On the Microscopic Spectra of the Massive Dirac Operator for Chiral Orthogonal and Chiral Symplectic Ensembles

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Abstract
Using Random Matrix Theory we set out to compute the microscopic correlators of the Euclidean Dirac operator in four dimensions. In particular we consider: the chiral Orthogonal Ensemble (χOE), corresponding to a Yang-Mills theory with two colors and fermions in the fundamental representation, and the chiral Symplectic Ensemble (χSE), corresponding to any number of colors and fermions in the adjoint representation. In both cases we deal with an arbitrary number of massive fermions. We use a recent method proposed by H. Widom for deriving closed formulas for the scalar kernels from which all spectral correlation functions of the χGOE and χGSE can be determined. Moreover, we obtain complete analytic expressions of such correlators in the double microscopic limit, extending previously known results of four-dimensional QCD at $\beta = 1$ and $\beta = 4$ to the general case with $N_f$ flavors, with arbitrary quark masses and arbitrary topological charge.
1 Introduction

One of the most successful and well-established physical applications of Random Matrix Theory (RMT) is the analysis of Quantum Chromodynamics (QCD) at low energies. In particular, the spectral statistical properties of the Euclidean Dirac operator in the infrared regime, can effectively be described within a RMT approach. In fact, there is a close correspondence between the finite-volume partition function of four dimensional QCD in the low-energy limit and the partition function of a RMT with the same global symmetries \[1\] \[2\] \[3\]. This theory is called chiral RMT (\(\chi\)RMT), because of its chiral content. The four-dimensional QCD Dirac operator with \(N_f\) fundamental fermions and gauge group \(SU(N_c = 2)\) or \(SU(N_c > 2)\), is described by the chiral Orthogonal (\(\chi\)OE) or chiral Unitary ensemble (\(\chi\)UE), respectively. The case of adjoint fermions and \(SU(N_c \geq 2)\) gauge group, corresponds to the chiral Symplectic ensemble (\(\chi\)SE) \[4\]. These agreements are valid only in the microscopic limit of \(\chi\)RMT, in which the universality-classes of the three chiral random matrix ensembles, manifest explicitly \[3\]. In this particular limit, one “magnifies” the Dirac operator spectra around the origin (zero virtuality) which, through the Banks-Casher relation \[6\], reflects the existence of a non-vanishing expectation value of the chiral condensate \(\Sigma = \langle \bar{\psi}\psi \rangle\), i.e. a chiral symmetry breaking at low energies. Indeed, Leutwyler and Smilga showed \[7\] that when \(\Sigma \neq 0\) suitable spectral sum-rules must hold. Such sum-rules can be written as integrals over the microscopic spectral density of the Dirac operator \[1\], given by

\[
\rho_s(z) = \lim_{V_4 \to \infty} \frac{1}{V_4 \Sigma} \rho\left(\frac{z}{V_4 \Sigma}\right),
\]

where \(V_4\) is the Euclidean space-time volume. The universality of \(\chi\)RMT in the microscopic limit (and therefore the universality of \(\rho_s(z)\)) supports the idea that the QCD Dirac spectra are universal in the large-volume scaling limit \[1\] \[3\] \[8\]. Several facts confirms this scenario, such as analytical results and extensive universality studies of microscopic spectral correlators \[9\] \[10\] \[11\] \[12\] \[13\] \[14\] as well as good agreement with numerical results obtained in Lattice QCD simulations for massless \[15\] and massive \[16\] fermions. For more details and a complete list of references we suggest some excellent reviews \[17\] \[18\], in which it is also possible to find information and references about non-chiral RMT and the corresponding three-dimensional QCD theory.

In the fundamental paper \[3\] the large-\(N\) behaviour of the orthogonal polynomials relevant for the chiral Unitary ensemble is proven to be universal near the origin, in the scaling limit \(x = N^2 \lambda\), where \(x\) is kept fixed. From this it follows that all microscopic correlators have the same universal behaviour. Also in the so-called double microscopic limit of \(\chi\)UE where both masses and eigenvalues are scaled at large-\(N\), one sees a similar universal be-
haviour of the orthogonal polynomials associated with $N_f$ massive flavors. Consequently all microscopic correlators are seen to be universal \cite{10}. In the $\chi$OE and $\chi$SE the question whether the microscopic correlators in general are universal or not is still open. By relating the massless kernels of $\chi$OE and $\chi$SE to the $\chi$RMT with complex elements ($\beta = 2$), which is known to be universal, it is shown in ref. \cite{13} \cite{14} that microscopic universality persists under certain smoothness assumptions. Though a similar universal behaviour in the double microscopic limit for $\beta = 1$ and $\beta = 4$ remains to be proven, it is widely recognized that universality in these two models is a reliable consequence as well. So far several results have been explicitly obtained for the case of massless flavors. For instance, the microscopic spectral density is known for all chiral ensembles in the case of an arbitrary number of massless flavors \cite{4} \cite{8} \cite{19} \cite{20}. Also, the cases concerning an arbitrary number of massless flavors and an even number of massive flavors in non-$\chi$UE have been derived in \cite{18} and \cite{21} respectively, and recently the cases of non-$\chi$OE and non-$\chi$SE have been solved in \cite{22}. In the case of massive flavors, the situation is as following. A general solution for the spectral correlators of the $\chi$UE is given in \cite{10} \cite{23}. However, computing the microscopic spectral density in the massive (non-) $\chi$OE and massive (non-) $\chi$SE poses many difficulties arising from the use of skew-orthogonal polynomials and their behaviour at large-$N$. In \cite{24} \cite{25} these ensembles have been studied and a solution is presented for doubly and $\beta$-fold degenerate masses. Also, the general result of \cite{12} could in principle give the spectral density for all the three ensembles with an arbitrary number of flavors and masses, but it seems technically difficult. Recently, H. Widom introduced a new technique in \cite{26} for dealing with OE and SE using the standard orthogonal polynomials of the UE and thereby avoiding skew-orthogonal polynomials. This technique has already been used successfully in \cite{22} for studying the massless non-$\chi$OE and non-$\chi$SE. In this paper we apply the same technique to the $\chi$OE and $\chi$SE with an arbitrary number of massive flavors. Applying this method presents several advantages. First of all this approach seems to be a promising alternative to the usual one with skew-orthogonal polynomials, and it gives a different point of view of the general problem of determining the scalar kernels of the OE and SE. Secondly, since this technique deals with standard orthogonal polynomials only, then it is possible in principle to get the microscopic limit of all the correlation functions from the already well-established results on the universal microscopic limit of orthogonal polynomials in the UE. Finally, the orthogonal polynomials for the general massive case naturally lead to a unifying notation which is very helpful when used with this new technique.

This paper is organized as follows. In the next Section we shortly address the definition of the chiral matrix model which is relevant for four-dimensional QCD. In Section 3 we
show how it is possible to compute correlation functions for the $\chi$SE and $\chi$OE by means of the technique of Widom. Such a technique requires as a basic ingredient, the explicit evaluation of orthonormal polynomials for the general massive case. These are explicitly determined in Section 4 (with their normalization factors) in a closed form\textsuperscript{[1]}. Within this formalism, the degenerate massive, and the massless case are obtained as particular cases of our general formulas. Moreover, we derive some equalities among orthogonal polynomials with different number of flavors. In Section 5 we apply the machinery described in Section 3 to the case of interest here, and finally in Section 6 we derive an expression for the scalar kernels of the massive $\chi$OE and massive $\chi$SE. The microscopic limit of our final formulas is discussed in Section 7.

Let us finally remark that in this paper we do not address the question of universality (which is actually considered here as a working hypothesis) of our results as well as explicit numerical calculations. We postpone both of them, and in the following we focus on the application of the new technique by Widom to the massive case of four-dimensional QCD.

2 Chiral ensembles

A chiral random matrix model with the same symmetries of the QCD partition function, can be set up by replacing the Dirac operator $\mathcal{D}$ with a suitable constant off-diagonal block random matrix. The off-diagonal structure stems from the anticommutation relation between $\mathcal{D}$ and $\gamma_5$. Thus $\chi$RMT for QCD in four dimensions is defined by the partition function [1] [4],

$$Z_{\nu}^{(N_f,\beta)}(\{m_f\}) = \int dW \prod_{f=1}^{N_f} \det (\mathcal{D} + m_f) e^{-\frac{N}{2} \text{Tr} V(W^\dagger W)} ,$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} ,$$

where the Dyson index $\beta = 1, 2, 4$ labels the different chiral ensembles. The rectangular matrix $W$ of size $N \times (N + |\nu|)$ is real, complex or quaternion real for $\beta = 1, 2, 4$, respectively. The integration measure $dW$ is the Haar measure of the group under consideration, the integer $\nu$ is related to the topological charge [2] [7] and $2N + |\nu|$ is the space-time volume. The polynomial potential $V$ in the exponential can be replaced by a pure quadratic term

\textsuperscript{[1]} The same set of polynomials have already been obtained in [10], but only in an recursive form.
\[ V(W^\dagger W) = W^\dagger W \] because of the universality of the matrix model in the microscopic large-\( N \) limit \([9] [13]\). Throughout this paper we will deal with an arbitrary topological charge \( \nu \).

The matrix integral can be rewritten, up to an irrelevant overall constant factor, in terms of the eigenvalues \( \lambda_i \) of the Hermitian positive-definite matrix \( W^\dagger W \), that is:

\[
Z^{(N_f, \beta)}(\{ m_f \}) = \int_0^\infty \prod_{i=1}^N d\lambda_i \prod_{f=1}^{N_f} (\lambda_i + m_f^2) \lambda_i^{\beta(\nu+1)/2-1} e^{-N^2 \beta \lambda_i} |\Delta(\{ \lambda_k \})|^{\beta} = \int_0^\infty \prod_{i=1}^N d\lambda_i w(\lambda_i) |\Delta(\{ \lambda_k \})|^{\beta},
\]

(2.3)

where \( \Delta(\{ \lambda_k \}) = \prod_{m>n}^N (\lambda_m - \lambda_n) \) is the Vandermonde determinant and

\[
w(\lambda) = \prod_{f=1}^{N_f} (\lambda + m_f^2) \lambda^{\beta(\nu+1)/2-1} e^{-N^2 \beta \lambda}
\]

(2.4)

is the weight function on \( \mathcal{I} = [0, \infty[ \). These two functions define all the spectral statistical properties of the matrix model in eq. (2.3). The corresponding properties for the model in eq. (2.1) are expressed in terms of the real eigenvalues \( \xi_i \) of the Dirac operator \( D \), by means of \( \xi_i^2 = \lambda_i \).

