Inscribed Matter Communication: Part II

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Abstract—This paper, combined with Part-I [1], provides a comprehensive information-theoretic treatment of the molecular communication problem at its finest grain – emission and detection of individual molecules, both identical (timing channel) or with embedded “inscribed matter” payloads. Part-I provides the overarching framework while this, Part-II, develops results which both extend previous timing channel results (“Bits Through Queues” [2]–[4]) to emission schedules with deadlines as well as providing analytic expressions for a quantity key to the analysis of the identical token timing channel – the “ordering entropy” $H(\Omega|\vec{S}, \vec{T})$. Expressions for $H(\Omega|\vec{S}, \vec{T})$ allow us to develop upper bounds for the mutual information between input and output of the identical token timing channel which are then used in Part-I to consider both the timing channel and the timing+payload channel.

I. INTRODUCTION

In Part-I [1] of this two-paper set we defined a signaling model and developed an information theoretic framework for evaluating the capacity and efficiency of channels which use molecules (or “tokens”) as information carriers. Here in Part-II we provide some necessary undergirding results which are interesting in their own right. In particular, we consider a timing channel similar but not identical to Anantharam’s and Verdú’s “Bits Through Queues” channel [2]–[4] wherein a mean launch time constraint is replaced by a launch deadline constraint. We derive closed forms for the optimizing distribution and the channel capacity for this older timing channel and then apply the results to the molecular communication problem. We also derive analytic expressions and bounds for a key quantity in our analysis – the ordering entropy $H(\Omega|\vec{S}, \vec{T})$ – first generally and then specifically for exponential first-passage time distribution. These results support the capacity bounds of Part-I [1], provide capacity results for a timing channel with an emission deadline under exponential first-passage, and also establish that unlike the mean-constrained timing channel, the worst case corruption is not exponential first-passage. Our analysis ends with the derivation of an upper bound on the token timing channel capacity.

II. BRIEF PROBLEM DESCRIPTION

A detailed discussion of the underlying molecular communication problem and its importance in both biology and engineering is provided in Part-I [1]. Here we assume basic familiarity with the concepts and provide only the mathematical description of the system. As a reader aid, key quantities are provided in TABLE I and in an identical table in Part-I.

Thus, consider a communication system in which $M$ identical tokens are released/launched at times $T_1, T_2, \cdots, T_M$ with no assumption that the $T_m$ are ordered in time. The duration of each token’s journey from transmitter to receiver is a random variable $D_m$ so that token $m$ arrives at time $S_m = T_m + D_m$. The $D_m$ are assumed independent and identically distributed (i.i.d.). In vector notation, we have $\vec{S} = \vec{T} + \vec{D}$. We denote the density of each $D_m$ as $f_{D_m}(d) = g(d), d \geq 0$ and the cumulative distribution function (CDF) as $F_{D_m}(d) = G(\cdot)$. Likewise, the complementary CDF (CCDF) is $\bar{G}(\cdot)$. The channel output is the time-sorted version of the $\{S_m\}$ which we denote as $\{\vec{S}_i\}, \vec{S}_i < \vec{S}_{i+1}$.

However, since the tokens are identical and their transit times are random, the receiver cannot unequivocally know which arrival, $S_i$ corresponds to which transmission $T_m$. That is, $\vec{S}$, the ordered arrival times are related to $\vec{S}$ through a

| Token | A unit released by the transmitter and captured by the receiver |
|-------|---------------------------------------------------------------|
| Payload | Physical information (inscribed matter) carried by a token |
| $\lambda$ | The average rate at which tokens are released/launched into the channel |
| $T$ | A vector of token release/launch times |
| First-Passage | The time between token release/launch and token capture at the receiver |
| $D$ | A vector of first-passage times associated with launch times $T$ |
| $G(\cdot)$ | The cumulative distribution function for first-passage random variable $D$ |
| $1/\mu$ | Average/mean first-passage time |
| $\rho$ | $\lambda/\mu$, a measure of system token “load” |
| $S$ | A vector of token arrival times, $S = T + D$ |
| $P_h(\vec{x})$ | A permutation operator which rearranges the order of elements in vector $\vec{x}$ |
| $\Omega$ | The “sorting index” which produces $\vec{S}$ from $\vec{S}$, i.e., $\vec{S} = P_h(\vec{S})$ |
| $\vec{S}$ | An ordered vector of arrival times obtained by sorting the elements of $\vec{S}$ (note, the receiver only sees $\vec{S}$ not $\vec{S}$) |
| $I(\vec{S}; \vec{T})$ | The mutual information between the launch times (input) and the arrival times (output) |
| $I(\vec{S}; \vec{T})$ | The mutual information between the launch times (input) and the ordered arrival times (output) |
| $h(\vec{S})$ | The differential entropy of the arrival vector $\vec{S}$ |
| $H(\Omega|\vec{S}, \vec{T})$ | The ordering entropy given the input $\vec{T}$ and the output $\vec{S}$ |
| $H^* (\vec{T})$ | An upper bound for $H(\Omega|\vec{S}, \vec{T})$ |
| $C_q$ and $C_t$ | The asymptotic per token and per unit time capacity between input and output |

TABLE I
Glossary of Useful Terms

$S = T + D$. We denote the density of each $D_m$ as $f_{D_m}(d) = g(d), d \geq 0$ and the cumulative distribution function (CDF) as $F_{D_m}(d) = G(\cdot)$. Likewise, the complementary CDF (CCDF) is $\bar{G}(\cdot)$. The channel output is the time-sorted version of the $\{S_m\}$ which we denote as $\{\vec{S}_i\}, \vec{S}_i < \vec{S}_{i+1}$.

However, since the tokens are identical and their transit times are random, the receiver cannot unequivocally know which arrival, $S_i$ corresponds to which transmission $T_m$. That is, $\vec{S}$, the ordered arrival times are related to $\vec{S}$ through a
permutation operation, \( P\Omega(\tilde{S}) = S \) and from the receiver’s perspective, \( \Omega \) is a random variable, \( \Omega = 1, 2, \ldots, M! \).

In the next section, we provide a sampling of results from Part-I upon which we will expand here in Part-II.

III. KEY RESULTS FROM THE COMPANION PAPER \[1\]

A good deal of effort was expended in Part-I quantifying the relationships between \( T, D, S \) and \( \tilde{S} \), and in developing a signaling discipline wherein the measure of communication efficacy is determined by the mutual information between \( S \) and \( T \), \( I(\tilde{S}; T) \). That is, we took care to make sure that channel coding theorem results \[3\] (chapts 8 & 10) could be applied by deriving a model in which channel uses were (asymptotically) independent.

Specifically, we assume sequential finite signaling intervals/epochs of duration \( \tau \) and then define the token intensity as \( \lambda = \frac{M}{\mu} \) as a proxy for transmitter power (each emission “costs” some fixed energy). In addition, we assume that the **mean first-passage time** exists with \( E[D] = 1/\mu \) so that tokens always (eventually) arrive at the receiver. It is important to note that finite first-passage time is important for information-theoretic patency of the analysis. As shown in Part-I \[1\], finite first-passage allows sequential signaling intervals (channel uses) to be derived which are, in the limit of long intervals, asymptotically independent. Infinite first-passage does not allow such asymptotically independent sequential intervals to be constructed so that mutual information \( I(\tilde{S}; T) \) is not necessarily the proper measure of information carriage for the system.

We note that transport processes such as free-space diffusion do not have finite first-passage. However, any physical system is limited in extent and therefore does have finite (though perhaps long) first-passage under an ergodic transport model. So, the analysis holds for situations where tokens eventually arrive at the receiver. Of course, as discussed in Part-I, there are situations where a token might never arrive at *any* time. Such situations include channels where the token “denatures” and becomes unrecognizable by the receiver or is “gathered” by agents in the channel which remove the token from circulation before detection \[6, 7\]. Such tokens do not contribute to intersymbol interference (earlier tokens corrupting a subsequent interval) so it is possible that slightly different first-passage time distributions could be used which still preserve asymptotically independent channel uses. However, since any such model produces a first passage density, \( g(\delta) \) with singularities, the specific analysis used in Part-I is not immediately applicable. The implications (and shortcomings) of the finite first-passage assumption are discussed more carefully in the Discussion & Conclusion section of Part-I.

