Bonnor-type Black Dihole Solution in Brans-Dicke-Maxwell Theory

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Abstract

It was originally thought that Bonnor’s solution in Einstein-Maxwell theory describes a singular point-like magnetic dipole. Lately, however, it has been demonstrated that indeed it may describe a black dihole, i.e., a pair of static, oppositely-charged extremal black holes with regular horizons. Motivated particularly by this new interpretation, in the present work, the construction and extensive analysis of a solution in the context of the Brans-Dicke-Maxwell theory representing a black dihole are attempted. It has been known for some time that the solution-generating algorithm of Singh and Rai produces stationary, axisymmetric, charged solutions in Brans-Dicke-Maxwell theory from the known such solutions in Einstein-Maxwell theory. Thus this algorithm of Singh and Rai’s is employed in order to construct a Bonnor-type magnetic black dihole solution in Brans-Dicke-Maxwell theory from the known Bonnor solution in Einstein-Maxwell theory. The peculiar features of the new solution including internal infinity nature of the symmetry axis and its stability issue have been discussed in full detail.

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I. Introduction

Not surprisingly, exact solutions of general relativity describing multiple black hole configurations are few since they would be highly non-trivial to find or construct. Although one would naturally expect that such configurations might in general involve a very complicated structure, there actually exist some simple solutions having remarkable properties. These include the Majumdar-Papapetrou solutions representing an arbitrary number of static, extremal charged black holes, all with charges of the same sign and the multi-Schwarzschild solution of Israel and Khan. Besides, the C and the Ernst metrics or the cosmological multi-black hole solutions of Kastor and Trachen, all describing the black holes in relative motion, may also fall into this category of multi-black hole solutions.

In the present work, we are particularly interested in Bonnor’s solution. Several years ago, using a technique to generate a static, axisymmetric solution of the Einstein-Maxwell theory from the stationary, axisymmetric Kerr solution of the vacuum Einstein theory, Bonnor constructed a solution describing a magnetic dipole. Bonnor’s solution was originally thought to describe a singular point-like dipole. Recently, the generalization of Bonnor’s solution in Einstein-Maxwell theory to its counterparts in Einstein-Maxwell-dilaton theories with the correct analysis of the conical singularity structure of the solutions was made by Davidson and Gedalin. Interestingly, the new identifications of these solutions with a pair of oppositely-charged extremal black holes (rather than with a conventional singular point-like dipole) have appeared only afterwards. To our knowledge, Gross and Perry were the first to find the Kaluza-Klein analog of the Bonnor’s solution and identify it with a “self-gravitating” pole-antipole configuration. Later on, it has been further refined by the work of Dowker et al. where a thorough analysis of the structure of the solutions and the introduction of the background magnetic field were carried out. Even in these works, the interpretation was made on the basis of the particular structure of the Kaluza-Klein theory, which cannot be applied to solutions with other values of the dilaton. It is also interesting to note that the Kaluza-Klein dipole solution of Gross and Perry has lately been uplifted to its counterpart in $d = 11$ M-theory by Sen, which, via the M/IIA string duality, can be interpreted as the $D6 - \bar{D}6$ pair solution in $d = 10$ IIA supergravity theory. That the magnetic dipole solution of Einstein-Maxwell-dilaton theories, including of course the Bonnor’s solution, actually describes a dihole, i.e., a pair of static, oppositely-charged...
extremal black holes with regular horizons was first demonstrated by Emparan \[13\]. Motivated particularly by this work of Emparan, in the present work, we would like to present the construction and extensive analysis of a solution in the context of the Brans-Dicke-Maxwell theory representing a pair of static, oppositely-charged extremal black holes. As such, it might be relevant to mention the current status of Brans-Dicke (BD) theory as a viable theory of gravity and describe briefly the strategy to be employed to construct a new solution of BD-Maxwell theory from a known solution of Einstein-Maxwell theory.

Perhaps the Brans-Dicke theory \[14\] is the most studied and hence the best-known of all the alternative theories of classical gravity to Einstein’s general relativity, . This theory can be thought of as a minimal extension of general relativity designed to properly accommodate both Mach’s principle \[15\] and Dirac’s large number hypothesis \[15\]. Namely, the theory employs the viewpoint in which the Newton’s constant $G$ is allowed to vary with space and time and can be written in terms of a scalar (“BD scalar”) field as $G = 1/\Phi$. Like in Einstein’s general relativity, to find the exact solutions of the highly non-linear BD field equations is a formidable task. For this reason, algorithms generating exact, new solutions from the known solutions of simpler situations either of the BD theory or of the conventional Einstein gravity have been actively looked for and actually quite a few have been found. To the best of our knowledge, methods thus far discovered along this line includes those of Janis et al., Buchdahl, McIntosh, Tupper \[16\], Tiwari and Nayak \[17\], and Singh and Rai \[18\]. In particular, Tiwari and Nayak \[17\] proposed an algorithm that allows us to generate stationary, axisymmetric solutions in vacuum BD theory from the known Kerr solution \[19\] in vacuum Einstein theory and later on Singh and Rai generalized this method to the one that generates stationary, axisymmetric, charged solutions in BD-Maxwell theory from the known Kerr-Newman (KN) solution \[19\] in Einstein-Maxwell theory. Thus in the present work, we shall employ the solution-generating algorithm of Singh and Rai in order to construct a Bonnor-type magnetic dipole solution in BD-Maxwell theory from the known Bonnor solution in Einstein-Maxwell theory and discuss its unfamiliar nature in full detail.

### II. Solution-generating algorithm of Singh and Rai

We begin by briefly reviewing the algorithm proposed first by Tiwari and Nayak \[17\] and generalized later by Singh and Rai \[18\]. Consider the BD-Maxwell theory described by the
action
\[
S = \int d^4x \sqrt{g} \left[ \frac{1}{16\pi} \left( \Phi R - \omega \frac{\nabla_\alpha \Phi \nabla^\alpha \Phi}{\Phi} \right) - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right]
\]

where \( \Phi \) is the BD scalar field and \( \omega \) is the generic parameter of the theory. Extremizing this action then with respect to the metric \( g_{\mu\nu} \), the BD scalar field \( \Phi \), and the Maxwell gauge field \( A_\mu \) (with the field strength \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \)) yields the classical field equations given respectively by

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\Phi} T^M_{\mu\nu} + 8\pi T^{BD}_{\mu\nu},
\]

where

\[
T^M_{\mu\nu} = F_\mu^\alpha F^\alpha_\nu - \frac{1}{4} g_{\mu\nu} F^\alpha_\beta F^{\alpha\beta},
\]

\[
T^{BD}_{\mu\nu} = \frac{1}{8\pi} \left[ \frac{\omega}{2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \right) + \frac{1}{\Phi} \left( \nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \Phi \right) \right]
\]

and

\[
\nabla_\alpha \nabla^\alpha \Phi = \frac{8\pi}{(2\omega + 3)} T^M_{\lambda\mu} = 0, \quad \nabla_\mu F^{\mu\nu} = 0, \quad \nabla_\mu \tilde{F}^{\mu\nu} = 0
\]

with the last equation being the Bianchi identity and \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}^{\alpha\beta} F^{\alpha\beta} \). And the Einstein-Maxwell theory is the \( \omega \to \infty \) limit of this BD-Maxwell theory. As mentioned in the introduction, the BD theory can be thought of as a minimal extension of general relativity designed to properly accommodate both Mach’s principle \[15\] and Dirac’s large number hypothesis \[15\]. Namely, the theory employs the viewpoint in which Newton’s constant \( G \) is allowed to vary with space and time and can be written in terms of a scalar (“BD scalar”) field as \( G = 1/\Phi \). Note also that in the action and hence in the classical field equations, there are no direct interactions between the BD scalar field \( \Phi \) and the ordinary matter, i.e., the Maxwell gauge field \( A_\mu \). Indeed this is the essential feature of the BD scalar field \( \Phi \) that distinguishes it from “dilaton” fields in other scalar-tensor theories such as Kaluza-Klein theories or low-energy effective string theories where the dilaton-matter couplings generically occur as a result of dimensional reduction. (Here we would like to stress that we shall work in the context of original BD theory format not some conformal transformation of it.) As a matter of fact, it is the original spirit \[14\] of BD theory of gravity in which the BD scalar field \( \Phi \) is prescribed to remain strictly massless by forbidding its direct interaction with matter fields. Now the algorithm of Tiwari and Nayak, and Singh and Rai goes as follows. Let the metric for a stationary, axisymmetric, charged solution to Einstein-Maxwell
field equations take the form
\[ ds^2 = -e^{2U_E}(dt + W_E d\phi)^2 + e^{2(k_E - U_E)}[(dx^1)^2 + (dx^2)^2] + h_E^2 e^{-2U_E} d\phi^2 \] (3)
while the metric for a stationary, axisymmetric, charged solution to BD-Maxwell field equations be
\[ ds^2 = -e^{2U_{BD}}(dt + W_{BD} d\phi)^2 + e^{2(k_{BD} - U_{BD})}[(dx^1)^2 + (dx^2)^2] + h_{BD}^2 e^{-2U_{BD}} d\phi^2 \] (4)
where \( U, W, k \) and \( h \) are functions of \( x^1 \) and \( x^2 \) only. The significance of the choice of the metric in this form has been thoroughly discussed by Matzner and Misner [20] and Misra and Pandey [21]. Tiwari and Nayak, and Singh and Rai first wrote down the Einstein-Maxwell and BD-Maxwell field equations for the choice of metrics in eq.(3) and (4) respectively. Comparing the two sets of field equations closely, they realized that stationary, axisymmetric solutions of the BD-Maxwell field equations are obtainable from those of Einstein-Maxwell field equations provided certain relations between metric functions hold.
That is, if \((W_E, k_E, U_E, h_E, A^E_\mu)\) form a stationary, axisymmetric solution to the Einstein-Maxwell field equations for the metric in eq.(3), then a corresponding stationary, axisymmetric solution to the BD-Maxwell field equations for the metric in eq.(4) is given by \((W_{BD}, k_{BD}, U_{BD}, h_{BD}, A^{BD}_\mu)\) where
\[ W_{BD} = W_E, \quad k_{BD} = k_E, \quad U_{BD} = U_E - \frac{1}{2} \log \Phi, \] (5)
\[ h_{BD} = [h_E]^{(2\omega - 1)/(2\omega + 3)}, \quad \Phi = [h_E]^{4/(2\omega + 3)}, \quad A^{BD}_\mu = A^E_\mu. \]
In the following subsections, we shall apply this solution-generating algorithm to the construction of the Brans-Dicke gravity Minkowski spacetime (i.e., the empty spacetime solution of the Brans-Dicke theory) and the Bonnor-type magnetic dipole solution of Brans-Dicke-Maxwell theory.