3  Spectral correlation functions for \( \beta = 1 \) and \( \beta = 4 \)

The \( m \)-point spectral correlation function for all three chiral ensembles is

\[
R^{(\beta)}_m(\lambda_1, \ldots, \lambda_m) \equiv \frac{N!}{(N-m)!} \langle \prod_{i=1}^m \text{Tr} \delta(\lambda_i - W^\dagger W) \rangle
\]

(3.1)

\[
= \det_{1 \leq i,j \leq m} [K_N^{(\beta)}(\lambda_i, \lambda_j)],
\]

(3.2)

where the expectation value in the first line, is understood with respect to the partition function in eq. (2.3) and the second line follows from a well-known result \([27] [28]\) which expresses all the correlation functions in terms of the determinant of the kernel of suitable polynomials\(^2\). For \( \beta = 2 \) such polynomials are orthogonal polynomials w.r.t. the weight

\(^2\)For \( \beta = 1 \) and \( \beta = 4 \) the determinant in eq. (3.2) is understood as the quaternion determinant of a matrix kernel (for details see [27]).
\( w(x) \) in eq. (2.4), whereas when \( \beta = 1 \) and \( \beta = 4 \) one usually chooses skew-orthogonal polynomials w.r.t. \( w(x) \). Indeed, one does not need to introduce skew-orthonormality since the matrix kernel is largely independent of the particular choice of the polynomials [29]. Therefore, if one defines

\[
\varphi_j(x) = p_j(x) \sqrt{w(x)}, \quad j = 0, 1, \ldots, 2N - 1, \quad \text{for } \beta = 4,
\]

\[
\varphi_j(x) = p_j(x) w(x), \quad j = 0, 1, \ldots, N - 1, \quad \text{for } \beta = 1,
\]

where \( p_j(x) \) are arbitrary polynomials of order \( j \), then the matrix kernels can be written as [30]:

\[
K_N^{(4)}(x, y) = \begin{pmatrix}
S_N^{(4)}(x, y) & S_N^{(4)} D(x, y) \\
IS_N^{(4)}(x, y) & S_N^{(4)}(y, x)
\end{pmatrix},
\]

\[
(3.5)
\]

and

\[
K_N^{(1)}(x, y) = \begin{pmatrix}
S_N^{(1)}(x, y) & S_N^{(1)} D(x, y) \\
IS_N^{(1)}(x, y) - \varepsilon(x - y) & S_N^{(1)}(y, x)
\end{pmatrix},
\]

\[
(3.6)
\]

where \( \varepsilon(x) = \text{sgn}(x)/2 \), \( S_N^{(\beta)}(x, y) \) equals the scalar kernel of a suitable operator \( \hat{S}_N^{(\beta)} \), and \( IS_N^{(\beta)}(x, y) \) and \( DS_N^{(\beta)}(x, y) \) are the kernels of the operators \( \hat{I} \hat{S}_N^{(\beta)} \), \( \hat{D} \hat{S}_N^{(\beta)} \) with \( \hat{I}, \hat{D} \) identified as the integration and differentiation operators, respectively. All the matrix elements in eq. (3.5) and eq. (3.6) are determined once the scalar kernel \( S_N^{(\beta)}(x, y) \) is known.

Indeed, the scalar kernel \( S_N^{(\beta)}(x, y) \) can be expressed in terms of \( \varphi_j \) functions only and it is a matter of fact that the choice of skew-orthogonal polynomials leads to the simplest possible expressions (see for instance [27]). However, skew-orthogonal polynomials are difficult to determine and furthermore they seem unsuitable for straightforward manipulations at large-\( N \) as opposed to the standard orthogonal polynomials. This fact has led some authors to use a different approach [24]. This approach uses orthogonal polynomials only and, without introducing skew-orthogonal polynomials, H. Widom derived general formulas for the kernels \( S_N^{(\beta)}(x, y) \) expressing it as corrections in addition to the unitary kernel \( K_N^{(2)}(x, y) \), with \( N \) replaced by \( 2N \) when \( \beta = 4 \) and \( w \) replaced by \( w^2 \) when \( \beta = 1 \). Because all quantities are expressed in terms of orthogonal polynomials, calculations and large-\( N \) asymptotics are straightforward in this approach.

In this paper we adopt the method developed by Widom in order to determine the functions \( S_N^{(\beta)}(x, y) \) and through that we analyze the original physical problem. In [22] the same technique has already been applied successfully to the case of three-dimensional QCD with

\[\text{[31]}\]
massless fermions. In the remaining part of this section we briefly sketch this technique. For more details we refer the reader to the original works \cite{26} \cite{29} and to \cite{22}.

The first step of this technique, is to build-up a Hilbert space $\mathcal{H}$ from the functions $\varphi_j$ defined as:

$$
\varphi_j(x) = p_j(x) \sqrt{w(x)} , \quad j = 0, 1, \ldots, N - 1 .
$$

(3.7)

According to eq. (3.3) or eq. (3.4), in what follows we always assume that $N$ should be replaced by $2N$ when $\beta = 4$ and $w$ should be replaced by $w^2$ when $\beta = 1$ (if not explicitly stated otherwise). Under suitable hypotheses on the measure $w(x) = e^{-V(x)}$, $\mathcal{H}$ is defined as the linear space spanned by the functions $\varphi_0, \ldots, \varphi_{N-1}$. In this picture the scalar kernel $K_N^{(2)}(x,y)$ can be considered as the kernel of the projection operator $\hat{\cal K}$ onto the Hilbert space $\mathcal{H}$, and it may be written as

$$
K_N^{(2)}(x,y) = \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y) = \frac{a_N}{x-y} (\varphi_N,\varphi_{N-1})_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_N \\ \varphi_{N-1} \end{pmatrix}_y ,
$$

(3.8)

where $a_N \equiv k_{N-1}/k_N$, with $k_N$ identified as the highest coefficient in $p_N(x)$. The kernel $S_N^{(2)}(x,y)$ can be written as the unitary scalar kernel $K_N^{(2)}(x,y)$ plus extra terms, whose number is independent of $N$ and closely related to the commutator $[\hat{D},\hat{K}]$. The kernel of the commutator $[\hat{D},\hat{K}]$ is \cite{20}:

$$
[D, K](x,y) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})K_N^{(2)}(x,y)
= a_N (\varphi_N,\varphi_{N-1})_x \begin{pmatrix} C(x)-C(y) \\ A(x)-A(y) \\ \frac{x-y}{x-y} \\ \frac{x-y}{x-y} \end{pmatrix} \begin{pmatrix} \varphi_N \\ \varphi_{N-1} \end{pmatrix}_y ,
$$

(3.9)

where

$$
A(x) = -A_N(x) - \frac{1}{2} \tilde{V}'(x), \quad B(x) = B_N(x), \quad C(x) = \frac{a_N}{a_{N-1}} B_{N-1}(x) ,
$$

(3.10)

are rational functions with the same poles as the ratio $w'(x)/w(x)$, and

$$
A_N(x) \equiv a_N \int_0^{+\infty} \varphi_N(z) \varphi_{N-1}(z) U(x,z) \, dz ,
$$

(3.11)

$$
B_N(x) \equiv a_N \int_0^{+\infty} \varphi_N(z)^2 U(x,z) \, dz ,
$$

(3.12)

\footnote{In particular $w'(x)/w(x)$ must be a rational function over $\mathcal{I}$.}

\footnote{For that it is crucial that the polynomials $\{p_j(x)\}$ are orthonormal.}
with
\[ U(x, z) \equiv \frac{\tilde{V}'(x) - \tilde{V}'(z)}{x - z}, \quad \tilde{V}'(x) = \chi(\beta)V'(x), \quad (3.13) \]
and
\[ \chi(\beta) = \begin{cases} 1 & \text{for } \beta = 4 \\ 2 & \text{for } \beta = 1 \end{cases}, \quad (3.14) \]
stemming from the change in notation. Let \( n_\infty \) and \( n_{x_i} \) denote the pole orders of \( w'(x)/w(x) \) at infinity and \( x_i \), respectively. Then the functions
\begin{align*}
&x^k \varphi_{N-1} ; \quad x^k \varphi_N \quad (0 \leq k < n_\infty), \\
&(x - x_i)^{-k-1} \varphi_{N-1} ; \quad (x - x_i)^{-k-1} \varphi_N \quad (0 \leq k < n_{x_i}),
\end{align*}
span a subspace \( \mathcal{H}_{\text{sub}} \subset \mathcal{H} \) and a corresponding subspace \( \mathcal{H}_{\text{sub}}^\perp \subset \mathcal{H}^\perp \) both of dimension \( n = n_\infty + \sum_i n_{x_i} \). The subspace \( \mathcal{H}_{\text{sub}} \) is determined by the condition that \( \mathcal{H}_{\text{sub}} \subset \mathcal{H} \), whereas the subspace \( \mathcal{H}_{\text{sub}}^\perp \) is determined by the \( n \) orthogonality conditions set up by the functions
\[ \varphi_{N-k} \quad (k < n_\infty) ; \quad \varphi_k \quad (k < n - n_\infty). \quad (3.17) \]
Let \( \psi_1, ..., \psi_n \in \mathcal{H} \) and \( \psi_{n+1}, ..., \psi_{2n} \in \mathcal{H}^\perp \) denote these \( 2n \) linearly independent functions, then one has
\[ [D, K](x, y) = \sum_{i,j=1}^{2n} \psi_i(x) A_{ij} \psi_j(y), \quad (3.18) \]
and once the \( \psi \)'s are chosen, the symmetric constant matrix \( A = [A_{ij}] \) is uniquely determined. Furthermore, the matrix \( A \) is always in a block off-diagonal form:
\[ A_{ij} = 0 \quad i, j \leq n \quad \text{or} \quad i, j > n. \quad (3.19) \]
One also defines the matrices:
\begin{align*}
&\quad B_{ij} = (\hat{\varepsilon} \psi_i, \psi_j) = \int_I \int_I dx dy \varepsilon(x - y) \psi_i(x) \psi_j(y), \\
&J = \begin{pmatrix} I_{n \times n} & \cdots & 0_{n \times n} \\
\cdots & \cdots & \cdots \\
0_{n \times n} & \cdots & 0_{n \times n} \end{pmatrix}_{2n \times 2n},
&\quad C = J + BA, \quad (3.21)
\end{align*}
with $0_{n\times n}$, $I_{n\times n}$ being the $n \times n$ null matrix and the identity matrix, respectively. The operator $\hat{\varepsilon}$ is defined as

$$(\hat{\varepsilon}\psi_i(x)) = \int_I dx \varepsilon(x - y)\psi_i(x).$$

Defining $A_0$, $C_0$ and $C_{00}$ as the matrices obtained by deleting from the corresponding matrices the last $n$ columns, the last $n$ rows and the last $n$ rows and columns respectively, one finally has the main result of [26], that is

$$S^{(1)}_N(x, y) = K^{(2)}_N(x, y) - 2\sum_{i \leq n, j = 1}^{2n} [AC(I - BAC)^{-1}]_{ij}\psi_i(x)\varepsilon\psi_j(y),$$

$$S^{(4)}_N(x, y) = K^{(2)}_N(x, y) - 2\sum_{i > n, j = 1}^{2n} [A_0C_{00}^{-1}C_0]_{ij}\psi_i(x)\varepsilon\psi_j(y),$$

remembering once again that $N \to 2N$ in eq. (3.24) and $w \to w^2$ in eq. (3.23).

A remarkable feature of eq. (3.23) and eq. (3.24) is that the scalar kernels $S^{(\beta)}_N(x, y)$ naturally are expressed as a finite number of corrections to the unitary kernel $K^{(2)}_N(x, y)$, a number which is independent of $N$. This fact will prove its importance in Section 7 where we study the microscopic limit of the quantities of interest here. Finally, inserting eq. (3.23) and eq. (3.24) into eq. (3.6) and eq. (3.5), one can calculate eq. (3.2) and through that all possible spectral correlation functions defined by eq. (3.1).

