The propagation of a pulse in the real strings and rods

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Abstract

We consider the elastic rod of a large mass $M$, the left end of which is fixed to a body of mass $m \ll M$ and the second body of mass $m$ is fixed to the right end of the rod. The force of the delta-function form is applied to the left side of the rod. We find the propagation of the pulse in the system. Our problem represents the missing problem in the Newton “Principia mathematica” and in any textbook on mechanics. The relation of our theory to the quark-string model of mesons is evident.

1 Introduction

It is interesting to consider the elastic rod of a large mass $M$, the left end of which is joined with mass $m \ll M$ and body of mass $m$ is fixed to the right end of the rod. Then, it is interesting to study the consequences of the application of the the force of the delta-function form to the left side of the rod. The delta-function is chosen for simplicity. This function can be replaced by the different functions. We show that the internal motion of the elastic rod medium is controlled by the wave equation. We derive the mathematical form of the mechanical motion of the considered string or rod. Our problem represents the missing problem in the Newton “Principia mathematica” [1] and in any textbook on mechanics. The relation of our theory to the quark-string model of mesons is evident.
2 Classical theory of interaction of particle with an impulsive force

We will first show that use of the impulsive force of the delta-function form is physically meaningful in a classical mechanics of a point particle. We idealize the impulsive force by the Dirac $\delta$-function.

Newton’s second law in the one-dimensional form for the interaction of a massive particle with mass $m$ with force $F$

$$ma = F$$

with $F$ being an impulsive force $P\delta(t)$ is as follows:

$$m\frac{d^2x}{dt^2} = P\delta(\alpha t),$$

where $P$ and $\alpha$ are some constants, with MKSA dimensionality $[P] = \text{kg.m.s}^{-2}$, $[\alpha] = \text{s}^{-1}$. We put $|\alpha| = 1$.

Using the Laplace transform \[2\] in the last equation, with

$$\int_0^\infty e^{-st}x(t)dt \overset{d}{=} X(s),$$

$$\int_0^\infty e^{-st}\dot{x}(t)dt = s^2X(s) - sx(0) - \ddot{x}(0),$$

$$\int_0^\infty e^{-st}\delta(\alpha t)dt = \frac{1}{\alpha},$$

we obtain:

$$ms^2X(s) - msx(0) - m\dot{x}(0) = P/\alpha.$$  

For a particle starting from the rest with $\dot{x}(0) = 0$, $x(0) = 0$, we get

$$X(s) = \frac{P}{ms^2\alpha}.$$  

Using the inverse Laplace transform, we obtain

$$x(t) = \frac{P}{m\alpha}t$$  

and

$$\dot{x}(t) = \frac{P}{m\alpha}.$$  

In case of the harmonic oscillator with the damping force and under influence of the general force $F(t)$, the Newton law is as follows:

$$m\frac{d^2x(t)}{dt^2} + b\dot{x}(t) + kx(t) = F(t).$$
After application of the Laplace transform (3) and with regard to the same initial conditions as in the preceding situation, $\dot{x}(0) = 0, x(0) = 0$, we get the following algebraic equation:

$$ms^2X(s) + bsX(s) + kX(s) = F(s),$$

(11)
or,

$$X(s) = \frac{F(s)}{m\omega_1\left(s + b/2m\right)^2 + \omega_1^2}$$

(12)

with $\omega_1^2 = k/m - b^2/4m^2$.

Using inverse Laplace transform denoted by symbol $\mathcal{L}^{-1}$ applied to multiplication of functions $f_1(s)f_2(s)$,

$$\mathcal{L}^{-1}(f_1(s)f_2(s)) = \int_0^t d\tau F_1(t-\tau)F_2(\tau),$$

(13)

we obtain with $f_1(s) = F(s)/m\omega_1, f_2(s) = \omega_1/((s + b/2m)^2 + \omega_1^2)$, $F_1(t) = F(t)/m\omega_1$, $F_2(t) = \exp(-bt/2m)\sin \omega_1 t$.

$$x(t) = \frac{1}{m\omega_1} \int_0^t F(t-\tau)e^{-b\tau/2m}\sin (\omega_1 \tau) d\tau.$$  

(14)

For impulsive force $F(t) = P\delta(at)$, we have from the last formula

$$x(t) = \frac{(P/\alpha)}{m\omega_1} e^{-bt/2m} \sin \omega_1 t.$$  

(15)

