MODULAR EQUATIONS AND LATTICE SUMS

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Abstract. We highlight modular equations discovered by Somos and Ramanujan, and use them to prove new relations between lattice sums and hypergeometric functions. We also discuss progress towards solving Boyd’s Mahler measure conjectures, and conjecture a new formula for $L(E, 2)$ of conductor 17 elliptic curves.

1. Introduction

Modular equations appear in a variety of number-theoretic contexts. Their connection to formulas for $1/\pi$ [14], Ramanujan constants such as $e^{\pi\sqrt{163}}$ [21], and elliptic curve cryptography is well established. In the classical theory of modular forms, an $n$th degree modular equation is an algebraic relation between $j(\tau)$ and $j(n\tau)$, where $j(\tau)$ is the $j$-invariant. For our purposes a modular equation is simply a non-trivial algebraic relation between theta (or eta) functions. In this paper we use modular equations to study four-dimensional lattice sums. The lattice sums are interesting because they arise in the study of Mahler measures of elliptic curves.

There are a large number of hypothetical relations between special values of $L$-series of elliptic curves, and Mahler measures of two-variable polynomials. The Mahler measures $m(\alpha)$, $n(\alpha)$, and $g(\alpha)$ are defined by

$$m(\alpha) := \int_0^1 \int_0^1 \log |y + y^{-1} + z + z^{-1} + \alpha| \, d\theta_1 d\theta_2,$$

$$n(\alpha) := \int_0^1 \int_0^1 \log |y^3 + z^3 + 1 - \alpha y z| \, d\theta_1 d\theta_2,$$

$$g(\alpha) := \int_0^1 \int_0^1 \log |(y + 1)(z + 1)(y + z) - \alpha y z| \, d\theta_1 d\theta_2,$$

where $y = e^{2\pi i \theta_1}$, and $z = e^{2\pi i \theta_2}$. Boyd conjectured that for all integral values of $k \neq 4$ [6]:

$$m(k) \equiv \frac{q}{\pi^2} L(E, 2),$$

where $E$ is an elliptic curve, $q$ is rational, and both $E$ and $q$ depend on $k$. He also discovered large numbers of formulas involving $g(\alpha)$ and $n(\alpha)$. In cases where $E$
has a small conductor, it is frequently possible to express $L(E, 2)$ in terms of four-dimensional lattice sums. Thus many of Boyd’s identities can be regarded as series acceleration formulas. The main goal of this paper is to prove new formulas for the lattice sum $F(b, c)$, defined in (12). So far there are at least 18 instances where $F(b, c)$ is known (or conjectured) to reduce to integrals of elementary functions. The modular equations of Somos and Ramanujan are the main tools in our analysis.

2. Eta function product identities

Somos discovered thousands of new modular equations by searching for linear relations between products of Dedekind eta functions. Somos refers to these formulas as “eta function product identities”. The existence of eta function product identities can be established by using fact that certain modular parameters (such as $j(\tau)$) equal rational expressions involving eta functions. By clearing denominators it is possible to rewrite classical modular equations as eta function product identities. A major surprise of Somos’s experimental approach, is that it turned up a large number of unexpectedly simple identities. In order to give an example, first consider the eta function with respect to $q$:

$$
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24},
$$

and adopt the short hand notation

$$e_j = \eta(q^j).$$

The following formula is the smallest eta function product identity in Somos’s list [19]:

$$e_2e_6e_{10}e_{30} = e_1e_{12}e_{15}e_{20} + e_3e_4e_5e_{60}. \quad (2)$$

Notice that all three monomials are products of four eta functions, and are essentially weight-two modular forms. There are no known identities between eta products of weight less than two, and (2) appears to be the only three-term linear relation between products of four eta functions. Many additional identities are known if the number of terms is allowed to increase, or if eta products of higher weight are considered. For additional examples see formulas (14), (22), (23), (26), and (29).

