Topological Structure in \( \hat{c} = 1 \) Fermionic String Theory

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ABSTRACT

\( \hat{c} = 1 \) fermionic string theory, which is considered as a fermionic string theory in two dimension, is shown to decompose into two mutually independent parts, one of which can be viewed as a topological model and the other is irrelevant for the theory. The physical contents of the theory is largely governed by this topological structure, and the discrete physical spectrum of \( \hat{c} = 1 \) string theory is naturally explained as the physical spectrum of the topological model. This topological structure turns out to be related with a novel hidden \( N = 2 \) superconformal algebra (SCA) in the enveloping algebra of the \( N = 3 \) SCA in fermionic string theories.
The remarkable feature of the two-dimensional string theory is that only the tachyon field is left as a continuous physical degree of freedom, while most of the excitation modes of the string are eliminated and there remains an infinite set of discrete physical states as their remnants. The drastic reduction of physical degrees of freedom makes the theory so simple that the two-dimensional string theory may provide a good testing ground for discussing the non-perturbative formulation and the dynamical principle of string theory.

The reduction of physical degrees of freedom suggests that the two-dimensional string theory can be described as a topological model, which may give us an alternative formulation of string theory. Indeed, several authors showed the correspondence of \( c = 1 \) string theory, which is considered as a bosonic string theory in two-dimensional space-time, to certain topological field theories. However they have not been able to clarify the role of the discrete states which we consider to be deeply connected with the topological nature of two-dimensional string theories.

In the recent work, a relation between \( c = 1 \) string theory and a topological sigma model was suggested, in which the former was shown to be a kind of realization, or ‘bosonization’, of the latter. In this formulation, the existence of the discrete states was crucial. Namely, the target space coordinate and the gauge ghost of the topological sigma model were realized by one of the ground ring generators and a discrete tachyon, respectively. Consequently, all the discrete states in \( c = 1 \) string theory were reproduced as the physical states of the topological sigma model.

In this letter, we apply the analysis performed on \( c = 1 \) string theory to the case of \( \hat{c} = 1 \) fermionic string theory, which is considered as a fermionic string theory in two dimension. We have found that \( \hat{c} = 1 \) string theory contains a subspace which can be viewed as a topological model. This topological model is essentially the same one as obtained in the \( c = 1 \) case. The physical spectrum of the topological model turns into the discrete physical spectrum in \( \hat{c} = 1 \) string theory. Therefore, one can say that \( \hat{c} = 1 \) string theory is largely governed by the topological model as for the physical contents of the theory. As we will show, the \( N = 3 \) superconformal symmetry in fermionic string theories plays an important role in this topological structure. Namely, we identify the (twisted) \( N = 2 \) superconformal algebra (SCA) of the topological model with a hidden \( N = 2 \) SCA in the enveloping algebra of the \( N = 3 \) SCA. Associated with this fact, we found a novel automorphism of \( N = 3 \) SCA in the enveloping algebra.
A notion that \( \hat{c} = 1 \) string theory can be regarded as a bosonization of a topological model was previously pointed out also by Distler [15]. The difference between his and our approaches shows up in the treatment of the diffeomorphism ghosts, i.e., upon bosonization, we combine the ghosts with the other fields into the form of the discrete states in \( \hat{c} = 1 \) string theory, while his approach leaves the ghosts unchanged. Therefore, the appearance of the discrete physical spectrum is more transparent from our point of view.

