G-Matrix Approach to Hyperon-Nucleus Systems

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The properties of the new YN · YY interaction model ESC08 are studied on the basis of the G-matrix theory. G-matrix calculations in nuclear matter are performed in the cases of ΛN-, ΣN-, ΞN- and ΛΛ-starting channels. In the ΛN case, the spin-spin and spin-orbit terms are shown to be of reasonable strengths. The folding-model analyses are done extensively for observed spectra of Λ hypernuclei. The experimental energy spectra are nicely reproduced. The shell-model analyses for typical Λ hypernuclei are performed with the G-matrix interactions. In the ΣN case, the repulsive values of $U_\Sigma$ are obtained owing to the effect of the quark Pauli-forbidden states which are taken into account phenomenologically in ESC08. In the ΞN case, the moderately attractive values of $U_\Xi$ are obtained mainly due to the strongly attractive contributions in $^3S_1 T = 1 \Xi N - \Lambda \Sigma - \Sigma \Sigma$ coupled states, which leads to extensive existence of $\Xi$ hypernuclei. In the ΛΛ case, the $B_{AA}$ values derived from ESC08 are found to be consistent with the experimental values.

§1. Introduction

The properties of baryon many-body systems, which contains not only nucleons but also hyperons with strangeness, link closely to the underlying hyperon (Y)-nucleon (N) interactions. Our goal is to reveal the entire picture of strong interactions among octet baryons through analyses not only for two-body processes but also for many-body systems in a unified way. SU(3)-invariant interaction models give useful guidance toward such an aim: In comparison with rich data on free-space NN interactions, the YN scattering data are extremely limited. Both data are utilized in a combined way in the parameter fitting of these interaction models. In spite of such an ingenuity, there remains the remarkable ambiguity in the obtained YN interaction due to lack of YN scattering data. Then, it is decisively important to test properties of interaction models by analyzing various hypernuclear phenomena. The G-matrix theory gives a good starting point for such an approach. Here, the correlations induced by short-range and tensor components, which are not taken into account in the model space, are renormalized into G-matrix interactions. In the cases of hyperonic many-body systems, the correlations induced by baryonic coupling interactions such as $\Lambda N$-$\Sigma N$ ones are also renormalized into single-channel parts of G-matrices. These G-matrix interactions are considered as effective interactions used in the model spaces. Thus, the hypernuclear phenomena and the underlying YN interaction models are linked through the models of hypernuclei and the YN G-matrix interactions, and the hypernuclear information gives a feedback to the interaction models.
The one-boson-exchange (OBE) models for YN interactions have been proposed by the Nijmegen group. In the earlier stage, they developed the hard-core models (NHC-D and -F) and the soft-core model (NSC89). After that, the trial started to take into account the G-matrix results in the modeling of YN interactions. As the first outcome of this approach, the NSC97 models were proposed, where the six versions $a$-$f$ were designed so as to be of different strengths of the $\Lambda N$ spin-spin parts. Then, the observed splitting energies of spin-doublet states in $\Lambda$ hypernuclei suggested that the spin-spin strengths of NSC97e and NSC07f were in a reasonable region.

Epoch-making development of the $SU(3)$-invariant interaction was accomplished by the Extended-Soft Core (ESC) model, in which two-meson and meson-pair exchanges are taken into account explicitly. In the OBE models these effects are implicitly and roughly described by meson-exchange, making the latter so-called ‘effective bosons’. After some trial versions (ESC96, ESC00 and ESC03), there appeared the specific versions ESC04a-$d$. The features of these versions are very different from those of the OBE models especially in $S = -2$ channels.

There remain some serious problems in NSC97 and ESC04 models, which can be found through G-matrix analyses. The first is that the derived values of $\Lambda$ spin-orbit splitting energies are too large in comparison with the experimental values. The second is that the derived $\Sigma$-nucleus potentials $U_\Sigma$ are attractive, whereas the experimental values are indicated to be repulsive. The third problem is related to the possible existence of $\Xi$ hypernuclei. Though they have not yet been observed, it is indicated experimentally that the $\Xi$-nucleus potentials $U_\Xi$ are rather attractive. The $U_\Xi$ values derived from the Nijmegen OBE models (NSC89/97 and NHC-F) are strongly repulsive except the special model NHC-D in which octet scalar mesons are not taken into account, and those for ESC04a/b are weakly repulsive. In the case of ESC04c/d, the obtained values of $U_\Xi$ are attractive, but their partial-wave contributions seem to be problematic as discussed later. These problems have been nicely solved in the new models ESC08a/b/a”. Here, the treatments for axial-vector and pair terms are improved, and the effects of the quark Pauli-forbidden states in repulsive-core representations are taken into account in $\Sigma^+ p(^3S_1, T = 3/2)$, $\Sigma N(^1S_0, T = 1/2)$ and $\Xi N(^1S_0, T = 1)$ states.

In this paper, we study comprehensively the properties of $\Lambda$, $\Sigma$ and $\Xi$ in nuclear medium on the basis of the G-matrix theory. Here, the features of NSC97f, ESC04a and ESC08a/b are compared. Another version ESC08a” is also investigated, which is improved from ESC08a such that the fitted values of $F/F+D$ ratio are understood better physics wise. In §2, the G-matrix formalism for hyperonic nuclear matter is recapitulated, and hyperon-nucleus folding potentials are expressed with $YN$ G-matrix interactions. In §3, $U_\Lambda$ values and $\Lambda N$ G-matrix interactions are obtained from ESC08 models, and their properties are discussed. $\Lambda$ single particle states in finite systems are calculated with G-matrix folding potentials, and compared with the experimental data. In §4, the shell-model analyses are performed for typical $\Lambda$ hypernuclei using the G-matrix interactions. We discuss here the $\Lambda$ single particle states in $^{89}_{\Lambda}Y$ and their spin-orbit splittings, and low-lying energy levels of $^{12}_{\Lambda}C$ and $^{11}_{\Lambda}B$. In §5, $U_\Sigma$ values are obtained from ESC08 models, and their repulsive natures
are discussed. Complex $\Sigma$ potentials in scattering states are investigated. In §6, $U_\Sigma$ values and $\Xi N$ $G$-matrix interactions are obtained from ESC08 models. On the basis of their properties, possible existence of $\Xi$ hypernuclei is investigated with use of $\Xi$-nucleus folding potentials. In §7, double-$\Lambda$ hypernuclei are studied using $\Lambda\Lambda$-core three-body models and $\Lambda\Lambda$ $G$-matrix interactions derived from ESC08 models. Summary of this paper is given in §8.

§2. $G$-matrix formalism

2.1. $G$-matrix interactions in nuclear matter

We start from the channel-coupled $G$-matrix equation for the baryon pair $B_1 B_2$ in nuclear matter,\textsuperscript{1)} where $B_1 B_2 = \Lambda N$, $\Sigma N$ and $\Xi N$, etc.:

$$G_{cc0} = v_{cc0} + \sum_{c'} v_{cc'} \frac{Q_y'}{\omega - \epsilon_{B_1'} - \epsilon_{B_2'} + \Delta_{yy'}} G_{c'c0}, \quad (2.1)$$

where $c$ denotes a $YN$ relative state $(y, T, L, S, J)$ with $y = (B_1 B_2)$. $S$ and $T$ are spin and isospin quantum numbers, respectively. Orbital and total angular momenta are denoted by $L$ and $J$, respectively, with $J = L + S$. Then, a two-particle state is represented as $2^S + 1L_J$ or $2^S + 1L_J + 1$. In Eq. (2.1), $\omega$ gives the starting energy in the starting channel $c_0$. $\Delta_{yy'} = M_{B_1} + M_{B_2} - M_{B_1'} - M_{B_2'}$ denotes the mass difference between two baryon channels. The Pauli operator $Q_y$ acts on intermediate nucleon states in a channel $y = (B_1 B_2) = (\Lambda N, \Sigma N$ and $\Xi N)$.

For a $y = (B_1 B_2)$ pair, the total and reduced masses and the relative and center-of-mass momenta are given as

$$M_y = M_{B_1} + M_{B_2}, \quad K_y = k_{B_1} + k_{B_2}, \quad \mu_y = M_{B_1} M_{B_2}/M_y, \quad k = (M_{B_2} k_{B_1} - M_{B_1} k_{B_2})/M_y.$$

Then the $G$-matrix equation (2.1) is represented in the coordinate space as follows:\textsuperscript{1)}

$$u_{cc1}(k; r) = \delta_{cc1} j_L(kr) + 4\pi \sum_{c_2} \int_0^\infty F_{c_1}(r, r') V_{c_1c_2}(r') u_{cc_2}(k; r') r'^2 dr', \quad (2.2)$$

$$F_{c}(r, r') = \frac{1}{2\pi^2} \int_0^\infty \frac{Q_y(k_{F}, q, K_{yy})}{\omega - \left(\frac{k_{y}^2}{2M_y} K_{yy}^2 + \frac{k_{y}^2}{2M_y} q^2 + U_{B_1}(k_{B_1}) + U_{B_2}(k_{B_2}) + \Delta_{yy'}\right)} dr', \quad (2.3)$$

where $k_{F}$ is a Fermi momentum of nuclear matter. The angle-averaged forms of the Pauli operator $Q_y$ and total momentum $K_y$, $\bar{Q}_y(k_{F}, q, K_{yy})$ and $\bar{K}_{yy}$, respectively, are given in Ref. 1). Similarly, the angle-averaged expressions $k_{B_1}$ and $k_{B_2}$ for single baryon momenta $k_{B_1} = (M_{B_1}/M_y) K + k$ and $k_{B_2} = (M_{B_2}/M_y) K - k$ can be derived, respectively. $U_{B_1}(k_{B_1})$ and $U_{B_2}(k_{B_2})$ are potentials of a baryon pair ($B_1$, $B_2$) in intermediate propagation.

In terms of the solution $u_{cc1}$ of Eq. (2.2), $G$-matrix elements are given as

$$G(k_y, \epsilon_Y, c_0, k) = 4\pi \sum_{c_1} \int_0^\infty j_L(kr) V_{c_0c_1}(r) u_{cc_1}(k; r) r'^2 dr. \quad (2.4)$$
The hyperon single particle (s.p.) energy $\epsilon_Y$ in nuclear matter is given by

$$\epsilon_Y(k_Y) = \frac{\hbar^2 k_Y^2}{2M_Y} + U_Y(k_Y) \ ,$$  \hspace{1cm} (2.5)

where $k_Y$ is the hyperon momentum. The potential energy $U_Y$ is obtained self-consistently in terms of the $G$-matrix as

$$U_Y(k_Y) = (1 - \kappa_N) \sum_{|k_N|} \langle k_Y k_N | G_{YN}(\omega = \epsilon_Y(k_Y) + \epsilon_N(k_N)) | k_Y k_N \rangle$$

$$= (1 - \kappa_N) \frac{(1 + M_N/M_Y)^3}{2\pi^2} \sum_{T_0 L_0 S_0 J_0} \frac{(2T_0 + 1)(2J_0 + 1)}{(2t_Y + 1)(2s_Y + 1)}$$

$$\times \int_0^{k_{max}} W(k; k_Y) G(k_Y, \epsilon_Y, YN, T_0, L_0, S_0, J_0, k) k^2 dk \ ,$$ \hspace{1cm} (2.6)

with $k_{max} = \frac{k_F + (M_N/M_Y) k_Y}{1 + M_N/M_Y}$. Here, $W(k; k_Y)$ is the statistical weight of $k$ for a value of $k_Y$, whose expression is given in Ref. 1).

The nucleon s.p. energy $\epsilon_N$ and potential $U_N$ are taken from the nuclear matter calculation. Then, the starting energy is taken as $\omega = \epsilon_{B_1} + \epsilon_{B_2}$ and obtained self-consistently from solved $G$-matrices.

In Eq. (2.6), the nucleon rearrangement effect is taken into account in an averaged way by the factor $(1 - \kappa_N)$, $\kappa_N$ being a nucleon correlation probability defined by $\kappa_N = -\sum_N \langle NN' | \frac{\partial G_{YN}(\omega)}{\partial \omega} | NN' - N'N \rangle$. The averaged values of $\kappa_N$ are calculated in nuclear matter with the NN interaction given by the ESC model, and parameterized as a function of $k_F$ as follows:

$$\kappa_N = \frac{.2755 k_F(1 - .9923 k_F + .3621 k_F^2)}{1} \ .$$ \hspace{1cm} (2.7)

In the cases of $\Sigma N$ and $\Xi N$ starting channels with $\Delta_{yy'} > 0$, we need to put the infinitesimal parameter $i\epsilon$ into the energy denominator, which assures the energy-conserving $y \rightarrow y'$ transition: There appear imaginary parts in $G$-matrices. The real and imaginary parts of the resulting complex s.p. potential are written as $U_Y$ and $W_Y^\Sigma$, respectively. Then the conversion width is given as $\Gamma_Y^\Sigma = -2W_Y^\Sigma$. $U_\Sigma$ and $U_\Xi$ are complex because of the $\Sigma N \rightarrow \Lambda N$ and $\Xi N \rightarrow \Lambda N$ conversion processes, respectively. There appears another type of imaginary part, when a hyperon is in a scattering state with a positive energy $E$ and the starting energy $\omega = E + \epsilon_N$ becomes positive, $\epsilon_N$ being a nucleon s.p. energy in nuclear matter. This type of imaginary part $W_Y^\Sigma$ is well known to appear in a nucleon optical potential based on the $G$-matrix approach. The corresponding quantity $\Gamma_Y^s = -2W_Y^s$ is called the scattering width.

The most ambiguous part in the $G$-matrix calculations is how to treat intermediate-state (off-shell) spectra, namely, $U_{B_1}$ and $U_{B_2}$ in the denominator of Eq. (2.3). Usually, they are treated in two ways: The simple one is the gap choice (QTQ prescription) which means that no potential term is taken into account. The other is the continuous choice where an off-shell potential is taken continuously from an
on-shell potential.\textsuperscript{10} Hereafter, the gap and continuous choices are denoted as GAP and CON, respectively. When the rearrangement effects are taken into account in the case of CON choices, the obtained $G$-matrices are denoted by CONr. In the case of usual nuclear matter, it was demonstrated that the gap and continuous choices lead to similar results when the calculations were performed up to the three-hole line contributions, and this higher-order result could be simulated by the lowest $G$-matrix one with the continuous choice better than that with the gap choice.\textsuperscript{11,12} In the case of hyperonic nuclear matter, however, there exists no corresponding higher-order calculation.

