Cosmic Confusion

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Abstract

We propose to minimise the cosmic confusion between Gaussian and non Gaussian theories by investigating the structure in the m’s for each multipole of the cosmic radiation temperature anisotropies. We prove that Gaussian theories are (nearly) the only theories which treat all the m’s equally. Hence we introduce a set of invariant measures of “m-preference” to be seen as non-Gaussianity indicators. We then derive the distribution function for the quadrupole “m-preference” measure in Gaussian theories. A class of physically motivated toy non Gaussian theories is introduced as an example. We show how the quadrupole m-structure is crucial in reducing the confusion between these theories and Gaussian theories.
1 Introduction

Several competing theories of the Early Universe have now been proposed. Overall, two major families have emerged, one centred around the idea of cosmic inflation \[1\], the other relying on topological defects formed in phase transitions \[2\] \[3\]. Crudely speaking, it was found that the first gives rise to a Universe with Gaussian features, whereas the latter should leave some sort of non-Gaussian imprint in the Universe. Producing a cosmic crucible has however been problematic. The discovery of temperature fluctuations in the cosmic microwave background (CMBR) opened doors to a new field with a chance for a decisive experiment. In this letter we examine the obstacles cosmic variance raises to the design of such an experiment \[4\] \[5\], and we propose a new approach to circumventing them.

Cosmic variance in the CMBR comes about because for the Universes predicted by every theory (and probably for our Universe too) there is only a probability distribution function for what the CMBR sky should look like. Different observers, located at different, uncorrelated, points of the Universe, observe different skies. These follow a distribution function which varies from theory to theory, but which is never so peaked that only one sky is possible. The distributions naturally overlap, and so cosmic confusion arises. Mere mortals, with access to only one sky, may be left with the problem that their sky could have derived from totally different theories. To be quantitative, let us define cosmic confusion as the percentage of common skies generated by two theories. Then, if a measurable quantity $Q$ is predicted to have distributions $F_1(Q)$ and $F_2(Q)$ in the two theories $T_1$ and $T_2$, the cosmic confusion in $Q$ between $T_1$ and $T_2$ is

$$C_Q(T_1, T_2) = \int_{all \ Q} dQ \ \min(F_1(Q), F_2(Q)).$$  \hspace{1cm} (1)

$C_Q(T_1, T_2)$ varies between 0 (no doubts) and 1 (totally confused), and is the confusion before $Q$ is actually measured. It is the quantity to be minimised (by means of an appropriate choice of $Q$) when projecting experiments which, one hopes, will be conclusive. Once $Q$ is measured, say, with an outcome $Q \in D_Q$, $D_Q = (Q_0 - \Delta Q^-, Q_0 + \Delta Q^+)$, the confusion becomes

$$\tilde{C}_Q(T_1, T_2) = \frac{2 \int_{D_Q} dQ \ \min(F_1, F_2)}{\int_{D_Q} dQ \ (F_1 + F_2)}$$ \hspace{1cm} (2)

and if $\int_{D_Q} dQ \ F_1 > \int_{D_Q} dQ \ F_2$, the probability of $T_1$ over $T_2$ is

$$P_Q(T_1, T_2) = \frac{\int_{D_Q} dQ \ F_1}{\int_{D_Q} dQ \ (F_1 + F_2)}.$$ \hspace{1cm} (3)

The attitude adopted in this letter is alarmingly anti-anarchist. We seek to minimise the cosmic confusion between Gaussian and non-Gaussian theories using quantities associated with the low order multipoles of $\delta T/T$ (as opposed to $l > 30$, where confusion is known to be small again). Rather than looking at the angular power spectrum $C^l$ or the quadrupole
intensity $Q_{\text{rms}}$, we investigate the structure in the m’s for each $l$ (we will actually concentrate on $l = 2$ in this letter). Gaussian theories are known to treat all m’s in the same way. In Section 3 we prove that such a feature is in fact peculiar to Gaussian theories, within a large class of theories. The physically most relevant theories, however, appear to be outside the class considered, but in Section 4 we show how “m-preference” still plays an important role as non-Gaussianity indicator in the general case. Hence quantities measuring m-preference appear as good candidates in our quest for low cosmic confusion. In Section 4 we write down the most general multipole invariants and identify a set of invariant measures of m-preference. We then specialise to the quadrupole (Section 5) and in Section 6 the distribution function for its measure of m-structure is derived for Gaussian theories. A class of physically motivated toy non Gaussian theories is introduced in Section 7. We exhibit one extreme case where the confusion in $Q_{\text{rms}}$ is 1, but where the confusion in the quadrupole m-structure is 0. We conclude with a brief summary of the results obtained.

