Efficiency of ENO-schemes for computation of linear waves in an elastic body

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Abstract. The efficiency of the UNO- and TVD-modifications of the Godunov method in computing one and two-dimensional linear waves in an elastic body is numerically studied. The body is simulated by an isotropic linear-elastic semi-space. Riemann problem of cylindrical discontinuity in hydrostatic pressure in a body and a problem of propagation of waves caused by the impulsive action on the free surface of a metal body are considered. It is shown that in all the cases both modifications resolve the waves in the body significantly better than the classic Godunov method. At that, in the case of presence of the pronounced extrema in the solution, the UNO-modification appears more preferable because the TVD-modification "cuts" the extrema to implement the TVD-property.

1. Introduction

Computations of elastic waves in bodies are necessary, in particular, when studying deformation and destruction of the body surfaces under the action of different factors. For example, if a multiphase gas-liquid medium with variable pressure contacts with a body, the surface of the body can be exposed to the action of the pressure pulses arising at cavitation bubble collapse. That is the reason the surfaces of structural elements contacting with liquid are subjected to cavitation erosion. Cavitation can destroy any material. Therefore, it is important to investigate the waves in a body in order to understand its internal processes to predict them and to determine the optimal operating conditions for the corresponding machines and mechanisms. To this end, one needs appropriate mathematical models and efficient methods of computation.

The classic Godunov method is widely used to solve many problems of the dynamics of continuous media [1, 2] due to its stability and monotonicity. Its disadvantages are low (first-order) accuracy and, as a result, strong smearing of jumps and contact discontinuities. In this paper, the efficiency of two modifications of the Godunov method in computing linear waves in an elastic body is examined. Those modifications belong to the class of ENO-schemes (ENO - Essentially Non Oscillatory) comprising UNO - and TVD-schemes of the second order (UNO - Uniformly Non Oscillatory, TVD - Total Variation Diminishing). They are similar to those used in [3] to compute nonlinear waves in gas and liquid. When constructing such schemes the time derivatives of the unknown functions are expressed in terms of their spatial derivatives. With the help of those expressions the values of the
unknown functions at the next temporal semi-layer in the center of the grid cells and on both sides of each of their faces are computed. The values on the faces are found by solving Riemann problem. Limiters are usually used to confine the values of spatial derivatives in a cell. The modifications considered in the present work differ in the form of the limiters. The TVD-limiter provides the diminishing of total variation of solution of the difference scheme (TVD-property). In extreme points the order of accuracy reduces to the first one. The UNO-scheme limiter includes approximations of derivatives of not only the first order but also the second one. The main advantage of the UNO-scheme is its second order of accuracy throughout the areas of smooth solutions. The efficiency of the considered UNO- and TVD-modifications is demonstrated using one- and two-dimensional problems on the propagation of linear waves in an elastic metal body.

2. Problem statement and computational technique
The efficiency of computing linear waves in an elastic body using the UNO- and TVD-modifications of the Godunov method is studied. The body is simulated by a semi-space. Its dynamics is described by a system of seven linear equations [4] in the two components of the velocity vector, the four deviator components of the stress tensor in Cartesian or cylindrical coordinates, and the all-round (hydrostatic) pressure. The waves in the body result from its initial state or the action on its surface. In the case of one-dimensional problems, the computational domain is a segment [0, R] the left end of which is assumed to be the surface of a body, the right end being an artificial boundary. In the case of two-dimensional problems, the computational domain is a square the upper and left sides of which are assumed to be the body surface and the symmetry axis, respectively, the right and the lower ones being artificial boundaries. In the problems of the present paper, the surface of the body is a free boundary with a specified pressure distribution. Artificial boundaries are usually introduced to limit the computational domain to the body area most interesting from a particular point of view. Non-reflecting conditions are posed on the artificial boundaries [5]. The computation domain is covered by the uniform grid. The time step is determined from the Courant condition.

The first of the three steps of the UNO- and TVD-algorithms computes the values of the unknowns at a semi-integer time layer. In this case, the values of the derivatives with respect to time are expressed in terms of the spatial derivatives. While computing the spatial derivatives, some limiters are used. The UNO- and TVD-schemes differ in the form of those limiters. The limiter of the UNO-scheme includes the approximations to both the first and second derivatives, while that of the TVD-scheme just utilizes the approximation to the first derivative.

