Convolution and correlation theorems for Wigner–Ville distribution associated with the quaternion offset linear canonical transform

Convolution and correlation theorems for WVD associated with the QOLCT

M. Younus Bhat · Aamir H. Dar

Received: 7 July 2021 / Revised: 15 September 2021 / Accepted: 27 October 2021 / Published online: 13 January 2022
© The Author(s), under exclusive licence to Springer-Verlag London Ltd., part of Springer Nature 2021

Abstract
The quaternion offset linear canonical transform (QOLCT) has gained much popularity in recent years because of its applications in many areas, including image and signal processing. At the same time, the applications of Wigner–Ville distribution (WVD) in signal analysis and image processing cannot be excluded. In this paper, we investigate the Wigner–Ville distribution associated with quaternion offset linear canonical transform (WVD-QOLCT). Firstly, we propose the definition of the WVD-QOLCT, and then, several important properties of newly defined WVD-QOLCT, such as reconstruction formula, orthogonality relation, are derived. Secondly, a novel canonical convolution operator and a related correlation operator for WVD-QOLCT are proposed. Based on the proposed operators, the corresponding generalized convolution and correlation theorems are studied. Moreover on the application part, detection of the linear frequency modulated signals is established in detail by constructing an example.

Keywords Quaternion algebra · Offset linear canonical transform · Quaternion offset linear canonical transform · Wigner–Ville distribution · Convolution · Correlation · Modulation

Mathematics Subject Classification 11R52 · 42C40 · 42C30 · 43A30

1 Introduction
In the time–frequency signal analysis, the classical Wigner–Ville distribution (WVD) or Wigner–Ville transform (WVT) has an important role to play. Eugene Wigner introduced the concept WVD while making his calculation of the quantum corrections. Later on, it was J. Ville who derived it independently as a quadratic representation of the local time–frequency energy of a signal in 1948. Numerous important properties of WVT have been studied by many authors. On replacing the kernel of the classical Fourier transform (FT) with the kernel of the LCT in the WVD domain, this transform can be extended to the domain of linear canonical transform [3–6,13–19].

On the other hand, the quaternion Fourier transform (QFT) is of the interest in the present era. Many important properties like shift, modulation, convolution, correlation, differentiation, energy conservation, uncertainty principle of QFT have been found. Many generalized transforms are closely related to the QFTs, for example, the quaternion wavelet transform, fractional quaternion Fourier transform, quaternion linear canonical transform, and quaternionic windowed Fourier transform. Based on the QFTs, one may also extend the WVD to the quaternion algebra while enjoying similar properties as in the classical case. Many authors generalized the classical WVD to quaternion algebra, which they called as the quaternion Wigner–Ville distribution (QWVD). For more details, we refer to [1,2,7–12].

Moving to the side of generalizations linear canonical transform (LCT), it is here worth mentioning that the LCT with four parameters \((a, b, c, d)\) has been generalized to a six-parameter transform \((a, b, c, d, u_0, w_0)\) known as offset linear canonical transform (OLCT). Due to the time shift-
ing $u_0$ and frequency modulation parameters, the OLCT has gained more flexibility over the classical LCT. Hence, wide applications in image and signal processing have been found. Another generalization of LCT, the quaternion offset linear canonical transform (QOLCT) has also gained much popularity in recent years on same grounds. At the same time, the applications of Wigner–Ville distribution (WVD) in signal analysis and image processing cannot be excluded. On the other side, the convolution has numerous important applications in various areas of Mathematics like linear algebra, numerical analysis and signal processing, whereas correlation like convolution is another important tool in signal processing, optics and detection applications. In the domains of LCT, WVD and OLCT, the convolution and correlation operations have been studied [7–10].

Motivated by the generalized LCT, the quaternion offset linear canonical transform and the Wigner–Ville distribution in this paper we studied the Wigner–Ville distribution associated with quaternion offset linear canonical transform (WVD-QOLCT). Firstly, we propose the definition of the WVD-QOLCT, and then, several important properties of newly defined WVD-QOLCT, such as nonlinearity, bounded, reconstruction formula, orthogonality relation and Plancherel formula, are derived. Secondly, a novel canonical convolution operator and a related correlation operator for WVD-QOLCT are proposed. Moreover, based on the proposed operators, the corresponding generalized convolution and correlation theorems are studied. The crux of our paper is that the convolution and correlation theorems of the quaternion Wigner–Ville distribution can be looked as a special case of our achieved results.

