Dirac Equation as a Bridge to the Gravitational Interaction of Antimatter: Antimatter Gravity and Charge Asymmetry

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We show that, because the Dirac equation describes both matter as well as antimatter composed of the same kind of particle simultaneously, it can be used as a “bridge” toward the description of gravitational interaction of antimatter. We find a symmetry, on the level of the gravitationally coupled Lagrangian, connecting the gravitational interaction terms for particles to those for antiparticles. As a result, we find that, on the level of canonical gravity, there is no room for a deviation of the gravitational interactions of matter from those of antimatter. A natural question arises: Given these observations, how do we interpret upcoming gravitational experiments with antimatter? We find that a very attractive possibility to reinterpret the tests consists in a connection to a potential charge excess in either matter or antimatter, a hypothesis discussed by Einstein in 1924 and by Lyttleton and Bondi in [Proc. Roy. Soc. London, Ser. A 252, pp. 313–333 (1959)]. A gravitational test of antimatter is shown to drastically limit the available parameter space for charge excess in the combined matter-antimatter system. This is because the charge excess in either hydrogen or antihydrogen, interacting with a potential excess charge of the entire Earth, is being compared to the gravitational force.

I. INTRODUCTION

It is common wisdom in atomic physics that the Dirac equation describes particles and antiparticles simultaneously, and that the negative-energy solutions of the Dirac equation have to be reinterpreted in terms of particles that carry the opposite charge as compared to particles, and whose numerical value of the energy $E$ is equal to the negative value of the physically observed energy $|E|$. Based on the Dirac equation, the existence of the positron was predicted, followed by its experimental detection in 1933, by Anderson [2].

If we did not reinterpret the negative-energy solutions of the Dirac equation, then the helium atom would be unstable against decay into a state where the two electrons perform quantum jumps into continuum states $|E|$. One of the electrons would jump into the positive-energy continuum, the other, into the negative-energy continuum, with the sum of the energies of the two continuum states being equal to the sum of the two bound-state energies of the helium atom from which the transition started [3, 4].

The absolute necessity to reinterpret the negative-energy solutions of the Dirac equation as antiparticle wave functions, i.e., the necessity to interpret positive-energy and negative-energy solutions of one single equation as describing two distinct particles, hints at the possibility to use the Dirac equation as a bridge to the description of the gravitational interaction of antimatter. Namely, if the Dirac equation is being coupled to a gravitational field, then, since it describes particles and antiparticles simultaneously, the Dirac equation offers us an additional dividend: In addition to describing the gravitational interaction of particles, the Dirac equation automatically couples the antiparticle, which is described by the same equation, to the gravitational field, too.

This program has been carried out in a series of recent publications, see Refs. [5–8]. The most pressing question is not only whether the dynamics of particles and antiparticles differ in a central, static, gravitational field, in first approximation, but also, if there are any small higher-order effects breaking the particle-antiparticle symmetry under the gravitational interaction. The first of these questions has been answered in Refs. [5–7], with the result being that the Dirac particle and antiparticle behave exactly the same in a central gravitational field, due to a perfect particle-antiparticle symmetry which extends to the relativistic and curved-space-time corrections to the equations of motion.

For the second question, it is necessary to perform the full particle→antiparticle symmetry transformation of the Dirac formalism, in an arbitrary (possibly dynamic) curved-space-time-background. This transformation is most stringently carried out on the level of the Lagrangian formalism. A preliminary result has recently been published in Ref. [8], where a relationship was established between the positive-energy and negative-energy solutions of the Dirac equation in an arbitrary dynamics curved-space-time-background. This derivation is ramified here on the basis of a transformation of the entire Lagrangian density, which can be expressed in terms of the charge-conjugated
(antiparticle) bispinor wave function. We anticipate that the conclusions of Ref. 8 will be confirmed, thus establishing the (weak) equivalence principle (the equality of the inertial and gravitational mass) for antiparticles, in an arbitrary curved space-time background.

The Dirac equation is an established tool in the analysis of atomic-physics processes, and it is hard to doubt the validity of the established formalism for the coupling of Dirac particles (and simultaneously antiparticles) which was developed during the Cold War period by both Russian as well as American scientists with mutually consistent results (for a necessarily incomplete list of references, see Refs. [9–16]).

Combining the symmetries of the Dirac equation with the solid footing on which the formalism of gravitational coupling of the Dirac equation stands, one might otherwise ask how the results of antimatter gravity should be interpreted [17–25], given that the Dirac equation implies the stringent prediction that the weak equivalence principle holds for antiparticles. Somewhat surprisingly, one such possibility consists in tests of a hypothetical charge excess in either matter or antimatter, which would mimic a small contribution to the gravitational force, at least within the nonrelativistic approximation. A very small charge excess in matter would not be capable of explaining the Earth’s magnetic field; this suggestion was given by “Monsieur Einstein” [sic!] at the 1924 Lucerne Meeting of the Swiss Physical Society, as was explicitly mentioned in Ref. [26, 27]. The latter reference describes a measurement of the charge asymmetry of matter by the gas efflux method. Likewise, a conceivable charge asymmetry would be able to explain the expansion of the Universe [28, 29]. While both hypotheses have meanwhile been ruled out by experiment, tests of the charge neutrality of matter (and antimatter) have recently attracted considerable attention [30–32]. A conceivable electric charge of the neutron has been investigated in Ref. [33]. If a putative residual electrostatic interaction exists, then, because of the different nature of the retardation corrections at long distances as compared to gravitational interactions, one could also speculate about possible explanations of apparent modifications of the gravitational law at large distances (dark matter). An excellent review on various theoretical models allowing for charge asymmetry is presented in Ref. [34].

