A Slowly Rotating Charged Black Hole in Five Dimensions

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Abstract

Black hole solutions in higher dimensional Einstein and Einstein-Maxwell gravity have been discussed by Tangherlini as well as Myers and Perry a long time ago. These solutions are the generalizations of the familiar Schwarzschild, Reissner-Nordstrom and Kerr solutions of four-dimensional general relativity. However, higher dimensional generalization of the Kerr-Newman solution in four dimensions has not been found yet. As a first step in this direction I shall report on a new solution of the Einstein-Maxwell system of equations that describes an electrically charged and slowly rotating black hole in five dimensions.

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I. INTRODUCTION

The study of black holes in higher dimensions is motivated by several reasons. First of all, it is related to the intrinsic properties of human nature; one knows the fundamental features of black holes in four dimensions, such as the equilibrium and uniqueness properties, astonishing thermodynamical properties, Hawking’s effect of evaporation of mini black holes and wonders what will happen when going over into higher dimensions. Strong motivation comes from developments in string/M-theory, which is believed to be the most consistent approach to quantum theory of gravity in higher dimensions. Black holes may have a crucial role in the analysis of dynamics in higher dimensions as well as in the compactification mechanisms. In particular, to test novel predictions of string/M-theory microscopic black holes may serve as good theoretical laboratories. As an example, one can recall the statistical-mechanical calculation of the Bekenstein-Hawking entropy for a class of supersymmetric black holes in five dimensions which is thought of as one of the remarkable results in string theory [1, 2].

Another strong motivation for the interest in higher dimensional black holes originates from the brane-world gravity theories, which predict a new fundamental scale of quantum gravity being of the order of TeV-scale [3]-[6]. One of the exciting signatures of these models is the possibility of TeV-size mini black hole production at future colliders [7]. Furthermore, it has been argued that if the radius of the event horizon is much smaller than the size-scale of extra dimensions ($r_+ \ll L$), these black holes to a good enough approximation can be described by the classical solutions of higher dimensional Einstein’s field equations. In this framework the most interesting black hole solutions are the Tangherlini solution [8] for a static black hole and the Myers-Perry solution [9] for a stationary black hole in higher dimensions. In recent developments the different physical properties of these solutions have been discussed in a number of papers [10]-[15] (see also Refs. [16], [17] for reviews). Furthermore, it has been shown that the Myers-Perry solution is not unique in five dimensions; there exists a rotating black ring solution with the horizon topology of $S^2 \times S^1$ which could have the same mass and spin as the Myers-Perry solution [18].

It is clear that black holes produced at colliders may in general have an electric charge as well as other type of charges. Therefore the study of charged black hole solutions in higher dimensions becomes of great importance. The first black hole solution to the higher dimensional Einstein-Maxwell equations was found by Tangherlini [8]. This solution is the general-
ization of the familiar Reissner-Nordstrom solution for a static and electrically charged black hole in ordinary general relativity. However, the case of charged rotating black holes has been basically discussed within certain supergravity theories [19], as well as in the context of string theory (see review papers [20]-[21]). As for the counterpart of the Kerr-Newman solution in higher dimensions, that is the charged generalization of the Myers-Perry solution in the Einstein-Maxwell gravity, it still remains to be found. Here I shall discuss a new solution of the Einstein-Maxwell field equations describing an electrically charged and slowly rotating black hole in five dimensions.

II. ROTATING BLACK HOLE WITH ELECTRIC CHARGE

As is known the Kerr metric for a stationary black hole in four dimensions is uniquely characterized by an axis of rotation consistent with the fact that there exists only one independent Casimir invariant of the rotation group $SO(3)$. However, in five dimensions the rotation group is $SO(4)$ which possesses two independent Casimir invariants. These two Casimir invariants are associated with two independent rotations of the system. Thus, a rotating black hole in five dimensions may have two distinct planes of rotation specified by appropriate azimuthal coordinates, rather than an axis of rotation. In accordance with this, the stationary and asymptotically flat Myers-Perry metric admits three commuting Killing vector fields

$$\xi_{(t)} = \partial/\partial t, \quad \xi_{(\phi)} = \partial/\partial \phi, \quad \xi_{(\psi)} = \partial/\partial \psi,$$

