Residues in group completions and the Čech cohomology of $BG$

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Let $G$ be a connected affine algebraic group over $\mathbb{C}$, $G \to X$ be an open immersion of $G$-varieties, $Z = X - G$ and $i : Z \to X$ be the inclusion. Let $\alpha \in H^*(G, \mathbb{C})$ be primitive. We give a method to compute the image of $\alpha$ in $H^*(Z, i^！\mathbb{C}_X)$, using a lift of $\alpha$ along the first edge map of the Čech spectral sequence for $H^*(BG, \mathbb{C})$. We apply it to the wonderful compactification of a centerless semisimple group $G$.

Introduction

Let $G$ be a connected affine algebraic group over $\mathbb{C}$, and let $X$ be an equivariant partial completion of $G$ - that is, a complex variety with a left $G$-action and an open immersion $j : G \hookrightarrow X$ such that the action of $G$ on $X$ restricts to the multiplication action of $G$ on itself. Let $Z = X - G$ and let $i : Z \hookrightarrow X$ be the inclusion. We want to relate the Gysin map $H^*(Z/G, i^！\mathbb{C}_{X/G}) \to H^*(X/G, \mathbb{C})$ and the residue map $H^*(G, \mathbb{C}) \to H^{*+1}(Z, i^！\mathbb{C}_X)$. Our main result (theorem 3.4) does this in a special case, via the first edge map of the Čech spectral sequence of $BG$. It is inspired by the degeneracy locus formula for Chern classes, and can be used to recover a weak form of that formula (example 1.8).

Section 1 explains theorem 3.4. Section 2 applies it, in the case where $X$ is the wonderful compactification of a centerless semisimple group $G$, to compute the residue of a primitive element $\alpha \in H^*(G)$. Section 3 proves it. The theorem is more useful if you can compute the 1$^{st}$ edge map. Section 4 explains how to do that, using Bott’s approach to the Chern-Weil homomorphism.

There are many powerful theorems comparing the cohomology of a $G$-variety $X$ with its equivariant cohomology (§ contains many examples), and the equivariant cohomology of $X$ was already used to compute a Gysin map in [9].

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1 Residues and equivariant cohomology

All schemes are over $\mathbb{C}$ and we always use the classical topology. If $A$ is a sheaf of abelian groups on $X$ there is an exact triangle $i_*i^！A \to A \to j_*j^{-1}A \to i_*i^！A[1]$ of sheaves on $X$. The associated long exact sequence is the Gysin sequence

$$\ldots \longrightarrow H^*(Z, i^！A) \longrightarrow H^*(X, A) \longrightarrow H^*(G, j^{-1}A) \overset{\text{res}}{\longrightarrow} H^{*+1}(Z, i^！A) \longrightarrow \ldots$$
We call \( \text{res} : H^*(G, j^*A) \to H^{*+1}(Z, i^!A) \) the residue map. For \( \alpha \in H^d(G, j^{-1}A) \) we call \( \text{res} \alpha \in H^{d+1}(Z, i^!A) \) the residue of \( \alpha \) along \( Z \).

**Notation 1.1.** If \( S \) is a scheme with left \( G \) action, write \( S/G \) for the stack quotient of \( S \) by \( G \). Write \( BG \) for \( pt/G \). Write \( S_\bullet \) for the simplicial scheme obtained as the nerve of the cover \( S \to S/G \). If \( S = pt \) we will write \( BG_\bullet \) instead of \( pt_\bullet \). If \( f : K \to S \) is a \( G \)-equivariant map of schemes with \( G \)-action we write \( f_p : K_p \to S_p \) for the induced map. We will often abuse notation and denote \( f : K \to S \) and its induced map \( K/G \to S/G \) by the same letter.

\( \mathbb{C}_X \) is the constant sheaf on \( X \) with stalk \( \mathbb{C} \). We are mostly interested in cohomology with \( \mathbb{C} \) coefficients and will write \( H^*(X) \) for \( H^*(X, \mathbb{C}) \).

If \( E^{p,q}_* \) is a first-quadrant, cohomologically graded spectral sequence abutting to a graded module \( L^\bullet \), write \( L = F_0L \supset F_1L \supset \ldots \) for the corresponding filtration and \( e_{n,q}^p(E) : F_pL^{p+q} \to E_{n,q} \) the edge map. We will often write \( e_{n,q}^p \), leaving \( E \) implicit.

**Definition 1.2.** There is a spectral functor \( \check{C}^{p,q}(X_\bullet, -) \) computing \( H^{p+q}(X/G, -) \) with \( \check{C}^{p,q}(X_\bullet, -) = H^q(X_p, -) \), called the Čech spectral sequence \([\text{Tag 06XJ}]\).

The differential on \( \check{C}_1(X_\bullet, -) \) is the alternating sum of pullbacks \( \sum (-1)^i \partial_i : H^q(X_p, p^{-1}_p) \to H^q(X_{p+1}, p^{-1}_{p+1}) \), where \( p_p : X_p \to X/G \) is the natural map.

**Definition 1.3.** Call \( \alpha \in H^*(G) \) primitive if \( \mu^* \alpha = \pi_0^* \alpha + \pi_1^* \alpha \) where \( \mu, \pi_0, \pi_1 : G \times G \to G \) are the multiplication and the two projections.

Note that \( \check{C}_2^{1,q}(BG_\bullet, \mathbb{C}) \) is the subspace \( H^q_{pt}(G) \subset H^q(G) \) consisting of the primitive elements. \( F_{p+1}H^{q+1}(BG) \) is the kernel of \( e_{n,q}^q \) for \( 0 < n \) and the codomain of \( e_{n,q}^q(\check{C}(BG_\bullet, \mathbb{C})) \) is 0 for \( 0 < n, q \). Thus \( F_1H^{q+1}(BG, \mathbb{C}) = H^{q+1}(BG, \mathbb{C}) \) and the edge map \( e_{2,q}^1 \) is defined on all of \( H^{q+1}(BG, \mathbb{C}) \). Schulman \([\text{11}]\) proved \( \check{C}(BG_\bullet, \mathbb{C}) \) degenerates on page 2, so \( e_{2,q}^1 : H^{q+1}(BG) \to H^q_{pt}(G) \) is surjective. Hopf proved that \( H^*(G) \) is a wedge algebra on \( H^*_{pt}(G) \) \([\text{10}]\) section 13.5.

Consider the diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow \pi & & \downarrow \pi \\
Z/G & \longrightarrow & X/G \\
\end{array}
\]

As \( \pi \) is smooth the diagram of Gysin sequences below commutes:

\[
\begin{array}{cccc}
H^{q-1}(G) & \longrightarrow & H^q(Z, i^!C_X) & \longrightarrow & H^q(X) & \longrightarrow & H^q(G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}(pt) & \longrightarrow & H^q(Z/G, i^!C_{X/G}) & \longrightarrow & H^q(X/G) & \longrightarrow & H^q(pt) \\
\end{array}
\]

For \( q > 0 \), \( H^q(pt) = 0 \) and there is an inverse map \( H^q(Z/G, i^!C) \leftrightarrow H^q(X/G) \). In section \([\text{3}]\) we will prove our main result:

**Theorem 3.4** Let \( q > 0 \). Then the diagram below commutes:

\[
\begin{array}{cccc}
H^{q+1}(Z/G, i^!C) & \longrightarrow & H^{q+1}(X/G) & \longrightarrow & H^{q+1}(BG) \\
\downarrow \pi_{2,q} & & \downarrow \pi_{2,q} & & \downarrow \pi_{2,q} \\
H^{q+1}(Z, i^!C) & \longrightarrow & H^{q+1}_{pt}(G) & \longrightarrow & H^{q+1}_{pt}(G) \\
\end{array}
\]
In order to apply this one wants to be able to compute $\epsilon_{q}^{2}(\hat{C}(BG, \mathbb{C}))$. Below we will avoid thinking hard about this by working only up to a scalar, but Bott has implicitly given an algorithm for it, which we describe section 4.

