On Galois equivariance of homomorphisms between torsion potentially crystalline representations

Yoshiyasu Ozeki

Abstract

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field. Let $(\pi_n)_{n \geq 0}$ be a system of $p$-power roots of a uniformizer $\pi = \pi_0$ of $K$ with $\pi_p^{n+1} = \pi_n$, and define $G_s$ (resp. $G_\infty$) the absolute Galois group of $K(\pi_s)$ (resp. $K_\infty := \bigcup_{n \geq 0} K(\pi_n)$). In this paper, we study $G_s$-equivariant properties of $G_\infty$-equivariant homomorphisms between torsion (potentially) crystalline representations.

Contents

1 Introduction 2

2 Preliminaries 4
  2.1 Basic notations ............................................... 4
  2.2 Kisin modules .................................................. 5
  2.3 $(\varphi, \hat{G})$-modules ...................................... 5
  2.4 $(\varphi, \hat{G})$-modules, Breuil modules and filtered $(\varphi, N)$-modules .................................................. 8
  2.5 Base changes for Kisin modules ................................. 9
  2.6 Base changes for $(\varphi, \hat{G})$-modules ......................... 9

3 Variants of free $(\varphi, \hat{G})$-modules 10
  3.1 Definitions .................................................... 10
  3.2 The functors $\Mod_{/\hat{G}}^{r, G_s, J} \to \Mod_{/\hat{G}}^{r, G_s} \to \Mod_{/\hat{G}}^{r, G_s} \to \Mod_{/\hat{G}}^{r, G_s} .................................................. 11
  3.3 Relations with crystalline representations ...................... 13

4 Variants of torsion $(\varphi, \hat{G})$-modules 14
  4.1 Full faithfulness for $\Mod_{/\hat{G}}^{r, G_s, J} .................................................. 14
  4.2 The category $\Rep_{tor}^{r, G_s, J}(G_s) ................................. 15
  4.3 Full faithfulness theorem for $\Rep_{tor}^{r, G_s, J}(G_s) .................. 16
  4.4 Proof of Theorem 1.2 .......................................... 18
  4.5 Proof of Theorem 1.4 .......................................... 20
  4.6 Galois equivariance for torsion semi-stable representations .......... 21
  4.7 Some consequences ............................................. 22

5 Crystalline lifts and $c$-weights 23
  5.1 General properties of $c$-weights ................................ 24
  5.2 Rank 2 cases .................................................. 24
  5.3 Extensions of $\mathbb{F}_p$ by $\mathbb{F}_p(1)$ and non-fullness theorems .......... 27

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN.
e-mail: yozeki@kurims.kyoto-u.ac.jp
Supported by the JSPS Fellowships for Young Scientists.
1 Introduction

Let \( p \) be a prime number and \( r \geq 0 \) an integer. Let \( K \) be a complete discrete valuation field of mixed characteristic \((0, p)\) with perfect residue field and absolute ramification index \( e \). Let \( \pi = \pi_0 \) be a uniformizer of \( K \) and \( \pi_n \) a \( p^n \)-th root of \( \pi \) such that \( \pi_{n+1}^p = \pi_n \) for all \( n \geq 0 \). For any integer \( s \geq 0 \), we put \( K_{(s)} = K(\pi_s) \). We also put \( K_\infty = \bigcup_{n \geq 0} K_{(n)} \). We denote by \( G_K, G_s \) and \( G_\infty \) absolute Galois groups of \( K, K_{(s)} \) and \( K_\infty \), respectively. By definition we have the following decreasing sequence of Galois groups:

\[
G_K = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_\infty.
\]

Since \( K_\infty \) is a strict APF extension of \( K \), the theory of fields of norm implies that \( G_\infty \) is isomorphic to the absolute Galois group of some field of characteristic \( p \). Therefore, representations of \( G_\infty \) have easy interpretations via Fontaine’s étale \( \varphi \)-modules. Hence it seems natural to have the following question:

**Question 1.1.** Let \( T \) be a representation of \( G_K \). For a “small” integer \( s \geq 0 \), can we reconstruct various information of the \( G_s \)-action on \( T \) from that of the \( G_\infty \)-action?

Nowadays there is an interesting insight of Breuil: he showed that representations of \( G_K \) arising from finite flat group schemes or \( p \)-divisible groups over the integer ring of \( K \) is “determined” by its restriction to \( G_\infty \). Moreover, for \( \mathbb{Q}_p \)-representations, Kisin proved the following theorem in [Kis] (which was a conjecture of Breuil): the restriction functor from the category of crystalline \( \mathbb{Q}_p \)-representations of \( G_K \) into the category of \( \mathbb{Q}_p \)-representations of \( G_\infty \) is fully faithful.

In this paper, we give some partial answers to Question 1.1 for torsion crystalline representations. Our first main result is as follows.

Let \( \text{Rep}_\text{tor}^{r, \text{ht}, \text{pcris}(s)}(G_K) \) be the category of torsion \( \mathbb{Z}_p \)-representations \( T \) of \( G_K \) which satisfy the following: there exist free \( \mathbb{Z}_p \)-representations \( L \) and \( L' \) of \( G_K \), of height \( \leq r \), such that

- \( L|_{G_s} \) is a subrepresentation of \( L'|_{G_s} \). Furthermore, \( L|_{G_s} \) and \( L'|_{G_s} \) are lattices in some crystalline \( \mathbb{Q}_p \)-representation of \( G_s \) with Hodge-Tate weights in \([0, r]\);
- \( T|_{G_s} \simeq (L'|_{G_s})/(L|_{G_s}) \).

**Theorem 1.2.** Suppose that \( p \) is odd and \( e(r-1) < p-1 \). Let \( T \) and \( T' \) be objects of \( \text{Rep}_\text{tor}^{r, \text{ht}, \text{pcris}(s)}(G_K) \). Then any \( G_\infty \)-equivariant homomorphism \( T \to T' \) is in fact \( G_s \)-equivariant.

We should remark that the condition \( e(r-1) < p-1 \) in the above does not depend on \( s \). We put \( \text{Rep}_\text{tor}^{r, \text{pcris}}(G_K) = \text{Rep}_\text{tor}^{r, \text{ht}, \text{pcris}(0)}(G_K) \). By definition, a torsion \( \mathbb{Z}_p \)-representation \( T \) of \( G_K \) is contained in this category if and only if it can be written as the quotient of lattices in some crystalline \( \mathbb{Q}_p \)-representation of \( G_K \) with Hodge-Tate weights in \([0, r]\). We call the objects in \( \text{Rep}_\text{tor}^{r, \text{pcris}}(G_K) \) torsion crystalline representations with Hodge-Tate weights in \([0, r]\). In the case \( r = 1 \), such representations are equivalent to finite flat representations. (Here, a torsion \( \mathbb{Z}_p \)-representation of \( G_K \) is finite flat if it arises from the generic fiber of some \( p \)-power order finite flat commutative group scheme over the integer ring of \( K \).) Combining Theorem 1.2 with results of [Kim], [La], [Li4] (the case \( p = 2 \)) we obtain the following full faithfulness theorem for torsion crystalline representations.

**Corollary 1.3** (Full Faithfulness Theorem). Suppose \( e(r-1) < p-1 \). Then the restriction functor \( \text{Rep}_\text{tor}^{r, \text{pcris}}(G_K) \to \text{Rep}_\text{tor}(G_\infty) \) is fully faithful.

Before this work, some results are already known. First, the full faithfulness theorem was proved by Breuil for \( e = 1 \) and \( r < p-1 \) via the Fontaine-Laffaille theory ([Br2], the proof of Théorème 5.2). He also proved the theorem under the assumptions \( p > 2 \) and \( r \leq 1 \) as a consequence of a study of the category of finite flat group schemes ([Br3, Theorem 3.4.3]). Later, his result was extended to the case \( p = 2 \) in [Kim], [La], [Li4] (proved independently). In particular, the full
faithfulness theorem for \( p = 2 \) is a consequence of their works. On the other hand, Abrashkin proved the full faithfulness in the case where \( p > 2, r < p \) and \( K \) is a finite unramified extension of \( \mathbb{Q}_p \) (Section 8.3.3). His proof is based on calculations of ramification bounds for torsion crystalline representations. In [Oz2], a proof of Corollary 5.13 under the assumption \( e < p - 1 \) is given via \((\varphi, \hat{G})\)-modules (which was introduced by Tong Liu [Li2] to classify lattices in semi-stable representations).

Our proof of Theorem 1.2 is similar to the proof for the main result of [Oz2], but we need more careful considerations for \((\varphi, \hat{G})\)-modules. Indeed we need special base change arguments to study some potential crystalline representations. In fact, we prove a full faithfulness theorem for torsion representations arising from certain classes of \((\varphi, \hat{G})\)-modules (cf. Theorem 4.13), which immediately gives our main theorem. In addition, our study gives a result as below which is the second main result of this paper (here, we define \( \log_p(x) := -\infty \) for any real number \( x \leq 0 \)).

**Theorem 1.4.** Suppose that \( p \) is odd and \( s > n - 1 + \log_p(r - (p - 1)/e) \). Let \( T \) and \( T' \) be objects of \( \text{Rep}^{r, \text{cris}}_\text{tor}(G_K) \) which are killed by \( p^n \). Then any \( G_\infty \)-equivariant homomorphism \( T \rightarrow T' \) is in fact \( G_s \)-equivariant.

For torsion semi-stable representations, a similar result was shown in Theorem 3 of [CL2], which was a consequence of a study of ramification bounds. The bound appearing in their theorem was \( n - 1 + \log_p(r) \). By applying our arguments given in this paper, we can obtain a generalization of their result; our refined condition is \( s > n - 1 + \log_p(r) \) (see Theorem 4.14). Some other consequences of our study are described in subsection 4.7. Motivated by the full faithfulness theorem (= Corollary 1.3) and Theorem 1.4, we raise the following question.

**Question 1.5.** Is any \( G_\infty \)-equivariant homomorphism in the category \( \text{Rep}^{r, \text{cris}}_\text{tor}(G_K) \) in fact \( G_s \)-equivariant for \( s > \log_p(r - (p - 1)/e) \)?

On the other hand, there exist counter examples of the full faithfulness theorem when we ignore the condition \( e(r - 1) < p - 1 \). Let \( \text{Rep}_\text{tor}(G_1) \) be the category of torsion \( \mathbb{Z}_p \)-representations of \( G_1 \).

**Theorem 1.6** (\( = \) Special case of Corollary 5.13). Suppose that \( K \) is a finite extension of \( \mathbb{Q}_p \), and also suppose \( e \mid (p - 1) \) or \( (p - 1) \mid e \). If \( e(r - 1) \geq p - 1 \), the restriction functor \( \text{Rep}^{r, \text{cris}}_\text{tor}(G_K) \rightarrow \text{Rep}_\text{tor}(G_1) \) is not full (in particular, the restriction functor \( \text{Rep}^{r, \text{cris}}_\text{tor}(G_K) \rightarrow \text{Rep}_\text{tor}(G_\infty) \) is not full).

In particular, if \( p = 2 \), then the full faithfulness never hold for any finite extension \( K \) of \( \mathbb{Q}_2 \) and any \( r \geq 2 \). Theorem 1.6 implies that the condition \( "e(r - 1) < p - 1" \) in Corollary 1.3 is the best possible for many finite extensions \( K \) of \( \mathbb{Q}_p \).

Now we describe the organization of this paper. In Section 2, we set up notations and summarize facts we need later. In Section 3, we define variant notions of \((\varphi, \hat{G})\)-modules and give some basic properties. They are needed to study certain classes of potentially crystalline representations and restrictions of semi-stable representations. In Section 4, we study technical torsion \((\varphi, \hat{G})\)-modules which are related with torsion (potentially) crystalline representations. The key result in this section is the full faithfulness result Theorem 1.3 on them, which allows us to prove our main results immediately. Finally, in Section 5, we calculate the smallest integer \( r \) for a given torsion representation \( T \) such that \( T \) admits a crystalline lift with Hodge-Tate weights in \([0, r]\). We mainly study the rank two case. We use our full faithfulness theorem to assure the non-existence of crystalline lifts with small Hodge-Tate weights. Theorem 1.6 is a consequence of studies of this section.

**Acknowledgements.** The author would like to thank Shin Hattori, Naoki Imai and Yuichiro Taguchi who gave him many valuable advice. This work was supported by JSPS KAKENHI Grant Number 25·173.

**Notation and convention:** Throughout this paper, we fix a prime number \( p \). Except Section 5, we always assume that \( p \) is odd.
For any topological group $H$, we denote by $\text{Rep}_{\text{tor}}(H)$ (resp. $\text{Rep}_{\text{Qp}}(H)$, resp. $\text{Rep}_{\text{Qp}}(H)$) the category of torsion $\mathbb{Z}_p$-representations of $H$ (resp. the category of free $\mathbb{Z}_p$-representations of $H$, resp. the category of $\mathbb{Q}_p$-representations of $H$). All $\mathbb{Z}_p$-representations (resp. $\mathbb{Q}_p$-representations) in this paper are always assumed to be finitely generated over $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$).

For any field $F$, we denote by $G_F$ the absolute Galois group of $F$ (for a fixed separable closure of $F$).

## 2 Preliminaries

In this section, we recall definitions and basic properties for Kisin modules and $(\varphi, \hat{G})$-modules. Throughout Section 2, 3 and 4, we always assume that $p$ is an odd prime.

### 2.1 Basic notations

Let $k$ be a perfect field of characteristic $p$, $W(k)$ the ring of Witt vectors with coefficients in $k$, $K_0 = W(k)[1/p]$, $K$ a finite totally ramified extension of $K_0$ of degree $e$, $K$ a fixed algebraic closure of $K$. Throughout this paper, we fix a uniformizer $\pi$ of $K$. Let $E(u)$ be the minimal polynomial of $\pi$ over $K_0$. Then $E(u)$ is an Eisenstein polynomial. For any integer $n \geq 0$, we fix a system $(\pi_n)_{n \geq 0}$ of a $p^n$-th root of $\pi$ in $K$ such that $\pi_{n+1} = \pi_n$. Let $R = \varprojlim \mathcal{O}_K/p$, where $\mathcal{O}_K$ is the integer ring of $F$ and the transition maps are given by the $p$-th power map. For any integer $s \geq 0$, we write $\pi_s := (\pi_{s+n})_{n \geq 0} \in R$ and $\pi_s := \pi_s \in R$. Note that we have $\pi_s p = \pi_s$

Let $L$ be the completion of an unramified algebraic extension of $K$ with residue field $k_L$. Then $\pi_L$ is a uniformizer of $L(\pi_L)$ and $L(\pi_L)$ is a totally ramified degree $e p^s$ extension of $L_0 := W(k_L)[1/p]$. We set $L_{\infty} := \bigcup_{n \geq 0} L(n)$. We put $G_{L,s} := G_{L,s} = \text{Gal}(\overline{L}/L(\pi_L))$ and $G_{L,\infty} := G_{L,\infty} = \text{Gal}(\overline{L}/L_{\infty})$. By definitions, we have $L = L(0)$ and $G_{L,0} = G_L$. Put $\mathfrak{g}_{L,s} = W(k_L)[\pi_s]$ (resp. $\mathfrak{g}_L = W(k_L)[\pi]$) with an indeterminate $u_s$ (resp. $u$). We equip a Frobenius endomorphism $\varphi$ of $\mathfrak{g}_{L,s}$ (resp. $\mathfrak{g}_L$) by $u_s \mapsto u_s^p$ (resp. $u \mapsto u^p$) and the Frobenius on $W(k_L)$. We embed the $W(k_L)$-algebra $W(k_L)[u_s]$ (resp. $W(k_L)[u]$) into $W(R)$ via the map $u_s \mapsto [\pi_s]$ (resp. $u \mapsto [\pi]$), where $[\pi]$ stands for the Teichmüller representative. This embedding extends to an embedding $\mathfrak{g}_{L,s} \hookrightarrow W(R)$ (resp. $\mathfrak{g}_L \hookrightarrow W(R)$). By identifying $u$ with $u_s^p$, we regard $\mathfrak{g}_L$ as a subalgebra of $\mathfrak{g}_{L,s}$. It is readily seen that the embedding $\mathfrak{g}_L \hookrightarrow \mathfrak{g}_{L,s} \hookrightarrow W(R)$ is compatible with the Frobenius endomorphisms. If we denote by $E_s(u_s)$ the minimal polynomial of $\pi_s$ over $K_0$, with indeterminate $u_s$, then we have $E_s(u_s) = E(u_s^p)$.

Therefore, we have $E_s(u_s) = E(u)$ in $\mathfrak{g}_{L,s}$. We note that the minimal polynomial of $\pi_s$ over $L_0$ is $E_s(u_s)$.

Let $S_{L,0,s}$ (resp. $S_{L,0,s}^{\int}$) be the $p$-adic completion of the divided power envelope of $W(k_L)[u_s]$ (resp. $W(k_L)[u]$) with respect to the ideal generated by $E_s(u_s)$ (resp. $E(u)$). There exists a unique Frobenius map $\varphi: S_{L,0,s} \rightarrow S_{L,0,s}^{\int}$ (resp. $\varphi: S_{L,0} \rightarrow S_{L,0}^{\int}$) and monodromy $N: S_{L,0,s}^{\int} \rightarrow S_{L,0,s}^{\int}$ defined by $\varphi(u_s) = u_s^p$ (resp. $\varphi(u) = u^p$) and $N(u_s) = -u_s$ (resp. $N(u) = -u$). Put $S_{L,0,s} = S_{L,0,s}^{\int}[1/p] = L_0 \otimes W(k_L) S_{L,0,s}^{\int}$ (resp. $S_{L,0} = S_{L,0}^{\int}[1/p] = L_0 \otimes W(k_L) S_{L,0}^{\int}$). We equip $S_{L,0,s}$ and $S_{L,0,s}$ (resp. $S_{L,0}^{\int}$ and $S_{L,0}$) with decreasing filtrations $\text{Fil}^s S_{L,0,s}$ and $\text{Fil}^s S_{L,0,s}$ (resp. $\text{Fil}^s S_{L,0,s}$ and $\text{Fil}^s S_{L,0,s}$) by the $p$-adic completion of the ideal generated by $E_s^j(u_s)/j!$ (resp. $E^j(u)/j!$) for all $j \geq 0$. The inclusion $W(k_L)[u_s] \hookrightarrow W(R)$ (resp. $W(k_L)[u] \hookrightarrow W(R)$) via the map $u_s \mapsto [\pi_s]$ (resp. $u \mapsto [\pi]$) induces $\varphi$-compatible inclusions $\mathfrak{g}_{L,s} \hookrightarrow S_{L,0,s}^{\int} \hookrightarrow A_{\text{cris}}$ and $S_{L,0,s} \hookrightarrow B_{\text{cris}}^+$. By these inclusions, we often regard these rings as subrings of $B_{\text{cris}}^+$. By identifying $u$ with $u_s$ as before, we regard $S_{L,0}$ (resp. $S_{L,0}$) as a $\varphi$-stable (but not $N$-stable) subalgebra of $S_{L,0}$ (resp. $S_{L,0,s}$). By definitions, we have $\hat{g}_L = \hat{S}_{L,0}$, $S_{L,0}^{\int} = S_{L,0}$ and $S_{L,0,s} = S_{L,0}$.

