On the spectrum of a class of quantum models

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Abstract – The spectrum of any quantum model whose eigenvalue equation reduces to a three-term recurrence, such as a displaced harmonic oscillator, the Jaynes-Cummings model, the Rabi model, and a generalized Rabi model, can be determined as zeros of a corresponding transcendental function $F(x)$. The latter can be analytically determined as an infinite series defined solely in terms of the recurrence coefficients.

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Introduction. – The present work deals with a class $\mathcal{R}$ of quantum models described by a Hamiltonian $\hat{H}$ characterized in that the eigenvalue equation

$$\hat{H}\phi = E\phi$$

(1)

in the Bargmann Hilbert space of analytical functions $\mathcal{B}$ [1,2] reduces to a three-term difference equation

$$c_{n+1} + a_n c_n + b_n c_{n-1} = 0 \quad (n \geq 0).$$

(2)

Here $\{c_n\}_{n=0}^{\infty}$ are the sought expansion coefficients of a physical state described by an entire function

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^n.$$  

(3)

The recurrence coefficients $a_n$ and $b_n$ are functions of model parameters and we require that $b_n \neq 0$. In what follows, models of the class $\mathcal{R}$ are characterized by the recurrence coefficients having an asymptotic power-like dependence

$$a_n \sim an^{\delta}, \quad b_n \sim bn^{\nu} \quad (n \to \infty),$$

(4)

where $2\delta > \nu$ and $\tau = \delta - \nu \geq 1/2$, which guarantees $\lim_{n \to \infty} b_n/a_n = 0$.

The conventional boson annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ satisfying commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ are represented in the Bargmann Hilbert space $\mathcal{B}$ as [1,2]

$$\hat{a} \to \frac{\partial}{\partial z}, \quad \hat{a}^\dagger \to z.$$  

(5)

Therefore, the class $\mathcal{R}$ comprises among others single-mode boson models represented by

$$\hat{H} = A_1 \hat{a}^\dagger \hat{a} + A_2 \hat{a}^\dagger + A_3 \hat{a} + A_4,$$  

(6)

where $A_j$ are in general matrix coefficients. Indeed, on comparing the same powers of $z$ on both sides of (1), the consecutive $\hat{a}^\dagger \hat{a}$, $\hat{a}^2$, and $\hat{a}$ terms in (6) lead to $nc_n$, $c_{n-1}$, and $(n+1)c_{n+1}$ terms in the recurrence (2), respectively. Then the eigenvalue equation (1) reduces to a coupled system of ordinary differential equations [1,3,4].

The model Hamiltonian (6) encompasses many prominent examples, such as a displaced harmonic oscillator [1], the Rabi model [1,5], the Jaynes and Cummings (JC) model proposed as an approximation to the Rabi model [6], and a generalized Rabi model [7]. The Rabi model describes the simplest interaction between a cavity mode with a bare frequency $\omega$ and a two-level system with a bare resonance frequency $\omega_0$. The model is characterized by the Hamiltonian $[1,5,7]$,

$$\hat{H}_R = \hbar \omega \hat{a}^\dagger \hat{a} + \lambda \sigma_1 (\hat{a}^\dagger + \hat{a}) + \mu \sigma_3,$$

(7)

where $\lambda$ is a coupling constant, and $\mu = \hbar \omega_0/2$. In what follows we assume the standard representation of the Pauli matrices $\sigma_j$ and set the Planck constant $\hbar = 1$.

Our main results is that the spectrum of any quantum model from $\mathcal{R}$ can be obtained as zeros of a transcendental function defined by infinite series

$$F(x) \equiv a_0 + \sum_{k=1}^{\infty} \rho_1 \rho_2 \cdots \rho_k$$

(8)

solely in terms of the coefficients of the three-term recurrence in eq. (2) [8]:

$$\rho_1 = \frac{b_1}{a_1}, \quad \rho_l = u_l - 1, \quad u_1 = 1, \quad u_l = \frac{1}{1 - u_{l-1} b_l/(a_l a_{l-1})}, \quad l \geq 2.$$  