**Notation 1.4.** Write $x \sim y$ if $x = ty$ for some $t \in \mathbb{C}$. 

**Proposition 1.5.** Let $H^{*}(Y)_{\text{red}}$ be the subspace of reducible elements of the cohomology ring of $Y$ under cup product. Suppose $\dim H^{2d}(BG)/H^{2d}_{\text{red}}(BG) = 1$, $\dim H^{2d-1}_{pr}(G) = 1$, $\beta \in H^{2d}(BG)$ is irreducible and $\alpha \in H^{2d-1}_{pr}(G)$. Then $\epsilon_{2}^{1,2d-1}(\beta) \sim \alpha$.

**Remark 1.6.** The conditions on $H^{2d}(BG)/H^{2d}_{\text{red}}(BG)$ and $H^{2d-1}_{pr}(G)$ are satisfied whenever $G$ is semisimple and not of type $D_{2n}$. They are also satisfied if $G = GL_{n}$ or if $G$ is of type $D_{2n}$ and $d \neq n$ [7] Section 3.7.

**Proof.** Since $\dim H^{2d-1}_{pr}(G) = 1$ it suffices to check that $\epsilon_{2}^{1,2d-1}(\beta) \neq 0$. The cup product on $BG$ lifts to a bigraded multiplication on $\hat{C}(BG, \mathbb{C})$, and $\hat{C}(BG, \mathbb{C})$ is generated as a ring by the $p = 1$ row, that is elements from $H^{q}(BG_{1}, \mathbb{C})$. Therefore $H^{2d}_{\text{red}}(BG) \subset F_{2}H^{*}(BG) = \ker \epsilon_{2}^{1,2d-1}$. Since $\dim H^{2d}(BG)/H^{2d}_{\text{red}}(BG) = 1$, if $\epsilon_{2}^{1,2d-1}(\beta) = 0$ then $\epsilon_{2}^{1,2d-1}$ is the zero map. But the Čech spectral sequence degenerates on page 2 so $\epsilon_{2}^{1,2d-1}$ is surjective, in particular nonzero. \(\square\)

We would like to use theorem 3.4 to compute res $\alpha$ for $\alpha \in H^{*}_{pr}(G)$. By “compute” we mean “express in terms of the intrinsic topology of $Z$”, ideally as a sum of products of fundamental classes. Our terminology is slightly nonstandard:

**Definition 1.7.** Let $i : K \hookrightarrow S$ be a closed immersion of smooth complex varieties of codimension $r$. The purity theorem gives a canonical isomorphism $i^{!}C_{S} \cong C_{K}[-2r]$. There is a corresponding isomorphism $H^{0}(K, \mathbb{C}) \cong H^{2r}(K, i^{!}C_{S})$. We call the image of 1 under this map the fundamental class of $K$, and denote it $[K]$.

If $S$ is smooth but $K$ is possibly singular, there is a unique element of $H^{r}(K, i^{!}C_{S})$ restricting to $[K] \in H^{r}(K^{sm}, (i^{sm})^{!}C_{S^{sm}})$ where $K^{sm}$ is the smooth locus of $K$ and $i : K^{sm} \to S - (K - K^{sm})$ is the restriction of $i$. We call that element the fundamental class of $K$ and denote it $[K]$.

If $K' \xrightarrow{a} K \xrightarrow{b} S$ is a chain of closed subspaces then the map

$$a_{!}(b^{!}a^{!})C_{S} = a_{!}^{!}b^{!}C_{S} \to b^{!}C_{S}$$

lets us interpret the fundamental class of $K'$ as an element of $H^{*}(K, b^{!}C_{S})$. We will sometimes write $[K']_{K}$ to emphasize that we mean the image of $[K']$ in $H^{*}(K, i^{!}C_{S})$, or $[K]_{S}$ to denote the image of $[K]$ in $H^{*}(S)$.

**Example 1.8.** We can use theorem 3.4 to recover a weak version of the degeneracy locus formula for the Chern classes of a globally generated vector bundle. Let $X$ be the space of $n \times n$ matrices and let $G = GL_{n}$. For $d = 1, \ldots, n$ let $U_{d} \subset X$ be the space of matrices whose top $d$ rows have rank $d$, and let $Z_{d}$ be the complement of $U_{d}$. We will show

**Proposition 1.9.** $H^{*}(X/G) \cong H^{*}(BGL_{n})$ is generated as a $\mathbb{C}$-algebra by the images of $[Z_{d}/G]$, $d = 1, \ldots, n$.

This is a weakening of the degeneracy locus formula for the Chern classes $c_{i}$ in the following sense: Let $E \to B$ be a rank-$n$ vector bundle on $B$, and suppose that it admits $n$ global sections
since the Čech spectral sequence degenerates on page 2 and the
is generated by

\[ \rho \]

\[ x \]

\[ d \]

\[ V \]

\[ \subset \]

\[ \Gamma \]

Write \[ [Z_d/G]_{X/G} \]

\[ \text{for the image of } [Z_d/G] \text{ in } H^*(X/G). \]

Let \( g_m : GL_m \hookrightarrow GL_n \)

be the map

\[ g_m(M) = \left[ \begin{array}{cc} I_{n-m} & 0 \\ 0 & M \end{array} \right] \]

\[ V_m := GL_n/g_m(GL_m) \]

is a Stiefel manifold. \( H^*(GL_n, \mathbb{C}) = \Lambda[x_1, \ldots, x_n] \)

where \( x_d \)

is a primitive element of degree \( 2d - 1 \), and \( x_d \)

is pulled back from \( H^*(V_{d-1}, \mathbb{C}) \). But \( GL_n \rightarrow V_{d-1} \)

factors through \( U_d \), so \( x_d \)

extends to a class \( \bar{x}_d \in H^*(U_d, \mathbb{C}) \).

\( Z_d \)

has codimension \( d \), and the Gysin sequence for the stratification

\[ Z_d \rightarrow X \leftarrow U_d \]

shows that \( res \bar{x}_d \sim [Z_d]_Z \in H^{2d}(Z_d, \mathbb{C}). \) \( X \)

is contractible, so the residue map is injective, so

\[ \text{theorem } [3.4] \text{ implies } \gamma_2^{1,2d-1}[Z_d/G]_{X/G} \sim x_d. \]

This implies the classes \([Z_d/G]_{X/G} \) generate \( H^*(X/G) \),

since the Čech spectral sequence degenerates on page 2 and the \( E_2 \) page of the Čech spectral sequence is generated by \( H^*_p(G) \) \[ [13] \]. \( \square \)

## 2 Application to the wonderful compactification

Let \( G \) be semisimple and centerless and let \( X \)

be the **wonderful compactification of \( G \)**. In this section

we will use theorem \[ [3.4] \] to compute \( res \alpha \) for \( \alpha \in H^*_p(G) \).