4 Orthogonal polynomials for $N_f$ massive fermions

In this section we apply the technique of Widom, described above, to the general massive chiral case with $N_f$ flavors. The first ingredient we are looking for are polynomials $P^{(N_f, \alpha)}_n(x; m_1, \ldots, m_{N_f})$ orthonormal with respect to the weight function

$$w^{(N_f, \alpha)}(x) \equiv \prod_{i=1}^{N_f} (x + m_i^2) x^\alpha e^{-x}$$

defined on the interval $\mathcal{I} = [0, +\infty[$, where $\alpha$ is a real non-negative constant. In this formalism, it is understood that $w^{(0, \alpha)}(x) \equiv x^\alpha e^{-x}$. The weight in eq. (1.1) is seen to be slightly different from the one in eq. (2.4), but if we identify $\alpha = \beta(\nu + 1)/2 - 1$ and $c = N\beta/2$, then we observe that
In order to determine the polynomials $P_{\nu}^{(N_f,\alpha)}$ explicitly, we fix some useful notation at first. We define the scalar product of two real functions $f(x), g(x)$ with respect to the weight function $w^{(N_f,\alpha)}(x)$ by

$$
\langle f, g \rangle_{N_f,\alpha} \equiv \int_{\mathcal{I}} dx \, w^{(N_f,\alpha)}(x) f(x) g(x) ,
$$

and furthermore we shall use also the following shortened and suggestive notation:

$$
\begin{align*}
P_{n}^{(N_f,\alpha)}(x; m) & \equiv P_{n}^{(N_f,\alpha)}(x; m_1, \ldots, m_{N_f}) , \\
P_{n}^{(N_f,\alpha)}(x; m, 0) & \equiv P_{n}^{(N_f,\alpha)}(x; m_1, \ldots, m_{N_f-1}, 0) , \\
P_{n}^{(N_f-1,\alpha)}(x; m_{\neq i}) & \equiv P_{n}^{(N_f-1,\alpha)}(x; m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{N_f}) .
\end{align*}
$$

Moreover, let $k_{n}^{(N_f,\alpha)}(x)$ denote the highest coefficient in $P_{n}^{(N_f,\alpha)} = k_{n}^{(N_f,\alpha)} x^n + \ldots$, and let us define the $(N_f + 1) \times (N_f + 1)$ matrix

$$
\Lambda_{n}^{(N_f,\alpha)}(x) \equiv \begin{pmatrix}
P_{n}^{(0,\alpha)}(x) & P_{n+1}^{(0,\alpha)}(x) & \cdots & P_{n+N_f}^{(0,\alpha)}(x) \\
P_{n}^{(0,\alpha)}(-m_1^2) & P_{n+1}^{(0,\alpha)}(-m_1^2) & \cdots & P_{n+N_f}^{(0,\alpha)}(-m_1^2) \\
\cdots & \cdots & \cdots & \cdots \\
P_{n}^{(0,\alpha)}(-m_{N_f}^2) & P_{n+1}^{(0,\alpha)}(-m_{N_f}^2) & \cdots & P_{n+N_f}^{(0,\alpha)}(-m_{N_f}^2) 
\end{pmatrix} .
$$

Finally, we identify $\Lambda_{n,i}^{(N_f,\alpha)}$, $i = 1, \ldots, N_f + 1$, as the $N_f \times N_f$ sub-matrix obtained by omitting the first row and the $i$-th column from $\Lambda_{n}^{(N_f,\alpha)}(x)$ in the definition (4.3). Notice that all $\Lambda$ matrices are expressed in terms of orthonormal polynomials $P_{n}^{(0,\alpha)}(x)$ with $N_f = 0$ flavors only. Since the polynomials $P_{n}^{(0,\alpha)}(x)$ are orthonormal w.r.t. the measure $w^{(0,\alpha)}(x)$, then they are necessarily proportional to the generalized Laguerre polynomials $L_{n}^{(\alpha)}(x)$ [32], i.e.

$$
P_{n}^{(0,\alpha)}(x) = L_{n}^{(\alpha)}(x) / \sqrt{h_{n}^{\alpha}} , \quad h_{n}^{\alpha} = \frac{\Gamma(n + \alpha + 1)}{n!} .
$$
Therefore, from $L_n^{(\alpha)}(x) = (-1)^n x^n/n! + \ldots$ one reads the highest coefficients

$$k_n^{(0,\alpha)} = \frac{(-1)^n}{\sqrt{\Gamma(n + \alpha + 1)n!}}. \quad (4.5)$$

Now, it is well-known that the orthonormality condition

$$\langle P_i^{(N_f,\alpha)}(x;m), P_j^{(N_f,\alpha)}(x;m) \rangle_{N_f,\alpha} = \delta_{ij} \quad (4.6)$$

uniquely determines polynomials $P_i^{(N_f,\alpha)}$, up to a relative sign. Under the hypothesis that all the masses $\{m_i\}$ are distinct, these polynomials can, according to Christoffel’s theorem [33], be represented in terms of $P_n^{(0,\alpha)}(x)$ as follows:

$$P_i^{(N_f,\alpha)}(x;m) = \frac{1}{\sqrt{h_{i}^{(N_f,\alpha)}(m)}} \frac{\det \Lambda_{N_f,\alpha}(x)}{\prod_{i=1}^{N_f}(x + m_i^2)}. \quad (4.7)$$

The normalization factor $h_i^{(N_f,\alpha)}$ is

$$h_i^{(N_f,\alpha)}(m) = (-1)^{N_f} k_n^{(0,\alpha)} \frac{k_n^{(0,\alpha)}}{k_{n+N_f}^{(0,\alpha)}} \det \Lambda_{N_f,\alpha} \Lambda_{n,N_f+1} \quad (4.8)$$

and the highest coefficient is

$$k_i^{(N_f,\alpha)}(m) = \left((-1)^{N_f} k_n^{(0,\alpha)} \frac{k_n^{(0,\alpha)}}{k_{n+N_f}^{(0,\alpha)}} \frac{\det \Lambda_{N_f,\alpha}}{\det \Lambda_{n,N_f+1}}\right)^{\frac{1}{2}}. \quad (4.9)$$

where $k_n^{(0,\alpha)}$ is the coefficient given in eq. (4.5). For an explicit derivation of eq. (4.8) and (4.9) we refer to appendix A. Eq. (4.7) is completely symmetric under permutations of the masses: although the determinant in the numerator is completely antisymmetric, the algebraic square root in the denominator is also completely antisymmetric (with $\sqrt{(-1)^2} = -1$). This observation is consistent with the fact that also the weight function in eq. (1.1) is completely symmetric under permutations of the masses. Moreover, the case of a degenerate mass $m_k$ is understood in a limit sense of eq. (4.7). For instance, if there are only two degenerate masses $m_1 = m_2$, we just take the limit of eq. (4.7) as $m_2 \to m_1$. Dividing both numerator and denominator by $m_2^2 - m_1^2$, and substituting the third row $\{P_i^{(0,\alpha)}(-m_2^2)\}$ with the combination $\{P_i^{(0,\alpha)}(-m_2^2) - P_i^{(0,\alpha)}(-m_1^2)\}$, in all the
where we obtain in the limit $m_1 \to m_2$, that the third row effectively is replaced by the first-order derivatives $dP_i^{(0,\alpha)}(y)/dy$ at $y = -m_2^2$. In the case of a degenerate mass $m_k$ of multiplicity $l$, $l > 1$, we replace in all the determinants of eq. (4.7) the corresponding rows $k + 1, k + 2, \ldots, k + l$ with the derivatives of order $0, 1, 2, \ldots, l - 1$ of the polynomials $P_i^{(0,\alpha)}(y)$ evaluated at $y = -m_2^2$. It is worthwhile to remind here that degenerate masses are definitely needed for the $\beta = 1$ case, because $w \to w^2$ effectively implies $N_f \to 2N_f$, i.e. $\{m_1, m_2, \ldots, m_{N_f}\} \to \{m_1, m_1, m_2, m_2, \ldots, m_{N_f}, m_{N_f}\}$.

Let us give two explicit examples where eq. (4.7) is applied. In the case $N_f = 1$, it reads

$$P_n^{(1,\alpha)}(x; m_1) = \sqrt{\frac{n!(n+1)!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}} \frac{L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(-m_1^2) - L_{n+1}^{(\alpha)}(x)L_n^{(\alpha)}(-m_1^2)}{(x+m_1^2)\sqrt{h_n^{(1,\alpha)}(m_1)}} \quad (4.10)$$

where

$$h_n^{(1,\alpha)}(m_1) = \frac{n!}{\Gamma(n+\alpha+2)} \left[ L_{n+1}^{(\alpha)}(-m_1^2)L_n^{(\alpha)}(-m_1^2) \right] . \quad (4.11)$$

In the case $N_f = 2$, with degenerate masses $m_1 = m_2 = m$, it reads

$$P_n^{(2,\alpha)}(x; m, m) = \frac{\det \begin{pmatrix} L_n^{(\alpha)}(x) & L_{n+1}^{(\alpha)}(x) & L_{n+2}^{(\alpha)}(x) \\ L_n^{(\alpha)}(-m^2) & L_{n+1}^{(\alpha)}(-m^2) & L_{n+2}^{(\alpha)}(-m^2) \\ L_n^{(\alpha)'}(-m^2) & L_{n+1}^{(\alpha)'}(-m^2) & L_{n+2}^{(\alpha)'}(-m^2) \end{pmatrix}}{(x+m^2)^2\sqrt{h_n^{(\alpha)}h_{n+1}^{(\alpha)}h_{n+2}^{(\alpha)}(m,m)}} \quad (4.12)$$

and

$$h_n^{(2,\alpha)}(m, m) = \frac{n!(n+1)!\det \left( \begin{array}{ccc} L_{n+1}^{(\alpha)}(-m^2) & L_{n+2}^{(\alpha)}(-m^2) \\ L_{n+1}^{(\alpha)'}(-m^2) & L_{n+2}^{(\alpha)'}(-m^2) \end{array} \right)}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+3)} \quad (4.13)$$

All the derivatives can easily be evaluated using the property $dL_n^{(\alpha)}(x)/dx = -L_{n+1}^{(\alpha)}$, $n > 0$ (iteratively, if needed). These examples are just two particular cases of a general fact: all the orthogonal polynomials for the general massive case are expressed nicely in terms of generalized Laguerre polynomials by means of eq. (4.7), (4.8) and (4.13).

Now, for the sake of future use we point out two remarkable properties of the polynomials $P_n^{(N_f,\alpha)}$.
1. Let us consider the set \( \{ m_1, \ldots, m_{N_f}, 0 \} \) consisting of \( N_f + 1 \) flavors. In this case eq. \( \text{(4.6)} \) reads

\[
\delta_{ij} = \langle P_i^{(N_f+1,\alpha)}(x; m, 0), P_j^{(N_f+1,\alpha)}(x; m, 0) \rangle_{N_f+1,\alpha} \bigg|_{m_{N_f+1}=0}
\]  

(4.14)

that is, the polynomials \( P_n^{(N_f+1,\alpha)}(x; m, 0) \) are orthonormal with respect to the measure \( x^{\alpha+1} \prod_{i=1}^{N_f} (x+m_i^2) e^{-x}dx \). On the other hand, also the polynomials \( P_n^{(N_f,\alpha+1)}(x; m) \) are orthonormal w.r.t. the same measure, i.e.

\[
\delta_{ij} = \langle P_i^{(N_f,\alpha+1)}(x; m), P_j^{(N_f,\alpha+1)}(x; m) \rangle_{N_f,\alpha+1} .
\]  

(4.15)

From uniqueness of orthogonal polynomials w.r.t. a given measure it follows

\[
P_n^{(N_f+1,\alpha)}(x; m, 0) = P_n^{(N_f,\alpha+1)}(x; m),
\]  

(4.16)

for every \( n \) and \( N_f \).

