Bifurcation of periodic solutions from a ring configuration in the vortex and filament problems

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Abstract

This paper gives an analysis of the movement of \( n + 1 \) almost parallel filaments or vortices. Starting from a polygonal equilibrium of \( n \) vortices with equal circulation and one vortex at the center of the polygon, we find bifurcation of periodic solutions. The bifurcation result makes use of the orthogonal degree in order to prove global bifurcation of periodic solutions depending on the circulation of the central vortex. In the case of the filament problem these solutions are periodic traveling waves.

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1 Introduction

Consider \( n \) point vortices or \( n \) almost parallel filaments turning at a constant speed in a plane around some central point. A relative equilibrium of this configuration is a stationary solution of the equations in the rotating coordinates.

In this paper, we give a complete study of the polygonal relative equilibrium where there are \( n \) identical point vortices arranged on a regular polygon and a central vortex, with a possibly different circulation. For this polygonal equilibrium, we give a full analysis for the bifurcation of periodic solutions. For the filament problem, we prove that there is a global bifurcation of periodic solutions of waves traveling in the vertical direction. In this problem, a study of periodic solutions in the vertical direction would lead to a small divisors setting.

There has been a renewed interest in point vortex problems in the last 30 years, as a model for fluid mechanics. We refer to [11], for an up-to-date study, or [10] for a more general reference. There are many papers on relative equilibria and some on application of KAM theory but few on periodic solutions. In the
case of nearly parallel filaments, we shall use the model proposed by Klein, Majda and Damodaran, see [8] and [7].

The linearization of the system at a critical point is a $n \times n$ matrix, which is non invertible, due to the rotational symmetry. These facts imply that the study of the spectrum of the linearization is not an easy task and that the classical bifurcation results for periodic solutions may not be applied directly. However, we shall use the change of variables proved in our previous paper, [4], in order to give not only this spectrum but also the consequences for the symmetries of the solutions.

The present paper is a continuation of [4], where we had a complete study of the bifurcation of relative equilibria. See also [9]. Thus, we shall use the results in that paper, but we shall recall all the important notions. In a parallel paper, [3], we study a similar problem for point masses. Although there are many similarities, in particular in the change of variables, the results are of a quite different nature.

The next section is devoted to the mathematical setting of the problem, with the symmetries involved. Then, we give, in the following two sections, the preliminary results needed in order to apply the orthogonal degree theory developed in [6], that is the global Liapunov-Schmidt reduction, the study of the irreducible representations, with the change of variables of [4], and the symmetries associated to these representations. In the next section, we prove our bifurcation results and, in the following section, we give the analysis of the spectrum, with the complete results on the type of solutions which bifurcate from the relative equilibrium, in the vortex and the filament cases.

## 2 Setting the problem

Let us denote by $q_j(t,s) \in \mathbb{R}^2$ the position of the $j$’th filament, where $s$ represents the vertical axis. Let us suppose that one filament has circulation $\kappa_0 = \mu$ and $n$ filaments have circulation $\kappa_j = 1$, for $j \in \{1, ..., n\}$. The dimensionless equations of $n+1$ almost parallel filaments in rotating coordinates, $u_j(t) = e^{-\omega J t} q_j(t)$, are given by

$$\kappa_j J \partial_t u_j + \kappa_j^2 \partial_{ss} u_j = \omega \kappa_j u_j - \sum_{i=0(i \neq j)}^{n} \kappa_i \kappa_j \frac{u_j - u_i}{\|u_j - u_i\|^2}$$

where $J$ is the canonical symplectic matrix.

Define the vector $u = (u_0, u_1, ..., u_n)^T$, the matrix of circulations $\mathcal{K} = \text{diag}(\mu I, I, ..., I)$ and the symplectic matrix $\mathcal{J} = \text{diag}(J, J, ..., J)$. Then, the equations of the filaments, in vectorial form, can be written as

$$\mathcal{K} \mathcal{J} u_t + \mathcal{K}^2 u_{ss} = \nabla V(u) \quad \text{with}$$

$$V(u) = \frac{\omega}{2} (u^T \mathcal{K} u) - \sum_{i < j} \kappa_i \kappa_j \ln(\|u_j - u_i\|).$$
The previous equations contain, as a particular case when the filaments $u(t, s)$ are constant on the vertical axis $s$, the vortex problem given by the equations

$$KJ \dot{u} = \nabla V(u).$$

(1)

In this paper we analyze two problems: The bifurcation of periodic solutions for the vortex problem, and bifurcation of traveling waves for the filament problem. In order to find traveling waves, one sets $u(t, s) = u(\gamma t + s)$, then the traveling waves become solutions of the ordinary differential equation

$$K^2 \ddot{u} + \gamma KJ \dot{u} = \nabla V(u).$$

(2)

Now, the critical points of the potential $V$ correspond to relative equilibria of the problem. Actually, the polygonal configuration

$$\bar{a} = (0, e^{i\zeta}, ..., e^{i(n-1)\zeta})$$

is a relative equilibrium when $\omega = s_1 + \mu$, with $s_1 = (n - 1)/2$. This fact is proven in [4], where the bifurcation of relative equilibria is analyzed using $\mu$ as a parameter.

**Remark 1** If, as we have done in [3], one replaces, in the change of coordinates, the term $e^{\omega \lambda}J$ with a complex factor $\phi(t)$ (taking $q_j$ and $u_j$ as complex functions instead of a planar vector), where $\phi$ satisfies the equation

$$i\phi' = -\omega \phi/|\phi|^2,$$

then the equations, for the vortex problem, become

$$|\phi|^2 K \dot{u} = \nabla V(u).$$

In particular, the stationary solutions of this system are the same solutions studied in [3]. However, the solutions of the equation for $\phi$ are $ce^{i\nu t}$, with $\nu = \omega/|c^2|$, that is circular orbits only. Thus, the vortex problem differs from the masses problem, where a similar argument gives all possible conical orbits.

In this paper we shall prove bifurcation of periodic solutions from the polygonal equilibrium $\bar{a}$. This is an analogous treatment to the bifurcation of periodic solutions for the $(n + 1)$-body problem in [3].

