ON SHARIFI’S CONJECTURE: EXCEPTIONAL CASE

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Abstract. In the present article, we study the conjecture of Sharifi on the surjectivity of the map $\pi_{\theta}$. Here $\theta$ is a primitive even Dirichlet character of conductor $Np$, which is exceptional in the sense of Ohta. After localizing at the prime ideal $p$ of the Iwasawa algebra related to the trivial zero of the Kubota–Leopoldt $p$-adic $L$-function $L_p(s, \theta^{-1} \omega^2)$, we compute the image of $\pi_{\theta,p}$ in local Galois cohomology groups and prove that it is an isomorphism. Also, we prove that the residual Galois representations associated to the cohomology of modular curves are decomposable after taking the same localization.

Contents

1. Introduction 1
2. Sharifi’s Conjecture 4
3. Galois cohomology and cohomology of modular curves 7
4. Subjectivity of $\pi_{\theta,p}$ 11
5. Galois representations attached to cohomology of modular curves 14
References 16

1. Introduction

Let $N$ be a positive integer, and let $p \geq 5$ be a prime number not dividing $N\phi(N)$, where $\phi(N)$ is the order of the group $(\mathbb{Z}/N\mathbb{Z})^\times$. Set $H := \varprojlim H^1_{\text{ét}}(X_1(Np^r)/\mathbb{Q}, \mathbb{Z}_p)_{\text{ord}}$, where $\text{ord}$ denotes the ordinary part for the Hecke operator $U_p^*$, and denote by $H^-$ the subgroup of $H$ on which the complex conjugation acts via $-1$. Let $h^*$ be the cuspidal Hecke algebra acting on $H$, and let $I^*$ be the Eisenstein ideal in $h^*$ generated by $T_l^* - l(l-1)^{-1}$ for all primes $l \nmid Np$ and by $U_q^* - 1$ for all primes $q \mid Np$. For a Dirichlet character $\theta$ of conductor $Np$, we denote by $H_\theta$ the $\theta$-eigenspace of $H$ and set $\Lambda_\theta := \mathcal{O}[T]$ for some extension $\mathcal{O}$ of $\mathbb{Z}_p$ containing all of the values of $\theta$. Let

$$\varpi : H^-(1) \to \varprojlim H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{p}], \mathbb{Z}_p(2))^+ := S$$

be the homomorphism constructed by Sharifi in [13, Section 5.3] (or see Section 2.1 for the definition). It was further conjectured by Sharifi (Conjecture 5.8 in loc. cit.) and proved

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by Fukaya–Kato ([6, Theorem 5.2.3]) that the kernel of \( \varpi \) contains the Eisenstein ideal \( I^* \). Combining this with a conjecture of McCallum–Sharifi [9], it is conjectured that the homomorphism \( \varpi \) induces a surjective homomorphism of \( \Lambda_\theta \)-modules

\[
\varpi_\theta : H_\theta(1)/I_\theta^* H_\theta(1) \twoheadrightarrow S_\theta
\]

for all primitive even Dirichlet characters \( \theta \) of conductor \( Np \). When \( \theta \) is not exceptional (in the sense of Ohta [12]), it was conjectured by Sharifi in loc. cit. that \( \varpi_\theta \) is indeed an isomorphism together with the inverse homomorphism \( \Upsilon_\theta \). Moreover, he also proved that there is a canonical isomorphism of \( \Lambda_\theta \)-modules \( S_\theta \cong X_\theta(1) \) (Lemma 4.11 in loc. cit.), where \( X \) is the Galois group of the maximal abelian unramified pro-\( p \) extension of \( \mathbb{Q}(\zeta_{Np}) \). Thus, the conjecture of \( \varpi_\theta \) being an isomorphism is a refinement of the Iwasawa main conjecture and describes the \( \Lambda_\theta \)-module structure of \( X_\theta \). Some partial results on this conjecture were proved by Fukaya–Kato loc. cit., Fukaya–Kato–Sharifi [7], and Wake–Wang-Erickson [15]. Their ideas are to prove that the homomorphism \( \Upsilon_\theta \) is an isomorphism and is the inverse of \( \varpi_\theta \). The requirement of \( \theta \) being not exceptional is essential to the construction of \( \Upsilon_\theta \) as in this case, one can apply a work of Ohta [12] to show that one has a short exact sequence of \( \Lambda_\theta[G_\mathbb{Q}] \)-modules

\[
0 \to H_\theta/I_\theta^* H_\theta \to H_\theta/I_\theta^* H_\theta \to H_\theta^+/I_\theta^* H_\theta^+ \to 0.
\]

When \( \theta \) is exceptional, there is no literature discussing the homomorphism \( \varpi_\theta \). One of the difficulties in this case is that it is not clear whether one of \( H_\theta/I_\theta^* H_\theta \) and \( H_\theta^+/I_\theta^* H_\theta^+ \) is \( G_\mathbb{Q} \)-stable, and hence, one can not construct \( \Upsilon_\theta \) following Sharifi’s construction.

One of the goals in this paper is to study the homomorphism \( \varpi_\theta \) when \( \theta \) is exceptional. We will prove that it is an isomorphism after taking localization at a certain height one prime of \( \Lambda_\theta \) corresponding to the trivial zero of the Kubota–Leopoldt \( p \)-adic L-function \( L_p(s, \theta^{-1} \omega^2) \) without constructing \( \Upsilon_\theta \) (Theorem 1.1). It is known that the leading coefficient of this \( p \)-adic L-function involves a certain \( L \)-invariant. Since the Sharifi’s conjecture is a refinement of the Iwasawa main conjecture, it is nature to ask how such an \( L \)-invariant is related to Sharifi’s conjecture. This question will be addressed in Theorem 1.2 below.

Another goal is to study the residual Galois representation attached to \( H_\theta = H_\theta^{-} \oplus H_\theta^{+} \) after taking the same localization (Theorem 1.3). An advantage of taking such localization is that the image of \( \varpi_\theta \) belongs to a certain local Galois cohomology so that we are able to compute it explicitly. Another advantage is that the cuspidal Hecke algebra is Gorenstein by a work of Betina–Dimitrov–Pozzi [1], which makes the study of the cohomology of modular curves modulo Eisenstein ideals easier (for example, see Proposition 3.5). These two advantages are essential in our study. For example, the former and the later respectively help us to show the surjectivity and the injectivity of \( \varpi_{\theta,p} \) in Theorem 1.1.

To state our results, we let \( \omega \) be the Teichmüller character, \( \kappa : \text{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_N)) \to \mathbb{Z}_p^\times \) be the \( p \)-adic cyclotomic character, \( \gamma \) be a topological generator of \( \text{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{Np})) \) and
θ be a primitive even Dirichlet character of conductor \(Np\) such that the character \(\chi := \theta \omega^{-1}\) is trivial on \((\mathbb{Z}/p\mathbb{Z})^\times\) and \(\chi((\mathbb{Z}/N\mathbb{Z})^\times) = 1\) (this is the definition of \(\theta\) being exceptional in the sense of Ohta). We denote by \(\mathcal{L}(\chi)\) the \(\mathcal{L}\)-invariant attached to \(\chi\) (for example, see [11, (15)] for the definition). Let \(p\) be the prime ideal of \(\Lambda_\theta\) generated by \(T + 1 - \kappa(\gamma)\), which is related to the trivial zero of the Kubota–Leopoldt \(p\)-adic \(L\)-function \(L_p(s, \chi^{-1}\omega)\) (see Section 3.2). We denote by \(\Lambda_{\theta,p}\) the localization of \(\Lambda_\theta\) at \(p\) with residue field \(k_{\theta,p} := \Lambda_{\theta,p}/p\), and for a \(\Lambda_\theta\)-module \(M\), we set \(M_p := M \otimes_{\Lambda_\theta} \Lambda_{\theta,p}\). The following is the first main result in this paper.

