The number of moduli of minimal surfaces in toric 3-folds

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Abstract

In this article we vary \( f \) over all nondegenerate Laurent polynomials with fixed 3-dimensional Newton polytope. Choosing an appropriate surrounding toric variety we get a family of minimal models of surfaces in a toric 3-fold. We compute the number of moduli of such a family. We illustrate our results at some examples of surfaces of general type.

1 Introduction

In this article we study natural families of minimal surfaces in toric 3-folds: We vary \( f \) over all nondegenerate Laurent polynomials \( U_{\text{reg}}(\Delta) \) with fixed 3-dimensional Newton polytope \( \Delta \). By the results from \([\text{Bat20}]\) there is a projective toric variety \( \mathbb{P}_\Sigma \) to a fan \( \Sigma \), such that the compactification \( Z_{\Sigma,f} \) of \( Z_f := \{ f = 0 \} \subset (\mathbb{C}^*)^3 \) in \( \mathbb{P}_\Sigma \) is smooth and has nef canonical divisor \( K_{Z_{\Sigma,f}} \), independently of \( f \in U_{\text{reg}}(\Delta) \). We say that \( Z_{\Sigma,f} \) is a minimal model of \( Z_f \).

We study the resulting natural family of toric hypersurfaces

\[
\mathcal{X} := \{(x, f) \in \mathbb{P}_\Sigma \times B | x \in Z_{\Sigma,f}\} \xrightarrow{pr_2} B, \quad B := \mathbb{P}U_{\text{reg}}(\Delta)
\]

and compute their number of moduli. Note the similarity to the natural family of smooth degree \( d \) projective hypersurfaces in some \( \mathbb{P}^n \).

Let \( F(\Delta) \) denote the Fine interior of \( \Delta \) (see \([\text{Bat20}]\)). Then the condition \( F(\Delta) \neq \emptyset \) ensures the existence of minimal models of \( Z_f \). We proceed in two
steps. First we assume that $\Delta = C(\Delta)$, where $C(\Delta)$ denotes the canonical closure of $\Delta$. For a large class of polytopes including all reflexive polytopes this condition is true. We prove (Theorem 7.1)

**Theorem 1.1.** Let $\Delta$ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and $C(\Delta) = \Delta$. Let $f \in U_{\text{reg}}(\Delta)$, $(\mathbb{P}, Y) := (\mathbb{P}_\Sigma, Z_{\Sigma,f})$ and let

$$\kappa : H^0(Y, N_{Y/X}) \to H^1(Y, T_Y)$$

be the Kodaira-Spencer map. Then

$$\dim \ker(\kappa) = \dim \text{Aut}(\mathbb{P})$$

We define the number of moduli of $X \to B$ as $\dim \text{Im}(\kappa)$. We write $Y$ and $\kappa$ instead of $Y_f$ and $\kappa_f$ since the dimension $\dim \ker(\kappa)$ actually does not depend on $f$. Since $N_{Y/X}$ is trivial of rank $l(\Delta) - 1$ we get as a consequence

$$\dim \text{Im}(\kappa) = l(\Delta) - 1 - \dim \text{Aut}(\mathbb{P}).$$

The Kodaira-Spencer map $\kappa$ is the connecting homomorphism in the tangent sheaf sequence for $Y \subset X$. It is known that $\dim \text{Aut}(\mathbb{P}) = 3 + \# R(N, \Sigma)$, where $R(N, \Sigma)$ are the roots of $\Sigma$ (see section 8). Let $\Sigma_\Delta$ and $\Sigma_{C(\Delta)}$ be the normal fans of $\Delta$ and $C(\Delta)$ and define $R(N, \Sigma_\Delta)$ and $R(N, \Sigma_{C(\Delta)})$ just as above. In Theorem 8.1 we prove

**Theorem 1.2.** In the situation of Theorem 1.1 but without the condition $\Delta = C(\Delta)$ we get at least for $f$ in a Zariski open subset of $U_{\text{reg}}(\Delta)$

$$\dim \ker(\kappa) = 3 + \# (R(N, \Sigma) \cap R(N, \Sigma_\Delta)).$$

Since $R(N, \Sigma) \subset R(N, \Sigma_{C(\Delta)})$ this really generalizes Theorem 1.1. We did not found an example were the Zariski open subset is not all of $U_{\text{reg}}(\Delta)$. To the proof of Theorem 1.1. We reduce the computation of $\ker(\kappa)$ to the one of $\ker(\kappa_\mathbb{P})$, where

$$0 \to H^0(Y, T_Y) \to H^0(Y, T_{\mathbb{P}|Y}) \to H^0(Y, N_{Y/\mathbb{P}}) \xrightarrow{\kappa_\mathbb{P}} H^1(Y, T_Y).$$

This is the first reduction step. In a second reduction step we prove that

$$h^0(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(\mathcal{O}(-Y))) = h^1(\mathbb{P}, T_{\mathbb{P}} \otimes \mathcal{O}(-Y)) = 0.$$
Knowing this we may deduce \( h^0(Y, T_Y) = h^0(\mathbb{P}, T_{\mathbb{P}}) \). Since \( H^0(\mathbb{P}, T_{\mathbb{P}}) \cong \text{Lie Aut}(\mathbb{P}) \) we are done. To prove \( (2) \) we use a slight generalization of a vanishing Theorem due to Mavlyutov.

We have a natural identification \( H^0(Y, N_Y) \cong \text{Lie Aut}(\mathbb{P}) \). Since \( H^0(\mathbb{P}, T_{\mathbb{P}}) \cong \text{Lie Aut}(\mathbb{P}) \) we are done. To prove \( (2) \) we use a slight generalization of a vanishing Theorem due to Mavlyutov.

Knowing a basis of \( \ker(\kappa) \) allows us also to compute the number of moduli of subfamilies of \( X \to B \) (Remark \([8,8]\)). What is known in the direction of Theorem \([1,1]\) in \([\text{Vo}i03\text{, Ch.6}]\) \( \dim \ker(\kappa) \) has essentially been computed for smooth degree \( d \) projective hypersurfaces \( Y \subset \mathbb{P}^n \).

In the dissertation \([\text{Koe}91]\) it is dealt with families of nondegenerate curves in toric surfaces. In this case the closure \( Z_\Delta \) is already smooth. Koelen proved his results first for \textit{maximal polygons} and then extended them to arbitrary polygons with \( \dim F(\Delta) = 2 \). A maximal polygon is a polygon \( \Delta \) which is maximal among all polygons containing \( F(\Delta) \) as their Fine interior. This maximality condition implies \( C(\Delta) = \Delta \) but not conversely.

