Degeneracy Breaking of Hydrogen Atom

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The three dimensional rotation group, SO(3), is a symmetry group of the normal hydrogen atom. Each reducible representation of this group can be associated with a degenerate energy level. If this atom is placed in an external magnetic field, the interaction between the orbital magnetic moment with this field will lead to a symmetry breaking where the symmetry group of the atom is a new group distinct from the SO(3) group. This phenomenon describes the normal Zeeman effect, where a degenerate energy level splits into several new energy levels. It is explicitly shown that each of the new energy levels can be associated with an irreducible representation of the new symmetry group.

Section II will explain about basic concept of group theory, especially the meaning of reducible representation and irreducible representation of continuous group or Lie group, which is useful to formulate symmetrical behavior of a system. Decomposition of a reducible representation into several irreducible representations is seemed as symmetry breaking. For a hydrogen atom, decomposition of reducible representation represented in three-dimensional rotation group into number of several irreducible representations will be useful to explain splitting of degenerate level into several new energy levels.

Section III clarifies Schrödinger and Heissenberg pictures associated to normal Zeeman effect. Schrödinger picture shows state vector dynamics and Heissenberg picture describes physical quantity associated to the hydrogen atom. In the case of hydrogen atom placed in an external magnetic field, the state vector is state vector of angular momentum operator and the physical quantity is angular momentum which is represented by generators of a three-dimensional rotation group.

Section IV builds structure of symmetry group for a hydrogen atom placed in an external magnetic field which is parallel to z axis. This symmetry group is shown as direct product from one-dimensional unitary group, U(1), with discrete group, $\mathcal{G} = \{E, I\}$, where $E$ represents identity operator and $I$ express inversion operator on $xy$-plane.

Section V provides conclusion emphasizing that group structure built for normal Zeeman effect can be developed to explain anomalous Zeeman effect. Normal Zeeman effect is resulted by interaction between orbital magnetic moment from electron under the influence of external magnetic field, while anomalous Zeeman effect is affected by the total magnetic moment, that is orbital magnetic moment added with spin magnetic moment, and external magnetic field. This kind of interaction causes hyperfine splitting of hydrogen atom spectrum.

I. INTRODUCTION

Generally, every system such as an atom or molecule, which has uniquely symmetrical behavior, can always be represented in a degenerate state; i.e., energy level of the system represents an eigenvalue of Hamiltonian operator for the system which has a set of eigenvector. The simplest example is a hydrogen atom.

According to Bohr model, hydrogen atom has a spherical symmetry property, i.e., Hamiltonian operator for the atom is invariant over three-dimensional group transformation, SO(3). This symmetrical property causes for every principal quantum number, $n=1,2,\ldots$, there is only one energy level for every orbital quantum number, $l=1, \ldots, n-1$, and and for every magnetic quantum number, $m=-l, -l+1, \ldots, l$. The energy level is given by:

$$E_{n,l,m} = -\frac{13.6}{n^2} \text{eV}$$

which is independent on $l$ and $m$.

If this hydrogen atom is placed in an external magnetic field, the spherical symmetry will be broken. This symmetry breaking causes hyperfine structure of energy level $E_{n,l,m}$ to the new several energy level. If one consider the interaction between orbital magnetic moment with the external magnetic field, so every energy level of $E_{n,l,m}$ will split into $(2l+1)$ of the new energy level, which are represented by:

$$E_{n,l,m} = -l, E_{n,l,m} = -l+1, \ldots, E_{n,l,m} = l$$

This paper proposes to formulate how hyperfine splitting of Hydrogen atom energy level placed in external magnetic field, in the form of three-dimensional rotation group representation.

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II. REPRESENTATION OF A GROUP SYMMETRY

