Non-unitary observables in the 2d critical Ising model

Louis-Pierre Arguin\textsuperscript{*}, Yvan Saint-Aubin\textsuperscript{†}
Centre de recherches mathématiques
and
Dépt. de mathématiques et de statistique
Université de Montréal
C.P. 6128, succ. centre-ville
Montréal, Québec
Canada H3C 3J7

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\textsuperscript{*}E-mail: arguinl@crm.umontreal.ca
\textsuperscript{†}E-mail: saint@crm.umontreal.ca
Abstract

We introduce three non-local observables for the two-dimensional Ising model. At criticality, conformal field theory may be used to obtain theoretical predictions for their behavior. These formulae are explicit enough to show that their asymptotics are described by highest weights $h_{pq}$ from the Kac table for $c = \frac{1}{2}$ distinct from those of the three unitary representations $(0, \frac{1}{16}, \frac{1}{2})$.

1 Introduction

It is widely agreed that the Ising model in two dimensions is described at its critical temperature by a conformal field theory at $c = \frac{1}{2}$. The spectrum of the transfer matrix is that of the operator $L_0 \oplus \overline{L_0}$ and is described by a linear combination of the squared norm of characters of the three unitary representations at this value of the central charge, namely those with highest weights 0, $\frac{1}{16}$, and $\frac{1}{2}$. The exact linear combination depends on the boundary conditions put on the geometry under consideration.

These highest weights belong to Kac table $h_{pq} = \frac{((m+1)p-mq)^2-1}{4m(m+1)}$ at $c = 1 - 6/m(m+1)$. For minimal models, the relevant (unitary) weights $h_{pq}$ are labeled by the integers $p, q$ with $1 \leq p < m$, $1 \leq q < m$ and $1 \leq p + q \leq m$. As for the three unitary representations at $c = \frac{1}{2}, m = 3$ (corresponding to $h_{00}, h_{1,2}$ and $h_{2,1}$), the Verma modules associated with the others $h_{pq}$ have singular vectors. The quotient by the subspace spanned by these vectors is irreducible but the inner product on the quotient space is not positive definite. Therefore these representations are not unitary and limits on field indices appearing in OPE are set to reject them. Prior to the work reported here, we did not know of any use for these non-unitary representations in the description of the Ising model, as for example in OPE's.

We describe in each of the following sections an observable for the Ising model. They are somewhat unconventional as they are non-local objects. Using the techniques of conformal field theory, we are able to give predictions for their behavior and calculate their asymptotic behavior. The latter is described, in the three cases, by exponents $h_{pq}$ from Kac table that lead to non-unitary representations.
2 Crossing probability on Ising clusters

The first observable is derived from one in percolation theory and we start by describing it in this context. In percolation by sites on a square lattice, each site is declared open (closed) with probability $p$ (resp. $(1 - p)$) independently of its neighbors. A configuration on a finite geometry is said to have a crossing between two disjoint intervals on the boundary if there is a cluster of open sites joining the two intervals. A common geometry is the rectangle, say of $m \times n$ sites, with the two disjoint intervals chosen to be the vertical sides. One then speaks of a horizontal crossing. A central quantity in percolation theory is the probability $\pi_{\text{perco}}(r)$ of such crossings when the number of sites goes to infinity, the aspect ratio $r = \text{width}/\text{height}$ being kept fixed. Cardy [3] gave a prediction for this function $\pi_{\text{perco}}(r)$ using conformal field theory and the agreement with numerical data is excellent [7].

For the Ising model, we define $\pi_h(r) = \pi_{\text{Ising}}(r)$ as the probability of crossing on clusters of $+$ spins, the limit on the number of sites being taken as above. Lapalme and one of the authors have adapted Cardy’s ideas to this case and obtained Monte-Carlo measurements to test their prediction [8]. Two steps in Cardy’s reasoning cannot be extended straightforwardly to the Ising model and Lapalme and Saint-Aubin had to resort to one basic property of $\pi_{\text{perco}}$ and $\pi_{\text{Ising}}$, their scale invariance. If $\pi_{\text{Ising}}$ is described by a four-point correlation function, it must be that of a field of vanishing conformal weight. They therefore chose to use the second singular vector of height 6 in the Verma module with $c = \frac{1}{2}$, $h = 0$. The ordinary differential equation obtained from this singular vector is of order 6 and the exponents are $0, \frac{1}{6}$ (twice degenerate), $\frac{1}{2}, \frac{5}{3}$ and $\frac{5}{2}$. If one is willing to ignore the constraints on $p$ and $q$, these exponents are precisely the conformal weights $h_{pq}$ of the fields appearing in the (naive) operator product expansion of $\phi_{23}$ of conformal weight $h_{23} = 0$.

