Unambiguous discrimination of coherent states

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Coherent states of the quantum electromagnetic field, the quantum description of ideal laser light, are a prime candidate as information carriers for optical communications. A large body of literature exists on quantum-limited parameter estimation and discrimination for coherent states. However, very little is known about practical realizations of receivers for unambiguous state discrimination (USD) of coherent states. Here we fill this gap and establish a theory of unambiguous discrimination of coherent states, with receivers that are allowed to employ: passive multimode linear optics, phase-space displacements, un-excited auxiliary input modes, and on-off photon detection. Our results indicate that these currently-available optical components are near optimal for unambiguous discrimination of multiple coherent states in a constellation.

I. INTRODUCTION

The fundamental indistinguishability of non-orthogonal quantum states of light is an important trait of quantum physics that underpins applications in quantum cryptography [1], and various other applications of quantum information processing. One goal of quantum information theory is to establish ultimate limits to the distinguishability of quantum states, and to devise effective measurements and associated practical receiver designs whose performance approaches the ultimate limits [2]. Improving discrimination of multiple non-orthogonal quantum states (compared to what is possible using conventional means) would pave the way to more efficient readout of information in quantum sensors [3], communications [4, 5], and probabilistic algorithms in computation [6].

There exist two main approaches to quantum state discrimination. In ambiguous state discrimination (ASD), the receiver is designed to always provide an output, with the goal of optimizing a metric that quantifies the performance of state discrimination on average. For example, the measurement that minimizes the average probability of error in discriminating a set of c non-orthogonal quantum states is an c-element projective measurement, which can be evaluated by solving the Yuen-Kennedy-Lax (YKL) conditions [7]. The minimum average error probability is also often referred to as the Helstrom limit. Despite straightforward computation of the Helstrom limit, optical receiver designs that achieve this performance is a very difficult task. Any two coherent states of an optical mode—the quantum description of ideal laser light—are non-orthogonal quantum states. Hence, coherent states cannot be discriminated with zero probability of error. There is an extensive body of research on receiver designs that distinguish coherent states [8–11]. The so-called Dolinar receiver [12], which employs a local-oscillator laser, and electro-optic modulator, shot-noise photon detection and electro-optic feedback, can achieve the Helstrom limit for discriminating any two coherent states. However, discriminating three or more non-orthogonal coherent states, an optimal ASD receiver design that achieves the Helstrom limit becomes exceedingly difficult, and may require the full power of quantum computing [13]. In contrast, for unambiguous state discrimination (USD), a decoder is designed to either yield an inconclusive output, or identify the true quantum state that the receiver was presented with, without error. The fundamental problem in USD measurements is to identify the measurement positive-operator valued measure (POVM), and a structured receiver design, which minimises the average probability of the inconclusive event given a set of quantum states on which USD measurement is to be performed [14, 15]. The analogue to the YKL conditions for USD are a theory for minimizing the average inconclusive-measurement probability, developed by Peres and Terno [16]. We refer to this as the Peres-Terno limit, and use it to benchmark performance of our novel receiver designs.

For specific applications, USD is preferred over ASD. This includes quantum key distribution [17–20] and quantum digital signatures [21], where an exact identification improves the transfer of secure information that cannot be forged or repudiated. For classical communications using coherent-state modulation, a joint quantum measurement acting on the received coherent-state code word that performs USD discrimination among all the code words in the code book, achieves the optimal communication capacity allowed for by quantum mechanics [22], known as the Holevo capacity [23]. Furthermore, when acting on a finite-length inner-code comprised of tensor product of coherent states, the USD measurement can even attain a higher channel capacity—Shannon capacity of the superchannel induced by the inner code and the receiver—compared to the optimal ASD mea-

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measurement that minimizes the average probability of error of choosing between the modulated-received inner code words [24, 25].

In general, a set of pure states can be unambiguously discriminated if and only if they are linear independent [26]. However, unless the state vectors are also mutually orthogonal, there is a fundamental non-zero probability of obtaining an inconclusive result. Peres and Terno identified a general construction for the abstract measurement that minimizes such probability [27].

This work focuses on the USD of Sudarshan-Glauber coherent states [28, 29]. This problem is of fundamental importance owing to the widespread use of laser light for optical communications given their intrinsic robustness against loss. A finite set of coherent states over m optical modes constitutes a set of linear independent vectors, unless any two have identical amplitudes. That is, any non-trivial finite set of coherent states can be unambiguously discriminated. However, the optimal measurement attaining this discrimination may require high non-linearities and may be out of reach even with access to the most advanced state-of-the-art optical technologies. While there is a vast literature on receiver designs that achieve or approach the optimal ASD of coherent states [30–37], very little is known about receiver designs for USD.

For c phase-shift encoded signals, the optimal USD measurement in the limit of small photon numbers consists of mode-wise displacement operations followed by photon-number-resolving detectors [19]. For c = 2, the so-called binary phase-shift-keying (BPSK) alphabet, this scheme is sufficient to attain the optimal USD performance without adaptive pre-detection displacements [38]. Post-selecting on the measurement result can further reduce the error rate for a fixed rate of inconclusive results [39]. For coherent state constellations with c > 2, there is a substantial gap between the optimal USD performance and non-adaptive receivers [19, 26, 40]. An adaptive receiver for QPSK has demonstrated an improvement to correct state identification [41] in an effort to close this gap.