Changing variables by $x(t) = u(t/\nu)$, the $2\pi/\nu$-periodic solutions of the differential equation for the filament become zeros of the bifurcation operator

$$f : H^2_{2\pi}(\mathbb{R}^{2(n+1)} \setminus \Psi) \rightarrow L^2_{2\pi}$$

$$f(x, \nu) = -\nu^2 K^2 \ddot{x} - \gamma \nu KJ \dot{x} + \nabla V(x),$$

where the set $\Psi = \{x \in \mathbb{R}^{2(n+1)} : x_i = x_j\}$ consists of the collision points, and the set

$$H^2_{2\pi}(\mathbb{R}^{2(n+1)} \setminus \Psi) = \{x \in H^2_{2\pi}(\mathbb{R}^{2(n+1)}) : x_i(t) \neq x_j(t)\}$$

consists of the collision-free orbits.
Remark 2 We shall concentrate the analysis on the filament problem. All the following statements apply to the vortex problem, except that one has the bifurcation operator \( f(x) = -\nu K J \dot{x} + \nabla V(x) \) defined in the space \( H^1_{2\pi}(\mathbb{R}^{2(n+1)} \setminus \Psi) \).

Definition 3 Let \( S_n \) be the group of permutations of \( \{1, \ldots, n\} \). One defines the action of \( S_n \) in \( \mathbb{R}^{2(n+1)} \) as

\[
\rho(\gamma)(x_0, x_1, \ldots, x_n) = (x_{\gamma(1)}, \ldots, x_{\gamma(n)}).
\]

The gradient \( \nabla V \) is \( S_n \)-equivariant, that is it commutes with the action of the group, because \( n \) vortices have the same circulation. Let \( \mathbb{Z}_n \) be the subgroup of permutations generated by \( \zeta(j) = j + 1 \) modulus \( n \), then the map \( f \) is \( \Gamma \times S^1 \)-equivariant with the abelian group

\[
\Gamma = \mathbb{Z}_n \times S^1.
\]

Now, the infinitesimal generators of \( S^1 \) and \( \Gamma \) are

\[
A x = \frac{\partial}{\partial \varphi} |_{\varphi=0} x(t+\varphi) = \dot{x} \text{ and } A_1 x = \frac{\partial}{\partial \theta} |_{\theta=0} e^{-J \theta} x = -J x.
\]

Since \( V \) is \( \Gamma \)-invariant, then the gradient \( \nabla V(x) \) must be orthogonal to the generator \( A_1 x \). As a consequence, the map \( f \) must be \( \Gamma \times S^1 \)-orthogonal, due to the equalities

\[
\langle f(x), \dot{x} \rangle_{L^2_0} = -\frac{\nu^2}{2} \| K \dot{x} \|^2_{L^2_0} - \gamma \nu \sum \kappa_j \langle J \dot{x}_j, \dot{x}_j \rangle_{L^2_{2\pi}} + V(x) \| \dot{x} \|^2_{L^2_0} = 0,
\]

\[
\langle f(x), J x \rangle_{L^2_{2\pi}} = \nu^2 \langle K \dot{x}, J K \dot{x} \rangle_{L^2_{2\pi}} - \frac{\gamma \nu}{2} \sum \| x_j \|^2_{L^2_0} + \int_0^{2\pi} \langle \nabla V, J x \rangle = 0.
\]

Define \( \mathbb{Z}_n \) as the subgroup of \( \Gamma \) generated by \( (\zeta, \zeta) \in \mathbb{Z}_n \times S^1 \) with \( \zeta = 2\pi/n \in S^1 \). Since the action of \( (\zeta, \zeta) \) leaves fixed the equilibrium \( \bar{a} \), then the isotropy group of \( \bar{a} \) is the group \( \Gamma_{\bar{a}} \times S^1 \) with

\[
\Gamma_{\bar{a}} = \mathbb{Z}_n.
\]

Thus, the orbit of \( \bar{a} \) is isomorphic to the group \( S^1 \). In fact, the orbit consists of the rotations of the equilibrium. As a consequence, the generator of the orbit \( A_1 \bar{a} = -J \bar{a} \) must be in the kernel of \( D^2 f(\bar{a}) \).

3 The Liapunov-Schmidt reduction

In order to apply the orthogonal degree of \([6]\), one needs to make a reduction of the bifurcation map to some finite space.

The bifurcation map \( f \) has Fourier series

\[
f(x) = \sum_{l \in \mathbb{Z}} (l^2 \nu^2 K^2 x_l - \gamma \nu (iJ) K x_l + gl) e^{ilt},
\]

4
where $x_l$ and $g_l$ are the Fourier modes of $x$ and $\nabla V(x)$. Since $l^2 \nu^2 K^2 - \gamma i l \nu (i J) K$ is invertible for all big $l$'s, then one may solve $x_l$ for $|l| > p$ from

$$l^2 \nu^2 K^2 x_l - \gamma l \nu (i J) K x_l + g_l = 0.$$ 

Actually, one may use the global implicit function theorem to perform the reduction globally.

In this way, the bifurcation operator $f$ has the same zeros as the bifurcation function

$$f(x_1, x_2(x_1, \nu), \nu) = \sum_{|l| \leq p} (l^2 \nu^2 K^2 x_l - \gamma l \nu (i J) K x_l + g_l) e^{ilt},$$

and the linearization of the bifurcation function at some equilibrium $\bar{a}$ is

$$f'(\bar{a}) x_1 = \sum_{|l| \leq p} (l^2 \nu^2 K^2 - \gamma l \nu (i J) K + D^2 V(\bar{a})) x_l e^{ilt}.$$ 

Since the bifurcation operator is real, the linearization of the bifurcation function is determined by blocks $M(l \nu)$ for $l \in \{0, \ldots, p\}$, where $M(\nu)$ is the matrix

$$M(\nu) = \nu^2 K^2 - \gamma \nu (i J) K + D^2 V(\bar{a}).$$

These blocks $M(l \nu)$ represent the Fourier modes of the linearized equation at the equilibrium.

**Remark 4** For the vortex problem one may prove a similar statement and get the Fourier modes

$$M(\nu) = -\nu (i J) K + D^2 V(\bar{a}).$$

**Remark 5** Actually, the orthogonal degree could be used to analyze the bifurcation of periodic solutions, in time and spatial $z$-coordinate, for the filaments. However, the Liapunov-Schmidt reduction cannot be performed without solving a small divisor problem.

## 4 Irreducible representations

In order to apply the orthogonal degree, one needs to find the irreducible representation subspaces for the action of $\Gamma_{\bar{a}} = \tilde{Z}_n$.

