PRICING EQUILIBRIUM OF TRANSPORTATION SYSTEMS WITH BEHAVIORAL COMMUTERS

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Abstract. We study Wardrop equilibrium in a transportation system with profit-maximizing firms and heterogeneous commuters. Standard commuters minimize the sum of monetary costs and equilibrium travel time in their route choice, while “oblivious” commuters choose the route with minimal idle time. Three possible scenarios can arise in equilibrium: A pooling scenario where all commuters make the same transport choice; A separating scenario where different types of commuters make different transport choices; A partial pooling scenario where some standard commuters make the same transport choice as the oblivious commuters. We characterize the equilibrium existence condition, derive equilibrium flows, prices and firms’ profits in each scenario, and conduct comparative analyses on parameters representing route conditions and heterogeneity of commuters, respectively. The framework nests the standard model in which all commuters are standard as a special case, and also allows for the case in which all commuters are oblivious as the other extreme. Our study shows how the presence of behavioral commuters under different route conditions affects equilibrium behavior of commuters and firms, as well as the equilibrium outcome of the transportation system.

1. Introduction. Modeling and predicting traffic flows in a transportation system is a practical challenge that warrants consideration of the psychology of commuters’ travel choices, as well as the heterogeneity of commuters’ reasoning tendencies in the population. Since the application of game theoretic modeling to transportation problems (Wardrop, 1952), a feature of equilibrium analysis is the underlying assumption that all commuters involved are perfectly rational in the game theoretic

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sense, meaning that they optimally respond to other commuters’ rational game theoretic behavior.

However, such assumptions may be considered too idealistic in real world situations, where often at least a subset of commuters may not incorporate strategic reasoning when making their choices. For example, visitors from out of town may know about the local travel route with the shortest distance, but they may not be aware of the typical travel time required on each route under traffic conditions, a factor which is well-taken into consideration by local commuters. Commuters who are especially busy or cognitively burdened, may heuristically know about the route that requires the least travel time based on their past experience, but may not have time to think carefully about how daily weather conditions or other external circumstances will affect the travel choices of other commuters in the population.

These examples share in common that there is a subset of commuters that are equilibrium players in the sense of Wardrop Equilibrium, while another subset of commuters are limited in their strategic considerations (see Camerer, Ho and Chong, 2004, Crawford, 2013, and Lien, Zhao and Zheng, 2019). The latter subset of commuters might consider some salient aspects of the travel routes, while neglecting to incorporate the behavior of other commuters who are using the transportation system into their decision. In this paper, we analyze the implication of such behavioral heterogeneity in the population of commuters on traffic flows across different routes, in a market where transportation firms compete with one another by providing substitutable routes to commuters.

In our setup, “oblivious” commuters not only ignore the traffic-contingent costs in the transport system in their decision, but they also do not pay the service price. This scenario corresponds to some plausible real world situations: For example, the oblivious commuters may not be aware of the requirement to pay the service price - they could be visitors choosing their route for the first time, oblivious to congestion considerations while neglecting to purchase a ticket; As another example, oblivious commuters may be exempt from paying the service price, or receive a deep price discount - for example, the case of children and the elderly, who often may be less strategically sensitive to congestion costs than ordinary commuters.

A substantial literature examines the role of boundedly rational commuters in transportation systems. Di and Liu (2016) provides a comprehensive survey of different approaches to bounded rationality utilized in the transportation literature. Our study differs from prior works by modeling a competitive market in the transportation network, whereas most of the prior literature focuses on the

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1 The cognitive hierarchy model by Camerer, Ho and Chong (2004) provides a non-equilibrium framework for heterogeneity in strategic thinking, as does the level-k class of models discussed in survey Crawford (2013). In the current paper, we maintain equilibrium concepts while allowing for some commuters to be unaffected by cost factors that are influenced by other commuters. However, one can easily infer that if such commuters took those cost factors into consideration in their utility but effectively ignored it in their decision-making, that equilibrium behavior in our analysis results in suboptimal utility realizations in a scenario where strategically determined payoff components in fact matter. In maintaining equilibrium concepts, our current approach bears resemblance to Lien, Zhao and Zheng (2019) which analyzes equilibria in contest games in which contestants have perception biases.

2 Another application is that some commuters may have priority to use the route for free - for example, certain government vehicles which are exempt from road service charges. From the decision-making standpoint, if such vehicles are exempt from congestion costs, such as the cases of police cars, fire trucks and ambulances, our model also applies without any behavioral labeling (ie. obliviousness). The welfare consequences however, would differ.
TRANSPORTATION SYSTEMS WITH BEHAVIORAL COMMUTERS

user (price-free) equilibrium. Our approach follows on that of Mazalov and Melnik (2016), Lien, Mazalov, Melnik and Zheng (2016), Kuang, Mazalov, Tang and Zheng (2020), and Kuang, Lian, Lien and Zheng (2020), in analyzing transport systems and traffic flows under duopoly competition between transport providers.

Among studies focusing on user equilibrium with boundedly rational commuters, Mahmassani and Chang (1987) provides an early framework and analysis for boundedly rational user equilibrium. Di, He, Guo and Liu (2014) study the conditions under which the Braess Paradox persists under a boundedly rational user equilibrium. Ye and Yang (2017) incorporate the boundedly rational user equilibrium into the rational behavior adjustment process, and examine the equilibrium existence conditions. Sun, Cheng and Ma (2018) use a boundedly rational user equilibrium approach to modeling travel time reliability. Our study is closely related in approach to Karakostas, Kim, Viglas and Xia (2011), which considers user equilibria with commuters who may be oblivious to congestion costs that are a result of other commuters’ choices.

On the topic of boundedly rational commuters, several empirical studies also address the issue. Among them, Jou, Hensher, Liu and Chiu (2010) use a stated choice approach to empirically examine boundedly rational transport mode switching behavior in Taipei. Zhao and Huang (2016) conduct a laboratory experiment to test boundedly rational choice of routes under a satisficing rule. Tang Luo and Liu (2016) conduct numerical simulations to show the increasing effects of driver bounded rationality on costs of travel.