In this work, we establish a theory of USD receiver design for coherent states using resources that can be readily implemented with current technology. This includes multi-mode linear passive optics, phase-space displacement operations, vacuum auxiliary modes, and mode-wise on-off photodetection [42]. Clearly, it is not possible to design an optimal USD measurement on c > 2 coherent states. This is because if that were possible, since USD measurement on a random coherent-state code achieves the Holevo capacity [22], a receiver built with linear optics, passive laser local oscillators, shot-noise-limited photon detection and electro-optic feedforward would achieve the Holevo capacity, which violates the no-go theorem proved in Ref. [43]. However, we show this set of receiver resources is sufficient to approach the ultimate bounds established by Peres and Terno. Specifically, for randomly picked coherent-state constellations, coherent-state USD receivers designed from linear optical resources are nearly optimal.

We start by introducing key concepts for linear optics in phase space in Section II, and in Section III we analyse photodetection of multimode coherent states. We then present in Section IV USD decoding schemes constructed with multi-mode linear passive optics, vacuum auxiliary modes, and mode-wise on-off photodetection. We then include phase-space displacement operations in Section V. We discuss the special case of pulse-position modulation (PPM) code words in Section VI, and compare the performance of our receivers to the globally optimal USD scheme with examples in Section VII. Conclusions and open questions are presented in Section VIII.

II. LINEAR OPTICS IN PHASE SPACE

Consider a collection of m bosonic modes, described by the annihilation and creation operators, $a_j, a_j^\dagger$, for $j = 1, \ldots, m$. A Sudarshan-Glauber coherent state on mode $j$ is defined by the property $a_j |\alpha\rangle_{a_j} = \alpha |\alpha\rangle_{a_j}$, where $\alpha$ is the amplitude. The expansion in the number basis is $|\alpha\rangle_{a_j} = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle_{a_j}$, where $|k\rangle_{a_j}$ is the state with $k$ photons on mode $j$.

We denote a multimode coherent state using bold Greek letters,

$$|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle_{a_1} |\alpha_2\rangle_{a_2} \cdots |\alpha_m\rangle_{a_m}. \quad (1)$$

Given a pair of multimode coherent states, $|\alpha\rangle$, $|\beta\rangle$, the Hilbert-space scalar product reads

$$\langle \alpha | \beta \rangle = \exp \left[ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \alpha^* \cdot \beta \right], \quad (2)$$

where $|\alpha|^2 = \sum_{j=1}^{m} |\alpha_j|^2$ and $\alpha^* \cdot \beta = \sum_{j=1}^{m} \alpha_j^* \beta_j$. The Hilbert space scalar product is invariant under unitary transformations in the multimode Hilbert space, and is the relevant structure underlying the general analysis of USD from Peres and Terno [27].

Since we focus on practical decoders, and not on the most general measurements, we consider an alternative notion of scalar product. We define the phase-space scalar product as

$$\langle \alpha, \beta \rangle := \alpha^* \cdot \beta = \sum_{j=1}^{m} \alpha_j^* \beta_j. \quad (3)$$

Linear optics passive (LOP) unitary transformations map the annihilation operators to themselves [44]:

$$U_{\text{LOP}}^\dagger a_j U_{\text{LOP}} = \sum_{k=1}^{m} U_{jk} a_k, \quad (4)$$

where $[U_{jk}]$ is an $m \times m$ unitary matrix. With this unitary, the multimode coherent state in Eq. (1) transforms into $U|\alpha\rangle = |\alpha_1^*\rangle_{a_1} |\alpha_2^*\rangle_{a_2} \cdots |\alpha_m^*\rangle_{a_m}$, with $\alpha_j^* = \sum_{k=1}^{m} U_{jk} \alpha_k$. 

It follows that the phase-space scalar product is invariant under LOP unitaries, \((\alpha, \beta) = (U\alpha, U\beta)\). The probability of detecting no photon is also invariant under LOP unitaries. Denoting as \(|0\rangle\) the vacuum state over \(m\) modes, the probability of detecting no photon is 
\[
|\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2} = e^{-|U\alpha|^2}.
\]

Related to this definition of scalar product in phase space, we introduce a notion of linear independence of coherent states in phase space. Consider a set of \(c\) coherent states over \(m\) modes, \(|\alpha^1\rangle, |\alpha^2\rangle, \ldots, |\alpha^c\rangle\), with amplitudes \(\alpha^j = (\alpha_{1}^j, \alpha_{2}^j, \ldots, \alpha_{m}^j)\). These coherent states are phase-space linear independent if and only if the matrix
\[
R = \begin{pmatrix}
\alpha_1^1 & \alpha_2^1 & \ldots & \alpha_m^1 \\
\alpha_1^2 & \alpha_2^2 & \ldots & \alpha_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^c & \alpha_2^c & \ldots & \alpha_m^c
\end{pmatrix}, \quad (5)
\]
has full rank. Note that this is possible only if \(c \leq m\).

### III. PROJECTION ON A NON-LOCAL MODE

Given two sets of bosonic modes, represented by the operators \(a_j, a_j^\dagger\) (for \(j = 1, \ldots, m\)) and \(b_i, b_i^\dagger\) (for \(i = 1, \ldots, c \leq m\)), and the multimode coherent state in Eq. (1), we determine the probability, \(P_\alpha(b_i)\), a photon is detected in mode \(b_i\)? We determine this probability by considering receivers with and without auxiliary modes.

