THE IVP FOR CERTAIN DISPERSION GENERALIZED OF THE ZK EQUATION IN THE CYLINDER SPACE

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Abstract. We establish well-posedness for the Cauchy problem associated to the dispersion generalized Zakharov-Kutneresov equation in the cylinder. Our main ingredient is a localized Strichartz estimate and an argument of compactness.

1. Introduction

We deal with the well-posedness of the initial value problem (IVP) in the cylinder:

\[
\begin{aligned}
\partial_t u - \partial_x \left(D_x^{1+\alpha} \pm D_y^{1+\beta}\right) u + uu_x &= 0 \quad (x, y) \in \mathbb{R} \times \mathbb{T}, \quad t \in \mathbb{R}, \\
\phi &= \phi \in H^s(\mathbb{R} \times \mathbb{T}),
\end{aligned}
\]

where \( \alpha > -1, \beta \geq 1, u = u(x, y, t) \) is a real-valued function, \( \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \) and \( D_y^\beta = (-\partial_y^2)^\beta \). The homogeneous fractional derivative in the variable \( y \) is defined via Fourier transform by \( (D_y^\beta f)(m, n) := |n|^\beta \hat{f}(m, n) \) (analogously is defined \( D_x^{\alpha} \) for the variable \( x \)). Moreover,

\[
M(u) = \int_{\mathbb{R} \times \mathbb{T}} u^2 \, dx \, dy
\]

and

\[
E(u) = \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}} \left( D_x^{1+\alpha} u \right)^2 \pm \left( D_y^{1+\beta} u \right)^2 - \frac{u^3}{3} \, dx \, dy,
\]

are conserved by the flow of the equation associated with the IVP (1.1). The equation in (1.1) with, \( \alpha = \beta = 1 \) and the positive sign, is the well-known Zakharov-Kuznetsov (ZK) equation, that is a natural bi-dimensional extension of the well-known Korteweg de Vries (KdV) equation and describes the propagation of nonlinear ion-sonic waves in a magnetized plasma (see [19] for more information to this respect). The well-posedness for the ZK equation in the cylinder \( \mathbb{R} \times \mathbb{T} \) was treated by Linares, Pastor and Saut in [12], where they adapt the method used by Ionescu and Kenig [6] to the KP-I equation in the same setting, obtaining local well-posedness in \( H^s(\mathbb{R} \times \mathbb{T}) \) for \( s > \frac{3}{2} \). Subsequently, this result was improve by Molinet and Pilod in [13] who proved global well-posed in \( H^1(\mathbb{R} \times \mathbb{T}) \), the main ingredient of the proof is a bilinear Strichartz estimate in the context of Bourgain’s spaces, which allows to control the high-low frequency interactions appearing in the nonlinearity of the equation. For \( \alpha = 0, \beta = 1 \) and the positive sign, the equation

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(1.1) coincides with the Benjamin-Ono-Zakharov-Kuznetsov (BOZK) equation that is a model for thin nano-conductors on a dielectric substrate in $[10]$ (see $[3, 4, 2]$ for information of well-posedness). However, excluding the above cases, the equation (1.1) does not seem to have known physical relevance. Therefore, the equation serves as a mathematical model to study the dispersive effects on the $x$ and $y$ directions. Other equations similar to (1.1) in $\mathbb{R}^2$ are the generalized Benjamin-Ono-Zakharov-Kuznetsov (gBOZK) equation,

$$\partial_t u - D_x^{1+\alpha} u_x + u_{xxy} = uu_x$$

and the fractional Zakharov-Kuznetsov (fZK) equation,

$$\partial_t u + \partial_x (-\Delta)^{\alpha/2} u = uu_x.$$  

The wellposedness for the gBOZK equation was established by F. Ribaudo and S. Vento in $[15]$ where they proved local well-posedness for $s > \frac{2}{\alpha} - \frac{3}{4}$ with $0 \leq \alpha \leq 1$, their prove consists in an energy method combined with linear and nonlinear estimates in the short-time Bourgain’s spaces. For the fZK equation (1.5) R. Shippa in $[16]$ proves local well-posedness for $s > \frac{5}{2} - \alpha$ with $1 \leq \alpha \leq 2$, as consequence of a short-time bilinear Strichartz estimate, join with perturbative and energy methods.

