Cavity QED and Quantum Computation in the

Strong Coupling Regime

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Abstract

In this paper we propose a Hamiltonian generalizing the interaction of the two–level atom and both the single radiation mode and external field · · · a kind of cavity QED. We solve the Schrödinger equation in the strong coupling regime by making use of rotating wave approximation under new resonance conditions containing the Bessel functions and etc, and obtain unitary transformations of four types corresponding to Rabi oscillations which perform quantum logic gates in Quantum Computation.

In this paper we consider a unified model of the interaction of the two–level atom and both the single radiation mode and external field (periodic usually) in a cavity. We deal with the external field as a classical one. As a general introduction to this topic in Quantum Optics see [1], [2], [3]. Our model is deeply related to the one of trapped ions in a cavity with the photon interaction (Cavity QED).

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In our model we are especially interested in the strong coupling regime, [9], [14], [15]. One of motivations is a recent very interesting experiment, [12]. See [3] and [10] as a general introduction.

In [9] and [14] we have treated the strong coupling regime of the interaction model of the two–level atom and the single radiation mode, and have given some explicit solutions under the resonance conditions and rotating wave approximations.

On the other hand we want to add some external field (like Laser one) to the above model which will make the model more realistic (for example in Quantum Computation). Therefore we propose the unified model.

We would like to solve our model in the strong coupling regime. Especially we want to show the existence of Rabi oscillations in this regime because the real purpose of a series of study ([14], [15], [18]) is an application to Quantum Computation (see [13] as a brief introduction to it).

In this paper we solve the Schrodinger equations in this regime by making use of rotating wave approximation under new resonance conditions containing the Bessel functions and etc, and obtain a unitary transformation in one qubit case and unitary transformations of four types in two qubit case, which perform quantum logic gates.

Our solutions might give a new insight into Quantum Optics or Condensed Matter Physics as well as Quantum Computation.

Let \{σ₁, σ₂, σ₃\} be Pauli matrices and 1₂ a unit matrix :

\[
σ₁ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad σ₂ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad σ₃ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1₂ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(1)

and \(σ₊ = (1/2)(σ₁ + iσ₁), σ₋ = (1/2)(σ₁ - iσ₁)\). Let \(W\) be the Walsh–Hadamard matrix

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W⁻¹,
\]

(2)

then we can diagonalize \(σ₁\) as \(σ₁ = Wσ₃W⁻¹ = Wσ₃W\) by making use of this \(W\). The
eigenvalues of $\sigma_1$ is $\{1, -1\}$ with eigenvectors

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad |\lambda\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \quad (3)$$

Let us consider an atom with 2 energy levels $E_0$ and $E_1$ (of course $E_1 > E_0$). Its Hamiltonian is in the diagonal form given as

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}. \quad (4)$$

This is rewritten as

$$H_0 = \frac{E_0 + E_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{E_1 - E_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \Delta_0 I_2 - \frac{\Delta}{2} \sigma_3, \quad (5)$$

where $\Delta = E_1 - E_0$ is an energy difference. Since we usually take no interest in constant terms, we set

$$H_0 = -\frac{\Delta}{2} \sigma_3. \quad (6)$$

We consider an atom with two energy levels which interacts with external (periodic) field with $g \cos(\omega_E t)$. In the following we set $\hbar = 1$ for simplicity. The Hamiltonian in the dipole approximation is given by

$$H = H_0 + g \cos(\omega_E t + \phi) \sigma_1 = -\frac{\Delta}{2} \sigma_3 + g \cos(\omega_E t + \phi) \sigma_1, \quad (7)$$

where $\omega_E$ is the frequency of the external field, $g$ the coupling constant between the external field and the atom. We note that to solve this model without assuming the rotating wave approximation is not easy, see [8], [11], [21], [22], [23].

In the following we change the sign in the kinetic term, namely from $-\Delta/2$ to $\Delta/2$, to set the model for other models. However this is minor.

Now we make a short review of the harmonic oscillator within our necessity. Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then we have

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \quad (8)$$
Let $\mathcal{H}$ be a Fock space generated by $a$ and $a^\dagger$, and $\{|n\rangle | n \in \mathbb{N} \cup \{0\}\}$ be its basis. The actions of $a$ and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle , \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle , \quad N|n\rangle = n|n\rangle$$ \hspace{1cm} (9)

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0 \rangle = 1$). From (9), states $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \left(a^\dagger\right)^n \sqrt{n!} |0\rangle .$$ \hspace{1cm} (10)

These states satisfy the orthogonality and completeness conditions

$$\langle m|n \rangle = \delta_{mn} , \quad \sum_{n=0}^{\infty} \langle n|n \rangle = 1 .$$ \hspace{1cm} (11)

Then the displacement (coherent) operator and coherent state are defined as

$$D(z) = e^{za^\dagger - za^\dagger}, \quad |z\rangle = D(z)|0\rangle \quad \text{for} \quad z \in \mathbb{C}.$$ \hspace{1cm} (12)

We consider the quantum theory of the interaction between an atom with two–energy levels and single radiation mode (a harmonic oscillator). The Hamiltonian in this case is

$$H = \omega_1 1_2 \otimes a^\dagger a + \Delta/2 \sigma_3 \otimes 1 + g \sigma_1 \otimes (a^\dagger + a)$$ \hspace{1cm} (13)

where $\omega$ is the frequency of the radiation mode, $g$ the coupling between the radiation field and the atom, see for example [3], [9].

Now it is very natural for us to include (7) into (13), so we present the following

**General Hamiltonian**

$$H = \omega_1 1_2 \otimes a^\dagger a + g_1 \sigma_1 \otimes (a^\dagger + a) + \frac{\Delta}{2} \sigma_3 \otimes 1 + g_2 \cos(\omega_E t + \phi) \sigma_1 \otimes 1 .$$ \hspace{1cm} (14)

Our Hamiltonian has two coupling constants. We note that our model is deeply related to the Cavity QED (trapped ions in a cavity with the photon interaction).
This Hamiltonian is also related to the one presented recently by Schönh and Cirac \cite{4}

\[ H = \frac{p^2}{2m} + \omega_0 1_2 \otimes a^\dagger a + g(x) \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger \right) + \frac{\omega_0}{2} \sigma_3 \otimes 1 + \frac{\Omega}{2} \left( e^{-i\omega_L t} \sigma_+ \otimes 1 + e^{i\omega_L t} \sigma_- \otimes 1 \right). \] (15)

For the meaning of several constants see \cite{4}. They have assumed the rotating wave approximation (see for example \cite{3}) and the resonance condition, and use a position-dependent coupling constant \( g(x) \), so their model is different from ours in these points.

