GROWTH OF PERMUTATIONAL EXTENSIONS

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ABSTRACT. We study the geometry of a class of group extensions, containing permutational wreath products, which we call “permutational extensions”. We construct for all $k \in \mathbb{N}$ a torsion group $K_k$ with growth function
\[ v_{K_k}(n) \sim \exp(n^{1-(1-\alpha)k}), \]
and a torsion-free group $H_k$ with growth function
\[ v_{H_k}(n) \sim \exp(\log(n)n^{1-(1-\alpha)k}). \]
These are the first examples of groups of intermediate growth for which the asymptotics of their growth function is known.

We construct a group of intermediate growth that contains the group of finitely supported permutations of a countable set as a subgroup. This gives the first example of a group of intermediate growth containing an infinite simple group as a subgroup.

1. Introduction

Let $G$ be a finitely generated group. One of its fundamental invariants is the group’s growth, studied since the 1950’s [26,35]: choose a finite set $S$ generating $G$ as a monoid, and define its growth function
\[ v_{G,S}(n) = \# \{ g \in G \mid g = s_1 \ldots s_m \text{ for some } m \leq n \text{ and } s_i \in S \}. \]
This function depends on $S$, but only mildly: given two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}_+$, we say that $g$ is asymptotically smaller than $f$, and write $g \preceq f$, if there exist $C > 0$ such $g(n) \leq f(Cn)$ for all sufficiently large $n$. We say that $f$ is asymptotically equivalent to $g$, and write $f \sim g$, if $f \preceq g$ and $g \preceq f$. The asymptotic equivalence class of $v_{G,S}$ is then independent of $S$, and is written $v_G$.

More information on growth of groups may be found in [22, Chapters VI–VIII] and in the monograph [30].

By a famous theorem of Gromov [21], the growth function is at most polynomial if and only if $G$ is virtually nilpotent. Furthermore, in that case, there exists an integer $d$ such that $v_G \sim n^d$; this fact is usually attributed to Guivarch and to Bass [8], see also [22, VII.26].

On the other hand, if $G$ is linear, or solvable, or word-hyperbolic, then, unless $G$ is virtually nilpotent, $v_G \sim \exp(n)$, so all such groups have asymptotically equivalent growth.

Milnor asked in [31] whether $v_G$ could take values strictly between polynomial and exponential functions; such groups would be called groups of intermediate growth. This question was answered positively by Grigorchuk in 1983 [16,17,19].

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Grigorchuk constructed a large class of groups of intermediate growth, showing in particular that for any subexponential function \( v(n) \) there exists a group of intermediate growth whose growth function is greater than \( v(n) \) for infinitely many \( n \). Essentially the same argument shows that for any subexponential \( v(n) \) there exists a group which is a direct sum of two Grigorchuk groups such that \( v_{G,S}(n) \geq v(n) \) for all sufficiently large \( n \) [13]. Another result of Grigorchuk is the existence of groups with incomparable growth functions.

There was, up to now, no group of intermediate growth for which the asymptotics of its growth function is known. On the other hand, it was known since the early 1970’s that semigroups can have intermediate growth, and some growth functions were computed explicitly: Govorov shows in [14] that the semigroup

\[
\langle x, y \mid x^i y^{-1} x^i = y^i x^{-1} y^i = 0 \quad \forall i \geq 2 \rangle_+
\]

has growth function \( \sim \exp(\sqrt{n}) \). The precise asymptotics of the growth of more semigroups (including \( 2 \times 2 \) matrix semigroups, and relatively free semigroups) were computed by Lavrik [27], Shmeerson [31], Okninski [33], and Reznikov and Sushchanskii [7]. The following growth types occur among these examples: \( n^{\log n} \), \( \exp(\sqrt{n}) \), and \( \exp(\sqrt{n/\log n}) \).

The most studied example of group of intermediate growth is the first Grigorchuk group \( G_{012} \). The best known estimates for this group’s growth are as follows: let \( \eta \) be the real root of the polynomial \( X^3 + X^2 + X - 2 \), and set \( \alpha = \log(2)/\log(2\eta) \approx 0.7674 \). Then

\[
\exp(n^{0.5153}) \lesssim v_{G_{012}} \lesssim \exp(n^{\alpha}).
\]

For the lower bound, see [2] and Leonov [28]; for the upper bound, see [1].

1.1. Main results. Up to now no growth function \( \not\sim n^d, \exp(n) \) had been determined. In this paper, we construct the first examples of groups of intermediate growth for which the asymptotics of the growth function is known:

**Theorem A.**

(1) For every \( k \in \mathbb{N} \) there exists a finitely generated torsion group \( K_k \) with growth

\[
v_{K_k} \sim \exp(n^{1-(1-\alpha)^k}).
\]

(2) For every \( k \in \mathbb{N} \) there exists a finitely generated torsion-free group \( H_k \) with growth

\[
v_{H_k} \sim \exp(\log(n)n^{1-(1-\alpha)^k}).
\]

In fact, the group \( H_1 \) that we consider had already been studied by Grigorchuk in [17], who showed that it is torsion-free and has subexponential growth, though he did not compute its growth function.

A classical result due to Higman, Neumann and Neumann states that any countable group can be embedded into a finitely generated group [23]. Given a countable group, it is natural to ask which extra properties that finitely generated group may possess; and, in the case that interests us, when that finitely generated group may be taken of subexponential growth. We show:

**Theorem B (= Theorem 6.1).** The group of finitely supported permutations of \( \mathbb{Z} \) embeds as a normal subgroup in a group of intermediate growth.
The theorem shows, in particular, that some groups of intermediate growth contain as a subgroup an infinite simple group. In our examples such infinite simple subgroups are not finitely generated. An open question due to Grigorchuk asks whether any infinite finitely generated simple group has exponential growth. All known example of finitely generated infinite simple groups are non-amenable.

**Question.** Does every countable group locally of subexponential growth embed in a finitely generated group of subexponential growth?

1.2. **Outline of the approach.** Our strategy may be summarized as follows. We consider groups $B, G$ with respective generating sets $R, S$, such that: $S$ is finite; $G$ acts on $B$; the action of $G$ on $B$ permutes $R$; and $R$ has finitely many $G$-orbits. Typically, $G$ will act on a set $X$, and $B$ will be either: a direct sum $\sum X A$ of a finitely generated group; a group of permutations of $X$; or a free Abelian, nilpotent etc. group generated by a set in bijection with $X$. We then take the semidirect product $W$ of $B$ with $G$. It turns out that the growth of $W$ is well controlled by the growth of “inverted orbits” of $G$ on $X$. By definition, the inverted orbit of a word $w = g_1 \ldots g_n$ over $G$ is $\{x_0 g_1 \ldots g_n, x_0 g_2 \ldots g_n, \ldots, x_0 g_n\}$ for a basepoint $x_0 \in X$.

The idea behind the fact that groups of Grigorchuk have intermediate growth is a certain contraction property. For some of Grigorchuk’s groups $G$, including the first one, this property states that there exist an injective homomorphism $\psi = (\psi_1, \ldots, \psi_d)$ from a finite index subgroup $H$ in $G$ to the direct product $G^d$, such that for some finite generating set $S$ in $G$, some $\eta < 1$, some $C > 0$, and some proper norm $| \cdot |$ on $G$, the following holds for all $h \in H$:

\[
\sum_{i=1}^{d} |\psi_i(h)| \leq \eta |h| + C. \tag{2}
\]

This implies that the growth of $G$ is bounded from above by $\exp(n^\beta)$ for a constant $\beta < 1$ depending only on $\eta$.

In this paper we introduce and study another contraction property, related to the sublinear growth of the inverted orbits for a group action. We prove that this property holds for the action on the boundary of many Grigorchuk groups, including the first Grigorchuk group. The property implies that not only these groups, but a large family of extensions of these groups have subexponential growth. We mention, however, that for some Grigorchuk groups the inverted orbit growth is linear (see Example 5.4).

The contracting property (2) in the case $d = 1$, deserves special mention; the group $G$ then admits a *dilatation*. This implies that the growth of $G$ is polynomial. Many nilpotent groups admit a dilatation. However, no nilpotent group may satisfy the “sublinear inverted orbit growth” property studied in this paper (see Remark 5.6).

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2. **Permutational wreath products and extensions**

We consider groups $A, G$ and a set $X$, such that $G$ acts on $X$ from the right. The *wreath product* $W = A \wr_X G$ is the semidirect product of $\sum X A$ with $G$. The
support \( \text{sup} f \) of a function \( f : X \to A \) consists of those \( x \in X \) such that \( f(x) \neq 1 \). We describe elements of \( \sum_X A \) as finitely supported functions \( X \to A \). The (left) action of \( G \) on \( \sum_X A \) is then defined by \((g \cdot f)(x) = f(xg)\); observe that for \( g_1, g_2 \) in \( G \)
\[
(g_1 g_2 \cdot f)(y) = f((g_1 g_2) y) = f(g_1 g_2) \cdot f(y) = (g_1 \cdot f)(g_2 \cdot f)(y) = (g_1 \cdot g_2 \cdot f)(y).
\]
We have in particular \( \text{sup}(g^{-1} \cdot f) = \text{sup}(f)g \). If \( A \) acts on a set \( Y \) from the right, then \( W \) naturally acts on \( X \times Y \) from the right, by \((x, y)f = (x, yf(x))\) and \((x, y)g = (xg, y)\).

