Orbifold Analysis of Broken Bulk Symmetries

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Abstract
In two-dimensional conformal field theory, we analyze conformally invariant boundary conditions which break part of the bulk symmetries. When the subalgebra that is preserved by the boundary conditions is the fixed algebra under the action of a finite group $G$, orbifold techniques can be used to determine the structure of the space of such boundary conditions. We present explicit results for the case when $G$ is abelian. In particular, we construct a classifying algebra which controls these symmetry breaking boundary conditions in the same way as the fusion algebra governs the boundary conditions that preserve the full bulk symmetry.
1. Boundary conditions and consistent chiral algebras

Conformal field theories on surfaces with boundaries have recently attracted renewed interest. It was known for quite some time that such theories play an important role in the analysis of condensed matter systems, like e.g. in the Kondo effect, as well as in critical percolation. An additional motivation to study this problem was given by the discovery [1] that string perturbation theory in the background of certain solitonic solutions that describe black D-branes can be described in terms of open strings with non-trivial boundary conditions. Thus by studying the space of conformally invariant boundary conditions for the conformal field theories that constitute string vacua one can obtain information about the possible solitonic sectors of string theory. Also, it is this space of boundary conditions (and the space of all possible crosscaps [2, 3]) on which the problem of tadpole cancellation should be considered.

Ideally, one would therefore like to study the space of all conformally invariant boundary conditions in any given conformal field theory model. Unfortunately, except for particularly simple models, this space does not seem to be tractable at the moment. To handle this classification problem in the general case, one should start by grouping the various boundary conditions in a coarse manner into subspaces, and then try a finer classification for each of these subspaces. As a reasonable approach to the first step, we propose to characterize these classes of boundary conditions by associating to each boundary condition the subalgebra $\mathfrak{A}$ of the chiral algebra $\mathfrak{A}$ of the theory that is preserved by the boundary condition. The requirement that the boundary condition has to be conformally invariant means that $\mathfrak{A}$ must contain the Virasoro subalgebra of $\mathfrak{A}$. Furthermore, the subalgebra $\mathfrak{A}$ has to be a consistent chiral algebra in the sense that the corresponding chiral blocks, as vector bundles over the moduli space of complex curves and insertion points, come with a Knizhnik–Zamolodchikov connection and obey suitable factorization rules.

The special case when the boundary conditions preserve the full chiral algebra $\mathfrak{A}$ already received attention long ago. As first argued by Cardy [4], in this case the consistent boundary conditions are in one-to-one correspondence with the (generalized) quantum dimensions of the theory, i.e. with the one-dimensional irreducible representations of the fusion algebra. Typically a chiral algebra $\mathfrak{A}$ will, however, possess very many, if not infinitely many, consistent subalgebras $\mathfrak{A}$. The first step towards a classification of all boundary conditions would be to classify all these subalgebras. This problem clearly depends largely on the specific bulk conformal field theory under consideration, and we will not have to say much about it in this letter.

The goal of this letter is, rather, to classify all those boundary conditions that preserve some prescribed subalgebra $\mathfrak{A}$. As long as $\mathfrak{A}$ is completely arbitrary, at present this problem is still too general to be tractable. We will therefore restrict our attention to a particular subclass of consistent subalgebras. Namely, we require that $\mathfrak{A}$ be the fixed algebra of some group $G$ of automorphisms of the chiral algebra $\mathfrak{A}$. In other words, $\mathfrak{A} = \mathfrak{A}^G$ is the chiral algebra of an orbifold of the theory that has chiral algebra $\mathfrak{A}$. The orbifold group $G$ need not necessarily be finite; it can even be a finite-dimensional Lie group. Still, for the purpose of the present letter we specialize further to the case when $G$
is a finite abelian group. This situation may seem rather special compared to the general problem sketched above, but it nevertheless covers a variety of examples of practical interest. Moreover, a number of physical insights can be gained, e.g. concerning the relation between boundary conditions that preserve subalgebras \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) of \( \mathfrak{A} \) which are contained in each other.

2. Examples

Let us present a number of concrete theories which realize the situation described in the introduction. Our first example is the \( c = 1 \) conformal field theory of a free boson \( X(z, \bar{z}) \) compactified on a circle. Its chiral algebra is generated by all polynomials in \( i\partial X(z) \) as well as, in the rational case, certain normal-ordered exponentials \( \exp(ikX(z)) \), where the momentum \( k \) lies on a lattice that depends on the compactification radius \( R \). The map \( \omega: X \mapsto -X \) induces a symmetry of this chiral algebra. The corresponding orbifold theory is the well-known \( Z_2 \)-orbifold of the free boson.\(^1\) One can recover the original compactified free boson theory by extending the chiral algebra of the orbifold theory by the field \( j = i\partial X \) which has quantum dimension one and conformal weight one. Correspondingly, we have two types of boundary conditions; conditions of the first type preserve the whole chiral algebra \( \mathfrak{A} \), while those of the second type preserve only the subalgebra \( \mathfrak{A}^{Z_2} \) that is fixed under \( \omega \). This distinction is again well known: the first type are Neumann boundary conditions, while the second are Dirichlet conditions, respectively the other way round (the two situations are exchanged by charge conjugation, i.e. T-duality).