Suppose $V$ is a $N$-dimensional linear vector space with basis vector $f_{\alpha, \alpha} = 1, 2, \ldots, N$. Consider a group $\mathcal{G}$ of linear transformation $T_k$, $k=1, 2, \ldots$, so that

$$T_k f_\alpha = D_{\beta \alpha}(T_k) f_{\beta}$$

Consequently, a set of $\mathcal{M}$ or $\mathcal{M} = \{D_{\beta \alpha}(T_k) f_{\beta}, k = 1, 2, \ldots\}$ of $N \times N$-dimensional matrices is resulted. Set of $\mathcal{M}$ defines a representation of group $\mathcal{G}$ that is, if

$$T_k = T_l T_m$$
only if

$$D_{\beta \alpha} = D_{\beta \gamma}(T_l) D_{\gamma \alpha}(T_m)$$

$V$ vector space is called a representation space, and vector $f_{\alpha, \alpha} = 1, 2, \ldots, N$, forms basis system for representation $D_{\beta \alpha}(T_k)$. Representation $D_{\beta \alpha}(T_k)$ is named $N$-dimensional representation, and usually called that $f_\alpha$ vector transformation is under representation $D_{\beta \alpha}(T_k)$ of group $\mathcal{G}$.

Furthermore, $D_{\beta \alpha}(T_k)$ representation will be written as $D(T)$. Two representation, $D_1(T)$ and $D_2(T)$ is called equivalent if there is a nonsingular matrix $P$ fulfill

$$D_1(T) = P^{-1} D_2(T) P$$

Suppose $N$-dimensional representation, $D(T)$, is equivalent to matrix $X(T)$, where

$$X(T) = \begin{pmatrix} X_1(T) & 0 & 0 & 0 \\ 0 & X_2(T) & 0 & 0 \\ 0 & 0 & \ddots & \ddots \\ 0 & 0 & \cdots & X_k(T) \end{pmatrix}$$

Subsequently, representation $D(T)$ is called an irreducible representation. In this case, representation $D(T)$ can be written as

$$D(T) = X_1(T) \oplus X_2(T) \oplus \cdots \oplus X_k(T)$$

If the dimension of matrices $X_1(T), X_2(T), \ldots, X_k(T)$ respectively are $N_1, N_2, \ldots, N_k$ so that

$$N = N_1 \oplus N_2 \oplus \cdots \oplus N_k$$

Representation $D(T)$ is called an irreducible representation if $D(T)$ is not a reducible representation.

$N$-dimensional Vector space $V$ is called reducible vector space if its basis vector, $f_{\alpha, \alpha} = 1, 2, \ldots, N$, transformation is under a reducible representation. If its vector basis transformation is under an irreducible representation, consequently it is called irreducible vector space.

The importance of representation in physics, especially in quantum mechanics, can be seen in several theorems as provided below.

Theorem 1 Suppose Hamilton operator $\mathcal{H}$ of a system is invariant respect to a transformation group $\mathcal{G}$. Let for every eigen vector of $\mathcal{H}$ associated with one eigenvalue so that the eigenfunctions are defined by a basis for representation of group $\mathcal{G}$.

Theorem 2 Suppose Hamilton operator $\mathcal{H}$ of a system is invariant respect to transformation group $\mathcal{G}$, so that every eigenvectors transforming under one irreducible representation of $\mathcal{G}$ will be associated with the equivalent energy level.

In Zeeman effect analysis, we only need compact group. Every representation of compact group are equivalent with unitary representation.

A compact group is a transformation group which is continuous with finite parameters; for example, a $N$-dimensional rotation group is a compact group due to the parameter is given by rotation angle between 0 and $2\pi$.

A non-compact group is a continuous transformation group with infinite parameter; for example, a $N$-dimensional translation group is a non-compact group because its parameter is given by the vector translation which is between 0 and $\infty$.

Representation of compact group and representation of non-compact group have different characteristics which is necessary to be understood from topology consideration, which is not mentioned in this paper.

III. SCHRODINGER AND HEISENBERG PICTURES OF NORMAL ZEEMAN EFFECT

Hamilton operator $\mathcal{H}$ for a hydrogen atom which is placed in an external magnetic field $\vec{B}$ is given by

$$\mathcal{H} = \mathcal{H}_0 - \frac{e}{2mc} \vec{B} \cdot \vec{L}$$

where

$$\mathcal{H}_0 = -\frac{\hbar}{2m} \nabla^2 + e\phi$$

with $\vec{L}$ represents angular momentum operator, $m$ is the mass of electron, $e$ is the charge of electron, $e\phi$ is electrostatic potential energy of electron, and $\mu_0 = \frac{e\hbar}{2mc} = 0.9273 \times 10^{-20}$ erg is electron orbital magnetic moment.

