CAUSAL FERMION SYSTEMS:
AN ELEMENTARY INTRODUCTION TO
PHYSICAL IDEAS AND MATHEMATICAL CONCEPTS

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Abstract. We give an elementary introduction to the theory of causal fermion systems, with a focus on the underlying physical ideas and the conceptual and mathematical foundations.

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1. The Challenge: Unifying Quantum Field Theory and General Relativity

One of the biggest problems of present-day theoretical physics is the incompatibility of Quantum Field Theory and General Relativity. While the standard model of elementary particle physics provides a quantum field theoretical description of matter together with its electromagnetic, weak and strong interactions down to atomic and
subatomic scales, General Relativity applies to gravitational phenomena on astrophysical or cosmological scales. Just as the standard model of elementary particle physics is well-confirmed by high-precision measurements, also the theoretical predictions of General Relativity agree with the experimental results to high accuracy. Nevertheless, when combining Quantum Field Theory and General Relativity on very small length scales, these theories become mathematically inconsistent, making physical predictions impossible.

The fact that combining Quantum Field Theory with General Relativity leads to inconsistencies, although each theory by itself provides excellent theoretical predictions, allows for different possible conclusions: While a convinced elementary particle physicist will refer to the overwhelming triumph of Quantum Field Theory and postulate the existence of a gravitational exchange particle, namely the graviton, thus forcing General Relativity into the setting of the standard model of elementary particle physics, a dedicated relativist, on the other hand, will question the mathematical formalism of Quantum Field Theory and instead refer to the aesthetics and mathematical clarity of the differential geometric approach to General Relativity. Undecided physicists, who are convinced of the concepts of both Quantum Field Theory and General Relativity, may argue that, instead of incorporating one theory in the other, one should try to find a new theory which reproduces both Quantum Field Theory and General Relativity in suitable limiting cases. Physicists skeptical of both theories will bring into play alternative approaches such as string theory or the theory of loop quantum gravity which are based on completely new assumptions.

Due to the lack of experimental evidence, most alternative approaches are mainly based on personal preferences and paradigms. They involve ad-hoc assumptions which are often detached from the well-established physical principles which were developed based on physical experiments. Since there are many ways to introduce new assumptions ad hoc, it is questionable whether these approaches will turn out to be successful. Therefore, we prefer to proceed differently as follows: We begin with a detailed and honest review of the concepts and principles which form the basis of Quantum Field Theory and General Relativity. Afterward, we select those principles which we consider to be essential (clearly, this is a subjective choice). Then we combine these principles in a novel mathematical setting, referred to as causal fermion systems. Working exclusively with the objects in this setting, we postulate new physical equations by formulating the so-called causal action principle. The causal action principle gives rise to additional objects and structures in space-time together with equations describing their dynamics. In this way, we obtain a new physical theory with predictive power.

2. Overview of Concepts and Mathematical Structures in Theoretical Physics

Following the above outline, this section is devoted to a review of the concepts and ideas, common beliefs as well as selected mathematical structures and objects used in contemporary theoretical physics. To sharpen the view for the few really fundamental principles underlying our present understanding and mathematical description of nature, we have decided to take a bird’s-eye perspective rather than a high-resolution examination of sophisticated mathematical constructions.

2.1. The Fabric of Spacetime. Before Einstein’s Special Theory of Relativity, physicists thought of space as being the geometric background in which physical processes
take place while time evolves. With this concept of space and time in mind, nobody could imagine that space itself might change while time evolves or – even more – that space-time as a whole participates in the physical interactions. After more than one hundred years of studying Einstein’s Theory of Relativity, however, our understanding of space and time has changed completely. Nowadays, we are used to the fact that space-time and its matter content cannot be considered independently, but rather form an inseparable unity interwoven by mutual interactions. This unity is sometimes referred to as the fabric of spacetime.

We now review the necessary concepts to capture and cast this intuitive notion in a formal mathematical framework as provided by differential geometry. In order to make this paper accessible to a broad readership, we also recall basic definitions which are clearly familiar to mathematicians.

2.1.1. Topological Manifolds as Models of Spacetime. At the most basic level, namely without considering any additional structures, spacetime is nothing but a set of points which locally – that is within the tiny snippet of the universe which is accessible to our everyday experience – looks like the familiar, three-dimensional Euclidean space. Including time as a fourth dimension naturally leads to the idea to model the fabric of spacetime as a four-dimensional topological manifold.

**Definition 2.1** (Topological Manifold). A topological manifold of dimension $d$ is a second-countable, topological Hausdorff space $(\mathcal{M}, \mathcal{O})$ which at every point $p \in \mathcal{M}$ has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^d$.

Here $\mathcal{O}$ denotes the family of all open subsets of $\mathcal{M}$. The reason why we do not consider a completely structureless set rather than the tuple $(\mathcal{M}, \mathcal{O})$ consisting of a set equipped with a topology, is needed in order to have a notion of continuity.

2.1.2. Establishing Smooth Structures in Spacetime. By modelling spacetime as a four-dimensional topological manifold, we have already implemented some of our knowledge about the general structure of our Universe. In order to describe smooth functions in spacetime and to be able to do calculus, one important ingredient is still missing and calls for the following definition:

**Definition 2.2** (Smooth Compatibility of Coordinate Charts). Let $(\mathcal{M}, \mathcal{O})$ be an $d$-dimensional topological manifold together with two coordinate charts $(U, \varphi)$ and $(V, \psi)$ such that the open sets $U, V \subseteq \mathbb{R}^n$ satisfy $U \cap V \neq \emptyset$.

The composition of the coordinate functions given by
$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called transition map from $\varphi$ to $\psi$. Two coordinate charts $(U, \varphi)$ and $(V, \psi)$ are smoothly compatible if the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

The definition of smoothly compatible coordinate charts allows us to introduce the notion of smooth atlases which in turn prepares the ground for defining smoothness of functions on manifolds.
Definition 2.3 (Smooth Atlas). Let \( \{(U_i, \varphi_i)\}_{i \in I} \) with \( I \subseteq \mathbb{N} \) be a family of charts of a topological manifold \((\mathcal{M}, \mathcal{O})\) with open sets \( U_i \subseteq \mathbb{R}^n \).

The family \( \{(U_i, \varphi_i)\}_{i \in I} \) of charts is called atlas, if the open sets \( U_i \) cover \( \mathcal{M} \). If in addition any two charts in the atlas are smoothly compatible, the atlas is referred to as smooth atlas.

A topological manifold equipped with a smooth atlas \( \mathcal{A} \) is referred to as a smooth manifold. We can now specify what we mean by smoothness of functions on a manifold.

Definition 2.4 (Smooth Functions on Manifolds). Let \((\mathcal{M}, \mathcal{A})\) be a smooth manifold.

A function \( f: \mathcal{M} \to \mathbb{R} \) on the manifold is called smooth if for every chart \((U, \varphi) \in \mathcal{A}\) the function \( f \circ \varphi^{-1} \) is smooth in the sense of functions being defined on open subsets of \( \mathbb{R}^d \).

2.1.3. Encoding the Lorentzian Geometry of Spacetime. From our everyday life within a small snippet of the universe, we are used to the properties of three-dimensional Euclidean space, especially its vector space character. In order not to lose these familiar and useful properties when modelling spacetime as a differentiable manifold, one introduces a vector space structure at every single point of the manifold. In order to avoid the misleading idea that spacetime is embedded in some higher-dimensional ambient space, we must work with an intrinsic characterization which only makes use of the already defined concepts of coordinate charts and smooth functions.

Definition 2.5 (Derivations and Tangent Space). Let \((\mathcal{M}, \mathcal{A})\) be an \( d \)-dimensional smooth manifold and \( p \) an element of \( \mathcal{M} \).

