Quark and Lepton Flavor Mixings in the SU(5) Grand Unification Theory

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Abstract

We explain the imbalance of the flavor mixing angles between the quark and the lepton sectors in the context of the SU(5) GUT with the see-saw mechanism. The quark masses and the CKM matrix elements are obtained by using, respectively, the Fritzsch and Branco–Silva-Marcos form for the up- and down-quark Yukawa matrices ($y^u$ and $y^d$) at the GUT scale. The charged-lepton Yukawa matrix ($y^e$) is the transpose of $y^d$, modified by the Georgi-Jarlskog factor. We show that the neutrino masses and mixing angles suggested by the recent solar and atmospheric neutrinos are then obtained from a simple texture of the neutrino Yukawa matrix ($y^\nu$) and a diagonal right-handed Majorana mass matrix at the GUT scale.

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1 Introduction

The Standard Model (SM) is a successful theory. Current high energy experiments are explained within the SM. However, the SM cannot predict fermion masses and their flavor mixing. It is generally expected that there is a more fundamental theory which gives the SM as its low-energy effective theory. The grand unified theory (GUT) is among the candidates of a more fundamental theory. It is also believed that fermion masses and flavor mixings are the keys to open the door to new physics beyond the SM.

Recent neutrino experiments [1]-[9] have provided evidences that there are neutrino masses and their flavor mixings. According to the atmospheric-neutrino observation [3]-[7], the lepton-flavor-mixing matrix, which we call the Maki-Nakagawa-Sakata (MNS) [10] matrix, has a large mixing angle $\sin^2 2\theta_{23} \simeq 1$, where $\theta_{23}$ is the mixing angle between the second and the third generations. This is in a clear contrast with the Cabibbo-Kobayashi-Maskawa (CKM) [11] quark-flavor-mixing matrix which does not exhibit such large mixings.

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In particular, $|V_{cb}|$ is smaller than $|V_{us}|$. The mixing angle between the second and the third generation is $O(1)$ in the lepton sector, whereas it is $O(10^{-2})$ in the quark sector. This imbalance may be a clue to obtain the theory of flavor. Many attempts have been made to explain this imbalance [12]-[14].

In the SU(5) GUT [15, 16], the Yukawa matrix of the charged leptons ($y^e$) is related to that of the down-type quarks ($y^d$). As a consequence, there are mass relations between the charged leptons and the down-type quarks at low energies. The b-quark $\tau$-lepton mass ratio has been reproduced in the original SU(5) model [15, 17] and in its supersymmetric version [16] [18]. A Yukawa matrix model which reproduces all the mass ratios between the down-type quarks and the charged leptons has also been found within the SU(5) model [19]. However, the imbalance between the quark-flavor-mixing matrix and the lepton-flavor-mixing matrix has not been understood within the SU(5) theory.

In this article, we study the possibility of naturally deriving the large flavor-mixing angle in the lepton sector by using suitable Yukawa matrices within the SU(5) GUT scheme. In particular, we examine the Fritzsch – Branco – Silva-Marcos (F-BS) type Yukawa matrices [20]-[22]. Texture of these matrices have generic forms within the SU(5) GUT. It has been known [22] that the Yukawa matrices with the F-BS texture at the GUT scale reproduce well the observed quark masses and the CKM matrix elements. If we adopt the see-saw mechanism [23] as a natural explanation of the tiny neutrino masses, we should discuss the neutrino Yukawa matrix ($y^\nu$) and the heavy right-handed Majorana neutrino mass matrix ($M_R$), in addition to the up-quark ($y^u$), down-quark ($y^d$) and charged-lepton ($y^e$) Yukawa matrices. The elements of the Yukawa matrices, $y^u$ and $y^d$ are constrained by the known quark masses and the CKM matrix elements, and those of $y^e$ is related to $y^d$ in the SU(5) theory. Without loosing generality we can take a basis where the Majorana mass matrix $M_R$ is diagonal. The $y^\nu$ elements are then constrained by the observed neutrino mass-squared differences. In our analysis, we assume that the neutrinos have mass hierarchies, $m_1 \ll m_2 \ll m_3$, so that each $m_i$ is constrained by the neutrino oscillation data. The MNS matrix elements can then be calculated by assuming the texture of $y^\nu$ at the GUT scale. With very simple textures of $y^\nu$, the diagonal form and Fritzsch form, we find consistent the MNS matrix elements at the weak scale, $|U_{e2}|$ is found to be very small for the diagonal $y^\nu$, whereas $|U_{\mu3}| \sim 0.7$, $|U_{e3}| < 0.2$. $|U_{e2}|$ is large when $y^\nu$ has the Fritzsch form. Both cases are consistent with the atmospheric neutrino-oscillation experiment and also with the solar-neutrino experiments, each corresponding to the small- and large-angle solution, respectively. We cannot explain the LSND experiment [24].

We are able to calculate the Jarlskog parameter of the lepton sector ($J_{\text{MNS}}$), because this parameter is related to that of the CKM matrix ($J_{\text{CKM}}$) at the GUT scale. In general, $J_{\text{MNS}}$ does not depend on the two additional Majorana phases of the MNS matrix. We find that $J_{\text{MNS}}$ and the magnitudes of the MNS matrix elements are sensitive to the $y^\nu$ texture at the GUT scale.

This article is organized as follows. In section 2, we give the definition and parameteri-
zation of the MNS lepton-flavor-mixing matrix. In section 3, we review recent experiments and show allowed regions of the MNS matrix elements and the neutrino mass differences. In section 4, we discuss properties of the renormalization group equations (RGE) for the MNS matrix elements and the dimension-five Majorana-mass operator. In section 5, we show the F-BS type Yukawa matrices at the GUT scale and discuss their properties. In section 6, we analyze the neutrino masses and the MNS matrix numerically by using the 1-loop RGE of the minimal supersymmetric standard model (MSSM). First, we study the case where the texture of $y^\nu$ is diagonal, and examine the sensitivity of the results to $M_R$, the right-handed neutrino decoupling scale. Next, we study the case where $y^\nu$ has the Fritzsch form by setting $M_R$ at the intermediate scale, $3 \times 10^{14}$ GeV. In section 7, we give summary and discussions.

2 The Maki-Nakagawa-Sakata Matrix from the See-Saw Mechanism

The Yukawa Lagrangian of leptons above the right-handed neutrino decoupling scale ($M_R$) is

$$L_{\text{yukawa}}^{\text{high}} = y_e^{ij} \phi_d L_i \cdot e_R^c + \frac{1}{2} \left( y_\nu^{ij} \phi_u L_i \cdot \nu_R^c + y_\nu^{ji} \phi_u \nu_R^c \cdot L_j + M_{Rij} \nu_R^c \cdot \nu_R^c \right) + h.c.,$$

(2.1)

where $y_\alpha^{ij}$ ($\alpha = \nu, e$) are the Yukawa matrix elements of the neutrino and the charged lepton. $\phi_u$ and $\phi_d$ are the SU(2)$_L$ doublet Higgs bosons that give Dirac masses to the up-type and the down-type fermions, respectively. $L_i$ is the $i$-th generation SU(2)$_L$ doublet lepton. The SU(2)$_L$ invariants are

$$\phi_d L_i = \begin{pmatrix} \phi_d^0 \\ \phi_d^- \end{pmatrix} \begin{pmatrix} \nu_{Li} \\ l_{Li} \end{pmatrix} = \phi_d^{0 Li} - \phi_d^{- Li} \nu_{Li},$$

$$\phi_u L_i = \begin{pmatrix} \phi_u^+ \\ \phi_u^0 \end{pmatrix} \begin{pmatrix} \nu_{Li} \\ l_{Li} \end{pmatrix} = \phi_u^{+ Li} - \phi_u^{0 Li} \nu_{Li}.$$  

(2.2)

$\nu_R^c$ and $e_R^c$ are the $i$-th generation right-handed neutrino and charged lepton, respectively. Majorana mass matrix $M_{Rij}$ is complex and symmetric. When the right-handed neutrinos

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1 Throughout this article, we use the spinor notation where the dot product denotes the scalar product of two left-handed Weyl spinors: $\xi \cdot \eta = \xi^\alpha \epsilon_{\alpha \beta} \eta^\beta = \eta \cdot \xi$ with $\epsilon_{12} = 1$. In terms of the 4-component Dirac or Majorana notation, one finds $L \cdot e_R^c = \overline{e}_R^c L$, $L \cdot \nu_R^c = \nu_R^c \cdot L = \overline{\nu}_R^c L$, $\nu_R^c \cdot \nu_R^c = \overline{\nu}_R^c P_L N$, where $\overline{\psi} = \psi^\dagger \gamma^0$, $\psi^c = C \overline{\psi}^T$, $P_L = (1 - \gamma_5)/2$, and $N = N^c = (\nu_R^c, \nu_R^c)^T$ is the 4-component Majorana representation of the right-handed neutrino.
\( \nu_{Ri} \) are integrated out at the scale \( M_R \), the effective Lagrangian becomes

\[
\mathcal{L}_{\text{yukawa}}^{\text{low}} = y_{eij} \phi_d L_i \cdot e_{Rj}^c - \frac{1}{2} \kappa_{ij} (\phi_u L_i) \cdot (\phi_u L_j) + \text{h.c.},
\]

with

\[
\kappa = y^\nu M_R^{-1} y^{\nu T}.
\]

The charged-lepton and neutrino mass-matrices are obtained as

\[
M_e^* = -y^e \langle \phi^0_d \rangle, \quad M_\nu^* = \kappa \left( \langle \phi^0_u \rangle \right)^2.
\]

The lepton mass terms can be expressed as

\[
\mathcal{L}_{\text{mass}} = - (M_e)^*_{ij} l_{Li} \cdot e_{Rj}^c - \frac{1}{2} (M_\nu)^*_{ij} \nu_{Li} \cdot \nu_{Lj} + \text{h.c.}
\]

\[
= - (M_e)^*_{ij} \tau_{Rj} l_{Li} - \frac{1}{2} (M_\nu)^*_{ij} \bar{n}_{Ri} n_{Lj} + \text{h.c.}
\]

\[
= - \tau_L M^*_{eR} - \tau_{Ri} M^*_{R} l_{Lj} - \frac{1}{2} (\bar{n}_L M_\nu n_R + \bar{n}_R M_\nu^* n_L),
\]

where we introduce the 4-component Majorana field for the light neutrinos, \( n_i = (\nu_{Li}, \nu_{cLi})^T \).