(1)

which reflect the time-translation invariance and bi-azimuthal symmetry of the metric in five dimensions. The explicit form of the metric written in Boyer-Lindquist type coordinates is given by

$$ds^2 = -dt^2 + \Sigma \left( \frac{r^2}{\Pi} dr^2 + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta \, d\phi^2 + (r^2 + b^2) \cos^2 \theta \, d\psi^2 + \frac{m}{\Sigma} \left( dt - a \sin^2 \theta \, d\phi - b \cos^2 \theta \, d\psi \right)^2,$$

(2)

where the functions $\Sigma$ and $\Pi$ are given as

$$\Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Pi = (r^2 + a^2)(r^2 + b^2) - m \, r^2,$$

(3)
$m$ is a parameter related to the physical mass of the black hole, the parameters $a$ and $b$ are associated with its two independent angular momenta.\footnote{The expression for $\Sigma$ in \cite{15} has a misprint. The parameters $a$ and $b$ must be interchanged.}

Earlier in \cite{15} we studied the electromagnetic properties of the Myers-Perry black assuming that it may possess a small enough electric charge, such that the spacetime is still well described by the unperturbed metric \cite{22}. It is remarkable that in this case the five-vector potential for the source-free Maxwell field can be constructed using the time-translation invariance property of the metric. Indeed, the Maxwell equations for the vector potential in the Lorentz gauge

$$A_\mu;\mu = 0,$$

have the form

$$A_\mu;\nu - R_\mu^\nu A_\nu = 0. \quad (5)$$

On the other hand, any Killing vector $\xi$ satisfies the equation

$$\xi_\mu;\nu + R_\mu^\nu \xi_\nu = 0. \quad (6)$$

Comparing these two equations we see that in a vacuum spacetime ($R_\mu^\nu = 0$), the Killing vector can serve as a vector potential for a test Maxwell field \cite{22}. Applying this fact to our case we can consider the 5-vector potential of the form

$$A_\mu = \alpha \xi_\mu^\mu(t) \quad (7)$$

where $\alpha$ is an arbitrary parameter which can be fixed from examining the Gauss integral for the electric charge of the black hole

$$Q = \frac{1}{4\pi^2} \oint F_\mu^\nu d^3\Sigma_{\mu\nu}. \quad (8)$$

The integral is taken over the three-sphere at spatial infinity

$$d^3\Sigma_{\mu\nu} = \frac{1}{3!} \sqrt{-g} \epsilon_{\mu\nu\alpha\beta\gamma} d x^\alpha \wedge d x^\beta \wedge d x^\gamma. \quad (9)$$

Taking this into account and requiring the vanishing behaviour of the potential at infinity we finally obtain

$$A = -\frac{Q}{2\Sigma} \left( dt - a \sin^2 \theta d \phi - b \cos^2 \theta d \psi \right). \quad (10)$$
Accordingly, the associated electromagnetic 2-form field can be written as

\[
F = \frac{Q}{\Sigma^2} \left( r \, d r \wedge d t + (b^2 - a^2) \sin \theta \cos \theta \, d \theta \wedge d t \right)
\]

\[
- \frac{Q \, a \sin \theta}{\Sigma^2} \left( r \sin \theta \, d r \wedge d \phi - (r^2 + a^2) \cos \theta \, d \theta \wedge d \phi \right)
\]

\[
- \frac{Q \, b \cos \theta}{\Sigma^2} \left( r \cos \theta \, d r \wedge d \psi + (r^2 + b^2) \sin \theta \, d \theta \wedge d \psi \right),
\]  

(11)

while the contravariant components of the electromagnetic field tensor are given by

\[
F^{tr} = \frac{Q (r^2 + a^2)(r^2 + b^2)}{r \Sigma^3}, \quad F^{t\theta} = \frac{Q (b^2 - a^2) \sin 2 \theta}{2 \Sigma^3},
\]

\[
F^{r\phi} = - \frac{Q a (r^2 + b^2)}{r \Sigma^3}, \quad F^{r\psi} = - \frac{Q b (r^2 + a^2)}{r \Sigma^3},
\]

\[
F^{\theta\phi} = \frac{Q \, a \cot \theta}{\Sigma^3}, \quad F^{\theta\psi} = - \frac{Q b \tan \theta}{\Sigma^3}.
\]  

(12)

Next, we suppose that the electric charge is no longer small that its electromagnetic field does influence the metric of space-time around the black hole.