The free boson at rational radius squared provides yet another example. One can see that actually only D-branes sitting at suitable roots of unity preserve the full rational symmetry of the theory (respectively of its \( Z_2 \)-orbifold). To obtain also D-branes at generic locations, one has to break the bulk symmetry in the following manner. Instead of including all the exponentials \( \exp(ikX) \) with \( k \) in the relevant lattice, one restricts the allowed values of \( k \) to a sublattice. The so obtained subalgebra \( \hat{\mathfrak{A}} \) of \( \mathfrak{A} \) is precisely the chiral algebra of a free boson theory whose compactification radius is an integral multiple \( MR \) of the original one; it can be described as the algebra \( \mathfrak{A}^{Z_M} \) that is invariant under the \( Z_M \) group of automorphisms generated by the shift \( X \mapsto X + 2\pi/M\sqrt{N} \), where \( N \) is the number of primary fields of the original theory.

Another example [5–7] is the three-state Potts model which has a \( \mathcal{W}_3 \)-symmetry. The boundary conditions which preserve the whole \( \mathcal{W}_3 \)-symmetry are the so-called fixed and mixed boundary conditions. The \( \mathcal{W}_3 \)-algebra has an automorphism \( \omega \) of order two that maps the spin-three current to minus itself; in the Potts model the fixed subalgebra with respect to \( \omega \) is just the Virasoro algebra; the boundary conditions which preserve only the Virasoro algebra are the free boundary condition as well as the new boundary condition discovered in [5]. A similar situation arises for all Virasoro minimal models with central charge \( c = 1 - 6/m(m+1) \) for \( m = 1 \) or 2 mod 4 and with modular invariant of extension type [6].

A different class of examples is provided by conformal field theories which are tensor

\(^1\) For \( d \) uncompactified bosons, the group \( Z_2 \) gets replaced by the Lie group \( O(d) \).
products of identical subtheories. Then there are boundary conditions which preserve only the subalgebra that is fixed under a cyclic group of permutations of the subtheories. Such boundary conditions can be analyzed by combining our results with the methods developed in [8, 9].

Finally we mention that when talking about boundary conditions one usually refers to the situation where the torus partition function is the charge conjugation modular invariant. T-duality, on the other hand, maps the boundary conditions for the true diagonal modular invariant that respect all bulk symmetries to those boundary conditions for the charge conjugation modular invariant that are twisted by charge conjugation. Applying the formalism developed in this paper to the orbifold by the \( \mathbb{Z}_2 \)-symmetry that is furnished by charge conjugation therefore allows in particular to determine boundary conditions for the true diagonal modular invariant.

3. Simple current extensions – a summary

The chiral algebra \( \mathfrak{A} \) can be decomposed into eigenspaces for the action of the finite abelian orbifold group \( G \). These eigenspaces are labelled by characters \( \Psi \) of \( G \), i.e. we have

\[
\mathfrak{A} = \bigoplus_{\Psi \in G^*} \mathfrak{A}_\Psi.
\]

(1)

The fixed algebra \( \mathfrak{A}_0 \) can be identified with the eigenspace for the trivial character \( \Psi_0 \in G^* \), \( \mathfrak{A}_0 = \mathfrak{A}_{\Psi_0} \); the other eigenspaces are modules of \( \mathfrak{A}_0 \). Inspection shows that all the examples presented above share another important feature: the spaces \( \mathfrak{A}_\Psi \) appearing in (1) are even irreducible \( \mathfrak{A} \)-modules. In fact, we are not aware of any abelian orbifold theory for which this property does not hold; accordingly we will from now on assume that indeed all \( \mathfrak{A}_\Psi \) are irreducible.

This assumption implies in particular that in the orbifold theory the fusion rules of the primary fields that correspond to the modules \( \mathfrak{A}_\Psi \) are given by the character group \( G^* \) of \( G \), which is again a finite abelian group. In other words, all these primary fields of the orbifold theory are simple currents \([10, 11] \), and the original chiral algebra \( \mathfrak{A} \) can be recovered from \( \mathfrak{A}_0 \) as a so-called integer spin simple current extension. This observation enables us to use simple current technology to investigate the problem. The rest of this section is devoted to a brief review of the properties of simple current extensions as established in [11, 12] that will be needed in the sequel.

Let us consider the following situation in chiral\(^2\) conformal field theory. We start with some chiral conformal field theory with chiral algebra \( \mathfrak{A}_0 \); the fusion algebra of this theory has the character group \( G^* \) as a subgroup, whose elements \( J \) correspond to integer spin simple currents. The theory obtained by extending \( \mathfrak{A}_0 \) by these simple currents is precisely the theory with chiral algebra \( \mathfrak{A} \). The simple currents \( J \in G^* \) act via the fusion product on the primary fields\(^3\) \( \lambda \) of the \( \mathfrak{A}_0 \)-theory; this action organizes them into orbits

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\(^2\) At the chiral level, where one deals with conformal field theory on a complex curve, there is no influence of boundaries at all [13]. The chiral conformal field theory structures considered here are thus logically independent of any boundary data; they have passed independent tests [12, 14] in the context of closed conformal field theory.