The point is that theorem \[ [3.4] \] lets us

work with equivariant cohomology, and the equivariant cohomology of \( X \)

is well understood thanks to \[ [13] \].

We recall some facts about the wonderful compactification from \[ [4], [13]. \]

Let \( \tilde{G} \) be the universal cover of \( G \) and \( \tilde{B} \subset \tilde{G} \)

a Borel subgroup. Let \( \lambda \) be a regular weight of \( \tilde{G} \)

and let \( V^\lambda \)

be the corresponding highest weight representation. There is a natural inclusion \( G \rightarrow \mathbb{P}End(V) \)

and \( X \)

is the closure of \( G \) under this embedding. \( X \)

possesses both a left \( G \)-action and a right \( G \)-action

extending the multiplication actions. We will let \( G^2 = G \times G \)

act from the left via \( (g_1, g_2) : x = g_1 x g_2^{-1}. \)

Let \( B \)

be the image of \( \tilde{B} \) in \( G \) and let \( \Delta = \{ \rho_1, \ldots, \rho_l \} \)

be a set of simple positive roots with respect to \( B \).

\( Z = X - G \)

is a normal crossings divisor with irreducible components \( D_\rho \)

labeled by the simple roots. For \( \Gamma \subset \Delta \)

let \( D_\Gamma = \bigcap_{\rho \in \Gamma} D_\rho \) (in particular \( X = D_\emptyset \)). Then \( D_\Gamma \)

is a smooth \( G^2 \)-orbit closure. Let \( P_\Gamma \subset G \)

be the parabolic subgroup generated by \( B \) together with all root

subgroups \( G_\rho \)

of roots \( \rho \in \Gamma \), and let \( P^-_\Gamma \)

be the opposite parabolic. Let \( G_1 \)

be the adjoint quotient of \( P_\Gamma \). Then \( D_\Gamma \)

is a fiber bundle over \( G/P_\rho \times P^-_\rho \) \( G \)

with fiber the wonderful compactification of \( G_\rho \).

In particular \( D_\Delta = G/B \times B^- \) \( G \)

and \( D_\Delta/G^2 = BB \times BB^- \).

Write \( H^*(BB \times BB^-) = \mathbb{C}[t \times \bar{t}] = \mathbb{C}[u_1, \ldots, u_l, v_1, \ldots, v_l] \)

where \( u_i \)

is the first chern class of the pullback of \( \rho_i \) to \( B \) and \( v_i \)

is the first Chern class of the pullback of \( \rho_i \) to \( B^- \).

Let

\[ x_i = u_i - v_i, \quad y_i = u_i + v_i \]

For \( \Gamma \subset \Delta \)

let \( W_\Gamma \subset W \)

be the subgroup generated by the reflections associated to the simple roots \( \rho \in \Gamma \), and let \( x^\Gamma = \prod_{\rho_i \in \Gamma} x_i \). \( \mathbb{C}[y_1, \ldots, y_l] \)

has an action of \( W \) by via the diagonal action on \( t \times t. \)
For $\Lambda \subset \Delta$ let $A_{\Lambda} \subset \mathbb{C}[u_1, \ldots, u_l, v_1, \ldots, v_l]$ be the span of the elements of the form

$$x^{r}q(x_1, \ldots, x_l)p(y_1, \ldots, y_l)$$

where $p$ is invariant under $W_{\Delta-(\Lambda \cup \Gamma)}$. Set $A = A_{\emptyset}$. For any graded ring $R$, let $\overline{R} \subset R$ be the ideal generated by all elements of strictly positive degree.

**Theorem 2.1.** (Strickland [13])

1. The pullback $H^*(D_{\Lambda}/G^2) \to H^*(D_{\Delta}/G^2)$ is an injection with image $A$.
2. Under this injection, $[D_{\Lambda}]$ maps to $x^\Lambda$.
3. The inclusion $\mathbb{C}[u, v]^{W \times W} = A_{\Lambda}^{W \times W} \hookrightarrow A_{\Lambda}$ corresponds to pullback along the projection $D_{\Lambda}/G^2 \to BG^2$.
4. $H^*(D_{\Lambda}/G^2) \to H^*(D_{\Lambda})$ is surjective with kernel generated by $\overline{H}^*(BG^2)$.

**Remark 2.2.** Strickland treats only the case $\Lambda = \emptyset$, but their arguments generalize without any extra work.

**Corollary 2.3.** $H^*(Z/G, i^!\mathbb{C}_{X/G^2})$ is canonically isomorphic to the cokernel of

$$\bigoplus_{i<j} A_{ij} \to \bigoplus_{k} A_k$$

$$(f_{ij}) \to \left(\sum_{i<k} x_i f_{ik} - \sum_{j>k} x_j f_{kj}\right)$$

$H^*(Z, i^!\mathbb{C})$ is canonically isomorphic to the cokernel of

$$\bigoplus_{i<j} (A_{ij}/\overline{A}_{ij}^{W \times W}) \to \bigoplus_{k} (A_k/\overline{A}_{k}^{W \times W})$$

given by the same formula.

**Proof.** Since $H^*(D_{\Lambda}/G^2, i^!\mathbb{C}_{X/G^2})$ vanishes in odd degrees, $H^*(Z/G^2, i^!\mathbb{C}_{X/G^2})$ is the cokernel of the map

$$\varphi: \bigoplus_{i<j} H^*(D_{(i,j)}/G^2, i^!\mathbb{C}_{X/G^2}) \to \bigoplus_{k} H^*(D_{(i)}/G^2, i^!\mathbb{C}_{X/G^2})$$

where $f \in H^*(D_{(i,j)}/G^2, i^!\mathbb{C}_{X/G^2})$ is sent to its image in $H^*(D_{(i,j)}/G^2, i^!\mathbb{C}_{X/G^2})$ minus its image in $H^*(D_{(i)}/G^2, i^!\mathbb{C}_{X/G^2})$.

We need only to describe $\varphi$ in terms of the $A_{ij}$. The isomorphism $H^*(D_{\Lambda}) \to A_{\Lambda}$ is obtained by pulling back all the way to $D_{\Delta}$, which factors through pullback to $D_{\Lambda}$. The composition

$$H^{d-2}(D_{(i,j)}/G^2) \cong H^d(D_{(i,j)}/G^2, i^!\mathbb{C}) \to H^d(D_{(i,j)}/G^2) \to H^d(D_{(i,j)}/G^2)$$

is multiplication by $[D_{(i,j)}]$ which implies the first claim.

The argument for $H^*(D_{(i,j)}, i^!\mathbb{C}_X)$ is similar, using parts (3) and (4) of theorem 2.1 to describe $H^*(D_{(i,j)})$ and $H^*(D_{(i)})$.

In particular there is a canonical surjection $\psi: \bigoplus_k A_{(k)} \to H^*(Z, i^!\mathbb{C}_X)$ and we know its kernel. Let $a \in H_{pr}^{2d-1}(G)$ and let $p(u_1, \ldots, u_l) \in \mathbb{C}[u_1, \ldots, u_l]^W$ be irreducible and homogeneous of degree $d$. Let $\beta = p(x_1 + y_1, \ldots, x_l + y_l) - (-1)^d p(x_1 - y_1, \ldots, x_l - y_l)$.
Proposition 2.4.

1. \( \beta = \sum_{k=1}^l x_k f_k(x, y) \) for some \( f_k(x, y) \in A(\{k\}) \).

