**Improved Diffuse Foreground Subtraction with the ILC Method: CMB Map and Angular Power Spectrum Using Planck and WMAP Observations**

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Abstract

We report an improved technique for diffuse foreground minimization from Cosmic Microwave Background (CMB) maps using a new multiphase iterative harmonic space internal-linear-combination (HILC) approach. Our method nullifies a foreground leakage that was present in the old and usual iterative HILC method. In phase 1 of the multiphase technique, we obtain an initial cleaned map using the single iteration HILC approach over the desired portion of the sky. In phase 2, we obtain a final CMB map using the iterative HILC approach; however, now, to nullify the leakage, during each iteration, some of the regions of the sky that are not being cleaned in the current iteration are replaced by the corresponding cleaned portions of the phase 1 map. We bring all input frequency maps to a common and maximum possible beam and pixel resolution at the beginning of the analysis, which significantly reduces data redundancy, memory usage, and computational cost, and avoids, during the HILC weight calculation, the deconvolution of partial sky harmonic coefficients by the azimuthally symmetric beam and pixel window functions, which in a strict mathematical sense, are not well defined. Using WMAP 9 year and Planck 2015 frequency maps, we obtain foreground-cleaned CMB maps and a CMB angular power spectrum for the multipole range 2 ≤ ℓ ≤ 2500. Our power spectrum matches the published Planck results with some differences at different multipole ranges. We validate our method by performing Monte Carlo simulations. Finally, we show that the weights for HILC foreground minimization have the intrinsic characteristic that they also tend to produce a statistically isotropic CMB map.

**Key words:** cosmic background radiation – cosmology: observations – diffuse radiation

1. Introduction

In recent years, after the observations of Planck (Planck Collaboration et al. 2016a, 2016b, 2016c, 2016d) and WMAP (Bennett et al. 2013; Hinshaw et al. 2013) and other Cosmic Microwave Background (CMB) experiments (Calabrese et al. 2013; Hou et al. 2014; Story et al. 2013), CMB radiation emerged as an important probe to disintegrate a wealth of information about the physics of our universe. The anisotropy field of the CMB intensity Stokes parameter has not only become a probe to determine the values of various cosmological parameters precisely (Hinshaw et al. 2013; Planck Collaboration et al. 2016c), but it can also constrain one of the fundamental concepts on which the current cosmology is based, the mechanism (Peiris et al. 2003; Planck Collaboration et al. 2016h) of the so-called inflation ( Guth 1981; Starobinsky 1982; Linde 1983)—an epoch in the very early universe that is defined by the stretching of any physical length scale in an increasing rate for some brief interval of time, as well as properties related to the geometry and topology of the three-dimensional space (Lachieze-Rey & Lumet 1995; Levin 2002; Comish et al. 2004; Bielewicz et al. 2012; Luminet 2016; Planck Collaboration et al. 2016g). The potential of the CMB to give rise to deep understanding about the physical nature of our universe is so enormous that it can even be used to shed some light on the very basic question: *Are different directions in space identical to each other?* (Copi et al. 2004; Eriksen et al. 2004, 2007a; Samal et al. 2008; Planck Collaboration et al. 2016f). Further, it has been argued that the anisotropy in the CMB polarization Stokes (Q and U) signal can be used to investigate physical problems that are complementary to the temperature-only case (Crittenden et al. 1993, 1995; Spergel & Zaldarriaga 1997), giving rise to additional information. Given the central role that the CMB plays in decoding the physics of our universe, it is important that the CMB signal is estimated by different independent research groups employing independent statistical methods. In this work, we desire to estimate a foreground-cleaned CMB map and its angular power spectrum using observations from the WMAP 9 year and Planck 2015 data release.

One of the important methods to isolate a cleaned signal of CMB anisotropy from the foreground contamination is the so-called internal-linear combination (ILC) approach (Tegmark & Efstathiou 1996; Bennett et al. 2003; Tegmark et al. 2003; Saha et al. 2006, 2008; Saha 2011; Saha & Aluri 2016). Bennett et al. (1992) apply the ILC approach to Cosmic Background Explorer data. Bennett et al. (2003), Eriksen et al. (2004), andGold et al. (2009) apply the ILC algorithm in pixel space to WMAP data. Tegmark & Efstathiou (1996) and Tegmark et al. (2003) propose and implement the ILC method in harmonic space on WMAP data. The harmonic space ILC method was later extended by Saha et al. (2006, 2008) for the CMB angular power spectrum estimation from the WMAP temperature data, removing the detector noise bias. Kim et al. (2008) apply the ILC method in harmonic space on WMAP temperature data to obtain a foreground-minimized CMB map. Samal et al. (2010) and Kim et al. (2009) apply the harmonic space ILC algorithm to WMAP polarization data. Delabrouille et al. (2009) use a needlet space ILC approach for the reconstruction of a CMB map. Basak & Delabrouille (2012, 2013) apply the needlet ILC method to WMAP temperature and polarization data. Rogers et al. (2016a) use the scale-discretized, directional wavelet ILC (SILC) for CMB component separation using Planck maps. Rogers et al. (2016b) use the spin-dependent SILC method for CMB polarization component estimation. Remazeilles et al. (2011a) use a
constrained ILC method to estimate CMB and Sunyaev–Zeldovich signal using simulations of Planck observations. Remazelles et al. (2011b) propose a generalized ILC approach for foreground component separation. Instead of using the usual standard deviation as the measure of contamination, Saha (2011) use a measure of non-Gaussianity given by the sample kurtosis to estimate a foreground-cleaned CMB map and its power spectrum from WMAP observations. The ILC method is unique in that it does not require modeling the foreground components in terms of any templates and relies only upon the assumption that the distribution of CMB photons follows a blackbody distribution along every direction of the sky, so that its temperature (anisotropy) is independent of the frequency. The method uses the principle of simple linear superposition of different frequency maps with certain amplitude terms associated with each frequency (called the weight factors). The ILC method has been extensively used in CMB signal processing for foreground subtraction and hence power spectrum estimation and reconstruction of various foreground components. In some recent publications, some of the authors of this paper have studied the bias properties of the ILC (in harmonic space) power spectrum extensively (Saha et al. 2008) and extended this method to jointly estimate the spectral properties of foregrounds and their templates along with the CMB signal using simulated observations of low resolution polarization Stokes $Q$ maps of the Planck and WMAP missions (Saha & Aluri 2016).

It is interesting to note that a related approach to that of Saha (2011), which minimizes kurtosis, is the Independent Component Analysis (ICA; Hyvärinen & Oja 2000) method. The main idea of the ICA method is to decompose an observed mixture of different signals into components that are maximally independent of each other. The ICA method has been applied in the context of CMB by Baccigalupi et al. (2004).

In the usual ILC method for reconstructing the CMB signal, when performed in spherical harmonic space in an iterative fashion, wherein each iteration is carried out to clean a given region of the sky, there is a leakage of foreground signal from the region of the sky that is not yet cleaned into the regions that is being currently cleaned at any given iteration. The leakage signal, if unaccounted for, leads to a biased reconstruction of the CMB signal. In this work, we improve the usual ILC approach by employing a technique that prevents the leakage phenomena completely. We further extend the usual ILC method so that during the iterative ILC technique in multipole space, the partial sky power spectrum is not required to be deconvoluted by dividing it by the Legendre transforms of the azimuthally symmetric beam functions of different frequency maps, which, strictly speaking, is not mathematically well defined in a rigorous sense. The method is also flexible enough, so that if one desires to mask off certain regions of the sky that may be strongly contaminated, one can do so at the onset of the foreground removal. Masking the contaminated regions to begin with leads to the important advantage that the ILC weights are not affected by the contaminated regions and provides a better foreground-minimized CMB map for the rest of the sky.

There have been many attempts in the literature to estimate a foreground-cleaned image of CMB fluctuations. Bunn et al. (1994) and Bouchet et al. (1999) developed a Weiner filtering approach. Bennett et al. (2003), Hinshaw et al. (2007), and Gold et al. (2009, 2011) used a template-cleaning approach. A Markov Chain Monte Carlo-based approach was used for all component reconstruction by Gold et al. (2009, 2011).

Reconstruction of CMB maps along with all other foreground components using the Gibbs sampling approach has been implemented by Eriksen et al. (2007b, 2008a, 2008b).

We organize our paper as follows. In Section 2, we define the basic problem and explain the motivation of the work in Section 3. We briefly review the old iterative ILC algorithm in Section 4. We describe the problems with the old algorithm in Section 5 and their remedy in Section 6. We describe the new ILC algorithm which improves the old approach by completely preventing the foreground leakage in Section 7. In Section 8, we highlight the advantages of the new ILC approach. In Section 9, we place the usual ILC weights in the more general context of estimating a statistically isotropic CMB signal by minimizing some suitable combination of the Bipolar Spherical Harmonics (Hajian & Souradeep 2003, 2006; Hajian et al. 2005), non-zero values of which are indicative of the breakdown of statistical isotropy. We describe the input frequency maps, beam window functions, and point source mask file in Section 10. In Section 11, we discuss the method of dividing the sky into several regions, taking the intensity level of foreground emissions as an indicator for multiple populations of different foreground-emitting sources. We describe the foreground removal and power spectrum estimation method in Section 12. We discuss our results in Section 13. In Section 14, we describe the simulation of the entire foreground removal and power spectrum estimation method. Finally, we conclude and discuss in Section 15.

2. Basic Problem

In this work, we focus on the problem of CMB map reconstruction and CMB angular power spectrum estimation using the multifrequency foreground- (and detector noise-) contaminated CMB maps as observed by the WMAP and Planck satellite missions. As discussed in Section 12, we estimate the CMB angular power spectrum using foreground-minimized CMB maps by employing an improved and iterative ILC algorithm in harmonic space.

3. Motivation

The foremost motivation of this work is to reconstruct the maximum resolution (diffuse) foreground-minimized CMB map and its power spectrum using the foreground-contaminated CMB maps of Planck and WMAP observations by employing an improved version of the ILC algorithm. Since as discussed in Section 1 both CMB maps and its angular power spectrum play a crucial role in decoding the physics of our universe, it is of utmost importance to reconstruct these observables using multiple algorithms and by independent science teams. Our work thus serves as a reconstruction of the CMB map and its angular power spectrum independent of the Planck science team. Overall, our reconstructed CMB map and its angular power spectrum match well the Planck published results.

