle f \circ h_t^\alpha, g \rangle| \leq C_r(\alpha) \| u \|_{r+1} \| g \|_X (1/t).
\]

Proof. The statement follows readily from Lemmas 8, 9 and 11.

Proof of Theorem 23. By ergodicity of the horocycle flow and of its time-changes, coboundaries with smooth transfer functions are dense in \( L^2(M,\text{vol}_\alpha) \). Thus it is enough to prove that their spectral measures are absolutely continuous. In fact, we will prove below that any coboundary \( f = U_\alpha u \) with smooth transfer function \( u \in W^{r+1}(M) \) has an absolutely continuous spectral measure with square-integrable density. Let \( \hat{\mu}_f \) be the Fourier transform of the spectral measure \( \mu_f \), which is the bounded function defined as follows:

\[
\hat{\mu}_f(t) := \int_{\mathbb{R}} e^{i\xi t} d\mu_f(\xi), \quad \text{for all } t \in \mathbb{R}.
\]

By definition of the spectral measures, for all \( t \in \mathbb{R} \) we have the following identity

\[
\hat{\mu}_f(t) = \langle f \circ h_t^\alpha, f \rangle_{L^2(M,\text{vol}_\alpha)} = \langle f \circ h_t^\alpha, \alpha f \rangle.
\]

By Lemma 24 we therefore have the following estimate: for any \( t > t_r(\alpha) \),

\[
|\hat{\mu}_f(t)| = |\langle f \circ h_t^\alpha, \alpha f \rangle| \leq C_r(\alpha) \| u \|_{r+1} \| \alpha f \|_X (1/t).
\]

Since \( \hat{\mu}_f \) is a bounded function and it is symmetric, that is, for all \( t \in \mathbb{R} \),

\[
\hat{\mu}_f(t) = \langle f \circ h_t^\alpha, f \rangle_{L^2(M,\text{vol}_\alpha)} = \langle f, f \circ h_t^\alpha \rangle_{L^2(M,\text{vol}_\alpha)} = \hat{\mu}_f(-t),
\]

and since the function \( 1/t \in L^2((t_r(\alpha), +\infty), dt) \), we have proved that the Fourier transform \( \hat{\mu}_f \in L^2(\mathbb{R}, dt) \), which readily implies that the spectral measure \( \mu_f \) is absolutely continuous with square-integrable Radon–Nikodym derivative. In fact, there exists a constant \( C'_r(\alpha) > 0 \) such that the following estimate holds:

\[
\left\| \frac{D\hat{\mu}_f}{D\xi} \right\|_{L^2(\mathbb{R},d\xi)} = \| \hat{\mu}_f \|_{L^2(\mathbb{R}, dt)} \leq C'_r(\alpha) \| u \|_{r+1} \| f \|_X.
\]

The proof of the theorem is complete.

6.3. Maximal spectral type. Let us now prove that the maximal spectral type of any smooth time-change of the horocycle flow is equivalent to Lebesgue.

Theorem 25. Let \( r > 11/2 \) and let \( \alpha \in W^r(M) \). The maximal spectral type of the time-change \( \{h_t^\alpha\} \) of the (stable) horocycle flow \( \{h_t^1\} \) with infinitesimal generator \( U_\alpha := U/\alpha \) is equivalent to Lebesgue.

The above theorem, our conclusive spectral result, will follow from Lemma 27 and Lemma 28 below.
**Lemma 26.** For any $\sigma \in (0, \sigma_r(\alpha))$ there exists a constant $C_{r, \sigma}(\alpha) > 0$ such that, for all $x \in M$, for all $t > t_r(\alpha)$ and for all functions $u \in C^1(M)$, we have

$$\left| \int_0^\sigma U_a u \circ h_t^\alpha \circ \phi_s^X(x) \, ds \right| \leq C_{r, \sigma}(\alpha) \max\{\|u\|_{L^\infty(M)}, \|Xu\|_{L^\infty(M)}\}(1/t).$$

