Solvable Nonlinear Volatility Diffusion Models with Affine Drift

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Abstract

We present a method for constructing new families of solvable one-dimensional diffusions with linear drift and nonlinear diffusion coefficient functions, whose transition densities are obtainable in analytically closed-form. Our approach is based on the so-called diffusion canonical transformation method that allows us to uncover new multiparameter diffusions that are mapped onto various simpler underlying diffusions. We give a simple rigorous boundary classification and characterization of the newly constructed processes with respect to the martingale property. Specifically, we construct, analyse and classify three new families of nonlinear diffusion models with affine drift that arise from the squared Bessel process (the Bessel family), the CIR process (the confluent hypergeometric family), and the Ornstein-Uhlenbeck diffusion (the OU family).

Introduction

A solvable continuous-time stochastic process can be defined as a process for which the transition probability density function is obtainable in analytically closed-form. This kind of solvability permits us to precisely simulate paths of the process from its exact sample distribution. Another important benefit is that some standard methods such as spectral expansions can be applied to obtain closed-form analytical formulas for the transition densities of regular as well as killed solvable diffusions. As a result of that, one can derive other fundamental quantities such as first-hitting time distributions as is done in \([13]\) and \([5]\).

The set of diffusion processes that are, on the one hand, tractable and applicable for mathematical modelling and, on the other hand, exactly solvable is not so vast. This set includes mostly linear diffusion processes or diffusions with power nonlinearity (see \([4]\) and \([12]\) for a comprehensive review of such diffusions). There are two main tools that allow us to construct new solvable diffusion processes. The first is related to a measure change on a chosen underlying diffusion and the second involves a change of variable or smooth monotonic mapping (the Itô formula). In recent years, a new approach that combines special measure changes with nonlinear variable transformations was introduced for uncovering
various new families of exactly solvable driftless diffusion models \([2, 3, 6, 11]\). The method has been coined as “diffusion canonical transformation”, partly in light of the fact that a diffusion process is reduced, or mapped, onto another underlying (simpler) diffusion process. In Subsection 1.1 we state the standard Green’s function methodology and provide the basic background formalism and general assumptions that define the underlying diffusions. Subsection 1.2 presents the diffusion canonical transformation method. Detailed discussions of other canonical aspects of the transformation methodology can be found in \([5]\).

In this paper we discover that a relatively simple change in the nonlinear variable transformation allows us to construct a whole new class of diffusions \((F_t)_{t \geq 0}\) with linear drift and nonlinear diffusion coefficient functions defined by the time-homogeneous stochastic differential equation (SDE) of the form \(dF_t = (a + bF_t)dt + \sigma(F_t)dW_t\). As a result of our study, we upgrade three main families of time-homogeneous diffusions named as Bessel, confluent hypergeometric, and Ornstein-Uhlenbeck families. These processes, which we also refer to as \(F\)-diffusions, arise via the diffusion canonical transformation method by respectively choosing the standard squared Bessel process, CIR process, and Ornstein-Uhlenbeck process as underlying diffusions (see Section 3). It is important that these three families include and recover all the corresponding families of driftless diffusions obtained previously in \([6]\) as special subfamilies. Moreover, the new affine models inherit some of the important salient properties of their driftless counterparts.

One immediate application of such diffusions is asset pricing in finance (when \(a = 0\) and \(b\) is a constant interest rate). In \([6]\), we show that these three families generate local volatility profiles with varied pronounced smiles and skews. Moreover, we prove that there exists a risk-neutral measure so that the driftless asset price process is a martingale. Option pricing under these models is straightforward and is well studied in \([5, 7, 8]\). Clearly, pricing of European options is reduced to the evaluation of a definite integral. Since the first-hitting time distributions for killed solvable diffusions is available in the form of a spectral expansion, no-arbitrage prices of barrier and lookback options are available in closed form as well. As well, discretely-monitored path-dependent options can be evaluated by using a path integral approach. In doing, so we use one important observation that the distribution of a bridge asset price process is reduced to the distribution of the corresponding underlying bridge diffusion (e.g. the Bessel, CIR, or Ornstein-Uhlenbeck bridge processes). Moreover, subfamilies of diffusions belonging to the Bessel and confluent hypergeometric families admit absorption at a boundary point, so they can naturally be used in derivatives pricing under credit risk. Last, but not least, the newly developed \(F\)-diffusions are also applicable to interest rate modelling. Although the diffusions with affine drift \(a + bF\) considered here are not mean-reverting, they still can be useful as short-term interest rate models thanks to the nonlinear local volatility function. Alternatively, by using a suitable change of variable, we are able to obtain models with nonlinear mean-reversion that permit closed-form solutions for valuing zero-coupon bonds.

One of the main contributions of this paper (in comparison with \([6]\) and references therein) is the analysis and classification of allowable mapping functions that lead to new diffusions with affine drift. Subsection 1.3 presents some results for proving the smooth monotonicity property of mappings that lead to \(F\)-diffusions with affine drift where \(b \neq 0\). Another main result, presented in Subsection 1.4, is the complete boundary classification of such newly constructed regular diffusions. Finally, we analyse whether or not such diffusions \((F_t)_{t \geq 0}\) “preserve” the drift rate, i.e. whether \(\frac{d}{dT}E[F_T | F_t] = E[a + bF_T | F_t], t \leq T,\) holds.
This property can be viewed as a generalization of the martingale property for *driftless* processes, where \( E[F_T \mid F_t] = F_t, 0 \leq t \leq T \), and, therefore, \( \frac{d}{dt}E[F_T \mid F_t] = 0 \) holds. Therefore, for the special case with \( a = 0 \) we prove that the discounted asset price process \( (e^{-bf}F_t) \) is a martingale. With the aim to provide a rigorous framework for the above characterizations of the families of \( F \)-diffusions, in Section 2 we present a theorem that state easy-to-implement limit (asymptotic) conditions for verifying the conservation of the expectation rate of the \( F \)-diffusions.

In Section 4 we apply the theory of Section 1 and Section 2 to the above-named three chosen underlying solvable diffusions defined in Section 3. In doing so, we give a construction, probabilistic classification and analysis of the resulting new three families of \( F \)-diffusion models. In Section 4.2 we discuss all the possible monotonic maps that lead to nonlinear \( F \)-diffusions with affine drift \( a + bF \) where \( b \neq 0 \). Some conclusions are drawn in Section 5. The Appendix contains useful asymptotic properties of certain fundamental solutions that characterise the three known underlying diffusions.

1 General Construction of Nonlinear Solvable Diffusions with Affine Drift

1.1 Underlying diffusion

Let \((X_t)_{t \geq 0}\) be a one-dimensional time-homogeneous regular diffusion on \( I \equiv (l, r), -\infty \leq l < r \leq \infty\), defined by its infinitesimal generator:

\[
G_x f \equiv (\mathcal{G} f)(x) \equiv \frac{1}{2} \nu''(x)f''(x) + \lambda(x)f'(x), \quad x \in I. \tag{1.1}
\]

The functions \(\lambda(x)\) and \(\nu(x)\) denote, respectively, the (infinitesimal) drift and diffusion coefficients of the process. Throughout we assume that the functions \(\lambda(x), \lambda'(x), \nu(x)\) and \(\nu''(x)\) are continuous on the open interval \(I\) and that \(\nu(x)\) is strictly positive on \(I\).

Let \(p_X(t; x_0, x)\) denote a transition probability density function (PDF) for the diffusion specified by the conditional probability that \(X_t\) is contained in an interval \(D \subseteq I\): \(P\{X_{s+t} \in D \mid X_s = x_0\} = \int_D p_X(t; x_0, x) \, dx\), \(\forall x_0 \in I, t, s \geq 0\). Since the drift and diffusion coefficients are assumed to have no explicit time dependence, then \(p = p_X(t; x_0, x)\) satisfies the time-homogeneous forward Kolmogorov equation \(\frac{\partial p}{\partial t} = \mathcal{L}_x p\) and backward Kolmogorov equation \(\frac{\partial p}{\partial s} = \mathcal{G}_x p\) subject to the Dirac delta function initial condition \(p(0+; x_0, x) = \delta(x - x_0)\). Here \(\mathcal{L}_x\) denotes the Fokker-Planck differential operator (see, e.g., [10]).

The diffusion \((X_t)_{t \geq 0}\) can also be defined by its *speed measure* \(M(dx)\) and *scale function* \(S(x)\) (see, e.g., [4]). For diffusions considered here, these characteristics are absolutely continuous with respect to the Lebesgue measure and have smooth derivatives. The speed and scale density functions \(m(x) = \frac{M(dx)}{dx}\) and \(s(x) = \frac{dS(x)}{dx}\) are as follows:

\[
s(x) = \exp \left( - \int^x 2\lambda(z) \nu^2(z) \, dz \right) \quad \text{and} \quad m(x) = \frac{2}{\nu^2(x)s(x)}. \tag{1.2}
\]

The functions \(s(x)\) and \(m(x)\) are hence continuous and strictly positive on \(I\). The infinites-
imal generator and Fokker-Planck differential operator can be re-written in compact form:
\[
(G \varphi)(x) = \frac{1}{m(x)} \left( \frac{f'(x)}{s(x)} \right) ', \quad (L \varphi)(x) = \left( \frac{1}{s(x)} \left( \frac{f(x)}{m(x)} \right) ' \right) '.
\]

As is well known, there are two linearly independent fundamental solutions \( \varphi^+_s \) and \( \varphi^-_s \) of the ordinary differential equation (ODE)
\[
(G \varphi)(x) = s \varphi(x), \ x \in \mathcal{I},
\]
such that for positive real values \( s = \rho > 0 \) the solutions \( \varphi^+_s(x) \) and \( \varphi^-_s(x) \) are correspondingly increasing and decreasing functions of \( x \) (see, e.g., [4]). The functions \( \varphi^\pm_\rho(x), \rho > 0 \), are convex and finite in the interior of the domain \( \mathcal{I} \).