Now, as a prelude to deriving channel capacity, we recall from Part-I \[1\] that if \( Q(x) \) is a hypersymmetric function, \( Q(x) = Q(P_k(x)) \) \( \forall k \) where \( P_k(\cdot) \) is a permutation operator and \( X \) is a hypersymmetric random vector whose PDF obeys \( f_X(x) = f_X(P_k(x)) \), then when \( \tilde{X} \) is the ordered version of random vector \( X \) we have

\[
E_{\tilde{X}}[Q(\tilde{X})] = E_X[Q(X)] \quad (1)
\]

This expression (Theorem 1 from Part-I) allows us to avoid deriving order distributions on potentially correlated random variables.

Next, the mutual information between the input \( T \) and the output \( \tilde{S} \) of the token timing channel is given by

\[
I(\tilde{S}; T) = h(\tilde{S}) - h(\tilde{S}|T) \quad (2)
\]

Then, if we assume that \( g(\cdot) \) does not contain singularities, we observe that the set of all \( S \) for which two or more elements are equal is of zero measure which allows us to “fold” the distribution on \( f_S(\cdot) \) to obtain a distribution on the ordered \( \tilde{S} \). If we then in addition assume hypersymmetric \( X \), we can write equation (2) as

\[
I(\tilde{S}; T) = h(S) - \log M! - h(\tilde{S}|T) \quad (3)
\]

Hypersymmetry of \( X \) and no singularity in \( g(\cdot) \) implies that we can ignore situations where one or more of the \( S_i \) are equal, which then implies an equivalence

\[
(\tilde{S}, \Omega) \Leftrightarrow S \quad (4)
\]

which leads to

\[
h(S|T) = h(\tilde{S}, \Omega|T) = H(\Omega|\tilde{S}, T) + h(\tilde{S}|T) \quad (5)
\]

where \( H(\Omega|\tilde{S}, T) \) is the **ordering entropy**, a measure of the uncertainty about which \( S_m \) correspond to which \( \tilde{S}_i \). Equation (5) allows us to write the equation (3) as

\[
I(\tilde{S}; T) = I(S; T) - \left( \log M! - H(\Omega|\tilde{S}, T) \right) \quad (6)
\]

And since we know asymptotically independent channel uses can be assured (Part-I Theorem 2, \[1\]), the channel capacity in bits/nats per channel use is

\[
C = \max_{f_T(\cdot)} I(S; T) - \left( \log M! - H(\Omega|\tilde{S}, T) \right)
\]

We then derived an upper bound for the ordering entropy \( H(\Omega|\tilde{S}, T) \) in Part-I as

\[
H(\Omega|\tilde{S}, t) \leq H^\dagger(t) \quad (7)
\]

and derived/defined \( H^\dagger(\cdot) \) as

\[
H^\dagger(t) = \sum_{\ell=1}^{M-1} \log(1 + \ell)
\]

\[
\times \sum_{m=\ell, |x|=\ell}^{M-1} \prod_{j=1}^{m} \tilde{G}^{x_j}(\tilde{t}_{m+1} - \tilde{t}_j)G^{1-x_j}(\tilde{t}_{m+1} - \tilde{t}_j) \quad (8)
\]

with \( \tilde{t} \) the size-ordered version of \( t \) and \( \bar{x} \) a binary \( m \)-vector with \( |x| \) defined as its number of non-zero entries. Then, through hypersymmetry arguments as in equation (1), we showed that

\[
H(\Omega|\tilde{S}, t) = E_t \left[ H(\Omega|\tilde{S}, t) \right] \leq E_t \left[ H^\dagger(t) \right] = H^\dagger(T) \quad (9)
\]

with equality iff the first passage time is exponential (Theorem 8, Part-I \[1\]).

Based on asymptotically independent channel uses, two key measures of channel capacity were derived. The first, \( C_q \), is the asymptotic per token capacity:

\[
C_q = \lim_{M \to \infty} \frac{1}{M} I(\tilde{S}; T) \quad (10)
\]

and the second is \( C_t = \lambda C_q \), the asymptotic per unit time capacity (Theorem 4, Part-I \[1\]).
In what follows we will seek to maximize \( h(S) \) under the deadline constraints on the \( T \), derive a variety of expressions for \( H^+(T) \) for general and then for exponential first-passage under both a deadline and also the mean launch constraint considered in “Bits Through Queues” \([2]\) and elsewhere \([3], [4]\). We follow with asymptotic results for \( H^+(T)/M^+ \) as \( M \to \infty \) again assuming exponential first-passage and close by providing upper bounds for \( C_q \) and \( C_I \).

IV. “Bits Through Queues” With a Deadline Constraint

A. Preliminaries

The award-winning paper, “Bits Through Queues” \([2]\) and others \([3], [4]\) derived capacity results for a timing channel under a mean launch time constraint. In this section we derive results for a similar single-token timing channel where instead of a mean constraint, the launch time \( T \) is limited to \([0, \tau] \) \([8], [9]\) and first-passage is exponential with parameter \( \mu \). Here we provide closed forms for both the capacity and for the capacity-achieving input density. However, unlike in \([2]\) we show that exponential first-passage is not the worst case corruption for the launch deadline-constrained channel.

Since \( T \) is independent of \( D \), the density of \( S = T + D \) is given by

\[
f_S(s) = \int_0^s f_T(t)f_D(s-t)dt \quad 0 \leq s
\]

and because \( T \) is constrained to \([0, \tau] \), we can divide \( f_S(s) \) into two regions: region I where \( s \in [0, \tau] \) and region II where \( s \in (\tau, \infty) \). We then have

\[
f_S(s) = \begin{cases} \sigma f_{S|I}(s) & 0 \leq s \leq \tau \\ (1-\sigma)f_{S|II}(s) & s > \tau \end{cases} \tag{11}
\]

where

\[
\sigma = \int_0^\tau f_S(s)ds
\]

with

\[
\sigma f_{S|I}(s) = \int_0^s f_T(t)f_D(s-t)dt \tag{12}
\]

and

\[
(1-\sigma)f_{S|II}(s) = \int_0^\tau f_T(t)f_D(s-t)dt \tag{13}
\]

For \( D \) exponential with parameter \( \mu \) we have

\[
f_S(s) = \int_0^s f_T(t)e^{-\mu(s-t)}dt \quad 0 \leq s \tag{14}
\]

and

\[
\sigma f_{S|I}(s) = \int_0^s f_T(t)e^{-\mu(s-t)}dt \tag{15}
\]

and

\[
(1-\sigma)f_{S|II}(s) = e^{-\mu s} \int_0^\tau f_T(t)e^{\mu t}dt \tag{16}
\]

The entropy of \( S \) is then

\[
h(S) = - \int_0^\infty f_S(s) \log f_S(s)ds = - \int_0^\tau \sigma f_{S|I}(s) \log (\sigma f_{S|I}(s)) ds\tag{17}
\]

\[
- \int_\tau^\infty (1-\sigma)f_{S|II}(s) \log ((1-\sigma)f_{S|II}(s)) ds = \sigma h(S|I) + (1-\sigma)h(S|II) + H_B(\sigma)
\]

where \( H_B(\cdot) \) is the binary entropy function. Notice that no particular care has to be taken with the integrals at \( s = T \) because \( f_S(s) \) cannot contain singularities – it is obtained by the convolution of two densities, one of which, \( f_D(\cdot) = g(\cdot), \) contains no singularities.

B. Maximization of \( h(S) \)

We observe of equation \((16)\) that the shape of the conditional density for \( s > \tau \) is completely determined – an exponential with parameter \( \mu \) as depicted in FIGURE 1.

Thus, selection of \( f_T(\cdot) \) does not affect \( f_{S|II}(\cdot) \) and we must have \( h(S|II) = 1 - \log \mu. \)

This observation suggests a three-step approach to maximizing \( h(S) \). In the first two steps, we completely ignore \( f_T(\cdot) \) and find the shape \( f_{S|II}(\cdot) \) and value of \( \sigma \) which maximize equation \((17)\). In step three, we determine that there indeed exists a density \( f_T(\cdot) \) which produces the optimizing \( f_S(\cdot) \).

**Step 1:** For fixed \( \sigma \) we see from equation \((17)\) that \( h(S) \) is maximized solely by our choice of \( f_{S|II}(\cdot) \). The uniform density maximizes entropy on a finite interval \([10]\). Thus, \( f_{S|II}(s) = \frac{1}{\sigma} \) and \( h(S|II) = \log \tau \) as depicted in FIGURE 2.