\(^3\) To be precise, on the corresponding generators \( \phi_\lambda \) of the fusion algebra.
We need the following additional data. To every primary field \( \lambda \) we associate its stabilizer
\[
S_\lambda := \{ J \in G^* | J \star \lambda = \lambda \}.
\]
Every \( S_\lambda \) is a subgroup of \( G^* \), and it is one and the same subgroup for fields on the same \( G^* \)-orbit. When \( J \in S_\lambda \), we say that \( \lambda \) is a fixed point of the simple current \( J \). Further, to every simple current \( J \in G^* \) and to every field \( \lambda \) one associates the monodromy charge
\[
Q_J(\lambda) := \Delta_\lambda + \Delta_J - \Delta_{J+\lambda} \mod \mathbb{Z}.
\]
Moreover, for every simple current \( J \in G^* \) we have a matrix \( S^J \) whose entries \( S^J_{\lambda, \mu} \) are non-vanishing only if both primaries \( \lambda \) and \( \mu \) of the \( \mathfrak{A} \)-theory are fixed points of \( J \), i.e. only if both \( J\lambda \equiv J \ast \lambda = \lambda \) and \( J\mu = \mu \). For the identity element \( 1 \in G^* \), \( S^1 = S \) is the ordinary modular S-matrix of the \( \mathfrak{A} \)-theory.

The restriction of \( S^J \) to the fixed points of \( J \) is unitary, and together with the restriction of the \( T \)-matrix it obeys the usual relations of the modular group; further, it satisfies the simple current relation
\[
S^J_{\lambda, \lambda} = e^{2\pi i Q_J(\lambda)} S^J_{\lambda, \lambda}^J
\]
for every simple current \( J' \in G^* \). As a matter of fact, in full generality the relation (4) only holds up to a certain two-cocycle on \( G^* \). This forces one to deal also with a subgroup of the stabilizer on which the cocycle vanishes, the untwisted stabilizer \([12, 14, 15]\), rather than only with the full stabilizer. In order to present our results without much additional notation, for the purposes of this letter we will ignore this important complication. For a detailed description, with full account of the untwisted stabilizer, we refer to a forthcoming publication \([28]\).

There is evidence \([14]\) that the matrix \( S^J \) coincides with the matrix that implements the modular transformation \( \tau \mapsto -1/\tau \) on the one-point chiral blocks on the torus with insertion \( J \). When the \( \mathfrak{A} \)-theory is a WZW model or a coset model, then the matrix \( S^J \) is the Kac–Peterson matrix of the relevant orbit Lie algebra, see \([16, 17]\). In the case of our present interest, for a large class of conformal field theories we can also use the result \([18]\) that under certain finiteness conditions one can associate to every descendant of the vacuum a representation of the modular group; this result is relevant here because in the extended theory with chiral algebra \( \mathfrak{A} \) the simple currents in \( G^* \) become descendants of the vacuum.

Under the restriction that all untwisted stabilizers equal the full stabilizers, the pertinent results of \([12]\) can be summarized as follows.

- The primary fields of the \( \mathfrak{A} \)-theory are (labelled by) \(([\lambda], \psi_\lambda)\), where \([\lambda] \) is a \( G^* \)-orbit with vanishing monodromy charge, \( Q_J(\lambda) = 0 \) for all \( J \in G^* \), \(^4\) and where \( \psi_\lambda \) is a character of the stabilizer, \( \psi_\lambda \in S^\lambda \).

\(^4\) Standard simple current relations imply that for every simple current \( J \) of integral conformal weight the monodromy charges \( Q_J(\lambda) \) defined by (3) are constant on \( G^* \)-orbits. As already mentioned, the same is true for the stabilizer subgroups. We therefore simplify the notation by writing \( Q(\lambda), \psi_\lambda, S^\lambda \), etc., in place of \( Q(\lambda) \) etc.
It follows in particular that an irreducible module $\mathcal{H}(\bar{\lambda}, \psi_\lambda)$ of the $\mathfrak{A}$-theory decomposes into irreducible $\bar{\mathfrak{A}}$-modules $\bar{\mathcal{H}}_{\bar{\lambda}}$, according to

$$\mathcal{H}(\bar{\lambda}, \psi_\lambda) = \bigoplus_{\bar{\lambda} \in G^*/S_{\bar{\lambda}}} \bar{\mathcal{H}}_{\bar{\lambda}}. \tag{5}$$

(In the special case where $\bar{\lambda} = \bar{\Omega}$ is the vacuum of the $\bar{\mathfrak{A}}$-theory, which has monodromy charge zero and is on a full $G^*$-orbit, this is nothing but (1).) Notice that for non-trivial stabilizer one and the same $\bar{\mathfrak{A}}$-module $\bar{\mathcal{H}}_{\bar{\lambda}}$ can appear in the decomposition of several distinct irreducible $\mathfrak{A}$-modules.

The modular matrix $S$ of the $\mathfrak{A}$-theory is given by

$$S(\bar{\lambda}, \psi_\lambda)(\bar{\mu}, \psi_{\mu}) = \frac{|G^*|}{|S_{\lambda}| |S_{\mu}|} \sum_{\bar{\lambda} \in G^*/S_{\bar{\lambda}}} \psi_{\lambda}(J) \psi_{\mu}(J)^* S_{\lambda,\bar{\mu}}. \tag{6}$$

4. The classifying algebra

By the requirement that $\bar{\mathfrak{A}}$ is a consistent chiral algebra, the chiral blocks of the orbifold theory satisfy the usual factorization rules. This allows us [19, 13] to analyze the factorization of bulk-bulk-boundary correlators [20–22] in the same manner as for boundary conditions that preserve all of $\mathfrak{A}$. This way one obtains [13] the reflection coefficients for a bulk field in the presence of any conformally invariant boundary condition from the one-dimensional irreducible representations of a certain algebra, the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$. The structure constants of $\mathcal{C}(\bar{\mathfrak{A}})$ can be expressed in terms of the operator product coefficients of the $\mathfrak{A}$-theory and of fusing matrices for the boundary blocks. Such fusing matrices exist because by assumption the chiral blocks of the $\bar{\mathfrak{A}}$-theory possess a Knizhnik–Zamolodchikov connection.