(1) Empty spacetime solution in Brans-Dicke theory

We begin with the construction and interpretation of the empty spacetime solution of Brans-Dicke theory which will serve as a reference background geometry for all the other spacetime solutions of Brans-Dicke-Maxwell theory. Then it will allow us eventually to establish the general physical interpretation of all the new solutions that result upon
performing the solution-generating algorithm presented above to the known solutions in general relativity. Consider the Minkowski spacetime metric which clearly is a solution to the vacuum Einstein equation

$$ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (6) $$

For later comparison with its counterpart in Brans-Dicke theory, we also point out that the Newton’s constant $G$ is indeed a constant in this Einstein theory. In order to set this Minkowski metric into the form well prepared for the solution-generating method given above, we first consider the change of radial coordinate $r = e^R$ and hence $dr = e^R dR$ which leads us to rewrite the eq.(6) into

$$ ds^2 = -dt^2 + e^{2R}(dR^2 + d\theta^2) + e^{2R} \sin^2 \theta d\phi^2. \quad (7) $$

Then identifying it with the standard form given in eq.(3), we can read off the metric components as

$$ e^{2U_E} = 1, \quad W_E = 0, \quad (8) $$
$$ e^{2k_E} = e^{2R}, \quad h^2_E = e^{2R} \sin^2 \theta. $$

Now using the solution-generating rule in eq.(5) in the algorithm by Tiwari and Nayak, and Singh and Rai, we can now construct the empty spacetime solution of BD theory as

$$ ds^2 = - \left( r^2 \sin^2 \theta \right)^{-2/(2\omega+3)} dt^2 + \left( r^2 \sin^2 \theta \right)^{2/(2\omega+3)} [dr^2 + r^2 d\theta^2]
+ \left( r^2 \sin^2 \theta \right)^{-2/(2\omega+3)} r^2 \sin^2 \theta d\phi^2, \quad (9) $$
$$ \Phi(r, \theta) = \frac{1}{G} \left( r^2 \sin^2 \theta \right)^{2/(2\omega+3)} $$

where we restored the “bare” Newton’s constant $G$ in the expression for the BD scalar field solution. Among other things, for $\omega \to \infty$, it reduces to the Minkowski spacetime metric with $\Phi = 1/G = \text{constant}$ in eq.(6) as it should since in this limit the Brans-Dicke theory goes over to the Einstein’s general relativity.

Clearly, the solution-generating algorithm of Singh and Rai breaks the spherical symmetry along the way from the Minkowski spacetime to the Brans-Dicke gravity empty spacetime as one can see in eq.(9) above. Close inspection reveals that this happens due to the emergence of non-trivial BD scalar field which essentially represents the inverse of the spacetime-dependent effective Newton’s constant as mentioned earlier, i.e., $\Phi(r, \theta) = \frac{1}{G_{\text{eff}}(r, \theta)}$. In
other words, since the inverse of the BD scalar field representing the strength of the gravitational interaction changes from point to point, both the spherical symmetry and the asymptotic flatness of the usual Minkowski spacetime are broken accordingly.

We now attempt to provide the physical interpretation of the nature of this empty spacetime solution of BD theory in a careful manner. This, however, cannot be achieved unless some particular finite value of the BD parameter is specified. Therefore we first need to appreciate the physical meaning of the dimensionless BD parameter $\omega$. In the BD gravity action given in eq.(1), the term $\sim \omega(\nabla_\alpha \Phi \nabla^{\alpha} \Phi / \Phi)$, like other terms in the action, should be finite. Thus large $\omega$ indicates the regime where the BD scalar field $\Phi$ remains nearly constant as the inverse of the present value of the Newton’s constant. Namely, this regime amounts to the Einstein’s general relativity limit. On the other hand, the small $\omega$ indicates the regime in which the BD scalar field varies sizably with space and time and hence the theory deviates largely from the general relativity. We have already noted that in the limit $\omega \to \infty$, the BD gravity empty spacetime in eq.(9) does reduce to the Minkowski spacetime metric with $\Phi = 1/G = \text{constant}$ as it should. Thus next we consider the nature of this spacetime solution in the other limit, namely for small $\omega$ values. We shall particularly pick the value of the $\omega$ parameter to be $\omega = -2$ which lies in the range of later interest, $-5/2 \leq \omega < -3/2$ (which has particular significance that will be explained in the next subsection where we shall discuss the Bonnor-type magnetic dipole solution in Brans-Dicke-Maxwell theory). For this value of the BD $\omega$ parameter, the BD gravity empty spacetime solution becomes

$$
g_{tt} = -(r^2 \sin^2 \theta)^2, \quad g_{rr} = g_{\theta\theta} = \frac{1}{(r^2 \sin^2 \theta)^2}, \quad g_{\phi\phi} = (r^2 \sin^2 \theta)^3,$$

$$\Phi(r, \theta) = \frac{1}{G} (r^2 \sin^2 \theta)^{-2}. \quad (10)$$

Among others, it is interesting to note that from the behavior of the metric function $g_{tt}$, one can realize that near the origin $r = 0$ and the symmetric axis $\theta = 0, \pi$, the infinite time delay occurs and hence all the dynamics (if exist) would be frozen there. This is reminiscent of what happens at the horizon of a black hole. Next, from the behavior of the metric functions $g_{rr}$ and $g_{\theta\theta}$, one can notice that both the origin and the symmetry axis are infinite proper distance away! Namely, they turn out to be internal infinities. Again this is very reminiscent of the feature of a black hole horizon which is indeed an internal infinity to a distant observer. We now attempt to interpret this peculiar geometric structure of the BD gravity empty spacetime solution in terms of the spacetime-dependent behavior.
of the BD scalar field which, as mentioned earlier, serves as the inverse of an effective Newton’s constant, \( \Phi(r, \theta) \equiv G_{\text{eff}}^{-1}(r, \theta) \). First, as we just pointed out, both the origin and the symmetry axis are \textit{infinite} proper distance away

\[
\int_r^0 \sqrt{g_{rr}} dr = \int_r^0 \frac{1}{r^2 \sin^2 \theta} dr \to \infty, \quad (11)
\]

\[
\int_0^\theta \sqrt{g_{\theta\theta}} d\theta = \int_0^\theta \frac{1}{r \sin^2 \theta} d\theta \to \infty \quad (12)
\]

while the asymptotic region \( r = \infty \) is \textit{finite} proper distance away

\[
\int_r^\infty \sqrt{g_{rr}} dr = \int_r^\infty \frac{1}{r^2 \sin^2 \theta} dr < \infty. \quad (13)
\]

It seems that this can be attributed to the spacetime-dependent behavior of the \textit{effective} Newton’s constant

\[
\Phi^{-1}(r, \theta) = G_{\text{eff}}(r, \theta) = G(r^2 \sin^2 \theta)^2. \quad (14)
\]

Namely, since the Newton’s constant is a measure of the strength of the gravitational interaction, this spacetime-dependent behavior of \( G_{\text{eff}} \) implies that the gravitational interaction vanishes at the origin and along the symmetry axis while it diverges asymptotically as \( r \to \infty \). To be a little more concrete, the regions such as the origin and the symmetry axis are essentially inaccessible as the gravitational interaction is effectively absent and hence no test object would get accelerated (or at least be in motion) there. Meanwhile the coordinate infinity \( r = \infty \) can be in a finite proper distance as the gravitational interaction grows as \( \sim r^4 \) and eventually diverges at \( r = \infty \) and thus any test particle there would get accelerated infinitely. Namely, the physical (proper) distance between the two spacetime points gets larger as the effective gravitational interaction there gets weaker whereas it gets smaller as the effective interaction there gets stronger. In order to support this type of interpretation in terms of the spacetime-dependent behavior of the effective Newton’s constant, one can take some other values of the BD \( \omega \) parameter and see if essentially the same interpretation still holds true. For instance if one takes, say, \( \omega = 1/2 \) or \( 5/2 \) which lie outside of the above-mentioned range, \( -5/2 \leq \omega < -3/2 \), just the other way around is the case. Namely, the origin and the symmetry axis are finite proper distance away (indeed instantly accessible) as the gravitational interaction strength diverges there while the asymptotic region \( r = \infty \) is infinite proper distance away as the gravitational interaction strength vanishes there. And of course in this analysis, the “backreaction” of the test particle to the background spacetime
that would modify the background spacetime structure itself is neglected. The peculiar features of the BD gravity empty spacetime geometry, particularly the fact that the symmetry axis actually is an internal infinity, in turn, leads to the occurrence of rather an embarrassing conical singularity and we now turn to this issue. Notice that the conical singularity, or more specifically, the deficit angle can be defined in the following manner. Take the ratio between the proper circumference and the proper radius of a small limiting circle around the symmetry axis \( \theta = 0, \pi \). If this ratio comes out to be exactly \( 2\pi \), then there is no deficit angle. If instead it turns out to be less than \( 2\pi \), then there exists non-vanishing deficit angle. Being guided by this criterion, we now compute the possible deficit angle in the BD gravity empty spacetime geometry.

