We investigate Floquet Topological Insulators in the presence of spatially-modulated light. We extend on previous work to show that light can be used to generate and control localized modes in the bulk of these systems. We provide examples of bulk modes generated through modulation of different properties of the light, such as its phase, polarization and frequency. We show that these effects may be realized in a variety of systems, including a zincblende and honeycomb models, and also provide a generalization of these results to three dimensional systems.

PACS numbers: 73.23. b, 03.65.Vf, 73.43. f

I. INTRODUCTION

Topological insulators have attracted great interest in recent years. These materials are predicted to display many interesting effects, such as fractionalized excitations and gapless boundary properties. Yet, despite intense experimental efforts, only a handful of realizations of intrinsic topological insulators are currently known\cite{1-10}. As a consequence, many recent proposals have focused on methods to engineer systems with topological properties.

One such proposal that has attracted significant attention was suggested by Lindner et. al\cite{11}, who demonstrated that time periodic perturbations can generate topological characteristics. This may be achieved, for example, by shining light on a conventional insulator. These systems, named “Floquet Topological Insulators” (FTIs), are predicted to display insulating behavior at the bulk that co-exists with metallic conductivity at the edges. In addition, FTIs are predicted to display many intriguing effects such as Dirac cones in three dimensions\cite{12} and Floquet Majorana fermions\cite{13} in superconductors. Proposals for FTIs include a wide range of solid state and atomic realizations\cite{11,14,15}. The direct observation of protected edge modes in photonic crystals\cite{16,17} has demonstrated that these proposals have experimental realizations which may lead to future practical applications.

In previous work\cite{18} we studied FTIs when the light is not uniform in space. We found that spatial modulation of the light can give rise to interesting effects in these so called “modulated FTIs”. For example, we studied a zincblende model driven at resonance by linearly polarized light, and found that domain walls and vortices in the phase of the light can give rise to localized modes and fractionalized excitations in the bulk of this system.

In this paper, we extend these results to much more general conditions than previously considered. In addition to configurations involving modulation in the phase of the light, we provide new schemes to induce localized modes in the bulk of FTIs where the frequency and the polarization of the radiating light vary in space. We establish these results for systems that are driven both on and off resonance by the light. In addition, we demonstrate that these effects not only apply to insulators of the zincblende type but can be also be generalized to semimetals like graphene. Furthermore, we provide the first example of a three-dimensional modulated FTI. These results demonstrate the versatility of modulated FTIs, and may have practical applications in photonic crystals and solid state devices.

This article is organized as follows: In Sec.\ II we briefly review the concept of FTI and provide a short description of the phenomenology of these phases. In Sec.\ III we summarize previous results for a square lattice zincblende model irradiated by on-resonance light, and extend the analysis to light with space dependent frequency. We also provide an analysis of the effect of particle-hole symmetry breaking. In Sec.\ IV we introduce modulated FTIs in graphene irradiated by light with on-resonance and off-resonance frequencies. In particular, we study the effect of light with space-dependent phase and polarization. In Sec.\ V we provide an extension of modulated FTI to a three dimensional model of a cubic lattice irradiated by on-resonance light with linear polarization.

Figure 1: $\hat{n}_k$, defined in Eq. (7), for the zincblende model, Eq. (10), with linearly polarized light, $\alpha = 0$. The arrows represent the $x$ and $y$ components of $\hat{n}_k$ and the color map shows the $z$ component. Note that $\hat{n}_k$ is in a hedgehog configuration, as it wraps the unit sphere exactly once. This corresponds to $C_F = 1$. 

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arXiv:1309.0203v2 [cond-mat.str-el] 17 Jan 2014
II. GENERAL DESCRIPTION

Let us begin by reviewing the concept of Floquet Topological Insulators (FTI). For simplicity, consider a $2 \times 2$ Bloch Hamiltonian in two dimensions

$$
\hat{H}_k = \tilde{d}_k \cdot \tilde{\sigma} + \epsilon_k I_{2 \times 2} \tag{1}
$$

where $\tilde{\sigma}$ denotes a vector of the Pauli matrices. Equation (1) has two energy bands for each $k$ value, with $E_k = \epsilon_k \pm |\tilde{d}_k|$. If $\tilde{d}_k \neq 0$ everywhere on the Brillouin zone, we can use the TKNN formula to define the topologically-invariant Chern number of the occupied (lower energy) band.

$$
C = \frac{1}{4\pi} \int_{BZ} d^2k \hat{d}_k \cdot \left( \partial_{k_x} \hat{d}_k \times \partial_{k_y} \hat{d}_k \right) \tag{2}
$$

Equation (2) counts the number of times $\hat{d}_k$ wraps around the unit sphere as $\hat{k}$ runs over the Brillouin zone. Physically speaking, $C$ gives the net number of chiral modes at the edge of the sample.

