Random Lattices and Random Sphere Packings: Typical Properties

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Abstract
We review results about the density of typical lattices in $\mathbb{R}^n$. They state that such density is of the order of $2^{-n}$. We then obtain similar results for random packings in $\mathbb{R}^n$: after taking suitably a fraction $\nu$ of a typical random packing $\sigma$, the resulting packing $\tau$ has density $C(\nu)2^{-n}$, with a reasonable $C(\nu)$. We obtain estimates on $C(\nu)$.

1 Introduction

The problem of filling the euclidean space $\mathbb{R}^n$ with equal non-overlapping spheres has a long and celebrated history. It was started by J. Kepler in 1610, who studied the hexagonal two-dimensional packing, and conjectured that the face-centered cubic three-dimensional one is the densest possible. Later on, I. Newton claimed that in $\mathbb{R}^3$ there cannot be more than 12 non-intersecting balls of unit radius touching a given unit ball. In the case when the centers of spheres form a lattice (i.e., a discrete additive subgroup of $\mathbb{R}^n$) the question of finding lattices with densities high enough was studied by C. F. Gauss. Starting from dimension 9, the highest density of the lattice packing is still unknown. Naturally, even less is known when the ball arrangement is not of a lattice nature. The problem constitutes an essential part of 18-th Hilbert problem. Whether or not the
best density is achieved on the lattice arrangements is a major open question in the field. Quite recently T.C.Hales and S.P.Ferguson seem to have proved that for \( n = 3 \) the largest density is that of the face-centered cubic lattice packing. The proof is very long and computer assisted. It is discussed in [Oe2].

Constructions of dense enough sphere packings are often based on rather subtle algebraic technique, sometimes using algebraic geometry and number theory methods (cf. [CS], [TV], [RT], [Oe1]).

In this paper we are not trying to solve any of the above problems. Instead, we address the question of the density of typical lattice packings and of typical random packings. Here, the word “typical” refers to the natural probability distributions, defined below. In essence, the probabilistic approach to the lattice packing problem is not new, and existing results on dense enough lattice packings are implicitly based on it (cf. [M, H, Sch]). So, the results about the typical lattices presented below, are rather of the review nature, though they are presented in a new way. The most important element in them is probably the concept of a typical lattice itself. The importance of these results for us is that they give the reference frame we need when we discuss the properties of random packings, the main topic of the present paper.

As it is usual in the probability theory, the reasonable results can be obtained only in the limit when the number of degrees of freedom of the system goes to infinity. For the case of lattices that means passing to the limit over the dimension \( n \to \infty \). On the other hand, for random packings we have infinitely many degrees of freedom already in the (infinite volume) finite dimensional euclidean space \( \mathbb{R}^n \), so no other limit is needed here.

Below we fix some notation. In the next section we present our results concerning the properties of the typical lattice packings. The last section deals with the random packings.

Let \( \sigma \) be a locally finite subset of \( \mathbb{R}^n \). Let \( V_N \subset \mathbb{R}^n \) denote a cube with the side \( 2N \), centered at the origin. By \( \sigma_N \) we denote the intersection \( \sigma \cap V_N \). Let \( d(\sigma_N) \) denote the minimal spacing between the points of \( \sigma_N \):

\[
d(\sigma_N) = \min_{x,y \in \sigma_N: x \neq y} |x - y|,
\]

while \( r(\sigma_N) = \frac{1}{2} d(\sigma_N) \). We define the geometric (or sphere packing) density of the configuration \( \sigma \) in the box \( V_N \) to be the number

\[
\Delta_N(\sigma) = \frac{\text{vol} \left[ \left( \bigcup_{x \in \sigma_N} B(x, r(\sigma_N)) \right) \cap V_N \right]}{(2N)^n}.
\]

Here \( B(x,r) \) is a ball of radius \( r \) centered at the point \( x \).

The sphere packing density of an infinite volume configuration \( \sigma \) is defined as

\[
\Delta(\sigma) = \limsup_{N \to \infty} \Delta_N(\sigma).
\]
This corresponds to the classical problem of packing equal non-overlapping spheres in \( \mathbb{R}^n \).

In the special case of \textit{lattice packings} the set \( \sigma \) is a discrete additive subgroup of \( \mathbb{R}^n \) and the density can be rewritten as

\[
\Delta (\sigma) = \frac{v_n r (\sigma)^n}{\det (\sigma)},
\]

where \( v_n (r) = r^n v_n (1) \) is the volume of the sphere of the radius \( r \) in \( \mathbb{R}^n \), \( v_n = v_n (1) \), and \( \det (\sigma) \) is the volume of the fundamental domain of \( \sigma \).

The classical question then is about the largest possible value of \( \Delta (\sigma) \) for any locally finite subset \( \sigma \subset \mathbb{R}^n \) or for any lattice \( \sigma \subset \mathbb{R}^n \), and also how to construct the corresponding \( \sigma \)-s.

In this paper we discuss a different problem. We are interested in the density of a typical (random) configuration \( \sigma \), and that of a typical (random) lattice. Thus, our main results deal with the distribution of the random quantity \( \Delta (\sigma) \). Namely, we show that in the lattice case the geometric density of a typical lattice is of the order of \( 2^{-n} \) (see Theorem 1 below). For the random sphere packing the geometric density of a typical realization of a corresponding random field is zero. The problem becomes interesting if we allow ourselves to decimate the configuration and to throw away a fraction of it, containing ‘bad’ sites. Then it turns out that the remaining random configuration has the geometric density of the same order as in the lattice case (see Theorem 3 below), for a proper choice of the bad set.

\section{Random lattices}

In this section we discuss the lattice case. In the first subsection we present the results concerning the case of very high dimension \( n \), which turn into a simple relation in the limit \( n \to \infty \). In the following subsection we consider the case \( n = 2 \), when again the results can be expressed by simple relations, due to explicit computations.

\subsection{The case of large dimensions}

Let \( \sigma \) is a lattice. Clearly, its geometric density remains the same after multiplication by a scalar. Therefore, we can consider here only unimodular lattices, i.e., those with \( \det (\sigma) = 1 \). The space of unimodular lattices in \( \mathbb{R}^n \) is naturally isomorphic to the symmetric space \( \Lambda_n = SL_n (\mathbb{R}) / SL_n (\mathbb{Z}) \). It is equipped with the Haar measure \( \mu_n \) and \( \mu_n (\Lambda_n) \) is finite. We therefore can normalize it so that \( \mu_n (\Lambda_n) = 1 \). After such choice the density \( \Delta (\sigma) \) becomes a random variable. As the following theorem shows, in order to get a nontrivial limiting distribution for it, one should normalize it by the factor \( 2^n \).