\[
\delta_{(0,\pi)} = 2\pi - \left| \frac{\Delta \phi d\sqrt{g_{\phi\phi}}}{\sqrt{g_{\theta\theta}}} \right|_{\theta=0,\pi} = 2\pi \tag{15}
\]

where \( \Delta \phi \) is the period of the azimuthal angle coordinate that will be chosen as \( 2\pi \) and we used \( g_{\theta\theta} = r^2/(r^2 \sin^2 \theta)^2 \) and \( g_{\phi\phi} = (r^2 \sin^2 \theta)^3 \) and hence \( (g_{\theta\theta})^{-1/2}d\sqrt{g_{\phi\phi}}/d\theta|_{\theta=0,\pi} = 0 \). This demonstrates that this seemingly pathological “\( 2\pi \)-angle deficit” is indeed a direct consequence of our earlier observation that the proper radius of a small limiting circle around the symmetry axis diverges, \( \int_0^\pi \sqrt{g_{\theta\theta}} d\theta \to \infty \) as the symmetry axis can be thought of as an internal infinity. Put differently, “\( 2\pi \)-angle deficit” is not a generic pathology but just another manifestation of the emergence of the internal infinity, i.e., the infinitely far symmetry axis.

(2) Bonnor-type magnetic dipole solution in Brans-Dicke-Maxwell theory

Next, we apply the same method to obtain the Bonnor-type magnetic dipole solution in BD-Maxwell theory from the known Bonnor solution \[6\] in Einstein-Maxwell theory. And to do so, again one needs some preparation which involves casting the Bonnor solution given in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) in the metric form in eq.(3) by performing a coordinate transformation (of \( r \) alone) suggested by Misra and Pandey \[21\]. Namely, we start with Bonnor’s magnetic dipole (dihole) solution of Einstein-Maxwell theory written in Boyer-Lindquist coordinates \[6, 13\].

\[
ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)^2 \left[ -dt^2 + \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \left( \frac{dr^2}{\Delta} + d\theta^2 \right) \right]
\]
\[ A = A_\mu dx^\mu = \frac{2aMr \sin^2 \theta}{\Delta + a^2 \sin^2 \theta}d\phi \]

where \( \Sigma = r^2 - a^2 \cos^2 \theta \) and \( \Delta = r^2 - 2Mr - a^2 \) with \( M \) denoting the ADM mass and \( a \) representing (roughly) the proper separation between the poles. Consider now the transformation of the radial coordinate

\[ r = e^R + M + \frac{(M^2 + a^2)}{4} e^{-R} \]

which gives \( dr^2/\Delta = dR^2 \). Indeed, this transformation has been deduced from the associated one introduced by Misra and Pandey \cite{21} for the case of single rotating (Kerr) black hole. Then Bonnor’s magnetic dipole solution can now be cast in the form in eq.(3), i.e.,

\[ ds^2 = - \left(1 - \frac{2ML}{\Sigma}\right)^2 dt^2 + \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \left(1 - \frac{2ML}{\Sigma}\right)^2 (dR^2 + d\theta^2) \]

\[ + \left(1 - \frac{2ML}{\Sigma}\right)^{-2} \Delta \sin^2 \theta d\phi^2 \]

with now \( \Sigma = L^2 - a^2 \cos^2 \theta \) and \( \Delta = L^2 - 2ML - a^2 \) where we set, as a short-hand notation, \( L \equiv e^R + M + \frac{(M^2 + a^2)}{4} e^{-R} \). Then identifying it with the standard form given in eq.(3), we can read off the metric components as

\[ e^{2U_E} = \left(1 - \frac{2ML}{\Sigma}\right)^2 = \left(\frac{\Delta + a^2 \sin^2 \theta}{\Sigma}\right)^2, \quad W_E = 0, \]

\[ e^{2k_E} = \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \left(1 - \frac{2ML}{\Sigma}\right)^4, \quad h^2_E = \Delta \sin^2 \theta. \]

Now using the solution-generating rule in eq.(5) in the algorithm by Tiwari and Nayak, and Singh and Rai, we can now construct the Bonnor-type magnetic dipole solution in BD-Maxwell theory as

\[ ds^2 = - \left(1 - \frac{2ML}{\Sigma}\right)^2 (L^2 - 2ML - a^2)^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)}(2\omega+3) \theta dt^2 \]

\[ + \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \left(1 - \frac{2ML}{\Sigma}\right)^2 (L^2 - 2ML - a^2)^{2/(2\omega+3)} \sin^{4/(2\omega+3)}(2\omega+3) \theta [dR^2 + d\theta^2] \]

\[ + \left(1 - \frac{2ML}{\Sigma}\right)^{-2} (L^2 - 2ML - a^2)^{(2\omega+1)/(2\omega+3)} \sin^{2(2\omega+1)/(2\omega+3)}(2\omega+3) \theta d\phi^2, \]

\[ \Phi(R, \theta) = (L^2 - 2ML - a^2)^{2/(2\omega+3)} \sin^{4/(2\omega+3)}(2\omega+3) \theta, \]

\[ A = \frac{2aML \sin^2 \theta}{\Delta + a^2 \sin^2 \theta} d\phi. \]
Finally by transforming back to the standard Boyer-Lindquist coordinates using eq.(7), we arrive at the Bonnor-type dipole solution in Boyer-Lindquist coordinates [22] given by

\[
ds^2 = \left(\frac{\Delta + a^2 \sin^2 \theta}{\Sigma}\right)^2 \left[-(\Delta \sin^2 \theta)^{-2/(2\omega+3)} dt^2 + (\Delta \sin^2 \theta)^{2/(2\omega+3)} \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \right]
\]

\[
\times \left(\frac{dy^2}{\Delta} + d\theta^2\right) + \left(\frac{\Sigma}{\Delta + a^2 \sin^2 \theta}\right)^2 (\Delta \sin^2 \theta)^{(2\omega+1)/(2\omega+3)} d\phi^2,
\]

\[
\Phi(r, \theta) = \frac{1}{G} (\Delta \sin^2 \theta)^{2/(2\omega+3)}, \quad A = \frac{2aMr \sin^2 \theta}{\Delta + a^2 \sin^2 \theta} d\phi,
\]

where we restored the “bare” Newton’s constant \(G\) in the expression for the BD scalar field solution. This solution is static and axisymmetric. It is interesting to note that this solution becomes the flat Minkowski spacetime asymptotically as \(r \to \infty\) or for \(M = 0\) and \(a = 0\) only as \(\omega \to \infty\) which amounts to the general relativity limit. And for finite \(\omega\), upon setting \(M = 0 = a\) as well as \(A = 0\), this solution reduces to the empty spacetime solution of BD theory given in eq.(9) as it should. Besides, this solution possesses singularities along the symmetry axis \(\theta = 0, \pi\) as well as at \(r = r_+ = M + \sqrt{M^2 + a^2}\) where \(\Delta = 0\) for finite \(\omega\). As we shall see later on, these are not curvature singularities but coordinate ones on which all the curvature invariants remain finite and indeed the symmetry axis of the geometry represented by this solution consists of the semi-infinite lines \(\theta = 0, \pi\) and the segment \(r = r_+\). And as we shall see in a moment, at each of the “poles” \((r = r_+, \theta = 0), (r = r_+, \theta = \pi)\), lies two extremal oppositely charged black holes of the BD-Maxwell theory. In addition, from the asymptotic \((r \to \infty)\) behavior of the magnetic vector potential given above, one can deduce that the solution involves the magnetic dipole moment of \(2Ma\). Thus the solution can be identified with a magnetic dipole solution. Associated with this last point, note that changing the sign of \(a\) amounts to reversing the orientation of the dipole, so we will consider, without any loss of generality, \(a \geq 0\). Another point worthy of note is that even if we take the limit \(a \to 0\), this Bonnor-type magnetic dipole solution fails to reduce to the single Brans-Dicke-Schwarzschild black hole solution. Noticing that the parameter, \(a\), can be regarded (for large \(a\)) as indicating the proper separation between the two opposite charges \([10, 12, 13]\), this observation reveals a peculiar feature that as \(a \to 0\), the pair of opposite magnetic charges (or black holes) never appear to merge and the limiting solution maintains axisymmetric structure. This point is reminiscent of the magnetic black dihole solutions in other Einstein-Maxwell-dilaton theories like Kaluza-Klein or low energy string theories as well as the general relativity coupled to the Maxwell theory \([13]\).
Originally, Bonnor’s magnetic dipole solution in Einstein-Maxwell theory was thought to describe a singular, point-like dipole. Recently, however, Emparan \[13\] demonstrated that it actually describes a “black dihole” which has regular horizons. Emparan’s argument for the black dihole interpretation of original Bonnor’s solution goes as follows. Bonnor’s solution is clearly asymptotically flat as \(r \to \infty\) and in this asymptotic region, the gauge field solution becomes that of a magnetic dipole. Next, although the solution exhibits apparent singularities at \(r = r_+\) (where \(\Delta = r^2 - 2Mr - a^2 = 0\)), they need to be treated more carefully. Note first that the axis of symmetry of the solution (namely the fixed point set of the Killing field \((\partial/\partial \phi)\)) consists of the semi-infinite lines, \(\theta = 0, \pi\) (running from \(r = r_+\) to \(r = \infty\)) and the segment \(r = r_+\) (running from \(\theta = 0\) to \(\theta = \pi\)). Then a crucial feature of the Bonnor’s solution is that at each of the poles, \((r = r_+, \theta = 0)\) and \((r = r_+, \theta = \pi)\), lies a distorted extremal Reissner-Nordstrom black holes.

