Source-Channel Matching for Sources with Memory

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Abstract—To be considered for an IEEE Jack Keil Wolf ISIT Student Paper Award. In this paper we analyze the probabilistic matching of sources with memory to channels with memory so that symbol-by-symbol code with memory without anticipation are optimal, with respect to an average distortion and excess distortion probability. We show achievability of such a symbol-by-symbol code with memory without anticipation, and we show matching for the Binary Symmetric Markov source (BSMS(p)) over a first-order symmetric channel with a cost constraint.

I. INTRODUCTION

In this paper we address the problem of Joint Source-Channel Coding JSCC based on symbol-by-symbol code transmission with memory without anticipation. Thus, at each instant of time $i$, we impose real-time transmission constraints on the encoder and decoder to process samples independently, with memory on past symbols, and without anticipation with respect to symbols occurring future times $j > i$. The aim is to match probabilistically the source to a channel, and evaluate its performance with respect to excess distortion probability.

For memoryless sources and channels, necessary and sufficient conditions for symbol-by-symbol transmission are given in [1] (see also [2]). However, extending these results to sources with memory is not a trivial task for the following two reasons. i) The optimal reproduction distribution of classical Rate Distortion Function (RDF), used during the realization procedure, to match the source to a channel is, in general noncausal (anticipative on future symbols); ii) the solution to the RDF is often unknown.

In this paper we consider a nonanticipative information RDF which is realizable in the above sense, and we proceed to obtain the expression of the optimal causal reproduction distribution. 1) We prove under certain conditions involving the nonanticipative information RDF, and the capacity of certain channels with memory and feedback, that symbol-by-symbol code with memory without anticipation is achievable. 2) we consider a BSMS(p) and we show that matching is possible over a symmetric channel with memory and cost constraint, 3) we evaluate the excess distortion probability and we show that convergence to zero, as the number of channel uses increases, establishing achievability.

II. SYMBOL-BY-SYMBOL CODES WITH MEMORY WITHOUT ANTICIPATION

Let $\mathbb{N} \triangleq \{0, 1, \ldots, n\}$, $\mathbb{N}^+ \triangleq \{1, 2, \ldots, n\}$. The spaces $\mathcal{X}, \mathcal{A}, \mathcal{B}, \mathcal{Y}$ denote the source output, channel input, channel output, and decoder output alphabets, respectively, which are assumed to be complete separable metric spaces (Polish spaces) to avoid excluding continuous alphabets. We define their product spaces by $\mathcal{X}_0, \mathcal{A}_0 \triangleq \times_{i=0}^{n} \mathcal{X}, \mathcal{B}_0, \mathcal{Y}_0 \triangleq \times_{i=0}^{n} \mathcal{B}, \mathcal{Y}_0 \triangleq \times_{i=0}^{n} \mathcal{Y}$, and associate them with their measurable spaces. Let $x^n \triangleq \{x_0, x_1, \ldots, x^n\} \in \mathcal{X}_0$ denote the source sequence of length $n + 1$, and similarly for the rest of the blocks. Next, we introduce the various distributions.

Definition II.1. (Source) The source is a sequence of conditional distributions defined by $P_{X_{i}}(dx^n) \triangleq \otimes_{i=0}^{n} P_{X_{i}}(dx_i|x_i^{-1})$.

Definition II.2. (Encoder) The encoder is a sequence of conditional distributions defined by $\hat{P}_{A_{i}|B_{i-1},X_{i}}(da_i|b_i^{-1}, x_i)$.

Thus, the encoder is nonanticipative in the sense that at each time $i \in \mathbb{N}$, $P_{A_i|A_{i-1},B_{i-1},X_{i}}(da_i|a_{i-1}^{-1}, b_{i-1}^{-1}, x_{i})$ is a measurable function of past and present symbols $x_i$ in $\mathcal{X}_{0,i}$ and past symbols $a_{i-1}^{-1} \in A_{0,i-1}$, $b_{i-1}^{-1} \in B_{0,i-1}$.