A comment is in order. Following \cite{4} the Hamiltonian (14) might be modified to

\[ H = \frac{p^2}{2m} + \omega_1 1_2 \otimes a^\dagger a + g_1(x) \sigma_1 \otimes (a^\dagger + a) + \frac{\Delta}{2} \sigma_3 \otimes 1 + g_2 \cos(\omega_E t + \phi) \sigma_1 \otimes 1. \] (16)

This model is a full generalization of (15), however we don’t consider this situation in the paper.

We have one question : Is the Hamiltonian (14) realistic or meaningful ? The answer is of course yes. Let us show one example. We consider the (effective) Hamiltonian presented by NIST group \cite{5, 6} which were used to construct the controlled NOT operation (see \cite{13} as an introduction).

\[ H = \omega_0 1_2 \otimes a^\dagger a + g \left( \sigma_+ \otimes e^{i\eta(a^\dagger + a)} + \sigma_- \otimes e^{-i\eta(a^\dagger + a)} \right) + \frac{\Delta}{2} \sigma_3 \otimes 1. \] (17)

We can show that under some unitary transformation the Hamiltonian (17) can be transformed to (14) with special coupling constants, \cite{7}. This is important, so we review and modify \cite{4}.

We set \( 2A = i\eta(a^\dagger + a) \) for simplicity, then

\[
\begin{align*}
\sigma_+ \otimes e^{i\eta(a^\dagger + a)} + \sigma_- \otimes e^{-i\eta(a^\dagger + a)} &= \\
&= \begin{pmatrix}
0 & e^{2A} \\
e^{-2A} & 0
\end{pmatrix} = \begin{pmatrix}
0 & e^{A} \\
e^{-A} & 0
\end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{A} \\
e^{-A} & 0
\end{pmatrix} \\
&= \begin{pmatrix}
0 & e^{A} \\
e^{-A} & 0
\end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\
&= \begin{pmatrix} 0 & e^{A} \\
e^{-A} & 0
\end{pmatrix} \equiv U(\eta)(\sigma_3 \otimes 1)U(\eta)^\dagger
\end{align*}
\] (18)
where
\[ U(\eta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{A} \\ e^{-A} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{A} & -e^{A} \\ e^{-A} & e^{-A} \end{pmatrix} \] (19)
and \( e^{A} = D(i\eta/2) \) where \( D(\beta) \) is a displacement (coherent) operator defined by (12).

Then it is not difficult to show
\[ U(\eta)^\dagger HU(\eta) = \frac{\omega_0 \eta^2}{4} 1 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (-ia^\dagger + ia) + g \sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1. \] (20)

To remove \( i \) in the term containing \( a \) we moreover operate the unitary one
\[ \left( 1_2 \otimes e^{i(\pi/2)N} \right) U(\eta)^\dagger HU(\eta) \left( 1_2 \otimes e^{-i(\pi/2)N} \right) \]
\[ = \frac{\omega_0 \eta^2}{4} 1_2 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (a^\dagger + a) + g \sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1, \] (21)

where we have used the well–known formula
\[ e^{i\theta N} a e^{-i\theta N} = e^{-i\theta} a, \quad e^{i\theta N} a^\dagger e^{-i\theta N} = e^{i\theta} a^\dagger, \]
see [16]. Since \( U(\eta) \) can be written as \( U(\eta) = (\sigma_+ \otimes e^{A} + \sigma_- \otimes e^{-A}) (W \otimes 1) \), so if we write
\[ T(\eta) \equiv U(\eta)(1_2 \otimes e^{-i(\pi/2)N}) = (\sigma_+ \otimes e^{A} + \sigma_- \otimes e^{-A})(W \otimes e^{-i(\pi/2)N}), \] (22)
then we have
\[ T(\eta)^\dagger HT(\eta) = \frac{\omega_0 \eta^2}{4} 1_2 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (a^\dagger + a) + g \sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1. \]

Here we have no interest in the constant term, so we finally obtain
\[ H = T(\eta) \left\{ \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (a^\dagger + a) + g \sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1 \right\} T(\eta)^\dagger. \] (23)

\( T(\eta) \) is just the unitary transformation required. We note that the last term is constant, which case is a special one.

At this stage we would like to make a further generalization of the Hamiltonian [14] to make wide applications to Quantum Computation.
Let \( \{K_+, K_-, K_3\} \) and \( \{J_+, J_-, J_3\} \) be a set of generators of unitary representations of Lie algebras \( su(1,1) \) and \( su(2) \). They are usually constructed by making use of two harmonic oscillators (two–photons) \( a_1, a_2 \) as

\[
\begin{align*}
  su(1,1) : & \quad K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_2 a_1, \quad K_3 = \frac{1}{2} \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right), \\
  su(2) : & \quad J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{1}{2} \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right).
\end{align*}
\]

Then we can make the similar arguments done for the Heisenberg algebra \( \{a^\dagger, a, N\} \), namely (8) \( \sim \) (12), see for example [16].

