Two–Dimensional Dilaton Gravity Black Hole Solution for

$N = 2$ Superstring Theory

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Abstract

We show that $N = 8$ self-dual supergravity theory, which is the consistent background for $N = 2$ closed superstring theory in $2 + 2$-dimensions, can accommodate the recently discovered two-dimensional dilaton gravity black hole solution, via appropriate dimensional reductions and truncations. Interestingly, the usual dilaton field in this set of solutions emerges from the scalar field in the $70$-dimensional representation of an intrinsic global $SO(8)$ group. We also give a set of exact solutions, which can be interpreted as the dilaton field on Eguchi-Hanson gravitational instanton background, realized in an $N = 1$ self-dual supergravity theory. This suggests that the $N = 2$ superstring has a close (even closer) relationship with the two-dimensional black hole solution, which was originally developed in the context of bosonic string and $N = 1$ superstring. Our result also provides supporting evidence for the conjecture that the $N = 2$ superstring theory is the “master theory” of supersymmetric integrable systems in lower-dimensions.

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1This work is supported in part by NSF grant # PHY-91-19746.

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1. Introduction

There has been recently developing interest in self-dual supersymmetric Yang-Mills and self-dual supergravity (SDSG) theories in $2 + 2$-dimensions motivated by the conjecture [1] that all the (bosonic) integrable systems in lower dimensions can be generated by self-dual Yang-Mills theory in four-dimensional space-time with two time and two spatial coordinates.\footnote{We denote this by $D = (2, 2)$, where in general $D = (t, s)$ denotes a space-time with $t$ time and $s$ spatial directions. The expression $D = 4$ is also used, when the signature is not important.}

Motivated by this development, we have presented in our recent papers [2-6] convenient formulations for self-dual supersymmetric Yang-Mills and SDSG theories. Remarkably, these self-dual supersymmetric theories turned out to be also the consistent target space-time backgrounds for $N = 2$ open and closed superstring [7,8]. Among them, the maximal SDSG theory has space-time $N = 8$ supersymmetry [8] with interesting features, such as the possibility of gauging a global $SO(8)$ symmetry similar to the ordinary $N = 8$ supergravity (SG) in $D = (1, 3)$.

Rather independently of this subject in $N = 2$ superstring, there has been interesting development in ordinary bosonic string and $N = 1$ superstring, about exact solutions for the background dilaton and graviton fields in two-dimensions [9]. It has been discovered that there is a class of exact solutions similar to the usual four-dimensional black hole solution that satisfies the background field equations in those string theories [9]. These solutions are also identified with a conformal sigma-model with a Wess-Zumino-Novikov-Witten (WZNW) term for the coset $SL(2, \mathbb{R})/U(1)$ [9].

In this paper, we show a new interesting link between these two originally independent developments, namely the $N = 2$ closed superstring theory and the dilaton gravity black hole solutions. We concentrate on the $D = (2, 2), N = 8$ SDSG which is the consistent background for the closed $N = 2$ superstring, and apply to this system the technique of simple dimensional reduction [10] into $D = 2$, in order to embed the field equations of dilaton gravity black hole solutions. Even though these solutions have been developed as the backgrounds for the ordinary bosonic or $N = 1$ superstring theories, our result will establish the new viewpoint, connecting it to the $N = 2$ superstring. Remarkably, we will see that the usual dilaton emerges from the scalars in the $70$-dimensional representation of the global $SO(8)$ symmetry in the original $N = 8$ SDSG theory. We also give another set of exact solutions, where the same scalar field can be interpreted as the dilaton field in the Eguchi-Hanson gravitational instanton background [11].
2. Bosonic Field Equations for Ungauged $N = 8$ SDSG Background

We start with reviewing the bosonic field part of the ungauged $N = 8$ SDSG background field lagrangian [8] for the $N = 2$ closed superstring. Its bosonic field content is the graviton (vierbein) $c_\mu^\hat{m}$, the Lorentz connection $\omega_\mu^\hat{m}\hat{n}$, the propagating multiplier field $\Lambda_\mu^\hat{m}\hat{n}$, the two vector fields $\hat{A}_{\mu AB}$, $\hat{B}_\mu^{AB}$ each in the 28-representation of $SO(8)$, and the scalar field $\hat{\phi}_{ABCD}$ in the 70-representation of the $SO(8)$

