A SHARPER SWISS CHEESE

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Abstract. It is shown that there exists a compact planar set $K$ such that the uniform algebra $R(K)$ is nontrivial and strongly regular. This settles an issue raised by Donald Wilken 54 years ago. It is shown that the set $K$ can be chosen such that, in addition, $R(K)$ is not weakly amenable. It is also shown that there exists a uniform algebra that has bounded relative units but is not weakly amenable. These results answer questions raised by Joel Feinstein and Matthew Heath 16 years ago. A key ingredient in our proofs is a bound we establish on the functions introduced by Körner to simplify McKissick’s construction of a normal uniform algebra.

1. Introduction

In this paper we answer several questions in the literature regarding strong regularity and weak amenability of uniform algebras. (These terms are defined later in this introduction.) Our main goal is to prove the following.

Theorem 1.1. There exists a compact set $K$ in the complex plane such that $R(K)$ is a nontrivial strongly regular uniform algebra.

Here, as usual, $R(K)$ denotes the uniform closure on $K$ of the holomorphic rational functions with poles off $K$.

The issue of whether $R(K)$, for $K$ a compact planar set, can be nontrivial and strongly regular (answered by the above theorem) was raised by Donald Wilken 54 years ago [20]. The question was reiterated 31 years ago by Joel Feinstein in the paper in which he constructed the first example of a strongly regular uniform algebra [7], and the question was reiterated again by Feinstein and Matthew Heath in [10].

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In fact, our main result gives more detailed information than the above theorem. Here, and throughout the paper, we denote the open unit disc in the complex plane by $D$, and given a disc $\Delta$, we denote the radius of $\Delta$ by $r(\Delta)$.

**Theorem 1.2.** For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$ and such that setting $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$, the uniform algebra $R(K)$ is nontrivial and strongly regular.

A set of the form $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$ with $\{D_k\}_{k=1}^{\infty}$ a sequence of open discs such that $\sum_{k=1}^{\infty} r(D_k) < \infty$ is called a Swiss cheese. (Sometimes in the literature a more restrictive definition of Swiss cheese is used.) Thus the set $K$ in Theorem 1.2 is a Swiss cheese. For such a set $K$, the following standard result gives a very useful criterion insuring that $R(K)$ is nontrivial [17, Lemma 24.1].

**Theorem 1.3.** Suppose that $\{D_k\}_{k=1}^{\infty}$ is a sequence of open discs in the complex plane such that $\sum_{k=1}^{\infty} r(D_k) < 1$, and set $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$. Then $R(K) \neq C(K)$.

The first example of a nontrivial normal uniform algebra was given by Robert McKissick [16]. His example is $R(K)$ for a certain Swiss cheese $K$. Theorem 1.2 is thus a sharpening of McKissick’s result.

For $K$ a compact planar set such that $R(K)$ is nontrivial, the set of nonpeak points has positive planar measure [17, Theorem 26.8]. Thus Theorem 1.1 gives an example of a strongly regular uniform algebra, on a metrizable space, with uncountably many nonpeak points. Furthermore, replacing $R(K)$ by its restriction to its essential set (see the beginning of the proof of Theorem 1.9) the theorem gives an example with a dense set of nonpeak points. All previously known examples of strongly regular uniform algebras on metrizable spaces had at most finitely many nonpeak points, and those on nonmetrizable spaces had at most finitely many points that were not generalized peak points [7]. Theorem 1.1 also gives the first example of a strongly regular uniform algebra with an infinite, in fact uncountable, Gleason part, since for $R(K)$ the Gleason part of every nonpeak point has positive planar measure [17, Corollary 26.13]. An example with a two-point Gleason part was given by Feinstein [9], and the same argument yields an example with an $n$-point Gleason part for any $n \in \mathbb{Z}_+$ [12].

In [11, Theorem 1.1], Feinstein and the author introduced a general method for constructing essential uniform algebras. Using this method they constructed an essential, natural, regular uniform algebra on the closed unit disc $\overline{D}$ [11, Theorem 1.2]. Repeating the proof of [11, Theorem 1.2] with the strongly regular uniform algebra of Theorem 1.2
above in place of McKissick’s normal uniform algebra shows that the result can be strengthened by replacing regularity by strong regularity.

**Theorem 1.4.** There exists an essential, natural, strongly regular uniform algebra on the closed unit disc $\mathbb{D}$.

In [10], Feinstein and Heath raised many questions including the following.

**Question 1.5.** [10, Question 5.5] Is there a uniform algebra that is strongly regular but is not weakly amenable?

**Question 1.6.** [10, Question 5.4] Is there a uniform algebra that has bounded relative units but is not weakly amenable?

**Question 1.7.** [10, Question 5.1] Is there a nontrivial weakly amenable uniform algebra?

We will answer the first two of these questions in the affirmative. The third question remains open.

**Theorem 1.8.** There exists a compact set $K$ in the complex plane such that $R(K)$ is strongly regular but not weakly amenable.

**Theorem 1.9.** There exists an essential uniform algebra $A$ on a compact metrizable space such that $A$ has bounded relative units, but $A$ is not weakly amenable.

To put these two theorems and their proofs in context, we recall some earlier examples. Feinstein [9] constructed a compact planar set $K$ such that $R(K)$ has no nonzero bounded point derivations but $R(K)$ is not weakly amenable, and he constructed a uniform algebra $A$ on a metrizable space such that every point of the maximal ideal space of $A$ is a peak point for $A$ but $A$ is not weakly amenable. Heath [14] showed that the constructions could be modified so as to obtain examples that are regular. In [10], Feinstein and Heath went further constructing an essential, regular uniform algebra on a metrizable space such that every point of the maximal ideal space $X$ of $A$ is a peak point while $A$ is not weakly amenable, and in addition, $A$ has bounded relative units at every point of a dense open subset of $X$. Our approach to proving Theorems [1.8] and [1.9] is essentially to combine ideas from our proof of Theorem [1.1] with the methods used by Feinstein and Heath to construct their earlier examples.

We present now notational conventions we will use, and we recall standard terminology some of which has already been used above. It is to be understood that all sequences, unions, and sums involving an index extend from 1 to $\infty$; thus for instance $\{D_k\}$ means $\{D_k\}_{k=1}^{\infty}$,
and $\bigcup D_k$ means $\bigcup_{k=1}^\infty D_k$. If $f$ is a function whose domain contains a subset $L$, we denote the restriction of $f$ to $L$ by $f|L$, and if $A$ is a collection of such functions, we denote the collection of restrictions of functions in $A$ to $L$ by $A|L$. We denote the interior of a set $N$ by $N^\circ$. The set of positive integers will be denoted by $\mathbb{Z}_+$. Given a positive real number $c$, we denote by $cD$ the closed disc of radius $c$ centered at the origin in the complex plane.

Recall that given a disc $\Delta$, we denote the radius of $\Delta$ by $r(\Delta)$. We will denote the distance from $\Delta$ to the coordinate axes $\mathbb{R} \cup i\mathbb{R}$ by $s_0(\Delta)$. More generally, given a point $a \in \mathbb{C}$, we will denote the distance from $\Delta$ to the union of the horizontal and vertical lines through $a$ by $s_a(\Delta)$. We will denote the distance from $\Delta$ to the boundary $\partial I^2$ of the closed unit square $I^2 = [0, 1] \times [0, 1]$ by $s(\Delta)$.

Throughout the paper all spaces will tacitly be required to be Hausdorff. Let $X$ be a compact space. We denote by $C(X)$ the algebra of all continuous complex-valued functions on $X$ with the supremum norm $\|f\|_\infty = \|f\|_X = \sup\{|f(x)| : x \in X\}$. A uniform algebra on $X$ is a closed subalgebra of $C(X)$ that contains the constants and separates the points of $X$. A uniform algebra $A$ on $X$ is said to be

(a) nontrivial if $A \neq C(X)$,

(b) essential if there is no proper closed subset of $E$ of $X$ such that $A$ contains every continuous complex-valued function on $X$ that vanishes on $E$,

(c) natural if the maximal ideal space of $A$ is $X$ (under the usual identification of a point of $X$ with the corresponding multiplicative linear functional),

(d) regular on $X$ if for each closed set $K_0$ of $X$ and each point $x$ of $X \setminus K_0$, there exists a function $f$ in $A$ such that $f(x) = 1$ and $f = 0$ on $K_0$,

(e) normal on $X$ if for each pair of disjoint closed sets $K_0$ and $K_1$ of $X$, there exists a function $f$ in $A$ such that $f = 1$ on $K_1$ and $f = 0$ on $K_0$,

(f) approximately regular on $X$ if for each closed set $K_0$ of $X$, each point $x$ of $X \setminus K_0$, and each $\varepsilon > 0$, there exists a function $f$ in $A$ such that $f(x) = 1$ and $|f| < \varepsilon$ on $K_0$,

(g) approximately normal on $X$ if for each pair of disjoint closed sets $K_0$ and $K_1$ of $X$, there exists a function $f$ in $A$ such that $|f - 1| < \varepsilon$ on $K_1$ and $|f| < \varepsilon$ on $K_0$. 


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The uniform algebra $A$ on $X$ is regular, normal, approximately regular, or approximately normal if $A$ is natural and is regular on $X$, normal on $X$, approximately regular on $X$, or approximately normal on $X$, respectively. In fact, every regular uniform algebra is normal [17, Theorem 27.2], and every approximately regular uniform algebra is approximately normal [19, Lemma 2.2]. Also, if a uniform algebra $A$ is normal on $X$, then $A$ is necessarily natural [17, Theorem 27.3]. In contrast a uniform algebra that is approximately normal on $X$ need not be natural; the disc algebra on the circle is a counterexample.

The essential set $E$ for a uniform algebra $A$ on $X$ is the unique smallest closed subset $E$ of $X$ such that $A$ contains every continuous function on $X$ that vanishes on $E$. For a proof of the existence of the essential set and other details, see [2, pp. 144–147]. Note that $A$ is essential if and only if the essential set for $A$ is $X$.

Let $A$ be a uniform algebra on $X$, and let $x \in X$. We define the ideals $M_x$ and $J_x$ by

$$M_x = \{ f \in A : f(x) = 0 \}$$

and

$$J_x = \{ f \in A : f^{-1}(0) \text{ contains a neighborhood of } x \text{ in } X \}.$$  

When it is necessary to indicate with respect to which algebra the ideals are taken, we will denote the ideals $J_x(A)$ and $M_x(A)$, respectively. The uniform algebra $A$ is strongly regular at $x$ if $J_x = M_x$, and $A$ is strongly regular if $A$ is strongly regular at every point of $X$. It was shown by Wilken that every strongly regular uniform algebra is normal [20, Corollary 1].

The point $x$ is said to be a peak point for $A$ if there is a function $f$ in $A$ such that $f(x) = 1$ and $|f(y)| < 1$ for every $y \in X \setminus \{x\}$. The point $x$ is said to be a generalized peak point if for every neighborhood $U$ of $x$ there exists a function $f$ in $A$ such that $f(x) = \|f\| = 1$ and $|f(y)| < 1$ for every $y \in X \setminus U$. When the space $X$ is metrizable, the notions of peak point and generalized peak point coincide.