We have considered the following three Hamiltonians in [14]:

\[
\begin{align*}
  (N) \quad H_N &= \omega_2 1_2 \otimes a_1^\dagger a_2 a + \frac{\Delta}{2} \sigma_3 \otimes 1 + g \sigma_1 \otimes (a_1^\dagger + a_2^\dagger a), \\
  (K) \quad H_K &= \omega_2 1_2 \otimes K_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_K + g \sigma_1 \otimes (K_+ + K_-), \\
  (J) \quad H_J &= \omega_2 1_2 \otimes J_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_J + g \sigma_1 \otimes (J_+ + J_-).
\end{align*}
\]

To deal with these three cases at the same time we set

\[
\{L_+, L_-, L_3\} = \begin{cases} 
(N) & \{a^\dagger, a, N\}, \\
(K) & \{K_+, K_-, K_3\}, \\
(J) & \{J_+, J_-, J_3\}
\end{cases}
\]

and

\[
H_L = \omega_2 1_2 \otimes L_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_L + g \sigma_1 \otimes (L_+ + L_-).
\]

Therefore the Hamiltonian that we are looking for is

**Unified Hamiltonian**

\[
\tilde{H}_L = \omega_2 1_2 \otimes L_3 + g_1 \sigma_1 \otimes (L_+ + L_-) + \frac{\Delta}{2} \sigma_3 \otimes 1_L + g_2 \cos(\omega_E t + \phi) \sigma_1 \otimes 1_L.
\]

By the way, from the lesson in [23] we can also consider a weak version of (29) with the last term being constant

\[
\tilde{H}_L = \omega_2 1_2 \otimes L_3 + g_1 \sigma_1 \otimes (L_+ + L_-) + \frac{\Delta}{2} \sigma_3 \otimes 1_L + g_2 \sigma_1 \otimes 1_L,
\]

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where we have reset \( g_2 \equiv g_2 \cos(\phi) \) for simplicity. This is not a bad model as seen from \( [23] \). This restricted model has been treated in \([15]\).

Now we would like to solve the Hamiltonian \( [29] \), especially in the strong coupling regime \((g_1 \gg \Delta)\).

Let us transform \( [29] \) into
\[
\tilde{H}_L = 1_2 \otimes \omega L_3 + \sigma_1 \otimes \{ g_1 (L_+ + L_-) + g_2 \cos(\omega_E t + \phi) 1_L \} + \frac{\Delta}{2} \sigma_3 \otimes 1_L \\
\equiv \tilde{H}_0 + \frac{\Delta}{2} \sigma_3 \otimes 1_L.
\]

The method to solve is almost identical to \([14]\), so we give only an outline. By making use of the Walsh–Hadamard matrix \([2]\),
\[
\tilde{H}_0 = (W \otimes 1_L) [1_2 \otimes \omega L_3 + \sigma_3 \otimes \{ g_1 (L_+ + L_-) + g_2 \cos(\omega_E t + \phi) 1_L \}] (W^{-1} \otimes 1_L)
\]
\[
= \sum_{\lambda = \pm 1} \left( |\lambda\rangle \otimes e^{-\frac{i}{\omega} (L_+ - L_-)} \right) \left\{ \Omega L_3 + \lambda g_2 \cos(\omega_E t + \phi) 1_L \right\} \left( \langle \lambda | \otimes e^{\frac{i}{\omega} (L_+ - L_-)} \right)
\]
where \(|\lambda\rangle\) is the eigenvectors of \( \sigma_1 \) defined in \( [3] \) and \( \Omega, \ x \) are given as
\[
(\Omega, \ x) = \begin{cases} 
(N) & \omega, \quad x = 2g_1/\omega, \\
(K) & \omega \sqrt{1 - (2g_1/\omega)^2}, \quad x = \tanh^{-1}(2g_1/\omega), \\
(J) & \omega \sqrt{1 + (2g_1/\omega)^2}, \quad x = \tan^{-1}(2g_1/\omega).
\end{cases}
\]
That is, we could diagonalize the Hamiltonian \( \tilde{H}_0 \). Its eigenvalues \( \{ E_n(t) \} \) and eigenvectors \( \{ |\{\lambda, n\}\rangle\} \) are given respectively
\[
(E_n(t), |\{\lambda, n\}\rangle) = \begin{cases} 
(N) & \Omega (-\frac{g_2}{\omega^2} + n) + \lambda g_2 \cos(\omega_E t + \phi), \quad |\lambda\rangle \otimes e^{-\frac{i}{\omega} (a^+_n - a)} |n\rangle, \\
(K) & \Omega (K + n) + \lambda g_2 \cos(\omega_E t + \phi), \quad |\lambda\rangle \otimes e^{-\frac{i}{\omega} (K_n - K)} |K, n\rangle, \\
(J) & \Omega (-J + n) + \lambda g_2 \cos(\omega_E t + \phi), \quad |\lambda\rangle \otimes e^{-\frac{i}{\omega} (J_n - J)} |J, n\rangle
\end{cases}
\]
for \( \lambda = \pm 1 \) and \( n \in \mathbb{N} \cup \{0\} \), where \( E_n(t) \equiv E_n + \lambda g_2 \cos(\omega_E t + \phi) \). Then \( \tilde{H}_0 \) above can be written as
\[
\tilde{H}_0 = \sum_{\lambda} \sum_n E_n(t) |\{\lambda, n\}\rangle \langle \{\lambda, n\}|.
\]

Next we would like to solve the following Schrödinger equation:
\[
i \frac{d}{dt} \Psi = \tilde{H} \Psi = \left( \tilde{H}_0 + \frac{\Delta}{2} \sigma_3 \otimes 1_L \right) \Psi.
\]
To solve this equation we appeal to the method of constant variation. First let us solve
\[ i \frac{d}{dt} \Psi = \tilde{H}_0 \Psi, \]
which general solution is given by \( \Psi(t) = U_0(t) \Psi_0 \), where \( \Psi_0 \) is a constant state and
\[ U_0(t) = \sum_{\lambda} \sum_n e^{-i(tE_n + \lambda(g_2/\omega_E)\sin(\omega_E t + \phi))} |{\lambda, n}\rangle \langle {\lambda, n}|. \] (34)