Our study is the first to our knowledge, to consider the role of boundedly rational commuters in the sense of being oblivious to other commuters’ influences on traffic flows, in a competitive transport system where prices motivate the equilibrium between all players in the market, firms and commuters. Transport firms compete with one another and set prices strategically, while commuters choose among the provided routes to maximize their utilities, which are influenced by the choices of other commuters through the congestion cost. We show that three possible scenarios can arise in equilibrium: A pooling scenario where all commuters make the same transport choice; A separating scenario where different types of commuters make different transport choices, A partial pooling scenario where some standard commuters make the same transport choice as the oblivious commuters. Our analysis provides conditions under which the flow of commuter traffic varies based on commuter sophistication and route followed, particularly as a function of the total traffic flow intensity and the fraction of oblivious commuters. In particular, we provide conditions under which each of the three scenarios can become an equilibrium.

The paper is organized as follows: Section 2 provides the model setup. Section 3 conducts equilibrium analysis. Section 4 provides a numerical example to illustrate the results. Section 5 concludes and discusses.

2. Model. We analyze a duopoly competition model between two transport service carriers in a transportation route system of parallel structure. Carriers set service prices to maximize their own profits, and commuters choose which carrier’s service to use in order to optimize their own objective functions, the form of which depends on the type of commuter, either standard or “oblivious”.

The standard commuters minimize their cost which is equal to the service price paid (the monetary cost) plus the actual travel time spent (the waiting cost), while
the oblivious commuters merely minimize the idle (unoccupied) travel time of the chosen route.

2.1. Setting. Consider a transport network composed of two parallel routes between origin A and destination B with the following linear BPR (Bureau of Public Roads) latency functions:  

\[ f_1(z_1, z_2) = t_1(1 + a_1 z_1 + b_1 z_2); \]
\[ f_2(z_1, z_2) = t_2(1 + a_2 z_2 + b_2 z_1). \]

In the above expressions, \( z_i \) represents the flow intensity of route \( i \) and \( (t_1, a_1, b_1); (t_2, a_2, b_2) \); are latency parameters such that \( t_i \) indicates the travel time on a completely unoccupied route \( i \), \( a_i \) denotes the internal effect of route \( i \)'s own flow on route \( i \), and \( b_i \) measures the external effect of route \(-i\)'s flow on route \( i \), where \( i = 1, 2 \). Without loss of generality, we suppose that the first route is faster than the second route, that is \( t_1 \leq t_2 \). As a reasonable setup, we also assume that the internal effects dominate the external effects, that is, \( min\{a_1, a_2\} \geq max\{b_1, b_2\} \).

Under such a route structure, commuters of flow intensity \( X \) travel from origin A to destination C, and each commuter must choose exactly one of the two routes. There are two carriers: Firm 1 takes charge of the first route and sets a service price of \( p_1 \); Firm 2 takes charge of the second route and sets a service price of \( p_2 \).

Of the coming flow of commuters \( X \), \( \alpha \) fraction of them, denoted by \( x_0 = \alpha X \), are oblivious commuters, where \( 0 \leq \alpha < 1 \). They do not pay the transport service price, and ignore congestion merely choosing the fastest route based on the unoccupied travel time (which can also be interpreted as congestion-independent route-specific characteristics such as geographical distance or road length).

The other commuters, of flow intensity \( (1-\alpha)X \), are standard commuters. They compare the sum of both monetary cost and waiting cost, namely \( p_1 + f_1(x) \) and \( p_2 + f_2(x) \), in using each route, and select the route with minimal total cost.

We use \( x = (x_0, x_1, x_2) \) to denote a profile of subflows among commuters, where the flow of standard consumers is divided into two subflows \( x_1 \) and \( x_2 \) such that \( x_i \) represents the flow of standard consumers using route \( i \), \( i = 1, 2 \). Thus we have \( x_0 = \alpha X \), \( x_1 + x_2 = (1-\alpha)X \).

Notice that the oblivious commuters will always choose the first route as long as \( t_1 < t_2 \). In the case where \( t_1 = t_2 \), as a tie breaking rule, we simply assume that route 1 will be chosen by all oblivious commuters. Therefore, the following conditions are satisfied:

\[ z_1 = x_0 + x_1, \quad z_2 = x_2. \]

Both carriers (firms 1 and 2) set their prices strategically and independently to maximize their profits, while their operating costs are simply normalized to zero. To be specific, firm \( i \)'s profit, \( H_i = x_i p_i \), equals the product of route \( i \) flow of standard commuters and firm \( i \)'s service price.

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3See also Lien, Mazalov, Melnik and Zheng (2016) which analyzes Wardrop Equilibrium under the BPR latency function with homogeneous commuters.

4Note that for a transport network of parallel structure with \( n \) routes, the number of external effect parameters is \( n(n-1) \). Thus for \( n > 2 \), we should use \( b_{ij} \) to measure the external effect of route \( j \)'s flow on route \( i \). For more details, see Kuang, Mazalov, Tang and Zheng (2020) which studies the transportation network with externalities.

5Such an assumption is also reasonable in the sense that in reality, routes are differentiable so that one route can be more salient than the other, therefore all oblivious commuters will choose the more salient one even when two routes share the same idle travel time.
The setting of the model can be illustrated by Figure 1.

![Figure 1](image)

**Figure 1.** 2-Route Transportation System with Duopoly Firms and Heterogeneous Commuters

2.2. Transformation and equivalence. When introducing externalities into the network pricing model, the intuition behind the results can be substantially different from that without externalities. Nonetheless, when the number of routes is 2, the two models are essentially the same through a mathematical transformation, which is due to the overall flow distribution being determined as long as one route’s flow is fixed. Such an equivalence between the cases with and without externalities is established in the following proposition.

**Proposition 1.** The 2-route heterogeneous-commuters linear transportation system with externalities, $(X; \alpha; x_0, x_1, x_2; t_1, t_2; a_1, a_2; b_1, b_2)$, is equivalent to another 2-route heterogeneous-commuters linear transportation system without externalities, $(X; \alpha; x_0, x_1, x_2; t'_1, t'_2; a'_1, a'_2, b'_1)$.

