Tensor gauge condition and tensor field decomposition

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(Dated: November 30, 2011)

We discuss various proposals of separating a tensor field into pure-gauge and gauge-invariant components. Such tensor field decomposition is intimately related to the effort of identifying the real gravitational degrees of freedom out of the metric tensor in Einstein’s general relativity. We show that, as for a vector field, the tensor field decomposition has exact correspondence to, and can be derived from, the gauge-fixing approach. The complication for the tensor field, however, is that there are infinitely many complete gauge conditions, in contrast to the uniqueness of Coulomb gauge for a vector field. The cause of such complication, as we reveal, is the emergence of a peculiar gauge-invariant pure-gauge construction for any gauge field of spin $\geq 2$. We make an extensive exploration of the complete tensor gauge conditions and their corresponding tensor field decompositions, regarding mathematical structures, equations of motion for the fields, and nonlinear properties. Apparently, no single choice is superior in all aspects, due to an awkward fact that no gauge-fixing can reduce a tensor field to be purely dynamical (i.e., transverse and traceless), as can the Coulomb gauge in a vector case.

PACS numbers: 11.15.-q, 04.20.Cv

I. INTRODUCTION

It is a familiar practice to separate a vector field $\vec{A}$ into transverse and longitudinal parts, $\vec{A} \equiv \vec{A}_\perp + \vec{A}_\parallel$, defined by $\vec{\partial} \cdot \vec{A}_\perp = 0$ and $\vec{\partial} \times \vec{A}_\parallel = 0$. One may naturally think of an analogous separation for a tensor field. Such field decompositions are well motivated in physics. For the vector case, e.g., it is the transverse field $\vec{A}_\perp$ that describes a real photon. The need of decomposing the tensor field is closely related to gravity. The Einstein equivalence principle dictates that gravitational effect is described by the metric tensor, which also characterizes the inertial effect associated with coordinate choice. But sometimes, one does face the necessity of identifying the real gravitational degrees of freedom out of the metric, e.g., in associating a meaningful energy to gravitational radiation, in analyzing canonical structure of gravitation, etc. The attempt to separate gravity from inertial effect dates back to the bimetric theory of Rosen in 1940 \cite{1}. This theory is yet \textit{formal} because Rosen does not give any actual prescription to separate a background metric from the experimentally measured total metric. After about 20 years, Arnowitt, Deser and Misner (ADM) proposed the famous transverse-traceless (TT) decomposition of a symmetric tensor in their canonical formulation of general relativity \cite{2}. Based upon the quantum action principle, Schwinger presently a slightly different TT decomposition in his attempt of quantizing the gravitational field \cite{3}. Later on, such tensor decomposition was further developed by Deser \cite{4}, and York \cite{5}, with the aim of going covariantly beyond the linear approximation. These decompositions are really \textit{operational} by giving explicit expression of the separated component in terms of the total tensor.

Separating the metric tensor is much more involved than separating $\vec{A}$ into $\vec{A}_\perp + \vec{A}_\parallel$, due to one more index and also the nonlinearity of gravity. Regarding nonlinearity, it should be mentioned that in the Yang-Mills theory separating the physical degrees of freedom from the gauge freedom is not so trivial either. This problem has revived recently in connection with the nucleon spin structure \cite{6}. A possible solution is presented by Chen \textit{et al.} \cite{7,8}, and is extended successfully to gravity by Chen and Zhu (CZ) \cite{9}.

The decompositions of ADM, Schwinger, Deser, York, and CZ are motivated from different aspects, and thus naturally contain notable differences. The CZ decomposition is specifically aimed at a clear separation of a pure-gauge background, while the others are closely associated with dynamical structure of gravity. In this paper, we explore and compare these decompositions from several perspectives. After a brief review of these decompositions, we first look at the relatively simple linearized gravity, for which we show that the aforementioned decompositions can all be conveniently rederived from a gauge-fixing approach. In so doing, we reveal an interesting mathematical structure that the decompositions of ADM, Schwinger, Deser and York contain doubly-nonlocal operation (namely, with the inverse Laplacian operator $\frac{1}{\Box}$ used quadratically), while that of CZ is the unique singly-nonlocal choice. We then go beyond the linear order and demonstrate that the CZ decomposition differs critically from those of Deser and York in their physical implications: The physical and pure-gauge (or inertial) effects are cleanly kept apart to all orders in the CZ formulation, but start to mix in the Deser and York

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approaches beyond linear order. We also explain how the Deser decomposition can be modified in the spirit of CZ formulation. Finally, we give some further discussion about how to analyze the physical and dynamical content of gravity, regarding the selection of constraints (or gauge-fixing) and the equations of motion. Apparently, there is no universally superior choice of gauge for a tensor field. The significant complication from the vector case to the tensor case (and virtually all higher-rank tensors) is revealed to arise from two facts: (i) It is possible to construct a peculiar gauge-invariant pure-gauge field, thus there are infinitely many ways to fix the gauge completely; (ii) however, no gauge-fixing can pick out directly the purely dynamical (i.e., transverse and traceless) component of a tensor, and extra extraction is needed.

II. VARIOUS TENSOR DECOMPOSITIONS

The original TT decomposition of ADM is linear and for a symmetric spatial tensor:

\[ h_{ij} \equiv h_{ij}^{TT} + h_{ij}^T + h_{ij}^L, \]  

(1)

where \( h_{ij}^{TT} \) is TT: \( h_{ij}^{TT} = h_{kk}^{TT} = 0 \), \( h_{ij}^T = \frac{1}{2}(\delta_{ij}h^T - \frac{1}{2}g_{ij}^T) \) is constructed to be transverse: \( h_{ij}^T = 0 \), and \( h_{ij}^L \equiv f_{ij} + f_{ji} \) is longitudinal (pure-gauge). (Notations: Greek indices run from 0 to 3, Latin indices run from 1 to 3, and repeated indices are summed over, even when they both appear raised or lowered. \( \partial \) or comma denotes ordinary derivative, and \( \nabla \) denotes covariant derivative.)

