THURSTON EQUIVALENCE OF TOPOLOGICAL POLYNOMIALS

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Abstract. We answer Hubbard’s question on determining the Thurston equivalence class of “twisted rabbits”, i.e. composita of the “rabbit” polynomial with n-th powers of the Dehn twists about its ears. The answer is expressed in terms of the 4-adic expansion of n.

We also answer the equivalent question for the other two families of degree-2 topological polynomials with three post-critical points.

In the process, we rephrase the questions in group-theoretical language, in terms of wreath recursions.

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\section{Introduction}

Consider the “rabbit” polynomial $f_R(z) \approx z^2 + (-0.1226 + 0.7449i)$, whose critical point 0 is on a periodic orbit of length 3. Up to affine transformations, there are exactly two other polynomials with same action on post-critical points (with the same ramification graph), called the “corabbit” $f_C \approx z^2 + (-0.1226 - 0.7449i)$ and the “airplane” $f_A \approx z^2 - 1.7549$. Furthermore, by a result of W. Thurston (see below), every branched covering with same ramification graph is equivalent to precisely one of $f_R, f_C, f_A$.

Consider now a Dehn twist $T$ of $C$ around the two non-critical values of the $f_R$-orbit of 0. The map $T^m f_R$ is again a branched covering, and it has the same ramification graph as $f_R$; therefore it is equivalent (i.e., conjugate up to homotopies) to one of $f_R, f_C, f_A$. Which one?

This question was asked by J. Hubbard, see [Pil03]. The answer, as we shall show (Theorem 4.7), is the following:

Write $m$ in base 4, as $m = \sum_{i=0}^{\infty} m_i 4^i$ with $m_i \in \{0, 1, 2, 3\}$ and almost all $m_i = 0$ if $m$ is non-negative, and almost all $m_i = 3$ if $m$ is negative. If one of the $m_i$ is 1 or 2, then $T^m f_R$ is equivalent to $f_A$. Otherwise, it is equivalent to $f_R$ if $m$ is non-negative, and to $f_C$ if $m$ is negative.

For example, $T^{-1} f_R$ and $T^{-4} f_R$ are equivalent to $f_C$, since $-1 = \ldots 333$ and $-4 = \ldots 3330$.

Consider now the polynomial $f_i = z^2 + i$, whose finite critical point has orbit $0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$. A branched covering with that ramification graph is either equivalent to $f_i$, or to $f_{-i}$, or is not equivalent to any rational map (it is obstructed). A. Douady and J. Hubbard ask ([DH93 page 293]) to determine, as a function of a Dehn twist $D$, whether $f_i \cdot D$ is obstructed, and if not, whether it is equivalent to $f_i$ or to $f_{-i}$. The answer (see Theorem 6.1) depends on the image of $D$ in a finite group of order 100.

Thurston’s theorem does not tell us when two obstructed maps are equivalent. We may however also answer that question: Corollary 6.11 shows that there are infinitely many inequivalent obstructed maps with the same ramification graph as $f_i$, and gives an algorithm to determine equivalence among obstructed maps.

The first construction of infinitely many non-equivalent Thurston maps with the same ramification graph was presented in [BBL+00 Proposition 2.12].

### 1.1. Thurston’s theorem

Consider more generally a branched covering $f$ of $S^2$, with set of critical points $C_f$, and let $P_f$ be its post-critical set: $P_f = \bigcup_{n \geq 1} f^n(C_f)$. Let us suppose that $P_f$ is finite, in which case $f$ is called post-critically finite, or a Thurston map. Two Thurston maps $f, g$ are equivalent if there exist orientation-preserving homeomorphisms $\phi_0, \phi_1 : (S^2, P_f) \rightarrow (S^2, P_g)$ such that $\phi_0$ and $\phi_1$ are
isotopic relative to $P_f$, and $\phi_0 f = g \phi_1$. Recall also that a branched covering is a topological polynomial if $f^{-1}(\infty) = \{\infty\}$.

A multicurve is a system $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of simple, closed, disjoint, non-homotopic, non-peripheral curves on $S^2 \setminus P_f$. Here a curve is called peripheral if one of the two parts into which it divides the sphere contains less than two post-critical points. The multicurve $\Gamma$ is stable if for all $\gamma \in \Gamma$, all non-peripheral elements of $f^{-1}(\gamma)$ are homotopic to elements of $\Gamma$. There is then an induced map

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma, \quad \gamma_i \mapsto \sum_{\delta \in f^{-1}(\gamma_i)} \frac{[\delta]}{\deg f|_\delta},$$

where $[\delta]$ is, if it exists, the element of $\Gamma$ homotopic to $\delta$, and is 0 otherwise.

**Theorem 1.1** (Thurston’s criterion; see [DH93]). A Thurston map $f$ with hyperbolic orbifold is equivalent to a rational function if and only if the spectral radius of $f_\Gamma$ is $< 1$ for all stable multicurves $\Gamma$.

In that case, the rational function equivalent to $f$ is unique up to conjugation by a Möbius transformation.

A stable multicurve $\Gamma$ is an obstruction if the spectral radius of $f_\Gamma$ is greater than or equal to 1. Therefore, Thurston’s Theorem says that a Thurston map is equivalent to a rational map if and only if there are no obstructions.

If the Thurston map $f$ is a topological polynomial, then the structure of obstructions is better understood. Namely, every Thurston obstruction of a topological polynomial contains a Levy cycle (see [BFH92]), i.e., a multicurve $\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ such that the only non-peripheral component of $f^{-1}(\gamma_i)$ is homotopic to $\gamma_{i-1}$ and the maps $f : \gamma_{i-1} \to \gamma_i$ are of degree 1 for every $i$ (here the indices $i$ are considered modulo $n$).

Thurston’s theorem does not, in principle, provide an algorithmic answer to the question when a Thurston map $f$ is equivalent to a rational function; nor does it construct the rational function. Many attempts were made to that end, notably [Pil03] and [Kam01]. The present paper may be seen as another step in this direction.

**1.2. Sketch of the method.** Given a post-critically finite branched covering $f$, we associate with it a finitely generated group acting faithfully on a rooted tree. Its action is given recursively by self-similar tree isometries. This group, the iterated monodromy group of $f$, is an invariant for the branched covering up to equivalence. In favourable (“contracting”) cases, the “nucleus” of its action is a finite-state automaton characterising the group. This gives an effective method to solve J. Hubbard’s question for any given $m$.

Let $P$ be the post-critical set of $f$. The (pure) mapping class group $\mathcal{G}_C$ of $C = \mathbb{C} \setminus P$ acts on the branched coverings with post-critical set $P$ by pre- and post-composition. The action of $\mathcal{G}_C$ can be pushed through the recursion to give a self-map $\psi$ (almost a homomorphism) of $\mathcal{G}_C$ such that $f \cdot g$ and $f \cdot \psi(g)$ are combinatorially equivalent. The map $\psi$ is contracting in the case of the “rabbit” polynomial: there is a finite set $L \subset \mathcal{G}_C$ such that for any $g \in \mathcal{G}_C$ we have $\psi^n(g) \in L$ for some $n$.

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1 For a definition of a hyperbolic orbifold see [DH93].

2 It is a complete invariant for combinatorial equivalence, if equipped with additional algebraic data — see [Nek05] Theorem 6.5.2] and Proposition 6.7.]
It remains to compute the iterated monodromy group of $f \cdot g$ for all $g \in L$ to obtain a general answer to Douady’s and Hubbard’s questions.

1.3. Outline of the paper. We explain in Section 2 the construction of iterated monodromy groups, the fundamental notion of “contracting” actions, and nuclei. We apply that construction to the study of Thurston equivalence and study the post- and pre-composition actions of the mapping class group on the covering in Section 3.

We consider in more detail in Section 4 the “rabbit” polynomial, for which we obtain the recursions as described above, and we specialise in Subsection 4.3 to Dehn twists around the rabbit’s ears, obtaining an answer to Hubbard’s question.

We re-express the solution in Section 5 in more classical terms of iterations of the pull-back map on Teichmüller and moduli spaces. This gives another approach to the problem, which is however less algorithmical.

We finally consider in Sections 6 and 7 the other two classes of degree-two topological polynomials with three finite post-critical points. Section 6 deals with the polynomials whose ramification graph is the same as that of $z^2 + i$ (period 2, preperiod 1). In particular, we classify the obstructed examples up to combinatorial equivalence. The last section considers the case of period 1 and preperiod 2.

1.4. A remark on notation. We compose in most cases transformations as if they acted on the right: in a product $f \cdot g$ the transformation $f$ acts before $g$. In particular, if $g_1$ and $g_2$ are elements of a group, then $g_2 g_1 = g_2 ^{-1} g_1 g_2$. However, we sometimes have to also consider left actions. Therefore, if we write $f \circ g$, then we mean left composition, in which $g$ acts before $f$.

2. Iterated monodromy groups

We give here an overview of techniques and results in self-similar actions and iterated monodromy groups. More details and proofs can be found in [Nek05].

2.1. Partial self-coverings and monodromy action. A covering is a continuous surjective map $f : C_1 \to C$ of topological spaces, with the property that for every $z \in C$ there exists an open neighborhood $U$ of $z$ such that $f^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeomorphically onto $U$ by $f$. We say that the covering $f$ is $d$-fold if there exists $d \in \mathbb{N}$ such that $|f^{-1}(z)| = d$ for all $z \in C$.

A partial self-covering is a covering map $f : C_1 \to C$ such that $C_1$ is an open subspace of a path connected and locally path connected space $C$.

Let $f : C_1 \to C$ be a $d$-fold partial self-covering. Then the $n$th iteration $f^n : C_n \to C$ of $f$ is a $d^n$-fold partial self-covering with domain $C_n$, usually smaller than $C_1$.

Choose a basepoint $t \in C$. Then the fundamental group $\pi_1(C, t)$ acts naturally by monodromy on each of the sets $f^{-n}(t)$. The action it induces on the disjoint union $\bigsqcup_{n \geq 0} f^{-n}(t)$ is called the iterated monodromy action.

The set $\bigsqcup_{n \geq 0} f^{-n}(t)$ has a natural structure of a rooted $d$-regular tree. The root of this tree is the point $t \in f^{-0}(t) = \{t\}$ and each vertex $z \in f^{-n}(t)$, for $n \geq 1$, is connected to the vertex $f(z) \in f^{-(n-1)}(t)$.

The iterated monodromy group of the partial self-covering $f$, denoted $\text{IMG}(f)$, is the quotient of the fundamental group $\pi_1(C, t)$ by the kernel of the iterated
monodromy action. This kernel, by definition, consists of the loops $\gamma \in \pi_1(\mathcal{C}, t)$ such that for every $n$ all $f^n$-preimages of $\gamma$ are also loops.

In other words, the iterated monodromy group of $f$ is the group of all automorphisms of the tree $\bigcup_{n \geq 0} f^{-n}(t)$ which are induced by elements of $\pi_1(\mathcal{C}, t)$.

The iterated monodromy group can be effectively computed using the following recursive formula. Choose an alphabet $X$ of $d$ letters. Then every rooted $d$-regular tree is isomorphic to the tree $X^*$ of finite words over the alphabet $X$. In this tree the empty word is the root and every word $v \in X^*$ is connected by edges to all the words of the form $vx$, where $x \in X$ is a letter.

We will use throughout the paper the following notation. If $\gamma$ is a path in $\mathcal{C}$ and $z$ is an $f$-preimage of the startpoint of $\gamma$, then

$$f^{-1}(\gamma)[z]$$

denotes the unique $f$-preimage of $\gamma$ starting at $z$.

Choose a bijection $\Lambda : X \longrightarrow f^{-1}(t)$ and a path $\ell_x$ in $\mathcal{C}$ starting at $t$ and ending in $\Lambda(x)$ for every $x \in X$. Extend the bijection $\Lambda : X \longrightarrow f^{-1}(t)$ to an isomorphism of the rooted trees $\Lambda : X^* \longrightarrow \bigcup_{n \geq 0} f^{-n}(t)$ inductively by the condition that

$$\Lambda(xv) = \text{the end of the path } f^{-|v|}(\ell_x)[\Lambda(x)].$$

It is not hard to prove that $\Lambda$ is a well defined isomorphism of the rooted trees (see [Nek05, Proposition 5.2.1]). We then have the following recursive description of the iterated monodromy action (see [Nek05, Proposition 5.2.2]).

**Proposition 2.1.** Define an action of $\pi_1(\mathcal{C}, t)$ on $X^*$ by conjugating the iterated monodromy action by $\Lambda$. Then for all $\gamma \in \pi_1(\mathcal{C}, t)$, $v \in X^*$ and $x \in X$ we have

$$(xv)^\gamma = y \left(\rho(\ell_x \gamma x \ell_x^{-1})\right),$$

where $\Lambda(y)$ is the end of the path $\gamma_x = f^{-1}(\gamma)[\Lambda(x)]$.

We multiply the paths in the natural order: in a product $\gamma_1 \gamma_2$ the path $\gamma_1$ is followed before the path $\gamma_2$. Therefore $\pi_1(\mathcal{C}, t)$ and $\text{IMG}(f)$ act from the right on the tree of preimages and on $X^*$. (Note that this is different from the convention in [Nek05].)

2.2. Wreath recursions. The recursive formula in Proposition 2.1 can be interpreted either as a wreath recursion, or as the description of an automaton.

A wreath recursion is a homomorphism $\Phi$ from a group $G$ to the wreath product $G \wr \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ is the symmetric group on $X$. Recall that the wreath product $G \wr \mathfrak{S}(X)$ is, by definition, the semidirect product $G^X \rtimes \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ acts on the direct power $G^X$ by permutation of the coordinates. If $X = \{1, \ldots, d\}$, we write the elements of the wreath product in the form $\langle g_1, g_2, \ldots, g_d \rangle \pi$, where $\langle g_1, g_2, \ldots, g_d \rangle$ is an element of the direct power $G^X$ and $\pi$ is an element of the symmetric group $\mathfrak{S}(X)$. The elements of the wreath product are multiplied according to the rule

$$\langle g_1, g_2, \ldots, g_d \rangle \rho \langle h_1, h_2, \ldots, h_d \rangle \tau = \langle g_1 h_1 \rho, g_2 h_2 \rho, \ldots, g_d h_d \rho \rangle \pi \tau.$$

We denote by $g|_x$ the $x$th coordinate of $\Phi(g)$, for $g \in G$ and $x \in X$. Inductively, we put

$$g|_{vx} = (g|_v)|_x$$

for all $v \in X^*$ and $x \in X$. 

We let $G$ act on $X$ by postcomposing $\Phi$ with the natural homomorphism $G \wr \mathfrak{S}(X) \to \mathfrak{S}(X)$. We extend this action to the associated action of $G$ on $X^*$, defined by the recursion

$$(xv)^g = x^g \left(v^g|_v\right).$$

The associated action and restrictions satisfy the following relations

$$\tag{2} (gh)|_v = g|_v h|_{v^h}, \quad g|_{vw} = g|_v w$$

for all $g, h \in G$ and $v, w \in X^*$.

If $X = \{1, 2, \ldots, d\}$, then Proposition 2.2 can be expressed in terms of wreath recursions in the following way:

**Proposition 2.2.** The action of $\pi_1(C,t)$ on $X^*$ is the action associated with the wreath recursion $\Phi : \pi_1(C,t) \to \pi_1(C,t) \wr \mathfrak{S}(X)$ given by

$$\Phi(\gamma) = \langle \ell_1 \gamma_1 \ell_1^{-1}, \ell_2 \gamma_2 \ell_2^{-1}, \ldots, \ell_d \gamma_d \ell_d^{-1} \rangle \rho,$$

where $\gamma_i = f^{-1}(\gamma)[\Lambda(i)]$, and $\Lambda(k_i)$ is the endpoint of $\gamma_i$, and $\rho$ is the permutation $i \mapsto k_i$.

**2.3. Virtual endomorphisms.** A wreath recursion $\Phi : G \to G \wr \mathfrak{S}(X)$ can be constructed using the associated virtual endomorphism. A virtual endomorphism $\phi : G \to G$ is a homomorphism $\text{Dom} \phi \to G$ from a subgroup of finite index into $G$. The subgroup $\text{Dom} \phi$ is called the domain of the virtual endomorphism $\phi$.

If $\Phi : G \to G \wr \mathfrak{S}(X)$ is a wreath recursion and $x_0 \in X$ is a letter, then the domain of the associated virtual endomorphism $\phi = \phi_{x_0}$ is the stabilizer of the letter $x_0$ (with respect to the action of $G$ on $X$); and the virtual endomorphism is defined as the restriction $\phi(g) = g|_{x_0}$.

It follows from (2) that $\phi : \text{Dom} \phi \to G$ is a homomorphism.

Suppose that the action of $G$ on $X = \{1, 2, \ldots, d\}$ is transitive, choose $x_0 = 1$, and choose some $r_i \in G$, for all $i \in X$, such that $1^{r_i} = i$. Write $h_i = r_i|_1$. Then the wreath recursion can be reconstructed by the formula

$$\Phi(g) = \langle h_1^{-1} \phi(r_1^{g^{-1}}) h_{k_1}, h_2^{-1} \phi(r_2^{g^{-1}}) h_{k_2}, \ldots, h_d^{-1} \phi(r_d^{g^{-1}}) h_{k_d} \rangle \rho,$$

where $h_i = r_i|_1$, and $\rho$ is the permutation $i \mapsto k_i$, and the indices $k_i$ are uniquely defined by the condition $r_i^{g^{-1}} \in \text{Dom} \phi$.