\( M_e \) is a general complex matrix, whereas \( M_\nu \) is a symmetric complex matrix in the generation space.

We give the definition and useful parameterization of the \( 3 \times 3 \) Maki-Nakagawa-Sakata (MNS) lepton-flavor-mixing matrix \( [10] \). The MNS matrix is defined analogously to the CKM matrix \( [11, 25] \), in terms of the unitary matrices \( U_e \) and \( U_\nu \) that transform the mass-eigenstates into the weak-current eigenstates:

\[
\begin{pmatrix}
  l_{L1} \\
  l_{L2} \\
  l_{L3}
\end{pmatrix} = U_e
\begin{pmatrix}
  e_L \\
  \mu_L \\
  \tau_L
\end{pmatrix}, \quad
\begin{pmatrix}
  \nu_{L1} \\
  \nu_{L2} \\
  \nu_{L3}
\end{pmatrix} = U_\nu
\begin{pmatrix}
  \nu_1 \\
  \nu_2 \\
  \nu_3
\end{pmatrix}.
\]

That is, these unitary matrices diagonalize the charged-lepton mass-matrix squared

\[
U^\dagger_e M_e U_e = \text{diag.}(m_e^2, m_\mu^2, m_\tau^2),
\]

and the Majorana-neutrino mass-matrix

\[
U^T_\nu M_\nu U_\nu = \text{diag.}(m_1, m_2, m_3),
\]

where \( 0 \leq m_1 \ll m_2 \ll m_3 \). The MNS matrix is then defined as:

\[
(V_{\text{MNS}})_{\alpha i} \equiv (U^\dagger_e U_\nu)_{\alpha i}, \quad \nu_{\alpha} = \sum_{i=1}^{3} (V_{\text{MNS}})_{\alpha i} \nu_i,
\]
where $\alpha$ and $i$ label the neutrino flavors ($\alpha = e, \mu, \tau$) and the mass eigenstates ($i = 1, 2, 3$).

In terms of the flavor-state $\nu_\alpha$ ($\alpha = e, \mu, \tau$), the leptonic charged-current interactions take the flavor-diagonal form

$$
\mathcal{L}_{\text{CC}} = -\frac{g}{\sqrt{2}} W^-_\mu \sum_{l=e,\mu,\tau} \bar{L}_l \gamma^\mu \nu_l + \text{h.c.}.
$$

(2.11)

The $3 \times 3$ MNS matrix has three mixing angles and three phases in general. We adopt the following parameterization

$$
V_{\text{MNS}} = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\varphi_2} & 0 \\
0 & 0 & e^{i\varphi_3}
\end{pmatrix},
$$

(2.12)

where the two Majorana phases, $\varphi_2$ and $\varphi_3$, are given explicitly. The remaining matrix $U$, which has three mixing angles and one phase, can be parameterized the same way as the CKM matrix. Because the present neutrino oscillation experiments constrain directly the elements, $U_{e2}, U_{e3},$ and $U_{\mu3},$ we find it most convenient to adopt the parameterization\cite{26, 27} where we take these three matrix elements in the upper-right corner of the matrix as the independent parameters. Without losing generality, we can take $U_{e2}$ and $U_{\mu3}$ to be real and non-negative. By allowing $U_{e3}$ to have a complex phase, $U_{e3} = |U_{e3}|e^{i\delta_{\text{MNS}}}$, the number of the independent parameters is four. All the other matrix elements are then determined by the unitary conditions:

$$
U_{e1} = \sqrt{1 - |U_{e3}|^2 - |U_{e2}|^2},
$$

(2.13a)

$$
U_{\tau3} = \sqrt{1 - |U_{e3}|^2 - |U_{\mu3}|^2},
$$

(2.13b)

$$
U_{\mu1} = -\frac{U_{e2}U_{\tau3} + U_{\mu3}U_{e1}U_{e3}^*}{1 - |U_{e3}|^2},
$$

(2.13c)

$$
U_{\mu2} = \frac{U_{e1}U_{\tau3} - U_{\mu3}U_{e2}U_{e3}^*}{1 - |U_{e3}|^2},
$$

(2.13d)

$$
U_{\tau1} = \frac{U_{e2}U_{\mu3} - U_{\tau3}U_{e1}U_{e3}^*}{1 - |U_{e3}|^2},
$$

(2.13e)

$$
U_{\tau2} = -\frac{U_{\mu3}U_{e1} + U_{e2}U_{\tau3}U_{e3}^*}{1 - |U_{e3}|^2}.
$$

(2.13f)

In this parameterization, $U_{e1}, U_{e2}, U_{\mu3},$ and $U_{\tau3}$ are real and non-negative numbers, and the other elements are complex numbers. In particular, we note

$$
| (V_{\text{MNS}})_{e2} | = U_{e2}, \quad | (V_{\text{MNS}})_{\mu3} | = U_{\mu3}.
$$

(2.14)
The two Majorana phases $\varphi_2$ and $\varphi_3$ do not contribute to the Jarlskog parameter \(^2\) of the MNS matrix:

\[
J_{\text{MNS}} = \text{Im} (V_{e1} V_{\tau 1}^* V_{\tau 3} V_{e3}^*) \\
= \text{Im} (U_{e1} U_{\tau 1}^* U_{\tau 3} U_{e3}^*) \\
= \frac{U_{e1} U_{e2} U_{\mu 3} U_{\tau 3}}{1 - |U_{e3}|^2} \text{Im} (U_{e3}) ,
\]

(2.15)

where $\text{Im} (U_{e3}) = |U_{e3}| \sin (\delta_{\text{MNS}})$.

3 Experimental constraints

In this section we review briefly the experimental constraints on the neutrino masses and the MNS matrix elements under the assumptions of the three neutrino flavors and the mass hierarchy $m_1 \ll m_2 \ll m_3$.

3.1 Constraints on the MNS matrix elements

The survival and transition probabilities of the neutrino oscillation in the vacuum take the following simple form \(^2\),

\[
P_{\nu_\alpha \to \nu_\beta} = 4 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \left( \frac{m_3^2 - m_2^2 L}{4E} \right) ,
\]

(3.1)

\[
P_{\nu_\alpha \to \nu_\alpha} = 1 - 4 |U_{\alpha 3}|^2 \left( 1 - |U_{\alpha 3}|^2 \right) \sin^2 \left( \frac{m_3^2 - m_2^2 L}{4E} \right) ,
\]

(3.2)

when the following condition is satisfied\(^3\):

\[
\frac{m_2^2 - m_1^2}{2E} L \ll 1 \sim \frac{m_3^2 - m_2^2}{2E} L .
\]

(3.3)

On the other hand, the probabilities take the following form,

\[
P_{\nu_\alpha \to \nu_\beta} = 2 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \\
- \left[ 4 \text{Re} (U_{\alpha 1} U_{\beta 1}^* U_{\beta 2} U_{\alpha 2}^*) \sin^2 \left( \frac{m_2^2 - m_1^2 L}{4E} \right) + 2 J_{\text{MNS}} \sin^2 \left( \frac{m_3^2 - m_2^2 L}{2E} \right) \right] ,
\]

(3.4)

\[
P_{\nu_\alpha \to \nu_\alpha} = 1 - 2 |U_{\alpha 3}|^2 \left( 1 - |U_{\alpha 3}|^2 \right) - 4 |U_{\alpha 1}|^2 |U_{\alpha 2}|^2 \sin^2 \left( \frac{\delta m_{12}^2}{4E} L \right) .
\]

(3.5)

\(^2\)See Appendix A for more details.
when the following conditions are satisfied:

\[
\frac{m_2^2 - m_1^2}{2E} L \sim 1 \ll \frac{m_3^2 - m_2^2}{2E} L. \tag{3.6}
\]

Below, we obtain constraints on the neutrino mass differences and the MNS matrix elements from the recent neutrino-oscillation experiments with the above approximations.

### 3.1.1 CHOOZ experiments

The CHOOZ experiment \([1]\) measured the survival probability of \(\nu_e\), and found

\[
\sin^2 2\theta_{\text{CHOOZ}} < 0.18, \quad \text{for} \quad \delta m_{\text{CHOOZ}}^2 > 1 \times 10^{-3}\text{eV}^2. \tag{3.7}
\]

By assuming \(m_2^2 - m_1^2 \ll 1 \times 10^{-3} \text{eV}^2\) to accommodate the solar-neutrino oscillation (see below), we can use eq.(3.2) to obtain the following constraint:

\[
|U_{e3}|^2 \left(1 - |U_{e3}|^2 \right) < 0.045, \tag{3.8a}
\]

and

\[
|U_{e3}|^2 \left(1 - |U_{e3}|^2 \right) > 1 \times 10^{-3}\text{eV}^2. \tag{3.8b}
\]

### 3.1.2 Solar-neutrino deficit

Deficits of solar neutrinos observed at several terrestrial experiments \([4, 5]\) have been successfully interpreted in terms of the \(\nu_e \to \nu_X (\nu_X \neq \nu_e, \overline{\nu}_e)\) oscillation in the following three scenarios \([2]\).