**Theorem 1.1** (Theorem 2.2). Suppose \(p \geq 5\) with \(p \nmid N\phi(N)\). If \(\theta\) is exceptional, then \(\varpi_\theta\) induces an isomorphism of \(k_{\theta,p}\)-vector spaces

\[
\varpi_{\theta,p} : H_{\theta,p}^* (1)/I_{\theta,p}^* H_{\theta,p}^* (1) \cong S_{\theta,p}.
\]

When \(\theta\) is not exceptional, assuming Greenberg’s conjecture, an analog result of the above theorem has been proved by Wake–Erickson-Wang [15, Corollary C]. Their idea is first to show that for each height one prime ideal \(q\) of \(\Lambda_\theta\), the map \(\Upsilon_{\theta,q}\) is an isomorphism and then, by a work of Fukaya–Kato [6], it is the inverse of \(\varpi_{\theta,q}\). When \(\theta\) is exceptional, their method can be adapted if one consider the localization at height one prime ideals \(q\) other than \(p\). In this case (or in general even without taking any localization), it is difficult to show that \(\varpi_{\theta,q}\) is an isomorphism without using \(\Upsilon_{\theta,q}\), since it is defined by the cup product of two cyclotomic units and is difficult to be computed.

A crucial point in the proof of Theorem 1.1 is that we are able to compute the image of \(\varpi_{\theta,p}\) via local explicit reciprocity law by showing that the image of \(\varpi_{\theta,p}\) is non-trivial in certain local cohomology groups. There are three steps in the proof: (1) Show that both \(H_{\theta,p}^* (1)/I_{\theta,p}^* H_{\theta,p}^* (1)\) and \(S_{\theta,p}\) are 1-dimensionlal \(k_{\theta,p}\)-vector spaces (Propositions 3.2 and 3.5). (2) Show that \((U_q^* - 1)(0, \infty)_{DM,\theta}\) is a basis of \(H_{\theta}^*/I_{\theta}^* H_{\theta}\) whose image under \(\varpi_{\theta}\) in \(S_{\theta}\) is the cup product \((q, 1 - \zeta_{Np^r})_{r \geq 1, \theta}\) for all primes \(q\mid Np\) (Corollary 3.4). Here \((0, \infty)\) is the modular symbol attached to the cusps 0 and \(\infty\). (3) Show that the cup product \((q, 1 - \zeta_{Np^r})_{r \geq 1, \theta}\) is non-zero for all big enough positive integers \(r\), which is the second main result in this article.

**Theorem 1.2** (Corollary 4.2). For each positive integer \(r\), we have

\[
(\ell, 1 - \zeta_{Np^r})_{r, \theta, p} = \frac{(p - 1) \log_p(\ell)}{p \phi(N)} \omega(N) \tau(\chi^{-1}) L(0, \chi) \in (\mathcal{O}/p^{r}\mathcal{O}) (1)
\]

for all \(\ell \nmid N\) and

\[
(p, 1 - \zeta_{Np^r})_{r, \theta, p} = -\frac{(p - 1)}{p \phi(N)} \omega(N) \tau(\chi^{-1}) \mathcal{L}(\chi) L(0, \chi) \in (\mathcal{O}/p^{r}\mathcal{O}) (1).
\]

In particular, \((q, 1 - \zeta_{Np^r})_{r, \theta, p}\) is non-zero for all \(r\) big enough and for all \(q\mid Np\).

By the second step of the proof of Theorem 1.1 mentioned above, one can show that

\[
H_{\theta,p}^*/I_{\theta,p}^* H_{\theta,p}^*\]

is \(G_\Sigma\)-stable by using a \(\Lambda_\theta\)-adic perfect pairing on \(H_\theta\) constructed by Ohta (see Section 5). We show that \(H_{\theta,p}^*/I_{\theta,p}^* H_{\theta,p}^*\) is also \(G_\Sigma\)-stable.
Theorem 1.3 (Theorem [5.1]). Let the notation be as above. Then the following short exact sequence of $k_{\theta,p}[G_Q]$-modules splits

$$0 \to H_{\theta,p}/I_{\theta,p}^*H_{\theta,p} \to H_{\theta,p}/I_{\theta,p}^*H_{\theta,p} \to H_{\theta,p}/I_{\theta,p}^*H_{\theta,p} \to 0.$$ 

If one considers the localization other than $p$ (or the case that $\theta$ is not exceptional), it was shown by Ohta [11, Section 3.3] (see [10, Section 5.3] for not exceptional case) that the corresponding short exact sequence of $G_Q$-modules in Theorem 1.3 does not split and the associated ordinary Galois representation is $p$-distinguished. Thus, one can construct an analog of the homomorphism $\Upsilon_{\theta}$ following Sharifi’s construction. This construction can not be adapted to the situation of Theorem 1.3 since it follows from the assumption $\chi(p) = 1$ that the associated Galois representation is not $p$-distinguished.

1.1. Outline. In Section 2 we briefly review the construction of $\varpi_{\theta}$ following [13] and sketch the strategy of the proof of Theorem 1.1.

In Section 3 we first show that $S_{\theta,p}$ is a 1-dimensional $k_{\theta,p}$-vector space (Proposition 3.2). This phenomena is opposite to the case of $\theta$ being not exceptional. In this case, it is known that $S_\theta \cong X_{K,\Sigma,\chi}(1)$ [13, Lemma 4.11]. Second, we construct some elements in $H_\theta^*$ by computing the congruence modules attached to cohomology of modular curves (Theorem 3.3). This plays an important role in computing the image of $\varpi_{\theta,p}$. When $\theta$ is not exceptional, this was done by Ohta [12, Section 3.5] by showing that the desired congruence module is essentially isomorphic to the congruence module attached to (3.5). His argument can not be adapted when the character is exceptional since in this case, it is not clear whether the short exact sequences (3.4.6) in loc. cit. split as Hecke-modules. Our idea is to use the Drinfeld–Manin modification and a result of Lafferty in [8]. Third, we will show that the source of $\varpi_{\theta,p}$ is a 1-dimensional $k_{\theta,p}$-vector space by a result of Betina–Dimitrov–Pozzi [1] on the Gorenstainess of the cuspidal Hecke algebra after localizing at $p$.

Section 4 is devoted to computing the image of $\varpi_{\theta,p}$ and completing the proof of Theorem 1.1. The main goal of Section 5 is to prove Theorem 1.3.

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2. Sharifi’s Conjecture

Throughout this paper, we will denote by $N$ a positive integer, denote by $p \geq 5$ a prime number not dividing $N\phi(N)$. The goal of this section is to review Sharifi’s conjecture on the map $\varpi$ following [13] and to state Theorem 1.1. We refer the reader to loc. cit. for more details on Sharifi’s conjecture.
2.1. The map $\varpi$. Set $H_r := H^1_{\text{ét}}(X_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)^{\text{ord}}$ and set $\bar{H}_r = H^1_{\text{ét}}(Y_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)$. One can identify $\bar{H}_r$ with a certain relative homology group by Poincaré duality and the comparison between Betti (co)homology groups and étale (co)homology groups. Namely, one has

$$\bar{H}_r(1) \cong H_1(Y_1(Np^r)(\mathbb{C}), C_1(Np^r), \mathbb{Z}_p),$$

where $C_1(Np^r)$ is the set of cusps for $\Gamma_1(Np^r)$ (see Sections 3.4 and 3.5 of loc. cit. for more details). We denote by $\bar{H}_r^+$ (resp. $\bar{H}_r^-$) the subgroup of $\bar{H}_r$ on which the complex conjugation acts trivially (resp. via $-1$) and denote by $H_\eta^+$ and $H_\eta^-$ in the same manner.

Form the discussion in Section 3.1 of loc. cit., the group $H_1(Y_1(Np^r)(\mathbb{C}), C_1(Np^r), \mathbb{Z}_p)^+$ is generated by the adjusted Manin symbols $[u, v]^+$ for all $u, v \in \mathbb{Z}/Np^r\mathbb{Z}$ with $(u, v) = 1$ satisfying the properties (3.3)-(3.7) in loc. cit.. We will identify these symbols as elements of $\bar{H}_r(1)^+$ via (2.1) without further notice. Let $\bar{H}_{r,0}(1)^+$ be the subgroup of $\bar{H}_r(1)^+$ generated by $[u, v]^+$ for all $u, v \in \mathbb{Z}/Np^r\mathbb{Z} = \{0\}$ with $(u, v) = 1$. Let $H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))^\circ$ be defined as at the end of p. 31 of loc. cit. It was proved in Proposition 5.7 of loc. cit. that there exists a homomorphism

$$\varpi_r : \bar{H}_{r,0}(1)^+ \to H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))^\circ,$$

sending $[u, v]^+$ to the cup product $(1 - \zeta_{Np^r}^u, 1 - \zeta_{Np^r}^v)^\circ$ and satisfying $\varpi_r \circ (j)_r = \sigma_j^{-1} \circ \varpi_r$ for all $j \in (\mathbb{Z}/Np^r\mathbb{Z})^\ast$. Here $(j)$ is the diamond operator and $\sigma_j \in \text{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q})$ satisfying $\sigma_j(\zeta_{Np^r}) = \zeta_{Np^r}^j$. Since $\bar{H}_r(1)^+ \cong \bar{H}_r^-$ and since $H^1_{\text{ét}}(X_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)^- (1)$ is contained in $\bar{H}_{r,0}^-(1)$, the homomorphism $\varpi_r$ induces via restriction a homomorphism

$$\varpi_r : H^1_{\text{ét}}(X_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)^-(1) \to H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))^\circ$$

whose image is contained in $H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{p}], \mathbb{Z}_p(2))^\circ$ by [6] Theorem 5.3.5.