The paper is organized as follows: In Sect. 2, we provide the definition of Wigner–Ville distribution associated with the quaternionic offset linear canonical transform (WVD-QOLCT). Then, we investigated several basic properties of the WVD-QOLCT which are important for signal representation in signal processing. In Sect. 3, we first define the convolution and correlation for the QOLCT. We then established the new convolution and correlation for the WVD-QOLCT. Moreover on the application part, detection of the linear frequency modulated (LFM) signals is established in detail.

## 2 Wigner–Ville Distribution associated with Quaternion Offset Linear Canonical Transform (WVD-QOLCT)

Since in practice most natural signals are non-stationary, in order to study a non-stationary signals the Wigner–Ville distribution has become a suite tool for the analysis of the non-stationary signals. Here, we provide the definition of Wigner–Ville distribution associated with the quaternionic offset linear canonical transform (WVD-QOLCT). Further some basic properties of the WVD-QOLCT, which are important for signal representation in signal processing, are investigated.

**Definition 1** Let $A_i = \begin{bmatrix} a_i & b_i & | r_i & 0 \end{bmatrix}$ be a matrix parameter such that $a_i, b_i, c_i, d_i, r_i, s_i \in \mathbb{R}$ and $a_id_i - b_ic_i = 1$, for $i = 1, 2$. The Wigner–Ville distribution associated with the two-sided quaternionic offset linear canonical transform (WVD-QOLCT) of signals $f, g \in L^2(\mathbb{R}^2, \mathbf{H})$, is given by

$$W_{f,g, A_1, A_2}^1(t, u) = \sqrt{\frac{d}{d_0} e^{i \left(\frac{2d}{2d_0} (u_1 - r_1)^2 + u_1 r_1\right)}} f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$K_{A_1}^0(n_1, u_1) f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$K_{A_2}^0(n_2, u_2) d_n,$n,$

$$b_1, b_2 \neq 0,$n,$

$$\sqrt{\frac{d}{d_0} e^{i \left(\frac{2d}{2d_0} (u_1 - r_1)^2 + u_1 r_1\right)}} f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$\bar{g} \left(t - \frac{n_1}{2}, t - \frac{n_2}{2}\right)$$

$$K_{A_1}^0(n_2, u_2),$$

$$b_1 = 0, b_2 \neq 0,$n,$

$$\sqrt{\frac{d}{d_0} e^{i \left(\frac{2d}{2d_0} (u_1 - r_1)^2 + u_1 r_1\right)}} f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$\bar{g} \left(t - \frac{n_1}{2}, t - \frac{n_2}{2}\right)$$

$$\bar{e}^{i \left(\frac{2d}{2d_0} (u_2 - r_2)^2 + u_2 r_2\right)}$$

$$b_1 = 0, b_2 = 0,$n,$

$$\sqrt{\frac{d}{d_0} e^{i \left(\frac{2d}{2d_0} (u_1 - r_1)^2 + u_1 r_1\right)}} f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$\bar{g} \left(t - \frac{n_1}{2}, t - \frac{n_2}{2}\right)$$

$$\bar{e}^{i \left(\frac{2d}{2d_0} (u_2 - r_2)^2 + u_2 r_2\right)}$$

where $t = (t_1, t_2), u = (u_1, u_2), n = (n_1, n_2)$ and $K_{A_1}^0(n_1, u_1)$ and $K_{A_2}^0(n_2, u_2)$ are the quaternion kernels.

**Notes 1** If $f = g$, then it is the auto WVD-QOLCT. Otherwise it is called cross WVD-QOLCT

Without loss of generality, we will deal with the case $b_i \neq 0, i = 1, 2$, as in other cases proposed transform reduces to a chirp multiplications. Thus for any $f, g \in L^2(\mathbb{R}^2, \mathbf{H})$, we have

$$W_{f,g, A_1, A_2}^2(t, u) = \int_{\mathbb{R}^2} K_{A_1}^0(n_1, u_1) f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right)$$

$$K_{A_2}^0(n_2, u_2) d_n$$

$$= g^{i,j} \left( f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right) \bar{g} \left(t - \frac{n_1}{2}, t - \frac{n_2}{2}\right) \right)$$

$$= g^{i,j} \left( f \left(t + \frac{n_1}{2}, t + \frac{n_2}{2}\right) \bar{g} \left(t - \frac{n_1}{2}, t - \frac{n_2}{2}\right) \right)$$