2. Suppose that \( G \) is not of type \( D \) or that \( 2d \neq 1 \). Then \( \psi(f_k) \sim \text{res}\alpha \).

Proof. Note that \( \beta \sim p(u_1, \ldots, u_l) \pm p(v_1, \ldots, v_l) \in \mathbb{C}[t \times t]^{W \times W} \). Therefore it descends to \( BG \times BG \), and lies in \( A \). On the other hand every term of \( \beta \) is divisible by some \( x_i \). This proves the first claim.

In fact the two summands \( p(u) \) and \( p(v) \) are individually \( W \times W \)-invariant, and therefore descend to \( p(u) \in B(G \times 1) \) and \( p(v) \in B(1 \times G) \) respectively. This implies that \( p(v) \) vanishes under pullback along \( X/G = X/(G \times 1) \to X/G^2 \). Thus the pullback of \( \beta \) to \( \beta' \in H^*(X/G) \) is a scalar multiple of the pullback of \( p(u) \).

By proposition 1.2 and the remark following, \( c_2^{l,2d-1}(p(u)) \sim \alpha \). By theorem 3.4 in order to compute \( \text{res}\alpha \) up to a scalar, it would suffice to express \( \beta' \) as the image of some \( \gamma' \in H^*(Z/G, i^!C) \) and compute the pullback of \( \gamma' \) to \( H^*(Z, i^!C) \).

The expression \( \beta = \sum_{k=1}^l x_k f_k(x, y) \) says that \( \beta \) is the image of \( \gamma \in H^*(Z/G^2, i^!C) \), where \( \gamma \) is the element represented by \( (f_k) \in \bigoplus_k A(\{k\}) \). The pullback of \( \gamma \) to \( H^*(Z, i^!C) \) is equal to the pullback of \( \gamma' \), so this completes the proof. \( \square \)

Example 2.5. Let \( G = \text{PGL}_2 \) and \( d = 2 \). Then

1. \( X = \mathbb{P} \text{Mat}_{2 \times 2} \cong \mathbb{P}^3 \), where \( \text{Mat}_{2 \times 2} \) denotes the vector space of \( 2 \times 2 \) matrices.

2. There is a single simple root \( \rho \), and \( D_\rho = Z \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset X \) is the Segre variety.

3. \( \mathbb{C}[t] = \mathbb{C}[u_1] \) is a polynomial ring in one variable.

Now let \( \alpha \in H_0^{\rho^*}(G) \) and let \( p(u_1) = u_1^2 \). This is an irreducible \( W \)-invariant polynomial. Then \( \beta = (x_1 + y_1)^2 - (x_1 - y_1)^2 = 4x_1y_1 \), so \( f_1 = 4y_1 = 4(u_1 + v_1) \in H^*(Z/G^2) \). Proposition 2.4 says that to compute \( \text{res}\alpha \) we need only pull back \( f_1 \) to \( H^*(Z) \).

Let \( \sigma \in H^*(\mathbb{P}^1) \) be the hyperplane section. Under the surjection \( H^*(Z/G^2) \to H^*(Z) = H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^1) \), \( u_1 \) is sent to \( \sigma \otimes 1 \) while \( v_1 \) is sent to \( 1 \otimes -\sigma \). (The apparent asymmetry arises because the second factor of \( G \times G \) acts via \((1, g) \cdot x \mapsto xg^{-1}\).) So \( \text{res}\alpha \sim \left[ \{0\} \times \mathbb{P}^1 \right] - \left[ \mathbb{P}^1 \times \{0\} \right]. \)

3 Proof of theorem 3.4

Notation 3.1. By a bicomplex we mean an array \( \mathcal{L} = \mathcal{L}^{\bullet \bullet} \) of objects with anticommuting morphisms \( \partial_1 : \mathcal{L}^{p,q} \to \mathcal{L}^{p+1,q} \) and \( \partial_1 : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1} \), and with \( \mathcal{L}^{p,q} = 0 \) for \( p \ll 0 \) or \( q \ll 0 \). \( \mathcal{L}[i,j] \) is the shifted bicomplex \( \mathcal{L}[i,j]^{p,q} = \mathcal{L}^{p+i,q+j} \). \( E_n^{p,q}(\mathcal{L}) \) is the spectral sequence associated to \( \mathcal{L} \) with the \( \partial_1 \) orientation, so in particular \( E_1^{p,q}(\mathcal{L}) = H^q(L^{p,*}) \). \( \text{Tot}(\mathcal{L}) \) is the total complex and \( \mathbb{H}(\mathcal{L}) \) is its cohomology. Write \( F_{i+1}^n \mathbb{H}(\mathcal{L}) \subseteq F_n^i \mathbb{H}(\mathcal{L}) \subseteq \cdots \subseteq F_0^i \mathbb{H}(\mathcal{L}) = \mathbb{H}(\mathcal{L}) \) for the filtration associated to the \( \partial_1 \) orientation and \( e_n^{p,q}(\mathcal{L}) : F_n^i \mathbb{H}^{p+q}(\mathcal{L}) \to E_n^{p,q}(\mathcal{L}) \) for the edge map.

Let \( \mathfrak{A} \) be an abelian category and let \( M \) be a left-exact additive functor from \( \mathfrak{A} \) to bounded-below complexes of abelian groups. If \( \mathfrak{A} \) is a bounded-below complex in \( \mathfrak{A} \) one obtains a first-quadrant bicomplex of abelian groups \( A = M(\mathfrak{A}) \) by the rule \( A^{p,q} = M(\mathfrak{A}^p) \), with \( \partial_1 : A^{p,q} \to A^{p+1,q} \) the differential of \( M(\mathfrak{A}) \) and \( \partial_1 : A^{p,q} \to A^{p,q+1} \) given by \((-1)^p M(\partial_1) \) (that is, up to a sign the degree-\( p \) component of the chain complex map \( M(\mathfrak{A}) \to M(\mathfrak{A}^{p+1}) \) induced functorially by the differential of \( \mathfrak{A} \)).
Suppose \( A^\bullet \) and \( B^\bullet \) are bounded-below complexes of objects from \( \mathfrak{A} \). Let \( \tilde{f} : A \to B \) be a map and \( C \) be its cone. Name the natural maps as shown:

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{f}} & B \\
\downarrow{\tilde{g}} & & \downarrow{k} \\
C & \to & A[1]
\end{array}
\]

Let \( A = M(A) \), \( B = M(B) \) and \( C = M(C) \). Then \( \tilde{f} \), \( \tilde{g} \) and \( k \) induce maps of bicomplexes

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{k} \\
C & \to & A[0, 1]
\end{array}
\]

Let \( P \) be the bicomplex

\[
P^{r,p} = \begin{cases} 
E_1^{p,q}(A) & r = 3m \\
E_1^{p,q}(B) & r = 3m + 1 \\
E_1^{p,q}(C) & r = 3m + 2 
\end{cases}
\]

with differential \( \partial^{p}_{II} \) inherited from the differentials \((-1)^{p} \partial_{I} \) of \( A, B \) and \( C \), and \( \partial^{p}_{II} \) identical to \( f, g \) or \( k \). On \( E_{2}(P) \) we obtain a differential

\[
-\phi : \frac{\ker : E_{2}^{p,q+1}(B) \to E_{2}^{p,q+1}(C)}{\operatorname{Im} : E_{2}^{p,q}(A) \to E_{2}^{p,q+1}(B)} \to \frac{\ker : E_{2}^{p,q+1}(A) \to E_{2}^{p,q+1}(B)}{\operatorname{Im} : E_{2}^{p,q}(C) \to E_{2}^{p,q+1}(A)}
\]

We will give a more concrete description of \( \phi \) in lemma 3.3.