The second step is the computation of the numerical fluxes across the cell sides using the solution of the Riemann problem.

On the third step, the unknown values of the new time layer are computed applying the explicit finite-difference formula.

A more detailed description of the present UNO- and TVD-algorithms can be found in [4].

3. Results
3.1 Cylindrical Riemann problem
At the initial time \( t = t_0 \), there is a discontinuity in the all-round pressure \( P \) in the computational domain of radius \( R \): \( P = 0 \) in the area \( r < 0.5R \) and \( P = p^* \) in the area \( r > 0.5R \) where \( R \) and \( p^* \) are known quantities.

Figure 1 demonstrates the solution of this problem on a uniform grid of 200 cells at some time \( t > t_0 \). It is seen that both schemes of the second order describe the fronts and extrema of the waves much better than the Godunov scheme. At that, the UNO scheme resolves the extrema better than the TVD scheme. For example, the deviations of the extremum value and its position computed by the TVD-scheme in the vicinity of \( r \approx 0.2R \) from those found by the UNO scheme are respectively 2% and 3%. The similar deviations of the results computed by the Godunov scheme are 7% and 22%.
Figure 1. Riemann problem on the cylindrical discontinuity in the all-round pressure $P$. The dashed curve is the initial distribution, the thick-solid, fine-solid and dotted curves are respectively the results by the Godunov method and its TVD- and UNO-modifications.

3.2. Dynamics of a body under the action on its surface

Two problems of propagation of waves inside an unperturbed body are considered. The waves are excited by the action on the free surface of the body located at $y = 0$. In the first case, the wave in the body results from the impulse action onto the whole surface of the body (the problem is one-dimensional), while in the second case the wave arises due to the action just onto a part of the body surface (the problem is two-dimensional). The time dependences of the pressure defining the corresponding impulse actions onto the body surface are shown in the insets of figure 2 and figure 3.

Figure 2. The distribution of the all-round pressure $P$ in the body at some instant of time in the case the wave is excited by the pressure pulse on the whole free surface $y = 0$ of the body (the boundary pulse is shown in the inset). The thick solid line is the exact analytical solution, the fine-solid, dashed and dotted lines correspond to the Godunov, TVD and UNO schemes, respectively.

Some results of computations of the first problem on a grid with a step $h = R/100$ are shown in figure 2. It can be seen that the UNO scheme solution is the closest to the exact solution. Its maximum

Figure 3. The dynamics of the body (in the case of impact onto a part of its free surface $y = 0$) in the vicinity of the impact spot: the stress intensity isolines (presented in fractions of $p^*$) in a fragment of the computational domain at two consecutive time moments $t_1$ and $t_2$ (marked in the inset where the boundary pulse is shown) computed by the Godunov (solid thick curves), TVD (dashed curves) and UNO (solid thin curves) schemes; the black stripes at the top are the impact spot.
deviation in the amplitude of the wave is 10%. The TVD scheme resolution is slightly worse. Its amplitude error of 24% is due to a decrease of the second order of accuracy of the TVD scheme to the first one at the extreme points of the solution. The amplitude error of the Godunov scheme is as large as 54%.

Some results of computations of the second problem on a uniform grid of 100 × 100 cells at two consecutive time moments \( t_1 \) and \( t_2 \) are shown in figure 3. It can be seen that again the second-order schemes give much more accurate numerical solutions than that by the Godunov scheme. And again, the UNO scheme resolves the extremes better than the TVD scheme.

4. Conclusion

Some results of a numerical investigation of the efficiency of the second-order accurate UNO- and TVD-modifications of the Godunov method in computing one- and two-dimensional linear waves in an elastic body are presented. Riemann problem of a cylindrical discontinuity in the all-round pressure inside a body and that of propagation of waves caused by the impulse action on the free surface of a body are considered. The exact solution and that computed by the Godunov method are used for comparison in the first and second cases, respectively. It is shown that in all the cases considered both modifications of the Godunov method describe the dynamics of the body considerably better than the Godunov method. Moreover, the UNO-modification is more preferable since the TVD-modification "cuts" the extremes to implement the TVD-property.

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