Let $I_r^+$ be the ideal of the Hecke algebra $\mathcal{H}_r^+$ (acting on of $\bar{H}_{r,0}$) generated by $T_i^+ - 1 - l^{-1}$ for all $l \nmid Np^r$ and $U_q^+ - 1$ for all $q | Np$, and denote by $I_r^+$ the image of $I_r^+$ in the cuspidal Hecke algebra $h_r^+$ acting on $H^1_{\text{ét}}(X_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)$ under the natural map $\mathcal{H}_r^+ \to h_r^+$. It was conjectured by Sharifi (Conjecture 5.8 in loc. cit.) and proved by Fukaya–Kato [6] Theorem 5.2.3] that the map $\varpi_r$ satisfies

$$\varpi_r(\eta x) = 0$$

for all $\eta \in I_r^+$ and $x \in \bar{H}_{r,0}(1)$. Moreover, they also proved that the following diagram commutes:

$$\begin{array}{ccc}
H^1_{\text{ét}}(X_1(Np^{r+1})\mathbb{Q}, \mathbb{Z}_p)^-(1) & \xrightarrow{\varpi_{r+1}} & H^2(\mathbb{Z}[\zeta_{Np^{r+1}}, \frac{1}{Np}], \mathbb{Z}_p(2))^\circ \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(X_1(Np^r)\mathbb{Q}, \mathbb{Z}_p)^-(1) & \xrightarrow{\varpi_r} & H^2(\mathbb{Z}[\zeta_{Np^r}, \frac{1}{Np}], \mathbb{Z}_p(2))^\circ.
\end{array}$$
where the left and right vertical maps are the trace map and the norm map, respectively. This diagram induces a map by taking projective limit
\[
\varpi : \lim_{\rightarrow} H^1_{et}(X_1(Np^\alpha)_{\mathbb{Q}}, \mathbb{Z}_p)^{-}(1) \to \lim_{\rightarrow} H^2(\mathbb{Z}[\zeta_{Np^\alpha}, \frac{1}{p}], \mathbb{Z}_p(2))^\varphi =: S.
\]
It follows from (2.2) that this map factors through the quotient by \( I^* \coloneqq \lim_{\rightarrow} I_r^* \). Since \( H^1_{et}(X_1(Np^\alpha)_{\mathbb{Q}}, \mathbb{Z}_p)/I_r^* H^1_{et}(X_1(Np^\alpha)_{\mathbb{Q}}, \mathbb{Z}_p) \) is isomorphic to \( H_r/I_r^* H_r \) for all \( r \in \mathbb{Z}_{\geq 1} \), we obtain a homomorphism
\[
\varpi : H^-(1)/I^* H^-(1) \to S.
\]
Set \( \Lambda := \mathbb{Z}_p[[\lim(\mathbb{Z}/Np^\alpha\mathbb{Z})^\times]] \cong \mathbb{Z}_p[[\mathbb{Z}/Np\mathbb{Z})^\times][(1+p\mathbb{Z}_p)] \). Both \( H^-(1) \) and \( S \) are \( \Lambda \)-modules on which \( \lim(\mathbb{Z}/Np^\alpha\mathbb{Z})^\times \) acts respectively via diamond operators and Galois actions. Moreover, the map \( \varpi \) satisfies \( \varpi \circ (j) = \sigma_j^{-1} \circ \varpi \) for all \( j \in (\mathbb{Z}/Np\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \). The following conjecture is the conjecture of McCallum-Sharifi in [9].

**Conjecture 2.1.** The \( \Lambda \)-module homomorphism \( \varpi : H^-(1)/I^* H^-(1) \to S \) is surjective.

### 2.2. Main result.

For a Dirichlet character \( \theta \) of modulus \( Np \), we denote by \( H_{\theta} \) the subgroup of \( H^* \) such that the action of \((\mathbb{Z}/Np\mathbb{Z})^\times \) is via \( \theta \) and denote by \( S_{\theta} \) in the same manner. Let \( \mathcal{O} \) be a finite extension of \( \mathbb{Z}_p \) containing values of \( \theta \) and set \( \Lambda_{\theta} := \Lambda \otimes_{\mathbb{Z}_p((\mathbb{Z}/N\mathbb{Z})^\times)} \mathcal{O} \). Let \( K = \mathbb{Q}(\zeta_{Np}) \). We fix a topological generator \( \gamma \) of \( \text{Gal}(K/\mathbb{Q}(\zeta_N)) \) and identify \( \Lambda_{\theta} \) with \( \mathcal{O}[T] \) via the map \( \gamma \mapsto 1 + T \). Denote by \( \kappa : \text{Gal}(K/\mathbb{Q}(\zeta_N)) \to \mathbb{Z}_p^* \) the universal cyclotomic character. For a height 1 prime \( p \) of \( \Lambda_{\theta} \), we will denote by \( \Lambda_{\theta, p} \) the localization at \( p \) and denote by \( k_{\theta, p} \) the residue field of \( \Lambda_{\theta, p} \).

It follows from the assumption \( p \nmid N\phi(N) \) that one has a decompositions \( H^-(1) = \bigoplus_{\theta} H_{\theta}^{-}(1) \) and \( S = \bigoplus_{\theta} S_{\theta} \). Here, the decompositions run through all even Dirichlet characters of modulus \( Np \). Thus, Conjecture (2.1) is equivalent to the surjectivity of
\[
(2.3) \quad \varpi_{\theta} : H_{\theta}^{-}(1)/I_{\theta} H_{\theta}^{-}(1) \to S_{\theta}
\]
for all even Dirichlet characters \( \theta \) of modulus \( Np \). The following theorem gives a partial result of the conjecture when \( \theta \) is exceptional.

**Theorem 2.2.** Suppose \( p \nmid N\phi(N) \). Let \( p \) be the height 1 prime of \( \Lambda_{\theta} \) generated by \( 1 + T - \kappa(\gamma) \), and let other notation be as above. If \( \theta \) is exceptional, then after taking localization at \( p \), the \( \Lambda_{\theta} \)-module homomorphism \( \varpi_{\theta} \) induces an isomorphism of \( \Lambda_{\theta, p} \)-modules
\[
(2.4) \quad \varpi_{\theta, p} : H_{\theta, p}^{-}(1)/I_{\theta, p} H_{\theta, p}^{-}(1) \cong S_{\theta, p}.
\]

**Proof.** We will prove in Proposition (3.2) and Proposition (3.5) that \( H_{\theta, p}^{-}(1)/I_{\theta, p} H_{\theta, p}^{-}(1) \) and \( S_{\theta, p} \) are 1-dimensional \( k_{\theta, p} \)-vector spaces. Moreover, we will prove in Corollary (4.3) that \( \varpi_{\theta, p} \) is non-trivial, and thus, it is an isomorphism. \( \blacksquare \)
3. Galois cohomology and cohomology of modular curves

From now on, we assume that the Dirichlet character \( \theta \) is exceptional. The aim of Section 3.1 is to show that \( k_{\theta,p} \)-vector space \( S_{\theta,p} \) is 1-dimensional. In Section 3.2 we construct elements in \( H_2(1) \) by computing congruence modules attached to \( S_{\theta,p} \). In addition, we show that the \( k_{\theta,p} \)-vector space \( H_{\theta,p}(1)/I_{\theta,p}H_{\theta,p}(1) \) is a 1-dimensional. Recall that \( \chi := \theta\omega^{-1} \) and that \( K = \mathbb{Q}(\zeta_{Np^\infty}) \).