**MSW small-mixing solution:**

\[
3 \times 10^{-3} < \sin^2 2\theta_{\text{SUN}} < 1.1 \times 10^{-2}, \tag{3.9a}
\]

and

\[
4 \times 10^{-6} < \delta m_{\text{SUN}}^2(\text{eV}^2) < 1.2 \times 10^{-5}. \tag{3.9b}
\]

**MSW large-mixing solution:**

\[
0.42 < \sin^2 2\theta_{\text{SUN}} < 0.74, \tag{3.10a}
\]

and

\[
8 \times 10^{-6} < \delta m_{\text{SUN}}^2(\text{eV}^2) < 3.0 \times 10^{-5}. \tag{3.10b}
\]

**Vacuum oscillation solution:**

\[
0.7 < \sin^2 2\theta_{\text{SUN}} < 1.0, \tag{3.11a}
\]

and

\[
6 \times 10^{-11} < \delta m_{\text{SUN}}^2(\text{eV}^2) < 1.1 \times 10^{-10}. \tag{3.11b}
\]

Under the mass-hierarchy assumption of \(m_1 \ll m_2 \ll m_3\), the solar-neutrino deficits should be explained by the oscillation of the lighter two neutrinos in both the MSW \([30]\) and the vacuum-oscillation \([31]\) scenarios. The \(\nu_e\) survival probability in the vacuum can then be expressed by eq.(3.5). Because the mass-squared difference \(m_3^2 - m_2^2\) suggested by the atmospheric neutrino observation is much larger than the differences eq.(3.9a) \(\sim\) eq.(3.11a),...
there appears an energy-independent deficit factor $-2|U_{e3}|^2 (1 - |U_{e3}|^2)$. This factor should be smaller than 9% by the CHOOZ experiment if $m_3^2 - m_1^2 > 1 \times 10^{-3}$ eV$^2$; see eq.(3.8). It is also constrained by the observation of lower energy solar neutrino [5]. Since we need only rough estimate of the allowed ranges of the MNS matrix elements, we ignore the small energy-independent deficit factor and interpret the results of the two-flavor analysis eq.(3.9a) $\sim$ eq.(3.11a) by using the following identifications:

$$\sin^2 2\theta_{\text{SUN}} = 4 |U_{e1}|^2 |U_{e2}|^2$$

$$= 4 \left(1 - |U_{e2}|^2 - |U_{e3}|^2 \right)|U_{e2}|^2 , \quad (3.12a)$$

$$\delta m_{\text{SUN}}^2 = m_2^2 - m_1^2 . \quad (3.12b)$$

### 3.1.3 Atmospheric-neutrino anomaly

The recent analysis of the atmospheric neutrino data from the Super-Kamiokande experiment finds [6].

$$0.7 < \sin^2 2\theta_{\text{ATM}} < 1 , \quad (3.13a)$$

and

$$3 \times 10^{-4} < \delta m_{\text{ATM}}^2 (\text{eV}^2) < 7 \times 10^{-3} , \quad (3.13b)$$

for $\nu_\mu \to \nu_X (\nu_X \neq \nu_\mu, \nu_e)$ oscillation. The $\nu_\mu \to \nu_e$ oscillation scenario is not only disfavored by the CHOOZ experiment eq.(3.7), but also disfavored by the Super-Kamiokande datum by itself. In our three-flavor analysis, the data should be interpreted with the $\nu_\mu \to \nu_\tau$ oscillation under the constraint eq.(3.3). By using eq.(3.2), we have the following identifications:

$$\sin^2 2\theta_{\text{ATM}} = 4 |U_{\mu3}|^2 \left(1 - |U_{\mu3}|^2 \right) , \quad (3.14a)$$

$$\delta m_{\text{ATM}}^2 = m_3^2 - m_2^2 . \quad (3.14b)$$

### 3.1.4 Neutrino masses

The neutrino oscillation experiments measure only the mass-squared differences of the three neutrinos. Under the assumption of the neutrino-mass hierarchies,

$$m_1 \ll m_2 \ll m_3 , \quad (3.15)$$

the absolute values of the neutrino masses can be constrained determined as follows:

$$m_3^2 \simeq m_3^2 - m_2^2 = \delta m_{\text{ATM}}^2 , \quad (3.16a)$$

$$m_2^2 \simeq m_2^2 - m_1^2 = \delta m_{\text{SUN}}^2 . \quad (3.16b)$$

The heaviest neutrino mass is then determined by eq.(3.13b)

$$m_3 = (0.02 \sim 0.08) \text{ eV} , \quad (3.17)$$

\[^3\text{Quantitative study of three-flavor oscillation effects will be reported elsewhere.}\]
and there are three possibilities for $m_2$:

$$m_2 = (0.002 \sim 0.003) \text{ eV}$$

(\text{MSW small-mixing solution}), \hspace{1cm} (3.18a)

$$m_2 = (0.003 \sim 0.005) \text{ eV}$$

(\text{MSW large-mixing solution}), \hspace{1cm} (3.18b)

$$m_2 = (8 \sim 10) \times 10^{-6} \text{ eV}$$

(Vacuum mixing solution). \hspace{1cm} (3.18c)

$m_1$ cannot be determined. Since its magnitude plays little role in the following analysis, we simply assume the hierarchy eq. (3.15):

$$m_1^2 < \frac{m_2^2}{10}. \hspace{1cm} (3.19)$$

### 3.2 Quark and lepton masses and the CKM matrix

In our analysis, we adopt the following quark and lepton masses at the Z-boson mass scale [25, 32] to constrain the Yukawa matrices elements:

$$m_t(m_Z) = 175 \pm 6 \text{ GeV},$$

$$m_c(m_Z) = 0.59 \pm 0.07 \text{ GeV},$$

$$m_u(m_Z) = 0.0022 \pm 0.0007 \text{ GeV}, \hspace{1cm} (3.20)$$

$$m_b(m_Z) = 3.02 \pm 0.19 \text{ GeV},$$

$$m_s(m_Z) = 0.077 \pm 0.011 \text{ GeV},$$

$$m_d(m_Z) = 0.0038 \pm 0.0007 \text{ GeV}, \hspace{1cm} (3.21)$$

$$m_\tau(m_Z) = 1746.5 \pm 0.3 \text{ MeV},$$

$$m_\mu(m_Z) = 102.7 \text{ MeV},$$

$$m_\epsilon(m_Z) = 0.487 \text{ MeV}. \hspace{1cm} (3.22)$$

The CKM matrix elements are determined by the magnitudes of the three off-diagonal elements [25]

$$|V_{us}| = 0.219 \sim 0.224, \hspace{1cm} (3.23a)$$

$$|V_{cb}| = 0.036 \sim 0.046, \hspace{1cm} (3.23b)$$

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.06 \sim 0.10, \hspace{1cm} (3.23c)$$
and one phase, which in our parameterization is related to the Jarlskog parameter as
\[
\sin \delta_{\text{CKM}} = \frac{(1 - |V_{ub}|^2)}{V_{ud}V_{us}V_{cb}V_{tb}|V_{ub}|} J_{\text{CKM}},
\]
(3.24)
where $\delta_{\text{CKM}}$ is the $CP$ violation angle. We adopt the same phase combination for the MNS ($U_{\alpha i}$) and the CKM matrices, and $V_{ud}, V_{us}, V_{cb}, V_{tb}$ are all real and positive. The remaining CKM matrix elements are determined by the unitarity conditions analogously to eq.(2.13).

Constraints on the neutrino masses and the MNS matrix elements depend on the three scenarios that explain the solar-neutrino experiments. In Table 1 we summarize the typical allowed ranges of these parameters under the assumption of $m_1 \ll m_2 \ll m_3$ for the three scenarios. We denote “MSW-S” for the MSW small-mixing scenario, “MSW-L” for the MSW large-mixing scenario, and “V-O” for the vacuum-oscillation scenario.

|       | $m_2$ (eV) | $m_3$ (eV) | $|U_{e2}|$ | $|U_{e3}|$ | $|U_{\mu 3}|$ |
|-------|-----------|-----------|----------|----------|-----------|
| MSW-S | 0.002 ~ 0.003 | 0.02 ~ 0.08 | 0.027 ~ 0.053 | < 0.22 | 0.48 ~ 0.88 |
| MSW-L | 0.003 ~ 0.005 | 0.02 ~ 0.08 | 0.35 ~ 0.50 | < 0.22 | 0.48 ~ 0.88 |
| V-O   | (8 ~ 10) $\times$ 10^{-6} | 0.02 ~ 0.08 | 0.48 ~ 0.71 | < 0.22 | 0.48 ~ 0.88 |

Table 1: Allowed ranges of the neutrino masses and the MNS matrix elements in the three scenarios under the assumption $m_1 \ll m_2 \ll m_3$.

4 Renormalization Group Equations

The RGE of the coefficient $\kappa$ of dimension-five operator in the effective Lagrangian eq.(2.3), which is formed by the see-saw mechanism [23], has been studied in Refs. [33]. Below the right-handed neutrino mass scale $M_R$, the matrix $\kappa$ satisfies the following RGE in the MSSM [33]:

\[
8\pi^2 \frac{d}{dt} \kappa = \left\{ \text{tr} \left( 3y^u y^{u\dagger} \right) - 4\pi \left( 3\alpha_2 + \frac{3}{5} \alpha_1 \right) \right\} \kappa + \frac{1}{2} \left\{ \left( y^e y^{e\dagger} \right) \kappa + \kappa \left( y^e y^{e\dagger} \right)^T \right\},
\]
(4.1)
where $t = \ln \mu$, $y^u$ is the up-quark Yukawa matrix and $y^e$ is the charged-lepton Yukawa matrix. The RGE’s of these matrices and the gauge couplings are summarized in Appendix B. When the contributions from the first-generation fermions can be neglected, the RGE of the lepton-flavor mixing angle $\theta_{23}$ ($|U_{\mu 3}| \sim \sin \theta_{23}$) takes a very simple form [33]:

\[
8\pi^2 \frac{d}{dt} \sin^2 2\theta_{23} = -\sin^2 2\theta_{23} \left( 1 - \sin^2 2\theta_{23} \right) \left( |y_\tau|^2 - |y_\mu|^2 \right) \frac{\kappa_{33} + \kappa_{22}}{\kappa_{33} - \kappa_{22}},
\]
(4.2)
where $y_\mu$ and $y_\tau$ are the second and third generation charged-lepton Yukawa couplings, respectively. It should be noted in eq.(4.2) that if $\kappa_{33} > \kappa_{22}$ holds, then $\sin^2 2\theta_{23} = 1$ is the infra-red fixed point of the equation [33], and hence it is possible to obtain $\sin^2 2\theta_{23} \sim 1$ at the $m_Z$ scale as long as its magnitude is large at the right-handed neutrino decoupling scale, $M_R$.

