INEQUALITIES SATISFIED BY THE ANDREWS SPT-FUNCTION

MADELINE LOCUS DAWSEY* AND RIAD MASRI

Abstract. In a recent paper, W. Y. Chen conjectured a number of inequalities involving the partition function \( p(n) \) and the Andrews smallest parts partition function \( \text{spt}(n) \). Here we prove these conjectures. Further, we strengthen one of the conjectures, and prove that for every \( \epsilon > 0 \) there is an effectively computable constant \( N(\epsilon) > 0 \) such that for all \( n \geq N(\epsilon) \), we have

\[
\frac{\sqrt{6}}{\pi} \sqrt{n} p(n) < \text{spt}(n) < \left( \frac{\sqrt{6}}{\pi} + \epsilon \right) \sqrt{n} p(n).
\]

To prove this theorem, we make use of classic work of Lehmer, combined with the first known effective error bounds for \( \text{spt}(n) \). Due to the conditional convergence of the Rademacher-type formula for \( \text{spt}(n) \), we must employ methods which are completely different from those used by Lehmer to estimate \( p(n) \). Instead, our methods rely on the distribution of Heegner points and the phenomenon in which \( p(n) \) and \( \text{spt}(n) \) are given in terms of finite sums of singular moduli.

1. Introduction and Statement of Results

The smallest parts partition function \( \text{spt}(n) \) of Andrews is defined for any integer \( n \geq 1 \) as the number of smallest parts among the integer partitions of size \( n \). For example, the partitions of \( n = 4 \) are (with the smallest parts underlined):

\[
4, \\
3 + \underline{1}, \\
2 + \underline{2}, \\
2 + \underline{1} + \underline{1}, \\
1 + \underline{1} + \underline{1} + \underline{1},
\]

and so \( \text{spt}(4) = 10 \). Andrews proved \([2]\) the following analogues of the well-known Ramanujan congruences for \( p(n) \):

\[
\text{spt}(5n + 4) \equiv 0 \pmod{5}, \\
\text{spt}(7n + 5) \equiv 0 \pmod{7}, \\
\text{spt}(13n + 6) \equiv 0 \pmod{13}.
\]

One can compute \( \text{spt}(n) \) by making use of the generating function

\[
\sum_{n=1}^{\infty} \text{spt}(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2 (q^{n+1}; q)_{\infty}},
\]

where \( (a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n) \). We use this generating function to compute the values of \( \text{spt}(n) \) required for this paper.

In analogy with the Hardy-Ramanujan asymptotic for \( p(n) \), there is an asymptotic formula for \( \text{spt}(n) \) which follows from work of Bringmann \([4]\). Namely, we have that

\[
\text{spt}(n) \sim \frac{1}{\pi \sqrt{8n}} e^{\pi \sqrt{\frac{n}{3}}}
\]

as \( n \to \infty \).

2010 Mathematics Subject Classification. 11P82, 11P99.

Key words and phrases. partition function, spt-function, Chen’s conjectures.

*This author was previously known as Madeline Locus.
Recently, Ahlgren and Andersen [1] gave the following Rademacher-type exact formula for the spt-function as a conditionally convergent infinite sum of $I$-Bessel functions and Kloosterman sums. For all $n \geq 1$, they proved that

\begin{equation}
\text{spt}(n) = \frac{\pi}{6} (24n - 1)^{1/2} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} \left( I_{1/2} - I_{3/2} \right) \left( \frac{\pi \sqrt{24n - 1}}{6c} \right).
\end{equation}

We will use a different formula for $\text{spt}(n)$ from [1], combined with some analytic number theory, to prove the following recent conjectures of Chen [6].

**Conjecture (Chen).**

1. For $n \geq 5$, we have
   \[ \frac{\sqrt{6}}{\pi} \sqrt{n} \cdot p(n) < \text{spt}(n) < \sqrt{n} \cdot p(n). \]

2. For $(a, b) \neq (2, 2)$ or $(3, 3)$, we have
   \[ \text{spt}(a) \cdot \text{spt}(b) > \text{spt}(a + b). \]

3. For $n \geq 36$, we have
   \[ \text{spt}(n)^2 > \text{spt}(n - 1) \cdot \text{spt}(n + 1). \]

4. For $n > m > 1$, we have
   \[ \text{spt}(n)^2 > \text{spt}(n - m) \cdot \text{spt}(n + m). \]

5. For $n \geq 13$, we have
   \[ \text{spt}(n - 1) \cdot \text{spt}(n) \left( 1 + \frac{1}{n} \right) > \frac{\text{spt}(n)}{\text{spt}(n + 1)}. \]

6. For $n \geq 73$, we have
   \[ \frac{\text{spt}(n - 1)}{\text{spt}(n)} \left( 1 + \frac{\pi}{\sqrt{24n^{3/2}}} \right) > \frac{\text{spt}(n)}{\text{spt}(n + 1)}. \]

**Remark.** Conjectures (1) and (2) are slight modifications of Chen’s original claims. The only difference is that (1) was conjectured to hold for all $n \geq 3$.

We prove the following theorem in Section 6.

**Theorem 1.1.** All of the conjectures are true.

There is a more precise version of Theorem 1.1 regarding Conjecture (1), which we will prove in Section 5. Theorem 1.1 (1) is the case where $\epsilon = 1 - \sqrt{6}/\pi$. In fact, the truth of this result for large $n$ can be seen from earlier work of Bringmann [4] on these asymptotics. However, to effectively determine the range of $n$ for all of these inequalities is a delicate matter which we discuss below.

**Theorem 1.2** (Refined Theorem 1.1 (1)). For each $\epsilon > 0$, there is an effectively computable constant $N(\epsilon) > 0$ such that for all $n \geq N(\epsilon)$, we have

\[ \frac{\sqrt{6}}{\pi} \sqrt{n} \cdot p(n) < \text{spt}(n) < \left( \frac{\sqrt{6}}{\pi} + \epsilon \right) \sqrt{n} \cdot p(n). \]

To prove Theorems 1.1 and 1.2 we make use of classic work of Lehmer [14] which gives effective bounds for the partition function, recent work of Desalvo and Pak [8] and Chen, Wang, and Xie [7], and a formula different from [1] by Ahlgren and Andersen [1] for $\text{spt}(n)$. Complications arise from the conditional convergence of the infinite sum in [1], which makes it difficult to find effective estimates for error terms. There are now different types of formulas for $p(n)$ and $\text{spt}(n)$. For example, Bruinier and Ono [5] proved that the coefficients of certain weight $-1/2$ harmonic Maass forms are essentially traces of singular moduli for weak Maass
forms, from which they obtained a formula for the partition function as a finite sum of algebraic numbers. More precisely, consider the weight zero weak Maass form for $\Gamma_0(6)$ defined by
\[
P(z) := -\left(\frac{1}{2\pi i} \frac{d}{dz} + \frac{1}{2\pi y}\right) g(z), \quad z = x + iy \in \mathbb{H}
\]
where $g(z)$ is the weight $-2$ weakly holomorphic modular form and $\Gamma_0(6)$ defined by
\[
g(z) := \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2}.
\]
Bruinier and Ono $[5]$ proved the following formula for $p(n)$.

**Theorem (Bruinier–Ono).** For all $n \geq 1$, we have
\[
p(n) = \frac{1}{24n-1} \sum_{\mathcal{Q} \in \mathcal{Q}_n^6} P(\tau_Q),
\]
where $\mathcal{Q}_n^6$ is any set of representatives of the equivalence classes of $\Gamma_0(6)$ acting on discriminant $-24n+1$ positive definite, integral binary quadratic forms $Q = [a,b,c]$ such that $6|a$ and $b \equiv 1 \pmod{12}$, and $\tau_Q$ is the Heegner point in the complex upper half-plane $\mathbb{H}$ for which $Q(\tau_Q, 1) = 0$.

Similarly, let $f(z)$ be the weight zero weakly holomorphic modular form for $\Gamma_0(6)$ defined by
\[
f(z) := \frac{1}{24} \frac{E_4(z) - 4E_4(2z) - 9E_4(3z) + 36E_4(6z)}{(\eta(z)\eta(2z)\eta(3z)\eta(6z))^2}.
\]
Ahlgren and Andersen $[1]$ proved the following analogue of (2) for $spt(n)$.

**Theorem (Ahlgren–Andersen).** For all $n \geq 1$, we have
\[
spt(n) = \frac{1}{12} \sum_{\mathcal{Q} \in \mathcal{Q}_n^6} (f(\tau_Q) - P(\tau_Q))
\]
In $[9]$, Dewar and Murty used (2) to re-prove the Hardy-Ramanujan asymptotic for $p(n)$. Their method lends itself to finding effective bounds for error terms. Using a similar approach, we carefully bound each term in the finite sum (4) to obtain the following effective asymptotic formula for $spt(n)$ which is analogous to the aforementioned bounds of Lehmer $[14]$ for $p(n)$.

**Theorem 1.3.** Let $\mu(n) := \pi \sqrt{24n - 1}/6$. Then for all $n \geq 1$, we have
\[
spt(n) = \frac{\sqrt{3}}{\pi \sqrt{24n - 1}} e^{\mu(n)} + E_s(n),
\]
where
\[
|E_s(n)| < (124000002265)(24n - 1)e^{\mu(n)/2}.
\]

**Remark.** The size of the constant in the error bound of Theorem 1.3 plays no essential role in our analysis.

This paper is organized as follows. In Section 2 we review some facts regarding quadratic forms and Heegner points. In Section 3 we give an effective asymptotic formula for the trace of $f(z)$. In Section 4 we discuss the asymptotic properties of $p(n)$ and $spt(n)$, and prove Theorem 1.3. In Section 5 we prove Theorem 1.2. Finally, in Section 6 we prove the remaining conjectures.

