Violations of Bell inequalities from random pure states

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We consider the expected violations of Bell inequalities from random pure states. More precisely, we focus on a slightly generalised version of the CGLMP inequality, which concerns Bell experiments of two parties, two measurement options and \( N \) outcomes and analyse their expected quantum violations from random pure states for varying \( N \), assuming the conjectured optimal measurement operators. It is seen that for small \( N \) the Bell inequality is not violated, while for larger \( N \) it is. Using techniques from random matrix theory this is obtained analytically for small and large \( N \) and numerically for intermediate \( N \). The results show a beautiful interplay of different aspects of random matrix theory, ranging from the Marchenko-Pastur distribution and fixed-trace ensembles to the \( O(n) \) model.

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Introduction – In quantum information theory “non-classical” properties of a quantum system are of importance for many of its applications \[1\]. There are different ways of classifying what is meant by a state of quantum system to be “non-classical”. Let us focus on a composite system \( AB \) made up of two sub-systems \( A \) and \( B \) which for simplicity we assume to have Hilbert spaces of equal dimension \( N \). One quantity classifying the “non-classical” correlations between \( A \) and \( B \) is entanglement entropy \[2\], measured by the von Neumann entropy of the reduced density matrix of either sub-system. In the above case that both of the sub-systems are \( N \)-dimensional, its maximum value is \( \log N \) and the state with this entropy is referred as the maximally entangled state. Another measure for “non-classicality” of a quantum system is given by the violations of Bell inequalities. Besides Bell’s original inequality \[3\], one of the most well-known Bell inequalities is the so-called CHSH inequality \[4\] which concerns a Bell experiment with two party system, Alice and Bob, each having two measurement options which can have two different outcomes. In the case of the CHSH inequality the maximal violation (under optimal measurement operators) is caused by the maximally entangled state, leading to an agreement between the both measures of “non-classicality” introduced above.

Recently there has been a considerable interest in random pure states (see \[5\] and references therein) which for example emerge due to noise in the preparation of the state or when the state is evolving in time under chaotic dynamics. Since random states arise from random density matrices their formulation involves techniques from random matrix theory (RMT) (see \[6\] and \[7\] Chapter 1) for an overview). One particularly interesting aspect of random pure states of a bipartite system is that the expected value of its entanglement entropy is given by \( \log N (1 - (2 \log N)^{-1}) \) \[8\] \[9\] (see also \[10\] for the full distribution function for large \( N \)), i.e. for large \( N \) it approaches the maximal value, associated to the maximally entangled state. In the context of the above discussion it is interesting to consider the violations of Bell inequalities under random pure states. This letter aims to give a first account of such an analysis.

Bell inequalities and their quantum violations from random pure states – A generalisation of the CHSH inequality in the case of \( N \) possible measurement outcomes is given by the CGLMP inequality \[11\]. It has a slightly generalised and simplified version \[12\],

\[
P_{LR}(A_2 < B_2) + P_{LR}(B_2 < A_1) + P_{LR}(A_1 < B_1) + P_{LR}(B_1 < A_2) \geq 1, \quad (1)
\]

where \( P_{LR}(A_a < B_b) \) denotes the probability, under local realism (LR), that the outcome of the Alice’s measurement, having chosen the measurement option \( a = 1, 2 \), is less than the outcome of Bob’s measurement, having chosen the measurement option \( b = 1, 2 \); both \( A_a \) and \( B_b \) taking values in the same set of \( N \) possible outcomes.

In the case of quantum mechanics (QM), as opposed to local realism, the corresponding probabilities are given by \( P_{QM}(A_a < B_b) = \sum_{k<l} \text{tr} (\rho A^k_a \otimes B^l_b) \), where \( \rho \) is the density matrix of the bipartite system \( AB \), while \( A^k_a \) and \( B^l_b \) are positive operators on the sub-systems \( A \) and \( B \) with \( \sum_{k=1}^{N} A^k_a = \mathbb{I} \) and \( \sum_{l=1}^{N} B^l_b = \mathbb{I} \); here the indices \( a, b \) label the two possible measurement options and \( k, l \) the \( N \) possible outcomes.