For applications to various hypernuclear problems, it is convenient to construct $k_F$-dependent effective local potentials $\mathcal{G}(r)$ which simulate the $G$-matrices. Here we parameterize them in a three-range Gaussian form

$$\mathcal{G}(r) = \sum_{i=1}^{3} \left( a_i + b_i k_F + c_i k_F^2 \right) \exp \left( -\frac{r^2}{\beta_i^2} \right).$$  \hspace{1cm} (2.8)

The parameters $(a_i, b_i, c_i)$ are determined so as to simulate the calculated $G$-matrix for each $(T, S, L, J)$ state. The procedures to fit the parameters are as follows: First, three-range Gaussian functions are determined for three chosen values of $k_F$ so that their matrix elements in momentum-space simulate corresponding $G$-matrix elements $\langle k|G|k \rangle$, and coefficients $(a_i, b_i, c_i)$ are obtained from the fitted values of Gaussian coefficients for three $k_F$ values. In this procedure, the outermost Gaussian strength for $\beta_3$ is fitted so as to reproduce the tail part of the bare interaction, and then it becomes independent on $k_F$ ($b_3 = c_3 = 0$). Next, The core parts corresponding to $\beta_1$ are fine-tuned to reproduce each partial-wave contribution to the potential energy $U_Y$.

2.2. Hyperon-nucleus folding potential

Hereafter, the $G$-matrix interactions in $S$- and $P$-states are considered as those in even- and odd-parity states, respectively. A hyperon-nucleus potential in a finite system is derived from $YN$ $G$-matrix interactions $G_{ST}^{\pm}(r)$ by the expression

$$U_Y(r, r') = U_{dr} + U_{ex},$$

$$U_{dr} = \delta(r - r') \int dr'' \rho(r'') V_{dr}(|r - r''|; k_F),$$

$$U_{ex} = \rho(r, r') V_{ex}(|r - r'|; k_F),$$  \hspace{1cm} (2.9)

$$\left( \begin{array}{c} V_{dr} \\ V_{ex} \end{array} \right) = \frac{1}{2(2t_Y + 1)(2s_Y + 1)} \sum_{TS} (2T + 1)(2S + 1) \left[ G_{ST}^{\pm} \pm G_{ST}^{\mp} \right],$$  \hspace{1cm} (2.10)

where $(\pm)$ denote party quantum numbers. Here, core nuclei are assumed to be spherical. In this paper, densities $\rho(r)$ and mixed densities $\rho(r, r')$ are obtained from Skyrme-HF wave functions. An important problem is how to treat $k_F$ values included in $G$-matrix interactions, which are related to $\rho$ values by $\frac{2k_F^3}{3\pi^2}$. The popular
treatment is based on the Local-Density Approximation (LDA), where the averaged values such as \((\rho(r) + \rho(r'))/2\) are used for \(k_F\) values included in G-matrix interactions. It is found, however, that the following Averaged-Density Approximation (ADA) is generally better than the LDA for reproducing observed \(\Lambda\) s.p. spectra: Here, an averaged value \(\langle \rho \rangle\) is calculated by \(\langle \phi_Y(r)|\rho(r)|\phi_Y(r) \rangle\) for each hyperon state \(\phi_Y(r)\), \(\langle k_F \rangle\) being obtained from \(\langle \rho \rangle\). Then, a value of \(\langle k_F \rangle\) is determined self-consistently for each hyperon state.

In general, a \(YN\) G-matrix is dependent on a hyperon energy \(\epsilon_Y\). This \(\epsilon_Y\) dependence is eliminated by the dispersion relation \(\epsilon_Y = \frac{\hbar^2 k_x^2}{2M_Y} + U_Y(k_Y, \epsilon_Y)\) in nuclear matter. When our localized G-matrix interaction \(G(r; k_F)\) is derived, the remained \(k_Y\)-dependence is eliminated by the averaging procedure. In the case of treating hyperon bound states in finite systems, it seems to be reasonable to use this interaction \(G(r; k_F)\) in the above expressions for \(U_Y(r, r')\). On the other hand, in the case of deriving hyperon-nucleus potentials in hyperon scattering states, the \(YN\) G-matrix should be dependent on the hyperon incident energy \(E_Y\). This \(E_Y\)-dependence is treated explicitly: Namely we have \(G(r; k_F, E_Y)\) and \(U_Y(r, r'; E_Y)\) in the above expressions. The procedure is the same as that in the derivation of nucleon optical potentials in the G-matrix approach.\(^{10}\)

\section{3. \(A\)-nucleus systems}

\subsection{3.1. \(UA\) and \(AN\) G-matrix interactions in nuclear matter}

Here, \(AN\) G-matrix calculations are performed for ESC08a/b/a”, ESC04a and NSC97f. In Table I we show the potential energies \(U_A\) for a zero-momentum \(A\) and their partial-wave contributions \(U_A(2S + 1L_J)\) at normal density \(\rho_0\) \((k_F=1.35\ \text{fm}^{-1})\) in the cases of GAP and CONr, where a statistical factor \((2J + 1)\) is included in \(U_A(2S + 1L_J)\). The calculated values of the \(A\) effective mass \(M_A^*\) are also given in Table I, which is defined by \(M_A^* = (1 + \frac{dU_A}{dT_A})^{-1}\), where \(T_A\) denotes \(A\) kinetic energy. The GAP results can be compared with those for ESC04a-d and NSC97e/f given in 7). Among ESC04a-d, the most similar one to ESC08a/b/a” (especially ESC08a) is ESC04a: The S-state contributions in ESC08a (ESC08b) are comparable to (slightly less attractive than) those in ESC04a. The \(P\)-state contributions in ESC08a/b/a” are very similar to each other, and they are less attractive than those in ESC04a.

Let us compare the CONr and GAP results in the cases of ESC08a/b. When the rearrangement effect is not taken into account, the values of \(U_A\) in the CON case are \(-42.2\) and \(-40.9\) MeV for ESC08a and ESC08b, respectively. The corresponding values in the GAP case are \(-35.8\) and \(-34.1\) MeV, respectively, being rather less attractive than those in the CON case. In the CONr case, the \(U_A\) values become \(-37.2\) and \(-36.0\) MeV, respectively, by multiplying the rearrangement factor \((1 - \kappa_N)\) on the values in the CON case. Namely, the simple GAP results become comparable roughly to the CONr ones, because the energy gain by taking the continuous energy spectrum is substantially canceled by the repulsive contribution of the rearrangement effect. The above values of \(U_A = -(34 - 39)\) MeV seem to be too attractive compared
Table I. Values of $U_A$ at normal density and partial wave contributions in $^{2S+1}L_J$ states for ESC08a/b/a" from the G-matrix calculations with the GAP and CONr prescriptions (in MeV). The value specified by $D$ gives the sum of $^{2S+1}D_J$ contributions. Contributions from S-state spin-spin interactions are given by $U_{\sigma\sigma} = (U(^3S_1) - 3U(^3S_0))/12$. $m_\Lambda$ is defined by $M_\Lambda/M_A$, $M_\Lambda$ being a $\Lambda$ effective mass.

|       | $^1S_0$ | $^3S_1$ | $^1P_1$ | $^3P_0$ | $^3P_1$ | $^3P_2$ | $D$   | $U_A$   | $U_{\sigma\sigma}$ | $m_\Lambda$ |
|-------|---------|---------|---------|---------|---------|---------|-------|---------|-------------------|-------------|
| GAP   |         |         |         |         |         |         |       |         |                   |             |
| ESC08a| -12.4   | -22.8   | 3.0     | 0.1     | 1.5     | -3.5    | -1.6  | -35.8   | 1.21              | 0.71        |
| ESC08b| -12.1   | -20.2   | 2.6     | -0.2    | 1.6     | -4.1    | -1.7  | -34.1   | 1.34              | 0.73        |
| ESC08a"| -12.8   | -25.6   | 2.9     | 0.2     | 1.4     | -3.2    | -1.6  | -38.6   | 1.06              | 0.70        |
| ESC04a| -13.7   | -20.5   | 0.6     | 0.2     | 0.5     | -4.5    | -1.1  | -38.5   | 1.73              | 0.81        |
| NSC97f| -14.3   | -22.4   | 2.4     | 0.5     | 4.0     | -0.7    | -1.2  | -31.8   | 1.71              | 0.66        |
| CONr  |         |         |         |         |         |         |       |         |                   |             |
| ESC08a| -11.6   | -24.4   | 2.4     | 0.0     | 1.2     | -3.3    | -1.5  | -37.2   | 0.88              | 0.75        |
| ESC08b| -11.4   | -22.4   | 2.1     | -0.2    | 1.2     | -3.8    | -1.6  | -36.0   | 0.99              | 0.78        |
| ESC08a"| -12.0   | -26.1   | 2.3     | 0.2     | 1.1     | -3.0    | -1.5  | -39.0   | 0.83              | 0.74        |
| ESC04a| -12.8   | -20.8   | 0.4     | 0.1     | 0.4     | -4.2    | -0.9  | -37.8   | 1.48              | 0.85        |
| NSC97f| -13.2   | -23.5   | 2.0     | 0.3     | 3.3     | -0.8    | -1.2  | -33.1   | 1.36              | 0.70        |

to the experimental value of about $-30$ MeV, which is the depth of the $\Lambda$ Woods-Saxon (WS) potential suitable to the data of $\Lambda$ hypernuclei. As discussed later, however, the WS depth strictly does not correspond to the $U_A$ value in normal-density nuclear matter.

The spin-dependent features of the $\Lambda N$ G-matrix interactions are very important, because they can be checked in experimental data of energy spectra of $\Lambda$ hypernuclei. For instance, the spin-spin and spin-orbit components come out in splitting energies of spin-doublet states observed in $\gamma$-ray experiments. The contributions to $U_A$ from S-state spin-spin components can be seen qualitatively in values of $U_{\sigma\sigma} = (U(^3S_1) - 3U(^3S_0))/12$. These values of $U_{\sigma\sigma}$ also are given in Table I. Various analyses suggest that the reasonable value of $U_{\sigma\sigma}$ is between those of NSC97e and NSC97f, which are 1.05 and 1.70 MeV, respectively, when they are calculated with the GAP choice. The $S$-state components of ESC08a/b/a" turn out to be of reasonable strengths.

In Ref. 13, the properties of ESC04a have been investigated by the shell-model analysis with the G-matrix interactions obtained in the harmonic oscillator space, and found to be reasonable in comparison with experimental data.

Let us here represent the $\Lambda N$ and $\Sigma N$ central parts of the G-matrix interactions in the three-range Gaussian forms whose coefficients are given as a function of $k_F$, as shown by Eq. (2.8). The obtained Gaussian parameters are shown in Table II, where the $G$-matrices are calculated with CONr in the cases of ESC08a/b/a". The solved $G$-matrices include not only $\Lambda N$ and $\Sigma N$ diagonal parts but also $\Lambda N-\Sigma N$ coupling parts, and it is possible to extract such coupling parts to treat $\Lambda N-\Sigma N$ mixing problems. The Gaussian-represented forms for $\Lambda N-\Sigma N$ coupling $G$-matrices are given in Appendix A.

The SLS interactions $G_{\text{SLS}}(r)$ are derived from $G$-matrices $G_{LL'}^{IS}(r)$ with $S = 1$...
by the relation
\[
G_{SLS}(r) = \frac{1}{2(2L+1)} \left[ - \frac{2L-1}{L} G_{LL}^{L-1,1}(r) - \frac{2L+1}{L(L+1)} G_{LL}^{L,1}(r) + \frac{2L+3}{L+1} G_{LL}^{L+1,1}(r) \right].
\]

Table II. Parameters of \( AN \) G-matrix interactions (CONr) represented by three-range Gaussian forms \( G(r,k_F) = \sum_a (a_i + b_i k_F + c_i k_F^2) \exp(-r/\beta_i)^2 \) in the cases of ESC08a/b/a".

| \( \beta_i \) | ESC08a | ESC08b | ESC08a" |
|-------------|--------|--------|---------|
| a           | 0.5    | 0.9    | 2.0     |
| 1E b        | -3144. | 368.0  | -1.467  |
| c           | -2478. | 394.5  | 0.0     |
| a           | -2734. | 316.8  | -1.044  |
| 3E b        | 5827.  | -901.6 | 0.0     |
| c           | -2404. | 395.8  | 0.0     |
| a           | 663.1  | 124.6  | -5606   |
| 1O b        | 1728.  | -50.97 | 0.0     |
| c           | -599.0 | 32.40  | 0.0     |
| a           | 810.6  | -182.7 | 0.0     |
| 3O b        | -703.2 | 118.1  | 0.0     |
| c           | 209.6  | -13.17 | 0.0     |

Table III. SLS and ALS G-matrix interactions (CONr) calculated at \( k_F = 1.0 \text{ fm}^{-1} \), which are represented in three-range Gaussian forms \( \sum_{i=1}^3 c_i \exp(-r/\beta_i)^2 \).

| \( \beta_i \) (fm) | 0.40 | 0.80 | 1.20 |
|-------------------|------|------|------|
| SLS               |      |      |      |
| ESC08a            | 1044.| -118.2| -2.701|
| ESC08b            | 350.0| -101.1| -2.680|
| ESC08a"           | 1270.| -122.8| -2.533|
| ALS               |      |      |      |
| ESC08a            | 1728. | 11.60 | 1.837|
| ESC08b            | 1810. | 7.703 | 2.085|
| ESC08a"           | 1136. | 24.59 | 1.813|

Table IV. Triplet-odd tensor G-matrix interactions (CONr) calculated at \( k_F = 1.0 \text{ fm}^{-1} \), which are represented in \( r^2 \)-Gaussian forms \( \sum_{i=1}^3 c_i r^2 \exp(-r/\beta_i)^2 \).

| \( \beta_i \) (fm) | 0.50 | 0.90 | 2.00 |
|-------------------|------|------|------|
| ESC08a            | 10.37| 0.0181| 0.0170|
| ESC08b            | 14.38| 0.0201| 0.0216|
| ESC08a"           | 7.926| 0.0205| 0.0168|

The diagonal tensor components of \( AN-AN \) G-matrices are derived from G-matrices \( G_{LL}^{JS}(r) \) with \( S = 1 \) by linear transformations similarly to the above case of SLS interactions. Here, they are represented in \( r^2 \)-Gaussian forms. The obtained parameters are given in Table IV.
In order to compare clearly the SLS and ALS components, it is convenient to derive the strengths of the \( \Lambda \) l-s potentials in hypernuclei. In the same way as in Refs. 5) and 7), we use the following expression derived with the Scheerbaum approximation,\(^{14}\)

\[
U_{\Lambda}^{ls}(r) = K_{\Lambda} \frac{1}{r} \frac{d}{dr} l \cdot s,
\]

\[
K_{\Lambda} = -\frac{\pi}{3} (S_{\text{SLS}} + S_{\text{ALS}}),
\]

\[
S_{\text{SLS,ALS}} = \frac{3}{\bar{q}} \int_0^{\infty} r^3 j_1(\bar{q}r) \mathcal{G}_{\text{SLS,ALS}}(r) dr,
\]

where \( \mathcal{G}_{\text{SLS}}(r) \) and \( \mathcal{G}_{\text{ALS}}(r) \) are SLS and ALS \( G \)-matrix interactions in configuration space, respectively, and \( \rho(r) \) is a nuclear density distribution. We take here \( \bar{q} = 0.7 \) fm\(^{-1}\).