If we change the elements $h_i$, or if we change $\{r_i\}$ to another left coset transversal, then we change $\Phi$ to its post-composition by an inner automorphism of $G \wr \mathfrak{S}(X)$, and therefore conjugate the associated actions of $G$ on $X^*$ by an automorphism of $X^*$. More precisely (see [Nek06] Sections 2.3 and 2.5), there will exist an automorphism $\Delta$ of the $d$-regular rooted tree, conjugating $G$ into the new action; and $\Delta$ will satisfy a recursion of the form

$$\Phi(\Delta) = \langle g_1 \Delta, \ldots, g_d \Delta \rangle \pi,$$

where $\langle g_1, \ldots, g_d \rangle \pi$ is the element defining the inner automorphism of $G \wr \mathfrak{S}(X)$. 
2.4. **Contraction.** A wreath recursion \( \Phi : G \rightarrow G \wr \mathfrak{S}(X) \) is contracting if there exists a finite set \( \mathcal{N} \subset G \) such that for every \( g \in G \) there exists \( n_0 \in \mathbb{N} \) with \( g|_v \in \mathcal{N} \) for all words \( v \in X^* \) of length greater than \( n_0 \). This property ensures that many calculations regarding arbitrary elements of \( G \) can be reduced, via the wreath recursion, to considerations on a finite set. For example, the “word problem” (determining if a given product of \( N \) generators is trivial) can be solved in polynomial time in contracting groups.

If \( G \) is generated by a finite symmetric set \( S = S^{-1} \), then a subset \( \mathcal{N} \subset G \) satisfies the above condition if and only if \( 1 \in \mathcal{N} \) and there exists \( n_0 \in \mathbb{N} \) such that
\[
(g^s)_v \in \mathcal{N}
\]
for all \( g \in \mathcal{N}, s \in S \) and words \( v \in X^* \) of length greater than \( n_0 \).

The smallest set \( \mathcal{N} \) satisfying the condition of these definitions is called the nucleus of the contracting action.

If \( \phi : G \rightarrow G \) is a virtual endomorphism of a finitely generated group, then its spectral radius is equal to
\[
r(\phi) = \limsup_{n \to \infty} \sqrt[n]{\limsup_{g \in \text{Dom} \phi^n} \frac{l(\phi^n(g))}{l(g)}}.
\]
where \( l(g) \) denotes the word length of \( g \) with respect to some fixed generating set of \( G \).

**Proposition 2.3** ([Nek05] Proposition 2.11.11). Let \( \Phi : G \rightarrow G \wr \mathfrak{S}(X) \) be a wreath recursion and let \( \phi \) be an associated virtual endomorphism.

If \( \Phi \) is contracting, then \( r(\phi) < 1 \). If the action of \( G \) on \( X^* \) is transitive on every level \( X^n \) (in particular, if it is transitive on \( X^1 \) and \( \phi \) is onto) and \( r(\phi) < 1 \), then the wreath recursion \( \Phi \) is contracting.

It is proved in [Nek05] Theorem 5.5.3] that if a partial self-covering \( f : C_1 \rightarrow C \) is expanding, then the associated wreath recursion \( \Phi_f \) on \( \pi_1(C) \), as defined in Proposition 2.2 is contracting.

2.5. **Automata.** It is convenient to describe wreath recursions and nuclei of contracting wreath recursions in terms of automata.

A subset \( A \subset G \) is state-closed if for every \( g \in A \) and \( x \in X \) we have \( g|_x \in A \). It is easy to see that the nucleus of a contracting wreath recursion is a state-closed set.

If \( A \) is a state-closed set, then we interpret it as an automaton, which, when it is in a state \( g \in A \) and it reads a letter \( x \in X \) on the input tape, prints the letter \( x^g \) on the output tape and goes to the state \( g|_x \). Then the automaton \( A \) with initial state \( g \) transforms any word \( v \in X^* \) to the word \( v^g \) and thus describes the associated action of \( G \) on \( X^* \).

We draw state-closed sets as graphs (Moore diagrams) with vertex set \( A \). The vertex \( g \in A \) is marked by its image in \( \mathfrak{S}(X) \), i.e., by the permutation it induces on \( X \), and for every \( g \in A \) and \( x \in X \) we draw an arrow from \( g \) to \( g|_x \) labeled by \( x \). Then the graph completely describes the restriction of the wreath recursion to \( A \).

We will always have in our paper \( X = \{0, 1\} \); then the symmetric group \( \mathfrak{S}(X) \) consists of two elements: 1 and \( \sigma = (0, 1) \). If an element of \( G^X \) or \( \mathfrak{S}(X) \) is trivial, then we usually do not write it, so that \( \langle g_0, g_1 \rangle = \{g_0, g_1\} \) and \( \sigma = \{1, 1\} \sigma \). The elements of \( G \wr \mathfrak{S}(X) \) are either written \( g = \langle g_0, g_1 \rangle \) (they are then called inactive),

\[
(0, 1) \in \mathfrak{S}(X)
\]
or \( g = \langle g_0, g_1 \rangle \sigma \) (and are then called active). All computations in \( G \wr \mathcal{S}(X) \) are based then on two rules:

\[
\langle g_0, g_1 \rangle \cdot \langle h_0, h_1 \rangle = \langle g_0 h_0, g_1 h_1 \rangle \quad \text{and} \quad \sigma \cdot \langle g_0, g_1 \rangle = \langle g_1, g_0 \rangle \sigma.
\]

In drawing Moore diagrams, we indicate active states by a grey dot and inactive states by a white dot.

### 3. Post-critically finite topological polynomials

#### 3.1. Homotopy in terms of wreath recursion

Let \( P_0 \subset \mathbb{C} \) be a finite set of complex numbers and suppose that a map \( f_P : P_0 \rightarrow P_0 \) and a point \( c_0 \in P_0 \) are given such that \( P_0 = \{ f_P^n(c_0) : n \geq 0 \} \). Let \( \mathcal{F} \) be the set of all degree-two orientation-preserving branched coverings \( f : \mathbb{C} \rightarrow \mathbb{C} \), with critical points \( c_0 \) and \( \infty \), whose restriction to \( P_0 \) coincides with \( f_P \) and \( f^{-1}(\infty) = \{ \infty \} \).

Let us denote by \( P \) the set of post-critical points of \( f \), i.e., \( P = \{ f^n(c_0) : n \geq 1 \} \cup \{ \infty \} \). Note that \( P = P_0 \cup \{ \infty \} \) if \( c_0 \) is periodic (i.e., if \( f_P^n(c_0) = c_0 \) for some positive \( n \)) and \( P = (P_0 \setminus \{ c_0 \}) \cup \{ \infty \} \) otherwise. The ramification graph of \( f \) is the directed graph with vertex set \( P \) and an arrow from \( p \) to \( f(p) \) for each \( p \in P \).

If \( f \in \mathcal{F} \), then \( \text{IMG}(f) \) is defined as the iterated monodromy group of \( f : C_1 \rightarrow C_\infty \), where \( C = \mathbb{C} \setminus P \) and \( C_1 = f^{-1}(C) \).

Let us denote by \( \mathfrak{S} \) the set of homotopy classes (within \( \mathcal{F} \)) of branched coverings \( f \in \mathcal{F} \); in other words, \( \mathfrak{S} \) is the set of path-connected components of the space \( \mathcal{F} \).

Choose a basepoint \( t \in C \), imagined close to infinity. Note that \( \pi_1(C, t) \) is a free group of rank \( |P| - 1 \); we can take its generators to be loops going around the finite post-critical points in the positive direction.

Let \( a \) be a small simple closed loop in \( \mathbb{C} \) going around \( \infty \) in the negative direction and based at \( t \) (on \( \mathbb{C} \) it is a big loop going in the positive direction around all the finite post-critical points). The loop \( a \) divides the sphere \( \mathbb{C} \) into two parts. One contains \( \infty \) and the other contains \( P \setminus \{ \infty \} \).

We may assume, after changing the map \( f \) to a homotopic one, that \( f(a) = a \). Then \( f \) maps \( a \) onto itself by a degree-two covering. We call \( a \) the circle at infinity.

We may also assume that \( f(t) = t \) (again after changing \( f \) to a homotopic map).

Let us denote by \( b \) the preimage of \( t \) different from \( t \) by \( -\infty \). The loop \( b \) has two \( f \)-preimages. One is the subcurve of \( a \) starting at \( +\infty \) and ending at \( -\infty \). The other is the curve starting at \( -\infty \) and ending at \( +\infty \).

Let the connecting path \( \ell_0 \) be the trivial path starting and ending in the basepoint \( +\infty \). Let \( \ell_1 \) be the \( f \)-preimage of \( a \) starting at \( +\infty \) and ending in \( -\infty \) (the upper semicircle at infinity). Let \( \Phi_f : \pi_1(C, t) \rightarrow \mathcal{S}(X) \wr \pi_1(C, t) \) be the wreath recursion defined by \( f \) and the given choice of connecting paths and let \( \Lambda_f : \pi_1(C, t) \rightarrow \text{Aut}(X^*) \) be the associated iterated monodromy action on the tree \( X^* \) (see Proposition 2.2 and the definition of the associated action before it).

Note that \( \Lambda_f(a) \) is always equal to the standard adding machine \( \tau = \langle \{1, \tau \} \rangle \sigma \).

**Proposition 3.1.** Two branched coverings \( f_0, f_1 \in \mathcal{F} \) are homotopic if and only if there exists \( n \in \mathbb{Z} \) such that for every \( \gamma \in \pi_1(C, +\infty) \) we have

\[
\Phi_{f_0}(\gamma) = \Phi_{f_1}(a^n \cdot \gamma \cdot a^{-n}).
\]

**Proof.** Suppose that the branched coverings \( f_0 \) and \( f_1 \) are homotopic. Let \( f_x \) be a homotopy between them, where \( x \) varies from 0 to 1. The basepoint \( t = +\infty \) has
two preimages under $f_x$. The first preimage $t_0(x)$ draws, as $x$ ranges over $[0, 1]$, a path $g_0$ starting at $+\infty$. The other preimage $t_1(x)$ draws a path $g_1$ starting at $-\infty$.

Either $t_0(1) = +\infty$ and $t_1(1) = -\infty$, and then $g_0, g_1$ are loops, or $t_0(1) = -\infty$ and $t_1(1) = +\infty$, and then $g_0$ goes from $+\infty$ to $-\infty$ and $g_1$ goes from $-\infty$ to $+\infty$. We have chosen $t = +\infty$ to be close to the point $\infty \in P$ and the homotopies must fix the point $\infty$, so $t_0(x)$ and $t_1(x)$ remain close to $\infty$. Consequently, if $g_0, g_1$ are loops, then $g_0 = a^n$ and $g_1 = \ell_1 a^m \ell_1^{-1}$ for some $n, m \in \mathbb{Z}$, and if $g_0, g_1$ are not loops, then they are of the form $a^n \ell_1$ and $\ell_1^{-1} a^m$, where $\ell_1$ is the lower semi-circle at infinity (we have $a = \ell_1 \ell_1^{-1}$). Given paths $b, c$, we write $b = c$ here and below to indicate that they are homotopic.

One of the $f_x$-preimages of the loop $a$ is a path $\ell(x, -) : [0, 1] \to C$ starting in $t_0(x)$ and ending in $t_1(x)$. We have $\ell(0, -) = \ell_1$. The path $\ell(1, -)$ is equal to $\ell_1$ if $g_0$ and $g_1$ are loops, and to the lower semicircle $\ell_1$ otherwise. Note also $\ell(-, 0) = g_0$ and $\ell(-, 1) = g_1$.

The path $\ell(x, y)$ depends continuously on $x$, and therefore defines a continuous map $\ell(x, y) : [0, 1] \times [0, 1] \to C$. It follows that if $g_0$ and $g_1$ are loops, then $g_0 = \ell_1 \cdot g_1 \cdot \ell_1^{-1}$, and if not, then $g_0 = \ell_1 \cdot g_1 \cdot (\ell_1)^{-1}$.

Consequently, if $g_0 = a^n$ then $g_1 = \ell_1^{-1} a^n \ell_1$, and if $g_0 = a^n \ell_1$ then $g_1 = \ell_1^{-1} a^n \ell_1$. The path $\ell_1^{-1} a^n \ell_1$ is homotopic to $a^n$.

Take an arbitrary loop $\gamma \in \pi_1(C, t)$, and let $\gamma_0 = f_0^{-1}(\gamma) + \infty$ and $\gamma_1 = f_0^{-1}(\gamma) - \infty$ denote their preimages starting at $+\infty$ and $-\infty$ respectively. The homotopy from $f_0$ to $f_1$ deforms the paths $\gamma_i$ continuously, giving paths $\gamma_{0,x} = f_x^{-1}(\gamma) [t_0(x)]$ and $\gamma_{1,x} = f_x^{-1}(\gamma) [t_1(x)]$.

Suppose first that $\gamma_0, \gamma_1$ and $g_0 = a^n, g_1 = \ell_1^{-1} a^n \ell_1$ are loops. Then we get a homotopy transforming $\gamma_0$ via $t_0([0, x])$ $\gamma_0, t_0([0, x])^{-1}$ to the loop

$$g_0 \gamma_0' g_0^{-1} = a^n \gamma_0' a^{-n},$$

where $\gamma_0' = f_1^{-1}(\gamma) [+\infty]$.

Similarly, the loop $\ell_1 \gamma_1 \ell_1^{-1}$ is homotopic to

$$\ell_1 g_1 \gamma_1' g_1^{-1} \ell_1^{-1} = a^n \gamma_1' \ell_1^{-1} a^{-n},$$

where $\gamma_1' = f_1^{-1}(\gamma) [-\infty]$.

Since we assumed $\gamma_i$ are loops, $\Phi_{f_1}(\gamma) = \langle \gamma_0', \ell_1 \gamma_1' \ell_1^{-1} \rangle$. We then compute

$$\Phi_{f_0}(\gamma) = \langle \gamma_0, \ell_1 \gamma_1' \ell_1^{-1} \rangle = \langle a^n \gamma_0' a^{-n}, a^n \gamma_1' \ell_1^{-1} a^{-n} \rangle$$

$$= \langle a^n, a^n \rangle \cdot \Phi_{f_1}(\gamma) \cdot \langle a^{-n}, a^{-n} \rangle = \Phi_{f_1} (a^{2n} \cdot \gamma \cdot a^{-2n}).$$

If on the other hand $\Phi_{f_1}(\gamma) = \langle \gamma_0', \gamma_1' \rangle \sigma$, then $\Phi_{f_1}(\gamma \cdot a) = \langle \gamma_0', \gamma_1' \rangle \sigma \langle a, 1 \rangle \sigma = \langle \gamma_0 a, \gamma_1' \rangle$. We then obtain $\Phi_{f_0}(\gamma a) = \Phi_{f_1} (a^{2n} \cdot \gamma \cdot a^{-2n})$, so again

$$\Phi_{f_0}(\gamma) = \Phi_{f_1} (a^{2n} \cdot \gamma \cdot a^{-2n}).$$

Consider now the case $g_0 = a^n \ell_1$ and $g_1 = \ell_1^{-1} a^{n+1}$. Take a loop $\gamma \in \pi_1(C, +\infty)$ such that its $f_1$-preimages are loops $\gamma_0$ and $\gamma_1$, so that we have $\Phi_{f_1}(\gamma) = \langle \gamma_0, \ell_1 \gamma_1 \ell_1^{-1} \rangle$. Let $\gamma_0'$ and $\gamma_1'$ be the $f_1$-preimages of $\gamma$, so that $\Phi_{f_1}(\gamma) = \langle \gamma_0', \ell_1 \gamma_1' \ell_1^{-1} \rangle$. Then, as before, $\gamma_0$ is homotopic to $g_0 \gamma_0' g_0^{-1} = a^n \ell_1 \gamma_1' \ell_1^{-1} a^{-n}$, and $\ell_1 \gamma_1' \ell_1^{-1}$ is homotopic to
\[\ell_1 \gamma_1 \gamma_0^{-1} \ell_1^{-1} = \ell_1 \ell_1^{-1} a^{n+1} \gamma_0 a^{-n-1} \ell_1 \ell_1^{-1} = a^{n+1} \gamma_0 a^{-n-1}.\] We then have

\[\Phi_{f_0} (\gamma) = \langle \gamma_0, \ell_1 \gamma_1 \ell_1^{-1} \rangle = \langle a^n, \ell_1 \gamma_1 \ell_1^{-1} a^{-n}, a^{n+1} \gamma_0 a^{-n-1} \rangle = \Phi_{f_1} (a^{n+1} \cdot \gamma \cdot a^{-2n-1}).\]

If \(\Phi_{f_1} (\gamma)\) is of the form \(\langle \gamma_0, \gamma_1 \ell_1^{-1} \rangle \sigma\), we reduce to the previous case by multiplying by \(a\), as before. We also obtain \(\Phi_{f_0} (\gamma) = \Phi_{f_1} (a^{2n+1} \cdot \gamma \cdot a^{-2n-1}).\)

Suppose now, in order to prove the proposition in the other direction, that there exists \(n \in \mathbb{Z}\) such that

\[
\Phi_{f_0} (\gamma) = \Phi_{f_1} (a^n \cdot \gamma \cdot a^{-n})
\]

holds for all \(\gamma \in \pi_1(C, +\infty).\)

For \(i \in \{0, 1\}\), denote by \(G_i \leq \pi_1(C, +\infty)\) the set of loops whose preimages under \(f_i\) are loops. The set \(G_i\) is an index-two subgroup, isomorphic both to \(\pi_1(f_i^{-1}(C), +\infty)\) and to \(\pi_1(f_i^{-1}(C), -\infty)\), where the isomorphisms \(G_i \to \pi_1(f_i^{-1}(C), \pm\infty)\) are the maps \(\gamma \mapsto f_i^{-1}(\gamma)[\pm\infty]\). Note that if \(\gamma \in G_i\), then

\[
\Phi_{f_i}(\gamma) = \langle f_i^{-1}(\gamma)[-\infty], \ell_1 \cdot f_i(\gamma)[-\infty] \cdot \ell_1^{-1} \rangle.
\]

Condition \(\phi\) implies that \(G_0 = G_1\) and that for all \(\gamma \in G_0 = G_1:\)

\[(4) \quad f_0^{-1}(\gamma)[+\infty] \text{ is homotopic to } f_1^{-1}(a^n \gamma a^{-n})[+\infty] \text{ in } C,\]

\[(5) \quad \ell_1 \cdot f_0^{-1}(\gamma)[-\infty] \cdot \ell_1^{-1} \text{ is homotopic to } \ell_1 \cdot f_1^{-1}(a^n \gamma a^{-n})[-\infty] \cdot \ell_1^{-1}.\]

The equality \(G_0 = G_1\) implies that there are homeomorphisms \(h_+, h_- : f_0^{-1}(C) \to f_1^{-1}(C)\) such that \(f_0 = f_1 \circ h_+\), where \(h_+\) fixes the preimages \(+\infty, -\infty of the base-point, and \(h_-\) permutes them.