In this paper we consider underlying diffusions \((X_t)_{t \geq 0}\) for which the endpoints \( l \) and \( r \) are either unattainable or regular instantaneously reflecting boundaries (see Feller’s classification in [4, 10]).

Let \( s_1, s_2 \in \mathbb{C} \) with positive real parts, then the functions \( \varphi^\pm \) satisfy the asymptotic relations
\[
\lim_{x \to l^+} \varphi^+_s(x) = 0 \quad \text{and} \quad \lim_{x \to r^-} \varphi^-_s(x) = \infty.
\]

Moreover, the square integrability conditions
\[
(\varphi^+_\rho, \varphi^+_\rho)(t,x) < \infty \quad \text{and} \quad (\varphi^-_\rho, \varphi^-_\rho)(t,x) < \infty
\]
hold for \( \rho > 0 \) and \( x \in \mathcal{I} \). Throughout this paper we conveniently define the inner product of two functions \( f, g \) with \( m \) on a closed interval \([a, b]\) as
\[
(f, g)_{[a, b]} \triangleq \int_a^b f(x)g(x)m(x)dx
\]
and \((f, g)_{(a,b]} \triangleq \lim_{\varepsilon \to a^+} (f, g)_{(x,b]}\), \((f, g)_{[a,b)} \triangleq \lim_{\varepsilon \to b^-} (f, g)_{[a,\varepsilon)}\), \(||f||_{(a,b]} \triangleq (f, f)_{(a,b]}\).

The Wronskian of the fundamental solutions can be computed as follows:
\[
W[\varphi^-_s, \varphi^+_s](x) \triangleq \varphi^-_s(x) d\varphi^+_s(x) dx - \varphi^+_s(x) d\varphi^-_s(x) dx = w_s \delta(x),
\]
where \( w_s \) is a constant w.r.t. \( x \) and \( w_\rho > 0 \) for real \( \rho > 0 \).

The time-independent Green’s function \( G(x, x_0, s) \) is defined by the Laplace transform of \( p_X(t; x_0, x) \) with respect to time \( t \). Consequently, a solution for the PDF \( p_X(t; x_0, x) \) follows by Laplace inversion, as given by the Bromwich integral. For an in-depth discussion of this inversion and how it leads to various spectral expansions for the transition PDF, see [12] and [5].

The Green’s function \( G \), for \( x, x_0 \in \mathcal{I} \), is written in terms of the functions \( \varphi^+_s, \varphi^-_s \), and \( m(x) \) in the standard form [4]:
\[
G(x, x_0, s) = w^{-1}_s m(x) \varphi^+_s(x_0) \varphi^-_s(x_0),
\]
where \( x_- \equiv \min\{x, x_0\} \) and \( x_+ \equiv \max\{x, x_0\} \). The boundary conditions for \( G \) follow from those satisfied by the appropriate solutions \( \varphi^+_s \) and \( \varphi^-_s \).
1.2 Diffusion Canonical Transformation

Consider a class of one-dimensional time-homogeneous regular diffusions \((X_t^{(\rho)})_{t \geq 0} \in \mathcal{I}\) with infinitesimal generator

\[
(G^{(\rho)} f)(x) \triangleq \frac{1}{2} \sigma^2(x) f''(x) + \left( \lambda(x) + \nu^2(x) \frac{\hat{u}_{\rho}(x)}{\hat{u}_{\rho}(x)} \right) f'(x),
\]

where \(\lambda(x)\) and \(\nu(x)\) are the infinitesimal drift and diffusion coefficients of the underlying process \(X_t\) defined by (1.1). A strictly positive generating function \(\hat{u} = \hat{u}_{\rho}(x)\) is a linear combination of the fundamental solutions \(\varphi_{\rho}^F\):

\[
\hat{u}_{\rho}(x) = q_1 \varphi_{\rho}^F(x) + q_2 \varphi_{\rho}^{-F}(x),
\]

with parameters \(q_1, q_2 \geq 0\) and at least one of them being strictly positive. Throughout, \(\rho\) is a real positive parameter. The speed and scale densities for an \(X^{(\rho)}\)-diffusion are given in terms of those for the underlying \(X\)-diffusion:

\[
m_{\rho}(x) = \hat{u}_{\rho}^2(x) m(x) \quad \text{and} \quad s_{\rho}(x) = \frac{\sigma(x)}{\hat{u}_{\rho}(x)}. \quad (1.11)
\]

By comparing the generators (1.1) and (1.9), observe that \(X^{(\rho)}\)-diffusions can also be viewed as arising from the underlying \(X\)-diffusion by the application of a measure change. In fact, the \(X^{(\rho)}\)-diffusion can be realized from the \(X\)-diffusion upon employing a so-called Doob-h transform, where \(h = \hat{u}_{\rho}\), or \(\rho\)-excessive transform. For more details on this and its connection to Girsanov’s Theorem, see [5]. Both processes are regular on the same state space \(\mathcal{I} = (l, r)\). A transition density \(p_X^{(\rho)}(t; x_0, x)\) for an \(X^{(\rho)}\)-diffusion is then related to that for the \(X\)-diffusion (in the same original \(P\)-measure) by

\[
p_X^{(\rho)}(t; x_0, x) = e^{-\rho t} \frac{\hat{u}_{\rho}(x)}{\hat{u}_{\rho}(x_0)} p_X(t; x_0, x), \quad x, x_0 \in \mathcal{I}, \quad t > 0. \quad (1.12)
\]

It is important to note that this relation (as well as equation (1.15) below) also holds for killed diffusions. In such cases, regular killing boundary conditions are (additionally) imposed at one or two interior points and the transition PDFs for the \(X\) and \(X^{(\rho)}\) killed diffusions are defined on the same subspace of \(\mathcal{I}\) (see [5] for theory and applications of various killed diffusions).

We now consider \(F\)-diffusions \(\{F_t \triangleq F(X_t^{(\rho)}), t \geq 0\}\) defined by strictly monotonic real-valued mapping \(F = F(x)\) with \(F', F''\) continuous on \(\mathcal{I}\) and having infinitesimal generator

\[
(G_F h)(F) \triangleq \frac{1}{2} \sigma^2(F) h''(F) + (a + bF) h'(F), \quad F \in \mathcal{I}_F = (F(l), F(r)). \quad (1.13)
\]

\((F_t)_{t \geq 0}\) is a regular diffusion on \(\mathcal{I}_F\) with endpoints \(F(l) = \min\{F(l+), F(r-)\}\) and \(F(r) = \max\{F(l+), F(r-)\}\). The diffusion coefficient function \(\sigma(F)\), as given by (1.23) below, is generally nonlinear and where \(a\) and \(b\) are arbitrary real constants such that \(b = 0\) implies \(a = 0\). The corresponding scale and speed densities are

\[
s_F(F) \triangleq \exp\left(-\int_{F}^{F'} \frac{2(a + bz)}{\sigma^2(z)} dz\right) \quad \text{and} \quad m_F(F) \triangleq \frac{2}{\sigma^2(F) s_F(F)}. \quad (1.14)
\]
Several families of analytically solvable state dependent volatility models can be derived from known underlying diffusion processes defined by \([1,11]\). We refer to this construction as the “diffusion canonical transformation” methodology. For driftless diffusions, see \([2,6,7]\). Extensions and applications to new families with linear drift \((a = 0, b \neq 0)\) are discussed in \([5]\).

The transition PDF \(p_F\) for an \(F\)-diffusion \((F_t)_{t \geq 0}\) is related to the transition PDF for the underlying \(X\) (or \(X^{(\nu)}\)) diffusion as follows:

\[
p_F(t; F_0, F) = \frac{\nu(X(F))}{\sigma(F)} p_X^{(\nu)}(t; X(F_0), X(F)) = \frac{\nu(X(F))}{\sigma(F)} \frac{\hat{a}_p(X(F))}{\sigma(X(F_0))} e^{-\mu t} p_X(t; X(F_0), X(F)).
\]

where \(F, F_0 \in \mathcal{I}_F, t > 0\). Here \(X \triangleq F^{-1}\) is the inverse map so that

\[|X'(F)| = \frac{\nu(X(F))}{\sigma(F)}.\]

This methodology was originally developed for driftless \(F\)-diffusions where \(a = b = 0\). For such cases, the volatility function has the form

\[\sigma(F) = \frac{\sigma_0 \nu(x) a(x)}{\hat{a}_\rho^2(x)}, \quad x = X(F), \quad \sigma_0 > 0.\]

The map \(F(x)\) (that solves equation \((1.18)\) below for the special case \(a = b = 0\)) admits the general quotient form:

\[F(x) = \frac{c_1 \varphi^+(x) + c_2 \varphi^-(x)}{q_1 \varphi^+(x) + q_2 \varphi^-(x)},\]

where \(c_1, c_2, q_1, q_2 \in \mathbb{R}\) are parameters such that \(q_1 c_2 - q_2 c_1 \neq 0\). Several families of \(F\)-diffusions arising from various choices of underlying diffusions (such as the squared Bessel, Ornstein-Uhlenbeck, CIR and Jacobi processes) are studied in \([2,6,5]\). In this paper, we expand the families constructed by finding new mapping functions \(F\) so that the newly obtained diffusions have generator in \([1,13]\), i.e. they satisfy an SDE with affine drift:

\[dF_t = (a + bF_t)dt + \sigma(F_t)dW_t.\]

Let \(F\) be a strictly monotonic twice continuously differentiable function. Applying the Itô formula, the process \(F_t = F(X_t^{(\nu)})\) satisfies the SDE

\[dF_t = (G^{(\nu)} F)(X(F_t))dt + \nu(X(F_t))|F'(X(F_t))|dW_t.\]

Clearly, by equating these two SDEs, the mapping function \(F(x)\) leads to a process with affine (linear) drift defined by the generator in \([1,13]\) and solves the following 2nd order linear nonhomogeneous ODE:

\[G^{(\nu)} F)(x) = a + bF(x).\]

The strictly positive diffusion coefficient is then \(\sigma(F) = \nu(X(F))|F'(X(F))|\), or

\[\sigma(F(x)) = \nu(x)|F'(x)|.\]
Lemma 1. Let $b \neq 0$ and $\rho, \rho + b > 0$ hold. Then the solution to equation (1.18) takes the general form

$$F(x) = -\frac{a}{b} + \frac{c_1\varphi_{\rho+b}^+(x) + c_2\varphi_{\rho+b}^-(x)}{q_1\varphi_{\rho}^+(x) + q_2\varphi_{\rho}^-(x)} \equiv -\frac{a}{b} + \frac{\hat{v}_{\rho+b}(x)}{\hat{u}_{\rho}(x)} \quad (1.20)$$

where $c_1$ and $c_2$ are arbitrary real constants.

