On the $L^2$-norm of periodizations of functions

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To the memory of Tom Wolff

Abstract

We prove that the $L^2([0,1]^d \times SO(d))$-norm of periodizations of a function from $L^1(\mathbb{R}^d)$ is equivalent to the $L^2(\mathbb{R}^d)$-norm of the function itself in higher dimensions. We generalize the statement for functions from $L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ in the spirit of the Stein-Tomas theorem.

0. Introduction.

Let $f$ be a function from $L^1(\mathbb{R}^d)$. Define a family of its periodizations with respect to a rotated integer lattice:

$$g_{\rho}(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

(1)

for all rotations $\rho \in SO(d)$. The main object of our study is $G$, the $L^2([0,1]^d \times SO(d))$-norm of the family of periodizations,

$$G^2 = \int_{\rho \in SO(d)} \int_{[0,1]^d} |g_{\rho}(x)|^2 dx \, d\rho$$

$$= \int_{\rho \in SO(d)} \|g_{\rho}\|_2^2 d\rho.$$  

(2)

The purpose of this work is to show how $G$ can give an estimate of the $L^2(\mathbb{R}^d)$-norm of a function from $L^1(\mathbb{R}^d)$ in higher dimensions. Some results on the Steinhaus tiling problem are related to Theorem 1 since periodizations naturally appear in the problem of Steinhaus. M. Kolountzakis [6]
proves that if a function \( f \in L^1(\mathbb{R}^2) \) and \( |x|^\alpha f \in L^1(\mathbb{R}^2) \), where \( \alpha > \frac{1}{3} \) and its periodizations are constants, then the function is continuous. Another result is obtained by M. Kolountzakis and T. Wolff ([4], Theorem 1). It says that if periodizations of a function from \( L^1(\mathbb{R}^d) \) are constants then the function is continuous provided that the dimension \( d \) is at least three.

The main theorems are the following.

**Theorem 1:** let \( d \geq 4 \) and let \( f \in L^1(\mathbb{R}^d) \). If periodizations of \( f \\
\begin{align*}
g_\rho(x) &= \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\end{align*}
are in \( L^2([0, 1]^d) \) for almost all rotations \( \rho \in SO(d) \) and \n\begin{align*}
G^2 &= \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty
\end{align*}
then \( f \in L^2(\mathbb{R}^d) \):
\begin{align*}
\|f\|_2 &\leq C(G + \|f\|_1),
\end{align*}
where \( C \) depends only on \( d \).

We also obtain the following inverse theorem.

**Theorem 1’** : let \( d \geq 5 \), let \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), and let \( g_\rho \) be periodizations of \( f \\
\begin{align*}
g_\rho(x) &= \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\end{align*}
then \( g_\rho \in L^2([0, 1]^d) \) for almost all rotations \( \rho \in SO(d) \) and \n\begin{align*}
\int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho &\leq C(\|f\|_2 + \|f\|_1)^2,
\end{align*}
where \( C \) depends only on \( d \).
We will generalize Theorems 1 and 1' in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).

**Theorem 2:** let \( d \geq 4 \), let \( 1 \leq p < \frac{2d}{d+2} \), and let \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \). If periodizations of \( f \)
\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]
are in \( L^2([0,1]^d) \) for almost all rotations \( \rho \in SO(d) \) and
\[
G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty
\]
then \( f \in L^2(\mathbb{R}^d) \):
\[
\|f\|_2 \leq C(G + \|f\|_p), \tag{4}
\]
where \( C \) depends only on \( d \) and \( p \).

We also obtain the following inverse theorem.

**Theorem 2':** let \( d \geq 5 \), let \( 1 \leq p < \frac{2d}{d+2} \), and let \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), and let \( g_\rho \) be periodizations of \( f \)
\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]
then \( g_\rho \in L^2([0,1]^d) \) for almost all rotations \( \rho \in SO(d) \) and
\[
\int_{\rho \in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_p)^2, \tag{5}
\]
where \( C \) depends only on \( d \) and \( p \).

The rest of the paper is concerned with the proofs of the theorems stated above.
1. Case $p = 1$.

Note that the constant $C$ below is not fixed and varies appropriately from one equality or inequality to another although such variations are not noted.

Proof of Theorem 1:

We will denote $\tilde{f}(x) = \bar{f}(-x)$ and $F(x) = f * \tilde{f}(x)$. Then $F \in L^1(\mathbb{R}^d)$ and

$$\|F\|_1 \leq \|f\|_1^2.$$  \hfill{(6)}

We will define the following functions $h, h_1, h_2 : \mathbb{R}^+ \to \mathbb{C}$

$$h(t) = \int |\hat{f}(\xi)|^2 d\sigma_t(\xi)$$ \hfill{(7)}

$$= \int_{\mathbb{R}^d} f * \tilde{f}(x)d\sigma_t(x)dx$$

$$= \int_{\mathbb{R}^d} F(x)d\sigma_t(x)dx,$$ \hfill{(8)}

$$h_1(t) = \int_{|x| \leq 1} F(x)d\sigma_t(x)dx,$$ \hfill{(9)}

$$h_2(t) = \int_{|x| > 1} F(x)d\sigma_t(x)dx.$$ \hfill{(10)}

Clearly $h = h_1 + h_2$.

Lemma 1 Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[1/2, 2]$, and let $b \in [0, 1)$. Define $H_1 : \mathbb{R} \to \mathbb{C}$

$$H_1(t) = \frac{1}{\sqrt{t + b}} h_1(\sqrt{t + b})q\left(\frac{\sqrt{t + b}}{N}\right).$$

Then for large enough $N$ we have
\[
\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C\|F\|_1}{N}
\]  

(11)

where \( C \) depends only on \( q \) and \( d \).