First, consider an LOP unitary transformation, without auxiliary modes such that
\[
b_i = \sum_{j=1}^{m} U_{ij} a_j, \quad (6)
\]
for some unitary matrix \([U_{ij}]\) of size \(m\). We write (1) in terms of the \(b\) modes:
\[
|\alpha\rangle = |\beta_1\rangle_{b_1} |\beta_2\rangle_{b_2} \cdots |\beta_m\rangle_{b_m}, \quad (7)
\]
where \(b_i |\alpha\rangle = \beta_i |\alpha\rangle\) and \(\beta_i = \sum_j U_{ij} \alpha_j\). From this we obtain
\[
P_\alpha(b_i) = 1 - \exp\left( -|\beta_i|^2 \right) = 1 - \exp\left( -|U_i \cdot \alpha|^2 \right), \quad (8)
\]
where we have defined \(U_i := (U_{i1}, U_{i2}, \ldots, U_{im})\), and \(U_i \cdot \alpha = \sum_{j=1}^{m} U_{ij} \alpha_j\).

The modes \(a\) and \(b\) need not be linked through a LOP unitary. We can account for losses by introducing \(m'\) auxiliary modes, \(a_{m+1}, a_{m+2}, \ldots, a_{m+m'}\), and extend the state in Eq. (1) by assuming that the auxiliary modes are not populated, i.e.,
\[
|\alpha'\rangle = |\alpha_1\rangle_{a_1} |\alpha_2\rangle_{a_2} \cdots |\alpha_m\rangle_{a_m} |\alpha_{m+1}\rangle_0 \cdots |\alpha_{m+m'}\rangle_0. \quad (9)
\]
The \(b\)-modes are defined as
\[
b_i = \sum_{j=1}^{m+m'} U_{ij} a_j = \sum_{j=1}^{m} M_{ij} a_j + \sum_{j=1}^{m'} N_{ij} a_{m+j}, \quad (10)
\]
for some unitary matrices \([U_{ij}]\) of size \(m + m'\), and we have defined the sub-matrices \(M\) and \(N\) such that \(M_{ij} = U_{ij}\) for \(j = 1, \ldots, m\), and \(N_{ij} = U_{ij}\) for \(j = m+1, \ldots, m+m'\). The probability of a photodetection event in the \(b_i\) mode is thus given by
\[
P_\alpha(b_i) = 1 - \exp\left( -|M_i \cdot \alpha|^2 \right), \quad (11)
\]
where \(M_i := (M_{i1}, M_{i2}, \ldots, M_{im})\), and \(M_i \cdot \alpha = \sum_{j=1}^{m} M_{ij} \alpha_j\).

The probability in Eq. (11) is the key quantity in our method for USD with limited resources. Before introducing our theory we need to clarify under which conditions a matrix \(M\) of size \(c \times m\) is the sub-matrix of a larger unitary matrix of size \(m + m'\). First, extend \(M\) into a square, \(m \times m\) matrix, \(M_0\), by appending \(m - c\) rows of zeros. Second, compute the singular value decomposition \(M_0 = UDV\),
\[
(12)
\]
where \(U\) and \(V\) are unitary matrices, and \(D\) is diagonal with non-negative entries.

For \(D \leq 1\), the following \(2m \times 2m\) matrix is unitary:
\[
\begin{pmatrix}
D & -\sqrt{I - D^2} \\
\sqrt{I - D^2} & D
\end{pmatrix}, \quad (13)
\]
where \(I\) is the identity matrix. By multiplying it by \(U\) and \(V\) we obtain another unitary matrix, which is a unitary extension of \(M_0\).
\[
U = \begin{pmatrix}
U & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix}
D & -\sqrt{I - D^2} \\
\sqrt{I - D^2} & D
\end{pmatrix} \begin{pmatrix}
V & 0 \\
0 & I
\end{pmatrix} = \begin{pmatrix}
M_0 & 0 \\
0 & -UVDV
\end{pmatrix}. \quad (14)
\]
As \(M_0\) is an extension of \(M\), it follows that \(U\) is a unitary extension of \(M\).

We conclude that a matrix \(M\) can be extended into a unitary if and only if its singular eigenvalues are not bigger than 1. This condition can be equivalently written as
\[
M_0 \leq I, \quad \text{or} \quad MM_0^\dagger \leq I. \quad (15)
\]

### IV. USD WITH LOP UNITARIES

We now present our first protocol for USD of multimode coherent states. It exploits LOP unitaries, vacuum auxiliary modes, and on-off photodetection.

Consider a set of \(c\), \(m\)-mode coherent states \(|\alpha^1\rangle, |\alpha^2\rangle, \ldots, |\alpha^c\rangle\), with amplitudes \(\alpha^j = (\alpha_{1}^j, \alpha_{2}^j, \ldots, \alpha_{m}^j)\), and prior probabilities \(p^1, p^2, \ldots, p^c\). Assuming that the phase-space vectors \(\alpha^1, \ldots, \alpha^c\) are linearly-independent (therefore, \(c \leq m\)), they define a subspace \(S\) of the phase space of dimensions \(c\). For each \(i\), we define \(v_i\) as the unique unit vector in \(S\) such that, for all \(j\),
\[
v_i \cdot \alpha^j = \delta_{ij} v_i \cdot \alpha^i. \quad (16)
\]
we introduce a second protocol that also exploits phase-space displacement. A phase-space displacement of $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ maps the code word $\alpha^i$ into $\beta^i = \alpha^i + \gamma$.

