Affirmative Action’s Cumulative Fractional Assignments*

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February 23, 2024

Abstract

The Central Educational Institutions (Reservation in Teachers’ Cadre) Act, 2019 provides for reserving teaching vacancies in India’s central educational institutions for beneficiaries of its affirmative action policy. Reservation of teaching vacancies had been a contentious issue, and the act was introduced to resolve it after the Supreme Court’s solution was met with protests from the Teachers’ Union. Our paper demonstrates an impossibility result in the Supreme Court’s solution and the act, which are flawed in reserving seats simultaneously at both the university and within its departments. To overcome this impossibility, we propose an alternative solution based on approximate implementation of fractional assignments, offering a promising middle-ground between the two disputed solutions practiced in India. This novel application demonstrates the practical relevance of the approximate implementation approach (Akbarpour and Nikzad (2020)) beyond the constraint structures examined in the literature.

Keywords: Market Design, Indivisibility, Random Allocation, Intersecting Constraints

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*We are deeply grateful to Tayfun Sönmez, M. Utku Üner, and M. Bumin Yenmez for their invaluable guidance and continuous support. We thank Abhijit Banerji, Inácio Bó, Umut Dur, Mehmet Ekmekci, Xiang Han, Gaoji Hu, Hideo Konishi, Morimitsu Kurino, Debasis Mishra, Thayer Morrill, William Phan, Uzi Segal, Rohit Vaish, Rakesh Vohra, Yongchao Zhang and seminars participants at North Carolina State University, Boston College, the 2022 SAET Conference at the Australian National University, Shenzhen University Public Policy and Experimental Economics Workshop, Indian Institute of Technology - Delhi, Indian Statistical Institute - Delhi, and Shanghai University of Finance and Economics for helpful comments and discussions. We are grateful to Yuan Gao and Xi Jin for helpful research assistance. All errors are our own.

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1 Introduction

1.1 Background

The 1950 Constitution of India provides a clear basis for positive discrimination in favor of disadvantaged groups, in the form of reservation policies. India’s reservation policies mandate exclusive access to a fixed percentage of government jobs and seats in publicly funded institutions to the members of Scheduled Castes (SC, 15%), Scheduled Tribes (ST, 7.5%), Other Backward Classes (OBC, 27%) and Economically Weaker Sections (EWS, 10%). For transparency, the number of reserved seats for each category is explicitly and publicly advertised before any admissions or recruitment cycle.

The procedures used to calculate the number of reserved seats in various settings are also explicit and public. However, they have nowhere been more contentious than in the case of universities. Unlike other government jobs, the eligibility and selection criteria change with the department for the same faculty position in a university (say, assistant professor). Thus, the faculty vacancies in different departments are not interchangeable across a university. Each faculty position, therefore, simultaneously represents two units, a department and the university, where each unit is subject to the reservation policy. This feature of faculty vacancies led to complications that made all three arms of the Indian government – the executive, the judiciary, and the legislative – intervene.

The Executive. In August 2006, the University Grants Commission (UGC) issued Guidelines for Strict Implementation of Reservation Policy of the Government in Universities to all government educational institutions in India. Through these guidelines, the UGC prohibited the practice of treating department as the unit for application of the reservation scheme, that is, for calculating the proportion of seats to be reserved (see clause 6(c) in the guidelines). Instead, UGC mandated university as the unit for reservation. That is, the vacancies in a university shall be clubbed together across departments as three separate categories: professors, associate professors (or readers), and assistant professors (or lecturers), for the application of the rule of reservation (see clause 8(a)(v) in the guidelines). However, UGC’s order was challenged in court.

The Judiciary. In April 2017, the Allahabad High Court allowed a petition demanding reservations in faculty vacancies treating the department as the unit and quashed clauses 6(c)
and 8(a)(v) of the UGC Guidelines of 2006. The court argued that treating the university as the unit “would be not only impracticable, unworkable but also unfair and unreasonable” for the following two reasons stated in the judgment:

Merely because the Assistant Professor, Reader, Associate Professor, and Professor of each subject or department are placed on the same pay scale, but their services are neither transferable nor they compete with each other. It is for this reason also that clubbing the posts for the same level and treating the University as a ‘Unit’ would be completely unworkable and impractical. It would be violative of Articles 14 and 16 of the Constitution.

If the University is taken as a ‘Unit’ for every level of teaching and applying the roster, it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates. Such a proposition again would be discriminatory and unreasonable. This, again, would be violative of Articles 14 and 16 of the Constitution.

Following the court order, universities advertised vacancies with a sharp fall in the number of reserved vacancies. This is apparent in the case of Banaras Hindu University, presented in Table 1, where the number of unreserved seats increased from 1188 under the government’s quashed solution to 1562 under the court’s proposed solution. The reason was that many departments had a small number of faculty vacancies (fewer than six). Given that each department followed the same fixed sequence in which categories take turns claiming a position, the court’s solution led to fewer vacancies for the reserved categories at the university level. This sparked a series of teachers’ union-led protests across India.

The Legislative. The protests compelled the government to file a petition in the Supreme Court against the Allahabad High Court verdict. “How can the post of professor of Anatomy be compared with the professor of Geography? Are you clubbing oranges with apples?” questioned the Supreme Court rejecting the appeal and terming the Allahabad high court judgment as “logical”. Facing a huge aggrieved vote bank, three days before the announcement of Lok Sabha election, in March 2019, the government promulgated an ordinance that considered the university as the unit. This ordinance is now an Act of Parliament.

3Judgement last accessed on 22 February 2024 at https://indiankanoon.org/doc/177500970/
4Last accessed on 22 February 2024 at https://indianexpress.com/article/explained/hrd-ministry-ordinance-teacher-quota-university-prakash-javadekar-5616157/
5See Figure 15 and Figure 16 for the sequence in which the beneficiary groups take turns in claiming a position in India.
6Last accessed on 22 February 2024 at https://main.sci.gov.in/supremecourt/2019/5495/5495_2019_Order_27-Feb-2019.pdf
Table 1: NUMBER OF RESERVED vacancies IN BANARAS HINDU UNIVERSITY

| Position          | University as a Unit (Government’s Solution) | Department as a Unit (Court’s Solution) |
|-------------------|---------------------------------------------|----------------------------------------|
|                   | General | SC | ST | OBC | Total | General | SC | ST | OBC | Total |
| Professor         | 197     | 38 | 18 | 0   | 253   | 250     | 3  | 0  | 0   | 253   |
| Associate Professor| 410     | 79 | 39 | 0   | 528   | 500     | 25 | 3  | 0   | 528   |
| Assistant Professor| 581     | 172| 86 | 310 | 1149  | 812     | 91 | 26 | 220 | 1149  |
| Total             | 1188    | 289| 143| 310 | 1930  | 1562    | 119| 220| 220 | 1930  |

Notes: Data shared in government’s Special Leave Petition filed in the Supreme Court of India.

and, therefore, the law in India.\(^7\)

Today, the university is the unit for applying the reservation scheme. The court’s objection that “it could result in some departments/subjects having all reserved candidates and some having only unreserved candidates” inspired us to write this paper.

1.2 Problem and contributions

The Indian affirmative action scheme, known as the reservation policy, sets a specific proportion of seats and jobs in publicly funded institutions for various beneficiary groups. Unlike its American counterpart, this policy mandates recruitment and admission advertisements to include information about the number of government vacancies reserved for protected groups. Meeting the prescribed percentage of seats is difficult in practice due to the indivisible nature of seats, which results in fractional seats that must be adjusted to whole numbers.

It is particularly challenging to meet the targeted quotas set by the affirmative action policy when it comes to filling teaching vacancies. Each teaching position comprises two units, the department and the university, both of which are subject to the reservation policy. Moreover, due to the growth of both departments and universities over time, keeping track of the cumulative percentage of seats reserved becomes imperative to comply with the affirmative action policy.\(^8\)

The problem of the university in each recruitment cycle can be summarized as a fractional assignment table \(X\). Its entries \(x_{i,j}\) signify the fraction of seats beneficiary \(j\) is entitled in

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\(^7\)Last accessed on 22 February 2024 at \[http://egazette.gov.in/WriteReadData/2019/206575.pdf\]

\(^8\)In Section 3, we detail how Indian public institutions use a tool called roster to keep track of the cumulative percentage of seats reserved. A commonly used roster is shown in Figure 15.
department $i$ as per the affirmative action policy. The rows represent the first subdivision of the university into departments. The columns accommodate several beneficiaries and, therefore, present a second subdivision. The university is assumed to be broken down, either way, providing department sizes as row sums and overall (university-level) beneficiary claims as column sums. The task is to find an integral assignment table, with seat allocations (whole numbers, not fractions) $\bar{x}_{i,j}$ summing row-wise to the pre-specified row sums while remaining “as near as may be” to the fractional assignments $x_{i,j}$.

We will search for solutions to the university’s problem that generate integral assignment tables deterministically or randomly for every possible fractional assignment table and satisfy the following four key properties.

(1) **Monotonic.** One of the primary concerns in practice is that as the university expands over various recruitment cycles, it is important to ensure that the cumulative integral assignment tables remain “close” to the prescribed cumulative fractional assignment tables at the department and university levels. This task is challenging due to the need to implement cumulative fractional assignments while adhering to a monotonicity constraint. The monotonicity constraint arises because once seats are reserved in a recruitment cycle, they cannot be unreserved later. This is an unarguably essential property that any proposed solution must fulfill.

(2) **Stays within department quota.** The fractional assignment tables would be the ideal seat allocations if only the seats were divisible. Therefore, integral assignment tables with entries rounded up or down to an adjacent integer of entries of the fractional assignment table can be considered the best integer solution. In this paper, we will search for integral assignments (seat allocations) $\bar{x}_{i,j}$ that consist of entries $x_{i,j}$ of the fractional assignment table rounded up or down to the nearest integer.

If all the fractional assignments corresponding to departments, given in internal entries of table $X$, are rounded up or down to the nearest integer, we say the solution stays within department quota. In other words, the fractional assignments in internal entries of table $X$ must be rounded up or down to the nearest integer for the solution to stay within department quota.

(3) **Stays within university quota.** If all the fractional assignments corresponding to the university, given in column sums of table $X$, are rounded up or down to the nearest integer, we say the solution stays within university quota. In other words, the fractional

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9It is useful to visualize such an integral assignment table at this point. See Figure 17 in Appendix B for a recent advertisement from Hindu College to recruit assistant professors.
assignments in the column sums of table $X$ must be rounded up or down to the nearest integer for the solution to stay within university quota.

(4) **Ex ante proportional.** Realizing the fractional assignment in expectation is considered a minimal “fairness” requirement for randomized methods (Grimmett (2004)). An ex ante proportional randomized solution to our problem would be that each category in each department and the university receives exactly their fractional assignment of seats in expectation.

The problem of the university in each recruitment cycle can be viewed as a problem of implementing a fractional assignment such that the seat allocations are obtained by rounding the fractional assignments up or down while preserving the additive structure of the assignments and adhering to a monotonicity constraint. The constraint structure induced by monotonicity, staying within department quota, and staying within university quota, combined, ceases to be a bihierarchy (as in Budish et al. (2013)). Moreover, there appears to be no way to incorporate the soft constraints into the deepest level of the bihierarchy (as in Akbarpour and Nikzad (2020)). This makes the implementation of cumulative fractional assignments both challenging and intriguing.

In Section 3, we describe the solutions practiced in India and analyze why and where they fail. Both solutions are monotonic and ensure that the cumulative integral assignments remain “close” to the cumulative fractional assignments. However, neither of the two solutions accounts for the interdependence of the departments and the university when calculating reserved seats (integral seat allocations). In particular, each solution either operates at the department level or the university level, but not at both simultaneously. If the solution operates at the department level, it fails to deliver the benefit of reservations at the university level. Whereas if it operates at the university level, the reserved vacancies could get allocated to merely a few departments in the university. Not surprisingly, these solutions are met with several petitions and protests leading to subsequent and frequent changes in the law.

There are four main theoretical results of our article presented in Section 4:

- We show that it is impossible to implement cumulative fractional assignments while adhering to the monotonicity constraint such that the seat allocations simultaneously stay within department and university quotas.