The classifying algebra $\mathcal{C}(\mathfrak{A})$ for those boundary conditions that preserve all of $\mathfrak{A}$ is just the fusion algebra of the $\mathfrak{A}$-theory. Accordingly a basis of $\mathcal{C}(\mathfrak{A})$ is given by the primary fields $([\lambda], \psi_\lambda)$ of the $\mathfrak{A}$-theory. On the other hand, in the presence of boundary conditions that preserve only a proper subalgebra $\bar{\mathfrak{A}}$ of the $\mathfrak{A}$-symmetry, different submodules $\bar{\mathcal{H}}_{\bar{\lambda}}$ in the decomposition (5) are reflected differently at the boundary. To take this behaviour into account, as a basis of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ we then take individual irreducible $\bar{\mathfrak{A}}$-modules rather than orbits of $\bar{\mathfrak{A}}$-modules. Nevertheless we also have to take the characters $\psi_\lambda$ into account, because one and the same irreducible $\bar{\mathfrak{A}}$-module is reflected differently when it appears in different $\mathfrak{A}$-modules $\mathcal{H}(\bar{\lambda}, \psi_\lambda)$. In short, the basis elements of $\mathcal{C}(\mathfrak{A})$ must be labelled by pairs $(\bar{\lambda}, \psi_\lambda)$, where $\bar{\lambda}$ is an $\bar{\mathfrak{A}}$-primary with vanishing monodromy charge and $\psi_\lambda$ is a character of the stabilizer $S_{\lambda}$. The set of these fields is closely related to the set of primaries in the untwisted sector of the orbifold theory based on $\bar{\mathfrak{A}}$; but it is not exactly the same, since we include multiplicities (encoded in the characters $\psi_\lambda$) for those fields in the untwisted sector that appear more than once in the $\mathfrak{A}$-theory.

To obtain the structure constants of the classifying algebra, one could in principle now proceed as described in [21, 13] and work out the factorization of bulk-bulk-boundary correlators. Unfortunately, except for a few special cases the required values of operator
product coefficients and fusing matrices are not known. However, we can circumvent this problem entirely by combining the information on \( \mathcal{C}(\mathfrak{A}) \) and its basis given above with our knowledge about simple current extensions.

This way we arrive at the following results (for details of the calculations, and also for a proper treatment of genuine untwisted stabilizers, see [28]). Let us first present the structure constants \( \tilde{N} \) of \( \mathcal{C}(\mathfrak{A}) \) with only lower indices; we have

\[
\tilde{N}(\lambda_1,\psi_1),(\lambda_2,\psi_2),(\lambda_3,\psi_3) = \frac{G^*}{|S_{\lambda_1},S_{\lambda_2},S_{\lambda_3}|} \tilde{N}(\lambda_1,\psi_1),(\lambda_2,\psi_2),(\lambda_3,\psi_3),
\]

where various quantities are introduced as follows. By \( S_{\lambda_1},S_{\lambda_2},S_{\lambda_3} \) we denote the subgroup of \( G^* \) that is generated by the three stabilizers \( S_{\lambda_i} \). The quantity \( \tilde{N}(\lambda_1,\psi_1),(\lambda_2,\psi_2),(\lambda_3,\psi_3) \) is the rank of a natural subsheaf of the bundle of chiral blocks of the \( \mathcal{A} \)-theory with insertions \((|\lambda_1|,\psi_1),(|\lambda_2|,\psi_2) \) and \((|\lambda_3|,\psi_3) \). More explicitly, \( \tilde{N} \) is given by the Verlinde-like formula [23]

\[
\tilde{N}(\lambda_1,\psi_1),(\lambda_2,\psi_2),(\lambda_3,\psi_3) = \sum_{J_1 \in S_{\lambda_1}} \sum_{J_2 \in S_{\lambda_2}} \sum_{J_3 \in S_{\lambda_3}} \delta_{J_1 J_2 J_3} \left( \prod_{i=1}^3 \psi_i(J_i) \right) \sum_{\tilde{\rho}} \frac{S^{1/2}_{\lambda_1} S^{1/2}_{\lambda_2} S^{1/2}_{\lambda_3}}{S_{\tilde{\rho}}},
\]

where \( N_{(1)} \) is the number of triples \((J_1,J_2,J_3) \in S_{\lambda_1} \times S_{\lambda_2} \times S_{\lambda_3} \) such that \( J_1 J_2 J_3 = 1 \), and where the matrices \( S^3 \) are those introduced in the previous section.