We shall offer two complementary possibilities of interpreting a gravitational antimatter experiment in terms of a test of charge neutrality. For the first of these interpretations, we assume that there is an ever so slight residual charge to be associated with a hydrogen atom, and that the charge asymmetry in antihydrogen is exactly the opposite of that found in hydrogen. We base this assumption on the following reasoning. Electrons and positrons constitute a particle-antiparticle pair and therefore, are described by the same Dirac equation, which predicts that the positron charge and the electron charge add up to zero. The same applies to protons and antiprotons. However, one observes that leptons and hadrons are still two completely different particle species, so that it is much easier to speculate about reasons why the electric neutrality of a hydrogen could be infinitesimally broken, in the sense that the electron and proton charges might not quite add up to zero, while the electron and positron charges do. In that case, if we assume charge conjugation symmetry within the same particle species, the residual charge of an antihydrogen atom would be the opposite of a hydrogen atom. In consequence, one would see a residual long-range repulsive electrostatic force between matter particles, which would of the same functional form as the gravitational force, at least within the nonrelativistic approximation. Also, there would exist a corresponding residual attractive force between matter and antimatter (particles and antiparticles), because of the opposite sign of the residual, infinitesimal charge of the matter and antimatter particles. If, now, a hydrogen atom interacts with the Earth’s gravitational field, then it will be subject to the hypothetical matter-matter repulsion due to the residual electrostatic repulsion, as described above. An antihydrogen atom, however, would be subject to residual electrostatic matter-antimatter attraction. Hence, an antimatter “gravity” experiment could be interpreted as a test of the electric neutrality of matter (and antimatter).

In the above reasoning, we have assumed that the residual charge of a hydrogen atom is the opposite of an antihydrogen atom. If we relax this assumption, then, for the second possible “electric” interpretation of the antimatter gravity experiment, as we shall show that the experiment can still constrain the allowed parameter space within the two-dimensional space of charge-neutrality violating parameters for matter and antimatter very significantly.

This paper is organized as follows. In order to properly address the particle-antiparticle symmetry of the gravitationally and electromagnetically coupled Dirac equation, we first need an explicit expression for the Dirac adjoint in curved space-times, which enters the Lagrangian. This is accomplished in Sec. III. We then proceed to show the matter-antimatter symmetries of the gravitationally and electromagnetically coupled Dirac equation on the level of the Lagrangian, in Sec. III. The connections between antimatter gravity experiments and matter and antimatter charge neutrality experiments are described in Sec. IV. Conclusions are reserved for Sec. V.

II. DIRAC ADJOINT FOR CURVED SPACE–TIMES

In order to properly write down the Lagrangian of a Dirac particle in a gravitational field, we first need to generalize the concept of the Dirac adjoint to curved space-times. We recall that the Dirac adjoint transforms with the inverse of the Lorentz transform as compared to the original Dirac spinor. A general spinor Lorentz transformation $S(\Lambda)$ is
given as follows,

\[ S(\Lambda) = \exp\left(-\frac{i}{4} \epsilon^{AB} \sigma_{AB}\right), \quad \sigma_{AB} = \frac{i}{2} [\gamma^A, \gamma^B]. \]  

(1)

Note that the generator parameters \( \epsilon^{AB} = -\epsilon^{BA} \), for local Lorentz transformations, can be coordinate-dependent. The (flat-space) Dirac matrices \( \gamma^A \) are assumed to be taken in the Dirac representation \([1, 35]\),

\[ \gamma^0 = \left( \begin{array}{cc} 1_{2\times2} & 0 \\ 0 & 1_{2\times2} \end{array} \right), \quad \gamma = \left( \begin{array}{cc} 0 & \sigma \\ -\sigma & 0 \end{array} \right). \]  

(2)

Here, the vector of Pauli spin matrices is denoted as \( \sigma \). We here adopt the conventions of Ref. \([35]\) and denote the coefficients in an anholonomic ("vierbein") basis by capital Latin indices. In consequence, the spin matrices \( \sigma_{AB} \) are the flat-space spin matrices. The spin matrices fulfill the commutation relations

\[ \frac{1}{2}\sigma^{CD}, \frac{i}{2}\delta^{EF} = i \left( g^{CF} \frac{1}{2}\sigma^{DE} + g^{DE} \frac{1}{2}\sigma^{CF} - g^{CE} \frac{1}{2}\sigma^{DF} - g^{DF} \frac{1}{2}\sigma^{CE} \right). \]  

(3)

These commutation relations, we should note in passing, are completely analogous to those fulfilled by the matrices \( \sigma_{AB} \) that generate (four-)vector local Lorentz transformations. The latter have the components (denoted by indices \( C \) and \( D \))

\[ (\mathcal{M}_{AB})^C_D = g^C_A g_{DB} - g^C_B g_{DA}. \]  