**Proof.** The numerator function $\hat{v} = \hat{v}_{\rho+b}(x)$, defined in (1.20), is a linear combination of $\varphi_{\rho+b}^\pm$ and hence solves $G\hat{v} = (\rho + b)\hat{v}$. The positive function $\hat{u} = \hat{u}_{\rho}(x)$ in the denominator solves $G\hat{u} = \rho\hat{u}$. Now differentiating and using the identity $G(\varphi) = Gf(x) + \nu^2(x)(\hat{u}'(x)/\hat{u}_{\rho}(x))f'(x)$, we readily have

$$G(\varphi)\hat{u} = \frac{1}{\hat{u}} \left( G\hat{v} - \frac{\hat{v}}{\hat{u}} G\hat{u} \right) = \frac{1}{\hat{u}} \left( (\rho + b)\hat{v} - \frac{\hat{v}}{\hat{u}} (\rho\hat{u}) \right) = b\frac{\hat{v}}{\hat{u}}.$$  

Hence, $\frac{\hat{v}}{\hat{u}}$ is a general solution to the corresponding homogeneous ODE (eq. (1.18) for $a = 0$) since the Wronskian

$$W\left[\frac{\varphi_{\rho}^+, \varphi_{\rho}^-}{\hat{u}_{\rho}}, \frac{\varphi_{\rho+b}^+, \varphi_{\rho+b}^-}{\hat{u}_{\rho+b}}\right](x) = \frac{W[\varphi_{\rho}^+, \varphi_{\rho}^-](x)}{\hat{u}_{\rho}(x)} \neq 0 \quad (1.7).$$

The constant function $F_p(x) = -a/b$ is a particular solution of (1.18).

The derivative of the mapping in (1.20) is simply

$$F'(x) = \frac{\hat{v}_{\rho+b}'(x)\hat{u}_{\rho}(x) - \hat{u}_{\rho}'(x)\hat{v}_{\rho+b}(x)}{\hat{u}_{\rho}^2(x)} = \frac{W(x)}{\hat{u}_{\rho}^2(x)}, \quad (1.21)$$

where we define the Wronskian

$$W(x) \equiv W(x; \rho, \rho + b) \triangleq W[\hat{u}_{\rho}, \hat{v}_{\rho+b}](x).$$

Assuming $F$ is strictly monotonic, and given an $X$-diffusion, the $F$-diffusion coefficient function $\sigma(F)$ is then given by substituting (1.21) into (1.19):

$$\sigma(F) = \frac{\nu(x)|W(x)|}{\hat{u}_{\rho}^2(x)}, \quad x = X(F), \; F \in \mathcal{F}. \quad (1.23)$$

We note that this expression holds for all parameter choices $a, b$ except when $a \neq 0$ and $b = 0$. For the latter special case (i.e. with constant nonzero drift function) equation (1.18) reads $G(\varphi) = a$ and hence simply reduces to a linear first order ODE in $F'$. Solving leads to various monotonic maps which in turn give rise to nonzero constant drift $F$-diffusions with various nonlinear specifications for the diffusion coefficients. In this paper, we shall not discuss the details of such special families as we focus on linear drift functions with $b \neq 0$.

### 1.3 Monotonic Maps

The map $F : \mathcal{I} \to \mathcal{I}_F$ in (1.20) does not generally satisfy $F'(x) \neq 0$. To guarantee that $(F_t)_{t \geq 0}$ is a regular diffusion on $\mathcal{I}_F = (F(0), F(t))$ the map $F$ has to be strictly monotonic. Then $F'(x) \neq 0$ and hence the diffusion coefficient function $\sigma(F)$ is strictly positive on $\mathcal{I}_F$. 

From \[\text{(1.21)}\] we observe that the sign of \(F'\) equals the sign of \(W(x)\). Using the representations of \(\hat{u}_\rho\) and \(\hat{v}_{\rho+b}\) in terms of \(\varphi^+_\rho\), \(\varphi^+_{\rho+b}\) gives

\[
W(x) = q_1c_1 W[\varphi^+_\rho, \varphi^+_\rho+b](x) + q_1c_2 W[\varphi^+_\rho, \varphi^-_{\rho+b}](x) + q_2c_1 W[\varphi^+_{\rho+b}, \varphi^+_\rho](x) + q_2c_2 W[\varphi^-_{\rho+b}, \varphi^-_{\rho+b}](x).
\]

(1.24)

There are two important cases where \(F\) is strictly monotonic. Recall that the fundamental solutions \(\varphi^+_\rho(x)\) and \(\varphi^-_{\rho}(x)\) are correspondingly strictly increasing and decreasing functions of \(x\). Therefore, the ratios \(\varphi^+_{\rho+b}(x) / \varphi^+_{\rho}(x)\) and \(\varphi^-_{\rho+b}(x) / \varphi^-_{\rho}(x)\) are strictly increasing and decreasing functions, respectively. In particular, we have the strict inequalities \(W[\varphi^+_\rho, \varphi^-_{\rho+b}](x) < 0\) and \(W[\varphi^-_{\rho+b}, \varphi^+_{\rho+b}](x) > 0\). Thus, the two choices of parameters \(c_2 = q_1 = 0\), \(c_1/q_2 = \pm c\) or \(c_1 = q_2 = 0\), \(c_2/q_1 = \pm c\) in equation \[\text{(1.20)}\] lead to dual subfamilies of strictly monotonic maps defined by \(F = F^{(1)}\).

\[
F^{(1)}(x) := -\frac{a}{b} + \epsilon \frac{\varphi^+_{\rho+b}(x)}{\varphi^+_{\rho}(x)},
\]

(1.25)

where \(\epsilon = \pm 1\) and \(c > 0\) is constant. Other parameter choices that lead to other families of monotonic maps are discussed in Section 4.2. The following propositions are useful in verifying whether or not \(W(x)\), and hence \(F'(x)\), changes sign in \(I\).

**Proposition 1.** The Wronskian \(W(x)\) in \[\text{(1.22)}\] satisfies the first order linear ODE \(\frac{1}{2}v^2(x)W'(x) + \lambda(x)W(x) = b \hat{u}_\rho \hat{v}_{\rho+b}\), and for any \(x, x_0 \in I\) the solution admits the following representation:

\[
\frac{W(x)}{s(x)} = \frac{W(x_0)}{s(x_0)} + b \int_{x_0}^{x} m(y) \hat{u}_\rho(y) \hat{v}_{\rho+b}(y) dy.
\]

(1.26)

**Proof.** The proof follows by direct verification upon using \(G \hat{u}_\rho = \rho \hat{u}_\rho\) and \(G \hat{v}_{\rho+b} = (\rho + b) \hat{v}_{\rho+b}\).

Note that for the driftless case, with \(a = b = 0\), the function \(\frac{W(x)}{s(x)}\) is constant and from \[\text{(1.7)}\] and \[\text{(1.21)}\]:

\[
W(x) = (c_1q_2 - c_2q_1)w_\rho s(x).
\]

Therefore, equation \[\text{(1.16)}\] defining \(\sigma(F)\) for all families of driftless \(F\)-diffusions is recovered as a particular case of the more general specification given by equation \[\text{(1.23)}\]. Moreover, the specification in \[\text{(1.23)}\] gives rise to a class of state dependent volatility models that generally can also have a dependence on the drift parameters \(a\) and \(b\).

**Proposition 2.** Assume that \(c_1\) and \(c_2\) in \[\text{(1.20)}\] are both nonnegative or nonpositive (with at least one of them being nonzero) and that \(W(x)\) in \[\text{(1.22)}\] preserves its sign as \(x\) approaches either endpoint \(l\) or \(r\); that is, \(\text{sign}(W(l+)) = \text{sign}(W(r-))\). Then \(\forall x \in I, W(x) \neq 0\), i.e. \(W(x)\) is either strictly positive or negative on \(I\).

**Proof.** Under the assumed conditions on \(c_1\) and \(c_2\), the function \(\hat{v}_{\rho+b}\) is either strictly positive or strictly negative on \(I\). Recall that \(\hat{u}_\rho\) is strictly positive. Hence, the function \(W(x)/s(x)\) given by \[\text{(1.26)}\] is monotonic in \(x\) and, since \(s(x) > 0\), \(W(x)\) has at most one zero in \(I\). Then \(\text{sign}(W(l+)) = \text{sign}(W(r-))\) implies \(W(x) \neq 0\), i.e. either \(W(x) > 0\) or \(W(x) < 0\) for all \(x \in I\).
Proposition 3. Assume that \(c_1\) and \(c_2\) for the map \(F\) in (1.20) are both nonzero and have opposite signs.

(i) If \(b > 0\), then \(W(x)\) is either strictly positive or negative on \(I\).

(ii) Let \(b < 0\) and \(x \in I\). If \(\text{sign}(W(l+)) = \text{sign}(W(r-)) = +1\) and \(c_1 > 0\), then \(W(x) > 0\). If \(\text{sign}(W(l+)) = \text{sign}(W(r-)) = -1\) and \(c_1 < 0\), then \(W(x) < 0\).