Table V. Strengths of \( \Lambda \) spin-orbit splittings given by CONr and GAP \( G \)-matrix interactions from various interaction models. See the text for the definitions of \( K_{\Lambda} \) and \( S_{\text{SLS,ALS}} \).

| Model  | \( S_{\text{SLS}} \) | \( S_{\text{ALS}} \) | \( K_{\Lambda} \) |
|--------|-----------------|-----------------|-----------------|
| CONr   | -20.8           | 16.2            | 4.4             |
| ESC08a | -22.1           | 16.3            | 5.5             |
| ESC08b | -19.8           | 15.6            | 5.0             |
| ESC08a’| -22.7           | 8.7             | 14.7            |
| ESC04a | -19.7           | 7.2             | 13.1            |
| NSC97f | -23.3           | 15.5            | 7.7             |
| GAP    | -24.3           | 15.5            | 8.8             |
| ESC08b | -22.5           | 14.8            | 8.1             |
| ESC08a’| -24.9           | 8.5             | 17.1            |
| ESC04a | -23.9           | 7.0             | 17.7            |

Table V shows the values of \( K_{\Lambda} \) and \( S_{\text{SLS,ALS}} \) obtained from the SLS and ALS \( G \)-matrix interactions calculated at \( k_F = 1.0 \) fm\(^{-1}\) in the cases of CONr and GAP. The obtained values for ESC08a/b/a’ can be compared with those for ESC04a-d and NSC97e/f given in Ref. 7). The ALS parts in the ESC08 models are found to be more repulsive than those in the older models. Thus, the smaller values of \( K_{\Lambda} \) are obtained in the cases of ESC08 models. In all cases, the obtained \( K_{\Lambda} \) values for CONr are rather smaller than those for GAP. The main reason is as follows: The difference of \( K_{\Lambda} \) values for CONr and GAP are due to that of \( S_{\text{SLS}} \). The SLS parts are given by the linear combination of the \( ^3P_J \) \( (J = 0, 1, 2) \) contributions, on which the reduction factor \((1 - \kappa_N)\) is multiplied. Namely, the SLS parts are reduced by the rearrangement effects. On the other hand, the \((1 - \kappa_N)\) factor is not related to the \( ^3P_0 - ^1P_0 \) coupling term by the ALS parts.

3.2. \( \Lambda \) single-particle states

Let us derive \( \Lambda \)-nucleus folding potentials from \( \Lambda N \) \( G \)-matrix interactions by using the expression (2.9), and calculate \( \Lambda \) s.p. energies in various \( \Lambda \) hypernuclei. Here, the folding potentials are derived from \( G \)-matrix interactions with ADA, because \( \Lambda \) energy spectra calculated with them are generally in better agreement with the experimental data than those with LDA.

The \( \Lambda \) energy spectra of medium-heavy hypernuclei are very useful to study the features of \( \Lambda \) s.p. states. Especially, the energy spectrum of \( ^{89}_\Lambda \)Y is of the most impor-
Table VI. $^{89}$Y with $G$-matrix folding model in Average-$k_F$ approximation with $k_F = \langle k_F \rangle + \Delta k_F$. E08aCr, E08bCr, E08a"Cr, E04aCr, N97fCr denote the CONr $G$-matrix interactions for ESC08a, ESC08b, ESC08a", ESC04a and NSC97f, respectively, and E08aG the GAP one for ESC08a.

| $\Delta k_F$ | E08aCr | E08bCr | E08a"Cr | E08aG | E04aCr | N97fCr | exp  |
|------------|--------|--------|---------|-------|--------|--------|------|
| $s_A$      | -24.04 | -23.76 | -24.32  | -24.48| -23.93 | -23.96 | -23.11 ± 0.10 |
|            | (-23.67) | (-22.33) | (-25.13) | (-21.93) | (-26.30) | (-21.20) |      |
| $p_A$      | -16.98 | -16.84 | -17.10  | -17.02| -17.15 | -17.03 | -17.10 ± 0.08 |
|            |         |        |         |       |        |        |      |
| $d_A$      | -9.91  | -9.97  | -9.84   | -9.72 | -10.06 | -9.96  | -10.32 ± 0.06 |
|            |         |        |         |       |        |        |      |
| $f_A$      | -3.30  | -3.54  | -3.06   | unbound | -3.24  | -3.19  | -3.13 ± 0.07 |

The result for E08aG seems to be slightly different from the others in spite of qualitative agreement with the experimental data: The energy spectrum for E08aG is of broader spacing than the experimental one quantitatively, even if the $\Delta k_F$
Fig. 1. Energy spectra of $^{13}\Lambda C$, $^{28}\Lambda Si$, $^{51}\Lambda V$, $^{89}\Lambda Y$, $^{139}\Lambda La$ and $^{208}\Lambda Pb$ are given as a function of $A^{-2/3}$, $A$ being mass numbers of core nuclei. Solid (dashed) lines show calculated values by the $G$-matrix folding model derived from ESC08a (the Skyrme-HF model). Open circles denote the experimental values taken from Ref. 17).

corrections are made. This is a common feature of GAP $G$-matrix interactions having the $k_F$ dependent properties different from the CONr ones. Furthermore, in the cases of GAP ones, iterations for $\langle k_F \rangle$ in our treatment give no converged solution for the $f_A$ state, because this state is too weakly bound or unbound. (In the E08aG case, the parameter $\Delta k_F$ is chosen by the $\chi^2$ fitting for $s_A$, $p_A$ and $d_A$ states.) Comparing the energy spectra for E08aCr and E08aG, the range of the $\Lambda$-nucleus potential for the latter is shorter than that for the former. The reason is because the density dependence of E08aCr is stronger than that of E08aG.

Now, we remark the $\Lambda s$-state energies with no $\Delta k_F$ correction given in parentheses, which depend straightforwardly on underlying interaction models and treatments of $G$-matrices (GAP or CONr, etc.). It is noted, here, that the $\Lambda$ s.p. energies in finite systems are not related simply to the $U_A(\rho_0)$ values given in Table I, when they are compared carefully. The $\Lambda$-nucleus folding potential depends not only on the strengths of $\Lambda N$ $G$-matrices but also on their $k_F$ dependences. Then, it is meaningless to consider the depth $U_{WS}$ of the phenomenological Woods-Saxon potential of $\Lambda$ as the $\Lambda$ potential depth in nuclear matter. Indeed, in the case of E08aCr where the experimental $\Lambda$ binding energies in $^{89}\Lambda Y$ are reproduced almost without the correction by $\Delta k_F$, the $U_A(\rho_0)$ value of $-37.2$ MeV is substantially deeper than
\[ U_{WS} \sim -30 \text{ MeV}. \]

Some comment is needed for the experimental value of the $s$-state energy. In Ref. 16, the double-peaked structures of the $p_A$, $d_A$ and $f_A$ states in the experimental spectrum of \(^{89}Y\)\(^{15}\) are explained by the effects of hole-excitations in the \(^{88}Y\) core. According to this understanding, the $s$-state has to be of a double-peak structure. However, the analysis for the experimental spectrum was performed by assuming a single-peak structure for the ground $s$-state. There still remains room to analyze it as an double-peak state, which might make the experimental value of the $s$-state energy slightly deeper.

Let us calculate the energy spectra of the other hypernuclei systematically (\(^{13}C\), \(^{28}Si\), \(^{51}V\), \(^{139}La\), \(^{208}Pb\))\(^{17}\) with the $G$-matrix interactions E08aCr adjusted for the observed energy spectrum of \(^{89}Y\) by taking $\Delta k_F = -0.014 \text{ fm}^{-1}$. In Fig. 1, the calculated values shown by solid lines are compared with the experimental values marked by open circles, where the horizontal axis is given as $A^{-2/3}$. Our $G$-matrix folding model for E08aCr turns out to reproduce the energy spectra of $\Lambda$ hypernuclei systematically almost with no free parameter.

In Fig. 1, we show also the result calculated by the Skyrme-Hartree-Fock (SkHF) model. Here, the Skyrme parameters are determined only by using the observed spectrum of \(^{89}Y\), as given in Appendix B. It should be noted that all the experimental spectra are reproduced almost completely by the SkHF model whose parameters are determined by \(^{89}Y\).

### §4. Applications to typical $\Lambda$-hypernuclear structure calculations

The $G$-matrix derived from the free $NN$ interaction was successfully used, for instance, by Kuo and Brown\(^{18}\) as the effective interaction in finite nuclear structure calculations. Here, we apply the nuclear-matter $YN$ $G$-matrix interactions to shell model analyses for typical $\Lambda$ hypernuclei. In this approach, $YN$ $G$-matrix interactions are calculated in nuclear matter, and therefore they depend on the nuclear Fermi momentum $k_F$ which restricts intermediate nucleon states after $YN$ scattering in medium: The obtained $G$-matrix interactions are represented as three-range Gaussian forms Eq. (2.8), whose parameters for ESC08 models are given in §3.1. When these $k_F$-dependent interactions are used in finite systems, it is natural that the $k_F$ value is different for each hypernuclear mass number and it also changes depending on the hypernuclear excitation energy. In the previous section, the adequate value of $k_F$ in each state is determined self-consistently under the average-$k_F$ approximation. Here, as another way, we take $k_F$ value as an adjustable parameter so as to reproduce a $\Lambda$ binding energy for each hypernuclear state most appropriately. Indeed, the successful results in the former approach indicate that the $k_F$ values determined in the latter approach are very similar to those done self-consistently.

#### 4.1. $\Lambda$ single-particle energy in heavy hypernuclei

As the first example, we demonstrate the $k_F$-dependent behavior of $\Lambda$ single-particle energies calculated with the ESC08a $G$-matrix interactions for the $^{90}Zr+\Lambda$ system. The $\Lambda(lj)$ energy is solved by diagonalization of the following Hamiltonian

\[ H = \begin{bmatrix}
    E_0 & G_{\Lambda NN} & G_{\Lambda NN} \\
    G_{\Lambda NN} & E_0 & G_{\Lambda NN} \\
    G_{\Lambda NN} & G_{\Lambda NN} & E_0
\end{bmatrix}
\]
Fig. 2. A single-particle energies (the lowest \(f\) and \(d\) orbits) at the \(^{90}\text{Zn}\) core calculated as a function of the nuclear Fermi momentum \(k_F\). The ESC08a interaction is used. Arrows show the observed energies.

Fig. 3. A single-particle energies (the lowest \(p\) and \(s\) orbits) at the \(^{90}\text{Zn}\) core calculated as a function of the nuclear Fermi momentum \(k_F\). The ESC08a interaction is used.

within the Harmonic oscillator model space \(\{|nlj >^{\text{HO}}; n = 0 - 4\}\):

\[
H = t_A + \sum_{N}^{\text{occupied}} V_{NA}.
\]  

(4.1)

Here \(t_A\) is the \(A\) kinetic energy of the relative motion with respect to the \(^{90}\text{Zr}_{50}\) nucleus which ground state is naturally assumed to have the shell model doubly-closed configuration: \([0s)^4(0p)^{12}(1s0d)^{24}(1p0f)^{40}(0g_{9/2})^n]^{10}\). For this mass region we employ the H.O. size parameter for nucleon \(b_N = 2.129\) fm which leads to \(b_Y = \sqrt{M_N/M_A} = 1.953\) fm.