Our goal in this work is to improve the local well-posedness in Sobolev space $H^s(\mathbb{R} \times T)$, $s > 2$ for (1.1), which is obtained from parabolic regularization. To improve this result we use localized Strichartz’s estimates, more accurately we adapt the method used to prove local well-posedness for the Cauchy problem associated to the third-order KP-I and fifth-order KP-I equations on $\mathbb{R} \times T$ proposed in $[6]$.

Our main results are established below,

**Theorem 1.1.** Let, $\alpha > -1$, $\beta \geq 1$, $\phi \in H^s(\mathbb{R} \times T)$ and $s > \frac{6-\alpha}{4} - \frac{1}{2(1+\beta)} + \frac{\lceil \beta \rceil}{2}$, where $\lceil \cdot \rceil$ is the ceiling function. Then, there exist $T = T(\|\phi\|_{H^s})$ and a unique solution of IVP (1.1), in the class $C([0, T] ; H^s(\mathbb{R} \times T)) \cap L^1([0, T] ; W^{1,\infty}(\mathbb{R} \times T)).$

Moreover, the map $\phi \in H^s(\mathbb{R} \times T) \mapsto u \in C([0, T] ; H^s(\mathbb{R} \times T))$ is continuous.

This result improve the local well-posedness obtained via parabolic regularization, for specific values of $\alpha$ and $\beta$ for example: If $\beta = 1$ and $\alpha > -1$, if $1 < \beta \leq 2$ and $\alpha > 1$, if $2 < \beta \leq 3$ and $\alpha > \frac{11}{4}$, if $3 < \beta \leq 4$ and $\alpha > \frac{11}{2}$. In general, if we aim to maintain $s < 2$, growth of $\beta$ implies growth of $\alpha$. The restriction on $\beta$ in the Theorem are necessary to guarantee the convergence both of the oscillatory integral in (2.7) bellow and of the integral (2.8) bellow.

The paper is organized as follows. In the following section we prove Localized Strichartz Estimate. Section 3, deals with Preliminary and Key estimates. Lastly, the main results are proved. First, some notation is necessary:

**Notation.**

- $a \lesssim b$ (resp. $a \gtrsim b$) means that there exists a positive constant $c$, such that, $a \leq cb$ (resp. $a \geq cb$).
- $a \sim b$, when $a \lesssim b$ and $a \gtrsim b$.
- $\lceil x \rceil = \min \{ k \in \mathbb{Z} \mid x \leq k \}$. 

$$\|u\|_{L^p_{x,t}} := \left( \int_{\mathbb{R}} \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}}$$
We observe that an equivalence for the Sobolev norms on $S'$, for integers $j, k \geq 0$, we define the operators $Q_x^j$ and $Q_y^k$ on $H^\infty(\mathbb{R} \times \mathbb{T})$ by

$$
\begin{aligned}
Q_x^0 g (\xi, n) &= \chi_{[0,1)} (|\xi|) \hat{g} (\xi, n) \\
Q_x^j g (\xi, n) &= \chi_{[2^j-1, 2^j)} (|\xi|) \hat{g} (\xi, n) \quad \text{if } j \geq 1
\end{aligned}
$$

and

$$
\begin{aligned}
Q_y^0 g (\xi, n) &= \chi_{[0,1)} (|n|) \hat{g} (\xi, n) \\
Q_y^k g (\xi, n) &= \chi_{[2^k-1, 2^k)} (|n|) \hat{g} (\xi, n) \quad \text{if } k \geq 1,
\end{aligned}
$$

where $\chi_I$ is the characteristic function of $I$.