By choosing $B$ in the direction of $z$ axis, the Hamilton operator will be in the form

$$\mathcal{H} = \mathcal{H}_0 - \mu_0 B_z L_z$$

where $L_z$ is an angular momentum operator in $z$ axis direction.

Perturbation potential, $\gamma \mu_0 B_z L_z$, will cause hyperfine splitting of energy level of this hydrogen atom. This effect is called normal Zeeman effect.
In order to explain Schrödinger and Heissenberg pictures which will be associated with the normal Zeeman effect, we need eigenvector $|\psi\rangle$ of angular momentum operator, $\vec{L}$, which is expressed in polar coordinate $(\theta, \phi)$. In polar coordinate $(\theta, \phi)$, angular momentum operator, $\vec{L}$, is given by:

$$\vec{L} = L_x \sin \theta \cos \phi + L_y \sin \theta \sin \phi + L_z \cos \theta$$

(13)

where

$$L_x = i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = i\hbar \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

(14)

$$L_z = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(15)

Consequently

$$\vec{L} = i\hbar \begin{pmatrix} 0 & -\cos \theta \sin \phi & \sin \theta \sin \phi \\ \cos \theta & 0 & -\sin \theta \cos \phi \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & 0 \end{pmatrix}$$

(16)

By solving eigenvalue problem

$$\vec{L} \ |\psi\rangle = \lambda |\psi\rangle$$

(17)

resulting $\lambda = \pm \hbar$. For $\lambda = +\hbar$, we can obtain a normalized eigen vector

$$|\psi\rangle = |\psi_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos \theta \cos \phi + i \sin \phi \\ -\cos \theta \sin \phi - i \cos \phi \\ \sin \theta \end{pmatrix}$$

(18)

Similarly, for $\lambda = -\hbar$

$$|\psi\rangle = |\psi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos \theta \cos \phi - i \sin \phi \\ -\cos \theta \sin \phi + i \cos \phi \\ \sin \theta \end{pmatrix}$$

(19)

Schrödinger picture formulates dynamics of $|\psi\rangle$, that is the change of $|\psi\rangle$ respect to time $t$, which is given by the equation

$$i \frac{\partial}{\partial t} |\psi\rangle = -\hbar \frac{eB_z}{2mc} L_z |\psi\rangle$$

(20)

By the use of $|\psi_+\rangle$, one can obtain

$$\frac{\partial}{\partial t} \begin{pmatrix} -\cos \theta \cos \phi + i \sin \phi \\ -\cos \theta \sin \phi - i \cos \phi \\ \sin \theta \end{pmatrix} = -\hbar \frac{eB_z}{2mc} \begin{pmatrix} \cos \theta \sin \phi + i \cos \phi \\ -\cos \theta \cos \phi + i \sin \phi \\ 0 \end{pmatrix}$$

which gives the equations below

$$\begin{align*}
\cos \theta \sin \phi \frac{d\phi}{dt} + \sin \theta \cos \phi \frac{d\theta}{dt} + i \cos \phi \frac{d\phi}{dt} &= -\hbar \frac{eB_z}{2mc} (\cos \theta \sin \phi + i \cos \phi) \\
-\cos \theta \cos \phi \frac{d\phi}{dt} + \sin \theta \sin \phi \frac{d\theta}{dt} + i \sin \phi \frac{d\phi}{dt} &= -\hbar \frac{eB_z}{2mc} (-\cos \theta \cos \phi + i \sin \phi) \\
\cos \theta \frac{d\theta}{dt} &= 0
\end{align*}$$

(21, 22, 23)

The above equations results

$$\theta = \text{constant}$$

(24)

$$\phi(t) = \phi(0) - \frac{eB_z}{2mc} t$$

(25)

where $\phi(0)$ is $\phi$ angle on time $t=0$. This result shows that state vector rotates respect to $z$ axis with angular velocity $\frac{eB_z}{2mc}$. The same conclusion can be obtained by the use of $|\psi_−\rangle$. 