Definition II.3. (Channel) The channel is a sequence of conditional distributions defined by $\hat{P}_{B_{i}|A_{i-1},X_{i}}(db_i|a_i^{-1}, x_i)$.

Thus the channel has memory, feedback and it is nonanticipative with respect to the source sequence.

Definition II.4. (Decoder) The decoder is a sequence of conditional distributions defined by $\hat{P}_{Y_{i}|B_{i}}(dy^n|b^n) \triangleq \otimes_{i=0}^{n} P_{Y_i|Y_{i-1},B_{i}}(dy_i|y_{i-1}^{-1}, b_i)$.

Definitions II.1, II.2 are general, since they allow memory and feedback without anticipation, hence we call the source-channel code symbol-by-symbol code with memory without
anticipation. Given the source, encoder, channel, decoder, we can define uniquely the joint measure by

\[
P_{X^n,A^n,B^n,Y^n}(dx^n, da^n, db^n, dy^n) = \bigotimes_{i=0}^n P_{Y_i|Y_{i-1},B^i}(dy_i|y_{i-1}, b^i) \bigotimes P_{B_i|B^{i-1},A^i,X^i}(db_i|b_{i-1}, a^i, x^i) \bigotimes P_{A_i|A^{i-1},B^{i-1},X^i}(da_i|a_{i-1}, b_{i-1}, x^i) \otimes P_{X_i|X_{i-1}}(dx_i|x_{i-1}),
\]

The previous equation implies the Markov Chains (MCs):

\[
(A^{i-1}, B^{i-1}, Y^{i-1}) \leftrightarrow X^{i-1} \leftrightarrow X_i, \quad \forall i \in \mathbb{N}^n
\]

\[
Y^{i-1} \leftrightarrow (A^{i-1}, B^{i-1}, X^i) \leftrightarrow A_i, \quad \forall i \in \mathbb{N}^n
\]

\[
Y^{i-1} \leftrightarrow (A^i, B^{i-1}, X^i) \leftrightarrow B_i, \quad \forall i \in \mathbb{N}^n
\]

\[
(A^i, X^i) \leftrightarrow (B^i, Y^{i-1}) \leftrightarrow Y_i, \quad \forall i \in \mathbb{N}^n.
\]

The distortion between the source and its reproduction is a measurable function \(d_{0,n}: \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n} \mapsto [0, \infty)\), and the cost of transmitting symbols over the channel is a measurable function \(c_{0,n}: \mathcal{A}_{0,n} \times \mathcal{Y}_{0,n-1} \mapsto [0, \infty)\) defined by

\[
d_{0,n}(x^n, y^n) = \sum_{i=0}^n \rho_{0,i}(T^i x^n, T^i y^n)
\]

\[
c_{0,n}(a^n, b^{n-1}) = \sum_{i=0}^n \gamma_{0,i}(a^i, b^{i-1}),
\]

where \((T^i x^n, T^i y^n)\) are the shift operations on \((x^n, y^n)\), respectively. For a single letter distortion function we take \(\rho_{0,i}(T^i x^n, T^i y^n) = \rho(x_i, y_i)\). Next, we shall state the definition of a symbol-by-symbol code (with memory without anticipation).

**Definition II.5. (Symbol-by-Symbol Code)** An \((n, d, \epsilon, P)\) symbol-by-symbol code for \((X_{0,n}, X_{0,n}, A_{0,n}, B_{0,n}, Y_{0,n}, P_{X^n}, \tilde{P}_{Y^n|X^n}, \mathcal{X}_{0,n}, \mathcal{Y}_{0,n}, \mathcal{A}_{0,n}, \mathcal{B}_{0,n}, \mathcal{D}_{0,n}, c_{0,n})\) is a code \(\{P_{A_i|A^{i-1},B^{i-1},X_i}, c_{0,n}, a_{0,n}\}\) such that \(P_{Y_i|Y_{i-1},B^i}(\cdot) : \forall i \in \mathbb{N}^n\) with excess distortion probability

\[
P_{\delta_{0,n}(x^n, y^n)}(x^n, y^n) > (n + 1)d \leq \epsilon, \quad \epsilon \in (0, 1), \quad d \geq 0,
\]

and transmission cost \(\frac{1}{n+1} \mathbb{E}\{c_{0,n}(A^n, B^{n-1})\} \leq P, \quad P \geq 0\).