We have given an even stronger result in Theorem 1. It is impossible to implement cumulative fractional assignments while adhering to the monotonicity constraint such that the seat allocations stay within university quota and deviations of the integral assignment from the fractional assignment table are bounded by a finite number.
These results perhaps justify the struggle in figuring out a solution in real-life practice, as discussed in Section 1.1 and Section 3.

- Since monotonicity, stays within department quota, and stays within university quota properties impose constraints that cannot be satisfied simultaneously, can a subset of these constraints be satisfied?

Theorem 2 shows that there exists a solution that is monotonic and stays within department quota. A similar solution exists that is monotonic and stays within university quota. Moreover, these solutions are ex ante proportional.

The proof of Theorem 2 involves constructing a randomized solution that stays within department quota and is ex ante proportional. The essential construction in the proof is that of a random roster.\textsuperscript{10} Given fractions of reservations, for every number of vacancies, a roster gives the number of vacancies reserved for each beneficiary. For a roster, staying within department quota constraints regulates the cumulative number of vacancies for each category. We show that the underlying constraint structure of a roster that stays within department quota is a bihierarchy, and therefore by Budish et al. (2013) it can be decomposed into integral assignments in an ex ante proportional manner.\textsuperscript{11} Since many rosters could stay within department quota, the procedure generates a random roster by assigning each solution roster a probability. Our solution assigns a roster to each department independently while adhering to the probabilities dictated by the procedure.

Constructing a random roster presented in the proof of Theorem 2 is of independent interest. In particular, the flows constructed (see Appendix A.2) would retain any provable properties even if the row constraints were fractional and if the fractions varied across the rows as well. It can, therefore, be generalized to capture other applications. For instance, it can serve as an alternative proof of the main result of Gölz et al. (2022) that shows the existence of a stochastic apportionment method that is ex ante proportional, house monotonic, and satisfies quota.\textsuperscript{12}

\textsuperscript{10}Inspired by the roster used in India, see Section 3 for details and see Figure 15 for a commonly used roster.

\textsuperscript{11}The procedure of constructing a random roster, detailed in Appendix A.2, is built around a network flow algorithm that takes a flow network as input and randomly constructs another flow network with fewer fractional flows as its output. By iterative application of this algorithm, a flow network with integral flows is generated. The random flow network has the following two properties: the expected value of each flow after the next iteration is the same as its current value, and each constraint (imposed by the stay within department quota property) remains satisfied. Since each flow network with integral flows can be mapped to a roster, this procedure generates a random roster.

\textsuperscript{12}We refer the reader to Gölz et al. (2022) for the setup, formal definitions, and their main result stated in Theorem 6.
• By **Theorem 1**, monotonicity, stays within department quota, and stays within university quota properties impose constraints that cannot be satisfied simultaneously. By **Theorem 2**, constraints imposed by monotonicity and stays within department quota properties are achievable. Can the other constraints imposed by stays within university quota property quota be satisfied *approximately*?

By approximately, we mean that the probability of violating these constraints exponentially decreases with the size of the constraint. Thus, the constraints introduced by stays within university quota property are treated as soft constraints, as goals rather than hard constraints.

**Theorem 3** gives an ex ante proportional randomized solution that is monotonic, stays within department quota, and almost stays within university quota with high probability.

We do so by applying the multiplicative form of Chernoff concentration bounds to our solution constructed in the proof of **Theorem 2**. Inspired by Akbarpour and Nikzad (2020)’s implementation of complex constraint structures, this result adds to the list of constraint structures where the approximate implementation approach is fruitful.\(^\text{13}\)

• Lastly, we show that if the cumulative aspect of the problem can be ignored to simplify the problem, one can get rid of the monotonicity constraint and solve a much simpler problem of implementing each recruitment cycle’s fractional assignment table independently.

Without the monotonicity constraint, the problem has an elegant solution based on *controlled rounding* that is simple enough to be implemented by hand. **Theorem 4** shows that this solution is ex ante proportional, stays within department quota, and stays within university quota.\(^\text{14}\)

Our four results can, therefore, be summarized as follows. **Theorem 1** shows that it is impossible for a solution to satisfy properties (1), (2), and (3). **Theorem 2** shows that a solution exists that satisfies properties (1), (2), and (4). **Theorem 3** shows that the solution constructed in **Theorem 2** also approximately satisfies property (3). **Theorem 4** shows that there is an elegant solution that satisfies properties (2), (3), and (4).

\(^\text{13}\)The result cannot be proven by simply using off-the-shelf results from Akbarpour and Nikzad (2020).

\(^\text{14}\)Cox (1987) introduced the technique to make slight perturbations in two-dimensional census data to ensure the confidentiality of aggregate statistics while maintaining a good approximation of the original data. Adaptation of Cox’s controlled rounding technique to our problem is summarized in proof of **Theorem 4**.
1.3 Related work

Hylland and Zeckhauser (1979) introduced the idea of implementing fractional assignments through a lottery over integral assignments. Subsequent studies delved into the analysis of ex ante and ex post considerations related to fairness, efficiency, and incentives within this framework (see Bogomolnaia and Moulin (2001), Abdulkadiroğlu and Sönmez (1998), Nesterov (2017), Han (2023), and Aziz et al. (2023)). Budish et al. (2013), Pycia and Ünver (2015), and Akbarpour and Nikzad (2020) broadened the scope of random allocation mechanisms to scenarios involving complex ex post constraints governing the feasibility of assignments.

Budish et al. (2013) and Akbarpour and Nikzad (2020) proposed implementation methods for fractional assignments subject to constraints by extending techniques of deterministic and randomized rounding developed in Edmonds (2003) and Gandhi et al. (2006). Our problem’s constraint structure closely resembles the one studied in these papers, but its multi-period aspect sets it apart, which imposes cumulative and monotonic constraints. These additional constraints make the constraint structure of the problem unique as it ceases to be a bihierarchy (as in Budish et al. (2013)), and there is no apparent way of re-writing the problem such that the soft constraints are contained at the deepest level of the bihierarchy (as in Akbarpour and Nikzad (2020)).

A common approach to accommodating complex constraint structures is treating a subset of constraints as “soft” constraints, as goals rather than hard constraints, allowing enough flexibility to attain desired properties (Akbarpour and Nikzad (2020)). Some excellent examples of this approach include: Budish (2011)’s combinatorial assignment, which treats course capacities as flexible constraints, Ehlers et al. (2014)’s deferred acceptance algorithm with adjustable group-specific lower and upper bounds soft bounds, Nguyen et al. (2016) and Nguyen and Vohra (2018)’s matching markets with complementarities that feature adjustable capacity constraints, and Nguyen and Vohra (2019)’s proportionality constraints that are adjusted to nearby integers in search of stable matchings. We also adopt this approach (in Theorem 3) after satisfaction of all constraints renders an impossibility result (in Theorem 1).

Proportional distribution of indivisible objects among a group of claimants in proportion to their claims, known as the apportionment problem, is the center point of the seminal work of Young (1995) and Balinski and Young (2010). Its two-dimensional version, the biphoproportional apportionment problem, gives rise to similar matrix problems as ours but has been investigated in the context of translating electoral votes into parliamentary seats (Maier et al. (2010), Lari et al. (2014), Pukelsheim (2017)). In that context, the cumulative constraints do not feature. The multi-period considerations make our problem a unique apportionment
problem that demands a new search for methods ensuring proportional representation in the face of monotonicity constraints.

Similar to our monotonicity constraint on randomized solutions is the house monotonicity constraint studied in stochastic apportionment problem (Grimmett (2004)). Gölz et al. (2022)’s cumulative rounding provides a randomized apportionment method that satisfies house monotonicity, quota, and ex ante proportionality. We believe the random roster constructed in the proof of Theorem 2 provides an alternative proof to their main result (Gölz et al. (2022), Theorem 6).

Our paper is also related to the application of rounding techniques. The controlled rounding procedure introduced in Cox (1987) to anonymize census data suffices for a restricted case of our problem (see Theorem 4). For fractional assignments with bihierarchical constraint structures, the pipage rounding technique by Gandhi et al. (2006) has been extended in Akbarpour and Nikzad (2020) to incorporate negative-correlation properties that facilitate the application of Chernoff concentration bounds. However, for our problem, the decomposition algorithm provided in Budish et al. (2013) suffices. An independent allocation in the construction allows us to apply Chernoff bounds. Other rounding methods that hold promise for future work is Lau et al. (2011)’s iterative rounding that has been used in Nguyen et al. (2016) and Nguyen and Vohra (2019). We speculate that this approach may lead to a solution allowing small additive bounds to violate some constraints that are shown as impossible to satisfy simultaneously (in Theorem 1).

Lastly, a considerable number of recent studies have offered practical alternatives for better implementation of nationwide affirmative action policies (see Abdulkadiroğlu and Sönmez (2003), Ehlers et al. (2014), Echenique and Yenmez (2015), Aygün and Turhan (2017), Dur et al. (2019), Aygun and Bó (2021), Aygün and Turhan (2023), Sönmez and Yenmez (2022) among others). Ours is another paper in this class. While the focus of the contemporary market design literature has been the design and analysis of assignment mechanisms given reserved seats and quotas, our paper focuses on implementing affirmative action’s prescribed fractional seats.

2 Formulation

A problem of reservation in three dimensions in period $t \in \mathbb{N}$ is a quadruple $\Lambda^t = (\mathcal{D}, \mathcal{C}, \alpha, (q^t_i)_{i=1}^m)$. $\mathcal{D}$ and $\mathcal{C}$ are finite sets of departments and categories where $m := |\mathcal{D}| \geq 2$ and $n := |\mathcal{C}| \geq 2$. The reservation scheme is defined by a vector of fractions $\alpha = [\alpha_j]_{j \in \mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0,1)$ is a rational number that represents the fraction of vacancies are to be reserved so that $\sum_{j \in \mathcal{C}} \alpha_j = 1$. $q^t = [q^t_i]_{i \in \mathcal{D}}$ represents
the **vector of vacancies** associated with the departments in period $s \in \{1, 2, \ldots, t\}$. Let $Q_t^i := \sum_{s \leq t} q_s^i$ denote **period-t (cumulative) sum of vacancies in department $i$**.

A **period-t (cumulative) fractional assignment table** for problem $\Lambda^t$ is a two-way table

$$X^t = \begin{pmatrix}
(x^t_{i,j})_{m \times n} & (x^t_{i,n+1})_{m \times 1} \\
(x^t_{m+1,j})_{1 \times n} & (x^t_{m+1,n+1})_{1 \times 1}
\end{pmatrix}$$

with rows indexed by $i \in D \cup \{m + 1\}$ and columns by $j \in C \cup \{n + 1\}$, such that internal entries $x^t_{i,j} = \alpha_j Q^t_i$ for any $i \in D$ and $j \in C$, row total entries $x^t_{i,n+1} = Q^t_i$ for any $i \in D$, column total entries $x^t_{m+1,j} = \alpha_j \sum_{i \in D} Q^t_i$ for any $j \in C$, and grand total entry $x^t_{m+1,n+1} = \sum_{i \in D} Q^t_i$. Fractional assignments specify the fraction of seats a category is entitled to receive as per the reservation scheme until period $t$. The internal entry $x^t_{i,j}$ represents the **period-t (cumulative) fractional assignment for category $j$ in department $i$**. The **period-t (cumulative) fractional assignment for a category $j$ in the university** is denoted by column total entry $x^t_{m+1,j}$. The row total entries $x^t_{i,n+1}$ represents the period-t (cumulative) sum of vacancies in department $i$. The grand total entry $x^t_{m+1,n+1}$ represents the (cumulative) sum of vacancies in the university. We denote the set of integral assignment tables by $X$.

