A regularity result for the nonlocal Fokker-Planck equation with Ornstein-Uhlenbeck drift

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Abstract. We are devoted to the study of nonlocal Fokker-Planck equations for a class of stochastic differential equations with non-Gaussian \(\alpha\)-stable Lévy motion in Euclidean space. More precisely, in cases of free Lévy motion and Lévy motion with Ornstein-Uhlenbeck drift, we prove that there exists a unique solution, and it is smooth in space for all \(0 < \alpha < 2\).

Keywords. Fractional Laplacian operator, Non-Gaussian Lévy noise, Nonlocal Fokker-Planck equation, free Lévy motion, Ornstein-Uhlenbeck drift.

1 Introduction

For a system described by a scalar stochastic differential equation with a random source denoted by \(\tilde{X}_t\), and a drift term given by a deterministic function \(f\),

\[
dX_t = f(X_t) dt + d\tilde{X}_t,
\]

the corresponding Fokker-Planck equation, for the probability density of \(X_t\), is

\[
\partial_t p = Ap - \nabla \cdot (f(x)p)_x.
\]

Both the Markov process \(X_t\) and its probability density function \(p\) have been widely known and well understood when \(\tilde{X}_t\) in equation (1.1), and \(A\) in equation (1.2) are the Gaussian process, such as the standard Brownian motion, and the differential operator, such as the usual Laplacian operator, respectively. However, many transport processes in physical, biological and social systems exhibit anomalous diffusion, resulting from non-Gaussian random sources (see [7, 12]). It is shown in [4] that for every \(\alpha \in (0, 2)\), when \(\tilde{X}_t\) in (1.1) is the \(\alpha\)-stable Lévy process (non-Gaussian process), then \(A\) in (1.2) is the fractional Laplacian operator (integral operator),

\[
\Delta^\alpha := -(-\Delta)^{\alpha, \text{P.V.}}u(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|x - y|^{d+\alpha}} dy \quad \text{for } \alpha \in (0, 2),
\]

where \(C_{d,\alpha}\) is a constant depending on \(d\) and \(\alpha\). Equivalently, it can also be defined by the Fourier transform

\[
\mathcal{F}(\Delta^{\alpha/2}u(t, \cdot))(\xi) = |\xi|^\alpha \mathcal{F}(u(t, \cdot))(\xi) \quad \text{for } \alpha \in (0, 2).
\]

This paper focuses on this nonlocal Fokker-Planck equation on the \(d\)-dimensional Euclidean space \(\mathbb{R}^d\), with the Ornstein-Uhlenbeck drift \(f(x) = -x\), and \(L^\infty\) bounded initial value. More precisely, we will show that
for all α in (0, 2), the solution is well-posed, and smooth for all x in \( \mathbb{R}^d \). Regarding the well-posedness and regularity of solutions on \( \mathbb{R}^d \) in case α is in (1, 2), there have been many results. In particular, Drouihou, Imbert, and others proved in [3, 10] that for some drift terms and proper initial data, the unique global solution has certain kind of smoothness. However, the Ornstein-Uhlenbeck drift, which does not satisfy their conditions on the drift term, is excluded (see Remark 2.3 for more details). In the case of α = 1, there are still regularity results, but the proof is much harder. Finally, when α is in (0, 1), the most recent regularity result is for the case of \( f(x) \equiv 0 \)(see Remark 2.4 for more details).

In an upcoming paper, we are going to show the existence of smooth solutions to equation (1.2) with general unbounded \( f(x) \), such as the double-well drift when \( f(x) = x - x^3 \). Moreover, not too much is known to the boundary value problem of the nonlocal Fokker-Planck equation, but in [9], the authors show the existence of a unique global solution in a weaker sense.

2 Regularity results for nonlocal Fokker-Plank equations

Through our this section, we will consider equation (1.2) with \( A \) being the fractional Laplacian operator (defined in (1.3)) on \( \mathbb{R}^d \). First, we consider the case of free Lévy motion,

\[
\begin{cases}
\partial_t u(t, x) = \Delta^{\alpha/2} u(t, x), & t > 0, x \in \mathbb{R}^d, \\
u(0, x) = u_0(x).
\end{cases}
\]  

(2.1)

Suppose that \( \hat{p}(t, x) \) is the fundamental solution to (2.1), then the solution of (2.1) can be denoted as

\[ u(t, x) = (\hat{p} * u_0)(x). \]

(2.2)

We show that the solution of (2.1) defined as in (2.2) is smooth in the following theorem.