The method of constant variation goes as follows. Changing like \( \Psi_0 \rightarrow \Psi_0(t) \), we have
\[ i \frac{d}{dt} \Psi_0 = \frac{\Delta}{2} U_0^\dagger (\sigma_3 \otimes 1_L) U_0 \Psi_0 \equiv \frac{\Delta}{2} \tilde{H}_F \Psi_0 \] (35)
after some algebra. We must solve this equation. \( \tilde{H}_F \) is
\[ \tilde{H}_F = \sum_{\lambda, \mu} \sum_{m, n} e^{i(tE_m - E_n + i(\lambda - \mu)(g_2/\omega_E)\sin(\omega_E t + \phi))} \langle \{ \lambda, m \} | (\sigma_3 \otimes 1_L) | \{ \mu, n \} \rangle \langle \{ \lambda, m \} | \{ \mu, n \} | \{ \lambda, n \} \rangle \langle \{ \lambda, n \} | \{ \mu, n \} \rangle \] (36)
where we have used \( \langle \lambda | \sigma_3 = - \lambda | n \rangle \) is respectively
\[ |n\rangle = \begin{cases} (N) & |n\rangle, \\ (K) & |K, n\rangle, \\ (J) & |J, n\rangle. \end{cases} \]

In the following we set for simplicity
\[ \Theta(t) \equiv g_2 \frac{\sin(\omega_E t + \phi)}{\omega_E}. \] (37)

Here we divide \( \tilde{H}_F \) into two parts \( \tilde{H}_F = \tilde{H}_F' + \tilde{H}_F'' \) where
\[ \tilde{H}_F' = \sum_{\lambda} \sum_{m, n} e^{2i\lambda \Theta(t)} \langle \{ m | e^{\lambda x(L_+ - L_-)} | n \} \rangle \langle \{ \lambda, m \} | \{ \mu, n \} \rangle \langle \{ \mu, n \} | \{ \lambda, m \} \rangle \] (38)
\[ \tilde{H}_F'' = \sum_{\lambda} \sum_{m, n} e^{i(t \Omega(m-n) + 2\lambda \Theta(t))} \langle \{ m | e^{\lambda x(L_+ - L_-)} | n \} \rangle \langle \{ \lambda, m \} | \{ -\lambda, n \} \rangle \] (39)

Noting \( \langle n | e^{x(L_+ - L_-)} | n \rangle = \langle n | e^{-x(L_+ - L_-)} | n \rangle \) by the results in section 3 of [14], \( \tilde{H}_F \) can be written as
\[ \tilde{H}_F = \sum_n \langle \{ n | e^{x(L_+ - L_-)} | n \} \rangle \sum_{\lambda} e^{2i\lambda \Theta(t)} \langle \{ \lambda, n \} | \{ \lambda, n \} \rangle \] (40)
Here we want to solve the equation $i(d/dt)\Psi_0 = \tilde{H}'_F\Psi_0$ completely, however it is very hard (see for example [8], [11], [21], [22]). Therefore let us appeal to a perturbation theory.

For simplicity we set $\phi = 0$ in (37), then we have the well–known formula

$$e^{2i\lambda\Theta(t)} = \sum_{\alpha \in \mathbb{Z}} J_\alpha(2\lambda g_2/\omega_E)e^{i\alpha\omega_E t} = J_0(2\lambda g_2/\omega_E) + \sum_{\alpha \neq 0} J_\alpha(2\lambda g_2/\omega_E)e^{i\alpha\omega_E t}, \quad (41)$$

where $J_\alpha(x)$ are the Bessel functions. For a further simplicity we set $2g_2/\omega_E = \Gamma$.

We decompose (40) as

$$\tilde{H}'_F = \tilde{H}'_0F + \tilde{H}'_1F;$$

where

$$\tilde{H}'_0F = \sum_n \langle \langle n|e^{x(L_+ - L_-)}|n\rangle \rangle J_0(\Gamma) \sum_\lambda |\{\lambda, n\}\rangle \langle \{-\lambda, n\}| \quad (42)$$

and

$$\tilde{H}'_1F = \sum_n \langle \langle n|e^{x(L_+ - L_-)}|n\rangle \rangle \sum_\lambda \sum_{\alpha \neq 0} J_\alpha(\lambda\Gamma)e^{i\alpha\omega_E t}|\{\lambda, n\}\rangle \langle \{-\lambda, n\}|. \quad (43)$$

Next let us transform (39).

$$\tilde{H}''_F = \sum_{m,n} e^{it\Omega(m-n)} \sum_\lambda e^{2i\lambda\Theta(t)} \langle \langle m|e^{\lambda x(L_+ - L_-)}|n\rangle \rangle |\{\lambda, m\}\rangle \langle \{-\lambda, m\}|$$

$$= \sum_{m,n} e^{it\Omega(m-n)} \sum_\lambda \sum_{\alpha \neq 0} J_\alpha(\lambda\Gamma)e^{i\alpha\omega_E t} \langle \langle m|e^{\lambda x(L_+ - L_-)}|n\rangle \rangle |\{\lambda, m\}\rangle \langle \{-\lambda, n\}|. \quad (44)$$

Now we define a new basis called Schrödinger cat states

$$|\{\sigma, \psi_n\}\rangle = \frac{1}{\sqrt{2}}(|\{1, n\}\rangle + \sigma|\{-1, n\}\rangle), \quad \sigma = \pm 1.$$
and moreover

\[
\tilde{H}_F'' = \sum_{m \neq n} e^{it\Omega(m-n)} \sum_{\alpha} e^{i\alpha\omega t} \sum_{\sigma} J_\sigma(\sigma \Gamma) \langle \langle m \mid e^{\sigma x(L_+ - L_-)} \mid n \rangle \rangle 
\times \\
\frac{1}{2} \left\{ \sum_{\lambda} \lambda \langle \lambda, \psi_m \rangle \langle \lambda, \psi_n \rangle - \sigma \sum_{\lambda} \lambda \langle \lambda, \psi_m \rangle \langle -\lambda, \psi_n \rangle \right\}.
\]

(47)

For simplicity in the following we set

\[
E_{\Delta, n, \lambda} = \frac{\Delta}{2} \lambda \langle \langle n \mid e^{x(L_+ - L_-)} \mid n \rangle \rangle J_0(\Gamma), \quad \Gamma = \frac{2g_2}{\omega_E}
\]

then

\[
E_{\Delta, n, \lambda} = \begin{cases} 
(N) & \frac{\Delta}{2} \lambda e^{-\frac{\kappa^2}{2}} L_n(\kappa^2) J_0(\Gamma) \quad \text{where } \kappa = x \\
(K) & \frac{\Delta}{2} \lambda \frac{n!}{(2K)^n} (1 + \kappa^2)^{-K-n} F_n(\kappa^2, 2K) J_0(\Gamma) \quad \text{where } \kappa = \sinh(x) \\
(J) & \frac{\Delta}{2} \lambda \frac{n!}{2J F_n} (1 - \kappa^2)^{J-n} F_n(\kappa^2, 2J) J_0(\Gamma) \quad \text{where } \kappa = \sin(x)
\end{cases}
\]

from the results in section 3.1 of [14].

