A WELL CONDITIONED METHOD OF FUNDAMENTAL SOLUTIONS

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Abstract. The method of fundamental solutions (MFS) is a numerical method for solving boundary value problems involving linear partial differential equations. It is well known that it can be very effective assuming regularity of the domain and boundary conditions. The main drawback of the MFS is that the matrices involved typically are ill-conditioned and this may prevent to achieve high accuracy.

In this work, we propose a new algorithm to remove the ill-conditioning of the classical MFS in the context of Laplace equation defined in planar domains. The main idea is to expand the MFS basis functions in terms of harmonic polynomials. Then, using the singular value decomposition and Arnoldi orthogonalization we define well conditioned basis functions spanning the same functional space as the MFS’s. Several numerical examples show that this approach is much superior to previous approaches, such as the classical MFS or the MFS-QR.

1. Introduction

The method of fundamental solutions (MFS) is a well known numerical method for the solution of boundary value problems with linear partial differential equations (PDEs) \([25, 32, 35]\). It was introduced by Kupradze and Aleksidze \([32]\) and has been applied in the context of problems arising from acoustics \([3, 8, 13]\), elasticity \([12, 14, 34]\), fluid dynamics \([6, 28, 36, 39]\) or electromagnetism \([20, 40]\).

It is a meshfree method for which the solution is approximated by a linear combination of shifts of the fundamental solution of the PDE to a set of points placed on an admissible source set. Since, by construction, the MFS approximation satisfies the PDE, we can focus just on the approximation of the boundary data, which is justified by density results \([1, 15]\). The method may present a very fast convergence, under some regularity assumptions and the simplicity and effectiveness make it a very appealing numerical method.

In the original formulation the application of the MFS was restricted to the numerical solution of homogeneous linear PDEs defined in smooth domains. Recent studies have extended the range of applications of the MFS, which includes the application to non-homogeneous PDEs \([7, 18]\), to non smooth boundaries \([4, 11]\) and non smooth boundary conditions \([5]\). There are two major issues in the MFS yet to be resolved: the choice of the
source points and the ill conditioning. The location of the source points has been widely addressed in the literature \([1, 2, 3, 13, 19, 23, 24, 26, 27, 33]\) and several choices have been advocated to be effective. Some works have proposed techniques to alleviate the ill conditioning if the MFS \([9, 10, 17, 37]\), but none of these approaches seem to completely solve the problem of ill conditioning.

In this work we propose a new technique to remove the ill conditioning in the MFS. The main idea is to consider an expansion of the MFS basis functions in terms of harmonic polynomials. Then, using the singular value decomposition and Arnoldi orthogonalization we define well conditioned basis functions spanning the same functional space as the original MFS basis functions. To the best of our knowledge, this is the first approach able to remove the ill conditioning of the classical MFS in the context of general planar domains.

2. THE CLASSICAL MFS

Let \(\Omega\) be a bounded and smooth planar domain. We will consider the numerical solution of the Dirichlet boundary value problem involving Laplace equation,

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]

for a given function \(g\) defined on \(\partial \Omega\).

As in \([9]\), we will use the terminology Direct-MFS to refer to the classical MFS approach that we briefly describe in this section.

Let \(\Phi\) be a fundamental solution of the operator \(-\Delta\),

\[
\Phi(x) = \frac{-1}{2\pi} \log |x|.
\]

This fundamental solution is analytic, except at the origin, where it has a singularity. The Direct-MFS approximation is a linear combination

\[
u^{\text{Direct}, N}(x) = \sum_{n=1}^{N} c_{n}^{\text{Direct}} \Phi(x - y_{n}),
\]

where each base function is a translation of the fundamental solution to some source point \(y_{n}\) placed on some admissible source set \(\hat{\Gamma}\) that does not intersect \(\Omega\).

The approximation can be mathematically justified by density results stating that

\[
\text{span} \left\{ \Phi(\bullet - y) |_{\Omega} : y \in \hat{\Gamma} \right\}
\]

is dense in \(L^{2}(\partial \Omega)\) \([1, 15]\). This Direct-MFS linear combination can be augmented with some extra basis functions. For example for ensuring completeness it is necessary to add the constant function \([1, 21]\) and to improve the MFS accuracy in the context of non smooth problems, some singular particular solutions may be added \([4, 5]\).
By construction, the MFS approximation (2.2) satisfies the Laplace equation in \( \Omega \) and the coefficients can be calculated by forcing the boundary conditions at some boundary points. We consider \( M \) (with \( M \geq N \)) collocation points \( x_i, i = 1, 2, ..., M \), and solve
\[
A_{\text{Direct}} \cdot c_{\text{Direct}} = G,
\]
where
\[
A_{\text{Direct}} = [\Phi(x_i - y_j)]_{M \times N}, \quad G = [g(x_i)]_{M \times 1}
\]
and \( c_{\text{Direct}} \) is a vector containing all the coefficients of the Direct-MFS linear combination (2.2). We all the numerical solutions of this work will consider oversampling, by fixing \( M = 2N \) and (2.3) is solved in the least-squares sense.

