A Note on Representations of N=2 SW-algebras

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Abstract

We investigate the representation theory of some recently constructed $N = 2$ super $\mathcal{W}$-algebras with two generators. Except for the central charges in the unitary minimal series of the $N = 2$ super Virasoro algebra we find no new rational models. However, from our results it is possible to arrange all known $N = 2$ super $\mathcal{W}$-algebras with two generators and vanishing self-coupling constant into four classes. For the algebras existing for $c \geq 3$ which can be understood by the spectral flow of the $N = 2$ super Virasoro algebra we find that the representations have quantized $U(1)$ charge.
1. Introduction

In the last few years the explicit construction of \( \mathcal{W} \)-algebras [1] has also reached the fast developing field of \( N = 2 \) supersymmetric conformal field theories (SCFTs) [2-7]. From a two dimensional quantum-gravity point of view \( N = 2 \) SCFTs are very important since on the one hand they describe the string world sheet CFT of \( N = 1 \) space-time supersymmetric heterotic string compactifications [8-10] and on the other hand the ghost sector of \( \mathcal{W} \)-gravity theories contains an \( N = 2 \) super \( \mathcal{W} \)-algebra [11]. In ref. [7] non-linear \( N = 2 \) super \( \mathcal{W} \)-algebras, namely extensions of the \( N = 2 \) super Virasoro algebra (\( N = 2 \) SVIR) by a pair of super-primary fields of opposite \( U(1) \)-charge, have been treated in a systematic way. The calculation of such objects is very involved but using a manifestly covariant approach one can simplify the problem tremendously. The extension of \( N = 2 \) SVIR by superprimaries of superconformal dimension \( \Delta_1, \ldots, \Delta_n \) is denoted by \( \mathcal{SW}(1, \Delta_1, \ldots, \Delta_n) \).

After presenting a further example, the \( \mathcal{SW}(1, 3) \) algebra with zero \( U(1) \)-charge and non-vanishing self-coupling constant, we investigate the representation theory of the former algebras focusing on the question whether these algebras admit rational models. To this end we evaluate Jacobi identities on the lowest weight states \( |h, q\rangle \) yielding restrictions on the allowed lowest weights \( (h, q) \) [12,13]. The results obtained by this method allow one to arrange all known \( N = 2 \) super \( \mathcal{W} \)-algebras with two generators and vanishing self-coupling constant into essentially four series. Only one series contains rational theories which can be explained by the classification of modular invariants of \( N = 2 \) SVIR [14,15]. In the case of \( N = 2 \) super \( \mathcal{W} \)-algebras with two generators there are many examples of algebras existing for \( c \geq 3 \). All these \( c \geq 3 \) theories have common features which can be explained by the existence of the spectral flow of \( N = 2 \) SVIR. Using this fact, we can describe these \( N = 2 \) super \( \mathcal{W} \)-algebras by a simple formula containing the well known algebras \( \mathcal{SW}(1, \frac{d}{2}) \) for \( Q = d, c = 3d \ (d \in \mathbb{N}) \) [16] as a subset. The latter algebras occur in the compactification of the heterotic string to \((10 - 2d)\) space-time dimensions [17,16].

2. Construction of \( N = 2 \) super \( \mathcal{W} \)-algebras: \( \mathcal{SW}(1,3) \)

In this section we use results of [18,19] about holomorphic \( N = 2 \) superconformal field theories, and in particular we will use the notation and formulae of section two of ref. [7]. We apply the formalism of [7] to construct the supplementary example \( \mathcal{SW}(1,3) \) with zero \( U(1) \)-charge and non-vanishing self-coupling constant. Firstly, we have to write down all super quasiprimary fields which can occur in the OPE \( \Phi_0(Z_1) \Phi_0^D(Z_2) \). We present only the fields of dimension 5 and \( \frac{11}{2} \). The fields with lower dimension have been presented in [7].