**Step 2:** Since for any \( \sigma, h(S|I) = \log \tau \), we have

\[
h(S) = \sigma \log \tau + (1-\sigma)(1-\log \mu) + H_B(\sigma) \tag{18}
\]

Taking the derivative of equation \((18)\) with respect to \( \sigma \) yields

\[
\log \tau - (1-\log \mu) - (1+\log \sigma) + (1+\log(1-\sigma))
\]

which we set to zero to obtain

\[
\log \mu \tau - \log \frac{\sigma}{1-\sigma} - 1 = 0
\]

We rearrange to obtain

\[
\mu \tau = \frac{e\sigma}{1-\sigma}
\]
which can generate the 
from which we deduce that the optimal \( \sigma \) is

\[
\sigma^* = \frac{\mu \tau}{e + \mu \tau}
\]

(19)

Returning to the entropy maximization we have

\[
\max_{f_T(\cdot)} h(S) \leq \sigma^* \log \tau + (1 - \sigma^*)(1 - \log \mu) + H_B(\sigma^*)
\]

which through substitution of \( \sigma^* \) according to equation (19) yields

\[
\max_{f_T(\cdot)} h(S) \leq \log \left( \frac{e + \mu \tau}{\mu} \right)
\]

(20)

with equality when

\[
f_S(s) = \begin{cases} 
\frac{\mu}{e + \mu \tau} & 0 \leq s < \tau \\
\frac{e}{e + \mu \tau} e^{-\mu(s-\tau)} & s \geq \tau
\end{cases}
\]

(21)

Step 3: All that remains is to ascertain whether \( \exists f_T(\cdot) \) which can generate the \( f_S(s) \) of equation (21). Since \( f_S(\cdot) \) is the convolution of \( f_D(\cdot) \) and \( f_T(\cdot) \) we can use Fourier transforms to obtain a candidate solution for \( f_T(\cdot) \). That is, the Fourier transform of \( f_D(\cdot) \) is \( \frac{\mu}{\mu + j2\pi f} \) so the Fourier transform of \( f_T(\cdot) \) is

\[
\mathcal{F} \{ f_T(\cdot) \} = \mathcal{F} \{ f_S(\cdot) \} \left( \frac{j2\pi f}{\mu} + 1 \right)
\]

Multiplication by \( j2\pi f \) implies differentiation so we must have

\[
f_T(t) = \frac{1}{\mu} \frac{d}{dt} f_S(t) + f_S(t)
\]

which implies via equation (21) that

\[
f_T(t) = \begin{cases} 
\frac{\mu}{e + \mu \tau} & 0 < t < \tau \\
\frac{1}{\mu} \delta(t) - \frac{1}{e + \mu \tau} \frac{e - 1}{e + \mu \tau} & \text{o.w.}
\end{cases}
\]

(22)

- a valid probability density function.

We can now state the maximum mutual information (capacity in bits per channel use) as

\[
\max_{f_T(\cdot)} I(S; T) = \log \left( \frac{e \mu \tau}{\mu} \right) - (1 - \log \mu) = \log \left( 1 + \frac{\mu \tau}{e} \right)
\]

(23)

which is achieved using the emission time density of equation

\[\text{(22)}\]

We summarize the result as a theorem:

Theorem 1: Maximum \( I(S; T) \) Under a Deadline Constraint:

If \( S = T + D \), where \( D \) is an exponential random variable with parameter \( \mu \) and \( T \in [0, \tau] \), then the mutual information between \( S \) and \( T \) obeys

\[
I(S; T) \leq \log \left( 1 + \frac{\mu \tau}{e} \right)
\]

with equality when

\[
f_T(t) = \begin{cases} 
\frac{\mu}{e + \mu \tau} & 0 < t < \tau \\
\delta(t) - \frac{1}{e + \mu \tau} \frac{e - 1}{e + \mu \tau} & \text{o.w.}
\end{cases}
\]

and

\[
f_S(s) = \begin{cases} 
\frac{\mu}{e + \mu \tau} e^{-\mu(s-\tau)} & 0 \leq s < \tau \\
\frac{e}{e + \mu \tau} & s \geq \tau
\end{cases}
\]

Proof: Theorem (1) See the development leading to the statement of equation (23).

The only remaining question is whether for interval-limited inputs, the exponential first-passage time density, to quote [2] “plays the same role ... that Gaussian noise plays in additive noise channels.” Unfortunately the answer is no, a result we state as a theorem:

Theorem 2: For \( T \) Constrained to \([0, \tau] \), the Minmax Mutual Information First-Passage Density Is NOT Exponential:

If \( g(\cdot) \) is a first passage density with mean \( 1/\mu \) and \( f_T(\cdot) \) can be nonzero only on \([0, \tau] \), then

\[
\arg \min_{g(\cdot)} \left[ \max_{f_T(\cdot)} I(S; T) \right] = g^*(s) \neq \mu e^{-\mu s} u(s)
\]

where \( u(\cdot) \) is the unit step function.

Proof: Theorem (2) Consider that

\[
I(S; T) = \int \int f_T(t) g(s-t) \cdot \frac{g(s-t)}{f_S(s)} ds \, dt
\]

(24)

is convex in \( q(\cdot) \). Since we constrain \( q(\cdot) \) to be non-negative with mean \( 1/\mu \) and unit integral, we can apply Euler-Lagrange variational techniques [11]. That is, we set \( q(x) = g(x) + \eta(x) \) where \( \eta(x) \) is any function defined on \([0, \infty) \), and look for the stationary point

\[
\frac{d}{dx} \left[ \int \int f_T(t) q(s-t) \cdot \frac{q(s-t)}{f_T(x) q(s-x)} \, ds \, dt + a \int s q(s) ds - \frac{1}{\mu} + b \left( \int q(s) ds - 1 \right) \right]_{x=0} = 0
\]

(25)

where \( a \) and \( b \) are (Lagrange) multipliers. Satisfaction of equation (25) for any possible \( \eta(\cdot) \) requires (after expansion and a change of coordinate systems in the double integral) that

\[
\log g(s) = \int_0^t f_T(t) \log f_S(s+t) dt + as + b
\]

(26)

for the \( g(\cdot) \) that minimizes equation (24).
Now, from Theorem 1 we know the form of the optimizing \( f_T(t), t \in [0, \tau] \) and the resulting \( f_S(s) \) were \( g(\cdot) \) exponential with mean 1/\( \mu \). We also know that \( I(S; T) \) is concave in \( f_T(t) \) [5]. Thus, were exponential \( g(\cdot) \) to minimize the maximum mutual information, the left hand side of equation (26) would be a linear function of \( s \). Thus, the integral term on the right would also need to be a linear function of \( s \) given \( f_S(s) \) as in equation (21) and \( f_T(t) \) as in equation (22).

For \( s \geq \tau \) we have \( f_S(s + t) = \frac{ue^{-\mu(s+t)\tau}}{e^{\tau\mu}+ue^{-\mu\tau}} \) and hence

\[
\int_0^\tau f_T(t) \log f_S(s + t)dt = \int_0^\tau \left( \frac{\delta(t) + \mu(\tau-t)(e-1)}{e + \mu \tau} - (s + \mu \tau)\right) dt
\]

which is indeed a linear function of \( s \). However, when \( 0 \leq s < \tau \) we obtain

\[
\int_0^\tau f_T(t) \log f_S(s + t)dt = \int_0^{\tau-s} \left( \frac{\delta(t)}{e + \mu \tau} + \frac{\mu}{e + \mu \tau} \right) \log \frac{\mu}{e + \mu \tau} dt + \int_{\tau-s}^{\tau} \left( \frac{\mu}{e + \mu \tau} + \frac{\delta(t)(e-1)}{e + \mu \tau} \right) \log \frac{\mu}{e + \mu \tau} dt
\]

which does not have the requisite form owing to the term in \( s^2 \).

Therefore, for \( T \) constrained to \([0, \tau] \), the minmax \( I(S; T) \)-first-passage density, \( g(\cdot) \), is not exponential.