Let \( \tilde{F}_{p} \mathbb{H}(A) \) be the preimage under \( f \) of \( F_{p} \mathbb{H}(B) \). Naturality of \( e_{n}^{p,q}(-) \) implies

Lemma 3.2.

1. The image of \( \left( \tilde{F}_{p+1} \mathbb{H}^{p+q+1}(A) \cap F_{p} \mathbb{H}^{p+q+1}(A) \right) \) under \( e_{n}^{p,q+1} \) is contained in \( \ker : E_{2}^{p,q+1}(A) \to E_{2}^{p,q+1}(B) \). In particular, \( e_{n}^{p,q+1} \) induces a map

\[
u_{n}^{p,q+1} : F_{p} \mathbb{H}^{p+q+1}(A) \cap \tilde{F}_{p+1} \mathbb{H}^{p+q+1}(A) \to \frac{\ker : E_{2}^{p,q+1}(A) \to E_{2}^{p,q+1}(B)}{\operatorname{Im} : E_{2}^{p,q}(C) \to E_{2}^{p,q+1}(A)}
\]

2. The image of \( \left( \tilde{F}_{p+1} \mathbb{H}^{p+q+1}(A) \right) \) under \( e_{n}^{p+1,q}(B) \circ f \) is contained in \( \ker : E_{2}^{p+1,q}(B) \to E_{2}^{p+1,q}(C) \). In particular, \( e_{n}^{p+1,q} \) induces a map

\[
u_{n}^{p+1,q} : f \left[ \tilde{F}_{p+1} \mathbb{H}^{p+q+1}(A) \right] \to \frac{\ker : E_{2}^{p+1,q}(B) \to E_{2}^{p+1,q}(C)}{\operatorname{Im} : E_{2}^{p+1,q}(A) \to E_{2}^{p+1,q}(B)}
\]

Lemma 3.3. The diagram commutes:

\[
\begin{array}{ccc}
\tilde{F}_{1} \mathbb{H}^{q+1}(A) & \xrightarrow{f} & \tilde{F}_{1} \mathbb{H}^{q+1}(A) \\
\downarrow{\nu_{2}^{0,q+1}} & & \downarrow{\nu_{2}^{0,q}} \\
E_{2}^{0,3q+3}(P) & \xleftarrow{\phi} & E_{2}^{1,3q+1}(P)
\end{array}
\]

Proof. If \( x \in A^{i,j} \), write \([x]_{1}\) for the class that \( x \) represents in \( E_{1}^{i,j}(A) \), \([x]_{2}\) for the class it represents in \( E_{2}^{i,j}(A) \), and \([x]_{3}\) for the class it represents in \( E_{2}^{i,j}(P) \) (assuming these classes are well-defined). Similarly for \( x \in B^{i,j} \) or \( x \in C^{i,j} \). Write \( x = i \) \( y \) for \([x]_{i} = [y]_{i}\).
Let \( a \in \widetilde{H}_i^{q+1}(A) \), and let \( (a^{ij}) \in \bigoplus_{i+j=n} A^{i,j} \) be a representative of \( a \) with \( a^{i,j} = 0 \) for \( i < 0 \). Then \( \epsilon^i_{2,q+1}(a) = [a^{i,q+1}]_2 \). We have \( f(a) \in \mathcal{F}_i(G) \) so
\[
0 = \epsilon^i_{1,q+1} f(a) = [f(a^{i,q+1})]_1
\]
This implies that there exists \( b \in B^{0,q} \) with \( \partial_I b = f(a^{0,q+1}) \). Then \( f(a) \) is represented by \( f(a^{ij}) - (\partial_I + \partial_{II}) b \), and
\[
\epsilon^2_{2,q} f(a) = [f(a^{1,q}) - \partial_I b]_2
\]
Thus our goal is to show \( \phi(f(a^{1,q}) - \partial_I b)_3 = [a^{0,q+1}]_3 \). We can compute \( \phi(f(a^{1,q}) - \partial_I b)_3 \) by lifting \( g[(a^{1,q}) - \partial_I b] \) to some \( \sigma \in E^{1,q}_1(C) \); then \( [k(\sigma)]_3 = \phi(f(a^{1,q}) - \partial_I b)_3 \).

We will prove the lemma by brute computation. Note that \( \mathcal{C}^{p,v} \) is identical to the cone of \( \mathcal{A}^{p,v} \to B^{p,v} \) and the maps \( g \) and \( k \) are the natural maps to and from the cone. Explicitly

1. \( C^{p,q} = A^{p,q+1} + B^{p,q} \) with differentials given by
\[
\partial_{II}(a, b) = (-\partial_{II} a, (-1)^p f(a) + \partial_{II} b) \quad \partial_I(a, b) = (\partial_I a, \partial_I b)
\]
2. \( g : B \to C \) and \( k : C \to A[0,1] \) are given by
\[
g(b) = (0, b) \quad k(a, b) = a
\]

We claim that \( \sigma = [(a^{0,q+1}, -b)]_1 \) is a lift of \( g[(a^{1,q}) - \partial_I b]_1 \). We must check two things:

a. \( (a^{0,q+1}, -b) \) is closed under the horizontal differential: By fact (1),
\[
\partial_{II}(a^{0,q+1}, b) = (\partial_{II} a^{0,q+1}, f(a^{0,q+1}) - \partial_{II} b)
\]
Since \( (a^{ij}) \) was a cocycle for \( \text{Tot}\mathcal{A} \) and is concentrated on the \( j \geq 0 \) rows, \( \partial_{II} a^{0,q+1} = 0 \). By choice of \( b \) we have \( \partial_{II} b = f(a^{0,q+1}) \).

b. \( \partial_I \sigma|_1 = g[(a^{1,q}) - \partial_I b]_1 \): Write \( \gamma = g[(a^{1,q}) - \partial_I b]_1 \). By fact (2), \( \gamma = (0, f(a^{1,q}) - \partial_I b) \). By (1), \( \partial_{II}(a^{1,q}, 0) = (-\partial_{II} a^{1,q}, -f(a^{1,q})) \). Then
\[
\gamma = (0, f(a^{1,q}) - \partial_I b) = (0, f(a^{0,q}) - \partial_I b + \partial_{II} (a^{1,q}, 0)) = (0, f(a^{1,q}) - \partial_I b + (-\partial_{II} a^{1,q}, -f(a^{1,q}))) = (-\partial_{II} a^{1,q}, -\partial_I b)
\]
Since \( a \) was a cocycle, \( -\partial_{II} a^{1,q} = \partial_I a^{0,q+1} \). Then \( \gamma = 1 \) \( (\partial_I a^{0,q+1}, -\partial_I b) = \partial_I \sigma \).

Finally, fact (2) says \( k(\sigma) = [a^{0,q+1}]_1 \), which proves the claim.