[[2]](Springer)
where \( h_{f,g}(t, n) = \frac{f(t + \frac{n}{2})}{g(t - \frac{n}{2})} \) is known as quaternion correlation product. Applying the inverse QOCLT to (2), we get

\[
{[h_{f,g}(t, n)]} = [C_{b}^{i,j}]^{-1}[W_{f,g}^{A_{1},A_{2}}(t, u)]
\]

which implies

\[
f(t + \frac{n}{2})g(t - \frac{n}{2}) = [C_{b}^{i,j}]^{-1}[W_{f,g}^{A_{1},A_{2}}(t, u)]
= \int_{R^2} K_{A_{1}}^{-j}(t_1, u_1)W_{f,g}^{A_{1},A_{2}}(t, u)K_{A_{2}}^{-j}(t_2, u_2)dw.
\]

(3)

Now, we discuss several basic properties of the WVD-QOCLT given by (1). These properties play important roles in signal representation.

The following theorem guarantees the reconstruction of the input quaternion signal from the corresponding WVD-QOCLT within a constant factor.

**Theorem 1** (Reconstruction formula). For \( f, g \in L^2(R^2, H) \) where \( g \) does not vanish at 0. We get the following inversion formula of the WVD-QOCLT:

\[
f(t) = \frac{1}{g(0)} \int_{R^2} K_{A_{1}}^{-j}(u_1, n_1)W_{f,g}^{A_{1},A_{2}}(t, u)K_{A_{2}}^{-j}(u_2, n_2)du
\]

(4)

**Proof** By (3), we have

\[
{[h_{f,g}(t, n)]} = [C_{b}^{i,j}]^{-1}[W_{f,g}^{A_{1},A_{2}}(t, u)]
\]

which implies

\[
f(t + \frac{n}{2})g(t - \frac{n}{2}) = \int_{R^2} K_{A_{1}}^{-j}(t_1, u_1)W_{f,g}^{A_{1},A_{2}}(t, u)K_{A_{2}}^{-j}(t_2, u_2)dw.
\]

Now let \( t = \frac{u}{2} \) and taking change of variable \( w = 2t \), we get

\[
f(w) = \frac{1}{g(0)} \int_{R^2} K_{A_{1}}^{-j}(u_1, n_1)W_{f,g}^{A_{1},A_{2}}\left(\frac{w}{2}, u\right)K_{A_{2}}^{-j}(u_2, n_2)du
\]

which completes the proof.

**Theorem 2** (Orthogonality relation). If \( f_1, f_2, g_1, g_2 \in L^2(R^2, H) \) are quaternion-valued signals. Then,

\[
\langle W_{f_1,g_1}^{A_1,A_2}(t, u), W_{f_2,g_2}^{A_1,A_2}(t, u) \rangle_H = \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle_H
\]

(5)

**Proof** By the definition of Wigner-Vila distribution associated with quaternion OLCT and linear product relation, we have

\[
\langle W_{f_1,g_1}^{A_1,A_2}(t, u), W_{f_2,g_2}^{A_1,A_2}(t, u) \rangle_H
= \int_{R^4} \left[ W_{f_1,g_1}^{A_1,A_2}(t, u)W_{f_2,g_2}^{A_1,A_2}(t, u) \right]_H dudt
= \int_{R^4} \left[ W_{f_1,g_1}^{A_1,A_2}(t, u) \right]_H dudt
\]

\[
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)f_2(t + \frac{n}{2})g_2(t - \frac{n}{2})K_{A_2}^{-j}(n_2, u_2)dn \right] dudt
\]

\[
\]

\[
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)W_{f_1,g_1}^{A_1,A_2}(t, u)K_{A_2}^{-j}(n_2, u_2)du \right] dudtn
\]

\[
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)W_{f_1,g_1}^{A_1,A_2}(t, u)K_{A_2}^{-j}(n_2, u_2)du \right] dudtn
\]

\[
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)W_{f_1,g_1}^{A_1,A_2}(t, u)K_{A_2}^{-j}(n_2, u_2)du \right] dudtn
\]

(6)

