Large monochromatic components in edge colored graphs with a minimum degree condition

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Abstract

It is well-known that in every $k$-coloring of the edges of the complete graph $K_n$, there is a monochromatic connected component of order at least $n/(k-1)$. In this paper we study an extension of this problem by replacing complete graphs by graphs of large minimum degree. For $k = 2$ the authors proved that $\delta(G) \geq \frac{2n}{k+1}$ ensures a monochromatic component with at least $\delta(G)+1$ vertices in every 2-coloring of the edges of a graph $G$ with $n$ vertices. This result is sharp, thus for $k = 2$ we really need a complete graph to guarantee that one of the colors has a monochromatic spanning subgraph. We conjecture that for larger values of $k$ the situation is different, graphs of sufficiently large minimum degree can replace complete graphs and still there is a monochromatic component of order at least $n/(k-1)$. More precisely, we conjecture that if $k \geq 3$ and $G$ is a graph of order $n$ such that $\delta(G) \geq (1 - \frac{k-1}{k^2})n$, then in any $k$-coloring of the edges of $G$ there is a monochromatic component of order at least $n/(k-1)$.

We prove two results in the direction of this conjecture. The first shows that we really do not need complete graphs to ensure a monochromatic component of order at least $n/(k-1)$ in every $k$-coloring, the assumption

$$\delta(G) \geq \left(1 - \frac{1}{128(k-1)^2}\right)n$$

suffices. Our second result is an improvement of this bound for $k = 3$: If the edges of $G$ with $\delta(G) \geq \frac{n}{10}$ are 3-colored, then there is a monochromatic component of order at least $n/2$.

1 Introduction

Erdős and Rado noticed that in every coloring of the edges of a complete graph with two colors there is a monochromatic spanning tree. This remark has been extended into many directions, a survey on the subject is [6]. For example, a well-known extension of the remark is that in every $k$-edge coloring of a complete graph on $n$ vertices there is a monochromatic component of order at least $n/(k-1)$. Recently there has been significant interest in extending classical Ramsey-type results to non-complete host graphs (e.g. [1], [2], [3], [4], [7], [9]). One such class is the graphs with appropriately large minimum degree. Along these lines, the authors obtained the following extension of the remark of Erdős and Rado.

Lemma 1. (Gyárfás, Sárközy [7]) Let $G$ be a graph with $n$ vertices and minimum degree $\delta(G) \geq \frac{2}{3}n$. If the edges of $G$ are colored with two colors, then there is a monochromatic connected subgraph that has at least $\delta(G)+1$ vertices. This bound is sharp.
The sharpness of Lemma 1 shows that we really need a complete graph to obtain a monochromatic spanning component. However, we conjecture that for $k \geq 3$ the situation changes, we can go down slightly with the minimum degree and we would still get the same result as in the case of the complete graph.

**Conjecture 1.** Let $G$ be a graph of order $n$ such that for some integer $k \geq 3$, $\delta(G) \geq (1 - \frac{k-1}{k^2})n$. If the edges of $G$ are $k$-colored then there is a monochromatic component of order at least $n/(k-1)$.

The bound in the conjecture cannot be improved when $k$ is a prime power and $n$ is divisible by $k^2$. Consider an affine plane of order $k$ and color pairs within the $i$th parallel class with color $i$ for $i = 1, 2, \ldots, k$. Then replace each point with a complete graph of order $t$ and color their edges arbitrarily. The graph obtained has $n = k^2t$ vertices and it is regular of degree

$$tk^2 - (k-1)t - 1 = \left(1 - \frac{k-1}{k^2}\right)n - 1.$$

Here we support the conjecture by showing that it is true for graphs with sufficiently large minimum degree.

**Theorem 1.** For every $k \geq 3$ there exists an $n_0 = n_0(k)$ such that the following is true. Let $G$ be a graph of order $n \geq n_0$ with $\delta(G) \geq \left(1 - \frac{1}{128(k-1)^2}\right)n$. If the edges of $G$ are $k$-colored then there is a monochromatic component of order at least $n/(k-1)$.

