A LAGRANGIAN SPHERE WHICH IS NOT A VANISHING CYCLE

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Abstract. We give examples of Calabi-Yau threefolds containing Lagrangian spheres which are not vanishing cycles of nodal degenerations, answering a question of Donaldson in the negative.

1. Introduction

The $n$-dimensional nodal singularity has a 1-parameter versal deformation

$$\left( \sum_{i=0}^{n} z_i^2 = t \right) \subset \mathbb{C}^{n+1} \times \mathbb{C}_t.$$  

The nearby fiber over $t = \epsilon > 0$ retracts onto an $n$-sphere, the vanishing cycle:

$$S^n \simeq \left( \sum_{i=0}^{n} x_i^2 = \epsilon \right),$$

which is Lagrangian with respect to the standard symplectic form $\omega = \sum dx_i \wedge dy_i$ ($z_i = x_i + iy_i$). A natural question, first raised by Donaldson [4], is whether all Lagrangian spheres arise in this way.

**Question 1.** Let $Z$ be a complex projective manifold, and $L \subset Z$ embedded sphere, Lagrangian with respect to a Kähler form on $Z$. Is $L$ always the vanishing cycle of a nodal degeneration of $Z$?

The answer is positive for curves, and unknown for surfaces. For K3 surfaces, the answer is positive [7] modulo a technical difficulty (Fukaya isomorphism implies Hamiltonian isotopy). For Horikawa surfaces, a positive answer would distinguish two particular deformation types as smooth manifolds [1]. We show that the answer to Question 1 is negative in general:

**Theorem 2.** There exists a rigid projective Calabi-Yau threefold $\hat{X}$ with a Lagrangian sphere $L \subset \hat{X}$ which is homologically non-trivial (essential).

Rigidity implies that any degeneration of $\hat{X}$ is isotrivial. We prove further that such a degeneration must have monodromy of order $\leq 6$ on $H_3(\hat{X})$. In particular, this rules out nodal degenerations with vanishing cycle $L$; their monodromy would be a Dehn twist by $[L] \in H_3(\hat{X})$, which has infinite order. It was known [8] that if an essential Lagrangian sphere existed on a rigid CY3, then it could not be the vanishing cycle of a nodal degeneration.

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2. The Construction

The counterexamples are among the Calabi-Yau threefolds considered by Schoen [6]. Consider the following pencil of cubics in \( \mathbb{P}^2 \):

\[(x + y)(y + z)(z + x) + txyz = 0.\]

Viewed as a family of curves over \( \mathbb{P}^1 \), the relatively minimal smooth model \( \nu: S \to \mathbb{P}^1 \) has 4 singular Kodaira fibers of types \( I_6, I_3, I_2, \) and \( I_1 \) over \( t = \infty, 0, 1, \) and \( -8 \), respectively. This is one of six semistable elliptic families over \( \mathbb{P}^1 \) (all extremal) with the minimum number of singular fibers, as constructed by Beauville [2], and it is isomorphic to the universal family over the compactified modular curve \( X_1(6) \).

Let \( \phi \) be a non-trivial automorphism of \( \mathbb{P}^1 \) which permutes \( \{\infty, 0, 1\} \), and note that \( \phi(-8) \neq -8 \). We form the Cartesian product

\[
\begin{array}{ccc}
X & \longrightarrow & S \\
\downarrow & & \downarrow \phi \circ \nu \\
S & \longrightarrow & \mathbb{P}^1.
\end{array}
\]

The result is a singular projective threefold with \( K_X = 0 \), fibered over \( \mathbb{P}^1 \) by abelian surfaces which are products of non-isogenous elliptic curves. There are 5 critical values: \( \infty, 0, 1, -8 \), and \( \phi(-8) \). The total space \( X \) has \( n \) conifold singularities in the fibers over \( \{\infty, 0, 1\} \), located at the product of two nodes in the elliptic fibers. For the different choices of \( \phi \), we get \( n = 33, 36, 40, 48 \).

There exists a projective small resolution \( \epsilon: \hat{X} \to X \) obtained by successive blow ups of the \( n \) Weil divisors which are irreducible components of the singular fibers. The resolution is crepant, so \( \hat{X} \) is a smooth Calabi-Yau threefold.