5 GUT scale Yukawa matrix texture

The Yukawa terms at the SU(5) GUT scale are

$$L_{\text{yukawa}}^{\text{GUT}} = y^{10}_{ij} \varepsilon_{abcde} (\phi^u)_a (\chi_i)_b \cdot (\chi_j)_d e + y^{5^*}_{ij} (\phi^d)_a (\chi_i)_b \cdot (\psi_j)^b + y^1_{ij} (\phi^u)_a (\chi_i)_b \cdot (\nu_{Rj})^a + \mathcal{M}_{Rij} (\nu_{Ri}) \cdot (\nu_{Rj})^b + h.c.,$$

(5.1)

where $\chi$, $\psi$ and $\nu_{Ri}$ are all left-handed fermions which transform as $10$, $5^*$ and $1$ representation of the SU(5) group, respectively. Indices $i, j (= 1 \sim 3)$ stand for generations and $a, b, \ldots (= 1 \sim 5)$ give the SU(5) indices. $\phi^u$ and $\phi^d$ are now $5$ and $5^*$ representation Higgs particles, respectively. The Yukawa matrices of the SM are related to those at the GUT scale as,

$$y^u_{ij} = y^{u}_{ji} = y^{10}_{ij},$$

(5.2a)

$$y^d_{ij} = y^{e}_{ji} = y^{5^*}_{ij},$$

(5.2b)

$$y^{\nu}_{ij} = y^1_{ij}.$$  

(5.2c)

We note here that the Yukawa matrix $y^u$ is symmetric and that the Yukawa matrix $y^e$ is the transpose on the Yukawa matrix $y^d$ in the flavor space:

$$y^u = (y^u)^T, \quad y^e = (y^d)^T,$$

(5.3)

at the GUT scale in the SU(5) theory [34]. These are direct consequences of the SU(5) representations of quarks and leptons:

$$10 : \chi_{ab} = u^c + Q + e^c,$$

$$5^* : \psi^a = d^c + L,$$

$$1 : \nu_{Ri},$$

(5.4)

where $u^c$ and $d^c$ are the charge-conjugation of the right-handed up and down quarks, respectively. $Q$ is the SU(2)$_L$ doublet left-handed quarks.

Both up- and down-type Yukawa matrices can be transformed into the nearest-neighbor-interaction (NNI) form by a weak-basis transformation without loosing generality [35]. Recently, many authors studied the texture of the Yukawa matrices at the $m_Z$ scale in the NNI basis [36]-[38]. E. Takasugi has further shown [39] that one of the up- or down-quark mass matrices can be transformed into either the Fritzsch form [20] or the BS form [21],
while keeping the other matrix in the NNI form. For instance, under the SU(5) constraint, eq.\ref{eq:5.2a}, we can take the Fritzsch texture for the up-quark Yukawa matrix,

\[
\overline{Q}_L y^u u_R = \overline{Q}_L \begin{pmatrix} 0 & a_u & 0 \\ a_u & 0 & b_u \\ 0 & b_u & c_u \end{pmatrix} u_R ,
\]

(5.5)

where \(a_u\) and \(b_u\) are real numbers and \(c_u\) is a complex number without losing generality. The down-quark Yukawa matrix (\(y^d\)) can still be parameterized by the NNI form,

\[
\overline{Q}_L y^d d_R = \overline{Q}_L \begin{pmatrix} 0 & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix} d_R ,
\]

(5.6)

in general. Here the nonzero five \(x_{ij}\) components are complex numbers. Three phases in eq.\ref{eq:5.6} can be removed by using the rephasing freedom of \(d_R\), and \(c_u\) can be made real by rephasing \(u_R\) once the GUT is broken. The parameterization is still most general, containing 10 real parameters for six quark masses and four independent elements of the CKM matrix.

If we adopt the parameterization of eq.\ref{eq:5.3} at the weak scale, three parameters are given analytically in terms of the up-quark-mass eigenvalues\footnote{When \(a_u, b_u,\) and \(c_u\) are non-negative real numbers, the determinant of \(y^u\) is negative. On the other hand, all up-quark-mass eigenvalues are real and positive. These two statements are consistent, because we can choose the unitary matrix for the right-handed fields as \(\text{det}(y^u) \text{det}(U_R^u) v_u^3 = m_u m_c m_t\), with \(\text{det}(U_R^u) = -1\).}

\[
m_u = \frac{v_u}{\sqrt{2}} \frac{a_u^2 c_u}{b_u^2} , \quad m_c = \frac{v_u}{\sqrt{2}} \frac{b_u^2}{c_u} , \quad m_t = \frac{v_u}{\sqrt{2}} c_u .
\]

(5.7)

Here \(v_u/\sqrt{2} = v/\sqrt{2} \cdot \sin \beta(> 0)\) is the vacuum expectation value (vev) of the \(\phi_u^0\) field, where

\[
\tan \beta = \frac{\langle \phi_u^0 \rangle}{\langle \phi_d^0 \rangle} = \frac{v_u}{v_d} ,
\]

(5.8)

and \(v_u^2 + v_d^2 = v^2 \simeq (246 \text{ GeV})^2\). The observed up-quark masses at the \(m_Z\) scale implies the hierarchy

\[
a_u : b_u : c_u = \sqrt{\frac{m_u m_c}{m_t}} : \sqrt{\frac{m_c}{m_t}} : 1 .
\]

(5.9)
On the other hand, the well-known empirical relations at the $m_Z$ scale,

$$|V_{us}| \sim \sqrt{\frac{m_d}{m_s}}, \quad |V_{cd}| \sim \frac{m_s}{m_b}, \quad |\frac{V_{ub}}{V_{cb}}| \sim \sqrt{\frac{m_u}{m_c}},$$

implies in the NNI basis of eq.(5.6) that the $y^d$ elements should satisfy

$$|x_{12}| \sim |x_{21}| \quad \text{and} \quad |x_{32}| \sim |x_{33}|.$$  \hspace{1cm} (5.11)

Hence at the weak scale, the matrix $y^d$ should approximately have the Branco–Silva–Marcos (BS) form

$$\bar{Q}_L y^d d_R = \bar{Q}_L \left( \begin{array}{ccc} 0 & a_d e^{i\phi_1} & 0 \\ a_d e^{-i\phi_1} & 0 & b_d e^{i\phi_2} \\ 0 & c_d & c_d \end{array} \right) d_R,$$  \hspace{1cm} (5.12)

where $a_d$, $b_d$, and $c_d$ are real positive numbers and $\phi_{1,2}$ are phases. We have made use of the rephasing freedom to obtain the above phase assignment. These textures of $y^u$ and $y^d$ are one of the simplest set of the quark Yukawa matrices at the $m_Z$ scale, which is consistent with all the experimental data. The down-quark mass eigenvalues are then related to the parameters $a_d$, $b_d$ and $c_d$ of eq.(5.12) as

$$m_d = \frac{v_d a_d^2}{\sqrt{2} b_d}, \quad m_s = \frac{v_d b_d}{2}, \quad m_b = v_d c_d,$$  \hspace{1cm} (5.13)

where $v_d/\sqrt{2} = v/\sqrt{2} \cdot \cos \beta (> 0)$ is the vev of the $\phi_d^0$ field.

Because the renormalization effect is not overwhelming between the GUT scale and weak scale in the MSSM, we adopt the parameterization eq.(5.5) for $y^u$ and eq.(5.12) for $y^d$ at the GUT scale. The elements eq.(5.7) and eq.(5.13) are then valid for effective “masses” at the GUT scale. The charged-lepton Yukawa matrix $y^e$ is the transpose of the down-quark one $y^d$ at the GUT scale. The asymmetry of the BS texture then leads to the sharp difference between the down-quark and the charged-lepton sectors in the squared Yukawa matrices that are diagonalized by the relevant Unitary matrices:

$$y^d y^d^\dagger = \left( \begin{array}{ccc} |a_d|^2 & 0 & a_d c_d e^{i\phi_1} \\ 0 & |a_d|^2 + |b_d|^2 & b_d c_d e^{i\phi_2} \\ a_d c_d e^{-i\phi_1} & b_d c_d e^{-i\phi_2} & 2 c_d^2 \end{array} \right) \simeq \frac{m_d^2}{v_d^2} \left( \begin{array}{ccc} |\alpha_d|^2 & 0 & \alpha_d \\ 0 & |\beta_d|^2 & \beta_d \\ \alpha_d^* & \beta_d^* & 2 \end{array} \right),$$

$$y^e y^e^\dagger = \left( \begin{array}{ccc} |a_d|^2 & 0 & a_d b_d e^{-i(\phi_1 + \phi_2)} \\ 0 & a_d^2 + c_d^2 & c_d^2 \\ a_d b_d e^{i(\phi_1 + \phi_2)} & c_d^2 & b_d^2 + c_d^2 \end{array} \right) \simeq \frac{m_e^2}{v_d^2} \left( \begin{array}{ccc} |\alpha_e|^2 & 0 & \alpha_e \beta_e \\ 0 & 1 + |\alpha_e|^2 & 1 \\ \alpha_e^* \beta_e^* & 1 & 1 + |\beta_e|^2 \end{array} \right).$$
where
\[ |\alpha_d| = \frac{2m_s}{m_b} \sqrt{\frac{m_d}{2m_s}}, \quad |\beta_d| = \frac{2m_s}{m_b}, \quad (5.15a) \]
\[ |\alpha_e| = \frac{2m_\mu}{m_\tau} \sqrt{\frac{m_e}{2m_\mu}}, \quad |\beta_e| = \frac{2m_\mu}{m_\tau}. \quad (5.15b) \]

When we omit the first generation contribution by setting \( a_u = a_d = 0 \), the mixing angle between the second and the third generation in the charged-lepton mixing matrix \( U_e \) of eq.(2.7) satisfies
\[ \tan 2\theta_{23}^e \simeq \frac{2}{|\beta_e|^2 - |\alpha_e|^2} \simeq \frac{1}{2} \left( \frac{m_\tau}{m_\mu} \right)^2 \gg 1, \quad (5.16) \]
where \( \sin \theta_{23}^e = (U_e)_{23} \). Therefore, if the neutrino-mixing matrix \( U_\nu \) of eq.(2.7) is not too much different from the diagonal matrix, then desired boundary condition of \( \sin^2 2\theta_{23} \simeq 1 \) at the GUT scale can be obtained. This is in sharp contrast with the corresponding mixing angle
\[ \tan 2\theta_{23}^d \simeq |\beta_d| = \frac{2m_s}{m_b} \ll 1, \quad (5.17) \]
in the down-quark sector. If the up-quark Yukawa matrix is approximately diagonal (as in the Fritzsch form eq.(5.3) with \( a_u \ll b_u \ll c_u \)), the corresponding CKM matrix element remains small. The two opposing results follows essentially from the same Yukawa matrix in the SU(5) theory.