4. The Old Iterative ILC Algorithm

4.1. Linear Superposition of Data and Weights

Let us assume that we have full sky observations of foreground-contaminated CMB maps at $n$ different frequency bands. Each of these maps is assumed to have the same HEALPix$^4$ pixel resolution $N_{\text{side}}$ parameter. At a given frequency band, $\nu$, where

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$^4$ Hierarchical Equal Area iso-Latitude Pixelization on sphere; see Górski et al. (2005), e.g., for details.
\[ a_{\ell m}^i = B_i^0 P_\ell (a_{\ell m}^c + a_{\ell m}^{f(i)}) + a_{\ell m}^{n(i)}, \]

where \( B_i^0 \) and \( P_\ell \) respectively, denote circularly symmetric beam and pixel window functions of the frequency map \( i \). The pixel window function is independent of frequency \( \nu_i \) since all input maps have the same pixel resolution. \( a_{\ell m}^c \) denotes the CMB signal. \( a_{\ell m}^{f(i)} \) and \( a_{\ell m}^{n(i)} \) respectively represent the net foreground and detector noise contamination at the frequency \( \nu_i \).

Using all the \( n \) available frequency maps, a cleaned map in harmonic space forming a linear superposition of all input frequency maps is defined:

\[ a_{\ell m}^{\text{Cleaned}} = \sum_{i=1}^n w_i B_i^0 P_\ell a_{\ell m}^i, \]

where \( B_i^0 \) denotes the beam window function of the highest resolution frequency map. We note that the pixel window function, \( P_\ell \), cancels in the above equation since all the input maps are assumed to have the same pixel resolution. The amplitude of superposition, \( w_i \), denotes the weight factor for the frequency map \( \nu_i \) for multipole \( \ell \). The weights, at each multipole, for all frequency maps are obtained by minimizing the corresponding angular power spectrum, \( C^{\ell}_{\text{Cleaned}} \), of the cleaned map. As shown in Saha et al. (2008), the choice of weights that minimizes the variance of the cleaned map at each multipole \( \ell \) is given by

\[ W_\ell = \frac{eC^+_{\ell}}{eC^-_{\ell}}. \]

where \( W_\ell = \{ w_1^\ell, w_2^\ell, w_3^\ell, \ldots, w_n^\ell \} \) is a \( 1 \times n \) row vector containing the amplitude of superposition for all frequency bands, \( e = \{ 1, 1, 1, \ldots, 1 \} \) is a \( 1 \times n \) array denoting the shape vector of the CMB in thermodynamic temperature units, \( ^t \) denotes the Moore–Penrose inverse, \( C \) denotes an \( n \times n \) cross-power (covariance) matrix of the \( n \) input frequency maps for multipole \( \ell \). The \( (i, j) \)th element of the covariance matrix is given by

\[ C_{\ell}^{ij} = \frac{(B_i^0)^2}{B_i^0 B_j^0} \sum_{m=-\ell}^{\ell} \frac{a_{\ell m}^c a_{\ell m}^j}{\ell + 1}. \]

The power spectrum of the cleaned map is given by

\[ C_{\ell}^{\text{Cleaned}} = \frac{1}{eC^-_{\ell}} e^{T}. \]

### 4.2. Iterations

In practice, the foreground spectral properties (e.g., synchrotron and thermal dust) depend on sky positions. The efficiency of ILC foreground removal increases if foreground removal is performed in an iterative fashion wherein each iteration cleans a given region of the sky. The iterative foreground removal method in multipole space has been described in Tegmark et al. (2003) and Saha et al. (2008). In this approach, one first divides the sky into a total of \( n_r \) disjoint regions. Starting with the initial set of \( n \) foreground-contaminated frequency maps (we call them initial maps), one then choses to perform foreground removal in a total of \( n_r \) iterations, wherein at iteration \( i \) the \( i \)th region is cleaned. At any given iteration, wherein one cleans any one of the \( n_r \) regions, one performs the following operations successively on the initial maps.

1. Choose the sky region to be cleaned and find the partial sky \( a_{\ell m}^i \) from this region for each of these maps. Use these partial sky \( a_{\ell m}^i \) to obtain weights for linear combination using the partial sky cross-power spectrum (covariance matrix).
2. Expand the full sky initial maps into \( a_{\ell m}^i \) and linearly superpose them in harmonic space using the weights for the currently chosen sky region. Convert the resulting \( a_{\ell m}^i \) to a full sky map, which is best cleaned in the current region.
3. Replace the current region of \( n \) initial maps by the corresponding region of the cleaned map obtained in step 2 above, after properly taking care of beam resolutions of different frequency bands. Use these \( n \) partially cleaned maps as the initial maps for the next iteration.

At the end of \( n_r \) iterations, all \( n_r \) regions becomes foreground cleaned and one obtains \( n \) foreground-cleaned maps for \( n \) frequency bands. The highest resolution cleaned map is the one corresponding to the highest resolution frequency map.

### 5. Problems with the Old Iterative ILC algorithm

The usual iterative ILC algorithm as described in Section 4 has two major problems.

1. First, the iterative ILC method may cause a foreground leakage from regions that are yet not cleaned into the region that is currently being cleaned. Intuitively, the leakage occurs since the \( a_{\ell m}^i \) on the right-hand side of Equation (2) are global quantities and therefore can cause foreground leakage into the region that is being cleaned from the rest of the uncleared region.
2. Second, division by the beam window function to deconvolve the effect of the instrumental beam (Equation (4)), assuming azimuthally symmetric beam functions in pixel space, is not a mathematically well-defined operation for partial sky analysis, since the partial sky \( a_{\ell m}^i \) are coupled due to masking effect and Legendre transform of the beam functions are applicable only to the spherical harmonics of the entire sky. This causes the weights as defined by Equation (3) to be sub-optimal when estimating them from a given sky region in the iterative ILC method.

We present below a rigorous analysis of the above shortcomings and describe our modifications to the usual iterative ILC algorithm so as to overcome both of these two problems. We call the modified algorithm the new iterative ILC algorithm. We first describe the second shortcoming since it is simpler than the first case.

### 6. Remedy of Old ILC Problems

#### 6.1. Beam Deconvolution

Before we discuss the remedy to the second shortcoming above, we first generalize the usual ILC method to handle frequency maps, each of which, in general, has a different \( N_{\text{side}} \) parameter. This can be achieved in the following way. Let the
ith frequency map have a pixel and beam window function $P_i^\ell$ and $B_i^\ell$, respectively. The finite beam and pixel resolution of the input maps will allow a spherical harmonic transform of each map up to a certain maximum multipole $\ell_{\text{max}}$, for the ith frequency map, to be performed. We first bring all the frequency maps to the common beam and pixel resolution of the highest resolution frequency map following

$$a_{i m}^{\ell} = \frac{B_i^0 P_i^0}{B_i^\ell P_\ell} a_{i m}^\ell,$$  \hspace{1cm} (6)

where $a_{i m}^\ell$ represents the spherical harmonic transform of the input map of pixel resolution $N_{\text{side}}$, and is given by Equation (1). $P_i^0$ represents the pixel window function of the highest pixel resolution map and $B_i^\ell$ represents the beam window function of the highest resolution frequency map. We note that the correction to the spherical harmonic coefficients by the ratio of the beam and pixel window functions (Equation (6)) is valid only up to $\ell = 2 \times N_{\text{side, min}}$, where $N_{\text{side, min}}$ represents the pixel resolution parameter of the lower resolution map, so that any inaccuracies resulting in the corrected spherical harmonic coefficients because of incorrectly trying to fill up the missing information due to coarse beam or pixel resolution can be avoided. In practice, to avoid these inaccuracies, for a frequency $\nu$, we perform these corrections up to multipoles given by $\ell_{\text{cut}}$ values, which are even less than $2 \times N_{\text{side, min}}$, as shown in Table 1. We convert the spherical harmonic coefficients of the left-hand side of the above equation to the maps with $N_{\text{side}} = 2048$, which is the largest pixel resolution parameter of the Planck and WMAP maps. In particular, during the (inverse) spherical harmonic transformation for frequency $\nu$, we use spherical harmonic coefficients up to $\ell_{\text{max}}$ (e.g., see Table 1). At this point, when all the input frequency maps have the same beam and pixel resolution functions, the usual ILC method is rendered general enough to handle original frequency maps that have a different pixel resolution parameter.\(^5\)

Once we obtain the sets of $a_{i m}^{\ell}$ that have same beam and pixel resolution for different input frequency maps, we form the cleaned mapping following

$$a_{i m}^{\text{Cleaned}} = \sum_{i} w_i a_{i m}^{\ell},$$  \hspace{1cm} (7)

where $w_i$ are given by

$$w_i = \int_{\ell_{\text{min}}}^{\ell_{\text{max}}} \frac{C_{ij}}{C_{jj}} d\ell,$$  \hspace{1cm} (8)

We note that in Equation (7), for any given $\ell$, the summation over $i$ runs only over those frequency maps for which the condition $\ell \leq \ell_{\text{cut}}$ is satisfied. One can find the cleaned CMB map as given by Equation (7) using the weights following Equation (3), where the elements of the cross-power covariance matrix are given by

$\text{C}^{ij}_{\ell} = \sum_{m = -\ell}^{\ell} a_{i m}^{\ell} a_{j m}^{\ell *}$.

How does our new ILC method remedy the second shortcoming? We notice that it does so since the weights that appear in Equation (7) can now be obtained from Equation (3) using the cross-power covariance matrix of Equation (8), which does not require any division by the beam window function. This is an important improvement over the usual iterative ILC method, since the weights determined at each individual multipole $\ell$ now has a well-defined meaning associated with it.

Apart from the remedy to the aforementioned shortcoming, bringing all maps to a common resolution at the beginning has the additional advantage that any strongly foreground-contaminated region may now be masked out, if desired so, before the foreground minimization starts. This has an important implication in our analysis since this enables us to mask out the localized resolved point sources at the beginning of our high resolution analysis, which leads to better efficiency in removing diffused foreground minimization.

6.2. Preventing Foreground Leakage

In this section, we aim to quantify the foreground leakage analytically. To better understand the leakage phenomenon, we first consider a simple case and discuss the origin of the leakage. This helps us to identify the leakage term. We then investigate the leakage that occurs in usual iterative ILC foreground removal method and propose a methodology to nullify it.