**Proof.** By definition of \(\hat{\nu}_{p+b}\), and the fact that \(\varphi^+\) and \(\varphi^-\) are respectively increasing and decreasing functions, we have \(\hat{\nu}'_{p+b}(x) > 0\) (or \(\hat{\nu}'_{p+b}(x) < 0\)) when \(c_1 > 0\), \(c_2 < 0\) (or \(c_1 < 0\), \(c_2 > 0\)), i.e. \(\hat{\nu}_{p+b}(x)\) is either a strictly increasing (or decreasing) function. Moreover, if follows from such monotonicity and the properties in (1.4) that \(\hat{\nu}_{p+b}(x)\) has exactly one zero, at \(x = \hat{x}_0 \in I\) where \(\varphi^+_{p+b}(\hat{x}_0)/\varphi^-_{p+b}(\hat{x}_0) = |c_2/c_1|\). Then, \(W(\hat{x}_0) = \hat{\nu}_{p}(\hat{x}_0)\varphi^+_{p+b}(\hat{x}_0)\) is accordingly strictly positive (or negative). Setting \(x_0 = \hat{x}_0\) in (1.20) now gives

\[
\frac{W(x)}{s(x)} = \frac{W(\hat{x}_0)}{s(\hat{x}_0)} + \epsilon b \int_{\min(\hat{x}_0,x)}^{\max(\hat{x}_0,x)} m(y)\hat{\nu}_{p}(y)|\varphi^+_{p+b}(y)| \, dy
\]

where \(\epsilon = +1(-1)\) if \(c_1 > 0\) (\(c_1 < 0\)). Hence, if \(b > 0\) then either \(W(x) > 0\) or \(W(x) < 0\), for all \(x \in I\), in the respective cases. If \(b < 0\), then \(W(x)\), as given in the last expression, can either have no zeros or at most two zeroes in \(I\). \(W(x)\) has no zeros if and only if \(\text{sign}(W(\hat{x}_0)) = \text{sign}(W(l+)) = \text{sign}(W(r-))\). This hence proves part (ii) for the respective cases.

As follows from the above propositions, to determine the monotonicity of a map \(F\), of interest is the asymptotic behaviour of \(W(x)\), as \(x\) approaches endpoint \(l\) or \(r\). Below we consider three families of \(F\)-diffusions arising from three separate underlying diffusions; namely, the squared Bessel (SQB) process, the Cox-Ingersoll-Ross (CIR) model, and the Ornstein-Uhlenbeck (OU) process. For all these families, the asymptotic properties of the fundamental solutions \(\varphi^\pm\) and of the corresponding Wronskian functions are presented in Appendix A.

### 1.4 Boundary Classification for \(F\)-Diffusions

Given an underlying \(X\)-diffusion and \(\rho > 0\), any regular diffusion \((X_t^{(\rho)})_{t \geq 0} \in (l,r)\) with generator \(G^{(\rho)}\) in (1.9) falls into one of three general families:

(i) \(\{q_1 = 0, q_2 > 0\}\) where \(\hat{\nu}_\rho(x) = q_2 \varphi^-_\rho(x)\),

(ii) \(\{q_1 > 0, q_2 = 0\}\) where \(\hat{\nu}_\rho(x) = q_1 \varphi^+_\rho(x)\),

(iii) \(\{q_1, q_2 > 0\}\) where \(\hat{\nu}_\rho(x) = q_1 \varphi^+_\rho(x) + q_2 \varphi^-_\rho(x)\).

**Lemma 2.** The above three families (i)-(iii) of regular diffusions \((X_t^{(\rho)})_{t \geq 0}\) on \((l, r)\) with generator \(G^{(\rho)}, \rho > 0\), defined by (1.9) and (1.10) have the following boundary classification:

(i) \(q_1 = 0, q_2 > 0\): The boundary \(l\) is attracting natural if \((\varphi^+_\rho, \varphi^-_\rho)_{l,t} = \infty\), exit if \((\varphi^+_\rho, \varphi^-_\rho)_{l,t} < \infty\) and \((\varphi^+_\rho, \varphi^-_\rho)_{l,t} = \infty\), and is otherwise regular when \((\varphi^-_\rho, \varphi^-_\rho)_{l,t} < \infty\). The boundary \(r\) is non-attracting natural if \((\varphi^+_\rho, \varphi^-_\rho)_{r,x} = \infty\) and is otherwise entrance.
(ii) $q_1 > 0, q_2 = 0$: The boundary $r$ is attracting natural if $(\varphi^+_p, \varphi^-_p)_{[x, r)} = \infty$, exit if $(\varphi^+_p, \varphi^-_p)_{[x, r]} < \infty$ and $(\varphi^+_p, \varphi^-_p)_{(r, \infty)} = \infty$, and is otherwise regular when $(\varphi^+_p, \varphi^-_p)_{[x, r]} < \infty$. The boundary $l$ is non-attracting natural if $(\varphi^+_p, \varphi^-_p)_{(l, x]} = \infty$ and is otherwise entrance.

(iii) $q_1 > 0, q_2 > 0$: The boundary $l$ has the same classification as in (i) and $r$ has the same classification as in (ii).

Proof. See [6].

According to this Lemma, the left and right boundary classification follows simply from the asymptotics of $m(x)\varphi^+_p(x)\varphi^-_p(x)$ as $x \to l+$ and $x \to r-$, $m(x)[\varphi^+_p(x)]^2$ as $x \to r-$ and $m(x)[\varphi^-_p(x)]^2$ as $x \to l+$.

**Theorem 1.** The boundary classification for an $F$-diffusion is equivalent to the corresponding $X^{(p)}$-diffusion.

Proof. This follows by the diffeomorphism $X^{(p)}_t \to F_t = F(X^{(p)}_t)$. See [6] for details.

## 2 Drift Rate Conservation and Martingale Property for F-Diffusions

For any time-homogeneous $F$-diffusion defined by (1.13), we introduce the rate of change of the conditional expectation:

$$
\frac{\partial}{\partial t} \mathbb{E}[F_{t+\tau} | F_t = Y] = \int_{F_{(t)}}^{F_{(r)}} F \frac{\partial}{\partial F} p_F(t; Y, F) \, dF, \quad Y \in \mathcal{F}_t, \ t > 0, \tau \geq 0.
$$

(2.1)

The PDF $p_F$ given by (1.15) satisfies the forward Kolmogorov equation

$$
\frac{\partial p_F}{\partial t} = \frac{\partial}{\partial F} \left( \frac{1}{s_F} \frac{\partial}{\partial F} \left( \frac{p_F}{m_F} \right) \right),
$$

(2.2)

with densities in (1.14) now defined in terms of those for the $X^{(p)}$-diffusion:

$$
m_F(F) = m_p(X(F)) |X'(F)| \quad \text{and} \quad s_F(F) = s_p(X(F)) |X'(F)|.
$$

(2.3)

Using (2.2) within (2.1), and applying integration by parts together with the derivative

$$
\frac{d}{dF} \left( \frac{1}{s_F(F)} \right) = (a + bF)m_F(F),
$$

(2.4)

gives the rate in (2.1) expressed as a sum:

$$
\frac{\partial}{\partial t} \mathbb{E}[F_{t+\tau} | F_t = Y] = \int_{F_{(t)}}^{F_{(r)}} (a + bF) p_F(t; Y, F) \, dF + \mathcal{E}(Y, t).
$$

The “bias” term $\mathcal{E}(Y, t)$ is given by the difference of two limits:

$$
\mathcal{E}(Y, t) = \left[ F \frac{\partial}{\partial F} \left( \frac{p_F(t; Y, F)}{m_F(F)} \right) - \frac{1}{s_F(F)} \frac{p_F(t; Y, F)}{m_F(F)} \right]_{F = F_{(t)}}^{F = F_{(r)}} - \frac{F(x)}{s_p(x)} \frac{p_p^{(p)}(t; y, x)}{m_p(x)} \left[ \frac{F'(x) \ p_p^{(p)}(t; y, x)}{s_p(x)} \right]_{x = l}\bigg|_{x = r-}.
$$

(2.5)
The last expression follows by changing variables \( x = X(F), y = X(Y) \) and by combining (2.3) and (1.15).

Consider the case where \( E(Y_t) = 0 \), for any \( F_t = Y \in \mathcal{F}_t, t > 0 \). Then, for such diffusions one obtains a simple representation for the rate of change in (2.4) as

\[
\frac{\partial}{\partial t} \mathbb{E}[F_{t+\tau} | F_t] = a \mathbb{E}[\mathbb{1}_{F_{t+\tau} \in \mathcal{F}_t} | F_t] + b \mathbb{E}[F_{t+\tau} | F_t],
\]

i.e. the rate of change of the conditional expectation equals the conditional expectation of the drift function for the process. If (i.e. the rate of change of the conditional expectation equals the conditional expectation of the drift function for the process). If \( \mathbb{E}[\mathbb{1}_{F_{t+\tau} \in \mathcal{F}_t} | F_t] \equiv \mathbb{P}\{F_{t+\tau} \in \mathcal{F}_t | F_t \} = 1 \) for all \( \tau \geq 0, t > 0 \), or if \( a = 0 \) holds, then \( E(t) \triangleq E[F_{t+\tau} | F_t = Y] \) satisfies the trivial linear ODE \( E'(t) = a + b E(t) \), subject to \( E(0) = Y \). Let \( a = 0 \), then there exists a well-known geometric drift solution \( E(t) = Ye^{bt} \), i.e. \( E[F_{t+\tau} | F_t = Y] = Ye^{bt} \) for all \( Y \in \mathcal{F}_t, \tau \geq 0, t > 0 \). In other words, the “discounted process” \( (e^{-bt}F_t)_{t \geq 0} \) is a martingale in case \( a = 0 \) and \( \mathcal{E} = 0 \). Note also that the discounted process is a strict supermartingale (submartingale) when \( a = 0 \) and \( \mathcal{E} < 0 \) (\( \mathcal{E} > 0 \)) for all \( t > 0 \). Setting \( \tau = 0 \) recovers the unconditional expectation \( E[F_t] = F_0e^{bt} \).

