The Klein-Gordon Equation for the Coulomb Potential in Non-commutative Space

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1 Abstract

In this paper the stationary Klein-Gordon equation is considered for the Coulomb potential in non-commutative space. The energy shift due to non-commutativity is obtained via the perturbation theory. Furthermore, we show that the degeneracy of the initial spectral line is broken in transition from commutative space to non-commutative space.

Keywords: Klein-Gordon equation; Coulomb potential; Non-commutative
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2 Introduction

Recently, there has been an increased interest in the study of the non-commutative field theory [1-2]. The most important motivation for studying these theories, comes mainly from the works that establish a connection between non-commutative geometry and string theory [3]. The investigation of these theories gives us the opportunity to understand interesting phenomena, such as non-locality and IR/UV mixing [4], new physics at very short distances [1-2], and possible implications of Lorentz violation [5-6]. Among these theories, the quantum mechanics is one of the simplest theories [7-8]. It is well-known that solutions of the relativistic wave equation play an essential role in the relativistic quantum mechanics for some physical potentials of interest [9-13]. Recently, there has been an increasing interest in finding exact
solutions of the Klein-Gordon (KG) equation [14-18]. In the past few years, exact solutions and energy eigenvalues of this equation have been presented for Scarf [19], Rosen-Morse type [20], Hulthen [21], Wood-saxon [22, 23], Posch-Teller [24], five-parameter exponential [25, 26], generalized symmetrical double-well [27], ring-shape harmonic oscillator [28], and pseudo harmonic oscillator [29] potentials, etc. In the above cited papers the scalar and vector potentials are almost taken to be equal in the relativistic framework. However, there is almost no explicit expression for the energy eigenvalues. Within the framework of non-commutativity, situation is more complicated and most models cannot be solved exactly. Accordingly, most of the available results are based upon perturbation theory [30-31]. This implies that a simple physical system in the commutative space may be changed into a complex theory within non-commutative framework.

Inclusion of non-commutativity into the quantum field theory can be achieved in two different ways: via Moyal product on the space of ordinary functions, or redefining the field theory on a coordinate operator space which is intrinsically non-commutative [32-33]. The equivalence between the two approaches has been described in references [34-35]. In the usual method, we introduce non-commutativity by means of non-commutative coordinates of position and momentum \((x, p)\) satisfying the following commutation relations

\[
[x_i , x_j] = i\hbar \epsilon_{ij}, \quad [x_i , p_j] = i\hbar \delta_{ij}, \quad [p_i , p_j] = 0, \quad i, j = 1, 2, 3
\]

where \(\hbar = \epsilon_{ij}\theta\), in which \(\epsilon_{ij}\) is Levicevita symbol and \(\theta\) is a parameter that measures the non-commutativity of coordinates. In the non-commutative space the ordinary product is replaced by Moyal product

\[
f(x) \star g(x) = \exp\left\{\frac{\hbar^2}{2} \theta^{jk} \partial_j^{(1)} \partial_k^{(2)} \right\} f(x_1) g(x_2) \big|_{x_1 = x_2 = x}
\]

where \(f(x)\) and \(g(x)\) are two arbitrary differentiable functions.

3 The Non-commutative Klein-Gordon Equation

In this section we consider the three dimensional Klein-Gordon equation for a long-range \(1/r\) interaction in the non-commutative space. For time
independent potentials, the KG equation for a particle of rest mass $M$ can be written as ($\hbar = c = 1$)

$$\{\nabla^2 + [V(r) - E]^2 - [S(r) + M]^2\} \psi(r) = 0 \quad (2)$$

in commutative space, where $E$ is the relativistic energy, $V(r)$ and $S(r)$ denote vector and scalar potentials, respectively. Recently, interest for considering of this equation with equal scalar and vector potentials has been increased [19-20]. Under assumption $V(r) = S(r)$, Eq. (2) takes the form

$$\{\nabla^2 + (E^2 - M^2) - 2(E + M)V(r)\} \psi(r) = 0. \quad (3)$$

By using the common separation of variables in the spherical polar coordinate $\psi(r) = Y(\theta, \phi) R(r)/r$, the radial part of this equation reads

$$\left\{ \frac{d^2}{dr^2} - [E_{eff} + V_{eff}(r)] \right\} R(r) = 0 \quad (4)$$

where

$$V_{eff}(r) = 2(M + E)V(r) + \ell(\ell + 1)/r^2, \quad E_{eff} = (M^2 - E^2). \quad (5)$$