The Direct-MFS may provide highly accurate approximation, even considering just a few a source points. For instance, assume that \( \Omega \) is the unitary disk and \( g \) is the trace of an entire function. If we define \( M = N \) collocation points uniformly distributed on \( \partial \Omega \) and the source points are placed on a circumference of radius equal to \( R \), we have exponential convergence (cf. [29, 30]) and there exists a constant \( C > 0 \) that does not depend on \( N \), such that
\[
\|g - u_N\|_{L^2(\partial \Omega)} \leq CR^{-N}.
\]
Moreover, this bound might suggest the convergence to be faster for large values of \( R \), that is, choosing the artificial boundary, where we place the source points, far from the boundary. However, under similar assumptions we know that the condition number also increases exponentially (cf. [31, 38]),
\[
\text{cond}_2(A_{\text{Direct}}) \sim \frac{1}{2} \log(R)R^{\frac{N}{2}}.
\]
Therefore, for large values of \( R \), the Direct-MFS is highly ill-conditioned, which affects the accuracy and prevents the exponential convergence to be achieved.

In the following section we will describe a different approach that reduces the problem of ill-conditioning of the Direct-MFS.

3. The MFS-QR

The MFS-QR was introduced in [9] as a technique to reduce the ill conditioning of the Direct-MFS. The source points are assumed to be placed on a circumference of radius \( \frac{1}{\epsilon} \),
\[
y_j = \frac{1}{\epsilon} (\cos(\alpha_j), \sin(\alpha_j)), \quad j = 1, ..., N, \quad \alpha_j = \frac{2\pi j}{N},
\]
where \( \epsilon \) is chosen such that \( \frac{1}{\epsilon} > R_\Omega := \max_{x \in \partial \Omega} \|x\| \).

Each source point can be written as \( \frac{1}{\epsilon} (\cos(\alpha), \sin(\alpha)) \), for some \( \alpha \in [0, 2\pi[ \) and dropping the constant \(-1/(2\pi)\), we have the following expansion of the
The MFS-QR involves the calculation of a QR corresponding MFS base function in polar coordinates \([9, 22]\),

\[
\psi(r, \theta) = \log \left( \sqrt{\left( r \cos(\theta) - \frac{1}{\epsilon} \cos(\alpha) \right)^2 + \left( r \sin(\theta) - \frac{1}{\epsilon} \sin(\alpha) \right)^2} \right) \\
= \log \left( \sqrt{r^2 + \frac{1}{\epsilon^2} - \frac{2r}{\epsilon} \cos(\theta - \alpha)} \right) \\
= \log \left( \frac{1}{\epsilon} \sqrt{\epsilon^2 r^2 + 1 - 2r\epsilon \cos(\theta - \alpha)} \right) \\
= - \log(\epsilon) + \log \left( \sqrt{\epsilon^2 r^2 + 1 - 2r\epsilon \cos(\theta - \alpha)} \right) \\
= - \log(\epsilon) - \sum_{m=1}^{\infty} \frac{r^m \epsilon^m}{m} \cos(m(\theta - \alpha)) \\
= - \log(\epsilon) - \sum_{m=1}^{\infty} \frac{r^m \epsilon^m}{m} \left( \cos(m\theta) \cos(m\alpha) + \sin(m\theta) \sin(m\alpha) \right).
\]

\((3.2)\)

Therefore, the MFS basis functions \(\psi_j(r, \theta)\) associated to \(N\) source points are given by

