Chaotic polynomial maps

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Abstract.
The existence of Smale horseshoes in a class of random polynomial maps on the plane is studied. The conditions under which there are invariant sets on which the maps are topologically conjugate with the fullshift on two symbols are obtained. The system is chaotic in the sense of both Li-Yorke and Devaney. The dynamical behavior of the random systems can be described by the product of a probability space and the symbolic space with the two-sided fullshift.

1. Introduction
The Hénon map is an important model in the study of the planar polynomial diffeomorphisms. Devaney and Nitecki investigated the conditions under which the real quadratic Hénon map has a hyperbolic invariant set on which it is topologically conjugate to the two-sided fullshift on two symbols [1]. Friedland and Milnor showed that the polynomial diffeomorphisms from the real or complex plane to themselves are either conjugate to a composition of generalized Hénon maps or dynamically trivial [2]. Dullin and Meiss studied the conditions which imply that the real cubic Hénon map has a hyperbolic invariant set [3].

The existence of Smale horseshoes in a class of planar random polynomial maps is studied. The conditions under which there are invariant sets on which the maps are topologically conjugate with the fullshift on two symbols. The systems are chaotic in the sense of both Li-Yorke and Devaney. The dynamics can be described by the product of the probability space and the symbolic space.

2. Chaotic dynamics
In this paper, the following planar polynomial diffeomorphism is considered:

\[ F_a : (x, y) \mapsto (y, a(y - b_1)(y - b_2) - \delta x), \]

where \( 0 < b_1 < b_2 \) and \( \delta \neq 0 \) are real constants, \( a \neq 0 \) is a parameter. For convenience, denote by \( p(y) = a(y - b_1)(y - b_2) \).

A simple example is provided to illustrate the random dynamical system considered here. Consider the twice coin toss model, where \( \{0, 1\} \) represents the \{“head”, “tail”\}, take

\[ \Omega = \{00, 01, 10, 11\}, \quad B := \text{set of all subsets of } \Omega. \]

So, the infinite coin toss model can be taken as

\[ \Omega = \{0, 1\}^\mathbb{Z} \setminus \{0\}, \]
so that a typical point $\omega$ of $\Omega$ is an infinite sequence
$$
\omega = (\ldots, \omega_{-2}, \omega_{-1}, \omega_1, \omega_2, \ldots), \quad \omega_i \in \{0, 1\}, \quad i \in \mathbb{Z} \setminus \{0\}.
$$

Consider the following random dynamical system:
$$
F^m_\omega = \begin{cases} 
F_{w_n} \circ \cdots \circ F_{w_1}, & \text{if } n > 0 \\
\text{Id}, & \text{if } n = 0 \\
F_{w_n}^{-1} \circ \cdots \circ F_{w_{-1}}, & \text{if } n < 0,
\end{cases}
$$
this map is simply denoted by $F_\omega$.

**Theorem 2.1.** Consider the following situations:

(i) $a < 0$ and $\delta > 0$;

(ii) $a < 0$ and $\frac{b_2}{b_1} > -\delta > 0$;

(iii) $a > 0$ and $\delta < -\frac{b_2}{b_1}$.

In the above situations, one has that for any fixed $\delta$, there is a sufficiently large number $N_\delta$, such that for any parameters $a_i$, $1 \leq i \leq m$, satisfy the above assumptions and $|a_i| \geq N_\delta$, $1 \leq i \leq m$. Consider the infinite sequence space $\Omega = \{a_1, \ldots, a_m\}^{\mathbb{Z} \setminus \{0\}}$. For any $\omega = (\ldots, \omega_{-2}, \omega_{-1}, \omega_1, \omega_2, \ldots) \in \Omega$, the random dynamical system $F_\omega$ has a Smale horseshoe and a uniformly hyperbolic invariant set $\Lambda_\omega$ such that $F_\omega : \Lambda_\omega \rightarrow \Lambda_\omega$ is topologically conjugate to $\sigma : \sum_2 \rightarrow \sum_2$.

To prove the theorem, several lemmas are needed.