Figure 2 shows the calculated \(\Lambda\) energies \(\epsilon_A(lj)\) for the \(f\) and \(d\) orbits as a function of \(k_F\) value adopted for the \(\Lambda N\) G-matrix interaction. Also the cases of \(p\) and \(s\) orbits are displayed in Fig. 3. First one sees that in general the \(\Lambda\) binding energy becomes smaller when the \(k_F\) value increases. The reason is that, in the nuclear matter calculation, the increase of \(k_F\) leads to the decrease of available nucleon states for the \(\Lambda N\) scattering and therefore this restriction makes the \(\Lambda N\)
Table VII. Fermi momentum $k_F$ to reproduce the empirical $\Lambda$ energy and calculated r.m.s. radii and spin-orbit splitting, $\delta(l) = \epsilon_\Lambda(j<) - \epsilon_\Lambda(j>)$.

| $\Lambda$ orbit | $\epsilon_\Lambda^{\text{exp}}$ | $k_F$ (ESC08a) | $k_F$ (ESC08b) | $\sqrt{\langle r^2 \rangle}$ (ESC08a) | $\sqrt{\langle r^2 \rangle}$ (ESC08b) | $\delta(l)$ (08a) | $\delta(l)$ (08b) |
|-----------------|-------------------------------|----------------|----------------|----------------------------------------|----------------------------------------|----------------|----------------|
| $f$             | $-3.13 \pm 0.07$              | 1.091          | 1.084          | 4.77                                   | 4.75                                   | 0.259         | 0.382         |
| $d$             | $-10.32 \pm 0.06$             | 1.153          | 1.139          | 4.19                                   | 4.14                                   | 0.201         | 0.296         |
| $p$             | $-17.10 \pm 0.08$             | 1.228          | 1.207          | 3.60                                   | 3.55                                   | 0.145         | 0.206         |
| $s$             | $-23.11 \pm 0.10$             | 1.321          | 1.292          | 2.89                                   | 2.84                                   | --            | --            |
|                 | [MeV]                         | [fm$^{-1}$]    | [fm$^{-1}$]    | [fm]                                   | [fm]                                   | [MeV]         | [MeV]         |

Correlation weaker. Experimentally the $\Lambda$ single-particle energies for $s$, $p$, $d$ and $f$ orbits are best known in the case of $^{89}\text{Y}$ by means of the $^{89}\text{Y}(\pi^+,K^+)$ reaction.$^{15}$ According to the careful DWIA analyses including core-excited configurations,$^{16}$ the $\Lambda$ spin-orbit partners ($j_\geq = l+1/2$ and $j_\leq = l-1/2$) are both in each left subpeak of the pronounced series of the ($\pi^+,K^+$) peaks. For example the $f_{7/2}$ and $f_{5/2}$ splitting has been analyzed to be less than 0.2 MeV. Thus we compare the spin-averaged $\Lambda$ energies $\epsilon_\Lambda(l)$ with the experimental ones which are indicated by arrows, respectively.

Here we find the appropriate $k_F$ value that reproduces each empirical $\Lambda$ single-particle energy,$^{15}$ and then it is interesting to see how they coincide or how they differ from each other. In Table VII, we list the calculated results for the ESC08a and ESC08b $\Lambda N$ interactions. When going from $s$ up to $f$ orbits, one sees that $k_F$ changes to smaller values by about 7% depending on the $\Lambda$ excitation energy. It is also notable that, once the $k_F$ parameter is fitted to reproduce the observed $\Lambda$ energy, both ESC08a and ESC08b interactions give almost similar bulk properties such as the root-mean-square radii. To compensate the hyperon oscillator size by which is smaller than $b_N$, the solved wave function includes higher H.O. components. For example, we write the $\Lambda$ $f$ orbit solution for ESC08a: $|f\rangle = \sqrt{0.9215}|0f\rangle^\text{HO} - \sqrt{0.0690}|1f\rangle^\text{HO} + \sqrt{0.077}|2f\rangle^\text{HO} - \sqrt{0.0019}|3f\rangle^\text{HO}$. This wave function gives the much

Fig. 4. $\Lambda$ spin-orbit splitting energies of $f_{7/2}$ and $f_{5/2}$ orbits calculated at the $^{90}\text{Zn}$ core. The results of NSC97 and ESC08 interaction models are compared. In each case, the $k_F$ value is chosen so that the averaged $f^4$ theoretical energy reproduces the experimental one.
larger root-mean-square radius than the corresponding nucleon’s one.

Next we are interested in the spin-orbit splittings deduced with the ESC08 G-matrix interactions. The results for the $f$ orbit are compared in Fig. 4 with those of NSC97 G-matrix interactions obtained with GAP treatments. The dotted lines show the $f_{5/2} - f_{7/2}$ splittings with the SLS interaction only. When the ALS component is switched on, then it reduces about 70% of the SLS interaction and the total splittings are calculated to be 0.26 (0.38) MeV for ESC08a(b), respectively. As the empirical splitting $\delta(f)$ is concluded to be around 0.2 MeV from the DWIA analyses, the ESC08a interaction model gives a better agreement. The calculated splittings for the other orbits, $\delta(d)$ and $\delta(p)$, are also listed in Fig. 4. In the cases of NSC97 models, the ALS component amounts to only 30–40% of the SLS interaction and the theoretical splittings remain to be much larger than the observed ones. Thus, the ESC08 models achieve remarkable improvement as far as the $\Lambda$ spin-orbit splittings are concerned.

4.2. $^{12}_1\Lambda$$C$ and $^{11}_1\Lambda$$B$ energy levels calculated with ESC08 G-matrices

In this subsection we apply the $\Lambda N$ G-matrices to typical $p$-shell hypernuclei and see whether the spin-spin component of the ESC08 interactions is consistent with the observed energy level order of the ground state doublet. Figure 5 shows the calculated energy levels of $^{12}_1\Lambda$$C$ for the NSC97a-f and ESC08a/b effective interactions. In the experiment, the ground state is known to be $J = 1^-$ member of the spin-doublet and the $J = 2^-$ partner lies at 0.161 MeV higher in energy. This fact tells that the spin-singlet $\Lambda N$ interaction is slightly more attractive than the spin-triplet interaction. As shown in Fig. 5, the experimental order is reproduced only by NSC97f, and the spin-triplet components of ESC08a/b interactions seem too strong. It is noted, however, that the even-state $\sigma \cdot \sigma$ interaction of ESC08 is almost consistent with the $J = 0^+ - 1^+$ energy level spacing known in $^4_1\Lambda$$H$.

Therefore we remark that the odd-state $\sigma \cdot \sigma$ interaction of ESC08 should be further improved, because both relative $s$ and $p$ interactions act well between $p$-shell nucleon and $s$-shell $\Lambda$. The role of the odd-state interaction in $p$-shell hypernuclei has been discussed also by Millener.

The calculated results for $^{11}_1\Lambda$$B$ is displayed in Fig. 6 together with the experimental energy levels. Here we focus on the energy level order of the ground-state doublet ($5/2^+$ and $7/2^+$), and we see again that only NSC97f can reproduce the observed level order which tells predominance of the spin-singlet attraction. As the $J = 3^+$ ground state of the nuclear core ($^{10}_1B$) is based on the stretched coupling ($L=2$)+(S=1), the resultant level spacing of the doublet due to the $\sigma \cdot \sigma$ interaction appears to be more enhanced than in the $^{12}_1\Lambda$$C$ case.

In order to find a possible way to improve the $\Lambda N$ parts of ESC08 interaction models, we compare in Table VIII the reduced matrix elements $\langle nl|V_c(r)|nl\rangle$ of the central interactions, with $nl$ denoting the $\Lambda N$ relative oscillator state. When compared with the most appropriate balance realized in NSC97f, the NCS97d version has negatively large $^3E$ contribution but the positive $^3O$ contribution is small, then spin-triplet state is favored to come to the lowest in energy (see $2^-$ in Fig. 5 and $7/5^+$ in Fig. 6). On the other hand, the reason for the level order inversion obtained...
with ESC08a/b is different from NCS97d. In both of ESC08 versions, we remark that the \( ^3O \) contribution is negative and this feature add to the negatively large \( ^3E \) interaction. As a result, the spin-triplet favored states such as \( 2^- \) of \(^{12}\Lambda C\) and \( 7/5^+ \) of \(^{11}\Lambda B\) are calculated to be the ground states, respectively. Thus we suggest that one of the possible ways of improving the ESC08 \( \Lambda N \) interactions is to modify the negative \( ^3O \) interaction to be positive.

§5. \( \Sigma \)-nucleus systems

5.1. \( U_\Sigma \) and \( \Sigma N \) \( G \)-matrix interactions in nuclear matter

Let us show the properties of \( \Sigma N \) \( G \)-matrix interactions. We solve here the \( \Sigma N \) starting channel \( G \)-matrix equation for ESC08a/b/a”, ESC04a and NSC97f with the GAP choice. In Table IX we show the potential energies \( U_\Sigma \) for a zero-momentum \( \Sigma \)

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Fig. 5. Low-lying energy levels of \(^{11}\Lambda B\) calculated with \( G \)-matrices derived from the NSC97 and ESC08 \( \Lambda N \) interaction models.

Fig. 6. Low-lying energy levels of \(^{11}\Lambda B\) calculated with \( G \)-matrices derived from the NSC97 and ESC08 \( \Lambda N \) interaction models.
Table VIII. Central interaction matrix elements $\langle nl|V_r|nl\rangle$ between $\Lambda N$ relative states of the lowest oscillator quanta, $nl = 0s$ and $0p$. In the parentheses the relative strengths of the spin-triplet matrix element are listed in % with respect to the spin-singlet strength.

|    | $\langle 0s|V^{(1)E}|0s\rangle$ | $\langle 0s|V^{(3)E}|0s\rangle$ | $\langle 0p|V^{(1)O}|0p\rangle$ | $\langle 0p|V^{(3)O}|0p\rangle$ |
|----|-------------------------------|-------------------------------|-----------------------------|-------------------------------|
| NSC97d | -4.02                        | -3.40 (84.6%)                | 1.40                        | 0.70 (50.0%)                 |
| NSC97f | -5.62                        | -2.90 (51.6%)                | 1.65                        | 1.09 (66.1%)                |
| ESC08a | -4.05                        | -3.09 (76.3%)                | 2.47                        | -0.67 (-27.1%)              |
| ESC08b | -4.04                        | -2.93 (72.5%)                | 2.06                        | -0.96 (-46.6%)              |

The conversion widths $\Gamma^c_\Sigma$ at normal density and partial wave contributions for ESC08a/b/a" (in MeV). $\Gamma^c_{\Sigma}$ denotes $\Sigma N$-$\Lambda N$ conversion width. $m^*_\Sigma$ is defined by $M^*_\Sigma/M_\Sigma$, $M^*_\Sigma$ being a $\Sigma$ effective mass.

| model   | $T$ | $^3S_0$ | $^3S_1$ | $^1P_1$ | $^3P_0$ | $^3P_1$ | $^3P_2$ | $D$ | $U_\Sigma$ | $m^*_\Sigma$ | $\Gamma^c_{\Sigma}$ |
|---------|-----|---------|---------|---------|---------|---------|---------|-----|------------|--------------|-------------------|
| ESC08a  | 1/2 | 11.3    | -23.6   | 1.7     | 1.9     | -5.0    | 0.0     | -0.7| +13.6      | 1.20          | 33.0              |
|         | 3/2 | -11.5   | 44.4    | -4.0    | -2.2    | 5.4     | -3.6    | -0.2| +19.8      | 1.33          | 34.5              |
| ESC08b  | 1/2 | 10.3    | -25.5   | 1.4     | 2.5     | -5.9    | 0.3     | -0.8| +19.8      | 1.33          | 34.5              |
|         | 3/2 | -10.4   | 52.4    | -3.0    | -2.7    | 5.9     | -4.4    | -0.1| +19.8      | 1.33          | 34.5              |
| ESC08a  | 1/2 | 11.3    | -17.6   | 2.0     | 1.8     | -5.3    | -0.5    | -0.7| +9.1       | 1.08          | 24.4              |
|         | 3/2 | -12.3   | 34.4    | -4.1    | -2.0    | 5.5     | -3.0    | -0.3| -36.5      | 1.08          | 16.8              |
| ESC04a  | 1/2 | 11.6    | -26.9   | 2.4     | 2.7     | -6.4    | -2.0    | -0.8| -36.5      | 1.08          | 16.8              |
|         | 3/2 | -11.3   | 2.6     | -6.8    | -2.3    | 5.9     | -5.1    | -0.2| -36.5      | 1.08          | 16.8              |
| NSC97f  | 1/2 | 14.9    | -8.3    | 2.1     | 2.5     | -4.6    | 0.5     | -0.5| -12.9      | 1.00          | 18.4              |
|         | 3/2 | -12.4   | -4.1    | -4.1    | -2.1    | 6.0     | -2.8    | -0.1| -12.9      | 1.00          | 18.4              |

and their partial-wave contributions $U_\Sigma^{(2S+1LJ,T)}$ at normal density. It should be noted here that the strongly repulsive values of $U_\Sigma$ can be obtained for ESC08a/b/a". In contrast, the $U_\Sigma$ values for NSC97f, ESC04a and the other NS/ESC models are attractive, as demonstrated in 7). In the cases of the repulsive values of $U_\Sigma$ for ESC08a/b/a", the most responsible is the contribution in $^3S_1$ $T = 3/2$ state, where the Pauli-forbidden state effect in this state is taken into account by strengthening the pomeron coupling in the ESC08 modeling. This approach has been motivated by the success of the quark-cluster model of YN-$YY$ interaction.\textsuperscript{21),22) Experimentally, the repulsive $\Sigma$-nucleus potentials are suggested in the observed $(\pi^-, K^-)$ spectra.\textsuperscript{23)-25) It is quite interesting that the repulsive values of $U_\Sigma$ can be realized under the specific modeling for the core part of the YN interaction.

The $\Sigma$ effective masses $M^*_\Sigma$ are also given in the Table IX, which are defined in the same way as $M^*_\Lambda$. The reason why the obtained values for ESC08a/b are substantially larger than 1 is because of the negative contributions of $\frac{dU_\Sigma}{dT_\Sigma}$, in which the $^3S_1$ $T = 3/2$ state contribution is especially large. In the case of NSC97f, for instance, the $^3S_1$ $T = 3/2$ contribution is weakly attractive ($U_\Sigma = -4.1$ MeV)\textsuperscript{7) and its $T_\Sigma$ derivative is slightly negative, because of which the obtained value of $M^*_\Sigma \sim 1.0$ is far smaller than those for ESC08a/b.

The conversion widths $\Gamma^c_{\Sigma}$ are estimated by the second-order perturbation, where the potentials are taken into account for nucleon and $\Lambda$ in intermediate states. The calculated values of $\Gamma^c_{\Sigma}$ are dominated by the strong $\Sigma N$-$\Lambda N$ coupling effects in $^3S_1$
$T = 1/2$ states. The $\Gamma^c_\Sigma$ values for ESC08a/b/a" are found to be rather large. For instance, that for NSC97f is given as 18.1 MeV. This means not necessarily that the $\Sigma N - \Lambda N$ coupling interactions in the formers are stronger than that in the latter, because the $\Gamma^c_\Sigma$ value depends on the $G$-matrix starting energy. If the positive value of $U_\Sigma$ similar to those for ESC08a/b is assumed for the starting energy in the $G$-matrix equation for NSC97f instead of the self-consistently obtained negative value, the obtained value of $\Gamma^c_\Sigma$ becomes comparable to those for ESC08a/b.