The degree bound of Theorem 1 is probably far from best possible. It would be desirable to close the gap between the degree bounds in Theorem 1 and Conjecture 1. For $k = 3$ the condition $\delta(G) \geq \left(1 - \frac{1}{2^7}\right)n$ of Theorem 1 is improved as follows.

**Theorem 2.** Let $G$ be a graph of order $n$ and with $\delta(G) \geq \frac{9}{10}n$. If the edges of $G$ are 3-colored then there is a monochromatic component of order at least $n/2$.

Note that Conjecture 1 claims that the bound of Theorem 2 can be improved to $\delta(G) \geq \frac{7}{5}n$.

Further extension of the problem would be to investigate graphs of smaller minimum degree, for example, extending Lemma 1 in this direction. For graphs of “very small” minimum degree this problem can be answered easily. Indeed, there is always a monochromatic star that have at least $\lceil \frac{\delta(G)}{2} \rceil + 1$ vertices, and this estimate is close to best possible if $\delta(G) < 2\sqrt{n}$. For instance, if $\delta(G)$ is even, one can partition $n$ vertices into disjoint copies of $K_2 \square K_2$ (where $\square$ denotes the Cartesian product) and color the edges between vertices in the same row blue, the edges in the same column red.
Thus the order of the largest monochromatic component (connected subgraph) we can guarantee decreases roughly from $\delta(G)$ to $\delta(G)/2$ when $\delta(G)$ decreases from $\frac{3}{4}n$ to $2\sqrt{n}$. It is natural to ask what happens in-between. Somewhat surprisingly in this range the order of the largest monochromatic component changes as a stepwise constant function in terms of $\delta(G)$. More precisely, the following holds.

**Theorem 3.** (White [11]) Let $G$ be a graph of order $n$ such that for some integer $m \geq 3$, $\delta(G) \geq \frac{2m-1}{m^2}n$. If the edges of $G$ are 2-colored then there is a monochromatic component of order at least $n=\left\lceil \frac{m-1}{m} \right\rceil$.

This result is basically implicit in the proof of Lemma 4.7 in White [11] (see also in [12]), however, it is not even stated there as a separate statement. Note that Theorem 3 is false for $m = 2$, then Lemma 1 gives the order of the largest monochromatic component. The bound on the minimum degree cannot be weakened as the following example shows.

**Example 1.** Let $G_b \square G_r$ be the Cartesian product of a blue $m$-clique $G_b$ and a red $m$-clique $G_r$, and substitute every vertex of $G_b \square G_r$ by an arbitrarily 2-colored $t$-clique, for any $t \geq 1$. We obtain a graph on $n = m^2t$ vertices, which has minimum degree $\delta = (2m - 1)t - 1 = \frac{2m-1}{m^2}n - 1$, and each monochromatic component has only $n/m$ vertices.

We believe that a similar phenomenon occurs for more than 2 colors.

**Conjecture 2.** Let $m \geq 3, k \geq 2, m \geq k$ be integers and let $G$ be a graph of order $n$ such that $\delta(G) \geq \frac{k(m-1)+1}{m^2}n$. If the edges of $G$ are $k$-colored then there is a monochromatic component of order at least $n/(m-1)$.

Again this would be best possible. For $k = 2$ we get Theorem 3. For $m = k \geq 3$ we get Conjecture 1.

# 2 Proof of Theorem 1

For a set $S$, $|S|$ denotes the cardinality of $S$, while for a real number $x$, $|x|$ denotes the absolute value of $x$.

Our starting point is the following lemma of the first author.

**Lemma 2.** (Gyárfás [5]) Let $t \geq 2$ be an integer and $G$ be a bipartite graph with partite sets of size $m$ and $n$. If $|E(G)| \geq \frac{mn}{t}$, then $G$ has a component of order at least $\lceil \frac{m+n}{t} \rceil$. 