Then we rewrite (35) as

\[
i \frac{d}{dt} \Psi_0 = (\Delta / 2) \{ \tilde{H}_0'F + (\tilde{H}_1'F + \tilde{H}_F'') \} \Psi_0.
\]

(50)

and appeal to a perturbation method. Namely, we treat \( \tilde{H}_0'F \) a unperturbed Hamiltonian and the remaining a perturbed one.

It is easy to solve the equation \( i(d/dt)\Psi_0 = (\Delta / 2) \tilde{H}_0'F \Psi_0 \), which solution is given by

\[
\Psi_0 = \sum_n \sum_{\lambda} e^{-itE_{\Delta, n, \lambda}} c_{n, \lambda} \langle \lambda, \psi_n \rangle,
\]

where \( \{c_{n, \lambda}\} \) are constant, so we can set an ansatz to solve (50) as

\[
\Psi_0 = \sum_n \sum_{\lambda} e^{-itE_{\Delta, n, \lambda}} c_{n, \lambda}(t) \langle \lambda, \psi_n \rangle
\]

(51)

and determine the coefficients \( \{c_{n, \lambda}(t)\} \) from (50). However the (infinite) equations are almost impossible to solve. On the other hand we are interested in Quantum Computation, so let us restrict the ansatz (51) which is enough for our purpose.

**One Qubit Case**
The ansatz is simple
\[ \Psi_0 = \sum_{\lambda \in \{1, -1\}} e^{-itE_{\Delta,n,\lambda}} c_{n,\lambda}(t) |\{\lambda, \psi_n\}\rangle, \] (52)
where \( n \) is fixed. Substituting this into (50) and after some algebras we obtain
\[
ie^{-itE_{\Delta,n,\lambda}} \frac{d}{dt} c_{n,\lambda}(t) = \]
\[
\frac{\Delta}{2} \lambda \langle n | e^{x(L_+ - L_-)} | n \rangle \sum_{\alpha \neq 0} \sigma J_\alpha(\sigma \Gamma) \left( e^{-itE_{\Delta,n,\lambda}} c_{n,\lambda}(t) - \sigma e^{-itE_{\Delta,n,-\lambda}} c_{n,-\lambda}(t) \right),
\]
so we have
\[
\frac{d}{dt} c_{n,\lambda}(t) =
- \frac{i \Delta}{2} \lambda \langle n | e^{x(L_+ - L_-)} | n \rangle \sum_{\alpha \neq 0} \sigma J_\alpha(\sigma \Gamma) \left( e^{i\alpha \omega_E t} c_{n,\lambda}(t) - \sigma e^{i(\alpha \omega_E - 2E_{\Delta,n,-\lambda})} c_{n,-\lambda}(t) \right).
\]
Now we set a resonance condition: for some \( \alpha \neq 0 \in \mathbb{Z} \)
\[ \alpha \omega_E - 2E_{\Delta,n,-1} = 0 \iff (-\alpha) \omega_E - 2E_{\Delta,n,1} = 0, \] (53)
because \( E_{\Delta,n,-\lambda} = -E_{\Delta,n,\lambda} \). Then the remaining terms might be neglected (a kind of rotating wave approximation), so we have
\[
\frac{d}{dt} c_{n,1}(t) = i \frac{\Delta}{2} \langle n | e^{x(L_+ - L_-)} | n \rangle \sum_{\sigma} \sigma J_\alpha(\sigma \Gamma) c_{n,-1}(t),
\]
\[
\frac{d}{dt} c_{n,-1}(t) = -i \frac{\Delta}{2} \langle n | e^{x(L_+ - L_-)} | n \rangle \sum_{\sigma} \sigma J_{-\alpha}(\sigma \Gamma) c_{n,1}(t).
\]
By the way, from the fact \( J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x) = J_\alpha(-x) \) we have
\[
\sum_{\sigma} \sigma J_{-\alpha}(\sigma \Gamma) = \sum_{\sigma} \sigma J_\alpha(-\sigma \Gamma) = \sum_{\sigma} (-\sigma) J_\alpha(\sigma \Gamma) = - \sum_{\sigma} \sigma J_\alpha(\sigma \Gamma).
\]
Therefore the equations above become
\[
\frac{d}{dt} \begin{pmatrix} c_{n,1} \\ c_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & i \frac{\mathcal{R}}{2} \\ i \frac{\mathcal{R}}{2} & 0 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ c_{n,-1} \end{pmatrix},
\] (54)
with
\[
\mathcal{R} = \Delta \langle n | e^{x(L_+ - L_-)} | n \rangle \sum_{\sigma} \sigma J_\alpha(\sigma \Gamma) = \Delta \langle n | e^{x(L_+ - L_-)} | n \rangle J_\alpha(\Gamma) \frac{1 - (-1)^\alpha}{2}. \] (55)
We note that this $\mathcal{R}$ is just the Rabi (flopping) frequency. The solution is given by

\[
\begin{pmatrix}
c_{n,1}(t) \\
c_{n,-1}(t)
\end{pmatrix} = \begin{pmatrix}
\cos\left(\frac{\mathcal{R}}{2} t\right) & i\sin\left(\frac{\mathcal{R}}{2} t\right) \\
i\sin\left(\frac{\mathcal{R}}{2} t\right) & \cos\left(\frac{\mathcal{R}}{2} t\right)
\end{pmatrix} \begin{pmatrix}
c_{n,1}(0) \\
c_{n,-1}(0)
\end{pmatrix}.
\] (56)

**Two Qubit Case**

We identify the two–qubit space with two excited states, $|5\rangle$, $|6\rangle$, $|14\rangle$.