The uniform algebra $A$ on $X$ satisfies Ditkin’s condition at $x \in X$ if for each $f \in M_x$ and for each $\varepsilon > 0$, there exists $g \in J_x$ such that $\|f - fg\|_X < \varepsilon$. The uniform algebra $A$ is a Ditkin algebra if it satisfies Ditkin’s condition at every point of $X$. We will say that $A$ satisfies the weak Ditkin condition at $x$ if there is a dense subset $S$ of $M_x$ such that for each $f \in S$ and for each $\varepsilon > 0$, there exists $g \in J_x$ such that $\|f - fg\|_X < \varepsilon$. We will say that $A$ is a weak Ditkin algebra if it satisfies the weak Ditkin condition at every point of $X$. Obviously Ditkin’s condition implies the weak Ditkin condition. The uniform algebra $A$ is
a strong Ditkin algebra if it is strongly regular and every point of $X$ is a peak point. The uniform algebra $A$ has bounded relative units at $x$ with bound $C \geq 1$ if for each compact subset $K$ of $X \setminus \{x\}$, there exists $f \in J_x$ such that $f|K = 1$ and $\|f\|_X \leq C$. If $A$ has bounded relative units at every point of $X$, then $A$ has bounded relative units. The uniform algebra $A$ is weakly amenable if there are no nonzero bounded derivations from $A$ into any commutative $A$-bimodule. As proved in [11], weak amenability of $A$ is equivalent to the statement that there are no nonzero bounded derivations from $A$ into the dual module $A^*$. It is standard that every trivial uniform algebra is weakly amenable.

Consider the following conditions on a uniform algebra $A$ on $X$:

(i) $A$ has bounded relative units.

(ii) $A$ is a strong Ditkin algebra.

(iii) $A$ is a Ditkin algebra.

(iv) $A$ is a weak Ditkin algebra.

(v) $A$ is strongly regular.

It is known that conditions (i) and (ii) are equivalent and that each condition implies the next. We refer the reader to [8]. (Some of the implications are obvious.) An example of Feinstein [7, Theorem 5.1] shows that (v) does not imply (ii), but it is unknown whether (v) implies (iii) or whether (iii) implies (ii). The following theorem, which we will prove at the end of the paper, asserts that for $R(K)$, $K$ a compact planar set, (v) implies (iv).

**Theorem 1.10.** Let $K$ be a compact set in the complex plane such that $R(K)$ is strongly regular. Then $R(K)$ is a weak Ditkin algebra.

In view of Theorem 1.10 it follows immediately that (iv) does not imply (ii), since no nontrivial $R(K)$, $K$ a compact planar set, can satisfy (ii) (equivalently (i)) because $R(K)$ is trivial whenever every point of $K$ is a peak point for $R(K)$ [17, Theorem 26.8].

We do not know whether (v) implies (iv) for arbitrary uniform algebras nor do we know whether (iv) implies (iii).

One can also consider conditions (i)–(v) at a particular point $x \in X$ (so that, for instance, condition (i) becomes that $A$ has bounded relative units at $x$). Our knowledge of which implications hold, which fail, and which are unresolved remains the same. In particular, we will prove the following sharper form of Theorem 1.10.

**Theorem 1.11.** Let $K$ be a compact set in the complex plane. If $R(K)$ is strongly regular at a point $x \in K$, then $R(K)$ satisfies the weak Ditkin condition at $x$. 
We now describe our approach to proving Theorem 1.2. Given a point \( x \in D \) and \( r > 0 \), using McKissick’s lemma [16, Lemma 2] (see also [17, Lemma 27.6] and [15, Lemma 1.2]), one can choose a sequence of open discs \( \{ D_k \} \) with \( \sum r(D_k) < r \) and such that \( R(D \setminus \bigcup D_k) \) is strongly regular at \( x \). By repeating this at a countable set of points, Chalice obtained an \( R(K) \) that is strongly regular at a countable dense set of points [3, pp. 302–303]. However, to achieve strong regularity at every point more seems to be needed. Specifically, if for a countable set \( \{ x_n \} \), we are to obtain the desired strong regularity from approximation of functions in \( M_{x_n} \) by functions in \( J_{x_n} \) for every \( n \in \mathbb{Z}^+ \), then we need some control over how large a disc about \( x_n \) is contained in the zero set of an approximating function. We will give a precise, general criterion (Lemma 3.1) on a compact planar set \( K \) that, given \( x \in K \) and \( s \in \mathbb{Z}^+ \), insures that \( J_x \supset M_x^s \). One might hope to apply McKissick’s lemma to construct a compact set satisfying the criterion at every point \( x \in K \) with \( s = 1 \) and thereby obtain strong regularity. However, the author was unable to do so.

A proof of McKissick’s lemma, simpler than the original one, was given by Körner [15]. To get the control needed to satisfy the criterion in Lemma 3.1 we will strengthen McKissick’s lemma by giving certain uniform bounds on the functions used by Körner. A surprising twist then arises in our argument. Our bounds do not seem to enable us to directly satisfy the criterion of Lemma 3.1 with \( s = 1 \). Instead we obtain a Swiss cheese satisfying the criterion with \( s = 2 \) (Theorem 3.5). Thus rather than a strongly regular uniform algebra, we obtain one in which \( J_x \supset M_x^2 \) for every point \( x \). Finally, to conclude the proof of Theorem 1.2 we combine our construction with Wermer’s construction [18, Theorem 1] of a Swiss cheese for which \( M_x^2 = M_x \) for every \( x \).

The condition that \( J_x \supset M_x^2 \) for every \( x \in K \) in the conclusion of Theorem 3.5 is actually equivalent to the condition that \( J_x = M_x^2 \) for every \( x \in K \). We digress from the main purpose of the paper to discuss this issue and in the process present some easy results of independent interest.

**Theorem 1.12.** Fix \( s \in \mathbb{Z}^+ \). If a uniform algebra \( A \) on a compact space \( K \) satisfies \( J_x \supset M_x^s \) for every \( x \in K \), then \( A \) is natural.

**Corollary 1.13.** Fix \( s \in \mathbb{Z}^+ \). If a uniform algebra \( A \) on a compact space \( K \) satisfies \( J_x \supset M_x^s \) for every \( x \in K \), then \( A \) is normal.

Theorem 1.12 can be proven by repeating Wilken’s proof [20, Lemma] that every strongly regular uniform algebra is natural, replacing \( f \) throughout by \( f^s \) (and noting that throughout Wilken’s proof \( dm \)
should be \( d\mu \). The corollary follows from the theorem since the hypothesis on \( A \) is easily seen to imply that \( A \) is regular on \( K \), and (as mentioned earlier) every uniform algebra that is regular on its maximal ideal space is normal \([17, \text{Theorem 27.2}]\).

Let \( A \) be a natural uniform algebra, and let \( I \) be an ideal in \( A \). Let \( E \) denote the hull of \( I \) (the common zero set of the functions in \( I \)). Set

\[
J(E) = \{ f \in A : f^{-1}(0) \text{ contains a neighborhood of } E \}.
\]

The ideal \( I \) is said to be **local** if \( I \supset J(E) \). The following result is standard \([5, \text{Proposition 4.1.20(iv)}]\).

**Theorem 1.14.** Every ideal in a normal uniform algebra is local.

As an immediate consequence of Corollary 1.13 and Theorem 1.14 we get the following result, and in particular, we get that in Theorem 3.5 \( \overline{J_x} = \overline{M^2_x} \) for every \( x \in K \).

**Corollary 1.15.** Fix \( s \in \mathbb{Z}_+ \). If a uniform algebra \( A \) on a compact space \( K \) satisfies \( \overline{J_x} \supset \overline{M^s_x} \) for every \( x \in K \), then \( \overline{J_x} = \overline{M^s_x} \) for every \( x \in K \).

Since the uniform algebra in Theorem 3.5 is \( R(K) \) for a certain compact planar set \( K \), an alternative approach to showing that in that algebra \( \overline{J_x} = \overline{M^s_x} \) for every \( x \in K \) is to invoke the following result which was found in collaboration with Feinstein. We present the proof of the result in the final section of the paper.

**Theorem 1.16.** Let \( K \subset \mathbb{C} \) be compact. Then every ideal in \( R(K) \) is local.

We will also show that a similar argument yields the following generalization of Theorem 1.14.

**Theorem 1.17.** Every ideal in an approximately regular uniform algebra is local.

Given that we obtained our set \( K \) such that \( R(K) \) is strongly regular by separately imposing the conditions that \( \overline{M^2_x} = M^2_x \) for every \( x \in K \) and that \( \overline{J_x} \supset \overline{M^2_x} \) for every \( x \in K \), it is natural to ask whether, for an \( R(K) \) (\( K \) compact planar), either of these conditions implies the other, and hence whether one of these conditions by itself is sufficient to insure that \( R(K) \) is strongly regular. The author will give an example showing that the first condition does not imply the second in a future paper. Conversely, it seems likely that the second condition does not imply the first. In fact, we make the following conjecture.
Conjecture 1.18. For each integer $s \geq 2$, there exists a compact set $K \subset \mathbb{C}$ such that in $R(K)$ we have $\overline{J_x} \supset M_x^s$ for every $x \in K$ but there is some $y \in K$ such that $\overline{J_y} \not\supset M_y^{s-1}$. Note that then by Corollary 1.15 (or alternatively by Theorem 1.16) $\overline{J_x} = M_x^s = M_x^{s+1} = M_x^{s+2} = \cdots$ for every $x \in K$ and $M_y \supset M_y^2 \supset \cdots \supset M_y = J_y$.

In the next section we establish our uniform bounds on Körner’s functions. These bounds seem likely to have further applications. In Section 3 we give our general criterion, on a compact planar set $K$, for $R(K)$ to satisfy $\overline{J_x} \supset M_x^s$ for $x \in K$, and we construct a Swiss cheese $K$ for which the criterion holds for every $x \in K$ with $s = 2$. In Section 4 we present two lemmas showing that certain inclusions of ideals in a uniform algebra $R(K)$ persist when we pass to a compact subset of $K$, and we conclude the proof of Theorem 1.2 giving the existence of a nontrivial strongly regular $R(K)$. The uniform algebras that are not weakly amenable in Theorems 1.8 and 1.9 are constructed in Sections 5 and 6 respectively. In Section 7 we prove the results on localness of ideals. In the concluding Section 8 we prove that for $R(K)$, strong regularity implies the weak Ditkin condition.

2. Bounds on the functions of Körner

As mentioned in the introduction, we will use the functions that were introduced by Körner [15] and used by him to simplify the proof of a lemma in McKissick’s construction [16] of a normal uniform algebra. In addition to the properties of these functions established by Körner, we will need certain uniform bounds on the functions. Actually the functions we will use are not quite the same ones used by Körner. Here is our modification of [15, Lemma 1.2].