Suppose now that \( A \) and \( G \) are finitely generated, and that the action of \( G \) on \( X \) is transitive. Fix generating sets \( S_A \) and \( S_G \) of \( A \) and \( G \) respectively, and fix a basepoint \( x_0 \in X \). The wreath product is generated by \( S = S_A \cup S_G \), in which we identify \( G \) with its image under the embedding \( g \to (1, g) \), and identify \( A \) with its image under the embedding \( a \to (f_a, 1) \); here \( f_a : X \to A \) is defined by \( f_a(x_0) = a \) and \( f(x) = 1 \) for all \( x \neq x_0 \). We call \( S \) the standard generating set of \( W \) defined by \( S_A, S_G \). Analogously, if the action of \( G \) on \( X \) has finitely many orbits, then \( W \) is finitely generated by \( S_G \cup (S_A \times (X/G)) \).

### 2.1. The Cayley graph of a permutational wreath product

The Cayley graph of \( W = (\sum_X A) \rtimes G \) with respect to the generating set \( S \) may be described as follows. Elements of \( W \) are written \( fg \) with \( f \in \sum_X A \) and \( g \in G \); multiplication is given by \((f_1 g_1)(f_2 g_2) = f_1(g_1 \cdot f_2) g_1 g_2\).

Consider a word \( v = s_1 s_2 \ldots s_\ell \), with all \( s_i \in S \), and write its value in \( W \) as \( f_v g_v \) with \( f \in \sum_X A \) and \( g \in G \). Set \( u = f_v g_v = s_1 s_2 \ldots s_\ell \). Here \( u, v \) belong to \( G \), and \( f_u, f_v : X \to A \).

We consider two cases, depending on whether \( s_\ell \in S_A \) or \( s_\ell \in S_G \). If \( s_\ell \in S_A \), we have an edge of “A” type from \( u \) to \( v \). The multiplication formula gives \( g_v = g_u \) and \( f_v(x) = f_u(x) \) for all \( x \neq x_0 g_u^{-1} \), while \( f_v(x_0 g_u^{-1}) = f_u(x_0 g_u^{-1}) s_\ell \). If \( s_\ell \in S_G \), we have an edge of “G” type from \( u \) to \( v \). Then \( f_v = f_u \), and \( g_v = g_u s_\ell \).

There is an alternative description of the edges in the Cayley graph, which appears if we write elements of \( (\sum_X A) \rtimes G \) in the form \( gf \) with \( g \in G \) and \( f \in \sum_X A \). Their product is then given by \((g_1 f_1)(g_2 f_2) = g_1 g_2 (g_2^{-1} \cdot f_1) f_2\).

In that notation, if there is an edge of “A” type from \( u = g_0 f_u \) to \( v = g_0 f_v \), then we have \( g_v = g_u \) and \( f_v = f_u \) except at \( x_0 \) where \( f_v(x_0) = f_u(x_0) s_\ell \). On the other hand, if there is an edge of “G” type from \( u \) to \( v \), then we have \( g_v = g_u s_\ell \) and \( f_v = g^{-1} f_u \).

For \( i = 1, \ldots, \ell \), set now \( g_i = s_i, a_i = 1 \) whenever \( s_i \in S_G \), and \( g_i = 1, a_i = s_i \) whenever \( s_i \in S_A \). Still writing \( v = g_v f_v \), we then have
\[
v = a_1 g_1 a_2 \ldots a_\ell g_\ell = a_1 \ldots a_\ell a_\ell^{-1} g_\ell \ldots a_2^{-1} g_2 \ldots a_1^{-1} g_1,
\]
so \( g_v = g_1 g_2 \ldots g_\ell \) and \( f_v = ((g_1 \cdot g_\ell^{-1} \cdot f_1)(g_2 \cdot g_\ell^{-1} \cdot f_2) \ldots (g_\ell^{-1} \cdot f_\ell)) \). We observe that the support of \( f_v \) is contained in \( \{x_0 g_\ell^{-1}, x_0 g_\ell^{-1} g_\ell, \ldots, x_0 g_1 g_2 \ldots g_\ell \} \). In other words, in order to understand the support of the configuration, we have to study inverted orbits of the action of \( G \) on \( X \) and the number of distinct points visited by these orbits.

**Remark 2.1.** In case \( G = X \), there is no difference between counting the number of points on the orbits or on the inverted orbits \((x_0 = 1, G \text{ acts on } X \text{ both from the right and from the left, and the inverted orbits for the right action are usual orbits for the left action). This is no longer the case if } X \neq G \).
Remark 2.2. We might wonder to which degree the geometry of the Cayley graphs of \( A \) and \( G \), and of the Schreier graph of \( X \) (the graph with vertex set \( X \) and an edge from \( x \) to \( xs \) for all \( x \in X \) and generator \( s \) of \( G \)), determine the geometry of the wreath product.

In contrast with the case \( X = G \) ("usual" wreath products), the Cayley graph of the permutational wreath product is in no way defined by the unmarked Cayley graphs of \( A \) and \( G \) and the Schreier graph of \( X \). We will see in the sequel that the following may happen: a group \( G \) acts on \( X_1 \) and \( X_2 \), the unmarked Schreier graphs of \( X_1 \) and \( X_2 \) are the same, but \( A \wr X_1 \) has exponential growth (for some finite group \( A \)), and \( A \wr X_2 \) has intermediate growth (see Example 5.5).

2.2. Inverted orbits. We formalize the discussion above as follows. Fix a group \( G \) acting on the right on a set \( X \); fix a set \( S \) generating \( G \) as a monoid; and fix a basepoint \( x_0 \in X \). Denote by \( S^* \) the set of words over \( S \). For a word \( w = w_1 \ldots w_\ell \in S^* \), its inverted orbit is

\[
O(w) = \{ x_0, x_0w_\ell, x_0w_{\ell-1}w_\ell, \ldots, x_0w_1w_2 \ldots w_\ell \}.
\]

Its inverted orbit growth is

\[
\delta(w) = |O(w)|.
\]

The inverted orbit growth function of \( G \) is the function

\[
\Delta(n) = \max\{|\delta(w)| \mid n = |w|\}.
\]

Clearly \( \Delta(n) \leq n + 1 \); and, if the orbit of \( x_0 \) is infinite, \( \Delta(n(n-1)/2) \geq n \), so \( \Delta(n) \geq \sqrt{n} \). Indeed, consider a word \( w = u_n \ldots u_1 \) in which \( u_i \) is a word of length \( \leq i \), chosen such that \( x_0u_i \not\in \{ x_0, x_0u_{i-1}, x_0u_{i-2}u_{i-1}, \ldots, x_0u_1 \ldots u_{i-1} \} \); then \( \delta(w) \geq n + 1 \).

The functions \( \delta \) and \( \Delta \) depend on the choice of \( x_0 \) and \( S \). However, it is easy to see that their asymptotics do not depend on the choice of the basepoint \( x_0 \) and the generating set \( S \):

Lemma 2.3. If \( G \) is finitely generated, then the \( \sim \)-equivalence class of \( \Delta \) does not depend on the choice of \( S \).

If \( G \) acts transitively on \( X \), then the \( \sim \)-equivalence class of \( \Delta \) does not depend on the choice of \( x_0 \).

Proof. Let \( S \) and \( S' \) be two finite generating sets for \( G \); write each element of \( S \) as a word over \( S' \); and let \( C \) be the maximum of such lengths. We temporarily write \( \Delta_S \) and \( \Delta_{x_0,S} \) to remember the dependence on the choices of \( x_0 \in X \) and \( S \subset G \).

Given \( w \in S^* \), let \( w' \) be the corresponding rewritten word over \( S' \). We have \( |w'| \leq C|w| \) and \( \delta_S(w) \leq \delta_{S'}(w') \), so \( \Delta_S(n) \leq \Delta_{S'}(Cn) \) for all \( n \in \mathbb{N} \), and \( \Delta_S \preceq \Delta_{S'} \). The reverse inequality gives \( \Delta_S \sim \Delta_{S'} \), and proves the first part of the lemma.