For instance, consider a problem $\Lambda^2 = (\{d_1, d_2\}, \{c_1, c_2\}, \alpha = [0.1, 0.9], (q^1, q^2) = ([9, 8], [17, 7]))$. **Figure 1** illustrates its period-1 and period-2 fractional assignment tables. There are two departments $D = \{d_1, d_2\}$, corresponding to rows in the tables, and two categories $C = \{c_1, c_2\}$, corresponding to columns. The reservation scheme reserves 10% vacancies in the university for members of category $c_1$. In period-1, department $d_1$ has 9, and department $d_2$ has 8 vacancies, represented by column 3 of $X^1$. In period-2, department $d_1$ has 17 and department $d_2$ has 7 vacancies. Therefore, period-2 cumulative sums of vacancies in departments $d_1$ and $d_2$ are 26 and 15, represented by column 3 of $X^2$. The first column of table $X^1$ ($X^2$) represents the period-1 (period-2) fractional assignments associated with the category $c_1$, and the second column represents the period-1 (period-2) fractional assignments associated with category $c_2$. The first row of $X^1$ ($X^2$) represents the period-1 (period-2) fractional assignments associated with the department $d_1$, and the second row represents the period-1 (period-2) fractional assignments associated with department $d_2$.

**Figure 1: FRACTIONAL ASSIGNMENT TABLES**

|       | $X^1$ |       | $X^2$ |
|-------|-------|-------|-------|
|       | 0.9   | 8.1   | 9     |
|       | 0.8   | 7.2   | 8     |
|       | 1.7   | 15.3  | 17    |
|       | 2.6   | 23.4  | 26    |
|       | 1.5   | 13.5  | 15    |
|       | 4.1   | 36.9  | 41    |

(a) **PERIOD-1 FRACTIONAL ASSIGNMENT**  (b) **PERIOD-2 FRACTIONAL ASSIGNMENT**
A period-t sequence of (cumulative) fractional assignment tables for the problem \( \Lambda_t \) is a sequence of two-way tables \( Y^t = (X^1, \ldots, X^t) \), where table \( X^s \) is the period-s fractional assignment table for any \( s \in \{1, 2, \ldots, t\} \). By \( \mathcal{Y}^t \), we denote the set of all period-t sequences of fractional assignment tables. Given a sequence of tables \( Y^t \), if \( Y^t = (Y^{t-1}, X^t) \) for some \( X^t \in \mathcal{X} \), then we say that \( Y^t \) follows \( Y^{t-1} \).

A two-way table is additive if entries add along the rows and columns to all corresponding totals. A period-t (cumulative) integral assignment table for the problem \( \Lambda_t \) is a \((m+1) \times (n+1)\) non-negative integer two-way table \( \bar{X}^t = (\bar{x}^t_{i,j}) \), with rows indexed by \( i \in D \cup \{m+1\} \) and columns by \( j \in C \cup \{n+1\} \), such that \( \bar{X}^t \) is additive and \( \bar{x}^t_{i,n+1} = x^t_{i,n+1} \) for any \( i \in D \). The internal entry \( \bar{x}^t_{i,j} \) represents the period-t (cumulative) integral assignment of category \( j \) in department \( i \). The period-t (cumulative) integral assignment of category \( j \) in the university is denoted by column total entry \( \bar{x}^t_{m+1,j} \). We denote the set of integral assignment tables by \( \bar{\mathcal{X}} \).

### 2.1 Deterministic solutions and properties

A deterministic solution \( R : \cup_{s \in \mathbb{N}} \mathcal{Y}^s \to \bar{\mathcal{X}} \) maps each sequence of fractional assignment tables to an integral assignment table, that is, for any \( Y^t \in \cup_{s \in \mathbb{N}} \mathcal{Y}^s \), \( R(Y^t) \) is a period-t integral assignment table. We will use \( R(y^t_{i,j}) \) to denote the entry \( \bar{x}^t_{i,j} \) of \( R(Y^t) \).

Bias of a deterministic solution \( R \) at \( Y^t \) is a two-way table \( \text{bias}(R(Y^t)) \), with each entry \( \text{bias}(R(y^t_{i,j})) := R(y^t_{i,j}) - \bar{x}^t_{i,j} \). The bias of a solution is the difference between the solution and the fractional assignment table.

- A deterministic solution \( R \) is monotonic if \( R(Y^t) \geq R(Y^{t-1}) \) for any \( Y^t \in \cup_{s \in \mathbb{N}} \mathcal{Y}^s \) that follow \( Y^{t-1} \).

The property incorporates the idea that reservations are irreversible. Going from period \( t \) to period \( t + 1 \), the cumulative integral assignments of each category can only weakly increase. This property is unquestionably essential and must be satisfied by any proposed method.

- A deterministic solution \( R \) stays within department quota if \( |\text{bias}(R(y^t_{i,j}))| < 1 \) for each internal entry \((1 \leq i \leq m, 1 \leq j \leq n)\) for any \( Y^t \in \cup_{s \in \mathbb{N}} \mathcal{Y}^s \).

The property formulates the idea that a deterministic solution should not deviate from its cumulative fractional assignment by more than one seat in each department.

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15The relation “is greater than or equal to”, denoted “\( \geq \)”, compares tables entry-wise; that is, \( X \geq X' \) if, for any \((1 \leq i \leq m+1, 1 \leq j \leq n+1)\), \( x_{i,j} \geq x'_{i,j} \).
Therefore, each category’s fractional assignments are rounded to an adjacent integer.\textsuperscript{16} This is essentially the ideal integer solution that can be achieved.

- A deterministic solution \( R \) \textbf{stays within university quota} if \( |\text{bias}(R(y_{m+1,j}^t))| < 1 \) for each column total entry \( (1 \leq j \leq n) \) for any \( Y^t \subseteq \bigcup_{s \in \mathbb{N}} \mathcal{Y}_s \).

The property formulates the same idea as the previous one but at the university level.

- A deterministic solution \( R \) has a \textbf{finite bias} if there exists a constant \( b > 0 \) such that \( |\text{bias}(R(y_{i,j}^t))| < b \) for each entry \( (1 \leq i \leq m + 1, \ 1 \leq j \leq n + 1) \) for any \( Y^t \subseteq \bigcup_{s \in \mathbb{N}} \mathcal{Y}_s \).

The property relaxes the previous two combined by requiring finite deviations from the fractional assignments.

Reconsider the problem depicted in Figure 1 for an instance of these properties. Figure 2 illustrates two possible monotonic deterministic solutions to the problem. Solution \( R_1 \) stays within both department and university quota. While solution \( R_2 \) stays within department quota only.

\begin{figure}[h]
\begin{center}
\begin{tabular}{c|c|c|c|c}
  & 0.9 & 8.1 & 9  \\
\hline
\( X^1 \) & 0.8 & 7.2 & 8 \\
\hline
& 1.7 & 15.3 & 17 \\
\hline
\end{tabular}
\hspace{1cm}
\begin{tabular}{c|c|c|c|c}
  & 2.6 & 23.4 & 26  \\
\hline
\( X^2 \) & 1.5 & 13.5 & 15 \\
\hline
& 4.1 & 36.9 & 41 \\
\hline
\end{tabular}
\end{center}
\caption{Two Deterministic Solutions}
\end{figure}

\begin{align*}
R_1(Y^1) &= \begin{pmatrix} 1 & 8 & 9 \\ 2 & 15 & 17 \end{pmatrix} & R_1(Y^2) &= \begin{pmatrix} 3 & 23 & 26 \\ 4 & 37 & 41 \end{pmatrix} \\
R_2(Y^1) &= \begin{pmatrix} 0 & 9 & 9 \\ 0 & 17 & 17 \end{pmatrix} & R_2(Y^2) &= \begin{pmatrix} 3 & 23 & 26 \\ 4 & 37 & 41 \end{pmatrix}
\end{align*}

\subsection{2.2 Randomized solutions and properties}

We are looking for a random process whose outcome pins down a deterministic solution that maps each sequence of fractional assignment tables to an integral assignment table.

\textsuperscript{16}For any \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) is the largest integer no larger than \( x \), the floor of \( x \). And \( \lceil x \rceil \) is the smallest integer no smaller than \( x \), the ceiling of \( x \).
A randomized solution is a probability distribution \( \phi \) over a finite set of deterministic solutions, where \( \phi(R) \) denotes the probability of deterministic solution \( R \). Moreover, for any \( Y^t \in \bigcup_{s \in \mathbb{N}} \mathcal{Y}^s \), \( Z^t := \phi(R(Y^t)) \) is a random variable (a table) specifying the integer assignment table, with entries denoted \( z^t_{i,j} \). We can consider a randomized solution \( \phi \) as a procedure initialized with a randomly chosen seed that takes \( Y^t \) as input and returns an integral assignment.\(^{17}\)

The following properties limit the random behavior of \( \phi \) and the consistency of deterministic solutions in its support across inputs \( Y^t \).

- A randomized solution \( \phi \) is **monotonic**, **stays within department quota**, and **stays within university quota** if all deterministic solutions in its support satisfy the respective properties.

- A randomized solution \( \phi \) is **ex-ante proportional** if, for any \( Y^t \in \bigcup_{s \in \mathbb{N}} \mathcal{Y}^s \),
  \[
  \mathbb{E}[\phi(R(Y^t))] = \mathbb{E}[Z^t] = X^t.
  \]

- A randomized solution \( \phi \) **almost stays within university quota with high probability** if, for any \( Y^t \in \bigcup_{s \in \mathbb{N}} \mathcal{Y}^s \), for any category \( j \in \mathcal{C} \) and for any \( b > 0 \) we have
  \[
  \Pr(z^t_{m+1,j} - x^t_{m+1,j} \geq b) < e^{-\frac{b^2}{2m}},
  \]
  \[
  \Pr(z^t_{m+1,j} - x^t_{m+1,j} \leq -b) < e^{-\frac{b^2}{2m}}.
  \]

Notice that the deviation of the outcome of randomized solution \( \phi \) for a category \( j \in \mathcal{C} \) in the university is \( z^t_{m+1,j} - x^t_{m+1,j} \). This random variable measures the deviation of the seat allocation at the university level from its fractional assignment. By this property, the probability of a randomized solution deviating from the university quota by a value greater than \( b \) decays exponentially with \( b^2 \).

### 3 Solutions from India and their shortcomings

Two solutions are seen in practice in India: the Government’s and the Court’s solutions. Both solutions are monotonic because they use a tool called roster to determine the number of vacancies to be reserved. Formally, a **roster** \( \sigma : \{1, 2, \ldots \} \to \mathcal{C} \) is an ordered list over

\(^{17}\)This is akin to the definition of randomized apportionment methods, where the randomness of the methods is determined by an implicit random seed (Götz et al. (2022)).
the set of categories $C$. A roster assigns each position to a category so that the number of vacancies to be reserved is clearly laid out for any number of total vacancies.

Note the following two principles that are followed for maintenance of rosters:\(^{18}\)

(f) The register / roster register shall be maintained in the form of a running account year after year. For example if recruitment in a year stops at point 6, recruitment in the following year shall begin from point 7.

(h) In case of cadres where reservation is given by rotation, fresh cycle of roster shall be started after completion of all the points in the roster.

Therefore, even though the publicly declared roster details the assignment of $k$ vacancies to various categories. The roster does not only decides the allocation of seats 1, 2, $\ldots$, $k$, but also the allocation of seats $k + 1, \ldots, 2k$, the allocation of seats $2k + 1, \ldots, 3k$, and so forth. A roster pins down an allocation of an infinite sequence of seats constructed as a concatenation of infinitely many finite seat sequences of length $k$. For instance, the number of reserved vacancies of the $q^1$ in period-1 is determined by $\sigma(1), \ldots, \sigma(q^1)$. The number of vacancies reserved of the $q^2$ in period-2 is determined by $\sigma(q^1 + 1), \ldots, \sigma(q^1 + q^2)$. So on and so forth.

Maintaining rosters is central to the implementation of reservations in India.\(^{19}\) It makes uniform and transparent implementation of the reservation policy across various government departments possible. Since only a few seats might arise every period, the objective of maintaining a roster is to ensure that each category gets its affirmative action policy prescribed percentage of seats over time. However, maintaining rosters for educational institutions raises additional complications. Does each department in a university maintain its roster? Or does the university as a whole maintain a roster? These questions gave rise to two solutions in India.

Before illustrating the solutions, we introduce an example that makes the solutions easier to comprehend. The example will also be sufficient to demonstrate the various shortcomings of the two solutions.\(^{20}\)

**Example 1.** Consider a problem $\Lambda^3 = (\{d_1, d_2, d_3, d_4\}, \{c_1, c_2\}, \alpha = [1/3, 2/3], (q^1, q^2, q^3) = ([2, 1, 2, 1], [2, 1, 2, 1], [2, 1, 2, 1]))$. Figure 3 illustrates its period-1, period-2, and period-3 frac-

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\(^{18}\)See page 1 of [https://dopt.gov.in/sites/default/files/Ch-05_2014.pdf](https://dopt.gov.in/sites/default/files/Ch-05_2014.pdf), last accessed on 22 February 2024.