**Proof of Lemma 1:**

First we will estimate derivatives of \( h_1(t) \)

\[
|h_1^{(k)}(t)| \leq Ct^{d-1}\|F\|_1
\]

(12)

where \( t \geq 1 \) and \( C \) depends only on \( k \) and \( d \). This follows from (11) by differentiating the last equality \( k \) times:

\[
h_1(t) = \int_{|x|\leq 1} F(x)\hat{\sigma}_t(x)dx
\]

\[
= t^{d-1}\int_{|x|\leq 1} F(x)\int_{|\xi|=1} e^{-i2\pi tx\cdot\xi}d\sigma(\xi)dx.
\]

We can easily prove by induction that

\[
\frac{d^k}{dt^k} \left( h_1(\sqrt{t+b}) \right) = \sum_{i=0}^{k} C_{i,k} \frac{h_1^{(i)}(\sqrt{t+b})}{(\sqrt{t+b})^{2k+1-i}}.
\]

(13)

It follows from (13) and (12) that when \( t \sim N^2 \) we have

\[
\left| \frac{d^k}{dt^k} \left( h_1(\sqrt{t+b}) \right) \right| \leq CN^{d-k-2}\|F\|_1
\]

(14)

with \( C \) depending only on \( k \) and \( d \).

Since \( q(\frac{\sqrt{t+b}}{N}) = q(\sqrt{t'+b'}) = \tilde{q}(t') \) with \( t' = \frac{t}{N^2} \) and \( b' = \frac{b}{N^2} \) and \( \tilde{q}(t') \) is a Schwartz function supported in \( t' \sim 1 \), we have

\[
\left| \frac{d^k}{dt^k} q(\frac{\sqrt{t+b}}{N}) \right| = N^{-2k} \left| \frac{d^k}{dt^k} \tilde{q}(t') \right| \leq CN^{-2k}
\]

(15)
with $C$ depending only on $k$ and $q$.

Since $q\left(\frac{\sqrt{t+b}}{N}\right)$ is supported in $t \sim N^2$ it follows from (14) and (15) that

$$\left| \frac{d^k}{dt^k} H_1(t) \right| = \left| \frac{d^k}{dt^k} \left( \frac{h_1(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N})}{\sqrt{t+b}} \right) \right| \leq C N^{d-2-k} \|F\|_1$$

with $C$ depending only on $k$, $d$ and $q$. Since $H_1(t)$ is also supported in $t \sim N^2$ we have

$$\|H_1^{(k)}\|_1 \leq C N^{d-k} \|F\|_1.$$  

Therefore

$$|\hat{H}_1(\nu)| \leq \frac{C}{|\nu|^k} \|H_1^{(k)}\|_1 \leq \frac{C N^{d-k} \|F\|_1}{|\nu|^k}$$

for every $\nu \neq 0$.

Summing (17) over all $\nu \neq 0$ and putting $k = d + 1$ we get our desired result

$$\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C \|F\|_1}{N}$$

where $C$ depends only on $q$ and $d$. \qed

In the next lemma we will use an approach related to (17). Lemma 1.1.

**Lemma 2** Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, and let $b \in [0, 1)$. Define $H_2 : \mathbb{R} \to \mathbb{C}$

$$H_2(t) = \frac{1}{\sqrt{t+b}} h_2(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough $N$ we have

$$\sum_{\nu \neq 0} |\hat{H}_2(\nu)| \leq \int_{|x| \geq 1} |F(x)| \cdot |D_N(x)|$$
where $D_N : \mathbb{R}^d \rightarrow \mathbb{C}$

$$|D_N(x)| \leq C \begin{cases} \left( \frac{N}{|x|} \right)^{\frac{d-2}{2}} & \text{if } |x| \geq \frac{N}{2} \\ \frac{1}{N} & \text{if } 1 \leq |x| \leq \frac{N}{2} \end{cases} \quad \text{(20)}$$

with $C$ depending only on $q$ and $d$.

Proof of Lemma 2:

We have

$$\hat{H}_2(\nu) = \int H_2(t)e^{-i2\pi \nu t} dt = 2e^{i2\pi \nu b} \int Nq(t)h_2(tN)e^{-i2\pi \nu (Nt)^2} dt = 2e^{i2\pi \nu b} \int Nq(t)e^{-i2\pi \nu (Nt)^2} \int \sigma_{Nt}(x) dx dt$$

$$= 2e^{i2\pi \nu b} \int F(x) \int Nq(t)e^{-i2\pi \nu (Nt)^2} (Nt)^{-1} \sigma_{Ntx} dt dx \quad \text{(21)}$$

We will use a well-known fact that $\hat{d}\sigma(x) = Re(B(|x|))$ with $B(r) = a(r)e^{i2\pi r}$ and $a(r)$ satisfying estimates

$$|a^k(r)| \leq \frac{C}{r^{\frac{d-1}{2}+k}} \quad \text{(22)}$$

with $C$ depending only on $k$ and $d$. Now we will need to estimate the inner integral in (21) with $B(|x|)$ instead of $\sigma(x)$

$$\int Nq(t)e^{-i2\pi \nu (Nt)^2} (Nt)^{d-1} a(N|x|t)e^{i2\pi N|x|t} dt = \frac{N^{d+1}}{|x|^{d-2}} \int q(t)e^{-i2\pi \nu (Nt)^2} t^{d-1} a(N|x|t)(N|x|) \frac{d-1}{2} e^{i2\pi N|x|t} dt$$

$$= \frac{N^{d+1}}{|x|^{d-2}} e^{i2\pi \frac{|x|^2}{4\nu}} \int q(t)a(N|x|t)(N|x|) \frac{d-1}{2} e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu})^2} dt$$

$$= \frac{N^{d+1}}{|x|^{d-2}} e^{i2\pi \frac{|x|^2}{4\nu}} \int \phi(t)e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu})^2} dt \quad \text{(23)}$$
where \( \phi(t) = q(t)a(N|x|t)(N|x|)^{d-1} \) is a Schwartz function supported in \([\frac{1}{2}, 2]\) whose derivatives and the function itself are bounded uniformly in \(t, x\) and \(N\) because of (23). Note that we used here the fact that \(N|x| \geq 1\). We can say even more. Note that in fact \( \phi(t) = \phi(t, |x|) \). Let \(|x| = c \cdot r\) where \(c \geq 2\) and \(r \geq \frac{1}{2}\). Then all partial derivatives of \(\phi(t, c \cdot r)\) with respect to \(t\) and \(r\) are also bounded uniformly in \(t, r, c\) and \(N\). The only place, where we will use that \(\phi(t)\) also depends on \(x\), is formula (63) from the proof of Lemma 4. Therefore, we will keep writing just \(\phi(t)\) until formula (63).