Now consider two points \( x_0, x_1 \in X \), and an element \( g \in G \) with \( x_0g = x_1 \), of length \( k \). Set \( S' = \{ s^g \mid s \in S \} \). It is clear that \( S' \) is a generating set of \( G \). Let \( w = w_1w_2 \ldots w_\ell \) be a given word, and consider the word \( w^g = (g^{-1}w_1g)(g^{-1}w_2g) \cdots (g^{-1}w_\ell g) \) over the alphabet \( S' \). Given \( i, j \in \{0, \ldots , \ell \} \), observe that \( x_0w_1w_{i+1} \ldots w_\ell = x_0w_jw_{j+1} \ldots w_\ell \) if and only if \( x_0w_1 \ldots w_\ell g \neq x_0w_1w_2 \ldots w_\ell g \), and only if

\[
x_1(g^{-1}w_1g) \cdots g^{-1}w_\ell g \neq x_1(g^{-1}w_jg) \cdots g^{-1}w_\ell g.
\]

This implies \( \delta_{x_0,S}(w) = \delta_{x_1,S'}(w^g) \). Therefore, we have \( \Delta_{x_0,S}(n) = \Delta_{x_1,S'}(n) \); and the first part of the lemma gives \( \Delta_{x_0,S} \sim \Delta_{x_1,S} \). \( \square \)
3. Self-similar groups

Below we recall the definition of some of Grigorchuk’s groups. They are groups acting on a rooted tree. The first Grigorchuk group belongs to a smaller class of self-similar groups. We fix our notation for such groups; for more information on self-similar groups, see Nekrashevych’s book [32]. Fix an integer \( d \geq 2 \) called the degree. Words \( q = q_1 \ldots q_n \in \{0, \ldots, d-1\}^* \) form the vertex set of a rooted regular tree \( T \), with root the empty word; and \( q_1 \ldots q_n \) connected by an edge to \( q_1 \ldots q_{n-1} \).

A self-similar group is, by definition, a group presented by a map, called the wreath recursion, \( \psi : G \to G \wr \mathfrak{S}_d \), from \( G \) to its permutational wreath product with the symmetric group \( \mathfrak{S}_d \). We write images under \( \psi \) in the form

\[
\psi(g) = \langle \langle g_0, \ldots, g_{d-1} \rangle \rangle \pi \quad \text{with} \quad (g_0, \ldots, g_{d-1}) \in G^d \quad \text{and} \quad \pi \in \mathfrak{S}_d.
\]

The wreath recursion \( \psi \) defines an action of \( G \) by isometries on \( T \), as follows. Consider \( g \in G \) and \( q = q_1 \ldots q_n \in T \). If \( n = 0 \), then \( qg = q \). Otherwise, compute \( \psi(g) = \langle \langle g_0, \ldots, g_{d-1} \rangle \rangle \pi \), and set inductively \( qg = \langle q_1^{g_0}(q_2 \ldots q_n)g_{q_1} \rangle \).

When a self-similar group is given by its wreath recursion, it is assumed that the action on \( T \) is faithful; namely, the group \( G \) defined by the wreath recursion \( \psi : \Gamma \to \Gamma \wr \mathfrak{S}_d \) is the quotient \( G \) of \( \Gamma \) by the kernel of \( \Gamma \)'s action on \( T \). We then drop ‘\( \psi \)’ from the notation, and write the wreath recursion on \( G \) in the form

\[
g = \langle \langle g_0, \ldots, g_{d-1} \rangle \rangle \pi \quad \text{or} \quad g = \pi \langle \langle g_0, \ldots, g_{d-1} \rangle \rangle.
\]

The boundary \( \partial T \) of \( T \) consists of infinite sequences; its elements are called rays. If \( G \) is a self-similar group acting on \( T \), then \( G \) also acts on \( \partial T \). Mainly, the action of \( G \) we will be interested in is that on a ray orbit \( \rho G \).

3.1. The first Grigorchuk group. An important example of self-similar group was extensively studied by Grigorchuk [20]. It may be defined by its wreath recursion as the 4-generated group \( G_{012} \) with generators \( \{a, b, c, d\} \); if \( \varepsilon \) denote the transposition \((0,1)\),

\[
\psi : a \mapsto \langle \langle 1,1 \rangle \rangle \varepsilon, \quad b \mapsto \langle \langle a, c \rangle \rangle, \quad c \mapsto \langle \langle a, d \rangle \rangle, \quad d \mapsto \langle \langle 1, b \rangle \rangle
\]

Grigorchuk proved in [15] that \( G_{012} \) is an infinite, finitely generated torsion group; and, in [16], that \( G_{012} \) is a group of intermediate growth.

A presentation of \( G_{012} \) by generators and relators was given by Lysionok [29]: define the endomorphism \( \sigma \) of \( \{a, b, c, d\}^\ast \) by

\[
\sigma : a \mapsto aca, \quad b \mapsto d, \quad c \mapsto b, \quad d \mapsto c.
\]

Then

\[
G_{012} = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^n([d, d^8]), \sigma^n([d, d^{(ac)^2a}]) \text{ for all } n \in \mathbb{N} \rangle.
\]

The notation \( G_{012} \) arises as follows: Grigorchuk defined a continuum of groups \( G_\omega \), for \( \omega \in \{0,1,2\}^\mathbb{N} \). The first Grigorchuk group is in fact the group \( G_{(012)} \) defined by a periodic sequence \( \omega \).

3.2. Another Grigorchuk group. By \( G_{01} \) we denote in the sequel the following Grigorchuk group. It is the group generated by \( a, b, c, d \) and given by the recursion

\[
\psi : a \mapsto \langle \langle 1,1 \rangle \rangle \varepsilon, \quad b \mapsto \langle \langle 1, c \rangle \rangle, \quad c \mapsto \langle \langle a, b \rangle \rangle, \quad d \mapsto \langle \langle a, d \rangle \rangle
\]

In contrast with the first Grigorchuk group, \( G_{01} \) contains elements of infinite order. Indeed, the subgroup generated by \( \{a, d\} \) is isomorphic to the infinite dihedral group. Furthermore, the infinite-order element \( ad \) acts freely on the boundary \( \partial T \).
of the tree on which \( G_{01} \) acts \cite{Linn} proof of Lemma 9.10]. If we denote by \( \rho \) the ray \( 1^\infty \), we then have for all integers \( m \neq n \)

\[
\rho(ad)^n \neq \rho(ad)^m.
\]

The group \( G_{01} \) has intermediate growth \cite{Grigorchuk}; the best known lower and upper bounds are, for all \( \epsilon > 0 \),

\[
\exp(n/\log^{2+\epsilon}(n)) \lesssim v_{G_{01}}(n) \lesssim \exp(n/\log^{1-\epsilon}(n)).
\]

4. Inverted orbit growth for Grigorchuk’s first group

We fix \( \rho = 1^\infty \) the ray in the binary tree \( T \). This ray is fixed by \( b, c, d \). We write \( \Omega = \{a, b, c, d\}^* \) the set of words over the standard generators. The length of \( w \in \Omega \) is written \(|w|\). The recursion \cite{Grigorchuk} gives rise to a map \( \Omega \to \mathcal{S}_2 \times \Omega \times \Omega \), defined by the same formulas. We still write it in the form \( w \mapsto \varepsilon \langle \langle u, v \rangle \rangle \).

We call a word \( w \in \Omega \) pre-reduced, if it does not contain two consecutive occurrences of \( b, c, d \); the pre-reduction of \( w \) is the word obtained from \( w \) by deleting consecutive \( bb, cc, dd \) and replacing \( be \) or \( cb \) by \( d \), \( cd \) or \( dc \) by \( b \), and \( db \) or \( bd \) by \( c \). These operations do not change the image of the word in \( G_{012} \). Recall that, for a word \( w = w_1 \ldots w_n \in \Omega \), we defined

\[
\delta(w) = \#\{\rho w_{i+1} \ldots w_n | i = 0, \ldots, n\}.
\]

Lemma 4.1. \( \delta(w) = \delta(\text{its pre-reduction}) \).

Proof. Consider a subword \( w_j w_{j+1} \) of \( w \) consisting only of \( b, c, d \)'s, and let \( u \) denote the shorter word obtained by replacing \( w_j w_{j+1} \) by its value. In computing \( \delta(w) \), either \( i \leq j \) or \( i > j + 1 \), in which case \( \rho w_i \ldots w_n = \rho u_i \ldots u_{n-1} \); or \( i = j + 1 \), in which case \( \rho w_i \ldots w_n = \rho w_{j+2} \ldots w_n = \rho u_{i+1} \ldots u_{n-1} \), because \( w_j \) and \( w_{j+1} \) fix \( \rho \).

Let \( \eta \approx 0.811 \) be the real root of the polynomial \( X^3 + X^2 + X - 2 \), and consider on \( \Omega \) the norm defined by

\[
\|a\| = 1 - \eta^3, \|b\| = \eta^3, \|c\| = 1 - \eta^2, \|d\| = 1 - \eta;
\]

namely, for a word \( w = w_1 \ldots w_n \in \Omega \) set \( \|w\| = \|w_1\| + \cdots + \|w_n\| \). The norm induced on \( G_{012} \) by \( \|\cdot\| \) was considered by the first author in \cite{Growth}. As we have already mentioned, the first Grigorchuk group satisfies the contraction property in Equation \cite{Grigorchuk}. Here as a word metric in \( G_{012} \) one can consider the word metric with respect to the generating set \( a, b, c, d \). The idea of \cite{Grigorchuk} was that if instead of the word metric we consider the norm as defined above, this leads to a better contraction coefficient \( \eta \) in \cite{Grigorchuk} and a better upper bound of the form \( \exp(n^\alpha) \) for the growth of \( G_{012} \). In this paper we use this norm in order to get upper bounds on the growth for extensions of \( G_{012} \).