The optimal choice of $\gamma$ is found as the solution to the following constrained optimisation:

$$\begin{align*}
\text{minimise} & \quad \sum_{i=1}^{c} p_i \exp \left[-k_i |v_i \cdot (\alpha^i + \gamma)|^2 \right], \\
\text{subject to} & \quad v_i \cdot (\alpha^i + \gamma) = \delta_{ij} v_i \cdot (\alpha^i + \gamma), \quad (25) \\
& \quad M_{ij} = \sqrt{k_i} v_{ij}, \quad M^\dagger M \leq I.
\end{align*}$$

The above applies only when the phase-space vectors $\beta^i = \alpha^i + \gamma$ are linearly independent. We now show that phase-space displacement can be used to map a dependent system $\alpha^i$ into an independent one. First assume $c \leq m$, and consider the matrix of coefficients in Eq. (5). The rank of the matrix $R$ equals the number of linearly independent code words. We will now show that phase-space displacement can be used to increase the rank by one. We therefore assume that the rank of $R$ is $c - 1$ and extend this matrix by adding one more row:

$$R_1 = \begin{pmatrix} \alpha^1 & \ldots & \alpha^c \\ \alpha^0 & 0 & \alpha^c \\ \vdots & \ddots & \vdots \\ \gamma & 0 & \gamma_{m+1} \end{pmatrix},$$

in such a way that $R_1$ has rank $c$. It follows that the set of displaced code words $\beta^i = \alpha^i + \gamma$ is linearly independent.

Consider now the case $c = m + 1$, with the matrix $R$ having rank $m$. We first add an additional column, and then add an additional row:

$$R_2 = \begin{pmatrix} \alpha^1 & 0 & \ldots & \ldots & 0 \\ \alpha^2 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha^c & 0 & \ldots & \ldots & 0 \\ \gamma & 0 & \gamma_{m+1} \end{pmatrix}.$$

As long as $\gamma_{m+1} \neq 0$, the matrix $R_2$ has rank $c = m + 1$. We can then introduce one auxiliary mode and obtain a set of $c$ phase-space linear independent code words over $m + 1$ modes, $|\beta^i\rangle = |\alpha^i + \gamma\rangle|\gamma_{m+1}\rangle$.

In conclusion, by allowing for phase-space displacements, phase-space linear dependent coherent states can be also discriminated, as long as the matrix $R$ in Eq. (5) has single degeneracy, i.e., its rank is $c - 1$.

A particular example of this construction is the following. Consider the pair $(c = 2)$ of coherent states, $|\alpha\rangle$, $|-\alpha\rangle$, over one mode ($m = 1$). These states are not linearly independent in phase-space. By adding an auxiliary mode in the vacuum state to yield $|\alpha\rangle|0\rangle$, $|-\alpha\rangle|0\rangle$, and by displacing the auxiliary mode by $\alpha$, we obtain the phase-space linearly independent code words $|\alpha\rangle|\alpha\rangle$, $|-\alpha\rangle|\alpha\rangle$. 

V. USD WITH LOP UNITARIES AND PHASE-SPACE DISPLACEMENTS

The USD protocol described above may be improved by enlarging the set of allowed resources. In this Section
VI. PULSE-POSITION MODULATION

A set of orthogonal vectors can always be discriminated perfectly. Since coherent states are not orthogonal, they can never be perfectly discriminated (though their discrimination can be improved by increasing their distance in phase space). In this section we explore a particular family of coherent state code books that share some formal features with orthogonal states. A pulse-position modulation (PPM) code book is a set of \( c = m \) coherent states over \( m \) modes, such that,

\[
\begin{align*}
|\alpha^1\rangle &= |\alpha\rangle |0\rangle \cdots |0\rangle, \\
|\alpha^2\rangle &= |0\rangle |\alpha\rangle \cdots |0\rangle, \\
&\quad \ldots . \\
|\alpha^m\rangle &= |0\rangle |0\rangle \cdots |\alpha\rangle.
\end{align*}
\]

Note that these code words are mutually orthogonal with respect to the phase-space scalar product, 
\[
(\alpha^i, \alpha^j) = |\alpha|^2 \delta_{ij}.
\]

In general, it may exist a gap between the general USD strategy of Peres and Terno [27] and the USD protocols with limited resources discussed here. However, for the case of PPM code book, it is possible to show that the gap vanishes and the PPM code can be optimally discriminated using on-off photodetection only. In fact, to unambiguously discriminate the PPM state, it is sufficient to apply mode-wise photodetection. If a click happens at mode \( j \), then we now the code word was \( |\alpha_j\rangle \) with no error. The inconclusive event is when no click is recorded, which happens with probability \( P_0 = e^{-|\alpha|^2} \). In turns out that this is the same inconclusive probability of the full Peres-Terno theory (see Appendix B). We now show that this property extends to a larger class of code books beyond PPM.