Lapalme and Saint-Aubin’s argument is not as convincing as Cardy’s which relies on better established ideas. The agreement of their prediction with numerical data is therefore welcome [8]. It is sufficient for the purpose of this letter to concentrate on the limiting behavior of $\pi_h(r)$ as $r \to 0$ and $r \to \infty$. One expects $\pi_h(r) \to 1$ and $\to 0$ in these two limits and the leading behavior prescribed by the ode is that of the exponent $\frac{1}{6}$. Because $\frac{1}{6}$ is twice degenerate, a logarithmic behavior is allowed but appears to be ruled out by numerical data. The behaviors that are seen in the
Figure 1: Asymptotic behaviors of $\pi_h$ as functions of $r$. The ordinates are $\log \pi_h(r)$ for the top graph, $\log(1 - \pi_h(r))$ for the bottom one.
simulations are
\[
\log \pi_h(r) \to ar + b, \quad r \to 0 \\
\log(1 - \pi_h(1/r)) \to \frac{c}{r} + d, \quad r \to \infty
\]
with \(\hat{a} = -0.1664\pi\) and \(\hat{c} = -0.1665\pi\). (The factor of \(\pi\) stems from the change of variables between the aspect ratio \(r\) and the variable used in the ode. To obtain \(\hat{a}\) and \(\hat{b}\), we used measurements of both horizontal and vertical crossings in \(\pi_v(r) = 1 - \pi_h(1/r)\) and fitted the asymptotic behaviors above to the values of \(\pi_h\) for the 20 smallest (largest) aspect ratios \(r\). See Figure 1.) The measurement errors are of a few units on the fourth digit. There is little doubt that \(\frac{1}{6} = h_{3,3}\) is the exponent describing these asymptotics.

3 Contours intersecting the boundary of a cylinder

Consider now the description of the spin \(\sigma\) at the boundary of a half-infinite cylinder. Let \(\theta_1, \theta_2, \ldots, \theta_{2n}\) be the angles along the boundary where spinflips occur. Using Onsager’s solution \([2]\) or conformal field theory, it is possible to calculate the probability density of configurations having precisely \(2n\) flips. For example, this density \(s(\theta_1, \theta_2, \theta_3, \theta_4)\) for the case \(n = 2\) is proportional to
\[
(sin \frac{1}{2}\theta_{12} sin \frac{1}{2}\theta_{34})^{-1} - (sin \frac{1}{2}\theta_{13} sin \frac{1}{2}\theta_{24})^{-1} \\
+ (sin \frac{1}{2}\theta_{14} sin \frac{1}{2}\theta_{23})^{-1}
\]

![Figure 2: Configurations contributing to different contour probabilities](image)
if $\theta_1, \theta_2, \theta_3, \theta_4$ appear in that order along the boundary and $\theta_{ij}$ is $\theta_i - \theta_j$. This density does not reveal however which pairs $(\theta_i, \theta_j)$ are actually joined by the contours between same-spin clusters. For $n = 1$ there is only one possible pairing but, for $n = 2$, there are already 2. Using conformal invariance, we represent the half-infinite cylinder by a disk (minus its center) and depict two possible distinct configurations (see Figure 2). Let $l(\theta_1, \theta_2, \theta_3, \theta_4)$ ($r(\theta_1, \theta_2, \theta_3, \theta_4)$) be the density probability for the pairing of the left (resp. right) configuration. Two requirements allow for the determination of $l$ and $r$ using conformal field theory. First their sum should reproduce the density when contours are ignored, namely $l + r = s$. Due to the singularity in $s$, it is natural to seek $l$ within the solution space of the ordinary differential equation that describes the 4-point correlation function of the field $\phi_{2,1}$ of conformal weight $\frac{1}{2}$. Second, when $\theta_{12} \to 0$, the probability $\text{Prob}(\theta_1 \theta_2 | \theta_3 \theta_4) = l(\theta_1, \theta_2, \theta_3, \theta_4)/s(\theta_1, \theta_2, \theta_3, \theta_4)$ should go to 1 and $\text{Prob}(\theta_2 \theta_3 | \theta_4 \theta_1) = r/s$ to 0.