The four unknowns, \( h^T \) and \( f_{ij} \), are solved as follows: \( h_{ij}^{T} = h_{ij,j} \) gives \( f_{i} \), then \( h_{kk}^{L} = h_{kk} \) gives \( h^{T} \). The results are:

\[ f_{i} = \frac{1}{\partial^2}(h_{ik,k} - \frac{1}{2}\partial h_{kl,kl}), \]  

(2)

\[ h^T = h_{kk} - \frac{1}{\partial^2}h_{kl,kl}. \]  

(3)

Inserting them into Eq. (1) gives the explicit form of the TT component:

\[ h_{ij}^{TT} = h_{ij} - \frac{1}{\partial^2}(h_{ik,k} - \frac{1}{2}\partial h_{kl,kl}) \]

\[ - \frac{1}{\partial^2}(h_{ik,kj} + h_{jk,ki} - \frac{1}{2}h_{kk,ij}) - \frac{1}{\partial^2}h_{kl,kl}. \]  

(4)

It can be easily checked that both \( h_{ij}^{TT} \) and \( h_{ij}^L \) are invariant under the linear gauge-transformation \( \delta h_{ij} = \xi_{i,j} + \xi_{j,i} \).\[ \text{Schwinger also formulated a TT decomposition, by a slightly different parametrization [3]:} \]

\[ h_{ij} = h_{ij}^{TT} + \frac{1}{2}(q_{ij} + q_{ji}) - \delta_{ij}q_{kk} + q_{ij}. \]  

(5)

Here the terms containing \( q_{i} \) are constructed to be “doubly transverse”: \( \frac{1}{2}(q_{ij} + q_{ji}) - \delta_{ij}q_{kk,k} + q_{ij,ij} = 0. \)

result, \( q \) is given by \( \partial^2 q = h_{ij,ij} \). Then by examining the trace and divergence of Eq. (5) one obtains

\[ q_{i} = \frac{1}{\partial^2}(2h_{ij,j} - \frac{1}{2}h_{ij,i} + \frac{3}{2}\partial^2 h_{jk,jk}). \]  

(6)

The TT component \( h_{ij}^{TT} \) is the same as that in Eq. (4).

Later, Deser presented a covariant decomposition [4]:

\[ h_{ij} = \psi_{ij}^{T} + \nabla_i V_j + \nabla_j V_i. \]  

(7)

Here \( \psi_{ij}^{T} \) is covariantly transverse: \( \nabla_i \psi_{ij}^{T} = 0 \). Deser does not extract further a TT part since the covariant extension of the linear \( h_{ij}^{TT} \) is rather involving. The unknown covariant vector \( V_{i} \) is to be solved iteratively, and the leading term is just \( f_{i} \) in (2).

By a clever parametrization, York obtained a covariant TT decomposition [5]:

\[ h_{ij} = h_{ij}^{TT} + (\nabla_i W_j + \nabla_j W_i - \frac{2}{3}g_{ij}\nabla k W^k) - \frac{1}{3}g_{ij}h_{kk}. \]  

(8)

Here the middle term is constructed to be traceless. Again, the unknown covariant vector \( W_{i} \) is to be solved iteratively. The leading term is

\[ W_{i} = \frac{1}{\partial^2}(h_{ij,j} - \frac{1}{4}h_{jj,i} - \frac{1}{4}\partial^2 h_{kk,jk}), \]  

(9)

and the leading TT component is the same as that in Eq. (4).

The CZ decomposition has a very different formulation [6]. It applies to space-time instead of just space, and is designed to be a clean separation of gravitational and pure-gauge degrees of freedom up to moderately strong field. The defining equations are

\[ g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \bar{g}_{\mu\nu}; \]  

(10a)

\[ \bar{R}_{\sigma\mu\nu\gamma}(g_{\alpha\beta}) = 0; \]  

(10b)

\[ g^{ij}\bar{\Gamma}_{ij} = 0. \]  

(10c)

Here \( \bar{g}_{\mu\nu} \) is intended as a pure-gauge background metric. Namely, it has an inverse \( \bar{g}^{\mu\nu} \) and can be used to define a background connection \( \bar{\Gamma}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}(\partial_{\mu}\bar{g}_{\sigma\nu} - \partial_{\nu}\bar{g}_{\sigma\mu} - \partial_{\sigma}\bar{g}_{\mu\nu}) \) with a vanishing Riemann curvature tensor \( \bar{R}_{\sigma\mu\nu\gamma} \equiv \partial_{\mu}\bar{\Gamma}_{\sigma\nu} - \partial_{\nu}\bar{\Gamma}_{\sigma\mu} + \bar{\Gamma}_{\alpha\mu}\bar{\Gamma}_{\sigma\nu} - \bar{\Gamma}_{\alpha\nu}\bar{\Gamma}_{\sigma\mu} = 0 \)

The intended physical term \( \bar{g}_{\mu\nu} \) satisfies a delicate constraint \( \bar{g}^{ij}\bar{\Gamma}_{ij} = 0 \). Here \( \bar{\Gamma}_{\mu\nu} = \Gamma_{\mu\nu} - \Gamma_{\mu}^{\rho}\Gamma_{\rho\nu} \) is not an affine connection, and \( \bar{g}_{\mu\nu} \equiv \bar{g}^{\mu\rho} - \bar{g}^{\mu\nu} \) is not the inverse of \( g_{\mu\nu} \).

The CZ decomposition is also to be solved iteratively. At linear order, \( g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \) with \( \eta_{\mu\nu} \) the Minkowski metric and \( |h_{\mu\nu}| \ll 1 \), an elegant, gauge-invariant expression is obtained after a fairly lengthy calculation [6]:

\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{\partial^2}(h_{\mu i,\nu} + h_{\nu i,\mu} - h_{i i,\mu\nu}). \]  

(11)

Its trace is of special use and worth recording:

\[ \bar{h}_{ii} = 2(h_{ii} - \frac{1}{\partial^2}h_{ij,ij}) = 2\frac{1}{\partial^2}\partial_j(h_{ij,j} - h_{ij,i}). \]  

(12)
Moreover, an elegant relation can be derived for $\hat{h}_{ij}^{CZ}$ and the (fairly complicated) $h_{ij}^{TT}$:

$$ h_{ij}^{TT} = \hat{h}_{ij}^{CZ} - \frac{1}{4}(\delta_{ij}\hat{h}_{kk} + \frac{1}{\partial^2}\hat{h}_{kk,ij}). $$

(13)

III. DERIVATION OF THE LINEAR DECOMPOSITION BY GAUGE-FIXING

Before discussing the highly tricky difference between Deser, York, and CZ at higher orders, we first take a closer look at their linear-order results, which we derive from a convenient gauge-fixing approach. The line of our derivation will show clearly the one-to-one correspondence between complete gauge conditions and gauge-covariant decompositions.