Note that the norm \( \|\cdot\| \) and the word length \(|\cdot|\) are equivalent. If \( w \) is pre-reduced of length \( n \), then it contains at least \( (n - 1)/2 \) times the letter ‘a’. We may therefore apply the argument in \cite{Grigorchuk} Proposition 4.2, which we reproduce here for completeness, with words in lieu of group elements:

Lemma 4.2 (see \cite{Grigorchuk} Proposition 4.2]). Consider \( w \in \Omega \) pre-reduced, and write \( w = \varepsilon \langle \langle u, v \rangle \rangle \) for \( u, v \in \Omega \) and \( s \in \{0, 1\} \). Define \( C = \eta \|a\| \). We then have

\[
\|u\| + \|v\| \leq \eta \|w\| + C.
\]
Lemma 4.3. Let \( \eta \) be defined in §2.2. We state the following general lemma:

\[ \Delta(n) = \max \{ \delta(w) \mid n \geq \|w\| \}; \]

This function is equivalent to that defined in §2.2. We state the following general lemma:

**Lemma 4.3.** Let \( \Delta : \mathbb{N} \to \mathbb{N} \) be a function. Let \( \eta \in (0,1) \) and \( C \) be such that, for all \( n \in \mathbb{N} \), there exists \( \ell, m \in \mathbb{N} \) with \( \ell + m \leq \eta n + C \) and \( \Delta(n) \leq \Delta(\ell) + \Delta(m) \). Set

\[ \alpha = \log(2)/\log(2/\eta). \]

Then we have for all \( n \in \mathbb{N} \)

\[ \Delta(n) \lesssim n^\alpha. \]

**Proof.** Define \( K = C/(2-\eta) \) and \( M = C/(1-\eta) \). We will prove, in fact, \( \Delta(n) \leq L(n-K)^\alpha \) for some constant \( L \) and all \( n \) large enough.

For that purpose, let \( L \) be large enough so that \( \Delta(n) \leq L(n-K)^\alpha \) for all \( n \leq M \). Set \( N = K/(1-\alpha) \), define \( \Delta^* \) by

\[ \Delta^*(n) = \begin{cases} L(n-K)^\alpha & \text{if } n \geq N, \\ 1 + (L(N-K)^\alpha - 1)n/N & \text{if } n \leq N, \end{cases} \]

and note that \( \Delta^* \) is the convex hull of 1 and \( L(n-K)^\alpha \); it is a monotone concave function satisfying \( \Delta(n) \leq \Delta^*(n) \) for all \( n \leq M \).

Consider now \( n > M \). We then have \( \ell, m < n \). By induction, we have \( \Delta(\ell) \leq \Delta^*(\ell) \) and \( \Delta(m) \leq \Delta^*(m) \); so, using concavity of \( \Delta^* \),

\[ \Delta(n) \leq \Delta^*(\ell) + \Delta^*(m) \leq 2\Delta^*(\frac{1}{2}(\ell + m)) \leq 2\Delta^*(\frac{n}{2}(n + \frac{c}{n})) \]

\[ = 2L\left(\frac{n}{2}(n + \frac{c}{n}) - K\right)^\alpha = 2L\left(\frac{n}{2}(n - K)\right)^\alpha = L(n-K)^\alpha = \Delta^*(n). \]

Therefore, \( \Delta(n) \lesssim \Delta^*(n) \sim (n-K)^\alpha \) for all \( n \in \mathbb{N} \).

**Proposition 4.4.** We have \( \delta(w) \lesssim \|w\|^\alpha \) for all \( w \in \Omega \). Equivalently, \( \Delta(n) \lesssim n^\alpha \).

**Proof.** We show that, for some \( C \) and all \( n \in \mathbb{N} \), there exists \( \ell, m \in \mathbb{N} \) with \( \ell + m \leq \eta n + C \) and \( \Delta(n) \leq \Delta(\ell) + \Delta(m) \); the claimed upper bound on \( \Delta \) will then follow by Lemma 4.3.

Consider a word \( w = w_1 \ldots w_n \) realizing the maximum in \( \Delta \), assumed without loss of generality to be pre-reduced. We will study the inverted orbit of \( w \) on \( X \). Write as above \( w = \varepsilon^a(u,v) \) with \( u = u_1 \ldots u_\ell \) and \( v = v_1 \ldots v_m \); then, as we will see, the inverted orbit of \( w \) is made of rays 0\( x \) for \( x \) in the inverted orbit of \( u \), and of rays 1\( x \) for \( x \) in the inverted orbit of \( v \). By Lemma 4.2, we have...
\[
\ell + m \leq \eta(n + \|a\|). \text{ A suffix } w' = w_{i+1} \ldots w_n \text{ has the form } w' = \varepsilon^s \langle \langle u', v' \rangle \rangle, \text{ in which } u', v' \text{ are respectively suffixes of } u, v. \text{ We have } \rho w' = 1\rho v' \text{ if } s' = 0, \text{ and } \rho w' = 0\rho v' \text{ if } s' = 1. \text{ Therefore, }
\]
\[
\Delta(n) = \#\{\rho w_{i+1} \ldots w_n \mid i = 0, \ldots, n\}
\leq \#\{0(\rho u_{j+1} \ldots u_i) \mid j = 0, \ldots, \ell\} + \#\{1(\rho v_j \ldots v_m) \mid j = 0, \ldots, m\}
\leq \Delta(\ell) + \Delta(m). \quad \square
\]

We now explore the range of \(\delta(w)\), for various words \(w \in \{a, b, c, d\}^n\). Let us insist that different words \(u, v \in \Omega\) which have the same value in \(G_{012}\) may very well have widely different \(\delta\)-values. The following result is not used in this text, but is included to stress the difference between direct and inverted orbit growth:

**Remark 4.5.** There exists for all \(n\) a word \(w \in \Omega\) of length \(2^n\), whose direct orbit on \(X\) has length \(2^n\), and such that \(\delta(w) \sim n\). In particular, \(w\) has length \(n\) in \(G_{012}\).

**Proof.** Consider again the substitution \(\sigma : \Omega \to \Omega\) given by (4), and the word \(w_n = \sigma^n(a(d))\). Let \(v\) be a suffix of \(w_n\).

Note that \(w_n = \langle \langle u, w_{n-1} \rangle \rangle\) for some word \(u\) over \(\{a, d\}\); therefore, the suffix \(v\) has the form \(\varepsilon^t \langle \langle u', v' \rangle \rangle\) for some \(t \in \{0, 1\}\) and suffixes \(u', v'\) of \(u, w_{n-1}\) respectively. Now either \(i = 1\), so \(pv = 0(\rho u') \in \{0\rho, 00\rho\}\), or \(i = 0\), and \(pv = 1(\rho v')\); by induction, \(pv\) can take at most \(2n\) values when \(v\) ranges over all suffixes of \(w_n\).

On the other hand, \(\rho w_n = 0^{n+1} \rho\) is at distance \(2^n - 1\) from \(\rho\), so the direct orbit of \(w_n\) traverses \(2^n\) points. \(\square\)

**Remark 4.6.** For comparison, let \(\delta'(w)\) denote the size of the direct orbit of \(w\); namely, \(\delta'(w) = \#\{\rho, \rho w_1, \rho w_1 w_2, \ldots, \rho w_1 \ldots w_n\}\) if \(w = w_1 \ldots w_n\). Then, for \(w = \langle \langle u, v \rangle \rangle\varepsilon^s\), we have the inequality \(\delta'(w) \leq 2\delta'(v)\), from which nothing can be deduced, instead of \(\delta(w) \leq \delta(u) + \delta(v)\). As soon as \(G\) is infinite, there is for all \(n \in \mathbb{N}\) a word of length \(n\) whose direct orbit visits \(n\) points.

We now show that the estimate in Proposition 4.4 is optimal:

**Proposition 4.7.** There exists a constant \(C\) such that, for all \(n \in \mathbb{N}\), there exists a word \(w_n \in \Omega\) of length at most \(C(2/\eta)^n\) with \(\delta(w) \geq 2^n\).

**Proof.** Write \(\Omega' = \{ab, ac, ad\}^* \subset \Omega\), consider the substitution \(\zeta : \Omega' \to \Omega'\) given by

\[
\zeta : ab \mapsto abadac, \quad ac \mapsto abab, \quad ad \mapsto acac,
\]
and consider the word \(w_n = \zeta^n(a(d))\). For example, we have \(w_0 = ad\), \(w_1 = acac\), \(w_2 = ababab\), \(w_3 = (abadac)^4\), . . .