Lemma 2.1. There exist a sequence of rational functions $\{f_n\}_{n=1}^{\infty}$ and a constant $C_1 > 0$ such that for every $0 < \varepsilon \leq 1$ there is a sequence of open discs $\{\Delta_k\}_{k=1}^{\infty}$ in the plane such that

(a) $\sum_{k=1}^{\infty} r(\Delta_k) \leq \varepsilon$.
(b) The poles of the $f_n$ lie in $\bigcup_{k=1}^{\infty} \Delta_k$.
(c) The sequence $\{f_n\}$ converges uniformly on $\mathbb{C} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ to a function $f = f_\varepsilon$ that is identically zero outside $D$ and zero free in $D \setminus \bigcup_{k=1}^{\infty} \Delta_k$.
(d) $\bigcup_{k=1}^{\infty} \Delta_k \subset \{z : 1/2 < |z| < 1\}$.
(e) The discs $\Delta_1, \Delta_2, \ldots$ are disjoint.
(f) $f(0) = 1$.
(g) $\|f_\varepsilon\|_{\infty} \leq C_1 \varepsilon^{-1}$.
Furthermore, given \( M \in \mathbb{Z}_+ \), the sequence \( \{f_n\} \) can be chosen such that the derivatives \( f^{(j)}(0) \) vanish for all \( j = 1, \ldots, M \).

In [15], the sequence \( \{f_n\} \) depends on \( \varepsilon \), and this enables Körner to arrange to have in place of condition (d) that \( \bigcup \Delta_k \subset \{ z : 1 - \varepsilon < |z| < 1 \} \). In the lemma above, the sequence \( \{f_n\} \) is independent of \( \varepsilon \). This has the advantage that any two limit functions \( f = f_\varepsilon \) and \( f_{\varepsilon'} \) agree where both are defined. In particular, \( f = f_\varepsilon \) is independent of \( \varepsilon \) on \( \{ z : 1/2 < |z| < 1 \} \). Another advantage is that this approach seems to yield a better dependence on \( \varepsilon \) of the bound in condition (g).

To obtain the nonamenable uniform algebras of Theorems 1.8 and 1.9 we will need to show that condition (a) above can be replaced by a more stringent condition. This will be done at the end of the section (see Lemma 2.5) so that the argument can be skipped by readers not concerned with nonamenability.

The proof of Lemma 2.1 proceeds via several lemmas. The following is [15, Lemma 2.1].

**Lemma 2.2.** If \( N \leq 2 \) is an integer and \( h_N(z) = 1/(1 - z^N) \) then

1. \( |h_N(z)| \leq 2 |z|^{-N} \) for \( |z|^N \geq 2 \),
2. \( |1 - h_N(z)| \leq 2 |z|^N \) for \( |z|^N \leq 2^{-1} \),
3. \( h_N(z) \neq 0 \) for all \( z \).

Furthermore, if \( (8 \log N)^{-1} > \delta > 0 \) then

4. \( |h_N(z)| \leq 2 \delta^{-1} \) provided only that \( |z - w| \geq \delta N^{-1} \) whenever \( w^N = 1 \).

The following is [15, Lemma 2.2] except for the introduction of the scaling factor \( \alpha \) in part (iv) and the addition of part (v).

**Lemma 2.3.** If in Lemma 2.2 we set \( N = n2^{2n} \) with \( n \) sufficiently large then

1. \( |h_N(z)| \leq (n + 1)^{-4} \) for \( |z|^N \geq 1 + 2^{-(2n+1)} \),
2. \( |1 - h_N(z)| \leq (n + 1)^{-4} \) for \( |z|^N \leq 1 - 2^{-(2n+1)} \),
3. \( h_N(z) \neq 0 \) for all \( z \),
4. \( |h_N(z)| \leq 2n^2 \alpha^{-1} \) provided only that \( |z - w| \geq n^{-3}2^{-2n} \alpha \) whenever \( w^N = 1 \),
5. \( |h_N(z)| \leq 2^{2n+1} n^4 \alpha^{-1} \) provided only that \( |z - w| \geq n^{-5}2^{-4n} \alpha \) whenever \( w^N = 1 \).

**Proof.** Parts (i), (ii), and (iii) are the corresponding parts of [15, Lemma 2.2]. Parts (iv) and (v) follows from Lemma 2.2 (iv) on setting \( \delta = n^{-2} \alpha \) and \( \delta = 2^{-2n} n^{-4} \alpha \), respectively. \( \square \)
Proof. The proof is essentially a repetition of the proof of [15, Lemma 2.3], but we include it for clarity. Let $N = n2^{2n}$, $\omega = \exp(2\pi i/N)$, and $g_n(z) = h_N((1 - 2^{-2n})z)$. (A typo occurs in the definition of $N$ in [15].) Let $A_\alpha(n)$ be the collection of discs with radii $n^{-3/2}2^{-2n}\alpha$ and centers $(1 - 2^{-2n})\omega^j$ for $j = 0, 1, \ldots, N - 1$. Then the required results are either trivial or follow directly from Lemma 2.3 on scaling by a factor of $1 - 2^{-2n}$.

Proof of Lemma 2.4 Choose $m$ sufficiently large that Lemma 2.4 applies to all $n \geq m$, set $f_n = \prod_{j=m}^n g_j$, let $\alpha = \varepsilon(\sum_{j=m}^\infty j^{-2})^{-1}$, and let $\{\Delta_k\}$ be an enumeration of the discs of $\bigcup_{j=m}^\infty A_\alpha(j)$. We may restrict attention to $0 < \varepsilon \leq \sum_{j=m}^\infty j^{-2}$ so that $0 < \alpha \leq 1$. Then conditions (a), (b), (d), (e), and (f) are easily verified.

Set $C = \prod_{j=1}^\infty [1 + (j + 1)^{-4}]$. A tedious computation (or in Körner’s words “a simple induction”) shows that for $z \notin \bigcup_{k=1}^\infty \Delta_k$ we have

\[ |f_n(z)| \leq 2m^2C\alpha^{-1} \quad \text{if } |z| \leq 1 - 2^{-2n-1} \]
\[ |f_n(z)| \leq n^{-2}\alpha^{-1} \leq 2m^2C\alpha^{-1} \quad \text{if } 1 - 2^{-(2n-1)} \leq |z| \leq 1 - 2^{-(2n+1)} \]
\[ |f_n(z)| \leq (n + 1)^{-4} \quad \text{if } 1 - 2^{-(2n+1)} \leq |z|. \]

Condition (c) can now be proven as in [15]. For condition (g), note that we have from above that $|f_n(z)| \leq 2m^2C\alpha^{-1}$ for all $z \notin \bigcup_{k=1}^\infty \Delta_k$, and so the same inequality holds with $f$ in place of $f_n$. Thus condition (g) holds with $C_1 = 2m^2C(\sum_{j=m}^\infty j^{-2})$.
The final assertion of the Lemma is evident since each \( f_n \) is a function of \( \varepsilon^{2m} \).

The rest of this section can be skipped by readers not concerned with amenability.

Recall that given a disc \( \Delta \), we denote the distance from \( \Delta \) to the coordinate axes \( \mathbb{R} \cup i\mathbb{R} \) by \( s_0(\Delta) \).

**Lemma 2.5.** In Lemma 2.1 we can replace condition (a) that \( \sum_{k=1}^{\infty} r(\Delta_k) \leq \varepsilon \) by

\[
(\alpha') \sum_{k=1}^{\infty} \frac{r(\Delta_k)}{s_0(\Delta_k)^2} \leq \varepsilon.
\]

Note that condition (\( \alpha' \)) is stronger than condition (a) since \( s_0(\Delta_k) \leq 1 \) for all \( k \).

The following is [14, Lemma 2.4] except for the introduction of the scaling factor \( \alpha \).

**Lemma 2.6.** In Lemma 2.4 we can replace conditions (i) and (v) by

\[
(i') \sum_{\Delta \in A_\alpha(n)} \frac{r(\Delta)}{s_0(\Delta)^2} \leq n^{-2} \alpha,
\]

\[
(v') |g_n(z)| \leq 2^{2n+1} n^{4} \alpha^{-1} \text{ for } z \notin \bigcup_{\Delta \in A_\alpha(n)} \Delta.
\]

**Proof.** The proof is essentially a repetition of the proof of [14, Lemma 2.4], but we repeat part of the argument for clarity. Let \( N = n2^{2n} \), \( \omega = \exp(2\pi i/N) \), \( \omega^{1/2} = \exp(\pi i/N) \), and \( g = h_N((\omega^{1/2})^{-1}(1 - 2^{-2n})^{-1}z) \).

Let \( A_\alpha(n) \) to be the collection of discs with radii \( n^{-5}2^{-4n} \alpha \) and centers \((1 - 2^{-2n})\omega^{1/2}z_j \) for \( j = 0, 1, \ldots, N - 1 \). Then, with the exception of (\( i' \)), results are either trivial or follow directly from Lemma 2.3 on “scaling” by a factor of \( \omega^{1/2}(1 - 2^{-2n}) \). For the proof of (\( i' \)) we refer the reader to [14] noting that of course one must now invoke Lemma 2.3 above in place of [14, Lemma 2.3]. (The argument in [14] contains the erroneous inequality

\[
\frac{(r + 1/2) \pi}{n 2^{2n+2}} - n^{-5}2^{-4n} \geq \left( \frac{1}{2} \right) \frac{2r + 1}{n 2^{2n}},
\]

but this does not matter since it is true that (for all \( n \) sufficiently large) the right hand side is greater than some fixed constant times the left hand side, and that suffices for the argument.)

**Proof of Lemma 2.5.** The proof is similar to the proof of Lemma 2.1. Choose \( m \) sufficiently large that Lemma 2.6 applies to all \( n \geq m \), set \( f_n = \prod_{j=m}^{n} g_j \), let \( \alpha = \varepsilon (\sum_{j=m}^{\infty} j^{-2})^{-1} \), and let \( \{\Delta_k\} \) be an enumeration of the discs of \( \bigcup_{j=m}^{\infty} A_\alpha(j) \) with \( A_\alpha(j) \) as in Lemma 2.6. We may restrict
attention to \( 0 < \varepsilon \leq \sum_{j=m}^{\infty} j^{-2} \) so that \( 0 < \alpha \leq 1 \). Then conditions (a), (b), (d), (e), and (f) are easily verified.