2 Structure in the m’s as a sign of non Gaussianity

In analyzing $\frac{\delta T}{T}(\theta, \phi)$ maps it is traditional to use the multipole expansion:

$$\frac{\delta T}{T} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_l^m Q_l^m.$$  

(4)

The real and complex spherical harmonics ($Q_l^m$ and $Y_l^m$) are related by

$$Y_0^0 = Q_0^0; \quad Y_{\pm m}^l = (\mp)^m \frac{Q_l^m \pm iQ_{-m}^l}{\sqrt{2}}$$

(5)

and similar relations apply to their coefficients $b_l^m$ and $a_l^m$. In this Section we assume that the $b_l^m$ are independent random variables. Under this condition we prove a rather surprising result: that only Gaussian theories are statistically spherically symmetric (SSS) and m-amorphous (i.e: treat all the $b_l^m$ in the same way for each $l$). The implication is that conversely any non-Gaussian theory is to some extent anisotropic, favouring particular directions in the sky and some m’s over the others.

We say that a theory is SSS if any two quantities in the sky related by an $O(3)$ transformation are predicted to have the same distribution function ($f(R(Q)) = f(Q)$). We also say that a theory is m-amorphous if for a given fixed $l$ the $b_l^m$ are independent random variables with the same distribution function, regardless of $m$ and the choice of axes. If this distribution is Gaussian we then say that the theory is Gaussian. Let us first prove the following Lemma:

Lemma 1 Let $x$ and $y$ be two independent continuous random variables with the same distribution $f$. Let $z = \alpha x + \beta y$ with $\alpha^2 + \beta^2 = 1$. Then, if $f$ is also the $z$ distribution, $f$ must be a Gaussian.
**Proof**: Let \( \phi \) be the characteristic function of \( f \) (e.g. [6]). Then
\[
\phi(t) = \phi(\alpha t) \phi(\beta t) .
\] (6)

Putting \( g = \log f \) this implies that
\[
g(t) = g(\alpha t) + g(\beta t) .
\] (7)

Now expand \( g \) in power series:
\[
g(t) = \sum g_n t^n .
\] (8)

Then (7) requires that
\[
g_n = (\alpha^n + \beta^n) g_n
\] (9)
and so \( g_n \neq 0 \) iff
\[
\alpha^n + \beta^n = 1 .
\] (10)

But we know that \( \alpha^2 + \beta^2 = 1 \), and by studying the curves (10) in the \((\alpha, \beta)\) plane for all the \( n \)’s one finds that they only intersect at \( \alpha = 0 \) or \( \beta = 0 \). Hence \( n = 2 \), and so \( g(t) \propto t^2 \) and \( \phi \propto e^{92t^2} \), which is the characteristic of a Gaussian distribution. (Q.E.D.)

We can now prove the central result of this Section.

**Theorem 1** For a theory in which all the \( b'_m \) are continuous independent random variables the following three properties are equivalent: SSS, m-amorphism, and Gaussianity.