Now we begin our analysis by giving some notations and conventions. \( \hat{c} = 1 \) fermionic string theory consists of two free scalars, the matter field \( \phi^M \) and the Liouville field \( \phi^L \), their superpartners \( \psi^M \) and \( \psi^L \), the gauge fixing ghosts \( b \) and \( c \) for the diffeomorphism invariance, and their superpartners \( \beta \) and \( \gamma \). The energy-momentum tensor \( T \) and the supercurrent \( G \) of \( \hat{c} = 1 \) string theory are given by

\[
T = T^M + T^L + T^G, \quad G = G^M + G^L + G^G,
\]

where the superscripts \( M, L \) and \( G \) denote the matter, the Liouville and the ghost sector respectively, while \( T^i \) and \( G^i \) \( (i = M, L, G) \) stand for the energy-momentum tensor and the supercurrent of each sector. In the above equation

\[
T^i = -\frac{1}{2}(\partial\phi_i)^2 + i\lambda^i\partial^2\phi^i - \frac{1}{2}\psi^i\partial\psi^i, \quad (i = M, L),
\]

\[
T^G = c\partial b + 2\partial cb - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\partial\gamma\beta,
\]

\[
G^i = i\partial\phi^i\psi^i + 2\lambda^i\partial\psi^i, \quad (i = M, L),
\]

\[
G^G = b\gamma - 3\partial c\beta - 2c\partial\beta,
\]

where \( \lambda^M = 0 \) and \( \lambda^L = -i \).

The BRST charge \( Q_{\hat{c}=1} \) of \( \hat{c} = 1 \) string theory is expressed as

\[
Q_{\hat{c}=1} = \oint dz j_{BRST} = \oint dz \left[ c \left( T^M + T^L + \frac{1}{2}T^G \right) - \frac{1}{2}\gamma \left( G^M + G^L + \frac{1}{2}G^G \right) \right].
\]

As usual, these fields have to be bosonized in order to treat different pictures [16]. We give the explicit form of the bosonization for \( \hat{c} = 1 \) string theory as follows:

\[
\psi^\pm = e^{\pm ih} I, \\
\gamma = \eta e^{\phi} I, \quad \beta = \partial \xi e^{-\phi} I,
\]

\[
h(z)h(w) \sim -ln(z-w) \sim \phi(z)\phi(w).
\]

\(^{1}\)Only the holomorphic part will be considered throughout this letter.
In the above equation, \( \psi^\pm = \frac{1}{\sqrt{2}} (\psi^M \pm i\psi^L) \), and \( h \) and \( \phi \) are free scalars. \((\xi, \eta)\) denotes a pair of the fermionic first-order fields with spin-(0,1). \( I \) is a cocycle factor which anti-commutes with \( bc \)-ghosts and commutes with the other fields. Then one can easily check that the bosonized form \( \mathfrak{8} \) of the fundamental fields in \( \hat{c} = 1 \) string theory have the correct mutual statistics.

The spin field \( \Sigma \) that creates the vacuum for the canonical Ramond sector from the \( SL(2, \mathbb{C}) \) invariant vacuum is expressed as \( \Sigma = e^{-\frac{i}{2} h} e^{-\frac{\phi}{2}} I_R \). Here, the cocycle factor \( I_R \) is determined so that the spin field has the definite statistics with respect to the BRST charge \( \mathfrak{1} \). The explicit form of \( I_R \) is given by

\[
I_R = e^{\frac{z_i}{2}(N_h - N_\phi)},
\]

where \( N_h = \frac{1}{2\pi i} \oint dz \partial h \), and \( N_\phi = -\frac{1}{2\pi i} \oint dz \partial \phi \). With this choice for \( I_R \), the statistics of \( \Sigma \) is bosonic with respect to the BRST charge.

We turn to the construction of a topological structure in \( \hat{c} = 1 \) fermionic string theory. Our starting point is a topological model consisting of the fields \( B, C, x \) and \( \bar{p} \), which is essentially a (twisted) supersymmetric first-order system. These fields satisfy the following operator product expansion (OPE)

\[
B(z)C(w) \sim \frac{1}{z - w} \sim C(z)B(w), \quad \bar{p}(z)x(w) \sim \frac{1}{z - w} \sim -x(z)\bar{p}(w).
\]

The other combinations vanish. The generators of the twisted \( N = 2 \) SCA which characterize this topological model are expressed as follows:

\[
T_{\text{top}} = \partial x \bar{p} - B \partial C, \quad G^+_{\text{top}} = -\partial C x, \quad G^-_{\text{top}} = B \bar{p}, \quad J_{\text{top}} = \bar{p} x.
\]