The homeomorphisms \(h_\pm\) fix then the points \(c_0, c_1, c_2\) and can be extended in a unique way to homeomorphisms \(\tilde{h}_\pm : C \to C\).

Suppose first that \(n\) is even. Then \(f_0^{-1}(a^n)[+\infty] = a^{n/2}\) and we have from \(\phi\)

\[
\tilde{h}_+(f_0^{-1}(\gamma)[+\infty]) = f_1^{-1}(\gamma)[+\infty] = f_0^{-1}(a^{-n} \gamma a^n)[+\infty] = a^{-n/2} f_0^{-1}(\gamma)[+\infty] a^{n/2},
\]

where all equalities are homotopies in \(C\). We then have

\[
\tilde{h}_+(\gamma) = a^{-n/2} \gamma a^{n/2}
\]

for all \(\gamma \in \pi_1(C, +\infty).\)

Suppose now that \(n\) is odd. Then \(f_0^{-1}(a^{-n})[-\infty] = \ell_1^{-1} a^{(-n+1)/2}.\) We have

\[
\tilde{h}_-(f_0^{-1}(\gamma)[+\infty]) = f_1^{-1}(\gamma)[-\infty]
\]

for every \(\gamma \in G_0.\) Let us identify the group \(\pi_1(C, +\infty)\) with \(\pi_1(C, -\infty)\), by identifying the loop \(f_1^{-1}(\gamma)[-\infty] \in \pi_1(C, -\infty)\) with the loop

\[
\ell_1 \cdot f_1^{-1}(\gamma)[-\infty] \ell_1^{-1} = \ell_1 f_0^{-1}(a^{-n} \gamma a^n)[-\infty] \ell_1^{-1} = a^{-(n-1)/2} f_0^{-1}(\gamma)[+\infty] a^{(n-1)/2}.
\]

We see that for every \(\gamma \in \pi_1(C, +\infty)\) the loop \(\tilde{h}_-(\gamma)\) is identified with \(a^{-(n-1)/2} \gamma a^{(n-1)/2}\).

Therefore in all cases we can find a homeomorphism \(h : C \to C\) satisfying \(f_0 = f_1 \circ h\) and such that the induced homomorphism \(h_*\) on the fundamental group \(\pi_1(C, +\infty)\) is inner. But if a homeomorphism of a surface induces an inner automorphism of the fundamental group, then this homeomorphism is isotopic to identity. Consequently, \(f_0\) and \(f_1\) are homotopic. \(\square\)
3.2. The set $\mathcal{F}$ as a bimodule over the mapping class group. Denote by $\mathcal{G}_C$ the (pure) mapping class group of $\mathcal{C} = \hat{\mathcal{C}} \setminus P$, i.e., the set of isotopy classes relative to $P$ of homeomorphisms $h : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ fixing $P$ pointwise. The group $\mathcal{G}_C$ is the quotient of the pure braid group on $|P| - 1$ strings by its centre. For instance, if $|P| = 4$, then $\mathcal{G}_C$ is a free group on two generators.

The set $\mathcal{F}$ has a natural structure of a permutational $\mathcal{G}_C$-bimodule, i.e. a set $\mathcal{F}$ equipped with commuting left- and right-actions of the group $\mathcal{G}_C$. If $f \in \mathcal{F}$ and $h \in \mathcal{G}_C$, then we just set $fh$ and $hf$ to be the corresponding compositions. (Here in the composition $fh$ the map $f$ acts before $h$.) It is easy to see that the left and the right actions of $\mathcal{G}_C$ on $\mathcal{F}$ are well defined and commute.

There exists a natural homomorphism $\mathcal{G}_C \rightarrow \text{Out}(\pi_1(\mathcal{C}))$ mapping every element $h \in \mathcal{G}_C$ to the automorphism $h^*$ of the fundamental group of $\mathcal{C}$ (which is defined uniquely up to an inner automorphism of $\pi_1(\mathcal{C})$). It is well known that this homomorphism is injective (see [ZVC80, Theorem 5.13.1]).

Using this fact and Proposition 3.1 one can describe the structure of the bimodule $\mathcal{F}$. Take an arbitrary $f \in \mathcal{F}$ and a homeomorphism $h : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$ acting trivially on $P$ (we will also denote by $f \in \mathcal{F}$ and $h \in \mathcal{G}_C$ the corresponding homotopy classes). Recall that $a$ denotes a small circle in the neighbourhood of $\infty$.

**Proposition 3.2.** For $f \in \mathcal{F}$ and $g, h \in \mathcal{G}_C$, the following conditions are equivalent:

(i) $fg = hf$;
(ii) there exists $n \in \mathbb{Z}$ such that

$$\Phi_f(g) = \Phi_f(\gamma a^n)^h$$

for all $\gamma \in \pi_1(\mathcal{C}, +\infty)$;
(iii) there exists $n \in \mathbb{Z}$ such that

$$\Phi_f(\gamma)^h = \Phi_f(\gamma a^n)$$

for all $\gamma \in \pi_1(\mathcal{C}, +\infty)$;
(iv) there exists $n \in \mathbb{Z}$ such that

$$\phi_f(\gamma)^h = \phi_f(\gamma a^n)$$

for all $\gamma \in \text{Dom} \phi_f$, where $\phi_f$ is the virtual endomorphism associated with the first coordinate of the wreath recursion $\Phi_f$.

**Proof.** It is obvious that (ii) is equivalent to (iii). The equivalence of (iii) and (iv) follows directly from the definition of the virtual endomorphism associated with a wreath recursion. The equivalence of (i) and (ii) follows directly from Propositions 3.1, 2.2 and the specific definition of $\Phi_f$ which we gave in Subsection 3.1. □

**Definition 3.1.** Two branched coverings $f_1, f_2 \in \mathcal{F}$ are combinatorially equivalent (also said Thurston equivalent) if there exists a homeomorphism $h$ of $\hat{\mathcal{C}}$ fixing $P$ pointwise, such that $f_1 \cdot h$ and $h \cdot f_2$ are homotopic.

In other words, two elements $f_1, f_2 \in \mathcal{F}$ are equivalent if there exists an element $h$ of the mapping class group of $\hat{\mathcal{C}} \setminus P$ such that $f_1 h = h f_2$ in $\mathcal{F}$ (recall that $\mathcal{F}$ is the set of homotopy classes of branched coverings).
Corollary 3.3. If two branched coverings $f_0, f_1 \in \mathcal{F}$ are combinatorially equivalent, then

$$\Lambda_{f_0}(\pi_1(C, t)) = \Lambda_{f_1}(\pi_1(C, t)),$$

as subsets of $\text{Aut}(X^*)$.

(Recall that $\Lambda_f : \pi_1(C, t) \to \text{Aut}(X^*)$ is the associated iterated monodromy action.)

Proof. Let first $\Phi : G \to G \wr \mathfrak{S}(X)$ be a wreath recursion, and let $h$ be an automorphism of $G$. Consider the wreath recursion $\Phi^h : G \to G \wr \mathfrak{S}(X)$ given by $\Psi^h(g) = \Phi(g^{h^{-1}})^h$, where $h$ acts on $G \wr \mathfrak{S}(X)$ by the diagonal action on $G^X$.

Let $\Lambda_\Psi$ denote the action of $G$ on $X^*$ defined by the recursion $\Psi$: if $g \in G$, then $\Lambda_\Psi(g)$ maps $x_1 \ldots x_n \in X^*$ to $y_1 \ldots y_n$, where $\Phi(g) = \langle g_1, \ldots, g_d \rangle \pi$ and $x_i^\tau = y_1$ and $\Lambda_\Psi(g_{x_1})$ maps $x_2 \ldots x_n$ to $y_2 \ldots y_n$.

We easily check by induction that $\Lambda_{\Psi^h}(g) = \Lambda_{\Psi^h}(g^h)$: the respective actions on the first level coincide, and the $x$th coordinate $g^h|_x$ of $\Phi^h(g^h)$ is equal to $(g_x)^h$, where $g_x$ is the $x$th coordinate of $\Phi(g)$.

Let now $h$ be an element of the mapping class group of $\hat{C} \setminus P_{f_0}$ such that $f_0 \cdot h$ and $h \cdot f_1$ are homotopic. Then by Proposition 3.1

$$\Phi_{f_0}(\gamma^{h^{-1}}) = \Phi_{f_1}(\gamma^{a^n})h^{-1}$$

for some $n$. Consequently, $\Phi_{f_0}^h(\gamma) = \Phi_{f_0}(\gamma^{h^{-1}})^h = \Phi_{f_1}(\gamma^{a^n})$. By the first two paragraphs, we have $\Lambda_{\Phi_{f_0}^h}(\gamma) = \Lambda_{\Phi_{f_0}}(\gamma^{h^{-1}})$.

The wreath recursion $\gamma \mapsto \Psi(\gamma) := \Phi_{f_1}(\gamma^{a^n})$ is the conjugate by $\Phi_{f_1}(a^n)$ of $\Phi_{f_1}$, so $\Lambda_{\Phi}(\gamma)$ is the conjugate of $\Lambda_{\Phi_{f_1}}(\gamma)$ by the automorphism $\Delta$ of $X^*$ given by the recursion

$$\Delta = \tau^n \cdot \langle \Delta, \Delta \rangle,$$

where $\tau = \langle 1, \tau \rangle \sigma = \Lambda_{f_1}(a)$ is the adding machine. Now the element $\tau^n \Delta$ satisfies the recursion

$$\tau^n \Delta = \tau^{2n} \langle \Delta, \Delta \rangle = \langle \tau^n \Delta, \tau^n \Delta \rangle$$

so $\tau^n \Delta = 1$ and therefore $\Delta$ equals $\tau^{-n}$.

Consequently, $\Lambda_{f_0}(\gamma^{h^{-1}}) = \Lambda_{f_1}(\gamma^{a^n})$, hence $\Lambda_{f_0}(\gamma) = \Lambda_{f_1}(\gamma^{h^{-1}}a^{-n})$, which implies that $\Lambda_{f_0}(\pi_1(C, t)) = \Lambda_{f_1}(\pi_1(C, t))$. \qed

4. Twisted rabbits

We consider now some concrete examples of bimodules $\mathfrak{F}$. We will consider the cases when $f$ is a quadratic polynomial whose set of finite post-critical points has size 3.

Let us consider first the “Douady rabbit” [Mil99, Figure 35]. It is the polynomial $f_R \approx z^2 + (-0.1226 + 0.7449i)$. The two other polynomials inducing the same permutation of their post-critical set are the “corabbit” $f_C \approx z^2 + (-0.1226 - 0.7449i)$ and the “airplane” $f_A \approx z^2 - 1.7549$.

We choose as usual $+\infty$ as the basepoint. Let $\alpha, \beta, \gamma$ be the loops going around, respectively, $c, c^2 + c$ and 0 in the positive direction and connected to the basepoint as it is shown on left-hand side part of Figure 4. Let $P = \{\infty, 0, c, c^2 + c\}$ be the post-critical set of $f_R$. 
The “rabbit”’s wreath recursion $\Phi_{f_R}$ is defined by
\[
\Phi_{f_R}(\alpha) = \langle \alpha^{-1} \beta^{-1}, \gamma \beta \alpha \rangle \sigma, \quad \Phi_{f_R}(\beta) = \langle \alpha, 1 \rangle, \quad \Phi_{f_R}(\gamma) = \langle \beta, 1 \rangle.
\]
The preimages of the paths $\alpha, \beta, \gamma$ are shown on the right-hand side part of Figure 1.

Note that $\tau = \gamma \beta \alpha = \langle 1, \gamma \beta \alpha \rangle \sigma$ is the standard adding machine.

4.1. The mapping class group action. The mapping class group $G_C$ is generated by the left-handed (counter-clockwise) Dehn twist $T$ about the curve encircling the points $c, c^2 + c$ and by the left-handed twist $S$ about the curve encircling the points $0, c^2 + c$ (see Figure 2).

The twists $T$ and $S$ are defined by their action on the fundamental group of the punctured plane by the rules
\[
\alpha^T = \alpha^\beta, \quad \beta^T = \beta^\alpha = \beta^\gamma, \quad \gamma^T = \gamma, \\
\alpha^S = \alpha, \quad \beta^S = \beta^\gamma, \quad \gamma^S = \gamma^\gamma = \gamma^\beta.
\]
Their inverses act by the rules
\[ \alpha^{-1} = \alpha^{-1}, \quad \beta^{-1} = \alpha^{-1}, \quad \gamma^{-1} = \gamma^{-1}. \]
\[ \alpha^0 = \alpha, \quad \beta^0 = \alpha^0, \quad \gamma^0 = \gamma^0. \]

**Proposition 4.1.** Let \( \psi \) be the virtual endomorphism of the group \( \mathcal{G}_C \) defined on the subgroup \( H = \langle T^2, S, ST \rangle \) of index 2 by
\[ \psi (T^2) = S^{-1}T^{-1}, \quad \psi (S) = T, \quad \psi (ST) = 1. \]

Consider the map
\[ \overline{\psi} : g \mapsto \begin{cases} \psi(g) & \text{if } g \text{ belongs to the domain of } \psi, \\ T\psi(gT^{-1}) & \text{otherwise.} \end{cases} \]

Then for every \( g \in \mathcal{G}_C \) the branched coverings \( f_{R} \cdot g \) and \( f_{R} \cdot \overline{\psi}(g) \) are combinatorially equivalent.

The subgroup \( H \) is generated by those Dehn twists about curves that encircle an even number of times the critical value \( c \). They are therefore those mapping classes that can be lifted through \( f_{R} \). The map \( \psi \) is precisely that lift.

**Proof.** We first claim that for all \( g \in H \) the maps \( f_{RG} \) and \( \psi(g)f_{R} \) are homotopic. It suffices to check this on the generators \( \{T^2, S, ST\} \) of \( H \), and this is done below.

Note then that
\[ f_{R} \cdot g = \psi(g) \cdot f_{R} = \psi(g) \cdot (f_{R} \cdot \psi(g)) \cdot \psi(g)^{-1}, \]
i.e., that \( f_{R} \cdot g \) and \( f_{R} \cdot \psi(g) \) are combinatorially equivalent.

If \( g \) does not belong to the domain of \( \psi \), then \( gT^{-1} \) does, and
\[ f_{R} \cdot g = f_{R} \cdot gT^{-1}T = \psi(gT^{-1}) \cdot f_{R} \cdot T = \psi(gT^{-1}) \cdot (f_{R} \cdot T \psi(gT^{-1})) \cdot \psi(gT^{-1})^{-1}, \]
i.e., \( f_{R} \cdot g \) and \( f_{R} \cdot \psi(g) \) are combinatorially equivalent.