A linear map \( X_p: C^\infty(\mathcal{M}, \mathbb{R}) \to \mathbb{R} \) is called derivation at \( p \in \mathcal{M} \) if it satisfies the Leibniz product rule

\[
\forall f, g \in C^\infty(\mathcal{M}, \mathbb{R}): X_p(fg) = f(p)X_p(g) + g(p)X_p(f)
\]

The set of all derivations at \( p \in \mathcal{M} \) forms a vector space under the operations

\[
(X + Y)_p(f) = X_p(f) + Y_p(f)
\]

\[
(\alpha X)_p(f) = \alpha X_p(f)
\]

which is referred to as the tangent space \( T_p\mathcal{M} \) at \( p \in \mathcal{M} \).

It can be shown that the tangent space is a \( d \)-dimensional real vector space. In order to better understand the similarities between the differential geometric formulation of Einstein’s General Theory of Relativity and the theory of causal fermion systems in the further course of this article, we shall introduce the bundle formulation.
Definition 2.6 (Tangent Bundle and Vector Fields). Let \((\mathcal{M}, \mathcal{A})\) be an \(d\)-dimensional smooth manifold with tangent spaces \(T_p\mathcal{M}\) at all points \(p \in \mathcal{M}\).

The tangent bundle \(T\mathcal{M}\) is defined as the disjoint union of the tangent spaces \(T_p\mathcal{M}\) at all points \(p \in \mathcal{M}\)

\[
T\mathcal{M} := \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}
\]

(endo\-unded with the product topology). A continuous function \(X \in C^0(\mathcal{M}, T\mathcal{M})\) is called vector field if it satisfies the condition

\[
\forall p \in \mathcal{M}: X(p) := X_p \in T_p\mathcal{M}
\]

Having defined tangent spaces, we are ready to add our knowledge about the geometric structure of spacetime to our model. In the familiar Euclidean geometry, the geometry is retrieved by computing lengths and angles between vectors. These quantities are encoded in a scalar product, being a positive definite bilinear form

\[
g_p: T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}
\]

In Special Relativity, the geometry is described again by a bilinear form, which however is no longer positive definite, but instead has signature \((1, 3)\):

Definition 2.7 (Lorentzian Manifold). Let \((\mathcal{M}, \mathcal{A})\) be an \(d\)-dimensional smooth manifold with tangent bundle \(T\mathcal{M}\).

A function \(g: T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}\) is called Lorentzian metric if the restriction

\[
g_p: T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}
\]

is a bilinear, symmetric and smooth mapping

\[
g(X,Y): \mathcal{M} \to \mathbb{R} \quad p \mapsto [g(X,Y)](p) := g_p(X_p, Y_p)
\]

of signature \((1, 3)\). A smooth manifold \((\mathcal{M}, \mathcal{A})\) equipped with a Lorentzian metric is referred to as Lorentzian manifold \((\mathcal{M}, g)\).

The Lorentzian signature implies that the inner product \(g_p(\xi, \xi)\) of a tangent vector \(\xi \in T_p\mathcal{M}\) with itself can be positive or negative. This gives rise to the following notion of causality. A tangent vector \(\xi \in T_p\mathcal{M}\) is said to be

\[
\begin{cases}
\text{timelike} & \text{if } g_p(\xi, \xi) > 0 \\
\text{spacelike} & \text{if } g_p(\xi, \xi) < 0 \\
\text{lightlike} & \text{if } g_p(\xi, \xi) = 0
\end{cases}
\]  

(2.1)

Lightlike vectors are also referred to as null vectors, and the term \(\text{non-spacelike}\) refers to timelike or lightlike vectors. The spacetime trajectory of a moving object is described by a curve \(\gamma(\tau)\) in \(\mathcal{M}\) (with \(\tau\) an arbitrary parameter). We say that the spacetime curve \(\gamma\) is timelike if the tangent vector \(\dot{\gamma}(\tau)\) is everywhere timelike. Spacelike, null and non-spacelike curves are defined analogously. Then the usual statement of causality that nothing can travel faster than the speed of light can be formulated as follows:
2.2. The Einstein Field Equations. After these preparatory considerations, we are now ready to formulate and investigate the significance of the Einstein field equations which are at the heart of General Relativity. They take the form

\[ \text{Ric} - \frac{1}{2} R g + \Lambda g = 8\pi\kappa T, \]

where Ric is the Ricci tensor, \( R \) is scalar curvature, \( \Lambda \) is the cosmological constant, \( \kappa \) is the gravitational coupling constant, and \( T \) is the energy-momentum tensor. These equations can be derived from an action principle. More precisely, metrics which solve the Einstein equations are critical points of the Einstein-Hilbert action

\[ S_{EH} = \int_M \left( \frac{1}{16\pi\kappa} \left( R - 2\Lambda g \right) + \mathcal{L}_{\text{matter}} \right) d\mu_M(x). \]  

The Einstein equations relate the curvature of spacetime (on the left side) to the matter distribution described by the energy-momentum tensor (on the right side). Combining the field equations with the equations of motion for the matter fields (like the geodesic equation, the Dirac equation, etc.), one gets a coupled system of partial differential equations. This coupled system can be understood in simple terms by the popular phrase that matter tells spacetime how to curve, and spacetime tells matter how to move.

Taking up the comparison between the brain-mind-relationship and the interplay of spacetime and physical processes therein, the Einstein field equations characterize this interrelation. In a similar way as our thinking shapes the brain structures which in turn have an influence on our thoughts, also the objects existing and processes happening in spacetime deform spacetime, which has a back effect on physical processes. Einstein’s revolutionary insight that spacetime together with its matter and energy content form an inseparable unity, is one of the cornerstones which the theory of causal fermion systems is built on.

2.3. Quantum Theory in a Classical Spacetime. The second groundbreaking discovery in the twentieth century besides General Relativity was Quantum Theory. The insight that certain physical quantities take discrete rather than continuous values revolutionized our understanding of Nature. This discovery triggered the development of Quantum Mechanics which is the appropriate framework to study the quantum behaviour of a single particle or a constant finite number of particles. Although the framework of Quantum Mechanics is appropriate to describe even arbitrarily large quantum systems of a fixed number of particles, it is in principle incapable to formalize processes involving a varying number of quantum particles. This limitation is overcome in Quantum Field Theory, where a quantum state can be a superposition of components involving a varying and arbitrarily large number of particles.

Relativistic Quantum Field Theory is usually formulated in Minkowski space, disregarding the gravitational field (see for example [2, 3, 29]). The fact that in Quantum Field Theory one deals with arbitrarily large number of particles which can have arbitrarily large momenta can be understood as the reason why divergences occur in the perturbative description. The renormalization program provides a systematic computational procedure for dealing with these divergences. The such-renormalized Quantum
Field Theory makes excellent physical predictions which have been confirmed experimentally to high precision. Nevertheless, it is often criticized that the renormalization program lacks a foundational justification. Also, it is not quite satisfying that the theory is well-defined only to every order in perturbation theory. But the perturbation series does not need to converge. Also, it is not clear whether there exists a mathematically meaningful non-perturbative formulation of Quantum Field Theory.

Most methods of Quantum Field Theory also apply to Quantum Field Theory in a fixed \textit{curved spacetime} (see for example [5] and the references therein). In other words, one considers Quantum Fields in the background of a \textit{classical} gravitational field without taking into account the backreaction of the quantum fields to classical gravity.