A. General Calculation of \( H^1(t) \)

To calculate \( H^1(t) \) we first define

\[
\Theta_{m,\ell}(\vec{t}) \equiv \sum_{|\bar{x}| = \ell} \prod_{j=1}^m \tilde{G}^{\bar{r}}(\vec{t}_{m+1} - \vec{t}_j)G^{\bar{r}}(\vec{t}_{m+1} - \vec{t}_j)
\]

which implies via equation (8) that

\[
H^1(t) = \sum_{\ell=1}^{M-1} \log(1 + \ell) \sum_{m=\ell}^{M-1} \Theta_{m,\ell}(\vec{t})
\]

In principle, we could derive \( H^1(T) \) by taking the expectation of equation (30) with respect to ordered emission times, \( \vec{t} \). However, direct analytic evaluation of \( H^1(T) \) requires we derive joint order densities on the underlying \( T \), a difficult task in general when the individual \( \{T_m\} \) are not necessarily independent.

So, we take a different approach. The sum over all permutations of binary vector \( \bar{x} \) in the definition of \( \Theta_{m,\ell}(\vec{t}) \) (equation (29)) renders it hypersymmetric in \( t_1, \cdots, t_m \) given the \((m+1)\)st smallest emission time \( t_{m+1} \). That is, \( \Theta_{m,\ell}(\vec{t}) = \Theta_{m,\ell}(\vec{F}(t_1, \cdots, t_m, t_{m+1})) \) for any permutation function \( k \) so long as \( t_{m+1} \) is fixed. In what follows we therefore drop the over-vector notation for the \( t_1, t_2, \cdots, t_m \), and assume all are less than \( t_{m+1} \).

Therefore, by equation (11) we can define \( E[\Theta_{m,\ell}] = \Theta_{m,\ell} \) as

\[
E_{T_{m+1}} \left[ E_{T_1, \cdots, T_m | T_{m+1}} \left[ \Theta_{m,\ell}(T_1, \cdots, T_m, T_{m+1}) \right] \right]
\]

Then, the CDF, \( F_{T_{m+1}}(t_{m+1}) \), of the \((m+1)\)st smallest emission time is

\[
F_{T_{m+1}}(t_{m+1}) = 1 - \sum_{k=0}^m \binom{M}{k} \int_0^{t_{m+1}} \int_k^{t_{m+1}} f_T(t)dt_M \cdots dt_1
\]

and likewise, the CDF, \( F_{T_1, \cdots, T_m | T_{m+1}}(t_1, \cdots, t_m | t_{m+1}) \), of the smallest unordered \( T_1, \cdots, T_m \) given \( t_{m+1} \) is

\[
F_{T_1, \cdots, T_m | T_{m+1}}(t_1, \cdots, t_m | t_{m+1}) = \frac{F_{T_1, \cdots, T_m}(t_1, \cdots, t_m)}{F_{T_1, \cdots, T_m}(t_1, \cdots, t_m | t_{m+1})}
\]

\forall t_j \leq t_{m+1} \text{ where } j = 1, \cdots, m

Therefore, by the hypersymmetry of \( \Theta_{m,\ell} \) in \( t_1, \cdots, t_m \) we may write \( \Theta_{m,\ell} \) as

\[
\int_0^{t_{m+1}} \int_0^{t_{m+1}} f_{T_{m+1}}(t_{m+1})f_{T_m}(t_m | t_{m+1})B(m, \ell, t) dt_M dt_{m+1}
\]
where \( T_m = \{T_1, \ldots, T_M\} \), \( t_m = \{t_1, \ldots, t_M\} \) and
\[
B(m, \ell, t) \equiv \left( m \atop \ell \right) \prod_{j=1}^{\ell} \bar{G}(t_{m+1-j}) \prod_{k=1}^{m} G(t_{m+1-k}) \quad (35)
\]
and thence
\[
H^\dagger(T) = \sum_{\ell=1}^{M-1} \log(1 + \ell) \sum_{m=\ell}^{M-1} \Theta_{m,\ell} \quad (36)
\]
In addition, if we define
\[
\Gamma_{M,\ell} = \sum_{m=\ell}^{M-1} \Theta_{m,\ell} \quad (37)
\]
and
\[
\Delta \Gamma_{M,\ell} = \Gamma_{M,\ell} - \Gamma_{M,\ell+1} \quad (38)
\]
then we can also express \( H^\dagger(T) \) as
\[
H^\dagger(T) = \sum_{\ell=1}^{M-1} \Delta \Gamma_{M,\ell} \log(\ell + 1)! \quad (39)
\]
The development starting in section \[\[\text{V-A}\] \] proves the following theorem:

**Theorem 3**: The General Form of \( H^\dagger(T) \):

If we define
\[
\Gamma_{M,\ell} = \sum_{m=\ell}^{M-1} \Theta_{m,\ell}
\]
and
\[
\Delta \Gamma_{M,\ell} = \Gamma_{M,\ell} - \Gamma_{M,\ell+1}
\]
where \( \Theta_{m,\ell} \) is as defined by equations \[\[\text{V-A}\] \] and equation \[\[\text{V-A}\] \], then we can express \( H^\dagger(T) \) as
\[
H^\dagger(T) = \sum_{\ell=1}^{M-1} \Delta \Gamma_{M,\ell} \log(\ell + 1)! \]

**Proof**: See the development starting in section \[\[\text{V-A}\] \] leading to the statement of Theorem 3.

This concludes our calculation of \( H^\dagger(T) \) for general input distributions \( f_T(\cdot) \). The key utility of our formulation is that it does not require joint order distributions for the \( \{T_m\} \), only the more easily calculable \( m \)th order distribution for \( T_m \). We now turn to the case where the \( T \) are i.i.d. - important because i.i.d. \( T \) increases entropy \( h(S) \).

**B. \( H(\Omega|\bar{S}, T) \) for General IID T**

With i.i.d. \( T \), we can use the definition of \( \Theta_{m,\ell}(\cdot) \) in equation \[\[\text{V-A}\] \] and the hypersymmetric result of equation \[\[\text{V-A}\] \] to obtain
\[
\bar{\Theta}_{m,\ell} = E_{F_{T_{m+1}}}
\]
which after rearranging as a telescoping sum simplifies to
\[
\sum_{k=0}^{m} (M-k) \binom{M}{k} f_T(t) F_k^T(t)(1 - F_T(t))^{M-k-1}
\]
and
\[
- \sum_{k=0}^{m-1} (k+1) \binom{M}{k+1} f_T(t) F_{k+1}^T(t)(1 - F_T(t))^{M-k-1}
\]
which further simplifies to
\[
(m+1) \binom{M}{m+1} f_T(t) F_T^m(t)(1 - F_T(t))^{M-m-1}
\]
We then define
\[
\phi(t) = \int_0^t f_T(x) G(t - x) dx \quad (44)
\]
and
\[
\int_0^t f_T(x)(1 - G(t - x)) dx = F_T(t) - \phi(t)
\]
which allows us to write
\[
\phi(T_{m+1}) = E_{T \leq T_{m+1}} \left[ \bar{G}(T_{m+1} - T) \right]
\]
and
\[
F_T(t) - \phi(T_{m+1}) = E_{T \leq T_{m+1}} \left[ 1 - \bar{G}(T_{m+1} - T) \right]
\]
which upon substitution into equation \[\[\text{V-A}\] \] allows us to write
\[
\bar{\Theta}_{m,\ell} = (m+1) \binom{M}{m+1} \binom{m}{\ell} \times \int_0^\infty \left[ \frac{f_T(t)(1 - F_T(t))^{M-m-1} \phi(t)}{(F_T(t) - \phi(t))^{m-\ell}} \right] dt \quad (45)
\]
and then as
\[
\bar{\Theta}_{m,\ell} = M \binom{M-1}{\ell} \binom{M-\ell-1}{m-\ell} \times \int_0^\infty \left[ \frac{f_T(t)(1 - F_T(t))^{M-m-1} \phi(t)}{(F_T(t) - \phi(t))^{m-\ell}} \right] dt \quad (46)
\]
To evaluate \( H^\dagger(T) \) in equation \[\[\text{V-A}\] \] we must first compute \( \Gamma_{M,\ell} = \sum_{m=\ell}^{M-1} \bar{\Theta}_{m,\ell} \) as
\[
\Gamma_{M,\ell} = M \binom{M-1}{\ell}
\]
and
\[
\times \int_0^\infty \left[ \frac{1 - F_T(t)(1 - F_T(t))^{M-1} \phi(t)}{(1 - F_T(t))^{M-1} \phi(t)} \right] dt \quad (47)
\]
which we rewrite as
\[
\Gamma_{M, \ell} = M \binom{M-1}{\ell} \times \int_0^\infty f_T(t) \left[ \left(1 - F_T(t)\right)^{M-1} \frac{\phi(t)}{F_T(t)-\phi(t)} \right] \ell \cdot \prod_{m=0}^{\ell-1} \left(1 - \frac{\phi(t)}{F_T(t)-\phi(t)}\right) \left(\frac{F_T(t)-\phi(t)}{1-F_T(t)}\right)^{M-\ell-1} dt \tag{48}
\]
We consolidate the binomial sum to obtain
\[
\Gamma_{M, \ell} = M \binom{M-1}{\ell} \times \int_0^\infty f_T(t) \left[ \left(1 - F_T(t)\right)^{M-1} \frac{\phi(t)}{F_T(t)-\phi(t)} \right] \ell \cdot \prod_{m=0}^{\ell-1} \left(1 - \frac{\phi(t)}{F_T(t)-\phi(t)}\right) \left(\frac{F_T(t)-\phi(t)}{1-F_T(t)}\right)^{M-\ell-1} dt \tag{49}
\]
which reduces to
\[
\Gamma_{M, \ell} = \int_0^\infty M \binom{M-1}{\ell} f_T(t) \phi(t) (1 - \phi(t))^{M-\ell-1} dt \tag{50}
\]
for \(\ell = 1, 2, \cdots, M-1\).