III. METRIC ANSATZ

It is clear that for an arbitrary electric charge of the black hole one must consistently solve the system of the Einstein-Maxwell equations

\[
R^{\mu \nu} = 8\pi G M^{\mu \nu},
\]  

(13)

\[
\partial_\nu (\sqrt{-g} F^{\mu \nu}) = 0,
\]  

(14)

where the source-term on the right-hand-side of equation (13) is given by

\[
M^{\mu \nu} = T^{\mu \nu} - \frac{1}{3} \delta^{\mu \nu} T.
\]  

(15)

Substituting into this expression the explicit form of the energy-momentum tensor for the electromagnetic field

\[
T^{\mu \nu} = \frac{1}{2 \pi^2} \left( F^{\mu \alpha} F_{\nu \alpha} - \frac{1}{4} \delta^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right)
\]  

(16)

and its trace \( T \) we put it in the form

\[
M^{\mu \nu} = \frac{1}{2 \pi^2} \left( F^{\mu \alpha} F_{\nu \alpha} - \frac{1}{6} \delta^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right).
\]  

(17)
For the sake of simplicity, let us suppose that the five-dimensional black hole possesses only one angular momentum (i.e. \( b = 0 \)) and consider an ansatz for the stationary metric in the Boyer-Lindquist type form, that is involving the only off-diagonal component with indices \( t \) and \( \phi \). Namely, we start with the metric ansatz

\[
\begin{align*}
 ds^2 &= -(1 - H) \, dt^2 + \frac{\Sigma}{\Delta} \, dr^2 + \Sigma \, d\theta^2 - 2 \, aH \sin^2 \theta \, dt \, d\phi \\
 &\quad + \left( r^2 + a^2 + Ha^2 \sin^2 \theta \right) \sin^2 \theta \, d\phi^2 + r^2 \cos^2 \theta \, d\psi^2 ,
\end{align*}
\]

(18)

where we have introduced an arbitrary scalar function \( H \) depending only on the coordinates \( r \) and \( \theta \) and the function \( \Delta \) is given by

\[
\Delta = r^2 + a^2 - H \Sigma .
\]

(19)

Performing straightforward calculations it is easy to show that the five-vector potential \([10]\) with \( b = 0 \) satisfies the Maxwell equation \([14]\) in the metric \([18]\) independently on the explicit form of the function \( H \). Thus, one can use the relations \([11]\) and \([12]\) to calculate the non-vanishing components of the source tensor \([17]\). We find that

\[
\begin{align*}
 M^t_t &= \frac{Q^2}{6\pi^2\Sigma^4} \left( 2\Sigma - 4r^2 - 3a^2 \right) , & M^r_r &= -\frac{Q^2}{6\pi^2\Sigma^4} \left( \Sigma + r^2 \right) , \\
 M^\theta_\theta &= \frac{Q^2}{6\pi^2\Sigma^4} \left( 2\Sigma - r^2 \right) , & M^\phi_\phi &= -\frac{Q^2}{6\pi^2\Sigma^4} \left( \Sigma - 2r^2 - 3a^2 \right) , \\
 M^\psi_\psi &= -\frac{Q^2}{6\pi^2\Sigma^4} \left( \Sigma - 2r^2 \right) , & M^t_\phi &= -\frac{Q^2 a}{2\pi^2\Sigma^4} = -\frac{M^t_\phi}{(r^2 + a^2) \sin^2 \theta} .
\end{align*}
\]

(20)

One can easily check that there exists the relation

\[
M^t_t + M^\phi_\phi + M^\psi_\psi = 0
\]

(21)

between the components of the source tensor.