2.4. **Incompatibility of General Relativity and Quantum Field Theory.** Quantum field theory in a classical spacetime has the shortcoming that classical and quantum objects coexist in a way which is conceptually not fully convincing. It would be desirable to describe all the objects on the same footing by “unifying” the theories. However, there is no consensus on how this “unification” should be carried out, or even on what “unification” should mean. Nevertheless, most physicists agree that serious problems arise, no matter which approach for “unification” is taken. In order not to take sides, we here merely list some of the most common arguments pointing towards the difficulties:

- The simplest method is to start from the Heisenberg Uncertainty Principle \( \Delta p \Delta x \geq \frac{\hbar}{2} \), which states that position and momentum of a point particle in quantum mechanics can be determined simultaneously only up to a fundamental uncertainty given by Planck’s constant \( \hbar \). In Quantum Field Theory, similar uncertainty relations hold for the field operators and the associated canonical momentum operators. In particular, acting with the local field operator \( \phi(x) \) on the vacuum state, the quantum state is localized in space, meaning that there is a large momentum uncertainty. This also gives rise to a large uncertainty in the corresponding energy. Intuitively speaking, we thus obtain large “fluctuations” of energy in a small spatial region. In General Relativity, on the other hand, high energy densities lead to the formation of black holes. Therefore, combining the principles of General Relativity and Quantum Field Theory in a naive way leads to the formation of microscopic black holes, implying that the concept of a spacetime being “locally Minkowski space” breaks down. The relevant length scale for such effects is the \textit{Planck length} \( \ell_P \approx 1.6 \times 10^{-35} \) m.

- The renormalization program only applies to a class of theories called renormalizable. It turns out that applying the canonical quantization methods to Einstein’s gravity, the resulting theory is not renormalizable. This shows that quantizing gravity with the present methods of perturbative Quantum Field Theory is not a fully convincing concept.

- It is sometimes argued that the problem of “unification” is rooted in shortcomings of present Quantum Field Theory. Indeed, the ultraviolet divergences of Quantum Field Theory suggest that the structure of spacetime should be modified for very small distances. A natural length scale for such modifications is given by the Planck length. In this way, the problem of the ultraviolet divergences seems to be intimately linked to gravity. Therefore, in order to resolve these problems, one should modify the structure of spacetime on the Planck scale.
2.5. A Step Back: Quantum Mechanics in Curved Spacetime. In order to avoid the just-described problems which arise when “unifying” General Relativity with Quantum Field Theory, it is a good idea to take a step back and return to the familiar and well-understood grounds of one-particle quantum mechanics. Indeed, formulating quantum mechanics in curved spacetime does not lead to any conceptual or technical problems. We now review a few basic concepts, which will also be our starting point for the constructions leading to causal fermion systems.

2.5.1. The Dirac Equation in Minkowski Space. In non-relativistic quantum mechanics, a particle is described by its Schrödinger wave-function \( \psi(t, \vec{x}) \). It has a probabilistic interpretation, meaning that its absolute square \( |\psi(t, \vec{x})|^2 \) is the probability density for the particle to be at the position \( \vec{x} \in \mathbb{R}^3 \). The relativistic generalization of the Schrödinger equation is the Dirac equation. In this case, the wave function \( \psi \) has four complex components, which describe the spin of the particle. In flat Minkowski space, the Dirac equation takes the form
\[
(i \gamma^k \frac{\partial}{\partial x^k} - m) \psi(x) = 0,
\]
where \( x = (t, \vec{x}) \in \mathcal{M} \) is a point of Minkowski space, \( m \) is the rest mass, and the so-called Dirac matrices \( \gamma^k \) are \( 4 \times 4 \)-matrices which are related to the Lorentzian metric by the anti-commutation relations
\[
2 g^{jk} \mathbbm{1} = \{ \gamma^j, \gamma^k \} \equiv \gamma^j \gamma^k + \gamma^k \gamma^j.
\]
The Dirac spinors at every spacetime point are endowed with an indefinite inner product of signature \((2, 2)\), which we call spin scalar product and denote by \( \langle \cdot | \cdot \rangle(x) \). To every solution \( \psi \) of the Dirac equation we can associate a vector field \( J \) by
\[
J^k = \langle \psi | \gamma^k \psi \rangle,
\]
referred to as the Dirac current. For solutions of the Dirac equation, this vector field is divergence-free. This is referred to as current conservation.

Current conservation is closely related to the probabilistic interpretation of the Dirac wave function. Indeed, as a consequence of current conservation, for a solution \( \psi \) of the Dirac equation, the spatial integral
\[
\langle \psi | \psi \rangle := 2\pi \int_{\mathbb{R}^3} \langle \psi | \gamma^0 \psi \rangle(t, \vec{x}) d^3x
\]
is time independent. Normalizing the value of this integral to one, its integrand gives the probability density of the particle to be at position \( \vec{x} \).

2.5.2. The Dirac Equation in Curved Spacetime. In curved spacetime, the Dirac equation is described most conveniently using vector bundles. Similar to the tangent bundle in Definition 2.6, the spinor bundle is obtained by “attaching” a vector space \( S_p \mathcal{M} \) to every spacetime point,
\[
S_{p \mathcal{M}} = \bigcup_{p \in \mathcal{M}} \{ p \} \times S_p \mathcal{M}.
\]
But now, the vector space \( S_p \mathcal{M} \), the so-called spinor space, is a four-dimensional complex vector space. This vector space is endowed with an indefinite inner product of signature \((2, 2)\) which, just as in Minkowski space, we refer to as the spin scalar product and denote by
\[
\langle \cdot | \cdot \rangle_p : S_p \mathcal{M} \times S_p \mathcal{M} \to \mathbb{C}.
\]
At each spacetime point \( p \), the Dirac wave function \( \psi \) takes a value in the corresponding spinor space \( S_p \mathcal{M} \). The Dirac operator \( D \) takes the form
\[
D := i \gamma^j \nabla_j ,
\]
where \( \nabla_j \) is a connection on the spinor bundle, and the Dirac matrices are related to the Lorentzian metric again by the anti-commutation relations
\[
\{ \gamma^j(p), \gamma^k(p) \} = 2 g^{jk}(p) \mathbf{1}_{S_p \mathcal{M}} .
\]
The Dirac equation in curved spacetime reads
\[
(D - m) \psi = 0.
\]
On solutions of the Dirac equation, one has the scalar product
\[
(\psi|\phi)_m := \int_{\mathcal{N}} \langle \psi | \nu^j \gamma_j \phi \rangle_p \, d\mu_N(p) ,
\]
where \( \nu \) is the future-directed normal on the Cauchy surface \( \mathcal{N} \), and \( d\mu_N \) is the induced measure. For mathematical completeness, we point out that we always assume that spacetime is globally hyperbolic, so that Cauchy surfaces exist. Moreover, in order for the integral in (2.4) to be finite, we restrict attention to wave functions of spatially compact support (i.e. to wave functions whose restriction to any Cauchy surface have compact support).

Due to current conservation, the scalar product (2.4) is independent of the choice of the Cauchy surface. Choosing \( \psi = \phi \) as a unit vector, the integrand of the above scalar product again has the interpretation as the quantum mechanical probability density.

3. Conceptual and Mathematical Foundations of Causal Fermion Systems

The theory of causal fermion systems is a novel approach to fundamental physics which is built on our conviction that, in order to resolve the incompatibility of General Relativity and Quantum Field Theory described above, one should modify the geometric structure of spacetime on microscopic scales. Having already surveyed our current way of modelling the fabric of spacetime and quantum wave functions therein, we now introduce the conceptual foundations of the theory of causal fermion systems.