2. Christoffel’s theorem eq. \( \text{(4.7)} \) can also be stated as

\[
P_n^{(N_f,\alpha)}(x; m) = c_{n,i}^{(N_f,\alpha)}(m) \frac{\det \begin{pmatrix} P_n^{(N_f-1,\alpha)}(x; m_{\neq i}) & P_n^{(N_f-1,\alpha)}(-m_i^2, m_{\neq i}) \\ P_{n+1}^{(N_f-1,\alpha)}(x; m_{\neq i}) & P_{n+1}^{(N_f-1,\alpha)}(-m_i^2, m_{\neq i}) \end{pmatrix}}{(x + m_i^2)},
\]

(4.17)

for \( i = 1, \ldots, N_f \), where the coefficient \( c_{n,i}^{(N_f,\alpha)} \) is easily determined by comparing the highest coefficients on the two sides of eq. \( \text{(4.17)} \), i.e.

\[
c_{n,i}^{(N_f,\alpha)}(m) = -\frac{k_n^{(N_f,\alpha)}(m)}{k_{n+1}^{(N_f-1,\alpha)}(m_{\neq i})} \frac{1}{P_n^{(N_f-1,\alpha)}(-m_i^2, m_{\neq i})}.
\]

(4.18)

And, by means of the Christoffel-Darboux formula \[33\], one can equivalently write eq. \( \text{(4.17)} \) as

\[
P_n^{(N_f,\alpha)}(x; m) = d_{n,i}^{(N_f,\alpha)}(m) \sum_{j=0}^{n} P_j^{(N_f-1,\alpha)}(-m_i^2, m_{\neq i}) P_j^{(N_f-1,\alpha)}(x; m_{\neq i}),
\]

(4.19)

\[\text{6A trivial check of formula} \text{ (4.16)} \text{ is for} \ N_f = 0. \text{ Indeed, if we put} \ m_1 = 0 \text{ in eq. (4.16) then we obtain exactly} \ P_n^{(0,\alpha+1)}(x), \text{ eq. (4.4). Similarly, if we put} \ m = 0 \text{ in eq. (4.12) then we obtain} \ P_n^{(2,\alpha)}(x; 0) = P_n^{(1,\alpha+1)}(x; 0) = P_n^{(0,\alpha+2)}(x).\]
for \( i = 1, \ldots, N_f \) where the coefficient \( d_{n,i}^{(N_f,\alpha)}(m) \) is determined as before by comparing the highest coefficients in eq. (4.19), i.e.

\[
d_{n,i}^{(N_f,\alpha)}(m) = \frac{k_{n}^{(N_f,\alpha)}(m)}{k_{n}^{(N_f-1,\alpha)}(m_{\neq i})} \frac{1}{P_{n}^{(N_f-1,\alpha)}(-m_{i}^{2}; m_{\neq i})}.
\]

(4.20)

The coefficients \( k_{n}^{(N_f,\alpha)} \)'s which appear both in eq. (4.18) and in eq. (4.20), are given explicitly in eq. (4.9).

These two considerations will be useful in the evaluation of some integrals appearing in next Section.

5 The Hilbert space \( \mathcal{H} \)

Once the orthonormal polynomials are known then the next step is to build-up a suitable Hilbert space according to the technique described in Section 3. In order to do that, from the orthonormal polynomials \( P_{n}^{(N_f,\alpha)}(x; m) \) we define the functions \( \varphi_{j}(x) \) as in eq. (3.7):

\[
\varphi_{j}(x) \equiv P_{j}^{(N_f,\alpha)}(x; m)\sqrt{w^{(N_f,\alpha)}(x)}
\]

(5.1)

The Hilbert space is \( \mathcal{H} = \text{span}\{\varphi_{0}, \varphi_{1}, \ldots, \varphi_{N_f-1}\} \), and the kernel of the projection operator \( \hat{K} \) onto \( \mathcal{H} \) and the kernel of the operator \([\hat{D}, \hat{K}]\) are given in eq. (3.8) and eq. (3.9), respectively. In particular, we notice that the former can be nicely written in a very compact form:

\[
K_{N_f}^{(2)}(x, y) = \sum_{j=0}^{N_f-1} P_{j}^{(N_f,\alpha)}(x; m)P_{j}^{(N_f,\alpha)}(y; m)\sqrt{w^{(N_f,\alpha)}(x)w^{(N_f,\alpha)}(y)}
\]

(5.2)

\[
= \frac{P_{N_f}^{(N_f+1,\alpha)}(x; m, \sqrt{-y})}{d_{N_f}^{(N_f+1,\alpha)}(m, \sqrt{-y})}\sqrt{w^{(N_f,\alpha)}(x)w^{(N_f,\alpha)}(y)},
\]

(5.3)

where we used eq. (4.19) in the second equality.

Since \( w^{(N_f,\alpha)}(x) \equiv e^{-V(x)} \), i.e.

\[
V(x) = x - \alpha \log x - \sum_{i=1}^{N_f} \log(x + m_{i}^{2})
\]

(5.4)
eq. (3.13) reads

\[ U(x, z) = \chi(\beta) \left[ \frac{\alpha}{xz} + \sum_{i=1}^{N_f} \frac{1}{(x + m_i^2)(z + m_i^2)} \right]. \quad (5.5) \]

where \( \chi(\beta) \) is defined in eq. (3.14). Therefore, substituting eq. (5.4) and (5.1) into eq. (3.9) we obtain the matrix elements in eq. (3.9):

\[ \frac{A(x) - A(y)}{x - y} = a_N \chi(\beta) \left[ \frac{\alpha}{xy} \left( I_0^{(N,N-1)} - \frac{1}{2a_N} \right) + \sum_{i=1}^{N_f} I_i^{(N,N-1)} - \frac{1}{2a_N} \right], \quad (5.6) \]

\[ \frac{B(x) - B(y)}{x - y} = B_N(x) - B_N(y) = a_N \chi(\beta) \left[ \frac{\alpha I_0^{(N,N)}}{xy} + \sum_{i=1}^{N_f} I_i^{(N,N)} \right], \quad (5.7) \]

\[ \frac{C(x) - C(y)}{x - y} = a_{N-1} \frac{B_{N-1}(x) - B_{N-1}(y)}{x - y}, \quad (5.8) \]

where \( a_N = k_{N-1}^{(Nf,\alpha)}(m)/k_{N}^{(Nf,\alpha)}(m) \) and the constants \( I_0^{(N,j)}, I_i^{(N,j)} \) are defined as

\[ I_0^{(N,j)} = \langle P_N^{(Nf,\alpha)}, \frac{P_j^{(Nf,\alpha)}}{z} \rangle_{Nf,\alpha}, \quad (5.9) \]

\[ I_i^{(N,j)} = \langle P_N^{(Nf,\alpha)}, \frac{P_j^{(Nf,\alpha)}}{z + m_i^2} \rangle_{Nf,\alpha}, \quad i = 1, 2, \ldots, N_f. \quad (5.10) \]

The above integrals are evaluated in appendix B using the previously stated properties of orthonormal polynomials for general \( N_f \). Thus we have

\[ I_0^{(N,N-1)} = d_{N,N+1}^{(Nf+1,\alpha-1)}(m, 0) P_{N-1}^{(Nf,\alpha)}(0; m) \]

\[ I_0^{(N,N)} = d_{N,N+1}^{(Nf+1,\alpha-1)}(m, 0) P_N^{(Nf,\alpha)}(0; m) \]

\[ I_i^{(N,N-1)} = d_{N,i}^{(Nf,\alpha)}(m) P_{N-1}^{(Nf,\alpha)}(-m_i^2; m) \]

\[ I_i^{(N,N)} = d_{N,i}^{(Nf,\alpha)}(m) P_N^{(Nf,\alpha)}(-m_i^2; m) \]

which appear in eq. (5.6), (5.7) and (5.8).

Now we can construct the \( \psi_i(x) \) functions. Since the logarithmic derivative of the weight is a rational function

\[ \frac{dw^{(Nf,\alpha)}(x)/dx}{w^{(Nf,\alpha)}(x)} = \frac{\alpha}{x} + \sum_{i=0}^{N_f} \frac{1}{x + m_i^2} - 1, \quad (5.11) \]
with no poles at infinity and simple poles at $x = 0$ (for $\alpha \neq 0$) and at $x = -m_i^2$, $i = 1, \ldots, N_f$, we have that the subspace $H_{\text{sub}}$ has dimension $n = N_f + 1$. The number of linearly independent functions $\psi_k \in \mathcal{H}$ we are looking for is thus equal to $n = N_f + 1$. According to eq. (3.15) and (3.16) we choose the following linear combinations:

$$
\psi_i(x) = \frac{\varphi_{N_i}^{(N_f, \alpha)}(x)}{x + \sigma_i^2} + D_i \frac{\varphi_{N-1}^{(N_f, \alpha)}(x)}{x + \sigma_i^2}, \quad i = 1, \ldots, N_f + 1,
$$

where we used the shortened notation $\sigma_i \equiv m_i$ for $i = 1, \ldots, N_f$ and $\sigma_{N_f+1} \equiv 0$. We determine all the coefficients $D_i$ by requiring that the functions $\psi_i$ are non-singular at $x = -\sigma_i^2$, since they have to belong to the Hilbert space $\mathcal{H}$. This means that the numerator in eq. (5.12) should be vanishing at $x = -\sigma_i^2$, i.e.

$$
D_i = -\frac{P_{N_f}^{(N_f, \alpha)}(-\sigma_i^2; m)}{P_{N_f}^{(N_f, \alpha)}(-\sigma_i^2; m)}, \quad i = 1, \ldots, N_f + 1.
$$

Thus all the functions $\psi_i \in \mathcal{H}$ are determined completely. In appendix C we obtain the following compact final expression:

$$
\psi_i(x) = k_{N_i}^{(N_f, \alpha)}(m) \frac{P_{N_f}^{(N_f+1, \alpha)}(x; m, \sigma_i)}{k_{N_f}^{(N_f+1, \alpha)}(m, \sigma_i)} \sqrt{w^{(N_f, \alpha)}(x)}, \quad i = 1, \ldots, N_f + 1.
$$

We now look for the functions $\psi_{N_f+2}, \ldots, \psi_{2N_f+2} \in \mathcal{H}^\perp$. We choose again to consider the combination

$$
\psi_{N_f+i+1}(x) = \frac{\varphi_{N_f}^{(N_f, \alpha)}(x)}{x + \sigma_i^2} + D_i^\perp \frac{\varphi_{N_f-1}^{(N_f, \alpha)}(x)}{x + \sigma_i^2}, \quad i = 1, \ldots, N_f + 1,
$$

where $D_i^\perp$ are coefficients to be determined. As discussed in Section B the space $S = \text{span}\{\varphi_0, \ldots, \varphi_{N_f}\}$ is a $n$-dimensional subspace of $\mathcal{H}$, hence we can fix all the $D_i^\perp$’s by imposing that each $\psi_{N_f+i+1} \in \mathcal{H}^\perp$ is orthogonal to all the $\{\varphi_0, \ldots, \varphi_{N_f}\}$, i.e.

$$
\int_0^{+\infty} \psi_{N_f+i+1}(x) \varphi_j(x) \, dx = 0,
$$

for $i = 1, \ldots, N_f + 1$ and $j = 0, \ldots, N_f$. When $i = 1, \ldots, N_f$, by substituting eq. (5.15) and (5.14) in the constraint equation (5.16), we see that all the integrals are of the form eq. (5.13)
or \((5.10)\), which have already been solved explicitly in eq. \((3.4)\) and \((B.5)\). Therefore, the constraint conditions \((5.16)\) can be written as:

\[
D^\perp_i = -\frac{d^{(N_i,\alpha)}_m}{d^{(N_{i-1},\alpha)}_{N-1,i}}(m), \quad i = 1, \ldots, N_f, \quad (5.17)
\]

with \(N \geq N_f + 1\). Notice that the bound \(N \geq N_f + 1\) is fully consistent with our purpose of taking the microscopic large-\(N\) limit (see Section \(7\)). When \(i = N_f + 1\) we have

\[
D^\perp_{N_f+1} = -\frac{d^{(N_f+1,\alpha-1)}_{N_f+1}(m,0)}{d^{(N_f+1,\alpha-1)}_{N_f+1}(m,0)}. \quad (5.18)
\]

In appendix \(C\) we obtain a more compact and explicit expression for \(\psi_{N_f+1+i}\), i.e.

\[
\psi_{N_f+1+i}(x) = k^{(N_f,\alpha)}_N(m) P^{(N_f-1,\alpha)}_N(x; m_{\neq i}) \sqrt{w^{(N_f,\alpha)}_N(x)} \left( x + m_i^2 \right), \quad (5.19)
\]

for \(i = 1, \ldots, N_f\) and

\[
\psi_{2N_f+2}(x) = k^{(N_f,\alpha)}_N(m) P^{(N_f,\alpha-1)}_N(x; m) \sqrt{w^{(N_f,\alpha)}_N(x)} x, \quad (5.20)
\]

for \(i = N_f + 1\).