\(^{19}\)See Figure 15 and Figure 16 for the rosters prescribed by Government of India.

\(^{20}\)An example with two categories and two departments is also sufficient to demonstrate the shortcomings. Example 1 is constructed to illustrate both solutions’ shortcomings and demonstrate the differences between the Court’s and the Government’s solutions.
tional assignment tables. The reservation scheme reserves 1/3 of the vacancies in the university for members of category \(c_1\). Each period, department \(d_1, d_2, d_3,\) and \(d_4\) have 2, 1, 2, and 1 vacancies, respectively. Therefore, period-2 cumulative sums of vacancies in departments are 4, 2, 4, and 2, respectively. And, period-3 cumulative sums of vacancies in departments are 6, 3, 6, and 3, respectively. The roster is

\[ \sigma(k) = \begin{cases} 
  c_1, & \text{if } k \text{ is a multiple of } 3 \\
  c_2, & \text{otherwise}
\end{cases} \]

Figure 3: FRACTIONAL ASSIGNMENT TABLES

\[
\begin{array}{c|c|c|c}
\hline
& X^1 & X^2 & X^3 \\
\hline
2/3 & 4/3 & 4/3 & 4/3 \\
1/3 & 2/3 & 2/3 & 2/3 \\
2/3 & 4/3 & 4/3 & 4/3 \\
1/3 & 2/3 & 4/3 & 4/3 \\
2 & 4 & 6 & 12 \\
\hline
\end{array}
\]

We will see that the choice of the roster in Example 1 is not the source of the shortcomings of the Government’s and Court’s solutions. The source of the problem is that they do not account for the interdependence of the departments and the university in calculating reserved seats.

3.1 Government’s solution

The Government’s solution treats the university as the unit. That is, vacancies across all departments are pooled together, and the roster is maintained at the university level.

For the problem in Example 1, in period-1, department \(d_1\) has two vacancies: The number of vacancies reserved for department \(d_1\) is determined by the 1st and 2nd vacancies in the roster (that is, \(\sigma(1) = c_2, \sigma(2) = c_2\)). Department \(d_2\) has one position: The number of vacancies reserved for department \(d_2\) is determined by the 3rd position in the roster (that is, \(\sigma(3) = c_1\)).\(^{21}\) Department \(d_3\) has two vacancies: The number of vacancies reserved for department \(d_3\) is determined by the 4th and 5th vacancies in the roster (that is, \(\sigma(4) = c_2, \sigma(5) = c_2\)). Department \(d_4\) has one position: The number of vacancies reserved for department \(d_4\) is determined by the 6th position in the roster (that is, \(\sigma(6) = c_1\)). The period-1 reservation table is illustrated by \(R_{G}(Y^1)\) in Figure 4.

\(^{21}\)When pooling vacancies across departments, a fixed order over departments is required to apply to the roster. In India, the alphabetic order over departments is used.
In period-2, department $d_1$ has two vacancies: The number of vacancies reserved for department $d_1$ is determined by the 7th and 8th vacancies in the roster (that is, $\sigma(7) = c_2$, $\sigma(8) = c_2$). Department $d_2$ has one position: The number of vacancies reserved for department $d_2$ is determined by the 9th vacancies in the roster (that is, $\sigma(9) = c_1$). Department $d_3$ has two vacancies: The number of vacancies reserved for department $d_3$ is determined by the 10th and 11th vacancies in the roster (that is, $\sigma(10) = c_2$, $\sigma(11) = c_2$). Department $d_4$ has one position: The number of vacancies reserved for department $d_4$ is determined by the 12th position in the roster (that is, $\sigma(12) = c_1$). The period-2 reservation table is illustrated by $R_G(Y^2)$ in Figure 4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_G(Y^3)$ in Figure 4.

![Figure 4: COURT'S AND GOVERNMENT'S SOLUTION](image)

Period-3 reservation for category $c_1$ in department $d_1$ and department $d_3$ is 0, however, the fractional assignment is 2 vacancies. Moreover, period-3 reservation for category $c_1$ in department $d_2$ and department $d_4$ is 3. However, the fractional assignment is 1 position. Therefore, the Government’s solution $R_G$ does not stay within department quota. Moreover, in Example 1, if the departments had the same number of vacancies for the next periods, de-
partment $d_1$ and department $d_3$ would not reserve any seats for category $c_1$, and department $d_2$ and department $d_4$ would not reserve any seats for category $c_2$.

The primary shortcoming of the Government’s solution $R_G$, as revealed by Example 1, is that $R_G$ allows for large deviations in seat allocations from fractional assignments at the department level. Essentially, Example 1 shows that treating the university as the unit can lead to outcomes that fail to follow the reservation policy at the department level.\footnote{In fact, in Theorem 1, we show that for any solution that stays within university quota, the deviations in seat allocations from fractional assignments at the department level can not be limited by a fixed number.}

### 3.2 Court’s solution

The Court’s solution treats department as the unit. That is, vacancies are not pooled across departments. Instead, each department independently maintains the roster.

For the problem in Example 1, in period-1, department $d_1$ has two vacancies: The number of vacancies reserved for department $d_1$ is determined by the 1st and 2nd vacancies in its roster (that is, $\sigma(1) = c_2$, $\sigma(2) = c_2$). Department $d_2$ has one position: The number of vacancies reserved for department $d_2$ is determined by the 1st position in its roster (that is, $\sigma(1) = c_2$). Department $d_3$ has two vacancies: The number of vacancies reserved for department $d_3$ is determined by the 1st and 2nd vacancies in its roster (that is, $\sigma(1) = c_2$, $\sigma(2) = c_2$). Department $d_4$ has one position: The number of vacancies reserved for department $d_4$ is determined by the 1st position in its roster (that is, $\sigma(1) = c_2$). The period-1 reservation table is illustrated by $R_C(Y^1)$ in Figure 4.

In period-2, department $d_1$ has two vacancies: The number of vacancies reserved for department $d_1$ is determined by the 3rd and 4th vacancies in its roster (that is, $\sigma(3) = c_1$, $\sigma(4) = c_2$). Department $d_2$ has one position: The number of vacancies reserved for department $d_2$ is determined by the 2nd vacancies in its roster (that is, $\sigma(2) = c_2$). Department $d_3$ has two vacancies: The number of vacancies reserved for department $d_3$ is determined by the 3rd and 4th vacancies in its roster (that is, $\sigma(3) = c_1$, $\sigma(4) = c_2$). Department $d_4$ has one position: The number of vacancies reserved for department $d_4$ is determined by the 2nd position in its roster (that is, $\sigma(2) = c_2$). The period-2 reservation table is illustrated by $R_C(Y^2)$ in Figure 4. We apply this solution for the next period. The period-3 reservation table is illustrated by $R_C(Y^3)$ in Figure 4.

Period-1 reservation for category $c_1$ in the university is 0. However, the fractional assignment is 2 vacancies. Moreover, period-2 reservation for category $c_1$ in the university is 2. However, the fractional assignment is 4. Therefore, the Court’s solution $R_C$ does not stay within university quota. Moreover, in Example 1, if there were 4 more departments $d_5$, $d_6$, $d_7$, and $d_8$, with the same number of vacancies as department $d_1$, $d_2$, $d_3$, and $d_4$, respectively,
period-1 reservation for category \( c_1 \) in the university would still be 0. And, period-2 reservation for category \( c_1 \) in the university would be 4, whereas the fractional assignment would be 8.

The primary shortcoming of the Court’s solution \( R_C \), as revealed by Example 1, is that \( R_C \) allows for large deviations in seat allocations from fractional assignments at the university level. Essentially, Example 1 shows that treating department as the unit can lead to outcomes that fail to follow the reservation policy at the university level.

4 Results

4.1 Impossibility of staying within department and university quota

Our first result shows that a monotonic solution cannot stay both within department quota and within university quota. It is further impossible for a solution to stay within university quota and allow for only finite deviations in seat allocations from the fractional assignments at the department level.

**Theorem 1.** There does not exist a deterministic solution \( R \) that is monotonic, stays within university quota, and has a finite bias.

**Corollary 1.** There does not exist a deterministic solution \( R \) that is monotonic, stays within department quota, and stays within university quota.

Let’s construct a counterexample to prove Theorem 1. A problem where the solution \( R \) can not have a finite bias. That is, for any constant \( b > 0 \), there exist a \( Y^t \) and an internal entry \( y_{t,i,j}^k \) such that \( |\text{bias}(R(y_{t,i,j}^k))| > b \).

**Example 2.** Consider a problem with three departments \( d_1, d_2, \) and \( d_3 \), two categories \( c_1, c_2 \), and reservation scheme vector \( \alpha = [0.5, 0.5] \). The departments \( d_1, d_2, \) and \( d_3 \) have \( q^1 = [0, 0, 1] \) vacancies in period-1 and \( q^2 = [1, 0, 0] \) vacancies in period-2.

Notice that staying within university quota is equivalent to reserving exactly \( k \) vacancies for \( c_1 \) and \( c_2 \) in every \( 2k \) cumulative sum of vacancies in the university, where \( k \in \mathbb{N} \). In period-1, department \( d_3 \) can reserve vacancies for either categories. Without loss of generality, we assume that it reserves 1 position for \( c_1 \). In period-2, since there are 2 cumulative vacancies in the university, there should be exactly 1 position reserved for \( c_1 \). Department \( d_1 \) should reserve 1 position for category \( c_2 \). The period-1 and period-2 integral assignment tables are shown in Figure 5.

If departments have \( q^3 = [0, 0, 1] \) vacancies in period-3, department \( d_3 \) can reserve its position to either category. These two cases are shown in Figure 6.
Case 1: Suppose that the solution is $R_1$. If the departments have $\mathbf{q}^4 = [1, 0, 0]$ vacancies in period-4, department $d_1$ should reserve 1 position for category $c_2$. Otherwise, the solution $R$ would violate staying within university quota property. Period-4 fractional assignment table and the period-4 reservation table are illustrated by $X^4$ and $R_1(X^4)$ in Figure 7.

Case 2: Suppose that the solution is $R_2$. If the departments have $\mathbf{q}^4 = [0, 1, 0]$ vacancies in period-4, department $d_2$ should reserve 1 position for category $c_1$. Otherwise, the solution $R$ would violate staying within university quota property. Period-4 fractional assignment table and the period-4 reservation table are illustrated by $X^4$ and $R_2(X^4)$ in Figure 7.

If departments have $\mathbf{q}^5 = [0, 0, 1]$ vacancies in period-3, department $d_3$ can reserve its position to either category. These two cases are shown in Figure 8.

For each case until now, period-5 reservation for category $c_1$ in department $d_1$ is 0, and period-5 reservation for category $c_2$ in department $d_2$ is 0. We can extend these examples analogously for more periods. The idea is the following. In each period, the university has only one position. Department $d_3$ always has one position in odd periods, and in the following period, either department $d_1$ or department $d_2$ has one position according to these following cases.

Case 3: If department $d_3$ reserves 1 position to category $c_1$, department $d_1$ has one position in the next period. Then, department $d_1$ should reserve 1 position for category $c_2$.  

---

**Case 1:** Suppose that the solution is $R_1$. If the departments have $\mathbf{q}^4 = [1, 0, 0]$ vacancies in period-4, department $d_1$ should reserve 1 position for category $c_2$. Otherwise, the solution $R$ would violate staying within university quota property. Period-4 fractional assignment table and the period-4 reservation table are illustrated by $X^4$ and $R_1(X^4)$ in Figure 7.

**Case 2:** Suppose that the solution is $R_2$. If the departments have $\mathbf{q}^4 = [0, 1, 0]$ vacancies in period-4, department $d_2$ should reserve 1 position for category $c_1$. Otherwise, the solution $R$ would violate staying within university quota property. Period-4 fractional assignment table and the period-4 reservation table are illustrated by $X^4$ and $R_2(X^4)$ in Figure 7.

If departments have $\mathbf{q}^5 = [0, 0, 1]$ vacancies in period-3, department $d_3$ can reserve its position to either category. These two cases are shown in Figure 8.