| \( \Delta \) | \( Q \) | quasi-primary fields |
|---|---|---|
| 5 | 0 | \( N_s(N_s(N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})\mathcal{L}), N_s(N_s(N_s(\mathcal{L}[\mathcal{D}, D]\mathcal{L})\mathcal{L})\mathcal{L}), N_s(N_s(\mathcal{L}\partial^2\mathcal{L})\mathcal{L}) \\ N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{D}\mathcal{L}), N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{D}\mathcal{L}), N_s(\mathcal{L}[\mathcal{D}, D]\partial^2\mathcal{L}) \\ N_s(N_s(\Phi\mathcal{L})\mathcal{L}), N_s(N_s(\Phi\partial\mathcal{L}), N_s(\Phi[D, D]L) \\ N_s(N_s(N_s(\mathcal{L}\mathcal{D}\partial\mathcal{L})\mathcal{L})\mathcal{L}), N_s(N_s(N_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L} D\mathcal{L}), N_s(N_s(\mathcal{L}\partial^2\mathcal{L})D\mathcal{L}) \\ N_s(N_s(\mathcal{L}[\mathcal{D}, D]\mathcal{L})\mathcal{D}\mathcal{L}), N_s(N_s(\mathcal{L}[\mathcal{D}, D]\mathcal{L})\mathcal{D}\mathcal{L}), N_s(N_s(\mathcal{L}\partial^2\mathcal{L})\mathcal{D}\mathcal{L}) \\ N_s(N_s(N_s(\mathcal{L}[\mathcal{D}, D]L)\mathcal{L})\mathcal{D}\mathcal{L}), N_s(N_s(\Phi\mathcal{D}\mathcal{L}), N_s(\Phi[D, D]L) |

Table 1: Quasi-primary fields of dimension 5 and \( \frac{11}{2} \) in \( \mathcal{SW}(1,3) \)
For completeness we give also the Kac-determinants in the vacuum sector:

| $\Delta$ | $det(D_\Delta) \sim$ |
|----------|------------------|
| 5        | $(c - 2)(c - 1)^5c^3(c + 1)(c + 6)^3(c + 12)(2c - 3)^3(5c - 9)(7c + 18)(c^2 + 26c - 75)$ |
| $\frac{11}{2}$ | $(c - 1)^6c^4(c + 1)^2(c + 6)^4(2c - 3)^4(5c - 9)^2(7c - 18)^2$ |

Table 2: Kac-determinants

The next step is to determine the structure constants $C^k_{ij}$, $\alpha_{ijk}$ for all normal ordered products which are rational functions in $c$ and one self-coupling constant $C_3^3 \alpha$. We skip their presentation and continue with the result that the Jacobi identities yield the following expression for the coupling constant $C_3^3 \alpha$:

$$
(C_3^3 \alpha)^2 = \frac{49c^3(7c - 18)^2(c^2 + 26c - 75)(3c^4 + 62c^3 - 129c^2 - 9c + 54)}{3(3 - 2c)(c - 2)(c - 1)(c + 1)(c + 6)(c + 12)(5c - 9)(3c^2 - 37c + 60)}.
$$

(1)

All Jacobi identities are satisfied only if the central charge takes the following rational numbers:

$$
c \in \left\{ \frac{18}{7}, \frac{10}{3}, -\frac{9}{5} \right\}.
$$

(2)

The corresponding values of the self-coupling constant are

$$
(C_3^3 \alpha)^2 \in \left\{ 0, \frac{268960000}{88803}, -\frac{106815267}{88000} \right\}.
$$

(3)

We conclude this section with a very brief review of the results of the construction of $N = 2$ super $\mathcal{W}$-algebras with two generators with vanishing self-coupling constant [7]. Using MATHEMATICA™ and a special C-program [20] we constructed $\mathcal{SW}(1, \Delta)$ algebras for $\Delta \in \left\{ \frac{3}{2}, 2, \frac{5}{2}, 3 \right\}$. In the following table we present the central charges $c$ and $U(1)$-charges $Q$ for which these $\mathcal{W}$-algebras exist.