6 The kernels \(S_N^{(\beta)}(x, y)\)

Now we have all the necessary ingredients to determine the symmetric \(2n \times 2n\) matrix \(A\), which is defined by eq. \((3.18)\). In \(26\) it is proven that the matrix \(A\) always has the block form

\[
A = \begin{pmatrix}
0_{n \times n} & \tilde{A}_{n \times n} \\
\cdots & \cdots & \cdots \\
\tilde{A}_{n \times n} & \vdots & 0_{n \times n}
\end{pmatrix}_{2n \times 2n}, \quad (6.1)
\]

where \(0_{n \times n}\) is the \(n \times n\) null matrix. With our choice of the \(\psi\) functions, the \(n \times n\) matrix \(\tilde{A}\) is always diagonal. Indeed, it is sufficient to prove that the off-diagonal terms \(\tilde{A}_{ij}\) are
identically zero. In fact, if such terms were not zero then they would be responsible for having mixed terms in \([D, K](x, y)\), eq. (3.18), of the type \(\psi_i(x)\psi_{j\neq i}(y)\), that is mixed terms of the type \(1/(x(y + m_i^2))\) or \(1/[(x + m_j^2)(y + m_i^2)]\) (\(i \neq j\)). But such mixed terms are definitely not there in eq. (5.4), (5.4) and (5.8) and therefore, through eq. (3.9), they do not appear at all in \([D, K](x, y)\). Finally, from eq. (3.18), we necessarily conclude that \(\bar{A}_{ij} = 0\) for \(i \neq j\).

Let us define \(\bar{A}_i \equiv \bar{A}_{ii}\) as the diagonal elements of \(\bar{A}\), \(i = 1, \ldots, N_f + 1\). These elements are uniquely determined by comparing eq. (3.9) and eq. (3.18): indeed, in our case it is sufficient to compare the highest coefficients in \(x\) and \(y\). In particular, in eq. (3.9) the terms proportional to \(P(N_f, \alpha)(N(x)) P(N_f, \alpha(N(y)))\) are contained in

\[
a_N P^{(N_f, \alpha)}(x) \frac{C(x) - C(y)}{x - y} P^{(N_f, \alpha)}(y) ,
\]

whereas in eq. (3.18) and (5.12) such terms are contained in

\[
2P^{(N_f, \alpha)}(x) \left[ \sum_{i=1}^{N_f} \frac{\bar{A}_i}{(x + m_i^2)(y + m_i^2)} + \frac{\bar{A}_{N_f+1}}{xy} \right] P^{(N_f, \alpha)}(y)
\]

(the factor 2 comes from the symmetricity of \(A\)). Comparing eq. (3.3) with eq. (6.2) and (6.3) term by term we identify

\[
2\bar{A}_i = -a_N^2 \chi(\beta) I_i^{(N-1,N-1)} = -a_N^2 \chi(\beta) d^{(N_f, \alpha)}_{N-1,1}(m) P^{(N_f, \alpha)}(-m_i^2; m) , \quad i = 1, \ldots, N_f
\]

\[
2\bar{A}_{N_f+1} = -a_N^2 \chi(\beta) I_{0}^{(N-1,N-1)} = -a_N^2 \chi(\beta) d^{(N_f+1, \alpha-1)}_{N-1,N_f+1}(m, 0) P^{(N_f, \alpha)}(0; m) ,
\]

from which, the matrix \(A\) is completely determined. Moreover, it is a result of Widom that the matrix \(A\) is uniquely determined by the given choice of \(\psi_i\) functions [26]. Therefore it is guaranteed that all the lower-order terms in eq. (3.9) and eq. (3.18) match exactly.

To calculate the corrections, we need the antisymmetric matrix \(B\) in eq. (3.20). Unfortunately enough, the matrix elements \(B_{ij}\) (which are a total number of \(2n(2n - 1)/2\) distinct elements) are much more difficult to evaluate than \(A_{ij}\), and therefore our final result for the corrections to the unitary kernel is given by eqs. (3.23), (3.24), and (3.20), with \(\psi\)'s explicitly given in eqs. (5.14), (5.19) and (5.20) and the matrix elements of \(A\) are given in (6.4).

Although such a final expression is not terribly simple, it is at least suitable for direct

\[\footnotetext{8}{Notice that with our choice of \(\psi\) functions, all the elements \(\bar{A}_i\) are determined through the function \(C(x)\) only.}\]
numerical evaluations. However, its major usefulness appears when one considers the microscopic limit of it. In fact, one should remember that the physical interest is in the microscopic limit, since it is in this limit that universal properties of $\chi RMT$ appear. In Section 7 we will study the kernel $S^{(\beta)}_N(x,y)$ exactly in this limit.

Before doing that, let us mention that all the results of the last three sections can be generalized easily to the case where the weight function is

$$w^{(N_f,\alpha,c)}(x) = \prod_{i=1}^{N_f} (x + m_i^2) x^\alpha e^{-cx},$$

i.e. where an additional parameter $c > 0$ has been introduced in the exponential. We do not need here to repeat all the calculations above with this new weight function, because it is possible to write down the $c$-dependence for any quantity, explicitly, just using scaling arguments. For instance, from the identity

$$1 = \int_0^\infty dx \left[ P_j^{(N_f,\alpha)}(x; \sqrt{cm}) \right]^2 x^\alpha \prod_{i=1}^{N_f} (x + cm_i^2) e^{-x}$$

$$= c^{\alpha + N_f + 1} \int_0^\infty dx \left[ P_j^{(N_f,\alpha)}(cx; \sqrt{cm}) \right]^2 x^\alpha \prod_{i=1}^{N_f} (x + m_i^2) e^{-cx},$$

we can read off the orthonormal polynomials w.r.t. the new weight function in eq. (6.5):

$$P_j^{(N_f,\alpha,c)}(x; m) = c^{\alpha + N_f + 1 / 2} P_j^{(N_f,\alpha)}(cx; \sqrt{cm}).$$

The same argument applies to all the other functions. In general, a quantity $g_{c=1}(x; m)$ defined with the measure $w^{(N_f,\alpha)}(x)dx$ turns out to be related to the same quantity $g_c(x; m)$ defined with the new measure $w^{(N_f,\alpha,c)}(x)dx$ according to this formula:

$$g_c(x; m) = c^{\gamma} g_{c=1}(cx; \sqrt{cm}),$$

where $\gamma$ is a suitable scaling exponent. The following table shows the scaling factor for the most important quantities in this paper.
The only exception of this scaling rule, is given by the potential $V(x)$ which, for $c \neq 1$ is,

$$V(x) = c - \alpha \log x - \sum_{k=1}^{N_f} (x + m_k^2).$$

We conclude this Section remembering once again that at the very end one has to perform an important substitution. Namely, it is a prescription of this method to replace $N \rightarrow 2N$ when $\beta = 4$ and $w \rightarrow w^2$ for $\beta = 1$. The latter substitution leads one to consider double degenerate masses $m_i$, which means $N_f \rightarrow 2N_f$, and to take $\alpha \rightarrow 2\alpha$, $c \rightarrow 2c$. We summarize these rules here:

$$\begin{align*}
\beta = 1 & : \quad N_f \rightarrow 2N_f, \quad \alpha \rightarrow 2\alpha, \quad c \rightarrow 2c \\
\beta = 4 & : \quad N \rightarrow 2N
\end{align*} \quad (6.9)$$

## 7 The microscopic limit

The physical interest in studying RMT relies on its universal properties which appear in the large-$N$ limit. In particular, we consider the double-microscopic limit of the scalar kernel:

$$\tilde{S}^{(\beta)}(\zeta_1, \zeta_2) = \lim_{N \to \infty} \frac{c}{N^2} S^{(\beta)}_N \left( \frac{c\zeta_1}{N^2}, \frac{c\zeta_2}{N^2} \right), \quad m_i = \frac{\mu_i \sqrt{c}}{N}, \quad c = \frac{\beta N}{2},$$

where $\zeta_1, \zeta_2,$ and $\mu_i$ are kept fixed. In general, the microscopic limit of a quantity $g_c(x, m)$ in this paper is:

$$\tilde{g}_c(\zeta, \mu) = \lim_{N \to \infty} N^6 c^\gamma g_{c=1}(\frac{c\zeta}{N^2}, \frac{\mu \sqrt{c}}{N}) = \left( \frac{\beta}{2} \right)^\gamma \lim_{N \to \infty} N^{\delta + \gamma} g_{c=1}(\frac{\beta \zeta}{2N}, \sqrt{\frac{\beta}{2N}} \mu) \quad (7.2)$$
where the exponent $\gamma$ can be read off from the table at the end of Section 6 and $\delta$ is a suitable exponent which is chosen such that a finite non-zero limit exists. According to eq. (7.2) we may obtain the microscopic limit of $g_c(x, m)$ simply by computing the microscopic limit of $g_{c=1}(x, m)$ with the scaling $x = \zeta/N$, $m = \mu/\sqrt{N}$, and then evaluating the result at the points $\zeta \to \tilde{\zeta} \equiv \zeta\beta/2$, $\mu \to \tilde{\mu} \equiv \sqrt{\beta/2}\mu$. Following this strategy when computing the microscopic limit of $S^{(\beta)}(x, y)$ in eq. (7.1), we consider $c = 1$ in the rest of this section, and then at the very end we substitute $\zeta \to \tilde{\zeta}$ and $\mu \to \tilde{\mu}$.

In eq. (3.23) and (3.24) the scalar kernel $S_N^{(\beta)}(x, y)$ is expressed in terms of the unitary kernel $K_N^{(2)}(x, y)$ and the $\psi_i$ functions which all are determined explicitly in Section 5. Hence we can now compute the microscopic large-$N$ limit of the scalar kernel $S_N^{(\beta)}(x, y)$. The basic ingredient is the set of the orthonormal polynomials $P_n^{(N_f, \alpha)}$ and their microscopic limit. Such a limit is straightforward analytically. In fact, the polynomials are defined completely in terms of determinants of matrices of the form as in eq. (1.3). If one naively takes the microscopic limit of such determinants, then it eventually ends up with an indeterminate form. The situation is actually analogous to the case of degenerate masses, i.e. one has to substitute rows or columns in the determinants with suitable linear combinations before taking the microscopic limit. For instance, the generic determinant can conveniently be written as (we use the shortened notation $t_0 = x, t_i = -m_i^2$):

$$\det[A^{(N_f, \alpha)}_N] = \det[P^{(0, \alpha)}_{N+l}(t_i)] = \det_{il}[L^{(\alpha)}_{N+l}(t_i)] \prod_{k=N}^{N+N_f} h_k^{\alpha} / \prod_{k=N}^{N+N_f} \sqrt{h_k^{\alpha}} \prod_{p=1}^{N_f} \left[-(N+p)\right]^{N_f-p+1},$$

with $i, l = 0, \ldots, N_f$. In the last equality we iteratively used the recurrence relation $xL_n^{(\alpha+1)}(x) = (n + \alpha + 1)L_n^{(\alpha)}(x) - (n + 1)L_{n+1}^{(\alpha)}(x)$. Now substituting the microscopic scaling $t_i \to z_i/N$, one obtains the following large-$N$ behaviour

$$\det[A^{(N_f, \alpha)}_N] \sim (-1)^{\frac{N_f(N_f+1)}{2}} \frac{N^{(N_f+1)(\alpha-N_f)/2}}{\prod_{i=1}^{N_f} J_{\alpha+i}(2\sqrt{z_i})}], \quad (7.3)$$

since $\lim_{N \to \infty} L_N^{(\alpha)}(x/N)/N^\alpha = J_\alpha(2\sqrt{x})/x^{\alpha/2}$. When the argument of the Bessel functions is complex (i.e., when $t_i = -m_i^2$), one will make use of $J_\nu(iz) = i^\nu I_\nu(z)$. Finally, from the previous simple arguments it follows that the microscopic limit of all the quantities evaluated in this paper, can be obtained using the determinant in eq. (7.3) instead of the determinant at finite-$N$ given in eq. (1.3).