\textbf{Theorem 1} Let

\[
F_n (x) = \mu_n \{ \sigma \in \Lambda_n : 2^n \Delta (\sigma) \leq x \}
\]
be the distribution function of the random variable $2^n \Delta(\sigma)$. Then
\[
\lim_{n \to \infty} F_n(x) = 1 - e^{-x/2}.
\]

This result follows from the following theorem of Schmidt:

**Theorem 2** [Sch] Let $n \geq 13$. Let $S$ be a Borel set in $\mathbb{R}^n$ such that $S \cap (-S) = \emptyset$. Suppose that $\text{vol}(S) \leq n - 1$. Then the measure
\[
\mu_n \{ \sigma \in \Lambda_n : \sigma \cap S = \emptyset \} = e^{-\text{vol}(S)} (1 - R_n),
\]
where
\[
|R_n| < 6 \left( \frac{3}{4} \right)^{n/2} e^{4\text{vol}(S)} + \text{vol}(S)^{n-1} n^{-n+1} e^{\text{vol}(S)+n}.
\]

**Proof of Theorem 2.** To derive Theorem 2 from Theorem 2 take $S = S(d)$ to be “one half of the ball” $B(0, d)$, in such a way as to satisfy the conditions $S \cap (-S) = \emptyset$ and $S \cup (-S) = B(0, d) \setminus \emptyset$. Then, for a lattice $\sigma$ the condition $\sigma \cap S = \emptyset$ is equivalent to the condition that $d(\sigma) > d$, so
\[
\mu_n \{ \sigma \in \Lambda_n : d(\sigma) > d \} = \mu_n \{ \sigma \in \Lambda_n : \sigma \cap S(d) = \emptyset \} = e^{-\text{vol}(S(d))} (1 - R_n).
\]

Let us put $x(\sigma) = v_n d(\sigma)^n$, then $2^n \Delta(\sigma) = x(\sigma)$, since $\det(\sigma) = 1$. Put $d = \left( \frac{x}{v_n} \right)^{1/n}$. Then, by definition
\[
F_n(x) = \mu_n \{ \sigma \in \Lambda_n : 2^n \Delta(\sigma) \leq x \} = \mu_n \{ \sigma \in \Lambda_n : x(\sigma) \leq x \} = 1 - \mu_n \{ \sigma \in \Lambda_n : d(\sigma) > d \} = 1 - \mu_n \{ \sigma \in \Lambda_n : \sigma \cap S(d) = \emptyset \}.
\]

Noting that $\text{vol}(S(d)) = \frac{n}{2}$, we have
\[
F_n(x) = 1 - \mu_n \{ \sigma \in \Lambda_n : \sigma \cap S(d) = \emptyset \} = 1 - e^{-\text{vol}(S(d))} (1 - R_{n,x}) = 1 - e^{-x/2} (1 - R_{n,x}),
\]
with
\[
|R_{n,x}| < 6 \left( \frac{3}{4} \right)^{n/2} e^{4x/2} + \left( \frac{x}{2} \right)^{n-1} n^{-n+1} e^{x/2+n} \to 0
\]
as $n \to \infty$, which proves our statement.
2.2 Two-dimensional case

For lattices in $\mathbb{R}^2$ the information is much more precise. Consider lattice packings in $\mathbb{R}^2$, i.e., lattices in $\mathbb{C}$. Instead of fixing their determinant, let us use the freedom of scalar multiplication and rotation to fix the shortest basis vector to be 1. Then we can take the other basis vector $z$ in the modular domain

$$ F = \{ x^2 + y^2 \geq 1, -\frac{1}{2} \leq x \leq \frac{1}{2} \}. $$

The corresponding lattice will be denoted by $L_z$. Its packing radius is $\frac{1}{\sqrt{2}}$. If $z = x + iy$, then the volume of the fundamental parallelepiped of $L_z$ is $y$. Therefore, its sphere packing density $\Delta(L_z)$ equals $\frac{\pi}{4}y$. The space of two-dimensional lattices $\Lambda_2 = \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z})$ is canonically homeomorphic to what we get from $F$ by identifying every boundary point $(x,y)$ with the boundary point $(-x,y)$.

We will use the well-known fact that the probability Haar measure on $\Lambda_2$ in the $(x,y)$ coordinates is given by $d\mu = \frac{3}{\pi} \frac{dx dy}{y^2}$. The sphere packing density $\Delta$ then becomes a random variable.

**Theorem 3** The random variable $\Delta$ has a density $p_{\Delta}(x)$, given by

$$ p_{\Delta}(x) = \begin{cases} \frac{12}{\pi} \left( 1 - 2 \sqrt{1 - (\frac{x}{\sqrt{3}})^2} \right) & \text{for } 0 \leq x \leq \frac{\pi}{4}, \\ 0 & \text{for } \frac{\pi}{4} \leq x \leq \frac{\pi}{\sqrt{12}}, \\ \frac{\pi}{\sqrt{12}} & \text{for } x \geq \frac{\pi}{\sqrt{12}}. \end{cases} $$

In particular, the mean value of $\Delta$ equals $\frac{3}{8} \log 3$, its variance equals $\left( \frac{\pi}{8 \sqrt{3}} - \frac{9}{16 \pi} (\log 3)^2 \right)$, and the maximal possible sphere packing density equals $\frac{\sqrt{3}}{\sqrt{12}}$.