From the method of stationary phase ([3], Theorem 7.7.3) it follows that if \(k \geq 1\) then
\[
| \int \phi(t) e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu N})^2} dt - \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{4}} \phi^{(2j)}(\frac{|x|}{2\nu N}) | \leq c_k(\nu N^2)^{-k-\frac{1}{4}} (24)
\]
where \(c_j\) are some constants.

Since \(\phi\) is supported in \([\frac{1}{2}, 2]\) we conclude from (24) that
\[
| \int \phi(t) e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu N})^2} dt | \leq \begin{cases} C(\nu N^2)^{-\frac{1}{2}} & \text{if } \nu \in [\frac{|x|}{4N}, \frac{|x|}{N}] \\ C_k(\nu N^2)^{-k-\frac{1}{2}} & \text{if } \nu \notin [\frac{|x|}{4N}, \frac{|x|}{N}] \end{cases} . (25)
\]

If \(\frac{|x|}{N} \leq \frac{1}{2}\), then there are no \(\nu\) in \([\frac{|x|}{4N}, \frac{|x|}{N}]\) and therefore if we sum (23) over all \(\nu \neq 0\) we will get
\[
| \int \phi(t) e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu N})^2} dt | \leq C_k N^{-2k-1} . (26)
\]

If \(\frac{|x|}{N} \geq \frac{1}{2}\) then the number of \(\nu\) in \([\frac{|x|}{4N}, \frac{|x|}{N}]\) is bounded by \(\frac{|x|}{N}\) and therefore if we sum (23) over all \(\nu \neq 0\) we will get
\[
\sum_{\nu \neq 0} | \int \phi(t) e^{-i2\pi \nu N^2(t-\frac{|x|}{2\nu N})^2} dt | \leq C_k \frac{|x|}{N} (|x|N)^{-\frac{1}{2}} + C_k N^{-2k-1} \leq C_k \frac{|x|}{N^2} . (27)
\]

Summing (23) over all \(\nu \neq 0\) and applying (26) or (27) we conclude
\[
\sum_{\nu \neq 0} | \int Nq(t) e^{-i2\pi \nu (Nt)^2} (Nt)^{d-1} B(N|x|t) dt | \leq \begin{cases} C_k \frac{N^{d-2}}{|x|^d} & \text{if } \frac{|x|}{N} \geq \frac{1}{2} \\ C_k N^{d-2} \frac{1}{|x|^d} & \text{if } \frac{|x|}{N} \leq \frac{1}{2} \end{cases} . (28)
\]
Replacing in \( \hat{d}\sigma(x) \) with \( \frac{B(|x|) + \tilde{B}(|x|)}{2} \), summing over all \( \nu \neq 0 \) and applying (28) with \( k \geq \frac{d+1}{4} \) we get the desired result

\[
\sum_{\nu \neq 0} |\tilde{H}_2(\nu)| \leq \int_{|x|\geq 1} |F(x)| \cdot |D_N(x)|
\]

where \( D_N : \mathbb{R}^d \to \mathbb{C} \)

\[
|D_N(x)| \leq C \begin{cases} \left( \frac{N}{|x|} \right)^{\frac{d-2}{2}} & \text{if } |x| \geq \frac{N}{2} \\ \frac{1}{N} & \text{if } 1 \leq |x| \leq \frac{N}{2} \end{cases}
\]

with \( C \) depending only on \( q \) and \( d \).

Now we are in a position to proceed with the proof of Theorem 1. From (1) it follows that

\[
\hat{g}_\rho(m) = \hat{f}(\rho m)
\]

for every \( m \in \mathbb{Z}^d \). By scaling we can assume that

\[
\hat{g}_\rho(m) = \hat{f}
\]

for every \( m \in \mathbb{Z}^d \). It follows that

\[
\|g_\rho\|_2^2 = \sum_{m \in \mathbb{Z}^d} |\hat{g}_\rho(m)|^2 = \sum_{m \in \mathbb{Z}^d} |\hat{f}(\frac{\rho m}{\sqrt{2}})|^2.
\]

Let \( r_d(n) \) denote the number of representations of an integer \( n \) as sums of \( d \) squares. It is a well-known fact from Number Theory that if \( d \geq 5 \) then

\[
r_d(n) \geq Cn^{\frac{d-2}{2}}
\]

and if \( d = 4 \) and \( n \) is odd then

\[
r_d(n) \geq Cn
\]

where \( C > 0 \) depends only on \( d \). See for example ([3], p.30, p.155, p.160).
Integrating (33) with respect to the Haar measure $d\rho$ and applying (2) we have

$$G^2 = \int_{\rho \in SO(d)} \sum_{m \in \mathbb{Z}^d} |\hat{f}(\frac{m}{\sqrt{2}})|^2 d\rho$$

$$= \int_{|\xi|=1} \sum_{m \in \mathbb{Z}^d} |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi)$$

$$= \sum_{n \geq 0} \sum_{|m|=n\xi=1} \int |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi)$$

$$\geq \sum_{n \geq 0} \sum_{|m|=2n+1\xi=1} \int |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi)$$

$$= \sum_{n \geq 0} r_d(2n+1) \int_{|\xi|=1} |\hat{f}(\sqrt{n + \frac{1}{2}}\xi)|^2 d\sigma(\xi)$$

$$= \sum_{n \geq 0} \frac{r_d(2n+1)}{(n + \frac{1}{2})^{\frac{d}{2}}} \int |\hat{f}(\xi)|^2 d\sigma \sqrt{n + \frac{1}{2}}(\xi). \quad (36)$$