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Our main tool will be a stability version of this lemma, i.e. either we have a slightly larger component than guaranteed by Lemma 2 or we are close to the extremal case which may be interesting on its own.

**Lemma 3.** For every integer \( t \geq 2 \) and \( \delta > 0 \) there is an \( n_0 = n_0(t, \delta) \) with the following properties. Let \( G \) be a bipartite graph with partite sets \( V_1, V_2 \) of size \( m \) and \( n \) with \( n_0 \leq \frac{n}{2t} \leq m \leq n \). If \( |E(G)| \geq (1 - \delta)\frac{mn}{t} \), then one of the following two cases holds:

1. \( G \) has a component of order at least \( \lceil \frac{m+n}{t} \rceil \).

2. In \( G \) there are \( t \) components \( C_i \) such that for each \( C_i, 1 \leq i \leq t \) we have the following properties:
   
   (a) \( |C_i| < \lceil \frac{m+n}{t} \rceil \),
   
   (b) \( ||C_i \cap V_1| - \frac{m}{t}| \leq 2t\sqrt{\frac{m}{t}} = 2\sqrt{\delta}m \),
   
   (c) \( ||C_i \cap V_2| - \frac{n}{t}| \leq 2t\sqrt{\frac{n}{t}} = 2\sqrt{\delta}n \).

**Proof of Lemma 3:** Let us assume indirectly that there is a bipartite graph \( G \) with partite sets \( V_1, V_2 \) of size \( m \) and \( n \) with \( n_0 \leq \frac{n}{2t} \leq m \leq n \) and we have

\[
|E(G)| \geq (1 - \delta)\frac{mn}{t},
\]

but neither 1. nor 2. is true in Lemma 3. This will lead to a contradiction. Thus we may assume that the size of each component is at most \( M = \lceil \frac{m+n}{t} \rceil - 1 \). Consider the \( t \) largest components \( C_i, 1 \leq i \leq t \). Let \( |C_i \cap V_1| = a_i \), \( |C_i \cap V_2| = b_i \). Then there exists a component \( C_i \) for some \( 1 \leq i \leq t \) which violates either 2(b) or 2(c). Accordingly we distinguish two cases.

**Case 1:** 2(b) is violated, i.e. there exists a component \( C_i \) for some \( 1 \leq i \leq t \) (wlog \( C_1 \)) for which we have

\[
\left| |C_i \cap V_1| - \frac{m}{t} \right| = |a_1 - \frac{m}{t}| > 2\sqrt{\delta}m.
\]

In this case we will show that the number of edges in \( G \) is less than \( (1 - \delta)\frac{mn}{t} \), contradicting (1). Let all the components be \( C_i, 1 \leq i \leq r \). Then the number of edges in \( G \) is at most

\[
S = \sum_{i=1}^{r} a_i b_i.
\]

Our next lemma gives an upper bound for \( S \).
Lemma 4 There exist non-negative integers $A_i, B_i, 1 \leq i \leq t+1$ such that $S \leq \sum_{i=1}^{t+1} A_i B_i$ and 

1. $\sum_{i=1}^{t+1} A_i = m, \sum_{i=1}^{t+1} B_i = n,$
2. $A_i + B_i = M$ for $i = 1, \ldots, t$,
3. $A_{t+1} + B_{t+1} \leq t$,
4. $|A_1 - \frac{m}{t}| \geq |a_1 - \frac{m}{t}|.$

Proof of Lemma 4: We define non-negative integers $A_i, B_i$ by iterating the general step below which makes some elementary changes that do not change the sum of the $A_i$-s and the sum of the $B_i$-s and do not decrease the actual value of $\sum A_i B_i$. A pair $A_i, B_i$ is saturated if $A_i + B_i = M$. Initially $A_i = a_i, B_i = b_i$ for $i = 1, \ldots, r$. For simplicity we keep the notation $A_i, B_i$ throughout the iteration.