Sections \([2]\) to \([8]\) aim to prove the Theorems \([1,1]\) and \([1,2]\). In section \([10]\) we illustrate our results by computing the number of moduli of some minimal surfaces of general type, known as \textit{Kanev} and \textit{Todorov surfaces} and we picture the corresponding Newton polytopes in Figure \([2]\).

### 2 Notation and background

Basically we use the same notation as in \([\text{Gie}21]\): We let \( M \) and \( N \) be dual lattice and \( T = (\mathbb{C}^*)^n \) the \( n \)-dimensional torus. We write \( \Delta \) for the \( n \)-dimensional Newton polytope of a Laurent polynomial \( f \) and \( f \in U_{\text{reg}}(\Delta) \) if \( f \) is nondegenerate with respect to \( \Delta \). For a polytope \( F \) whose vertices have rational coordinates we write \( \mathbb{P}_F \) for the projective toric variety defined via the normal fan \( \Sigma_F \) of \( F \). More generally for a complete fan \( \Sigma \) we write \( \mathbb{P}_\Sigma \) for the toric variety associated to \( \Sigma \). We denote by \( \Sigma[1] \) the rays of \( \Sigma \).

Given \( f \in U_{\text{reg}}(\Delta) \) let \( Z_f := \{ f = 0 \} \subset (\mathbb{C}^*)^n \) and for an \( n \)-dimensional toric
variety $\mathbb{P}_\Sigma$ let $Z_\Sigma$ be the closure of $Z_f$ in $\mathbb{P}_\Sigma$. If $\Sigma = \Sigma_F$ is the normal fan to some polytope $F$ we write $Z_F$ instead of $Z_{\Sigma_F}$.

Taking $\Delta$ as given, we work with the modifications $F(\Delta)$, $C(\Delta)$ and their Minkowski sum

$$\tilde{\Delta} := C(\Delta) + F(\Delta)$$

(see [Gie21, Section 3] and [Bat20]). The normal fan of $\tilde{\Delta}$ refines the normal fans of $C(\Delta)$ and $F(\Delta)$ and $\dim \tilde{\Delta} = \dim C(\Delta)$. Thus there is a birational morphism $\rho : \mathbb{P}_{\tilde{\Delta}} \to \mathbb{P}_{C(\Delta)}$.

**Theorem 2.1.** ([Bat20, Cor.4.5])

Assume $F(\Delta) \neq \emptyset$. Then there is a fan $\Sigma$ refining $\Sigma_{\tilde{\Delta}}$ such that

- $\mathbb{P}_\Sigma$ has $\mathbb{Q}$-factorial terminal singularities
- The adjoint divisor $K_{\mathbb{P}_\Sigma} + Z_\Sigma$ is nef.

Let $\pi : \mathbb{P}_\Sigma \to \mathbb{P}_{\tilde{\Delta}}$ be the birational morphism associated to the refinement $\Sigma$ of $\Sigma_{\tilde{\Delta}}$. Then $\pi$ is crepant ([Bat20, Cor.4.5]). We have to pass to the birational model $Z_{C(\Delta)}$ to obtain a partial resolution of singularities $\rho \circ \pi : \mathbb{P}_\Sigma \to Z_{C(\Delta)}$ with $Z_\Sigma$ a minimal model of $Z_f$. But often $C(\Delta)$ equals $\Delta$.

**Corollary 2.2.** ([Bat20, Thm.5.4])

$Z_\Sigma \subset \mathbb{P}_\Sigma$ has at most terminal singularities and $K_{Z_\Sigma}$ is nef. We say that $Z_\Sigma$ is a minimal model of $Z_f$.

The Kodaira dimension of $Z_\Sigma$ is given by (compare [Bat20, Thm.6.2])

$$\kappa(Z_\Sigma) = \min(n - 1, \dim F(\Delta)).$$

We write $L(\Delta)$ for the $\mathbb{C}$ vector space spanned by all lattice points on $\Delta$ and $l(\Delta) = \#(\Delta \cap M)$ for its dimension. Likewise we write $L^*(\Delta)$ for the $\mathbb{C}$ vector space spanned by all interior points of $\Delta$ and $l^*(\Delta)$ for its dimension.

**Remark 2.3.** Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Let $(\mathbb{P}, Y) := (\mathbb{P}_\Sigma, Z_\Sigma)$. Then by ([Bat20 Prop.5.1] and [CLS11 Prop.6.1.20])

$$Z_{\tilde{\Delta}} = \rho^*(Z_{C(\Delta)}), \quad Y = \pi^*(Z_{\tilde{\Delta}})$$

and thus to $Y$ is associated the polytope $C(\Delta)$, that is

$$H^0(\mathbb{P}, \mathcal{O}(Y)) \cong L(C(\Delta)).$$
Proposition 2.4. Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$, $Z := Z_\Delta$ and $K := K_{\tilde{\Delta}}$. Then $2Z + K$ is ample.

Proof. On $\mathbb{P}_{C(\Delta)}$ we have the ample $\mathbb{Q}$-Cartier divisor $Z_{C(\Delta)}$ and on $\mathbb{P}_{F(\Delta)}$ the ample $\mathbb{Q}$-Cartier divisor $D_{F(\Delta)}$ associated to the polytope $F(\Delta)$. Their pullbacks $\rho^*(Z_{C(\Delta)})$ and $\theta^*(D_{F(\Delta)})$ are big and nef and have intersection number 0 exactly with the $\rho$ or $\pi$ exceptional curves. Since the normal fan of $\hat{\Delta}$ is the coarsest refinement of the normal fans of $C(\Delta)$ and $F(\Delta)$ there are no common exceptional curves of $\rho$ and $\pi$. Let

$$D := \rho^*(Z_{C(\Delta)}) + \theta^*(D_{F(\Delta)}).$$

By the toric Nakai-Moishezon criterion (Lin03, Thm.5.3) $D$ is ample since it has positive intersection with every curve. By (Gie21, section 5) and since $\pi$ is crepant we have

$$\theta^*(mD_{F(\Delta)}) = m(K + Z),$$

whereas $\rho^*(Z_{C(\Delta)}) = Z$. Thus

$$mD = m(2 \cdot Z + K).$$

Remark 2.5. At least if $\Delta \neq C(\Delta)$ then $Z_\Delta$ might pass through a torus fixed point $p \in \mathbb{P}_\Delta$. But $Z_\Sigma = \pi^*(Z_\Delta)$ and since $Z_\Sigma$ is irreducible there is no ray $\nu \in \Sigma[1]$ refining the $n$-dimensional cone of $\Sigma_{\tilde{\Delta}}$ to $p$. Thus in constructing $\mathbb{P}_\Sigma$ we may skip rays $\nu \in \Sigma[1]$ refining $n$-dimensional cones of $\Sigma_{\tilde{\Delta}}$. Omitting them from $\Sigma[1]$ we still have $K_{\tilde{\Sigma}} + Z_\Sigma$ nef, since $\pi$ is crepant, and $Z_\Sigma$ gets a minimal model of $Z_f$.