The BRST current of this model is nothing but the supercurrent \( G^+_{\text{top}} \). Hence the BRST charge \( Q_{\text{top}} \) of the model is expressed as

\[
Q_{\text{top}} = \oint dz (-\partial C x).
\]

This topological model is previously shown [13] to yield \( c = 1 \) bosonic string theory by appropriately identifying the fields with those in the \( c = 1 \) Fock space [13]. In [13], \( x \) and
were realized by the discrete operators of \( c = 1 \) string theory, and the discrete physical spectrum of \( c = 1 \) string theory was derived from the physical spectrum of the topological model. One can say that the existence of the discrete physical spectrum is intimately related to the existence of the corresponding topological model. We can expect that the similar program is also possible to carry out in the present case, since the structure of the discrete states is almost the same as that in the \( c = 1 \) case. As we will show below, this is the case.

First, we introduce a set of four fields, the explicit form of which is given by

\[
B_{st} = \left( b\Sigma^{(1/2,1/2)} - \sqrt{2}\partial \xi \Sigma^{(-1/2,-1/2)} \right) e^{\frac{4X^+}{\sqrt{2}}} I_R^{-1},
\]

\[
C_{st} = c\Sigma^{(-1/2,-1/2)} e^{-\frac{4X^+}{\sqrt{2}}} I_R, \tag{14}
\]

\[
x_{st} = \left[ (\sqrt{2}i\partial X - bc) \Sigma^{(1/2,1/2)} - \sqrt{2}c\partial \xi \Sigma^{(-1/2,-1/2)} \right.
\]

\[
+ \frac{1}{\sqrt{2}} b\eta \Sigma^{(-1/2,3/2)} + \Sigma^{(-3/2,1/2)} \bigg] e^{\frac{4X^+}{\sqrt{2}}} I_R^{-1} \tag{15}
\]

\[
\bar{p}_{st} = \frac{1}{2} \left( \Sigma^{(-1/2,-1/2)} - \sqrt{2}c\partial \xi \Sigma^{(1/2,-3/2)} \right) e^{-\frac{4X^+}{\sqrt{2}}} I_R. \tag{16}
\]

The fields \( C_{st} \) and \( x_{st} \) are physical with respect to the BRST charge \( \Pi \) of \( \hat{c} = 1 \) string theory. \( C_{st} \) is one of the discrete tachyon while \( x_{st} \) is the ground ring generator \( \Pi \) (The subscript ‘\( st \)’ means that these fields are defined in the Fock space of \( \hat{c} = 1 \) string theory).

It is clear from the appearance of the cocycle factor \( I_R \) that the above fields are in the Ramond sector (Note that the ground ring generators appear in the Ramond sector for \( \hat{c} \leq 1 \) fermionic string theory \( \Pi \)). For fermionic string theories, it is well-known that there exists the picture-changing isomorphism \( \Pi \) of the physical operators. We have chosen the picture of \( x_{st} \) and \( B_{st} \) as \(+1/2\) and that of \( C_{st} \) and \( \bar{p}_{st} \) as \(-1/2\)\( \Pi \).

These fields realize, in the \( \hat{c} = 1 \) Fock space, the fields \( \Pi \) of the topological model. In fact, one can show that these fields satisfy the following OPE

\[
B_{st}(z)C_{st}(w) \sim \frac{1}{z - w} \sim C_{st}(z)B_{st}(w),
\]

\[
\bar{p}_{st}(z)x_{st}(w) \sim \frac{1}{z - w} \sim -x_{st}(z)\bar{p}_{st}(w),
\]  

which is exactly the same form as in eq.\( \Pi \). If we identify these fields with the fields \( \Pi \) of the topological model, we can also realize the twisted \( N = 2 \) SCA \( \Pi \) in the \( \hat{c} = 1 \) Fock space. In the case of \( c = 1 \) bosonic string theory \( \Pi \), this procedure yielded