6.2.1. A Simple Case

Let us consider a subdivision of the entire sky into two disjoint regions, region 1 and region 2. The sky signal can be written as $f(\theta, \phi) = f_1(\theta, \phi) + f_2(\theta, \phi)$, where $f_i$ ($i = 1, 2$) is

| Set | K1 | 30 GHz | Q1 | W1 | W2 | V1 | 70 GHz | 100 GHz | 143 GHz | 217 GHz | 353 GHz |
|-----|----|--------|----|----|----|----|--------|--------|--------|--------|--------|
| $S_1$ | (27M, 27S) | 512 | 1024 | 512 | 512 | 512 | 2048 | 2048 | 2048 | 2048 | 2048 |
| $N_{\text{side}}$ | | 290 | 490 | 540 | 740 | 740 | 790 | 2038 | 2440 | 3240 | 3990 |
| $P_{\text{cut}}$ | | | | | | | | | | 3990 | |
| $S_2$ | (28M, 28S) | 512 | 1024 | 512 | 512 | 512 | 2048 | 2048 | 2048 | 2048 | 2048 |
| $N_{\text{side}}$ | | 390 | 490 | 540 | 740 | 740 | 790 | 2038 | 2440 | 3240 | 3990 |
| $P_{\text{cut}}$ | | | | | | | | | | 3990 | |

Note. Planck and WMAP frequency maps used in our work along with their specifications. The top panel shows the specifications for the set $S_1$ and the bottom panel shows the same for the set $S_2$. $DS1$ and $DS2$ respectively represent detector set 1 and detector set 2 for Planck frequency bands, wherever applicable. The second row of each panel shows the the HEALPix pixel resolution parameter $N_{\text{side}}$ of the native WMAP and Planck frequency maps. The third row of each panel shows the maximum values of $\ell$, given by the $\ell_{\text{cut}}$ variable for different input frequency maps for sets $S_1$ and $S_2$.

\(^5\) We note that if one is interested in ILC analysis in low pixel resolution, one can as well choose a lower $N_{\text{side}}$ parameter, instead of the largest one, and the corresponding pixel window function $P_\ell^0$ in Equation (6).
non-zero only for the region \( i \) and zero otherwise. Such a decomposition is valid since the two regions are disjoint and covers the entire sky. We can rewrite this equation in the harmonic space as \( a_{\ell m} = \tilde{a}_{\ell m} + \tilde{a}_{\ell m} \), where \( \tilde{a}_{\ell m} \) and \( \tilde{a}_{\ell m} \) are the spherical harmonic coefficients of \( f_1 \) and \( f_2 \), respectively. That this result in harmonic space is true can be verified in the following way.

1. First, generate a pure CMB map \( f(\theta, \phi) \) from the Planck best-fit LCDM model at \( N_{\text{side}} = 2048 \).
2. Mask \( f \) with the two complementary masks and obtain \( f_1 \) and \( f_2 \).
3. Obtain \( \tilde{a}_{\ell m}^1 \) and \( \tilde{a}_{\ell m}^2 \) from \( f_1 \) and \( f_2 \) up to \( \ell_{\text{max}} = 4096 \).
4. Obtain \( a_{\ell m} = \tilde{a}_{\ell m} + \tilde{a}_{\ell m} \) for all \( \ell \leq \ell_{\text{max}} \).
5. Convert the \( a_{\ell m} \) above to a pixel map \( f'(\theta, \phi) \) again for \( \ell \leq \ell_{\text{max}} \).
6. Verify that \( f = f' \).

We show the difference map \( f(\theta, \phi) - f'(\theta, \phi) \) in the left panel of Figure 1, which is consistent with zero everywhere except at the locations near the poles where spherical harmonic transformations becomes somewhat erroneous due to the fewer number of HEALPix pixels on the HEALPix isolatitude rings.

To introduce the leakage now, we follow steps 1–3 as previously. We now form \( \tilde{a}_{\ell m}^2 = B_i \tilde{a}_{\ell m} \), where \( B_i \) is some filter function that may as well represent the weight factors in the context of the ILC foreground removal algorithm. If we make a map of \( \tilde{a}_{\ell m}^2 \), the resulting map will not only completely occupy region 2, but it will also extend to region 1. We call this leakage. Depending upon the exact form of the filter function, \( B_i \), the leakage may be quite different on a case-to-case basis.

We show the leakage by first forming a map of the spherical harmonic coefficient \( \tilde{a}_{\ell m}^1 + B_i \tilde{a}_{\ell m}^2 \), then subtracting map \( f \) from this map, and finally masking the difference map by the mask corresponding to region 1. Mathematically, the masked difference map in multipole space can be modeled as \( \tilde{a}_{\ell m}^1 + K_{\ell m'} B_i \tilde{a}_{\ell m'}^2 \), where a summation over the repeated indices \( \ell', m' \) are implied. \( K_{\ell m'} \) represents the mode–mode coupling matrix corresponding to region 1, as defined in Hivon et al. (2002). It is easy to note that the term \( K_{\ell m'} B_i \tilde{a}_{\ell m'}^2 \) represents the leakage that occurs from region 2 to region 1. A pictorial representation of the leakage map (from region 2 to region 1) is shown in the right panel of Figure 1 for \( B_i \) computed from a Gaussian beam function of FWHM = 300'.

### 6.2.2. Leakage in the Old Iterative ILC Method

How does foreground leakage occur in the usual and old iterative ILC method? To understand this, let \( a_{\ell m}^{ij} \) denote the spherical harmonic coefficients of the initial map corresponding to frequency \( \nu_j \) at the beginning of the iteration \( i \). The \( a_{\ell m} \) of the full sky cleaned map obtained at step 2 of usual ILC iteration \( i \) is now given by

\[
a_{\ell m}^{ci} = \sum_{j=1}^n w_j^{ij} a_{\ell m}^{ij},
\]

We note that the \( a_{\ell m}^{ij} \) coefficients on the right-hand side are partially cleaned for \( i \geq 2 \) and totally uncleaned for \( i = 1 \). The spherical harmonic coefficients on the left-hand side are cleaner than those of the right-hand side because of the cleaning operation of the current iteration. The partial sky cleaned spherical harmonic coefficients corresponding to the \( i \)th region are given by

\[
\tilde{a}_{\ell m}^{ci} = K_{\ell m'}^{i} a_{\ell m'}^{ci},
\]

where \( K_{\ell m'}^{i} \) represents the mode–mode coupling matrix for the \( i \)th region, and a summation over repeated indices is assumed. The spherical harmonic coefficients of the initial maps at the beginning of iteration \( i \) are composed of three classes. First, the partial sky cleaned spherical harmonic coefficients from the already cleaned regions (counting a total of \( i - 1 \) of them). Second, the spherical harmonic coefficients \( a_{\ell m}^{ci} \) of the region to be cleaned in the current iteration, and third, the spherical harmonic coefficients of the rest of the regions together \( \tilde{a}_{\ell m}^{rest}, \tilde{a}_{\ell m}^{rest}, \) which are to be cleaned in \((i + 1)\)th iteration onwards. Note that the uncleaned partial sky spherical harmonic coefficients depend upon the frequency index \( j \), whereas the spherical harmonic coefficients corresponding to the cleaned region are independent of it. Mathematically, spherical harmonic coefficients of initial maps at the beginning of iteration \( i \) are given by

\[
a_{\ell m}^{ci} = \tilde{a}_{\ell m}^1 + \tilde{a}_{\ell m}^2 + ... + \tilde{a}_{\ell m}^{i-1} + a_{\ell m}^i + \tilde{a}_{\ell m}^{rest},
\]

where the first \( i - 1 \) terms of the right-hand side represent the already cleaned spherical harmonic coefficients. Using
Equation (11) in Equation (9), we obtain
\[ a_{i,m} = \tilde{a}_{i,m}^1 + \tilde{a}_{i,m}^2 + \ldots + \tilde{a}_{i,m}^{i-1} + \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^i + \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{rest,ij}, \]

(12)

where we have used the fact that \( \sum_{j=1}^{n} w_{ij} = 1 \) for each multipole \( \ell \) for any region \( i \). Using Equation (12) in Equation (10), we obtain
\[ \tilde{a}_{i,m} = K_{i,m}^{\text{ij}} \left( \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^i + \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{rest,ij} \right), \]

(13)

where the summation over the repeated indices \( \ell', m' \) are assumed. The spherical harmonic coefficients on the first term inside the bracket contains contributions from CMB (\( \tilde{a}_{i,m}^\text{CMB} \)) and foreground \( \tilde{a}_{i,m}^\text{FG} \) components (and of course, detector noise also, which we ignore for the time being since we are primarily interested in leakage caused by foreground components); hence, \( \tilde{a}_{i,m} = \tilde{a}_{i,m}^\text{CMB} + \tilde{a}_{i,m}^\text{FG} \). Assuming a perfect estimation of weights from the \( i \)th regions so that the term \( \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{ij} \) can be ignored, we obtain from Equation (13)
\[ \tilde{a}_{i,m} = \tilde{a}_{i,m}^\text{CMB} + K_{i,m}^{\text{ij}} \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{rest,ij}. \]

(14)

Clearly, the second term of the right-hand side represents the leakage term. Considering Equation (14), we immediately draw the following inferences.

1. Leakage occurs only from all regions with indices \( > i \) to the current region \( i \), and no leakage occurs from regions \( < i \) to the current region.
2. At any given ILC iteration, when we are cleaning the \( i \)th region of the sky, the leakage is induced only into the \( i \)th region and into no other region.
3. After a region has been cleaned, no modifications to this region is done at any subsequent stage of cleaning.
4. Although \( \tilde{a}_{i,m}^\text{rest,ij} \) contains both the CMB and foreground signal from the uncleaned region, due to the blackbody nature of the CMB on all sky regions, only the uncleaned foreground portion contributes to the leakage since \( K_{i,m}^{\text{ij}} \left( \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{CMB,ij} \right) = 0 \), as the \( i \)th sky region and the rest of the region to be cleaned after the \( i \)th iterations are disjoint. Thus, no CMB signal takes part in the leakage phenomenon.

From the last conclusion above, one can make an important observation. If \( \tilde{a}_{i,m}^\text{rest,ij} = \tilde{a}_{i,m}^\text{rest,ij} \), i.e., if the partial sky spherical harmonics from the uncleaned region are independent of frequency \( i \), then \( K_{i,m}^{\text{ij}} \left( \sum_{j=1}^{n} w_{ij} \tilde{a}_{j,m}^\text{CMB,ij} \right) = 0 \), and hence the leakage will be completely nullified.\(^*\) Based upon this observation, we propose that the new multiphase iterative ILC algorithm be performed in two phases.

7. The New Multiphase Iterative Algorithm

1. **Phase 1.** Perform a single iteration ILC map reconstruction over the sky region desired to be cleaned in harmonic space, using the input frequency maps.
2. **Phase 2.** Now perform the old iterative ILC foreground removal method on the same set of input frequency maps as in the first phase, with one modification in step 2 (e.g., see steps for the old ILC as mentioned in Section 4). Before obtaining the spherical harmonics of the cleaned map over the full sky at any iteration, replace the uncleaned regions of the initial maps with the corresponding region of the single cleaned map obtained in phase 1 above. Since the cleaned map obtained in the first phase is independent of frequency, no foreground leakage will occur in the iterative cleaning during phase 2.