\[
\begin{bmatrix}
\psi_1(r, \theta) \\
\psi_2(r, \theta) \\
\vdots \\
\psi_N(r, \theta)
\end{bmatrix} =
\begin{bmatrix}
-1 & -\cos(\alpha_1) & -\sin(\alpha_1) & -\cos(2\alpha_1) & -\sin(2\alpha_1) & -\cos(3\alpha_1) & -\sin(3\alpha_1) & \ldots \\
-1 & -\cos(\alpha_2) & -\sin(\alpha_2) & -\cos(2\alpha_2) & -\sin(2\alpha_2) & -\cos(3\alpha_2) & -\sin(3\alpha_2) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
-1 & -\cos(\alpha_N) & -\sin(\alpha_N) & -\cos(2\alpha_N) & -\sin(2\alpha_N) & -\cos(3\alpha_N) & -\sin(3\alpha_N) & \ldots
\end{bmatrix}
\begin{bmatrix}
\log(\epsilon) \\
\epsilon \\
\epsilon^2 \\
\epsilon^3 \\
\epsilon^4 \\
\vdots
\end{bmatrix}
\begin{bmatrix}
1 \\
r \cos(\theta) \\
r \sin(\theta) \\
r^2 \cos(\theta) \\
r^2 \sin(\theta) \\
r^3 \cos(\theta) \\
r^3 \sin(\theta) \\
\vdots
\end{bmatrix}
\]

\((3.3)\)

and after truncating this expansion, considering the sum in \((3.2)\) just up to \(m = p\) (such that \(2p + 1 > N\)) we obtain a factorization

\[
\Theta(r, \theta) = B \cdot D \cdot F(r, \theta).
\]

\((3.4)\)

The MFS-QR involves the calculation of a QR factorization of the matrix \(B\),

\[
B = Q \cdot R,
\]

and we define a new set of functions spanning the same functional space as Direct-MFS basis functions given by

\[
\Psi(r, \theta) = \tilde{R} \cdot F(r, \theta),
\]
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defining

\[ \tilde{R} = \tilde{T} \circ R, \]

where

\[ \tilde{T} = \begin{bmatrix}
1 & \epsilon/\log(\epsilon) & \epsilon^2/(2\log(\epsilon)) & \epsilon^2/(2\log(\epsilon)) & \ldots & \epsilon^p/(p\log(\epsilon)) & \epsilon^p/(\log(\epsilon)p)
\end{bmatrix} \]

and \( \circ \) denotes the Hadamard product of matrices,

\[ (A \circ B)_{i,j} = (A)_{i,j} \cdot (B)_{i,j}. \]

The MFS-QR approximation is a linear combination

\[ u_{QR}^N (r, \theta) = \sum_{n=1}^{N} c_n^{QR} \Psi_n (r, \theta) \quad (3.5) \]

and the solution of the boundary value problem \( (2.1) \) is obtained through the solution of the linear system

\[ A^{QR} \cdot c^{QR} = G, \quad (3.6) \]

where \( A^{QR} = [\Psi(x_i)]^T \) or equivalently

\[ [F(x_i)]^T \cdot \tilde{R}^T \cdot c^{QR} = G. \]

As was reported in [9], the MFS-QR removes completely the ill conditioning when \( \partial \Omega \) and the artificial boundary are concentric circumferences. However, if the domain is not a disk, though at a lower rate than the Direct-MFS but the condition number also grows exponentially as the number of basis functions increase and this can be prohibitive for a large values of \( N \). The main reason for this growth is the increasing powers of \( r \) in the harmonic polynomials in \( F(r, \theta) \) that are highly ill-conditioned.

Moreover, by construction, the MFS-QR assumes the artificial boundary to be a circumference and this may be too restrictive when applying the MFS to some geometries, such as domains that are elongated in one direction or domains with re-entrant regions for which, in general, placing the source points on a circumference may not be suitable choice.

Next, we will introduce a new technique to remove the ill conditioning of the MFS in the context of general planar domains and artificial boundaries, provided the latter satisfy a geometric constraint defined in next section.