3.1. Localization of \( S_{\theta} \) at \( p \). Let \( \Sigma \) be the set of the finite places of \( \mathbb{Q}(\zeta_{Np^\infty}) \) above \( p \), and let \( X_K \) (resp. \( X_{K,\Sigma} \)) be the Galois group of the maximal abelian unramified pro-\( p \) extension of \( K \) (resp. in which all primes above those in \( \Sigma \) split completely).

By [13] Lemma 2.1, We have the following exact sequence

\[
0 \to X_{K,\Sigma} \to \lim_{\rightarrow \Sigma} H^2(Z[\zeta_{Np^\infty}, 1/p], \mathbb{Z}_p(1)) \to \bigoplus_{\nu \in \Sigma} H^2(\mathbb{Q}(\zeta_{Np^\infty})_\nu, \mathbb{Z}_p(1)) \overset{\text{inv}}{\to} \mathbb{Z}_p \to 0.
\]

Note that one has

\[
\lim_{\rightarrow \Sigma} H^2(Z[\zeta_{Np^\infty}, 1/p], \mathbb{Z}_p(2))_\theta = \lim_{\rightarrow \Sigma} H^2(Z[\zeta_{Np^\infty}, 1/p], \mathbb{Z}_p(1))_\chi(1)
\]

and

\[
H^2(\mathbb{Q}(\zeta_{Np^\infty})_\nu, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p
\]

for all \( \nu \in \Sigma \). Denote by \((\oplus_{\nu \in \Sigma} \mathbb{Z}_p)_\theta\) the kernel of inv in \((3.1)\) under the identification \((3.2)\). After taking \( \chi \) components and Tate twist on the exact sequence \((3.1)\), we have the following short exact sequence

\[
0 \to X_{K,\Sigma,\chi}(1) \to S_{\theta} \to (\oplus_{\nu \in \Sigma} \mathbb{Z}_p)_\chi(1) \to 0.
\]

The following lemma shows that the first term in \((3.3)\) is trivial after localizing at \( p \).

**Lemma 3.1.** Let the notation be as above. Then, the characteristic ideal of \( X_{K,\chi}/X_{K,\Sigma,\chi} \) is \( (T) \). Moreover, one has \( X_{K,\Sigma,\chi}(1)_p = 0 \).

**Proof.** From the discussion in the last paragraph of p. 99 in [3] Section 4], one can see that \( X_{K,\chi}/X_{K,\Sigma,\chi} \) is a direct summand of the Pontryagin dual of the direct limit of \( D_n \) (see the definition in loc. cit.) whose characteristic ideal is a power of \( (T) \). Since \( \theta \) is exceptional, it was also shown in loc. cit. that \( (T) \) divides the characteristic ideal of \( X_{K,\chi} \) exactly once. Hence, the characteristic ideal of \( X_{K,\chi}/X_{K,\Sigma,\chi} \) is \( (T) \). Thus, \( X_{K,\Sigma,\chi} \otimes_{\Lambda_{\theta}} \Lambda_{\theta,(T)} = 0 \) which implies the second assertion by taking Tate twist. \( \square \)

**Proposition 3.2.** We have isomorphisms of \( \Lambda_{\theta,p} \)-modules

\[
S_{\theta,p} \cong ((\oplus_{\nu \in \Sigma} \mathbb{Z}_p)_\chi(1))_p \cong \Lambda_{\theta,p}/p.
\]
Proof. The first isomorphism follows from Lemma 3.1 and 3.3. We now prove the second isomorphism. Note that as \( \mathbb{Z}_p[\chi] \)-module, \((\oplus_{v \in S} \mathbb{Z}_p)_{\chi}\) is isomorphic to \( \mathbb{Z}_p[\chi] \). For \( \sigma \in \text{Gal}(\mathbb{Q}(z)/\mathbb{Q}) \), it acts on \((\oplus_{v \in S} \mathbb{Z}_p)_{\chi}(1)\) via \( \omega(\sigma) \). Moreover, the topological generator \( \gamma \) of \( \text{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{Np})) \) acts on \((\oplus_{v \in S} \mathbb{Z}_p)_{\chi}(1)\) via \( \kappa(\gamma) \). Therefore, one has

\[ (\oplus_{v \in S} \mathbb{Z}_p)_{\chi}(1) \cong \Lambda_{\theta}/p, \]

and hence, the assertion follows after taking localization at \( p \).  

3.2. Cohomology of modular curves. The \( \Lambda_{\theta} \)-adic Eisenstein series \( E(\theta, \mathbb{Z}) \) is defined by

\[ 2^{-1}G_{\theta}(T) + \sum_{n=1}^{\infty} \left( \sum_{d|n, d(p)=1} \theta(d)(1 + T)^{s(d)d} \right) e^{2\pi i n z}, \]

where \( s(d) := \frac{\log_p(d)}{\log_p(\gamma)} \) and \( G_{\theta}(T) \in \Lambda_{\theta} \) is the power series expression of the Kubota–Leopoldt \( p \)-adic \( L \)-function \( L_{p}(s, \theta \omega^2) \) satisfying

\[ G_{\theta}(\gamma^s - 1) = L_{p}(-s - 1, \theta \omega^2). \]

Recall that the Kubota–Leopoldt \( p \)-adic \( L \)-function \( L_{p}(s, \psi) \) satisfies the interpolation property

\[ L_{p}(1-k, \psi) = (1 - \psi \omega^{-k}(p)) L(1-k, \psi \omega^{-k}) \]

for all \( k \in \mathbb{Z}_{\geq 0} \) and for all Dirichlet character \( \psi \). Set \( \xi_{\theta}(T) := G_{\theta-1}((1 + T)^{-1} - 1) \). Then, it follows from the assumption \( \theta \omega^{-1}(p) = 1 \) that \( T + 1 - \kappa(\gamma) \) is a zero of \( \xi_{\theta}(T) \), called the trivial zero.

Let \( M_{\text{ord}}(N, \theta; \Lambda_{\theta}) \) be the space of ordinary \( \Lambda_{\theta} \)-adic modular forms of type \( \theta \), and let \( S_{\text{ord}}(N, \theta; \Lambda_{\theta}) \) be the subspace of \( M_{\text{ord}}(N, \theta; \Lambda_{\theta}) \) consisting of \( \Lambda_{\theta} \)-adic cusp forms. Recall that we denote by \( C_{1}(Np^r) \) the set of cusps for \( \Gamma_{1}(Np^r) \). It was proved by Ohta [12] (2.4.4) that one has the following short exact sequence of free \( \Lambda_{\theta} \)-modules

\[ 0 \to S_{\text{ord}}(N, \theta; \Lambda_{\theta}) \to M_{\text{ord}}(N, \theta; \Lambda_{\theta}) \xrightarrow{\text{Res}} e \cdot \Lambda_{\theta}[C_{\infty}] \to 0, \]

where \( e = \lim_{n \to \infty} U_{p}^n \) is the ordinary projector and \( \Lambda_{\theta}[C_{\infty}] := \varprojlim \Lambda_{\theta}[C_{1}(Np^r)] \) for which the projective limit is with respect to the natural projection \( C_{1}(Np^r) \to C_{1}(Np^{s}) \) for \( r \geq s \).