Identities such as (2) can be proved almost effortlessly with the theory of modular forms. A typical proof involves checking that the first few Fourier coefficients of a presumed identity vanish. Sturm’s Theorem furnishes an upper bound on the number of coefficients that need to be examined [13]. We note that it is often possible, but usually more difficult, to prove such identities via elementary $q$-series methods. Ramanujan was a master $q$-series manipulator, and his notebooks are filled with various modular equations and their corollaries. We conclude this section by providing a Ramanujan-style proof of (2). The main news is that (2) can be derived from modular equations known to Ramanujan.

Theorem 1. The following formula is true:

$$e_2e_6e_{10}e_{30} = e_1e_{12}e_{15}e_{20} + e_3e_4e_5e_{60}. \quad (3)$$
Proof. Before proving (3) we need to define a small amount of notation. Let us denote the usual theta functions by

\begin{equation}
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \\
\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\end{equation}

Furthermore define \( u_j \) and \( z_j \) by

\[ u_j := 1 - \frac{\varphi^4(-q^{j^2})}{\varphi^4(q^{j^2})}, \quad z_j := \varphi^2(q^{j^2}). \]

Notice that Ramanujan uses a slightly different notation [2]. He typically sets \( \alpha = u_1 \), and he says that “\( \beta \) has degree \( j \) over \( \alpha \)” when \( \beta = u_j \). Certain values of the \( \eta \) function can be expressed in terms of \( u_1 \) and \( z_1 \) [2, p. 124]. We have

\begin{align}
\eta(q) &= 2^{-1/6} u_1^{1/24} (1 - u_1)^{1/6} \sqrt{z_1}, \\
\eta(q^2) &= 2^{-1/3} \{u_1(1 - u_1)\}^{1/12} \sqrt{z_1}, \\
\eta(q^4) &= 2^{-2/3} u_1^{1/6} (1 - u_1)^{1/24} \sqrt{z_1}.
\end{align}

Now we prove (3). By (6) the left-hand side of the identity becomes

\[ e_{2\epsilon_0} e_3 e_{10} e_{30} = 2^{-4/3} \{u_1 u_3 u_5 u_{15} (1 - u_1)(1 - u_3)(1 - u_5)(1 - u_{15})\}^{1/12} \sqrt{z_1 z_3 z_5 z_{15}}. \]

By (5) and (7), the right-hand side of the identity becomes

\[ e_1 e_{12} e_{15} e_{20} + e_3 e_4 e_5 e_{60} = 2^{-5/3} \left(\{u_3 u_5 (1 - u_1)(1 - u_{15})\}^{1/6} \{u_1 u_{15} (1 - u_3)(1 - u_5)\}^{1/24} \right. \\
\left. + \{u_1 u_{15} (1 - u_3)(1 - u_5)\}^{1/6} \{u_3 u_5 (1 - u_1)(1 - u_{15})\}^{1/24} \right) \sqrt{z_1 z_3 z_5 z_{15}}. \]

Combining the last two formulas shows that (3) is equivalent to

\begin{align}
2^{1/3} \{u_1 u_3 u_5 u_{15} (1 - u_1)(1 - u_3)(1 - u_5)(1 - u_{15})\}^{1/24} \\
= \{u_3 u_5 (1 - u_1)(1 - u_{15})\}^{1/8} + \{u_1 u_{15} (1 - u_3)(1 - u_5)\}^{1/8}.
\end{align}

It is sufficient to show that (8) can be deduced from Ramanujan’s modular equations.