First we note that, since the phase-space scalar product is invariant under LOP unitaries, it immediately follows that LOP unitaries and on-off photodetection are sufficient to optimally discriminate any set of coherent states such that

\[
(\alpha^i, \alpha^j) = |\alpha|^2 \delta_{ij},
\]

where \([G_{ij}] = [(\alpha^i, \alpha^j)]\) is the Gram matrix of the code book with respect to the phase-space scalar product.

If the Gram matrix is not diagonal, we may try to make it diagonal by applying a phase-space displacement \( \beta^j = \alpha^j + \gamma \). The displacement changes the Gram matrix into

\[
(\beta^i, \beta^j) = (\alpha^i, \alpha^j) + (\alpha^i, \gamma) + (\gamma, \alpha^j) + (\gamma, \gamma).
\]

We then obtain that a code book of \( c \) coherent states \(|\alpha^i\rangle\) over \( m \) modes can be optimally discriminated with LOP unitaries, displacements, and photodetection, if there exist \( \gamma \) and \( \tau > 0 \) such that

\[
(\alpha^i, \alpha^j) + (\alpha^i, \gamma) + (\gamma, \alpha^j) + (\gamma, \gamma) = \tau \delta_{ij}.
\]

This is a system of \( c^2 \) independent real equations and \( 2m + 1 \) real unknowns (the components of the complex displacement vector \( \gamma \) and \( \tau \)). Therefore, in general we expect this system of equations to have solutions for \( c^2 \leq 2m + 1 \).

As we have seen above, sometimes the use of an auxiliary mode can improve the effectiveness of USD. In fact, adding an auxiliary mode allows us to introduce one additional real degree of freedom, i.e., \(|\gamma|_{m+1}^2 \). The new system of equations reads

\[
(\alpha^i, \alpha^j) + (\alpha^i, \gamma) + (\gamma, \alpha^j) + (\gamma, \gamma) + |\gamma|_{m+1}^2 = \tau \delta_{ij},
\]

and comprises \( c^2 \) equations and \( 2m + 2 \) unknowns. Therefore, we expect this to typically admit a solution for \( c^2 \leq 2m + 2 \). Note that there is no benefit in adding more than one auxiliary mode.

VII. PERFORMANCE OF LOP USD RECEIVER

We explore the performance of our linear optical USD receivers relative to the optimal Peres-Terno scheme, for different code books and figures of merits. We illustrate that, despite its simplicity, decoders constructed using only linear components and on-off photodetection can typically generate near optimal unambiguous state discrimination. This typicality result is obtained by testing our receivers on randomly generated coherent states.

In general, we find that LOP unitaries and phase-space displacement are necessary to achieve near optimal performances on typical code books. Otherwise, if only LOP unitaries are used, we observe a notable gap with the optimal Peres-Terno scheme.

The situation appears to be different for non-typical code book. In fact, our decoders works poorly on degenerate code books, whose rank is smaller than \( c - 1 \). For these highly degenerate code books, we illustrate a recipe for achieving improved performance based on multimode detection.

For completeness, the USD protocol of Peres and Terno is reviewed in Appendix A.

A. Random codes

We consider random code books comprised of \( c = 3 \) coherent states, \(|\alpha^1\rangle|, |\alpha^2\rangle|, |\alpha^3\rangle|, \) over \( m = 3 \) modes. For a given mean photon number \( n \), a 3-mode coherent state is identified by a point on a complex sphere of radius \( \sqrt{n} \). For simplicity, we restrict amplitudes to real values and sample the coherent states from the uniform distribution on the sphere.

We apply our schemes for USD (with and without displacement) to these random codes, determining the minimised value for the probability of inconclusive outcomes.
FIG. 1: Comparative performance of different USD receivers on random code books with different photon numbers: the Peres-Terno theory; our scheme with multi-mode LOP, vacuum auxiliary modes, and mode-wise on-off photodetection; and our scheme with added phase-space displacement operations. Fig. 1a: The solid lines represent the average minimal $P_0$ for random codes, obtained by sampling over $N = 500$ random codes for each value of $n$. The shaded regions account for data within one standard deviation. LOP without displacement is the red line on the top. LOP with displacement is the blue line in the middle. Peres-Terno is the brown line at the bottom. Fig. 1b: For a particular value of $n$, this shows the frequency distribution of the minimal inconclusive probability $P_0$, for random codes with fixed photon number, $n = 0.6$. This shows a sample of $N = 6600$ random codes. LOP without displacement is shown by the red bars on the right. LOP with displacement by the blue bars in the middle. Peres-Terno by the brown bars on the left.

$P_0$. These values are then compared with the globally optimal inconclusive probability determined using the theory of Peres and Terno [27]. For a specific photon number, we repeat the optimisation over $N$ random codes to determine the statistics of $P_0$.

Fig. 1a illustrates $P_0$ attained from the three schemes. Notice that as the photon number increases, the code words become increasingly distinguishable leading to a monotonically decreasing value for $P_0$, whereas all methods yield a unit $P_0$ as the photon number of the code words tends to zero. Our results show that the use of displacement operations significantly improves the performance of the receiver to near optimal values. To see this clearly, Fig. 1b illustrates the distribution of $P_0$ values for $n = 0.6$ with $N = 6600$ random codes. The distribution of the inconclusive probability for our scheme with displacement closely follows the Peres-Terno one. The difference between the two schemes slightly increases at smaller $P_0$.

