Discrete Gauge Symmetries and the Weak Gravity Conjecture

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Abstract: In theories with discrete Abelian gauge groups, avoiding stable remnants leads to an upper bound on the product of a charged particle’s mass and the cutoff scale above which the effective description of the theory breaks down. To the extent that discrete gauge symmetries can arise at low energies from the spontaneous breaking of continuous ones, this suggests that a residual of the Weak Gravity Conjecture may persist in the Higgs phase. Here, we take a step towards making this expectation more precise by studying $\mathbb{Z}_N$ and $\mathbb{Z}_2^N$ gauge symmetries realised via Abelian Higgs models. In this setting, considering the effects of discrete hair on black holes reproduces existing bounds when the cutoff scale is identified with the scale of spontaneous symmetry breaking, and provides a mechanism through which discrete hair can be lost without modifying the gravitational sector. Moreover, applying the electric form of the conjecture to a dual description of the Abelian Higgs model leads to constraints consistent with existing bounds on discrete gauge groups. We explore the possible implications of these arguments for understanding the smallness of the weak scale compared to $M_{Pl}$. 

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1 Introduction

The Weak Gravity Conjecture (WGC) [1] states that a $U(1)$ gauge theory consistently coupled to gravity must contain a particle with mass $m$ and charge $g$ satisfying

$$m \lesssim g M_{Pl}.$$ (1.1)

Because this charge-to-mass-ratio is larger than that of an extremal black hole (BH), the corresponding state is referred to as being ‘super-extremal’. Since the electromagnetic force between two such particles would overcome the gravitational attraction between them, the conjecture is sometimes recast as the statement that ‘gravity is the weakest force’.

Motivation for the conjecture stems from the same type of considerations that question the viability of exact global symmetries in a theory of quantum gravity [2–6], namely the potential presence of a large number of stable Planckian remnants at the end of the evaporation process if BHs cannot lose their charge [7]. Unlike for global symmetries, for which the number of such remnants could be infinite, for a local theory this number is finite once charge quantization is taken into account, therefore making the problem less severe. For a $U(1)$ gauge theory with electric charge quantized in units of $g$, the number of different Planck-sized BH solutions potentially unable to shed their charge is of order $\sim 1/g$. In the regime of weak coupling, $g \ll 1$, this number could be very large, leading to a situation similar in spirit to that for global symmetries. The presence in the spectrum of an elementary particle satisfying Eq.(1.1) allows BHs to lose their charge as they evaporate, alleviating the problem. In particular, it implies that extremal BHs will be able to decay.\footnote{It is not clear which version of the WGC should be applied in the far infrared regime of effective field theories [8], and, in principle, the state satisfying Eq.(1.1) may not carry the unit of charge. Such considerations, however, will not be relevant for the work presented here, and in the following we stick to the ‘unit-charge’ version of the conjecture for simplicity of presentation.}
The same arguments can be applied to BHs carrying magnetic, rather than electric, charge, with the caveat that in a theory containing weakly coupled electrically charged particles we expect magnetic monopoles to be extended, solitonic objects. If the typical size of the monopole solution carrying the unit of magnetic charge is $L \sim \Lambda^{-1}$, demanding that it satisfies the super-extremality condition leads to an upper bound \[1\]^2

$$\Lambda \lesssim g \Lambda^2 .$$  \hspace{1cm} (1.2)

It was advocated in \[1\] that \(\Lambda\) ought to be interpreted as a cutoff scale beyond which an effective description of the \(U(1)\) gauge theory needs to be extended in order to account for the monopoles’ structure. For instance, if the \(U(1)\) is embedded into a broken non-Abelian gauge group, the scale \(\Lambda\) corresponds to the appearance of new degrees of freedom, specifically the gauge bosons associated to the broken directions.

The WGC is further supported by the absence of counter-examples in non-trivial string constructions, as well as by additional low energy arguments involving BHs, and an apparent connection to cosmic censorship (see \[1, 9–18\]). There exist many variants of the conjecture \[19–27\], and attempts to find a proof have been the focus of recent literature \[28–31\].

The existence of asymptotic observables clearly plays a crucial role in arguments motivating the WGC from infrared considerations involving BH physics as originally discussed in \[1\]. For a \(U(1)\) gauge theory, the presence of two such observables, electric and magnetic charge, leads to two versions of the conjecture, corresponding to Eq. (1.1) and (1.2) respectively. If the \(U(1)\) is spontaneously broken, the absence of a long-range classical force \textit{a priori} precludes similar arguments to survive in the broken theory.