Using (7) and (34) or (35) we conclude from (36) that

$$\sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} h(\sqrt{n + \frac{1}{2}}) \leq CG^2. \quad (37)$$

Let $q : \mathbb{R} \rightarrow \mathbb{R}$ be a fixed non-negative Schwartz function supported in $[\frac{1}{2}, 2]$ such that

$$q(x) + q(x/2) = 1$$

when $x \in [1, 2]$. It follows that

$$\sum_{j \geq 0} q(\frac{x}{2^j}) = 1 \quad (38)$$

when $x \geq 1$. 

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Applying the Poisson summation formula to

\[
H(t) = \frac{1}{\sqrt{t + \frac{1}{2}}} h(\sqrt{t + \frac{1}{2}}) q(\sqrt{\frac{t + \frac{1}{2}}{N}})
\]

\[
= \frac{1}{\sqrt{t + \frac{1}{2}}} h_1(\sqrt{t + \frac{1}{2}}) q(\sqrt{\frac{t + \frac{1}{2}}{N}}) + \frac{1}{\sqrt{t + \frac{1}{2}}} h_2(\sqrt{t + \frac{1}{2}}) q(\sqrt{\frac{t + \frac{1}{2}}{N}})
\]

\[
= H_1(t) + H_2(t)
\]

we have

\[
\sum_n H(n) = \sum_\nu \hat{H}(\nu) = \hat{H}(0) + \sum_{\nu \neq 0} \hat{H}_1(\nu) + \sum_{\nu \neq 0} \hat{H}_2(\nu). \tag{39}
\]

Note that

\[
\hat{H}(0) = \int \frac{1}{\sqrt{t + \frac{1}{2}}} h(\sqrt{t + \frac{1}{2}}) q(\sqrt{\frac{t + \frac{1}{2}}{N}}) dt
\]

\[
= 2 \int h(t) q\left(\frac{t}{N}\right) dt. \tag{40}
\]

Substituting (40) into (39) we get that

\[
2 \int h(t) q\left(\frac{t}{N}\right) dt \leq \sum_n H(n) + \sum_{\nu \neq 0} |\hat{H}_1(\nu)| + \sum_{\nu \neq 0} |\hat{H}_2(\nu)| \leq \sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} h(\sqrt{n + \frac{1}{2}}) q(\sqrt{\frac{n + \frac{1}{2}}{N}}) + C \|F\|_1 + \int |F(x) D_N(x)| dx \tag{41}
\]

where the last inequality follows from Lemma 1 and Lemma 2.
From the definition of \(D_N(x)\) in (20), it follows that
\[
\sum_{j \geq 0} |D_2^j(x)| = \sum_{2^j \leq |x|} |D_2^j(x)| + \sum_{2^j > |x|} |D_2^j(x)| \\
\leq \sum_{2^j \leq |x|} C \left( \frac{2^j}{|x|} \right)^{\frac{d-2}{2}} + \sum_{2^j > |x|} \frac{C}{2^j} \leq C
\]  \hspace{1cm} (42)
for every \(|x| \geq 1\).

Putting \(N = 2^j\) in (41), summing over all \(j \geq 0\) and applying (38) we get by Lebesgue Monotone Convergence Theorem
\[
2 \int_1^\infty h(t)dt \leq \sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} \left( n + \frac{1}{2} \right)^{\frac{d-2}{2}} + C\|F\|_1 + C \int_{|x| \geq 1} |F(x)|dx \\
\leq C(G^2 + \|F\|_1) \hspace{1cm} (43)
\]
where the last inequality follows from (37). From the definition of \(h(t)\) in (7) it follows that
\[
h(t) \leq C\|f\|_2^2
\]  \hspace{1cm} (44)
for \(t \leq 1\). Therefore we have
\[
\int |\hat{f}(x)|^2dx = \int_0^\infty |\hat{f}(\xi)|^2d\sigma_t(\xi)dt = \int_0^\infty h(t)dt \leq C(G^2 + \|f\|_2^2) \hspace{1cm} (45)
\]
where the last inequality is obtained from (43), (44) and (6). From (45) it follows that \(f \in L^2\) and
\[
\|f\|_2 \leq C(G + \|f\|_1)
\]
with \(C\) depending only on \(d\).
\(\square\)
If \( d \geq 5 \) then

\[
r_d(n) \leq C n^{\frac{d-2}{2}}
\]  

(46)

where \( C > 0 \) depends only on \( d \). See for example ([3], p.155, p.160). An argument similar to one used to get (36), but without scaling, shows that

\[
G^2 = \int_{\rho \in SO(d)} \sum_{m \in \mathbb{Z}^d} |\hat{f}(\rho m)|^2 d\rho
\]

\[
= |\hat{f}(0)|^2 + \sum_{n \geq 1} r_d(n) \int_{|\xi| = 1} |\hat{f}(\sqrt{n} \xi)|^2 d\sigma(\xi)
\]

\[
= |\hat{f}(0)|^2 + \sum_{n \geq 1} r_d(n) \frac{1}{n^{\frac{d-2}{2}}} \int |\hat{f}(\xi)|^2 d\sigma(\sqrt{n} \xi).
\]  

(47)

Using (7) and (46) we conclude from (47) that

\[
G^2 \leq \|f\|_1^2 + C \sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}).
\]  

(48)

Repeating arguments which we used to obtain (13) we get

\[
\sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}) \leq 2 \int_0^\infty h(t) dt + C \|F\|_1
\]

\[
\leq C(\|\hat{f}\|_2^2 + \|f\|_1^2).
\]  

(49)

Hence we can formulate an inverse theorem to Theorem 1:

**Theorem 1'**: let \( d \geq 5 \) and let \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and let \( g_{\rho} \) be periodizations of \( f \)

\[
g_{\rho}(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]  

(50)

then \( g_{\rho} \in L^2([0,1]^d) \) for almost all rotations \( \rho \in SO(d) \) and

\[
\int_{\rho \in SO(d)} \|g_{\rho}\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_1)^2
\]  

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where $C$ depends only on $d$.