| $\Delta$ | $Q$ | $c$ |
|----------|-----|-----|
| $\frac{3}{2}$ | 0 | $\frac{9}{4}, \frac{3}{5}, -\frac{3}{2}$ |
| | $\pm 3$ | 9 |
| | $\pm 1$ | 3 |
| | $\pm 3$ | $\frac{15}{4}$ |
| 2 | 0 | $\frac{12}{5}, -3$ |
| | $\pm 4$ | 12, $-9, -21$ |
| | $\pm \frac{3}{2}$ | $\frac{15}{4}$ |
| | $\pm 3$ | $\frac{10}{3}$ |
| | $\pm \frac{3}{2}$ | $\frac{21}{4}$ |

Table 3: Central charges of $\mathcal{SW}(1, \Delta)$ algebras with vanishing self-coupling constant
3. Representations of $N = 2$ super $\mathcal{W}$-algebras

In this section we study the possible lowest weight representations in the Neveu-Schwarz (NS) sector of the algebras presented in table 3. To this end one evaluates Jacobi-identities on the lowest weight states yielding necessary conditions for the quantum numbers of the allowed representations. According to the results obtained this way, we propose an arrangement of $N = 2$ super $\mathcal{W}$-algebras with two generators and vanishing self-coupling constant into the following four classes:

1. $c = \frac{6\Delta}{2\Delta + 1}, \ Q = 0, \ 2\Delta \in \mathbb{N}$

These values of $c$ lie in the unitary minimal series $c(k) = \frac{3k}{k+2}, \ k \in \mathbb{N}$ of $N = 2$ SVIR with $k = 4\Delta$. The lowest weights in the NS sector are given by (see e.g. [15]):

$$h_{l,m} = \frac{l(l+2) - m^2}{4(k+2)}, \ q_m = \frac{m}{k+2}, \ l = 0, \ldots, k \quad \text{and} \quad m = -l, -l+2, \ldots, l. \quad (4)$$

The primary field corresponding to the lowest weight $(h_{k,0}, 0)$ is a $\mathbb{Z}_2$ simple current of conformal dimension $\Delta$ for $k = 4\Delta$. The classification of modular invariant partition functions [15] yields off-diagonal invariants for $k \equiv 0, 2 \pmod{4}$ which can be viewed as diagonal invariants of the extended model with symmetry algebra $\mathcal{SW}(1, \Delta)$. The explicit calculation of the lowest weights of the $\mathcal{SW}(1, \Delta)$ at $c(4\Delta)$ yields exactly those lowest weights which one expects from the off-diagonal invariant. We stress that these models are the only rational ones among the central charges given in table 3 above. The NS part of the non-diagonal modular invariant partition function for $k \equiv 2 \pmod{4}$ is given by [15]:

$$Z_{NS} = \sum_{l=0,2, \ldots, \frac{k}{2} - 2, m=-l, \ldots, -l+2, \ldots, l} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2 + \sum_{l=\frac{k}{2}+1, \ldots, k-2, k, m \in \mathbb{Z}} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2 + \sum_{l=\frac{k}{2}+1, \ldots, k-2, k, m > (k-2)} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2. \quad (5)$$

The $\chi_m^l$ are the characters of the irreducible lowest weight representations of $N = 2$ SVIR with lowest weight $(h_{l,m}, q_m)$. For $k \equiv 0 \pmod{4}$ it reads [15]:

$$Z_{NS} = \sum_{l=0,2, \ldots, \frac{k}{2} - 2, m=-l, \ldots, -l+2, \ldots, l} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2 + \sum_{m=-\frac{k}{2}, -\frac{k}{2} + 2, \ldots, \frac{k}{2}} 2|\chi_m^\frac{k}{2}|^2 + \sum_{l=\frac{k}{2}+1, \ldots, k-2, k, m \in \mathbb{Z}} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2 + \sum_{l=\frac{k}{2}+1, \ldots, k-2, k, m > (k-2)} \frac{1}{2} |\chi_m^l + \chi_m^{k-l}|^2. \quad (6)$$