As shown in Ref.11, even in cases where $C = 0$, it is possible to induce topological properties by perturbing the system in a time-periodic fashion,

$$
H(t) = \hat{H}_k + V(t), \tag{3}
$$

where $V(t + \tau) = V(t)$. The time evolution is then given by the Floquet theorem, which states that the solutions of the Schrödinger equation can be written as $\psi(t) = \sum_a e^{i \varepsilon_a t} \varphi_a(t)$, with $\varphi_a(t) = \varphi_a(t + \tau)$. The quasi-energies, $\varepsilon_a$, are conserved quantities that are defined modulo $\omega = \frac{2\pi}{\tau}$. They describe the evolution of states over a full cycle.

The states $\varphi_a$ and their corresponding quasi-energies $\varepsilon_a$ satisfy the eigenvalue problem,

$$
H_F \varphi_a(t) = \varepsilon_a \varphi_a(t) \tag{4}
$$

where the “Floquet Hamiltonian” $H_F$ is defined as $C_F$

$$
e^{-iH_F\tau} \equiv U(t + \tau, t) \tag{5}
$$

and $U(t + \tau, t)$ is the time evolution operator over a full cycle,

$$
U(t + \tau, t) = T \left\{ \exp \left( -i \int_{t}^{t+\tau} H(t') dt' \right) \right\}. \tag{6}
$$

Here, $T$ is the time ordering operator. In this paper, we evaluate the time evolution operator numerically by discretizing the time interval $t \in [0, \tau]$, $\tau = \frac{2\pi}{\omega}$ and computing the time-ordered product of $e^{-iH(t)\Delta t}$ over the sub-intervals $\Delta t$. We then extract the Floquet Hamiltonian by computing the logarithm of the operator $U$.

For two level systems, the Floquet Hamiltonian can be written most generally as

$$
H_F = \tilde{\sigma} \epsilon_k I_{2 \times 2} \tag{7}
$$

Provided that $\epsilon_k$ does not vanish over the Brillouin zone, we can define a new topological invariant $C_F$ associated with $H_F$,

$$
C_F = \frac{1}{4\pi} \int_{BZ} d^2k \hat{d}_k \cdot \left( \partial_{k_x} \hat{d}_k \times \partial_{k_y} \hat{d}_k \right) \tag{8}
$$

A Floquet topological insulator is characterized by non-vanishing $C_F$. Similarly to time-independent topological insulators, a non-zero value of $C_F$ implies the existence of topologically-protected chiral states at the edge of the sample.

We note that, whereas $C_F \neq 0$ is a sufficient condition for a system to be topological, it is not necessary in the case of Floquet insulators. The value of $C_F$ assigned to a given band counts the difference between the number of right and left moving modes in a given edge above the band, minus the difference between right and left moving modes in the same edge below that band. In the Floquet spectrum, quasi-energy is periodic. Then, for example, it is possible for every band in the system to have a right moving mode both above and below itself. In this situation, every band has $C_F = 0$, but the system is nevertheless topological. The full characterization of edge states in two-dimensional FTIs requires the introduction of an extra topological invariant in addition to $C_F$, as discussed in detail in Ref.12 where the new invariant is called $W$. In this paper we do not consider situations where only $W$ is modulated in space (and $C_F$ is not), although it would be interesting to consider such situations.

III. ZINCBLENDE MODEL

A. Previous results

In this section we give an explicit example of an FTI. The results presented here can be found in Refs.11 and 13. Consider the Hamiltonian

$$
H_k = \begin{pmatrix} \tilde{H}_k & 0 \\ 0 & \tilde{H}_{-k}^* \end{pmatrix}. \tag{9}
$$

where $\tilde{H}_k$ is given by Eq. (1), with

$$
\tilde{d}_k = (A \sin k_x, A \sin k_y, M + 2B (\cos k_x + \cos k_y - 2)) \tag{10}
$$

and $A, B, M$ are constants. This model can describe, for example, HgTe/CdTe quantum wells. In this case $\tilde{H}_k \left( \tilde{H}_{-k}^* \right)$ acts on the subspace spanned by the $J_z =$
\[
\left(\frac{1}{2}, \frac{3}{2}\right) \text{ and } \left(\frac{3}{2}, \frac{1}{2}\right)\]
states respectively and \(\vec{d}_k\) describes the dispersion of the bands including the effects of spin-orbit interaction.