Heissenberg picture formulates the dynamics of angular momentum operator $L_x$, $L_y$, and $L_z$, that is the change of those operators respect to time $t$, which are
represented by the equations

\[
\begin{align*}
\frac{idL_x}{dt} &= [L_x, \mathcal{H}'] \\
\frac{idL_y}{dt} &= [L_y, \mathcal{H}'] \\
\frac{idL_z}{dt} &= [L_z, \mathcal{H}']
\end{align*}
\]

where

\[
\mathcal{H}' = -\frac{eBL_z\hbar}{2mc} L_z
\]

Due to the relation

\[
[L_z, L_y] = i\hbar L_x, [L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y
\]

subsequently, the equations result

\[
\begin{align*}
\frac{dL_x}{dt} &= \frac{e\hbar B_z}{2mc} L_y \\
\frac{dL_y}{dt} &= -\frac{e\hbar B_z}{2mc} L_x \\
\frac{dL_z}{dt} &= 0
\end{align*}
\]

so that

\[
L_z = \text{constant}
\]

\[
\begin{align*}
\frac{d^2L_x}{dt^2} &= -\frac{e^2\hbar^2 B^2}{4m^2c^2} L_x \\
\frac{d^2L_y}{dt^2} &= -\frac{e^2\hbar^2 B^2}{4m^2c^2} L_y
\end{align*}
\]

which have the solutions

\[
\begin{align*}
L_x(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\
L_y(t) &= Ce^{i\omega t} + De^{-i\omega t}
\end{align*}
\]

where

\[
\omega = \frac{e\hbar B_z}{2mc}
\]

and A, B, C and D respectively are a constant. It is clear that angular momentum in z axis direction is also a constant, while angular momentum in the direction of x axis and y axis respectively has precision with velocity magnitude \(\frac{e\hbar B_z}{2mc}\).

**IV. SYMMETRY GROUP OF NORMAL ZEEMAN EFFECT**

In order to explain hyperfine splitting in normal Zeeman effect, we will use the well-known theorem in group theory and also its application is widely used in quantum mechanics.

**Theorem 3** Suppose Hamilton operator \(\mathcal{H}\) of a system is given by

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'
\]

where \(\mathcal{H}'\) represents a perturbation operator. Let \(\mathcal{H}, \mathcal{H}_0\) and \(\mathcal{H}'\) are invariant respect to transformation group \(\mathcal{G}\) which its generators are expressed by \(X_{\alpha}, \alpha=1,2,...,\), that is

\[
[\mathcal{H}, X_{\alpha}] = 0, \ [\mathcal{H}_0, X_{\alpha}] = 0, \ [\mathcal{H}', X_{\alpha}] = 0
\]

If set of \(\psi_k\), \(k=1,2,...,n\) are eigenvectors of \(\mathcal{H}_0\) with one eigenvalue \(\lambda\), that is

\[
\mathcal{H}_0|\psi_k\rangle = \lambda|\psi_k\rangle
\]

and if \(|\psi_k\rangle\) are transformed under representation

\[
D = D^{(1)} \oplus D^{(2)} \oplus D^{(3)} \oplus ...D^{(k)}
\]

where \(D^{(k)}\) represents an irreducible component of representation \(D\), subsequently the number of new energy levels which is resulted from energy hyperfince splitting which is represented by eigenvalue \(\lambda\) is \(n\). Every \(D^{(k)}\) is associated with one new energy level.

This theorem helps us to understand hyperfine splitting of a degeneracy energy level to several new energy levels. Hamilton operator

\[
\mathcal{H}_0 = -\frac{\hbar^2}{2m} \nabla^2 - \frac{e^2}{r}
\]

of a hydrogen atom is invariant respect to rotation group \(SO(3)\).

Because the perturbation operator

\[
\mathcal{H}' = -\frac{e\hbar}{2mc} B_z L_z
\]

is only invariant respect to rotation in z axis direction, that is

\[
[\mathcal{H}', L_x] \neq 0, \ [\mathcal{H}', L_y] \neq 0, \ [\mathcal{H}', L_z] = 0
\]

consequently, new symmetry group of the system is direct product group which is given by \(U(1) \otimes E, I\), where \(U(1)\) is a one-dimensional unitary group representing rotation in z axis direction, \(E\) is identity operator and \(I\) is inversion operator which is defined by

\[
I: (x, y, z) \rightarrow (-x, -y, -z)
\]