It has been well-known that the SU(5) constraint, eq.(5.21), on the lepton and down-quark Yukawa matrices leads to the unacceptable mass relations: \( m_d/m_e = m_s/m_\mu = m_b/m_\tau \). A possible solution to this problem has been proposed by Georgi and Jarlskog [19] where a 45* Higgs boson gives masses to the first two generation down quarks and charged-leptons\(^5\). By adopting their idea, we find that the lepton Yukawa matrix of the form
\[ \mathcal{L}_L y^e e_R = \mathcal{L}_L \left( \begin{array}{ccc} 0 & a_d e^{-i\phi_1} & 0 \\ a_d e^{i\phi_1} & 0 & c_d \\ 0 & -3b_d e^{i\phi_2} & c_d \end{array} \right) e_R, \quad (5.18) \]
reproduces the desired mass formulas
\[ m_e = \frac{v_d}{\sqrt{2}} a_3^2 \frac{a_d^2}{3b_d}, \quad m_\mu = \frac{v_d}{3} \frac{3b_d}{2}, \quad m_\tau = v_d c_d. \quad (5.19) \]

The Yukawa matrices, eq.(5.12) for \( y^d \) and eq.(5.18) for \( y^e \), can be obtained at the GUT scale if the element \( y^d_{23} (y^e_{32}) \) is dominated by the coupling to the 45* Higgs, while the

\(^5\) An alternative, but similar, solution in the supersymmetric SU(5) model is found e.g. in ref.[14].
other elements come from the couplings to the $5^*$ Higgs. The Yukawa matrices eq. (5.3), eq. (5.12) and eq. (5.18) can e.g. be obtained from the SU(5) Lagrangian

$$L_{\text{yukawa}}^\text{GUT} = y_{ia3}^{10} \varepsilon^{abcde} (\phi^u)_a (\chi_{ia})_{bc} \cdot (\chi_{ja})_{de} + y_{ia3}^{5^*} (\phi^d)_a (\chi_{ia})_{ab} \cdot (\psi_{jd})^b \\
+ y_{23}^{45^*} (\sigma)^{ab} (\chi_2)_{ab} \cdot (\psi_3)^c + \text{neutrino terms} + h.c.,$$

(5.20)

where, $\sigma$ is the $45^*$ Higgs particle. All Yukawa elements are set to zero except $(i_u, j_u) = (1, 2), (2, 1), (2, 3), (3, 2), (3, 3),$ and $(i_d, j_d) = (1, 2), (2, 1), (3, 2), (3, 3).$ It should be noted here that although the NNI texture of eqs. (5.3), (5.12), and (5.18) are achieved by using the weak-basis transformation of the $y$ matrix in eq.(5.18) that consist our assumptions [22].

Let us clarify our assumptions for the Yukawa matrices at the GUT scale. It has been known that the zeros of the F-BS textures are automatically obtained by making a proper weak-basis transformation, because they are special cases of the generic NNI form [33]. In the case of the SU(5) GUT, this statement needs clarification because the left and right components of the up-quarks belong to the same representation and hence they cannot be rotated independently. We find, however, that as long as we restrict ourselves to the symmetric Yukawa matrix for the up-quark, the Yukawa matrix $y^u$ can be transformed into the Fritzsch form while retaining the general NNI form for the down-quark Yukawa matrix. The proof is obtained by using the weak-basis transformation of the $\chi(10)$ and $\psi(5^*)$ fields, similarly to that of Ref. [33]. Thus, the textures of our Yukawa matrices $y^u$ and $y^d(y^e)$ are still general. It is the restriction of the down-quark Yukawa matrix to the BS form in eq. (5.12) and the Georgi-Jarlskog modification for the charged-lepton Yukawa matrix in eq. (5.18) that consist our assumptions [22].

In addition, we have the Dirac-type Yukawa matrix for the neutrino ($y^{\nu}$) and the Majorana type Yukawa matrix for the right-handed neutrino ($\mathcal{M}_R$) at the SU(5) GUT scale. We can transform $\mathcal{M}_R$ to the diagonal form in general, but $y^{\nu}$ cannot be transformed to the NNI form at the same time. This is because in the SU(5) GUT, only three types of the matter fields, that transform as $10, 5^*$, and $1$, are rotated independently in the flavor space. The degrees of freedom associated with the weak-basis transformation in the flavor space can be used to make $y^u$ and $y^d$ in the NNI form and $\mathcal{M}_R$ diagonal, but there is no freedom left to simplify the fourth matrix $y^{\nu}$. Therefore, we need to assume the texture of $y^{\nu}$ in order to make predictions on the MNS matrix elements.

In summary, the matrices $y^u$ and $y^d(y^e)$, are transformed to the F-BS form and $\mathcal{M}_R$ to the diagonal form by using the weak-basis transformation degrees of freedom. However, the three eigenvalues of $\mathcal{M}_R$ and the texture of $y^{\nu}$ are not known. In the following section, we study consequences of a few simple textures of $y^{\nu}$ while taking $\mathcal{M}_R$ to be $M_R \times I$, where $M_R$ is the right-handed neutrino decoupling scale and $I$ is the $3 \times 3$ unit matrix. We note here that the light-neutrino mass matrix $\kappa = y^{\nu} \mathcal{M}_R^{-1} y^{\nu T}$ takes its most general form with $\mathcal{M}_R = M_R \times I$, as long as the matrix $y^{\nu}$ is taken general. It is worth noting here
that when the original Majorana matrix $\mathcal{M}_R$ contains $CP$ violating phases, the phases appear in the matrix $y^\nu$ in this basis. By taking $y^\nu$ to be real in the following analysis, we assume $CP$ invariance in the $\nu_R$ mass matrix. This assumption affects our predictions for the Majorana phases $\varphi_2$ and $\varphi_3$ of the MNS matrix elements at the weak scale.

6 The texture of $y^\nu$ and the MNS matrix

In this section, we analyze the Maki-Nakagawa-Sakata (MNS) lepton-flavor mixing matrix by using the 1-loop RGE of the MSSM from the GUT scale ($M_{GUT} = 1.7 \times 10^{16}$ GeV) to the $m_Z$ scale.

We report the results of our exploratory studies where we examine consequences of a few simple textures of the matrix $y^\nu$. We first examine the case of the simplest texture, the diagonal $y^\nu$, and study consequences of the two choices of the right-handed neutrino decoupling scale, $M_R = M_{GUT}$ and $M_R = 3 \times 10^{14}$ GeV. In the latter part, we study consequences of the Fritzsch-type texture.

6.1 Diagonal $y^\nu$ texture

We first study consequences of diagonal $y^\nu$. The Yukawa matrices take the following forms at the SU(5) GUT scale

$$y^u = \begin{pmatrix} 0 & a_u & 0 \\ a_u & 0 & b_u \\ 0 & b_u & c_u \end{pmatrix}, \quad y^\nu = \begin{pmatrix} a_\nu & 0 & 0 \\ 0 & b_\nu & 0 \\ 0 & 0 & c_\nu \end{pmatrix},$$

$$y^d = \begin{pmatrix} 0 & a_d e^{i\phi_1} & 0 \\ a_d e^{-i\phi_1} & 0 & b_d e^{i\phi_2} \\ 0 & c_d & c_d \end{pmatrix}, \quad y^e = \begin{pmatrix} 0 & a_d e^{-i\phi_1} & 0 \\ a_d e^{i\phi_1} & 0 & c_d \\ 0 & -3b_d e^{i\phi_2} & c_d \end{pmatrix}. \quad (6.1)$$

Here we assume for brevity that all elements of $y^\nu$, $a_\nu$, $b_\nu$, and $c_\nu$ are real.

We have 11 parameters in the Yukawa matrices. On the other hand, there are 22 observables: 12 fermion masses, 3 angles and 1 phase each for the CKM matrix and the MNS matrix, and the 2 more Majorana phases of the MNS matrix. Thus, there are 11 predictions, whereas two Majorana phases $\varphi_2$ and $\varphi_3$ are unobservable in the neutrino oscillation experiments.\footnote{They are observable in the lepton-number violating processes e.g. in the neutrino-less double beta decays.}

The generation hierarchy among the $y^\nu$ elements,

$$a_\nu \ll b_\nu \ll c_\nu, \quad (6.2)$$

follows from our assumption of $m_1 \ll m_2 \ll m_3$, while the following relations

$$a_u \ll b_u \ll c_u, \quad a_d \ll b_d \ll c_d, \quad (6.3)$$
are needed to reproduce the known quark and lepton masses.

We set $\tan \beta = 3$ in our numerical evaluation of the 1-loop RGE’s. The gauge couplings are set at $\alpha_1(m_Z) = 0.017$ and $\alpha_2(m_Z) = 0.034$. If $\tan \beta$ is significantly smaller than 2, the Yukawa coupling of the top quark blows up below the GUT scale. The fits with the experimental values of the CKM matrix elements become worse as $\tan \beta$ increases [22].

The parameters $a_u$, $b_u$, $c_u$ and $a_d$, $b_d$, $c_d$ are fixed, respectively, by the central values of the up-type-quark and the charged-lepton masses, as shown in eq.(3.20) and eq.(3.22). $a_\nu$, $b_\nu$ and $c_\nu$ are fixed by the neutrino masses which are chosen in the allowed range of Table 1. Phases in the $y^d$ ($y^e$) are constrained by the CKM matrix elements $|V_{us}|$ and $|V_{cb}|$. The three down-type quark masses, the CKM matrix element $|V_{ub}|$, $J_{CKM}$, and all the six independent parameters of the MNS matrix can be predicted.

By numerical analysis of the RGE, we obtain the down-type quark masses
\[
\begin{align*}
    m_b(m_Z) &= 3.3 \text{ GeV}, \\
    m_s(m_Z) &= 0.081 \text{ GeV}, \\
    m_d(m_Z) &= 0.0032 \text{ GeV}.
\end{align*}
\]
These values are roughly consistent with the experimental values of eq.(3.21), reconfirming the validity of the Georgi-Jarlskog scenario [19]. The CKM matrix elements are
\[
V_{us} = 0.22, \quad V_{cb} = 0.046, \quad \frac{|V_{ub}|}{V_{cb}} = 0.10, \\
J_{CKM} = 3.6 \times 10^{-5} \quad (\delta_{CKM} = 78.1^\circ),
\]
when we take the parameters in Table 2. They are also consistent with the experimental constraint of eq.(3.23). The results on the quark masses and the CKM matrix elements are almost common in all the analyses below.