For each case until now, period-5 reservation for category $c_1$ in department $d_1$ is 0, and period-5 reservation for category $c_2$ in department $d_2$ is 0. We can extend these examples analogously for more periods. The idea is the following. In each period, the university has only one position. Department $d_3$ always has one position in odd periods, and in the following period, either department $d_1$ or department $d_2$ has one position according to these following cases.

**Case 3:** If department $d_3$ reserves 1 position to category $c_1$, department $d_1$ has one position in the next period. Then, department $d_1$ should reserve 1 position for category $c_2$.  

---
Figure 7: TWO CASES FOR PERIOD-4 ASSIGNMENT TABLES

|   |   |   |   |
|---|---|---|---|
| 1 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 2 |
| 2 | 2 | 4 | 4 |

(a) CASE 1: PERIOD-4 FRACTIONAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 0 | 0 | 0 | 0 |
| 2 | 0 | 2 | 2 |
| 2 | 2 | 4 | 4 |

(b) CASE 1: PERIOD-4 INTEGRAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0.5 | 0.5 | 1 | 1 |
| 0.5 | 0.5 | 1 | 1 |
| 1 | 1 | 2 | 2 |
| 2 | 2 | 4 | 4 |

(c) CASE 2: PERIOD-4 FRACTIONAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 2 | 2 |
| 2 | 2 | 4 | 4 |

(d) CASE 2: PERIOD-4 INTEGRAL ASSIGNMENT

Figure 8: TWO CASES FOR PERIOD-5 ASSIGNMENT TABLES

|   |   |   |   |
|---|---|---|---|
| 1 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 |
| 1.5 | 1.5 | 3 | 3 |
| 2.5 | 2.5 | 5 | 5 |

(a) PERIOD-5 FRACTIONAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 0 | 0 | 0 | 0 |
| 3 | 0 | 3 | 3 |
| 3 | 2 | 5 | 5 |

(b) PERIOD-5 INTEGRAL ASSIGNMENT

|   |   |   |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 2 | 3 |
| 2 | 3 | 5 |

(c) PERIOD-5 INTEGRAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0.5 | 0.5 | 1 | 1 |
| 0.5 | 0.5 | 1 | 1 |
| 1.5 | 1.5 | 3 | 3 |
| 2.5 | 2.5 | 5 | 5 |

(d) PERIOD-5 FRACTIONAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 2 | 3 | 3 |
| 2 | 3 | 5 | 5 |

(e) PERIOD-5 INTEGRAL ASSIGNMENT

|   |   |   |   |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 0 | 0 | 0 | 0 |
| 2 | 1 | 3 | 3 |
| 2 | 3 | 5 | 5 |

(f) PERIOD-5 INTEGRAL ASSIGNMENT

Otherwise, the solution would violate the staying university quota property.

**Case 4:** If department $d_3$ reserves 1 position to category $c_2$, department $d_2$ has one position in the next period. then, department $d_2$ should reserve 1 position for category $c_1$, otherwise, solution would violate staying university quota property.

Example 2 shows that if a solution stays within university quota, departments can grow in size without giving a seat to one category; that is, the solution does not have a finite bias.$^{23}$

**Remark 1.** On the other hand, a solution that stays within department quota would still have finite bias given that there are only a finite number of deviations from fractional assignments.

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$^{23}$An example for any number of categories and departments can be constructed similarly. Example 2 is constructed so that it not only illustrates the failure but also demonstrates any solution can fail to have a finite bias for all categories.
at the department level that sum up to the university level. Our next example shows that there is no clever way to eliminate deviations from fractional assignments at the university level due to the monotonicity constraint.\footnote{In fact, under any solution that stays within department quota, the worst deviation from fractional assignments at the university level for any category $j$ is $m \cdot (1 - (1/q))$, where $q$ is such that $\alpha_j = r/q$ is an irreducible fraction. For example, consider a university with 100 departments and 3 vacancies in each department. A solution that stays within departments quota may give 2 seats in each department to a category with $\alpha_j = r/q = 7/20 = 0.35$. This adds up to 200 seats for category $j$ at the university level. However, at the university level the fractional assignment of category $j$ is just $300 \times 0.35 = 105$. That is, there is a deviation from the fractional assignment of $100 \times (1 - (1/20)) = 95$.}

Example 3. Consider a problem with hundred departments $d_1, \ldots, d_{100}$, two categories $c_1, c_2$, and reservation scheme vector $\alpha = [0.5, 0.5]$. The departments have one vacancy each in period-1.

Notice that staying within department (or university) quota is equivalent to reserving exactly $k$ vacancies for $c_1$ and $c_2$ in every $2k$ cumulative sum of vacancies in the department (or university), where $k \in \mathbb{N}$.

In period-1, to eliminate deviations from fractional assignments at the university level, 50 departments must reserve vacancies for $c_1$, and the other 50 for $c_2$.

In period-2, if only those departments that reserved vacancy for $c_1$ in period-1 have an additional vacancy each. Then, to stay within department quota, they must all reserve the additional vacancy for $c_2$. However, reserving the 50 additional vacancies just for $c_2$ leads to a deviation from fractional assignments of 25 at the university level.

4.2 Staying within department or university quota

In the presence of only staying within department (or university) quota constraints, we will show that a solution to the problem exists. This is due to a manageable bihierarchical constraint structure that can yield cumulative integral assignments for any sequence of cumulative fractional assignments. This leads to our first positive result for the problem.

Theorem 2. There exists a randomized solution $\phi$ that is ex ante proportional, monotonic, and stays within department quota.

Since the sequence of cumulative fractional assignments is unknown, the solution must work for any possible sequence. The essential technique in the upcoming proof is to design a procedure that takes the fractions of reservations and generates a roster for a single department. Recall from Section 3 that a roster $\sigma : \{1, 2, \ldots \} \rightarrow C$ is an ordered list over the set of categories $C$. Thus for any number of vacancies in the department, the roster decides the seat(s) allocated (integral assignment) to the categories. Staying within department quota
Proof of Theorem 2. In this proof, we will construct a set of rosters for departments that ensure the integral assignment stays within department quota for any sequence of fractional assignments.

**Part 1: Constructing a roster for departments.**

Let \( k \) be the lowest common denominator of the reservation fractions in \( \alpha \). Since \( \alpha_j \)'s are rational numbers, such \( k \) exists. Let \( P \) represent the given reservation scheme as a \( k \times n \) two-way table, where the rows denote the index of the seats and the columns denote the categories. The internal entry \( p_{ij} \) equals to \( \alpha_j \) for every \((i, j)\).

We want to transform this table of fractional entries into an integral table. The integral table would assign each of the \( k \) vacancies (in rows) to a category (in columns). That is, the integral table would generate a unique roster.

We introduce some new definitions to define the constraints that monotonicity and staying within department quota impose on the table \( P \). An **element** \((i, j)\) denotes an entry in the row \( i \) and column \( j \) of the \( k \times n \) two-way table \( P \). Let \( E \) be the set of all elements in \( P \). A **constraint** is a subset \( K \subseteq E \) of elements \((i, j)\) with a floor [\( \sum_{(i,j) \in K} p_{ij} \)] and a ceiling \( \lceil \sum_{(i,j) \in K} p_{ij} \rceil \). A **constraint set** is a set of constraints. A **constraint structure** is a set of subsets of \( E \).

We are looking for integral tables \( P' \) that satisfy the following three types of constraints. Appendix A.1 gives an example to help understand the constraint structure better.

1. Internal constraints ensure that each internal entry can be 1 or 0.
   That is, \( 0 \leq p'_{ij} \leq 1 \) for every \((i, j)\).
   Let \( \mathcal{K}_I \) be the constraint structure associated with internal constraints.

2. Row sums are required to be one since every position is assigned to exactly one category.
   That is, \( \sum_{j \in \mathcal{C}} p'_{ij} = 1 \) for every \( i \).
   Let \( \mathcal{K}_R \) be the constraint structure associated with row constraints.

3. Column constraints ensure that the difference between the cumulative sum of vacancies given to a category and cumulative fractional assignments is less than one.
   That is, \( \lfloor \sum_{i \leq l} p_{ij} \rfloor \leq \sum_{i \leq l} p'_{ij} \leq \lceil \sum_{i \leq l} p_{ij} \rceil \) for every \( 2 \leq l \leq m \) and \( j \in \mathcal{C} \).
   Let \( \mathcal{K}_C \) be the constraint structure associated with column constraints.

---

\textsuperscript{25}We are using new notation because this two-way table \( P \) must not be confused with the fractional assignment table \( X' \). Moreover, we use \( p_{ij} \) instead of \( p_{i,j} \) because the focus is only the internal entries of the table.
Therefore, we will consider tables $P'$ that satisfy, for each $K \in K_C \cup K_I \cup K_R$,

$$
\lfloor \sum_{(i,j) \in K} p_{ij} \rfloor \leq \sum_{(i,j) \in K} p'_{ij} \leq \lceil \sum_{(i,j) \in K} p_{ij} \rceil.
$$

Notice that each of the constraint structures associated with the constraint sets $K_C \cup K_I$ and $K_R \cup K_I$ is a hierarchy. Recall from Budish et al. (2013), a constraint structure $K$ is a hierarchy if for every pair $E$ and $E'$ in $K$, we have that $E \subset E'$ or $E' \subset E$ or $E \cap E' = \emptyset$. Therefore, due to Edmonds (2003) and Budish et al. (2013), an ex ante proportional randomized solution exists that randomizes over integral assignment tables that satisfy the abovementioned constraints. The technique to construct such a solution is further detailed in Appendix A.2 and Appendix A.3.

In particular, by Theorem 1 of Budish et al. (2013), there exist positive numbers $\lambda_1, \ldots, \lambda_N$, which sum up to one, and integral assignments $\bar{P}_1, \ldots, \bar{P}_N$, which are feasible under constraints in $K_C \cup K_I \cup K_R$, such that

$$
P = \sum_{h=1}^{N} \lambda_h \bar{P}_h.
$$

Generating rosters: Recall that a roster pins down an allocation of an infinite sequence of starts constructed as a concatenation of infinitely many finite seat sequences of length $k$.

Generate a unique roster $\sigma_h$ for each $\bar{P}_h$ such that $\overline{p}_{ij} = 1$ implies $\sigma_h(qk + i) = c_j$ for every non-negative integer $q$.

**Part 2: Assigning rosters to departments independently.**

Independently assigning each department a roster $\sigma_h$ with probability $\lambda_h$ generates the required randomized solution $\phi^*$. Each department then reserves vacancies in every period according to the assigned roster. For example, if roster $\sigma$ is realized for department $d$ then the number of vacancies reserved in department $d$ in period-1 is determined by $\sigma(1), \ldots, \sigma(q_1d)$. The number of vacancies reserved in department $d$ in period 2 is determined by $\sigma(q_1d + 1), \ldots, \sigma(q_1d + q_2d)$. So on and so forth.

By construction, the randomized solution $\phi^*$ stays within department quota; that is, ex-post fractional assignments (for departments) are rounded up or down. The solution is monotonic because each department follows a roster that can only weakly increase each category’s cumulative integer assignment. These assignments sum up at the university level, making the solution monotonic even at that level. Moreover, $\phi^*$ is ex ante proportional. This is because the expected number of vacancies reserved to category $j$ in department $i$ until period-$t$ is $\mathbb{E}(z_{i,j}^t) = \sum_{s \leq t} q_s^j \alpha_j$. The internal entry $x_{i,j}^t$ of fractional assignment table $X^t$ also equals to $\sum_{s \leq t} q_s^j \alpha_j$. Thus, $\mathbb{E}[Z^t] = X^t$. That is, the randomized solution $\phi^*$ is ex
ante proportional. This proves the theorem.

Remark 2. Replacing part 2 in the proof by assigning roster $\sigma_h$ with probability $\lambda_h$ to the university (instead of departments) generates a randomized solution $\phi'$ that is ex ante proportional, monotonic, and stays within university quota.

4.3 Approximately staying within university quota

By Theorem 1, monotonicity, staying within department and university quota as a hard constraint is not achievable. By Theorem 2, a part of the constraint, monotonicity and staying within department quota, is achievable. In this subsection, we will treat the other part, staying within university quota, as a soft constraint, a goal rather than a hard constraint.