From these expressions one can directly read off the lowest weights of $\mathcal{SW}(1, \Delta)$ in the NS-sector. The following picture shows the allowed lowest weights $(h, q)$ for the first member of the series ($k = 6$). A cross represents a representation module with non-degenerate lowest weight, i.e. the zero modes of bosonic components act trivial in the NS sector. The
rectangles denote a representation module with doubly degenerate lowest weight, i.e. one has $|h, -\frac{1}{2}\rangle \sim \psi_0 |h, \frac{1}{2}\rangle$.  

$$
\begin{array}{c|cccc}
\hline
& 1/8 & 3/8 & & \\hline
-1 & - & - & \times & \times \\
-3/4 & - & - & \times & \times \\
-1/2 & - & - & \times & \times \\
0 & - & - & \times & \times \\
1/2 & - & - & \times & \times \\
3/4 & - & - & \times & \times \\
1 & - & - & \times & \times \\
\hline
\end{array}
$$

The Ramond sector is determined by the spectral flow

$$
L'_n = L_n + \eta J_n + \frac{c}{6} \eta^2 \delta_{n,0}, \quad J'_n = J_n + \frac{c}{3} \eta \delta_{n,0}
$$

$$
\varphi'_r = \varphi_r + \eta Q, \quad \varphi \in \{G, \overline{G}, \text{components of } \Phi\}
$$

for $\eta = \pm \frac{1}{2}$. Note that in the Ramond sector the bosonic fields $\psi$ and $\overline{\psi}$ of dimension 2 carry half-integer modes.

(2a) $c = 3 \left(1 - \frac{2}{\Delta + 1}\right), Q = 0$ for $\Delta \in \mathbb{N} + \frac{1}{2}$

In particular, the values $c = \frac{3}{5} (\Delta = \frac{3}{2})$ and $c = \frac{2}{7} (\Delta = \frac{5}{2})$ are contained in this series. These models have infinitely many lowest weight representations with infinitely degenerate lowest weight. One obtains for the minimal lowest weight $h_{\min} = \frac{c - 3}{24}$, i.e. one has $c_{\text{eff}} = c - 24h_{\min} = 3$. However, there exist finitely many representations with finitely degenerate lowest weight. Note that the $N = 0 (c_{\text{eff}} = 1)$ and $N = 1 (c_{\text{eff}} = \frac{3}{2})$ analogues are rational theories (see e.g. [21-23]).

For $\Delta \in \mathbb{N}$ there are two other series with $c_{\text{eff}} = 3$.

(2b) $c = 3 \left(1 - 2\Delta\right), Q = 2\Delta$ and $c = 3 \left(1 - 4\Delta\right), Q = 2\Delta$ for $\Delta \in \mathbb{N}$

The space of representations looks similar to the case (2a).

---

1) The components $\phi, \psi, \overline{\psi}, \chi$ of a super field $\Phi$ are defined according to $\Phi(Z) = \phi(z) + \frac{1}{\sqrt{2}} (\theta\overline{\psi}(z) - \overline{\theta}\psi(z)) + \theta\theta\chi(z)$. 


As examples we present the $c = \frac{3}{5}$ ($\Delta = \frac{3}{2}$) and $c = -9$ ($\Delta = 2$) models:

$\text{SW}(1, 3/2)$

| $c$ | $Q = 0$ |
|-----|---------|
| $3/5$ | $\times$ |
| $1/2$ | $\times$ |
| $2/5$ | $\times$ |
| $3/10$ | $\times$ |

Explicitly we obtained the following representations for $\text{SW}(1, 3/2)$ and $c = 3/5$:

$$
\langle h, q\mid \psi_0^+ \psi_0^* | h, q \rangle = 0
\quad \langle h, q\mid \psi_0^+ \psi_0^* | h, q \rangle \sim (q - 3)^2(q - 1)^2
\quad \langle h, q\mid \phi_0^+ \phi_0^* | h, q \rangle \sim (q + 3)^2(q + 1)^2
$$