Note that \(\hat{H}_k\) and \(\hat{H}_{-k}\) are related by a time reversal (TR) transformation, such that Eq. \(9\) is TR invariant. This implies that the overall Chern number is zero. However, it is still possible to obtain a TR protected topological phase in which the Chern number assigned to the \(2 \times 2\) block \(\hat{H}_k\) is nonzero. Explicit calculation yields that \(C = \frac{1}{2} (1 + \text{sign} \left(\frac{\omega}{M}\right))\). We work in the parameter space \(M > 0, B < 0\), for which \(\hat{H}_k\) is trivial, and we add a time-periodic potential in order to induce the topology.

As the time dependent potential, we use perturbations that do not connect the two Hamiltonian blocks and perform the analysis for the \(2 \times 2\) block \(\hat{H}_k\). We choose

\[
H(t) = \vec{d}_k \cdot \vec{\sigma} + \epsilon_k I_{2 \times 2} + \vec{V}_k \cdot \vec{\sigma} \cos(\omega t + \alpha) \tag{11}
\]

where \(\alpha\) is the delay phase of the external perturbation and \(\omega\) is its frequency. For simplicity, we take \(\vec{V}_k = V_0 \hat{z}\) in what follows. Equation \(11\) can describe, for example, the effect of linearly polarized light in HgTe/CdTe valence and conduction bands of the Floquet problem.

The analogy is imperfect; while in an actual superconductor in the \(\frac{\omega}{\pi} \geq M\) region a gap opens as a result. Hence, the spectrum of Eq. \(12\) will match that of the corresponding pSC, but the Hilbert spaces are different. The nature of the states in the two cases is related by a particle-hole transformation.

Notice that in Eq. \(12\), the delay phase \(\alpha\) plays the role of the superconducting phase. Correspondingly, space modulation of \(\alpha\) leads to many intriguing effects. For example, in a domain wall configuration, in which the phase \(\alpha\) shifts by \(\pi\) at \(y = 0\), localized modes with zero quasi energy appear in the vicinity of the domain wall. Similarly, in a vortex configuration, in which the phase \(\alpha\) winds by \(2\pi\) about a point, a state with zero quasi-energy and fractional charge is localized at the vortex core. The quasi-stationary modes are analogous to the well known zero modes of \(\pi\) junctions and pSC vortices.

The above results can be reproduced with circularly polarized light, which breaks TR explicitly. In this case, the light can be described by a vector potential \(\vec{A}(t) = A_0 (\cos(\omega t + \alpha), \pm \sin(\omega t + \alpha), 0)\), where \(\pm(+)\) denotes left-handed (right-handed) polarization. \(\vec{A}(t)\) is then implemented in the Hamiltonian by the minimal substitution \(\vec{k} \rightarrow \vec{k} + \vec{A}\). The resulting low-energy Floquet Hamiltonian is identical to Eq. \(12\), up to a constant shift in the initial phase of the light. We therefore conclude that the presence of gapless modes is unaffected by breaking of TRS, provided that the two Hamiltonian blocks in Eq. \(9\) do not mix.

**B. Position-Dependent frequency**

As we now demonstrate, localized bulk modes can also be generated by light whose frequency is varying in space. We first consider a simple situation in which two frequencies \(\omega\) and \(\omega'\) are present on the two halves of the system, and take \(\omega/\omega'\) to be a rational number, such that the Floquet theorem can be used. As a first example, we take \(\omega = 2.7\) and \(\omega' = \frac{\omega}{2} = 1.35\) with time-independent parameters \(A = -B = 0.05, M = 1, V_0 = 1\). In this case \(\omega\) is on resonance and \(\omega'\) is nominally off-resonance. However, \(\omega'\) can induce two-photon resonances between valence and conduction band states, and we find that in the \(\omega'\) region a gap opens as a result. Hence, the system is a Floquet insulator throughout. Furthermore, we find that \(C_F = 1\) on both regions of the sample. Yet, despite the fact that \(C_F\) is constant, we obtain a pair of modes with zero quasi-energy that are localized at the interface, see Fig. 2.