4. A NEW TECHNIQUE TO REMOVE THE ILL CONDITIONING OF THE DIRECT-MFS

In this section, we will describe a new technique to remove the ill conditioning of the Direct-MFS. We assume that \( \Omega \) is a smooth star shaped
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planar domain and for simplicity, we assume that it contains the origin. We will denote by \( \hat{\Omega} \) an open set for which \( \bar{\Omega} \subset \hat{\Omega} \) and will take the boundary of \( \hat{\Omega} \) as artificial curve for the MFS. We assume that \( \hat{\Omega} \) is chosen in such a way that there exists a ball \( B^\Omega \) for which we have \( \Omega \subset B^\Omega \subset \hat{\Omega} \), as illustrated in Figure 1. We will denote by \((r, \theta)\) the polar coordinates of an arbitrary point \( x \in \mathbb{R}^2 \), that is
\[
x = r(\cos(\theta), \sin(\theta))
\]
and given a sample of \( N \) source points placed outside \( \Omega \) we calculate their polar coordinates, which we assume to be of the form \( \left( \frac{1}{\epsilon_j}, \alpha_j \right) \) where \( \alpha_j \in [0, 2\pi] \), for \( j = 1, \ldots, N \). Thus, using the fact that for any \( z \in \mathbb{C} \), we have
\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2},
\]
the MFS base functions corresponding to a source point \( y_j \) has the following expansion, directly obtained from (3.2),
\[
\psi_j(r, \theta) = -\log(\epsilon_j) - \sum_{m=1}^{\infty} \frac{r^m \epsilon_j^m}{m} \frac{e^{im(\theta-\alpha)} + e^{-im(\theta-\alpha)}}{2}
\]
\[
= -\log(\epsilon_j) - \sum_{m=1}^{\infty} \frac{r^m \epsilon_j^m}{m} \frac{e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2}
\]
\[
= -\log(\epsilon_j) - \sum_{m=1}^{\infty} \left( \frac{r}{R_\Omega} \right)^m (\epsilon_j R_\Omega)^m \frac{e^{ima} e^{-ima} + e^{-ima} e^{ima}}{2m} (4.1)
\]
where for convenience we multiplied and divided each term in the sum by \( R_\Omega^m \). Note that, by definition of \( R_\Omega \), for all points in \( \hat{\Omega} \) we have \( \frac{r}{R_\Omega} \leq 1 \) and the assumption that there exists a ball \( B^\Omega \) for which we have \( \Omega \subset B^\Omega \subset \hat{\Omega} \) ensures that
\[
\max_{j=1,\ldots,N} (\epsilon_j R_\Omega) < 1
\]

**Figure 1.** The setting for the numerical approach. We assume that there exists a ball \( B^\Omega \) such that \( \Omega \subset B^\Omega \subset \Omega \).
which implies convergence of the series. Formally, we can expand all the MFS basis functions defined for \((r, \theta) \in \Omega\) in terms of harmonic polynomials as follows

\[
\psi(r, \theta) = \mathbf{M}_\infty \cdot \mathbf{F}_\infty(r, \theta)
\]

where

\[
\psi(r, \theta) = \begin{bmatrix} \psi_1(r, \theta) & \psi_2(r, \theta) & \ldots & \psi_N(r, \theta) \end{bmatrix}^T,
\]

\[
\mathbf{M}_\infty = \begin{bmatrix}
- \log(\epsilon_1) & - \frac{(\epsilon_1 R_\Omega) e^{-i\alpha_1}}{2} & - \frac{(\epsilon_1 R_\Omega)^2 e^{-2i\alpha_1}}{4} & \ldots & - \frac{(\epsilon_1 R_\Omega)^p e^{2^p i\alpha_1}}{4} \\
- \log(\epsilon_2) & - \frac{(\epsilon_2 R_\Omega) e^{-i\alpha_2}}{2} & - \frac{(\epsilon_2 R_\Omega)^2 e^{-2i\alpha_2}}{4} & \ldots & - \frac{(\epsilon_2 R_\Omega)^p e^{2^p i\alpha_2}}{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- \log(\epsilon_N) & - \frac{(\epsilon_N R_\Omega) e^{-i\alpha_N}}{2} & - \frac{(\epsilon_N R_\Omega)^2 e^{-2i\alpha_N}}{4} & \ldots & - \frac{(\epsilon_N R_\Omega)^p e^{2^p i\alpha_N}}{4}
\end{bmatrix}
\]

and

\[
\mathbf{F}_\infty(r, \theta) = \begin{bmatrix} 1 & \frac{r}{R_\Omega} e^{i\theta} \left( \frac{r}{R_\Omega} \right)^2 e^{2i\theta} & \ldots & \frac{r}{R_\Omega} e^{-i\theta} \left( \frac{r}{R_\Omega} \right)^2 e^{-2i\theta} & \ldots \end{bmatrix}^T.
\]

Note that since \(R_\Omega\) is just a constant, the (infinite) matrix \(\mathbf{M}\) depends just on the source points while \(\mathbf{F}(r, \theta)\) depends just on \(\Omega\). The expansions can be truncated in such a way that the residual is smaller than machine precision, for instance considering the expansion \([11]\) just up to \(m = p_0\), provided \(2p_0 + 1 \geq N\). Actually, we can determine \(p_0\) taking into account that

\[
\psi_j(r, \theta) = - \log(\epsilon_j) - \sum_{m=1}^{\infty} \left( \frac{r}{R_\Omega} \right)^m \frac{(\epsilon_j R_\Omega)^m e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2m} \\
= - \log(\epsilon_j) - \sum_{m=1}^{p_0} \left( \frac{r}{R_\Omega} \right)^m \frac{(\epsilon_j R_\Omega)^m e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2m} \\n+ \sum_{m=p_0+1}^{\infty} \left( \frac{r}{R_\Omega} \right)^m \frac{(\epsilon_j R_\Omega)^m e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2m}
\]