(4)

The vector local Lorentz transformation \( \Lambda \) with components \( \Lambda^C_D \) is obtained as the matrix exponential

\[ \Lambda^C_D = \left( \exp \left[ \frac{1}{2} \epsilon^{AB} \mathcal{M}_{AB} \right] \right)^C_D. \]  

(5)

The algebra fulfilled by the \( \mathcal{M} \) matrices is well known to be

\[ [\mathcal{M}^{CD}, \mathcal{M}^{EF}] = g^{CF} \mathcal{M}^{DE} + g^{DE} \mathcal{M}^{CF} - g^{CE} \mathcal{M}^{DF} - g^{DF} \mathcal{M}^{CE}. \]  

(6)

The two algebraic relations \([9]\) and \([10]\) are equivalent if one replaces

\[ \mathcal{M}^{CD} \rightarrow -\frac{i}{2} \sigma^{CD}, \]  

(7)

which exactly leads from Eq. \([1]\) to Eq. \([5]\). Under a local Lorentz transformation, a Dirac spinor transforms as

\[ \psi'(x') = S(\Lambda) \psi(x). \]  

(8)

In order to write the Lagrangian, one needs to define the Dirac adjoint in curved space-time. In order to address this question, one has to remember that in flat-space-time, the Dirac adjoint \( \overline{\psi}(x) \) is defined in such a way that is transforms with the inverse of the spinor Lorentz transform as compared to \( \psi(x) \),

\[ \overline{\psi}'(x') = \overline{\psi}(x) S(\Lambda)^{-1} = \overline{\psi}(x) [S(\Lambda)]^{-1}. \]  

(9)

The problem of the definition of \( \overline{\psi}(x) \) in curved space-time is sometimes treated in the literature in a rather cursory fashion \([12]\). Let us see if in curved space-time, we can use the ansatz

\[ \overline{\psi}(x) = \overline{\psi}^+(x) \gamma^0, \]  

(10)

with the same flat-space \( \gamma^0 \) as is used in the flat-space Dirac adjoint. In this case,

\[ \overline{\psi}(x') = \overline{\psi}^+(x') S^+(\Lambda) \gamma^0 = \left( \overline{\psi}^+(x') \gamma^0 \right) \left[ \gamma^0 S^+(\Lambda) \gamma^0 \right], \]  

(11)

To first order in the Lorentz generators \( \epsilon_{AB} \), we have indeed,

\[ \gamma^0 S^+(\Lambda) \gamma^0 = 1 + \frac{i}{4} \epsilon^{AB} \gamma^0 \sigma_{AB} \gamma^0 = 1 + \frac{i}{4} \epsilon^{AB} \sigma_{AB} = [S(\Lambda)]^{-1} \]  

(12)

where we have used the identity

\[ \sigma_{AB}^+ = -\frac{i}{2} [\gamma_B^+, \gamma_A^-] = -\frac{i}{2} \gamma^0 [\gamma^0_{AB} \gamma^0, \gamma^0_{BA} \gamma^0] \gamma^0 \]

\[ = -\frac{i}{2} \gamma^0 [\gamma_B, \gamma_A] \gamma^0 = -\gamma^0 \sigma_{BA} \gamma^0 = \gamma^0 \sigma_{AB} \gamma^0. \]  

(13)

It is easy to show that Eq. \([12]\) generalizes to all orders in the \( \epsilon^{AB} \) parameters, which justifies our ansatz given in Eq. \([10]\). Indeed, the flat-space \( \gamma^0 \) matrix can be used in curved space, too, in order to construct the Dirac adjoint. The Dirac adjoint spinor transforms with the inverse spinor representation of the Lorentz group [see Eq. \([9]\)].
III. LAGRANGIAN AND CHARGE CONJUGATION

Equipped with an appropriate form of the Dirac adjoint in curved space-time, we start from the Lagrangian density

$$\mathcal{L} = \overline{\psi}(x) \left[\gamma^\mu \left\{i (\partial_\mu - \Gamma_\mu) - e A_\mu\right\} - m_I\right] \psi(x),$$

and attempt to derive the particle-antiparticle symmetry on the level of a transformation of the Lagrangian. The Lagrangian is Hermitian, and so

$$\mathcal{L} = \mathcal{L}^+ = \overline{\psi}(x) \left[(\gamma^\mu)^+ \left\{-i \overline{\partial}_\mu - e A_\mu\right\} - (-i) (\Gamma_\mu)^+ (\gamma^\mu)^+ - m_I\right] \left[\overline{\psi}(x)\right]^+.\quad (15)$$

An insertion of γ0 matrices under use of the identity (γ0)2 = 1 leads to the relation

$$\mathcal{L}^+ = \psi^+(x) \gamma^0 \left[\gamma^0 (\gamma^\mu)^+ \gamma^0 \left\{-i \overline{\partial}_\mu - e A_\mu\right\}\right] + i \left\{\gamma^0 (\Gamma_\mu)^+ \gamma^0 (\gamma^\mu)^+ \gamma^0 - m_I\right\} \gamma^0 \left[\overline{\psi}(x)\right]^+.\quad (16)$$