The purely bosonic part of the lagrangian in the $N = 8$ SDSG that interests us is

$$\hat{\mathcal{L}}_B^{N=8} = \frac{1}{2} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{c}_{\hat{\mu}\hat{n}} \hat{\Lambda}_{\hat{\nu}\hat{n}} \hat{T}_{\hat{\rho}\hat{\sigma} \hat{n}} + \frac{1}{16} \hat{G}_{\hat{\mu}\hat{\nu}AB} \hat{F}^{\hat{\mu}\hat{\nu}AB}$$

$$- \frac{1}{2304} \varepsilon^{ABCDEFGH} \left[ 2 \hat{g}^{\hat{\mu}\hat{\nu}} (\hat{\partial}_{\hat{A}CD})(\hat{\partial}_{EFGH}) + 3 \hat{\phi}_{ABCD} \hat{F}_{\hat{\mu}\hat{\nu}EF} \hat{F}_{\hat{\mu}\hat{\sigma}GH} \right] , \quad (2.1)$$

in an appropriate normalization of fields. The $\varepsilon^{ABCDEFGH}$ is the totally antisymmetric constant invariant tensor for $SO(8)$. If all the fermionic fields are ignored,

$$\hat{T}_{\hat{\mu}\hat{\nu} \hat{m}} \equiv \hat{\partial}_{\hat{\mu}} \hat{c}_{\hat{\nu}\hat{m}} - \hat{\partial}_{\hat{\nu}} \hat{c}_{\hat{\mu}\hat{m}} + \hat{\omega}_{\hat{\mu}\hat{m}\hat{n}} \hat{c}_{\hat{\nu}\hat{n}} - \hat{\omega}_{\hat{\nu}\hat{m}\hat{n}} \hat{c}_{\hat{\mu}\hat{n}} \quad (2.2)$$

is the usual torsion tensor with $\hat{\omega}_{\hat{\mu}\hat{m}\hat{n}}$ regarded as an independent field [8]. This $\hat{\omega}_{\hat{\mu}\hat{m}\hat{n}}$ and the propagating multiplier field satisfy the self-duality and anti-self-duality conditions, respectively:

$$\hat{\omega}_{\hat{\mu}\hat{m}\hat{n}} = \frac{1}{2} \varepsilon_{\hat{m}\hat{n}\hat{r}\hat{s}} \hat{\omega}_{\hat{\mu}\hat{r}\hat{s}} , \quad (2.3)$$

$$\hat{\Lambda}_{\hat{m}\hat{n}} = -\frac{1}{2} \varepsilon_{\hat{m}\hat{n}\hat{r}\hat{s}} \hat{\Lambda}_{\hat{r}\hat{s}} . \quad (2.4)$$

The $\hat{F}_{\hat{\mu}\hat{\nu}AB}$ and $\hat{G}_{\hat{\mu}\hat{\nu}AB}$ are the field-strengths of the vector fields $\hat{A}_{\hat{\mu}AB}$ and $\hat{B}_\mu^{AB}$. Since the $SO(8)$ symmetry is global, the non-Abelian terms are absent in these field strengths. The $\hat{G}_{\hat{\mu}\hat{\nu}AB}$ satisfies the anti-self-duality condition:

$$\hat{G}_{\hat{m}\hat{n}AB} = -\frac{1}{2} \varepsilon_{\hat{m}\hat{n}\hat{r}\hat{s}} \hat{G}_{\hat{r}\hat{s}AB} . \quad (2.5)$$

Eqs. (2.3) - (2.5) are not field equations, but are built-in conditions by definition, which become clearer in the spinorial notation in ref. [8]. The symbol $(\pm)$ on $\hat{F}_{\hat{\mu}\hat{\nu}AB}$ denotes the self-dual part of $\hat{F}_{\hat{\mu}\hat{\nu}AB}$. Since $\hat{\omega}_{\hat{\mu}\hat{m}\hat{n}}$ is self-dual, the corresponding Riemann tensor

