ANALYTICAL MODELS OF EXOPLANETARY ATMOSPHERES. I. ATMOSPHERIC DYNAMICS VIA THE SHALLOW WATER SYSTEM

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Received 2014 January 29; accepted 2014 June 2; published 2014 July 18

ABSTRACT

Within the context of exoplanetary atmospheres, we present a comprehensive linear analysis of forced, damped, magnetized shallow water systems, exploring the effects of dimensionality, geometry (Cartesian, pseudo-spherical, and spherical), rotation, magnetic tension, and hydrodynamic and magnetic sources of friction. Across a broad range of conditions, we find that the key governing equation for atmospheres and quantum harmonic oscillators are identical, even when forcing (stellar irradiation), sources of friction (molecular viscosity, Rayleigh drag, and magnetic drag), and magnetic tension are included. The global atmospheric structure is largely controlled by a single key parameter that involves the Rossby and Prandtl numbers. This near-universality breaks down when either molecular viscosity or magnetic drag acts non-uniformly across latitude or a poloidal magnetic field is present, suggesting that these effects will introduce qualitative changes to the familiar chevron-shaped feature witnessed in simulations of atmospheric circulation. We also find that hydrodynamic and magnetic sources of friction have dissimilar phase signatures and affect the flow in fundamentally different ways, implying that using Rayleigh drag to mimic magnetic drag is inaccurate. We exhaustively lay down the theoretical formalism (dispersion relations, governing equations, and time-dependent wave solutions) for a broad suite of models. In all situations, we derive the steady state of an atmosphere, which is relevant to interpreting infrared phase and eclipse maps of exoplanetary atmospheres. We elucidate a pinching effect that confines the atmospheric structure to be near the equator. Our suite of analytical models may be used to develop decisively physical intuition and as a reference point for three-dimensional magnetohydrodynamic simulations of atmospheric circulation.

Key words: hydrodynamics – methods: analytical – planets and satellites: atmospheres

Online-only material: color figures

1. INTRODUCTION

1.1. Motivation

With the atmospheres of exoplanets now available for astronomical scrutiny, there is motivation to understand the basic physics governing their structure. Since highly irradiated exoplanets are most amenable to atmospheric characterization, a growing body of work has focused on hot Earths/Neptunes/Jupiters, ranging from analytical models to simulations of atmospheric circulation (e.g., Dobbs-Dixon & Lin 2008; Showman et al. 2009, 2013; Heng et al. 2011; Thrastarson & Cho 2011; Showman & Polvani 2011; Rauscher & Menou 2013; Batygin et al. 2013; Rogers & Showman 2014) in one, two, and three dimensions. The path to full understanding requires the construction of a hierarchy of theoretical models of varying sophistication (Held 2005). In this context, analytical models have a vital role to play, since they provide crisp physical insight and are immune to numerical issues (e.g., numerical viscosity, sub-grid physics, spin-up).

Atmospheres behave like heat engines. Sources of forcing (e.g., stellar irradiation) induce atmospheric motion, and are eventually damped out by sources of friction (e.g., viscosity, magnetic drag). It is essential to understand the atmospheric dynamics, as it sets the background state of velocity, temperature, density, and pressure that determines the spectral and temporal appearance of an atmosphere. It also determines whether an atmosphere attains or is driven away from chemical, radiative, and thermodynamic equilibrium. Even if an atmosphere is not in equilibrium, it must be in a global state of equi- poise—sources of forcing and friction negate one another (e.g., Goodman 2009).

In this study, our over-arching physical goal is to analytically derive the global steady state of an exoplanetary atmosphere (the “exoclimе”) in the presence of forcing, friction, rotation, and magnetic fields.

Shallow water models are a decisive way of studying exoclimes. The term “shallow water” comes from their traditional use in meteorology and oceanography (Matsuno 1966; Lindzen 1967; Longuet-Higgins 1967, 1968; Gill 1980) and refers to the approximation that the horizontal extent modeled far exceeds the vertical/radial one. They have been used to understand the solar tachocline (Gilman 2000; Zaqarashvili et al. 2007), the atmospheres of neutron stars (Spitkovsky et al. 2002; Heng & Spitkovsky 2009), and exoplanetary atmospheres (Showman & Polvani 2011; Heng & Kopparla 2012). Our over-arching technical goal is to perform a comprehensive theoretical survey across dimensionality (one and two dimensions), geometry (Cartesian, pseudo-spherical, and spherical), and sources of friction (molecular viscosity, Rayleigh drag, and magnetic drag) for forced, damped, rotating magnetized systems. To retain algebraic tractability, we study the limits of purely vertical/radial or horizontal/toroidal background magnetic fields.

Our main finding is that the global structure of exoplanetary atmospheres is largely controlled by a single, key parameter—at least in the shallow water approximation—which we denote with \(\alpha\). In the hydrodynamic limit, \(\alpha\) is directly related to the Rossby and Prandtl numbers. In forced, damped atmospheres with non-ideal magnetohydrodynamics (MHD), \(\alpha\) encapsulates the effects of molecular viscosity, Rayleigh drag, forcing, magnetic tension, and magnetic drag.
Due to the technical nature of this study, we find it useful to concisely summarize a set of terminology that we will use throughout the paper.

From a set of perturbed equations, we will obtain wave solutions for the velocity, height, and magnetic field perturbations. We assume that the waves have a temporal component of the form $\exp(-i\omega t)$, where $\omega$ is the wave frequency. Generally, the wave frequency has real and imaginary components ($\omega = \omega_R + i\omega_I$), which describe the oscillatory and growing or decaying parts of the wave, respectively. For each model, we will obtain a pair of equations describing $\omega_R$ and $\omega_I$, which we call the “oscillatory dispersion relation” and the “growth/decay dispersion relation,” respectively. We will refer to them collectively as “dispersion relations.” A “balanced flow” corresponds to the situation when $\omega_I = 0$. The “steady state” of an atmosphere has $\omega_R = \omega_I = 0$.

We will refer to molecular viscosity, Rayleigh drag, and magnetic drag collectively as “friction.” If we are only considering molecular viscosity and Rayleigh drag, we will use the term “hydrodynamic friction.” We will use the terms “friction” and “damping” interchangeably, but not qualify the latter as being hydrodynamic or MHD in nature. We will refer to systems as being “free” if forcing and friction are absent. Similarly, we will use the terms “fast” and “rapid” interchangeably when referring to rotation. The zeroth order effect of including a magnetic field introduces a term each into the momentum and magnetic induction equations—we will refer to the effects of these ideal-MHD terms as “magnetic tension.” When non-ideal MHD is considered, we include a resistive term in the induction equation that mathematically resembles diffusion—we will refer to its influence as “magnetic drag.”

We will examine an approximation for including the effects of non-constant rotation across latitude on a Cartesian grid, known traditionally as the “\(\beta\)-plane approximation” (e.g., Vallis 2006 and references therein). Note that this is not the same as a departure from solid body rotation. Rather, it is an approximation to include the dynamical effects of sphericity. There are two flavors of this approximation: the simpler version solves for waves that are oscillatory in both spatial dimensions (simply called “\(\beta\)-plane”), while a more sophisticated version allows for an arbitrary functional dependence in latitude (“equatorial \(\beta\)-plane”). Since the equatorial \(\beta\)-plane treatment more closely approximates the situation on a sphere, we will refer to it as being “pseudo-spherical.”

In constructing the mathematical machinery in this paper, we often have to evaluate long, complex expressions. To this end, we find it convenient to separate out the real and imaginary components using a series of “separation functions,” which we denote by $\zeta_\pm$, $\zeta_\times$, $\xi_\pm$, $\xi_\times$, and $\zeta_0$. The definitions of these dimensionless quantities vary from model to model. We note that $\zeta_0$ also functions like a generalized friction that includes magnetic tension.

1.3. Layout of Paper

In Section 2, we state the governing equations and derive their linearized, perturbed forms. In Section 3, we review and extend the one-dimensional (1D) models. We extend our models to two-dimensional (2D) Cartesian geometry in Section 4 and begin to consider the effects of sphericity in Section 5. In Section 6, we present results for 2D models in spherical geometry. Applications to exoplanetary atmospheres are described in Section 7. We summarize our findings in Section 8. Table 1 lists the most commonly used quantities and the symbols used to denote them. Table 2 compares our study with previous analytical work. Table 3 summarizes the salient lessons learned from studying each shallow water model. Figure 1 provides a graphical summary of our technical achievements.
2. GOVERNING EQUATIONS OF THE SHALLOW WATER SYSTEM

2.1. General Equations

By including molecular viscosity, Rayleigh drag, and magnetic tension in the conservation of linear momentum, we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -g \nabla h - 2\Omega \times \mathbf{v} + \nu \nabla^2 \mathbf{v} - \frac{\nu}{t_{\text{drag}}} \mathbf{v} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi \rho},$$

(1)

where $\mathbf{v}$ is the 2D horizontal velocity vector, $t$ denotes the time, $\Omega$ is the planetary rotation vector (pointing north), $\nu$ is the kinematic viscosity, $t_{\text{drag}}$ is a constant drag timescale, $\mathbf{B}$ is the magnetic field strength, and $\rho$ is the mass density. Generally, the momentum equation contains a Lorentz force term, which is $\propto (\nabla \times \mathbf{B}) \times \mathbf{B}$ and consists of two terms representing magnetic tension and magnetic pressure. We have assumed that magnetic pressure (associated with the term $-\nabla P_B/\rho$, where $P_B = B^2/8\pi$) is negligible compared to thermal pressure. The remaining magnetic term represents magnetic tension. Note that Equation (1) is only meaningful in the horizontal directions. It uses the fact that when $\rho$ is constant, hydrostatic equilibrium yields a linear relationship between $h$ and the pressure $P$,

$$P = P_0 + \rho g (h - z),$$

(2)

with $P_0$ being the reference pressure defined at $z = h$ and $z$ being the vertical coordinate.

A convenient hallmark of the shallow water model is that the mass continuity and thermodynamic equations are replaced by an equation for the shallow water height $h$,

$$\frac{\partial h}{\partial t} + \nabla \cdot (hv) = Q,$$

(3)

where the preceding equation is also only meaningful in the horizontal directions. The term $Q$ mimics the effect of radiative heating,

$$Q = \frac{h_{\text{eq}} - h}{t_{\text{rad}}},$$

(4)

where $h_{\text{eq}}$ is the “equilibrium shallow water height,” attained in the event of radiative equilibrium. Such an approach is often termed “Newtonian relaxation.” When $Q = 0$, Equation (3) is the 2D analogue of the mass continuity equation in three dimensions.

When magnetic fields are included, one needs to consider the three-dimensional (3D) magnetic induction equation. We assume the terms associated with ambipolar diffusion and the Hall effect to be negligible, at least for highly irradiated atmospheres (Perna et al. 2010). To render the problem analytically tractable, we assume that the magnetic diffusivity/resistivity ($\eta$) has no spatial dependence ($\nabla \eta = 0$). With these simplifications, we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times [\eta (\nabla \times \mathbf{B})]$$

$$= v (\nabla \cdot \mathbf{B}) - B (\nabla \cdot \mathbf{v}) + B \nabla v - v \nabla B$$

$$+ \eta \nabla^2 \mathbf{B} - \eta \nabla (\nabla \cdot \mathbf{B}) - \nabla \eta \times (\nabla \times \mathbf{B})$$

$$= v (\nabla \cdot \mathbf{B}) + B \nabla v + \eta \nabla^2 \mathbf{B}. $$

(5)

The $-\mathbf{B} (\nabla \cdot \mathbf{v})$ term always vanishes for a shallow water system due to the condition of incompressibility, while the $-\mathbf{v} \cdot \nabla \mathbf{B}$ term may be neglected for a system perturbed from rest and if the background magnetic field is constant (as it is then second order in magnitude). The terms involving $\eta$ describe the effect of magnetic drag. As a first approximation, we consider only the $\eta \nabla^2 \mathbf{B}$ term among the ones involving $\eta$. 

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**Figure 1.** Schematic describing the key governing equation in shallow water systems, both on the equatorial $\phi$-plane (pseudo-spherical geometry) and in full spherical geometry. The key quantity to solve for is the meridional (north–south) velocity, from which the zonal (east–west) velocity, shallow water height perturbation, and magnetic field perturbations straightforwardly follow. (A color version of this figure is available in the online journal.)

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**Table 2**

Comparison to Previous Analytical Work

| Reference            | Spherical Geometry? | HD: Forcing+Friction? | MHD: Free? | MHD: Forcing+Friction? |
|----------------------|---------------------|-----------------------|------------|------------------------|
| Matsuno (1966)       | N                   | Y                     | N          | N                      |
| Lindzen (1967)       | N                   | Y                     | N          | N                      |
| Longuet-Higgins (1968)| Y                   | N                     | N          | N                      |
| Gill (1980)          | N                   | Y                     | N          | N                      |
| Spitkovsky et al. (2002)| N               | Y                     | N          | N                      |
| Holton (2004)        | N                   | N                     | N          | N                      |
| Kudru & Cohen (2004) | N                   | N                     | N          | N                      |
| Vallis (2006)        | N                   | N                     | N          | N                      |
| Zakharov et al. (2006)| Y                 | N                     | Y          | N                      |
| Heng & Spitkovsky (2009)| Y               | N                     | Y          | N                      |
| Showman & Polvani (2011)| N              | Y                     | N          | N                      |
| Heng & Workman (current work) | Y         | Y                     | Y          | Y                      |

Note. HD: hydrodynamic. MHD: magnetohydrodynamic.
In atmospheres where a balance between collisional ionization and recombination is attained, $\eta$ has an exponential dependence on the temperature and may vary by orders of magnitude (Perna et al. 2010). Nevertheless, idealized models with constant $\eta$ provide a starting point for MHD shallow water investigations and are useful for elucidating the phase behavior of magnetic drag.

To simplify the problem, we will make separate approximations for the first and second derivatives of the components of the magnetic field strengths. The exact form of these approximations depends on whether we are considering a purely vertical/radial or horizontal background field.

### 2.2. Linearized Perturbed Equations in Cartesian Coordinates

We perturb the system about a state of rest $v_{x,y} = v'_{x,y}$ and $h = H + h'$, where $H$ is a constant and $H \gg h'$. In Cartesian coordinates, the equation for the shallow water height becomes

$$\frac{\partial h'}{\partial t} + H \left( \frac{\partial v'}{\partial x} + \frac{\partial v'}{\partial y} \right) = S - \omega_{rad} h',$$

(6)

where $\omega_{rad} \equiv 1/r_{rad}$ and the source term is $S \equiv (h_0 - H)\omega_{rad}$. We assume $S = S_0 h'$ and define $F_0 \equiv S_0 - \omega_{rad}$, as will be made clear in Section 3.1.5.
The forms of the momentum and magnetic induction equations depend on whether one is considering the idealized situation of a purely vertical or horizontal background magnetic field. We perturb the magnetic field about its constant background vertical or horizontal value. Even for an initial field that is purely vertical in nature, disturbances to it introduce finite horizontal perturbations.

2.2.1. Vertical Background Magnetic Field

We begin by denoting the components of the total magnetic field strength by $B_{x,y,z}$. We make the following approximations:

$$\frac{\partial v_{x,y}}{\partial z} = 0,$$
$$\frac{\partial B_{x,y}}{\partial x} \ll \frac{\partial B_z}{\partial y} \ll \frac{\partial B_z}{\partial z},$$
$$\frac{\partial B_{x,y}}{\partial z} = -\frac{B_{x,y}}{H}, \quad \frac{\partial B_z}{\partial z} = \frac{B_z}{H}, \quad \nabla^2 B_{x,y} \neq 0. \quad (7)$$

The first statement in Equation (7) is a property of the shallow water system. These approximations ensure that the dominant magnetic term in the momentum equation produces a restoring force that depends on the background vertical field (Heng & Spitkovsky 2009), while the induction equation retains a term that is the magnetic analogue of molecular diffusion. The negative signs associated with the horizontal field components are intended to provide a restoring, rather than a runaway, force to any perturbation of the magnetic field by fluid motion.

We represent the constant background magnetic field strengths with $B_{x,y}$. We perturb about the horizontal field strengths,

$$B_{x,y} = \bar{B}_{x,y} + b_{x,y}. \quad (8)$$

The background state is simple: at rest, with $\bar{B}_{x,y} = 0$ and $B_z = \bar{B}_z$. The vertical field is not perturbed.

With these assumptions, the linearized momentum equations read

$$\frac{\partial v_x'}{\partial t} = -\bar{g} \frac{\partial h'}{\partial x} + f v_y' + \nu \left( \frac{\partial^2 v_x'}{\partial x^2} + \frac{\partial^2 v_x'}{\partial y^2} \right)$$
$$- \omega_{\text{drag}} v_x' - \frac{\bar{B}_x}{4\pi \rho H} b_y',$$
$$\frac{\partial v_y'}{\partial t} = -\bar{g} \frac{\partial h'}{\partial y} - f v_x' + \nu \left( \frac{\partial^2 v_y'}{\partial x^2} + \frac{\partial^2 v_y'}{\partial y^2} \right)$$
$$- \omega_{\text{drag}} v_y' + \frac{\bar{B}_y}{4\pi \rho H} b_x', \quad (9)$$

where we have defined a drag frequency $\omega_{\text{drag}} \equiv t^{-1}_{\text{drag}}$. The quantity $f$ generalizes upon $2\Omega$ and is the first step toward including the effects of sphericity (e.g., Vallis 2006): $f = f_0 + \beta y$, \quad (10)

where $f_0 \equiv -2\Omega \cos \theta$ and $\theta$ denotes the co-latitude (or the polar coordinate in standard spherical coordinates). The quantity $\beta = 2\Omega \sin \theta / R$ is the coefficient of the first-order term in the series expansion of $f$. Physically, it is the gradient of the Coriolis term. At the poles, we recover $f = f_0$, known as the “f-plane” model. At the equator, we have $f = \beta y$. General consideration of $f$ with both terms is known as the “β-plane” approximation, which itself comes in two varieties. The first approximation is to seek sinusoidal solutions in both directions, including for $y$ (see Section 4). A better approximation is to seek sinusoidal solutions only in the $x$-direction and solve for the wave amplitudes as general functions of $y$, which we explore in Section 5.

For the linearized magnetic induction equation, the $\nu(\nabla \cdot B)$ term has a non-zero contribution, since a purely vertical field implies a magnetic monopole,

$$\frac{\partial b_x'}{\partial t} = \frac{\hat{B}_x}{H} v_y' + \eta \left( \frac{\partial^2 b_x'}{\partial x^2} + \frac{\partial^2 b_x'}{\partial y^2} \right). \quad (11)$$

While magnetic monopoles have never been seen in nature, this approximation provides a convenient way to study the effect of localized patches of atmospheres where a vertical magnetic field may exist.

Physically, the system starts out from a state of rest with a purely vertical background field that is coupled with the flow. Any motion of the flow in the horizontal direction induces horizontal magnetic field perturbations that resist this movement. Since atmospheres are hardly expected to be perfect conductors, this process is expected to be diffusive ($\nu \neq 0$).

2.2.2. Horizontal Background Magnetic Field

For a purely horizontal magnetic field strength, we assume $\nabla \cdot B = 0$. We set

$$\frac{\partial v_{x,y}}{\partial z} = 0, \quad (12)$$

which is, as stated previously, a property of the shallow water system. We set $\bar{B}_z = 0$. The background state satisfies

$$\bar{B}_x \frac{\partial \bar{B}_{x,y}}{\partial x} + \bar{B}_y \frac{\partial \bar{B}_{x,y}}{\partial y} = 0,$$
$$\frac{\partial \bar{B}_{x,y}}{\partial t} = \eta \nabla^2 \bar{B}_{x,y}. \quad (13)$$

Analogous expressions for the velocity would exist if one was perturbing about a moving state.

The linearized momentum equations read

$$\frac{\partial v_x'}{\partial t} = -\bar{g} \frac{\partial h'}{\partial x} + f v_y' + \nu \left( \frac{\partial^2 v_x'}{\partial x^2} + \frac{\partial^2 v_x'}{\partial y^2} \right)$$
$$- \omega_{\text{drag}} v_x' + \frac{\bar{B}_x}{4\pi \rho H} b_y',$$
$$\frac{\partial v_y'}{\partial t} = -\bar{g} \frac{\partial h'}{\partial y} - f v_x' + \nu \left( \frac{\partial^2 v_y'}{\partial x^2} + \frac{\partial^2 v_y'}{\partial y^2} \right)$$
$$- \omega_{\text{drag}} v_y' + \frac{\bar{B}_y}{4\pi \rho H} b_x', \quad (14)$$

The linearized magnetic induction equation is derived from the $B \cdot \nabla v$ term,

$$\frac{\partial b_x'}{\partial t} = \bar{B}_x \frac{\partial v_{x,y}'}{\partial x} + \bar{B}_y \frac{\partial v_{x,y}'}{\partial y} + \eta \left( \frac{\partial^2 b_x'}{\partial x^2} + \frac{\partial^2 b_x'}{\partial y^2} \right). \quad (15)$$

2.3. Equations in Spherical Coordinates

We employ the $(r, \theta, \phi)$ coordinate system, where $r$ is the radial coordinate, $\theta$ is the co-latitude, and $\phi$ is the longitude.
It is instructive to begin by stating the full equations in spherical coordinates,

\[
\begin{align*}
Dv_\theta \over Dt - \frac{\nu^2}{r \tan \theta} &= - \frac{g}{r \tan \theta} + 2\Omega v_\phi \cos \theta \\
+ v^2 \left[ \nabla^2 v_\theta - \frac{1}{r^2 \sin^2 \theta} \left( v_\theta + 2 \cos \theta \frac{\partial v_\theta}{\partial \phi} \right) \right] \\
- \omega_{\text{drag}} v_\theta + \frac{1}{4\pi \rho} \left[ B \nabla B_\theta + \frac{1}{r} \left( B_r B_\theta - \frac{B_\phi^2}{\tan \theta} \right) \right],
\end{align*}
\]

\[
\begin{align*}
Dv_\phi \over Dt + v_\theta \frac{\nu^2}{r \tan \theta} &= - \frac{g}{r \sin \theta} - 2\Omega v_\phi \cos \theta \\
+ v^2 \left[ \nabla^2 v_\phi - \frac{1}{r^2 \sin^2 \theta} \left( 2 \cos \theta \frac{\partial v_\phi}{\partial \phi} - v_\phi \right) \right] \\
- \omega_{\text{drag}} v_\phi + \frac{1}{4\pi \rho} \left[ B \nabla B_\phi + \frac{B_\phi}{r} \left( B_r + \frac{B_\theta}{\tan \theta} \right) \right].
\end{align*}
\]

We have made the approximation that the terms involving the radial velocity \(v_r\) and its gradients are sub-dominant and may be neglected. Myriad geometric terms appear from taking the momentum equations vanish for a system perturbed from rest, magnetic field. Similarly, the geometric terms appearing in the\(^{19}\) equation for the shallow water height perturbation may be stated without knowledge of the magnetic field geometry,

\[
\begin{align*}
\partial h' \over \partial t &+ \frac{H}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{\partial v_\phi'}{\partial \phi} \right] = F_0 h',
\end{align*}
\]

where we have evaluated the radial coordinate at the (constant) radius of the exoplanet \((r = R)\).

\[2.3.1. \text{Radial Background Magnetic Field}\]

We employ the same procedure as in the case of Cartesian coordinates. We first represent the components of the total magnetic field by \(B_{r,\theta,\phi}\) and make the following approximations:

\[
\begin{align*}
\frac{\partial v_{\phi,\phi}}{\partial r} &= 0, \\
\frac{\partial B_{r,\theta,\phi}}{\partial \theta} &\ll \frac{\partial B_r}{\partial r}, \frac{\partial B_{\theta,\phi}}{\partial \phi} \ll \frac{\partial B_r}{\partial r}, \\
\frac{\partial B_{\theta,\phi}}{\partial r} &= - \frac{B_\phi}{H}, \frac{\partial B_r}{\partial r} = \frac{B_r}{H}, \frac{\partial B_{\theta,\phi}}{\partial \phi} \neq 0. \quad (18)
\end{align*}
\]

We now represent the components of the background magnetic field by \(\bar{B}_{r,\theta,\phi}\). The background state is again at rest and with \(B_r = B_r\) and \(B_{\theta,\phi} = 0\). The radial field is not perturbed. The perturbed equations are

\[
\begin{align*}
\frac{\partial \bar{B}_{\phi,\phi}}{\partial t} &= \frac{\bar{B}_r v_\theta}{H} - \frac{\eta \bar{b}_{\phi,\phi}}{R^2 \sin^2 \theta} \\
&+ \frac{\eta}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial \bar{b}_{\phi,\phi}}{\partial \theta} + \frac{\partial^2 \bar{b}_{\phi,\phi}}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \bar{b}_{\phi,\phi}}{\partial \phi^2} \right), \\
\frac{\partial \bar{v}_\phi}{\partial t} &= - \frac{g}{R} \frac{\partial h'}{\partial \theta} + 2\Omega \bar{v}_\phi \cos \theta - \omega_{\text{drag}} \bar{v}_\phi \\
&+ \frac{v}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial \bar{v}_\phi}{\partial \theta} + \frac{\partial^2 \bar{v}_\phi}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \bar{v}_\phi}{\partial \phi^2} \right) \\
&- \frac{v}{R^2 \sin^2 \theta} \left( v_\phi' + 2 \cos \theta \frac{\partial \bar{v}_\phi}{\partial \phi} \right) - \frac{B_\phi}{4\pi \rho H}, \\
\frac{\partial \bar{v}_\phi}{\partial t} &= - \frac{g}{R \sin \theta} \frac{\partial \bar{b}_{\phi,\phi}}{\partial \phi} - 2\Omega \bar{v}_\phi \cos \theta - \omega_{\text{drag}} \bar{v}_\phi \\
&+ \frac{v}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial \bar{v}_\phi}{\partial \theta} + \frac{\partial^2 \bar{v}_\phi}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \bar{v}_\phi}{\partial \phi^2} \right) \\
&+ \frac{v}{R^2 \sin^2 \theta} \left( 2 \cos \theta \frac{\partial \bar{v}_\phi}{\partial \phi} - v_\phi' \right) - \frac{B_\phi}{4\pi \rho H}. \quad (19)
\end{align*}
\]

Several terms vanish because \(1/R \ll 1/H\).

\[2.3.2. \text{Horizontal Background Magnetic Field}\]

We set

\[
\frac{\partial v_{\phi,\phi}}{\partial r} = 0. \quad (20)
\]

The background state satisfies

\[
\begin{align*}
\bar{B}_r \sin \theta \frac{\partial \bar{B}_r}{\partial \theta} + \bar{B}_\phi \frac{\partial \bar{B}_r}{\partial \phi} &= \bar{B}_r^2 \cos \theta, \\
\bar{B}_r \sin \theta \frac{\partial \bar{B}_\phi}{\partial \theta} + \bar{B}_\phi \frac{\partial \bar{B}_\phi}{\partial \phi} &= -\bar{B}_r \bar{B}_\phi \cos \theta, \\
\frac{\partial \bar{B}_{\theta,\phi}}{\partial t} &= \eta \nabla^2 \bar{B}_{\theta,\phi} - \eta \bar{B}_{\theta,\phi}. \quad (21)
\end{align*}
\]

The first two expressions reduce to their Cartesian counterparts at the equator \((\theta = 90')\).