**Proof.** The statement follows from Lemma 8 and Lemma 11.

**Lemma 27.** Let $r > 11/2$ and let $\alpha \in W^r(M)$. Assume that the maximal spectral type of the time-change $|h_t^\alpha|$ is not Lebesgue. There exists a smooth nonzero function $w \in L^2(\mathbb{R}, dt)$ such that for all $x \in M$, for all $\sigma \in (0, \sigma_r(\alpha))$ and for all functions $u \in C^1(M)$ the following holds:

$$\int_\mathbb{R} w(t) \int_0^\sigma U_a u \circ h_t^\alpha \circ \phi_s^X(x) \, ds \, dt = 0.$$

**Proof.** If the maximal spectral type is not Lebesgue, then by Theorem 23 the Lebesgue measure is not absolutely continuous with respect to the maximal spectral measure. Thus, there exists a compact set $A \subset \mathbb{R}$ such that $A$ has measure zero with respect to the maximal spectral measure of the flow $\{h_t^\alpha\}$, hence with respect to all its spectral measures, but $A$ has strictly positive Lebesgue measure.

Let $w \in L^2(\mathbb{R})$ be the complex conjugate of the Fourier transform of the characteristic function $\chi_A$ of the set $A \subset \mathbb{R}$. For any pair of functions $f$, $g \in L^2(M)$ let $\mu_{f,g}$ denote the joint spectral measure. By Theorem 23 the measure $\mu_{f,g}$ is absolutely continuous with respect to Lebesgue. Whenever $\mu_{f,g}$ has square-integrable density we have

$$(26) \quad \int_\mathbb{R} w(t) \langle f \circ h_t^\alpha, g \rangle_{L^2(M, \text{vol}_\mu)} \, dt = \int_\mathbb{R} \chi_A(\xi) d\mu_{f,g}(\xi) = 0.$$  

It follows from Lemma 24 that, whenever $f = U_a u$ is a coboundary with transfer function $u \in C^1(M)$, then the Fourier transform of the spectral measure $\mu_{f,g}$, hence its density, is square-integrable, so that the identity in formula (26) holds.

From formula (26) by translation under the geodesic flow and by integration we derive that, for any $\sigma > 0$ and for any function $g \in W^r(M) \subset \text{dom}(X)$,

$$(27) \quad \int_\mathbb{R} \int_0^\sigma w(t) \langle U_a u \circ h_t^\alpha \circ \phi_s^X, g \rangle_{L^2(M, \text{vol}_\mu)} \, ds \, dt = 0.$$  

By Lemma 24 the double integral in formula (27) is absolutely convergent, hence

$$(28) \quad \langle \int_\mathbb{R} w(t) \int_0^\sigma U_a u \circ h_t^\alpha \circ \phi_s^X(\cdot) \, ds \, dt, g \rangle_{L^2(M, \text{vol}_\mu)} = 0.$$  

It follows from Lemma 26 that the function

$$\int_\mathbb{R} w(t) \int_0^\sigma U_a u \circ h_t^\alpha \circ \phi_s^X(\cdot) \, ds \, dt$$  

is bounded on $M$, hence it vanishes identically by formula (28) and by density of the subspace $W^r(M) \subset L^2(M)$.

\[\square\]
Let us prove the above commutation identity. For $S(34)$ of the flow-box map. The second formula follows from the first of the above formulas is an immediate consequence of the definition, all $(\geodesic segment, hence for any $\sigma$ Since the horocycle flow has no periodic orbits, it never returns to any given proof. Let us fix $R$. For all $(\geodesic segment, hence for any $\sigma$ Since the horocycle flow has no periodic orbits, it never returns to any given

**Lemma 28.** Let $w \in L^2(\mathbb{R}, dt)$ be a smooth function. Assume that for some $x \in M$, for some $\sigma \in (0, \sigma(x))$ and for all functions $u \in C^1(M)$,

$$
\int_{\mathbb{R}} w(t) \int_{0}^{\sigma} U_\alpha u \circ h^\alpha_t \circ \phi^X_t(x) \, ds \, dt = 0.
$$

It follows that $w$ vanishes identically.