Set \( C = \prod_{j=1}^{\infty} [1 + (j + 1)^{-4}] \). A tedious computation shows that for \( z \not\in \bigcup_{k=1}^{\infty} \Delta_k \) we have

\[
|f_n(z)| \leq 2^{2m+1} m^4 C \alpha^{-1} \quad \text{if } |z| \leq 1 - 2^{-(2n-1)} \\
|f_n(z)| \leq 2^{-2n} 2^{4m+5} \alpha^{-1} \quad \text{if } 1 - 2^{-(2n-1)} \leq |z| \leq 1 - 2^{-(2n+1)} \\
|f_n(z)| \leq [(n + 1)!]^{-4}(m!)^4 \quad \text{if } 1 - 2^{-(2n+1)} \leq |z|.
\]

For notational convenience set \( \tilde{C} = \max\{2^{2m+1} m^4 C, 2^{4m+5}\} \), so that these inequalities become

\[
|f_n(z)| \leq \tilde{C} \alpha^{-1} \quad \text{if } |z| \leq 1 - 2^{-(2n-1)} \\
|f_n(z)| \leq \tilde{C} \alpha^{-1} \quad \text{if } 1 - 2^{-(2n-1)} \leq |z| \leq 1 - 2^{-(2n+1)} \\
|f_n(z)| \leq [(n + 1)!]^{-4}(m!)^4 \quad \text{if } 1 - 2^{-(2n+1)} \leq |z|.
\]

As in Körner [15], apply the trivial inequality

\[
|f_{n+1}(z) - f_n(z)| = |f_n(z)| |1 - g_{n+1}(z)|.
\]

This gives

\[
|f_{n+1}(z) - f_n(z)| \leq \tilde{C} \alpha^{-1} (n + 1)^{-4} \quad \text{if } |z| \leq 1 - 2^{-(2n+1)}
\]

\[
|f_{n+1}(z) - f_n(z)| \leq [(n + 1)!]^{-4}(m!)^4[1 + 2^{2n+3}(n + 1)^4] \quad \text{if } 1 - 2^{-(2n-1)} \leq |z| \leq 1 - 2^{-(2n+1)}
\]

Consequently, for all \( n \) sufficiently large, we have \( |f_{n+1}(z) - f_n(z)| \leq \tilde{C} \alpha^{-1} (n + 1)^{-4} \) for all \( z \not\in \bigcup_{k=1}^{\infty} \Delta_k \). Therefore, by the Weierstrass M-test, \((f_n)\) converges uniformly to some function \( f \).

That \( f(z) = 0 \) for \( |z| \geq 1 \) is evident. The proof that \( f(z) \neq 0 \) for \( |z| < 1 \) is the same as in Körner [15]. The remainder of the proof is essentially the same as in the proof of Lemma 2.1.

\[\Box\]

### 3. The main construction

Our goal in this section is to prove the existence of a Swiss cheese \( K \) such that in the uniform algebra \( R(K) \) we have \( \overline{f_x} \supset M^2_x \) for every \( x \in K \). We will begin with a lemma that provides a general criterion, given \( s \in \mathbb{Z}_+ \), for the inclusion \( \overline{f_x} \supset M^s_x \) to hold. We will then prove a technical lemma that will enable us to satisfy that criterion at every
point with $s = 2$. Finally using the two lemmas, we will construct the desired Swiss cheese.

**Lemma 3.1.** Let $K \subset \mathbb{C}$ be compact, let $s \in \mathbb{Z}_+$, and let $x \in K$. Suppose that for every $\sigma > 0$ and $\eta > 0$ there is an open disc $\Delta$ that contains $x$ and has radius $r(\Delta) \leq \sigma$, and such that, denoting the center of $\Delta$ by $a$, there is a function $g$ in $R(K)$ such that $g$ is identically zero on $\Delta \cap K$ and $\|(z-a)^s - g\|_K < \eta$. Then in the uniform algebra $R(K)$, we have $J_x \supset M_x^s$.

**Proof.** Since the function $(z - x)^s$ generates the ideal $M_x^s$, it suffices to show that $(z - x)^s$ is in $J_x$. Let $\sigma > 0$ and $\eta > 0$ be arbitrary. Let $\Delta$, $a$, and $g$ be as in the statement of the lemma. Let $m = \sup_{z \in K} |(d/dz)(z^s)|$. Then

$$\|(z - x)^s - (z - a)^s\|_K \leq m\sigma.$$ 

Thus

$$\|(z - x)^s - g\|_K \leq \|(z - x)^s - (z - a)^s\|_K + \|(z - a)^s - g\|_K < m\sigma + \eta.$$ 

Since $g$ is in $J_x$, and $\sigma > 0$ and $\eta > 0$ are arbitrary, this shows that $(z - x)^s$ is in $J_x$, as desired. \hfill \square

Recall that the restriction to $(1/2)D$ of the function $f = f_{\varepsilon}$ in Lemma 2.1 is independent of $\varepsilon$. Thus, in particular, $\|f''\|_{(1/4)D}$ is a well-defined number.

**Lemma 3.2.** Let $C_1$ and $f = f_{\varepsilon}$ be as in Lemma 2.1. Set

$$C = \min\{2^{1/2}\|f''\|_{(1/4)D}^{-1/2}, 2^{-5/2}C_1^{-1/2}\}.$$ 

Then for $0 < \varepsilon \leq 1$ and $\eta > 0$ and

$$\sigma = \sigma(\varepsilon, \eta) = C\eta^{1/2}\varepsilon^{1/2},$$

there is a sequence of disjoint open discs $\{D_k\}_{k=1}^\infty$ contained in the annulus $\{z : \sigma < |z| < 2\sigma\}$ such that

(a) $\sum_{k=1}^\infty r(D_k) < 4\sigma\varepsilon$, and

(b) there is a function $h$ on $\mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ that is a uniform limit of rational functions with poles in $\bigcup_{k=1}^\infty D_k$ such that $h = 0$ on $\{|z| \leq \sigma\}$ and such that for all $z \in \mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ we have $|z^2 - z^2h(z)| \leq \eta$. 

Proof. Given $0 < \varepsilon \leq 1$, choose a sequence of disjoint open discs
\{\Delta_k\} as in Lemma [2.1]. Set $M = \|f''\|_{(1/4)\mathcal{D}}$. By the final assertion
of Lemma [2.1] we may assume that $f'(0) = 0$. Then,
\begin{equation}
|f_\varepsilon(w) - 1| = |f_\varepsilon(w) - f_\varepsilon(0)| \leq (M/2)|w|^2 \quad \text{for } |w| \leq 1/4.
\end{equation}
Set $h(z) = h_\varepsilon(z) = f_\varepsilon(\sigma/z)$ (and $h(0) = 0$). Let $D_k$ be the image of
\Delta_k under the map $z \mapsto \sigma/z$. Since each $\Delta_k$ lies in $\{z : 1/2 < |z| < 1\}$,
we have $\sum r(D_k) \leq 4\sigma \sum r(\Delta_k) < 4\sigma \varepsilon$. The function $h$ is defined on
$\mathbb{C} \setminus \bigcup D_k$ and is a uniform limit there of rational functions with poles
in $\bigcup D_k$. Furthermore, $h = 0$ on \{|z| \leq \sigma\}. Also inequality (1) gives
\begin{equation}
|h(z) - 1| \leq (M/2)\sigma^2|z^{-2}| \quad \text{for } |z| \geq 4\sigma.
\end{equation}
It remains to be shown that
\[ |z^2 - z^2 h(z)| = |z^2| |h(z) - 1| \leq \eta \quad \text{for all } z \in \mathbb{C} \setminus \bigcup D_k. \]
For $|z| \geq 4\sigma$, we have by inequality (2)
\[ |z^2| |h(z) - 1| \leq (M/2)\sigma^2 = (M/2)C^2\eta \leq \eta \varepsilon \leq \eta. \]
For $|z| \leq 4\sigma$ (and $z \notin \bigcup D_k$), applying condition (g) of Lemma [2.1]
yields
\begin{align*}
|z^2| |h(z) - 1| &\leq (4\sigma)^2 (\|h\|_\infty + 1) \\
&= (4\sigma)^2 (\|f_\varepsilon\|_\infty + 1) \\
&\leq (4\sigma)^2 (C_1 \varepsilon^{-1} + 1) \\
&\leq (4\sigma)^2 (2C_1 \varepsilon^{-1}) \\
&\leq \eta.
\end{align*}
The lemma is proved. \qed

A simple translation argument yields the following immediate corollary.

**Corollary 3.3.** Let $\sigma = \sigma(\varepsilon, \eta)$ be as in Lemma [3.2]. Given $a \in \mathbb{C}$,
there is a sequence of open discs $\{D_k\}_{k=1}^\infty$ contained in the annulus
\{z : \sigma < |z - a| < 2\sigma\} such that
\begin{enumerate}
\item[(a)] $\sum_{k=1}^\infty r(D_k) < 4\sigma \varepsilon$, and
\item[(b)] there is a function $h$ on $\mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ that is a uniform limit
of rational functions with poles in $\bigcup_{k=1}^\infty D_k$ such that $h = 0$ on
\{|z - a| \leq \sigma\} and such that for all $z \in \mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ we have
\[ |(z - a)^2 - (z - a)^2 h(z)| \leq \eta. \]
\end{enumerate}
Observation 3.4. Given $c > 0$, there is a number $M \leq 9/c^2$ such that the square $[-1, 1] \times [-1, 1]$, and hence the disc $\overline{D}$, can be covered by $M$ open discs of radius $c$. To see this, note that $[-1, 1] \times [-1, 1] \supset \overline{D}$ can be expressed as the union of $\left([2/c]\right)^2 \leq 9/c^2$ squares of side length $c$, and each such square is contained in the open disc of radius $c$ with center the center of the square.

We can now prove the main result of this section.

**Theorem 3.5.** For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$ and such that setting $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$ we have, in the uniform algebra $R(K)$, that $J_x \supset M_x^2$ for every $x \in K$.

In fact, as discussed in the introduction, the algebra $R(K)$ in the theorem actually satisfies $J_x = M_x^2$ for every $x \in K$.

**Proof.** Fix $r > 0$. Let $\sigma = \sigma(\varepsilon, \eta)$ and $C$ and $C_1$ be as in Lemma 3.2. Note that

$$\varepsilon/\sigma = C^{-1} \eta^{-1/2} \varepsilon^{1/2}.$$  

It follows trivially that setting $\eta_n = 1/n$, choosing a sequence $\{\varepsilon_n\}$ going to zero fast enough, and setting $\sigma_n = \sigma(\varepsilon_n, \eta_n)$, we can arrange to have $36 \sum \varepsilon_n/\sigma_n < r$ and $\sigma_n \to 0$ as $n \to \infty$.

By Observation 3.4 for each $n = 1, 2, \ldots$, we can cover $\overline{D}$ by a collection $\mathcal{D}_n$ of $M_n \leq 9/\sigma_n^2$ open discs of radius $\sigma_n$. Given a disc $\Delta$ in $\mathcal{D}_n$, let $a$ denote the center of $\Delta$ so that $\Delta = \{|z - a| \leq \sigma_n\}$. Apply Corollary 3.3 to choose a sequence of open discs $\{D^*_k\}$ such that $\sum r(D^*_k) < 4\sigma_n \varepsilon_n$ and there is a function $h$ as in the corollary with $\sigma = \sigma_n$ (and $\eta = \eta_n$). Do this for each disc in each $\mathcal{D}_n$. Let $\{D_k\}$ be an enumeration of all the discs so chosen. Then $\sum r(D_k) < 36 \sum \varepsilon_n/\sigma_n < r$.

Set $K = D \setminus \bigcup D_k$. Applying Lemma 3.1 then shows that $J_x \supset M_x^2$ for all $x \in K$. \hfill \Box

4. **Proof of the main theorem**

In this section we complete the proof of Theorem 1.2 by combining Theorem 3.5 with Wermer’s theorem [18] on the existence of a Swiss cheese $K$ such that the uniform algebra $R(K)$ has no nonzero bounded point derivations. It is well known (and easy to show) [2, p. 64] that for a uniform algebra $A$ on a compact space $K$ and a point $x \in K$, there exists a bounded point derivation on $A$ at $x$ if and only if $M_x^2$ is not dense in $M_x$. Thus Wermer’s result can be restated as follows.
Theorem 4.1. For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty r(D_k) < r$ and such that setting $K = \overline{D} \setminus \bigcup_{k=1}^\infty D_k$ we have, in the uniform algebra $R(K)$, that $\overline{M_x^2} = M_x$ for all $x \in K$.