We observe that an equivalence for the Sobolev norms on $\mathbb{R} \times \mathbb{T}$,

$$
\|g\|^2_{L^2(\mathbb{R} \times \mathbb{T})} \sim \sum_{k,j \geq 0} \|Q_x^k Q_y^j g\|^2_{L^2(\mathbb{R} \times \mathbb{T})}
$$

and

$$
\|J_x^j g\|^2_{L^2(\mathbb{R} \times \mathbb{T})} + \|J_y^k g\|^2_{L^2(\mathbb{R} \times \mathbb{T})} \sim \sum_{k \geq 0, j \geq 1} (2^j)^{2s} \|Q_x^j Q_y^k g\|^2_{L^2(\mathbb{R} \times \mathbb{T})} + \sum_{k \geq 1, j \geq 0} (2^k)^{2s} \|Q_x^j Q_y^k g\|^2_{L^2(\mathbb{R} \times \mathbb{T})} + \sum_{k, j \geq 0} \|Q_x^j Q_y^k g\|^2_{L^2(\mathbb{R} \times \mathbb{T})}
$$
2. Localized Strichartz Estimates

In this section, we prove Strichartz estimates, localized in frequency and time. Let \( \{ W_0(t) \}_t \) be the group associated with the linear part of the equation (1.1), that is,

\[
W_0(t) \phi = \int_{\xi \in \mathbb{R}} \sum_{n \in \mathbb{Z}} \hat{\phi}(\xi) e^{i(\xi x + \xi(\xi^1 + n|n| + \beta))},
\]

where \( \xi = (\xi, n), x = (x, y) \) and \( \xi x = \xi + ny \).

**Theorem 2.1.** Let \( \alpha > -1, \beta \geq 1 \) and \( \phi \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \), then for any \( \epsilon > 0 \),

\[
\| W_0 (.) Q_y^k Q_z^j \|_{L^2 \rightarrow L^{\infty}} \lesssim 2^{-(\frac{\epsilon}{2} + \frac{k}{12})} \| Q_y^k Q_z^j \phi \|_{L^2(\mathbb{R} \times \mathbb{T})}. \tag{2.1}
\]

First we recall the following lemmas;

**Lemma 2.2** (Poisson Summation Formula). If \( f, \hat{f} \) are in \( L^1(\mathbb{R}^n) \) and satisfy the condition

\[
|f(x)| + |\hat{f}(x)| \leq C (1 + |x|)^{-\delta}
\]

for some \( C, \delta > 0 \). Then, \( f \) and \( \hat{f} \) are continuous and

\[
\sum_{m \in \mathbb{Z}^n} \hat{f}(m) = \sum_{k \in \mathbb{Z}^n} f(2\pi k)
\]

**Proof.** See [11] Theorem 3.1.17. \( \square \)

**Lemma 2.3** (Van der Corput). Let \( p \geq 2, I = [a, b] \), \( \varphi \in C^p (I) \) is a real function such that \( |\varphi^{(p)}(x)| \geq \lambda > 0 \), \( \psi \in L^\infty (I) \) and \( \psi' \in L^1 (I) \), then,

\[
\left| \int_I e^{i\varphi(x)} \psi(x) \, dx \right| \leq C_p \lambda^\frac{1}{p} (\| \psi \|_{L^\infty} + \| \psi' \|_{L^1}).
\]

**Proof.** See [18] Chapter 8. \( \square \)

**Lemma 2.4.** Let \( f \in C_0^\infty ([a, b]) \) and \( \varphi'(x) \neq 0 \) for any \( x \in [a, b] \). Then,

\[
I (\lambda) = \int_a^b e^{i\lambda \varphi(x)} f(x) \, dx = O (\lambda^{-k}),
\]

as \( \lambda \to \infty \), for any \( k \in \mathbb{Z}^+ \),

**Proof.** See [11] Chapter 1. \( \square \)

**Proof of Theorem 2.1.** Let \( \psi_1 : \mathbb{R} \to [0, 1] \) denote a smooth even function, supported in \( \{ r \mid |r| \in [\frac{1}{2}, 2] \} \) and \( \psi_1 \equiv 1 \) in \( \{ r \mid |r| \in [\frac{1}{2}, 2] \} \), \( \psi_0 : \mathbb{R} \to [0, 1] \) denote a smooth even function, supported in \( [-2, 2] \) and \( \psi_0 \equiv 1 \) in \( [-1, 1] \), and \( a(\xi) = \left( Q_y^k Q_z^j \phi \right)(\xi) \). As, \( \psi_1 \left( \frac{x}{2^j} \right) \psi_0 \left( \frac{n}{2^k} \right) = 1 \), in \( [-2^{j+1}, -2^{j-1}] \times [-2^k, 0] \cup [2^{j-1}, 2^{j+1}] \times [0, 2^k] \), then,