Consider the linear gauge-transformation:

$$ h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu. $$

We show that the gauge parameter $\xi$ that brings $h'_{\mu\nu}(x)$ into the “generalized” transverse gauge,

$$ \partial_\mu h'_{\nu0} + a \partial_\nu h'_{\mu0} = 0, \quad (14a) $$

$$ \partial_\mu h'_{\nu j} + b \partial_\nu h'_{\mu j} = 0, \quad (14b) $$

is essentially unique. Here $a, b$ can take any value except $b = -1$, as Eq. (12) indicates that $(h_{ji,i} - h_{ii,j})$ has a gauge-invariant divergence, thus cannot be used to define a gauge. Various combinations of $a, b$ have been studied in the literature. The gauge $a = -\frac{1}{2}, b = 0$ was employed by ADM [2], who also suggested an important choice of $a = -\frac{1}{2}, b = -\frac{1}{3}$, which agrees with the Dirac gauge at linear approximation [11]. The gauge $a = -\frac{2}{3}, b = -\frac{1}{4}$ was encountered by Weinberg and termed “too ugly to deserve a name” [12], while the special properties of $a = b = -\frac{1}{2}$ was recently revealed by CZ [13].

We cast Eqs. (13) into the equations for $\xi$:

$$ \partial^2 \xi_0 + (1 + 2a) \partial_\mu \partial_\mu \xi_0 = \partial_\mu h_{\mu0} + a \partial_\nu h_{\nu0}, \quad (15a) $$

$$ \partial^2 \xi_j + (1 + 2b) \partial_\mu \partial_\mu \xi_j = \partial_\mu h_{\mu j} + b \partial_\nu h_{\nu j}. \quad (15b) $$

To solve, act on both sides of (15b) with $\partial_j$ and sum over $j$, we get

$$ \xi_{i,i} = \frac{1}{2(1+b)}(bh_{ii} + \frac{1}{\partial^2}h_{ii,ij}). \quad (16) $$

We remind that inversion of the Laplacian operator $\partial^2$ implies a vanishing boundary value at infinity. Substituting $\xi_{i,i}$ back into Eqs. (15), we get

$$ \xi_0 = \frac{1}{\partial^2}[h_{0i,i} + \frac{2a-b}{2(1+b)}h_{i0,0} - \frac{1+2a}{2(1+b)} h_{ik,ik0}] \quad (17a) $$

$$ \xi_j = \frac{1}{\partial^2}[h_{ij,i} + \frac{b}{2(1+b)}h_{ii,j} - \frac{1+2b}{2(1+b)} h_{ik,ikj}] \quad (17b) $$

These are the unique $\xi$ that bring $h'_{\mu\nu}$ into the gauge $[13]$. The above derivation also show clearly that Eqs. (13) fix the gauge completely: If $h_{\mu\nu}$ is already in this gauge, then we get $\xi_{\mu} = 0$. Namely, no gauge-transformation can preserve Eqs. (13).

The uniqueness of $\xi_{\mu}$ suggests the definition of a gauge-invariant tensor, $\hat{h}_{\mu\nu}^{(ab)} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, which satisfies

$$ \hat{h}_{00}^{(ab)} + a h_{0i,0}^{(b)} = 0 \quad (\text{for } a, b \text{ used for clarity}.) $$

The explicit expression of $\hat{h}_{\mu\nu}^{(ab)}$ is

$$ \hat{h}_{\mu\nu}^{(ab)} = h_{\mu\nu} - \frac{1}{\partial^2}(h_{kk,00} - h_{kk,k0}) \quad (18a) $$

$$ \hat{h}_{0\mu}^{(ab)} = h_{0\mu} - \frac{1}{\partial^2}(h_{kk,0k} + h_{k,jk} - h_{kk,0j}) \quad (18b) $$

$$ \hat{h}_{i\mu}^{(b)} = h_{ij} - \frac{1}{\partial^2}(h_{kk,ki} + h_{kj,ki} - h_{kk,ki}) \quad (18c) $$

We have organized $\hat{h}_{\mu\nu}^{(ab)}$ in a form which displays evident gauge-invariance. By Eqs. (18), the tensor $h_{\mu\nu}$ is separated into a gauge-invariant part, $\hat{h}_{\mu\nu}^{(ab)}$, plus a pure-gauge part, $\hat{h}_{\mu\nu} = h_{\mu\nu} - \hat{h}_{\mu\nu}^{(ab)} = \xi_{\mu,\nu} + \xi_{\nu,\mu}$. Notice that this is an operational separation since $\hat{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}^{(ab)}$ are both expressed explicitly in terms of the given $h_{\mu\nu}$. We thus see that each complete gauge condition $h_{0i,i} + ah_{ii,0} = 0, h_{ij,i} + bh_{ii,j} = 0$ (specified by $a, b$) corresponds to one way of decomposing $h_{\mu\nu}$ into an invariant part plus a pure-gauge part.

The expressions for $\hat{h}_{\mu\nu}^{(ab)}$ look rather complicated, but the spatial trace $\hat{h}_{ii}^{(b)}$ is remarkably simple for all choice of $b$:

$$ \hat{h}_{ii}^{(b)} = \frac{1}{2(1+b)}(h_{ii} - \frac{1}{\partial^2}h_{ii,ij}) = \frac{1}{2(1+b)} \hat{h}_{ij}^{CZ}. \quad (19) $$

The gauge-invariant $\hat{h}_{\mu\nu}^{(ab)}$ relates to the gauge-invariant $h_{\mu\nu}^{CZ}$ by

$$ \hat{h}_{\mu\nu}^{(ab)} = h_{\mu\nu}^{CZ} - \frac{1}{2(1+b)} \frac{1}{\partial^2}(h_{kk}^{CZ})_{,00}, \quad (20a) $$

$$ \hat{h}_{0\mu}^{(ab)} = h_{0\mu}^{CZ} - \frac{1+a+b}{2(1+b)} \frac{1}{\partial^2}(h_{kk}^{CZ})_{,0j}, \quad (20b) $$

$$ \hat{h}_{i\mu}^{(b)} = h_{ij}^{CZ} - \frac{1+2b}{2(1+b)} \frac{1}{\partial^2}(h_{kk}^{CZ})_{,ij}, \quad (20c) $$

Another important relation is between $h_{ij}^{TT}$ and $\hat{h}_{ij}^{(b)}$:

$$ h_{ij}^{TT} = \hat{h}_{ij}^{(b)} - \frac{1}{2} \delta_{ij} h_{kk}^{(b)} + \frac{1}{2} 3b \frac{1}{\partial^2} h_{kk,ij}^{(b)}. \quad (21) $$

This agrees with Eq. (4) in the gauge $h_{ij,i} + bh_{ii,j} = 0$.\]
For comparison, we recall the parallel construction for a vector field $A^\mu$ with the gauge-transformation $A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x)$. The parameter $\Lambda$ that brings $A'_\mu$ into the Coulomb gauge, $\partial_\mu A'_\mu = 0$, is also unique: $\Lambda = \frac{1}{\partial^2} A_{i,i}$. This says that Coulomb gauge is a complete gauge for a vector field. We can again define a gauge-invariant quantity, $A_\mu = A_\mu - \partial_\mu \frac{1}{\partial^2} A_{i,i}$; and make the decomposition $A_\mu = \bar{A}_\mu + \mu_\mu$, with $\partial_\mu \frac{1}{\partial^2} A_{i,i}$ being a pure-gauge.