(9)
The function $F(x)$ is different from the $G$-functions of Braak [7]. The latter have only been obtained in the case of the Rabi model by making explicit use of the parity symmetry. In contrast to Braak’s result [7], our case of the Rabi model by making explicit use of the Braak [7]. The latter have only been obtained in the equation (2) explicitly. All what is needed to determine a discrete symmetry or to solve the three-term difference equation (2) degenerates for $n=0$ into an equation involving mere two terms.

\[ \frac{m_{n+1}}{m_n} \sim -\frac{b}{a n^\tau} \to 0 \quad (n \to \infty) \quad (11) \]

in virtue of (4) and $\tau > 1/2 > 0$. On substituting the minimal solution for the $c_n$’s in eq. (3), $\phi(z)$ automatically becomes an entire function. In what follows, the entire function generated by the minimal solution of the $n \geq 1$ part of (2) will be denoted as $\phi_m(z)$. Now (11) presumes (up to the terms reducing to a $O(1)$ term in (11)) an asymptotic behaviour

\[ m_n \sim k^n / \Gamma(n+1)^\tau \quad (12) \]

with $k = -b/a$. A sufficient condition for a minimal solution to generate $\phi_m(z) \in B$ can be read off from asymptotic properties of the generalized Mittag-Leffler function $E_{\tau,\gamma}(z)$ (cf. sect. 3 of [17]), an entire function with the expansion coefficients $1/\Gamma(\tau n + \gamma)$, where $\tau$ and $\gamma$ are real constants. In brief $\phi_m(z) \in B$ if the leading-order decay of $m_n$ for $n \to \infty$ is not slower than $\propto k^n/\Gamma(\tau n + \gamma)$, where $k$ is a constant and $\tau > 1/2$. For $\tau = 1/2$ the otherwise unrestricted constant $k$ has to satisfy $|k| < 1/\sqrt{2}$. Up to an irrelevant proportionality constant,

\[ \Gamma(n+1)^\tau \approx (n)^{\tau/2}(\tau^\tau)^n \Gamma(\tau n+1). \quad (13) \]

Therefore, when $\Gamma(\tau n+1)$ had been substituted by $[\Gamma(n+1)]^\tau$ in $E_{\tau,\gamma}(z)$, one would have arrived at substantially the same conclusions, but with $k$ replaced by $k \tau^{\tau}$. For $\tau = 1/2$ the latter yields $k/\sqrt{2}$, and hence the sufficient condition $|k| < 1$ for $\phi_m(z)$ to belong to $B$ (cf. eq. (1.4) of ref. [2]).

Now it is important to realize that only the ratios of subsequent terms $m_{n+1}/m_n$ of the minimal solution are related to infinite continued fractions (cf. Theorem 1.1 due to Pincherle in ref. [16])

\[ r_n = \frac{m_{n+1}}{m_n} = \frac{-b_{n+1}}{a_{n+1}} - \frac{b_{n+2}}{a_{n+2}} - \frac{b_{n+3}}{a_{n+3}} - \ldots. \quad (14) \]
Any minimal solution is, up to a multiplication by a constant, unique [16]. This has two immediate consequences. First, \( \phi_m(z) \) is, up to a multiplication constant, unique. Second, the ratio \( r_0 = m_1/m_0 \) of the first two terms of a given minimal solution is unambiguously fixed. Note that the ratio \( r_0 = m_1/m_0 \) involves \( m_0 \), although it takes into account the recurrence (2) only for \( n \geq 1 \) [16]. The remaining \( n = 0 \) part of the recurrence (2) imposes another condition (cf. eq. (10)) on the ratio \( r_0 = m_1/m_0 \). The condition (10) can be translated into the boundary condition on the logarithmic derivative of \( \phi_m(z) \),

\[
   r_0 = \frac{\phi'_m(0)}{\phi_m(0)} = -a_0. \tag{15}
\]

Obviously, one has \( r_0 \neq -a_0 \) in general. As a rule it is impossible to find a minimal solution satisfying an arbitrarily prescribed initial condition on the ratio \( m_1/m_0 \) of its first two terms. (This point was not made clear by either Schweber [1] or later on by Braak [7].) If one defines the function \( F = a_0 + r_0 \), with \( r_0 \) given by the continued fraction in (14), the zeros of \( F \) would correspond to those points in a parameter space where the condition (15) is satisfied. Then the coefficients of \( \phi_m(z) \) satisfy the recurrence (2) including \( n = 0 \), and \( \phi_m(z) \) belongs to the spectrum. Analogous to the Schrödinger equation, the boundary condition (15) enforces quantization of energy levels.