An exception to the above statement arises if a discrete subgroup is left unbroken, the simplest example being a \(U(1)\) gauge theory broken down to a \(Z_N\) subgroup as a result of the non-zero vacuum expectation value (vev) of a Higgs field carrying charge \(gN\), with \(N > 1\). In the \(Z_N\) phase, electric fields are screened, decaying exponentially away from the source, whereas magnetic flux remains confined into flux tubes, or cosmic strings. In an Abelian Higgs model, the amount of flux confined into strings is quantized in units of \(2\pi/(gN)\), and the tension of the string carrying the unit of flux is \(T_s \sim v^2\), with \(v\) the Higgs vev. Despite there being no massless fields associated with a discrete gauge symmetry, and therefore no long-range classical force, the \(Z_N\) theory nevertheless features long-range interactions, namely Aharonov-Bohm scattering of cosmic strings with matter \[32, 33\]. By means of Aharonov-Bohm scattering experiments, discrete electric charge can be measured from far away, and becomes an asymptotic observable.

As a result, BHs may carry discrete charge as hair, a possibility first discussed in \[34\], with \[35, 36\] extending the argument to the non-Abelian case. (Notice this type of hair is purely quantum, crucially depending on both charge quantization and the associated Aharonov-Bohm scattering, and therefore falls outside of the scope of classical no-hair theorems.) BHs carrying discrete electric charge therefore qualify for the same type of thought experiments that led to the WGC for continuous gauge symmetries, as first discussed by

\footnote{The same result follows by demanding that the unit-charge magnetic monopole is not itself a BH.}
Dvali and collaborators [37–39]. In a theory with $\mathbb{Z}_N$ gauge group, if the state $\psi$ that carries the unit of discrete charge has mass $m$, then the requirement that there be no remnants left that are stabilized by discrete charge leads to an upper bound relating $m$ and the scale $\Lambda$ beyond which an effective description of the $\mathbb{Z}_N$ gauge theory is expected to no longer be valid, of the form [37, 39]

$$m \cdot \Lambda \lesssim \frac{M_{Pl}^2}{N}. \quad (1.3)$$

The same bound as in Eq.(1.3) also applies if the gauge group is $\mathbb{Z}_N^2$ [37, 38], under the assumption that the states carrying the unit of electric charge under each of the $\mathbb{Z}_2$ factors all have mass $m$. However, the situation in this case may in detail be different, with a variety of arguments suggesting individual upper bounds on both $m$ and $\Lambda$, of the form [37, 38]

$$m, \Lambda \lesssim \frac{M_{Pl}}{\sqrt{N}}. \quad (1.4)$$

The results of [37–39] suggest that a non-trivial version of the WGC survives in the Higgsed phase of a $U(1)$ gauge theory so long as a discrete subgroup is left unbroken. We take a step towards making this expectation more precise by studying scenarios where the discrete symmetry is realised through the spontaneous breaking of a $U(1)$ gauge symmetry, focusing for simplicity on the regime in which the separation of scales between the vector mass and the Higgs vev is not too large. This leads us to the following observations in Abelian Higgs models that realize discrete symmetries in the infrared:

- Basic scaling properties of the effect of discrete hair on black holes [40] reproduce Eq.(1.3) for models of spontaneous symmetry breaking that give either $\mathbb{Z}_N$ and $\mathbb{Z}_N^2$ gauge groups at low energies, with $\Lambda$ corresponding to the Higgs vev $v$.

- For an Abelian Higgs model giving rise to a discrete gauge theory in the infrared, the potentially large impact of cosmic string loop nucleation could provide a mechanism for allowing small black holes to lose their discrete hair without requiring the scale $\Lambda$ to imply new physics in the gravitational sector.

- The upper bound in Eq.(1.3), again with $\Lambda \sim v$, also follows from applying the electric form of the WGC in an equivalent version of the theory in terms of two $U(1)$ gauge groups coupled through a topological term, with the additional $U(1)$ factor emerging out of a duality transformation [6, 40, 41].

- The application of the electric form of the WGC to the above dual picture leads to individual bounds on $m$ and $\Lambda \sim v$ that are consistent with Eq.(1.3), for both $\mathbb{Z}_N$ and $\mathbb{Z}_N^2$ gauge groups. In the latter case, the multi-field version of the WGC [22] further allows us to recover Eq.(1.4).

The dual picture of the Abelian Higgs model illuminates key features of the bounds in Eqs.(1.3)-(1.4). In particular, electric charge under the emergent $U(1)$ factor corresponds to the units of magnetic flux carried by cosmic strings in the Higgs description, and the
appearance of both $m$ and $\Lambda \sim v$ in Eqs.(1.3)-(1.4) admits a simple physical interpretation: light charged particles must be part of the spectrum for BHs to decay, whereas an upper bound on the scale of spontaneous symmetry breaking ensures that there exists a mechanism through which discrete charge can be lost, namely the non-zero electric field that is generated when a virtual cosmic string loop wraps around the BH horizon.