**Proof.** The lattice $L_z$ has density $\Delta$ if and only if $y = y(\Delta) = \frac{x}{\sqrt{3}}$. The parameter $z = x + iy$ is therefore uniquely determined by the pair $(x,\Delta)$. So it can be taken as the coordinate system on $F$. We have $dy(\Delta) = -\frac{\pi}{2} \frac{dx}{\sqrt{3}}$. The measure $\mu$ can be rewritten in coordinates $(x,\Delta)$ as

$$ d\mu = \frac{3}{\pi} \frac{dx dy}{y^2} = \frac{3}{\pi} \frac{dx \left( \frac{2}{\sqrt{3}} \Delta \right)}{\left( \frac{\pi}{\sqrt{3}} \right)^2} = \frac{12}{\pi^2} dx d\Delta. $$

Let $F_b = F \cap \{ y = b \}$. The density $p_{\Delta}(a)$ is then equal to $\frac{12}{\pi^2} \text{mes} \left( F_{y(a)} \right)$, where $\text{mes}(\cdot)$ stands for the one-dimensional Lebesgue measure. For $y(a) \geq 1$, i.e., for $a \leq \frac{\pi}{\sqrt{3}}$, we have $\text{mes} \left( F_{y(a)} \right) = 1$; for $y(a) \leq \frac{\sqrt{3}}{2}$, i.e., for $a \geq \frac{\pi}{2 \sqrt{3}}$, we have $\text{mes} \left( F_{y(a)} \right) = 0$. For $\frac{\sqrt{3}}{2} \leq y(a) \leq 1$ the total length $\text{mes} \left( F_{y(a)} \right)$ equals $2 \left( \frac{1}{2} - \sqrt{1 - y(a)^2} \right) = 1 - 2 \sqrt{1 - \left( \frac{\pi}{\sqrt{3}} a \right)^2}$, which proves the first statement. The
statement about the maximal value of density follows immediately. The mean value is given by the elementary integral

\[ \int_0^\infty x p_\Delta(x) \, dx = \frac{12}{\pi^2} \left( \int_0^{\frac{\sqrt{\pi}}{2}} x \, dx - 2 \int_{\frac{\sqrt{\pi}}{2}}^\infty x \sqrt{1 - \left( \frac{\pi}{4x} \right)^2} \, dx \right), \]

and a similar equality defines the variance. 

\section{Random packings}

In this section instead of lattices we will consider configurations \( \sigma \) of the point random fields, and we will try to solve for them the same questions we were discussing above for lattices.

By a point random field in \( \mathbb{R}^n \) we mean a probability measure \( P \) on the set \( S \) of all countable locally finite subsets of \( \mathbb{R}^n \). The simplest example of such measure is the Poisson random field. To define it we first introduce for every \( V \subset \mathbb{R}^n \) the notation \( S_V \) for the set of all locally finite subsets of \( V \), and for every \( \sigma \in S \) we denote by \( \sigma_V \in S_V \) the intersection \( \sigma \cap V \). We denote by \( |\sigma_V| \) the cardinality of the set \( \sigma_V \).

A random field \( P^\lambda \) is called a \textit{Poisson random field with intensity} \( \lambda > 0 \) if and only if
i) for every two disjoint subsets \( V, W \subset \mathbb{R}^n \), \( V \cap W = \emptyset \), the random configurations \( \sigma_V, \sigma_W \) are independent;
ii) for every finite subset \( V \subset \mathbb{R}^n \) the conditional distribution of \( \sigma_V \) under the condition that \( |\sigma_V| = m \) is just the Lebesgue measure on \( V^m \), normalized by the factor \( \frac{1}{(\text{vol}(V))^m} \), while the probability of the event \( |\sigma_V| = m \) is given by

\[ P \{ \sigma : |\sigma_V| = m \} = e^{-\lambda \text{vol}(V)} \frac{(\lambda \text{vol}(V))^m}{m!}. \]

(We recall that the independence property means that for every two events \( A \subset S_V, B \subset S_W \) we have \( P \{ \sigma : \sigma_V \in A, \sigma_W \in B \} = P \{ \sigma : \sigma_V \in A \} P \{ \sigma : \sigma_W \in B \} \) provided \( V, W \) are disjoint.) In what follows we will denote by \( P \) the Poisson random field with intensity \( \lambda = 1 \), and we will omit \( \lambda \) from our notation, as well as the adjective “Poisson”.

More general random fields can be treated by the methods presented below. These are called Gibbs random fields corresponding to the interactions \( U \). Here \( U = U(x, y) \) is some given function interpreted as the strength of interaction between two particles situated at locations \( x \neq y \in \mathbb{R}^n \). A random field \( P^U \) is called Gibbs random field with interaction \( U \) if and only if its conditional distribution inside the finite box \( V \subset \mathbb{R}^n \), given the configuration \( \sigma_{\mathbb{R}^n \setminus V} \) outside it, has density with respect to the measure \( P^\lambda \) proportional to the “Gibbs factor”

\[ \exp \left\{ - \sum_{\substack{x \in \sigma_V, \ y \in \sigma_V \cup \sigma_{\mathbb{R}^n \setminus V} \atop y \neq x}} U(x, y) \right\}. \]
(For a general function $U$ the Gibbs random field lacks the independence property $i$) above.) To make it into a well-defined object, one has to put some restrictions on the function $U$. The most general one is called superstability condition. We will not formulate it, but just note that it is satisfied if $U$ is repulsive, which just means that $U \geq 0$. For further details see \cite{R} or \cite{D}. The Poisson fields above correspond to zero interaction.

Let now $\sigma$ be our point random field. Then it is easy to see that for every $\epsilon > 0$

$$P(\Delta_N (\sigma) > \epsilon) \to 0 \text{ as } N \to \infty.$$ 

Since our goal is to construct subsets with positive (and even as big as possible) geometric density, this is unsatisfactory. To save the situation we allow to decimate the configuration $\sigma$, by considering only “a fraction $\nu$” of our collection of points $\sigma$, where $\nu \in (0, 1)$. To this end we introduce the quantity

$$\Delta_{N,\nu} (\sigma) = \max_{\tau \subseteq \sigma_N : \nu |\sigma_N| - |\sigma_N|^{1/2+\epsilon} \leq |\tau| \leq \nu |\sigma_N| + |\sigma_N|^{1/2+\epsilon}} \Delta_N (\tau), \quad (1)$$

where $\epsilon$ is some fixed small number. This fraction is so designed that for any $\tau$ satisfying the restrictions in (1) we have $\frac{|\tau|}{|\sigma_N|} \to \nu$ as $N \to \infty$ for $\sigma$ typical. As the following statement shows, this improves the situation.