For $m < n$ the ansatz is

\[
\Psi_0 = \sum_{\lambda \in \{1, -1\}} e^{-it\Delta_{m,\lambda}} c_{m,\lambda}(t) |\lambda, \psi_m\rangle + \sum_{\lambda \in \{1, -1\}} e^{-it\Delta_{n,\lambda}} c_{n,\lambda}(t) |\lambda, \psi_n\rangle.
\] (57)

Substituting this into (56) and after long algebras we obtain

\[
ie^{-it\Delta_{m,\lambda}} \frac{d}{dt} c_{m,\lambda}(t) =
\]

\[
\Delta \lambda \langle \langle m | e^{i(L_+ - L_-)} | m \rangle \rangle \sum_{\alpha \neq 0} e^{i\alpha \sigma_E t} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \left( e^{-it\Delta_{m,\lambda}} c_{m,\lambda}(t) - \sigma e^{-it\Delta_{m,\lambda}} c_{m,-\lambda}(t) \right) +
\]

\[
\Delta \lambda e^{i\Omega(m-n)} \sum_{\alpha \neq 0} e^{i\alpha \sigma_E t} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \left( \langle \langle m | e^{i\sigma (L_+ - L_-)} | n \rangle \rangle \left( e^{-it\Delta_{n,\lambda}} c_{n,\lambda}(t) - \sigma e^{-it\Delta_{n,\lambda}} c_{n,-\lambda}(t) \right) \right),
\]

\[
ie^{-it\Delta_{n,\lambda}} \frac{d}{dt} c_{n,\lambda}(t) =
\]

\[
\Delta \lambda \langle \langle n | e^{i(L_+ - L_-)} | n \rangle \rangle \sum_{\alpha \neq 0} e^{i\alpha \sigma_E t} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \left( e^{-it\Delta_{n,\lambda}} c_{n,\lambda}(t) - \sigma e^{-it\Delta_{n,\lambda}} c_{n,-\lambda}(t) \right) +
\]

\[
\Delta \lambda e^{i\Omega(n-m)} \sum_{\alpha \neq 0} e^{i\alpha \sigma_E t} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \langle \langle n | e^{i\sigma (L_+ - L_-)} | m \rangle \rangle \left( e^{-it\Delta_{m,\lambda}} c_{m,\lambda}(t) - \sigma e^{-it\Delta_{m,\lambda}} c_{m,-\lambda}(t) \right).
\]

Then we have

\[
i \frac{d}{dt} c_{m,\lambda}(t) =
\]

\[
\Delta \lambda \langle \langle m | e^{i(L_+ - L_-)} | m \rangle \rangle \sum_{\alpha \neq 0} e^{i\alpha \sigma_E t} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \left( e^{i\alpha \sigma_E t} c_{m,\lambda}(t) - \sigma e^{i(\alpha \sigma_E - 2\Delta_{m,-\lambda})} c_{m,-\lambda}(t) \right) +
\]

\[
\Delta \lambda \sum_{\alpha} \langle \langle m | e^{i\sigma (L_+ - L_-)} | n \rangle \rangle \times
\]

\[
\left\{ e^{i\sigma (\alpha \sigma_E + \Omega(m-n) + \Delta_{m,\lambda} - \Delta_{n,\lambda})} c_{n,\lambda}(t) - \sigma e^{i\sigma (\alpha \sigma_E + \Omega(m-n) + \Delta_{m,\lambda} - \Delta_{n,\lambda})} c_{n,-\lambda}(t) \right\}.
\]

13
\[ i \frac{d}{dt} c_{n,\lambda}(t) = \]
\[ \frac{\Delta}{2} \lambda \langle \langle n | e^{i \sigma x (L_+ - L_-)} | n \rangle \rangle \sum_{\sigma \neq 0} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \left( e^{i \sigma \omega_E t} c_{n,\lambda}(t) - \sigma e^{i t(\alpha \omega_E - 2E_{\Delta,n,\lambda})} c_{n,-\lambda}(t) \right) + \]
\[ \frac{\Delta}{2} \lambda \sum_{\alpha} \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \langle \langle n | e^{\sigma x (L_+ - L_-)} | m \rangle \rangle \times \]
\[ \left\{ e^{it(\alpha \omega_E + \Omega(n-m)+E_{\Delta,n,\lambda}-E_{\Delta,m,\lambda})} c_{m,\lambda}(t) - \sigma e^{it(\alpha \omega_E + \Omega(n-m)+E_{\Delta,n,\lambda}-E_{\Delta,m,\lambda})} c_{m,-\lambda}(t) \right\} \]

Now we can set four (possible) resonance conditions:

1. a resonance condition: for some \( \alpha \in \mathbb{Z} \)

\[ \alpha \omega_E + \Omega(n - m) + E_{\Delta,m,1} - E_{\Delta,n,1} = 0 \]
\[ \iff (-\alpha) \omega_E + \Omega(n - m) + E_{\Delta,n,1} - E_{\Delta,m,1} = 0. \]  \tag{58}

Then by using the rotating wave approximation we obtain the equations

\[ \frac{d}{dt} \begin{pmatrix} c_{m,1} \\ c_{m,-1} \\ c_{n,1} \\ c_{n,-1} \end{pmatrix} = \begin{pmatrix} 0 & -i \frac{\mathcal{R}}{2} \\ 0 & 0 \\ -i \frac{\mathcal{R}}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_{m,1} \\ c_{m,-1} \\ c_{n,1} \\ c_{n,-1} \end{pmatrix} \]  \tag{59}

with

\[ \mathcal{R} = \Delta \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \langle \langle m | e^{\sigma x (L_+ - L_-)} | n \rangle \rangle \]
\[ = \Delta J_\alpha(\Gamma) \frac{1}{2} \left\{ \langle \langle m | e^{\sigma x (L_+ - L_-)} | n \rangle \rangle + (-1)^\alpha \langle \langle m | e^{-\sigma x (L_+ - L_-)} | n \rangle \rangle \right\}. \]  \tag{60}