\[
W_0(t) Q_y^k Q_z^j \phi(x) = \int_{\xi \in \mathbb{R}} \sum_{n \in \mathbb{Z}} a(\xi) \psi_1 \left( \frac{\xi}{2^j} \right) \psi_0 \left( \frac{n}{2^k} \right) e^{i(\xi x + F(\xi)t)} \, d\xi,
\]
where,
\[ F(\xi) = \xi |\xi|^{1+\alpha} \pm \xi |n|^{1+\beta}. \]

Using the same argument as in the proof of Theorem 9.3.2 of [6], it is enough to prove that,
\[ \left| \int_{\xi \in \mathbb{R}} \sum_{n \in \mathbb{Z}} \psi_1^2 \left( \frac{\xi}{2^j} \right) e^{i(\xi x + t\xi|\xi|^{1+\alpha})} \psi_0^2 \left( \frac{n}{2^j} \right) e^{i(ny + t\xi|\xi|^{1+\beta})} d\xi \right| \lesssim 2^{l-(\frac{3}{2} + \frac{1}{4} \pi)} 2^j, \tag{2.2} \]
for any \( x \in \mathbb{R}, y \in [0, 2\pi) \) and \( |t| \in [2^{-l}, 2^{-l+2}] \) and \( l \geq \lfloor \beta \rfloor k + j \). The inequality (2.2), in the case \( j = 0 \) is immediate. For \( j > 0 \), the Poisson summation formula Lemma 2.2 transform the expression inside module on the left-hand side of (2.2) to,
\[ \int_{\xi \in \mathbb{R}} \psi_1^2 \left( \frac{\xi}{2^j} \right) e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} d\eta. \]

The integral in \( A(\xi, t, y) \), is transformed using integration by parts. That is,
\[ \int_{\mathbb{R}} \psi_0^2 \left( \frac{\eta}{2^k} \right) e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} d\eta \tag{2.3} \]
\[ = \int_{|\eta| \leq 2k+1} \psi_0^2 \left( \frac{\eta}{2^k} \right) e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} \frac{d}{d\eta} \left( e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} \right) d\eta \]
\[ = i \int_{|\eta| \leq 2k+1} \left( \frac{2 \cdot 2^{-k} \psi_0^2}{(y-2\pi\nu) \pm (1+\beta) \text{sgn}(\eta) |\eta|^{\beta} \xi t} \right) e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} d\eta. \]

For the second term of the right-hand side of (2.3) in the previous integral, we have that,
\[ (1+\beta) |\eta|^{\beta-1} \xi t \leq 2^{\beta+1} 2^{k(\beta-1)2^{-l} + \lfloor \beta \rfloor k} \lesssim_\beta 2^{-k}, \]
\[ (1+\beta) |\eta|^{\beta} \xi t \lesssim_\beta 2^{k\beta} 2^{-j} \lesssim_\beta 1, \]
\( y \in [0, 2\pi) \) and \( |\nu| > 100 \), then \( |(y-2\pi\nu) \pm (1+\beta) \text{sgn}(\eta) |\eta|^{\beta} \xi t|^2 \sim |\nu|^2 \). We get,
\[ \left| \int_{|\eta| \leq 2k+1} \frac{\psi_0^2(1+\beta) |\eta|^{\beta-1} \xi t}{(y-2\pi\nu) \pm (1+\beta) \text{sgn}(\eta) |\eta|^{\beta} \xi t} e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} d\eta \right| \lesssim_\beta \frac{2^{2-k}}{|\nu|^2} \lesssim_\beta \frac{1}{|\nu|^2}. \tag{2.4} \]