In the following sections we shall assume that $n > 2$, since the case $n = 2$ is different and will be treated at the end of the paper.

**Definition 6** For $k \in \{2, \ldots, n - 2, n\}$, we define the isomorphisms $T_k : \mathbb{C}^2 \to W_k$ as

$$T_k(w) = (0, n^{-1/2} e^{(ikl+J)\zeta} w, \ldots, n^{-1/2} e^{n(ikl+J)\zeta} w)$$

with $W_k = \{(0, e^{(ikl+J)\zeta} w, \ldots, e^{n(ikl+J)\zeta} w) : w \in \mathbb{C}^2\}$. 
For $k \in \{1, n-1\}$, we define the isomorphism $T_k : \mathbb{C}^3 \to W_k$ as

$$T_k(\alpha, w) = (v_k \alpha, n^{-1/2}e^{(ikI+J)\zeta}w, ..., n^{-1/2}e^{n(ikI+J)\zeta}w)$$

with

$$W_k = \{(v_k \alpha, e^{(ikI+J)\zeta}w, ..., e^{n(ikI+J)\zeta}w) : \alpha \in \mathbb{C}, w \in \mathbb{C}^2 \},$$

where $v_1$ and $v_{n-1}$ are the vectors

$$v_1 = 2^{-1/2}(1, i) \text{ and } v_{n-1} = 2^{-1/2}(1, -i).$$

In the paper [4], we have proven that the subspaces $W_k$ are irreducible representations. Also, we showed that the action of $(\zeta, \zeta) \in \tilde{Z}_n$ on the space $W_k$ is given by

$$\rho(\zeta, \zeta) = e^{ik\zeta}.$$

Since the subspaces $W_k$ are orthogonal, then the linear map

$$Pw = \sum_{j=1}^{n} T(w_k)$$

is orthogonal, where $w = (w_1, ..., w_n)$, with $w_k \in \mathbb{C}^3$ for $k = 1, n-1$ and $w_k \in \mathbb{C}^2$ for the other $k$'s.

Since the map $P$ rearranges the coordinates of the irreducible representations, one has, from Schur’s lemma, that

$$P^{-1}D^2V(\bar{a})P = \text{diag}(B_1, ..., B_n),$$

where $B_k$ are matrices which satisfy $D^2V(\bar{a})T_k(w) = T_k(B_kw)$. In the paper [4], we have found the blocks $B_k$: they satisfy $B_{n-k} = \bar{B}_k$ and

$$B_k = \text{diag}(2(\mu + s_1) - s_k, s_k) \text{ for } k \in \{2, ..., n-2, n\}, 	ext{ and}$$

$$B_1 = \begin{pmatrix}
\mu (s_1 + \mu) & -(n/2)^{1/2} \mu & -(n/2)^{1/2} \mu i \\
-(n/2)^{1/2} \mu & s_1 + 2\mu & 0 \\
(n/2)^{1/2} \mu i & 0 & s_1
\end{pmatrix},$$

where

$$s_k = k(n-k)/2.$$

For the linearization of the equation one has that

$$P^{-1}M(\nu)P = \text{diag}(m_1(\nu), ..., m_n(\nu)).$$

Thus, we find the matrices $m_k(\nu)$ in terms of the blocks $B_k$.

**Proposition 7** The matrices $m_k(\nu)$ satisfy $m_k(\nu) = \bar{m}_{n-k}(-\nu)$ with

$$m_k(\nu) = \nu^2 I - 2\gamma \nu (iJ) + B_k \text{ for } k \in \{2, ..., n-2, n\}, \text{ and}$$

$$m_1(\nu) = \nu^2 \text{diag}(\mu^2, I) - 2\gamma \nu \text{diag}(\mu, iJ) + B_1.$$
Proof. For \( k \in \{2, ..., n - 2, n\} \), the matrix \( K \) in the space \( W_k \) is \( KT_k(z) = T_k(z) \), and the matrix \( iJ \) satisfies \( JT_k(z) = T_k(Jz) \). The matrix \( K \) in \( W_1 \) is \( KT_1(z) = T_1(diag(\mu, 1, 1)z) \), and since \( (iJ)v_1 = v_1 \), then the matrix \( iJ \) satisfies \( (iJ)T_1(z) = T_1(diag(1, iJ)z) \). From these facts we conclude the statements.

Finally, using that \( (iJ)v_2 = v_2 \), one has

\[
m_{n-1}(\nu) = \nu^2 diag(\mu^2, I) - 2\gamma \nu diag(-\mu, iJ) + B_{n-1}.
\]

Then, from the equality \( B_{n-k} = \tilde{B}_k \), one obtains the equality \( m_{n-k}(\nu) = \tilde{m}_k(-\nu) \).

Remark 8 Analogously, for the vortex problem one has that \( m_k(\nu) = \tilde{m}_{n-k}(-\nu) \) with

\[
m_k(\nu) = -\nu(iJ) + B_k \quad \text{for} \quad k \in \{2, ..., n - 2, n\}, \quad \text{and}
\]

\[
m_1(\nu) = -\nu diag(\mu, iJ) + B_1.
\]

The action of \( (\zeta, \zeta, \varphi) \in \tilde{Z}_n \times S^1 \) on \( W_k \) is \( \rho(\zeta, \zeta, \varphi) = e^{ik\zeta}e^{i\varphi} \). Therefore, the isotropy group of the space \( W_k \) is

\[
Z_n(k) = \langle (\zeta, \zeta, -k\zeta) \rangle.
\]

5 Bifurcation theorem

The fixed point subspace of the isotropy group \( \Gamma_\alpha \times S^1 \) corresponds to the block \( m_n(0) = B_n \). Since the generator of the kernel is \( A_1 \alpha = T_n(-n^{1/2}e_2) \), then \( e_2 \) must be in the kernel of \( m_n(0) \).

Following [6], one defines \( \sigma \) to be the sign of \( m_n(0) \) in the orthogonal subspace to \( e_2 \). Since \( B_n = 2 diag(\omega, 0) \), then

\[
\sigma = \text{sgn}(e_1^T B_n e_1) = \text{sgn}(\omega).
\]

We have proven, in [4], that \( m_k(0) = B_k \) is invertible except for a point \( \mu_k \) for \( k = 1, ..., n - 1 \): \( \mu_k = s_k/2 - s_1 \) for \( k = 2, ..., [n/2] \) and \( \mu_1 = s_1^2 \). Therefore, the hypotheses of [6] apply for \( \mu \neq \mu_k \). In fact, in [4], we proved a global bifurcation of stationary solutions from each \( \mu_k \). See also [9].

