Global characteristic problem for Einstein vacuum equations with small initial data: (I)
The initial data constraints.

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Abstract

We show how to prescribe the initial data of a characteristic problem satisfying the constraints, the smallness, the regularity and the asymptotic decay suitable to prove a global existence result. In this paper, the first of two, we show in detail the construction of the initial data and give a sketch of the existence result. This proof, which mimics the analogous one for the non characteristic problem in [Kl-Ni], will be the content of a subsequent paper.

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1 Introduction

1.1 Statement of the problem

Recently Sergiu Klainerman and one of the present authors (F.N.) gave a new proof of the stability result for the Minkowski spacetime, [Kl-Ni], previously obtained by D.Christodoulou and S.Klainerman, [Ch-Kl]. The proof refers to the spacetime outside the domain of influence of a compact region with initial data on a spacelike hypersurface $\Sigma_0$. The global existence is proved via a bootstrap mechanism in a way which avoids the choice of a specific set of coordinates as, for instance, the harmonic ones. In the present paper we show that similar techniques can be used to solve a characteristic problem for the vacuum Einstein equations.

As in the non characteristic case, the problem is naturally split in two parts. In the first one the goal is showing how the initial data have to be assigned, which of them can be given freely and which are determined by the constraint equations which, in the characteristic case, are “ordinary differential” transport equations. In the second part we prove, given the initial data, a (global) existence theorem. In this paper we focus our attention on the first part and give only a broad sketch of the existence proof. Therefore while we discuss in a great detail the way of obtaining the initial data satisfying the “characteristic problem” we postpone to the second paper a detailed discussion on the minimal regularity assumptions required to prove the maximal development existence.

The characteristic problem has been treated by various authors, see for instance F.Cagnac, [Ca], H.Muller Zum Hagen, [Mu], H.Muller Zum Hagen and H.J.Seifert, [Mu-Se] and the series of papers by Dossa, see [Do] and references therein, but, in our opinion, the more significant paper is the one by A.Rendall, [Ren], where a thorough examination is done to show how to obtain initial data satisfying the constraint equations and the harmonic conditions. In the present work, as our existence theorem will be an adaptation of the result in [Kl-Ni] which is, basically, coordinate independent, we discuss the constraint equations for the initial data without having to worry about the harmonic conditions. This, we believe, is an advantage and a simplification with respect to the work of A.Rendall. Moreover as our existence result is a global result, in a sense we are going to specify later on, it uses, as in [Kl-Ni], a bootstrap mechanism based on some a priori estimates for a family of generalized “energy integral norms”. Therefore we have to impose that the analogous norms, written in terms of the initial data as $L^2$ integrals on the null initial hypersurface, be finite (and small). This requires that our initial data, in addition to satisfy the constraints, must have appropriate regularity and asymptotic decay.

To apply the bootstrap program and prove the global existence result, we need a separate local existence proof; we can use, to provide it, the H.Muller Zum Hagen, H.Muller Zum Hagen and H.J.Seifert result, [Mu-Se]. Their results provide, nevertheless, a local existence proof for the vacuum Einstein equations written in harmonic coordinates. Therefore, to use our initial data in proving these local existence results we have, at least locally, to reexpress them in the
harmonic gauge. This is done in the second part of the present paper.

Finally we have to specify the geometry of the null hypersurface on which we prescribe the initial data. In this paper we consider as null initial hypersurface a portion of an “outgoing null cone”, $C_0$, and a portion of an “incoming null cone”, $\overline{C}_0$. $^1$ such that their intersection, $S_0$, is a two dimensional surface $S^2$-diffeomorphic, $S_0 = C_0 \cap \overline{C}_0$. This particular initial hypersurface allows us to prove the existence of a maximal Cauchy development of $C_0 \cup \overline{C}_0$, using the same techniques developed in [Kl-Ni]. If, nevertheless, we restrict our goal only to proving the existence of initial data satisfying the constraints it turns out that the procedure developed here can be also used to obtain initial data on a single outgoing cone truncated near the vertex. $^2$ Finally let us point out that in the paper by H.Muller Zum Hagen and H.J.Seifert, [Mu-Se] and also in the one by A.Rendall, [Ren] the choice of the initial hypersurface is more general. There in fact the analogous of $C_0$ and $\overline{C}_0$ are two null hypersurfaces, denoted $N_1$, $N_2$, intersecting on a codimension 2 surface $S$ which is not necessarily diffeomorphic to $S^2$. In particular $N_1$ and $N_2$ can be diffeomorphic to null hyperplanes. $^3$

We believe that our approach can also deal, with minor modifications, the case where $N_1$ and $N_2$ are diffeomorphic to null hyperplanes.

1.2 Structure equations

Let $(\mathcal{M}, g)$ a manifold with a Lorentzian metric. If we introduce in it a null orthonormal frame, the so called “rep`ere mobile” of Cartan, see [Sp], Vol.2, it is immediate to show that some differential relations between the covariant derivatives of the vector fields forming the null frame are automatically satisfied. These relations, called “structure equations”, have the following expressions. Let us write the null frame and its dual basis as

$$\{e_\alpha\} = \{e_1, e_2, e_3, e_4\}, \quad \{\theta^\alpha\} = \{\theta^1, \theta^2, \theta^3, \theta^4\}$$

and define

$$D_{e_\alpha} e_\beta \equiv \Gamma^\gamma_{\alpha\beta} e_\gamma, \quad R(e_\alpha, e_\beta)e_\gamma \equiv R^\delta_{\gamma\alpha\beta} e_\delta,$$  \hspace{1cm} (1.1)

where $R$ is the Riemann tensor of the manifold $\mathcal{M}$. The connection coefficients are the components $\Gamma^\gamma_{\alpha\beta}$. The connection 1-form and curvature 2-form are

$$\omega^\gamma_{\beta} \equiv \Gamma^\gamma_{\beta\gamma}, \quad \Omega^\gamma_{\beta\delta} \equiv \frac{1}{2} R^\gamma_{\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta.$$  \hspace{1cm} (1.2)

They satisfy the first and the second structure equations, see [Sp]$^4$,

$$d\theta^\alpha = - \omega^\alpha_{\gamma} \wedge \theta^\gamma, \quad d\omega^\alpha_{\gamma} = - \omega^\alpha_{\beta} \wedge \omega^\beta_{\gamma} + \Omega^\alpha_{\gamma}.$$  \hspace{1cm} (1.3)

$^1$The definition of “outgoing null cone” and “incoming null cone” are made precise in subsection 1.3 where we specify the properties of $C_0$ and $\overline{C}_0$.

$^2$The truncation is to avoid the singularity problem at the cone vertex.

$^3$In [Mu-Se] the case where only one of the two hypersurfaces is null is also considered.

$^4$With the obvious modifications due to the Lorentzian metric.
Although the structure equations are valid in a more general framework, let us assume that it is possible to foliate the manifold \((\mathcal{M}, g)\) with a double null foliation, that is by two families of null hypersurfaces (the analogous of the null cones in the Minkowski spacetime) we denote \(\{C(\lambda)\}\) and \(\{C(\nu)\}\). Moreover we assume that each two dimensional surface \(S(\lambda, \nu) \equiv C(\lambda) \cap C(\nu)\) is diffeomorphic to \(S^2\) and that the family of these surfaces defines a foliation of \((\mathcal{M}, g)\). The null moving frame we consider is a moving frame adapted to this foliation in the sense that the connection coefficients, are the first structure equations, see 1.3, written explicitly in terms of the connections, \(\chi_{ab} = g(D_{ea} e_b, e_a)\) and \(\chi_{ab} = g(D_{ea} e_3, e_b)\), the relations hold

\[
e_4 = 2\Omega L, \quad e_3 = 2\Omega L.
\]

Under these assumptions the connection coefficients have the following expressions,

\[
\begin{align*}
\chi_{ab} &= g(D_{ea} e_4, e_b) \quad \chi_{ab} = g(D_{ea} e_3, e_b) \\
\xi_a &= \frac{1}{2} g(D_{e_3 e_4}, e_a) = \frac{1}{2} e_3 (\log \Omega) g(e_3, e_a) = 0 \\
\xi_a &= \frac{1}{2} g(D_{e_4 e_3}, e_a) = \frac{1}{2} e_4 (\log \Omega) g(e_4, e_a) = 0 \\
\omega &= \frac{1}{4} g(D_{e_3 e_4}, e_4) = -\frac{1}{2} D_4 (\log \Omega) \\
\eta_a &= \frac{1}{4} g(D_{e_4 e_3}, e_3) = -\frac{1}{2} D_3 (\log \Omega) \\
\zeta_a &= -\zeta_a + \nabla_a \log \Omega, \quad \eta_a = \zeta_a + \nabla_a \log \Omega \\
\zeta_a &= \frac{1}{2} g(D_{e_a e_4}, e_3).
\end{align*}
\]

The first structure equations, see 1.3, written explicitly in terms of the connection coefficients, are

\[
\begin{align*}
D_a e_b &= \nabla_a e_b + \frac{1}{2} \chi_{ab} e_3 + \frac{1}{2} \chi_{ab} e_4 \\
D_a e_3 &= \chi_{ab} e_b + \zeta_a e_3, \quad D_a e_4 = \chi_{ab} e_b - \zeta_a e_4 \\
D_3 e_a &= \nabla_3 e_a + \eta_a e_3, \quad D_4 e_a = \nabla_4 e_a + \eta_a e_4 \\
D_3 e_3 &= (D_3 \log \Omega)e_3, \quad D_3 e_4 = -(D_3 \log \Omega)e_4 + 2\eta_b e_b \\
D_4 e_4 &= (D_4 \log \Omega)e_4, \quad D_4 e_3 = -(D_4 \log \Omega)e_3 + 2\eta_b e_b.
\end{align*}
\]

Keeping in mind that the structure equations are automatically satisfied in any Lorentzian manifold, \((\mathcal{M}, g)\), they can be interpreted simply as a way of

\[^5\text{Hereafter } D_{ea} = D_a \text{ and } D_{e(3,4)} = D_{(3,4)}.\]
rewriting the first order covariant derivatives and the Riemann tensor in terms of the connection coefficients and their first derivatives. Equations 1.6 show it in a clear way. In the same way the second set of equations in 1.3 expresses the Riemann tensor in terms of the connection coefficients and their derivatives.

These relations acquire, nevertheless, a different meaning if we consider them in a vacuum Einstein spacetime, namely in a manifold \((M, g)\) where the Ricci part of the Riemann tensor is identically zero. In this case a subset of the second set of equations in 1.3 can be interpreted as a way of writing the vacuum Einstein equations in terms of the connection coefficients and their first derivatives,\(^6\)

As our goal is posing and solving a class of initial data characteristic problems for the vacuum Einstein equations, we will consider initial data given on a null hypersurface (a “null outgoing cone” or the union of a “null outgoing cone” and an “incoming” one). Instead of writing the initial data in terms of the metric tensor components and their partial derivatives we will use as initial data the metric tensor components and the connection coefficients restricted to the initial hypersurface.\(^7\) Therefore the subset of the structure equations defined above will play the role of constraint equations that our initial data have to satisfy.

To better clarify this point let us recall that chosen a specific set of coordinates the initial data of the characteristic problem can be assigned prescribing the various components of the Lorentzian metric and their first partial derivatives. The characteristic nature of the problem implies that these quantities cannot be given freely, but have to satisfy some constraints. To associate these constraints to the structure equations for the connection coefficients we observe that, once we have given on the null initial hypersurface a moving frame and a foliation, all the first derivatives of the metric components or their Christoffel symbols can be expressed in terms of the connection coefficients and of the derivatives of the moving frame vector fields (this will be discussed in any detail later on). Therefore the constraint equations for the first derivatives of the metric components are the immediate consequence of this subset of the structure equations we have previously introduced.

This approach has the advantage of presenting the constraint equations in a more covariant way, without requiring a specific set of coordinates. Moreover the choice of a “gauge” for our problem is associated to specifying a foliation on the initial hypersurface and, subsequently, in the whole spacetime instead of the more usual choice of “harmonic coordinates”.

In the remaining part of this section we give the explicit form of the structure equations which will be interpreted as constraint equations and in the following section we give the definition of the initial Cauchy characteristic problem. The remaining sections are devote to showing how one can explicitly obtain a family of initial data satisfying the characteristic constraint equations and some general “regularity smallness conditions” which will allow to prove a global existence result.

\(^6\)As we shall show in the sequel of this section these equations are of two types: evolution equations along the null directions \(e_3, e_4\) and elliptic equations of the Hodge type along the two dimensional surfaces \(\Sigma(\lambda, \nu)\).

\(^7\)The possibility of doing this will be discussed in the next sections.
1.2.1 Structure equations for vacuum Einstein manifolds

We recall here, without derivation, see, for more details, [Kl-Ni], Chapter 3, the form of the Einstein equations written as a subset of the structure equations.

Denoting \( \mathbf{R}_{\alpha \beta} = \text{Ricci}(e_\alpha, e_\beta) \), we have:

i) \( \mathbf{R}_{44} = 0. \)

\[
\mathbf{R}_{44} = D_4 \text{tr} \chi + \frac{1}{2}(\text{tr} \chi)^2 + 2\omega \text{tr} \chi + |\hat{\chi}|^2 = 0 \tag{1.7}
\]

ii) \( \mathbf{R}_{33} = 0. \)

\[
\mathbf{R}_{33} = D_3 \text{tr} \chi + \frac{1}{2}(\text{tr} \chi)^2 + 2\omega \text{tr} \chi + |\hat{\chi}|^2 = 0 \tag{1.8}
\]

iii) \( \mathbf{R}_{4a} = 0. \)

\[
\mathbf{R}_{4a} = D_4 \zeta + \frac{3}{2} \text{tr} \chi \zeta + \zeta \text{div} \chi + \frac{1}{2} \nabla \text{tr} \chi + D_4 \nabla \log \Omega = 0 \tag{1.9}
\]

iv) \( \mathbf{R}_{3a} = 0. \)

\[
\mathbf{R}_{3a} = D_3 \zeta + \frac{3}{2} \text{tr} \chi \zeta + \zeta \text{div} \chi - \frac{1}{2} \nabla \text{tr} \chi - D_3 \nabla \log \Omega = 0 \tag{1.10}
\]

v) \( \mathbf{R}_{ab} = 0. \) This equation is

\[
\mathbf{R}_{ab} = -\frac{1}{2}(\mathbf{R}_{a3b4} + \mathbf{R}_{b3a4}) + \mathbf{R}_{bac} = 0, \tag{1.11}
\]

and, from explicit computation we obtain:

\[
-\frac{1}{2}(\mathbf{R}_{a3b4} + \mathbf{R}_{b3a4}) = D_4 \chi_{ab} + \frac{1}{2} (\chi \cdot \chi)_{ab} + \chi \cdot \chi_{ba} + (D_4 \log \Omega) \chi_{ab} - ((\nabla \eta)_{ab} + (\nabla \eta)_{ba} - ((\eta \eta)_{ab} + (\eta \eta)_{ba}) \tag{1.12}
\]

\[
\mathbf{R}_{bac} = (2)\mathbf{R}_{bac} = \frac{1}{2} \left( \text{tr} \chi \chi_{ab} + \text{tr} \chi \chi_{ba} - (\chi \cdot \chi)_{ab} - (\chi \cdot \chi)_{ba} \right) \tag{2.1}
\]

Therefore equation 1.11 becomes

\[
D_4 \chi_{ab} + \frac{1}{2} ((\chi \cdot \chi)_{ab} + (\chi \cdot \chi)_{ba}) + (D_4 \log \Omega) \chi_{ab} - ((\nabla \eta)_{ab} + (\nabla \eta)_{ba} - ((\eta \eta)_{ab} + (\eta \eta)_{ba}) \tag{1.13}
\]

Decomposing this equation in its trace and traceless part, with respect to the \( a, b \) indices, we obtain, for the trace part:

\[
D_4 \text{tr} \chi + \chi \cdot \chi + (D_4 \log \Omega) \text{tr} \chi - 2(\text{div} \eta) - 2|\eta|^2 = -\frac{1}{2} (\mathbf{R} - \text{tr} \chi \chi + \text{tr} \chi) \tag{1.14}
\]
which we rewrite, recalling that $(2)R = 2K$, twice the scalar curvature of the leaves $S(\lambda, \nu)$ of the null hypersurface $C(\lambda)$,

$$D_4 tr \chi + tr \chi tr \chi - 2\omega tr \chi - 2(|\nabla v| - 2|\eta|^2 + 2K = 0. \quad (1.15)$$

The equation associated to the traceless part is:

$$D_4 \tilde{\chi}_{ab} + \frac{1}{2} tr \tilde{\chi}_{ab} + \frac{1}{2} tr \tilde{\chi}_{ab} - 2\omega \tilde{\chi}_{ab} - 2(\nabla \otimes \eta)_{ab} - 2(\eta \tilde{\otimes} \eta)_{ab} = 0. \quad (1.16)$$

Equations 1.15, 1.16 correspond to $R_{ab} = 0$. They can also be written in a similar way as evolution equations along the incoming cones, namely:

$$D_3 tr \chi + tr \chi tr \chi - 2\omega tr \chi - 2(|\nabla v| - 2|\eta|^2 + 2K = 0, \quad (1.17)$$

$$D_3 \tilde{\chi}_{ab} + \frac{1}{2} tr \tilde{\chi}_{ab} + \frac{1}{2} tr \tilde{\chi}_{ab} - 2\omega \tilde{\chi}_{ab} - 2(\nabla \otimes \eta)_{ab} - 2(\eta \tilde{\otimes} \eta)_{ab} = 0. \quad (1.18)$$

Interpreting these equations as constraint equations it is clear that their expressions 1.15, 1.16 are suitable for the initial data on the outgoing part of the null initial hypersurface, while those provided by 1.17, 1.18 will be used to assign $tr \chi$ and $\tilde{\chi}$ on the incoming part of the null initial hypersurface.

vi) $R_{34} = 0$. This equation can be written as

$$R_{34} = \frac{1}{2} R_{4343} + \sum_c R_{c3c4} = 0 \quad (1.19)$$

and the explicit expressions of $R_{4343}$ and $R_{c3c4}$ are:

$$\frac{1}{4} R_{4343} = [D_3 \omega + D_4 \omega - 4\omega - 3|\zeta|^2 + |\nabla \log \Omega|^2]$$

$$\sum_c R_{c3c4} = - \left[ D_4 tr \chi + \frac{1}{2} tr \chi tr \chi - 2\omega tr \chi + \tilde{\chi} \cdot \tilde{\chi} - 2(|\nabla v| - 2|\eta|^2 \right]$$

so that, finally, we obtain

$$D_3 \omega + D_4 \omega - 4\omega - 3|\zeta|^2 + |\nabla \log \Omega|^2 - \frac{1}{2} \left[ D_4 tr \chi + \frac{1}{2} tr \chi tr \chi - 2\omega tr \chi + \tilde{\chi} \cdot \tilde{\chi} - 2(|\nabla v| - 2|\eta|^2 \right] = 0. \quad (1.20)$$

We also list without derivations the remaining structure equations which depend on the conformal part of the Riemann tensor. They will be used when we require the appropriate regularity and smallness conditions for the initial data. To do it we first define the various conformal Riemann components with respect to the null orthonormal frame,

$$\alpha(C)(X, Y) = C(X, e_4, Y, e_4)$$

$$\beta(C)(X) = \frac{1}{2} C(X, e_4, e_3, e_4)$$
\[ \rho(C) = \frac{1}{4} C(e_3, e_4, e_3, e_4) \] (1.21)
\[ \sigma(C) = \frac{1}{4} \rho(\star C) = \frac{1}{4} \cdot C(e_3, e_4, e_3, e_4) \]
\[ \beta(C)(X) = \frac{1}{2} C(X, e_3, e_3, e_4) \]
\[ \alpha(C)(X, Y) = C(X, e_3, Y, e_3) \]

with \( X, Y \) arbitrary vectors tangent to \( S \) at \( p \) and the left Hodge dual of \( C \):

\[ \star C_{\alpha \beta \gamma \delta} = \frac{1}{2} \epsilon_{\alpha \beta \mu \nu} C_{\mu \nu}^{\gamma \delta}, \] (1.22)

where \( \epsilon^{\alpha \beta \gamma \delta} \) are the components of the volume element in \( M \).

### 1.2.2 The remaining structure equations.

The remaining structure equations, due to their dependance on the conformal part of the Riemann tensor, do not play the role of constraint equations for the initial data. They are automatically satisfied as they are just the explicit expression of the conformal part of the Riemann tensor in terms of the connection coefficients and their first derivatives. We list all of them.

\[ D_4 \hat{\chi} + \text{tr} \chi \hat{\chi} - (D_4 \log \Omega) \hat{\chi} = -\alpha \] (1.23)
\[ D_4 \zeta + 2 \chi \cdot \zeta + D_4 \nabla \log \Omega = -\beta \] (1.24)
\[ \nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi = \beta \] (1.25)

\[ D_4 \text{tr} \chi + \frac{1}{2} \text{tr} \chi \text{tr} \chi + (D_4 \log \Omega) \text{tr} \chi + \hat{\chi} \cdot \hat{\chi} + 2 \text{div} \zeta - 2 \Box \log \Omega \]
\[ -2|\zeta|^2 - 4 \zeta \cdot \nabla \log \Omega - 2|\nabla \log \Omega|^2 = 2\rho \] (1.26)

\[ D_3 \text{tr} \chi + \frac{1}{2} \text{tr} \chi \text{tr} \chi + (D_3 \log \Omega) \text{tr} \chi + \hat{\chi} \cdot \hat{\chi} - 2 \text{div} \zeta - 2|\zeta|^2 \]
\[ -2 \Box \log \Omega - 4 \zeta \cdot \nabla \log \Omega - 2|\nabla \log \Omega|^2 = 2\rho \] (1.27)

\[ K + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} = -\rho \] (1.28)

\[ \text{curl} \zeta - \frac{1}{2} \hat{\chi} \wedge \hat{\chi} = \sigma \] (1.29)

\[ D_3 \zeta + 2 \chi \cdot \zeta - D_3 \nabla \log \Omega = -\beta \] (1.30)
\[ \nabla \text{tr} \chi - \text{div} \chi + \zeta \cdot \chi - \zeta \text{tr} \chi = -\beta \]  
\[ D_3 \hat{\chi} + \text{tr} \hat{\chi} \hat{\chi} - (D_3 \log \Omega) \hat{\chi} = -\alpha \]  

It is important to observe that while \( \beta, \rho, \sigma, \beta \) can be expressed in two different ways in terms of the connection coefficients and their derivatives, this is not true in the case of \( \alpha \) and \( \alpha \).

1.3 The initial data for the characteristic problem

In this subsection we present a definition of the initial data for the Einstein vacuum characteristic Cauchy problem from the point of view discussed before, without prescribing a specific choice of coordinates. This will be done in three steps, first we specify a foliation on \( \mathcal{C} \) and we define a degenerate metric “adapted” to this foliation, second we introduce as initial data some tensor fields on \( \mathcal{C} \) which later have to be interpreted as the restriction of the connection coefficients on the initial hypersurface and, third, we define the constraints they have to satisfy. Finally we discuss the relation between these initial data and the more usual ones expressed in terms of partial derivatives of the components of the metric tensor.

1.3.1 Foliation of the initial data hypersurface

Let us consider two three-dimensional manifolds with an edge, we denote \( (C_0, g_0) \) and \( (C_0, g_0) \), endowed with two metric tensors \( g_0 \) and \( g_0 \). Let \( (C_0, g_0) \) and \( (C_0, g_0) \) have the following properties:

i) The metrics \( g_0 \) and \( g_0 \) are degenerate.\(^8\) This means the following: for any \( p \in C_0 \) there exist on \( T_p C_0 \) a vector \( N_0 \) such that

\[ g_0 (N_0, Y) = 0 \quad \forall \ Y \in T_p C_0 . \]  

and the same definition for the metric \( g_0 \) of \( C_0 \).

ii) The intersection of \( C_0 \) and \( C_0 \) be a two-dimensional surface diffeomorphic to \( S^2 \), \( S_0 = C_0 \cap C_0 \) and on \( S_0 \) the restrictions of \( g_0 \) and of \( g_0 \) coincide.

We will call these two manifolds the “initial (null) outgoing cone” \( C_0 \) and the “initial (null) incoming cone” \( C_0 \).\(^9\)

iii) The null geodesics on \( C_0 \) and \( C_0 \) have past end points on \( S_0 \).

Their union defines the initial hypersurface, \( \mathcal{C} = C_0 \cup C_0 \), of our characteristic problem. Once \( \mathcal{C} \) is given, to prescribe the initial data for the characteristic problem we have to specify some other properties of our initial outgoing and incoming (truncated) “cones”.

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\(^8\)\( g_0 \) and \( g_0 \) are sufficiently regular to make all next properties meaningful.

\(^9\)More precisely, as it will be discussed in detail later on, \( C_0 \) and \( C_0 \), when immersed in the Einstein spacetime, are truncated portions of an outgoing null hypersurface and an incoming null hypersurface, the analogue of null cones of the Minkowski spacetime, see also [Kl-Ni].
i) Properties of \( C_0 \):

a) \( C_0 \) is foliated by a family of two dimensional surfaces \( S_0(\nu) \) diffeomorphic to \( S^2 \) with \( \nu \in [\nu_0, \infty) \), defined as the level surfaces of a scalar function \( \eta(p) \) defined on \( C_0 \):

\[
S_0(\nu) = \{ p \in C_0 | \eta(p) = \nu \}.
\]

(1.34)

with \( S_0(\nu_0) = S_0 \). The function \( \eta(p) \), whose level surfaces are \( S_0(\nu) \), is defined in the following way: let \( \gamma(\nu) \) be the null geodesic on \( C_0 \) the affine parameter \( \nu \) being such that \( L = 2\frac{\partial}{\partial \nu} \); let \( \gamma(\nu) \) starting on \( S_0 \) with affine parameter \( \nu = 0 \) and passing through \( p \) when \( \nu = \nu(p) \), then we define

\[
\eta(p) = \nu_0 + \int_0^{\nu(p)} (4\Omega)^{-2}(\gamma(\nu'))d\nu'.
\]

(1.35)

and the scalar function \( \Omega \) will be specified later on.\(^{10}\)

Once defined the leaves \( S_0(\nu) \) of the foliation we introduce on \( C_0 \) a null orthonormal frame, \( \{e_4, e_1, e_2\} \), choosing \( e_4 = 2\Omega L \), with \( L \) the null geodesic vector field on \( C_0 \), and \( \{e_1, e_2\} \) an orthonormal frame tangent to \( S_0(\nu) \).

We define \( (\eta, \theta, \phi) \) as adapted coordinates on \( C_0 \), \( \theta(p), \phi(p) \) being the angular coordinates of the point \( p_0 \in S_0 = S_0(\nu_0) \) such that the null geodesic starting from \( p_0 \) reaches the point \( p \). With this definition an arbitrary point \( p \) on \( S_0(\nu) \) has coordinates \( (\nu, \theta(p), \phi(p)) \).

b) On each \( S_0(\nu) \) we define the metric tensor \( \gamma_{ab} = \gamma(\nu)_{ab} \), restriction to \( S_0(\nu) \) of the metric tensor \( g_0 \), \( \gamma(\nu)_{ab} = (g_0|S_0(\nu))_{ab} \). The second null fundamental form \( \chi \) is defined through the Lie derivative of \( g_0 \) with respect to \( N = 2\Omega L \),\(^{11}\)

\[
2\Omega \chi = \mathcal{L}_Ng_0
\]

(1.36)

which, in the \( (\nu, \theta, \phi) \) coordinates, takes the form

\[
\chi_{ab} = \frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \nu},
\]

(1.37)

and we require that \( \gamma_{ab} \) be such that \( \text{tr} \chi > 0 \) on the whole \( C_0 \). We define next, starting from the function \( \Omega \), the scalar function \( \omega \),

\[
\omega = -\frac{1}{2\Omega} \frac{\partial \log \Omega}{\partial \nu}.
\]

(1.38)

We introduce on \( C_0 \) some other quantities: a one form \( \zeta \) tangent to the leaves \( S_0(\nu) \) and two more quantities, a scalar function, denoted by \( \omega \), and a symmetric tensor, \( \chi \), whose trace and traceless parts we denote \( \text{tr} \chi \) and \( \tilde{\chi} \).

\(^{10}\) The value of \( \eta \) at \( S_0 \) is somewhat arbitrary, but in the following we will choose to connect it to the “radius” of \( S_0 \), \( r_0 \equiv \sqrt{4\pi^{-1}|S_0|_0} \), posing \( \nu_0 \in [c_1 \tilde{r}_0, c_2 \tilde{r}_0] \) with \( c_1, c_2 \) approximately equal to one. The precise bounds on \( c_1, c_2 \) will be given in Section 2.

\(^{11}\) The vector field \( N = 2\Omega L \) defines a diffeomorphism \( \Phi_N \) such that \( \Phi_N(\delta)[S_0(\nu)] = S_0(\nu + \delta) \).
\(\zeta, \omega, \chi\) do not have a direct geometrical meaning, but they will acquire it when \(C_0\) becomes a null hypersurface embedded in the vacuum Einstein spacetime \((\mathcal{M}, g)\).

Finally we assume that the quantities we have introduced are sufficiently regular so that on \(C_0\), together with \(\chi, \omega, \zeta, \chi\), the derivatives\(^{12}\)

\[
\nabla^k \chi, \nabla^k_\chi, \nabla^k \omega, \nabla^k \zeta, \nabla^{k-1} D_4 \hat{\chi}
\]

are well defined with \(k \in [1, q]\) and \(\nabla\) the covariant derivative associated to the metric \(\gamma_{ab}\). The choice of \(q\) will be specified in the final version of the existence theorem in subsection 3.4.

As we said these quantities have to satisfy some constraints. More precisely we require that the tensor fields \(\chi, \omega, \zeta, \chi\) satisfy the following (constraint) equations:\(^{13}\)

\[
\begin{align*}
D_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 + 2 \text{tr} \chi + |\chi|^2 &= 0 \quad (1.40) \\
D_4 \zeta + \chi + \text{tr} \chi \zeta - \triangle \chi + \nabla \text{tr} \chi + D_4 \nabla \log \Omega &= 0 \quad (1.41) \\
D_4 \text{tr} \chi + \text{tr} \chi \text{tr} \chi - 2 \text{tr} \chi \log \Omega - 2 |\eta|^2 + 2K &= 0 \quad (1.42) \\
D_4 \tilde{\chi} + \frac{1}{2} \text{tr} \chi \tilde{\chi} + \frac{1}{2} \text{tr} \tilde{\chi} + (D_4 \log \Omega) \tilde{\chi} + \nabla \tilde{\chi} (\zeta - \nabla \log \Omega) \\
&- (\zeta - \nabla \log \Omega) \tilde{\chi} (\zeta - \nabla \log \Omega) &= 0 \quad (1.43) \\
D_4 \omega - 2 \omega \omega - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + 1 \frac{1}{2} |\nabla \log \Omega|^2 - (\text{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \chi \cdot \chi) &= 0 \quad (1.44)
\end{align*}
\]

The nature of these equations is clear when the hypersurface \(C\) is considered embedded in the Einstein spacetime \((\mathcal{M}, g)\). In fact they are, restricted to \(C\), part of the vacuum Einstein equations previously written, see 1.7., 1.20., \(\zeta, \omega, \chi\) are connection coefficients and their constraint equations are some of the structure equations defined in \((\mathcal{M}, g)\). On the other side observe that we can define the constraints that the initial data have to satisfy in a formulation “intrinsic” to \(C\), which does not require to consider \(C\) embedded in a Lorentzian manifold.\(^{15}\) To do it we observe that equations 1.40...1.44 can be rewritten as transport equations for scalar quantities along \(C_0\), requiring that \(\{e_A\}\) be a Fermi transported orthonormal frame.\(^{16}\) With this frame \(\{e_A\}\) the previous equations can be rewritten as, see [KL-Ni] Chapter 4,

\[
\frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega \text{tr} \chi}{2} + 2 \text{tr} \omega \text{tr} \chi + \Omega |\chi|^2 = 0
\]

\(^{12}\)The assumptions on \(D_4 \nabla^{k-1} \chi\) are used in subsection 2.3.

\(^{13}\)The last equation has been written in a different way using the expression of the commutator \([D_4, D_4]\).

\(^{14}\)In fact the covariant derivative \(D_4\) is associated to the four dimensional metric \(g\).

\(^{15}\)In principle both formulations are possible, the intrinsic one is more general, but subsequently, the explicit construction of the initial data will be obtained following the second formulation.