Now consider the integrand of the difference \(\Gamma_{M, \ell} - \Gamma_{M, \ell+1}\) where we drop the \(t\) dependence for notational convenience
\[
\Gamma_{M, \ell} - \Gamma_{M, \ell+1} = M \binom{M}{\ell} \phi^\ell (1 - \phi)^{M-\ell-1} \left[ \frac{M \binom{M}{\ell}}{M \binom{M}{\ell+1}} \frac{\phi(t)}{\phi(t)+1} \right]
\]
We can rewrite this expression as
\[
M \phi^\ell \left[ \binom{M}{\ell} - \binom{M}{\ell+1} (1 - \phi)^{M-\ell-2} \right] + \sum_{r=1}^{M-1} (-1)^r \phi^r \left[ \binom{M}{r} \binom{M-\ell-1}{r} \right]
\]
which after consolidating terms becomes
\[
M \phi^\ell \left[ \binom{M}{\ell} \sum_{r=0}^{\ell-1} (-1)^r \binom{M-\ell-1}{r} (1 + \ell + 1) \phi^r \right]
\]
Extending the sum to \(r = 0\) and subtracting the \(r = 0\) term produces
\[
\left( M + 1 \right) \sum_{r=0}^{M-\ell-1} (-1)^r \binom{M-\ell-1}{r} (\ell + r + 1) \phi^r
\]
which can be recognized as
\[
\frac{d}{d\phi} \left[ \left( M + 1 \right) \sum_{r=0}^{M-\ell-1} (-1)^r \binom{M-\ell-1}{r} \phi^{r+\ell} \right]
\]
and then reduced to
\[
\left( M + 1 \right) \frac{d}{d\phi} \left[ \phi^{\ell+1} (1 - \phi)^{M-\ell-1} \right]
\]
so that we have \(\Delta \Gamma_{M, \ell}\) as
\[
\left( M + 1 \right) \sum_{r=0}^{M-\ell-1} (-1)^r \binom{M-\ell-1}{r} (\ell + r + 1) E[\phi^{r+\ell}(t)] \tag{51}
\]
where \(E[\cdot]\) is the expectation using \(f_T(t)\).

The previous development of section \(V-B\) proves the following theorem:

**Theorem 4**: An Upper Bound for Ordering Entropy
\[
H(\Omega | \bar{S}, T) \leq H^*(T) = \sum_{\ell=1}^{M-1} \Delta \Gamma_{M, \ell} \log(\ell + 1)!
\]

**Proof**: See the development of section \(V-B\) leading to the statement of Theorem 4.

**C. \(H(\Omega | \bar{S}, T)\) Special Case IID \(T\)**

Here we derive expressions for \(H(\Omega | \bar{S}, T)\) when the i.i.d. input distribution is that which maximizes \(I(S; T)\). We consider the following cases:

1) **Exponential Transit Times with a Mean Constraint**: For exponential first-passage times with mean \(1/\mu\), the probability density of \(T\) that maximizes \(h(S)\) subject to a mean constraint \(E[\sum_m T_m] \leq M\tau\) is i.i.d. with marginal
\[
f_{T_m}(t) = a \delta(t) + \mu a (1 - a) e^{-\mu at} u(t) \tag{52}
\]
where \(a = 1/(\mu + 1)\) and \(u(t)\) is the unit step function. For exponential transit we have
\[
\bar{G}(t) = e^{-\mu t} u(t)
\]
and thereby
\[
\phi(t) = \int_0^t f_T(x) \bar{G}(t-x) dx = ae^{-\mu at} u(t)
\]
We then require an expression for \(E_T[\phi^k(T)]\). Remembering that \(f_T(t)\delta(t) u(t) dt = 1/k + 1\) we obtain
\[
E_T[\phi^k(T)] = \int_0^\infty f_T(t) a e^{-\mu at} u(t) dt = \frac{a^k}{k + 1}
\]
so that equation (51) becomes
\[
\Delta \Gamma_{M, \ell} = \left( M + 1 \right) \sum_{r=0}^{M-\ell-1} (-1)^r \binom{M-\ell-1}{r} a^{r+\ell}
\]
which reduces to
\[
\Delta \Gamma_{M, \ell} = \left( M + 1 \right) a^\ell (1 - a)^{M-\ell-1} \tag{53}
\]
for \( \ell = 1, 2, \cdots, M - 1 \).

With \( a = \frac{1}{\mu + \tau} \) we can write \( H(\Omega|\vec{S}, T) \) as
\[
(\mu + 1)^m \sum_{k=0}^m \log(k!) \left( \frac{\mu \tau}{\mu \tau + 1} \right)^{M-k} \left( \frac{1}{\mu \tau + 1} \right)^k
\]
which is the expectation of \( (\mu + 1) \log K! \) for a binomial random variable \( K \) with parameters \( M \) and \( \frac{1}{\mu \tau + 1} \), or
\[
H(\Omega|\vec{S}, T) = (\mu + 1)E_K [\log K!]
\]
and then finally,
\[
E_T [\phi^k(T)] = \left( \frac{1}{e + \mu \tau} \right)^{k+1} \left[ \mu \tau + \frac{e^{k+1}}{k+1} \right]
\]
so that \( \Delta \Gamma_{M,\ell} \) in equation (51) becomes
\[
\left( M \right)_{\ell+1} \sum_{r=0}^{M-\ell-1} \left( \frac{1}{e + \mu \tau} \right)^{r+1} \left[ \mu \tau + \frac{e^{r+1}}{r+1} \right]
\]
which reduces to
\[
\left( M \right)_{\ell+1} \sum_{r=0}^{M-\ell-1} \left( \frac{1}{e + \mu \tau} \right)^{r+1} \left[ \mu \tau + \frac{e^{r+1}}{r+1} \right]
\]
and then to
\[
\left( M \right)_{\ell+1} \sum_{r=0}^{M-\ell-1} \left( \frac{1}{e + \mu \tau} \right)^{r+1} \left[ \mu \tau + \frac{e^{r+1}}{r+1} \right]
\]

We restate this result as a theorem:

**Theorem 5:** \( H(\Omega|\vec{S}, T) \) for Exponential First-Passage with a Mean Constraint (\( E[T] = \tau \)):

For \( T \) distributed as equation (52) and exponential first-passage with parameter \( \mu \), we have
\[
H(\Omega|\vec{S}, T) = (\mu + 1)E_K [\log K!]
\]
where \( K \) is a binomial random variable with parameters \( M \) and \( \frac{1}{\mu \tau} \).