\footnote{We assign picture \(-1/2\) to the canonical Ramond sector and 0 to the \( SL(2, \mathbb{C}) \) vacuum.}
a twisted $N = 2$ SCA in the $c = 1$ Fock space with the BRST current and the total energy-momentum tensor as the generators $G^+$ and $T$ respectively. More specifically, by substituting an appropriate expression for the fields of the topological model into eq.(11), the energy-momentum tensor $T_{\text{top}}$ in (11) of the topological model turned into the total energy-momentum tensor of $c = 1$ string theory and the supercurrent, or the BRST current, $G_{\text{top}}$ in (11) into the BRST current of string theory. The fact that the energy-momentum tensor of the topological model yields the one of $c = 1$ string theory indicates that the topological model is realized not in a part of but in the entire Fock space of the string theory. On the other hand, the correspondence of the BRST currents implies the equivalence of the physical spectra of both models. These two facts enabled us to regard $c = 1$ string theory as a realization of the topological model (10). We can expect that this topological model describes $\hat{c} = 1$ string theory as for, in particular, its BRST charge and physical spectrum, in the same manner. However, the direct application of this procedure fails; $Q_{\hat{c}=1} \neq \oint (-\partial C_{st} x_{st})$.

The situation can be remedied by replacing the usual derivative $\partial$ with a ‘modified’ one $L_{-1}$, namely we can obtain the following expression

$$Q_{\hat{c}=1} = \oint dz (-L_{-1} C_{st} x_{st}) .$$

(18)

In the above equation, $L_{-1} C_{st} x_{st}$ is the finite part of the OPE of $L_{-1} C_{st}$ with $x_{st}$. The operator $L_{-1}$ is defined as $(-1)$-mode of a ‘modified’ energy-momentum tensor $L(z)$ which is given by

$$L(z) = \sum_n L_{-n} z^{n-2} = \{Q_{\hat{c}=1}, b^*(z)\} ,$$

(19)

where $b^*$ takes the following form

$$b^* = \frac{1}{2} \left( b + \sqrt{2} \beta \left( \psi^+ - \psi^- \right) + 2c\beta^2 \right)$$

(20)

$$= \frac{1}{2} \left( b + \sqrt{2} \partial \xi \left( e^{ih-\phi} - e^{-ih+\phi} \right) + 2c\beta^2 \xi \partial \xi e^{-2\phi} \right) ,$$

and has a vanishing OPE with itself. The above equation (19) is considered as a modified version of the well-known relation

$$T(z) = \{Q_{\hat{c}=1}, b(z)\} .$$

(21)

Actually, by simple but tedious calculation, one can show that the field $L(z)$ satisfies the Virasoro algebra with vanishing central charge; a necessary property for an energy-momentum tensor of a topological model. This is the reason we call the field in eq.(19) as
a modified energy-momentum tensor. Accordingly the \((-1)\)-mode of $\mathcal{L}$ can be considered as a modified derivative. As is shown in eq.\((18)\), it is necessary to replace the ordinary derivative $\partial$ with the modified one $\mathcal{L}_{-1}$ in order to account for the physical contents of $\hat{c} = 1$ string theory in terms of the topological model.

This energy-momentum tensor $\mathcal{L}$ describes a part of degrees of freedom in $\hat{c} = 1$ string theory. In order to see this, let us decompose the total energy-momentum tensor \((1)\) of string theory as follows:

$$ T = \mathcal{L} + \tilde{\mathcal{L}}. $$

(22)

Since $b^*$ is primary with respect to the total energy-momentum tensor $T$, it follows that, from eq.\((19)\), $\mathcal{L}$ is primary with respect to $T$ (Note that $T$ commutes with the BRST charge $Q_{\hat{c}=1}$). From this and the fact that $\mathcal{L}$ satisfies the Virasoro algebra, one can show that $\tilde{\mathcal{L}}$ also satisfies the Virasoro algebra with vanishing central charge and commutes with $\mathcal{L}$:

$$ \mathcal{L}(z)\mathcal{L}(w) \sim \frac{2\mathcal{L}(w)}{(z-w)^2} + \frac{\partial \mathcal{L}(w)}{z-w}, $$

