I. INTRODUCTION

The properties of phase-transitions in a pure system can be modified due to quenched disorder. This problem has been studied in most details at a second-order transition point\cite{12}, but much less is known when the transition is of first order\cite{3}. When the disorder is coupled to the local energy density, such as for bond disorder, there is a general tendency that the latent heat at the transition point is reduced\cite{2}. In two-dimensional systems with nearest-neighbour (or short-range (SR)) interactions any amount of bond disorder is enough to turn the transition into second order\cite{10}. The new universality class of the problem, however, remains unknown and numerical investigations are needed to identify the properties of the emergent random fixed point\cite{8,10}. In three- and higher dimensional SR systems, however, weak disorder is generally irrelevant, thus the phase-transition stays discontinuous and only for strong enough disorder will it turn to a second-order one. This type of problem has been numerically studied for the q-state Potts model with \( q > 2 \)\cite{11,19}. In particular a mapping between the random-field Ising model (RFIM) and the Potts model in the \( q \rightarrow \infty \) limit has been used to predict some tricritical exponents of the latter random model\cite{19}.

Homogeneous, i.e. nonrandom systems with long-range (LR) interactions could have an ordered phase\cite{30}, and a first-order transition, too, even if the system is one-dimensional. This happens, among others for the \( q \)-state Potts chain\cite{30} with power-law interactions

\[
J(r) \approx J r^{-(1+\sigma)}, \quad (1)
\]

where \( r \) is the distance between the sites and the exponent is \( \sigma > 0 \) to have extensive total energy (for \( \sigma < 0 \) one should divide \( J \) by \( L^\sigma \)). According to numerical results\cite{15} the transition in the LR Potts chain is of first order for sufficiently large values of \( q \), where the limiting value \( q_c = q_c(\sigma) \) is an increasing function of \( \sigma \). On the other hand the transition for \( q < q_c(\sigma) \) is of second order.

Low-dimensional LR models with power-law interactions became the subject of intensive research recently, after it has been noticed that the decay exponent, \( \sigma \) in the problems plays the role of some kind of effective dimensionality of the analogous SR model. Among the classical problems studied so far we mention the non-random Ising model in one- and two-dimensions\cite{19,23}, the non-random Potts chain\cite{15}, the Ising spin-glass model\cite{23,27} and the RFIM in one dimension\cite{23,33}. For quantum models we mention investigations of the transverse-field Ising model both with pure\cite{50,59} and random couplings\cite{17,50} and the Anderson localization problem\cite{31}, for reaction-diffusion type models the contact process and similar models wit\cite{35} and without\cite{26,34} quenched disorder.

The critical properties of LR models are often unusual. Here we mention that the classical Ising chain for \( \sigma = 1 \), as well as other one-dimensional discrete spin models with LR interaction have a so-called mixed-order (MO) phase transition\cite{50,84}, at which point the order-parameter has a jump, but at the same time the correlation length is divergent. We note that recently MO transitions have been observed in other problems, too\cite{20,26,63}.

In the present paper we consider LR models, having a first-order transition in their non-random version and study the effect of quenched disorder on the phase-transition properties of the system. To be specific, we consider the LR Potts model in one dimension for large values of \( q \) (actually we consider the \( q \rightarrow \infty \) limit), when the transition of the pure model is of first order for all values of the decay exponents, \( \sigma > 0 \). We have random nearest-neighbour couplings with a variance \( \Delta^2 \), but the long-range forces are non-random and follow the behaviour in Eq. (1). We study the phase-transition of the system for different values of the effective dimensionality (\( \sigma \)) and the strength of disorder (\( \Delta \)). The free energy and the magnetization of a given random sample is calculated exactly by a computer algorithm, which works in polynomial time\cite{59}. We follow the temperature dependence of the average magnetization in relatively large finite samples and the location of the phase-transition point and its properties are analyzed by finite-size extrapolation methods.

The structure of the paper is the following. The model and some results are summarized in Sec[II]. Numerical
results at different points of the phase diagram are presented in Sec [II] and analyzed by finite-size scaling. We close our paper with a discussion in Sec [IV].