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4These fields are respectively called $e_m^{\alpha\beta}$, $\omega_m^{\alpha\beta}$, $\omega_m^{\alpha\beta'}$, $A_{mab}$, $B_m^{ab}$, and $\phi_{abcd}$ in ref. [8] in its own notation. In our notation, all the fields and indices with hats are for $D = 4$, to be distinguished from $D = 2$ quantities later. The indices $\hat{\mu}, \hat{\nu}, \cdots = 0, \cdots, 3$ are for the $D = 4$ curved indices, while $\hat{m}, \hat{n}, \cdots = (0), \cdots, (3)$ are for the local Lorentz indices in $D = 4$. Relevantly, we have the signatures $(\eta_{\hat{m}\hat{n}}) = \text{diag.}(+, -, +, -)$ for the Minkowskian flat metric, and $\tilde{c}^{0123} = +1$. This convention is slightly different from that in our previous papers [2-6]. Additionally, the indices $A, B, \cdots = 1, \cdots, 8$ are for the 8-representation of $SO(8)$ which correspond to $a, b, \cdots$ in ref. [8]. Our anti-symmetrization symbols $[ \ ]$ are always normalized.
automatically satisfies the self-duality condition in this formulation [8]. A comparison with the ordinary \( N = 8 \) SG in \( D = (1, 3) \) elucidates the peculiar property of this \( N = 8 \) SDSG system due to the propagating multiplier field \( \hat{\Lambda}_{\mu \dot{\nu} \dot{m}} \), which maintains the self-duality of the Lorentz connection, while making the lagrangian formulation possible. This special role played by this multiplier field is similar to the \( N = 4 \) self-dual supersymmetric Yang-Mills system [8], as described in our previous paper [5].

The field equations for the lagrangian, with all fermionic fields suppressed, are

\[
2\bar{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{D}_{\hat{\nu}} \hat{\Lambda}_{\hat{\rho} \hat{\sigma} \hat{m}} - \bar{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{\Lambda}_{\hat{\rho} \hat{\sigma} \hat{m}} + \frac{1}{576} \bar{\epsilon}_{ABCDEFGH} \left[ \bar{\epsilon}_{\hat{m} \hat{\mu}} (\hat{D}_{\hat{\nu}} \hat{\phi}_{ABCD}) (\hat{D}_{\hat{\sigma}} \hat{\phi}_{EFGH}) - 2 (\hat{D}_{\hat{m}} \hat{\phi}_{ABCD}) (\hat{D}_{\hat{\mu}} \hat{\phi}_{EFGH}) \right] + \frac{1}{384} \bar{\epsilon}_{ABCDEFGH} \hat{\phi}_{ABCD} \left[ \hat{\epsilon}_{\hat{m} \hat{\mu}} \hat{F}^{(+)}_{\hat{\rho} \hat{\sigma} \hat{EF} \hat{GH}} - 4 \hat{F}^{(+)}_{\hat{m} \hat{\sigma} \hat{EF} \hat{GH}} \right] = 0 \ , \tag{2.6}
\]

\[
\bar{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \left[ \hat{e}_{\hat{\rho}} [\hat{m}] \hat{T}^{\hat{\nu} \hat{\lambda} \hat{\sigma} \hat{\mu}] - \frac{1}{2} \hat{e}^{\hat{m} \hat{\lambda} \hat{\mu} \hat{\nu}} \hat{e}_{\hat{\rho}} \hat{T}^{\hat{\lambda} \hat{\mu} \hat{\sigma} \hat{\rho}} \right] = 0 \ , \tag{2.7}
\]

\[
\bar{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \left[ \hat{\Lambda}_{\hat{\rho} \hat{\sigma}} [\hat{m}] \hat{e}_{\hat{\lambda} \hat{\mu}] - \frac{1}{2} \hat{e}^{\hat{m} \hat{\lambda} \hat{\mu} \hat{\nu}} \hat{\Lambda}_{\hat{\rho} \hat{\sigma}} \hat{e}_{\hat{\lambda} \hat{\mu} \hat{\nu}} \right] = 0 \ , \tag{2.8}
\]

\[
4 \hat{D}^{2} \hat{\phi}_{ABCD} - 3 \hat{F}^{(+)}_{\hat{m} \hat{\sigma} \hat{EF} \hat{GH}} \hat{F}^{(+) \hat{m} \hat{\sigma} \hat{EF} \hat{GH}} = 0 \ , \tag{2.9}
\]

\[
\hat{D}_{\hat{\nu}} \hat{G}^{\hat{\mu} \hat{\nu} \hat{AB}} - \frac{1}{24} \hat{e}^{ABCDEFGH} \hat{D}_{\hat{\nu}} \hat{\phi}_{CDEF \hat{G}^{(+) \hat{EF}}} = 0 \ , \tag{2.10}
\]

\[
\hat{D}_{\hat{\nu}} \hat{F}^{\hat{\mu} \hat{\nu} \hat{AB}} = 0 \ , \tag{2.11}
\]