It is well known that quantum mechanics can violate the above Bell inequality which occurs when \( A_N = P_{QM}(A_2 < B_2) + P_{QM}(B_2 < A_1) + P_{QM}(A_1 < B_1) + P_{QM}(B_1 < A_2) \) is less than one and the maximal violation corresponds to the minimal value of \( A_N \), which we often refer to as the minimal target value. Quantum violations of the above Bell inequality and the original CGLMP inequality were studied numerically for \( N > 2 \) \[12\] \[13\] (see also \[14\] for an exact result for \( N \to \infty \)), where it was
seen that the maximal violation is obtained for a pure state under earlier conjectured best measurement operators [11, 15]. However, for $N > 2$ the maximal violation is not caused by the maximally entangled state, while the optimal measurement operators remain the same for both the optimal and maximally entangled state. Let us therefore assume that the operators $A_k$ and $B_l$ are given by the conjectured best measurement operators and that $\rho = |\psi\rangle\langle\psi|$ is pure. It is useful to introduce the Schmidt decomposition of the pure state

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} \langle i \rangle_A \otimes |i\rangle_B,$$

where the so-called Schmidt coefficients $\{\lambda_i\}$ are non-negative and normalised to $\sum_i \lambda_i = 1$. Recall that the maximal entangled state has $\lambda_i = 1/N$ for all $i = 1,...,N$. It was shown in [12] that under the above assumptions one has

$$\mathcal{A}_N(\{\lambda_i\}) = \sum_{i,j=1}^{N} M_{ij} \sqrt{\lambda_i \lambda_j},$$

with $M_{ij} = 2\delta_{ij} - \frac{1}{N} p_{ij}$ and $p_{ij} = \sec \left( \frac{(i-j)\pi}{2N} \right)$.

Our aim is to analyse the quantum violations of the above Bell inequalities (1) for random pure states. Let us now summarise the main results which are derived above. Assuming that the best measurements are the same as for the maximally entangled and optimal state, we can analyse the quantum violations of the Bell inequality (1) by means of expression (3). As we will see in the next section for a random pure state (rs) the Schmidt coefficients $\{\lambda_i\}$ in (2) and (3) can be related to the square singular values of random Wishart matrices which are constrained to have $\sum_i \lambda_i = 1$. By using techniques from RMT we are able to calculate the expected value $\langle \mathcal{A}_N \rangle_{rs}$ of (3), where the expectation value is taken with respect to the measure $P_{rs}$ of random pure states. This is done analytically for both $N = 2$ and large $N$, while for intermediate values of $N$, we compute $\langle \mathcal{A}_N \rangle_{rs}$ numerically. The results are summarised in Figure 1. Shown is the expected minimal target value $\langle \mathcal{A}_N \rangle_{rs}$ under a random pure state in comparison to the minimal target value $A_N$ for the maximally entangled state, for $N$ varying from 2 to 500. The numerical data shows that the maximally entangled state always violates the Bell inequality [12], while for the random pure state one observes that the Bell inequality is only violated for $N \geq 8$. Using techniques from RMT we obtain analytically the value for $N = 2$ given by $\langle \mathcal{A}_2 \rangle_{rs} = 3/2 - 3\pi/(16\sqrt{2}) \approx 1.083$, as well as the asymptotic value as $N \to \infty$ given by $\langle \mathcal{A}_\infty \rangle_{rs} = 2 - 1024G/(9\pi^4) \approx 0.93$, with $G$ being Catalan’s constant. Below we also indicate how to analytically obtain $\langle \mathcal{A}_N \rangle_{rs}$ at finite $N$ together with the $k$-th moments. At this point we only note that the variance of $A_N$ under the measure of pure random states goes to zero as $N \to \infty$.

**Random matrix theory and random pure states** – For a bipartite system it is natural to define random pure states using the Fubini-Study measure which is essentially a random unitary matrix acting on some fixed reference state. In [9] it was shown that such a random pure state has Schmidt coefficients distributed according to the joint probability distribution function (jpdf),

$$P_{rs}(\{\lambda_i\}) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^2 \delta \left( \sum_{i=1}^{N} \lambda_i - 1 \right)$$

where the partition function $Z_N$ is the normalisation such that $\int_0^\infty P(\{\lambda_i\}) \prod_i d\lambda_i = 1$. One way to arrive at the above expression is to note that the Fubini-Study measure implies that an (unnormalised) random pure state is given by $|\psi\rangle = \sum_{i=1}^{N} X_{ij} |i\rangle_A \otimes |j\rangle_B$ where $X$ is a complex Wishart matrix [16], i.e. real and imaginary parts of all entries are i.i.d. normal random variables. It is then simple to show that the reduced density matrix for either sub-system takes the form,