II. MODEL AND SOME RESULTS

We consider the ferromagnetic $q$-state Potts-model in a one-dimensional periodic lattice with long-range interactions defined by the Hamiltonian:

$$\mathcal{H} = -\sum_i J_i \delta(s_i, s_{i+1}) - \sum_{i<j} J_{ij} \delta(s_i, s_j).$$

(2)

Here $s_i = 1, 2, \ldots, q$ is a Potts-spin variable at site $i = 1, 2, \ldots, L$ and the long-range interaction, $J_{ij}$, has a power-law dependence as in Eq. [II] with $r = \min(|L - |i - j|, |i - j|)$. The nearest neighbour couplings, $J_i \equiv J_{i,i+1}$, are random variables. For simplicity we take $J_i$ from a bimodal distribution, being either $J_- = J - \Delta$ or $J_+ = J + \Delta$ with equal probability. In the following we set the energy-scale to $J = 1$ and restrict ourselves to $0 < \Delta < 1$.

A. The large-$q$ limit

In this paper we consider the $q \rightarrow \infty$ limit of the model, when the reduced free-energy in the Fortuin-Kasteleyn representation is dominated by a single graph, $G$, and given by:

$$-\beta f L = \max c W(G), \quad W(G) = \left[c(G) + \beta \sum_{ij \in G} J_{ij} \right].$$

(3)

Here $c(G)$ stands for the number of connected components of $G$ and $\beta = 1/(T \ln q)$, with the temperature $T$.

In the homogeneous nonrandom model with $\Delta = 0$ there are only trivial optimal graphs as shown in appendix for any $\sigma$. In the low-temperature phase, $T < T_c$, it is the fully connected graph with $W(G_c) = -\beta L f_{\hom} = 1 + L \beta Z_L(1 + \sigma)$, with

$$Z_L(1 + \sigma) = \frac{1}{L^{\frac{L}{2}}} \sum_{i=1}^{L/2} (L + 1 - i)^{-(1+\sigma)} = \left(1 + \frac{1}{L}ight) \zeta_{L/2}(1+\sigma) - \frac{1}{L} \zeta_{L/2}(\sigma),$$

(4)

where we have assumed that $L$ is even. Here $\zeta_{L/2}(\alpha) = \sum_{i=1}^{L/2} i^{-\alpha}$ and for $L \rightarrow \infty$ we have the Riemann-zeta function, $\zeta(\alpha)$. In the high-temperature phase, $T > T_c$, the optimal graph is the empty graph with $W(G_e) = -\beta L f_{\hom} = L$. The phase-transition point in the thermodynamic limit is given by $\beta_c = 1/\zeta(1 + \sigma)$ where the phase transition is of first order having the maximal jump in the magnetization.

In the limit where $\sigma$ goes to infinity one recovers the disordered SR Potts chain. In that case and for finite size $L$, there are non-trivial optimal sets. But in the thermodynamical limit, the magnetization still jumps from zero to one for the bimodal distribution. This is shown in the appendix.

B. Stability analysis of the random model

Here we start with weak disorder, $\Delta \ll 1$, and estimate the characteristic function of non-homogeneous optimal graphs. First let us consider an island of $l + 1 \leq \frac{L}{2}$ consecutive sites, which are fully connected within the sea of isolated points. The corresponding characteristic function is given by:

$$W(G_1) = L - l + \beta \left[(l+1)\zeta(1+\sigma) - \zeta(\sigma)\right] + \beta \Delta \epsilon(l),$$

(5)

where $\epsilon(n)$ is the sum of $n$ random numbers with mean zero and variance unity, thus $\epsilon(n) \sim \sqrt{n}$ for large $n$. At the transition point of the pure system, $\beta = \beta_c = 1/\zeta(1 + \sigma)$, the new diagram is the optimal set, i.e. $W(G_1) = W(G_c)$, provided: $\Delta > [(\zeta(1+\sigma) - \zeta(\sigma)) + \zeta(\sigma)/\epsilon(l-1)$. For large-$l$ the r.h.s. of this inequality scales as: $l^{1-\sigma} / l^{1/2} \sim l^{1/2 - \sigma}$, thus we have the condition

$$\Delta > C l^{1/2 - \sigma}, \quad l \gg 1.$$  

(6)

Consequently for a decay exponent $\sigma > 1/2$ there is a new, non-homogeneous optimal set and the (phase-transition) properties of the system are modified by any small amount of disorder, at least in the thermodynamic limit. On the contrary for $\sigma < 1/2$ the transition, at least for small $\Delta$ stays first order and it could be changed only by strong enough disorder, i.e. for large $\Delta$.