In the following discussions, fix $\delta$ and a constant $\lambda > \max\{|\delta|^{1/2}, 1\}$. Denote
$$
M_0 := \max\{\lambda + |\delta|, 1 + \lambda |\delta|\}, \quad S := [b_1, b_2] \times [b_1, b_2].
$$

Let the vertices of the rectangle $U_1 = (b_1, b_1)$, $U_2 = (b_2, b_1)$, $U_3 = (b_2, b_2)$, and $U_4 = (b_1, b_2)$. Set
$$
[U_1, U_2] := [b_1, b_2] \times \{b_1\}, \quad [U_2, U_3] := \{b_2\} \times [b_1, b_2],
\quad [U_3, U_4] := [b_1, b_2] \times \{b_2\}, \quad [U_4, U_1] := \{b_1\} \times [b_1, b_2].
$$

It is easy to obtain that in Cases (i)-(iii),
$$
F_a([U_1, U_2]) \cap S = \emptyset, \quad F_a([U_3, U_4]) \cap S = \emptyset,
$$
$$
F_a^{-1}([U_1, U_4]) \cap S = \emptyset, \quad F_a^{-1}([U_2, U_3]) \cap S = \emptyset.
$$

**Lemma 2.1.** In Cases (i)-(iii) of Theorem 2.1, for fixed $\delta$ and sufficiently large $|a|$, one has that

(a) $F_a(S) \cap S$ and $F_a^{-1}(S) \cap S$ have two nonempty connected components respectively;

(b) for any $z = (x, y) \in S \cap F_a^{-1}(S)$, $|p'(y)| > M_0$;

and for any $z = (x, y) \in S \cap F_a(S)$, $|p'(x)| > M_0$. 


Proof. Take \( y_0, y'_0 \in (b_1, b_2) \) with \( y_0 < \frac{b_1 + b_2}{2} < y'_0 \) and \( p(y_0) = p(y'_0) \), so \( |p'(y)| > 0 \) for all \( y \in (b_1, y_0] \cup [y'_0, b_2) \). Set \( m_0 := (y_0 - b_1)(b_2 - y_0) \).

Now, we show (a).

For Case (i), let \( |a| > \frac{(1+\delta)b_2}{m_0} \); for Case (ii), let \( |a| > \frac{b_2 + \delta b_0}{m_0} \); for Case (iii), let \( |a| > \frac{|b_1 + \delta b_2|}{m_0} \), then \( F_a(S) \cap S \) and \( F_{a-1}^{-1}(S) \cap S \) have two non-empty connected components, respectively.

Now, we prove (b).

Since \( |p'(y)| > 0 \) for all \( y \in (b_1, y_0] \cup [y'_0, b_2) \), there is \( N_\delta > 0 \) such that (b) holds, where we need to suppose that \( |a| \) is large enough such that (a) holds.

The whole proof is complete. \( \square \)

For any \( \omega \in \Omega \), denote by

\[
\Lambda_\omega = \left( \bigcap_{-\infty}^{+\infty} F_\omega^j(S) \right).
\]

By induction, one has that \( F_\omega: \Lambda_\omega \rightarrow \Lambda_\omega \) is topologically semi-conjugate to \( \sigma: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \) for any \( a \) satisfying the above assumptions in Cases (i)-(iii), respectively.

For the given constant \( \lambda > 1 \) and any point \( z \in \Lambda \), the cones \( C^u(z) \) and \( C^s(z) \) are defined as follows:

\[
C^u(z) = \{ (\xi, \eta) \in T_z\mathbb{R}^2 : |\xi| \leq \lambda^{-1} |\eta| \},
\]

\[
C^s(z) = \{ (\xi, \eta) \in T_z\mathbb{R}^2 : |\eta| \leq \lambda^{-1} |\xi| \}.
\]

For any vector \( (\xi, \eta) \in T_z\mathbb{R}^2 \), the norm \( |(\xi, \eta)|_* = \max\{|\xi|, |\eta|\} \) is used, which measures the larger component of the vector.

**Lemma 2.2.** For the given \( \lambda \) and \( a \) satisfying the above assumptions, the following statements are true:

(a) for all \( z \in S \cap F_a^{-1}(S) \) and \( v \in C^u(z) \), one has that \( DF_a(z)v \in C^u(F_a(z)) \) and \( |DF_a(z)v|_* \geq \lambda |v|_* \);

(b) for all \( z \in S \cap F_a(S) \) and \( v \in C^s(z) \), one has that \( DF_a^{-1}(z)v \in C^s(F_a^{-1}(z)) \) and \( |DF_a^{-1}(z)v|_* \geq \lambda |v|_* \).