Because

\[
K_{A_1}^{-j}(n_1, u_1) = K_{A_1}^{-j}(u_1, n_1) = K_{A_1}^{-j}(u_1, n_1)
\]

\[
K_{A_2}^{-j}(n_2, u_2) = K_{A_2}^{-j}(n_2, u_2) = K_{A_1}^{-j}(u_1, n_1)
\]

Now by using (3) in (6), we have

\[
\langle W_{f_1,g_1}^{A_1,A_2}(t, u), W_{f_2,g_2}^{A_1,A_2}(t, u) \rangle_H
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(u_1, n_1)W_{f_1,g_1}^{A_1,A_2}(t, u)K_{A_2}^{-j}(n_2, u_2)du \right] dudtn
\]

\[
= \int_{R^4} \left[ \int_{R^2} W_{f_1,g_1}^{A_1,A_2}(t, u)K_{A_2}^{-j}(n_2, u_2)du \right] dudtn
\]

\[
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)f_2(t + \frac{n}{2})g_2(t - \frac{n}{2})K_{A_2}^{-j}(n_2, u_2)dn \right] dudtn
\]

(7)

Using the change of variables \( t + \frac{n}{2} = \omega, t - \frac{n}{2} = \xi \), the equation becomes

\[
\langle W_{f_1,g_1}^{A_1,A_2}(t, u), W_{f_2,g_2}^{A_1,A_2}(t, u) \rangle_H
= \int_{R^4} \left[ \int_{R^2} K_{A_1}^{-j}(n_1, u_1)f_2(\omega)g_2(\xi)K_{A_2}^{-j}(n_2, u_2)dn \right] dudtn
\]

(8)
\[
\int_{\mathbb{R}^2} \left[ f_1(\omega)g_1(\xi)g_2(\xi)g_2(\omega) \right] \, d\omega d\xi \\
= \left[ \int_{\mathbb{R}^2} f_1(\omega)g_2(\omega) d\omega \int_{\mathbb{R}^2} g_2(\xi)g_1(\xi) d\xi \right] \mathbf{H}
\]
which completes the proof. \qed

**Consequences of Theorem 2.**

1. If \( g_1 = g_2 = g \), then
\[
\langle W_{f_1, g}^{A_1, A_2} (t, u), W_{f_2, g}^{A_1, A_2} (w, u) \rangle = \| g \|_{L^2(\mathbb{R}^2)}^2 \langle f_1, f_2 \rangle. \tag{7}
\]

2. If \( f_1 = f_2 = f \), then
\[
\langle W_{f, g_1}^{A_1, A_2} (t, u), W_{f, g_2}^{A_1, A_2} (w, u) \rangle = \| f \|_{L^2(\mathbb{R}^2)}^2 \| g_1 \|_{L^2(\mathbb{R}^2)}^2 \| g_2 \|_{L^2(\mathbb{R}^2)}^2. \tag{8}
\]

3. If \( f_1 = f_2 = f \) and \( g_1 = g_2 = g \), then
\[
\langle W_{f, g}^{A_1, A_2} (t, u), W_{f, g}^{A_1, A_2} (w, u) \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W_{f, g}^{A_1, A_2} (t, u)|^2 \, du dt
\]
\[
= \| f \|_{L^2(\mathbb{R}^2)}^2 \| g \|_{L^2(\mathbb{R}^2)}^2. \tag{9}
\]

Now we move forward toward our main section that is convolution and correlation theorems for Wigner–Ville distribution associated with quaternion offset linear canonical transform.

### 3 Convolution and correlation theorem for WVD-QOLCT

The convolution and correlation are fundamental signal processing algorithms in the theory of linear time-invariant (LTI) systems. In engineering, they have been widely used for various template matchings. In the following, we first define the convolution and correlation for the QOLCT. They are extensions of the convolution definition from the OLCT (see [16]) to the QOLCT domain. We then establish the new convolution and correlation for the WVD-QOLCT. The convolution and correlation theorems will open new gates to investigate the sampling and filtering theorems of the WVD-QOLCT.

**Definition 2** For any two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbf{H}) \), we define the convolution operator of the QOLCT as
\[
(f * g)(t) = \int_{\mathbb{R}^2} \Psi(z_1, t_1) f(z) g(t - z) \Psi(z_2, t_2) \, dz \tag{10}
\]
where \( \Psi(z_1, t_1) \) and \( \Psi(z_2, t_2) \) are known as weight functions.