The first modular equation we require can be recovered by multiplying entries 11.1 and 11.2 in [2, p. 383]:

\[ \left( (u_1 u_{15})^{1/8} + \{(1 - u_1)(1 - u_{15})\}^{1/8} \right) \left( (u_3 u_5)^{1/8} + \{(1 - u_3)(1 - u_5)\}^{1/8} \right) = 1. \]

Rearranging yields an identity for the right-hand side of (8):

\begin{align}
\{u_3 u_5 (1 - u_1)(1 - u_{15})\}^{1/8} + \{u_1 u_{15} (1 - u_3)(1 - u_5)\}^{1/8} \\
= 1 - \{u_1 u_3 u_5 u_{15} \}^{1/8} - \{(1 - u_1)(1 - u_3)(1 - u_5)(1 - u_{15})\}^{1/8}.
\end{align}

By entry 11.14 in [2, p. 385], it is clear that

\begin{align}
1 - \{u_1 u_3 u_5 u_{15} \}^{1/8} - \{(1 - u_1)(1 - u_3)(1 - u_5)(1 - u_{15})\}^{1/8} \\
= 2^{1/3} \{u_1 u_3 u_5 u_{15} (1 - u_1)(1 - u_3)(1 - u_5)(1 - u_{15})\}^{1/24}.
\end{align}
The theorem follows from combining (9) and (10) to recover (8).

It appears to be very difficult to explain why identities such as (3) exist. Our initial motivation for constructing an elementary proof of (3), was to find a method for generating more identities. It would be interesting if a systematic method for generating weight-two eta product identities could be discovered. This is a surprisingly important question for studying lattice sums and the Beilinson conjectures.

3. Lattice Sums

In this section we investigate four-dimensional lattice sums. Many of these sums appear in the study of Mahler measures of elliptic curves. Let us define

\[ F(a, b, c, d) := (a + b + c + d)^2 \times \sum_{n=-\infty}^{\infty} \frac{(-1)^{n_1 + n_2 + n_3 + n_4}}{(a(6n_1 + 1)^2 + b(6n_2 + 1)^2 + c(6n_3 + 1)^2 + d(6n_4 + 1)^2)^2}. \]

The four-dimensional series is not absolutely convergent, so it is necessary to employ summation by cubes [5]. Notice that Euler’s pentagonal number theorem can be used to represent \( F(a, b, c, d) \) as a useful integral

\[ F(a, b, c, d) = -\frac{(a + b + c + d)^2}{24^2} \int_0^1 \eta(q^a)\eta(q^b)\eta(q^c)\eta(q^d) \log q \frac{dq}{q}. \]

We also use the shorthand notation

\[ F(b, c) := F(1, b, c, bc), \]

since we are primarily interested in cases where \( a = 1, d = bc, \) and \( b \) and \( c \) are rational.

The interplay between values of \( F(b, c) \), Boyd’s Mahler measure conjectures, and the Beilinson conjectures is outlined in [16]. If \( (b, c) \in \mathbb{N}^2 \) and \((1+b)(1+c)\) divides 24, then \( F(b, c) = L(E, 2) \) for an elliptic curve \( E \). Formulas are now rigorously proved relating each of those eight cases to Mahler measures such as \( m(\alpha) \) [22]. Because Mahler measures often reduce to generalized hypergeometric functions, many of Boyd’s identities can be regarded as series transformations [15], [12]. It is known that

\[ m(\alpha) = \text{Re} \left[ \log(\alpha) - \frac{2}{\alpha^2} F_3 \left( \frac{3}{2}, \frac{1}{2}, 1, 1 ; \frac{16}{\alpha^2} \right) \right], \text{ if } \alpha \neq 0, \]

\[ n(\alpha) = \text{Re} \left[ \log(\alpha) - \frac{2}{\alpha^{3/2}} F_3 \left( \frac{5}{4}, \frac{5}{4}, 1, 1 ; \frac{27}{\alpha^2} \right) \right], \text{ if } |\alpha| \text{ is sufficiently large}, \]

\[ 3g(\alpha) = n \left( \frac{\alpha + 4}{\alpha^{2/3}} \right) + 4n \left( \frac{\alpha - 2}{\alpha^{1/3}} \right), \text{ if } |\alpha| \text{ is sufficiently large}. \]

