From vacuum field equations on principal bundles to Einstein’s equations with fluids

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Abstract. In the present work we show that the Einstein equations on $M$ without cosmological constant and with perfect fluid as source, can be obtained from the field equations for vacuum with cosmological constant on the principal fibre bundle $P\left(\frac{1}{4}M, U(1)\right)$, $M$ being the space-time and $I$ the radius of the internal space $U(1)$.

Resumen. Mostramos que las ecuaciones de Einstein sobre $M$ sin constante cosmológica y con fluido perfecto como fuente, pueden obtenerse a partir de las ecuaciones de campo para vacío con constante cosmológica sobre el haz fibrado principal $P\left(\frac{1}{4}M, U(1)\right)$, donde $M$ es el espacio-tiempo e $I$ el radio del espacio interno $U(1)$.

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1. Introduction

In a recent work [1] it has been shown that vacuum solutions in scalar-tensor theories are equivalent to solutions of general relativity with imperfect fluid as source. The above models have the defect that the scalar fields do not arise from a natural framework of unification, but they are put by hand as in the inflationary models [2] and are therefore artificial fields in the theory. On the other hand, we know that the geometric formalism of principal fibre bundles [3,4] is a natural scheme to unify the general relativity theory with gauge field theories (Abelian and Non-Abelian). If the principal fibre bundle $P(\tilde{M},U(1))$ is endowed with a metric “dimensionally reducible” to $\tilde{M}$ by means of the reduction theorem [5], i.e., if the metric can be built out from quantities defined only on $M$, then the scalar fields arise in a natural way. Therefore, it is important to study the above model in the context of [1] for the particular principal fibre bundle $P(\tilde{M},U(1))$, $M$ being the space-time and $I$ the scalar field. This paper is organized as follows: in the next section we review the geometric formalism of principal fibre bundles while in Sect. 3 we deduce the Einstein equations without cosmological constant and perfect fluid as source from the field equations on $P(\tilde{M},U(1))$ for vacuum and cosmological constant. We give an example in Sect. 4 when $\tilde{M}$ is conformally $FRW$. Finally we summarize the results in Sect. 5.

2. The geometry

The actual version of the Kaluza-Klein theories is based on the mathematical structure of principal fibre bundles [5,6]. In this scheme, the unification of the general relativity theory with the gauge theories is a natural fact. Moreover, the reduction theorem provides a metric on (right) principal fibre bundles $P(\tilde{M},G)$ which is right-invariant under the action of the structure group $G$ on the whole space $P$. In the trivialization of the bundle this metric
reads [5,6]
\[ \tilde{g} = \tilde{g}_{\alpha\beta} \, dx^\alpha \otimes dx^\beta + \xi_{mn} \left( \omega^m + A^m_\alpha \, dx^\alpha \right) \otimes \left( \omega^n + A^n_\beta \, dx^\beta \right), \]  

where the metric of the base space \( \tilde{M} \) (generally identified with the space-time of general relativity) is \( \tilde{g}_{\alpha\beta} \, dx^\alpha \otimes dx^\beta \) while the metric on the fibre \( (x^\alpha = \text{const.}) \) is \( \xi_{mn} \, \omega^m \otimes \omega^n \) and \( \{ \omega^m \} \) is a basis of right-invariant 1-forms on \( G \). The quantities \( \tilde{g}_{\alpha\beta}, \xi_{mn} \) and \( A^n_\alpha \) depend only on the coordinates on \( \tilde{M} \) and the \( A^n_\alpha \) correspond to Yang-Mills potentials in the gauge theory while the \( \xi_{mn} \) are the scalar fields.

In particular, the principal fibre bundle \( P(\tilde{M},U(1)) \) has the metric
\[ \hat{g} = \tilde{g}_{\alpha\beta} \, dx^\alpha \otimes dx^\beta + I^2 \left( d\psi + A_\alpha \, dx^\alpha \right) \otimes \left( d\psi + A_\beta \, dx^\beta \right), \]

where the scalar field \( I \) correspond to the radius of the internal space \( U(1) \) and \( \psi \) is the coordinate on \( U(1) \) too. However, the magnitude of the internal radius \( I \) depends on the particular cases; cosmological or astrophysical models (for details on units and magnitude on the scalar field \( I \) see Refs. [6,7]). For vanishing electromagnetic potential, \( A_\alpha = 0 \), we obtain the unification of \( \tilde{g}_{\alpha\beta} \) with the scalar field \( I \)
\[ \hat{g} = \tilde{g}_{\alpha\beta} \, dx^\alpha \otimes dx^\beta + I^2 \, d\psi^2. \]