**Theorem 4** With $P$-probability 1 the limit

$$D(\nu) = \lim \inf_{N \to \infty} \Delta_{N,\nu} (\sigma)$$

does not depend on $\sigma$, for every $\nu$, $0 < \nu < 1$.

i) Let us introduce the function $\nu_1 (d) = 1 - \frac{1}{2}v_n (d)$. Then for every $d > 0$ we have the lower bound

$$D(\nu_1 (d)) \geq 2^{-n}v_n (d) \nu_1 (d), \quad (2)$$

provided of course that $\nu_1 (d) > 0$. In particular, for $d = d_n$, where $d_n$ satisfies $v_n (d_n) = 1$, we get $\nu_1 (d_n) = \frac{1}{2}$, and so

$$D\left(\frac{1}{2}\right) \geq 2^{-n-1}.$$

ii) Introducing the function $\nu_2 (d) = 1 - \frac{1}{2}v_n (d) e^{-2v_n (d)}$, we have for every $d > 0$

$$D(\nu_2 (d)) \leq 2^{-n}v_n (d) \nu_2 (d). \quad (3)$$

iii) Introducing the function $\nu_3 (d) = 1 - \frac{1}{2}v_n (d) - \frac{1}{6}v_n (d) e^{-v_n (d)}$, we have for every $d > 0$

$$D(\nu_3 (d)) \geq 2^{-n}v_n (d) \nu_3 (d). \quad (4)$$

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Note 1. The relations (3)-(4) give implicit bounds on the function $D(\nu)$. The relation (2) can easily be rewritten in explicit form: $D(\nu) \geq 2^{-n+1}\nu (1-\nu)$.

The remaining relations cannot be rewritten so easily.

Note 2. The relation (4) is an improvement of (2), see Fig. 1.

Note 3. Because of the properties of the function $\nu_2(d)$ the relation (3) gives an upper estimate on $D(\nu)$ only for the values $\nu > 1 - 4e^{-1} \approx 0.908$.

Note 4. Further improvements of the relations (2)-(4) can also be obtained, see the proof of the theorem below.

Note 5. The above theorem tells us that for almost every configuration $\sigma$ and for every $N$ large enough we can find a subconfiguration $\sigma'_N \subset \sigma_N$ with about $\nu |\sigma_N|$ points in it and with the geometric density close to $D(\nu)$. However, the configurations $\sigma'_N$ might not converge as $N \to \infty$.

3.1 Proof of the Theorem 4

We impose periodic boundary conditions in the box $V_N$, which wraps it into a torus. For a finite subset of points $\sigma \subset V_N$, a point $x \in V_N$ and a number $d > 0$
we define

\[ m(x, d, \sigma) = \left| B(x, d) \cap \{ \sigma \setminus \{x\} \} \right|. \]

This is just the number of points in \( \sigma \), different from \( x \), which are at a distance not bigger than \( d \) from \( x \). Let us also introduce the measure \( \delta_\sigma \) on \( V_N \) by putting a \( \delta \)-measure at each point of \( \sigma \):

\[ \delta_\sigma = \sum_{y \in \sigma} \delta_y. \]

The relevant quantity to look on is now the sum

\[ M(d, \sigma) = \frac{1}{2} \int_{V_N} m(x, d, \sigma) \, \delta_\sigma (dx). \]

\( M(d, \sigma) \) is the number of pairs of points in \( \sigma \) at a distance not bigger than \( d \).

The reason we introduce the quantity \( M(d, \sigma) \) is the following. Let us consider the graph \( G_d(\sigma) \) with vertices at the points of \( \sigma \) and with edges connecting any two vertices at distance \( \leq d \), then \( M(d, \sigma) \) is precisely the number of edges in \( G_d(\sigma) \). Clearly, in the case \( M(d, \sigma) = M \), we can find a subset \( \tau \subset \sigma \) with

\[ |\tau| = |\sigma| - M, \]

such that \( d(\tau) \geq d \), so the graph \( G_d(\tau) \) has no edges and therefore

\[ \Delta_N(\tau) \geq \frac{\nu_n \left( \frac{4}{d^2} \right) (|\sigma| - M)}{(2N)^n}. \tag{5} \]

Since evidently for \( \nu = \frac{|\tau|}{|\sigma|} \) we have

\[ \Delta_{N,\nu}(\sigma) \geq \Delta_N(\tau), \tag{6} \]

the rhs of the estimate (5) provides the lower estimate for the quantity we are interested in.

We introduce now other important quantities, which allow us to obtain better estimates on the number of points in the subset \( \tau \subset \sigma \) with the property that \( d(\tau) \geq d \). Let \( M_1(d, \sigma) \) be the number of isolated edges of the graph \( G_d(\sigma) \). Then for every \( \tau \) with \( d(\tau) \geq d \) we have

\[ |\tau| \leq |\sigma| - M_1. \tag{7} \]

Likewise, we introduce the quantity \( M_3(d, \sigma) \) to be the number of subgraphs of \( G_d(\sigma) \) which are maximal connected components of \( G_d(\sigma) \) with 3 vertices and 3 edges (i.e. just isolated triangles), and \( M_2(d, \sigma) \) to be the number of maximal connected components of \( G_d(\sigma) \) with 3 vertices and 2 edges. Then we have for all \( \tau \) with \( d(\tau) \geq d \) that

\[ |\tau| \leq |\sigma| - M_1 - M_2 - 2M_3. \]
On the other hand, because of the Theorem 8 below, we can claim the existence of the subset \( \tau \subset \sigma \) with \( d(\tau) \geq d \) and such that
\[
|\tau| \geq |\sigma| - M_1 - \frac{2}{3} (M - M_1)
\] (8)
(see estimate (24) below), and even
\[
|\tau| \geq |\sigma| - M_1 - M_2 - 2M_3 - \frac{3}{5} (M - M_1 - 2M_2 - 3M_3)
\]
(see estimate (25) below). One can proceed further with such estimates, introducing the quantities \( M_i \) with higher \( i \)-s; any information we have on the behavior of the random variables \( M_i(d, \sigma) \) provides us with some answer to the question we are interested in.

We start by the study of the random variable \( M(d, \sigma) \).