The spatial components, $\bar{A}$ and $\bar{A}$, are nothing but the transverse field $\vec{A}_\perp$ and the longitudinal field $\vec{A}_\parallel$.

Eqs. (17) and (18) are far more complicated than their counterparts for vector field, not only by more indices, but by a doubly-nonlocal operation $\frac{1}{\partial^2}$ which also appears in (2), (4), (6) and (9).

Some special choices of $a, b$ are particularly interesting: (i) $a = b = -\frac{1}{2}$ kills the doubly-nonlocal terms in Eqs. (17) and (18), and gives the CZ decomposition in Eq. (11), with the pure-gauge part:

$$\tilde{h}^{\text{CZ}}_{\mu\nu} = \frac{1}{\partial^2} (h_{\mu,i,iv} + h_{\nu,i,\mu} - h_{i,i,\mu})$$

$$\equiv \xi_{\mu,i}^{\text{CZ}} + \xi_{\nu,i}^{\text{CZ}},$$

$$\xi_{\mu,i}^{\text{CZ}} = \frac{1}{\partial^2} (h_{\mu,i,i} - \frac{1}{2} h_{i,i,i}).$$

This is also the linear-order result of the covariant Deser decomposition in Eq. (7). The choice $b = 0$ is not as convenient as one may expect, as the doubly-nonlocal structure remains. This is similar to the familiar fact in gravity that the gauge $\partial^\mu h_{\mu,\nu} = 0$ is not convenient, and a cleverer choice is the harmonic gauge $h_{\mu,\nu} - \frac{1}{2} h_{\mu,\mu,\nu} = 0$.

The two pure-gauge terms $\tilde{h}^{\text{ADM}}_{ij}$ and $\tilde{h}^{\text{CZ}}_{ij}$ differ by another pure-gauge term:

$$\tilde{h}^{\text{ADM}}_{ij} = \tilde{h}^{\text{CZ}}_{ij} + \frac{1}{2} \frac{1}{\partial^2} (h_{i,k,k},ij),$$

$$\tilde{h}^{\text{ADM}}_{ij} = \tilde{h}^{\text{CZ}}_{ij} + \frac{1}{2} \frac{1}{\partial^2} (h_{i,k,k},ij).$$

(iii) $b = -\frac{1}{2}$ gives a gauge-transformation parameter $\xi_i$ equal to $W_i$ in Eq. (9), and thus the pure-gauge part of York decomposition at linear order:

$$\tilde{h}^{\text{York}}_{ij} = \frac{1}{\partial^2} (h_{ik,kj} + h_{jk,ki} - \frac{1}{2} h_{kk,kl,ij}$$

$$\equiv \xi_{j,i}^{\text{York}} + \xi_{i,j}^{\text{York}},$$

$$\xi_{i}^{\text{York}} = W_i = \frac{1}{\partial^2} (h_{ik,k} - \frac{1}{2} h_{kk,i} - \frac{1}{4} \frac{1}{\partial^2} h_{kk,kl,ij}).$$

$\tilde{h}^{\text{York}}_{ij}$ differs from $\tilde{h}^{\text{CZ}}_{ij}$ by yet another pure-gauge term:

$$\tilde{h}^{\text{York}}_{ij} = \tilde{h}^{\text{CZ}}_{ij} + \frac{1}{4} \frac{1}{\partial^2} (h_{i,k,k},ij)$$

When applied to general relativity, the pure-gauge term $\tilde{h}_{\mu\nu} = \xi_{\mu,i} + \xi_{\nu,i}$ relates to the inertial effect associated with coordinate choice. They differ in the decompositions of ADM (Deser), York, and CZ. Accordingly, the gauge-invariant parts in these decompositions, which are essentially the tensors defined in the gauges $h_{i,i,j} = 0$, $h_{j,i,i} - \frac{1}{2} h_{i,i,j} = 0$, and $h_{j,i,i} - \frac{1}{2} h_{i,i,j} = 0$, respectively, also differ. We will discuss further the properties of these different gauge-invariant quantities in later sections.

As far as gauge transformation is concerned, the CZ gauge with $a = b = -\frac{1}{2}$ is the most convenient one: It relates to all gauge configurations by just a singly-nonlocal manipulation, while doubly-nonlocal manipulation may be needed when transiting between two other gauges. The choice $b = -\frac{1}{3}$, however, possesses a most important feature that it kills the non-local term in Eq. (21), thus simplifies the extraction of $h_{TT}^{ij}$ to be just algebraic. This feature leads to various important applications:

(iv) Weinberg combines $b = -\frac{1}{3}$ with $a = -\frac{1}{2}$, which simplifies $h_{TT}^{ij}$ and kills the doubly-nonlocal term in Eq. (18a). By this choice Weinberg was able to demonstrate quantum Lorentz invariance for the gravitational coupling [12].

(v) The linearized Dirac gauge is to combine $b = -\frac{1}{3}$ with $a = -\frac{1}{2}$. It simplifies $h_{TT}^{ij}$ and kills the doubly-nonlocal term of the time-time component in Eq. (18a), which closely relates to the Newtonian potential. This gauge is thus especially advantageous in post-Newtonian dynamics [14, 15], and is also referred to as the ADM-TT gauge. It should be clarified that the term “TT gauge” here just means that in this gauge the TT component $h_{ij}^{TT}$ relates to $h_{ij}$ locally: $h_{ij}^{TT} = h_{ij} - \frac{1}{3} \delta_{ij} h_{kk}$, not that $h_{ij}$ is purely TT.

IV. BEYOND THE LINEAR ORDER

The difference between Deser, York, and CZ beyond the linear order, on the other hand, is much more drastic. The delicate formulation of CZ via the Riemann curvature guarantees that $\tilde{g}_{\mu\nu}$ is a pure-gauge background to all orders. However, in the Deser or York formulation, $\nabla_i V_j + \nabla_j V_i = 0$, $V_i W_j + \nabla_i W_j$ is not a pure-gauge beyond linear order, namely, it does not give a vanishing Riemann curvature. A correct parametrization of a pure-gauge background metric can be obtained through coordinate transformation $x'^\mu = x^\mu + \xi^\mu$ from the Minkowski metric $\eta_{\mu\nu}$:

$$\tilde{g}_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \eta_{\rho\sigma}$$

$$= \eta_{\mu\nu} + \eta_{\rho\sigma} \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu} + \eta_{\rho\sigma} \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\sigma}{\partial x^\nu}.$$ (27)
The Riemann curvature of this metric is identically zero: \( R^\mu_{\sigma\rho\nu}(g_{\alpha\beta}) \equiv 0 \). In principle, one may use such a parametrization to formulate a clean separation of a pure-gauge background, equivalent to what CZ achieve:

\[
g_{\mu\nu} \equiv g^T_{\mu\nu} + \eta_{\mu\nu} + \eta_{\rho\nu} \xi_\rho^T + \eta_{\rho\mu} \xi_\rho^T + \eta_{\rho\sigma} \xi_\rho^\sigma \xi_\sigma^T, \quad \nabla^i \psi^T_{ij} = 0. \quad (29a)\]

\[
\nabla^i \psi^T_{ij} = 0. \quad (29b)\]

It can be seen, however, that this formulation is no less demanding than that of CZ, since (except at linear order) the equations for \( \xi^T \) are even more involving than those for \( \psi^T \).