Definition 9 Following [6], we define

\[
\eta_k(\nu_0) = \sigma \{ n_k(\nu_0 - \rho) - n_k(\nu_0 + \rho) \},
\]

where \( n_k(\nu) \) is the Morse index of \( m_k(\nu) \).

This number corresponds to the jump of the orthogonal index at \( \nu_0 \). Then, from the results of [6], we can state the following theorem for \( \mu \neq \mu_1, ..., \mu_{n-1} \).

The orthogonal degree is defined for orthogonal maps that are non-zero on the boundary of some open bounded invariant set. The degree is made
of integers, one for each orbit type, and it has all the properties of the usual
Brouwer degree. Hence, if one of the integers is non-zero, then the map has
a zero corresponding to the orbit type of that integer. In addition, the degree
is invariant under orthogonal deformations that are non-zero on the boundary.
The degree has other properties such as sum, products and suspensions, for
instance, the degree of two pieces of the set is the sum of the degrees.

Now, if one has an isolated orbit, then its linearization at one point of the
orbit \( x_0 \) has a block diagonal structure, due to Schur’s lemma, where the isotropy
subgroup of \( x_0 \) acts as \( \mathbb{Z}_n \) or as \( \mathbb{S}_1 \). Therefore, the orthogonal index of the orbit
is given by the signs of the determinants of the submatrices where the action
is as \( \mathbb{Z}_n \), for \( n = 1 \) and \( n = 2 \), and the Morse indices of the submatrices
where the action is as \( \mathbb{S}_1 \). In particular, for problems with a parameter, if
the orthogonal index changes at some value of the parameter, one will have
bifurcation of solutions with the corresponding orbit type. Here, the parameter
is the frequency \( \nu \).

**Theorem 10** If \( \eta_k(\nu_k) \) is different from zero, then the polygonal equilibrium
has a global bifurcation of periodic solutions from \( 2\pi/\nu \) with isotropy group
\( \tilde{\mathbb{Z}}_n(k) \).

The solutions with isotropy group \( \tilde{\mathbb{Z}}_n(k) \) must satisfy the symmetries
\[
u_j(t) = e^{-it\zeta}u_{j_0}(t - k\zeta).
\]

For the \( n \) elements with equal circulation, if we use the notation \( u_j = u_{j+kn} \) for
\( j \in \{1, \ldots, n\} \), then \( \zeta(j) = j + 1 \), and the \( n \) elements satisfy
\[
u_{j+1}(t) = e^{ij\zeta}u_1(t + jk\zeta).
\]

On the other hand, the central element remains at the origin if \( k \) and \( n \) have a
common factor and, if they are relatively prime, then \( u_0(t) = e^{ik^{-1}\zeta}u_0(t + \zeta) \),
where \( k^{-1} \) is such that \( k^{-1}k = 1 \), modulo \( n \).

By global bifurcation, we mean that the branch goes to infinity in norm
or period, or goes to the collision set (in these three cases, we say that the
bifurcation is non-admissible) or, if none of the above happens, then the sum
of the above jumps, over all the bifurcation points, is zero. See [3] for this kind of
arguments.

A complete description of these solutions may be found in the paper [3] and
in [2].

### 6 Spectral analysis

The \( m_k(\nu) \) have real eigenvalues, so the matrices \( m_{n-k}(\nu) \) and \( m_k(-\nu) \) have
the same spectrum due to the equality \( m_{n-k}(\nu) = m_k(\nu) \). As a consequence,
the Morse numbers satisfy
\[
n_{n-k}(\nu) = n_k(-\nu).
\]

In order to analyze easily the spectrum of these problems, one may use the
parameter \( \mu \) or equivalently \( \omega = \mu + s_1 \).
6.1 Vortex problem

6.1.1 Blocks \( k \in \{2, ..., n - 2, n\} \)

In this case, the blocks are given by

\[ B_k = 2 \text{diag} (\omega - \omega_k, \omega_k), \]

with \( \omega_k = s_k/2 \).

Proposition 11 Define \( \nu_k \) as

\[ \nu_k = \left[ 4 \omega_k (\omega - \omega_k) \right]^{1/2}. \]

For \( \omega > \omega_k \), the matrix \( m_k(\nu) \) changes its Morse index at the positive value \( \nu_k \) with

\[ \eta_k(\nu_k) = -1. \]

For \( \omega < \omega_k \), the matrix \( m_k(\nu) \) is always invertible.

Proof. The determinant of \( m_k(\nu) \) is \( d_k(\nu) = -\nu^2 + 4 \omega_k (\omega - \omega_k) \), and it changes sign only at \( \pm \nu_k \) for \( \omega > \omega_k \). The trace of \( m_k(\nu) \) is \( T_k(\nu) = 2 \omega \), and \( \omega_k \) is positive, then \( T_k(0) > 0 \) for \( \omega > \omega_k \). As a consequence, one has that \( n_k(0) = 0 \) and \( n_k(\infty) = 1 \), because \( d_k(0) > 0 \) and \( d_k(\infty) < 0 \). Therefore, \( \eta(\nu_k) = 0 - 1 \), since \( \sigma = 1 \), for \( \omega > \omega_k \).

Since \( \omega_n = 0 \), then the value \( \nu_n = 0 \) is not a bifurcation point for \( k = n \).

Theorem 12 For each \( k \in \{2, ..., n - 2\} \) and \( \omega > \omega_k \), the polygonal equilibrium has a global bifurcation of periodic solutions from \( 2\pi/\nu_k \), with symmetries \( \tilde{Z}_{n-k} \).

Since the bifurcation points have the same sign \( \eta_k = -1 \), then these branches cannot return to the same equilibrium. Therefore, the bifurcation branch is non-admissible or goes to another equilibrium.

Blocks \( k \in \{1, n - 1\} \)

Due to the equality \( n_{n-1}(\nu) = n_1(-\nu) \), one may analyze the spectrum of the block \( m_1(\nu) \) in \( \mathbb{R}^+ \), instead of the spectrum of \( m_{n-1}(\nu) \) in \( \mathbb{R}^+ \).