**Proposition 3.** The Picard group of \( \hat{X} \) has rank \( n \).

**Proof.** The specialization of \( \pi: \hat{X} \to \mathbb{P}^1 \) to the generic point \( \eta \in \mathbb{P}^1 \) gives a split short exact sequence

\[0 \to A \to \text{Pic}(\hat{X}) \to \text{Pic}(X_\eta) \to 0,\]

where \( A \) is the span of the \( n \) divisor classes supported over \( \{\infty, 0, 1\} \). They satisfy 2 relations using the rational equivalence over \( \mathbb{P}^1 \). The generic fibers of \( \nu \) and \( \phi \circ \nu \) are non-isogenous elliptic curves, so

\[\text{Pic}(X_\eta) \simeq \text{Pic}(S_\eta) \oplus \text{Pic}(S_\eta).\]

The specialization of \( \nu: S \to \mathbb{P}^1 \) to \( \eta \) gives a split short exact sequence

\[0 \to B \to \text{Pic}(S) \to \text{Pic}(S_\eta) \to 0,\]

where \( B \) is the span of the 12 curves classes supported over \( \{\infty, 0, 1, -8\} \). They satisfy 3 relations using the rational equivalence over \( \mathbb{P}^1 \). Since \( \rho(S) = 10 \), we find that \( \text{Pic}(S_\eta) \) has rank 1. In fact, the torsion Mordell-Weil group of \( S_\eta \) is known [5]:

\[\text{Pic}(S_\eta) \simeq \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.\]

\[\square\]
Proposition 4. The threefold $\hat{X}$ is rigid in the sense that $H^1(T_{\hat{X}}) = 0$.

Proof. The cup product gives an isomorphism $H^1(T_{\hat{X}}) \cong H^1(\hat{X})^\vee$ since $\hat{X}$ is Calabi-Yau. As $X$ is fibered by tori, its topological Euler characteristic is determined by the singular fibers $I_b \times I_{b'}$. This gives
\[
\chi_{\text{top}}(X) = n
\]
\[
\chi_{\text{top}}(\hat{X}) = 2n,
\]
since the small resolution replaces each conifold point with a $\mathbb{P}^1$. On the other hand, the Calabi-Yau property gives
\[
\chi_{\text{top}}(\hat{X}) = 2(h^{1,1}(\hat{X}) - h^{1,2}(\hat{X}))
\]
and $H^{1,1}(\hat{X}) \cong \text{Pic}(\hat{X})$, so we are done by Proposition 3. \qed

There is a conifold transition relating $\hat{X}$ to the more standard Schoen Calabi-Yau:
\[
Y := S \times_{\mathbb{P}^1} S'.
\]
Here, $S' \to \mathbb{P}^1$ is a generic rational elliptic surface. By deforming $S'$, we see that $Y$ degenerates to $X$. Schematically,
\[
\begin{array}{ccc}
\hat{X} & \rightarrow & Y \\
\downarrow & & \downarrow \\
Y & \sim & X
\end{array}
\]

The topological description of the conifold transition in terms of vanishing 3-spheres and exceptional 2-spheres allows us to compute the Betti numbers of $X$ and $\hat{X}$. Using $\chi_{\text{top}}(Y) = 0$, it is not hard to check that $h^{1,1}(Y) = h^{1,2}(Y) = 19$. Let $r$ be the homological rank of the vanishing 3-spheres in $H_3(Y)$. We have the following:
\[
\begin{align*}
b_2(Y) &= 19 & b_2(X) &= 19 & b_2(\hat{X}) &= 19 + (n - r) \\
b_3(Y) &= 40 & b_3(X) &= 40 - r & b_3(\hat{X}) &= 40 - 2r \\
b_4(Y) &= 19 & b_4(X) &= 19 + (n - r) & b_4(\hat{X}) &= 19 + (n - r).
\end{align*}
\]
Since $b_3(\hat{X}) = 2$, we deduce that $r = 19$.

Proposition 5. $\hat{X}$ contains a 3-sphere $L$ that is Lagrangian with respect to a Kähler form.