**Proof**: It suffices to show that SSS implies Gaussianity. (It looks as if we are proving more than the equivalence of m-amorphism and Gaussianity because it is not immediately obvious that a SSS theory is also m-amorphous. This is because given a basis \( \{Q^l_m\} \) its elements cannot all be \( O(3) \)-transformed into each other.) We start by noting that by rotating an angle \( \phi \) around \( z \), for a given fixed \( m \neq 0 \), the elements \( Q^l_m \) and \( Q^l_{-m} \) transform like
\[
Q^l_m \rightarrow \tilde{Q}^l_m = \alpha Q^l_m - \beta Q^l_{-m}
\]
\[
Q^l_{-m} \rightarrow \tilde{Q}^l_{-m} = \beta Q^l_m + \alpha Q^l_{-m}
\] (11)
with \( \alpha = \cos \phi \), and \( \beta = \sin \phi \). Choosing \( \phi = \pi/2 \) we then know that \( f(b'_m) = f(b'_m) = f \).

Since also \( f(b'_m) = f \) for any \( \phi \), using the Lemma we know that \( f \) has to be a Gaussian.

Now, let us apply the most general rotation to \( Q^l_m \). Any element of its orbit will have the distribution function \( f \). The subspace spanned by the orbit is not the whole space, but because the representation is irreducible, it will not be an invariant subspace. This means that vectors \( V \) outside the subspace exist which, although not obtainable directly by a rotation of \( Q^l_m \), can still be obtained by rotating a linear combination of rotations of \( Q^l_m \):
\[
V = R[\sum_n \alpha_n R_n(Q^l_m)] .
\] (12)

By choosing \( \sum \alpha_n^2 = 1 \) one finds that \( f(V) = f \). \( V \) and the orbit of \( Q^l_m \) still do not span the whole space, but the procedure described has allowed us to extend non trivially the subspace spanned by the elements for which the distribution is \( f \). By iteration one can therefore show that for all \( m \) the distribution \( f(b'_m) \) is a Gaussian distribution function \( f \) with a variance which can only depend on \( l \). (Q.E.D.)
3 SSS non-Gaussian theories

The theories for which all the $b_l^m$ are independent form a large class. Within this class m-structure is a certain indicator of non-Gaussianity. However only non-SSS non-Gaussian theories can belong to this class. While violations of SSS have appeared in proposed models (such as the $\delta T$ brought about by any anisotropic cosmological model), one should beware of these violations as they can never be tested experimentally. Furthermore most practical applications concern $\delta T$ due to perturbations in FRW models, for which SSS is always satisfied. If one takes SSS as a starting point, then one must confront the following corollary of Theorem 1.

**Corollary 1** For SSS non-Gaussian theories the $b_l^m$ can never be all independent random variables.

Once one allows the $b_l^m$ to be dependent the rest of Theorem 1 breaks down (see the end of Section 6 for an example). For SSS theories m-structure may or may not be associated with non-Gaussianity. In some cases m-structure is still the hallmark of non-Gaussianity (see Section 7), but there is also a class of non-Gaussian m-amorphous theories.

However we now show that if a non-Gaussian theory is m-amorphous and sufficiently distant in functional space from Gaussian theories, then the $C_l$ must be distinctly non-Gaussian. The power spectrum appears to save the day when m-structure is useless. We show this fact with the aid of the concept of usefulness of a variable. Let the distance between two theories be

$$D(T_1, T_2) = \int |F_1 - F_2| \in (0, 2) .$$

(13)

For any subset of variables $S$ we have

$$1 - \frac{D}{2} \leq C_S(T_1, T_2) \leq 1 ,$$

(14)

showing that the minimal possible confusion is $C_{\text{min}} = 1 - \frac{D}{2}$. If $D = 0$, the two theories are the same, and no wonder we are totally confused whatever set of variables we look at. For $D \neq 0$, however, one can assess the usefulness of the variables $S$ with the quantity

$$U_S = 1 - \frac{C_S - C_{\text{min}}}{1 - C_{\text{min}}}. $$

(15)

$U_S$ varies between 0 (utterly useless) to 1 (perfect choice). Obviously if $S$ contains all the variables, $U_S = 1$, but what we should be after is maximising $U$ with the smallest number of variables. Now if $T_1$ and $T_2$ are two m-amorphous theories (Gaussian or not) we have $F(b_l^m) = f(C_l)$ and so

$$D = \int (\prod_{lm} db_l^m)|F_1(b_l^m) - F_2(b_l^m)| = \int (\prod_{l} dC_l)|F_1(C_l) - F_2(C_l)| .$$