On the other hand, the non-vanishing components of the Ricci tensor calculated for the metric \([18]\) have the form

\[
\begin{align*}
 R^t_t &= \frac{r^2 + a^2}{2\Sigma} \left( H_{rr} + \frac{H_{\theta \theta}}{\Delta} + \frac{\Sigma}{\Delta^2} H^2_{\theta} + \frac{3r^2 + a^2}{r^2 + a^2} \frac{H_{r}}{r} \right) - \frac{a^2 H^2}{\Delta \Sigma} \left( \frac{2 \cos 2\theta - \frac{a^2 H \sin^2 2\theta}{2\Delta}}{2\Sigma} \right) \\
 &\quad + \frac{H_{\theta}}{\Delta \Sigma} \left[ (r^2 + a^2) \cot 2\theta - a^2 H \sin 2\theta \left( \frac{1}{2} + \frac{r^2 + a^2}{\Delta} \right) \right] ,
\end{align*}
\]

(22)
\[ R^r_r = \frac{1}{2} \left( H_{rr} - \frac{H_{\theta \theta}}{\Delta} - \frac{\Sigma}{\Delta^2} H_\theta^2 + \frac{\Sigma + 2r^2}{\Sigma} H_r - \frac{H_\theta}{\Delta} \right) - \frac{H_\theta}{\Delta} \left( \cot 2\theta - \frac{r^2 + a^2}{\Delta \Sigma} a^2 \sin 2\theta \right) \\
+ \frac{a^2 H}{\Delta^2 \Sigma} \left[ (r^2 + a^2) \left( 1 + 3 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{2\Sigma} \right) + 2H^2 \Sigma^2 \cos^2 \theta \right] \\
- \frac{2a^2 H^2}{\Delta^2} \left( 1 + 2 \cos 2\theta + \frac{3a^2 \sin^2 2\theta}{4\Sigma} \right), \quad (23) \]

\[ R^\theta_\theta = \frac{r}{\Sigma} H_r - \frac{H_\theta}{2\Delta^2} \left( \Sigma H_\theta - 2a^2 H \sin 2\theta \right) + \frac{2r^2 H}{\Delta^2 \Sigma^2} \left[ (r^2 + a^2)^2 + \Sigma^2 H^2 \right] \\
- \frac{H^2}{2\Delta^2 \Sigma} \left[ 8r^2 (r^2 + a^2) - a^4 \sin^2 2\theta \right], \quad (24) \]

\[ R^\phi_\phi = -\frac{a^2 \sin^2 \theta}{2\Sigma} \left( H_{rr} + \frac{H_{\theta \theta}}{\Delta} + \frac{\Sigma}{\Delta^2} H_\theta^2 \right) - \frac{2r^2 - a^2 \sin^2 \theta}{2\Sigma} H_r - \frac{a^2 H^2 \sin 2\theta}{2\Delta^2} H_\theta \\
- \frac{a^2 H_\theta \tan \theta}{2\Delta^2} \left[ \frac{r^2 + a^2}{\Sigma} \left( 2 + 3 \cos 2\theta \right) - H \left( 3 + 4 \cos 2\theta + \frac{3a^2 \sin^2 2\theta}{2\Sigma} \right) \right] \\
- \frac{2H^2}{\Delta^2 \Sigma} \left[ a^4 \sin^2 \theta \left( 1 + \cos^2 \theta \right) + 2r^2 \left( r^2 + a^2 + a^2 \sin^2 \theta \right) \right] \\
+ \frac{2H (r^2 + a^2)}{\Delta^2} \left( r^2 + 2a^2 \sin^2 \theta + r^2 H^2 \right), \quad (25) \]

\[ R^\phi_t = \frac{a}{2\Sigma} \left[ \left( H_{rr} + \frac{H_r}{r} + \frac{H_{\theta \theta}}{\Delta} + \frac{\Sigma}{\Delta^2} H_\theta^2 \right) - \frac{2H}{\Delta^2} \left( r^2 + a^2 + r^2 H^2 - H(a^2 + 2r^2) \right) \right] \\
+ \frac{aH_\theta}{\Delta^2 \sin 2\theta} \left[ \frac{r^2 + a^2}{\Sigma} \left( 1 + 2 \cos 2\theta \right) - H \left( 2 + 3 \cos 2\theta + \frac{3a^2 \sin^2 2\theta}{2\Sigma} \right) \right] \\
+ \frac{aH^2 \cot \theta}{\Delta^2}, \quad (26) \]

\[ R^t_\phi = -(r^2 + a^2) \sin^2 \theta R^\phi_t + \frac{aH \sin 2\theta}{2\Sigma} \left( \frac{a^2 H \sin 2\theta}{\Sigma} - H_\theta \right), \quad (27) \]