**Lemma 5** The mean value
\[
\mathbb{E}(M(d, \sigma)) = \frac{v_n(d)}{2} (2N)^n.
\]

**Proof.** We first use the identity
\[
\mathbb{E}(M(d, \sigma)) = \mathbb{E}\left[\mathbb{E}(M(d, \sigma) | \{|\sigma| = k\})\right].
\]
The conditional expectation \( \mathbb{E}(M(d, \sigma) | \{|\sigma| = k\}) \) can be computed in the following way. Denote by \( \chi_d(x, y) \) the function
\[
\chi_d(x, y) = \begin{cases} 1 & \text{if } |x - y| \leq d, \\ 0 & \text{if } |x - y| > d, \end{cases}
\]
\( x, y \in V_N \). Let \( \xi_1, ..., \xi_k \) be \( k \) independent random points in \( V_N \), distributed uniformly according to the Lebesgue measure on \( V_N \). Then
\[
\mathbb{E}(M(d, \sigma) | \{ |\sigma| = k \}) = \mathbb{E}\left( \sum_{1 \leq i < j \leq k} \chi_d(\xi_i, \xi_j) \right) = \frac{k(k-1)}{2} \frac{v_n(d)}{(2N)^n}.
\]
Hence
\[
\mathbb{E}(M(d, \sigma)) = \frac{v_n(d)}{2(2N)^n} \sum_{k \geq 1} k(k-1) \frac{(2N)^nk}{k!} e^{-(2N)^n} = \frac{v_n(d)}{2} (2N)^n.
\]

In the same way one can compute the expectation \( \mathbb{E}(M_1(d, \sigma_N)) \). A particle \( x \in \sigma_N \) contributes to \( M_1(d, \sigma_N) \) if there is another particle \( y \in \sigma_N \) with \( |x - y| \leq d \), and there are no other particles in the union \( B(x, d) \cup B(y, d) \).
Since \( \text{vol}(B(x, d) \cup B(y, d)) \in [v_n(d), 2v_n(d)] \), we obtain
\[
\frac{v_n(d)}{2} (2N)^n e^{-2v_n(d)} \leq \mathbb{E}(M_1(d, \sigma_N)) \leq \frac{v_n(d)}{2} (2N)^n e^{-v_n(d)}.
\]
Before dealing with the variance, we will make a slight generalization. Namely, let \( \phi (x, \sigma) \) be a 'local observable', that is, a function which depends only on the intersection \( B \left( x, \tilde{d} \right) \cap \{ \sigma \setminus x \} \), for some \( \tilde{d} = \tilde{d} (\phi) \). We introduce a random variable \( \Phi (\sigma) \) by

\[
\Phi (\sigma) = \int_{V_N} \phi (x, \sigma_N) \delta_\sigma (dx).
\]

(9)

For example, the choice \( \phi (x, \sigma) = m (x, d, \sigma) \) corresponds to \( \Phi (\sigma) = M (d, \sigma) \).

In case

\[
\phi (x, \sigma) = \begin{cases} 
1, & \text{if } m (x, d, \sigma) = 1, \text{ and for the unique } y \in \sigma \text{ with } |x - y| \leq d, \ m (y, d, \sigma) = 1, \\
0 & \text{in all other cases}
\end{cases}
\]

we obtain \( \Phi (\sigma) = M_1 (d, \sigma) \). We will impose the following restriction on \( \phi \) : there exists a constant \( c > 0 \), such that for all \( x, y \in \mathbb{R}^n \), all \( \sigma \subset \mathbb{R}^n \)

\[
|\phi (x, \sigma \cup y) - \phi (x, \sigma)| \leq c.
\]

(10)

It clearly holds in the above examples, with \( c = 1 \).

**Lemma 6**  The variance \( \mathbb{D} (\Phi (\sigma_N)) \) satisfies the estimate

\[
\mathbb{D} (\Phi (\sigma_N)) \leq C (2N)^n,
\]

with \( C = C (\phi) \).

**Proof.** We start with the identity

\[
\mathbb{D} (\xi) = \mathbb{D} [\mathbb{E} (\xi | \eta)] + \mathbb{E} [\mathbb{D} (\xi | \eta)],
\]

valid for any two random variables. We are going to apply it for the case of \( \xi = \Phi (\sigma) \), while \( \eta \) will be the restriction of \( \sigma \) to certain subsets \( K \subset V_N \), which are called “corridors” and which are defined as follows. Consider the partition \( \Pi \) of the box \( V_N \), formed by the cubic subvolume \( V_l \) together with all its shifts by the vectors of the lattice \( 2l \mathbb{Z}^n \). The number \( l \) is chosen to be equal to \( \tilde{d} + D \), where the number \( D \) has to be of the order of \( \tilde{d} : D \in \left[ \frac{\tilde{d} k (n)}{2}, \frac{2 \tilde{d} k (n)}{2} \right] \), with the integer \( k (n) \) depending only on the dimension \( n \). For such a partition to exist, we need the ratio \( \frac{N}{\tilde{d} + D} \) to be an integer; clearly, such a choice of the number \( D \) is possible. The union of the corridors \( K_1 \) is now defined as the \( d \)-neighborhood of the union of the boundaries of all the boxes of \( \Pi \). In other words, it consists of the strips of the width \( 2d \), which are parallel to the various coordinate planes, with the spacing between the two consecutive ones to be equal to \( 2D \). (The subscript in the notation \( K_1 \) is needed due to the fact that later we will have to consider other corridors.) The reason to introduce the corridors \( K_1 \)
is that the random variable $\Phi (\sigma)$, conditioned by the value $\eta$ of the restriction $\sigma|_{K_1}$, turns into the sum of independent random variables.

Let us start with the second term of (11), $E_{\sigma} \left[ D (\xi | \eta) \right]$. We first estimate the contribution to the inner variance, coming from a single box $V_l$ of $\Pi$. That is, we need to consider the box $V_{l-d}$, the Poisson random field $\sigma$ inside it, together with the fixed configuration $\eta_{K(V_l)}$ in the corridor $K(V_l) = V_l \setminus V_{l-d}$. The random variable we should study is the sum

$$\phi^{V_l} (\sigma | \eta_{K(V_l)}) = \int_{V_{l-d}} \phi (x, \sigma) \delta_{\sigma} (dx) + \int_{V_{l-d}} [\phi (x, \sigma \cup \eta) - \phi (x, \sigma)] \delta_{\sigma} (dx).$$

(12)

To estimate its variance we will use the evident relation:

$$\mathbb{D} (\zeta + \chi) = \mathbb{D} (\zeta) + \mathbb{D} (\chi) + 2 (E (\zeta \chi) - E (\zeta) E (\chi))$$

(13)

with $\zeta$ to be the first term in (12), and $\chi$ - the second. The first variance is just a constant:

$$R_0 \equiv R_0 (l, \phi) = \mathbb{D} \left( \int_{V_{l-d}} \phi (x, \sigma) \delta_{\sigma} (dx) \right).$$