Following similar consideration for the first term of the right-hand side of (2.3).
We can conclude that, if \( |\nu| > 100 \),
\[ \left| \int_{\mathbb{R}} \psi_0^2 \left( \frac{\eta}{2^k} \right) e^{i((y-2\pi\nu)\eta \pm t\xi|\xi|^{1+\beta})} d\eta \right| \leq \frac{C}{|\nu|^3} + \frac{C}{|\nu|^2} \leq \frac{C}{|\nu|^2}. \]
Then,
\[
A(\xi, t, y) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\psi_0^2}{\eta^2} \left( \frac{\eta}{2^k} \right) e^{i(y-2\pi \nu \eta \pm t \xi \eta | \eta |^{1+\beta})} d\eta \\
= \sum_{|n| \leq 100} \int_{\mathbb{R}} \frac{\psi_0^2}{\eta^2} \left( \frac{\eta}{2^k} \right) e^{i((y-2\pi \nu \eta) \pm t \xi \eta | \eta |^{1+\beta})} d\eta + O(1) .
\] (2.5)

The term \(O(1)\) in (2.2) imply that only we need estimate one integral, which is dominated by the right-hand side of (2.2). Thus, we replace the sum over \(n\) with a sum of \(C\) integrals, modulo acceptable error, then (2.2) reduces to the proof of

\[
\left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\xi}{2^k} \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} \int_{\mathbb{R}} \psi_0^2 \left( \frac{\eta}{2^k} \right) e^{i((y \eta' + \xi \eta | \eta |^{1+\beta})} d\eta d\xi \right| \lesssim 2^{l-(\frac{3}{2} + \frac{1}{1+\beta})} j \quad (2.6)
\]
\(\xi \sim 2^l, \ |t| \sim 2^{-l}\) and \(\eta \sim 2^k\). We write,

\[
\int_{\mathbb{R}} \psi_0^2 \left( \frac{\eta}{2^k} \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} d\eta = \int_{\mathbb{R}} 2^k \widehat{\psi_0^2} \left( 2^k \eta \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} (\eta) d\eta
\]

\[
= \frac{C}{(t\xi)^{\frac{1}{1+\beta}}} \int_{\mathbb{R}} \widehat{\psi_0^2}(\lambda) H \left( \frac{\eta}{(t\xi)^{\frac{1}{1+\beta}}} \right) d\lambda \quad (2.7)
\]

\(H(\eta) = \int_{\mathbb{R}} e^{i(\eta \pm \eta | \eta |^{1+\beta})} d\eta, H \in L^\infty(\mathbb{R})\) if \(\beta \geq 1\) (see [8, 17]). Then, substituting (2.7) into (2.6), it suffices to prove that

\[
\left| \int_{\mathbb{R}} \frac{1}{(t\xi)^{\frac{1}{1+\beta}}} \psi_1^2 \left( \frac{\xi}{2^l} \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} d\xi \right| \lesssim 2^{l-(\frac{3}{2} + \frac{1}{1+\beta})} j \quad (2.8)
\]

where \(|t| \sim 2^{-l}\) and \(\xi \sim 2^l\). The Van der Corput lemma implies that,

\[
\left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\xi}{2^l} \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} d\xi \right| \lesssim 2^{l-(l-\alpha)}/2 \quad (2.9)
\]

and thus,

\[
\left| \int_{\mathbb{R}} \psi_1^2 \left( \frac{\xi}{2^l} \right) e^{i((y \eta + \xi \eta | \eta |^{1+\beta})} d\xi \right| \lesssim 2^{l-(l-\alpha)}/2 \quad (2.10)
\]

because \(\frac{\alpha}{1+\beta} \geq \frac{3}{4}\) if \(\beta \geq 1\), then we obtain (2.8).

\[\square\]

3. Preliminary and Key estimates

As a consequence of the Strichartz inequality Theorem 2.1 we obtain:

**Lemma 3.1.** Let \(\alpha > -1, \beta \geq 1, u \in C([0, T]; H^\infty(\mathbb{R} \times \mathbb{T})) \cap C^1([0, T]; H^\infty(\mathbb{R} \times \mathbb{T}))\) and \(f \in C([0, T]; H^\infty(\mathbb{R} \times \mathbb{T}))\), \(T \in [0, 1]\) such that

\[
\partial_t u - \partial_x \left( D^{1+\alpha} x \pm D^{1+\beta} y \right) u = \partial_x f.
\]