By Laplace transforming (2.5) with respect to time \( t \), changing the order of integration and differentiation, while employing the assumptions formulated in Subsection [1.1] and using (1.8), (1.12) and (1.21), the following theorem is readily proven. A similar proof for driftless \( F \)-diffusions is given in [6].

**Theorem 2.** Let \( F \) be a monotonic map given by (1.29) for \( b \neq 0 \) and by (1.17) for \( a = b = 0 \). The diffusion \( F_t = F(X_{t}^{(r)}), t \geq 0 \), with \( (X_{t}^{(r)})_{t \geq 0} \) defined by (1.9), conserves the expectation rate, i.e. the relation (2.6) is true if and only if the following boundary conditions hold:

\[
\lim_{x \to -t+} \left[ F(x) \frac{W[u_{r}, \varphi_{r+s}^{+}](x)}{s(x)} - \frac{W[\hat{u}_{r}, \hat{\varphi}_{r+s}^{+}](x)}{s(x)} \right] = 0, \tag{2.7}
\]

\[
\lim_{x \to -r-} \left[ F(x) \frac{W[u_{r}, \varphi_{r+s}^{+}](x)}{s(x)} - \frac{W[\hat{u}_{r}, \hat{\varphi}_{r+s}^{+}](x)}{s(x)} \right] = 0, \tag{2.8}
\]

for all complex-valued \( s \) such that \( Re s > c \), where \( c \) is some real constant.

## 3 Three Choices of Underlying Solvable Diffusions

### 3.1 The Squared Bessel Process

We specifically consider an underlying \( \lambda_0 \)-dimensional squared Bessel (SQB) process obeying the SDE \( dX_t = \lambda_0 dt + \nu_0 \sqrt{X_t} dW_t \) with constants \( \nu_0 > 0 \) and \( \lambda_0 > 0 \). The scale and speed densities for the diffusion are \( s(x) = x^{-\mu-1} \) and \( m(x) = \frac{2}{\nu_0^2} x^\mu \), where \( \mu = \frac{2\lambda_0}{\nu_0^2} - 1 \). Here we take \( \lambda_0 > \nu_0^2/2 \), hence \( \mu > 0 \), where the SQB process is defined on \( \mathcal{I} = (0, \infty) \). The point \( x = 0 \) is an entrance and \( x = \infty \) is an attracting natural boundary. For the regular diffusion on \( \mathcal{I} \) the transition density function is

\[
p_X(t; x_0, x) = \left( \frac{x}{x_0} \right)^{\frac{\mu}{2}} e^{-2(x+x_0)/\nu_0^2 t} I_{\mu} \left( \frac{4\sqrt{x_0}}{\nu_0^2 t} \right). \tag{3.1}
\]
The generating function \( \hat{u}_\mu(x) \) is a linear combination of
\[
\varphi_\mu^+(x) = x^{-\mu/2} I_\mu(2\sqrt{2px}/\nu_0) \quad \text{and} \quad \varphi_\mu^-(x) = x^{-\mu/2} K_\mu(2\sqrt{2px}/\nu_0),
\]
where \( I_\mu(z) \) and \( K_\mu(z) \) are the modified Bessel functions (of order \( \mu \)). The first and second kind, respectively (for definitions and properties see [1]). The pair \( \varphi_\mu^+(x) \) satisfies (1.7) and where \( w_s = 1/2. \) For \( s_1, s_2 \in \mathbb{C}, \) all Wronskians \( W[\varphi_{s_1}^\pm, \varphi_{s_2}^\pm](x) \) and \( W[\varphi_{s_1}^\pm, \varphi_{s_2}^\pm](x) \) are readily obtained using differential recurrences \( z I'_\mu(z) = \mu I_\mu(z) + z I_{\mu+1}(z) \) and \( z K'_\mu(z) = \mu K_\mu(z) - z K_{\mu+1}(z). \)

3.2 The CIR Process

Consider the Cox-Ingersoll-Ross (CIR) process [9] \((X_t)_{t \geq 0} \in \mathcal{I} = (0, \infty)\) solving the SDE
\[
dX_t = (\lambda_0 - \lambda_1 X_t)dt + \nu_0 \sqrt{X_t}dW_t,
\]
where \( \lambda_0 > 0, \lambda_1 > 0, \) and \( \nu_0 > 0. \) Here we take \( \mu \equiv \frac{2\lambda_0}{\nu_0^2} - 1 > 0. \) The endpoint \( x = \infty \) is non-attracting natural and the origin is an entrance boundary. The respective scale and speed densities are \( s(x) = x^{-\mu+1}e^{\kappa x} \) and \( m(x) = \frac{2}{\nu_0^2} x^{\mu}e^{-\kappa x}, \) where \( \kappa = \frac{2\lambda_1}{\nu_0} > 0. \) For the regular diffusion on \( \mathcal{I} \) the transition PDF is
\[
p_X(t; x_0, x) = c_t e^{\lambda_1 t} \left( \frac{x e^{\lambda_1 t}}{x_0} \right)^{\mu/2} e^{-c_t(xe^{\lambda_1 t}+x_0)} I_\mu \left( 2c_t \sqrt{x_0xe^{\lambda_1 t}} \right),
\]
where \( c_t \equiv \kappa/(e^{\lambda_1 t} - 1). \)

The generating function \( \hat{u}_\mu(x) \) is a linear combination of the functions
\[
\varphi_\mu^+(x) = M(v, \mu + 1, \kappa x) \quad \text{and} \quad \varphi_\mu^-(x) = U(v, \mu + 1, \kappa x)
\]
where \( v \equiv \rho/\lambda_1 > 0. \) The functions \( M(a, b, z) \) and \( U(a, b, z) \) are confluent hypergeometric functions, i.e. the standard Kummer functions (see [1]). The pair \( \varphi_\mu^+(x) \) satisfies (1.7) and where \( w_s = \kappa - \mu \frac{\Gamma(\mu+1)}{\Gamma(s/\lambda_1)} \) For \( s_1, s_2 \in \mathbb{C}, W[\varphi_{s_1}^\pm, \varphi_{s_2}^\pm](x) \) and \( W[\varphi_{s_1}^\pm, \varphi_{s_2}^\pm](x) \) are obtained using differential recurrences \( \frac{d}{dz} M(a, b, z) = (a/b) M(a+1, b+1, z) \) and \( \frac{d}{dz} U(a, b, z) = -aU(a+1, b+1, z). \)

3.3 The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck (OU) process solves the SDE \( dX_t = (\lambda_0 - \lambda_1 X_t)dt + \nu_0 dW_t, \) where \( \lambda_0, \lambda_1 > 0. \) Both boundaries, \( l = -\infty \) and \( r = \infty, \) of the state space \( \mathcal{I} = (-\infty, \infty) \) are non-attracting natural for all choices of parameters. Without loss in generality we set \( \lambda_0 = 0. \) Otherwise we can consider the shifted process \( Y_t = X_t - \frac{\lambda_0}{\lambda_1}, \) and the formulas follow by simply shifting \( x \rightarrow x - \frac{\lambda_0}{\lambda_1}, x_0 \rightarrow x_0 - \frac{\lambda_0}{\lambda_1}. \) The speed and scale densities are \( s(x) = e^{\kappa x^2/2} \) and \( m(x) = (2/\nu_0^2)e^{-\kappa x^2/2}. \) As in Subsection 3.2, we define the positive constants \( \kappa = \frac{2\lambda_1}{\nu_0}, \) \( \nu = \frac{\lambda_0}{\lambda_1}. \) The regular OU diffusion on \( \mathcal{I} \) has the transition PDF
\[
p_X(t; x_0, x) = \frac{\kappa}{2\pi(1 - e^{-2\lambda_1 t})} \exp \left( -\frac{\kappa(x - x_0 e^{-\lambda_1 t})^2}{2(1 - e^{-2\lambda_1 t})} \right).
\]
The generating function $\hat{u}_\rho(x)$ is a linear combination of
\[
\varphi^-_\rho(x) = e^{\kappa x^2/4} D_{-\nu}(\sqrt{\kappa} x) \quad \text{and} \quad \varphi^+_\rho(x) = \varphi^-(x) (-x)
\] (3.6)
where $D_{-\nu}(x)$ is Whittaker’s parabolic cylinder function (see [1] for definitions and properties). The Wronskian constant in equation (1.7) is given by $w_s = \sqrt{2\pi} \Gamma(s/\lambda_1)$. For $s_1, s_2 \in \mathbb{C}$, $W[\varphi^-_{s_1}, \varphi^+_s](x)$ and $W[\varphi^-_{s_2}, \varphi^+_s](x)$ are obtained using differential recurrences $\frac{d}{dz} D_{-\nu}(z) = -\left(\frac{z}{2}\right) D_{-\nu}(z) - \nu D_{-\nu-1}(z)$.