The function \( m(\alpha) \) also reduces to a \( 3F_2 \) function if \( \alpha \in \mathbb{R} \) [11], [16]. The first author and Zudilin [17] recently proved that

\[ F(3, 5) = \frac{4\pi^2}{15} m(1) = \frac{\pi^2}{15} 3F_2 \left( \frac{1}{3}, \frac{1}{3}, 1 ; \frac{1}{16} \right). \]
Equation (13) is equivalent to a formula that was conjectured by Deninger [8], and which helped motivate Boyd’s seminal paper [6]. One of the main results in [16], is that it is also possible to find formulas for values such as \( F(1, 4) \) and \( F(2, 2) \). These cases are probably not related to elliptic curve \( L \)-values. As a result it was hypothesized that it should be possible to “sum up” \( F(b, c) \) for arbitrary values of \( b \) and \( c \).

3.1. Lacunary cases. It is typically very difficult to prove formulas such as (13). The proof of (13) is a \( q \)-series proof which utilizes the integral representation (11). In general the difficulty of dealing with a lattice sum depends on whether it is *lacunary* or *non-lacunary*. The lacunary values can be reduced to two-dimensional sums, which are (almost) always easier to deal with than four-dimensional sums. It is often difficult to determine whether or not a particular sum is lacunary. Cases such as \( F(1, 1), F(1, 2), \) and \( F(1, 3) \) can be equated to \( L \)-values of CM elliptic curves, and their lacunarity follows from the CM hypothesis. Values such as \( F(1, 4) \) and \( F(2, 2) \) have no arithmetic interpretation, however they easily reduce to two-dimensional sums via classical theta series results. The usual method for detecting lacunarity, is to expand the associated cusp form in an infinite series. If one writes

\[
\eta(q^a)\eta(q^b)\eta(q^c)\eta(q^d) = q^{(a+b+c+d)/24} \left( a_0 + a_1 q + a_2 q^2 + \ldots \right),
\]

then the non-vanishing \( a_i \)'s should have zero-density. For the cases discussed herein it is usually necessary to compute thousands of coefficients to observe lacunarity. Additional non-obvious lacunary values include \( F(2, 9) \) and \( F(4, 7, 7, 28) \). It is necessary to employ eta function product identities to deal with these last two cases. By a result of Ramanujan [3, p. 210, Entry 56], we have

\[
3e_1e_2e_9e_{18} = -e_1^2 e_2^2 + e_1^3 \frac{e_9^2}{e_9} + e_2^3 \frac{e_9^2}{e_{18}}.
\]

Substituting classical theta expansions for \( e_1^3, e_2^2/e_1, \) and \( e_1^2/e_2 \) [10, pg. 114-117], leads to

\[
3\eta(q)\eta(q^2)\eta(q^9)\eta(q^{18}) = -\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n (2n + 1) q^{(2n+1)^2 + (2k+1)^2} + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n (2n + 1) q^{(2n+1)^2 + 9(2k+1)^2} + \sum_{n=0}^{\infty} (-1)^{n+k} (2n + 1) q^{(2n+1)^2 + 9(2k)^2}.
\]

Because \( e_1e_2e_9e_{18} \) is a finite linear combination of two-dimensional theta series, it must be a lacunary eta product. Formula (15) is the main ingredient needed to relate \( F(2, 9) \) to hypergeometric functions and Mahler measures.
Theorem 2. Let \( t = \sqrt{12} \), then the following identity is true:

\[
\frac{144}{25\pi^2} F(2, 9) = -3m(4i) + 2m \left( \frac{1}{\sqrt{2}} (4 - 2t - 2t^2 + t^3) \right) + m \left( 4i (7 + 4t + 2t^2 + t^3) \right).
\]