Then,

\[
\|u\|_{L^1_t L^\infty_x (\mathbb{R} \times \mathbb{T})} \leq s_1 s_2 T^\frac{1}{2} \left( \|J_1^x f\|_{L^\infty_t L^2_x \mathbb{T}} + \|J_2^y f\|_{L^\infty_t L^2_y \mathbb{T}} \right) ,
\] (3.1)

for any \(s_1 > \frac{2-\alpha}{4} - \frac{1}{2(1+\beta)}\) and \(s_2 > \frac{[\beta]}{2}\)
Proof of Lemma 3.1. We partition the interval \([0, T]\) into \(2^{[β/(k+j)]}\) equal intervals of length \(T^{-([β/(k+j)])}\), denote by \([a_{k,m}, a_{k,(m+1)}]\), \(m = 0, 1, 2, \ldots 2^{[β/k]}\). We observe that, Cauchy–Schwarz inequality implies that,

\[
\|Q^k_y Q^j_x u\|_{L^1_y L^\infty_T} \leq \sum_{m=1}^{2^{[β/(k+j)]}} \left\| X[a_{k,m}, a_{k,(m+1)}] (t) Q^k_y Q^j_x u \right\|_{L^1_y L^\infty_T} \tag{3.2}
\]

Duhamel’s formula,

\[
u(t) = W^α_0 (t - a_{k,m}) (u(a_{k,m})) + \int_{a_{k,m}}^{t} W^α_0 (t - s) (∂_x f (s)) ds, \tag{3.3}
\]

for \(t \in [a_{k,m}, a_{k,(m+1)}]\), join with (3.2) and Theorem 2.1 imply that,

\[
\|Q^k_y Q^j_x u\|_{L^1_y L^\infty_T} \leq 2^{-([β/(k+j)])} T^\frac{β}{2} \left( 2^{-([β/(k+j)])} \sum_{m=1}^{2^{[β/(k+j)]}} \left( \|J^s_x J^s_y u (a_{k,m})\|_{L^2_y} \right) \right.
\]

\[
\left. + \|J^s_x f\|_{L^1_y L^2_T} \right).
\]

To obtain the energy estimate, we need the cylinder version of the Kato–Ponce commutator,

Proposition 3.2. Let \(s \geq 1\) and \(f, g \in H^\infty (\mathbb{R} \times \mathbb{T})\). Then,

\[
\| J^s_y (fg) - f J^s_y g \|_{L^2_T} \leq C_s \| f \|_{L^\infty_T} \| g \|_{L^\infty_T} + (\| f \|_{L^\infty_T} + \| f \|_{L^\infty_T}) \| J^s_y g \|_{L^2_T},
\]

\(C_s\) is a constant.

Proof. By Lemma 9.A.1 in [6],

\[
\| J^s_y (fg) - f J^s_y g \|_{L^2_T} = \left\| \| J^s_y (fg) - f J^s_y g \|_{L^2_T} \right\|
\]

\[
\leq C_s \| J^s_y f \|_{L^2_T} \| g \|_{L^\infty_T} + (\| f \|_{L^\infty_T} + \| f \|_{L^\infty_T}) \| J^s_y g \|_{L^2_T}
\]

\[
\leq C_s \| J^s_y f \|_{L^2_T} \| g \|_{L^\infty_T} + (\| f \|_{L^\infty_T} + \| f \|_{L^\infty_T}) \| J^s_y g \|_{L^2_T}.
\]

Lemma 3.3 (Energy estimate). Let \(α > -1, β \geq 1\) and \(u\) solution of IVP (1.1) with \(φ \in H^\infty (\mathbb{R} \times \mathbb{T})\). Then,

\[
\sup_{0 < t < T} \| u \|_{H^s} \leq e \left( \| u \|_{L^1_y L^\infty_T} + \| φ \|_{H^s} \right) \| φ \|_{H^s},
\]

for any \(s \geq 1, T \in [0, 1]\).