Table 4: $\text{SW}(1, 3/2)$ at $c = \frac{3}{5}, Q = 0$

The relations for $h = -\frac{1}{10}$ imply that for $(\pm q) \in \left\{\frac{1}{5}, \frac{2}{5}, \ldots, \frac{1}{10}\right\}$ the degeneracy of the lowest weight is partially removed but is nevertheless infinite. That is to say that e.g. the states $|-\frac{1}{10}, q\rangle$ and $|\frac{1}{10}, q\rangle$ are not in the same representation module as it is the case for $|-\frac{1}{10}, q\rangle$ and $|\frac{1}{10}, q \pm 1\rangle$ if $q$ is generic.

For $\text{SW}(1, 2)$ and $c = -9$ we obtained the following results:

$\text{SW}(1, 2)$

| $c = -9$ | $Q = 4$ |
|---------|---------|
| $|h, -2/3 - \phi_0^* | h, 2\rangle, h > 0$ |

$$
\langle h, q\mid \psi_0^+ \psi_0^* | h, q \rangle \sim \left(q^2 - 3q + 3\right)(q^2 - 4q + 4)
\quad \langle h, q\mid \phi_0^+ \phi_0^* | h, q \rangle \sim \left(q^2 + 3q + 3\right)(q^2 + 4q + 4)
$$

Table 5: $\text{SW}(1, 2)$ at $c = -9, Q = 4$

The implication of the relations for $h = -\frac{1}{2}$ is similar to the previous case. Note that the representations with positive $h$ are doubly degenerate.
(3) \( c = 3(1 - \Delta), Q = 0, \quad 2\Delta \in \mathbb{N} \)

These models are pathological because their spectrum is not bounded from below.

\[
\text{SW}(1, 3/2) \\
c = -3/2 \quad Q = 0 \\
\langle \frac{1}{4}, -\frac{1}{2} \rangle \sim \overline{\psi}_0 | \frac{1}{4}, +\frac{1}{2} \rangle
\]

To be explicit, from our calculations we could not exclude the following representations:

\[
\begin{array}{|c|c|}
\hline
(h, q) & \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = 0 \\
0, 0, (-\frac{1}{3}q^2 - \frac{1}{6}, q) & \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = 0, \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle \neq 0 \\
(\frac{1}{4}, +\frac{1}{2}) & \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle \neq 0, \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle = 0 \\
(\frac{1}{4}, -\frac{1}{2}) & \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle \neq 0, \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle = 0 \\
\hline
\end{array}
\]

Table 6: \( \text{SW}(1, 3/2) \) at \( c = -\frac{3}{2}, Q = 0 \)

(4) \( c \geq 3, Q \) rational

For these algebras there exists no known analogue in the theory of \( N = 0 \) and \( N = 1 \) \( \mathcal{W} \)-algebras with two generators (see e.g. [1] and references therein). All representations of these algebras have a similar structure. The theories are not rational and have the property that infinitely many representations with fixed \( U(1) \)-charge \( q \) exist, where \( q \) takes finitely many rational values. The representations with fixed \( h \) have a finitely degenerate lowest weight. Furthermore, in all cases the field \( \mathcal{N}_s(\Phi \mathcal{L}) \) vanishes identically.

For the subset \( c = 3d \ (d \in \mathbb{N}) \) it has been shown explicitly in [16] that the extension of the \( N = 2 \) SVIR by the local chiral primaries \((\Delta, Q) = (c/6, \pm c/3)\) exists. In [16] the null field \( \mathcal{N}_s(\Phi \mathcal{L}) \) has been used to find the unitary representations for \( c = 3d \). Using the methods described above we were able to reproduce these results without exploiting the null field:
The following tables show the concrete results in more detail. Note that we give only the unitary irreducible representations implying the condition $h \geq \frac{1}{2}|q|$.