B. Rank-deficient codes with single degeneracy

In this Section, we compare the performance of the different USD receivers in distinguishing a particular coherent state code [45]:

$$|\alpha^1\rangle = |\alpha\rangle |\alpha\rangle,$$

$$|\alpha^2\rangle = |\alpha\rangle |-\alpha\rangle,$$

$$|\alpha^3\rangle = |-\alpha\rangle |\alpha\rangle,$$

with $|\alpha|^2$ mean photons per mode.

Note that these code words are not linear independent in phase space according to our definition. This implies that they cannot be discriminated using LOP only. However, as they have rank 2 (any pair of code words is linear independent), they can be discriminated with phase-space displacement. This is shown in Fig. 2, where we illustrate $P_0$ (solid curves) as a function of the photon number per mode. A comparison with the globally optimally scheme of Peres and Terno suggests that our receiver is nearly optimal, especially for small $\alpha$.

The gap between our scheme and Peres-Terno can be further reduced for alternative figures of merit. Consider for example the communication capacity associated to the given input code words and a given detection strategy (either our scheme with displacement or Peres-Terno). The code words in Eqs. (35)-(37) can be used for a communication protocol where the sender (Alice) uses these coherent states to encode a random variable $X$ that takes values $x = 1, 2, 3$ with associated probabilities $p_X(x)$. The receiver (Bob) decodes this information by applying either our USD decoder or the USD decoder of Peres-Terno. The outcome of the receiver is described by a random variable $Y$ that takes four possible values, $y = 0, 1, 2, 3$, with probability $p_Y(y)$, where $y = 0$ is the inconclusive event. The maximum asymptotic communication rate achievable in this way is given by the Shannon capacity [46]

$$C = \max_{p_X} H(Y) - H(Y|X),$$

where

$$H(Y) = -\sum_y p_Y(y) \log p_Y(y)$$
is the Shannon entropy of $Y$, and
\[ H(Y|X) = - \sum_x p_X(x) \sum_y p_Y(y|x) \log p_Y(y|x) \] (40)
is the conditional entropy, where $p_Y(y|x)$ is the conditional probability of $Y = y$ given $X = x$.

Since the decoders are un-ambiguous, $P_0(x) := p_Y(y = 0|x)$ is the probability of inconclusive event for given input, and $p(y|x) = \delta_{yx}[1 - P_0(x)]$ for $y = 1, 2, 3$. The key quantity that determines the capacity is thus the conditional inconclusive probability $P_0(x)$. For our scheme with displacement this is given by
\[ P_0(x) = \exp[-k_\gamma (|x| \cdot |\alpha^x + \gamma|)], \] (41)
and a similar quantity can be computed for the Peres-Terno theory (see Appendix A). From this we obtain
\[ C = \max_{p_X} \left[ -P_0 \log P_0 - \sum_x p_X(x) \log p_X(x) \right. \]
\[ \left. + \sum_x p_X(x) P_0(x) \log [p_X(x) P_0(x)] \right], \] (42)
where $P_0 = \sum_x p_X(x) P_0(x)$ is the average inconclusive probability. Figure 2 illustrates the communication capacity (dashed lines) for our scheme (with LOP and displacement) compared with Peres-Terno. The two schemes yield nearly equal capacities, the gap being too small to be visualised in the scale of the plot.

In addition to the inconclusive probability and the Shannon capacity, we explore the optimality of our receiver using the finite communication block length rate, $F$. This rate is the communication rate attainable when both the code length, $L$, is finite, and there is a block error probability threshold, $\epsilon$, imposed on the communication \cite{allocation}. The normal approximation to the finite block length rate is given by \cite{estimation}
\[ F(L, \epsilon) = C - \sqrt{\frac{V}{L}} Q^{-1}(\epsilon), \] (43)
where $Q(x) = 1/\sqrt{2\pi} \int_x^\infty dt \exp(-t^2/2)$, and $V$ denotes the variance of the information transition probabilities of the channel,
\[ V = \sum_{x,y} p_Y(y)p_Y(y|x) \left( \log_2 \left[ \frac{p_Y(y|x)}{\sum_z p_Y(z)p_Y(z|x)} \right] - \bar{X} \right)^2, \] (44)
\[ \bar{X} = \sum_{x,y} p_Y(y)p_Y(y|x) \log_2 \left[ \frac{p_Y(y|x)}{\sum_z p_Y(z)p_Y(z|x)} \right]. \] (45)
We maximise the finite block length rate over feasible values for $k_j$ that define a unitary extension of $M$, and the prior code book probabilities. Fig. 3 illustrates this maximisation with different code lengths $L$ for difference receivers. We find our receiver matches the finite-code-length performance generated using the Peres-Terno scheme. Note that in the asymptotic regime of large code block lengths, the finite rate tends towards the channel capacity with $\epsilon = 0$, that are illustrated in horizontal lines.