Let us look at some explicit examples. For sake of simplicity, we consider $N_f$ non-degenerate masses first. The microscopic limit of the normalization factor $h_N^{(N_f, \alpha)}$ in eq. (1.8) is:

$$h_N^{(N_f, \alpha)}(\mu) = [\Delta^{(\alpha)}(\mu)]^2, \quad (7.4)$$
where \( \mathbf{\mu} \equiv \{ \mu_1, \mu_2, \ldots, \mu_{N_f} \} \) and
\[
\Delta^{(\alpha)}(\mu_1, \mu_2, \ldots, \mu_{N_f}) \equiv \det \left[ \mu_p^q \frac{2}{\alpha} I_{\alpha+q}(2\mu_p) \right],
\]
(7.5)
where \( p = 1, \ldots, N_f, \ q = 0, \ldots, N_f - 1 \). In this definition, we assume that the range of \( p \) and \( q \) is determined by the number of arguments of \( \Delta^{(\alpha)} \). We can also compute the microscopic limit of the highest coefficient \( k^{(N_f, \alpha)}_N(\mathbf{m}) \) in an analogous way:
\[
\tilde{K}^{(N_f, \alpha)}(\mathbf{\mu}) = 1,
\]
(7.6)
because the ratio of the two determinants in eq. (4.20) equals unity in the microscopic large-\( N \) limit. From eq. (4.7), (7.6) and (7.4) we immediately read off the microscopic limit of the orthonormal polynomials for \( N_f \) massive fermions, that is:
\[
\tilde{P}^{(N_f, \alpha)}(\zeta; \mathbf{\mu}) = \frac{\Delta_1^{(\alpha)}(\zeta, \mathbf{\mu})}{\Delta^{(\alpha)}(\mathbf{\mu}) \prod_{i=1}^{N_f}(\zeta + \mu_i^2)}.
\]
(7.7)
where \( \Delta_1^{(\alpha)}(\zeta, \mathbf{\mu}) \) is
\[
\Delta_1^{(\alpha)}(\zeta, \mathbf{\mu}) \equiv \det \left( \begin{array}{cccc}
\zeta^{-\frac{1}{2}} J_\alpha(2\sqrt{\zeta}) & -\zeta^{\frac{1}{2}} J_{\alpha+1}(2\sqrt{\zeta}) & \cdots & (-1)^{N_f} \zeta^{\frac{1}{2}} J_{\alpha+N_f}(2\sqrt{\zeta}) \\
\mu_1^{-\alpha} I_\alpha(2\mu_1) & \mu_1^{-\alpha} I_{\alpha+1}(2\mu_1) & \cdots & \mu_1^{\alpha-N_f} I_{\alpha+N_f}(2\mu_1) \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{N_f}^{-\alpha} I_\alpha(2\mu_{N_f}) & \mu_{N_f}^{-\alpha} I_{\alpha+1}(2\mu_{N_f}) & \cdots & \mu_{N_f}^{\alpha-N_f} I_{\alpha+N_f}(2\mu_{N_f})
\end{array} \right).
\]
(7.8)
The microscopic limit of the coefficients \( d^{(N_f, \alpha)}_{N_j} (\mathbf{m}) \) in eq. (4.20) is
\[
d^{(N_f, \alpha)}_i (\mathbf{\mu}) = \frac{\Delta^{(\alpha)}(\mathbf{\mu}, \zeta) \prod_{j \neq i}^{N_f} (\mu_j^2 - \mu_i^2)}{(-1)^{i+1} \Delta^{(\alpha)}(\mathbf{\mu})}.
\]
(7.9)
The microscopic limit of the weight \( w^{(N_f, \alpha)}(x) \) is given by
\[
\tilde{w}(\zeta) = \zeta^\alpha \prod_{i=1}^{N_f}(\zeta + \mu_i^2),
\]
(7.10)
which appear in the following expression for the microscopic unitary kernel (from eq. (5.2)):
\[
\tilde{K}^{(2)}(\zeta_1, \zeta_2) = \frac{\tilde{P}^{(N_f+1, \alpha)}(\zeta_1; \mathbf{\mu}, \sqrt{-\zeta_2}) \sqrt{\tilde{w}(\zeta_1) \tilde{w}(\zeta_2)}}{d^{(N_f+1, \alpha)}_{N_f+1}(\mathbf{\mu}, \sqrt{-\zeta_2})} \frac{(-1)^{N_f} \Delta_1^{(\alpha)}(\zeta_1; \mathbf{\mu}, \sqrt{-\zeta_2})}{\Delta^{(\alpha)}(\mathbf{\mu})(\zeta_1 - \zeta_2) \sqrt{\prod_{j=1}^{N_f}(\zeta_1 + \mu_j^2)(\zeta_2 + \mu_j^2)}}.
\]
(7.11)
The microscopic limit of the $\tilde{\psi}_i$ functions is:

$$
\tilde{\psi}_i(\zeta) = \frac{\Delta_1^{(\alpha)}(\zeta, \mu, \sigma_i)}{\Delta^{(\alpha)}(\mu, \sigma_i)(\zeta + \sigma_i^2)\sqrt{\prod_{j=1}^{N_f}(\zeta + \mu_j^2)}}\sqrt{\zeta}, \quad i = 1, \ldots, N_f + 1,
$$

$$
\tilde{\psi}_{N_f+1+j}(\zeta) = \frac{\Delta_1^{(\alpha)}(\zeta, \mu_{\neq j})}{\Delta^{(\alpha)}(\mu_{\neq j})\sqrt{\prod_{j=1}^{N_f}(\zeta + \mu_j^2)}}\sqrt{\zeta}, \quad j = 1, \ldots, N_f,
$$

$$
\tilde{\psi}_{2N_f+2}(\zeta) = \frac{\Delta_1^{(\alpha-1)}(\zeta, \mu)}{\Delta^{(\alpha-1)}(\mu)\sqrt{\prod_{j=1}^{N_f}(\zeta + \mu_j^2)}}\sqrt{\zeta}^{-2},
$$

where $\sigma_i \equiv m_i$ for $i = 1, \ldots, N_f$ and $\sigma_{N_f+1} \equiv 0$. The microscopic limit of the diagonal elements of the matrix $\tilde{A}$ is (from eq. (3.4)):

$$
2\tilde{A}_i = -\chi(\beta)\tilde{d}_i^{N_f,0}(\mu)\tilde{P}^{(N_f,0)}(-\mu_i^2; \mu), \quad i = 1, \ldots, N_f, \quad (7.12)
$$

$$
2\tilde{A}_{N_f+1} = -\alpha\chi(\beta)\tilde{d}_{N_f+1}^{(\alpha-1)}(\mu, 0)\tilde{P}^{(N_f,0)}(0; \mu)
$$

$$
= -\alpha\chi(\beta)\frac{\Delta_1^{(\alpha-1)}(\mu)}{\Delta^{(\alpha-1)}(\mu)} \frac{\Delta_1^{(\alpha)}(0; \mu)}{\Delta^{(\alpha)}(0; \mu)}. \quad (7.13)
$$

In equation (7.12) the orthogonal polynomials $\tilde{P}^{(N_f,0)}$ are evaluated at $\zeta = -\mu_i^2$. As we have already discussed for the degenerate massive case, this situation should be considered in a limiting sense of eq. (7.7), that is $\zeta \to -\mu_i^2$. In particular, in the ratio $\Delta_1^{(\alpha)}(\zeta, \mu)\prod_{j=1}^{N_f}(\zeta + \mu_j^2)$ one substitutes each element of the first line of the matrix in eq. (7.8) with its derivative with respect to $\zeta$, evaluated at $\zeta = -\mu_i^2$.

Finally, the elements of the matrix $\tilde{B}$ and the terms $\tilde{\psi}_i$ are given in eq. (3.20) and eq. (3.22), respectively. Let $\tilde{B}_{ij}$ and $\tilde{\psi}_i$ be their microscopic large-$N$ limit, respectively. By means of eq. (3.21), one obtains also $\tilde{C}$, $\tilde{C}_{00}$ and $\tilde{C}_0$. Putting together all of the terms above, we obtain the microscopic limit of the scalar kernel $S_N^{(\beta)}(x, y)$. It is very important at this point to note that the number of corrections to the unitary kernel $K_N^{(2)}(x, y)$ appearing in $S_N^{(\beta)}(x, y)$ is indeed independent of $N$, because it depends only on the potential $V(x)$, and thus in the double-microscopic large-$N$ limit the scalar kernel $S_N^{(\beta)}(x, y)$ is still obtained through the formulas (3.23), (3.24). Namely, in the orthogonal case $\beta = 1$, the scalar kernel is:

$$
\tilde{S}^{(1)}(\zeta_1, \zeta_2) = \tilde{K}^{(2)}(\zeta_1, \zeta_2) - \sum_{i \leq n,j=1}^{2n} [\tilde{A}\tilde{C}(I - \tilde{B}\tilde{A}\tilde{C})^{-1}]_{ij}\tilde{\psi}_i(\zeta_1)\tilde{\psi}_j(\zeta_2). \quad (7.14)
$$
In fact, the required substitution $c \to 2c$, which effectively means $\beta \to 2\beta$ in this case, is compensated by the $\beta$-scaling required for recovering the general case $c \neq 1$, as discussed at the beginning of this section. Moreover we have to put $\alpha \to 2\alpha$ and $N_f \to 2N_f$, the latter meaning that all the masses are double degenerate. Indeed the degenerate-masses case does not present any peculiar problem, and by applying the same technique as described after eq. (4.9), the only consequence is that one has to consider higher-order derivatives of Bessel functions in all the determinants.

In the symplectic case $\beta = 4$, one has to apply the substitution $N \to 2N$. Accordingly, the microscopic limit of the unitary kernel reads:

$$\lim_{N \to \infty} N^\delta K^{(2)}_{2N}(\frac{\zeta_1}{2N}, \frac{\zeta_2}{2N}) = \bar{K}^{(2)}(2\zeta_1, 2\zeta_2),$$

(7.15)

and the scalar kernel is

$$\bar{S}^{(4)}(\zeta_1, \zeta_2) = \left[2\bar{K}^{(2)}(2\zeta_1, 2\zeta_2) - 2 \sum_{i>n, j=1}^{2n} [\bar{A}_0 \bar{C}_{00}\bar{C}_0]_{ij} \bar{\psi}_i(2\zeta_1) \bar{\psi}_j(2\zeta_2) \right] \mu \to \sqrt{2\mu},$$

(7.16)

where once again the final $\beta$-scaling is for recovering $c = N\beta/2 \neq 1$. The only elements in the previous formulas which require further attention are the large-$N$ quantities $\bar{B}_{ij}$ and $\bar{\varepsilon}\bar{\psi}_j$. In order to obtain them, we need to evaluate the large-$N$ behaviour of the integrals defining the matrix elements $B_{ij}$ and $\varepsilon\psi_i$. When doing that, it could be advantageous to exchange the integrals with the large-$N$ microscopic limit, i.e. substituting the microscopic expansion of the functions $\psi_i$ into the integrals. Such an interchange can be done under suitable smoothness assumptions. In [13] this issue has been investigated carefully for the massless case. It turns out that for $\beta = 4$ the procedure holds without any problems, whereas for $\beta = 1$ one has to consider also an additional contribution coming from the soft-edge of the spectrum. Since the case of one mass taken to zero or to infinity in our formulas reproduces results with one additional massless flavor and one massless flavor less, respectively, we shall assume that the interchanging procedure, when $\beta = 4$, is permitted in the intermediate mass region as well. Thus, in the $\beta = 4$ case we can write

$$\bar{B}_{ij} = \int_0^\infty \int_0^\infty dx dy \varepsilon(x - y) \bar{\psi}_i(x) \bar{\psi}_j(y),$$

(7.17)

$$\bar{\varepsilon}\bar{\psi}_j(y) = \int_0^\infty dx \varepsilon(x - y) \bar{\psi}_j(x),$$

(7.18)

with $i, j = 1, \ldots, 2N_f + 2$. The question whether such a simplification is allowed for $\beta = 1$ case, can be addressed by means of numerical investigations. Nevertheless, with or without
exchanging the integrals and limits, from the very general result in eq. (7.14) and eq. (7.16) one obtains analytic expressions for the microscopic limit of all the correlation functions of the general massive chiral ensembles for $\beta = 1$ and $\beta = 4$. For instance, the microscopic spectral density is the simplest one and it is given by $\rho^{(\beta)}(x) = S^{(\beta)}(x, x)$. Switching to the real eigenvalues $\xi_i$ of the Dirac operator, by means of $\xi^2 = x$, one obtains the microscopic spectral density of the Dirac operator $\rho^{(\beta)}_D(\xi) = 2|\xi|\rho^{(\beta)}(\xi^2)$. If universality arguments apply to correlation functions of the general massive Chiral Ensembles, then the expressions obtained from eq. (7.14) and eq. (7.16), through eq. (3.1), are the very natural candidates.