The rest of this article is organized as follows: In section 2 we briefly review the arguments of [37–39] leading to Eqs.(1.3)-(1.4), and show how similar upper bounds can be obtained by considering the effect of discrete hair on BHs when the discrete symmetry is realised through Higgsing. A UV completion into a model of spontaneous symmetry breaking also provides a potential mechanism for small black holes to lose their discrete hair without requiring a parametrically lower cutoff in the gravitational sector. In section 3, we review how a discrete gauge symmetry written in terms of an Abelian Higgs model is dual to a theory containing two $U(1)$ gauge groups, and show how applying the WGC to the dual theory leads to non-trivial constraints consistent with those of section 2. Section 4 addresses ways in which the WGC applied to a $Z_N$ gauge theory can lead to novel constraints on effective field theory parameters, with particular relevance to the apparent fine-tuning of the weak scale. We present our conclusions in section 5.

2 No remnants stabilized by discrete charge

2.1 Non-perturbative black hole arguments

We now review the argument presented in [37, 39] leading to the upper bound in Eq.(1.3) for theories with gauge group $Z_N$. In order to simplify the discussion, we assume that the only degree of freedom charged under the discrete symmetry is a state $\psi$, with mass $m$, which carries the unit of $Z_N$ charge. Below some cutoff scale, the effective description of the theory is that of $\psi$ particles with $Z_N$-preserving interactions.

We start by considering a BH carrying $\sim N$ units of discrete charge, with initial size much larger than the Compton wavelength of the unit-charge particle. This ensures that the initial temperature of the BH is $T \ll m$, so that as the BH evaporates it does so without losing any of its charge (Hawking evaporation into particles with mass larger than the BH temperature is exponentially suppressed). However, if we want there to be no remnants stabilized by discrete hair at the end of the evaporation process, the BH must be able to shed its charge when it reaches some size of order $\Lambda^{-1}$, with $\Lambda$ an energy scale somewhere in the range $m \lesssim \Lambda \lesssim M_{Pl}$. At this stage, the mass of the BH is $M \sim M_{Pl}^2/\Lambda$, which must be larger than $N$ times the mass of the unit-charge quantum. By imposing this basic kinematic requirement one obtains the upper bound in Eq.(1.3), i.e.

$$M \sim \frac{M_{Pl}^2}{\Lambda} \gtrsim N \cdot m \quad \Rightarrow \quad m \cdot \Lambda \lesssim \frac{M_{Pl}^2}{N}.$$ 

The physical effect of discrete electric charge on BH properties, first studied in [40], is exponentially suppressed in the regime where the theory can be described in terms of effective $Z_N$-preserving interactions among light degrees of freedom. So long as this effective description is valid, the behaviour of a BH carrying discrete charge is approximately
the same as for the Schwarzschild solution, as noted in [37, 39]. In particular, up to exponentially small corrections, no electric field exists outside the BH horizon, and Hawking evaporation proceeds equally into $\psi$ and $\bar{\psi}$ quanta. As a result, in this regime, the BH will not be able to lose its discrete hair. The scale $\Lambda$ must therefore be identified with a cutoff scale beyond which such an effective description is no longer valid [37, 39]. Specifically, new effects must become relevant at the scale $\Lambda$ that allow the BH to shed its discrete hair, significantly deviating from its previous Schwarzschild behaviour. In [39] a stronger constraint on $\Lambda$ is advocated for by making the additional assumption that the hair loss mechanism is intrinsically thermal, and that a BH of size $R \sim \Lambda^{-1}$ radiates preferentially into modes with energies of order $\Lambda$. This leads to the bound $\Lambda \lesssim M_{Pl}/\sqrt{N}$, which with the implicit constraint $m \lesssim \Lambda$ combines with Eq.(1.3) to give an identical bound on $m$. Although this assumption may in principle hold in other UV completions of a discrete gauge symmetry, it appears unjustified in the context of an Abelian Higgs model, as we discuss in section 2.2. We therefore regard Eq.(1.3) as the only solid upper bound applicable to $Z_N$ gauge theories.