**Definition II.6. (Minimum Excess Distortion)** The minimum excess distortion achievable by a symbol-by-symbol code \((n, d, \epsilon, P)\) is defined by

\[
D^*(n, \epsilon, P) = \inf \{\delta: \exists(n, d, \epsilon, P) \text{ symbol-by-symbol code}\}
\]

Our definition of symbol-by-symbol code is randomized, hence it embeds deterministic codes as a special case.

**III. NONANTICIPATIVE RDF**

The necessary conditions for transmitting a symbol-by-symbol code (they also hold for memoryless sources and channels) is the following.

1) Computation of the RDF and that of the optimal reproduction distribution so that probabilistic matching of the source and channel is feasible;

2) Realization of the optimal reproduction distribution of lossy compression with fidelity by an encoder-channel-decoder scheme, processing information causally.

Therefore, to facilitate the matching we introduce the RDF. Given a source distribution \(P_{X^n}(\cdot)\) and a reproduction distribution \(P_{Y^n|X^n}(\cdot)\) the average fidelity set is

\[
\mathcal{Q}_{0,n}(D) = \left\{ P_{Y^n|X^n} : \frac{1}{n+1} \int d_{0,n}(x^n, y^n)(P_{Y^n|X^n} \otimes P_{X^n})(dx^n, dy^n) \leq D \right\}.
\]

It is known that for stationary ergodic sources, the OPTA is defined by the RDF \(\mathcal{Q}_{0,n}(D) = \lim_{n \to \infty} \mathcal{R}_{0,n}(D)\). Given a source distribution \(P_{X^n}(\cdot)\) and a causal conditional distribution defined by

\[
P_{X^n} \rightarrow Y^n \rightarrow \mathcal{P}_{Y^n|X^n} \rightarrow P_{X^n} \rightarrow \mathcal{P}_{Y^n|X^n}
\]

we introduce the information measure

\[
I_{P_{X^n}}(X^n \rightarrow Y^n) \geq \mathcal{P}_{X^n \rightarrow Y^n} \times \mathcal{P}_{X^n \rightarrow Y^n}
\]

Consider the fidelity set defined by

\[
\mathcal{Q}_{0,n}(D) = \left\{ \mathcal{P}_{X^n \rightarrow Y^n} : \frac{1}{n+1} \int d_{0,n}(x^n, y^n)(P_{Y^n|X^n} \otimes P_{X^n})(dx^n, dy^n) \leq D \right\}.
\]

**Definition III.1. (Nonanticipative Information RDF)** Given \(\mathcal{Q}_{0,n}(D)\) the nonanticipative information RDF is defined by

\[
\mathcal{R}_{0,n}(D) = \inf \mathcal{P}_{Y^n|X^n \in \mathcal{Q}_{0,n}(D)} \mathcal{P}_{X^n \rightarrow Y^n}(P_{X^n} \times \mathcal{P}_{Y^n|X^n})
\]

and its rate by \(R_{\text{opt}}(D) = \lim_{n \to \infty} \mathcal{R}_{0,n}(D)\) provided infimum and the limit exist.

Clearly, if the minimum of \(\mathcal{R}_{0,n}(D)\) exists the optimal reproduction distribution is nonanticipative, and hence realizable.