Recall the construction of \( \check{C}(X_\ast, \mathcal{F}) \) from [12][Tag 06X2]. Let \( M \) be the functor from sheaves of abelian groups on \( X/G \) to complexes of abelian groups, sending \( \mathcal{G} \) to
\[
\ldots \to \Gamma(X_p, \pi_p^{-1} \mathcal{G}) \to \Gamma(X_{p+1}, \pi_{p+1}^{-1} \mathcal{G}) \to \ldots
\]
Then \( \check{C}(X_\ast, \mathcal{F}) \) is the spectral sequence associated to the double complex \( M(\mathcal{I}) \), where \( \mathcal{I}_* \) is an injective resolution of \( \mathcal{F} \); from the first page it no longer depends on the choice of \( \mathcal{I}_* \).
Theorem 3.4. Let $q > 0$, and let $s : X/G \to BG$ be the quotient of $X \to pt$. Then the diagram below commutes:

$$
\begin{array}{c}
H^{q+1}(Z/G, i^!C_{X/G}) & \to & H^{q+1}(X/G) & \to & H^{q+1}(BG) \\
\uparrow & & \uparrow & & \downarrow \epsilon_2^{q+1} \\
H^{q+1}(Z, i^!C_X) & \to & H^q(G) & \uparrow \text{res} & \text{res} & \downarrow H^q(G)
\end{array}
$$

Proof. The edge maps $\epsilon_2^{0,q+1}$ are natural transformations, and $\epsilon_2^{0,q+1}(BG_\ast) : H^{q+1}(BG) \to H^{q+1}(pt)$ is the zero map, so $H^{q+1}(BG) \to H^{q+1}(X/G)$ factors through $F_1H^{q+1}(X/G)$. Therefore it would suffice to show commutativity of

$$
\begin{array}{c}
\tilde{F}_1H^{q+1}(Z/G, i^!C) & \to & F_1H^{q+1}(X/G) & \to & H^{q+1}(BG) \\
\uparrow & & \uparrow & & \downarrow \epsilon_2^{q+1} \\
H^{q+1}(Z, i^!C) & \to & C_2^{1,q}(X_\ast, i^!C_X) & \to & \text{res} & \downarrow H^q(G)
\end{array}
$$

Let $\mathcal{A}$ be a complex of injectives quasi-isomorphic to $i_\ast i^!C_{X/G}$ and $\mathcal{B}$ be a complex of injectives quasi-isomorphic to $C_{X/G}$. Let $\tilde{f} : \mathcal{A} \to \mathcal{B}$ be the natural map $i_\ast i^!C_{X/G} \to C_{X/G}$ and let $\mathcal{C}$ be its cone. Then $\mathcal{C}$ is a complex of injectives quasi-isomorphic to $j_\ast j^{-1}C_{X/G}$. We will apply lemma 3.3 to $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$. Note that in this case $E_2^{i,j}(C) = 0$ unless $i = j = 0$. In particular the domain of $\phi$ is a quotient of $C_2^{1,q}(X_\ast, C_X)$ and the codomain of $\phi$ is a subspace of $C_0^{0,q+1}(Z_\ast, i^!C_X) \subset H^{q+1}(Z, i^!C_X)$. The commuting diagram of lemma 3.3 implies commutativity of the simpler diagram

$$
\begin{array}{c}
\tilde{F}_1H^{q+1}(Z/G, i^!C_{X/G}) & \to & F_1H^{q+1}(X/G) \\
\uparrow & & \uparrow \\
H^{q+1}(Z, i^!C_X) & \to & \tilde{C}_2^{1,q}(X_\ast, C) \\
\phi & & \text{ker} \left[ C_2^{0,q+1}(Z_\ast, i^!C_X) \to C_2^{0,q+1}(X_\ast, C_X) \right] & \to & H^{q+1}(Z, i^!C_X)
\end{array}
$$

where we have written $\phi$ for the composition

$$
\tilde{C}_2^{1,q}(X_\ast, C) \to \tilde{C}_2^{1,q}(X_\ast, i^!C_X) \to \phi \left[ C_2^{0,q+1}(Z_\ast, i^!C_X) \to C_2^{0,q+1}(X_\ast, C_X) \right] \to H^{q+1}(Z, i^!C_X)
$$

It suffices to show that res : $H^q(G) \to H^{q+1}(Z, i^!C)$ is equal to the composition

$$
H^q(G) \xrightarrow{=} \tilde{C}_2^{1,q}(BG_\ast, C) \xrightarrow{s_i} \tilde{C}_2^{1,q}(X_\ast, C) \xrightarrow{\varphi} \tilde{C}_2^{0,q+1}(X_\ast, i^!i^!C)
$$

Here $s_\ast : X_\ast \to BG_\ast$ is the map induced by $s$, so in particular $s_1 : G \times X \to G$ is the projection.

Let $\alpha \in H^q_{pr+1}(G)$. We will compute $\varphi(s_1^*\alpha)$ referring to the diagram below:

$$
\begin{array}{c}
H^q(G \times Z, i^!C) & \to & H^q(G \times X) & \to & H^q(G \times G) & \to & H^{q+1}(G \times Z, i^!C) & \to & H^{q+1}(G \times X) \\
\uparrow & & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
H^q(Z, i^!C) & \to & H^q(X) & \to & H^q(G) & \to & \text{res} & H^{q+1}(Z, i^!C) & \to & H^{q+1}(X) \\
0 & = & 0 & = & 0 & = & 0 & = & 0
\end{array}
$$
The class \( \varphi(s_1^*\alpha) \) is represented by any \( \beta \in H^{q+1}(Z, i^!\mathbb{C}) \) that lifts to \( \sigma \in H^q(G) \) with \( \partial \sigma = j_1^*s_1^*\alpha \), so we need to show that \( \beta = \text{res} \alpha \) works. Then \( \beta \) lifts to \( \alpha \in H^q(G) \), and \( \partial \alpha = \mu^*\alpha - \pi_1^*\alpha \), where \( \mu : G \times G \to G \) is the multiplication and \( \pi_0, \pi_1 : G \times G \to G \) are the projections. Since \( \alpha \) is primitive \( \mu^*\alpha = \pi_0^*\alpha + \pi_1^*\alpha \) and \( \partial \alpha = \pi_0^*\alpha \). As \( s_1 \circ j_1 = \pi_0 \) this proves the claim. \( \square \)

4 Computation of the edge map for \( BG \)

Here we explain how to compute the edge map \( c_{2,q}^i : H^{q+1}(BG) \to H^q(G) \) coming from the Čech spectral sequence \( \hat{C}_n^{p,q}(BG_\bullet, \mathbb{C}) \). This is a repackaging of work of Bott \[1\]. We assume that \( G \) is connected and semisimple. Let \( \mathfrak{g} \) be the Lie algebra of \( G \). Our explicit models of \( H^*(BG) \) and \( H^*(G) \) are respectively the ring of invariant polynomials \( \mathbb{C}[\mathfrak{g}]^G = (\text{Sym}^*\mathfrak{g}^\vee[2])^G \) and the cohomology of the Chevalley-Eilenberg complex \( \mathcal{C}(\mathfrak{g}^\vee) \) of \( \mathfrak{g} \). The precise identifications are explained in definition \[4, 9\] and theorem \[4, 9\].

**Definition 4.1.** Denote by \( \Omega^q_X \) the sheaf of smooth complex-valued \( q \)-forms on \( X \). The Čech-de Rham bicomplex \( \hat{\Omega}_0^{\bullet, \bullet}(X_\bullet, \mathbb{C}) \) is the bicomplex with \( \hat{\Omega}_0^{p,q}(X_\bullet, \mathbb{C}) = \Gamma(X_p, \Omega^q) \). The differential \( \partial_{II} \) is \((-1)^p\) times the de Rham differential of \( X_p \), and the differential \( \partial_I \) is the alternating sum of pullbacks along the projections \( X_{p+1} \to X_p \). Write \( \hat{\Omega}_\bullet(X_\bullet, \mathbb{C}) \) for the associated spectral sequence.