and we can estimate the residual by

\[
\left| \sum_{m=p_0+1}^{\infty} \left( \frac{r}{R_\Omega} \right)^m \frac{(\epsilon_j R_\Omega)^m e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2m} \right| \leq \\
\sum_{m=p_0+1}^{\infty} \left| \left( \frac{r}{R_\Omega} \right)^m \frac{(\epsilon_j R_\Omega)^m e^{im\theta} e^{-ima} + e^{-im\theta} e^{ima}}{2m} \right| \leq \\
\sum_{m=p_0+1}^{\infty} \frac{(\max_j \epsilon_j R_\Omega)^m}{m} \leq \left( \max_j \epsilon_j R_\Omega \right)^{p_0+1} \Phi_{HL}(\max_j \epsilon_j R_\Omega, 1, p_0 + 1),
\]

where \(\Phi_{HL}\) is the Hurwitz-Lerch transcendent function. Thus, we define \(p_0\) to be the smallest integer such that

\[
\left( \max_j \epsilon_j R_\Omega \right)^{p_0+1} \Phi_{HL}(\max_j \epsilon_j R_\Omega, 1, p_0 + 1) \leq \epsilon.
\]
After determining \( p := \max(p_0, \left\lceil \frac{N-1}{2} \right\rceil) \) we truncate the expansion and obtain

\[
\psi(r, \theta) = M \cdot F(r, \theta),
\]

where \( M \) is a \( N \times (2p+1) \) matrix and \( F(r, \theta) \), a vector-valued function with \( 2p + 1 \) components that are the truncated versions of \( M_{\infty} \) and \( F_{\infty}(r, \theta) \), respectively. Note that there is a slight abuse of notation in (4.2) in the sense that the equality shall be understood as approximation with accuracy at machine precision level.

We will propose a new technique to remove the ill conditioning in the Direct-MFS by performing a suitable change of basis. For this purpose, an important remark is that in (4.2) we can multiply from the left any non singular matrix. This corresponds to applying a change of the basis functions without modifying the functional space spanned by them. Thus, an interesting question is to identify a suitable matrix to be applied from the left in (4.2).

Given a set of collocation points and their polar coordinates

\[
x_j = (r_j, \theta_j), \quad j = 1, ..., P,
\]

the (transpose of the) matrix of the Direct-MFS system \( \mathbf{A}^{\text{Direct}} \), is obtained directly from evaluating (4.2) at the collocation points. At this point we can identify two possible sources for the ill-conditioning of the Direct-MFS:

- The matrix \( M \) is ill conditioned
- the increasing powers of \( \frac{j}{r_0} \) in \( F(r, \theta) \) also generate ill conditioning.

We will propose a technique to tackle these two sources of the ill-conditioning respectively in next two sections.

### 4.1. reducing the effect of the ill conditioning of matrix \( M \).

For now, we will assume that we can write the functions in \( F(r, \theta) \) in terms of a well conditioned basis,

\[
F(r, \theta) = K \cdot J(r, \theta),
\]

where \( K \) is an invertible matrix and \( J(r, \theta) \) is a vector-valued function built by a set of well conditioned basis functions that are particular solutions of Laplace equation. Note that from (4.2) we can write

\[
\psi(r, \theta) = \underbrace{M \cdot K \cdot J(r, \theta)}_{:=M_1} := M_1 \cdot J(r, \theta).
\]

We start by calculating the singular value decomposition of matrix \( M_1 \). This allows to obtain the factorization

\[
M_1 = U \cdot S \cdot V^*,
\]

where \( U \) and \( V \) are unitary and \( S \) is diagonal with non negative entries. Thus, multiplying the matrix \( U^* \) from the left we obtain

\[
U^* \cdot M_1 = \underbrace{U^* \cdot U}_{I} \cdot S \cdot V^* = S \cdot V^*.
\]
We know that $S$ has the same dimensions as $M_1$ and since we took $2p+1 \geq N$ we can write

$$S = \begin{bmatrix} S_1 & 0 \end{bmatrix},$$

where $S_1$ is a diagonal square matrix and 0 denotes a block matrix with all entries equal to zero and we have

$$S \cdot V^* = \begin{bmatrix} S_1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = S_1 \cdot V_1^* + 0 \cdot V_2^* = S_1 \cdot V_1^*,$$

where $V_1^*$ is the matrix composed by the first $N$ rows of $V^*$. Now, multiplying from the left the matrix $S_1^{-1}$ in (4.5) we obtain

$$S_1^{-1} \cdot U^* \cdot M_1 = S_1^{-1} \cdot S \cdot V^* = S_1^{-1} \cdot S_1 \cdot V_1^* = V_1^*.$$