(23)

$$ \tilde{\mathcal{L}}(z)\tilde{\mathcal{L}}(w) \sim \frac{2\tilde{\mathcal{L}}(w)}{(z-w)^2} + \frac{\partial \tilde{\mathcal{L}}(w)}{z-w}, $$

(24)

$$ \mathcal{L}(z)\tilde{\mathcal{L}}(w) \sim 0. $$

(25)

Hence, the field $\tilde{\mathcal{L}}$ can also be regarded as an energy-momentum tensor independent of $\mathcal{L}$. The decomposition \((23)\) of the total energy-momentum tensor implies that the $\hat{c} = 1$ Fock space $\mathcal{F}_{\hat{c}=1}$ can be viewed as a direct product of two Fock spaces, which we denote as $\mathcal{F}$ and $\tilde{\mathcal{F}}$, corresponding to two mutually independent energy-momentum tensors $\mathcal{L}$ and $\tilde{\mathcal{L}}$:

$$ \mathcal{F}_{\hat{c}=1} = \mathcal{F} \otimes \tilde{\mathcal{F}}. $$

(26)

Since $\mathcal{L}$ and $\tilde{\mathcal{L}}$ have vanishing central charge, both of these two sectors are considered to be topological.

The fact that the modified derivative $\mathcal{L}_{-1}$ appears in eq.\((18)\) instead of the ordinary one $\partial$ suggests that the energy-momentum tensor we should use in order to keep the structure of the topological model is not the usual but the modified one \((19)\). In other words, the topological model \((11)\) is realized in the subspace $\mathcal{F}$, which is characterized by $\mathcal{L}$, of the $\hat{c} = 1$ Fock space. In fact, by identifying the derivative in the topological model with this modified one $\mathcal{L}_{-1}$, the $N = 2$ SCA \((11)\) in the topological model yields a twisted
\[
N = 2 \text{ SCA with } \mathcal{L} \text{ as the energy-momentum tensor:}
\]

\[
T_{st} = \mathcal{L}_{-1} x_{st} \bar{p}_{st} - B_{st} \mathcal{L}_{-1} C_{st} = \mathcal{L},
\]

\[
G^+_{st} = -\mathcal{L}_{-1} C_{st} x_{st} = j_{BRST} - \partial \left( \frac{1}{2} \partial c + c \partial \phi^L + \frac{i}{2} \gamma \psi^L - \frac{1}{4} c \beta \gamma \right),
\]

\[
G^-_{st} = B_{st} \bar{p}_{st} = b^* , \quad J_{st} = \bar{p}_{st} x_{st} = -b c - \beta \gamma + \partial \phi^L .
\]

Thus we have obtained a topological structure in \( \hat{c} = 1 \) string theory which can be regarded as a realization of the topological model (10). Although this topological structure is realized in the subspace \( \mathcal{F} \) rather than the entire Fock space of string theory, the physical contents of string theory is largely governed by this structure. The physical spectrum of the topological model (10) consists of zero modes of \( x \) and \( C \). Through the realization (13)-(16) of the topological model, this physical spectrum gives rise to that of the topological model realized in \( \mathcal{F} \) (27), which is expressed by the zero modes of \( x_{st} \) and \( C_{st} \). However, these are nothing but the discrete physical operators, and are physical not only in \( \mathcal{F} \) but also in the entire \( \hat{c} = 1 \) Fock space. Hence, the discrete physical spectrum of \( \hat{c} = 1 \) string theory is explained in terms of the topological model through the topological structure in \( \mathcal{F} \) (27). One can say that, at least for the discrete spectrum, the subspace \( \tilde{\mathcal{F}} \) in the decomposition (26) is irrelevant, and that the physical contents of the theory is governed by the topological structure in \( \mathcal{F} \) (27) or the topological model (10) itself.