By using Eq. (3) we compute the Ricci tensor
\[ \hat{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta} - I^{-1} I_{\alpha\beta}, \]
\[ \hat{R}_{\alpha4} = 0, \]
\[ \hat{R}_{44} = -I \, \Box I, \]
where greek indices run on 0,1,2,3 and the label “4” corresponds to the fifth dimension.

Usually the base space \( \tilde{M} \) of \( P(\tilde{M},G) \) is identified as the space-time , in this paper we adopt the version where the base space \( \tilde{M} \) of \( P(\tilde{M},U(1)) \) is conformally the space-time \( M \)
of general relativity, i.e., $\bar{M} = \frac{1}{I} M$. That is to say, we start with the metric (compare Ref. [8])

$$\hat{g} = \frac{1}{I} g_{\alpha\beta} \, dx^\alpha \otimes dx^\beta + I^2 \, d\psi^2,$$

(7)

where $g_{\alpha\beta} \, dx^\alpha \otimes dx^\beta$ is the space-time metric. Then by using Eq. (7) we obtain the Ricci tensor

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \frac{1}{2} \left( I^{-1} \Box I - I_{,\lambda} I^{,\lambda} \right) g_{\alpha\beta} - \frac{3}{2} I^{-2} I_{,\alpha} I_{,\beta},$$

(8)

$$\hat{R}_{\alpha 4} = 0,$$

(9)

$$\hat{R}_{44} = -I^2 \Box I + I_{,\lambda} I^{,\lambda}.$$  

(10)

In the next we use the signature (−,+,+,+,+) for the space-time metric on $M$.

3. Perfect fluid structure

The field equations on $P(\frac{1}{I} M, U(1))$ in vacuum with cosmological constant $\Lambda$ are given by

$$\hat{R}_{AB} - \frac{2}{3} \hat{g}_{AB} = \Lambda \hat{g}_{AB}$$

or in equivalent form

$$\hat{R}_{AB} = -\frac{2}{3} \Lambda \hat{g}_{AB},$$

(11)

where $A,B$ run on greek indices $\alpha$ and 4.

By using Eqs. (8)-(10) and (11) we obtain

$$R_{\alpha\beta} = I^{-2} \left( \frac{1}{2} I_{,\lambda} I^{,\lambda} g_{\alpha\beta} + \frac{3}{2} I_{,\alpha} I_{,\beta} \right) - I^{-1} \left( \frac{1}{2} \Box I + \frac{2}{3} \Lambda \right) g_{\alpha\beta},$$

(12)

$$\Box I = \frac{1}{I} I_{,\lambda} I^{,\lambda} + \frac{2}{3} \Lambda.$$  

(13)

By substituting the field equation for $I$ [Eq. (13)] into the Ricci tensor [Eq. (12)] we obtain the next equivalent equations system

$$R_{\alpha\beta} = \frac{3}{2} I^{-2} I_{,\alpha} I_{,\beta} - I^{-1} \Lambda g_{\alpha\beta},$$

(14)

$$\Box I = \frac{1}{I} I_{,\lambda} I^{,\lambda} + \frac{2}{3} \Lambda.$$  

(15)
On the other hand, by using the Einstein equations without cosmological constant,

\[ R_{\alpha \beta} = T_{\alpha \beta} - \frac{T}{2} g_{\alpha \beta} \]  

and Eq. (14), we can define the energy-momentum tensor associated with the scalar field \( I \)

\[
T_{\alpha \beta} = \frac{3}{2} I^{-2} I_{;\alpha} I_{;\beta} + \left( -\frac{3}{4} I^{-2} I_{;\lambda} I^{\lambda} + I^{-1} \Lambda \right) g_{\alpha \beta}.
\]  

