An attractor dynamics in a non-Hermitian two-level system

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Exceptional point in non-Hermitian system possesses fascinating properties. We present an exactly solvable attractor dynamics for the first time from a two-level time dependent non-Hermitian Hamiltonian. It allows a way to evolve to the coalescence state from a pure or mixed initial state through varying the imaginary parameter along a specific diabatic passage. Contrast to a chaotic attractor that is ultrasensitive to the initial condition, the design attractor is insensitive to the initial conditions. The attractor-like behavior still exists for several adiabatic processes.

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The non-Hermiticity is described by external imaginary parameters, which can be imaginary potentials [1–4] or nonreciprocal couplings [5]. The non-Hermiticity leads to considerably unusual features even in simple systems. These include $PT$ phase transition [6–10], unidirectional and anomalous transport [11–15], asymmetric reflection-less [16], and loss induced large nonlinearity [17, 18]. Exceptional point (EP) is an exclusive critical point in non-Hermitian systems, at which pairs of eigenstates coalesce and exotic features occur. Such as invisible defects [19, 21], coherent absorption [22] and self sustained emission [23, 27], loss-induced revival of lasing [28], laser-mode selection [29, 30], as well as $PT$ chaos [31]. In contrast to degenerate eigenstates, coalesced state is immune from tunneling between coalescence eigenstates, stabilizing the target quantum state [32].

The EPs possess fascinating properties, the states in a two-level system switch when circling an EP after one circle; moreover, the geometric phase accumulated is circling direction dependent [33, 35]. Dynamically encircling an EP was non-adiabatic, the energy transfer is nonreciprocal [36, 38]. The dissipative dynamics in the non-Hermitian system were discussed through master equation approach [39, 40]. Arbitrary control of pair polarization was achieved in complex birefringent metamaterials, the orthogonal polarizations is generated from nonorthogonal pairs of initial state through dynamical evolution [41]. These works open up the possibility of exploring other EPs related dynamical effects. It is interesting to investigate how a quantum state evolves when a system tends to EP.

In this letter, we propose an exactly solvable time-dependent two-level system. The system can behave as an attractor when it tends to or crosses an EP. An attractor can be a point, a curve, or a surface in the phase space of the system to which orbits are attracted. Typical observations are an infinite number of unstable orbits embedded in a chaotic attractor. The chaotic dynamics is sensitive to the initial conditions, points on two arbitrarily close trajectories may have entirely distinct dynamics, efforts have been made to obtain improved performance and multiple uses [42]. Nonlinear dissipative system under driven may exhibit chaotic behavior [43], which were extensively investigated [44–46]. Recently, periodic or chaotic dynamics were demonstrated for both the optical and mechanical modes caused by the optomechanical coupling induced nonlinearity [47]. Here, an attractor dynamics is presented for the first time from the two-level non-Hermitian Hamiltonian. The time evolution is studied when the system approaches its EP. We show that both pure and mixed states evolve to the coalescence state through varying the imaginary parameter along a specific diabatic passage. Contrast to a chaotic attractor that is ultrasensitive to the initial condition, we propose an attractor in a time-dependent non-Hermitian system without nonlinearity. The attractor is a limit circle, where the dynamics is insensitive to the initial condition, the evolution of any state finally converges to one fix orbit. Numerical simulation shows that the attractor dynamics is applicable to several adiabatic processes.

Time-dependent two-level system. In a Hermitian two-level system, the transition dynamics is governed by the Landau–Zener formula [48, 51], giving the probability of a diabatic transition between the two energy states. In a non-Hermitian system, EP is quite different from degenerate states since two eigenstate coalesce into one eigenstate. Before the construction of a general theory for the dynamics of the time-dependent system, we first present an exact solvable time-dependent passage.