Next we study the stability of the fully connected graph $G_c$ and consider a diagram, $G_d$, in which a fully connected sea of points there are $l$ disconnected sites. Its characteristic function is given by:

$$W(G_2) = l + 1 + \beta L E_L(1 + \sigma)$$

$$-\beta \left[(2\zeta_{L/2}(1+\sigma) - \zeta(1+\sigma)) + \zeta(\sigma)\right] + \beta \Delta \epsilon(l+1).$$

(7)

At $\beta = \beta_c$ we have $W(G_2) < W(G_c)$, at least for weak disorder for any value of $\sigma > 0$. This means, that considering the stability of the two trivial optimal sets of the pure system at $\beta = \beta_c$, these are not symmetric. For $\sigma > 1/2$ the empty diagram is unstable, while the fully connected graph is stable for weak disorder. We note that in the SR model both graphs become unstable at the same value of the dimensionality: $d \leq 2$.

In the LR model in the modified transition regime, $\sigma > 1/2$, we can define a breaking-up length:

$$l^* \sim \Delta^{1/(1/2 - \sigma)}, \quad \sigma > 1/2,$$

(8)

which is the typical size of connected clusters. This means, that in a finite system one should have $L > l^*$.
in order to be able to observe a new type of transition, otherwise there is a pseudo-first-order transitions in the finite system.

C. Relation with the RFIM

The previous stability analysis is based on the properties of an interface separating the two trivial optimal graphs and analogous reasoning due to Imry and Ma work for the RFIM, in which case the interface separates the ordered and disordered regions of the model. This mapping has been observed by Cardy and Jacobsen and can be generalized for LR interactions in which case the RFIM in a one-dimensional lattice is defined by the Hamiltonian:

$$\mathcal{H}_{RFIM} = -\sum_i B_i S_i - \sum_{i<j} J_{ij} S_i S_j ,$$

in terms of $S_i = \pm 1$. Here $B_i$ is a random variable with zero mean and variance $\Delta^2$ and $J_{ij}$ is in the same form as in Eq. (1). The critical behavior of $\mathcal{H}_{RFIM}$ has been studied in the literature and $\sigma$-dependent properties are found, which are summarized in the following.

There is a ferromagnetic ordered phase in the system for $0 < \sigma < 1/2$ (which corresponds to phase-coexistence, i.e. first-order transition in the RBPM) and there is no spontaneous ordering for $\sigma > 1/2$ (which is analogous to the absence of first-order transition in the RBPM). The transition to the ferromagnetic ordered phase is mean-field (MF) type in the region $0 < \sigma < 1/3$, where the critical exponents are the MF ones: $\alpha_{RF} = 0, \beta_{RF} = 1/2, \gamma_{RF} = 1$ and $\nu_{RF} = 1/\sigma$. On the contrary for $1/3 < \sigma < 1/2$ the transition is non-MF: the critical exponent $\nu_{RF}$ is not known exactly, but we have the relations:

$$\frac{2-\alpha_{RF}}{\nu_{RF}} = 1 - \sigma, \quad \frac{\beta_{RF}}{\nu_{RF}} = \frac{1}{2} - \sigma, \quad \frac{\gamma_{RF}}{\nu_{RF}} = \sigma$$

Cardy and Jacobsen has conjectured relations between the magnetization exponents of the RFIM and the tricritical exponents in the energy sector of the RBPM, at least for SR models. If we assume the validity of these relations for LR interactions, too, we have for the correlation-length exponent of the RBPM at the tricritical point:

$$\nu = \frac{\nu_{RF}}{\beta_{RF} + \gamma_{RF}} .$$

Thus the conjectured results are $\nu = \frac{2}{3\sigma}$ and $\nu = 2$ in the MF-region and in the non-MF region, respectively.