This energy-momentum tensor is covariantly conserved, \( T^\alpha_{\; ;\beta} = 0 \) as follows from the field equation for \( I \). Finally, by comparing the above energy-momentum tensor associated with the scalar field \( I \) with that of an imperfect fluid

\[
T_{\alpha \beta} = \rho U_\alpha U_\beta + 2 q_{(\alpha} U_{\beta)} + p h_{\alpha \beta} + \pi_{\alpha \beta},
\]  

where \( \rho \) is the energy density of fluid, \( U_\alpha \) the velocity, \( q_\alpha \) the heat flux vector, \( p \) the pressure, \( \pi_{\alpha \beta} \) the anisotropic stress tensor and

\[
h_{\alpha \beta} = g_{\alpha \beta} + U_\alpha U_\beta,
\]  

is the projection orthogonal to the velocity, we conclude [1]

\[
q_\alpha = 0, \quad \pi_{\alpha \beta} = 0, \quad \rho = -\frac{3}{4} I^{-2} I_{;\lambda} I^{\lambda} - \frac{\Lambda}{T}, \quad p = -\frac{3}{4} I^{-2} I_{;\lambda} I^{\lambda} + \frac{\Lambda}{T},
\]  

where the velocity has been choosen in the form [1]

\[
U_\alpha = \frac{I_\alpha}{\sqrt{-I_{;\lambda} I^{\lambda}}}.
\]  

That is to say, Eqs. (20)-(23) implies that Eq. (17) has the structure corresponding to a perfect fluid. Moreover, if \( \Lambda = 0 \) then Eqs. (17) and (20)-(23) correspond to the so called “Zeldovich ultrastiff matter” fluid, \( p = \rho \) (see Ref. [1]).
4. Example: the p.f.b. $P^{(1)}_{FRW, U(1)}$

We start from the metric
\[
\hat{g} = \frac{1}{I(t)} \left[ -dt^2 + R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right) \right] + I^2(t) \, d\psi^2,
\] (25)

where $I = I(t)$ on account of the isotropy and homogeneity of the FRW metric. In this case the Eqs. (14)-(15) read
\[
-3 \left( \frac{\ddot{R}}{R} \right) = \frac{3}{2} \left( \frac{i}{T} \right)^2 + \left( \frac{1}{I} \right) \Lambda,
\] (26)
\[
2 \left( \frac{k}{R^2} \right) + 2 \left( \frac{\dot{R}}{R} \right)^2 + \left( \frac{\ddot{R}}{R} \right) = - \left( \frac{1}{I} \right) \Lambda,
\] (27)
\[
3 \left( \frac{\dot{R}}{R} \right) \left( \frac{i}{T} \right) - \left( \frac{\dot{R}}{R} \right)^2 + \left( \frac{i}{T} \right)^2 = - \frac{2}{3} \left( \frac{1}{I} \right) \Lambda,
\] (28)

where dot means derivation with respect to the cosmological time $t$. These equations are equivalent to the Einstein equations for FRW with perfect fluid as source, provided that
\[
\rho = \frac{3}{4} \left( \frac{i}{T} \right)^2 - \left( \frac{1}{I} \right) \Lambda,
\] (29)
\[
p = \frac{3}{4} \left( \frac{i}{T} \right)^2 + \left( \frac{1}{I} \right) \Lambda.
\] (30)

By the way, the field equation for $I$ [Eq. (28)] is the covariant conservation of $T_{\alpha\beta}$,

\[
T^{\alpha\beta}_{\; ; \beta} = 0
\]

\[
\dot{\rho} + 3 \left( \frac{\dot{R}}{R} \right) (\rho + p) = 0.
\] (31)

5. Conclusion

We have shown that the field equations with cosmological constant $\Lambda$ on the principal fibre bundle $P^{(1)}_{1M, U(1)}$ are equivalent to the Einstein equations without cosmological
constant on $M$ and with perfect fluid as source. In order to show it we start from the field equations on $P(\frac{1}{4}M, U(1))$, $\hat{R}_{AB} = -\frac{2}{3} \Lambda \hat{g}_{AB}$ and separate them in their 4-dimensional and five dimensional parts. We have found that from the 4-dimensional part of these equations it is possible to define an effective energy-momentum tensor $T_{\alpha\beta}$ and that it is covariantly conserved, being $T^{\alpha\beta} ;_\beta = 0$ equivalent to the field equation for $I$. Finally, we applied the above result to the particular bundle $P(\frac{1}{4}FRW, U(1))$.

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