8. Advantages of the New Method

Why should we prefer the new iterative ILC method over the old one? The main benefit of the new method is that it stops foreground leakage completely. Moreover, in the new method, we consider all input frequency maps to be on equal footing as far as resolution is concerned, although initially they might have different beam and pixel resolutions. To convert the resolutions of different input frequency maps to a common resolution, we first upgrade the pixel resolution of all input frequency maps to the highest pixel resolution of \( N_{\text{side}} = 2048 \) and then bring them to the same instrumental beam resolution as the highest resolution map. A detailed description of how to actually do this is described in Section 6.1. Bringing all maps to the common beam and pixel resolution has the advantage that only one cleaned map at the highest beam resolution needs to be formed during each iteration of phase 2. Working with a single pixel resolution also makes it possible to use only a single mask file at the chosen pixel resolution encoding information about different sky regions. These result in a reduction in the net memory requirement and execution time of the code, which is extremely helpful for performing massive Monte Carlo simulations of the method. Moreover, since all frequency maps are in the same beam and pixel resolution at the beginning of the foreground cleaning, we can use Equation (8) instead of Equation (4) to estimate the cross-power covariance matrix in the expression of weights as given by Equation (3) for any region \( i \) of the sky. This avoids any need for the deconvolution of partial sky spherical harmonic coefficients by the beam window function. Finally, bringing all maps to a common resolution to begin with allows one to mask out positions of strongly contaminated regions, if desired. This results in better performance of foreground removal from the rest of the regions since now the weights are not affected by the undesired strongly contaminated region.

9. Relationship of ILC Weights and the Concept of Isotropy

Although Equation (3) was derived (e.g., see Saha et al. 2008) requiring that the angular power spectrum of the cleaned map is minimized subject to the constraint that the CMB is projected out completely, assuming its blackbody spectrum, in this section we show that the usual ILC weights in
harmonic space, as given by Equation (3), which minimize foreground residuals, also have an interesting property. They tend to produce a statistically isotropic (or minimally statistically anisotropic) CMB map by minimizing a suitably chosen measure of statistical anisotropy. Although it is a harder problem than simple variance measurement to define any unique quantitative measure of statistical isotropy, a set of statistics measuring statistical anisotropy as given has been proposed by Hajian & Souradeep (2003, 2006) and Hajian et al. (2005) using Bipolar Spherical Harmonics. The authors of the above references define a set of estimators of statistical anisotropy following

\[ \hat{N}_L = \sum_{\ell_i, \ell_2} \vert \hat{A}_{\ell_i, \ell_2}^{L,M} \vert^2, \]  

(15)

where

\[ \hat{A}_{\ell_i, \ell_2}^{L,M} = \sum_{m_i, m_2} a_{\ell_i m_i} a_{\ell_2 m_2}^* C_{\ell_i m_i, \ell_2 m_2}^{L,M}, \]  

(16)

where \( C_{\ell_i m_i, \ell_2 m_2}^{L,M} \) represents the usual Clebsch–Gordon coefficients. Using Equation (15), one can define a measure of statistical anisotropy, \( \hat{N} \), which can be described by a single scalar number for the entire map following

\[ \hat{N} \equiv \sum_L \hat{N}_L. \]  

(17)

Using the orthogonality of the Clebsch–Gordon coefficients, one can show that

\[ \hat{N} = \sum_{\ell_i, \ell_2} \hat{C}_{\ell_i, \ell_2}, \]  

(18)

where

\[ \hat{C}_{\ell_i, \ell_2} = \sum_{m_i, m_2} \vert a_{\ell_i m_i} a_{\ell_2 m_2}^* \vert^2. \]  

(19)

What is the choice of weights that minimizes the statistic defined in Equation (17) estimated from the cleaned map, which is defined by Equation (7)? Using Equations (7) and (19) in Equation (18), we obtain

\[ \hat{N} = \sum_{\ell_i, \ell_2, m_i, m_2, i,j,p,q} w_i^j w_k^l a_{\ell_i m_i} a_{\ell_2 m_2}^* w_h^p w_s^q a_{\ell_i m_i}^* a_{\ell_2 m_2}^q. \]  

(20)

After some algebra, the above equation can be written as

\[ \hat{N} = \left( \sum_{\ell} (2\ell + 1) W_{\ell} C_{\ell} W_{\ell}^T \right)^2, \]  

(21)

where \( C_{\ell} \) denotes the cross-power covariance matrix as defined by Equation (8). Clearly, minimizing our measure of statistical anisotropies (\( \hat{N} \)), is equivalent to minimizing the positive definite terms \( W_{\ell} C_{\ell} W_{\ell}^T \), which is nothing but the power spectrum of the cleaned map at a multipole \( \ell \). Minimizing the angular power spectrum of the cleaned map at each multipole subject to the constraint that the CMB signal is preserved, we find that the weights that minimize \( \hat{N} \) are given by the ILC weights as given by Equation (3).

10. Input Data Set

10.1. Frequency Maps

We use a set of Planck LFI and HFI detector and detector set maps and all 10 WMAP differencing assembly (DA) maps in our analysis. For the 44 GHz case, we have not used individual detector set maps since the detector set 2 map (LFI_SkyMap_044-25-26_1024_R2.01_full.fits) contains some missing pixels. Since the two sets of our input frequency maps need to have independent detector noise properties, we also cannot use detector set 1 map (LFI_SkyMap_044-24-1024_R2.01_full.fits), which does not have any missing pixels, in our method. Individual detector maps, 25M, 25S, 26M, and 26S, also contain some missing pixels, so we avoid them in our work. Using the LFI 24M and 24S maps for the 44 GHz case, we get reduced power near the acoustic peak and beyond, as shown in Figure 12. We therefore do not use 44 GHz maps altogether in the current work. The actual reason for such low power is unknown to us. However, a systematic effect of some sort in at least one of the 24M and 24S Planck 44 GHz input frequency maps may result in such an effect. In the current work, we have focused on the CMB component estimation, removing diffuse contamination using the improved ILC algorithm in harmonic space. Since the CMB is a weaker signal than the strong thermal dust contaminations in the Planck 545 and 857 GHz maps, we do not use these frequency maps in the current work. However, it would be interesting to investigate the performance of our method in the presence of such strong foreground contamination. We defer a more detailed study of our method using all Planck frequency bands to future work. Since the primary aim of our current work is to reconstruct an improved foreground-cleaned CMB signal though the iterative ILC method, as well as an estimation of the CMB angular power spectrum, which we desire should be free from detector noise bias, instead of merely producing a single foreground-cleaned CMB map, we reconstruct two foreground-cleaned CMB maps —each of which having detector noise properties independent of the other. The cleaned CMB maps with independent detector noise properties can then be used for CMB angular power spectrum estimation by a MASTER type (Hivon et al. 2002) cross-power spectrum estimation, which removes detector noise bias (e.g., Saha et al. 2006, 2008). We form two cleaned maps with independent detector noise properties by using linear superposition within two disjoint sets of input frequency maps, which do not contain any common detector or detector set map.

We label these sets as S1 and S2. We list all WMAP and Planck frequency maps used in our method in Table 1. In the top and bottom panel of this table, we indicate whether the detector or detector set map indicated by the first rows of the two panels belong to set S1 or S2. For Planck 217 GHz, we do not use the detector or detector set map. Instead, we form input maps corresponding to S1 and S2 by averaging respectively detector 1, 2 and detector 3, 4 maps within each pair.

10.2. Beam Window Function

For WMAP maps, we use WMAP 9 year published beam window functions corresponding to different DAs.7 For Planck LFI frequency maps, we use beam window functions for the 30 GHz frequency map and individual beam window functions applicable

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7 Available from https://lambda.gsfc.nasa.gov/product/map/dr5/beam_xfer_get.cfm.
for detector set 1 and detector set 2 for 70 GHz. LFI beam window functions were extracted from the “LFI_RIMO_R2.50.fits” file. For HFI maps, we use beam window functions corresponding to different detector and detector set maps as extracted from the file “HFI_RIMO_Beams-100pc_R2.00.fits.” Since for 217 GHz we average the detector set 1, 2 and 3, 4 maps separately, we also average the beam window functions corresponding to these detector maps for use with the averaged maps. A plot of the beam window functions $B_\ell$ for different frequency maps of our analysis is shown in Figure 2.

10.3. Point Source Mask

We use the WMAP point source mask, Planck LFI point source mask, as well as Planck HFI point source masks at the 100, 143, 217, and 353 GHz frequency bands to form a composite point source mask for use in our analysis. We first upgrade the WMAP point source mask at $N_{side} = 2048$. We then multiply all masks at $N_{side} = 2048$ to form our composite point source mask (henceforth PSMask) at $N_{side} = 2048$. The “PSMask” covers 89.5% of the area of the entire sky. This mask is shown in Figure 3.

11. Masks that Define the Sky Regions

11.1. Background

Since the population of the foreground sources varies in their innate nature depending upon where they are located in our Galaxy (e.g., galactic center, disk, or halo region), the ILC foreground removal method performs better when the sky containing the emission from the Milky Way is divided into a number of regions depending upon the nature of the population of these sources. In earlier publications, some of the authors of this article (Saha et al. 2006, 2008) presented an approach for sky divisions that is based upon the magnitudes of the emission levels of foreground sources in the Milky Way. As discussed in Saha et al. (2008), for an ideal experiment that is noiseless, all foregrounds can be completely removed if all of the following conditions can be met simultaneously: (1) foregrounds follow a rigid frequency scaling all over the sky, at least over the entire frequency window of observation, (2) the total number of free parameters to model the frequency variation of all foreground components is one less than the total number of detector maps available for linear combination, and (3) any chance correlation between the CMB and other foreground components must be ignorable. Even if there is only one foreground component all over the sky and conditions (1) and (3) are satisfied, a complete foreground removal will be impossible if the foreground component under consideration has some angular variation of its spectral index, e.g., synchrotron emission and thermal dust emission. In this case, the foreground component with varying spectral index can be thought of as a superposition of several foreground components, each with a fixed and different spectral index all over the sky (Bouchet & Gispert 1999; Saha & Aluri 2016). Such a representation of a component with a varying index indicates that a foreground component with a varying index can never be completely removed from the sky (on account of the violation of condition 2 above, even if one guarantees that conditions (1) and (3) are satisfied).

From the foregoing discussion, one can infer that the efficiency of ILC foreground removal improves if one is able to find a region on the sky where different foreground components have approximately constant spectral indices. Finding sky regions with approximately constant spectral indices may appear relatively simpler if there were only one foreground component on the sky. However, in the presence of spectral index variation of more than one component (e.g., synchrotron and thermal dust), a reliable solution to this problem becomes almost impossible to attain, since different foreground components may have a completely different morphological pattern of spectral index variation. Even in the case of spectral index variation due to only a single component, the variation may be on a very small angular scale on the sky (e.g., two different but spatially close by populations of localized sources). In principle, an optimal choice of sky region
on which the spectral index remains roughly constant for a single component may even become invisible without further zooming in the desired region of the sky. These requirements demand that one perform foreground removal in increasing the number of sky divisions, which increases the total number of iterations required for foreground removal, making the ILC method computationally very expensive.