The rest of the proof follows upon an application of the Euler theorem, which transforms the continued fraction defining \( r_0 \) in eq. (14) into an infinite series (cf. eqs. (4.4), (4.5) on p. 43 of [16] forming the basis of the “third” method of Gautschi [16] at the special case of his \( N = 0 \) that is also used in our numerical implementation [9]). The latter leads directly to eqs. (8) and (9) for \( F \) [8]. The series in (8) converges whenever the continued fraction in (14) converges for \( n = 0 \), which occurs if and only if \( \phi_m(0) = m_0 \neq 0 \) (cf. Theorem 1.1 due to Pincherle in ref. [16]).

**Illustration of some basic properties.** — In the case of a displaced harmonic oscillator, which is the special case of \( \hat{H}_R \) in (7) for \( \mu = 0 \), the recurrence (2) becomes (cf. eq. (A.17) of ref. [1])

\[
   c_{n+1} + \frac{n-x}{(n+1)\kappa} c_n + \frac{1}{n+1} c_{n-1} = 0, \tag{16}
\]

where dimensionless parameter \( \kappa = \lambda/\omega \) reflects the coupling strength and \( x = \epsilon = E/\omega \) is a dimensionless energy parameter. Energy levels satisfy [1]

\[
   \epsilon_l = E_l/\omega = l - \kappa^2, \tag{17}
\]

where \( l \in \mathbb{N} \) is a non-negative integer (including zero). As demonstrated in fig. 1, \( F_{dho}(x) \) defined by eqs. (8) and (9) displays for \( \omega = 1 \) and \( \kappa = 0.7 \) a series of discontinuous branches extending monotonically between \( +\infty \) and \( -\infty \), which intersect the \( x \)-axis at \( x_l = l - 0.49 \), \( l = 0, 1, \ldots \) The zeros correspond exactly to the position of the energy levels (17).

In the present example one can also explicitly illustrate that the difference equation (2) with the condition (10) imposed as an initial condition of the recurrence, and then solved upwardly, always possesses a solution. Obviously, such a solution would be generally a dominant solution. Consequently, the function \( \phi(z) \) defined by power series expansion (3) with the expansion coefficients \( \{d_l\}_{l=0}^\infty \) would exhibit (typically branch-cut [1]) singularities in the \( z \)-complex plane and would not belong to \( \mathcal{B} \). Indeed, the recurrence (16) is solved for \( n \geq 0 \) with

\[
   c_n = \kappa^{a-n} F_n^{(a-n)}(\kappa^2), \tag{18}
\]

where \( \alpha = x + \kappa^2 \) and \( L_n^\beta \) are associated Laguerre polynomials [18] (note different sign of \( \kappa \) compared to eq. (2.16) of Schweber [1]). The substitution (18) transforms (16) into a 3-point rule that is identically satisfied by the associated Laguerre polynomials [18]. The Rodrigues formula [18]

\[
   L_n^{(\alpha-n)}(z) = \frac{z^{-\alpha} e^z}{n!} \frac{d^n}{dz^n} \left( e^{-z} z^{\alpha+n} \right) \tag{19}
\]

implies \( L_1^{(0)}(z) = 1 \), \( L_1^{(1)}(z) = -z + 1 + \alpha \), and in virtue of (18)

\[
   c_0 = \kappa^{a} L_0^{(a)}(\kappa^2) = \kappa^a, \tag{20}
\]

\[
   c_1 = \kappa^{a-1} L_1^{(a-1)}(\kappa^2) = \kappa^{a-1} x. \tag{21}
\]

Thus, \( c_1/c_0 = x/\kappa \), which is exactly the \( n = 0 \) part of (16).