Proof. We adapt a construction from \cite{8}: let $\gamma : [0, 1] \to \mathbb{P}^1$ be a smooth path missing $\{\infty, 0, 1\}$ with $\gamma(0) = -8$ and $\gamma(1) = \phi(-8)$. Using the horizontal distribution symplectically orthogonal to the vertical tangent spaces, there is a symplectic parallel transport along $\gamma$. Choose a Kähler form which is a product form on the fibers $E \times E'$ over points $\gamma(s)$. Let $\ell_0 \subset \pi^{-1}(\gamma(\frac{1}{2}))$ be a vanishing loop in $E$ for the flow toward 0, and $\ell_1 \subset \pi^{-1}(\gamma(\frac{1}{2}))$ a vanishing loop in $E'$ for the flow toward 1. The parallel transport sweeps out a Lagrangian:
\[
L = \bigcup_{s \in [0,1]} (\ell_0)_s \times (\ell_1)_s,
\]
diffeomorphic to $S^3$ fibered by 2-tori, with $S^1$ caps on either side. It is smooth at the caps because it is locally the product of a Lefschetz thimble with $S^1$. \qed
3. Elliptic Modular Surfaces

The classical modular curves $X_1(N)$ are constructed by compactifying quotients of
the upper half plane $\mathbb{H}$ by the congruence subgroups $\Gamma_1(N) \subset SL_2(\mathbb{Z})$:

$$X_1(N) := \mathbb{H}^*/\Gamma_1(N)$$
$$\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}).$$

The action of $\Gamma_1(N)$ on $\mathbb{P}^1(\mathbb{Q})$ has finitely many orbits, which become cusps in the
modular curve. The stabilizer of a point in $\mathbb{P}^1(\mathbb{Q})$ is a parabolic subgroup of $\Gamma_1(N)$
generated by a conjugate of

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{N}$ is called the width of the corresponding cusp. There is a universal
family over $\mathbb{H}/\Gamma_1(N)$ whose fiber at $\tau$ is the elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. This family
admits a compactification over $X_1(N)$ by adding a Kodaira fiber of type $I_6$ over
each cusp. For $1 \leq N \leq 10$ and $N = 12$, the curve $X_1(N)$ has genus 0, and
the Hauptmodul $j_N : X_1(N) \to \mathbb{P}^1$ is an isomorphism defined over $\mathbb{Q}$. For our
application, we specialize to the case $N = 6$ where the elliptic modular surface is
isomorphic to our example $\nu : S \to \mathbb{P}^1$.

Toward our ultimate goal of describing the homology of the Calabi-Yau threefold
$\hat{X}$, we record the monodromy representation associated to $\nu : S \to \mathbb{P}^1$.

**Proposition 6.** For a chosen base point $* \in \mathbb{P}^1 - \{\infty, 0, 1, -8\}$, the monodromy

$$\mu : \pi_1(\mathbb{P}^1 - \{\infty, 0, 1, -8\}, *) \to SL(H_1(\nu^{-1}(\ast), \mathbb{Z})) \simeq SL_2(\mathbb{Z})$$

sends the simple loops $\gamma_{-8}$, $\gamma_{10}$, and $\gamma_7$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix}, \begin{pmatrix} -5 & 2 \\ -18 & 7 \end{pmatrix}, \begin{pmatrix} -5 & 3 \\ -12 & 7 \end{pmatrix},$$

respectively. The loops are arranged so that $\gamma_{-8}\gamma_{10}\gamma_7\gamma_0 \sim 1$.

**Proof.** The action of $\Gamma_1(6)$ on $\mathbb{H}$ has a fundamental domain with cusps at $\tau =
ix, \frac{1}{3}, \frac{1}{2}$. The stabilizer of each cusp is generated by the corresponding matrix
above. The isomorphism $j_6 : \mathbb{H}^*/\Gamma_1(6) \to \mathbb{P}^1$ allows us to identify the cusps with
critical values of the pencil via $j_6(\tau) = t$. From the widths of the cusps, we see that
$j_6(\infty) = -8$, $j_6(0) = \infty$, $j_6(\frac{1}{3}) = 1$, and $j_6(\frac{1}{2}) = 0$. \( \square \)

**Proposition 7.** For any path in $\mathbb{P}^1$ between two critical values of $\nu$, the vanishing
loops are non-homologous.