Given the complexity described above in defining the sky regions with constant spectral indices, we follow an approach that is similar to the foreground-intensity-based sky division approach in Saha et al. (2006, 2008). In the following, we discuss the procedure for sky divisions based upon the intensity levels of the net foreground emission.

11.2. Sky Regions

For our analysis at $N_{side} = 2048$, we make a mask that defines different sky regions, which is determined primarily by the emission pattern of foreground components at the low frequency side of the WMAP and Planck observation window. For this purpose, we take the Planck 70 GHz DS1 and DS2 maps and downgrade their pixel resolution to $N_{side} = 256$. We then smooth these maps by a Gaussian beam function of FWHM $= 360^\circ$ in harmonic space after first deconvolving their spherical harmonic coefficients by their individual beam window function. We average these smoothed maps to form a single map at 70 GHz. In a similar fashion, we downgrade the pixel resolution of the WMAP K1 band frequency map to $N_{side} = 256$ and smooth this map by a Gaussian beam window of FWHM $= 360^\circ$ after first deconvolving by its original beam function. We subtract the 70 GHz smoothed averaged map from the K1 band smoothed map to form a map free from CMB signal and dominated by the low frequency foreground components. We show the resulting difference map in the top panel of Figure 4 and it has a maximum and minimum value, respectively, of 16974.539 and 14.454618 $\mu K$. We make nine sky regions from this map with pixel temperature values $\Delta T(p)$ lying in the following ranges: $\Delta T(p) < 100$, $100 \leq \Delta T(p) < 200$, $200 \leq \Delta T(p) < 400$, $400 \leq \Delta T(p) < 800$, $800 \leq \Delta T(p) < 1600$, $1600 \leq \Delta T(p) < 3200$, $3200 \leq \Delta T(p) < 6400$, $6400 \leq \Delta T(p) < 10,000$, and finally $\Delta T(p) \geq 10,000 \mu K$. The resulting sky regions contain some isolated very small regions over different parts of the sky. In Table 2, we reassociate these regions with the bigger sky regions that surround them, keeping the total number of disjoint sky regions to nine. The resulting sky-region definition map contains pixel values from one to nine starting from 1 to 9.

![Figure 4](image.png)

**Figure 4.** Top: pixel temperature distribution (in $\mu K$ thermodynamic temperature units) for low frequency foreground components. A histogram equalized color scale is used to take into account the wide dynamical range of the image. Bottom: all nine sky regions indexed from one to nine, depending upon the level of foreground contamination for high resolution analysis of this work.

### Table 2

| Sky Location, $(\theta, \phi)$ | Index for Initial Sky Region | Index for Final Sky Region |
|-----------------------------|-----------------------------|----------------------------|
| $55^\circ > \phi \geq 45^\circ$ | 2                           | 3                           |
| $140^\circ > \phi \geq 130^\circ$ | 3                           | 4                           |
| $300^\circ > \phi \geq 280^\circ$ | 7                           | 8                           |
| $30^\circ > \theta \geq 20^\circ$ | …                           | …                           |
| $70^\circ > \phi \geq 60^\circ$ | 8                           | 9                           |
| $60^\circ > \theta \geq 40^\circ$ | …                           | …                           |

**Note.** List of very small sky patches redirected to a neighboring bigger sky region in the intensity-based sky division scheme. The first column shows the location of the sky patches in the usual spherical polar coordinates $(\theta, \phi)$. The second column indicates the initial sky region to which the patch originally belonged. The third column shows the sky region to which a small sky region with the corresponding specifications given in the first and second columns is merged after the redistribution.

12. Methodology

All the WMAP DA maps are provided by the WMAP science team in $mk$ thermodynamic temperature units, whereas the Planck detector and detector set maps are provided in Kelvin thermodynamic temperature units. We first convert all the input frequency maps as listed in Table 1 in $\mu K$ (thermodynamic) temperature units. As listed in this table, each of the two sets, $S_1$ and $S_2$, in our analysis, consists of a total of 11 maps. The highest resolution frequency maps for sets $S_1$ and $S_2$ are

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8 We chose 70 GHz since it is overall the least foreground-contaminated frequency band for the WMAP and Planck observations.
faster than those of high resolution maps (e.g., see Figure 2), we apply a suitably chosen cutoff on the maximum ℓ(ℓmax)cut up to which spherical harmonic transformation is performed in Equation (6) for each input frequency map. The choice of these maximum ℓ values help avoid division by numerically very small numbers in Equation (6). The ℓmax values for different WMAP and Planck frequency maps are listed in Table 1. We convert the smoothed spherical harmonic coefficients to upgraded resolution input maps at Nside = 2048, up to ℓmax values as listed in Table 1, by performing a backward spherical harmonic transformation using the HEALPix supplied facility synfast. Once all maps have been upgraded in their beam and pixel resolution, we apply our point source mask (PSMask) to each of them to mask out the locations of bright sources. Masking out the locations of bright sources at the very beginning of the analysis has the advantage that the weights for different regions are not affected by these sources and ensures better performance for the removal of diffuse foreground emission.

Using point source masked frequency maps at Nside = 2048, we perform foreground removal in phase 1 over the point source masked sky in a single iteration for both sets S1 and S2 and obtain two cleaned maps C1P1 and C2P1. In phase 2, we follow the new iterative ILC algorithm. Here we clean the point source masked sky in a total of nine iterations over nine different sky regions (shown in Figure 4), starting successively from the most contaminated region. As discussed in Section 11, the different sky regions of Figure 4 indicate regions of the sky where the population of the foreground is likely to differ, resulting in a different spectral behavior. While estimating the spherical harmonic coefficients of the full sky cleaned map at each iteration for sets S1 or S2 in phase 2 analysis, we replace the yet uncleaned regions of the partially cleaned frequency maps by the corresponding region of cleaned map C1P1 where i = (1, 2). As discussed in Section 6.2, this replacement nullifies the foreground leakage from the uncleaned regions into the region that is being cleaned at any given iteration. For both phase 1 and phase 2 analyses, we restrict any forward and backward spherical harmonic transformations up to ℓmax values as specified in Table 1. We denote the two cleaned maps corresponding to sets S1 and S2, obtained at the end of phase 2 analysis by C1P2 and C2P2, respectively.

How much improvement do we achieve in the new iterative ILC method over the old one? To see this, we also perform foreground removal using frequency maps from sets S1 and S2 using the old ILC method. We call this phase 3 analysis. Unlike cleaned maps obtained from phase 2 analysis, the two cleaned maps C1P3 and C2P3 obtained from phase 3 analysis are independent of phase 1 cleaned maps C1P1 and C2P1.

13. Results from Planck and WMAP Observations

13.1. Weights

How do the weights vary with sky regions and multipole ℓ? We show this variation in Figure 5 for the lower side (2 ≤ ℓ < 300) of the entire range of multipoles which are used to produce the cleaned map C1P2, obtained from the set S1 input map of the phase 2 analysis. From the topmost left to the bottom right, the plots of this figure show the weights for the most to least contaminated regions of the sky. As we see from this figure, weights show different types of variation with respect to the multipole moments for each region, which indicates that the foreground spectra varies from region to region. At the high multipoles where only two Planck frequency bands, 217 and 353 GHz, have comparable resolution, all weights tend to the Planck 217 GHz frequency maps since it has the lower detector noise level between these bands.

13.2. Cleaned Maps

We obtain two cleaned maps, C1P2 and C2P2, at the end of nine iterations of phase 2 analysis using the frequency maps from sets S1 and S2, respectively. Since we mask out the positions of the known point sources by applying “PSMask” at the beginning of both phase 1 and phase 2 foreground analyses, our final cleaned maps are also masked out at these positions. Both these maps are produced at the maximum resolution within a given set (i.e., C1P2 has the beam resolution of detector 1 of the 353 GHz frequency map and cleaned map C2P2 has the beam resolution of detector set 2 of the 353 GHz frequency map). We have shown these cleaned maps for sets S1 and S2, respectively, at the top and middle panels of Figure 6. In the bottom panel of the figure, we have shown the difference between the top and middle panel figures. As clearly visible from this figure, the difference map contains no visible signature of foreground emissions and is dominated by the detector noise. This indicates that both maps are cleaned in a similar level. We note that the resolution of cleaned map C2P2 is slightly better than the resolution of cleaned map C1P2 (e.g., see Figure 2). We therefore interpret the difference map primarily for the visual inspection of any residual foreground contamination left over in any one of the two cleaned maps. The presence of the detector scan pattern as induced in the difference map shows that it is the detector noise and not the residual foregrounds that contributes dominantly to the residuals of cleaned maps C1P2 and C2P2.

Since we obtain the cleaned map in phase 1 in a single iteration over the point source masked sky, it contains some residual contamination compared to the phase 2 cleaned map. We show the difference between the phase 1 and phase 2 cleaned maps in the top panel of Figure 7. The left figure of the top panel shows the difference between the cleaned maps for the set S1 (i.e., C1P1−C1P2) whereas the right panel shows the difference between the cleaned maps for the set S2 (i.e., C2P1−C2P2). The imperfect foreground removal of the phase 1 cleaned maps compared to the phase 2 cleaned map is clearly reflected in both difference maps in the top panel of this figure. The bottom panel of Figure 7 shows the difference between the phase 3 and phase 2 cleaned maps for sets S1 and S2, respectively. The features of this map is dominated by the foreground leakage signal present in the phase 3 cleaned map.

13.3. Power Spectrum

We generate the CMB angular power spectrum by cross-correlating the cleaned maps C1P2 and C2P2 using the MASTER approach as described in Hivon et al. (2002). We define an “M6” mask (shown in Figure 8) that excludes all regions near the galactic plane defined by regions 1–6 of Figure 4 along with the position of known point sources. We obtain the power spectrum, Cℓm, of this mask and use it to obtain the mode–mode coupling matrix, Mℓm, following

\[
M_{ℓm} = \frac{2ℓ′ + 1}{4π} \sum ℓ'' (2ℓ'' + 1) C^{ℓm}_ℓ(ℓ 0 0 ℓ'' 0)^2,
\]

(22)
where the last term on the right-hand side denotes the Wigner-3j symbol. We apply the M6 mask to the $C_{12}$ and $C_{22}$ maps and obtain the partial cross-power spectrum, $\hat{C}_\ell$. We convert the partial sky cross-power spectrum to the full sky estimates of the CMB angular power spectrum, $C_\ell$, by first inverting the mode-mode coupling matrix and then computing

$$\hat{C}_\ell = \frac{M_{\ell', \ell}^{-1}}{P_{\ell}^2 B_1^2 B_2^2} \hat{C}_\ell',$$

(23)

where $P_{\ell}$ denotes the pixel window function and $B_1^2$ and $B_2^2$ are, respectively, the beam window function of the 353 GHz maps corresponding to sets $S_1$ and $S_2$. We denote this power spectrum by “CL0.”