where \( \hat{D} \) is the general covariant derivative with the Christoffel symbol \( \{ \hat{\mu} \hat{\nu} \} \). The familiar \( \pm 2 \) helicities of the graviton in the ordinary \( D = (1, 3) \) are now shared by \( \hat{\omega}_{\mu \dot{m} \dot{n}} \) and \( \hat{\Lambda}_{\mu \dot{m} \dot{n}} \), where especially the latter is a propagating multiplier field, as seen in (2.6) and (2.7). The last two lines in (2.6) are the source terms for the \( \hat{\Lambda}_{\mu \dot{m} \dot{n}} \)-field equation, whose important significance will be seen later.

### 3. Dilaton Gravity Black Hole Solution

We first give some ansätze for classical background solutions to satisfy eqs. (2.6) - (2.11). First of all, we put simply

\[
\hat{\Lambda}_{\mu \dot{m} \dot{n}} = 0 \ , \tag{3.1}
\]

which satisfies (2.8) trivially, and makes the first two terms of (2.6) vanish. Note that it also obeys (2.4) as well. In a way similar to the simple dimensional reduction scheme by Scherk-Schwarz [10], we specify the vierbein as

\[
(\hat{e}_{\dot{\mu}} {\dot{m}}) = \begin{pmatrix} e^{-\varphi} e_{\mu} m & 0 \\ 0 & \delta_{\alpha}^{\beta} \end{pmatrix} \ , \quad (\hat{g}_{\dot{\mu} \dot{\nu}}) = \begin{pmatrix} e^{-2\varphi} g_{\mu \nu} & 0 \\ 0 & \eta_{\alpha \beta} \end{pmatrix} \ , \tag{3.2}
\]

where $e_\mu^m$ and $g_{\mu\nu}$ are respectively the zweibein and the metric in the resultant $D = 2$. For our purpose of embedding the black hole system [9], we have to further specify $e_\mu^m$ and $g_{\mu\nu}$. The simplest choice is

$$
(e_\mu^m) = \frac{1}{\sqrt{2}} \left( e^\varphi \begin{pmatrix} 1 & -e^\varphi \\ e^\varphi & e^{2\varphi} \end{pmatrix} \right), \quad (g_{\mu\nu}) = \begin{pmatrix} 0 & e^{2\varphi} \\ e^{2\varphi} & 0 \end{pmatrix}.
$$

(3.3)

Here we are mainly using the notation in ref. [10], namely the curved indices $\mu, \nu, \cdots = 0, 1$ and the local Lorentz indices $m, n, \cdots = (0), (1)$ for $D = (1, 1)$ with the Minkowskian metric $(\eta_{mn}) = \text{diag.} (+, -)$, into which our dimensional reduction is performed, while the curved ones $\alpha, \beta, \cdots = 2, 3$ and the local Lorentz ones $a, b, \cdots = (2), (3)$ for the extra two-dimensions $E = (1, 1)$ with the Minkowskian metric $(\eta_{ab}) = \text{diag.} (+, -)$. All the fields with hats are in the original $D = 4$, while others without hats are in the $D = (1, 1)$, to be distinguished from the former. We have simply truncated the $\hat{e}_\mu^\alpha$-components just for simplicity. As usual in the simple dimensional reduction [10], we require on all the fields their independence of the coordinates $x^2$ and $x^3$, namely $\partial_\alpha = 0$. After this dimensional reduction, the original Lorentz symmetry $SO(2, 2)$ is eventually reduced to $SO(1, 1) \otimes SO(1, 1)$.

Due to the opposite signs in the exponents in (3.2) and in (3.3), it is clear that $\hat{e}_\mu^\hat{m}$ is the flat vierbein. At first glance, this sounds like eliminating all the physically important freedom of the system, but as we will see shortly, this will result in highly non-trivial embedding of the black hole solution into the $D = 4$ SDSG system.

