Spectrum of light scattered from a "deformed" Bose–Einstein condensate

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Abstract

The spectrum of light scattered from a Bose–Einstein condensate is studied in the limit of particle-number conservation. To this end, a description in terms of deformed bosons is invoked and this leads to a deviation from the usual predicted spectrum’s shape as soon as the number of particles decreases.

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The recent achievements of Bose–Einstein condensate (BEC) with a gas of atoms confined by a magnetic trap [1] has stimulated renewed interest in the question as to what signatures Bose–Einstein condensation imprints in the spectrum of light scattered from atoms in such a condensate [2, 3].

As well known, to deal with the dynamics of BEC gas the Bogolubov approximation in quantum many-body theory [4] is an efficient approach, in which the creation and annihilation operators for condensed atoms are substituted by c-numbers. One shortcoming of this method is that the total atomic particle-number may not be conserved after the approximation. Or a symmetry may be broken. To remedy this default, Gardiner [5] suggested a modified Bogolubov approximation by introducing phonon operators which conserve the total atomic particle number \( N \) and obey the bosonic commutation relation in the case of \( N \to \infty \). In this sense, this phonon operator approach gives an elegant infinite atomic particle-number approximation theory for BEC taking into account the conservation of the total atomic number.

Along this line, the case of finite number of particle has been recently investigated [6], and the algebraic method of treating the effects of finite particle number in the atomic BEC has been developed. It results a physical and natural realization of the quantum group theory [7] in the BEC systems, whose possibility was already suggested in [8], thought in a different manner.

Here, we shall use the deformed algebra to study the response of a condensate with finite number of atoms to the laser light and focus our attention on steady-state excitation.

We consider a system of weakly interacting Bose gas in a trap and a classical radiation field interacting with these two-level atoms, where \( b^\dagger, b \) denote the creation and annihilation operators for the atoms in the excited state; \( a^\dagger, a \), the creation and annihilation operators for the atoms in the ground state. These operators satisfy the
usual bosonic commutation relations. The Hamiltonian of the model reads

\[ H = \hbar \omega b^\dagger b + \hbar \left[ g(t)b^\dagger a + g^*(t)ba^\dagger \right] , \] (1)

where \( g(t) \) is a time-dependent coupling coefficient for the (classical) laser field coupled to those two states with level difference \( \hbar \omega \). Usually, the time dependence of \( g(t) \) is given by \( g \exp(-i\Omega t) \), with \( \Omega \) being the frequency of the laser beam.

Note that, with the above Hamiltonian, the total atomic particle number \( N = b^\dagger b + a^\dagger a \) is conserved. In the thermodynamic limit \( N \to \infty \), the Bogolubov approximation is usually applied, in which the ladder operators \( a^\dagger, a \) of the ground state are replaced by a c-number \( \sqrt{N_c} \), where \( N_c \) is the number of the initial condensed atoms. As a result, Eq. (1) becomes the Hamiltonian of a forced harmonic oscillator.

Moreover, we have to consider the bath of photon modes, beside the classical driving field, so that the total Hamiltonian will be

\[ H = \hbar \omega b^\dagger b + \hbar \sqrt{N_c} \left[ g(t)b^\dagger + h.c. \right] + \sum_k \Omega_k c_k^\dagger c_k + \hbar \sqrt{N_c} \sum_k \xi(k) \left[ b^\dagger c_k + h.c. \right] , \] (2)

where \( c_k \) represent radiation modes (of frequency \( \Omega_k \)) which constitute the bath and \( \xi(k) \) is the coupling coefficient pertaining to the internal atomic states.