### 6.1.1 $M_R = M_{GUT}$

We first study the simplest case of $M_R = M_{GUT}$, where the light neutrino Yukawa matrix $\kappa$ takes the following form
\[
\kappa = y^\nu M_R^{-1} y^\nu T = \frac{1}{M_{GUT}} \begin{pmatrix} a_\nu^2 & 0 & 0 \\ 0 & b_\nu^2 & 0 \\ 0 & 0 & c_\nu^2 \end{pmatrix},
\]

| $a_u$ | $b_u$ | $c_u$ | $a_d$ | $b_d$ | $c_d$ | $\phi_1$ | $\phi_2$ |
|-------|-------|-------|-------|-------|-------|-----------|-----------|
| $1.1 \times 10^{-4}$ | $4.0 \times 10^{-2}$ | 0.93 | $1.0 \times 10^{-4}$ | $5.9 \times 10^{-4}$ | $1.5 \times 10^{-2}$ | 60° | 253° |

Table 2: The input parameters at the GUT scale.
at the GUT scale. When the neutrino mass $m_2$ is fitted to the MSW small mixing solution of eq. (3.18a), $(m_1, m_2, m_3) = (0.0003, 0.003, 0.03)\text{eV}$ for definiteness, the MNS matrix elements become

$$U_{e2} = 0.058, \quad |U_{e3}| = 0.058, \quad U_{\mu 3} = 0.70.$$ (6.7)

These values are consistent with the corresponding experimental constraints that are summarized in Table 1. The values of $a_\nu$, $b_\nu$ and $c_\nu$ at the GUT scale are

$$a_\nu = 0.54, \quad b_\nu = 1.7, \quad c_\nu = 5.4.$$ (6.8)

The results are summarized in Table 3 in the first row (MSW-S).

|       | $a_\nu$ | $b_\nu$ | $c_\nu$ | $m_1(\text{eV})$ | $m_2(\text{eV})$ | $m_3(\text{eV})$ | $U_{e2}$ | $|U_{e3}|$ | $U_{\mu 3}$ |
|-------|---------|---------|---------|-----------------|-----------------|----------------|----------|-----------|------------|
| MSW-S | 0.54    | 1.7     | 5.4     | 0.0003          | 0.003           | 0.03           | 0.058    | 0.058     | 0.70       |
| MSW-L | 0.70    | 2.2     | 4.4     | 0.0005          | 0.005           | 0.02           | 0.058    | 0.058     | 0.70       |
| V-O   | 0.031   | 0.10    | 4.4     | $1 \times 10^{-6}$ | $1 \times 10^{-5}$ | 0.02           | 0.058    | 0.058     | 0.70       |

Table 3: The input parameters $(a_\nu, b_\nu, c_\nu)$ and the predictions when we take $y^\nu$ diagonal and $M_R = M_{\text{GUT}}$.

The results for the MSW large-mixing solution (MSW-L) and the vacuum-oscillation solution (V-O) are also shown in Table 3. Because we use the diagonal $y^\nu$ texture, the MNS matrix elements are essentially determined by the matrix $y^e y^{e\dagger}$. The large $U_{\mu 3}$ and small $U_{e2}$ and $|U_{e3}|$ then follow almost independently of the input neutrino mass values. Summing up, with the diagonal $y^\nu$ texture, we can reproduce the MSW-S solution but not the other two solutions.

The $CP$ violating parameters are listed in the Table 4. The magnitude of the $CP$

|       | $J_{\text{MNS}}$ | $\varphi_2$ | $\varphi_3$ |
|-------|-----------------|-------------|-------------|
| MSW-S | $-3.4 \times 10^{-11}$ | $133.0^\circ$ | $-7.8 \times 10^{-9}$ |
| MSW-L | $-8.5 \times 10^{-11}$ | $133.0^\circ$ | $-2.0 \times 10^{-8}$ |
| V-O   | $-1.2 \times 10^{-13}$ | $133.0^\circ$ | $-3.9 \times 10^{-11}$ |

Table 4: The predicted values of the $CP$ violating parameters of the MNS matrix, $J_{\text{MNS}}$, $\varphi_2$ and $\varphi_3$ when we take $y^\nu$ diagonal and $M_R = M_{\text{GUT}}$.

violation parameter $J_{\text{MNS}}$ remains small when $M_R = M_{\text{GUT}}$. The remaining two angles, $\varphi_2$ and $\varphi_3$ in the MNS matrix can also be predicted. Because we neglect the Majorana phases of the $\nu_R$ mass matrix $M_R$, by taking $y^\nu$ to be real, they are determined essentially

\footnote{See Appendix C for more details.}
by the unitary matrix $U_e$ that diagonalize $y^e y^{e\dagger}$; see eqs (C.7) and (C.8) in Appendix C. The magnitudes of $\varphi_2$ for all solutions are large and the magnitudes of $\varphi_3$ for all solutions are small, reflecting the phases structure of eq.\((5.14b)\) at the GUT scale.

6.1.2 $M_R = 3 \times 10^{14}$ GeV

When $M_R$ is lower than the $M_{\text{GUT}}$, the RGE’s including the Yukawa matrix $y^\nu$, eq.\((B.1)\) of Appendix B, apply in the region $M_R < \mu < M_{\text{GUT}}$, while the terms proportional to $y^\nu$ decouple below $\mu = M_R$, and the RGE of $\kappa$, eq.\((4.1)\), takes over. The magnitudes of the MNS matrix elements are only slightly different from the $M_R = M_{\text{GUT}}$ case:

$$U_{e2} = 0.057, \quad |U_{e3}| = 0.058, \quad U_{\mu3} = 0.71,$$

when $m_2$ is fitted to the MSW small mixing solution. No improvements are found for the other two scenarios; the elements $|U_{e2}|$ and $|U_{e3}|$ remain too small for the large mixing solutions. The magnitudes of the input parameters at the GUT scale change significantly:

$$a_\nu = 0.071, \quad b_\nu = 0.23, \quad c_\nu = 0.74.$$

$c_\nu$ and $c_u$ (see Table 2) are now comparable in magnitude. The results are summarized in Table 5.

|       | $a_\nu$ | $b_\nu$ | $c_\nu$ | $m_1$(eV) | $m_2$(eV) | $m_3$(eV) | $U_{e2}$ | $|U_{e3}|$ | $U_{\mu3}$ |
|-------|--------|--------|--------|----------|----------|----------|---------|---------|----------|
| MSW-S | 0.071  | 0.23   | 0.74   | 0.0003   | 0.003    | 0.03     | 0.057   | 0.058   | 0.71     |
| MSW-L | 0.093  | 0.30   | 0.60   | 0.0005   | 0.005    | 0.02     | 0.057   | 0.058   | 0.71     |
| V-O   | 0.0043 | 0.013  | 0.60   | $1 \times 10^{-6}$ | $1 \times 10^{-5}$ | 0.02 | 0.057 | 0.058 | 0.71 |

Table 5: The input parameters ($a_\nu$, $b_\nu$, $c_\nu$) and the predictions when we take $y^\nu$ diagonal and $M_R = 3 \times 10^{14}$ GeV.

The predictions for the $CP$ violating parameters are listed in Table 6. The magnitude

|       | $J_{\text{MNS}}$ | $\varphi_2$ | $\varphi_3$ |
|-------|-----------------|-------------|-------------|
| MSW-S | $-8.6 \times 10^{-8}$ | 133.0° | $-1.5° \times 10^{-3}$ |
| MSW-L | $-6.0 \times 10^{-8}$ | 133.0° | $-3.1° \times 10^{-3}$ |
| V-O   | $-1.1 \times 10^{-6}$ | 133.2° | $-9.9° \times 10^{-5}$ |

Table 6: Predictions for the $CP$ violating parameters of the MNS matrix, $J_{\text{MNS}}$, $\varphi_2$, $\varphi_3$, for diagonal $y^\nu$ and $M_R = 3 \times 10^{14}$ GeV.
in Table 6. This is because the matrix $y^\nu$ acquires phases by the RGE effects at $M_R < \mu < M_{GUT}$. The magnitude of the one of the Majorana phases $\varphi_3$ is also found to be bigger than the $M_R = M_{GUT}$ case, reflecting the renormalization effect on $y^e$ at $M_R < \mu < M_{GUT}$. On the other hand, the magnitude of $\varphi_2$ is not sensitive to the $M_R$, because the phase, $\varphi_2$, is essentially determined by $\text{arg}(U_e^2)$.

### 6.2 Fritzsch-type $y^\nu$ texture

In this subsection, we study consequence of the Fritzsch form $y^\nu$. The Yukawa matrix of $y^\nu$ takes the following form at the SU(5) GUT scale,

$$y^\nu = \begin{pmatrix} 0 & a_{\nu} & 0 \\ a_{\nu} & 0 & b_{\nu} \\ 0 & b_{\nu} & c_{\nu} \end{pmatrix}, \quad (6.11)$$

where all elements of $y^\nu$, $a_{\nu}$, $b_{\nu}$, and $c_{\nu}$ are real for brevity. We set the right-handed neutrino decoupling scale, $M_R$, to be $3 \times 10^{14}$ GeV.

Here also we have 11 parameters in the Yukawa matrices and hence 11 predictions. The 8 parameters of the Yukawa matrices $y^u$ and $y^d$ are fixed by the three up-quark masses, the three charged-lepton masses, and the two CKM matrix elements, $V_{us}$ and $V_{cb}$, as in the previous case. The magnitudes of the input values, $a_u$, $b_u$, $c_u$, $a_d$, $b_d$, $c_d$, $\phi_1$ and $\phi_2$ hence take the same values as those listed in Table 3. The predictions for the three down-type quark masses and the remaining CKM matrix elements are also the same.