Let \( e_{(\theta, 1)} \in e \cdot \Lambda_{\theta}[C_{\infty}] \) be defined in [8], Proposition 3.2.1] such that \( \Lambda_{\theta} \cdot e_{(\theta, 1)} \) is a direct summand of \( e \cdot \Lambda_{\theta}[C_{\infty}] \) and \( \text{Res}(\mathcal{E}(\theta, 1)) = G_{\theta}(T) \cdot e_{(\theta, 1)} \). Let \( M_{L_{\theta}} \) be the preimage of \( \Lambda_{\theta} \cdot e_{(\theta, 1)} \) under the map \( \text{Res} \), and set \( S_{\Lambda_{\theta}} := S_{\text{ord}}(N, \theta; \Lambda_{\theta}) \). Then we obtain a short exact sequence of free \( \Lambda_{\theta} \)-modules

\[ 0 \to S_{\Lambda_{\theta}} \to M_{\Lambda_{\theta}} \xrightarrow{\text{Res}} \Lambda_{\theta} \cdot e_{(\theta, 1)} \to 0. \]

Let \( \mathcal{H}_{\theta} \) and \( h_{\theta} \) be the Hecke algebras acting on \( M_{\Lambda_{\theta}} \) and \( S_{\Lambda_{\theta}} \), respectively. Also, let \( \mathcal{I}_{\theta} := \text{Ann}_{\Lambda_{\theta}}(\mathcal{E}(\theta, 1)) \) be the Eisenstein ideal, and let \( I_{\theta} \) be the image of \( \mathcal{I}_{\theta} \) under the natural homomorphism \( \mathcal{H}_{\theta} \to h_{\theta} \). Denote by \( s' : M_{\Lambda_{\theta}} \otimes_{\Lambda_{\theta}} Q(\Lambda_{\theta}) \to S_{\Lambda_{\theta}} \otimes_{\Lambda_{\theta}} Q(\Lambda_{\theta}) \) the unique Hecke
equivariant splitting map and set $M_{\Lambda_\theta,DM'} := s'(M_{\Lambda_\theta})$. Here $Q(\Lambda_\theta)$ denotes the quotient field of $\Lambda_\theta$. It was proved in Section 3 of loc. cit. that one has isomorphisms of $\Lambda_\theta$-modules

\[(3.6)\quad M_{\Lambda_\theta,DM'}/S_{\Lambda_\theta} \cong \Lambda_\theta/(G_\theta(T)) \cong h_\theta/I_\theta.\]

Recall that in Section 2, we set $H_\theta := \lim \to H^1_{et}(X_1(Np^r)_{/\mathbb{Q}}, \mathbb{Z}_p)_{\theta}^{\text{ord}}$. By [10, (4.3.12)], one has the short exact sequence of free $\Lambda_\theta[\mathbb{G}_\mathbb{Q}]$-modules

\[(3.7)\quad 0 \to H_\theta \to \lim \to H^1_{et}(Y_1(Np^r)_{/\mathbb{Q}}, \mathbb{Z}_p)_{\theta}^{\text{ord}} \to e \cdot \Lambda_\theta[\mathbb{C}_\infty](-1) \to 0.\]

Let $\widetilde{H}_\theta \subset \lim \to H^1_{et}(Y_1(Np^r)_{/\mathbb{Q}}, \mathbb{Z}_p)_{\theta}^{\text{ord}}$ be the preimage of $\Lambda \cdot e_{(\theta,1)}$. Then, one obtains from (3.7) a short exact sequence of free $\Lambda_\theta[\mathbb{G}_\mathbb{Q}]$-modules

\[(3.8)\quad 0 \to H_\theta \to \widetilde{H}_\theta \to \Lambda_\theta \cdot e_{(\theta,1)}(-1) \to 0.\]

We denote by $\mathcal{H}_\theta^*$ (resp. $h_\theta^*$) the Hecke algebra acting on $\widetilde{H}_\theta$ (resp. on $H_\theta$). Also, we denote by $I_\theta^*$ and $I_\theta$ the corresponding Eisenstein ideals in $\mathcal{H}_\theta^*$ and $h_\theta^*$, respectively. Note that the last map in (3.8) commutes the action of $\mathcal{H}_\theta^*$ on $\widetilde{H}_\theta$ and the action of $\mathcal{H}_\theta$ on $\Lambda_\theta \cdot e_{(\theta,1)}(-1)$. Moreover, by (3.10), we obtain an isomorphism of $\Lambda_\theta$-modules

\[(3.9)\quad h_\theta^*/I_\theta^* \cong \Lambda_\theta/(\xi_\theta)\]

induced by the canonical isomorphism $h_\theta \cong h_\theta^*$ sending $U_q \to U_q^*$ for all $q|Np$, $T_i$ to $T_i^*$ and $(l)$ to $(l)^{-1}$ for all $l \nmid Np$.

The Drinfeld–Manin modification $\widetilde{H}_{\theta,DM}$ of $\widetilde{H}_\theta$ is defined as $\widetilde{H}_\theta \otimes_{h_\theta^*} h_\theta^*$. For each $r \in \mathbb{Z}_{>1}$, we denote by $s^r : H^1(Y_1(Np^r), \mathbb{Q}_p) \to H^1(X_1(Np^r), \mathbb{Q}_p)$ the Drinfeld–Manin splitting which induces a splitting map $s : \lim \to H^1(Y_1(Np^r), \mathbb{Q}_p) \to \lim \to H^1(X_1(Np^r), \mathbb{Q}_p)$. Set $\widetilde{H}_{\theta,DM'} = s(\widetilde{H}_{\theta})$.

**Proposition 3.3.** Let the notation be as above. We have $\Lambda_\theta$-modules

\[(3.10)\quad \widetilde{H}_{\theta,DM}/H_\theta \cong \Lambda_\theta/(\xi_\theta)\]

and

\[(3.11)\quad \widetilde{H}_{\theta,DM'}/H_\theta \cong \Lambda_\theta/(\xi_\theta)\]

In particular, we have an isomorphism of $\Lambda_\theta$-modules $\widetilde{H}_{\theta,DM'} \cong \widetilde{H}_{\theta,DM}$, and the $\Lambda_\theta/(\xi_\theta)$-module $\widetilde{H}_{\theta,DM}/H_\theta$ is generated by $\{0, \infty\}_{\theta,DM}$, where $\{0, \infty\}$ is the modular symbol attached to the cusps 0 and $\infty$.

**Proof.** We first prove the isomorphism (3.10). Note that one has a natural isomorphism of $\Lambda_\theta$-modules $\mathcal{H}_\theta^*/I_\theta^* \cong \mathcal{H}_\theta/I_\theta \mathcal{H} \cong \Lambda_\theta \cdot e_{\theta,1}$. Thus, the sequence (3.8) yields the following short exact sequence of free $\Lambda_\theta$-modules

\[0 \to H_\theta \to \widetilde{H}_\theta \to \mathcal{H}_\theta^*/I_\theta^* \to 0.\]
By tensoring $h_\phi^*$ over $H_\phi^*$ on the above sequence, one obtains

$$0 \rightarrow H_\theta \rightarrow \tilde{H}_{\theta,DM} \rightarrow h_\theta^*/I_\theta^* \rightarrow 0,$$

which yields (3.10) by (3.11).

Next, we prove the isomorphism (3.11). It follows from [10] Lemma 1.1.4] that the congruence module $\tilde{H}_{\theta,DM'}/H_\theta$ is isomorphic to $\Lambda_\theta/(f)$ for some $f \in \Lambda_\theta$. The Drinfeld–Manin splitting induces a surjective homomorphism of $\Lambda_\theta$-modules $\tilde{H}_{\theta,DM} \rightarrow \tilde{H}_{\theta,DM'}$ (see the proof of [13] Lemma 4.1]), which yields another surjective homomorphism of $\Lambda_\theta$-modules $H_{\theta,DM}/H_\theta \rightarrow H_{\theta,DM'}/H_\theta$.

It follows from (3.10) that $f|_{\xi_\theta}$. We claim that one also has $\xi_\theta|f$, which implies (3.11) by the above discussion. It was proved by Ohta that one has a surjective homomorphisms $\tilde{H}_\theta \rightarrow M_\Lambda$ and $H_\theta \rightarrow S_\Lambda$ on which the action of $T_i^*$ and $U_q^*$ on the left commute with $T_i$ and $U_q$ on the right for all $l \mid Np$ and for all $q|Np$. They induce surjective homomorphisms $\tilde{H}_{\theta,DM} \rightarrow M_{\Lambda,DM'}$ and $\tilde{H}_{\theta,DM'}/H_\theta \rightarrow M_{\Lambda,DM'}/S_\Lambda$. Thus, the claim follows from (3.10). The isomorphism $\tilde{H}_{\theta,DM'} \cong \tilde{H}_{\theta,DM}$ follows from (3.10), (3.11), and the Snake lemma.

Finally, we note that $\{0, \infty\}_\theta$ is part of a basis of $\lim\overset{\leftarrow}{H}^1(Y_1(Np^r)_{et}, \mathbb{Q}, \mathbb{Z})$ whose image under the boundary map is in $\Lambda_\theta \cdot e_{(\theta,1)}$. Therefore, $\{0, \infty\}_\theta$ is a part of basis of $\tilde{H}_{\theta,DM}$, and hence, $\tilde{H}_{\theta,DM}/H_\theta$ is generated by $\{0, \infty\}_\theta$.