3 The pulse propagating in a rod

In this section we will solve the motion of a string or rod with the massive ends (the body with mass $m$ is fixed to the every end of the string) on the assumption that the tension in the string is linear and the applied force is of the Dirac delta-function. First, we will derive the Euler wave equation from the Hook law of tension and then we will give the rigorous mathematical formulation of the problem. Linearity of the wave equation enables to solve this problem by the Laplace transform method. We follow [3] and the author preprint [4] where this method was used to solve the Gassendi model of gravity. Although Gassendi [5] is known in physics as the founder of the modern atomic theory of matter, his string model of gravity was not accepted. The Newton reaction to this model was empirical. He said: “Hypotheses non fingo”. It seems that Gassendi ideas was applied later by Faraday in his theory of electromagnetism. We know also that Gassendi was independent thinker and he was persecuted. Every independent thinker is persecuted in any society.

The present problem can be also defined as a central collision of two bodies (balls). While in the basic mechanics the central collision is considered as a contact collision of the two balls, here, the collision is mediated by the string, or, rod.

To our knowledge, the present problem is not involved in the textbooks of mathematical physics or in the mathematical journals. This problem was not possible to define and solve in the Newton period, because the method of solution is based on the Euler partial
wave equation, the Laplace transform, The Riemann-Mellin transform, the Bromwich integral and Bromwich contour and other ingredients of the operator calculus which was elaborated after the Newton period. So, this is why the problem is not involved in the Newton “Principia Mathematica” [1].

Now, let us consider the rod (or string) of the length $L$, the left end of which is joined with mass $m$ and the right end is joined with mass $m$. The force of the delta-function form is applied to the left end and the initial state of the rod is the state of equilibrium. The deflection of the rod element $dx$ at point $x$ and time $t$ let be $u(x, t)$ where $x \in (0, L)$.

The differential equation of motion of string elements can be derived by the following way [3]. We suppose that the force acting on the element $dx$ of the string is given by the law:

$$T(x, t) = ES \left( \frac{\partial u}{\partial x} \right),$$  \hspace{1cm} (16)

where $E$ is the modulus of elasticity, $S$ is the cross section of the string. We easily derive that

$$T(x + dx) - T(x) = ESu_{xx}dx.$$ \hspace{1cm} (17)

The mass $dm$ of the element $dx$ is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt} = ESu_{xx}dx.$$ \hspace{1cm} (18)

So, we get

$$\frac{1}{c^2} u_{tt} - u_{xx} = 0; \hspace{0.5cm} c = \left( \frac{E}{\varrho} \right)^{1/2}.$$ \hspace{1cm} (19)

Now, we get the problem of the mathematical physics in the form:

$$u_{tt} = c^2 u_{xx}$$ \hspace{1cm} (20)

with the initial conditions

$$u(x, 0) = 0; \hspace{0.5cm} u_t(x, 0) = 0$$ \hspace{1cm} (21)

and with the boundary conditions

$$mu_{tt}(0, t) = au_x(0, t) + P\delta(\alpha t); \hspace{0.5cm} mu_{tt}(L, t) = au_x(L, t),$$ \hspace{1cm} (22)

where we have put

$$a = -ES; \hspace{0.5cm} P = \text{some constant}.$$ \hspace{1cm} (23)

The delta-function can be approximatively realized by the strike of the hammer to the left end of the rod.

The equation (20) with the initial and boundary conditions (21) and (22) represents one of the standard problems of the mathematical physics and can be easily solved using the Laplace transform [2]:
\( \hat{L}u(x, t) \overset{d}{=} \int_{0}^{\infty} e^{-pt}u(x, t)dt \overset{d}{=} \varphi(x, p). \)  

(24)

Using (24) and (20) we get:

\( \hat{L}u_{tt}(x, t) = p^2\varphi(x, p) - pu(x, 0) - u_t(x, 0) = p^2\varphi(x, p), \)

(25)

\( \hat{L}u_{xx}(x, t) = \varphi_{xx}(x, p) \); \( \hat{L}\delta(\alpha, t) = 1/\alpha. \)

(26)

After elementary mathematical operations we get the differential equation for \( \varphi \) in the form

\[ \varphi_{xx}(x, p) - k^2\varphi(x, p) = 0; \quad k = p/c. \]

(27)

with the boundary condition in eq. (22).