5.2. $U_\Sigma$ in $\Sigma$ scattering states

Because the $\Sigma N$ interactions are repulsive in average, $\Sigma$ hypernuclei in bound states are not considered to be exist except $^4\Lambda$He having no contribution of the $^3S_1$ $T = 3/2$ repulsion. Then, studies on $\Sigma$-nucleus potentials in scattering states are very important to supplement the limited information obtained from the $\Sigma N$ scattering data. Indeed, the experimental indications of the repulsive $\Sigma$-nucleus potentials were obtained by analyzing the final state interactions in $N(\pi,K)\Sigma$ reactions on nuclear targets. In future, it is expected to perform $\Sigma$-nucleus scattering experiments at J-PARC, as well as $\Sigma N$ ones. Then, measurements of cross sections and analyzing powers will give valuable information on $\Sigma$-nucleus potentials.

It has been quite successful to derive real and imaginary parts of nucleon optical potentials from free-space $NN$ interactions on the basis of the $G$-matrix theory, in which complex $G$-matrices are calculated as a function of an incident-nucleon energy $E$ and a Fermi momentum $k_F$ in nuclear matter. In the same framework, let us try to derive complex $\Sigma$-nucleus potentials, whose real and imaginary parts are denoted as $U_\Sigma(E_\Sigma; k_F)$ and $W_\Sigma(E_\Sigma; k_F)$, respectively, starting from the $\Sigma N$ interaction models. For comparison, we derive complex $\Lambda$-nucleus potentials $U_\Lambda(E_\Lambda; k_F) + iW_\Lambda(E_\Lambda; k_F)$ in the same framework. Then, the important thing is that there appear the two origins of the imaginary part of the $\Sigma$-nucleus potential: One is the $\Sigma N - \Lambda N$ conversion and the other is the $\Sigma N - \Sigma N$ scattering, both of which occur in medium and are treated reasonably in the $G$-matrix framework. The latter is of the same origin as the nucleon-nucleus imaginary potential derived from $NN$ scatterings in medium. In the case of the $\Lambda$-nucleus potential, there occurs no $\Lambda N - \Sigma N$ conversion unless the incident-$\Lambda$ energy is high enough.

Here, we compare the results obtained from ESC08a and ESC04a giving repulsive and attractive values of $U_\Sigma(\rho_0)$, respectively. In the left panel of Fig. 2 the real parts of $U_\Sigma(E_\Sigma; k_F)$ are drawn as a function of $\Sigma$ incident energy $E_\Sigma$ at $k_F = 1.35$ fm$^{-1}$, where the solid and dashed curves are for ESC08a and ESC04a, respectively. The $\Sigma$ potentials derived from these two interaction models are found to be extremely different from each other. The dotted curve shows the corresponding $\Lambda$-nucleus potential $U_\Lambda(E_\Lambda; k_F)$ ESC08a. The $\Lambda$-nucleus potential becomes shallow with increase of incident energy, which is similar to the well known energy dependence of the nucleon-$\Lambda$ potential. On the other hand, the energy derivative of the $\Sigma$ potential has a negative sign in the region of $E_\Sigma < \sim 100$ MeV in contrast to the nucleon or $\Lambda$ potentials. This feature of $\Sigma$ potentials is related to the fact that the $\Sigma$ effective mass is larger than the $\Sigma$ mass: As understood from $\frac{M^*\Sigma}{M_\Sigma} = \left(1 + \frac{dU_\Sigma}{dT_\Sigma}\right)^{-1}$,
Fig. 7. In the left panel, real parts $U_\Sigma(E_\Sigma; k_F)$ are drawn for ESC08a (solid curve) and ESC04a (dashed curve), and $U_A(E_A; k_F)$ for ESC08a is drawn by the dotted curve. In the right panel, imaginary parts $W_\Sigma(E_\Sigma; k_F)$ and $W_A(E_A; k_F)$ are drawn correspondingly. The dot-dashed curve shows the conversion part of $W_\Sigma(E_\Sigma; k_F)$, that is $W^c_\Sigma(E_\Sigma; k_F)$.

$$\frac{M^*_\Sigma}{M_\Sigma} > 1 \text{ means } \frac{dU_\Sigma}{dE_\Sigma} \sim \frac{dU_A}{dT^c} < 1.$$  

The imaginary parts $W_\Sigma(E_\Sigma; k_F)$ for ESC08a (solid curve) and ESC04a (dashed curve) and $W_A(E_A; k_F)$ for ESC08a (dotted curve) are given in the right panel of Fig. 2. The $\Sigma$ imaginary potentials turn out to be remarkably stronger than the $\Lambda$ one. We can understand the reason why imaginary parts in $\Sigma$-nucleus potentials are so strong as follows: $\Sigma N$ interactions have a strong spin- and isospin-dependences, and their $(S, T)$-state contributions to real parts cancel strongly with each other. However, $\Sigma$-nucleus imaginary parts are dominated by $\Sigma N-\Sigma N$ second-order processes contributing coherently. In the figure, the imaginary potential $W^c_\Sigma$ derived from the $\Sigma N-\Lambda N$ conversion is drawn by the dot-dashed curve in the case of ESC08a. It is found that the $W^c_\Sigma$ part is of weak $E_\Sigma$-dependence and its contribution to the total imaginary potential $W_\Sigma$ become relatively small with increase of $E_\Sigma$.

§6. $\Xi$-nucleus systems

6.1. $U_\Xi$ and $\Xi N$ G-matrix interactions in nuclear matter

We calculate here $\Xi$ potential energies $U_\Xi$ and derive $\Xi N$ G-matrix interactions in nuclear matter. Then, imaginary parts of $G$-matrices in $^1S_1$ and $^3P_J$ $T = 0$ states appear due to energy-conserving transitions from $\Xi N$ to $\Lambda \Lambda$ channels. These processes are treated in the second-order perturbation, where potentials are taken into account for two $\Lambda$'s in intermediate states. The conversion width $\Gamma^c_{\Xi}$ is obtained from the imaginary part of $U_\Xi$ by multiplying $-2$, and dominated by the contribution of $\Xi N-\Lambda \Lambda$ coupling interaction in $^1S_0$ $T = 0$ state.
Table X. $U_\Xi(\rho_0)$ and partial wave contributions for ESC08a/b and ESC04c/d (ESC08a") calculated with the GAP (CONr) treatments. $\Gamma_\Xi$ denotes $\Xi N - \Lambda \Lambda$ conversion width. All entries are in MeV.

| model    | $T$ | $^1S_0$ | $^3S_1$ | $^3P_0$ | $^3P_1$ | $^3P_2$ | $U_\Xi$ (MeV) | $\Gamma_\Xi$ (MeV) |
|----------|-----|---------|---------|---------|---------|---------|--------------|------------------|
| ESC08a   | 0   | 6.1     | -0.9    | -0.3    | -2.8    | 1.4     | -1.0         | -9.0             |
|          | 1   | 21.8    | -31.7   | 2.5     | 0.3     | -3.7    | -0.6         | 7.6              |
| ESC08b   | 0   | 2.1     | 2.0     | -0.6    | -1.3    | -0.1    | -0.7         |                  |
|          | 1   | 26.6    | -40.3   | 3.0     | -0.6    | -3.7    | -1.3         | -14.7            |
| ESC08a" (CONr) | 0   | 3.6     | -6.7    | 0.1     | -4.4    | 1.7     | -0.6         | -14.7            |
|          | 1   | 13.0    | -13.6   | 2.3     | 0.8     | -2.1    | 1.0          | 5.9              |
| ESC04c   | 0   | 5.9     | -15.7   | 1.2     | -0.1    | -1.8    | -1.2         | -5.5             |
|          | 1   | 6.8     | 1.9     | -0.8    | 0.1     | -0.3    | -1.7         | 13.4             |
| ESC04d   | 0   | 6.4     | -19.6   | 1.1     | 1.2     | -1.3    | -2.0         | -18.7            |
|          | 1   | 6.4     | -5.0    | -1.0    | -0.6    | -1.4    | -2.8         | 14.0             |

For ESC08a/b and ESC04c/d, $G$-matrix calculations are performed with the GAP choice, which give rise to the negative values of $U_\Xi(\rho_0)$. Then, it should be remarked that the values of $U_\Xi(\rho_0)$ with the GAP treatment are less attractive by about 10 MeV than those with the CONr treatment, being far larger than the difference of the corresponding values of $U_A(\rho_0)$. Though such a large difference between the GAP and CONr treatments might be considered as an ambiguity in our $\Xi N$ $G$-matrix, we use both treatments in a practical way depending on the occasion: In the case of ESC08a", the value of $U_\Xi(\rho_0)$ obtained with the GAP treatment is repulsive, and then we adopt the CONr treatment. For simplicity of this treatment, we take into account only intermediate potentials of nucleon and $\Lambda$, omitting those of $\Xi$ and $\Sigma$. Another approximation in the ESC08a" case is to omit the coupling to $\Sigma \Sigma$ states in the $^3P_0$ $T = 0$ $\Lambda \Lambda - \Xi N - \Sigma \Sigma$ coupled states, the reason of which is because this coupling induces problematic behaviors of $\Xi N$ $G$-matrices.

Let us show the results for ESC08a/b and ESC04c/d (ESC08a") with the GAP (CONr) choice. Table X gives the potential energies $U_\Xi$ and their partial-wave contributions. The $U_\Xi$ values are found to be substantially attractive in the cases of these interactions. However, the partial-wave contributions to $U_\Xi$ are distinctly different between ESC08a/b and ESC04c/d models: In the former (latter) case, the attractive contributions to $U_\Xi$ are dominated by those in the $^3S_1$ $T = 1$ ($T = 0$) state. In the case of ESC08a", the property is rather intermediate: the strengths of $^3S_1$ attractions in $T = 0$ and $T = 1$ states are found to be comparable, considering the statistical factor $(2T + 1)$ on partial-wave contributions. As discussed in Ref. 7), the strong attraction in the $^3S_1$ $T = 0$ state in the case of ESC04d is because the contributions of vector and axial-vector meson exchanges are strongly cancelled in this channel. The strong attractions in $^3S_1$ $T = 1$ states in the case of ESC08 models are caused by the strong $\Xi N - \Lambda \Sigma - \Sigma \Sigma$ coupling interactions, where these strengths come from the pair terms dominantly. The $\Xi N - \Lambda \Sigma - \Sigma \Sigma$ triplet in the $T = 1$ state belongs to the baryon-baryon decuplet-state $\{10\}$ together with the $np$ $T = 0$ and $\Lambda N - \Sigma N$ $T = 1/2$ pair. It is interesting that the $\Xi N - \Lambda \Sigma - \Sigma \Sigma$ coupling tensor interactions in ESC08 models work similarly with the $np$ and $\Lambda N - \Sigma N$ tensor interactions. If the $\Xi N - \Lambda \Sigma - \Sigma \Sigma$ tensor coupling interactions are switched off in $\Xi N$ $G$-matrix calcu-
lations with ESC08 models, contributions in the \( ^3S_1 \) \( T = 1 \) state become repulsive. There appear similar situations, when \( np \) (\( AN-\Sigma N \)) tensor interactions are switched off in \( NN \) (\( AN \)) G-matrix calculations.

The calculated values of conversion widths \( \Gamma^c_\Xi(\rho_0) \) are also given in Table X, the contributions of which come dominantly from the \( \Lambda\Lambda-\Sigma \Sigma \) coupling interactions in \( ^1S_0 \) \( T = 0 \) states. Here, it is found that the values of \( \Gamma^c_\Xi \) for ESC08 models are substantially smaller than that for ESC04 models. Among ESC08 models, the \( \Gamma^c_\Xi \) value for ESC08b is found to be rather smaller than those for ESC08a/a”. Here, \( \Xi N-\Lambda \Lambda \) coupling interactions in ESC08a/b/a” are noted to be comparable. The difference comes from the fact that the innermost part of the \( ^1S_0 = 0 \) \( \Xi N \) potential in ESC08b is repulsive and those for ESC08a/a” are attractive, and then the latter induces the stronger \( \Xi N-\Lambda \) mixing than the former.

The effective mass defined by
\[
m^*_\Xi = \frac{M^*_\Xi}{M^*_\Xi} = \left(1 + \frac{dU^\Xi}{dT^\Xi}\right)^{-1}
\]
can be obtained in the same way as \( m^*_\Lambda \) and \( m^*_\Sigma \). The calculated values of \( m^*_\Xi \) are 0.95, 0.95 and 1.08 for ESC08a, ESC08b and ESC08a”, respectively. In the cases of ESC08a/b, the values of \( m^*_\Xi \) are found to be between those of \( m^*_\Lambda \) and \( m^*_\Sigma \). On the other hand, the values of \( m^*_\Xi \) are similar to each other in the case of ESC08a”.

For applications to finite \( \Xi \) systems, \( \Xi N-\Xi N \) central parts of complex G-matrix interactions for ESC08a/b/a” are represented in Gaussian forms whose coefficients are given as a function of \( k_F \), as shown by Eq. (2.8). In the cases of ESC08a/b, the local potentials in coordinate space are parameterized in a three-range Gaussian form in \( ^3S_1 \) \( T = 0 \), \( ^1S_0 \) \( T = 1 \) and \( ^3S_1 \) \( T = 1 \) states. Those in the other states are parameterized in a two-range Gaussian form, because it is difficult to obtain smooth \( k_F \) dependences of three-ranged potential forms. In the case of ESC08a”, the potentials in all states are parameterized in a two-range Gaussian form. The parameters are shown in Table XI, where the G-matrices are calculated with GAP(CONr) in the cases of ESC08a/b (ESC08a”).