**Notation 4.2.** If \( M^{p,q} \) is a cosimplicial complex (where \( p \) is the “simplicial coordinate” and \( q \) is the “chain complex coordinate”), with differential \( \delta : M^{p,q} \to M^{p,q+1} \) and coface maps \( \phi_i^{p,q} : M^{p,q} \to M^{p+1,q} \), \( i = 0, 1, \ldots, p \), let \( kM \) be the associated (anticommuting) bicomplex defined by \((kM)^{p,q} = M^{p,q} \) with differentials

\[
\partial_I^{p,q}(m) = \sum_{i=0}^{p} (-1)^i \phi_i^{p,q}(m) \quad \partial_{II}^{p,q}(m) = (-1)^p \delta(m)
\]

Write \( H^*M \) for the cohomology of the total complex \( \text{Tot}(kM) \) of \( kM \). We identify cosimplicial modules with cosimplicial complexes concentrated in degree 0.

Let \( \Gamma_{\Delta}(X_\bullet, -) \) be the functor sending sheaves of abelian groups on \( X_\bullet \) to cosimplicial modules by the rule \( \Gamma_{\Delta}(X_\bullet, \mathcal{F})_p = \Gamma_{\Delta}(X_p, \mathcal{F}) \).

Since Bott’s results use the Čech-de Rham spectral sequence rather than the Čech spectral sequence, we need to compare them. We will do so via a third spectral sequence studied by Friedlander \[5\], Proposition 2.4. The following lemma and corollary are probably well known.

**Lemma 4.3.** Let \( \mathcal{I} \) be an injective resolution of \( \mathbb{C}_{X/G} \). There is a bicomplex \( Q^{p,q} \) with maps \( \hat{C}_0(X_\bullet, \mathcal{I}) \to Q \leftarrow \hat{\Omega}_0(X_\bullet, \mathbb{C}) \) inducing isomorphisms \( \hat{C}_n(X_\bullet, \mathcal{I}) \to E_n(Q) \leftarrow \hat{\Omega}_n(X_\bullet, \mathbb{C}) \) for all \( n \geq 1 \).

**Proof.** Following \[5\], we work with sheaves on the simplicial scheme \( X_\bullet \). For \( q \) fixed, the collection of sheaves \( \hat{\Omega}_n(X_p, \mathbb{C}) \) defines such a sheaf. \( \Omega^q_{X_\bullet} \) defines a complex of sheaves on \( X_\bullet \) resolving the constant sheaf \( \mathbb{C}_{X_\bullet} \). Similarly \( \pi_1^{-1}\mathcal{I}^q \) defines a complex of sheaves on \( X_\bullet \) resolving \( \mathbb{C}_{X_\bullet} \).

Let \( J^\bullet \) be an injective complex of sheaves on \( X_\bullet \) resolving \( \mathbb{C}_{X_\bullet} \), so that \( \Gamma_{\Delta}(X_\bullet, J) \) defines a cosimplicial complex. Let \( Q^{p,q} = k\Gamma_{\Delta}(X_\bullet, J) \). The identity morphism of \( \mathbb{C}_{X_\bullet} \) induces maps of resolutions \( x' : \Omega^q_{X_\bullet} \to J \) and \( y' : \pi_1^{-1}\mathcal{I}^q \to J \), and taking global sections in each simplicial degree we obtain \( x \) and \( y \).
Each $J_p^\bullet$ is an injective resolution of $X_{C_p}$ (see the first paragraph of the proof of [5] Proposition 2.4). $\pi_p^{-1}T^\bullet$ is an injective resolution of $\mathbb{C}X$ (see the proof of [12] Tag 06XF) and $\Omega^\text{X}_{C_p}$ is a $\Gamma(X_p, -)$-acyclic resolution of $\mathbb{C}X$, so $x'$ and $y'$ induce quasi-isomorphisms $\Gamma(X_p, \pi_p^{-1}T) \to \Gamma(X_p, J_p)$ and $\Gamma(X_p, \Omega^\bullet_{C_p}) \to \Gamma(X_p, J_p)$. In other words, $x$ and $y$ induce isomorphisms on $E_1$. Therefore they induce isomorphisms on all later pages.

\begin{corollary}
The edge maps $c_{n, q}^p : F_p H^{p+q}(X/G, \mathbb{C}) \to H^q(X_p, \mathbb{C})$ obtained from the Čech spectral sequence and the Čech-de Rham spectral sequence are identical for $n \geq 1$.
\end{corollary}

This reduces the problem to computing the edge map for the Čech-de Rham spectral sequence.

\begin{notation}
If $M$ is a module, let $CM$ be the cosimplicial module $C^p M = M^{\oplus p+1}$, where the coface maps are defined by inserting zeros (for example $(m_0, \ldots, m_p) \to (m_0, \ldots, m_i, 0, m_{i+1}, \ldots, m_p)$). Let $\Sigma M \subset CM$ be the sub-cosimplicial module defined by $\Sigma^p M = \{(m_0, \ldots, m_p)| \sum m_i = 0\}$.

Let $\delta : g^\vee \to \wedge^2 g^\vee$ be the dual of the Lie bracket. This extends using the Leibniz rule to $\delta : \wedge^d g^\vee \to \wedge^{d+1} g^\vee$. The Chevalley-Eilenberg complex $\mathcal{C}g^\vee$ of $g$ is defined by $\mathcal{C}^d g^\vee = \wedge^d g^\vee$ with differential $\delta$. We consider $\mathcal{C}$ to be a covariant functor from Lie coalgebras to chain complexes.

There is a natural cosimplicial Lie coalgebra structure on $\Sigma g^\vee$. Applying $\mathcal{C}$ we obtain a cosimplicial chain complex $\mathcal{C}(\Sigma g^\vee)$.

\begin{definition}
Identifying elements of $\mathcal{C}^q(\Sigma^p g^\vee)$ with left-invariant $q$-forms on $BG$, [1] decomposition lemma] defines a map of cosimplicial chain complexes $\tau : \mathcal{C}^\bullet(\Sigma^\bullet g^\vee) \to \Omega(BG, \mathbb{C})$.
\end{definition}

\begin{lemma}
(Bott) $\tau$ induces an isomorphism $H^\ast \mathcal{C}(\Sigma g^\vee) \to H^\ast \Omega(BG, \mathbb{C})$.
\end{lemma}

\begin{remark}
Lemma 4.7 is the decomposition lemma of [1] plus the discussion immediately afterwards. Bott works with the restriction of $\tau$ to the $G$-invariants of $\mathcal{C}(\Sigma g^\vee)$. Since $G$ is semisimple and connected $H^\ast \mathcal{C}(\Sigma g^\vee)^G \to H^\ast \mathcal{C}(\Sigma g^\vee)$ is an isomorphism, and it will be convenient for us to use this less-refined version of Bott’s decomposition lemma.