The mechanism behind this result is similar to that described Ref. \(13\) to explain the phase domain modes. Since momentum \(k_z\) in the direction parallel to the domain wall is a good quantum number, the system can be analyzed for each \(k_z\) value separately and one can define a \(k_z\)-dependent topological invariant, \(C_{k_z}'\), as the winding number of \(\vec{n}_k\) in the \((n_x, n_y)\) plane. We find by direct calculation that \(C_{k_z}'\) has opposite signs in the \(\omega\) and \(\omega'\) regions. The localized states are therefore topologically
protected, provided particle-hole and reflection symmetries are present. It is interesting to ask whether these effects can be seen in a scenario where the frequency of the light varies continuously with position. In practice, such a configuration could be realized, for example, by passing a broad spectrum light source through a prism. To test this, we chose a position-dependent frequency of the form \( \omega(x) = \frac{\omega_0}{2} + \frac{\omega_0}{2} \tanh \left( \frac{x}{\lambda} \right) \), such that the frequency interpolates between \( \omega \) and \( \omega' \) smoothly over a region of width \( \lambda \). In this case, the Floquet theorem no longer is valid, since the system is only quasi-periodic and quasi-energies are no longer conserved. However, it is still possible to evaluate the time evolution \( U(T,0) \) for a very long time \( T \) and search for localized eigenstates of \( U \). In all cases that we examined, we found that the localized interface modes survive when \( \lambda \) is smaller than the lattice constant, but that for larger \( \lambda \) these modes can no longer be discerned, indicating that in practice this effect can only be seen for sharp jumps in frequency. By contrast, by this procedure we find that the edge modes are robust even when \( \lambda \) is comparable with the system size.

### C. Particle-hole symmetry breaking

The existence of quasi-stationary modes in the bulk of the zincblende model was found to rely on the presence of PHS. However, in real systems PHS is only approximate. Thus, it is natural to ask how these results are affected by breaking of this symmetry. To answer this question, we add a PHS breaking term

\[
\epsilon_k = -\epsilon_{ph}(\cos k_x + \cos k_y - 2)
\]

and consider \( \epsilon_{ph} \leq 2|B| \), such that the time-independent system remains gapped. Figure 3(a) shows that localized sub-gap states survive the breaking of this symmetry, and that the system remains topologically non-trivial. The edge modes remain gapless in this case. However, a small gap opens in the Floquet spectrum of the domain-wall modes. This gap is too small to be seen in Fig. 3(a). The dependence of the gap on \( \epsilon_{ph} \) is shown in the inset of Fig. 3(a). This small gap may be overridden by thermal fluctuations or a small external bias. Thus, experiments carried out at temperatures above this gap will not be sensitive to PHS breaking.

The vortex core state in the vortex configuration shows higher degree of robustness to PHS breaking than the domain wall modes. This robustness is a result of the vortex core state being well separated in energy, as well as in space, from the remaining of the spectrum. Specifically, we find that weak PHS breaking shifts the quasi-energy of the bound mode away from zero. However, the mode remains a mid-gap state that is well separated from the remaining of the spectrum. Moreover, the topological properties of this excitation, such as its charge, remain fractional and are unaltered by breaking of PHS.

### IV. HONEYCOMB LATTICE MODEL

The generation of FTI is not limited to band-insulators. In particular, previous work argued that on-resonance and off-resonance light can both induce topological properties in graphene and in photonic crystals based on honeycomb lattices. From an experimental point of view, Floquet topological insulators have been realized in photonic crystals for a honeycomb lattice with helical wave guides, see Ref. 15.

We will now demonstrate that the modulated FTI can be realized in a honeycomb lattice. We model the system by the tight binding approximation. The Hamiltonian is given by Eq. (1) with \( \epsilon_k = 0 \) and \( \delta_k = t(-\Re f_k, \Im f_k, 0) \), where \( f_k = 1 + 2e^{\pm i k_x a} \cos \left( \frac{\sqrt{3}}{2} k_y a \right) \). Here, \( a \) is the interatomic distance and \( t \) is the hopping parameter. The spectrum consists of conduction and valence bands with
Dirac cones that intersect at two Fermi points, \( K_{\pm} = \left( \frac{\pi}{\sqrt{3} a}, \pm \frac{\pi}{3\sqrt{3} a} \right) \). \( K_{\pm} \) are commonly referred to as “valleys”, and are related to each other by time reversal. Note that \(|f_{\pm}|^2 \leq 3t\), such that the bandwidth is \(6t\). The system is invariant under particle-hole, spatial inversion, and time reversal symmetries\(^{25}\).

The generation of topological insulators requires the formation of a gap in the energy spectrum. In the studied model, the two Fermi points are protected by spatial inversion and time-reversal symmetries. The formation of FTI therefore requires the breaking of either of these symmetries. As a result, linearly polarized light which does not break TR nor spatial inversion cannot induce a gap or generate topological properties. By contrast, circularly polarized light induces a gap of opposite masses on each of the two Dirac cones, thus creating a topological Floquet spectrum\(^{14,15,26–29}\).