Thus it is of equal interest to study if the Bonnor-type dipole solution in BD-Maxwell theory constructed in the present work can describe dihole geometry as well and if so, under what condition. Among others, the bottomline condition for possible dihole interpretation concerns the particular values of the BD \(\omega\) parameter. Note that some time ago it has been pointed out \[23\] that in order for the BD-Reissner-Nordstrom solution (or more generally, for the BD-Kerr-Newman solution) to describe a non-trivial, \textit{regular}, charged black hole spacetime, the \(\omega\)-parameter of the BD theory should take values in the range, \(-\frac{5}{2} \leq \omega < -\frac{3}{2}\). Here, by “regular” it means that the black hole spacetime possesses regular Killing horizons at which the invariant curvature polynomials such as \(R, R_{\mu \nu}R^{\mu \nu},\) and \(R_{\mu \nu \alpha \beta}R^{\mu \nu \alpha \beta}\) remain nonsingular. Thus hereafter we shall assume that the BD \(\omega\) parameter takes values in this particular range.

Again, our eventual objective is the dihole interpretation of the BD-Bonnor solution in eq.(21). Prior to this, however, it seems necessary to discuss the peculiar geometric structure of this BD-Bonnor solution that appears to inherit essentially the same generic features of the BD gravity empty spacetime solution we studied earlier in the previous subsection. Firstly for finite \(\omega\) value, this BD-Bonnor solution in eq.(21) fails to be asymptotically flat for the following reason. Namely, the solution-generating algorithm of Singh and Rai \textit{breaks} the asymptotic flatness along the way essentially due to the emergence of non-trivial BD scalar field representing the inverse of the spacetime-dependent effective Newton’s constant. This is a generic nature of the Singh and Rai’s solution-generating algorithm that we first
realized in the construction of the BD gravity empty spacetime solution discussed in the previous subsection. Next, the symmetry axis again turns out to be some kind of internal infinity. To see this, consider the BD-Bonnor metric solution for the value \( \omega = -2 \) which lies in the range, \(-5/2 \leq \omega < -3/2\),

\[
\begin{align*}
g_{tt} & \sim - (\Delta \sin^2 \theta)^2, \\
g_{rr} &= g_{\theta\theta}/\Delta \sim \frac{1}{(\Delta \sin^2 \theta)^2}, \\
g_{\phi\phi} &\sim (\Delta \sin^2 \theta)^3, \\
\Phi(r, \theta) &= \frac{1}{G} (\Delta \sin^2 \theta)^{-2}. \quad (22)
\end{align*}
\]

From this, we can readily realize that the symmetry axis is infinite proper distance away

\[
\int_\theta^0 \sqrt{g_{\theta\theta}} d\theta \sim \int_\theta^0 \frac{1}{\sin^2 \theta} d\theta \rightarrow \infty \quad (23)
\]

while the asymptotic region \( r = \infty \) is finite proper distance away

\[
\int_r^\infty \sqrt{g_{rr}} dr \sim \int_r^\infty \frac{1}{\Delta \sin^2 \theta} dr < \infty. \quad (24)
\]

Like before, this can be attributed to the spacetime-dependent behavior of the effective Newton’s constant

\[
\Phi^{-1}(r, \theta) = G_{\text{eff}}(r, \theta) = G(\Delta \sin^2 \theta)^2. \quad (25)
\]

Namely, since the Newton’s constant is a measure of the strength of the gravitational interaction, this spacetime-dependent behavior of \( G_{\text{eff}} \) implies that the gravitational interaction vanishes on the symmetry axis while it diverges asymptotically as \( r \rightarrow \infty \). More specifically, the region such as the symmetry axis is essentially inaccessible as the gravitational interaction is effectively absent there whereas the coordinate infinity \( r = \infty \) can be in a finite proper distance as the gravitational interaction grows as \( \sim r^4 \) and eventually diverges at \( r = \infty \). That is, the physical (proper) distance between the two spacetime points gets larger as the effective gravitational interaction there gets weaker whereas it gets smaller as the effective interaction there gets stronger. Again, this is indeed the same generic feature of any spacetime solution of the BD gravity theory that we first encountered in the case of the BD gravity empty spacetime solution before. Then next, we are led to anticipate that the internal infinity nature of the symmetry axis we just observed would likewise lead to the occurrence of the conical singularity in the geometry described by this BD-Bonnor solution. Thus we turn now to the study of possible conical singularity structure of this BD-Bonnor
solution. First observe that the rotational Killing field $\psi^\mu = (\partial / \partial \phi)^\mu$ possesses vanishing norm, i.e., $\psi^\mu \psi_\mu = g_{\alpha \beta} \psi^\alpha \psi^\beta = g_{\phi \phi} = \left[ \Sigma / (\Delta + \alpha^2 \sin^2 \theta) \right]^2 (\Delta \sin^2 \theta)^{(2 \omega + 1) / (2 \omega + 3)} = 0$ at the locus of $r = r_+$ as well as along the semi-infinite lines $\theta = 0, \pi$ for $-5/2 \leq \omega < -3/2$. This implies that $r = r_+$ can be thought of as a part of the symmetry axis of the solution. Namely unlike the other familiar axisymmetric solutions, for the case of the BD-Bonnor solution under consideration (and of course for the original Bonnor solution as well), the endpoints of the two semi-axes $\theta = 0$ and $\theta = \pi$ do not come to join at a common point. Instead, the axis of symmetry is completed by the segment $r = r_+$. And as $\theta$ varies from 0 to $\pi$, one moves along the segment from $(r = r_+, \theta = 0)$ where a black hole is situated to $(r = r_+, \theta = \pi)$ where the other hole with opposite charge is placed. Obviously, then the natural question to be addressed is whether or not the conical singularities arise on different portions of the symmetry axis. Thus once again we define the notion of conical angle deficit (or excess). Consider that ; if $C$ is the proper length of the circumference around the symmetry axis and $R$ is its proper radius, then the occurrence of a conical angle deficit (or excess) $\delta$ would manifest itself if $(dC/dR)|_{R \to 0} = 2\pi - \delta$. We now proceed to evaluate this conical deficit (or excess). The conical angle deficit along the axes $\theta = 0, \pi$ and along the segment $r = r_+$ are given respectively by

$$
\delta_{(0, \pi)} = 2\pi - \left. \frac{\Delta \phi d \sqrt{g_{\phi \phi}}}{\sqrt{g_{\theta \theta} d\theta}} \right|_{\theta = 0, \pi} = 2\pi,
$$

$$
\delta_{(r=r_+)} = 2\pi - \left. \frac{\Delta \phi d \sqrt{g_{\phi \phi}}}{\sqrt{g_{rr} dr}} \right|_{r=r_+} = 2\pi
$$

where, of course, we used the BD-Bonnor metric solution given above in eq.(21) for $\omega$-parameter of the BD theory taking values in the range, $-5/2 \leq \omega < -3/2$. Just as we expected, we encounter the seemingly pathological “$2\pi$-angle deficit” again. This, of course, is no surprise as it is not a generic pathology but just another manifestation of the emergence of the internal infinity, i.e., the symmetry axis. Namely, as the symmetry axis is infinite proper distance away, we, as a consequence, have $(g_{\theta \theta})^{-1/2} d \sqrt{g_{\phi \phi} / d\theta}|_{\theta = 0, \pi} = 0$ which, in turn, results in $\delta_{(0, \pi)} = 2\pi$. Once again, this conical singularity structure is not a separate pathology but a generic feature that appears to inherit essentially the same thing of the BD gravity empty spacetime solution we studied earlier in the previous subsection.

Note that these peculiar features are the generic characteristics of the static solutions of the BD-Maxwell theory that appear as long as we confine our interest to the case of the black “dihole” interpretation, namely, as long as we assume $-5/2 \leq \omega < -3/2$ and are
really somethings that do not show up in the case of the Einstein-Maxwell theory or other scalar-tensor theories such as the minimal Kaluza-Klein theory or the bosonic sector of the low energy effective string theories [13].