To estimate the covariance term, corresponding to $E (\zeta \chi) - E (\zeta) E (\chi)$ in (13), we rewrite the difference $[\phi (x, \sigma \cup \eta) - \phi (x, \sigma)]$ as

$$\phi (x, \sigma \cup \eta) - \phi (x, \sigma) = \sum_{i=1}^{n} \psi (x, \sigma \cup \eta_{i-1}, y_i),$$

where

$$\psi (x, \kappa, y) = \phi (x, \kappa \cup y) - \phi (x, \kappa).$$

Here we use some enumeration $\eta_{K(V_l)} = \{y_1, y_2, ..., y_n\}$, while $\eta_i = \{y_1, y_2, ..., y_i\}$. Then

$$E (\zeta \chi) - E (\zeta) E (\chi) =$$

$$= \sum_{i=1}^{n} \{E^\sigma \left( \int_{V_{l-d}} \phi (x, \sigma) \delta_{\sigma} (dx) \int_{V_{l-d}} \psi (x, \sigma \cup \eta_{i-1}, y_i) \delta_{\sigma} (dx) \right) -$$

$$- E^\sigma \left( \int_{V_{l-d}} \phi (x, \sigma) \delta_{\sigma} (dx) \right) E^\sigma \left( \int_{V_{l-d}} \psi (x, \sigma \cup \eta_{i-1}, y_i) \delta_{\sigma} (dx) \right) \}.$$  

(14)

(The symbol $E^\sigma$ means taking expectation in $\sigma$; $\eta$ is here a fixed parameter.)

Note that

$$\psi (x, \sigma \cup \eta_{i-1}, y_i) \equiv \phi (x, \sigma \cup \eta_{i-1} \cup y_i) - \phi (x, \sigma \cup \eta_{i-1}) = 0$$

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unless \( x \in B \left( y_i, \hat{d} \right) \), in which case

\[
\psi (x, \sigma \cup \eta_{i-1}, y_i) = \psi \left( x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i \right),
\]

and

\[
\phi (x, \sigma) = \phi \left( x, \sigma_B(y_i, 2\hat{d}) \right).
\]

Therefore

\[
E^\sigma \left( \int_{V_{i-\hat{d}}} \phi(x, \sigma) \, \delta_\sigma (dx) \int_{V_{i-\hat{d}}} \psi(x, \sigma \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right) -
\]

\[
- E^\sigma \left( \int_{V_{i-\hat{d}}} \phi(x, \sigma) \, \delta_\sigma (dx) \right) E^\sigma \left( \psi(x, \sigma \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right)
\]

\[
= E^\sigma \left( \int_{V_{i-\hat{d}}} \phi(x, \sigma_B(y_i, 2\hat{d})) \, \delta_\sigma (dx) \int_{V_{i-\hat{d}}} \psi(x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right) -
\]

\[
- E^\sigma \left( \int_{V_{i-\hat{d}}} \phi(x, \sigma_B(y_i, 2\hat{d})) \, \delta_\sigma (dx) \right) E^\sigma \left( \int_{V_{i-\hat{d}}} \psi(x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right).
\]

Because of (14), the random variable \( |\psi(x, \sigma \cup \eta_{i-1}, y_i)| \equiv c(\psi) \), while \( |\phi(x, \sigma_B(y_i, 2\hat{d})| \leq c(\phi) |\sigma_B(y_i, 2\hat{d})| \), so \( E^\sigma \left( |\phi(x, \sigma_B(y_i, 2\hat{d})| \right) \) is bounded as well. Therefore we can continue in (13) by

\[
E(\zeta_\chi) - E(\zeta) E(\chi) \leq \frac{1}{2} R_1 |\eta_{K(V_i)}|,
\]

where \( R_1 \equiv R_1 (\phi) \), thus having

\[
D^\sigma \left( \phi^{V_i}(\sigma | \eta_{K(V_i)}) \right) \leq R_0 + R_1 |\eta_{K(V_i)}| +
\]

\[
+ D^\sigma \left( \sum_{i=1}^n \left( \int_{V_{i-\hat{d}}} \psi(x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right) \right).
\]

Two random variables: \( A_i = \left( \int_{V_{i-\hat{d}}} \psi(x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right) \) and \( A_j = \left( \int_{V_{j-\hat{d}}} \psi(x, \sigma_B(y_j, 2\hat{d}) \cup \eta_{j-1}, y_j) \, \delta_\sigma (dx) \right) \) - are independent, provided \( |y_i - y_j| > 2\hat{d} \). Hence, applying again (13),

\[
D^\sigma \left( \sum_{i=1}^n \left( \int_{V_{i-\hat{d}}} \psi(x, \sigma_B(y_i, 2\hat{d}) \cup \eta_{i-1}, y_i) \, \delta_\sigma (dx) \right) \right) = D^\sigma \left( \sum_{i=1}^n A_i \right) \quad (16)
\]

\[
= \sum_{i=1}^n D^\sigma A_i + 2 \sum_{i \neq j, |y_i - y_j| < 2\hat{d}} \left[ E(A_i A_j) - E(A_i) E(A_j) \right].
\]
The first term in (16) can be estimated from above by
\[ R_2 (\phi) \left| \eta_{K(V_i)} \right| , \quad (17) \]
while the second term – by
\[ R_3 (\phi) \int_{K(V_i)} m \left( y, 2\tilde{d}, \eta \right) \delta_\eta (dy) . \quad (18) \]
Putting together (15), (16), (17) and (18), we get
\[ \mathbb{D} (\phi^{V_i} (\sigma | \eta_{K(V_i)})) \leq R_0 + R_4 \left| \eta_{K(V_i)} \right| + R_3 \int_{K(V_i)} m \left( y, 2\tilde{d}, \eta \right) \delta_\eta (dy) . \]

Hence the total variance
\[ \mathbb{D} (\xi | \eta) \equiv \sum_{V \in \Pi} \mathbb{D} (\phi^V (\sigma_V | \eta_{K(V)})) \leq \left( \frac{N}{d + D} \right)^n R_0 + R_4 \int_{K_1} \delta_\sigma (dy) + R_3 \int_{K_1} m \left( y, 2\tilde{d}, \sigma \right) \delta_\sigma (dy) \]
\[ < \left( \frac{N}{d + D} \right)^n R_0 + R_4 \int_{V_N} \delta_\sigma (dx) + R_3 \int_{V_N} m \left( x, 2\tilde{d}, \sigma \right) \delta_\sigma (dx) , \]
and the expectation \( \mathbb{E} [\mathbb{D} (\xi | \eta)] \) is indeed less than \( C_1 (2N)^n \) for some \( C_1 \), according to the Lemma 5.