It can be shown that \(\mathcal{R}_{0,n}(D)\) is equal to the nonanticipatory ε-entropy introduced by Gorbunov and Pinsker in [4], via

\[
\mathcal{R}_{0,n}(D) = \inf \mathcal{P}_{Y^n|X^n \in \mathcal{Q}_{0,n}(D)} \mathcal{P}_{X^n \rightarrow Y^n}(I(X^n; Y^n))
\]

The MC in (9) implies that the reproduction distribution which minimizes (9) can be realized via an encoder-channel-decoder, using nonanticipative operations (causal).

Under the conditions in [4], or assuming the solution of \(\mathcal{R}_{0,n}(D)\) is stationary, which implies \(\mathcal{P}_{Y^n|X^n}(dy^n|x^n)\) is a stationary conditional distribution, we have the following theorem [5].
Theorem III.2. Suppose there exist an interior point of the fidelity set, and the infimum over $\hat{Q}_{0,n}(D)$ in $[3]$ is attained by

$$
\hat{P}_{Y_i|X_i}^*(dy^n|x^n) = \bigotimes_{i=0}^n \frac{e^{sp(T_i^x, T_i^y)|x_i^n|} P_{Y_i|X_i}^*(dy_i|y_i^{n-1})}{\int_{X_i} e^{sp(T_i^x, T_i^y)|x_i^n|} P_{Y_i|X_i}^*(dy_i|y_i^{n-1})}
$$

(10)

where $s \leq 0$ is the Lagrange multiplier associated with the constraint which is satisfied with equality, and

$$
R_{0,n}^a(D) = sD - \frac{1}{n+1} \sum_{i=0}^n \int_{X_i \times Y_i^{n-1}} \log \left( \int_{Y_i} e^{sp(T_i^x, T_i^y)|x_i^n|} P_{Y_i|X_i}^*(dy_i|y_i^{n-1}) \right)
\otimes P_{X_i|Y_i}^*(dx_i|y_i^{n-1}, y_i^{n-1})
$$

(11)

where $P_{X_i|Y_i}^*(\cdot, \cdot) = \hat{P}_{Y_i|X_i}^*(\cdot|\cdot) \otimes P_{X_i}^*(\cdot)$.

Proof: The derivation is given in [6].

Clearly, (10) is nonanticipative, and as we show in the next section, easy to compute, even for sources with memory.

IV. CODING THEOREM

In this section we show achievability of symbol-by-symbol code. First, we define the probabilistic realization of optimal reproduction distribution.

Definition IV.1. (Realization) Given a source $\{P_{X_i|X_i-1} (dx_i|x_i^{n-1}) : \forall i \in \mathbb{N}^n\}$, a general channel $\{P_{Y_i|X_i-1, X_i} (dy_i|y_i^{n-1}, x_i) : \forall i \in \mathbb{N}^n\}$ is a realization of the optimal reproduction distribution $\{P_{Y_i|X_i-1, X_i} (dy_i|y_i^{n-1}, x_i) : \forall i \in \mathbb{N}^n\}$ of theorem III.2 if there exists a pre-channel encoder $\{P_{A_i|X_i-1, B_i, A_i, X_i} (da_i|a_i^{n-1}, b_i^{n-1}, x_i) : \forall i \in \mathbb{N}^n\}$ and a post-channel decoder $\{P_{X_i|Y_i, B_i} (dx_i|y_i^{n-1}, b_i) : \forall i \in \mathbb{N}^n\}$ such that

$$
\hat{P}_{Y_i|X_i}^*(dy^n|x^n) = \bigotimes_{i=0}^n P_{Y_i|X_i-1, X_i} (dy_i|y_i^{n-1}, x_i)
$$

(12)

where the joint distribution from which (12) is obtained is given precisely by (10). Moreover we say that $R_{0,n}^a(D)$ is realizable if in addition the realization operates with average distortion $D$ and $I_{P_{X^n}} (P_X^n, \hat{P}_{Y^n|X^n}) = R_{0,n}^a(D)$

If the optimal reproduction distribution is realizable (see Definition IV.1), then the data processing inequality holds:

$$
I_{X^n \rightarrow Y^n} (P_{X^n}, \hat{P}_{Y^n|X^n}) \leq I(X^n \rightarrow B^n), \quad \forall n \in \mathbb{N}
$$

(13)

If $R_{0,n}^a(D)$ is realizable according to Definition IV.1 then the source is not necessarily matched to the channel. Next, we prove (under certain conditions) achievability.