(16)

Therefore $U_{C_l} = 1$, implying that if $T_1$ is Gaussian and $D(T_1, T_2) \approx 2$ then $C_{C_l}(T_1, T_2) \approx 0$, that is, $C_l$ for $T_2$ must be distinctly non-Gaussian.
The important converse statement of the last paragraph is that a very non-Gaussian theory with a very Gaussian $C_l$ must display a distinctly non-Gaussian m-structure. This clarifies the role of m-structure for SSS theories: m-structure may or may not work for them as a non-Gaussianity indicator, but it always works when $C_l$ does not.

4 Multipole invariants and structure in the m’s

These results prompt us to quantify m-preference as m-preference measures will also measure non-Gaussianity. For this purpose it will be useful to first recall the isomorphism between the space spanned by the real spherical harmonics of degree $l$ and the space of the real traceless symmetric cartesian tensors of rank $l$ ($Q_{i\ldots j} = Q_{(i\ldots j)}$, $Q_{ii\ldots k} = 0$). This isomorphism can best be established by writing the functions $Q_{lm}^{\theta, \phi}$ in cartesian coordinates. The resulting polynomial of degree $l$ has the form

$$Q_{lm}^{i\ldots j} x_i \ldots x_j$$

identifying a set of traceless symmetric tensors of rank $l$:

$$\{Q_{lm}^{i\ldots j}(\theta, \phi)\} \leftrightarrow \{Q_{i\ldots j}^{lm}\}. \quad (17)$$

The isomorphism can then be extended to the rest of the space using linearity. The basis $\{Q_{lm}^{i\ldots j}\}$ is known as an irreducible tensor set and one can check that indeed both spaces are $2l + 1$ dimensional (see for instance [7]).

Now for whatever combination of the $a_{lm}^l$ (or $b_{lm}^l$) one considers it is important to demand spherical invariance, so as to factor out the artifacts introduced by our inevitable choice of axes. Hence we look for multipole invariants. Under an axes transformation defined by the Euler angles $(\psi, \theta, \phi)$ the $a_{lm}$ transform according to

$$a_{lm} \rightarrow \tilde{a}_{lm} = \sum_{m'} D_{lm}^{l'm} a_{lm'}, \quad (18)$$

where $D_{lm}^{l'm}(\psi, \theta, \phi)$ is the Wigner matrix. The tensors $Q_{i\ldots j}$, on the other hand, transform just like any other cartesian tensor. The fact that the matrices $D_{lm}^{l'm}$ form an irreducible representation of the 3-dimensional rotation group requires, by means of Schur’s Lemma, that the only matrices they commute with be multiples of the identity. As a result, the only independent bilinear invariant one can construct out of the $a_{lm}^l$ (or $b_{lm}^l$) is

$$C_l = \sum_{m=-l}^{l} |a_{lm}|^2 = \sum_{m=-l}^{l} |b_{lm}|^2 \quad (19)$$

more proverbially known as the angular power spectrum. However, by simply counting degrees of freedom, one is easily convinced that there are in fact $2l - 2$ invariants for each $l \geq 2$. What Schur’s Lemma states is that $2l - 3$ of them will not be bilinear (that is, they will be of higher order). Therefore, the invariants are in general multilinear forms

$$I^{(n)} = \sum_{i\ldots j=1}^{2l+1} I_{i\ldots j}^{(n)} a_{i}^l \ldots a_{j}^l \quad (20)$$

5
satisfying the condition

\[ I_{\nu...\nu}' = D^l_{\nu\nu'}...D^l_{j\nu} I^{(n)}_{i...j}. \]  

(21)

These are very difficult to find for a general \( l \). Since the rotation matrices \( D^l_{m'm} \) are also a subgroup of \( SO(2l + 1) \) we know that \( l \) of these invariants will be the fundamental representations of the Casimir operators of \( SO(2l + 1) \). Still, even these have a rather unpleasant form. We have found it easier to write the invariants in terms of the tensors \( \{ Q_{i...j} \} \) instead. They then appear as the independent contractions of a symmetric traceless tensor of rank \( l \) in 3 dimensions. We cannot produce a general formula for these, but this approach has proved to be more manageable case by case.