The perturbed equations are

\[
\begin{align*}
\frac{\partial \bar{b}_{\phi,\phi}}{\partial t} &= \frac{1}{R} \left( \bar{B}_r \frac{\partial \bar{v}_\phi}{\partial \theta} + \bar{B}_\phi \frac{\partial \bar{v}_\phi}{\partial \phi} - \frac{\bar{B}_\phi}{\tan \theta} \bar{v}_\phi' \right) \\
&+ \frac{\eta}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial \bar{b}_{\phi,\phi}}{\partial \theta} + \frac{\partial^2 \bar{b}_{\phi,\phi}}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \bar{b}_{\phi,\phi}}{\partial \phi^2} \right) \\
&- \frac{\eta}{R^2 \sin^2 \theta} \left( b_\phi' + 2 \cos \theta \frac{\partial b_{\phi,\phi}}{\partial \phi} \right),
\end{align*}
\]
\[ \frac{\partial b_\phi}{\partial t} = \frac{1}{R} \left( \tilde{B}_\phi \frac{\partial v_\phi}{\partial \theta} + \tilde{B}_\phi \frac{\partial t}{\sin \theta} + \tilde{B}_\phi v_\phi \frac{\partial \theta}{\tan \theta} \right) 
+ \frac{\eta}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial b_\phi}{\partial \theta} + \frac{\partial^2 b_\phi}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 b_\phi}{\partial \phi^2} \right) 
+ \frac{\eta}{R^2 \sin^2 \theta} \left( 2 \cos \theta \frac{\partial b_\phi}{\partial \phi} - b_\phi \right), \]
\[ \frac{\partial v'_0}{\partial t} = \frac{g}{R} \frac{\partial h'}{\partial \theta} + 2\Omega \nu_x \cos \theta - \omega_{\text{drag}} v'_0 
+ \frac{\nu}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial v'_0}{\partial \theta} + \frac{\partial^2 v'_0}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 v'_0}{\partial \phi^2} \right) 
- \frac{\nu}{R^2 \sin^2 \theta} (v'_0 + 2 \cos \theta \frac{\partial v'_0}{\partial \phi} - \frac{2 \tilde{B}_\phi b'_\phi}{\tan \theta}), \]
\[ \frac{\partial v'_\phi}{\partial t} = \frac{-g}{R \sin \theta} \frac{\partial h'}{\partial \phi} - 2\Omega \nu_x \cos \theta - \omega_{\text{drag}} v'_\phi 
+ \frac{\nu}{R^2} \left( \frac{1}{\tan \theta} \frac{\partial v'_\phi}{\partial \theta} + \frac{\partial^2 v'_\phi}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 v'_\phi}{\partial \phi^2} \right) 
+ \frac{\nu}{R^2 \sin^2 \theta} (2 \cos \theta \frac{\partial v'_\phi}{\partial \phi} - v'_\phi) 
+ \frac{1}{4\pi R} \left( \tilde{B}_\phi \frac{\partial b'_\phi}{\partial \theta} + \tilde{B}_\phi \frac{\partial b'_\phi}{\partial \phi} \right) 
+ \frac{1}{4\pi R \tan \theta} (\tilde{B}_\phi b'_\phi + \tilde{B}_\phi b'_\phi). \tag{22} \]

### 2.4. Seeking Wave Solutions

#### 2.4.1. Cartesian and Pseudo-Cartesian Geometries

For a perturbed quantity \( X \), we will generally seek wave solutions of the following form:
\[ X = X_0 \Psi_0 \Psi, \tag{23} \]
where \( X_0 \) is an arbitrary normalization factor and we have defined
\[ \Psi \equiv \exp [i(k_x x - \omega t)] = \exp (\omega t) \exp [i(k_x x - \omega_R t)], \tag{24} \]
where \( k_x \) is the wavenumber in the \( x \)- or zonal direction. For 1D models, we set \( X_0 = 1 \). For 2D Cartesian models, we set \( X_0 = \exp (ik_x y) \), where \( k_y \) is the wavenumber in the \( y \)- or meridional direction. For 2D pseudo-spherical models, we will allow \( X_0 \Psi_0 \) to retain a general functional dependence on \( y \) and solve for this dependence.

#### 2.4.2. Spherical Geometry

On a sphere, we define
\[ \Psi \equiv \exp [i(m\phi - \omega_0 t)], \tag{25} \]
where \( m \) is the azimuthal or zonal wavenumber, and \( \omega \) and \( \omega_0 \equiv 2\Omega \) are the wave frequency and time, respectively, in dimensionless units. Analogous to the pseudo-spherical case, we solve for the \( \theta \)-dependence of \( X_0 \Psi_0 \).

### 3. 1D MODELS

Due to the relative simplicity of the algebra involved, 1D models can provide clear insights into the effects of each piece of physics. Our main finding in this section is that some of these effects may couple in non-intuitive ways.

In order to compare the 1D and 2D models, we have cast the velocity and height perturbations in terms of the velocity amplitude \( (v_{x0}) \). As we will see in Section 5 and Section 6, it is more natural to use \( v_{x0} \) rather than the height perturbation amplitude \( (h_0) \), because the key governing equation is for the meridional velocity.

#### 3.1. Hydrodynamics

##### 3.1.1. Basic

To develop physical intuition, we review the most basic model: 1D with no rotation, viscosity, drag, or magnetic fields. Since this is a free system (with no sources of forcing or friction), the wave frequency is real, i.e., \( \omega = \omega_R \). The velocity amplitude is
\[ v_{x0} = \frac{g k_x h_0}{\omega_R} = \frac{\omega_R h_0}{k_x^2 H}. \tag{26} \]
The velocity perturbation has the solution
\[ v'_x = v_{x0} \cos (k_x x - \omega_R t), \tag{27} \]
while the height perturbation takes on the form
\[ h' = \frac{k_x}{\omega_R} \frac{H v_{x0}}{v_{x0}} \cos (k_x x - \omega_R t). \tag{28} \]
Since no sources of dissipation exist, the velocity and height perturbations are perfectly in phase. The dispersion relation is
\[ \omega_R = \pm (g H)^{1/2} k_x. \tag{29} \]
Only gravity waves exist in this most basic of systems and they have the special property of being non-dispersive (i.e., they possess the same group velocity regardless of wavelength).

##### 3.1.2. Kinematic (Molecular) Viscosity

For the second simplest model, we add a source of friction: kinematic viscosity \( (\nu) \). It is usually associated with molecular viscosity, although it has sometimes been used to mimic the presence of large-scale turbulent viscosity (e.g., Dobbs-Dixon & Lin 2008). Strictly speaking, the \( \nu \nabla^2 v \) term is non-negligible only when the Reynolds number is of the order of unity (or less).

The dispersion relation now has real and imaginary parts,
\[ i \omega^2 - \nu \omega^2 k_x^2 = i g H k_x^2. \tag{30} \]
To properly evaluate it, we need to substitute \( \omega = \omega_R + i \omega_1 \) into the preceding expression, which yields separate equations for the oscillatory \( (\omega_R) \) and decaying \( (\omega_1) \) parts of the wave solutions. The former has the solution
\[ \omega_R = \sqrt{g H k_x^2 - \frac{v k_x^2}{2}}^{1/2}, \tag{31} \]
implying that viscosity damps the wave frequency. Viscosity also tends to act on smaller scales, since
\[ \omega_1 = -\frac{v k_x^2}{2}, \tag{32} \]
on a characteristic viscous timescale of \( t_v \sim |\omega_I|^{-1} \). In fact, from examining the expression for \( \omega_R \), one may derive a viscous length scale \( (L_v) \) and Reynolds number \( (\mathcal{R}) \) at which gravity waves are completely damped out,

\[
L_v = \frac{\pi v}{c_0 R} \equiv \frac{c_0 L_v}{v} = \pi, \tag{33}
\]

where \( c_0 \equiv (g H)^{1/2} \).

The wave solutions are

\[
v_x' = v_{0x} \exp (\omega_I t) \cos (k_x x - \omega_R t),
\]

\[
h' = \frac{v_{0h}}{g k_x} \exp (\omega_I t) \times \left[ \omega_R \cos (k_x x - \omega_R t) - \frac{v k_x}{2} \sin (k_x x - \omega_R t) \right]. \tag{34}
\]

We will use this form for \( v_x' \) throughout Section 3. In the presence of viscosity, the velocity and height perturbations are now out of phase. Specifically, there is a viscous term that is 90° out of phase with the non-viscous terms.

A “hyper-viscosity” has sometimes been used in simulations of atmospheric circulation of exoplanets (e.g., Heng et al. 2011). It asserts that the viscous term in the momentum equation can be expressed as \( v \nabla^2 v \) with \( v \) having to take on the appropriate physical units for it to retain the physical units of acceleration. It follows that

\[
\omega_I = \frac{(-1)^{n_v/2} v k_x n_v}{2}, \quad n_v \in 2\mathbb{Z}. \tag{35}
\]

Oddly enough, it means that certain values of \( n_v \) result in growing wave solutions that are unstable, e.g., \( n_v = 8 \) (a value often used in 3D simulations). Odd values of \( n_v \) are unphysical, at least for the shallow water system, because they produce spurious oscillatory modes.

### 3.1.3. Rayleigh Drag

The term \(-v/t_{drag}\) is often inserted into the momentum equation to mimic large-scale friction in atmospheres. It is called “Rayleigh drag.” On Earth, it mimics the effect of the planetary boundary layer, where the atmosphere transitions from a “free slip” to a “no slip” boundary condition. In atmospheric circulation simulations of exoplanets, it has been used to mimic the effects of magnetic drag (Perna et al. 2010; Rauscher & Menou 2013). A key difference between Rayleigh drag and molecular viscosity is that the former acts equally on all scales, since

\[
\omega_I = -\frac{1}{2t_{drag}} = -\frac{\omega_{1drag}}{2}. \tag{36}
\]

Like viscosity, Rayleigh drag damps the frequency of gravity waves,

\[
\omega_R = \pm \left[ g H k_x^2 - \left( \frac{\omega_{1drag}}{2} \right)^2 \right]^{1/2}. \tag{37}
\]

Analogous to the viscous length scale, there is a drag length scale on which gravity waves are completely damped out,

\[
L_{drag} = 4\pi c_0 t_{drag}. \tag{38}
\]

The Taylor number is then \( T = (t_{drag} c_0 / L_{drag})^2 = 1/16\pi^2 \ll 1 \), as expected for flows under strong drag.

Again, analogous to the case of viscosity, Rayleigh drag damps the wave amplitudes and introduces an out-of-phase component to the wave solutions, since

\[
h' = \frac{v_{0h}}{g k_x} \exp (\omega_I t) \times \left[ \omega_R \cos (k_x x - \omega_R t) - \frac{\omega_{drag}}{2} \sin (k_x x - \omega_R t) \right], \tag{39}
\]

with the difference being that \( \omega_{drag} \) has no \( k_x \)-dependence.

### 3.1.4. Molecular Viscosity and Rayleigh Drag

When both viscosity and Rayleigh drag are included, the mathematical form of the dispersion relation and wave solutions are identical, except that one replaces \( v k_x^2 \) or \( \omega_{drag} \) by

\[
\omega_r \equiv v k_x^2 + \omega_{drag}. \tag{40}
\]

Friction acting upon the flow now has two components: a scale-free part and one directed strongly at small scales.

### 3.1.5. Forcing

In the presence of external forcing (e.g., stellar irradiation), the equation for the shallow water height has a non-zero source term \( \mathcal{Q} \). In its perturbed form, it involves two terms: \( S - \omega_r h' \). In seeking wave solutions, we adopt

\[
S = S_0 h_0 \Psi, \tag{41}
\]

where \( S_0 \) is a dimensionless number. As expected, forcing results in a growing wave solution, since

\[
\omega_r = \frac{S_0 - \omega_{rad}}{2} \equiv \frac{F_0}{2}. \tag{42}
\]

We have specifically defined \( F_0 \equiv S_0 - \omega_{rad} \) as the “forcing.” It vanishes when heating is balanced by radiative cooling \((S_0 = \omega_{rad})\).

Forcing introduces an out-of-phase component to the wave solution,

\[
h' = \frac{v_{0h}}{g k_x} \exp (\omega_I t) \times \left[ \omega_R \cos (k_x x - \omega_R t) - \frac{F_0}{2} \sin (k_x x - \omega_R t) \right]. \tag{43}
\]

Non-intuitively, forcing decreases the wave frequency, since

\[
\omega_R = \pm \left( g H k_x^2 - \frac{F_0^2}{4} \right)^{1/2}. \tag{44}
\]

Gravity waves are most strongly affected by forcing on a length scale

\[
L_{F} = \frac{4\pi c_0}{F_0}. \tag{45}
\]

No stable solution for the flow exists unless \( F_0 = 0 \).

### 3.1.6. Forcing with Molecular Viscosity and Rayleigh Drag

With the framework developed in Sections 3.1.1–3.1.5, we can now explore a 1D hydrodynamic forced system with friction. This system admits both growing and decaying wave solutions, since

\[
\omega_r = \frac{F_0 - \omega_{rad}}{2}. \tag{46}
\]
Globally, irradiated atmospheres are in a state of equipoise—forcing balances friction, such that there is stability. In such a balanced flow, we have \( \omega_I = 0 \) and \( F_0 = \omega_c \). Globally, a stable atmosphere needs to be damped exactly at the same frequency at which it is being forced. In 3D nonlinear systems, equipoise might be absent at local scales. In our simplified 1D linear model, we assume equipoise to be both a local and a global condition. By enforcing \( F_0 = \omega_c \), the wave frequency becomes

\[
\omega_R = \pm \left[ gH k_x^2 - \left( \frac{F_0 + \omega_c}{2} \right)^2 \right]^{1/2}.
\]

(47)

Forcing and friction combine to damp the wave frequency more strongly than if either was acting alone. In a balanced flow, we have

\[
h' = \frac{v_{\text{in}}}{g k_x} [\omega_R \cos(\omega_c x - \omega_R t) - \omega_c \sin(\omega_c x - \omega_R t)].
\]

(48)

### 3.2. Magnetohydrodynamics

#### 3.2.1. Basic (Ideal MHD)

In one dimension, the simplest instance of an MHD system is one in which all forms of friction (molecular viscosity, Rayleigh drag, magnetic drag) are absent. (By definition, rotation is a 2D phenomenon.) For a vertical background field, the dispersion relation reads

\[
\omega_R = \pm \left[ gH k_x^2 + \left( \frac{v_A}{H} \right)^2 \right]^{1/2},
\]

(49)

where \( v_A = \sqrt{B_0/2\mu_0\rho} \) is the Alfvén speed. Evidently, the presence of a magnetic field enhances the wave frequency relative to the hydrodynamic case. These are called “magneto-gravity waves.” Their wave solutions are

\[
h' = \frac{k_xH v_{\text{in}}}{\omega_R} \cos(\omega_c x - \omega_R t),
\]

\[
b'_x = -\frac{k_x B_0 v_{\text{in}}}{\omega_R} \sin(\omega_c x - \omega_R t).
\]

(50)

The height and velocity perturbations remain in phase, but they are out of phase with the magnetic field by 90°.

The qualitative effects of a horizontal background field are equivalent, except that they act more strongly at smaller scales, and the Alfvén speed is now \( v_A = \sqrt{B_0/2\mu_0\rho} \).

\[
\omega_R = \pm \left[ gH k_x^2 + (v_A k_x)^2 \right]^{1/2},
\]

\[
h' = \frac{k_xH v_{\text{in}}}{\omega_R} \cos(\omega_c x - \omega_R t),
\]

\[
b'_x = -\frac{k_x B_0 v_{\text{in}}}{\omega_R} \cos(\omega_c x - \omega_R t).
\]

(51)

#### 3.2.2. Magnetic Drag

A 1D system with magnetic drag (\( \eta \neq 0 \)) is the first example where an analytical solution for \( \omega_I \) cannot be obtained, although the equation describing it may be stated (for a vertical background field),

\[
8\omega_I^3 + 8\eta k_x^2 \omega_I^2 + 2\omega_I \left[ gH k_x^2 + \left( \frac{v_A}{H} \right)^2 + (\eta k_x^2) \right]
\]

\[
+ \eta \left( k_x v_A \right)^2 = 0.
\]

(52)

By examining the preceding equation for small and large \( \omega_I \), one may infer that two physically meaningful solutions of \( \omega_I < 0 \) exist. The oscillatory part of the wave solution has the frequency

\[
\omega_R = \pm \left[ gH k_x^2 + \left( \frac{v_A}{H} \right)^2 \right]^{1/2} \left( 3\omega_I^2 + 2\eta k_x^2 \right)^{1/2}.
\]

(53)

Even without obtaining an analytical solution for \( \omega_I \), we see that in a balanced flow (\( \omega_I = 0 \)) the 1D free MHD limit is obtained—the expression for \( \omega_R \) does not involve \( \eta \).

For a horizontal background field, the governing equations for \( \omega_I \) and \( \omega_R \) are similar,

\[
8\omega_I^3 + 8\eta k_x^2 \omega_I^2 + 2\omega_I \left[ gH k_x^2 + (v_A k_x)^2 + (\eta k_x^2) \right]
\]

\[
+ \eta (v_A k_x^2)^2 = 0,
\]

\[
\omega_R = \pm \left[ gH k_x^2 + (v_A k_x)^2 + \omega_I \left( 3\omega_I + 2\eta k_x^2 \right) \right]^{1/2}.
\]

(54)

In other words, a balanced flow may only exist when \( \eta = 0 \). Our derivation is somewhat unsatisfactory, because it does not allow us to study the effects of \( \eta \) on the phases of the wave solutions.

#### 3.2.3. Forcing

When forcing is added to the 1D MHD system, a similar situation occurs when magnetic drag is present. A cubic equation for \( \omega_I \) ensues (for a vertical background field):

\[
8\omega_I^3 - 8F_0 \omega_I^2 + 2\omega_I \left[ F_0^2 + gH k_x^2 + \left( \frac{v_A}{H} \right)^2 \right]
\]

\[- F_0 gH k_x^2 = 0.
\]

(55)

When \( F_0 = 0 \), the solution for \( \omega_I \) is unphysical (\( \omega_I^2 < 0 \)) and may be disregarded. The other two solutions, at small and large \( \omega_I \), are positive, implying growing modes as expected. The wave frequency is

\[
\omega_R = \pm \left[ gH k_x^2 + \left( \frac{v_A}{H} \right)^2 \right]^{1/2} \left( 3\omega_I - 2F_0 \right)^{1/2}.
\]

(56)

No balanced flow exists unless \( F_0 = 0 \). Again, we caution that we are interpreting this to be both a local and a global condition for our simplified 1D linear models.

With a horizontal background field, we reach a similar conclusion, since

\[
8\omega_I^3 - 8F_0 \omega_I^2 + 2\omega_I \left[ F_0^2 + gH k_x^2 + (v_A k_x)^2 \right]
\]

\[- F_0 gH k_x^2 = 0,
\]

\[
\omega_R = \pm \left[ gH k_x^2 + (v_A k_x)^2 + \omega_I \left( 3\omega_I - 2F_0 \right) \right]^{1/2}.
\]

(57)

#### 3.2.4. Molecular Viscosity and Rayleigh Drag

Before we examine the 1D MHD case with forcing and all three forms of friction, it is instructive to examine the
basic magnetic case with just hydrodynamic friction present. Specifically, we set \( F_0 = \eta = 0 \). The wave solutions are decaying, since (for a vertical background field)

\[
\omega_1 = -\frac{\omega_v}{2},
\]

while the wave frequencies are damped,

\[
\omega_R = \pm \left[ g H k_x^2 + \left( \frac{v_A}{H} \right)^2 - \left( \frac{\omega_v}{2} \right)^2 \right]^{1/2}.
\]

The wave solutions are

\[
\begin{align*}
    h' &= \frac{k_x H v_{x0}}{g H k_x^2 + (v_A k_x)^2} \exp(\omega_1 t) \\
    &\quad \times \left[ \omega_R \cos(k_x x - \omega_R t) - \frac{\omega_v}{2} \sin(k_x x - \omega_R t) \right], \\
    b'_I &= -\frac{k_x B_z v_{x0}}{g H k_x^2 + (v_A k_x)^2} \exp(\omega_1 t) \\
    &\quad \times \left[ \omega_R \cos(k_x x - \omega_R t) + \omega_v \sin(k_x x - \omega_R t) \right].
\end{align*}
\]

For a horizontal background field, we have

\[
\omega_1 = -\frac{\omega_v}{2},
\]

\[
\omega_R = \pm \left[ g H k_x^2 + (v_A k_x)^2 - \left( \frac{\omega_v}{2} \right)^2 \right]^{1/2},
\]

\[
\begin{align*}
    h' &= \frac{k_x H v_{x0}}{g H k_x^2 + (v_A k_x)^2} \exp(\omega_1 t) \\
    &\quad \times \left[ \omega_R \cos(k_x x - \omega_R t) - \frac{\omega_v}{2} \sin(k_x x - \omega_R t) \right], \\
    b'_I &= -\frac{k_x B_z v_{x0}}{g H k_x^2 + (v_A k_x)^2} \exp(\omega_1 t) \\
    &\quad \times \left[ \omega_R \cos(k_x x - \omega_R t) + \omega_v \sin(k_x x - \omega_R t) \right].
\end{align*}
\]

The preceding equation has the curious property that when \( \omega_R = 0 \), one obtains an expression relating \( v \) and \( \omega_v \), suggesting that magnetic drag and hydrodynamic friction are balancing out each other. In a balanced flow, one obtains

\[
\omega_R^2 = \frac{g H k_x^4 \eta}{\eta k_x^2 + \omega_v},
\]

\[
\begin{align*}
    h' &= \frac{k_x H v_{x0}}{\omega_R} \cos(k_x x - \omega_R t), \\
    b'_I &= -\left( \frac{B_z v_{x0}}{H} \right) \omega_R \cos(k_x x - \omega_R t) \times \left[ \eta k_x^2 \cos(k_x x - \omega_R t) - \omega_R \sin(k_x x - \omega_R t) \right].
\end{align*}
\]

As anticipated, there is an in-phase component to the magnetic field perturbation (due solely to magnetic drag) and an out-of-phase component that depends on all forms of friction. In other words, magnetic and hydrodynamic drag possess different phase signatures, producing magnetic field perturbations that negate each other. Such a finding suggests that one should not use hydrodynamic drag as a proxy for magnetic drag (e.g., Perna et al. 2010).