4 The Bessel, Confluent Hypergeometric, and Ornstein-Uhlenbeck Families of Affine Diffusions

4.1 Three Main Families of Affine Drift Diffusions: Classification and Properties

Using the general construction presented in Section 1 and the three underlying $X$-diffusions from Section 3, we can construct three new families of $F$-diffusions with affine drift as defined by (1.13) and (1.23). For all such new families, the transition PDF $p_F$ is given in analytically closed form for regular as well as killed $F$-diffusions. This partly explains why these processes are also termed “solvable” diffusions. Generally, a transition PDF $p_F$ is given by (1.15), in terms of the corresponding transition PDF $p_X$ of the underlying $X$-diffusion, where the monotonic map $F(x)$ (with its inverse map $X(F)$) is given in (1.20) and the generating function $\hat{u}_\rho$ is given in (1.10). By choosing the SQB, CIR, or OU diffusion as underlying $X$-diffusion, we respectively obtain the Bessel, confluent hypergeometric, or Ornstein-Uhlenbeck families of $F$-diffusions with affine drift. Some properties and classification of the corresponding driftless diffusions are discussed in [6]. For regular diffusions on the respective state spaces $\mathcal{I} = (0, \infty)$ or $\mathcal{I} = (-\infty, \infty)$, the transition PDF $p_X$ is given by equation (3.1), (3.3), or (3.5), respectively. For corresponding diffusions with imposed killing at one or two interior points of the state space, the transition densities $p_X$ are given in terms of closed-form spectral expansions and these, in turn, lead to closed-form spectral expansions for transition PDFs $p_F$, as well as other fundamental quantities such as first hitting (passage) time densities for $F$-diffusions (see [5]). The fundamental solutions $\varphi^\pm$ used in the map and generating function are given by either (3.2), (3.4), or (3.6), accordingly.

Each $F$-diffusion is described by the set of parameters that are inherited from the chosen underlying diffusion. In addition to that set, the nonnegative parameters $q_1, q_2,$ and $\rho > 0$ are added thanks to the measure change $X \to X^{(\rho)}$. Two parameters $a$ and $b$ describe the affine drift coefficient in (1.13). Finally, up to two other parameters $c_1$ and $c_2$ are used in the map function. As observed from the diffusion coefficient function $\sigma(F)$ in (1.23), the combination of all such parameters make the new diffusions quite flexible for modelling various stochastic processes. Different choices of monotonic maps $F = F^{(i)}_\pm$, $i = 1, \ldots, 5$, lead to different $F$-diffusions. In any case, the diffusion function is specified by equation (1.23).

The choice $F = F^{(1)}_\pm$ leads to $F$-diffusions that have applications in asset pricing in finance. In particular, the dual maps defined by (1.25) with $\epsilon = +1$ give rise to sets of dual
subfamilies (i) and (ii) of affine drift $F$-diffusions with respective volatility specification:

$$
\sigma(F) = cv(x) \begin{cases} 
\frac{W[\varphi^+_{\rho,b} \varphi^+_{\rho,b}] (x)}{|\varphi^+_{\rho,b} (x)|^2} & (i) \\
\frac{-\varphi^+_{\rho,b} (x) \frac{\partial}{\partial x} \varphi^-_{\rho,b} (x) + \varphi^+_{\rho,b} (x)}{|\varphi^-_{\rho,b} (x)|^2} & (ii) 
\end{cases}
$$

(4.1)

where $x = X(F) = F^{-1}(F)$ is the unique inverse map for the respective subfamilies (i) $F(x) = -\frac{a}{b} + c \frac{\varphi^+_{\rho,b}(x)}{\varphi^+_{\rho,b}(x)}$ and (ii) $F(x) = -\frac{a}{b} + c \frac{\varphi^+_{\rho,b}(x)}{\varphi^+_{\rho,b}(x)}$. Both subfamilies have regular state space $F \in (-\frac{a}{b}, \infty)$. Computing the Wronskians in equation (4.1) gives the diffusion coefficient function for three dual subfamilies as follows:

$$
\sigma(F) = c \sqrt{2} \begin{cases} 
\sqrt{b} I_u \left( \frac{\lambda}{\sqrt{2}} \right) K_{\mu+1} \left( \frac{\rho}{\sqrt{2}} \right) + \sqrt{b} \sqrt{2} \mu+1 \left( \frac{\rho}{\sqrt{2}} \right) K_{\mu+1} \left( \frac{\rho}{\sqrt{2}} \right) & (i) \\
\sqrt{b} K_{\mu} \left( \frac{\lambda}{\sqrt{2}} \right) I_{\mu+1} \left( \frac{\rho}{\sqrt{2}} \right) + \sqrt{b} I_{\mu} \left( \frac{\rho}{\sqrt{2}} \right) & (ii)
\end{cases}
$$

(4.2)

for the Bessel family,

$$
\sigma(F) = cK_{\nu_0} \sqrt{x} \begin{cases} 
v \lambda \left( v + \frac{\lambda}{\sqrt{2}} \right) \mu+1 \kappa \xi U \left( v + \frac{1}{\nu+1}, \mu+1, \kappa \xi \right) + (v+\frac{1}{\nu+1}) \lambda \left( v + \frac{1}{\nu+1}, \mu+1, \kappa \xi \right) & (i) \\
v U \left( v + \frac{1}{\nu+1}, \mu+1, \kappa \xi \right) \lambda \left( v + \frac{1}{\nu+1}, \mu+1, \kappa \xi \right) + (v+\frac{1}{\nu+1}) \lambda \left( v + \frac{1}{\nu+1}, \mu+1, \kappa \xi \right) & (ii)
\end{cases}
$$

(4.3)

for the confluent hypergeometric family, and

$$
\sigma(F) = c \sqrt{\frac{2}{\lambda_1}} \begin{cases} 
\frac{(v+b) \lambda_{-v} \frac{1}{\nu+1} \xi (v+b) \lambda_{-v} \frac{1}{\nu+1} \xi} {\left( v \lambda_{-v} \frac{1}{\nu+1} \xi \right) D_{-v} (\sqrt{x})} + \frac{\rho \lambda_{-v} \frac{1}{\nu+1} \xi \left( \sqrt{x} \right) D_{-v} \left( v+b \lambda_{-v} \frac{1}{\nu+1} \xi \right) \left( \sqrt{x} \right)} {\left( v \lambda_{-v} \frac{1}{\nu+1} \xi \right) D_{-v} (\sqrt{x})} & (i) \\
\frac{(v+b) \lambda_{-v} \frac{1}{\nu+1} \xi (v+b) \lambda_{-v} \frac{1}{\nu+1} \xi} {\left( v \lambda_{-v} \frac{1}{\nu+1} \xi \right) D_{-v} (\sqrt{x})} + \frac{\rho \lambda_{-v} \frac{1}{\nu+1} \xi \left( \sqrt{x} \right) D_{-v} \left( v+b \lambda_{-v} \frac{1}{\nu+1} \xi \right) \left( \sqrt{x} \right)} {\left( v \lambda_{-v} \frac{1}{\nu+1} \xi \right) D_{-v} (\sqrt{x})} & (ii)
\end{cases}
$$

(4.4)

for the OU family. In all cases, $x = X(F) = F^{-1}(F)$ using the respective map for (i) or (ii). Subsets of these families with $a = 0, b \neq 0$ have regular state space $F \in (0, \infty)$ and are useful for modelling asset prices in finance. Figures [1] and [2] display some computed curves of the local volatility function $\sigma_{loc}(F) := \sigma(F)/F$ for the subfamilies in equations (4.2), (4.3)i and (4.4) when $a = 0, b \neq 0$. We note that the dual OU subfamilies (i) and (ii) coalesce when $a = 0$. As seen in Figures [1] and [2] by adjusting parameters, the models are all readily calibrated to attain a prescribed level of local volatility for a given value of $F$. The respective sets of freely adjustable positive parameters for the dual Bessel, Confluent and OU subfamilies are: $(\rho, b, \mu, c, \nu_0)$, $(\rho, b, \kappa, c, \nu_0)$ and $(\rho, b, \kappa, c, \nu_0)$.

The following two theorems describe the properties of the Bessel, confluent hypergeometric, and Ornstein-Uhlenbeck families of $F$-diffusions with affine drift. These theorems generalize the results obtained previously for the driftless case (see [3]).

**Theorem 3.** The Bessel, Confluent and OU families of regular diffusions $F_t = F(X^\rho_t)$, with affine drift and monotonic map defined as in Theorem 4, have the following boundary classification.
**Bessel and Confluent families:** The endpoint $F = F(0^+)$ is entrance if $q_2 = 0$; is exit if $q_2 > 0$ and $\mu \geq 1$; is a regular killing boundary if $q_2 > 0$ and $0 < \mu < 1$. The endpoint $F = F(\infty)$ is non-attracting natural if $q_1 = 0$; is attracting natural if $q_1 > 0$.

**OU family:** The endpoint $F = F(-\infty)$ is non-attracting natural if $q_2 = 0$ and is attracting natural if $q_2 > 0$. The endpoint $F = F(\infty)$ is non-attracting natural if $q_1 = 0$ and is attracting natural if $q_1 > 0$.

**Proof.** The results follow from Lemma 2 and Theorem 1 and the asymptotic relations given in Appendix A. For the SQB and CIR diffusions: $(\varphi^-_{\rho}, \varphi^+_{\rho})(0, x] < \infty$, $(\varphi^-_{\rho}, \varphi^+_{\rho})(0, x] < \infty$ ($= \infty$) if $\mu \in (0, 1)$ ($\mu \geq 1$) and $(\varphi^+_{\rho}, \varphi^+_{\rho})(\infty, x] = (\varphi^+_{\rho}, \varphi^+_{\rho})(x, \infty] = \infty$. For the OU diffusion: $(\varphi^-_{\rho}, \varphi^-_{\rho})(-\infty, x] = (\varphi^+_{\rho}, \varphi^-_{\rho})(-\infty, x] = (\varphi^+_{\rho}, \varphi^+_{\rho})(x, \infty] = (\varphi^+_{\rho}, \varphi^-_{\rho})(x, \infty] = \infty$. Finally, we show that for the Bessel and confluent families of $F$-diffusions (with $q_2 > 0$ and $\mu \in (0, 1)$) the regular boundary $F(0^+)$ is killing. By Theorem 1 we need only prove that the regular boundary $l = 0$ of the corresponding process $X^\rho_t$ is killing. Now notice that $\psi^\pm_s(x) = \frac{\varphi^\pm_{\rho+s}(x)}{\tilde{u}_s(x)}$, Re $s > 0$, are fundamental solutions for $X^\rho_t$-diffusions. Using the asymptotic properties for the SQB and CIR diffusions from Appendix A, we have

$$\lim_{x \to 0^+} \psi^+_s(x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{s\rho(x)} \frac{d\psi^+_s(x)}{dx} = \lim_{x \to 0^+} \frac{W[\tilde{u}_{\rho}, \varphi^+_{\rho+s}(x)]}{s(x)} = \text{Const} \neq 0.$$ 

Therefore, $l = 0$ is a killing boundary for $X^\rho_t$.