Let now \( \phi = \phi_{f_{R}} \) be the virtual endomorphism associated with the first coordinate of the wreath recursion \( \Phi_{f_{R}} \). We have
\[ \alpha = \phi(\beta), \quad \beta = \phi(\gamma), \quad \gamma = \phi(\alpha^0), \]
\[ \gamma^0 = \phi(\alpha^2), \quad \phi(\alpha) = 1, \quad \phi(\gamma) = 1, \]
from which we compute
\[ \phi(\beta T^2) = \phi(\alpha^0) = \phi(\alpha^0) = \alpha^0, \]
\[ \phi(\gamma T^2) = \phi(\gamma) = \beta, \]
\[ \phi(\alpha^0 T^2) = \phi(\alpha^0) = \phi(\alpha^0) = \gamma^0. \]

We see that for every \( \delta \in \pi_1(\mathcal{C}, +\infty) \) we have \( \phi(\delta T^2) = \phi(\delta)^h \), where the automorphism \( h \) is given on the generators by
\[ \alpha^h = \alpha^0, \quad \beta^h = \beta, \quad \gamma^h = \gamma^0. \]

(6)
The automorphism $h$ is equal to the product $S^{-1}T^{-1}a$, where $a$ is conjugation by $\gamma\beta\alpha$:

\[
\begin{align*}
\alpha & \mapsto \alpha^{-1} \quad T \mapsto \alpha^{\beta^{-1}} \quad a \mapsto \alpha^{\beta^{-1}\gamma\beta\alpha}, \\
\beta & \mapsto \beta^{\gamma^{-1}} \quad T \mapsto \beta^{\alpha^{-1}\beta^{-1}\gamma^{-1}} \quad a \mapsto \beta, \\
\gamma & \mapsto \gamma^{\beta^{-1}\gamma^{-1}} \quad T \mapsto \gamma^{\beta^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} \quad a \mapsto \gamma^{\beta\alpha\beta^{-1}}.
\end{align*}
\]

Consequently, by Proposition 3.2 (iv) with $n = 1$,

\[f_R \cdot T^2 = S^{-1}T^{-1} \cdot f_R.
\]

We have next

\[
\begin{align*}
\phi(\beta^S) &= \phi(\beta^1) = \alpha^{\beta\alpha}, \\
\phi(\gamma^S) &= \phi(\gamma^1) = \beta^\alpha, \\
\phi\left(\left(\alpha^{2\beta^{-1}\gamma^{-1}}\right)^S\right) &= \phi\left(\alpha^{2\beta^{-1}\gamma^{-1}}\right) = \gamma,
\end{align*}
\]

so

\[f_R \cdot S = T \cdot f_R.
\]

Let us finally compute the action of $ST$ on the generators $\{\alpha, \beta, \gamma\}$ of the fundamental group:

\[
\begin{align*}
\alpha & \mapsto \alpha^{\beta^{-1}} \quad S \mapsto \alpha^{\beta^{-1}\gamma^{-1}} \quad T \mapsto \alpha^{\beta\alpha\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} = \alpha^{\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} = \alpha^{\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}}, \\
\beta & \mapsto \beta^{\alpha^{-1}\beta^{-1}} \quad S \mapsto \beta^{\gamma\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} \quad T \mapsto \beta^{\alpha\gamma\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} = \beta^{\alpha\gamma\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}} = \beta^{\alpha\gamma\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\gamma^{-1}}, \\
\gamma & \mapsto \gamma^{\beta^{-1}\gamma^{-1}} \quad S \mapsto \gamma^{\beta\alpha\beta^{-1}} \quad T \mapsto \gamma^{\beta\alpha\beta^{-1}} = \gamma^{\alpha^{-1}\beta\alpha}.
\end{align*}
\]

Let us apply the virtual endomorphism $\phi$ to the conjugators:

\[
\begin{align*}
\phi(\gamma^{-1} \cdot \alpha^{-1} \beta^{-1} \cdot \alpha \cdot \alpha^{-1} \beta\alpha) &= \beta^{-1} \beta = 1, \\
\phi(\alpha \gamma^{-1} \cdot \alpha^{-1} \beta^{-1} \cdot \alpha \cdot \alpha^{-1} \beta\alpha) &= \beta^{-1} \beta = 1, \\
\phi(\alpha^{-1} \beta\alpha) &= 1.
\end{align*}
\]

Consequently,

\[
\phi\left(\left(\alpha^{2}\right)^{T^{-1}ST}\right) = \phi\left(\alpha^2\right), \quad \phi\left(\beta^{T^{-1}ST}\right) = \phi(\beta), \quad \phi\left(\gamma^{T^{-1}ST}\right) = \phi(\gamma)
\]

which implies that

\[f_R \cdot ST = f_R. \quad \square
\]

**Proposition 4.2.** The map $\overline{\psi} : G_C \rightarrow G_C$ is contracting: for every $g \in G_C$ there exists $n \in \mathbb{N}$ such that $\overline{\psi}^n(g) \in \{1, T, T^{-1}\}$.

**Proof.** Consider the wreath recursion for $G_C$ given by

(7) \[\Phi(T) = \langle 1, S^{-1}T^{-1} \rangle, \quad \Phi(S) = \langle T, 1 \rangle.\]

It is straightforward to check that $\overline{\psi}$ may be recovered from the recursion as follows: if $\Phi(g) = \langle g_0, g_1 \rangle$, then $\overline{\psi}(g) = g_0$; if $\Phi(g) = \langle g_0, g_1 \rangle \sigma$, then $\overline{\psi}(g) = Tg_0$. Since $T|_0 = 1$, we obtain inductively for all $n \in \mathbb{N}$

\[
\overline{\psi}^n(g) = g|_v \text{ or } Tg|_v \text{ for some } v \in X^n.
\]
We first claim that the recursion \( \psi \) is contracting. For that purpose, it suffices to compute the nucleus of \( \psi \). A simple calculation shows that it is
\[
\mathcal{N} = \{1, S, T, TS, S^{-1}, T^{-1}, S^{-1}T^{-1}\}.
\]
It follows that for all \( g \in \mathcal{G}_C \) we have \( \overline{\psi}^n \in \mathcal{N} \cup T\mathcal{N} \) for all sufficiently large \( n \), and therefore that \( \overline{\psi}(g) \) lands on a \( \overline{\psi} \)-cycle. Direct computations then show that the only \( \overline{\psi} \)-periodic elements in \( \mathcal{N} \cup T\mathcal{N} \) are the fixed points 1, \( T \) and the cycle \( T^{-1} \to T^2S \to S^{-1} \to T^{-1} \).

\[\square\]

4.2. **Contraction along the subgroup \( \langle T \rangle \).** Every integer \( m \) has a unique 4-adic expansion
\[
m = \sum_{k=0}^{\infty} m_k 4^k,
\]
with \( m_k \in \{0, 1, 2, 3\} \), and almost all \( m_k = 0 \) if \( m \) is non-negative, and almost all \( m_k = 3 \) if \( m \) is negative.\(^3\)

**Proposition 4.3.** If the 4-adic expansion of the number \( m \) has digits 1 or 2, then the branched covering \( f_R \cdot T^m \) is equivalent to the \( f_R \cdot T \). Otherwise it is equivalent to \( f_R \) for non-negative \( m \) and to \( f_R \cdot T^{-1} \) for negative \( m \).

**Proof.** Let us iterate the map \( \overline{\psi} \) on the cyclic subgroup \( \langle T \rangle \). We have
\[
\overline{\psi}^3(T^{4k}) = \overline{\psi}^2(S^{-1}T^{-1})^{2k} = \overline{\psi} T^{-1} \cdot S^{-1} \cdot T^{-2})^k = \overline{\psi}(T^{-1} \cdot 1 \cdot TS)^k = \overline{\psi}(S)^k = T^k,
\]
so \( f_R \cdot T^{4k} \) is equivalent to \( f_R \cdot T^k \). Similarly
\[
\overline{\psi}^3(T^{4k+1}) = \overline{\psi}^2(T\psi(T^{4k})) = \overline{\psi}^2(T(S^{-1}T^{-1})^{2k})
= \overline{\psi}(T\psi(T(S^{-1}T^{-1})^{2k}T^{-1})) = \overline{\psi}(T\psi((TS^{-1}T^{-1} \cdot S^{-1} \cdot T^{-2})^k))
= \overline{\psi}(T(1 \cdot T^{-1} \cdot TS)^k) = \overline{\psi}(TS^k)
= T(TS^kT^{-1}) = T,
\]
so all branched coverings \( f_R \cdot T^{4k+1} \) are equivalent to \( f_R \cdot T \). Next
\[
\overline{\psi}^3(T^{4k+2}) = \overline{\psi}^2(S^{-1}T^{-1})^{2k+1} = \overline{\psi}(T\psi((S^{-1}T^{-1})^{2k+1}T^{-1}))
= \overline{\psi}(T\psi((S^{-1} \cdot S^{-T} \cdot T^{-2} \cdot S^{-1} \cdot T^{-2}))
= \overline{\psi}(T(T^{-1} \cdot 1 \cdot TS)^k \cdot T^{-1} \cdot TS) = \overline{\psi}(TS^{k+1})
= T\psi(TS^{k+1}T^{-1}) = T,
\]
so all branched coverings \( f_R \cdot T^{4k+2} \) are equivalent to \( f_R \cdot T \). Finally
\[
\overline{\psi}^3(T^{4k+3}) = \overline{\psi}^2(T\psi(T^{4k+2})) = \overline{\psi}^2(T(S^{-1}T^{-1})^{2k+1})
= \overline{\psi}^2(TS^{-1}T^{-1} \cdot (S^{-1} \cdot S^{-T} \cdot T^{-2})^k) = \overline{\psi}(1 \cdot (T^{-1} \cdot 1 \cdot TS)^k)
= \overline{\psi}(S^k) = T^k,
\]
\(^3\)For example, \( m = -1 \) corresponds to \( m_k = 3 \) for all \( k \).
so $f_R \cdot T^{4k+3}$ is equivalent to $f_R \cdot T^k$. The statement of the proposition now easily follows.

4.3. **Solving the problem for all** $m \in \mathbb{Z}$. The wreath recursions for the “airplane” polynomial $f_A$ is given by

\begin{align*}
\Phi_{f_A}(\alpha) = \left\langle \alpha^{-1}, \gamma \alpha \right\rangle \sigma, \\
\Phi_{f_A}(\beta) = \left\langle \alpha^\beta, 1 \right\rangle, \\
\Phi_{f_A}(\gamma) = \left\langle 1, \beta^{-1} \right\rangle.
\end{align*}

Here $\alpha, \beta, \gamma$ are loops going in the positive direction around $c, c^2 + c, (c^2 + c)^2 + c$ respectively, as shown on the left part of Figure 3.

Let us identify the planes of $f_A$ and $f_C$ with the plane of $f_R$ by identifying their respective loops $\alpha, \beta, \gamma$ (the definition of $\alpha, \beta, \gamma$ in the plane of $f_R$ is given on the left-hand side of Figure 3). Note that for $f = f_A, f_C$, just as for $f = f_R$, we have $\Phi_f(\gamma \alpha) = (1, \gamma \alpha) \sigma$.

Let us identify the planes of $f_A$ and $f_C$ with the plane of $f_R$ by identifying their respective loops $\alpha, \beta, \gamma$ in the plane of $f_R$ is given on the left-hand side of Figure 3.

Let $T$ denote, as before, the left Dehn twist about the curve around the points $c, c^2 + c$ in the plane of $f_R$. Then, from $T$’s definition, we get the following wreath recursion for the standard iterated monodromy action for $T^m \cdot f_R$:

\begin{align*}
\Phi_{T^m \cdot f_R}(\alpha) &= \Phi_{f_R}(\alpha)^{T^{-m}} = \left\langle \alpha^{-1} \beta^{-1}, \gamma \beta \alpha \right\rangle \sigma, \\
\Phi_{T^m \cdot f_R}(\beta) &= \Phi_{f_R}(\beta)^{T^{-m}} = \left\langle \alpha^\beta, 1 \right\rangle, \\
\Phi_{T^m \cdot f_R}(\gamma) &= \Phi_{f_R}(\gamma)^{T^{-m}} = \left\langle 1, \beta^{-1} \right\rangle,
\end{align*}

where $T^{-m}$ acts on $\pi_1(C, +\infty) \wr \mathbb{S}(X)$ by the diagonal action:

\[
(\langle x, y \rangle \sigma^k)^{T^{-m}} = \left\langle x^{T^{-m}}, y^{T^{-m}} \right\rangle \sigma^k.
\]

Corollary 3.3 makes it possible to solve Hubbard’s question algorithmically for every given $m$ in the following way.
Thurston’s Theorem 1.1 implies that $T^m \cdot f_R$ is combinatorially equivalent to exactly one polynomial in the set \{f_R, f_A, f_C\}. There are no obstructions, since the only obstructions for polynomials are Lévy cycles, which cannot exist in the case of a periodic critical point. Corollary 3.3 then tells us that $\Lambda_{T^m \cdot f_R}(\sigma_1(C))$ coincides with the iterated monodromy group of the associated polynomial. One can prove that these groups are different (as sets), and therefore, if we prove that $\Lambda_{T^m \cdot f_R}(\sigma_1(C))$ coincides with a given group $\text{IMG}(f_*)$ for $* \in \{R,A,C\}$, then we can conclude that $T^m \cdot f_R$ is equivalent to the respective $f_*$. We therefore prove that the $\text{IMG}(f_*)$ are all distinct. This is done by computing their nuclei, and checking that they are distinct as finite automata; this is done in Figure 4.

**Figure 4.** Nuclei of the “rabbit” (top), the “corabbit” (right) and the “airplane” (bottom)

**Proposition 4.4.** The group $\text{IMG}(T \cdot f_R) = \Lambda_{T \cdot f_R}(\sigma_1(C))$ coincides with $\text{IMG}(f_A)$. Indeed the homeomorphism $h = TS^{-1}a$ conjugates $T \cdot f_R$ with $f_A$, if the planes of $f_R, f_A$ are identified as above.

**Proof.** Let $\alpha, \beta, \gamma$ be the generators of $\text{IMG}(f_A)$. They are defined now as the automorphisms of $X^*$ satisfying the recursion (compare with (8))

$$\alpha = \langle \langle \alpha^{-1}, \gamma \alpha \rangle \sigma, \beta = \langle \langle \alpha, 1 \rangle, \gamma = \langle \langle 1, \beta \gamma^{-1} \rangle \rangle.$$

Let $\alpha_1, \beta_1$ and $\gamma_1$ be the generators of $\text{IMG}(T \cdot f_R)$. They are given by the recursion (10):

$$\alpha_1 = \langle \langle \alpha_1^{-1}, \beta_1^{-1}, \gamma_1 \beta_1 \alpha_1 \rangle \sigma, \beta_1 = \langle \langle \alpha_1 \beta_1^{-1}, 1 \rangle, \gamma_1 = \langle \langle \beta_1^{-1} \alpha_1 \gamma_1^{-1}, 1 \rangle \rangle,$$

We claim that

$$\alpha_1 = \alpha^h = \alpha^{\beta \gamma^{-1} \alpha \gamma \alpha}, \beta_1 = \beta^h = \beta^{\gamma^{-1} \alpha \gamma \alpha}, \gamma_1 = \gamma^h = \gamma^a.$$
For that purpose, it suffices to show that the left-hand sides of these equalities satisfy the same recursions as \( \alpha_1, \beta_1, \gamma_1 \). Note that \( \gamma^h \beta^h \gamma^h = (\gamma \beta \alpha)^h = \gamma \beta \alpha \). We have
\[
\beta^{-1} \alpha \gamma \beta \alpha = \langle \alpha, 1 \rangle \langle \alpha^-1, \gamma \alpha \rangle \sigma \langle 1, \gamma \beta \alpha \rangle \sigma = \langle \gamma \beta \alpha, \gamma \alpha \rangle,
\]
hence
\[
\alpha^{\gamma^{-1}} \alpha \gamma \beta \alpha = \langle \langle \alpha^{-1}, \gamma \alpha \rangle \sigma \langle \gamma \beta \alpha, \gamma \alpha \rangle \rangle = \langle \alpha^{-1} \beta^{-1} \gamma^{-1} \alpha^{-1} \gamma \alpha, \alpha^{-1} \gamma^{-1} \alpha \gamma \beta \alpha \rangle \sigma
\]
\[
= \langle \alpha^{-1} \beta^{-1} \gamma^{-1} \alpha^{-1} \gamma \alpha, \gamma \beta \alpha \rangle \sigma = \langle \langle \gamma \beta \alpha \rangle^{-h} \gamma^h, (\gamma \beta \alpha)^h \rangle \sigma
\]
\[
= \langle \alpha^{-h} \beta^{-h}, \gamma^h \beta^h \alpha^{-h} \rangle \sigma.
\]
\[
\beta^{-1} \alpha \gamma \beta \alpha = \beta^{-1} \beta^{-1} \alpha \gamma \beta \alpha = \langle \langle \alpha, 1 \rangle \langle \gamma \beta \alpha, \gamma \alpha \rangle \rangle = \langle \alpha \gamma \beta \alpha, 1 \rangle
\]
\[
= \langle \langle \langle \alpha \beta \gamma^{-1} \gamma \beta \alpha \rangle^{-1} \alpha \gamma \beta \alpha \rangle^{-h} \gamma^h, 1 \rangle = \langle \langle \alpha \beta^h \gamma^{-h} \beta, 1 \rangle, \gamma^h \beta^h \alpha^{-h} \rangle \sigma.
\]
\[
\gamma^\alpha = \langle \langle \alpha^{-1} \beta^{-1} \gamma^{-1} \alpha \gamma \beta \alpha \rangle^{-h} \alpha \gamma \beta \alpha \rangle^{-h} \gamma^{-1} \alpha \gamma \beta \alpha, 1 \rangle
\]
\[
= \langle \langle \alpha \beta^{-1} \gamma^{-1} \alpha \gamma \beta \alpha, 1 \rangle, \alpha \beta^{-1} \gamma^{-1} \alpha \gamma \beta \alpha \rangle^{-h} \alpha \gamma \beta \alpha, 1 \rangle.
\]

This shows the required relations between \( \alpha, \beta, \gamma \) and \( \alpha_1, \beta_1, \gamma_1 \), so \text{IMG} (f_A) = \text{IMG} (T \cdot f_R). \text{Proposition 4.3} now implies that \( T \cdot f_R \) is homotopic to \( h^{-1} \cdot f_A \cdot h \). \qed

**Proposition 4.5.** The group \( \text{IMG} (T^{-1} \cdot f_R) = \Lambda_{T^{-1} \cdot f_R} (\pi_1 (C)) \) coincides with \text{IMG} (f_C). Moreover, \( T^{-1} \cdot f_R \) and \( f_C \) are homotopic if the planes of \( f_R \) and \( f_C \) are identified as above.