3.1. Guiding Principles of the Theory of Causal Fermion Systems. Following Einstein’s celebrated insight that “one cannot solve problems with the same level of thinking that created them,” the theory of causal fermion systems does not try to force obviously incompatible concepts into an already existing setting, but instead provides a new mathematical framework which is inspired by carefully selected concepts from contemporary theoretical physics. The main guiding principles of the theory of causal fermion systems are the following ideas:

- Unified description of spacetime and the objects therein
  The General Theory of Relativity impressively demonstrates that seemingly disparate concepts such as the motion of matter and the metric tensor structure of spacetime are closely related and cannot be considered independent of each other. This surprising insight illustrates the geometric character, high degree of complexity and interconnectedness of the Universe. The interdependence of matter distributions and the shape of spacetime which reacts on local changes as a whole, strongly suggests to take a unified point of view when developing new
physical theories. This fundamental conviction is implemented in the theory of causal fermion systems in that spacetime, together with all objects therein (such as particles, fields, etc.), are determined dynamically as a whole by minimizing the so-called causal action.

- **Equivalence principle**
  In General Relativity, the equivalence principle is implemented mathematically by working with geometric objects on a Lorentzian manifold. In particular, the Einstein-Hilbert action is diffeomorphism invariant. Allowing for a nontrivial microscopic structure, in the setting of causal fermion systems spacetime does not necessarily need to be a smooth manifold. Consequently, instead of diffeomorphisms, one must allow for more general transformations of spacetime. The causal action is invariant under these more general transformations, thereby generalizing the equivalence principle.

- **Principle of causality**
  The principle of causality plays a crucial role in our understanding of the structure of physical interactions in spacetime. A guiding conception in the development of causal fermion systems was that the causal structure of spacetime is not given a-priori, but that it is determined dynamically when solving the physical equations. For a better comparison, we recall that in General Relativity, the causal structure is encoded in the Lorentzian metric (as explained after (2.1)). Therefore, when varying the metric in the Einstein-Hilbert action (2.2), also the causal structure changes. Only after a critical point of the Einstein-Hilbert action has been found, the corresponding metric determines the causal structure of spacetime. Similarly, in the theory of causal fermion systems, the causal structure of spacetime is determined only after a critical point of the causal action has been found. The principle of causality is implemented in the form that points with spacelike separation are not related to each other in the Euler-Lagrange equations corresponding to the causal action principle.

- **Local gauge principle**
  In classical electrodynamics, the local gauge principle means the freedom \( A \rightarrow A + d\Lambda \) in changing the electromagnetic potential \( A \) by the derivative of a scalar function \( \Lambda \). This observation was the starting point for the development of gauge theories, which have been highly successful in describing all the bosonic interactions in the standard model. In Quantum Theory, local gauge transformations correspond to generalized local phase transformations of the wave functions

\[
\psi(x) \rightarrow U(x) \psi(x),
\]

where \( U(x) \) is an isometry on the fibres of the spinor bundle. The theory of causal fermion systems incorporates this principle in that the causal action is invariant under such local transformations.

- **Microscopic spacetime structure**
  The ultraviolet divergences in Quantum Field Theory suggest that one should modify the microscopic structure of spacetime. In order to include these microscopic features of spacetime, the theory of causal fermion systems does not assume physical spacetime to be continuous down to smallest scales, but instead allows for a nontrivial, possibly discrete microstructure of spacetime.
Fermionic building blocks
From high energy physics we have a quite clear and consistent picture of the elementary building blocks of Nature which is formalized in the Standard Model of Particle Physics. In particular, we know that the fundamental matter particles are fermions while the forces are mediated by bosons. Inspired by Dirac’s concept that in the Minkowski vacuum a whole “sea” of fermions is present, we consider the fermions as being more fundamental, whereas bosons appear in our approach merely as a device to describe the interaction of the fermions.

Causal fermion systems evolved in the attempt to combine the above principles in a simple and compact mathematical setting.

In the following sections we enlarge on each of the guiding principles and explain how they are formalized within the mathematical framework of the theory of causal fermion systems.

3.2. Unified Description of Spacetime and the Objects Therein. The basic conceptual idea underlying the theory of causal fermion systems consists in the belief that a successful unified theory must provide a unified description of the Universe in the sense that it does not treat spacetime separate from its matter and energy content. This central conception is inspired by the inseparable unity of spacetime and its matter-energy content as described by Einstein’s field equations. In much the same way as the Einstein-Hilbert action singles out those metric tensors which are critical points of the action and declares them to be the physically admissible choices, also the theory of causal fermion systems is based on a variational principle. Before we can formulate such a variational principle, we give the general definition of a causal fermion system and explain it afterward.

Definition 3.1 (Causal Fermion System). A causal fermion system of spin dimension $n \in \mathbb{N}$ is a triple $(\mathcal{H}, \mathcal{F}, \rho)$ consisting of the following three mathematical structures:

1. $\mathcal{H}$ is a complex, separable Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$.
2. $\mathcal{F}$ is the subset of the Banach space $(L(\mathcal{H}), \| \cdot \|)$ comprising all self-adjoint operators on $\mathcal{H}$ of finite rank, which – counted with multiplicity – have at most $n \in \mathbb{N}$ positive and at most $n \in \mathbb{N}$ negative eigenvalues.
3. $\rho$ is a positive Borel measure $\rho : \mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}$ on $\mathcal{F}$ (where $\mathcal{B}$ is the $\sigma$-algebra generated by all open subsets of $\mathcal{F}$). The measure $\rho$ is referred to as the universal measure.

The connection of this definition to physics is not obvious. In order to convey a better, more intuitive understanding of this definition, let us have a detailed view on the individual ingredients. The structure of a complex Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ is a commonly used structure both in mathematics as well as in theoretical physics and should therefore need no further explanation. In contrast to this, the set $\mathcal{F}$ as well as the measure $\rho$ – although familiar to mathematicians – are not commonly used in theoretical physics. In order to make the theory of causal fermion systems easier accessible to interested physicists, we now explain these structures in greater detail.

3.2.1. The Measure Space $(\mathcal{F}, \mathcal{B}, \rho)$. In contrast to what one might expect from the ordering in the above definition, the central structure of a causal fermion system is
not the Hilbert space $\mathcal{H}$ itself but rather the measure $\rho$. Measures appear in physics mainly as integration measures, like for example the measure $d\mu = d^3x$ in the three-dimensional integral

$$\int_{\mathbb{R}^3} f(x) \, d\mu(x)$$

(of a, say, continuous and compactly supported function $f$). In mathematics, the measure $\mu$ is a mapping which to a subset $\Omega \subset \mathbb{R}^3$ associates its volume,

$$\mu : \Omega \mapsto \mu(\Omega) := \int_{\Omega} d^3x.$$ 

A central conclusion from measure theory is that it is mathematically not sensible to associate a measure to every subset of $\mathbb{R}^3$. Instead, one must distinguish a class of sufficiently “nice” subsets as being measurable. The measurable sets form a $\sigma$-algebra, meaning that applying any finite or countable number of set operations on measurable sets gives again a measurable set. Here it suffices to always work with the Borel algebra, defined as the smallest $\sigma$-algebra where all open sets are measurable.

A difference to usual integration measures is that the universal measure $\rho$ is a measure on linear operators. The starting point is the the Banach space $L(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ together with the operator norm

$$\|x\| := \sup \left\{ \|xu\|_{\mathcal{H}} \mid \|u\|_{\mathcal{H}} = 1 \right\}. \tag{3.1}$$

The set $\mathcal{F}$ is by definition a subset of $L(\mathcal{H})$. We point out that $\mathcal{F}$ is not a subspace of $\mathcal{L}(\mathcal{H})$, because linear combinations of operators in $\mathcal{F}$ will in general have rank greater than $2n$. But, being a closed subset of $L(\mathcal{H})$, it is a complete metric space with the distance function

$$d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+,$$

$$d(x,y) := \|x - y\|.$$ 

We remark that $\mathcal{F}$ is not a manifold, even if $\mathcal{H}$ is finite-dimensional. However, the subset of all operators of maximal rank

$$\mathcal{F}^\text{reg} := \{ x \in \mathcal{F} \mid x \text{ has rank } 2n \}$$

is dense in $\mathcal{F}$ and indeed a smooth manifold of dimension

$$\dim \mathcal{F}^\text{reg} = 4n(\dim \mathcal{H} - n).$$

(for details see \cite{25} Proposition 2.4.4)). Since in physical applications the dimension of $\mathcal{H}$ is very large, $\mathcal{F}$ should be regarded as a subset of $L(\mathcal{H})$ of very high dimension.