What is the meaning of the \( I^{(n)} \)? The invariant \( C^l \) is obviously a measure of the overall intensity of the multipole. It treats all \( m \)'s equally, so it does not assess \( m \)-preference at all. The other \( I^{(n)} \), however, treat the various \( m \)'s differently (see Section 5 for an example). For all invariants of order \( m > 2 \) one should then define the ratios:

\[ r^{(n)} = \frac{I^{(n)}}{(C^l)^{m/2}}. \]  

(22)

The shape factors \( r^{(n)} \) reveal how some \( m \)'s are preferred over others, or, in other words, how much and what type of anisotropy there is in the multipole, but only in so far as there is a rotationally invariant meaning to the concept. Demanding rotational invariance is important since otherwise our anisotropy measures would reflect not only the directionality pertaining to the CMBR but also the directionality imparted by our choice of axes. To make the point clear consider the case \( l = 1 \). Using (17) the dipole can be seen as a vector, for which the only invariant is \( C^1 \) (the dipole has no shape). Clearly, any dipole has a direction, so it could never distinguish between an isotropic and an anisotropic theory.

There are also invariants which combine different \( l \)'s. By counting degrees of freedom we find that, for each pair \( l, l' \geq 2 \), there are 3 inter-\( l \) invariants which do not depend on the invariants for each \( l \). We may wish to look at \( I^{(n)} \) as generalised eigenvalues. For a general \( l \) one can define the eigenvectors of \( Q_{i...j} \) as the set of axes where three independent prechosen components of \( Q_{i...j} \) are set to zero (for \( l = 2 \), \( Q_{ij} = 0 \) for \( i \neq j \)), and its eigenvalues as the values of the remaining components in that basis (for \( l = 2 \) the diagonal components). The inter-\( l \) invariants can then be interpreted as the Euler angles of one set of eigenvectors with respect to the other. They are uniformly distributed in Gaussian theories, but not in defect theories, where they reveal the correlations between successive generations of defects 8.

5 Quadrupole variables and invariants

We now specialise to \( l = 2 \). Using the isomorphism (17) a quadrupole can be seen as a traceless symmetric real matrix \( Q_{ij} \). This can be fully characterised by the Euler angles \((\psi, \theta, \phi)\) specifying the rotation required to diagonalize \( Q_{ij} \), together with its diagonal form \( \hat{Q}_{ij} \). The matrix \( Q_{ij} \) has two invariants under \( O(3) \) transformations, which can be
parameterized in a variety of ways. The eigenvalues \( \hat{Q}_{ij} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) (subject to \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \)), if considered in modulus and in unordered triplets, constitute one such parameterization (recall that improper rotations permutate the eigenvalues). One can also expand \( \hat{Q}_{ij} \) in multipoles:

\[
\hat{Q}_{ij} = a\hat{Q}_{ij}^0 + b\hat{Q}_{ij}^2.
\]

Again, \( a \) and \( b \) are not left unchanged by improper rotations, but transform according to

\[
\begin{align*}
(1) & \quad a \rightarrow -a \\
(2) & \quad b \rightarrow -b \\
(3) & \quad a \rightarrow \frac{\sqrt{3}b - a}{2} \\
& \quad b \rightarrow \frac{\sqrt{3}a + b}{2}.
\end{align*}
\]