**Acknowledgements**

The authors would like to thank Ken Ono for many valuable suggestions regarding this work. They would also like to thank Adrian Barquero-Sanchez, Sheng-Chi Liu, Wei-Lun Tsai, and Matt Young for very helpful discussions, Michael Griffin and Lea Beneish for computing the values of $spt(n)$ which were required to prove the conjectures, and Karl Mahlburg for comments which helped to improve the exposition of the paper.
2. Quadratic forms and Heegner points

Let $N$ be a positive integer and $Q_{-D,N}$ be the set of positive definite, integral binary quadratic forms

$$Q(X,Y) = [a_Q, b_Q, c_Q](X,Y) = a_QX^2 + b_QXY + c_QY^2$$

of discriminant $b_Q^2 - 4a_Qc_Q = -D < 0$ with $a_Q \equiv 0 \pmod{N}$. There is a (right) action of $\Gamma_0(N)$ on $Q_{-D,N}$ defined by

$$Q \mapsto Q \circ \sigma = [a_Q^\sigma, b_Q^\sigma, c_Q^\sigma],$$

where for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ we have

$$a_Q^\sigma = a_Q\alpha^2 + b_Q\alpha\gamma + c_Q\gamma^2,$$

$$b_Q^\sigma = 2a_Q\alpha\beta + b_Q(\alpha\delta + \beta\gamma) + 2c_Q\gamma\delta,$$

$$c_Q^\sigma = a_Q\beta^2 + b_Q\beta\delta + c_Q\delta^2.$$

Given a solution $r \pmod{2N}$ of $r^2 \equiv -D \pmod{4N}$, we define the subset of forms

$$Q_{-D,N,r} := \{Q = [a_Q, b_Q, c_Q] \in Q_{-D,N} : b_Q \equiv r \pmod{2N}\}.$$

Then the group $\Gamma_0(N)$ also acts on $Q_{-D,N,r}$.

To each form $Q \in Q_{-D,N}$, we associate a Heegner point $\tau_Q$ which is the root of $Q(X,1)$ given by

$$\tau_Q = \frac{-b_Q + \sqrt{-D}}{2a_Q} \in \mathbb{H}.$$  

The Heegner points $\tau_Q$ are compatible with the action of $\Gamma_0(N)$ in the sense that if $\sigma \in \Gamma_0(N)$, then

$$\sigma(\tau_Q) = \tau_{Q\circ\sigma^{-1}}.$$

3. Asymptotics for the trace of $f(z)$

In this section, we give an effective asymptotic formula for the trace of the $\Gamma_0(6)$-invariant function $f(z)$ defined by $\mathbb{H}$ which appears in the algebraic formula $\mathbb{H}$ for $spt(n)$. This asymptotic formula will be used crucially in the proof of Theorem 1.3.

Following $[\mathbb{H}]$, we let $Q_n^1$ denote a set of primitive, $SL_2(\mathbb{Z})$-reduced forms representing the classes in $Q_{-24n+1,6}/SL_2(\mathbb{Z})$. The number of such forms equals the class number $h(-24n + 1)$ of the imaginary quadratic field $Q(\sqrt{-24n + 1})$. Also, we let $Q_n^6$ denote a set of primitive forms representing the classes in $Q_{-24n+1,6,1}/\Gamma_0(6)$.

The group $\Gamma_0(6)$ has index 12 in $SL_2(\mathbb{Z})$. We choose the following 12 right coset representatives:

$$\gamma_\infty := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\gamma_{1/3,r} := \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad r = 0,1$$

$$\gamma_{1/2,s} := \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s = 0,1,2$$

$$\gamma_{0,t} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t = 0,1,2,3,4,5.$$

We denote this set of coset representatives by $C_6$. The matrices $\gamma \in C_6$ are scaling matrices for the 4 cusps $\{\infty, 1/3, 1/2, 0\}$ of the modular curve $X_0(6)$, which have widths 1, 2, 3, and 6, respectively. In particular, we have $\gamma_\infty(\infty) = \infty$, $\gamma_{1/3,r}(\infty) = 1/3$, $\gamma_{1/2,s}(\infty) = 1/2$, and $\gamma_{0,t}(\infty) = 0.$
For each $Q \in \mathbb{Q}_n^1$, there is a unique choice of coset representative $\gamma_Q \in \mathcal{C}_6$ such that $[Q \circ \gamma_Q^{-1}] \in \mathbb{Q}_n^0$. This induces a bijection $\mathbb{Q}_n^1 \to \mathbb{Q}_n^0$ defined by $Q \mapsto [Q \circ \gamma_Q^{-1}]$; see the proposition on page 505 in [11], or more concretely, [9, Lemma 3], where an explicit list of the matrices $\gamma_Q \in \mathcal{C}_6$ is given.

Define the trace of $f(z)$ by

$$S(n) := \sum_{Q \in \mathbb{Q}_n^0} f(\tau_Q).$$

By virtue of the bijection $\mathbb{Q}_n^1 \to \mathbb{Q}_n^0$ and [11], the trace can be written as

$$S(n) = \sum_{Q \in \mathbb{Q}_n^1} f(\gamma_Q \tau_Q).$$

Therefore, to study the asymptotic distribution of $S(n)$, we need the Fourier expansion of $f(z)$ with respect to the scaling matrices $\gamma_{\infty}, \gamma_{1/3, r}, \gamma_{1/2, s}, \gamma_{0, t}$.

First, observe that the Fourier expansion of $f(z)$ at the cusp $\infty$ is given explicitly by (see [11, equation (4.6)])

$$f_{|\infty} \gamma_\infty(z) = e(-z) - e(-z) + a(0) + \sum_{m=1}^\infty \frac{1}{\sqrt{m}} \{a(m)e(mz) + a(-m)e(-mz)\}, \quad e(z) := e^{2\pi i z}$$

where

$$a(0) = 4\pi^2 \sum_{\ell \mid 6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \mod(6/\ell) \atop (c, \ell) = 1}} \frac{S(-7, 0; c)}{(\sqrt{\ell c})^2},$$

$$a(m) = 2\pi \sum_{\ell \mid 6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \mod(6/\ell) \atop (c, \ell) = 1}} \frac{S(-7, m; c)}{\sqrt{\ell c}} I_1 \left( \frac{4\pi \sqrt{m}}{\sqrt{\ell c}} \right), \quad m > 0,$$

$$a(m) = 2\pi \sum_{\ell \mid 6} \mu(\ell) \sum_{\substack{0 < c \equiv 0 \mod(6/\ell) \atop (c, \ell) = 1}} \frac{S(-7, m; c)}{\sqrt{\ell c}} J_1 \left( \frac{4\pi \sqrt{m}}{\sqrt{\ell c}} \right), \quad m < 0.$$

Here $\mu(\ell)$ is the Möbius function, $S(a, b; c)$ is the Kloosterman sum

$$S(a, b; c) := \sum_{d \equiv 0 \mod c \atop (d, c) = 1} e\left( \frac{ad + bd}{c} \right),$$

and $I_1, J_1$ are the usual Bessel functions (note that $d$ is the multiplicative inverse of $d \mod c$).

Since $f(z)$ is an eigenfunction for the Atkin-Lehner involutions of level 6, the Fourier expansion of $f(z)$ with respect to the scaling matrices $\gamma_{1/3, r}, \gamma_{1/2, s}, \gamma_{0, t}$ can be determined from the Fourier expansion at $\infty$. In particular, if $z_6 := e(1/6)$ is a primitive sixth root of unity, we have (see for example [9, (3.2)])

$$f_{|\infty} \gamma_{1/3, r}(z) = (-1)^r \left[ e(-z/2) - e(-z/3) \right] + a(0) + \sum_{m=1}^\infty a_{m, r}(z),$$

$$f_{|\infty} \gamma_{1/2, s}(z) = z_6^{3-2s} \left[ e(-z/3) - e(-z/6) \right] - a(0) + \sum_{m=1}^\infty b_{m, s}(z),$$

$$f_{|\infty} \gamma_{0, t}(z) = z_6^{t} \left[ e(-z/6) - e(-z/3) \right] + a(0) + \sum_{m=1}^\infty c_{m, t}(z),$$

where

$$a_{m, r}(z) := \frac{(-1)^r}{\sqrt{m}} \left[ a(m)e(mz/2) + a(-m)e(-mz/2) \right],$$
\[ b_{m,s}(z) := \frac{1}{\sqrt{m}} \left[ \zeta_6^{3+2ms} a(m)e(mz/3) + \zeta_6^{3-2ms} a(-m)e(-m\overline{z}/3) \right], \]

\[ c_{m,t}(z) := \frac{1}{\sqrt{m}} \left[ \zeta_6^{mt} a(m)e(mz/6) + \zeta_6^{-mt} a(-m)e(-m\overline{z}/6) \right]. \]

More generally, given a form \( Q \in \mathcal{Q}_n^1 \) and corresponding coset representative \( \gamma_Q \in C_6 \), we let \( h_Q \in \{1, 2, 3, 6\} \) be the width of the cusp \( \gamma_Q(\infty) \) and \( \zeta_Q \) be the sixth root of unity such that

\[ f_{|Q} \gamma_Q(z) = \zeta_Q [e(-z/h_Q) - e(-\overline{z}/h_Q)] \pm a(0) + \sum_{m=1}^{\infty} C_{m,Q}(z), \]

where

\[ C_{m,Q}(z) := \frac{1}{\sqrt{m}} \left[ \zeta_Q^{-1} a(m)e(mz/h_Q) + \zeta_Q a(-m)e(-m\overline{z}/h_Q) \right]. \]

In the following lemma we give effective bounds for the Fourier coefficients \( a(m) \).