We will need two simple lemmas showing that certain inclusions of ideals in a uniform algebra $R(K)$ persist when we pass to a compact subset $L$ of $K$.

Lemma 4.2. Given compact sets $L \subset K \subset \mathbb{C}$, given $s \in \mathbb{Z}_+$, and given a point $x \in L$, if $J_x(R(K)) \supset M_x(R(K))^s$, then $J_x(R(L)) \supset M_x(R(L))^s$.

Proof. Since the function $(z - x)^s$ generates the ideal $M_x(R(L))^s$, it suffices to show that $(z - x)^s$ is in $J_x(R(L))$. Fix $\varepsilon > 0$ arbitrary. By hypothesis, $(z - x)^s$ is in $J_x(R(K))$. Thus there is a function $f$ in $R(K)$ that vanishes on a neighborhood of $x$ in $K$ such that $\|f - (z - x)^s\|_K < \varepsilon$. Then the function $f|L$ is in $R(L)$, vanishes on a neighborhood of $x$ in $L$, and satisfies $\|f - (z - x)^s\|_L < \varepsilon$. Consequently, $(z - x)^s$ is in $J_x(R(L))$, as desired.

Lemma 4.3. Given compact sets $L \subset K \subset \mathbb{C}$ and given a point $x \in L$, if $\overline{M_x(R(K))^2} = M_x(R(K))$, then $\overline{M_x(R(L))^2} = M_x(R(L))$.

Proof. This can be proven by an argument analogous to the one just given for Lemma 4.2. Alternatively, the lemma follows immediately from Hallstrom’s criterion \cite[Theorem 1]{Hallstrom} for the existence of a nonzero bounded point derivation on $R(K)$.

We can now finish the proof of Theorem 4.1.

Proof of Theorem 4.1. Without loss of generality assume that $r < 1$. By Theorem 3.5, there exists a sequence of open discs $\{\Delta_k^1\}_{k=1}^\infty$ such that $\sum r(\Delta_k^1) < r/2$ and such that setting $K_1 = \overline{D} \setminus \bigcup \Delta_k^1$ we have $J_x(R(K_1)) \supset M_x(R(K_1))^2$ for all $x \in K_1$. By Theorem 4.1, there exists a sequence of open discs $\{\Delta_k^W\}_{k=1}^\infty$ such that $\sum r(\Delta_k^W) < r/2$ and such that setting $K_2 = \overline{D} \setminus \bigcup \Delta_k^W$ we have that $M_x(R(K_2))^2 = M_x(R(K_2))$ for all $x \in K_2$. Let $\{D_k\}$ be an enumeration of the collection of discs $\{\Delta_k^1\} \cup \{\Delta_k^W\}$. Set $K = K_1 \cap K_2 = \overline{D} \setminus \bigcup D_k$. Then the uniform algebra $R(K)$ is nontrivial by Theorem 4.3. Furthermore, Lemmas 4.2 and 4.3 yield that $J_x(R(K)) \supset M_x(R(K))^2 = M_x(R(K))$ for every $x \in K$. Thus $R(K)$ is strongly regular.
5. Strong regularity without weak amenability

In this section we establish Theorem 1.8. The construction of the desired compact set $K$ is similar to the construction of the set $K$ in Theorem 1.2 but somewhat more elaborate. Our approach to proving that our uniform algebra is not weakly amenable is the same as in the papers [9, 10, 14] of Feinstein and Heath. Given a compact planar set $K$ denote by $R(K_0)$ the set of restrictions to $K$ of the holomorphic rational functions with poles off $K$. Suppose that $\mu$ is a measure on $K$ such that the bilinear form defined on $R(K_0) \times R(K_0)$ by

$$(f, g) \mapsto \int_K f'g \, d\mu$$

is bounded. Then as noted in [6], we can extend the form by continuity to $R(K) \times R(K)$ and obtain a bounded derivation $D : R(K) \to R(K)^*$ such that for $f$ and $g$ in $R_0(K)$,

$$(Df)(g) = \int_K f'g \, d\mu.$$  

Such a derivation is the zero derivation if and only if $\mu$ annihilates $R(K)$.

We will prove the following theorem.

**Theorem 5.1.** For each $C > 0$ there exists a compact planar set $K$ obtained by deleting from the closed unit square $I^2$ a countable union of open discs such that the boundary $\partial I^2$ of $I^2$ is contained in the essential set for $R(K)$, $R(K)$ is strongly regular, but for all $f$ and $g$ in $R_0(K)$,

$$\left| \int_{\partial I^2} f'(z)g(z) \, dz \right| \leq C\|f\|_K\|g\|_K.$$

(3)

Theorem 1.8 is an immediate consequence, for taking $K$ to be as in Theorem 5.1 the above discussion shows that there is a nonzero bounded derivation $D : R(K) \to R(K)^*$ such that for $f$ and $g$ in $R_0(K)$,

$$(Df)(g) = \int_{\partial I^2} f'(z)g(z) \, dz.$$  

**Remark 5.2.** Given a uniform algebra $A$ on a compact Hausdorff space $X$ and a closed subset $E$ of $X$, there is an obvious algebraic isomorphism of the restriction algebra $A|E$ with the quotient algebra $A/I(E)$, where $I(E)$ denotes the ideal of functions in $A$ vanishing on $E$. Thus $A|E$ can be regarded as a Banach algebra using the quotient norm on $A/I(E)$. Applying an argument of Feinstein [9, p. 2393] then shows that, with $K$ the compact planar set of the above theorem, any
uniform algebra with a restriction isomorphic to \( R(K) \mid \partial I^2 \) must fail to be weakly amenable.

The proof of Theorem 5.1 will use several preliminary lemmas. The first of these is a minor modification of [9, Lemma 2.1] of Feinstein. Its proof is essentially identical to that of Feinstein’s lemma and hence omitted.

Recall the notations \( r(\Delta) \), \( s_0(\Delta) \), \( s_a(\Delta) \), and \( s(\Delta) \) introduced in the introduction.

**Lemma 5.3.** Let \( \{D_k\}_{k=1}^\infty \) be a sequence of open discs in the complex plane whose closures are contained in the interior of the unit square \( I^2 \). Set \( K = I^2 \setminus \bigcup_{k=1}^\infty D_k \). Let \( f \) and \( g \) be in \( R_0(I^2) \). Then
\[
\left| \int_{\partial I^2} f'(z)g(z) \, dz \right| \leq 4\pi \|f\|_K \|g\|_K \sum_{k=1}^\infty \frac{r(D_k)}{s(D_k)^2}.
\]

**Lemma 5.4.** Let \( \{D_k\}_{k=1}^\infty \) be a sequence of open discs in the complex plane whose closures are contained in the interior of the unit square \( I^2 \). Set \( K = I^2 \setminus \bigcup_{k=1}^\infty D_k \). Suppose that \( \sum r(D_k)/s(D_k) < \infty \). Then \( \partial I^2 \) is contained in the essential set for \( R(K) \).

**Proof.** For convenience set \( r_k = r(D_k) \) and \( s_k = s(D_k) \). Choose \( N \) such that \( \sum_{k=N}^\infty r_k/s_k < 1/2 \). Set \( \delta = \min\{s_1, \ldots, s_{N-1}\} \). Let \( E \) be an arbitrary closed disc of radius \( r(E) < \delta/2 \) that is contained in \( I^2 \) and intersects \( \partial I^2 \). We will show that \( R(E \setminus \bigcup D_k) \neq C(E \setminus \bigcup D_k) \). Since every (relatively) open set of \( I^2 \) that intersects \( \partial I \) contains such a disc \( E \), the lemma follows.

By Theorem 1.3 and a trivial scaling argument, to show that \( R(E \setminus \bigcup D_k) \neq C(E \setminus \bigcup D_k) \) it suffices to show that \( \sum_{\{k : D_k \cap E \neq \emptyset\}} r_k < r(E) \).

Now note that
\[
\sum_{\{k : D_k \cap E \neq \emptyset\}} r_k/r(E) \leq \sum_{\{k : s_k \leq 2r(E)\}} r_k/r(E)
\leq \sum_{\{k : s_k \leq 2r(E)\}} 2r_k/s_k
\leq 2 \sum_{k=N}^\infty r_k/s_k
< 1.
\]

**Lemma 5.5.** Let \( K \) be a compact set in the complex plane, and let \( x \in K \). If there is a neighborhood \( U \) of \( x \) in \( K \) such that \( R(U) \) has no
nonzero bounded point derivations at \(x\), then \(R(K)\) also has no nonzero bounded point derivations at \(x\).

**Proof.** This is immediate from Hallstrom’s criterion ([13, Theorem 1]) for the existence of a nonzero bounded point derivation on \(R(K)\). \(\square\)

**Theorem 5.6.** For each \(r > 0\), there exists a sequence of open discs \(\{D_k\}_{k=1}^{\infty}\) such that \(\sum_{k=1}^{\infty} r(D_k)/s(D_k)^2 < r\) and such that setting \(K = I^2 \setminus \bigcup_{k=1}^{\infty} D_k\) we have, in the uniform algebra \(R(K)\), that \(M_x^2 = M_x\) for every \(x \in K\).

**Proof.** Choose a countable collection \(\{U_j\}\) of open discs that covers \((I^2)^\circ\) with the closure of each \(U_j\) contained in \((I^2)^\circ\). By Theorem 4.1 we can choose, for each \(j\), a sequence of open discs \(\{\Delta^j_i\}_{i=1}^{\infty}\) contained in \(U_j\) such that \(\sum_{i=1}^{\infty} r(\Delta^j_i) < r s(U_j)^2/2^j\) and such that in the uniform algebra \(R(\overline{U}_j \setminus \bigcup_{i=1}^{\infty} \Delta^j_i)\) we have \(M_x^2 = M_x\) for every \(x \in \overline{U}_j \setminus \bigcup_{i=1}^{\infty} \Delta^j_i\). Let \(\{D_k\}\) be an enumeration of the collection \(\{\Delta^j_i : j = 1, 2, \ldots, l = 1, 2, \ldots\}\). Then \(\sum_{k=1}^{\infty} r(D_k)/s(D_k)^2 < r\). Set \(K = I^2 \setminus \bigcup_{k=1}^{\infty} D_k\). Then the equality \(M_x^2 = M_x\) holds for \(x \in K \cap (I^2)^\circ\) by Lemma 5.5 and holds for \(x \in \partial I^2\) because each point of \(\partial I^2\) is a peak point for \(R(K)\). \(\square\)

Recall that the restriction to \((1/2)\overline{D}\) of the function \(f = f_\varepsilon\) in Lemmas 2.4 and 2.5 is independent of \(\varepsilon\). Thus, in particular, \(\|f''\|_{(1/4)\overline{D}}\) is a well-defined number.