**Theorem 2.1.** Let \( u_0 \in L^\infty(\mathbb{R}^d) \) and \( u_0(x) \) be continuous almost everywhere. The function \( u(t, x) \) defined in (2.2) is the solution to (2.1). Furthermore, for \( T > 0 \) or \( T = \infty \), we have

\[ u(t, \cdot) \in C^\infty(\mathbb{R}^d), \quad \forall t \in (0, T]. \]

(2.3)

**Remark 2.1.** When \( \alpha \in (1, 2) \) and \( u_0 \in C_0(\mathbb{R}^d) \), it is shown that the solution to (2.1) is \( C^\infty \) (in \( (t, x) \)) in [10] Prop 1. And when \( \alpha \in (0, 2) \) and \( u_0 \in L^2(\mathbb{R}) \), it is shown that the solution to (2.1) is \( L^\infty(\mathbb{R}) \) for any given \( t > 0 \) in [2].

**Remark 2.2.** The conditions of the initial data \( u_0 \) are natural. Suppose the solution \( u(t, x) \) is considered as the density of a particle at position \( x \) and time \( t \), \( u_0 \) is then the probability profile of its initial position, e.g., Gaussian \( \langle u_0(x) = \sqrt{\frac{M}{\pi}} e^{-\frac{M}{\pi} x^2} \rangle \) or uniform \( \langle u_0(x) = \frac{1}{2} I_{[x < 1]} \rangle \) (see [5]).

Second, we consider the nonlocal Fokker-Plank equation with the Ornstein-Uhlenbeck drift \( f(x) = -x \),

\[
\begin{cases}
\partial_t u = \Delta^{\alpha/2} u - \nabla \cdot (-xu), & t > 0, x \in \mathbb{R}^d, \\
u(0, x) = u_0(x).
\end{cases}
\]

(2.4)

Suppose that \( p(t, x) \) is the fundamental solution to (2.4), then the solution of (2.4) can be denoted as

\[ u(t, x) = (p * u_0)(x). \]

(2.5)

To overcome the difficulties due to the unbounded function \( f(x) \), we apply the following relation (see [6, 8, 11])

\[ p(t, x, y) = e^{\frac{\alpha t}{2}} \hat{p} \left( \frac{e^{\alpha t} - 1}{\alpha}, e^t x, y \right) \]

(2.6)
where \( \hat{p} \) and \( p \) are the fundamental solutions of (2.3) and (2.4), respectively. Thanks to (2.6), the solution of (2.4) has the same regularity as that of (2.1), as shown in the following theorem.

**Theorem 2.2.** Let \( u_0 \in L^\infty(\mathbb{R}^d) \) and \( u_0(x) \) is continuous almost everywhere, The function \( u \) defined in (2.3) is the solution to (2.4). Furthermore, for \( T > 0 \) or \( T = \infty \),

\[
   u(t, \cdot) \in C^\infty(\mathbb{R}^d), \quad \forall t \geq 0. \tag{2.7}
\]

**Remark 2.3.** When \( 1 < \alpha < 2 \), the fractional Laplacian operator was proved to have certain smoothing effect. For example, the author in [3, Theorem 1.1] showed that the solution is \( L^\infty \) continuous, if the initial data \( u_0 \) is the solution to (2.3). Theorem 2.2 shows the smoothing effect of the fractional Laplacian operator was proved to have certain smoothing effect. For example, the author in [3, Theorem 1.1] showed that the solution is \( L^\infty \) continuous, if the initial data \( u_0 \) is the solution to (2.3). Theorem 2.2 shows the smoothing effect of the fractional Laplacian operator was proved to have certain smoothing effect. For example, the author in [3, Theorem 1.1] showed that the solution is \( L^\infty \) continuous, if the initial data \( u_0 \) is the solution to (2.3).

To prove Theorem 2.1, the following properties of \( \hat{p}(t, x) \) are essential (see [1, 2, 3, 10] for more details).