Any quantum state, being either pure state or mixed state, can be depicted by a density matrix $\rho = \sum_{i,j} p_{ij} |i\rangle \langle j|$, where $\{ |i\rangle \}$ denotes a complete orthonormal set, $\langle i | j \rangle = \delta_{ij}$. For an arbitrary Hamiltonian, including Hermitian and non-Hermitian ones, the time evolution of the density matrix obeys the equation

$$\frac{\partial}{\partial t} \rho = [H_+, \rho] + [H_-, \rho]$$

(1)

where we denote $H_\pm = (H + H^\dagger)/2$. Here, the square brackets denote the commutator and the curly brackets denote the anticommutator, respectively. In principle, the dynamics of a mixed state can be obtained from the solution of the equation. However, exact analytical solution is rare, especially for the time-dependent non-Hermitian system with $H(t) \neq H^\dagger(t)$. 
where $\gamma(t)$ is the strength of the coupling and $\gamma(t)$ is the gain or loss of each cavity. All parameters are dependent of time $t$. The time varying quantities $\gamma(t)$ and $\kappa(t)$ satisfy

$$
\gamma(t) = \gamma_n(t) = t^2 - 2n - 1/2, \\
\kappa(t) = \kappa_n(t) = 1/2 - \gamma_n(t),
$$

where $n = 0, 1, 2, \ldots$. According to the Eq. (1), we have

$$
i \frac{\partial}{\partial t} R \rho = [RH_{AB}, R^{-1} \rho] + \{RH_{AB}, R^{-1} \rho\}
$$

for the density matrix $\rho$ of $H_{AB}$, where a rotation transformation

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
$$

changes $H_{AB}$ into

$$H_n(t) = \begin{pmatrix} 0 & 1 \\ \omega_n^2(t) & 0 \end{pmatrix},
$$

Note that $\omega_n^2(t) = 2n + 1 - t^2$, $(n = 0, 1, 2, \ldots)$, and $\omega_n(t)$ can be real or imaginary. For real $\omega_n$, the system has balanced gain and loss. $H_n(t)$ becomes a Jordan-block at $t_c = \pm \sqrt{2n + 1} \ (\omega_n^2(t_c) = 0)$ and two eigenvectors coalescence to $(1, 0)^T$. This is the EP of the two-level system.

Equation (1) has exact solution. For the matrix $H_n(t)$, the density matrix can be expressed as

$$\rho_n = R \rho = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},
$$

with the elements $p_{ij}$ satisfying the coupled differential equations

$$\begin{cases}
    i \dot{p}_{11} = p_{21} - p_{12}, \\
    i \dot{p}_{12} = p_{22} - \omega_n^2 p_{11}, \\
    i \dot{p}_{21} = \omega_n^2 p_{12} - p_{22}, \\
    i \dot{p}_{22} = \omega_n^2 (p_{21} - p_{22}).
\end{cases}
$$

A special solution of above equations is

$$p_{11} = [x_n(t)]^2, p_{22} = [y_n(t)]^2 = [\dot{x}_n(t)]^2, \\
p_{12} = (p_{21})^* = i \dot{x}_n(t) x_n(t)
$$

with

$$x_n(t) = (2^n n! \sqrt{\pi})^{-1/2} e^{t^2/2} h_n(t) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2}$$

where $h_n(t)$ is the Hermite polynomial. The solution above is still valid for the original Hamiltonian $H_{AB}$ under an inverse transformation of Eq. (5). The density operator $\rho_n$ is the density matrix of the pure state $(x_n, y_n)^T$, which is the solution of a quantum harmonic oscillator $\ddot{x} + \omega_n^2 x = 0$. The eigen functions behave distinctly in three different regions: In the center region, the eigen function is a standing wave inside and an evanescent wave outside. This solution governs the dynamics when the system varies through the passage $\omega_n^2(t)$, also provides the exact evolved wave function for a series of initial wave functions as the system varies along a fixed passage. In other word, for a given $\omega_n^2(t)$, the initial and final states are two points $[x_n(t_1), y_n(t_1)]$ and $[x_n(t_2), y_n(t_2)]$ on the curve $\{x_n(t), y_n(t)\}$, respectively, where $t_1$ and $t_2$ are initial and final instants, respectively.