**Corollary:** complex interpolation between the trivial $p = 1$ and $p = 2$ gives us the following result for $1 < p < 2$: let $d \geq 5$ and let $f \in L^p(\mathbb{R}^d) \cap L^{\frac{p}{2}}(\mathbb{R}^d)$ and let $g_\rho$ be periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

then $g_\rho \in L^p([0, 1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\left(\int_{\rho \in SO(d)} \|g_\rho\|_p^{p'} d\rho \right)^{1/p'} \leq C\|f\|_p^{2-p}(\|f\|_p + \|f\|_{\frac{p}{2}})^{p-1} = C\|f\|_p(1 + \frac{\|f\|_{\frac{p}{2}}}{\|f\|_p})^{p-1}$$

where $C$ depends only on $d$.

If $p'$ is an even integer then $|\hat{f}|^{p'} = \hat{F}$ where $\|F\|_1 \leq \|f\|_1^{p'}$. Using the same proof as for $p' = 2$ we get for $d \geq 4$

$$\|\hat{f}\|_{p'}^{p'} \leq C\left(\int_{\rho \in SO(d)} \|\hat{g}_\rho\|_{p'}^{p'} d\rho + \|f\|_{p'}^{p'}\right)$$

and for $d \geq 5$

$$\int_{\rho \in SO(d)} \|\hat{g}_\rho\|_{p'}^{p'} d\rho \leq C(\|\hat{f}\|_{p'}^{p'} + \|f\|_{p'}^{p'})$$

2. Case $1 \leq p < \frac{2d}{d+2}$.

We will generalize Theorems 1 and 1' in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).

**Theorem 2:** let $d \geq 4$ and let $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$. If periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

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are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_{\rho}\|^2_2 d\rho < \infty$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_p) \quad (51)$$

where $C$ depends only on $d$ and $p$.

It will follow from the proof (see (36)) that we can replace $\int_{\rho \in SO(d)} \|g_{\rho}\|^2_2 d\rho$ with $\int_{\rho \in SO(d)} \|g_{\rho} - \hat{g}(0)\|^2_2 d\rho$ in Theorem 2, which is the norm of $g$ in the quotient space $L^2([0,1]^d \times SO(d))$ modulo constants. We will also obtain an inverse theorem.

**Theorem 2'**: let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $1 \leq p < \frac{2d}{d+2}$ and let $g_{\rho}$ be periodizations of $f$

$$g_{\rho}(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

then $g_{\rho} \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_{\rho} - \hat{g}_{\rho}(0)\|^2_2 d\rho \leq C(\|f\|_2 + \|f\|_p)^2 \quad (52)$$

where $C$ depends only on $d$ and $p$.

Since Schwartz functions are dense in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ it follows from **Theorem 2'** that we can define periodizations $g_{\rho}$ of $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ for a.e. $\rho \in SO(d)$ as elements of the quotient space of $L^2([0,1]^d)$ modulo constants.

We say that $f \in L^p(\mathbb{R}^d)$ has periodizations $g$ in the quotient space $L^2([0,1]^d \times SO(d))$ modulo constants if there exists a sequence of Shwartz
functions $f_k$ converging to $f$ in $L^p(\mathbb{R}^d)$ and such that $g_k \to g$ in the quotient space $L^2([0, 1]^d \times SO(d))$ modulo constants. From Theorem 2 we conclude that $f \in L^2(\mathbb{R}^d)$ and $f_k \to f$ in $L^2(\mathbb{R}^d)$. It follows from Theorem 2' that $g$ is a well-defined element of the quotient space $L^2([0, 1]^d \times SO(d))$ modulo constants.

Remarks: 1. As the following example shows, we can not replace
\[
\int_{\rho \in SO(d)} \|g_\rho - \hat{g}(0)\|_2^2 d\rho \text{ with } \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho \text{ in Theorem 2' when } p > 1. \]
Let $\phi : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function supported in $B(0, 1)$ such that $\phi(0) = 1$. Put $\hat{f}(x) = \phi(\frac{x}{\epsilon})$. Then
\[g_\rho = \hat{f}(0) = 1\]
but
\[\|f\|_p = \epsilon^d \|\hat{\phi}\|_p.\]

2. The next example from (E, Chapter 6.3) shows that $p$ can not be greater than $\frac{2d+2}{d+3}$ in Theorem 2'. Put
\[\hat{f}(x_1, \ldots, x_d) = \phi\left(\frac{x_1 - 1}{\epsilon^2}, \frac{x_2}{\epsilon}, \ldots, \frac{x_d}{\epsilon}\right) \quad (53)\]
where $\phi : \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function supported in $B(0, 2)$ such that $\phi = 1$ in $B(0, 1)$. Then
\[
\int_{\rho \in SO(d)} \|g_\rho - \hat{g}(0)\|_2^2 d\rho = 2d \int_{|\xi| = 1} |\hat{f}(\xi)|^2 d\sigma(\xi) \geq C \epsilon^{d-1}
\]
but
\[\|f\|_p^2 = \epsilon^{\frac{2d+2}{d+3}} \|\hat{\phi}\|_p^p.\]
It is an open question whether Theorems 2 and 2' are valid when $\frac{2d}{d+2} \leq p < \frac{2d+2}{d+3}$. We discuss this further in Remark 2 at the end of the paper.