III. NUMERICAL CALCULATION

A. Preliminaries

As for systems with quenched disorder one should perform two averages: first, the thermal average for a given realization of disorder and second, averaging over the disorder realizations. For a given random sample of length $L$ the thermal average is obtained through the solution of the optimization problem given in Eq. (2). Having the optimal graph of the sample, we have the free-energy as well as the structure of connected clusters in this graph. The magnetization of the sample, $m$, is given by the number of sites in the largest cluster, $N_{\text{max}}$, as $m = N_{\text{max}}/L$. The optimization process for a given sample is solved exactly by a combinatorial optimization algorithm which works polynomially in time. This makes us possible to treat relatively large samples up to $L = 1024$ in some cases up to $L = 2048$. In the latter case the typical computational time of a sample in the complete temperature range is about 6-7 hours in a 2.4 GHz processor. A drawback of the calculation, that the possible graphs in the present problem are fully connected, having $L(L-1)/2$ possible edges and the algorithm needs so many iterations, which increases the computational time accordingly. In the second step of the averaging process we have considered several independent random samples, their typical number being a few 10000, for $L = 1024$ a few 1000.

B. Magnetization profiles

Before entering in details to study the phase-diagram of the system we have made a rough estimate of the domain, in which the transition is very strongly first order.

FIG. 1: Schematic phase-diagram of the LR Potts chain with random nearest neighbour couplings in the $q \to \infty$ limit together with points of the phase diagram studied numerically. The border of the strongly 1st order regime in a finite system (dashed line) is calculated with $L = 256$ and with $N^R = 600$ samples, see text. The plus sign refers to long range first order, the cross sign refers to mixed order, the circle to short range first order, and the square to second order transitions, respectively.
For this purpose we have analysed the phase-transition of $N^\# = 600$ samples of length $L = 256$. In the shaded area of the $\sigma - \Delta$ phase diagram in Fig. 1, in all samples the transition is between the fully connected graph and the empty graph, thus the transition is maximally 1st order, as in the homogeneous system. Then, we have chosen a several points outside the strongly first-order regime, which are indicated in Fig. 1. The selected points can be divided into two groups: a set of points with relatively weak disorder, $\Delta = 0.2$ and $\Delta = 0.5$ and another set with quite strong disorder $\Delta = 0.75$. At each point the calculation of the optimal graph is performed in the complete temperature range: we have monitored the temperature dependence of the magnetization and focused to its possible singular behaviour. These calculations are performed in finite systems with $L = 64, 128, \ldots, 1024$ and the actual properties of the singularity, thus the form of the phase transition is analyzed by finite-size extrapolation.

1. Weak and intermediate disorder regimes: $\Delta = 0.2$ and $\sigma = 1$

For weak disorder with $\Delta = 0.2$ we have studied the point of the phase-diagram with $\sigma = 1$ (square on Fig. 1), i.e. at border of the LR regime, the magnetization profiles are shown in Fig. 2. It is seen that due to disorder the first-order transition in the pure system is rounded: the jump in the magnetization is decreasing with increasing size and in the thermodynamic limit the jump is expected to disappear, $\Delta(L) \sim L^{-\beta/\nu}$, so that the limiting curve $\lim_{L \to \infty} m(L,T) = m(T)$ is continuous. However, its derivative at $T = T_c$ is expected to be divergent, so that $m(T) - m(T_c) \sim |T - T_c|^{\beta}$. The finite-size transition points are shifted as $T_c(L) - T_c \sim L^{-1/\nu}$. Note, that for $T < T_c$ ($T > T_c$) the profiles satisfy $m(L_1,T) > m(L_2,T)$ ($m(L_1,T) < m(L_2,T)$) for $L_1 < L_2$. We have studied also the temperature dependence of the average energy-density, which is shown in Fig. 3. At the transition point in small finite systems there is a discontinuity of the energy-density, which seems to disappear in the thermodynamic limit, but its first derivative, the specific heat is divergent: $C(T) \sim |T - T_c|^{-\alpha}$. Consequently the transition according to the Ehrenfest classification is of second-order.

For intermediate disorder, $\Delta = 0.5$, two points are considered with $\sigma = 0.75$ and $\sigma = 1.0$, the calculated average magnetization profiles are presented in Figs. 4 and 5. In both cases the transition seems to be of second-order, which is in agreement with the temperature dependence of the energy-densities.
2. Strong disorder regime: $\Delta = 0.75$

At the disorder parameter $\Delta = 0.75$ we have studied different regimes by varying the decay exponent $\sigma$.

a. $\sigma \lesssim 0.5$: LR first-order transitions (plus signs on Fig.1) At the point $\sigma = 0.4$ in Fig.6 the average magnetization has a finite jump of $\Delta m \approx 1$ for all finite systems. The finite-size transition points, which are identified with the position of the jump, $T_c(L)$, are shifted such that $T_c(L_1) < T_c(L_2)$ for $L_1 < L_2$. Furthermore the distance from the true transition point is well described by the asymptotic behaviour in the non-random system: $\Delta T_c = T_c - T_c(L) \sim L^{-\sigma}$, since $T_c(L) \propto \text{cst} \, \zeta_L(1 + \sigma)$. Thus the scaling exponent associated to lengths is $\nu \approx 1/\sigma$. At this point, and in more general in the regime $\sigma \lesssim 0.5$ there is a random first-order transition due to LR forces.