We repeat the following general step until at most one pair $A_i, B_i$ is unsaturated. At the beginning of every step the pairs $A_i, B_i$ are reindexed so that the $A_i$-s are decreasing or increasing and this depends only on the sign of $a_1 - \frac{m}{t}$: if it is non-negative then we make the $A_i$-s decreasing otherwise we make the $A_i$-s increasing. Thus at every step we make the same type of reordering and this ensures that Property 4. is maintained provided that the general step does not decrease (increase) $A_1$ in case of a decreasing (increasing) order.

General step.

Suppose $A_i, B_i$ and $A_j, B_j$ are unsaturated pairs, $A_i \leq A_j$. If $B_i, B_j$ is ordered the same way as $A_i, A_j$, i.e. $B_i \leq B_j$, then we can find non-negative integers $x, y$ such that by changing $A_i, B_i$ to $A_i - x, B_i - y$ and $A_j, B_j$ to $A_j + x, B_j + y$, either $A_j, B_j$ becomes saturated or $A_i, B_i$ becomes the 0, 0 pair. Thus either there is one less unsaturated pair or the number of pairs decrease. Clearly, the sums of the $A_i$-s and $B_i$-s do not change and the sum of $A_i B_i$-s does not decrease. If $B_i, B_j$ is ordered in the opposite way as $A_i, A_j$, i.e. $A_i \leq A_j, B_i > B_j$, then we can find a non-negative integer $z$ such that changing $A_i, B_i$ to $A_i, B_i - z$ and $A_j, B_j$ to $A_j, B_j + z$, either $A_j, B_j$ becomes saturated or $B_i - z \leq B_j + z$ and this allows to apply the previous case. (Again the sums of the $A_i$-s and $B_i$-s do not change and the sum of $A_i B_i$-s does not decrease.)

In case of $i = 1$, $A_1$ is not decreased if we have a decreasing order of $A_i$-s and $A_1$ is not increased if we have an increasing order. It may happen that the decreasing (increasing) order is not maintained. Then we reindex the pairs to get a decreasing (increasing) order of the $A_i$-s, where $A_1$ is still a largest (smallest) among the $A_i$s. Thus Property 4. is maintained.

End of general step.
After each step either some pair $A_i, B_i$ becomes the 0,0 pair and the number of terms decrease by one or one pair $A_i, B_i$ becomes saturated. Clearly we end up with $t+1$ terms if there is one unsaturated pair. If no unsaturated pair remains, we can consider it as a degenerate unsaturated pair with $A_{t+1} = B_{t+1} = 0$.

Properties 1,2 are obviously maintained and we showed the same for Property 4. Since $m + n = t([m+n] - 1) + A_{t+1} + B_{t+1}$ we have

$$A_{t+1} + B_{t+1} - t = m + n - t \left\lfloor \frac{m+n}{t} \right\rfloor \leq 0$$

i.e. $A_{t+1} + B_{t+1} \leq t$, proving Property 3. □

Continuing the proof of Lemma 3, from Lemma 4 we get the following upper bound for the number of edges in $G$.

$$S \leq \sum_{i=1}^{t} A_i B_i + A_{t+1} B_{t+1} = \frac{1}{2} \left( \sum_{i=1}^{t} (A_i + B_i)^2 - \sum_{i=1}^{t} A_i^2 - \sum_{i=1}^{t} B_i^2 \right) + A_{t+1} B_{t+1} =$$

$$\frac{1}{2} \left( \sum_{i=1}^{t} (A_i + B_i)^2 + \sum_{i=1}^{t} (A_i^2 - B_i^2) \right) - \sum_{i=1}^{t} A_i^2 + A_{t+1} B_{t+1} =$$

$$\sum_{i=1}^{t} (A_i + B_i)A_i - \sum_{i=1}^{t} A_i^2 + A_{t+1} B_{t+1} = M(m - A_{t+1}) - \sum_{i=1}^{t} A_i^2 + A_{t+1} B_{t+1} \quad (4)$$

(using Properties 1 and 2 in Lemma 4 in the last equality).