These two requirements determine uniquely $l, r$ and $\text{Prob}(\theta_1 \theta_2 | \theta_3 \theta_4)$ [2]. For the latter one gets

$$\text{Prob}(\theta_1 \theta_2 | \theta_3 \theta_4) = \frac{1}{2} - \frac{9}{20} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})^2 f^{(0)}(z)} \left( f^{(5/3)}(z) - \frac{z}{1-z} f^{(5/3)}(1-z) \right)$$

where the anharmonic ratio is chosen to be $z = (\sin \frac{1}{2} \theta_{12} \sin \frac{1}{2} \theta_{34})/(\sin \frac{1}{2} \theta_{13} \sin \frac{1}{2} \theta_{24})$ and where $f^{(0)}(z) = 1 - z + z/(1 - z)$ and $f^{(5/3)}(z) = z^{\frac{5}{3}} 2F1(\frac{1}{3}, \frac{4}{3}, \frac{8}{3}, z)/(1-z)$. The behavior of this probability as $\theta_{12} \to 0$ ($z \to 0$) is

$$1 - \frac{10}{9} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})} z^{5/3} + \mathcal{O}(z^2).$$

The exponent $\frac{5}{3}$ is the highest weight $h_{31}$ of Kac table.

The function $\text{Prob}(\theta_1 \theta_2 | \theta_3 \theta_4)$ is plotted on Figure 3 together with Monte-Carlo measurements of this probability on a cylinder whose length is twice as long as its circumference. The four dots close to $z = 0$ and the four close to $z = 1$ were measured on a cylinder with 32000 sites and samples were larger than $10^6$ configurations. Statistical errors on these 8 points are smaller than the size of the dots on the figure. (Details will be given in [3].)
Figure 3: The function \(\text{Prob}(\theta_1\theta_2|\theta_3\theta_4)\) as function of \(z \in [0,1]\) with Monte-Carlo data.

4 Homology class of Fortuin-Kasteleyn clusters

The last observable describes the Ising model on a geometry without boundary. Denote by \(\alpha (\beta)\) a non-trivial cycle in the horizontal (vertical) direction on a torus of modulus \(\tau\). Let \(a,b \in \mathbb{Z}\) be two integers with \(\gcd(a,b) = 1\) and let \(\pi(\{a,b\})\) be the probability that a spin configuration has a Fortuin-Kasteleyn cluster wrapping precisely \(a\) times around \(\alpha\) and \(b\) around \(\beta\). (Configurations having simultaneously clusters of type \(\{1,0\}\) and \(\{0,1\}\) are not included in the computation of \(\pi(\{a,b\})\). These configurations are said to contain a cross.) In Figure 4 the Fortuin-Kasteleyn clusters of three configurations on a square torus \((\tau = i)\) are shown. The original signs of the Ising spins are depicted in black (+) or in white (−). Only the leftmost configuration in this Figure contributes to \(\pi(\{1,0\})\).

Using the Coulomb gas representation \([4, 9]\) it is possible to write an explicit formula for \(\pi(\{a,b\})\) (see \([1]\) where this expression is given for various Potts models and where Monte-Carlo simulations checking it are reported). In particular for the Ising model

\[
\pi_{\tau}(\{(1,0)\}) = \frac{1}{|\eta(q^2)|} \frac{\theta_2(\frac{i}{3}\tau_i) - |\theta_3(\frac{i}{3}\tau_i) - \theta_4(\frac{i}{3}\tau_i)|}{|\theta_2(\tau)| + |\theta_3(\tau)| + |\theta_4(\tau)|}
\]

where \(\tau_i = \text{Im} \, \tau, \, q = e^{i\pi \tau}\), the \(\theta_i\) are the elliptic theta functions and \(\eta(q)\) is Dedekind function \(\eta(q) = q^{1/24} \prod_{n=1}^\infty (1 - q^n)\). For \(\text{Re} \, \tau = 0\) and \(\tau_i \to \infty\), that is for a torus
Figure 4: Fortuin-Kasteleyn clusters for three configurations. They contribute to \( \pi \{(1,0)\} \), \( \pi \{(1,-1)\} \) and \( \pi \text{(cross)} \) respectively.

represented by a very narrow and tall rectangle, any configuration will almost surely contain a horizontal cluster and no vertical one. In that limit one expects that all \( \pi_\tau(\{(a,b)\}) \) will vanish, including \( \pi_\tau(\text{cross}) \), except for \( \pi_\tau(\{(1,0)\}) \). For that case \( q = e^{-\pi \tau_i} \) and one gets indeed as \( \tau_i \to \infty \)

\[
\pi_\tau(\{(1,0)\}) \to 1 - (q^2)^{\frac{1}{8}} f_1(q^2) - (q^2)^{\frac{1}{13}} f_2(q^2) - \ldots
\]

where the \( f_i \) are real analytic in their argument in a neighborhood of 0. The two leading terms are recognized to be twice the weights \( h_{12} \) and \( h_{33} \). (The doubling of highest weights in expansions in \( q^2 \) is the natural thing to expect and accounts for the contributions of holomorphic and antiholomorphic sectors of the theory.)