Thus, we obtain a new set of basis functions directly from (4.2) (formally) multiplying from the left the matrix $(S_1^{-1} \cdot U^*)$,

$$\phi(r, \theta) = \begin{bmatrix} \phi_1(r, \theta) & \phi_2(r, \theta) & \ldots & \phi_N(r, \theta) \end{bmatrix}^T$$

defined by

$$\phi(r, \theta) = (S_1^{-1} \cdot U^*) \psi(r, \theta) = V_1^* \cdot J(r, \theta). \quad (4.6)$$

Note that from (4.6) instead of the multiplication of the matrix $(S_1^{-1} \cdot U^*)$ to perform the change of basis, for practical purposes we simply calculate the product $V_1^* \cdot J(r, \theta)$ in order to have the new set of basis functions.

We will call MFS-SVD to the approximation made by a linear combination of the new basis functions of $\phi(r, \theta)$,

$$u_{SVD}^N(r, \theta) = \sum_{n=1}^{N} c_{n}^{SVD} \phi_n(r, \theta) \quad (4.7)$$

and the coefficients can be determined by solving

$$[\phi(x_i)]^T \cdot C^{SVD} = G. \quad (4.8)$$

Each component of the vector valued function $J(r, \theta)$ satisfies the Laplace equation. Thus, each component of $\phi(r, \theta)$, which is a linear combination of the components of $J(r, \theta)$ is also a particular solution of the Laplace equation. Thus, since $u$ and $u_{SVD}^N$ are both harmonic functions, by the maximum principle we have

$$\|u - u_{SVD}^N\|_{L^\infty(\Omega)} \leq \|u - u_{SVD}^N\|_{L^\infty(\partial\Omega)} = \|g - u_{SVD}^N\|_{L^\infty(\partial\Omega)}. \quad (4.9)$$

In all the numerical simulations we will measure the error by estimating $\|g - u_{SVD}^N\|_{L^\infty(\partial\Omega)}$ evaluating the error at 10001 boundary points.

To illustrate the numerical technique to remove the ill conditioning of matrix $M$ developed in this section we consider the unit disk and $\hat{\Omega}$ is the domain with boundary defined by

$$\Gamma = \{(4\gamma(t) \cos(t) - 1, 4\gamma(t) \sin(t) - 1) \in \mathbb{R}^2 : \ t \in [0, 2\pi]\}.$$
where \( \gamma(t) = e^{\sin(t)} \sin^2(2t) + e^{\cos(t)} \cos^2(2t) \). Note that the components of \( F(r, \theta) \) are orthogonal functions in the unit disk, which means that in this case we can assume that the matrix \( K \) in (4.3) is simply the identity matrix.

Figure 2 shows the collocation points on the boundary of the unit disk, marked with · and the source points on \( \Gamma \) marked with ○. Figure 3-left shows the \( L^\infty \) norm of the error on the boundary of the numerical approximations given by the Direct-MFS and MFS-SVD, as a function of the number of basis functions, \( N \). In this example we took the boundary data given by \( g(x, y) = x^2 y^3 \). The right plot of the same Figure shows the condition number of the matrix of the linear system. We can observe that for \( N \leq N_0 \approx 30 \) we obtain similar errors with the Direct-MFS and the MFS-SVD because the basis functions of both approaches span the same functional space. For \( N > N_0 \) the convergence of the Direct-MFS breaks down due to ill conditioning, while the errors of the MFS-SVD decrease until we reach values close to machine precision, keeping the condition number of order 1, independently of \( N \).

**Figure 2.** Collocation points on the boundary of the unit disk, marked with · and source points on \( \Gamma \) marked with ○.

### 4.2. An Arnoldi-SVD Method of Fundamental Solutions

In this section we introduce a technique to avoid the increasing powers of \( \frac{1}{r^k} \) in \( F(r, \theta) \), through expansion (4.3), for suitable matrices \( K \) and vector valued function \( J(r, \theta) \). The coefficients of the MFS-SVD are calculated by solving the linear system (4.8) involving the matrix

\[
[\phi(r_i, \theta_i)]^T = [H(r_i, \theta_i)]^T \cdot (V_1^*)^T.
\]