\(^{16}\)\(\{e_A\}\) is Fermi transported if \(D_4 e_a = 0\). This implies that the components of \(e_A\) satisfy the equations \(\partial_a e_a + \Omega \chi^a e^b A = 0\), see [KL-Ni], appendix to Chapter 3, which are intrinsically defined on \(C_0\). Other choice for transporting the \(e_A\)’s along \(C_0\) could also be adopted.
\[
\frac{\partial \zeta_A}{\partial \nu} + \Omega \chi_{AB} \zeta_B + \Omega \chi \chi_A - \Omega (\delta \nu \chi)_A + \Omega \nabla_A \chi + \frac{\partial \nabla_A \log \Omega}{\partial \nu} = 0 \\
\frac{\partial \chi}{\partial \nu} + \Omega \chi \frac{\partial \chi}{\partial \nu} - 2 \Omega \omega \chi - 2 \Omega (\delta \nu \eta) - 2 \Omega |\eta|^2 + 2 \Omega K = 0 \\
\frac{\partial \chi_{AB}}{\partial \nu} + \frac{\Omega \chi_{AB}}{2} + \frac{\partial \chi_{AB}}{\partial \nu} + (\frac{\partial \log \Omega}{\partial \nu}) \chi_{AB} + \Omega \nabla_A \nabla_B (\chi - \nabla \log \Omega)_B - \Omega (\zeta - \nabla \log \Omega)_B = 0 \\
- \frac{\partial \omega}{\partial \nu} - 2 \Omega \omega + \Omega \chi \nabla \log \Omega + \frac{1}{2} [-3 \Omega |\chi|^2 + \Omega |\nabla \log \Omega|^2] \\
+ \Omega \left( K + \frac{1}{4} \chi \cdot \nabla - \frac{1}{2} \chi \cdot \chi \right) = 0 .
\]

Written in this way these equations appear as ordinary differential equations on \( C_0 \) for some scalars functions and, therefore, can be interpreted as the intrinsically defined (constraint) equations for the initial data.

ii) Properties of \( C_0 \):

The properties we require for \( C_0 \) are of the same type as those for \( C_0 \).

a) \( C_0 \) is foliated by a family of two dimensional surfaces \( S_0(\lambda) \) diffeomorphic to \( S^2 \) with \( \lambda \in [\lambda_1, \lambda_0] \), \( \lambda_0 < 0 \), defined as the level surfaces of a scalar function \( u(p) \) defined on \( C_0 \):

\[
S_0(\lambda) = \{ p \in C_0 | u(p) = \lambda \}. 
\]

b) We require that \( S_0(\lambda_1) = S_0(\nu_0) \).

The construction of the foliation proceeds as in the previous case and we do not repeat it here. We only observe that a function \( \Omega \) is assigned on \( C_0 \), analogous to the function \( \Omega \) defined on the “outgoing cone”, and \( u(p) \) is defined as \( ^{18} \)

\[
u(p) = \lambda_1 + \int_0^{\nu(p)} (4 \Omega)^{-2} (\gamma(v')) dv' .
\]

where \( \gamma \) is the null geodesic starting at the point \( p_0 \) in \( S_0 \) which reaches the point \( p \). \( ^{19} \)

Once we have defined the leaves \( S_0(\lambda) \) of this foliation we introduce, as before, a null orthonormal frame \( \{ e_3, e_1, e_2 \} \) on \( C_0 \) choosing \( e_3 = 2 \Omega L \), with \( L \) the null geodesic vector field on \( C_0 \), and \( \{ e_1, e_2 \} \) an orthonormal frame tangent to \( S_0(\lambda) \).

\( ^{18} \) It is enough to require that \( |\lambda_0| \) is upper bounded.

\( ^{19} \) When immersed in the spacetime the null hypersurface \( C_0 \) is an “incoming null hypersurface” and its foliation is defined starting from the two dimensional surface \( S_0(\lambda_1) \), \( \nu = 0 \) is associated to the points on \( S_0(\lambda_1) \) and the negative values of \( v \) correspond to two dimensional surfaces of smaller radius.

\( ^{19} \) As for \( \eta \) the choice of the value for \( u \) at \( S_0 \) is somewhat arbitrary. Nevertheless in Section 2 we will connect it to the “radius” of \( S_0, r_0 = \sqrt{4\pi^{-1} |S_0|^2} \), posing \( |\lambda_1| \in [c_1 r_0, c_2 r_0] \) with \( c_1, c_2 \) approximately equal to one.
As done before on $C_0$ we choose $(u, \theta, \phi)$ as adapted coordinates on $C_0$, $\theta(p), \phi(p)$ being defined as the angular coordinates of the point $p_0 \in S_0(\lambda_1)$ such that the null geodesic starting from it reaches the point $p$. Therefore an arbitrary point $p$ on $S_0(\lambda)$ has coordinates $(\lambda, \theta(p), \phi(p))$.

b) As before we define on each $S_0(\lambda)$ the metric tensor $\gamma_{ab}$, restriction of the metric tensor $g_0$, $\gamma(\lambda)_{ab} = (g_0|_{S_0(\lambda)})_{ab}$. To the metric tensor $g_0$ is associated the second null fundamental form relative to $e_3$,

$$2\Omega \chi_{ab} = L_N \gamma,$$

where $N = 2\Omega^2 L$ which in the $(\lambda, \theta, \phi)$ coordinates has the form

$$\chi_{ab} = \frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \lambda}.$$ (1.49)

Here we require $\text{tr} \chi|_{S_0(\lambda_1)} < 0$ (which justifies the name “incoming cone” for $C_0$).

We define on $C_0$, starting from the function $\Omega$, a scalar function $\omega$:

$$\omega = -\frac{1}{2\Omega} \frac{\partial \log \Omega}{\partial \lambda},$$ (1.50)

which again will acquire a geometrical meaning once the characteristic initial value problem is stated. As in the $C_0$ case, we define on $C_0$ the “torsion” one form $\zeta$, a symmetric tensor, $\chi$ on $C_0$ whose trace and traceless parts we denote $\text{tr} \chi$ and $\hat{\chi}$ and a scalar function $\omega$. Again on $C_0$ $\chi$ and $\omega$ do not have a direct geometrical meaning, but will acquire it when $C_0$ becomes a null hypersurface embedded in the vacuum Einstein spacetime $(M, g)$.

Also on $C_0$ the introduced quantities have to satisfy some constraints. More precisely we require that the tensor fields $\chi, \omega, \zeta, \chi, \omega$ satisfy on $C_0$ the following constraint equations, again written in a way intrinsic to $C_0$:

$$\frac{\partial \text{tr} \chi}{\partial \lambda} + \frac{\Omega}{2} \text{tr} \chi + 2 \Omega \omega \text{tr} \chi + \Omega \chi^2 = 0$$

$$\frac{\partial \chi_A}{\partial \lambda} + \frac{\Omega}{2} \chi_{AB} \zeta_B + \frac{\Omega}{2} \chi_A + \Omega (\text{div} \chi)_A - \Omega \nabla_A \text{tr} \chi - \frac{\partial \nabla_A \log \Omega}{\partial \lambda} = 0$$

$$\frac{\partial \chi_{AB}}{\partial \lambda} + \frac{\Omega}{2} \chi_{AB} + \frac{\Omega}{2} \chi_{AB} + \frac{\partial \log \Omega}{\partial \lambda} \chi_{AB} - \Omega \nabla_A (\zeta + \nabla \log \Omega)_B$$

$$- \Omega (\zeta_A + \nabla_A \log \Omega) \zeta (\zeta + \nabla \log \Omega)_B = 0$$

$$\frac{\partial \omega}{\partial \lambda} - 2 \Omega \omega + \Omega \chi_C \nabla_C \log \Omega + \frac{1}{2} \left[ -3 \Omega \chi^2 + \Omega \nabla \log \Omega^2 ight.$$

$$\left. + \Omega (K + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \zeta \cdot \zeta) \right] = 0.$$ (1.51)

20The vector field $N = 2\Omega^2 L$ defines a diffeomorphism $\Phi_N$ such that $\Phi_N(\delta)|_{S_0(\lambda)} = S_0(\lambda + \delta)$. 

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Finally we assume that these quantities are sufficiently regular so that on $C_0$, together with $\chi, \chi, \omega, \omega, \zeta$, the derivatives

$$\nabla^k \chi, \nabla^k \omega, \nabla^k \chi, \nabla^k \omega, \nabla^k \zeta, \nabla^{k-1} \hat{D} \hat{\chi}$$

are well defined with $k \in [1, q]$ and $\nabla$ the covariant derivative associated to the metric $\gamma_{ab}$.

**Remarks:**

a) It is appropriate to observe that the definitions and intrinsic properties we have given for $C_0$ and $C_0$ do not specify completely their relative position when they “become” embedded hypersurfaces in the Einstein spacetime. Nevertheless the way we define explicitly the hypersurface $C$, to solve the constraint equations, as an embedded hypersurface in a $R^4$ manifold implies that $C_0$ and $C_0$ have to be interpreted as approximate null cones with their vertices lying on the vertical axis passing through the origin. See also the remark at the beginning of subsection 3.5.

b) We have written the constraint equations in such a way that there is an exact symmetry between the equations on $C_0$ and those on $C_0$. To pass from the first to the second group it is enough to interchange the underlined quantities with the non underlined ones, $\nu$ with $\lambda$ and finally $\zeta$ with $-\zeta$, a detailed discussion is in [Kl-Ni], Chapter 3. This is, nevertheless, a somewhat delicate point: one could be tempted to believe that the coordinates $(\nu, \theta, \phi)$ on $C_0$ and $(\lambda, \theta, \phi)$ on $C_0$ are the restriction, on $C_0$ and $C_0$ respectively, of a set of coordinates for the Einstein vacuum spacetime. In fact, as discussed in [Kl-Ni], Chapter 3 for the non characteristic case, when the existence result is achieved we have a vacuum Einstein spacetime $(M, g)$ and $C_0, C_0$ are embedded null hypersurfaces part of a double null foliation made by family of null hypersurfaces $\{C(\lambda)\}, \{C(\nu)\}$, namely $i(C_0) = C(\lambda_1), i(C_0) = C(\nu_0)$. Also in this case we can use $u, u, \theta, \phi$ as coordinates, but it is not true that in this set of coordinates we have:

$$N = \frac{\partial}{\partial u}, \quad \bar{N} = \frac{\partial}{\partial u},$$

as this will imply $[N, \bar{N}] = 0$, which is false in a curved spacetime. The conclusion is that the introduction of a unique set of coordinates destroys the symmetry of the previous coordinate independent formulation. This will be even more clear when, to prove the existence of initial data satisfying the constraints, we will consider $C$ immersed in a Lorentz spacetime and we will assigne to it a specific set of coordinates. The same loss of symmetry will appear when, at the end of this section, we show how to connect our approach to a more standard one and express the connection coefficients in terms of partial derivatives of the metric components.

---

21This remark holds even if the initial hypersurface $C$ is embedded in a Lorentz spacetime.
**Definition 1.1** The initial data set of the characteristic problem consists in an initial data set relative to the “null outgoing cone” and an initial data set relative to the “null incoming cone”:

\[ \{ C_0 : \gamma_{ab}, \Omega, \zeta, \chi, \omega \} \cup \{ C_0' : \gamma_{ab}, \Omega, \zeta, \chi, \omega \} . \] (1.53)

We denote \( \{ S(\nu) \} \) and \( \{ S(\lambda) \} \) the leaves of the foliations of \( C_0 \) and \( C_0' \), defined through the functions \( \Omega \) and \( \Omega' \), with \( \nu \in [\nu_0, \infty), \lambda \in [\lambda_1, \lambda_0 < 0] \) and such that \( S(\lambda_1) = S(\nu_0) = S_0 = C_0 \cap C_0' \). On \( C_0 \), assuming \( \{ e_A \} \) a Fermi transported orthonormal frame,

\[ \chi, \omega, \zeta, \chi_{ab} = \frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \nu} \] (1.54)

satisfy the following “constraint equations”:

\[
\begin{align*}
\frac{\partial \text{tr}_X}{\partial \nu} + \frac{\Omega \text{tr}_X}{2} \text{tr}_X + 2\Omega \text{tr}_Y + \Omega |\chi|^2 &= 0, \\
\frac{\partial \zeta_A}{\partial \nu} + \Omega \chi_{AB} \zeta_B + \Omega \text{tr}_X \zeta_A - \Omega (\text{div}_X A) + \Omega \chi_{AB} \text{tr}_Y + \frac{\partial \chi_{AB} \log \Omega}{\partial \nu} &= 0, \\
\frac{\partial \text{tr}_X}{\partial \nu} + \Omega \text{tr}_X \text{tr}_Y - 2\Omega \text{tr}_X - 2\Omega (\text{div}_Y \eta) - 2\Omega |\eta|^2 + 2\Omega K &= 0, \\
\frac{\partial \chi_{AB}}{\partial \nu} + \frac{\Omega \text{tr}_X \chi_{AB}}{2} + \frac{\Omega \text{tr}_Y \chi_{AB}}{2} + \frac{(\partial \log \Omega)}{\partial \nu} \chi_{AB} + \Omega \chi_{AB} \text{tr}_Y (\zeta - \nabla \log \Omega)_B \\
- \Omega (\zeta_A + \chi_{AB} \log \Omega) \text{tr}_Y (\zeta - \nabla \log \Omega)_B &= 0 \quad (1.55) \\
\frac{\partial \omega}{\partial \nu} - 2\Omega \omega - \Omega \zeta_C \chi_{C} \log \Omega + \frac{1}{4} [-3\Omega |\chi|^2 + \Omega |\nabla \log \Omega|^2] \\
+ \{ K + \frac{1}{4} \frac{\text{tr}_X \chi_{AB}}{2} - \frac{1}{2} \chi \cdot \chi \} &= 0 .
\end{align*}
\]

Moreover on \( C_0' \), assuming \( \{ e_A \} \) a Fermi transported orthonormal frame,

\[ \chi, \omega, \zeta, \chi_{ab} = \frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \lambda} \]

satisfy the constraint equations:

\[
\begin{align*}
\frac{\partial \text{tr}_X}{\partial \lambda} + \frac{\Omega \text{tr}_X}{2} \text{tr}_X + 2\Omega \text{tr}_Y + \Omega |\chi|^2 &= 0, \\
\frac{\partial \zeta_A}{\partial \lambda} + \Omega \chi_{AB} \zeta_B + \Omega \text{tr}_X \zeta_A - \Omega (\text{div}_X A) - \Omega \chi_{AB} \text{tr}_Y - \frac{\partial \chi_{AB} \log \Omega}{\partial \lambda} &= 0, \\
\frac{\partial \text{tr}_X}{\partial \lambda} + \Omega \text{tr}_X \text{tr}_Y - 2\Omega \text{tr}_X - 2\Omega (\text{div}_Y \eta) - 2\Omega |\eta|^2 + 2\Omega K &= 0, \\
\frac{\partial \chi_{AB}}{\partial \lambda} + \frac{\Omega \text{tr}_X \chi_{AB}}{2} + \frac{\Omega \text{tr}_Y \chi_{AB}}{2} + \frac{\partial \log \Omega}{\partial \lambda} \chi_{AB} - \Omega \chi_{AB} \text{tr}_Y (\zeta + \nabla \log \Omega)_B \\
- \Omega (\zeta_A + \chi_{AB} \log \Omega) \text{tr}_Y (\zeta + \nabla \log \Omega)_B &= 0 \quad (1.56) \\
\frac{\partial \omega}{\partial \lambda} - 2\Omega \omega - \Omega \zeta_C \chi_{C} \log \Omega + \frac{1}{4} [-3\Omega |\chi|^2 + \Omega |\nabla \log \Omega|^2] \\
+ \Omega \{ K + \frac{1}{4} \frac{\text{tr}_X \chi_{AB}}{2} - \frac{1}{2} \chi \cdot \chi \} &= 0 .
\end{align*}
\]
Remark: When assigning the initial data to prove a global existence result we have to require on $C$, the regularity of $\chi, \bar{\chi}, \omega, \bar{\omega}, \zeta, \bar{\zeta}$, together with their tangential derivatives, 

$$\nabla^k \chi, \nabla^k \omega, \nabla^k \zeta, \nabla^k \bar{\chi}, \nabla^k \bar{\omega},$$

(1.57)

and of 

$$\nabla^{k-1} D_{\chi}, \nabla^{k-1} D_{\omega}$$

(1.58)

on $C_0$ and $\bar{C}_0$ respectively. $k \in [1, q]$ and $\nabla$ is the covariant derivative associated to the metric $\gamma_{ab}$. The amount of regularity needed for the existence proof will be specified later on.

1.4 The characteristic initial value problem.

Starting with the previous definitions of the “initial data” we state the characteristic initial value problem in the following way:

**Definition 1.2 (The characteristic initial value problem)** To solve the characteristic initial value problem for the vacuum Einstein equations with initial data set

$$\{C_0 : \gamma_{ab}, \zeta_a, \chi_{ab}, \omega \} \cup \{\bar{C}_0 : \bar{\gamma}_{ab}, \bar{\zeta}_a, \bar{\chi}_{ab}, \bar{\omega} \}$$

(1.59)

means to find a four dimensional manifold $M$, a Lorentz metric $g$ satisfying the vacuum Einstein equations as well as an imbedding

$$i : \{C_0 : \gamma_{ab}, \zeta_a, \chi_{ab}, \omega \} \cup \{\bar{C}_0 : \bar{\gamma}_{ab}, \bar{\zeta}_a, \bar{\chi}_{ab}, \bar{\omega} \} \to M$$

(1.60)

such that:

a) $i(C_0) = C(\lambda_1)$, $i(\bar{C}_0) = C(\nu_0)$, $i(S_0(\lambda_1)) = i(S_0(\nu_0)) = C(\lambda_1) \cap C(\nu_0)$, where $C(\lambda_1)$ and $C(\nu_0)$ are two null hypersurfaces embedded in $M$.

b) $M$ is the maximal future development of $C(\lambda_1) \cup C(\nu_0)$.

c) On $C(\lambda_1)$ with respect to the initial data foliation, we have

$$i^*(\gamma) = \gamma, \ i^*(\Omega) = \Omega, \ i^*(\zeta) = \zeta$$
$$i^*(\chi) = \chi, \ i^*(\bar{\chi}) = \bar{\chi}, \ i^*(\omega) = \omega, \ i^*(\bar{\omega}) = \bar{\omega}.$$  

(1.61)

d) On $C(\nu_0)$ with respect to the initial data foliation, we have

$$i^*(\gamma) = \gamma, \ i^*(\Omega) = \Omega, \ i^*(\zeta) = \zeta$$
$$i^*(\chi) = \chi, \ i^*(\bar{\chi}) = \bar{\chi}, \ i^*(\omega) = \omega, \ i^*(\bar{\omega}) = \bar{\omega}.$$  

(1.62)
where $\chi, \chi, \omega, \omega, \zeta$ are the restriction to $C(\lambda_1)$ and $C(\nu_0)$ of the connection coefficients of the spacetime $\mathcal{M}$,

$$
\chi_{ab} = g(De_a e_4, e_b), \quad \chi_{ab} = g(De_a e_3, e_b)
$$

$$
\omega = \frac{1}{4} g(de_3, e_4) = -\frac{1}{2} \frac{D_3}{\Omega} \log \Omega 
$$

$$
\omega = \frac{1}{4} g(De_4, e_4) = -\frac{1}{2} \frac{D_4}{\Omega} \log \Omega 
$$

$$
\zeta_a = \frac{1}{2} g(De_a e_4, e_3). 
$$

with $\{e_4, e_3, e_a\}$ a null orthonormal frame of $\mathcal{M}$. $\Omega$ is the “null” lapse function restricted to $C(\lambda_1) \cup C(\nu_0)$, $\gamma$ is the metric tensor restricted to the leaves $S_0(\nu)$ and $S_0(\lambda)$ of $C(\lambda_1), C(\nu_0)$ respectively.

e) The constraint equations 1.55, 1.56 are the pull back of (some of) the structure equations of $\mathcal{M}$ restricted to $C(\lambda_1)$ and $C(\nu_0)$.

Remarks:

a) The previous definition of the characteristic initial value problem allows to visualize $C(\lambda_1)$ as a portion of an outgoing truncated (before its lower vertex) cone “starting” from $S_0(\nu_0)$ and going up indefinitely and $C(\nu_0)$ as a portion of an incoming cone truncated (before its upper vertex) at the two dimensional surface $S_0(\lambda_0)$ and, going “backward” in time, up to the two dimensional surface $S_0(\lambda_1) = S_0(\nu_0)$.

b) The reason why we consider $C_0$ truncated with $S_0(\lambda_0)$ is a technical one associated to the need of avoiding the problems connected to the “vertices” of the null cones. We specify in the next sections in which sense this characteristic problem can be interpreted as a global existence problem.

1.4.1 The initial data set in terms of the connection coefficients

The initial data set given in Definition 1.1 is equivalent to the more standard definition of initial data set made by the metric components and their partial derivatives. To prove it we show that all the first order partial derivatives of the metric tensor can be expressed in terms of the connection coefficients. To do it we consider $\mathcal{C}$ as an hypersurface immersed in a four dimensional manifold $R^4$ endowed with a Lorentzian metric $\tilde{g}$. As we want to express the initial data as the metric components and their first partial derivatives we need to choose a coordinate set. We introduce, therefore, the coordinates $\{u, u, \omega^a\}$ and require that $\tilde{g}$ has the following expression:

$$
\tilde{g} = \tilde{X}^2 d u^2 - 2 \tilde{X}^2 (d u d u + d \omega^a d \omega^a) - \tilde{X} (d u d \omega^a + d \omega^a d u) + \tilde{g}_{ab} d \omega^a d \omega^b. 
$$

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With this choice we define $C_0$ as (a portion of) a level set of the function $u(p)$ and $C_p$ as (a portion of) a level set of the function $u(p)$, namely:

$$C_0 \equiv C(\lambda_1) = \{ p \in \mathbb{R}^4 | u(p) = \lambda_1, \nu \in [\nu_0, \infty) \}$$

$$C_p \equiv C(\nu_0) = \{ p \in \mathbb{R}^4 | u(p) = \nu_0, \lambda \in [\lambda_1, \lambda_0] \} . \quad (1.65)$$

From the explicit form of the metric tensor $\tilde{g}$ it is immediate to recognize that $C(\lambda_1)$ is a portion of a truncated outgoing null cone in $\mathbb{R}^4$ and, analogously, $C(\nu_0)$, is a portion of a truncated incoming null cone. Moreover the null geodesics generating $C(\lambda_1)$ and $C(\nu_0)$ have tangent vector fields

$$L = \frac{1}{2\Omega^2} \frac{\partial}{\partial u} , \quad L = \frac{1}{2\Omega^2} \left( \frac{\partial}{\partial u} + \tilde{X} \right) \quad (1.66)$$

with $\tilde{X} = \tilde{X}^a \frac{\partial}{\partial x^a}$.

We require that the restriction of the various components of the metric on $\mathcal{C}$, be equal to the quantities defined before in a more abstract way,

$$\gamma |_{C_{ab}} = \gamma_{ab}, \quad \tilde{X} |_{C_{aa}} = X_a, \quad \tilde{\Omega} |_{C_0} = \Omega$$

$$\gamma |_{C_{ab}} = \gamma_{ab}, \quad \tilde{X} |_{C_{aa}} = X_a, \quad \tilde{\Omega} |_{C_0} = \Omega \quad (1.67)$$

From equations 1.66 it follows immediately that the functions $u(p)$ and $u(p)$ restricted to $C_0$ and $C_p$ satisfy, respectively,

$$u(p) = \nu_0 + \int_0^{u(p)} (4\Omega)^{-2}(\gamma(\nu')) d\nu'$$

$$u(p) = \lambda_1 + \int_0^{u(p)} (4\Omega)^{-2}(\gamma(\nu')) d\nu' . \quad (1.68)$$

where, denoted $(u, \theta, \phi)$ the coordinates of the point $p$, $\gamma$ is the null geodesic along $C_0$ starting on $S_0 = C_0 \cap \mathcal{C}_0$, at the point $q$ of angular coordinates $(\theta, \phi)$. $\gamma'$ is the null geodesic along $C_p$ starting on $S_0 = C_0 \cap \mathcal{C}_0$ and reaching the point $p$ when its affine parameter has the value $v = v(p)$. It is important to observe that from the explicit expression of $L$, 1.66, it follows that moving along $\gamma'$ the angular coordinates $\omega^a$ vary differently from what happens along $\gamma$. This is, again, the effect of the definition of the $\omega^a$ coordinates associated to the choice of the metric expression 1.64.24 Moreover due to the fact that, as discussed before, the vector fields $e_3$ and $e_4$ do not commute we have here from the explicit expression of the commutator that on $\mathcal{C}$ the following relation holds:25

$$\zeta_a = -\frac{1}{4\Omega^2} \frac{\partial}{\partial u} \frac{\partial X^b}{\partial u} . \quad (1.69)$$

23This follows immediately observing that the inverse metric $g^{-1}$ has the following components: $g^{\omega a} = (2\Omega^2)^{-1}$, $g^{\omega a} = (2\Omega^2)^{-1}X^a$. $g^{ab} = \gamma_{ab}$, $g^{a} = \tilde{\omega} = g^{a} = 0$. Therefore both the functions $u(p)$ and $u(p)$ satisfy the eikonal equation $g^{\mu
u} \partial_{\mu} \omega_{\nu} = 0$.

24In [Ki-Ni] the different ways of defining the $\omega^a$ coordinates are carefully discussed, see equations (3.1.58), (3.1.59), the second choice would correspond to a metric expression of the following kind: $\tilde{g} = \tilde{Y}^2 d\tilde{u} - 2\tilde{u}^2 (d\tilde{u} + d\tilde{v}) - Y_\theta (d\omega^a + \omega^a d\theta) + \tilde{z}_{ab} d\omega^a d\omega^b$.

25Of course it holds in the whole $\{ R^4, \tilde{g} \}$. A detailed discussion is in [Ki-Ni], Chapter 3.
The null frames \( \{e_4,e_A\} \) defined on \( C_0 \) and \( \{e_3,e_A\} \) on \( C_0^* \) can now be extended adding a vector field \( e_3 \) on \( C_0 \) and a vector field \( e_4 \) on \( C_0^* \) such that \( \mathbf{g}(e_4,e_3) = -2, \mathbf{g}(e_4,e_A) = 0, \mathbf{g}(e_3,e_A) = 0 \). Therefore we have defined on \( C \) a null frame \( \{e_4,e_3,e_A\} \).

Finally we require that the tensor fields defined on \( C, X, \zeta \) and \( \omega \) for \( C_0, X, \zeta \) and \( \omega \) for \( C_0^* \) are the corresponding connection coefficients of \((R^4, \mathbf{g})\) when restricted to \( C \).

We can now express all the connection coefficients in terms of the metric components and its Christoffel symbols and look for the reciprocal relations. One easily proves the following relations:

\[
\begin{align*}
\chi_{AB} &= e^a_A e^b_B \left[ \Omega^{-1} \left( -\Gamma^u_{au} X_b + \Gamma^u_{bu} \gamma_{cb} \right) \right] \\
\dot{\chi}_{AB} &= e^a_A e^b_B \left[ \Omega^{-1} \left( -\Gamma^u_{au} X_b + \Gamma^u_{bu} \gamma_{cb} + \partial_a X^c \gamma_{cb} + X^d \Gamma^u_{ad} \gamma_{cb} \right) \right] \\
\eta_A &= e^a_A \left( \Gamma^u_{au} + X^b \Gamma^u_{ba} \right), \quad \dot{\eta}_A = e^a_A \Gamma^u_{au} \\
\zeta_a &= e^a_A \left( \partial_a \log \Omega - \Gamma^u_{au} - (2\Omega^2)^{-1} |X|^2 \Gamma^u_{uu} \right) \\
\omega &= -\frac{1}{2\Omega} \partial_a \log \Omega + \frac{1}{2\Omega} \Omega^{-1} \left( \Gamma^u_{au} + X^a \Gamma^u_{ua} \right) \\
\omega &= -\frac{1}{2\Omega} \partial_a \log \Omega - \frac{1}{2\Omega^2} X^a \partial_a \log \Omega - \frac{1}{4\Omega^2} \left( 2\Omega^2 \Gamma^u_{uu} + X_a \Gamma^u_{au} + X^a \Gamma^u_{au} - |X|^2 \Gamma^u_{uu} \right) .
\end{align*}
\] (1.70)

In the chosen metric the Christoffel symbols have the following expressions:

\[
\begin{align*}
\Gamma^u_{uu} &= 2\partial_u \log \Omega + (4\Omega^2)^{-1} \partial_u |X|^2 \\
\Gamma^u_{ua} &= \partial_u \log \Omega - (4\Omega^2)^{-1} \partial_u X_a \\
\Gamma^u_{ab} &= (4\Omega^2)^{-1} \partial_u \gamma_{ab} \\
\Gamma^u_{uu} &= 2\partial_u \log \Omega \\
\Gamma^u_{ab} &= (4\Omega^2)^{-1} X^a X^b \partial_u \gamma_{ab} + (4\Omega^2)^{-1} X^a \partial_u |X|^2 \\
\Gamma^u_{uu} &= -X^a \partial_u \log \Omega - (4\Omega^2)^{-1} X_a \partial_u X^a \\
\Gamma^u_{ab} &= -(4\Omega^2)^{-1} \partial_a |X|^2 - (4\Omega^2)^{-1} X^b \partial_u \gamma_{ab} + (4\Omega^2)^{-1} X^b (\partial_u X_b - \partial_b X_u) \\
\Gamma^u_{uu} &= \partial_u \log \Omega + (4\Omega^2)^{-1} \gamma_{ab} \partial_u X^b \\
\Gamma^u_{ab} &= (4\Omega^2)^{-1} (\partial_u X_b + \partial_b X_a + \partial_a \gamma_{ab}) - (4\Omega^2)^{-1} X^c (\partial_u \gamma_{ca} + \partial_a \gamma_{cb} - \partial_c \gamma_{ab}) \\
\Gamma^u_{uu} &= -\gamma_{ab} \partial_u X_b - \frac{1}{2} \gamma_{ab} \partial_u |X|^2 + 2X^a \partial_u \log \Omega + (4\Omega^2)^{-1} X^a \partial_u |X|^2 \\
\Gamma^u_{uu} &= -\frac{1}{2} \gamma_{ab} \partial_u X_b + 2\Omega^2 \gamma_{ab} \partial_b \log \Omega \\
\Gamma^u_{ab} &= \frac{1}{2} \gamma_{ac} \partial_u \gamma_{cb} + \frac{1}{2} \gamma_{ac} (\partial_c X_b - \partial_b X_c) + X^a \partial_b \log \Omega - (4\Omega^2)^{-1} X^a \partial_u X_b \\
\Gamma^u_{ab} &= \frac{1}{2} \gamma_{ac} \partial_u \gamma_{cb} , \quad \Gamma^u_{bc} = (2) \Gamma^u_{bc} + (4\Omega^2)^{-1} X^a \partial_u \gamma_{bc} \\
\Gamma^u_{uu} &= \Gamma^u_{uu} = \Gamma^u_{au} = \Gamma^u_{ua} = 0
\end{align*}
\]
where \((2)^{\Gamma}_{bc}\) are the Christoffel symbols associated to the metric \(\gamma_{ab}\).