Proof of Theorem 2:
The proof is quite similarly to that of Theorem 1. We will replace Lemma 1 with
Lemma 3 Let $q: \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ and let $b \in [0, 1)$. Define $H_1: \mathbb{R} \to \mathbb{C}$

$$H_1(t) = \frac{1}{\sqrt{t+b}} h_1(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough $N$ we have

$$\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq C \frac{\|f\|_p^2}{N}$$

(54)

where $C$ depends only on $q$ and $d$.

Proof of Lemma 3:

The only difference in the proof is how to obtain an inequality analogous to (12). Using Young’s inequality we have $\|f \ast \tilde{\hat{f}}\|_q \leq \|f\|_p^2$ where $1 + \frac{1}{q} = \frac{2}{p}$. Therefore $|\int (f \ast \tilde{\hat{f}})(x) w(x) dx| \leq \|f\|_p^2 \|w\|_q$. Substituting derivatives of $\hat{\sigma_t}(x) \chi_{\{|x| \leq 1\}}$ with respect to $t$ instead of $w$, we get the desired inequality

$$|h_1^{(k)}(t)| \leq Ct^{d-1} \|f\|_p^2$$

(55)

where $t \geq 1$ and $C$ depends only on $k$ and $d$. \hfill \square

The main difficulty is to prove a lemma analogous to Lemma 2:

Lemma 4 Let $q: \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ and let $b \in [0, 1)$. Define $H_{2,N}: \mathbb{R} \to \mathbb{C}$

$$H_{2,N}(t) = \frac{1}{\sqrt{t+b}} h_2(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then we have

$$\sum_{\nu \neq 0} \sum_{j \geq 0} |\hat{H}_{2,2j}(\nu)| \leq C \|f\|_p^2$$

(56)

with $C$ depending only on $p$, $q$ and $d$. 17
Proof of Lemma 4:

Recall from (21) that

\[ \hat{H}_{2,N}(\nu) = 2 \int (f * \tilde{f})(x)D_{N,\nu}(x)dx \]

where

\[ D_{N,\nu}(x) = \chi_{\{|x|>1\}}e^{i2\pi\nu b} \int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}d\sigma(Ntx)dt. \]  

(57)

Denote by

\[ K_{\nu}(x) = \sum_{l \geq 0} D_{2^l,\nu}(x). \]  

(58)

Then

\[ \left| \sum_{l \geq 0} \hat{H}_{2,2^l}(\nu) \right| = 2 \left| \int (f * \tilde{f})(x) \sum_{l \geq 0} D_{2^l,\nu}(x)dx \right| \]

\[ = 2 \left| \int \tilde{f}(x)(K_{\nu} * f)(x)dx \right| \]

\[ \leq 2 \|f\|_p \|K_{\nu} * f\|_{p'}. \]  

(59)

If \( p' = \infty \) or \( p' = 2 \) we have

\[ \|K_{\nu} * f\|_\infty \leq \|K_{\nu}\|_\infty \|f\|_1 \]

\[ \|K_{\nu} * f\|_2 \leq \|\hat{K}_{\nu}\|_\infty \|f\|_2. \]

First we will show that

\[ \|K_{\nu}\|_\infty \leq \|\sum_{l \geq 0} |D_{2^l,\nu}|(x)\|_\infty \]

\[ \leq C\nu^{-\frac{d}{2}}. \]  

(60)

It follows from (22) that

\[ |D_{N,\nu}(x)| \leq \frac{N^{d+1}}{|x|^{\frac{d+1}{2}}} \begin{cases} C(\nu N^2)^{-\frac{1}{2}} & \text{if } N \in \left[ \frac{|x|}{\nu}, \frac{|x|}{\nu} \right] \\ C_k(\nu N^2)^{-k-\frac{1}{2}} & \text{if } N \notin \left[ \frac{|x|}{\nu}, \frac{|x|}{\nu} \right] \end{cases}. \]  

(61)
If \( \nu > 0 \) then the number of diadic \( N \in \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \) is at most 3. If \( \nu < 0 \) then there are no \( N \) in \( \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \). Therefore choosing \( k \geq \frac{d-1}{2} \) and summing (61) over all diadic \( N \) we have

\[
\sum_{l \geq 0} |D_{2^l, \nu}(x)| \leq C|\nu|^{-\frac{d}{2}}
\]

with \( C \) depending only on \( d \) and \( q \).

Now we will show that

\[
\|\hat{K}_\nu\|_\infty \leq \|\sum_{l \geq 0} |\hat{D}_{2^l, \nu}|(y)\|_\infty \leq C. \tag{62}
\]

Since \( \text{supp } \phi \in \left[ \frac{1}{2}, 2 \right] \) we can re-write (24) for a stronger version of the method of stationary phase ([5], Theorems 7.6.4, 7.6.5, 7.7.3)

\[
\left| \int \phi(t)e^{-i2\pi\nu N^2\left(t - \frac{|x|}{2\nu N}\right)^2} dt - \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}\left( \frac{|x|}{2\nu N} \right) \right| \leq \frac{c_k(\nu N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8N\nu})^k}
\]

where \( c_j \) are some constants. Therefore

\[
D_{N, \nu}(x) = \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^d} e^{2\pi |x|^2(\nu N^2)^{-\frac{1}{2}}} \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}\left( \frac{|x|}{2\nu N} \right) + \phi_k(x) \tag{63}
\]

where \( |\phi_k(x)| \leq \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^d} \frac{c_k(\nu N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8N\nu})^k} \). Choosing \( k \geq \frac{d+2}{2} \) we have

\[
\|\hat{\phi}_k\|_\infty \leq \|\phi_k\|_1 \leq \int_{|x| \leq 8\nu N} |\phi_k| dx + \int_{|x| > 8\nu N} |\phi_k| dx \leq \frac{C}{N} \tag{64}
\]