| $(h, q)$       | $(h, -1), (h, 0), (h, +1)$ for $\frac{|q|}{2} \leq h < \infty$ | $\langle h, q|\psi_0^+\psi_0^-|h, q\rangle = 0$ |
|---------------|-------------------------------------------------|-------------------------------------------------|
| $(\frac{1}{2}, +1)$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ |
| $(\frac{1}{2}, -1)$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ |

Table 7: $\text{SW}(1, \frac{3}{2})$ at $c = 9, Q = 3$

| $(h, q)$       | $(h, 0), (h, \pm 1)$ for $\frac{|q|}{2} \leq h < \infty$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ |
|---------------|-------------------------------------------------|-------------------------------------------------|
| $(1, -2)$     | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle \neq 0$ |
| $(1, +2)$     | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle \neq 0$ | $\langle h, q|\phi_0^+\phi_0^-|h, q\rangle = 0$ |

Table 8: $\text{SW}(1, 2)$ at $c = 12, Q = 4$

For the new algebras with $c \geq 3$ we obtained a very similar structure for the representations. As for the $c = 3d$ algebras the theories are not rational and we also obtain a quantization of the $U(1)$–charge $q$. 

| $\text{SW}(1, 5/2)$ | $\text{SW}(1, 5/2)$ | $\text{SW}(1, 5/2)$ |
|---------------------|---------------------|---------------------|
| $c = 9/2$           | $c = 9/2$           | $c = 9/2$           |
| $Q = 2$             | $Q = 2$             | $Q = 2$             |
| $|\frac{1}{4}, -\frac{3}{4}\rangle \sim \psi_0^-|\frac{1}{4}, \frac{3}{4}\rangle$ | $|\frac{1}{4}, -\frac{3}{4}\rangle \sim \psi_0^-|\frac{1}{4}, \frac{3}{4}\rangle$ | $|\frac{1}{4}, -\frac{3}{4}\rangle \sim \psi_0^-|\frac{1}{4}, \frac{3}{4}\rangle$ |
| $|h, -\frac{1}{2}\rangle \sim \psi_0^-|\frac{1}{2}, \frac{1}{2}\rangle, h > \frac{1}{2}$ | $|h, -\frac{1}{2}\rangle \sim \psi_0^-|\frac{1}{2}, \frac{1}{2}\rangle, h > \frac{1}{2}$ | $|h, -\frac{1}{2}\rangle \sim \psi_0^-|\frac{1}{2}, \frac{1}{2}\rangle, h > \frac{1}{2}$ |
The explicit data are given in the following two tables:

| \( (h, q) \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle = \langle h, q | \psi_0 \psi_0^+ | h, q \rangle = 0 \) |
| --- | --- | --- |
| \( (\frac{1}{2}, \pm 1), (h, 0) \) for \( 0 \leq h < \infty \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = (4h - 1)^2, \langle h, q | \psi_0 \overline{\psi}_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = (4h - 1)^2, \langle h, q | \psi_0 \psi_0^+ | h, q \rangle = 0 \) |
| \( (h, \frac{1}{2}) \) for \( \frac{1}{4} \leq h < \infty \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = 0, \langle h, q | \psi_0 \overline{\psi}_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = 0, \langle h, q | \psi_0 \psi_0^+ | h, q \rangle = 0 \) |
| \( (h, -\frac{1}{2}) \) for \( \frac{1}{4} \leq h < \infty \) | \( \langle h, q | \psi_0 \overline{\psi}_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle = 0 \) |
| \( (\frac{3}{4}, \pm \frac{3}{2}) \) | \( \langle h, q | \psi_0 \overline{\psi}_0 | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = 0 \) |
| \( (\frac{3}{4}, -\frac{3}{2}) \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) |

Table 9: \( SW(1, \frac{3}{2}) \) at \( c = \frac{9}{2}, Q = 2 \)

| \( (h, q) \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = (m - 1)^2, \langle h, q | \psi_0 \psi_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) |
| --- | --- | --- |
| \( (h, 0) \) for \( 0 \leq h < \infty \) | \( \langle h, q | \overline{\psi}_0 \psi_0 | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) |
| \( (\frac{1}{6}, \pm \frac{1}{3}), (\frac{1}{3}, \pm \frac{1}{3}), (\frac{1}{3}, \pm \frac{1}{3}), (\frac{1}{3}, \pm 1) \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) | \( \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = \langle h, q | \overline{\psi}_0 \psi_0^+ | h, q \rangle = 0 \) |

Table 10: \( SW(1, \frac{3}{2}) \) at \( c = 3, Q = 1 \)

In the following we will argue that all models with \( c \geq 3 \) can be understood from the existence of the spectral flow in \( N = 2 \) superconformal theories (see eq. (7)).