**Proof:** (Theorem 4) See the development leading to the statement of Theorem 4 and direct application of Theorem 3.

2) Exponential Transit Times with a Deadline: Theorem 1 states that if \( T \) is constrained to \([0, \tau]\) then the \( f_T(t) \) that maximizes \( h(S) \) (and therefore \( h(\vec{S}) \) when in i.i.d. form) is
\[
f_T(t) = \frac{1}{e + \mu \tau} \delta(t) + \frac{\mu \tau}{e + \mu \tau} + \frac{e - 1}{e + \mu \tau} \delta(t - \tau)
\]
for \( t \in [0, \tau] \) and zero otherwise.

To obtain the corresponding \( H(\Omega|\vec{S}, T) = H^{(\ell)}(T) \) we calculate \( \phi(t) \) as
\[
\int_0^t f_T(x)e^{-\mu(t-x)}dx = \begin{cases} \frac{1}{e + \mu \tau} & 0 \leq t \leq \tau \\ \frac{\mu \tau}{e + \mu \tau} e^{-\mu(t-\tau)} & t > \tau \end{cases} \text{ o.w.}
\]
Once again, we require an expression for the integral \( \int_0^\infty f_T(t)\phi^k(t)dt \), and again remembering that \( \int_0^\infty \delta(t)\phi^k(t)dt = \frac{1}{k+1} \) we obtain \( E_T [\phi^k(T)] \) as
\[
\left( \frac{1}{e + \mu \tau} \right)^{k+1} \int_0^\infty \delta(t)\phi^k(t)dt = \mu \tau + \frac{e^{k+1}}{k+1} \]
which reduces to
\[
\left( \frac{1}{e + \mu \tau} \right)^{k+1} \left[ \frac{1}{k+1} + \mu \tau + \sum_{r=0}^{k} \left( \frac{k}{r} \right) \frac{1}{r+1} (e - 1)^{r+1} \right]
\]
which further reduces to
\[
\left( \frac{1}{e + \mu \tau} \right)^{k+1} \left[ \frac{1}{k+1} + \mu \tau + \frac{e^{k+1}}{k+1} - \frac{1}{k+1} \right]
\]
and then finally,
\[
E_T [\phi^k(T)] = \left( \frac{1}{e + \mu \tau} \right)^{k+1} \left[ \mu \tau + \frac{e^{k+1}}{k+1} \right]
\]
Now if we define random variables \( K_1 \) to be binomial with parameters \( M \) and \( p_1 \), the following theorem results from direct application of Theorem 3.

**Theorem 6:** \( H(\Omega|\vec{S}, T) \) for Exponential First-Passage with a Launch Deadline (\( T \in [0, \tau]\)):

For \( T \) distributed as equation (55) we have
\[
H(\Omega|\vec{S}, T) = \begin{cases} E_{K_1} [\log K_1] + \frac{\mu \tau}{p_1} E_{K_2} [2\log K_2] \end{cases} \text{ o.w.}
\]
where \( K_1 \) is a binomial random variable with parameters \( M \) and \( \frac{e}{\mu \tau} \) and \( K_2 \) is a binomial random variable with parameters \( M \) and \( \frac{1}{\mu \tau + \tau} \).

**Proof:** (Theorem 6) See the development leading to the statement of Theorem 6 and direct application of Theorem 5.
D. Asymptotic $H(\Omega|\bar{S}, T)/M$ For Exponential First-Passage

We are interested in asymptotic values of $H(\Omega|\bar{S}, T)/M$ owing to our definition of capacity per token in equation (11) (see also in Part-I [1]). To that end, recall that $\lambda \tau = M$ and we define $\rho = \lambda / \mu$, a measure of system token “load” (also a proxy for power expenditure in units of energy per passage time), so that

$$\frac{1}{1 + \mu M/\lambda} = \frac{1}{1 + M/\rho}$$

and likewise

$$\frac{e}{e + \mu M/\lambda} = \frac{e}{e + M/\rho}$$

and

$$\frac{1}{e + \mu M/\lambda} = \frac{1}{e + M/\rho}$$

Now, remember the binomial distribution for fixed $k$ and large $M$ is approximated by

$$\binom{M}{k} p^k (1-p)^{M-k} \approx \frac{M^k}{k!} p^k (1-p)^{M-k}$$

So, for any finite $k$ it is easily seen that for $M \to \infty$

$$\left( \frac{M}{k} \right) \left( \frac{1}{1 + \frac{M}{\rho}} \right)^k \left( 1 - \frac{1}{1 + \frac{M}{\rho}} \right)^{M-k} \to e^{-\rho} \frac{1}{k!} p^k$$ (60)

and

$$\left( \frac{M}{k} \right) \left( \frac{1}{e + \frac{M}{\rho}} \right)^k \left( 1 - \frac{1}{e + \frac{M}{\rho}} \right)^{M-k} \to e^{-pe} \frac{1}{k!} e^k$$ (61)

and

$$\left( \frac{M}{k} \right) \left( \frac{e}{e + \frac{M}{\rho}} \right)^k \left( 1 - \frac{e}{e + \frac{M}{\rho}} \right)^{M-k} \to e^{-pe} \frac{1}{k!} e^k$$ (62)

and we note that all these limiting distributions are Poisson.

Equation (55) and equation (59) can then be combined with equation (60), equation (61) and equation (62) to produce the following two theorems:

**Theorem 7: Asymptotic $H(\Omega|\bar{S}, T)/M$ for Exponential First-Passage with a Mean Constraint ($E[T] = \tau$):**

For exponential first-passage with $E[T] = \tau$ and $f_T(\cdot)$ as given in equation (52), $H(\Omega|\bar{S}, T)$ is given by

$$\lim_{M \to \infty} \frac{H(\Omega|\bar{S}, T)}{M} = e^{-\rho} \sum_{k=2}^{\infty} \rho^{k-1} \log k! \cdot \frac{1}{\rho} = E[\log \frac{1}{\rho}]$$ (63)

where the final expectation is for $k$ a Poisson random variable with parameter $\rho$.

**Proof: Theorem 7** See Theorem 6 and the development leading up to the statement of Theorem 7.

**Theorem 8: Asymptotic $H(\Omega|\bar{S}, T)/M$ for Exponential First-Passage with a Deadline Constraint ($T \in [0, \tau]$):**

For exponential first-passage with $T \in [0, M/\rho]$ and $f_T(\cdot)$ as given in equation (56), $H(\Omega|\bar{S}, T)$ is given by

$$\lim_{M \to \infty} \frac{H(\Omega|\bar{S}, T)}{M} = E[\log (k - 1) \log k!]$$ (64)

where the final expectation is for $k$ a Poisson random variable with parameter $\rho$.

**Proof: Theorem 8** See Theorem 6 and the development leading up to the statement of Theorem 7.

VI. UPPER BOUND FOR $I(\bar{S}; T)$

With analytic bounds for $H(\Omega, T)$, we can now consider bounds on the mutual information, $I(\bar{S}; T)$. In Part-I (using results from this, Part-II) lower bounds were derived. Here we consider an upper bound. To begin, however, we must find an upper bound for $H(\Omega|\bar{S}, T)$.

A. A Useful Upper Bound On $H(\Omega|\bar{S}, T)$

We state the bound as a theorem with proof.

**Theorem 9: An Upper Bound for $H(\Omega|\bar{S}, T)$:**

Given

$$Q(\cdot) = \bar{G}(\cdot | \cdot)$$ (65)

where $\bar{G}(\cdot)$ is the CCDF of the passage time, and defining

$$\gamma_T = E_T [Q(T_1 - T_2)]$$ (66)

we have

$$H(\Omega|\bar{S}, T) \leq E_T [H^1(T)] \leq M \log \left( 1 + \frac{M - 1}{2 \rho} \right)$$ (67)