To see why the above proposition holds, note that the latency functions with externalities can be re-written as

$$f_1(x) = t_1(1 + a_1(x_0 + x_1) + b_1 x_2) = t_1(1 + a_1(x_0 + x_1) + b_1(X - (x_0 + x_1))) =$$

$$= t'_1(1 + a'_1(x_0 + x_1));$$

$$f_2(x) = t_2(1 + a_2 x_2 + b_2(x_0 + x_1)) = t_2(1 + a_2 x_2 + b_2(X - x_2) =$$

$$= t'_2(1 + a'_2 x_2),$$

where

$$t'_1 = t_1(1 + b_1 X), \quad a'_1 = \frac{a_1 - b_1}{1 + b_1 X},$$

$$t'_2 = t_2(1 + b_2 X), \quad a'_2 = \frac{a_2 - b_2}{1 + b_2 X}.$$
commuters into one with standard commuters only. Note that
\[ f_1(x) = t_1(1 + a_1(x_0 + x_1) + b_1 x_2) = t_1(1 + a_1(\alpha X + x_1) + b_1 x_2) = \\
= t_1'(1 + a''_1 x_1 + b''_1 x_2); \]
\[ f_2(x) = t_2(1 + a_2 x_2 + b_2(x_0 + x_1)) = t_2(1 + a_2 x_2 + b_2(\alpha X + x_1) = \\
= t_2'(1 + a''_2 x_2 + b''_2 x_1), \]
where
\[ t''_1 = t_1(1 + \alpha a_1 X), \quad a''_1 = \frac{a_1}{1 + \alpha a_1 X}, \quad b''_1 = \frac{b_1}{1 + \alpha a_1 X}, \]
\[ t''_2 = t_2(1 + \alpha b_2 X), \quad a''_2 = \frac{a_2}{1 + \alpha b_2 X}, \quad b''_2 = \frac{b_2}{1 + \alpha b_2 X}. \]

This gives us the following equivalence proposition.

**Proposition 2.** The 2-route heterogeneous-commuters linear transportation system with externalities, \((X; \alpha; x_0, x_1, x_2; t_1, t_2; a_1, a_2; b_1, b_2)\), is equivalent to another 2-route standard-commuters linear transportation system with externalities, \((X; \alpha; x_1, x_2; t'_1, t'_2; a''_1, a''_2; b''_1, b''_2)\).

Finally, we can further convert the heterogeneous-commuters linear transportation system with externalities into a standard-commuters system without externalities, through the following transformation:
\[ f_1(x) = t_1(1 + a_1(x_0 + x_1) + b_1 x_2) = t_1(1 + a_1(\alpha X + x_1) + b_1((1 - \alpha)X - x_1)) = \\
= t_1'(1 + a'_{1} x_1); \]
\[ f_2(x) = t_2(1 + a_2 x_2 + b_2(x_0 + x_1)) = t_2(1 + a_2 x_2 + b_2(X - x_2) = \\
= t_2'(1 + a'_{2} x_2), \]
where
\[ t'_1 = t_1(1 + (\alpha a_1 + (1 - \alpha)b_1)X), \quad a'_1 = \frac{a_1 - b_1}{1 + (\alpha a_1 + (1 - \alpha)b_1)X}, \]
\[ t'_2 = t_2(1 + b_2 X), \quad a'_2 = \frac{a_2 - b_2}{1 + b_2 X}. \]

This leads to the third equivalence proposition in which the transformation is achieved from removing both commuters’ heterogeneity and cross-route externalities.

**Proposition 3.** The 2-route heterogeneous-commuters linear transportation system with externalities, \((X; \alpha; x_0, x_1, x_2; t_1, t_2; a_1, a_2; b_1, b_2)\), is equivalent to another 2-route standard-commuters linear transportation system without externalities, \((X; \alpha; x_1, x_2; t'_1, t'_2; a'_1, a'_2)\).

Comparing the above three equivalence results, it is easy to verify that when there are no oblivious commuters \((\alpha = 0)\), we have \( t'_i = t''_i \) and \( a'_i = a''_i \), \( i = 1, 2 \). Also, when there are no externalities \((b_i = 0, i = 1, 2)\), we have \( t'_i = t''_i \), \( a'_i = a''_i \), and \( 0 = b''_i \), \( i = 1, 2 \).

3. **Equilibrium analysis.** In this section, we characterize the equilibrium of the heterogeneous-commuters 2-carrier 2-route transportation system introduced in Section 2.

Recall that due to the assumption \( t_1 \leq t_2 \), the oblivious commuters will always follow the route 1. Thus, there are in total three possible scenarios regarding
how standard commuters (and oblivious commuters) are allocated between the two routes:

1) **Concentration on One Route**: In this pooling scenario, all commuters, regardless of whether they are oblivious or standard, travel through the first route.

2) **Competition for Standard Commuters**: In this partial pooling scenario, oblivious commuters travel through the first route and the standard commuters are distributed between the first and second routes.

3) **Separation between Oblivious and Standard Commuters**: In this separating scenario, the oblivious commuters travel through the first route and the standard commuters travel through the second route.

In the following subsections, we first discuss each of these scenarios and derive flows, prices and firms’ profits, and then characterize the equilibrium depending on the values of model parameters \((X; \alpha; t_1; t_2; a_1, a_2; b_1, b_2)\).

### 3.1. Concentration on one route.

In general, the oblivious commuters will surely follow the first route since \(t_1 \leq t_2\), but the standard commuters make their decision by comparing the sum of both monetary cost and waiting cost in using each route. In order for all standard commuters to choose the first route, it must be that the travel time of the first route with the flow \(\alpha X\) of oblivious commuters is no more than the travel time of the unoccupied second route, that is,

\[
t'_1 \leq t'_2
\]

or

\[
t_1 (1 + (\alpha a_1 + (1 - \alpha)b_1)X) \leq t_2(1 + b_2 X). \tag{1}
\]

Furthermore, the first route should also be more attractive for the flow \((1 - \alpha)X\) of standard commuters in the sense that even if the price of the second route is \(p_2 = 0\), these standard commuters will weakly prefer the first route. Thus, the following condition must be satisfied for \(x = (x_0, x_1, x_2) = (\alpha X, (1 - \alpha)X, 0)\):  

\[
p_1 + f_1(x) \leq f_2(x),
\]

or

\[
p_1 + t'_1 (1 + a'_1 (1 - \alpha)X) \leq t'_2.
\]

Consequently, the optimal price for firm 1 is given by the expression

\[
p^*_1 = t'_2 - t'_1 (1 + a'_1 (1 - \alpha)X) = t_2 - t_1 - (t_1 a_1 - t_2 b_2) X,
\]

and the price for firm 2 should be set at \(p^*_2 = 0\). We observe that the optimal price for firm 1 is not dependent on \(\alpha\). This is natural, since firm 2’s optimal price is determined by the difference between the travel time of the unoccupied second route and the travel time of the fully occupied first route, where all commuters, regardless of whether they are oblivious or standard, have the same effect upon the travel time.