6.2. Possible existence of \( \Xi \) hypernuclei

Although there is almost no information on \( \Xi N \) interactions, the BNL-E885 experiment\(^ {27} \) suggests that a \( \Xi^- \) s.p. potential in \( ^{11}_\Xi \) Be is given by the attractive Wood-Saxon potential with the depth \( \sim -14 \text{ MeV} \) (called WS14). The \( \Xi \) s-state binding energy in this case is \(-2.2 \text{ MeV} \) without the Coulomb interaction. Similar \( \Xi \) well depths are suggested in some emulsion events producing a twin of \( \Lambda \) hypernuclei.\(^ {28} \) It is found that the values of \( U_\Xi(\rho_0) \) for ESC08a/b/a” in Table X are comparable to or less than the depth of WS14, though the latter should not be compared strictly with the former quantities. It is interesting that these \( \Xi N \) interactions give rise to \( \Xi \) bound states. Here, we study the possible existence of \( \Xi \) bound states (\( \Xi \) hypernuclei) using the G-matrix folding models derived from ESC08a/b/a”.

The observed spectra of \( \Lambda \) hypernuclei are described successfully with the \( \Lambda \)-nucleus folding potentials derived from the \( AN \) G-matrix interactions, as seen in Fig. 1. Here, the same method is applied to \( \Xi \) hypernuclei, where the imaginary parts of \( \Xi N \) interactions and \( \Xi^-p \) Coulomb interactions are taken into account. As well as the treatments for \( \Lambda \) hypernuclei, nuclear cores are assumed to be spherically
Table XI. Parameters of $\Xi N$ $G$-matrix interactions represented by three-range Gaussian forms $G(r; k_F) = \sum_i (a_i + b_i k_F + c_i k_F^2) \exp(-r/\beta_i^2)$, obtained with the GAP (CONr) treatments in the cases of ESC08a/b (ESC08a”). $\Xi N$ two body states are denoted by $(2S+1)(2T+1)^E$ and $(2S+1)(2T+1)^O$ for parity-even and -odd states, respectively.

| $\beta_i$ | ESC08a (GAP) | ESC08b (GAP) | ESC08a” (CONr) |
|-----------|--------------|--------------|----------------|
|           | 0.50 | 0.90 | 2.00 | 0.50 | 0.90 | 2.00 | 0.50 | 0.90 | 2.00 |
| real      |      |      |      |      |      |      |      |      |      |
| $^{11}E$  |      |      |      |      |      |      |      |      |      |
| a         | 1.20 | 1.50 | 2.00 | 0.00 | 0.00 | 0.00 | 143.0 | 61.62 | 0.00 |
| b         | 0.00 | 0.00 | 0.00 | 417.8 | 263.2 | 0.00 | 104.0 | 0.00 | 0.00 |
| c         | 0.00 | 0.00 | 0.00 | 192.1 | 74.75 | 0.00 | 45.06 | 0.00 | 0.00 |
| $^{31}E$  |      |      |      |      |      |      |      |      |      |
| a         | 192.1 | 18.32 | 0.00 | 358.3 | 74.75 | 0.00 | 45.06 | 0.00 | 0.00 |
| b         | 0.00 | 0.00 | 0.00 | 413.6 | 263.2 | 0.00 | 104.0 | 0.00 | 0.00 |
| c         | 0.00 | 0.00 | 0.00 | 5.99 | 34.06 | 0.00 | 45.06 | 0.00 | 0.00 |
| $^{11}O$  |      |      |      |      |      |      |      |      |      |
| a         | 0.00 | 0.00 | 0.00 | 76.33 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| b         | 0.00 | 0.00 | 0.00 | 12.89 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| c         | 0.00 | 0.00 | 0.00 | 18.32 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| $^{31}O$  |      |      |      |      |      |      |      |      |      |
| a         | 0.00 | 0.00 | 0.00 | 137.1 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| b         | 0.00 | 0.00 | 0.00 | 191.6 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| c         | 0.00 | 0.00 | 0.00 | 18.32 | 5.99 | 0.00 | 45.06 | 0.00 | 0.00 |
| imag      |      |      |      |      |      |      |      |      |      |
| $^{11}E$  |      |      |      |      |      |      |      |      |      |
| a         | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 110.5 | 0.00 | 0.00 |
| b         | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 110.5 | 0.00 | 0.00 |
| c         | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 110.5 | 0.00 | 0.00 |

symmetric and density $\rho(r)$ and mixed density $\rho(r, r')$ are constructed from SkHF wave functions. The ADA is used for $k_F$ dependences of $G$-matrix interactions. The isospin-dependence of $G_{ST}(±)(r; k_F)$ leads to the Lane term. In this work, only the diagonal parts of the $t_{\Xi} \cdot T_c$ term are taken into account.

In Table XII, we show calculated values of $\Xi^-$ s.p. energies $E_{\Xi^-}$, conversion widths $\Gamma_{\Xi}$ and r.m.s radii $\sqrt{\langle r^2 \Xi \rangle}$ of solved $\Xi^-$ wave functions in the case of $^{12}$Be, where $\Delta E_L$ and $\Delta E_C$ are contributions from Lane terms and Coulomb interactions, respectively. The result for WS14 is also given in Table XII. Then, the $G$-matrix folding potentials obtained from ESC08a/b (ESC08a”) are found to be more (less)
Table XII. Calculated values of $\Xi^-$ single particle energies $E_{\Xi^-}$ and conversion widths $\Gamma^c_{\Xi^-}$ for $^{12}_\Xi$Be for ESC08a/b/a”. $\Delta E_L$ and $\Delta E_C$ are contributions from Lane terms and Coulomb forces, respectively. The values of $E_{\Xi^-}$, $\Gamma^c_{\Xi^-}$, $\Delta E_L$ and $\Delta E_C$ are given in MeV. Values of $\sqrt{\langle r^2_{\Xi^-} \rangle}$ are mean square radii of $\Xi^-$ wave functions in fm. The marks * means that there is no nuclear bound state. The correction $\Delta k_F = 0.08$ fm$^{-1}$ in ADA for ESC08a is taken so as to reproduce the s-state value of $E_{\Xi^-}$ similarly to that for WS14.

| model    | $E_{\Xi^-}$ | $\Delta E_L$ | $\Delta E_C$ | $\Gamma^c_{\Xi^-}$ | $\sqrt{\langle r^2_{\Xi^-} \rangle}$ |
|----------|-------------|--------------|--------------|--------------------|----------------------------------|
| ESC08a   | s           | -6.09        | -0.23        | -2.84              | 3.51                             | 2.61                             |
|          | p           | -2.10        | -0.08        | *                  | 2.22                             | 3.70                             |
| ESC08b   | s           | -8.16        | -0.35        | -3.00              | 1.93                             | 2.35                             |
|          | p           | -3.40        | -0.18        | -2.51              | 1.40                             | 3.14                             |
| ESC08a’  | s           | -3.62        | +0.10        | -2.44              | 2.64                             | 3.32                             |
|          | p           | -0.77        | +0.06        | *                  | 2.30                             | 6.17                             |
| ESC08a   | s           | -4.89        | -0.19        | -2.70              | 3.20                             | 2.85                             |
| $\Delta k_F = 0.08$ | p           | -1.49        | -0.05        | *                  | 1.83                             | 4.24                             |
| WS14     | s           | -4.89        | -2.71        |                    |                                  |                                  |
|          | p           | -0.23        | *            |                    |                                  |                                  |

attractive than WS14. In the same way as the $A$ folding potentials, let us modify the $G$-matrix interaction from ESC08a by taking the averaged value $\langle k_F \rangle + \Delta k_F$ in ADA. The result for $\Delta k_F = 0.08$ fm$^{-1}$ in the case of ESC08a is given also in Table XII, which turns out to be very similar to the result for WS14.

In order to demonstrate the features of spin- and isospin-averaged $\Xi N$ interactions, we perform calculations for simple systems composed of spin- and isospin-saturated nuclear cores attached by a $\Xi^-$ particle; $^{12}_\Xi C + \Xi^-$, $^{16}_\Xi O + \Xi^-$, $^{28}_\Xi Si + \Xi^-$, $^{40}_\Xi Ca + \Xi^-$, $^{90}_\Xi Zr + \Xi^-$, where Coulomb interactions between $\Xi^-$ and nuclear cores are taken into account. In Fig. 3, full circles connected by solid lines show the s.p. energies of $\Xi^-$-bound states calculated with $G$-matrix folding potentials derived from ESC08a with $\Delta k_F = 0.08$ fm$^{-1}$. The horizontal axis is taken as a function of $A^{-2/3}$, $A$ being a mass number of a core nucleus. The “error bars” for full circles present the calculated values of conversion widths $\Gamma^c_{\Xi^-}$. Open circles connected by dotted lines are $\Xi^0$ s-state energies, for which “error bars” are omitted for easiness to see. In these results, there is no contribution from the Lane term except the case of $^{90}_\Xi Zr$ core. It should be noted that the Coulomb contributions to $\Xi^-$ binding energies are substantial in large mass-number region. When a $\Xi$ particle can be bound without an assist from a $\Xi$-nucleus Coulomb interaction, we define it as a $\Xi$-nuclear bound state. In Fig. 3, $p$-states in $^{12}_\Xi C$ and $^{16}_\Xi O$, $d$-states in $^{28}_\Xi Si$ and $^{40}_\Xi Ca$, $f$- and $g$-states in $^{90}_\Xi Zr$ are so-called Coulomb-assisted bound states (cab). Namely, these $\Xi$ states become unbound, when Coulomb interactions are switched off. Their wave functions deviate substantially from pure Coulomb ones, and not so different from $\Xi$-nuclear bound states qualitatively.

In Fig. 4, let us compare the results for ESC08a/b/a”. The calculations for $^{12}_\Xi C + \Xi^-$, $^{16}_\Xi O + \Xi^-$, $^{28}_\Xi Si + \Xi^-$, $^{40}_\Xi Ca + \Xi^-$ and $^{90}_\Xi Zr + \Xi^-$ are performed including Coulomb interactions. The corrections by $\Delta k_F$ are not made in these calculations so that the difference among ESC08a/b/a” is purely demonstrated. In the figure, $\Xi^-$
Fig. 8. $\Xi^-$ single particle energies for $^{12}$C+$\Xi^-$, $^{16}$O+$\Xi^-$, $^{28}$Si+$\Xi^-$, $^{40}$Ca+$\Xi^-$ and $^{90}$Zr+$\Xi^-$ are shown by full circles for ESC08a with $\Delta k_F = 0.08$ fm$^{-1}$. The “error bars” show calculated values of conversion widths. Open circles are show $\Xi^0$ s-state energies, where “error bars” are omitted.

s.p. energies obtained by ESC08a/b/a” are shown by full circles, asterisks and open circles. Calculated energies of $s$-, $p$- and $d$-states are connected by solid, dashed and dotted lines.

Next, let us study $\Xi^-$ hypernuclei more realistically produced by $p(K^-, K^+)$ reactions on available nuclear targets. In Table XIII, we list the results for some targets ($^{10}$B, $^{12}$C, $^{16}$O, $^{28}$Si, $^{89}$Y) in the cases of ESC08a/b/a”, where the calculated values of $\Xi^-$ s.p. energies $E_{\Xi^-}$, contributions $\Delta E_C$ from Coulomb interactions, conversion widths $\Gamma^c_{\Xi^-}$ and mean square radii of $\Xi^-$ wave functions are shown. In the case of ESC08a, the value of $\Delta k_F = 0.08$ fm$^{-1}$ is chosen so that the obtained value of $\Xi^-$ s-state energy in $^{12}$Be is comparable to the corresponding value for WS14. The values in parentheses are $\Xi$ s.p. energies obtained with switching off Coulomb interactions.

Now, $\Xi$-nucleus interactions derived from ESC08a/b/a” turn out to be attractive enough to produce $\Xi$ hypernuclear states extensively, which are realized by the strong
Fig. 9. $\Xi^-$ single particle energies for $^{12}\text{C}+\Xi^-, \, ^{16}\text{O}+\Xi^-, \, ^{28}\text{Si}+\Xi^-, \, ^{40}\text{Ca}+\Xi^-$ and $^{90}\text{Zr}+\Xi^-$ are shown for ESC08a (full circle), ESC08b (asterisk) and ESC08a" (open circle), where Coulomb interactions are included. Calculated energies of $s$-, $p$- and $d$-states are connected by solid, dashed and dotted lines.

$\Xi N$ attractions in the $^3S_1 \, T=1$ states. It is interesting here that there still exist a $\Xi$-nuclear bound state in a light system such as $^{10}\Xi^-\text{Li}$ even for ESC08a". Then, one should notice that the value of $U_{\Xi}(\rho_0)$ in nuclear matter is only $-4.9$ MeV, as shown in Table X: The reason is because the strong $k_F$ dependence makes $G$-matrix interactions attractive in low-$k_F$ regions, and this effect is remarkable in the cases of ESC08 models.