As described in Section 14 (also see Saha et al. 2008), the ILC power spectrum contains a negative bias that is strongest at the lowest multipoles. We estimate the negative bias in our power spectrum estimated from the sky region, which survived after masking by the “M6” mask, using Monte Carlo simulations. We correct for this negative bias in “CL0” cross-spectra for the multipole range $2 \leq \ell < 500$. The ILC method in harmonic space leaves a residual in the CMB power spectrum, which is significant at the high multipole regime, due to unresolved point sources (Saha et al. 2006, 2008). A detailed discussion about the different types of sources that cause residual point source contamination at the WMAP and Planck frequency maps is given by Planck Collaboration et al. (2016e, 2014). Since in the current work we are primarily motivated to estimate the CMB angular power spectrum rather than modeling individual source components, we do not attempt to model residuals due to each type of unresolved point source. Instead, we simply model the net residuals due to all types of unresolved point sources in the “CL0” spectrum by fitting the excess with respect to a fiducial CMB power spectrum (Planck Collaboration et al. 2016e) with the simple phenomenological model

$$C_\ell' = C \left(\frac{\ell}{2000}\right)^{\alpha} + C' \left(\frac{\ell}{2000}\right)^{\gamma},$$

(24)

where $C_\ell'$ denotes the excess power at multipole $\ell$ with the factor $\ell (\ell + 1)/(2\pi)$ included and $C$, $C'$, $\alpha$, $\gamma$ are constants obtained by minimizing the $\chi^2$ statistic defined by

$$\chi^2 = \sum_{\ell=500}^{\ell=3000} \left(\frac{\hat{C}_\ell - C_{\ell,d}}{\sigma_\ell}\right)^2,$$

(25)
where $\hat{C}_\ell$ is given by Equation (23) and $\sigma_\ell$ represents the standard deviation of the power spectrum “$C_{10}$,” which we estimate from the diagonal elements of the multipole space covariance matrix. From the fit we obtain $C = 84.082 \pm 12.82$, $\alpha = 1.7793 \pm 0.4659$, $C' = 44.9187 \pm 11.48$, and $\gamma = 7.83475 \pm 0.5524$, with a value of reduced $\chi^2 = 5.77$. We use these best-fit parameter values in Equation (24) and correct the “C10” cross-power spectrum by subtracting the excess as determined by this equation for the multipole range $500 \leq \ell \leq 3000$. We compare our residual unresolved point source contamination as given by Equation (24) with the corresponding residuals of the Planck results as shown in Figure E.4 of Planck Collaboration et al. (2016i). From this figure, we see that the binned CMB angular power spectrum estimated from the Planck COMMANDER CMB map at $\ell \sim 1950$ contains approximately 64 $\mu K^2$ residual unresolved point source contamination compared to our unresolved point source contamination of $\sim 115$ $\mu K^2$ at the same multipole value. Also, from Figure 2 of Planck Collaboration et al. (2016e), we see that the Planck 217 GHz CMB cross-power spectrum has a residual point source contamination of about 140 $\mu K^2$ at $\ell = 2500$. On the other hand, at $\ell = 2500$, our model predicts a residual unresolved point source contamination of about 383 $\mu K^2$.

Using the bias-corrected “C10” power spectrum, we produce our final CMB angular power spectrum, “CFinal,” as follows. For $2 \leq \ell \leq 29$, we take the power spectrum at each multipole $\ell$. For $2500 \geq \ell > 29$, we perform a binning identical to the Planck 2015 results. The resulting power spectrum is shown in Figure 9 with brown-colored points, along with the Planck 2015 binned power spectrum (shown in green points). We have also shown in the figure the Planck 2015 best-fit theoretical CMB power spectrum (Planck Collaboration et al. 2016e) for comparison. In Figure 10, we have shown the difference of the “CFinal” and Planck 2015 binned power spectrum for the bin-middle values starting from $\ell = 47$, along with the error bars as computed from the Monte Carlo simulations. We see from this figure that “CFinal” contains less power for multipoles $47 \leq \ell \leq 300$.

We estimate the full multipole space covariance matrix of the bias-corrected “C10” angular power spectrum using Monte Carlo simulations. The error bar on the unbinned and binned power spectra at each multipole $\ell$ and at each bin is given by the square root of the diagonal elements of such covariance matrices. We show the correlation matrix of the power spectrum shown in Figure 9 in Figure 11. Bin indices between
0 and 28 of this figure represent multipoles $\ell = 2$ to $\ell = 29$. For bin indices between 29 and 110, the correlation between binned spectra for different Planck bins are represented.

It is interesting to note that using Planck 24M and 24S respectively in sets $S_1$ and $S_2$ of the input frequency maps, we recover a deficit of power near and beyond the first acoustic peak (e.g., see Figure 12). The actual cause behind the origin of such a power deficit is unknown to us at present and beyond the scope of the current work. One possible cause for such a power deficit, however, could be the presence of a residual systematic effect in at least one of the two maps (24M, 24S) of 44 GHz used as input maps in our method.

14. Simulations

We validate the methodology of the new iterative ILC algorithm by performing Monte Carlo simulations of the entire foreground removal and power spectrum estimation pipeline. Using the results from the simulations, we understand foreground leakage, and ILC negative bias (Saha et al. 2008) and positive bias due to residual foregrounds in the final CMB cross-power spectrum. Further, we estimate the covariance matrix of the final CMB power spectrum in multipole space using the simulations. We use the diagonal elements of the covariance matrix to estimate the error bar on our final power spectrum. Below, we first describe the different components used in the simulations. We then discuss the results of our simulations.

14.1. CMB Component

We generate 160 random realizations of the CMB temperature anisotropy maps compatible with the Planck LCDM power spectrum (Planck Collaboration et al. 2016e) at $N_{side} = 2048$. We downgrade the pixel resolution of this parent
set of CMB maps to the corresponding pixel resolution of each map belonging to sets $S_1$ and $S_2$ as mentioned in Table 1. Each of these maps is then smoothed by the instrumental beam function (e.g., see Figure 2) of the corresponding detector (or detector set) map.

14.2. Foreground Components

The galactic foreground model used in our simulations consists of three major foreground components, namely, synchrotron, free–free, and thermal dust. We use the Planck 2015 foreground model (Planck Collaboration et al. 2015) and their published templates (provided by the Planck science team at $N_{\text{side}}=256$ and beam resolution $1^\circ$) to estimate the emission levels of these foreground components at different Planck and WMAP frequencies.

One of the major concerns in the reliable removal of foreground components is the variation of the spectral indices of synchrotron and thermal dust components over the sky. Although Planck Collaboration et al. (2015) provided a spectral index map for the thermal dust component, we do not have any template map for the spectral index of the synchrotron component from a joint analysis of WMAP and Planck observations. To generate the synchrotron spectral index map at various WMAP and Planck frequencies, we therefore first create a synchrotron spectral index map using WMAP K1 and Planck 30 GHz frequency maps.

14.2.1. Synchrotron Spectral Index Map

To form the synchrotron spectral index map, we take WMAP 23 GHz (K1 band, $N_{\text{side}}=512$) and Planck 30 GHz ($N_{\text{side}}=1024$) frequency maps, convert their pixel temperature values into $\mu$K (thermodynamic) units, and downgrade their pixel resolution to $N_{\text{side}}=256$. We smooth these low pixel resolution maps by a Gaussian beam function of FWHM = $1^\circ$ by multiplying their spherical harmonic coefficients by the ratio of the beam window functions of $1^\circ$ Gaussian window to the window function of each map’s native resolution beam function. We also downgrade the pixel resolution of the Planck COMMANDER CMB map to $N_{\text{side}}=256$. Since the COMMANDER CMB map is provided in a Gaussian beam resolution of $5^\prime$, we smooth the low pixel resolution CMB map to $1^\circ$ beam resolution by multiplying its spherical harmonic coefficients by the beam window function of a Gaussian beam of FWHM = $\sqrt{60^2 - 5^2} = 59.79$ and convert it to $\mu$K temperature units. We subtract from the 23 and 30 GHz frequency maps at $N_{\text{side}}=256$ the COMMANDER CMB map, and the Planck best-fit thermal dust and free–free maps derived at these two frequencies. Ignoring spinning dust emission, the two resulting maps at 23 and 30 GHz contain only the synchrotron component. We further smooth these two maps by the beam function of a Gaussian window of FWHM = $\sqrt{900^2 - 60^2}$ to bring them to a resolution of $15^\circ$. We denote the resulting low resolution maps at 23 and 30 GHz, respectively, as $M_{23}$ and $M_{30}$. We form the synchrotron spectral index map, $\beta_\nu(p)$, following

$$
\beta_\nu(p) = -\ln \frac{a_{23}M_{23}}{a_{30}M_{30}} \times \ln \frac{30}{23},
$$

where $a_{23}$ and $a_{30}$ are the antennas to the thermodynamic temperature conversion factors at the indicated frequencies. Since the natural logarithm of zero or a negative number is undefined, in principle one should be restricted to only those pixels for which both maps $M_{23}$ and $M_{30}$ are positive definite. We, however, note that due to the relatively larger smoothing window present in these maps, they are already synchrotron signal dominated and hence we find no zero or negative pixels in these maps. The maximum and minimum values of the spectral index map are, respectively, $-1.933$ and $-11.475$. Our final synchrotron spectral index map, $\beta_{256}(p)$, at $N_{\text{side}}=256$ is obtained by replacing pixels of this map less than $-3.50$ by $-3.50$. We have shown the synchrotron spectral index map in Figure 13.

To generate the synchrotron emission map, $S_{\nu}(p)$, at different WMAP and Planck frequencies at $N_{\text{side}}=256$, we first extrapolate the Planck synchrotron template (provided at

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Figure 12. Deficit of the estimated power spectrum at the end of phase 2 analysis when using Planck 44 GHz 24S and 24M frequency maps in our analysis, shown in orange line. The green line shows the corresponding results excluding Planck 44 GHz frequency maps. We note that no bias (neither the negative bias at low $\ell$ nor the positive bias due to residual unresolved point sources at high $\ell$) has been corrected for from these power spectra. A detailed discussions on how to generate the thermal dust and free–free emissions at different Planck and WMAP frequencies are mentioned later in this section.
Figure 13. Synchrotron spectral index map, $\beta_{256}(p)$, at $N_{\text{obs}} = 256$ used to estimate synchrotron emission levels at the various WMAP and Planck frequencies of this work.

\[ \nu = 408 \text{ MHz}, \text{ in } \mu\text{K antenna temperature units} \) at 23 GHz using a constant spectral index, $\beta_s = -2.80$, all over the sky. We call this template $S_{23}(\nu)$. This new template is now extrapolated at all WMAP and Planck frequencies following

\[ S_{\nu}(p) = S_{23}(\nu)\left(\frac{\nu}{23.0}\right)^{\beta_{256}(p)}, \]  

(27)

where $\beta_{256}$ represents the spectral index map described above. Thus, our synchrotron emission maps consist of a variable spectral index model at WMAP and Planck frequencies.