$$\rho_A = \rho_B = \frac{XX^\dagger}{\text{tr}(XX^\dagger)}.$$

Since $\rho_A = \sum_i \lambda_i |i\rangle_A \langle i|_A$ and $\rho_B = \sum_i \lambda_i |i\rangle_B \langle i|_B$, we identify the (unnormalised) Schmidt coefficients with the square singular values of $X$ which leads to (4).
Our primary interest is in how $\langle A_N \rangle_{rs}$ varies with $N$. We begin writing down the expectation value of $A_N$ with respect to the pdf, from (3) one gets
\[
\langle A_N \rangle_{rs} = 2 - \frac{1}{N} - \frac{2}{N} \sum_{i<j} P_{ij} \langle \sqrt{\lambda_i \lambda_j} \rangle_{rs}.
\]
We first consider the case $N = 2$, which from the above discussion has the integral representation,
\[
\langle A^2 \rangle_{rs} = \frac{3}{2} - \frac{1}{Z_{rs}} \sum_{i<j} P_{ij} \int_0^1 d\lambda (\lambda_1 - \lambda_2)^2 \sqrt{\lambda_1 \lambda_2}.
\]
where we note that due to the permutation symmetry of the jpdf we may factor the integral out of the sum over $i$ and $j$. Evaluating this expression yields,
\[
\langle A^2 \rangle_{rs} = \frac{3}{2} - \frac{3\pi}{16\sqrt{2}}
\]
as announced above. The behaviour of $\langle A_N \rangle_{rs}$ as $N \to \infty$ can be found by a saddle point analysis of (4). Central to this approach is the eigenvalue density $\rho(\lambda, N) = \langle \delta(\lambda - \lambda_i) \rangle_{rs}$ of the pdf (4) for large $N$. From the discussion surrounding equation (5) one sees that the spectral density is exactly the Marchenko-Pastur distribution \cite{7} in the case of square matrices. Another route to obtain the eigenvalue density is to start from the partition function $Z_N$ associated to (1) and write the constraint as a Lagrange multiplier in the effective action. The spectral density can then be found using a saddle point analysis of the resulting Coulomb gas model. Yet another alternative is to note that the pdf (4) is known in the literature as a fixed-trace ensemble and has been studied in the works \cite{6,9,18,19}. By considering a fixed trace $\sum_i \lambda_i = t$, taking the Laplace transform of expectation values with respect to $t$, rescaling the eigenvalues, transforming back and setting $t = 1$, one finds the relation,
\[
\langle \prod_{i=1}^r \lambda_i^{\eta_i} \rangle_{rs} = \frac{\Gamma(N^2)}{\Gamma(N^2 + \eta)} N^\eta \left( \prod_{i=1}^r \lambda_i^{\eta_i} \right)_{\text{LUE}}
\]
where $\eta_i \in \mathbb{R}$, $\eta = \sum_{i=1}^r \eta_i$ and the expectation on the right-hand-side is with respect to the Laguerre ensemble defined by the jpdf,
\[
\mathbb{P}_{\text{LUE}}(\{\lambda_i\}) = \frac{1}{Z_{\text{LUE}}} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N \sum_i \lambda_i}.
\]
The spectral density of the Laguerre ensemble is well known and is easily obtained using loop equation techniques. This together with (9) then gives,
\[
\rho(\lambda, N) = N \mu(N\lambda),
\]
and $\mu(x) = 0$ otherwise (see also \cite{9}). Let us remark that the fact $\rho(\lambda, N) \approx N \mu(N\lambda)$ shows the spacing between eigenvalues is of order $1/N$ as expected.