Analogously, when decomposing a Yang-Mills field: \( A^\mu = \bar{A}^\mu + \hat{A}^\mu \), it is not really easy to parameterize the pure-gauge field \( \hat{A}^\mu \) explicitly as \( U \partial^\mu U^{-1} \) and try to solve \( U = e^{i\omega^\mu T^\mu} \), compared to defining \( \bar{A}^\mu \) implicitly by a vanishing field strength \( \bar{F}^\mu_{\nu}(\bar{A}^\mu) = 0 \) and solving \( \bar{A}^\mu \) as Chen et al. do \[8]. Though being tedious, the field decomposition in Refs. \[7–9\] can indeed be solved straightforwardly to any desired order.

We point out that the Deser decomposition can be modified to pick out a clean pure-gauge background to all orders, following the construction line of CZ:

\[
g_{\mu\nu} = \psi^T_{\mu\nu} + \eta_{\mu\nu}; \quad \bar{R}^\mu_{\sigma\rho\nu}(g_{\alpha\beta}) = 0, \quad \nabla^i \psi^T_{ij} = 0. \quad (30a)\]

At linear order, Eqs. \(29\) or \(30\) give the same \( \psi^T_{ij} \) as by Eq. \(7\). But beyond the linear order, Eqs. \(29\) or \(30\) can guarantee that \( g_{\mu\nu} - \psi^T_{\mu\nu} \) is a pure-gauge, while Eq. \(7\) cannot.

\[\] V. EQUATIONS OF MOTION FOR THE GAUGE-INvariant FIELDS

To look into the physical implications of the tensor decompositions, we derive here the equations of motion for the various gauge-invariant quantities constructed by ADM, Deser, York, and CZ. As we will see, this provides a very illuminating perspective on the dynamics of general relativity, and sheds important light on the question of what are the most appropriate physical variables (or equivalently, the most appropriate gauge) for gravitational field.

The unconstrained (gauge-dependent) \( h_{\mu\nu} \) satisfies the linearized Einstein equation

\[
\square h_{\mu\nu} - \partial_\mu \partial_\nu h_{\rho\rho} - \partial_\rho \partial_\rho h_{\mu\nu} + \partial_\mu \partial_\nu h_{\rho\rho} = -S_{\mu\nu}. \quad (31)\]

Here \( \square \equiv \partial^\mu \partial_\mu \), \( S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^\rho_{\rho} \), and we put \( 16\pi G = 1 \). Of the gauge-invariant quantities built out of \( h_{\mu\nu} \), \( h_{\mu\nu}^{(ab)} \) stands a primary role. Other quantities like \( h_{\mu\nu}^{CZ} \) or \( \psi^T_{\mu\nu} \) are just special cases of \( h_{\mu\nu}^{(ab)} \) for certain \( a,b \).

We first display the relevant equations for \( h_{\mu\nu}^{(ab)} \), then explain their derivations and physical meanings:

\[
\partial^2 h_{\mu\nu}^{(ab)} = -S_{\mu\nu} + \frac{1 + 2a}{1 + b} \frac{1}{\partial^2} T_{00,00} \quad (32a)\]

\[
\partial^2 h_{\mu i}^{(ab)} = -S_{\mu i} + \frac{1 + a + b}{1 + b} \frac{1}{\partial^2} T_{00,0i} \quad (32b)\]

\[
\partial^2 h_{ij,i}^{(b)} = -\frac{b}{1 + b} T_{00,j} = \partial^2 h_{ij,i}\big|_{(h_{ij,i} + bh_{ii,j}) = 0} \quad (32c)\]

\[
\Box h_{ij}^{(b)} = -\delta h_{ij}^{(b)} = \nabla h_{ij}\big|_{(h_{ij,i} + bh_{ii,j}) = 0}. \quad (32d)\]

The source term \( S_{ij}^{(b)} \) is built with \( S_{ij} \) in the same way as \( h_{ij}^{(b)} \) is built with \( h_{ij} \) in Eq. \(18c\):

\[
S_{ij}^{(b)} = S_{ij} - \frac{1}{\partial^2} (S_{ik,kj} + S_{kj,ki} - S_{kk,ij}) - \frac{2 + 2b}{1 + b} \frac{1}{\partial^2} (S_{ik} - \delta S_{kk,kl},ij). \quad (33)\]

The reason is that the equation for the gauge-invariant quantity \( h_{ij}^{(b)} \) must be gauge-invariant, therefore can be derived in any convenient gauge. Eqs. \(32\) and \(33\) are direct consequence of the equation in the harmonic gauge, \( \Box h_{\mu\nu} = -S_{\mu\nu} \).

We have organized Eqs. \(32\) to show an important and delicate dual relation between gauge-invariant equations and gauge-fixed equations: If there exists an \( X \) gauge in which a gauge-invariant quantity \( Y_{\mu\nu} = h_{\mu\nu} \), then the gauge-invariant equations for the gauge-invariant \( Y_{\mu\nu} \) take the same form as the gauge-dependent equations for the gauge-dependent \( h_{\mu\nu} \) in the specific \( X \) gauge. This dual relation can be very handy in deriving equations. E.g., to seek the equations of motion for \( h_{\mu\nu} \) in the gauge \( h_{\mu\nu} + ah_{\mu,0} + bh_{ii,0} = 0, h_{ij,i} + bh_{ii,j} = 0 \), we can instead look at the equations of motion for the gauge-invariant quantity \( h_{\mu\nu}^{(ab)} \) in Eqs. \(15\), which then can be derived in the convenient harmonic gauge. Even more conveniently, we see in Eqs. \(20\) that the expression of \( h_{\mu\nu}^{(ab)} \) via \( h_{\mu\nu}^{CZ} \) is fairly simple, so we can derive the equations of motion for \( h_{\mu\nu}^{(ab)} \) via the equations of motion for \( h_{\mu\nu}^{CZ} \), which in turn are just the equations of motion for \( h_{\mu\nu} \) in the gauge.
\( h_{ij,\mu} - \frac{1}{2} h_{ij,\mu} = 0 \), as we recently derive in Ref. 13:

\[
\begin{align*}
\partial^2 h_{ij}^{\text{CZ}} &= -S_{ij} - \partial^2 h_{ij} \bigg|_{(h_{ij,i} = 0, h_{ij,\mu} = 0)}, \quad (34a) \\
\partial^2 h_{ij}^{\text{CZ}} &= -2T_{00} - \partial^2 h_{ij} \bigg|_{(h_{ij,i} = 0, h_{ij,\mu} = 0)}, \quad (34b) \\
\partial^2 h_{ij}^{\text{CZ}} &= -T_{00,j} - \partial^2 h_{ij,j} \bigg|_{(h_{ij,i} = 0, h_{ij,\mu} = 0)}, \quad (34c) \\
\Box h_{ij}^{\text{CZ}} &= -\delta S_{ij}^{\text{CZ}} - \Box h_{ij} \bigg|_{(h_{ij,i} = 0, h_{ij,\mu} = 0)}. \quad (34d)
\end{align*}
\]

Like \( \delta S_{ij}^{(b)} \), \( \delta S_{ij}^{\text{CZ}} \) is built with \( S_{ij} \) in the same way as \( h_{ij}^{\text{CZ}} \) is built with \( h_{ij} \); and \( \delta S_{ij}^{(b)} \) relates to \( \hat{S}_{ij}^{\text{CZ}} \) in the same way as \( h_{ij}^{(b)} \) relates to \( h_{ij} \):

\[
\begin{align*}
\hat{S}_{ij}^{\text{CZ}} &= S_{ij} - \frac{1}{2} \partial^2 (S_{kk,ij} + S_{jk,k,i} - S_{kk,ij}), \quad (35a) \\
\tilde{S}_{ij}^{(b)} &= \hat{S}_{ij}^{\text{CZ}} - \frac{1 + 2b}{2(1 + b)} \frac{1}{\partial^2} \hat{S}_{ij}^{\text{CZ}}, \quad (35b)
\end{align*}
\]

For a consistency check: as \( a = b = -\frac{3}{2} \), Eqs. (32) reduce to Eqs. (34), which are just the special case of Eqs. (32) with the simplest source terms.

The instantaneous Laplacian operator \( \Box \) in Eqs. (32a-32d) means that the gauge-invariant quantities \( h_{ij}^{(ab)} \), \( h_{ij}^{(b)} \), and \( h_{ij,j}^{(b)} \) are non-dynamical and non-propagating. Equivalently, in the gauge \( h_{0,i} + ah_{ii,0} = 0 \), \( h_{ij,i} + bh_{ii,j} = 0 \), the component \( h_{0ij} \), the spatial trace \( h_{ij} \) and the spatial divergence \( h_{ij,j} \) are non-dynamical. We give a note on the equations for the spatial trace and divergence. They are certainly consistent with, and can be carefully reorganized from the apparently “propagating-looking” equations in Eqs. (32a-32c). But a more elucidating way to reveal the non-dynamical character of \( h_{ij} \) and \( h_{ij,j} \) is through Eq. (32b) and the constraint \( h_{0,i} + ah_{ii,0} = 0 \), \( h_{ij,i} + bh_{ii,j} = 0 \): First, \( h_{0,i} + ah_{ii,0} = 0 \) says that the trace \( h_{ii} \) has the same property as \( h_{0ij} \), which is non-dynamical by Eq. (32d). Then, \( h_{ij,i} + bh_{ii,j} = 0 \) says that the spatial divergence \( h_{ij,j} \) is non-dynamical as well.

Since \( \hat{h}_{ij}^{(b)} \) and \( \hat{h}_{ij,j}^{(b)} \) are non-dynamical, the truly dynamical component of \( h_{ij}^{(b)} \) is its TT part. The TT part of \( \hat{h}_{ij}^{(b)} \) actually equals \( h_{ij}^{TT} \) in Eq. (3), because \( \hat{h}_{ij}^{(b)} \) and \( h_{ij} \) differ by a pure-gauge term which does not contribute TT component. The equation of motion for \( h_{ij}^{TT} \) is thus of special importance:

\[
\Box h_{ij}^{TT} = -S_{ij}^{\text{TT}}. \quad (36)
\]

The source term \( S_{ij}^{\text{TT}} \) is the TT part of \( S_{ij} \). It relates to \( S_{ij} \) and \( S_{ij}^{(b)} \) in the same way as \( h_{ij}^{TT} \) relates to \( h_{ij} \) and \( h_{ij}^{(b)} \):

\[
S_{ij}^{TT} = S_{ij} - \frac{1}{2} \delta_{ij} (S_{kk} - \frac{1}{2} \partial^2 S_{kk,kl,kl})
- \frac{1}{\partial^2} (S_{kk,ij} + S_{jk,k,i} - \frac{1}{2} S_{kk,ij} - \frac{1}{2} \partial^2 S_{kk,kl,ij})
= \delta S_{ij} - \frac{1}{2} \frac{3b}{2} \delta_{ij} S_{kk}^{(b)} + \frac{1}{2} \partial^2 S_{kk,ij}^{(b)}, \quad (37)
\]

It should be noted that the equation for the TT component \( h_{ij}^{TT} \) does not have a gauge-fixing dual. The reason is that \( h_{ij} \) cannot in general be reduced to contain only TT component, except for a pure wave without source 13. In this regard, the tensor gauge field behaves rather peculiar that there are infinitely many gauges which can remove all nonphysical (gauge) degrees of freedom, but there is no gauge which can directly pick out the dynamical (TT) component. (As we explained in Section III, one can at best simply the extraction of \( h_{ij}^{TT} \) to be algebraic and local.) Here one should notice the difference between “nonphysical” and “non-dynamical”. E.g., in electrodynamics, the instantaneous Coulomb potential is non-dynamical but physical, and must be included into the total Hamiltonian of the system. Similarly, for gravity the instantaneous Newtonian interaction (contributed by \( h_{0ij} \) and \( h_{ii} \)) is non-dynamical but physical, and must be included into the total Hamiltonian, especially in a quantum theory 12.