Also, we recall that γ0 (Γμ)^+ γ0 = −Γμ, because

$$\Gamma_\mu^+ = -\frac{i}{4} \omega^A_\mu \sigma^+_{AB} = -\frac{i}{4} \omega^A_\mu \gamma^0 \sigma_{AB} \gamma^0 = -\gamma^0 \Gamma_\mu \gamma^0.\quad (17)$$

So, the adjoint of the Lagrangian is

$$\mathcal{L}^+ = \psi^+(x) \gamma^0 \left[\gamma^0 \left\{i \overline{\partial}_\mu - e A_\mu\right\} - i \Gamma_\mu \gamma^0 - m_I\right] \gamma^0 \left[\overline{\psi}(x)\right]^+.\quad (18)$$

Now, we use the relations ψ^+(x) γ0 = ψ(x) and γ0 [ψ(x)]^+ = ψ(x), and arrive at the form

$$\mathcal{L}^+ = \overline{\psi}(x) \left[\gamma^\mu \left\{-i \overline{\partial}_\mu - e A_\mu\right\} - i \Gamma_\mu \gamma^0 - m_I\right] \psi(x).\quad (19)$$

Because \(\mathcal{L}\) is a scalar, a transposition again does not change the Lagrangian, and we have

$$(\mathcal{L}^+)^T = \psi^T(x) \left[(\gamma^\mu)^T \left\{-i \overline{\partial}_\mu - e A_\mu\right\} - i (\gamma^\mu)^T (\Gamma_\mu)^T - m_I\right] \left[\overline{\psi}(x)\right]^T.\quad (20)$$

A insertion of the charge conjugation matrix C = iγ^2 γ^0 leads to

$$(\mathcal{L}^+)^T = \psi^T(x) C^{-1} \left[C (\gamma^\mu)^T C^{-1} \left\{-i \overline{\partial}_\mu - e A_\mu\right\}\right] - i C (\gamma^\mu)^T C^{-1} C \Gamma_\mu C^{-1} - m_I\right] C \left[\overline{\psi}(x)\right]^T.\quad (21)$$

we use the identities C (Γμ)T C−1 = −Γμ, and C (Γμ)T C−1 = −Γμ. The latter of these can be shown as follows,

$$C \Gamma_\mu C^{-1} = \frac{i}{4} \left\{\frac{i}{2} \omega^A_\mu C \left[\gamma^B, \gamma^A\right] C^{-1}\right\} = \frac{i}{4} \left\{\frac{i}{2} \omega^A_\mu \left[-\gamma^B, -\gamma^A\right]\right\} = -\Gamma_\mu.\quad (22)$$

The result is the expression

$$(\mathcal{L}^+)^T = \psi^T(x) C^{-1} \left[(-\gamma^\mu) \left\{-i \overline{\partial}_\mu - e A_\mu\right\} - i (-\gamma^\mu) (-\Gamma_\mu) - m_I\right] C \left[\overline{\psi}(x)\right]^T.\quad (23)$$

Now we express the result in terms of the charge-conjugate spinor \(\psi^C(x)\) and its adjoint \(\overline{\psi}^C(x)\) (further remarks on this point are presented in Appendix B),

$$\psi^C(x) = C \left[\overline{\psi}(x)\right]^T, \quad \overline{\psi}^C(x) = -\psi^T(x) C^{-1},$$

where we use the identity C−1 = −C (see also Appendix A). The Lagrangian becomes

$$\mathcal{L} = (\mathcal{L}^+)^T = -\overline{\psi}^C(x) \left[\gamma^\mu \left\{i \overline{\partial}_\mu + e A_\mu\right\} - i \gamma^\mu \Gamma_\mu - m_I\right] \psi^C(x) = -\psi^C(x) \left[\gamma^\mu \left\{i (\partial_\mu - \Gamma_\mu) + e A_\mu\right\} - m_I\right] \psi^C(x).\quad (25)$$
The Lagrangian given in Eq. (25) differs from (15) only with respect to the sign of electric charge, as is to be expected, and with respect to the replacement of the Dirac spinor $\psi(x)$ by its charge conjugation $\psi^C(x)$. The overall minus sign is physically irrelevant as it does not influence the variational equations derived from the Lagrangian; besides, it finds a natural explanation in terms of the reinterpretation principle.

Namely, there is a connection of the spatial integrals

$$ J = \int d^3r \overline{\psi}(x)\psi(x) = \int d^3r \overline{\psi}(t, \vec{r})\psi(t, \vec{r}) = \int d^3r \overline{\psi^+}(t, \vec{r})\gamma^0\psi(t, \vec{r}) $$

and

$$ J^C = \int d^3r \overline{\psi^C}(x)\psi^C(x) = \int d^3r \overline{(\psi^C(t, \vec{r}))^+}\gamma^0\psi(t, \vec{r}) $$

and the energy eigenvalue of the Dirac equation in the limit of time-independent fields (see Appendices A and B). One can show that the energy eigenvalues of Dirac eigenstates $\psi$, in the limit of weak potentials and states composed of small momentum components, exactly correspond to the integrals $J$ and $J^C$ (up to a factor $m_I$). In turn, the dominant term in the Lagrangian in this limit is

$$ \mathcal{L} \rightarrow -\overline{\psi}(x) m_I \psi(x) = \overline{\psi^C(x)} m_I \psi^C(x). $$