**Proof.** The vanishing cycle class in $H_1(\nu^{-1}(\ast), \mathbb{Z})$ for a path from $*$ to a critical
value $t$ is fixed by the associated monodromy matrix $\mu(\gamma_t)$. With the labels from
Proposition 6, the vanishing cycles are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$  

These vectors represent elements of $(\mathbb{Z}/6\mathbb{Z})^2$ which lie in different orbits under
$\Gamma_1(6)$, so no two vanishing cycles are conjugate under the monodromy action. \( \square \)

Another way of stating Proposition 7 is that there are no matching cycles in the
pencil. This observation is used in the Mayer-Vietoris computations of Section 4.
Lemma 8. For two unequal subsets \( \{ \gamma, \gamma' \} \) and \( \{ \tilde{\gamma}, \tilde{\gamma}' \} \) of \( \{ \gamma_\infty, \gamma_0, \gamma_1 \} \), the matrices 
\[ \mu(\gamma)^{\pm 1} \mu(\gamma')^{\pm 1} \] and \( \mu(\tilde{\gamma})^{\pm 1} \mu(\tilde{\gamma}')^{\pm 1} \)
are hyperbolic and have no common eigenvalues.

Proof. This is a straightforward exercise using the matrices from Proposition 6. \( \square \)

4. Homology class of \( L \)

The goal of this section is to prove that \( [L] \neq 0 \in H_3(\tilde{X}) \) by explicit computation. A key input is the monodromy representation for the elliptic fibration \( \nu : S \to \mathbb{P}^1 \). We will use Mayer-Vietoris (MV) sequences to understand the rational homology groups \( H_*(X, \mathbb{Q}) \) by gluing together neighborhoods of the critical values in \( \mathbb{P}^1 \).

Let \( U_1, U_2 \subset \mathbb{P}^1 \) be overlapping disks which contain \(-8\) and \( \phi(-8) \), respectively, and no other critical values. Set \( U = U_1 \cup U_2 \) and \( U_{12} = U_1 \cap U_2 \). Since \( \pi^{-1}(U_{12}) \) retracts to a product of elliptic curves, it is easy to compute the restriction morphisms
\[ \alpha_i : H_i(\pi^{-1}(U_{12})) \simeq \wedge^i H_i(\pi^{-1}(U_{12})) \to H_i(\pi^{-1}(U_1)) \oplus H_i(\pi^{-1}(U_2)). \]

Lemma 9. The rational homology of \( X_U = \pi^{-1}(U) \) is given by
\[ H_*(X_U, \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^3, \mathbb{Q}^4, \mathbb{Q}). \]

Proof. Since \( \pi^{-1}(U_1) \) retracts onto the singular fiber, the Künneth formula gives
\[ H_*(\pi^{-1}(U_1), \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^3, \mathbb{Q}^4, \mathbb{Q}^3, \mathbb{Q}), \]
\[ H_*(\pi^{-1}(U_{12}), \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^4, \mathbb{Q}^6, \mathbb{Q}^4). \]
The MV sequence for \( X_U = \pi^{-1}(U_1) \cup \pi^{-1}(U_2) \) is short exact in degrees 0 and 4, and elsewhere \( H_*(X_U, \mathbb{Q}) \) is determined by the facts that \( \alpha_1 \) is injective, and \( \alpha_2 \) has 1-dimensional kernel. The class \([L] \neq 0 \in H_3(X_U)\) because \( \partial_3[L] \) spans the kernel of \( \alpha_2 \). A basis for \( H_3(X_U, \mathbb{Q}) \simeq \mathbb{Q}^3 \) is given by \([L], [T_1], [T_2]\), where \( T_i \) is the distinguished 3-torus in \( \pi^{-1}(U_i) \). \( \square \)

Let \( W_1, W_2 \subset \mathbb{P}^1 \) be overlapping disks, each containing one critical value in \( \{ \infty, 0, 1 \} \) denoted \( t \) and \( t' \). It is always possible to pick \( \{ t, t' \} \) such that \( \{ t, t' \} \neq \{ \phi(t), \phi(t') \} \).

Set \( W = W_1 \cup W_2 \) and \( W_{12} = W_1 \cap W_2 \). Since \( \pi^{-1}(W_{12}) \) retracts to a product of elliptic curves, one computes the restriction morphisms
\[ \alpha_i : H_i(\pi^{-1}(W_{12})) \simeq \wedge^i H_i(\pi^{-1}(W_{12})) \to H_i(\pi^{-1}(W_1)) \oplus H_i(\pi^{-1}(W_2)) \]
using Proposition 7. To set notation, the central fiber of \( W_1 \) is of type \( I_0 \times I_0 \), and the central fiber of \( W_2 \) is of type \( I_0 \times I_0 \).