The following corollary will be used in Section 4.

**Corollary 3.4.** The elements $\xi_\theta\{0, \infty\}_{DM, \theta}$ and $(U_q^*-1)\cdot\{0, \infty\}_{DM, \theta}$ for all $q|Np$ are in $H_\theta$. Moreover, we have

$$\varpi_\theta((U_q^*-1)\cdot\{0, \infty\}_{DM, \theta}) = (q, 1 - \zeta_{Np^r})_{r \geq 1, \theta} \in S_\theta$$

for all $q|Np$.

**Proof.** Note that $U_q^*-1$ is in $I_\theta^*$ for all $q|Np$. By Proposition 3.3, the proof of the first assertion is essentially the same as the proof of [13] Lemma 4.8]. For $q = p$, (3.12) was proved in [6] Section 10.3] so it reminds to deal with the case $q|N$. Given any $r \in \mathbb{Z}_{>0}$, by the definition of the Hecke actions on the set of cusps (see [11] Section 2.1] for example), we have $(1 - U_q^*) \cdot\{0, \infty\}_r = \sum_{i=1}^{q-1} \frac{1}{q^i} \cdot\{0, \infty\}_r$. From the discussion in [13] Section 3.1] and using (3.1)-(3.3) in loc. cit., one can write the above modular symbols as Manin symbols. Namely, one obtains $\sum_{i=1}^{q-1} \{q, \infty\}_r = \sum_{i=1}^{q-1} [Np^r, 1]_r$, where $N = N/q$. Furthermore, by the definition of $\varpi$ (see Section 2.1), one has

$$\varpi\left(\sum_{i=1}^{q-1} [Np^r, 1]_{r \geq 1}\right) = \sum_{i=1}^{q-1} \left(1 - \zeta_{q^i}, 1 - \zeta_{Np^r}\right)_{r \geq 1} = \left(\prod_{i=1}^{q-1} (1 - \zeta_q^i), \zeta_{Np^r}\right)_{r \geq 1} = \{q, 1 - \zeta_{Np^r}\}_{r \geq 1}.$$

Then, by taking $\theta$-components and taking Drinfeld–Manin modification, (3.12) for $q|N$ follows.
To close this section, we next show that $H^+_{\theta,p}/I^+_{\theta,p}H^-_{\theta,p}$ is a 1-dimensional vector space over $k_{\theta,p}$. It was proved by Ferrero–Greenberg [3] that the trivial zero of Kubota–Leopoldt $p$-adic $L$-functions is a simple zero if it exists. Therefore, $I^+_{\theta,p}$ is the maximal ideal of $h^+_{\theta,p}$ by (3.31) and hence, it coincides with $\mathfrak{p}$, since it was proved by Betina–Dimitrov–Pozzi [1] that $h^+_{\theta,p} = \Lambda_{\theta,p}$. One can deduce from this that $H^+_{\theta,p}$ is a free $h^+_{\theta,p}$-module of rank 2, since it is known that $H^1_{\theta}$ is a torsion free $h^+_{\theta}$-module and $H_{\theta} \otimes_{h^+_{\theta}} Q(h^+_{\theta})$ is a 2-dimensional vector space over $Q(h^+_{\theta})$, where $Q(h^+_{\theta})$ is the quotient field of $h^+_{\theta}$. Therefore, both $H^+_{\theta,p}$ and $H^-_{\theta,p}$ are free $h^+_{\theta,p}$-modules of rank 1. The following proposition follows from the above discussion immediately.

**Proposition 3.5.** One has $\dim_{k_{\theta,p}} H^+_{\theta,p}/I^+_{\theta,p}H^-_{\theta,p} - 1 = \dim_{k_{\theta,p}} H^+_{\theta,p}/I^+_{\theta,p}H^+_{\theta,p}$.

4. Surjectivity of $\varpi_{\theta,p}$

The goal of this section is to show that the image of $\varpi_{\theta,p}$ is non-trivial, which completes the proof of Theorem [22].

We first recall some useful formulas for later use. For a positive integer $a < N$, the partial zeta function associated to $a$ is defined by

$$\zeta_a(Np^r)(s) := \sum_{n \equiv a \mod Np^r} n^{-s}.$$  

It absolutely converges when $\Re(s) > 1$ and admits an holomorphic continuation on $\mathbb{C} - \{1\}$ which has a simple pole at $s = 1$ with residue $1/N$ [5, Section 1.3.1].

**Lemma 4.1.** Let the notation be as above. Then the following assertions hold.

1. If $a = a'p^r$ for some positive integer $a' < N$, then $\zeta_a(Np^r)(s) = p^{-rs}\zeta_{a'}(N)(s)$.
2. For any $g_N \in \mathbb{Z}/N\mathbb{Z}^\times$ and $g_{p^r} \in \mathbb{Z}/p^r\mathbb{Z}^\times$, we have

$$\frac{\zeta_{g_N}^{g_{p^r}}}{\zeta_{g_N}^{g_{p^r}} - 1} = - \sum_{a \equiv g_N \mathbb{Z}/Np^r \mathbb{Z}} \zeta_a(Np^r)(0)(\zeta_{g_N}^{g_{p^r}})^a.$$  

3. For $r \in \mathbb{Z}_{\geq 1}$, one has

$$\sum_{i \in \mathbb{Z}/p^r\mathbb{Z}^\times} \zeta_i^i = \begin{cases} -1 & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Note that for each positive integer $n$ with $n \equiv a \mod Np^r$, one can write $n$ as $a + iNp^r$ for some $i \in \mathbb{Z}_{\geq 0}$. Suppose $a = a'p^r$ for some positive integer $a' < N$. Then, one has

$$\zeta_a(Np^r)(s) = \sum_{n \equiv a'p^r \mod Np^r} n^{-s} = \sum_{i=0}^{\infty} (a'p^r + iNp^r)^{-s} = p^{-rs} \sum_{i=0}^{\infty} (a' + iN)^{-s} = \zeta_{a'}(N)(s).$$

This proves the assertion (1). The assertion (2) follows from [5 Lemma 1.3.15(1)] by taking $r = 1$ and $t = \zeta_{g_N}^{g_{p^r}}$. To prove the assertion (3), it is known that $\sum_{i=0}^{r-1} \zeta_i^i = 0$. When $r = 1$,
this implies that \( \sum_{i=1}^{p-1} \zeta_p^i = -1 \). If \( r \in \mathbb{Z}_{>1} \), then one has
\[
\sum_{p|\ell, 1 \leq i \leq p^{r-1}-1} \zeta_p^i = \sum_{i=1}^{p^{r-1}-1} \zeta_p^i = -1.
\]
which implies \( \sum_{i \in (\mathbb{Z}/p^{r}\mathbb{Z})^*} \zeta_p^i = 0 \).

For each \( r \in \mathbb{Z}_{\geq 1} \) and each \( q|N \), the cup product \( (q, 1 - \zeta_N^{p^{r}} \zeta_{p^{r}})_{r, \theta, p} \) can be identified with the local Hilbert symbol via the first isomorphism in Proposition 3.2. In the following theorem, those Hilbert symbols will be computed by local explicit reciprocity law.

**Theorem 4.2.** For each integer \( r \) and each prime \( \ell|N \), we have
\[
(\ell, 1 - \zeta_N^{p^{r}} \zeta_{p^{r}})_{r, \theta, p} = \frac{(p - 1) \log_p(\ell)}{p \phi(N)} \cdot (\chi(1)^{-1})L(0, \chi) \in (\mathcal{O}/p^{r}\mathcal{O})(1).
\]

*Proof.* Note that the action of \( (\mathbb{Z}/p\mathbb{Z})^* \) on the group of \( p^{r} \)th root of unity is given by \( a \cdot \zeta_{p^{r}} = \zeta_{p^{r}}^{a} \) for all \( a \in (\mathbb{Z}/p\mathbb{Z})^* \). Then, one has
\[
(\ell, 1 - \zeta_N^{p^{r}} \zeta_{p^{r}})_{r, \theta, p} = \phi(Np)^{-1} \sum_{g_N \in \mathbb{D}_p} \chi(g_N^{-1}) \omega(g_p^{-1})(\ell, 1 - \zeta_N^{p^{r}} \zeta_{p^{r}})_{r, \theta, p}.
\]
Here \( g_N \) and \( g_p \) run through all elements in \( (\mathbb{Z}/N\mathbb{Z})^* \) and \( (\mathbb{Z}/p\mathbb{Z})^* \), respectively. Set \( G_N = \mathbb{Z}/p^{r} \mathbb{Z} \). Using the assumption that \( \chi(p) = 1 \), one can rewrite the above summation as
\[
\phi(Np)^{-1} \sum_{G_N, \mathbb{D}_p} \chi(G_N^{-1}) \omega(g_p^{-1})(\ell, 1 - \zeta_N^{p^{r}} \zeta_{p^{r}})_{r, \theta, p}.
\]
Here \( G_N \) runs through all elements in \( (\mathbb{Z}/N\mathbb{Z})^* \), since it follows from the assumption \( p \not| \phi(N) \) that \( p^{r} \in (\mathbb{Z}/N\mathbb{Z})^* \).