Going now to the first term, \( \mathbb{D} [\mathbb{E} (\xi | \eta)] \), in (11), we will slightly abuse the notation, denoting by \( \sigma \) the Poisson field outside the corridor \( K_1 \), with \( \eta \) denoting the Poisson field on \( K_1 \). We have
\[ \mathbb{E}^\sigma (\Phi (\sigma \cup \eta) | \eta) = \left( \frac{N}{d + D} \right)^n \mathbb{E}^\sigma \left( \int_{V_{i-d}} \phi (x, \sigma) \delta_\sigma (dx) \right) + \]
\[ \mathbb{E}^\sigma \left( \int_{V_N} [\phi (x, \sigma \cup \eta) - \phi (x, \sigma)] \delta_{\sigma \cup \eta} (dx) \right) . \]
The first expectation is just a constant:
\[ R_5 (\phi) = \mathbb{E} \left( \int_{V_{i-d}} \phi (x, \sigma) \delta_\sigma (dx) \right) , \]
and so does not contribute to the variance. Therefore we need to study the variance of the random variable
\[ \Phi^1 (\eta) = \mathbb{E}^\sigma \left( \int_{V_N} [\phi (x, \sigma \cup \eta) - \phi (x, \sigma)] \delta_{\sigma \cup \eta} (dx) \right) . \quad (19) \]
We want to argue that this function can be represented similarly to (9):

$$
\Phi^1(\eta) = \int_{K_1} \phi^1(y, \eta) \delta_y(dy),
$$

(20)

with the function $\phi^1(x, \eta)$ satisfying the analog of relation (10). To show this let us first fix an enumeration $\eta_{K_1} = \{y_1, y_2, ..., y_n\}$ for every configuration $\eta_{K_1}$. (This even can be done in a measurable way, but since this enumeration will not be needed in the final analysis, we will not elaborate on this point.) Define $\eta_i = \{y_1, y_2, ..., y_i\}$ and rewrite (19) as

$$
\Phi^1(\eta) = \sum_{i=1}^{n} \mathbb{E}^\sigma \left[ \int_{V_N} \phi(x, \sigma \cup \eta_i) \delta_{\sigma \cup \eta_i}(dx) - \int_{V_N} \phi(x, \sigma \cup \eta_{i-1}) \delta_{\sigma \cup \eta_{i-1}}(dx) \right].
$$

We define now the function

$$
\tilde{\phi}^1(y, \eta) = \mathbb{E}^\sigma \left[ \phi(y, \sigma \cup \eta_{i-1} \cup y) + \left( \int_{V_N} \phi(x, \sigma \cup \eta_{i-1} \cup y) - \phi(x, \sigma \cup \eta_{i-1}) \right) \delta_{\sigma \cup \eta_{i-1}}(dx) \right].
$$

(21)

It is almost what we need. Indeed, the relation (21) is straightforward, while the estimate (10) follows from the same estimate for $\phi$. The only drawback is that the function $\tilde{\phi}^1(y, \eta)$ depends not only on $\eta$, but also on the ordering we choose. However, if we now pass from $\tilde{\phi}^1(y, \eta)$ to its symmetrization $\phi^1(y, \eta)$, obtained by averaging over all possible orderings on $\eta$, we get what we want.

The net result of the above discussion is that the estimate of the variance of the random variable $\Phi(\sigma_N)$ is reduced with the help of the identity (11) to the estimate of the variance of the random variable $\Phi^1(\eta)$. The key advantage of passing from $\Phi$ to $\Phi^1$ is that the latter random variable depends only on restriction $\eta$ of $\sigma$ to "$(n-1)$-dimensional" subset $K_1$ of $\mathbb{R}^n$. For example, in case $n = 1$ the set $K_1$ splits into union of disjoint segments, and the random variable $\Phi^1(\eta)$ is then the sum of independent random variables, corresponding to these segments. Hence, the variance of this sum is just the sum of the variances. For general values of $n$ we have to repeat the above scheme for $(n-1)$ more times, reducing after each step the "dimension" of the corridors by one.

So we proceed as follows. To estimate the variance of $\Phi^1(\eta)$ we again apply the identity (11), with the following choices: $\xi = \Phi^1(\sigma|_{K_1})$, while $\eta$ is the restriction of $\sigma$ to the "$(n-2)$-dimensional" corridor $K_2 \subset K_1$, defined by the following construction. Let $w_1 \subset V_i$ be the subset of $2n$ points of the cube $V_i$, consisting of the centers of all its $(n-1)$-dimensional faces. (The previous subset $w_0 \subset V_i$ contained just one point: the center of $V_i$ itself.) Then

$$
K_2 = K_1 \setminus \bigcup_{b \in \{w_1 + 2d, d\}} (V_i - 2d + b).
$$

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(Again, for \( n = 2 \) the set \( K_2 \) splits into disjoint union.) The key feature of thus defined corridor is that the random variable \( \Phi^1 (\sigma | K_1) \) conditioned by the value of the restriction \( \sigma | K_2 \) splits into the sum of independent random variables, so we can repeat the above arguments. The rest of the proof is just this repetition and is omitted. ■

Let \( \xi \) be any random variable. The Chebychev inequality claims that for any \( \varepsilon > 0 \)

\[
P \{ |\xi - \mathbb{E}(\xi)| \geq \varepsilon \} \leq \frac{\mathbb{D}(\xi)}{\varepsilon^2}.
\]

Applying it to \( M (d, \sigma_N) \) we get the following

**Proposition 7** Let \( \varepsilon > 0 \) and \( \delta > 1/2 \) be fixed. Then

\[
P \left\{ \left| \frac{M (d, \sigma_N) - \nu_n(d) (2N)^n}{((2N)^{n})^{2/3}} \right| > \varepsilon \right\} \to 0
\]

as \( N \to \infty \). ■

**End of the proof of Theorem 6.** Let us fix some value of the distance \( d \). Let \( N \) be large enough, \( \varepsilon \) be fixed, and \( \sigma_N \) be a typical configuration in \( V_N \). Then it follows from the last proposition that with probability \( P \) going to 1 as \( N \to \infty \) the configuration \( \sigma_N \) has the properties:

\[
\left| \frac{M (d, \sigma_N) - \nu_n(d) (2N)^n}{((2N)^{n})^{2/3}} \right| < \varepsilon
\]

(22)

and

\[
\left| \frac{|\sigma_N| - (2N)^n}{((2N)^{n})^{2/3}} \right| < \varepsilon
\]

(23)

(again by Chebychev).