In view of the asymptotic [18]

\[
   L_n^{(\alpha-n)}(z) \approx e^z \left( \frac{\alpha}{n} \right), \quad (n \to \infty) \tag{22}
\]

one finds from (18)

\[
   \frac{c_{n+1}}{c_n} \sim \frac{1}{\kappa} \frac{\alpha - n}{n+1} \to -\frac{1}{\kappa}, \quad (n \to \infty) \tag{23}
\]

whenever \( \alpha \notin \mathbb{N} \). Therefore, unless \( \alpha \) is a non-negative integer, the solution of eq. (18) is the dominant solution, the power series solution (3) has only a finite radius of convergence and thus does not belong to \( \mathcal{B} \). Only if the condition (17) is satisfied, \( \alpha = x + \kappa^2 \) is an integer, the dominant solution changes smoothly into the minimal solution of the recurrence (18), and the corresponding \( F_{dho} \) vanishes.

In the case of the Rabi model, the recurrence (2) becomes

\[
   c_{n+1} - \frac{f_n(x)}{(n+1)} c_n + \frac{1}{n+1} c_{n-1} = 0, \tag{24}
\]

where \( f_n(x) \) is given by

\[
   f_n(x) = 2\kappa + \frac{1}{2\kappa} \left( n - x - \Delta^2 \right), \tag{25}
\]
\( \kappa = \lambda / \omega \), as in eq. (16), and dimensionless \( \Delta = \mu / \omega \) (cf. eq. (A8) of Schweber [1], which has mistyped sign in front of his \( b_{n-1} \), and eqs. (4) and (5) of [7]). At first glance the recurrence (24) does not reduce to (16) for \( \Delta = 0 \) as one would expect from (7). The equivalence of eqs. (16) and (24) for \( \Delta = 0 \) is disguised by the fact that the recurrence (24) has been obtained after the unitary transformation of \( \hat{H}_R \) in (7) induced by \( D = \exp[\kappa (\hat{a}^\dagger - \hat{a}) \sigma_1] \) [1]. Thereby the dimensionless energy parameter \( x \) in eqs. (24), (25) is \( x = (E/\omega) + \kappa^2 \).

Both examples (16) and (24) represent the special Poincaré case characterized in that the respective \( a_n \) and \( b_n \) in (4) have finite limits \( \hat{a} \) and \( \hat{b} \) = 0 [1,16]1. The recurrences (16) and (24) correspond to the choice of \( \kappa = 1/\kappa \) in (16) and \( \kappa = -1/(2 \kappa) \) in (24), whereas \( \hat{b} = 1 \) in both examples. Linearly independent solutions of (24) can be written for \( n \gg 1 \) as

\[
\nu_n = \begin{cases} 
1/(n \kappa^n), & \text{dominant,} \\
\kappa^n / \Gamma(n + 1), & \text{minimal,}
\end{cases}
\]

in agreement with (A.29a,b) of Schweber [1] and eq. (12) (note that \( \kappa^2 \) leads to \( O(1) \) term in (11)). In order to establish asymptotic of \( \phi_m(z) \) for \( z \to \infty \), one defines

\[
\chi(z) = \sum_{n=0}^{\infty} \nu_n z^n,
\]

where the minimal solution (26) has been substituted for \( \nu_n \). Obviously, \( \phi_m(z) \sim \chi(z) \) for \( z \to \infty \). The asymptotic of \( \chi(z) \) can be determined by the saddle point of the Euler-Maclaurin integral representation of \( \chi(z) \) [17,19,20]. For \( z \to \infty \) and \( \arg z \leq \pi/2 \), which comprises the positive real axis with the largest growth of \( \chi(z) \), one finds a unique saddle point in the cut complex plane that completely governs the asymptotic behaviour of \( \chi(z) \) [17,19,20],

\[
\chi(z) \sim h(z, x) e^{\kappa z + p (1/2)} \ln[\kappa z + x - (1/2)] + O(|z|^p),
\]

where \( h(z, x) = \sqrt{\kappa x + x - (1/2)} \) and \( p \) is real constant satisfying \(-1/2 < p < 0\). As expected, \( \chi(z) \) behaves essentially as \( e^{\kappa z} \), because the asymptotic of the minimal solution (26) is only a slightly perturbed version of \( 1/\Gamma(n + 1) \). Consequently, \( \phi_m(z) \in B \) for all model parameters. The leading order behaviour of the recurrence coefficients in (16) for \( n \to \infty \) resembles that of the Rabi model and the above conclusions apply also to the case of the displaced harmonic oscillator.