For a horizontal background field, we get similar results:

\[
\omega_R^2 = \frac{g H k_x^4 \eta}{\eta k_x^2 + \omega_v},
\]

\[
\begin{align*}
    h' &= \frac{k_x H v_{x0}}{\omega_R} \cos(k_x x - \omega_R t), \\
    b'_I &= -\left( \frac{B_z v_{x0}}{H} \right) \omega_R \cos(k_x x - \omega_R t) \times \left[ \eta k_x^2 \cos(k_x x - \omega_R t) + \omega_v \sin(k_x x - \omega_R t) \right].
\end{align*}
\]

Here, both components in the magnetic field perturbation are out of phase with the height and velocity perturbations.

In summary, magnetic and hydrodynamic drag may negate each other in a balanced flow.

### 3.2.5. Friction (Molecular Viscosity, Rayleigh Drag, and Magnetic Drag)

When both hydrodynamic (viscosity and Rayleigh drag) and magnetic sources of friction are included, something curious happens. The wave frequency becomes (for a vertical background field)

\[
\omega_R^2 = g H k_x^2 + \left( \frac{v_A}{H} \right)^2 + \omega_v \eta_k^2 + \omega_v \left[ 2 (\eta k_x^2 + \omega_v) + 3 \omega_1 \right],
\]

implying that an extra contribution \((\omega_v \eta k_x^2)\) is present even for a balanced flow \((\omega_v = 0)\). For this term to be non-vanishing, both magnetic and hydrodynamic drag have to be present. The growing or decaying part of the wave solution is described by

\[
8 \omega_1^3 + 8 (\eta k_x^2 + \omega_v) \omega_1^2 + 2 \left[ g H k_x^2 + \left( \frac{v_A}{H} \right)^2 + \omega_v \eta_k^2 + (\eta k_x^2 + \omega_v)^2 \right] \omega_1 + (\eta k_x^2 + \omega_v) \left[ g H k_x^2 + \left( \frac{v_A}{H} \right)^2 + \omega_v \eta_k^2 \right] - g H k_x^4 \eta = 0.
\]
In the limit of \( F_0 = \omega_1 = \omega_0 = 0 \), one may also verify that a balanced flow only exists when \( \eta = 0 \), since \( \eta k_x^2 (g H k_x^2 - \omega_0^2) = 0 \). The second expression in Equation (66) yields the value of the fourth parameter, via

\[
g H k_x^2 (\omega_1 - F_0) + \left[ \frac{v_A}{H} \right]^2 + \omega_1 \eta k_x^2 \left( \omega_1 + \eta k_x^2 \right) = F_0 (\omega_1 + \eta k_x^2) (\omega_1 + \eta k_x^2 - F_0) = 0. \tag{67}
\]

Algebraic amenability suggests that it is easiest to specify \( F_0 \), \( \omega_0 \), and \( \eta \) and use the preceding expression to solve for \( v_A \).

The wave solutions are

\[
h' = \frac{k_x H v_{so}}{\omega_R^2 + F_0^2} \left[ \omega_R \cos (k_x x - \omega_R t) - F_0 \sin (k_x x - \omega_R t) \right],
\]

\[
b_x' = -\frac{B_x v_{so}}{H \left[ (\eta k_x^2)^2 + \omega_R^2 \right]} \times \eta k_x^2 \cos (k_x x - \omega_R t) - \omega_R \sin (k_x x - \omega_R t))]. \tag{68}
\]

For a horizontal background field, the algebra is somewhat more tractable. The oscillatory part of the wave solutions is described by

\[
\omega_R^2 = k_x^2 \left[ g H + v_A^2 + \eta (\omega_1 - F_0) \right] - F_0 \omega_0 + \omega_0 \left[ 3 \omega_0 + 2 (\eta k_x^2 + \omega_0 - F_0) \right], \tag{69}
\]

from which it is possible to obtain an expression for \( \omega_R \) for a balanced flow. Requiring \( \omega_R^2 \geq 0 \) sets a condition on the strength of forcing,

\[
F_0 \leq \frac{k_x^2 \left[ g H + v_A^2 + \eta \omega_0 \right]}{\eta k_x^2 + \omega_0}. \tag{70}
\]

If balanced flow is interpreted as a local condition, then the preceding equation specifies the maximum amount of forcing that can be applied to maintain it. An equation for \( \omega_R \) may be obtained,

\[
8 \omega_R^3 + 8 (\omega_0 + \eta k_x^2 - F_0) \omega_R^2 + 2k_x^2 \left[ g H + v_A^2 + \eta (\omega_1 - F_0) \right] \omega_R + 2 \left[ (\omega_0 + \eta k_x^2 - F_0)^2 - F_0 \omega_0 \right] \omega_R + (\omega_0 + \eta k_x^2 - F_0) \left[ k_x^2 \left[ g H + v_A^2 + \eta (\omega_1 - F_0) \right] \right] - F_0 \omega_0 \left( \omega_0 + \eta k_x^2 - F_0 \right) - g H k_x^4 \eta = 0. \tag{71}
\]

Even in a balanced flow (\( \omega_1 = 0 \)), the preceding expression demonstrates that non-vanishing terms remain because of the various couplings between forcing and friction. The wave solutions are

\[
h' = \frac{k_x H v_{so}}{\omega_R^2 + F_0^2} \left[ \omega_R \cos (k_x x - \omega_R t) - F_0 \sin (k_x x - \omega_R t) \right],
\]

\[
b_x' = -\frac{k_x B_x v_{so}}{H \left[ (\eta k_x^2)^2 + \omega_R^2 \right]} \times \omega_R \cos (k_x x - \omega_R t) + \eta k_x^2 \sin (k_x x - \omega_R t). \tag{72}
\]

## 4. 2D MODELS (CARTESEAN)

Rotation is intrinsically a 2D effect and exerts non-trivial effects on the atmospheric flow. It generally modifies the condition corresponding to a balanced flow away from its 1D counterpart. In this section, we seek wave solutions with \( \Psi_0 = \exp(i k_y y) \) (see Section 2.4). This approach of seeking sinusoidal solutions in both the \( x \)- and \( y \)-directions is less general than solving for the \( y \)-dependence of each quantity (see Section 5). We shall simply refer to it as the “\( \beta \)-plane approximation.” Our main goal in this section is to extract dispersion relations in the presence of rotation, in anticipation of the more general treatments in Sections 5 and 6.

Generally, we find that the hydrodynamic and MHD systems are described by dispersion relations with the same mathematical forms, but with the wave frequency generalized to complex frequencies involving forcing, friction, and magnetic field strength.

### 4.1. Forcing with Hydrodynamic Friction

A useful mathematical trick is to is to differentiate the equation for \( v' \) with respect to \( y \) before seeking wave solutions (Heng & Spitkovsky 2009),

\[
\frac{\partial^2 v'}{\partial t^2} = -g \frac{\partial^2 h'}{\partial x \partial y} + \beta v' + f \frac{\partial v'}{\partial y} + v \left( \frac{\partial^3 v'}{\partial x^2 \partial y} + \frac{\partial^3 v'}{\partial y^3} \right) - \omega_{\text{drag}} \frac{\partial v'}{\partial y}, \tag{73}
\]

where we note that \( \partial f/\partial y = \beta \). Along with the other two perturbed equations, we seek wave solutions and arrange them into the following form:

\[
\hat{M} \begin{pmatrix} v_{so} \\ v_{yo} \\ h_0 \end{pmatrix} = 0.
\]

The preceding expression is non-trivial only if the determinant of the matrix,

\[
\hat{M} = \begin{pmatrix} k_x \omega_0 - \beta + i f k_y & -g k_x k_y \\ f & -i \omega_0 & i g k_x \\ k_x H & k_y H & -\omega_F \end{pmatrix},
\]

is zero. To express the matrix more compactly, we have defined

\[
\begin{align*}
\omega_0 & \equiv v k^2 + \omega_{\text{drag}}, \\
\omega_0 & \equiv \omega + i \omega_v, \\
\omega_F & \equiv \omega - i F_0,
\end{align*}
\]

where \( k^2 = k_x^2 + k_y^2 \).

When one sets \( \det \hat{M} = 0 \), it follows that

\[
\begin{align*}
\omega_0^2 + \omega_0 \left[ g H k_x^2 + f^2 + \omega_0 (\omega_0 - 2 F_0) - 3 \omega_0^2 \right] + \beta f \omega_0 \frac{\omega_0}{k_y} - F_0 (\omega_0^2 + f^2) + g H k_x^2 \omega_0 & = 0, \\
\omega_0^2 - \omega_0 \left[ g H k_x^2 + f^2 + \omega_0 (\omega_0 - 2 F_0) + 3 \omega_0^2 \right] - 2 (2 \omega_0 - F_0) \omega_R \omega_1 - g H k_x \beta - \frac{\beta f (F_0 - \omega_0)}{k_y} & = 0. \tag{75}
\end{align*}
\]

As in one dimension, the first equation describes how to obtain the conditions for a balanced flow (\( \omega_0 = 0 \)). The second equation
is the dispersion relation for $\omega_R$. In general, it is challenging to solve this pair of coupled equations, but we are primarily interested in the $\omega = 0$ limit, for which the first expression in Equation (75) becomes

$$ F_0 \omega_R^2 + \frac{\beta f \omega_R}{k_y} + F_0 \left[ (F_0^2 + f^2) - g H k_y^2 \right] = 0. \quad (76) $$

An important implication of the preceding expression is that $F_0 = \omega_R$ is generally not a consequence of $\omega_R = 0$. However, when one additionally sets $k_z = 1$, one obtains $\beta f \omega_R = 0$, which implies that $F_0 = \omega_R$ is a consequence of $\omega_R = 0$ either at the poles ($\beta = 0$, since $\beta = 2Q \sin \theta / R$) or when rotation is absent ($f = 0$). These inferences are also obtained in free systems, since setting $F_0 = 0$ yields $\beta f \omega_R = 0$.

In general, given specified values of $F_0$ and $\omega_R$, one may solve the coupled pair of equations in Equation (75) for the values of $\omega_R$ and $\omega_I$. Such an approach yields solutions for flows that are generally unbalanced. In this four-dimensional space, there exist solutions for balanced flows ($\omega_R = 0$). An easier way to proceed is to assume a balanced flow by setting $\omega_R = 0$ and solving the equations in Equation (75). One obtains the usual three values of $\omega_R$ for the eastward- and westward-propagating Poincaré waves and the westward-propagating Rossby waves (e.g., Kundu & Cohen 2004). Poincaré waves are rotationally modified gravity waves, while Rossby waves arise from non-constant rotation across latitude ($\beta \neq 0$). Balanced flow exists for appropriate combinations of $F_0$ and $\omega_R$ values. We will see that in the pseudo-spherical (Section 5) and spherical (Section 6) cases, the algebra is intractable for evaluating balanced flows and one must instead resort to solving for the steady state of an atmosphere ($\omega_R = \omega_I = 0$).

### 4.2. Magnetohydrodynamics (Vertical Background Field)

We employ the same mathematical trick with one of the perturbed equations,

$$ \frac{\partial^2 v_y'}{\partial t \partial y} = \frac{\partial^2 h'}{\partial x \partial y} + \beta v_y' + f \frac{\partial v_y'}{\partial y} - \frac{\bar{B}_z}{4\pi \rho_H} \frac{\partial b_z'}{\partial y} + v \left( \frac{\partial^3 v_y'}{\partial x^2 \partial y} + \frac{\partial^3 v_y'}{\partial y^3} \right) - \omega_{\text{drag}} \frac{\partial v_y'}{\partial y}. \quad (77) $$

Seeking wave solutions and constructing the $\hat{\mathcal{M}}$ matrix as before, we obtain

$$ \hat{\mathcal{M}} = \begin{pmatrix} \kappa_x \omega_{B_0} & - (\beta + i \kappa_y) & -g k_x k_y & \frac{\bar{B}_z}{2\pi \rho_H} \frac{\partial b_z'}{\partial y} \\ \kappa_y \omega_{B_0} & i \omega_{B_0} & i g k_y & -\omega_{\text{drag}} \frac{\partial v_y'}{\partial y} \end{pmatrix}, $$

where we have defined

$$ \omega_H \equiv \omega + i \eta k^2, $$

$$ \omega_B \equiv \omega + i \omega_{B_0}, $$

$$ \omega_{BH} \equiv \omega + i \omega_{B_0}. \quad (78) $$

Setting $\det \hat{\mathcal{M}} = 0$ yields

$$ i k_x \omega_{B_0}^2 \omega_R - g H k_y (i^2 \omega_{B_0} + k_x \beta - \beta \omega_{B_0} f - i f^2 k_y, \omega = 0. \quad (79) $$.  

To proceed, we find it convenient to first write

$$ \omega_B = \zeta_R + i \zeta_I, \quad (80) $$

en route to obtaining a pair of equations for $\omega_R$ and $\omega_I$,

$$ \omega_R^3 - 3 \omega_R^2 \omega_I - (2 \zeta_R - F_0) (\omega_R^2 - \omega_I^2) + 4 \zeta_0 \omega_R \omega_I \omega_l \left[ \omega_R \left( F_0 - \zeta_R \right) + \zeta_I^2 \right] - 2 \zeta_0 \omega_R \omega_I \left( F_0 - \zeta_0 \right) \omega_I + g H k_y^2 (\omega_0 + \zeta_R - \beta f \omega_R \omega_I) + \frac{\partial b_z'}{\partial y}, \omega = 0, $$

$$ \omega_R^3 - 3 \omega_R^2 \omega_I - 2 (2 \zeta_R - F_0) \omega_R \omega_I - 2 \zeta_0 (\omega_R^2 - \omega_I^2) + \omega_R \left[ \omega_R \left( F_0 - \zeta_R \right) + \zeta_I^2 \right] - 2 \zeta_0 \omega_R \omega_I \left( F_0 - \zeta_0 \right) \omega_I + g H k_y^2 (\omega_0 - \omega_R) - f^2 \omega_R \omega_I \left( F_0 - \omega_I \right), $$

$$ - 2 \zeta_0 \omega_R \omega_I \left( F_0 - g H k_y \beta = 0. \quad (81) $$

It can then be shown, after the fact, that

$$ \zeta_R \equiv \omega_0 + \frac{(\mathcal{V}_A)^2}{\omega_R^2} \frac{\eta k^2 + \omega_0}{\omega_R^2 + (\eta k^2 + \omega_0)^2}, $$

$$ \zeta_I \equiv \omega_R \left( \frac{\mathcal{V}_A}{\omega_R} \right)^2 \frac{\omega_0 \omega_R}{\omega_R^2 + (\eta k^2 + \omega_0)^2}. \quad (82) $$

by substituting the expression for $\omega_0$ into $\omega_B$ and separating out the real and imaginary components.

At this point, the algebra is becoming intractable, since the pair of expressions in Equation (81) no longer takes the form of a cubic equation.

### 4.3. Magnetohydrodynamics (Horizontal Background Field)

For a horizontal background field, the equation for the velocity perturbation in the $x$-direction reads

$$ \frac{\partial^2 v_x'}{\partial t \partial y} = -g \frac{\partial^2 h'}{\partial x \partial y} + \beta v_x' + f \frac{\partial v_x'}{\partial y} + \frac{\bar{B}_z}{4\pi \rho_H} \frac{\partial b_z'}{\partial y} + \frac{\bar{B}_z}{4\pi \rho_H} \frac{\partial^2 b_x'}{\partial y^2} + v \left( \frac{\partial^3 v_x'}{\partial x^2 \partial y} + \frac{\partial^3 v_x'}{\partial y^3} \right) - \omega_{\text{drag}} \frac{\partial v_x'}{\partial y}. \quad (83) $$

Seeking wave solutions, we find that the $\hat{\mathcal{M}}$ matrix is identical to the one derived for the vertical background field, except that

$$ \mathcal{V}_A \mathbf{k} = \frac{\bar{B}_z \kappa_x + \bar{B}_x \kappa_y}{2(\pi \rho_H)^{1/2}}, $$

$$ \omega_B \equiv \omega, \quad \omega_{BH} \equiv \omega + i \omega_{B_0}. \quad (84) $$

The quantities $\omega_H$ and $\omega_{BH}$ have the same definitions as in the case of a vertical background field. Again, we first write $\omega_B = \zeta_R + i \zeta_I$ and evaluate $\det \hat{\mathcal{M}} = 0$. Since $\hat{\mathcal{M}}$ is mathematically identical to the previous situation, we recover the pair of expressions in Equation (81), although the definitions for $\zeta_R$ and $\zeta_I$ differ slightly:

$$ \zeta_R \equiv \omega_0 + \frac{(\mathcal{V}_A \mathbf{k})^2}{\omega_R^2} \frac{(\eta k^2 + \omega_0)}{\omega_R^2 + (\eta k^2 + \omega_0)^2}, $$

$$ \zeta_I \equiv \frac{\omega_B (\mathcal{V}_A \mathbf{k})^2}{\omega_R^2 + (\eta k^2 + \omega_0)^2}. \quad (85) $$
5.2D PSEUDO-SPHERICAL MODELS
(EQUATORIAL β-PLANE)

The first step toward considering the effects of sphericity is to allow for the physical quantities to have an arbitrary, rather than a sinusoidal, functional dependence on the latitude (y). Specifically, we solve for the functional dependence of \( X_0 \Psi_0 \) on \( y \) (see Section 2.4). This approximation captures the essence of being near the equator on a sphere without actually working in full spherical coordinates. We shall refer to this approach as the “equatorial β-plane approximation.”

In this section, we choose to non-dimensionalize our quantities by defining the following characteristic length \( (L_0) \) and time \( (t_{\text{dyn}}) \) scales as

\[
L_0 = \frac{c_0}{\beta}, \quad t_{\text{dyn}} = (c_0\beta)^{-1/2},
\]

following Matsuno (1966). The characteristic velocity is \( c_0 \equiv (gH)^{1/2} \). Furthermore, the shallow water height perturbation is normalized by \( H \). Near the equator, we have \( f \approx \beta y \).

Generally, we find that to obtain the dispersion relations, one needs to evaluate the square root of a complex quantity, \( \zeta_R + i\zeta_I \), which requires the use of De Moivre’s formula (e.g., Arfken & Weber 1995). The expressions for \( \zeta_R \) and \( \zeta_I \) in terms of the other quantities, are generally tedious. To avoid introducing more notation than is necessary, we will generally make use of the separation functions \( \zeta_R \) and \( \zeta_I \) when invoking De Moivre’s formula, but we note that their exact definitions will vary between models.

5.1. Relationship to the Quantum Harmonic Oscillator

Consider the general equation for the function \( F = F(z) \),

\[
\frac{d^2 F}{d z^2} - (A z^2 - B z - C) F = 0,
\]

where the coefficients \( A \), \( B \), and \( C \) may be complex. By completing the square, this equation may be written as

\[
\frac{d^2 F}{d z^2} - A \left( z - \frac{B}{2A} \right)^2 F + \left( \frac{B^2}{4A} + C \right) F = 0.
\]

The next step is to rescale \( z \). It turns out that the choice of rescaling is important. If we pick

\[
z \to 2^{1/2} A^{1/4} \left( z - \frac{B}{2A} \right),
\]

then Equation (88) becomes one of the Weber equations,

\[
\frac{d^2 F}{d z^2} - \frac{z^2 F}{4} + A' F = 0.
\]

However, if we instead pick

\[
z \to A^{1/4} \left( z - \frac{B}{2A} \right),
\]

then Equation (88) becomes the governing equation for the quantum harmonic oscillator,

\[
\frac{d^2 F}{d z^2} - z^2 F + A' F = 0,
\]

provided that

\[
A' \equiv \left( \frac{B^2}{4A^{3/2}} + \frac{C}{A^{1/2}} \right)
\]

is discretized/quantized.

The subtle difference between these two equations may be emphasized by considering the equation

\[
\frac{d^2 F}{d z^2} + (2n + 1 - D_1 z^2) F = 0,
\]

where \( n \) is an integer. Consider the proposed solution,

\[
F \propto \exp \left( \frac{-D_2 z^2}{2} \right) \mathcal{H}_n,
\]

where \( \mathcal{H}_n = \mathcal{H}_n(z) \) is the Hermite polynomial. The function \( F \) is the parabolic cylinder function. Using the recurrence relations for Hermite polynomials,

\[
\frac{d^2 \mathcal{H}_n}{d z^2} = 2n (2z \mathcal{H}_{n-1} - \mathcal{H}_n),
\]

we may reduce Equation (94) to

\[
4n z \mathcal{H}_{n-1}(1 - D_2) + \mathcal{H}_n z^2 (D_2 - D_1) - D_2 + 1 = 0.
\]

For \( z \neq 0 \), the equality holds only if \( D_1 = D_2 = 1 \). Thus, the proposed solution in Equation (95) is true only if the governing equation follows the same form as that for the quantum harmonic oscillator. However, since all three equations are related by the appropriate transformations, one may always cast the problem in terms of a quantum harmonic oscillator and transform back to the appropriate form.

Generally, a shallow water model on the β-plane, with forcing and friction, is described by Equation (87) with \( B = 0 \). For a free system, the coefficients \( A \) and \( C \) are real; for a forced or damped system, they are generally imaginary.

The recognition that the meridional velocity obeys the same governing equation as a quantum harmonic oscillator, for a free system, is not novel (Matsuno 1966). However, the insight that the governing equation for the meridional velocity may always be transformed into the quantum harmonic oscillator equation, even in the presence of forcing, friction, and magnetic fields for time-dependent systems, is novel and constitutes an improvement over the time-independent (stationary) analyses of Matsuno (1966) and Gill (1980). Furthermore, the use of De Moivre’s formula to derive the dispersion relations is novel.