Proposition 13 The matrix \( m_1(\nu) \) changes its Morse index only at the curves \( \mu = 0 \) for \( \nu \in \mathbb{R} \), \( \nu_0(\mu) = \mu + s_1 \) for \( \mu \in \mathbb{R} \), and

\[ \nu_k(\mu) = \pm \sqrt{s_1^2 - \mu} \text{ for } \mu \in (-\infty, s_1^2). \]

Moreover, the Morse number of \( m_1(\nu) \) in the eight regions are: \( n_1 = 0 \) in the regions \((0x)\), \( n_1 = 1 \) in the regions \((1x)\), \( n_1 = 2 \) in the regions \((2x)\).
Figure 1: Graph $d_1(\mu, \nu) = 0$.

Proof. The block $m_1(\nu)$ is

$$m_1(\nu) = \begin{pmatrix}
\mu (-\nu + s_1 + \mu) & - (n/2)^{1/2} \mu & - (n/2)^{1/2} \nu \\
- (n/2)^{1/2} \mu & s_1 + 2\mu & i\nu \\
(n/2)^{1/2} \mu & -i\nu & s_1
\end{pmatrix}.$$

Since $n = 2s_1 + 1$, then the determinant is

$$d_1(\nu) = \mu (\nu - (\mu + s_1)) (\nu^2 - (s_1^2 - \mu)).$$

Thus, the matrix $m_1(\nu)$ changes its Morse index only at $\mu = 0$, $\nu_0(\mu)$ and $\nu_{\pm}(\mu)$. Moreover, the curves $\nu_0(\mu)$ and $\nu_{\pm}(\mu)$ intersect at the point

$$(\mu_0, \nu_0) = (-2s_1 - 1, -s_1 - 1).$$

Therefore, the plane $(\mu, \nu)$ is divided in eight regions as shown in the graph.

For $\nu$ big enough, the Morse index of the matrices $m_1(\nu)$ and $-\nu \text{diag}(\mu, iJ)$ are the same. Thus, the Morse indices are $n_1(\infty) = 2$ and $n_1(-\infty) = 1$ for $\mu > 0$, and $n_1(\infty) = 1$ and $n_1(-\infty) = 2$ for $\mu < 0$. Therefore, one gets that $n_1 = 2$ in the regions (2a) and (2b), and $n_1 = 1$ in the regions (1a) and (1b).

The region (1d) lies between the curves $\nu_0(\mu)$ and $\nu_{-}(\mu)$ for $\mu \in (-\infty, -2s_1 - 1)$. Since the determinant $d_1$ is negative in (1d), then the Morse index satisfies $n_1 \in \{1, 3\}$. Now, the trace of $m_1(\nu)$ is

$$T_1(\mu, \nu) = 2\mu + 2s_1 + \mu (\mu + s_1 - \nu).$$

Setting $\tilde{\nu} = (\nu_0 + \nu_-)/2$, one has that $\tilde{\nu} = \mu/2 + o(\mu)$ and $T_1(\tilde{\nu}) = \mu^2/2 + o(\mu^2)$ when $\mu \to \infty$. Since $T_1(\tilde{\nu})$ is positive when $\mu \to -\infty$, then $n_1 \neq 3$. Therefore, the Morse index must be $n_1 = 1$ in the region (1d).
For \( \nu = 0 \), the determinant is
\[
d_1(\mu) = -\mu(\mu + s_1) (\mu - s_1^2).
\]
Hence, \( d_1 > 0 \) in the region (2c), \( d_1 < 0 \) in the region (1c), and \( d_1 > 0 \) in the region (0a). We conclude that \( n_1 \in \{1, 3\} \) in (1c) and \( n_1 \in \{0, 2\} \) in (0a). Since \( m_1(\mu, \nu) \) is continuous and \( m_1(0, 0) = \text{diag}(0, s_1, s_1) \), then \( n_1 \leq 1 \) for \((\mu, \nu)\) near \((0, 0)\). Thus, one has that \( n_1 = 1 \) in the region (1c), and \( n_1 = 0 \) in the region (0a).

Since \( d_1 > 0 \) in (2c), then \( n_1 \in \{0, 2\} \). At the point \((\mu_0, \nu_0)\) the trace \( T_1 = -2(s_1 + 1) \) is negative. Since \( T_1(\mu, \nu) \) is continuous, then \( T_1 < 0 \) for \((\nu, \mu)\) near \((\mu_0, \nu_0)\). Thus, \( n_1 \neq 0 \), and the Morse index must be \( n_1 = 2 \) in the region (2c).

In the previous section we have found that \( n_1(\mu, \nu) \) changes its Morse index on the curves \( \nu_*(\mu) \), for \(* \in \{0, +, -\} \). Using the equality \( n_{n-1}(\nu) = n_1(-\nu) \), we get the following result.

**Theorem 14** For each \( k \in \{1, n-1\} \) such that \( \mu \in (-\infty, s_1^2) \), the polygonal relative equilibrium has a global bifurcation of periodic solutions starting from \( 2\pi/\nu_+ \) with symmetries \( \mathbb{Z}_n(k) \). Moreover, for \( k = 1 \) with \( \mu \in (-s_1, \infty) \) and for \( k = n-1 \) with \( \mu \in (-\infty, -s_1) \), there is another bifurcation of periodic solutions with symmetries \( \mathbb{Z}_n(k) \).

Actually, one may find all the numbers \( \eta_k(\nu) \), for \( k \in \{1, n-1\} \), using that \( \sigma = \text{sgn}(\omega) \). Thus, one may see that the bifurcations with symmetries \( \mathbb{Z}_n(1) \) have all the same index for \( \mu \in (-\infty, -s_1) \cup (s_1^2, \infty) \), and also with the symmetries \( \mathbb{Z}_n(n-1) \) for \( \mu \in (-s_1, s_1^2) \). Therefore, these bifurcating branches cannot return to the equilibrium \( \bar{a} \), and must be non-admissible or go to another equilibrium.

For \( \mu = 0 \), the two blocks are
\[
m_1(\nu) = \begin{pmatrix} s_1 & i\nu \\ -i\nu & s_1 \end{pmatrix} = m_{n-1}(\nu).
\]
Thus, the determinant \( \det m_1(\nu) = s_1^2 - \nu^2 \) is zero at \( \pm s_1 \) with \( \eta_1(s_1) = \eta_{n-1}(s_1) = -1 \). Therefore, there is only one global bifurcation of periodic solutions from \( 2\pi/s_1 \) with symmetries \( \mathbb{Z}_n(k) \) for \( k \in \{1, n-1\} \). The branches are non-admissible or go to another equilibrium.