We assume
\[
\Psi(z_1, t_1) = e^{-i \frac{\pi}{2} (z_1^2(t_1 - z_1))},
\]
and
\[
\Psi(z_2, t_2) = e^{-i \frac{\pi}{2} (z_2^2(t_2 - z_2))}. \tag{11}
\]

As a consequence of the above definition, we get the following important theorem, which states how the convolution of two quaternion-valued functions interacts with their QOLCTs.

**Theorem 3** (WVD-QOLCT Convolution). For any two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbf{H}) \), the following result holds
\[
W_{f, g}^{A_1, A_2} (t, u) = \sqrt{2\pi} e^{-\frac{\pi}{2} (d_2(w^2 + r^2) - 2w_2d_2w_1)} \int_{\mathbb{R}^2} e^{-i \frac{\pi}{2} (4w_2(t_1 - w_1))} W_{f, g}^{A_1, A_2} (w, u) W_{f, g}^{A_1, A_2} (w - u, u) \, dw \\
\times \sqrt{2\pi} e^{-\frac{\pi}{2} (d_2(w^2 + r^2) - 2w_2d_2w_1)} \, dw \tag{12}
\]

**Proof** Applying the definition of the WVD-QOLCT, we have
\[
W_{f, g}^{A_1, A_2} (t, u) = \int_{\mathbb{R}^2} K_{A_1}^j (n_1, u_1) \left[ \left( f * g \right)(t + \frac{n_2}{2}) \right] \left( \mathcal{F} \ast \mathcal{G} \left( t - \frac{n_2}{2} \right) \right) K_{A_2}^j (n_2, u_2) \, dn. \tag{13}
\]

Now using Definition 2 in (13), we have
\[
W_{f, g}^{A_1, A_2} (t, u) = \int_{\mathbb{R}^2} K_{A_1}^j (n_1, u_1) \left[ \int_{\mathbb{R}^2} \mathcal{F} \Psi_1 \left( z_1, t_1 + \frac{n_1}{2} \right) f(z) \right. \\
g \left( t + \frac{n_2}{2} - z \right) \left. \Psi_2 \left( z_2, t_2 + \frac{n_2}{2} \right) \, dz \right. \\
\times \int_{\mathbb{R}^2} \mathcal{F} \Psi_1 \left( y_1, t_1 - \frac{n_1}{2} \right) f(y) g \left( t - \frac{n_2}{2} - y \right) \Psi_2 \left( y_2, t_2 - \frac{n_2}{2} \right) \, dy \\
\left. \Psi_2 \left( y_2, t_2 - \frac{n_2}{2} \right) K_{A_2}^j (n_2, u_2) \, dn \right]. \tag{14}
\]
For simplicity, let us denote

\[
K_{A_1}^{j}(t_1, u_1) = K_{A_1}^{j} e^{\frac{\pi}{2}(a_1 r_1^2 + 2a_1 r_1 u_1 - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
K_{A_1}^{j} = \frac{1}{2\pi b_1 i} e^{\frac{\pi}{2} b_1^2 r_1^2}
\]

(15)

and

\[
K_{A_2}^{j}(t_2, u_2) = K_{A_2}^{j} e^{\frac{\pi}{2}(a_2 r_2^2 + 2a_2 r_2 u_2 - 2a_2 (d_2 r_2 - b_2 s_2) + d_2 u_2^2)}
\]

\[
K_{A_2}^{j} = \frac{1}{2\pi b_2 i} e^{\frac{\pi}{2} b_2^2 r_2^2}
\]

(16)

Now using (15) and (16) in (14), we have

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u) = \int_{\mathbb{R}^6} K_{A_1}^{i} e^{\frac{\pi}{2}(a_1 r_1^2 + 2a_1 r_1 u_1 - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
e^{-\frac{\pi}{4} b_1^2 (t_1 + u_1 - z_1)}
\]

\[
\times f(z) g \left( t + \frac{n}{2} - z \right) e^{-\frac{\pi}{2} b_2^2 (t_2 + u_2 - z_2)}
\]

\[
e^{-\frac{\pi}{4} b_2^2 (t_2 + u_2 - z_2)}
\]

\[
\times f(\gamma) g \left( t - \frac{n}{2} - \gamma \right) e^{-\frac{\pi}{2} b_2^2 (t_2 + u_2 - \gamma - z_2)}
\]

\[
\times K_{A_2}^{j} e^{\frac{\pi}{2}(a_2 r_2^2 + 2a_2 r_2 u_2 - 2a_2 (d_2 r_2 - b_2 s_2) + d_2 u_2^2)}
\]

\[dz\,dy\,dn\]