Consider the following average cost set defined by

$$
P_{0,n}(P) \triangleq \left\{ (X^n, A^n) : \frac{1}{n+1} \mathbb{E} [c_{0,n}(A^n, B^n-1)] \leq P \right\}.
$$

Since we consider the general scenario that $[3]-[5]$ hold, then we define the information channel capacity as follows [7].

$$
C_{0,n}(P) \triangleq \sup_{(X^n, A^n) \in P_{0,n}(P)} \frac{1}{n+1} I(X^n \rightarrow B^n)
$$

and its rate (provided sup is finite and the limit exists) by $C(P) = \lim_{n \rightarrow \infty} C_{0,n}(P)$.

Next, we prove achievability of a symbol-by-symbol code.

Theorem IV.2. (Achievability of Symbol-by-Symbol Code). Suppose the following conditions hold.

1. $R_{0,n}^a(D)$ has a solution, and the optimal reproduction distribution is stationary of the form $\{P_{Y_i|X_i-1, X_i} : \forall i = 0, 1, \ldots, n\}$;

2. $C_{0,n}(P)$ has a solution, the maximizing processes are stationary, and the encoder is of the form $\{P_{A_i|X_i-1, X_i} : \forall i = 0, 1, \ldots, n\}$;

3. The optimal reproduction distribution $\hat{P}_{Y_i|X_i}^*(dy^n|x^n) \; \forall (D)$ given by Theorem III.2 is realizable, and $R_{0,n}^a(D)$ is also realizable.

4. For a given $D$ there exists a $P$ such that $R_{0,n}^a(D) = C(P)$.

If

$$
\mathbb{P} \left\{ \sum_{i=0}^n \rho_i (T^n_i X^n, T^n_i Y^n) > (n+1)d \right\} \leq \epsilon
$$

(14)

where $\mathbb{P}$ is taken with respect to $P_{Y^n, X^n} (dy^n, dx^n) = \hat{P}_{Y^n|X^n} (dy^n|x^n) \otimes P_{X^n} (dx^n)$ induced by matching, then there exists an $(n, d, \epsilon, P)$ symbol-by-symbol code with memory without anticipation.

Proof: The derivation is similar to [11]. If conditions (1), (3) hold then the optimal reproduction distribution is realizable, and this realization achieves $R_{0,n}^a(D)$. By (4) the source is matched to the channel so that the excess distortion probability of a symbol-by-symbol code with memory without anticipation satisfies [13].

A. Existence of Symbol-by-Symbol Codes

Next, we give sufficient conditions so that the conditions of Theorem IV.2 (1), (2) hold, i.e., establishing existence of a symbol-by-symbol encoder $\{P_{A_i|X_i-1, B_i, A_i, X_i} : \forall i = 0, 1, \ldots, n\}$.

Suppose the following conditions hold.

(A1) $\rho_i (T^n_i x^n, T^n_i y^n) = \rho_i (x_i, T^n_i y^n), \forall i \in \mathbb{N}^n$;

(A2) $P_{X_i|X_i-1} (x_i|x_i^{n-1}) = P_{X_i|X_i-1} (x_i|x_i^{n-1}), \forall i \in \mathbb{N}^n$;

(A3) $P_{B_i|A_i, B_i-1, A_i, X_i} (db_i|b_i^{n-1}, a_i, x_i), \forall i \in \mathbb{N}^n$.