We are looking for the solution of eq. (27) in the form

\[ \varphi(x, p) = c_1 \cosh kx + c_2 \sinh kx. \]

(28)

We get from the boundary conditions in eq. (22)

\[ c_1 = \frac{1}{p} \frac{ac(P/\alpha) \cosh(pL/c) - (P/\alpha)mpc^2 \sinh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)}, \]

(29)

\[ c_2 = -\frac{(P/\alpha)c}{ap} + \frac{(P/\alpha)mac^2 \cosh(pL/c) - (P/\alpha)pmc^3 \sinh(pL/c)}{a \sinh(pL/c)(a^2 - m^2c^2p^2)}. \]

(30)

The corresponding \( \varphi(x, p) \) is of the form:

\[ \varphi(x, p) = \frac{1}{p} \frac{ac(P/\alpha) \cosh(pL/c) - (P/\alpha)mpe^2 \sinh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)} \cosh(px/c) + \]

\[ \left[ -\frac{(P/\alpha)c}{ap} + \frac{a(P/\alpha)mac^2 \cosh(pL/c) - bpcm^2c^3 \sinh(pL/c)}{a \sinh(pL/c)(a^2 - m^2c^2p^2)} \right] \sinh(px/c). \]

(31)

The corresponding function \( u(x, t) \) follows from the theory of the Laplace transform as the mathematical formula (res is residuum)[2]:

\[ u(x, t) = \frac{1}{2\pi i} \int e^{pt} \varphi(x, p)dp = \sum_{p=p_n} \text{res} e^{pt} \varphi(x, p) = \]

\[ \sum_{p=p_n} \text{res} e^{pt} \frac{1}{p} \frac{ac(P/\alpha) \cosh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)} \cosh(px/c) - \]

\[ \sum_{p=p_n} \text{res} e^{pt} \frac{(P/\alpha)mce^2}{(a^2 - m^2p^2c^2)} \cosh(px/c) - \]

\[ \sum_{p=p_n} \text{res} e^{pt} \left[ \frac{(P/\alpha)c}{ap} \right] \sinh(px/c) + \]

\[ \sum_{p=p_n} \text{res} e^{pt} \left[ \frac{m(P/\alpha)c^2 \cosh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)} \right] \sinh(px/c) - \]
\[ \sum \text{res} e^{pt} \left[ \frac{(P/\alpha)pm^2c^3}{a} \frac{1}{(a^2 - m^2c^2p^2)} \right] \sinh(px/c) = u_1 - u_2 - u_3 + u_4 - u_5, \]  

\( (32) \)

where

\[ u_j = \sum \text{res} \frac{A_j}{B_j}; \quad j = 1, 2, 3, 4, 5 \]

\( (33) \)

and

\[ A_1 = ac(P/\alpha) \cosh(pL/c) \cosh(px/c); \quad B_1 = p \sinh(pL/c)(a^2 - m^2p^2c^2) \]  

\( (34) \)

\[ A_2 = (P/\alpha)mc^2 \cosh(px/c); \quad B_2 = (a^2 - m^2p^2c^2) \]  

\( (35) \)

\[ A_3 = (P/\alpha)c \sinh(px/c); \quad B_3 = ap \]  

\( (36) \)

\[ A_4 = (P/\alpha)mc^2 \cosh(pL/c) \sinh(px/c); \quad B_4 = \sinh(pL/c)(a^2 - m^2p^2c^2) \]  

\( (37) \)

\[ A_5 = (P/\alpha)pm^2c^3 \sinh(px/c); \quad B_5 = a(a^2 - m^2p^2c^2). \]  

\( (38) \)

We know from the theory of the complex functions that if the pole of some function \( f(z)/g(z) \) is simple and it is at point \( a \), then the residuum is as follows [2]:

\[ \text{residuum} = \frac{f(a)}{g'(a)}. \]  

\( (39) \)

If the pole at point \( a \) of the function \( f(z) \) is multiply of the order \( m \), then the residuum is defined as follows:

\[ \text{residuum} = \frac{1}{(m - 1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]. \]  

\( (40) \)

Let us first determined the function

\[ u_1 = \sum \text{res} \frac{A_1}{B_1}. \]  

\( (41) \)