Using these relations the connection coefficients in terms of the metric components and their derivatives are:

\[
\eta_a = \partial_a \log \Omega - (4\Omega^2)^{-1} \gamma_{ab} \partial_a X^b \\
\eta_a = \partial_a \log \Omega + (4\Omega^2)^{-1} \gamma_{ab} \partial_a X^b \\
\zeta_a = -(4\Omega^2)^{-1} \gamma_{ab} \partial_a X^b \\
\chi_{ab} = (2\Omega)^{-1} \partial_a \gamma_{ab} \\
\chi_{ab} = (2\Omega)^{-1} \partial_a \gamma_{ab} + (2\Omega)^{-1} \nabla_b X_a + (2\Omega) X_b \\
\omega = -(2\Omega)^{-1} \partial_a \log \Omega \\
\omega = -(2\Omega)^{-1} (\partial_a \log \Omega + X^a \partial_a \log \Omega)
\]

where \((2)^{\nabla}_b\) is the covariant derivative associated to the \(\gamma\) metric tensor of the leaves \(S(\lambda_1, \nu)\) and \(S(\lambda, \nu_0)\). Through relations 1.71 and 1.72 it is possible to write all the Christoffel symbols in terms of the connection coefficients, the metric components and their derivatives. The Christoffel symbols, different from zero, have the following expressions:

\[
\begin{align*}
\Gamma^u_{uu} &= -4\omega_a + (2\Omega)^{-1} \gamma_{ab} X^a X^b - 2X^a (\partial_a \log \Omega + \zeta_a) \\
\Gamma^u_{ua} &= -(2\Omega)^{-1} \gamma_{ab} X^b + (\partial_a \log \Omega + \zeta_a) \\
\Gamma^u_{ab} &= (2\Omega)^{-1} \gamma_{ab} \\
\Gamma^b_{aa} &= -4\omega_a \\
\Gamma^b_{ab} &= (2\Omega)^{-1} \gamma_{ab} X^b \\
\Gamma^b_{ab} &= -(\partial_a \log \Omega - \zeta_a) \\
\Gamma^a_{bb} &= (2\Omega)^{-1} \gamma_{ab} X^b \\
\Gamma^a_{bb} &= -(\partial_a \log \Omega - \zeta_a) \\
\Gamma^b_{bb} &= (2\Omega)^{-1} \gamma_{ab} \\
\Gamma^a_{uv} &= -\Omega \gamma^{ab} \chi_{bc} X^c + \gamma^{ab} (\partial_b \log \Omega + \zeta_b) \\
\Gamma^a_{ub} &= \Omega \gamma^{ac} \chi_{cb} - (2\omega^a_b X^c - (2\Omega)^{-1} \chi_{bc} X^c X^a + X^a (\partial_b \log \Omega + \zeta_b) \\
\Gamma^b_{ab} &= \Omega \gamma^{ac} \chi_{cb} \\
\Gamma^a_{bc} &= (2)^{\Gamma}_{bc} + (2\Omega)^{-1} \chi_{bc} X^a \\
\Gamma^a_{uu} &= -\gamma^{ab} \partial_u X_b - \frac{1}{2} \gamma^{ab} \partial_b |X|^2 - X^a \left[4\omega_a - (2\Omega)^{-1} \chi_{be} X^b X^c + 2X^b (\partial_b \log \Omega + \zeta_b) \right]
\end{align*}
\]

All the Christoffel symbols, with the exception of the last one, \(\Gamma^a_{uu}\), can be written in terms of the connection coefficients the metric components and their tangential derivatives. \(\Gamma^a_{uu}\) is somewhat different for the presence of the term \(\partial_u X_b\). Nevertheless, we will show later on, see subsection 2.1.2, that also this

\footnote{This is connected to the fact that the vector field \(X\) is tied to the choice of the coordinates.}
term can be obtained once we prescribe on \( \mathcal{C} \) the metric components and the connection coefficients.

All this justifies the statement that the initial data are specified once we give on \( \mathcal{C} \) the connection coefficients together with the metric components \( \Omega \) and \( \gamma_{ab} \).

## 2 The construction of the initial data

Once we have defined the initial data set of the characteristic Cauchy problem and the constraint equations they have to satisfy, we have to prove that these initial data do exist. This problem is somewhat analogous to the problem of solving the constraint equations

\[
\nabla^j k_{ij} - \nabla_i \text{tr} k = 0
\]

\[
R - |k|^2 + (\text{tr} k)^2 = 0
\]

(2.1)

of the non characteristic problem. This section is devoted to the explicit construction of the initial data. Moreover, as done for the problem 2.1, see [Ch-Kl], [Kl-Ni], while trying to solve the characteristic constraint problem we will also prescribe the initial data decay along the null hypersurfaces \( C_0 \) and \( \mathcal{C}_0 \) when their coordinate \( \nu \) or \( |\lambda| \) tend to \( \infty \). The decay rate of some initial data quantities is basically an immediate consequence of their geometric meaning (and some smallness assumptions). This is the case for \( \text{tr} \chi \) and \( \text{tr} \bar{\chi} \). For the other quantities, \( \chi, \bar{\chi}, \omega, \bar{\omega}, \zeta \), we require a decay such that some “energy type” integral norms defined on \( C_0 \) and \( \mathcal{C}_0 \) are bounded. In fact to apply the techniques used in [Kl-Ni] to prove our global existence result, we need that some “flux integrals”, \( Q \), are bounded. In the non characteristic case this implies that these integral norms, \( Q \), see their definition in [Kl-Ni], Chapter 3, are bounded in terms of the corresponding (finite) norms on the initial hypersurface \( \Sigma_0 \).

In the characteristic case the \( Q \) flux norms have to be bounded in terms of the same flux norms defined on the initial hypersurface \( \mathcal{C} \). The boundedness of these norms requires that the various null Riemann components involved decay sufficiently fast along the initial cones, \( C_0 \) and \( \mathcal{C}_0 \). In particular if the \( Q \) norms we use are those defined in [Kl-Ni] it follows immediately that on \( C_0 \) and \( \mathcal{C}_0 \) we need the following decay, with \( \delta > 0 \):

\[
\alpha = O(r^{-\left(\frac{5}{2} + \delta\right)}) , \quad \beta = O(r^{-\left(\frac{7}{2} + \delta\right)}) , \quad (\rho - \bar{\rho}, \sigma) = O(r^{-3}|\lambda|^{-(\frac{3}{2} + \delta)})
\]

\[
\overline{\alpha} = O(r^{-2}|\lambda|^{-(\frac{3}{2} + \delta)}) , \quad \overline{\beta} = O(r^{-1}|\lambda|^{-(\frac{1}{2} + \delta)})
\]

(2.2)

In the next subsections we prove the existence of characteristic initial data such that the decays 2.2 are satisfied.

\[27\] From their decay also the decay of the metric components follows.
2.1 The solution of the characteristic constraint problem for the initial data on the outgoing cone, with a prescribed decay rate

We start, as described in the previous subsection, considering $C$ as an hypersurface immersed in the four dimensional manifold $\mathbb{R}^4$ endowed with the Lorentzian metric 1.64,

$$\tilde{g} = |\tilde{X}|^2 du^2 - 2\tilde{\Omega}^2 (d\tilde{u}d\tilde{u} + d\tilde{u}d\tilde{u}) - \tilde{X}_a (d\tilde{u}d\omega^a + d\omega^a du) + \tilde{\gamma}_{ab} d\omega^a d\omega^b .$$

With this choice $C_0$ is (a portion of) a level set of the function $u(p)$, see 1.65.

We choose the function $\tilde{\Omega}$ of the metric $\tilde{g}$ in such a way that its restriction on $C_0$ is an assigned function $\Omega$,

$$\tilde{\Omega}|_{C_0} = \Omega .$$ (2.3)

Once $\Omega$ is given we define the “$\Omega$-foliation” of $C_0 = C(\lambda_1)$ as done before. The leaves of the foliation are the surfaces $S_0(\nu) = \{ p \in C(\lambda_1) | u(p) = \nu \}$ we denote also $S(\lambda_1, \nu_0)$ and $u(p)$ has been defined in 1.35. On each leave $S_0(\nu) = S(\lambda_1, \nu)$ we can define two orthonormal vector fields $\tilde{e}_A$ tangent to it and Fermi transported along the cone,

$$\tilde{e}_A = \tilde{e}_A^a \partial / \partial \omega^a .$$

It is important to remark that the metric $\tilde{\gamma}$ introduced in 1.64 is not the metric which we want to prescribe as initial data; therefore it does not satisfy the “characteristic” constraints. We use it as a background metric from which to construct the final one satisfying the constraints together with its derivatives.

Given $\Omega$ we define $\omega$ as

$$\omega = -\frac{1}{2\Omega} \frac{\partial}{\partial \nu} \log \Omega ,$$ (2.4)

then we assign on $C_0$ a symmetric traceless tensor $\tilde{\chi}$ tangent to each $S_0(\nu)$, $\nu \in [\nu_0, \infty)$, which we will consider as the traceless part of a tensor $\chi$.

The function $\text{tr} \chi$ and the “initial data metric” $\gamma_{ab}$ are determined requiring that they satisfy the following system of differential equations:

$$\frac{\partial \gamma_{ab}}{\partial \nu} - \Omega \text{tr} \chi \gamma_{ab} - 2\Omega \tilde{\chi}_{ab} = 0$$

$$\frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega \text{tr} \chi}{2} \text{tr} \chi + 2\Omega \omega \text{tr} \chi + |\tilde{\chi}|^2 = 0$$ (2.5)

The solution of system 2.5 is the main part of the following lemma. The estimates appearing there are pointwise estimates or $| \cdot |_{p, S}$ norms, where

$$|f|_{p, S} = \left( \int_S |f|^p \mu_\gamma \right)^{1/p} .$$
Lemma 2.1 Assign a symmetric tensor $\hat{\chi}$, tangent to the leaves $S_0(\nu)$ satisfying, together with $\text{tr}\hat{\chi}$, the trace of the null outgoing second fundamental form associated to $\hat{\gamma}$, the following equations

\[
\frac{\partial \hat{\gamma}_{ab}}{\partial \nu} - \Omega \text{tr}\hat{\chi} \hat{\gamma}_{ab} = 0 \\
\frac{\partial \text{tr}\hat{\chi}}{\partial \nu} + \frac{\Omega \text{tr}\hat{\chi}}{2} + 2\Omega \omega \text{tr}\hat{\chi} = 0.
\]

Assume that, with $\varepsilon > 0$,

$$
\nu_0 \in [c_1\hat{r}_0, c_2\hat{r}_0] \quad \text{with} \quad |c_{1,2} - 1| = O(\varepsilon).
$$

Given a symmetric tensor $\hat{\chi}$ on $C_0$, assume that, with $\varepsilon > 0$, the following decays hold,

\[
|\Omega - \frac{1}{2}| = O(\varepsilon r^{-1}) \quad , \quad |\nabla \log \Omega| = O(\varepsilon r^{-2+\delta}) \quad , \quad |\nu| = O(\varepsilon r^{-2}) \quad , \quad |\hat{\chi}|_\gamma = O(\varepsilon r^{-\left(\frac{3}{2}+\delta\right)}) \quad , \quad |\mathcal{D}_4 \hat{\chi}|_\gamma = O(\varepsilon r^{-\left(\frac{3}{2}+\delta\right)}) \quad , \quad |\mathcal{D}_4 \nabla \log \Omega|_\gamma = O(\varepsilon r^{-2+\delta}),
\]

with $\delta > 0$.\(^{32}\) $\hat{r}$ defined from the relation $4\pi \hat{r}^2(\nu) = |S_0(\nu)|_\gamma$ and $\nabla$ the covariant derivative associated to the metric $\hat{\gamma}$. Assume that on $S_0(\nu_0)$,

$$
|\hat{r}^3 - \hat{r} \nabla \text{tr}\chi|_{p,S} = O(\varepsilon),
$$

then there exists on $C_0$ a metric tensor $\gamma$ tangent to the leaves of the same foliation such that, denoted $\chi$ the second fundamental form of the $S_0(\nu)$ with respect to this metric, we have:

\[
\frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \nu} = \chi_{ab} = \hat{\chi}_{ab} + \frac{\gamma_{ab}}{2} \text{tr}\chi,
\]

with $\gamma|_{S_0} = \hat{\gamma}|_{S_0}$ and $\text{tr}\chi > 0$ satisfies the constraint equation

\[
\frac{\partial \text{tr}\chi}{\partial \nu} + \Omega \text{tr}\chi + 2\Omega \omega \text{tr}\chi + |\hat{\chi}|^2 = 0,
\]

with $\text{tr}\chi|_{S_0} = \text{tr}\hat{\chi}|_{S_0}$. Moreover the following estimates hold, with $p \in [2,4]$,

\[
|\Omega - \frac{1}{2}| = O(\varepsilon r^{-1}) \quad , \quad |\nabla \log \Omega| = O(\varepsilon r^{-2+\delta}) \quad , \quad |\nu| = O(\varepsilon r^{-2})
\]

\[
|\text{tr}\chi - \frac{2}{p}| = O(\varepsilon r^{-2+\delta}) \quad , \quad |\hat{\chi}|_\gamma = O(\varepsilon r^{-\left(\frac{3}{2}+\delta\right)})
\]

\[
|r^{(\frac{3}{2}+\delta)} \mathcal{D}_4 \hat{\chi}|_{p,S} = O(\varepsilon) \quad , \quad r^{(\frac{3}{2}+\delta)} \nabla \hat{\chi}|_{p,S} = O(\varepsilon)
\]

\[
|r^{(\frac{3}{2}+\delta)} \mathcal{D}_4 \nabla \log \Omega|_{p,S} = O(\varepsilon) \quad , \quad r^{3} \hat{r} \nabla \text{tr}\chi|_{p,S} = O(\varepsilon),
\]

\(^{32}\)The reason of this requirement will be clear when we discuss the constraints to impose to the one form $\zeta$. 

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with \( r \) defined by the relation
\[
4\pi r^2(\nu) = |S_0(\nu)|_\gamma .
\] (2.13)

**Proof:** The proof is in the appendix.  

Once we have solved equations 2.5 we have assigned on the whole \( C_0, \gamma, \Omega, \omega \) and \( \chi \). \( \gamma \) is the (initial data) metric relative to the leaves \( S_0(\nu) \), \( \chi \) is the second (null outgoing) fundamental form of the leaves \( S_0(\nu) \) relative to the same metric. In the explicit expression of \( \bar{g}, \bar{\Omega} \) is the extension of the quantity previously defined on \( C_0 \)
\[
\Omega = \bar{\Omega}|_{C_0}
\] (2.14)
and \( \bar{\gamma} \) is a new extension to \( R^4 \) of the metric \( \gamma \) obtained in Lemma 2.1,  
\[
\bar{\gamma}_{ab}|_{C_0} = \gamma_{ab} .
\] (2.15)

The restriction to \( C_0 \) of the metric components \( |\bar{X}|^2 \) and \( \bar{X}_a \) is obtained later on, after we introduce the one form \( \zeta \).  

To obtain \( \zeta \) along \( C_0 \) we have to use its evolution equation 1.41 and its value on \( S_0(\nu_0) \).  

We want \( \zeta \) defined on the whole \( C_0 \) decaying as \( O(r^{-2}) \). This is the result of the following lemma.

**Lemma 2.2** Assume that, on \( C_0 \),  
\[
|r^{(3+\delta)-\frac{2}{\delta}}D_1 \nabla \log \Omega|_{p,S} = O(\varepsilon) ,
\]
\[
|r^{-\frac{2}{\delta}}\nabla \text{tr} \chi|_{p,S} = O(\varepsilon)
\] (2.16)
assume that on \( S_0(\nu_0) = \mathcal{S}_0(\lambda_1) \),
\[
|r^{-2-\frac{2}{\delta}}\zeta|_{p,S}(\lambda_1,\nu_0) \leq c\varepsilon .
\] (2.17)

Let \( \zeta \) be the solution along \( C_0 \) of the constraint equation 1.41,
\[
D_4 \zeta + \zeta \chi + \text{tr} \chi \zeta = \text{div} \chi - \nabla \text{tr} \chi - D_4 \nabla \log \Omega ,
\]
then, on \( C_0 \),
\[
|r^{-2-\frac{2}{\delta}}\zeta|_{p,S}(\lambda_1,\nu) \leq c\varepsilon .
\] (2.18)

---

33Assumptions 2.8 are needed to prove Lemma 2.1, other more stringent conditions on the regularity of these quantities will be needed to prove the existence of the initial data.
34Remark that the choice of the null hypersurfaces \( C_0 \) and \( C_0' \) does not depend on the choice of the metric \( \gamma_{ab} \).
35Observe that the leaves of the \( \Omega \)-foliation are not changed. This follows from the fact that the scalar function \( \Omega \) on \( C_0 \) is not changed. Moreover the null geodesics on \( C_0 \) with respect the metric \( \gamma \) starting from the point \((\nu_0, \theta, \phi)\) of \( S_0 = S_0(\nu_0) \) are the same as those relative to the metric \( \bar{\gamma} \).
36The way to obtain the one form \( \zeta \) on \( S_0 \) is discussed when we obtain the initial data on the incoming cone, see later on.
37Assumptions 2.16 is a condition we impose on \( \Omega \) on \( C_0 \).
Proof: The evolution equation for $\zeta$ can be written as
\[
\mathcal{D}_4 \zeta + \frac{3}{2} \text{tr} \chi \zeta = -\chi \zeta + F(\nabla \chi, \nabla \log \Omega) \tag{2.19}
\]

where \( F(\nabla \chi, \nabla \log \Omega) = [\text{div} \chi - \frac{1}{2} \chi \text{tr} \chi - \mathcal{D}_4 \nabla \log \Omega] \).

Using the assumptions, whose validity follows from the previous lemma, we control the right hand side \( F(\nabla \chi, \nabla \log \Omega) \), choosing \( \{e,A\} \) Fermi transported and using Gronwall’s Lemma, see Chapter 4 of [Kl-Ni], we obtain
\[
|\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu) \leq c \left( |\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu_0) + \int_{\nu_0}^{\nu} |\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu') d\nu' \right) \tag{2.20}
\]

Dividing the left hand side by \( r(\lambda_1, \nu) \) and observing that on \( C_0 \),
\[
r^{-1}(\lambda_1, \nu) \leq r^{-1}(\lambda_1, \nu') \leq r^{-1}(\lambda_1, \nu_0) \tag{2.21}
\]
we rewrite 2.20 as
\[
|\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu) \leq c \left( |\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu_0) + \frac{1}{r(\lambda_1, \nu)} \int_{\nu_0}^{\nu} |\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu') d\nu' \right) . \tag{2.22}
\]

The integrand on the right hand side behaves, due to the assumptions on the terms \( \mathcal{D}_4 \nabla \log \Omega \) and \( \nabla \text{tr} \chi \), as \( O(\varepsilon) \). This implies that the integral can be bounded by \( c(\lambda_1) r(\lambda_1, \nu) \varepsilon \) and the \( r(\lambda_1, \nu) \) factor is exactly compensated by the denominator. This allows us to conclude that
\[
|\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu) \leq c \left( |\mathcal{D}_4 \zeta|_{p,S}(\lambda_1, \nu_0) + \varepsilon \right) \tag{2.23}
\]
and using assumption 2.17 the result follows.

Finally we have to specify \( \chi \) and \( \omega \) on \( C_0 \). Again we decompose \( \chi \) in its trace part and its traceless part, \( \text{tr} \chi \) and \( \hat{\chi} \); also these quantities cannot be assigned freely, but have to satisfy some constraint equations. Moreover we have to control their decay along \( C_0 \). Let us start considering \( \text{tr} \chi \). As discussed in subsection 1.2, it has to satisfy the constraint equation
\[
\mathcal{D}_4 \text{tr} \chi + \text{tr} \chi \text{tr} \chi - 2 \omega \text{tr} \chi - 2(\text{div} \eta) - 2|\eta|^2 + 2\mathcal{K} = 0 , \tag{2.24}
\]
where \( \eta \) has to be read everywhere as \( (-\zeta + \nabla \log \Omega) \) and \( \text{tr} \chi \) can be assigned freely on \( S_0 \).

To obtain \( \hat{\chi} \) and its decay we anticipate that \( \hat{\chi} \) will be given freely on \( C_0 \) and it has to satisfy along \( C_0 \) the constraint equation,
\[
\mathcal{D}_4 \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} - 2\omega \hat{\chi} - \nabla \hat{\eta} - \eta \hat{\eta} = 0 . \tag{2.25}
\]
We use evolution equations 2.24 and 2.25 to obtain, on \( C_0 \), \( \text{tr} \chi \) and \( \hat{\chi} \) and to specify their decay in \( r \). We collect these results in the following lemma:
Lemma 2.3 Assuming the results of Lemma 2.1 and Lemma 2.2 there exists a symmetric tensor $\chi$ whose trace part and traceless part satisfy the evolution equations 2.24, 2.25. Moreover they satisfy the following estimates:

$$|r^{1-\frac{2}{p}}\text{tr}_\chi|_{p,S} \leq c, \quad |r^{1-\frac{2}{p}}\chi|_{p,S} \leq ce^\varepsilon.$$ \hspace{1cm} (2.26)

Remark: Again these estimates can be improved to pointwise ones, assuming enough regularity for the initial data. \(^3^8\)

Proof: First we determine $\text{tr}_\chi$ solving equation 2.24. This is easy as all the coefficients and the inhomogeneous terms appearing in the equation have been already obtained together with their decay. To prove that $\text{tr}_\chi$ decays as $O(r^{-1})$ we proceed as in the case of the one form $\zeta$ of Lemma 2.1 as we have to control the second term in the right hand side can be bounded by a constant so that, finally,

$$(\text{tr}_\chi)^2 \leq c.$$ \hspace{1cm} (2.27)

where

$$H(\eta, \nabla \eta, K) \equiv -2K + 2(\nabla \eta) + 2|\eta|^2.$$ Proceeding as done before for $\zeta$, we obtain

$$|r^{2-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu) \leq e \left( |r^{2-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu_0) + \int_{r_0}^{\nu} |r^{2-\frac{2}{p}}H(\eta, \nabla \eta, K)|_{p,S}(\lambda_1, \nu') d\nu' \right).$$

Dividing the left hand side by $r(\lambda_1, \nu)$ and recalling 2.21 we obtain

$$|r^{1-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu) \leq c \left( |r^{1-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu_0) + \frac{1}{r(\lambda_1, \nu)} \int_{r_0}^{\nu} |r^{2-\frac{2}{p}}H(\eta, \nabla \eta, K)|_{p,S}(\lambda_1, \nu') d\nu' \right).$$

As, due to the presence of the Gauss curvature $K$, $H(\eta, \nabla \eta, K) = O(r^{-2})$, the second term in the right hand side can be bounded by a constant so that, finally,

$$|r^{1-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu) \leq c \left( |r^{1-\frac{2}{p}}\text{tr}_\chi|_{p,S}(\lambda_1, \nu_0) + 1 \right).$$ \hspace{1cm} (2.30)

Once we have obtained $\text{tr}_\chi$, $\hat{\chi}$ is obtained immediately applying Gronwall’s Lemma to equation 2.25. \(^3^9\)

Remark: More precisely, to prove Lemma 2.3 we need an extended version of Lemma 2.1 as we have to control $\nabla \zeta$. This is achieved writing the evolution equation for $\nabla \zeta$:

$$D_a \nabla_a \zeta_b = -2 \text{tr}_\chi \nabla_a \zeta_b - (\hat{\chi}_{ac} \nabla_c \zeta_b + \hat{\zeta}_{bc} \nabla_a \zeta_c) - \eta_b \chi_{ac} \zeta_c + \chi_{ab} (\eta \cdot \zeta) - \frac{3}{2} \zeta_b \nabla_a \text{tr}_\chi - \zeta_b \nabla_a \hat{\chi} + \nabla_a (\nabla \chi) b - \frac{1}{2} \nabla_a \nabla_b \text{tr}_\chi - \nabla_a D_4 \nabla_b \log \Omega.$$  

\(^3^8\)The decay estimate for $\hat{\chi}$ on $C_0$ will be stronger in $|\lambda|$.

\(^3^9\)Observe that due to the fact that we are integrating forward in time along $C_0$ we cannot improve the decay for $\hat{\chi}$. 

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From it the control of \( \nabla \zeta \) proceeds in the same way as the one for \( \zeta \), but it requires the control of \( \nabla^2 \chi \) and of \( \nabla^2 \mathcal{D}_4 \log \Omega \). The terms \( \nabla_a \nabla_b \text{tr} \chi \), \( \nabla_a \nabla_b \hat{\chi} \) and \( \nabla^2 \mathcal{D}_4 \log \Omega \) have to satisfy the estimates

\[
|r|^{(\frac{3}{2})} \frac{1}{2} \nabla_a \nabla_b \nabla \text{tr} \chi |_{p,S} \leq \varepsilon \epsilon, \quad |r|^{(\frac{3}{2})} \frac{1}{2} \nabla^2 \chi |_{p,S} \leq \varepsilon \epsilon
\]

(2.31)

Therefore estimates 2.31 have to be included in the initial assumptions of Lemma 2.3. In other words they must be part of the results of an easily extended version of Lemma 2.1 and Lemma 2.2.

We are left with determining the function \( \omega \). In the Einstein vacuum spacetime it will be related to the function \( \Omega \) by the relation

\[
\omega = -\frac{1}{2\Omega} \frac{\partial \log \Omega}{\partial \lambda}.
\]

(2.32)

Again the Einstein equations imply a constraint equation for it on \( C_0 \). In the introduction we have shown that \( R_{44} = 0 \) implies for \( \omega \) and \( \bar{\omega} \) the following equation:

\[
\mathcal{D}_4 \omega + \mathcal{D}_4 \bar{\omega} - 4|\bar{\omega}|^2 = \rho(\chi, \eta, \eta)
\]

where, see 1.28,

\[
\rho(\chi, \eta, \eta) = -\left[ K + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \chi \cdot \hat{\chi} \right].
\]

(2.34)

This equation can be written in a slightly different way in the Einstein vacuum spacetime, \( (\mathcal{M}, g) \). In fact, there, the following relation holds:

\[
\mathcal{D}_4 \omega = \mathcal{D}_4 \bar{\omega} + (\bar{\eta} - \eta) \cdot \nabla \log \Omega
\]

(2.35)

and 2.33 can be rewritten as

\[
\mathcal{D}_4 \bar{\omega} - 2\omega \bar{\omega} = \left[ \chi \cdot \nabla \log \Omega + \frac{3}{2} |\xi|^2 - \frac{1}{2} |\nabla \log \Omega|^2 + \frac{1}{2} \rho(\chi, \eta, \eta) \right]
\]

(2.36)

This evolution equation is the constraint equation that the scalar quantity \( \omega \) has to satisfy on \( C_0 \). All the terms in the right hand side of 2.36 have already been obtained, therefore from this equation we determine \( \omega \) on \( C_0 \), once we specify it on \( S_0 \).

**Remark:** We can choose the initial data for \( \omega \) on \( S_0 \) in such a way that

\[
\omega(\lambda_1, \nu_0) = -\int_{\lambda_1}^{\infty} \left[ \chi \cdot \nabla \log \Omega + \frac{3}{2} |\xi|^2 - \frac{1}{2} |\nabla \log \Omega|^2 + \frac{1}{2} \rho(\chi, \eta, \eta) \right].
\]

(2.37)

40Equation 2.35 holds in the Einstein spacetime, but cannot be derived intrinsically on \( C_0 \). On the other side on \( C_0 \) we can choose as constraint equation either 2.33 or 2.36.

41Specifying the value of \( \omega \) on \( S_0 \) amounts to assigne \(-\frac{1}{2} \frac{\partial \log \Omega}{\partial \lambda} \) on \( S_0 \). Recall that on \( \mathcal{C}_0 \) we have: \( u(v, \theta, \phi) = \int_{0}^{v} \frac{1}{\nu'}(\bar{v}', \theta, \phi) d\nu' \). From the change of variable \( u = u(v, \theta, \phi) \) it follows that \( \frac{du}{dv} = (\frac{dv}{ds})^{-1} \) which implies \( \frac{dv}{ds} = \frac{dv}{du} \frac{du}{ds} = \Omega^2 \frac{dv}{du} \). Therefore imposing \( \omega |_{S_0} = k_0 \) is equivalent to require that \( \Omega \) satisfies on \( S_0 \) \(-\frac{1}{2} \frac{\partial \log \Omega}{\partial \lambda} = k_0 \).
The choice \(2.37\) for \(\omega(\lambda_1, \nu_0)\) implies that \(\omega\) decays as \(O(r^{-2})\) along \(C_0\), a condition which will be used in the proof of the global existence result.\(^{42}\)

### 2.1.1 The determination of the vector field \(X\)

The subject of this subsection is somewhat separated from the previous ones. In fact the choice of \(X\) is connected to the choice of the \(\omega\) coordinates. From the knowledge of \(\zeta\) on \(C_0\), using the relation \(1.69\)

\[
\zeta_a = -\frac{1}{4\Omega^2} \gamma^{ab} \frac{\partial X^b}{\partial \nu},
\]

and requiring that \(X^a\) takes a well defined value on \(S_0\), we can obtain \(X^a\) on the whole \(C_0\) such that

\[
|X| = O(r^{-1}).
\]  

(2.38)

To prove this last estimate we have to assume that \(X^b\) goes to zero, as \(r \to \infty\), on \(C_0\). In this case from \(1.69\) we have:

\[
X^a|_{S_0} = X^a(\lambda_1; \nu_0, \theta, \phi) = \int_{\nu_0}^{\infty} 4\Omega^2 \gamma^{ac} \zeta_c(\lambda_1; \nu', \theta, \phi) d\nu' \quad (2.39)
\]

and from it

\[
X^a(\lambda_1; \nu, \theta, \phi) = \int_{\nu}^{\infty} 4\Omega^2 \gamma^{ac} \zeta_c(\lambda_1; \nu', \theta, \phi) d\nu'. \quad (2.40)
\]

As the integrand decays as \(O(1/r)\) it follows that \(X^b = O(r^{-2})\) and \(|X| = \sqrt{\gamma_{ab}X^aX^b} = O(r^{-1})\). Observe that the right hand side of \(2.39\) does not depend on \(X\), therefore, once we have obtained \(\zeta\), choosing \(X^a\) on \(S_0\) as in \(2.39\) implies the result.

#### 2.1.2 The control of \(\frac{\partial X^a}{\partial u}\) on \(C_0\).

In subsubsection 1.4.1 it was said that, knowing the connection coefficients on \(C_0 \cup \overline{C_0}\) and the orthonormal moving frame, we can obtain, with respect to the \(\{u, \overline{u}, \omega^a\}\) coordinates, the Christoffel symbols and the first derivatives of the metric tensor 1.64 restricted to the initial hypersurface \(\mathcal{C} = C_0 \cup \overline{C_0}\). Nevertheless the issue was not complete as one Christoffel symbol, \(\Gamma^a_{uu}\), could not be expressed directly as a combination of the connection coefficients and the metric components. This is due to the presence, in its explicit expression, of the term \(\frac{\partial X^a}{\partial u}\). To prove our statement we have, therefore, also to express this quantity in terms of the connection coefficients and the moving frame. This will be obtained observing that \(\frac{\partial X^a}{\partial u}\) cannot be assigned freely along \(C_0\), but has to satisfy a transport equation.

\(^{42}\)The same procedure is not needed for obtaining \(\omega\) along \(\overline{C_0}\) due to the fact that \(\overline{C_0}\) is an incoming cone.

\(^{43}\)If \(|r^2 - \Phi|_{p,S}(\lambda_1, \nu)\) is bounded it follows that \(|\zeta^a| = O(r^{-3})\) and \(|X^a| = O(r^{-2})\).
Lemma 2.4 Given on $C_0$ the components of the metric $\tilde{g}$, the connection coefficients $\zeta, \chi, \omega$ and $\partial_a X_b$ on $S_0$, we can obtain $\partial_a X_b$ on the whole $C_0$.

Proof: the result is obtained writing a transport equation for $\partial_a X^a$ which specifies it on the whole $C_0$. This transport equation is derived starting from the relation:

$$\frac{\partial}{\partial u} \left( \frac{\partial X^a}{\partial u} \right) = \frac{\partial}{\partial u} \left( \frac{\partial X^a}{\partial u} \right) = \frac{\partial}{\partial u} \left( 4\Omega^2 \zeta^a \right) = 8\Omega^2 \frac{\partial \log \Omega}{\partial u} \zeta^a + 4\Omega^2 \frac{\partial}{\partial u} \zeta^a$$

The first term in the right hand side is known as it is expressed in terms of $\Omega, \omega$ and $\zeta$; the second term can be rewritten as

$$\frac{1}{\Omega} \frac{\partial}{\partial u} \zeta^a = \partial_a \zeta^a - \Omega^{-1} \partial_X \zeta^a = (D_{a3} \zeta)^a - \Omega^{-1} (D_X \zeta)^a - e^{a}_{b \rho} \Gamma^a_{\mu \rho} \zeta^\mu + \Omega^{-1} X^c \Gamma^a_{c b} \zeta^b \, . \, (2.41)$$

Observe that the components $\zeta^a$ different from zero are $(\zeta^b, \zeta^u)$ therefore all the Christoffel symbols written in the right hand side of 2.41 are different from $\Gamma^a_{u u}$ and can be expressed in terms of the connection coefficients, see 1.73. The term $(D_X \zeta)^a$ is also known in terms of the previous quantities, therefore the only thing left is $(D_{a3} \zeta)^a$ which is known once we know $(D_{a3} \zeta)_a$. The knowledge of this last quantity is, nevertheless a consequence of the structure equations. In fact equation $R_{3a} = 0$, which must be satisfied on $C_0$, is

$$D_3 \zeta = -2 \chi \cdot \zeta + D_3 \nabla \log \Omega + \nabla \tr \chi - \delta \nabla \chi + \zeta \cdot \chi - \zeta \tr \chi \cdot \chi \, . \, (2.42)$$

therefore we know explicitly $D_3 \zeta$ as the right hand side is expressed in terms of quantities already obtained. Substituting this expression in 2.41 we end with a linear transport equation, where all terms in the right hand side are known,

$$\frac{\partial}{\partial u} \left( \frac{\partial X^a}{\partial u} \right) = H \left( \Omega, \nabla \Omega, \zeta, \nabla \zeta, \chi, \nabla \chi, \nabla \chi, \omega, \nabla \omega \right) \, . \, (2.43)$$

2.2 The solution of the characteristic constraint problem for the initial data on the incoming cone, with a prescribed decay rate.