If (A1) holds, then by Theorem III.2 the optimal stationary reproduction distribution is $P_{Y_i|X_i-1, X_i} = P_{Y_i|X_i-1, X_i}, \forall i \in \mathbb{N}^n$, and hence the form of the optimal reproduction distribution in Theorem IV.2 (1) holds. Moreover, if (A2), (A3) hold, then maximizing directed information $I(X^n \rightarrow B^n)$ over non-Markov encoders $\{P_{A_i|X_i-1, B_i, A_i, X_i} : \forall i = 0, 1, \ldots, n\}$ is equivalent to maximizing it over encoders $\{P_{A_i|B_i-1, X_i} : \forall i = 0, 1, \ldots, n\}$, and similarly, maximizing $I(X^n \rightarrow B^n)$ over non-Markov deterministic encoders $\{c_i(x_i, a_i^{n-1}, b_i^{n-1}) : \forall i = 1, \ldots, n\}$ is equivalent to the maximization with respect to
The form of the channel is motivated by the form of the $P_{Y_i|X_i,Y_i-1}$ (as in the IID Bernoulli source is matched via a binary symmetric channel). The state of the channel is defined as the modulo2 addition of the current input and previous output symbol, $s_i = a_i \oplus b_{i-1}$. Then we may transform the channel to its equivalent form defined by $P_{B_i|A_i,s_i}(b_i|a_i,s_i)$. This channel is called binary state symmetric channel, since given the state the channel it is binary symmetric. We introduce a cost constraint on the channel that has the following physical interpretation. Assume $\alpha_1 > \beta_1 \geq 0.5$. Then the capacity of the state zero channel $(1-H(\alpha_1))$, is greater than the capacity of the state one channel $(1-H(\beta_1))$. With “abuse” of terminology, we interpret the $(BSC(1-\alpha_1))$ as the “good channel” and the $(BSC(1-\beta_1))$ as the bad channel. It is further reasonable to assume that the we pay a larger fee to use the “good channel” and a smaller fee to use the “bad channel”. We quantify this policy by assigning a binary pay off to each of the channels. Hence, we assign a cost equal to 1 for the good channel, and a cost equal to 0 for the bad channel, defined by

$$c(a_i, b_{i-1}) = \begin{cases} 1 & \text{if } a_i = b_{i-1}, \text{ or } s_i = 0 \\ 0 & \text{if } a_i \neq b_{i-1}, \text{ or } s_i = 1 \end{cases}$$

hence the average cost constraint is

$$\mathbb{E}\{c(a_i, b_{i-1})\} = P_{A_i,b_{i-1}}(0,0) + P_{A_i,b_{i-1}}(1,1) = P_S(0)$$

Note that $c(a_i, b_{i-1})$ is not required to be binary and can be easily upgraded to more complex forms. We know that for the $BSSC(\alpha_1, \beta_1)$ feedback does not increase the capacity. The definition of the constrained capacity without feedback is defined by

$$C_{\text{fbs}}(k) = \lim_{n \to \infty} \max_{P_{X^n}, \sum_{i=0}^{n} \mathbb{E}\{c_0,(x_i,y_{i-1})\}=k} \frac{1}{n+1} I(X^n \to Y^n) \quad (16)$$

**Proposition V.2.** The capacity of the $BSSC(\alpha_1, \beta_1)$, with or without feedback, subject to the average cost constrain $\mathbb{E}\{c(a_i, b_{i-1})\} = k$, where $k = \text{constant}$, given by

$$C(\kappa) = H(\alpha_1\kappa+(1-\beta_1)(1-\kappa))-\kappa H(\alpha_1)-(1-\kappa)H(\beta_1) \quad (17)$$

The optimal input distribution without feedback is given by

$$P_{A_i|A_i-1}(a_i|a_{i-1}) = \begin{bmatrix} 1-\kappa-\gamma & \kappa-\gamma \\ 1-2\gamma & 1-\kappa-\gamma \end{bmatrix}$$

where $\gamma = \alpha_1 \kappa + \beta_1 (1-\kappa)$.