FIG. 5: Magnetisation for $\Delta = 0.5$ and at the border $\sigma = 1$

FIG. 6: First-order transition due to LR forces at $\Delta = 0.75$ and $\sigma = 0.4$.

FIG. 7: The jump in the magnetisation is rounded due to disorder at $\Delta = 0.75$ and $\sigma = 0.5$.

FIG. 8: Mixed-order transition at $\Delta = 0.75$ and $\sigma = 0.625$

FIG. 9: Mixed-order transition at $\Delta = 0.75$ and $\sigma = 0.75$
At the borderline value of $\sigma = 0.5$ the magnetization profiles in Fig[4] show still a jump, at least for smaller finite systems. With increasing $L$, however, the jump in the magnetization is going to be rounded, so that the transition could be continuous in the thermodynamic limit. With the finite-size results at hand we can not discriminate between these scenarios. The shift of the finite-size transition points are characterized by an exponent: $\nu \approx 2 = 1/\sigma$, in this case, too.

b. $0.5 < \sigma \leq 1.0$: Mixed-order transitions (crosses in Fig[1]). In this regime we have a series of points with $\sigma = 0.625$, 0.75, 0.875 and 1.0 and the corresponding profiles are shown in Figs[8][11]. The new feature of the profiles, that for different sizes they cross each other, so that for $T < T_c$ ($T > T_c$) the profiles satisfy $m(L_1,T) < m(L_2,T)$ ($m(L_1,T) > m(L_2,T)$) for $L_1 < L_2$. Furthermore at the transition point in the thermodynamic limit the magnetization has a finite limiting value: $\lim_{L \to \infty} m(L,T_c^-) = m^- > 0$, which is different from the limit $\lim_{L \to \infty} m(L,T_c^+) = m^+$. Consequently at the transition point there is a jump in the magnetization: $\Delta m = m^- - m^+$. We also expect that the actual value of $m^+$ is (close to) zero for strong disorder (large $\Delta$) and it is increasing for smaller value of $\Delta$. In the thermodynamic limit for $T < T_c$ the magnetization is expected to follow a singular temperature dependence: $m(T) - m^- \sim (T_c - T)^\beta$. This can be checked in finite systems by defining finite-size transition points as the crossing points of the profiles $m(L_1,T)$ and $m(L_2,T)$:

$$m(L_1,T_c(L_1,L_2)) = m(L_2,T_c(L_1,L_2)) \equiv m^-(L_1,L_2).$$

According to scaling theory the differences should behave asymptotically as: $T_c(L_1,L_2) - T_c \sim (L_1L_2)^{-1/\nu}$ and $m^- - m^-(L_1,L_2) \sim (L_1L_2)^{-\beta/\nu}$. Due to strong finite-size corrections we could make an estimate for the critical exponents only in the case $\sigma = 0.875$ with the result: $1/\nu \approx 1.27$ and $\beta/\nu \approx 0.78$. This means that in this point, or more generally in the $0.5 < \sigma \leq 1.0$ part of the phase diagram (with $\Delta = 0.75$) there is a mixed-order phase-transition in the system: the magnetization has a jump at the transition point, but the correlation length is divergent at $T_c$.

Comparing the magnetization profiles at different values of $\sigma$, one can notice, that its limiting value, $m^-$, and thus the jump $\Delta m$ is an increasing function of $\sigma$ in the given range. Increasing $\sigma$ over the upper limit, $\sigma = 1$, the form of the singularity changes ones more.

c. $\sigma > 1.0$: SR first-order transitions (circles on Fig[1]). The magnetization profiles at $\sigma = 1.25$ and 1.5 in Fig[12][13] show similar features: a jump is developed for large $L$, the asymptotic position of which is at $T_c(\sigma)/(1 + \sigma) < 1$, which ratio is decreasing with increasing $\sigma$ and in the true SR model with $\sigma \to \infty$ this ratio is just $1 - \Delta$. Thus in this region the transition is of first order due to SR interactions. Comparing the finite-size transition temperatures, $T_c(L)$, defined
as the inflection point of the profiles, we observe the
asymptotic behaviour: \( T_c(L) - T_c \sim L^{-1} \), characteristic
for SR forces. We note that spontaneous order in the
LR Potts chain for \( \sigma > 1 \) can be observed only in the
\( q \to \infty \) limit. For any finite value of \( q \) due to thermal
fluctuations there is no ordered phase, thus the SR
first-order transition regime is absent.