**Convention:** For simplicity, if $L = K$, then we often omit the subscript “$L$” from various notions (e.g. $G_K = G_s$, $G_{K,\infty} = G_{\infty}$, $\hat{g}_K = \hat{g}_s$, $\hat{g}_{K,s} = \hat{g}_s$).
2.2 Kisin modules

Let \( r, s \geq 0 \) be integers. A \( \varphi \)-module over \( \mathcal{S}_{L,s} \) is an \( \mathcal{S}_{L,s} \)-module \( \mathcal{M} \) equipped with a \( \varphi \)-semilinear map \( \varphi : \mathcal{M} \rightarrow \mathcal{M} \). A morphism between two \( \varphi \)-modules \((\mathcal{M}_1, \varphi_1)\) and \((\mathcal{M}_2, \varphi_2)\) over \( \mathcal{S}_{L,s} \) is an \( \mathcal{S}_{L,s} \)-linear map \( \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) compatible with \( \varphi_1 \) and \( \varphi_2 \). Denote by \( \text{Mod}^r_{/\mathcal{S}_{L,s}} \), the category of \( \varphi \)-modules \((\mathcal{M}, \varphi)\) over \( \mathcal{S}_{L,s} \) of height \( \leq r \) in the sense that \( \mathcal{M} \) is of finite type over \( \mathcal{S}_{L,s} \) and the cokernel of \( 1 \otimes \varphi : \mathcal{S}_{L,s} \otimes \varphi, \mathcal{S}_{L,s} \mathcal{M} \rightarrow \mathcal{M} \) is killed by \( E_s(u_s)^r \).

Let \( \text{Mod}^r_{/\mathcal{S}_{L,s}} \) be the full subcategory of \( \text{Mod}^r_{/\mathcal{S}_{L,s}} \) consisting of finite free \( \mathcal{S}_{L,s} \)-modules. We call an object of \( \text{Mod}^r_{/\mathcal{S}_{L,s}} \) a free Kisin module of height \( \leq r \) (over \( \mathcal{S}_{L,s} \)).

Let \( \text{Mod}^{t, r}_{/\mathcal{S}_{L,s}} \) be the full subcategory of \( \text{Mod}^r_{/\mathcal{S}_{L,s}} \) consisting of finite \( \mathcal{S}_{L,s} \)-modules which are killed by some power of \( p \) and have projective dimension 1 in the sense that \( \mathcal{M} \) has a two term resolution by finite free \( \mathcal{S}_{L,s} \)-modules. We call an object of \( \text{Mod}^{t, r}_{/\mathcal{S}_{L,s}} \) a torsion Kisin module of height \( \leq r \) (over \( \mathcal{S}_{L,s} \)).

For any free or torsion Kisin module \( \mathcal{M} \) over \( \mathcal{S}_{L,s} \), we define a \( \mathbb{Z}_p[G_{L,\infty}] \)-module \( T_{\mathcal{E}_{L,s}}(\mathcal{M}) \) by

\[
T_{\mathcal{E}_{L,s}}(\mathcal{M}) := \begin{cases} 
\text{Hom}_{\mathcal{E}_{L,s}}(\mathcal{M}, W(R)) & \text{if } \mathcal{M} \text{ is free,} \\
\text{Hom}_{\mathcal{E}_{L,s}}(\mathcal{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathbb{Z}_p W(R)) & \text{if } \mathcal{M} \text{ is torsion.}
\end{cases}
\]

Here a \( G_{L,\infty} \)-action on \( T_{\mathcal{E}_{L,s}}(\mathcal{M}) \) is given by \( (\sigma, g)(x) = \sigma(g(x)) \) for \( \sigma \in G_{L,\infty}, g \in T_{\mathcal{E}}(\mathcal{M}), x \in \mathcal{M} \).

**Convention:** For simplicity, if \( L = K \), then we often omit the subscript “\( L \)” from various notations (e.g. \( \text{Mod}^r_{/\mathcal{E}_{K,s}} = \text{Mod}^r_{/\mathcal{E}_{s}} \), \( T_{\mathcal{E}_{K,s}} = T_{\mathcal{E}_{s}}(\sigma) \)). Also, if \( s = 0 \), we often omit the subscript “\( s \)” from various notations (e.g. \( \text{Mod}^r_{/\mathcal{E}_{L,0,\infty}} = \text{Mod}^r_{/\mathcal{E}_{L,\infty}}, T_{\mathcal{E}_{L,s}} = T_{\mathcal{E}_{L}}, \text{Mod}^r_{/\mathcal{E}_{K,0,\infty}} = \text{Mod}^r_{/\mathcal{E}_{\infty}}, T_{\mathcal{E}_{K,s}} = T_{\mathcal{E}_{s}}(\sigma) \)).

**Proposition 2.1.** (1) ([Kis] Corollary 2.1.4 and Proposition 2.1.12) The functor \( T_{\mathcal{E}_{L,s}} : \text{Mod}^r_{/\mathcal{E}_{L,s}} \rightarrow \text{Rep}_{G_{\infty}}(G_{\infty}) \) is exact and fully faithful.

(2) ([Coh] Corollary 2.1.6, 3.3.10 and 3.3.15) The functor \( T_{\mathcal{E}_{L,s}} : \text{Mod}^{t, r}_{/\mathcal{E}_{L,s}} \rightarrow \text{Rep}_{tor}(G_{\infty}) \) is exact and faithful. Furthermore, it is full if \( er < p - 1 \).

2.3 \((\varphi, \hat{G})\)-modules

The notion of \((\varphi, \hat{G})\)-modules are introduced by Tong Liu in [Li2] to classify lattices in semi-stable representations. We recall definitions and properties of them. We continue to use same notations
as above.

Let $L_p^\infty$ be the field obtained by adjoining all $p$-power roots of unity to $L$. We denote by $\hat{L}$ the composite field of $L_\infty$ and $L_p^\infty$. We define $H_L := \text{Gal}(\hat{L}/L_\infty)$, $H_{L,\infty} := \text{Gal}(K/L)$ $G_{L,p^\infty} := \text{Gal}(\hat{L}/L_p^\infty)$ and $\hat{G}_L := \text{Gal}(\hat{L}/L)$. Furthermore, putting $L_{(s),p^\infty} = L_{(s)}L_p^\infty$, we define $\hat{G}_{L,s} = \text{Gal}(\hat{L}/L_{(s)})$ and $G_{L,s,p^\infty} := \text{Gal}(\hat{L}/L_{(s),p^\infty})$.

![Galois groups of field extensions](Figure 2: Galois groups of field extensions)

Since $p > 2$, it is known that $L_{(s),p^\infty} \cap L_\infty = L_{(s)}$ and thus $\hat{G}_{L,s} = G_{L,s,p^\infty} \times H_{L,s}$. Furthermore, $G_{L,s,p^\infty}$ is topologically isomorphic to $\mathbb{Z}_p$.

**Lemma 2.2.** A natural map $G_{L,s,p^\infty} \to G_{K,s,p^\infty}$ defined by $g \mapsto g|_K$ is bijective.

**Proof.** By replacing $L_s$ with $L$, we may assume $s = 0$. It suffices to prove $\hat{K} \cap L_p^\infty = K_p^\infty$. Since $G_{K,p^\infty}$ is isomorphic to $\mathbb{Z}_p$, we know that any finite subextension of $\hat{K}/K_p^\infty$ is of the form $K_{(s),p^\infty}$ for some $s \geq 0$. Assume that we have $\hat{K} \cap L_p^\infty \neq K_p^\infty$. Then we have $K_{(1)} \subset \hat{K} \cap L_p^\infty \subset L_p^\infty$. Thus $\pi_1$ is contained in $L_p^\infty \cap L_\infty = L$. However, since $L$ is unramified over $K$, this contradicts the fact that $\pi$ is a uniformizer of $L$. \qed

We fix a topological generator $\tau$ of $G_{K,p^\infty}$. We also denote by $\tau$ the pre-image of $\tau \in G_{K,p^\infty}$ for the bijection $G_{L,p^\infty} \simeq G_{K,p^\infty}$ of the above lemma. Note that $\tau^{p^r}$ is a topological generator of $G_{L,s,p^\infty}$.

For any $g \in G_K$, we put $\varphi(g) = g(\pi)/\pi \in R$, and define $\varphi := \varphi(\hat{\tau})$. Here, $\hat{\tau} \in G_K$ is any lift of $\tau \in G_K$ and then $\varphi(\hat{\tau})$ is independent of the choice of the lift of $\tau$. With these notation, we also note that we have $g(u) = [\varphi(g)]u$ (recall that $\varphi$ is embedded in $W(R)$). An easy computation shows that $\tau(\varphi)/\varphi = \tau^{p^r}(\pi)/\pi = 1$. Therefore, we have $\varphi(u)/u = \tau^{p^r}(u)/u = \varphi$. We put $t = -\log(\varphi) \in A_{\text{cris}}$. Denote by $\nu: W(R) \to W(K)$ a unique lift of the projection $R \to K$, which extends to a map $\nu: B^{\text{cris}}_+ \to W(K)[1/p]$. For any subring $A \subset B^{\text{cris}}_+$, we put $I_{\lambda, A} = \text{Ker}(\nu)$ on $B^{\text{cris}}_+ \cap A$. For any integer $n \geq 0$, let $t^{(n)} := t^{\nu^{(n)}(\varphi)/p}$ where $n = (p-1)\varphi(n) + r(n)$ with $\varphi(n) \geq 0$, $0 \leq r(n) < p-1$ and $\gamma_i(x) = x^{p^i}$ is the standard divided power. We define a subring $R_{L_{n,s}}$ (resp. $R_{L_{n}}$) of $B^{\text{cris}}_+$ as below:

$$R_{L_{n,s}} := \left\{ \sum_{i=0}^{\infty} f_i t^{(i)} | f_i \in S_{L_{n,s}} \text{ and } f_i \to 0 \text{ as } i \to \infty \right\}$$

(resp. $R_{L_{n}} := \left\{ \sum_{i=0}^{\infty} f_i t^{(i)} | f_i \in S_{L_{n}} \text{ and } f_i \to 0 \text{ as } i \to \infty \right\}$).
Put $\hat{R}_{L,s} = R_{L,0,s} \cap W(R)$ (resp. $\hat{R}_L = R_{L,0} \cap W(R)$) and $I_{+,L,s} = I_+ \hat{R}_{L,s}$ (resp. $I_{+,L} = I_+ \hat{R}_L$). By definitions, we have $R_{L,0} = R_0$, $\hat{R}_L = R_L$ and $I_{+,L,0} = I_{+,L}$. Lemma 2.2.1 in [L2] shows that $\hat{R}_{L,s}$ (resp. $R_{L,0,s}$) is a $\varphi$-stable $\mathcal{S}_{L,s}$-subalgebra of $W(R)$ (resp. $B^{\text{cris}}_L$), and $\nu$ induces $R_{L,0,s} / I_+ R_{L,0,s} \simeq L_0$ and $\hat{R}_{L,s} / I_+ \hat{R}_{L,s} \simeq \tilde{S}^\text{int}_{L_0,s} / I_+ \tilde{S}^\text{cris}_{L_0,s} \simeq \mathcal{S}_{L,s} / I_+ \mathcal{S}_{L,s} \simeq W(k_L)$. Furthermore, $\hat{R}_{L,s} / I_+ \hat{R}_{L,s}, R_{L,0,s}$ and $I_+ R_{L,0,s}$ are $G_{L,s}$-stable, and $G_{L,s}$-actions on them factors through $G_{L,s}$.

For any torsion Kisin module $\mathcal{M}$ over $\mathcal{S}_{L,s}$, we equip $\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$ with a Frobenius by $\varphi_{\hat{R}_{L,s}} \otimes \varphi_{\mathcal{M}}$. It is known that the natural map $\mathcal{M} \to \hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$ given by $x \mapsto 1 \otimes x$ is an injection (cf. [Dz1] Corollary 2.12)). By this injection, we regard $\mathcal{M}$ as a $\varphi(\mathcal{S}_{L,s})$-stable submodule of $\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$.

**Definition 2.3.** A free (resp. torsion) $(\varphi, \hat{G}_{L,s})$-module of height $\leq r$ over $\mathcal{S}_{L,s}$ is a triple $\hat{\mathcal{M}} = (\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G}_{L,s})$ where

1. $(\mathcal{M}, \varphi_{\mathcal{M}})$ is a free (resp. torsion) Kisin module of height $\leq r$ over $\mathcal{S}_{L,s}$,
2. $\hat{G}_{L,s}$ is an $\hat{R}_{L,s}$-semilinear $\hat{G}_{L,s}$-action on $\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$ which induces a continuous $G_{L,s}$-action on $W(FrR) \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$ for the weak topology,
3. the $\hat{G}_{L,s}$-action commutes with $\varphi_{\hat{R}_{L,s}} \otimes \varphi_{\mathcal{M}}$,
4. $\mathcal{M} \subset (\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M})^{\hat{G}_{L,s}}$,
5. $\hat{G}_{L,s}$ acts on the $W(k_L)$-module $(\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}) / I_+ \hat{R}_{L,s}(\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M})$ trivially.

A morphism between two $(\varphi, \hat{G}_{L,s})$-modules $\hat{\mathcal{M}}_1 = (\mathcal{M}_1, \varphi_1, \hat{G}_1)$ and $\hat{\mathcal{M}}_2 = (\mathcal{M}_2, \varphi_2, \hat{G}_2)$ is a morphism $f: \hat{\mathcal{M}}_1 \to \hat{\mathcal{M}}_2$ of $\varphi$-modules over $\mathcal{S}_{L,s}$ such that $\hat{R}_{L,s} \otimes f: \hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}_1 \to \hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}_2$ is $\hat{G}_{L,s}$-equivariant. We denote by $\text{Mod}^{r,\hat{G}_{L,s}}_L$ (resp. $\text{Mod}^{t,\hat{G}_{L,s}}_L$) the category of free (resp. torsion) $(\varphi, \hat{G}_{L,s})$-modules of height $\leq r$ over $\mathcal{S}_{L,s}$. We often regard $\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$ as a $G_{L,s}$-module via the projection $G_{L,s} \twoheadrightarrow \hat{G}_{L,s}$.

For any free or torsion $(\varphi, \hat{G}_{L,s})$-module $\hat{\mathcal{M}}$ over $\mathcal{S}_{L,s}$, we define a $\mathbb{Z}_p[G_{L,s}]$-module $\hat{T}_{L,s}(\hat{\mathcal{M}})$ by

$$\hat{T}_{L,s}(\hat{\mathcal{M}}) = \begin{cases} \text{Hom}_{\hat{R}_{L,s}, \varphi}(\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}, W(R)) & \text{if } \mathcal{M} \text{ is free}, \\ \text{Hom}_{\hat{R}_{L,s}, \varphi}(\hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}, \mathbb{Q}_p / \mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathcal{M} \text{ is torsion.} \end{cases}$$

Here, $G_{L,s}$ acts on $\hat{T}_{L,s}(\hat{\mathcal{M}})$ by $(\sigma, f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G_{L,s}$, $f \in \hat{T}_{L,s}(\hat{\mathcal{M}})$, $x \in \hat{R}_{L,s} \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M}$.

Then, there exists a natural $G_{L,\infty}$-equivariant map

$$\theta_{\varphi, \mathcal{S}_{L,s}}: T_{\varphi, \mathcal{S}_{L,s}}(\hat{\mathcal{M}}) \to \hat{T}_{L,s}(\hat{\mathcal{M}})$$

defined by $(\theta(f))(a \otimes x) = a \varphi(f(x))$ for $f \in T_{\varphi, \mathcal{S}_{L,s}}(\hat{\mathcal{M}})$, $a \in \hat{R}_{L,s}, x \in \mathcal{M}$. We have

**Theorem 2.4** ([L2] Theorem 2.3.1 (1), [CL2] Theorem 3.1.3 (1)). The map $\theta_{\varphi, \mathcal{S}_{L,s}}$ is an isomorphism of $\mathbb{Z}_p[G_{L,\infty}]$-modules.

**Convention:** For simplicity, if $L = K$, then we often omit the subscript “$L$” from various notations (e.g. “a $(\varphi, \hat{G}_{K,s})$-module” = “a $(\varphi, \hat{G}_{s})$-module”, $\text{Mod}^{r,\hat{G}_{K,s}}_L = \text{Mod}^{r,\hat{G}_{s}}_s$, $\text{Mod}^{r,\hat{G}_{K,s}}_{L,\infty} = \text{Mod}^{r,\hat{G}_{s}}_{s,\infty}$, $T_{K,s} = \hat{T}_s$, $\theta_{K,s} = \theta_s$). Furthermore, if $s = 0$, we often omit the subscript “$s$” from various notations (e.g. $\text{Mod}^{r,\hat{G}_{L,0}}_L = \text{Mod}^{r,\hat{G}_{L}}_s$, $\text{Mod}^{r,\hat{G}_{L,0}}_{L,\infty} = \text{Mod}^{r,\hat{G}_{L}}_{s,\infty}$, $\hat{T}_{L,0} = \hat{T}_L$).

---

1 This continuity condition is needed to assure that the $G_{L,s}$-action on $\hat{T}_{L,s}(\hat{\mathcal{M}})$, defined after Definition 2.3 is continuous (here, note that $\hat{T}_{L,s}(\hat{\mathcal{M}})$ is the dual of $(W(FrR) \otimes_{\varphi, \mathcal{S}_{L,s}} \mathcal{M})^{\varphi=1}$; see Corollary 3.12 of [Dz1]).
\[
\text{Moq}_{/\Theta^{K,0}}^{\hat{G}, \theta} = \text{Moq}_{/\Theta}^{\hat{G}, \theta}, \quad \text{"a} (\varphi, \hat{G}, K, 0)\text{-module} = \text{"a} (\varphi, \hat{G})\text{-module"}, \quad \hat{T}_{K, 0} = \hat{T}, \quad \theta_{K, 0} = \theta.
\]

Let \( \text{Rep}_{\Theta}^{\text{st}}(G_{L, s}) \) (resp. \( \text{Rep}_{\Theta}^{\text{cris}}(G_{L, s}) \), resp. \( \text{Rep}_{\Theta}^{\text{st}}(G_{L, s}) \), resp. \( \text{Rep}_{\Theta}^{\text{cris}}(G_{L, s}) \)) be the categories of semi-stable \( \mathbb{Q}_p \)-representations of \( G_{L, s} \) with Hodge-Tate weights in \([0, r]\) (resp. the categories of crystalline \( \mathbb{Q}_p \)-representations of \( G_{L, s} \) with Hodge-Tate weights in \([0, r]\), resp. the categories of lattices in semi-stable \( \mathbb{Q}_p \)-representations of \( G_{L, s} \) with Hodge-Tate weights in \([0, r]\), resp. the categories of lattices in crystalline \( \mathbb{Q}_p \)-representations of \( G_{L, s} \) with Hodge-Tate weights in \([0, r]\)).

There exists \( t \in W(R) \setminus pW(R) \) such that \( \varphi(t) = pE(0)^{-1}E(u)t \). Such \( t \) is unique up to units of \( \mathbb{Z}_p \) (cf. \cite{Li2} Example 2.3.5]). Now we define the full subcategory \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \) (resp. \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \)) of \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \) (resp. \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \)) consisting of objects \( \mathcal{M} \) which satisfy the following condition: \( \tau(x) - x \in w^p \varphi(t)W(R) \otimes_{\varphi, \Theta} \mathcal{M} \) for any \( x \in \mathcal{M} \).

The following results are important properties for the functor \( \hat{T}_{L, s} \).

**Theorem 2.5.** (1) (\cite{Li2} Theorem 2.3.1 (2))) The functor \( \hat{T} \) induces an anti-equivalence of categories between \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \) and \( \text{Rep}_{\Theta}^{\text{cris}}(G_{K}) \).

(2) (\cite{GLS} Proposition 5.9, \cite{Oz2} Theorem 19)) The functor \( \hat{T} \) induces an anti-equivalence of categories between \( \text{Mod}_{/\Theta}^{\hat{G}, \text{cris}} \) and \( \text{Rep}_{\Theta}^{\text{cris}}(G_{K}) \).

(3) (\cite{Oz2} Corollary 2.8 and 5.34) The functor \( \hat{T}_{L, s} : \text{Mod}_{/\Theta_{L, \infty}}^{\hat{G}_{L, s}} \to \text{Rep}_{\Theta_{\infty}}(G_{L, s}) \) is exact and faithful. Furthermore, it is full if \( es < p - 1 \).