Next we define a matrix \( \tilde{C} \) with entries

\[
\tilde{C}(\lambda_1,\psi_1),(\lambda_2,\psi_2) := \tilde{N}(\lambda_1,\psi_1),(\lambda_2,\psi_2).\Omega.
\]

One can show that, up to a normalization, this matrix is a conjugation; concretely,

\[
\tilde{C}(\lambda_1,\psi_1),(\lambda_2,\psi_2) = \frac{G^*}{|S_{\lambda_1}|} C^{(\tilde{\lambda})}_{\lambda_1,\psi_1,\lambda_1,\psi_2} \delta_{\tilde{\lambda} \tilde{\lambda}},
\]

where \( C^{(\tilde{\lambda})}_{\lambda_1,\psi_1,\lambda_1,\psi_2} \) is the conjugation on resolved fixed points that has been defined in [12]. In particular, \( \tilde{C} \) is invertible; we define the structure constants of the classifying algebra by using the inverse \( \tilde{C}^{-1} \) as a metric to raise the third index.

In the \( \mathcal{A} \)-theory only the fields in the untwisted sector of the orbifold theory appear; in terms of the torus, one only has the twisting by \( 1 \in G \) in the ‘space’ direction, but projections in the ‘time’ direction. A modular \( S \)-transformation exchanges ‘space’ and ‘time’, thus yielding also the twist sectors; they come without insertion in time direction, so the fields are not projected and we have to consider orbits rather than individual fields. Accordingly, an important tool in the investigation of the classifying algebra \( \mathcal{C}(\mathfrak{A}) \) is a matrix \( \tilde{S} \) whose row index takes values in the set of basis elements of \( \mathcal{C}(\mathfrak{A}) \), while the set for the column indices consists of pairs \( (|\tilde{\rho}|,\psi_{\tilde{\rho}}) \), where \( |\tilde{\rho}| \) is any \( G^* \)-orbit of primary fields of the \( \mathcal{A} \)-theory and \( \psi_{\tilde{\rho}} \) is a character of the (untwisted) stabilizer \( S_{\tilde{\rho}} \) of that orbit. This matrix \( \tilde{S} \) takes over the role that the modular matrix \( S \) of the \( \mathcal{A} \)-theory plays for the \( \mathcal{A} \)-preserving boundary conditions. We emphasize that all orbits of the \( \mathcal{A} \)-theory
appear, not just the ones with vanishing monodromy charge. Explicitly, \( \tilde{S} \) is given by an expression similar to (6),

\[
\tilde{S}(\lambda, \psi, A, (\bar{\rho}, \psi_r)) = \frac{|G^*|}{|S_\lambda| |S_r|} \sum_{J \in S_\rho} \psi_J(J) \psi_r(J)^* S^J_{\lambda, \bar{\rho}}.
\]

One can see that \( \tilde{S} \) is invertible; the inverse is the matrix with entries

\[
(\tilde{S}^{-1})(\lambda, \psi, A, (\bar{\rho}, \psi_r)) = \frac{|S_\lambda|}{|G^*|} \tilde{S}^*_{(\lambda, \psi, A, (\bar{\rho}, \psi_r))}.
\]

In particular \( \tilde{S} \) is a square matrix, which implies the sum rule

\[
\sum_{\lambda \in \mathcal{G}(\lambda) \in \tilde{S}} |S_\lambda| = \sum_{\rho \in \tilde{S}} |S_\rho|.
\]

In words, the number of untwisted fields of the \( \tilde{\mathcal{A}} \)-theory equals the number of all \( G^* \)-orbits of fields when both are counted with multiplicities given by the number of elements in the stabilizer.

Combining the previous formulae one checks that the matrix \( \tilde{S} \) diagonalizes the matrices of structure constants of the classifying algebra. Put differently, the structure constants of \( \mathcal{C}(\tilde{\mathcal{A}}) \) with three lower indices obey the Verlinde-like formula

\[
\tilde{N}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) = \sum_{\rho \in \tilde{S}} \frac{\tilde{S}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_2, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_3, \psi, A, (\bar{\rho}, \psi_r))}{\tilde{S}(\lambda, \psi, A, (\bar{\rho}, \psi_r))}.
\]

Also, the conjugation \( \tilde{C} \) can be expressed through \( \tilde{S} \) as

\[
\tilde{C}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) = \sum_{\rho \in \tilde{S}} \tilde{S}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_2, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_3, \psi, A, (\bar{\rho}, \psi_r)),
\]

so that after raising the third index we have

\[
\tilde{N}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) = \sum_{\rho \in \tilde{S}} \frac{|S_\lambda|}{|G^*|} \frac{\tilde{S}(\lambda_1, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_2, \psi, A, (\bar{\rho}, \psi_r)) \tilde{S}(\lambda_3, \psi, A, (\bar{\rho}, \psi_r))}{\tilde{S}(\lambda, \psi, A, (\bar{\rho}, \psi_r))}.
\]

Together these results imply that the classifying algebra \( \mathcal{C}(\tilde{\mathcal{A}}) \) is commutative and associative, and that the vacuum \( \tilde{\Omega} \) is a unit element. Since \( \mathcal{C}(\tilde{\mathcal{A}}) \) is also endowed with a conjugation \( \tilde{C} \) which is a (weighted) evaluation on the identity, it is semi-simple. It follows in particular that all irreducible \( \mathcal{C}(\tilde{\mathcal{A}}) \)-representations are one-dimensional. They are labelled by the pairs \( (\bar{\rho}, \psi_r) \) and can be neatly expressed in terms of the matrix \( \tilde{S} \):

\[
R((\bar{\rho}, \psi_r))(\phi(\lambda, \psi)) = \frac{\tilde{S}(\lambda, \psi, A, (\bar{\rho}, \psi_r))}{\tilde{S}(\lambda, \psi, A, (\bar{\rho}, \psi_r))}.
\]
To summarize: the conformally invariant boundary conditions preserving \( \mathfrak{A} \) are in one-to-one correspondence with the pairs \( ([\tilde{\rho}], \psi_{[\rho]}) \), and the reflection coefficients for any boundary condition are expressible in terms of the matrix \( \tilde{S} \) as in (17).