**Lemma 3.1.** The following bounds hold.

1. For \( m = 0 \) we have

\[ |a(0)| \leq \frac{8}{3} \pi^4. \]

2. For \( m < 0 \) we have

\[ |a(m)| \leq C_1 |m|, \]

where

\[ C_1 := 16\sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 \zeta^2(3/2). \]

3. For \( m > 0 \) we have

\[ |a(m)| \leq C_2 m \exp(4\pi \sqrt{m}), \]

where

\[ C_2 := \frac{32}{3} \sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 e \zeta^2(3/2). \]

**Proof.** We first estimate \( |a(0)| \). Since \((\ell, c) = 1\), we can evaluate the Ramanujan sum as

\[ S(-7, 0; c) = \sum_{d \mod c \atop (d,c)=1} e\left(-\frac{\ell d}{c}\right) = \sum_{d \mod c \atop (d,c)=1} e\left(\frac{7d}{c}\right) = \mu(c), \]

where the last equality follows from [13 equation (3.4)]. Therefore,

\[ |S(-7, 0; c)| = |\mu(c)| \leq 1, \]

and we get

\[ |a(0)| \leq 4\pi^2 \sum_{\ell \mid 6} \sum_{0 < c \equiv 0 \mod(6/\ell) \atop (c,\ell)=1} |\mu(\ell)| \frac{|S(-7, 0; c)|}{|\sqrt{\ell c}|^2} \]

\[ \leq 4\pi^2 \sum_{\ell \mid 6} \sum_{0 < c \equiv 0 \mod(6/\ell) \atop (c,\ell)=1} \frac{1}{c^2} \]

\[ \leq 16\pi^2 \sum_{c=1}^{\infty} \frac{1}{c^2} = 16\pi^2 \zeta(2) = \frac{8}{3} \pi^4. \]
We next estimate $|a(m)|$ for $m < 0$. Using the series
\[ J_1(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k+1)!}, \quad x > 0 \]
we find that
\[ |J_1(x)| \leq \frac{x}{2} \quad \text{for} \quad 0 < x < 1. \tag{8} \]

On the other hand, by [16, page 3] we have the uniform asymptotic formula
\[ J_1(x) = \sqrt{2} \pi x \left\{ \cos(x - 3\pi/4) \left[ 3 + R_1^J(x, 1) \right] - \sin(x - 3\pi/4) \left[ \frac{3}{8x} - R_3^J(x, 1) \right] \right\}, \]
where the error terms satisfy the bounds (see [16, equation (1.30)])
\[ \left| R_2^J(x, 1) \right| < 1, \quad \left| R_3^J(x, 1) \right| < 1 \quad \text{for} \quad x \geq 1. \tag{9} \]

Hence
\[ |J_1(x)| \leq 6 \sqrt{\frac{2}{\pi}} \sqrt{x} \quad \text{for} \quad x \geq 1. \tag{9} \]

Let $M = 4\pi \sqrt{|m|/\sqrt{\ell}}$. Then using the Weil bound
\[ |S(a, b; c)| \leq \tau(c) (a, b, c)^{1/2} \]
and the estimates (8) and (9), we get
\[ |a(m)| \leq 2\pi \sum_{\ell|6} |\mu(\ell)| \sum_{0 < c \equiv \ell \pmod{6/\ell}} \frac{|S(-\ell, m; c)|}{\sqrt{\ell c}} |J_1(M/c)| \]
\[ \leq 6 \sqrt{2} |m|^{-1/4} S_1 + 4\pi^2 |m|^{1/2} S_2, \]
where
\[ S_1 := \sum_{\ell|6} \ell^{-1/4} \sum_{0 < c < M \atop (c, \ell) = 1} \tau(c) \left( -\ell, m, c \right)^{1/2} \]
and
\[ S_2 := \sum_{\ell|6} \ell^{-1} \sum_{c \geq M \atop (c, \ell) = 1} \frac{\tau(c) \left( -\ell, m, c \right)^{1/2}}{c^{3/2}}. \]

For all $\epsilon > 0$ we have the following effective bound for the divisor function,
\[ \tau(n) \leq C(\epsilon) n^\epsilon \tag{11} \]
where
\[ C(\epsilon) := \prod_{p < 2^{1/\epsilon}} \max_{a > 0} \frac{a + 1}{p^a \epsilon}. \]

For future reference, we note that
\[ C(1/2) = \frac{4\sqrt{6}}{e^2 \log(2) \log(3)}. \]

Then using (11) with $\epsilon = 1/2$ we get
\[ |S_1| \leq C(1/2) |m|^{1/2} \sum_{\ell|6} \ell^{-1/4} \sum_{0 < c < M \atop (c, \ell) = 1} c^{1/2} \leq \frac{8}{3} (4\pi)^{3/2} C(1/2) |m|^{5/4}. \]
Also, we have
\[ |S_2| \leq 4|m|^{1/2} \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{1/2}} = 4|m|^{1/2} \zeta(3/2). \]

Then after combining estimates, we get
\[ |a(m)| \leq C_1|m|, \quad m < 0 \]
where
\[ C_1 := 16\sqrt{2}(4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 \zeta^2(3/2). \]

Finally, we estimate \(|a(m)|\) for \(m > 0\). Using the inequality (see [15, (6.25)])
\[ |I_1(x)| < \frac{1}{2} \left( 1 + \exp(-2x) \right) \frac{x}{2} \exp(x) \quad \text{for } x > 0 \]
we find that
\begin{equation}
|I_1(x)| \leq \frac{e}{2} x \quad \text{for } 0 < x < 1.
\end{equation}

On the other hand, by [16, equation (A.4)] we have
\[ I_1(x) = \mp i K_1(x e^{\mp i}) \pm i e^{\mp \pi} K_1(x), \quad x > 0 \]
where \(K_1\) is the usual Bessel function. Then
\[ |I_1(x)| \leq \frac{1}{\pi} |K_1(x e^{-\pi})| + \frac{1}{\pi} |K_1(x e^{\pi})| + \frac{2}{\pi} |K_1(x)|. \]

Now, by [16, page 26] we have the uniform asymptotic formulas
\[ K_1(x e^{\mp i}) = \pm \sqrt{\frac{\pi}{2x}} \exp(x) \left[ 3 + R_1^{(K)}(x e^{\mp i}, 1) \right] \]
and
\[ K_1(x) = \sqrt{\frac{\pi}{2x}} \exp(-x) \left[ 3 + R_1^{(K)}(x, 1) \right], \]
where the error terms satisfy the bounds (see [16, equations (1.26) and (B.1)])
\[ |R_1^{(K)}(x e^{\mp i}, 1)| \leq \sum_{r \geq 1} |\Lambda_1(2x e^{\mp i} r)| \leq |1| \quad \text{for } x \geq 1 \]
and (see [16, equation (1.30)])
\[ |R_1^{(K)}(x, 1)| < 1 \quad \text{for } x \geq 1. \]

Here for \(p > 0\) and \(\omega \neq 0\), we have
\[ \Lambda_p(\omega) := \omega^p e^{\omega} \Gamma(1 - p, \omega) \]
where \(\Gamma(1 - p, \omega)\) is the incomplete Gamma function. Hence we get
\begin{equation}
|I_1(x)| \leq 2 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} (\exp(x) + \exp(-x)) \leq 4 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \exp(x) \quad \text{for } x \geq 1.
\end{equation}

Again, let \(M = 4\pi \sqrt{m/\sqrt{\ell}}\). Then using the estimates [10], [12] and [13], we get
\[ |a(m)| \leq 2\pi \sum_{\ell|6} |\mu(\ell)| \sum_{0 < \ell \equiv 0 \pmod{6(\ell)}} \sum_{(c, \ell) = 1} \frac{|S(-7, m; c)| |I_1(M/c)|}{\sqrt{\ell c}} \leq 4\sqrt{2} m^{-1/4} S_3 + 4\pi^2 e m^{1/2} S_4, \]
where
\[ S_3 := \sum_{\ell \mid 6} \ell^{-1/4} \sum_{\substack{0 < \ell, m, c \leq M \\ell \ell + 1}} \tau(c) \left( -\ell, m, c \right)^{1/2} \exp(4\pi\sqrt{m}/\sqrt{\ell}c) \]
and
\[ S_4 := \sum_{\ell \mid 6} \ell^{-1} \sum_{\substack{c \geq M \\ell \ell + 1}} \tau(c) \left( -\ell, m, c \right)^{1/2} \exp(3/2). \]

As before, using (11) with \( \epsilon = 1/2 \) we get
\[ |S_3| \leq C(1/2)m^{1/2} \exp \left( 4\pi\sqrt{m} \right) \sum_{\ell \mid 6} \ell^{-1/4} \sum_{\substack{0 < \ell, m, c \leq M \\ell \ell + 1}} c^{1/2} \]
and
\[ |S_4| = |S_2| \leq 4m^{1/2} \sum_{c=1}^{\infty} \frac{\tau(c)}{c^{3/2}} = 4m^{1/2} \zeta^2(3/2). \]

Then combining estimates yields
\[ |a(m)| \leq C_2 m \exp \left( 4\pi\sqrt{m} \right), \quad m > 0 \]
where
\[ C_2 := \frac{32}{3} \sqrt{2} (4\pi)^{3/2} \cdot \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} + 16\pi^2 e \zeta^2(3/2). \]

In the following theorem we give an effective asymptotic formula for \( S(n) \).