**Proof.** Let \( \alpha, \beta, \gamma \) be the generators of the iterated monodromy group of the “corabbits”. They are defined by the recursion (compare with (4))
\[
\alpha = \langle \langle \alpha^{-1} \beta^{-1}, \gamma \beta \alpha \rangle \sigma, \beta = \langle \alpha \beta \alpha, 1 \rangle, \gamma = \langle \beta, 1 \rangle, \rangle.
\]

Let \( \alpha_{-1}, \beta_{-1} \) and \( \gamma_{-1} \) be the generators of \( \text{IMG} (T^{-1} \cdot f_R) \). They are given by the recursion (10):
\[
\alpha_{-1} = \langle \langle \alpha^{-1} \beta^{-1}, \gamma \beta_{-1} \alpha^{-1} \rangle \sigma, \beta_{-1} = \langle \alpha \beta^{-1} \gamma \alpha, 1 \rangle, \gamma_{-1} = \langle \beta, 1 \rangle, \rangle.
\]

Since these two recursions are the same, we have \( \alpha_{-1} = \alpha, \beta_{-1} = \beta, \gamma_{-1} = \gamma \). \text{Proposition 4.3} now implies that \( f_C \) and \( T^{-1} \cdot f_R \) are homotopic. \qed

**Corollary 4.6.** The branched coverings \( T \cdot f_R \) and \( f_R \cdot T \) are equivalent to \( f_A \) and the branched coverings \( T^{-1} \cdot f_R \) and \( f_R \cdot T^{-1} \) are equivalent to \( f_C \).

The last corollary together with Proposition 4.3 prove the following solution of the “twisted rabbit question”.

**Theorem 4.7.** If the \( 4 \)-adic expansion of the number \( m \) has digits 1 or 2, then the branched covering \( f_R \cdot T^m \) is equivalent to the “airplane” \( f_A \). Otherwise it is equivalent to the “rabbit” \( f_R \) for non-negative \( m \) and to the “corabbit” \( f_C \) for negative \( m \).

The general case of any element of the mapping class group \( G_C \) is treated using Proposition 4.3 in the following theorem.
Theorem 4.8. Let $g \in G_C$ be an arbitrary homeomorphism fixing the post-critical set pointwise. Let $\varphi : G_C \to G_C$ be the map defined in Proposition 4.1. The orbit of $g$ under the iteration of $\varphi$ will land either on 1, or on $T$, or on $T^{-1}$. In the first case $g \cdot f_R$ is equivalent to the “rabbit”, in the second case it is equivalent to the “airplane” and in the last to the “corabbit”.

This theorem gives an algorithm solving the general “twisted rabbit question”. Note that due to the fact that $\varphi$ is contracting on $G_C$, this algorithm has linear complexity with respect to the word-length of elements of $G_C$.

Note also that in Theorem 4.7 the typical answer is “airplane”, and exponentially few values of $m$ yield “rabbits” or “corabbits”. This seems to happen quite often on cyclic subgroups of $G_C$. On the other hand, on the cyclic subgroup $\langle ST^2 \rangle$ all twists are “rabbits”; and on the subgroup $\langle ST \rangle$ there are roughly as many “rabbits” as “airplanes”: $(ST)^m \cdot f_R$ is equivalent to

- $f_R$ if $m < -2$, $m = 2n - 3$, and $n$’s 4-adic expansion has only 0’s and 3’s,
- $f_C$ if $m > -2$, $m = 2n - 1$, and $n$’s 4-adic expansion has only 0’s and 3’s,
- $f_T$ if $m \equiv 0 \pmod{2}$,
- $f_A$ if $m \equiv 1 \pmod{2}$ in the cases not covered above.

This is because $\varphi((ST)^{2m}) = S^{-mT^{-1}}$ and $\varphi((ST)^{2m+1}) = T^2S^{-m}$; calculations are similar to those of Proposition 4.1.

These phenomena can ultimately be traced to the following reason: the subgroup $\mathrm{Dom} \psi^3$ of $G_C$ has index 8:

$$\mathrm{Dom} \psi^3 = \langle S^4, T^4, ST^2, S^{-2}T^2S, S^{-1}T^2S^2, ST, S^{T^{-1}}, STS, S^{T^{-1}S} \rangle.$$  

The map $\psi^3$ is defined on these generators by $\psi^3(S^4) = S$, $\psi^3(T^4) = T$, and all other generators are mapped to 1.

5. Dynamics on moduli space

We have computed in the previous sections a virtual endomorphism $\psi$ of the mapping class group $G_C$ associated with the bimodule $\mathfrak{F}$ of branched coverings with the ramification graph of the “rabbit” polynomial. This virtual endomorphism is given by

$$\psi(T^2) = S^{-1}T^{-1}, \quad \psi(S) = T, \quad \psi(ST) = 1.$$  

This virtual endomorphism is associated to a self-similar action of $G_C$ on the binary tree. Let us show that this action coincides with the iterated monodromy action associated with the rational function $F(w) = 1 - \frac{1}{w^*}$.

The critical points of $1 - \frac{1}{w^*}$ are 0 and $\infty$. Their orbit is

$$0 \mapsto \infty \mapsto 1 \mapsto 0.$$  

Let us take the fixed point $t \approx 0.8774 + 0.7449i$ as our basepoint. The fundamental group of $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is generated by the loops going in the negative direction around 0 and 1. Let us denote the first by $X$ and the second by $Y$ (see the solid lines on the left- and right-hand sides of Figure 4, respectively). Let us choose the connecting paths $\ell_0$ and $\ell_1$, so that $\ell_0$ is the trivial path at $t$ and $\ell_1$ connects $t$ to $-t$ passing above the puncture 0, as shown in the right-hand side of Figure 5.

We then get the following recursion for the iterated monodromy group of $F$:

$$X = \langle Y, 1 \rangle, \quad Y = \langle 1, X^{-1}Y^{-1} \rangle \sigma.$$
Next $Y^2 = \langle X^{-1}Y^{-1}, X^{-1}Y^{-1} \rangle$ and $Y^{-1}XY = \langle 1, Y \rangle$. We see that the virtual endomorphism associated with the wreath recursion is given by

$$X \mapsto Y, \quad Y^2 \mapsto X^{-1}Y^{-1}, \quad Y^{-1}XY \mapsto 1.$$ 

It is therefore precisely the virtual endomorphism $\psi$ associated to $F$, if we identify $X$ with $S$ and $Y$ with $T$.

5.1. **Moduli and Teichmüller space.** This coincidence has a nice explanation in terms of Teichmüller theory. Let $P \subset S^2$ be a finite subset of the sphere. Then the **Teichmüller space** $T_P$ modelled on $(S^2, P)$ is the space of homeomorphisms $\tau : S^2 \to \hat{C}$, where $\tau_1$ and $\tau_2$ are identified if there exists a biholomorphic isomorphism $\Theta : \hat{C} \to \hat{C}$ (i.e., an element of the Möbius group) such that $\Theta \circ \tau_1 = \tau_2$ on $P$ and $\Theta \circ \tau_1$ is isotopic to $\tau_2$ relative to $P$.

The **moduli space** $M_P$ of $(S^2, P)$ is the space of all injective maps $P \hookrightarrow \hat{C}$ modulo post-compositions with elements of the Möbius group. The Teichmüller space $T_P$ is the universal cover of the moduli space $M_P$, where the covering map is the restriction map of $\tau \in T_P$ to $P$.

Let now $f : S^2 \to S^2$ be a branched covering with post-critical set $P$. Then for every $\tau \in T_P$ there exists a unique element $\tau' \in T_P$ such that we have a commutative diagram

$$
\begin{array}{ccc}
S^2 & \xrightarrow{f} & S^2 \\
\downarrow{\tau'} & & \downarrow{\tau} \\
\hat{C} & \xrightarrow{f'} & \hat{C}
\end{array}
$$

and $f' = (\tau')^{-1} \cdot f \cdot \tau : \hat{C} \to \hat{C}$ is a rational function. Let us write $\tau' = \sigma_f(\tau)$. The map $\sigma_f : T_P \to T_P$ is analytic and weakly (i.e. non-uniformly) contracting (see [Roy71] or [DH93]).

Let us return to the case when $P$ is the post-critical set of the “rabbit” polynomial. The moduli space $M_P$ is the set of maps $\tau|_P : \{0, c, c^2 + c, \infty\} \hookrightarrow \hat{C}$ modulo post-compositions with Möbius transformations. We may assume, applying an appropriate element of the Möbius group, that 0 is mapped by $\tau$ to 0, $c$ to 1 and $\infty$ to $\infty$. 

**Figure 5.**
to $\infty$. Then the points of moduli space are uniquely determined by the value of $\tau|_{P}(c^{2}+c) = w$.

We have $w \notin \{0, 1, \infty\}$, since the map $\tau|_{P}$ is injective.

Therefore, the moduli space is isomorphic to the punctured sphere $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$, where the point $w \in \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ corresponds to the element $\tau|_{P}$ such that $\tau|_{P}(\infty) = \infty$, $\tau|_{P}(0) = 0$, $\tau|_{P}(c) = 1$ and $\tau|_{P}(c^{2}+c) = w$.

Let $f \in \mathcal{F}$ be arbitrary. Recall that $\mathcal{F}$ is the set of degree-two branched coverings of $\hat{\mathbb{C}}$ with critical points $0, \infty$ whose ramification graph coincides with that of the “rabbit” polynomial.

Let $\tau$ be arbitrary. Suppose that the projection of $\tau' = \sigma_{f}(\tau)$ on moduli space is given by the point $w_{0} \in \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and the projection of $\tau$ is given by $w_{1} \in \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Then the rational function $f_{\tau}$ in the diagram (11) is a degree-two map having critical points at $0$ and $\infty$ and satisfying

$$f_{\tau}(\infty) = \infty, \quad f_{\tau}(0) = 1, \quad f_{\tau}(1) = w_{1}, \quad f_{\tau}(w_{0}) = 0,$$

since $f_{\tau}|_{P} = (\tau'|_{P})^{-1} \cdot f|_{P} \cdot \tau|_{P}$.

We conclude that $f_{\tau}$ is a quadratic polynomial. It is of the form $az^{2}+1$, since $0$ is critical and $f_{\tau}(0) = 1$. We get therefore

$$a + 1 = w_{1}, \quad aw_{0}^{2} + 1 = 0,$$

hence $a = -\frac{1}{w_{0}}$, so that

$$w_{1} = 1 - \frac{1}{w_{0}^{2}}.$$

We have thus obtained the following description of the action of the pull-back map $\sigma_{f}$ on moduli space.

**Proposition 5.1.** The correspondence $\sigma_{f}(\tau) \mapsto \tau$ on Teichmüller space is projected on moduli space $\mathcal{M}_{P} = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ to the rational function

$$F : w \mapsto 1 - \frac{1}{w^{2}}.$$

Suppose now that $h \in \mathcal{G}_{C}$ is an arbitrary element of the mapping class group of $C = \hat{\mathbb{C}} \setminus P$. It defines an automorphism of Teichmüller space by pre-composition: $\tau \mapsto h \cdot \tau$. The mapping class group $\mathcal{G}_{C}$ is the fundamental group of the moduli space $\mathcal{M}_{P}$ and the action of $h$ on $\mathcal{T}_{P}$ coincides with the corresponding deck transformation. Note that if we identify elements of $\mathcal{T}_{P}$ with paths in $\mathcal{M}_{P}$, then the action of $\mathcal{G}_{C}$ by deck transformations is given by pre-composition of paths; therefore both actions of $\mathcal{G}_{C}$ are left actions.

Let $f = f_{h}$ denote the “rabbit” polynomial. The corresponding point of moduli space is given by the identical map $\{0, c, c^{2}+c, \infty\} \rightarrow \hat{\mathbb{C}}$. After normalization, we see that the corresponding point of $\mathcal{M}_{P} = \hat{\mathbb{C}} \setminus \{0, 1\}$ is the fixed point $\frac{c^{2}+c}{c} = t \approx 0.8774 + 0.7449i$ of the rational function $1 - \frac{1}{w^{2}}$.

Let $\tau_{0} \in \mathcal{T}_{P}$ be the point of Teichmüller space given by the identity map $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. It is projected onto the point $t$ of the moduli space $\mathcal{M}_{P}$. Every point $\tau \in \mathcal{T}_{P}$ can be identified with the homotopy class of a path $\ell_{\tau}$ in $\mathcal{M}_{P}$ starting at $t$, ending at the projection of $\tau$ and equal to the image of a path in $\mathcal{T}_{P}$ starting at $\tau_{0}$ and ending at $\tau$. The homotopy class $\ell_{\tau}$ is uniquely defined and we have $\ell_{h \cdot \tau} = \gamma_{h} \cdot \ell_{\tau}$, where $\gamma_{h} \in \pi_{1}(\mathcal{M}_{P}, t)$ is the loop corresponding to $h \in \mathcal{G}_{C}$. 
Proposition 5.2. For all $\tau \in \mathcal{T}_P$, $h \in \mathcal{G}_C$ and $f \in \mathcal{F}$ the following equalities hold:

$$\sigma_{h, f}(\tau) = h \cdot \sigma_f(\tau), \quad \sigma_{f, h}(\tau) = \sigma_f(h \cdot \tau).$$

If $\tau \in \mathcal{F}$ corresponds to a path $\ell_\tau$ in $\mathcal{M}_P$, then $\sigma_{f, h}(\tau)$ is represented by the path $F^{-1}(\ell_\tau)[t]$, where $F(w) = 1 - \frac{1}{w^2}$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{S}^2 & \xrightarrow{h} & \mathbb{S}^2 \\
\downarrow h \cdot \sigma_f(\tau) & \downarrow \sigma_f(\tau) & \downarrow \tau \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} & \xrightarrow{f} & \mathbb{C} & \xrightarrow{id} & \mathbb{C}
\end{array}$$

It implies that $\sigma_{h, f}(\tau) = h \cdot \sigma_f(\tau)$ and $\sigma_{f, h}(h^{-1} \cdot \tau) = \sigma_f(\tau)$ for all $h \in \mathcal{G}_C$ and $\tau \in \mathcal{T}_P$. The last equality implies that $\sigma_{f, h}(\tau) = \sigma_f(h \cdot \tau)$ for all $\tau \in \mathcal{T}_P$.

The second statement of the proposition is a direct corollary of Proposition 5.1. \qed

Consider now an arbitrary element $h \in \mathcal{G}_C$ and the composition $h \cdot f_R$. It is known (see [DH93]) that the orbit of $\tau_0$ under iteration of $\sigma_{h, f_R}$ will converge to a point $\tau$ such that $\sigma_{h, f_R}(\tau) = \tau$, and the polynomial $f_R$ in the diagram (11) for $f = h \cdot f_R$ is the polynomial which is Thurston equivalent to $h \cdot f_R$.

Let $\gamma_h$ be the loop in $\mathcal{M}_P$ corresponding to $h \in \mathcal{G}_C$. By Proposition 5.2 we have $\sigma_{h, f_R}(\tau_0) = h \cdot \sigma_{f_R}(\tau_0) = h \cdot \tau_0$, hence the path representing $\sigma_{h, f_R}(\tau)$ is $\gamma_h$.

If $\ell_n$ is the path representing $\sigma_{h, f_R}(\tau_0)$, then the path representing the point

$$\sigma_{h, f_R}^{(n+1)}(\tau_0) = h \cdot \sigma_{f_R}(\sigma_{h, f_R}(\tau_0))$$

is $\gamma_h \cdot F^{-1}(\ell_n)[t]$, by Proposition 5.2. Consequently, the path representing the limit point $\tau$ is

$$\ell_\tau = \gamma_h^{(0)} \cdot \gamma_h^{(1)} \cdot \gamma_h^{(2)} \cdot \gamma_h^{(3)} \cdots,$$

where $\gamma_h^{(0)} = \gamma_h$ and $\gamma_h^{(n)}$ is the preimage of $\gamma_h$ under $F^{on}$ which starts at the end of $\gamma_h^{(n-1)}$.

The endpoint of the path $\ell_\tau$ is one of the three fixed points of $1 - \frac{1}{w^2}$. It is easy to see that the fixed point $t \approx 0.8774 + 0.7449i$ corresponds to the “rabbit”, the point $\tilde{t} \approx 0.8774 - 0.7449i$ corresponds to the “corabbit” and the point $\approx -0.7549$ corresponds to the “airplane”.

See for instance Figure 3 where the paths $\ell_\tau$ for $h = Y = T$ and $h = Y^{-1} = T^{-1}$ are indicated.

One can see that in the first case the path $\ell_\tau$ converges to the fixed point corresponding to the “airplane” and in the second case it converges to $\tilde{t}$, which corresponds to the “corabbit”. This can be shown after a detailed analysis of the dynamics of the Fatou components of the rational function $F$.

Proposition 5.1 can also be interpreted in these terms: namely, we have $\psi = \phi_F$. If $h$ belongs to the domain of the virtual endomorphism $\psi$, then the corresponding loop $\gamma_h \in \pi_1(\mathcal{M}_P, t)$ belongs to the domain of the virtual endomorphism $\phi_F$ associated with $F$. We have then that the path converging to the fixed point of $\sigma_{h, f_R}$ is of the form

$$\gamma_h \cdot F^{-1}(\gamma_h)[t] \cdot F^{-2}(\gamma_h)[t_1] \cdot F^{-3}(\gamma_h)[t_2] \cdots.$$
Figure 6. The path $\ell_\tau$ for $h = T$ and $T^{-1}$

where $t_n$ is the endpoint of $F^{-n}(\gamma_h)[t_{n-1}]$. This path is equal to

$$\gamma_h \gamma_g \cdot F^{-1}(\gamma_g) [t] \cdot F^{-2}(\gamma_g) [t_1'] \cdot F^{-3}(\gamma_g) [t_2'] \cdots,$$

where $g = \phi_F(h) = F^{-1}(h)[t]$ and $t'_n$ is the endpoint of $F^{-n}(\gamma_g)[t_{n-1}]$. This proves that $h \cdot f_R$ and $\psi(h) \cdot f_R$ are combinatorially equivalent. Similar arguments work also in the case when $h$ does not belong to the domain of $\psi$ (see [Nek05, Theorem 6.6.3]).