1. Off-Resonance Light

Let us first examine the effect of uniform light with off-resonance frequency (\(\omega > 6t\))\(^{14}\). We solve for the Floquet Hamiltonian with periodic boundary conditions. Figure 4 shows that for left circularly polarized light, \(\hat{n}_k\) points towards the north pole at the \(K\)–valley and towards the south pole at the \(K\)\(^+\) valley. Surrounding these points, the \(x\) and \(y\) components of \(\hat{n}_k\) form a vortex and an anti-vortex, respectively. This indicates that the Dirac points pick up opposite masses, and that the Chern number is \(C_F = 1\). Similarly, for right-polarized light \(C_F = -1\), as expected from TRS invariance.

The topological nature of the system is reflected in the presence of chiral edge modes, as illustrated in Figure 4 which shows that the spectrum for a ribbon in the armchair configuration. The spectrum includes two quasi-stationary modes, localized at each of the two edges. Similar results are obtained for a ribbon in a zig-zag configuration.

Motivated by earlier results, we searched for quasi-stationary states both at a domain wall and also at a vortex in the phase of the light. However, we found through numerical simulations that when the light is off-resonance, neither one of these configurations induce quasi-stationary states in the bulk.

In order to explain these results, we evaluate the Floquet Hamiltonian analytically near the Dirac points. The low energy Hamiltonian of the honeycomb lattice model consists of two blocks, each describing the system near a valley (\(\tau_z = \pm 1\)),

\[
H_k = -v_f \left( \sigma_x k_x + \sigma_y \tau_z k_y \right) \tag{14}
\]

where \(v_f = \frac{2\pi}{3}ta\) is the Fermi velocity. We describe the effect of the radiating light by the minimal substitution, \(H_{\tilde{k}} \rightarrow H_{\tilde{k} + \tilde{A},}\) where \(\tilde{A}(t) = A_0 (\cos (\omega t + \alpha), \sin (\omega t + \alpha), 0)\) for LH polarized light.

The effect of off-resonance light (sec15) can be described by a static Floquet Hamiltonian, with a Floquet spectrum described in terms of a “dressed” energy spectrum. In this case, the RWA can not be used to estimate the Floquet Hamiltonian and a different approach is required. We use perturbation theory and expand the time evolution operator, Eq. 6, as a series in the small parameter \(\frac{A_0^2}{\omega^2}\). For uniform light, this procedure yields

\[
H_F = H_0 + \frac{1}{\omega} [H_1, H_{-1}] + \frac{1}{\omega^2} \left( e^{i\alpha} [H_0, H_1] - e^{-i\alpha} [H_0, H_{-1}] \right) + \mathcal{O} \left( \frac{A_0^2}{\omega^2} \right) \tag{15}
\]

where \(H_n = \frac{1}{\omega} \int_0^n e^{-i\omega t} H(t) dt\) is the \(n^{th}\) Fourier coefficient of the Hamiltonian. Thus \(H_0\) is given by Eq. 14 and

\[
H_{\pm 1} = A_0 (\sigma_x \pm i\sigma_y) \tag{16}
\]

Thus, the Floquet Hamiltonian is

\[
H_F = -v_f \left( \sigma_x k_x + \tau_z \sigma_y k_y \right) + \tau_z \sigma_z m \tag{17}
\]

\[\text{Figure 4: (a) \(\hat{n}_k\) over the first Brillouin zone for the honeycomb lattice model irradiated by left handed polarized light with off-resonance frequency (\(\omega = 6.5t\)). The colors denote the magnitude of \(\hat{n}_x\) and the arrows are the direction of \(\hat{n}_x\) and \(\hat{n}_y\). The X markers denote north poles and the circles denote south poles. There is one north pole, with vorticity} 1 \text{ and one south pole with vorticity} -1. \text{This corresponds to} C_F = 1. \text{(b) The Floquet spectrum. The blue (dashed green) lines correspond to states localized at the right (left) edge. The inset gives an illustration of a honeycomb lattice in an Armchair configuration.}\]
where \( m = v_F^2 \frac{\Delta^2}{\omega} \) is the effective mass at the two valleys and \( \Sigma_{k,\alpha} = v_F A_0 (k_x \cos \alpha + k_y \sin \alpha) \) is a linear term which can be absorbed into an \( \alpha \)-dependent shift in the position of the Dirac points. Note that the Floquet Hamiltonian is otherwise independent of \( \alpha \). As a result, no localized modes are generated by space modulation of the phase \( \alpha \). In a similar manner, since modifying \( \omega \) does not affect the sign of \( m \) provided \( \omega \) is off-resonance, no quasi-stationary modes are induced by light radiation with space-modulated frequency.