**Lemma 5.7.** Let \(C_1\) and \(f = f_\varepsilon\) be as in Lemma 2.4. Set
\[
C = \min\{6^{1/3}\|f''\|_{(1/4)\overline{D}}^{-1/3}, 2^{-7/3}C_1^{-1/3}\}.
\]
Then for \(0 < \varepsilon \leq 1\) and \(\eta > 0\) and
\[
\rho = \rho(\varepsilon, \eta) = C\eta^{1/3}\varepsilon^{1/3}
\]
there is a sequence of disjoint open discs \(\{D_k\}_{k=1}^{\infty}\) contained in the annulus \(\{z : \rho < |z| < 2\rho\}\) such that
(a) \(\sum_{k=1}^{\infty} r(D_k)/s_0(D_k)^2 < 4\varepsilon/\rho\), and
(b) there is a function \(h\) on \(\mathbb{C} \setminus \bigcup_{k=1}^{\infty} D_k\) that is a uniform limit of rational functions with poles in \(\bigcup_{k=1}^{\infty} D_k\) such that \(h = 0\) on \(\{|z| \leq \rho\}\) and such that for all \(z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} D_k\) we have
\[
|z^3 - z^3 h(z)| \leq \eta.
\]

**Proof.** The proof is similar to the proof of Lemma 3.2. Given \(0 < \varepsilon \leq 1\), choose a sequence of disjoint open discs \(\{\Delta_k\}\) as in Lemma 2.4 such
that condition (a') of Lemma 2.3 holds. Set $M = \| f'' \|_{(1/4)\Pi}$. By the final assertion of Lemma 2.1 we may assume that $f''(0) = 0$. Then,

$$\text{(4) } |f_\varepsilon(w) - 1| = |f_\varepsilon(w) - f_\varepsilon(0)| \leq (M/6)|w|^3 \text{ for } |w| \leq 1/4.$$

Set $h(z) = h_\varepsilon(z) = f_\varepsilon(\rho/z)$ (and $h(0) = 0$). Let $D_k$ be the image of $\Delta_k$ under the map $z \mapsto \rho/z$. Since each $\Delta_k$ lies in $\{ z : 1/2 < |z| < 1 \}$, we have $r(D_k) \leq 4\rho r(\Delta_k)$. From the geometry of the conformal map $z \mapsto \rho/z$ we see that $s(D_k) \geq \rho s(\Delta_k)$. Consequently, $\sum r(D_k)/s_0(D_k)^2 \leq 4 \sum r(\Delta_k)/\rho s_0(\Delta_k)^2 < 4\varepsilon/\rho$. The function $h$ is defined on $\mathbb{C} \setminus \bigcup D_k$ and is a uniform limit there of rational functions with poles in $\bigcup D_k$. Furthermore, $h = 0$ on $\{ |z| \leq \rho \}$. Also inequality (4) gives

$$\text{(5) } |h(z) - 1| \leq (M/6)\rho^3|z^{-3}| \text{ for } |z| \geq 4\rho.$$

It remains to be shown that

$$|z^3 - z^3 h(z)| = |z^3| |h(z) - 1| \leq \eta \text{ for all } z \in \mathbb{C} \setminus \bigcup D_k.$$

For $|z| \geq 4\rho$, we have by inequality (5)

$$|z^3| |h(z) - 1| \leq (M/6)\rho^3 = (M/6)C^3 \eta \leq \eta \varepsilon \leq \eta.$$

For $|z| \leq 4\rho$ (and $z \notin \bigcup D_k$), applying condition (g) of Lemma 2.1 and Lemma 2.3 yields

$$|z^3| |h(z) - 1| \leq (4\rho)^3 (\| f_\varepsilon \|_\infty + 1) \leq (4\rho)^3 (2C_1 \varepsilon^{-1}) \leq \eta.$$

The lemma is proved.

A simple translation argument yields the following immediate corollary.

**Corollary 5.8.** Let $\rho = \rho(\varepsilon, \eta)$ be as in Lemma 5.7. Let $a \in \mathbb{C}$ be fixed. Then there is a sequence of open discs $\{ D_k \}_{k=1}^\infty$ contained in the annulus $\{ z : \rho < |z - a| < 2\rho \}$ such that

(a) $\sum_{k=1}^\infty r(D_k)/s_a(D_k)^2 < 4\varepsilon/\rho$, and

(b) there is a function $h$ on $\mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ that is a uniform limit of rational functions with poles in $\bigcup_{k=1}^\infty D_k$ such that $h = 0$ on $\{ |z - a| \leq \rho \}$ and such that for all $z \in \mathbb{C} \setminus \bigcup_{k=1}^\infty D_k$ we have

$$|(z - a)^3 - (z - a)^3 h(z)| \leq \eta.$$

**Theorem 5.9.** For each $r > 0$, there exists a sequence of open discs $\{ D_k \}_{k=1}^\infty$ contained in the open unit square $(I^2)^o$ such that $\sum_{k=1}^\infty r(D_k)/s(D_k)^2 < 4\varepsilon/\rho$.
and such that setting $K = I^2 \setminus \bigcup_{k=1}^{\infty} D_k$ we have, in the uniform algebra $R(K)$, that $\overline{x} \supset \overline{M}^3_x$ for every $x \in K$.

In fact, as discussed in the introduction, the algebra $R(K)$ in the theorem actually satisfies $\overline{x} = \overline{M}^3_x$ for every $x \in K$.

Proof. Fix $r > 0$. The square $I^2$ is the union of three (disjoint) sets: the interior of $I^2$, the boundary of $I^2$ minus the set of corners, and the set of corners. We will work in turn on each of these three sets.

Let $\sigma = \sigma(\varepsilon, \eta)$ and $C$ and $C_1$ be as in Lemma 3.2. Note that

$$\varepsilon/\sigma = C^{-1}\eta^{-1/2}\varepsilon^{1/2}.$$  

It follows trivially that setting $\eta_n = 1/n$, choosing a sequence $\{\varepsilon_n\}$ going to zero fast enough, and setting $\sigma_n = \sigma(\varepsilon_n, \eta_n)$, we can arrange to have $36 \sum n^2 \varepsilon_n/\sigma_n < r/3$ and $\sigma_n \to 0$ as $n \to \infty$.

For each $n \in \mathbb{Z}_+$ such that $\frac{1}{n} + 4\sigma_n < 1$, let $Q_n$ be the square

$$Q_n = \left[-1 + \left(\frac{1}{n} + 4\sigma_n\right), 1 - \left(\frac{1}{n} + 4\sigma_n\right)\right]^2.$$  

By Observation 3.4 we can cover $Q_n$ by a collection $\mathscr{D}^1_n$ of $M_n \leq 9/\sigma_n^2$ open discs of radius $\sigma_n$. We may assume that each disc in $\mathscr{D}^1_n$ intersects $Q_n$.

Given a disc $\Delta$ in $\mathscr{D}^1_n$, let $a$ denote the center of $\Delta$ so that $\Delta = \{z - a| \leq \sigma_n\}$. Choose a sequence of open discs $\{D_k^n\}$ as in Corollary 3.3 (with $\varepsilon = \varepsilon_n$, $\eta = \eta_n$, and $\sigma = \sigma_n$), and note that each disc $D_k^n$ chosen satisfies $s(D_k^n) > 1/n$. Carry out this procedure for each disc $\Delta$ in each $\mathscr{D}^1_n$. Let $\{D_k^n\}$ be an enumeration of all the discs so chosen. Then

$$\sum r(D_k^n)/s(D_k^n)^2 < \sum (\sigma_n^2/(4\varepsilon_n))(4\sigma_n\varepsilon_n)\varepsilon_n^2 = 36 \sum n^2 \varepsilon_n/\sigma_n < r/3.$$  

Now let $\rho = \rho(\varepsilon, \eta)$ and $C$ and $C_1$ be as in Lemma 5.7. Note that

$$\varepsilon/\rho^2 = C^{-2}\eta^{-2/3}\varepsilon^{1/3}.$$  

It follows trivially that setting $\eta_n = 1/n$, choosing a sequence $\{\varepsilon_n\}$ going to zero fast enough, and setting $\rho_n = \rho(\varepsilon_n, \eta_n)$, we can arrange to have $8 \sum \varepsilon_n/\rho_n^2 < r/3$ and $\rho_n \to 0$ as $n \to \infty$.

For each $n \in \mathbb{Z}_+$ such that $8\rho_n < 1$, let $L_n$ be the union of the four line segments obtained by deleting from $\partial I^2$ the four discs of radius $8\rho_n$ whose centers are the four corners of $I^2$. We can cover $L_n$ by a collection $\mathscr{D}^2_n$ of $N_n \leq 2/\rho_n$ open discs of radius $\rho_n$ with center in $L_n$.

Given a disc $\Delta$ in $\mathscr{D}^2_n$, let $a$ denote the center of $\Delta$ so that $\Delta = \{|z - a| \leq \rho_n\}$. Choose a sequence of open discs $\{D_k^n\}$ as in Corollary 5.8 (with $\varepsilon = \varepsilon_n$, $\eta = \eta_n$, and $\rho = \rho_n$), and note that each disc $D_k^n$ chosen satisfies $s(D_k^n) \geq s_a(D_k^n)$. Carry out this procedure for each disc $\Delta$ in each $\mathscr{D}^2_n$. Let $\{D_k^n\}$ be an enumeration of all the discs so chosen. Then

$$\sum r(D_k^n)/s(D_k^n)^2 < \sum (2/\rho_n)(4\varepsilon_n/\rho_n) = 8 \sum \varepsilon_n/\rho_n^2 < r/3.$$  


Again setting $\eta_n = 1/n$, choose a new sequence $\{\varepsilon_n\}$ such that again setting $\rho_n = \rho(\varepsilon_n, \eta_n)$, we have $16\sum \varepsilon_n/\rho_n < r/3$ and $\rho_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{Z}_+$, let $\mathcal{D}_n^3$ be the collection whose members are the four discs of radius $\rho_n$ whose centers are the four corners of $I^2$.

Given a disc $\Delta$ in $\mathcal{D}_n^3$, let $a$ denote the center of $\Delta$ so that $\Delta = \{|z-a| \leq \rho_n\}$. Choose a sequence of open discs $\{D_k^3\}$ as in Corollary 5.8 (with $\varepsilon = \varepsilon_n$, $\eta = \eta_n$, and $\rho = \rho_n$), and note that each disc $D_k^3$ chosen satisfies $s(D_k^3) = s_a(D_k^3)$. Carry out this procedure for each disc $\Delta$ in each $\mathcal{D}_n^3$. Let $\{D_k^3\}$ be an enumeration of all the discs so chosen. Then $\sum r(D_k^3)/s(D_k^3)^2 < \sum 4(4\varepsilon_n/\rho_n) = 16 \sum \varepsilon_n/\rho_n < r/3$.

Let $\{D_k\}$ be an enumeration of those discs in $\{D_k^3\} \cup \{D_k^2\} \cup \{D_k^1\}$ that are contained in $(I^2)^c$. Then $\sum r(D_k)/s(D_k)^2 < r$. Set $K = \overline{D} \setminus \bigcup D_k$.