8 Conclusions

In this paper we considered the problem of computing the correlation functions for massive Dirac spectra of four-dimensional QCD. Starting from the fact that RMT is just a simple and effective tool for calculating actual spectral correlations in the infrared regime (which in principle also can be obtained from direct calculations in terms of finite-volume partition function as in [34]) we consider a matrix model with matrices either in the $\chi$GOE or in the $\chi$GSE. The former corresponds to four-dimensional QCD with $N_f$ fermions in the fundamental representation and $SU(2)$ gauge theory, the latter corresponds to four-dimensional QCD with $N_f$ fermions in the adjoint representation and $SU(N_c \geq 2)$ gauge theory. Matrix models with orthogonal and symplectic matrices naturally leads to the application of skew-orthogonal polynomials, which have the drawback that at present they are difficult to determine and handle in the microscopic large-$N$ limit. Therefore we investigated whether the recent technique proposed by H. Widom for dealing with OE and SE using standard orthogonal polynomials of the UE, could be applied effectively in our actual case. We have succeeded in deriving the orthonormal polynomials for the general massive fermion case, in an explicit and closed form. We found explicit formulas for the $\psi$ functions which are the basic ingredients for computing the scalar kernels $S_N^{(\beta)}(x, y)$ for massive fermions, when $\beta = 1$ and $\beta = 4$. In particular, such scalar kernels are expressed as the unitary kernel $K_N^{(2)}(x, y)$ plus a finite number of corrections, which depend only on one-dimensional and two-dimensional integrals involving the functions $\psi_i$. Cases with degenerate masses, or massless fermions can then be obtained just as simple limits of the formulas we have derived. We obtained fully analytical formulas for the scalar kernels $\tilde{S}^{(1)}$.

9In general, the $k$-point correlation function $\rho^{(\beta)}_D(\xi_1, \ldots, \xi_k)$ of the spectrum of the Dirac operator $D$ is given by the change of variables $\xi_j^2 = \lambda_j$ in eq. (3.1).
and $\tilde{S}^{(4)}$ in the double microscopic large-$N$ limit. From such formulas one can in principle derive all the microscopic correlation functions of the Chiral Orthogonal and Chiral Symplectic Ensembles, with an arbitrary number of flavors, arbitrary masses and arbitrary topological charge.

We emphasize that the issue of universality has not been considered in the present paper. However, the fact that the method of Widom can be applied to four-dimensional QCD with massive fermions is indeed encouraging and provides a new general framework where it might be possible to analyze some of the still open issues. We believe that the present framework is the most suitable setting for proving universality of these massive cases.

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A Normalization factors

The orthonormality condition of polynomials is an essential requirement for the proper use of Widom’s method. Therefore, an explicit expression for the normalization factors of the orthogonal polynomials for the general $N_f$-flavor case is of primary importance. Even though it is a classical result, we choose to present in this appendix a derivation of the normalization factors $h_n^{(N_f,\alpha)}$, and of the highest coefficients $k_0^{(N_f,\alpha)}$, i.e. eq. (4.8) and eq. (4.9), respectively. Substituting Christoffel’s formula eq. (4.7) into the orthonormality condition eq. (4.6) evaluated at $i = j = n$, yields:

$$\sqrt{h_n^{(N_f,\alpha)}(m)} = \int_0^{+\infty} dx x^\alpha e^{-x} P_n^{(N_f,\alpha)}(x; m) \det \left[ \Lambda_n^{(N_f,\alpha)}(x) \right]$$

$$= \int_0^{+\infty} dx x^\alpha e^{-x} P_n^{(N_f,\alpha)}(x; m) \sum_{k=0}^{N_f} P_{n+k}^{(0,\alpha)}(x) (-1)^k \det \left[ \Lambda_{n+k+1}^{(N_f,\alpha)} \right], \quad (A.1)$$

where we in the second equation used the so-called Laplace expansion of the determinant of a matrix with respect to its first row. Since $\int_0^{+\infty} dx x^\alpha e^{-x} x^n P_j^{(0,\alpha)}(x) = \delta_{nj} k_n^{(0,\alpha)}$, for $n \leq j$, one has that the only non-vanishing term in the sum in eq. (A.1) is when $k = 0.$
Therefore we obtain:

\[
\sqrt{h_n^{(N_f,\alpha)}}(m) = \frac{k_n^{(N_f,\alpha)}(m)}{k_n^{(0,\alpha)}} \det \left[ \Lambda_{n,1}^{(N_f,\alpha)} \right],
\]  

(A.2)

with the coefficient \(k_n^{(0,\alpha)}\) given in eq. (1.3). The still unknown highest coefficients \(k_n^{(N_f,\alpha)}\) can easily be determined as follows:

\[
k_n^{(N_f,\alpha)}(m) = \lim_{x \to +\infty} \frac{P_n^{(N_f,\alpha)}(x;m)}{x^N} = \lim_{x \to +\infty} \frac{\sum_{k=0}^{N_f} P_n^{(0,\alpha)}(x)(-1)^k \det \left[ \Lambda_{n,k+1}^{(N_f,\alpha)} \right]}{x^N \sqrt{h_n^{(N_f,\alpha)}}(m) \prod_{i=1}^{N_f} (x + m_i^2)}
\]

\[
= \frac{k_n^{(0,\alpha)}}{\sqrt{h_n^{(N_f,\alpha)}}(m)} (-1)^{N_f} \det \left[ \Lambda_{n,N_f+1}^{(N_f,\alpha)} \right],
\]  

(A.3)

In the first line we used Christoffel’s formula eq. (1.7) and the Laplace expansion of the determinant, whereas the second line comes from the observation that only the last term in the sum contributes in the large-\(x\) limit. Combining eq. (A.3) with eq. (A.2) one finally has

\[
h_n^{(N_f,\alpha)}(m) = (-1)^{N_f} k_n^{(0,\alpha)} k_n^{(0,\alpha)} \det \left[ \Lambda_{n,1}^{(N_f,\alpha)} \right] \det \left[ \Lambda_{n,N_f+1}^{(N_f,\alpha)} \right],
\]  

(A.4)

\[
k_n^{(N_f,\alpha)}(m) = \sqrt{(-1)^{N_f} k_n^{(0,\alpha)} k_n^{(0,\alpha)} \det \left[ \Lambda_{n,N_f+1}^{(N_f,\alpha)} \right] \det \left[ \Lambda_{n,1}^{(N_f,\alpha)} \right]},
\]  

(A.5)

where the coefficients \(k_n^{(0,\alpha)}\) are given in eq. (1.3).

**B  Some integrals**

In this appendix we evaluate the integrals in eq. (5.9) and eq. (5.10), that is:

\[
\begin{align*}
I_0^{(N,j)} &= \langle P_N^{(N_f,\alpha)}(z;m), \frac{P_N^{(N_f,\alpha)}(z;m)}{z} \rangle_{N_f,\alpha}, \\
I_i^{(N,j)} &= \langle P_N^{(N_f,\alpha)}(z;m), \frac{P_N^{(N_f,\alpha)}(z;m)}{z+m_i^2} \rangle_{N_f,\alpha} \quad \text{for } i = 1, 2, \ldots, N_f
\end{align*}
\]  

(B.1)

\[26\]
Using eq. (4.16) (i.e. adding $m_{N_f+1} = 0$) the integral $I_0^{(N,j)}$ can be written as

$$I_0^{(N,j)} = \langle P_N^{(N_f+1,\alpha-1)}(z; \mathbf{m}, 0), P_j^{(N_f+1,\alpha-1)}(z; \mathbf{m}, 0) \rangle_{N_f+1,\alpha-1} ;$$

therefore it can be obtained from $I_i^{(N,j)}$ in eq. (B.1), by applying the substitutions $N_f \rightarrow N_f + 1$, $\alpha \rightarrow \alpha - 1$, $\{\mathbf{m}\} \rightarrow \{\mathbf{m}, 0\}$ and $i \rightarrow N_f + 1$. The integral $I_i^{(N,j)}$ can be evaluated as follows. First we apply eq. (4.19) and then we use the orthonormality condition (4.6), that is

$$\int \Phi_{N,i}^2 = \left( \sum_{j=0}^{\min(N,j)} \Phi_{N,j} \Phi_{N,j}^* \right)_{N_f-1,\alpha}.$$

Using eq. (4.19) again, we finally end up with

$$I_i^{(N,j)} = d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) \sum_{r=0}^{N_f} \sum_{s=0}^{N_f} P_r^{(N_f-1,\alpha)}(-m_i^2; \mathbf{m} \neq i) \times P_s^{(N_f-1,\alpha)}(z; \mathbf{m} \neq i) P_s^{(N_f-1,\alpha)}(z; \mathbf{m} \neq i)_{N_f-1,\alpha} = d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) \sum_{r=0}^{\min(N,j)} \left[ P_r^{(N_f-1,\alpha)}(-m_i^2; \mathbf{m} \neq i) \right]^2 .$$

Using eq. (4.19) again, we finally end up with

$$I_i^{(N,j)} = d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) d_{N,i}^{(N_f,\alpha)}(\mathbf{m}) P_{\min(N,j)}^{(N_f,\alpha)}(-m_i^2; \mathbf{m}) = d_{\max(N,j),i}^{(N_f,\alpha)}(\mathbf{m}) P_{\min(N,j)}^{(N_f,\alpha)}(-m_i^2; \mathbf{m}) .$$

Therefore the integral $I_0^{(N,j)}$ is

$$I_0^{(N,j)} = d_{\max(N,j),N_f+1}^{(N_f+1,\alpha-1)}(\mathbf{m}, 0) P_{\min(N,j)}^{(N_f+1,\alpha-1)}(0; \mathbf{m}, 0) = d_{\max(N,j),N_f+1}^{(N_f,\alpha)}(\mathbf{m}, 0) P_{\min(N,j)}^{(N_f,\alpha)}(0; \mathbf{m}) .$$

A final comment: one could wonder about the meaning of the “degenerate” expressions $P_n^{(N_f+1,\alpha)}(0; \mathbf{m}, 0)$ and $P_n^{(N_f,\alpha)}(-m_i^2; \mathbf{m})$. But as in the case of degenerate masses, they should be understood in the limit sense $x \rightarrow 0$ or $x \rightarrow -m_i^2$, respectively. Such a limit is easily evaluated from eq. (4.7) by subtracting the $i$-th row from the first row of the matrix $\Lambda_n^{(N,f,\alpha)}$ (eq. (4.3)), and dividing it by the term $(x + m_i^2)$ stemming from the product in the denominator of eq. (4.7). Therefore, the limit $x \rightarrow -m_i^2$ of eq. (4.7) is equivalent to substituting all the polynomials $P_j^{(0,\alpha)}(x)$ in the first row of $\Lambda_n^{(N_f,\alpha)}$, with the corresponding derivatives $P_j^{(0,\alpha)}'(x)$ evaluated at $x = -m_i^2$.  