The construction of the initial data on $C_0$ is very similar to what has been done on $C_0$, basically the role of the underlined and not underlined quantities is interchanged, but there are also some other relevant differences and in this subsection we will focus mainly on them.

The first remark is that, as already discussed, the symmetry between the structure equations along the outgoing or the incoming cone is true when we consider the equations written in a coordinate independent way, see for instance subsection 1.2. This symmetry is lost if, vice versa, we write the equations in a coordinate dependant way, choosing, for instance, the coordinates $\{u, \mathbf{u}, \omega^a\}$ such that the metric has the expression $1.64$,

$$\bar{g} = |\hat{X}|^2 du^2 - 2\Omega^2 (du \mathbf{u} + d\mathbf{u} du) - \hat{X}_a (du \omega^a + d\omega^a du) + \gamma_{ab} d\omega^a d\omega^b \, . \, 30$$
In this case, in fact, the expression for the geodesic vector fields $L$ and $\mathcal{L}$ are different, see 1.66. Therefore, while all the estimates for the connection coefficients along $\mathcal{C}_0$ can be obtained exactly as before, just interchanging the underlined and the not underlined coefficients and taking into account that we are considering here a portion of the incoming “cone”, the estimates relative to the metric components $\gamma_{ab}$ depend necessarily on the coordinates choice. It follows that the proof of Lemma 2.5, the analogue of Lemma 2.1, if done in the same chart is slightly different, while it will be completely equivalent if we choose a different set of coordinates such that $\mathcal{L} = \partial/\partial u$ and $L = \partial/\partial u + Y$, see the discussion in subsection 1.4.1 and footnote 20. It is easy to realize that in most of our work the choice of a set of coordinates is not relevant and that both strategies to prove Lemma 2.5 are equivalent. The only part of this work where we have to use a specific set of coordinates is when we look for a connection with harmonic coordinates, which is needed only to use a local existence proof needed to start our argument. This will be discussed in more detail in subsections 4.1 and 3.5.

We have defined $\mathcal{C}_0$ as the level surface, see 1.65, 

$$\mathcal{C}_0 \equiv C(\nu_0) = \{ p \in \mathbb{R}^4 | u(p) = \nu_0, \lambda \in [\lambda_1, \lambda_0] \}$$

in the manifold $(\mathbb{R}^4, \tilde{g})$. Choosing the previous coordinates the metric $\tilde{g}$ has the expression 1.64,

$$\tilde{g} = |\tilde{X}|^2 du^2 - 2\tilde{\Omega}^2(dudu + dudu) - \tilde{X}_a(dud\omega^a + d\omega^a du) + \tilde{\gamma}_{ab}d\omega^a d\omega^b$$

where $\tilde{\Omega}, \tilde{X}$ and $\tilde{\gamma}$ can be thought as extensions to $\mathbb{R}^4$ of the quantities we have obtained on $C_0$,

$$X_a = \tilde{X}_a|_{C_0}, \quad \Omega = \tilde{\Omega}|_{C_0}, \quad \gamma_{ab} = \tilde{\gamma}_{ab}|_{C_0}.$$ 

As we said these extended components are arbitrary in the interior region between $C_0$ and $\mathcal{C}_0$. The construction of the appropriate initial data on $\mathcal{C}_0$, will determine, as before, the restriction of these metric components on $\mathcal{C}_0$.

We start specifying the restriction of the function $\Omega$ on $\mathcal{C}_0$, $\Omega|_{\mathcal{C}_0} = \Omega_0$.

$\Omega$ is assigned freely except for the constraint on $S_0$ following from the condition 2.37 required to have the correct decay of $\omega$ on $C_0$,

$$D_3 \log \Omega = -2\omega|_{S_0}. \tag{2.44}$$

Once $\Omega$ is given we define, as done before for $C_0$, the “$\Omega$-foliation” of $\mathcal{C}_0$. The leaves of the foliation are the surfaces $S_0(\lambda) = S(\lambda, \nu_0) = \{ p \in \mathcal{C}(\nu_0)| u(p) = \lambda \}$. Denoting $v$ the affine parameter of the null geodesics on $\mathcal{C}_0$, we have:

$$u(p) = \lambda_1 + \int_0^{\nu(p)} (4\Omega)^{-2}(\gamma(v))dv. \tag{2.45}$$

44In this case differently from $C_0$, denoted $(u, \theta, \phi)$ the coordinates of the point $p$, $\gamma$ is the null geodesic starting on $S_0$ at the point $q$, whose angular coordinates are different from $(\theta, \phi)$, such that $\gamma(\nu(p)) = p$. 

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As in the $C_0$ case the vector field $N$ is the equivariant vector field associated to this foliation; denoting $\phi_\delta$, the diffeomorphisms generated by $N$ on $C(\nu_0)$ it follows

$$\phi_\delta[S(\lambda, \nu_0)] = S(\lambda + \delta, \nu_0).$$

From now on the estimates on $C_0$ are performed assuming on $(R^4, \tilde{g})$ a different set of coordinates such that $\tilde{g}$ has the following expression, see the discussion at the beginning of subsection 1.4.1 and footnote 19,

$$\tilde{g} = |\tilde{Y}|^2 du^2 - 2\tilde{Y}^2 (dudu + dudu) - \tilde{Y}_a (dudu + dudu) + \tilde{\gamma}'_{ab} du^a du^b \quad (2.46)$$

On each leaf $S_0(\lambda) = S(\lambda, \nu_0)$ we define two orthonormal 45 vector fields $\tilde{e}_A$ tangent to it,

$$\tilde{e}_A = \tilde{e}_a A \frac{\partial}{\partial \omega^a}$$

and require that the orthonormal frame $\{e_A\}$ be Fermi transported along $C_0$.

Again the metric $\tilde{\gamma}'$ in 2.46 is not the metric we want to give as initial data on $C_0$, but we use it as a background metric from which to construct the final one satisfying, with its derivatives, the “characteristic” constraints.

To perform the construction of the initial data, proceeding as in the previous section, we start defining

$$\omega = -\frac{1}{2\Omega} \frac{\partial}{\partial \log \Omega} \Omega \quad (2.47)$$

Then we assign a symmetric traceless tensor $\tilde{\chi}$ on $S_0(\lambda), \lambda \in [\lambda_1, \lambda_0]$, which we will consider as the traceless part of a tensor $\chi$. We require that the scalar functions $\Omega, \omega$ and the tensor $\tilde{\chi}$ have the following asymptotic behaviours:

$$|\Omega - \frac{1}{2}| = O(\varepsilon \tilde{r}^{-1}) \quad , \quad |\tilde{\nabla} \log \Omega| = O(\varepsilon \tilde{r}^{-(2+\delta)})$$

$$|\omega| = O(\varepsilon \tilde{r}^{-2}) \quad , \quad |\tilde{\chi}|_\gamma = O(\varepsilon \tilde{r}^{-1} |\lambda|^{-(\frac{3}{2}+\delta)})$$

$$|\tilde{\nabla} \tilde{\chi}|_\gamma = O(\varepsilon \tilde{r}^{-2} |\lambda|^{-(\frac{3}{2}+\delta)}) \quad , \quad |\tilde{\nabla}_3 \tilde{\chi}|_\gamma = O(\varepsilon \tilde{r}^{-1} |\lambda|^{-(\frac{3}{2}+\delta)}$$

(2.48)

with $\delta > 0$ and $\tilde{r}$ is defined from the relation $4\pi \tilde{r}^2(\lambda) = |S_0(\lambda)|_{\tilde{\gamma}'}$.

We require that the function $\text{tr}_\gamma$ and the initial metric components $\gamma'_{ab}$, we denote hereafter again $\gamma_{ab}$, assigned on $S_0 = S_0(\lambda_1) = S_0(\nu_0)$, satisfy the following system of differential equations:

$$\frac{\partial \gamma_{ab}}{\partial \lambda} - \Omega \text{tr}_\gamma \gamma_{ab} - 2\Omega \tilde{\gamma}_{ab} = 0$$

$$\frac{\partial \text{tr}_\gamma}{\partial \lambda} + \frac{\Omega \text{tr}_\gamma}{2} \text{tr}_\gamma + 2\Omega \omega \text{tr}_\gamma + |\tilde{\chi}|^2 = 0 \quad (2.49)$$

\[45\text{Orthonormal with respect to the } \tilde{\gamma} \text{ metric.}\]
Lemma 2.5 Assign on the null hypersurface $\mathcal{C}_0$, a $\Omega$-foliation with leaves $\mathcal{S}_0(\lambda)$, a metric tensor $\tilde{\gamma}$, tangent to the leaves $\mathcal{S}_0(\lambda)$ satisfying, together with $\text{tr}_{\tilde{\gamma}}$, the trace of the null outgoing second fundamental form associated to $\tilde{\gamma}$, the equations

$$
\frac{\partial \tilde{\gamma}_{ab}}{\partial \lambda} - \Omega \text{tr}_{\tilde{\gamma}} \tilde{\gamma}_{ab} = 0 \quad (2.50)
$$

$$
\frac{\partial \text{tr}_{\tilde{\gamma}}}{\partial \lambda} + \frac{\Omega \text{tr}_{\tilde{\gamma}}}{2} = \text{tr}_{\tilde{\gamma}} + 2 \Omega \omega \text{tr}_{\tilde{\gamma}} = 0 .
$$

Assume that, given $\varepsilon > 0$,

$$
|\lambda_1| \in [c_1 \tilde{r}_0, c_2 \tilde{r}_0] \quad \text{with} \quad |c_{1,2} - 1| = O(\varepsilon) . \quad (2.51)
$$

Given a tensor $\hat{\chi}$ on $\mathcal{C}_0$, assume that, with $\varepsilon > 0$, the following decays hold

$$
|\Omega - \frac{1}{2}| = O(\varepsilon \tilde{r}^{-1}) , \quad |\nabla \log \Omega| = O(\varepsilon \tilde{r}^{-(2+\delta)})
$$

$$
|\omega| = O(\varepsilon \tilde{r}^{-1}|\lambda|^{-1}) , \quad |\hat{\chi}|_{\tilde{\gamma}} = O(\varepsilon \tilde{r}^{-1}|\lambda|^{-(\frac{2}{3}+\delta)})
$$

$$
|\nabla \hat{\chi}|_{\tilde{\gamma}} = O(\varepsilon \tilde{r}^{-2}|\lambda|^{-(\frac{5}{3}+\delta)}) , \quad |\nabla_\lambda \hat{\chi}|_{\tilde{\gamma}} = O(\varepsilon \tilde{r}^{-1}|\lambda|^{-(\frac{4}{3}+\delta)})
$$

$$
|\nabla_\lambda \log \Omega| = O(\varepsilon \tilde{r}^{-(3+\delta)})
$$

with $\delta > 0$, and $\tilde{r}$ defined through the relation $4\pi \tilde{r}^2(\lambda, \nu_0) = |\mathcal{S}_0(\lambda)|_{\tilde{\gamma}}$. Assume that on $\mathcal{S}_0(\lambda_1)$,

$$
|r_0^{\frac{3}{2}} | \nabla \text{tr}_{\tilde{\gamma}}|_{\nu_0} = O(\varepsilon) ,
$$

then there exists on $\mathcal{C}_0$, a metric tensor $\gamma$ tangent to the leaves of the same foliation such that, denoted $\chi$ the second fundamental form of the $\mathcal{S}_0(\lambda)$ with respect to this metric, an orthonormal frame tangent to the $\mathcal{S}_0(\lambda)$ with respect to $\gamma$ we have:

$$
\chi_{ab} = \frac{1}{2\Omega} \frac{\partial \gamma_{ab}}{\partial \lambda} = \hat{\chi}_{ab} + \gamma_{ab} \frac{1}{2} \text{tr}_{\gamma} ,
$$

with $\gamma|_{\mathcal{S}_0} = \tilde{\gamma}|_{\mathcal{S}_0}$ and $\text{tr}_{\gamma} < 0$ satisfies the constraint equation

$$
\frac{\partial \text{tr}_{\gamma}}{\partial \lambda} + \frac{\Omega \text{tr}_{\gamma}}{2} = \text{tr}_{\gamma} + 2 \Omega \omega \text{tr}_{\gamma} + |\chi|_{\gamma}^2 = 0 \quad (2.55)
$$

with $\text{tr}_{\gamma}|_{\mathcal{S}_0} = \text{tr}_{\tilde{\gamma}}|_{\mathcal{S}_0}$. Moreover the following estimates hold, with $p \in [2, 4]$,

$$
|\Omega - \frac{1}{2}| = O(\varepsilon \tilde{r}^{-1}) , \quad |\nabla \log \Omega| = O(\varepsilon \tilde{r}^{-(2+\delta)}) , \quad |\omega| = O(\varepsilon \tilde{r}^{-1}|\lambda|^{-1})
$$

$$
|\text{tr}_{\gamma} + \frac{2}{r}| = O(\varepsilon \tilde{r}^{-2} \log r) , \quad |\chi|_{\gamma} = O(\varepsilon \tilde{r}^{-1}|\lambda|^{-(\frac{4}{3}+\delta)}) ,
$$

$$
|r^{2-\frac{2}{3}}| \lambda|^{\frac{2}{3}+\delta} |\nabla\chi|_{\nu_0} = O(\varepsilon) , \quad |r^{1-\frac{2}{3}}| \lambda|^{\frac{5}{3}+\delta} |\nabla_\lambda \chi|_{\nu_0} = O(\varepsilon)
$$

$$
|r^{2+\delta-\frac{2}{3}}| \lambda|^{\frac{5}{3}+\delta} |\nabla_\lambda \log \Omega|_{\nu_0} = O(\varepsilon) , \quad |r^{3-\frac{5}{3}} \nabla \text{tr}_{\gamma}|_{\nu_0} = O(\varepsilon)
$$

and $r$ defined by the relation: $4\pi r^2(\lambda, \nu_0) = |\mathcal{S}_0(\lambda)|_{\gamma}$.  

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**Proof:** We do not report the proof of Lemma 2.5 as it is obtained exactly as the one of Lemma 2.1.

Once solved equations 2.49 we have assigned on the whole \( C_0 \), \( \gamma \), \( \Omega \), \( \omega \) and \( \chi \). \( \gamma \) is the (initial data) metric relative to the leaves \( S_0(\lambda) \), \( \chi \) is the second fundamental form of the leaves \( S_0(\lambda) \) relative to this metric.

Let us discuss the way to fulfill the remaining constraint equations which have to be satisfied on \( C_0 \). For \( \zeta \) we use the evolution equation 1.9,

\[
\mathcal{D}_t \zeta + \frac{3}{2} \text{tr} \chi \zeta = -\hat{\chi} \cdot \zeta + [-d\nabla \chi + \nabla \text{tr} \chi + \mathcal{D}_3 \nabla \log \Omega] \ .
\] (2.57)

We want that \( r^2 \zeta \) be bounded and small, in the appropriate norm, on the whole \( C_0 \). In this case we integrate going down along the \( C_0 \) cone, namely moving toward the increasing radius of the leaves, as in the case of \( C_0 \). The result is in the following lemma.

**Lemma 2.6** Assume that, on \( C_0 \),

\[
|r^{(2+\delta)} - \frac{2}{\hat{s}}| \mathcal{D}_t \nabla \log \Omega_{|p,S} = O(\varepsilon) \ , \quad |r^{3-\frac{2}{\hat{s}}} \nabla \text{tr} \chi|_{p,S} = O(\varepsilon)
\] (2.58)

Let \( \zeta \) be the solution along \( C_0 \) of the constraint equation 2.57 then for \( p \in [2,4] \), we have the following bound:

\[
|r^{2-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda, \nu_0) \leq c \left( |r^{2-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda_0, \nu_0) + \varepsilon \right) \ .
\] (2.59)

**Proof:** The evolution equation for \( \zeta \) can be written as

\[
\mathcal{D}_t \zeta + \frac{3}{2} \text{tr} \chi \zeta + \hat{\chi} \cdot \zeta = \mathcal{F}(\nabla \chi, \nabla \log \Omega)
\] (2.60)

where

\[
\mathcal{F}(\nabla \chi, \nabla \log \Omega) = [-d\nabla \chi + \frac{1}{2} \nabla \text{tr} \chi + \mathcal{D}_3 \nabla \log \Omega] .
\]

Using standard techniques, see Chapter 4 of [Kl-Ni], we obtain, as we have already proved that \( |\hat{\chi}| = O(r^{-1} |\lambda|^{-(2+\delta)}) \),

\[
|r^{3-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda, \nu_0) \leq c \left( |r^{3-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} |r^{3-\frac{2}{\hat{s}}} \mathcal{F}(\nabla \chi, \nabla \log \Omega)|_{p,S}(\lambda', \nu_0) d\lambda' \right) \ .
\] (2.61)

As on \( C_0 \),

\[
r^{-1}(\lambda, \nu_0) \leq r^{-1}(\lambda', \nu_0) \leq r^{-1}(\lambda_0, \nu_0) \ ,
\] (2.62)

from 2.61 it follows

\[
|r^{2-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda, \nu_0) \leq c \left( |r^{2-\frac{2}{\hat{s}}} \zeta|_{p,S}(\lambda_0, \nu_0) + \frac{1}{r(\lambda, \nu_0)} \int_{\lambda_0}^{\lambda} |r^{3-\frac{2}{\hat{s}}} \mathcal{F}(\nabla \chi, \nabla \log \Omega)|_{p,S}(\lambda', \nu_0) d\lambda' \right) .
\] (2.63)
The integrand of the right hand side behaves, due to the terms \(D_3 \nabla \log \Omega\) and \(\nabla \text{tr} \chi\), as \(O(\varepsilon)\). Therefore the integral can be bounded by \(c(\lambda_0)r(\lambda, \nu_0)\) and the \(r(\lambda, \nu_0)\) factor is exactly compensated by the denominator. In conclusion, if we assume \(|r^{-2} \tilde{\lambda}|_{p,S}(\lambda_0, \nu_0) \leq c\varepsilon\),

\[
|r^{-2} \tilde{\lambda}|_{p,S}(\lambda_0, \nu_0) \leq c \left(|r^{-2} \tilde{\lambda}|_{p,S}(\lambda_0, \nu_0) + \varepsilon\right) \leq c\varepsilon .
\]  

(2.64)

The knowledge of \(\zeta\) on \(S_0(\lambda_0) = S(\lambda_0, \nu_0)\), allows to obtain an estimate for \(\zeta\) on \(S_0 = S(\lambda_1, \nu_0)\). Therefore the value on \(S_0\) which was used to obtain and control \(\zeta\) on the whole \(C_0\) can be still considered arbitrary. This will turn out important when, in the next subsection, we introduce the integral norms \(Q\) associated to the initial data.

The tensor quantity \(\chi\) on \(C_0\) is obtained in the same way as we obtained \(\chi\) on \(C_0\). The trace and the traceless part of \(\chi\), \(\text{tr} \chi\) and \(\hat{\chi}\), have to satisfy on \(C_0\) the constraint equations analogous to 2.24 and 2.25, namely

\[
D_3 \text{tr} \chi + \text{tr} \chi = 2(\text{div} \chi) - 2(|\xi|)^2 + 2K = 0 ,
\]

(2.65)

\[
D_3 \hat{\chi} + \frac{1}{2} \text{tr} \hat{\chi} + \frac{1}{2} \text{tr} \chi + 2(\text{div} \hat{\chi}) - \text{div} \eta - \nabla \otimes \eta = 0 .
\]

(2.66)

Proceeding as before, we prove the following lemma analogous to Lemma 2.3.

**Lemma 2.7** Assuming the results of Lemma 2.5 and Lemma 2.6 there exists a symmetric tensor \(\chi\) whose trace part and traceless part satisfy the evolution equations 2.65, 2.66. Moreover they satisfy the following estimates:

\[
|r^{-2} \text{tr} \chi|_{p,S} \leq c \varepsilon , \quad |r^{-2} \hat{\chi}|_{p,S} \leq c\varepsilon .
\]

(2.67)

**Proof:** To estimate \(\text{tr} \chi\) we proceed exactly as done before for \(\text{tr} \chi\) but integrating \(\text{tr} \chi\) on \(C_0\) backward, starting from \(S(\lambda_0, \nu_0)\).

(2.69)

\[
G(\text{tr} \chi, \hat{\chi}, \eta, \nabla \eta) = \left[\frac{1}{2} \text{tr} \hat{\chi} - \nabla \otimes \eta - \eta \otimes \eta\right]
\]

(2.69)

satisfies

\[
|r^{-2} G(\text{tr} \chi, \hat{\chi}, \eta, \nabla \eta)|_{p,S}(\lambda, \nu_0) \leq c \left(\frac{1}{r(\lambda, \nu_0)|\lambda|^{\frac{3}{2} + \delta}} + \frac{1}{r(\lambda, \nu_0)^2}\right).
\]

(2.70)

\[\text{We will choose tr} \chi\] on \(S_0(\lambda_0) = S(\lambda_0, \nu_0)\) in a way consistent with the initial value of \(\text{tr} \chi\) on \(S_0(\nu_0) = S_0(\lambda_1)\), specified in Lemma 2.1.
This implies the following bound of $\hat{\chi}$ along $C_0$ (for all the $\lambda$ values such that $|\lambda| < r(\lambda, \nu_0)$),

$$|r^{2-\frac{2}{p}} \hat{\chi}|_{p,S} \leq c\varepsilon,$$

proving the lemma. Again this estimate can be improved to a pointwise one, assuming enough regularity for the initial data.

We are left with determining the function $\omega$ on $C_0$. Again this function cannot be assigned freely but has to satisfy on $C_0$ the constraint equation 2.33:

$$D_3\omega + D_4\hat{\omega} - 4\omega\hat{\omega} - 3|\zeta|^2 + |\nabla \log \Omega|^2 = \rho(\chi, \chi, \eta, \eta)$$

(2.72)

which, proceeding as done for $\omega$ on $C_0$, can be written as:

$$D_3\omega - 2\omega\hat{\omega} = G(\zeta, \nabla \log \Omega, \rho)$$

(2.73)

where

$$G(\zeta, \nabla \log \Omega, \rho) = -\zeta \cdot \nabla \log \Omega + \frac{3}{2}|\zeta|^2 - \frac{1}{2}|\nabla \log \Omega|^2 + \frac{1}{2}\rho(\chi, \chi, \eta, \eta).$$

(2.74)

All the terms in the right hand side of the evolution equation 2.74 are known, therefore this equation can be used to determine $\omega$ on $C_0$ once we know it on $S_0$.

**Remark:** Observe that, differently from the estimate of $\omega$ on $C_0$, in this case equation 2.74 allows to obtain the decay for $\omega$ on $C_0$ without requiring a specific choice of $\omega$ on $S_0$. In fact observing that $\omega$ is integrable along $C_0$ and that $G(\zeta, \nabla \log \Omega, \rho) = O(r^{-3})$, we conclude, integrating forward in time, that, assuming on $S_0$, $|r^2 D_4 \log \Omega| = O(\varepsilon)$, the following bound holds on $C_0$, with $\delta > 0$,

$$|||\lambda|^{\delta} e^{(2-\delta)-\frac{2}{p}} \omega||_{p,S} \leq c\varepsilon.$$  

(2.75)

**Remark:** As discussed at the beginning of this subsection the initial data quantities on $C_0$ have been obtained in a way competely symmetric with the one used to obtain the initial data on $C_0$. This implies for the estimates of $\gamma_{ab}$ in Lemma 2.5 that we used a set of coordinates different from the coordinates $\{u, u_a, \omega\}$ used in Lemma 2.1. As we already said we could also use the original set of coordinates and the difference would be that equations 2.49 have to be substituted by the equations, see 1.72,

$$\frac{\partial \gamma_{ab}}{\partial \lambda} - \Omega \text{tr}_X \gamma_{ab} + (\nabla^2 \chi_a X_a + \nabla^a X_b) - 2\Omega \chi_{ab} = 0$$

(2.76)

$$\frac{\partial \text{tr}_X}{\partial \lambda} + \frac{\Omega \text{tr}_X}{2} + 2\Omega \omega \text{tr}_X + X^a \nabla_a \text{tr}_X + |\hat{\chi}|^2 = 0.$$

It is easy to realize that Lemma 2.5 can be proved also starting from equations 2.76. This requires to impose some conditions on $X_a$ allowed as $X_a$ can be assigned freely on $C_0$.

47In the Einstein vacuum spacetime it will be connected to the function $\Omega$ by the relation $\omega = -\frac{1}{2} \frac{\partial \log \Omega}{\partial \lambda}$.

48We require some decay for $\nabla^2 \chi_a X_a$ and that the condition $X^a \nabla_a \text{tr}_X = 0$ be satisfied.
2.3 The decay of the null Riemann components on $C_0$

In this subsection we show how to deduce from the initial data and their asymptotic behaviour the decay conditions for the various null Riemann components. Let us explain, first of all, why we need this result.

In the global existence proof of the characteristic Cauchy problem we use, mimicking the strategy developed in [Kl-Ni], some integral norms along the null outgoing and null incoming “cones” which are $L^2$ integrals of the conformal part of the Riemann tensor. In particular we prove that these norms are bounded in terms of the same norms defined on the initial hypersurface $C$. Therefore it is crucial that the initial data be such that these norms are bounded (and small). To fulfill this request the initial data must have appropriate decays along $C_0$ and $C_{in}$.\footnote{The request of a decay along $C_{in}$ has to be interpreted as we always consider a truncated finite portion of it. Nevertheless in the existence proof there is not an upper bound on its size, this is the sense in which our result is a global one, see the discussion in Section 3.}

Let us start to consider the initial data on $C_0$. It is important to emphasize that the various null components of the Riemann tensor whose decay along $C_0$ we want to control refer to the Riemann tensor of the Einstein vacuum spacetime (whose global existence we want to prove). Nevertheless, given the initial data satisfying the characteristic constraints, they can be defined, all except one,\footnote{But this will not be harmful.} as specific combinations, still denoted $\alpha, \beta, \rho, \sigma, \beta_{\bar{0}}$, of the initial data defined on $C_0$. Once solved the existence problem, $C_0$ becomes an embedded hypersurface, $C(\lambda_1)$, in $(\mathcal{M},g)$ and the initial data tensor fields are the various connection coefficients restricted to $C(\lambda_1)$ and the various combinations $\alpha, \beta, ...$ become the various null components of the Riemann tensor expressed in terms of the connection coefficients.

More precisely, see for instance [Kl-Ni], Chapter 3, the various null Riemann components in the vacuum Einstein spacetime $(\mathcal{M},g)$ are defined as\footnote{In a vacuum Einstein spacetime the Riemann tensor coincides with its conformal part.}

\begin{align*}
\alpha(R)(e_A, e_B) &= R(e_A, e_4, e_B, e_4) \\
\beta(R)(e_A) &= \frac{1}{2} R(e_A, e_4, e_3) \\
\rho(R) &= \frac{1}{4} R(e_3, e_4, e_3, e_4) \\
\sigma(R) &= \frac{1}{4} \rho(R^* R) = \frac{1}{4} R(e_3, e_4, e_3, e_4) \\
\beta_{\bar{0}}(R)(e_A) &= \frac{1}{2} R(e_A, e_3, e_3, e_4) \\
\omega(R)(e_A, e_B) &= R(e_A, e_3, e_B, e_3).
\end{align*}

On the other side, as discussed in Section 1 the structure equations give the explicit expression of the conformal Riemann tensor $C$ in terms of the connection coefficients. Therefore we can write the null components $\alpha, \beta, \rho, \sigma, \beta_{\bar{0}}$ (restricted
to \( C_0 \) in terms of the initial data, see subsubsection 1.2.2, as \(^{52}\)

\[
\begin{align*}
\alpha &= -\left[ D_4 \dot{\chi} + \text{tr} \chi \dot{\chi} - (D_4 \log \Omega) \dot{\chi} \right] \\
\beta &= \nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi \\
\rho &= -\left[ K + \frac{1}{4} \text{tr} \chi \dot{\chi} - \frac{1}{2} \dot{\chi} \cdot \dot{\chi} \right] \\
\sigma &= \text{curl} \chi - \frac{1}{2} \dot{\chi} \wedge \dot{\chi}
\end{align*}
\]

(2.78)

These expressions can also be interpreted as the restriction of the Riemann tensor of the Lorentzian manifold \((R^4, \tilde{g})\) where \( \tilde{g} \) is the metric \(^{53}\) where \( \tilde{\Omega}, \tilde{X} \) and \( \tilde{\gamma} \) are extensions to \( R^4 \) of the quantities we have obtained on \( C_0 \). Therefore in \((R^4, \tilde{g})\) the Bianchi equations, when restricted to \( C_0 \), have exactly the same expression as in the vacuum Einstein manifold. In other words, as a consequence of the constraint equations, the Ricci tensor of \((R^4, \tilde{g})\) restricted to \( C_0 \), is identically zero.\(^{54}\)

This allows us to use the Bianchi equations to show that the initial data imply the appropriate decay for the null components \( \beta, \rho, \sigma, \beta_0 \).

This discussion and the decay estimates we are looking for are summarized in the following propositions:

**Proposition 2.1** Let \( \{\gamma_{ab}, \Omega, \zeta_a, \chi_{ab}, \omega\} \) be the initial data on \( C_0 \) satisfying the \(| \cdot |_{p,S} \) norm bounds of Lemma 2.1 and Lemma 2.2, then the combinations of the (derivatives of the) initial data \( \beta, \rho, \sigma, \beta_0 \), defined in 2.78, satisfy on \( C_0 \) the following evolution equations, with \( \eta = -\zeta + \nabla \log \Omega \):

\[
\begin{align*}
D_4 \beta + 2 \text{tr} \chi \beta &= \text{div} \alpha - [2 \omega \beta - (2 \zeta + \eta) \alpha] \\
D_4 \rho + \frac{3}{2} \text{tr} \chi \rho &= \text{div} \beta - \left[ \frac{1}{2} \dot{\chi} \cdot \alpha - \zeta \cdot \beta - 2 \eta \cdot \beta \right] \\
D_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma &= -\text{div} \beta + \left[ \frac{1}{2} \dot{\chi} \cdot * \alpha - \zeta \cdot * \beta - 2 \eta \cdot * \beta \right] \\
D_4 \beta_0 + \text{tr} \chi \beta_0 &= -\nabla \rho + [2 \omega \beta + 2 \dot{\chi} \cdot \beta + \nabla \sigma - 3(\eta \rho - * \eta \sigma)]
\end{align*}
\]

(2.79)

**Proof:** This explicit expression of the Bianchi equations has been derived, for instance, in [Ch-Kl], see also [Kl-Ni], Chapter 3.

**Remarks:**

a) Observe that the Bianchi equations do not provide an evolution equation along the outgoing null cones for the null component \( \alpha \). Therefore its decay along \( C_0 \) has to be obtained in a different way.

---

\(^{52}\)Observe that \( \alpha \) cannot be expressed in terms of the initial data and their derivatives along \( C_0 \), see 1.32.

\(^{53}\)Obviously \((R^4, \tilde{g})\) is not an Einstein vacuum spacetime as the extensions of the metric components outside \( C_0 \) do not satisfy the Einstein equations.

\(^{54}\)The structure equations 1.7,...,1.20 which on the right hand side are equal to zero, in a general manifold would be equal to the Ricci part of the Riemann tensor. Requiring that the constraints equations are satisfied implies, therefore, that on \( C_0 \) the Ricci tensor is zero.
b) We are not interested on the decay along $C_0$ of the null component $\alpha$. This is due to the fact that the $Q$ flux-norms we will use along the outgoing cones do not depend on $\alpha$, see their definition in [Kl-Ni], subsection 3.5.1.