### 2.4. \((\varphi, \hat{G})\)-modules, Breuil modules and filtered \((\varphi, N)\)-modules

We recall some relations between Breuil modules and \((\varphi, \hat{G})\)-modules. Here we give a rough sketch only. For more precise information, see \cite{Br1} Section 6, \cite{Li1} Section 5 and the proof of \cite{Li2} Theorem 2.3.1 (2)).

Let \( \mathcal{M} \) be a free \((\varphi, \hat{G}_{L, s})\)-module over \( \Theta_{L, s} \). If we put \( \mathcal{D} := S_{L, s} \otimes_{\varphi, \Theta_{L, s}} \mathcal{M} \), then \( \mathcal{D} \) has a structure of a Breuil module over \( S_{L, s} \) which corresponds to the semi-stable representation \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{L, s}(\mathcal{M}) \) of \( G_{L, s} \) (for the definition and properties of Breuil modules, see \cite{Br1}). Thus \( \mathcal{D} \) is equipped with a Frobenius \( \varphi_D := \varphi_{S_{L, s}} \otimes \varphi_{\mathcal{M}} \), a decreasing filtration \( (\text{Fil}^n \mathcal{D})_{n \geq 0} \) of \( S_{L, s} \)-submodules of \( \mathcal{D} \) and a \( L_0 \)-linear monodromy operator \( N : \mathcal{D} \to \mathcal{D} \) which satisfies certain properties (for example, Griffiths transcendentalities).

Putting \( D := D/L_0, S_{L, s}D \), we can associate a filtered \((\varphi, N)\)-module over \( L_0, s \) as following:

\[
\varphi_D := \varphi_D \mod I + S_{L, s}D, \quad N_D := N_D \mod I + S_{L, s}D \quad \text{and} \quad \text{Fil}^n D := f_{\gamma}(\text{Fil}^n(D)).
\]

Here, \( f_{\gamma} : \mathcal{D} \to D/L_0 \) is the projection defined by \( \mathcal{D} \to D/\text{Fil}^n S_{L, s}D \simeq D/L_0 \). Proposition 6.2.1.1 of \cite{Br1} implies that there exists a unique \( \varphi \)-compatible section \( s : D \hookrightarrow \mathcal{D} \to \mathcal{D} \). By this embedding, we regard \( D \) as a submodule of \( \mathcal{D} \). Then we have \( N_D|D = N_D \) and \( N_D = N_{S_{L, s}} \otimes \text{Id}_D + \text{Id}_{S_{L, s}} \otimes N_D \). Under the identification \( D = S_{L, s} \otimes L_0 \), \( D \).

The \( \hat{G}_{L, s} \)-action on \( \hat{R}_{L, s} \otimes_{\varphi, \Theta_{L, s}} \mathcal{M} \) extends to \( B_{\text{cris}}^{+} \otimes \hat{R}_{L, s} \otimes_{\varphi, \Theta_{L, s}} \mathcal{M} \simeq B_{\text{cris}}^{+} \otimes S_{L, s} \mathcal{D} \) of \( \mathcal{D} \).

This action is in fact explicitly written as follows:

\[
g(a \otimes x) = \sum_{i=0}^{\infty} g(a) \gamma_i(- \log(g(a)_{\mathbb{Z}_p} / [x])) \otimes N_D^i(x) \quad \text{for} \quad g \in G_{L, s}, a \in B_{\text{cris}}^{+}, x \in D. \tag{2.4.1}
\]

By this explicit formula, we can obtain an easy relation between \( N_D \) and \( \tau^{p^r} \)-action on \( \mathcal{M} \) as follows: first we recall that \( t = - \log \tau(x) / [x] \) (resp. \( \log \tau^{p^r}(x) / [x] \)). By the formula, for any \( n \geq 0 \) and \( x \in D \), an induction on \( n \) shows that we have

\[
(\tau^{p^r} - 1)^n(x) = \sum_{m=n}^{\infty} \left( \sum_{\substack{i_1 + \cdots + i_n = m, i_j \geq 0 \atop t_1! \cdots t_n!}} \right) g_m(t) \otimes N_D^m(x) \in B_{\text{cris}}^{+} \otimes S_{L, s} \mathcal{D}
\]
and in particular we see \( \frac{(\tau^p-1)^n}{n} (x) \to 0 \) p-adically as \( n \to \infty \). Hence we can define
\[
\log(\tau^p)(x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau^p-1)^n}{n} (x) \in B_{\text{crys}}^+ \otimes \mathcal{S}_{L,s} \mathcal{D}.
\]
It is not difficult to check the equation
\[
\log(\tau^p)(x) = t \otimes N_D(x).
\]

2.5 Base changes for Kisin modules

Let \( \mathfrak{M} \) be a free or torsion Kisin module of height \( \leq r \) over \( \mathfrak{S}_L \) (resp. over \( \mathfrak{S} \)). We put \( \mathfrak{M}_{L,s} = \mathfrak{S}_{L,s} \otimes F \mathfrak{M} \) (resp. \( \mathfrak{S}_L = \mathfrak{S}_L \otimes \mathfrak{M} \) and equip \( \mathfrak{M}_{L,s} \) (resp. \( \mathfrak{M}_L \)) with a Frobenius by \( \varphi = \varphi_{\mathfrak{S}_{L,s}} \otimes \varphi_{\mathfrak{M}} \) (resp. \( \varphi = \varphi_{\mathfrak{S}_L} \otimes \varphi_{\mathfrak{M}} \)). Then it is not difficult to check that \( \mathfrak{M}_{L,s} \) (resp. \( \mathfrak{M}_L \)) is a free or torsion Kisin module of height \( \leq r \) over \( \mathfrak{S}_{L,s} \) (resp. over \( \mathfrak{S}_L \)) (here we recall that \( \mathfrak{E}_L(u_s) = E(u^p) = E(u) \)).

Hence we obtained natural functors
\[
\text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}} \quad \text{and} \quad \text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}}.
\]
By definition, we immediately see that we have \( T_{\mathfrak{S}_L}(\mathfrak{M}) \simeq T_{\mathfrak{S}_{L,s}}(\mathfrak{M}_{L,s}) \) (resp. \( T_{\mathfrak{S}_L}(\mathfrak{M}) \simeq T_{\mathfrak{S}_{L,s}}(\mathfrak{M}_{L,s}) \)). In particular, it follows from Proposition 2.1 (1) that the following holds:

**Proposition 2.6.** The functor \( \text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}} \) is fully faithful.

2.6 Base changes for \((\varphi, \breve{G})\)-modules

Let \( \breve{\mathfrak{M}} \) be a free or torsion \((\varphi, \breve{G}_L)\)-module (resp. \((\varphi, \breve{G})\)-module) of height \( \leq r \) over \( \mathfrak{S}_L \) (resp. over \( \mathfrak{S} \)). The \( G_{L,s} \) action on \( \breve{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} \) (resp. the \( G_L \) action on \( \breve{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} \)) extends to \( \breve{\mathcal{R}}_{L,s} \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_{L,s} \) (resp. \( \mathcal{R}_L \otimes_{\breve{\mathcal{R}}, \mathfrak{S}_L} \mathfrak{M}_{L,s} \)) (resp. \( \mathcal{R}_L \otimes_{\breve{\mathcal{R}}, \mathfrak{S}_L} \mathfrak{M}_{L,s} \)), which factors through \( \breve{G}_{L,s} \) (resp. \( \breve{G}_L \)). Then it is not difficult to check that \( \mathfrak{M}_{L,s} \) (resp. \( \mathfrak{M}_L \)) has a structure of a \((\varphi, \breve{G}_{L,s})\)-module (resp. \((\varphi, \breve{G}_L)\)-module). Hence we obtained natural functors
\[
\text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}} \quad \text{and} \quad \text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}}.
\]
By definition, we immediately see that we have \( \mathfrak{T}_{\mathfrak{S}_L}(\mathfrak{M}) \simeq \mathfrak{T}_{\mathfrak{S}_{L,s}}(\mathfrak{M}_{L,s}) \) (resp. \( \mathfrak{T}(\mathfrak{M}) \simeq \mathfrak{T}(\mathfrak{M}_{L,s}) \)). Similar to Proposition 2.6, we can prove the following:

**Proposition 2.7.** The functor \( \text{Mod}^r_{\mathfrak{S}_L} \to \text{Mod}^r_{\mathfrak{S}_{L,s}} \) is fully faithful.

The proposition immediately follows from the full faithfulness property of Theorem 2.5 (1) and the lemma below.

**Lemma 2.8.** Let \( K' \) be a finite totally ramified extension of \( K \). Then the restriction functor from the category of semi-stable \( \mathbb{Q}_p \)-representations of \( G_K \) into the category of semi-stable \( \mathbb{Q}_p \)-representations of \( G_{K'} \) is fully faithful.

**Proof.** Let \( V \) and \( V' \) be semi-stable \( \mathbb{Q}_p \)-representations of \( G_K \) and let \( f : V \to V' \) be a \( G_{K'} \)-equivariant homomorphism. Considering the morphism of filtered \((\varphi, N)\)-modules over \( K' \) corresponding to \( f \), we can check without difficulty that \( f \) is in fact a morphism of filtered \((\varphi, N)\)-modules over \( K \). This is because \( K' \) is totally ramified over \( K_0 \) as same as \( K \). This gives the desired result. □
3 Variants of free \((\varphi, \hat{G})\)-modules

In this section, we define some variant notions of \((\varphi, \hat{G})\)-modules. We continue to use same notation as in the previous section. In particular, \(p\) is odd.

3.1 Definitions

We start with some definitions which are our main concern in this and the next section.

Definition 3.1. We define the category \(\text{Mod}^{r}_{/\Theta_{L}}(\hat{G}_{L,s})\) (resp. \(\text{Mod}^{T}_{/\Theta_{L}}(\hat{G}_{L,s})\)) as follows. An object is a triple \(\mathcal{M} = (\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G}_{L,s})\) where

1. \((\mathcal{M}, \varphi_{\mathcal{M}})\) is a free Kisin module of height \(\leq r\) over \(\Theta_{L}\),
2. \(\hat{G}_{L,s}\) is an \(\hat{R}_{L}\)-semilinear \(\hat{G}_{L,s}\)-action on \(\hat{R}_{L} \otimes_{\Theta_{L}} \mathcal{M}\) (resp. an \(\hat{R}_{L}\)-semilinear \(\hat{G}_{L,s}\)-action on \(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M}\)) which induces a continuous \(\hat{G}_{L,s}\)-action on \(W(F_{r}R) \otimes_{\Theta_{L}} \mathcal{M}\) for the weak topology,
3. the \(\hat{G}_{L,s}\)-action commutes with \(\varphi_{\hat{R}_{L}} \otimes \varphi_{\mathcal{M}}\) (resp. \(\varphi_{\hat{R}_{L,s}} \otimes \varphi_{\mathcal{M}}\)),
4. \(\mathcal{M} \subset (\hat{R}_{L} \otimes_{\Theta_{L}} \mathcal{M})^{H_{L}}\) (resp. \(\mathcal{M} \subset (\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M})^{H_{L}}\)),
5. \(\hat{G}_{L,s}\) acts on the \(W(k_{L})\)-module \((\hat{R}_{L} \otimes_{\Theta_{L}} \mathcal{M})/I_{+L}(\hat{R}_{L} \otimes_{\Theta_{L}} \mathcal{M})\) (resp. \((\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M})/I_+L,s(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M}))\) trivially.

Morphisms are defined by the obvious way. By replacing “free” of (1) with “torsion\(^{\mathbf{3}}\)” we define the category \(\text{Mod}_{/\Theta_{L,\infty}}^{r}(\hat{G}_{L,s})\) (resp. \(\text{Mod}_{/\Theta_{L,\infty}}^{T}(\hat{G}_{L,s})\)).

For any object \(\mathcal{M}\) of \(\text{Mod}_{/\Theta_{L}}^{r}(\hat{G}_{L,s})\) or \(\text{Mod}_{/\Theta_{L}}^{r}(\hat{G}_{L,s})\), we define a \(Z_{p}[G_{L,s}]\)-module \(\hat{T}_{L,s}(\mathcal{M})\) by

\[
\hat{T}_{L,s}(\mathcal{M}) = \begin{cases} 
\text{Hom}_{\hat{R}_{L,s}}(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M}, W(R)) & \text{if } \mathcal{M} \text{ is free}, \\
\text{Hom}_{\hat{R}_{L,s}}(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M}, Q_{p}/Z_{p} \otimes W(R)) & \text{if } \mathcal{M} \text{ is torsion}.
\end{cases}
\]

Here, \(G_{L,s}\) acts on \(\hat{T}_{L,s}(\mathcal{M})\) by \((\sigma, f)(x) = \sigma(f(\sigma^{-1}(x)))\) for \(\sigma \in G_{L,s}\), \(f \in \hat{T}_{L,s}(\mathcal{M})\), \(x \in \hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{M}\). Similar to the above, for any object \(\mathcal{N}\) of \(\text{Mod}_{/\Theta_{L,\infty}}^{r}(\hat{G}_{L,s})\) or \(\text{Mod}_{/\Theta_{L,\infty}}^{T}(\hat{G}_{L,s})\), we define a \(Z_{p}[G_{L,s}]\)-module \(\hat{T}_{L,s}(\mathcal{N})\) by

\[
\hat{T}_{L,s}(\mathcal{N}) = \begin{cases} 
\text{Hom}_{\hat{R}_{L,s}}(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{N}, W(R)) & \text{if } \mathcal{N} \text{ is free}, \\
\text{Hom}_{\hat{R}_{L,s}}(\hat{R}_{L,s} \otimes_{\Theta_{L}} \mathcal{N}, Q_{p}/Z_{p} \otimes W(R)) & \text{if } \mathcal{N} \text{ is torsion}.
\end{cases}
\]

On the other hand, we obtain functors \(\text{Mod}_{/\Theta_{L}}^{r}(\hat{G}_{L}) \to \text{Mod}_{/\Theta_{L}}^{r}(\hat{G}_{L,s}) \to \text{Mod}_{/\Theta_{L}}^{T}(\hat{G}_{L,s})\) and \(\text{Mod}_{/\Theta_{L,\infty}}^{r}(\hat{G}_{L}) \to \text{Mod}_{/\Theta_{L,\infty}}^{r}(\hat{G}_{L,s}) \to \text{Mod}_{/\Theta_{L,\infty}}^{T}(\hat{G}_{L,s})\) by natural manners and it is readily seen that these functors are compatible with \(\hat{T}_{L}\) and \(\hat{T}_{L,s}\). In particular, the essential images of the functors \(\hat{T}_{L,s}\) on \(\text{Mod}_{/\Theta_{L}}^{T}(\hat{G}_{L,s})\) and \(\text{Mod}_{/\Theta_{L}}^{T}(\hat{G}_{L,s})\) has values in \(\text{Rep}_{Z_{p}}^{st}(G_{L,s})\) since we have an equivalence of categories \(\hat{T}_{L,s}: \text{Mod}_{/\Theta_{L}}^{T}(\hat{G}_{L,s}) \sim \text{Rep}_{Z_{p}}^{st}(G_{L,s})\) by Theorem \(\text{2.5}\).

In the rest of this section, we study free cases. We leave studies for torsion cases to the next section.

\(^{3}\)As explained in the footnote of Definition \(\text{2.3}\), we may ignore the continuity condition of (2) in the case where \(\mathcal{M}\) is torsion.
**Conventions:** For simplicity, if $L = K$, then we often omit the subscript “$L$” from various notations (e.g. $\text{Mod}^{r,\hat{G}_{K,L}}_\mathfrak{O}_K = \text{Mod}^{r,\hat{G}_K}_\mathfrak{O}_K$, $\text{Mod}^{r,\hat{G}_{0,L}}_\mathfrak{O} = \text{Mod}^{r,\hat{G}_0}_\mathfrak{O}$). Furthermore, if $s = 0$, we often omit the subscript “$s$” from various notations (e.g. $\text{Mod}^{r,\hat{G}_{L,0}}_\mathfrak{O} = \text{Mod}^{r,\hat{G}_L}_\mathfrak{O}$).

### 3.2 The functors $\text{Mod}^{r,\hat{G}}_\hat{r} \to \text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_\mathfrak{O} \to \text{Mod}^{r,\hat{G}}_s$

Now we consider the functors $\text{Mod}^{r,\hat{G}}_\hat{r} \to \text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_\mathfrak{O} \to \text{Mod}^{r,\hat{G}}_s$. At first, by Proposition 2.8, we see that the functor $\text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_s$ is fully faithful. It follows from this fact and Theorem 2.5(1) that the functor $\tilde{T}_s: \text{Mod}^{r,\hat{G}}_s \to \text{Rep}_{\mathfrak{Z}_p}(G_s)$ is fully faithful. It is clear that the functor $\text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_s$ is fully faithful and thus so is $\tilde{T}_s: \text{Mod}^{r,\hat{G}}_s \to \text{Rep}_{\mathfrak{Z}_p}(G_s)$. Combining this with Theorem 2.5(1) and Lemma 2.8, we obtain that the functor $\text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_s$ is also fully faithful. Furthermore, we prove the following.

**Proposition 3.2.** The functor $\tilde{T}_s: \text{Mod}^{r,\hat{G}}_s \to \text{Rep}_{\mathfrak{Z}_p}(G_s)$ is an equivalence of categories.

Summary, we obtained the following commutative diagram.

\[
\begin{array}{ccc}
\text{Mod}^{r,\hat{G}}_\mathfrak{O} & \xrightarrow{\sim} & \text{Mod}^{r,\hat{G}}_s \\
\downarrow \tilde{T} & & \downarrow \tilde{T}_s \\
\text{Rep}_{\mathfrak{Z}_p}(G_K) & \xrightarrow{\text{restriction}} & \text{Rep}_{\mathfrak{Z}_p}(G_s).
\end{array}
\]

**Remark 3.3.** The functor $\tilde{T}_s: \text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_s$ may not be possibly essentially surjective. In fact, under some conditions, there exists a representation of $G_K$ which is crystalline over $K_s$ but not of finite height. For more precise information, see [Li2, Example 4.2.3].