**Discrete symmetries.**—The Rabi Hamiltonian \( \hat{H}_R \) (eq. (7)) is known to possess a discrete \( \mathbb{Z}_2 \)-symmetry corresponding to the constant of motion, or parity, \( P = \exp(i \pi \hat{J}) \) [3,7], where

\[
\hat{J} = \hat{a}^\dagger \hat{a} + 1/2 (1 + \sigma_3)
\]

is the familiar operator known to generate a continuous \( U(1) \) symmetry of the JC model [6,7]. In contrast to ref. [7], our approach does not necessitate any active use of any underlying discrete symmetry. This has been demonstrated in a recent comment [8], where the regular spectrum of the Rabi model has been reproduced as zeros of a corresponding \( F_{RL}(x) \) based on the recurrence (24). Obviously it was not possible then to determine what is the parity of a state corresponding to a given zero of \( F_{RL}(x) \).

Nevertheless, not only any discrete symmetry can be easily incorporated in our approach, this can be accomplished more straightforwardly and more easy than in the approach of Braak [7]. Indeed, the Rabi Hamiltonian \( \hat{H}_R \) is an example of a general Fulton and Gouterman Hamiltonian of a two-level system [21]

\[
\hat{H}_{FG} = A \mathbb{1} + B \sigma_1 + C \sigma_3,
\]

with

\[
A = \omega \hat{a}^\dagger \hat{a}, \quad B = \lambda (\hat{a}^\dagger + \hat{a}), \quad C = \mu.
\]

The Fulton and Gouterman symmetry operation \( \hat{g} \) [21] is realized by reflections

\[
\hat{a} \to -\hat{a}, \quad \hat{a}^\dagger \to -\hat{a}^\dagger,
\]

which leave the boson number operator \( \hat{a}^\dagger \hat{a} \) invariant. Because \( [\hat{g}, A] = [\hat{g}, C] = \{\hat{g}, B\} = 0 \), \( \hat{g} \sigma_3 \) is the symmetry of \( \hat{H}_R \) [21]. To any cyclic \( \mathbb{Z}_2 \) operator, such as \( \hat{g} \sigma_3 \), one can associate a pair of projection operators

\[
P^\pm = \frac{1}{2} \left( 1 \pm \hat{g} \sigma_3 \right), \quad (P^\pm)^2 = P^\pm.
\]

However, because of \( \hat{g} \sigma_3 \), the projectors \( P^\pm \) do not mix the upper and lower components of a wave function \( \phi \) (conventional Pauli representation of \( \sigma_1 \) and \( \sigma_3 \) is assumed). In order to employ the Fulton and Gouterman reduction in the positive- and negative-parity spaces, wherein one component of \( \phi \) is generated from the other by means of the operator \( \hat{g} \), one is forced to work in a unitary equivalent *single-mode spin-boson picture*

\[
\hat{H}_R = \omega \hat{a}^\dagger \hat{a} + \mu \sigma_1 + \lambda \sigma_3 (\hat{a}^\dagger + \hat{a}).
\]

The transformation is accomplished by means of the unitary operator \( U = (\sigma_1 + \sigma_3) / \sqrt{2} = U^{-1} \). The transformation interchanges the expressions for \( B \) and \( C \) in (31), resulting in \( \hat{g} \sigma_1 \) becoming the symmetry of \( \hat{H}_R \).