In electrodynamics, the Coulomb gauge is the unique constraint to pick out the two physical (and at the same time dynamical) components \( \hat{A}_i \). The above peculiar feature of tensor gauge field leads to an embarrassing fact that among the many infinite many complete gauge conditions, \( h_{0,i} + ah_{ii,0} = 0 \), \( h_{ij,i} + bh_{ii,j} = 0 \), no single choice is superior in all aspects. Equations (32) indicate that by \( a = b = -\frac{1}{2} \) we obtain the simplest form of equations for all components of \( h_{\mu\nu} \). However, we still have to extract the dynamical component \( h_{ij}^{TT} \), and Eq. (21) indicates that \( b = -\frac{1}{2} \) leads to the simplest expression for \( h_{ij}^{TT} \). In Ref. 12, Weinberg chooses \( b = -\frac{1}{2} \) together with \( a = -\frac{3}{2} \), which simplifies the equation for \( h_{ij} \), but leaves a complicated equation for \( h_{0ij} \). Since \( h_{0ij} \) is closely related to the Newtonian potential, post-Newtonian dynamics favors \( a = -\frac{3}{2} \) which leads to the simplest equation for \( h_{0ij} \); but then one must choose between the convenience in dealing with \( h_{0ij} \) (which favors \( b = -\frac{1}{2} \)) or the dynamical component \( h_{ij}^{TT} \) (which favors \( b = -\frac{1}{3} \)).

VI. ANOTHER PERSPECTIVE ON GAUGE CONDITION AND GAUGE-FIELD DECOMPOSITION

To understand better the trickiness in a tensor gauge theory, in this section we approach the problem of gauge condition and gauge-field decomposition from another perspective. In the decomposition of gauge fields into an invariant part plus a pure-gauge part,

\[
A_i \equiv \hat{A}_i + \hat{h}_i, \quad h_{ij} \equiv \hat{h}_{ij} + \hat{h}_{ij}, \quad (38)
\]

we had essentially followed in Sec. III a line of constructing the pure-gauge fields \( \hat{A}_i \) and \( \hat{h}_{ij} \). We now proceed by asking: What are the possible means of constructing the invariant quantities \( \hat{A}_i \) and \( \hat{h}_{ij} \)? For the vector
field, we know that the fundamental gauge-invariant local quantity is the field strength $F_{ij}(A_i) = A_{ij} - A_{i,j}$. (For simplicity, we consider the linear theory, and first look at the Euclidean space with positive-definite metric. The extension to Minkowski space-time with indefinite metric requires just a little bit caution.) The gauge-invariant vector $\hat{A}_i$ must be built with $F_{ij}$, and the only possible structure with the same dimension as $A_i$ is $\frac{1}{\sqrt{2}} F_{ij,i}$. For the tensor field, the fundamental gauge-invariant local quantity is the linearized Riemann curvature, $R_{ijkl}(\hat{h}_{ij}) = h_{ik,jl} - h_{ij,k} - h_{ij,l} + h_{l,jk}$, with which one can build the gauge-invariant tensor $h_{ij}$. One finds now two possible structures with the same dimension as $h_{ij}$: $\frac{1}{\sqrt{2}} R_{ikjk}$ and $\frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$.

To assist the analysis of decompositions in Eq. (38), we introduce the terms competent invariant and competent pure-gauge. A competent invariant is the gauge-invariant part of a gauge field, which gives the same field strength (or curvature) and contain all the gauge degrees of freedom (namely, it transforms in the same as does the full gauge field). For a gauge field, the rest part of a competent invariant must be a competent pure-gauge, and the rest part of a competent pure-gauge must be a competent invariant. In Eq. (38), since $\hat{A}_i$ and $\hat{h}_{ij}$ are required to be pure-gauge, the gauge-invariant $A_i$ and $h_{ij}$ must contribute the whole field strength or curvature, and thus are necessarily competent invariants.

We now show that the gauge-invariant quantities, $\frac{1}{\sqrt{2}} F_{ij,i}, \frac{1}{\sqrt{2}} R_{ikjk}$, and $\frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$, are all competent invariants. Using the Bianchi identities,

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0,$$  
$$R_{ijkl,m} + R_{ijlm,k} + R_{ijkm,l} = 0,$$

and using the symmetry properties of $F_{ij}$ and $R_{ijkl}$, a slight algebra can prove that $\frac{1}{\sqrt{2}} F_{ij,i}$ gives the same field strength as that of $A_i$, and that both $\frac{1}{\sqrt{2}} R_{ikjk}$ and $\frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$ give the same curvature as that of $h_{ij}$:

$$F_{ij} \left( \frac{1}{\sqrt{2}} F_{ij,i} \right) = \left( \frac{1}{\sqrt{2}} F_{kj,k} \right)_i - \left( \frac{1}{\sqrt{2}} F_{ki,k} \right)_j = \frac{1}{\sqrt{2}} (F_{k,j} + F_{i,k})_j = \frac{1}{\sqrt{2}} F_{ij,k}.$$

for $R_{ijkl}$, we see that $\frac{1}{\sqrt{2}} R_{ikjk}$ is just $A_i^t$, $\frac{1}{\sqrt{2}} R_{ikjk}$ is just $\hat{h}_{ij}^{CZ}$, and $\frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$ is just $\hat{h}_{ij}^{(0)}$.

One can now understand, in two ways, why a tensor gauge field cannot be uniquely decomposed into an invariant part plus a pure-gauge:

(i) Since $\frac{1}{\sqrt{2}} R_{ikjk}$ and $\frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$ are both competent invariants, so is the weighted combination, $\alpha \frac{1}{\sqrt{2}} R_{ikjk} + (1 - \alpha) \frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl}$, with $\alpha$ an arbitrary parameter.

(ii) The difference of two competent invariants makes a peculiar gauge-invariant pure-gauge:

$$\frac{1}{\sqrt{2}} R_{ikjk} - \frac{1}{\sqrt{2}} \frac{1}{2} R_{ikjl,kl} = \frac{1}{\sqrt{2}} (h_{kk} - \frac{1}{\sqrt{2}} h_{kl,kl})_{ij} = (\Xi, j) + (\Xi, i),$$

where $\Xi = \frac{1}{\sqrt{2}} (h_{kk} - \frac{1}{\sqrt{2}} h_{kl,kl})$ (42)