Staying within university quota requires that the sum of the fractional assignments across departments must be rounded up or down for any sequence of cumulative fractional assignments. When the staying within department quota constraint and monotonicity of the solution is combined with the staying within university quota constraint, the constraint structure on the cumulative fractional assignments ceases to be a bihierarchy. Akbarpour and Nikzad (2020) showed that if such constraints (corresponding to staying within university quota property) are treated as soft constraints and are in the deepest level of the bihierarchy, then they are approximately implementable.\(^{26}\) However, there is no apparent way of rewriting the problem such that staying within the university is at the deepest level of the constraint structure induced by monotonicity and the department quota. Therefore, we cannot establish approximate satisfaction of this constraint structure using results of Akbarpour and Nikzad (2020).

However, we will next see that in addition to being monotonic, ex ante proportional, and staying within department quota, the randomized solution $\phi^*$ constructed in the proof of Theorem 2 almost stays within university quota with high probability. Theorem 3 follows from a Chernoff-type probabilistic concentration bounds. It shows that the randomized solution $\phi^*$ is such that it hardly ever round up (or round down) most entries in each column of $X^t$.

Theorem 3. There exists a randomized solution $\phi$ that is ex ante proportional, monotonic, stays within department quota, and almost stays within university quota with high probability.

Proof of Theorem 3. We will prove that randomized solution $\phi^*$ in Theorem 2 almost stays within university quota with high probability. We show this by proving two lemmas. First, we prove that entries of each column of $Z^t$ are independent. Next, we show that the almost

\(^{26}\)See page 10 of Akbarpour and Nikzad (2020) for the definition.
stays within university quota with high probability property follows from the application of Chernoff concentration bounds.

**Lemma 1.** For any subset of $S \subset \{1, 2, \ldots, m\}$ and any $j \in \{1, 2, \ldots, n\}$, we have

\[
Pr\left[ \bigwedge_{i \in S} z_{ij}^t = \lceil x_{ij}^t \rceil \right] = \prod_{i \in S} Pr\left[ z_{ij}^t = \lceil x_{ij}^t \rceil \right],
\]

\[
Pr\left[ \bigwedge_{i \in S} z_{ij}^t = \lfloor x_{ij}^t \rfloor \right] = \prod_{i \in S} Pr\left[ z_{ij}^t = \lfloor x_{ij}^t \rfloor \right].
\]

**Proof.** Notice that the random roster is assigned to each department independently. Consequently, for any pair of departments $i, i'$, random variables $z_{ij}^t$ and $z_{i'j}^t$ are independent, which proves the lemma. ■

We next recall a result of Chernoff et al. (1952), demonstrating that the independence property has the following large deviations result. Chernoff bounds are well-known concentration inequalities that limit the deviation of a weighted sum of Bernoulli random variables from their mean. Here, we use the multiplicative form of the Chernoff concentration bound.

**Lemma 2.** Chernoff bound: Let $A_1, A_2, \ldots, A_m$ be $m$ independent random variables taking values in $\{0, 1\}$. Let $\mu = \sum_{i=1}^m E[A_i]$. Then, for any epsilon with $0 \leq \epsilon \leq 1$ we have

\[
Pr\left[ \sum_{i=1}^m A_i \geq (1 + \epsilon)\mu \right] \leq e^{-\mu \frac{\epsilon^2}{2}},
\]

\[
Pr\left[ \sum_{i=1}^m A_i \leq (1 - \epsilon)\mu \right] \leq e^{-\mu \frac{\epsilon^2}{2}}.
\]

**Lemma 3.** Suppose that the random variables $z_{ij}^t$ with $(1 \leq i \leq m, 1 \leq j \leq n)$ are such that (i) $E[z_{ij}^t] = x_{ij}^t$, and (ii) can take only two values, either $\lceil x_{ij}^t \rceil$ or $\lfloor x_{ij}^t \rfloor$. Then any $j \in \mathcal{C}$, and for any $0 \leq b < m$ we have

\[
Pr\left[ \sum_{i=1}^m (z_{ij}^t - x_{ij}^t) > b \right] \leq e^{-\frac{b^2}{2\gamma}},
\]

\[
Pr\left[ \sum_{i=1}^m (z_{ij}^t - x_{ij}^t) < -b \right] \leq e^{-\frac{b^2}{2\gamma}},
\]

where $\gamma = \sum_{i=1}^m (x_{ij}^t - \lfloor x_{ij}^t \rfloor)$.

**Proof.** The random variables $z_{ij}^t$ with $(1 \leq i \leq m, 1 \leq j \leq n)$ can take two values, either $\lceil x_{ij}^t \rceil$ or $\lfloor x_{ij}^t \rfloor$. Therefore $A_i := z_{ij}^t - \lfloor x_{ij}^t \rfloor$ is a Bernoulli variable. By Lemma 1,
random variables $A_i$ with $(1 \leq i \leq m)$ are independent. Moreover, $\mathbb{E}[A_i] = x_{ij}^t - \lfloor x_{ij}^t \rfloor$ and $z_{ij}^t - x_{ij}^t = A_i - \mathbb{E}[A_i]$.

Replacing value of $A_i$ in Lemma 2 we get,

$$\Pr \left[ \sum_{i=1}^{m} (A_i - \mathbb{E}[A_i]) > \epsilon \gamma \right] \leq e^{-\gamma \epsilon^2 m} ,$$

that can be rewritten as,

$$\Pr \left[ \sum_{i=1}^{m} (z_{ij}^t - x_{ij}^t) > b \right] \leq e^{-\frac{b^2}{2m}} .$$

Similarly, replacing the value of $A_i$ in Lemma 2 we can derive the second part of the statement.

For a solution that stays within department quota, the deviation from fractional assignments at the university level is bounded above, as $|z_{m+1,j}^t - x_{m+1,j}^t| \leq \sum_{i=1}^{m} |z_{ij}^t - x_{ij}^t| \leq \sum_{i=1}^{m} \max \{ x_{ij}^t - \lfloor x_{ij}^t \rfloor, \lceil x_{ij}^t \rceil - x_{ij}^t \} < m$. Therefore, for $b \geq m$,

$$\Pr \left[ |z_{m+1,j}^t - x_{m+1,j}^t| \geq b \right] = 0 .$$

Since $\gamma < m$, Lemma 2 shows that a solution that is ex-ante proportional and stays within department quota also almost stays within university quota with high probability.\footnote{Notice that the theorem has demonstrated a better bound than the one required by our property. This is because for any category $j$, $\gamma \leq \gamma_j$ where $\gamma_j = m \times (1 - (1/q))$ is such that $\alpha_j = p/q$ is an irreducible fraction.}

That is, for $0 \leq b < m$,

$$\Pr (z_{m+1,j}^t - x_{m+1,j}^t \geq b) < e^{-\frac{b^2}{2m}} ,$$

$$\Pr (z_{m+1,j}^t - x_{m+1,j}^t \leq -b) < e^{-\frac{b^2}{2m}} .$$

### 4.4 Simpler solution for a simplified problem

Maintaining a roster for each department is already a demanding task. Our solutions add to the difficulty of this task by requiring each department to maintain an independent roster. Therefore, the question arises whether there is a simpler solution. The answer is yes but to a simpler problem.
Ignoring the multi-period aspect of the reservation problem in three dimensions, one can much simplify the problem. By doing so, the problem gets restricted to a single period problem, and consequently, the monotonicity constraint vanishes. In that case, the impossibility result established in Theorem 1 does not hold.

**Theorem 4.** There exists a randomized solution $\phi$ that is ex ante proportional, stays within department quota, and stays within university quota.

One way to prove this result is by utilizing the network flow approach in decomposing the bihierarchical constraint structure of this problem. However, for practical purposes, we prove this result in Appendix A.4 using Cox (1987)'s controlled rounding procedure that is simple enough to be implemented by hand.

It is useful to remember that treating each period’s problem independently can lead to adverse outcomes over time. In particular, since integral assignments differ from fractional assignments in every period, accumulating these differences can result in sizeable deviations from cumulative fractional assignments over time.

## 5 Conclusion and future work

Implementing affirmative action’s cumulative fractional assignments has been challenging for practitioners in India. This paper demonstrates an impossibility result rooted in this struggle and proposes an alternative solution based on the approximate implementation approach of Akbarpour and Nikzad (2020). This alternative solution offers a promising middle ground between the two existing solutions in India.

The unique additive and monotonicity requirements of our problem suggest several promising alternative approaches that merit further investigation. Speculatively, one potential direction involves exploring an iterative rounding approach, which may yield solutions allowing only minor deviations from cumulative fractional assignments (Nguyen and Vohra (2019)). Another possibility lies in formulating a constrained optimization problem to measure deviations of integral assignment tables from fractional ones while preserving their additive structure and adhering to the monotonicity constraint (Ricca et al. (2012)). Exploring the feasibility of solutions to this problem could uncover new avenues of inquiry. Additionally, drawing inspiration from ex post fairness and efficiency guarantees in fair division literature may offer insights into developing richer constraint structures that can coexist with monotonicity (Aziz et al. (2023)). Lastly, the multi-period considerations introduced in this paper could be worth exploring in the classic biproportional apportionment problem context of translating electoral votes into parliamentary seats (Pukelsheim (2017)).
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Appendices for Online Publication

A Examples and proofs

A.1 An example for Theorem 2: Explaining constraints

To understand the constraint structure of the problem better, take the following example. Consider a problem with two categories \( C = \{c_1, c_2\} \) and the reservation scheme is \( \alpha = (\alpha_1, \alpha_2) = (1/3, 2/3) \). Suppose we wish to implement the reservation scheme in a problem of reservation in three dimensions. We represent the given reservation scheme as a two-way table \( P \), where the rows denote the index of the vacancies and the columns denote the categories. Each internal entry \( p_{ij} = \alpha_j \). Take \( k = 3 \), for \( k \times n \) table \( P \), as 3 is the lowest common denominator.

Figure 9 illustrates the constraint structure. Column constraints are \( C_{31} = \{k_{11}, k_{21}, k_{31}\} \), \( C_{21} = \{k_{11}, k_{21}\} \), \( C_{32} = \{k_{12}, k_{22}, k_{32}\} \), and \( C_{22} = \{k_{12}, k_{22}\} \), and row constraints are \( R_1 = \{k_{11}, k_{12}\} \), \( R_2 = \{k_{21}, k_{22}\} \), and \( R_3 = \{k_{31}, k_{32}\} \).

Figure 9: CONSTRAINT STRUCTURE OF THE EXAMPLE \( P \)
A.2 Decomposition to construct rosters in Theorem 2

For practical purposes, we provide a method that generates rosters in polynomial time.

Part 1. Revisiting the constraints.

Recall that in the proof of Theorem 2 we were looking for integral tables $P'$ that satisfy three types of constraints:

1. Internal constraints ensure that each internal entry can be 1 or 0.
   That is, $0 \leq p'_{ij} \leq 1$ for every $(i, j)$.
   Let $\mathcal{K}_I$ be the constraint structure associated with internal constraints.
   Let $k_{ij} := \{(i, j)\}$ denote such constraint.

2. Row sums are required to be one since every position is assigned to exactly one category.
   That is, $\sum_{j \in \mathcal{C}} p'_{ij} = 1$ for every $i$.
   Let $\mathcal{K}_R$ be the constraint structure associated with row constraints.
   Let $R_i := \{(i, j) | j \in \mathcal{C}\}$ denote such constraint.

3. Column constraints ensure that the difference between the cumulative sum of vacancies given to a category and cumulative fractional assignments is less than one.
   That is, $\lfloor \sum_{i \leq l} p_{ij} \rfloor \leq \sum_{i \leq l} p'_{ij} \leq \lceil \sum_{i \leq l} p_{ij} \rceil$ for every $2 \leq l \leq m$ and $j \in \mathcal{C}$.
   Let $\mathcal{K}_C$ be the constraint structure associated with column constraints.
   Let $C_{lj} := \{(i, j) | i \leq l\}$ denote such constraint.

Part 2. Map table $P$ to a flow network.