Modulo these transformations, \( a \) and \( b \) also parameterize the \( Q_{ij} \) invariants. Finally, one can do away with any residual transformations by considering the invariants:

\[
\begin{align*}
I_1 &= \text{Tr} Q^2 = Q_{ij} Q^{ij} = \sum \lambda^2_i = \frac{30}{16\pi} \sum b_i^2 = \frac{30}{16\pi} (a^2 + b^2) \\
I_2 &= \frac{1}{3} |\text{Tr} Q^3| = |\det Q| = \prod \lambda_i = \left( \frac{15}{16\pi} \right)^{3/2} \left| \begin{array}{ccc}
\frac{2a}{\sqrt{3}} & b_1 & b_{-1} \\
b_1 & \frac{b_0}{\sqrt{3}} + b_2 & -b_{-2} \\
b_{-1} & b_{-2} & \frac{b_0}{\sqrt{3}} - b_2
\end{array} \right| = \left( \frac{15}{16\pi} \right)^{3/2} \left| \frac{2a}{\sqrt{3}} \left( \frac{a^2}{3} - b^2 \right) \right|.
\end{align*}
\]

We should note that since in a SSS theory \( \{\lambda_i\} \) and \( (a, b) \) are subjected to residual symmetries interconnecting them they can never be independent random variables. In fact their joint distribution functions, subjected to the same symmetries, can never factorize (cf. \((35)\)). Therefore, for statistical applications, they are not good parameterizations of the quadrupole invariants. Also, one can prove that

\[
\frac{|\lambda_1 \lambda_2 \lambda_3|}{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{3/2}} \leq \frac{1}{3\sqrt{6}}
\]

and so we have the constraint

\[
I_2 \leq \frac{I_1^{3/2}}{3\sqrt{6}}.
\]

Consequently, even though \( I_1 \) and \( I_2 \) are not interconnected by residual symmetries (and so their joint distribution function does factorize), their ranges of variation are dependent \((I_1 \in (0, \infty), I_2 \in (0, \frac{1}{3\sqrt{6}}))\). Hence \( I_1 \) and \( I_2 \) are in fact dependent random variables, a fact manifest in that the integrated distribution function of \( I_2 \), \( \check{F}(I_2) = \int F(I_1, I_2)dI_1 \), is different from the factor \( F_2(I_2) \) appearing in \( F(I_1, I_2) = F_1(I_1)F_2(I_2) \). For this reason we
shall characterize the quadrupole $Q_{ij}$ by the Euler angles $(\psi, \theta, \phi)$, the intensity $I_1$ (known as $Q_{rms}$), and the ratio $r$ (to be called the “quadrupole shape”):

\[ r = 3\sqrt{6} \frac{I_2}{I_1^{3/2}} \]  

(29)

which varies in the range $r \in (0,1)$. These five quantities are statistically independent: their joint distribution function factorizes and their ranges of variation are independent. The shape factor $r$ is an invariant measure of how axis-symmetric the quadrupole is. It varies from maximal symmetry for $r = 1$ (two equal $\lambda$; $b = 0$ possible) to maximal symmetry breaking for $r = 0$ (a null $\lambda$; $a = 0$ possible).

6 The distribution of the quadrupole shape in Gaussian theories

It turns out to be easier to find first the distribution function $F(a, b, \psi, \theta, \phi)$. Since Gaussian theories are SSS we know that $F$ has to have the form

\[ F(a, b, \psi, \theta, \phi) = F(a, b) \frac{\sin \theta}{8\pi^2} \]  

(30)

where $F(a, b)$ is further invariant under the transformations \(\mathbb{D}\). Then start by writing:

\[ F(a, b, \psi, \theta, \phi) = \frac{1}{(2\pi)^{5/2}\sigma^2} \left| \frac{\partial}{\partial (a, b, \psi, \theta, \phi)} \right| \]

(31)