Of the three observables discussed in this note, this one (\( \pi_\tau(\{(1,0)\}) \)) is probably the less compelling. There is no differential equation here to dictate a finite set of exponents. The “...” in the above asymptotic expansion contains other exponents, namely other integral linear combinations of \( \frac{1}{8} \) and \( \frac{1}{13} \). Not all these combinations however occur in Kac table. (We showed that the exponents are restricted to the set \( \{\frac{n}{8}, n = 0, 1, \ldots, 7\} \cup \{\frac{1}{13} + \frac{n}{8}, n = 0, 1, \ldots, 7\} \) but we did not prove that all of these do occur.)

Despite these comments, the fact that the highest weights \( h_{12} \) and \( h_{33} \) describe the leading behavior seems remarkable.

5 Conclusion

What are the possible exponents of the Kac table at \( m = 3 \) if one identifies \( h_{pq} \) and \( h_{p'q'} \) whenever \( h_{pq} - h_{p'q'} \in \mathbb{Z} \)? This amounts to asking the simple number theoretic question of which integers modulo 48 have a square root. The answer to the first question is the following list of \( h \)'s: the three unitary 0, \( \frac{1}{16} \) and \( \frac{1}{2} \) and the non-unitary
$\frac{5}{2}, \frac{1}{6}, \frac{35}{48}$ and $-\frac{1}{48}$. The question of whether other observables can be constructed that are ruled by $\frac{15}{16}, \frac{35}{16}$ and $-\frac{1}{48}$ is more difficult.

More to the point are the following questions. Which linear space of states must be considered so that non-local observables may be taken into account in the framework of conformal field theory? Why do statistical models seem to prefer unitary representations even when mutually non-local fields are considered (like the pair $\sigma-\mu$ in the Ising model)? And why, when they do step out of the unitary representations, do they remain in Kac table?

NOTE

After the research reported here was completed, one of us (YSA) learned from Duplantier that various exponents for the Ising model have been introduced by him and his colleagues that do not belong to the small set $\{0, \frac{1}{16}, \frac{1}{2}\}$.

Here are two representative examples discussed in their work. For $O(n)$ models, consider the probability density of having $L$ loops between two points $x$ and $y$ in the plane. Duplantier [5] shows that it decays as $|x-y|^{-2x_L}$ with $x_L = 2h_L/2.0$. Not only do these exponents miss the unitary set $\{0, \frac{1}{16}, \frac{1}{2}\}$ but, for $L$ odd, they are out of Kac table. This fact raises questions beyond the present letter.

More recently [6], Duplantier obtained the fractal dimensions of three properties of Potts clusters, namely of the external perimeter ($D_{EP}$), of the set of outer boundary sites (dimension $D_H$ of the hull) and of the singly connecting sites that appear close under the scaling limit ($D_{SC}$). For the Ising model he gets $D_{EP} = \frac{11}{8}$, $D_H = \frac{5}{3}$ and $D_{SC} = \frac{13}{24}$. It would be hard to believe that the $\frac{5}{3}$ is a coincidence but the other two dimensions are out of Kac table. The role of these exponents for the argument presented here is unclear to us.

The most intriguing connection with the present work has appeared recently. Read and Saleur [10] consider nonlinear sigma models whose fields take values in supersymmetric coset spaces. They argue that, for the target space $S^{2n|2n}$ (a supersymmetric generalization of the sphere), the model shows a Ising-like transition. The spectrum of its conformal weights is described by their formula (3.7–8) with $e_0f = \frac{1}{6}$ and $gf^2 = \frac{1}{3}$ and includes the set $\{0, \frac{1}{16}, \frac{1}{2}\}$ and the exponents observed here: $\frac{5}{2}$ and $-\frac{1}{48}$. The relationship between their models and the observables discussed here remains to be established.
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