Lemma 10. The rational homology of \( X_W = \pi^{-1}(W) \) is given by
\[ H_*(X_W, \mathbb{Q}) \simeq (\mathbb{Q}, 0, \mathbb{Q}^{b+\delta+\delta'}-2, \mathbb{Q}^{b+\delta+\delta'}-2, \mathbb{Q}^{b+\delta+\delta'}-2, \mathbb{Q}^{b+\delta+\delta'}-1). \]

Proof. Since \( \pi^{-1}(W_j) \) retracts onto the singular fiber, the Künneth formula gives
\[ H_*(\pi^{-1}(W_1), \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^{b+\delta+1}, \mathbb{Q}^{b+\delta}, \mathbb{Q}^{b\delta}) \]
\[ H_*(\pi^{-1}(W_2), \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^{b'+\delta'+1}, \mathbb{Q}^{b'+\delta'}, \mathbb{Q}^{b'\delta'}) \]
\[ H_*(\pi^{-1}(W_{12}), \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}^4, \mathbb{Q}^6, \mathbb{Q}^4, \mathbb{Q}). \]
The MV sequence for $X_W = \pi^{-1}(W_1) \cup \pi^{-1}(W_2)$ is short exact in degrees 0 and 4, and elsewhere $H_\ast(X_W, \mathbb{Q})$ is determined by the facts that $\alpha_1$ is injective, and $\alpha_2$ has 2-dimensional kernel. □

The fibers of $\pi$ over the boundary circle of $W \subset \mathbb{P}^1$ together form a closed 5-manifold $M$, a bundle of 4-tori over $S^1$. Lemma 8 forces its homology to be as small as possible:

**Lemma 11.** $H_\ast(M, \mathbb{Q}) \simeq (\mathbb{Q}, \mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^2, \mathbb{Q}, \mathbb{Q})$.

**Proof.** The homology of a fiber bundle over $S^1$ can be computed from the monodromy operator $T$ on the homology of the fiber $F$, which in this case is a product of elliptic curves:

$$0 \to \text{coker} \, (T - 1)|_{H_4(F)} \to H_4(M) \to \ker \, (T - 1)|_{H_{i-1}(F)} \to 0.$$  

Now $(T|_{H_4(F)})$ is a block matrix with $\mu(\gamma_l)^{\pm 1} \mu(\gamma_r)^{\pm 1}$ and $\mu(\gamma_l)^{\pm 1} \mu(\gamma_r)^{\pm 1}$ as the blocks, and $(T|_{H_4(F)}) = \wedge^i(T|_{H_1(F)})$. The claim now follows from Lemma 8 □

Let $V$ be a disk containing the last remaining critical value $t''$ of type $I_{\nu''}$, overlapping with $U$. The critical value $\phi(t'')$ is of type $I_{\nu''}$.

**Lemma 12.** The rational homology of $X_{U \cup V} = \pi^{-1}(U \cup V)$ is given by

$$H_\ast(X_{U \cup V}, \mathbb{Q}) \simeq (\mathbb{Q}, 0, \mathbb{Q}^{b''+\tilde{b}'}, \mathbb{Q}^{b''+\tilde{b}' + 1}, \mathbb{Q}^{b''})$$

**Proof.** Once again, the Kunneth formula gives

$$H_\ast(\pi^{-1}(V), \mathbb{Q}) \simeq (\mathbb{Q}, 0, \mathbb{Q}^{b''+\tilde{b}'}, \mathbb{Q}^{b''+\tilde{b}' + 1}, \mathbb{Q}^{b''})$$

The MV sequence for $X_{U \cup V} = \pi^{-1}(U) \cup \pi^{-1}(V)$ is short exact in degree 0 and 4, and elsewhere $H_\ast(X_{U \cup V}, \mathbb{Q})$ is determined by the facts that $\alpha_1$ is injective, and $\alpha_2$ has 2-dimensional kernel. The class $[L] \neq 0 \in H_3(X_{U \cup V})$ because the image of $\alpha_3$ does not contain $([L], 0)$. Indeed, the first component of the image of $\alpha_3$ is supported on the singular fibers of $\pi^{-1}(U)$, spanned by $[T_1]$ and $[T_2]$. □

We are now prepared to understand the global topology of $X$ in terms of the gluing $(U \cup V) \cup W = \mathbb{P}^1$. By the theory of conifold transitions, we already know that

$$H_\ast(X, \mathbb{Q}) = (\mathbb{Q}, 0, \mathbb{Q}^{19}, \mathbb{Q}^{21}, \mathbb{Q}^{n}, 0, \mathbb{Q}),$$

where $n = \tilde{b}b + b\tilde{b}' + b''\tilde{b}''$ is the number of conifolds.