Because the integral $\int d^3r \mathcal{L}$ equals $-J$ (or $+J^C$), the sign change becomes evident: it is due to the fact that the states $\psi^C$ describe antiparticle wave functions where the sign of the energy flips in comparison to particles. The matching of $m_I$ to the gravitational mass can be performed in a central, static field \[ 2, 8 \], and results in the identification $m_I = m_G$, where $m_G$ is the gravitational mass. The gravitational covariant derivative $\partial_\mu - \Gamma_\mu$ has retained its form in going from (15) to (25), in agreement with the perfect particle-antiparticle symmetry of the gravitational interaction. Because the above demonstration is general and holds for arbitrary (possibly dynamic) space-time background $\Gamma$, there is no room for a deviation of the gravitational interactions of antiparticles (antimatter) to deviate from those of matter. This has been demonstrated here on the basis of Lagrangian methods, supplementing a recent preliminary result \[ 8 \].

IV. CHARGE SYMMETRY AND GRAVITY

A. Orientation

In accordance with Ref. [30], we parameterize a putative charge excess in matter and antimatter as follows,

$$ q_e = -|e|, \quad q_p = |e|(1 + \epsilon_{p-e}), \quad \epsilon_{p-e} = \frac{q_e + q_p}{|e|}, \quad \epsilon_n = \frac{q_n}{|e|}. $$

Here, $q_e = e$ is the electron charge and $|e|$ its modulus, while $q_p$ and $q_n$ are the proton and neutron charges. If we assume, with Ref. [30], charge conservation in the $\beta$ decay of the neutron, then the charge-neutrality violating parameter for the neutron becomes

$$ \epsilon_n = \epsilon_{p-e} \equiv \epsilon. $$

If a body containing with $Z$ protons and electrons, as well as $N$ neutrons is measured as being neutral with sensitivity $\delta q$, one has

$$ |Z\epsilon_{p-e} + N\epsilon_n| = (Z + N)|\epsilon_n| \leq \delta q, \quad |\epsilon_n| \leq \frac{\delta q}{(Z + N)|e|}. $$

One should note that the acoustic method used in Ref. [30] in order to determine limits on $|\epsilon_n|$ is not free from pitfalls and requires a considerable additional mathematical formalism in the evaluation of the experiment. For example, in a note to Table I of Ref. [30], data published in previous work [37] may exhibit inconsistencies. A paper which initially claimed an accuracy on the level of $10^{-23}$ for $|\epsilon_n|$ (Ref. [38]) has recently been questioned in Ref. [30], with the claim that their result on $|\epsilon_n|$ could not be considered to be better than $10^{-19}$ if all inaccuracies and neglected systematic effects in the paper [38] are properly taken into account. The paper [30] also points out some additional
rectification of the analysis of the resonant modes in the gas-filled capacitor used in previous experiment \[38\]. Table I of Ref. \[30\] contains a comprehensive compilation of previous measurements of $\epsilon_q$. We will use their result, given in an unnumbered equation on the last-but-one page of Ref. \[30\],

$$
\epsilon_q = (-0.1 \pm 1.1) \times 10^{-21},
$$

(33)

for matter particles (both a hydrogen atom as well as the constituent atoms of the Earth). Limits on charge asymmetry of matter have also been derived on the basis of model-dependent astrophysical methods \[39, 40\]. Separate limits have been set on the neutrality of the neutrino by astrophysical methods \[41–45\].

Tests on the charges of positrons and antiprotons, derived from measurements of their cyclotron resonance frequencies and from spectroscopic data \[46\]. The most recent direct test \[32\] reveals a limit

$$
|\tau_q| \leq 7.1 \times 10^{-10}
$$

(34)

for antimatter where the limit is given at the 1$\sigma$ level (68.3% confidence level). Here, the parameters for antiparticles are given by

$$
q_\pi = |e|, \quad q_\bar{\pi} = -|e| (1 + \epsilon_{\bar{\pi} - e}),
$$

$$
\epsilon_{\bar{\pi} - e} = \frac{q_\pi + q_\bar{\pi}}{-|e|}, \quad \epsilon_{\bar{\pi}} = \frac{q_\bar{\pi}}{-|e|}.
$$

Here, $\pi$, $\bar{\pi}$ and $\bar{\pi}$ stand for the positron, antiproton, and antineutron, respectively. We shall also make the assumption that charge is conserved in the $\beta$ decay of the antineutron and write

$$
\epsilon_{\bar{\pi}} = \epsilon_{\bar{\pi} - e} \equiv \tau_q.
$$

(37)

**B. Lagrangian and Matter–Antimatter Charge Neutrality**

Gravity and the electromagnetic interaction are the only long-range fundamental forces known to us. In the quantized picture, they are both mediated by the exchange of zero-mass quanta, namely, gravitons and photons. Let us write a slight modification of the quantum electrodynamical (QED) Lagrangian,

$$
\mathcal{L} = -\sum_i e \bar{\psi}_e \gamma^\mu \psi_e A_\mu - (1 + \epsilon_{p - e}) |e| \bar{\psi}_p \gamma^\mu \psi_p A_\mu - e_n |e| \bar{\psi}_n \gamma^\mu \psi_n A_\mu.
$$

(38)

Here, the electron-positron field operator is $\psi_e$, while the composite spin-1/2 operator for the proton-antiproton field is $\psi_p$, and the neutron-antineutron field could be described by a spin-1 generalization of the Dirac equation (see Ref. [47]). As a side remark, we note that the current conservation implies that

$$
\epsilon_{p - e} = 2\epsilon_u + \epsilon_d
$$

(39)

where the valence quark couplings for the up and the down quark are $\epsilon_u$ and $\epsilon_d$, respectively.