Proof. The most difficult portion of the calculation is to find a two-dimensional theta series for \( e_1 e_2 e_9 e_{18} \). This task has been accomplished via an eta function product identity. The remaining calculations parallel those carried out in [16]. Integrating (15) leads to

\[
\frac{3}{25} F(2, 9) + F(1, 2) = 4 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n(2n + 1)}{((2n + 1)^2 + 9(2k + 1)^2)^2}
\]

(17)

\[
+ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}(2n + 1)}{((2n + 1)^2 + 9(2k)^2)^2}.
\]

There are two possible formulas for \( F(1, 2) \) [15]:

\[
F(1, 2) = \frac{\pi^2}{8} m \left( 2\sqrt{2} \right) = \frac{\pi^2}{16} m(4i).
\]

By the formula for \( F(1, 2)(3) \) in [16, Eq. 115], we also have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}(2n + 1)}{((2n + 1)^2 + 9(2k)^2)^2} = \frac{\pi^2}{48} m \left( 4i (7 + 4t + 2t^2 + t^3) \right),
\]

(19)

where \( t = \sqrt{12} \). Next we evaluate the remaining term in (17). Notice that for \( x > 0 \)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(2n + 1)}{((2n + 1)^2 + x(2k + 1)^2)^2}
\]

\[
= \frac{\pi^2}{16} \int_0^{\infty} u \left( \sum_{n=0}^{\infty} (-1)^n (2n + 1) e^{-\pi(n+1/2)^2 u} \right) \left( \sum_{k=0}^{\infty} e^{-\pi x(k+1/2)^2 u} \right) du.
\]

By the involution for the weight-3/2 theta function

\[
\sum_{n=0}^{\infty} (-1)^n (2n + 1) e^{-\pi(n+1/2)^2 u} = \frac{1}{u^{3/2}} \sum_{n=0}^{\infty} (-1)^n (2n + 1) e^{-\pi(n+1/2)^2 u},
\]

(18)

(19)
this becomes
\[
\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{((2n+1)^2 + x(2k+1)^2)^2}
\]
\[
= \frac{\pi^2}{16} \sum_{n=0}^{\infty} (-1)^n(2n+1) \int_0^\infty u^{-1/2} e^{-\pi ((n+1/2)^2 + x(k+1/2)^2)u} \, du
\]
\[
= \frac{\pi^2}{16\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n(2n+1) \frac{e^{-\pi\sqrt{x}(2n+1)(2k+1)}}{(2k+1)}
\]
\[
= \frac{\pi^2}{16\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n(2n+1) \log \left( \frac{1 + e^{-\pi\sqrt{x}(n+1/2)}}{1 - e^{-\pi\sqrt{x}(n+1/2)}} \right).
\]

Applying formulas (1.6), (1.7), and (2.9) in [12], we have
\[
= \frac{\pi^2}{32\sqrt{x}} m \left( \frac{4}{\sqrt{\alpha_{x/4}}} \right) - m \left( \frac{4i\sqrt{1 - \alpha_{x/4}}}{\sqrt{\alpha_{x/4}}} \right)
\]
\[
= \frac{\pi^2}{32\sqrt{x}} m \left( 4 \left( 1 - \sqrt{1 - \alpha_{x/4}} \right) \frac{1 - \sqrt{1 - \alpha_{x/4}}}{1 + \sqrt{1 - \alpha_{x/4}}} \right),
\]
where \( \alpha_x \) is the singular modulus (recall that \( \alpha_x = 1 - \varphi^4(\varphi \sqrt{x})/\varphi^4(e^{-\pi\sqrt{x}}) \)). The second degree modular equation shows that
\[
\frac{1 - \sqrt{1 - \alpha_{x/4}}}{1 + \sqrt{1 - \alpha_{x/4}}} = \sqrt{\alpha_x},
\]
and hence we obtain
\[
(20) \quad \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{((2n+1)^2 + x(2k+1)^2)^2} = \frac{\pi^2}{32\sqrt{x}} m (4\sqrt{\alpha_x}).
\]