By contrast, if we make a domain wall in the polarization of the light, that is, an interface between left and right polarizations, this results in two chiral modes at the interface that propagate in the same direction. This can simply be understood in terms of the change in Chern number between the two domains, as shown in Fig. 5. Such a configuration can be created, for instance, in a photonic crystal, by using wave guides with opposite helicities.

In Ref.\(^{18}\) we found that imposing a slowly-varying phase twist results in a current analogous to the Josephson effect, \( j = \rho_0 \nabla \alpha \). This result also holds here, where we find numerically that a slow modulation in space of \( \alpha \) yields a DC current proportional to \( \nabla \alpha \).

### 2. On-Resonance Light

We will now extend the analysis to on-resonance light. For concreteness we will consider light frequencies in the range \( 3t < \omega < 6t \), that is, frequencies that only allow a single photon resonance. The frequencies of the radiating light in this scenario are small compared to the off-resonance case. This enables the use of less energetic photons in experiments, thus opening a possibility for additional realizations of our results in condensed matter systems. For example, recent work\(^{27}\) demonstrated that resonant light can induce controlled gapless modes in graphene.

The Chern number for left-polarized light is \( C_F = 3 \), as seen from the winding number of \( \tilde{n}_k \) in Figure 6. The change in Chern number relative to the off-resonance case is due to the folding in energy for states with momentum on a circle around the \( \Gamma = (0,0) \) point. We demonstrate this topological nature by evaluating the spectrum on a ribbon in an armchair configuration. Figure 6 shows that three quasi-stationary modes are now localized at each boundary. Focusing on one edge, there are two right-moving modes with quasi-energy \( \frac{\omega}{2} \) and one left-moving mode with zero quasi energy.

We now allow the light to vary in space. As a first example, we examine an interface between right and left handed polarized light. As in the off-resonance case, the spectrum of this setup includes gapless domain modes, which are a direct result of the change in Chern numbers at the interface. In this case, we find three right movers and three left movers at the interface, see Fig. 6.

Next, we consider a vortex configuration, with \( \alpha = \arctan (\frac{y}{x}) \). In this configuration, the phase \( \alpha \) winds by \( 2\pi \) about the origin, which is taken to be at the center of the sample. Numerical simulations show that in addition to the edge modes, the Floquet spectrum now in-
corresponds to a $\pi$ phase shift in the light, without a change in the polarization. As in the vortex configuration, the spectrum includes localized states with finite energy gap. If instead the domain wall is smoothened, $A = A_0 \tanh \left( \frac{t}{\lambda} \right) \left( \cos \omega t, \sin \omega t, 0 \right)$, the energy gap decays exponentially with $\lambda$, as shown in Fig. 8. For example, for $\lambda = 14a$ we find $E_{\text{domain-wall}} = 6 \times 10^{-10}$.

In order to explain these results we evaluate the Floquet Hamiltonian around the $\Gamma$ point. For resonant light, we can use the RWA for this purpose. We find that up to a unitary transformation ($\psi \to e^{ik_0 \sigma^y \psi}$) the Floquet Hamiltonian is (in the $\left( \hat{z}, \hat{x}, \hat{y} \right)$ coordinate basis)

$$H^\Gamma_F = 3t \left( \frac{z^2 - \mu}{\Delta_k e^{-i\omega t}} - \frac{\hat{x}^2}{2m} + \mu \right)$$

where $\Delta_k = \frac{\lambda}{m} \left( k_x - i k_y \right)^2$, $m^{-1} = \frac{1}{4} \left( \frac{\lambda}{\pi} - 3 \right)$ and $\mu = 1 - \frac{\lambda}{2}$. Equation (19) is analogous to a $d + id$ SC. For $\omega < 6t$, the chemical potential is positive and $H^\Gamma_F$ is topological. It supports two topologically protected chiral modes at the system edges. These modes have quasi-energy $\omega/2$, as opposed to the mode arising due to the gap at the valleys, which has quasi-energy 0. When $\omega > 6t$, the chemical potential becomes negative and Eq. (19) corresponds to a trivial insulator. Then, only the mode due to the gap at the valleys survive.