For \( \eta \in \mathbb{Z} \) the Neveu-Schwarz sector flows to itself. Starting with the vacuum state \( (h, q) = (0, 0) \) and applying successively the spectral flow with \( \eta = \pm 1 \) one obtains the following tower of \( N = 2 \) Virasoro primary states

\[
\begin{align*}
    h_m &= \frac{c - 1}{6} m^2 - \frac{(m - 1)^2}{2}, \quad q_m = \pm \left( \frac{mc}{3} - m + 1 \right), \quad m \in \mathbb{N}.
\end{align*}
\] (8)

Note that this spectral flow does in general not map lowest weight states onto lowest weight states but these subtleties can be easily taken care of. For \( c = 3d \) it was shown in [16] that the vacuum representation of the extension of the \( N = 2 \) SVIR by the local chiral primaries \( (h, q) = (h_1 = c/6, q_1 = \pm c/3) \) is the direct sum of an infinite number of \( N = 2 \) Virasoro representations with lowest weights \( (h_m, q_m) \). This implies that the extension of the \( N = 2 \) SVIR by the two fields \( (h, q) = \langle h_m, \pm q_m \rangle \) also exists for \( c = 3d \) where the generators can be written as normal ordered products of the fundamental fields \( (h, \pm |q_1|) \). Furthermore, in all these theories the field \( \mathcal{N}_s(\Phi \mathcal{L}) \) vanishes.
Application of these arguments to $d = 1$ implies, for instance, that all $SW(1, \Delta)$ algebras with $\Delta = \mathbb{Z} + 1/2$ and $Q = 1$ exist for $c = 3$ being in agreement with our calculations. Assuming that the construction of [16] can be generalized to arbitrary rational values of $c \geq 3$, yielding generically a parafermionic fundamental algebra [24], we conjecture the $SW(1, \Delta)$ algebra to exist for

$$c = \frac{6}{m^2} \left( \Delta + \frac{(m - 1)^2}{2} \right), \quad q = \pm \frac{1}{m} (2\Delta - m + 1)$$

(9)

where $m \in \{1, \ldots, [\Delta + \frac{1}{2}]\}$.

We emphasize that the results of table 3 are in perfect agreement with this conjecture.

4. Conclusion

Investigating the representation theory of $N = 2$ $\mathcal{W}$-algebras with two generators and vanishing self-coupling constant we found that all known algebras can be arranged into four series. Only the algebras existing for central charges in the unitary minimal series of $N = 2$ SVIR yield rational models which can be explained by the non-diagonal modular invariants of $N = 2$ SVIR. The $\mathcal{W}$-algebras existing for $c \geq 3$ can be interpreted by the existence of the spectral flow for $N = 2$ supersymmetric CFTs. Every extension of the $N = 2$ super Virasoro algebra by the spectral flow operators implies a whole hierarchy of $\mathcal{W}$-algebras existing for the same central charge. From our results we expect that these algebras do not admit rational models. However, one may expect a $U(1)$-charge quantization of the representations of these algebras because this happened to be true in all cases considered so far. In order to find new rational $N = 2$ SCFTs one has to investigate $N = 2 \mathcal{W}$-algebras with more generators. We rephrase that the only rational models known so far lie in the unitary minimal series of $N = 2$ SVIR. However, the question if unitarity is a necessary condition for an $N = 2$ supersymmetric CFT to be rational lies beyond the scope of this letter but should be investigated in the near future.

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