**Proof: Theorem 9** $H^1(t)$, defined in equation (6) derived in Part-I [1], [9] is an upper bound for $H(\Omega|\bar{S}, T)$. The bound is satisfied with equality if the first-passage density is exponential [1], [9]. For a given $m$, let us define $G_k = G(\bar{t}_{m+1} - \bar{t}_k)$ and $G_k$ in a corresponding way. Then, consider the sum of the following $2^m$ terms

$$G_m G_{m-1} G_{m-2} \cdots G_3 G_2 G_1$$

$$G_m G_{m-1} G_{m-2} \cdots G_3 G_2 G_1$$

$$\vdots$$

$$G_m G_{m-1} G_{m-2} \cdots G_3 G_2 G_1$$

$$G_m G_{m-1} G_{m-2} \cdots G_3 G_2 G_1$$

Taken pairwise it is easy to see that this sum telescopes to 1 since $G_i + G_j = 1$ so that the ensemble of terms is a PMF. Furthermore, since $m = 1, 2, \cdots, M$, the complete ensemble of the terms, $\prod_{j=1}^{m} G_{X_j} (\bar{t}_{m+1} - \bar{t}_j) G_1^{1-X_j} (\bar{t}_{m+1} - \bar{t}_j)$, $m = 1, 2, \cdots, M$, sums to $M$. So, we can define

$$P_{\ell|\ell, m} = \sum_{|\ell| = \ell} P_{\ell|\ell, m} = \sum_{|\ell| = \ell} \prod_{j=1}^{m} G_{X_j} (\bar{t}_{m+1} - \bar{t}_j) G_1^{1-X_j} (\bar{t}_{m+1} - \bar{t}_j)$$ (68)

and then

$$P_{\ell|\ell, m} = \sum_{|\ell| = \ell} \sum_{|\ell| = \ell} \prod_{j=1}^{m} G_{X_j} (\bar{t}_{m+1} - \bar{t}_j) G_1^{1-X_j} (\bar{t}_{m+1} - \bar{t}_j)$$ (69)
for \( \ell = 0, 1, \ldots, M - 1 \). We can use Jensen’s inequality to write
\[
H^\dagger(t) = E_{\ell[t]}[\log(1 + \ell)] \leq M \log(E[\ell[t]] + 1) \tag{70}
\]
Now consider that
\[
E[\ell[t]] = \frac{1}{M} \sum_{m=0}^{M-1} E[\ell[t, m]]
\]
and the explicit expansion of \( E[\ell[t, m]] \) is
\[
\sum_{\ell=0}^{m} \left( \sum_{|k| = \ell} \prod_{j=1}^{m} G_{\ell,j}(\vec{t}_{m+1} - \vec{t}_j) G^{1-\delta_j}(\vec{t}_{m+1} - \vec{t}_j) \right) \tag{71}
\]
Then consider that \( E[\ell[t, m]] \) has the terms
\[
0 \times \left[ G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \right] \text{ 1 term}
\]
\[
1 \times \left[ \begin{array}{cccc}
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \\
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \\
\vdots \\
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1
\end{array} \right] \text{ 1 term}
\]
\[
2 \times \left[ \begin{array}{cccc}
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \\
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \\
\vdots \\
G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1
\end{array} \right] \text{ 2 term}
\]
with final term
\[
m \times \left[ G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \right] \text{ 1 term}
\]
Then consider the term \( G_{m} G_{m-1} G_{m-2} \cdots G_2 G_1 \) and group together the other \( 2^{m-1} - 1 \) different terms that contain \( G_1 \). The sum of all these terms is \( G_1 \). We can do a corresponding grouping for each of the \( m \) terms in which \( G_1 \) appears exactly once.

Thus, by expanding and regrouping the inner product terms of equation \( \text{(71)} \) we can show that
\[
E[\ell[t, m]] = \sum_{j=1}^{m} G(\vec{t}_{m+1} - \vec{t}_j)
\]
which results in
\[
H^\dagger(t) \leq M \log \left( 1 + \frac{1}{M} \sum_{m=1}^{M-1} \sum_{j=1}^{m} G(\vec{t}_{m+1} - \vec{t}_j) \right)
\]
via equation \( \text{(69)} \) and equation \( \text{(70)} \), remembering that \( E[\ell[t, m = 0] = 0 \). Taking the expectation in \( T \) yields
\[
H^\dagger(T) \leq M \log \left( 1 + \sum_{m=1}^{M-1} \sum_{j=1}^{m} E \left[ G(\vec{t}_{m+1} - \vec{t}_j) \right] / M \right) \tag{72}
\]
We then note that all ordered differences between the \( T_i \) are accounted for in equation \( \text{(72)} \). For any given \( T \) there are \( M(M-1)/2 \) ordered terms. Thus, we can rewrite equation \( \text{(72)} \) as
\[
E_T \left[ H^\dagger(T) \right] \leq M \log \left( 1 + \sum_{i,j,i \neq j} E \left[ G(T_i - T_j) \right] / 2M \right) \tag{73}
\]
where the factor of \( 1/2 \) is introduced to account for terms \( T_i < T_j \) which would not appear in the ordered case of equation \( \text{(72)} \). Finally, hypersymmetry of \( T \) requires that \( E \left[ G(\vert T_i - T_j \vert) \right] = \gamma_T \), a constant for \( i \neq j \) so that
\[
H(\Omega[S, T]) \leq E_T \left[ H^\dagger(T) \right] \leq M \log \left( 1 + \frac{M-1}{2} \gamma_T \right)
\]
which matches the result stated in Theorem \( \text{(7)} \) and thus proves the theorem. \( \bullet \)

B. Maximizing \( h(S) + M \log(1 + \gamma_S(M - 1)) \)

We now have the rudiments of an upper bound for \( I(\hat{S}, T) \)
\[
\max_{f_T()} \frac{I(\hat{S}, T)}{M} \leq \frac{h(S)}{M} + \log(1 + \gamma_S(M - 1)) - \frac{\log M!}{M} - h(D) \tag{74}
\]
However, the upper bound equation \( \text{(67)} \) is in terms of \( f_T() \) whereas \( h(S) \) is a function(al) of \( f_S() \). Therefore, we must develop a relationship between \( \gamma_T = E[Q(T_1 - T_2)] \) and \( \gamma_S = E[Q(S_1 - S_2)] \). This relationship allows us to fix \( \gamma_S \) and maximize \( h(S) \) while still maintaining an upper bound on \( H(\Omega[S, T]) \). From here onward we assume exponential first-passage of tokens.

Theorem 10: \( \gamma_T \) versus \( \gamma_S \) for Exponential First-Passage:

If the first-passage density \( f_D() \) is exponential then
\[
E[Q(S_1 - S_2)] \geq \frac{1}{2} E[Q(T_1 - T_2)]
\]
or
\[
\gamma_S \geq \frac{1}{2} \gamma_T \tag{75}
\]

Proof: Theorem \( \text{(10)} \) Let \( \Delta = T_1 - T_2 \) and \( D = D_2 - D_1 \). Then \( \Delta + D = S_1 - S_2 \). For the i.d. \( D \) exponential we have \( G(d) = e^{-\lambda d}, d \geq 0 \). Thus, \( Q() = e^{-\lambda|\cdot|} \). We then note that \( |a + b| \leq |a| + |b| \) so that
\[
E[Q(\Delta + D)] = E[e^{-\lambda |\Delta + D|}] \geq E[e^{-\lambda |\Delta| - \lambda |D|}] = E[Q(\Delta)] E[Q(D)]
\]
because \( \Delta \) and \( D \) are independent. Then consider that the density of \( D \) is \( f_D() = \frac{\lambda}{2} e^{-\lambda |\cdot|} \) so that \( E[Q(D)] = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda |z|} e^{-\lambda |z|} dz = \frac{1}{2} \) which completes the proof. \( \bullet \)

Now, suppose we fix \( E[Q(S_1 - S_2)] = \gamma_S \). Then, owing to hypersymmetry we have \( E[Q(S_i - S_j)] = \gamma_S \forall i, j, i \neq j \). Using standard Euler-Lagrange optimization \( \text{(12)} \), we can find the density \( f_S() \) which maximizes \( h(S) \) as
\[
f^*_S(s) = \frac{1}{A(\beta)} e^{\sum_{i,j,i \neq j} Q(s_i - s_j)}
\]
with respect to \( S \) we have from the definition of \( \gamma_S(\beta) \) that

\[
M(M-1)\gamma_S(\beta) = E \left[ \sum_{i \neq j} Q(s_i - s_j) \right] = 1
\]

Then

\[
\Gamma_S'(\beta) = E \left[ \frac{1}{2} \left( \sum_{i \neq j} Q(s_i - s_j) \right)^2 - E^2 \left( \sum_{i \neq j} Q(s_i - s_j) \right) \right]
\]

which is a variance and therefore greater than or equal to zero. Thus, \( \gamma_S'(\beta) \geq 0 \). And since \( 0 \leq \gamma_S(\beta) \leq 1 \), we must also have \( \gamma_S'(\beta) \rightarrow 0 \) in the limits \( \beta \rightarrow \pm \infty \).