**Proposition 2.2** Let $\{\gamma_{ab}, \Omega, \zeta_a, \chi_{ab}\}$ be the initial data on $C_0$ satisfying the $|\cdot|_{p, S}$ norm bounds of Lemma 2.1 and Lemma 2.2 and analogous estimates for the derivatives of the initial data assume that on $S_0 = S(\lambda_1, \nu_0)$ the following conditions are satisfied:

\[
\begin{align*}
\nabla \text{tr} \chi + \zeta \text{tr} \chi &= O(\varepsilon r_0^{-(\frac{7}{2} + \delta)}) \\
K - K + \frac{1}{4} (\text{tr} \chi - \text{tr} \chi) &= O(\varepsilon r_0^{-(\frac{7}{2} + \delta)}) \\
c \text{curl} \zeta - \frac{1}{2} (\hat{\chi} \wedge \hat{\chi}) &= O(\varepsilon r_0^{-(\frac{7}{2} + \delta)}) \\
\nabla \text{tr} \chi - \zeta \text{tr} \chi &= O(\varepsilon r_0^{-(\frac{7}{2} + \delta)}) \; ,
\end{align*}
\]

then the combinations $\alpha, \beta, \rho, \sigma, \beta$ of the (derivatives of the) initial data defined in 2.78 satisfy the following bounds, with $\delta > 0$:

\[
\begin{align*}
|r_{(\frac{7}{2} + \delta)} \alpha|_{C_0} &\leq c \varepsilon , \; |r_{(\frac{7}{2} + \delta)} - \frac{1}{2} \beta|_{p, S} \leq c \varepsilon , \; |r_{(3 + \delta)}(\rho - \sigma)|_{p, S} \leq c \varepsilon \\
|r_{3 - \frac{2}{3}} \rho|_{p, S} &\leq c \varepsilon , \; |r_{(2 + \delta)} - \frac{1}{2} \beta|_{p, S} \leq c \varepsilon \; ,
\end{align*}
\]

**Proof:** We start looking at $\alpha$. From its definition in 2.78,

\[
\alpha = -[D_4 \hat{\chi} + \text{tr} \chi - (D_4 \log \Omega) \hat{\chi}] \; ,
\]

and the results of Lemma 2.1, see 2.12, the correct decay for $\alpha$ follows immediately.

To determine the asymptotic behaviour of $\beta$, through the expression

\[
\nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi = \beta 
\]

we use the evolution equation

\[
D_4 \beta + 2 \text{tr} \chi \beta = 2 \omega \beta - [\text{div} \alpha - (\zeta + \nabla \log \Omega) \alpha] 
\]

Proceeding as we did for $\zeta$ we obtain, using the Gronwall’s lemma, the following estimate, with $G(\alpha, \nabla \alpha, \zeta, \nabla \log \Omega) \equiv [\text{div} \alpha - (\zeta + \nabla \log \Omega) \alpha],

\[
|r_{4 - \frac{2}{3}} \beta|_{p, S}(\lambda_1, \nu) \leq c \left( |r_{4 - \frac{2}{3}} \beta|_{p, S}(\lambda_1, \nu_0) + \int_{\nu_0}^{\nu} |r_{4 - \frac{2}{3}} G(\alpha, \nabla \alpha, \zeta, \nabla \log \Omega)|_{p, S}(\lambda_1, \nu') d\nu' \right) \; .
\]

55This is consistent with our setting as to express $\alpha$ in terms of initial data on $C_0$ we would need the derivative along $u$ of the $\chi$ tensor which is not provided by the initial data (on $C_0$).

56In the following remark we discuss how much regularity for the initial data is needed to prove Proposition 2.2.
Dividing both sides by \( r^{\left(\frac{7}{2} - \delta\right)}(\lambda_1, \nu) \) using inequality 2.62 we rewrite 2.82 as

\[
|r^{\left(\frac{7}{2} + \delta\right) - \frac{2}{7}}\beta|_{p, S}(\lambda_1, \nu) \leq c \left(r^{\left(\frac{7}{2} + \delta\right) - \frac{2}{7}}\beta|_{p, S}(\lambda_1, \nu_0) + \varepsilon\right) + \frac{1}{r^{\left(\frac{7}{2} - \delta\right)}(\lambda_1, \nu)} \int_{r_0}^{r} |r^{d - \frac{7}{2}} G(\alpha, \nabla \alpha, \zeta, \nabla \log \Omega)|_{p, S}(\lambda_1, \nu') d\nu'.
\]  

(2.83)

From the previous estimate on \( \alpha, G = O(r^{-\left(\frac{7}{2} + \delta\right)}) \) and the integral in the right hand side can be estimated by \( c(\lambda_1)r^{\left(\frac{7}{2} - \delta\right)}(\lambda_1, \nu)\varepsilon, \) we conclude that

\[
|r^{\left(\frac{7}{2} + \delta\right) - \frac{2}{7}}\beta|_{p, S}(\lambda_1, \nu) \leq c \left(r^{\left(\frac{7}{2} + \delta\right) - \frac{2}{7}}\beta|_{p, S}(\lambda_1, \nu_0) + \varepsilon\right).
\]  

(2.84)

Finally to prove that \( |r^{\left(\frac{7}{2} + \delta\right) - \frac{2}{7}}\beta|_{p, S}(\lambda_1, \nu_0) \leq c\varepsilon \) uniformly in \( r_0 = r(\lambda_1, \nu_0) \) we need that

\[
\beta|_{S_0} = (\nabla \text{tr} \chi - \nabla \chi - \nabla \log \Omega)_{|S_0} = O(\varepsilon r^{-\left(\frac{7}{2} + \delta\right)}).
\]

This follows, recalling the behaviour of \( \hat{\chi}, \) from the assumption, see 2.80: \(^58\)

\[
\nabla \text{tr} \chi|_{S_0} + \zeta|_{S_0} \text{tr} \chi|_{S_0} = O(\varepsilon r^{-\left(\frac{7}{2} + \delta\right)}).
\]  

(2.85)

To determine the asymptotic decay of \( \rho \) and \( \rho - \tilde{\rho} \) we proceed in the same way. Again the initial data on \( S_0 \) have to be such that

\[
K - \tilde{K} + \frac{1}{4}\left(\text{tr} \chi \text{tr} \chi - \text{tr} \chi \text{tr} \chi\right) = O(\varepsilon r^{-\left(\frac{7}{2} + \delta\right)}).
\]  

(2.86)

To prove that \( \sigma \) has the correct decay we must require, after a simple application of the Gronwall’s Lemma to its evolution equations, that on \( S_0 \) the following relation holds:

\[
c \omega \text{tr} \chi - \frac{1}{2}\tilde{\chi} \wedge \hat{\chi} = O(\varepsilon r^{-\left(\frac{7}{2} + \delta\right)}).
\]  

(2.87)

Finally proceeding in the same way for \( \beta, \) using Gronwall’s Lemma and the decay of the previous null components we obtain the expected result provided on \( S_0 \) the following estimate holds:

\[
\nabla \text{tr} \chi|_{S_0} - \zeta|_{S_0} \text{tr} \chi|_{S_0} = O(\varepsilon r^{-\left(\frac{7}{2} + \delta\right)}).
\]  

(2.88)

Remark: To complete the proof we have to specify the exact regularity of the initial data required to prove Proposition 2.2. To control \( \beta \) we use the first transport equation in 2.79, which requires the control of \( \nabla \alpha, \) to control \( \rho \) and \( \sigma \) we need to control \( \nabla \beta \) which at its turn requires the control of \( \nabla^2 \alpha \) and finally to control \( \beta \) we require the control of \( \nabla \rho \) which finally implies the control of

\(^{57}\)Observe that the estimate for \( \beta \) can also be a pointwise estimate if we assume enough regularity for the initial data.

\(^{58}\)This quantity plays the role of the \( \psi \) form defined in [Kl-Ni], equation (4.3.5).
\( \nabla^3 \alpha \) Recalling the explicit expression of \( \alpha \), this implies that we need to control \( |r(\hat{\omega}^{\delta}) - \hat{\omega}^3 D_3 \hat{\chi}|_{p,S} \), \( |r(\hat{\omega}^{\delta}) - \hat{\omega}^3 \hat{\chi}|_{p,S} \) and \( |r^{(4+\delta)} - \hat{\omega}^3 \hat{\omega}|_{p,S} \). The needed regularity is also higher if we require pointwise norms for all the Riemann components.

This completes the existence proof for the initial data satisfying the characteristic constraints and the appropriate decay on \( C_0 \). To achieve our result we have to implement the analogous program for \( C_0 \).

2.4 The decay of the null Riemann components on the incoming null cone.

The discussion about the null Riemann components on \( C_0 \) goes exactly as in the case of \( C_0 \) and we do not repeat it here. The various combinations of connection coefficients which describe, once the global existence problem is solved, the restriction to \( C_0 \) of the Riemann tensor are the same as in 2.78, with the expression for \( \alpha \), instead of the one for \( \alpha \),

\[
\alpha = -\left[ D_3 \hat{\chi} + \text{tr} \hat{\chi} - (D_3 \log \Omega) \hat{\chi} \right].
\]

(2.89)

Again in \((R^4, g)\) the Bianchi equations, restricted to \( C_0 \), have exactly the same expression as in the vacuum Einstein manifold. Therefore we prove the following propositions:

**Proposition 2.3** Let \( \{ \gamma_{ab}, \Omega, X^a, \zeta_a, \chi_{ab} \} \) be the initial data on \( C_0 \) satisfying the \( | \cdot |_{p,S} \) norm bounds of Lemma 2.5 and Lemma 2.6, then \( \beta, \rho, \sigma, \beta \), combinations of the (derivatives of the) initial data defined in 2.78 satisfy on \( C_0 \) the following evolution equations, with \( \eta = \zeta + \nabla \log \Omega \):

\[
\begin{align*}
D_3 \beta + 2 \text{tr} \hat{\chi} \beta &= -\text{div} \alpha - \left[ 2 \omega \beta + (-2 \zeta + \eta) \cdot \alpha \right] \\
D_3 \rho + \frac{3}{2} \text{tr} \chi \rho &= -\text{div} \beta - \left[ \frac{1}{2} \hat{\chi} \cdot \alpha - \zeta \cdot \beta + 2 \eta \cdot \beta \right] \\
D_3 \sigma + \frac{3}{2} \text{tr} \chi \sigma &= -\text{div} \nu \beta + \left[ \frac{1}{2} \hat{\chi} \cdot \nu \alpha - \zeta \cdot \nu \beta - 2 \eta \cdot \nu \beta \right] \\
D_3 \beta + \text{tr} \chi \beta &= \nabla \rho + \left[ 2 \omega \beta + \nabla \sigma + 2 \hat{\chi} \cdot \beta + 3(\eta \rho + \nu \sigma) \right]
\end{align*}
\]

(2.90)

**Remark:** The Bianchi equations do not provide an evolution equation along the incoming null cones for \( \alpha \). Therefore its decay along \( C_0 \) has to be obtained in a different way.

**Proposition 2.4** Let \( \{ \gamma_{ab}, \Omega, X^a, \chi_{ab} \} \) be the initial data on \( C_0 \) satisfying the \( | \cdot |_{p,S} \) norm bounds of Lemma 2.5 and 2.6 then, provided that on \( S_0 = S(\lambda_1, \nu_0) \)

\[
59 \text{In the global existence proof the boundedness of the } Q \text{ norms defined on } C \text{ requires the boundedness of the } L^2(C_0) \text{ and } L^2(C_0) \text{ norms of the various Riemann components and their derivatives up to second order.}
\]

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the following conditions are satisfied:

\[
\nabla \text{tr}\chi + \zeta \text{tr}\chi = O(\varepsilon r_0^{-(\frac{4}{5} + \delta)})
\]

\[
K - K + \frac{1}{4}(\text{tr}\chi \text{tr}\chi - \text{tr}\chi \text{tr}\chi) = O(\varepsilon r_0^{-(\frac{4}{5} + \delta)})
\]

\[
\nabla \text{curl}\zeta - \frac{1}{2}\hat{\chi} \wedge \hat{\chi} = O(\varepsilon r_0^{-(\frac{4}{5} + \delta)})
\]

where \(|\hat{\chi}| = O(\varepsilon r_0^{-(\frac{4}{5} + \delta)})\). The following estimate, with \(G(\alpha, \nabla \alpha, \zeta, \nabla \log \Omega) = -[\text{div}\alpha - (\zeta - \nabla \log \Omega)\alpha]\),

\[
|\begin{bmatrix} \lambda \hat{\chi} \end{bmatrix} |^2_{p,S}(\lambda, \nu_0) \leq c\left[|\lambda|^{\frac{1}{\delta}}|\hat{\chi}|^2_{p,S}(\lambda_1, \nu_0) + \int_{\lambda_1}^{\lambda} |\lambda|^{\frac{1}{\delta}}|\hat{\chi}|^2_{p,S}(\lambda', \nu_0) d\lambda\right] \quad (2.93)
\]

Consider now the function \(u(p)\) not exactly equal to \(\nu_0 - 2r\), but the argument holds also in that case.
Multiplying both sides by $\frac{|\lambda'|^{\frac{2}{3}+\delta'}}{r^2(\lambda',\nu_0)}$, we rewrite 2.93 as

$$|r^{2-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|c_{p,S}(\lambda,\nu_0)| \leq c \left| r^{2-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|c_{p,S}(\lambda_1,\nu_0) + \int_{\lambda_1}^{\lambda} |\lambda'||\lambda'|^{\frac{2}{3}+\delta'} r^{2-\frac{2}{3}} G(\alpha, \nabla \alpha, \zeta, \nabla \log \Omega)|_{p,S}(\lambda',\nu_0) d\lambda' \right|$$

(2.96)

and as, from the previous estimates on $\alpha$, the integrand in the right hand side can be estimated by

$$c(\lambda') |\lambda'||\lambda'|^{-\frac{2}{3}+\delta'} r^2(\lambda',\nu_0) |\lambda'|^{-(\frac{2}{3}+\delta')} \leq c(\lambda') \frac{1}{|\lambda'|^{1+(\delta-\delta')}} \varepsilon,$$

we conclude again that

$$|r^{2-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|c_{p,S}(\lambda,\nu)| \leq c \left( |r^{2-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|c_{p,S}(\lambda_1,\nu_0) + \varepsilon \right).$$

(2.97)

Finally to prove that $|r^{2-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|c_{p,S}(\lambda_1,\nu_0) \leq c\varepsilon$, uniformly in $r_0 = r(\lambda_1,\nu_0)$ we require that

$$\beta|S_0 = (\nabla \text{tr} \chi - \delta \nabla \chi + \zeta \cdot \chi - \zeta \text{tr} \chi)|_{S_0} = O(r_0^{-2}|\lambda|^{-(\frac{2}{3}+\delta')}) = O(r_0^{-\frac{2}{3}+\delta'}).$$

This estimate, recalling the assumption on $\chi$, is satisfied from the assumptions of Proposition 2.4:

$$\nabla \text{tr} \chi|_{S_0} - \zeta|_{S_0} \text{tr} \chi|_{S_0} = O(r_0^{-\frac{2}{3}+\delta'}).$$

Observe that the estimate for $\beta$ can also be a pointwise estimate if we assume enough regularity for the initial data.

To prove that, given the initial data, $\rho$, $\sigma$ and $\beta$ satisfy the right bounds along $S_0$ is a simple application of the Gronwall’s Lemma to their evolution equations, 2.90, which can be estimated starting from $S_0$ plus the assumptions of Proposition 2.4.

**Remark:** It is important to observe that to obtain the expected bound for $\beta$ we have to use, together with the evolution equation for $\rho$, 2.90, an improved estimate for $\nabla \rho$, namely $|r^{1-\frac{2}{3}}|\lambda||\lambda'|^{\frac{2}{3}+\delta'}|\nabla \rho|_{p,S} \leq c\varepsilon$. The same happens when we estimate $\rho,$ $\sigma$, and $\beta$.

Proposition 2.2 and Proposition 2.4 complete the construction of initial data satisfying the appropriate decay, provided we show that conditions 2.91 can be satisfied on $S_0$. This is possible as the one form $\zeta$ can be assigned freely on $S_0$, as previously discussed. Therefore we can proceed in the following way: Given $\hat{\chi}$ and $\hat{\chi}$ on $S_0$ we solve the equation

$$c\nabla \nabla \zeta - \frac{1}{2} \hat{\chi} \wedge \hat{\chi} = 0.$$  

(2.98)
Then with this $\zeta$ we solve the equations
\[
\begin{align*}
\nabla \chi|_{S_0} + \zeta|_{S_0} \chi|_{S_0} &= 0 \\
\nabla \chi|_{S_0} - \zeta|_{S_0} \chi|_{S_0} &= 0.
\end{align*}
\]
(2.99)
Finally we verify that with these solutions on $S_0$ the estimate
\[
K|_{S_0} - \overline{K}|_{S_0} + \frac{1}{4} (\chi|_{S_0} \chi|_{S_0} - \overline{\chi|_{S_0} \chi|_{S_0}}) = O(\varepsilon r_0^{-(\frac{3}{2} + \delta)})
\]
(2.100)
can be satisfied.

More precisely we recall that in the initial data construction we were free to assign on $S_0$, $\chi|_{S_0}$, $\overline{\chi|_{S_0}}$, $\zeta$ and $\tilde{\gamma}_{ab}$. We can therefore assign them from the beginning requiring they satisfy equations 2.98, 2.99 and the estimate 2.100. Then we proceed in the construction of the initial data with the appropriate decay and, finally, we prove that, due to the asymptotic behaviour of the restriction of the null Riemann components, the $Q$ norms on $C_0$ and $C_0$ are finite and bounded by $\varepsilon$.

3 The global existence theorem, strategy of the proof.

In this section we state the global existence theorem we want to prove and describe the main steps of the proof. The following sections are devoted to their implementation.

3.1 The regularity of the initial data

To state and prove an existence theorem we have to specify the function spaces of the initial data, in other words their regularity. Let us focus first on the initial data on $C_0$, an analogous discussion is done later for the initial data on $C_0$.

3.1.1 The regularity of the initial data on $C_0$

In Section 2 we have shown how to obtain the initial data on $C_0$ satisfying the constraints and some appropriate decay rates. We want to investigate here their regularity with respect to the tangential derivatives, $\nabla$. It is clear that the amount of regularity we need will be fixed by the construction of the solution to the characteristic problem. Here we want to show how, due to the constraints, the regularities of different terms of the initial data are related. To see it in more detail let us go through the various steps done in Section 2 to obtain the initial data. Let us first introduce the following definition:

**Definition 3.1** let $f$ be a covariant tensor defined on $C_0$ and at each point tangent to the leaves $S_0(\nu)$ of the $\Omega$-foliation. We say $f \in C^q(S)$ if, for any component of $f$, $f_{\alpha_1, \ldots, \alpha_q}(\nu, \omega) \in C^q(S^2)$ for any $\nu \in [\nu_0, \infty)$.

61 Assuming the appropriate derivability in $\nu$. 

44
Given this definition, following the strategy of Section 2 we prescribe the initial data starting from \( \hat{\chi} \) and \( \Omega \). Let us assume, therefore, that \( \hat{\chi} \), \( \Omega \), \( \mathcal{D}_4 \hat{\chi} \), \( \mathcal{D}_4 \Omega \in C^q(S) \), then from Lemma 2.1 it follows that \( \text{tr}\chi \) and \( \gamma \) also belong to \( C^q(S) \). The connection coefficient \( \zeta \), obtained solving equation 1.12 in Lemma 2.2 is, therefore, in \( C^{q-1}(S) \) as \( \nabla \log \Omega \). Moreover from its definition \( \omega \in C^q(S) \) and \( X \), due to its relation with \( \zeta \), see equation 1.69, belong also to \( C^{q-1}(S) \).

On \( C_0 \) the initial data are completely assigned once we prescribe also the connection coefficients \( \chi \) and \( \omega \). To do it we have to solve equations 2.24, 2.25 and 2.36. From their inspection one sees immediately that \( \chi \in C^{q-2}(S) \). The proof that \( \omega \in C^{q-2}(S) \) has, viceversa, to be postponed after the determination of the regularity of \( \rho(\chi, \chi, \eta, \eta) \). Summarizing we have

\[
\begin{align*}
\Omega, \gamma, \chi, \omega, \mathcal{D}_4 \hat{\chi} & \in C^q(S) \\
X, \zeta, \nabla \log \Omega & \in C^{q-1}(S) \\
\chi, \omega & \in C^{q-2}(S)
\end{align*}
\]  

Once the regularity of the metric components and the connection coefficients is stated we can also see which regularity this implies for the various components of the Riemann tensor. From equations 2.78 we have immediately

\[
\alpha \in C^q(S), \beta \in C^{q-1}(S), (\rho, \sigma) \in C^{q-2}(S), \beta \in C^{q-3}(S), \]

and from the regularity of \( \rho \) that one of \( \omega \) follows, using equation 2.36.

### 3.1.2 The regularity of the initial data on \( C_0 \)

We do not repeat for \( C_0 \) the previous argument and we only report the result obtained exactly in the same way:

\[
\begin{align*}
\Omega, \gamma, \chi, \omega, \mathcal{D}_4 \hat{\chi} & \in C^q(S) \\
X, \zeta, \nabla \log \Omega & \in C^{q-1}(S) \\
\chi, \omega & \in C^{q-2}(S)
\end{align*}
\]  

and

\[
\alpha \in C^q(S), \beta \in C^{q-1}(S), (\rho, \sigma) \in C^{q-2}(S), \beta \in C^{q-3}(S).
\]  

### 3.1.3 The “loss of derivatives” of the initial data

It is well known, see for instance H.Muller Zum Hagen, [Mu], that to solve the characteristic problem we need to control on the initial null hypersurfaces the initial data and their derivatives along the “normal” direction. On \( C_0 \), for instance, we need an estimate of the derivatives along the \( e_3 \) direction. On the other side, due to the costraints associated to the characteristic problem, these data cannot be given freely and, therefore, are related to the data on the initial hypersurface and their tangential derivatives. This has the effect that to control the derivatives along the “normal” direction up to order, say, \( k \), the
order of tangential derivatives of the initial data which have to be controlled is greater than \( k \). This is what we call the “loss of derivatives” of the initial data. The aim of this section is to see how this fact appears in our formalism. In this way we can specify in the subsequent paper, where we state and prove in a precise way the existence result, which is the regularity required to the initial data. In [Mu], see Lemma 4.3 and Lemma 4.4, it is shown that the control of the first derivative normal to a null hypersurface requires the control of second tangential derivatives and in general that of a normal derivative of order \( k \) requires \( 2k \) tangential derivatives. To recognize the same phenomenon in our case we proceed in the following way.

First of all in our formalism we do not consider the partial derivatives of the metric components, but the connection coefficients associated to the \( \Omega \) foliation. Let us consider for the moment the \( C^0 \) part of the initial data null hypersurface, to select the connection coefficients associated to the “normal” null direction \( e_3 \) relative to \( C_0 \). We use the notion of signature introduced by D. Christodoulou and S. Klainerman in [Ch-Kl], see also [Kl-Ni], paragraph 3.1.24. Extending this notion also to the connection coefficients it follows that the signature of the various connection coefficients is

\[
\text{sig}(\chi) = +1, \quad \text{sig}(\omega) = +1, \quad \text{sig}(\zeta) = 0, \quad \text{sig}(\omega) = -1, \quad \text{sig}(\chi) = -1 \quad (3.5)
\]

moreover each derivative along \( e_3 \) decreases the signature by 1 and any derivative along \( e_4 \) increases it by 1. Any decrease of -1 in the corresponds, in some sense, to a derivative along the \( e_3 \) direction.

Let us recall that if \( \gamma \in C^q(S) \) then \( \chi \in C^{q-2}(S) \), consistent with the loss of derivatives we discussed. To substantiate this result for higher order derivatives, let us consider the derivative along \( e_3 \) of \( \chi \), \( D_3 \chi \). To control \( D_3 \chi \) on \( C_0 \) we have to look at the evolution equation of it along \( C_0 \) which can be obtained applying \( D_3 \) to the evolution equations 2.24 and 2.25. Let us consider, to have a concrete example, the second equation,

\[
D_4 \chi + \frac{1}{2} \text{tr} \chi \chi + \frac{1}{2} \text{tr} \chi - 2 \omega \chi - \nabla \otimes \eta - \eta \otimes \eta = 0 .
\]

Applying \( D_3 \) to this equation, neglecting the commutator between \( D_3 \) and \( D_4 \), we obtain an evolution equation along \( C_0 \), formally of this type

\[
D_4 (D_3 \chi) = \cdots + \nabla (D_3 \zeta) + \cdots ,
\]

where dots denote the less relevant terms, and using for \( D_3 \zeta \) the transport equation 1.30, \( D_3 \zeta + 2 \chi \cdot \zeta - D_3 \nabla \log \Omega = -\beta \), we can write

\[
D_4 (D_3 \chi) = \cdots - \nabla \beta + \cdots
\]

which implies, recalling 3.2, that \( D_3 \chi \in C^{q-4}(S) \), again one derivative \( D_3 \) implying the loss of two tangential derivatives. Iterating the argument, with the same simplifications, we have immediately, using the Bianchi equations,

\[
D_4 (D_3^2 \chi) = \cdots - \nabla D_3 \beta + \cdots = \cdots - \nabla^2 \beta + \cdots
\]

\(^{62}\)We use the “ ” for the adjective normal as, in fact, \( g(e_3, \tau_4) = -2. \)
where
\[ \alpha = -\left[ \mathcal{D}_3 \hat{\chi} + \text{tr} \hat{\chi} \hat{\chi} - (\mathcal{D}_3 \log \Omega) \hat{\chi} \right], \tag{3.9} \]
\( \nabla^2 \hat{\alpha} \in C^{q-6}(S) \) as expected. Finally for an arbitrary order \( k \) of \( \mathcal{D}_3 \) derivatives we substitute in 3.8 the explicit expression 3.9, obtaining
\[ \mathcal{D}_3^2 \hat{\chi} = \nabla^2 \mathcal{D}_3 \hat{\chi} \tag{3.10} \]
implying that each extra \( \text{"D}_3 \text{ regularity" requires the regularity of two more tangential derivatives. These considerations will imply the well known phenomenon that the regularity of the solution will turn out to be lower than the one of (some of) the initial data. In fact in the present case \( \hat{\chi} \) will be inside the spacetime \( C^{q-4}(S) \) while on \( C_0 \) it is \( C^{q-2}(S) \).

An analogous argument can be done for \( \text{tr} \hat{\chi} \) and \( \omega \) and we do not report it here.

3.2 The integral \( \mathcal{Q} \) norms on the initial hypersurface

In Chapter 3 of [Kl-Ni], section 3.5.1 we have introduced a family of integral norms, denoted \( \text{"}\mathcal{Q}\text{-integral norms"} \), see also [Ch-Kl], \( L^2 \) integrals made along the incoming and outgoing cones (or portions of them). The control of these norms is a crucial step in the global existence proof as from them it is possible to control the family of norms relative to the Riemann tensor null components. This allows to start the bootstrap mechanism discussed in detail in [Kl-Ni] which we do not repeat here. To control these norms means to prove that they can be bounded by the same norms on the initial hypersurface. As the \( \mathcal{Q} \) norms are \( L^2 \) weighted integral norms of the various Riemann null components expressed on the initial hypersurface in terms of the initial data, 2.78, the regularity and the decay of the initial data have to be prescribed to satisfy the existence of these norms. From the inspection of the explicit expression of the \( \mathcal{Q} \) norms, one easily sees that the assumptions on the decay of the initial data along the \( \text{"cones"} \), are such that the \( \mathcal{Q} \) norms are finite, provided that also the derivatives of the initial data are sufficiently regular and decay in a consistent way. More precisely we have the following expressions for the \( \mathcal{Q} \) norms:

\[ \mathcal{Q}(\lambda, \nu) = \mathcal{Q}_1(\lambda, \nu) + \mathcal{Q}_2(\lambda, \nu) \]
\[ \mathcal{Q}(\lambda, \nu) = \mathcal{Q}_1(\lambda, \nu) + \mathcal{Q}_2(\lambda, \nu) \tag{3.11} \]

where,

\[ \mathcal{Q}_1(\lambda, \nu) = \int_{C(\lambda) \cap V(\lambda, \nu)} \mathcal{Q}(\hat{\mathcal{L}}_T \mathcal{R})(\bar{K}, \bar{K}, e_4) \]
\[ + \int_{C(\lambda) \cap V(\lambda, \nu)} \mathcal{Q}(\hat{\mathcal{L}}_T \mathcal{R})(\bar{K}, \bar{K}, e_4) \]
\[ \mathcal{Q}_2(\lambda, \nu) = \int_{C(\lambda) \cap V(\lambda, \nu)} \mathcal{Q}(\hat{\mathcal{L}}_T \mathcal{R})(\bar{K}, \bar{K}, e_4) \]
\[ + \int_{\mathcal{C}(\lambda) \cap V(\lambda, \nu)} Q(\mathcal{L}_O R)(\tilde{K}, \tilde{K}, T, e_4) \]
\[ + \int_{\mathcal{C}(\lambda) \cap V(\lambda, \nu)} Q(\mathcal{L}_S \mathcal{T} R)(\tilde{K}, \tilde{K}, e_4) \]

\[ Q_0(\lambda, \nu) = \sup_{V(\lambda, \nu) \subseteq C} |r^2 \beta|^2 + \int_{\mathcal{C}(\nu) \cap V(\lambda, \nu)} Q(\mathcal{L}_T R)(\tilde{K}, \tilde{K}, e_3) \]
\[ + \int_{\mathcal{C}(\nu) \cap V(\lambda, \nu)} Q(\mathcal{L}_O R)(\tilde{K}, \tilde{K}, T, e_3) \]
\[ Q_2(\lambda, \nu) = \int_{\mathcal{C}(\nu) \cap V(\lambda, \nu)} Q(\mathcal{L}_O \mathcal{T} R)(\tilde{K}, \tilde{K}, e_3) \]
\[ + \int_{\mathcal{C}(\nu) \cap V(\lambda, \nu)} Q(\mathcal{L}_O^2 R)(\tilde{K}, \tilde{K}, T, e_3) \]
\[ + \int_{\mathcal{C}(\nu) \cap V(\lambda, \nu)} Q(\mathcal{L}_S \mathcal{T} R)(\tilde{K}, \tilde{K}, e_3) \] (3.13)

with

\[ Q(R)(K_0, K_0, T, e_4) = \frac{1}{4} u^4 |\alpha|^2 + \frac{1}{2} (u^4 + 2u^2 u^2) |\beta|^2 + \frac{1}{2} (u^4 + 2u^2 u^2)(\rho^2 + \sigma^2) \]
\[ + \frac{1}{2} u^4 |\beta|^2 \]

\[ Q(R)(K_0, K_0, T, e_3) = \frac{1}{4} u^4 |\alpha|^2 + \frac{1}{2} (u^4 + 2u^2 u^2) |\beta|^2 + \frac{1}{2} (u^4 + 2u^2 u^2)(\rho^2 + \sigma^2) \]
\[ + \frac{1}{2} u^4 |\beta|^2 \] (3.14)

\[ Q(R)(K_0, K_0, K_0, e_4) = \frac{1}{4} u^6 |\alpha|^2 + \frac{3}{2} u^4 u^2 |\beta|^2 + \frac{3}{2} u^4 u^2(\rho^2 + \sigma^2) + \frac{1}{2} u^6 |\beta|^2 \]

\[ Q(R)(K_0, K_0, K_0, e_3) = \frac{1}{4} u^6 |\alpha|^2 + \frac{3}{2} u^4 u^2 |\beta|^2 + \frac{3}{2} u^4 u^2(\rho^2 + \sigma^2) + \frac{1}{2} u^6 |\beta|^2 \] (3.15)

and \( V(\lambda, \nu) \) is the part of \( J^- (S(\lambda, \nu)) \) above \( C \). \( \mathcal{L}_O, \mathcal{L}_T, \mathcal{L}_S \) are some “modified” Lie derivatives, \( T, O, S, K \) are vector fields, Killing or conformal Killing in the Minkowski spacetime, “near” Killing or conformal Killing in the general case, see for a detailed discussion on these definitions [Kl-Ni] Chapter 3. From these expressions we see that the norms depend on the second Lie derivatives of the null Riemann components \( \alpha, \beta, \rho, \sigma, \bar{\alpha}, \bar{\beta} \). Expressing the Lie derivatives in terms of the covariant derivatives we conclude that we have to control the null components of the Riemann tensor field and their \( \mathcal{V}, \mathcal{D}_3, \mathcal{D}_4 \) derivatives up to second order. On \( C_0 \) the \( \mathcal{D}_4 \) derivatives of the null components can be expressed in terms of the tangential \( \mathcal{V} \) derivatives\(^{\text{63}} \) using the Bianchi equations. This, as discussed in the previous subsection, implies the so called “loss of derivatives”.