Notice that the price for firm 1 should be non-negative, which requires

\[
X \leq \frac{t_2 - t_1}{t_1 a_1 - t_2 b_2}. \tag{1}
\]

\(^7\)This implies \((t_1 (\alpha a_1 + (1 - \alpha) b_1) - t_2 b_2) X < t_2 - t_1\). Since by assumption \(a_1 > b_1\), we have \(t_1 (\alpha a_1 + (1 - \alpha) b_1) - t_2 b_2 \leq t_1 a_1 - t_2 b_2\), suggesting that this inequality is a weaker condition than Inequality (1), which will be specified below.

\(^8\)Otherwise, firm 2 can attract some flows of standard commuters to the second route by setting a price which is just above zero.

\(^9\)Otherwise, firm 1 can earn more profit by raising its price slightly above \(p^*_1\), which violates the optimality condition.
Inequality (1) presents a necessary condition under this pooling scenario, which suggests that the total flow intensity cannot be too high. However, it does not mean that all commuters will always choose the first route in equilibrium under this condition, which would be considered a sufficient condition. As we confirm in the subsequent analysis, the equilibrium cutoff condition for this scenario is indeed more strict, and essentially depends on the parameter \( \alpha \). Furthermore, when this scenario becomes an equilibrium outcome, the limit of \( X \) allowed is an increasing function of \( \alpha \).

Using \( H_i^{(k)} \) to denote firm \( i \)'s profit in Scenario \( k \), where \( i = 1, 2 \) and \( k = 1, 2, 3 \), in this scenario (Scenario 1) the payoffs of the two firms are given by

\[
H_1^{(1)} = p_1^*(1 - \alpha)X = (t_2' - t_1'(1 + a'_1(1 - \alpha)X))(1 - \alpha)X =
\]

\[
(t_2 - t_1 - (t_1a_1 - t_2b_2)X)(1 - \alpha)X,
\]

\[
H_2^{(1)} = 0.
\]

It is obvious that such a fully pooling scenario is good for firm 1 but not for firm 2, as no standard commuters have any incentive to use the second route.

3.2. Competition for standard commuters. In the partial pooling competitive scenario, both firms charge positive prices \( p_1 > 0, p_2 > 0 \), and the traffic flow \((1 - \alpha)X\) of standard commuters is split into two subflows, \( x_1 > 0 \) and \( x_2 > 0 \), which satisfy the following equations according to the indifference condition of the standard commuters:

\[
p_1 + t'_1(1 + a'_1x_1) = p_2 + t'_2(1 + a'_2x_2), \quad x_1 + x_2 = (1 - \alpha)X. \tag{2}
\]

The two firms strategically set their own prices to maximize their profits

\[
H_1 = p_1x_1, \quad H_2 = p_2x_2.
\]

We now derive the equilibrium by solving the pricing game above. First, we express the subflows \((x_1, x_2)\) in terms of prices \((p_1, p_2)\) by using equations (2), and then substitute them into the firms’ profit functions, as shown below.

\[
H_1(p_1, p_2) = p_1 \frac{p_2 - p_1 + t'_2(1 + a'_2(1 - \alpha)X) - t'_1}{a'_1t'_1 + a'_2t'_2},
\]

\[
H_2(p_1, p_2) = p_2 \frac{p_1 - p_2 + t'_1(1 + a'_1(1 - \alpha)X) - t'_2}{a'_1t'_1 + a'_2t'_2}.
\]

Then, fixing \( p_2 \) we find the best response function of firm 1 through the first order condition of \( H_1(p_1, p_2) \) with respect to \( p_1 \), which yields

\[
p_1 = \frac{1}{2} \left( p_2 + t'_2 - t'_1 + a'_2t'_2(1 - \alpha)X \right).
\]

Similarly, by fixing \( p_1 \), we calculate the best response of firm 2 and obtain

\[
p_2 = \frac{1}{2} \left( p_1 + t'_1 - t'_2 + a'_1t'_1(1 - \alpha)X \right).
\]

From these equations, we find the equilibrium prices

\[
p_1^* = \frac{1}{3} \left( t'_2 - t'_1 + (a'_1t'_1 + 2a'_2t'_2)(1 - \alpha)X \right), \tag{3}
\]

\[
p_2^* = \frac{1}{3} \left( t'_1 - t'_2 + (2a'_1t'_1 + a'_2t'_2)(1 - \alpha)X \right). \tag{4}
\]
Substituting the notation of parameters \( t', \alpha' \) into expressions (3) and (4) we obtain the equilibrium prices

\[
p_1^* = \frac{1}{3} (t_2 - t_1 - (a_1 t_1 - b_2 t_2)X + 2(1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i X),
\]

\[
p_2^* = \frac{1}{3} (t_1 - t_2 + (a_1 t_1 - b_2 t_2)X + (1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i X).
\]

Thus, we can derive the following equilibrium traffic flows

\[
x_1^* = \frac{t_2 - t_1 + (a_1 t_1 + 2a_2 t_2)(1 - \alpha)X}{3(a_1 t_1' + a_2 t_2')} = \frac{p_1^*}{3(a_1 t_1' + a_2 t_2')} = \frac{p_1^*}{3 \sum_{i=1}^{2} (a_i - b_i) t_i}, \tag{5}
\]

\[
x_2^* = \frac{t_1 - t_2 + (a_2 t_2 + 2a_1 t_1)(1 - \alpha)X}{3(a_1 t_1' + a_2 t_2')} = \frac{p_2^*}{3(a_1 t_1' + a_2 t_2')} = \frac{p_2^*}{3 \sum_{i=1}^{2} (a_i - b_i) t_i}. \tag{6}
\]

Consequently, the equilibrium profits of the firms in this scenario (Scenario 2) are

\[
H_1^{(2)} = p_1^* x_1^* = \frac{(p_1^*)^2}{3(a_1 t_1' + a_2 t_2')} = \frac{\sum_{i=1}^{2} (a_i - b_i) t_i}{3 \sum_{i=1}^{2} (a_i - b_i) t_i},
\]

\[
H_2^{(2)} = p_2^* x_2^* = \frac{(p_2^*)^2}{3(a_1 t_1' + a_2 t_2')} = \frac{\sum_{i=1}^{2} (a_i - b_i) t_i}{3 \sum_{i=1}^{2} (a_i - b_i) t_i}.
\]

The competitive scenario takes place only if all optimal values for prices and flows, that is \( x_1^*, p_1^*, i = 1, 2 \), are non-negative, which implies the following condition:

\[
(a_1 t_1 - b_2 t_2 - 2(1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i X \leq t_2 - t_1 \leq (a_1 t_1 - b_2 t_2 + (1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i) X.
\]

\[
\tag{7}
\]

A more informative way to rewrite Inequality (7) is as follows:

\[
X \geq \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 + (1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i} \quad \text{if} \quad \alpha \leq \hat{\alpha},
\]

and

\[
\frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 + (1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i} \leq X \leq \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 - 2(1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i}
\]

if \( \alpha > \hat{\alpha} \), where \( \hat{\alpha} = 1 - \frac{a_1 t_1 - b_2 t_2}{2 \sum_{i=1}^{2} (a_i - b_i) t_i} \in (\frac{1}{2}, 1) \). \(^{10}\)

Inequality (7) represents a necessary condition under the competitive scenario. Again, this is not a sufficient condition. \(^{11}\) It is worth noting that the presence of oblivious commuters influences the flow, prices and firms’ profits of both routes, as exhibited by the parameter \( \alpha \) in the cutoff value expressions of Inequality (7).