To estimate $\sum_{i=1}^{t} A_i^2$ from below we will use the “defect form” of the Cauchy-Schwarz inequality (as in [10] or in [8]): if

$$\sum_{i=1}^{k} A_i = \frac{k}{t} \sum_{i=1}^{t} A_i + \Delta \quad (k \leq t),$$

then

$$\sum_{i=1}^{t} A_i^2 \geq \frac{1}{t} \left( \sum_{i=1}^{t} A_i \right)^2 + \frac{\Delta^2 t}{k(t-k)}.$$ 

Indeed, we will use this with $k = 1$. Then from (2) and Properties 3, 4 in Lemma 4 we get

$$|\Delta| = \left| A_1 - \frac{m}{t} + \frac{A_{t+1}}{t} \right| \geq \left| A_1 - \frac{m}{t} \right| - \left| \frac{A_{t+1}}{t} \right| \geq \left| a_1 - \frac{m}{t} \right| - \frac{A_{t+1}}{t} >$$

$$2\sqrt{\delta m} - \frac{A_{t+1}}{t} \geq 2\sqrt{\delta m} - 1 \geq \frac{3}{2} \sqrt{\delta m},$$

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(using the triangle inequality and \( m \geq n_0(t, \delta) \)) and thus \( \Delta^2 > \Delta^2 > \frac{9}{4} \delta m^2 \). Thus continuing the estimation in (4), the number of edges in \( G \) is less than

\[
M(m - A_{t+1}) - \frac{(m - A_{t+1})^2}{t} - \frac{9}{4} \delta m^2 + A_{t+1}B_{t+1} \leq \\
\frac{m + n}{t} (m - A_{t+1}) - \frac{m^2 - 2mA_{t+1} + A_{t+1}^2}{t} - \frac{9}{4} \delta m^2 + A_{t+1}B_{t+1} \leq \\
\frac{mn}{t} - \frac{nA_{t+1}}{t} + \frac{mA_{t+1}}{t} - \frac{9}{4} \delta m^2 + A_{t+1}B_{t+1} \leq \frac{mn}{t} - \frac{9}{8} \frac{mn}{t} + \delta \frac{mn}{t} \leq (1 - \delta) \frac{mn}{t},
\]

indeed a contradiction with (1) (here in the last line we used \( A_{t+1}, B_{t+1} \leq t \) (Property 3 in Lemma 4) and \( m \geq \frac{n}{2t} \geq n_0(t, \delta) \)).

**Case 2:** 2(c) is violated, i.e. there exists a component \( C_i \) for some \( 1 \leq i \leq t \) (wlog \( C_1 \)) for which we have

\[
\left| C_1 \cap V_2 \right| - \frac{n}{t} = \frac{n_1}{t} > 2\sqrt{\delta n}.
\]

The proof is symmetric to Case 1. In place of (4) we get

\[
M(n - B_{t+1}) - \sum_{i=1}^{t} B_i^2 + A_{t+1}B_{t+1},
\]

which we have to estimate similarly. \( \square \)

**Proof of Theorem 1:** Let \( k \geq 3 \) be an integer and let \( G \) be a graph of order \( n \geq n_0 \) with \( \delta(G) \geq \left( 1 - \frac{1}{128(k-1)} \right) n \) and a \( k \)-coloring on the edges. Put \( t = k - 1 \) (then \( t \geq 2 \)) and let \( \delta' = \frac{1}{128(2t)} \). We have to show that there is a monochromatic component of order at least \( \frac{2}{t} \). Assume indirectly that this is not the case. Consider the largest monochromatic component (say it is red) and denote the set of vertices in this component by \( V_1, V_2 = V(G) \setminus V_1, |V_1| = m' \) and \( |V_2| = n - m' = n' \). We may clearly assume the following

\[
n' \geq \frac{n}{t} > m' \geq \frac{n'}{2t}.
\]