**Theorem 13.** The class $[L] \neq 0 \in H_3(\hat{X})$.

**Proof.** In the MV sequence for $X = X_{U \cup V} \cup X_W$, we have already determined all the groups, so we deduce properties of the maps:

$$0 \longrightarrow H_4(M) \longrightarrow H_4(X_{U \cup V}) \oplus H_4(X_W) \longrightarrow H_4(X)$$

$$\longrightarrow H_3(M) \longrightarrow H_3(X_{U \cup V}) \oplus H_3(X_W) \longrightarrow H_3(X)$$

$$0 \longrightarrow H_2(M) \longrightarrow H_2(X_{U \cup V}) \oplus H_2(X_W) \longrightarrow H_2(X)$$

$$\longrightarrow H_1(M) \longrightarrow 0.$$
We see that $H_3(X) \simeq H_3(X_U \cup V) \oplus H_3(X_W)$, so $[L] \neq 0 \in H_3(X)$. By applying the pushforward $\epsilon_* : H_3(\hat{X}) \to H_3(X)$ to a cycle representing $L \subset \hat{X}$, we see that $[L] \neq 0 \in H_3(\hat{X})$ as well.

After the small resolution surgery, we have

$$H_3(\hat{X}_{\cup V}, \mathbb{Q}) \simeq H_3(\hat{X}, \mathbb{Q}) \simeq \mathbb{Q}^2.$$

We can produce a pair of Lagrangian spheres $L, L' \subset \hat{X}_{U \cup V}$ with nonzero intersection number in $\hat{X}$, which immediately proves Theorem 13. Construct $L'$ as in Proposition 5, but using the other path $\gamma'$ between $-8$ and $\phi(-8)$. The signed intersection of $L$ with $L'$ is a nonzero multiple of 6, using the monodromy matrices of Proposition 6. An example of $X_{U \cup V}$ is pictured below, with $b'' = 2$ and $\hat{b}'' = 6$. 
5. Degenerations of $\hat{X}$

Suppose that $\hat{X}$ admits a Kähler degeneration. That is, $\hat{X}$ is isomorphic to a fiber of a proper holomorphic family over a curve:

$$f : \mathcal{X} \to B,$$

and $f^{-1}(0)$ is singular for some $0 \in B$. By Proposition 4, $\hat{X}$ has no moduli so $f$ is holomorphically locally trivial (isotrivial) away from its critical values, by the Fischer-Grauert theorem. If we choose a small complex disk $\Delta$ centered at 0, the fiber bundle

$$f^{-1}(\Delta^*) \to \Delta^*$$

has monodromy valued in $\text{Aut}(\hat{X})$. In other words,

$$\mathbb{Z} \cong \pi_1(\Delta^*) \to \text{Aut}(\hat{X}) \to \text{Sp}(H_3(\hat{X}, \mathbb{Z})) \cong SL_2(\mathbb{Z}).$$

To control potential isotrivial degenerations, we prove that $\text{Aut}(\hat{X})$ is finite.

**Proposition 14.** Every automorphism preserves the fibration $\pi : \hat{X} \to \mathbb{P}^1$.

**Proof.** Let $\varphi : \hat{X} \to \hat{X}$ be an automorphism, and let $A$ be a general fiber of $\pi$. If $\varphi$ does not preserve the fibration, then $\varphi(A)$ surjects onto $\mathbb{P}^1$. The image of $\varphi(A)$ in $S$ cannot be all of $S$ because complex tori only surject onto projective spaces and complex tori [3]. Thus, $\varphi(A)$ maps onto a curve $C$ in $S$, and the generic fiber is an elliptic curve. There must be singular fibers because $C$ surjects to $\mathbb{P}^1$, which contradicts the fact that $\chi_{\text{top}}(A) = 0$. □

The Mordell-Weil group of $X_\eta$ is finite (it is $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$), so $\text{Aut}(\hat{X})$ is finite. Therefore, the image of monodromy is a finite subgroup of $SL_2(\mathbb{Z})$, so it is abelian of order $\leq 6$. In particular, $L$ is not the vanishing cycle of a nodal degeneration.

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