\(^{10}\)To see why \( \hat{\alpha} \in (\frac{1}{2}, 1) \), note that \( \sum_{i=1}^{2} (a_i - b_i) t_i = a_1 t_1 - b_2 t_2 + a_2 t_2 - b_1 t_1 > a_1 t_1 - b_2 t_2 \).

The last inequality holds since \( \min \{a_1, a_2\} \geq \max \{b_1, b_2\} \) and \( t_2 \geq t_1 \), implying \( a_2 t_2 > b_1 t_1 \).

\(^{11}\)Notice that the lower bound expression of Inequality (7) for \( X \) is less than the cutoff expression in Inequality (1), that is \( \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 + (1 - \alpha) \sum_{i=1}^{2} (a_i - b_i) t_i} < \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2} \), since \( a_i > b_i \) for \( i = 1, 2 \). This implies that there is a potentially overlapping range of parameters such that both scenarios 1 and 2 can be possible. In the subsection of Equilibrium Characterization, we will further consider firms’ available pricing strategies and solve for the unique equilibrium which results in one of the possible scenarios.
3.3. Separation between oblivious and standard commuters. We now consider the last scenario, in which the oblivious commuters of flow $\alpha X$ travel through the first route while all standard commuters of flow $(1 - \alpha) X$ travel through the second route. This means $x_1 = 0$ and $x_2 = (1 - \alpha) X$. Following a logical argument similar to that in the first scenario, we can show that such a separating scenario is possible only if

$$
t'_2(1 + a'_2(1 - \alpha)X) \leq t'_1,
$$

which implies

$$\alpha > \frac{a_2 t_2 - b_1 t_1}{\sum_{i=1}^{\alpha}(a_i - b_i)t_i},$$

and

$$X \geq \frac{t_2 - t_1}{-(a_2 t_2 - b_1 t_1) + \alpha \sum_{i=1}^{\alpha}(a_i - b_i)t_i}.\quad (9)$$

Inequality (8) implies that in order for all standard commuters to choose the second route, the fraction of oblivious commuters must be sufficiently high, and Inequality (9) provides another necessary condition which is that the total flow intensity cannot be too low. Also notice that Inequalities (7) and (9) have an overlapping parameter range so neither of them are sufficient conditions for scenarios 2 and 3, respectively.\(^\text{12}\)

In this separating scenario, firm 2 can optimally set the price for travel service, which is equal to the difference between the travel time of the first route with all oblivious commuters and the travel time of the second route with all standard commuters, as below

$$p^*_2 = t'_1 - t'_2(1 + a'_2(1 - \alpha)X) = t_1 - t_2 + (a_1 t_1 - b_2 t_2)\alpha X - (a_2 t_2 - b_1 t_1)(1 - \alpha)X,$$

and the price for firm 1 should be set at $p^*_1 = 0$. Note that, contrary to firm 1’s optimal price being independent of $\alpha$ in the first scenario, in the current scenario, firm 2’s optimal price increases in $\alpha$. This result is intuitive since the more oblivious commuters there are, the longer it takes to travel through the first route, and the less attractive the first route is compared to the second route, hence the higher the price that firm 2 can set to attract all standard commuters.

Having derived firms’ optimal prices, we can easily express the profits of the firms as

$$H_1^{(3)} = 0, \quad H_2^{(3)} = p^*_2((1 - \alpha)X) = (t'_1 - t'_2(1 + a'_2(1 - \alpha)X))(1 - \alpha)X.$$ 

3.4. Equilibrium characterization. Having analyzed the optimal pricing strategy under each scenario, we can now completely characterize the equilibrium of the pricing game between the two firms.

In the first scenario, where the total traffic flow $X$ is sufficiently low such that Inequality (1) holds, the situation is fully determined by firm 1. Specifically, firm 1 announces the price $p^*_1 = t_2 - t_1 - (t_1 a_1 - t_2 b_2)X$ and uses this pricing strategy if the profit $H_1^{(1)}$ is greater than the profit under the competitive scenario $H_1^{(2)}$. That

\[^{12}\text{The upper bound expression of Inequality (7) for } X \text{ is greater than the cutoff expression in Inequality (9), that is, } \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 - 2(1 - \alpha) \sum_{i=1}^{\alpha}(a_i - b_i)t_i} > \frac{t_2 - t_1}{-a_2 t_2 - b_1 t_1 + \alpha \sum_{i=1}^{\alpha}(a_i - b_i)t_i} - \frac{t_2 - t_1}{a_1 t_1 - b_2 t_2 - 2(1 - \alpha) \sum_{i=1}^{\alpha}(a_i - b_i)t_i} = (1 - \alpha) \sum_{i=1}^{\alpha}(a_i - b_i)t_i > 0. \text{ The last inequality holds since } a_i > b_i \text{ for } i = 1, 2.]
is,
\[(t_2' - t_1'(1 + a_1'(1 - \alpha)X))(1 - \alpha)X = H^{(1)}_1 \geq H^{(2)}_1 = \frac{(t_2' - t_1' + (a_1't_1' + 2a_2't_2')(1 - \alpha)X)^2}{27(a_1't_1' + a_2't_2')}\] (10)

In the third scenario where the fraction of oblivious commuters \(\alpha\) and the total traffic flow \(X\) are both sufficiently high such that Inequalities (8) and (9) hold, the situation is fully controlled by firm 2. Specifically, firm 2 sets the price \(p^*_2 = t_1 - t_2 + (a_1t_1 - b_2t_2)\alpha X - (a_2t_2 - b_1t_1)(1 - \alpha)X\) if the profit \(H^{(3)}_2\) is greater than the profit under the competitive scenario \(H^{(2)}_2\), that is,
\[(t_1' - t_2' - a_2't_2'(1 - \alpha)X)(1 - \alpha)X = H^{(3)}_2 \geq H^{(2)}_2 = \frac{(t_1' - t_2' + (2a_1't_1' + a_2't_2')(1 - \alpha)X)^2}{27(a_1't_1' + a_2't_2')}\] (11)

Based on the previous analyses and the above discussion, we summarize the results by presenting the equilibrium characterization in the following proposition.