Set \( \beta_{Np} = \zeta_N^{G_N \zeta_N^{\omega(g_p)}} - 1 \). The the Coleman power series \( g_{\beta_{Np}}(T) \) associated to \( \beta_{Np} \) is defined by
\[
g_{\beta_{Np}}(T) = \zeta_N^{G_N}(1 + T)^{\omega(g_p)} - 1.
\]
Then, one has \( g_{\beta_{Np}}(\zeta_{p^{r}} - 1) = \beta_{Np} \). Set \( \delta g_{\beta_{Np}} := (1 + T)^{\frac{dg_{\beta_{Np}}(T)}{dT}} \times g_{\beta_{Np}}(T)^{-1} \). Then, one has
\[
\delta g_{\beta_{Np}}(\zeta_{p^{r}} - 1) = \omega(g_p) \frac{\zeta_N^{G_N \zeta_N^{\omega(g_p)}}}{\zeta_N^{G_N \zeta_N^{\omega(g_p)}} - 1}.
\]
By local explicit reciprocity law [4, Theorem 8.18] or [2, Theorem 1.4.2] (taking \( \lambda(X) = 1 + X \)) one can write (4.1) as
\[
\frac{1}{p^r \phi(Np)} \sum_{G_N, \mathbb{D}_p} \chi(G_N^{-1}) \omega(g_p^{-1}) \cdot \text{Tr}_{\mathbb{Q}_p(\mu_{p^{r}})/\mathbb{Q}_p} \left( \log_p(\ell) \cdot \omega(g_p) \frac{\zeta_N^{G_N \zeta_N^{\omega(g_p)}}}{\zeta_N^{G_N \zeta_N^{\omega(g_p)}} - 1} \right).
\]
One can simplify (4.2) by computing the trace as
\[
\frac{1}{p^r \phi(Np)} \sum_{G_N, \mathbb{D}_p} \chi(G_N^{-1}) \sum_{g_{p^{r}} \in (\mathbb{Z}/p^{r}\mathbb{Z})^*} \frac{\zeta_N^{G_N \zeta_N^{\omega(g_p)}}}{\zeta_N^{G_N \zeta_N^{\omega(g_p)}} - 1}.
\]

Note that for a fixed \( g_p \in (\mathbb{Z}/p^r\mathbb{Z})^\times \), \( \omega(g_p) \cdot (\mathbb{Z}/p^r\mathbb{Z})^\times = (\mathbb{Z}/p^r\mathbb{Z})^\times \). Hence, by setting \( G_{p^r} = \omega(g_p)g_{p^r} \), one can simplify \( 4.3 \) as

\[
(4.4) \quad \frac{(p-1) \log_p(\ell)}{p^r \phi(Np)} \sum_{G_N} \chi(G_N^{-1}) \sum_{G_{p^r} \in (\mathbb{Z}/p^r\mathbb{Z})^r} \frac{\zeta_N^{G_N} \zeta_{p^r}^{G_{p^r}}}{\zeta_N^{G_N} \zeta_{p^r}^{G_{p^r}} - 1}.
\]

By Lemma 4.1(3), one can write \( 4.4 \) as

\[
(4.5) \quad \frac{(p-1) \log_p(\ell)}{p^r \phi(Np)} \sum_{G_N, G_{p^r}} \chi(G_N^{-1}) \sum_{a \in \mathbb{Z}/Np^r\mathbb{Z}} \zeta_a(n_{p^r}) \zeta_a^a = \sum_{G_N} \chi(G_N^{-1}) \zeta_a(n_{p^r}) \zeta_a^a = 0,
\]

since \( \sum_{G_N} \chi(G_N^{-1}) = 0 \).

Now, we assume that \( a \) is not divisible by \( N \). By setting \( G_N^p = G_N \cdot a \), we have

\[
\sum_{G_N, G_{p^r}} \chi(G_N^{-1}) \zeta_a(n_{p^r}) \zeta_a^a = \chi(a) \zeta_a(n_{p^r}) \left( \sum_{G_N} \chi(G_N^{-1}) \zeta_a^0 \right) \left( \sum_{G_{p^r}} \zeta_a^a \right) = \chi(a) \zeta_a(n_{p^r}) \tau(\chi^{-1}) \sum_{G_{p^r}} \zeta_a^a.
\]

- If \( a \) is divisible by \( p^i \) for some \( 0 \leq i \leq r-1 \), by Lemma 4.1(5), one has \( \sum_{G_N} \zeta_a^a = 0 \).
- If \( a \) is divisible by \( p^r \), namely, \( a = np \) for some \( 1 \leq i \leq N-1 \), then by Lemma 4.1(1) and using the assumption that \( \chi(p) = 1 \), one has

\[
\chi(a) \zeta_a(n_{p^r}) \tau(\chi^{-1}) \sum_{G_{p^r}} \zeta_a^a = \chi(p^r) \tau(\chi^{-1}) \chi(i) \zeta_i(N)(0).
\]

It is known that the Dirichlet \( L \)-function \( L(s, \chi) \) satisfies

\[
L(s, \chi) = \sum_{a=1}^{N-1} \chi(a) \zeta_a(N)(s)
\]

and has analytic continuation to whole complex plane as \( \chi \) is not a trivial character (for example, see [16, Ch. 4]). From the above discussion, one can simplify \( 4.3 \) as

\[
\frac{\phi(p^r)(p-1) \log_p(\ell)}{p^r \phi(Np)} \tau(\chi^{-1}) \sum_{n \in \mathbb{Z}/N\mathbb{Z}} \chi(a) \zeta_a(n)(0) = \frac{(p-1) \log_p(\ell)}{p\phi(N)} \tau(\chi^{-1}) L(0, \chi).
\]

**Corollary 4.3.** We have

\[
(4.6) \quad (\ell, 1 - \zeta_{Np^r})_{r, p, \theta} = \frac{(p-1) \log_p(\ell)}{p\phi(N)} \omega(N) \tau(\chi^{-1}) L(0, \chi) \in (O/p^rO)(1)
\]
for all $\ell | N$ and

\[
(p, 1 - \zeta_{Np^r})_{r, \theta, p} = \frac{(p-1)}{p\phi(N)} \omega(N) \tau(\chi^{-1}) L(\chi) L(0, \chi) \in (\mathcal{O}/p^r \mathcal{O})(1).
\]

In particular, for all prime $q | Np$, $(q, 1 - \zeta_{Np^r})_{r, \theta, p}$ is non-zero for all $r$ big enough, and hence, the map $\varpi_{\theta, p}$ is surjective.