Before a proof of Proposition 3.2, we give an explicit formula such as (2.4.1) for an object of $\text{Mod}^{r,\hat{G}}_\mathfrak{O}$. The argument below follows the method of [Li2]. Let $\mathfrak{M}$ be an object of $\text{Mod}^{r,\hat{G}}_\mathfrak{O}$. Let $\mathfrak{M}_s$ be the image of $\mathfrak{M}$ for the functor $\tilde{T}_s: \text{Mod}^{r,\hat{G}}_s \to \text{Mod}^{r,\hat{G}}_s$. Put $D = S_{K_0} \otimes_{\mathfrak{O},G} \mathfrak{M}$ and also put $D_s = S_{K_0,s} \otimes_{\mathfrak{O},G} \mathfrak{M}_s = S_{K_0,s} \otimes_{S_{K_0}} D$. Then $D_s$ has a structure of a Breuil module and also $D = D_s/I_s S_{K_0,s} D_s$ has a structure of a filtered $(\varphi,N)$-module corresponding to $Q_p \otimes \mathfrak{Z} T_s(\mathfrak{M}_s)$ (see subsection 2.4.1), which is isomorphic to $D/I_s S_{K_0} D$ as a $\varphi$-module over $K_0$. By [Li1, Lemma 7.3.1], we have a unique $\varphi$-compatible section $D \hookrightarrow D$ and we regard $D$ as a submodule of $D \subset D_s$ by this section. Then we have $D = S_{K_0} \otimes_{K_0} D$ and $D_s = S_{K_0,s} \otimes_{K_0} D$. By the explicit formula (2.4.1) for $\mathfrak{M}_s$, we know that

\[G_s(D) \subset (K_0[I]) \cap R_{K_0,s} \otimes_{K_0} D.\]

Hence, taking any $K_0$-basis $e_1, \ldots, e_d$ of $D$, there exist $A_s(t) \in M_{d \times d}(K_0[[t]])$ such that $\tau^p(t) = (e_1 \cdots e_d) A_s(t)$. Since $A_s(0) = I_d$, we see that $\log(A_s(t)) \in M_{d \times d}(K_0[[t]])$ is well-defined. On the other hand, choose $g_0 \in G_s$ such that $\chi_p(g_0) \neq 1$, where $\chi_p$ is the $p$-adic cyclotomic character. Since $g_0 \tau^p(t) = (\tau^p(t))^{\chi_p(g_0)}$, we have $A_s(\chi_p(g_0)t) = A_s(t)^{\chi_p(g_0)}$ and thus we also have $\log(A_s(\chi_p(g_0)t)) = \chi_p(g_0) \log(A_s(t))$. Since $\log(A_s(0)) = \log(I_d) = 0$, we can write $\log(A_s(t))$ as $t B(t)$ for some $B(t) \in M_{d \times d}(K_0[[t]])$. Then we have $\chi_p(g_0)t B(\chi_p(g_0)t) = \chi_p(g_0) t B(t)$, that is,
Since \( \tilde{\chi}_p(g_0) \neq 1 \) implies that \( B(t) \) is a constant. Putting \( N_s = B(t) \in M_{d \times d}(K_0) \), we obtain

$$\tau^p(e_1, \ldots, e_d) = (e_1, \ldots, e_d)(\sum_{i \geq 0} N_s^i \gamma_i(t)).$$

Now we define \( N_D : D \to D \) by \( N(e_1, \ldots, e_d) = (e_1, \ldots, e_d)p^{-s}N_s \) and also define \( N_D := N_{S_{K_0}} \otimes \text{Id}_D + \text{Id}_{S_{K_0}} \otimes N_D \). (Note that we have \( N_D \varphi_D = p \varphi_D N_D \) and thus \( N_D \) is nilpotent.) It is a routine work to check the following:

$$g(ab) = \sum_{i=0}^{\infty} g(a)\gamma_i(\log([\log(\tau)])) \otimes N_D^i(b) \quad \text{for } g \in R, a \in B^+_\text{cris}, b \in B^+_\text{cris}.$$  \( (3.2.1) \)

Since we have

$$g(f) = \sum_{i \geq 0} \gamma_i(\log([\log(\tau)]))N_{S_{K_0}}^i(f)$$  \( (3.2.2) \)

for any \( g \in G_K \) and \( f \in S_{K_0} \), we obtain the following explicit formula:

$$g(ab) = \sum_{i=0}^{\infty} g(a)\gamma_i(\log([\log(\tau)])) \otimes N_D^i(b) \quad \text{for } g \in R, a \in B^+_\text{cris}, b \in B^+_\text{cris}.$$  \( (3.2.3) \)

In particular, as in subsection 2.3, we can show that

$$\log(\tau^p)(x) = p^t \otimes N_D(x)$$  \( (3.2.4) \)

for any \( x \in D \).

**Proof of Proposition 3.2.** We continue to use the above notation. It suffices to prove that the \( G_s \)-action on \( \hat{R}_s \otimes_{\varphi, \psi} \mathfrak{M} \) preserves \( R \otimes_{\varphi, \psi} \mathfrak{M} \). Take any \( g \in G_s \). We know that \( g(\mathfrak{M}) \subset \hat{R}_s \otimes_{\varphi, \psi} \mathfrak{M} \subset W(R) \otimes_{\varphi, \psi} \mathfrak{M} \). Hence it is enough to prove that \( g(D) \in \hat{R}_s \otimes_{\varphi, \psi} \mathfrak{M} \). Let \( s \in S_{K_0}^{\text{int}} \) and \( y \in D \) and put \( x = s \otimes y \in S_{K_0}^{\text{int}} \otimes_{W(k)} D = D \). By \( (3.2.1) \) or \( (3.2.3) \), we have

$$g(x) = \sum_{i \geq 0} \sum_{0 \leq j \leq i} \gamma_i(\log([\log(\tau)])) \binom{i}{j} N_{S_{K_0}}^j(s) \otimes N_D^i(y).$$

On the other hand, we know that \( N_D \) is nilpotent, that is, there exists \( j_0 > 0 \) such that \( N_D^{j_0} = 0 \). Then we obtain

$$g(x) = \sum_{0 \leq j \leq j_0} \sum_{i=0}^{\infty} \gamma_i(\log([\log(\tau)])) \binom{i}{j} N_{S_{K_0}}^j(s) \otimes N_D^i(y).$$

Therefore, it suffices to show that \( \sum_{i=0}^{\infty} \gamma_i(\log([\log(\tau)])) \binom{i}{j} N_{S_{K_0}}^j(s) \) is contained in \( R_{K_0} \) for each \( 0 \leq j \leq j_0 \). Taking \( \alpha(g) \in \mathbb{Z}_p \) such that \( \log([\log(\tau)])) = -\alpha(g)t \), we have

$$\sum_{i=0}^{\infty} \gamma_i(\log([\log(\tau)])) \binom{i}{j} N_{S_{K_0}}^j(s) = \sum_{i=0}^{\infty} (\alpha(g))^i \frac{q(i)!}{i!} \binom{i}{j} N_{S_{K_0}}^j(s) \ell(i).$$

Since \( \frac{q(i)!}{i!} \ell(i) \to 0 \) (p-adically) as \( i \to \infty \), we finish a proof.
3.3 Relations with crystalline representations

We know that $\mathcal{M}$ is semi-stable over $K_s$ for any object $\mathcal{M}$ of $\text{Mod}^{r,G_s}/\mathcal{O}$ or $\text{Mod}^{r,G_s}/\mathcal{O}$. This subsection is devoted to prove a criterion, for $\mathcal{M}$, that describes when $\mathcal{M}$ becomes crystalline.

Following [Fo2, Section 5] we set $I^{[m]}B^+_{\text{c}} := \{ x \in B^+_{\text{c}} \mid \varphi^n(x) \in \text{Fil}^m B^+_{\text{c}} \text{ for all } n \geq 0 \}$. For any subring $A \subset B^+_{\text{c}}$ we put $I^{[m]}A := A \cap I^{[m]}B^+_{\text{c}}$. Furthermore, we also put $I^{[m]}A = I^{[m]}A_A A$ (here, $I_A A$ is defined in Subsection 2.2.3). By [Fo2] Proposition 5.1.3 and the proof of [GL2] Lemma 3.2.2, we know that $I^{[m]}W(R)$ is a principal ideal which is generated by $\varphi(t)^m$.

Now we recall Theorem [2.4] (2): if $\mathcal{M}$ is an object of $\text{Mod}^{r,G_s}/\mathcal{O}$, then $\mathcal{M}$ is crystalline if and only if $\tau^p(x) - x \in u^p \langle I^{[1]} W(R) \otimes_{\varphi, \mathcal{O}} \mathcal{M} \rangle$ for any $x \in \mathcal{M}$. However, if such $\mathcal{M}$ descends to a Kisin module over $\mathcal{O}$, then we can show the following.

**Theorem 3.4.** Let $\mathcal{M}$ be an object of $\text{Mod}^{r,G_s}/\mathcal{O}$ or $\text{Mod}^{r,G_s}/\mathcal{O}$. Then the following is equivalent:

1. $\mathcal{M}$ is crystalline,
2. $\tau^p(x) - x \in u^p \langle I^{[1]} W(R) \otimes_{\varphi, \mathcal{O}} \mathcal{M} \rangle$ for any $x \in \mathcal{M}$,
3. $\tau^p(x) - x \in I^{[1]} W(R) \otimes_{\varphi, \mathcal{O}} \mathcal{M}$ for any $x \in \mathcal{M}$.

**Proof.** (1) $\Rightarrow$ (2): The proof here mainly follows that of [GLS] Proposition 4.7. We may suppose $\mathcal{M}$ is an object of $\text{Mod}^{r,G_s}/\mathcal{O}$. Put $D = S_{K_0} \otimes_{\mathcal{O}} \mathcal{M}$ and $D = D/\mathcal{I}_s S_{K_0} \mathcal{D}$ as in the previous subsection. We fix a $\varphi(\mathcal{O})$-basis $(e_1, \ldots, e_d)$ of $\mathcal{M} \subset D$ and denote by $(e_1, \ldots, e_d)$ the image of $(e_1, \ldots, e_d)$ for the projection $D \rightarrow D$. Then $(e_1, \ldots, e_d)$ is a $K_0$-basis of $D$. As described before the proof of Proposition [2.2], we regard $D$ as a $\varphi$-stable submodule of $D$, and we have $N_D: D \rightarrow D$ and $N_D: D \rightarrow D_D$.

Now we consider a matrix $X \in GL_{d \times d}(S_{K_0})$ such that $(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X$. We define $\tilde{S} = W(k)[u^p, u^{p^{[p]}}]$ as in Section 4 of [GLS], which is a sub $W(k)$-algebra of $S_{K_0}$ with the property $N_{S_{K_0}}(\tilde{S}) \subset u^p \tilde{S}$. By an easy computation we have $U = X^{-1}BX = X^{-1}N_{S_{K_0}}(X)$. Here, $B \in M_{d \times d}(S_{K_0})$ and $U \in M_{d \times d}(S_{K_0})$ are defined by $N_D(e_1, \ldots, e_d) = (e_1, \ldots, e_d)B$ and $N_D(e_1, \ldots, e_d) = (e_1, \ldots, e_d)U$. By the same proof as in the former half part of the proof of [GLS] Proposition 4.7, we obtain $X, X^{-1} \in M_{d \times d}(\tilde{S}[1/p])$. On the other hand, let $\mathcal{M}$ be the image of $\mathcal{M}$ for the functor $\text{Mod}^{r,G_s}/\mathcal{O} \rightarrow \text{Mod}^{r,G_s}/\mathcal{O}$. Now we recall that $D_s = S_{K_0,s} \otimes_{\mathcal{O}} \mathcal{M}$ has a structure of the Breuil module corresponding to $\mathcal{Q}_p \otimes_{\mathcal{O}} T_s(\mathcal{M})$. Denote by $N_{D_D}$ its monodromy operator. By the formula [2.4] for $\mathcal{M}$, and the formula [2.2] for $\mathcal{M}$, we see that $p^s N_D = N_{D_D}$ on $D$. Therefore, $\mathcal{Q}_p \otimes_{\mathcal{O}} T_s(\mathcal{M})$ is crystalline if and only if $N_{D_D}$ mod $I_s S_{K_0,s} D_s$ is zero, which is equivalent to say that $N_D = (N_D \mod I_s S_{K_0,D})$ is zero, that is, $B = 0$. Therefore, the latter half part of the proof [GLS] Proposition gives the assertion (2).

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Suppose that (3) holds. We denote by $\mathcal{M}_s$ the image of $\mathcal{M}$ for the functor $\text{Mod}^{r,G_s}/\mathcal{O} \rightarrow \text{Mod}^{r,G_s}/\mathcal{O}$. As above, we claim that, for any $x \in \mathcal{M}_s$, we have $\tau^p(x) - x = I^{[1]} W(R) \otimes_{\varphi, \mathcal{O}} \mathcal{M}_s$. Let $x = a \otimes y \in \mathcal{M}_s = \mathcal{S}_s \otimes_{\mathcal{O}} \mathcal{M}$ where $a \in \mathcal{S}_s$ and $y \in \mathcal{M}$. Then

$$\tau^p(x) - x = \tau^p(\varphi(a))(\tau^p(y) - y) + (\tau^p(\varphi(a)) - \varphi(a))y$$

and thus it suffices to show $\tau^p(\varphi(a)) - \varphi(a) \in I^{[1]} W(R)$. This follows from the lemma below and thus we obtained the claim. By the claim and Theorem [2.2] (2), we know that $\mathcal{Q}_p \otimes_{\mathcal{O}} T_s(\mathcal{M}_s) \simeq \mathcal{Q}_p \otimes_{\mathcal{O}} T_s(\mathcal{M})$ is crystalline.

**Lemma 3.5.** (1) We have $I^{[1]} W(R) \cap u^r B^+_{\text{c}} = u^r I^{[1]} W(R)$ for $r \geq 0$.
(2) We have $g(a) - a \in u I^{[1]} W(R)$ for $g \in G$ and $a \in \mathcal{S}$.  

13
Proof. This is due to [GLS, the proof of Proposition 7] but we write a proof here.
(1) Take \( x = u'y \in I^{[1]}W(R) \) with \( y \in B^+_{\text{cris}} \). By Lemma 3.2.2 of [14] we have \( y \in W(R) \). Now we remark that \( u \in \text{Fil}^pW(R) \) with \( z \in W(R) \) implies \( z \in \text{Fil}^pW(R) \) since \( u \) is a unit of \( B^+_{\text{cris}} \). Hence \( u'y \in I^{[1]}W(R) \) implies \( y \in I^{[1]}W(R) \).
(2) By the relation (3.2.2), we see that \( \phi(a - a) \in I^{[1]}W(R) \). On the other hand, if \( i > 0 \), we can write \( N^r_{\text{Fil}^i}(a) = ub_i \) for some \( b_i \in \mathfrak{S} \). Thus by the relation (3.2.2) again we obtain \( g(a - a) \in uB^+_{\text{cris}} \).
Then the result follows from (1).

\( \square \)

4 Variants of torsion \((\varphi, \hat{G})\)-modules

In this section, we mainly study full subcategories of \( \widetilde{\text{Mod}}_{\varphi, \hat{G}} \) defined below and also study representations associated with them. As a consequence, we prove theorems in Introduction. We use same notation as in Section 2 and 3. In particular, \( p \) is odd. In below, let \( v_\mathfrak{p} \) be the valuation of \( R \) normalized such that \( v_\mathfrak{p}(\pi) = 1/e \) and, for any real number \( x \geq 0 \), we denote by \( m^x_\mathfrak{p} \) the ideal of \( R \) consisting of elements \( a \) with \( v_\mathfrak{p}(a) \geq x \).

Let \( J \) be an ideal of \( W(R) \) which satisfies the following conditions:

- \( J \not\subseteq pW(R) \),
- \( J \) is a principal ideal,
- \( J \) is \( \varphi \)-stable and \( G_\varphi \)-stable in \( W(R) \).

By the above first and second assumptions for \( J \), the image of \( J \) under the projection \( W(R) \to R \) is of the form \( m^{c_J}_\mathfrak{p} \) for some real number \( c_J \geq 0 \).

Definition 4.1. We denote by \( \text{Mod}_{\varphi, \hat{G}, J} \) the full subcategory of \( \text{Mod}_{\varphi, \hat{G}} \) consisting of objects \( \mathfrak{M} \) which satisfy the following condition:

\[ \tau^J(x) - x \in JW(R) \otimes_{\varphi, \hat{G}} \mathfrak{M} \quad \text{for any } x \in \mathfrak{M}. \]

Also, we denote by \( \text{Rep}_{\text{tor}}(G_s) \) the essential image of the functor \( \hat{T}_s : \text{Mod}_{\varphi, \hat{G}} \to \text{Rep}_{\text{tor}}(G_s) \) restricted to \( \text{Mod}_{\varphi, \hat{G}, J} \).

By definition, we have \( \text{Mod}_{\varphi, \hat{G}, J} \subset \text{Mod}_{\varphi, \hat{G}, J'} \) and \( \text{Rep}_{\text{tor}}(G_s) \subset \text{Rep}_{\text{tor}}(G_s) \) for \( J \subset J' \).

4.1 Full faithfulness for \( \text{Mod}_{\varphi, \hat{G}, J} \)

For the beginning of a study of \( \text{Mod}_{\varphi, \hat{G}, J} \), we prove the following full faithfulness result.

Proposition 4.2. Let \( r \) and \( r' \) be non-negative integers with \( c_J > pr/(p - 1) \). Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be objects of \( \text{Mod}_{\varphi, \hat{G}, J} \) and \( \text{Mod}_{\varphi, \hat{G}, J} \), respectively. Then we have \( \text{Hom}(\mathfrak{M}, \mathfrak{N}) = \text{Hom}(\mathfrak{M}, \mathfrak{N}) \).
(Here, two \( \mathfrak{Hom}'s \) are defined by obvious manners.)

In particular, if \( c_J > pr/(p - 1) \), then the forgetful functor \( \text{Mod}_{\varphi, \hat{G}, J} \to \text{Mod}_{\varphi, \hat{G}, J} \) is fully faithful.

Proof. A very similar proof of [O2, Lemma 7] proceeds, and hence we only give a sketch here. Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be objects of \( \text{Mod}_{\varphi, \hat{G}, J} \) and \( \text{Mod}_{\varphi, \hat{G}, J} \), respectively. Let \( f : \mathfrak{M} \to \mathfrak{N} \) be a morphism of Kisin modules over \( \mathfrak{S} \). Put \( f = W(R) \otimes f : W(R) \otimes_{\varphi, \hat{G}} \mathfrak{M} \to W(R) \otimes_{\varphi, \hat{G}} \mathfrak{N} \). Choose any lift of
\(\tau \in \tilde{G}\) to \(G_K\); we denote it also by \(\tau\). Since the \(\tilde{G}\)-action for \(\tilde{M}\) is continuous, it suffices to prove that \(\Delta(1 \otimes x) = 0\) for any \(x \in M\) where \(\Delta := \tau^{p^r} \circ \tilde{f} - \tilde{f} \circ \tau^{p^r}\). We use induction on \(n\) such that \(\pi^n\tilde{M} = 0\).

Suppose \(n = 1\). Since \(\Delta = (\tau^{p^r} - 1) \circ \tilde{f} - \tilde{f} \circ (\tau^{p^r} - 1)\), we obtain the following:

\[(0): \text{ For any } x \in M, \ \Delta(1 \otimes x) \in m_{\tilde{M}}^{c(0)}(R \otimes_{\varphi, \tilde{G}} \tilde{\Omega})\]

where \(c(0) = c_J\). Since \(M\) is of height \(\leq r\), we further obtain the following for any \(i \geq 1\) inductively:

\[(i): \text{ For any } x \in M, \ \Delta(1 \otimes x) \in m_{\tilde{M}}^{c(i)}(R \otimes_{\varphi, \tilde{G}} \tilde{\Omega})\]

where \(c(i) = pc(i - 1) - pr = (c_J - pr/(p - 1))^p + pr/(p - 1)\). The condition \(c_J > pr/(p - 1)\) implies that \(c(i) \rightarrow \infty\) as \(i \rightarrow \infty\) and thus \(\Delta(1 \otimes x) = 0\).

Suppose \(n > 1\). Consider the exact sequence \(0 \rightarrow \text{Ker}(p) \rightarrow M \xrightarrow{\beta} \tilde{M} \xrightarrow{\phi} \Omega(\varphi) \rightarrow 0\) of \(\varphi\)-modules over \(\Omega\). It is not difficult to check that \(\Omega' := \text{Ker}(p)\) and \(\Omega'' := \phi\Omega\) are torsion Kisin modules of height \(\leq r\) over \(\Omega\) (cf. \cite[Lemma 2.3.1]{Li2}). Moreover, we can check that \(\tilde{M}'\) and \(\tilde{M}''\) have natural structures of objects of \(\text{Mod}_{/\phi}^{\tilde{G}}\) (which are denoted by \(\tilde{M}'\) and \(\tilde{M}''\), respectively) such that the sequence \(0 \rightarrow M' \rightarrow M \xrightarrow{\beta} M'' \rightarrow 0\) induces an exact sequence \(0 \rightarrow \tilde{M}' \rightarrow \tilde{M} \rightarrow \tilde{M}'' \rightarrow 0\). By the lemma below, we know that \(\tilde{M}'\) and \(\tilde{M}''\) are in fact contained in \(\text{Mod}_{/\phi}^{\tilde{G}, J}\). By the induction hypothesis, we see that \(\Delta(1 \otimes x)\) has values in \((W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega})\). Since \(\tilde{M}' = 0\), an analogous argument in the case \(n = 1\) proceeds and we have \(\Delta(1 \otimes x) = 0\) as desired.

\[\square\]

**Lemma 4.3.** Let \(0 \rightarrow \tilde{M}' \rightarrow \tilde{M} \rightarrow \tilde{M}'' \rightarrow 0\) be an exact sequence in \(\text{Mod}_{/\phi}^{\tilde{G}, J}\). Suppose that \(\tilde{M}\) is an object of \(\text{Mod}_{/\phi}^{\tilde{G}, J}\). Then \(\tilde{M}'\) and \(\tilde{M}''\) are also objects of \(\text{Mod}_{/\phi}^{\tilde{G}, J}\).