**Theorem 3.2.** For all \( n \geq 1 \), we have
\[ S(n) = 2\pi(24n - 1)^{1/4} I_{1/2} \left( \pi\sqrt{24n - 1}/6 \right) + E(n), \]
where
\[ |E(n)| < (124000000940)(12n - 1) \exp \left( \pi\sqrt{24n - 1}/12 \right). \]

**Proof.** By (6) and (7) we have
\[ S(n) = \sum_{Q \in Q_h^1} f(\gamma_Q(\tau_Q)) = \sum_{Q \in Q_h^1} f_{[0]} \gamma_Q(\tau_Q) = \sum_{Q \in Q_h^1} \zeta(Q \left[ e(-\tau_Q/h_Q) - e(-\tau_Q/h_Q) \right]) + E_1(n), \]
where
\[ E_1(n) := \pm a(0) h(-24n + 1) + \sum_{m=1}^{\infty} \sum_{Q \in Q_h^1} C_{m,Q}(\tau_Q). \]

We first give an effective upper bound for \( |E_1(n)| \). The class number bound
\[ h(-24n + 1) \leq 12n - 1 \]
and Lemma 3.1 give
\[ |a(0) h(-24n + 1)| \leq \frac{8}{3} \pi^2(12n - 1). \]

Next, observe that
\[ e(m\tau_Q/h_Q) = e^{-h_Q m} \exp \left( -\pi m\sqrt{24n - 1}/2a_Q h_Q \right) \]
where $\zeta_{2a_Q h_Q}$ is the primitive $2a_Q h_Q$-th root of unity

$$\zeta_{2a_Q h_Q} := e\left(\frac{1}{2a_Q h_Q}\right).$$

Since $Q \in \mathcal{Q}_n^1$ is reduced, the corresponding Heegner point $\tau_Q$ lies in the standard fundamental domain $\mathcal{F}$ for $SL_2(\mathbb{Z})$. In particular, we have

$$\text{Im}(\tau_Q) = \frac{\sqrt{24n - 1}}{2a_Q} \geq \sqrt{3}/2,$$

which implies that

$$a_Q \leq \frac{\sqrt{24n - 1}}{\sqrt{3}}.$$  

Hence

$$-\frac{\pi m \sqrt{24n - 1}}{2a_Q h_Q} \leq -\frac{\sqrt{3}}{2} \pi m.$$

Then using (15) and (14), we get

$$\left| \sum_{Q \in \mathcal{Q}_n^1} e(m \tau_Q / h_Q) \right| \leq \sum_{Q \in \mathcal{Q}_n^1} \exp\left(-\pi m \sqrt{24n - 1}/2a_Q h_Q\right) \leq h(-24n + 1) \exp\left(-\pi \sqrt{3m}/2\right) \leq (12n - 1) \exp\left(-\pi \sqrt{3m}/2\right).$$

Similarly, since

$$e\left(-m \tau_Q / h_Q\right) = \zeta_{2a_Q h_Q}^{b_{Qm}} \exp\left(-\pi m \sqrt{24n - 1}/2a_Q h_Q\right),$$

the same argument gives

$$\left| \sum_{Q \in \mathcal{Q}_n^1} e(-m \tau_Q / h_Q) \right| \leq (12n - 1) \exp\left(-\pi \sqrt{3m}/2\right).$$

Hence by Lemma 3.1, we have

$$\left| \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{Q}_n^1} C_{m, Q}(\tau_Q) \right| \leq (12n - 1) \left\{ C_1 \sum_{m=1}^{\infty} m^{1/2} \exp\left(-\pi \sqrt{3m}/2\right) + C_2 \sum_{m=1}^{\infty} m^{1/2} \exp\left(4\pi \sqrt{m} - \pi \sqrt{3m}/2\right) \right\}.$$  

Combining the preceding estimates yields

$$|E_1(n)| \leq |a(0)h(-24n + 1)| + \left| \sum_{m=1}^{\infty} \sum_{Q \in \mathcal{Q}_n^1} C_{m, Q}(\tau_Q) \right| \leq \frac{8}{3} \pi^2 (12n - 1) + (12n - 1) \left\{ C_1 \sum_{m=1}^{\infty} m^{1/2} \exp\left(-\pi \sqrt{3m}/2\right) + C_2 \sum_{m=1}^{\infty} m^{1/2} \exp\left(4\pi \sqrt{m} - \pi \sqrt{3m}/2\right) \right\}.$$  

We now estimate the infinite sums. First, we have

$$\sum_{m=1}^{\infty} m^{1/2} \exp\left(-\pi \sqrt{3m}/2\right) \leq \int_0^{\infty} x^{1/2} \exp\left(-\pi \sqrt{x}/2\right) dx = \left(\frac{2}{\pi \sqrt{3}}\right)^{3/2} \Gamma(3/2) = \left(\frac{2}{\pi \sqrt{3}}\right)^{3/2} \frac{\sqrt{\pi}}{2}. $$
Next, write
\[
4\pi \sqrt{m} - \frac{\pi \sqrt{3m}}{2} = -m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{m}} \right),
\]
and observe that
\[
\frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{m}} > 0 \quad \iff \quad m > \frac{64}{3}.
\]
Let \( \alpha := \lfloor 64/3 \rfloor = 22 \). Then if \( m \geq \alpha \), we have
\[
\frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{m}} \geq \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{\alpha}} > 0
\]
so that
\[
-m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{m}} \right) \leq -m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{\alpha}} \right).
\]
Now, we split the sum into appropriate ranges and use the preceding bound to get
\[
\sum_{m=1}^{\infty} m^{1/2} \exp \left( 4\pi \sqrt{m} - \frac{\pi \sqrt{3m}}{2} \right)
= \sum_{m=1}^{\alpha-1} m^{1/2} \exp \left( 4\pi \sqrt{m} - \frac{\pi \sqrt{3m}}{2} \right) + \sum_{m=\alpha}^{\infty} m^{1/2} \exp \left( -m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{m}} \right) \right)
\leq \sum_{m=1}^{\alpha-1} m^{1/2} \exp \left( 4\pi \sqrt{m} - \frac{\pi \sqrt{3m}}{2} \right) + \sum_{m=\alpha}^{\infty} m^{1/2} \exp \left( -m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{\alpha}} \right) \right).
\]
A calculation shows that
\[
\sum_{m=1}^{\alpha} m^{1/2} \exp \left( 4\pi \sqrt{m} - \frac{\pi \sqrt{3m}}{2} \right) \leq \int_{1}^{\alpha} x^{1/2} \exp \left( 4\pi \sqrt{x} - \frac{\pi \sqrt{3x}}{2} \right) dx < 2.38 \times 10^7.
\]
Also, if we let
\[
\beta := \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{\alpha}}
\]
then
\[
\sum_{m=\alpha}^{\infty} m^{1/2} \exp \left( -m \left( \frac{\pi \sqrt{3}}{2} - \frac{4\pi}{\sqrt{\alpha}} \right) \right) \leq \int_{\alpha}^{\infty} x^{1/2} \exp (-\beta x) dx = \beta^{-3/2} \Gamma(3/2, \alpha \beta).
\]
Combining things, we have shown that
\[
|E_1(n)| \leq C(\alpha, \beta)(12n - 1),
\]
where \( C(\alpha, \beta) \) is the explicit positive constant defined by
\[
C(\alpha, \beta) := \frac{8}{3} \pi^4 + C_1 \left( \frac{2}{\pi \sqrt{3}} \right)^{3/2} \frac{\sqrt{\pi}}{2} + C_2 \left[ 2.38 \times 10^7 + \beta^{-3/2} \Gamma(3/2, \alpha \beta) \right].
\]
We now find an explicit upper bound for \( C(\alpha, \beta) \). We have
\[
\frac{8}{3} \pi^4 \approx 259.75 < 260.
\]
Also, since
\[
C(1/2) = \frac{4\sqrt{6}}{e^2 \log(2) \log(3)} \approx 1.74 < 2
\]
and \( \zeta(3/2) \approx 2.61 < 3 \), we have
\[
C_1 = 16\sqrt{2}(4\pi)^{3/2}C(1/2) + 16\pi^2 \zeta^2(3/2) < 32\sqrt{2}(4\pi)^{3/2} + 144\pi^2 \approx 3437.17 < 3438,
\]
so that
\[ C_1 \left( \frac{2}{\pi \sqrt{3}} \right)^{3/2} \frac{\sqrt{\pi}}{2} < 3438 \left( \frac{2}{\pi \sqrt{3}} \right)^{3/2} \frac{\sqrt{\pi}}{2} \approx 678.94 < 679. \]
Similarly, we have
\[ C_2 = \frac{32}{3} \sqrt{2(4\pi)^{3/2}C(1/2)} + 16\pi^2 \kappa^2(3/2) < \frac{64}{3} \sqrt{2(4\pi)^{3/2}} + 144\pi^2 \kappa \approx 5207.25 < 5208, \]
and \( \Gamma(3/2, \alpha \beta) \approx 0.54 < 1, \) so that
\[ C_2 \left[ 2.38 \times 10^7 + \beta^{3/2} \Gamma(3/2, \alpha \beta) \right] < 5208 \left[ 2.38 \times 10^7 + 119 \right] < 1.24 \times 10^{11}. \]
After combining estimates, we get
\[ C(\alpha, \beta) < 260 + 679 + (1.24 \times 10^{11}) = 939 + (1.24 \times 10^{11}) = 124000000939. \]
Summarizing, we have shown that
\[ |E_1(n)| < (124000000939)(12n - 1). \]
It remains to analyze the main term. For any form \( Q = [aQ, bQ, cQ] \in \mathcal{Q}_n^1, \) we have
\[ aQhQ \equiv 0 \pmod{6}. \]
Now, by [9 (4.2)] there are exactly 4 forms \( Q \in \mathcal{Q}_n^1 \) with \( aQhQ = 6, \) and these are given by
\[ Q_1 = [1, 1, 6n], \quad Q_2 = [2, 1, 3n], \quad Q_3 = [3, 1, 2n], \quad Q_4 = [6, 1, n]. \]
Moreover, the corresponding coset representatives \( \gamma_{Q_i} \in \mathcal{C}_6 \) such that \( [Q_i, \gamma_{Q_i}] \in \mathcal{Q}_n^6 \) are given by
\[ \gamma_{Q_1} = \gamma_{0, 1}, \quad \gamma_{Q_2} = \gamma_{1/2, -1}, \quad \gamma_{Q_3} = \gamma_{1/3, 0}, \quad \gamma_{Q_4} = \gamma_{\infty}. \]
For all other forms \( Q = [aQ, bQ, cQ] \in \mathcal{Q}_n^1 \) with \( Q \neq Q_i, \) we have \( aQhQ \geq 12. \)
Write the main term as
\[ \sum_{Q \in \mathcal{Q}_n^1} \zeta_Q [e(-\tau_Q/hQ) - e(-\tau_Q/hQ)] = \sum_{i=1}^{4} 4 \zeta_{Q_i} [e(-\tau_{Q_i}/hQ) - e(-\tau_{Q_i}/hQ)] + E_2(n), \]
where
\[ E_2(n) := \sum_{Q \in \mathcal{Q}_n^1, Q \neq Q_i} \zeta_Q [e(-\tau_Q/hQ) - e(-\tau_Q/hQ)]. \]
A short calculation gives
\[ e(-\tau_Q/hQ) - e(-\tau_Q/hQ) = \zeta_{Q, aQhQ} \left[ \exp(\pi \sqrt{24n-1}/aQhQ) - \exp(-\pi \sqrt{24n-1}/aQhQ) \right]. \]
Then since \( aQhQ \geq 12 \) for \( Q \neq Q_i, \) using (14) we get
\[ |E_2(n)| \leq \sum_{Q \in \mathcal{Q}_n^1, Q \neq Q_i} \left[ \exp(\pi \sqrt{24n-1}/aQhQ) - \exp(-\pi \sqrt{24n-1}/aQhQ) \right] \leq \sum_{Q \in \mathcal{Q}_n^1, Q \neq Q_i} \exp(\pi \sqrt{24n-1}/aQhQ) \leq h(-24n + 1)\exp(\pi \sqrt{24n-1}/12) \leq (12n - 1)\exp(\pi \sqrt{24n-1}/12). \]
Next, using the identity
\[ I_{1/2}^2(z) = \sqrt{\frac{2}{\pi z}} \left( \exp(z) - \exp(-z) \right) \]
we have
\[ e(-z) - e(-\overline{z}) = 2\pi \sqrt{\gamma} I_{1/2}(2\pi y)e(-x), \quad z = x + iy. \]