**Fixed point of evolution.** From the above results, $[x_n(t), y_n(t)]$ is not the instantaneous eigenstate of $H_n(t)$. Thus, the time evolution dynamics is not a adiabatic process, but a diabatic one. To characterize the process, we employ the normalized Bloch vector $(\sigma_x, \sigma_y, \sigma_z)$ which is defined as $\sigma_x = (x_n y_n + x_n y_n^*)/\Omega$,
\[ \sigma_y = -i(x_n y_n - x_n y_n)/\Omega, \quad \sigma_z = (x_n^2 - |y_n|^2)/\Omega, \text{ with } \Omega = 2(|x_n|^2 + |y_n|^2). \] 

It directly indicates that \( \sigma_x = 0 \) and \( \sigma_y^2 + \sigma_z^2 = 1 \), which means that the trajectories of time evolution for any \( n \) are the same fixed longitude line on the Bloch sphere. This line is an asymptotic attractor for any nontrivial initial state. The dynamics is insensitive to the initial states, not only pure states but also mixed states will evolve to this orbit.

In fact, the solution tells us that \( [x_n(t), y_n(t)]^T \rightarrow [0, 0]^T \) as \( t \rightarrow \pm \infty \). This indicates that a state probability decays in the time evolution, similar as a complete absorption; occurring in certain non-Hermitian systems at the spectral singularity. Inversely, the time-reversal process is similar as a laser emission. However, the probability increasing rate differs, the probability gain rate is a function of time square. We demonstrate the features of the dynamics from two limit cases of \( n = 0 \) and \( n \gg 1 \). We show that in both cases, the dynamics is diabatic process and the state \( [1, 0]^T \) can be dynamically prepared, the preparation efficiency is \( n \) dependent.

When \( n = 0 \), we have

\[
\begin{align*}
x_0(t) &= e^{-t^2/2}, \\
y_0(t) &= -it e^{-t^2/2}.
\end{align*}
\]

At time \( t = t_c = \pm 1 \), we have \( \omega^2_0(t_c) = 0 \) and \( [x_0(t_c), y_0(t_c)]^T = e^{-1/2} [1, \pm i]^T \), which leads to the EP of the matrix \( H_0(t) \). \( H_0(t) \) is defective and reduces to a \( 2 \times 2 \) Jordan block, its instantaneous coalesced eigenstate is \( [1, 0]^T \). Obviously, the evolved state \( [x_0(t_c), y_0(t_c)]^T \) is not the instantaneous coalesced eigenstate. However, at the instant \( t = 0 \), we have the evolved state \( [x_0(0), y_0(0)]^T = [1, 0]^T \), which is the coalescing state, while the matrix \( H_0(0) = \sigma_x \) is Hermitian with eigenstates \( 2^{-1/2} [1, \pm 1]^T \). The initial state is \( [x_0(0), y_0(0)]^T \) at \( t = 0 \), the final state at \( t \rightarrow \infty \) approaches zero vector. On the other hand, there is an initial state \( |\phi(t_0)\rangle = (1 + t_0^2)^{-1/2} [i, t_0]^T \) at \( t_0 \rightarrow -\infty \), which can evolve to state \( [1, 0]^T \) at \( t = 0 \) but with infinite amplitude, the normalized state is \( |\phi(0)\rangle / ||\phi(0)\rangle = [1, 0]^T \). Moreover, any initial state can have this feature since for an arbitrary state \( |\phi\rangle = \cos \theta, e^{-i\theta} \sin \theta \rangle \), the Dirac inner product \( \langle \phi | \phi(t_0) \rangle = (1 + t_0^2)^{-1/2} (i \cos \theta + t_0 e^{i\theta} \sin \theta) \) is always nonzero when \( t_0 \rightarrow -\infty \), which indicates that any state has the component of \( |\phi(t_0)\rangle \). This is crucial for the application of an attractor, i.e., any unknown state evolves to state \( [1, 0]^T \). Thus, robust state preparation via dynamical evolution is possible.