Counting the number of occurrences of \(ab, ac, ad\) in the word \(w_n\), we get

\[
|w_n| = \left( \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right)^n = \left( \begin{array}{c} 1 \\ 2 \\ 0 \end{array} \right)^n;
\]

the characteristic polynomial of the \(3 \times 3\) matrix is \(X^3 - X^2 - 2X - 4\). This polynomial has a positive real root \(2/\eta\), and two conjugate complex roots of a smaller absolute value. Therefore, there exists a constant \(C\) such that the length of \(w_n\) is at most \(C(2/\eta)^n\) for all \(n\). (In fact, it is also \(\geq C'(2/\eta)^n\) for another constant \(C'\), because the polynomial is irreducible over \(\mathbb{Q}\)).
Consider \( w = \varepsilon s \langle \langle u, v \rangle \rangle \) and \( w' = \varepsilon s' \langle \langle u', v' \rangle \rangle \). We write \( w \approx w' \) to mean that \( s = s' \), that \( u, u' \) have the same pre-reduction, and that \( v, v' \) have the same pre-reduction. Under the wreath recursion, we have
\[
\zeta(ab) = \varepsilon \langle \langle aba, c1d \rangle \rangle = \varepsilon \langle \langle ab, b \rangle \rangle,
\quad \zeta(ac) = \langle \langle ca, ac \rangle \rangle,
\quad \zeta(ad) = \langle \langle da, ad \rangle \rangle.
\]
It follows that, for any word \( av \in \Omega' \), we have
\[
\zeta(av) \equiv \begin{cases} 
\varepsilon \langle \langle ava, av \rangle \rangle & \text{if } \zeta(av) \text{ contains an odd number of } a's,
\langle \langle va, av \rangle \rangle & \text{if } \zeta(av) \text{ contains an even number of } a's.
\end{cases}
\]
In particular, let us denote by \( w'_{n-1} := a^{-1}w_{n-1}a \) the word obtained from \( w_{n-1} \) by deleting its initial \( a \); then \( w_n \equiv \langle \langle w'_{n-1}, w_{n-1} \rangle \rangle \).

Let now \( u \) be a suffix of \( w_{n-1} \). There exists then a suffix \( v \) of \( w_n \) such that \( v \equiv \langle \langle *, w \rangle \rangle \). Similarly, let \( u' \) be a suffix of \( w'_{n-1} \). There exists then a suffix \( v' \) of \( w_n \) such that \( v' \equiv \varepsilon \langle \langle u', * \rangle \rangle \). Now \( \rho v = 1puu' \), and \( \rho v' = 0pu'a \); so \( \mathcal{O}(w_n) \supseteq 1\mathcal{O}(w_{n-1}) \cup 0\mathcal{O}(w'_{n-1}) \), and
\[
\delta(w_n) \geq \delta(w_{n-1}) + \delta(w'_{n-1}).
\]
A similar reasoning shows \( \delta(w'_{n-1}) \geq 2\delta(w'_{n-1}) \). Since \( \delta(w_0) = 2 \) and \( \delta(w'_0) = 1 \), we get
\[
\delta(w_n) \geq 2^n + 1,
\quad \delta(w'_n) \geq 2^n.
\]
(These inequalities are in fact equalities). \( \square \)

**Corollary 4.8.** There exist constants \( C_1, C_2 > 0 \) such that, for any \( \ell \in \mathbb{N} \),
\begin{enumerate}
\item for all words \( w \) of length \( \ell \), we have \( \delta(w) \leq C_1 \ell^\alpha \);
\item there exists a word \( w \) of length \( \ell \) with \( \delta(w) \geq C_2 \ell^\alpha \).
\end{enumerate}

We will also need to control the total number of possibilities \( \Sigma(n) \) for the inverted orbit of a word of length \( \leq n \). Since the inverted orbit has cardinality at most \( \sim n^\alpha \), and lies in the ball of radius \( n \) in the Schreier graph \( \rho G \), which contains at most \( 2n+1 \) points (see Figure[1]), we have \( \Sigma(n) \lesssim \left\lceil \frac{2n+1}{n^\alpha} \right\rceil \); but this estimate is too crude for our purposes. We improve it as follows:

**Lemma 4.9.** Set \( \Sigma(n) = \#\{\mathcal{O}(w) \mid n \geq |w|\} \). Then \( \Sigma(n) \lesssim \exp(n^\alpha) \).

**Proof.** Recalling that the norms \( \| \cdot \| \) and \( | \cdot | \) are equivalent, we consider \( w \) with \( |w| \leq n \), assumed without loss of generality to be pre-reduced (because the inverted orbit is invariant under pre-reduction). We write \( w = \varepsilon s \langle \langle u, v \rangle \rangle \) with \( |u| = \ell, |v| = m \), and recall that a suffix \( w' \) of \( w \) has the form \( w' = \varepsilon s' \langle \langle u', v' \rangle \rangle \) for suffixes \( u', v' \) of \( u, v \) respectively. As in Proposition[4.4] we then have \( \mathcal{O}(w) = \mathcal{O}(u) \cup \mathcal{O}(v) \varepsilon \); we get
\[
\Sigma(n) \leq \sum_{\ell + m \leq \eta(n + |u|)} \Sigma(\ell) \Sigma(m).
\]
We reuse the notation of Proposition[4.4] and take as Ansatz
\( \Sigma(n) \leq \exp(\Delta^*(n)) \eta/4n \),
for a function \( \Delta^* \) as in (8), with a large enough constant \( L \) that (9) holds whenever \( n \leq M \). Then, because \( \exp(\Delta^*(n))/n \) is log-concave, we get as before
\[
\Sigma(n) \leq \eta n (\exp(K(n^\alpha/2)\alpha) / (2n))^2 = \exp(Kn^\alpha) \eta/4n
\]
and (9) holds by induction for all \( n \in \mathbb{N} \). \( \square \)
5. Groups of intermediate growth

Recall that $G_{012}$ denotes the first Grigorchuk group, and that $X$ denotes the orbit of $\rho = 1^{\infty}$ under the right $G_{012}$-action on the (boundary of the) binary tree $T = \{0, 1\}^\ast$.

It is convenient, when considering groups of intermediate growth, to write their growth function in the form $\exp(n/\phi(n))$, for an unbounded function $\phi$.

**Lemma 5.1.** Let $A$ be a non-trivial group having growth $\sim \exp(n/\phi(n))$, and assume that $n/\phi(n)$ is concave. Consider the wreath product $W = A \wr X G_{012}$. Then the growth of $W$ is $\sim \exp(n/\phi(n^{1-\alpha}))$.

**Proof.** We begin by the lower bound. For $n \in \mathbb{N}$, consider a word $w$ of length $n$ with $\delta(w) \sim n^\alpha$, which exists by Proposition 5.1. Write $O(w) = \{x_1, \ldots, x_k\}$ for $k \sim n^\alpha$. Choose then $k$ elements $a_1, \ldots, a_k$ of length $\leq n^{1-\alpha}$ in $A$. Define $f \in \sum_X A$ by $f(x_i) = a_i$, all unspecified values being 1. Then $w \in W$ may be expressed as a word of length $n + |a_1| + \cdots + |a_k| \sim 2n$ in the standard generators of $W$.

Furthermore, different choices of $a_i$ yield different elements of $W$; and there are $\sim \exp(n^{1-\alpha}/\phi(n^{1-\alpha})) \sim \exp(n/\phi(n^{1-\alpha}))$ choices for all the elements of $A$. This proves the lower bound.

For the upper bound, consider a word $w$ of length $n$ in $W$, and let $f \in \sum_X A$ denote its value in the base of the wreath product. The support of $f$ has cardinality at most $\delta(w) \lesssim n^\alpha$ by Proposition 5.4, and may take at most $\sim \exp(n^\alpha)$ values by Lemma 4.19.

Write then $\text{sup}(f) = \{x_1, \ldots, x_k\}$ for some $k \lesssim n^\alpha$, and let $a_1, \ldots, a_k \in A$ be the values of $w$ at its support; write $\ell_i = |a_i|$. We now consider two cases. If $A$ is finite, then each of the $a_i$ may be chosen among, at most, $|A|$ possibilities, so there are $\sim \exp(n^\alpha)$ possibilities in total for the element $f$.

Assume now that $A$ is infinite, so that $v_A(n) \gtrsim n$. Since $\sum \ell_i \leq n$, the lengths of the different elements on the support of $f$ define a composition of a number not greater than $n$ into at most $n^\alpha$ summands; such a composition is determined by $n^\alpha$ “marked positions” among $n + n^\alpha$, so there are at most $\binom{n}{\ell_i} \sim \exp(\log(n)n^\alpha)$ such compositions. Each of the $a_i$ is then chosen among $v_A(\ell_i)$ elements, and there are $\sim \exp(\ell_i/\phi(\ell_i))$ such choices for each $i$.