Therefore, since \( a_Q h_Q = 6 \) for \( i = 1, 2, 3, 4 \), we get
\[
\sum_{i=1}^{4} \zeta_Q \left[ e(-\tau_Q / h_Q) - e(-\overline{\tau_Q} / h_Q) \right] = \left[ 2\pi \left( \frac{24n - 1}{12} \right)^{1/4} I_{1/2}(\pi \sqrt{24n - 1}/6) \right] \cdot \exp(\pi i/6) \cdot \sum_{i=1}^{4} \zeta_Q.
\]

Also, from the Fourier expansions of \( f(z) \) with respect to \( \gamma_{0,1}, \gamma_{1/2,-1}, \gamma_{1/3,0} \), and \( \gamma_{\infty} \) given previously, we have
\[
\zeta_{Q_1} = \zeta_6^{-1}, \quad \zeta_{Q_2} = \zeta_6^{3-2(-1)}, \quad \zeta_{Q_3} = (-1)^0, \quad \zeta_{Q_4} = 1.
\]

Hence
\[
\exp(\pi i/6) \cdot \sum_{i=1}^{4} \zeta_Q = \exp(\pi i/6) \cdot \left( \zeta_6^{-1} + \zeta_6^{3-2(-1)} + (-1)^0 + 1 \right) = 4 \cos(\pi/6) = 2\sqrt{3}.
\]

Then simplifying yields
\[
\sum_{i=1}^{4} \zeta_Q \left[ e(-\tau_Q / h_Q) - e(-\overline{\tau_Q} / h_Q) \right] = 2\pi (24n - 1)^{1/4} I_{1/2}(\pi \sqrt{24n - 1}/6).
\]

Finally, by combining the preceding results we conclude that
\[
S(n) = 2\pi (24n - 1)^{1/4} I_{1/2}(\pi \sqrt{24n - 1}/6) + E(n),
\]
where \( E(n) := E_1(n) + E_2(n) \) with
\[
|E(n)| \leq |E_1(n)| + |E_2(n)| < (12n - 1)\exp(\pi \sqrt{24n - 1}/12) + (124000000939)(12n - 1)
< (124000000940)(12n - 1)\exp(\pi \sqrt{24n - 1}/12).
\]

\[ \square \]

4. Effective bounds for \( \text{spt}(n) \)

To prove our results, we will require an effective asymptotic formula for \( p(n) \) due to Lehmer [14]. For convenience, define
\[
\mu(n) := \frac{\pi}{6} \sqrt{24n - 1}.
\]

Inspired by the Hardy-Ramanujan asymptotic for \( p(n) \), Rademacher [17] obtained the following exact formula:
\[
p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{3/2} \left( \frac{\mu(n)}{c} \right),
\]
where \( A_c(n) \) is the Kloosterman sum
\[
A_c(n) := \sum_{d \pmod{c}} e^{2\pi i (d, c)} e^{-2\pi i d n / c},
\]
and \( s(d, c) \) is the Dedekind sum
\[
s(d, c) := \sum_{r=1}^{c-1} \frac{1}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right).
\]

Using Rademacher’s formula, Lehmer [14] proved the following result.
Theorem 4.1 (Lehmer). For all \( n \geq 1 \), we have
\[
p(n) = \frac{\sqrt{12}}{24n - 1} \sum_{c=1}^{N} \frac{A_c(n)}{\sqrt{c}} \left\{ \left( 1 - \frac{c}{\mu(n)} \right) e^{\mu(n)/c} + \left( 1 + \frac{c}{\mu(n)} \right) e^{-\mu(n)/c} \right\} + R_2(n, N),
\]
where
\[
|R_2(n, N)| < \frac{N^{-2/3} \pi^2}{\sqrt{3}} \left\{ \frac{N^3}{2\mu(n)^3} \left( e^{\mu(n)/N} - e^{-\mu(n)/N} \right) + \frac{1}{6} - \frac{N^2}{\mu(n)^2} \right\}.
\]

We first use Theorem 4.1 to quickly deduce the following effective bound.

Lemma 4.2. For all \( n \geq 1 \), we have
\[
p(n) = \frac{2\sqrt{3}}{24n - 1} \left( 1 - \frac{1}{\mu(n)} \right) e^{\mu(n)} + E_p(n),
\]
where
\[
|E_p(n)| \leq (1324)e^{\mu(n)/2}.
\]

Proof. Using the identity
\[
I_{3/2}(x) = \frac{1}{2} \sqrt{\frac{2}{\pi x}} \left[ \left( 1 - \frac{1}{x} \right) e^x + \left( 1 + \frac{1}{x} \right) e^{-x} \right],
\]
we may write Theorem 4.1 (with the choice \( N = 2 \)) as
\[
p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{c=1}^{2} \frac{A_c(n)}{c} I_{3/2} \left( \frac{\mu(n)}{c} \right) + R_2(n, 2),
\]
where
\[
|R_2(n, 2)| < \frac{\pi^2}{\sqrt{3} 2^{2/3}} \left[ \left( \frac{2}{\mu(n)} \right)^3 \left\{ e^{\mu(n)/2} - e^{-\mu(n)/2} \right\} + 1/6 - \left( \frac{2}{\mu(n)} \right)^2 \right].
\]

Now, using (18) and (17) we get
\[
p(n) = \frac{2\pi}{(24n - 1)^{3/4}} I_{3/2} (\mu(n)) + \frac{2\pi}{(24n - 1)^{3/4}} \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) + R_2(n, 2)
\]
\[
= \frac{2\sqrt{3}}{24n - 1} \left( 1 - \frac{1}{\mu(n)} \right) e^{\mu(n)} + E_p(n),
\]
where
\[
E_p(n) := \frac{2\pi}{(24n - 1)^{3/4}} \left[ \frac{1}{\pi \mu(n)} \left( 1 + \frac{1}{\mu(n)} \right) e^{-\mu(n)} + \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) \right] + R_2(n, 2).
\]

By [15] (6.24), we have the bound
\[
I_{3/2}(x) < \left( \frac{x}{2} \right)^{3/2} \frac{1}{\Gamma(5/2)} \frac{1}{2} (1 + e^{-x}) e^{x} \leq x^{3/2} e^{x}, \quad x > 0.
\]

Then an estimate using the trivial bound
\[
|A_c(n)| < c
\]
and the bound (19) yields
\[
\left| \frac{2\pi}{(24n - 1)^{3/4}} \frac{A_2(n)}{2} I_{3/2} \left( \frac{\mu(n)}{2} \right) \right| \leq 12e^{\mu(n)/2}.
\]

Similarly, two straightforward estimates yield
\[
\left| \frac{2\pi}{(24n - 1)^{3/4}} \frac{1}{2} \sqrt{\frac{2}{\pi \mu(n)}} \left( 1 + \frac{1}{\mu(n)} \right) e^{-\mu(n)} \right| \leq 16e^{\mu(n)/2}
\]
and
\[ |R_2(n, 2)| < (1296)e^{\mu(n)/2}. \]
Hence
\[ |E_p(n)| \leq (1324)e^{\mu(n)/2}. \]