**Proof.** The fact \(\tilde{M}'' \in \text{Mod}_{/\phi}^{\tilde{G}, J}\) is clear. Take any \(x \in M\). Then we have \(\tau^{p^r}(x) - x \in (JW(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega})\). Since \(J\) is a principal ideal which is not contained in \(pW(R)\), we obtain \(\tau^{p^r}(x) - x \in JW(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega}\) by Lemma 6 of [Oz2]. This implies \(\tilde{M}' \in \text{Mod}_{/\phi}^{\tilde{G}, J}\).

\[\square\]

### 4.2 The category \(\text{Rep}_{/\text{tor}}^{\tilde{G}, J}(G_s)\)

In this subsection, we study some categorical properties of \(\text{Rep}_{/\text{tor}}^{\tilde{G}, J}(G_s)\). Let \(\tilde{M}\) be an object of \(\text{Mod}_{/\phi}^{\tilde{G}}\). Following Section 3.2 of [Li2] (note that arguments in [Li2] is the “free case”), we construct a map \(\tilde{i}_s\) which connects \(\tilde{M}\) and \(\bar{T}_s(\tilde{M})\) as follows. Observe that there exists a natural isomorphism of \(\mathbb{Z}_p[G_s]\)-modules

\[\bar{T}_s(\tilde{M}) \simeq \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))\]

where \(G_s\) acts on \(\text{Hom}_{W(R), \varphi}(W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))\) by \((\sigma f)(x) = \sigma f(\sigma^{-1}(x))\) for \(\sigma \in G_s, f \in \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))\), \(x \in W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega} = W(R) \otimes_{\mathbb{Z}_p} (\mathbb{R}_{\varphi, \tilde{G}} \tilde{\Omega})\). Thus we can define a morphism \(\tilde{i}'_s: W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega} \rightarrow \text{Hom}_{\mathbb{Q}_p}(\bar{T}_s(\tilde{M}), \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))\) by

\[x \mapsto (f \mapsto f(x)), \quad x \in W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega}, \quad f \in \bar{T}_s(\tilde{M})\].

Since \(\bar{T}_s(\tilde{M}) \simeq \oplus_{i \in \mathbb{Z}_p/p^n\mathbb{Z}_p} \tilde{Z}_p\) as \(\mathbb{Z}_p\)-modules, we have a natural isomorphism \(\text{Hom}_{\mathbb{Q}_p}(\bar{T}_s(\tilde{M}), \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) \simeq W(R) \otimes_{\mathbb{Z}_p} \bar{T}_s(\tilde{M})\) where \(\bar{T}_s(\tilde{M}) = \text{Hom}_{\mathbb{Q}_p}(\bar{T}_s(\tilde{M}), \mathbb{Q}_p/\mathbb{Z}_p)\) is the dual representation of \(\bar{T}_s(\tilde{M})\). Composing this isomorphism with \(\tilde{i}'_s\), we obtain the desired map

\[\tilde{i}_s: W(R) \otimes_{\varphi, \tilde{G}} \tilde{\Omega} \rightarrow W(R) \otimes_{\mathbb{Z}_p} \bar{T}_s(\tilde{M})\].
It follows from a direct calculation that \( i_s \) is \( \varphi \)-equivariant and \( G_s \)-equivariant. If we denote by \( \hat{\mathcal{M}}_s \) the image of \( \mathcal{M} \) for the functor \( \text{Mod}_{/\hat{G}_s} \to \text{Mod}_{/G_s} \) (cf. Section 5.2), then the above \( i_s \) is isomorphic to “\( i \) for \( \hat{\mathcal{M}}_s \) in Section 4.1 of [Oz1]”. Hence Lemma 4.2 (4) in loc. cit. implies that

\[
W(\text{Fr} R) \otimes i_s: W(\text{Fr} R) \otimes_{W(R)} (W(R) \otimes_{\mathcal{M}} \mathcal{M}) \to W(\text{Fr} R) \otimes_{W(R)} (W(R) \otimes_{\mathcal{M}} \hat{G}_s) (\mathcal{M})
\]

is bijective.

**Proposition 4.4.** Let \((\mathcal{M}, \hat{\mathcal{M}}_s) \in \text{Mod}_{/\hat{G}_s, J} \) such that \( \hat{T}_s(\mathcal{M}) \simeq T \). Then there exists an exact sequence \((M): 0 \to \hat{\mathcal{M}}'' \to \hat{\mathcal{M}} \to \hat{\mathcal{M}}' \to 0 \) in \( \text{Mod}_{/\hat{G}_s, J} \) such that \( \hat{T}_s((M)) \simeq (R) \).

**Proof.** The same proof as [Oz1] Theorem 4.5, except using not \( i \) in the proof of loc. cit. but \( i_s \) as above, gives an exact sequence \((M): 0 \to \hat{\mathcal{M}}'' \to \hat{\mathcal{M}} \to \hat{\mathcal{M}}' \to 0 \) in \( \text{Mod}_{/G_s} \) such that \( \hat{T}_s((M)) \simeq (R) \). Therefore, Lemma 4.3 gives the desired result.

**Corollary 4.5.** The full subcategory \( \text{Rep}_{/\hat{G}_s, J} \) of \( \text{Rep}_{/G_s} \) is stable under subquotients.

Let \( L \) be as in Section 2, that is, the completion of an unramified algebraic extension of \( K \) with residue field \( k_L \). We prove the following base change lemma.

**Lemma 4.6.** Assume that \( J \supseteq \text{ur}^1 \text{I}(1)^{[1]} W(R) \) or \( L \) is a finite unramified extension of \( K \). If \( T \) is an object of \( \text{Rep}_{/\hat{G}_s, J} \) (\( G_s \)), then \( T|_{G_{L,s}} \) is an object of \( \text{Rep}_{/G_{L,s}, J} \).

By an obvious way, we define a functor \( \text{Mod}_{/\hat{G}_s} \to \text{Mod}_{/G_{L,s}} \). The underlying Kisin module of the image of \( \hat{\mathcal{M}} \in \text{Mod}_{/\hat{G}_s} \) for this functor is \( \mathcal{M}_L = \mathcal{S}_L \otimes \mathcal{M} \). Lemma 4.6 immediately follows from the lemma below.

**Lemma 4.7.** Assume that \( J \supseteq \text{ur}^1 \text{I}(1)^{[1]} W(R) \) or \( L \) is a finite unramified extension of \( K \). Then the functor \( \text{Mod}_{/\hat{G}_s} \to \text{Mod}_{/G_{L,s}} \) induces a functor \( \text{Mod}_{/\hat{G}_s, J} \to \text{Mod}_{/G_{L,s}, J} \).

**Proof.** Let \( \hat{\mathcal{M}} \) be an object of \( \text{Mod}_{/\hat{G}_s} \) and let \( \mathcal{M}_L \) be the image of \( \mathcal{M} \) for the functor \( \text{Mod}_{/\hat{G}_s, J} \to \text{Mod}_{/G_{L,s}, J} \). In the rest of this proof, to avoid confusions, we denote the image of \( x \in \mathcal{M}_L \) in \( W(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \) by \( 1 \otimes x \). Recall that we abuse notations by writing \( \tau \) for the pre-image of \( \tau \in G_{K,p} \) via the bijection \( G_{L,p} \simeq G_{K,p} \) of lemma 2.3. Then \( \tau^p \) is a topological generator of \( G_{L,s,p} \). It suffices to show the following: if \( \hat{\mathcal{M}} \) is an object of \( \text{Mod}_{/\hat{G}_s, J} \), then we have \( \tau^p (1 \otimes x) - (1 \otimes x) \in JW(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \) for any \( x \in \mathcal{M}_L \). Now suppose \( \hat{\mathcal{M}} \in \text{Mod}_{/\hat{G}_s} \). Take any \( a \in \mathcal{S}_L \) and \( x \in \mathcal{M} \). Note that we have \( \tau^p (1 \otimes ax) - (1 \otimes ax) = \tau^p (\varphi(a)) (\tau^p (1 \otimes x) - (1 \otimes x)) \in JW(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \). Since \( \hat{\mathcal{M}} \) is an object of \( \text{Mod}_{/\hat{G}_s, J} \), we have \( \tau^p (\varphi(a)) (\tau^p (1 \otimes x) - (1 \otimes x)) \in JW(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \). Therefore, it is enough to show \( \tau^p (\varphi(a)) (\tau^p (1 \otimes x) - (1 \otimes x)) \in JW(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \). This follows from Lemma 6.1 immediately in the case where \( J \supseteq \text{ur}^1 \text{I}(1)^{[1]} W(R) \). Next we consider the case where \( L \) is a finite unramified extension of \( K \). Let \( c_1, \ldots, c_r \in W(k_L) \) be generators of \( W(k_L) \) as a \( W(k) \)-module. Then we have \( \mathcal{S}_L = \sum_{j=1}^r c_j \mathcal{S} \) and thus we can write \( a = \sum_{j=1}^r a_j c_j \) for some \( a_j \in \mathcal{S} \). Hence it suffices to show \( \tau^p (\varphi(a_j)) (\tau^p (1 \otimes x) - (1 \otimes x)) \in JW(R) \otimes_{\mathcal{M}_L} \mathcal{M}_L \) but this in fact immediately follows from the equation \( (\tau^p (\varphi(a_j)) (\tau^p (1 \otimes x) - (1 \otimes x))) = (\tau^p (1 \otimes a_j x) - (1 \otimes a_j x)) \).
4.3 Full faithfulness theorem for $\text{Rep}_{\text{tor}}^{s, J}(G_s)$

Our goal in this subsection is to prove the following full faithfulness theorem, which plays an important role in our proofs of main theorems.

**Theorem 4.8.** Assume that $J \supset u^pI[1]W(R)$ or $k$ is algebraically closed. If $p^{s+2}/(p-1) \geq c_J > pr/(p-1)$, then the restriction functor $\text{Rep}_{\text{tor}}^{s, J}(G_s) \rightarrow \text{Rep}_{\text{tor}}(G_{\infty})$ is fully faithful.

First we give a very rough sketch of the theory of maximal models for Kisin modules (cf. [CL1]). For any $\mathfrak{M} \in \text{Mod}_{\hat{\mathfrak{r}}}^{s, J}$, put $\mathfrak{M}[1/u] = S[1/u] \otimes S \mathfrak{M}$ and denote by $F^s_0(\mathfrak{M}[1/u])$ the (partially) ordered set (by inclusion) of torsion Kisin modules $\mathfrak{R}$ of height $\leq r$ which are contained in $\mathfrak{M}[1/u]$ and $\mathfrak{M}[1/u] = \mathfrak{M}[1/u]$ as $\varphi$-modules. The set $F^s_0(\mathfrak{M}[1/u])$ has a greatest element (cf. loc. cit., Corollary 3.2.6). We denote this element by $\text{Max}^s(\mathfrak{M})$. We say that $\mathfrak{M}$ is maximal of height $\leq r$ (or, maximal for simplicity) if it is the greatest element of $F^s_0(\mathfrak{M}[1/u])$. The implication $\mathfrak{M} \rightarrow \text{Max}^s(\mathfrak{M})$ defines a functor “Max” from the category $\text{Mod}_{\hat{\mathfrak{r}}}^{s, J}$ of torsion Kisin modules of height $\leq r$ into the category $\text{Max}_{\hat{\mathfrak{r}}}^{s, J}$ of maximal Kisin modules of height $\leq r$. The category $\text{Max}_{\hat{\mathfrak{r}}}^{s, J}$ is abelian (cf. loc. cit., Theorem 3.3.8). Furthermore, the functor $T_{\hat{\mathfrak{r}}}: \text{Max}_{\hat{\mathfrak{r}}}^{s, J} \rightarrow \text{Rep}_{\text{tor}}(G_{\infty})$, defined by $T_{\hat{\mathfrak{r}}}(\mathfrak{M}) = \text{Hom}_{\hat{\mathfrak{r}}, \varphi}(\mathfrak{M}, Q_p/Z_p \otimes Z_p W(R))$, is exact and fully faithful (cf. loc. cit., Corollary 3.3.10). It is not difficult to check that $T_{\hat{\mathfrak{r}}}(\text{Max}^s(\mathfrak{M}))$ is canonically isomorphic to $T_{\hat{\mathfrak{r}}}(\mathfrak{M})$ as representations of $G_{\infty}$ for any torsion Kisin module $\mathfrak{M}$ of height $\leq r$.

**Definition 4.9 ([CL1] Section 3.6.1).** Let $d$ be a positive integer. Let $n = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ be a sequence of non-negative integers of smallest period $d$. We define a torsion Kisin module $\mathfrak{M}(n)$ as below:

- as a $k[u]$-module, $\mathfrak{M}(n) = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} k[u]e_i$;
- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\varphi(e_i) = u^{n_i}e_{i+1}$.

We denote by $S_{\text{max}}^s$ the set of sequences $n = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ of integers $0 \leq n_i \leq \min\{er, p-1\}$ with smallest period $d$ for some integer $d$ except the constant sequence with value $p-1$ (if necessary). By definition, we see that $\mathfrak{M}(n)$ is of height $\leq r$ for any $n \in S_{\text{max}}^s$. Putting $r_0 = \max\{r^d \in \mathbb{Z}_{\geq 0}; cr^d - 1 < p - 1\}$, we also see that $\mathfrak{M}(n)$ is of height $\leq r_0$ for any $n \in S_{\text{max}}^s$. It is known that $\mathfrak{M}(n)$ is maximal for any $n \in S_{\text{max}}^s$ ([CL1] Proposition 3.6.7). If $k$ is algebraically closed, then $\mathfrak{M}(n)$ is simple in $\text{Max}_{\hat{\mathfrak{r}}}^{s, J}$ for any $n \in S_{\text{max}}^s$ (cf. loc. cit., Propositions 3.6.7 and 3.6.12) and furthermore, the converse holds; any simple object in $\text{Max}_{\hat{\mathfrak{r}}}^{s, J}$ is of the form $\mathfrak{M}(n)$ for some $n \in S_{\text{max}}^s$ (cf. loc. cit., Propositions 3.6.8 and 3.6.12).

**Lemma 4.10.** Assume that $p^{s+2}/(p-1) \geq c_J$. Let $d$ be a positive integer. Let $n = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ be a sequence of non-negative integers of smallest period $d$. If $\mathfrak{M}(n)$ is of height $\leq r$, then $\mathfrak{M}(n)$ has a structure of an object of $\text{Mod}_{\hat{\mathfrak{r}}}^{s, J}$.

**Proof.** Choose any $(p^d-1)$-th root $\eta \in R$ of $z$. Since $[\eta] \cdot \exp(t/(p^d-1))$ is a $(p^d-1)$-th root of unity, it is of the form $[a]$ for some $a \in F_p^\times$. Replacing $na^{-1}$ with $\eta$, we obtain $[\eta] = \exp(-t/(p^d-1)) \in \hat{R}^\times$. Put $x_i = [\eta]^{n_i} \in \hat{R}^\times$ and $x_i = [\eta]^{n_i} \in \hat{R}^\times$ for any $i \in \mathbb{Z}/d\mathbb{Z}$, where $m_i = \sum_{j=0}^{i-1} n_{i+j}p^{d-j}$. We see that $x_i - 1$ is contained in $I_i\hat{R}$. In the rest of this proof, to avoid confusions, we denote the image of $x \in \mathfrak{M}(n)$ in $\hat{R} \otimes_{\varphi, \hat{\mathfrak{r}}} \mathfrak{M}(n) \subset R \otimes_{\varphi, \hat{\mathfrak{r}}} \mathfrak{M}(n)$ by $1 \otimes x$. Now we define a $\hat{G}$-action on $\hat{R} \otimes_{\varphi, \hat{\mathfrak{r}}} \mathfrak{M}(n)$ by $\hat{G}^{\prime} \otimes (1 \otimes x) := x^{\hat{G}^{\prime}} (1 \otimes x)$ for the basis $\{e_i\}_{i \in \mathbb{Z}/d\mathbb{Z}}$ of $\mathfrak{M}(n)$ as in Definition 4.9. It is not difficult to check that $\mathfrak{M}(n)$ has a structure of an object of $\text{Mod}_{\hat{\mathfrak{r}}}^{s, J}$ via this $\hat{G}$-action; we denote it by $\mathfrak{M}(n)$. It suffices to prove that $\mathfrak{M}(n)$ is in fact an object of $\text{Mod}_{\hat{\mathfrak{r}}}^{s, J}$.
$$v_p(p) = 1.$$ Note that we have $$v_R(\xi - 1) = p/(p - 1)$$ and $$v_R(1) = 1/(p - 1)$$ (here, the latter equation follows from the relation $$\varphi(t) = pE(0)^{-1}E(u)t$$). We see that
\[
v_R(\xi^{p^s} - 1) = p^{s+v_p(m_i)} \cdot p/(p - 1) \geq p^{s+2}/(p - 1).
\]
Since $$p^{s+2}/(p - 1) \geq c_J$$ and the image of $$J$$ in $$R$$ is $$\mathfrak{m}_R^{\geq c_J}$$, we obtain
\[
\tau^p \left(1 \otimes e_i\right) - (1 \otimes e_i) \in \mathfrak{m}_R^{\geq c_J} R \otimes_{\varphi,k[u]} \mathfrak{M}(n) \simeq JW(R) \otimes_{\varphi,k} \mathfrak{M}(n).
\]
Finally we have to show that $$\tau^p \left(1 \otimes a e_i\right) = (1 \otimes a e_i) \in \mathfrak{m}_R^{\geq c_J} R \otimes_{\varphi,k[u]} \mathfrak{M}(n)$$ for any $$a \in k[u]$$.

Since $$\tau^p \left(1 \otimes a e_i\right) - (1 \otimes a e_i) = \tau^p (\varphi(a)) \left(\tau^p \left(1 \otimes e_i\right) - (1 \otimes e_i)\right) + (\tau^p(\varphi(a)) - \varphi(a)) (1 \otimes e_i)$$, it suffices to show $$\tau^p(\varphi(a)) - \varphi(a) \in \mathfrak{m}_R^{\geq c_J}$$. Write $$\varphi(a) = \sum_{i \geq 0} a_i u^{p^i}$$ for some $$a_i \in k$$. Then we have
\[
\tau^p(\varphi(a)) - \varphi(a) = \sum_{i \geq 1} a_i (\xi^{p^{i+1}} - 1) u^{p^i}.
\]
Since we have
\[
v_R((\xi^{p^{i+1}} - 1) u^{p^i}) = p^{s+1} v_R(\xi^{i} - 1) + v_R(u^{p^i}) > p^{s+2}/(p - 1) \geq c_J
\]
for any $$i \geq 1$$, we have done.  

Recall that $$r_0 = \max\{r' \in \mathbb{Z}_{\geq 0}; 0 < r' - 1 < p - 1\}$$. Put $$r_1 := \min\{r, r_0\}$$.

**Corollary 4.11.** Assume that $$p^{s+2}/(p - 1) \geq c_J$$. If $$n \in \mathcal{S}_{\text{max}}^c$$, then $$\mathfrak{M}(n)$$ has a structure of an object of $$\text{Mod}^I_{\varphi, G_s, J}$$ for any $$r' \geq r_1$$. Furthermore, if $$c_J > pr_1/(p - 1)$$, it is uniquely determined. We denote this object by $$\mathfrak{M}(n)$$.