where \( C \) depends only on \( d \) and \( q \). We can assume that \( \nu > 0 \) since \( D_{N, \nu}(x) = \phi_k(x) \) for \( \nu < 0 \). We can also ignore \( \chi_{\{|x|>1\}} \) in front of the sum in (63) because
if $|x|/2νN \in [\frac{1}{2}, 2]$, then $|x| \geq νN ≥ 1$. We will consider only the zero term in the sum. The other terms can be treated similarly. The Fourier transform of

$$
N \frac{d+1}{2} (2νN)^{d+1} (νN^2)^{-\frac{d}{2}} \psi(\frac{|x|}{2νN})
$$

at point $y$ is equal to

$$
N^{d+1} (2νN)^{d+1} (νN^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \psi(|x|) e^{i2πνN^2|y|^2} e^{-i2πνNx \cdot y} dx = C(νN^2)^{\frac{d}{2}} e^{-i2πνN^2|y|^2} \int_{\mathbb{R}^d} \psi(|x|) e^{i2πνN^2|y|^2} dx
$$

(65)

where $\psi(t) = φ(t, 2νNt)^{-\frac{d+1}{2}}$ is a Schwartz function supported in $[\frac{1}{2}, 2]$ whose derivatives and the function itself are bounded uniformly in $t$, $ν$ and $N$ (see the remarks after (23)). The same is true about partial derivatives of $\psi(|x|)$. Applying the stationary phase method for $\mathbb{R}^d$ (4, Theorem 7.7.3) we get

$$
|\int_{\mathbb{R}^d} \psi(|x|) e^{i2πνN^2|y|^2} dx| \leq \begin{cases} C(νN^2)^{-\frac{d}{2}} & \text{if } N ∈ [\frac{|y|}{2}, 2|y|] \\ C_k(νN^2)^{-k} & \text{if } N /∈ [\frac{|y|}{2}, 2|y|] \end{cases}
$$

(66)

Therefore the absolute value of (65) can be bounded from above by:

$$
\leq \begin{cases} C & \text{if } N ∈ [\frac{|y|}{2}, 2|y|] \\ C_k(νN^2)^{-k} & \text{if } N /∈ [\frac{|y|}{2}, 2|y|] \end{cases}
$$

(67)

Similar inequalities hold for Fourier transforms for the rest of the terms in the sum in (63). The number of diadic $N ∈ [\frac{|y|}{2}, 2|y|]$ is bounded by $3$. Using (64), choosing $k \geq 1$ in (67) and summing over all diadic $N$ we get

$$
\sum_{l ≥ 0} |\hat{D}_{2l,ν}(y)| \leq C
$$

(68)

with $C$ depending only on $d$ and $q$. Using (60) and (62) and interpolating between $p = 1$ and $p = 2$, we obtain

$$
\|K_ν * f\|_{p'} \leq Cν^{-α_p}\|f\|_p
$$
where $\alpha_p = \frac{d^2 - p}{p}$. We have $\alpha_p > 1$ if $p < \frac{2d}{d+2}$. Summing (59) over all $\nu \neq 0$, we get the desired inequality

$$\sum_{\nu \neq 0} | \sum_{j \geq 0} \hat{H}_{2j}(\nu)| \leq C \|f\|_p^2.$$  

□

Now we are in a position to proceed with the proof of Theorem 2. The proof is almost the same as the one of Theorem 1. We also need to replace inequality (44) with the inequality

$$\int_0^1 h(t) dt = \int_{|y| \leq 1} |\hat{f}(y)|^2 dy \leq C \|\hat{f}\|_{p'}^2 \leq C \|f\|_p^2$$

where $p \leq 2$ and $C$ depends only on $d$. An argument similar to the one used to get (41), (43) and (45) yields the desired inequality

$$\int |\hat{f}(x)|^2 dx = \int_0^\infty |\hat{f}(\xi)|^2 d\sigma_t(\xi) dt = \int_0^\infty h(t) dt \leq C(G^2 + \|f\|_p^2)$$

with $C$ depending on $d$ and $p$. Note that the interchange of summation by $\nu$ and $N$ is not a problem. □

The proof of Theorem 2′ is the same (see the argument before Theorem 1′). The important thing is that now we exclude $\hat{g}_\rho(0) = \hat{f}(0)$ in (44) now.
**Final remarks:** 1. We can further generalize Theorem 2'. Fix some $q \in \left[1, \frac{2d}{d+2}\right)$. Applying complex interpolation between the trivial $p = 1$ and $p = 2$, we obtain the following result for $1 < p < 2$: let $d \geq 5$ and let $f \in L^{\frac{2q}{q+2}}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and let $g_\rho$ be periodizations of $f$

\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]

then $g_\rho \in L^p([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

\[
\left( \int_{\rho \in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_p^{p'} d\rho \right)^{\frac{1}{p'}} \leq C \|f\|_p^{2-p} (\|f\|_p + \|f\|_q^{p'})^{p-1}
\]

where $C$ depends only on $d$ and $q$. Choosing $q = \frac{2}{p}$ in the above inequality we obtain the following generalization of Theorem 1': if $\frac{d+2}{d} < p \leq 2$ and $d \geq 5$ then

\[
\left( \int_{\rho \in SO(d)} \|g_\rho\|_p^{p'} d\rho \right)^{\frac{1}{p'}} \leq C_p (\|f\|_p + \|f\|_1).
\]