Now to consider this equation in the non-commutative space, let us introduce the non-commuting coordinates in terms of the commuting coordinates and their momenta

$$\begin{cases} \hat{x}_i = x_i + \frac{1}{2} \theta_{ij} p_j, \\ \hat{p}_i = p_i. \end{cases} \quad (6)$$

Under these transformations a radial form potential takes the form

$$V(\hat{r}) = V(|\vec{r} - \frac{\vec{p}}{2}|)$$

$$= V\left(\sqrt{(x_i - \frac{1}{2} \theta_{ij} p_j)(x_i - \frac{1}{2} \theta_{ij} p_j)}\right)$$

$$= V(r) + \frac{1}{2}(\vec{\theta} \times \vec{p}) \cdot \vec{\nabla} V(r) + O(\theta^2)$$

$$= V(r) - \frac{\vec{\theta} \cdot \vec{E}}{2r} \frac{\partial V}{\partial r} + O(\theta^2)$$

$$\approx V(r) - \frac{\vec{\theta} \cdot \vec{E}}{2r} \frac{\partial V}{\partial r} \quad (7)$$
up to the first order of $\theta$, where $r = \sqrt{x_i x_i}$ and $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum operator.

By replacement of the ordinary product with Moyal, Eq. (6) takes the following form

$$\left\{ \frac{d^2}{dr^2} - [E_{\text{eff}} + V_{\text{eff}}(r)] \right\} \ast R_{n\ell}(r) = 0$$

in the non-commutative space, or equivalently

$$\left\{ \frac{d^2}{dr^2} - [E_{\text{eff}} + V_{\text{eff}}(|\vec{r} - \frac{1}{2}\vec{p}|)] \right\} R_{n\ell}(r) = 0. \quad (8)$$

Comparing Eq. (6) with Eq. (8) indicates that under the Moyal product the only modification in the radial part of the KG equation appears in the effective potential term. By substituting Coulomb potential $V(r) = -\frac{Ze^2}{r}$ into relation (5) and using effective potential (7) the last equation can be rewritten as

$$\left\{ \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} + 2(E + M)\frac{Ze^2}{r} - E_{\text{eff}} - \frac{(\vec{\theta} \cdot \vec{L})}{2r} \left( \frac{2\ell(\ell + 1)}{r^3} - 2(E + M)\frac{Ze^2}{r^2} \right) \right\} R(r) = 0. \quad (9)$$

By introducing dimensionless new variable $\rho = 2r\sqrt{E_{\text{eff}}}$, Eq. (9) is transformed into the following form

$$\left\{ \frac{d^2}{d\rho^2} - \frac{\ell(\ell + 1)}{\rho^2} + \frac{\varsigma}{\rho} - \frac{1}{4} - (\vec{\theta} \cdot \vec{L}) \left( \frac{4\ell(\ell + 1)}{\rho^3} E_{\text{eff}} - 2(1 + E/M)\sqrt{E_{\text{eff}}} \frac{Ze^2}{\rho^2} \right) \right\} R(\rho) = 0 \quad (10)$$

where $\varsigma = \frac{Ze^2}{M}\sqrt{1 + \frac{2E}{M-E}}$.

4 The Solution

The last equation has not yet been solved exactly in the presence of the last two terms, whereas in their absence, its exact solution is available [36]. To obtain the solution, we choose $\theta = 0$, and get

$$\left\{ \frac{d^2}{d\rho^2} - \frac{\ell(\ell + 1)}{\rho^2} + \frac{\varsigma}{\rho} - \frac{1}{4} \right\} R^{(0)}(\rho) = 0. \quad (11)$$