3 Reflexive and torsion free sheaves

Definition 3.1. A coherent sheaf $\mathcal{F}$ on a normal variety $X$ is called reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, where $\mathcal{F}^{**}$ is the double dual of the sheaf $\mathcal{F}$. If $\mathcal{F} \to \mathcal{F}^{**}$ is injective $\mathcal{F}$ is called torsion free. $\mathcal{F}^{**}$ is called the reflexive hull of $\mathcal{F}$.
Remark 3.2. ([CLS12, Prop.8.0.1, Thm.8.0.4])

Given an open subset $j : U \subset X$ with $\text{Codim}(X \setminus U) \geq 2$, a reflexive sheaf is uniquely determined by its restriction to $U$, that is

$$\mathcal{F} \cong j_*(\mathcal{F}|_U). \quad (4)$$

Further a coherent sheaf $\mathcal{F}$ of rank 1 as defined in (4) for $\text{Codim}(X \setminus U) \geq 2$ and $\mathcal{F}|_U$ a line bundle is reflexive, and the dual of a coherent sheaf on a normal variety is always reflexive. In particular the reflexive hull of a coherent sheaf is reflexive.

Remark 3.3. The tensor product $\mathcal{F} \otimes \mathcal{G}$ of two reflexive sheaves need not be reflexive, but we may solve this problem by defining the reflexive tensor product of two sheaves as

$$\mathcal{F} \otimes_r \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{**}.$$ 

In this sense given two Weil divisors $D, D'$ the rank one reflexive sheaf $\mathcal{O}(D + D')$ is isomorphic to

$$\mathcal{O}(D) \otimes_r \mathcal{O}(D').$$

If $\mathcal{F}$ is reflexive and $\mathcal{L}$ a line bundle then $\mathcal{F} \otimes \mathcal{L}$ remains reflexive. This may be checked stalkwise using ([Hart77, Ch.3, Prop.6.8]).

Remark 3.4. (Torsion free sheaves)

If a coherent sheaf $\mathcal{F}$ on $X$ fails to be torsion free, then there is an open subset $U \subset X$, a section $r \in \mathcal{F}(U)$ and $f \in \mathcal{O}_{X,x}$ for some $x \in U$ with $f.r_x = 0$, where $r_x$ denotes the restriction of $r$ to the stalk $\mathcal{F}_x$. Thus $r$ is locally around $x$ supported on $\{f = 0\}$ and $\mathcal{F}|_U$ has a section which is supported on a proper subvariety. Conversely the existence of such a section implies that $\mathcal{F}$ fails to be torsion free.

If $\sigma : X \to Y$ is a proper morphism between normal varieties and $\mathcal{F}$ a torsion free sheaf on $X$, then $\sigma_\ast \mathcal{F}$ remains torsion free, since

$$\sigma_\ast \mathcal{F}(U) = \mathcal{F}(\sigma^{-1}(U))$$

for $U$ an open subset of $Y$ and a torsion section of $\sigma_\ast \mathcal{F}$ would yield one of $\mathcal{F}$.

Lemma 3.5. Let $\dim \Delta = n$, $\mathbb{P} := \mathbb{P}_\Sigma$, and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}$, which is locally free outside of the singular locus $\mathbb{P}_{\text{sing}}$ of $\mathbb{P}$. Then we have an injection

$$H^{n-2}(\mathbb{P}, \mathcal{F}^{**}) \to H^{n-2}(\mathbb{P}, \mathcal{F}) \quad (5)$$
and an isomorphism

\[ H^i(\mathbb{P}, \mathcal{F}^{**}) \cong H^i(\mathbb{P}, \mathcal{F}) \quad i \geq n - 1. \]

Proof. We have exact sequences

\[
0 \to I \to \mathcal{F} \to \mathcal{H} \to 0 \\
0 \to \mathcal{H} \to \mathcal{F}^{**} \to J \to 0
\]

where \( I \) and \( J \) are sheaves supported on \( \mathbb{P}_{\text{sing}} \). \( I \) is called the torsion of \( \mathcal{F} \) and \( J \) the cotorsion of \( \mathcal{F} \). Since \( \mathbb{P} = \mathbb{P}_\Sigma \) has terminal singularities by ([KM98, Cor.5.18]) the singular locus of \( \mathbb{P} \) is of codimension \( \geq 3 \). In effect we get

\[
H^i(\mathbb{P}, I) = 0, \quad H^i(\mathbb{P}, J) = 0, \quad i \geq n - 2
\]

and thus

\[
H^i(\mathbb{P}, \mathcal{F}) \cong H^i(\mathbb{P}, \mathcal{H}), \quad i \geq n - 2 \\
H^i(\mathbb{P}, \mathcal{F}^{**}) \to H^i(\mathbb{P}, \mathcal{H}),
\]

where the second map is an isomorphism for \( i \geq n - 1 \) and injective for \( i = n - 2 \).

4 Tangent sheaf and sheaves of differential p-forms

Definition 4.1. For \( X \) a normal \( n \)-dimensional algebraic variety we define

\[
\Omega^p_X := \iota_* \Omega^p_U, \quad 1 \leq p \leq n, \\
T_X := (\Omega^1_X)^*,
\]

where \( \iota : U \to X \) denotes the inclusion of the smooth locus of \( X \).