It is well known that \( \alpha_n \) can be expressed in terms of class invariants if \( n \in \mathbb{Z} \):
\[
\alpha_n = \frac{1}{2} \left( 1 - \sqrt{1 - 1/G_n^{24}} \right),
\]
and the values of \( G_n \) have been extensively tabulated, read: [4, p. 188]. Setting \( n = 9 \) yields
\[
\alpha_9 = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{\sqrt{2}}{\sqrt{3} + 1} \right)^8} \right)
\]
\[
= \frac{1}{2} \left( 1 - 4t + t^3 \right)
\]
\[
= \frac{(4 - 2t - 2t^2 + t^3)^2}{32},
\]
where $t = \sqrt{12}$. It follows immediately that
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n (2n+1)}{((2n+1)^2 + 9(2k+1)^2)^2} = \frac{\pi^2}{96} \left( \frac{1}{\sqrt{2}} \left( 4 - 2t - 2t^2 + t^3 \right) \right).
\]
The proof of (16) can be completed by combining (17), (18), (19), and (21). □

In order to avoid tedious calculations we have chosen to omit the explicit formula for $F(4,7,7,28)$ from this paper\footnote{The formula is available in previous versions of this paper on the Arxivs}. It suffices to say that the sum reduces to an extremely complicated expression involving hypergeometric functions and Meijer G-functions. The key modular equation used to prove lacunarity is due to Somos [20, Entry $q_{28,9,35}$]:
\[
28 e_4 e_7 e_{28} = -7 e_1 e_7^3 - \frac{e_1^5 e_{14}}{e_2^2 e_7} + 8 \frac{e_2^5 e_7}{e_1^2}.
\]
By classical theta expansions [10, pg. 114-117], the identity can be rewritten as
\[
28 \eta(q^4) \eta^2(q^7) \eta(q^{28}) = -7 \sum_{n=-\infty}^{\infty} (-1)^{n+k} (2k+1) q^{(6n+1)^2 + 21(2k+1)^2} 24
\]
\[- \sum_{n=0}^{\infty} (6n+1) q^{(6n+1)^2 + 21(2k+1)^2} 24
\]
\[+ 8 \sum_{n,k=-\infty}^{\infty} (-1)^{n+k} (3n+1) q^{4(3n+1)^2 + 7(6k+1)^2} 12,
\]
and as a result it is easy to see that $e_4 e_7 e_{28}$ is lacunary.

Apart from $F(2,9)$, $F(4,7,7,28)$, and the examples discussed in [16], we are not aware of any additional lacunary values of $F(a,b,c,d)$ (although they probably do exist). It is also possible to find two-dimensional reductions for certain linear combinations of lattice sums, however these formulas are generally less interesting than the previous examples. To give a single case let us briefly consider the following modular equation [20, Entry $x_{50,6,81}$]:
\[
5e_1 e_2 e_{25} e_{50} + 2e_1^2 e_2 e_{50} + 2e_1 e_2^2 e_{25} = -e_1^2 e_2^2 + e_1^3 e_{50}^2 + e_1^2 e_{25} + e_2 e_{50}^2.
\]
All three eta quotients on the right-hand side of (23) have two-dimensional theta series expansions. As a result it is possible to prove that
\[
\frac{5}{13} F(2,25) + \frac{2}{9} F(1,1,2,50) + \frac{2}{5} F(1,2,2,25)
\]
\[= \frac{\pi^2}{80} \left( -5m(4i) + 2m(4\sqrt{a_{25}}) + m \left( 4i \sqrt{\frac{1 - \alpha_{25}}{\alpha_{25}}} \right) \right),
\]
where $\alpha_{25} = \frac{1}{3^5} (\sqrt{5} - 1)^8 (\sqrt{5} - 1)^8$. There are many additional results along the lines of (24) which we will not discuss here.