By our concavity assumption, there are at most $\sim \prod \exp(\ell_i/\phi(\ell_i)) \lesssim \exp(n/\phi(n^{1-\alpha}))$ choices for the elements in $A$.

We have now decomposed $w$ into data that specify it uniquely, and we multiply the different possibilities for each of the pieces of data. First, there are $\lesssim \exp(n^\alpha)$ possibilities for the value of $w$ in $G_{012}$, by the upper bound (1). There are $\lesssim \exp(n^\alpha)$ possibilities for the support of $w$. There are $\lesssim \exp(\log(n)n^\alpha) \exp(n/\phi(n^{1-\alpha}))$ possibilities for the values of $w$ at its support, the first factor counting the number of compositions of $n$ as a sum of $n^\alpha$ terms and the second factor counting the number of elements in $A$ of these lengths. Altogether, we get

$$v_W(n) \lesssim \exp(n^\alpha + n^\alpha + n^\alpha \log(n) + n/\phi(n^{1-\alpha})) \sim \exp(n/\phi(n^{1-\alpha})), $$

and we have obtained the claimed upper bound.

We are ready to prove the first part of Theorem A.

**Theorem 5.2.** Consider the following sequence of groups: $K_0 = \mathbb{Z}/2\mathbb{Z}$, and $K_{k+1} = K_k \wr X G_{012}$. Then every $K_k$ is a finitely generated infinite torsion group,
with growth function
\[ v_{K_1}(n) \sim \exp(n^{1-(1-\alpha)^k}). \]

Proof. We start by \[ \phi_0 = n; \] then \[ \phi_{k+1}(n) = \phi_k(n^{1-\alpha}), \] so \[ \phi_k(n) = n^{(1-\alpha)^k}. \]

**Theorem 5.3.** Consider the following sequence of groups: \[ L_0 = \mathbb{Z}, \] and \[ L_{k+1} = L_k \wr_X G_{012}. \] Then their growth functions satisfy
\[ v_{L_k}(n) \sim \exp( (\log(n)) n^{1-(1-\alpha)^k}). \]

Proof. We start by \[ \phi_0 = n/\log(n); \] then \[ \phi_{k+1}(n) = \phi_k(n^{1-\alpha}). \] Now \[ \log(n^{1-\alpha}) \sim \log(n), \] so we get \[ \phi_k(n) \sim n^{(1-\alpha)^k}/\log(n). \]

**Example 5.4.** The inverted orbits of \[ G_{01}, \] for its action on the orbit \( X \) of the rightmost ray \( \rho = 11\ldots \), followed by \( 0\rho, 00\rho, 10\rho, \ldots \)

Let \( A \) be a finite group containing at least two elements. The unmarked Schreier graph of \( (G, X) \) is the same as the unmarked Schreier graph of \( (G', X') \).

However, the growth of the wreath product \( A \wr_X G \) is subexponential, whereas the growth of \( A \wr_X G' \) is exponential.

Indeed, observe that \[ W = A \wr_X G = A \wr_X G_{012} \times G_{01}. \] By Theorem 5.2 we know that \( K_1 = A \wr_X G_{012} \) has intermediate growth. We see that \( W \) is a direct sum of two groups of intermediate growth, and hence the growth of this group is intermediate.

On the other hand \( W' = A \wr_X G' = A \wr_X G_01 \times G_{012}, \) and already the first factor has exponential growth, see Example 5.4.

The unmarked Schreier graph of \((G_{012}, X)\), as well as the Schreier graph of \((G_{01}, X')\), are rays, in which every second edge edge has been duplicated, a loop has been added at each vertex, and three loops are drawn at the origin (see Figure 1).

The unmarked Schreier graphs of \((G, X)\) and \((G', X')\) are obtained from that graph by drawing four additional loops at each vertex.

**Remark 5.6.** Let \( N \) be a finitely generated nilpotent group, acting transitively on an infinite set \( X \). Then the inverted orbits for this action have linear growth: that is, there exists \( C > 0 \) such that for any \( n > 1 \) there exist a word \( w_n \) of length \( n \) such that its inverted orbit for the action on \( X \) visits at least \( Cn \) points.
Proof. Take $A = \mathbb{Z}/2\mathbb{Z}$ and let $G$ be the wreath product of $A$ with $(N, X)$. Observe that $G$ is an extension of an Abelian group by a nilpotent group, so $G$ is solvable. Since $N$ and $A$ are finitely generated, so is $G$. We know that $G$ contains as a subgroup $\sum X A$. Since $G$ contains an infinitely generated subgroup, we conclude that $G$ is not virtually nilpotent. Therefore, $G$ has exponential growth. However, $N$ has subexponential growth, so if it also had sublinear inverted orbit growth then $G$ would have subexponential growth. □

5.1. Torsion-free examples. Grigorchuk constructed in [18] §5 a torsion-free group $H$ of intermediate growth. We recall the basic steps: start by

$$H_0 = \langle a, b, c, d \mid [a^2, b], [a^2, c], [a^2, d], [b, c], [b, d], [c, d] \rangle,$$

and define a wreath recursion $\psi : H_0 \to H_0 \wr \mathbb{S}_2$ by the same formula (3) as for Grigorchuk’s first group, namely

$$\psi : a \mapsto \langle 1, 1 \rangle \varepsilon, \ b \mapsto \langle a, c \rangle, \ c \mapsto \langle a, d \rangle, \ d \mapsto \langle 1, b \rangle.$$  

Set $K_0 = 1$ and inductively $K_{n+1} = \psi^{-1}(K_n \times K_n)$. Define then $H = H_0/\bigcup_{n \geq 0} K_n$, and use the same notation for the generators $a, b, c, d$ of $H$ and the induced homomorphism $\psi : H \to H_0 \wr \mathbb{S}_2$. Note that $H$ is the largest quotient of $H_0$ such that the restriction of $\psi$ to $(b, c, d)$ is injective.

We view the group $H$ in terms of permutational extensions, and compute its growth function by adapting Lemma 5.1.

Note first that the natural map $\xi : a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ defines a homomorphism from $H$ to $G_{012}$. Consider the subgroup $C = \langle b^2, c^2, d^2, bcd \rangle$ of $H$; then, for instance using the presentation (5) of $G_{012}$, the normal closure $\langle a^2, C \rangle^H$ equals $\ker \xi$.

Grigorchuk proves in [18] pages 199–200 that $C \cong \mathbb{Z}^3$, and that $\langle a^2 \rangle \cong \mathbb{Z}$. Because $a^2$ is central in $H$, we have exact sequences

$$1 \to \langle a^2 \rangle \to H \to H/\langle a^2 \rangle \to 1,$$

$$1 \to C^H \to H/\langle a^2 \rangle \to G_{012} \to 1.$$

Let $\psi_0, \psi_1$ be the coordinates of $\psi$, namely, the set-maps defined by the projections $H \to H \wr \mathbb{S}_2 \to H \times H \to H$, as in $\psi(g) = \langle \psi_0(g), \psi_1(g) \rangle \varepsilon^\tau$. Note that $\psi_0, \psi_1$ are not homomorphisms, but their restriction to $B := \langle b, c, d \rangle$ is a homomorphism: $\psi_0$ maps to $\langle a \rangle$, while $\psi_1$ permutes cyclically $b, c, d$. Consider $\tau = \tau_1 \tau_2 \cdots \in \{0, 1\}^\infty$ a ray in $T$. Given $g \in H$, it is easy to see (see [18] page 200) that $\psi_{\tau_n} \cdot \psi_{\tau_{n-1}} \cdots \psi_{\tau_1}(g)$ belongs to $B$ for all $n$ large enough. Recall the endomorphism $\sigma$ from (4); it induces an automorphism of $B$ permuting cyclically $b, c, d$, so we have $\sigma(\psi_1(g)) = g$ for all $g \in B$. The sequence $a^n \psi_{\tau_n} \cdot \psi_{\tau_{n-1}} \cdots \psi_{\tau_1}(g)$ eventually stabilizes, and we call its limit $g_\tau \in B$ the germ of $g$ at $\tau$. Note that $g_\tau = 1$ unless $\tau$ is in the $H$-orbit of $\rho = 1$. For an element $g \in B$, its germ is $g_\rho = g$ and $g_\tau = 1$ for all $\tau \neq \rho$. Similarly, for $g \in B$ and $x \in H$, the germ of the conjugate $x^{-1}gx = g^x \in B^x$ are $(g^x)_\rho = g$ and $(g^x)_\tau = 1$ for all $\tau \neq \rho_\tau$.

Lemma 5.7. An element of $H$ is determined by its projection to $G_{012}$, its $a$-exponent sum, and its germs:

$$\ker \xi = \langle a^2 \rangle \times \sum_{x \in (G_{012})_\rho \setminus G_{012}} C^x \cong \mathbb{Z} \times \sum_X \mathbb{Z}^3.$$
Proof. On the one hand, \( a^2 \) is central, and generates a split copy of \( \mathbb{Z} \). As we noted above, \( \ker \xi \) is generated by \( a^2 \) and conjugates of \( C \).