Lastly, therefore, we need to ask, prior to its black dihole interpretation, whether or not the BD-Bonnor solution given in eq. (21) is qualified at all to be endowed with any meaning as a physical spacetime solution. To answer this, recall that for a value of the BD $\omega$ parameter in the range of interest $-5/2 \leq \omega < -3/2$, the BD gravity empty spacetime solution was given an acceptable physical meaning or interpretation in terms of the spacetime-dependent behavior of the effective Newton’s constant. Next, since we keep employing the same solution-generating algorithm of Singh and Rai, the Bonnor-type solution of BD-Maxwell theory can be thought of as the one that results when one puts a pair of dipole (or oppositely-charged black holes as we shall demonstrate shortly) on opposite sides along the symmetry axis of the empty spacetime. As can be expected, however, the generic peculiar features of the spacetime such as the internal infinity nature of the symmetry axis and its consequence, namely the conical singularity still survive the addition process of the dipole content. Thus the BD-Bonnor solution deserves essentially the same physical meaning or interpretation as that of the BD gravity empty spacetime solution we studied earlier. The only delicate issue to be addressed would be whether or not the dipole lying along the symmetry axis is really physically meaningful when the symmetry axis is an internal infinity. At least it seems to us that it should nevertheless be of enough physical interest as long as we take the BD gravity empty spacetime solution seriously as the one that embodies some spirit of Mach’s principle, namely that the Newton’s constant may take different values in different spacetime points. This issue also can be viewed from a different angle. Here we are worried about whether or not a pair of black holes lying along an infinitely far symmetry axis can be of any meaning. Along this line we need to recall an essentially analogous (single) black hole case in which the horizon (namely, the black hole itself) is infinite proper distance away to an asymptotic observer. In this sense, the BD-Bonnor spacetime is indeed nearly as weird as the familiar black hole solutions.

Since we accepted the BD-Bonnor solution as having a physical meaning, we now proceed to elaborate on the magnetic black dihole spacetime interpretation of this solution in some more detail. Particularly, we would like to show explicitly that at each of the poles, $(r = r_+, \theta = 0)$ and $(r = r_+, \theta = \pi)$, lies a distorted extremal BD-Reissner-Nordstrom black holes in a man-
ner similar to what happens in the Bonnor’s magnetic dihole solution in Einstein-Maxwell theory first demonstrated by Emparan. To this end, we begin by performing the change of coordinates from \((r, \theta)\) to \((\rho, \bar{\theta})\) given by the following transformation law [10, 12, 13]

\[
r = r_+ + \frac{\rho}{2}(1 + \cos \bar{\theta}), \quad \sin^2 \theta = \frac{\rho}{\sqrt{M^2 + a^2}}(1 - \cos \bar{\theta})
\]

where \(\Delta(r_+) = 0\), namely \(r_+ = M + \sqrt{M^2 + a^2}\) as introduced earlier. Then one can realize that upon changing the coordinates as given above and taking \(\rho\) to be much smaller than any other length scale involved so as to get near each pole, the Bonnor-type magnetic dipole solution in BD-Maxwell theory given earlier becomes

\[
ds^2 \simeq g^2(\bar{\theta}) \left[-(\rho^2 \sin^2 \bar{\theta})^{-2/(2\omega + 3)} \left(\frac{q}{\rho}\right)^2 dt^2 + (\rho^2 \sin^2 \bar{\theta})^{-2/(2\omega + 3)} \left(\frac{q}{\rho}\right)^2 (d\rho^2 + \rho^2 d\bar{\theta}^2) \right] + g^{-2}(\bar{\theta})(\rho^2 \sin^2 \bar{\theta})^{-2/(2\omega + 3)} \left(\frac{q}{\rho}\right)^2 \rho^2 \sin^2 \bar{\theta} d\phi^2,
\]

\[
\Phi(\rho, \bar{\theta}) \simeq \frac{1}{G}(\rho^2 \sin^2 \bar{\theta})^{2/(2\omega + 3)},
\]

\[
A \simeq q \frac{a}{\sqrt{M^2 + a^2}} g^{-1}(\bar{\theta})(1 - \cos \bar{\theta})d\phi
\]

where \(q \equiv \frac{Mr}{\sqrt{M^2 + a^2}}\) and \(g(\bar{\theta}) \equiv [\cos^2(\bar{\theta}) + a^2/(M^2 + a^2) \sin^2(\bar{\theta})]\). Certainly, this limit of the solution can be identified with the “near-horizon limit” \((r \to r_+, \rho \to 0)\) of a distorted extremal charged black hole in BD-Maxwell theory with the distortion being described by the factor \(g(\bar{\theta})\). And due to this distortion factor \(g(\bar{\theta})\), the geometry is not spherically symmetric, rather it is elongated along the axis in a prolate shape. And of course this deformation of the horizon geometry (to a prolate spheroid) is due to the field created by the other hole and possibly by the conical defect to which we shall come back in a moment.

The limiting geometry above were valid for arbitrary value of “a” as long as we remain close enough to (one of the) poles. If instead we consider the limit of very large “a”, while keeping \((r - r_+)\) and \(a \sin^2 \theta\) finite, the Bonnor-type magnetic dipole solution reduces, in this time, to

\[
ds^2 \simeq - \left(1 + \frac{q}{\rho}\right)^{-2} (\rho^2 \sin^2 \bar{\theta})^{-2/(2\omega + 3)} dt^2 + \left(1 + \frac{q}{\rho}\right)^2 \left[(\rho^2 \sin^2 \bar{\theta})^{2/(2\omega + 3)} (d\rho^2 + \rho^2 d\bar{\theta}^2) + (\rho^2 \sin^2 \bar{\theta})^{-2/(2\omega + 3)} \rho^2 \sin^2 \bar{\theta} d\phi^2 \right],
\]

\[
\Phi(\rho, \bar{\theta}) \simeq \frac{1}{G}(\rho^2 \sin^2 \bar{\theta})^{2/(2\omega + 3)},
\]

\[
A \simeq q(1 - \cos \bar{\theta})d\phi
\]
with \( q = \frac{Mr_+}{\sqrt{M^2 + a^2}} \to M \) as \( a \to \infty \).

As we shall demonstrate below, this limit of the solution can be recognized as representing the extremal BD-Reissner-Nordstrom black hole (i.e., the charged single black hole solution in BD-Maxwell theory). Indeed, this result was rather expected since physically, taking the limit \( a \to \infty \), amounts to pushing one of the poles to a large distance and studying the geometry of the remaining pole. We now demonstrate in an explicit manner that the above limit of the BD-Bonnor solution indeed represents the BD-Reissner-Nordstrom black hole spacetime. To this end, we consider the change of the radial coordinate (which takes \( \rho \) to the standard Boyer-Lindquist radial coordinate \( R \)) given by

\[
R = (\rho + q).
\]

Then one ends up with

\[
ds^2 = \left[ R^2 \left( 1 - \frac{q}{R} \right)^2 \sin^2 \tilde{\theta} \right]^{-2/(2\omega + 3)} \left[ - \left( 1 - \frac{q}{R} \right)^2 dt^2 + R^2 \sin^2 \tilde{\theta} d\phi^2 \right] + \left[ R^2 \left( 1 - \frac{q}{R} \right)^2 \sin^2 \tilde{\theta} \right]^{2/(2\omega + 3)} \left[ - \left( 1 - \frac{q}{R} \right)^{-2} dR^2 + R^2 d\tilde{\theta}^2 \right].
\]

Certainly, this can be identified with the extremal, charged non-rotating black hole solution in BD-Maxwell theory found in [23]. To check this, one only needs to take the non-rotating limit of the BD-Kerr-Newman black hole solution there and set \( r \to R \), \( \theta \to \tilde{\theta} \) and \( e \to q \) (see the appendix). Then one gets the BD-Reissner-Nordstrom black hole solution which is, in our present coordinates \((t, R, \tilde{\theta}, \phi)\), given by eq.(31).

To summarize, based on these observations, we may conclude that the BD-Bonnor solution given above indeed describes a black dihole geometry.