When \( n \gg 1 \), the range of the oscillating regions can be estimated from the leftmost or rightmost maximum of the amplitude \( x_n(t) \), where we have

\[ \frac{d}{dt} x_n(t_b) = 0, x_n(t_b) \approx \pm x_{n-1}(t_b), \]

From the recursion identities

\[
\begin{align*}
x_n &= \sqrt{\frac{2}{n+1}} x_{n-1} - \sqrt{\frac{n+1}{2}} x_{n+1}, \\
x_n &= \sqrt{\frac{2}{n+1}} x_{n-1} + \sqrt{\frac{n+1}{2}} x_{n+1},
\end{align*}
\]

we have

\[
\begin{align*}
0 &\approx x_{n-1} - x_{n+1}, \\
\pm t_b x_{n-1}(t_b) &\approx \sqrt{2} (x_{n-1} + x_{n+1}),
\end{align*}
\]

which leads to \( t_b \approx \pm \sqrt{2n} \). We note that at instance \( t_b \), the solution is the state \( [1, 0]^T \), i.e., \( x_n(t_b) \) reaches the maximum, while \( y_n(t_b) \) as the velocity of \( x_n \) vanishes.

When we apply the solution to the dynamics of state, similar things happen as that in \( n = 0 \) case. If the initial state is \( [1, 0]^T \) at \( t = t_b \), the final state at \( t \rightarrow \infty \) approaches zero vector. On the other hand, the inverse process happens if we take an initial state \( [x_n(t_0), y_n(t_0)] \) at \( t_0 \rightarrow -\infty \). It can evolve to state \( [1, 0]^T \) at \( t = -t_b \) but with infinite amplitude. Comparing to the \( n = 0 \) case, this diabatic process is faster. In Fig. 3, we plot the expression of \( x_n(t) \) for different \( n \) to demonstrate this point. Similarly, we note that \( \omega_0^2(t_b) = 1 \), while \( \omega_n^2(t_b) = 0 \) with \( t_c = \sqrt{2n+1} \).

So far, we conclude that there exist a series of diabatic passages which can dynamically prepare the coalescing state \( [1, 0]^T \) from an unknown (arbitrary/any) initial state, including mixed state. The duration time monotonously depends on \( n \), i.e., larger \( n \) leading to faster process.

To demonstrate our conclusions, numerical simulations are performed for the evolutions of pure and mixed states. For small time increment \( \Delta t \), the Schrödinger equation for density matrix becomes

\[ \rho(t + \Delta t) \approx \rho(t) - i [\{H_+(t), \rho(t)\} - i \{H_-(t), \rho(t)\}] \Delta t, \]

which is employed to compute the time evolution of density matrix numerically. A normalized state is depicted by a Bloch vector \( \mathbf{a} \), which is defined as \( \rho = \frac{1}{2} (I + \mathbf{a} \cdot \sigma) \), \( I \) is unitary matrix and \( \sigma \) are Pauli matrices. Therefore, the trajectory of \( \mathbf{a} \) in the Bloch sphere can describe the time evolution of a state.

We depict the dynamics of initial states for \( n = 0 \) and \( n = 1 \) in Fig. 1. From the trajectories, all initial states coincide to the analytical solution finally. The final state is \( [1, 0]^T \) for \( n = 0 \) and \( [0, 1]^T \) for \( n = 1 \). These imply that when \( n \in (0, 1) \), there is a possibility that we can dynamically prepare any pure state from an unknown (arbitrary/any) initial state. The numerical results have displayed this tendency to us.