We generate a free–free signal at different frequencies using the model prescribed in the third column of Table 4 of Planck Collaboration et al. (2015). We note that both emission measure and electron temperature $T_e$ are functions of locations on the sky in this model. For thermal dust component, we use the variable spectral index and dust temperature model as given in the Table 4 of Planck Collaboration et al. (2015).

All three components for the different frequencies are now smoothed by the beam window functions of the different detector and detector set of the WMAP and Planck observations. We upgrade the pixel resolution of the smoothed maps to the pixel resolutions as mentioned in the last column of Table 1. We add the three components at these new pixel resolutions and convert the resulting net foreground emission maps at different frequencies to thermodynamic ($\mu$K) temperature units.

14.3. Detector Noise

To generate the noise maps at WMAP frequencies we use the noise per observation ($\sigma_0$) values for Stokes I observations as given in Table 5 of Hinshaw et al. (2013) along with the total number of observations for a pixel $p$, $N_{\text{obs}}(p)$, as provided by the WMAP science team with the individual detector set maps.\(^{10}\) The noise at pixel $p$ at a frequency map $\nu_i$ is given by

\[ n_i(p) = \frac{\sigma_0}{\sqrt{N_{\text{obs}}}} G, \]  

(28)

where $G$ denotes a Gaussian deviate with zero mean and unit variance, and $\sigma_0$ represents the noise level per observation for the DA with frequency $\nu_i$. For both LFI and HFI Planck maps, we create noise realization by using the intensity noise variance values given in the fifth columns of the respective frequency band map files. We create a total of 160 noise realizations for each set $S_1$ and $S_2$. The noise properties of the different detector and detector set maps are uncorrelated with each other for all realizations. For WMAP DA maps following the order from K1 to W4, the mean angular power spectrum estimated from these 160 noise maps and and averaged over all multipoles are respectively 7.181E$-3$, 7.254E$-3$, 13.887E$-3$, 12.495E$-3$, 21.726E$-3$, 17.206E$-3$, 45.381E$-3$, 56.398E$-3$, 63.109E$-3$, and 59.686E$-3$ $\mu$K$^2$. For Planck, the 30, 70, 100, 143, 217, and 353 GHz corresponding mean angular power spectrum of the noise maps are, respectively, 3.220E$-3$, 3.359E$-3$, 5.094E$-4$, 9.410E$-5$, 1.865E$-4$, and 2.0153E$-3$ $\mu$K$^2$. Thus, Planck 143 GHz has the least noisy observations. Among the two highest frequency maps, 217 GHz has a lower noise level than the 353 GHz maps.

14.4. Simulated Frequency Maps

Once CMB, foreground, and detector noise realizations have been created, we simply add them to form 160 realizations of WMAP and Planck detector and detector set maps at the native pixel resolution of the different maps corresponding to sets $S_1$ and $S_2$. These realizations, henceforth labeled as set “Sim0,” represent the simulations of the signal (in thermodynamic $\mu$K temperature units) that is actually measured by the WMAP and Planck satellite missions.

Since we perform our analysis after converting all WMAP and Planck frequency maps to $N_{\text{side}} = 2048$, we form another set of frequency maps for all 160 realizations, labeled as “Sim2048,” by transforming the “Sim0” maps to spherical harmonic space and then using Equation (6) up to $\ell_{\text{max}}$ values as listed in Table 1. This brings all the maps of set “Sim2048” to the same beam and pixel resolution up to the chosen $\ell_{\text{max}}$ values. Since point sources dominate on small angular scale on CMB maps, we further mask the locations of the resolved point sources from the maps of the set “Sim2048” using “PSMask.” This is justified since we are interested in removing diffuse foregrounds. Further, masking out the positions of the resolved point sources guarantees that the weights are not affected by these sources, which leads to better efficiency in diffuse foreground removal. We label the set of point source masked realizations as “Sim2048m.”

14.5. Results and Analysis

Using the input maps of set “Sim2048m,” we perform 160 Monte Carlo simulations of the entire pipeline of the foreground removal and power spectrum estimation technique for both phase 1 and phase 2 analyses, following the same procedure as followed for the case of real data. We also perform 160 Monte Carlo simulations of the old iterative ILC foreground removal and power spectrum estimation as in phase 3 analysis of the data. Since the input frequency maps for all simulations contain varying spectral indices for both synchrotron and thermal dust components, and moreover, each frequency map contains detector noise contamination, foreground removal in ILC method cannot be completely efficient (Saha et al. 2008). This leads to residual contamination. Further, the old iterative ILC method suffers from the problem of foreground leakage as discussed in Section 6.2.

To understand the residuals present in the cleaned CMB maps obtained at the end of different phases of analysis, we form difference maps by subtracting the input CMB maps from...
the cleaned maps obtained from the simulation. We make an average difference map using all such difference maps for each set $S_1$ and $S_2$ and for each phase using a total of 160 Monte Carlo simulations. Since foreground removal in phase 1 involves only one iteration over the entire point source masked sky, and the spectral indices of both synchrotron and thermal dust vary with sky positions, there exist some foreground residuals due to the imperfect removal of foregrounds in the averaged cleaned map $C_1 F_1$ and $C_2 F_1$. In the top row of Figure 14, we show the residual foreground present in the cleaned map of phase 1 analysis. The top-left figure of the top panel shows the residual for the cleaned map $C_1 F_1$ and the top-right figure shows the residual for the cleaned map $C_2 F_1$. In both maps of the top panel, we see the residuals present along the galactic plane and along the north polar spur. We note that since in phase 1 analysis the entire point source masked region was cleaned in a single iteration, foreground leakage is absent in these plots.

In the phase 3 foreground removal, we have simply followed the old ILC algorithm in an iterative fashion in multipole space. The cleaned maps from different Monte Carlo simulations now contain foreground residuals that arise due to imperfect foreground cleaning because of the variation of the foreground spectral indices within each region plus the foreground leakage as described in Section 6.2. Contributions due to both these types of residuals are manifested in the galactic region. Such residuals are shown in the middle panel of Figure 14 for phase 3, set $S_1$ (left), and phase 2, set $S_2$ cleaned map (right).

Following the discussion of Section 6.2 in the phase 2 cleaned maps, the foreground leakage completely stops. The foreground residuals in the phase 2 cleaned maps then arise due to imperfect foreground cleaning because of spectral index variation within any given region. Plots of such residuals are shown in the average difference of the output and input CMB maps corresponding to sets $S_1$ and $S_2$ computed from our Monte Carlo simulations at the bottom panel of Figure 14. Comparing the middle and bottom panels of this figure, we see improvements achieved by extending the usual iterative ILC approach in multipole space to the two-phase approach as proposed in this work. The difference of the middle and bottom panel maps in Figure 14 will contain foreground leakage signal.

We quantify the foreground residuals that originate due to imperfect foreground removal as well as foreground leakage by estimating the angular power spectrum of the difference maps, which are shown in Figure 14. We show the power spectrum of the set $S_1$ residual maps for all three phases in Figure 15. As shown from the top-left and top-right panels of this figure, the phase 1 (set $S_1$) cleaned map contains more residuals in general for multipoles $\ell \gtrsim 400$ compared to both phase 2 and phase 3 cleaned maps. This clearly demonstrates the advantage of performing foreground removal in multiple iterations over a single iteration. At a large scale on the sky, for multipoles $7 \lesssim \ell \lesssim 14$, the phase 1 (set $S_1$) cleaned map contains less foreground residuals than both phase 2 and phase 3 cleaned maps. Comparing results from the phase 2 and phase 3 analyses, we see that our new method performs better than the old iterative ILC method in multipole space for almost all multipoles. For $\ell \gtrsim 20$ (e.g., see the top and bottom panels at the left side of Figure 15), the new method performs significantly better than the old iterative ILC method. Moreover, the phase 2 results have less foreground residuals than the single iteration phase 1 analysis for multipoles $\ell > 20$.

What is the total bias in the ILC cross-power spectra obtained from the analysis of the three individual phases? The ILC power spectrum contains a negative bias, which is most dominant at the lowest multipoles as discussed in Saha et al. (2008). Since the bias due to foreground residuals becomes positive, one has to be cautious when interpreting the net bias in the ILC power spectrum. A net positive bias may not indicate the presence of only residual foreground bias; similarly, a net negative bias may not merely indicate the actual magnitude of the negative bias at any given $\ell$. Since we do not know a priori at which multipole a foreground bias will be present, and neither do we know the exact analytical form of the negative ILC bias for the iterative method of the ILC foreground removal, it becomes difficult to estimate the magnitude of the individual positive and negative biases at any multipole $\ell$. However, one can find from a plot of net bias different multipole ranges where individual positive and negative biases dominate. In the left panel of Figure 16, we have shown the average cleaned cross-power spectra of the two sets at any given phase for all three phases of analysis of this paper compared against the average CMB input power spectrum. All of the spectra of this plot have both pixel and beam effect present. All four cross-spectra of this plot are estimated from the portion of the sky after masking by the “PSMask.” In the middle panel of this figure, we have shown the negative of the net bias in the phase 1, 2 and 3 cross-spectra, which are plotted in the left panel. At the multipole range $\ell \lesssim 15$, we see that the bias in phase 2 and 3 cross-spectra are more negative than the phase 1 cross-power spectrum. This indicates that the phase 1 cross-power spectrum in this multipole range contains a foreground residual, making the net bias in it less negative. In fact, it is likely that the foreground residual exists in phase 1 cross-spectrum for all multipoles up to $\ell \lesssim 300$. Since the negative bias is correlated with the CMB angular power spectrum, there exists some negative bias at the first acoustic peak for all, phase 1, phase 2, and phase 3 cross-power spectra. In the middle panel of this

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11 In Saha et al. (2008), the authors provided an expression of the negative bias only for a full sky single iteration analysis.
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Figure 15. Left: cross-power spectrum, $C_\ell = \ell (\ell + 1) C_\ell$, of the foreground leakage signal from pairs of cleaned maps obtained from the different phases of the analysis as described in the text, for the multipole range $2 \leq \ell \leq 50$. The lower part shows the difference in the power spectrum between the phase 3 and phase 2 analyses, indicating foreground leakage in phase 3, but no foreground leakage in phase 2 analysis. Right: same as the left panel, but for a higher multipole range, $50 \leq \ell \leq 3500$.