Now, consider all terms as functions of \( \beta \) as in

\[
I(\bar{S}; T) = \log A(\beta) - \beta M(M-1)\gamma_S(\beta) + M \log (1 + \gamma_S(\beta)(M-1)) - h(S|T) - \log M!
\]

We can find extremal points by differentiating equation (79) with respect to \( \beta \) to obtain the first derivative

\[
M(M-1)\gamma_S(\beta) \left( -\beta + \frac{1}{1 + \gamma_S(\beta)(M-1)} \right)
\]

and the second derivative

\[
M(M-1)\gamma_S''(\beta) \left( -\beta + \frac{1}{1 + \gamma_S(\beta)(M-1)} \right) +
- M(M-1)\gamma_S'(\beta) \left( 1 + (M-1) \frac{\gamma_S'(\beta)}{1 + \gamma_S(\beta)(M-1)} \right)
\]

which when the first derivative is zero reduces to

\[
- M(M-1)\gamma_S'(\beta) \left( 1 + (M-1) \frac{\gamma_S'(\beta)}{1 + \gamma_S(\beta)(M-1)} \right) \leq 0
\]

We then have

\[
\gamma_S^* = \gamma_S(\beta^*) = \frac{1 - \beta^*}{(M-1)\beta^*}
\]

and note that equation (80) requires \( \frac{1}{M} \leq \beta^* \leq 1 \) since \( 0 \leq \gamma_S(\beta) \leq 1 \). In addition, there is at most one solution to equation (80) since \( \gamma_S''(\beta) \) monotonically decreases in \( \beta \) while \( \gamma_S'(\beta) \) monotonically increases in \( \beta \). Since the second derivative at the extremum is non-positive, the unique point defined by equation (80) is a maximum.

Unfortunately, solutions to equation (80) have no closed form and numerical solutions for asymptotically large \( M \) are impractical. Nonetheless, the constraints on \( \beta^* \) will allow an oblique approach to deriving a bound.

We note again that \( \Gamma_S'(\beta) \) is the variance of \( \sum_{i \neq j} Q(s_i - s_j) \) and must decrease monotonically in \( \beta \) since as previously discussed, increased \( \beta \) concentrates \( f_S(\cdot) \) around larger values of \( \sum_{i \neq j} Q(s_i - s_j) \). Thus,

\[
\Gamma_S'(\beta) \leq \Gamma_S'(0)
\]

and

\[
\Gamma_S(\beta) \leq \beta \Gamma_S'(0) + \Gamma_S(0)
\]

Assuming exponential first-passage, \( Q(x) = e^{-\mu|x|} \), we can calculate both \( \gamma_S(0) \) and \( \gamma_S^*(0) \) in closed form as

\[
\gamma_S(0) = \frac{2}{(\mu \tau)^2} (\mu \tau + e^{-\mu \tau} - 1)
\]

respectively. Defining \( M = \lambda \tau \) and taking the limit for large \( M \) yields

\[
\lim_{M \to \infty} M \gamma_S(0) = \frac{2 \lambda}{\mu} = 2 \rho
\]

and

\[
\lim_{M \to \infty} (M-1)\gamma_S(0) = 8 \frac{\lambda^2}{\mu^2} + 2 \frac{\lambda}{\mu} = 8 \rho^2 + 2 \rho
\]

where once again \( \rho = \frac{\lambda}{\tau} \).

Remembering that \( \Gamma_S(\beta) = M(M-1)\gamma_S(\beta) \) and utilizing equation (82) we have

\[
\gamma_S(0) \leq \gamma_S(\beta^*) \leq \gamma_S(0) \beta^* + \gamma_S(0)
\]

Thus, the \( \gamma \)-intercept of the monotonically decreasing \( \frac{1-\beta}{(M-1)\beta} \) with the right hand side of equation (87) must yield a value at least as large as \( \gamma(\beta^*) \). To solve for this intercept we set

\[
\frac{1 - \beta}{(M-1)\beta} = \gamma_S(0) \beta^* + \gamma_S(0)
\]

so that in the limit of large \( M \) we have

\[
\beta = \frac{1}{8 \rho^2 + 2 \rho + 2 \rho} \left( 1 + \frac{1}{4 \rho + 1} \right)
\]

which results in

\[
(M-1)\gamma(\beta^*) \leq \frac{1}{4 \rho + 1} (8 \rho^2 + 2 \rho) + 2 \rho = 4 \rho
\]

so that for large \( M \) we have

\[
I(\bar{S}; T) \leq \log A(\beta^*) - \beta^* M(M-1)\gamma_S(\beta^*) + M \log (1 + 4 \rho) - h(S|T) - \log M!
\]
To complete the mutual information bound, we could then derive upper bounds on $A(\beta^*) - \beta^* M (M - 1) / 2$, $\gamma / \gamma^*$. However, in the limit of large $M = \tau / \lambda$, the density on $S$ is effectively constrained to $(0, \tau)$ [1, 13] which constrains $h(S) \leq M \log \tau$. Then, since $h(S/T) = M (1 - \log \mu)$ for exponential first-passage, equation (90) produces mutual information per token

$$I(\bar{S}; T) / M \leq \log \tau - (1 - \log \mu) + \log (1 + 4\rho) - \frac{M!}{M} \log M + 1$$

(91)

Application of Stirling’s approximation for large $M$

$$\log M! / M \approx \log M - 1$$

(92)

in combination with equation (91) produces our main theorem:

**Theorem 11:** An Upper Bound on the Asymptotic Capacity per Token, $C_q$:

For exponential passage with mean first-passage time $1/\mu$ and token emission intensity $\lambda$, an upper bound for the asymptotic capacity per token is given by

$$C_q = \max_{\mu(\cdot)} \lim_{M \to \infty} \frac{1}{M} I(\bar{S}; T) \leq \log \left( \frac{1}{\rho} + 4 \right)$$

(93)

where $\rho = \lambda / \mu$.

**Proof:** Substitution of equation (92) and $\tau = M/\lambda$ into equation (91) completes the proof.

VII. DISCUSSION & CONCLUSION

The timing channel [2]–[4], [14] is a building block upon which the information theory of the identical molecule/token timing channel is built. In this paper we considered a version of the timing channel where a single emission is restricted to an interval $[0, \tau]$ and we derived closed form expressions for the channel capacity under exponential first-passage as well as the optimal input (emission) distribution. We also established that unlike for the mean-constrained channel, exponential first-passage is not the worst case corruption.

Building block though the single emission channel is, the identical molecule timing channel differs from previous models because which emission corresponds to which arrival is ambiguous expressly because travel time from sender to receiver is random and the molecules are identical. This ambiguity is captured by a quantity we define as the “ordering entropy” $H(\Omega | \bar{S}, T)$ and understanding its properties is critical to understanding the capacity of not only the molecular timing channel, but also channels where tokens/molecules may themselves carry information payloads – portions of messages to be strung together at the receiver [1].

In the Part-I companion to this paper [1], we carefully explored the information theory formulation of the problem to establish that the usual information $I(S; T)$ is indeed the proper measure of information flow over this channel and its relationship to $H(\Omega | \bar{S}, T)$. In this paper, Part-II, we carefully explored the properties of $H(\Omega | \bar{S}, T)$, showing how it can be calculated without deriving full order distributions and deriving closed form expressions for cases where the emission times $T$ are i.i.d. random variables. We then derived closed form expressions for the special cases of the input distribution being that which achieves capacity for the mean-constrained and the deadline-constrained timing channel with exponential first-passage and the asymptotic behavior $\lim_{M \to \infty} H(\Omega | \bar{S}, T)$. Our understanding of $H(\Omega | \bar{S}, T)$ then allowed derivation of lower bounds on timing channel capacity for exponential first passage (Part-I, Theorem 14) and here in Part-II, an upper bound for the molecular timing channel capacity.

Although the machinery necessary to consider a mean-constrained version of the identical token timing channel was derived, capacity results were not pursued owing to our inability to derive an appropriate sequential channel use model with asymptotic independence. However, if physically parallel channels were used (so as to avoid corruption of one channel by arrivals from another), the results of [2] combined with Theorem 5 might be used to derive upper and lower bounds analogous to those provided here and in Part-I [1]. This might prove interesting since the mean-constraint seems analytically simpler than the deadline constraint with respect to both the single-token entropy and capacity as well as the ordering entropy.

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