Equations (4.12) and (4.13) of [21] then yield

\[
[\omega \hat{a}^\dagger \hat{a} + \lambda (\hat{a}^\dagger + \hat{a}) \pm \mu \hat{g}] |\phi^\pm\rangle = E^\pm |\phi^\pm\rangle,
\]

where the superscripts \( \pm \) denote the positive- and negative-parity eigenstates of \( P^\pm \) (with \( \sigma_3 \) being replaced by \( \sigma_1 \) in (33)). Working in the Bargmann space,

\[
[\omega z \hat{a}^\dagger + \lambda (z + \hat{a}) \pm \mu \hat{g}] |\phi^\pm\rangle = E^\pm |\phi^\pm\rangle.
\]

Equations (36) are equivalent to a coupled system of first-order equations (10) of the on-line supplementary material of [7]. In contrast to [7], one does not need...
any ill-motivated substitution to arrive at the symmetry-resolved spectrum of the Rabi model. Assuming power series expansions \( \phi^{\pm}(z) = \sum_{n=0}^{\infty} c^{\pm}_n z^n \) for the positive- and negative-parity states, one arrives directly at the following three-term recurrence:

\[
\begin{align*}
    c^{\pm}_{n+1} + \frac{1}{\kappa(n+1)} \left[ n - x \pm (-1)^n \Delta \right] c^{\pm}_n \\
    + \frac{1}{n+1} c^{\pm}_{n-1} &= 0, \\
\end{align*}
\]

where \( x = E^{\pm}/\omega \) and \( \kappa = \lambda/\omega \), as in eq. (16), and \( \Delta = \mu/\omega \), as in eq. (24). In arriving at eqs. (37) we have merely used that

\[
    \tilde{g} \phi^{\pm}(z) = \phi^{\pm}(-z) = \sum_{n=0}^{\infty} (-1)^n c^{\pm}_n z^n. 
\]

Upon defining corresponding \( F^{\pm}_R(x) \) by means of eqs. (8) and (9), one can not only recover the regular spectrum of the Rabi, but also distinguish different parity eigenstates. This is demonstrated in fig. 2. Figure 3 demonstrates that on setting \( \Delta = 0 \) the parity eigenvalues become doubly-degenerate eigenvalues corresponding to those of a displaced harmonic oscillator. That was to be expected, because the three-term recurrence (37) then reduces to the three-term recurrence (16).

Discussion. – In his recent letter [7], Braak claimed to have solved the Rabi model analytically (see also the viewpoint by Solano [22]). He suggested that a regular spectrum of the Rabi model was given by the zeros of the transcendental functions \( G^{\pm}_+(x) \) in the variable \( x = (E/\omega) + \kappa^2 \) [7],

\[
    G^{\pm}_+(x) = \sum_{n=0}^{\infty} K_n(x) \left[ 1 \mp \frac{\Delta}{x - n} \right] \kappa^n. 
\]

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    G^{\pm}_+(x) = \sum_{n=0}^{\infty} K_n(x) \left[ 1 \mp \frac{\Delta}{x - n} \right] \kappa^n. 
\]

Here the coefficients \( K_n(x) \) were obtained recursively by solving the Poincaré difference equation (24) upwardly starting from the initial condition (10). Braak needed one page of arguments (between eq. (10) on p. 2 and the end of the first paragraph on p. 3 of the on-line supplement to [7]) to arrive at his \( G^{\pm}_+(x) \). The arguments involved an ill-motivated substitution and that a sufficient condition for the vanishing of an analytic function \( G^{\pm}_+(x; z) \) (defined by eq. (16) of his supplement) for all \( z \in \mathbb{C} \) if it vanishes at a single point \( z = 0 \). The argument is essential to arrive at (39). However, as an example of any homogeneous polynomial shows, the argument is obviously invalid. One needs additionally that \( (d/dz)G^{\pm}_+(x; z) = 0 \) at \( z = 0 [23] \). At zeros of \( G^{\pm}_+(x) \) the coefficients \( K_n(x) \) in (39) have to be the minimal solutions. The difficulty of calculating minimal solutions can be illustrated by an attempt to calculate the Bessel functions of the first kind \( J_n(x) \) for fixed \( x = 1 \) by an upward three-term recurrence from the initial values \( J_0(x) \) and \( J_1(x) \). It turns out that all digits calculated in single precision came out illusory already for \( n = 7 \) (cf. table 1 of [16]).