4.1. **Proof of Theorem 1.3.** Using (2), the formula (4) can be written as
\[ \text{spt}(n) = \frac{1}{12} [S(n) - (24n - 1)p(n)]. \]
Now, a straightforward calculation using (10) shows that
\[ 2\pi(24n - 1)^{1/4}I_{1/2}(\mu(n)) = 2\sqrt{3}e^{\mu(n)} + E_3(n), \]
where
\[ |E_3(n)| \leq (12n - 1)e^{\mu(n)/2}. \]
Hence the asymptotic formula in Theorem 4.2 can be written as
\[ S(n) = 2\sqrt{3}e^{\mu(n)} + \tilde{E}(n), \]
where \( \tilde{E}(n) := E(n) + E_3(n) \) with
\[ |\tilde{E}(n)| \leq |E(n)| + |E_3(n)| < (124000000941)(12n - 1)e^{\mu(n)/2}. \]

Then using (20), (21), and Lemma 4.2 another straightforward calculation yields
\[ \text{spt}(n) = \frac{\sqrt{3}}{\pi\sqrt{24n - 1}}e^{\mu(n)} + E_s(n), \]
where the error term
\[ E_s(n) := \frac{\tilde{E}(n)}{12} - \frac{24n - 1}{12}E_p(n) \]
satisfies the bound
\[ |E_s(n)| \leq (124000002265)(24n - 1)e^{\mu(n)/2}. \]

As pointed out by Bessenrodt and Ono [3], it is straightforward to obtain from Theorem 4.1 that
\[ \frac{\sqrt{3}}{12n} \left(1 - \frac{1}{\sqrt{n}}\right) e^{\mu(n)} < p(n) < \frac{\sqrt{3}}{12n} \left(1 + \frac{1}{\sqrt{n}}\right) e^{\mu(n)} \]
for all \( n \geq 1 \). We will use Theorem 1.3 to prove the following analogous statement for \( \text{spt}(n) \), where \( \sqrt{n} \) is replaced by any positive integral power of \( n \).

**Theorem 4.3.** For each \( l \in \mathbb{Z}^+ \) and \( k \in \mathbb{Z}^+ \), there is an effective positive integer \( B_k(l) \) such that for all \( n \geq B_k(l) \), we have
\[ \frac{\sqrt{3}}{\pi\sqrt{24n - 1}} \left(1 - \frac{1}{ln^k}\right) e^{\mu(n)} < \text{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n - 1}} \left(1 + \frac{1}{ln^k}\right) e^{\mu(n)}. \]

**Proof.** By Theorem 1.3 we have the bounds
\[ \frac{\sqrt{3}}{\pi\sqrt{24n - 1}}e^{\mu(n)} - |E_s(n)| < \text{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n - 1}}e^{\mu(n)} + |E_s(n)|, \]
where
\[ |E_s(n)| < (124000002265)(24n - 1)e^{\mu(n)/2}. \]
Clearly, there is an effective positive integer \( B_k(\lambda) \) such that the inequality
\[ (1240000002265)(24n - 1)e^{\mu(n)/2} < \frac{\sqrt{3}}{\pi\sqrt{24n - 1}} \cdot \frac{1}{ln^k} e^{\mu(n)} \]
holds for all \( n \geq B_k(\lambda) \). For instance, if \( \lambda = k = 1 \) then \( B_1(1) = 1468 \). This completes the proof. 

\[ \square \]

5. Proof of Theorem 1.2

By Theorem 1.3 and Lemma 4.2, we may write

\[ \text{spt}(n) = \alpha(n)e^{\mu(n)} + E_s(n) \]

and

\[ p(n) = \beta(n)e^{\mu(n)} + E_p(n), \]

where

\[ \alpha(n) := \frac{\sqrt{3}}{\pi\sqrt{24n-1}}, \quad \beta(n) := \frac{2\sqrt{3}}{24n-1} \left( 1 - \frac{6}{\pi\sqrt{24n-1}} \right). \]

Also, for \( \epsilon > 0 \) we define

\[ \gamma(n) := \frac{\sqrt{6}}{\pi} \sqrt{n}, \quad \gamma(n, \epsilon) := \left( \frac{\sqrt{6}}{\pi} + \epsilon \right) \sqrt{n}. \]

We must prove that there exists an effective positive constant \( N(\epsilon) > 0 \) such that for all \( n \geq N(\epsilon) \), we have

\[ \gamma(n)p(n) < \text{spt}(n) < \gamma(n, \epsilon)p(n). \]

First, using (22) and (23) we find that the lower bound in (24) is equivalent to

\[ c_1(n)e^{\mu(n)} > \gamma(n)E_p(n) - E_s(n), \]

where \( c_1(n) := \alpha(n) - \beta(n)\gamma(n) \). Now, the error bounds in Lemma 4.2 and Theorem 1.3 imply that

\[ |\gamma(n)E_p(n) - E_s(n)| \leq c_2(n)e^{\mu(n)/2}, \]

where

\[ c_2(n) := (1324)\gamma(n) + (124000002265)(24n-1). \]

Then noting that \( c_1(n) > 0 \) for all \( n \geq 1 \), we find that (25) is implied by the bound

\[ e^{\mu(n)/2} > c_3(n) := \frac{c_2(n)}{c_1(n)}, \]

or equivalently, the bound

\[ n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_3(n)) \right)^2 + 1 \right]. \]

A calculation shows that (26) holds for all \( n \geq N := 1297 \).

Similarly, using (22) and (23) we find that the upper bound in (24) is equivalent to

\[ c_4(n, \epsilon)e^{\mu(n)} > E_s(n) - \gamma(n, \epsilon)E_p(n), \]

where \( c_4(n, \epsilon) := \beta(n)\gamma(n, \epsilon) - \alpha(n) \). The error bounds in Lemma 4.2 and Theorem 1.3 imply that

\[ |E_s(n) - \gamma(n, \epsilon)E_p(n)| \leq c_5(n, \epsilon)e^{\mu(n)/2}, \]

where

\[ c_5(n, \epsilon) := (1324)\gamma(n, \epsilon) + (124000002265)(24n-1). \]
Moreover, there exists an effective positive constant \( N_1(\epsilon) > 0 \) such that \( c_4(n, \epsilon) > 0 \) for all \( n \geq N_1(\epsilon) \). Then arguing as above, we find that if \( n \geq N_1(\epsilon) \), the bound (27) is implied by the bound

\[
(28) \quad n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_6(n, \epsilon)) \right)^2 + 1 \right],
\]

where \( c_6(n, \epsilon) := c_5(n, \epsilon)/c_4(n, \epsilon) \). Clearly, there exists an effective positive constant \( N_2(\epsilon) \geq N_1(\epsilon) \) such that (28) holds for all \( n \geq N_2(\epsilon) \).

Let \( N(\epsilon) := \max\{N, N_2(\epsilon)\} \). Then the inequalities (24) hold for all \( n \geq N(\epsilon) \).

\[
\square
\]

6. Proof of Theorem 1.1

6.1. Proof of Conjecture (1). Let \( \epsilon = 1 - \sqrt{6}/\pi \) in Theorem 1.2. We need to determine the constant \( N(1 - \sqrt{6}/\pi) \). A short calculation shows that the inequality \( c_3(n, 1 - \sqrt{6}/\pi) > 0 \) holds if and only if

\[
(29) \quad \frac{1}{\sqrt{n}} \left( \frac{\alpha(n)}{\beta(n)} - \gamma(n) \right) < 1 - \frac{\sqrt{6}}{\pi}.
\]

Another calculation shows that (29) holds if \( n \geq N_1(1 - \sqrt{6}/\pi) \) for \( N_1(1 - \sqrt{6}/\pi) = 4 \). Next, we need to find the smallest positive integer \( N_2(1 - \sqrt{6}/\pi) \geq 4 \) such that the bound

\[
(30) \quad n > \frac{1}{24} \left[ \left( \frac{12}{\pi} \log(c_6(n, 1 - \sqrt{6}/\pi)) \right)^2 + 1 \right]
\]

holds for all \( n \geq N_2(1 - \sqrt{6}/\pi) \). A calculation shows that this constant is given by \( N_2(1 - \sqrt{6}/\pi) = 1111 \). We now have

\[
N(1 - \sqrt{6}/\pi) := \max\{N, N_2(1 - \sqrt{6}/\pi)\} = \max\{1297, 1111\} = 1297.
\]

Therefore, the inequalities

\[
\frac{\sqrt{6}}{\pi} \sqrt{np(n)} < \text{spt}(n) < \sqrt{np(n)}
\]

hold for all \( n \geq 1297 \). Finally, one can verify with a computer that these inequalities also hold for \( 5 \leq n < 1297 \).

\[
\square
\]

6.2. Proof of Conjecture (2). We follow closely the proof of [3, Theorem 2.1]. By taking \( l = k = 1 \) in Theorem 1.3 (recall that \( B_1(1) = 1468 \)), we find that

\[
(31) \quad \frac{\sqrt{3}}{\pi/24n - 1} \left( 1 - \frac{1}{n} \right) e^{\mu(n)} < \text{spt}(n) < \frac{\sqrt{3}}{\pi/24n - 1} \left( 1 - \frac{1}{n} \right) e^{\mu(n)}
\]

for all \( n \geq 1468 \). One can verify with a computer that (31) also holds for \( 1 \leq n < 1468 \).