This is a second order differential equation and can be easily solved via Nikiforov-Uvarov (NU) mathematical method. In this method a second order linear differential equation is reduced to a generalized equation of hypergeometric type whose exact solutions are expressed in terms of special orthogonal functions [37], as well as corresponding eigenvalues are obtained.
To apply this method for Eq. (11), we compare this equation with the generalized hyper-geometric type equation
\[
\left\{ \frac{d^2}{d\rho^2} + \frac{\tilde{\tau}(\rho)}{\sigma(\rho)} \frac{d}{d\rho} + \frac{\tilde{\sigma}(\rho)}{\sigma^2(\rho)} \right\} R^{(0)}(\rho) = 0 \tag{12}
\]
and get
\[
\tilde{\tau}(\rho) = 0, \quad \sigma(\rho) = 2\rho, \quad \tilde{\sigma}(\rho) = -4\ell(\ell + 1) - \rho^2 + 4\varsigma \rho. \tag{13}
\]
Using these functions it is straightforward to show that the exact solution of Eq. (11) is [19]
\[
R^{(0)}(\rho) = N \rho^{\ell+1} \frac{(n-\ell-1)!}{(n+\ell)!} (2\ell+1)! L^{2\ell+1}_{n-\ell-1}(\rho)e^{-\frac{2}{\rho}}, \quad n = 0, 1, 2, \ldots \tag{14}
\]
where \( L^{2\ell+1}_{n-\ell-1}(\rho) \) denotes the generalized Laguerre polynomials and \( N \) is normalization constant
\[
N = \sqrt{\frac{(n+\ell)!}{2|E^{(0)}|n(n-\ell-1)! (2\ell+1)!}} \tag{15}
\]
in which \( E^{(0)} \) is the energy eigenvalues and is given by
\[
E^{(0)} = \left\{ \frac{(Z\alpha)^2 - (n-\ell)^2M^2}{(Z\alpha)^2 + (n-\ell)^2M^2} \right\} M, \quad n = 0, 1, 2, \ldots \tag{16}
\]
Now, to obtain the modification of energy levels as a result of the last two terms in Eq. (10) due to the non-commutativity, we use perturbation theory. For simplicity, first of all we take \( \theta_3 = \theta \) and assume that the other \( \theta \)-components are zero (by rotation or redefinition of coordinates), such that \( \vec{\theta} \cdot \vec{L} = \ell \theta \). In addition, we use
\[
<nlm|L_z|nl'm'> = m\delta_{mm'}, \quad -l \leq m \leq l
\]
and also the fact that in the first order perturbation theory the expectation value of \( \rho^{-3} \) and \( \rho^{-4} \) with respect to the exact solution of Eq. (11), are given by [38]
\[
< n|\rho^{-3}|n > = \int \{ R^{(0)}(\rho) \}^2 \rho^{-1} d\rho = \frac{1}{2|E^{(0)}|} \frac{1}{\ell(2\ell + 1)(2\ell + 2)}
\]
\[
< n|\rho^{-4}|n > = \int \{ R^{(0)}(\rho) \}^2 \rho^{-2} d\rho = \frac{1}{n|E^{(0)}|} \frac{\Gamma(2\ell - 1)}{\Gamma(2\ell + 4)} [3n^2 - \ell(\ell + 1)].
\]
Putting these results together, one gets

$$\Delta E_{NC} = \frac{m\theta}{4(2\ell+1)|E^0|} \left\{ \frac{(3n^2 - \ell(\ell + 1))}{n(2\ell - 1)(2\ell + 3)} - \frac{2(n - \ell)^2 Z\alpha}{\ell(\ell + 1)(n - \ell)^2 + (Z\alpha/M)^2} \right\}, \quad n = 0, 1, 2, \ldots$$

This is energy shift due to the additional last two terms of Eq. (10). The appearance of the magnetic quantum number $m$ in this expression explicitly indicates the splitting of states with the same orbital angular momentum into the corresponding components. In fact each level $\ell$ splits into $2\ell + 1$ sublevels and subsequently breaks the degeneracy of the initial spectral line. The lifting of degeneracy is due to the emergence of a magnetic field associated with the non-commutative space in transition from commutative space into non-commutative space. This behavior is similar to the Zeeman effect. In addition, it is worth noting that the correction terms containing $\vec{\theta} \cdot \vec{L}$ are very similar to that of the spin orbit coupling, in which the non-commutative parameter $\vec{\theta}$ plays the role of the spin.

5 Conclusion

In this paper, we have investigated the Klein-Gordon equation for the Coulomb potential in the non-commutative space. The energy shift, due to the non-commutativity, is obtained via first order perturbation theory. It is explicitly shown that the degeneracy of the initial spectral line is broken in transition from commutative space into non-commutative space by splitting states into the corresponding components. This behavior is similar to the Zeeman effect in which a magnetic field is applied to the system. In this space the non-commutative parameter $\vec{\theta}$ plays the role of the spin.

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