**III. Bonnor-type magnetic dipole solution with Melvin-type universe content in Brans-Dicke-Maxwell theory**

In the preceding section, we have constructed the Bonnor-type magnetic dipole solution in BD-Maxwell theory and demonstrated that indeed it can be identified with a “black dihole” solution particularly for values of the BD \( \omega \)-parameter in the range \(-5/2 \leq \omega < -3/2 \) representing a configuration in which a pair of oppositely charged extreme regular black holes is placed at each of the poles, \((r = r_+, \theta = 0)\) and \((r = r_+, \theta = \pi)\) respectively.
However, since we failed to address the detailed nature of conical singularities along the symmetry axis, $\theta = 0, \pi$ and along the segment, $r = r_+$ for the reason stated in detail earlier, we are still away from the satisfactory understanding of how the two oppositely charged black holes can be sustained in an (un)stable equilibrium against the combined gravitational (same mass) and gauge or electromagnetic (opposite charge) attractions. Usually, if there are finite conical angle deficits along the semi-infinite axes, $\theta = 0, \pi$ (as is the case with generalized Bonnor solutions in Einstein-Maxwell-dilaton theories [13]), one interprets these angle deficits as the presence of cosmic strings providing tension that pulls the pole and the antipole at the endpoints apart. Namely, in this interpretation, the tension generated by the cosmic strings counterbalances the combined gravitational and gauge attractions and holds the dipole (or dihole) apart against the collapse. Indeed, the emergence of the conical singularities signals the instability of the configuration and by the recourse to the presence of cosmic strings with proper tension, one achieves the stability as the line $r = r_+, 0 < \theta < \pi$ joining the poles can be made to be completely non-singular then. Alternatively, perhaps another very relevant approach toward the stabilization of the system one can turn to would be to introduce an external magnetic field, aligned with the axis joining the dipole, to counterbalance the combined gravitational and gauge attractive forces by pulling them apart. By properly “tuning” the strength of the magnetic field, the internal stresses along the axis would be rendered to vanish. And this can be checked in an explicit manner if the introduction of the magnetic field with “right” strength leads to the vanishing of the finite conical singularities along the symmetry axis, $\theta = 0, \pi$ and along the segment, $r = r_+$ at the same time. Particularly for the case at hand in which the cosmic string interpretation is not available for the reason stated earlier, this second option involving the introduction of external magnetic field looks quite appropriate. Note also that the conical singularity structure of the generalized Bonnor solution in Einstein-Maxwell-dilaton theories [13] and its cure via the introduction of external magnetic field is reminiscent of the Ernst’s prescription [4] for the elimination of conical singularities of the charged C-metric in Einstein-Maxwell theory. In this section, therefore, we attempt to construct the Bonnor-type magnetic dipole solution of Brans-Dicke-Maxwell theory containing the magnetic field content that asymptotes to the “Melvin-type” magnetic universe which is a flux tube that provides the best possible approximation to a self-gravitating uniform magnetic field. And to this end, we shall employ
the Ehlers-Harrison-type transformation generalized to the BD-Maxwell theory which, in the Einstein-Maxwell theory, has been known to generate an axisymmetric solution with a magnetic field content from a known axisymmetric solution without. It, however, is interesting to note that there indeed are two different ways of achieving this. The first route is to start with the Bonnor-type magnetic dipole solution obtained in the preceding section and apply to it the generalized Ehlers-Harrison transformation to arrive at our destination, i.e., the BD-Bonnor solution with magnetic Melvin-type universe content. Meanwhile, the second route is to start instead with the Bonnor solution with magnetic Melvin universe content in Einstein-Maxwell theory (which is already known in the literature \[13\]) obtained from the original Bonnor solution via the Ehlers-Harrison transformation and then apply to it the Singh and Rai’s algorithm which is known as a solution-generating method that allows one to generate a stationary axisymmetric charged solutions in BD-Maxwell theory from their counterparts in Einstein-Maxwell theory. As we shall see in a moment, these two methods yield the same final solution and this indicates, among others, that the two separate solution-generating algorithms, the Singh and Rai’s one and the Ehlers-Harrison transformation indeed commute. And in the following we shall discuss both two routes.

(1) Route 1

In this subsection, therefore, we construct the BD-Bonnor solution with magnetic Melvin-type universe content via the first route stated above. Consider the BD gravity theory coupled to Maxwell field described by the action

\[
S = \int d^4 x \sqrt{g} \left[ \frac{1}{16 \pi} \left( \Phi R - \omega g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi \right) - \frac{1}{4} g^{\mu \alpha} g^{\nu \beta} F_{\mu \nu} F_{\alpha \beta} \right]. \tag{32}
\]

We now consider the Weyl-rescaling given by

\[
g_{\mu \nu} = \Omega^2(x) \tilde{g}_{\mu \nu}, \quad \Phi = M_{pl}^2 \Psi / \Psi_0 \tag{33}
\]

with \( \Omega^2(x) = \frac{M_{pl}^2}{\Phi} \), \( \Psi_0^2 = \frac{M_{pl}^2}{16 \pi} (2 \omega + 3) \)

where \( M_{pl} \) denotes the Planck mass. Under this Weyl-rescaling, the action above transforms to

\[
\tilde{S} = \int d^4 x \sqrt{\tilde{g}} \left[ \frac{1}{16 \pi} \left( \tilde{R} - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \tilde{\Psi} \partial_\nu \tilde{\Psi} \right) - \frac{1}{4} \tilde{g}^{\mu \alpha} \tilde{g}^{\nu \beta} F_{\mu \nu} F_{\alpha \beta} \right]. \tag{34}
\]
Namely, upon this Weyl-rescaling, the action of the BD gravity coupled to the Maxwell field takes the form of that of the Einstein-Maxwell-dilaton theory in which the dilaton field is a minimally-coupled, massless (Weyl-rescaled) BD scalar field.

Then we are now ready to perform the “generalized” Ehlers-Harrison transformation (already known in the literature [13]) which takes an axisymmetric solution of the Einstein-Maxwell-dilaton theory to another such solution containing particularly the (asymptotically) Melvin-type magnetic universe content. It is given by

\[
\begin{align*}
\tilde{g}_{ij}' &= \lambda^2 \tilde{g}_{ij} \ (i, j \neq \phi), \quad \tilde{g}_{\phi\phi}' = \lambda^{-2} \tilde{g}_{\phi\phi}, \\
\Psi' &= \Psi, \quad A'_{\phi} = -\frac{2}{B\lambda} \left[ 1 + \frac{1}{2} BA_{\phi} \right] + \frac{2}{B}, \\
\end{align*}
\]

\[\lambda \equiv \left[ 1 + \frac{1}{2} BA_{\phi} \right]^2 + \frac{1}{4} B^2 \tilde{g}_{\phi\phi}.
\]

Indeed, the virtue of the above Weyl-rescaling on the action of BD-Maxwell theory was to make use of the generalized Ehlers-Harrison transformation already known in the literature. Since we achieved this goal, now the remaining task is to translate this generalized Ehlers-Harrison transformation given in terms of the Weyl-rescaled fields \( (\tilde{g}_{\mu\nu}, A_{\mu}, \Psi) \) back into that given in the original fields \( (g_{\mu\nu}, A_{\mu}, \Phi) \), which now reads

\[
\begin{align*}
g_{ij}' &= \lambda^2 g_{ij} \ (i, j \neq \phi), \quad g_{\phi\phi}' = \lambda^{-2} g_{\phi\phi}, \\
\Phi' &= \Phi, \quad A'_{\phi} = -\frac{2}{B\lambda} \left[ 1 + \frac{1}{2} BA_{\phi} \right] + \frac{2}{B}, \\
\end{align*}
\]

\[\lambda \equiv \left[ 1 + \frac{1}{2} BA_{\phi} \right]^2 + \frac{1}{4} B^2 \Phi M^2_{\text{pl}} g_{\phi\phi}.
\]

Finally, upon performing this generalized Ehlers-Harrison transformation (22) on the BD-Bonnor solution given earlier in eq.(11), one gets the BD-Bonnor solution with magnetic Melvin-type universe content which is given by

\[
\begin{align*}
ds^2 &= \Lambda^2 \left[ -(\Delta \sin^2 \theta)^{-2/(2\omega+3)} dt^2 + (\Delta \sin^2 \theta)^{2/(2\omega+3)} \frac{\Sigma^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \left( \frac{dr^2}{\Delta} + d\theta^2 \right) \right] \\
&\quad + \Lambda^{-2} (\Delta \sin^2 \theta)^{(2\omega+1)/(2\omega+3)} d\phi^2, \\
\Phi(r, \theta) &= \frac{1}{G} (\Delta \sin^2 \theta)^{2/(2\omega+3)}, \\
A &= \frac{1}{2\Lambda \Sigma} \left\{ 4Mr a + B[(r^2 - a^2)^2 + \Delta a^2 \sin^2 \theta] \right\} d\phi, \\
\end{align*}
\]

\[\lambda \equiv \left( \frac{\Delta + a^2 \sin^2 \theta}{\Sigma} \right) \lambda = \frac{1}{\Sigma} \left\{ (\Delta + a^2 \sin^2 \theta) + 2BaMr \sin^2 \theta + \frac{1}{4} B^2 \sin^2 \theta [(r^2 - a^2)^2 + \Delta a^2 \sin^2 \theta] \right\}.
\]
(2) Route 2

Next, we attempt to construct the BD-Bonnor solution with magnetic Melvin-type universe content via the second route described before. And to this end, we start with the Bonnor solution with magnetic Melvin universe content in Einstein-Maxwell theory discussed in [13] and then apply to it the Singh and Rai’s algorithm. Again in Boyer-Lindquist-type coordinates, the Bonnor solution involving Melvin’s magnetic universe content is given by

$$ds^2 = \Lambda^2 \left[ -dt^2 + \frac{\sum^4}{\Delta + (M^2 + a^2) \sin^2 \theta} \left( \frac{dr^2}{\Delta} + d\theta^2 \right) \right] + \Lambda^{-2} \Delta \sin^2 \theta d\phi^2,$$

$$A = \frac{1}{2\Lambda \Sigma} \left\{ 4Mr \alpha + B[(r^2 - a^2)^2 + \Delta a^2 \sin^2 \theta] \right\} d\phi,$$

with \( \Lambda = \frac{1}{\Sigma} \left\{ (\Delta + a^2 \sin^2 \theta) + 2BaMr \sin^2 \theta + \frac{1}{4} B^2 \sin^2 \theta [(r^2 - a^2)^2 + \Delta a^2 \sin^2 \theta] \right\} \).

From this point on, the procedure to generate the BD-Bonnor solution with magnetic Melvin-type universe content by applying the Singh and Rai’s algorithm to this solution is quite straightforward and essentially the same as that to generate the BD-Bonnor solution from the Bonnor solution in Einstein-Maxwell theory we discussed earlier in section II. To briefly sketch the procedure, consider, again, the coordinate transformation given in eq.(17) which gives \( dr^2/\Delta = dR^2 \). Then the Bonnor solution with magnetic Melvin universe content above can be cast into the form in eq.(3) from which we can read off

$$e^{2U_E} = \Lambda^2, \quad W_E = 0,$$

$$e^{2k_E} = \frac{\sum^4}{(\Delta + (M^2 + a^2) \sin^2 \theta)^3} \Lambda^4, \quad h^2_E = \Delta \sin^2 \theta.$$

Then using the solution-generating rule in eq.(5) in the algorithm by Tiwari and Nayak, and Singh and Rai, and then transforming back to the standard Boyer-Lindquist coordinates using eq.(17), finally we can construct the BD-Bonnor solution with magnetic Melvin-type universe content which turns out to be precisely the same as the one obtained earlier via the “route 1”, namely eq.(37).