The existence of a gauge-invariant pure-gauge also explains the non-uniqueness of complete tensor gauge condition. Since a competent pure-gauge carries all the gauge degrees of freedom, a constraint would qualify as a complete gauge condition were it able to set a competent pure-gauge to be identically zero. For a vector gauge field, the competent pure-gauge is unique: $\hat{A}_i = A_i^1 = (\frac{1}{\sqrt{2}} A_{ij,j})_i$. This can be set identically zero by requiring $A_{i,j} = 0$, which is the unique constraint that can fix the (vector) gauge completely. For a tensor gauge field, however, the existence of infinitely many competent pure gauges (with the freedom of adding a gauge-invariant pure-gauge in Eq. (42) multiplied by an arbitrary factor) leads to infinitely many complete tensor gauge conditions. If employing our earlier derivations, the competent pure-gauge can be put in the form $\hat{h}_{ij} = \xi_{ij} + \xi_{ji}$, where $\xi_i$ takes the same expression as in Eq. (17d) with a free parameter $b$. $\xi_j$ (and thus $\hat{h}_{ij}$) can be made identically zero by the constraint $h_{ij,j} + bh_{jj,j} = 0$, which then qualifies as a complete tensor gauge condition (in Euclidean space).
In Minkowski space-time, one needs to pay attention that only the Laplacian operator has a well-defined inverse, while the d’Alembert operator $\Box = \partial^2_t - \partial^2_\vec{x}$ does not. (More exactly, inverting the Laplacian operator requires just boundary conditions, while inverting the d’Alembert operator would also require initial conditions which are unavailable.) Therefore, $\hat{A}_\mu$ is to be constructed with $\frac{1}{\sqrt{g}} F_{\mu,j}$, instead of $\frac{1}{\sqrt{g}} F_{\mu,\nu}$. Analogously, $\hat{h}_{\mu\nu}$ is to be constructed with $\frac{1}{\sqrt{g}} R_{\mu k \nu l}$, and $\frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} R_{\mu k \nu l; \nu l}$, instead of $\frac{1}{\sqrt{g}} R_{\mu \nu \rho \sigma}$ and $\frac{1}{\sqrt{g}} \frac{1}{\sqrt{g}} R_{\mu \nu \rho \sigma; \rho \sigma}$. Another fact worth noting is that $\hat{h}_{\mu\nu}$ can possess two free parameters, not just one. The reason is that $\hat{h}_{\mu\nu}$ can be added by a gauge-invariant pure-gauge $h_{\mu\nu} \equiv \xi_{\mu,\nu} + \xi_{\nu,\mu}$, where $\xi_0 = \alpha \Xi_j$ and $\xi_j = \beta \Xi_j$, with $\Xi$ in Eq. (12) and $\alpha, \beta$ being two free parameters. Accordingly, the complete tensor gauge conditions in Minkowski space-time also contain two free parameters, as formulated equivalently by $a$ and $b$ in Eqs. (14).

VII. SUMMARY AND DISCUSSION

In this paper we carefully examined and demonstrated why decomposing a tensor is much more tricky than decomposing a vector, even for the linear case. Concerning mathematical structures, it is the CZ decomposition that has exact correspondence to the simple (and unique) vector decomposition. Namely, by a (singly) nonlocal construction, a (uniquely) gauge-invariant field $\hat{h}^{CZ}_{\mu\nu}$ can be built out of $h_{\mu\nu}$. Like the transverse vector current $\vec{j}_\perp = j - \vec{\partial} \frac{1}{\sqrt{g}} \vec{\partial} \cdot \vec{j}$, the source for $\hat{h}^{CZ}_{\mu\nu}$ also contains at most a singly nonlocal structure.

If doubly nonlocal constructions are allowed, however, more nontrivial possibilities emerge for a tensor (but not for a vector). Especially, a peculiar quantity which is gauge-invariant but at the same time a pure-gauge can be constructed. It is essentially this peculiar quantity that leads to infinitely many ways of fixing the gauge completely, or equivalently, of decomposing the gauge field into invariant and pure-gauge parts. (From our demonstration, it is clear that such phenomena would arise for all gauge fields of spin $\geq 2$.)

A further and vital complication for the tensor gauge field is that its truly dynamical component (the TT part) does not show up automatically after all gauge degrees of freedom have been removed; and extraction of the TT part favors a different gauge than that of CZ. Our observation reveals vividly that tensor coupling is indeed extraordinarily nontrivial, and requires further careful studies, even at linear approximation.

Beyond the linear order, separation of a pure-gauge background requires very careful formulation. Moreover, the nonlinear Einstein equations are so hard to solve that, for the purpose of achieving certain sort of convenience, one may have to invent some particularly delicate field variables and gauge conditions [16-18]: some gauges are even too complicated to put into explicit forms [18]. It should be remarked, however, that the issue of gauge choice may not be just a matter of convenience. As Weinberg elaborated in Ref. [12], quantum Lorentz invariance for gravitation might not be guaranteed with an arbitrary gauge-fixing.

This work is supported by the China NSF Grants 10875082 and 11035003. X.S.C. is also supported by the NCET Program of the China Ministry of Education.

[1] N. Rosen, Phys. Rev. 57, 147 (1940).
[2] R. Arnowitt, S. Deser, and C.W. Misner, in Gravitation, L. Witten ed. (Wiley, New York, 1962), Chapter 7 (posted as arXiv:gr-qc/0405109); and references therein.
[3] J. Schwinger, Phys. Rev. 130, 1253 (1963).
[4] S. Deser, Ann. Inst. Henri Poincaré 7, 149 (1967).
[5] J.W. York, J. Math. Phys. 14, 456 (1973); see also a recent discussion by H.P. Pfeiffer and J.W. York, Phys. Rev. D 67, 044022 (2003).
[6] For a recent review of the nucleon spin problem, see, e.g., F. Myhrer and A.W. Thomas, J. Phys. G 37, 023101 (2010).
[7] X.S. Chen, X.F. Lü, W.M. Sun, F. Wang, and T. Goldman, Phys. Rev. Lett. 100, 232002 (2008).
[8] X.S. Chen, W.M. Sun, X.F. Lü, F. Wang, and T. Goldman, Phys. Rev. Lett. 103, 062001 (2009).
[9] X.S. Chen and B.C. Zhu, Phys. Rev. D 83, 084006 (2011).
[10] C. Misner, K. Thorne, and J. Wheeler, Gravitation (Freeman, San Francisco, 1973), pp. 948-949 (Box 35.1).
[11] P.A.M. Dirac, Phys. Rev. 114, 924 (1959). See also the Lecture notes by É. Gourgoulhon, arXiv:gr-qc/0703035.
[12] S. Weinberg, Phys. Rev. 138, B988 (1965).
[13] X.S. Chen and B.C. Zhu, Phys. Rev. D 83, 061501 (2011).
[14] P. Jaranowski and G. Schaefer, Phys. Rev. D 57, 7274 (1998) [arXiv:gr-qc/9712075].
[15] T. Damour, P. Jaranowski, G. Schaefer, Phys. Lett. B 513, 147 (2001) [arXiv:gr-qc/0105038].
[16] B. Kol, arXiv:1009.1876.
[17] T. Damour and G. Schaefer, Gen. Rel. Grav. 17, 879 (1985).
[18] T. Ohta and T. Kimura, Prog. Theor. Phys. 79, 819 (1988); ibid. 76, 329 (1986); T. Ohta, H. Okamura, T. Kimura, and K. Hiida, ibid. 51, 1598 (1974); ibid. 50, 492 (1973).