Now, assume that \( 1 < a \leq b \), and let \( b = Ca \) where \( C \geq 1 \). From (31) we get the inequalities

\[
\text{spt}(a)spt(Ca) > \frac{3}{\pi^2/24a - 1/\sqrt{24Ca} - 1} \left( 1 - \frac{1}{a} \right) \left( 1 - \frac{1}{Ca} \right) e^{\mu(a) + \mu(Ca)}
\]

and

\[
\text{spt}(a + Ca) < \frac{\sqrt{3}}{\pi/24(a + Ca) - 1} \left( 1 + \frac{1}{a + Ca} \right) e^{\mu(a + Ca)}.
\]

Hence, for all but finitely many cases, it suffices to find conditions on \( a > 1 \) and \( C \geq 1 \) such that

\[
(32) \quad e^{\mu(a) + \mu(Ca) - \mu(a + Ca)} > \frac{\pi/24a - 1/\sqrt{24Ca} - 1}{\sqrt{3}/24(a + Ca) - 1} \frac{1 + \frac{1}{a + Ca}}{(1 - \frac{1}{a})(1 - \frac{1}{Ca})}.
\]
For convenience, define
\[ T_a(C) := \mu(a) + \mu(Ca) - \mu(a + Ca) \quad \text{and} \quad S_a(C) := \frac{1 + \frac{1}{a + Ca}}{(1 - \frac{1}{a}) (1 - \frac{1}{Ca})}. \]

Then by taking logarithms, we find that (32) is equivalent to
\[
T_a(C) > \log \left( \frac{\pi \sqrt{24a - 1} \sqrt{24Ca - 1}}{\sqrt{3} \sqrt{24(a + Ca) - 1}} \right) + \log S_a(C).
\]

As functions of $C$, it can be shown that $T_a(C)$ is increasing and $S_a(C)$ is decreasing for $C \geq 1$, so we have that
\[ T_a(C) \geq T_a(1) \]
and
\[ \log(S_a(1)) \geq \log(S_a(C)). \]

Hence it suffices to show that
\[
T_a(1) > \log \left( \frac{\pi \sqrt{24a - 1}}{\sqrt{3}} \right) + \log(S_a(1)).
\]

Moreover, since
\[
\frac{\sqrt{24Ca - 1}}{\sqrt{24(a + Ca) - 1}} \leq 1
\]
for all $C \geq 1$ and all $a > 1$, it suffices to show that
\[
T_a(1) > \log \left( \frac{\pi \sqrt{24a - 1}}{\sqrt{3}} \right) + \log(S_a(1)).
\]

By computing the values $T_a(1)$ and $S_a(1)$, we find that (34) holds for all $a \geq 5$.

To complete the proof, assume that $2 \leq a \leq 4$. For each such integer $a$, we calculate the real number $C_a$ for which
\[ T_a(C_a) = \log \left( \frac{\pi \sqrt{24a - 1}}{\sqrt{3}} \right) + \log(S_a(C_a)). \]

The values $C_a$ are listed in the table below.

| $a$ | $C_a$ |
|-----|-------|
| 2   | 27.87. . . |
| 3   | 3.54. . .  |
| 4   | 1.79. . .  |

By the discussion above, if $b = Ca \geq a$ is an integer for which $C > C_a$, then (33) holds, which in turn gives the theorem in these cases. Only finitely many cases remain, namely the pairs of integers where $2 \leq a \leq 4$ and $1 \leq b/a \leq C_a$. We compute spt($a$), spt($b$), and spt($a + b$) in these cases to complete the proof. 

\[ \square \]

6.3. Proof of Conjecture (3). We require some lemmas and a proposition analogous to those of Desalvo and Pak \[8\] in order to prove the remaining conjectures.

The following is \[8\] Lemma 2.1.

**Lemma 6.1.** Suppose $h(x)$ is a positive, increasing function with two continuous derivatives for all $x > 0$, and that $h'(x) > 0$ is decreasing, and $h''(x) < 0$ is increasing for all $x > 0$. Then for all $x > 0$, we have
\[ h''(x - 1) < h(x + 1) - 2h(x) + h(x - 1) < h''(x + 1). \]
By Theorem 1.3, we may write
\[ spt(n) = f(n) + E_s(n) \]
where
\[ f(n) := \frac{\sqrt{3}}{\pi \sqrt{24n - 1}} e^{\mu(n)} \quad \text{and} \quad |E_s(n)| \leq (124000002265)(24n - 1)e^{\mu(n)/2}. \]

**Lemma 6.2.** Let
\[ F(n) := 2\log(f(n)) - \log(f(n + 1)) - \log(f(n - 1)). \]
Then for all \(n \geq 4\), we have
\[ \frac{24\pi}{(24(n + 1) - 1)^{3/2}} - \frac{1}{n^2} < F(n) < \frac{24\pi}{(24(n - 1) - 1)^{3/2}} - \frac{288}{(24(n + 1) - 1)^2}. \]

**Proof.** We can write \(f(n)\) from (35) as
\[ f(n) = \frac{1}{\sqrt{3}\mu(n)} e^{\mu(n)}, \]
so that
\[ \log(f(n)) = \mu(n) - \log(\mu(n)) - \log(2\sqrt{3}). \]

Then we have
\[ F(n) = 2\mu(n) - \mu(n + 1) - \mu(n - 1) - 2\log(\mu(n)) + \log(\mu(n + 1)) + \log(\mu(n - 1)). \]
Since the functions \(\mu(x)\) and \(\tilde{\mu}(x) := \log(\mu(x))\) satisfy the hypotheses of Lemma 6.1 we get
\[ -\mu''(n + 1) + \tilde{\mu}''(n - 1) < F(n) < -\mu''(n - 1) + \tilde{\mu}''(n + 1). \]
Computing derivatives gives
\[ \frac{24\pi}{(24(n + 1) - 1)^{3/2}} - \frac{288}{(24(n + 1) - 1)^2} < F(n) < \frac{24\pi}{(24(n - 1) - 1)^{3/2}} - \frac{288}{(24(n + 1) - 1)^2} \]
for all \(n \geq 2\), from which we deduce that
\[ \frac{24\pi}{(24(n + 1) - 1)^{3/2}} - \frac{1}{n^2} < F(n) < \frac{24\pi}{(24(n - 1) - 1)^{3/2}} - \frac{288}{(24(n + 1) - 1)^2} \]
for all \(n \geq 4\). \(\square\)

**Lemma 6.3.** Define the functions \(y_n := |E_s(n)| / f(n)\),
\[ M(n) := 2\sqrt{3}\mu(n)(124000002265)(24n - 1)e^{-\mu(n)/2}, \]
and
\[ g(n) := \frac{M(n)}{1 - M(n)}. \]
Then for all \(n \geq 2\), we have
\[ \log\left[ \frac{(1 - y_n)^2}{(1 + y_{n+1})(1 + y_{n-1})} \right] > -2g(n) - M(n + 1) - M(n - 1) \]
and
\[ \log\left[ \frac{(1 + y_n)^2}{(1 - y_{n+1})(1 - y_{n-1})} \right] < 2M(n) + g(n + 1) + g(n - 1). \]
Proof. First observe that for all $n \geq 1$, we have
\begin{equation}
0 < y_n := \frac{|E_n(n)|}{f(n)} < \frac{(124000002265)(24n - 1) e^{\mu(n)/2}}{243\sqrt{\mu(n)}} = M(n).
\end{equation}
Note also that $y_n < 1$ for all $n \geq 1$. Then using (36) and the inequalities
\[\log(1 - x) \geq -\frac{x}{1 - x} \quad \text{for} \quad 0 < x < 1\]
and
\[\log(1 + x) < x \quad \text{for} \quad x > 0,\]
we get
\[
\log \left[ \frac{(1 - y_n)^2}{(1 + y_{n+1})(1 + y_{n-1})} \right] = 2\log(1 - y_n) - \log(1 + y_{n+1}) - \log(1 + y_{n-1}) > -2 \frac{y_n}{1 - y_n} - y_{n+1} - y_{n-1} > -2g(n) - M(n + 1) - M(n - 1)
\]
for all $n \geq 2$. Similarly, we get
\[
\log \left[ \frac{(1 + y_n)^2}{(1 - y_{n+1})(1 - y_{n-1})} \right] = 2\log(1 + y_n) - \log(1 - y_{n+1}) - \log(1 - y_{n-1}) < 2y_n + \frac{y_{n+1}}{1 - y_{n+1}} + \frac{y_{n-1}}{1 - y_{n-1}} < 2M(n) + g(n + 1) + g(n - 1)
\]
for all $n \geq 2$. \hfill \square

**Proposition 6.4.** Let
\[spt_2(n) := 2\log(spt(n)) - \log(spt(n + 1)) - \log(spt(n - 1)).\]
Then we have
\[\frac{1}{(24n)^{3/2}} < spt_2(n) < \frac{2}{n^{3/2}},\]
where the lower bound holds for all $n \geq 1865$ and the upper bound holds for all $n \geq 1804$.