We have advanced an alternative approach which does not require any explicit solution of any difference equation and which applies to an entire class of models. A symmetry is not required to characterize a quantum model in terms of a suitable \( F(x) \). Contrary to ref. [7], \( F(x) \) can be obtained directly from the recurrence coefficients characterizing an eigenvalue equation of a given model. In the case of the Rabi model, nothing but the elementary relation (38) has been employed in arriving from (36) to (37), and the corresponding \( F^{\pm}_R(x) \) were unambiguously determined from (37) through eqs. (8) and (9). The resulting transcendental functions \( F^{\pm}_R(x) \) are different from \( G^{\pm}_+(x) \) considered by Braak [7]. With the exception of the displaced harmonic oscillator and the JC model, the zeros of \( F(x) \) have to be still determined numerically.
Therefore we disagree with Solano’s view [22] that a mere characterization of a model by a function such as $G_{\pm}(x)$ of ref. [7], or our $F(x)$, is tantamount to solving the model analytically in a closed form.

Our results rejuvenate the Schweber quantization criterion $r_0 + a_0 = 0$ known in the case when $r_0$ remained to be expressed in terms of continued fractions (e.g., eq. (14)) [1]. The Schweber criterion was deemed impractical and has not been employed to solve for the quantized energy levels of any quantum model [1,3,4,7]. The criterion was either regarded to require a numerical diagonalization in a truncated Hilbert space (pp. 4–5 of the on-line supplementary material of ref. [7]), or simply refuted as being identically valid irrespective of the value of an energy parameter $x$ (see for instance p. 4 of the on-line supplementary material of ref. [7]). With the help of the Euler theorem, which transforms the continued fractions into an infinite series, our approach turned the Schweber quantization criterion into an efficient computational tool.

Our approach has been demonstrated on the examples of a displaced harmonic oscillator and the Rabi model. However, the outlined approach would work for any of the models of the class $R$. For instance, the JC model and the single-mode spin-boson form of a generalized Rabi model introduced in [7],

$$H_{RB} = \omega \hat{a}^\dagger \hat{a} + \lambda \sigma_3 (\hat{a}^\dagger + \hat{a}) + \theta \sigma_3 + \mu \sigma_1,$$  

(40)

where $\theta$ is a deformation parameter. Another example is a modified Rabi model with the interaction Hamiltonian $\hat{H}_I = i\hbar \lambda \sigma_1 (\hat{a}^\dagger - \hat{a})$. The latter arises if the dipole interaction with a Fabry-Pérot cavity light mode is replaced by the interaction with a single-mode plane-wave field.

We have characterized our class $R$ of models implicitly in that their eigenvalue equation can be reduced to a difference equation (2) with its coefficients satisfying eq. (4). An interesting problem is to provide an explicit characterization of the models. Eventually note that whereas the condition $\tau \geq 1/2$ ensures $\phi_m(z) \in B$, one needs only $\tau > 0$ to define $F(x)$ by eq. (8) if $m_0 \neq 0$.

**Conclusions.** – A general formalism has been developed which allows to determine the spectrum of an entire class of quantum models as zeros of a corresponding transcendental function $F(x)$. The function can be analytically determined as an infinite series defined solely in terms of recurrence coefficients. The class of quantum models comprises the displaced harmonic oscillator, the Jaynes-Cummings (JC) model, the Rabi model, and a generalized Rabi model. Applications of the Rabi model range from quantum optics and magnetic resonance to solid state and molecular physics. The model plays a prominent role in cavity QED and circuit QED, and can be experimentally realized in Cooper-pair boxes, flux q-bits, in Josephson junctions or using trapped ions, and is of importance for various approaches to quantum computing. Therefore, our results could have implications for further theoretical and experimental work that explores the interaction between light and matter, from weak to strong interactions. The ease in obtaining the spectrum is of importance regarding recent experimental advances in preparing (ultra)strongly interacting quantum systems, which can no longer be reliably described by the exactly solvable JC model. The relevant computer code has been made freely available on-line [9].

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