**Proof.** We should remark that $$\mathfrak{M}(n)$$ is of height $$\leq r_1$$ for any $$n \in \mathcal{S}_{\text{max}}^c$$. The uniqueness assertion follows from Proposition 4.2.

**Lemma 4.12.** The functor from tamely ramified $$\mathbb{Z}_p$$-representations of $$G_K$$ to $$\mathbb{Z}_p$$-representations of $$G_\infty$$, obtained by restricting the action of $$G_K$$ to $$G_\infty$$, is fully faithful.

**Proof.** The result immediately follows from the fact that $$G_K$$ is topologically generated by $$G_\infty$$ and the wild inertia subgroup of $$G_K$$.

We remark that any semi-simple $$\mathbb{F}_p$$-representation of $$G_K$$ is automatically tame.

**Lemma 4.13.** Assume that $$J \supset u^p I^{[1]} W(R)$$ or $$k$$ is algebraically closed. Let $$T \in \text{Rep}_{\text{tor}}(G_s)$$ and $$T' \in \text{Rep}_{\text{tor}}(G_s)$$.

Suppose that $$T$$ is tame, $$\varphi T = 0$$ and $$T|_{G_s} \simeq T_{\varphi}(\mathfrak{M})$$ for some $$\mathfrak{M} \in \text{Mod}^I_{\varphi, \mathfrak{M}}$$.

Furthermore, we suppose $$p^{s+2}/(p - 1) \geq c_J > pr_1/(p - 1)$$. Then all $$G_\infty$$-equivariant homomorphisms $$T \to T'$$ are in fact $$G_s$$-equivariant.

**Proof.** Let $$L$$ be the completion of the maximal unramified extension $$K^\infty$$ of $$K$$. By identifying $$G_L$$ with $$G_{K^\infty}$$, we may regard $$G_L$$ as a subgroup of $$G_K$$. Note that $$L(\xi) = K(\xi)$$ is the completion of the maximal unramified extension of $$K(\xi)$$, and $$G_s$$ is topologically generated by $$G_{L,s}$$ and $$G_\infty$$.

Consider the following commutative diagram:
\[
\begin{array}{ccc}
\text{Hom}_{G_{L,s}}(T, T') & \longrightarrow & \text{Hom}_{G_{L,s}}(T, T') \\
\downarrow & & \downarrow \\
\text{Hom}_{G_s}(T, T') & \longrightarrow & \text{Hom}_{G_s}(T, T').
\end{array}
\]

Since $$T'|_{G_{L,s}}$$ is contained in $$\text{Rep}_{\text{tor}}(G_{L,s})$$ if $$J \supset u^p I^{[1]} W(R)$$ (cf. Lemma 4.6), the above diagram allows us to reduce a proof to the case where $$k$$ is algebraically closed. In the rest of this proof, we assume that $$k$$ is algebraically closed. Under this assumption, an $$\mathbb{F}_p$$-representation of $$G_s$$ is tame if and only if it is semi-simple by Maschke’s theorem. Thus we may also assume that $$T$$ is
irreducible (here, we remark that any subquotient of $T$ is tame and, also remark that the essential image of $T_0 : \text{Mod}^r_{G_\infty} \to \text{Rep}_{tor}(G_\infty)$ is stable under subquotients in $\text{Rep}_{tor}(G_\infty)$. We claim that $T_0|_{G_\infty}$ is also irreducible. If not, there exists a non-zero irreducible $\mathbb{F}_p[G_\infty]$-submodule $W$ of $T_0|_{G_\infty}$. Let $K_{(s)}$ be the maximal tamely ramified extension of $K(s)$ and $I_{p,s} := \text{Gal}(K/K_{(s)})$ the wild inertia subgroup of $G_s$. We see that $K_{(s)} \cap K_\infty = K(s)$. Since $G_\infty \supset I_{p,s}$ acts on $W$ trivially, the $G_\infty$-action on $W$ extends to $G_s$ via the composition map $G_s \to \text{Gal}(K_{(s)}/K(s)) \simeq G_\infty/(G_\infty \cap I_{p,s})$. Thus we can regard $W$ as an irreducible $\mathbb{F}_p[G_s]$-module. By Lemma 4.12 we see that $W$ is a sub $\mathbb{F}_p[G_s]$-module of $T$. This contradicts the irreducibility of $T$ and the claim follows.

By the assumption on $T$, we have $T_0|_{G_\infty} \simeq T_0(\mathfrak{M}) \simeq T_0(\text{Max}^r(\mathfrak{N}))$ for some $\mathfrak{M} \in \text{Mod}^r_{G_\infty}$. Since $T_0|_{G_\infty}$ is irreducible and $T_0 : \text{Max}^r_{G_\infty} \to \text{Rep}_{tor}(G_\infty)$ is exact and fully faithful, we know that $\text{Max}^r(\mathfrak{N})$ is a simple object in the abelian category $\text{Max}^r_{G_\infty}$. Therefore, since $k$ is algebraically closed, we have $\text{Max}^r(\mathfrak{N}) \simeq \mathfrak{M}(n)$ for some $n \in S^r_{max}$ (cf. [CL1] Propositions 3.6.8 and 3.6.12).

Let $\mathfrak{M}(n)$ be the object of $\text{Mod}^r_{G_\infty}$ as in Corollary 4.11. We recall that $T_0(\mathfrak{M}(n))$ is isomorphic to $\hat{T}_s(\mathfrak{M}(n))|_{G_\infty}$ (see Theorem 2.5 (1)), and hence we have an isomorphism $T_0|_{G_\infty} \simeq \hat{T}_s(\mathfrak{M}(n))|_{G_\infty}$. Here, we note that $T$ and $\hat{T}_s(\mathfrak{M}(n))$ are irreducible as representations of $G_s$ (cf. [CL1] Theorem 3.6.11). Applying Lemma 4.12 again, we obtain an isomorphism $T \simeq \hat{T}_s(\mathfrak{M}(n))$ as representations of $G_s$. On the other hand, we can take $\mathfrak{N} = (\mathfrak{N}', \varphi, \hat{G}_s) \in \text{Mod}^r_{G_\infty}$ such that $T' \simeq \hat{T}_s(\mathfrak{N}')$. We consider the following commutative diagram:

Here, $\text{Hom}(\mathfrak{M}', \mathfrak{M}(n))$ (resp. $\text{Hom}(\mathfrak{M}', \mathfrak{M}(n))$, resp. $\text{Hom}(\text{Max}^r(\mathfrak{N}'), \mathfrak{M}(n))$) is the set of morphisms $\mathfrak{M}' \to \mathfrak{M}(n)$ (resp. $\mathfrak{M}' \to \mathfrak{M}(n)$, resp. $\text{Max}^r(\mathfrak{N}') \to \mathfrak{M}(n)$) in $\text{Mod}^r_{G_\infty}$ (resp. $\text{Mod}^r_{G_\infty}$, resp. $\text{Max}^r_{G_\infty}$). The first bottom horizontal arrow is bijective by Theorem 2.5 (3) and so is the second (this follows from the fact that $\mathfrak{M}(n)$ is maximal by [CL1] Proposition 3.6.7]). Since the right vertical arrow is bijective, the top horizontal arrow must be bijective.

Now we are ready to prove Theorem 4.8

**Proof of Theorem 4.8** Let $T$ and $T'$ be objects of $\text{Rep}_{tor}(G_s)$. Take any Jordan-Hölder sequence $0 = T_0 \subset T_1 \subset \cdots \subset T_n = T$ of $T$ in $\text{Rep}_{tor}(G_s)$. By Corollary 4.5 we know that $T_i$ and $T_i/T_{i-1}$ are contained in $\text{Rep}_{tor}(G_s)$ for any $i$. By Corollary 4.5 again, the category $\text{Rep}_{tor}(G_s)$ is an exact category in the sense of Quillen ([QR] Section 2]). Hence short exact sequences in $\text{Rep}_{tor}(G_s)$ give rise to exact sequences of Hom’s and Ext’s in the usual way. (This property holds for any exact category.) On the other hand, by Lemma 4.13 if an exact sequence $0 \to T' \to V \to T_i/T_{i-1} \to 0$ in $\text{Rep}_{tor}(G_s)$ splits as representation of $G_\infty$, then it splits as a sequence of representations of $G_s$. Therefore, comparing exact sequences of Hom’s and Ext’s arising from $0 \to T_{i-1} \to T_i \to T_i/T_{i-1} \to 0$ in the category $\text{Rep}_{tor}(G_s)$ with that in the category $\text{Rep}_{tor}(G_\infty)$, we obtain the following implication (here, use Lemma 4.13 again): if we have $\text{Hom}_{G_s}(T_{i-1}, T') = \text{Hom}_{G_\infty}(T_{i-1}, T')$, then it gives the equality $\text{Hom}_{G_s}(T_i, T') = \text{Hom}_{G_\infty}(T_i, T')$. Hence a dévissage argument works and the desired full faithfulness follows.

**4.4 Proof of Theorem 1.2**

Now we are ready to prove our main theorems. First we prove Theorem 1.2
Recall that $\text{Rep}_{\text{tor}}^{r,\text{ht-peris}(s)}(G_K)$ is the category of torsion $\mathbb{Z}_p$-representations $T$ of $G_K$ which satisfy the following: there exist free $\mathbb{Z}_p$-representations $L$ and $L'$ of $G_K$, of height $\leq r$, such that

- $L|_{G_s}$ is a subrepresentation of $L'|_{G_s}$. Furthermore, $L|_{G_s}$ and $L'|_{G_s}$ are lattices in some crystalline $\mathbb{Q}_p$-representation of $G_s$ with Hodge-Tate weights in $[0,r]$;
- $T|_{G_s} \cong (L'|_{G_s})/(L|_{G_s})$.

We apply our arguments given in previous subsections with the following $J$:

\[ J = u^p I^{[1]} W(R) = u^p \varphi(t) W(R). \]

Then we have $c_J = p/e + p/(p-1)$ and thus the inequalities $ps/(p-1) \geq c_J > pr/(p-1)$ are satisfied if $e(r-1) < p-1$. Therefore, Theorem 4.12 is an easy consequence of the following proposition and Theorem 4.8.

**Proposition 4.14.** If $T$ is an object of $\text{Rep}_{\text{tor}}^{r,\text{ht-peris}(s)}(G_K)$, then $T|_{G_s}$ is contained in $\text{Rep}_{\text{tor}}^{r,G_s,J}(G_s)$. 

**Proof.** Take free $\mathbb{Z}_p$-representations $L$ and $L'$ of $G_K$, of height $\leq r$, such that

- $L|_{G_s}$ is a subrepresentation of $L'|_{G_s}$. Furthermore, $L|_{G_s}$ and $L'|_{G_s}$ are lattices in some crystalline $\mathbb{Q}_p$-representation of $G_s$ with Hodge-Tate weights in $[0,r]$;
- $T|_{G_s} \cong (L'|_{G_s})/(L|_{G_s})$.

By Theorem 2.5 (1), there exists an injection $\hat{\mathcal{E}}' \hookrightarrow \mathcal{E}$ of $(\varphi, G_s)$-modules over $\mathfrak{S}_s$ which corresponds to the injection $L|_{G_s} \hookrightarrow L'|_{G_s}$. On the other hand, there exist $\mathfrak{M}$ and $\mathfrak{M}'$ in $\text{Mod}_{\hat{\mathfrak{S}}_s}$ such that $T_{\mathfrak{M}}(\mathfrak{M}) \cong L|_{G_s}$ and $T_{\mathfrak{M}'}(\mathfrak{M}') \cong L'|_{G_s}$. Then Proposition 2.1 (1) implies that $\mathfrak{M}_s \otimes_{\varphi} \mathfrak{M} \cong L$ and $\mathfrak{M}_s \otimes_{\varphi} \mathfrak{M}' \cong L'$ as $\varphi$-modules over $\mathfrak{S}_s$. Therefore, we see that $\mathfrak{M}$ and $\mathfrak{M}'$ have structures of objects of $\text{Mod}_{\hat{\mathfrak{S}}_s}^{r,G_s}$; denote them by $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}'$, respectively. Since the functor $\text{Mod}_{\hat{\mathfrak{S}}_s}^{r,G_s} \to \text{Mod}_{\hat{\mathfrak{S}}_s}^{r,G_s}$ is fully faithful (cf. Section 3.1), the injection $\hat{\mathcal{E}}' \hookrightarrow \mathcal{E}$ descends to an injection $\hat{\mathfrak{M}} \hookrightarrow \hat{\mathfrak{M}}$. Now we put $\mathfrak{M} = \hat{\mathfrak{M}} \otimes_{\varphi,G_s} \mathfrak{N}$ by a natural isomorphism $\mathfrak{N}_s \otimes_{\varphi,G_s} \mathfrak{M}_s \cong (\mathfrak{N}_s \otimes_{\varphi,G_s} \mathfrak{M}_s)/((\mathfrak{N}_s \otimes_{\varphi,G_s} \mathfrak{N}_s))$. Then we see that $\mathfrak{M}$ has a structure of an object of $\text{Mod}_{\hat{\mathfrak{S}}_s}^{r,G_s,J}$; denote it by $\mathfrak{M}$. Moreover, Theorem 3.4 implies that $\mathfrak{M}$ is in fact contained in $\text{Mod}_{\hat{\mathfrak{S}}_s}^{r,G_s,J}$. By a similar argument to the proof of Lemma 3.1.4 of [CL2], we have an exact sequence $0 \to T_s(\mathfrak{M}) \to T_s(\mathfrak{M}') \to T_s(\mathfrak{M}) \to 0$ in $\text{Rep}_{\text{tor}}(G_s)$, which is isomorphic to $0 \to L|_{G_s} \to L'|_{G_s} \to T|_{G_s} \to 0$. This finishes a proof. □

**4.5 Proof of Theorem 4.3**

We give a proof of Theorem 4.3. If $s \geq n-1$, then we put

\[ J = u^p I^{[p^{s-n+1}]} W(R) = u^p \varphi(t)^{p^{s-n+1}} W(R). \]

Note that we have $c_J = p/e + p^{s-n+2}/(p-1)$ and thus the inequalities $p^{s-n+2}/(p-1) \geq c_J > pr/(p-1)$ are satisfied if $s > n-1 + \log_p(r/e)(p-1))$.

**Proposition 4.15.** Suppose $s \geq n-1$. If $T$ is an object of $\text{Rep}_{\text{tor}}^{r,\text{cris}}(G_K)$ which is killed by $p^n$, then $T|_{G_s}$ is contained in $\text{Rep}_{\text{tor}}^{r,G_s,J}(G_s)$.

**Proof.** Let $L$ be an object of $\text{Rep}_{\text{tor}}^{r,\text{cris}}(G_K)$. Take a $(\varphi, G)$-module $\mathcal{E}$ over $\mathfrak{S}$ such that $L \cong \hat{T}(\mathcal{E})$. It is known that $(\tau-1)^i(x) \in u^p I^{[i]} W(R) \otimes_{\varphi,G} \mathcal{E}$ for any $i \geq 1$ and any $x \in \mathcal{L}$ (cf. the latter half part
of the proof of \cite[Proposition 4.7]{GLS}). Take any \(x \in \mathfrak{L}\). Since \((\tau^p - 1)(x) = \sum_{i=1}^{p^r} (p^i)(r - 1)^i(x)\), we obtain that
\[
(\tau^p - 1)(x) \in \sum_{i=1}^{p^r} p^{s-v_p(i)} u^p J^i W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{L}.
\] (4.5.1)

Now let \(T\) be an object of \(\text{Rep}^\text{r,cris}(G_K)\) which is killed by \(p^n\). Take an exact sequence \((R): 0 \to L_1 \to L_2 \to T \to 0\) of \(\mathbb{Z}_p\)-representations of \(G_K\) with \(L_1, L_2 \in \text{Rep}^\text{r,cris}(G_K)\). By Theorem 3.1.3 and Lemma 3.1.4 of \cite{CL2}, there exists an exact sequence \((M): 0 \to \mathfrak{L}_2 \to \mathfrak{L}_1 \to \mathfrak{M} \to 0\) of \((\varphi, G)\)-modules over \(\mathfrak{S}\) such that \(\hat{T}(\mathfrak{M}) \simeq (R)\). By 4.5.1, we see that
\[
(\tau^p - 1)(x) \in \sum_{i=1}^{p^r} p^{s-v_p(i)} u^p J^i W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{M}
\]
for any \(x \in \mathfrak{M}\). Since \(\mathfrak{M}\) is killed by \(p^n\) and \(s \geq n - 1\), we have
\[
\sum_{i=1}^{p^r} p^{s-v_p(i)} u^p J^i W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{M} = \sum_{i=1, \ldots, p^r, s-v_p(i) < n}^{n-1} p^{s-v_p(i)} u^p J^i W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{M}
\]
\[
= \sum_{i=0}^{n-1} p^i u^p J^{[s-1]} W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{M}
\]
\[
\subset u^p J^{[s-n+1]} W(R) \otimes_{\varphi, \mathfrak{M}} \mathfrak{M}.
\]
Therefore, we obtained the desired result.

**Proof of Theorem 4.16** By Corollary 1.3 we may suppose \(\log_p(r - (p - 1)/e) \geq 0\), that is, \(e(r - 1) \geq p - 1\). Suppose \(s > n - 1 + \log_p(r - (p - 1)/e)\). Note that the condition \(s \geq n - 1\) is now satisfied.

Let \(T, T'\) be as in the statement of Theorem 4.1. Let \(f: T \to T'\) be a \(G_\infty\)-equivariant homomorphism. Denote by \(L\) the completion of \(K\) and identify \(G_L\) with the inertia subgroup of \(G_K\). We note that \(T|_{G_L}\) and \(T'|_{G_L}\) are objects of \(\text{Rep}^\text{r,cris}(G_L)\). By Proposition 4.1.5, \(T|_{G_L}\) and \(T'|_{G_L}\) are objects of \(\text{Rep}^\text{r,cris}(G_\infty)\). Hence we have that \(f\) is \(G_L\)-equivariant by Theorem 4.1. Since \(G_\infty\) is topologically generated by \(G_L\), we see that \(f\) is \(G_\infty\)-equivariant.

**4.6 Galois equivariance for torsion semi-stable representations**

In this subsection, we prove a Galois equivariance theorem for torsion semi-stable representations. A torsion \(\mathbb{Z}_p\)-representation \(T\) of \(G_K\) is **torsion semi-stable with Hodge-Tate weights in \([0, r]\)** if it can be written as the quotient of lattices in some semi-stable \(\mathbb{Q}_p\)-representation of \(G_K\) with Hodge-Tate weights in \([0, r]\). We denote by \(\text{Rep}^\text{r,st}(G_K)\) the category of them. Note that \(\text{Rep}^\text{0,cris}(G_K) = \text{Rep}^\text{0,cris}(G_K)\). Similar to Theorem 4.1.4 we show the following, which is the main result of this subsection.

**Theorem 4.16.** Suppose that \(s > n - 1 + \log_p r\). Let \(T\) and \(T'\) be objects of \(\text{Rep}^\text{r,st}(G_K)\) which are killed by \(p^n\). Then any \(G_\infty\)-equivariant homomorphism \(T \to T'\) is in fact \(G_s\)-equivariant.

If \(s \geq n - 1\), then we put
\[
J = J^{[s-n+1]} W(R) = \varphi(t)^{s-n+1} W(R).
\]
Then we have \(c_J = p^{s-n+2}/(p - 1)\). To show Theorem 4.16 we use similar arguments to those in the proof of Theorem 4.1.4.