Applying Lemma 3.1 then shows that $\overline{f_x} \supset M_x^1$ for every $x \in K$ (and in fact $\overline{f_x} \supset M_x^2$ for every $x \in K \cap (I^2)^c$).

**Proof of Theorem 1.8.** By Theorem 5.9 there exists a sequence of open discs $\{\Delta^1_k\}_{k=1}^{\infty}$ such that $\sum r(\Delta^1_k)/s(\Delta^1_k)^2 < C/4\pi$ and such that setting $K_1 = I^2 \setminus \bigcup \Delta^1_k$ we have $J_x(R(K_1)) \supset M_x(R(K_1))^3$ for all $x \in K_1$. By Theorem 5.6 there exists a sequence of open discs $\{\Delta^W_k\}_{k=1}^{\infty}$ such that $\sum r(\Delta^W_k)/s(\Delta^W_k)^2 < C/4\pi$ and such that setting $K_2 = I^2 \setminus \bigcup \Delta^W_k$ we have that $M_x(R(K_2))^2 = M_x(R(K_2))$ for all $x \in K_2$. Let $\{D_k\}$ be an enumeration of the collection of discs $\{\Delta^1_k\} \cup \{\Delta^W_k\}$. Set $K = K_1 \cap K_2 = I^2 \setminus \bigcup D_k$. Then Lemmas 4.2 and 4.3 yield that $R(K)$ is strongly regular.

(Note that the condition $M_x^2 = M_x$ implies $M_x^1 = M_x$.) Lemma 5.3 shows that inequality 3 holds, with $C = 4\pi \sum r(D_k)/s(D_k)^2$, for all $f$ and $g$ in $R_0(K)$. Lemma 5.4 shows that $\partial I^2$ is contained in the essential set for $R(K)$.

6. **Bounded relative units without weak amenability**

In this section we prove Theorem 1.9 by applying Cole’s method of root extensions to the uniform algebra given by Theorem 1.8. We begin by recalling some aspects of Cole’s construction [4] (see also [17, Section 19]).

Let $A$ be a uniform algebra on a compact space $X$, and let $\mathcal{F}$ be a (nonempty) subset of $A$. Endow $C^G$ with the product topology. Let $p_1 : X \times C^G \to X$ and $p_f : X \times C^G \to C$ denote the projections given by $p_1(x, (z_g)_{g \in \mathcal{F}}) = x$ and $p_f(x, (z_g)_{g \in \mathcal{F}}) = z_f$. Define $X_{\mathcal{F}} \subset X \times C^G$ by

$$X_{\mathcal{F}} = \{ y \in X \times C^G : (p_f(y))^2 = f(p_1(y)) \text{ for all } f \in \mathcal{F} \},$$

and let $A_{\mathcal{F}}$ be the uniform algebra on $X_{\mathcal{F}}$ generated by the set of functions $\{ f \circ p_1 : f \in A \} \cup \{ p_f : f \in \mathcal{F} \}$. On $X_{\mathcal{F}}$ we have $p_f^2 = f \circ p_1$.
for every \( f \in \mathcal{F} \). Set \( \pi = p_1|X_{\mathcal{F}} \), and note that \( \pi \) is surjective. There is an isometric embedding \( \pi^*: A \to A_{\mathcal{F}} \) given by \( \pi^*(f) = f \circ \pi \).

We call the uniform algebra \( A_{\mathcal{F}} \) or the pair \( (A_{\mathcal{F}}, X_{\mathcal{F}}) \), the \( \mathcal{F} \)-extension of \( A \), and we call \( \pi \) the associated surjection. Note that if \( X \) is metrizable and \( \mathcal{F} \) is countable, then \( X_{\mathcal{F}} \) is metrizable also.

Given \( x \in X \), if \( \mathcal{F} \) is contained in \( M_x \), then the set \( \pi^{-1}(x) \) consists of a single point.

To prove Theorem 1.9, we will iterate the above extension process to obtain an infinite sequence of uniform algebras and then take a direct limit to obtain the desired uniform algebra.

We will need the following lemma of Feinstein and Heath [10, Lemma 4.3].

**Lemma 6.1.** Let \( A \) be a uniform algebra on \( X \) and \( x \in X \). Suppose that, for each compact subset \( E \) of \( X \setminus \{x\} \), there exists a neighborhood \( U \) of \( x \) and a function \( f \in A \) such that

1. \( f|U = 1 \),
2. \( f|E = 0 \),
3. For each \( k \in \mathbb{N} \) there is a function \( g \in A \) with \( g^{2k} = f \).

Then \( A \) has bounded relative units at \( x \).

The next lemma, whose elementary proof we omit, is a modification of a lemma of Feinstein [7, Lemma 3.5].

**Lemma 6.2.** Let \( A \) be a normal uniform algebra on a compact metrizable space \( X \), and let \( \mathcal{F} \) be a closed subset of \( X \). Then there exists a countable subset \( \mathcal{F} \) of \( A \) consisting of functions each vanishing identically on a neighborhood of \( \mathcal{F} \) such that for each point \( x \in X \setminus \mathcal{F} \), and for each compact subset \( E \) of \( X \setminus \{x\} \), there exists a neighborhood \( U \) of \( x \), and a function \( f \in \mathcal{F} \) such that \( f|U = 1 \) and \( f|E = 0 \).

**Proof of Theorem 1.9.** Let \( K \) be the compact planar set given by Theorem 1.8. Let \( X_0 \) be the essential set for \( R(K) \). Strong regularity of \( R(K)|X_0 \) follows trivially from strong regularity of \( R(K) \). Furthermore, \( R(K)|X_0 = R(X_0) \) (by [2] Lemma 3.2.5 for instance). Thus \( R(X_0) \) is essential and strongly regular. Note also that \( R(X_0)|\partial I^2 = R(K)|\partial I^2 \).

We will construct a sequence of uniform algebras \( \{A_m\}_{m=0}^{\infty} \). First set \( A_0 = R(X_0) \), and set \( F_0 = \partial I^2 \). By Lemma 6.2, there is a countable subset \( \mathcal{F}_0 \) of \( A_0 \) consisting of functions each vanishing identically on a neighborhood of \( F_0 \) such that for each point \( x \in X_0 \setminus F_0 \), and for each compact subset \( E \) of \( X_0 \setminus \{x\} \), there exists a neighborhood \( U \) of \( x \), and a function \( f \in \mathcal{F}_0 \) such that \( f|U = 1 \) and \( f|E = 0 \). Let \( (A_1, X_1) \) be the \( \mathcal{F}_0 \)-extension of \( A_0 \), and let \( \pi_1 : X_1 \to X_0 \) be the associated surjection. Because each member of \( \mathcal{F}_0 \) is identically zero on \( F_0 \), the
map $\pi_1$ takes $\pi_1^{-1}(F_0)$ homeomorphically onto $F_0$. Let $F_1 = \pi_1^{-1}(F_0)$.

By [22 Theorem 2.4], $A_1$ is normal.

We then iterate this process to obtain a sequence $\{A_m, X_m, \pi_m, F_m, \mathcal{F}_m\}_{m=0}^\infty$ where each $(A_m, X_m)$ is the $\mathcal{F}_m$-extension of $(A_{m-1}, X_{m-1})$, each $\pi_m : X_m \to X_{m-1}$ is the associated surjection, $F_m = \pi_m^{-1}(F_{m-1})$, and $\mathcal{F}_m$ is a countable subset of $A_m$ consisting of functions each vanishing identically on a neighborhood of $F_m$ such that each function in $\mathcal{F}_{m-1}$ has a square root in $\mathcal{F}_m$ and such that for each point $x \in X_m \setminus F_m$, and for each compact subset $E$ of $X_m \setminus \{x\}$, there exists a neighborhood $U$ of $x$, and a function $f \in \mathcal{F}_m$ such that $f|U = 1$ and $f|E = 0$. Because each member of $\mathcal{F}_{m-1}$ is identically zero on $F_{m-1}$, the map $\pi_m$ takes $F_m$ homeomorphically onto $F_{m-1}$. Finally we take the direct limit of the system of uniform algebras $\{A_m\}$. Explicitly, we set

$$X_\omega = \left\{ (x_j)_{j=0}^\infty \in \prod_{j=0}^\infty X_j : \pi_{m+1}(x_{m+1}) = x_m \text{ for all } m = 0, 1, 2, \ldots \right\},$$

and letting $q_m : X_\omega \to X_m$ be the restriction of the canonical projection $\prod_{j=0}^\infty X_j \to X_m$, we let $A_\omega$ be the closure of $\bigcup_{m=0}^\infty \{ h \circ q_m : h \in A_m \}$ in $C(X_\omega)$. Set $F_\omega = \{(x_j)_{j=0}^\infty \in X_\omega : x_0 \in F_0\}$. Set $\pi = q_0$. Then $\pi$ maps $F_\omega = \pi^{-1}(F_0)$ homeomorphically onto $F_0$.

Note that $X_\omega$ is metrizable.

To prove that $A_\omega$ is essential, we first prove by induction that $A_m$ is essential for each $m = 0, 1, 2, \ldots$. Recall that $A_0 = R(X_0)$ is essential. Assume as the induction hypothesis that $A_{m-1}$ is essential. Denote by $\mathbb{Z}_2$ the multiplicative group consisting of 1 and $-1$ and set $\mathcal{F} = \mathcal{F}_{m-1}$. The product group $\mathbb{Z}_2^\mathcal{F}$ acts as a topological transformation group on $X_m$ as follows: Given $\gamma \in \mathbb{Z}_2^\mathcal{F}$ and $(x, z) \in X_m \subset X_{m-1} \times \mathbb{C}^\mathcal{F}$, let $\gamma(x, z)$ be the point $(x, z')$ where the $f$th coordinate of $z'$ is $\gamma_f z_f$. The uniform algebra $A_m$ is invariant under this action in the sense that the function $(x, z) \mapsto f(\gamma(x, z))$ is in $A_m$ for each $\gamma \in \mathbb{Z}_2^\mathcal{F}$ whenever $f$ is in $A_m$. Consequently, the essential set for $A_m$ must also be invariant under the action of $\mathbb{Z}_2^\mathcal{F}$ on $X_m$. Since $\mathbb{Z}_2^\mathcal{F}$ acts transitively on each fiber of $\pi_m$, it follows that if $A_m$ is not essential, then there must be an open set $U$ of $X_{m-1}$ such that $\pi_{m-1}^{-1}(U)$ lies outside the essential set for $A_m$. It is standard (see [17 Theorem 19.1]) that given a uniform algebra $A$ on a space $X$, a subset $\mathcal{F}$ of $A$, the $\mathcal{F}$-extension $A'$ of $A$ with associated surjection $\pi$, and a function $g \in C(X)$ such that the function $g \circ \pi$ is in $A'$, then the function $g$ must be in $A$. Consequently, the condition that $\pi_m^{-1}(U)$ lies outside the essential set for $A_m$ implies that $U$ lies outside the essential set for $A_m$, a contradiction. Thus $A_m$ must be essential.
If \( A_\omega \) is not essential, then there must be some \( n \in \mathbb{Z}_+ \) and some open set \( W \) in \( X_n \) such that \( q_n^{-1}(W) \) lies outside the essential set for \( A_\omega \). It is standard that if \( f \in C(X_n) \) is such that \( f \circ q_n \) is in \( A_\omega \), then \( f \) is in \( A_n \). (This can be derived from [17, Lemma 19.3] noting that discarding the terms \( A_0, \ldots, A_{n-1} \) from the system of uniform algebras \( \{A_m\} \) does not affect the direct limit \( A_\omega \).) Consequently, the condition that \( q_n^{-1}(W) \) lies outside the essential set for \( A_\omega \) implies that \( W \) lies outside the essential set for \( A_n \), a contradiction. Thus \( A_\omega \) must be essential.