In order to define a measure $\rho$ on this set of operators, we must construct a $\sigma$-algebra. The simplest choice (which also covers all cases of present physical interest) is to take the Borel algebra $\mathcal{B}$, i.e. the $\sigma$-algebra generated by all open subsets of $\mathcal{F}$, with respect to the topology induced by the operator norm \eqref{3.1}. The measure $\rho$ makes it possible to integrate a continuous (or Borel) function $f : \mathcal{F} \to \mathbb{R}_0^+$ over $\mathcal{F}$,

$$\int_{\mathcal{F}} f(x) \, d\rho(x) \in [0, \infty].$$ 

All familiar concepts from integration theory in $\mathbb{R}^3$ also apply here. However, one should keep in mind that we integrate over a set of operators of the Hilbert space (in other words, the integration variable $x$ is operator-valued).
3.2.2. The Causal Action Principle. We are now in the position to define the causal Lagrangian and the causal action. For any $x, y \in \mathcal{F}$, the product $xy$ is an operator of rank at most $2n$. We denote its non-trivial eigenvalues counting algebraic multiplicities by $\lambda_{xy}^1, \ldots, \lambda_{xy}^{2n} \in \mathbb{C}$ (more specifically, denoting the rank of $xy$ by $k \leq 2n$, we choose $\lambda_{xy}^1, \ldots, \lambda_{xy}^k$ as all the non-zero eigenvalues and set $\lambda_{xy}^{k+1}, \ldots, \lambda_{xy}^{2n} = 0$).

**Definition 3.2** (Causal Lagrangian and Causal Action). The causal Lagrangian is a function defined as

$$L: \mathcal{F} \times \mathcal{F} \to \mathbb{R}^+_0 \quad (x, y) \mapsto L(x, y) := \frac{1}{4n} \sum_{i,j=1}^{2n} (|\lambda_{xy}^i| - |\lambda_{xy}^j|)^2.$$  (3.2)

where $|\lambda_{xy}^i|$ denote the absolute values of the eigenvalues $\lambda_{xy}^i$ of the operator product $xy$.

The causal action is obtained by integrating the Lagrangian with respect to the universal measure,

$$S(\rho) := \iint_{\mathcal{F} \times \mathcal{F}} L(x, y) \, d\rho(x) \, d\rho(y).$$

Having defined the causal action, we can introduce the variational principle, which is the core of the theory of causal fermion systems:

**Definition 3.3** (Causal Action Principle and Constraints). The causal action principle is to minimize $S$ by varying the universal measure under the following constraints:

- **volume constraint:** $\rho(\mathcal{F}) = \text{const}$  (3.3)
- **trace constraint:** $\int_{\mathcal{F}} \text{tr}(x) \, d\rho(x) = \text{const}$  (3.4)
- **boundedness constraint:** $\mathcal{T}(\rho) := \iint_{\mathcal{F} \times \mathcal{F}} \sum_{i=1}^{2n} |\lambda_{xy}^i|^2 \, d\rho(x) \, d\rho(y) \leq C$.  (3.5)

Here $C$ is a given parameter (and tr denotes the trace of a linear operator on $\mathcal{F}$). The constraints are needed in order to obtain a well-posed variational principle without trivial minimizers.

Although the mathematical structure of the causal action principle can be understood from general considerations (as will be outlined below), its detailed form is far from obvious. It is the result of many computations and long considerations, which we cannot review here. Instead, we note that the causal action was first proposed in [7, Section 3.5], based on considerations outlined in [7, Sections 5.5 and 5.6]. The significance of the constraints became clear in the later mathematical analysis [9].

Note that the universal measure $\rho$ is the basic object in the theory of causal fermion systems. It is a unified theory in the sense that all spacetime structures are encoded in and must be derived from this measure. In other words, the measure $\rho$ describes our universe as a whole. This explains the name universal measure.
3.3. The Equivalence Principle. Let \((\mathcal{H}, \mathcal{F}, \rho)\) be a causal fermion system of spin dimension \(n\) which minimizes the causal action, respecting the constraints.

**Definition 3.4 (Spacetime).** Spacetime \(M\) is defined as the support of the universal measure,

\[
M := \text{supp} \rho \subset \mathcal{F}.
\]

Here the support of a measure is defined as the complement of the largest open set of measure zero, i.e.

\[
\text{supp} \rho := \mathcal{F} \setminus \bigcup \{ \Omega \subset \mathcal{F} \mid \Omega \text{ is open and } \rho(\Omega) = 0 \}.
\]

Thus the space-time points are symmetric linear operators on \(\mathcal{H}\). On \(M\) we consider the topology induced by \(\mathcal{F}\) (generated by the sup-norm \((3.1)\) on \(L(\mathcal{H})\)). Moreover, the universal measure \(\rho|_M\) restricted to \(M\) can be regarded as a volume measure on space-time. This makes space-time to a topological measure space.

Let \(\Phi : M \to M\) be a homeomorphism of spacetime. Given a Borel set \(\Omega \subset \mathcal{F}\), the preimage \(\Phi^{-1}(\Omega \cap M)\) is a Borel set of \(M\). Therefore, we can define a new Borel measure \(\tilde{\rho}\) on \(\mathcal{F}\) by \(\tilde{\rho}(\Omega) := \rho(\Phi^{-1}(\Omega \cap M))\). This is the so-called push-forward measure denoted by

\[
\tilde{\rho} = \Phi_* \rho.
\]

The causal action as well as all the constraints are invariant under the transformation

\[
M \to \Phi(M), \quad \rho \mapsto \tilde{\rho}.
\]

This invariance generalizes the diffeomorphism invariance of General Relativity. In this sense, the equivalence principle is implemented in the theory of causal fermion systems.

3.4. Principle of Causality. For any \(x, y \in M\), the product \(xy\) is an operator of rank at most \(2n\). Exactly as defined at the beginning of Section 3.2.2 we denote its non-trivial eigenvalues (counting algebraic multiplicities) by \(\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}\).

**Definition 3.5 (Causal Structure).** The points \(x, y \in M\) are said to be

\[
\begin{align*}
\text{timelike separated} & \quad \text{if all the } \lambda_j^{xy} \text{ have the same absolute value} \\
\text{spacelike separated} & \quad \text{if the } \lambda_j^{xy} \text{ are all real and do not all have} \\
& \quad \text{the same absolute value} \\
\text{lightlike separated} & \quad \text{otherwise}.
\end{align*}
\]

This “spectral definition” of causality indeed gives back the causal structure of Minkowski space or a Lorentzian manifold in the corresponding limiting cases (for more details see Section 4 below). At this stage, one sees at least that our definition of the causal structure is compatible with the Lagrangian in the following sense. Suppose that two points \(x, y \in \mathcal{F}\) are spacelike separated. Then the eigenvalues \(\lambda_j^{xy}\) all have the same absolute value, implying that the Lagrangian vanishes. Working out the corresponding Euler-Lagrange equations (for details see [23]), one finds that pairs of points with spacelike separation again drop out. This can be seen in analogy to the
usual notion of causality where points with spacelike separation cannot influence each other. In this sense, the principle of causality is built into the theory of causal fermion systems.