\(^{\text{63}}\)Except for \( \bar{\alpha} \) which does not appear in the \( Q \) norms along the outgoing cones.
This does not happen, on \( C_0 \), for the \( D_1 \) derivatives, while on \( C_0 \), the role of \( D_3 \) and \( D_4 \) are interchanged.

The detailed examination of the \( Q \) norms tells us the amount of regularity in the tangential derivatives we have to require. We do not discuss it in detail here as it will be done in the second part of this work. We only show how the argument goes in the case of the null component \( \alpha \) for the \( Q \) norms along \( C_0 \).

Starting with the explicit expression of the \( Q \) norms it will follow that in the existence proof we have to control a weighted \( L^2(C_0) \) norm of \( D_2^0 \alpha \). From the Bianchi equations, 2.79, 2.90 we have that \( D_2^0 \alpha = -\nabla \beta \) and \( D_3^0 \alpha = -\nabla^2 \rho \). On the other side from 3.2 it follows that if \( \alpha \in C^q \) then \( D_3^0 \alpha \) is a smooth \( C^q-4 \) showing the loss of derivatives previously discussed. From the requirement that \( D_3^0 \alpha \in L^2(C_0) \) it follows, therefore, that \( \nabla^2 \rho \in L^2(C_0) \). This condition implies, at its turn, that \( \rho \in C^0(S) \) which requires on \( C_0 \), \( q = 2 \) for the initial data defined in 3.1.

### 3.3 Global initial data smallness condition

We say that the initial data on \( C = C_0 \cup C_0 \) are small if the following quantity, \( J^{(q)}_{C_0 \cup C_0} \), is small: 64

\[
J^{(q)}_{C_0 \cup C_0} = J^{(q)}_{C_0} \left[ \tau_{ab}, \bar{\Omega}^a, \bar{\chi}_a, \bar{\omega} \right] + J^{(q)}_{C_0} \left[ \tau_{ab}, \bar{\Omega}^a, \bar{\chi}_a, \bar{\omega} \right] \leq \varepsilon, \tag{3.16}
\]

where, with \( p \in [2, 4] \) and \( \delta > 0 \),

\[
J^{(q)}_{C_0} \left[ \tau_{ab}, \bar{\Omega}^a, \bar{\chi}_a, \bar{\omega} \right] = \sup_{C_0} \left( \frac{r}{2} |\bar{\Omega}^a - \frac{1}{2} |^2 + \frac{r^2}{\log r} |\text{tr} \bar{\chi} - \frac{2}{r} |^2 + r(\delta + \frac{\hat{\theta}}{\bar{\Omega}^a}) \right) + \\
\sup_{C_0} \left[ \left( \sum_{l=1}^{q} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=1}^{q} |r^{(2+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-1} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{log} \bar{\Omega}^a/p, S \right| \right. \\
+ \left. \sum_{l=0}^{q-1} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-2} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-2} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| .
\]

\[J^{(q)}_{C_0} \left[ \tau_{ab}, \bar{\Omega}^a, \bar{\chi}_a, \bar{\omega} \right] = \sup_{C_0} \left( \frac{r}{2} |\bar{\Omega}^a - \frac{1}{2} |^2 + \frac{r^2}{\log r} |\text{tr} \bar{\chi} + \frac{2}{r} |^2 + r(\delta + \frac{\hat{\theta}}{\bar{\Omega}^a}) \right) + \\
\sup_{C_0} \left[ \left( \sum_{l=1}^{q} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=1}^{q} |r^{(2+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-1} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{log} \bar{\Omega}^a/p, S \right| \right. \\
+ \left. \sum_{l=0}^{q-1} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-2} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| + \sum_{l=0}^{q-2} |r^{(2+l+\frac{\hat{\theta}}{\bar{\Omega}^a})} \text{tr} \bar{\chi}^P/p, S \right| .
\]

64Observe that in Definition 1.1 of the initial data the quantities \( X \) or \( \bar{X} \) are not present. In fact although we can obtain it on \( C_0 \) and prescribe on \( C_0 \), they are not needed to build the initial data \( Q \) norms. Nevertheless in the local part of the existence proof we need to write our initial data in terms of the harmonic coordinates. This requires to connect our \( \{u, \bar{u}, \omega\} \) coordinates with the harmonic coordinates and this implies the knowledge of the vector fields \( X \) or \( \bar{X} \). Motivated by this argument we also add \( \bar{X} \) as initial data and specify its appropriate norms.
Moreover, let the initial data be given in the forthcoming paper when we discuss the local existence will imply the smallness of 3.16. The explicit form of the “restricted condition” will be enough to impose that the norms of the quantities which are assigned freely on \( C_0 \) and \( C'_0 \) be small, together with the smallness of some norms relative to \( S_0 = S'_0 = C_0 \cap C'_0 \). These restricted conditions plus the transport equations 1.55, 1.56 along \( C_0 \) and \( C'_0 \), respectively, will imply the smallness of 3.16. The explicit form of the “restricted condition” will be given in the forthcoming paper when we discuss the local existence solution.

### 3.4 Statement of the characteristic global existence theorem.

**Theorem 3.1 (characteristic global existence theorem)** Let the initial data

\[
\{ C_0 : \nabla_{ab} \Omega, \zeta_a, \sum_{a b} \Sigma \} \cup \{ C'_0 : \nabla_{ab} \Omega', \zeta_a', \sum_{a b} \Sigma' \}
\]

(3.17)

be assigned together with their partial tangential derivatives to a fixed order specified by the integer \( q \geq 7 \).\(^{65}\) Let us assume they satisfy the smallness conditions:

\[
J_{C_0}^{(q)\infty} = J_{C'_0}^{(q)\infty} \left[ \nabla_{ab} \Omega, \zeta_a, \sum_{a b} \Sigma \right] + J_{C'_0}^{(q)\infty} \left[ \nabla_{ab} \Omega', \zeta_a', \sum_{a b} \Sigma' \right] \leq \varepsilon
\]

(3.18)

Then there exists and it is unique a vacuum Einstein spacetime \( \{ M, g \} \) solving the characteristic initial value problem with initial data 3.17. \( M \) is the maximal future development of \( C(\lambda_1) \cup C(\nu_0) \),\(^{66}\)

\[
M = \lim_{\nu_1 \to \infty} J^+(S(\lambda_1, \nu_0)) \cap J^-(S(\lambda_0, \nu_1))
\]

(3.19)

Moreover \( M \) is endowed with the following structures:

---

\(^{65}\) The minimum value of \( q \) will be discussed in a forthcoming paper. The techniques in [Kl-Ni] to obtain a “global” solution would require \( q \geq s = 4 \), but the local existence requires a stronger condition, namely \( q \geq 2s - 1 = 7 \), see for instance [Mu-Se].

\(^{66}\) Sometimes to avoid any confusion we write explicitly \( C(\lambda_1; [\nu_0, \nu_1]) \cup C(\omega; [\lambda_1, \lambda_0]) \).
a) \( \mathcal{M} \) is foliated by a “double null canonical foliation” \( \{ C(\lambda) \}, \{ \mathcal{C}(\nu) \} \), with \( \lambda \in [\lambda_1, \lambda_0] \), \( \nu \in [\nu_0, \nu_1] \).\(^{67}\) Double canonical foliation means that the null hypersurfaces \( C(\lambda) \) are the level hypersurfaces of a function \( u(p) \) solution of the eikonal equation
\[
g^{\mu \nu} \partial_\mu w \partial_\nu w = 0 ,
\]
with initial data a function \( u_*(p) \) defining the foliation of the “final” incoming cone \( \mathcal{C}(\nu_0) \),\(^{68}\) while the null hypersurfaces \( \mathcal{C}(\nu) \) are the level hypersurfaces of a function \( u(p) \) solution of the eikonal equation with initial data a function \( u_0(p) \) defining a canonical foliation of the initial outgoing cone \( C(\lambda_1) \).\(^{69}\) The family of two dimensional surfaces \( \{ S(\lambda, \nu) \} \), where \( S(\lambda, \nu) \equiv C(\lambda) \cap \mathcal{C}(\nu) \), defines a two dimensional foliation of \( \mathcal{M} \).

b) \( i(C_0) = C(\lambda_1), \ i(C_\nu) = C(\nu_0), \ i(S_0(\lambda_1)) = i(S_\nu(\nu_0)) = i(S_\nu(\lambda_1)) = C(\lambda_1) \cap \mathcal{C}(\nu_0) \).

c) On \( C(\lambda_1) \) with respect to the initial data foliation,\(^{70}\) we have
\[
i^*(\gamma') = \gamma , \ i^*(\Omega') = \Omega , \ i_*^{-1}(X') = X
\]
\[
i^*(\chi') = \chi , \ i^*(\omega') = \omega , \ i^*(\zeta') = \zeta
\]
(3.20)
together with their tangential derivatives up to \( q \).

d) On \( C(\nu_0) \) with respect to the initial data foliation, we have
\[
i^*(\gamma') = \gamma , \ i^*(\Omega') = \Omega , \ i_*^{-1}(X') = X
\]
\[
i^*(\chi') = \chi , \ i^*(\omega') = \omega , \ i^*(\zeta') = \zeta
\]
(3.21)
where \( \gamma', \Omega', X', \chi', \omega', \zeta' \) are the metric components and the connection coefficients in a neighbourhood of \( C(\lambda_1) \) and \( \mathcal{C}(\nu_0) \).\(^{71}\)

e) The constraint equations 1.55,1.56 are the pull back of (some of) the structure equations of \( \mathcal{M} \) restricted to \( C(\lambda_1) \) and \( \mathcal{C}(\nu_0) \).\(^{72}\)

f) The double null canonical foliation and the associated two dimensional one, \( \{ S(\lambda, \nu) \} \), implies different foliations of \( C(\lambda_1) \) and \( \mathcal{C}(\nu_0) \): We can define on the whole \( \mathcal{M} \) a null orthonormal frame \( \{ e_a, e_\nu, e_\lambda \} \) adapted to the double null canonical foliation, the (Lorentzian) metric components, \( \gamma_{ab}, \Omega, X^a \) with

\(^{67}\)To avoid any misunderstanding “double canonical foliation” refers to a foliation of the spacetime \( (\mathcal{M}, g) \), while with canonical foliation we denote a specific foliation of the initial data on \( C_0 \). Of course the first is related to the second.

\(^{68}\)A detailed discussion of the function \( u_*(p) \) when \( \nu_1 \to \infty \) is in [Kl-Ni], Chapter 8.

\(^{69}\)It is important to remark that the “canonical” foliation on the portion \( C(\lambda_1) \) of the initial hypersurface is not the one given when the initial data are specified. Nevertheless it can be proved, see the discussion during the global existence proof, that given the initial data foliation of \( C(\lambda_1) \), it is possible to build on \( C(\lambda_1) \) a “canonical” foliation. Its precise definition and the way for doing it will be made clear in the course of the proof. See also [Kl-Ni], Chapter 3.

\(^{70}\)Observe that while \( \mathcal{M} \) is globally foliated by a double null canonical foliation, it is not possible to foliate it with a double foliation solution of the eikonal equation with initial data the \( \Omega \)-foliation of \( C_0 \) and \( \mathcal{C}_\nu \). This is, nevertheless possible in a small neighbourhood of \( C_0 \) and \( \mathcal{C}_\nu \).

\(^{71}\)These quantities denoted here \( \gamma', \Omega', X' \dots \) were denoted in the previous sections \( \gamma, \Omega, X, \chi, \omega, \zeta \). In the sequel these notations are referred to the corresponding quantities relative to the double canonical foliation of \( \mathcal{M} \) and their restrictions to \( C_0 \cup \mathcal{C}_\nu \).

\(^{72}\)With respect to the foliations of the initial data.
respect to the adapted coordinates \( \{u, u, \theta, \phi\} \) and the corresponding connection coefficients

\[
\chi_{ab} = g(D_e^a e_4, e_b) , \quad \chi_{ab} = g(D_e^a e_3, e_b) \\
\omega = \frac{1}{4} g(D_e^a e_3, e_4) = -\frac{1}{2} D_3 \log \Omega \\
\omega = \frac{1}{4} g(D_e^a e_4, e_3) = -\frac{1}{2} D_4 \log \Omega \\
\zeta_a = \frac{1}{2} g(D_e^a e_4, e_3)
\]

Moreover \( X^a \) is a vector field defined in \( M \) such that denoting \( N \) and \( N_0 \) two null vector fields equivariant with respects to the \( S(\lambda, \nu) \) surfaces, the following relations hold: \( N = \Omega e_4, N_0 = \Omega e_3 \) and in the \( \{u, u, \theta, \phi\} \) coordinates

\[
N = 2\Omega^2 \frac{\partial}{\partial u} , \quad N_0 = 2\Omega^2 \left( \frac{\partial}{\partial u} + X^a \frac{\partial}{\partial \omega^a} \right)
\]

\( g \) The null geodesics of \( M \) along the outgoing and incoming null direction \( e_4, e_3 \) are defined for all \( \nu \in [\nu_0, \infty) \) and all \( \lambda \in [\lambda_1, \lambda_0] \) respectively. Finally the existence result is uniform in \( \lambda_1 (< \lambda_0 < 0) \).

### 3.5 The general structure of the proof.

The proof is made by two different parts whose structure is, basically, the same.

**First part:** We prove the existence of the spacetime

\[
M' = J^+(S(\lambda_1, \nu_0)) \cap J^-(S(\lambda_0, \nu_1 = \lambda_0 + 2\rho_0))
\]

where

\[
2\rho_0 = \nu_0 - \lambda_1
\]

**Second part:** We prove the existence of the spacetime \( M \supset M' \),

\[
M = \lim_{\nu_1 \to \infty} J^+(S(\lambda_1, \nu_0)) \cap J^-(S(\lambda_0, \nu_1))
\]

**Remark:** It is appropriate now to give an intuitive picture of the spacetime we are building. As our result is a small data result we expect that our spacetime stays near to (a portion of) the Minkowski spacetime. This implies that the functions \( \underline{u}(p) \) and \( u(p) \) do not differ much, written in spherical Minkowski coordinates, from \( t + r \) and \( t - r \), respectively. Moreover the radius \( r(\lambda, \nu) \) defined as proportional to the square root of the area of \( S(\lambda, \nu) \), see 2.13, will stay near to the \( r \) spherical coordinate. The initial cones \( C_0 \) and \( C_0 \) approximate two minkowskian cones, one outgoing and one incoming, with their vertices on the origin vertical axis. The assumption that \( \nu_0 \) and \( |\lambda_1| \) are approximately equal to

---

\(^{73}\)The precise way the coordinate \( \theta \) and \( \phi \) are defined will be discussed elsewhere. See, anyway [Kl-Ni], paragraph 3.1.6.
\( r_0 \) just pictures the two dimensional surface \( S_0 \) as lying (approximately) on the hyperplane \( t = 0 \) where \( t \) is “near” to \( \frac{1}{2}(u + u) \).

Proof of the first part: We denote \((K(\tau), g)\) a solution of the “characteristic Cauchy problem” with the following properties:

i) \((K(\tau), g)\) is foliated by a double null canonical foliation \( \{C(\lambda)\}, \{C(\nu)\} \) with \( \lambda \in [\lambda_1, \lambda] \), \( \nu \in [\nu_0, \nu] \). Moreover

\[
i(C_0) \cap K(\tau) = C(\lambda_1; [\nu_0, \nu]) ; \quad i(C_0) \cap K(\tau) = C(\nu_0; [\lambda_1, \lambda])
\]

where

\[
\overline{\nu} + \overline{\lambda} = 2\tau ; \quad \overline{\nu} - \overline{\lambda} = 2\rho_0 .
\]

ii) Denoted \( \{e_3, e_4, e_a\} \) the null orthonormal frame adapted to the double null canonical foliation. We introduce a family of norms \( R, O \) for the null components of the Riemann curvature tensor and for the connection coefficients respectively, as done in [Kl-Ni], Chapter 3.

iii) Given \( \epsilon_0 > 0 \) sufficiently small, but larger than \( \epsilon \), the norms \( R, O \) satisfy the following inequalities

\[
R \leq \epsilon_0 , \quad O \leq \epsilon_0 . \tag{3.24}
\]

iv) Denoted by \( T \) the set of all values \( \tau \) for which the spacetime \( K(\tau) \) does exist, we define \( \tau_* \) as the sup over all the values of \( \tau \in T \):

\[
\tau_* = \sup\{\tau \in T\} . \tag{3.25}
\]

There are, now, two possibilities:

\[ \tau_* = \lambda_0 + \rho_0 \quad \text{or} \quad \tau_* < \lambda_0 + \rho_0 \]

If \( \tau_* = \lambda_0 + \rho_0 \) then

\[
K(\tau_*) = J^+ (S(\lambda_1, \nu_0)) \cap J^- (S(\lambda_0, \lambda_0 + 2\rho_0)) \tag{3.26}
\]

and the first part is proved. If, viceversa, \( \tau_* < \lambda_0 + \rho_0 \), we show that in \( K(\tau_*) \) the norms \( R \) and \( O \) satisfy the following estimates

\[
R \leq \frac{1}{2} \epsilon_0 ; \quad O \leq \frac{1}{2} \epsilon_0 . \tag{3.27}
\]

v) Using the inequalities 3.27 restricted to \( C(\nu_*) \) it is possible to prove that we can extend the spacetime \( K(\tau_*) \) to a spacetime \( K(\tau_* + \delta) \). This contradicts

\[\text{Although the picture defined here is the more natural, with a redefinition of the functions} u, v, r \text{ in terms of the standard coordinates one could also treat in a similar way the case where the two “initial” cones \( C_0 \) and \( C_0' \) are shifted one respect to the other and their vertices do not lie even approximately on the same vertical axis.}\]

\[\text{As the spacetime we obtain is “near” to a portion of the Minkowski spacetime, condition 3.23 can be visualized as the requirement that} S(\overline{\lambda}, \overline{\nu}) \text{ coincides with a vertical (time) translation of} S_0 .\]

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the fact that \( \tau_\ast \) is the supremum defined in 3.25, unless \( \tau_\ast = \lambda_0 + \rho_0 \). In fact in this case \( K(\tau_\ast) \) describes the maximal region \( K(\tau) \) which can be determined from the initial conditions assigned on \( \{ C(\nu_0; [\lambda_1, \lambda_0]) \} \cup C(\lambda_1; [\nu_0, \nu_1]) \) even if \( R \leq \frac{1}{2} \epsilon_0; \quad O \leq \frac{1}{2} \epsilon_0 \).

vi) Observe that, due to the fact, which will be evident in the course of the proof, that the estimates for \( R \) and \( O \) do not depend on the magnitude of the interval \( [\lambda_1, \lambda_0] \), also the proof of the first part can be considered a global existence result. In other words fixed \( \lambda_0 \), we can choose \( \rho_0 = \frac{1}{2}(\nu_0 - \lambda_1) \) arbitrarily large implying that \( \tau_\ast = \lambda_0 + \rho_0 \) can be chosen arbitrarily large.

In conclusion the result of the first part is achieved if we can prove that:

i) The set \( T \) is not empty.

ii) in \( K(\tau_\ast) \) the norms \( R \) and \( O \) satisfy the following estimates

\[
R \leq \frac{1}{2} \epsilon_0; \quad O \leq \frac{1}{2} \epsilon_0 .
\]

(3.28)

iii) If \( \tau_\ast < \lambda_0 + \rho_0 \) we can extend \( K(\tau_\ast) \) to \( K(\tau_\ast + \delta) \) with \( \delta > 0 \).

The proofs of i), ii) and iii) are the technical parts of the result. Their main lines are sketched in the next subsection. The detailed proofs are in a forthcoming paper.

**Proof of the second part:** It is, basically, a corollary of the proof of the first part. We consider in this case the spacetimes \( M(\nu) \) solutions of the “characteristic Cauchy problem” with the following properties:

i) \( (M(\nu), g) \) is foliated by a double null canonical foliation \( \{ C(\lambda) \}, \{ C(\nu) \} \) with \( \lambda \in [\lambda_1, \lambda_0] , \nu \in [\nu_0, \nu_1] \). Moreover

\[
i(C_0) \cap M(\nu) = C(\lambda; [\nu_0, \nu_1]) ; \quad i(C_0) \cap M(\nu) = C(\nu; [\lambda_1, \lambda_0])
\]

ii) Denoted \( \{ e_3, e_4, e_a \} \) the null orthonormal frame adapted to the double null canonical foliation, we introduce a family of norms \( O, R \) for the connection coefficients and for the null components of the Riemann curvature tensor respectively, as done in [Kl-Ni], Chapter 3.

iii) Given \( \epsilon_0 > 0 \) sufficiently small, but larger than \( \epsilon \), the norms \( R, O \) satisfy the following inequalities

\[
R \leq \epsilon_0; \quad O \leq \epsilon_0 .
\]

(3.29)

iv) Denoted by \( N \) the set of all values \( \nu \) for which the spacetime \( M(\nu) \) does exist, we define \( \nu_\ast \) as the sup over all the values of \( \nu \in N \):

\[
\nu_\ast = \sup \{ \nu \in N \} .
\]

(3.30)

Again there are two possibilities: if \( \nu_\ast = \infty \) the result is achieved, therefore we are left to show that the second possibility, \( \nu_\ast < \infty \), leads to a contradiction. In fact we prove that in \( M(\nu_\ast) \) the norms \( R \) and \( O \) satisfy the following estimates

\[
R \leq \frac{1}{2} \epsilon_0; \quad O \leq \frac{1}{2} \epsilon_0 .
\]

(3.31)
v) Using the inequalities 3.31 restricted to \( C(\nu_\ast; [\lambda_1, \lambda_0]) \) it is possible to prove that we can extend the spacetime \( M(\nu_\ast) \) to a spacetime \( M(\nu_\ast + \delta) \) which contradicts the definition of \( \nu_\ast \) unless \( \nu_\ast = \infty \).

**Remark:** Observe that in the second part of the result we do not have to prove that the set \( N \) is not empty as, due to the first part of the proof, it contains at least the element \( \overline{\nu} = \lambda_0 + 2\rho_0 \).

### 3.6 Thechnical parts of the proof, a broad sketch.

**First part:**

i) To prove that the set \( T \) is not empty we have to show that a spacetime \( K(\tau) \) exists, possibly with small \( \tau \). This requires a local existence theorem and we can use the results of A.Rendall, [Ren] or of H.Muller Zum Hagen, H.Muller Zum Hagen and H.J.Seifert, [Mu], [Mu-Se]. The main difficulty in adapting these results to our case is that they are proved using harmonic coordinates while the “gauge” we use for \( K(\tau) \) is the one associated to the double null canonical foliation. This requires a precise connection between the initial data written in the two different gauges in such a way that we can reexpress their results in our formalism. This will be achieved in the next section.

ii) Once we have proved that \( T \) is not empty we can define \( K(\tau_\ast) \) and prove that in this spacetime inequalities 3.27 hold. This requires, see [Kl-Ni], Chapter 3 for a detailed discussion, that the double null foliation of \( K(\tau_\ast) \) be canonical which, at its turn, implies proving the existence of specific foliations, denoted again “canonical” on \( C(\lambda) \) and on \( C(\nu_\ast) \). These “canonical” foliations will play the role of the “initial data” for the solutions of the eikonal equations

\[
g^{\mu\nu} \partial_\mu w \partial_\nu w = 0
\]

whose level hypersurfaces define the double null canonical foliation \( \{ C(\lambda) \}, \{ C(\nu_\ast) \} \) of \( K(\tau_\ast) \), see also [Ni].

iii) The central part of the proof consists in showing that we can extend \( K(\tau_\ast) \) to \( K(\tau_\ast + \delta) \) with \( \delta > 0 \). To achieve it we have to implement the following steps:

1) First of all we have to prove an existence theorem in the strips \( \Delta_1 K, \Delta_2 K \)

\[
\begin{align*}
\Delta_1 K &= J^+(S(\lambda_1, \nu_s) \cap J^- (S(\lambda(\nu_s), \nu_s + \delta)) \\
\Delta_2 K &= J^+(S(\lambda(\nu_s), \nu_0) \cap J^- (S(\lambda(\nu_s + \delta), \nu_s))
\end{align*}
\]

whose boundaries are:

\[
\partial \Delta_1 K = C(\lambda_1; [\nu_s, \nu_s + \delta]) \cup \overline{C(\nu_s + \delta; [\lambda_1, \lambda(\nu_s)])} \cup C(\lambda(\nu_s); [\nu_s, \nu_s + \delta]) \cup \overline{C(\nu_s + \delta; [\lambda_1, \lambda(\nu_s)])}
\]

The existence of this foliation has to be proved also for the (local) spacetime of i). The proof one gives for \( K(\tau_\ast) \) can be easily adapted to this case. The canonical foliation is crucial to obtain on \( C(\nu_\ast) \) all the connection coefficients to be used again as initial data, without any loss of derivatives.

---

\[76\]
\[ \partial \Delta_2 \mathcal{K} = \mathcal{C}(\nu_0; [\overline{\lambda}(\nu_0), \overline{\lambda}(\nu_0 + \delta)]) \cup \mathcal{C}(\overline{\lambda}(\nu_0 + \delta); [\nu_0, \nu_0]) \cup \mathcal{C}(\nu_0; [\overline{\lambda}(\nu_0), \overline{\lambda}(\nu_0 + \delta)]) \cup \mathcal{C}(\overline{\lambda}(\nu_0 + \delta); [\nu_0, \nu_0]) , \] (3.34)

where
\[ \overline{\lambda}(\nu) = -2\rho_0 + \nu . \]

2) We are then left with proving the (local) existence of a diamond shaped spacetime \( \Delta_3 \mathcal{K} \),
\[ \Delta_3 \mathcal{K} = J^+(S(\overline{\lambda}(\nu_0), \nu_0) \cap J^-(S(\overline{\lambda}(\nu_0 + \delta), \nu_0 + \delta)) , \]
(3.35)
specified by the initial data
\[ \mathcal{C}(\overline{\lambda}(\nu_0); [\nu_0, \nu_0 + \delta]) \cup \mathcal{C}(\nu_0; [\overline{\lambda}(\nu_0), \overline{\lambda}(\nu_0 + \delta)]) . \] (3.36)

3) Finally on the portion of the boundary of
\[ \mathcal{K}(\tau_0 + \delta) = \mathcal{K}(\tau_0) \cup \Delta_1 \mathcal{K} \cup \Delta_2 \mathcal{K} \cup \Delta_3 \mathcal{K} , \]
made by \( \mathcal{C}(\nu_0 + \delta; [\lambda_1, \overline{\lambda}(\nu_0 + \delta)]) \) \(^{77}\) a new canonical foliation has to be constructed which stays near to the one obtained extending the double null canonical foliation of \( \mathcal{K}(\tau_0) \) up to \( \mathcal{C}(\nu_0 + \delta; [\lambda_1, \overline{\lambda}(\nu_0 + \delta)]) \) which will be considered, in this case, as the background foliation.

This completes the description of the steps needed to prove our result. The detailed proof will be given in a subsequent paper.

**Second part:**

In this case, as we said, we already know that the set \( \mathcal{N} \) is not empty. To prove that \( \nu_0 = \infty \) we have to show that, if \( \nu_0 \) were finite, the spacetime \( \mathcal{M}(\nu_0) \) could be extended to a spacetime \( \mathcal{M}(\nu_0 + \delta) \) with the same properties. To prove it we need an existence theorem for the strip
\[ J^+(S(\lambda_1, \nu_0)) \cap J^-(S(\lambda_0, \nu_0 + \delta)) \]
Again, as in the case of \( \Delta_1 \mathcal{K} \) and \( \Delta_2 \mathcal{K} \), this existence theorem is a non local result as the “length” of the strip is not required to be small in the \( e_3 \) direction. Therefore as in previous cases the proof requires a bootstrap mechanism.

Finally as at the end of the proof of the first part we have to show that \( \mathcal{M}(\nu_0 + \delta) \) can be endowed with a double canonical foliation whose existence can be proved once we prove the existence of a canonical foliation on the “last slice”, see [Kl-Ni] Chapter 3, on \( \mathcal{C}(\nu_0 + \delta; [\lambda_1, \lambda_0]) \).

\(^{77}\)The canonical foliation on \( \mathcal{C}(\lambda_1; [\nu_0, \nu_0 + \delta]) \) was already proved and it does not change.
4 The “harmonic” initial data

In this section we use the results of A.Rendall, [Ren] and of H.Muller Zum Hagen [Mu] to provide the local existence result that we need to start the bootstrap mechanism for our problem.

The main difficulty in adapting these results to our case is that they have been proved using harmonic coordinates while we want that the “gauge” for $K(\tau)$ be the one associated to the double null canonical foliation. This requires stating the connection between the initial data written in the two different gauges so that we can reexpress their results in our formalism.

4.1 The relation between the harmonic and the $\Omega$-foliation gauges

4.1.1 The reduction of the Einstein equations to hyperbolic equations in the characteristic case

As it is well known the standard procedure to solve locally the Einstein equations has been to find a “gauge” which reduces them to hyperbolic P.D.E. equations.\textsuperscript{78}

The “reduction” mechanism for the Einstein equations is based on the fact that if the initial data satisfy, besides the constraint equations, the conditions

$$\Gamma^\mu = g^{\rho\sigma} \Gamma^\mu_{\rho\sigma} = 0,$$

then it can be proved that the equality $\Gamma^\mu = 0$ holds in the whole spacetime $(\mathcal{M}, g)$ where $g$ is solution of the “reduced” Einstein equations, (the Einstein equations where $\Gamma^\mu$ is posed, ab initio, equal to zero). This fact is relevant as the reduced Einstein equations are of (quasilinear) hyperbolic type.

The strategy to find a “gauge”, that is a change of coordinates, such that on the initial hypersurface $\Gamma^\mu = 0$, is well known in the non characteristic case. This “gauge” is usually called “harmonic” gauge. A similar approach can be also used in the characteristic case. In particular this has been investigated in great detail by A.Rendall, [Ren].

Here we show how to express the initial data in a different family of gauges in terms of data expressed in the harmonic gauge and viceversa.

More precisely, specified our initial data null hypersurface, we define on it the “$\Omega$-foliation”, described in the previous sections. Then we introduce another foliation of the same null hypersurface, whose leaves are different from those of the $\Omega$-foliation, but still diffeomorphic to $S^2$ surfaces, which has the property of being the foliation associated to the “harmonic gauge”. This foliation will be called the “harmonic foliation”.

The advantage of this approach is that it keeps separated in a clear way the two main difficulties connected with the choice of the initial data for the characteristic problem.

\textsuperscript{78}Different approaches have been used to solve the equations globally, see [Ch-Kl], [Kl-Ni], [Ro-Li].
The first one, discussed at length in the previous sections, is that not all the initial data can be freely assigned on the initial hypersurface; some of them have to satisfy “transport equations” and therefore depend, through these equations, on the initial data assigned on the specific leaf $S_0$. This problem is not related to the harmonic gauge and to the reduction problem; whatever “gauge” we choose, not all the initial data can be assigned freely. In particular the “normal” derivatives with respect to the null hypersurfaces of the components of the metric tensor have to satisfy some transport equations.\footnote{See the previous sections. This has also been discussed in detail by A.Rendall, [Ren] in a more coordinate dependent way.}

The second difficulty is associated to the “reduction problem”. Namely to solve the Einstein equations in the “reduced form”, the “reduction problem” one has to implement the “harmonic” condition $\Gamma^\mu = 0$.

Once we are able to compare in a precise way the “$\Omega$-foliation” and the “harmonic” foliation, we can express the norms of data for the “harmonic” situation in terms of the norms of the “$\Omega$-foliation” data and viceversa. This is important for us as we will choose the initial data in the “$\Omega$-foliation” setting such that the corresponding data expressed in the harmonic gauge are suitable for using the local existence theorems for the characteristic problem in the harmonic gauge, see H.Muller Zum Hagen, [Mu] and M.Dossa, [Do].