On the other hand, the magnitudes of the input parameters, $a_{\nu}$, $b_{\nu}$, and $c_{\nu}$ at the GUT scale should be bigger than those of the diagonal $y^e$ case in order to obtain the same neutrino masses. For instance, $c_{\nu}$ is now bigger than $c_u$ see Table 7 and Table 2. By comparing Table 7 and Table 5, we find that by choosing the Fritzsch form the “hierarchy” between $b_{\nu}$ and $c_{\nu}$ is very weak, $b_{\nu}/c_{\nu} \simeq 0.4$, for the MSW solutions. This has significant consequences in the predictions for the MNS matrix elements:

$$U_{e2} = 0.054, \quad |U_{e3}| = 0.033, \quad U_{\nu3} = 0.42, \quad (\text{MSW-S}), \quad (6.12a)$$
\[
U_{e2} = 0.45, \quad |U_{e3}| = 0.029, \quad U_{\mu 3} = 0.45, \quad (\text{MSW-L}),
\]
\[
U_{e2} = 0.45, \quad |U_{e3}| = 0.054, \quad U_{\mu 3} = 0.66, \quad (\text{V-O}).
\]  

We can now reproduce vacuum-oscillation solution with the Fritzsch form \( y' \). However, the predictions for the \( U_{\mu 3} \) element are slightly smaller than the experimental constraint of Table 1 for the MSW solutions. In the diagonal \( y' \) case, the MNS matrix elements are essentially determined by the matrix \( y'y^\dagger \), and large \( U_{\mu 3} \) results as a consequence of large \( \tan 2\theta_{23} \) in eq.(5.10). The prediction holds with the Fritzsch form for the V-O solution because the hierarchy \( b_{\nu} \ll c_{\nu} \) in the third row of Table 1 implies essentially diagonal \( y'y^\dagger \). The predicted values of the \( U_{\mu 3} \) element reduces significantly for the MSW solutions because the absence of the hierarchy, \( b_{\nu}/c_{\nu} \approx 0.4 \), implies significantly non-diagonal \( y'y^\dagger \). By choosing large \( m_3 \) and small \( m_2 \) within the allowed ranges of Table 1, \( U_{\mu 3} \) can increase up to 0.42 for MSW-S and 0.45 for MSW-L solutions, slightly below the range allowed by Super-Kamiokande in Table 1.

The matrix element \( |U_{e3}| \) remains small in all three solutions. The element \( U_{e2} \) can also be made barely consistent with the allowed ranges of Table 1 for all the solutions. Because the \( U_{e2} \) element from the diagonalization of \( y'y^\dagger \) is rather small (\( U_{e2} = 0.057 \) in Table 3 when \( y' \) is diagonal) large \( U_{e2} \) results from the diagonalization of \( y'y^\dagger \). We find that \( U_{e2} \) is proportional to \( \sqrt{m_1/m_2} \), and \( U_{e2} = 0.45 \) in Table 4 results when \( m_1/m_2 = 1/10 \). In order to accommodate small \( U_{e2} \) in the MSW-S solution, \( m_1/m_2 \) should be chosen small. The minimal of \( U_{e2} \) is found to be \( U_{e2} = 0.054 \) at \( m_1/m_2 = 7 \times 10^{-6} \).

The magnitude of the parameter \( J_{\text{MNS}} \) in Table 8 is now found to be much bigger than that of the diagonal \( y' \) case in Table 3 for all three cases. Predicted magnitudes of \( J_{\text{MNS}} \) are now bigger than that of \( J_{\text{CKM}} \) in eq.(5.7). This is because the only non-real element of \( y'y^\dagger \) in eq.(5.14b) is in the \((1,3)\) element and has small magnitude. Large \( |J_{\text{MNS}}| \) results only with significantly non-diagonal \( y' \). We also note that the magnitude of the Majorana phase \( \varphi_3 \) is now larger than that of the diagonal \( y' \) case and the sign of \( \varphi_3 \) is changed in the MSW solutions. Their magnitudes remain smaller than 1°, however. We find significantly different predictions for \( \varphi_2 \) as compared to the diagonal \( y' \) case; \( \varphi_2 \approx 133^\circ \) in Table 4. Significant fraction of the contribution from \( y'y^\dagger \) is canceled by the non-diagonality of \( y' \). The magnitude of \( \varphi_2 \) in the “MSW-S” case is large \( \varphi_2 \approx 90^\circ \) in Table 8) because of our

|           | \( J_{\text{MNS}} \)       | \( \varphi_2 \) | \( \varphi_3 \) |
|-----------|-----------------------------|----------------|----------------|
| MSW-S     | \(-4.3 \times 10^{-4}\)     | 89.7°          | 0.014°         |
| MSW-L     | \(-4.2 \times 10^{-3}\)     | 5.8°           | 0.26°          |
| V-O       | \(-8.4 \times 10^{-3}\)     | 5.0°           | -0.057°        |

Table 8: Predictions for the \( CP \) violating parameters of the MNS matrix, \( J_{\text{MNS}}, \varphi_2, \varphi_3 \), for the Fritzsch-type \( y' \) (at the GUT scale) and \( M_R = 3 \times 10^{14}\text{GeV} \).
choice of very small $a_\nu$ to accommodate small $U_{e2}$. If we take $a_\nu = 0.11$ for $m_1 = m_2/10$, the predictions are

$$U_{e2} = 0.45, \quad |U_{e3}| = 0.027, \quad U_{\mu 3} = 0.48,$$

(6.13)

and

$$J_{\text{MNS}} = -4.8 \times 10^{-3}, \quad \varphi_2 = 5.7^\circ, \quad \varphi_3 = 0.18^\circ.$$

(6.14)

Therefore large $U_{e2}$ and $U_{\mu 3}$, large $|J_{\text{MNS}}|$ and small $\varphi_2$, $\varphi_3$ are natural consequences of the Fritzsch type $y^\nu$ in our model. In order to accommodate small $U_{e2}$ for the MSW small-mixing solution, we should assume $m_1/m_2 \simeq 7 \times 10^{-6}$. Note, however, that our predictions for the Majorana phases $\varphi_2$ and $\varphi_3$ assume $CP$-invariance in the right-handed neutrino mass matrix $M_R$.

Finally, we report our finding for $y^\nu = y^u$, i.e., when we impose the following conditions

$$a_u = a_\nu, \quad b_u = b_\nu, \quad c_u = c_\nu,$$

(6.15)

at the GUT scale. In this case, we have three more predictions. In particular, the neutrino masses can be predicted for a given value of the right-handed-neutrino decoupling scale $M_R$. By choosing $M_R = 1.7 \times 10^{14}$ GeV, we find

$$m_1 = 4.8 \times 10^{-12} \text{ eV},$$

$$m_2 = 3.4 \times 10^{-7} \text{ eV},$$

$$m_3 = 0.08 \text{ eV}.$$

(6.16)

The mass squared differences are

$$O(m_3^2 - m_2^2) \simeq 6 \times 10^{-3}\text{eV}^2,$$

$$O(m_2^2 - m_1^2) \simeq 1 \times 10^{-13}\text{eV}^2.$$

(6.17)

The difference $m_2^2 - m_1^2$ is too small even for the V-O solution of the solar-neutrino experiments.

7 Summary

In this article, we explain the imbalance of the flavor-mixing-angles between the quark and the lepton sectors suggested by the recent neutrino-oscillation experiments [1]-[9] in the supersymmetric SU(5) GUT with the see-saw mechanism. Especially, we look for the reason why the matrix element $|U_{\mu 3}|$ is much larger than the corresponding $|V_{cb}|$ of the CKM matrix.

We use two tools for analyzing the cause of the imbalance between the CKM matrix and the corresponding MNS matrix for lepton-flavor mixing [10, 11]. One is the RGE of the MSSM between the GUT scale and the weak scale. The RGE of the neutrino-mass matrix [33] generated by the see-saw mechanism implies that a large mixing between the
second and third generation leptons is obtained at the $m_Z$ scale as long as the mixing is large at the GUT scale. The other is the SU(5) constraints on the Yukawa matrices at the GUT scale. In particular, $y^u = y^u T$ and $y^d = y^e T$ follow if the Higgs doublets of the MSSM belong to 5 and $5^*$ representations of SU(5) \cite{34}. By choosing an appropriate weak basis at the GUT scale, we can take $y^u$ to have the symmetric Fritzsch form \cite{20} while $y^d$ should take the generic NNI form \cite{39}. The observed quark masses and the CKM matrix elements then forces $y^d$ to take the asymmetric BS form \cite{21} where the largest elements appear in the bottom row of the second and the third column \cite{40}. The SU(5) relation then forces $y^e$ to have the largest elements in the third column at the second and the third row. We show that this asymmetry between $y^d$ and $y^e$ leads to the asymmetry in the unitary matrices that diagonalize the down-quark and the charged-lepton mass matrices. In particular, $|V_{cb}| \ll 1$ and $|U_{\mu 3}| \sim 0.7$ can be obtained at the same time if the neutrino mass matrix is approximately diagonal. We present a few examples by adopting simple textures for the neutrino Yukawa matrix $y^\nu$, in the basis where the heavy right-handed-neutrino Majorana mass matrix $M_R$ is diagonal.

The favorable MNS matrix elements are thus obtained together with the acceptable $m_\tau/m_b$ ratio from the simplest SU(5) Yukawa sector. As has been well-known, however, this model gives unacceptable predictions for the other ratios, $m_\mu/m_s$ and $m_e/m_d$. In our actual numerical calculation, we modify the SU(5) Yukawa sector by introducing a $45^*$ Higgs boson coupling between the second generation decuplet (10) and the third generation quintet ($5^*$), in order to accommodate the Georgi-Jarlskog mass relations \cite{19} $m_\mu/m_s = m_d/m_e = 3$ at the GUT scale. Acceptable quark and lepton masses then follow for not too large tan $\beta$ \cite{22}. The mixing between the second and third generation leptons, $|U_{\mu 3}|$, is insensitive to this modification, while the others, $|U_{e 2}|$ and $|U_{e 3}|$, are affected significantly.