Now, by eliminating the heat-bath variables, in view of the Markov approximation, in the case of Hamiltonian (2), it is possible to obtain a quantum stochastic differential equation that describes the dynamics of the \( b \) mode in the Heisenberg picture

\[ \partial_t b(t) = -i \Delta b(t) - ig \sqrt{N_c} - \Gamma b(t) + \sqrt{2\Gamma} b_{in}(t) , \] (3)

where \( \Delta = \omega - \Omega \), and \( \Gamma \) is the damping rate. Roughly, the latter is given by \( \Gamma = \gamma \sqrt{N_c} \), where \( \gamma \) is the one-atom linewidth. Finally, \( b_{in}(t) \) is the vacuum noise operator

\[ \langle b_{in}^\dagger(t) b_{in}(t') \rangle = \langle b_{in}(t) b_{in}(t') \rangle = 0 , \]

\[ \langle b_{in}(t) b_{in}^\dagger(t') \rangle = \delta(t-t') . \] (4)
The solution of Eq. (3) is well known [9], and in the steady-state regime it becomes
\[ \langle b(t) \rangle \equiv \beta = \frac{-ig\sqrt{N_c}}{\Gamma + i\Delta}, \] (5)
\[ \delta b(\omega) = \frac{\sqrt{2\Gamma}}{\Gamma + i\Delta} b_{in}(\omega), \] (6)
where the semiclassical approximation \( b(t) = \beta + \delta b(t) \) has been used. In (6), \( \delta b(\omega) \) is the Fourier component of the operator \( \delta b(t) \).

The spectrum of the light scattered from the atoms is given by the correlation of the operators \( b^\dagger(t) \) and \( b(t) \) [3]. Hence, in the steady state, the spectrum of fluctuations \( \langle \delta b^\dagger(\omega) \delta b(\omega') \rangle \) results zero everywhere, by virtue of (6) and (4). This means that in the long time limit, only the equal time correlations survive.

Let us now come back to the Bogolubov approximation [4]. It destroys the symmetry of Hamiltonian (1), i.e., the conservation of the total particle number is violated because \([N, H] \neq 0\). Then, to preserve the property of the initial model, it is possible to determine the following phonon operators [5]
\[ B = \frac{1}{\sqrt{N}} a^\dagger b, \quad B^\dagger = \frac{1}{\sqrt{N}} a b^\dagger. \] (7)
These operators obey a deformed algebra [6]. In fact, a straightforward calculation leads to the following commutation relation
\[ [B, B^\dagger] = 1 - 2\eta b^\dagger b, \] (8)
where we have introduced a small operator parameter \( \eta = 1/N \), which for sufficiently large number of atoms is considered as c-number. The algebra defined by Eq.(8) belongs to the \( f \)-deformed algebra [10], where in general the deformed operator is related to the undeformed one through an operator valued function \( f \) as
\[ B = bf(b^\dagger b). \] (9)
In our particular case, we have
\[ f(b^\dagger b) = \sqrt{1 - \eta(b^\dagger b - 1)}, \] (10)
and for small deformation we get

\[ B \approx b \left[ 1 - \frac{\eta}{2} (b^\dagger b - 1) \right]. \tag{11} \]

With the above in mind, the total Hamiltonian (2) should be rewritten as

\[
H = \hbar \omega b^\dagger b + \hbar \sqrt{N} \left[ g(t) B^\dagger + h.c. \right] + \sum_k \Omega_k c_k^\dagger c_k + \hbar \sqrt{N} \sum_k \xi(k) \left[ B^\dagger c_k + h.c. \right]. \tag{12}
\]

Since now \( N \) is a conserved quantity, we can consider it as a \( c \)-number (we suppose that it coincide with \( N_c \), i.e., all the atoms are initially in the condensate). Essentially, Eq. (12) describes the damped dynamics of a deformed oscillator. This is a rather cumbersome problem to deal with, as shown in [11]. Here, we simplify the treatment with the following argumentations: in the second term of r.h.s. of Eq.(12), the nonlinear character of \( B \) must be taken into account, since it is evidenciate by the radiation-field amplitude \( g \); instead, in the last term of r.h.s. of Eq.(12), such nonlinear character can be neglected due to the weak-coupling assumption with the heat bath. Hence, the resulting effective Hamiltonian, in a frame rotating with the laser frequency, is

\[
H_{eff} = \hbar \Delta b^\dagger b + \hbar \sqrt{N} g \left( b^\dagger + b \right) - \hbar \frac{\sqrt{N} g \eta}{2} \left( b^\dagger b^2 + b^2 b^\dagger \right) + \sum_k \Omega_k c_k^\dagger c_k + \hbar \sqrt{N} \sum_k \xi(k) \left[ b^\dagger c_k + bc_k^\dagger \right], \tag{13}
\]

where we have expressed \( B \) in terms of \( b \) by means of Eq.(11).