**Theorem 4.** The Bessel, Confluent and OU families of regular diffusions $F_t = F(X^\rho_t)$, with affine drift and monotonic map defined as in Theorem 3, conserve the expectation rate, i.e., the relation (2.6) holds and hence $(e^{-bt} F_t)_{t \geq 0}$ is a martingale when the drift parameter $a = 0$, under the following conditions:

**Bessel and Confluent families:** $c_2 = 0$ or $c_2 \neq 0$ & $q_2 \neq 0$ & $F(0^+) = 0$ holds;

**OU family:** the relation (2.6) holds for every choice of parameters.

**Proof.** According to Theorem 2 we need only show that there exists some constant $c$ such that the boundary conditions (2.7) and (2.8) hold when Re $s > c$. Using the respective asymptotic relations in Appendix A for the SQB, CIR or OU processes, one readily verifies that (2.7) and (2.8) are satisfied for Re $s > b$. 

\[\Box\]
4.2 Classification of Monotonic Maps

In addition to the set of monotonic maps $F^{(1)}$ defined by (1.25), we introduce the following four classes of maps:

\[
F^{(2)}_{\pm}(x) = \frac{a}{b} + \epsilon\frac{\varphi_{\rho+b}^{\pm}(x)}{\varphi_{\rho}^{\pm}(x)},
\]

(4.5)

\[
F^{(3)}_{\pm}(x) = \frac{a}{b} + \epsilon\frac{c_1 \varphi_{\rho+b}^{\pm}(x) + c_2 \varphi_{\rho+b}^{\pm}(x)}{\varphi_{\rho}^{\pm}(x)},
\]

(4.6)

\[
F^{(4)}_{\pm}(x) = \frac{a}{b} + \epsilon\frac{\varphi_{\rho+b}^{\pm}(x)}{q_1 \varphi_{\rho}^{\pm}(x) + q_2 \varphi_{\rho}^{\pm}(x)},
\]

(4.7)

\[
F^{(5)}(x) = \frac{a}{b} + \epsilon\frac{c_1 \varphi_{\rho+b}^{\pm}(x) - c_2 \varphi_{\rho+b}^{\pm}(x)}{q_1 \varphi_{\rho}^{\pm}(x) + q_2 \varphi_{\rho}^{\pm}(x)},
\]

(4.8)

where $\rho, \rho + b, c, c_1, c_2 > 0$, $\epsilon = \pm 1$, and $q_1, q_2 > 0$ with the only exception of $F^{(5)}$ for which one of $q_1, q_2$ may be zero.

As follows from Proposition 2, a function $F$, given by equation (1.25), (4.5), (4.6), or (4.7), defines a monotonic map if and only if the Wronskian $W(x) = W[\varphi_{\rho+b}^{\pm}(x)]$ has the same sign in neighbourhoods of both endpoints $l$ and $r$. For the map $F^{(5)}$ in (4.8), monotonicity follows from Proposition 3 when $b > 0$. If $b < 0$, then we again need to analyse the asymptotic behaviour of $W(x)$, and hence of $W[\varphi_{\rho+b}^{\pm}(x)]$, as $x \to l+$ and $x \to r-$. The asymptotics of such Wronskians are given in Proposition 4 of Appendix A.

The following lemma summarizes the monotonicity properties for all families of maps presented so far.

**Lemma 3.** Let $\varphi^{\pm}$ be the fundamental solutions for the SQB, CIR, or OU diffusion process. The maps $F^{(l)}$, $l = 1, 2, \ldots, 5$, defined in (1.25), (4.5), (4.6), (4.7), and (4.8), respectively, are strictly monotonic under the following conditions:

- the maps $F^{(1,2)}_{\pm}$ are strictly monotonic for all choices of parameters;
- the map $F^{(3)}_{\pm}$ is strictly monotonic if $b < 0$;
- the map $F^{(4)}_{\pm}$ is strictly monotonic if $b > 0$;
- the map $F^{(5)}$ is strictly monotonic if and only if $b > 0$.

The derivative $dF^{(l)}_{\pm}/dx$, $l = 1, 3, 4$, has the sign $\pm \epsilon$. For cases 2 and 5 we have: $\text{sign}(dF^{(2)}_{\pm}/dx) = \pm \epsilon \text{sign}(b)$ and $\text{sign}(dF^{(5)}/dx) = \epsilon$.

**Proof.** The proof follows directly from Propositions 2, 3, and 4.

5 Conclusions

By applying the diffusion canonical transformation method which combines a special class of variable and measure changes on solvable underlying diffusion processes, this paper has
developed new families of exactly solvable multiparameter diffusion models with affine drift
and nonlinear diffusion coefficient functions. As an application of our general formulation,
this paper has presented three new main families of time-homogeneous affine diffusions,
named as Bessel, confluent hypergeometric, and Ornstein-Uhlenbeck families. These pro-
cesses can be used to model asset prices, interest rates and other financial quantities. In
particular, analytically exact closed-form expressions for transition densities and state-price
densities are obtained for these families of nonlinear local volatility models in terms of
known special functions (i.e., modified Bessel, confluent hypergeometric, hypergeometric).
The present formulation extends and includes all of the more special families of driftless
diffusions that were obtained in previous literature with the use of a diffusion canonical
transformation methodology (e.g., see [2, 6]). Subfamilies of these new models encompass
the CEV model and other related models as special cases, and have been shown to exhibit
a wide range of implied volatility surfaces with pronounced smiles and skews (see [6, 7, 8]
for further details).

By making use of closed-form spectral expansions for the densities of the underlying
simpler processes, the theory of the present paper can also be used to further derive new
classes of analytically exact probability densities for first-hitting times and the extrema
(maximum and minimum) of the new affine drift families of nonlinear local volatility diffusion
processes. This then also leads to analytical pricing formulas for various lookback and barrier
options. These applications are the subject of currently related research (see [5]).

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A. Large and Small Argument Asymptotics of the Fundamental Solutions and Wronskian Functions

Let $s$, $s_1$ and $s_2$ be complex-valued parameters with positive real part.

A.1. The SQB process

(i) Asymptotic forms of the fundamental solutions:

\[
\begin{align*}
\varphi_{s}^{+} (x) & \sim \frac{2x/\nu}{\Gamma(\mu+1)} e^{2\sqrt{2sx/\nu_0}} x^{\mu/2+1/4}, & \varphi_{s}^{-} (x) & \sim \frac{2\nu_0/\nu}{\sqrt{2sx/\nu_0}} e^{-2\sqrt{2sx/\nu_0}} x^{\mu/2+1/4}, & \text{as } x \to 0, \\
\varphi_{s}^{+} (x) & \sim \frac{(2s/\nu_0)^{\mu/2}}{\Gamma(\mu+1)} e^{2\sqrt{2sx/\nu_0}} x^{\mu/2+1/4}, & \varphi_{s}^{-} (x) & \sim \frac{2\nu_0/\nu}{\sqrt{2sx/\nu_0}} e^{-2\sqrt{2sx/\nu_0}} x^{\mu/2+1/4}, & \text{as } x \to \infty.
\end{align*}
\]
(ii) Asymptotic forms of the Wronskian functions as $x \to 0$:

$$W[\varphi_1^+, \varphi_2^+](x) \sim \left( \frac{2\sqrt{s_1 s_2}}{v_0^2} \right) \frac{2(s_2 - s_1)}{\nu_0^2 \Gamma(\mu + 1) \Gamma(\mu + 2)} x^\mu,$$

$$W[\varphi_1^-, \varphi_2^-](x) \sim -\left( \frac{s_1}{s_2} \right)^{\mu/2} x^{-\mu - 1},$$

$$W[\varphi_1^+, \varphi_2^-](x) \sim \begin{cases} \frac{\Gamma(\mu)\Gamma(\mu - 1)}{s_1 s_2} \frac{2(2\sqrt{\pi} s_2)}{v_0^2} x^{-2 - \mu} \ln \left( \frac{x}{s_1} \right) & \text{if } \mu > 1, \\ \frac{\Gamma(\mu)\Gamma(1 - \mu)}{s_1 s_2} \frac{s_1^\mu - s_2^\mu}{(s_1 s_2)^{\mu/2}} x^{-\mu - 1} & \text{if } \mu = 1, \\ \frac{\Gamma(\mu)\Gamma(1 - \mu)}{s_1 s_2} \frac{s_1^\mu - s_2^\mu}{(s_1 s_2)^{\mu/2}} x^{-\mu - 1} & \text{if } \mu \in (0, 1). \end{cases}$$

(iii) Asymptotic forms of the Wronskian functions as $x \to \infty$:

$$W[\varphi_1^+, \varphi_2^+](x) \sim \frac{\pi}{4\nu_0} \frac{\sqrt{s_2} - \sqrt{s_1}}{(s_1 s_2)^{1/4}} x^{-\mu - 1} e^{2(\sqrt{s_1} + \sqrt{s_2}) \sqrt{2x/v_0}},$$

$$W[\varphi_1^+, \varphi_2^-](x) \sim -\frac{\pi}{4\nu_0} \frac{\sqrt{s_1} + \sqrt{s_2}}{(s_1 s_2)^{1/4}} x^{-\mu - 1} e^{2(\sqrt{s_1} - \sqrt{s_2}) \sqrt{2x/v_0}},$$

$$W[\varphi_1^-, \varphi_2^-](x) \sim \frac{\pi}{4\nu_0} \frac{\sqrt{s_1} - \sqrt{s_2}}{(s_1 s_2)^{1/4}} x^{-\mu - 1} e^{-2(\sqrt{s_1} + \sqrt{s_2}) \sqrt{2x/v_0}}.$$

Note that for multiple-valued power function of the form $s^a$ with $\Re s > 0$ and $a > 0$ the principle value is used.