The existence of the quasi-stationary modes can be understood through the $d + id$ SC analogy. In the vortex configuration, a vortex core state is analogous to the $d + id$ SC Caroli-de-Gennes-Matricon vortex states (see Ref.\textsuperscript{[59]}) with an energy gap that is proportional to $\frac{1}{\lambda}$.

In the domain wall configuration, the BdG equations have four quasi-stationary solutions. These modes are analogous to the $d$-wave $\pi$-junction modes, see Ref.\textsuperscript{[61]} with linear low energy dispersion. We find that in the step function configuration these states have a finite gap, while in the smooth domain wall setup the gap decays exponentially with $\lambda$.

V. THREE DIMENSIONAL SYSTEMS

In this section we generalize our results to a three dimensional model. In the case of 3d FTI, the localized surface modes form an odd number of Dirac cones. Following Ref.\textsuperscript{[22]}, we first present a brief description of the formalism for a 3D FTI. Consider a general $4 \times 4$ trivial insulator, described by the Hamiltonian

$$H_k = D^\mu_k \gamma^\mu + \epsilon_k I \ .$$

Here, $D^\mu_k$ is a 4-vector that depends on the 3d lattice momentum $k$, and $\gamma^\mu = \left( \gamma^1, \gamma^2, \gamma^3, \gamma^4 \right)$ is a vector composed of the four Dirac matrices, given by $\gamma_i = \sigma_i \otimes \tau_z$ for $i = 1, 2, 3$ and $\gamma_4 = I \otimes \tau_z$. In addition, we define $\gamma_5 = I \otimes \tau_y$. This Hamiltonian describes an insulator provided that $D^\mu_k$ does not vanish anywhere on the Brillouin zone. We take $D^\mu_k = \left( \hat{d}_k, \hat{d}_k^\dagger \right)$ with $\hat{d}_k$ odd and $\hat{d}_k^\dagger$ even.
In the low energy limit, Eq. (20) is invariant under both time reversal and space inversion symmetries. For simplicity we first consider the situation where there is particle-hole symmetry, such that $\varepsilon_k = 0$.

According to the topological classification of 3d systems, time-reversal topological insulators are characterized by a $\mathbb{Z}_2$ topological invariant. In the case of Eq. (20), a natural topological invariant is given by a generalization of the TKNN formula to higher dimensions. Here, the Brillouin zone is a three dimensional torus, $T^3$, and $D_k$ is a four component unit vector that lies on $S^3$. $D_k$ can therefore be described as a map, $D : T^3 \rightarrow S^3$. We define the topological invariant $\chi$ as the degree of this map, which is the number of times $D_k$ wraps around the unit sphere $S^3$ as $k$ runs over the Brillouin zone. When PH and TR symmetries are both present, $\chi$ is only defined modulo 2.

As an example of a 3d FTI, we consider a cubic lattice model, with

$$D_k^4 = M + 2B (\cos k_x + \cos k_y + \cos k_z - 3).$$

Similarly to its two dimensional counterpart, discussed in Sec. II, the topology of the unperturbed system depends on the choice of parameters. We take $M/B < 0$, in which case the time-independent system is a trivial insulator with a doubly degenerate spectrum.

In order to demonstrate that on-resonance light can induce topological properties in this system, we consider linearly polarized radiation, described by the scalar potential term $V(t) = V_0 \gamma_4 \cos(\omega t + \alpha)$, where $V_0$, $\alpha$ and $\omega$ are constants. We place the system in a toroidal geometry, with open boundaries in one direction and periodic in the remaining two. The inset of Figure 9(b) shows that a single Dirac cone now exists at each boundary of the system, in agreement with Ref. 22. These states are topologically protected. In particular, we find numerically that the edge modes are robust against weak breaking of PH symmetry.

As a result, we find that the topological invariant is preserved, even under spatial inversion. Then, Eq. (20) is invariant under both time reversal and space inversion symmetries. For simplicity we first consider the situation where there is particle-hole symmetry, such that $\varepsilon_k = 0$.

According to the topological classification of 3d systems, time-reversal topological insulators are characterized by a $\mathbb{Z}_2$ topological invariant. In the case of Eq. (20), a natural topological invariant is given by a generalization of the TKNN formula to higher dimensions. Here, the Brillouin zone is a three dimensional torus, $T^3$, and $D_k$ is a four component unit vector that lies on $S^3$. $D_k$ can therefore be described as a map, $D : T^3 \rightarrow S^3$. We define the topological invariant $\chi$ as the degree of this map, which is the number of times $D_k$ wraps around the unit sphere $S^3$ as $k$ runs over the Brillouin zone. When PH and TR symmetries are both present, $\chi$ is only defined modulo 2.