To clarify energy hyperfine splitting, let us observe again a representation of three-dimensional rotation group. Due to the algebra structure of this group is given by:

\[
[L_{\alpha}, L_{\beta}] = i\epsilon_{\alpha\beta\gamma} L_{\gamma}
\]

where \(a, b, c=1,2,3\), and \(\epsilon_{\alpha\beta\gamma}\) represents a three-dimensional permutation symbol, that is

\[
\epsilon_{\alpha\beta\gamma} = \begin{cases} 
+1 \text{ if } abc \text{ is even permutation} \\
0 \text{ if two or more of } a, b, c \text{ are same} \\
-1 \text{ if } abc \text{ is odd permutation}
\end{cases}
\]
Generally, representation of three-dimensional rotation group is described by following theorem

**Theorem 4** Suppose $D[g(\alpha)]$ is a matrix associated with an element $g(\alpha)$ of a three-dimensional rotation group where $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ is a parameter which is very small or $\alpha \approx 0$. Consequently, set of all matrices

$$D[g(\alpha)] = e^{i\hat{L} \cdot \alpha} \quad (45)$$

defines a representation of the group $\hat{L}$.

**Proof**

Consider $D[g(\alpha)] = e^{i\hat{L} \cdot \alpha}$ and $D[g(\beta)] = e^{i\hat{L} \cdot \beta}$, where $\alpha$ and $\beta$ express very small parameters. If

$$g(\alpha)g(\beta) = g(\alpha + \beta) \quad (46)$$

it can be shown that

$$D[g(\alpha)]D[g(\beta)] = D[g(\alpha + \beta)] \quad (47)$$

Note that

$$D[g(\alpha)]D[g(\beta)] = e^{i\hat{L} \cdot \alpha} e^{i\hat{L} \cdot \beta} = (1 + i \hat{L} \cdot \alpha)(1 + i \hat{L} \cdot \beta) = 1 + i \hat{L} (\alpha + \beta) = e^{i\hat{L} \cdot (\alpha + \beta)}$$

which means

$$D[g(\alpha)]D[g(\beta)] = D[g(\alpha + \beta)]$$

representing that set of matrices $D[g(\alpha)]$ define a representation of the three-dimensional rotation group.

Consider

$$L_z|\phi\rangle = \lambda|\phi\rangle$$

$$(L_x + i L_y)|\phi\rangle = |\phi_+\rangle$$

$$(L_x - i L_y)|\phi\rangle = |\phi_-\rangle$$

subsequently $|\phi_+\rangle$ is an eigenvector of $L_z$ with eigenvalue $(\lambda + 1)$, and $|\phi_-\rangle$ is an eigenvector of $L_z$ with eigenvalue $(\lambda - 1)$. Suppose $l$ is maximum eigenvalue of $L_z$ with eigenvector $|\phi_l\rangle$, where $\langle \phi_l|\phi_l\rangle = 1$. If $L_x + i L_y|\phi_l\rangle = \mu_l|\phi_l\rangle$ with $\langle \phi_{l-1}|\phi_{l-1}\rangle = 1$, one can obtain $L_z|\phi_{l-1}\rangle = (l - 1)|\phi_{l-1}\rangle$. Furthermore, $(L_x - i L_y)|\phi_{l-1}\rangle = \mu_{l-1}|\phi_{l-1}\rangle$, with $\langle \phi_{l-2}|\phi_{l-2}\rangle = 1$, thus $L_z|\phi_{l-2}\rangle = (l - 2)|\phi_{l-2}\rangle$ can be determined. Generally, it can be shown that $L_z|\phi_{l-k}\rangle = \mu_{l-k}|\phi_{l-k}\rangle$ with $\langle \phi_{l-k}|\phi_{l-k}\rangle = 1$, consequently $L_z|\phi_m\rangle = (l - m)|\phi_m\rangle$, where $k = 0, 1, 2, \ldots$.

Because

$$(L_x - i L_y)|\phi_{m}\rangle = \mu_m|\phi_{m-1}\rangle \quad (49)$$

means $\mu_m = 0$. Furthermore, consider step-up operator $(L_x + i L_y)|\phi_{l-1}\rangle = \frac{2\mu_l}{\mu_{l+1}}|\phi_l\rangle$. Thus, the vector of $(L_x + i L_y)|\phi_{l-1}\rangle$ is equal to $|\phi_l\rangle$ vector, which means that there is a number $\alpha_l > 0$ which suffices $(L_x + i L_y)|\phi_{l-1}\rangle = \alpha_l|\phi_{l}\rangle$. Similarly, it can be shown that $(L_x + i L_y)|\phi_{m}\rangle$ is a vector which is equivalent to $|\phi_{m+1}\rangle$ meaning that there is a number $\alpha_{m+1} > 0$ which fulfills $(L_x + i L_y)|\phi_{m}\rangle = \alpha_{m+1}|\phi_{m+1}\rangle$.