Setting \( z_i = w_i + \frac{p_i}{2}, \gamma_i = w_i - \frac{p_i}{2}, i = 1, 2 \) we get

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u) = \int_{\mathbb{R}^6} K_{A_1}^{i} e^{\frac{\pi}{2}(a_1 r_1^2 + 2a_1 r_1 u_1 - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
e^{-\frac{\pi}{4} b_1^2 (w_1 + \frac{p_1}{2} - (w_1 + \frac{p_1}{2}))}
\]

\[
\times f \left( w + \frac{p}{2} \right) g \left( t + \frac{n}{2} - \left( w + \frac{p}{2} \right) \right)
\]

\[
e^{-\frac{\pi}{2} b_2^2 (w_2 + \frac{p_2}{2}) (t_2 + \frac{n}{2} - (w_2 + \frac{p_2}{2}))}
\]

\[
\times f \left( w - \frac{p}{2} \right) g \left( t - \frac{n}{2} - \left( w - \frac{p}{2} \right) \right)
\]

\[dw\,du\,\sqrt{2\pi b_1}\]

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u)
\]

and \( n_i = p_i + q_i, i = 1, 2 \) we obtain

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u) = \int_{\mathbb{R}^6} K_{A_1}^{i} e^{\frac{\pi}{2}(a_1 (p_1 + q_1)^2 + 2(p_1 + q_1)(r_1 - u_1) - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
e^{-\frac{\pi}{4} b_1^2 (4w_1 (r_1 - u_1) - 4w_1 (t_1 - w_1))}
\]

\[\times f \left( w + \frac{p}{2} \right) f \left( w - \frac{p}{2} \right) g \left( t - w + \frac{q}{2} \right)
\]

\[g \left( t - w - \frac{q}{2} \right)
\]

\[
e^{-j \frac{\pi}{2} (4w_1 (t_1 - w_1))} e^{-j \frac{\pi}{2} p_1 q_1}
\]

\[
\times K_{A_2}^{j} e^{\frac{\pi}{2}(a_2 r_2^2 + 2a_2 r_2 u_2 - 2a_2 (d_2 r_2 - b_2 s_2) + d_2 u_2^2)}
\]

\[dpdqdw\]

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u)
\]

Now multiplying Eq. (17) both sides by

\[
K_{A_1}^{j} e^{\frac{\pi}{2}(a_1 r_1^2 + 2a_1 r_1 u_1 - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u)
\]

(18)

Now using (15) and (16) in (18), we get

\[
\mathcal{W}_{f_{x,y}}^{A_1,A_2}(t, u) = \sqrt{2\pi b_1} e^{\frac{\pi}{2}(a_1 r_1^2 + 2a_1 r_1 u_1 - 2a_1 (d_1 r_1 - b_1 s_1) + d_1 u_1^2)}
\]

\[
\times \int_{\mathbb{R}^6} e^{-j \frac{\pi}{2} (4w_1 (t_1 - w_1))} \mathcal{W}_{f_{x,y}}^{A_1,A_2}(w, u)
\]

\[\times e^{-j \frac{\pi}{2} (4w_2 (t_2 - w_2))}
\]

\[dw.
\]

which completes the proof. \( \square \)

**Consequences of theorem 3.**
1. Changing parameter \( A_i = \left[ \begin{array}{c|c} a_i & b_i \\ \hline c_i & d_i \end{array} \right], i = 1, 2 \) to \( A_i = \left[ \begin{array}{c|c} a_i & b_i \\ \hline 0 & 1 \end{array} \right], i = 1, 2 \), then Theorem 2 reduces to convolution theorem of the WVD-QLCT.

2. Changing parameter \( A_i = \left[ \begin{array}{c|c} a_i & b_i \\ \hline 1 & 0 \end{array} \right], i = 1, 2 \), then Theorem 2 reduces to convolution theorem of the WVD in Quaternion domain.