### 4.1.2 The “harmonic” null frame.

Let us recall that the null initial hypersurface $C$ has been thought as immersed in the four dimensional manifold $\mathbb{R}^4$ endowed with a Lorentzian metric $\tilde{g}$. We introduced in $\mathbb{R}^4$ the coordinates $\{\tilde{u}, u, \omega^a\}$ and we required that $\tilde{g}$ has, in these coordinates, the following expression, see 1.64:

$$
\tilde{g} = |\tilde{X}|^2 du^2 - 2\tilde{\Omega}^2 (d\tilde{u} + du) - \tilde{X}_a (du d\omega^a + dw^a du) + \tilde{\gamma}_{ab} d\omega^a d\omega^b.
$$

(4.1)

With this choice of coordinates the restriction on $C = C_0 \cup C_0'$ of the various components of the metric $\tilde{g}$ has been specified in the previous sections during the initial data construction, see 1.67. Moreover adapted to the foliation specified by $\Omega$ and $\bar{\Omega}$, see 1.68 a null orthonormal frame is specified $\{e_4, e_3, e_a\}$,

$$
e_4 = 2\Omega L, \quad e_3 = 2\Omega \bar{L}, \quad e_a = e_a^b \frac{\partial}{\partial \omega^b}.
$$

where\footnote{Here $X$ is $X$ when defined on $\tilde{C}_0$.}

$$
L = \frac{1}{2\bar{\Omega}^2} \frac{\partial}{\partial \bar{u}}, \quad \bar{L} = \frac{1}{2\tilde{\Omega}^2} \left( \frac{\partial}{\partial u} + X \right)
$$

(4.2)

and $X = X^a \frac{\partial}{\partial \omega^a}$. Therefore

$$
e_4 = \Omega^{-1} N = \Omega^{-1} \frac{\partial}{\partial \bar{u}}
$$
\[ e_3 = \Omega^{-1} N = \Omega^{-1} \left( \frac{\partial}{\partial u} + X \right) \]  
\[ e_A = e^A_{\alpha} \frac{\partial}{\partial \omega^\alpha} \]  

We introduce now on \( C \) a different foliation, we call “harmonic foliation”, tied to the harmonic coordinates, denoted \( \{x^1, x^2, x^a\} \). The harmonic coordinates as functions of the \( \{u, \omega^a\} \) coordinates, \( x^\mu(u, \omega^a) \), satisfy the wave equations

\[ \Box g x^\mu = 0 \]  

where the D’Alembertian \( \Box g \) is written in the coordinates \( \{u, \omega^a\} \) and \( g \) is the metric \( \ref{metric} \). Explicitly, equation \( \ref{wave_eq} \) has the following form:

\[ \tilde{g}^{\alpha\sigma} \partial_\alpha \partial_\sigma x^\mu - \tilde{g}^{\alpha\sigma} \Gamma^\lambda_{\rho\sigma} \partial_\lambda x^\mu = 0 \]  

\( \tilde{g} \) are the components of the D’Alembertian in the coordinates \( \{u, \omega^a\} \). To define the “harmonic foliation” we need that the coordinates \( \{x^1, x^2, x^a\} \) satisfy, on \( C_0 \), the argument for the hypersurface \( C_0 \) is analogous and will be done later on. The new harmonic coordinates, (they will be made to satisfy \( \ref{wave_eq} \)), are denoted \( \{x^1, x^2, x^3, x^4\} \) following Rendall, \([\text{Ren}]\). We assume that the null outgoing “cone” \( C_0 \), in these coordinates, is the level hypersurface \( \{x^2 = 0\} \left( C_0 \right) \), the level hypersurface \( \{x^1 = 0\} \) and we require that the metric tensor has, restricted to \( C_0 \), the following expression:

\[ \tilde{g} \big|_{C_0} = g'_{22} dx^2 dx^2 + g'_{12} (dx^1 dx^2 + dx^2 dx^1) + g'_{2a} (dx^2 dx^a + dx^a dx^2) + \gamma'_{ab} dx^a dx^b . \]  

We specify the coordinate \( x^1 \) on \( C_0 \) as the affine parameter of the null geodesics generating \( C_0 \). Therefore, denoting \( L \) their tangent vector fields, it follows that \( L = \frac{\partial}{\partial x^1} \).

We introduce a foliation on \( C_0 \), we call “harmonic-foliation”, defining the leaves of the foliation, \( \{S'_0(\nu')\} \), in the following way:

\[ S'_0(\nu') = \{p \in C_0 | x^1(p) = \nu'\} . \]  

We define the coordinates \( x^a, a \in \{3, 4\} \), as adapted to these leaves (the analogues of the \( \omega^a \) in the \( \Omega \)-foliation) and an orthonormal frame tangent to each leave \( S'_0(\nu') \),

\[ e'_A = e^A_{\alpha} \frac{\partial}{\partial x^\alpha} . \]  

We consider the null frame \( \{L, e'_A\} \) relative to the outgoing cone \( C_0 \) and we extend it to a null frame \( \{L, L^*, e'_A\} \), we call “harmonic null frame”, satisfying the conditions

\[ g(L, L^*) = -1 , \quad g(L^*, L^*) = 0 , \quad g(L^*, e'_A) = 0 . \]  

\( \text{\footnotesize[81]} \)It is important to recognize that, differently from the case of the \( \{u, \omega^a\} \) coordinates, the other metric components in the harmonic coordinates are different from zero when not restricted to \( C_0 \), for instance the component \( g'_{11} \) is different from zero. Moreover the expression of the metric when restricted to \( C_0 \) will be different. This will be discussed in more detail later on.
The expression of the metric \(4.6\), with \(g'_{2a} \neq 0\), and the condition \(g(L, L^*) = -1\) imply that \(L^*\) has to be written as
\[
L^* = \sigma \frac{\partial}{\partial x^2} + X' 
\] (4.10)
with \(X' = X'_A e'_A\). The remaining conditions \(4.9\) determine the relations between \(\sigma, X'_a = \gamma'_{ab} X'^b\) and the various components of the metric \(4.6\):
\[
\begin{align*}
2\sigma g'_{2a} X'^a &= - (\sigma^2 g'_{22} + |X'|^2) , \quad \sigma g'_{12} = -1 \\
\sigma g'_{2a} &= -X'_a , \quad \sigma^2 g'_{22} = |X'|^2 .
\end{align*}
\] (4.11)

The metric tensor \(\tilde{g}|_{C_0}\), in the harmonic coordinates, can be rewritten as
\[
g|_{C_0} = \sigma^{-2} |X'|^2 dx^2 dx^2 - \sigma^{-1}(dx^1 dx^2 + dx^2 dx^1) \\
- \sigma^{-1} X'_a (dx^a dx^2 + dx^2 dx^a) + \gamma'_{ab} dx^a dx^b 
\] (4.12)
where \(\sigma\) and \(X'_a\) can be thought as extensions in \(R^4\) of the quantities defined on \(C_0\).

Summarizing, the “harmonic null frame” \(\{L, L^*, e'_A\}\), on \(C_0\), written in the “harmonic” coordinates, has the following expression:
\[
\begin{align*}
L &= \frac{\partial}{\partial x^1} \\
L^* &= \sigma \frac{\partial}{\partial x^2} + X' \\
e'_A &= e'_A \frac{\partial}{\partial x^a}.
\end{align*}
\] (4.13)

To justify the name of the harmonic null frame for \(\{L, L^*, e'_A\}\) we have to impose that the \(\{x^1, x^2, x^{3,4}\}\) be harmonic coordinates. To relate these coordinates to the \(\{u, u^a, \omega^a\}\) ones it turns out convenient to look first at the relation between the \(\Omega\)-null frame and the harmonic null frame. Therefore we express the “harmonic null frame” in terms of the \(\Omega\)-null frame \(\{e_4, e_3, e_A\}\) writing
\[
\begin{align*}
L &= (2\Omega)^{-1} e_4 \\
L^* &= \alpha e_3 + \beta e_4 + \delta_A e_A \\
e'_A &= \gamma e_A + \rho_A e_4 + \sigma_A e_3 .
\end{align*}
\] (4.14)

where all the coefficients are functions defined on \(C\). As both frames are null orthonormal the following relations hold:
\[
\alpha = \Omega , \quad \beta = \frac{|\delta|^2}{4\Omega} , \quad \gamma = 1 , \quad \rho_A = \frac{\delta_A}{2\Omega} , \quad \sigma_A = 0 
\] (4.15)

and, therefore,
\[
\begin{align*}
L &= (2\Omega)^{-1} e_4 \\
L^* &= \Omega e_3 + \frac{|\delta|^2}{4\Omega} e_4 + \delta_A e_A \\
e'_A &= e_A + \frac{\delta_A}{2\Omega} e_4
\end{align*}
\] (4.16)

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From 4.5, 4.13, 4.16 we obtain immediately the explicit expressions of the first derivatives of the functions \(x^\mu(u, u, \omega^a)\). The relation between \(L\) and \(e_4\) gives immediately

\[
\frac{\partial}{\partial u} = 2\Omega^2 \frac{\partial}{\partial x_1}.
\]  

(4.17)

Let us consider the relation

\[
e_3 = \Omega^{-1}\left(\frac{\partial}{\partial u} + X\right) = \Omega^{-1}\left(\frac{|\delta|^2}{2} \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_2} + (X^a - \delta^a) \frac{\partial}{\partial x^a}\right)
\]

(4.18)

where \(\delta^a = \delta^A e_A^a\). From it follows

\[
\frac{\partial}{\partial u} = \sigma \frac{\partial}{\partial x_2} + \left(\frac{|\delta|^2}{2} + \delta \cdot X\right) \frac{\partial}{\partial x_1} + (\Delta_A - \delta_A) e_A^a \frac{\partial}{\partial x^a}.
\]

(4.19)

where \(\Delta_A \equiv X_A' - X_A\). Finally the relation

\[
e_A^a \frac{\partial}{\partial \omega^a} = -\delta_A \frac{\partial}{\partial x_1} + e_A^a \frac{\partial}{\partial x^a}
\]

(4.20)

implies

\[
\frac{\partial}{\partial \omega^a} = -\theta_A^a \delta_A \frac{\partial}{\partial x_1} + \theta_A^a e_A^c \frac{\partial}{\partial x^c}
\]

(4.21)

where \(\theta_A = \theta_A^a d\omega^a\) is the one form associated to the vector \(e_A\), \(\theta_A(e_B) = \delta^A_B\).

Collecting all these results we have

\[
\frac{\partial}{\partial u} = 2\Omega^2 \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial u} = \left(\frac{|\delta|^2}{2} + \delta \cdot X\right) \frac{\partial}{\partial x_1} + \sigma \frac{\partial}{\partial x_2} + (\Delta_A - \delta_A) e_A^a \frac{\partial}{\partial x^a},
\]

(4.22)

and, immediately,

\[
\frac{\partial x_1}{\partial u} = 2\Omega^2, \quad \frac{\partial x_2}{\partial u} = 0, \quad \frac{\partial x^a}{\partial u} = 0
\]

\[
\frac{\partial x_1}{\partial u} = \left(\frac{|\delta|^2}{2} + \delta \cdot X\right), \quad \frac{\partial x^2}{\partial u} = \sigma, \quad \frac{\partial x^a}{\partial u} = (\Delta_A - \delta_A) e_A^a
\]

\[
\frac{\partial x_1}{\partial \omega^b} = -\theta_A^a \delta_A, \quad \frac{\partial x_2}{\partial \omega^b} = 0, \quad \frac{\partial x^a}{\partial \omega^b} = \theta_A^a e_A^a.
\]

(4.23)

The null orthonormal frame \(\{L, L^*, e_A^a\}\) is harmonic if the coordinates \(\{x_1, x_2, x^a\}\) are harmonic, that is if the functions \(x^\mu(u, u, \omega^a)\) satisfy the wave equation 4.4,

\[
\hat{\gamma}^{\rho\sigma} \partial_{\rho} \partial_{\sigma} x^\mu - \hat{\gamma}^{\rho\sigma} \Gamma_{\rho\sigma}^\lambda \partial_\lambda x^\mu = 0.
\]

(4.24)
The components of the inverse metric $g^{\rho\sigma}$ different from zero are

$$g^{uu} = -(2\Omega^2)^{-1}, \quad g^{au} = -\frac{X^a}{2\Omega^2}, \quad g^{ab} = \gamma^{ab}$$

(4.25)

and, therefore, 4.24 takes the form

$$\frac{1}{2\Omega^2} \partial_{\rho} \partial_{\sigma} x^\mu - \frac{X^a}{2\Omega^2} \partial_{\rho} \partial_{a} x^\mu + \frac{1}{2} \gamma^{ab} \partial_{a} x^\mu$$

$$+ \frac{1}{2\Omega^2} \left( \Gamma^u_{\rho\mu} \partial_{u} x^\mu + \Gamma^a_{\rho\mu} \partial_{a} x^\mu + \Gamma^{\mu} \partial_{\mu} x^\mu \right)$$

$$+ \frac{X^a}{2\Omega^2} \left( \Gamma^u_{a\mu} \partial_{u} x^\mu + \Gamma^a_{a\mu} \partial_{a} x^\mu + \Gamma^{\mu} \partial_{a} x^\mu \right)$$

$$= 0$$

(4.26)

$$\frac{1}{2} \gamma^{ab} \left( \Gamma^u_{ab} \partial_{u} x^\mu + \Gamma^a_{ab} \partial_{a} x^\mu + \Gamma^{b} \partial_{b} x^\mu \right) = 0$$

The Christoffel symbols $\Gamma^\rho_{\mu\sigma}$ relative to the coordinates $u, u, \omega^a$ have been previously expressed in terms of the connection coefficients, see subsection 1.4.1. Therefore, once the initial data in the “$\Omega$-foliation” are assigned, all the factors which, in equation 4.26, multiply the first and second partial derivatives are known (on $C_0^0$).

Writing the explicit expressions for equations 4.26 for each value of $\mu$, using 4.23 and the expressions of the connection coefficients given in 1.73, we obtain:

$$\frac{\partial}{\partial u} T + \frac{\Omega^{tr} X}{2} - \frac{\eta}{2} (\delta) + \Omega^2 (\nabla A) \delta + \frac{\Omega^{tr} X}{2} \cdot \delta - \left( \nabla \log \Omega - \Omega^2 (\Omega^{tr} X) \right) = 0$$

$$\frac{\partial}{\partial u} \sigma + \Omega^{tr} X = 0$$

$$\frac{\partial}{\partial u} W^a + \frac{\Omega^{tr} X}{2} W^a = \left[ \Omega^2 A X^a - \left( \frac{\Omega^{tr} X}{2} X^c + \eta^c \right) \partial_c X^a \right] = 0,$$

where

$$T = \frac{1}{2} \delta^2 + \delta \cdot X,$$  \quad $$W^a = (\Delta A - \delta A) e^a_A$$

$$\delta = \delta^a \frac{\partial}{\partial \omega^a}, \quad X = X^a \frac{\partial}{\partial \omega^a}, \quad X^a = X^a \frac{\partial}{\partial x^a}$$

$$\Delta A = X^a_A - X_A.$$

A simplification of 4.27 is obtained if we choose $e^a_A = e^a_A$. In this case $W^a = (X' - \delta^a) - X^a$. The terms in brackets of the right hand side of 4.27 are known terms as they depend on the initial data associated to the $\Omega$-foliation. Therefore their decay is known. More precisely we have

$$\left[ \nabla \log \Omega - \Omega^2 (\Omega^{tr} X) \right] = O\left( \frac{1}{r} \right)$$

$$\left[ \Omega^2 A X^a - \left( \frac{\Omega^{tr} X}{2} X^c + \eta^c \right) \partial_c X^a \right] = O\left( \frac{\varepsilon}{r^3} \right).$$

(4.29)

Observe that the only Christoffel symbol which cannot be directly expressed in terms of the connection coefficients, namely $\Gamma^u_{uu}$, see subsections 1.4.1, 2.1.2, does not appear.
Using these inequalities we prove the following lemma,

**Lemma 4.1** Assume we control the norm \(| \cdot |_{p,S} , p \in [2, 4], \) for the connection coefficients relative to the \( \Omega \)-foliation and their first derivatives, then, assuming \( \delta |_{\mathcal{S}_0} = 0, \sigma |_{\mathcal{S}_0} = 2 \Omega^2, \) \( X^\nu |_{\mathcal{S}_0} = 0, \) equations 4.27 have a solution along the whole \( C_0 \) such that the following estimates hold

\[
\sigma = O\left( \frac{C}{r} \right), \quad \delta^a = O\left( \frac{\varepsilon}{r^2} \right), \quad \nabla^a = O\left( \frac{\varepsilon}{r^2} \right), \quad (X^\nu - \delta^a) = O\left( \frac{\varepsilon}{r^2} \right) \quad (4.30)
\]

Moreover \( \delta \cdot X = O\left( \frac{\varepsilon}{r} \right) \).

**Proof:** Observe that \( \delta \cdot X \) satisfies \( | \delta \cdot X | \leq c \varepsilon r^{-1} | \delta |. \) An application of Gronwall’s Lemma to the first equation in 4.27 gives

\[
| r^{1-\frac{2}{p}} T |_{p,S}(\lambda_1, \nu) \leq c | r^{1-\frac{2}{p}} T |_{p,S}(\lambda_1, \nu_0) + \int_{v_0}^{v} | r^{1-\frac{2}{p}} (\eta(\delta) - (\nabla_A \delta) A - \frac{\Omega \text{tr} X \cdot \delta}) |_{p,S} \nonumber \\
+ \int_{v_0}^{v} | r^{1-\frac{2}{p}} \left[ \nabla_X \log \Omega - \Omega^2 (\Omega \text{tr} X) \right] |_{p,S}. \quad (4.31)
\]

As

\[
\left[ \nabla_X \log \Omega - \Omega^2 (\Omega \text{tr} X) \right] = O \left( \frac{1}{r} \right),
\]

the last integral is bounded by \( c \varepsilon (\lambda_1, \nu) \). Dividing both members by \( r(\lambda_1, \nu) \) and observing that \( r^{-1}(\lambda_1, \nu) \leq r^{-1}(\lambda_1, \nu') \leq r^{-1}(\lambda_1, \nu_0) \) we can write

\[
| T |_{p,S}(\lambda_1, \nu) \leq c \left( \frac{| r^{1-\frac{2}{p}} T |_{p,S}(\lambda_1, \nu_0)}{r(\lambda_1, \nu)} + \frac{1}{r(\lambda_1, \nu)} \int_{v_0}^{v} | r^{1-\frac{2}{p}} (| \eta(\delta) | + | (\nabla_A \delta) A |) \right. \\
\quad \left. + | \text{tr} X \cdot \delta | |_{p,S} + 1 \right) \quad (4.32)
\]

Observe that \( \eta_\delta = O(\varepsilon r^{-1}), \delta^a \leq c \varepsilon r^{-1} | \delta |, \) it follows that \( \eta(\delta) = O(\varepsilon r^{-2} | \delta |), \) \( | \text{tr} X \cdot \delta | \leq c O(\varepsilon r^{-2} | \delta |), \) therefore

\[
| r^{1-\frac{2}{p}} \eta (\delta) |_{p,S} \leq c \frac{\varepsilon}{r^2} | r^{1-\frac{2}{p}} | \delta | |_{p,S}, \quad | r^{1-\frac{2}{p}} \text{tr} X \cdot \delta |_{p,S} \leq c \frac{\varepsilon}{r^2} | r^{1-\frac{2}{p}} | \delta | |_{p,S}. \quad (4.33)
\]

Moreover, as we are free to assign \( \nabla \delta \) along \( C_0, \) we require

\[
| r^{1-\frac{2}{p}} (\nabla_A \delta) A |_{p,S} \leq c \frac{\varepsilon}{r^2} | r^{1-\frac{2}{p}} | \delta | |_{p,S}.
\]

In conclusion we expect that \( \left( \frac{d^2}{dr^2} + \delta \cdot X \right) \) stays bounded along \( C_0. \) In fact

\[
| T |_{p,S}(\lambda_1, \nu) \leq c \left( 1 + \frac{1}{r(\lambda_1, \nu)} \left( | r^{1-\frac{2}{p}} T |_{p,S}(\lambda_1, \nu_0) + \varepsilon \log r(\lambda_1, \nu) \right) \right)
\]

\[83\)To pass from the \( | \cdot |_{p,S} \) norm estimates to the pointwise ones we need the analogous estimates for the first tangential derivatives which can be obtained exactly in the same way, provided we have sufficient norm estimates for the derivatives of the connection coefficients relative to the \( \Omega \) foliation.\]
As $|\delta \cdot X| \leq |\delta||X| \leq |\delta|^2 + O(\varepsilon^2 r^{-2})$ it follows that $|\delta|^2 = O(1)$ which implies $\delta^a = O(r^{-1})$ and $\delta \cdot X = O(\varepsilon r^{-1})$.

The estimate $\sigma = O(r^{-1})$ follows immediately from the second equation. Finally, writing for the third equation an estimate analogous to 4.32, we obtain, recalling 4.29, denoting $|W|^2 = \gamma_{\alpha\beta}W^\alpha W^\beta$ and proceeding as in Chapter 4 of [Kl-Ni],

$$|W|_{\mu,S}(\lambda_1,\nu) \leq c \left( \frac{|r^{1-\frac{3}{2}}|W||_{\mu,S}(\lambda_1,\nu_0)}{r(\lambda_1,\nu)} \right)^{\frac{1}{2}} + \frac{1}{r(\lambda_1,\nu)} \int_{\nu_0}^{\nu} |r^{1-\frac{3}{2}} \left[ \Omega^2 \Delta X^a - \left( \frac{\Omega_{\mu} \chi}{2} X^c + \eta^c \right) \partial \mu X^a \right]|_{\mu,S} \right) \leq c \frac{1}{r(\lambda_1,\nu)} \left( |r^{1-\frac{3}{2}} |W||_{\mu,S}(\lambda_1,\nu_0) + \varepsilon \int_{\nu_0}^{\nu} |r^{1-\frac{3}{2}} r^{-3}|_{\mu,S} \right).$$

This estimate together with the analogous one for the first tangential derivative of $W$ implies that $|W| = |\gamma_{\alpha\beta}W^\alpha W^\beta| = O(\varepsilon r^{-1})$.

Therefore $W^\alpha = O(r^{-2})$ and as $X^a = O(\varepsilon r^{-2})$ and $\delta^a = O(\varepsilon r^{-1})$, it follows that $(X^a - \delta^a) = O(r^{-2})$.

Once we have solved equations 4.27 we have obtained, on $C_0$, the first derivatives of the harmonic coordinates written in terms of the $\Omega$-coordinates, see 4.41. This implies that we have the metric components in the harmonic coordinates, once we have the corresponding quantities written in the $\Omega$-coordinates. To have the whole initial data required to solve locally the characteristic problem in the harmonic coordinates we need also the first derivatives of the metric along the null direction “orthogonal” to $C_0$. In principle one can rely, for instance, on the Muller Zum Hagen result, see [Mu], where these derivatives are obtained using the constraint equations, but in the present case this would be a useless repetition as we have already done it in the $\Omega$-coordinates. Therefore the only thing to do is to obtain the second derivatives of the harmonic coordinates with respect to the “$\Omega$” ones. We achieve this result proceeding as done before for the first derivatives. We start deriving equations 4.24

$$\tilde{g}^{\alpha\beta} \partial_\alpha \partial_\beta (\partial_\tau x^\mu) - \tilde{g}^{\alpha\beta} \Gamma^\lambda_\alpha \partial_\lambda (\partial_\tau x^\mu) - (\partial_\tau \tilde{g}^{\alpha\beta}) \Gamma^\lambda_\alpha \partial_\lambda x^\mu - \tilde{g}^{\alpha\beta} (\partial_\tau \Gamma^\lambda_\alpha) \partial_\lambda x^\mu = 0 \quad (4.35)$$

and observing that, from equations 4.41 we have

$$\frac{\partial^2 x^1}{\partial u^2} = -8\Omega^3 \omega , \quad \frac{\partial^2 x^1}{\partial \omega^2} = \frac{\partial T}{\partial u} , \quad \frac{\partial^2 x^1}{\partial u^2} = \frac{\partial T}{\partial u}$$

$$\frac{\partial^2 x^1}{\partial u^2} = \frac{\partial^2 T}{\partial u^2} = \frac{\partial^2 x^1}{\partial \omega^2} = \frac{\partial^2 \delta_b}{\partial \omega^a} , \quad \frac{\partial^2 x^1}{\partial \omega^2} = \frac{\partial T}{\partial \omega^a}$$

$$\frac{\partial^2 x^2}{\partial u^2} = 0 , \quad \frac{\partial^2 x^2}{\partial u^2} = -(\Omega \partial \chi) \sigma , \quad \frac{\partial^2 x^2}{\partial u^2} = \frac{\partial \sigma}{\partial u} \sigma$$

\[84\text{In fact even more complicated, see for instance } \text{[Ren].}\]
\[
\frac{\partial^2 x^2}{\partial \omega^a \partial u^b} = 0,
\frac{\partial^2 x^2}{\partial \omega^a \partial \omega^b} = 0,
\frac{\partial^2 x^2}{\partial \omega^a \partial u} = \frac{\partial \sigma}{\partial u},
\frac{\partial^2 x^2}{\partial \omega^a \partial \omega^b} = 0,
\frac{\partial^2 x^2}{\partial u^a \partial u^b} = \frac{\partial W^a}{\partial u},
\frac{\partial^2 x^2}{\partial \omega^a \partial u^b} = 0,\]

All these second derivatives are easily obtained with the exception of

\[
\frac{\partial^2 x^1}{\partial u^2} = \frac{\partial T}{\partial u},
\frac{\partial^2 x^2}{\partial u^2} = \frac{\partial \sigma}{\partial u},
\frac{\partial^2 x^a}{\partial u^a} = \frac{\partial W^a}{\partial u}.
\]

To obtain and control these derivatives we have to use equation 4.35 with \(\tau = u\). It is easy to see that they provide, for these quantities, evolution equations along \(C_0\) of the same type as equations 4.27 which can be solved in the same way. This completes the knowledge of the second derivatives \(\frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma}\), where \(\{y^\tau\} = \{u, \omega^a\}\).

Once we know these second derivatives we can express the first partial derivatives of the metric components with respect the \(\{u, \omega^a\}\) coordinates in terms of the first partial derivatives of the metric components \(g'_{\mu\nu}\) with respect the harmonic coordinates:

\[
\frac{\partial}{\partial y^\lambda} g_{\rho\sigma} = \left(\frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma} + \frac{\partial^2 x^\nu}{\partial y^\rho \partial y^\tau} \frac{\partial x^\tau}{\partial y^\lambda}\right) g'_{\mu\nu} + \left(\frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma} \frac{\partial x^\tau}{\partial y^\lambda}\right) \left(\frac{\partial}{\partial x^\tau} g'_{\mu\nu}\right) \quad (4.36)
\]

As \(\{\frac{\partial x^\mu}{\partial y^\rho} \frac{\partial x^\nu}{\partial y^\sigma}\}\) is an invertible matrix \(64 \times 64\) we can solve the linear system 4.36 obtaining the initial data for the characteristic problem in the harmonic coordinates, the derivatives along the direction “orthonormal” to \(C_0\) satisfying the constraints.

**Remark:** A detailed examination along the same lines of Lemma 4.1 would allow to control the decay of these quantities along \(C_0\). Nevertheless this is not relevant here as we will use the harmonic coordinates only to prove the local existence of the vacuum Einstein characteristic problem in a small region.

### 4.1.3 The harmonic coordinates on the incoming cone.

We define on \(C_0\) the metric components, following Rendall, [Ren],

\[
g|_{C_0} = g'_{11} dx^1 dx^1 + g'_{12}(dx^1 dx^2 + dx^2 dx^1) + g'_{1a}(dx^1 dx^a + dx^a dx^1) + \gamma'_{ab} dx^a dx^b. \quad (4.37)
\]

Proceeding as before, the analogous of equations 4.13 are

\[
\begin{align*}
L &= \frac{\partial}{\partial x^2} \\
L^* &= \sigma \frac{\partial}{\partial x^1} + X \\
e'_A &= c_A \frac{\partial}{\partial x^a}.
\end{align*}
\]

\[65\]
and for consistency with the previous case we require on $S_0$, $\sigma = 2\Omega^2$, $\bar{X} = 0$. The relation between this null frame and the $\Omega$-frame is

$$L = (2\Omega)^{-1}e_3$$
$$L^* = \Omega e_4 + \frac{\sqrt{d^2}}{4\Omega} e_3 + \bar{\sigma}A e_A$$

(4.39)

$$e'_A = e_A + \frac{\delta_A}{2\Omega} e_3$$

The analogous of relations 4.22 are

$$\frac{\partial}{\partial u} = (2\Omega^2 - X_A \delta A) \frac{\partial}{\partial x^2} - X_A e'_A \frac{\partial}{\partial x^a}$$
$$\frac{\partial}{\partial u} = \sigma \frac{\partial}{\partial x^1} + \frac{\sqrt{d^2}}{2} \frac{\partial}{\partial x^2} + (X_A - \delta A) e'_A \frac{\partial}{\partial x^a}$$

(4.40)

$$\frac{\partial}{\partial \omega^a} = -\theta^A_A \delta A \frac{\partial}{\partial x^2} + \theta^A_A e'_A \frac{\partial}{\partial x^a}$$

and immediately,

$$\frac{\partial x^1}{\partial u} = \sigma , \quad \frac{\partial x^2}{\partial u} = \frac{\sqrt{d^2}}{2} , \quad \frac{\partial x^a}{\partial u} = (X^a - \delta^a)$$

$$\frac{\partial x^1}{\partial u} = 0 , \quad \frac{\partial x^2}{\partial u} = (2\Omega^2 - X_A \delta A) , \quad \frac{\partial x^a}{\partial u} = -X^a$$

$$\frac{\partial x^1}{\partial \omega^b} = 0 , \quad \frac{\partial x^2}{\partial \omega^b} = -\theta^A_A \delta A , \quad \frac{\partial x^a}{\partial \omega^b} = \theta^A_A e'_A .$$

(4.41)

Equations 4.26 take the form

$$\left( \frac{\partial}{\partial u} + X \right) T + \frac{\Omega \operatorname{tr} \chi}{2} T + \left[ \eta \bar{\sigma} + \Omega^2 (\nabla A \bar{\sigma}) \right] A + \Omega \operatorname{tr} \chi (\Omega^2 - X \cdot \delta) = 0$$

$$\left( \frac{\partial}{\partial u} + X \right) \bar{T} + \frac{\Omega \operatorname{tr} \chi}{2} \bar{T} = 0$$

(4.42)

$$\left( \frac{\partial}{\partial u} + X \right) W^a + \frac{\Omega \operatorname{tr} \chi}{2} W^a + \Omega^2 \gamma^{cb} (\bar{T}^{a}_{cb} - \eta^a) = 0$$

where

$$T = \frac{\sqrt{d^2}}{2} , \quad W^a = (X'_A - \bar{\sigma}_A)e'_a$$

(4.43)

As in the case of the outgoing cone we can control the norms of the solutions of 4.42. The main difference is that in this case moving forward in time along $C_0$, the radius of the leaves is decreasing, therefore we do not have the analogous of the previous decay estimates. Moreover, as discussed in the previous section,
where the strategy for the global existence proof is discussed, the harmonic coordinates and the initial data expressed through them are needed only to obtain a local existence proof of the characteristic problem, therefore the way the harmonic null frame deviates from the $\Omega$-null frame moving along the whole cones $C_0$ and $\tilde{C}_0$ is not relevant here.

**Lemma 4.2** Let us assume we control the norm $| \cdot |_{p,S}$, $p \in [2,4]$, for the connection coefficients relative to the $\Omega$-foliation and their first derivatives. On $S_0$ we require that

$$
\delta|_{S_0} = 0, \quad \sigma|_{S_0} = 2\Omega_0^2, \quad X'_{a}|_{S_0} = 0,
$$

Then equations 4.42 have a solution along the whole $\tilde{C}_0$ such that the following estimates hold:

$$
\sigma = O\left(\frac{c_0}{r}\right), \quad \delta^a = O\left(\frac{c_0}{r}\right), \quad W^a = O\left(\frac{c_0}{r^2}\right), \quad (X^{a'} - \delta^a) = O\left(\frac{c_0}{r^2}\right)
$$

Moreover $\overline{\delta} \cdot X = O\left(\frac{c_0}{r}\right)$ and $(\nabla_A \delta)_A = 0$.

**Remark:** Here $c_0$ depends on $r_0 = r(\lambda_1, \nu_0)$.

**Proof:** The proof goes basically as in the previous Lemma 4.1 and we do not report it here. We only note that in the $\Omega$-null frame $(\frac{\partial}{\partial u} + X) = \Omega e_3$ and the transport equations along $\tilde{C}_0$ and their estimates are of the same type as in Chapter 4 of [Kl-Ni].