5. Automorphism types

As already mentioned, the monodromy charges (3) are constant on \( G^* \)-orbits. This allows us to associate to every \( G^* \)-orbit \( [\lambda] \) of the \( \mathfrak{A}-\)theory a function \( Q_{[\lambda]} : G^* \to \mathbb{C} \) given by

\[
Q_{[\lambda]}(J) = \exp(2\pi i Q_{\lambda}(\lambda)).
\]

The functions \( Q_{[\lambda]} \) are actually characters on \( G^* \), i.e., elements of the character group \((G^*)^*\), which can be naturally identified with the orbifold group, \( Q_{[\lambda]}(G) = G \). Thus we can associate to every boundary condition \( ([\tilde{\rho}], \psi_{[\rho]}) \) an element \( Q_{[\lambda]} \) of the orbifold group. We now show that this group element constitutes the \textit{automorphism type} \([13, 19]\) of the boundary condition. This follows from the fact that for every \( J \in G^* \) we have

\[
R_{([\tilde{\rho}], \psi_{[\rho]})}((\phi_{[\lambda], \psi_{\lambda}})) = \frac{\tilde{S}_{[\lambda], \psi_{\lambda}, [\rho], \psi_{[\rho]}}}{\tilde{S}_{[\lambda], ([\rho], \psi_{[\rho]})}} = Q_{[\lambda]}(J) \frac{\tilde{S}_{[\lambda], \psi_{\lambda}, ([\rho], \psi_{[\rho]})}}{\tilde{S}_{[\lambda], ([\rho], \psi_{[\rho]})}} = Q_{[\lambda]}(J) R_{([\tilde{\rho}], \psi_{[\rho]})}((\phi_{[\lambda], \psi_{\lambda}})).
\]

In particular, the boundary blocks for fields on full orbits contribute to the boundary states with a relative phase \( Q_{[\lambda]}(J) \), given by the value of the character \( J \in G^* \) on the group element \( Q_{[\lambda]} \), which can be expressed by saying that the reflection of a bulk field at the boundary is twisted by the action of the group element \( Q_{[\lambda]} \in G \). It follows that indeed to any boundary condition one can associate an automorphism of \( \mathfrak{A} \), namely the one that multiplies the subspace \( \mathcal{H}_{[\lambda]} \subseteq \mathcal{H} \) by \( Q_{[\lambda]}(J) \). We stress that this statement arises as a \textit{result} of our analysis rather than being an ad hoc input.

Also note that twisted boundary conditions in the \( \mathfrak{A} \)-theory are in a natural correspondence with the twist sectors of the orbifold theory \( \mathfrak{A} \). (In other words, boundary operators which change the automorphism type correspond to the twist fields of the orbifold.) By taking appropriate ideals of \( C(\mathfrak{A}) \), one can associate an individual classifying algebra \( C_Q(\mathfrak{A}) \) to each automorphism type \( Q \in G \). In particular, for the trivial automorphism type \( 1 \in G \) one recovers the fusion algebra of \( \mathfrak{A} \). Individual classifying algebras for non-trivial automorphism types were discussed in [13]; they were used in [6] to classify all boundary conditions of the critical three-state Potts model and to discuss the boundary conditions for other minimal and WZW models with extension modular invariants. In the special case of the \( D_{\text{even}} \) (i.e., \( \mathbb{Z}_2 \)-extension) type \( \mathfrak{sl}(2) \) WZW theories at level \( k \in 4\mathbb{Z} \), there is a simple closed formula for the matrix \( \tilde{S} \), and the classifying algebra for non-trivial automorphism type can be shown to be isomorphic to the fusion algebra of the \((\frac{k}{2}+1, 2)\) non-unitary Virasoro minimal models. Incidentally, in this particular case we can also show that the total classifying algebra \( C(\mathfrak{A}) \) is isomorphic to the Pasquier [24–26] algebra, even though the natural basis for \( C(\mathfrak{A}) \) arising here differs from the one used in Pasquier’s context (where the diagonalising matrix is taken to be unitary). It follows in particular that, as also advocated in [7, 27], the conformally invariant boundary conditions of the unitary Virasoro minimal models are controlled by the representation theory of a semi-simple classifying algebra.

\footnote{To be precise, at least those which do not correspond to complex Chan–Paton charges, compare [22].}
6. Annulus coefficients

Now that we know the reflection coefficients, we would like to compute the annulus amplitudes. They are linear combinations of characters. One can show that for an annulus with boundary conditions \((\overline{p}_1, \psi_1)\) and \((\overline{p}_2, \psi_2)\), the characters that appear are those of an integer spin simple current extension of the \(\hat{S}\)-theory by the subgroup

\[
H' \equiv H'_{\rho_1, \rho_2} := \{ J \in G^* \mid Q_3(\rho_1) = 0 = Q_3(\rho_2) \}
\]

of \(G^*\). The characters of the primary fields in this extension are

\[
\chi'_{(\overline{p}', \psi')} := \sum_{J \in H'/S'_\rho} \chi_{(\overline{p}, \psi)} = \frac{1}{|S'_\rho|} \sum_{J \in H'} \chi_{(\overline{p}, \psi)},
\]

where we introduced \(S'_\rho := S_\rho \cap H'\) and where \(\psi' \in (S'_\rho)^*\).