From the form of the Lagrangian (38), ramifying the considerations of Sec. II it follows that electrons (and consequently positrons) carry a charge $\pm e$, while protons (and antiprotons) carry a charge $\pm (1 + \epsilon_{p - e}) |e|$. This results in a hydrogen atom having a charge $\epsilon_{p - e} |e|$, while antihydrogen atoms carry a charge $(-\epsilon_{p - e} |e|)$. For the first interpretation of an antimatter gravity experiment delineated in Sec. II we shall thus assume that

$$
\epsilon_q = -\tau_q.
$$

(40)

This assumption leads to an effective interaction between hydrogen and antihydrogen atoms of

$$
V_{H\bar{H}}(R) = -\frac{e^2 e^2}{4\pi \epsilon_0 R},
$$

(41)

where $R$ is the distance between the hydrogen and the antihydrogen atoms. (We here ignore the van-der-Waals interaction which goes as $1/R^6$; one can show that it is negligible at long distances which greatly exceed the Bohr radius. It is interesting to note, though, that a closed inspection reveals that the van der Waals interaction between hydrogen and antihydrogen has exactly the same coefficients and the same, attractive, sign, as compared to the
interaction between two hydrogen atoms.) For the interaction of hydrogen and other hydrogen atoms, the interaction potential is the opposite,

\[ V_{HH}(R) = \frac{\varepsilon_q^2 e^2}{4\pi\varepsilon_0 R} . \] (42)

The two interaction potentials (41) and (42) mimic the gravitational (Newtonian) interaction in the nonrelativistic limit. At the same time, the quantity \( \varepsilon_{p-e} \) is a measure for the electric neutrality of the hydrogen atom. A matter-antimatter gravitational experiment allows us to compare the putative deviation of the quantity \( \varepsilon_{p-e} \) from zero, to gravitational interactions.

We now proceed to discuss the precision of the limits on the parameter \( \varepsilon_q \) achievable by measuring the acceleration due to gravity on antihydrogen atoms in the Earth’s gravitational field. We will derive our estimates by considering the residual interaction of a single hydrogen (antihydrogen) atom with the Earth.

In order to treat the residual interaction of the hydrogen (or antihydrogen) atom with the Earth, we shall thus treat the Earth as being composed of neutral atoms, i.e., in terms of a uniformly distributed (in space) collection of electrons and protons, across the volume of the Earth. Because the gravitational and the putative residual electrostatic interaction have the same functional form, it suffices to compare the prefactors of the potentials at the Earth’s surface. By Gauss’s theorem, this amounts to a comparison of the parameters \( \xi_G \) and \( \xi_E \),

\[ \xi_G = G m_p \, M_\oplus , \quad \xi_E = \frac{\varepsilon_q^2 e^2 N_p}{4\pi\varepsilon_0} . \] (43)

Here, \( N_p \) is the number of protons and neutrons in the Earth, taken into account with the assumption given in Eq. (37) in mind,

\[ N_p \approx \frac{M_\oplus}{m_p} = 3.57 \times 10^{51} . \] (44)

Hence,

\[ \xi_G \approx 6.67 \times 10^{-13} \text{Nm}^2 , \quad \xi_E \approx \varepsilon_q^2 \times 8.24 \times 10^{23} \text{Nm}^2 . \] (45)
We observe that, under the assumptions delineated above, \( \xi_E \) flips sign as we go from hydrogen to antihydrogen. The ratio of the sum of the gravitational and residual electrostatic force, to the gravitational force, is given by \( (\xi_G - \xi_E)/\xi_G \) for matter (because of the repulsive nature of the residual electrostatic interaction), and \( (\xi_G + \xi_E)/\xi_G \) for antimatter (because of the attractive nature of the residual electrostatic interaction). Let us assume that an experiment establishes that the attractive forces on hydrogen and antihydrogen atoms do not differ by a fraction of more than \( \chi \). This means that

\[
\left| \frac{\xi_G + \xi_E}{\xi_G} - \frac{\xi_G - \xi_E}{\xi_G} \right| = \frac{2\xi_E}{\xi_G} \leq \chi. \tag{46}
\]

This will set a limit on the charge neutrality of both matter and antimatter determined by the relation

\[
|\epsilon_q| = |\tau_q| \leq 6.4 \times 10^{-19} \sqrt{\chi}. \tag{47}
\]

Let us assume that, in the future, antimatter gravity could be measured with the same precision at which the (local) acceleration due to gravity is currently being determined \([48]\), namely, \(10^{-9}\). In this case, the measurement of the acceleration due to gravity of matter and antimatter could simultaneously set limits on \(|\epsilon_q|\) and \(|\tau_q|\), which would be so precise that the limit on \(|\epsilon_q|\) would surpass the accuracy of limits from matter experiments alone \([30]\).