Because of the flatness of the $D = 4$ metric, we can simply put the ansatz for $\hat{\omega}_\mu^{\hat{r}\hat{s}}$ as

$$
\hat{\omega}_\mu^{\hat{r}\hat{s}} = 0,
$$

(3.4)

satisfying also (2.3). Accordingly we get

$$
\hat{F}_{\hat{\mu}\hat{\nu}^{\hat{m}}} = 0,
$$

(3.5)

and we have now satisfied eq. (2.7). To solve the remaining equations, we further set up our ansatz:

$$
\hat{\phi}_{1234} = \phi(u, v),
$$

(3.6)

$$
\begin{align*}
\hat{F}_{mn, 12} &= \hat{F}_{mn, 34} = f_{mn}, \\
\hat{F}_{ab, 12} &= \hat{F}_{ab, 34} = -f_{ab},
\end{align*}
$$

(3.7a)

$$
\hat{G}_{\hat{\mu}\hat{\nu}^{AB}} = 0,
$$

(3.7b)

\textit{Notice, however, that $\hat{g}_{\mu\nu}$ does not coincide with the $SO(1, 1)$-invariant metric $\eta_{mn}$, which can be reached only after the coordinate transformation $x^{0'} \equiv (x^0 + x^1)/\sqrt{2}$, $x^{1'} \equiv (x^0 - x^1)/\sqrt{2}$.}

\textit{The $\hat{F}_{\mu\nu}^{AB}$ in (3.7a) satisfies the self-duality condition. However, we can also put non-zero anti-self-dual part, still satisfying all the field equations.}
and all other independent components of $\hat{\phi}_{ABCD}$ and $\hat{F}_{\mu\nu AB}$ are zero. The $f$ is a constant to be fixed later, and $\epsilon^{\mu\nu}$ and $\epsilon^{\alpha\beta}$ are respectively the $D = 2$ and $E = 2$ Levi-Civita tensors such that $\epsilon^{01} = -\epsilon^{10} = +1$, $\epsilon^{23} = -\epsilon^{32} = +1$. From now on, we use $u \equiv x^0$, $v \equiv x^1$ for the $D = 2$ coordinates. Obviously (3.6) and (3.7) break the original global $SO(8)$ symmetry down to a product of $U(1)$. Note also that eq. (3.7b) obeys (2.5), as desired. These topological ansätze will play an important role for a constant term in the “dilaton” equation, as we will see later.

It is crucial to see how eq. (2.6) is now satisfied, because its last two lines left over after (3.1) are now vanishing due to (3.6) and (3.7). Eq. (2.11) now holds because of (3.7). The satisfaction of (2.10) is also easily seen, because its last term is vanishing under (3.6) and (3.7). The only non-trivial equation left over is (2.9) equivalent to

$$\partial_u \partial_v \phi = \frac{3}{2} f^2 ,$$

which is nothing but the standard Poisson equation for the dilaton field in the $D = 2$ dilaton gravity black hole system [9]. By the appropriate identification

$$\phi = e^{-2\varphi} ,$$

eq (3.8) is rewritten as

$$2e^{-2\varphi} [\partial_u \partial_v \varphi - 2(\partial_u \varphi)(\partial_v \varphi)] + \frac{3}{2} f^2 = 0 .$$

To accord with the familiar normalization [9], we put

$$f = \frac{2}{\sqrt{3}} .$$

(3.11)

The only non-trivial consistency check now is the satisfaction of the $D = 2$ gravitational equation for $e_m^\mu$ in (3.3), because we are using the same $\varphi$ both for (3.3) and (3.10). In fact, by computing the Ricci tensor in $D = (1, 1)$ for (3.3), we get\footnote{Note that this Ricci tensor is computed purely in $D = (1, 1)$, which should not be confused with $\hat{R}_{\mu\nu}$ in $D = (2, 2)$, etc.}

$$R_{\mu\nu} = g_{\mu\nu} D^2 \varphi = 2 D_\mu D_\nu \varphi ,$$

(3.12)

which coincides with the dilaton gravity black hole gravitational equation [9]. The first equality can be understood as the natural result for the Weyl rescaling for a metric in $D = 2$ as in (3.2). The second equality is due to the relations $D^2 \varphi = D^2_1 \varphi = 0$ for (3.3). Eq. (3.10) with (3.11) can be put into a general covariant form, by the use of the relation $D^2_\mu \varphi = 2e^{-2\varphi} \partial_u \partial_v \varphi$ under (3.3):

$$D^2_\mu \varphi - 2(D_\mu \varphi)^2 + 2 = 0 .$$

(3.13)
After all, our independent covariant field equations are (3.12) and (3.13), coinciding with the familiar dilaton gravity black hole system, which has the solutions [9]

\[
(g_{\mu \nu}) = \frac{1}{2} \frac{1}{uv - 1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (e_\mu^m) = \frac{1}{2} \frac{1}{\sqrt{uv - 1}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (3.14)
\]

\[
\varphi = -\frac{1}{2} \ln \left[ 2(uv - 1) \right]. \quad (3.15)
\]