3.2. Non-lacunary cases. In instances where $F(a, b, c, d)$ does not reduce to a two-dimensional sum, the calculations become far more difficult. The recent proofs of formulas for $F(1, 5), F(2, 3)$ and $F(3, 5)$ are all based upon new types of $q$-integral transformations [17], [18]. The fundamental transformation used to prove a formula for $F(2, 3)$ is

$$\int_0^1 q^{1/2} \psi(q) \varphi(-q^x) \log \frac{q}{q} \psi \left( \frac{1}{4} \right) \frac{\psi(q^{12x})}{\psi(q^{6x})} \frac{dq}{q},$$

where $\omega = e^{2\pi i/3}$. When $x = 1$ the left-hand side equals $4F(2, 3)$ (to see this use $q^{1/8} \psi(q) = \eta^2(q^2)/\eta(q)$ and $\varphi(-q) = \eta^2(q)/\eta(q^2)$), and the right-hand side becomes an extremely complicated elementary integral. The most difficult portion of the calculation is to reduce the elementary integral to hypergeometric functions. It was proved with difficulty that

$$F(2, 3) = \frac{\pi^2}{6} \Gamma(2) = \frac{\pi^2}{12} 3 F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \right).$$

Boyd’s numerical work was instrumental in the calculation, because it allowed for the final formula to be anticipated in advance.

There are many cases where non-lacunary lattice sums reduce to elementary integrals, but where the integrals are extremely difficult to deal with. We recently used the method from [17] to find a formula for $F(1, 8)$:

$$F(1, 8) = \frac{9\sqrt{2}}{128} \int_0^1 \frac{(1 - k)^2 + 2 \sqrt{2(k + k^3)}}{(1 + k)(k + k^3)^{3/4}} \log \left( \frac{1 + 2k - k^2 + 2\sqrt{k - k^3}}{1 + k^2} \right) dk.$$

We checked this monstrous identity to more than 100 decimal places by calculating $F(1, 8)$ with (11). We can only speculate that the integral should reduce to something along the lines of (16).

Occasionally eta function identities provide shortcuts for avoiding integrals like (25). We have already demonstrated that linear dependencies exist between lattice sums (see (24)). In certain cases it is possible to relate new lattice sums to well-known examples. Consider a forty-fifth degree modular equation due to Somos [20, Entry $x_{45,4,12}$]:

$$6 e_1 e_5 e_9 e_{45} = -e_1^2 e_5^2 - 2 e_3^2 e_{15}^2 - 9 e_9^2 e_{45}^2 + e_3^4 + 5 e_{15}^4.$$

We were unable to prove (26) by elementary methods. Integrating (26) leads to a linear dependency between three lattice sums. We have

$$9F(5, 9) = 45F(1, 1) - 50F(1, 5).$$
Both $F(1, 1)$ and $F(1, 5)$ equal values of hypergeometric functions [15], [17]. Since the difficult task of evaluating $F(1, 5)$ is accomplished in [17], we easily obtain the following theorem.

**Theorem 3.** Recall that $n(\alpha)$ is defined in (1). We have

\begin{equation}
\frac{108}{5\pi^2} F(5, 9) = 8n \left( 3\sqrt{2} \right) - 9n \left( 2\sqrt{4} \right).
\end{equation}

There are various additional formulas which follow from Boyd’s Mahler measure conjectures. A proof of Boyd’s conductor 30 conjectures would lead to closed forms for both $F(2, 15)$ and $F(2, 5/3)$. To make this explicit we use two relations. First consider a four term modular equation which Somos highlighted in [19]:

\begin{equation}
e_1 e_3 e_5 e_{15} + 2e_2 e_6 e_{10} e_{30} = e_1 e_2 e_{15} e_{30} + e_3 e_5 e_6 e_{10}.
\end{equation}

Integrating (29), and then using the evaluation $F(3, 5) = 4\pi^2 m(1)/15$ from [18], leads to

\begin{equation}
F(2, 15) + 4F \left( 2, \frac{5}{3} \right) = \frac{8\pi^2}{5} m(1).
\end{equation}