Consider next \( y, z \in G \), and \( g \in H \); we show that \( y \) and \( z \) commute. Write \( g = \langle g', g'' \rangle \) for \( g', g'' \in H \) and \( s \in \{0, 1\} \). Write also \( y = \langle a^{2k}, y' \rangle \) and \( z = \langle a^{2l}, z' \rangle \). If \( s = 1 \), then \( z = \langle (z')^{a^s}, (a^{2l})^{a^s} \rangle = \langle (z')^{a^s}, a^{2l} \rangle \) using the relations in \( H_0 \); so \( [y, z'] = \langle (z')^{a^s}, a^{2l} \rangle \). If \( s = 0 \), then \( z = \langle (a^{2k})^{a^s}, (z')^{a^s} \rangle = \langle a^{2k}, (z')^{a^s} \rangle ; \) so \( [y, z] = \langle [a^{2k}, a^{2l}], [a^{2k}, (z')^{a^s}] \rangle = \langle 1, [y', (z')^{a^s}] \rangle \), and now, because \( g'' \) is shorter than \( g \), we eventually have \( g \in B \), so by induction \( [y, z] = 1 \).

Consider finally \( y \in C \) and \( h \in H \) whose image \( \xi(h) \) in \( G_{012} \) fixes \( \rho \); we show that \( y \) and \( h \) commute. Write again \( y = \langle a^{2k}, y' \rangle \) and \( h = \langle h', h'' \rangle \); then \( [y, h] = \langle 1, [y', h''] \rangle \), and by induction eventually \( h \in B \) so \( [y, h] = 1 \). It now follows that \( \ker \xi \) is a quotient of \( \mathbb{Z} \times \sum X \mathbb{Z}^3 \).

On the other hand, consider an element \( h \) of \( H \) of the form \( y_1^{a_1} \ldots y_{a_t}^{a_t} \) for some distinct \( y_i \in (G_{012})_0 \setminus G_{012} \) and \( y_i \in C \). Consider some \( i \in \{1, \ldots, \ell\} \); then the germ \( h_{y_i} \) equals \( y_i \), so no relations occur among the elements of \( \mathbb{Z} \times \sum X \mathbb{Z}^3 \) when they are mapped to \( \ker \xi \).

The difference between \( H \) and the wreath product \( \mathbb{Z}^3 \times G_{012} \) is twofold: \( H \) is not a split extension of \( \sum X \mathbb{Z}^3 \); and the generator \( a \in G_{012} \) lifts to an infinite-order element of \( H \). We nevertheless show that \( H \) and \( \mathbb{Z}^3 \times G_{012} \) have the same asymptotic growth:

**Proposition 5.8.** The group \( H \) has growth \( \sim \exp(\log(n)n^\alpha) \).

Proof. We define a set-theoretic splitting \( \nu \) of \( \xi : H \to G_{012} \) by the condition that, for all \( g \in G_{012} \), the germs \( \nu(g)_a \), all belong to \( \{1, b, c, d\} \), and that the total exponent sum \( |\nu(g)|_a \) of \( a \) in \( \nu(g) \) is 0 or 1.

By Lemma 5.7, elements \( h \in H \) can, and will, be put in the form \( a^{2k} f \nu(g) \), with \( g \in G_{012} \), \( f : X \to C \) finitely supported, and \( k \in \mathbb{Z} \). We consider the effect of left-multiplying such an expression by a generator \( t \in \{a, b, c, d\}_\pm \). First, consider \( t = a^k \) for \( k \in \{1, -1\} \). Write \( |\nu(g)|_a + k = n + 2m \), with \( n \in \{0, 1\} \). Then

\[
\nu(g) = a^{2\ell} f (t \cdot f) \nu(g)
\]

Consider next \( t \in \{b, c, d\}_\pm \). Then

\[
\nu(g) = a^{2\ell} f (t \cdot f) \nu(g)
\]

now, in \( \{b, c, d\} \), write \( t \nu(g)_a = zr \) for \( z \in C \) and \( r \in \{1, b, c, d\} \). Denote still by \( z \) the function \( X \to C \) which takes value \( z \) at \( r \) and is trivial everywhere else. We then have

\[
\nu(g) = a^{2\ell} (t \cdot f) z \nu(g)
\]

It follows that the action of a generator on an element of \( H \), written in the form \( a^{2\ell} f \nu(g) \), is by translation of \( f \) (just as in the wreath product \( \mathbb{Z}^3 \times G_{012} \)), possibly followed by a multipication at \( \rho \) of \( f \) by a generator of \( C \) or its inverse.

More pedantically, the computation above shows that the cocycle \( \eta(g, h) := \nu(gh)^{-1} \nu(g) \nu(h) \) associated with the extension \( 1 \to C^H \to H \to G_{012} \to 1 \) is controlled in the following manner: if \( |g|, |h| \leq n \), then \( \nu(g, h) : X \to C \) is supported on a set of cardinality \( n^\alpha \), and takes values bounded in \( \{-n, \ldots, n\} \).

The remainder of the growth computation follows closely the argument in Lemma 5.3.
Consider the representations as $a^{2k}f_{\nu}(g)$ of elements $h \in H$ of norm at most $n$. The element $g = \xi(h)$ belongs to $G_{012}$ and has norm at most $n$, so may take at most $\exp(Dn^\alpha)$ values, for a predefined constant $D$. The function $f$ is supported on a set of cardinality at most $Cn^\alpha$, for another predefined constant $C$, and takes values in $\{-n, \ldots, n\}$; so there are at most $\exp(\log(2n+1)Cn^\alpha)$ possibilities for $f$. Finally $|\ell| \leq n$. In total, there are $\leq \exp(\log(n)n^\alpha)$ values for $h$.

For the lower bound, consider a word $w = w_1 \ldots w_n$ of length $n$ over $\{a, b, c, d\}$ with $\delta(w) \sim n^\alpha$, which exists by Lemma 4.7. Write $O(w) = \{x_1, \ldots, x_k\}$ for $k \sim n^\alpha$, and let $i_1, \ldots, i_k \in \{1, \ldots, n\}$ be such that $\rho w_i \ldots w_n = x_j$ for $j = 1, \ldots, k$. Choose then $k$ numbers $a_1, \ldots, a_k$ in $\mathbb{Z} \cap [1, n^{1-\alpha}]$. Insert $(bcd)^{a_j}$ before position $i_j$ in $w$, and call the resulting word $v(a_1, \ldots, a_k)$.

First, the length of $v(a_1, \ldots, a_k)$ is at most $n + 3n^\alpha n^{1-\alpha} = 4n$. Then, the germ at $x_j$ of $v(a_1, \ldots, a_k)$ belongs to $\{b, c, d\}(bcd)^{a_j}$, so all $v(a_1, \ldots, a_k)$ are distinct. It follows that there are at least $(n^{1-\alpha})n^\alpha$ elements of length $4n$ in $H$.

**Proof of Theorem A second part.** For $k = 0$, consider the group $H_0 = \mathbb{Z}$; for $k = 1$, consider the group $H_1 = H$ from Proposition 5.8. For $k > 1$, consider inductively $H_k = H_{k-1} \wr \mathbb{Z}$. They are torsion-free, as extensions of a torsion-free group by a torsion-free group.

### 5.2. Orbits on pairs of rays

We gather here some results from [6]. Consider the ray $\rho = 1^\infty$, the ray in the binary tree $\mathcal{T}$, and its orbit $X := \rho G_{012}$. The group $G_{012}$ acts on $X$, and therefore acts (diagonally) on $X \times X$.

Because $G_{012}$ acts transitively on $X$, the $G_{012}$-orbits on $X \times X$ are in bijection with the orbits of the stabilizer $P = (G_{012})_\rho$ on $X$, and also in bijection with the double cosets $PYP$ of $P$ in $G_{012}$.

The set of orbits of $G_{012}$ on $X \times X \setminus \{(x, x) \mid x \in X\}$ may be readily described. A pair of distinct points $(x, y) \in X \times X$ determines a bi-infinite path in $\mathcal{T}$, namely the path $\gamma$ that starts from $x$, goes to the root of $\mathcal{T}$, and leaves towards $y$. Let $\mathcal{T}$ denote the geodesic path (without backtracking) associated with $\gamma$, and let $(x|y) \in \mathbb{N}$ denote the minimal distance of this geodesic to the root of $\mathcal{T}$. The action of $G_{012}$ on $\mathcal{T}$ and on $\partial T$ induces an action on bi-infinite geodesics in $\mathcal{T}$, so $(x|y)$ is preserved by the $G_{012}$-action. We now show that pairs $(x, y)$ and $(x', y')$ belong to the same $G_{012}$-orbit if and only if $(x|y) = (x'|y')$.