At first sight, our initial ansätze (3.2) and (3.3) appeared to be trivial, because the metric in \( D = (2, 2) \) became flat. However, this is not trivial at all from the following consideration closely related to the peculiar feature of the SDSG system. In the ordinary system with gravity in \( D = (1, 3) \), we can not always put the metric to be exactly flat, keeping other matter fields propagating, due to the presence of the energy-momentum tensor of the matter fields, which does not equate the vanishing Einstein tensor in the left-hand side of the gravitational field equation. This is physically equivalent to the fact that even a “free” field will generate gravity around it. If we review eq. (2.6), we see that the vanishing of the first two terms for the flat metric does not pose this sort of problem, because the remaining \( \tilde{\phi} \)-dependent terms can vanish, thanks to the peculiar quadratic terms in \( \tilde{\phi} \) or \( \tilde{F} \) contracted by the \( \epsilon^{A_1 \cdots A_8} \)-tensor, even when the field \( \tilde{\phi} \) satisfies its free field equation. In other words, in the SDSG system the scalar field will not create any gravitational field around itself! This is the very feature that enables the SDSG system to embed such \( D = 2 \) dilaton gravity black hole solutions in a non-trivial way, and thus strongly suggests that the \( N = 2 \) superstring theory is the possible “master theory” of such lower-dimensional systems.

We mention another possible class of solutions with non-vanishing values for \( \tilde{G}_{\mu \nu}^{AB} \). For example, instead of (3.7b) we can put

\[
\tilde{G}_{mn}^{AB} = \epsilon_{mn} g^{AB}, \quad \tilde{G}_{ab}^{AB} = \epsilon_{ab} g^{AB}, \quad (3.16)
\]

with arbitrary constants \( g^{AB} \), such that the anti-self-duality (2.5) is maintained. Since \( \tilde{G}_{\mu \nu}^{AB} \) appears in field equations only in (2.10), this can provide a generalized class of non-trivial solutions for the \( N = 8 \) SDSG background. The \( SO(8) \) symmetry is global, so that any of \( g^{AB} \) can be non-zero constants, with each component providing an \( U(1) \) monopole-type solution. We stress that this special feature of non-zero \( \tilde{G}_{\mu \nu}^{AB} \) solutions without disturbing other field equations is again peculiar to the \( N = 8 \) SDSG.

4. Exact Solutions of Dilaton with Gravitational Instanton

\[8\text{This is also related to the fact that the self-dual Riemann tensor is automatically Ricci-flat.}\]
Our result so far suggests the possibility of other sets of exact solutions including a dilaton field embedded into the 70-dimensional scalars. We now show that this is indeed the case by presenting exact solution of the dilaton field on the Eguchi-Hanson gravitational background [11]. To this end, we temporarily forget about the dimensional reduction we have used so far, and treat the system in the total $D = (2, 2)$. Our result is very similar to the exact solution given in ref. [12], where the $N = 1$ SDSG is coupled to self-dual tensor multiplet and the self-dual Yang-Mills multiplet on the Eguchi-Hanson background.

A self-dual Lorentz connection for the Eguchi-Hanson metric [11] can be easily obtained as [12]

$$
\hat{\omega}_2^{(1)(2)} = \hat{\omega}_2^{(3)(4)} = \frac{1}{2} \sqrt{1 - \frac{1}{r^4}} \sin \psi , \quad \hat{\omega}_2^{(1)(4)} = \hat{\omega}_2^{(2)(3)} = -\frac{1}{2} \sqrt{1 - \frac{1}{r^4}} \cos \psi ,
$$

$$
\hat{\omega}_3^{(1)(2)} = \hat{\omega}_3^{(3)(4)} = -\frac{1}{2} \sqrt{1 - \frac{1}{r^4}} \sinh \vartheta \cos \psi , \quad \hat{\omega}_3^{(2)(3)} = \hat{\omega}_3^{(1)(4)} = -\frac{1}{2} \sqrt{1 - \frac{1}{r^4}} \sinh \vartheta \sin \psi ,
$$

$$
\hat{\omega}_3^{(1)(3)} = \hat{\omega}_3^{(2)(4)} = \frac{1}{2} \left(1 + \frac{1}{r^4}\right) \cosh \vartheta ,
$$

$$
\hat{\omega}_4^{(1)(3)} = \hat{\omega}_4^{(2)(4)} = \frac{1}{2} \left(1 + \frac{1}{r^4}\right) , \quad (4.1)
$$