If $p'$ is an even integer then $|\hat{f}|^{p'} = \hat{F} \ast \hat{F}$ where $\|F\|_r \leq \|f\|_{\frac{p'}{q}}^{p'}$ with $\frac{p'}{2} - 1 + \frac{1}{p} = \frac{p'/2}{q}$. Repeating the same arguments as for $p' = 2$ we obtain for $d \geq 4$

\[
\|\hat{f}\|_{\frac{p'}{q}}^{p'} \leq C (\|f\|_q^{p'} + \int_{\rho \in SO(d)} \|\hat{g}_\rho\|_p^{p'} d\rho)
\]

and for $d \geq 5$

\[
\int_{\rho \in SO(d)} \|\hat{g}_\rho - \hat{g}_\rho(0)\|_p^{p'} d\rho \leq C (\|f\|_q^{p'} + \|\hat{f}\|_{\frac{p'}{q}}^{p'})
\]

where $1 \leq q < \frac{d+2}{2-1+\frac{d+2}{d}}$ and $C$ depends only on $d$ and $q.
2. Conditionally on the exponent pair conjecture ([8], Chapter 4, Conjecture 2) we can clarify what happens when $\frac{2d}{d+2} \leq p < \frac{2d+2}{d+3}$. In our case the conjecture says that

$$\left| \sum_{n \leq \nu \leq m} e^{i \frac{\nu^2}{x}} \right| \leq C_\epsilon |x|^\frac{1}{2} \left| n \right| \left( \frac{1}{2} \right)^{\frac{1}{2}}$$

(70)

where $m \leq 2n$ and $|x| \geq n$. Let $\beta(x) = \max(1, |x|)$.

**Proposition 1.** Theorems 2 and 2' hold if we replace $\|f\|_p$ with $\|\beta^\epsilon f\|_p$ and if $p < \frac{2d+2}{d+3}$, provided the conjecture (70) is valid.

Using the example (53) we can show that the Proposition 1 is sharp up to $\epsilon$ in the range of $p$ for the estimate (52).

**Proof of Proposition 1:**
The main issue is to improve the result of Lemma 4. Denote by

$$L_j(x) = \sum_{2^j \leq \nu \leq 2^{j+1}} K_\nu(x).$$

(71)

Using summation by parts we obtain from (53), (58) and (70) that

$$|L_j(x)| = \left| \sum_{2^j \leq \nu \leq 2^{j+1}} K_\nu(x) \right|$$

$$\leq C_\epsilon |x|^2 \left| 2^{\frac{j+1}{2}} \right|.$$ 

We will deal with the following expression instead of (59)

$$\left| \int \tilde{f}(x) (L_j * f)(x) dx \right| = \left| \int \tilde{f}(x) \beta^\epsilon \frac{(L_j * f)(x)}{\beta^\epsilon} dx \right|$$

$$\leq \|\beta^\epsilon f(x)\|_\nu \left\| \frac{(L_j * f)}{\beta^\epsilon} \right\|_{\nu'}.$$ 

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If $p' = \infty$ or $p' = 2$ we have

$$\left\| \frac{(L_j * f)}{\beta^c} \right\|_\infty \leq \left\| \frac{\int_{|y| \leq |x|} |L_j(x-y)| \cdot |f(y)|dy + \int_{|y| \geq |x|} |L_j(x-y)| \cdot |f(y)|dy}{\beta^c(x)} \right\|_\infty$$

$$\leq \left\| \frac{L_j}{\beta^c} \right\|_\infty \| \beta^c f \|_1$$

$$\leq C \epsilon 2^{-\frac{-d+1}{2-j}} \| \beta^c f \|_1,$$

$$\left\| \frac{(L_j * f)}{\beta^c} \right\|_2 \leq \left\| L_j * f \right\|_2$$

$$\leq \left\| \tilde{L}_j \|_\infty \| f \|_2$$

$$\leq C2^j \| f \|_2.$$

Interpolating between $p = 1$ and $p = 2$, we obtain

$$\left\| \frac{(L_j * f)}{\beta^c} \right\|_{p'} \leq C \epsilon 2^{-j \alpha_p} \| \beta^c f \|_p$$

where $\alpha_p = \frac{d+1}{p} - \frac{d+3}{2}$. We have $\alpha_p > 0$ if $p < \frac{2d+2}{d+3}$. □

3. Concerning the lower dimensional cases we can use the following results from the Number Theory:

$$r_3(n) \leq Cn^{\frac{1}{2}} \ln n \ln \ln n,$$

$$r_4(n) \leq Cn \ln \ln n.$$

See for example ([1]). There is an infinite arithmetic progression, e.g. $n = 8k + 1$, such that

$$r_3(n) \geq C \epsilon n^{\frac{1}{2}}.$$

See for example ([4]). Then Theorem 2 holds when $d = 3$ and Theorem 2' holds when $d = 3$ or $d = 4$ if we replace

$$\| g_\rho \|_2^2 = \sum_{m \in \mathbb{Z}^d} |\hat{g}_\rho(m)|_2^2$$

with

$$\sum_{m \in \mathbb{Z}^d} |m|^c |\hat{g}_\rho(m)|_2^2,$$

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\[
\sum_{m \in \mathbb{Z}^d, |m| > 3} \frac{\hat{g}_\rho(m)^2}{\ln |m| \ln \ln |m|}
\]
or
\[
\sum_{m \in \mathbb{Z}^d, |m| > 3} \frac{\hat{g}_\rho(m)^2}{\ln \ln |m|}
\]
correspondingly.

Using a technique similar to the one in the proof of Proposition 1, second remark we also obtain the following results in lower dimensions:

\[
\| \beta \cdot \hat{f} \|_2 \lesssim \| f \|_p + \| g \|_2 \text{ when } d = 3 \text{ and } \beta(x) = \min(1, \frac{1}{|x|^p}),
\]

\[
\| g - \hat{g}(0) \|_2 \lesssim \| \beta \cdot f \|_p + \| \beta \cdot \hat{f} \|_2 \text{ when } d = 3 \text{ and } \beta(x) = \max(1, \ln_+ |x| \ln_+ (\ln_+ |x|)),
\]

\[
\| g - \hat{g}(0) \|_2 \lesssim \| \beta \cdot f \|_p + \| \beta \cdot \hat{f} \|_2 \text{ when } d = 4 \text{ and } \beta(x) = \max(1, \ln_+ (\ln_+ |x|)),
\]

and \(1 \leq p < \frac{2d}{d+2}\) in all three cases.

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