We would like to know the annulus amplitude in the open string channel. To this end we have to identify the modular matrix that implements the transformation of the characters (21) under \(\tau \mapsto -1/\tau\); this is the modular matrix \(S'\) of the \(H'\)-extension as constructed in [12] (compare also section 3). Afterwards we define the annulus coefficients as the multiplicities of the characters of the \(H'\)-extension in the annulus amplitude. We find that

\[
A_{(\overline{p}_1, \psi_1) (\overline{p}_2, \psi_2)}(t) = \sum_{\overline{p}'} \sum_{\psi' \in (S'_\lambda)^*} A_{(\overline{p}', \psi')}^{(\overline{p}_1, \psi_1)(\overline{p}_2, \psi_2)} \chi'^{(\overline{p}', \psi')} \left( \frac{H'}{G^*} \right)
\]

with

\[
A_{(\overline{p}', \psi')}^{(\overline{p}_1, \psi_1)(\overline{p}_2, \psi_2)} = \sum_{Q(\lambda) = 0} \sum_{\psi_\lambda \in S_\lambda^*} \left( \tilde{S}_{(\lambda, \psi_\lambda), (\overline{p}_1, \psi_1)} \tilde{S}_{(\overline{p}_1, \psi_1), (\overline{p}_1, \psi_1)} \right)^* \cdot \left( \tilde{S}_{(\lambda, \psi_\lambda), (\overline{p}_2, \psi_2)} \tilde{S}_{(\overline{p}_1, \psi_1), (\overline{p}_1, \psi_1)} \right)
\]

\[
\cdot \frac{|S_\lambda|}{|G^*|} \frac{1}{S'_{(\overline{p}', \psi_\lambda')(\overline{p}_1, \psi_1)}} S'_{(\overline{p}', \psi_\lambda')(\overline{p}_1, \psi_1)}
\]

\[
= \sum_{Q(\lambda) = 0} \sum_{\psi_\lambda \in S_\lambda^*} \left[ \frac{|S_\lambda|}{|G^*|} \frac{\tilde{S}_{(\lambda, \psi_\lambda), (\overline{p}_1, \psi_1)} \tilde{S}_{(\overline{p}_1, \psi_1), (\overline{p}_1, \psi_1)} S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}}{S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}} \right].
\]

Here the two factors on the right hand side of the first line are products of reflection coefficients and normalizations of vacuum boundary fields, while the factors in the second line come from the normalization of the Ishibashi boundary states and from the modular transformation of the characters \(\chi_{(\lambda, \psi_\lambda)}\), respectively. \(\psi_\lambda' \in (S'_\lambda)^*\) is the restriction of the \(S_\lambda\)-character \(\psi_\lambda\) to \(S'_\lambda\).

A crucial property of the annulus multiplicities (23) is that they are non-negative integers; this is required in order to have an interpretation of the annulus amplitude as a partition function. Indeed, up to a factor one can write the numbers (23) as a sum of fusion rule coefficients \(N'\) in the \(H'\)-extension,

\[
A_{(\overline{p}', \psi')}^{(\overline{p}_1, \psi_1)(\overline{p}_2, \psi_2)} = \frac{|H'|}{|G^*|} \frac{|S'_\rho|}{|S_{\rho_1}| \cdot |S_{\rho_2}|} \sum_{J \in G^*/H'} \frac{N'}{(\overline{p}', \psi_\lambda')(\overline{p}_1, \psi_1)} \cdot J_{(\overline{p}_1, \psi_1), (\overline{p}_2, \psi_2)}
\]

\[
\cdot \sum_{J \in H'/S_\rho} \chi'^{(\overline{p}', \psi')} \left( \frac{H'}{G^*} \right)
\]

\[
\cdot \frac{|S_\lambda|}{|G^*|} \frac{1}{S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}} S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}
\]

\[
\cdot \left[ \frac{|S_\lambda|}{|G^*|} \frac{\tilde{S}_{(\lambda, \psi_\lambda), (\overline{p}_1, \psi_1)} \tilde{S}_{(\overline{p}_1, \psi_1), (\overline{p}_1, \psi_1)} S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}}{S'_{(\overline{p}', \psi_\lambda'), (\overline{p}_1, \psi_1)}} \right].
\]
where $H'' := S_{\rho_1} \cdot S_{\rho_2} \cdot S_\sigma \cdot H'_{\rho_1 \rho_2}$; the prefactor can be shown to be integral (for details, see [28]). When both boundary conditions preserve the full bulk symmetry, formula (24) reduces to the well-known result that for such boundary conditions the annulus multiplicities just coincide with fusion rule coefficients of the $\mathcal{A}$-theory.