Now let us appreciate the meaning of this agreement once again. If we recall the nature of these two methods, the “route 1” involves Singh and Rai’s solution-generating method followed by the Ehlers-Harrison transformation while the “route 2” involves actions of the reversed order. This indicates, among others, that the two separate solution-generating methods, that of Singh and Rai and that of Ehlers-Harrison indeed commute. Although
this realization may not be so surprising, at least the lesson we learned from it is the fact that the Ehlers-Harrison-type transformation we derived in eq.(36) in the context of the BD-Maxwell theory was indeed correct and hence can be applied to other cases of interest. Now upon successfully constructing the BD-Bonnor solution with magnetic Melvin-type universe content, we now turn to the question asking if this solution can describe dihole geometry as well. In order to address this issue, we, as before, perform the change of coordinates from \((r, \theta)\) to \((\rho, \bar{\theta})\) given in eq.(27) and examine the solution for very small value of \(\rho\) to get

\[
\begin{align*}
\Phi(\rho, \bar{\theta}) &\approx \frac{1}{G} \frac{a}{\sqrt{M^2 + a^2}} + Bq \tilde{g}^{-1}(\bar{\theta})(1 - \cos \bar{\theta})d\phi^2,
A &\approx q \left[ \frac{a}{\sqrt{M^2 + a^2}} + Bq \right] \tilde{g}^{-1}(\bar{\theta})(1 - \cos \bar{\theta})d\phi, \tag{40}
\end{align*}
\]

where \(q \equiv \frac{M \rho}{\sqrt{\omega^2 + a^2}}\) and \(\tilde{g}(\bar{\theta}) \equiv [\cos^2(\frac{\bar{\theta}}{2}) + (\frac{a}{\sqrt{M^2 + a^2}} + Bq)^2 \sin^2(\frac{\bar{\theta}}{2})].\) And lastly, the magnetic charge of the solution is

\[
Q_m = q \left[ \frac{a}{\sqrt{M^2 + a^2}} + Bq \right]^{-1} \left( \frac{\Delta \phi}{2\pi} \right) \tag{41}
\]

where \(\Delta \phi\) denotes the yet unspecified period of the azimuthal angle \(\phi\). It is the “physical charge” of a single pole computed using Gauss’ law \(Q_m = \frac{1}{4\pi} \int_{S^2} F_{[2]}\) with \(S^2\) being any topological sphere surrounding each pole and \(F_{[2]} = dA\). Again, this limit of the solution can be identified with the “near-horizon limit” \((r \to r_+\) or \(\rho \to 0\)) of a distorted extremal charged black hole in BD-Maxwell theory with the distortion now being described by the factor \(\tilde{g}(\bar{\theta})\). And the horizon, i.e., the \(\rho = 0\) surface lacks spherical symmetry, instead, it is a prolate spheroid.

Lastly, we turn to the conical singularity structure of this BD-Bonnor solution with magnetic Melvin-type universe content. Using the metric solution given above in eq.(37) for \(\omega\)-parameter of the BD theory taking values in the range, \(-5/2 \leq \omega < -3/2\), the conical angle deficit along the axes \(\theta = 0, \pi\) and along the segment \(r = r_+\) are given respectively by

\[
\delta_{(0, \pi)} = 2\pi - \left| \frac{\Delta \phi d\bar{\theta} \sqrt{g_{\phi\phi}}}{\sqrt{g_{\phi\phi} d\bar{\theta}}} \right|_{\theta = 0, \pi} = 2\pi, \tag{42}
\]
\[ \delta_{(r=r_+)} = 2\pi - \left. \frac{\Delta \phi d\sqrt{g_{\phi\phi}}}{\sqrt{g_{rr}dr}} \right|_{r=r_+} = 2\pi. \]

To our dismay, we are still left with the same embarrassing results as before when the magnetic field content was absent. Once again, this seemingly pathological result, namely the “2\pi-angle deficits” along both the semi-infinite lines \( \theta = 0, \pi \) and the segment \( r = r_+ \) can be attributed to the fact that the metric structure of eq.(37) leading to the emergence of the “coordinate singularities” along \( \theta = 0, \pi \) and \( r = r_+ \) essentially remained the same (even if we have introduced the external magnetic field content into the BD-Bonnor solution) as the one of eq.(21) which, as stressed several times, exhibits the internal infinity nature of the symmetric axis. Earlier, we mentioned that particularly for the case at hand in which the cosmic string interpretation is not available, perhaps this second option involving the introduction of external magnetic field could be quite an appropriate approach toward the stabilization of the dihole system. And this can be checked in an explicit manner if the introduction of the magnetic field with proper strength leads to the vanishing of the conical singularities along the symmetry axis, \( \theta = 0, \pi \) and along the segment, \( r = r_+ \) at the same time. Apparently, however, this hope of ours failed again. As a result, any attempt to remove the conical singularities along the symmetry axis and along the segment, \( r = r_+ \) in terms of the cosmic string interpretation or by introducing an external (Melvin-type) magnetic field of proper strength failed in the present case of the Bonnor-type spacetime solution in BD-Maxwell theory. And we know that this failure can essentially be attributed to the internal infinity nature of the symmetry axis to begin with. Nevertheless, we anticipate that, although it cannot be checked in an explicit fashion for the reason stated above, the introduction of external magnetic field, aligned with the axis joining the dihole, can in principle counterbalance the combined gravitational and gauge attractive forces and hence eventually stabilize the configuration.

IV. Summary and Discussions

In the present work, the construction and extensive analysis of a solution in the context of the BD-Maxwell theory representing a pair of static, oppositely-charged extremal black holes have been performed. To this end, the algorithm of Singh and Rai’s is employed which is known to generate stationary, axisymmetric, charged solutions in BD-Maxwell theory from the known such solutions in Einstein-Maxwell theory. Indeed it was originally thought
that Bonnor’s solution in Einstein-Maxwell theory describes a singular point-like magnetic dipole. Lately, however, it has been demonstrated that it instead may describe a black dihole. Thus it is of equal interest to examine if the Bonnor-type dipole solution in BD-Maxwell theory constructed in this work can describe dihole geometry as well. Particularly for values of the BD $\omega$-parameter in the range $-5/2 \leq \omega < -3/2$, it has been demonstrated that indeed it can be identified with a black dihole configuration. The supporting argument for this conclusion is that although both the BD gravity empty spacetime and the BD-Bonnor solution exhibit some peculiar geometric structures such as the internal infinity nature of the symmetry axis, they should not be viewed as a pathology but instead should be thought of as the manifestation of the Mach’s principle that the BD theory itself attempts to embody. Then followed the discussion of the nature of this new solution particularly concerning its stability issue. Obviously, the configuration described by this black dihole solution involves an instability arising from the combined gravitational (same mass) and electromagnetic (opposite charge) attractions and it indeed is represented by the emergence of conical singularities. As we realized in the text, however, any attempt to address the nature of possible conical singularity structure of the BD-Bonnor solution along the symmetry axis, $\theta = 0, \pi$ and along the segment, $r = r_+$ failed essentially due to the internal infinity nature of the symmetry axis and the coordinate singularity nature at $r = r_+$ as long as we confined our interest to the case of the black dihole interpretation, namely, as long as $-5/2 \leq \omega < -3/2$. As a result, the cosmic string interpretation of the conical singularities was not available and therefore we turned to the second option involving the introduction of the Melvin’s magnetic universe content, i.e., the external magnetic field. Along the way, we also noticed that actually there are two possible ways of achieving this and the “route 1” involves Singh and Rai’s solution-generating method followed by the Ehlers-Harrison transformation while the “route 2” involves actions of the reversed order. It, then, has been realized that the two methods yield the same final solution which indicates that the two separate solution-generating methods, that of Singh and Rai and that of Ehlers-Harrison indeed commute. Unfortunately, however, this second approach toward the stabilization of the dihole system failed again as the metric structure in eq.(37) leading to the emergence of the “coordinate singularities” along $\theta = 0, \pi$ and $r = r_+$ essentially remained the same even if we have introduced the external magnetic field content into the BD-Bonnor solution. This failure manifests itself since one cannot determine for the case at hand the proper strength
of the magnetic field that can eliminate the conical singularities along the symmetry axis, \( \theta = 0, \pi \) and along the segment, \( r = r_+ \) at the same time. Despite this technical difficulties, however, it may be anticipated that *in principle* the introduction of external magnetic field, aligned with the axis joining the dihole, can counterbalance the combined gravitational and electromagnetic attractive forces and eventually stabilize the configuration. Lastly, since the reinterpretation of Bonnor’s solution in Einstein-Maxwell theory as a magnetic black dihole configuration, its generalizations in other Einstein-Maxwell-dilaton theories such as the Kaluza-Klein or the low-energy string theories have been discussed in the literature [13]. Thus the only remaining viable scalar-tensor theory of gravity (coupled to Maxwell theory) left untouched along this line was the Brans-Dicke-Maxwell theory. And precisely it is this issue that we would like to address in the present work.

**Appendix : Regular black hole solutions in Brans-Dicke-Maxwell theory**

In this appendix, we shall briefly review the characteristics of the non-trivial, regular black hole solutions of the BD-Maxwell theory constructed and studied in [23] and particularly discuss its BD-Reissner-Nordstrom black hole solution limit which is of some relevance to the present work.