3.5. Local Gauge Principle. The fact that spacetime points of a causal fermion system are operators in $\mathcal{F}$ gives rise to additional structures. In particular, there is an inherent notion of spinors and wave functions, as we now explain.

**Definition 3.6** (Spin Spaces). For every $x \in M$ we define the spin space $S_x$ by $S_x = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension at most $2n$. On $S_x$ we introduce an inner product $\langle ., . \rangle_x$ by

$$\langle u | v \rangle_x = -\langle u \mid xv \rangle_{\mathcal{H}},$$

referred to as the spin scalar product.

Since $x$ has at most $n$ positive and at most $n$ negative eigenvalues, the spin scalar product is an indefinite inner product of signature $(p_x, q_x)$ with $p_x, q_x \leq n$ (for textbooks on indefinite inner product spaces see [4, 28]). In this way, to every spacetime point $x \in M$ we associate a corresponding indefinite inner product space $(S_x, \langle ., . \rangle_x)$. If the signature of the spin spaces is constant in spacetime, we thus obtain the structure of a topological vector bundle (for more details in this direction see [20]). However, in contrast to a vector bundle, all the spin spaces are subspaces of the same Hilbert space $\mathcal{H}$; see Figure 1.

The vectors in $\mathcal{H}$ can be represented as wave functions in spacetime:

**Definition 3.7** (Physical Wave Function). For a vector $u \in \mathcal{H}$ one introduces the corresponding physical wave function $\psi^u$ as

$$\psi^u : M \to \mathcal{H}, \quad \psi^u(x) = \pi_x u \in S_x,$$

where $\pi_x : \mathcal{H} \to S_x$ denotes the orthogonal projection on the subspace $S_x \subset \mathcal{H}$.

This definition is illustrated in Figure 2.

A local gauge principle becomes apparent once we choose basis representations of the spin spaces and write the wave functions in components. Denoting the signature of $(S_x, \langle ., . \rangle_x)$ by $(p_x, q_x)$, we choose a pseudo-orthonormal basis $(e_\alpha(x))_{\alpha = 1, \ldots, p_x+q_x}$ of $S_x$, i.e.

$$\langle e_\alpha(x) | e_\beta(x) \rangle_x = s_\alpha \delta_\beta^\alpha$$
Figure 2. The physical wave function

with $s_1 = \ldots = s_{p_x} = 1$ and $s_{p_x+1} = \ldots = s_{p_x+q_x} = -1$. Then a physical wave function $\psi^u$ can be represented as

$$\psi^u(x) = \sum_{\alpha=1}^{p_x+q_x} \psi^\alpha(x) e^\alpha(x)$$

with component functions $\psi(x)^1, \ldots, \psi(x)^{p_x+q_x}$. The freedom in choosing the basis $(e^\alpha)$ is described by the group $U(p_x, q_x)$ of unitary transformations with respect to an inner product of signature $(p_x, q_x)$. This gives rise to the transformations

$$e^\alpha(x) \to \sum_{\beta=1}^{p_x+q_x} U^{-1}(x)_{\beta}^\alpha e^\beta(x) \quad \text{and} \quad \psi^\alpha(x) \to \sum_{\beta=1}^{p_x+q_x} U(x)_{\beta}^\alpha \psi^\beta(x) \quad (3.7)$$

with $U \in U(p_x, q_x)$. As the basis $(e^\alpha)$ can be chosen independently at each space-time point, one obtains local gauge transformations of the wave functions, where the gauge group is determined to be the isometry group of the spin scalar product.

The causal action is gauge invariant in the sense that it does not depend on the choice of spinor bases. This connection becomes clearer if the Lagrangian is expressed in terms of the physical wave functions. This can be accomplished as follows.

**Definition 3.8 (Kernel of the Fermionic Projector).** For any $x, y \in M$ we define the kernel of the fermionic projector $P(x, y)$ by

$$P(x, y) = \pi_x y|_{S_y} : S_y \to S_x \quad (3.8)$$

This definition is illustrated in Figure 3. We remark that this definition harmonizes with the definition of the spin scalar product (3.6) in the sense that the kernel of the fermionic projector is symmetric with respect to the spin scalar product,

$$\langle u | P(x, y) v \rangle_x = -\langle u | x P(x, y) v \rangle_{2x} = -\langle u | xy v \rangle_{3x} = -\langle y u x v \rangle_{xy} = \langle P(y, x) u | v \rangle_{y}$$

(where $u \in S_x$ and $v \in S_y$). Taking the trace of (3.8) in the case $x = y$, one finds that $\text{tr}(x) = \text{Tr}_{S_x}(P(x, x))$ (where tr and Tr are the traces on $H$ and the spin space, respectively), making it possible to express the integrand of the trace constraint (3.4) in terms of the kernel of the fermionic projector. In order to also express the eigenvalues of the operator $xy$ in terms of the kernel of the fermionic projector, we introduce the closed chain $A_{xy}$ as the product

$$A_{xy} = P(x, y) P(y, x) : S_x \to S_x \quad (3.9)$$
Computing powers of the closed chain, one obtains
\[ A_{xy} = (\pi y x)(\pi y x) | S_x = \pi x y x | S_x , \quad (A_{xy})^p = \pi x (y x)^p | S_x . \]
Taking the trace, one sees in particular that
\[ \text{Tr}_{S_x} (A_{xy}^p) = \text{tr} ((y x)^p) = \text{tr} ((x y)^p) \]
(where the last identity simply is the invariance of the trace under cyclic permutations).

As a consequence\(^1\), the eigenvalues of the closed chain coincide with the non-trivial eigenvalues \(\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}\) of the operator product \(xy\). This makes it possible to express both the Lagrangian (3.2) and the integrand of the boundedness constraint (3.5) in terms of \(A_{xy}\). The main advantage of working with the kernel of the fermionic projector is that the closed chain (3.9) is a linear operator on a vector space of dimension at most \(2n\), making it possible to compute the \(\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}\) as the eigenvalues of a finite matrix.

The kernel of the fermionic projector can be expressed in terms of the physical wave functions as follows. Choosing an orthonormal basis \((e_i)\) of \(\mathcal{H}\) and using the completeness relation as well as (3.6), one obtains for any \(\phi \in S_y\)
\[ P(x, y) \phi = \pi x y | S_y \phi = \sum_i \pi x e_i \langle e_i | y \phi \rangle_{\mathcal{H}} = - \sum_i \psi^e_i (x) \langle \psi^e_i (y) | \phi \rangle_{S_y} , \]
showing that \(P(x, y)\) is indeed composed of all the physical wave functions, i.e. in a bra/ket notation
\[ P(x, y) = - \sum_i \langle \psi^e_i (x) | \psi^e_i (y) \rangle . \]  
(3.10)

Finally, choosing again bases \((e_\alpha(x))_{\alpha=1,\ldots,p_x+q_x}\) of the spin spaces, the kernel \(P(x, y)\) is expressed by a \((p_x + q_x) \times (p_y + q_y)\)-matrix. According to (3.7), this matrix behaves under gauge transformations as
\[ P(x, y)_{\alpha}^\beta \rightarrow \sum_{\gamma=1}^{p_x+q_x} \sum_{\delta=1}^{p_y+q_y} U(x)^\alpha_\gamma P(x, y)^\gamma_\delta \left( U(y)^* \right)^\delta_\beta , \]
where the star denotes the adjoint with respect to the spin scalar product. Since \(U(y) \in U(p_x, q_x)\) is unitary with respect to the spin scalar product, the gauge transformation

\(^1\)More precisely, since all our operators have finite rank, there is a finite-dimensional subspace \(I\) of \(\mathcal{H}\) such that \(xy\) maps \(I\) to itself and vanishes on the orthogonal complement of \(I\). Then the non-trivial eigenvalues of the operator product \(xy\) are given as the zeros of the characteristic polynomial of the restriction \(xy|_I : I \rightarrow I\). The coefficients of this characteristic polynomial (like the trace, the determinant, etc.) are symmetric polynomials in the eigenvalues and can therefore be expressed in terms of traces of powers of \(A_{xy}\).
at $y$ drops out when forming the closed chain, i.e.

$$(A_{xy})^a_\beta \to \sum_{\gamma, \delta = 1} U(x)^\alpha_\gamma \gamma (A_{xy})^\gamma_\delta (U(x)^*)^\delta_\beta.$$ 

Since $U(x) \in U(p_x, q_x)$ is unitary, the eigenvalues of the closed chain do not depend on the choice of the gauge.