It can also be checked that the annulus multiplicities fulfil further consistency relations of the usual form. These look most transparent if one works with $\tilde{\mathcal{A}}$-characters $\chi_{(\bar{s}, \psi)}$ in place of the extended characters $\chi'(n, \psi)$ of equation (21). It turns out that the corresponding coefficients $\tilde{\Lambda}$ in the annulus amplitude depend only on the $G^*$-orbit of $\bar{s}$ and are given by

$$\tilde{\Lambda}[[\bar{s}], \psi] = \frac{S'_1}{S_2} \Lambda[[\bar{s}], \psi'], \quad (25)$$

($\psi'$ denotes again the restriction of $\psi$ to $S'_\sigma$). We then find that, first, the coefficients $\tilde{\Lambda}$ furnish a matrix representation of some algebra,

$$\sum_{[\bar{s}_1]} \sum_{\psi_1 \in S_{\psi_1}} \tilde{\Lambda}[[\bar{s}_1], \psi_1] \tilde{\Lambda}[[\bar{s}_2], \psi_1] = \sum_{[\bar{s}_2]} \sum_{\psi_2 \in S_{\psi_2}} \tilde{\Lambda}[[\bar{s}_2], \psi_2] \tilde{\Lambda}[[\bar{s}_1], \psi_2], \quad (26)$$

with the structure constants of that algebra again given by the $\tilde{\Lambda}$,

$$\tilde{M}[[\bar{s}_1], \psi_1], [[\bar{s}_2], \psi_2] = \tilde{\Lambda}[[\bar{s}_1], \psi_1, \psi_2], \quad (27)$$

(In particular, according to (24) up to a prefactor the structure constants $\tilde{M}$ are nothing but sums of fusion rule coefficients of the $H'$-extension of the $\tilde{\mathcal{A}}$-theory.) Note that the algebra with structure constants $\tilde{M}$ involves orbits $[\bar{s}]$ of arbitrary monodromy charge; the monodromy charge actually provides a grading of the algebra, with the grade-zero subalgebra being just the fusion algebra of the $\tilde{\mathcal{A}}$-theory.

Second, the annulus coefficients are ‘associative’ in the sense that

$$\sum_{[\bar{s}]} \sum_{\psi \in S_{\psi}} \tilde{\Lambda}[[\bar{s}], \psi] \tilde{\Lambda}[[\bar{s}^+, \psi^+]] = \sum_{[\bar{s}]} \sum_{\psi \in S_{\psi}} \tilde{\Lambda}[[\bar{s}], \psi] \tilde{\Lambda}[[\bar{s}^+, \psi^+]], \quad (28)$$

In view of (27), the two identities (28) and (26) are merely different manifestations of one and the same relationship.

Finally we mention that, as seen by comparing the result (23) with formula (16) for the structure constants $N$, up to a factor the annulus coefficients are the ‘opposite structure constants’ for $\mathcal{C}(\tilde{\mathcal{A}})$, i.e. those obtained when summing over the other index of the non-symmetric diagonalizing matrix $\tilde{S}$.

7. Outlook

To conclude this letter we summarize the structure we found and then speculate about possible generalizations of this structure. We have seen that if we require boundary conditions to preserve only the symmetries in an abelian orbifold subalgebra of the chiral
algebra, then the boundary conditions can be obtained with the help of a natural classifying algebra. Moreover, using structures in the corresponding orbifold theory, we could derive rather than assume that each boundary condition comes with a specific automorphism type. The Chan-Paton types [13] for a given automorphism type correspond to simple current orbits in the relevant twist sector of the orbifold theory.

It is reasonable to expect that these features will persist for orbifold subalgebras under a non-abelian group $G$. For a general consistent subalgebra $\mathfrak{A}$ of $\mathfrak{A}$ which is not given as an orbifold subalgebra, we expect that a classifying algebra can be determined once the following two pieces of information are available:
- the decomposition of $\mathfrak{A}$-modules in terms of irreducible $\mathfrak{A}$-modules;
- an expression of the chiral blocks of the $\mathfrak{A}$-theory in terms of linear combinations of quotient sheaves of the sheaves of chiral blocks of the $\mathfrak{A}$-theory.

Another insight is that for any inclusion $\mathfrak{A} \hookrightarrow \mathfrak{A}$ of preserved bulk symmetry algebras, we have a projection of the corresponding classifying algebras: the classifying algebra for $\mathfrak{A}$ is a quotient of the one for $\mathfrak{A}$. Thus the following picture emerges: the set $\mathcal{M}$ of all consistent subalgebras of a given chiral algebra $\mathfrak{A}$ is partially ordered by inclusion. It is reasonable to expect that it is even an inductive system, i.e. given any two consistent subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, one can find a consistent subalgebra $\mathfrak{A}_3$ that is contained in their intersection, $\mathfrak{A}_3 \subset \mathfrak{A}_1 \cap \mathfrak{A}_2$. Assuming that also in general for $\mathfrak{A}_1 \subset \mathfrak{A}_2$ the classifying algebra for $\mathfrak{A}_2$ is a quotient of the one for $\mathfrak{A}_1$, we will obtain a projective system of classifying algebras. Taking the projective limit over this system, we obtain a universal classifying algebra which gives all conformally invariant boundary conditions. This universal classifying algebra can be explicitly displayed in simple cases, e.g. for the free boson compactified on a circle or for the $\mathbb{Z}_2$-orbifold of these theories. We are planning to come back to a detailed study of this algebra in the future.

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