(For the last inequality we may consider the monochromatic stars from any vertex of \( V_1 \) to vertices of \( V_2 \).) We consider the bipartite graph \( G^b \) induced by \( G \) between \( V_1 \) and \( V_2 \). Using the minimum degree condition in \( G \), the number of edges in \( G^b \) is at least

\[
m'(n' - \delta n) \geq m'(n' - 2\delta n') = (1 - 2\delta')m'n',
\]

(6)
(using \( n' \geq \frac{n}{2} \)). We cannot have a red edge in \( G^b \) and thus the number of colors used on the edges is at most \( t = k - 1 \). Denote the monochromatic bipartite graphs induced by the \( t \) colors with \( G^b_1, \ldots, G^b_t \). Then for each \( 1 \leq i \leq t \) we have \(|E(G^b_i)| < \frac{m'\cdot n'}{t}\), since otherwise we are done by applying Lemma 2 to \( G^b_i \), we have a monochromatic component of order at least \( n/t \) in color \( i \). This implies that for each \( 1 \leq i \leq t \) we have

\[
|E(G^b_i)| \geq (1 - 2t\delta') \frac{m'\cdot n'}{t}.
\]

(7)

Indeed, otherwise the number of edges in \( G^b \) would be less than

\[
(1 - 2t\delta') \frac{m'\cdot n'}{t} + (t-1) \frac{m'\cdot n'}{t} = (1 - 2\delta') m'\cdot n',
\]

a contradiction with (6).

Using (7), we can apply Lemma 3 for each \( G^b_i, 1 \leq i \leq t \) with \( m = m', n = n' \) and \( \delta = 2t\delta' \). Note that (5) and (7) imply that the conditions of the lemma are satisfied. Since we cannot have 1. in Lemma 3, we must have the \( t \) components described in 2. of Lemma 3 for each \( G^b_i \).

Consider the first color \( G^b_1 \) (say blue). Thus we have the \( t \) blue components satisfying 2(a), (b) and (c). Consider the remaining set of vertices \( R \) not covered by these \( t \) components and put \( R_1 = V_1 \cap R \) and \( R_2 = V_2 \cap R \). We have

\[
|R_1| \leq 2t\sqrt{2t\delta'm'}, |R_2| \leq 2t\sqrt{2t\delta'n'}.
\]

(8)

By 2(a), either \( R_1 \) or \( R_2 \) is non-empty, say we have \( v \in R_1 \) (it is symmetric in the other case). Consider the edges between \( v \) and \( V_2 \setminus R_2 \). These edges cannot be blue (and red of course), so they may be one of only \( (t-1) \) colors. This implies using the minimum degree condition in \( G \) and (8) that in one of these \( (t-1) \) colors (say green), there is a green star from \( v \) to \( V_2 \setminus R_2 \) of size at least

\[
\frac{n' - \delta'n - 2t\sqrt{2t\delta'n'}}{t-1} \geq \frac{1 - 4t\sqrt{2t\delta'}}{t-1} n'.
\]

However, in green we also have the \( t \) components satisfying 2(a), (b) and (c). But the above green star from \( v \) will connect two green components, which is a contradiction, if we have

\[
\frac{1 - 4t\sqrt{2t\delta'}}{t-1} - 2t\sqrt{2t\delta'} > \frac{1}{t} + 2\sqrt{2t\delta'}.
\]

This in turn is true if

\[
\frac{1 - 4t\sqrt{2t\delta'}}{t-1} \geq \frac{1}{t} + 4t\sqrt{2t\delta'},
\]
which is true if
\[1 - 4t\sqrt{2t\delta'} \geq 1 - \frac{1}{t} + 4t^2\sqrt{2t\delta'}.
\]
Finally, this is true if
\[1 \geq 1 - \frac{1}{t} + 8t^2\sqrt{2t\delta'},
\]
\[\frac{1}{t} \geq 8t^2\sqrt{2t\delta'},
\]
\[\frac{1}{128t^3} \geq \delta',
\]
which is true by our choice of \(\delta'\). \(\square\)

3 Proof of Theorem 2

Our starting point is the following lemma.