**Proposition 4.** Suppose \(t_1 \leq t_2, \ min\{a_1, a_2\} \geq max\{b_1, b_2\}\), and \(t_1a_1 > t_2b_2\). There exist the following possible equilibrium scenarios depending on the parameters of the transportation system \((X; \alpha; t_1, t_2; a_1, a_2; b_1, b_2);\):

1) Under Condition (1), if either Condition (7) fails or both Conditions (7) and (10) hold, then all traffic follows the first route: \(x_0^* = \alpha X, x_1^* = (1 - \alpha)X, x_2^* = 0, p_1^* = t_2 - t_1 - (t_1a_1 - t_2b_2)X, p_2^* = 0\).

2) Under Conditions (8) and (9), if either Condition (7) fails or both Conditions (7) and (11) hold, then oblivious commuters follow the first route and standard commuters follow the second route: \(x_0^* = \alpha X, x_1^* = 0, x_2^* = (1 - \alpha)X, p_1^* = 0, p_2^* = t_1 - t_2 + (a_1t_1 - b_2t_2)\alpha X - (a_2t_2 - b_1t_1)(1 - \alpha)X\).

3) In all other cases, standard commuters are split between the first route and the second route, where \(x_0^* = \alpha X, x_i^*, i = 1, 2\) are determined by Equations (5) and (6), and \(p_i^*, i = 1, 2\) are determined by Equations (3) and (4).

Proposition 3.1 confirms that the pooling scenario in which all commuters concentrate on one route (Scenario 1) cannot occur in equilibrium if the total traffic flow is very high. This result is intuitive because with a large \(X\), the route followed by all commuters will become too crowded and eventually some commuters will be attracted by the other route. The proposition also tells us that the separating scenario of different types of commuters using different routes (Scenario 3) cannot occur in equilibrium if either the total traffic flow or the fraction of oblivious commuters is very low. The intuition is that a small \(X\) or a small \(\alpha\) implies that the traffic flow by oblivious commuters cannot be too large, which means the delay caused by those commuters will not be too large, thus enhancing the comparative advantage of the faster route against the slower route in terms of attracting the standard commuters.

4. **Numerical example.** In this section, we provide a simple numerical example to help illustrate the results established in the previous section. Assume that the latency parameters are given by \(t_1 = 2, a_1 = 3, b_1 = 1; t_2 = 3, a_2 = 2, b_2 = 1\), which leads to the following specification of the latency functions:
\[f_1(x) = 2(1 + 3x_1 + x_2), \quad f_2(x) = 3(1 + 2x_2 + x_1)\]
By the equivalence result in Proposition 2.3, the new parameters are such that
\[ t'_1 = 2(1 + (3\alpha + (1 - \alpha))X) = 2(1 + (1 + 2\alpha)X), \quad a'_1 = \frac{2}{1 + (1 + 2\alpha)X}, \]
\[ t'_2 = 3(1 + X), \quad a'_2 = \frac{1}{1 + X}, \]
and the new latency functions are
\[ f_1(x) = t'_1(1 + a'_1 x_1), \quad f_2(x) = t'_2(1 + a'_2 x_2). \]

We now compare the three scenarios discussed previously: 1) All commuters follow the first route; 2) Standard commuters follow both routes; 3) Standard commuters follow only the second route.

**Case 1.** All commuters follow the first route.
In this case, \[ x_1 = (1 - \alpha)X, \quad x_2 = 0, \] and optimal prices are given by \[ p^*_1 = 1 - 3X, \quad p^*_2 = 0. \]
Since the prices should be non-negative, we obtain the following expression for Inequality (1):
\[ X \leq \frac{1}{3}. \]
The profits of the two firms are \[ H^{(1)}_1 = (1 - 3X)(1 - \alpha)X \] and \[ H_2 = 0, \] respectively.

**Case 2.** Both routes are followed by standard commuters. By Equations (3)-(6), the optimal prices and flows in this case are
\[ p^*_1 = \frac{1}{3}(1 + 11X - 14\alpha X), \quad p^*_2 = \frac{1}{3}(-1 + 10X - 7\alpha X); \]
\[ x^*_1 = \frac{1}{63}(1 + 11X - 14\alpha X), \quad x^*_2 = \frac{1}{63}(-1 + 10X - 7\alpha X). \]
Note that the competitive case takes place only when the optimal values of \( p_1, p_2, x_1, x_2 \) are non-negative. This requirement is characterized by Inequality (7), now with the following form
\[ X \geq \frac{1}{10 - 7\alpha} \quad \text{if} \quad \alpha \leq \frac{11}{14}; \]
\[ \frac{1}{10 - 7\alpha} \leq X \leq \frac{1}{14\alpha - 11} \quad \text{if} \quad \alpha > \frac{11}{14}. \]
The firms’ profits are given by
\[ H^{(2)}_1 = \frac{1}{189}(1 + 11X - 14\alpha X)^2, \quad H^{(2)}_2 = \frac{1}{189}(-1 + 10X - 7\alpha X)^2. \]

**Case 3.** Standard commuters follow only the second route.
In the last case, oblivious commuters follow the first route and standard commuters follow the second route, which means \( x_1 = 0 \) and \( x_2 = (1 - \alpha)X \). The optimal prices are
\[ p^*_1 = 0, \quad p^*_2 = -1 + (7\alpha - 4)X. \]

Previous analyses in Section 3. require that Inequalities (8) and (9) must hold in this case:
\[ \alpha > \frac{4}{7}, \quad X \geq \frac{1}{7\alpha - 4}. \]
The firms’ profits are given by
\[ H^{(3)}_1 = 0, \quad H^{(3)}_2 = (-1 + (7\alpha - 4)X)(1 - \alpha)X. \]
**Equilibrium Characterization.** Now we can solve for firms’ optimal strategies by comparing their profits in these scenarios.