We analyze the Maki-Nakagawa-Sakata (MNS) \cite{10} lepton-flavor mixing matrix and neutrino masses numerically by using the 1-loop RGE of the MSSM. First, we study the case where the texture of $y^\nu$ is diagonal, and examine the sensitivity of the results to the right-handed-neutrino decoupling scale $M_R$, by setting $M_R = M_{GUT}$ or $M_R = 3 \times 10^{14}$GeV. We can reproduce the MSW small-mixing solution but not the other solutions of the solar-neutrino deficit with the diagonal $y^\nu$, almost independent of $M_R$. The magnitudes of the $CP$ violating parameters of the MNS matrix, $J_{MNS}$ and $\varphi_3$ are very small, while large $\varphi_2$ can arise even when the heavy Majorana mass matrix $M_R$ does not have a $CP$ violating phase. We also study the case where $y^\nu$ has the texture of Fritzsch form, by setting $M_R = 3 \times 10^{14}$GeV. We find the tendency to have both $|U_{\mu 3}|$ and $|U_{e 2}|$ large when $m_1/m_2 \simeq 1/10$. MSW large-angle solution and the vacuum-oscillation solution can hence be reproduced. In order to accommodate the MSW small-angle solution, we need to fine-tune the mass ratio to be $m_1/m_2 \simeq 7 \times 10^{-6}$. The magnitude of the $CP$ violating parameter $J_{MNS}$ can now be bigger than that of $J_{CKM}$, while the Majorana phases $\varphi_2$ and $\varphi_3$ are small for the large angle solutions. As a special case, we examine the consequence of $y^u = y^\nu$ at
the GUT scale and find that the ratios of the three neutrino masses are inconsistent with the observation.

We can naturally explain the imbalance of the mixing angles between the lepton and quark sectors within the SU(5) GUT. We hope that our finding may shed light on the Yukawa sector of the supersymmetric GUT theories.

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A Vacuum oscillation probabilities and the MNS matrix.

The transition probability $P_{\nu_\alpha \rightarrow \nu_\beta}$, ($\alpha \neq \beta$) is

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{j=1}^{3} (V_{\text{MNS}})_{\beta j} \exp \left( -\frac{i m_j^2}{2 E} L \right) \left| (V_{\text{MNS}}^\dagger)_{j\alpha} \right|^2$$

$$= \left| U_{\beta 1} U_{\alpha 1}^* + U_{\beta 2} \exp \left( -\frac{i \delta m_{12}^2}{2 E} L \right) U_{\alpha 2}^* + U_{\beta 3} \exp \left( -\frac{i \delta m_{13}^2}{2 E} L \right) U_{\alpha 3}^* \right|^2,$$  \hspace{1cm} (A.1)

where $\delta m_{ij}^2 = m_j^2 - m_i^2$, and we used the identity

$$(V_{\text{MNS}})_{\beta j} \left( V_{\text{MNS}}^\dagger \right)_{j\alpha} = U_{\beta j} U_{\alpha j}^*$$  \hspace{1cm} (A.2)

which follows directly from our parameterization eq.(2.12) of the MNS matrix. It reduces to simple forms when the conditions

$$\delta m_{12}^2 \ll m_{13}^2$$

(A.3)

holds. there are two cases.

When the conditions

$$\frac{\delta m_{12}^2}{2 E} L \ll 1 \sim \frac{\delta m_{13}^2}{2 E} L$$

(A.4)

are satisfied, eq.(A.1) can be simplified significantly:

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \left| -U_{\beta 3} U_{\alpha 3}^* + U_{\beta 3} \exp \left( -\frac{i \delta m_{13}^2}{2 E} L \right) U_{\alpha 3}^* \right|^2$$

(A.5)
\[ P_{\nu_\alpha \rightarrow \nu_\beta} = 2 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 - 4 \text{Re}(U_{\alpha 1} U_{\beta 1}^* U_{\beta 2}^* U_{\alpha 2}) \sin^2 \left( \frac{\delta m_{12}^2}{4E} L \right) + 2 J_{\text{MNS}} \sin^2 \left( \frac{\delta m_{12}^2}{2E} L \right) \] 

(A.7)

where \( J_{\text{MNS}} = \text{Im}(U_{\alpha 1} U_{\beta 1}^* U_{\beta 2}^* U_{\alpha 2}) \).

The survival probability is,

\[ P_{\nu_\alpha \rightarrow \nu_\alpha} = \left| \sum_{j=1}^3 (V_{\text{MNS}})_{\alpha j} \exp \left( -\frac{i\delta m_{12}^2}{2E} L \right) (V_{\text{MNS}}^*)_{j\alpha} \right|^2 \]

(A.8)

Under the condition (A.4), we find

\[ P_{\nu_\alpha \rightarrow \nu_\alpha} = \left| 1 - U_{\alpha 3}^* U_{\alpha 3} + U_{\alpha 3} \exp \left( -\frac{i\delta m_{12}^2}{2E} L \right) U_{\alpha 3}^* \right|^2 \]

(A.9)

When the condition (A.6) applies, we find

\[ P_{\nu_\alpha \rightarrow \nu_\alpha} = 1 - 2 |U_{\alpha 3}|^2 \left( 1 - |U_{\alpha 3}|^2 \right) - 4 |U_{\alpha 1}|^2 |U_{\alpha 2}|^2 \sin^2 \left( \frac{\delta m_{12}^2}{4E} L \right) . \] 

(A.10)

B Renormalization Group Equations of Yukawa matrices and gauge couplings

In the MSSM, the RGE of the Yukawa matrices in the 1-loop level are [42]

\[ \frac{d}{dt} y^u = \frac{1}{(4\pi)^2} \left[ \text{tr} \left( 3 y^u y^u + y^d y^d \right) + 3 y^u y^u + y^d y^d - 4 \left( \frac{16}{3} \alpha_3 + 3 \alpha_2 + \frac{13}{15} \alpha_1 \right) \right] y^u , \]
\[
\frac{d}{dt} y^d = \frac{1}{(4\pi)^2} \left[ \text{tr} \left( 3y^d y^d + y^e y^e \right) + 3y^d y^d + y^u y^u - 4\pi \left( \frac{16}{3} \alpha_3 + 3\alpha_2 + \frac{7}{15} \alpha_1 \right) \right] y^d,
\]

\[
\frac{d}{dt} y^e = \frac{1}{(4\pi)^2} \left[ \text{tr} \left( 3y^e y^e + y^e y^e \right) + 3y^e y^e + y^e y^e - 4\pi \left( 3\alpha_2 + \frac{9}{5} \alpha_1 \right) \right] y^e.
\]

\[
\frac{d}{dt} y^\nu = \frac{1}{(4\pi)^2} \left[ \text{tr} \left( 3y^\nu y^\nu + y^\nu y^\nu \right) + 3y^\nu y^\nu + y^e y^e - 4\pi \left( 3\alpha_2 + \frac{3}{5} \alpha_1 \right) \right] y^\nu,
\]

where \( t \) is the logarithm of the renormalization scale \( \mu \):

\[
t = \ln \mu.
\]

The gauge couplings satisfy the RGE's

\[
\frac{d}{dt} \alpha_i = b_i \left( \frac{\alpha_i}{\pi} \right)^2,
\]

where \( \alpha_i \) are

\[
\alpha_i = \frac{g_i^2}{4\pi}, \quad (i = 1, 2, 3),
\]

and

\[
g_1^2 = \frac{5}{3} g_2^2.
\]

\( g' \), \( g_2 \) and \( g_3 \) are gauge coupling constant of the \( U(1)_Y \), \( SU(2)_L \) and \( SU(3)_C \), respectively, and the coefficient \( b_i \) are

\[
b_1 = n_g + \frac{3}{20} n_H, \\
b_2 = n_g + \frac{1}{4} n_H - 3, \\
b_3 = n_g - \frac{9}{2}.
\]

The factors \( n_H \) and \( n_g \) are, respectively, the number of Higgs doublets and that of fermion generations. In this article, we set \( n_H = 2 \) and \( n_g = 3 \). Below the right-handed neutrino decoupling scale \( (\mu = M_R) \), the matrix \( y^{\nu} \) decouples and the RGE of the effective dimension-five operator, eq.(A.1), takes over.

\section{The phases in the MNS matrix}

We define the eigenvalues of \( \kappa \) as

\[
\kappa_i = |\kappa_i e^{i\xi_i}|, \quad (|\kappa_1| < |\kappa_2| < |\kappa_3|)
\]
at the $m_Z$ scale. They are obtained by the unitary transformation
\begin{equation}
U_\nu^T \kappa^* U_\nu = \text{diag.}(\kappa_1^*, \kappa_2^*, \kappa_3^*). \tag{C.2}
\end{equation}
On the other hand, the unitary matrix $U_\nu$ that gives real positive neutrino masses obtain
\begin{equation}
\text{diag.}(m_1, m_2, m_3) = U_\nu^T M_\nu U_\nu \\
= U_\nu^T \kappa^* U_\nu v_u^2 \\
= \text{diag.}(|\kappa_1|, |\kappa_2|, |\kappa_3|) v_u^2. \tag{C.3}
\end{equation}
Hence the matrix $U_\nu$ is obtained from $U_\nu$ by
\begin{equation}
U_\nu = U_\nu P, \tag{C.4}
\end{equation}
where $P$ is a diagonal phase matrix. This phase matrix is obtained as
\begin{align*}
U_\nu^T \kappa^* U_\nu &= PU_\nu^T \kappa^* U_\nu P \\
&= P \text{diag.}(\kappa_1^*, \kappa_2^*, \kappa_3^*) P \\
&= \text{diag.}(|\kappa_1|, |\kappa_2|, |\kappa_3|), \tag{C.5}
\end{align*}
and hence
\begin{equation}
P = \begin{pmatrix}
e^{i \frac{\varphi_2}{2}} & 0 & 0 \\
0 & e^{i \frac{\varphi_3}{2}} & 0 \\
0 & 0 & e^{i \frac{\varphi_3}{2}}
\end{pmatrix}. \tag{C.6}
\end{equation}
The MNS matrix is rewritten by using eq. (C.4)
\begin{equation}
V_{\text{MNS}} = U_\nu^\dagger U_\nu \\
= U_\nu^\dagger U_\nu P \\
= U_{\text{MNS}} P' P, \tag{C.7}
\end{equation}
where an additional phase matrix
\begin{equation}
P' = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i \varphi_2} & 0 \\
0 & 0 & e^{i \varphi_3}
\end{pmatrix}, \tag{C.8}
\end{equation}
makes the $U_{e2}$ and $U_{\mu3}$ elements real and non-negative in our phase convention for the MNS matrix. The Majorana phases of the MNS matrix, $\varphi_2$ and $\varphi_3$, are now obtained as:
\begin{align*}
\varphi_2 &= \varphi_2' + \frac{s_2 - s_1}{2}, \\
\varphi_3 &= \varphi_3' + \frac{s_3 - s_1}{2}. \tag{C.9}
\end{align*}
These phases are observable, and hence are independent of phase convention, in lepton-number violating processes.
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