**Remark 15** We have proven that \( m_k(\nu) \), for \( k \in \{2, ..., n-2\} \), is non-invertible at two points only if \( \mu > \mu_k \). Also, we proved that \( m_k(\nu) \), for \( k \in \{1, n-1\} \), is non-invertible at three points only if \( \mu < \mu_1 \). Moreover, the determinant of \( m_n(\nu) \) has a double zero at \( \nu = 0 \), due to the symmetries. Since the \( \mu_k \)'s are increasing for \( k \in \{2, ..., [n/2]\} \), the determinant of the matrix \( M(\nu) \) has \( 2(n+1) \) zeros counted with multiplicity for
\[
\mu \in (\mu_{[n/2]}, \mu_1).
\]
From the previous fact, one may conclude, analogously to the \( n \)-body problem, that the polygonal relative equilibrium is spectrally stable only if \( \mu \in (\mu_1/2, \mu_1) \), where \( \mu_1 = (n-1)^2/4 \) and \( \mu_k = (-k^2 + nk - 2n + 2)/4 \). This fact is proved in the paper [1].

6.2 Filaments

Next we wish to analyze the spectrum of traveling waves in the filaments. We have two parameters: the traveling wave velocity \( \gamma \) and the frequency \( \omega \).

Blocks \( k \in \{2, \ldots, n-2, n\} \)

**Proposition 16** Define \( \nu_+ \) and \( \nu_- \) as

\[
\nu_\pm = \left[ (2\gamma^2 - \omega) \pm \sqrt{(2\gamma^2 - \omega)^2 - 4\omega_k(\omega - \omega_k)} \right]^{1/2}. 
\]

For \( \omega < \omega_k \), and any \( \gamma \in \mathbb{R} \), the matrix \( m_k(\nu) \) changes its Morse index at the positive value \( \nu_+ \), with

\[
\eta_k(\nu_+) = \text{sgn}(\omega). 
\]

For \( \omega > \omega_k \), and any \( \gamma^2 > \omega/2 + (\omega_k(\omega - \omega_k))^{1/2} \), the matrix \( m_k(\nu) \) changes its Morse index at the positive values \( \nu_\pm \), with

\[
\eta_k(\nu_\pm) = \pm 1. 
\]

**Proof.** The block \( m_k(\nu) \) is given by

\[
m_k(\nu) = \nu^2 - 2\gamma\nu i J + 2 \text{diag}(\omega - \omega_k, \omega_k). 
\]

Thus, the trace is \( T_k(\mu) = 2(\nu^2 + \omega) \) and the determinant is

\[
d_k(\nu) = \nu^4 - 2(2\gamma^2 - \omega)\nu^2 + 4\omega_k(\omega - \omega_k) \\
= (\nu^2 - \nu_+^2)(\nu^2 - \nu_-^2). 
\]

Therefore, the determinant is zero at the four roots \( \pm \nu_\pm \).

For \( \omega < \omega_k \) and \( \gamma \in \mathbb{R} \), only the value \( \nu_+ \) is positive. Since \( d_k(0) = 4\omega_k(\omega - \omega_k) < 0 \), then \( n_k(0) = 1 \). Since all eigenvalues of \( m_k(\nu) \) are positive for big \( \nu \), then \( n(\infty) = 0 \). Hence, the change in the Morse index is \( \eta(\nu_+) = \sigma(1-0) \), with \( \sigma = \text{sgn}(\omega) \).

For \( \omega > \omega_k \) and \( \gamma^2 > \omega/2 + (\omega_k(\omega - \omega_k))^{1/2} \), the two values \( \nu_\pm \) are positive. Since \( \omega_k \) is positive, then \( \sigma = \text{sgn}(\omega) = 1 \). Since \( d_k(0) > 0 \) and \( T_k(0) > 0 \), then \( n_k(0) = 0 \), and as before, one has that \( n_k(\infty) = 0 \). Moreover, using that \( \det m(\nu) \) is negative between \( \nu_- \) and \( \nu_+ \), then \( n_k(\nu) = 1 \) for \( \nu \in (\nu_-, \nu_+) \). Thus, we conclude that \( \eta(\nu_-) = -1 \) and \( \eta(\nu_+) = 1 \). ■

Since \( \omega_n = 0 \), then \( \nu_- = 0 \) for \( k = n \). Thus, in the following theorem there is no bifurcation from \( 2\pi/\nu_- \), for \( k = n \).
Theorem 17: For each \( k \in \{2, ..., n-2, n\} \) such that \( \omega < \omega_k \), the polygonal equilibrium has a global bifurcation of traveling waves from \( 2\pi/\nu_+ \), for each \( \gamma \in \mathbb{R} \), with symmetries \( \tilde{Z}_n(k) \). For each \( k \in \{2, ..., n-2, n\} \) such that \( \omega > \omega_k \), there are two bifurcations from \( 2\pi/\nu_+ \) and \( 2\pi/\nu_- \), for each \( \gamma^2 > \omega/2 + (\omega_k(\omega - \omega_k))^{1/2} \), with symmetries \( \tilde{Z}_n(k) \).

The bifurcating branches cannot return to the polygonal equilibrium for \( \omega < \omega_k \).

Blocks \( k \in \{1, n-1\} \)

Here the block \( m_1(\nu) \) is

\[
m_1(\nu) = \begin{pmatrix}
\mu (\mu \nu^2 - 2\gamma \nu + s_1 + \mu) & -(n/2)^{1/2} \mu & -(n/2)^{1/2} \mu i \\
-(n/2)^{1/2} \mu & \nu^2 + s_1 + 2\mu & 2\gamma \nu i \\
(n/2)^{1/2} \mu i & -2\gamma \nu i & \nu^2 + s_1
\end{pmatrix}.
\]

The determinant of this matrix is a polynomial of degree six in \( \nu \), without an explicit factorization. So, analytically one is able to analyze only some special cases.