Definition 4.2. Let \( \dim \Delta = 3 \) and \((\mathbb{P}, Y) := (\mathbb{P}_\Sigma, Z_\Sigma)\). Then \( Y \subset \mathbb{P} \) is a Cartier divisor. We define the normal sheaf of \( Y \) in \( \mathbb{P} \) as

\[
N_{Y/\mathbb{P}} := \mathcal{O}_\mathbb{P}(Y)|_Y.
\]
Remark 4.3. The sheaf $N_{Y/P}$ is locally free. It follows from the exact sequence
\[ 0 \to \mathcal{O}_P \to \mathcal{O}_P(Y) \to N_{Y/P} \to 0 \]
and from the vanishing $H^1(P, \mathcal{O}_P) = 0$ due to Demazure that
\[ H^0(Y, N_{Y/P}) \cong H^0(P, \mathcal{O}_P(Y))/H^0(P, \mathcal{O}_P). \]
With Remark 2.3 this may be rewritten as
\[ H^0(Y, N_{Y/P}) \cong L(C(\Delta))/C \cdot f \cong T_{P|L(C(\Delta)),f}. \]

Remark 4.4. There is a different method for the construction of the sheaf $T_X$ for $X$ a normal variety: Let $\Omega^p_{X,\text{Kähl}}$ denote the sheaf of Kähler $p$-differentials on $X$ and $T_{X,\text{Kähl}} := (\Omega^1_{X,\text{Kähl}})^*$. Then $\Omega^1_{X,\text{Kähl}}$ is coherent and thus by Remark 3.2 its dual $T_{X,\text{Kähl}}$ is reflexive and coincides with $T_X$ on the smooth locus $U$ of $X$. Since both sheaves are reflexive, we get
\[ T_{X,\text{Kähl}} \cong T_X. \]

The advantage of $T_{X,\text{Kähl}}$ is that we have
\[ H^0(X, T_{X,\text{Kähl}}) \cong \text{Lie} \left( \text{Aut}(X) \right), \quad (6) \]
where $\text{Lie}(\text{Aut}(X))$ denotes the Lie algebra of the automorphism group of $X$ (see [Bri18, Remark 2.4(iii)]). In particular if $\text{Aut}(X)$ is a finite group then $h^0(X, T_X) = 0$. But for $P$ an arbitrary normal projective toric variety $\text{Aut}(P)$ is an algebraic group ([Cox95, Prop.4.3]), whereas for $Y$ a normal projective variety it might happen that $h^0(Y, T_Y) = 0$ but $\text{Aut}(Y)$ is a discrete space with infinitely many components.

Example 4.5. Take the Fermat quartic $Y \subset \mathbb{P}^3$
\[ 0 = x_0^4 + \ldots + x_3^4, \]
which defines a K3-surfaces with Picard number 20 and thus $h^0(Y, T_Y) = 0$ but $Y$ has infinite automorphism group (see [Huy16, Ch.1;Ch.3, 2.6; Ch.15, Remark after Example 2.11]).
5 Mavlyutov’s vanishing Theorem

Construction 5.1. (Frobenius splitting) ([Fuj06, 2.5, Prop.3.2], [CLST1, Lemma 9.2.6, Proof of Thm. 9.3.1])
Let $\mathbb{P}$ be a normal projective toric variety. There is a useful construction to reduce computations of cohomology groups of $\mathbb{Q}$-Cartier Weil divisors $D$ to the cohomology of multiples $mD$ of $D$, which are Cartier. Namely for $l \in \mathbb{N}_{\geq 1}$ the map $\Phi_l : \mathbb{P} \to \mathbb{P}$. $\Phi_l$ induces an injection

$$H^p(\mathbb{P}, \Omega^q_{\mathbb{P}} \otimes \mathcal{O}(D)) \to H^p(\mathbb{P}, \Omega^q_{\mathbb{P}} \otimes \mathcal{O}(lD)) \cong H^p(\mathbb{P}, \Omega^q_{\mathbb{P}} \otimes \mathcal{O}(lD)), \quad p \geq 0,$$

where the last isomorphism follows from Remark 3.3 since $\mathcal{O}(lD)$ is Cartier.

With this Construction we get the following generalization of Mavlyutov’s vanishing Theorem. In case $\mathbb{P} = \mathbb{P}_{\Sigma}$ and $p \geq n-1$ by Lemma 3.5 we may work with the usual tensor product.

Theorem 5.2. ([CLST1, Thm.9.3.3])
Let $\mathbb{P}$ be an $n$-dimensional complete simplicial toric variety. If $D$ is a nef $\mathbb{Q}$-Cartier divisor on $\mathbb{P}$, then

$$H^p(\mathbb{P}, \Omega^q_{\mathbb{P}} \otimes \mathcal{O}(D)) = 0$$

for $p > q$ or $q > p + \dim P_D$ with $P_D$ the polytope associated to $D$.

6 The Kodaira-Spencer map

Construction 6.1. Let $\dim \Delta = 3$, $\mathbb{P} := \mathbb{P}_{\Sigma}$ and let $B$ be the image of $U_{\text{reg}}(\Delta)$ in $\mathbb{P}L(\Delta)$. Consider the following natural family

$$\mathcal{X} := \{(x, f) \in \mathbb{P} \times B | x \in Y_f\} \xrightarrow{pr_2} B,$$

where $Y_f := Z_{\Sigma,f}$. Note that $B$ is as a Zariski open subset of $\mathbb{P}L(\Delta)$ irreducible. By Ehresmann’s fibration theorem ([Voi02, Ch.9]) all fibres of this family are diffeomorphic. Consider the tangent sheaf sequence

$$0 \to T_Y \to T_{\mathcal{X}|Y} \to N_{Y/\mathcal{X}} \to 0 \quad (7)$$
Remark 6.2. Since $Y_f \subset \mathcal{X}$ is the pullback of the point $\{f\} \in B$ under $pr_2$, the normal sheaf $N_{Y_f/\mathcal{X}} = \mathcal{O}_\mathcal{X}(Y_f)|_{Y_f}$ is trivial of rank $\dim B = l(\Delta) - 1$, that is $$N_{Y_f/\mathcal{X}} = pr_2^*(T_{B,f}) = T_{B,f} \otimes \mathcal{O}_{Y_f}.$$ 

Definition 6.3. The coboundary map in the long exact cohomology sequence to (7) $$\kappa_f: T_{B,f} \rightarrow H^1(Y_f, T_{Y_f})$$ is called the Kodaira-Spencer map.

We will see in Theorem 7.1 that if $\Delta = C(\Delta)$ then the dimension of $\text{Im}(\kappa_f)$ is constant for $f \in U_{\text{reg}}(\Delta)$. If $\Delta \neq C(\Delta)$ the dimension of $\text{Im}(\kappa_f)$ is constant at least for $f$ in a Zariski open subset of $U_{\text{reg}}(\Delta)$. To ease the notation we write $\kappa$ for $\kappa_f$.

Definition 6.4. We define the number of moduli of the family $\mathcal{X} \to B$ as $\dim \text{Im}(\kappa)$.