Next, we will derive the correlation theorem in the WVD-QLCT. Let us define the correlation for the QOLCT.

**Definition 3** For any two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), we define the correlation operator of the QOLCT as:

\[
(f \circ g)(t) = \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (z^1(t+1) + n^1)} f(z) g(z + t) e^{i \frac{\pi}{2} (z^2(z^2 + t^2))} dz
\]

Now, we reap a consequence of the above definition below which gives the tightness between the correlation of two quaternion functions and the QOLCT.

**Theorem 4** (WVD-QLCT Correlation). For any two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), the following result holds

\[
\mathcal{W}_{f \circ g}^{A_1, A_2}(t, u) = \sqrt{2 \pi b_1} e^{i \frac{\pi}{2} [d_1(u_1^2 + r_1^2) + 2u_1(d_1r_1 - b_1s_1)]} \times \left\{ \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (4u_1(t_1 + u_1))} \mathcal{W}_{f \circ g}^{A_1, A_2}(w, -u) \right\}
\]

\[
\mathcal{W}_{f \circ g}^{A_1, A_2}(t, u) = \sqrt{2 \pi b_2} e^{i \frac{\pi}{2} [d_2(u_2^2 + r_2^2) + 2u_2(d_2r_2 - b_2s_2)]} \times \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (4u_2(t_2 + u_2))} \right\}
\]

**Proof** Applying the definition of the WVD-QLCT, we have

\[
\mathcal{W}_{f \circ g}^{A_1, A_2}(t, u) = \int_{\mathbb{R}^2} K_{A_1}^{i} (n_1, u_1) \left[ (f \circ g) \left( t + \frac{n_1}{2} \right) \right] g \left( z + t + \frac{n_1}{2} \right) e^{i \frac{\pi}{2} (z^1(t^1(t + \frac{n_1}{2}) + n^1))} f(z) g(z + t) e^{i \frac{\pi}{2} (z^2(z^2 + t^2))} dz
\]

Now using definition 3 in (21), we have

\[
\mathcal{W}_{f \circ g}^{A_1, A_2}(t, u) = \int_{\mathbb{R}^2} K_{A_1}^{i} (n_1, u_1) \left[ \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (z^1(t^1(t + \frac{n_1}{2}) + n^1))} f(z) g(z + t) e^{i \frac{\pi}{2} (z^2(z^2 + t^2))} dz \right] \times \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (z^1(t^1(t + \frac{n_1}{2}) + n^1))} f(z) g(z + t) e^{i \frac{\pi}{2} (z^2(z^2 + t^2))} dz
\]

Now put \( n_i = q_i - p_i, i = 1, 2 \) and on following the same procedure as followed in Theorem 3, we have from (23)

\[
\mathcal{W}_{f \circ g}^{A_1, A_2}(t, u) = \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} e^{i \frac{\pi}{2} (a_1p_1^2 - 2p_1(p_1 - u_1))} f(w + \frac{P}{2}) f(w - \frac{P}{2}) e^{i \frac{\pi}{2} (a_2p_2^2 - 2p_2(p_2 - u_2))} dp \right]
\]

\[
\times \left[ \int_{\mathbb{R}^2} K_{A_1}^{i} e^{i \frac{\pi}{2} (a_1q_1^2 + 2q_1(q_1 - u_1))} f(w + \frac{Q}{2}) f(w - \frac{Q}{2}) e^{i \frac{\pi}{2} (a_2q_2^2 - 2q_2(q_2 - u_2))} dq \right]
\]
On multiplying (24) both sides by $K_{A_1}^{-1}e^{\frac{\phi_1}{2\pi} [d_1(t_1-r_1)+2\sigma_1(d_1r_1-b_1s_1)]} + \frac{1}{2\pi}i\int_{\mathbb{R}^2} e^{\frac{\phi_1}{2\pi} [4w_1(t_1-w_1)]} W_{A_1,A_2}^{A_1,A_2}(t,w,u) \times e^{i\frac{\phi_2}{2\pi} [4w_2(t_2-w_2)]} dw$.