Here we have used the following identity: neglecting \( \Delta \) in \( \mathcal{R} \)

\[ \mathcal{R} = \left\{ \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \langle \langle m | e^{\sigma x (L_+ - L_-)} | n \rangle \rangle \right\}^\dagger = \sum_{\sigma} \frac{J_\alpha(\sigma \Gamma)}{2} \langle \langle n | e^{-\sigma x (L_+ - L_-)} | m \rangle \rangle \]
\[ = \sum_{\sigma} \frac{J_\alpha(-\sigma \Gamma)}{2} \langle \langle n | e^{\sigma x (L_+ - L_-)} | m \rangle \rangle = \sum_{\sigma} \frac{J_{-\alpha}(\sigma \Gamma)}{2} \langle \langle n | e^{\sigma x (L_+ - L_-)} | m \rangle \rangle. \]

For the explicit value for matrix element \( \langle \langle m | e^{\sigma x (L_+ - L_-)} | n \rangle \rangle (\sigma = \pm 1) \) see \[14\].
The solution is

\[
\begin{pmatrix}
  c_{m,1}(t) \\
  c_{m,-1}(t) \\
  c_{n,1}(t) \\
  c_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
  \cos\left(t\frac{|R|}{2}\right) & -i\frac{R}{|R|}\sin\left(t\frac{|R|}{2}\right) \\
  1 & 1 \\
  -i\frac{R}{|R|}\sin\left(t\frac{|R|}{2}\right) & \cos\left(t\frac{|R|}{2}\right)
\end{pmatrix}
\begin{pmatrix}
  c_{m,1}(0) \\
  c_{m,-1}(0) \\
  c_{n,1}(0) \\
  c_{n,-1}(0)
\end{pmatrix},
\] (61)

(2) a resonance condition: for some \(\alpha \in \mathbb{Z}\)

\[
\alpha \omega_E + \Omega(m - n) + E_{\Delta,m,-1} - E_{\Delta,n,-1} = 0
\]

\[
\iff (-\alpha) \omega_E + \Omega(n - m) + E_{\Delta,n,-1} - E_{\Delta,m,-1} = 0.
\] (62)

Then by using the rotating wave approximation we obtain the equations

\[
\frac{d}{dt} \begin{pmatrix}
  c_{m,1} \\
  c_{m,-1} \\
  c_{n,1} \\
  c_{n,-1}
\end{pmatrix} =
\begin{pmatrix}
  0 & i\frac{R}{2} \\
  0 & 0 \\
  i\frac{R}{2} & 0
\end{pmatrix}
\begin{pmatrix}
  c_{m,1} \\
  c_{m,-1} \\
  c_{n,1} \\
  c_{n,-1}
\end{pmatrix},
\] (63)

with

\[
R = \Delta \sum_{\sigma} \frac{J_{\alpha}(\sigma \Gamma)}{2} \langle\langle m | e^{i \sigma (L_+ - L_-)} | n \rangle\rangle.
\] (64)

The solution is

\[
\begin{pmatrix}
  c_{m,1}(t) \\
  c_{m,-1}(t) \\
  c_{n,1}(t) \\
  c_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
  1 & \cos\left(t\frac{|R|}{2}\right) & i\frac{R}{|R|}\sin\left(t\frac{|R|}{2}\right) \\
  i\frac{R}{|R|}\sin\left(t\frac{|R|}{2}\right) & 1 & \cos\left(t\frac{|R|}{2}\right)
\end{pmatrix}
\begin{pmatrix}
  c_{m,1}(0) \\
  c_{m,-1}(0) \\
  c_{n,1}(0) \\
  c_{n,-1}(0)
\end{pmatrix},
\] (65)

(3) a resonance condition: for some \(\alpha \in \mathbb{Z}\)

\[
\alpha \omega_E + \Omega(m - n) + E_{\Delta,m,1} - E_{\Delta,n,-1} = 0
\]

\[
\iff (-\alpha) \omega_E + \Omega(n - m) + E_{\Delta,n,-1} - E_{\Delta,m,1} = 0.
\] (66)
Then by using the rotating wave approximation we obtain the equations

\[
\frac{d}{dt} \begin{pmatrix}
c_{m,1} \\
c_{m,-1} \\
c_{n,1} \\
c_{n,-1}
\end{pmatrix} = \begin{pmatrix}
0 & i \frac{R}{2} \\
0 & 0 \\
-i \frac{R}{2} & 0 \\
i \frac{R}{2} & 0
\end{pmatrix} \begin{pmatrix}
c_{m,1} \\
c_{m,-1} \\
c_{n,1} \\
c_{n,-1}
\end{pmatrix}
\]  

(67)

with

\[
\mathcal{R} = \Delta \sum_{\sigma} \sigma \frac{J_{\alpha}(\sigma \Gamma)}{2} \langle \langle m | e^{i\sigma x(L^+ - L^-)} | n \rangle \rangle
\]

\[
= \Delta J_{\alpha}(\Gamma) \frac{1}{2} \left\{ \langle \langle m | e^{i x(L^+ - L^-)} | n \rangle \rangle - (-1)^{\sigma} \langle \langle m | e^{-i x(L^+ - L^-)} | n \rangle \rangle \right\}.  
\]  

(68)