5.2. Hydrodynamics

5.2.1. Free System

For the most basic equatorial β-plane model (i.e., no molecular viscosity or Rayleigh drag), the dimensionless governing equations are

\[
\begin{align*}
\frac{\partial v_x'}{\partial t} &= -\frac{\partial h'}{\partial x} + y v_y', \\
\frac{\partial v_y'}{\partial t} &= -\frac{\partial h'}{\partial y} - y v_x', \\
\frac{\partial h'}{\partial t} &= -\frac{\partial v_x'}{\partial x} - \frac{\partial v_y'}{\partial y}.
\end{align*}
\]

These are the “physicists’ Hermite polynomials,” where \( \mathcal{H}_1 = 2z \) rather than \( z \).
By seeking wave solutions, one obtains three expressions for \( v_{x0} \),
\[
v_{x0} = \omega^{-1}(k_x h_0 + i y v_{y0}),
\]
\[
v_{x0} = y^{-1}(i \omega v_{y0} - \frac{\partial h_0}{\partial y}),
\]
\[
v_{x0} = k_x^{-1} \left( \omega h_0 + i \frac{\partial v_{x0}}{\partial y} \right).
\]

Differentiating the third equation in Equation (99) yields
\[
\frac{\partial v_{x0}}{\partial y} = k_x^{-1} \left( \omega \frac{\partial h_0}{\partial y} + i \frac{\partial^2 v_{x0}}{\partial y^2} \right).
\]

Clearly, one needs expressions for \( \partial v_{x0}/\partial y \) and \( \partial h_0/\partial y \) in terms of \( h_0 \) and \( v_{y0} \). Differentiating the first equation in Equation (99) gives \( \partial v_{x0}/\partial y \). Combining the first and third equations yields
\[
h_0 = i \left( \frac{1}{k_x} \frac{\partial v_{y0}}{\partial y} - \frac{y v_{y0}}{\omega} \right) \left( \frac{k_x}{\omega} - \frac{\omega}{k_x} \right)^{-1}.
\]

Combining the first and second equations gives
\[
\left( \frac{k_x}{\omega} - \frac{\omega}{k_x} \right) \frac{\partial h_0}{\partial y} = iv_0 \left( \frac{\omega - y^2}{\omega} \right) \left( \frac{k_x}{\omega} - \frac{\omega}{k_x} \right) - \frac{i y \partial v_{y0}}{\omega} + \frac{i y^2 k_x v_{y0}}{\omega^2}.
\]

Putting it all together, we get
\[
\frac{\partial^2 v_{y0}}{\partial y^2} + \left( \omega^2 - k_x^2 - \frac{k_x}{\omega} - \frac{\omega}{k_x} \right) v_{y0} = 0.
\]

Equation (103) is exactly the equation for the quantum harmonic oscillator if the following combination of quantities is discretized:
\[
\omega^2 - k_x^2 - \frac{k_x}{\omega} - \frac{\omega}{k_x} = 2n + 1,
\]

where \( n \) is the meridional wavenumber and takes on integer values. Physically, \( k_x \) and \( n \) behave like the \( m \) and \( l \) quantum numbers (Matsuno 1966)—an analogy we will formalize in Section 6—except that \( k_x \) is free to take on non-integer values. We note that Equation (103) differs from Equation (6) of Matsuno (1966), because in that work, free solutions with \( \Psi \propto \exp (i \omega t) \), rather than \( \Psi \propto \exp (-i \omega t) \), were sought.

While it is clear that the imaginary part of the wave frequency vanishes for a free system, we find it instructive to consider \( \omega = \omega_R + i \omega_I \), which produces two expressions from the dispersion relation,
\[
\omega_I \left[ \omega^2 - 3 \omega_R^2 + (2n + 1 + k_x^2) \right] = 0,
\]
\[
\omega_R^3 - 3 \omega_R \omega_I^2 - (2n + 1 + k_x^2) \omega_R - k_x = 0.
\]

One of the solutions of the first expression is \( \omega_I = 0 \), which corresponds to balanced flow in a free system. The other two solutions for \( \omega_I \) are not physically meaningful, because they permit finite or imaginary values of \( \omega_I \). The second expression yields three solutions for \( \omega_R \), which correspond to the expected three types of waves that exist in free hydrodynamic systems: eastward- and westward-propagating Poincaré waves and westward-propagating Rossby waves (Matsuno 1966; Longuet-Higgins 1967, 1968; Gill 1980). In other words, the first expression describes how a balanced flow may be obtained, while the second expression describes how the waves oscillate. In a balanced flow (\( \omega_I = 0 \)), the dispersion relation becomes (Matsuno 1966)
\[
\omega_R^3 - (2n + 1 + k_x^2) \omega_R - k_x = 0.
\]

By solving Equation (103) and using Equation (101) and the first expression in Equation (99), we obtain the amplitudes of the perturbations,
\[
v_{y0} = v_0 \exp \left( -\frac{y^2}{2} \right) H_n,
\]
\[
h_0 = i v_0 \exp \left( -\frac{y^2}{2} \right) \left( \frac{1}{k_x} \frac{\omega_R - \omega}{\omega_R} \right)^{-1} \times \left[ \frac{2n H_n - y}{k_x} \frac{1}{\omega_R} \right] H_n,
\]
\[
v_{x0} = \frac{i v_0}{k_x^2 - \omega_R^2} \exp \left( -\frac{y^2}{2} \right) \times \left[ 2n k_x H_n - y (\omega_R + k_x) H_n \right].
\]

where \( v_0 \) is an arbitrary normalization factor. As realized by Matsuno (1966), the \( \omega_R = k_x \) solution is rejected when \( n = 0 \).

The perturbations are obtained from taking the real parts of the wave solutions,
\[
u_y' = v_0 \exp \left( -\frac{y^2}{2} \right) H_n \cos (k_x x - \omega_R t),
\]
\[
h' = v_0 \exp \left( -\frac{y^2}{2} \right) \left( \frac{k_x}{\omega_R} \right)^{-1} \sin (k_x x - \omega_R t) \times \left[ \frac{1}{k_x} \frac{1}{\omega_R} \right] H_n - \frac{2n H_n - 1}{k_x} H_n - \frac{2n k_x H_n - 1}{k_x} H_n - 1.
\]

In deriving Equations (107) and (108), we find that a useful recurrence relation to use is \( d H_n/dy = 2n H_{n-1} \).

For a steady-state system (\( \omega_R = 0 \)), we obtain
\[
v_y' = v_0 \exp \left( -\frac{y^2}{2} \right) H_n \cos (k_x x),
\]
\[
h' = \frac{v_0}{k_x} \exp \left( -\frac{y^2}{2} \right) H_n \sin (k_x x),
\]
\[

5.2.2. Forcing with Hydrodynamic Friction

When forcing and friction (both molecular viscosity and Rayleigh drag) are included, the dimensionless governing equations are
\[
\frac{\partial v_x'}{\partial t} = -\frac{\partial h'}{\partial x} + y v_y' + \frac{1}{R} \left( \frac{\partial^2 v_x'}{\partial x^2} + \frac{\partial^2 v_x'}{\partial y^2} \right) - \omega_{\text{drag}} v_x',
\]
\[
\frac{\partial v_y'}{\partial t} = -\frac{\partial h'}{\partial y} - y v_x' + \frac{1}{R} \left( \frac{\partial^2 v_y'}{\partial x^2} + \frac{\partial^2 v_y'}{\partial y^2} \right) - \omega_{\text{drag}} v_y',
\]
\[
\frac{\partial h'}{\partial t} = -\frac{\partial v_x'}{\partial x} - \frac{\partial v_y'}{\partial y} + F_0 h',
\]
where $\omega_{\text{drag}}$ and $F_0$ have been normalized by $t_0^{-1}$. The Reynolds number is $R = \epsilon^{3/2}/\nu^{1/2} = \epsilon_0 L_0/\nu$. As before, we define

$$\omega \equiv \omega_{\text{drag}} + \frac{k^2}{R}. \quad (111)$$

We further define the dimensionless quantities,

$$\omega_0 \equiv \omega + i \omega_i, \quad \omega_F \equiv \omega - i F_0. \quad (112)$$

The three expressions for $v_{y0}$ become

$$v_{y0} = \omega_0^{-1} \left( k_x h_0 + i y v_{y0} + \frac{i}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \right), \quad (113a)$$

$$v_{y0} = \omega_0^{-1} \left( i \omega_i \nu v_{y0} - \frac{\partial h_0}{\partial y} + \frac{1}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \right), \quad (113b)$$

$$v_{y0} = k_x^{-1} \left( h_0 \omega_F + \frac{i}{R} \frac{\partial v_{y0}}{\partial y} \right). \quad (113c)$$

The expression for $h_0$ becomes

$$h_0 = \frac{i}{\omega_0} \left( \frac{1}{k_x} \frac{\partial v_{y0}}{\partial y} - \frac{i y v_{y0}}{\omega_0} - \frac{1}{\omega_0 \nu} \frac{\partial^2 v_{y0}}{\partial y^2} \right) \left( \frac{k_x}{\omega_0} - \frac{\omega_F}{k_x} \right)^{-1}. \quad (114)$$

Furthermore, we have

$$i v_{y0} \left( \frac{\omega_0^2}{\omega} \right) \left( \frac{k_x}{\omega_0} - \frac{\omega_F}{k_x} \right) - \frac{i y \partial v_{y0}}{\omega_0} + \frac{i y^2 k_x v_{y0}}{\omega_0} + \frac{1}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \left( \frac{k_x}{\omega_0} - \frac{\omega_F}{k_x} \right) \frac{\partial h_0}{\partial y}. \quad (115)$$

Putting it all together, we get

$$\frac{\partial^2 v_{y0}}{\partial y^2} \left[ 1 + \frac{i}{R} \left( \frac{k_x}{\omega_0} - \frac{\omega_F}{k_x} \right) \right] + \left( \omega_0 \omega_F - k_x^2 - \frac{k_x}{\omega_0} - \frac{\nu^2 \omega_F}{\omega_0} \right) v_{y0} - \frac{1}{\omega_0 \nu} \left( \nu \omega_F \frac{\partial^2 v_{y0}}{\partial y^2} + k_x \frac{\partial v_{y0}}{\partial y} \right) = 0. \quad (116)$$

We see that the challenging terms are those explicitly involving $R$ in Equation (116), which prevent the equation from being cast in the form of a quantum harmonic oscillator. To proceed analytically, we perform a trick: we take the limit as $R \to \infty$, while allowing it to retain a finite value within $\omega_i$. Physically, this amounts to assuming that molecular viscosity has a scale dependence across longitude, but not across latitude. The governing equation for $v_{y0}$ becomes

$$\frac{\partial^2 v_{y0}}{\partial y^2} + \left( \omega_0 \omega_F - k_x^2 - \frac{k_x}{\omega_0} - \frac{\nu^2 \omega_F}{\omega_0} \right) v_{y0} = 0. \quad (117)$$

Written in this form, it is easy to see that when forcing and friction are absent, we obtain $\omega = \omega_0 = \omega_F$ and we recover Equation (103). Since $\omega_0$ and $\omega_F$ are both complex in general, it is easy to see that Equation (117) can be cast in terms of a quantum harmonic oscillator equation with complex coefficients,

$$\frac{\partial^2 v_{y0}}{\partial y^2} + \left( \omega_0 \omega_F - k_x^2 - \frac{k_x}{\omega_0} \right) \left( \frac{\omega_0}{\omega_F} \right)^{1/2} - \left( \frac{\omega_0}{\omega_F} \right)^{1/2} v_{y0} = 0, \quad (118)$$

where the transformed latitude is

$$\tilde{y} \equiv \alpha y, \quad \alpha \equiv \left( \frac{\omega_F}{\omega_0} \right)^{1/4}. \quad (119)$$

The dispersion relation follows from considering

$$\left( \omega_0 \omega_F - k_x^2 - \frac{k_x}{\omega_0} \right) \left( \frac{\omega_0}{\omega_F} \right)^{1/2} = 2n + 1, \quad (120)$$

from which two expressions again follow,

$$\omega_0^3 - 3 \omega_0^2 \omega_F + \omega_0^2 \omega_F^2 = (F_0 - 2 \omega_i), \quad (121a)$$

$$+ \omega_0 \left( k_x^2 + \omega_i (\omega_i - 2 F_0) \right) + \omega_i \left( k_x^2 - \omega_i F_0 \right) + (2n + 1) \left( \frac{\zeta - \xi R^2}{2} \right)^{1/2} = 0, \quad (121b)$$

$$\omega_0^3 - 3 \omega_0^2 \omega_F^2 + 2 \omega_0 \omega_F (F_0 - 2 \omega_i) + \omega_0 \left( \omega_i (2 F_0 - 2 \omega_i) - k_x^2 \right) - k_x \left( \frac{\zeta + \xi R^2}{2} \right)^{1/2} = 0, \quad (121c)$$

where we have defined the separation functions,

$$\zeta_R \equiv \omega_0^2 - \omega_i^2 + \omega_i (F_0 - 2 \omega_i) + \omega_i F_0, \quad \zeta \equiv \omega_R (2 \omega_0 + \omega_0 - F_0), \quad \zeta = \left( \frac{\zeta_R}{\xi} + \xi^2 \right)^{1/2}. \quad (122)$$

The expressions in Equation (121) make use of De Moivre’s formula,

$$(\omega_0 \omega_F)^{1/2} = (\zeta + i \xi)^{1/2} = \xi^{1/2} \left( \frac{\cos \psi \Delta}{2} + i \sin \frac{\psi \Delta}{2} \right), \quad (123)$$

where the cosine and sine terms may be expressed in terms of the separation functions,

$$\sin \frac{\psi \Delta}{2} = \frac{1 - \xi R / \xi}{2}, \quad \cos \frac{\psi \Delta}{2} = \frac{1 + \xi R / \xi}{2}. \quad (124)$$

The complex quantity $(\omega_0 \omega_F)^{1/2}$ is generally double-valued; sin($\psi \Delta/2$) and cos($\psi \Delta/2$) are mathematically associated with ± signs. However, we pick the positive sign on the physical ground that Equation (121) needs to reduce to Equation (105) in the limit of a free hydrodynamic system.

In the limit of $R \to \infty$, the amplitudes of the wave perturbations are
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\[ v_\nu = v_0 \exp \left( -\frac{y^2}{2} \right) \tilde{H}_n, \]
\[ h_0 = i v_0 \exp \left( -\frac{y^2}{2} \right) \left( \frac{k_x}{\omega_0} - \frac{\omega_\nu}{k_x} \right)^{-1} \times \left\{ \frac{2n}{k_x} \left( \frac{\omega_\nu}{\omega_0} \right)^{1/4} \tilde{H}_{n-1} - \tilde{H}_n \left\{ \frac{\omega_\nu}{k_x} \left( \frac{\omega_\nu}{\omega_0} \right)^{1/4} + \frac{y}{\omega_0} \right\} \right\}, \]
\[ v_\nu = -\omega_0^{-1} \left( k_x h_0 + i y v_\nu \right), \quad (125) \]

where \( \tilde{H}_{n-1} \equiv \tilde{H}_{n-1}(\tilde{y}) \) and \( \tilde{H}_n \equiv \tilde{H}_n(\tilde{y}) \). Since \( \tilde{y} \) is generally a complex quantity, one cannot easily write down the wave solutions in a closed form. Instead, one has to multiply these amplitudes by \( \Psi \) and take the real parts of the resulting expressions.

It is possible, however, to solve for the steady state of a forced, damped hydrodynamic atmosphere in a closed form, as already demonstrated by Matsuno (1966), Gill (1980), and Showman & Polvani (2011). We find that we arrive at solutions equivalent to these previous works only if we allow for \( F_0 \) to be negative,

\[ F_0 = -|F_0|, \quad (126) \]

Physically, one expects \( F_0 < 0 \) because a positive height perturbation then leads to cooling, while a negative one leads to heating, just as one would expect for Newtonian cooling. In a steady state, such a forcing allows for the key controlling parameter to assume a simple and real form,

\[ \alpha \equiv \left( \frac{|F_0|}{\omega_0} \right)^{1/4}, \quad (127) \]

and avoids the mathematical complication of having to evaluate \( \tilde{y} \) using De Moivre’s formula. As realized by Showman & Polvani (2011), the quantity \( 1/\alpha^4 \) is the Prandtl number on the sphere, as \( \alpha \) generally involves the Rossby number as well. It follows that the steady-state solutions are

\[ v_\nu = v_0 \exp \left( -\frac{y^2}{2} \right) \tilde{H}_n \cos(k_x x), \]
\[ h' = \frac{v_0 k_x \omega_\nu}{k_x^2 + \omega_\nu |F_0|} \exp \left( -\frac{y^2}{2} \right) \times \left[ \frac{\alpha}{k_x} (2n \tilde{H}_{n-1} - \alpha y \tilde{H}_n) \cos(k_x x) - \frac{y \tilde{H}_n}{\omega_0} \sin(k_x x) \right], \]
\[ v_\nu' = \frac{v_0}{k_x^2 + \omega_\nu |F_0|} \exp \left( -\frac{y^2}{2} \right) \times \left[ \frac{y}{2} |F_0| \tilde{H}_n \cos(k_x x) \right. \]
\[ + k_x \alpha (2n \tilde{H}_{n-1} - \alpha y \tilde{H}_n) \sin(k_x x) \left]. \quad (128) \]

5.3. Magnetohydrodynamics (Vertical Background Field)

MHD shallow water systems on the \( \beta \)-plane, with a vertical background magnetic field, are generally described by five dimensionless parameters: the reciprocal of the aspect ratio \( \epsilon \), the strength of hydrodynamic versus magnetic wave propagation \( \Gamma \), the forcing \( F_0 \), the hydrodynamic friction \( \omega_\nu \), and the magnetic Reynolds number \( \mathcal{R}_B \). However, since \( \epsilon \) and \( \Gamma \) always appear as products of each other in the dispersion relations and wave solutions, only four parameters are independent.

5.3.1. Free System, Ideal MHD

Free MHD systems with vertical/radial background fields have previously been considered by Heng & Spitkovsky (2009), both in the equatorial \( \beta \)-plane and spherical geometries. We expand upon their analysis and also cast the problem in terms of more intuitive notation. For a purely vertical background magnetic field, the dimensionless governing equations are

\[ \frac{\partial v_\nu'}{\partial t} = -\frac{\partial h'}{\partial x} + y v_\nu' - \Gamma b_x', \]
\[ \frac{\partial v_\nu'}{\partial t} = -\frac{\partial h'}{\partial y} - y v_\nu' - \Gamma b_y', \]
\[ \frac{\partial h'}{\partial t} = -\frac{\partial v_\nu'}{\partial x} - \frac{\partial v_\nu'}{\partial y}, \]
\[ \frac{\partial b_x'}{\partial t} = \epsilon v_\nu', \quad (129) \]

The reciprocal of the aspect ratio is

\[ \epsilon \equiv \frac{g^{1/4}}{\beta^{1/2} H^{3/4}} = \frac{L_0}{H}, \quad (130) \]

while the other dimensionless quantity is

\[ \Gamma \equiv \frac{\bar{B}_z^2}{4 \pi \rho \beta^{1/2} H^{7/4}} = \epsilon \left( \frac{v_A}{\epsilon_0} \right)^2, \quad (131) \]

such that their product is the square of the ratio of dynamical to Alfvén timescales,

\[ \epsilon \Gamma = \left( \frac{t_{dyn}}{t_A} \right)^2, \quad (132) \]

if we write \( t_A = H/v_A \).

The three expressions for \( v_\nu' \) require defining an additional dimensionless frequency,

\[ \omega_{B_\nu} \equiv \omega - \frac{\epsilon \Gamma}{\omega} = \omega - \frac{1}{\epsilon} \left( \frac{t_{dyn}}{t_A} \right)^2. \quad (133) \]

It is important to recognize that \( \omega_{B_\nu} \) is real in the free MHD limit. The expressions for \( v_\nu' \) are very similar to the basic hydrodynamic situation,

\[ v_{\nu i} = \omega_{B_\nu}^{-1} (k_x h_0 + i y v_\nu), \]
\[ v_{\nu x} = y^{-1} \left( i \omega_{B_\nu} v_\nu - \frac{\partial h_0}{\partial y} \right), \]
\[ v_{\nu y} = k_x^{-1} \left( h_0 \omega + i \frac{\partial v_\nu}{\partial y} \right). \quad (134) \]

The expression for \( h_0 \) becomes

\[ h_0 = i \left( \frac{1}{k_x} \frac{\partial v_\nu}{\partial y} - \frac{y v_\nu}{\omega_{B_\nu}} \right) \left( \frac{k_x}{\omega_{B_\nu}} - \frac{\omega}{k_x} \right)^{-1}. \quad (135) \]

The expression for \( \partial h_0/\partial y \) becomes

\[ \left( \frac{k_x}{\omega_{B_\nu}} - \frac{\omega}{k_x} \right) \frac{\partial h_0}{\partial y} = i v_\nu \left( \frac{y^2}{\omega_{B_\nu}^2} \left( \frac{k_x}{\omega_{B_\nu}} - \frac{\omega}{k_x} \right) \right. \]
\[ \left. - i y \frac{\partial v_\nu}{\partial y} + i y^2 k_x v_\nu \right). \quad (136) \]
Assembling all of the different parts yields
\[
\frac{\partial^2 v_{\eta}}{\partial y^2} + \left( \frac{\omega \omega_{B_0} - k_x^2}{\omega_{B_0}} - \frac{k_x}{\omega_{B_0}} - \frac{y^2 \omega}{\omega_{B_0}} \right) v_{\eta} = 0, \tag{137}
\]
which is identical to Equation (103) when \(\omega = \omega_{B_0}\). The coefficients in the preceding equation remain real.