**Proposition 4.17.** Suppose \(s \geq n - 1\). If \(T\) is an object of \(\text{Rep}^\text{r,cris}(G_K)\) which is killed by \(p^n\), then \(T|_{G_s}\) is contained in \(\text{Rep}^\text{r,cris}(G_s)\).
Proof. Let $L$ be a lattice in a semi-stable $\mathbb{Q}_p$-representation of $G_K$ with Hodge-Tate weights in $[0,r]$. Take a $(\varphi, \mathcal{G})$-module $\mathcal{E}$ over $\mathcal{E}$ such that $L \simeq \hat{T}(\mathcal{E})$. It is known that $(\tau - 1)^i(x) \in I[i]W(R) \otimes \varphi \otimes \mathcal{E}$ for any $i \geq 1$ and any $x \in \mathcal{E}$ (cf. the proof of Proposition 2.4.1]). Thus the same proof proceeds as that of Proposition 4.15.

Proof of Theorem 4.16. We have the equality $\text{Rep}_{\text{tor}}^{0, \text{st}}(G_K) = \text{Rep}_{\text{tor}}^{0, \text{cris}}(G_K)$ and thus Theorem 4.14 for $r = 0$ is an easy consequence of Corollary 4.3. Hence we may assume $r \geq 1$. The rest of a proof is similar to the proof of Theorem 4.14.

4.7 Some consequences

In this subsection, we generalize some results proved in Section 3.4 of [Br3]. First of all, we show the following elementary lemma, which should be well-known to experts, but we include a proof here for the sake of completeness.

Lemma 4.18. The full subcategories $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$ and $\text{Rep}_{\text{tor}}^{r, \text{st}}(G_K)$ of $\text{Rep}_{\text{tor}}(G_K)$ are stable under formation of subquotients, direct sums and the association $T \mapsto T^\vee(r)$. Here $T^\vee = \text{Hom}_{\mathbb{Z}_p}(T, \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of $T$.

Proof. We prove the statement only for $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$. Let $T \in \text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$ be killed by $p^n$ for some $n > 0$. Assertions for quotients and direct sums are clear. We prove that $T^\vee(r)$ is contained in $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$. There exist lattices $L_1 \subset L_2$ in some crystalline $\mathbb{Q}_p$-representation of $G_K$ and an exact sequence $0 \rightarrow L_1 \rightarrow L_2 \rightarrow T \rightarrow 0$ of $\mathbb{Z}_p[G_K]$-modules. This exact sequence induces an exact sequence $0 \rightarrow T \rightarrow L_1/p^nL_1 \rightarrow L_2/p^nL_2 \rightarrow T \rightarrow 0$ of finite $\mathbb{Z}_p[G_K]$-modules. By duality, we obtain an exact sequence $0 \rightarrow T^\vee \rightarrow (L_1/p^nL_1)^\vee \rightarrow (L_2/p^nL_2)^\vee \rightarrow T^\vee \rightarrow 0$ of finite $\mathbb{Z}_p[G_K]$-modules. Then we obtain a $G_K$-equivariant surjection $L_1^\vee \rightarrow T^\vee$ by the composite $L_1^\vee \rightarrow L_1^\vee/p^nL_1^\vee \rightarrow (L_1/p^nL_1)^\vee \rightarrow T^\vee$ of natural maps (here, for any free $\mathbb{Z}_p$-representation $L$ of $G_K$, $L^\vee := \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$ stands for the dual of $L$). Therefore, we obtain $L_1^\vee(r) \rightarrow T^\vee(r)$ and thus $T^\vee(r) \in \text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$. Finally, we prove the stability assertion for subobjects. Let $T'$ be a $G_K$-stable submodule of $T$. We have a $G_K$-equivariant surjection $f : L_1^\vee \rightarrow (T')^\vee$. Let $L_2$ be a free $\mathbb{Z}_p$-representation of $G_K$ such that its dual is the kernel of $f$. We have an exact sequence $0 \rightarrow (L_2)^\vee \rightarrow L_1^\vee \rightarrow (T')^\vee \rightarrow 0$ of $\mathbb{Z}_p[G_K]$-modules. Repeating the construction of the surjection $L_1^\vee \rightarrow T^\vee$, we obtain a $G_K$-equivariant surjection $L_2' = (L_2)^\vee \rightarrow (T')^\vee = T'$ and thus we have $T' \in \text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$.

In the case where $r = 1$, the assertion (1) of the following corollary was shown in Theorem 3.4.3 of [Br3].

Corollary 4.19. Let $T$ be an object of $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$ which is killed by $p^n$ for some $n > 0$. Let $T'$ be a $G_\infty$-stable subquotient of $T$.

(1) If $e(r - 1) < p - 1$, then $T'$ is $G_\infty$-stable (with respect to $T$).

(2) If $s > n - 1 + \log_p(r - (p - 1)/e)$, then $T'$ is $G_s$-stable (with respect to $T$).

Proof. By the duality assertion of Lemma 4.18 it is enough to show the case where $T'$ is a $G_\infty$-stable submodule of $T$. Take any sequence $T' = T_0 \subset T_1 \subset \cdots \subset T_m = T$ of $G_\infty$-stable submodules of $T$ such that $T_i/T_{i-1}$ is irreducible for any $i$. As explained in the proof of Proposition 4.13 the $G_\infty$-action on $T_i/T_{i-1}$ can be (uniquely) extended to $G_K$. By Theorem 5.3 given in the next section, we know that $T_i/T_{i-1}$ is an object of $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$ where $r_0 := \max\{r' \in \mathbb{Z}_{\geq 0} : e(r' - 1) < p - 1\}$.

(1) We may suppose $r = r_0$. The $G_\infty$-equivariant projection $T = T_m \rightarrow T_{m-1} = T_m/T_{m-1}$ is $G_K$-stable by the full faithfulness theorem (= Corollary 1.3). Thus we know that $T_{m-1}$ is $G_K$-stable in $T$, and also know that $T_{m-1}$ is contained in $\text{Rep}_{\text{tor}}^{r, \text{cris}}(G_K)$ by Lemma 4.18. By the same argument for the $G_\infty$-equivariant projection $T_{m-1} \rightarrow T_{m-2}$, we know that $T_{m-2}$ is $G_K$-stable in $T$, and also
know that $T_{m-2}$ is contained in $\text{Rep}_{\text{tor}}^{r,\text{cris}}(G_K)$. Repeating this argument, we have that $T' = T_0$ is $G_K$-stable in $T$.

(2) Put $J = u^m J_{(i^{-n+1})} W(R)$. By (1) we may assume $e(r-1) \geq p-1$. Under this assumption we have $r \geq r_0$ and $s > n - 1 + \log_p (r - (p - 1)/e) \geq n - 1$. In particular, $T|_{G_s}$ and $(T_i/T_i-1)|_{G_s}$, for any $i$, are contained in $\text{Rep}_{\text{tor}}^{r,G_s,J} (G_s)$ by Proposition 4.15. First we consider the case where $k$ is algebraically closed. By Theorem 4.8, the $G_\infty$-equivariant projection $T = T_m \twoheadrightarrow T_m/T_{m-1}$ is $G_s$-stable. Thus we know that $T_{m-1}$ is $G_s$-stable in $T$, and also know that $T_{m-1}$ is contained in $\text{Rep}_{\text{tor}}^{r,G_s,J} (G_s)$ by Corollary 4.19. By the same argument for the $G_\infty$-equivariant projection $T_{m-1} \twoheadrightarrow T_{m-2}$, we know that $T_m$ is $G_s$-stable in $T$, and also know that $T_{m-1}$ is contained in $\text{Rep}_{\text{tor}}^{r,G_s,J} (G_s)$. Repeating this argument, we have that $T' = T_0$ is $G_s$-stable in $T$. Next we consider the case where $k$ is not algebraically closed. Let $L$ be the completion of the maximal unramified extension $K_{\text{ur}}$ of $K$, and we identify $G_L$ with the inertia subgroup of $G_K$. Clearly $T|_{G_s}$ is contained in $\text{Rep}_{\text{tor}}^{r,\text{cris}}(G_L)$ and $T' = G_{L,s}$-stable submodule of $T$. We have already shown that $T'$ is $G_{L,s}$-stable in $T$. Since $G_s$ is topologically generated by $G_{L,s}$ and $G_\infty$, we conclude that $T'$ is $G_s$-stable in $T$.

Now let $V$ be a $\mathbb{Q}_p$-representation of $G_K$ and $T$ a $\mathbb{Z}_p$-lattice of $V$ which is stable under $G_\infty$. Then we know that $T$ is automatically $G_s$-stable for some $s \geq 0$. Indeed we can check this as follows. Take any $G_K$-stable $\mathbb{Z}_p$-lattice $T'$ of $V$ which contains $T$, and take an integer $n > 0$ with the property that $p^n T' \subset T$. Furthermore, we take a finite extension $K'$ of $K$ such that $G_{K'}$ acts trivially on $T'/p^n T'$. Then $T'/p^n T'$ is $G_{\infty}$-stable and also $G_{K'}$-stable. If we take any integer $s \geq 0$ with the property $K' \cap K_{\infty} \subset K(s)$, we know that $T'/p^n T'$ is $G_s$-stable. This implies that $T$ is $G_s$-stable in $T'$.

The following corollary, which was shown in Corollary 3.4.4 of [Br3] in the case where $r = 1$, is related with the above property.

**Corollary 4.20.** Let $V$ be a crystalline $\mathbb{Q}_p$-representation of $G_K$ with Hodge-Tate weights in $[0,r]$ and $T$ a finitely generated $\mathbb{Z}_p$-submodule of $V$ which is stable under $G_\infty$. If $e(r-1) < p-1$, then $T$ is stable under $G_K$.

**Proof.** We completely follow the method of the proof of [Br3] Corollary 3.4.4. Take any $G_K$-stable $\mathbb{Z}_p$-lattice $T'$ of $V$ which contains $T$. Since $T'/p^n T'$ is contained in $\text{Rep}_{\text{tor}}^{r,\text{cris}}(G_K)$ for any $n > 0$, Corollary 4.19 (1) implies that any $G_\infty$-stable submodule of $T'/p^n T'$ is in fact $G_{K'}$-stable. Thus $(T + p^n T')/p^n T'$ is $G_K$-stable in $T'/p^n T'$. Therefore, we obtain $g(T) \subset \bigcap_{n>0} (T + p^n T') = T$ for any $g \in G_K$.

## 5 Crystalline lifts and c-weights

We continue to use the same notation except for that we may allow $p = 2$. We remark that a torsion $\mathbb{Z}_p$-representation of $G_K$ is torsion crystalline with Hodge-Tate weights in $[0,r]$ if there exist a lattice $L$ in some crystalline $\mathbb{Q}_p$-representation of $G_K$ with Hodge-Tate weights in $[0,r]$ and a $G_K$-equivariant surjection $f: L \twoheadrightarrow T$. We call $f$ a crystalline lift (of $T$) of weight $\leq r$. Our interest in this section is to determine the minimum integer $r$ (if it exists) such that $T$ admits crystalline lifts of weight $\leq r$. We call this minimum integer the c-weight of $T$ and denote it by $w_c(T)$. If $T$ does not have crystalline lifts of weight $\leq r$ for any integer $r$, then we define the c-weight $w_c(T)$ of $T$ to be $\infty$. Motivated by [L2 Question 5.5], we raise the following question.

**Question 5.1.** For a torsion $\mathbb{Z}_p$-representation $T$ of $G_K$, is the c-weight $w_c(T)$ of $T$ finite? Furthermore, can we calculate $w_c(T)$?
5.1 General properties of c-weights

We study general properties of c-weights. At first, by ramification estimates, it is known that c-weights may have infinitely large values (CL2 Theorem 5.4); for any $c > 0$, there exists a torsion $\mathbb{Z}_p$-extension $T$ of $G_K$ with $w_c(T) > c$. In this paper, we mainly consider representations with “small” c-weights. If c-weights are “small”, they are closely related with tame inertia weights.

Now we recall the definition of tame inertia weights. Let $I_K$ be the inertia subgroup of $G_K$. Let $T$ be a $d$-dimensional irreducible $\mathbb{F}_p$-representation of $I_K$. Then $T$ is isomorphic to

$$\mathbb{F}_p(\theta^{n_1}_d \cdots \theta^{n_d}_d)$$

for some sequence of integers $0$ and $p - 1$, periodic of period $d$. Here, $\theta_d, \ldots, \theta_{d, d}$ are the fundamental characters of level $d$. The integers $n_d/e, \ldots, n_d/e$ are called the tame inertia weights of $T$. For any $\mathbb{F}_p$-representation $T$ of $G_K$, the tame inertia weights of $T$ are the tame inertia weights of the Jordan–Hölder quotients of $T|_{I_K}$.

Let $\chi_p: G_K \to \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic character and $\chi_p: G_K \to \mathbb{Z}_p^\times$ the $p$ cyclotomic character. It is well-known that $\chi_p|_{I_K} = \theta^e_1$ where $\theta_1: I_K \to \mathbb{Z}_p^\times$ is the fundamental character of level $1$. In particular, denoting by $K^{ur}$ the maximal unramified extension of $K$, we have $[K^{ur}(\mu_p)^{r} : K^{ur}] = (p - 1)/\gcd(e, p - 1)$.

Proposition 5.2. (1) Minimum c-weights are invariant under finite unramified extensions of the base field $K$.
(2) The c-weight of an unramified torsion $\mathbb{Z}_p$-representation of $G_K$ is 0.
(3) Put $\nu = (p - 1)/\gcd(e, p - 1)$. If $\nu(s - 1) < w_c(T) \leq \nu s$, then we have $\nu(s - 1) < w_c(T^\nu) \leq \nu s$.
(4) Let $L$ be an $\mathbb{F}_p$-representation of $G_K$ and $i$ be the largest tame inertia weight of $T$. Then we have $w_c(T) \geq i$.

Proof. (1) Let $T$ be a torsion $\mathbb{Z}_p$-representations of $G_K$. Let $K'$ be a finite unramified extension of $K$. It suffices to prove that $T$ has crystalline lifts of weight $\leq r$ if and only if $T|_{G_{K'}}$ has crystalline lifts of weight $\leq r$. The “only if” assertion is clear and thus it is enough to prove the “if” assertion. Let $f: L \to T|_{G_{K'}}$ be a crystalline lift of $T|_{G_{K'}}$ of weight $\leq r$. Since $K'/K$ is unramified, $\text{Ind}^G_{G_{K'}} L$ is a lattice in some crystalline $\mathbb{Z}_p$-representation of $G_K'$ with Hodge-Tate weights in $[0, r]$. Furthermore, the map

$$\text{Ind}^G_{G_{K'}} L = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} L \to f, \quad \sigma \otimes x \mapsto \sigma(f(x))$$

is a $G_K$-equivariant surjection and hence we have done.

(2) The result follows from (1) immediately.

(3) Taking a finite unramified extension $K'$ of $K$ with $[K^{ur}(\mu_p)^{r} : K^{ur}] = [K'(\mu_p)^{r} : K']$, it follows from Lemma 4.13 that we have $\nu(s - 1) < w_c(T|_{G_{K'}}) \leq \nu s$ if and only if we have $\nu(s - 1) < w_c((T^\nu)|_{G_{K'}}) \leq \nu s$. Thus the result follows from the assertion (1).

(4) If $c w_c(T) \geq p - 1$, then there is nothing to prove, and thus we may suppose that $c w_c(T) < p - 1$.

Let $L \to T$ be a crystalline lift of $T$ of weight $\leq w_c(T)$. Since the tame inertia polygon of $L$ lies on the Hodge polygon of $L$ (CN Théorème 1), the largest slope of the former polygon is less than or equal to that of the latter polygon. This implies $w_c(T) \geq i$.

Theorem 5.3. Let $T$ be a tamely ramified $\mathbb{F}_p$-representation of $G_K$. Let $i$ be the largest tame inertia weight of $T$. Then we have $w_c(T) = \min\{h \in \mathbb{Z}_{\geq 0}; h \geq i\}$.

Proof. The proof below is essentially due to Caruso and Liu (CL2 Theorem 5.7), but we give a proof here for the sake of completeness. Put $i_0 = \min\{h \in \mathbb{Z}_{\geq 0}; h \geq i\}$. By Proposition 5.2 (4), we have $w_c(T) \geq i_0$. Thus it suffices to show $w_c(T) \leq i_0$. We note that $T|_{I_K}$ is semi-simple. Any irreducible component $T_0$ of $T|_{I_K}$ is of the form $\mathbb{F}_p(\theta^{n_1}_d \cdots \theta^{n_d}_d)$ for some sequence of integers
between 0 and $p-1$, periodic of period $d$. We decompose $n_j = em_j + n'_j$ by integers $0 \leq m_j \leq i_0$ and $0 \leq n'_j < e$. Now we define an integer $k_{j,\ell}$ by

$$k_{j,\ell} := \begin{cases} e & \text{if } 1 \leq \ell \leq m_j, \\ n'_j & \text{if } \ell = m_j + 1, \\ 0 & \text{if } \ell > m_j + 1. \end{cases}$$

Note that we have $n_j = \sum_{\ell=1}^{i_0} k_{j,\ell}$, and also have an $I_K$-equivariant surjection

$$T_0 = \mathbb{F}_p\langle \theta_{d,1}^{n_1} \cdots \theta_{d,d}^{n_d} \rangle = \bigotimes_{\ell=1, \ldots, i_0} \mathbb{F}_p\langle \theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}} \rangle \twoheadrightarrow \bigotimes_{\ell=1, \ldots, i_0} \mathbb{F}_p\langle \theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}} \rangle.$$

By a classical result of Raynaud, each $\mathbb{F}_p\langle \theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}} \rangle$ comes from a finite flat group scheme defined over $K^w$. We should remark that such a finite flat group scheme is in fact defined over a finite unramified extension of $K$. Since any finite flat group scheme can be embedded in a $p$-divisible group, the above observation implies the following: there exist a finite unramified extension $K'$ over $K$, a lattice $L$ in some crystalline $\mathbb{Q}_p$-representation of $G_{K'}$ with Hodge-Tate weights in $[0, i_0]$ and an $I_K$-equivariant surjection $f: L \to T$. The map $f$ induces an $I_K$-equivariant surjection $\tilde{f}: L/pL \to T$. Since $L/pL$ and $T$ is finite, we see that $\tilde{f}$ is in fact $G_{K'}$-equivariant for some finite unramified extension $K''$ over $K'$, and then so is $f$. Therefore, we obtain $w_c(T|_{G_{K''}}) \leq i_0$. By Proposition 5.2 (1), we obtain $w_c(T) \leq i_0$. \hfill \Box

5.2 Rank 2 cases

We give some computations of $c$-weights related with torsion representations of rank 2. We prove the following lemma by an almost identical method with [GLS] Lemma 9.4.

**Lemma 5.4.** Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $E$ be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Let $i$ and $\nu$ be integers such that $\nu$ is divisible by $[K(\mu_p) : K]$. Suppose that $T$ is an $\mathbb{F}$-representation of $G_K$ which sits in an exact sequence $(\ast): 0 \to \mathbb{F}(i) \to T \to \mathbb{F} \to 0$ of $\mathbb{F}$-representations of $G_K$. Then there exist a ramified degree at most 2 extension $E'$ over $E$, with integer ring $\mathcal{O}_{E'}$, and an unramified continuous character $\chi: G_K \to \mathbb{F}^\times$ with trivial reduction such that $(\ast)$ is the reduction of some exact sequence $0 \to \mathcal{O}_{E'}(\chi_p^{1+i}) \to \Lambda \to \mathcal{O}_{E'} \to 0$ of free $\mathcal{O}_{E'}$-representations of $G_K$. Furthermore, we have the followings:

1. If $i + \nu = 1$ or $\chi_p^{1-i} \neq 1$, then we can take $E' = E$ and $\chi = 1$.
2. If $i + \nu = 0$ and $T$ is unramified, then we can take $E' = E$, $\chi = 1$ and $\Lambda$ to be unramified.