Remark 6.5. If $H^0(Y, T_Y) = 0$ for $Y$ a minimal algebraic surface then there exists a universal deformation $\mathcal{Y} \to S$ of $Y$ ([Cat11, Thm.8]). The base space $S$ is called the Kuranishi space for $Y$. The dimension of $S$ could be bounded by $$h^1(Y, T_Y) - h^2(Y, T_Y) \leq \dim S \leq h^1(Y, T_Y)$$ ([Cat11, Remark after Cor.9]). If $Y$ is of general type then usually the dimension of $S$ is called the number of moduli of $Y$ ([Cat06, Def.20.1]). There is a unique holomorphic map $\phi: B \to S$ such that the family $\mathcal{X} \to B$ is induced by pulling back $\mathcal{Y} \to S$ via $\phi$. At least if $S$ is smooth, for example if $H^2(Y, T_Y) = 0$, then $H^1(Y, T_Y)$ is the tangent space of $S$ at $Y$ and $\kappa$ the differential of $\phi$ ([Cat11, Thm.8, Remark after Cor. 9]). Thus by the constant rank Theorem applied to the open subset of $B$ where $\phi$ has maximal rank $r$, that is those $f \in B$ with $\dim \text{Im}(\kappa_f) = r$ maximal, we have $\dim \text{Im}(\phi) = r$. This relates our Definition 6.4 to the general notion of the number of moduli.

7 The number of moduli if $\Delta = C(\Delta)$

Theorem 7.1. Let $\Delta$ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and $C(\Delta) = \Delta$. Let $f \in U_{\text{reg}}(\Delta)$, $(\mathbb{P}, Y) := (\mathbb{P}_\Sigma, Z_{\Sigma,f})$ and let $$\kappa: H^0(Y, N_{Y/\mathcal{X}}) \to H^1(Y, T_Y)$$
be the Kodaira-Spencer map. Then

\[ \dim \ker(\kappa) = \dim \text{Aut}(\mathbb{P}) \]  

(8)

Our proof is similar to ([Koe91, Ch.2.1] and [Voi03, Lemma 6.15]).

Proof. \(Y\) and \(X\) are smooth and \(Y\) does not pass through the singular locus of \(\mathbb{P}\). Thus there are two tangent sheaf sequences

\[
0 \to T_Y \to T_{X|Y} \to N_{Y/X} \to 0
\]

\[
0 \to T_Y \to T_{\mathbb{P}|Y} \to N_{Y/\mathbb{P}} \to 0
\]

These exact sequences are related via the differential

\[
(pr_1)_* : T_{X|Y} \to T_{\mathbb{P}|Y},
\]

of the first projection. \((pr_1)_*\) restricts to the identity on \(T_Y\) and the map

\[
N_{Y/X} \cong T_{B,f} \otimes \mathcal{O}_Y \cong H^0(Y, N_{Y/\mathbb{P}}) \otimes \mathcal{O}_Y \to N_{Y/\mathbb{P}}
\]

is just the map given by multiplication of sections, where we have used Remark 4.3, Remark 6.2 and that \(C(\Delta) = \Delta\). In effect we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & H^0(Y, T_Y) & \to & H^0(Y, T_{X|Y}) & \to & H^0(Y, N_{Y/X}) & \xrightarrow{\kappa} & H^1(Y, T_Y) \\
\downarrow{id} & & \downarrow{(pr_1)_*} & & \downarrow{\cong} & & \downarrow{id} & \\
0 & \to & H^0(Y, T_Y) & \to & H^0(Y, T_{\mathbb{P}|Y}) & \to & H^0(Y, N_{Y/\mathbb{P}}) & \xrightarrow{\kappa_\mathbb{P}} & H^1(Y, T_Y)
\end{array}
\]

where the third vertical homomorphism is an isomorphism, and we may compute \(\dim \ker(\kappa)\) also by the lower exact sequence

\[
\dim \ker(\kappa) = h^0(Y, T_{\mathbb{P}|Y}) - h^0(Y, T_Y).
\]

We prove in section 9 independently that \(H^0(Y, T_Y) = 0\). By Remark 4.3 it remains to show that \(h^0(Y, T_{\mathbb{P}|Y}) = h^0(\mathbb{P}, T_\mathbb{P})\). For this we use the following ideal sheaf sequence

\[
0 \to H^0(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}(-Y)) \to H^0(\mathbb{P}, T_\mathbb{P}) \to H^0(Y, T_{\mathbb{P}|Y}) \\
\to H^1(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}(-Y))
\]

and conclude with the following Lemma.
Lemma 7.2.

\[ h^0(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}(-Y)) = h^1(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}(-Y)) = 0. \]

Proof. The vanishing of the \( h^0 \) term could be derived from the Euler exact sequence ([CLS11, Thm.8.1.6])

\[ 0 \rightarrow Cl(\mathbb{P}) \otimes \mathcal{O}_\mathbb{P} \rightarrow \bigoplus_{\nu \in \Sigma[1]} \mathcal{O}_{\mathbb{P}}(D_\nu) \rightarrow T_\mathbb{P} \rightarrow 0. \]

\( Y \subset \mathbb{P} \) is nef and big, thus \( H^1(\mathbb{P}, \mathcal{O}(-Y)) = 0 \) ([CLS11, Thm.9.2.7]). Further \( D_\nu - Y \) is the negative of a strictly effective divisor, thus \( H^0(\mathbb{P}, \mathcal{O}(D_\nu - Y)) = 0 \). For the \( h^1 \) term we use that \( T_\mathbb{P} \otimes \mathcal{O}(-Y) \) is reflexive by Remark 3.3 and coincides with \( \Omega^2_\mathbb{P} \otimes \mathcal{O}(-Y - K_\mathbb{P}) \) on the smooth locus of \( \mathbb{P} \). Thus by Lemma 3.5 we have an injection

\[ H^1(\mathbb{P}, T_\mathbb{P} \otimes \mathcal{O}(-Y)) \rightarrow H^1(\mathbb{P}, \Omega^2_\mathbb{P} \otimes \mathcal{O}(-Y - K_\mathbb{P})). \]

By ([CLS11, Proof of Thm. 9.2.10]) \( \Omega^2_\mathbb{P} \) is maximally Cohen Macaulay, that is we may apply Serre duality in the following form

\[ H^1(\mathbb{P}, \Omega^2_\mathbb{P} \otimes \mathcal{O}(-Y - K_\mathbb{P})) \cong H^2(\mathbb{P}, \Omega^1_\mathbb{P} \otimes \mathcal{O}(Y + K_\mathbb{P})). \]

The right hand side vanishes by Theorem 5.2. \( \square \)

Remark 7.3. Using reflexive sheaves allows us to apply Serre duality. But more than this we think that reflexive sheaves build an appropriate tool to deal with the higher dimensional case \( \dim \Delta \geq 4 \).