The conversion widths obtained from ESC08a/b/a" are not large so that level structures seem to be observed. Especially, ESC08b turns out to give deeper binding energies and narrower conversion widths than ESC08a/a". High-lying states are found to be Coulomb-assisted bound states. The conversion widths of these states are considerably smaller than those of low-lying states, because overlaps between extended $\Xi^-$ wave functions and nucleon ones become small. On the other hand, mean square radii of wave functions of these high-lying $\Xi^-$ states are not so large in comparison with those of low-lying states. The interesting point is that the values of
Table XIII. Calculated values of $\Xi^-$ single particle energies $E_{\Xi^-}$, conversion widths $\Gamma_{\Xi^-}^c$ and Coulomb-force contributions $\Delta E_C$ are given in MeV. Values of $\sqrt{\langle r^2_{\Xi^-} \rangle}$ are mean square radii of $\Xi^-$ wave functions in fm. The results are given for ESC08a/b/a", where $\Delta k_F = 0.08$ fm$^{-1}$ is taken in the case of ESC08a. The values in parentheses are energies obtained by switching off Coulomb interactions. Coulomb-assisted bound states are marked by $\text{cab}$.

|                  | $\text{ESC08a} (\Delta k_F = 0.08)$ | $\text{ESC08b}$ | $\text{ESC08a}^\prime$ |
|------------------|-------------------------------------|-----------------|------------------------|
|                  | $E_{\Xi^-}$ $\Gamma_{\Xi^-}^c$ $\sqrt{\langle r^2_{\Xi^-} \rangle}$ | $E_{\Xi^-}$ $\Gamma_{\Xi^-}^c$ $\sqrt{\langle r^2_{\Xi^-} \rangle}$ | $E_{\Xi^-}$ $\Gamma_{\Xi^-}^c$ $\sqrt{\langle r^2_{\Xi^-} \rangle}$ |
| $^{10}_{\Xi^-}\text{Li}$ | $s$ $-5.69$ $3.67$ $2.59$ | $s$ $-7.84$ $2.08$ $2.31$ | $s$ $-2.71$ $2.75$ $3.59$ |
|                  | $p$ $-1.48$ $2.35$ $3.91$ | $p$ $-2.88$ $1.52$ $3.14$ | $p$ $-0.82$ |
| $^{10}_{\Xi^-}\text{C}$ | $s$ $-7.41$ $3.91$ $2.62$ | $s$ $-9.75$ $2.04$ $2.38$ | $s$ $-4.95$ $3.01$ $3.15$ |
|                  | $p$ $-3.52$ $2.63$ $3.47$ | $p$ $-5.09$ $1.60$ $3.08$ | $p$ $-1.70$ |
| $^{28}_{\Xi^-}\text{Mg}$ | $s$ $-10.5$ $3.56$ $2.68$ | $s$ $-13.5$ $1.69$ $2.46$ | $s$ $-8.06$ $2.60$ $3.03$ |
|                  | $p$ $-6.79$ $2.28$ $3.39$ | $p$ $-8.90$ $1.27$ $3.11$ | $p$ $-4.72$ $1.99$ $3.99$ |
|                  | $d$ $-2.86$ $1.55$ $4.20$ | $d$ $-4.21$ $0.95$ $3.73$ | $d$ $-1.66$ $1.97$ $5.59$ |
| $^{89}_{\Xi^-}\text{Sr}$ | $s$ $-21.2$ $3.88$ $2.87$ | $s$ $-25.7$ $1.61$ $2.74$ | $s$ $-17.4$ $2.87$ $3.08$ |
|                  | $p$ $-17.0$ $2.43$ $3.68$ | $p$ $-20.9$ $1.11$ $3.49$ | $p$ $-13.6$ $1.86$ $4.03$ |
|                  | $d$ $-13.2$ $1.62$ $4.30$ | $d$ $-16.3$ $0.82$ $4.05$ | $d$ $-10.1$ $1.41$ $4.76$ |
|                  | $f$ $-9.33$ $1.17$ $4.81$ | $f$ $-11.8$ $0.64$ $4.51$ | $f$ $-6.88$ $1.25$ $5.44$ |
|                  | $g$ $-5.43$ $0.91$ $5.33$ | $g$ $-7.26$ $0.52$ $4.94$ | $g$ |

Mean square radii of these high-lying $\Xi^-$ states are rather similar to those of proton states which are converted by $p(K^-, K^+)\Xi^-$ reactions: These $\Xi^-$ states are expected to be observed clearly by $(K^-, K^+)$ reactions. Then, it might be determined which one has the most reasonable attraction among ESC08a/b/a".

§7. Double-Λ systems

7.1. Double-Λ hypernuclei observed in emulsion

Observed double-Λ hypernuclei provide reliable information on ΛΛ interactions. Historically, in the 1960s, there appeared two reports on the observation of double-Λ hypernuclei, $^{16}_{\Lambda}\text{Be}$ and $^{16}_{\Lambda}\text{He}$, but the validity of the latter case was considered doubtful. Two decades later the emulsion-counter hybrid technique was applied in the KEK-E176 experiment, and a new double-Λ hypernucleus $^{13}_{\Lambda}\text{B}$ was found. These emulsion data indicated $\Lambda\Lambda$ bond energies $\Delta B_{\Lambda\Lambda} = B_{\Lambda\Lambda} - 2B_{\Lambda} \approx 4$ MeV, $B_{\Lambda\Lambda}$ and $B_{\Lambda}$ being $\Lambda\Lambda$- and $\Lambda$-binding energies in double-Λ and single-Λ hypernuclei,
and obtained the modified values of $B_{34}$ for the most probable interpretation for the Demachi-Yanagi event, where the production of $^{6}\Lambda\Lambda$ hypernuclei, the most epoch-making is the Nagara event. Recently, the mass of $\Xi^{-}$ has been modified by 0.4 MeV in the PDB. With the use of this new value, they have re-analyzed the double-$\Lambda$ events and obtained the modified values of $B_{AA}$. Hereafter, we use these values. Among the observed events of double-$\Lambda$ hypernuclei, the most epoch-making is the Nagara event, where the production of $^{6}\Lambda\Lambda$He was uniquely identified with no ambiguity and a precise value of $\Delta B_{AA}$ was obtained with a small error bar. The extracted value of the $\Lambda\Lambda$ bond energy $\Delta B_{AA}(^{6}\Lambda\Lambda)$He = 0.67 ± 0.17 MeV is far smaller than the above old value of $\simeq$ 4 MeV, which suggests that $\Lambda\Lambda$ interactions are substantially weaker than those believed before. In the present, the Nagara event provides us the most reliable information on $\Lambda\Lambda$ interaction models. In order to understand the importance of the Nagara event, one should notice the following situation: Generally, it is difficult to identify neutral objects such as neutrons and $\gamma$ ray in emulsion, which leads to serious uncertainties inherent in interpretations of emulsion events. The most important is that the Nagara event was free from this kind of uncertainty. Thus, any interpretation contradictory to the Nagara event should be rejected.

Hiyama et al. have given a nice interpretation for the data of $^{10}_{AA}\Lambda\Lambda$Be: The most probable interpretation for the Demachi-Yanagi event is a bound state of $^{10}_{AA}\Lambda\Lambda$Be with $B_{AA} = 11.90 \pm 0.13$ MeV. Experimentally, however, it could not be determined whether or not this double-$\Lambda$ nucleus is in a ground state. If this observed system is assumed to be in the ground state, the resulting $\Lambda\Lambda$ bond energy becomes repulsive and contradicts the Nagara event. Thus, the Demachi-Yanagi event can be interpreted as the $2^{+}$ excited state of $^{10}_{AA}\Lambda\Lambda$Be. Then, the calculated value of $B_{AA}$ in the ground state is 14.74 MeV. This value is very different from the earlier observed data 17.7 ± 0.4 MeV in Ref. 29) obtained by interpreting the emulsion event as the pionic decay of $^{10}_{AA}\Lambda\Lambda$Be(0$^{+}$) $\rightarrow$ $^{9}_{A}\Lambda\Lambda$Be(1/2$^{+}$) + $p$ + $\pi^{-}$. In Ref. 29), however, they suggested also another interpretation as the decay $^{10}_{AA}\Lambda\Lambda$Be(0$^{+}$) $\rightarrow$ $^{9}_{A}\Lambda\Lambda$Be(3/2$^{+}$, 5/2$^{+}$) + $p$ + $\pi^{-}$ with $B_{AA}^{\exp}(^{10}_{AA}\Lambda\Lambda$Be(0$^{+}$)) = 14.7 ± 0.6 MeV. Then, this interpretation turns out to be justified by the above calculated value.

Also the $^{13}_{A}\Lambda\Lambda$ event observed in the E176 should be re-interpreted so as to be consistent with the Nagara event: The revised value of $B_{AA}$ is 23.3 ± 0.7 MeV.

7.2. Core+$2\Lambda$ three-body model

Our model wave function is composed of two $\Lambda$ particles and a frozen-core nucleus. Coordinates of these particles $r_1$ and $r_2$ are taken from the center-of-mass of the core, and a s.p. angular-momentum state is denoted by $(lsj)$. Here, the core+$\Lambda$ three-body state is represented as a superposition of Gaussian functions as follows:

$$\Psi_{JM}(r_1, r_2) = \sum_{l_1,j_1,l_2,j_2} C_{l_1,j_1,l_2,j_2} \psi_{l_1,l_2;j_1,j_2;JM}(r_1, r_2) \cdot \Phi_{\text{core}}, \quad (7.1)$$

$$\psi_{l_1,l_2;j_1,j_2;JM}(r_1, r_2) = \sum_{\nu_1,\nu_2} c_{\nu_1,\nu_2} \psi_{l_1,l_2;j_1,j_2;JM}(r_1, r_2), \quad (7.2)$$
\[ \psi_{l_1j_1,l_2j_2;JM}(r_1, r_2) = \left[ \phi_{l_1}^{(\nu_1)}(r_1) \times \chi_{1/2,j_1} \times \phi_{l_2}^{(\nu_2)}(r_2) \times \chi_{1/2,j_2} \right]_{JM}, \quad (7.3) \]

with a Gaussian function \( \phi_{l}^{(\nu)}(r) = N_l(\nu) r^l \exp(-\nu r^2) \) normalized by \( N_l(\nu) \), and a spin function \( \chi_{1/2} \). Coefficients \( c_{l_1l_2} \) are determined by diagonalization procedures.

The Hamiltonian is given by

\[
H = H_{\text{core}} + H_{AA},
\]

\[
H_{AA} = T_{A_1} + T_{A_2} - T_{AA,cm} + U_{A_1} + U_{A_2} + V_{AA}. \quad (7.5)
\]

Here, \( T_i \) \((i = A_1, A_2)\) is the kinetic energy operator of each particle with respect to the core nucleus, and \( T_{AA,cm} \) is a center-of-mass kinetic energy operator of a \( \Lambda \Lambda \) pair. \( U_i \) \((i = A_1, A_2)\) are potential energies between respective particles and a core nucleus, which are chosen phenomenologically so as to reproduce threshold energies reasonably. \( V_{AA} \) is a \( \Lambda \Lambda \) interaction.

It is well known that three-body correlation are fully taken into account by taking the Jacobi coordinates on the basis of free-space interactions. When \( G \)-matrix interactions are used in such a model space, there occurs a sort of double counting for high-lying pair correlations renormalized into \( G \)-matrices. On the other hand, coordinates in our three-body model are essentially the same as a shell model, called as a cluster-orbital shell model. In this model space, high-momentum components of relative motions between two \( \Lambda \)'s are not included and \( \Lambda \Lambda \) \( G \)-matrix interactions can be used safely without double counting.

### 7.3. \( \Lambda \Lambda \) \( G \)-matrix interactions

We solve the coupled-channel \( G \)-matrix equations in \( ^1S_0 \) and \( ^3P_J \) \( T = 0 \) states for \( \Lambda \Lambda \), \( \Xi N \) and \( \Sigma \Sigma \) pairs in symmetric nuclear matter with the GAP choice, where the starting channel is the \( \Lambda \Lambda \) state. The effects of \( \Lambda \)-\( \Xi N \) and \( \Lambda \)-\( \Sigma \Sigma \) couplings are renormalized into \( \Lambda \Lambda \)-\( \Lambda \Lambda \) diagonal \( G \)-matrices.

The coordinate-space \( G \)-matrix interactions are represented by Gaussian functions in the following procedure: First we calculate the momentum-space matrix elements \( \langle k | G_{\Lambda \Lambda,\Lambda \Lambda} | k \rangle \). Next, we assume effective local potentials for the diagonal \( \Lambda \Lambda \)-\( \Lambda \Lambda \) interaction \( G_{\Lambda \Lambda,\Lambda \Lambda} \), simulating the respective calculated \( G \)-matrix elements in three-range Gaussian forms. Gaussian coefficients are determined for each value of \( k_F \) so that \( \langle k | G_{\Lambda \Lambda,\Lambda \Lambda} | k \rangle \) simulates the corresponding \( G \)-matrix elements \( \langle k | G_{\Lambda \Lambda,\Lambda \Lambda} | k \rangle \).

The \( k_F \)-dependences in the \( G \)-matrix interactions are parameterized as a second-order polynomials of \( k_F \), as given by Eq. (2-8). The parameters for ESC08a/b are given in Table XIV, where the interaction in the \( ^3O_J \) state is given by averaging those in \( ^3O_J \) states with a statistical weight \( (2J + 1) \).

When this \( \Lambda \Lambda \) \( G \)-matrix interaction \( G(r; k_F) \) is used as the interaction \( V_{\Lambda \Lambda} \) in Eq. (7.5), the \( k_F \)-dependent part is treated by ADA: The averaged value of \( \langle k_F \rangle \) is defined as by the expectation value of averaged density by the \( \text{core} + 2\Lambda \) wave functions \( \langle \Psi_{JM}(r_1, r_2) | \frac{1}{2} (\rho(r_1) + \rho(r_2)) | \Psi_{JM}(r_1, r_2) \rangle \). Then, \( \langle k_F \rangle \) values are determined self-consistently for each double-\( \Lambda \) state.
Table XIV. Parameters of Gaussian-represented $ΛΛ$ G-matrix interactions as a function of $k_F$: $G(r;k_F) = \sum (a_i + b_i k_F + c_i k_F^2) \exp(-(r/\beta_i)^2)$ in the cases of ESC08a/b.

| $\beta_i$ | ESC08a       | ESC08b       |
|-----------|--------------|--------------|
|           | 0.5          | 0.9          | 1.5          | 0.5          | 0.9          | 1.5          |
| $1^E$     | 2464.        | -594.1       | 19.83        | 1890.        | -446.0       | 7.525        |
| $2O$      | 1534.        | -373.3       | 25.83        | 910.1        | -227.8       | 13.18        |

7.4. $Λ$-core effective potentials

Our three-body calculations are performed for some double-$Λ$ hypernuclei observed in emulsion: $^6A\Lambda$He, $^{10}A\Lambda$Be, $^{11}A\Lambda$Be, $^{12}A\Lambda$Be and $^{13}A\Lambda$B. In our three-body model, then, it is necessary to define interactions $V_{ΛX}$ between a $Λ$ particle and a nuclear core $X$ together with $ΛΛ$ G-matrix interactions: Interactions $V_{ΛX}$ are determined phenomenologically so as to reproduce the binding energy between $Λ$ and $X$. For $V_{Λα}$ in $^6A\Lambda$He, we use the two-range Gaussian potentials given in Ref. 40, whose parameters are determined so as to reproduce $B_{Λ}(^5\Lambda$He). For interactions between $Λ$ and cores with ($A=8,9,10,11$), we use Woods-Saxon potentials with radii $R = 1.17A^{1/3}$ fm and diffuseness parameters $a = 0.65$ fm. The depth parameters are taken as $-26.27$, $-28.87$, $-29.71$ and $-30.26$ MeV, respectively, for $A=8,9,10,11$. The depth for $A=8$ is taken so as to reproduce the experimental value $6.71$ MeV for $B_{Λ}(^9\Lambda$Be). The $B_{Λ}(^{10}\Lambda$Be) is considered to be given by a weighted sum of the values for ground $1^-$ and excited $2^-$ states of $^{10}\Lambda$Be so that there is no contribution of the $ΛN$ spin-spin interaction in the double-$Λ$ state. The similar procedure is done for $B_{Λ}(^{12}\Lambda$Be). Then, the depth for $A=9$ ($A=11$) is taken so as to reproduce the $B_{Λ}$ value $8.97$ ($11.28$) MeV. In the case of $A=10$, we use the experimental value $10.24$ MeV of $B_{Λ}(^{11}\Lambda$Be) instead of $B_{Λ}(^{11}\Lambda$Be), because $^{11}\Lambda$Be is not observed experimentally.