Next we require an unproven relation. Boyd conjectured\(^2\) that for a conductor 30 elliptic curve

\begin{equation}
L(E_{30}, 2) \equiv \frac{2\pi^2}{15} g(3),
\end{equation}

where $g(\alpha)$ is defined in (1). The modularity theorem guarantees that $L(E_{30}, 2) = L(f_{30}, 2)$, where $f_{30}(e^{2\pi i \tau})$ is a weight-two cusp form on $\Gamma_0(30)$. Somos has calculated a basis for the 1-dimensional space of cusp forms on $\Gamma_0(30)$, and consequently the cusp form associated with conductor 30 elliptic curves is given by

\begin{equation}
f_{30}(q) = \eta(q^3) \eta(q^5) \eta(q^6) \eta(q^{10}) - \eta(q) \eta(q^2) \eta(q^{15}) \eta(q^{30}).
\end{equation}

Upon integrating this cusp form, Boyd’s conjecture becomes

\begin{equation}
F \left( 2, \frac{5}{3} \right) - \frac{1}{4} F(2, 15) \equiv \frac{2\pi^2}{15} g(3).
\end{equation}

Combining (30) and (31) leads to a pair of conjectural evaluations.

**Conjecture 4.** Recall that $m(\alpha)$ and $g(\alpha)$ are defined in (1). The following formulas are numerically true:

\begin{equation}
\frac{15}{4\pi^2} F(2, 15) \equiv 3m(1) - g(3),
\end{equation}

\begin{equation}
\frac{15}{\pi^2} F \left( 2, \frac{5}{3} \right) \equiv 3m(1) + g(3).
\end{equation}

Tracking backwards shows that a solution of either (32) or (33) would settle Boyd’s conductor 30 Mahler measure conjectures. Proofs remain out of reach, however we are optimistic that both identities may eventually be established using Eisenstein series identities contained in [1].

\(^2\)See Table 2 in [6]. In our notation, Boyd’s entries correspond to values of $g(2 - k)$.
4. Conclusion: Conductor 17 elliptic curves

A strong connection exists between lattice sums and Mahler measures, however this relationship has limitations. While our ultimate goal is to “sum up” $F(b, c)$ for arbitrary values of $b$ and $c$, it is important to realize that this would only settle a small portion of the conjectures in Boyd’s paper [6]. Conductor 17 curves are the first cases in Cremona’s list [7], where $L(E, 2)$ probably does not reduce to values of $F(b, c)$. If we let $E_{17}$ denote a conductor 17 curve (we used $y^2 + xy + y = x^3 - x^2 - x$), then

$$\frac{17}{2\pi^2} L(E_{17}, 2) = m \left( \frac{(1 + \sqrt{17})^2}{4} \right) - m \left( \sqrt{17} \right).$$

We discovered (34) via numerical experiments involving elliptic dilogarithms. The cusp form associated with conductor 17 curves is stated in [9]. We have

$$f_{17}(q) = \frac{\eta(q)\eta^2(q^4)\eta^5(q^{34})}{\eta(q^2)\eta(q^{17})\eta^2(q^{68})} - \frac{\eta^5(q^2)\eta(q^{17})\eta^2(q^{68})}{\eta(q)\eta^2(q^4)\eta(q^{34})}.$$

Since $L(E_{17}, 2) = L(f_{17}, 2)$, formula (34) can be changed into a complicated (!) elementary identity. There does not seem to be an easy way to relate $L(E_{17}, 2)$ to Mahler measures of rational polynomials. We surmise that this is the reason conductor 17 curves never appear in Boyd’s paper [6]. Given the complexity of $f_{17}(q)$, we feel confident to conjecture that $L(E_{17}, 2)$ is linearly independent from values of $F(b, c)$ over $\mathbb{Q}$.

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