Taking into account (4.3) we have

\[
[F(r_i, \theta_i)]^T = [H(r_i, \theta_i)]^T \cdot K^T
\]
and defining
\[ z_i := \frac{r_i}{R_{i\Omega}} e^{i\theta_i} \quad \text{and} \quad w_i := \frac{r_i}{R_{i\Omega}} e^{-i\theta_i} \]
we have
\[
[F(r_i, \theta_i)]^T = \begin{bmatrix}
1 & z_1 & z_1^2 & \cdots & z_1^p & w_1 & w_1^2 & \cdots & w_1^p \\
1 & z_2 & z_2^2 & \cdots & z_2^p & w_2 & w_2^2 & \cdots & w_2^p \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & z_M & z_M^2 & \cdots & z_M^p & w_M & w_M^2 & \cdots & w_M^p \\
\end{bmatrix} := \begin{bmatrix} Z & W \end{bmatrix},
\]
where the block \( Z \) is a Vandermonde matrix. The matrix \( W \) can also be obtained from a Vandermonde matrix \( \tilde{W} \) after excluding the first column. As pointed out in [16], the columns of a Vandermonde matrix, such as \( Z \), can be regarded as vectors \( q_0, \ X^1 Z q_0, \ X^2 Z q_0, \ldots \), where \( q_0 = [1 \cdots 1]^T \) and \( X^Z = \text{diag}(z_1, \ldots, z_M) \). The Arnoldi process allows to orthogonalize at each step leading to a sequence of orthogonal vectors \( q_0, q_1, q_2, \ldots \) spanning the same space as the columns of \( Z \). After \( n \) steps, we obtain \( n + 1 \) orthogonal vectors \( q_0, \ldots, q_n \) and an \( (n + 1) \times n \) Hessenberg matrix \( H_Z \) that verifies
\[ X^Z \cdot Q_{-Z} = Q_Z \cdot H_Z, \]
where \( Q_Z \) is a matrix whose columns are the orthogonal vectors \( q_0, \ldots, q_n \) and \( Q_{-Z} \) is obtained from \( Q_Z \) after removing the last column. The matrices \( Z \) and \( Q_Z \) are related through an upper triangular matrix \( R_Z \), by
\[ Z = Q_Z \cdot R_Z. \]
In a similar fashion, for the matrix \( \tilde{W} \), we obtain
\[ \tilde{W} = Q_{\tilde{W}} \cdot \tilde{R}_{\tilde{W}} \]
and
\[ W = Q_W \cdot R_W, \]
where $\mathbf{R}_W$ is obtained from $\tilde{\mathbf{R}}_W$ after excluding the first column.

Therefore,

$$[\mathbf{F}(r_i, \theta_i)]^T = [\mathbf{Z} \mid \mathbf{W}] = [\mathbf{Q}_Z \cdot \mathbf{R}_Z \mid \mathbf{Q}_W \cdot \mathbf{R}_W] = [\mathbf{Q}_Z \mid \mathbf{Q}_W] \cdot [\mathbf{R}_Z \ 0 \ 0 \ \mathbf{R}_W]$$

and this last equality can be regarded as the evaluation of (4.3) at the boundary points $(r_i, \theta_i)$,

$$\mathbf{F}(r_i, \theta_i) = \begin{bmatrix} \mathbf{R}_Z \\ 0 \\ \mathbf{R}_W \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{Q}_Z^T \\ \mathbf{Q}_W \end{bmatrix}$$

The evaluation of $\mathbf{J}(r, \theta)$ at a general point $(r, \theta)$ can be performed using the Hessenberg matrices $\mathbf{H}_Z$ and $\mathbf{H}_W$, as in the routine `polyval` of reference [16] and the MFS-SVD solution is evaluated using (4.6).

The whole numerical technique that we developed in this work is summarized in Algorithm 1.
centered at the origin and radius equal to 2. Figure 4 shows the collocation points on the boundary and the source points. Figure 5 shows results obtained with the Direct-MFS, the MFS-QR and the MFS-SVD. The left plot shows the $L^\infty$ norm of the error, measured at 10001 points on the boundary, as a function of $N$, while the right plot of the same Figure shows the condition number of the matrices of the linear systems. We can observe that the numerical results obtained with the three approaches are similar for $N \leq N_0 \approx 120$ because the functional spaces are the same although defined through different basis functions. For $N > N_0$, the convergence of the Direct-MFS breaks down due to ill conditioning. Note that the condition numbers corresponding to the Direct-MFS grow exponentially. The condition numbers corresponding to the MFS-QR also grow exponentially, but at a lower rate than the Direct-MFS. This growth is due to the increasing powers of the harmonic polynomials of the MFS-QR expansion. The numerical results of the MFS-QR and MFS-SVD are similar for $N \leq N_1 \approx 290$. For $N > N_1$ the convergence of the MFS-QR also breaks down due to ill conditioning, while the MFS-SVD converges until we reach accuracy close to machine precision, keeping the condition number approximately equal to 1.65, independently of $N$.