6. Preperiod 1, period 2

There are three families of quadratic topological polynomials with three post-critical points: the first contains the “rabbit” and the “airplane”, and its ramification graph is a cycle of length three. The next family has ramification graph with preperiod 1 and period 2: it contains the polynomial $f_i(z) = z^2 + i$ and $f_{-i} = z^2 - i$, as well as obstructed topological polynomials. The last family has ramification graph with preperiod 2 and a fixed post-critical point; it contains the polynomials $\approx z^2 - 1.5434$ and $\approx z^2 - 0.2282 + 1.1151i$, and is dealt with in Section 7.

6.1. The iterated monodromy group. Let us consider first the polynomial $f_i(z) = z^2 + i$. The dynamics of $f_i$ on its post-critical set is

$$i \mapsto -1 + i \mapsto -i \mapsto -1 + i.$$

Let us compute the iterated monodromy group of $f_i$. We again take $+\infty$ as the basepoint. Let $\alpha$ be the loop going around $i$ in the positive direction and connected to the basepoint by the external ray $R_{1/6}$ and the arc $[0, 1/6]$ of the circle at infinity. The loops $\beta$ and $\gamma$ go around the points $-1 + i$ and $-i$ and are connected to the circle at infinity by the rays $R_{i/3}$ and $R_{2/3}$ and to the point $+\infty$ by the arcs $[0, 1/3]$ and $[0, 2/3]$, respectively. The connecting paths $\ell_0$ and $\ell_1$ are, as usual, the trivial
path and the upper semicircle. See the loops $\alpha, \beta, \gamma$ and their preimages in Figure 7.

Computation of the wreath recursion gives

$$
\Phi_{f_i}(\alpha) = \langle \alpha^{-1} \beta^{-1}, \beta \alpha \rangle \sigma, \quad \Phi_{f_i}(\beta) = \langle \alpha, \gamma \rangle, \quad \Phi_{f_i}(\gamma) = \langle \beta, 1 \rangle.
$$

![Figure 7. Computation of IMG ($z^2 + i$)](image)

We see that the corresponding elements $\alpha, \beta, \gamma$ of IMG ($f_i$) are of order two.

Let $\mathcal{F}$ be the set of homotopy classes of branched coverings which have the same ramification graph as $f_i$. Namely, $\mathcal{F}$ is the set of homotopy classes of degree-two topological polynomials $f$ such that the finite post-critical set of $f$ has three different points $c_1, c_2, c_3$ which are mapped in the following way:

$$
c_1 \mapsto c_2 \mapsto c_3 \mapsto c_2
$$

(where we assume that the same points $c_1, c_2, c_3$ are chosen for all elements of $\mathcal{F}$).

Among quadratic polynomials $f(z) = z^2 + c$ only $z^2 + i$ and $z^2 - i$ have this ramification graph, since the set of the roots of the equation $f^3(c) = f(c)$ is $\{0, -1, i, -i, -2\}$, but $z^2, z^2 - 1$ and $z^2 - 2$ have different post-critical dynamics.

Note, however, that there exist obstructed topological polynomials in $\mathcal{F}$. A way to construct one is shown in Figure 8. It describes the post-critical points $c_1, c_2, c_3$ and curves connecting them to infinity on the right-hand side of the figure. The left-hand side shows the preimages of the points and curves. The map folds the horizontal line in two and maps the critical point $f^{-1}_i(c_1)$ to $c_1$. It is a homeomorphism from each of the upper and lower half-planes to the complement of the line connecting $c_1$ to infinity.

Consider the simple closed curve $\Gamma$ around the points $c_2$ and $c_3$. It has two $f_i$-preimages. One is peripheral, and the other is homotopic to $\Gamma$. The map $f_*$ is of degree 1 on the non-peripheral preimage of $\Gamma$, so the curve $\Gamma$ is an obstruction.

Let us compute the iterated monodromy group of the map $f_*$. Choose the basepoint $t$ on the circle at infinity, as shown on Figure 8 and let the connecting paths from $t$ to its preimages be the trivial path $\ell_0$ at $t$ and the semicircle $\ell_1$ going in the positive direction from $t$ to its other preimage. Let $\alpha, \beta, \gamma$ be generators of $\pi_1(\hat{C} \setminus P_{f_*}, t)$ following the circle at infinity and the arcs towards the points $c_1, c_2, c_3$ respectively, encircling the point in the positive direction, and returning back to $t$, as shown on the right-hand side of Figure 8.
It is easy to see from Figure 8 that the wreath recursion is

$$\Phi_{f_\ast}(\alpha) = \langle \alpha^{-1}, \alpha \rangle \sigma, \quad \Phi_{f_\ast}(\beta) = \langle \alpha, \gamma \rangle, \quad \Phi_{f_\ast}(\gamma) = \langle 1, \beta \gamma^{-1} \rangle.$$  

Direct computation shows that the corresponding elements \(\alpha, \beta, \gamma\) of the iterated monodromy group are of order two and that \(\beta\) and \(\gamma\) commute. Conjugating \(IMG(f_\ast)\) by \(\Delta = \langle \alpha \Delta, \Delta \rangle\), we see that the generators of \(IMG(f_\ast)\) can be defined by the recursion

$$\sigma = \alpha, \quad \beta = \langle \alpha, \gamma \rangle, \quad \gamma = \langle 1, \beta \rangle.$$  

This is one of the Grigorchuk groups \(G_\omega\) from [Gri85], namely that given by \(\omega = (01)^\infty\). One of the ways to see that it is not the iterated monodromy group of a rational function (and thus that the map \(f_\ast\) is not combinatorially equivalent to a polynomial) is to show that its limit orbispace (see [Nek05, Section 4.6]) has an isotropy group isomorphic to the Klein group \(C_2 \times C_2\), namely \(\langle \beta, \gamma \rangle\).

Note also that the composition of \(f_\ast\) with a power of the Dehn twist about the curve \(\Gamma\) is also obstructed (with the same obstruction \(\Gamma\)).

### 6.2. Moduli space

Let us compute the virtual endomorphism \(\psi\) on the mapping class group \(G_C\) of the punctured plane \(\mathbb{C} \setminus P = \mathbb{C} \setminus \{i, i-1, -i\}\), via the moduli space approach.

Let \(\mathcal{F}\) be as before the set of homotopy classes of branched coverings having the same ramification graph as \(z^2 + i\). The moduli space of \(\hat{C} \setminus P\) is again a plane with two punctures. We normalize the mappings \(P \hookrightarrow \hat{C}\) so that \(\infty\) is mapped to \(\infty\), 0 is the critical value of \(f \in \mathcal{F}\) and \(f(0) = 1\). Let us compute the action of the inverse of \(\sigma_f\) on moduli space in this case. We have

$$f_{\tau}(0) = 1, \quad f_{\tau}(1) = w_1, \quad f_{\tau}(w_0) = 1.$$  

We also have that 0 is the critical value of the quadratic polynomial \(f_\tau\). Hence \(f_\tau\) is of the form \((az - 1)^2\), where \(1/a\) is its critical point. The last equality implies that \(aw_0 - 1 = 1\) (as \(aw_0 - 1 = -1\) would give \(w_0 = 0\)), hence \(a = 2/w_0\), so

$$w_1 = f_{\tau}(1) = \left(\frac{2 - w_0}{w_0}\right)^2.$$
Therefore, the correspondence $\sigma_F(\tau) \mapsto \tau$ is projected to the rational function $F(z) = \left(\frac{2-w}{w}\right)^2$ on moduli space (compare with Proposition 5.1).

The fixed points of the rational function $\left(\frac{2-w}{w}\right)^2$ are $w = 1$ and $w = \pm 2i$. The point $1$ belongs to the post-critical set and is the puncture of the moduli space. Thus, it does not correspond to any quadratic polynomial (and, as we will see, corresponds to obstructed topological polynomials). The point $2i$ corresponds to the polynomial
\[
\frac{4}{(2i)^2} \left(z - \frac{2i}{2}\right)^2 = -(z - i)^2,
\]
which is conjugate to $z^2 + i$. The point $-2i$ corresponds to $-(z + i)^2$, which is conjugate to $z^2 - i$.

The critical points of the rational function $\left(\frac{2-w}{w}\right)^2$ are $w = 2, w = 0$. We have the following dynamics on the post-critical orbit:
\[
2 \mapsto 0 \mapsto \infty \mapsto 1 \mapsto 1.
\]

Therefore, the post-critical set of $F(w) = \left(\frac{2-w}{w}\right)^2$ is $\{0, 1, \infty\}$. The Julia set of $F$ is the whole sphere, since there are no superattracting cycles. Actually, the Thurston orbifold $O_F$ of $F$ is $(2, 4, 4)$, i.e., the orbifold of the action on $\mathbb{C}$ of the group of affine transformations $\{z \mapsto i^k z + a + bi : k, a, b \in \mathbb{Z}\}$. Moreover, $F$ is induced on the orbifold $O_F$ by the expanding map $z \mapsto (1 + i)z$ of $\mathbb{C}$.

To show this explicitly, consider the Weierstrass function
\[
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}\right]
\]
associated to the lattice $\mathbb{Z}[i]$.

It follows from its definition that $\wp$ is an even function and that, by the choice of the lattice $\mathbb{Z}[i]$, we have $\wp(iz) = -\wp(z)$. Consequently, $i\wp'(iz) = -i\wp'(z)$ and thus
\[
\wp'(iz) = i\wp'(z).
\]

It is known that
\[
(\wp'(z))^2 = 4\wp^3(z) - 2g_2\wp(z) - g_3,
\]
with $g_2 = 60s_4$ and $g_3 = 140s_6$, where $s_m = \sum_{\omega \in \mathbb{Z}[i], \omega \neq 0} \omega^{-m}$ are the Eisenstein series. It is clear that $s_6 = 0$ because $\mathbb{Z}[i]$ has a 4-fold symmetry, so $g_3 = 0$.

Another classical fact is the addition formula
\[
\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4} \left(\wp'(z_1) - \wp'(z_2)\right)^2.
\]

Let us then compute $\wp((i + 1)z)$:
\[
\wp(iz + z) = -\wp(iz) - \wp(z) + \frac{1}{4} \left(\wp'(iz) - \wp'(z)\right)^2 = \frac{1}{4} \left(\frac{(i - 1)\wp'(z)}{-2\wp(z)}\right)^2
\]
\[
= -\frac{i}{8} \cdot \frac{4\wp^2(z) - g_2}{\wp(z)}.
\]

The function $\wp : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ realizes the universal covering of the orbifold $(2, 2, 2, 2)$. The group of deck transformations of this covering is the group of all affine transformations of the form $z \mapsto z + c$, with $c \in \mathbb{Z}[i]$. In other words this group is the group of holomorphic automorphisms $\alpha : \mathbb{C} \rightarrow \mathbb{C}$ such that $\wp(\alpha(z)) = \wp(z)$.
It follows that the function $\varphi^2 : \mathbb{C} \to \hat{\mathbb{C}}$ realizes the universal covering of the orbifold $(2, 4, 4)$ for which the group $G$ of deck transformations is the group of affine transformations of the form $z \mapsto i^k z + c$, with $c \in \mathbb{Z}[i]$.

We have

$$\varphi^2 ((i + 1)z) = -\frac{1}{64} \left( 4\varphi^2(z) - g_2 \right)^2 = -\frac{\left( \varphi^2(z) - \frac{g_2}{4} \right)^2}{4\varphi^2(z)}.$$ 

It follows that the map $z \mapsto (i + 1)z$ on the universal cover $\mathbb{C}$ induces on the orbifold $G \setminus \hat{\mathbb{C}}$ the rational map $t \mapsto -\left( t - \frac{g_2}{4} \right)^2$. Performing the change of variables $t = \frac{g_2}{4 - w}$ we see that this rational map is conjugate to $w \mapsto \left( \frac{2 - w}{w} \right)^2$.

We conclude that the iterated monodromy group of the rational function $\left( \frac{2 - w}{w} \right)^2$ is isomorphic to the group of affine transformations $z \mapsto i^k z + c$, $c \in \mathbb{Z}[i]$ (see [Nek05, Subsection 6.3.2.2] and [BGN03, Section 5]).

This group is the group of the orientation-preserving automorphisms of the tiling of the plane by triangles shown on Figure 9. The triangles are fundamental domains of the group action. We assume that the vertices of the grid coincide with the Gaussian integers $\mathbb{Z}[i]$.

![Figure 9. The fundamental group of the orbifold (2, 4, 4)](image)

Let us cut the Riemann sphere $\hat{\mathbb{C}}$ along the ray $[0, +\infty]$ consisting of the non-negative reals and infinity. This ray will then contain the post-critical set of $F(z) = \left( \frac{2 - w}{w} \right)^2$. It is also easy to see that the preimage of this cut in the universal cover $\hat{\mathbb{C}}$ of the orbifold $O_F$ is precisely the union of the lines of the tiling in Figure 9. In particular, the preimage of $\hat{\mathbb{C}} \setminus [0, \infty]$ is the disjoint union of the open triangles of the tiling.

6.3. The iterated monodromy group on moduli space. Let us compute the recursion defining IMG ($F$). We take $t = 2i$ as basepoint. The iterated monodromy group is generated by the loops $a$ and $b$ going in the positive direction around 0 and 1 respectively, as shown in Figure 10. The loops $a$ and $b$ correspond to the right Dehn twists about the curves going around $i$, $-i$ and $-1 + i$, $-i$, respectively (see Figure 11).

4Here $\varphi^2(z) = (\varphi(z))^2$. 
The preimages of $2i$ under $F$ are $2i$ and $\frac{4-2i}{5}$. Let $\ell_0$ be the trivial path at $2i$ and let $\ell_1$ be the path connecting $2i$ to $\frac{4-2i}{5}$ passing between 1 and 2 as shown in Figure 12.
The preimages of the loops $a$ and $b$ are shown on Figure 12. We see that the wreath recursion is

$$\Phi(a) = \sigma, \quad \Phi(b) = \langle \langle b^{-1}a^{-1}, b \rangle \rangle.$$  

The virtual endomorphism associated with the first coordinate of the wreath recursion $\Phi$ on the mapping class group is given by

$$\psi(a^2) = 1, \quad \psi(b) = b^{-1}a^{-1}, \quad \psi(b^a) = b,$$

since

$$\Phi(a^2) = 1, \quad \Phi(b) = \langle \langle b^{-1}a^{-1}, b \rangle \rangle, \quad \Phi(b^a) = \langle \langle b, b^{-1}a^{-1} \rangle \rangle.$$  

This virtual endomorphism has the property that $f \cdot g = \psi(g) \cdot f$ for every $g$ in the index-two subgroup $\text{Dom}\psi = \langle a^2, b, b^a \rangle$ of the mapping class group $\mathcal{G}_C$. The proof is the same as in the case of the “rabbit” polynomial.

Let us see now what happens on the universal cover of the orbifold $O_F$. For every path $\gamma$ in the moduli space $M_P$ and for every preimage $\zeta \in \mathbb{C}$ of the startpoint of $\gamma$ in the universal cover of the orbifold $O_F$, there exists a unique preimage of $\gamma$ in $\mathbb{C}$ which starts at $\zeta$.

Let $t \in \mathbb{C}$ be the preimage of $2i$ under the covering map $\mathbb{C} \rightarrow O_F$ which belongs to the triangle $\Delta$ with vertices $0, 1, (1 + i)/2$. Note that $0$ and $1 \in \mathbb{C}$ are preimages of $1 \in O_F$, that $(1 + i)/2$ is a preimage of $\infty \in O_F$, and that $1/2$ is a preimage of $0 \in O_F$.

Then the preimages of the loops $a$ and $b$ starting at $t$ go in the positive direction around the points $1/2$ and $1$, if we assume that the universal covering map $\mathbb{C} \rightarrow O_F$ is orientation-preserving (see Figure 13). We use here the fact that the segment $[1/2, 1]$ is a preimage of the interval $[0, 1] \subset O_F$. It follows that the elements $a$ and $b$ of the fundamental group of $M_P$ act on the plane $\mathbb{C}$ via the affine maps

$$A(z) = -z + 1, \quad B(z) = iz + 1 - i,$$

respectively. Recall that this has to be a left action.

![Figure 13. Loops $a$ and $b$ in the universal cover $\mathbb{C}$](image)
The element $b^{-1}a^{-1}$ of the fundamental group $\pi_1(M, 2i)$ is a small loop around $\infty$ connected to the basepoint by a curve disjoint from the positive ray. It follows that the preimage of the loop $b^{-1}a^{-1}$ in the triangle $\Delta$ is the top curve shown in Figure 13.

We have

$$B^{-1}A^{-1}(z) = iz + 1, \quad B^A(z) = iz.$$ 

The preimage of the loop $b^a$ in the triangle $\Delta$ goes around the vertex 0.

Let $T_P$ be the Teichmüller space modeled on $(\hat{\mathbb{C}}, P)$ for $P = \{i, -1 + i, -i, \infty\}$.