**5 Appendix**

5.1 The construction of the background metric $\tilde{\gamma}_{ab}$.

In this subsection we define in a explicit way the background metric $\tilde{\gamma}_{ab}$ which is used as the starting point for the $B$ map in Lemma 2.1 and allows also to define explicetly the norm we use in the proof of the $B$-fixed point existence.

Let us start defining $C_0$ as an embedded hypersurface in $\{R^4, \tilde{\mathbf{g}}\}$ in the following way: we introduce, in $R^4$, the coordinates $\{\bar{u}, u, \theta, \phi\}$, such that the hypersurface $C_0$ is defined by $u = \lambda_1$,

$$
C_0 = \{p \in R^4 | u(p) = \lambda_1\}.
$$

We assume that, in these coordinates, the metric $\tilde{\mathbf{g}}$ restricted to $C_0$ has the form:

$$
\tilde{\mathbf{g}}|_{C_0} = -\frac{1}{2}(d\bar{u}du + dud\bar{u}) + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.1)
$$

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we denote $\tilde{g}'_{\mu\nu}$ the components of $\tilde{g}|_{C_0}$ in these coordinates and $\{\omega'^a\} = \{\theta, \phi\}$. Therefore
\begin{equation}
\tilde{g}'_{uu} = \tilde{g}'_{uv} = \tilde{g}'_{va} = \tilde{g}'_{ua} = 0
\end{equation}
\begin{equation}
\tilde{g}'_{uv} = -\frac{1}{2}, \quad \tilde{g}'_{\theta\theta} = r'^2, \quad \tilde{g}'_{\phi\phi} = r'^2 \sin^2 \theta; \quad r' = \frac{1}{2}(\nu - u).
\end{equation}
The inverse metric satisfies
\begin{equation}
\tilde{g}'_{uu} = \tilde{g}'_{uv} = \tilde{g}'_{va} = \tilde{g}'_{ua} = 0
\end{equation}
\begin{equation}
\tilde{g}'_{uv} = -2, \quad \tilde{g}'_{\theta\theta} = r'^{-2}, \quad \tilde{g}'_{\phi\phi} = r'^{-2} \sin^{-2} \theta.
\end{equation}
which implies that the eikonal equation $g^{\mu\nu} \partial_\mu w \partial_\nu w = 0$ is satisfied by the functions $w(p) = u$ and $w(p) = \nu$. Therefore the level surface $C_0$ is a null hypersurface and the tangent vector field of the null geodesics generating $C_0$ is
\begin{equation}
L = 2 \frac{\partial}{\partial v}.
\end{equation}
Analogously on $C_0$, the tangent field for the incoming null geodesics is
\begin{equation}
L = 2 \frac{\partial}{\partial u}.
\end{equation}
Given these vector fields we define a null orthonormal frame on $C_0$, $\{e'_3, e'_4, e_A\}$ as
\begin{equation}
e'_3 = L , \quad e'_4 = L , \quad e'_A = e'_A a \frac{\partial}{\partial \omega'^a}.
\end{equation}
Let us define on $C_0$ the foliation made by the two dimensional surfaces
\begin{equation}
S'_0(\nu') = \{ p \in C_0 | w(p) = \nu' \}.
\end{equation}
It follows immediately that the second null (outgoing) fundamental form of the $S'_0(\nu')$ surfaces is
\begin{equation}
\tilde{\chi}'_{ab} = \tilde{g}|_{C_0}(D_{\omega'^a} e'_4, \frac{\partial}{\partial \omega'^b}) = \frac{\partial \mu_{ab}}{\partial \omega'^a} = \frac{1}{2} \text{tr} \chi'_{\mu} \mu_{ab} = \frac{1}{r'} \mu_{ab}
\end{equation}
where $\mu$ is the standard metric of $S^2$:
\begin{equation}
\mu_{ab} d\omega'^a d\omega'^b = r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\end{equation}
From 5.7 it follows that $\text{tr} \tilde{\chi}'$ satisfies the following equation, with $\Omega_0 = \frac{1}{2}$,
\begin{equation}
\frac{\partial \text{tr} \tilde{\chi}'}{\partial \omega'} + \frac{1}{4} \text{tr} \tilde{\chi}' = \frac{\partial \text{tr} \tilde{\chi}'}{\partial \omega'} + \frac{\Omega_0}{2} \text{tr} \tilde{\chi}' = 0.
\end{equation}
Let us introduce on $C_0$ a scalar function $\Omega = \Omega(\nu, \theta, \phi)$, and assume that, on $C_0$, $\Omega$ stays near to $\Omega_0$, that is
\begin{equation}
\sup_{C_0} |(1 + \nu^2)^2 (\Omega - \Omega_0)| \leq \epsilon.
\end{equation}
We define on $C_0$ the function $\mathcal{u}(p) = \mathcal{u}(u, \theta, \phi)$ in the following way:

$$\mathcal{u}(u, \theta, \phi) = \nu_0 + \int_0^u \frac{1}{4\Omega^2(u', \theta, \phi)} \, du'$$

(5.11)

and on $C_0$ we define the following leaves, different from the previous $S'_0(\nu')$,

$$S_0(\nu) = \{ p \in C_0 | \mathcal{u}(p) = \nu \}.$$  

(5.12)

Let us introduce on $C_0$ the new coordinates $\{ u, u_a, \omega^a \}$, with $\{ \omega^a \} = \{ \omega^{\alpha} \} = \{ \theta, \phi \}$ and write in these coordinates the restriction $\tilde{g}|_{C_0}$ of the metric $\tilde{g}$ of $R^4$. It is clear that in this case we have some arbitrariness as the components of the metric transform, under a change $\psi$ of coordinates, through relations which depend on the partial derivatives $\partial_x \psi$, of the transformation.

Let us consider the following change of coordinates in a neighborhood $V \subset R^4$ containing $C_0$, $^86$

$$\begin{align*}
\mathcal{u} &= \mathcal{u}(u, \theta, \phi) + \psi^0(u, \theta, \phi) \\
u &= \nu \\
\omega^a &= \omega^a 
\end{align*}$$

(5.13)

which reduces, on $C_0$, to changing only $\mathcal{u}$ into $\tilde{\mathcal{u}}$, if $\psi^0|_{C_0} = 0$. Nevertheless as we do not require that the first derivatives of $\psi$ be zero on $C_0$ the expression of the various components of $\tilde{g}|_{C_0}$ in the $\{ u, u_a, \omega^a \}$ coordinates, $g_{\mu \nu}$, depend on $\psi$.

We ask then that the change of coordinates in $\mathcal{U}$ be such that on $C_0$ the vector fields $L^\mu = -g^{\mu \nu} \partial_{\nu} u$ and $L^\mu = -g^{\mu \nu} \partial_{\nu} \tilde{u}$ be null vector fields. Moreover we require that $L$ be proportional to $\partial_{\mathcal{u}}$. These conditions imply that

$$\tilde{g}_{uu} = \tilde{g}_{uu} = \tilde{g}_{uu} = 0.$$  

Moreover, requiring $\tilde{g}_{uu} = 0$, it follows that $\tilde{g}_{uu} = \bar{g}_{uu}$. $^87$ The change of coordinates 5.13 satisfy this last condition if

$$\frac{\partial \nu}{\partial u} = -\frac{1}{4} \tilde{g}_{ab} \frac{\partial \nu}{\partial \omega^a} \frac{\partial \nu}{\partial \omega^b}.$$  

(5.14)

Therefore with this change of coordinates we have,$^88$

$$\begin{align*}
\tilde{g}_{uu} &= -\frac{1}{2} \frac{\partial \nu}{\partial \omega^a} = -2\Omega^2 \\
\tilde{g}_{uu} &= -\frac{1}{2} \frac{\partial \nu}{\partial \omega^a} = -\Omega^2 \int_0^u \frac{1}{2\Omega^2} \partial_a \log \Omega \, du' \equiv -\tilde{X}_a \\
\tilde{g}_{uu} &= -\frac{\partial \nu}{\partial u} = \tilde{g}_{ab} \tilde{g}_{uu} \tilde{g}_{bu} = |\tilde{X}|^2
\end{align*}$$

(5.15)

$^86$We can choose many other changes of coordinates bringing to the same conclusions.

$^87$In fact the following relation holds $g_{ab} = -\gamma^{ba} g_{uu} g_{uu}$ which with the relation $g_{uu} g_{uu} = -g_{uu} g_{uu}$ following from $g_{uu} = 0$, implies the relation.

$^88$The second equation in 5.15 is obtained deriving equation 5.11, where $\nu = \nu(\nu, \theta, \phi)$, $\nu = \mathcal{u}(\nu, \theta, \phi), \theta, \phi) = \nu_0 + \int_0^u \frac{1}{4\Omega^2(u', \theta, \phi)} \, du'$.  

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and the metric has, on $C_0$, the following expression:
\[ \tilde{g}_{|C_0}(\cdot, \cdot) = |\tilde{X}|^2 du^2 - 2\Omega^2 (dud\nu + d\nu du) - \tilde{X}_a (dud\omega^a + d\omega^a du) + \tilde{\gamma}_{ab} d\omega^a d\omega^b \]
(5.16)
where $\tilde{\gamma}_{ab} = \tilde{g}_{ab}$ and $\tilde{g}_{ab} = \tilde{\gamma}_{ab}$. With these definitions we have on $C_0$
\[ L = \frac{1}{2\Omega^2} \frac{\partial}{\partial u}, \quad L = \frac{1}{2\Omega^2} \left( \frac{\partial}{\partial u} + \tilde{X} \right) \]
(5.17)
and the null orthonormal frame \{\(e_4, e_3, e_A\)\}, with
\[ e_4 = 2\Omega L, \quad e_3 = 2\Omega L, \quad e_A = e^a_A \frac{\partial}{\partial \omega^a} \]
(5.18)
and $\tilde{\gamma}_{ab} e^a_A e^b_B = \delta_{AB}$. Observe also that once we have performed the coordinate change, the vector field $\frac{\partial}{\partial \omega^a}$ has to be interpreted as a tangent vector to the surface $S_0(\nu)$, which implies that the partial derivative is made keeping $u$ and $\nu$ constants. The result is that $\tilde{g}_{|C_0} \left( \frac{\partial}{\partial \omega^a}, N \right) = 0$, but $L \frac{\partial}{\partial \omega^a} = [N, \frac{\partial}{\partial \omega^a}] \neq 0$, with $N = (\frac{\partial}{\partial \nu} + \tilde{X})$.

The global effect is that the second null fundamental form of $S_0(\nu)$ relative to $e_4$, $\tilde{\chi}_{ab}$, is connected in a simple way to $\tilde{\chi}'_{ab}$:
\[ \tilde{\chi}_{ab} = 1 \frac{2\Omega}{\partial u} \frac{\partial \tilde{\gamma}_{ab}}{\partial u} = 1 \frac{2\Omega}{4\Omega^2} \frac{\partial \tilde{\gamma}_{ab}}{\partial u} = 2\Omega \frac{\partial \tilde{\gamma}_{ab}}{\partial u} = 2\Omega \tilde{\chi}'_{ab} \]
(5.19)
From the evolution equation satisfied by $\tilde{\chi}'_{ab}$ it follows immediately the equation satisfied by $\text{tr}\tilde{\chi}$:
\[ \frac{\partial \text{tr}\tilde{\chi}}{\partial u} + \frac{\Omega}{2} (\text{tr}\tilde{\chi})^2 + 2\Omega \omega \text{tr}\tilde{\chi} = 0 \]
(5.20)
where $\omega$ is defined as
\[ \omega = -\frac{1}{2\Omega} \frac{\partial \log \Omega}{\partial u} \]

5.2 Proof of Lemma 2.1
The result is obtained in two steps: first we prove the existence of a solution of the system 2.5 we rewrite in the following way
\[ \frac{\partial \gamma}{\partial \nu} - \Omega \text{tr}\gamma\gamma - 2\Omega \tilde{\chi} = 0 \]
(5.21)
\[ \frac{\partial}{\partial \nu} \left( \Omega^{-1} \text{tr}\chi \right) + \frac{\Omega^2}{2} (\Omega^{-1} \text{tr}\chi)^2 + \Omega^{-1} |\tilde{\chi}|^2 = 0 \]

Observe that while $L$ is the tangent vector field of the null geodesics generating $C_0$, $L = (2\Omega)^{-1} (\frac{\partial}{\partial u} + \tilde{X})$ is a null vector field defined on $C_0$ different from the vector field $L$ defined in 5.4. This is due to the fact that now $L$ is orthogonal to the vector fields tangent to the leaves $S_0(\nu)$ instead of $S_0'(\nu')$. Of course $L$ can be considered as the vector field of the null incoming geodesics starting on $C_0$.

The one relative to $e_3$ has a more complicated expression.
which satisfies the following inequalities

$$
\| \tilde{r}^{-2} - \gamma - \tilde{r}^{-2} \gamma \| \leq c\varepsilon \ , \ \| \text{tr} \chi - \text{tr} \tilde{\chi} \| \leq c\varepsilon \log \tilde{r} \over \tilde{r}^2 ,
$$

(5.22)

with $\text{tr} \tilde{\chi}$ the trace of the second fundamental form relative to the metric $\tilde{\gamma}$ and the norm $\| \cdot \|$ is defined, for a covariant tensor $f$, as

$$
\| f \| \equiv \sup_{\nu \in C_0} \left( \sup_{\{a_1 \ldots a_k\}} | f_{\{a_1 \ldots a_k\} \} \right).
$$

(5.23)

Second we show that, given the solution $(\gamma, \text{tr} \chi)$, equations 5.21 provide also an estimate for the Sobolev $\| \cdot \|_{p,S}$ norms of $\gamma$ and $\text{tr} \chi$, relative to the metric $\tilde{\gamma}$, 91

$$
| f |_{p,S(\nu)} \equiv \left( \int_{S_0(\nu)} | f|^p d\mu_\gamma \right)^{1 \over p}.
$$

Proof of the first step: To solve 5.21 we define a map $\mathcal{B}$ bringing $(\gamma', \text{tr} \chi')$ into $(\gamma, \text{tr} \chi)$:

The map $\mathcal{B}$: Given $(\gamma', \text{tr} \chi')$, $(\gamma, \text{tr} \chi) \equiv \mathcal{B}([\gamma', \text{tr} \chi'])$ is a solution of the equations

$$
\frac{\partial \gamma}{\partial \nu} - \Omega \text{tr} \chi' \gamma - 2 \Omega \tilde{\chi} = 0
$$

(5.24)

$$
\frac{\partial}{\partial \nu} (\Omega^{-1} \text{tr} \chi) + \frac{\Omega^2}{2} (\Omega^{-1} \text{tr} \chi)^2 + \Omega^{-1} | \tilde{\chi}' |^2 = 0 .
$$

with initial conditions 92

$$
\gamma(\nu_0)(\cdot, \cdot) = \tilde{\gamma}(\nu_0)(\cdot, \cdot) = \tilde{r}^2(\nu_0)(d\theta^2 + \sin^2 \theta d\phi^2) \ , \ \text{tr} \chi(\nu_0) = \text{tr} \tilde{\chi}(\nu_0) .
$$

(5.25)

The solution of 5.21 is obtained proving the existence of a fixed point for the map $\mathcal{B}$. To do it we show first that $\mathcal{B}$ sends the points $(\gamma', \text{tr} \chi')$ of a closed set $\mathcal{U}$ in other points of the same set, where, with $\mu$ the standard metric of $S^2$,

$$
\mathcal{U} = \left\{ (\gamma, \text{tr} \chi) \left\| \tilde{r}^{-2} \gamma - \mu \right\| \leq \delta_2 \varepsilon \ , \ \left\| \text{tr} \chi - \frac{2}{\tilde{r}} \right\| \leq \delta_1 \varepsilon \log \tilde{r} \over \tilde{r}^2 , \right\}
$$

(5.26)

and fixed $\delta_1, \delta_2 > 0$.

Second we show that a contraction holds, namely that, with $\sigma < 1$,

$$
\left\| \tilde{r}^{-2} (\gamma_n - \gamma_{n-1}) \right\| \leq \sigma \left\| \tilde{r}^{-2} (\gamma_{n-2} - \gamma_{n-3}) \right\|
$$

and

$$
\left\| \text{tr} \chi_{n-1} - \text{tr} \chi_{n-2} \right\| \leq \sigma \left\| \text{tr} \chi_{n-3} - \text{tr} \chi_{n-4} \right\| .
$$

(5.27)

Let us start proving that $\text{tr} \chi \in \mathcal{U}$ assuming $\gamma' \in \mathcal{U}$. We look at the second equation of 5.24

$$
\frac{\partial}{\partial \nu} (\Omega^{-1} \text{tr} \chi) + \frac{\Omega^2}{2} (\Omega^{-1} \text{tr} \chi)^2 + \Omega^{-1} | \tilde{\chi}' |^2 = 0 .
$$

91 $| f | = | f |_{\mathcal{B}} = f_{a b c d e} \delta^{a b} \delta^{c d} \delta^{e f}$

92 On $S_0(\nu_0), \tilde{r}(\nu_0) = r(\nu_0)$. Recall that on $C_0 \tilde{r}(\nu)$ is defined, see Lemma 2.1, through the equation $4\pi \tilde{r}(\nu)^2 = |S_0(\nu)|$. 71
This equation is still a non-linear one. To look for a solution of it we define
\[ u = (\Omega^{-1}\text{tr}\chi)^{-1} \] and rewrite the previous equation as,
\[
\frac{\partial u}{\partial \nu} = \frac{\Omega^2}{2} + \Omega^{-1}|\hat{\chi}|^2 u^2 .
\] (5.28)

This equation can be solved again by a fixed point method. Define the map \( T \) in the following way
\[
u' \rightarrow u \equiv T(u')
\] (5.29)
where \( u \) is the solution of the equation
\[
\frac{\partial u}{\partial \nu} = \frac{\Omega^2}{2} + \Omega^{-1}|\hat{\chi}|^2 u^2 ,
\] (5.30)
with initial conditions at \( \nu_0 \). The ball in the function space where \( u \) remains after the application of \( T \), is defined, with a given \( \delta_0 > 0 \), as
\[
O_{\delta_0} = \left\{ f \in C^4([\nu_0, \infty)) \mid \sup_{\nu \in C_0} |\log \hat{v}^{-1}(f(\nu) - \Omega^2\hat{\nu}(\nu))| \leq \delta_0 \varepsilon, f(\nu_0) - \Omega^2\hat{\nu}(\nu_0) = c_0 \varepsilon \right\}
\] (5.31)
In fact let \( u' \in O_{\delta_0} \) then, solving 5.30, we obtain
\[
|u(\nu) - \Omega^2\hat{\nu}(\nu)| \leq (c_0 + c_1 + \hat{c})\varepsilon \log \hat{\nu}(\nu) + \left| \int_{\nu_0}^{\nu} O(\varepsilon^2) - \varepsilon \right| 
\leq (c_0 + c_1 + \hat{c})\varepsilon \log \hat{\nu}(\nu) + C\varepsilon^2
\] (5.32)
where \( c_1 \) and \( \hat{c} \) satisfy
\[
\sup_{\nu \in C_0} |\log \hat{v}^{-1}(\hat{\nu}(\nu) - \frac{\nu - \lambda_1}{2})| \leq c_1 \varepsilon, \sup_{\nu \in C_0} |\hat{v}(\frac{1}{4} - \Omega^2)| \leq \hat{c} \varepsilon
\] (5.33)
and choosing \( \varepsilon \) sufficiently small and \( \delta_0 > c_0 + c_1 + \hat{c} \) the right hand side is less than \( \delta_0 \varepsilon \log \hat{\nu}(\nu) \), as needed. The contraction mapping is proved in the same way. Integrating the equation for the difference \( u_k - u_{k-1} \), we obtain
\[
\|u_k - u_{k-1}\| \equiv \sup_{\nu \in [\nu_0, \infty)} |u_k - u_{k-1}(\nu)| \leq \int_{\nu_0}^{\nu} O(\varepsilon^2) |u_k - u_{k-1}||u_k - u_{k-2}| 
\leq \left( c\varepsilon^2 \int_{\nu_0}^{\nu} \frac{1}{\nu^{1+\sigma}} \right) \|u_k - u_{k-1} - u_{k-2}\| \leq \sigma \|u_k - u_{k-1} - u_{k-2}\| ,
\] (5.34)
with \( \sigma < 1 \). All this amounts to conclude that, given \( \gamma' \in U \) the solution of the second equation of 5.24 satisfies
\[
\text{tr} \chi \leq \frac{\Omega^{-1}}{\hat{\nu}(\nu)} \left( 1 + \delta_0 \varepsilon \frac{\log \hat{\nu}(\nu)}{\hat{\nu}(\nu)} \right) \leq \frac{2}{\hat{\nu}(\nu)} \left( 1 + (c_2 + \delta_0)\varepsilon \frac{\log \hat{\nu}(\nu)}{2\hat{\nu}(\nu)} \right)
\] (5.35)
where \( c_2 \) is defined through the inequality
\[
|\Omega^{-1} - 2| \leq \frac{c_2\varepsilon}{\hat{\nu}} .
\] (5.36)
Therefore, \( \text{tr} \chi \in \mathcal{U} \), choosing \( 2\delta_1 > c_2 + \delta_0 \). The proof that \( \gamma \in \mathcal{U} \) once \( (\gamma', \text{tr}'\gamma') \in \mathcal{U} \) is immediate. In fact
\[
\frac{\partial}{\partial \nu}(\tilde{r}^{-2}\gamma) = \tilde{r}^{-2}\frac{\partial \gamma}{\partial \nu} - 2\tilde{r}^{-3}\frac{\partial}{\partial \nu}\tilde{r}\gamma = \tilde{r}^{-2}\frac{\partial \gamma}{\partial \nu} - \tilde{r}^{-2}\text{tr}\gamma (\tilde{r})^{-2} \quad (5.37)
\]

As \( \gamma \in \mathcal{U} \), \( \text{(tr} \chi \text{)} - \text{tr} \chi \leq 2\delta_1 \tilde{r}^{-2} \log \tilde{r} \) an application of Gronwall’s Lemma gives
\[
|\tilde{r}^{-2}(\gamma_{ab})(\nu) - (\tilde{r}^{-2}\gamma_{ab})(\nu_0)| \leq c_0 \tilde{r} \quad (5.38)
\]

which, assuming \( \delta_2 > c_0 \), implies that \( \gamma \in \mathcal{U} \). To prove the contraction for the \( \gamma_n \)’s we write the equations satisfied by \( \gamma_n \):
\[
\frac{\partial}{\partial \nu}(\gamma_n - \gamma_{n-1}) - \text{tr}\chi_{n-1}(\gamma_n - \gamma_{n-1}) - (\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})\gamma_{n-1} = 0
\]

which we rewrite, as an equation for \( \tilde{r}^{-2}(\gamma_n - \gamma_{n-1}) \) in the following way:
\[
\frac{\partial}{\partial \nu}(\tilde{r}^{-2}(\gamma_n - \gamma_{n-1})) - (\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})\tilde{r}^{-2}(\gamma_n - \gamma_{n-1}) = 0
\]

As \( \gamma_n - \gamma_{n-1}(\nu_0) = 0 \), proceeding as before we obtain
\[
|\tilde{r}^{-2}(\gamma_n - \gamma_{n-1})(\nu)| \leq c \int^{\nu}_{\nu_0} \tilde{r}^2(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})\nu \quad (5.40)
\]

and from it, recalling that \( \Omega \) is bounded, with a different constant \( c \),
\[
\|\tilde{r}^{-2}(\gamma_n - \gamma_{n-1})\| \leq c\|\Omega^{-1}\tilde{r}^2(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})\| \quad (5.41)
\]

The difference \( \Omega^{-1}(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2}) \) satisfies the following equation:
\[
\frac{\partial}{\partial \nu}[\Omega^{-1}(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})] - \Omega^{-1}(\text{tr}\chi_{n-1} + \text{tr}\chi_{n-2})[\Omega^{-1}(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})] = - \left( \Omega^{-1}|\hat{\chi}|^2_{\gamma_{n-2}} - \Omega^{-1}|\hat{\chi}|^2_{\gamma_{n-3}} \right) \quad (5.42)
\]

As \( \text{tr}\chi_{n} \) and \( \text{tr}\chi_{n-1} \) belong to \( \mathcal{U} \) we can rewrite the previous equation, denoting \( V_n \equiv [\Omega^{-1}(\text{tr}\chi_{n-1} - \text{tr}\chi_{n-2})] \), as
\[
\frac{\partial}{\partial \nu}V_n + \text{tr}\chi V_n + \left[ \frac{\Omega}{2}(\text{tr}\chi_{n-1} + \text{tr}\chi_{n-2}) - \text{tr}\chi \right]V_n = - \left( \Omega^{-1}|\hat{\chi}|^2_{\gamma_{n-2}} - \Omega^{-1}|\hat{\chi}|^2_{\gamma_{n-3}} \right) \quad (5.43)
\]

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From it immediately
\[
\frac{\partial}{\partial \nu}(\tilde{r}^2 V_n) + O(\varepsilon \log \tilde{r})(\tilde{r}^2 V_n) + c\tilde{r}^2(\tilde{\chi}^2_{\nu_{2-2}} - |\tilde{\chi}|^2_{\nu_{2-3}}) = 0 \quad (5.44)
\]

Applying Gronwall’s Lemma we obtain
\[
[(\tilde{r}^2 V_n)(\nu)](\gamma_{n-2ac} + \gamma_{n-3ac})(\gamma_{n-2bd} + \gamma_{n-3bd}) + (\gamma_{n-2ac} - \gamma_{n-3ac})(\gamma_{n-2bd} + \gamma_{n-3bd})\]
\[
\leq \frac{c}{2} \int_{\nu_0}^{\nu} \tilde{r}^4 (\gamma_{n-2ab})^2 |\tilde{\gamma}_{n-2ab} - \tilde{\gamma}_{n-3ab}| \]
\[
\leq \left( \frac{c}{2} \int_{\nu_0}^{\nu} \tilde{r}^4 (\gamma_{n-2ab})^2 |\tilde{\gamma}_{n-2ab} - \tilde{\gamma}_{n-3ab}| \right) \cdot\tilde{r}^{-2} \gamma_{n-2ab} - \gamma_{n-3ab} \]
\[
\leq \left( \frac{c}{2} \int_{\nu_0}^{\nu} \tilde{r}^6 \frac{\tilde{r}^2}{V_{n}} \right) \cdot\tilde{r}^{-2} \gamma_{n-2ab} - \gamma_{n-3ab} \]
\[
\leq \left( \frac{c}{2} \int_{\nu_0}^{\nu} \tilde{r}^6 \frac{\tilde{r}^2}{V_{n}} \right) \cdot\tilde{r}^{-2} \gamma_{n-2ab} - \gamma_{n-3ab} \]
\[
\leq \frac{c}{2} \tilde{r}^2 \gamma_{n-2ab} - \gamma_{n-3ab} \]

Substituting in 5.41 we obtain
\[
\|\tilde{r}^{-2}(\gamma_{n-2ab} - \gamma_{n-3ab})\| \leq c\|\Omega^{-1}\tilde{r}^2 (\text{tr} \chi_{n-2ab} - \text{tr} \chi_{n-3ab})\|
\leq c\varepsilon^2 \|\tilde{r}^{-2}(\gamma_{n-2ab} - \gamma_{n-3ab})\| \quad (5.46)
\]

Therefore we have proved the following inequalities with \( \sigma < 1 \), choosing \( \varepsilon \) sufficiently small,
\[
\|\tilde{r}^{-2}(\gamma_{n-2ab} - \gamma_{n-3ab})\| \leq \sigma\|\tilde{r}^{-2}(\gamma_{n-2ab} - \gamma_{n-3ab})\|
\|
\text{tr} \chi_{n-1} - \text{tr} \chi_{n-2} \| \leq \sigma\|\text{tr} \chi_{n-3} - \text{tr} \chi_{n-4} \| \quad (5.47)
\]

They imply existence of a fixed point for the map \( \mathcal{B} \) and from it a solution of 5.21.

Let us prove the second step of our result. The estimate for \( \gamma \) satisfying 5.21 is of the following type:
\[
|\tilde{\gamma}^{(-2 - \frac{2}{\tilde{r}^2})} \gamma|_{p,S}(\lambda_1, \nu) \leq c_0 \left( |\tilde{\gamma}^{(-2 - \frac{2}{\tilde{r}^2})} \gamma|_{p,S}(\lambda_1, \nu_0) + \int_{\nu_0}^{\nu} |\tilde{\gamma}^{(-2 - \frac{2}{\tilde{r}^2})} \gamma|_{p,S} d\nu \right)
\leq c_0 |\tilde{\gamma}^{(-2 - \frac{2}{\tilde{r}^2})} \gamma|_{p,S}(\lambda_1, \nu_0) + c_1 \varepsilon \quad (5.48)
\]

This is obtained rewriting the first equation of 5.21 as
\[
\frac{\partial \gamma_{ab}}{\partial \nu} - \Omega \text{tr} \chi \gamma_{ab} + \left[ \Omega (\text{tr} \chi - \text{tr} \chi) \gamma_{ab} - 2\Omega \delta_{ab} \right] = 0 \quad (5.49)
\]

and observing that the term \( \Omega (\text{tr} \chi - \text{tr} \chi) \) behaves, due to the results of step one, as \( O(\varepsilon \log \tilde{r} \tilde{r}^{-2}) \). To get an estimate for \( \text{tr} \chi \) we use the second equation in 5.21 which we rewrite as
\[
\frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega}{2} \text{tr} \chi + \frac{\Omega}{2} (\text{tr} \chi - \text{tr} \chi) \text{tr} \chi + |\tilde{\chi}|^2 = 0 \quad (5.50)
\]
Again the term \(2\Omega + \frac{\Omega}{2}(\text{tr} - \tilde{\text{tr}})\) behaves, due to the results of step one, as \(O(\varepsilon \log \tilde{r} \tilde{r}^{-2})\). From it, applying standard techniques and Gronwall’s Lemma we obtain:

\[
|\hat{r}^{(1-\frac{2}{\hat{p}})}\text{tr}x|_{p,S}(\lambda_1, \nu) \leq c_0 \left|\hat{r}^{(1-\frac{2}{\hat{p}})}\text{tr}x|_{p,S}(\lambda_1, \nu_0) + \int_{\nu_0}^{\nu} |\hat{r}^{(1-\frac{2}{\hat{p}})}|_{p,S}^2 d\nu'\right|
\]

\[
\leq c_0|\hat{r}^{(1-\frac{2}{\hat{p}})}\text{tr}x|_{p,S}(\lambda_1, \nu_0) + c_1 \varepsilon^2.
\]

(5.51)

with \(c_0 = 1 + c\varepsilon\). From the second inequality in 5.22, defining \(r(\nu) \equiv |x_{S(\nu)}|\), it follows immediately that

\[
\frac{dr}{d\tilde{r}} = \frac{dr}{d\nu} \left(\frac{d\tilde{r}}{d\nu}\right)^{-1} = 1 + \frac{1}{\Omega \text{tr}x} O(\text{tr}x - \tilde{\text{tr}}x)
\]

and from it

\[
\left|\frac{dr}{d\tilde{r}}\right| \leq 1 + c\varepsilon \log \tilde{r} \tilde{r}^{-1}
\]

(5.52)

implying that there exist constants \(c_1, c_2\) bounded by \(1 + c\varepsilon\), such that

\[
c_1 \tilde{r} \leq r \leq c_2 \tilde{r}.
\]

(5.53)

From 5.22 and 5.54 inequalities 2.12 follow. Moreover we also obtain that on \(C_0\) the following estimate holds:

\[
|\text{tr}x - \tilde{\text{tr}}x| = O(\varepsilon) \frac{\log r}{r^2}.
\]

(5.55)

Finally to prove the last estimate of 2.12

\[
|r^{3-\frac{2}{\hat{p}}} \nabla x|_{p,S} = O(\varepsilon),
\]

we write the evolution equation along \(C_0\) for \(\nabla x\text{tr}x\), see [Kl-Ni], Chapter 4, equation (4.3.4), and obtain the result from the assumptions of Lemma 2.1 and an application of Gronwall’s lemma.\(^{93}\)

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\(^{93}\)The \(\delta > 0\) in the decay assumptions for \(\nabla \log \Omega\) and \(\nabla \log \Omega\) is crucial for this result.
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