Now let $D_{mn}^2(\psi, \theta, \phi)$ be the Wigner matrix of the rotation taking $\hat{Q}_{ij}$ to $Q_{ij}$. Then

\[ a_m^2 = aD_{m0}^2 + bH_m^{2+} \]  

where $H_m^{2\pm} = (D_{m2}^2 \pm D_{m-2}^2)/\sqrt{2}$. The matrices $D_{mn}^2$ can be seen as generalised spherical harmonics satisfying differential equations similar to the spherical harmonics equations \(\mathbb{H}\). In particular:

\[ \frac{\partial D_{m0}^2}{\partial \phi} = -imD_{m0}^2 \]
\[ \frac{\partial D_{m0}^2}{\partial \psi} = 0 \]
\[ \frac{\partial D_{m0}^2}{\partial \theta} = m \cot \theta D_{m0}^2 + \sqrt{(2 - m + 1)(2 + m)} \cdot e^{i\phi} \cdot D_{m-10}^2 \]  

(32)

and

\[ \frac{\partial H_m^{2\pm}}{\partial \phi} = -imH_m^{2\pm} \]
\[
\frac{\partial H_m^{2+}}{\partial \psi} = -2iH_m^{2+}
\]
\[
\frac{\partial H_m^{2+}}{\partial \theta} = 2\cos \theta H_m^{2+} - \frac{m}{\sin \theta}H_m^{2-}.
\]

These relations imply that the determinant in (31) must have the form
\[
\left| \frac{\partial (a_0, a_1, a_{-1}, a_2, a_{-2})}{\partial (a, b, \psi, \theta, \phi)} \right| = a^2b\phi_1(\psi, \theta, \phi) + ab^2\phi_2(\psi, \theta, \phi) + b^3\phi_3(\psi, \theta, \phi)
\]
and its exact form can then be found from the symmetries. In fact (30) implies that \(\phi_1, \phi_2, \phi_3 \propto \sin \theta\) and (24) leads to:
\[
F(a, b, \psi, \theta, \phi) = C \sin \theta \exp \left(\frac{-a^2 + b^2}{2\sigma_2^2}\right) b(3a^2 - b^2).
\]

Having found \(F(a, b, \psi, \theta, \phi)\) one can now write \(F(I_1, I_2, \psi, \theta, \phi)\) as
\[
F(I_1, I_2, \psi, \theta, \phi) = \sum_{8 \text{ branches}} F(a, b, \psi, \theta, \phi) \left| \frac{\partial (a, b)}{\partial (I_1, I_2)} \right|.
\]

The branches referred to under the summation sign are the 8 branches generated by the symmetries (24) which give the same values \((I_1, I_2)\). It can be proved that not only \(F(a, b, \psi, \theta, \phi)\) but also the determinant \(\left| \frac{\partial (a, b)}{\partial (I_1, I_2)} \right|\) are invariant under (24). Hence all 8 branches give the same result which can be more easily evaluated for the \(a > \sqrt{3}b > 0\) branch. One then gets
\[
F(I_1, I_2, \psi, \theta, \phi) = C' \sin \theta \frac{I_1}{8\pi^2} e^{-\frac{I_1}{2\sigma_2^2}}
\]
with \(\sigma_2 = \sigma_2 \sqrt{\frac{20}{16\pi}}\), and from it one finally obtains
\[
F(I_1, r, \psi, \theta, \phi) = e^{-\frac{I_1}{2\sigma_2^2}I_1^{3/2}} \frac{\sin \theta}{3\sqrt{2\pi\sigma_2^3}}
\]
where the proportionality constant was computed from a normalization condition. We have thus determined the distribution of the 5 independent variables of the quadrupole in a Gaussian theory. The quadrupole intensity \(I_1\) is a \(\chi_5^2\) variable (as well known), its shape \(r\) is uniformly distributed in \((0, 1)\) and its axes variables \((\psi, \theta, \phi)\) have a uniform distribution. We have confirmed these results with a Monte-Carlo simulation.