We will now map table $P$ with the above constraint structure to a flow network. The set of vertices consists of the source, the sink, vertices for each $k_{ij} \in \mathcal{K}_I$, each $R_i \in \mathcal{K}_R$, and for each $C_{lj} \in \mathcal{K}_C$. The following rules govern the placement of directed edges:

1. A directed edge from source to $C_{mj}$ for every $j \in \mathcal{C}$.
2. A directed edge from $C_{ij}$ to $k_{lj}$ and $C_{l-1j}$ for every $l \geq 3$ and $j \in \mathcal{C}$.
3. A directed edge from $C_{2j}$ to $k_{2j}$ and $k_{1j}$ for every $j \in \mathcal{C}$.
4. A directed edge from $k_{ij}$ to $R_i$ for every $(i, j)$.
5. A directed edge from $R_i$ to sink for every $i$.

We next associate flow with each edge. Notice that there is only one incoming edge for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$. And, there is only one outgoing edge for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. This is because of the hierarchical constraint structure. Therefore, it is sufficient to associate
incoming flows for each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$ and outgoing flows for each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$. For each vertex $K \in \mathcal{K}_C \cup \mathcal{K}_I$, the incoming flow is equal to $\sum_{(i,j) \in K} p_{ij}$. For each vertex $K \in \mathcal{K}_R \cup \mathcal{K}_I$, the outgoing flow is equal to $\sum_{(i,j) \in K} p_{ij}$. Furthermore, the flow association ensures that the amount of incoming flow equals the amount of outgoing flow for each vertex.

We have mapped table $P$ with the constraint structures to a flow network. The mapping is injective; as long as the constraints are still satisfied after the transformation, every transformation in the flow network can be mapped back to table $P$.

**Part 3. Generate integral assignments for rosters.**

We need a new definition before proceeding. We call the pair of tables $(P^1, P^2)$ a decomposition of table $P$, if

1. there exists $\beta \in (0, 1)$ such that $P = \beta P^1 + (1 - \beta) P^2$,
2. for each constraint $K$, $\lfloor \sum_{(i,j) \in K} p_{ij} \rfloor \leq \sum_{(i,j) \in K} p_{ij} l \leq \lceil \sum_{(i,j) \in K} p_{ij} \rceil$ for $l = 1, 2$, and
3. table $P^1$ and $P^2$ have more number of integral entries than table $P$.

**Decomposition Algorithm**

If the flow network contains a fractional edge:

**Step 1:** Choose any edge that has fractional flow. Since the total inflow equals the total outflow for each vertex, there will be an adjacent edge with fractional flow.\(^{28}\) Continue to add new edges with fractional flows until a cycle is formed.\(^{29}\)

**Step 2:** Modify the flows in the cycle in two ways to create $P^1$ and $P^2$:

1. First way: the flow of each forward edge\(^{30}\) is increased and the flow of each backward edge is decreased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as $d_-$. Map back the resulting flow network to a two-way table. Denote the table as $P^1$.
2. Second way: the flow of each forward edge is decreased, and the flow of each backward edge is increased at the same rate until at least one flow reaches an integer value. Record the amount of adjustment as $d_+$. Map back the resulting flow network to a two-way table. Denote the table as $P^2$.
3. Set $\beta = \frac{d_-}{d_- + d_+}$.
4. The pair of tables $(P^1, P^2)$ is a decomposition of table $P$, where $P = \beta P^1 + (1 - \beta) P^2$.

\(^{28}\)The reason is that an integer cannot be written as the sum of integers and a proper fraction.

\(^{29}\)The number of vertexes is finite so we can find a cycle.

\(^{30}\)We say an edge in the cycle is a forward edge if its direction is the same as the cycle path, the backward edge is defined conversely.
Generate the integral assignment table $\bar{P}_1, \ldots, \bar{P}_N$ with probability $\lambda_1, \ldots, \lambda_N$ in the following way. Decompose the fractional assignment table $P$ using the decomposition algorithm into a convex combination $P = \beta P^1 + (1 - \beta) P^2$ of two assignment tables, each of which has at least one more integer-valued element $(i, j)$. Then, generate a random number and, with probability $\beta$ continue by similarly decomposing $P^1$, while with probability $1 - \beta$, continue by decomposing $P^2$. Stop at an integer (integral) assignment.

This generates the integral two-way tables with the same constraint structure as table $P$. Moreover, the (compound) probability of arriving at each integral two-table is such that the expected assignment table equals $P$.

A.3 Example for Theorem 2 continued: Explaining decomposition

The example’s table $P$ is

\[
P = \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ 1/3 & 2/3 & 1 \\ 1 & 2 & 3 \end{array}
\]

The two-way table $P$ with the constraints is then represented as a network flow. Starting from the source, the flows pass through the sets in column constraints, arranged in descending order of set inclusion. That is, for example, $C_{31} \supset C_{21} \supset k_{11}$. This explains the flow network on the left side of Figure 10, where the numbers on the edges represent the flows. The flows then proceed along the directed edges representing the set-inclusion tree, eventually reaching the singleton sets. That is, for example, $k_{11} \subset R_1$. This explains the flow network on the right side of Figure 10.

In the flow network, note that the flow associated with each edge reflects the totals of elements in the corresponding set. And the flow arriving at each vertex equals the flow leaving that vertex. Now we are ready to present the algorithm. The algorithm will conserve these two properties while constructing a new flow network with fewer fractional elements.

We first identify a cycle of edges with fractional flows. Choosing any fractional edge, say $(C_{31}, k_{31})$, we find another fractional edge that is neighbor to $k_{31}$. If a vertex has a fractional edge, it has to have another fractional edge. Since total inflow equals outflow for every vertex (except source and sink), we would have a contradiction. We continue to add new fractional edges until we form a cycle. In our example, the cycle of fractional edges is $C_{31} \rightarrow^{1/3} k_{31} \rightarrow^{1/3} R_3 \leftarrow^{2/3} k_{32} \leftarrow^{2/3} C_{32} \rightarrow^{4/3} C_{22} \ldots \leftarrow^{2/3} C_{31}$. We illustrate this cycle in Figure 11 with dashed lines.
Next, we alter the cycle’s edge flows. We first increase each forward edge’s flow while simultaneously decreasing each backward edge’s flow until at least one flow reaches an integer value. A table $P_1$ is created due to the resulting network flow. In the example, flows along all forward edges increase from $\frac{2}{3}$ to $1$, $\frac{1}{3}$ to $\frac{2}{3}$, and $\frac{4}{3}$ to $\frac{5}{3}$, while flows along all backward edges decrease from $\frac{1}{3}$ to $0$ and $\frac{2}{3}$ to $\frac{1}{3}$. The adjustment is $d_+ = \frac{1}{3}$. Next, the flows of the edges in the cycle are readjusted in the opposite direction, increasing those with backward edges and lowering those with forward edges analogously, resulting in a new table $P_2$. In the example, flows along all forward edges decrease from $\frac{2}{3}$ to $\frac{1}{3}$, $\frac{1}{3}$ to $0$, and $\frac{4}{3}$ to $1$, while flows along all backward edges increase from $\frac{1}{3}$ to $\frac{2}{3}$ and $\frac{2}{3}$ to $1$. The adjustment is $d_- = \frac{1}{3}$.

Now, we can decompose $P$ into these two tables, that is, $P = \frac{d_-}{d_- + d_+}P_1 + \frac{d_+}{d_- + d_+}P_2 = \frac{1}{2}P_1 + \frac{1}{2}P_2$. The algorithm picks $P_1$ with probability $0.5$ and $P_2$ with probability $0.5$. We reiterate the decomposition process until no fractions are left.

At least one fraction in $P$ is converted to an integer at each iteration, while all current integers remain constant. Each fraction must appear in at least one iteration. As a result, the process must converge to an integer table in fewer iterations than the initial number of
fractions in table $P$.

Since only the fractions along one cycle in the flow network are modified in each iteration, the expected change at this iteration for entries not on this cycle is 0, so the expected change in corresponding entries in $P$ is 0. For those fractional edges that are modified, the probabilities are picked so that the expected adjustment in each iteration is 0.

Fractional edges that are adjusted multiple times will have a variety of intermediate adjustment probabilities. Still, because our procedure keeps the expected change at 0 in each iteration, the compound probabilities will also keep the expected change at 0.

**A.4 Proof of Theorem 4**

*Proof of Theorem 4.* Without monotonicity, the sequence of tables does not matter for the solution, and the problem is much simplified. That is, the problem earlier characterized using $Y^t$, is now reduced to $X^t$. We can even ignore the superscript and consider the problem of rounding a cumulative fractional assignment table $X$ to an integral one $\bar{X}$. We will use Cox (1987)'s controlled rounding algorithm that takes a fractional assignment table as input and generates an integral assignment table as output such that each fractional entry is rounded
up or down to an adjacent integer. The probability of arriving at an integer assignment table will be such that the solution is ex ante proportional. To make the algorithm easier to understand, after each step, we demonstrate the algorithm on an example depicted in Figure 12.

**Cox’s controlled rounding algorithm**

**Step 0:** Given a fractional assignment table $X$, we construct an extended table $V$ by adding an extra row to table $X$. The last row of $V$ is generated by subtracting the fractional part of the column totals of table $X$ from 1, that is, $1 - (x_{m+1,j} - \lfloor x_{m+1,j} \rfloor)$.

In our example, shown in Figure 12, table $V$ is equivalent to table $X$ except for the last row. Adding this extra row makes the column totals integers.

**Figure 12: STEP 1 OF PROCEDURE**

|          | 0.5 | 0.5 | 1 | 2 |
|----------|-----|-----|---|---|
| $X$      | 0.25| 0.25| 0.5| 1 |
|          | 0.75| 0.75| 1.5| 3 |
|          | 1.5 | 1.5 | 3 | 6 |

|          | 0.5 | 0.5 | 1 | 2 |
|----------|-----|-----|---|---|
| $V$      | 0.25| 0.25| 0.5| 1 |
|          | 0.75| 0.75| 1.5| 3 |
|          | 0.5 | 0.5 | 0 | 1 |
|          | 2   | 2   | 3 | 7 |

(a) FRACTIONAL ASSIGNMENT TABLE  

(b) EXTENDED TABLE

The rounding procedure involves iterative adjustment of the fractional internal entries of table $V$ until integers have replaced all fractions.

**Step 1:** If table $V$ contains no fractions, then skip to Step 7.

**Step 2:** Choose any fraction $v_{ij}$ in table $V$. At $(i, j)$, begin an alternating row-column (or column-row) path of fractions $C$. Because the total entries of $V$ are integers, each row or column containing fractions must contain at least two. Therefore, we may assume that the path of fractions contains at least four entries. At some stage, $C$ either becomes a cycle containing $(i, j)$ or else $C$ can be terminated the first time it returns to any row or column previously visited, in which case a cycle is formed that begins at the first intersection of this row or column with $C$ and ends at $C$’s terminal entry. Let $L$ denote the first cycle formed from $C$.

In our example, shown in Figure 13, a cycle of fractions is $(i_1, j_1) \rightarrow (i_1, j_2) \rightarrow (i_2, j_2) \rightarrow (i_2, j_3) \rightarrow (i_3, j_3) \rightarrow (i_3, j_2) \rightarrow (i_4, j_2) \rightarrow (i_4, j_1) \rightarrow (i_1, j_1)$.

**Step 3:** Choose any $(i_1, j_1)$ on $L$. Denote the members of $L$ by $(i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots$.
Figure 13: STEP 2 OF PROCEDURE

\[
V = \begin{pmatrix}
0.5 & 0.5 & 1 & 2 \\
0.25 & 0.25 & 0.5 & 1 \\
0.75 & 0.75 & 1 & 3 \\
0.5 & 0.5 & 0 & 1 \\
2 & 2 & 3 & 7
\end{pmatrix}
\]

Let \((i_1, j_1), \ldots, (i_k, j_{k+1}) = (i_k, j_1)\). Let
\[
d_- = \min_{1 \leq q \leq k} \left[ v_{i_q, j_q}, 1 - v_{i_q, j_{q+1}} \right]
\]
\[
d_+ = \min_{1 \leq q \leq k} \left[ 1 - v_{i_q, j_q}, v_{i_q, j_{q+1}} \right]
\]
Note that both \(d_-\) and \(d_+\) lie strictly between 0 and 1.