Although the geometrical meaning of the decomposition of the Fock space (24), or the energy-momentum tensor (23), has not yet been clarified, we do have the algebraic meaning of \( \mathcal{L}(z) \), i.e. it can be characterized by a twisted \( N = 3 \) SCA in \( \hat{c} = 1 \) string theory.

As was discussed in [14], any fermionic string theory has a twisted \( N = 3 \) superconformal symmetry. In the following, we will give an explicit representation of the twisted \( N = 3 \) SCA in \( \hat{c} = 1 \) fermionic string theory. The three supercurrents \( (G^+, G^3, G^-) \) form a \( SU(2) \) triplet which consists of the BRST current with a suitable total derivative modification, \( N = 1 \) supercurrent, and the anti-ghost \( b \), respectively. Namely,

\[
G^+ = \sqrt{2} \left[ c \left( T^M + T^L + \frac{1}{2} T^G \right) - \frac{1}{2} \gamma \left( G^M + G^L + \frac{1}{2} G^G \right) \right. \\
+ \partial \left\{ \frac{1}{2} \partial c + \frac{1}{4} c \beta \gamma - q_1 \left( 2 c i \partial \phi^M - \psi^M \gamma \right) \right. \\
- q_2 \left( 2 c i \partial c + 2 c i \partial \phi^L - \psi^L \gamma \right) \left\} \right],
\]
\[ G^3 = G^M + G^L + G^G, \quad \text{(29)} \]
\[ G^- = \sqrt{2}b. \quad \text{(30)} \]

Here and hereafter \( q_1 = \pm (i q_2 - \frac{1}{2}) \).

The energy-momentum tensor of this algebra is that of \( \hat{c} = 1 \) fermionic string theory \((\mathbb{I})\). The \( SU(2) \) currents \( (J^+, J^3, J^-) \) are identified with the \( G^3 \)-super partner of the modified BRST current \( G^+ \), the modified ghost number current, and the \( G^3 \)-super partner of \( G^- \), respectively:

\[ J^+ = -c \left( G^M + G^L + b \gamma - \partial c \beta \right) + \frac{1}{2} \beta \gamma^2 + \partial \gamma \\
-2q_1 \left( \gamma i \partial \phi^M - \partial c \psi^M - 2c \partial \psi^M \right) \\
+2q_2 \left( \partial c \psi^L + 2c \partial \psi^L - \gamma i \partial \phi^L - 2i \partial \gamma \right), \quad \text{(31)} \]
\[ J^3 = -bc - \beta \gamma + 2q_1 i \partial \phi^M + 2q_2 i \partial \phi^L, \quad \text{(32)} \]
\[ J^- = -2 \beta. \quad \text{(33)} \]

There is also an additional fermion which is the \( G^3 \)-super partner of the \( U(1) \) current \( J^3 \):

\[ \Psi = -\beta c + 2q_1 \psi^M + 2q_2 \psi^L. \quad \text{(34)} \]

The central charge of this \( N = 3 \) SCA is equal to \( c^{N=3} = 3 - 12i q_2 \). In the following, we shall show that the twisted \( N = 2 \) SCA in (27) is nothing but a hidden \( N = 2 \) algebra in the enveloping algebra of this twisted \( N = 3 \) SCA.

First, note that the supercurrent \( G^+_{st} \) and the \( U(1) \) current \( J_{st} \) of the topological structure (27) are equal to those of the twisted \( N = 3 \) SCA given above, the parameters \( q_1 \) and \( q_2 \) of which are set to be 0 and \(-i/2\) respectively:

\[ \sqrt{2} G^+_{st}(z) = G^+(z), \quad \text{(35)} \]
\[ J_{st}(z) = J^3(z). \quad \text{(36)} \]