Now using (15) and (16) in (25) we obtain,

$$W_{f,g}^{A_1,A_2}(t,u) = \sqrt{2\pi b_1} e^{\frac{\phi_1}{2\pi} [d_1(t_1+r_1)+2\sigma_1(d_1r_1-b_1s_1)]} \times \left\{ \int_{\mathbb{R}^2} e^{\frac{\phi_1}{2\pi} [4w_1(t_1+w_1)]} W_{f,f}^{A_1,A_2}(w, -u) \right\} \times e^{i\frac{\phi_2}{2\pi} [4w_2(t_2+w_2)]} dw \right.$$
which is a quaternion chirp (LFM) signal in \((t, u)\) plane. Thus, it is clear from the above illustration that we can detect non-stationery LMF quaternion-valued signal with WVD-QOLCT.

**Acknowledgements** This work was supported by the UGC-BSR Research Start Up Grant (No. F.30-498/2019(BSR)) provided by UGC, Govt. of India.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Bahri, M., Ashino, R., Vaillancourt, R.: Convolution theorems for quaternion Fourier transform: properties and applications, Abst. Appl Anal. 2013, Article ID 162769
2. Bahri, M., Hitzer, E.S.M., Hayashi, A., Ashino, R.: An uncertainty principle for quaternion Fourier transform. Comp. Maths with Appl. 56(9), 2398–2410 (2008)
3. Bahri, M.: Correlation theorem for Wigner–Ville distribution. Far East J. Math. Sci. 80(1), 123–133 (2013)
4. Bai, R. F., Li, B. Z., Cheng, Q. Y., Wigner-Ville distribution associated with the linear canonical transform, J. Appl. Maths, 2012, Article ID 740161
5. Debnath, L., Shankara, B. V., Rao, N.: On new two-dimensional Wigner-Ville nonlinear integral transforms and their basic properties. Int. Trans. Sp. Funct. 21(3), 165–174 (2010)
6. Gao, W. B., Li, B. Z.: Convolution and correlation theorems for the windowed offset linear canonical transform arxiv: 1905.01835v2 [math.GM](2019)
7. Guanlei, X., Xiaotong, W., Xiaogang, X.: Uncertainty inequalities for linear canonical transform. IET Signal Process. 3(5), 392–402 (2009)
8. El Haoui Y.S., Fahlaoui, S.: Generalized Uncertainty Principles associated with the Quaternionic Offset Linear Canonical Transform, arxiv: 1807.04068v1
9. Hitzer, E.M.S.: Quaternion Fourier transform on quaternion fields and generalizations. Adv. Appl. Clifford Algs 17(3), 497–517 (2007)
10. Huo, H.Y., Sun, W.C., Xiao, L.: Uncertainty principles associated with the offset linear canonical transform Mathl. Methods Appl. Scis. 42(2), 466–474 (2019)
11. Kou, K. I., Jian-Yu Ou, Morais, J.: On uncertainty principle for quaternionic linear canonical transform, Abstr. Appl. Anal., 2013 (Article ID 725952) (2013) 14pp
12. Kou, K.I., Morais, J., Zhang, Y.: Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis. Math. Methods Appl. Sci. 36(9), 1028–1041 (2013)
13. Li, Y. G., Li, B. Z., Sun, H. F.: Uncertainty principle for Wigner-Ville distribution associated with the linear canonical transform, Abstr. Appl. Anal., 2014, Article ID 470459
14. Song, Y.E., Zhang, X.Y., Shang, C.H., Bu, H.X., Wang, X.Y.: The Wigner-Ville distribution based on the linear canonical transform and its applications for QFM signal parameters estimation, J. App. Maths (2014) 8 pages
15. Urynbassarova, D., Li, B.Z., Tao, R.: The Wigner–Ville distribution in the linear canonical transform domain. IAENG Int. J. Appl. Maths. 46(4), 559–563 (2016)
16. Urynbassarova, D., Zhao, B., Tao, R.: Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with the Offset Linear Canonical Transform. Int. J. Light Elect. Optics 10.1016/j.ijleo.2017.08.099
17. Wei, D., Ran, Q., Li, Y.: A convolution and correlation theorem for the linear canonical transform and its application. Circuits Syst. Signal Process. 31(1), 301–312 (2012)
18. Wei, D., Ran, Q., Li, Y.: New convolution theorem for the linear canonical transform and its translation invariance property. Optik. 123(16), 1478–1481 (2012)
19. Zhang, Z.C.: Sampling theorem for the short-time linear canonical transform and its applications. Signal Proces. 113138–146 (2015)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.