We can now, as a side remark, provide the example required in Section 3. Note that for \(l = 2\), SSS (equivalent to \(F(I_1, r, \psi, \theta, \phi) = f(I_1, r)\frac{\sin \theta}{8\pi^2}\)) implies that \(F(b_0, b_1, b_{-1}, b_2, b_{-2}) \propto f(I_1, r)I_1^{3/2}\). On the other hand, m-amorphism (equivalent to \(F(b_0, b_1, b_{-1}, b_2, b_{-2}) = f(I_1))\) implies that \(F(I_1, r, \psi, \theta, \phi) \propto f(I_1)I_1^{3/2}\sin \theta\). These general formulae show that, unless we require that \(F(b_0, b_1, b_{-1}, b_2, b_{-2})\) factorizes, SSS does not imply m-amorphism, and neither implies Gaussianity.
We now produce a toy non-Gaussian theory in order to show how m-structure measures can minimize the confusion between Gaussian and non-Gaussian theories. Imagine a theory which imprints a quadrupole in $\delta T$ such that there is always an eigenvector basis where

$$\tilde{Q}_{ij} = aQ^{(20)}_{ij},$$

(39)

but the eigenvectors’ orientation is uniformly distributed. This toy model can be physically motivated. It is a very good approximation to the quadrupole brought about by a B-field with a very large coherence length (see [10] or [11]). It is also a good approximation to the quadrupole predicted in a texture scenario with scaling and with a small average number of defects per horizon volume ($<N> \ll 1$). Reasoning in accordance with [12] one should expect the quadrupole in this scenario to result mostly from the gravitational effects of the last defect. If most of the collapses are reasonably spherical the quadrupole will then be approximated by (39).

The quadrupole invariants for this theory are

$$I_1 = \frac{30}{16\pi}a^2$$

$$r = 1.$$  

(40)

Consider then the extreme situation in which $a$ is such that $I_1$ is $\chi_5^2$-distributed. The other quadrupole variables follow the distributions

$$f(r) = 2\delta(r - 1)$$

$$f(\psi, \theta, \phi) = \frac{\sin \theta}{8\pi^2}.$$  

(41)

Hence, the cosmic confusion in the quadrupole invariants between this theory ($T_2$) and a Gaussian theory ($T_1$) with the same $I_1$ is

$$C_{I_1}(T_1, T_2) = 1$$

$$C_r(T_1, T_2) = 0.$$  

(42)

The distributions $f_{T_2}(b_n)$ are somewhat complicated (they will be derived and used in [8]), but it should be obvious that they do not reduce the confusion to 0. The invariant $r$ is clearly the appropriate variable to thoroughly make out the difference between the two theories. Using the concept of usefulness $U_{r^2} = 0$ and $U_r = 1$.

Naturally one should not expect to find such an extreme situation, say, in the texture scenario. First, the confusion in $I_1$ is probably not as big, and second $r$ will peak around 1 but not so drastically. The current all-sky simulations have not yet produced a sufficient number of skies for distribution functions to be predicted with any statistical relevance (only 8 skies were produced in [2]). In [8] we shall describe a computational shortcut which will allow a prediction for $F_{tex}(I_1)$ and $F_{tex}(r)$. 

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8 Summary and planned developments

In this letter we proposed yet another non-Gaussianity indicator: m-structure (see [13] for a summary of other options). We gave general arguments linking m-structure and non-Gaussianity, and set up the general framework for quantifying m-structure. We wrote down explicit measures for the quadrupole m-structure, and derived their distribution in Gaussian theories. We finally displayed physically motivated examples of theories for which only m-structure can bring out non-Gaussian nature.

The signal to noise ratio in current all-sky data is not yet good enough to allow any meaningful measurement of $r$ (still we hope to apply our approach to COBE data analysis). Nevertheless, an accurate measurement of $r$ meets no obstacles of principle, with the possible exclusion of the effects of galactic obscuration. It is not clear how much of a problem this is, but we hope to generalise our work to $l > 2$, so as to minimise this effect.

In work currently in progress we also seek to derive an approximation to $f(r)$ in various defect theories. Preliminary results seem to generally link the quadrupole with one individual defect (the closest defect). That being the case, the defect morphology has an imprint on the quadrupole shape, as the defect symmetries impose selection rules on the spherical harmonics expansion. A rich peak structure in $f(r)$, depending on the defect type, ensues. This supports the claim that $r$, if measurable, is a low confusion variable when confronting defect theories among themselves and against Gaussian theories.

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