**Proof.** Let us fix $x \in M$ and $\sigma > 0$. For any given $T > 0$ and $\rho > 0$, let $E^T_{p,\sigma}$ be the flow-box for the time-change $(h^\alpha_t)$ defined as follows:

$$
E^T_{p,\sigma}(r, s, t) = (h^\alpha_t \circ \phi^X_t \circ h^Y_t)(x), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T).
$$

Since the horocycle flow has no periodic orbits, it never returns to any given geodesic segment, hence for any $\sigma > 0$ and any $T > 0$ there exists $\rho > 0$ such that $E^T_{p,\sigma}$ is an embedding. For any $\chi \in C^\infty_0(-1, 1)$ and any $\psi \in C^\infty_0(-T, T)$, let

$$
\tilde{u}(r, s, t) := \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \psi(t), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T).
$$

Let $u \in C^\infty(M)$ be the function defined as $u = 0$ on $M \sim \operatorname{Im}(E^T_{p,\sigma})$ and as

$$
(u \circ E^T_{p,\sigma})(r, s, t) := \tilde{u}(r, s, t), \quad \text{for all } (r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T),
$$

on $\operatorname{Im}(E^T_{p,\sigma})$. We claim that the following formulas hold:

$$
(U_\alpha u) \circ E^T_{p,\sigma}(r, s, t) := \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \frac{d \psi}{dt}(t),
$$

$$
(Xu) \circ E^T_{p,\sigma}(r, s, t) := \sigma \chi\left(\frac{r}{\rho}\right) \frac{d \chi}{ds}(\frac{s}{\sigma}) \psi(t) - v_t(h^Y_t(x), s) \chi\left(\frac{r}{\rho}\right) \chi\left(\frac{s}{\sigma}\right) \frac{d \psi}{dt}(t).
$$

In fact, the above formulas are immediate consequence of the identities below. Let $\mathcal{T}_{r,s}: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the unique solution of the parametric Cauchy problem

$$
\begin{align*}
\frac{\partial \mathcal{T}_{r,s}}{\partial S}(t, S) &= -v_{\mathcal{T}_{r,s}(t, S)}(h^Y_t(x), s + S), \\
\mathcal{T}_{r,s}(t, 0) &= t.
\end{align*}
$$

For any $(r, s, t) \in (-\rho, \rho) \times (-\sigma, \sigma) \times (-T, T)$ there exists $(\tau_0, S_0) \in \mathbb{R}^+$ such that, for all $(r, S) \in (-\tau_0, \tau_0) \times (-S_0, S_0)$, the following holds:

$$
\begin{align*}
&h^\alpha_t \circ E^T_{p,\sigma}(r, s, t) = E^T_{p,\sigma}(r, s, t + \tau), \\
&\phi^X_t \circ E^T_{p,\sigma}(r, s, t) = E^T_{p,\sigma}(r, s + S, \mathcal{T}_{r,s}(t, S)).
\end{align*}
$$

The first of the above formulas is an immediate consequence of the definition, in formula (29), of the flow-box map. The second formula follows from the commutation relation:

$$
\phi^X_t \circ h^\alpha_t \circ \phi^X_t \circ h^Y_t(x) = h_{\mathcal{T}_{r,s}(t, S)} \circ \phi^X_{r+S,S} \circ h^Y_t(x), \quad \text{for all } (t, s, S) \in \mathbb{R}^3.
$$