Proof. We apply \(J^s_y\) to equation in (1.1), multiply by \(J^s_y u\) and integrate, to obtain

\[
\int_{\mathbb{R} \times \mathbb{T}} J^s_y u \partial_x J^s_y u dx dy - \int_{\mathbb{R} \times \mathbb{T}} J^s_y \partial_x D^s_y 1+α u J^s_y u dx dy + \int_{\mathbb{R} \times \mathbb{T}} J^s_y \partial_x D^s_y 1+β u J^s_y u dx dy
\]
that, integrating by parts and applying the Proposition\textsuperscript{[3.2]} transform this equality in
\[
\frac{1}{2} \frac{d}{dt} \| J_y^s u \|_{L_y^{2}}^2 \lesssim \| \partial_x u \|_{L_x^{\infty}} \| J_y^s u \|_{L_y^{2}} + \| J_y^s u \|_{L_y^{2}} \| \partial_x u \|_{L_x^{\infty}} u \| J_y^s, u \|_{L_y^{2}}^2
\]
\[
+ \| \partial_x u \|_{L_x^{\infty}} \| J_y^s u \|_{L_y^{2}} + \left( \| u \|_{L_y^{\infty}} + \| \partial_y u \|_{L_y^{\infty}} \right) \| J_y^{s-1} \partial_x u \|_{L_y^{2}}.
\]

A similar argument shows that
\[
\frac{1}{2} \frac{d}{dt} \| J_x^s u \|_{L_x^{2}}^2 \lesssim \| \partial_u u \|_{L_x^{\infty}} \| J_x^s u \|_{L_x^{2}} + \| J_x^s u \|_{L_x^{2}} \| \partial_u u \|_{L_x^{\infty}}
\]

Gronwall’s inequality implies the result. \hfill \square

Now, we remember some facts that we need for the next result.

**Theorem 3.4.** [Leibniz rule] Let \( 1 < p < \infty \), \( 0 < s < 1 \) and \( f, g \in H^{\infty}(\mathbb{R} \times T) \). Then,
\[
\| J_x^s (f g) \|_{L^p} \leq C \left( \| J_x^s f \|_{L^\infty} \| g \|_{L^p} + \| f \|_{L^\infty} \| J_x^s g \|_{L^p} \right)
\]

**Proof.** By Kenig, Ponce and Vega in \textsuperscript{[9]} we get,
\[
\| J_x^s (f g) \|_{L^p} \leq \left\| \| J_x^s (f g) \|_{L^p} \right\|_{L_x^p}
\]
\[
\leq C \left( \left\| \left( \left\| J_x^s f \|_{L^\infty} \| g \|_{L^p} + \| f \|_{L^\infty} \| J_x^s g \|_{L^p} \right) \right\|_{L_x^p}
\]
\[
\leq C \left( \left\| J_x^s f \|_{L^\infty} \| g \|_{L^p} + \| f \|_{L^\infty} \| J_x^s g \|_{L^p} \right) \right)
\]

\hfill \square

**Lemma 3.5.** For every \( 0 < s < 1 \) there exists a constant \( C \) such that for every \( u \in L_{xy}^{\infty} \), satisfying \( \partial_x u \in L_{xy}^{\infty} \), one has
\[
\| J_x^s u \|_{L_{xy}^{\infty}} \leq C \left( \| u \|_{L_{xy}^{\infty}} + \| \partial_x u \|_{L_{xy}^{\infty}} \right)
\]

**Proof.** See Molinet, Saut and Tzvetkov in \textsuperscript{[14]} \hfill \square

As a consequence of Lemma \textsuperscript{[3.1]}, we obtain:

**Proposition 3.6.** Let \( \alpha > -1 \), \( \beta \geq 1 \), \( u \) be a solution of IVP \textsuperscript{[1.1]} with \( \phi \in H^{\infty}(\mathbb{R} \times T) \). Then, for any \( s > \frac{6-\alpha}{4} - \frac{1}{2(1+\beta)} + \left[ \frac{\beta}{2} \right] \), there exists \( T = T(\| \phi \|_{s}, s) \) and a constant \( C_T(\| \phi \|_{s}, s) \) such that,
\[
g(T) := \int_0^T \left( \| u \|_{L_{xy}^{\infty}(\mathbb{R} \times T)} + \| u_x \|_{L_{xy}^{\infty}(\mathbb{R} \times T)} + \| u_y \|_{L_{xy}^{\infty}(\mathbb{R} \times T)} \right) dt' \leq C_T \quad (3.4)
\]