**Lemma 5.** Assume that \(G = [A, B]\) is a bipartite graph with \(n\) vertices, \(|A| = \alpha n, |B| = \beta n\) and \(\alpha \leq \beta\). Set \(\rho = \min\{\alpha, \frac{\beta}{2}\}\). If any vertex of \(G\) is non-adjacent to less than \(\rho n\) vertices then \(G\) is connected.

**Proof of Lemma 5:** Let \(x, y \in A\). Since \(\rho n < \frac{|B|}{2}\), the neighbors of \(x, y\) intersect in \(B\). Also, because \(\rho n < |A|\), every vertex of \(B\) has a neighbor in \(A\). Thus any two vertices of \(G\) can be connected by a path (of length at most four). \(\square\)

Let \(G\) be a graph of order \(n\) such that every vertex is non-adjacent to less than \(\rho n\) vertices and consider a 3-coloring on its edges. Let \(v\) be an arbitrary vertex and let \(N_i\) denote its neighbors in color \(i\). We may assume that \(|N_1| \geq |N_2| \geq |N_3|\), let \(C_1, C_2\) be the monochromatic components in colors 1, 2 containing \(N_1, N_2\). Assume that \(|C_1|, |C_2| < \frac{n}{2}\). We are going to prove that there is a monochromatic component of size at least \(\frac{n}{2}\) in color 3. Set

\[M = C_1 \cap C_2, A_1 = C_1 \setminus M, A_2 = C_2 \setminus M, X = V(G) \setminus (C_1 \cup C_2).
\]

Observe that all edges of the bipartite graphs \([A_1, A_2], [M, X]\) are colored with color 3. We claim that the larger of them is connected and this proves the theorem since the larger must have at least \(\frac{n}{2}\) vertices.

**Case 1.** \([A_1, A_2]\) has at least \(\frac{n}{2}\) vertices. We may assume that \(|A_1| \leq |A_2|\) (otherwise the argument is symmetric). Since \(\delta(G) \geq (1 - \rho)n\), the choice of \(C_1, C_2\) implies that

\[|A_1| = |(C_1 \cup C_2) \setminus A_2| \geq \frac{2n}{3} (1 - \rho) - \frac{n}{2} = \frac{n(1 - 4\rho)}{6}.
\]
Therefore if we select $\rho$ to satisfy

$$\rho = \frac{(1 - 4\rho)}{6}$$

i.e. $\rho = \frac{1}{10}$, then $|A_1| \leq \frac{n}{10}$ and thus

$$\frac{|A_2|}{2} \geq \frac{n}{4} - \frac{n}{20} = \frac{n}{5} > |A_1|.$$ 

Therefore Lemma 5 implies that (the color 3) bipartite graph $[A_1, A_2]$ is connected.

**Case 2.** $[M, X]$ has at least $\frac{n}{2}$ vertices. Since $|C_1|, |C_2| \leq \frac{n}{2}$,

$$n = |C_1| + |C_2| - |M| + |X| \leq n - |M| + |X|,$$

we have $|M| \leq |X|$. Also, from the choice of $C_1, C_2$, $|C_1 \cup C_2| \geq \frac{2n}{3}(1 - \rho)$, therefore $|X| \leq \frac{n}{3}(1 + 2\rho)$ thus

$$|M| \geq \frac{n}{2} - \frac{n}{3}(1 + 2\rho) = \frac{n(1 - 4\rho)}{6},$$

giving the same inequality for $|M|$ as we had before for $|A_1|$. Since $|M| \leq |X|$, the same proof as in Case 1. works here as well.

Thus if $\rho = \frac{1}{10}$ (i.e. $\delta(G) \geq \frac{9n}{10}$) we have a monochromatic component of size at least $\frac{n}{2}$ in every 3-coloring of the edges of $G$. \qed
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