- for any \( t > 0 \),
  \[
  \|\hat{p}(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1 \text{ for all } t > 0. \tag{2.8}
  \]
- \( \hat{p}(t, x, y) \) is \( C^\infty \) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\) for each \( t > 0 \);
- for \( t > 0 \), \( x, y \in \mathbb{R}^d, x \neq y \), the sharp estimate of \( \hat{p}(t, x) \) is
  \[
  \hat{p}(t, x, y) \approx \min \left( \frac{t}{|x - y|^{d+\alpha}}, t^{-\frac{d}{\alpha}} \right); \tag{2.9}
  \]
- for \( t > 0 \), \( x, y \in \mathbb{R}^d, x \neq y \), the estimate of the first order derivative of \( \hat{p}(t, x) \) is
  \[
  |\nabla_x \hat{p}(t, x, y)| \approx |y - x| \min \left\{ \frac{t}{|y - x|^{d+2+\alpha}}, t^{-\frac{d+2}{\alpha}} \right\}. \tag{2.10}
  \]

The notation \( f(x) \approx g(x) \) means that there is a number \( 0 < C < \infty \) independent of \( x \), i.e. a constant, such that for every \( x \) we have \( C^{-1} f(x) \leq g(x) \leq C f(x) \). The estimate (2.9) for the fundamental solution \( \hat{p}(t, x) \) plays a key role to get the regularity results. And the estimate (2.10) for the first order derivative of \( \hat{p}(t, x) \) was derived in [1, Lemma 5]. More generally, we apply the similar idea as in the derivation of (2.10), and get the estimate of the \( m \)-th order derivative of \( \hat{p}(t, x) \) by induction.

**Lemma 2.1.** For any \( m \geq 0 \), we have

\[
  \partial_x^m \hat{p}(t, x) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} C_n |x - y|^{m-2n} \min \left\{ \frac{t}{|x - y|^{d+\alpha+2(m-n)}}, t^{-\frac{d+2(m-n)}{\alpha}} \right\}, \tag{2.11}
\]

where \( \lfloor \frac{m}{2} \rfloor \) means the largest integer that is less than \( \frac{m}{2} \).
Proof. By (2.9), it is equivalently to show that for any $x_j (1 \leq j \leq d)$ and $m \geq 0$,

$$\partial_x^m \hat{p}(t, x) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} C_n x_j^{m-2n} \hat{p}^{(d+2(m-n))}(t, x^{(d+2(m-n))}),$$

(2.12)

where $x \in \mathbb{R}^d$, $x^{(k)} \in \mathbb{R}^k$, and $\hat{p}(t, x)$, $\hat{p}^{(k)}(t, x^{(k)})$ are the probability density functions in the corresponding state spaces. Then we prove (2.12) by induction. First, by (2.9) and (2.10), (2.12) holds in the cases of $m = 0$, and $m = 1$, respectively. Second, suppose (2.12) is true for $m = 2k$, that is

$$\partial_x^{2k} \hat{p}(t, x) = \sum_{n=0}^{k} C_n x_j^{(2k-2n)} \hat{p}^{(d+2(2k-n))}(t, x^{(d+2(2k-n)))}.$$ 

(2.13)

Let $g(t, x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ be the Gaussian kernel, and $\eta(t, u)$ be the density function of the $\alpha/2$-stable subordinator at time $t$, which has the following properties for all $u > 0$,

$$\eta(1, u) \leq C_1 u^{-1-\frac{\alpha}{2}}, \text{ and } \eta(t, u) \leq C_1 tu^{-1-\frac{\alpha}{2}}.$$

For $x \in \mathbb{R}^d \setminus \{0\}$, by the subordination formula (see [2]), we have

$$\hat{p}(t, x) = \int_0^\infty g(u, x) \eta(t, u) du.$$ 

Hence

$$\hat{p}^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}) = \int_0^\infty g^{(d+2(2k-n))}(u, x^{(d+2(2k-n))}) \eta(t, u) du.$$ 

Note that $g^{(d+2(2k-n))}(t, \cdot)$ is $C^\infty(\mathbb{R}^{d+2(2k-n)})$. Then by the mean value theorem and dominate convergence theorem, it is not hard to see that

$$\partial_x^j \hat{p}^{(d+2(2k-n))}(t, x^{(d+2(2k-n))}) = -2\pi x_j \hat{p}^{(d+2(2k-n)+2)}(t, x^{(d+2(2k-n)+2)}).$$

(2.14)

Together with (2.14) and (2.13), we have for $m = 2k + 1$,

$$\partial_x^{2k+1} \hat{p}(t, x) = \partial_x^j \sum_{n=0}^{k} C_n x_j^{(2k+1-2n)} \hat{p}^{(d+2(2k-n))}(t, x^{(d+2(2k-n)))}$$

$$= \sum_{n=0}^{k} C_n x_j^{(2k+1-2n)} \hat{p}^{(d+2(2k+1-n))}(t, x^{(d+2(2k+1-n)))}. $$