We have the following commutative diagram

$$\begin{array}{ccc}
T_P & \rightarrow & \mathbb{C} \\
\downarrow & & \downarrow \\
M_P & \hookrightarrow & O_F
\end{array}$$

where $T_P \rightarrow M_P$ is the natural projection, $M_P \hookrightarrow O_F = \hat{\mathbb{C}}$ is the identical embedding and $\mathbb{C} \rightarrow O_F$ is the universal covering map. The map $T_P \rightarrow \mathbb{C}$ exists, since $T_P$ is the universal cover of $M_P$. It is defined uniquely up to an element of the group $G = \{i^k z + c : c \in \mathbb{Z}[i]\}$ acting on $\mathbb{C}$.

Let $\tau_0 \in T_P$ be the original complex structure on $\hat{\mathbb{C}}$. It is projected to $2i \in M_P$ and we may assume that its image $\zeta$ in $\mathbb{C}$ belongs to the triangle $\Delta$. This fixes uniquely the map $T_P \rightarrow \mathbb{C}$.

We know that the correspondence $\tau \mapsto \sigma_f(\tau)$ on $T_P$ is semi-conjugated via the map $T_P \rightarrow \mathbb{C}$ to an affine map $\Sigma : z \mapsto i^k \frac{z}{1 + i} + \xi$ for some $k \in \mathbb{Z}$ and $\xi \in \mathbb{C}$ (since $F$ is induced by multiplication by $(1 + i)$ on $\mathbb{C}$). The point $\zeta$ is fixed by $\Sigma$, since $\tau_0$ is fixed under $\sigma_f$.

The preimage of the positive ray under the action of $F$ is the whole real line. The preimage of the negative ray in the universal cover $\mathbb{C}$ divides the triangle $\Delta$ into two similar triangles, see Figure 14. This figure also displays the preimages of the curves $F^{-1}(a)$ and $F^{-1}(b)$.

We know from the wreath recursions that the $F$-preimage of $b$ which starts at $2i$ coincides with $b^{-1}a^{-1}$, which is a loop around infinity. This shows that the point $\zeta$ belongs to the right-hand side triangle on Figure 14.

It follows that

$$\Sigma(z) = \frac{i - 1}{2} z + 1$$

and, therefore,

$$\zeta = \frac{1 + i}{2 + i} = \frac{3 + i}{5}.$$
6.4. The answer. We are now in a position to determine the combinatorial equivalence class of the Thurston maps considered in the non-obstructed case. The obstructed maps will be considered later.

Let \( \pi \) be the homomorphism from the mapping class group \( \mathcal{G}_C \) onto the affine group \( G = \{ z \mapsto ikz + c : k \in \mathbb{Z}, c \in \mathbb{Z}[i] \} \) defined on the generators by
\[
\pi(a) = A : z \mapsto -z + 1, \quad \pi(b) = B : z \mapsto iz + 1 - i.
\]
We have shown above that \( \pi \) is the canonical epimorphism of \( \mathcal{G}_C \) onto the iterated monodromy group \( \text{IMG} \left( \left( \frac{2}{2n+1} \right)^2 \right) \). Note that multiplication in the affine group \( G \) corresponds to left action, i.e., \( \pi(g_1g_2) = \pi(g_1) \circ \pi(g_2) \).

**Theorem 6.1.** Let \( h \in \mathcal{G}_C \) be an element of the mapping class group. Write \( \pi(h) \) as the affine transformation \( \pi(h)(z) = ikz + c \). Then:
- if \( k = 0 \) and \( c \equiv 0 \pmod{2} \), then \( f_i \cdot h \) is equivalent to \( f_i \);
- if \( k = 1 \) and \( c \equiv 0 \pmod{2} \), then \( f_i \cdot h \) is equivalent to \( f_{-i} \);
- in all other cases, \( f \) is obstructed.

Let \( Q \) be the group \( \mathbb{Z}[i]/5\mathbb{Z}[i] \times \mathbb{Z}/4 \) of order 100, where the action of \( \mathbb{Z}/4 \) is by multiplication by \( i \). Then the answer (in \{\( f_i, f_{-i}, \) obstructed\}) depends only on the image of \( h \) in \( Q \) under the homomorphism \( \mathcal{G}_C \rightarrow \mathbb{Q}_C \) mapping \( a \) to \((-1,2)\) and \( b \) to \((1-i,1)\).

Therefore, unlike in the “rabbit/airplane” case, the answer is periodic. See Figure 15 for dependence of the answer on the element \( \pi(h) \) of the group \( G \). The lower left black triangle is the origin (it corresponds to the trivial element of \( G \)). For every other element \( h \) of the picture there is a unique element \( f \) of the original triangle to \( \Delta \). If \( \Delta \) is black, then \( f_i \cdot h \) is equivalent to \( z^2 + i \), if \( \Delta \) is grey, then \( f_i \cdot h \) is equivalent to \( z^2 - i \). White triangles correspond to obstructed maps.

The picture is periodic with period 5 in both directions.

**Proof of Theorem 6.1.** By Proposition 5.2, the map \( \Sigma \circ \pi(h) \) of \( C \). The fixed point of \( \Sigma \circ f_i \), if it exists, is mapped to the fixed point of \( \Sigma \circ H(z) = \frac{i-1}{2}(ikz + c) + 1 \in \mathbb{C} \), which is
\[
\xi = \frac{\frac{i}{2}-c+1}{1 - \frac{i}{2}ik} = -\frac{c-i-1}{i+1+ik}
\]
Let us consider the possible values for \( k \) in turn.

- **if** \( k = 0 \): then \( \xi = -\frac{c-i-1}{i+2} \), which is in the same \( G \)-orbit as \( \frac{c+i}{i+2} \). The possible residues modulo \( i+2 \) are 0, \pm 1, \pm 2, which, modulo multiplication by powers of \( i \) are either 0 or 1. Consequently, if \( \frac{c+i}{i+2} \in \mathbb{Z}[i] \), then \( \xi \) belongs to the \( G \)-orbit of 0, otherwise it belongs to the \( G \)-orbit of \( \frac{c+i}{i+2} \), which coincides with the \( G \)-orbit of \( \zeta = \frac{3+i}{2} \).
  - If \( \xi \) belongs to the orbit of 0, then the map \( f_i \cdot h \) is obstructed. If it belongs to the orbit of \( \zeta \), then \( f_i \cdot h \) is equivalent to \( f_i(z) = z^2 + i \).
- **if** \( k = 1 \): then \( \xi = -\frac{c-i-1}{2i+2} \), which is in the same \( G \)-orbit as \( \frac{c+i}{i+2} \). Here again, if \( \frac{c+i}{i+2} \in \mathbb{Z}[i] \), then \( f_i \cdot h \) is obstructed, otherwise it is equivalent to \( z^2 - i \).
- **if** \( k = 2 \): then \( \xi = -\frac{c-i-1}{i} \), which is integral, and we get an obstructed map.
- **if** \( k = 3 \): then \( \xi = -c+i+1 \), and we also get an obstructed map.

\( \square \)
6.5. **The obstructed cases.** No description of combinatorial classes of obstructed Thurston maps follows directly from Thurston’s Theorem 1.1; therefore we have to treat the obstructed cases separately.

Let $F$ be the set of quadratic topological polynomials $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with finite critical value $i$ and such that $f(i) = -1 + i$, $f(-1+i) = -i$ and $f(-i) = -1 + i$. Let $\mathfrak{F}$ be the set of homotopy classes of elements of $F$ relative to $P = \{ \infty, i, -1+i, -i \}$. Let $\mathcal{G}_{\mathbb{C}}$ be the mapping class group of $\hat{\mathbb{C}} \setminus P$. Recall that $\mathfrak{F}$ is a $\mathcal{G}_{\mathbb{C}}$-bimodule by pre- and post-composition.

**Proposition 6.2.** For every $f \in \mathfrak{F}$ there exist elements $h$ and $g \in \mathcal{G}_{\mathbb{C}}$ such that $h \cdot f \cdot g$ is equal to the polynomial $f_i(z) = z^2 + i$.

In the terminology of [Nek05] this means that the $\mathcal{G}_{\mathbb{C}}$-bimodule $\mathfrak{F}$ is irreducible.

**Proof.** Let $\mathcal{T}_P$ and $\mathcal{M}_P$ denote the Teichmüller and the moduli space of $\hat{\mathbb{C}} \setminus P$, respectively. We identify, as above, $\mathcal{M}_P$ with $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ so that the correspondence $\sigma_f(\tau) \mapsto \tau$ on $\mathcal{T}_P$ is projected to the rational function $F(w) = \left(\frac{2-w}{w}\right)^2$ on $\mathcal{M}_P$.

Let $\zeta = 2i$ be the fixed point of $F$ corresponding to the polynomial $z^2 + i$. Let $\tau_i$ be an arbitrary preimage of $\zeta$ in $\mathcal{T}_P$. Then the image of $\sigma_f(\tau_i)$ in $\mathcal{M}_P$ is equal to one of the $F$-preimages of $\zeta$, i.e., either to $2i$ or to $\frac{2-2i}{3}$. 

**Figure 15.**
The path $\sigma_f(a)$ is a lift to $T_P$ of an $F$-preimage of the loop $a$ in $M_P$. We know, by (14), that both $F$-preimages of $a$ are paths starting at one of the points $2i, \frac{2k-2i}{3}$ and ending at the other.

Consequently, if the image of $\sigma_f(\tau_i)$ in $M_P$ is equal to $\frac{2k-2i}{3}$, then the image of $\sigma_{f,a}(\tau_i) = \sigma_f(a \cdot \tau_i)$ is equal to $2i$ (see Proposition 5.2).

Hence, either $\sigma_f(\tau_i)$ or $\sigma_{f,a}(\tau_i)$ belongs to the $\pi_1(M_P)$-orbit of $\tau_i$, i.e., there exists a homeomorphism $h \in G_C$ such that $\sigma_{h,f}(\tau_i) = h \cdot \sigma_f(\tau_i) = \tau_i$, or $\sigma_{h,f,a}(\tau_i) = h \cdot \sigma_{f,a}(\tau_i) = \tau_i$. But this implies that either $h \cdot f$ or $h \cdot f \cdot a$ is combinatorially equivalent to $f_i$.

**Corollary 6.3.** For every branched covering $f \in \mathfrak{F}$ there exists $g \in G_C$ such that $f = f_i \cdot g$.

**Proof.** This follows from Proposition 5.2 and the fact that the associated virtual endomorphism $\psi$ is onto, see 15. □

It will be more convenient for us to proceed with the branched covering $f_i \cdot a$, which is obstructed by Theorem 6.1. We now show that it is equivalent to the branched covering $f_\ast$ shown on Figure 8.

For that purpose, note that the twist $a$ acts on the generators $\alpha, \beta, \gamma$ of the fundamental group of the punctured plane in the same way as the element $h$ in (10). Applying that twist to the wreath recursion $\Phi_f$, we see that the wreath recursion $\Phi_{f,a}$ is given by (13), so that the maps $f_\ast a$ and $f_\ast$ are equivalent, by Proposition 5.1.

Let $\phi$ denote the virtual endomorphism of $G_C$ such that $f_\ast g = \phi(g) \cdot f_\ast$. It follows from the wreath recursion $\Phi$ defined by (13) that $\phi$ is the virtual endomorphism associated with the second coordinate of the wreath recursion $\Phi$ and thus has domain $H = (a^2, b, b^a)$ and is given by

$$
\phi(a^2) = 1, \quad \phi(b) = b, \quad \phi(b^a) = b^{-1} a^{-1}.
$$

Let us introduce the following subgroups $E_n \triangleleft G_C$ (see Nek05 Subsection 2.13.1). We set $E_0 = \{1\}$ and

$$
E_n = \{ g \in H \triangleleft G_C : \phi(g) \text{ and } \phi(g^a) \text{ belong to } E_{n-1} \}.
$$

In other words, $E_n$ is the kernel of the wreath recursion $\Phi^n : G_C \rightarrow G_C \wr \mathfrak{S}(X^n)$ describing the action and the restrictions of $G_C$ on words of length $n$.

It follows that $E_n$ are normal subgroups and that $E_n \geq E_{n-1}$. Let us denote $E_\infty = \bigcup_{n \geq 0} E_n$.

**Lemma 6.4.** If $g \in E_\infty$, then for all $f \in \mathfrak{F}$ the branched coverings $f \cdot g$ and $f$ are combinatorially equivalent.

**Proof.** We have $f = f_\ast \cdot h$ for some $h \in G_C$, by Corollary 5.2. Let us prove the statement by induction on $n$ such that $g \in E_n$. The statement is trivial for $n = 0$. We have

$$
f_\ast \cdot h g = f_\ast \cdot h g h^{-1} \cdot h = \phi(h g h^{-1}) \cdot f_\ast \cdot h = \phi(h g h^{-1}) \cdot (f_\ast \cdot h \phi(h g h^{-1})) \cdot \phi(h g h^{-1})^{-1}.
$$

We have $h g h^{-1} \in E_n$, hence $\phi(h g h^{-1}) \in E_{n-1}$, and by the inductive hypothesis, $f_\ast \cdot h \phi(h g h^{-1})$ is combinatorially equivalent to $f_\ast \cdot h$, which shows that $f_\ast \cdot h g$ is combinatorially equivalent to $f_\ast \cdot h$. □
Corollary 6.5. The combinatorial equivalence class of a branched covering \( f \cdot g \) depends only on the image of \( g \) in the quotient group \( G = G_C / E_\infty \).

It is easy to see that the wreath recursion \( \Phi \) induces a well-defined wreath recursion (which we will also denote \( \Phi \)) on \( G = G_C / E_\infty \), given by the same formula on the generators. We also denote by \( \phi \) the virtual endomorphism induced on \( G \) by the virtual endomorphism \( \phi \) of \( G \).

We have shown earlier that the iterated monodromy group of \( F(w) = \left( \frac{2-w}{w} \right)^2 \) is the quotient of \( G_C \) given by the presentation
\[
\text{IMG} (F) = G = \mathbb{Z}[i] \rtimes \mathbb{Z}/4 = \langle A, B | A^2, (AB)^4, B^4 \rangle.
\]

The virtual endomorphism \( \phi \) induces a virtual endomorphism of \( G \) (still denoted \( \phi \)), which is contracting on \( G \).

In \( G_C \) we have the equalities
\[
\phi(a^2) = 1, \quad \phi((ab)^4) = b^{-1}a^{-2}b, \quad \phi(a^{-1}(ab)^4a) = a^{-2},
\]
which imply that \( a^2 \in \mathcal{E}_1 \) and \( (ab)^4 \in \mathcal{E}_2 \).

The following lemma computes the nucleus of the wreath recursion on \( \tilde{G} \).

Lemma 6.6. For every \( g \in \tilde{G} \) there exists \( n \) such that for every word \( v \in X^n \) of length greater than \( n \) the restriction \( g|_v \) (computed with respect to the wreath recursion \( \Phi \)) belongs to the set
\[
\mathcal{N} = \{ a, ab, ab^2, ab^3, a^b, b^a, (ab)^2, b^{-2}ab, abab^2 \}^{\pm 1} \cup \{ b^k \}_{k \in \mathbb{Z}}.
\]

Proof. It is sufficient to show that \( \mathcal{N} \) is state-closed, and that there exists \( m \) such that modulo the relations \( a^2 = (ab)^4 = 1 \) obtained in 13 we have \( A|_v \subseteq \mathcal{N} \) for all words \( v \) of length \( m \), where
\[
A = \mathcal{N} \cdot \{ 1, a, b, b^{-1} \}.
\]

Figure 16 shows the Moore diagram of the set \( \mathcal{N} \).