### C. Charge Neutrality of Matter and Antimatter

For the second possible interpretation of an antimatter “gravity” experiment in terms of charge asymmetry, we shall relax the assumption \([40]\) and write the matter-antimatter potential as follows,

\[
V_{HH}(R) = \frac{\epsilon_q \tau_q e^2}{4\pi \epsilon_0 R}, \tag{48}
\]

while the hydrogen-hydrogen interaction retains the form \((41)\). The difference is

\[
V_{HH}(R) - V_{HH}(R) = \frac{\epsilon_q (\epsilon_q - \tau_q) e^2}{4\pi \epsilon_0 R} \approx -\frac{\epsilon_q \tau_q e^2}{4\pi \epsilon_0 R}, \tag{49}
\]

where we use the fact that the bound \([33]\) for antimatter is much less strict and allows for much larger \(\tau_q\) parameters as compared to the bound \([33]\) which we have for matter, as parameterized by the coefficient \(\epsilon_q\).

We define the alternative parameter

\[
\xi_E \rightarrow \bar{\xi}_E \approx |\tau_q \epsilon_q| \times 8.24 \times 10^{23} \text{ N m}^2. \tag{50}
\]

Equation \([46]\) is thus replaced by

\[
\left| \frac{\xi_G + \bar{\xi}_E}{\xi_G} - \frac{\xi_G - \xi_E}{\xi_G} \right| \approx \frac{\bar{\xi}_E}{\xi_G} \leq \chi. \tag{51}
\]

This will set a limit on \(|\tau_q \epsilon_q|\) determined by the solution of the equation

\[
\bar{\xi}_E \leq \chi \xi_G, \quad |\tau_q| |\epsilon_q| \leq 8.1 \times 10^{-37} \chi. \tag{52}
\]

For \(\chi = 1\), as evident from Fig. 1, the condition \([52]\) eliminates 99.9986% of the parameter space otherwise allowed by the conditions \([33]\) and \([34]\). Note that the case \(\chi = 1\) corresponds to a pessimistic view of the accuracy achievable for an antimatter gravity experiment; namely, it would correspond to a scenario where all that one could say is that the difference of the gravitational interactions of antimatter and matter, with the Earth, is not larger than the total gravitational interaction of matter, i.e., not larger than the acceleration due to gravity experienced by matter.

### V. CONCLUSIONS

In this paper, we have analyzed attractive alternative possibilities for interpreting upcoming antimatter gravity experiments. We have approached the underlying questions in the following way. First, we have asked the question...
of whether there is any room for a genuine violation of the equivalence principle for antiparticles, based on the gravitationally coupled Dirac equation alone (Secs. II and III). On the basis of Lagrangian methods, we can deny such a possibility (Sec. III). Along the course of the discussion, we also discuss, within a necessary detour, how to define the Dirac adjoint (Sec. II) for curved space-times, with the principle idea that the Dirac adjoint bispinor should transform under the inverse of the spinor Lorentz transformation as compared to the original Dirac bispinor (Sec. II). The results of Sec. III imply that the equivalence principle holds for antimatter. They ramify recent conclusions made in Ref. [8] made on the basis of the properties of the solutions of the Dirac equation. The proof is completed here on the basis of much more general, and elegant, Lagrangian methods.

We then proceed to ask the question of the best possible interpretation of upcoming gravitational antimatter experiments. Surprisingly, we find that such experiments could provide us with the very good limits on the electric neutrality of the hydrogen atom, detecting a charge asymmetry between electron and proton (or positron and antiproton). We advocate two possible alternative ways to interpret upcoming antimatter gravity experiments, depending on which additional assumptions are being made. First, in line with many other articles on the subject of charge asymmetry, we assume charge conservation in nuclear beta decay, which enables us to define the parameters $\epsilon_q$ and $\chi_q$ in Eqs. (31) and (37).

Within the first advocated interpretation, we make the additional assumption that the charge asymmetry parameters for matter and antimatter fulfill the relation $\epsilon_q = -\tau_q$, i.e., that the asymmetry parameters for matter and antimatter are opposite. Under this additional assumption, if we can set a limit $\chi$ on the relative difference of the gravitational forces experienced by matter and antimatter, we can derive the bound given in Eq. (47) for the uniform charge asymmetry parameter $|\epsilon_q| = |\tau_q|$.

If we keep $\epsilon_q$ and $\tau_q$ as independent parameters, then we conclude that an antimatter gravity experiment leads to bounds on the product $|\epsilon_q \tau_q|$ given by the inequality [31]. An antimatter gravity experiment which compares gravity and antimatter acceleration due to gravity up to a (pessimistic) relative difference $\chi = 1$ [see Eq. (51)] leads to additional bounds within the $(\epsilon_q, \tau_q)$ plane shown in Fig. 1.

A final remark concerns the charge asymmetry parameters connected with the neutron. Throughout this work, we make the assumptions [31] and [37] connected with charge conservation in nuclear beta decay. We should point out that questions regarding the coupling constants $\epsilon_n = \epsilon_u + 2\epsilon_d$ and $\tau_n = \tau_u + 2\tau_d$ could be answered by performing an experiment with antideuterium, in comparison to antihydrogen, testing the assumptions [31] and [37] and mapping out a more diversified parameter space, with additional parameters $\epsilon_n$ and $\tau_n$.