8 A basis for \( \ker(\kappa) \) and the case \( \Delta \neq C(\Delta) \)

Let

\[ R(N, \Sigma) := \{ \alpha \in M \mid \langle \alpha, n(\alpha) \rangle = 1 \text{ for some } n(\alpha) \in \Sigma[1] \}
\text{and } \langle \alpha, n_j \rangle \leq 0 \text{ for } n_j \in \Sigma[1] \setminus \{ n(\alpha) \} \]

denote the roots of \( \Sigma \). Likewise we define \( R(N, \Sigma_{C(\Delta)}) \) and \( R(N, \Sigma_{\Delta}) \). These roots and the Lie algebra of the torus \( T \) allow us to describe \( \text{Lie}(\text{Aut}(\mathbb{P})) \) (see [Oda88, Prop.3.13]). But we want to see how \( H^0(\mathbb{P}, T_\mathbb{P}) \cong \text{Lie}(\text{Aut}(\mathbb{P})) \) maps to \( L(C(\Delta)) \).
Following ([BG99] Def.2.1) we call a vector \( v \in M \setminus \{0\} \) a column vector for \( C(\Delta) \) if there is a facet \( \Gamma_v \) of \( C(\Delta) \) such that for \( x^m \in L(C(\Delta)) \setminus L(\Gamma_v) \) we have \( x^m + v \in L(C(\Delta)). \) Assume \((0, 0, 0) \in \mathbb{L}^*(\Delta)\). Then for \( \alpha \in R(N, \Sigma C(\Delta)) \) we have \(-\alpha \in C(\Delta)\). Further the vectors \(-\alpha \in -R(N, \Sigma C(\Delta))\) are exactly the column vectors for \( C(\Delta) \) (see [BG99, Remark before Lemma 5.2]).

![Figure 1: A lattice polytope with a column vector \( v \) and \( \text{ht}_v(x^m) = 2 \).](image)

**Definition 8.1.** ([BG99] section 3])

Let \( S_{C(\Delta)} := \text{Cone}(C(\Delta) \times \{1\}) \cap (M \times \mathbb{Z}) \), \( \mathbb{C}[S_{C(\Delta)}] \) be the semigroup algebra of \( S_{C(\Delta)} \) and \( \alpha \in R(N, \Sigma C(\Delta)) \). Define 

\[
\text{ht}_{-\alpha}(x^m) := \max\{k \in \mathbb{N}_{\geq 0} | m - k \cdot \alpha \in S_{C(\Delta)}\}, \quad \text{for } x^m \in S_{C(\Delta)} \cap M.
\]

For \( \lambda \in \mathbb{C} \) define a graded automorphism \( e^\lambda_{-\alpha} : \mathbb{C}[S_{C(\Delta)}] \to \mathbb{C}[S_{C(\Delta)}] \) by 

\[
e^\lambda_{-\alpha}(x^m) := x^m \cdot (1 + \lambda x^{-\alpha})^{\text{ht}_{-\alpha}(m)}.
\]

Note that \( \mathbb{P} \mathbb{C}(\Delta) \cong \text{Proj}(\mathbb{C}[S_{C(\Delta)}]) \). Thus by the functoriality of Proj the graded automorphism \( e^\lambda_{-\alpha} \) yields an automorphism of \( \mathbb{P} \mathbb{C}(\Delta) \), which we still denote by \( e^\lambda_{-\alpha} \).

**Theorem 8.2.** ([BG99, Lemma 3.1, Thm.3.2b, Thm.5.4])

Let \( r := \#R(N, \Sigma C(\Delta)) \). Then there is an embedding of algebraic groups \( \phi : \mathbb{C}^r \to \text{Aut}(\mathbb{P} \mathbb{C}(\Delta)) \)

\[
\phi : (\lambda_1, ..., \lambda_r) \mapsto e^{\lambda_1}_{-\alpha_1} \circ ... \circ e^{\lambda_r}_{-\alpha_r}
\]

\( T \) and \( \phi(\mathbb{C}^r) \) generate the identity component \( \text{Aut}^0(\mathbb{P} \mathbb{C}(\Delta)) \) and \( \dim \text{Aut}^0(\mathbb{P} \mathbb{C}(\Delta)) = 3 + r \).

Since \( \text{Lie}(T) \) has basis \( x_1 \frac{\partial}{\partial x_1}, ..., x_3 \frac{\partial}{\partial x_3} \) we get

\[13\]
Corollary 8.3. \( \text{Lie}(\text{Aut}(\mathbb{P}_{C(\Delta)})) \) has as basis the derivations

\[
x_i \frac{\partial}{\partial x_i} : x^m \mapsto m_i \cdot x^m, \quad i = 1, \ldots, 3,
\]

\[
\frac{\partial e^{-\alpha}}{\partial \lambda} \bigg|_{\lambda=0} : x^m \mapsto ht_{-\alpha}(m) \cdot x^{m-\alpha}, \quad \alpha \in R(N, \Sigma_{C(\Delta)}).
\]

Remark 8.4. Let \((\mathbb{P}, Y) := (\mathbb{P}_{\Sigma}, Z_{\Sigma})\). Note that \(H^0(\mathbb{P}, T_{\mathbb{P}}) \subset H^0(\mathbb{P}_{C(\Delta)}, T_{\mathbb{P}_{C(\Delta)}})\).

For \((\rho \circ \pi)_* T_{\mathbb{P}}\) is torsion free by Remark 3.4 and has as reflexive hull \(T_{\mathbb{P}_{C(\Delta)}}\). Thus we get \(R(N, \Sigma) \subset R(N, \Sigma_{C(\Delta)})\).