The Brans-Dicke-Kerr-Newman (BDKN) solution of the BD-Maxwell theory in Boyer-Lindquist coordinates is given by [23]

\[
\begin{align*}
    ds^2 &= \left( \Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[ -\left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\
    &\quad + \left( \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2 \right] + \left( \Delta \sin^2 \theta \right)^{2/(2\omega+3)} \left[ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \right], \\
    \Phi(r, \theta) &= \frac{1}{G} \left( \Delta \sin^2 \theta \right)^{2/(2\omega+3)}, \quad A_{\mu} = -\frac{e r}{\Sigma} \left[ \delta_{\mu}^t - a \sin^2 \theta \delta_{\mu}^\phi \right]
\end{align*}
\]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \) and \( \Delta = r^2 - 2Mr + a^2 + e^2 \) with \( M, a, \) and \( e \) denoting the ADM mass, angular momentum per unit mass, and the electric charge respectively. Notice that the quantities \( \Sigma \) and \( \Delta \) above in eq.(43) are defined differently from those with the same names appeared in the BD-Bonnor solution in eq.(21) in the text. Indeed, they are related by \( a \to ia \). Just like the way how the BD-Bonnor solution has been derived in the text of the present work, this BDKN solution in BD-Maxwell theory has been constructed starting
from the Kerr-Newman (KN) solution in Einstein-Maxwell theory and then applying to it the Singh and Rai’s solution-generating algorithm as well. Note also that the BDKN solution above has possible coordinate singularities not only at the outer event horizon where $\Delta = 0$ but also along the symmetry axis $\theta = 0, \pi$. Thus in order to explore the nature of this singularity along the symmetry axis, the computation of invariant curvature polynomials is necessary. And it is a straightforward matter to realize that the two invariant curvature polynomials $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ become finite both on the horizon candidate at which $\Delta = 0$ and along the symmetry axis $\theta = 0, \pi$ provided the generic BD $\omega$-parameter takes values in the range $-5/2 \leq \omega < -3/2$.

Particularly note that if one takes the non-rotating limit of this BDKN black hole solution by setting $a$ (i.e., the angular momentum per unit mass) to zero one ends up with the BD-Reissner-Nordstrom black hole solution

$$
 ds^2 = \left[ r^2 \left( 1 - \frac{e}{r} \right)^2 \sin^2 \theta \right]^{-2/(2\omega+3)} \left[ -\left( 1 - \frac{e}{r} \right)^2 dt^2 + r^2 \sin^2 \theta d\phi^2 \right] + \left[ r^2 \left( 1 - \frac{e}{r} \right)^2 \sin^2 \theta \right]^{2/(2\omega+3)} \left[ -\left( 1 - \frac{e}{r} \right)^{-2} dr^2 + r^2 d\theta^2 \right]
$$

which, upon setting $r \rightarrow R$, $\theta \rightarrow \bar{\theta}$ and $e \rightarrow q$, coincides with eq.(31) in the text.

Now it seems relevant to explore thermodynamics and causal structure of this non-trivial BDKN black hole solution in eq.(43) in some more detail. Firstly, these BDKN black hole solutions have vanishing surface gravity at the event horizon, $\kappa_+ = 0$ and hence zero Hawking temperature, $T_H = \kappa_+/2\pi = 0$ provided $-5/2 \leq \omega < -3/2$. In other words, they do not radiate and hence are completely “dark and cold”. Certainly, this is a very bizarre feature in sharp contrast to evaporating black holes in general relativity. Next, we turn to their causal structure. The two Killing horizons, i.e., the outer event horizon and the inner Cauchy horizon turn out to occur precisely at the same locations (i.e., same coordinate distances) as those of KN black hole solutions in Einstein-Maxwell theory, i.e., at $r_\pm = M \pm (M^2 - a^2 - e^2)^{1/2}$. Also it is interesting to note that the proper area of the event horizon at $r = r_+$,

$$
 A = \int_{r_+} d\theta d\phi (g_{\theta\theta}g_{\phi\phi})^{1/2} = 4\pi (r_+^2 + a^2)
$$

is again exactly the same as that of standard KN black hole spacetime. In addition, its
angular velocity at the event horizon coincides with that of standard KN solution as well

\[ -W_{BD}^{-1}(r_+) = \frac{a}{r_+^2 + a^2} = -W_{E}^{-1}(r_+) . \]  

(46)

Next, observe that the norm of the time translational Killing field

\[ \xi_{\mu} \xi^{\mu} = g_{tt} = -(\Delta \sin^2 \theta)^{-2/(2\omega + 3)} \left[ \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right] \]  

(47)

goes like negative \((r_− < r < r_+)\) \(\rightarrow\) positive \((r_+ < r < r_\Sigma)\) \(\rightarrow\) negative \((r > r_\Sigma)\) with \(r_\Sigma = M + (M^2 - a^2 \cos^2 \theta - e^2)^{1/2} > r_+\) being the larger root of \(\xi_{\mu} \xi^{\mu}\), indicating that \(\xi^{\mu}\) behaves as timelike \(\rightarrow\) spacelike \(\rightarrow\) timelike correspondingly. And particularly the region in which \(\xi^{\mu}\) stays spacelike extends outside hole’s event horizon. This region is the so-called “ergoregion” and its outer boundary on which \(\xi^{\mu}\) becomes null, i.e., \(r = r_\Sigma\) is called “static limit” since inside of which no observer can possibly remain static. Thus if we recall the location of the static limit in standard KN black hole solution, we can realize that even the locations of ergoregions in two black hole spacetimes, KN and BDKN, are the same as well. Namely in two theories, i.e., the BD-Maxwell theory and the Einstein-Maxwell theory, rotating, charged black hole solutions turn out to possess \textit{identical} causal structure (i.e., the locations of ring singularities, two Killing horizons and static limits are the same) and hence exhibit the same global topology. Thus actually what distinguishes the BDKN black hole spacetime from its general relativity’s counterpart, i.e., the KN black hole is the local geometry alone such as the curvature characterized by the specific \(\omega\)-values, \(-5/2 \leq \omega < -3/2\).

Next, we would like to comment on the divergent behavior of the BD scalar field solution on the horizon which, in addition to the null Hawking radiation mentioned earlier, is another peculiar feature of the solution. As we mentioned earlier, the BD theory is an alternative theory to Einstein gravity and the BD scalar field represents spacetime-varying effective Newton’s constant, not a matter. Thus the divergent behavior of the BD scalar field in BD theory essentially represents the vanishing effective Newton’s constant in a certain region of spacetime. Besides, since the energy density of the BD scalar field \(T_{\mu \nu}^{BD} \xi^{\mu} \xi^{\nu}\) vanishes and hence satisfies the weak energy condition on the horizon at which \(\Delta = 0\) (of course for \(-5/2 \leq \omega < -3/2\)) \[23\], we do not worry too much about the divergent behavior of the BD scalar field there. For more detailed account of the nature of the BDKN solution, we refer
the reader to [23].

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References

[1] S. D. Majumdar, Phys. Rev. 72, 390 (1947) ; A. Papapetrou, Proc. Roy. Irish. Acad. A51, 191 (1947).
[2] W. Israel and K. A. Khan, Nuovo Cim. 33, 331 (1964).
[3] W. Kinnersley and M. Walker, Phys. Rev. D2, 1359 (1970).
[4] F. J. Ernst, J. Math. Phys. 17, 515 (1976).
[5] D. Kastor and J. Traschen, Phys. Rev. D47, 5370 (1993).
[6] W. B. Bonnor, Z. Phys. 190, 444 (1966).
[7] A. Davidson and E. Gedalin, Phys. Lett. B339, 304 (1994).
[8] D. J. Gross and M. J. Perry, Nucl. Phys. B226, 29 (1983).
[9] F. Dowker, J. P. Gauntlett, G. W. Gibbons, and G. T. Horowitz, Phys. Rev. D53, 7115 (1996).
[10] A. Sen, J. High Energy Phys. 9710, 002 (1997).
[11] A. Chattaraputi, R. Emparan, and A. Taormina, Nucl. Phys. B573, 291 (2000).
[12] H. Kim, J. High Energy Phys. 0301, 080 (2003) ; Nucl. Phys. B651, 143 (2003).
[13] R. Emparan, Phys. Rev. D61, 104009 (2000).
[14] C. Brans and C. H. Dicke, Phys. Rev. 124, 925 (1961).
[15] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[16] A. I. Janis, D. C. Robinson, and J. Winicour, Phys. Rev. 186, 1729 (1969) ; H. A. Buchdahl, Int. J. Theor. Phys. 6, 407 (1972) ; C. B. G. McIntosh, Commun. Math. Phys. 37, 335 (1974) ; B. O. J. Tupper, Nuovo Cimento 19B, 135 (1974).
[17] R. N. Tiwari and B. K. Nayak, Phys. Rev. D14, 2502 (1976); J. Math. Phys. 18, 289 (1977).

[18] T. Singh and L. N. Rai, Gen. Rel. Grav. 11, 37 (1979).

[19] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963); E. J. Newman, E. Couch, K. Chinapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).

[20] R. A. Matzner and C. W. Misner, Phys. Rev. 154, 1229 (1967).

[21] R. M. Misra and D. B. Pandey, J. Math. Phys. 13, 1538 (1972).

[22] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

[23] H. Kim, Phys. Rev. D60, 024001 (1999).