IV. DISCUSSION

We have studied numerically the phase-diagram of
the ferromagnetic LR Potts chain with random nearest-
neighbour couplings in the \( q \to \infty \) limit. Depending on
the strength of disorder, \( \Delta \), and the decay exponent, \( \sigma \),
different type of phase-transitions are found: first-order
due to LR interactions, first-order due to SR interactions,
second-order and mixed-order transitions. A schematic
phase-diagram is depicted in Fig.1

For small values of \( \sigma < \sigma_c(\Delta) \leq 0.5 \) the long-range
interactions are dominant over quenched disorder and the
transition is of first order, as in the non-random system.
For large values of \( \sigma > 1 \), the transition is also of first-
order, however now due to short-range interactions. We
note, that for finite-values of \( q \) in this region there is
no ferromagnetic order in the system. For intermediate
values of the decay exponent: \( \sigma_c(\Delta) < \sigma < 1 \). quenched
disorder is going to change the order of the transition.
For weaker disorder the transition turns to second-order,
which is manifested by a divergent specific heat and by a
divergent correlation length, however the magnetization
at the critical point is continuous and has a finite value.
For strong disorder the transition turns to be of mixed-
order. At the transition point the correlation length is
divergent, but there is a finite jump in the magnetization,
as well as in the energy-density. The finite-size scaling
behaviour of the magnetization profiles are also different
in the SO and the MO transitions.

The different type of transitions are connected with the
geometric properties of the optimal graphs. At first-order
transitions the optimal graphs are different at the two
sides of the transition points: in the ferromagnetic phase
there is a giant cluster, whereas in the high-temperature
phase the clusters have finite mass and extent. At the
second-order transition at both sides there is a giant clus-
ter, however at the transition a hole in this giant cluster
is developed, the size of which as well as its mass is di-
vergent. This hole in the SO transition point is a fractal,
therefore the average magnetization is continuous. Sim-
ilar process takes place at a mixed-order transition, too,
with the difference, that in this case the “hole” in the
high-temperature phase is a compact object having a fi-
nite density of mass. This leads to a jump in the mag-
etization in the thermodynamic limit. For large enough
\( \Delta \) this hole is going to disconnect the giant cluster, so
that the density of its mass, being the magnetization has
a vanishing value in the thermodynamic limit.

We expect that the results summarized in the phase-
diagram in Fig.1 remain qualitatively correct for another,
more general models, too. First we mention, that the
LR forces in Eq. (1) can be (weakly) random, too, which
means that in Eq. (1) the prefactor is modified as \( J \to J_i \),
and the \( J_i > 0 \) are random variables. Another set of mod-
els are obtained if the parameter \( q \) is a large, but finite
value. As noted before this model for \( \sigma > 1 \) has no or-
dered phase, however similar phase-diagram is expected
to hold in the regime \( 0 < \sigma < 1 \). This conjecture is
based on the known results in the SR models, in which
the properties of the phase transitions in different dimen-
sions are found to be a smooth function of \( q \), so that the
\( q \to \infty \) limit is not singular. Further numerical work
is needed to clarify, if similar relation holds also for the
LR model.

Appendix

In this appendix we give the solution of the optimal
cooperation problem on the two lines \( \Delta = 0 \) and \( \sigma \to \infty \)
of the phase diagram. Let’s recall that an optimal set is
a set of edges which maximizes the objective function

\[
 f(S; \beta) = c(S) + \beta \sum_{e \in S} J(e) 
\]

where \( c(S) \) is the number of connected components of \( S \)
and \( \beta \) is the inverse temperature.