As an example of a 3d FTI, we consider a cubic lattice model, with

$$D_k^4 = M + 2B (\cos k_x + \cos k_y + \cos k_z - 3).$$

Similarly to its two dimensional counterpart, discussed in Sec. II, the topology of the unperturbed system depends on the choice of parameters. We take $M/B < 0$, in which case the time-independent system is a trivial insulator with a doubly degenerate spectrum.

In order to demonstrate that on-resonance light can induce topological properties in this system, we consider linearly polarized radiation, described by the scalar potential term $V(t) = V_0 \gamma_4 \cos(\omega t + \alpha)$, where $V_0$, $\alpha$ and $\omega$ are constants. We place the system in a toroidal geometry, with open boundaries in one direction and periodic in the remaining two. The inset of Figure 9(b) shows that a single Dirac cone now exists at each boundary of the system, in agreement with Ref. 22. These states are topologically protected. In particular, we find numerically that the edge modes are robust against weak breaking of PH symmetry.

We now replace the uniform light by a domain wall configuration in which the potential changes sign along the $z$ direction. Figure 9(a) shows that in addition to the edge modes, the spectrum now includes a pair of Dirac cones at each boundary of the system.

Figure 9: (a) The Floquet spectrum of the studied three dimensional model in a domain wall configuration. The dashed red line denotes modes that are localized at the domain wall, while the blue line denotes modes that are localized at the edges. The spectrum is doubly degenerate. Results are for $A = 0.2$, $B = -0.2$, $M = 1$, $\omega = 2.7$, $V_0 = 0.5$, $L = 300$. Note that for $L = 300$, a small mixing between the edge and domain wall modes exists. This mixing decays exponentially with the system size. (b) The amplitude of the localized modes for $L = 1000$. The inset denotes the Floquet spectrum when the radiating light is uniform.

In Summary, we have demonstrated that spatially modulated light can induce dramatic effects in Floquet
topological insulators and provided various schemes to generate these effects. For example, we established that domain walls in the frequency of the light and domain walls and vortices in the phase of the light may lead to localized modes with zero quasi-energy in the bulk of a system described by the studied zincblende model. In addition, we found that similar effects can be realized in a honeycomb lattice such as graphene, by domain wall and vortex configurations of the phase of the light and by an interface between light beams with different polarizations. We also provided a generalization of these results to a three dimensional model. Our work illustrates the great potential and versatility of modulated FTI as topological phases of matter.

Let us now briefly discuss the physical manifestations of our results in realistic systems. Currently, the most promising proposal for the application of modulated FTIs is in the field of optics. Recent work[16,17] demonstrated that FTIs can be realized in engineered photonic systems. The phenomenology of these setups include protected boundary modes. For example, in Ref[18] helical waveguides were arranged on a honeycomb lattice in order to create an FTI with a chiral edge mode and no backscattering. By simple modifications of the waveguide configuration, it may be possible to implement our results in these experiments. In this case, the helicity of the light plays the role of circularly polarized light, and by rotating helices about their screw axis, one can control the phase of the external perturbation locally. Thus, for example, by arranging the helices in a vortex configuration it may be possible to simulate the Caroli-de Gennes-Matricon vortex core states of a superconductor[20].

Applications of our results in solid state systems may also be possible, but some issues must be resolved for this to be accomplished, especially in cases where the driving frequency is on-resonance. Mainly, in this case it is difficult to determine what will be the occupation of states for realistic systems. Early work in this field demonstrated that in certain special cases, resonantly-driven systems may achieve a steady state with occupation that is given by a Fermi-Dirac distribution of the quasi-energies, with the effective temperature determined by the interaction and phonon relaxation rates.[19,21,22] In addition, some experimental work on the occupation of electronic states in radiated systems also exist.[23,24] An important result in this context is the direct observation of an energy gap in a Floquet system[25]. Yet, a general understanding of the occupation of states in systems driven on-resonance is still lacking. On the other hand, systems driven off-resonance are much easier to understand and perhaps hold the greatest potential for solid state realizations.[14,15,26] For example, the conductivity of these systems when connected to external leads has been shown to behave as expected for topological phases with gapless modes[27,28]. Our results suggest that it is possible to induce anisotropic conductivity in these systems by using space modulated light.

VII. ACKNOWLEDGMENTS

We would like to thank Adi Stern for the suggestion to consider frequency modulation. This research was supported by the Israel Science Foundation and by the E.U. under grant agreement No. 276923 – MC–MOTIPROX.

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