This explains in particular why the Lagrangian is invariant under local gauge transformations of the physical wave functions. Such computations were helpful for formulating the causal action principle (for details see [7, Chapter 3]).

3.6. Fermionic Building Blocks. In the above formulas, the physical wave functions play a dominant role. Indeed, according to (3.10), the ensemble of all these wave functions determines the kernel of the fermionic projector, which, forming the closed chain and computing its eigenvalues, gives rise to all the quantities needed in the causal action principle. In this way, the causal variational principle can be formulated directly in terms of the ensemble of all physical wave functions. Minimizing the causal action amounts to finding an “optimal” configuration of the physical wave functions. In other words, the causal action principle can be understood as a variational principle which determines the collective behavior of all physical wave functions.

As will be worked out in detail in Section 4 below, in concrete examples the physical wave functions go over to solutions of the Dirac equation. More specifically, describing the Minkowski vacuum as a causal fermion system (see Section 4.3), the ensemble of all physical wave functions correspond to all the negative-frequency solutions of the Dirac equation. In this way, Dirac’s original concept of the Dirac sea is realized. The fact that Dirac wave functions describe fermionic particles is the motivation for the name “causal fermion system.”

3.7. Microscopic Spacetime Structure. In the theory of causal fermion systems, spacetime defined as the support of the universal measure $\rho$ (see Definition 3.4) does not need to be a differentiable manifold. Instead, it could be discrete on a microscopic scale or could have another nontrivial microstructure. Exactly as explained above for the causal structure, also the microscopic structure of spacetime is not given a-priori, but it is determined dynamically by the causal action principle. The analysis of simple model examples reveals that minimizing measures of the causal action principles are typically discrete (for details see [27, 1] or the survey in [15, Section 3]). Although it is an open problem whether these discreteness results also hold for general causal fermion systems, these results suggest that the concept of smooth spacetime structures should be modified on small scales, typically thought of as the Planck scale. The theory of causal fermion systems provides a mathematical setting in which such generalized spacetimes can be described and analyzed.

4. Modelling a Lorentzian Spacetime by a Causal Fermion System

4.1. General Construction in Curved Spacetimes. We return to the setting of the Dirac equation in curved spacetime in Section 2.5. We now explain how to describe this spacetime by a causal fermion system. We denote the Hilbert space of solutions of the Dirac equation with the scalar product (2.4) by $(\mathcal{H}_m, (\cdot, \cdot)_m)$ (more precisely, we take the completion of all smooth solutions with spatially compact support). Next, we choose a closed subspace $\mathcal{H} \subset \mathcal{H}_m$ of the solution space of the Dirac equation. The
induced scalar product on $\mathcal{H}$ is denoted by $\langle .| . \rangle_{\mathcal{H}}$. There is the technical difficulty that the wave functions in $\mathcal{H}$ are in general not continuous, making it impossible to evaluate them pointwise. For this reason, we need to introduce an ultraviolet regularization, described mathematically by a linear

\[ R : \mathcal{H} \rightarrow C^0(M, S^M) \]

We postpone the discussion of the physical significance of the regularization operator to Section 4.2. Mathematically, the simplest method to obtain a regularization operator is by taking the convolution with a smooth, compactly supported function on a Cauchy surface or in spacetime (for details see [26, Section 4] or [12, Section §1.1.2]).

Given $R$, for any space-time point $p \in M$ we consider the bilinear form $b_p : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, $b_p(\psi, \phi) = -\langle (R\psi)(p) | (R\phi)(p) \rangle_p$.

This bilinear form is well-defined and bounded because $R$ maps to the continuous wave functions and because evaluation at $p$ gives a linear operator of finite rank. Thus for any $\phi \in \mathcal{H}$, the anti-linear form $b_p(., \phi) : \mathcal{H} \rightarrow \mathbb{C}$ is continuous. By the Fréchet-Riesz theorem, there is a unique $\chi \in \mathcal{H}$ such that $b_p(\psi, \phi) = \langle \psi | \chi \rangle_{\mathcal{H}}$ for all $\psi \in \mathcal{H}$. The mapping $\phi \mapsto \chi$ is linear and bounded, giving rise to the following linear operator:

\[ F(p) : \mathcal{H} \rightarrow \mathcal{H} \]

\[ (\psi | F(p) \phi) = -\langle (R\psi)(p) | (R\phi)(p) \rangle_p \quad \text{for all } \psi, \phi \in \mathcal{H}. \quad (4.1) \]

Taking into account that the inner product on the Dirac spinors at $p$ has signature $(2,2)$, the local correlation operator $F(p)$ is a symmetric operator on $\mathcal{H}$ of rank at most four, which (counting multiplicities) has at most two positive and at most two negative eigenvalues. Varying the space-time point, we obtain a mapping

\[ F : M \rightarrow \mathcal{F} \subset L(\mathcal{H}) \]

where $\mathcal{F}$ denotes all symmetric operators of rank at most four with at most two positive and at most two negative eigenvalues. Finally, we introduce the

\[ \text{universal measure} \quad d\rho := F_* d\mu_M \quad (4.2) \]

as the push-forward of the volume measure on $M$ under the mapping $F$ (thus $\rho(\Omega) := \mu_M(F^{-1}(\Omega))$). We thus obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two.

We close with a few comments on the underlying physical picture. The vectors in the subspace $\mathcal{H} \subset \mathcal{H}_m$ have the interpretation as those Dirac wave functions which are realized in the physical system under consideration. If we describe for example a system of one electron, then the wave function of the electron is contained in $\mathcal{H}$. Moreover, $\mathcal{H}$ includes all the wave functions which form the so-called Dirac sea (for an explanation of this point see for example [10]).

According to (4.1), the local correlation operator $F(p)$ describes densities and correlations of the physical wave functions at the space-time point $p$. Working exclusively with the local correlation operators and the corresponding push-forward measure $\rho$ means in particular that the geometric structures are encoded in and must be retrieved from the physical wave functions. Since the physical wave functions describe the distribution of matter in space-time, one can summarize this concept by saying
that *matter encodes geometry*. Going one step further, one can also say that matter and geometry form an inseparable unity.

4.2. **Physical Significance of the Regularization Operator.** The regularization operator requires a detailed explanation. We first convey the underlying physical picture. The regularization operators should leave the wave functions unchanged on macroscopic scales (i.e. scales much larger than the Planck length). Thus on macroscopic length scales, the Dirac equation still holds, giving agreement with the common physical description. However, on a microscopic scale $\varepsilon$, which can be thought of as the Planck scale, the regularization may change the wave functions completely. As a consequence, also the universal measure $\rho$ in (4.2) is changed, which means that the microscopic structure of spacetime is modified. Therefore, in contrast to the renormalization program in Quantum Field Theory, in the theory of causal fermion systems the regularization is not just a technical tool, but it realizes our concept that we want to allow for a nontrivial microstructure of spacetime. With this in mind, we always consider the regularized quantities as those having mathematical and physical significance. Different choices of regularization operators realize different microscopic spacetime structures.