**Proof.** Suppose $i + \nu = 1$ (resp. $\chi_p^{1-i} \neq 1$). Then the map $H^1(K, \mathcal{O}_E(i + \nu)) \to H^1(K, \mathbb{F}(i))$ arising from the exact sequence $0 \to \mathcal{O}_E(i + \nu) \twoheadrightarrow \mathcal{O}_E(i + \nu) \to \mathbb{F}(i) \to 0$ is surjective since $H^2(K, \mathcal{O}_E(1)) \cong \mathcal{O}_E$ (resp. $H^2(K, \mathcal{O}_E(i + \nu)) = 0$), where $\twoheadrightarrow$ is a uniformizer of $E$. Hence we obtained a proof of (1). The assertion (2) follows immediately from the fact that the natural map $H^1(G_K/I_K, \mathcal{O}_E) \to H^1(G_K/I_K, \mathbb{F})$ is surjective.

In the rest of this proof, we always assume that $i + \nu \neq 1$ and $\chi_p^{1-i} = 1$. Let $L \in H^1(K, \mathbb{F}(i))$ be a 1-cocycle corresponding to $(\ast)$. We may suppose $L \neq 0$. For any unramified continuous character $\chi: G_K \to \mathbb{F}^\times$ with trivial reduction, we denote by

$$\delta^1_\chi: H^1(K, \mathbb{F}(i)) \to H^2(K, \mathcal{O}_E(\chi_p^{1+i}))$$

(resp. $\delta^0_\chi: H^0(K, E/\mathcal{O}_E(\chi_p^{-1-i}) \to H^1(K, \mathbb{F})$)

the connection map arising from the exact sequence $0 \to \mathcal{O}_E(\chi_p^{1+i}) \twoheadrightarrow \mathcal{O}_E(\chi_p^{1+i}) \to \mathbb{F}(i) \to 0$ (resp. $0 \to E/\mathcal{O}_E(\chi_p^{-1-i}) \twoheadrightarrow E/\mathcal{O}_E(\chi_p^{-1-i}) \to 0$) of $\mathcal{O}_E[G_K]$-modules. Consider the following commutative diagram:
The result follows immediately from Proposition 5.2 and Lemma 5.4. 

Since any \( \chi \) the relation \( \chi \equiv 1 \mod \varpi^n \) (such \( n \) exists since \( \chi_p^{-1} = 1 \) and \( 1 - i - \nu \neq 0 \)). We define \( \alpha \chi : G_K \to \mathcal{O}_E \) by the relation \( \chi^{-1} \chi_p^{1-i-\nu} = 1 + \varpi^n \alpha \chi \), and denote \( (\alpha \chi \mod \varpi) : G_K \to \mathbb{F} \) by \( \bar{\alpha} \chi \). By definition, \( \bar{\alpha} \chi \) is a non-zero character of \( H^1(G, \mathbb{F}) \), and it is not difficult to check that the image of \( \delta^i \) is contained in \( H \) for some \( \epsilon \), we are done. Suppose this is not the case.

Suppose that \( H \) is not contained in the unramified line in \( H^1(G, \mathbb{F}) \). We claim that we can choose \( \chi \) such that \( \bar{\alpha} \chi \) is ramified. Let \( m \) be the largest integer with the property that \((\chi^{-1} \chi_p^{1-i-\nu})|_{I_K} = 1 + 2\alpha \). Clearly, we have \( m \geq n \). If \( m = n \), then we are done and thus we may assume \( m > n \). Fix a lift \( \hat{g} \in G_K \) of the Frobenius of \( K \). We see that \( \alpha \chi \neq 0 \). Let \( \chi' \) be the unramified character sending \( g \) \( 1 + \varpi^n \alpha \chi \). Then \( \chi' \) has trivial reduction. After replacing \( \chi \) with \( \chi' \chi \), we reduce the case where \( m = n \) and the claim follows. Suppose \( \bar{\alpha} \chi \) is ramified. Then there exists a unique \( \bar{x} \in \mathbb{F}^\times \) such that \( \bar{\alpha} \chi \bar{x} \in H \) where \( \bar{x} : G_K \to \mathbb{F} \) is the unramified character sending \( g \) to \( \bar{x} \). Denote by \( \chi'' \) the unramified character sending \( g \) to \( 1 + \varpi^n \alpha \chi \). Replacing \( \chi \) with \( \chi'' \), we have done.

Suppose that \( H \) is contained in the unramified line in \( H^1(G, \mathbb{F}) \) (thus \( H \) and the unramified line coincide with each other). By replacing \( E \) with \( E(\sqrt{\varpi}) \), we may assume that \( n > 1 \). Let \( \chi_0 \) be a character defined by \( \chi \) times the unramified character sending our fixed \( g \) \( 1 + \varpi^n \alpha \chi_0 \). Then \( \chi_0 \) has trivial reduction. After replacing \( \chi \) with \( \chi \chi \), we replace \( \chi \) with \( \chi'' \). Since \( \epsilon \), we have done.

Let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let \( \chi : G_K \to \mathbb{F}^\times \) an unramified character. Since any \( E \)-representation of \( G_K \) which is an extension of \( E \) by \( E(\chi \chi_p) \) is automatically crystalline, we obtain the following.

**Proposition 5.5.** Suppose \( p > 2 \). Let \( K \) be a finite unramified extension of \( \mathbb{Q}_p \). Let \( T \in \text{Rep}_{\text{tor}}(G_K) \) be killed by \( p \) and sit in an exact sequence \( 0 \to \mathbb{F}_p(i) \to T \to \mathbb{F}_p \to 0 \) of \( \mathbb{F}_p \)-representations of \( G_K \). Then we have the followings:

1. If \( i = 0 \) and \( T \) is unramified, then we have \( w_e(T) = 0 \).
2. If \( i = 0 \) and \( T \) is not unramified, then we have \( w_e(T) = p - 1 \).
3. If \( i = 2, \ldots, p - 2 \), then we have \( w_e(T) = i \).

**Proof.** (1), (2) By Lemma 5.4, it suffices to prove that \( T \) is not torsion crystalline with Hodge-Tate weights in \([0, p - 2]\) if \( T \) is not unramified. Let \( K_T \) be the definition field of the representation \( T \) of \( G_K \) and put \( G = \text{Gal}(K_T/K) \). Let \( G^1 \) be the upper numbering \( j \)-th ramification subgroup of \( G \). Since \( T \) is not unramified and killed by \( p \), we see that \( K_T \) is a totally ramified degree \( p \) extension over \( K \). Thus \( G^1 \) is the wild inertia subgroup of \( G \) and \( G^1 \) acts on \( T \). Since \( T \) is not torsion crystalline with Hodge-Tate weights in \([0, 2p - 2]\), then \( G^1 \) acts on \( T \) trivially for any \( j \) \( (p - 2)/(p - 1) \).

(3) The result follows immediately from Proposition 5.4(4) and Lemma 5.2.

**Corollary 5.6.** Let \( K \) be a finite unramified extension of \( \mathbb{Q}_p \). Then any 2-dimensional \( \mathbb{F}_p \)-representation of \( G_K \) is torsion crystalline with Hodge-Tate weights in \([0, 2p - 2]\).
Proof. If $T$ is irreducible, the result follows from Theorem 5.3. Assume that $T$ is reducible. Since $K$ is unramified over $\mathbb{Q}_p$, any continuous character $G_K \to \mathbb{F}_p^\times$ is of the form $\chi_p^i$ for some unramified character $\chi$ and some integer $i$. Replacing $K$ with its finite unramified extension, we may assume that $T$ sits in an exact sequence $0 \to \mathbb{F}_p(i) \to T \to \mathbb{F}_p(j) \to 0$ of $\mathbb{F}_p$-representations of $G_K$, where $i$ and $j$ are integers in the range $[0, p - 2]$ (we remark that $w_e(T)$ is invariant under unramified extensions of $K$ by Proposition 5.2 (1)). It follows from Lemma 5.3 that $w_e(T(-j)) \leq p$. Therefore, we obtain $w_e(T) = w_e(T(-j) \otimes_{\mathbb{F}_p} \mathbb{F}_p(j)) \leq w_e(T(-j)) + w_e(\mathbb{F}_p(j)) \leq p + (p - 2) = 2p - 2$. □

5.3 Extensions of $\mathbb{F}_p$ by $\mathbb{F}_p(1)$ and non-fullness theorems

By Lemma 5.4, we know that the $c$-weight $w_e(T)$ of a lattice in some $\mathbb{F}_p$-representation $T$ of $G_K$ which sits in an exact sequence $0 \to \mathbb{F}_p(1) \to T \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$-representations of $G_K$, is less than or equal to $p$. Let us calculate $w_e(T)$ for such $T$ more precisely. We should remark that such $T$ is written as $p$-torsion points of a Tate curve. Hence we consider torsion representations coming from Tate curves.

Let $v_K$ be the valuation of $K$ normalized such that $v_K(K^\times) = \mathbb{Z}$, and take any $q \in K^\times$ with $v_K(q) > 0$. Let $E_q$ be the Tate curve over $K$ associated with $q$ and $E_q[p^n]$ the module of $p^n$-torsion points of $E_q$ for any integer $n > 0$. It is well-known that there exists an exact sequence of $\mathbb{Z}$-modules.

\[
\begin{array}{c}
\mathbb{Z} & \to & \mu_{p^n} & \to & E_q[p^n] & \to & \mathbb{Z}/p^n\mathbb{Z} & \to & 0 \\
\end{array}
\]

\[
\text{of } \mathbb{Z}[G_K]\text{-modules. Here, } \mu_{p^n} \text{ is the group of } p^n\text{-th roots of unity in } \overline{K}. \text{ Let } x_n : G_K \to \mu_{p^n} \text{ be the 1-cocycle defined to be the image of 1 for the connection map } H^0(K, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(K, \mu_{p^n}) \text{ arising from the exact sequence } (\#). \text{ Then } x_n \text{ corresponds to } q \pmod{(K^\times)^p} \text{ via the isomorphism } K^\times/(K^\times)^p \simeq H^1(K, \mu_{p^n}) \text{ of Kummer theory. Thus the exact sequence } (\#) \text{ splits if and only if } q \in (K^\times)^p.
\]

First we consider the case $p \mid v_K(q)$ (i.e. $\text{peu ramifié}$ case).

Lemma 5.7. Let $K$ be a finite extension of $\mathbb{Q}_p$. If $p \nmid v_K(q)$, then $E_q[p]$ is the reduction modulo $p$ of a lattice in some 2-dimensional crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0, 1]$.

Proof. Since $p \nmid v_K(q)$, there exists $q' \in K^\times$ such that $v_K(q' - 1) > 0$ and $q \equiv q' \pmod{(K^\times)^p}$. Considering the exact sequence $0 \to \mathbb{Z}_p(1) \to L \to \mathbb{Z} \to 0$ of $\mathbb{Z}_p$-representations of $G_K$ corresponding to $q'$ via the isomorphism $H^1(K, \mathbb{Z}_p(1)) \simeq \lim_{\leftarrow n} K^\times/(K^\times)^p$ of Kummer theory, we obtain the desired result. □

Corollary 5.8. Suppose that $K$ is a finite extension of $\mathbb{Q}_p$, $(p - 1) \nmid e$ and $p \mid v_K(q)$. Then we have $w_e(E_q[p]) = 1$.

Proof. By the assumption $(p - 1) \nmid e$, we know that the largest tame inertia weight of $E_q[p]$ is positive. Thus Proposition 5.2 (4) shows $w_e(E_q[p]) \geq 1$. The inequality $w_e(E_q[p]) \leq 1$ follows from Lemma 5.7. □

Next we consider the case $p \nmid v_K(q)$ (i.e. $\text{très ramifié}$ case).

Proposition 5.9. If $e(r - 1) < p - 1$ and $p \nmid v_K(q)$, then $E_q[p^n]$ is not torsion crystalline with Hodge-Tate weights in $[0, r]$ for any $n > 0$.

Remark 5.10. If $e = 1$, the fact that $E_q[p^n]$ is not torsion crystalline with Hodge-Tate weights in $[0, p - 1]$ immediately follows from the theory of ramification bound as below. We may suppose $n = 1$. Suppose $E_q[p]$ is torsion crystalline with Hodge-Tate weights in $[0, p - 1]$. Then the upper numbering $j$-th ramification subgroup $G_K^j$ of $G_K$ (in the sense of [Se]) acts trivially on $E_q[p]$ for any $j > 1$ (see [AB1] Section 6, Theorem 3.1). However, this contradicts the fact that the upper bound of the ramification of $E_q[p]$ is $1 + 1/(p - 1)$.
Proof of Proposition 5.9. We may suppose $n = 1$. We choose any uniformizer $\pi'$ of $K$. Putting $v_K(q) = m$, we can write $q = (\pi')^m x$ with some unit $x$ of the integer ring of $K$. Since $m$ is prime to $p$, we have a decomposition $x = \zeta q^m$ in $K^\times$ for some $\ell > 0$ prime to $p$ and $y \in K$ with $v_K(y - 1) > 0$. Here $\zeta$ is a (not necessary primitive) $\ell$-th root of unity. Since $\ell$ is prime to $p$, we have $\zeta = \zeta^s$ for some integer $s$. We put $\pi = \pi' y$. This is a uniformizer of $K$. Choose any $p$-th root $\pi_1$ of $\pi$ and put $q_1 = \zeta^s \pi_1^m \in K(\pi_1)^\times$. Then we have $q = q_1^e \in (K(\pi_1)^\times)^p$ and in particular, the exact sequence $(\#)$ (for $n = 1$) splits as representations of $\Gal(K/K(\pi_1))$. Now assume that $E_q[p]$ is torsion crystalline with Hodge-Tate weights in $[0, \tau]$. Then $(\#)$ (for $n = 1$) splits as representations of $G_K$ by Corollary 5.14. This contradicts the assumption $p \mid v_K(q)$ (and hence $q \notin (K^\times)^p$).

Now we put $r'_0 = \min\{r \in \mathbb{Z}_{\geq 0}; e(r - 1) \geq p - 1\}$. Recall that we have $[K^\ur(\mu_p) : K^\ur] = (p-1)/\gcd(e, p-1)$.

Lemma 5.11. Let $K$ be a finite extension of $\mathbb{Q}_p$. Then $E_q[p]$ is torsion crystalline with Hodge-Tate weights in $[0, 1 + (p-1)/\gcd(e, p-1)]$.

Proof. Taking a finite unramified extension $K'$ of $K$ such that $[K^\ur(\mu_p) : K^\ur] = [K'(\mu_p) : K']$, we obtain $w_c((E_q[p])/G_{K'}) \leq 1 + (p-1)/\gcd(e, p-1)$ by Lemma 5.4. Thus we have $w_c(E_q[p]) \leq 1 + (p-1)/\gcd(e, p-1)$ by Proposition 5.5.$\square$

Corollary 5.12. Suppose that $K$ is a finite extension of $\mathbb{Q}_p$, and also suppose $e (p-1)$ or $(p-1) | e$. We further suppose that $p \nmid v_K(q)$. Then we have $w_c(E_q[p]) = r'_0$.

Proof. We have $w_c(E_q[p]) \leq r'_0$ by Lemma 5.11. In addition, we also have $w_c(E_q[p]) \geq r'_0$ by Proposition 5.5.$\square$

Lemma 5.14 gives some non-fullness results on torsion crystalline representations.

Corollary 5.13. Suppose that $K$ is a finite extension of $\mathbb{Q}_p$. If $r \geq 1 + (p-1)/\gcd(e, p-1)$, then the restriction functor $\Rep_{\tor, \cris}(G_K) \to \Rep_{\tor}(G_1)$ is not full.

Proof. Two representations $E_q[p]$ and $F_p(1) \oplus F_p$ are objects of $\Rep_{\tor}(G_K)$ by Lemma 5.11. They are not isomorphic as representations of $G_K$ but isomorphic as representations of $G_1$. Thus the desired non-fullness follows.$\square$

Corollary 5.14. Suppose that any one of the following holds:

- $p = 2$ and $K$ is a finite extension of $\mathbb{Q}_2$ (in this case $r'_0 = 2$);
- $K$ is a finite unramified extension of $\mathbb{Q}_p$ (in this case $r'_0 = p$);
- $K$ is a finite extension of $\mathbb{Q}_p(\mu_p)$ (in this case $r'_0 = 2$).

Then the restriction functor $\Rep_{\tor, \cris}(G_K) \to \Rep_{\tor}(G_1)$ is not full.$\square$

References

[Ab1] Victor Abrashkin, Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1135–1182 (Russian), Math. USSR-Izv. 35, no. 3, 469–518 (English).

[Ab2] Victor Abrashkin, Group schemes of period $p > 2$, Proc. Lond. Math. Soc. (3), 101 (2010), 207–259.

[Br1] Christophe Breuil, Représentations $p$-adiques semi-stables et transversalité de Griffiths, Math. Ann. 307 (1997), 191–224.
[Br2] Christophe Breuil, *Une application du corps des normes*, Compos. Math. **117** (1999) 189–203.

[Br3] Christophe Breuil, *Integral p-adic Hodge theory*, in Algebraic geometry 2000, Azumino (Hokkaido), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, 2002, 51–80.

[Ca] Xavier Caruso, Représentations semi-stables de torsion dans le cas $e r < p - 1$, J. Reine Angew. Math. **594** (2006), 35–92.

[CL1] Xavier Caruso and Tong Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France, **137** (2009), no. 2, 185–223.

[CL2] Xavier Caruso and Tong Liu, *Some bounds for ramification of $p^n$-torsion semi-stable representations*, J. Algebra, **325** (2011), 70–96.

[CS] Xavier Caruso and David Savitt, *Polygons de Hodge, de Newton et de linérité modéré des représentations semi-stables*, Math. Ann. **343** (2009): 773–789.

[Fo1] Jean-Marc Fontaine, *Schémas propres et lisses sur $\mathbb{Z}$*, Proceedings of the Indo-French Conference on Geometry Bombay, 1989, (Hindustan Book Agency 1993), 43–56.

[Fo2] Jean-Marc Fontaine, *Le corps des périodes $p$-adiques*, Astérisque (1994), no. 223, 59–111, With an appendix by Pierre Colmez, Périodes $p$-adiques (Bures-sur-Yvette, 1988).

[GLS] Toby Gee, Tong Liu and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), 389-435.

[Kis] Mark Kisin, *Crystalline representations and $F$-crystals*, Algebraic geometry and number theory, Progr. Math. **253**, Birkhäuser Boston, Boston, MA (2006), 459–496.

[Kim] Wansu Kim, *The classification of $p$-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. **19** (2012), no. 1, 121–141.

[La] Eike Lau, *A relation between Dieudonné displays and crystalline Dieudonné theory*, preprint, arXiv:1006.2720v3

[Li1] Tong Liu, *Torsion $p$-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 633–674.

[Li2] Tong Liu, *A note on lattices in semi-stable representations*, Math. Ann. **346** (2010), 117–138.

[Li3] Tong Liu, *Lattices in filtered $(\varphi, N)$-modules*, J. Inst. Math. Jussieu 2, Volume 11, Issue 03 (2012), 659-693.

[Li4] Tong Liu, *The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$*, appear at Journal de Théorie des Nombres de Bordeaux.

[Qu] Daniel Quillen, *Higher algebraic K-theory: I*, in Algebraic K-theory, I: Higher K-theories (Seattle, 1972), Lecture Notes in Math. **341**, Springer-Verlag, New York, 1973, 85–147.

[Se] Jean-Pierre Serre, *Corps locaux*, Hermann, Paris, 1968.

[Oz1] Yoshiyasu Ozeki, *Torsion representations arising from $(\varphi, \hat{G})$-modules*, J. Number Theory **133** (2013), 3810–3861.

[Oz2] Yoshiyasu Ozeki, *Full faithfulness theorem for torsion crystalline representations*, preprint, arXiv:1206.4751v4