This Lorentz connection can reproduce the self-dual Riemann tensor for the Eguchi-Hanson metric [5,12]:

$$
ds^2 = \frac{1}{1 - \frac{1}{r^4}} dr^2 + \frac{r^4}{4} \left(1 - \frac{1}{r^4} \cosh^2 \vartheta\right) d\varphi^2 - \frac{1}{4} r^2 d\vartheta^2 + \frac{r^2}{4} \left(1 - \frac{1}{r^4}\right) d\psi^2
$$

$$
+ \frac{r^2}{2} \left(1 - \frac{1}{r^4}\right) \cosh \vartheta d\varphi d\psi , \quad (4.2)
$$

for our indefinite signature of $D = (2, 2)$ with the coordinate-labeling $(\hat{x}^\mu) = (r, \vartheta, \varphi, \psi)$.

We next use the ansatz for the dilaton field similar to (3.6) embedded into $\phi_{ABCD}$:

$$
\hat{\phi}_{1234} \equiv \phi(r) , \quad (4.3)
$$

only with the $r$-dependence. For simplicity, we put all other fields to be zero. Eventually the only non-trivial field equation to be solved is (2.9) for the dilaton

$$
\phi''(r) + \frac{3r^4 + 1}{r(r^4 - 1)} \phi'(r) = 0 , \quad (4.4)
$$

where each prime denoting the derivative $d/dr$. This equation is exactly the same as eq. (3.17) of ref. [12] for a dilaton field coupled to the $N = 1$ SDSG up to a non-homogeneous term from a $\hat{F}\hat{F}$-term in the latter, and is solved by

$$
\phi = a \ln \left(\frac{r^2 - 1}{r^2 + 1}\right) + b , \quad (4.5)
$$
with arbitrary constants \( a, b \). The difference of this set of exact solutions from the previous black hole solution is the absence of the non-vanishing \( \hat{F}_{\hat{\mu}\hat{\nu}AB} \)-field.

Our result here suggests that the maximal \( N = 8 \) SDSG has other SDSG with fewer supersymmetries as its sub-theories. We stress that this is just a simple example with the dilaton field on non-trivial self-dual gravitational background, and the universal feature of our methods used in this Letter can be applied to get more interesting solutions embedded into the \( N = 8 \) SDSG.

5. Concluding Remarks

In this paper we have shown that the massless bosonic fields in \( N = 8 \) SDSG, as the consistent background for the \( N = 2 \) closed superstring, has the \( D = 2 \) dilaton gravity black hole solution. The interesting point is that the exact solution, which were originally designed for the backgrounds in the bosonic or \( N = 1 \) superstring, turned out to be an important solution also for the \( N = 2 \) superstring theory.

We have utilized the simple dimensional reduction scheme with appropriate truncation, to get a set of exact solutions for the backgrounds of the \( N = 2 \) superstring. In our previous papers [13,14] we have taken a separate approach to the dimensional reduction of \( N = 2 \) open superstring to generate supersymmetric integrable models [13] and topological field theories [14]. The results in the present Letter have different interesting aspects, due to the exact solutions closely related to the bosonic and \( N = 1 \) superstrings. Our result provides encouragement for the trial of finding other non-trivial solutions as a generalization of the dilaton gravity black hole solution.

In our embedding, the dilaton field is embedded into one component of the original 70-dimensional representation of \( SO(8) \). This is also the reflection of the fact that in the \( N = 8 \) SDSG there is a priori no particular single scalar playing a role of dilaton, but all the scalars fit into the irreducible 70-representation. Also to be stressed is the interesting feature about the topological solution for the \( \hat{F}_{\hat{\mu}\hat{\nu}AB} \), which played an important role to supply the constant term in the \( \phi \)-field equation (3.8). We have also seen the important aspect of the SDSG allowing the propagating \( \hat{\phi}_{ABCD} \)-field, while the metric is kept exactly flat. It is interesting to note that the black hole solution emerges out of a simple scalar field equation.

We have also presented another set of exact solutions with a scalar field embedded into \( \hat{\phi}_{ABCD} \) on the Eguchi-Hanson gravitational instanton background [11], which shares the same exact solution as the dilaton field in what we call self-dual tensor multiplet coupled to
$N = 1$ SDSG system [12]. This result suggests that the maximal $N = 8$ SDSG contains other SDSG as its sub-theories via appropriate truncations.