We summarize the results:

**Lemma 5.9 ([6, Lemma 9.10]).** The orbits of the stabilizer $P = (G_{012})_\rho$ of $\rho$ on $X \setminus \{\rho\}$ are described as follows:

$$O = \{1^k01^* \rho \mid k \in \mathbb{N}\}.$$  

In particular, there are infinitely many orbits of $G_{012}$ on $X \times X$. For $x \neq y \in X$, we denote by $(x|y) \in \mathbb{N}$ the length of the maximal common prefix of $x, y$; it is the distance to the root of the geodesic in $\mathcal{T}$. We also set $(x|x) = \infty$. The orbit of $(x, y)$ is then completely determined by $(x|y) \in \mathbb{N} \cup \{\infty\}$.

Recall the endomorphism $\sigma$ from [4]. The set $T = \{\sigma^n(a) \mid n \in \mathbb{N}\}$ is a set of non-trivial double coset representatives of $P$. 

A generating set for $P$ has also been computed:
5.3. Presentations for wreath products. We recall the notion of \(L\)-presentation, introduced in \([8]\). A group \(G\) is \emph{finitely \(L\)-presented} if there exists a finitely generated free group \(F = \langle S \rangle\), a finite set \(\Phi\) of endomorphisms of \(F\), and finite subsets \(Q, R\) of \(F\), such that \(G \cong F/(Q \cup \bigcup_{\phi \in \Phi} \phi(R))^F\). The expression \(\langle S | Q, \Phi | R \rangle\) is the corresponding \emph{finite \(L\)-presentation}.

In particular, the first Grigorchuk group is finitely \(L\)-presented as
\[
G_{012} = \langle a, b, c, d || \sigma | a^2, b, c, d, [a, d^a], [d, d^{(ac)^a}] \rangle.
\]

**Proposition 5.11.** Let \(A\) be a finitely \(L\)-presented group. Then \(A \wr_X G_{012}\) is finitely \(L\)-presented.

**Proof.** Cornulier characterizes in \([10]\) when permutational wreath products are finitely presented. A permutational wreath product \(A \wr_X G\), for \(G\) acting transitively on a transitive \(G\)-set \(X = P \backslash G\), is presented as follows: as generators, take those of \(A\) and \(G\). As relations, take: those of \(A\) and \(G\); the relation \([a, u]\) for every generator \(a\) of \(A\) and every \(u\) in a generating set \(U\) of \(P\); and the relations \([a, b]\) for every generators \(a, b\) of \(A\) and \(q\) in a set of double coset representatives of \(P\) in \(G\): namely \(t \in T\) with \(G = P \cup \bigcup_{p \in T} PgP\).

In the case of the first Grigorchuk group, a generating set for \(P = (G_{012})^0\), and a set of double coset representatives, have been computed in \([8]\); see Lemmata 5.9 and 5.10. Let \(\langle S | Q, \Phi | R \rangle\) be a finite \(L\)-presentation of \(A\). A finite \(L\)-presentation for \(A \wr_X G_{012}\) is then \(\langle S, a, b, c, d | Q, \Phi' \cup \{\sigma'\} | R \cup R' \rangle\), with \(\Phi'\) the endomorphisms in \(\Phi\), extended by fixing \(a, b, c, d\);
\[
\sigma' = \sigma \text{ on } \{a, b, c, d\}, \text{ and fixing } S;
\]
\[
R' = \{[s, b], [s, d^a], [s, (ac)^4], [s, \sigma([a, b])^a], [s, \sigma^2([a, b])^a] | s \in S\}
\]
\[
\cup \{[s', s^a] | s, s' \in S\}.
\]

**Corollary 5.12.** The groups \(K_k\) from Theorem 5.2 and \(L_k\) from Theorem 5.3 are finitely \(L\)-presented.

**Example 5.13.** A recursive presentation for the group \(K_1 = \mathbb{Z}/2\mathbb{Z} \wr_X G_{012}\) is
\[
K_1 = \langle a, b, c, d, s \mid a^2, b^2, c^2, d^2, s^2, bcd, \sigma^n(r_1), \ldots, \sigma^n(r_8) \text{ for all } n \in \mathbb{N} \rangle
\]
for \(\sigma\) the same endomorphism as in \([4]\), extended by \(\sigma(s) = s\), and iterated relations
\[
\begin{align*}
    r_1 &= [d, d^a], & r_2 &= [d, d^{(ac)^a}], \\
    r_3 &= [s, s^a], & r_4 &= [s, b], \\
    r_5 &= [s, d^a], & r_6 &= [s, (ac)^4], \\
    r_7 &= [s, \sigma([a, b])^a], & r_8 &= [s, \sigma^2([a, b])^a].
\end{align*}
\]

6. Embeddings of the group of finitely supported permutations

**Theorem 6.1.** There exists a group \(H\) of intermediate growth that contains as a subgroup the group \(\mathfrak{S}_\infty\) of finitely supported permutations of an infinite countable set.
Moreover, the group $H$ can be chosen in such a way that its growth function satisfies
\[ \exp(n^\alpha) \preceq v_H(n) \preceq \exp(\log(n)n^\alpha). \]

Proof. If $G$ acts on a set $X$, then it acts on the group of finitely supported permutations of $X$; for $\sigma : X \to X$ with $\sigma(x) = x$ for all $x \in X$ except finitely many, $g^{-1}\sigma g$ is still finitely supported for all $g \in G$.

Let $X$ denote the orbit of the ray $p = 1^\infty$ under the action of the first Grigorchuk group $G_{012}$, and let $\mathfrak{S}_\infty$ denote the group of finitely supported permutations of $X$. Set $H = \mathfrak{S}_\infty \rtimes G_{012}$. Take as generating set for $H$ the union of the generating set $\{a, b, c, d\}$ of $G$ with the involution $\rho$ and $\rho_0$. By Proposition 4.4 there are $\preceq n^\alpha$ possibilities for the first, and in view of Lemma 2.3 also for the second of these two points; so the support of the permutation has cardinality $\preceq n^\alpha$. An element of $H$ of length $\sim n$ may therefore be described by: an element of $G_{012}$ of length $\preceq n$; a subset of $X$ of cardinality $\preceq n^\alpha$; and a permutation of that subset. There are at most, respectively, $v_{G_{012}}(n)$, $\binom{n}{\alpha}$, and $(n^\alpha)!$ choices for each of these pieces of data. We get
\[ v_H(n) \preceq \exp(n^\alpha) \exp(\log(n)n^\alpha)(n^\alpha)! \sim \exp(\log(n)n^\alpha). \]

On the other hand, consider the word $w_n = g_1 \ldots g_\ell$ given by Proposition 4.7. Set inductively $S_0 = \emptyset$ and $S_k = S_{k-1} \cup \{pg_k \ldots g_\ell, 0pg_k \ldots g_\ell\}$; and select positions $k_1, k_2, \ldots, k_\ell$ such that $S_k \neq S_{k-1}$. Consider then the $2^{2^n}$ elements of $H$ obtained by inserting, at each position $k_i$ in $w_n$, the word $s^{e_i}$ for all choices of $e_i \in \{0, 1\}$.

An easy induction shows that all these elements are distinct in $H$: given such an element, expressed as $\sigma g$ with $\sigma \in \mathfrak{S}_\infty$ and $g = w_n \in G_{012}$, we recover the $e_i$ as follows. Let $x \in X$ be in the support of $\sigma$, and in $S_{k_i} \setminus S_{k_i-1}$ for maximal $k_i$. This determines $e_i = 1$. Right-divide then by $s^{e_i}g_{k_i+1} \ldots g_{\ell}$, and proceed inductively. All other $e_j$’s are 0.

This gives $v_H(2|w_n|) \geq 2^{2^n}$, proving the lower bound. $\square$

The first examples of groups of intermediate growth that are not residually finite are constructed in [12]. Theorem 6.1 gives new examples of this kind:

**Corollary 6.2.** There exist finitely generated groups of growth $\preceq \exp(\log(n)n^\alpha)$ that contain an infinite simple group as a (normal) subgroup. In particular, such groups provide new examples of non residually finite groups of intermediate growth.

Proof. We recall that the group of even permutations is a characteristic subgroup of the group $\mathfrak{S}_\infty$ of finitely supported permutations, and is simple. $\square$

**Remark 6.3.** It is shown in [11] that there exist groups of intermediate growth which admit a non-degenerate measure with non-trivial Poisson-Furstenberg boundary. Kaimanovich shows in [24] that the group of finitely supported permutations of a countable set admits a symmetric measure with non-trivial boundary. This provides an example of a measure with non-trivial boundary on the group $H$ from Theorem 6.1. However, the measures obtained in this way are degenerate.
It can be shown that the group $H$ considered in the proof of Theorem 6.1 as well as other groups constructed in this paper, also admit non-degenerate measures with non-trivial Poisson-Furstenberg boundary. We will study random walks on permutational extensions, not restricted to those considered in this paper, in [4]. Some of the groups that we will be treated in [4] lead to new phenomena in boundary behavior.

**Remark 6.4.** After this paper was completed, further developments on the growth of groups appeared in the preprints [9, 25, 5].

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