**Proof:** see [10].

**B. Symbol-By-Symbol Joint Source Channel Matching**

Recall that symbol-by-symbol joint source channel matching is achievable if $R^{\text{as}}(D) = C(\kappa)$ and if there exists an encoder decoder scheme for $d \geq D$, such that

$$\mathbb{P}\left\{ \sum_{i=0}^{n} T_i X^n, T_i Y^n > (n+1)d \right\} \leq \epsilon \quad (18)$$
By setting \( \kappa = m, \alpha_1 = \alpha, \beta_1 = \beta \), then \( \frac{1 - \kappa - x}{2\gamma} = p \),
\[
C(\kappa) = H(\beta_1(1-\kappa)+(1-\alpha_1)\kappa) - \kappa H(\alpha_1) - (1-\kappa)H(\beta_1) = H(\beta_1(1-m)+(1-\alpha)\gamma) - \kappa H(\alpha_1) - (1-\kappa)H(\beta_1) = H(p) - mH(\alpha) - (1-m)H(\beta) = R^\alpha(D)
\]

Moreover, the optimal input distribution is given by
\[
P_{A_i|A_{i-1}}(a_i|a_{i-1}) = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}
\]

Since the optimal input distribution is identical to the probability distribution of the source, then no encoder is required. Next, we check whether the average distortion is satisfied in the absence of a decoder. The average distortion between the source symbols and the reproduction symbols, \( \Delta \), is equal to
\[
\Delta = \mathbb{E}[d(X_i, Y_i)] = \mathbb{E}[d(A_i, B_i)] = \sum_{A_i, B_i} d(A_i, B_i) P_{B_i|A_i, B_{i-1}}(b_i|a_i, b_{i-1}) P_{A_i|B_{i-1}}(a_i|b_{i-1}) P_{B_i}(b_{i-1}) = (1 - \beta)(1 - m) + (1 - \alpha)m = D
\]

Thus, we established source channel matching of a BSMS(\( p \)) with Hamming fidelity constraint over a BSSC(\( \alpha_1, \beta_1 \)) subject to cost constraint, in the spirit of [11]. A realization of the described scheme is illustrated in Fig. 2 where it is shown that as the number of channel uses \( n \) is increased, the single letter distortion between the source symbol sequence and the reproduction sequence converges to the average distortion \( D \).

Next, we bound the excess distortion probability of Theorem IV.2 by applying an extension of Hoeffding’s inequality for MCs [11], to the Markov process \( \{Z_i : Z_i \triangleq (Y_i, X_i) : \forall i \in \mathbb{N}\} \) (this is easily shown to hold). Set \( \rho(x, y) = x \oplus y \) and let \( S_n \triangleq \sum_{i=0}^{n} \rho(X_i, Y_i) \). Let \( d \triangleq \delta + \frac{2[S_n]}{n+1}, \delta > 0 \). By Hoeffding’s inequality [11], the excess distortion probability is bounded by
\[
P\{\frac{S_n}{n} > (n + 1)d\} \leq \exp\left(-\frac{\lambda^2((n + 1)\delta - 2\|f\|m/\lambda)^2}{2(n + 1)\|f\|^2m^2}\right)
\]

where \( \|f\| = \sup\{y_i : i = 0, 1, \ldots \} = 1, m = 1, \lambda = \min\{p, 1 - p\} \min\{a, \beta, 1 - a, 1 - \beta\} \), for \( n > 2\|f\|m/\lambda \). This bound is illustrated in Fig. 3. Although, this bound is not tight and holds for \( n \) large enough, it shows the achievability of Markov sources via uncoded transmission. It might be possible to compute the excess distortion probability in closed form to get tighter bounds.

VI. Conclusions

This paper discusses General Source-Channel Matching for symbol-by-symbol. Using the nonanticipative RFD it shows achievability of a symbol-by-symbol code with respect to average and excess distortion probability. Then it considers the BSMS(\( p \)), it computes the nonanticipative RFD with respect to Hamming distortion, and shows that is is matched, uncoded, over a BSSC(\( \alpha_1, \beta_1 \)) subject to cost constraint but without feedback.

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