Let us prove the above commutation identity. For $S = 0$ it holds for all $(r, s, t) \in \mathbb{R}^3$. For all $(r, s) \in \mathbb{R}^2$ let $x_{r,s} := (\phi^X_s \circ h^Y_t)(x)$. By Lemma 1, by differentiation of
By assumption and by formula (31) we have
\[
\lim_{\rho} T_{\rho, \sigma} > 0 \text{ be defined as follows:}
\]
\[
T_{\rho, \sigma} := \min \{|t| > T| \bigcup_{s=-\sigma, \sigma} (h^{\alpha}_t \circ \phi^X_s(x)) \cap \text{Im}(E^T_{\rho, \sigma}) \neq \emptyset\}.
\]
Since the horocycle flow never returns to any given geodesic segment, for every fixed \(\sigma > 0\), the following holds:
\[
\lim_{\rho \to 0^+} T_{\rho, \sigma} = +\infty.
\]
By assumption and by formula (31) we have
\[
\chi(0) \left( \int_0^\sigma \frac{\chi(s)}{\sigma} ds \right) \left( \int_{-T}^T w(t) \frac{d\psi}{dt}(t) dt \right)
+ \int_{\mathbb{R}^2}_{T_{\rho, \sigma}, T_{\rho, \sigma}} w(t) \int_0^\sigma U_\alpha u \circ h^{\alpha}_t \circ \phi^X_s(x) ds dt = 0.
\]
We claim that the following holds: for any fixed \(\sigma \in (0, \sigma_0(\alpha))\) and \(T > 0\),
\[
\lim_{\rho \to 0^+} \int_{\mathbb{R}^2}_{T_{\rho, \sigma}, T_{\rho, \sigma}} w(t) \int_0^\sigma U_\alpha u \circ h^{\alpha}_t \circ \phi^X_s(x) ds dt = 0.
\]
Since the function \(u \in C^{\infty}(M)\), by Lemma 26, combined with a trivial estimate for \(0 \leq t \leq t_\eta(\alpha)\), there exists a constant \(C'_{r, \sigma}(\alpha) > 0\) such that, for all \(x \in M\) and for all \(\sigma \in (0, \sigma_0(\alpha))\), we have
\[
\left\| \int_0^\sigma U_\alpha u \circ h^{\alpha}_t \circ \phi^X_s(x) ds \right\|_{L^2(\mathbb{R}, dt)} \leq C'_{r, \sigma}(\alpha) \max \{|u|_{L^\infty(M)}, \|X u\|_{L^\infty(M)}, \|U_\alpha u\|_{L^\infty(M)}\}.
\]
By the definition of the function \(u \in C^{\infty}(M)\) (see formula (30), by the trivial estimate \(|v_\ell| \leq C_\alpha t \leq C_\alpha T\) and by the estimates in formula (31) we also have
\[
\max \{|u|_{L^\infty(M)}, \|X u\|_{L^\infty(M)}, \|U_\alpha u\|_{L^\infty(M)}\} \leq C'_{r, \sigma}(\alpha) \max \{1, T\}
\times \max \left\{ \chi \|X\|_{L^\infty(\mathbb{R})}, \|X'\|_{L^\infty(\mathbb{R})} \right\} \max \left\{ \|\psi\|_{L^\infty(\mathbb{R})}, \|\psi'\|_{L^\infty(\mathbb{R})} \right\}.
\]
In particular the above bound is uniform with respect to \(\rho > 0\). Thus the limit in formula (38) follows from the uniform \(L^2\) bound given by formulas (39) and (40).
Since formulas (37) and (38) hold for all functions \( \chi \in C^\infty_0(\mathbb{R}) \), for all \( T > 0 \) and for all functions \( \psi \in C^\infty(\mathbb{R}) \), it follows that
\[
\int_{\mathbb{R}} w(t) \frac{d\psi}{dt}(t) \, dt = 0,
\]
for all \( \psi \in C^\infty_0(\mathbb{R}) \), hence the function \( w \in L^2(\mathbb{R}, dt) \) is a constant (necessarily equal to zero).

\[ \square \]

Proof of Theorem 25. The statement follows from Lemmas 27 and 28.

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