The ill conditioning of the Direct-MFS is related to the fact that all the basis functions become almost linearly dependent on $\partial \Omega$. This effect is even more evident if we take the artificial boundary far from $\partial \Omega$. Figure 6 shows the plots of $N = 8$ Direct-MFS basis functions, normalized to have unit $L^\infty$ norm, for source points equally spaced on the boundary of the disk with radius equal to 10. All the basis functions become almost undistinguishable at this scale. Figure 7 shows the real part (left plot) and imaginary part (right plot) of the (well conditioned) basis functions corresponding to MFS-SVD spanning the same functional space as the Direct-MFS basis functions.
In the second numerical example we take the domain with boundary defined by
\[ \{ r_1(t)(\cos(t), \sin(t)), \ t \in [0, 2\pi] \}, \]
where
\[ r_1(t) = \frac{6}{5} + \frac{\cos(6t)}{5} + \frac{\cos(3t)}{10}, \]
the artificial boundary parametrized by
\[ r(t) = 2 + \frac{\cos(6t)}{5} + \frac{\cos(3t)}{10}, \ t \in [0, 2\pi[ \]
and the same boundary condition defined by the function \( g(x, y) = x^2y^3. \) Figure 8 shows the collocation and source points in this second numerical example.
Figure 7. Plots of the real part (left plot) and imaginary part (right plot) of the (well conditioned) basis functions corresponding to MFS-SVD spanning the same functional space as the Direct-MFS basis functions plotted in Figure 6.

Figure 8. Collocation points on the boundary of the unit disk, marked with · and source points marked with o.

Figure 9 shows the plots of the $L^\infty$ norms of the error and condition numbers, both as a function of $N$. Again, we can observe that the Direct-MFS converges, as we increase $N$ until some $N_0 \approx 150$. Then, the convergence breaks down due to ill conditioning, while the MFS-SVD converges until an accuracy close to machine precision. The condition number is approximately equal to 1.89, independently of $N$.

Next, we change the boundary condition to a more oscillatory function

$$g(x, y) = \cos(10x) \sin(10y).$$

In Figure 10 we plot the boundary data $g(r_1(t) \cos(t), r_1(t) \sin(t)), t \in [0, 2\pi]$. Figure 11 shows the plots of the $L^\infty$ norm of the error of the approximations obtained with Direct-MFS and MFS-SVD, as a function of $N$. Figure 12 shows the plot of the solution of the boundary value
Figure 9. Plot of the $L^\infty$ norm of the error of the approximations obtained with Direct-MFS and MFS-SVD, as a function of $N$ in the second numerical example (left plot) and plot of the condition number of the matrices of these two approaches. The boundary condition is defined by the function $g(x, y) = x^2y^3$.

Figure 10. Plot of the boundary data $g(r_1(t) \cos(t), r_1(t) \sin(t))$, $t \in [0, 2\pi]$.

problem of the second numerical example.

6. Conclusions and future work

We proposed a new algorithm for generating a set of functions spanning the same functional space as the Direct-MFS basis functions, but which are much better conditioned. This approach allows to remove the ill conditioning of the Direct-MFS for general star shaped domains. The artificial boundary for the MFS was assumed to satisfy a certain geometric constraint, defined in section 4 for ensuring convergence of the series expansions considered.
Figure 11. Plot of the $L_{\infty}$ norm of the error of the approximations obtained with Direct-MFS and MFS-SVD, as a function of $N$ in the second numerical example for $g(x, y) = \cos(10x)\sin(10y)$.

Figure 12. Plot of the solution of the boundary value problem of the second numerical example with boundary condition defined by the function $g(x, y) = \cos(10x)\sin(10y)$.

The extension to the general case, where we drop this constraint will be considered in a future work.

As was mentioned in the Introduction, in our opinion there are two major issues in the MFS to be resolved: a clear criteria for placing the source points and the ill conditioning. Actually both issues are related in the sense that the optimal locations for the source points advocated in the literature were proposed in the context of the application of the Direct-MFS, where the ill
conditioning limits the accuracy that can be achieved and imposes a trade-off between accuracy and conditioning. It is our belief that the MFS-SVD or any other technique to perform a change of basis will allow to explore other choices for artificial boundaries without the constraints imposed by the ill conditioning. In particular, we believe that it would be interesting to revisit the problem of the location of the source points in the context of a better conditioned basis for the MFS for which the ill conditioning is not an issue.

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