We first derive the necessary and sufficient condition for Scenario 1 to be an equilibrium outcome. The first scenario becomes an equilibrium outcome if (i) Condition (1) $X \leq \frac{2}{3}$ holds and (ii) whenever the second scenario is possible Condition (10) $H_1^{(1)} \geq H_1^{(2)}$ holds. In our example $H_1^{(1)} \geq H_1^{(2)}$ has the following specification:

$$(1 - 3X)(1 - \alpha)X \geq \frac{1}{189}(1 + 11X - 14\alpha X)^2.$$ 

The above inequality is equivalent to condition $X_1 \leq X \leq X_2$ where

$$X_1 = \frac{167 - 161\alpha - 21\sqrt{57}(1 - \alpha)}{2(688 - 875\alpha + 196\alpha^2)}, \quad X_2 = \frac{167 - 161\alpha + 21\sqrt{57}(1 - \alpha)}{2(688 - 875\alpha + 196\alpha^2)}.$$

Note that these values $X_1(\alpha), X_2(\alpha)$ are increasing functions of $\alpha$, tending to $1/3$ when $\alpha \to 1$. This implies that $X_2 \leq \frac{2}{3}$ for $\alpha \in [0, 1]$.

Recall that the second scenario takes place only if Condition (7) holds. That is, either $X \geq \frac{1}{10 - 7\alpha}$ and $\alpha \leq \frac{11}{14}$, or $\frac{1}{10 - 7\alpha} \leq X \leq \frac{1}{14\alpha - 11}$ and $\alpha > \frac{11}{14}$. Thus, for any flow $X$ less than $\frac{1}{10 - 7\alpha}$ the second scenario is impossible, which implies that the first scenario with $X < \frac{1}{10 - 7\alpha}$ is an equilibrium outcome since $\frac{1}{10 - 7\alpha} \leq \frac{2}{3}$, suggesting Condition (1) always holds in this case. Also, we have $X_2 \leq \frac{2}{3} \leq \frac{1}{14\alpha - 11}$, so the upper bound of $X$ for the first scenario to be an equilibrium outcome should be $X_2$. Lastly, notice that $X_1 \leq \frac{1}{10 - 7\alpha} \leq \frac{2}{3}$ for $\alpha \in [0, 1]$, so $X_1$ is irrelevant in determining the equilibrium condition for the first scenario.

Based on the above results, we obtain the condition under which the first scenario is optimal in equilibrium:

$$0 \leq X \leq X_2 = \frac{167 - 161\alpha + 21\sqrt{57}(1 - \alpha)}{2(688 - 875\alpha + 196\alpha^2)}.$$ 

\[ (12) \]

**Figure 2.** The regions of optimal behavior of commuters and firms

The area in which Condition (12) holds is filled with vertical blue lines and marked as “Case 1” in Figure 2, where the horizontal axis measures the fraction
of oblivious commuters $\alpha$ and the vertical axis measures the total traffic flow $X$. The cutoff condition specifying the border of equilibrium Cases 1 and 2, namely $X = X_2 = \frac{167 - 161\alpha + 21\sqrt{77} (1 - \alpha)}{2(688 - 875\alpha + 196\alpha^2)}$, is denoted **Condition 1-2.**

Next, we derive the necessary and sufficient condition for **Scenario 3** to be an equilibrium outcome. The third scenario becomes an equilibrium outcome if (i) both Condition (8) $\alpha > \frac{7}{4}$ and Condition (9) $X \geq \frac{1}{7\alpha - 4}$ hold, and (ii) whenever the second scenario is possible, Condition (11) $H_2^{(3)} \geq H_2^{(2)}$ holds. In our example $H_2^{(3)} \geq H_2^{(2)}$ has the following specification:

$$(-1 + (7\alpha - 4)X)(1 - \alpha)X \geq \frac{1}{189}(-1 + 10X - 7\alpha X)^2.$$  

The above inequality leads to the following sensible condition:

$$X \geq X_3 \quad \text{if} \quad 0.635 \approx \frac{317 - 9\sqrt{57}}{392} < \alpha < \frac{317 + 9\sqrt{57}}{392} \approx 0.982;$$

$$X_3 \leq X \leq X_4 \quad \text{if} \quad 0.982 \approx \frac{317 + 9\sqrt{57}}{392} \leq \alpha \leq 1,$$

where

$$X_3 = \frac{-169 + 175\alpha - 21\sqrt{57}(1 - \alpha)}{2(856 - 2219\alpha + 1372\alpha^2)}, \quad X_4 = \frac{-169 + 175\alpha + 21\sqrt{57}(1 - \alpha)}{2(856 - 2219\alpha + 1372\alpha^2)}.$$

Recall that the second scenario takes place only if either $X \geq \frac{1}{10 - 7\alpha}$ and $\alpha \leq \frac{11}{14}$, or $\frac{1}{10 - 7\alpha} \leq X \leq \frac{1}{14\alpha - 11}$ and $\alpha > \frac{11}{14}$. Thus, for any flow $X$ greater than $\frac{1}{14\alpha - 11}$ for $\alpha > \frac{11}{14}$, the second scenario is impossible, which implies that the third scenario with $X > \frac{1}{14\alpha - 11}$ and $\alpha > \frac{11}{14}$ is an equilibrium outcome, since $\frac{1}{14\alpha - 11} \leq \frac{1}{7\alpha - 4}$ and $\frac{11}{14} > \frac{7}{4}$, suggesting Conditions (8) and (9) always hold in this case. Also, we have $X_3 \geq \frac{1}{10 - 7\alpha} \geq \frac{1}{14\alpha - 11}$ for $\alpha > \frac{4}{7}$, so the lower bound of $X$ for the third scenario to be an equilibrium outcome should be $X_3$. Lastly, notice that $X_4 \geq \frac{175\alpha - 9\sqrt{57}}{392}$ for $\alpha > \frac{317 + 9\sqrt{57}}{392} > \frac{11}{14}$, so $X_4$ is irrelevant in determining the equilibrium condition for the third scenario.

Based on the above results, we obtain the condition under which the third scenario is optimal in equilibrium:

$$X \geq X_3 = \frac{-169 + 175\alpha - 21\sqrt{57}(1 - \alpha)}{2(856 - 2219\alpha + 1372\alpha^2)} \quad \text{and} \quad \alpha > \frac{317 - 9\sqrt{57}}{392} \approx 0.635. \quad (13)$$

The area in which **Condition (13)** holds is filled with horizontal red lines and marked as “Case 3” in Figure 2. The cutoff condition specifying the border of equilibrium scenarios 2 and 3, namely $X = X_3 = \frac{-169 + 175\alpha - 21\sqrt{57}(1 - \alpha)}{2(856 - 2219\alpha + 1372\alpha^2)}$, is denoted **Condition 2-3.** The area where neither Condition (12) nor Condition (13) holds is left blank and marked as “Case 2”, which represents the competitive scenario where standard commuters use both routes.