The dynamics of a \( \mathcal{PT} \) dimer has two fixed points except for the \( \mathcal{PT} \) transition point, where two fixed points coincide. In exact \( \mathcal{PT} \) phase, the orbits are closed circles, which transform nonorthogonal pair states to orthogonal states using complex birefringent material.
and (i)

consider two passages with

different initial states (including pure and mixed states, which are identified by solid and dash lines).

In broken $\mathcal{PT}$ phase, the orbits start from the source fixed point to the sink fixed point. The attractor in this letter has different dynamics, there exists only one fixed point, the pure states on the sphere surface and mixed states inside the sphere both evolve to the fixed point.

Adiabatic process. Until now, we have proved that the specific forms of $\gamma(t)$ and $\kappa(t)$ in $H_{\text{AB}}$ can lead to the dynamics which has similar phenomena for the complete absorption and laser emission, i.e., implication of the attractor-like behavior. So a natural question is whether an adiabatic passage can accomplish the same task. For example, we consider two passages with $\kappa$ is always a real constant 1 and (i)

$$\gamma(t) = t, \quad (16)$$

(ii)

$$\gamma(t) = 1 - t^2. \quad (17)$$

Unlike the situation of Hermitian systems, there is no well established theory to describe the process. On the other hand, we have no idea about the tunneling between two coalescing levels. Nevertheless, the diabatic solution implies the possibility of the amplification of amplitude. In this situation, numerical simulation is a better way to follow a quasi-adiabatic passage. It is obvious that although the position of EP is different in the two cases, both of them can go through(or reach) the EP during the time evolution from a $-t$ satisfied $\kappa^2 - \gamma^2(-t) \ll 0$ to $t = 0$. Then according to the simulation, the dynamics of these two cases are same as what we found before, which implies that once the model can get close to the EP during the time evolution from the broken area($\kappa^2 - \gamma^2(t) \ll 0$ at the beginning of the evolution), there is always a fixed point in this dynamical process. The simulation result is showed in Fig. 2. It indicates that although the speed of the evolution and the position of the fixed point changed, the asymptotic line, which likes the one in Fig. 1 still exists even in the adiabatic process with different forms of $\kappa(t)$ and $\gamma(t)$.

Experimental realization. At last we talk something about the realization of our model. In fact, there is no scheme has been proposed to realize such a system that both $\kappa(t)$ and $\gamma(t)$ can change with the time in specific forms. But in practice, the two-level non-Hamiltonian system likes

$$H = \left( \begin{array}{cc} i\gamma(t) & \kappa \\ \kappa & -i\gamma(t) \end{array} \right) \quad (18)$$

which requires that $\kappa$ is independent of $t$, has been realized to study the special feature of the EP point. Such as the emergence of multiple EPs in the coupled acoustic cavity resonators [54] and the properties associated with encirclement of an EP [58]. In optics, this kind of system can be also realized by using waveguides or a series of varying dichroic birefringent plates. We can just let $\gamma(t) = t$ for convenient, which is also easy to achieve in experiment. Based on the numerical result before, the dynamics of this system is similar with the original system we proposed in Eq. (6). So one can build a simple platform described by Eq. (18) to investigate the phenomena of the attractor, i.e., a fixed point during the time evolution.

Summary. In summary, we have proposed a time-dependent non-Hamiltonian two-level system, in which the dynamics has an exactly solvable passage. By varying the imaginary parameter along a specific diabatic passage, the two-level system behaves as an attractor, which is insensitive to the initial conditions. An attractor-like behavior is found for the first time from a non-Hermitian two-level system. We have shown that arbitrary pure and mixed state can evolve to the coalescence state. Moreover, the numerical results indicated two other things: (i) Except the coalescing state, there is a possibility that we can dynamically prepare several target states from an unknown (arbitrary/any) initial state. (ii) The same phenomenon still exists even in adiabatic passages with different forms of parameters of the cavity. Finally, we give a practical model which can possibly realize the dynamics of our model in the experiment.
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