**Proof.** For the proof please refer to [4]. \( \square \)

Fix any \( \omega \in \Omega \), and for each \( z \in \Lambda_\omega \), it follows from Lemma 2.2 that for any \( n > 0 \), the set

\[
\left( \bigcap_{j=1}^{n} \left( \prod_{j=1}^{n} DF_{\omega_{-j}}(F_{\omega_{-j}}^{-1}(z))C^u(F_{\omega_{-j}}^{-1}(z)) \right) \right) \bigcap C^u(z)
\]

is a nested set of cones at \( z \). So, the infinite intersection is a nonempty cone, that is,

\[
E_{\omega,z}^u := \left( \bigcap_{j=1}^{+\infty} \left( \prod_{j=1}^{+\infty} DF_{\omega_{-j}}(F_{\omega_{-j}}^{-1}(z))C^u(F_{\omega_{-j}}^{-1}(z)) \right) \right) \bigcap C^u(z) \neq \emptyset.
\]

With a similar discussion, one has

\[
E_{\omega,z}^s := \left( \bigcap_{j=1}^{+\infty} \left( \prod_{j=1}^{+\infty} DF_{\omega_{-j}}^{-1}(z)C^s(F_{\omega_{-j}}^{-1}(z)) \right) \right) \bigcap C^s(z) \neq \emptyset.
\]

**Lemma 2.3.** Fix any \( \omega \in \Omega \), for any \( z = (x, y) \in \Lambda_\omega \), \( E_{\omega,z}^u \) and \( E_{\omega,z}^s \) are lines.
Proof. By applying a similar argument in [4], we could show the statement. □

By Lemmas 2.2 and 2.3, one has the following result.

**Lemma 2.4.** The connected components of $\Lambda_\omega$ are points.

**Proof.** Let $r > 0$ be a sufficiently small constant and $B_z(r) = \{ z' \in \mathbb{R}^2 : |z' - z| < r \}$. Set

$$W^s_r(z) := \left( \bigcap_{j=1}^{\infty} F^{-1}_{\omega_j} \circ \cdots \circ F^{-1}_{\omega_1} (B_{F^j_{\omega_j}(z)}(r)) \right) \cap B_z(r)$$

and

$$W^u_r(z) := \left( \bigcap_{j=1}^{\infty} F^{-1}_{\omega_{-1}} \circ \cdots \circ F^{-1}_{\omega_{-j}} (B_{F^{-j}_{\omega_{-j}}(z)}(r)) \right) \cap B_z(r).$$

Denote by $\text{comp}_z(U)$ the connected component of $U$ containing $z$. By Lemma 2.2, we see that

$$\text{diam}\left\{ \text{comp}_z \left( W^s_r(z) \cap \bigcap_{-n}^n F^j(S') \right) \right\} \leq 2r\lambda^{-n} \to 0 \text{ as } n \to \infty,$$

$$\text{diam}\left\{ \text{comp}_z \left( W^u_r(z) \cap \bigcap_{-n}^n F^j(S) \right) \right\} \leq 2r\lambda^{-n} \to 0 \text{ as } n \to \infty.$$ 

Therefore, for $z \in \Lambda_\omega$, the connected components of $\Lambda_\omega$ are points. This shows that $\Lambda_\omega$ is a Cantor set. □

**Lemma 2.5.** Fix any $\omega \in \Omega$, the map $F_\omega : \Lambda_\omega \to \Lambda_\omega$ is topologically conjugate to $\sigma : \sum_2 \to \sum_2$.

**Proof.** This assertion can be verified by the similar method used in Theorem 4.1 of [4, Page 253]. □

By Lemmas 2.1-2.5, we could obtain the results in Theorem 2.1.

**Remark 2.1.** Similar results could be obtained for general polynomial maps with two distinct positive roots via similar arguments.

**Theorem 2.2.** For the random dynamical systems satisfying the assumptions of Theorem 2.1, the dynamical behavior of the random system can be described by the product of the probability space and the two-sided fullshift on two symbols, i.e., $\Omega \times \sum_2$.

**References**

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