Construction 8.5. In the following we assume \((0, 0, 0) \in L^*(\Delta)\) and include the case \(\Delta \neq C(\Delta)\). The proof of Theorem 7.1 still works if \(\Delta \neq C(\Delta)\) but the homomorphisms in the proof of Theorem 7.1 induced by \((pr_1)_*\) are just injective and we have to compute their images:

The image of \(H^0(Y, N_{Y/X})\) in \(H^0(Y, N_{Y/\mathbb{P}})\) is \(L(\Delta)/\mathbb{C} \cdot f\). By definition of the tangent sheaf sequence the homomorphism

\[
j : H^0(\mathbb{P}, T_{\mathbb{P}}) \cong H^0(Y, T_{\mathbb{P}/Y}) \to H^0(Y, N_{Y/\mathbb{P}})
\]

is given by applying the derivations from Corollary 8.3 to \(f = \sum_{m \in \Delta \cap M} a_m x^m \in U_{\text{reg}}(\Delta)\) and restricting to \(Y\). From the proof of Theorem 7.1 we get an isomorphism

\[
H^0(Y, T_{X/Y}) \cong j(H^0(\mathbb{P}, T_{\mathbb{P}})) \cap L(\Delta)/\mathbb{C} \cdot f.
\]

Thus we obtain the generators \(x_1 \cdot \frac{\partial f}{\partial x_1}, \ldots, x_3 \cdot \frac{\partial f}{\partial x_3}\) and additionally

\[
w_{-\alpha}(f) := \sum_{m \in \Delta \cap M} ht_{-\alpha}(m) \cdot a_m \cdot x^{m-\alpha}, \quad \alpha \in R(N, \Sigma) \subset R(N, \Sigma_{C(\Delta)}).
\]

Remark 8.6. If \(\Delta \neq C(\Delta)\) there might be roots \(\alpha \in R(N, \Sigma)\) with \(\alpha \notin R(N, \Sigma_{\Delta})\). In this case \(-\alpha\) is no column vector for \(\Delta\), that is there is \(m \in \Delta \cap M, m \notin \Gamma_{-\alpha}\) with \(m - \alpha \notin \Delta\). Thus if \(a_m \neq 0\) then \(w_{-\alpha}(f)\) does not belong to \(L(\Delta)\). Conversely if \(-\alpha \in R(N, \Sigma) \cap R(N, \Sigma_{\Delta})\) or \(a_m = 0\) for all such \(m\) then \(w_{-\alpha}(f) \in L(\Delta)\). In particular
Theorem 8.7. In the situation of Theorem 7.1 but without the condition $\Delta = C(\Delta)$ we get for $f \in U_{reg}(\Delta)$ generic
\[
\dim \ker(\kappa) = 3 + #(R(N, \Sigma) \cap R(N, \Sigma_\Delta)).
\]
\[\square\]

Remark 8.8. Construction 8.5 allows us also to compute the number of moduli of subfamilies of $U_{reg}(\Delta)$ where we vary just some of the coefficients. For example take $g \in U_{reg}(C(\Delta))$ and let $A \subset C(\Delta)$, then
\[
f := g + \sum_{m \in A \cap M} c_mx^m
\]
is nondegenerate with respect to $C(\Delta)$ for $c_m$ small. We start with $#(A \cap M)$ many parameters (if $g$ is 0 and $f \in U_{reg}(C(\Delta))$ then we start with $#(A \cap M) - 1$ many parameters for then we may quotient by $\mathbb{C}^*$. With Construction 8.5 we may check if
\[
x_i \cdot \frac{\partial f}{\partial x_i} = g + \sum_{m \in A \cap M} d_mx^m
\]
for some $d_m \in \mathbb{C}$, $i = 1, 2, 3$ and the same for $w_{-\alpha}(f)$, where $\alpha \in R(N, \Sigma)$. In this way we see which of these Laurent-polynomials reduce the number of moduli.

9 The vanishing of $H^0(Y, T_Y)$

Theorem 9.1. Let $\Delta$ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then for $Y := Z_\Sigma$
\[
H^0(Y, T_Y) = 0.
\]

Proof. Let
\[
(\mathbb{P}, Y, K) := (\mathbb{P}_\Sigma, Z_\Sigma, K_{\mathbb{P}_\Sigma}), \quad (\mathbb{P}', Y', K') := (\mathbb{P}_\Delta, Z_\Delta, K_{\mathbb{P}_\Delta}).
\]
First by Serre duality we have
\[
H^0(Y, T_Y)^* \cong H^2(Y, \Omega^1_Y \otimes K_Y).
\]
Away from the finitely many singular points \( \mathbb{P}_{\text{sing}} \) there are two exact sequences ([EsVi92, Prop.2.3b,c]) the first one being the Poincaré residue sequence.

\[
0 \to \Omega^2_{\mathbb{P}} \to \Omega^2_{\mathbb{P}}(\log Y) \to \Omega^1_Y \to 0 \quad (9)
\]

\[
0 \to \Omega^2_{\mathbb{P}}(\log Y) \to \Omega^2_{\mathbb{P}} \otimes \mathcal{O}(Y) \to K_Y \otimes N_{Y/P} \to 0 \quad (10)
\]

Tensoring with \( \mathcal{O}(K + Y) \) the sequence (9) stays exact up to finitely many points. Since we just work with cohomology groups \( H^k \) on \( \mathbb{P} \) where \( k \geq 2 \) or \( H^k \) on \( Y \), where \( k \geq 1 \) we do not have to care about this codimension \( \geq 3 \) subset. It is enough to show that

\[
H^2(\mathbb{P}, \Omega^2_{\mathbb{P}}(\log Y)) \otimes \mathcal{O}(Y + K) = 0, \quad H^3(\mathbb{P}, \Omega^2_{\mathbb{P}} \otimes \mathcal{O}(Y + K)) = 0.
\]

The second term vanishes by Mavlyutov’s vanishing Theorem 5.2.

To the first term: Tensor the exact sequence (10) with \( \mathcal{O}(K + Y) \). The divisor \( K_Y \otimes N_{Y/P} = (Y + Y + K)_Y \) is nef and big, thus

\[
H^1(Y, K_Y \otimes (K_Y \otimes N_{Y/P})) = 0
\]

by the Kawamata-Viehweg vanishing Theorem. It remains to show

\[
H^2(\mathbb{P}, \Omega^2_{\mathbb{P}} \otimes \mathcal{O}(2Y + K)) = 0.
\]

By Construction 5.1 we are reduced to show

\[
H^2(\mathbb{P}, \Omega^2_{\mathbb{P}} \otimes \mathcal{O}(2mY + mK)) = 0,
\]

for some \( m \geq 1 \). By the Leray spectral sequence it is enough to show

\[
E_2^{2-i,i} := H^{2-i}(\mathbb{P}', R^i\pi_*(\Omega^2_{\mathbb{P}} \otimes \mathcal{O}(2mY + mK))) = 0, \quad i = 0, 1, 2.
\]