In Table XV, we show the calculated values of the three-body binding energy $B_{ΛΛ}$ in the cases of using the $ΛΛ$ G-matrix interactions derived from ESC08a/b, which are compared with the observed values in emulsion, where the values in parentheses are the values of $\langle k_F \rangle$ obtained self-consistently in ADA. The $^{10}A\Lambda$Be data is from the old emulsion data. The Demachi-Yanagi event is considered to be the $2^+$ excited state of $^{10}A\Lambda$Be, and the calculated value for its ground state is quite similar to the value in Table XV. The Hida event is not uniquely identified: There is two possibilities of $^{11}A\Lambda$Be and $^{12}A\Lambda$Be. Both cases are calculated. The experimental value of $B_{ΛΛ}(^{13}A\Lambda$B) is the revised one for the E176 event.

The calculated results are found to be nicely consistent with the experimental data. Thus, we can say that the $ΛΛ$ interactions in ESC08a/b are of reasonable strengths. The similar results are obtained also in the case of ESC08a”. In cases of the other Nijmegen models, we obtain the values of $B_{ΛΛ}(^6A\Lambda$He) 7.6, 7.0 and 6.8 MeV, respectively, for ESC04a, ESC04d and NSC97f.
Table XV. Calculated values of $B_{ΛΛ}$ (in MeV) with the core + 2Λ models with ΛΛ G-matrix interactions derived from ESC08a/b. Values in parentheses are $\langle k_F \rangle$ in fm$^{-1}$ obtained self-consistently.

|            | ESC08a   | ESC08b   | EXP            |
|------------|----------|----------|----------------|
| $^6_{ΛΛ}$He | 7.1 (0.85) | 7.2 (0.86) | 6.91±0.16$^{43,45}$ |
| $^{10}_{ΛΛ}$Be | 14.2 (1.01) | 14.5 (1.01) | 14.7±0.6$^{20}$ |
| $^{11}_{ΛΛ}$Be | 18.7 (1.07) | 19.1 (1.07) | 20.83±1.27$^{25}$ |
| $^{12}_{ΛΛ}$Be | 21.2 (1.11) | 21.6 (1.12) | 22.54±1.21$^{25}$ |
| $^{13}_{ΛΛ}$B  | 23.2 (1.16) | 23.6 (1.17) | 23.3±0.7$^{32,38}$ |

§8. Summary

There remains the remarkable ambiguity in the theoretical models of $YN$ and $YY$ interactions due to lack of two-body scattering data. Our aim is to test these interaction models applying them to various studies for hypernuclear phenomena on the basis of the G-matrix theory. In this paper, we have studied the various properties of $ΛN$, $ΣN$, $ΞN$ and $ΛΛ$ G-matrix interactions derived from ESC08 models (ESC08a/b/a”). The G-matrix calculations have been performed with the continuous choice (CON) for intermediate spectra in the case of $ΛN$ starting channels, and the $ω$-rearrangement effects have been taken into account (CONr). The obtained $ΛN$ G-matrix interactions are similar to those calculated with the gap choice (GAP) for intermediate spectra, because the energy gain from GAP to CON is rather canceled by the repulsive $ω$-rearrangement effect. In the cases of $ΣN$-, $ΞN$- and $ΛΛ$-starting channels, the G-matrix calculations are performed with the GAP treatments for ESC08a/b. $ΞN$ G-matrices for ESC08a” are obtained with the CONr treatment. For applications to finite systems, the $k_F$-dependent G-matrix interactions are represented in three- or two-range Gaussian forms.

In the $ΛN$ case, the properties of G-matrix interactions derived from ESC08a/b/a” are similar to each other: The strengths of S-state spin-spin interactions are in a reasonable region. The spin-orbit interactions are substantially smaller than those in the previous Nijmegen models owing to the more repulsive ALS interactions in ESC08 models. $Λ$-nucleus folding potentials are derived from the $k_F$-dependent $ΛN$ G-matrix interactions by folding them into SkHF wave functions for nuclear cores, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. With these folding potentials, energy spectra of observed $Λ$ hypernuclei are calculated, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. The obtained energy spectra of observed $Λ$ hypernuclei are calculated, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. The obtained $Λ$ hypernuclei are calculated, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. The obtained $Λ$ hypernuclei are calculated, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. The obtained $Λ$ hypernuclei are calculated, where the averaged $k_F$ approximation is used for the $k_F$-dependent terms. Then, the observed energy spectra from $^{13}_{Λ}C$ to $^{208}_{Λ}Pb$ are reproduced extensively. The spin-dependent terms of the ESC08 G-matrix interactions can be studied by the shell-model analyses for typical $Λ$ hypernuclei: The spin-orbit splittings of the $Λ$ single particle states in $^{89}_{Λ}Y$ are reproduced fairly well by the SLS and ALS interactions in ESC08a/b/a” contrastively to those in the older Nijmegen models such as NSC97. On the other hand, ground spin-doublet states in $^{12}_{Λ}C$ and $^{14}_{Λ}B$ calculated by the ESC08a/b G-matrix interactions are of opposite orders compared to the experimental ones in spite of reasonable strengths of S-state interactions.
spin-spin components. This fact suggests that the odd-state spin-spin components in ESC08 models should be further improved.

In the $\Sigma N$ case, the $G$-matrices are calculated with the GAP treatments. Most important is that the calculated values of $U_\Sigma$ at normal density for ESC08a/b/a" are repulsive in contrast to the attractive values in the cases of the previous Nijmegen soft-core models NSC89/97 and ESC04. This feature of ESC08 models is due to the fact that the repulsive effect by the quark Pauli-forbidden states are taken into account phenomenologically by strengthening the pomeron-baryon coupling constants in the relevant states. Because of the repulsive nature of $U_\Sigma$, nuclear bound states of $\Sigma$ are not expected generally. In order to study features of $\Sigma N$ interactions in $\Sigma$-nucleus scattering states, the complex $\Sigma$ potentials are calculated for the $\Sigma$ with positive energies in nuclear matter. They are peculiarly energy-dependent and very different from simple optical potentials. Especially, their imaginary parts are noted to be very strong.

In the $\Xi N$ case, the $G$-matrix calculations are performed with the GAP (CONr) treatments for ESC08a/b (ESC08a""). The obtained values of $U_\Xi$ for ESC08a/b/a" are attractive so that $\Xi$ hypernuclei exist extensively. The attractive value of $U_\Xi$ is realized dominantly owing to the strongly attractive contribution by the $\Xi N-\Lambda \Sigma-\Sigma \Sigma$ tensor coupling in the $^3S_1$ $T=1$ states. This feature works favorably to make $T=1$ $\Xi^-$ bound states produced by $(K^-, K^+)$ reactions on $T=0$ nuclear targets. The strengths of $\Xi$-nucleus attractions for ESC08a/b/a" are rather different from each other. Even in the case of the weakest model ESC08a"$, there exist light $\Xi$ hypernuclei as nuclear bound states. In heavier systems including a $\Xi^-$ hyperon, $\Xi^-$ bound states appear comparable to or more than corresponding $\Lambda$ ones owing to assistance of strong Coulomb attractions, though the $\Xi N$ interactions in average considerably weaker than $\Lambda N$ ones. Then, Coulomb-assisted bound states are expected to be observed in $(K^-, K^+)$ reactions. The $\Xi$ conversion widths are not so large in the cases of ESC08a/b/a"$, which are dominated by the $\Lambda \Xi N-\Sigma \Sigma$ coupling interactions in $^1S_0$ $T=0$ states.

In the $\Lambda \Lambda$ case, $\Lambda \Lambda G$-matrix interactions are derived from ESC08a/b with the GAP choice, which are used in core+$\Lambda$ three-body models of double-$\Lambda$ hypernuclei. The obtained values of $B_{\Lambda \Lambda}$ are consistent with emulsion data of $^6_{\Lambda \Lambda}$He and the other data for $^{10}_{\Lambda \Lambda}$Be, $^{11}_{\Lambda \Lambda}$Be, $^{12}_{\Lambda \Lambda}$Be and $^{13}_{\Lambda \Lambda}$B.

Thus, the ESC08 interaction models are found to be nicely consistent with various hypernuclear data in the present, giving us an important guidance for coming experimental data at J-PARC.

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Appendix A

--- ΛN-ΣN Coupling Interaction ---

The ΛN-ΣN coupling interaction represented in a Gaussian form is determined so that its matrix elements in $k$ space simulate the corresponding $G$-matrix elements at $k_F = 1.0$ fm$^{-1}$ and its radial form tends to that of the bare interaction in the outermost region. In Table XVI, the parameters are given for ESC08a/b/a", where the central parts of ΛN-ΣN and ΣN-ΣN interactions in $^1S_0$ and $^3S_1$ states are given in three-range Gaussian forms. These interactions can be used for ΛN-ΣN mixing problems together with the ΛN-ΛN central interactions given in the text. In Table XVII, the tensor parts of ΛN-ΣN and ΛN-ΛN interactions in $^3S_1$ states are given in three-range $r^2$-Gauss forms.

Table XVI. Central coupling parts of $G$-matrix interactions at $k_F = 1.0$ fm$^{-1}$ for ESC08a/b/a", represented in a Gaussian form $\sum_{i=1}^{3} c_i \exp\left(-\frac{(r/\beta_i)^2}{2}\right)$.

| $\beta_i$ (fm) | 0.50 | 0.90 | 2.00 |
|----------------|------|------|------|
| ΛN-ΣN $^1S_0$ |      |      |      |
| ESC08a         | 137.7| 11.75| 8.550|
| ESC08b         | 48.92| 45.27| 7.605|
| ESC08a"        | 200.1| 4.853| 8.602|
| ΣN-ΣN $^1S_0$ |      |      |      |
| ESC08a         | 410.2| 115.2| 8.370|
| ESC08b         | 424.0| 83.03| 10.27|
| ESC08a"        | 358.8| 125.8| 8.443|
| ΛN-ΣN $^3S_1$ |      |      |      |
| ESC08a         | −382.5| 9.615| −2.779|
| ESC08b         | −598.0| 10.80| −2.377|
| ESC08a"        | 26.31 | −177.5| −2.792|
| ΣN-ΣN $^3S_1$ |      |      |      |
| ESC08a         | 998.3 | −186.3| −3.960|
| ESC08b         | 1004. | −207.2| −4.613|
| ESC08a"        | 938.1 | −157.3| −4.123|

Table XVII. Tensor coupling parts of the $G$-matrix interactions at $k_F = 1.0$ fm$^{-1}$ for ESC08a/b/a", represented in a form of $r^2$-Gaussian form $\sum_{i=1}^{3} c_i r^2 \exp\left(-\frac{(r/\beta_i)^2}{2}\right)$.

| $\beta_i$ (fm) | 0.50 | 0.90 | 2.00 |
|----------------|------|------|------|
| ΛN-ΣN $^3S_1$ |      |      |      |
| ESC08a         | −1233. | −87.82| −0.7492|
| ESC08b         | −955.5 | −70.90| −0.6564|
| ESC08a"        | −1164. | −85.97| −0.7528|
| ΛN-ΛN $^3S_1$ |      |      |      |
| ESC08a         | −830.1 | 0.9359| −0.0225|
| ESC08b         | −1474. | 1.643 | −0.0253|
| ESC08a"        | 328.3  | −8.398| −0.0223|
Appendix B

ΛN Skyrme Parameters

In the SkHF model, the Skyrme-type ΛN interaction is given as

\[
V_{\Lambda N}(r_\Lambda - r_N) = t_0 \delta(r_\Lambda - r_N) + \frac{1}{2} t_1 \left[ k'^2 \delta(r_\Lambda - r_N) + \delta(r_\Lambda - r_N) k^2 \right] + t_2 k' \delta(r_\Lambda - r_N) \cdot k + \frac{3}{8} t_3 \delta(r_\Lambda - r_N) \rho^\alpha \left( \frac{r_\Lambda + r_N}{2} \right).
\]

The parameters \(t_0, t_1, t_2\) and \(t_3\) are fitted phenomenologically as follows: First, the values of \(t_3\) is assumed appropriately referring to the density dependence of the \(\Lambda N\) \(G\)-matrix.\(^\text{41}\) Next, remained parameters \((t_0, t_1\) and \(t_2)\) are determined so as to minimize the \(\chi^2\) values for the energy spectrum of \(^{89}\Lambda Y\). It should be noted that this Skyme parameters are determined only by using the observed spectrum of \(^{89}\Lambda Y\).

In the parameter fitting, the parameter \(\alpha\) specifying the density dependence is taken so as to simulate roughly the density dependence of the \(\Lambda N\) \(G\)-matrices. However, it is possible to find a \((t_0, t_1, t_2, t_3)\) set for a different value of \(\alpha\), which gives similar quality of fitting. For instance, we can give a set with \(\alpha = 1\) corresponding to a \(\delta\)-function type three-body force.

| Table XVIII. |
|--------------|
| \(t_0\)      | \(t_1\)      | \(t_2\)      | \(t_3\)      | \(\alpha\)     |
| -1056.2      | 96.248       | 8.743        | 2811.2       | 0.125          |

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