**Proof of Proposition 3.7.** We apply Lemma \textsuperscript{[3.1]} with \( s_1 > \frac{2\beta}{4} - \frac{1}{2(1+\beta)} \) and \( s_2 > \left[ \frac{\beta}{2} \right] \) in \( u, \partial_x u, \partial_y u \) and respectively by \( f = \frac{1}{4} u^2, \frac{1}{4} \partial_x u^2, \frac{1}{4} \partial_y u^2 \). We get,
\[
\| u \|_{L_{xy}^{\infty}} + \| u_x \|_{L_{xy}^{\infty}} + \| u_y \|_{L_{xy}^{\infty}}
\]
\[
\lesssim \left( 2^{\frac{\beta}{2}} \right) \left( \| J_x^s J_y^s u \|_{L_{xy}^{\infty}} + \| J_x^s J_y^s \partial_x u \|_{L_{xy}^{\infty}} + \| J_x^s J_y^s \partial_y u \|_{L_{xy}^{\infty}} \right)
\]
\[
+ \| J_x^s u^2 \|_{L_{xy}^{\infty}} + \| J_x^s \partial_x (u^2) \|_{L_{xy}^{\infty}} + \| J_x^s \partial_y (u^2) \|_{L_{xy}^{\infty}}
\]


The first three terms on the right-hand side of the above inequality are estimate respectively by Young’s inequality, since $\frac{s_1}{s_1 + s_2 + 1} + \frac{s_2}{s_1 + s_2 + 1} + \frac{s_1 + 1}{s_1 + s_2 + 1} = 1$. The last three can be estimated by applying Leibniz’s product rule Theorem 3.4. Then, we obtain the inequality,

$$g(T) \lesssim \|\phi\|_s e^{g(T)} (1 + g(T)).$$

An argument of continuity complete the proof, if $T \leq T_0 C_T(\|\phi\|_s, s)$ is small enough.

□

In this point, we can use standard compactness arguments as in Kenig [7] for proving Theorem 1.1.

4. PROOF OF THEOREM 1.1

Let $s > \frac{6-a}{4} - \frac{1}{2(1+\beta)} + \frac{\beta}{2}$, $\phi \in H^s(\mathbb{R} \times T)$, $\phi_\gamma \in H^\infty(\mathbb{R} \times T)$, such that

$$\lim_{\gamma \to \infty} \|\phi_\gamma - \phi\|_{H^s(\mathbb{R} \times T)} = 0$$

and

$$\|\phi_\gamma\|_{H^\infty(\mathbb{R} \times T)} \leq C \|\phi\|_{H^\infty(\mathbb{R} \times T)}$$

and the solutions $\{u_\gamma\}$ of

$$u_{\gamma,0} = \phi_\gamma$$

associated to the initial data $\{\phi_\gamma\}$ such that $u_\gamma \in C([0, T']; H^s(\mathbb{R} \times T))$, $T' > 0$ guaranteed by the local well-posedness of (1.1) in $H^s(\mathbb{R} \times T)$ for $s > 2$. We can extend $u_\gamma$ on a time interval $[0, T]$, $T = T(\|\phi\|_{H^s}, s)$ by Proposition 3.3 and also show that there is a constant $C_T$ such that,

$$\int_0^T \left( \|u_\gamma\|_{L^2_y} + \|\partial_x u_\gamma\|_{L^\infty_y} + \|\partial_y u_\gamma\|_{L^\infty_y} \right) dt \leq C_T$$

(4.1)

We deduce from energy estimate Lemma 3.3 and previous inequality (4.1) that,

$$\sup_{0 < t < T} \|u_\gamma\|_{H^s(\mathbb{R} \times T)} \leq C_T$$

(4.2)

by using Gronwall’s inequality and inequality (4.1) we get,

$$\lim_{\gamma, \mu \to \infty} \sup_{0 < t < T} \|u_\gamma - u_\mu\|_0 = 0$$

(4.3)

Now, an interpolation argument together with the inequality (4.2) and (4.3), allow us to find $u \in C\left([0, T]; H^s(\mathbb{R} \times T) \cap L^\infty(\mathbb{R} \times T)\right)$ with $s' < s$, such that $u_\gamma \to u$ in $C\left([0, T]; H^{s'}(\mathbb{R} \times T)\right)$.

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