If \( 2mY' + mK' \) is Cartier then by the projection formula

\[
R^i\pi_*(\Omega^2_{\mathbb{P}} \otimes \mathcal{O}(2mY + mK)) = R^i\pi_*(\Omega^2_{\mathbb{P}}) \otimes \mathcal{O}(2mY' + mK').
\]

The sheaves \( R^i\pi_*\Omega^2_{\mathbb{P}} \) are coherent. Choosing \( m \gg 0 \) sufficiently large and divisible by Serre’s vanishing Theorem we get

\[
H^{2-i}(\mathbb{P}', R^i\pi_*(\Omega^2_{\mathbb{P}}) \otimes \mathcal{O}(2mY' + mK')) = 0, \quad i = 0, 1
\]

since \( 2mY' + mK' \) is ample by Proposition 2.4. But by Remark 2.5 we may assume that \( \pi \) has fibres of dimension \( \leq 1 \) thus \( R^2\pi_*(\Omega^2_{\mathbb{P}}) = 0 \).

\( \square \)
10 Examples

Proposition 10.1. Let $\Delta$ be a 3-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and let $Y := Z_\Sigma$ be a minimal model of $Z_f$. Then

$$q(Y) := h^0(Y, \Omega^1_Y) = 0.$$ 

Proof. By the Hodge decomposition it is enough to show that $H^1(Y, \mathbb{C}) = 0$. By ([DK86, Prop.3.4]) we have $H^1(Z_\Delta, \mathbb{C}) = 0$, with $Z_\Delta$ the closure of $Z_f$ in $\mathbb{P}_\Delta$.

Choose a partial toric resolution of singularities $\sigma : \mathbb{P}' \to \mathbb{P}_\Delta$ modifying just the 1-dimensional orbits of $\mathbb{P}_\Delta$. Then $\sigma : Z' \to Z_\Delta$ resolves the singularities and for $E$ a $\sigma$-exceptional irreducible curves on $Z'$ we have $E \cong \mathbb{P}^1$. It follows with the Mayer-Vietoris sequence that $h^1(Z', \mathbb{C}) = 0$. But $Z'$ is gotten by blowing up $Z_\Sigma$ at several points, thus $H^1(Z_\Sigma, \mathbb{C}) = 0$. 

Example 10.2. (Kanev and Todorov surfaces)
A Kanev surface is a minimal surface $Y$ with $p_g(Y) = 1$, $K_Y^2 = 1$.

A Todorov surface is a minimal surface with $p_g(Y) = 1$, $K_Y^2 = 2$, $q(Y) = 0$.

For $\Delta$ a 3-dimensional lattice polytope with $\dim F(\Delta) = 3$ and $l^*(\Delta) = 1$ the minimal surface $Y := Z_\Sigma$ gets a Kanev or Todorov surface: We have $p_g(Y) = l^*(\Delta) = 1$. By Proposition [10] $q(Y) = 0$. $K_Y^2$ could be deduced from the computation of $e(Y)$ in ([Sch18 Appendix A.3]) and Noether’s formula. Among 49 such polytopes in the list ([Sch18 Appendix A.3]) there are 5 maximal polytopes $a), b), c), d), e)$ with respect to the inclusion of sets (see Figure [2] and

$$a), b) : l(\Delta) = 18, \quad c), d), e) : l(\Delta) = 15.$$ 

In $a)$ and $b)$ we get $K_Y^2 = 1$ whereas in $c), d)$ and $e)$ $K_Y^2 = 2$. We find 2 roots of $\text{Aut}(\mathbb{P})$ in $a)$, one root in $c)$ and no root in $b), d)$ and $e)$. Thus we get for the number of moduli

$$a) : \dim \text{Im}(\kappa) = 12, \quad b) : \dim \text{Im}(\kappa) = 14, \quad c) : \dim \text{Im}(\kappa) = 10, \quad d), e) : \dim \text{Im}(\kappa) = 11.$$
Example 10.3. Consider the polytope

\[ \Delta = \langle \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 10 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rangle \]

We have \( l^*(\Delta) = 3 \), \( F(\Delta) \) is 1-dimensional and \( C(\Delta) \) has the additional vertex \((0, -1, 0)\). There are 7 roots

\[ R(N, \Sigma) = \{ \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \} \]

and one root \((-1, 0, -1)\) not belonging to \( R(N, \Sigma_\Delta) \). The column vector \(-\alpha = (1, 0, 1)\) belongs to the facet \( \Gamma_{-\alpha} \) spanned by \((5, 1, 3), (-1, 10, 0), (-1, -1, 0)\). The vertex \((-1, -1, -1)\) does not lie on \( \Gamma_{-\alpha} \) and \(-\alpha + (-1, -1, -1) \notin \Delta \). Since \((-1, -1, -1)\) is a vertex of \( \Delta \) the coefficient of \( f \) at \((-1, -1, -1)\) is \( \neq 0 \) and thus only 6 of the roots in \( R(N, \Sigma) \) reduce the number of moduli. We did not find an example of a lattice polytope \( \Delta \) with necessarily \( \Delta \neq C(\Delta) \), with \((0, 0, 0) \in L^*(\Delta)\), \( R(N, \Sigma) \not\subseteq R(N, \Sigma_\Delta) \) and an \( \alpha \in R(N, \Sigma) \setminus R(N, \Sigma_\Delta) \), such that \( x^{m-\alpha} \in L(\Delta) \) for all vertices of \( \Delta \) not on \( \Gamma_{-\alpha} \).
Figure 2: The maximal polytopes in the classes a), b), c), d) and e).

\[ a) \Delta = \langle a = (2, 1, -2), b = (2, 0, 1), d = (2, 2, 1), p = (-4, -2, 1) \rangle \]

\[ b) \Delta = \langle a = (2, 3, 1), b = (2, -1, 2), c = (2, 0, 1), d = (2, -1, 0), p = (-4, 3, -2) \rangle \]

\[ c) \Delta = \langle a = (2, 1, 5), p = (-2, -1, -3), b = (2, 0, 1), d = (2, 2, 1) \rangle \]

\[ d) \Delta = \langle a = (2, -1, 3), b = (2, 0, 1), c = (2, -1, -1), d = (2, -2, 1), p = (-2, 1, -1) \rangle \]

\[ e) \Delta = \langle a = (2, 0, 1), b = (2, 1, -1), c = (2, 4, -3), d = (2, 1, 1), p = (-2, -2, 1) \rangle \]
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