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Appendix A: Sign Change of $\overline{\psi} \psi$ under Charge Conjugation

With the charge conjugation matrix $C = i\gamma^2\gamma^0$ (superscripts denote Cartesian indices), and the Dirac adjoint $\overline{\psi} = \psi^+ \gamma^0$, we have

$$\psi^C = C \overline{\psi}^T = i\gamma^2 \psi^0 \psi^* = i\gamma^2 \psi^*.$$  \hspace{1cm} (A1)

We recall that the $\gamma^2$ (contravariant index, no square) matrix in the Dirac representation matrix is

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma^2)^+ = \sigma^2, \quad (\gamma^2)^+ = -\gamma^2.$$  \hspace{1cm} (A2)

The Dirac adjoint of the charge conjugate is

$$\overline{\psi}^C = (\psi^C)^+ \gamma^0 = \psi^T(-i)(\gamma^2)^+ \gamma^0 = \psi^T(-i)(-\gamma^2)\gamma^0 = \psi^T i \gamma^2 \gamma^0.$$  \hspace{1cm} (A3)

This leads to a verification of the sign flip of the mass terms in the gravitationally coupled Lagrangian for antimatter, given in Eq. (25) [see also Eqs. (26) and (27)],

$$\overline{\psi}^C \psi^C = (\psi^T i \gamma^2) \gamma^0 (i \gamma^2 \psi^*) = -(i)^2 \psi^T (\gamma^2)^2 \gamma^0 \psi^* = -\psi^T \gamma^0 \psi^* = \overline{\psi} \psi.$$  \hspace{1cm} (A4)
Two useful identities \((i)\) \(\gamma^0 C^+ \gamma^0 = C\) and \((ii)\) \(C^{-1} = -C\) have been used in Sec. III. These will be derived in the following. The explicit form of the \(\gamma^2\) matrix in the Dirac representation implies that \((\gamma^2)^+ = -\gamma^2\). Based on this relation, we can easily show that

\[
C^+ = (i \gamma^2 \gamma^0)^+ = -i \gamma^0 (\gamma^2)^+ = i \gamma^0 \gamma^2 = -i \gamma^2 \gamma^0 = -C.
\]

The first identity \(\gamma^0 C^+ \gamma^0 = C\) can now be shown as follows,

\[
\gamma^0 C^+ \gamma^0 = [i \gamma^2 \gamma^0] \gamma^0 = -i \gamma^0 \gamma^2 = i \gamma^2 \gamma^0 = C.
\]

Furthermore, one has

\[
CC^+ = C(-C) = i \gamma^2 \gamma^0 i \gamma^0 \gamma^2 = -(\gamma^2)^2 = -(-\mathbb{1}_{4 \times 4}) = \mathbb{1}_{4 \times 4},
\]

so that

\[
C^{-1} = C^+ = -C,
\]

which proves, in particular, that \(C^{-1} = -C\).

Appendix B: General Considerations

A few illustrative remarks are in order. These concern the following questions: \((i)\) To which extent do gravitational and electrostatic interactions differ for relativistic particles? This question is relevant because, in the nonrelativistic limit, in a central field, both interactions are described by potentials of the same functional form (“1/R potentials”). \((ii)\) Also, we should clarify why the integrals \((26)\) and \((27)\) represent the dominant terms in the evaluation of the Dirac particle energies, in the nonrelativistic limit.

After some rather deliberate and extensive considerations, one can show \([7]\) that, up to corrections which combine momentum operators and potentials, the general Hamiltonian for a Dirac particle in a combined electric and gravitational field is

\[
H_D = \vec{\alpha} \cdot \vec{p} + \beta \{ m(1 + \phi_G) \} + e\phi_C,
\]

where \(\phi_G\) is the gravitational, and \(\phi_C\) is the electrostatic potential. Also, \(\vec{\alpha}\) is the vector of Dirac \(\alpha\) matrices, \(\vec{p}\) is the momentum operator, and \(\beta = \gamma^0\) is the Dirac \(\beta\) matrix. After a Foldy–Wouthuysen transformation \([49]\), one sees that the gravitational interaction respects the particle-antiparticle symmetry, while the Coulomb potential does not, commensurate with the opposite sign of the charge for antiparticles. Question \((i)\) as posed above can thus be answered with reference to the fact that, in leading approximation, the gravitational potential enters the Dirac equation as a scalar potential, modifying the mass term, while the electrostatic potential can be added to the free Dirac Hamiltonian \(\vec{v} \alpha \cdot \vec{p} + \beta m\) by covariant coupling \([1]\).

The second question posed above is now easy to answer: Namely, in the nonrelativistic limit, one has

\[
\vec{\alpha} \cdot \vec{p} \to 0,
\]

and furthermore, the gravitational and electrostatic potentials can be assumed to be weak against the mass term, at least for non-extreme Coulomb fields \([50]\). Under these assumptions, one has \(H_D \to \beta m\), and the matrix element \(\langle \psi | H_D | \psi \rangle\) assumes the form \(\int d^3r \psi^+(\vec{r}) \gamma^0 m \psi(\vec{r})\) [see Eq. \((20)\)].

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