This concept immediately raises the question how the “physical regularization” should look like. Generally speaking, the regularized spacetime should look like Lorentzian spacetime down to distances of the scale $\varepsilon$. For distances smaller than $\varepsilon$, the structure of space-time may be completely different, in a way which cannot be guessed or extrapolated from the structures of Minkowski space. Since experiments on the length scale $\varepsilon$ seem out of reach, it is completely unknown what the microscopic structure of space-time is. Within the theory of causal fermion systems, the above question could be answered in principle by minimizing the causal action over all possible regularization operators. However, this approach turns out to be very difficult and at present is out of reach (for a first step in this direction see [8]). In view of these difficulties, the only available method is the so-called method of variable regularization: Instead of trying to determine the microstructure experimentally or with mathematical analysis, the strategy is to a-priori include all conceivable regularizations and, with hindsight, to eliminate those which are in conflict with well-established physical facts. The remaining regularizations which comply with all experimental constraints should be treated as equally admissible, because at present there is no criterion to distinguish between different choices or to favor one regularization over the others.

For the method of variable regularization to be sensible and to retain the predictive power of the theory, the detailed form of the microstructure must have no influence on the effective physical equations which are valid on the energy scales accessible to experiments. More precisely, the picture is that the general structure of the effective physical equations should be independent of the microstructure of spacetime. Values of mass ratios or coupling constants, however, may well depend on the microstructure (a typical example is the gravitational constant, which is closely tied to the Planck length). In more general terms, the unknown microstructure of spacetime should enter the effective physical equations only by a finite (hopefully small) number of free parameters, which can then be taken as empirical free parameters of the effective macroscopic theory. In [12] it was shown that these conditions are indeed satisfied.
4.3. Concrete Example: the Minkowski Vacuum. We now make the construction of Section 4.1 more explicit by working out the example of the Minkowski vacuum with the simplest possible regularization. We proceed in the following steps:

- Choosing the Hilbert space of all negative-frequency solutions
  Our starting point are the plane-wave solutions of the Dirac equation in Minkowski space (2.3), which we write as
  \[ \psi_{\vec{p}a}^\pm(x) = \frac{1}{\sqrt{2\pi}} e^{\pm i\omega x + i\vec{p}\cdot\vec{x}} \chi_{\vec{p}a}^\pm \quad \text{with} \quad \omega = \omega(p) := \sqrt{\|\vec{p}\|^2 + m^2}. \]
  Here the spinor \( \chi_{\vec{p}a}^\pm \) solves the algebraic equation
  \[ (\gamma^k p_k - m \mathbb{1}) \chi_{\vec{p}a}^\pm = 0, \]
  where \((p_k) = (\omega, \vec{p})\) denotes the four-momentum. Negative-frequency wave packets of the form
  \[ \psi_f(x) := \int_{\mathbb{R}^3} \psi_{\vec{p}a}^-(x) f(\vec{p}) \, d^3p \quad \text{with} \quad f \in C^\infty_0(\mathbb{R}^3, \mathbb{C}) \quad (4.3) \]
  span a subspace of \( \mathcal{H}_m \). We choose the Hilbert space \( \mathcal{H} \) of the causal fermion system as the closure of this subspace. This choice realizes the concept of the Dirac sea vacuum.

- Constructing the local correlation operators
  The simplest method to choose regularization operators consists in inserting a convergence-generating factor \( e^{-\varepsilon\omega} \) into the wave packet (4.3), i.e.
  \[ (\mathcal{R}\psi_f)(x) := \int_{\mathbb{R}^3} e^{-\varepsilon\omega} \psi_{\vec{p}a}^-(x) f(\vec{p}) \, d^3p. \]
  Now we can define the local correlations operators by (4.1) and construct the universal measure according to (4.2). We thus obtain a causal fermion system \((\mathcal{H}, \mathcal{F}, \rho)\).

In this example, one can compute the objects of the causal fermion system explicitly (for details see [12, Section 1.2]). One finds that in the limit \( \varepsilon \to 0 \), the inherent structures of the causal fermion system go over to the usual objects and relations in Minkowski space. More specifically, mapping a point \( p \in \mathcal{M} \) to the corresponding local correlation operator \( F(p) \) gives a one-to-one correspondence between Minkowski space \( \mathcal{M} \) and the spacetime \( M := \text{supp} \rho \) of the causal fermion system. Moreover, the causal structure of Definition 3.5 gives back the causal structure of Minkowski space, and the spin space \( S_x \) of Definition 3.6 can be identified with the space of Dirac spinors \( S_x \mathcal{M} \). Under these identifications, the physical wave functions of Definition 3.7 agree with the regularized Dirac wave functions of negative frequency.

5. Results of the Theory and Further Reading

Let us explain in which sense and to which extent the goal of unifying Quantum Field Theory and General Relativity has been achieved. Causal fermion systems provide a mathematically consistent theory which gives General Relativity and Quantum Theory as limiting cases. The causal action principle has well-defined minimizers in the case of a finite-dimensional Hilbert space and finite total volume (see [9]; more general cases are presently under investigation). The reason why the inconsistencies of Quantum Field Theory and General Relativity as described in Section 2.4 as well as
the divergences of Quantum Field Theory disappear is that we modified the structure of spacetime on the Planck scale. In more technical terms, in a causal fermion system one works with the regularized objects. Thus we consider the regularized objects as the fundamental physical objects. This concept could be implemented coherently because the causal action principle is formulated purely in terms of these regularized objects.

Causal fermion systems are a unified theory in the sense that spacetime and all objects therein are described by a single object: the universal measure. The causal action principle singles out those measures which describe physically admissible spacetimes. The Euler-Lagrange equations corresponding to the causal action principle describe the spacetime dynamics.

Clearly, in this short review we could only cover certain aspects of the theory from a particular perspective. Therefore, in order to help the interested reader to get a more complete picture, we now outline a few other directions and give references for further study. For other review articles with a somewhat different focus we refer to [10, 17, 21].

(a) A causal fermion system also provides topological (topological spinor bundle) and geometric objects (parallel transport and curvature). We refer the interested reader to [16, 20] or the introduction [14].

(b) The limiting case $\varepsilon \downarrow 0$, when the ultraviolet regularization is removed, is worked out in detail in [12]. In this limiting case, the so-called continuum limit, the causal action principle gives rise to the interactions of the standard model and gravity, on the level of classical bosonic fields interacting with a second-quantized fermionic field.

(c) An important concept for more recent developments are surface layer integrals, which generalize surface integrals to the setting of causal fermion systems. Symmetries of causal fermion systems give rise to conservation laws which can be expressed in terms of surface layer integrals [22].

(d) Another concept which has turned out to be fruitful for the analysis of the causal action principle are linearized solutions [23]. Similar to linearized gravitational waves, linearized solutions can be understood as linear perturbations of the measure $\rho$ which preserve the Euler-Lagrange equations of the causal action principle. As shown in [23, 24], linearized solutions come with corresponding conserved surface layer integrals, in particular the symplectic form and the surface layer inner product.

(e) Generally speaking, the conservation laws for surface layer integrals give rise to objects in space which evolve dynamically in time. This concept was worked out for linearized solutions in [6], where it is proven under general assumptions that the Cauchy problem for linearized solutions is well-posed and that the solutions propagate with finite speed.

(f) A first connection to Quantum Field Theory has been made in [11], however based on the classical field equations obtained in the continuum limit. Deriving Quantum Field Theory as a limiting case of causal fermion systems without referring to the continuum limit is a major objective of present research: The perturbation theory for the universal measure is worked out in [13]. For interacting bosonic fields, the constructions in [19] give rise to a description of the dynamics in terms of a unitary time evolution on bosonic Fock spaces. The generalization of these constructions to include fermionic fields is currently under investigation [18].
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