In regard to the dilaton field hidden among scalars in the maximal SG, we can find a analogous situation in the usual $D = (1, 3), N = 8$ SG [15]. We have shown in our previous paper [15] that two components out of the 70 scalars in the usual $N = 8$ SG in $D = (1, 3)$ can be separated from the other components systematically, as the dilaton and an axion fields. Therefore, by following a procedure similar to ref. [15], we may be able to re-formulate the $N = 8$ SDSG in a more suitable way, such that the dilaton and axion fields are treated separately from the outset, at the expense of the manifest global $SO(8)$ symmetry.

Even though we have dealt only with the bosonic field equations in this Letter, we stress the important role played by supersymmetry in the $N = 8$ SDSG. The peculiar $\hat{\phi} F \hat{F}$-type coupling was required only by supersymmetry, but not strongly motivated by any other purely bosonic self-dual theory. It is plausible that other coupling in the lagrangian (2.1) will lead to more interesting exact solutions.

Our result in this Letter is natural from the viewpoint that the $N = 2$ superstring has great chance to be the “master theory” of supersymmetric integrable models in lower-dimensions [2-6,13,14,16]. Our result gives other supporting evidence for this conjecture for the closed $N = 2$ superstring, in addition to the recent examples [13,14,16] for open $N = 2$ superstring. This applies to any other possible exact solutions that may lead to conformal field theories in $D = 2$ [16], as well as topological field theories in $D = 3$ [14]. It is highly plausible that the $N = 2$ superstring theory generate a wider class of other unknown solutions in lower dimensions in such a comparatively easy way as by the free scalar field equation we have seen.

The original mathematical conjecture [1] that all the integrable systems are generated by $D = (2, 2)$ self-dual Yang-Mills theory, applied only to bosonic integrable systems. The recent development for $N = 2$ superstring [7,8] in physics has further promoted this conjecture to a more general one that all the space-time supersymmetric integrable systems in lower-dimension are generated by $N = 2$ superstring. It is amusing to note that the structure of $N = 2$ superstring is elaborate enough to generate those supersymmetric integrable systems as the most fundamental underlying “master theory”.

One of the motivation and advantage of embedding exact solutions into the “master theory” is the possibility of describing non-perturbative deformation of integrable systems. Suppose $D = (2, 2)$ “compactifies” into $[D = (2, 0)] \otimes [E = (0, 2)]$, where both manifolds are compact, such as the product of two tori: $T^2 \otimes T^2$, and consider the possible non-zero solution for $\tilde{G}_{\hat{\mu} \hat{\nu}}^{AB}$ like (3.16). Then the anti-self-duality condition (2.5) for the 2-form
\[ \hat{G} \text{ yields } \int_{D=(2,0)} \hat{G} = -\int_{E=(0,2)} \hat{G}, \tag{5.1} \]

where \( \hat{G} \) in both sides have the common \( AB \)-components. Interestingly (5.1) relates the two kinds of quantized \( U(1) \) monopole charges on \( D = (2,0) \) and \( E = (0,2) \). We can find many solutions describing different vacua, which may be connected to each other by “quantum tunneling”.

In our recent paper [16], we have shown that the \( N = 4 \) self-dual supersymmetric Yang-Mills theory, which is consistent background for open \( N = 2 \) superstring, can generate \( N = (1,1) \) and \( N = (1,0) \) WZNW models on coset \( G/H \) in \( D = 2 \). We have found that the supergeometry of a superconformal WZNW model has universal composite superpotential, which satisfies the super-integrability condition \( F_{AB} = 0 \) in superspace \([13,14,16]\). This result is also natural according to our past experience that SG usually has the aspects of supersymmetric Yang-Mills theories as its built-in features. It is also plausible that the dilaton gravity black hole solution can have a link to the self-dual Yang-Mills theory via a \( \sigma \)-model based on the coset \( SL(2,\mathbb{R})/U(1) \) \([9]\). Considering these results, it seems that the open and closed \( N = 2 \) superstring theory is really playing key roles for generating lower-dimensional integrable systems with or without curved backgrounds. We expect much more development in this direction related to the self-dual supersymmetric systems in the future.

We are indebted to S.J. Gates, Jr. and W. Siegel for valuable suggestions.
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