Here and hereafter the generators without subscripts are those of the twisted \( N = 3 \) SCA. Next let us examine the modified anti-ghost \( b^* = G^-_{st} \) in the topological structure. One can easily find that this can be written in terms of the generators of the twisted \( N = 3 \) SCA, namely,

\[ \sqrt{2} G^-_{st}(z) = \frac{1}{2} \left( G^-(z) + \sqrt{2} \Psi J^-(z) \right). \quad \text{(37)} \]
Finally, we can read off the expression of the energy-momentum tensor $L(z)$ of the topological structure in terms of the $N = 3$ SCA from the single pole of the OPE $G^+_sl(z)G^c_{st}(w)$:

$$L(z) = \frac{1}{2} \left( T(z) + \frac{1}{2} J^{-} J^{+}(z) + G^{3} \Psi(z) \right). \quad (38)$$

Thus, the $N = 2$ SCA (27) in the subspace $\mathcal{F}$ has a simple expression in terms of the twisted $N = 3$ SCA in $\hat{c} = 1$ fermionic string theory.

The above expressions (35)-(38) suggest the existence of an $N = 2$ SCA in the enveloping algebra of a generic $N = 3$ SCA. In fact, apart from $\hat{c} = 1$ string theory, we can show that in general there exists a hidden $N = 2$ SCA in the enveloping algebra of an $N = 3$ SCA, the generators of which read

$$\tilde{T} = \frac{1}{1 + \frac{\alpha}{\sqrt{2}}} \left[ T + \frac{\alpha}{2\sqrt{2}} (J^{-} J^{+} + 2G^{3} \Psi) \right], \quad (39)$$

$$\tilde{G}^{+} = G^{+}, \quad (40)$$

$$\tilde{G}^{-} = \frac{1}{1 + \frac{\alpha}{\sqrt{2}}} \left( G^{-} + \alpha \Psi J^{-} \right), \quad (41)$$

$$\tilde{J}^{\pm} = J^{\pm}, \quad (42)$$

where $\alpha = -\frac{3\sqrt{2}}{c}$ and the central charge of this algebra equals to $c$. In the above equation, all the generators are expressed in the twisted form for both a hidden $N = 2$ SCA and the $N = 3$ SCA. One can easily verify that the case of $\hat{c} = 1$ string theory (35)-(38) corresponds to $\alpha = \sqrt{2}$; the $N = 2$ SCA in (27), which realizes the topological model in $\hat{c} = 1$ string theory, is the hidden $N = 2$ SCA in the enveloping algebra of the $N = 3$ SCA.

Since an $N = 3$ superconformal algebra has a symmetry under the replacements $T \to T$, $G^{\pm} \to G^{\mp}$, $G^{3} \to -G^{3}$, $J^{3} \to -J^{3}$, $J^{\pm} \to J^{\mp}$, and $\Psi \to \Psi$, we can also obtain another $N = 2$ SCA with these replacements. Associated with this fact, we can obtain an automorphism in the enveloping algebra of an $N = 3$ SCA:

- Energy-momentum tensor: $\tilde{T}(z) = T(z)$,
- Supercurrents: $\tilde{G}^{\pm}(z) = G^{\pm}(z) + \alpha \Psi J^{\pm}(z)$, $\tilde{G}^{3}(z) = G^{3}(z) + \sqrt{2} \alpha \Psi J^{3}(z)$,
- $SU(2)$ currents: $\tilde{J}^{\pm}(z) = J^{\pm}(z)$, $\tilde{J}^{3}(z) = J^{3}(z)$,
- Fermion: $\tilde{\Psi}(z) = -\Psi(z),$

where $\alpha = -\frac{3\sqrt{2}}{c}$ and the central charge of this algebra equals to $c$, and all the generators are of untwisted form.
To summarize, we have shown that $\hat{c} = 1$ string theory consists of two parts $\mathcal{F}$ and $\tilde{\mathcal{F}}$, one of which ($\mathcal{F}$) can be regarded as a realization of a topological model. The physical spectrum of the topological model turns into the discrete physical spectrum of $\hat{c} = 1$ string theory in this realization. Hence, we can say that the other part $\tilde{\mathcal{F}}$ is irrelevant and $\hat{c} = 1$ string theory is largely governed by the topological model as for the physical contents of the theory. We have also found that this topological structure in the string theory can be understood as a hidden $N = 2$ SCA in the enveloping algebra of the $N = 3$ SCA in the fermionic string theory.