Hence, (2.12) follows. By (2.9), the proof is concluded. \qed

Proof of Theorem 2.3

Proof. First, we show that $u(t, \cdot) \in C^\infty(\mathbb{R}^d)$ for all $t \in (0, T]$. Since $u_0 \in L^\infty(\mathbb{R}^d)$, and $\hat{p}(t, \cdot) \in L^1(\mathbb{R}^d)$ for $t > 0$, $\hat{p}(t, \cdot) * u_0$ is well-defined. By the Yong’s inequality for the convolution in (2.2), we have

$$\forall (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad \| \hat{p}(t, \cdot) \ast u_0(x) \|_{L^1(\mathbb{R}^d)} \leq \| u_0 \|_{L^\infty(\mathbb{R}^d)}. $$

(2.15)

By the smoothness of $\hat{p}(t, x, y)$, to show (2.2), it is sufficient to show that

$$| \nabla_x^m \hat{p}(t, x - y) | \in L^1(\mathbb{R}^d), \quad \forall m \geq 0, t_0 < t < T, \text{ and } x, y \in \mathbb{R}^d.$$
Indeed, by Lemma 2.1 we have
\[
\int_{\mathbb{R}^d} |\nabla_x \hat{m}(t, x - y)| |dy| \leq \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \int_{\mathbb{R}^d} |x - y|^{-2n} \min \left\{ \frac{t}{|x - y|^{d + \alpha + m}} \right\} |dy|
\]
\[
= \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \int_{B(x, r)} |x - y|^{-2n} \frac{t}{|x - y|^{d + \alpha + m}} |dy| + \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \int_{B(x, r)^c} |x - y|^{-2n} \frac{t}{|x - y|^{d + \alpha + m}} |dy|
\]
\[
\leq \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{t}{|x - y|^{d + \alpha + m}} \int_{B(x, r)} |x - y|^{-2n} |dy| + \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} T \int_{B(x, r)^c} |x - y|^{-2n} |dy|
\]
\[< \infty.\]

Sine \( u_0 \in L^\infty(\mathbb{R}^d) \) and \( \tilde{p}(t, \cdot) \in C^\infty(\mathbb{R}^d) \), the theorem of continuity under the integral sign gives (2.4).

Second, we show that \( u(t, x) \) defined in (2.1) is the unique solution to (2.1). It is easily to see that \( u(t, x) = (\hat{p} * u_0)(x) \) satisfies the equation in (2.1) for \( x \in \mathbb{R}^d \) and \( t \in (0, T) \). And the uniqueness follows automatically from the Fourier Transform. In fact, it can be seen that the solution to (2.1) is
\[
\hat{u}(t, \xi) = \hat{u}_0(\xi)e^{-t|\xi|^\alpha}. \tag{2.16}
\]

Hence, to finalize the proof, we only need to show that for \( x^0 \) at which \( u_0(x) \) is continuous, we have
\[
\lim_{t \to 0, x \to x_0} u(t, x) = u_0(x^0). \tag{2.17}
\]

Fix \( x^0 \in \mathbb{R}^d, \epsilon > 0 \). Suppose \( u_0(x) \) is continuous at \( x_0 \). Choose \( \delta > 0 \) such that
\[
|u_0(y) - u(x^0)| < \epsilon \quad \text{if} \quad |y - x^0| < \delta, \quad \text{and} \quad y \in \mathbb{R}^d.
\]

Then if \( |x - x^0| < \frac{\delta}{2} \), we have
\[
|u(t, x) - u_0(x^0)| \leq \int_{\mathbb{R}^d} \hat{p}(t, x - y)u_0(y) - u_0(x^0) |dy|
\]
\[
= \int_{B(x^0, \delta)} \hat{p}(t, x - y)u_0(y) - u_0(x^0) |dy| + \int_{B(x^0, \delta)^c} \hat{p}(t, x - y)u_0(y) - u_0(x^0) |dy|
\]
\[
\leq \epsilon + 2\|u_0\|_{L^\infty} \int_{B(x^0, \delta)^c} \frac{t}{|x - y|^{d + \alpha}} |dy|
\]

In the second term, we have \( |y - x^0| > \delta \). Hence
\[
|y - x^0| \leq |y - x| + |x^0 - x| < |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|.
\]

Thus \( |y - x| \geq \frac{1}{2}|y - x^0| \). Consequently, the second term in the last line tends to 0 as \( t \to 0^+ \). Hence, if \( |x - x^0| < \frac{\delta}{2} \), and \( t > 0 \) is small enough, \( |u(t, x) - u_0(x^0)| < 2\epsilon \). \( \square \)