C. Rank-deficient codes with double degeneracy

Single mode code words find applications in quantum communications and sensing where a single measurement is preferred at a time. We consider the following rank-deficient single mode code for USD discrimination
\[ |\alpha^1\rangle = |\alpha\rangle, |\alpha^2\rangle = |0\rangle, |\alpha^3\rangle = |\alpha\rangle. \] (46)
Note that this code has rank 1, that is, no pair of coherent states is linear independent in phase space. Therefore, it is not possible to achieve USD with the decoders introduced in Sections IV-V, not even with the help of phase-space displacement operations. Nevertheless, we now show that there exists a different kind of receiver, still based on linear optics and on-off photodetection, that achieve nearly optimal USD. In contrast with the receivers discussed above, we now consider a decoding strategy based on multiple photodetection events happening on multiple modes.

The receiver is illustrated in Fig. 4. It requires two ancillary modes, one in the vacuum state and one prepared in a coherent state of amplitude $\alpha/\sqrt{2}$. The modes are mixed at two beam splitters of transmissivity $\eta = 1/2$. The two modes after the first beam splitter are in a coherent state $|\beta_1^j|\beta_2^j)$ of amplitudes $\beta_1^j = \alpha j/\sqrt{2}$, $\beta_2^j = -\alpha j/\sqrt{2}$. After the second beam splitter, the three modes are in the coherent state $|\gamma_1^j|\gamma_2^j|\gamma_3^j)$, with $\gamma_1^j = \beta_1^j = \alpha j/\sqrt{2}$, $\gamma_2^j = (\alpha - \alpha j)/2$, $\gamma_3^j = -(\alpha + \alpha j)/2$.

Overall, the final coherent state amplitudes, as function of the input, is given in Table I, which explicitly shows that the code words can be unambiguously discriminated if two joint detection event are recorded in two different modes. For example, a joint detection on modes one and three identifies the input code word $|\alpha\rangle$ without error. From this table of output amplitudes, we determine the transition probability matrix $P(k|j) = \prod_{i \in k} (1 - \exp[-|\gamma_i^j|^2])$ for input $j$ and output $k \in \{(1,2), (1,3), (2,3)\}$, which identify the corresponding vectors are linearly independent. The theory of state discrimination (USD) of pure states is known to be possible if and only if the corresponding vectors are linearly independent. This seems to be the case for the USD of the ubiquitous Sudarshan-Glauber coherent states, which are the prime candidates as information carriers in quantum optical technologies.

Quantum mechanics forbids us to perfectly discriminate non-orthogonal quantum states. Notwithstanding, it is still possible to discriminate them unambiguously if one allows for a non-zero probability of inconclusive discrimination. Unambiguous state discrimination (USD) of pure states is known to be possible if and only if the corresponding vectors are linearly independent. The theory of Peres and Terno [27] provided a constructive way to obtain a formal mathematical description of a globally optimal measurement for USD. Whereas a mathematically description of the measurement is known, this does not mean that the measurement can be implemented experimentally. This could be a challenging or even impossible task, given the currently available techniques. This seems to be the case for the USD of the ubiquitous Sudarshan-Glauber coherent states, which are the prime candidates as information carriers in quantum optical technologies.

Motivated by practical experimental considerations, we have established a theory of USD for multimode coherent states under a restricted set of allowed physical resources, which includes linear passive optics (LOP) unitaries, phase-space displacement, auxiliary vacuum

| Signal | Mode 1  | Mode 2  | Mode 3 |
|--------|--------|--------|--------|
| $\alpha$ | $\alpha/\sqrt{2}$ | 0 | $-\alpha$ |
| 0      | 0      | $\alpha/2$ | $-\alpha/2$ |
| $-\alpha$ | $-\alpha/\sqrt{2}$ | $\alpha$ | 0 |

TABLE I: Input-output coherent state amplitudes through our receiver with joint detection.
modes, and on-off mode-wise photodetection. Numerical investigations show that these restricted resources can be surprisingly good at discriminating coherent states. In fact, they allow us to nearly close the gap with the globally optimal measurement of Peres and Terno. This suggests that high order non-linearities or more advanced quantum technologies may only improve USD by a relatively small amount. Furthermore, these resources define a minimal set with this property, as we have shown the existence of coherent states that cannot be discriminated at all if phase-space displacements are not included in the pool of allowed resources.

A number of questions remain open. First, our numerical investigations are not sufficient to determine whether our scheme is only nearly optimal or if it exactly saturates the ultimate bound of Peres and Terno, at least in some regime. This answer may be addressed by developing a theory that gives an analytical expression for the minimal inconclusive probability with restricted resources. This would also allow us to extend the analysis to arbitrary number of modes and code words. Second, a more general theory is needed to deal with the case of highly degenerate codes, which here we have only discussed for a specific example. Finally, our theory can be naturally reformulated as a resource theory, in a similar way as done by Refs. [49, 50] in the framework of ambiguous state discrimination. This approach may reveal fruitful to gain insight and to compare different sets of allowed resources, for example by including homodyne detection, photon addition and subtraction, or some mild non-linear interactions.