As in the hydrodynamic case, we find it instructive to first consider both \(\omega_{R}\) and \(\omega_{I}\) to be non-vanishing. By discretizing the expression
\[
\left( \frac{\omega_{B_0} \omega - k_x^2}{\omega_{B_0}} \right) \left( \frac{\omega_{B_0}}{\omega} \right)^{1/2} = 2n + 1, \tag{138}
\]
we obtain
\[
\omega_{I} \left[ \zeta - (\zeta + 2\zeta_{+}) \omega_{R}^2 - \zeta^2 \omega_{I}^{2} \right] - \zeta k_x^2 \omega_{I} - (2n + 1) \left( \frac{\zeta - \zeta_{R}}{2} \right)^{1/2} = 0,
\]
\[
\omega_{R} \left[ \zeta^2 \omega_{R}^2 - \zeta + (2\zeta_{+} + \zeta_{-}) \omega_{I}^{2} \right] - \zeta - k_x \omega_{R} - k - (2n + 1) \left( \frac{\zeta + \zeta_{R}}{2} \right)^{1/2} = 0, \tag{139}
\]
where we have defined the separation functions,
\[
\zeta_{\pm} = 1 \pm \frac{\epsilon \Gamma}{\omega_{R}^2 + \omega_{I}^{2}}, \quad \zeta_{R} = \zeta_{-} \omega_{R}^2 - \zeta \omega_{I},
\]
\[
\zeta = (\zeta^2 + \zeta_{I}^2)^{1/2}. \tag{140}
\]

For ease of evaluating Equation (138), we have written
\[
\omega_{B_0} = \zeta_{-} \omega_{R} + i \epsilon \omega_{I}. \tag{141}
\]

The expression \((\omega_{B_0} \omega_{R})^{1/2}\) is again evaluated using De Moivre’s formula; the double-valued nature of this quantity is eliminated by ensuring that Equation (105) is obtained in the free hydrodynamic limit \((\zeta_{\pm} = 1)\). In this limit, we previously showed that \(\omega_{I} = 0\) is a solution of the growth/decay dispersion relation. We are unable to show that this is generally true for the first expression in Equation (139). Nevertheless, we expect \(\omega_{I} = 0\) on physical grounds, since we are dealing with a free MHD system, which yields the dispersion relation
\[
\zeta^2 \omega_{R}^2 - \zeta_{-} k_x \omega_{R} - \frac{\epsilon}{\zeta_{-}^2} (2n + 1) \omega_{R} - k_{x} = 0. \tag{142}
\]

Physically, the quantity \(\zeta\) controls the effect of “magnetic pinching” (Heng & Spitkovsky 2009). It is contained within the transformed latitude,
\[
\tilde{\gamma} \equiv \gamma \zeta_{n}, \quad \alpha \equiv \frac{1}{\zeta_{-}^{1/4}}. \tag{143}
\]

Since \(1/\zeta_{-} = \omega_{R}^2 / (\omega_{R}^2 - \epsilon \Gamma) \geq 1\), we have \(\alpha \geq 1\) and the transformed latitude is always longer than the actual latitude (i.e., \(\tilde{\gamma} \geq \gamma\)) when a magnetic field is present, implying that the term \(\exp(-\tilde{\gamma}^2/2)\) becomes smaller. This has the effect that waves will be more concentrated across latitude; it becomes more pronounced as the magnetic field strength increases (larger \(\Gamma\)). In terms of the transformed latitude, the equation for \(v_{\eta}\), is again that of a quantum harmonic oscillator and yields the following solutions for the wave amplitudes:
\[
v_{\eta} = v_{0} \exp \left( -\frac{\tilde{\gamma}^2}{2} \right) \tilde{H}_{n}, \tag{144}
\]
\[
h_{0} = i v_{0} \exp \left( -\frac{\tilde{\gamma}^2}{2} \right) \left( \frac{k_x}{\zeta_{-} \omega_{R}} - \frac{\omega_{R}}{k_x} \right)^{-1} \times \left[ \frac{2 n \tilde{H}_{n-1}}{\zeta_{-}^{1/4} k_x} - \frac{y \omega_{R}^{1/2}}{\zeta_{-}^{1/4}} \right] \tilde{H}_{n}, \tag{145}
\]
which reduce to Equation (107) in the free hydrodynamic limit. Again, it is possible to state the general wave solutions for a free system,
\[
v'_{\eta} = v_{0} \exp \left( -\frac{\tilde{\gamma}^2}{2} \right) \tilde{H}_{n} \cos (k_x x - \omega_{R} t),
\]
\[
h' = v_{0} \exp \left( -\frac{\tilde{\gamma}^2}{2} \right) \left( \frac{k_x}{\zeta_{-} \omega_{R}} - \frac{\omega_{R}}{k_x} \right)^{-1} \times \left[ \frac{2 n \tilde{H}_{n-1}}{\zeta_{-}^{1/4} k_x} - \frac{y \omega_{R}^{1/2}}{\zeta_{-}^{1/4}} \right] \tilde{H}_{n}.
\]

5.3.2. Forcing with Hydrodynamic Friction

Some insight is gained by considering a situation of intermediate complexity, namely, that involving forcing and only hydrodynamic friction (molecular viscosity and Rayleigh drag). Magnetic drag is omitted for now. The dimensionless governing equations are
\[
\frac{\partial v'}{\partial t} = -\frac{\partial h'}{\partial x} + y v' + \frac{1}{\mathcal{R}} \left( \frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right)
\]
\[
- \omega_{\text{drag}} v' - \Gamma b', \tag{146}
\]
\[
\frac{\partial v'}{\partial t} = -\frac{\partial h'}{\partial y} - y v' + \frac{1}{\mathcal{R}} \left( \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial x^2} \right)
\]
\[
- \omega_{\text{drag}} v' - \Gamma b',
\]
\[
\frac{\partial b'}{\partial t} = -\frac{\partial v'}{\partial x} - \frac{\partial v'}{\partial y} + F_{0} h',
\]
\[
\frac{\partial h'}{\partial t} = \epsilon v_{x,y}. \tag{146}
\]

In the three expressions for \(v_{\eta}\),
where an additional dimensionless frequency is defined, dynamic situation with forcing and friction, except that

$$v_{x0} = \omega_{B0}^{-1} \left( k_x h_0 + iy v_{y0} + \frac{i}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \right),$$

$$v_{x0} = y^{-1} \left( i \omega_B v_{y0} - \frac{\partial h_0}{\partial y} + 1 \frac{\partial^2 v_{y0}}{R \partial y^2} \right),$$

$$v_{x0} = k_x^{-1} \left( h_0 \omega_B + i \frac{\partial v_{y0}}{\partial y} \right),$$

the expression for $\omega_{B0}$ is generalized to

$$\omega_{B0} = \omega + i \omega_B,$$  \hspace{1cm} (148)

where an additional dimensionless frequency is defined,

$$\omega_B = \omega + i \upsilon \frac{R}{\nu}. \hspace{1cm} (149)$$

One may verify that Equation (148) reduces to Equation (133) in the absence of hydrodynamic friction ($\omega_B = 0$).

The expression for $h_0$ is structurally identical to the hydrodynamic situation with forcing and friction, except that $\omega_0$ is replaced by $\omega_{B0}$,

$$h_0 = i \left( \frac{1}{k_x} \frac{\partial v_{y0}}{\partial y} - \frac{y v_{y0}}{\omega_{B0}} - \frac{\partial^2 v_{y0}}{\omega_{B0} R \partial y^2} \right) \left( \frac{k_x}{\omega_{B0}} - \frac{\omega\nu}{\omega_B} \right)^{-1}. \hspace{1cm} (150)$$

The same applies for the expression for $\partial h_0/\partial y$,

$$i v_{y0} \left( \omega_{B0} - \frac{y^2}{\omega_{B0}} \right) \left( \frac{k_x}{\omega_B} - \frac{\omega\nu}{\omega_B} \right) - i y \frac{\partial v_{y0}}{\partial y} + \frac{y^2 k_x v_{y0}}{\omega_B} \frac{\partial^2 v_{y0}}{\partial y^2} + \frac{1}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \left( k_x \frac{\partial h_0}{\partial y} \right) = 0. \hspace{1cm} (151)$$

Putting it all together, the general governing equation for $v_{y0}$ is

$$\frac{\partial^2 v_{y0}}{\partial y^2} \left[ 1 + i \frac{R}{\omega_{B0}} \left( \frac{k_x}{\omega_B} - \frac{\omega\nu}{\omega_B} \right) \left( \frac{k_x}{\omega_B} - \frac{\omega\nu}{\omega_B} \right) \right] + \left( \omega_{B0} \omega\nu \omega_B - k_x^2 - \frac{k_x}{\omega_{B0}} \frac{y^2 \omega\nu}{\omega_B} \right) v_{y0} - \frac{1}{\omega_{B0} R} \left( \frac{\omega\nu}{\omega_B} \frac{\partial^2 v_{y0}}{\partial y^2} + \frac{\partial^2 v_{y0}}{\partial y^2} \right) = 0. \hspace{1cm} (152)$$

Again, if we make the approximation that molecular viscosity has a scale dependence across longitude but not latitude, the governing equation becomes amenable to an analytical solution,

$$\frac{\partial^2 v_{y0}}{\partial y^2} + \left( \omega_{B0} \omega\nu \omega_B - k_x^2 - \frac{k_x}{\omega_{B0}} \frac{y^2 \omega\nu}{\omega_B} \right) v_{y0} = 0. \hspace{1cm} (153)$$

The key point is that when a vertical background magnetic field is added to a system with forcing and hydrodynamic friction, one only needs to replace $\omega_0$ in the equations and solutions by its magnetic counterpart ($\omega_{B0}$). Both dimensionless frequencies are generally complex. This intermediate case also shows the progressive generalization of $\omega_B$ and $\omega_{B0}$.

### 5.3.3. Forcing with Friction

When magnetic drag is added to a system with forcing, hydrodynamic friction, and magnetic fields, the dimensionless induction equations become

$$\frac{\partial b_{x,y}}{\partial t} = \epsilon v_{x,y} + \frac{1}{R_B} \left( \frac{\partial^2 b_{x,y}}{\partial x^2} + \frac{\partial^2 b_{x,y}}{\partial y^2} \right), \hspace{1cm} (154)$$

where the magnetic Reynolds number is defined as

$$R_B \equiv \frac{c_0 L_0}{\beta^{1/2} \eta} = \frac{c_0 L_0}{\eta}. \hspace{1cm} (155)$$

A final additional dimensionless frequency is needed,

$$\omega_B \equiv \omega + i \epsilon \Gamma \frac{R}{\nu}. \hspace{1cm} (156)$$

The definition for $\omega_B$ is generalized,

$$\omega_B \equiv \omega \frac{R}{\nu} + i \epsilon \Gamma \frac{R}{\nu}, \hspace{1cm} (157)$$

while still retaining $\omega_{B0} \equiv \omega + i \omega_B$. The expressions for $v_{x0}$ pick up extra contributions involving $R_B$ and $\omega_B$,

$$v_{x0} = \omega_{B0}^{-1} \left( k_x h_0 + iy v_{y0} + \frac{i}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \right), \hspace{1cm} (158)$$

The expressions for $h_0$,

$$h_0 = i \left( \frac{1}{k_x} \frac{\partial v_{y0}}{\partial y} - \frac{y v_{y0}}{\omega_{B0}} - \frac{1}{R} \frac{\partial^2 v_{y0}}{\partial y^2} \right) \left( \frac{k_x}{\omega_{B0}} - \frac{\omega\nu}{\omega_B} \right)^{-1}, \hspace{1cm} (159)$$

and its derivative,

$$i v_{y0} \left( \omega_{B0} - \frac{y^2}{\omega_{B0}} \right) \left( \frac{k_x}{\omega_B} - \frac{\omega\nu}{\omega_B} \right) - \frac{y v_{y0}}{\omega_{B0}} \frac{\partial^2 v_{y0}}{\partial y^2} + \frac{i \epsilon \Gamma}{R_B} \frac{\partial b_{x,y}}{\partial y^2} \left( \frac{k_x}{\omega_{B0}} - \frac{\omega\nu}{\omega_B} \right) \frac{\partial b_{x,y}}{\partial y^2} = 0. \hspace{1cm} (160)$$

Also pick up extra contributions.
In the $\beta$-plane approximation, the most general governing equation for $\nu_y$ is

$$\frac{\partial^2 \nu_y}{\partial y^2} \left[ 1 + i \frac{k_x}{\omega_B} \right] = \frac{1}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

with Equation (153), but with a more general definition of $\omega_B$. We again find the algebra to be more tractable if we write

$$\frac{1}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

$$\equiv \frac{1}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

$$\equiv \frac{1}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

$$\equiv \frac{1}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

$$+ \frac{i}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

$$+ \frac{i}{\omega_B} \left( \frac{\partial^2 \nu_y}{\partial y^2} - \frac{k_x}{\omega_B} \frac{\partial \nu_y}{\partial y} \right) \nu_y$$

To proceed analytically, both molecular viscosity and magnetic drag are assumed to have scale dependences only across longitude. Mathematically, we set $\mathcal{R}, \mathcal{R}_B \to \infty$ wherever they appear explicitly in Equation (161), while allowing them to retain finite values within $\omega_n$ and $\omega_o$. In this limit, one ends up with Equation (153), but with a more general definition of $\omega_B$.

The dispersion relations are obtained from discretizing the expression

$$\left( \omega_B, \omega_F \right)^{1/2} = \left( \omega_B, \omega_F \right) \left( \omega_B, \omega_F \right)^{1/2} = 2n + 1. \quad (162)$$

We again find the algebra to be more tractable if we write

$$\left( \omega_B, \omega_F \right)^{1/2} = \left( \omega_B, \omega_F \right) \left( \omega_B, \omega_F \right)^{1/2} = 2n + 1. \quad (163)$$

and

$$\omega_B = \zeta - \omega_R + i (\omega_0 + \zeta + \omega_n), \quad (164)$$

where we have defined the separation functions,

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

$$\zeta_0 = \omega_B + \frac{\epsilon \Gamma k_x^2}{R_B \left[ \omega_B^2 + (k_x^2/R_B + \omega_n)^2 \right]}, \quad (166)$$

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

It follows that

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

$$\zeta = \frac{\epsilon \Gamma}{\omega_B^2 + (k_x^2/R_B + \omega_n)^2}, \quad (165)$$

$$\zeta_0 = \omega_B + \frac{\epsilon \Gamma k_x^2}{R_B \left[ \omega_B^2 + (k_x^2/R_B + \omega_n)^2 \right]}, \quad (166)$$

In the preceding expressions, we have again used De Moivre’s formula and picked the positive root, such that the dispersion relations reduce to Equation (105) in the free hydrodynamic limit.

In the limit of $\mathcal{R}, \mathcal{R}_B \to \infty$, the amplitudes of the wave perturbations are

$$v_y = v_0 \exp \left( -\frac{\gamma^2}{2} \right) \tilde{H}_n, \quad (167)$$

$$h_0 = i v_0 \exp \left( -\frac{\gamma^2}{2} \right) \left( \frac{k_x}{\omega_B} - \frac{\omega_F}{k_x} \right)^{-1} \times \left\{ \begin{array}{l} 2n \left( \frac{\omega_F}{\omega_B} \frac{1}{4} / \tilde{H}_{n-1} \right) \\ \frac{\gamma^2}{k_x} \left( \frac{\omega_F}{\omega_B} \right)^{1/4} + y \end{array} \right\},$$

where we have

$$\gamma = \alpha y, \quad (168)$$

Again, it is easier to write down the steady-state solutions (with $F_0 = -|F_0|$),

$$v_y = v_0 \exp \left( -\frac{\alpha^2 y^2}{2} \right) \tilde{H}_n \cos (k_n x), \quad (170)$$

$$h_0 = -\frac{v_0 \alpha y}{k_x + \omega_0} \exp \left( -\frac{\alpha^2 y^2}{2} \right) \times \left\{ \begin{array}{l} \frac{\alpha}{k_x} (2n \tilde{H}_{n-1} - \alpha y \tilde{H}_n) \cos (k_n x) \\ -\frac{\gamma^2}{\tilde{H}_n} \sin (k_n x) \end{array} \right\},$$

where, in the steady-state limit, we have

$$\alpha = \frac{|F_0|^{1/4}}{\tilde{H}_n}, \quad (170)$$

$$\zeta_0 = \omega_B + \frac{k_x^2}{R_B} + \frac{\epsilon \Gamma k_x^2}{R_B \left[ \omega_B^2 + (k_x^2/R_B + \omega_n)^2 \right]}, \quad (166)$$

The “generalized friction” $\zeta_0$ contains all of the physical effects that act like sources of friction, including magnetic tension. It has the property that molecular viscosity acts predominantly on small scales, while magnetic tension and magnetic drag act collectively and preferentially on large scales.

As is evident, the parameters $\epsilon$ and $\Gamma$ are degenerate and appear only as products of each other. The purely hydrodynamic case is recovered when $\epsilon \Gamma = 0$. Magnetic drag vanishes in the limit of $\mathcal{R}_B \to \infty$. 

$$\zeta_0 = \omega_B + \frac{k_x^2}{R_B} + \frac{\epsilon \Gamma k_x^2}{R_B \left[ \omega_B^2 + (k_x^2/R_B + \omega_n)^2 \right]}, \quad (166)$$

$$\epsilon \Gamma = 0. \quad \text{Magnetic drag vanishes in the limit of } \mathcal{R}_B \to \infty. \quad (170)$$
5.4. Magnetohydrodynamics (Horizontal Background Field)

5.4.1. Free System, Ideal MHD

For a purely horizontal background magnetic field, the dimensionless governing equations are

\[
\begin{align*}
\frac{\partial v'_x}{\partial t} &= -\frac{\partial h'}{\partial x} + yv'_y + \Gamma_x \frac{\partial h'}{\partial x} + (\Gamma_x, \Gamma_y) \frac{1}{\sqrt{\gamma}} \frac{\partial b'_y}{\partial y}, \\
\frac{\partial v'_y}{\partial t} &= -\frac{\partial h'}{\partial y} - yv'_x + (\Gamma_x, \Gamma_y) \frac{1}{\sqrt{\gamma}} \frac{\partial b'_x}{\partial x} + \Gamma_y \frac{\partial h'}{\partial y}, \\
\frac{\partial h'}{\partial t} &= -\frac{\partial v'_y}{\partial y} - \frac{\partial v'_x}{\partial x}, \\
\frac{\partial b'_y}{\partial t} &= \frac{\partial v'_x}{\partial x} + \epsilon \frac{\partial v'_y}{\partial y}, \\
\frac{\partial b'_x}{\partial t} &= 1 - \frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y},
\end{align*}
\]

where the definitions for \( \epsilon \) and \( \Gamma \) (now separated into two components) have changed. The former is now the ratio of magnetic field strengths,

\[
\epsilon \equiv \frac{\dot{B}_x}{\dot{B}_z},
\]

while the latter is the ratio of Alfvén to dynamical velocities,

\[
\Gamma_x, y \equiv \left( \frac{v_{\lambda,x}}{c_0} \right)^2 = \left( \frac{t_{\text{dyn}}}{t_\lambda} \right)^2,
\]

where \( v_{\lambda,x} \equiv \dot{B}_{x,y}/2\sqrt{\pi \rho} \).

The three expressions for the velocity amplitude in the \( x \)-direction are

\[
v_{s0} = \omega_{B_0}^{-1} (k_x h_0 + iy v_{y0}) - \frac{i k_x}{\omega_{B_0}} [e \Gamma_x + (\Gamma_x, \Gamma_y)]^{1/2} \frac{\partial v_{y0}}{\partial y},
\]

\[
\omega_{B_0} = \frac{\omega_R}{\omega_{\text{drag}}},
\]

where the definition of \( \omega_{B_0} \) has changed:

\[
\omega_{B_0} \equiv \omega - k_x^2 \Gamma_x / \omega.
\]

It is apparent that the preceding set of equations cannot be solved in the usual way. However, if we demand that the poloidal background field vanishes (\( B_y = 0 \)) and do not consider any perturbation of the poloidal magnetic field (\( \delta B_y = 0 \)), the algebra becomes tractable, since we end up with

\[
\begin{align*}
v_{s0} &= \omega_{B_0}^{-1} (k_x h_0 + iy v_{y0}), \\
v_{y0} &= y^{-1} \left( i \omega v_{y0} - \frac{\partial h_0}{\partial y} \right), \\
v_{x0} &= k_x^{-1} \left( h_0 \omega + \frac{\partial v_{y0}}{\partial y} \right).
\end{align*}
\]

Note that these expressions are almost identical to the free MHD case with a vertical background field (Equation (134)), except that the second expression has \( \omega \) instead of \( \omega_{B_0} \). Employing the usual mathematical machinery, we obtain

\[
\frac{\partial^2 v_{y0}}{\partial y^2} + \left( \omega^2 - \frac{k_y^2 \omega}{\omega_{B_0}^2} - \frac{k_x}{\omega \omega_{B_0}} \left( \frac{\omega_{B_0}}{\omega} \right)^{1/2} - y^2 \right) v_{y0} = 0,
\]

where we have defined

\[
\tilde{y} \equiv \alpha y, \\
\alpha \equiv \left( \frac{\omega}{\omega_{B_0}} \right)^{1/4}.
\]

To obtain the dispersion relations, we again find it convenient to write \( \omega_{B_0} = \zeta - \omega R + i \zeta \omega_{\text{on}} \), from which it follows that

\[
\begin{align*}
\omega_R (\zeta - \omega_{\text{R}}^2 - \zeta \omega_{\text{on}}^2) - \omega_{\text{R}}^2 \omega_R (\zeta + \zeta_R) &= 0, \\
k_y^2 \omega_{\text{on}} (\zeta + \zeta_R) + \omega_R (\omega_{\text{R}}^2 - \zeta \omega_{\text{on}}^2) &= 0, \\
- k_y^2 \omega_{\text{on}} (\zeta + \zeta_R) &+ \omega_R \left( \zeta - \omega_{\text{R}} \right)^{1/2} = 0,
\end{align*}
\]

where we have defined

\[
\zeta_\pm \equiv 1 \pm \frac{k_y^2 \Gamma_x}{\omega_{\text{R}} + \omega_{\text{on}}}, \\
\zeta_R \equiv \zeta - \omega_{\text{R}}^2 - \zeta \omega_{\text{on}}^2, \\
\zeta_1 \equiv (\zeta + \zeta_R) \omega_{\text{on}} \omega_R, \\
\zeta = (\zeta_R + \epsilon^2)^{1/2}.
\]

It turns out that the wave amplitudes and solutions are identical to those previously stated in Equations (144) and (145), respectively, but with different definitions of \( \omega_{B_0} \) and \( \zeta_\pm \).

5.4.2. Forcing with Friction

When all sources of friction are added, the dimensionless governing equations are

\[
\begin{align*}
\frac{\partial v'_x}{\partial t} &= -\frac{\partial h'}{\partial x} + yv'_y + \Gamma_x \frac{\partial b'_y}{\partial x} + (\Gamma_x, \Gamma_y) \frac{1}{\sqrt{\gamma}} \frac{\partial b'_y}{\partial y}, \\
\frac{\partial v'_y}{\partial t} &= -\frac{\partial h'}{\partial y} - yv'_x + (\Gamma_x, \Gamma_y) \frac{1}{\sqrt{\gamma}} \frac{\partial b'_x}{\partial x} + \Gamma_y \frac{\partial h'}{\partial y}, \\
\frac{\partial h'}{\partial t} &= -\frac{\partial v'_y}{\partial y} - \frac{\partial v'_x}{\partial x}, \\
\frac{\partial b'_y}{\partial t} &= \frac{\partial v'_x}{\partial x} + \epsilon \frac{\partial v'_y}{\partial y}, \\
\frac{\partial b'_x}{\partial t} &= 1 - \frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y},
\end{align*}
\]

where we have defined

\[
\omega_{\text{drag}} = \omega_{\text{drag}}(\zeta, \omega_{\text{on}}).
\]
\[
\frac{\partial h'}{\partial t} = -\frac{\partial v'_x}{\partial x} - \frac{\partial v'_y}{\partial y} + F_0 h',
\]
\[
\frac{\partial b'_x}{\partial t} = \frac{\partial v'_x}{\partial x} + i\omega v - \frac{1}{\mathcal{R} B} \left( \frac{\partial^2 b'_x}{\partial x^2} + \frac{\partial^2 b'_y}{\partial y^2} \right),
\]
\[
\frac{\partial b'_y}{\partial t} = \frac{1}{\epsilon} \frac{\partial v'_y}{\partial y} + \frac{1}{\mathcal{R} B} \left( \frac{\partial^2 b'_y}{\partial x^2} + \frac{\partial^2 b'_y}{\partial y^2} \right),
\]
from which it follows that
\[
v_{x 0} = \omega^{-1}_B (k_x h_0 + i y v_{y 0})
- \frac{i k_x}{\omega_B \omega_R} \left[ y \left( \frac{\partial^2 b_{y 0}}{\partial y^2} \right) + \left( \frac{\partial^3 b_{y 0}}{\partial y^3} \right) \right],
\]
\[
\frac{1}{\omega_B} \left[ \frac{\partial^2 b_{y 0}}{\partial y^2} \right] + \mathcal{R} B \left[ \frac{\partial^3 b_{y 0}}{\partial y^3} \right] \right], \quad \text{(181)}
\]

Again, to make the algebra tractable, we assume \( \tilde{B}_y = b'_y = 0 \) and take the \( \mathcal{R}, \mathcal{R}_B \to \infty \) limit, which yields
\[
v_{x 0} = \omega^{-1}_B (k_x h_0 + i y v_{y 0}),
\]
\[
v_{x 0} = y^{-1} \left( i \omega v_{y 0} - \frac{\partial h_0}{\partial y} \right),
\]
\[
v_{x 0} = k^{-1}_x \left( h_{0 \omega \epsilon} + i \frac{\partial v_{y 0}}{\partial y} \right), \quad \text{(183)}
\]

where the definitions for some of the dimensionless frequencies have changed,
\[
\omega_{\eta} \equiv \omega + \frac{i k_x^2}{\mathcal{R} B},
\omega_B \equiv \omega + \frac{i k_x^2 \Gamma_x}{\omega_R}, \quad \text{(184)}
\]

while retaining \( \omega_0 \equiv \omega_{\text{drag}} + \frac{k_x^2}{\mathcal{R}}, \omega_{\text{oh}} \equiv \omega + i \omega_R, \) and \( \omega_{\text{oh}} \equiv \omega + i \omega_R \) as in the case of a vertical background magnetic field. Performing the same mathematical procedure, we obtain
\[
\frac{\partial^2 v_{y 0}}{\partial y^2} + \left( \frac{\omega \omega_R - \omega_{\text{drag}}}{\omega_B} - \frac{k_x}{\omega_B} \right) \left( \frac{\omega_B}{\omega_{\text{drag}}} \right) \left( \frac{\omega_B}{\omega} \right)^{1/2} - \tilde{y}^2 \right) v_{y 0} = 0,
\]
\[
\text{where we have } \tilde{y} \equiv \alpha y \text{ and } \alpha \equiv (\omega_{\text{drag}}/\omega_B)^{1/4}. \quad \text{(185)}
\]

In deriving the dispersion relations, we again find it useful to first write \( \omega_{\text{drag}} = \omega_R + i (\xi_0 + \xi_{\omega R}) \) and evaluate \( (\omega_{\text{drag}}/\omega_{\text{drag}})^{1/2} = (\zeta_{\omega R} + i \xi_0)^{1/2} \) using De Moivre’s formula. It follows that
\[
\zeta_{\omega R} \left( \frac{\omega_{\text{drag}}}{\omega_{\text{drag}}^2} - \omega_{\text{drag}} \right) - 2 \omega \omega_R (\omega + \xi_{\omega R})
- \zeta_\omega (\omega - F_0) \omega_{\text{drag}} - \omega_R (\omega - F_0) (\xi_0 + \xi_{\omega R})
+ \omega_{\text{drag}} \zeta_{\omega R} - k^2 \omega_{\text{drag}} - k - (2n + 1) \left( \frac{\xi_{\omega R} + \xi_{\omega R}^2}{2} \right)^{1/2} = 0,
\]
\[
2 \zeta - \omega_{\text{drag}}^2 (\xi_0 + \xi_{\omega R}) \left( \xi_{\omega R}^2 - \omega_{\text{drag}}^2 \right) + \zeta_\omega (\omega - F_0) (\xi_0 + \xi_{\omega R})
- \omega_{\text{drag}} \omega_{\text{drag}} (\xi_0 + \xi_{\omega R}) (\xi_0 + \xi_{\omega R}) F_{\omega_{\text{drag}}}
- k^2 (\omega - F_0) - (2n + 1) \left( \frac{\xi_{\omega R} + \xi_{\omega R}^2}{2} \right)^{1/2} = 0, \text{ (186)}
\]

where we have defined the separation functions,
\[
\zeta_{\omega R} \equiv \frac{1}{2} \left( \frac{\Gamma_x k_x^2}{\omega_{\text{drag}}^2} \right)^{1/2},
\zeta_{\omega} \equiv \frac{1}{2} \left( \frac{\Gamma_x k_x^2}{\omega_{\text{drag}}^2} \right)^{1/2}, \text{ (187)}
\]

The wave amplitudes and steady-state solutions are identical to the vertical-field situation, as given in Equations (167) and (169), respectively, except that the definition of \( \zeta_0 \) has changed:
\[
\zeta_0 \equiv \omega_{\text{drag}} + \frac{k_x^2}{\mathcal{R}} + \Gamma_x \mathcal{R}_B. \quad \text{(188)}
\]

We retain \( \alpha = (F_0/\zeta_0)^{1/4} \). The major difference is that magnetic tension and magnetic drag operate equally on all scales in a collective manner.