We now proceed by utilising the invariance of the jpdf under permutations of the eigenvalues which allows (6) to be written as,
\[
\langle A_N \rangle_{rs} = 2 - \frac{1}{N} +
\]
\[
- \left( \frac{2}{N^2} \sum_{i<j} P_{ij} \right) \left( \sum_{k=1}^N \frac{1}{N-1} \sum_{i \neq k} N \sqrt{\lambda_k \lambda_i} \right)_{rs}
\]
For $N$ large we make the following approximations,
\[
\frac{1}{N^2} \sum_{i<j} P_{ij} \approx \int_0^1 dx \int_0^x dy \sec \left( \frac{\pi(x-y)}{2} \right) = \frac{8G}{\pi^2}
\]
where $G$ is Catalan’s constant and
\[
\frac{1}{N^2} \sum_k \sum_l \sum_{i \neq j} N \sqrt{\lambda_k \lambda_i} \approx \int_0^\infty \int_0^\infty \mu(x) \mu(y) \sqrt{xy} dxdy = \frac{64}{9\pi^2}.
\]
This leads to the result, already announced above,
\[
\langle A_N \rangle_{rs} = 2 - \frac{1024G}{9\pi^4}.
\]
Finally let us briefly sketch an approach to computing $\langle A_N \rangle_{rs}$ at finite $N$ together with the $k$-th moments $\langle A_N^k \rangle_{rs}$. Consider the $k$-th moment $\langle A_N^k \rangle_{rs}$. From (3) it takes the form,
\[
\langle A_N^k \rangle_{rs} = \sum_{i_1, i_2, \ldots, i_{2k}} \prod_{j=1}^k M_{i_{2j-1}, i_{2j}} \left( \prod_{j=1}^k \lambda_{i_{2j-1}}^{i_{2j}} \right)_{rs}.
\]
Let $P_r$ denote the set of partitions of the set $1, 2, \ldots, r$ and $\sim$ the associated equivalence relation. Then we may write,
\[
\langle A_N^k \rangle_{rs} = \sum_{p \in P_{2k}} \left( \sum_{(i_1, \ldots, i_{2k}) \in \Omega_p} \prod_{j=1}^k M_{i_{2j-1}, i_{2j}} \right) \times
\]
\[
\times \left( \prod_{i=1}^r \lambda_{i_{r/2}}^{i_{r/2}} \right)_{rs}.
\]
Here, $\Omega_p = \{(i_1, \ldots, i_{2k}) : i_l = i_{l'} \iff l \sim l'\}$ and $p_i$ is $i$-th part of the partition $p$. Note we have used the invariance of (4) under permutations of the eigenvalues. By using (9) and making the change of variables $\lambda_i = \zeta_i^2$, we have,
\[
\langle \prod_{i=1}^r \lambda_i^{\eta_i} \rangle_{rs} = \frac{\Gamma(N^2)}{\Gamma(N^2 + k)} N^k \left( \prod_{i=1}^r \lambda_i^{\eta_i} \right)_{O}
\]
where the second expectation value is computed in the \(O(-2)\) model with the restriction that \(\zeta_j > 0\). The general \(O(n)\) model has been studied in \cite{21, 22} where it was shown that all products of resolvents could be computed in a recursive scheme known as topological recursion. We can place our problem within this framework by again using the invariance under permutation to rewrite,

\[
\left\langle \prod_{i=1}^{N} \zeta_{p_i} \right\rangle_O = \frac{(N - |p|)!}{N!} \left\langle \sum_{j_1 \neq j_2 \neq \ldots \neq j_{|p|}=1}^{N} \prod_{j_i \neq j_2 \neq \ldots \neq j_{|p|}=1}^{N} \zeta_{p_i} \right\rangle_O .
\]  

(19)

By a standard inclusion-exclusion argument one can write the right hand side as products of traces. For example, if \(|p| = 3\), we would have,

\[
\begin{align*}
\left\langle \sum_{j_1 \neq j_2 \neq j_3} \zeta_{p_1} \zeta_{p_2} \zeta_{p_3} \right\rangle_O & = \left\langle \sum_{j_1, j_2, j_3=1}^{N} \zeta_{p_1} \zeta_{p_2} \zeta_{p_3} \right\rangle_O - \left\langle \sum_{j_1, j_2, j_3=1}^{N} \zeta_{p_1} \zeta_{p_2} \zeta_{p_3} \right\rangle_O
- \left\langle \sum_{j_1, j_2, j_3=1}^{N} \zeta_{p_1} \zeta_{p_2} \zeta_{p_3} \right\rangle_O
+ 2 \left\langle \zeta_{p_1} \zeta_{p_2} \zeta_{p_3} \right\rangle_O .
\end{align*}
\]

(20)

Finally we note that products of traces can be obtained directly from the asymptotic behaviour near infinity of products of resolvents.

Numerical simulations – The numerical analysis is based on \cite{5}. For a fixed value of \(N\) we sample \(i.i.d.\) normal random variables as entries of the \(N \times N\) complex Wishart matrix \(X\) and then numerically calculate the eigenvalues of \(XX^\dagger/\text{tr}(XX^\dagger)\). From this we obtain a sample of \(\{\lambda_i\}\) which is used to compute \cite{4}. Repeating this 1000 times and taking the sample average yields an estimate for \(\langle A_N \rangle_{\text{rs}}\). This procedure is done for \(N\) in the range of 2 to 500 in exponentially increasing increments, resulting in the data points in Figure \ref{fig:1}. It is checked that for \(N = 2\) and \(N = 500\) the values agree with the analytical predictions \cite{8} and \cite{5} respectively. Furthermore, for \(N = 100\) we also use the 1000 samples of \(\{\lambda_i\}\) to numerically check the functional form of the eigenvalue density \cite{11}. The above numerical analysis centred around \cite{5} is particularly useful to obtain numerical estimates of \(\langle A_N \rangle_{\text{rs}}\) for intermediate values of \(N\), alternatively for large \(N\) one can also simulate the Coulomb gas \cite{4} using the Metropolis algorithm.