Proof. Let $p_{r, N}$ be the inverse of $p^r$ in $(\mathbb{Z}/N\mathbb{Z})^\times$. For each $r \in \mathbb{Z}_{\geq 1}$, we write $\zeta_{Np^r} = \exp(2\pi i / N p^r)$ and $\zeta_{N}^{p^r} \zeta_{p^r} = \exp(2\pi i p_{r, N} / N + 2\pi i / p^r)$. Then, for each $\ell | N$, one has

\[
(\ell, 1 - \zeta_{Np^r})_{r, \theta, p} = (\ell, 1 - (\zeta_{N}^{p^r} \zeta_{p^r})^{1/(p_{r, N}p^r + N)})_{r, \theta, p} = \theta(p_{r, N}p^r + N)^{-1} (\ell, 1 - (\zeta_{N}^{p^r} \zeta_{p^r}))_{r, \theta, p}.
\]

Note that one can simplify $\theta(p_{r, N}p^r + N)^{-1}$ as $\omega(N)$. Thus, (4.6) follows from Theorem 4.2 and hence, it is non-zero for all $r$ big enough. (4.7) is obtained by (3.12) and by a consequence of [1] Proposition 5.7, which asserts that for all prime $\ell$ divides $N$, one has

\[
(U_p^* - 1) / (U_\ell^* - 1) = -L(\chi) \log_p(\ell)^{-1} \in \text{h}_{\theta, p}^* / I_{\theta, p}^*,
\]

where $L(\chi) = 0$ is the $L$-invariant attached to $\chi$. \hfill \ensuremath{\blacksquare}

5. Galois representations attached to cohomology of modular curves

Throughout this section, we set $h_\theta^* := h_{\theta, p}^*$ and $I_\theta^* := I_{\theta, p}^*$ for simplicity. Let $\Lambda_\theta^\#$ be $\Lambda_\theta$ on which $G_\mathbb{Q}$ acts as follows. For $\sigma \in G_\mathbb{Q}$, $\sigma$ acts as the multiplication by $\theta^{-1}(\sigma)(\sigma)^{-1}$, where $\{\sigma\} := \{a\} \in \Lambda_\theta$ for some $a \in 1 + p\mathbb{Z}_p$ satisfying $\sigma(\zeta_{p^r}) = \zeta_{p^r}^a$ for all $r \geq 1$. Recall that we have seen in Section 3.2 that $h_\theta^* = \Lambda_\theta^\#, I_\theta^* = \mathfrak{p}$, and both $H_{\theta, p}^*$ and $H_{\theta, p}^\#$ are free $h_\theta^*$-modules of rank 1. It was shown in the proof of [1] Proposition 2.1 that the Galois representation attached to $H_{\theta, p}^* / H_{\theta, p}^\#$ is reducible modulo $(I_\theta^*)^2$. Therefore, by fixing a basis of $H_{\theta, p}^* \oplus H_{\theta, p}^\#$ over $h_\theta^*$, this Galois representation modulo $I_\theta^*$ can be realized as either an upper triangular matrix, a lower triangular matrix, or a diagonal matrix. The goal of this section is to determine which one is the case.

Since the Galois representation attached to $H_{\theta, p}^* / I_\theta^* H_{\theta, p}^* \oplus H_{\theta, p}^\# / I_\theta^* H_{\theta, p}^\#$ is reducible, at least one of $H_{\theta, p}^* / I_\theta^* H_{\theta, p}^*$ and $H_{\theta, p}^\# / I_\theta^* H_{\theta, p}^\#$ is stable under the action of $G_\mathbb{Q}$. We start by showing that the minus part is Galois stable. Let

\[
(\cdot, \cdot)_{\Lambda_\theta} : H_\theta \times H_\theta \to \Lambda_\theta
\]

be the perfect pairing constructed by Ohta satisfying

\[
(\sigma x, \sigma y) = \kappa(\sigma)^{-1} \theta^{-1}(\sigma)(\sigma)^{-1}(x, y)
\]

for all $\sigma \in G_\mathbb{Q}$ (see [6] Section 1.6.3 for the definition). Recall that we have shown in Corollary 3.4 that $\xi_\theta(0, \infty)_{\theta, DM}$ is in $H_\theta$. By the same argument as in Section 6.3.8 of loc. cit., the above pairing induces a $\Lambda_\theta[G_\mathbb{Q}]$-module homomorphism

\[
(\cdot, \xi_\theta(0, \infty)_{\theta, DM})_{\Lambda_\theta} : H_\theta / I_\theta^* H_\theta \to \Lambda_\theta^\# / (\xi_\theta).
\]
Let \( \mathcal{P} \) be the kernel of the above homomorphism after localizing at \( p \), and let

\[
Q := (H_{\theta,p}/I^*H_{\theta,p})/\mathcal{P} \cong \Lambda_{\theta,p}^\#/(\xi_{\theta,p}) \cong \Lambda_{\theta,p}^\#/p
\]

be the quotient. Indeed, one has \( \mathcal{P} \cong (H_{\theta,p}/I^*H_{\theta,p})(-1) \) and \( Q \cong H_{\theta,p}^+/I^*H_{\theta,p}^+ \) as \( k_{\theta,p}[G_Q] \)-modules, since \( \xi_{\theta,p}(0,\infty) \) is a basis of \( H_{\theta,p}/I^*H_{\theta,p} \) over \( k_{\theta,p} \). Thus, we obtain a short exact sequence of \( k_{\theta,p}[G_Q] \)-modules

\[
0 \to H_{\theta,p}^+/I^*H_{\theta,p}^+ \to H_{\theta,p}/I^*H_{\theta,p} \to H_{\theta,p}^+/I^*H_{\theta,p} \to 0.
\]

As in [6, Section 9.6], we have an exact sequence of \( k_{\theta,p}[G_Q] \)-modules

\[
0 \to Q \to \frac{H_{DM,\theta,p}}{(\ker: H_{\theta,p} \to Q)} \to H_{DM,\theta,p} \to 0
\]

which gives an extension class in \( H^1(Z[1/\Lambda_{\theta,p}^\#/(\xi_{\theta,p}))(1)) \) by (3.8) and (3.10). Since \( I^* = p \) is a principle ideal, one obtains from (5.3) by tensoring \( I^*/(I^*)^2 \) an exact sequence of \( k_{\theta,p}[G_Q] \)-modules

\[
0 \to I^*H_{\theta,p}^+/I^2H_{\theta,p}^+ \to (I^*H_{\theta,p}^+ \oplus H_{\theta,p}^+)/I^*(I^*H_{\theta,p}^+ \oplus H_{\theta,p}^+) \to H_{\theta,p}^+/I^*H_{\theta,p} \to 0.
\]

The following theorem describes the Galois representations attached to (5.2) and (5.4).

**Theorem 5.1.** The short exact sequence (5.4) does not split as \( k_{\theta,p}[G_Q] \)-modules, and the short exact sequence (5.2) splits as \( k_{\theta,p}[G_Q] \)-modules.

**Proof.** To prove the first assertion, by the above discussion, it suffices to show that the extension class (5.3) is non-trivial. It is known [6, Theorem 9.6.3] that the desired extension class coincides with the image of the family \((1 - N_{\eta^r})_{\epsilon_1} \theta_{p} \) under the canonical homomorphism \( \lim_{\epsilon_1} (Z[1/\Lambda_{\theta,p}^\#/(\xi_{\theta,p}))(1)) \) induced by the short exact sequence

\[
0 \to \Lambda_{\theta,p}^\#(1) \to \Lambda_{\theta,p}(1) \to \Lambda_{\theta,p}^\#/\xi_{\theta,p}(1) \to 0
\]

and the fact that \( \lim_{\epsilon_1} (Z[1/\Lambda_{\theta,p}^\#/(\xi_{\theta,p}))(1)) \) is not trivial. Let \( U_{\theta} \) be the \( \theta \) eigenspace of the project limit of local unit and \( C_{\theta} \) be the image of cyclotomic unit in \( U_{\theta} \). By [14, Proposition 5.2(a)(ii)] , we know that \( U_{\theta,p}/p \cong H^1(Q[1], \Lambda_{\theta,p}^\#/(\xi_{\theta}))(1)) \) is not trivial. Note that both of these vector spaces are dimension two. By Theorem 3.1(2) in loc. cit., the image of \( C_{\theta} \) in \( U_{\theta,p}/p \) is not trivial, which proves the first assertion.

To prove the second assertion, let \((e_-, e_+) \) be a basis of \( H_{\theta,p}^+ \oplus H_{\theta,p}^+ \). Recall that the maximal ideal \( p = I^* \) is generated by \( f = T + 1 - \kappa(\gamma) \in \Lambda_{\theta,p} \). Then, it is clear that \((f e_+, e_-) \) is a basis of \( I^*H_{\theta,p}^+ \oplus H_{\theta,p}^+ \). Thus, the second assertion follows from the first assertion as we have seen that the Galois representation attached to \( I^*H_{\theta,p}^+ \oplus H_{\theta,p}^+ \) is reducible modulo \((I^*)^2\).
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