Plugging in these data into estimates (6), (7), we find that for \( \nu = 1 - \frac{1}{3} \nu_n \) (d)

\[
\Delta_{N, \nu} (\sigma) \geq \nu_n \left( \frac{d}{2} \right) \left( 1 - \frac{1}{2} \nu_n (d) \right) + o \left( N^{-1/4} \right)
\]

uniformly over configurations \( \sigma_N \), satisfying (22), (23), which proves (2). The proof of the rest of the Theorem follows by similar arguments, applied to relations (6), (7). ■
3.2 Vertex covering number of a graph

In this subsection we prove a result from the graph theory, which was used above. The Theorem 8 below contains in fact a result slightly stronger than what was needed.

Let $G = (V(G), E(G))$ be a finite connected graph without loops and multiple edges. Let $v(G) = |V(G)|$ be the number of its vertices and $e(G) = |E(G)|$ the number of its edges. A set $A \subseteq V(G)$ is called a covering vertex set if any vertex of $G$ is a neighbor of a vertex of $A$. Another way to look at this property of $A$ is to say that if we cross out all the vertices of $A$ together with all edges having at least one end in $A$, no edges are left. The vertex covering number $\alpha(G)$ is defined as the smallest number of elements in a vertex covering set.

**Theorem 8** Let $G$ be as above, with $e(G) > 0$, then

$$\alpha(G) \leq \frac{e(G) + 1}{2},$$

and, in particular, for $e(G)$ even

$$\alpha(G) \leq \frac{1}{2} e(G).$$

Therefore, if $e(G) \geq 2$, then

$$\alpha(G) \leq \frac{2}{3} e(G),$$

and if $e(G) \geq 4$, then

$$\alpha(G) \leq \frac{3}{5} e(G).$$

Also,

$$\max_{G, v(G) = v} \frac{\alpha(G)}{e(G)} = \begin{cases} \frac{1}{2} + \frac{1}{v} & \text{for } v \text{ odd}, \\ \frac{1}{2} + \frac{1}{v-1} & \text{for } v \text{ even}, \end{cases}$$

where the maximum is taken over all graphs $G$ of the described type such that $v(G) = v$.

We say that a vertex is an end vertex if there is exactly one edge adjacent to it.

**Lemma 9** Let $G$ be as above, with $v(G) \geq 3$. Then there exists a non-end vertex such that if we cross it out together with all its edges, and then cross out all resulted isolated vertices, the resulting graph will be connected.
Proof. Suppose the assertion to be false. Denote by $V_{ne} \ (G)$ the set of all non-end vertices of $G$. Then for every vertex $s \in V_{ne} \ (G)$ the result of crossing it out together with all its edges, followed by crossing out all isolated vertices, is a graph $G_s$ with several components: $G_s = G_{s,1} \cup G_{s,2} \cup \cdots \cup G_{s,m_s}$ with $m_s \geq 2$. Let $e_{\min} \ (s)$ be the minimum number of edges in $G_{s,i}$, $i = 1, 2, \ldots, m_s$. Define $s_0 \in V_{ne} \ (G)$ to be a minimizer of the function $e_{\min} \ (\cdot)$. Fix a component $G_{s_0}^0$ with $e_{\min} \ (s_0)$ edges. Note that $G - G_{s_0}^0$ is connected. Take any vertex $v_1 \in G_{s_0}^0$ which is a neighbor of $s_0$. Clearly, $s_1 \in V_{ne} \ (G)$. Let us cross it out from $G$. By assumption, $G_{s_1}$ also has several components, the one containing $G - G_{s_0}^0$ is not the smallest one, hence the smallest one lies in $G_{s_0}^0$ and has therefore less edges than $e_{\min} \ (s_0)$, which brings us to contradiction. 

Proof of the Theorem 8. We argue by induction on the number of edges, and we use the notation of the proof of the Lemma 9 above. The case of one edge is clear. By Lemma 9 we can find a non-end vertex $s \in V_{ne} \ (G)$, such that the cross-out graph $G_s$ remains connected; since $\alpha (G) \leq \alpha (G_s) + 1$ and $e (G_s) \leq e (G) - 2$, we get the first four statements.

To prove (26) we first prove the upper bound. We have: $\frac{\alpha (G)}{\alpha (G)} \leq \frac{1}{2} + \frac{1}{2 (v (G) - 1)}$, while $e (G) \geq v (G)$ and $\frac{\alpha (G)}{\alpha (G)} \leq \frac{1}{2} + \frac{1}{2 (v (G) - 1)}$ whenever there is a cycle in $G$. To prove that the left hand side of (26) is less than or equal to the right hand side, the only case left is when $G$ is a tree with $v (G)$ odd. This is done by induction on $v$. For $v = 3$ the statement is clear. If $v > 3$, fix a vertex $s_0 \in V \ (G)$ and take an end vertex $s_1$ most distant from $s_0$. The only vertex $s_2$ adjacent to it has the graph $G_{s_2}$ connected. We have $\alpha (G) \leq \alpha (G_{s_2}) + 1$, $e (G) = v (G) - 1$ and $e (G_{s_2}) = v (G_{s_2}) - 1$ (since both $G$ and $G_{s_2}$ are trees). If $v (G_{s_2}) = v (G) - 2$, then by the induction hypothesis

$$\alpha (G) \leq \alpha (G_{s_2}) + 1 \leq \frac{e (G_{s_2})}{2} + \frac{e (G_{s_2})}{2 v (G_{s_2})} + 1$$

$$= \frac{1}{2} (v (G_{s_2}) - 1 + \frac{v (G_{s_2}) - 1}{v (G_{s_2})} + 2)$$

$$= \frac{1}{2} (v (G_{s_2}) - \frac{1}{v (G_{s_2})} + 2)$$

$$\leq \frac{1}{2} (v (G) - \frac{1}{v (G)}) = \frac{e (G)}{2} \left( 1 + \frac{1}{v (G)} \right).$$

If $v (G_{s_2}) \leq v (G) - 3$, then similarly

$$\alpha (G) \leq \alpha (G_{s_2}) + 1 \leq \frac{e (G_{s_2})}{2} + \frac{e (G_{s_2})}{2 (v (G_{s_2}) - 1)} + 1$$

$$= \frac{1}{2} (v (G_{s_2}) + 2)$$

$$\leq \frac{1}{2} (v (G) - \frac{1}{v (G)}) = \frac{e (G)}{2} \left( 1 + \frac{1}{v (G)} \right).$$

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Since the estimate (26) is achieved on the line graph (for $v$ even) and on the circle graph (for $v$ odd), the proof is complete. ■

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