Poles of \( B_1 \) are at points \( p = 0 \), this is pole of the order 2, \( p = +a/mc, p = -a/mc \) and \( p_n = +i\pi nc/L, p_n = -i\pi nc/L, n = 1, 2, 3, \ldots \). So, the function \( u_1 \) is as follows:

\[ u_1 = \frac{(P/\alpha)c^2}{La} t - \frac{(P/\alpha)c}{a} \cosh \left( \frac{aL}{mc^2} \right) \cosh \left( \frac{ax}{mc^2} \right) \sinh \left( \frac{at}{mc} \right) + \]

\[ \sum_{n=1}^{\infty} \frac{2a(P/\alpha)c}{\pi n} \frac{L^2}{\pi n^2 a^2 L^2 + m^2 \pi^2 n^2 c^4} \cos \left( \frac{\pi nx}{L} \right) \sin \left( \frac{\pi nct}{L} \right). \]  

\( (42) \)
For the function $u_2$ we get:

$$u_2 = \left( -\frac{(P/\alpha)c}{a} \right) \sinh \left( \frac{at}{mc} \right) \cosh \left( \frac{ax}{mc^2} \right). \quad (43)$$

$$u_3 = 0. \quad (44)$$

For $u_4$ and $u_5$ we get:

$$u_4 = \left( -\frac{(P/\alpha)c}{a} \right) \coth \left( \frac{aL}{mc^2} \right) \sinh \left( \frac{ax}{mc^2} \right) \sinh \left( \frac{at}{mc} \right) \quad (45)$$

$$u_5 = \left( -\frac{(P/\alpha)c}{a} \right) \sinh \left( \frac{ax}{mc^2} \right) \sinh \left( \frac{at}{mc} \right) \quad (46)$$

The dimensionality of $u$ is $[u] = m$ and $u(x,0) = 0$. The momentum of a left particle $p = mu(0,t)$, or right particle $p = mu(L,t)$ is not conserved. Only the total momentum of a system is conserved.

4 Discussion

Our problem is the modification of some problems involved in the textbooks on mathematical physics. However, our approach is pedagogically original in the sense that we use the initial force of a delta-function form to show the internal motion of the string, or, rod. The delta-function form of electromagnetic pulse was used by author in [6] and [7] to discuss the quantum motion of an electron in the laser pulse. We have considered here the real strings and rods in the real space and we do not use extra-dimensions and unrealistic strings. The M-dimensional geometrical object cannot be realized in N-dimensional space for $M > N$ [8]. The mathematical theory of unrealistic strings is well known as the string theory in particle physics. Our problem with the real strings and rods can be generalized for the two-dimensional and three-dimensional situation. It can be also generalized to the situation with the dissipation of waves in the strings and rods. In this case it is necessary to write the wave equation with the dissipative term and then to solve this problem “ab initio”.

While we have solved the problem for the situation where the pulse was generated by the force of the delta-form, we give some general ideas following from the wave equation. It is well known that the solution of this equation is in general in the form [9]:

$$f \left( t - \frac{x}{c} \right); \quad g \left( t + \frac{x}{c} \right) \quad (47)$$

where functions $f, g$ are general. It means it involves also the function of the delta-form. For the wave propagating from the left side to the right side, we take function $f$. The corresponding tension in the rod is

$$T = E Su_x(x,t) = ES f' \left( t - \frac{x}{c} \right) \left( \frac{-1}{c} \right). \quad (48)$$

We easily see that $T(x=0, t=0) = T(x=L, t=L/c)$, and it means that when the pulse force is created at the left end of the rod then it propagates in the rod and after time $L/c$ it is localized in the right end of the rod.
However, we have seen that the pulse force generated in the system with the massive ends of strings or rods develops in time according to laws of the mathematical physics of strings and rods and cannot be intuitively predicted. Only rigorous solution of the dynamics of the system can give the answer on the real motion of tension in the string.

There is no information in the Newton “Principia mathematica” [1] and in any textbook on mechanics on the central collision of two particles where the force is mediated by string or rod. Similarly, there is not the solution of our problem in the famous monograph by Pars [10]. So, This is the missing problem in the textbooks on mechanics.

The propagation of a pulse in one direction was confirmed experimentally by author using the heavy elastic rod (the segment of a rail). The delta-form force (tension) was generated approximately by the strike of hammer. The experiment was performed as the table experiment and it can be repeated by any theorist.

The proposed model with the string with massive ends can be also related in the modified form to the problem of the radial motion of quarks bound by a strings, and used to calculate the excited states of such system. The resent analysis of such problem was performed by Lambiase and Nesterenko [11] and Nesterenko and Pirozhenko [12], and others. So, can we hope that our approach and their approach will be unified to generate the new revolution of the string theory of matter and space-time? Why not?

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