6.2D MODELS (SPHERICAL)

Generally, a shallow water model on a sphere may be described by five parameters: the forcing \( (F_0) \), the hydrodynamic friction \( (\omega_R) \), the magnetic Reynolds number \( (\mathcal{R}_B) \), the Rossby number \( (\mathcal{R}_B) \), and the ratio of dynamical to Alfvén timescales \( (\tau_{\text{dyn}}/\tau_A) \). One also needs to specify the characteristic length scales of interest via the zonal \( (m) \) and meridional \( (l) \) wavenumbers, analogous to the quantum numbers for the quantum harmonic oscillator.

6.1. Hydrodynamics

6.1.1. Free System

We revisit the classical work of Longuet-Higgins (1968), who studied free hydrodynamic shallow water systems on a sphere, as a basis for generalizing to forced, damped magnetized systems in spherical geometry. We note that Heng & Spitkovsky (2009) has previously rederived the work of Longuet-Higgins (1968) in a condensed form, but we still find it useful to introduce self-consistent notation and also reconcile some differences between the present work and that of Longuet-Higgins (1968).
In the free hydrodynamical limit, the governing equations are
\[
\frac{\partial v_\phi'}{\partial t} = \frac{g}{R} \frac{\partial h'}{\partial \theta} + 2\Omega v_\phi' \cos \theta, \\
\frac{\partial v_\theta'}{\partial t} = -\frac{g}{R \sin \theta} \frac{\partial h'}{\partial \phi} - 2\Omega v_\phi' \cos \theta, \\
\frac{\partial h'}{\partial t} + \frac{H}{R \sin \theta} \left[ \frac{\partial}{\partial \theta} (v_\phi' \sin \theta) + \frac{\partial v_\phi'}{\partial \phi} \right] = 0. \tag{189}
\]
These equations differ from those of Longuet-Higgins (1968), presented in Equations (2.1)–(2.3) of that paper, for a simple reason: Longuet-Higgins (1968) calls his polar coordinate the “co-latitude”; if we denote this by \( \theta_{\text{LH}} \), then it is related to our co-latitude by \( \theta = \pi - \theta_{\text{LH}} \). If we assume a unit sphere \( (R = 1) \) and substitute \( \sin \theta = \sin \theta_{\text{LH}}, \cos \theta = -\cos \theta_{\text{LH}}, \) and \( \partial/\partial \theta = -\partial/\partial \theta_{\text{LH}} \) into the preceding equations, we (nearly) recover Equations (2.1)–(2.3) of Longuet-Higgins (1968). We note a typographical error in Equation (2.2) of that work, where the \( 2\Omega v_\phi' \sin \theta_{\text{LH}} \) term should instead be \( 2\Omega v_\phi' \cos \theta_{\text{LH}} \), otherwise one does not recover Equation (7.2) of the same work when seeking wave solutions.\(^7\) This error does not propagate into the rest of the results in Longuet-Higgins (1968).\(^8\)

To proceed, we first need to recast the equations using the following transformations (Margules 1893; Longuet-Higgins 1968):
\[
v_{\theta,\phi}' \equiv v_{\theta,\phi}' \sin \theta, \\
h' \equiv \frac{gh'}{2\Omega R}, \\
\mu \equiv \cos \theta, \\
\hat{D} \equiv -\sin \theta \frac{\partial}{\partial \theta} = (1 - \mu^2) \frac{\partial}{\partial \mu}, \\
t_0 \equiv 2\Omega t. \tag{190}
\]
Several aspects of these transformations are worth emphasizing. Unlike for the \( \beta \)-plane models, the velocities retain their dimensional form. The quantity \( h' \) has the physical units of velocity, rather than length. In defining the dimensionless time \( (t_0) \), we automatically allow for the wave frequency \( (\omega) \) to be cast in dimensionless units. By taking the characteristic length scale to be the radius of the exoplanet, we define the Rossby number,
\[
R_0 \equiv \frac{c_0}{2\Omega R}, \tag{191}
\]
and write Lamb’s parameter as (Longuet-Higgins 1968)
\[
\xi \equiv \frac{1}{R_0^2}. \tag{192}
\]
With these transformations, the governing equations become
\[
-i \frac{\partial}{\partial t_0} (iv_\phi'') - \mu v_\phi'' \hat{D} h''_\phi = 0, \\
i \mu (iv_\phi'') - \frac{\partial v_\phi''}{\partial t_0} - \frac{\partial h''_\phi}{\partial \phi} = 0, \\
(1 - \mu^2) \frac{\partial h''_\phi}{\partial t_0} + R_0^2 \left[ i \hat{D} (iv_\phi'') + \frac{\partial v_\phi''}{\partial \phi} \right] = 0. \tag{193}
\]
This somewhat peculiar way of writing the governing equations comes from the desire to seek solutions for the following quantities:
\[
i v_\phi'' = \nu_0 \Psi, \quad v_\phi'' = v_\phi^0 \Psi, \quad h''_\phi = h_\phi^0 \Psi, \tag{194}
\]
where \( \Psi \equiv \exp \left[ i(m \phi - \omega t_0) \right] \) and \( m \) is the zonal wavenumber.

By seeking wave solutions, one obtains the equations for the wave amplitudes,
\[
\omega v_0 \nu + \mu v_\phi^0 + \hat{D} h_\phi^0 = 0, \\
\mu v_\phi^0 + \omega v_\phi^0 - m h_\phi^0 = 0, \\
\hat{D} v_\phi^0 + m v_\phi^0 - \omega \xi (1 - \mu^2) h_\phi^0 = 0, \tag{195}
\]
where we have used Lamb’s parameter instead of the Rossby number in order to write the preceding expressions in a more compact form. These expressions differ from those in Equation (7.2) of Longuet-Higgins (1968) due to the \( \theta \rightarrow \theta_{\text{LH}} \) transformation previously described.

We next employ a series of mathematical steps first described in Longuet-Higgins (1968). From the second equation in Equation (195), we obtain
\[
v_\phi^0 = \omega^{-1} (m h_\phi^0 - \mu v_\phi^0). \tag{196}
\]
Substituting this expression into the first equation in Equation (195) yields
\[
(\omega \hat{D} + \mu m) h_\phi^0 = (\mu^2 - \omega^2) v_\phi^0. \tag{197}
\]
Substituting the same expression into the third equation in Equation (195) gives
\[
(\omega \hat{D} + \mu m) \left[ \frac{(\omega \hat{D} - \mu m) v_\phi^0}{\omega^2 \xi (1 - \mu^2) - m^2} \right] + (\omega^2 - \mu^2) v_\phi^0 = 0. \tag{199}
\]
Combining Equations (197) and (198) to eliminate \( h_\phi^0 \) yields
\[
(\omega \hat{D} + \mu m) \left[ \frac{(\omega \hat{D} - \mu m) v_\phi^0}{\omega^2 \xi (1 - \mu^2) - m^2} \right] + (\omega^2 - \mu^2) v_\phi^0 = 0. \tag{200}
\]
the provisional governing equation for \( v_\phi^0 \). Equation (199) is in agreement with Equation (7.5) of Longuet-Higgins (1968) because Equations (197) and (198) each possess a sign flip resulting from the \( \theta \rightarrow \theta_{\text{LH}} \) transformation, rendering the provisional governing equation invariant to it.

To proceed, we need to generalize an identity described in Heng & Spitkovsky (2009). For an arbitrary function \( G \), one may show that
\[
(\omega_1 \hat{D} + \mu m)(G(\omega_2 \hat{D} - \mu m)v_\phi^0) = G(\omega_1 \hat{D} + \mu m)(\omega_2 \hat{D} - \mu m)v_\phi^0 + \omega_1(DG)(\omega_2 \hat{D} - \mu m)v_\phi^0, \tag{200}
\]
where \( \omega_1 \) and \( \omega_2 \) are arbitrary constants. In the free hydrodynamic limit, we use the identity with \( \omega_1 = \omega_2 = \omega \). In our case, we have
\[
G \equiv \frac{1}{\omega^2 \xi (1 - \mu^2) - m^2}, \tag{201}
\]
from which using the identity in Equation (199) yields
\[ G(\omega \dot{D} + \mu m)(\omega \dot{D} - \mu m)v_{\theta 0} + \omega(DG)(\omega \dot{D} - \mu m)v_{\theta 0} + (\omega^2 - \mu^2)v_{\theta 0} = 0. \] (202)

It is useful to note that
\[ \dot{D}G = \frac{2\omega^2 \xi \mu (1 - \mu^2)G}{\omega^2 \xi (1 - \mu^2) - m^2}. \] (203)

Following through on the algebra, one obtains the general governing equation for the meridional velocity amplitude,
\[ \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial v_{\theta 0}}{\partial \mu} - m \frac{v_{\theta 0}}{\omega} - m^2 v_{\theta 0} \right] + \xi (\omega^2 - \mu^2)v_{\theta 0} + \frac{2\omega \xi \mu}{\omega^2 \xi (1 - \mu^2) - m^2} (\omega \dot{D} - \mu m)v_{\theta 0} = 0, \] (204)
in agreement with Equation (7.8) of Longuet-Higgins (1968) and Equation (B3) of Heng & Spitkovsky (2009). This general equation is amenable to an analytical solution only in the limits of slow or fast rotation.

In the limit of slow rotation ($\xi \to 0$), the governing equation for $v_{\theta 0}$ reduces to the associated Legendre equation (Abramowitz & Stegun 1970; Arfken & Weber 1995),
\[ (1 - \mu^2) \frac{\partial^2 v_{\theta 0}}{\partial \mu^2} - 2\mu \frac{\partial v_{\theta 0}}{\partial \mu} - m \left[ \frac{1}{\omega} + \frac{m}{(1 - \mu^2)} \right] v_{\theta 0} = 0, \] (205)
if the following quantity is discretized in terms of an integer $l$,
\[ \frac{m}{\omega} = l(l + 1). \] (206)

Here, $l$ is the meridional wavenumber. The dispersion relations follow immediately,
\[ \omega_R = \frac{m}{l(l + 1)}, \]
\[ \omega_l = 0, \] (207)
and they describe the slow, undamped Rossby waves in the system, as expected.$^9$

The solution to Equation (205) is
\[ v_{\theta 0} = v_0 P_l^m, \] (208)
where $P_l^m$ is the associated Legendre function. We then use Equation (198) and the following recurrence relation for associated Legendre functions,
\[ \hat{D}P_l^m = (l + 1)\mu P_l^m - (l + m + 1)P_{l+1}^m, \] (209)
from which we obtain
\[ h_{v0} = \frac{v_0}{m} \left[ \omega_R(l + 1)P_{l+1}^m - \mu[\omega_R(l + 1) - m]P_l^m \right]. \] (210)

Finally, using Equation (196), we obtain
\[ v_{\phi 0} = \frac{v_0}{m} \left[ (l - m + 1)P_{l+1}^m - \mu(l + 1)P_l^m \right]. \] (211)

It is worth noting that, short of a normalization factor, $P_l^m \exp(imo\phi)$ are spherical harmonics.

The wave solutions are
\[ v_{\phi} = \frac{v_0 P_l^m}{\sin \theta} \sin (m \phi - \omega_R t_0), \]
\[ h_{\phi} = \frac{v_0}{m} \left[ \frac{\omega_R(l - m + 1)P_{l+1}^m - \mu[\omega_R(l + 1) - m]P_l^m}{\cos (m \phi - \omega_R t_0)} \right] \]
\[ v_{\phi} = \frac{v_0}{m} \sin \theta \left[ (l - m + 1)P_{l+1}^m - \mu(l + 1)P_l^m \right] \cos (m \phi - \omega_R t_0). \] (212)

from which the steady-state solutions naturally follow,
\[ v_{\phi} = \frac{v_0 P_l^m}{\sin \theta} \sin (m \phi), \]
\[ h_{\phi} = \frac{v_0}{m} \left[ \frac{\omega_R(l - m + 1)P_{l+1}^m - \mu[\omega_R(l + 1) - m]P_l^m}{\cos (m \phi)} \right] \]
\[ v_{\phi} = \frac{v_0}{m} \sin \theta \left[ (l - m + 1)P_{l+1}^m - \mu(l + 1)P_l^m \right] \cos (m \phi). \] (213)

At first sight, one may be concerned that the solutions blow up when $\theta = 0^\circ$, since $\sin \theta = 0$. However, we also have $P_l^m = 0$ when $\theta = 0^\circ$ ($\mu = 1$). Using the following recurrence relation,
\[ \frac{P_l^m}{(1 - \mu^2)^{1/2}} = \frac{P_{l+1}^m + [(l + 1)(l - m - 1)]P_{l-1}^m}{2m\mu}, \] (214)
one realizes that $P_l^m / \sin \theta = 0$ when $\theta = 0^\circ$ (for $m \neq 0$), so the solutions for $v_{\phi}$ and $v_{\phi}'$ simply vanish at the poles. These free hydrodynamic solutions on a sphere in the slowly rotating limit have previously been described by Margules (1893) and Hough (1898).

In the rapidly rotating limit ($\xi \to \infty$), Longuet-Higgins (1968) has previously shown that for the solutions to satisfy the boundary conditions at $\mu \to \pm 1$ (i.e., be finite), the first four terms in Equation (204) need to be retained. However, Longuet-Higgins (1968) then made the approximation that $\mu \approx 1$ near-equator solutions, which causes these solutions to formally diverge at $\mu = \pm 1$, as we will see. Nevertheless, we follow this approach to obtain approximate solutions on a sphere in the rapidly rotating limit, while being aware that our solutions will break down near the poles. Much of the mathematical machinery for rapid rotators on a sphere has already been constructed during our $\beta$-plane analysis. By making the transformation,
\[ \bar{\mu} = \alpha \mu, \]
\[ \alpha \equiv \xi^{1/4}, \] (215)
we obtain the governing equation for $v_{\theta 0}$ in the rapidly rotating limit,
\[ \frac{\partial^2 v_{\theta 0}}{\partial \bar{\mu}^2} + \left[ \left( \xi \omega^2 - m^2 - \frac{m}{\omega} \right) \frac{1}{\xi^{1/2} - \bar{\mu}^2} \right] v_{\theta 0} = 0. \] (216)
which is again the quantum harmonic oscillator equation if the
following quantity is quantized:

\[ (\xi \omega^2 - m^2 - \frac{m}{\omega} \xi^{-1/2} = 2l + 1. \]  

(217)

Unlike on the \( \beta \)-plane, we already have \( \alpha \neq 1 \) for a free
hydrodynamic system on the sphere, because \( \xi \neq 1 \). It follows
that the dispersion relations are

\[ \omega_1 \left[ \xi \omega^2 - 3\xi \omega^2 + m^2 + (2l + 1) \xi^{-1/2} \right] = 0, \]

\[ \xi \omega^3 - 3\xi \omega \omega^2 - \omega_1 [m^2 + (2l + 1) \xi^{-1/2}] - m = 0. \]  

(218)

As before (on the \( \beta \)-plane), \( \omega_1 = 0 \) is a solution of the dispersion
relations. The equations for the wave amplitudes are

\[ v_{\theta 0} = v_0 \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_l, \]

\[ h_{\theta 0} = \frac{v_0}{\omega \xi - m} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]

\[ \times [2l \omega \alpha \tilde{H}_{l-1} - \tilde{H}_l (\omega \alpha \tilde{\mu} + \mu m)], \]

\[ v_{\phi 0} = \frac{v_0}{\omega \xi - m} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]

\[ \times [2l \alpha \tilde{H}_{l-1} - \tilde{H}_l (\tilde{\mu} \mu + \mu \omega \xi)], \]  

(219)

where \( \tilde{H}_l \equiv H_l(\tilde{\mu}) \). The wave solutions are

\[ v_\theta = \frac{v_0}{\sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_l \sin (m \phi - \omega R_0 t), \]

\[ h_r \approx \frac{v_0}{\omega \xi - m} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \cos (m \phi - \omega R_0 t) \]

\[ \times [2l \omega \alpha \tilde{H}_{l-1} - \tilde{H}_l (\omega \alpha \tilde{\mu} + \mu m)], \]

\[ v_\phi = \frac{v_0}{\omega \xi - m} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \cos (m \phi - \omega R_0 t) \]

\[ \times [2l \alpha \tilde{H}_{l-1} - \tilde{H}_l (\tilde{\mu} \mu + \mu \omega \xi)]. \]  

(220)

The steady-state solutions follow naturally,

\[ v_\theta = \frac{v_0}{\sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_l \sin (m \phi), \]

\[ h_r \approx \frac{v_0 \mu}{m} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_l \cos (m \phi) \]

\[ v_\phi = \frac{v_0 \alpha}{m \sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_l \tilde{\mu} \tilde{H}_{l-1} \cos (m \phi). \]  

(221)

It is worth noting that the Hermite polynomials do not vanish as \( \tilde{\mu} \to \pm 1 \), unlike for the associated Legendre functions, implying that the solutions for the velocities do blow up at the poles. This is an artifact of the \( (1 - \mu^2) \approx 1 \) approximation.

6.1.2. Forcing with Hydrodynamic Friction

When hydrodynamic friction is added, we find that the problem is analytically tractable only when the \( \nu \partial / \partial \theta \) and \( \nu \partial^2 / \partial \theta^2 \) terms are neglected, which is equivalent to assuming that molecular viscosity acts uniformly across latitude. Note that this assumption does not apply to Rayleigh drag. Additionally, we ignore the geometric terms associated with \( \nu \) to render the algebra tractable. The governing equations become

\[ -i \frac{\partial}{\partial \tau} (iv_0''') - \mu v_0''' - \hat{D} v_\phi''' - i \omega_{\text{drag}} (iv_\phi) \]

\[ + \frac{i}{\mathcal{R}(1 - \mu^2)} \frac{\partial}{\partial \phi} (iv_\phi') = 0, \]

\[ i \mu (iv_0') - \frac{\partial v_0''}{\partial t} - \frac{\partial h_\phi''}{\partial \phi} - \omega_{\text{drag}} v_\phi' + \frac{1}{\mathcal{R}(1 - \mu^2)} \frac{\partial^2 v_\phi''}{\partial \phi^2} = 0, \]

\[ (1 - \mu^2) \left( \frac{\partial h_\phi'}{\partial t} - F_0 h_\phi' \right) + \mathcal{R}_0 \left[ i \hat{D} (iv_\phi') + \frac{\partial v_\phi''}{\partial \phi} \right] = 0, \]  

(222)

where \( \omega_{\text{drag}} \) and \( F_0 \) have been cast in dimensionless units
(normalized by \( 2 \Omega \)). We have defined the Reynolds number as

\[ \mathcal{R} \equiv \frac{2 \Omega R^2}{\nu}, \]  

(223)

where we have assumed that the characteristic velocity is \( \Omega R \).
Note that this implies that more rapidly rotating exoplanets have faster wind speeds, which is not necessarily the case. When modeling a specific object, there is no confusion as long as one specifies \( \Omega \), \( R \), and \( \nu \) and then use it to construct \( \mathcal{R} \).

The wave amplitudes are

\[ \omega_0 v_{\theta 0} + \mu v_{\phi 0} + \hat{D} h_{\phi 0} = 0, \]

\[ \mu v_{\theta 0} + \omega_0 v_{\phi 0} - m h_{\phi 0} = 0, \]

\[ \hat{D} v_{\theta 0} + m v_{\phi 0} - \omega_0 \xi (1 - \mu^2) h_{\phi 0} = 0, \]  

(224)

where \( \omega_0 \equiv \omega + i \omega_r \), \( \omega_r \equiv \omega - i \omega_l \), and

\[ \omega_0 \equiv \omega_{\text{drag}} + \frac{m^2}{\mathcal{R}(1 - \mu^2)}. \]  

(225)

Using the same mathematical machinery as outlined in Section 6.1.1, we derive

\[ (\omega_0 \hat{D} + \mu m) v_{\theta 0} = (\mu^2 - \omega_0^2) v_{\theta 0}, \]

\[ (\omega_0 \hat{D} - \mu m) v_{\phi 0} = [\omega_0 \omega_r \xi (1 - \mu^2) - m^2] h_{\phi 0}, \]  

(226)

from which we obtain

\[ (\omega_0 \hat{D} + \mu m) \frac{[\omega_0 \hat{D} - \mu m] v_{\theta 0}}{\omega_0 \omega_r \xi (1 - \mu^2) - m^2} + (\omega_0^2 - \mu^2) v_{\phi 0} = 0. \]  

(227)

To utilize the identity in Equation (200), we require that

\[ \omega_0 \approx \omega_{\text{drag}} + \frac{m^2}{\mathcal{R}}. \]  

(228)

Physically, we are seeking near-equator solutions. Using Equation (200) and noting that

\[ \hat{D} \tilde{G} = \frac{2 \omega_0 \omega_r \xi \mu (1 - \mu^2) \tilde{G}}{\omega_0 \omega_r \xi (1 - \mu^2) - m^2}, \]  

(229)

we obtain

\[ \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial}{\partial \mu} \right] v_{\theta 0} - m v_{\theta 0} - m^2 v_{\phi 0} \]

\[ + \xi \omega_r \left( \omega_0 - \frac{\mu^2}{\omega_0} \right) v_{\phi 0} \]

\[ + \frac{2omega_0 \xi \mu}{\omega_0 \omega_r \xi (1 - \mu^2) - m^2} \omega_0 \hat{D} - \mu m) v_{\phi 0} = 0. \]  

(230)
In the slowly rotating limit \((\xi \to 0)\), the governing equation for \(v_0\) is again the associated Legendre equation,

\[
(1 - \mu^2) \frac{\partial^2 v_0}{\partial \mu^2} - 2\mu \frac{\partial v_0}{\partial \mu} - m \left[ \frac{1}{\omega_0} + \frac{m}{(1 - \mu^2)} \right] v_0 = 0. \tag{231}
\]

The dispersion relations are

\[
\omega_R = -\frac{m}{l(l + 1)},
\]
\[
\omega_I = -\omega_v. \tag{232}
\]

An immediate, curious inference is that forcing does not appear to affect the wave solutions. A forced, damped hydrodynamic atmosphere behaves like a purely damped one in the slowly rotating limit. We will confirm this finding by explicitly deriving \(v_0', h_0',\) and \(v_\phi'.\)

Using the same procedure described in Section 6.1.1, we obtain the solutions for the wave amplitudes,

\[
v_0 = v_0 P_l^m, \\
h_0 = \frac{v_0}{m} \left[ \omega_0 \left( l(l + 1) P_{l+1}^m - \mu (l + 1) P_l^m \right) \right], \\
v_\phi = \frac{v_0}{m} \left[ (l - m + 1) P_{l+1}^m - \mu (l + 1) P_l^m \right]. \tag{233}
\]

The steady-state solutions are

\[
v_\theta' = \frac{v_0 P_l^m}{\sin \theta}, \sin (m\phi),
\]
\[
h'_\theta = \frac{v_0 m P_l^m}{\sin \theta} \cos (m\phi),
\]
\[
v_\phi = \frac{v_0}{m} \left[ (l - m + 1) P_{l+1}^m - \mu (l + 1) P_l^m \right] \cos (m\phi). \tag{234}
\]

One immediately sees that the solutions for \(v_\theta'\) and \(v_\phi'\) are exactly the same as in the hydrodynamic limit. Hydrodynamic friction only affects \(h_\theta'\) and introduces an out-of-phase component to the solution. Forcing is completely absent from these solutions.