The nonlinear quantum stochastic differential equation [9] describing the dynamics of the \( b \)-mode is now derived from Eq. (13)

\[
\partial_t b(t) = i \frac{\sqrt{N} g \eta}{2} \left( b^2(t) + 2 b^\dagger(t) b(t) \right) - i \Delta b(t) - ig \sqrt{N} - \Gamma b(t) + \sqrt{2 \Gamma} b_{in}(t). \tag{14}
\]

It obviously reduces to linear equation (3) as soon as \( \eta \to 0 \).

The steady state value of the field is given by the solution of the following equation

\[
0 = i \frac{\sqrt{N} g \eta}{2} \left( |\beta|^2 + 2 |\beta|^2 \right) - i \Delta \beta - ig \sqrt{N} - \Gamma \beta. \tag{15}
\]
Of course, the solution of the above equation will be different from that of Eq. (3) (we refer to the latter as $\beta_\infty$), but they approach each other as soon as $N$ increases, as can be seen in Fig. 1.

The dynamics of the small fluctuations is given by

$$\partial_t \delta b(t) = A \delta b(t) + B \delta b^\dagger(t) + \sqrt{2\Gamma} b_n(t),$$

(16)

where

$$A = -i\Delta - \Gamma + i\sqrt{Ng} \eta (\beta + \beta^*),$$

(17)

$$B = i\sqrt{Ng} \eta \beta.$$  

(18)

In this case, the solution takes the form

$$\delta b(\omega) = \frac{1}{\Xi(\omega)} \left\{ [i\omega - A^*] b_n(\omega) + B b^\dagger_n(\omega) \right\},$$

(19)

where

$$\Xi(\omega) = |A|^2 - |B|^2 - \omega^2 - i\omega (A + A^*).$$

(20)

Finally, the spectrum, by means of Eqs. (19), (20) and (4), reads

$$S(\omega) = \int d\omega' \langle \delta b^\dagger(\omega) \delta b(\omega') \rangle = \frac{|B|^2}{|\Xi(\omega)|^2}.$$  

(21)

It is shown in Fig. 2 as a function of $N$ (besides $\omega$). It trivially vanishes for $\eta \to 0$ (i.e., $N \to \infty$), otherwise it gives a signature of finite number of particles. More precisely, we may see that it shows a central peak (typical of the Lorentian shape) by decreasing the number of particles. On the other hand, it would be interesting to study the transition from this structure to the characteristic Mollow triplet [12] of a single trapped atom. Unfortunately, our approximations are no longer valid for very few atoms, and one should devise a technique to solve completely the problem or investigate it numerically. This is a plan for the future study.

Summarizing, we have seen that the particle-number conservation in BEC requires a deformation of the bosonic field, hence the introduction of nonlinearity [10], which
may lead (in the limiting case of small number of particles) to observable effects on a probe light field. Beyond the oversimplified model used, we retain the measurement of the light spectrum in presence of few condensed atoms a promising experimental challenge. On the other hand, the use of a BEC with small number of atoms would be the subject of next generation experiments [13].

The same aim could be pursued in elementary particle field as well. In fact, the BEC may also describe the final state of pions in high-energy-heavy-ion collisions [14, 15].

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FIGURE CAPTIONS

Fig. 1. A plot of the quantity $||\beta| - |\beta_\infty||$ as a function of $N$. Values of the parameters are: $\Delta = 0$ and $g = 2.5 \gamma$. Furthermore, $\arg[\beta] = \arg[\beta_\infty] = \pi/2 \forall N$.

Fig. 2. The spectrum $S$ as a function of $\omega$ and $N$. Values of parameters as in Fig. 1.
Fig. 1 S. Mancini and V. I. Man'ko
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Fig. 2 S. Mancini and V. I. Man'ko
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