For any sample, the optimal set for zero temperature
is the set of all the bonds, while for high temperature
the optimal set is empty. Between these two limits the
optimal set changes at a finite number \( n_T \) of tempera-
tures (\( n_T < L \)). We call these temperature breaking
temperatures. If there is only one breaking temperature
(\( n_T = 1 \)) the model is maximally first order since the
magnetization jumps from zero to one.
Let us first consider the case $\Delta = 0$, therefore a non-disordered model. We show below that for any decreasing weight function of the distance (as for example $d^{-\alpha}$) there is a single breaking temperature for any size $L$. Note first that if a bond of length $d$ belongs to an optimal set, then there is an optimal set to which all bonds of length $d$ are present. Indeed the permutation of the sites $i \rightarrow i + 1$ preserve the length of the bonds and therefore any bond of length $d$ belongs to some optimal set. Since the union of two optimal sets is also an optimal set, we deduce that there is an optimal set to which all bonds of length $d$ belong. Suppose now that the bond between site 0 and site $d$ belongs to the optimal set. Then the bond between the sites $d$ and $2d$ also belongs to the optimal set and consequently the site 0, $d$, $2d$ belongs to the same cluster. More generally all the bonds between $\alpha d$ and $(\alpha + 1)d$ also belong to the optimal set and consequently all sites $\alpha d$, where the product is modulo $L$ and $\alpha$ an arbitrary integer, belong to the same cluster. If $L$ is a prime number, then all the sites will be attained, and therefore the optimal set contain all bonds if it contains any one, which proves the results. Note that in this special case of $L$ being prime we did not use the fact that the weight function is decreasing. To sketch the results in the case where $L = \ln n$ is not a prime number we introduce the sets of edges $\Gamma_n(k)$ induced by the vertex sets \{\{k, l + k, 2l + k, \ldots, (n - 1)l + k\}. It is clear that every optimal set is of the form $R(n) = \bigcup_{k=0}^{L-1} \Gamma_n(k)$ and is therefore characterized by a divisor of $L$. Showing that the transition is maximally first order amounts to showing that the optimal set is characterized by only either 1 or $L$. To this end, let’s introduce the sets of edges $\Gamma_n(nk)$ induced by the set of vertices \{\{k, k + 1, \ldots, k + (n - 1)\}. A union $S(n) = \bigcup_{k=0}^{L-1} \Gamma_n(nk)$ is in general not a optimal set. However comparing the objective function for $S(n)$ and $R(n)$ and using the fact that $J$ is a decreasing function of the distance, we find that only $S(1)$ and $S(L)$ can be the optimal set. This proves that the model is maximally first order also when $L$ is not prime.

Now we turn to the case $\sigma \rightarrow \infty$, ie when only the short range disordered bonds are present. In the general case the couplings constant can take $n$ values $0 < J_0 \leq J_1 \leq \cdots \leq J_{n-1}$ the breaking temperatures $T_k = \frac{L}{L-1} \sum_{i=0}^{L-1} J_i$ for $2 \leq k \leq L$. Using this relation in the case of bimodal distribution with an equal number of strong $(1 + \Delta)$ and weak $(1 - \Delta)$ bonds, one gets $1 - \Delta = J_0 = \cdots = J_{\frac{L}{2}} < J_{\frac{L}{2} + 1} = \cdots = J_{L-1} = 1 + \Delta$ from which the $T_k$ are easily deduced. After some algebra one gets that if $\Delta \leq \frac{1}{L-1}$ the model is maximally first order with a breaking temperature $\frac{L-1}{L-1}$, while if $\frac{1}{L-1} < \Delta$ there are two breaking temperatures $T_1 = \frac{L}{2} (1 - \Delta)$ and $T_2 = 1 + \Delta$. In the intermediate regime $T_1 \leq T \leq T_2$ the free energy is $f(T, L) = \frac{1}{2} + \frac{1}{2} (1 + \Delta)$, and we have numerically observed that magnetization scales as $L^{-0.82}$. So in the thermodynamical limit the magnetization jumps from 0 to 1 at $T_2$.

Note finally that all realizations have exactly the same behavior. Therefore in some sense, the model is not disordered.

Acknowledgments

This work was supported by the Hungarian Scientific Research Fund under grant No. K109577 and K115959. J-Ch AdA extends thanks to the "Theoretical Physics Workshop" and FI to the Université Joseph Fourier for supporting their visits to Budapest and Grenoble, respectively.

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* Electronic address: igloi.ferenc@wigner.mta.hu

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