**A.2 The CIR process**

(i) Asymptotic forms of the fundamental solutions:

$$\varphi_1^+(x) \sim 1 \quad \varphi_2^-(x) \sim \frac{\Gamma(\mu)}{\Gamma(\nu)} (\kappa x)^{-\mu} \quad \text{as } x \to 0,$$

$$\varphi_1^+(x) \sim \frac{\Gamma(\mu + 1)}{\Gamma(\nu)} e^{\kappa x (\kappa x)^{\nu - \mu - 1}} \quad \varphi_2^-(x) \sim (\kappa x)^{-\nu} \quad \text{as } x \to \infty,$$

where in Subsections A.2 and A.3 we define $\nu \equiv s/\lambda_1$.

(ii) Asymptotic forms of the Wronskian functions as $x \to 0$:

$$W[\varphi_1^+, \varphi_2^+](x) \sim \frac{(v_2 - v_1) \kappa}{\mu + 1},$$

$$W[\varphi_1^-, \varphi_2^-](x) \sim -\frac{\Gamma(\nu + 1)}{\Gamma(v_2)} \kappa^{-\mu} x^{-\mu - 1},$$

$$W[\varphi_1^-, \varphi_2^-](x) \sim \begin{cases} \frac{\Gamma(\mu)\Gamma(\mu - 1)}{\Gamma(v_1)\Gamma(v_2)} \kappa^{-2\mu - 1 + (v_1 - v_2)} x^{-2\mu} & \text{if } \mu > 1, \\ \frac{(v_1)\Gamma(v_1 - \mu)}{\Gamma(v_1)\Gamma(v_2)} \frac{\kappa^{-\mu} \Gamma(1 - \mu)}{\Gamma(v_1 - \mu)} \frac{\Gamma(1 - \mu)}{\Gamma(v_2 - \mu)} x^{-\mu - 1} & \text{if } \mu = 1, \\ \frac{(v_1)\Gamma(v_1 - \mu)}{\Gamma(v_1)\Gamma(v_2)} x^{-\mu - 1} \ln \left( \frac{1}{x} \right) & \text{if } \mu \in (0, 1). \end{cases}$$

where in Subsections A.2 and A.3 we define $v_i \equiv s_i/\lambda_1$, $i = 1, 2$. 
(iii) Asymptotic forms of the Wronskian functions as \( x \to \infty \):

\[
\begin{align*}
W[\varphi^+_1, \varphi^+_2](x) & \sim \frac{\kappa(v_2 - v_1)\Gamma(\mu + 1)^2}{\Gamma(v_1)\Gamma(v_2)} e^{2\kappa x} (\kappa x)^{v_1 + v_2 - 2\mu - 3}, \\
W[\varphi^+_1, \varphi^-_2](x) & \sim -\frac{\kappa\Gamma(\mu + 1)}{\Gamma(v_1)} e^{\kappa x} (\kappa x)^{v_1 - v_2 - \mu - 1}, \\
W[\varphi^-_1, \varphi^-_2](x) & \sim \kappa(v_1 - v_2)(\kappa x)^{-v_1 - v_2 - 1}.
\end{align*}
\]

A.3 The Ornstein-Uhlenbeck process

(i) Asymptotic forms of the fundamental solutions:

\[
\begin{align*}
\varphi^+_s(x) & \sim \frac{\sqrt{2\pi}}{\Gamma(v)} (\sqrt{\kappa} |x|)^{v-1} e^{\kappa x^2/2} \quad \text{as } x \to \pm \infty, \\
\varphi^+_s(x) & \sim (\sqrt{\kappa} |x|)^{-v} \quad \text{as } x \to \mp \infty.
\end{align*}
\]

(ii) Asymptotic forms of the Wronskian functions as \( x \to -\infty \):

\[
\begin{align*}
W[\varphi^+_1, \varphi^+_2](x) & \sim (v_2 - v_1) \sqrt{\kappa} (\sqrt{\kappa} |x|)^{v_2 - v_1 - 1}, \\
W[\varphi^+_1, \varphi^-_2](x) & \sim -\frac{\sqrt{2\pi\kappa}}{\Gamma(v_2)} (\sqrt{\kappa} |x|)^{v_2 - v_1} e^{\kappa x^2/2}, \\
W[\varphi^-_1, \varphi^-_2](x) & \sim (v_1 - v_2) \frac{2\pi\sqrt{\kappa}}{\Gamma(v_1)\Gamma(v_2)} (\sqrt{\kappa} |x|)^{v_1 + v_2 - 3} e^{\kappa x^2}.
\end{align*}
\]

(iii) Asymptotic forms of the Wronskian functions as \( x \to \infty \):

\[
\begin{align*}
W[\varphi^+_1, \varphi^+_2](x) & \sim (v_2 - v_1) \frac{2\pi\sqrt{\kappa}}{\Gamma(v_1)\Gamma(v_2)} (\sqrt{\kappa} x)^{v_1 + v_2 - 3} e^{\kappa x^2}, \\
W[\varphi^+_1, \varphi^-_2](x) & \sim -\frac{\sqrt{2\pi\kappa}}{\Gamma(v_1)} (\sqrt{\kappa} x)^{v_2 - v_1} e^{\kappa x^2/2}, \\
W[\varphi^-_1, \varphi^-_2](x) & \sim (v_1 - v_2) \sqrt{\kappa} (\sqrt{\kappa} x)^{-v_1 - v_2 - 1}.
\end{align*}
\]

A.4 Asymptotic Properties of Wronskians of the Fundamental Solutions

Proposition 4. Let \( \rho_1 \) and \( \rho_2 \) be positive real-valued parameters. For the three underlying diffusions, namely, the SQB, CIR, and OU diffusions defined in Sections 3.1-3.3, we have
the following limits for the Wronskians of the fundamental solutions:

\[
W[\varphi^+_{\rho_1, \rho_2}](x) \to \begin{cases} 
\text{sign}(\rho_2 - \rho_1)C, & \text{as } x \to l^+, \\
\text{sign}(\rho_2 - \rho_1)\infty, & \text{as } x \to r^-, 
\end{cases}
\]

\[
W[\varphi^-_{\rho_1, \rho_2}](x) \to \begin{cases} 
\text{sign}(\rho_1 - \rho_2)\infty, & \text{as } x \to l^+, \\
\text{sign}(\rho_1 - \rho_2)0, & \text{as } x \to r^-, 
\end{cases}
\]

\[
W[\varphi^+_{\rho_1, \rho_2}](x) \to -\infty, \text{ as } x \to l^+, 
\]

\[
W[\varphi^-_{\rho_1, \rho_2}](x) \to -\infty, \text{ as } x \to r^-, \text{ for the CIR and OU}
\]

\[
W[\varphi^+_{\rho_1, \rho_2}](x) \to \begin{cases} 
-0 & \text{if } \rho_1 < \rho_2, \\
-\infty & \text{if } \rho_1 \geq \rho_2, 
\end{cases} \text{ as } x \to r^-, \text{ for the SQB}
\]

with constant \(C > 0\) for the SQB and CIR diffusions and \(C = 0\) for the OU diffusion. Here \(\text{sign}(x) = +1(-1)\) for \(x > 0\) \((x < 0)\) and we define \(\text{sign}(0) = 0\) as well as \(0 \cdot \infty = 0\). The convergences are monotonic with notation \(W(x) \to +0\) and \(W(x) \to +\infty\) \((\text{or } W(x) \to -0\) and \(W(x) \to -\infty)\) meaning that \(W(x) > 0\) \((\text{or } W(x) < 0)\) holds as \(x\) approaches an endpoint.

**Proof.** The limits follow from the asymptotics of the Wronskian functions presented in Subsections \(A.1\)–\(A.3\) where \(s_i = \rho_i\), \(l = 0\) for SQB and CIR, \(l = -\infty\) for OU and \(r = \infty\) for all three diffusions. For the CIR diffusion with \(\mu \in (0, 1)\), the proof for the limiting value of \(W[\varphi^+_{\rho_1, \rho_2}](x)\) follows from Proposition 5. □

**Proposition 5.** Let \(0 < a < 1\) be a real constant. Then:

(i) \(S(x; a) = \frac{1}{x^a} \frac{\Gamma(x+a)}{\Gamma(x)}\) is an increasing function of \(x > 0\).

(ii) \(R(x; a) = \frac{\Gamma(x)}{\Gamma(x-a)}\) is an increasing function of \(x > 0\).

**Proof.** The first result is proved by [14]. The other is its immediate corollary. Indeed, we have

\[
\frac{\Gamma(x)}{\Gamma(x-a)} = \frac{\Gamma(x+1)}{\Gamma(x-a+1)} \frac{x-a}{x} = \frac{\Gamma(y+a)}{\Gamma(y)} \frac{y^a(y-1)}{y^a - (1-a)} = S(y; a) \frac{y^a(y-1)}{y^a - (1-a)},
\]

where \(y = x+1-a > 1-a\). It is easy to verify that the quotient \(\frac{y^a(y-1)}{y^a - (1-a)}\) is an increasing function of \(y\). Hence, \(R(x; a)\) is a product of two increasing functions. □
Figure 1: Sample local volatility curves for the drifted Bessel-K model with drift parameters $a = 0, b \neq 0$ and other parameters calibrated such that $\sigma_{loc}(F) = 0.25$ at $F = 100$.

Figure 2: Sample local volatility curves for the Confluent-\textit{U} (a) and OU (b) models with drift parameters $a = 0, b \neq 0$ and other parameters calibrated such that $\sigma_{loc}(F) = 0.20$ at $F = 100$. 