In the rapidly rotating limit \((\xi \to \infty)\), the governing equation for \(v_0\) is

\[
\frac{\partial^2 v_0}{\partial \mu^2} + \left( \xi \omega_0 \omega_F - m^2 - \frac{m}{\omega_0} \right) \left( \frac{\omega_0}{\xi \omega_F} \right)^{1/2} - \tilde{\mu}^2 \right] v_0 = 0, \tag{235}
\]

where we have defined the transformed (cosine of the) colatitude as

\[
\tilde{\mu} = \alpha \mu, \\
\alpha = \left( \frac{\xi \omega_F}{\omega_0} \right)^{1/4}. \tag{236}
\]

Via the usual use of De Moivre’s formula, we obtain the dispersion relations,

\[
\begin{align*}
\xi \omega_R^2 - 3\xi \omega_R \omega_I^2 + 2\xi \omega_R \omega_I (F_0 - 2\omega_v) &+ \omega_R [\xi \omega_I (2F_0 - \omega_v) - \omega_v^2] - m^2 - m \\
&- (2l + 1) \left( \frac{\xi + \xi_R}{2} \right)^{1/2} = 0,
\end{align*} \tag{237}
\]

where the separation functions are

\[
\begin{align*}
\zeta_R &\equiv \xi \left[ \omega_R^2 - \omega_v^2 - \omega_v (\omega_v - F_0) + \omega_v F_0 \right], \\
\zeta_I &\equiv \xi \omega_I (2\omega_v + \omega_v - F_0), \\
\zeta &\equiv \left( \frac{\zeta_R + \zeta_I}{2} \right)^{1/2}. \tag{238}
\end{align*}
\]

The wave amplitudes are

\[
\begin{align*}
v_0 &\equiv v_0 \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) R_l, \\
h_0 &\equiv \frac{v_0}{\omega \omega_0 \xi \omega_F - m^2} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \times [2\omega_0 \omega_F H_{l+1} - H_l (\omega_0 \omega_F \mu + \mu m)], \\
v_\phi &\equiv \frac{v_0}{\omega \omega_0 \xi \omega_F - m^2} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \times [2m l \omega \omega_0 \xi \omega_F H_{l+1} - H_l (\mu \omega_0 \mu + \mu m \omega_F \xi)]. \tag{239}
\end{align*}
\]

In the steady-state limit, we set \(F_0 = -|F_0|\) and obtain

\[
\alpha = \left( \frac{\xi |F_0|}{\omega_v} \right)^{1/4}. \tag{240}
\]

Unlike in the \(\beta\)-plane treatment, \(\alpha\) contains an extra factor of \(\xi\), implying that \(\alpha\) is related to both the Prandtl and Rossby numbers in the forced, damped hydrodynamic limit. The steady-state solutions are

\[
\begin{align*}
v_\theta' &\equiv \frac{v_0}{\sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) R_l \sin (m\phi), \\
h'_\theta &\equiv \frac{v_0}{\omega_v |F_0| |\xi + m \omega_v|} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \times \left[ \mu m R_l \cos (m\phi) + \mu \omega_v (2l H_{l+1} - \tilde{\mu} H_l) \sin (m\phi) \right], \\
v_\phi' &\equiv -\frac{v_0}{(\omega_v |F_0| |\xi + m \omega_v|) \sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \times \left[ \mu m (2l H_{l+1} - \tilde{\mu} H_l) \cos (m\phi) + \mu |F_0| \xi R_l \sin (m\phi) \right]. \tag{241}
\end{align*}
\]

6.2. Magnetohydrodynamics (Radial Background Field)

We consider forcing, magnetic tension, and all forms of friction. For a forced, dragged shallow water system in spherical geometry with a purely radial background magnetic field, we find that the problem is analytically tractable only when the \(\eta \partial / \partial \theta\) and \(\eta \partial^2 / \partial \theta^2\) terms are neglected. We apply the same
reasoning to the molecular viscosity. We ignore the geometric terms associated with both molecular viscosity and magnetic drag. With these simplifications, the governing equations become

\[
-i \frac{\partial}{\partial t_0} (i \nu_0^v) - \mu \nu_0^v \equiv D h^v_0 - i \omega_{\text{drag}} (i \nu_0^v) + \frac{i}{R(1 - \mu^2)} \frac{\partial^2 v_0^v}{\partial \phi^2} (i \nu_0^v) + \frac{B_b b_0^v}{8 \pi \rho H \Omega} = 0,
\]

\[
i \mu (i \nu_0^\phi) - \frac{\partial v_0^\phi}{\partial t_0} - \frac{\partial h^\phi_0}{\partial \phi} - \omega_{\text{drag}} v_0^\phi + \frac{1}{R(1 - \mu^2)} \frac{\partial^2 v_0^\phi}{\partial \phi^2} = 0,
\]

\[
- \frac{\hat{B}_b b_0^\phi}{8 \pi \rho H \Omega} = 0,
\]

\[
\frac{\partial h^\phi_0}{\partial t_0} + \frac{\partial^2 b_0^\phi}{\partial \xi^\phi} + \frac{\partial^2 b_0^\phi}{\partial \phi^2} \Omega^2 H = \frac{1}{R(1 - \mu^2)} \frac{\partial^2 b_0^\phi}{\partial \phi^2} = 0,
\]

\[
\frac{\partial b_0^\phi}{\partial t_0} - \frac{\hat{B}_b b_0^\phi}{8 \pi \rho H \Omega} = 0.
\]

where we again have \( R \equiv 2 \Omega R^2 / \nu \) and the magnetic Reynolds number is

\[
R_B \equiv 2 \Omega R^2 / \eta.
\]

We have also defined

\[
b_0^\phi \equiv b_0^\phi \sin \theta.
\]

By seeking wave solutions, we find the wave amplitudes to be

\[
\omega_{B0} v_0^\nu + \mu v_0^\phi \equiv D h_0^v = 0,
\]

\[
\mu v_0^\nu + \omega_{B0} v_0^\phi - m h_0^\nu = 0,
\]

\[
\hat{D} v_0^\nu + m v_0^\phi - \omega F (1 - \mu^2) h_0^\nu = 0,
\]

where we additionally define

\[
\omega_\eta \equiv \omega + \frac{im^2}{R(1 - \mu^2)} \approx \omega + \frac{im^2}{R_B},
\]

\[
\omega_B \equiv \omega_{\nu} + \frac{i}{\omega_\eta} \frac{\nu_A^2}{2 \Omega H} = \omega_{\nu} + \frac{i}{\omega_\eta} \left( \frac{t_\text{dyn}}{t_\text{A}} \right)^2,
\]

and \( t_\text{dyn} \equiv 1 / 2 \Omega \). The quantity \( \omega_{\nu} \) is the same as defined in Equation (225). We again have \( \omega_{B0} \equiv \omega + i \omega_{\eta} \). All of these generalized frequencies are dimensionless, as is \( \omega \). The approximation associated with \( \omega_{\eta} \) is made such that the identity in Equation (200) can again be applied; we do the same for \( \omega_{\nu} \) via Equation (228). Physically, we are seeking near-equator solutions.

Since the expressions in Equation (245) are identical to those in Equation (224), except for the substitution \( \omega_{\nu} \rightarrow \omega_{B0} \), we may immediately write down the general governing equation for \( v_0^\nu \),

\[
\frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial}{\partial \mu} \right] v_{\nu} - \frac{m v_{\nu}}{\omega_{B0}} - \frac{m^2 v_{\nu}}{1 - \mu^2} + \xi \omega_F \left( \omega_{B0} - \frac{\mu^2}{\omega_{B0}} \right) v_0^\nu + \frac{2 \omega_F \xi \mu}{\omega_{B0} \omega_F \xi (1 - \mu^2) - m^2} \omega_{B0} (\hat{D} - \mu \mu) v_0^\nu = 0.
\]

In the slowly rotating limit \( (\xi \rightarrow 0) \), the governing equation for \( v_0^\nu \) is the same as Equation (231), except that \( \omega_0 \) is replaced by \( \omega_{B0} \). Making use of the separation functions and writing \( \omega_{B0} = \xi \omega_{\nu} + i (\zeta_0 + \zeta \omega_{\nu}) \) as in the case of the \( \beta \)-plane treatment, we derive the dispersion relations

\[
\omega_{\nu} = - \frac{m}{(l + 1) \zeta_0},
\]

\[
\omega_{\nu} = - \frac{\zeta_0}{\zeta},
\]

where we have defined

\[
\zeta_0 \equiv 1 \pm \left( \frac{\nu_A}{2 \Omega H} \right)^2 \omega_{\nu}^2 + \frac{1}{(\omega_{\nu}^2 + m^2 / R_B)^2},
\]

\[
\zeta_0 \equiv \omega_{\nu} + \left( \frac{\nu_A}{2 \Omega H} \right)^2 \frac{m^2}{R_B \zeta} (\omega_{\nu}^2 + m^2 / R_B)^2.
\]

The equations for the wave amplitudes are

\[
v_{0\nu} = v_0 \nu \frac{\nu_{\nu}{\mu}}{m^2},
\]

\[
h_0^v = \frac{v_0}{m^2} \left\{ \omega_{B0} (l - m + 1) \nu_{\nu}{\mu} - \mu \left[ \omega_{B0} (l - m + 1 - m) \nu_{\nu}{\mu} \right] \right\},
\]

\[
v_{0\phi} = \frac{v_0}{m^2} \left\{ (l - m + 1) \nu_{\nu}{\mu} - \mu (l + 1) \nu_{\nu}{\mu} \right\}.
\]

In the steady-state limit, we have \( \omega_{B0} = i \zeta_0 \) and the generalized friction becomes

\[
\zeta_0 = \omega_{\nu} + \left( \frac{\nu_A}{2 \Omega H} \right)^2 \frac{R_B}{m^2}.
\]

The steady-state solutions are

\[
v_{0\nu} = v_0 \nu \frac{\nu_{\nu}{\mu}}{\sin \theta} \sin (m \phi),
\]

\[
h_0^v = v_0 \nu \frac{\nu_{\nu}{\mu}}{\cos (m \phi)} \left[ (l - m + 1) \nu_{\nu}{\mu} - \mu (l + 1) \nu_{\nu}{\mu} \right] \sin (m \phi),
\]

\[
v_{0\phi} = v_0 \frac{\nu_{\nu}{\mu}}{m^2 \sin \theta} \left[ (l - m + 1) \nu_{\nu}{\mu} - \mu (l + 1) \nu_{\nu}{\mu} \right] \cos (m \phi).
\]

One can immediately see the justification for calling \( \xi_0 \) the “generalized friction”; it replaces \( \omega_{\nu} \) in the equation for \( h_0^v \) and includes the effects of hydrodynamic friction, magnetic tension, and magnetic drag. The velocities are unaffected by the generalized friction; forcing is again absent from the solutions in the slowly rotating limit.

In the rapidly rotating limit \( (\xi \rightarrow \infty) \), the governing equation for \( v_0^\nu \) is

\[
\frac{\partial^2 v_0^\nu}{\partial \mu^2} + \left[ \left( \xi \omega_{B0} \omega_F - \frac{m^2}{\omega_{B0}} \right) \left( \omega_{B0} \xi \omega_F \right)^{1/2} \right] v_0^\nu = 0,
\]

where we have defined

\[
\tilde{\mu} \equiv \alpha \mu,
\]

\[
\alpha \equiv \left( \frac{\xi \omega_F}{\omega_{B0}} \right)^{1/4}.
\]
The dispersion relations are
\[ \xi \omega_k \zeta^2 - \xi \omega_R (\zeta_0 + \omega \zeta_+)^2 - 2 \xi \zeta \omega_R (\omega_1 - F_0) (\zeta_0 + \omega \zeta_+) - m^2 \zeta - \omega_R - m - (2l + 1) \left( \frac{\xi + \xi_R}{2} \right)^{1/2} = 0, \]
\[ 2 \xi \zeta \omega_R^2 (\zeta_0 + \omega \zeta_+) + \xi \zeta \omega_R^2 (\omega_1 - F_0) - \xi (\omega_1 - F_0) (\zeta_0 + \omega \zeta_+) - m^2 (\zeta_0 + \omega \zeta_+) - (2l + 1) \left( \frac{\xi - \xi_R}{2} \right)^{1/2} = 0, \]
where we have defined
\[ \zeta_R \equiv \xi \left[ \omega_R \zeta - (\omega_1 - F_0) (\zeta_0 + \omega \zeta_+) \right], \]
\[ \zeta_1 \equiv \xi \omega_R (\zeta_0 - (\omega_1 - F_0) + \zeta_0 + \omega \zeta_+), \]
\[ \zeta = \left( \xi R + \zeta_1 \right)^{1/2}. \]

The wave amplitudes follow directly from the forced, damped hydrodynamic case, with a \( \omega_0 \to \omega_{B0} \) transformation,

\[ v_{0i} = v_0 \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_i, \]
\[ h_{0i} = \frac{v_0}{\omega_{B0} \alpha \phi - m^2} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]
\[ \times [2 \omega_{B0} \alpha \tilde{H}_{i-1} - \tilde{H}_i (\omega_{B0} \alpha \mu + \mu m)], \]
\[ v_{0\phi} = \frac{v_0}{\omega_{B0} \alpha \phi - m^2} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]
\[ \times [2 \mu \alpha \tilde{H}_{i-1} - \tilde{H}_i (\mu \alpha \mu + \mu \omega \phi)], \]

The steady-state solutions follow,

\[ v_0' = \frac{v_0}{\sin \theta} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \tilde{H}_i \sin (m \phi), \]
\[ h_0' = \frac{v_0}{\xi_0 |F_0| \xi + m^2} \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]
\[ \times [\mu \alpha \tilde{H}_i \cos (m \phi) + \alpha \xi_0 (2 \tilde{H}_{i-1} - \tilde{H}_i) \sin (m \phi)], \]
\[ v_0' = \frac{v_0}{\xi_0 |F_0| \xi + m^2} \sin \theta \exp \left( -\frac{\alpha^2 \mu^2}{2} \right) \]
\[ \times [\mu \alpha (2 \tilde{H}_{i-1} - \tilde{H}_i) \cos (m \phi) + \mu |F_0| \xi \tilde{H}_i \sin (m \phi)], \]

where we have \( \alpha = (\xi |F_0| / \xi_0)^{1/2} \). In the rapidly rotating limit, these steady-state solutions are identical to the forced, damped hydrodynamic case, except for a \( \omega_0 \to \xi_0 \) transformation, once again illustrating why \( \xi_0 \) is termed the “generalized friction.”

### 6.3. Magnetohydrodynamics (Toroidal Background Field)

In evaluating a forced, damped shallow water system with a horizontal background magnetic field on a sphere, we encounter an obstacle already elucidated on the \( \beta \)-plane: we need to set \( B_0 = b_0 = 0 \) to proceed. We ignore all geometric terms. It follows that

\[ -i \frac{\partial}{\partial t_0} (v_0'' - \mu v_\phi'') \phi - \dot{D} h_0' - i \omega_{diss} (v_0'') + i \frac{\partial}{\partial t_0} \phi \right) = 0, \]
\[ + \frac{i}{\mathcal{R}(1 - \mu^2)} \frac{\partial^2}{\partial \phi^2} (v_0'') = 0, \]
\[ i \mu (v_0'') - \frac{\partial v_0''}{\partial t_0} - \frac{\partial h_0'}{\partial \phi} - \omega_{diss} v_\phi'' + \frac{1}{\mathcal{R}(1 - \mu^2)} \frac{\partial^2 v_\phi''}{\partial \phi^2} + \frac{\overline{B}_0}{8 \pi \rho \mathcal{R} \Omega \sin \theta \phi} \frac{\partial^2 v_\phi''}{\partial \phi^2} = 0, \]
\[ (1 - \mu^2) \left( \frac{\partial h_0'}{\partial t_0} - F_0 h_0' \right) + \mu \Omega \left( i \dot{D} (v_0'') + \frac{\partial v_0''}{\partial \phi} \right) = 0, \]
\[ \frac{\partial b_\phi''}{\partial t_0} - \frac{\overline{B}_0}{2 \Omega \mathcal{R} \sin \theta \phi} - \frac{1}{\mathcal{R} \mathcal{B}(1 - \mu^2)} \frac{\partial^2 b_\phi''}{\partial \phi^2} = 0, \]

from which we obtain

\[ \omega_0 v_{ih} + \mu v_{ih} + \dot{D} h_{0i} = 0, \]
\[ \mu v_{ih} + \omega_{B0} v_{ih} - mh_0 = 0, \]
\[ \dot{D} v_{ih} + m v_{ih} - \omega_0 \xi (1 - \mu^2) h_{0i} = 0, \]

where the definitions for \( \omega_0 \), \( \omega_{B0} \), and \( \omega_{diss} \) are identical to the radial-field case, while the definition for \( \omega_{B0} \) has changed,

\[ \omega_{B0} \equiv \omega_0 + \frac{im^2}{\omega_0 (1 - \mu^2)} \left( \frac{v_A}{2\Omega \mathcal{R}} \right)^2 \approx \omega_A + \frac{im^2}{\omega_0} \left( \frac{t_{\text{en}}}{t_A} \right)^2, \]

where \( v_A \equiv \overline{B}_0 / 2 \sqrt{\pi \mathcal{R}} \). We are again seeking near-equator solutions.

By manipulating the equations for the wave amplitudes using the usual procedure, we obtain

\[ (\omega_{B0} \dot{D} + \mu \omega_{B0}) h_{0i} = (\mu^2 - \omega_0 \omega_{B0}) v_{ih}, \]
\[ (\omega_{B0} \dot{D} - \mu \omega_{B0}) v_{ih} = [\omega_0 \omega_{B0} \xi (1 - \mu^2) - m^2] h_{0i}, \]

from which the general governing equation for \( v_{ih} \) follows,

\[ \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial}{\partial \mu} \right] v_{ih} = \frac{m v_{ih}}{\omega_{B0}} - \frac{m^2 v_{ih} \omega_0}{(1 - \mu^2) \omega_{B0}} \]
\[ + \xi \omega_{B0} \left( \omega_0 - \frac{\mu^2}{\omega_{B0}} \right) v_{ih} \]
\[ + 2 \omega_0 \xi \mu \frac{\omega_{B0} \omega_{diss} (1 - \mu^2)}{\omega_0 \omega_{B0} \xi (1 - \mu^2) - m^2} (\omega_{B0} \dot{D} - \mu \omega_{B0}) v_{ih} = 0. \]

In the slowly rotating limit (\( \xi \to 0 \)), the governing equation for \( v_{ih} \) is slightly different from the case with a radial background magnetic field,

\[ (1 - \mu^2) \frac{\partial^2 v_{ih}}{\partial \mu^2} - 2 \mu \frac{\partial v_{ih}}{\partial \mu} - \frac{m v_{ih}}{\omega_{B0}} - \frac{m^2 v_{ih}}{(1 - \mu^2) \omega_{B0}} = 0, \]

where

\[ \tilde{m} \equiv \left( \frac{\omega_0}{\omega_{B0}} \right)^{1/2} m. \]
The dispersion relations take exactly the same form as in Equation (248), but with different definitions for the separation functions,

\[ \xi_\pm = 1 \pm \left( \frac{v_A}{2\Omega R} \right)^2 \frac{m^2}{\omega_R^2 + (\omega + m^2/R_B)^2}, \]

\[ \zeta_0 = \omega + \left( \frac{v_A}{2\Omega R} \right)^2 \frac{R_B}{\omega_R^2 + (\omega + m^2/R_B)^2}. \]  

(266)

Formally, the usual analytical solutions are obtained from Equation (264) only when \( m \) is both an integer and real. We immediately specialize to steady-state solutions, for which

\[ \tilde{m} = \left( \frac{\omega}{\omega_R + R_B (v_A/2\Omega R)^2} \right)^{1/2} m. \]  

(267)

In the steady-state limit, \( \tilde{m} \) is real but will not generally take on integer values. However, if \( v_A/2\Omega R \ll 1 \), we may assume \( \tilde{m} \approx m \). In this limit, the steady-state solutions take the same form as in Equation (252).

In the rapidly rotating limit (\( \xi \to \infty \)), the governing equation for \( \omega_{th} \) is

\[ \frac{d^2 \omega_{th}}{d\mu^2} + \left( \xi \omega_R \omega_R - \omega_{th} - \frac{m}{\omega_{th}} - \frac{m^2}{\omega_{th}} \right)^{1/2} \frac{\omega_{th}}{\xi \omega_R^2} - \tilde{\mu} = 0, \]  

where we have again defined \( \tilde{\mu} \equiv \alpha \mu \) and \( \alpha \equiv (\xi \omega_{th}/\omega_{th})^{1/4} \). Again using the technique of writing \( \omega_{th} = \omega_R \zeta + i(\zeta_0 + \omega \zeta_r) \), we obtain the dispersion relations,

\[ \xi \omega_R^3 \zeta_\theta \zeta_\phi \omega_R (\omega_R - F_0) (\omega_R + \omega_r) \]

\[ -\xi \omega_R (\zeta_0 + \omega \zeta_r) (2\omega_R + \omega_R - F_0) - m^2 \omega_R \\
- m - (2\iota + 1) \left( \frac{\zeta + \omega_R}{2} \right)^{1/2} = 0, \]

\[ \xi \omega_R^2 (\zeta_0 + \omega \zeta_r) - \xi (\zeta_0 + \omega \zeta_r) (\omega_R - F_0) (\omega_R + \omega_r) \\
+ \xi \omega_R \omega_{th} (2m^2 + \omega_R - F_0) - m^2 (\omega_R + \omega_r) \\
- (2\iota + 1) \left( \frac{\zeta - \omega_R}{2} \right)^{1/2} = 0, \]  

(269)

where we have defined

\[ \xi_R \equiv \xi \left[ \omega_R \zeta_\theta \zeta_\phi - (\omega_R - F_0)(\zeta_0 + \omega \zeta_r) \right], \]

\[ \zeta_1 \equiv \xi \omega_R [\zeta_\theta (\omega_R - F_0) + \zeta_0 + \omega \zeta_r], \]

\[ \zeta \equiv \left( \xi_R + \zeta_1 \right)^{1/2}. \]  

(270)

The wave amplitudes and steady-state solutions are identical to the expressions for the radial-field case, as given by Equations (257) and (258), respectively, except that the expression for the generalized friction has changed,

\[ \zeta_0 = \omega + R_B \left( \frac{v_A}{2\Omega R} \right)^2. \]  

(271)

7. APPLICATIONS TO EXOPLANETARY ATMOSPHERES

The state of the art of characterizing exoplanetary atmospheres has advanced to the point where 2D infrared maps of the atmosphere may now be obtained, albeit in a non-unique manner (de Wit et al. 2012; Majeau et al. 2012a, 2012b). These astronomical observations provide motivation for better understanding the global structure of atmospheres. Here, we use our shallow water models to elucidate some general theoretical trends. Since we are mostly interested in the global structure of exoplanetary atmospheres, we examine models with \( n = k_x = 1 \) (pseudospherical geometry) and \( l = m = 1 \) (spherical geometry).