**Proof.** We first bound $spt(n)$ by
\[f(n) \left( 1 - \frac{|E_n(n)|}{f(n)} \right) < spt(n) < f(n) \left( 1 + \frac{|E_n(n)|}{f(n)} \right).
\]
Then recalling that $y_n := \frac{|E_n(n)|}{f(n)}$, we take logarithms in the preceding inequalities to get
\[F(n) + \log \left[ \frac{(1 - y_n)^2}{(1 + y_{n+1})(1 + y_{n-1})} \right] < spt_2(n) < F(n) + \log \left[ \frac{(1 + y_n)^2}{(1 - y_{n+1})(1 - y_{n-1})} \right].\]
It follows immediately from Lemmas 6.2 and 6.3 that for all $n \geq 4$, we have
\[spt_2(n) > \frac{24\pi}{(24n + 1 - 1)^{3/2}} - \frac{1}{n^2} - 2g(n) - M(n + 1) - M(n - 1)\]
and
\begin{equation}
\text{(37)} \quad spt_2(n) < \frac{24\pi}{(24n - 1 - 1)^{3/2}} - \frac{288}{(24n + 1 - 1)^2} + 2M(n) + g(n + 1) + g(n - 1).
\end{equation}
Finally, a calculation shows that
\[
\frac{24\pi}{(24n + 1 - 1)^{3/2}} - \frac{1}{n^2} - 2g(n) - M(n + 1) - M(n - 1) > \frac{1}{(24n)^{3/2}}.
\]
for all $n \geq 1865$ and

$$\frac{24\pi}{(24(n - 1) - 1)^{3/2}} - \frac{288}{(24(n + 1) - 1)^2} + 2M(n) + g(n + 1) + g(n - 1) < \frac{2}{n^{3/2}}$$

for all $n \geq 1804$. This completes the proof. \hfill \Box

To prove Conjecture (3), it suffices to show that

$$\text{spt}(n)^2 > \text{spt}(n - 1)\text{spt}(n + 1).$$

Taking logarithms, we see that this is equivalent to $\text{spt}_2(n) > 0$. By the lower bound in Proposition 6.4, we have $\text{spt}_2(n) > 0$ for all $n \geq 1865$. Finally, one can verify with a computer that $\text{spt}_2(n) > 0$ for all $36 \leq n < 1865$. This completes the proof. \hfill \Box

6.4. Proof of Conjecture (4). We follow closely the proof of [8, Theorem 5.1]. We have proved that the sequence $\text{spt}(n)$ is log-concave, that is

$$\text{spt}(n)^2 - \text{spt}(n - 1)\text{spt}(n + 1) > 0$$

for all $n \geq 36$. It is known that log-concavity implies strong log-concavity

$$\text{spt}(k)\text{spt}(\ell) \leq \text{spt}(\ell - i)\text{spt}(k + i),$$

for all $0 \leq k \leq \ell \leq n$ and $0 \leq i \leq \ell - k$ (see e.g. [18]). In particular, we take $k = n - m$, $\ell = n + m$, and $i = m$ to obtain

$$\text{spt}(n)^2 - \text{spt}(n - m)\text{spt}(n + m) > 0$$

for all $n > m > 1$ with $n - m > 36$.

We next consider the case $n > m > 1$ with $1 \leq n - m \leq 36$. Since $n \geq m + 1$ we have

$$\text{spt}(n)^2 \geq \text{spt}(m + 1)^2.$$  \hfill (38)

Moreover, since $n - m \leq 36$ we have

$$\text{spt}(n - m) < \text{spt}(36) < 90000,$$  \hfill (39)

and thus

$$\text{spt}(36)\text{spt}(36 + 2m) \geq \text{spt}(n - m)\text{spt}(n + m).$$  \hfill (40)

Now, we claim that

$$\text{spt}(m + 1)^2 \geq \text{spt}(36)\text{spt}(36 + 2m)$$

for all $m \geq 6290$. Taking logarithms in (41), we see that it suffices to prove that

$$2\log(\text{spt}(m + 1)) - \log(\text{spt}(36)) - \log(\text{spt}(36 + 2m)) > 0$$

for all $m \geq 6290$. By [12] and [10], respectively, we have the lower and upper bounds

$$\frac{e^2\sqrt{m}}{2\pi me^{1/6m}} < p(m) < \frac{\sqrt{12m}}{\pi e^{(2/3)m}}.$$  \hfill (42)

Then by Theorem 1.2 with the choice $\epsilon = (2 - \sqrt{2})\sqrt{3}/\pi$, we get

$$\frac{\sqrt{6m}}{\pi} \cdot \frac{e^2\sqrt{m}}{2\pi me^{1/6m}} < \text{spt}(m) < \frac{\sqrt{12m}}{\pi e^{(2/3)m}}.$$  \hfill (43)

Using the inequalities (39) and (43), we see that (42) is implied by the lower bound

$$2\log\left(\frac{\sqrt{6(m + 1)}}{2\pi^2(m + 1)e^{1/6(m + 1)}}\right) + 4\sqrt{m + 1} - \log(90000) - \log\left(\frac{\sqrt{12(36 + 2m)}}{\pi}\right) - \pi\sqrt{2(36 + 2m)} > 0$$

for all $m \geq 6290$. A calculation shows this is true for the given range of $m$.

Finally, it follows from (38), (40), and (41) that

$$\text{spt}(n)^2 \geq \text{spt}(m + 1)^2 \geq \text{spt}(36)\text{spt}(36 + 2m) \geq \text{spt}(n - m)\text{spt}(n + m)$$

for all $1 \leq n - m \leq 36$ with $m \geq 6290$. On the other hand, one can verify with a computer that

$$\text{spt}(n)^2 - \text{spt}(n - m)\text{spt}(n + m) > 0$$
for all $1 \leq n - m \leq 36$ with $m < 6290$. This completes the proof.

6.5. **Proof of Conjecture (5)**. Taking logarithms, we find that Conjecture (5) is equivalent to

$$spt_2(n) < \log \left(1 + \frac{1}{n}\right)$$

for all $n \geq 13$. By the upper bound in Proposition 6.4 and some straightforward estimates, we have

$$spt_2(n) < \frac{2}{n^{3/2}} < \frac{1}{n+1} < \log \left(1 + \frac{1}{n}\right)$$

for all $n \geq 1804$. Finally, one can verify with a computer that the conjectured inequality also holds for all $13 \leq n < 1804$. This completes the proof.

6.6. **Proof of Conjecture (6)**. We follow closely the proof of [7, Conjecture 1.3]. Taking logarithms, we find that Conjecture (6) is equivalent to

$$spt_2(n) < \log \left(1 + \frac{\pi}{\sqrt{24n^{3/2}}}\right)$$

for all $n \geq 73$. By (37) we have

$$spt_2(n) < \frac{24\pi}{(24(n-1)-1)^{3/2}} - \frac{288}{(24(n+1)-1)^2} + 2M(n) + g(n+1) + g(n-1)$$

for all $n \geq 4$. On the other hand, by [7, (2.3)] we have

$$\frac{24\pi}{(24(n+1)-1)^{3/2}} < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24\pi)^{3/2}}\right)^2 + \frac{3}{2n^{5/2}}$$

for all $n \geq 50$, and by the second inequality following [7, (2.23)] we have

$$-\frac{288}{(24(n+1)-1)^2} < \frac{1}{6n^{5/2}} - \frac{1}{2n^{2}}$$

for all $n \geq 50$. Therefore, for all $n \geq 50$ we have

$$spt_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24\pi)^{3/2}}\right)^2 + \frac{5}{3n^{5/2}} - \frac{1}{2n^{2}} + 2M(n) + g(n+1) + g(n-1).$$

Now, a calculation shows that

$$\frac{5}{3n^{5/2}} - \frac{1}{2n^{2}} + 2M(n) + g(n+1) + g(n-1) < 0$$

for all $n \geq 2187$. Hence

$$spt_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24\pi)^{3/2}}\right)^2 = \frac{24\pi}{(24n)^{3/2}} \left(1 - \frac{24\pi}{(24n)^{3/2}}\right)$$

for all $n \geq 2187$. Then using the inequality

$$x(1-x) < \log(1+x) \quad \text{for} \quad x > 0,$$

we get

$$spt_2(n) < \log \left(1 + \frac{24\pi}{(24n)^{3/2}}\right) = \log \left(1 + \frac{\pi}{\sqrt{24n^{3/2}}}\right)$$

for all $n \geq 2187$. Finally, one can verify with a computer that this inequality also holds for all $73 \leq n < 2187$. This completes the proof.
References

[1] S. Ahlgren and N. Andersen, *Algebraic and transcendental formulas for the smallest parts function*. Adv. Math. 280 (2016), 411–437.

[2] G. E. Andrews, *The number of smallest parts in the partitions of n*. J. Reine Angew. Math. 624 (2008), 133–142.

[3] C. Bessenrodt and K. Ono, *Maximal multiplicative properties of partitions*. Ann. Comb. 20 (2016), 59–64.

[4] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*. Duke Math. J. 144 (2008), 195–233.

[5] J. H. Bruinier and K. Ono, *Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms*. Adv. Math. 246 (2013), 198–219.

[6] W. Y. Chen, *The spt-function of Andrews*. arXiv:1707.04369

[7] W. Y. Chen, L. X. Wang, and G. Y. Xie, *Finite differences of the logarithm of the partition function*. Math. Comp. 85 (2015), 825–847.

[8] S. Desalvo and I. Pak, *Log-concavity of the partition function*. Ramanujan J. 38 (2015), 61–73.

[9] M. Dewar and R. Murty, *A derivation of the Hardy-Ramanujan formula from an arithmetic formula*. Proc. Amer. Math. Soc. 141 (2012), 1903–1911.

[10] P. Erdős, *On an elementary proof of some asymptotic formulas in the theory of partitions*. Ann. Math 43 (1942), 437–450.

[11] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II*. Math. Ann. 278 (1987), 497–562.

[12] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatorial analysis*. Proc. London Math. Soc. 17 (1918), 75–115.

[13] H. Iwaniec and E. Kowalski, *Analytic number theory*. American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004. xii+615 pp.

[14] D. H. Lehmer, *On the remainders and convergence of the series for the partition function*. Trans. Amer. Math. Soc. 46 (1939), 362–373.

[15] Y. L. Luke, *Inequalities for generalized hypergeometric functions*. J. Approx. Theory 5 (1972), 41–65.

[16] G. Nemes, *Error bounds for the large-argument asymptotic expansions of the Hankel and Bessel functions*. arXiv:1606.07961v2

[17] H. Rademacher, *On the expansion of the partition function in a series*. Ann. of Math. 44 (1943), 416–422.

[18] B. E. Sagan, *Inductive and injective proofs of log concavity results*. Discret. Math. 68 (1988), 281–292.

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322
E-mail address: madeline.locus@emory.edu

Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843-3368
E-mail address: masri@math.tamu.edu