In the usual practice, an ILC foreground minimization is performed in multipole space to take into account the variation of the foreground spectral properties (or the variation of the detector noise level) with multipoles. Foreground spectral properties, however, not only depend upon the multipoles but also on the sky regions under consideration, since the populations of foreground-emitting sources vary in their intrinsic characteristics with their locations in the Milky Way galaxy and hence with the directions on the sky. One therefore chooses to perform the usual ILC technique in multipole space successively over different regions of the sky, which are so created that the nature of the foreground properties remains approximately similar over any given region. This so-called multiscale and multiregion approach of sky, however, suffers from the major limitation of leakage of foreground signal from the uncleaned region to the region that is currently being cleaned at any given iteration. In this work, we have for the first time discovered the leakage signal by formulating an analytical framework. We have extended the usual ILC approach into a new two-phase analysis so that the leakage signal is completely stopped in the new method.

We have applied the new approach to the Planck 2015 data sets and WMAP 9 year frequency maps to estimate a pair of diffuse foreground-minimized CMB temperature anisotropy maps with the independent detector noise properties from the regions of the sky left after masking the positions of the known point sources. Both the cleaned maps obtained by us have higher resolution than the Planck 2015 foreground-cleaned CMB maps. We estimate the angular cross-power spectrum of these two foreground-minimized maps to estimate the CMB angular power spectrum from the region of the sky defined by the so-called “M6” mask. Our final power spectrum, “Clinal,” matches well with the published Planck 2015 power spectrum (Planck Collaboration et al. 2016e), with some differences at different multipole regions.

We have performed a detailed Monte Carlo analysis with Planck foreground templates for synchrotron, free–free, and thermal dust components along with the detector noise levels compatible with the Planck and WMAP observations. The foreground models used in the simulations take into account complex scenario such as spectral index variations of the synchrotron and thermal dust components with the angular positions on the sky. The Monte Carlo analysis confirms the foreground leakage signal for the case of the old iterative ILC algorithm in multipole space, and shows that the new multiphase iterative ILC algorithm gets rid of the foreground leakage signal as proved in Section 6.2. We use the Monte Carlo simulations to estimate the biases in different phases of the ILC algorithm, and show that the new method has less net bias (due to the ILC negative bias and the positive foreground bias) than the old, at all angular scales, before the biases for both methods become negligible at high $\ell$. We use the Monte Carlo simulations to estimate the error bar of the final power spectrum estimated from the Planck and WMAP data.

Removing the foreground leakage signal in the new multiphase algorithm will enable us to perform even more reliable analysis of the CMB in terms of its power spectrum. The improvements of this work are expected to have a very important implication for CMB polarization analysis, where the effect of the leakage is expected to be more significant, since the polarized diffuse emissions, specifically near the galactic plane, may be way stronger than the weak primordial CMB signal.

The new ILC algorithm as implemented in this paper has some additional advantages also. It can now handle different input frequency maps with equal footing by first converting them to the resolution function of the highest resolution input.
frequency map up to certain maximum multipoles \( \ell_{\text{max}} \) which are decided by the resolution of each input frequency map, and simultaneously upgrading them to the highest pixel resolution of all available frequency maps. This avoids any need to deconvolve partial sky cross-power spectra covariance matrices by the beam window function during the weight estimation stage of the ILC algorithm. Since division of the partial sky cross covariance matrix by the beam functions for deconvolution purposes is not a mathematically well-defined operation, we consider avoiding such operations as an important improvement to the ILC algorithm. We bring the beam and pixel resolution of all input frequency maps to the highest resolution at the very beginning of the analysis. This allows the foreground removal on the part of the sky, after masking off any undesired region for foreground cleaning, which may contain very strong foreground emissions that they cannot be cleaned effectively for the purpose of estimation of the cosmological signal, to be performed. This helps remove foregrounds from the rest of the unmasked sky portions with better efficiency, since the weights are now no longer affected by the strongly contaminated regions.

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References

Baccigalupi, C., Perrotta, F., de Zotti, G., et al. 2004, MNRAS, 354, 55
Basak, S., & Delabrouille, J. 2012, MNRAS, 419, 1163
Basak, S., & Delabrouille, J. 2013, MNRAS, 435, 18
Bennett, C. L., Hill, R. S., Hinshaw, G., et al. 2003, ApJS, 148, 97
Bennett, C. L., Larson, D., Weiland, J. L., et al. 2013, ApJS, 208, 20
Bennett, C. L., Smoot, G. F., Hinshaw, G., et al. 1992, ApJL, 396, L7
Bielewicz, P., Banday, A. J., & Górski, K. M. 2012, MNRAS, 421, 1064
Bouchet, F. R., & Gispert, R. 1999, NewA, 4, 443
Bouchet, F. R., Prunet, S., & Sato, K. S. 1999, MNRAS, 302, 663
Bunn, E. F., Fisher, K. B., Hoffman, Y., et al. 1994, ApJL, 432, L75
Calabrese, E., Hlozek, R. A., Battaglia, N., et al. 2013, PhRvD, 87, 103012
Copi, C. J., Huterer, D., & Starkman, G. D. 2004, PhRvD, 70, 043515
Cornish, N. J., Spergel, D. N., Starkman, G. D., & Komatsu, E. 2004, PhRvL, 92, 201302
Crittenden, R. G., Davis, R. L., & Steinhardt, P. J. 1993, ApJL, 417, L13
Crittenden, R. G., Coulson, D., & Turok, N. G. 1995, PhRvD, 52, 5440
Delabrouille, J., Cardoso, J.-F., Le Jeune, M., et al. 2009, A&A, 493, 835
Eriksen, H. K., Banday, A. J., Górski, K. M., Hansen, F. K., & Lilje, P. B. 2007a, ApJL, 660, L81
Eriksen, H. K., Banday, A. J., Górski, K. M., & Lilje, P. B. 2004, ApJ, 612, 633
Eriksen, H. K., Dickinson, C., Jewell, J. B., et al. 2008a, ApJL, 672, L87
Eriksen, H. K., Hansen, F. K., Banday, A. J., Górski, K. M., & Lilje, P. B. 2004, ApJ, 605, 14
Eriksen, H. K., Huey, G., Saha, R., et al. 2007b, ApJL, 656, 641
Eriksen, H. K., Jewell, J. B., Dickinson, C., et al. 2008b, ApJL, 676, 10
Gold, B., Bennett, C. L., Hill, R. S., et al. 2009, ApJS, 180, 265
Gold, B., Odegard, N., Weiland, J. L., et al. 2011, ApJS, 192, 15
Górski, K. M., Hivon, E., Banday, A. J., et al. 2005, ApJ, 622, 759
Guth, A. H. 1981, PhRvD, 23, 347
Hajian, A., & Souradeep, T. 2004, ApJL, 597, L5
Hajian, A., & Souradeep, T. 2006, PhRvD, 74, 123521
Hajian, A., Souradeep, T., & Cornish, N. 2005, ApJ, 618, L63
Hinshaw, G., Larson, D., Komatsu, E., et al. 2013, ApJS, 208, 19
Hinshaw, G., Nolta, M. R., Bennett, C. L., et al. 2007, ApJS, 170, 288
Hivon, E., Górski, K. M., Netterfield, C. B., et al. 2002, ApJL, 567, 2
Hou, Z., Reichardt, C. L., Story, K. T., et al. 2014, ApJL, 782, 74
Hyvärinen, A., & Oja, E. 2000, NN, 13, 411
Kim, J., Naselsky, P., & Christensen, P. R. 2008, PhRvD, 77, 103002
Kim, J., Naselsky, P., & Christensen, P. R. 2009, PhRvD, 79, 023003
Lachieze-Rey, M., & Linde, A. D. 1995, PhPh, 254, 135
Levin, J. 2002, PhRv, 365, 251
Linde, A. D. 1983, PhLB, 129, 177
Luminet, J.-P. 2016, Univ, 2, 1
Peiris, H. V., Komatsu, E., Verde, L., et al. 2003, ApJS, 148, 213
Planck Collaboration, Adam, R., Ade, P. A. R., et al. 2015, arXiv:1502.01588
Planck Collaboration, Adam, R., Ade, P. A. R., et al. 2016a, A&A, 594, A1
Planck Collaboration, Adam, R., Ade, P. A. R., et al. 2016b, A&A, 594, A9
Planck Collaboration, Adam, R., Ade, P. A. R., et al. 2016c, A&A, 594, A8
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2014, A&A, 571, A16
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016d, A&A, 594, A2
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016e, A&A, 594, A13
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016f, A&A, 594, A16
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016g, A&A, 594, A18
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016h, \textit{A&A}, 594, A20
Planck Collaboration, Aghanim, N., Arnaud, M., et al. 2016i, \textit{A&A}, 594, A11
Remazeilles, M., Delabrouille, J., & Cardoso, J.-F. 2011a, \textit{MNRAS}, 410, 2481
Remazeilles, M., Delabrouille, J., & Cardoso, J.-F. 2011b, arXiv:1103.1166
Rogers, K. K., Peiris, H. V., Leistedt, B., McEwen, J. D., & Pontzen, A. 2016a, \textit{MNRAS}, 460, 3014
Rogers, K. K., Peiris, H. V., Leistedt, B., McEwen, J. D., & Pontzen, A. 2016b, \textit{MNRAS}, 463, 2310
Saha, R. 2011, \textit{ApJL}, 739, L56
Saha, R., & Ahuri, P. K. 2016, \textit{ApJ}, 829, 113
Saha, R., Jain, P., & Souradeep, T. 2006, \textit{ApJL}, 645, L89
Saha, R., Prunet, S., Jain, P., & Souradeep, T. 2008, \textit{PhRvD}, 78, 023003
Samal, P. K., Saha, R., Delabrouille, J., et al. 2010, \textit{ApJ}, 714, 840
Samal, P. K., Saha, R., Jain, P., & Ralston, J. P. 2008, \textit{MNRAS}, 385, 1718
Spergel, D. N., & Zaldarriaga, M. 1997, \textit{PhRvL}, 79, 2180
Starobinsky, A. A. 1982, \textit{PhLB}, 117, 175
Story, K. T., Reichardt, C. L., Hou, Z., et al. 2013, \textit{ApJ}, 779, 86
Tegmark, M., de Oliveira-Costa, A., & Hamilton, A. J. 2003, \textit{PhRvD}, 68, 123523
Tegmark, M., & Efstathiou, G. 1996, \textit{MNRAS}, 281, 1297