It is clear that

$$(L_x + i L_y)|\phi_{m}\rangle = \frac{2\mu_{l-1}}{\mu_{l+1}}|\phi_{l+1}\rangle + \frac{\alpha_{m+2}}{\mu_{m+1}}(L_x - i L_y)|\phi_{m+2}\rangle \quad (50)$$

and by using equations (48) and (49), one can obtain

$$2(m + 1) + (\mu_{m+2}|\alpha_{m+2}|) = \mu_{m+1} \quad (51)$$

Because of $(L_x + i L_y)|\phi_{l}\rangle = 0$ thus $\alpha_l = 0$ for $l = m$. Furthermore, because $\langle \phi_{m+1}|\phi_{m}\rangle = \langle \phi_{m-1}|\phi_{m-1}\rangle = 1$ and $(L_x + i L_y)|\phi_{l}\rangle = (L_x - i L_y)|\phi_{l}\rangle$ where $(L_x + i L_y)|\phi_{l}\rangle$ represents hermitian conjugate of $(L_x + i L_y)$, subsequently $\alpha_m|\phi_{m}|\phi_{m-1}|\phi_{m-1}\rangle = m|\phi_{m}|\phi_{m-1}|\phi_{m-1}\rangle$ which means $\alpha_m = \mu_m$.

Equation (51) is in the form

$$\mu^2_{m} - \mu^2_{m+1} = 2m \quad (52)$$

By adding from $\alpha = m$ to $\alpha = +l$, that is

$$\sum_{\alpha=m}^{l} (\mu_{\alpha}^2 - \mu_{\alpha+1}^2) = \sum_{\alpha=m}^{l} 2\alpha$$

thus, following equation can be obtained

$$\mu^2_{m} = (l + m)(l - m + 1) \quad (53)$$

Because $l - (l) = 2l$ so $l$ can be natural numbers $0, 1, 2, \ldots$ or odd multiplicant of $\frac{1}{2}$ that is $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ . For Zeeman effect case, $l$ is arbitrary positive natural numbers.

One can see that for every $l$ value, there is $2(l + 1)$ type of $m$ value associated with $(2l + 1)$ orthornormal eigenvector of $L_z$.

This result shows that every reducible representation, $D$, of a three-dimensional rotation group can be explained by the sum of irreducible representations $D^{(l)} \oplus D^{(l+1)} \oplus \ldots \oplus D^{(l)}$.

Symmetry group of Hamilton operator $H$, where

$$H = \mathcal{H} = \mathcal{H}_0 - \frac{e^2 \hbar B_z}{2mc} L_z \quad (54)$$

which is represented by $U(1) \otimes \{E, I\}$ is a commutative group so that all irreducible representation of this group are one-dimensionl representation. Representation of group $U(1)$ is $e^{im\theta}$, $m=0, \pm 1, \pm 2, \ldots$, where $\theta$ express rotation angle respect to $z$ axis. It is clear that irreducible representation of $U(1) \otimes \{E, I\}$ will be given by $\pm e^{im\theta}$. 
Character of irreducible representation is given by
\[ \chi = \sum_{m=-l}^{l} e^{im\theta} = e^{-il\theta} + \ldots + e^{il\theta} \quad (55) \]

Similar with the case of finite group, the character shows hyperfine splitting of degeneracy energy level into \((2l+1)\) energy levels. This result, which is given by equation (55), can be basically concluded from the previous results which were given from equation (48) to equation (51).

V. CONCLUSION

All symmetry group which are used to explain normal Zeeman effect are compact group. Representation of compact group and representation of finite group have similar properties. All irreducible representation of compact group are equivalent with unitary representation. Likewise, character of irreducible representation will give dimensions of this irreducible representation.

Representation space of compact group should be a Hilbert space which is necessary to define orthonormal vectors. This type of vector space is called separable Hilbert space which was extensively used in Section III.

Finally, symmetry group, which is used to explain normal Zeeman effect as shown in this paper, can be developed to construct symmetry group explaining anomalous Zeeman effect.

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