It can also be seen from Figure 2 that the area below the dashed line $X = \frac{1}{5}$ indicates where Condition (1) holds, the area above the dashed curve $\frac{1}{10 - 7\alpha}$ and below the dashed curve $\frac{1}{14\alpha - 11}$ indicates where Condition (7) holds, and the area above the dashed curve $\frac{1}{7\alpha - 4}$ with $\alpha > \frac{7}{4}$ indicates where Conditions (8) and (9) hold.
5. Conclusion. In this paper we study a transportation system with competing firms, and commuters that vary in terms of their strategic considerations in the congestion problem. The model setup is a reasonable representation of many actual transportation markets in the sense that firms can be reasonably expected to strategically maximize profit, while only some commuters in the population are fully strategic in their transport choices. That is, a subset of commuters may not be expected to be behaving strategically in their transport route choice, due to lack of familiarity with the route, or special permissions given by the government. The absence of strategic considerations may often coincide with a substantially discounted price provided to those commuters compared with the standard strategic commuters, coinciding with our model setup.

In such a model, we show that three scenarios (pooling, partial pooling, and separating) can exist in equilibrium, depending on the parameter values in the model. Our numerical example demonstrates that when overall traffic flows are sufficiently low, regardless of the fraction of oblivious commuters in the population, all traffic will flow through the shorter of the two possible routes. On the other hand, when overall traffic flows are sufficiently high, and the fraction of oblivious commuters is sufficiently high, commuters will separate across the two possible routes, based on their degree of sophistication. That is, oblivious commuters take the shorter route, and standard commuters take the longer route. In the third scenario, oblivious commuters follow the shorter route, while standard commuters travel through both routes. The result may be intuitive to observed scenarios in transportation systems, where a subset of commuters are price sensitive but not so time sensitive. In such situations, when traffic volume is high, those consumers may tend to opt for the route which appears in all aspects less cost incurring, although in fact the real cost when considering congestion factors may be quite high.

There are several potential directions for further development of this framework. Firstly, our analysis has considered only two types of commuters, oblivious and standard. Future work can consider commuters with varying intermediate degrees of obliviousness. Secondly, in the current framework, we have for simplicity modeled the oblivious commuters similarly to the concept of noise traders in the finance literature, in the sense that they do not incur nor pay attention to the monetary price of the transport route, or at least less so compared to the standard commuters. Future work can consider relaxing this assumption to allow oblivious commuters to be more closely incorporated into the pricing interaction in the transport system. Finally, it may be possible to gain an empirical assessment of how prevalent oblivious consumers are in the real world using survey methods, experimental techniques, or transport system electronic data similar to that used in Ford, Lien, Mazalov and Zheng (2019), for example. Such assessments can allow our model to have further policy implications in analyzing transportation systems by simulating the traffic flow consequences of heterogeneous consumer types.

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REFERENCES

[1] C. F. Camerer, T. H. Ho and J. K. Chong, A cognitive hierarchy model of games, *Quart. J. Econom.*, 119 (2004), 861–898.
[2] V. P. Crawford, Boundedly rational versus optimization-based models of strategic thinking and learning games, *J. Econom. Lit.*, 51 (2013), 512–527.
[3] X. Di, X. He, X. Guo and H. X. Liu, Braess paradox under the boundedly rational user equilibria, *Trans. Res. Part B*, 67 (2014), 86–108.
[4] X. Di and H. X. Liu, Boundedly rational route choice behavior: A review of models and methodologies, *Trans. Res. Part B*, 85 (2016), 142–179.
[5] W. Ford, J. W. Lien, V. V. Mazalov and J. Zheng, Riding to Wall Street: Determinants of commute time using Citibike, *Int. Journal of Logistics: Research and Applications*, 22 (2019), 473–490.
[6] R. Jou, D. A. Hensher, Y. Liu and C. Chi, Urban commuters’ mode-switching behaviour in Taipai, with an application of the bounded rationality principle, *Urban Studies*, 47 (2010), 650–665.
[7] G. Karakostas, N. Kim, A. Viglas and H. Xia, On the degradation of performance for traffic networks with oblivious users, *Trans. Res. Part B*, 45 (2011), 364–371.
[8] Z. Kuang, V. V. Mazalov, X. Tang and J. Zheng, Transportation network with externalities, *J. Comp. Appl. Math.*, (2020).
[9] Z. Kuang, Z. Lian, J. W. Lien and J. Zheng, Serial and parallel duopoly competition in two-part transportation routes, *Trans. Res. Part E*, 133 (2020), 101821.
[10] J. W. Lien, V. V. Mazalov, A. V. Melnik and J. Zheng, Wardrop equilibrium for networks with the BPR latency function, *Lecture Notes in Computer Science*, 9869 (2016), 37–49.
[11] J. L. Lien, H. Zhao and J. Zheng, Perception bias in Tullock contests, *Working Paper*, (2019).
[12] H. S. Mahmassani and G. Chang, On boundedly rational user equilibrium in transportation systems, *Trans. Sci.*, 21 (1987), 89–99.
[13] V. V. Mazalov and A. V. Melnik, Equilibrium prices and flows in the passenger traffic problem, *Int. Game Theory Rev.*, 18 (2016).
[14] C. Sun, L. Cheng and J. Ma, Travel time reliability with boundedly rational travelers, *Transportmetrica A: Transport Science*, 14 (2018), 210–229.
[15] T. Tang, X. Luo and K. Ma, Impacts of the driver’s bounded rationality on the traffic running cost under the car-following model, *Physica A*, 457 (2016), 316–321.
[16] J. Wardrop, Some theoretical aspects of road traffic research, *Proceedings of the Institution of Civil Engineers, Part II* (1952), 325–278.
[17] H. Ye and H. Yang, Rational Behavior adjustment process with boundedly rational user equilibrium, *Trans. Sci.*, 51 (2017), 968–980.
[18] C. Zhao and H. Huang, Experiment of bounded rational route choice behavior and the model under satisficing rule, *Trans. Res. Part C*, 68 (2016), 22–37.

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