**Step 4.** Select \(d_-\) with probability \(p_- = \frac{d_-}{d_- + d_+}\) or select \(d_+\) with probability \(p_+ = \frac{d_+}{d_- + d_+}\).

**Step 5a.** If \(d_+\) is selected, move \(d_+\) units into the \((i_1, j_1)\) cell around \(L\) on an alternating (+, −) path, namely,
\[
\begin{align*}
&\text{transform } v_{i_q, j_q} \text{ to } v_{i_q, j_q} + d_+, \\
&\text{transform } v_{i_q, j_{q+1}} \text{ to } v_{i_q, j_{q+1}} - d_+.
\end{align*}
\]
Return to Step 1.

This amounts to raising the odd edges and reducing the even edges at the same rate until at least one edge reaches an integer value. In our example, the odd edges rise by 0.5 and even edges reduce by 0.5 (\(d_+ = 0.5\)). The resulting table \(V_+\) is shown in Figure 14.

**Step 5b.** If \(d_-\) is selected, move \(d_-\) units out of the \((i_1, j_1)\) cell around \(L\) on an alternating (−, +) path, namely, along \(L\)
\[
\begin{align*}
&\text{transform } v_{i_q, j_q} \text{ to } v_{i_q, j_q} - d_-, \\
&\text{transform } v_{i_q, j_{q+1}} \text{ to } v_{i_q, j_{q+1}} + d_-.
\end{align*}
\]
Return to Step 1.

This amounts to raising even edges and reducing odd edges at the same rate until at least one edge reaches an integer value. In our example, the even edges rise by 0.25, and odd edges reduce by 0.25 (\(d_- = 0.25\)). The resulting table \(V_-\) is shown in Figure 14.

Table \(V\) is thus decomposed into table \(V_+\) and table \(V_-\) where \(V = \frac{1}{3} V_+ + \frac{2}{3} V_-\).

There are fewer fraction elements in both tables.

**Step 6:** Reiterate Step 1-5.

**Step 7:** Transform \(V\) back to \(X\) by deleting the extra row added in Step 0, and subtracting its integer entries from the corresponding column totals. Report \(X\)
Figure 14: STEP 5 OF PROCEDURE

\[
\begin{array}{cccc|c}
1 & 0 & 1 & 2 \\
0.25 & 0.75 & 0 & 1 \\
V_+ = & 0.75 & 0.25 & 2 & 3 \\
0 & 1 & 0 & 1 \\
\hline
2 & 2 & 3 & 7
\end{array}
\quad
\begin{array}{cccc|c}
0.25 & 0.75 & 1 & 2 \\
0.25 & 0 & 0.75 & 1 \\
V_- = & 0.75 & 1 & 1.25 & 3 \\
0.75 & 0.25 & 0 & 1 \\
\hline
2 & 2 & 3 & 7
\end{array}
\]

as the algorithm’s outcome.

The algorithm must end in finite steps (at most, the number of fractions in table \( V \)). For any fractional assignment \( X \), Cox’s algorithm outputs integral tables with probabilities such that the induced randomized solution is ex ante proportional, stays within department quota, and stays within university quota. Let’s see why.

**Lemma 4.** The outcome of Cox’s rounding algorithm stays within department and university quota.

**Proof.** In Step 5, the row and column sums remain the same after each adjustment. Moreover, after adjustments, every element \( v_{ij} \) in table \( V \) always remains less than or equal to \( \lceil v_{ij} \rceil \) and greater than or equal to \( \lfloor v_{ij} \rfloor \). Therefore, the algorithm’s outcome will stay within the department and university quota.

**Lemma 5.** Cox’s rounding algorithm ensures that the expected adjustment \( E(d) \) to each entry of \( V \) at each iteration is 0.

**Proof.** Notice that in each iteration of Step 1-5, the probability of raising \( v_{ij} \) by \( d_+ \) is \( d_-/(d_- + d_+) \). While the probability of reducing \( v_{ij} \) by \( d_- \) is \( d_+/(d_- + d_+) \). Therefore, the expected adjustment is \( d_+ \cdot \frac{d_-}{d_- + d_+} - d_- \cdot \frac{d_+}{d_- + d_+} = 0 \).

Lemma 5 proves that in each iteration, entries of the fractional assignment table \( X \) are adjusted so that (ex ante) positive and negative deviations from fractional assignments balance to yield zero deviation in expectation. Moreover, the compound probabilities for rounding fractional assignments up or down under Cox’s rounding algorithm are unique (Cox (1987)). Therefore, Lemma 4 and Lemma 5 prove Theorem 4.
## B  Tables and Figures

Figure 15: 200-point Roster prescribed by Government of India

FOR DIRECT RECRUITMENT

Model Roster of Reservation with reference to posts for Direct recruitment on All India Basis by Open Competition

| Sl. No. of Post | Share of Entitlement | Category for which the posts should be earmarked |
|----------------|----------------------|-------------------------------------------------|
|                | SC @15% | ST @7.5% | OBC @27% | EWS @10% |                      |
| 1              | 0.15    | 0.08     | 0.27     | 0.10     | UR                   |
| 2              | 0.30    | 0.15     | 0.54     | 0.20     | UR                   |
| 3              | 0.45    | 0.23     | 0.81     | 0.30     | UR                   |
| 4              | 0.60    | 0.30     | 1.08     | 0.40     | OBC-1                |
| 5              | 0.75    | 0.38     | 1.35     | 0.50     | UR                   |
| 6              | 0.90    | 0.45     | 1.62     | 0.60     | UR                   |
| 7              | 1.05    | 0.53     | 1.89     | 0.70     | SC-1                 |
| 8              | 1.20    | 0.60     | 2.16     | 0.80     | OBC-2                |
| 9              | 1.35    | 0.68     | 2.43     | 0.90     | UR                   |
| 10             | 1.50    | 0.75     | 2.70     | 1.00     | EWS-1                |
| 11             | 1.65    | 0.83     | 2.97     | 1.10     | UR                   |
| 12             | 1.80    | 0.90     | 3.24     | 1.20     | OBC-3                |
| 13             | 1.95    | 0.98     | 3.51     | 1.30     | UR                   |
| 14             | 2.10    | 1.05     | 3.78     | 1.40     | ST-1                 |
| 15             | 2.25    | 1.13     | 4.05     | 1.50     | SC-2                 |
| 16             | 2.40    | 1.20     | 4.32     | 1.60     | OBC-4                |
| 17             | 2.55    | 1.28     | 4.59     | 1.70     | UR                   |
| 18             | 2.70    | 1.35     | 4.86     | 1.80     | UR                   |
| 19             | 2.85    | 1.43     | 5.13     | 1.90     | OBC-5                |
| 20             | 3.00    | 1.50     | 5.40     | 2.00     | SC-3                 |
| 21             | 3.15    | 1.58     | 5.67     | 2.10     | EWS-2                |
| 22             | 3.30    | 1.65     | 5.94     | 2.20     | UR                   |
| 23             | 3.45    | 1.73     | 6.21     | 2.30     | OBC-6                |
| 24             | 3.60    | 1.80     | 6.48     | 2.40     | UR                   |
| 25             | 3.75    | 1.88     | 6.75     | 2.50     | UR                   |
| 26             | 3.90    | 1.95     | 7.02     | 2.60     | OBC-7                |
| 27             | 4.05    | 2.03     | 7.29     | 2.70     | SC-4                 |
| 28             | 4.20    | 2.10     | 7.56     | 2.80     | ST-2                 |
| 29             | 4.35    | 2.18     | 7.83     | 2.90     | UR                   |
| 30             | 4.50    | 2.25     | 8.10     | 3.00     | OBC-8                |
| 31             | 4.65    | 2.33     | 8.37     | 3.10     | EWS-3                |

Source: [https://dopt.gov.in/sites/default/files/ews28fT.PDF](https://dopt.gov.in/sites/default/files/ews28fT.PDF)
Figure 16: 13-point Roster prescribed by Government of India

| Cadre Strength | Initial Recruitment | Replacement No. |
|----------------|---------------------|-----------------|
|                | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th | 9th | 10th | 11th | 12th | 13th |
| 1  | UR  | UR  | UR  | OBC | UR  | UR  | SC  | OBC | UR  | EWS  | UR  | OBC | SC  | ST  |
| 2  | UR  | OBC | UR  | UR  | SC  | OBC | UR  | EWS | UR  | OBC  | SC  | ST  |
| 3  | UR  | OBC | UR  | UR  | SC  | OBC | UR  | EWS | UR  | OBC  | SC  | ST  |
| 4  | OBC | UR  | UR  | SC  | OBC | UR  | EWS | UR  | OBC  | SC  | ST  |
| 5  | UR  | UR  | SC  | OBC | UR  | EWS | UR  | OBC | SC  | ST  |
| 6  | UR  | SC  | OBC | UR  | EWS | UR  | OBC | SC  | ST  |
| 7  | SC  | OBC | UR  | EWS | UR  | OBC | SC  | ST  |
| 8  | OBC | UR  | EWS | UR  | OBC | SC  | ST  |
| 9  | UR  | EWS | UR  | OBC | SC  | ST  |
| 10 | EWS | UR  | OBC | SC  | ST  |
| 11 | UR  | OBC | SC  | ST  |
| 12 | OBC | SC  | ST  |
| 13 | SC  | ST  |

Source: https://dopt.gov.in/sites/default/files/ews28fT.PDF
Online applications are invited in the prescribed Application form at web link https://colrec.uod.ac.in from the eligible candidates for appointment to the post of Assistant Professors in the Academic Pay Level 10 with rationalized entry pay of Rs. 57,700/- as per 7th CPC pay matrix in addition to other usual allowances. The last date for receipt of application is 1.10.2022 or two weeks from the date of publication of advertisement in the Employment News. For details please visit the College Website (www.hinducollege.ac.in) as well as University of Delhi Website (www.du.ac.in).

Persons with Benchmark Disability (PwBD) candidates may approach the Helpdesk set up at the College in case they require any assistance in filling up application form.

| S. No. | Department                  | Total Vacancy | Nature of Post for Assistant Professor |
|--------|----------------------------|---------------|---------------------------------------|
|        |                            | UR | SC | ST | OBC | EWS | PwBD |
| 1      | Botany                     | 04 | 01 | 01 | 01  | -   | -    |
| 2      | Chemistry                  | 07 | 03 | 01 | -   | 02  | 01   |
| 3      | Commerce                   | 04 | 02 | 01 | 01  | -   | -    |
| 4      | Economics                  | 02 | 01 | -  | -   | 01  | -    |
| 5      | English                    | 06 | 02 | 01 | 01  | 01  | 01– (VI) |
| 6      | Environmental Science      | 02 | 01 | -  | -   | 01  | -    |
| 7      | Hindi                      | 04 | 01 | -  | -   | 03  | -    |
| 8      | History                    | 05 | 01 | 02 | 01  | 01  | -    |
| 9      | Mathematics                | 04 | -  | 01 | 02 | 01   | -    |
| 10     | Philosophy                 | 04 | -  | 01 | -   | 01  | 02   |
| 11     | Asstt. Prof. in Physical Education | 01 | 01 | -  | -   | -   | -    |
| 12     | Physics                    | 11 | 04 | 02 | 01 | 03  | 01   |
| 13     | Political Science          | 03 | 01 | -  | 01  | -    | 01– (LD) |
| 14     | Sanskrit                   | 02 | 01 | -  | -   | 01  | -    |
| 15     | Sociology                  | 02 | 01 | -  | 01  | -    |       |
| 16     | Statistics                 | 03 | 02 | -  | 01  | 01   | 01– (LD) |
| 17     | Zoology                    | 05 | 01 | 01 | 03  | -   |       |
|        | Total                      | 69 | 23 | 11 | 05 | 21  | 06   | 03 |

Abbreviations used: UR: Unreserved, OBC: Other Backward Classes, SC: Scheduled Caste, ST: Scheduled Tribe, EWS: Economically Weaker Section, PwBD: Persons with Benchmark Disability, VI: Visual Impairment including Blindness & low Vision, LD: Locomotor Disability including leprosy cured, dwarfism, acid attack victims, Cerebral Palsy and muscular dystrophy, HH: Hard of Hearing.

Any addendum/corrigendum shall be posted only on the College Website

Date 13.9.2022

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