7.1. Effects of Stellar Irradiation and Hydrodynamic Friction

Figure 2 shows examples of maps computed using our forced, damped hydrodynamic shallow water models in pseudospherical geometry, which are a consistency check with the work of Matsuno (1966), Gill (1980), and Showman & Polvani (2011). Due to a difference in the mathematical machinery used to arrive at the same solutions (see Section 5.2.2), our solutions are shifted in longitude (\( \xi \)), but this is of no consequence since they are periodic in \( \xi \). Specifically, Matsuno (1966), Gill (1980), and Showman & Polvani (2011) start with a set of stationary (time-independent) equations, derive a key governing equation involving derivatives of the forcing, and then perform a series expansion of the forcing to obtain their solutions. We assume a functional form for the forcing from the beginning, but solve the time-dependent key governing equation and obtain the stationary state as a final step.

For a free system (\( |F_0| = \omega_r = 0 \)), we recover the solution of Matsuno (1966, see his Figure 7). For forced solutions with a moderate strength of friction present (\( |F_0| = 1, \omega_r = 0.1 \) in dimensionless units\(^{10} \)), we recover the familiar chevron-shaped feature published by Matsuno (1966), Gill (1980), and Showman & Polvani (2011). A noteworthy feature of our approach is that the transition from free to forced, damped solutions is smooth with no translation in \( \xi \). A curious feature of the forced, damped solutions is a “pinching” effect, which confines the solutions to be closer to the equator for either stronger forcing or weaker friction. The same effect is seen in the solutions of Showman & Polvani (2011, see their Figure 3).

The pronounced nature of this pinching effect is an artifact of the \( \beta \)-plane approximation, partly because the Rossby number (or Lamb’s parameter) does not explicitly appear in the solutions. If one instead examines the solutions in full spherical geometry, one will see that the pinching becomes less pronounced (Figure 3). The structure of the exoplanetary atmosphere is confined to be near the equator only when the Rossby number is less than unity. The chevron-shaped feature, witnessed in 3D simulations of atmospheric circulation, is especially prominent when \( R_0 = 1 \); strong forcing further accentuates it.

7.2. Effects of Radial Magnetic Fields and Magnetic Drag

Next, we examine forced, damped, magnetized atmospheres. We define our fiducial hydrodynamic model as having \( |F_0| = 5 \) and \( \omega_r = 0.1 \) (again in dimensionless units). These parameter values were arbitrarily chosen to emphasize the chevron-shaped feature. We then examine the effects of adding magnetic tension and magnetic drag on the atmospheric structure in both pseudospherical and spherical geometries.

A fundamental parameter involved is the (square of the) ratio of dynamical to Alfvén timescales, which we estimate to be

\[ \left( \frac{t_{dyn}}{t_A} \right)^2 \sim 1-10^2 \text{ (vertical/ radial),} \]

\[ 10^{-6}-10^{-4} \text{ (horizontal),} \]  

(272)

\(^{10} \) Normalized by the reciprocal of the dynamical timescale (\( t_{dyn}^{-1} \)).
Figure 2. Montage of plots of velocity (arrows) and water height (contours) perturbations for steady-state hydrodynamic systems in the equatorial β-plane approximation for $n = k_x = 1$. The different panels are for different strengths of forcing ($|F_0|$) and hydrodynamic friction ($\omega_0$). All quantities are computed in terms of an arbitrary velocity normalization ($v_0$). Bright and dark colors correspond to positive and negative height perturbations, respectively.

(A color version of this figure is available in the online journal.)

where we have adopted parameter values appropriate to hot Jupiters: $L_0 = R \sim 10^{10}$ cm, $H \sim 10^7$ cm, $v_A \sim 10^2$–$10^3$ cm s$^{-1}$, and $\epsilon_0 \sim 10^5$ cm s$^{-1}$. These values of $v_A$ correspond to field strengths of $\sim$1–10 G and temperatures of $\sim$1500 K at $\sim$1 bar. This diverse range of parameters stems from the difference between the characteristic horizontal and vertical/radial length scales involved and produces a rich variety of atmospheric structures. These estimates already show that unless unrealistic field strengths are adopted, magnetized atmospheres with purely toroidal magnetic fields resemble their hydrodynamic counterparts. For this reason, we will only examine models with purely vertical/radial background magnetic fields. Algebraic intractability prevents us from exploring purely poloidal fields, which may produce markedly different structures.

In Figure 4, we examine examples of magnetized atmospheres with three field strengths: $\sim$1 G, $\sim$3 G, and $\sim$10 G. For the spherical models, we set the Rossby number to be unity. Magnetic tension and magnetic drag are degenerate effects when specified via $\alpha$, so it is sufficient to hold the magnetic Reynolds number fixed ($R_B = 1$) and vary $t_{\text{dyn}}/t_A$. Generally, we see that the steady state of the atmosphere looks qualitatively different from its hydrodynamic counterpart. The pseudo-spherical and spherical solutions are shifted in longitude by some amount, due to the slightly different mathematical approaches used to arrive at the steady-state solutions, but otherwise the computed maps are in qualitative agreement. The familiar chevron-shaped feature seen in Figure 3 is diluted by the enhanced presence of generalized friction ($\zeta_0 \sim 1$–100). We again see that the pronounced nature of this pinching effect is an artifact of the β-plane approximation—it is diluted in the spherical models, even though we have set $R_0 = 1$. As the magnetic field strength increases, the flow transitions from being predominantly zonal (and possessing the chevron-shaped feature) to being predominant meridional. When the field strength is $\sim$10 G, the height perturbation field—a proxy for the temperature field—resembles the irradiation profile. An analogous transition has been witnessed in 3D hydrodynamic simulations, which elucidate the transition from jet- to drag-dominated regimes (Showman et al. 2013). The key difference is that, in our shallow water models, the flow converges at the substellar point, opposite from what the 3D models of Showman et al. (2013) find.

While we have elucidated trends using specific values of parameters, our formalism shows that forcing, hydrodynamic friction, magnetic drag, and magnetic tension are degenerate effects that combine to determine the global structure of an exoplanetary atmosphere, at least with the approximations we have taken in deriving our analytical solutions. This finding informs us that infrared phase curves alone will not suffice to uniquely distinguish between these different effects.

7.3. Effects of Rotation

In the slowly rotating limit, forced, damped atmospheres in the shallow water approximation behave like purely damped ones in the absence of friction. The velocity field is unaffected
Figure 3. Montage of plots of velocity perturbations (arrows) and $h_\nu$ (contours) for steady-state hydrodynamic systems in full spherical geometry for $l = m = 1$, exploring the effects of rotation (via variation of the Rossby number) and forcing. All quantities are computed in terms of an arbitrary velocity normalization ($v_0$). Bright and dark colors correspond to positive and negative height perturbations, respectively. (A color version of this figure is available in the online journal.)

by all forms of friction, including magnetic tension, while the water height (a proxy for the temperature) is shifted in longitude. Figure 5 shows examples of free and forced, damped shallow water models on a sphere, where this phenomenon is clearly seen for both hydrodynamic and magnetized systems. From the estimates made in Equation (272), we consider systems with purely toroidal magnetic fields to be uninteresting, since they behave mostly like hydrodynamic systems. Therefore, we consider only systems with purely radial background magnetic fields (but with horizontal field perturbations present) in Figure 5. The basic conclusion is that, when rotation is unimportant, all forms of friction simply introduce a phase shift to the shallow water height perturbation.

Next, we “turn on” rotation by examining models in the rapidly rotating limit. We start by focusing on free hydrodynamic models on a sphere. When rotation becomes rapid, vortices start to appear in the velocity field (Figure 6). When the Rossby number is of the order of unity ($R_0 \approx 0.5$ in our example), the solution resembles that on the $\beta$-plane (Figure 2). At $R_0 = 0.05$, rotation becomes rapid enough that the atmospheric structure is confined to being near the equator (rotational pinching).

7.4. Why Hydrodynamic Friction and Magnetic Drag are Fundamentally Different

Among the sources of friction explored, Rayleigh drag is the easiest to incorporate into any model, as it acts equally on all length scales and does not vary across either latitude or longitude. These properties make it attractive to use Rayleigh drag to mimic magnetic drag, especially when adapting 3D
Figure 4. Montage of plots of velocity (arrows) and water height perturbations (β-plane) or $h'$ (spherical), shown as contours, for steady-state MHD systems in both the equatorial β-plane approximation (left column; for $n = k_x = 1$) and full spherical geometry (right column; for $l = m = 1$) with purely vertical/radial background magnetic fields. We adopt fixed values of the forcing ($|F_0| = 5$) and hydrodynamic friction ($\omega_\nu = 0.1$). The different panels are for $\bar{B}_z \sim 1$ G, $\sim 3$ G, and $\sim 10$ G. All quantities are computed in terms of an arbitrary velocity normalization ($v_0$). Bright and dark colors correspond to positive and negative height perturbations, respectively.

(A color version of this figure is available in the online journal.)

hydrodynamic simulations of atmospheric circulation to include magnetic drag (e.g., Perna et al. 2010). To some extent, this approach is reasonable within the context of our shallow water models, as all sources of friction, including magnetic tension, are included within the generalized friction ($\zeta_0$). However, even when specifying $\zeta_0$, magnetic drag and magnetic tension act collectively and their overall scale dependence depends on the assumed field geometry. Furthermore, our 1D models have demonstrated that hydrodynamic friction and magnetic drag have different phase signatures of damping. Due to the approximations taken, we are unable to explore this property further for our 2D pseudo-spherical and spherical models, but it suggests that hydrodynamic friction and magnetic drag will alter the atmospheric flow in fundamentally different ways in a full-fledged numerical calculation (Batygin et al. 2013; Rogers & Showman 2014).

7.5. Qualitatively Altering the Structure of Irradiated Exoplanetary Atmospheres

Our study has shown that under a wide range of conditions, the chevron-shaped feature believed to be present in the highly irradiated atmospheres of tidally locked exoplanets is generic and robust, due to the near-universality of the quantum
Figure 5. Montage of plots of velocity perturbations (arrows) and $h'_v$ (contours) for steady-state, hydrodynamic, and MHD systems in full spherical geometry for $l = m = 1$. In the slowly rotating limit, forcing has no effect on the atmospheric structure, for both hydrodynamic and magnetized systems. All quantities are computed in terms of an arbitrary velocity normalization ($v_0$). Bright and dark colors correspond to positive and negative height perturbations, respectively. (A color version of this figure is available in the online journal.)

Figure 6. Plots of velocity perturbations (arrows) and $h'_v$ (contours) for steady-state hydrodynamic systems in full spherical geometry for $l = m = 1$ in the rapidly rotating limit. All quantities are computed in terms of an arbitrary velocity normalization ($v_0$). Bright and dark colors correspond to positive and negative height perturbations, respectively. (A color version of this figure is available in the online journal.)

The harmonic oscillator equation governing the meridional velocity. Conversely, our study has also shown that the key governing equation does not follow that of the quantum harmonic oscillator if the following conditions occur:

1. Molecular viscosity or magnetic drag acts non-uniformly across latitude.
2. A poloidal magnetic field is present.

(See Equations (116), (152), and (161) for the first statement. See Sections 5.4.1 and 6.3 for the second statement.) That the meridional velocity ceases to be described by the quantum harmonic oscillator equation under these conditions suggests that the chevron-shaped feature will be qualitatively altered in the presence of these effects.

Hints of this behavior have already been seen in the non-ideal MHD simulations of Batygin et al. (2013), who assumed a constant magnetic resistivity and a shallow atmosphere (we have adopted both assumptions) in the Boussinesq approximation (2D). The conclusion regarding the importance of the poloidal magnetic field was also reached by Rogers & Showman (2014).
via performing 3D non-ideal MHD simulations in the anelastic approximation.

8. DISCUSSION AND SUMMARY

8.1. Summaries

1. Near-universality. Atmospheres in the shallow water approximation are fundamentally described by the quantum harmonic oscillator equation, even when they are forced, rotating, magnetized, and possess both hydrodynamic and magnetic sources of friction. This near-universality is broken when either molecular viscosity or magnetic drag acts non-uniformly across latitude; it is also broken in the presence of a poloidal magnetic field.

2. Key controlling parameter. The global structure of an exoplanetary atmosphere is essentially controlled by a single, dimensionless number that we call the “key controlling parameter” ($\alpha$). In the hydrodynamic limit, $\alpha$ is directly related to the Prandtl and Rossby numbers. When magnetic fields are present, $\alpha$ additionally involves magnetic tension and magnetic drag as sources of friction. In pseudo-spherical geometry, it was previously realized that $1/\alpha^4$ is the Prandtl number in the hydrodynamic limit. We demonstrate that in full spherical geometry, this description is incomplete, as $\alpha$ generally involves the Rossby number as well.

3. Global structure of exoplanetary atmospheres. We are able to solve for the steady state of an atmosphere in the presence of forcing, friction, rotation, and magnetic fields. We use our analytical solutions to elucidate the manifestation of each effect in 2D thermal maps. Generally, there is degeneracy between the various effects and it will require multi-wavelength data across multiple epochs to disentangle them.

4. Hydrodynamic friction versus magnetic drag. Molecular viscosity acts predominantly on small scales, while Rayleigh drag acts equally on all scales. Magnetic tension and magnetic drag act collectively—whether they favor large scales or are collectively scale-free depends on the field geometry. Using Rayleigh drag to mimic magnetic drag is akin to asserting that it acts preferentially on a scale that is germane to the problem. Hydrodynamic friction and magnetic drag possess dissimilar phase signatures and will generally alter the atmospheric flow in qualitatively different ways.

5. Rotation. Rotation, an intrinsically 2D phenomenon, modifies the balance between forcing and friction in a non-trivial manner. When rotation is unimportant, forced atmospheres in the shallow water approximation behave like purely damped ones, as if forcing was absent. In spherical geometry, rapid rotation acts to confine the global structure of the atmosphere to be near the equator.

6. Pinching effect. In the shallow water approximation, atmospheres experience a pinching effect that is caused by faster rotation, stronger forcing, or weaker friction, because all of these effects cause the key controlling parameter ($\alpha$) to take on higher values. In pseudo-spherical geometry, this pinching effect is more pronounced—it is an artifact of the equatorial $\beta$-plane approximation, partly because it does not explicitly involve the Rossby number.

7. Coupling of physical effects. Forcing, rotation, magnetic fields, and sources of friction couple in various ways to modify the frequencies and structures of waves in shallow water systems.

8.1.1. Physical Summary

1. Broad theoretical survey. Our survey of shallow water models covers a broad range of technical possibilities, exploring dimensionality (one and two dimensions); geometry (Cartesian, pseudo-spherical, and spherical); free, forced, and damped systems; hydrodynamic and MHD systems; and hydrodynamic and magnetic sources of friction. Generally, shallow water systems may be described by five parameters, albeit with a series of assumptions and caveats (see “Obstacles to analytical solutions”); the forcing ($F_0$), the hydrodynamic friction ($\omega_H$), the magnetic Reynolds number ($R_B$), the Rossby number ($R$, the ratio of dynamical to Alfvén timescales ($t_{\text{dyn}}/t_A$).

2. Generalized, complex frequencies. When generalizing from free hydrodynamic systems to forced MHD systems with friction, the governing equations, dispersion relations, and wave solutions contain complex frequencies that are generalizations of the wave frequency, which is strictly real in free systems. Four complex frequencies are needed: $\omega_F$ (forcing), $\omega_H$ (hydrodynamic friction), $\omega_M$ (magnetic drag), and $\omega_T$ (magnetic tension and friction). With each generalization, the mathematical equations retain the same form, except that the wave frequency ($\omega$) in various places is substituted with one of these four complex frequencies. Generally, $\omega_T$ contains a quantity denoted by $\zeta_0$, which we term the “generalized friction,” as it involves all sources of friction, including magnetic tension.

3. Slowly versus rapidly rotating limits. In full spherical geometry, the key governing equation has two limiting forms with analytical solutions (Figure 1). In the slowly rotating limit, it is the associated Legendre equation, which yields spherical harmonics for the wave solutions. In the rapidly rotating limit, it is the quantum harmonic oscillator equation, which yields Hermite polynomials as solutions for the wave amplitudes. The solutions on the equatorial $\beta$-plane mirror the spherical solutions when the Rossby number is of the order of unity.

4. Dispersion relations. For each system, we obtain the dispersion relations, a pair of equations for the real ($\omega_R$) and imaginary ($\omega_I$) components of the wave frequency, which describe the oscillatory and growing or decaying parts of the wave, respectively. The $\beta$-plane and spherical derivations mirror each other, except that Lamb’s parameter ($\xi$) is generally not unity for the latter. Deriving the dispersion relations requires the use of De Moivre’s formula and a set of separation functions ($\xi_\beta$, $\xi_\beta$, $\xi_\zeta$, $\xi_\zeta$, and $\xi_0$).

5. Obstacles to analytical solutions. On the $\beta$-plane, we are unable to proceed analytically unless we ignore the poloidal background magnetic field and its perturbations. On a sphere, we further restrict ourselves to near-equator solutions. In all of the models, we require molecular viscosity and magnetic drag to act uniformly across latitude.

Table 3 provides an executive summary of the lessons learned from each shallow water model, in order of increasing sophistication, and lists progressively the approximations needed to render the problem amenable to analytical solution.
8.2. Comparison to Previous Analytical Work

To place our current study in context, we provide a comparison to previous analytical work in Table 2. Generally, the early works on the shallow water system were inspired by studies of the terrestrial atmosphere or ocean and tend to focus on free or forced, damped hydrodynamic systems (Matsuno 1966; Lindzen 1967; Longuet-Higgins 1967; Gill 1980), including the monographs of Holton (2004), Kundu & Cohen (2004), and Vallis (2006). Most works did not compute the shallow water system in spherical geometry and instead utilized the β-plane approximation, with the seminal work of Longuet-Higgins (1968) being a notable exception.

Later works that were inspired by the study of the Sun (Gilman 2000; Zaqarashvili et al. 2007) or neutron stars (Spitkovsky et al. 2002; Heng & Spitkovsky 2009) tend to focus on free magnetized systems. Little attention has been paid to studying forced, magnetized systems with friction, since this is an unfamiliar regime for the atmospheric dynamics of these objects. The first generalization of the shallow water system to exoplanets considered forced, damped hydrodynamic systems on the β-plane (Showman & Polvani 2011). Furthermore, there has been no previous generalization to consider both vertical/radial and horizontal magnetic fields, as well as both slow and rapid rotation, within a single study. We have written down the governing equations, dispersion relations, and wave solutions for every system, which was previously not done even for forced, damped hydrodynamic systems.

As already mentioned, while it was realized that the meridional velocity is governed by the quantum harmonic oscillator equation in the limit of a free hydrodynamic shallow water system (either in pseudo-spherical or spherical geometry), it was previously not demonstrated that this property extended to forced, damped, magnetized time-dependent systems.

8.3. Relevance to Atmospheric Circulation Simulations

Since the most easily characterizable exoplanets are the large, highly irradiated gas giants, there has been intense interest in understanding the physics of ~1000–3000 K partially ionized atmospheres. Initial work in the field has focused on adapting 3D general circulation models, traditionally used for studying the relatively quiescent and neutral atmosphere of Earth, toward understanding hot Jupiters (e.g., Showman et al. 2009). Besides the formidable problem of working in unfamiliar physical and chemical regimes, several concerns have been raised about the numerical issues surrounding such studies, ranging from the ambiguity associated with numerical dissipation (Heng et al. 2011; Thrastarson & Cho 2011) to the possible non-uniqueness of solutions due to differing initial conditions (Thrastarson & Cho 2010; Liu & Showman 2013). Additionally, shocks are expected to exist in these highly irradiated atmospheres (Dobbs-Dixon & Lin 2008; Li & Goodman 2010; Heng 2012). No simulation has succeeded in including non-ideal MHD and shocks in a 3D general circulation model. The analytical solutions in this study provide a point of reference and a suite of tests for building up to such a simulation, although it should be noted that shallow water systems do not include shocks by definition.

8.4. Correspondence Between the Shallow Water and Isothermal Euler Equations

The shallow water system has a direct correspondence to the mass continuity and isothermal Euler equations in two dimensions. Consider \( \rho = \rho_0 + \rho' \), where \( \rho_0 \) is the background mass density and \( \rho' \) is the perturbation of the mass density. When we let \( h \to \rho \) in Equation (3), it becomes the mass continuity equation in two dimensions. Using the ideal gas law \( P = \rho R_{\text{gas}} T_0 \), one can show that \( g \) needs to be replaced by \( R_{\text{gas}} T_0 / \rho_0 \) in the momentum equation, where \( R_{\text{gas}} \) is the specific gas constant and \( T_0 \) is a constant value of the temperature. The square of the characteristic velocity is then \( c_0^2 = R_{\text{gas}} T_0 \), which is the specific energy.

In practice, this correspondence means that one may adapt numerical fluid dynamical solvers to mimic shallow water systems and compare them with the analytical solutions presented in this study.

8.5. Temporal Behavior of Forced, Damped, Magnetized Systems: Power Spectra of Exoplanets and Brown Dwarfs

In this study, we have derived a suite of dispersion relations, which describe how the frequencies of Poincaré and Rossby waves are modified in the presence of forcing, friction, and magnetic fields. Previous mathematical techniques allowed for the dispersion relations to be derived only in the free (unforced) limit (Matsuno 1966; Lindzen 1967; Longuet-Higgins 1968; Gill 1980). These dispersion relations need to be solved numerically, which is beyond the scope of this study. Their solution allows for time-dependent wave solutions to be constructed, which may aid the interpretation of simulations. Such solutions are relevant for studying the temporal behavior of exoplanets (Showman et al. 2009; Agol et al. 2010) and brown dwarfs (Artigau et al. 2009; Showman & Kaspi 2013; Robinson & Marley 2014). One may also derive the dispersion relations and wave solutions near the poles (Longuet-Higgins 1968; Heng & Spitkovsky 2009); in this study, we have focused on near-equator solutions.

K.H. acknowledges financial, secretarial, and logistical support from the Center for Space and Habitability (CSH) and the Space Research and Planetary Sciences (WP) Division of the University of Bern, as well as grants from the Swiss National Science Foundation (SNSF) and the Swiss-based MERAC Foundation for the Exoclimes Simulation Platform (www.exoclimes.org). We are grateful to Sébastien Fromang for useful conversations.

APPENDIX

USEFUL EXPRESSIONS AND IDENTITIES

Some useful, commonly used expressions include

\[
\omega^2 = \omega_R^2 - \omega_I^2 + 2i\omega_R \omega_I,
\]

\[
\omega^3 = \omega_R^3 - i \omega_I^3 - 3\omega_R \omega_I^2 + 3i\omega_I^2 \omega_R,
\]

\[
\hat{D}_{\nu \theta_0} = \nu_0 \alpha \exp \left( \frac{-\mu^2}{2} \right) (2i\tilde{H}_{\nu \theta_0} - \mu \tilde{H}_{\theta_0}). \tag{A1}
\]

The expression involving \( \hat{D}_{\nu \theta_0} \) makes the approximation that \((1 - \mu^2) \approx 1\).

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