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in Appendix A: Peres-Terno theory: Unambiguous quantum state discrimination

In this Appendix we review the theory of Peres and Terno \cite{Peres1998} for the USD of a set of linearly independent vectors \( |u_j\rangle \) with \( j = 1, \ldots, c \), associated with prior probabilities \( p^j \) (satisfying \( \sum_{j=1}^{c} p^j = 1 \)). These vectors span a \( c \)-dimensional Hilbert space \( \mathcal{H}_c \). We define a unique set of (not necessarily normalized) vectors \( |v_j\rangle \in \mathcal{H}_c \) such that

\[
\langle v_i | u_j \rangle = \delta_{ij} \langle v_i | u_j \rangle .
\]  

(A1)
We use these vectors to define a POVM with $n$ elements

$$A_j = k_j^2 |v_j⟩⟨v_j| ,$$

(A2)

for $j = 1, \ldots, n$. The POVM element corresponding to an inconclusive event is

$$A_0 = I - \sum_{j=1}^{c} k_j^2 |v_j⟩⟨v_j|. $$

(A3)

The parameters $k_j$ are chosen in such a way to ensure $A_0 \geq 0$, and $I$ is the identity matrix in the space $\mathcal{H}_c$. For a suitable choice of the parameters $k$'s, this POVM allows for the unambiguous discrimination of the code words. The corresponding probability of an inconclusive outcome is

$$P_0 = \sum_{j=1}^{c} p^j⟨u_j|A_0|u_j⟩ = 1 - \sum_{j=1}^{c} p^j k_j^2 ⟨u_j|v_j⟩^2. $$

(A4)

A globally optimal unambiguous state discrimination corresponds to one that minimises $P_0$ subject to the positivity of $A_0$. This is obtained as the solution to the constrained maximization problem:

$$\max_{k_1, \ldots, k_n} \sum_{j=1}^{n} p^j k_j^2 ⟨u_j|v_j⟩^2,$$

s.t. $\sum_{j=1}^{n} k_j^2 |v_j⟩⟨v_j| \leq I.$$

(A5)

**Appendix B: PPM codes**

In this Appendix we explicitly show that mode-wise on-off photodetection is the globally optimal USD strategy for the PPM code. We show this by computing the minimal un conclusive probability from the theory of Peres and Terno, and showing that it matches our scheme. We can write the PPM code words as follows

$$|\alpha^j⟩ = \sqrt{p}|0⟩ + \sqrt{1-p}|j⟩ ,$$

(B1)

where

$$p = e^{-|\alpha|^2} ,$$

(B2)

$$|j⟩ = \frac{e^{-|\alpha|^2/2}}{\sqrt{1-e^{-|\alpha|^2}}} \sum_{n=1}^{\infty} \frac{(\alpha a^j)^n}{n!}|0⟩ .$$

(B3)

We define the following (non-normalised) vectors, for $j = 1, \ldots, c$,

$$|v_j⟩ = [a + (c - 1)b] |0⟩ + a \sqrt{1-p}|j⟩ + b \sqrt{1-p} \sum_{i\neq j} |i⟩ .$$

(B4)

The orthogonality condition reads (for $i \neq j$

$$⟨\alpha^i|v_j⟩ = [a + (c - 1)b]p + b(1-p) = 0 ,$$

(B5)

from which we obtain

$$b = - \frac{ap}{1 + (c-2)p} .$$

(B6)

Therefore

$$|v_j⟩ = \left[a - (c - 1)p \frac{ap}{1 + (c-2)p} \right] \sqrt{p}|0⟩ + a \sqrt{1-p}|j⟩ - \frac{ap}{1 + (c-2)p} \sqrt{1-p} \sum_{i\neq j} |i⟩ .$$

(B7)

For example, we can fix the value of $a$ to obtain

$$|v_j⟩ = \sqrt{p(1-p)}|0⟩ + [1 + (c-2)p]|j⟩ - p \sum_{i\neq j} |i⟩ .$$

(B8)

From this we compute

$$⟨\alpha^2|v_j⟩ = [1 + (c - 1)p] \sqrt{1-p} ,$$

(B9)

$$⟨v_j|v_j⟩ = p(1-p) + [1 + (c-2)p]^2 + (c-1)p^2 ,$$

(B10)

and, for $i \neq j,$

$$⟨v_i|v_j⟩ = p(1-p) - 2[1 + (c-2)p]p + (c-2)p^2 .$$

(B11)

The Peres-Terno construction requires to consider the operator $\sum_{j} k_j^2 |v_j⟩⟨v_j|$. First of all, by symmetry, we can put $k_j = k$ for all $j$. We are then interested in finding the largest eigenvalue of the operator $\sum_{j} |v_j⟩⟨v_j|$. Note that this operator has the same spectrum of the Gram matrix $G_{ij} = ⟨v_i|v_j⟩ = (t - u)δ_{ij} + u$, where $t = ⟨v_j|v_j⟩$, and $u = ⟨v_i|v_j⟩$. The eigenvalues of $G$ are: $e_1 = t + (c-1)u$ (with multiplicity 1), and $e_2 = t - u$ (with multiplicity $c-1$). As $u < 0$, we obtain that

$$k^2 = \min \left\{ \frac{1}{e_1}, \frac{1}{e_2} \right\} = \frac{1}{e_2} = \frac{1}{t-u}.$$  

(B12)

$$= \frac{1}{[1 + (c-1)p]^2} .$$

(B13)

This finally yields

$$P_0 = 1 - \sum_{j} p_j k_j^2 |⟨\alpha^j|v_j⟩|^2$$

(B14)

$$= 1 - k^2 |⟨\alpha^j|v_j⟩|^2$$

(B15)

$$= 1 - \frac{[1 + (c-1)p]p^2(1-p)}{t-u}$$

(B16)

$$= p = e^{-|\alpha|^2}.$$  

(B17)