\[ R^r_\theta = \Delta R^\theta_r = \frac{1}{2} \left( H_{r \theta} + \frac{H_\theta}{r} \right) - \frac{a^2 \sin 2\theta}{2\Sigma} \left( H_r + \frac{\Sigma - 2r^2}{r \Sigma} H \right), \quad (28) \]

\[ R^\psi_\psi = \frac{1}{r \Sigma} \frac{\partial}{\partial r} (\Sigma H), \quad (29) \]
where the indices $r$ and $\theta$ at the scalar function $H$ denote its differentiation with respect to the coordinates $r$ and $\theta$. Substituting the quantities (20) and (22)-(29) into Einstein’s equation (13) we see that one can easily solve the equation with $R^\phi_\psi$ and $M^\phi_\psi$, however the solution does not satisfy the remaining set of equations. Thus, there is no scalar function $H$ fulfilling both equations (13) and (14) simultaneously. Earlier this fact was also noted in [9] within the Kerr-Schild type metric ansatz.

IV. SLOW ROTATION

Let us now consider the case when the rotation of the black hole occurs slowly enough that we can expand all the expressions above in powers of the rotation parameter restricting ourselves only to the linear order in $a$ terms. Then we arrive at the following set of the field equations

\[
H_{rr} + \frac{3H_r}{r} + \frac{H_{\theta \theta}}{\Delta} + \frac{r^2}{\Delta^2} H^2 + \frac{2 \cot 2\theta}{\Delta} H_{\theta} = -\frac{16G}{3\pi} \frac{Q^2}{r^6},
\]

\[
H_{rr} + \frac{3H_r}{r} - \frac{H_{\theta \theta}}{\Delta} - \frac{r^2}{\Delta^2} H^2 - \frac{2 \cot 2\theta}{\Delta} H_{\theta} = -\frac{16G}{3\pi} \frac{Q^2}{r^6},
\]

\[
H_r - \frac{r^3}{2\Delta^2} \left[H^2 - 4H(1-H)^2\right] = \frac{4G}{3\pi} \frac{Q^2}{r^5},
\]

\[
H_r + \frac{2r^3H}{\Delta^2} (1-H)^2 = \frac{4G}{3\pi} \frac{Q^2}{r^5},
\]

\[
H_{rr} + \frac{H_r}{r} + \frac{H_{\theta \theta}}{\Delta} + \frac{r^2}{\Delta^2} H^2 - \frac{4r^2H}{\Delta^2} (1-H)^2
\]

\[
+ \frac{2r^2(1-H)}{\Delta^2 \sin 2\theta} H_{\theta} \left(1 + 2 \cos 2\theta - 2H \cos^2 \theta\right) = \frac{8G}{\pi} \frac{Q^2}{r^6},
\]

\[
H_r + \frac{2}{r} H = \frac{4G}{3\pi} \frac{Q^2}{r^5},
\]

\[
H_{r\theta} + \frac{H_{\theta}}{r} = 0.
\]

The solution to these equations has the simple form

\[
H = \frac{m}{r^2} - \frac{q^2}{r^4},
\]
where in addition to the mass parameter, we have also introduced the charge parameter

\[ q = \pm \sqrt{\frac{2G}{3\pi}} Q. \]  

Finally, substituting the expression (31) into equations (18) and (19), up to terms linear in \( a \), we obtain the metric for the charged and slowly rotating black hole in five dimensions

\[
\begin{aligned}
ds^2 &= - \left( 1 - \frac{m}{r^2} + \frac{q^2}{r^4} \right) dt^2 + \left( 1 - \frac{m}{r^2} + \frac{q^2}{r^4} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2 \right) \\
&\quad - \frac{2a}{r^2} \left( m - \frac{q^2}{r^2} \right) dt d\phi,
\end{aligned}
\]

while the associated electromagnetic field is given by the 2-form

\[
F = \frac{Q}{r^3} \left[ d r \wedge d t - a \sin \theta \left( \sin \theta d r - r \cos \theta d \theta \right) \wedge d \phi \right].
\]

This metric generalizes the Schwarzschild-Tangherlini solution to the case of a five-dimensional slowly rotating black hole with one angular momentum. In the similar way, one can also write down a corresponding five-dimensional solution with two independent angular momenta. This will be done in a future work.

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