We see that \( \mathcal{N} \) is state-closed. Therefore it is sufficient to check our condition for \( A = \mathcal{N} \cdot \{ a, b, b^{-1} \} \setminus \mathcal{N} \). Moreover, multiplication by \( a \) from either side does not change the restrictions, therefore, we may take \( A \) equal to
\[
\mathcal{N} \cdot \{ b, b^{-1} \} \setminus \mathcal{N} = \{ ab^4, abab^{-1}, b^{-2}ab^2, abab^3, b^{-1}ab^{-1}, b^{-2}ab^{-1},
\[
\quad b^{-3}ab, b^{-3}ab^{-1}, ab^{-1}ab, ab^{-1}ab^{-1},
\[
\quad b^{-1}ab^3, b^{-2}ab^{-1}ab, b^{-2}ab^{-1}ab^{-1} = b^{-1}aba \}
\]

Taking into account again that multiplication by \( a \) does not change the restrictions, and replacing some of the elements by their inverses (as \( \mathcal{N} = \mathcal{N}^{-1} \)), we reduce our checking to the set
\[
A = \{ bab^{-1}, b^{-2}ab^2, bab^3, b^{-1}ab^3, b^{-2}ab^{-1}ab \}.
\]

But we have
\[
\begin{align*}
bab^{-1} &= \langle b^{-1}ab^{-1}, bab \rangle \sigma = \langle (ab)^2a, a(ab)^2 \rangle \sigma \\
b^{-2}ab^2 &= \langle abab^3, b^{-3}ab^{-1}a \rangle \sigma = \langle a \cdot bab^3, (a \cdot bab^3)^{-1} \rangle \sigma \\
bab^3 &= \langle b^{-1}ab^3, bab \rangle \sigma = \langle b^{-1}ab^3, a(ab)^2 \rangle \sigma \\
b^{-1}ab^3 &= \langle a \cdot b^4, b^{-1}ab \rangle \sigma \\
b^{-2}ab^{-1}ab &= \langle (ab)^2 \cdot a, b^{-2} \rangle,
\end{align*}
\]
hence we may take \( m = 4 \), since the longest chain staying outside \( \mathcal{N} \) is \( b^{-2}ab^2 \mapsto abab^3 \mapsto b^{-1}ab^3 \mapsto ab^4 \mapsto b^4 \).

\[ \boxed{\text{Lemma 6.7.} \quad \text{Let } P = \bigcap_{n \geq 1} \text{Dom } \phi^n \text{ be stabilizer of the path 111... in the group } \tilde{G}. \text{ For every } g \in P \text{ there exists } n \text{ such that } \phi^n(g) \in \langle b \rangle.} \]

Proof. Let \( g \in P \) stabilize the path 111... There exists \( n \) such that \( \phi^n(g) \in \mathcal{N} \), where \( \mathcal{N} \) is given by Lemma 6.6. The sequence \( (\phi^m(g))_{m \geq n} \) must pass through inactive (white) states of the Moore diagram of \( \mathcal{N} \) (Figure 16) and follow arrows labeled by 1. The only such infinite paths in the Moore diagram are the loops attached to the powers of \( b \). Consequently, \( \mathcal{N} \cap P = \langle b \rangle \). \[ \boxed{\text{Proposition 6.8.} \quad \text{Every branched covering } f \in \mathfrak{F} \text{ is combinatorially equivalent either to } z^2 + i, \text{ or to } z^2 - i, \text{ or to } f_* \cdot b^k \text{ for some } k \in \mathbb{Z}, \text{ where } b \text{ is as before the Dehn twist about the curve } \Gamma \text{ encircling the points } -1 + i \text{ and } -i. \text{ The branched coverings } f_* \cdot b^k \text{ are obstructed, with obstruction } \Gamma.} \]

Proof. Consider the map \( \bar{\phi} : \tilde{G} \rightarrow \tilde{G} \) defined by

\[
\bar{\phi} : g \mapsto \begin{cases} 
\phi(g) & \text{if } w \text{ belongs to the domain of } \phi, \\
a\phi(ga) & \text{otherwise.}
\end{cases}
\]
Following the line of reasoning used for the “rabbit” polynomial (see Proposition 4.1), and using Corollary 6.5, we check the identity \( f_\ast \cdot g = \psi(g) \cdot f_\ast \). It then follows that the branched coverings \( f_\ast \cdot g \) and \( f_\ast \cdot \overline{\phi}(g) \) are combinatorially equivalent for all \( g \in \tilde{G} \).

Lemma 6.6 and the argument of Proposition 6.2 show that it is sufficient to study the dynamics of the map induced by \( \overline{\phi} \) on the set \( \mathcal{N} \cup a\mathcal{N} \subset \tilde{G} \).

Direct computations show that the map \( \overline{\phi} \) on \( \mathcal{N} \cup a\mathcal{N} \) has fixed points \( a \) and \( b^k \) for \( k \in \mathbb{Z} \) and cycles \( ab \leftrightarrow b^{-a} \) and \( a^b \rightarrow b^{-2a} \rightarrow ab \rightarrow a^b \).

This shows that for every \( g \in \tilde{G} \) there exists \( n \) such that \( \overline{\phi}^n(g) \in \{a, ab, a^b\} \cup \{b^k\}_{k \in \mathbb{Z}} \). Consequently, by Corollary 6.5 every element \( f \) in \( \mathcal{F} \) is combinatorially equivalent either to \( f_\ast \cdot a = f_\ast \cdot a^2 \), or to \( f_\ast \cdot ab = f_\ast \cdot a^2b \), or to \( f_\ast \cdot b^{-1}ab = f_\ast \cdot ab^{-1}ab \), or to \( f_\ast \cdot b^k \) for some \( k \in \mathbb{Z} \).

The branched coverings \( f_\ast \cdot a^2, f_\ast \cdot a^2b \) and \( f_\ast \cdot ab^{-1}ab \) are combinatorially equivalent to, respectively, \( f_i, f_{-i} \) and \( f_i \), by Theorem 6.1. The branched coverings \( f_\ast \cdot b^k = f_i \cdot ab^k \) are obstructed, also by Theorem 6.1. □

The following proposition completely describes the group \( \tilde{G} = G_\mathcal{C}/\mathcal{E}_\infty \).

**Proposition 6.9.** \( \tilde{G} \) is a (non-split) extension of \( G \) by the \( G \)-module \( K = \mathbb{Z}[[B]]/G \). It may be given by the presentation

\[
\tilde{G} = \langle a, b | a^2, (ab)^4, [b^4, b^4 w] \text{ for all } w \in \tilde{G} \rangle.
\]

We will only need the fact that the order of \( b \) in \( \tilde{G} \) is infinite, so we give only a sketch of the proof.

**Proof.** Let \( N \) denote the normal closure in \( G_\mathcal{C} \) of \( \langle a^2, (ab)^4, [b^4, b^4 w] \rangle \) for all \( w \in G_\mathcal{C} \). We wish to prove \( N = \mathcal{E}_\infty \).

By comparing \( G_\mathcal{C}/N \) with the presentation \( (17) \), we see that the map \( a \mapsto A, b \mapsto B \) defines a natural epimorphism \( G_\mathcal{C}/N \rightarrow \tilde{G} \), whose kernel \( K = \langle b^{4w} : w \in G_\mathcal{C} \rangle \cap N \) is abelian, isomorphic as a \( G \)-module to \( \mathbb{Z}[[B]]/G \).

A straightforward argument shows that the wreath recursion \( \Phi \) on \( G_\mathcal{C} \) induces a wreath recursion (still denoted \( \Phi \)) on the group \( G_\mathcal{C}/N \). Furthermore, the virtual endomorphism \( \phi : \langle B, B^A \rangle \rightarrow G \) is a bijection, and induces a bijection \( \Phi : K \rightarrow K \times K \), given by

\[
\Phi(b^{4w}) = \begin{cases} 
(1, b^{4w_1}) & \text{if } \Phi(w) = (w_0, w_1), \\
(b^{4w_1}, 1) & \text{if } \Phi(w) = (w_0, w_1) \sigma.
\end{cases}
\]

We first prove \( N \subseteq \mathcal{E}_\infty \). We already checked in (18) that the first two generators of \( N \) lie in \( \mathcal{E}_\infty \). We may view Equation (20) as a relation in \( \tilde{G} \); Lemma 6.6 then implies that for every \( w \in \tilde{G} \) there exists \( n_0(w) \in \mathbb{N} \) such that if \( n \geq n_0(w) \) then all coordinates of \( \Phi^n(b^{4w}) \in \tilde{G}^{2^n} \) are trivial, except for one, which is equal to \( b^4 \).

Therefore, the elements \( b^{4w_1} \) and \( b^{4w_2} \) commute for all \( w_1, w_2 \in \tilde{G} \).

We next prove \( \mathcal{E}_\infty \subseteq N \). Assume for contradiction that \( \mathcal{E}_n \supseteq N \) for some minimal \( n \); and choose \( r \in \mathcal{E}_n \setminus N \). Then clearly \( r \in K \), because the action of \( r \) on \( X^* \) is trivial, so the image of \( r \) in \( G \) is trivial. We have \( \Phi(r) = (r_0, r_1) \), with \( r_0, r_1 \in \mathcal{E}_{n-1} \).

Since \( \Phi \) is a bijection \( K \rightarrow K \times K \), one of \( r_0, r_1 \) is non-trivial in \( G_\mathcal{C}/N \), and we have contradicted the minimality of \( n \). □

**Proposition 6.10.** None of the maps \( f_\ast \cdot b^k \), for \( k \in \mathbb{Z} \), are equivalent to each other.
Proof. We may work inside the group $\tilde{G}$, by Corollary 6.5.

Suppose that $f_3 \cdot b^r$ and $f_4 \cdot b^s$ are equivalent. Then there exists $h \in \tilde{G}$ such that $h \cdot f_3 \cdot b^r = f_4 \cdot b^s$ in $\mathfrak{F}$. We get $hb^r \cdot f_3 = f_4 \cdot b^s h$, which implies $hb^r = \phi(b^s h)$, or
\begin{equation}
\phi^a(h) = b^{-sn} h b^r.
\end{equation}

by definition of $\phi$ and since the left $\mathcal{G}_C$-action on the bimodule $\mathfrak{F}$ is free (which follows, for instance, from the fact that the $\mathcal{G}_C$-bimodule $\mathfrak{F}$ is isomorphic to the bimodule associated with the self-covering $F(z) = (\frac{2z}{1-z})^2$ of moduli space).

By induction we get from (21) that for every $n$
\begin{equation}
\phi^a(h) = b^{-sn} h b^r.
\end{equation}

By Lemma 6.7 there exists $n$ such that $\phi^n(h) \in \langle b \rangle$. Consequently, $h = b^m$ for some $m \in \mathbb{Z}$, hence $b^{-sn} h b^r = h$ and $b^{-s} = 1$. But this implies that $r = s$, since the order of $b$ is infinite. □

Corollary 6.11. Let $f$ be a degree-two topological polynomial with preperiod of length 1 and period of length 2. Then $f$ is combinatorially equivalent to precisely one of $f_3$, $f_{-3}$, or $f_4 \cdot b^r$ for some $r \in \mathbb{Z}$.

Theorem 6.4 describes the equivalence classes of the polynomials $f_3$ and $f_{-3}$. If $f_3 \cdot g = f_1 \cdot ag$ is an obstructed topological polynomial, then it is equivalent to $f_3 \cdot b^a$ if and only if there exists $m \in \mathbb{N}$ such that $\overline{\sigma^m(g)} = b^a$ in $\tilde{G}$, where $\overline{\sigma} : \tilde{G} \rightarrow \tilde{G}$ is given by (16) and (17).

7. Preperiod 2, period 1

7.1. Iterated monodromy groups. We next consider the degree-two topological polynomials with ramification graph $c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_3$, where $c_1$ is the critical value. The calculations are close to those for the “rabbit” and “airplane” polynomials, so will be given in a more condensed manner.

If $f(z) = z^2 + c$ is a polynomial whose ramification graph has preperiod 2 and period 1, then the parameter $c$ must be a root of the polynomial $x^3 + 2x^2 + 2x + 2$, i.e. one of approximately
\begin{align*}
-0.2282 + 1.1151i, & \quad -0.2282 - 1.1151i, \quad -1.5437.
\end{align*}

The corresponding points of the Mandelbrot set are the landing points of the external rays at angles $1/4, 3/4$ and $5/12$, respectively. The last angle is particular because under doubling we have $5/12 \rightarrow 5/6 \rightarrow 2/3 \rightarrow 1/3 \rightarrow 2/3$; but the dynamical rays with angles $1/3$ and $2/3$ land at the same point of the Julia set. Let us denote the corresponding polynomials by $f_{1/4}, f_{3/4}$ and $f_{5/12}$.

The wreath recursions are given by
\begin{align*}
\Phi_{f_{1/4}}(\alpha) &= \langle \alpha^{-1} \beta^{-1}, \beta \alpha \rangle, \quad \Phi_{f_{1/4}}(\beta) = \langle \alpha, 1 \rangle, \quad \Phi_{f_{1/4}}(\gamma) = \langle \gamma, \beta \rangle; \\
\Phi_{f_{3/4}}(\alpha) &= \langle \beta^{-1} \alpha^{-1}, \alpha \beta \rangle, \quad \Phi_{f_{3/4}}(\beta) = \langle 1, \alpha \rangle, \quad \Phi_{f_{3/4}}(\gamma) = \langle \gamma, \beta \rangle; \\
\Phi_{f_{5/12}}(\alpha) &= \langle \alpha^{-1} \gamma^{-1}, \gamma \alpha \rangle, \quad \Phi_{f_{5/12}}(\beta) = \langle \alpha, 1 \rangle, \quad \Phi_{f_{5/12}}(\gamma) = \langle \gamma^a, \beta \rangle.
\end{align*}

Here the $\alpha$ of $f_3$ is a small loop going positively around the critical value and connected to the circle at infinity by the external ray at angle $t$, the $\beta$ is connected by the ray at angle $2t$ and the $\gamma$ at angle $0$ (for $f_{1/4}$ and $f_{3/4}$) or $2/3$ (for $f_{5/12}$). See for example the loops $\alpha, \beta, \gamma$ for the polynomial $f_{1/4}$ on Figure 17.
Their respective nuclei are:

\[ N_{1/4} = \{ 1, \alpha, \beta, \gamma^\alpha, \alpha \beta, \beta \alpha, (\beta \alpha \gamma)^{\pm 1} \}, \]

\[ N_{3/4} = \{ 1, \alpha, \beta, \gamma^\beta, \alpha \beta, \beta \alpha, (\alpha \beta \gamma)^{\pm 1} \}, \]

\[ N_{5/12} = \{ 1, \alpha, \beta, \gamma^\gamma, \beta \gamma, \alpha \gamma, \gamma \alpha, (\beta \gamma \alpha)^{\pm 1} \}. \]

Direct verification shows that the nuclei are different as automata. It is not hard to prove that there is no possible obstruction, so every degree-two branched covering with the given post-critical dynamics is equivalent to precisely one of \( f_{1/4}, f_{3/4} \) or \( f_{5/12} \).

7.2. The iterated monodromy group on moduli space. As before, we force the post-critical sets to be of the form \( \{ 0, 1, w \} \), so that \( w \in \hat{\mathbb{C}} \setminus \{ 0, 1, \infty \} \) represents a point in moduli space. A polynomial with critical value 0 and orbit 0 \( \mapsto 1 \mapsto w \) is of the form \( f(z) = (az - 1)^2 \); then we have

\[ (a - 1)^2 = w_1, \quad (aw_0 - 1)^2 = w_1, \]

so \( aw_0 - 1 = -a + 1 \), hence

\[ a = \frac{2}{w_0 + 1} \]

and

\[ F(w_0) = w_1 = \left( \frac{w_0 - 1}{w_0 + 1} \right)^2. \]

The fixed points of \( F \) are approximately

\[ -0.6478 + 1.7214i, \quad -0.6478 - 1.7214i, \quad 0.2956. \]

The fixed point \( w \) corresponds to the polynomial \( \left( \frac{2z}{w+1} - 1 \right)^2 \), which is conjugate to \( z^2 - \frac{2}{1+w} \). Direct computation shows that the above fixed points correspond to the polynomials \( f_{1/4}, f_{3/4} \) and \( f_{5/12} \), respectively.

Let us consider the first point \( t \approx -0.6478 + 1.7214i \), and let \( a \) and \( b \) be small loops going around the points 0 and 1 in the positive direction and connected to \( t \) by straight segments. The \( F \)-preimages of \( t \) are \( t \) itself and \( 1/t \approx -0.1915 - 0.5089i \). Take the connecting path \( \ell_0 \) to be the trivial path at \( t \), and let \( \ell_1 \) be a path connecting \( t \) to \( 1/t \), intersecting the real line only once, with the intersection point strictly between 0 and 1.

Then the iterated monodromy group of \( F \) is given by the recursions

\[ \Phi(a) = \langle 1, b \rangle \sigma, \quad \Phi(b) = \langle b^{-1}a^{-1}, a \rangle. \]
This wreath recursion is contracting on the free group \( \langle a, b \rangle \), and has nucleus

\[ N = \{1, a, b, ab, a^{-1}b\}^{\pm 1}. \]

Let \( \psi \) denote the virtual endomorphism of \( G_C \) corresponding to projection on the first coordinate, i.e.

\[ \psi(a^2) = b, \quad \psi(b) = b^{-1}a^{-1}, \quad \psi(aba^{-1}) = a. \]

Define \( \overline{\psi} : G_C \rightarrow G_C \) by

\[
(22) \quad \overline{\psi} : g \mapsto \begin{cases} 
\psi(g) & \text{if } g \text{ belongs to the domain of } \psi, \\
(a\psi(ga^{-1}) & \text{otherwise}.
\end{cases}
\]

**Lemma 7.1.** For every \( g \in G_C \), there is \( n \in \mathbb{N} \) such that \( \overline{\psi}^n(g) \in \{1, a, a^{-1}b\} \).

**Proof.** It suffices to compute the orbits of \( \overline{\psi} \) on \( N \cup aN \). There are fixed points 1 and \( a \), and a cycle \( ab^{-1}a \leftrightarrow a^{-1}b \). All the other elements of \( N \cup aN \) are attracted to these cycles. \( \square \)

The generators \( a \) and \( b \) of the mapping class group \( G_C = \pi_1(\hat{C} \setminus \{0,1,\infty\}) \) correspond to the right Dehn twists shown in Figure 18, since they correspond to the fixed post-critical point \( c_3 \) moving in the positive direction around the critical value \( c_1 \) and the other post-critical point \( c_2 \), respectively.

![Figure 18. Twisting \( f_{1/4} \)](image_url)

Computation of the iterated monodromy groups of the maps \( f_{1/4} \cdot a \) and \( f_{1/4} \cdot a^{-1}b \) and their nuclei show that the first branched covering is equivalent to \( f_{5/12} \) and the second to \( f_{3/4} \).

**Theorem 7.2.** Let \( g \in \langle a, b \rangle \) be an arbitrary element of the mapping class group. Then \( f_{1/4} \cdot g \) is equivalent to \( f_{1/4}, f_{3/4} \) or \( f_{5/12} \) if and only if the orbit of \( g \) under iteration of \( \overline{\psi} \) (given by (22)) eventually reaches the fixed point 1, \( a \), or the cycle \( ab^{-1}a \leftrightarrow a^{-1}b \), respectively.

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