In [13], $c = 1$ string theory is shown to be related with a topological model. Since $\hat{c} = 1$ string theory has a richer symmetry than $c = 1$ string theory, one may naively consider that the topological model corresponding to $\hat{c} = 1$ string theory possesses an extended symmetry compared with the $c = 1$ case. In contrast with this expectation, we have shown that a topological model for the $\hat{c} = 1$ case is essentially the same one as obtained in the $c = 1$ case. The effect of the additional symmetry appears as the modification of the energy-momentum tensor, or the derivative. Indeed, this modification is possible because of the existence of the $N = 3$ SCA, which results from the world-sheet supersymmetry in $\hat{c} = 1$ string theory.

Although we have obtained the algebraic meaning of the modified energy-momentum tensor $\mathcal{L}$, it is still unclear what this modification implies in the context of string theory and how the topological model relates with $\hat{c} = 1$ string theory. In order to answer to these questions, we need understand the role of the subspace $\tilde{\mathcal{F}}$. This sector is independent of the relevant one $\mathcal{F}$ and seems not to contribute to the physical contents of the model. Moreover, the energy-momentum tensor $\tilde{\mathcal{L}}$ for this sector has vanishing central charge. This situation reminds us of the $G/H$ coset model [17, 18]. The $G/H$ coset model consists of a $G$-current algebra, the gauge field and the gauge ghosts. The entire Fock space is split into two parts; one is the $G/H$ part and the other corresponds to the $H$-current algebra, the gauge fields and the gauge ghosts. The latter is topological and gauged away by making use of the BRST charge for gauge fixing. This structure is similar to the present case if we regard $\mathcal{F}$ as the $G/H$ sector and $\tilde{\mathcal{F}}$ as the topological sector which is gauged away. If this is the case, the topological structure we have found is a string theory with a certain gauge symmetry, and the fact that the $\tilde{\mathcal{F}}$ is irrelevant for the physical contents is naturally explained from the point of view of gauge fixing. In order to establish this point of view for the topological structure in $\hat{c} = 1$ string theory, we need more careful analysis.
for the irrelevant sector \( \tilde{F} \), in particular identification of the BRST charge for the gauge fixing.

Another possible interpretation of the irrelevant sector \( \tilde{F} \) is concerned with the embedding of bosonic string theories into a fermionic string theory. As was mentioned above, the topological structure (27) we have observed in \( \hat{c} = 1 \) string theory is essentially the same one as obtained in the \( c = 1 \) case [13]. In this sense, this topological structure can be viewed as an embedding of \( c = 1 \) bosonic string theory into \( \hat{c} = 1 \) fermionic string theory. Recently, it has been shown [19, 20] that all the bosonic string theories are considered to be equivalent with a particular class of fermionic string theories. More precisely, the Fock space of this fermionic string theory is possible [20] to decompose into the tensor product of that of the bosonic string theory and a topological sector which is irrelevant for the physical contents of the theory. In other words, bosonic string theories can be embedded into fermionic string theories. Since \( c = 1 \) string theory is one of the bosonic string theories, we can apply this procedure to \( c = 1 \) string theory and embed it into a fermionic string theory. Thus we have two embeddings of \( c = 1 \) string theory into a fermionic string theory; one is our topological structure (27) and the other is based on the method in [13, 20]. It is highly plausible that these two embeddings are actually identical and that \( \hat{c} = 1 \) string theory belongs to the particular class of fermionic string theories obtained in [13]. Although these two kinds of fermionic string theories are apparently distinct from each other, it may be possible to relate them via, say, a similarity transformation as utilized in [20].

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