27
C  The functions $\psi_j(x)$

In this appendix we find a compact and useful expression for the functions $\psi$. We will show the remarkable fact that every $\psi$ function is proportional to a polynomial $P^{(N_f,\alpha)}_n$. Let us first consider the functions in the Hilbert space $\mathcal{H}$. Substituting the coefficients $D_i$ eq. (5.13) into the linear combination eq. (5.12) we obtain:

$$
\psi_i(x) = \frac{P^{(N_f,\alpha)}_{N-1}\left(-\sigma^2_i; m\right)P^{(N_f,\alpha)}_N(x; m) - P^{(N_f,\alpha)}_{N-1}\left(-\sigma^2_i; m\right)P^{(N_f,\alpha)}_N(x; m)}{(x + \sigma^2_i)P^{(N_f,\alpha)}_{N-1}\left(-\sigma^2_i; m\right)} \sqrt{w^{(N_f,\alpha)}(x)}
$$

$$
= -\frac{P^{(N_f+1,\alpha)}_{N-1}(x, m, \sigma_i)}{P^{(N_f,\alpha)}_{N-1}(-\sigma^2_i; m)\ c^{(N_f+1,\alpha)}_{N-1,N_f+1}(m, \sigma_i)} \sqrt{w^{(N_f,\alpha)}(x)} ,
$$

(C.1)

where $\sigma_i \equiv m_i$ for $i = 1, \ldots, N_f$, $\sigma_{N_f+1} \equiv 0$ and in the second line we used the formula (4.17). From eq. (4.18) we obtain:

$$
\psi_i(x) = k^{(N_f,\alpha)}_N(m) \frac{P^{(N_f+1,\alpha)}_{N-1}(x, m, \sigma_i)}{k^{(N_f+1,\alpha)}_{N-1}(m, \sigma_i)} \sqrt{w^{(N_f,\alpha)}(x)} , \quad i = 1, \ldots, N_f + 1 .
$$

(C.2)

Also the $\psi$ functions in the orthogonal Hilbert space $\mathcal{H}^\perp$ can be written in a simplified form. In fact, let us first notice that by isolating the last term of the sum in eq. (4.19) and using again eq. (4.13) on the remaining sum, one has:

$$
P^{(N_f,\alpha)}_N(x; m) = d^{(N_f,\alpha)}_{N,\bar{i}}(m) \left(\frac{P^{(N_f,\alpha)}_{N-1}(x; m)}{d^{(N_f,\alpha)}_{N-1,\bar{i}}(m)} + P^{(N_f-1,\alpha)}_N(-m^2_i; m_{\bar{i}})P^{(N_f-1,\alpha)}_N(x; m_{\bar{i}})\right) ,
$$

(C.3)

that is

$$
P^{(N_f,\alpha)}_N(x; m) - d^{(N_f,\alpha)}_{N,\bar{i}}(m)P^{(N_f,\alpha)}_{N-1}(x; m) = d^{(N_f,\alpha)}_{N,\bar{i}}(m)P^{(N_f-1,\alpha)}_N(-m^2_i; m_{\bar{i}})P^{(N_f-1,\alpha)}_N(x; m_{\bar{i}}) .
$$

(C.4)

From eq. (5.13) and (5.17) (or (5.18)), we exactly obtain the l.h.s. of last formula, therefore

$$
\psi_{N_f+1+i}(x) = d^{(N_f,\alpha)}_{N,\bar{i}}(m)P^{(N_f-1,\alpha)}_N(-m^2_i; m_{\bar{i}})P^{(N_f-1,\alpha)}_N(x; m_{\bar{i}})
$$

$$
= \sqrt{w^{(N_f,\alpha)}(x)} \frac{P^{(N_f-1,\alpha)}_N(x; m_{\bar{i}})}{k^{(N_f-1,\alpha)}_N(m_{\bar{i}})} \frac{\sqrt{w^{(N_f,\alpha)}(x)}}{(x + m^2_i)} ,
$$

(C.5)
for $i = 1, \ldots, N_f$ and
\[
\psi_{2N_f+2}(x) = k^{(N_f,\alpha)}(m) P_N^{(N_f,\alpha-1)}(x; m) \frac{\sqrt{w^{(N_f,\alpha)}(x)}}{x},
\]
(C.6)
for $i = N_f + 1$.

Finally, let us remark that such compact forms eq. (C.5) and eq. (C.6) are quite useful and effective in calculating the microscopic limit of $\psi$ functions.

References

[1] E. V. Shuryak and J. J. M. Verbaarschot, *Random matrix theory and spectral sum rules for the Dirac operator in QCD*, Nucl. Phys. A560 (1993) 306, hep-th/9212088.

[2] M. A. Halasz and J. J. M. Verbaarschot, *Effective Lagrangians and chiral random matrix theory*, Phys. Rev. D 52 (1995) 2563, hep-th/9502096.

[3] A. Smilga and J. J. M. Verbaarschot, *Spectral sum rules and finite volume partition function in gauge theories with real and pseudoreal fermions*, Phys. Rev. D 51 (1995) 829, hep-th/9404031.

[4] J. J. M. Verbaarschot, *The Spectrum of the QCD Dirac operator and chiral random matrix theory: The Threefold way*, Phys. Rev. Lett. 72 (1994) 2531, hep-th/9401059; *Spectral sum rules and Selberg’s integral*, Phys. Lett. B 329 (1994) 351, hep-th/9402008; *The Spectrum of the Dirac operator near zero virtuality for $N(c) = 2$ and chiral random matrix theory*, Nucl. Phys. B426 (1994) 559, hep-th/9401092.

[5] J. J. M. Verbaarschot, *Universal behavior in Dirac spectra*, hep-th/9710114.

[6] T. Banks and A. Casher, *Chiral Symmetry Breaking In Confining Theories*, Nucl. Phys. B169 (1980) 103.

[7] H. Leutwyler and A. Smilga, *Spectrum of Dirac operator and role of winding number in QCD*, Phys. Rev. D 46 (1992) 5607.

[8] J. J. M. Verbaarschot and I. Zahed, *Spectral density of the QCD Dirac operator near zero virtuality*, Phys. Rev. Lett. 70, (1993) 3852, hep-th/9303012.
[9] G. Akemann, P. H. Damgaard, U. Magnea and S. Nishigaki, *Universality of random matrices in the microscopic limit and the Dirac operator spectrum*, Nucl. Phys. B487, (1997) 721, [hep-th/9609174].

[10] P. H. Damgaard and S. M. Nishigaki, *Universal Spectral Correlators and Massive Dirac Operators* Nucl. Phys. B518, (1998) 495-512, [hep-th/9711023].

[11] J. C. Osborn, D. Toublan and J. J. Verbaarschot, *From chiral random matrix theory to chiral perturbation theory*, Nucl. Phys. B540 (1999) 317, [hep-th/9806110]; P. H. Damgaard, J. C. Osborn, D. Toublan and J. J. M. Verbaarschot, *The microscopic spectral density of the QCD Dirac operator*, Nucl. Phys. B547 (1999) 305, [hep-th/9811212]; D. Toublan and J. J. Verbaarschot, *The spectral density of the QCD Dirac operator and patterns of chiral symmetry breaking*, Nucl. Phys. B560 (1999) 259, [hep-th/9904199].

[12] S. M. Nishigaki, P. H. Damgaard and T. Wettig, *Smallest Dirac eigenvalue distribution from random matrix theory*, Phys. Rev. D 58 (1998) 087704, [hep-th/9803007]; P. H. Damgaard and S. M. Nishigaki, *Distribution of the k-th smallest Dirac operator eigenvalue*, Phys. Rev. D 63 (2001) 45012, [hep-th/0006111].

[13] M. K. Sener and J. J. M. Verbaarschot, *Universality in chiral random matrix theory at β = 1 and β = 4*, Phys. Rev. Lett. 81 (1998) 248, [hep-th/9801043].

[14] B. Klein and J. J. M. Verbaarschot, *Spectral universality for real chiral random matrix ensembles* Nucl. Phys. B588 (2000) 483 [hep-th/0004119].

[15] M. E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. J. M. Verbaarschot and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146; P. H. Damgaard, U. M. Heller and A. Krasnitz, *Microscopic spectral density of the Dirac operator in quenched QCD*, Phys. Lett. B 445 (1999) 366, [hep-lat/9810006]; R. G. Edwards, U. M. Heller, J. Kiskis and R. Narayanan, *Quark spectra, topology and random matrix theory*, Phys. Rev. Lett. 82 (1999) 4188, [hep-th/9902117].

[16] M. E. Berbenni-Bitsch, S. Meyer and T. Wettig, Phys. Rev. D 58 (1998) 71502, [hep-lat/9804030]; P. H. Damgaard, U. M. Heller, R. Niclasen and K. Rummukainen, *Eigenvalue distributions of the QCD Dirac operator*, Phys. Lett. B 495 (2000) 263, [hep-lat/0007041].

[17] J. J. M. Verbaarschot and T. Wettig, [hep-ph/0003017], in Ann. Rev. Nucl. Part. Sci. (2000); J. J. M. Verbaarschot, Lectures given at APCTP-RCNP Joint International
School on Physics of hadrons and QCD, Osaka, Japan, and the 1998 YITP Workshop on QCD and Hadron Physics, Kyoto, Japan, (1998), [hep-ph/9902394].

[18] J. J. M. Verbaarschot and I. Zahed, *Random matrix theory and QCD in three-dimensions*, Phys. Rev. Lett. **73** (1994) 2288, [hep-th/9405005].

[19] T. Nagao and M. Wadati, *Correlation Functions of Random Matrix Ensembles Related to Classical Orthogonal Polynomials*, J. Phys. Soc. Jpn. **60**, (1991) 1943.

[20] T. Nagao and P. J. Forrester, *Asymptotic Correlations at the Spectrum Edge of Random Matrices*, Nucl. Phys. **B435** (1995) 401.

[21] P. H. Damgaard and S. M. Nishigaki, *Universal massive spectral correlators and QCD(3)*, Phys. Rev. D **57** (1998) 5299, [hep-th/9711096].

[22] C. Hilmoine and R. Niclasen, *The microscopic spectral density of the Dirac operator derived from Gaussian orthogonal and symplectic ensembles*, Phys. Rev. D **62** (2000) 096013, [hep-th/0004081].

[23] T. Wilke, T. Guhr and T. Wettig, *The microscopic spectrum of the QCD Dirac operator with finite quark masses*, Phys. Rev. D **57** (1998) 6486, [hep-th/9711054].

[24] T. Nagao and S. M. Nishigaki, *Massive chiral random matrix ensembles at β = 1 and 4 : QCD Dirac operator spectra*, Phys. Rev. D **62** (2000) 065007, [hep-th/0003009].

[25] G. Akemann and E. Kanzieper, *Spectra of massive and massless QCD Dirac operators : A novel link*, Phys. Rev. Lett. **85** (2000) 1174, [hep-th/0001188].

[26] H. Widom, *On the relation between orthogonal, symplectic and unitary matrix ensembles*, J. Statist. Phys. **94**, (1999) 347, [solv-int/9804005].

[27] M. L. Mehta, *Random Matrices*, 2nd Edition, Academic Press (1991).

[28] G. Mahoux and M. L. Metha, J. Phys. I (France) (1991) 1093.

[29] C. A. Tracy and H. Widom, *Fredholm determinants, differential equations and matrix models*, Commun. Math. Phys. **163**, (1994) 33, [hep-th/9306042].
[30] C. A. Tracy and H. Widom, *Correlation functions, cluster functions and spacing distributions for random matrices*, J. Stat. Phys. 92 (1998) 809-835, hep-th/9804004.

[31] B. Eynard, *Asymptotics of skew orthogonal polynomials*, cond-mat/0012046.

[32] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover Publications, New York (1974).

[33] G. Szegö, *Orthogonal Polynomials*, Am. Math. Soc., Providence, RI, 1939.

[34] P. H. Damgaard, *Dirac operator spectra from finite-volume partition functions*, Phys. Lett. B 424 (1998) 322, hep-th/9711110. G. Akemann and P. H. Damgaard, *Microscopic spectra of Dirac operators and finite-volume partition functions*, Nucl. Phys. B 528 (1998) 411, hep-th/9801133. G. Akemann and P. H. Damgaard, *Consistency conditions for finite-volume partition functions*, Phys. Lett. B 432 (1998) 390, hep-th/9802174.