Discussion – In this letter we obtain the expected quantum violations of the Bell inequality \cite{1} for \(N\)-dimensional random pure states, assuming the conjectured best measurement operators. The results, as summarised in Figure \ref{fig:4} show that for small values of \(N\) the Bell inequality is not violated, while for \(N \geq 8\) it is.

Using techniques from RMT, in particular the relation to fixed-trace ensembles, we obtain this analytically for \(N = 2\) and \(N \to \infty\) and numerically for an intermediate range of \(N\). Besides the expected value at large \(N\), using a relation to the \(O(n)\) model with \(n = -2\), we also indicate how to obtain the corresponding result at finite \(N\) together with the higher moments. Details of this calculation will be presented elsewhere. Let us finally comment on experimental implementations. Recent experiments already analyse the quantum violations of the Bell inequality \cite{1} for \(N = 3\) \cite{23}. When moving to larger \(N\) and having noise in the preparation of the state, the here presented results become potentially relevant.

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1. M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
2. C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53(4), 2046 (1996), quant-ph/9511030.
3. J. S. Bell, Physics 1, 195 (1964).
4. J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23(15), 880 (1969).
5. K. Zyczkowski, K. A. Penson, I. Nechita, B. Collins J. Math. Phys. 52, 062201 (2011), 1010.3570.
6. M. L. Mehta, Random Matrices (Academic Press, Third Edition, London 2004);
7. G. Akemann, J. Baik, P. Di Francesco, (Ed.) The Oxford Handbook of Random Matrix Theory, (Oxford University Press, 2011).
8. E. Lubkin, J. Math. Phys. 19, 1028 (1978); S. Lloyd and H. Pagels, Ann. Phys. (N.Y.) 188, 186 (1988).
9. D. N. Page, Phys. Rev. Lett. 71 1291 (1993).
10. C. Nadal, S. N. Majumdar, and M. Vergassola, Phys. Rev. Lett. 104, 110501 (2010), 0911.2844.
11. D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88(4), 040404 (2002), quant-ph/0106024.
12. S. Zohren and R. Gill, Phys. Rev. Lett. 100, 120406 (2008), quant-ph/0612020.
13. A. Acin, R. Gill, and N. Gisin, Phys. Rev. Lett. 95, 210402 (2005), quant-ph/0506225; A. Acín, T. Durt, N. Gisin, and J. Latorre, Phys. Rev. A 65, 052325 (2002), quant-ph/0111143; J.-L. Chen, C. Wu, L.C. Kwek, C.H. Oh, and M.-L. Ge, Phys. Rev. A 74, 032106 (2006), quant-ph/050722.
14. S. Zohren, P. Reska, R. D. Gill, W. Westra, Europhysics Lett. 90, 10002 (2010), 1003.0616.
15. T. Durt, D. Kaszlikowski, M. Żukowski, Phys. Rev. A 64, 024101 (2001), quant-ph/0101084.
[16] J. Wishart, Biometrika **20A**, 32 (1928); Erratum ibid. **20A**, 424 (1928).

[17] V. A. Marchenko and L. A. Pastur, Math. Sb. **72**, 507 (1967).

[18] H.-J. Sommers and K. Zyczkowski, J. Phys. A: Math. Theor. **37**, 8457 (2004); H. Kubotani, S. Adachi, and M. Toda, Phys. Rev. Lett. **100**, 240501 (2008); S. Adachi, M. Toda, and H. Kubotani, Ann. Phys. **324**, 2278 (2009).

[19] G. Akemann, P. Vivo, J. Stat. Mech. **1105** P05020 (2011),1103.5617; G. Akemann, G. M. Cicuta, L. Molinari, G. Vernizzi, Phys. Rev. E **60** 5287 (1999), cond-mat/9904446; Phys. Rev.E **59** 1489 (1999), cond-mat/9809270.

[20] S. N. Majumdar, O. Bohigas, and A. Lakshminarayan, J. Stat. Phys. **131**, 33 (2008); M. Znidaric, J. Phys. A: Math. Theor. **40**, F105 (2007); Y. Chen, D.-Z. Liu, and D.-S. Zhou, J. Phys. A: Math. Theor. **43**, 315303 (2010).

[21] G. Borot, B. Eynard, J. Stat. Mech. **2011** P01010, (2011), 0910.5896.

[22] B. Eynard, C. Kristjansen, Nucl. Phys. B **455** 577, (1995), hep-th/9506193.

[23] C. Bernhard, B. Bessire, A. Montina, M. Pfaffhauser, A. Stefanov, S. Wolf, (2014), 1402.5026.