Bott then studies the spectral sequence of $\mathbb{C}(\Sigma g^\vee)$ in the $\partial_I$ orientation (rather than the $\partial_H$ orientation used to get the Čech-de Rham spectral sequence). He shows it degenerates on page 1, and (using the Eilenberg-Zilbur theorem) shows that the Alexander-Whitney map identifies its total cohomology with $\mathbb{C}[g]^G$ [1] Lemma 3.1. Together with lemma 4.7 this recovers the Chern-Weil isomorphism.

\begin{theorem}
1. (Bott, [1], Theorem 1) $\tau$ induces an isomorphism $\mathcal{C}[g]^G \to H^\ast(BG)$

2. (Chevalley-Eilenberg, [2], Theorem 15.2) $\tau^{1, \bullet}$ induces an isomorphism $\mathcal{C}g^\vee \to H^\ast(G)$.
\end{theorem}

\begin{remark}
To get theorem 4.9 both Bott and Chevalley-Eilenberg require that $G$ be compact. Since $G$ is semisimple we can reduce to that case by choosing a maximal compact $K \subset G$, whose Lie algebra $\mathfrak{k}$ is a real form of $\mathfrak{g}$ (see [3] 3.4, 3.5 and 4.5). Then $K \to G$ is a deformation retract, $BK \to BG$ is a homotopy equivalence, and the Chevalley-Eilenberg complex of $\Sigma g$ is the base change to $\mathbb{C}$ of the (real) Chevalley-Eilenberg complex of $\mathfrak{k}^\ast := \text{Hom}_R(\mathfrak{k}, \mathbb{R})$. So if these statements hold for $K$ then they hold for $G$ as well.

\end{remark}
We can now describe an algorithm for the edge map. The input is a homogeneous invariant polynomial \( a \in \left( \text{Sym}^d g^\vee[2] \right)^G \) of degree \( 2d \).

**Algorithm 4.11.**

1. Identify \( a \) with some \( \bar{a} \in \bigotimes^d g^\vee \) under the inclusion \( \text{Sym}^d g^\vee \hookrightarrow \bigotimes^d g^\vee \).

2. Use the inverse Alexander-Whitney map (see [14] 8.5.4 for the simplicial version) to identify \( \bar{a} \) with some \( a^{d,d} \in \mathbb{k}(\Sigma g^\vee)^{\otimes d} \).

3. We now construct an element \( (a^{p,q}) \in \bigoplus_{p+q=2d} \mathcal{C}^q(\Sigma g^\vee)^p \) that represents the class of \( H^* \mathcal{C}(\Sigma g^\vee) \) corresponding to \( a \) under the isomorphism of theorem 4.9. Set \( a^{p,q} = 0 \) for \( p > d \). We got \( a^{d,d} \) in the previous step. Now choose the remaining \( a^{p,q} \) to satisfy the recurrence \( \partial_I a^{p,q} = \partial_I a_{p-1,q+1} \). This is solvable since the \( \partial_I \)-cohomology of \( \mathbb{k}^\bullet \mathcal{C}(\Sigma g^\vee) \) is concentrated in degree \( q \).

4. Finally, \( a^{1,2d-1} \in \wedge^{2d-1} g^\vee = \mathcal{C}^{2d-1} g^\vee \) represents \( \epsilon_1^{1,2d-1}(a) \).

As an example we show

**Proposition 4.12.** Let \( G = \text{PGL}_2 \), and let \( a \in \mathbb{C}[g]^G \) be the determinant polynomial. Let \( \eta \in \wedge^3 g^\vee \) be the form \( \eta(u,v,w) = \langle [u,v],w \rangle \) where \( \langle -,- \rangle \) is the Killing form, and let \( [\eta] \in H^3(\mathcal{C}g^\vee) \) be the class it represents. Then \( \epsilon_1^{3,3}(a) = \frac{1}{2} [\eta] \).

**Proof.** Choose coordinates \( x,y,z \in g^\vee \) given by \( \begin{bmatrix} x & y & z \end{bmatrix} \). In other words \( x,y,z \) are dual to the basis \( (h,e,f) \) where

\[
\begin{align*}
  h &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, & e &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & f &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{align*}
\]

Let \( a = -x^2 - yz \in (\text{Sym}^2 g^\vee)^G \), the determinant polynomial. Then \( \bar{a} = -x \otimes x - \frac{1}{2} (y \otimes z + z \otimes y) \). Under the identification \( H^1(\Sigma g^\vee) \otimes H^1(\Sigma g^\vee) \cong g^\vee \otimes g^\vee \), \( \bar{a} \) is represented by

\[
\begin{bmatrix} -x \\ x \end{bmatrix} \otimes \begin{bmatrix} x \\ -x \end{bmatrix} + \frac{1}{2} \begin{bmatrix} y \\ -y \end{bmatrix} \otimes \begin{bmatrix} z \\ -z \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -z \\ z \end{bmatrix} \otimes \begin{bmatrix} y \\ -y \end{bmatrix} \in C^1 g^\vee \otimes C^1 g^\vee
\]

To keep the notation under control, write \( x_1 = \begin{bmatrix} x \\ 0 \end{bmatrix}, \  y_2 = \begin{bmatrix} 0 \\ y \end{bmatrix}, \text{ etc. } \) and identify \( u \wedge v = \frac{1}{2} (u \otimes v - v \otimes u) \). Then the inverse Alexander-Whitney map sends this representative of \( \bar{a} \) to \( a^{2,2} \in \wedge^2 C^2 g^\vee \) given by

\[
a^{2,2} = 2 (x_2 \wedge x_1 + x_3 \wedge x_2 + x_1 \wedge x_3) + z_2 \wedge y_1 + y_1 \wedge z_3 + y_3 \wedge z_2 + z_3 \wedge y_3 + y_2 \wedge z_1 + z_1 \wedge y_3 + z_3 \wedge y_2 + y_3 \wedge z_3
\]

Now we compute the image of \( a^{2,2} \) under the Chevalley-Eilenberg differential \( \partial_{II} \). Note that

\[
\partial_{II} y = 4x \wedge y \quad \partial_{II} z = 4z \wedge x \quad \partial h x = 2z \wedge y
\]
Therefore \( \partial_h y_3 = 4x_3 \wedge y_3 \), and similarly for the other variables. Writing \( x_1 y_2 z_3 := x_1 \wedge y_2 \wedge z_3 \) etc. one obtains by the Leibniz rule

\[
\frac{1}{4} \partial_I a^{2,2} = x_1 y_2 z_2 - x_2 y_1 z_1 + x_2 y_3 z_3 - x_3 y_2 z_2 + x_3 y_1 z_1 - x_1 y_3 z_3
+ z_2 x_2 y_1 - z_2 x_1 y_1 + x_1 y_1 z_1 - y_1 z_1 x_3 + x_3 y_3 z_2 - y_3 z_2 x_2
+ x_2 y_2 z_1 - y_2 z_1 x_2 + z_1 x_1 y_3 - z_1 x_3 y_3 + z_3 x_3 y_2 - z_3 x_2 y_2 \in \bigwedge^3 C^2 g^\vee
\]

In our coordinates \( \eta = 8 x \wedge y \wedge z \), which we identify with the element \( 8 (x_1 - x_2) \wedge (y_1 - y_2) \wedge (z_1 - z_2) \in \bigwedge^3 \Sigma^1 g^\vee \). We compute

\[
\partial_I \frac{1}{8} \eta = (x_1 - x_2) \wedge (y_1 - y_2) \wedge (z_1 - z_2)
- (x_1 - x_3) \wedge (y_1 - y_3) \wedge (z_1 - z_3)
+ (x_2 - x_3) \wedge (y_2 - y_3) \wedge (z_2 - z_3)
\]

Sage [3] verifies that this equals \( \partial_I a^{2,2} \), so \( a^{1,3} = \frac{1}{4} \eta \) solves the recurrence. \( \square \)

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