For \( \mu \in (-s_1, 0) \cup (s_1, \infty) \), the determinant at \( \nu = 0 \) is negative,

\[
d_1(0) = -\mu(\mu + s_1)(\mu - s_1^2) < 0.
\]

Moreover, since the determinant \( d_1(\nu) \) is positive for big \( |\nu| \), then the matrix \( m_1(\nu_k) \) must change its Morse index at least at some values \( \nu_1 \) and \( -\nu_{n-1} \). Thus, one has that \( \eta_k(\nu_k) \neq 0 \) for \( k \in \{1, n-1\} \).

Theorem 18: For each \( k \in \{1, n-1\} \) such that \( \mu \in (-s_1, 0) \cup (s_1, \infty) \), the polygonal relative equilibrium has at least one global bifurcation of traveling waves for each velocity \( \gamma \in \mathbb{R} \), starting from the period \( 2\pi/\nu_k \), and with symmetries \( \tilde{Z}_n(k) \).

Now, we analyze completely the particular case \( \mu = 1 \), that is when all the \( n+1 \) filaments have the same circulation, with \( \omega = s_1 + 1 \).

Proposition 19: Define \( \bar{\nu}_\pm \) and \( \nu_\pm \) as

\[
\bar{\nu}_\pm = \gamma \pm (\gamma^2 - \omega)^{1/2} \quad \text{and} \quad \nu_\pm = \left(-b \pm \sqrt{b^2 - c}\right)^{1/2},
\]

where \( b = \omega - 2\gamma^2 \) and \( c = \omega^2 - 2\omega > 0 \).

(a) For \( (\omega + \sqrt{c})/2 < \gamma^2 < \omega \), the matrix \( m_1(\nu) \) changes its Morse index at \( \pm \nu_\pm \) with \( \eta_1(\nu_\pm) = \pm 1 \).
(b) For \( \omega < \gamma^2 \), the matrix \( m_1(\nu) \) changes its Morse index at \( \bar{\nu}_\pm \) and \( \pm \nu_\pm \), with

\[
\eta_1(\bar{\nu}_\pm) = \pm 1 \quad \text{and} \quad \eta_1(\nu_\pm) = \pm 1.
\]

**Proof.** For \( \mu = 1 \) one finds that the eigenvalues of \( m_1(\nu) \) are

\[
\lambda_0 = \nu^2 - 2\gamma \nu + \omega \quad \text{and} \quad \lambda_\pm = \nu^2 + \omega \pm \sqrt{4\gamma^2 \nu^2 + 2\omega}.
\]

The eigenvalue \( \lambda_0 \) is zero only at \( \bar{\nu}_\pm \) for \( \gamma^2 > \omega \), and the eigenvalue \( \lambda_+ \) remains always positive.

The eigenvalue \( \lambda_- \) is zero at the solutions of \( \nu^4 + 2b\nu^2 + c = 0 \), that is at \( \nu^2 = -b \pm \sqrt{b^2 - c} \). Since \( c \) is positive for \( n \geq 3 \), then \( \lambda_- \) is zero at \( \pm \nu_\pm \) only if \( b < 0 \) and \( b^2 - c > 0 \). For \( \gamma^2 > (\omega + \sqrt{c})/2 \), one has that \( b < 0 \) and \( b^2 - c = (2\gamma^2 - \omega)^2 - c > 0 \), then the eigenvalue \( \lambda_- \) is zero at \( \pm \nu_\pm \) only for \( \gamma^2 > (\omega + \sqrt{c})/2 \).

Since \( \omega = \frac{s_1}{2} + 1 \), then \( \sigma = \text{sgn}(\omega) = 1 \). For (a), the eigenvalues \( \lambda_0 \) and \( \lambda_+ \) are positive, and \( \lambda_- \) is negative only when \( |\nu| \in (\nu_-, \nu_+) \). Therefore, one has that \( \eta_1(\nu_-) = -1 \) and \( \eta_1(\nu_+) = 1 \).

For (b), one has that \( \lambda_+ \) is positive, \( \lambda_0 \) is negative when \( \nu \in (\bar{\nu}_-, \bar{\nu}_+) \), and \( \lambda_- \) is negative when \( |\nu| \in (\nu_-, \nu_+) \). Since

\[
\lambda_-(\bar{\nu}_\pm) = 2\gamma \bar{\nu}_\pm - \sqrt{4\gamma^2 \bar{\nu}_\pm^2 + 2\omega},
\]

then \( \lambda_-(\bar{\nu}_\pm) \) is negative and \( (\bar{\nu}_-, \bar{\nu}_+) \subset (\nu_-, \nu_+) \). Therefore, one has that \( \eta_1(\nu_-) = 0 - 1 \), \( \eta_1(\bar{\nu}_-) = 1 - 2 \), \( \eta_1(\nu_+) = 2 - 1 \) and \( \eta_1(\bar{\nu}_+) = 1 - 0 \).

**Theorem 20** For \( k \in \{1, n - 1\} \) and \( \mu = 1 \), the polygonal equilibrium has two global bifurcation of periodic traveling waves with velocity \( \gamma^2 > (\omega + \sqrt{c})/2 \), starting from the periods \( 2\pi/\nu_+ \) and \( 2\pi/\nu_- \), and with symmetries \( Z_n(k) \). In addition, there are two global bifurcation from \( 2\pi/\bar{\nu}_+ \) and \( 2\pi/\bar{\nu}_- \), for each \( \gamma^2 > \omega \), with symmetries \( \bar{Z}_n(k) \).

For \( \mu = 0 \), the block \( m_1(\nu) \) is given by

\[
m_1(\nu) = \begin{pmatrix} \nu^2 + s_1 & 2\gamma \nu i \\ -2\gamma \nu i & \nu^2 + s_1 \end{pmatrix}.
\]

Setting \( m_1(\nu) = \nu^2 - 2\gamma \nu i J + 2\text{diag} (\omega - \bar{\omega}_1, \bar{\omega}_1) \), with \( \omega = s_1 \) and \( \bar{\omega}_1 = s_1/2 \), one may analyze the block as we did for the blocks \( m_k(\nu) \). In this way one gets that \( m_1(\nu) \) changes its Morse index at \( \pm \nu_\pm \) for \( \gamma^2 > s_1 \), with \( \eta_1(\nu_\pm) = \pm 1 \), where

\[
\nu_\pm^2 = (2\gamma^2 - s_1) \pm \sqrt{4\gamma^2 (\gamma^2 - s_1)}.
\]

Therefore, for \( \mu = 0 \) and \( k \in \{1, n - 1\} \), the polygonal equilibrium has two global bifurcations of traveling waves with velocity \( \gamma^2 > s_1 \), starting from the periods \( 2\pi/\nu_+ \) and \( 2\pi/\nu_- \), and with symmetries \( \bar{Z}_n(k) \). These bifurcations are non-admissible or go to another bifurcation point.
6.3 The case \( n = 2 \)

The irreducible representations for \( n = 2 \) are different from those for larger \( n \), due to the action of \( \mathbb{Z}_2 \). This fact gives a change in the representation for \( k = 1 \), but the change of variables for \( k = 2 \) remains the same as for larger \( n \)'s. In particular the bifurcation results for the case \( k = n = 2 \) are similar to the ones already given.