Proof of Theorem 2.2

Proof. Set \( \tilde{t} = \frac{e^{\alpha t} - 1}{\alpha} \) and \( \tilde{x} = e^{\alpha t} \). By (2.4), we have
\[
\nabla_x^m p(t, x, y) = e^{\alpha t} e^{-\alpha t} \nabla_x^m \hat{p}(\tilde{t}, \tilde{x}, y), \tag{2.18}
\]
Hence by Lemma 2.1 we have
\[
\int_{\mathbb{R}^d} |\nabla_x p(t, x - y)| \, dy \leq e^{(d+m)t} \sum_{n=0}^{n=\lfloor m/2 \rfloor} \int_{\mathbb{R}^d} |\tilde{x} - y|^{m-2n} \min \left\{ \frac{\tilde{t}}{|\tilde{x} - y|^{d+\alpha}} \right\} \, dy
\]
\[
= e^{(d+m)t} \sum_{n=0}^{n=\lfloor m/2 \rfloor} \int_{B(\tilde{x}, \tilde{r})} |\tilde{x} - y|^{m-2n} \tilde{y}^{-\frac{d+2(m-n)}{m}} \, dy + \sum_{n=0}^{n=\lfloor m/2 \rfloor} \int_{B(\tilde{x}, \tilde{r})^c} \frac{\tilde{t}}{|\tilde{x} - y|^{d+\alpha}} \, dy
\]
\[
\leq e^{(d+m)t} \sum_{n=0}^{n=\lfloor m/2 \rfloor} \tilde{t}^{-\frac{d+2(m-n)}{m}} \int_{B(\tilde{x}, \tilde{r})} |\tilde{x} - y|^{m-2n} \, dy + \sum_{n=0}^{n=\lfloor m/2 \rfloor} \tilde{t} \int_{B(\tilde{x}, \tilde{r})^c} \frac{1}{|\tilde{x} - y|^{d+\alpha}} \, dy
\]
\[
< \infty
\]
for any $0 < t_0 < t < T$. The rest of the proof is similar to that of Theorem 2.1. Hence, we omit it. \(\square\)

References

[1] Bogdan, Krzysztof; Jakubowski, Tomasz. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. Comm. Math. Phys. 271 (2007), no. 1, 179-198.

[2] Bogdan, Krzysztof; Stós, Andrzej; Sztonyk, Pawe. Harnack inequality for stable processes on d-sets. Studia Math. 158 (2003), no. 2, 163-198.

[3] Droniou, Jérome; Gallouët, Thierry; Vovelle, Julien. Global solution and smoothing effect for a non-local regularization of a hyperbolic equation. J. Evol. Equ. 3 (2003), no. 3, 499-521.

[4] Duan, Jinqiao. An Introduction to Stochastic Dynamics. Cambridge University Press, New York, 2015.

[5] Gao, Ting; Duan, Jinqiao; Lí, Xiaofan. Fokker-Plank equations for stochastic dynamical systems with symmetric Lévy motions.

[6] Gentil, Ivan; Imbert, Cyril. The Lévy-Fokker-Planck equation: Φ-entropies and convergence to equilibrium. Asymptot. Anal. 59 (2008), no. 3-4, 125-138.

[7] Gentil, Ivan; Imbert, Cyril. Logarithmic Sobolev inequalities: regularizing effect of Lévy operators and asymptotic convergence in the Lévy-Fokker-Planck equation. Stochastics 81 (2009), no. 3-4, 401-414.

[8] Garbaczewski, Piotr; Olkiewicz, Robert. Ornstein-Uhlenbeck-Cauchy process. J. Math. Phys. 41 (2000), no. 10, 6843-6860.

[9] He, Jinchun; Duan, Jinqiao; Gao, Hongjun. A nonlocal Fokker-Plank equation for non-Gaussian stochastic dynamical systems. To appear in Applied Math. Lett., 2015.

[10] Imbert, Cyril. A non-local regularization of first order Hamilton-Jacobi equations, J. Differential Equations 211 (2005), no. 1, 218-246.

[11] Jakubowski, Tomasz. The estimates of the mean first exit time from a ball for the α-stable Ornstein-Uhlenbeck processes. Stochastic Process. Appl. 117 (2007), no. 10, 1540-1560.

[12] West, Bruce J.; Grigolini, Paolo; Metzler, Ralf; Nonnenmacher, Theo F.. Fractional diffusion and Lévy stable processes. Phys. Rev. E (3) 55 (1997), no. 1, part A, 99106.