Here we have used the following identity: neglecting \( \Delta \) in \( \mathcal{R} \)

\[
\tilde{\mathcal{R}} = \left\{ \sum_{\sigma} \sigma \frac{J_{\alpha}(\sigma \Gamma)}{2} \langle \langle m | e^{i\sigma x(L^+ - L^-)} | n \rangle \rangle \right\}^{\dagger} = \sum_{\sigma} \sigma \frac{J_{\alpha}(\sigma \Gamma)}{2} \langle \langle n | e^{-i\sigma x(L^+ - L^-)} | m \rangle \rangle
\]

\[
= \sum_{\sigma} (-\sigma) \frac{J_{-\alpha}(\sigma \Gamma)}{2} \langle \langle n | e^{i\sigma x(L^+ - L^-)} | m \rangle \rangle = - \sum_{\sigma} \sigma \frac{J_{-\alpha}(\sigma \Gamma)}{2} \langle \langle n | e^{i\sigma x(L^+ - L^-)} | m \rangle \rangle,  
\]  

(69)

so

\[
\sum_{\sigma} \sigma \frac{J_{-\alpha}(\sigma \Gamma)}{2} \langle \langle n | e^{i\sigma x(L^+ - L^-)} | m \rangle \rangle = - \tilde{\mathcal{R}}.  
\]  

(70)

The solution is

\[
\begin{pmatrix}
c_{m,1}(t) \\
c_{m,-1}(t) \\
c_{n,1}(t) \\
c_{n,-1}(t)
\end{pmatrix} = \begin{pmatrix}
\cos\left(t \frac{|\mathcal{R}|}{2}\right) & i \frac{R}{|\mathcal{R}|} \sin\left(t \frac{|\mathcal{R}|}{2}\right) \\
0 & 1 \\
i \frac{R}{|\mathcal{R}|} \sin\left(t \frac{|\mathcal{R}|}{2}\right) & \cos\left(t \frac{|\mathcal{R}|}{2}\right)
\end{pmatrix} \begin{pmatrix}
c_{m,1}(0) \\
c_{m,-1}(0) \\
c_{n,1}(0) \\
c_{n,-1}(0)
\end{pmatrix}.  
\]  

(71)

(4) a resonance condition : for some \( \alpha \in \mathbb{Z} \)

\[
\alpha \omega_E + \Omega(m - n) + E_{\Delta,m,-1} - E_{\Delta,n,1} = 0
\]

\[
\iff (-\alpha) \omega_E + \Omega(n - m) + E_{\Delta,n,1} - E_{\Delta,m,-1} = 0.  
\]  

(72)

Then by using the rotating wave approximation we obtain the equations

\[
\frac{d}{dt} \begin{pmatrix}
c_{m,1} \\
c_{m,-1} \\
c_{n,1} \\
c_{n,-1}
\end{pmatrix} = \begin{pmatrix}
0 & -i \frac{R}{2} \\
0 & 0 \\
i \frac{R}{2} & 0 \\
i \frac{R}{2} & 0
\end{pmatrix} \begin{pmatrix}
c_{m,1} \\
c_{m,-1} \\
c_{n,1} \\
c_{n,-1}
\end{pmatrix}
\]  

(73)
with
\[ \mathcal{R} = \Delta \sum_{\sigma} J_{\alpha}(\sigma \Gamma) \langle m | e^{\sigma (L_+ - L_-)} | n \rangle. \] (74)

The solution is
\[
\begin{pmatrix}
    c_{m,1}(t) \\
    c_{m,-1}(t) \\
    c_{n,1}(t) \\
    c_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
    1 & \cos\left(\frac{\mathcal{R}}{2} t\right) & -i \frac{\mathcal{R}}{|\mathcal{R}|} \sin\left(\frac{\mathcal{R}}{2} t\right) \\
    -i \frac{\mathcal{R}}{|\mathcal{R}|} \sin\left(\frac{\mathcal{R}}{2} t\right) & \cos\left(\frac{\mathcal{R}}{2} t\right) & 1
\end{pmatrix}
\begin{pmatrix}
    c_{m,1}(0) \\
    c_{m,-1}(0) \\
    c_{n,1}(0) \\
    c_{n,-1}(0)
\end{pmatrix}. \] (75)

On the ansatz (57) we solved the Schrödinger equation (50) in the strong coupling regime (!) under the resonance conditions and rotating wave approximations, and obtained the unitary transformations of four types which are a generalization of [14]. They will play an important role in not only Quantum Computation but also Quantum Optics or Condensed Matter Physics.

It is very interesting that each Rabi frequency \( \mathcal{R} \) contains some Bessel functions \( \{ J_\alpha(\Gamma) \} \).

See [12] for an interesting experiment which the Bessel function \( J_0 \) appeared. See also [9].

By the way, we considered the case of one atom with two–level, so we would like to generalize our method to the case of \( n \) atoms (with two–level) interacting both the single radiation mode and external periodic fields like (\( n \) atoms trapped in a cavity with the photon interaction)

\[
\begin{pmatrix}
    |0\rangle & |0\rangle & |0\rangle \\
    |1\rangle & |1\rangle & |1\rangle
\end{pmatrix}
\]

Then the Hamiltonian may be
\[
\hat{H}_{nL} = \omega 1_M \otimes L_3 + g_1 \sum_{j=1}^{n} \sigma_1^{(j)} \otimes (L_+ + L_-) + \frac{\Delta}{2} \sum_{j=1}^{n} \sigma_3^{(j)} \otimes 1_L + g_2 \sum_{j=1}^{n} \cos(\omega_j t + \phi_j) \sigma_1^{(j)} \otimes 1_L,
\] (76)
where $M = 2^n$ and $\sigma_k^{(j)} (k = 1, 3)$ is

$$\sigma_k^{(j)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_k \otimes 1_2 \otimes \cdots \otimes 1_2 (j - \text{position}).$$

See [24] as another model similar to this (its model assumes the RWA from the starting point).

In the near future we will attempt an attack to this model. We would like to construct C-NOT operators for each pair of atoms (this is a very important subject in realistic Quantum Computation).

By the way, according to increase of the number of atoms (we are expecting at least $n = 100$ in the realistic quantum computation) we meet a very severe problem called Decoherence, see for example [10] and its references. We unfortunately don’t know how to control this at the present.

A generalization of the model to N–level system (see for example [15], [17], [19], [20]) is now under consideration and will be published in a separate paper$^1$.

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$^1$The author believes that it is important for us to consider the N–level system to prevent (or lessen) the decoherence problem.
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