For \( k = 1 \), define the isomorphism \( T_1: \mathbb{C}^4 \rightarrow W_1 \) as

\[
T_1(v, w) = (v, 2^{-1/2}w, 2^{-1/2}w) \text{ with }
W_1 = \{(v, w, w) : v, w \in \mathbb{C}^2\}.
\]

It is not difficult to compute the Hessian of the potential \( V \). The matrix \( B_2 \) is the same as before, but \( B_1 \) is now a \( 4 \times 4 \) complex matrix. One has that \( s_1 = 1/2 \).

6.3.1 Vortices

The matrix \( m_1(\nu) \) is

\[
\begin{pmatrix}
\mu (\mu + 5/2) & -i\nu \mu & -\sqrt{2}\mu & 0 \\
-i\nu \mu & \mu (\mu - 3/2) & 0 & \sqrt{2}\mu \\
-\sqrt{2}\mu & 0 & 2\mu + 1/2 & -i\nu \\
0 & \sqrt{2}\mu & i\nu & 1/2
\end{pmatrix}.
\]

Figure 2: Graph of \( d_1(\mu, \nu) = 0 \).

**Proposition 21** Let

\[
\nu_0(\mu) = |\mu + 1/2| \quad \text{and} \quad \nu_1(\mu) = \sqrt{3}(\mu - 5/4)^{1/2},
\]

15
then the determinant of $m_1(\nu)$ is zero only on the curves: $\mu = 0$ for $\nu \in \mathbb{R}$, $\pm \nu_0$ for $\mu \in \mathbb{R}$ and $\pm \nu_1$ for $\mu \in (-\infty, -\frac{5}{4})$. Furthermore, dividing the semiplane $\nu > 0$ in the seven regions separated by these curves, the Morse number of $m_1(\nu)$ in the regions (1x) is $n_1 = 1$, in the regions (2x) it is $n_1 = 2$ and in the regions (3x) it is $n_1 = 3$.

**Proof.** Since the determinant $\det m_1(\nu)$ is

$$\det m_1 = \mu^2 (\nu^2 - (\mu + 1/2)^2) (\nu^2 + 3(\mu + 5/4)),$$

then this determinant is zero on the above curves, which intersect only at $\mu = -2$.

For $\nu$ large, the Morse number of $m_1(\nu)$ is the same as the one of the matrix $-\nu \text{diag}(\mu_1, iJ)$. Hence, $n_1 = 2$ in the regions (2a) and (2b). Now, the matrix $m_1(\mu, 0)$ has eigenvalues

$$\frac{1}{4}(2\mu^2 - 3\mu + 1) \pm \frac{1}{4}\sqrt{4\mu^4 - 12\mu^3 + 37\mu^2 + 6\mu + 1} \quad \text{and}$$

$$\frac{1}{4}(2\mu^2 + 9\mu + 1) \pm \frac{1}{4}\sqrt{4\mu^4 + 4\mu^3 + 29\mu^2 - 2\mu + 1}.$$

Then, $n_1 = 2$ in (2c), $n_1 = 3$ in (3a) and $n_1 = 1$ in (1a) and (1b).

In order to find the Morse number in (1c), one sees that the matrix $m_1(\mu, \mu + 1/2)$ has eigenvalues $0$, $3\mu$ and

$$\mu^2 + 1/2 \pm \sqrt{\mu^2 + 1/2)^2 + \mu (\mu + 2)(2\mu + 1)}.$$

Since $m_1(\mu, \mu + 1/2)$ has two positive eigenvalues for $\mu < -2$, then $n_1 \leq 2$ in (1c). But, since $\det(m_1)$ is negative in (1c), then $n_1 = 1$ there.

Note that the polygonal equilibrium is stable for $\mu < -5/4$.

Thus, there is a global bifurcation of periodic orbits, starting from the period $2\pi/\nu_0$ and, for $\mu < -5/4$, starting form the period $2\pi/\nu_1$, and symmetry $Z_1(1)$, that is $u_0(t + \pi) = -u_0(t)$ for the central element, and $u_2(t) = -u_1(t + \pi)$ for the two elements with circulation 1.

### 6.3.2 Filaments

The matrix $m_1(\nu)$ is

$$
\begin{pmatrix}
\mu (\mu \nu^2 + \mu + 5/2) & -2i\nu \gamma \mu & -\sqrt{2} \mu & 0 \\
2i\nu \gamma \mu & \mu (\mu \nu^2 + \mu - 3/2) & 0 & \sqrt{2} \mu \\
-\sqrt{2} \mu & 0 & \nu^2 + 2\mu + 1/2 & -2i\nu \gamma \\
0 & \sqrt{2} \mu & 2i\nu \gamma & \nu^2 + 1/2
\end{pmatrix}.
$$

In this case the determinant is a polynomial of degree 8, with no easy factorization. In the simple case of $\nu = 0$, the determinant is
\[ \det m_1(0) = -3\mu^2 (\mu + 1/2)^2 (\mu + 5/4), \]

Then, the determinant \( \det m_1(\mu, 0) \) is negative for \( \mu > -5/4 \), but \( \det m_1(\mu, \nu) \) is positive for \( \nu \) large. Thus, for \( \mu > -5/4 \), there is a value \( \nu_1 \) where \( \det m_1(\mu, \nu) \) changes sign. Hence, \( \eta_1(\nu_1) \neq 0 \) for \( \mu > -5/4 \). That is, for \( \mu > -5/4 \) and any \( \gamma \), there is a global bifurcation of periodic traveling waves starting at the period \( 2\pi/\nu_1 \), wave velocity \( \gamma \) and symmetry \( \mathbb{Z}_1(1) \), as before.

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