To prove that \( A_\omega \) has bounded relative units, we consider separately points in the fibers of \( \pi \) lying over \( X_0 \setminus F_0 \) and over \( F_0 \). At points over \( X_0 \setminus F_0 \) we apply Lemma [6.1]. Consider a point \( x \in X_\omega \) lying over \( X_0 \setminus F_0 \) and a compact subset \( E \) of \( X_\omega \setminus \{x\} \). For some \( n \in \mathbb{Z}_+ \), the point \( q_n(x) \) is outside \( q_n(E) \). Then there is a neighborhood \( V \) of \( q_n(x) \) in \( X_n \) and a function \( h \in \mathcal{F}_n \) such that \( h|U = 1 \) and \( h|(q_n(E)) = 0 \). Then the neighborhood \( U = q_n^{-1}(V) \) and the function \( f = h \circ q_n \) satisfy the conditions in Lemma [6.1]. Consequently, \( A_\omega \) has bounded relative units at \( x \).

Now consider a point \( x \) lying over \( F_0 \). To show that \( A_\omega \) has bounded relative units at \( x \) we first note that by a theorem of Feinstein [8, Theorem 1.5] it is enough to show that \( x \) is a peak point for \( A_\omega \) and that \( A_\omega \) is strongly regular at \( x \). That \( x \) is a peak point is immediate since given a function \( h \) in \( A_0 \) that peaks at \( \pi(x) \), the function \( h \circ \pi \) peaks at \( x \), since \( x \) is the only point in the fiber \( \pi^{-1}(\pi(x)) \). Because the set \( \bigcup_{m=0}^{\infty} \{h \circ q_m : h \in A_m\} \) is dense in \( A_\omega \), to show that \( A_\omega \) is strongly regular at \( x \), it suffices to show that \( A_m \) is strongly regular at \( q_m(x) \) for each \( m = 0, 1, 2, \ldots \). Recall that \( A_0 \) is strongly regular. Assume for the purpose of induction that \( A_{m-1} \) is strongly regular at \( q_{m-1}(x) \). Every function in \( A_m \) that vanishes at \( q_m(x) \) can be uniformly approximated by a function \( f \) of the form

\[
f = \pi_m^*(f_0) + \sum_{u=1}^{s} \pi_m^*(f_u)g_u
\]

where \( f_0, f_1, \ldots, f_s \in A_{m-1} \), the function \( f_0 \) vanishes at \( q_{m-1}(x) \), and each \( g_u \) is a product of functions of the form \( p_f \) for \( f \in \mathcal{F}_{m-1} \). Because every function in \( \mathcal{F}_{m-1} \) vanishes identically on a neighborhood of \( q_{m-1}(x) \), the sum \( \sum_{u=1}^{s} \pi_m^*(f_u)g_u \) vanishes identically on a neighborhood of \( q_m(x) \). Because \( A_{m-1} \) is strongly regular at \( q_{m-1}(x) \), the function \( f_0 \) can be uniformly approximated by a function in \( A_{m-1} \) vanishing identically on a neighborhood of \( q_{m-1}(x) \), and hence \( \pi_m^*(f_0) \) can
be uniformly approximated by a function in $A_m$ vanishing identically in a neighborhood of $q_m(x)$. Therefore, $A_m$ is strongly regular at $q_m(x)$.

Finally we show that $A_\omega$ is not weakly amenable. There is a standard norm-decreasing linear operator $T: A_\omega \to A_0$ such that $T(f \circ \pi) = f$ for every $f \in A_\omega$. (See [17, Lemma 19.3] for instance.) It is easily seen from the way this operator $T$ is defined that for each point $x$ of $X_0$ whose fiber $\pi^{-1}(x)$ consists of a single point, we have for each function $g \in A_\omega$ that $(T(g))(x) = g(\pi^{-1}(x))$. (The operator $T$ is essentially given by averaging over the fibers of $\pi$.) Consequently, for each $g \in A_\omega$ we have $g|F_\omega = (T(g)|F_0)\circ \pi$. It follows that the restriction algebras $A_\omega|F_\omega$ and $A_0|F_0$ are isomorphic, and thus $A_\omega$ fails to be weakly amenable by Remark 5.2.

7. PROOFS OF THE RESULTS ON LOCALNESS OF IDEALS

Proof of Theorem 1.16. Let $J$ be an ideal in $R(K)$. Let $E$ denote the hull of $J$. Suppose $g \in R(K)$ vanishes identically on a neighborhood of $E$ in $K$. We are to show that $g$ lies in $J$.

Let $M$ be a closed neighborhood of $E$ on which $g$ vanishes identically, and let $N$ be a closed neighborhood of $E$ contained in the interior of $M$. Let $L = K \setminus N^\circ$. Then $M$ and $L$ are closed subsets of $K$ whose interiors cover $K$.

On the compact set $L$, the functions in $J$ have no common zeros. Consequently, the subset $J|L$ of $R(L)$ consisting of the restrictions of functions in $J$ to $L$ is contained in no maximal ideal of $R(L)$. Therefore, there exist functions $f_1, \ldots, f_m$ in $J$ and functions $h_1, \ldots, h_m$ in $R(L)$ such that

$$h_1f_1 + \cdots + h_mf_m = 1 \quad \text{on } L.$$  

Define functions $g_1, \ldots, g_m$ on $K$ by

$$g_j = \begin{cases} 
gh_j & \text{on } L \\
0 & \text{on } M.
\end{cases}$$

Note that this yields well-defined continuous functions on $K$. Note also that each $g_j$ is in $R(K)$ by the localization theorem for rational approximation [17, Theorem 26.1]. By equation (6) we have

$$g = gh_1f_1 + \cdots + gh_mf_m \quad \text{on } L$$

and we have

$$g = g_1f_1 + \cdots + g_mf_m \quad \text{on } M.$$
because there both sides of the equation are 0. Therefore,
\[ g = g_1 f_1 + \cdots + g_m f_m \quad \text{on } K. \]

Thus \( g \) is in \( J \), as desired. \( \square \)

**Proof of Theorem 1.17.** The proof is similar to the proof of Theorem 1.16 but differs in some details, so we present the argument in full. Given an ideal \( J \) in \( A \) and a function \( g \) in \( A \) that vanishes identically on a neighborhood of the hull \( E \) of \( J \), we are to show that \( g \) lies in \( J \).

Denote the maximal ideal space of \( A \) by \( X \). Let \( M \) be a closed neighborhood of \( E \) in \( X \) on which \( g \) vanishes identically, and let \( N \) be a closed neighborhood of \( E \) contained in the interior of \( M \). Let \( L = X \setminus N^o \). Then \( M \) and \( L \) are closed subsets of \( X \) whose interiors cover \( X \).

On the compact set \( L \), the functions in \( J \) have no common zeros. Furthermore, because \( A \) is approximately regular, every subset of \( X \) is \( A \)-convex. Thus the maximal ideal space of the uniform algebra \( A|L \) is \( L \). Consequently, the subset \( J|L \) of \( A|L \) is contained in no maximal ideal of \( A|L \). Therefore, there exist functions \( f_1, \ldots, f_m \) in \( J \) and functions \( h_1, \ldots, h_m \) in \( A|L \) such that
\[
(7) \quad h_1 f_1 + \cdots + h_m f_m = 1 \quad \text{on } L.
\]

Define functions \( g_1, \ldots, g_m \) on \( K \) by
\[
g_j = \begin{cases} gh_j & \text{on } L \\ 0 & \text{on } M. \end{cases}
\]

Note that this yields well-defined continuous functions on \( X \).

For each \( j = 1, \ldots, m \), choose a sequence \( \{h_j^{(n)}\}_{n=1}^\infty \) of functions in \( A \) such that \( h_j^{(n)}|L \to h_j \) uniformly on \( L \) as \( n \to \infty \). For \( j = 1, \ldots, m \) and \( n = 1, 2, \ldots, \), set \( g_j^{(n)} = gh_j^{(n)} \). Then \( g_j^{(n)} \to g_j \) uniformly on \( X \) as \( n \to \infty \), and hence, each \( g_j \) is in \( A \).

By equation (7) we have
\[
g = gh_1 f_1 + \cdots + gh_m f_m \quad \text{on } L
\]
\[
= g_1 f_1 + \cdots + g_m f_m \quad \text{on } L,
\]

and we have
\[
g = g_1 f_1 + \cdots + g_m f_m \quad \text{on } M
\]

because there both sides of the equation are 0. Therefore,
\[
g = g_1 f_1 + \cdots + g_m f_m \quad \text{on } K.
\]

Thus \( g \) is in \( J \), as desired. \( \square \)
8. Strongly Regular Implies Weak Ditkin for $R(K)$

Proof of Theorem 1.11. Suppose that $R(K)$ is strongly regular at $x$. We will show that for every rational function $f$ in $M_x$ and for every $\varepsilon > 0$ there exists $g \in J_x$ such that $\|f - fg\|_K < \varepsilon$. Since the rational functions in $M_x$ are dense in $M_x$, this will establish the theorem.

We consider first the case $f = z - x$. Note that the set $(z - x)J_x = \{ (z - x)g : g \in J_x \}$ is an ideal in $R(K)$. Since $R(K)$ is strongly regular at $x$, the hull of the ideal $J_x$ is $\{ x \}$, and hence the same is true of the ideal $(z - x)J_x$. Consequently, $(z - x)J_x \supset J_x$, since every ideal in $R(K)$ is local by Theorem 1.16. By hypothesis $J_x = M_x$. Thus $(z - x)J_x \supset J_x = M_x$. In particular, $z - x$ is in $(z - x)J_x$, or equivalently, for every $\varepsilon > 0$ there exists $g \in J_x$ such that $\|(z - x) - (z - x)g\|_K < \varepsilon$.

Now let $f$ be an arbitrary nonzero rational function in $M_x$. Then $f = (z - x)q$ for some rational function $q$ in $R(K)$. Given $\varepsilon > 0$, we proved in the preceding paragraph the existence of a $g \in J_x$ such that $\|(z - x) - (z - x)g\|_K < \varepsilon/\|q\|_K$. Now

$$\|f - fg\|_K = \|(z - x)q - (z - x)qg\|_K$$

$$\leq \|q\|_K \|(z - x) - (z - x)g\|_K$$

$$< \varepsilon,$$

as desired. □

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