The same general argument applies when the gauge group is $Z_N^2$ instead of $Z_N$, leading to the same upper bound as in Eq.(1.3), under the assumption that all the states $\psi_i$ carrying the unit of charge under the $N$ individual $Z_2$ factors have the same mass $m$ [37, 38]. In this case, the argument can be taken a step further by making the observation that in theories containing $N$ stable species the gravitational cutoff is expected to be lowered down to $M_{Pl}/\sqrt{N}$ [38]. The requirement that both $m$ and $\Lambda$ are below the gravitational cutoff of the theory therefore leads to individual upper bounds of the form

$$m, \Lambda \lesssim \frac{M_{Pl}}{\sqrt{N}},$$

as in Eq.(1.4).\(^3\)

The authors of [37–39] further speculate that in both cases the UV completion required at the scale $\Lambda$ must be necessarily linked to the gravitational sector. Although there might be realisations of a discrete gauge symmetry where this is the case, if the theory is implemented through an Abelian Higgs model, a more plausible expectation is that $\Lambda$ may just correspond to the scale of spontaneous symmetry breaking, as we discuss next.

2.2 Effect of discrete hair on black holes

The effect of electric $Z_N$ charge on BH thermodynamics was first explored by Coleman, Preskill, and Wilczek in [40], where the discrete gauge symmetry is realised through a model of spontaneous symmetry breaking. It is argued in [40] that the leading effect of discrete hair on BHs is due to a process whereby a cosmic string loop is nucleated on the horizon surface, grows to envelope the BH, and re-annihilates at the antipodal point. In the Euclidean path integral approach to BH thermodynamics, this process is expressed through the existence of instantons, specifically vortex solutions on the $r - \tau$ plane ($\tau$ being the periodic Euclidean time coordinate). Different units of magnetic flux wrapping...
the BH in the space-time picture correspond to vortices with different winding numbers of the scalar field in the Euclidean formalism.

When computing the partition function relevant for a description of the system in the context of a given thermodynamic ensemble, one must therefore also sum over sectors with different scalar field vorticities. In the canonical ensemble, where both temperature $\beta^{-1}$ and charge $Q$ are fixed, the partition function factors into a discrete sum of the form [40]

$$Z(\beta, Q) \sim \sum_{k=-\infty}^{+\infty} e^{i2\pi Qk/N} Z_k(\beta, Q),$$

where $Z_k$ represents the partition function in a sector with winding number $k$, and appears in the sum weighted by a phase factor $e^{i2\pi Qk/N}$, corresponding to the Aharonov-Bohm phase that a string carrying $k$ units of flux picks up as it envelops a BH with charge $Q$.

Field configurations with higher winding number have higher action costs, and, as a result, Eq.(2.1) can be evaluated in a semiclassical expansion. The sum is dominated by the $k = 0$ configuration, and contributions from the sectors with $k = \pm 1$ provide the leading charge-dependent correction. To leading order, the effect of $Z_N$ charge on the thermodynamic partition function can be written as, parametrically [40]

$$\log \left( \frac{Z(\beta, Q)}{Z(\beta)} \right)^{-1} \sim \left[ 1 - \cos \left( \frac{2\pi Q}{N} \right) \right] e^{-\Delta S_v},$$

where $\Delta S_v$ represents the action of the $k = 1$ vortex, and the correction vanishes whenever $Q$ is an integer multiple of $N$, as expected.

Of particular interest is the status of the electric field generated by a BH carrying discrete hair. At the classical level, i.e. for $k = 0$, the electric field vanishes. At the quantum level this is no longer true, and a non-zero expectation value of the electric field in the radial direction arises when taking into account the contribution from sectors of non-zero winding number. As before, this can be computed in a semiclassical expansion, and, to leading order, one obtains an expression of the form [40]:

$$\langle E_r(\tau) \rangle \sim \sin \left( \frac{2\pi Q}{N} \right) F_{rr}(\tau) e^{-\Delta S_v},$$

where $F_{rr}(\tau)$ corresponds to the Euclidean magnetic field component for the unit-flux vortex. This expression is consistent with the space-time picture put forward in [40]: as the string wraps around the BH, the moving magnetic flux inside the string generates an electric field in the radial direction. This electric field is only non-zero when a string is nucleated, and therefore its expectation value is suppressed by a factor related to the corresponding tunnelling rate, with such suppression provided by the last factor in Eq.(2.3).

The validity of the semiclassical expansion remains so long as $\Delta S_v \gg 1$, and there are two limiting regimes in which an analytic expression for the vortex action can be obtained: the thin- and thick-string limits, corresponding to the thickness of the string being much smaller or much larger than the size of the BH. \footnote{Both limits were discussed in [40], with [44] and [45] providing important extra considerations in the thin- and thick-string regimes respectively.}
respectively, for BH sizes much bigger or much smaller than the Compton wavelength of the vector. In particular, the vortex action in the thin-string limit is given by, parametrically

$$\Delta S_v \sim r_+^2 v^2,$$

with $r_+$ the horizon radius. This expression intuitively corresponds to the tension of the string $T_s \sim v^2$ times the corresponding world-sheet area. This is the regime of interest for large BHs, with sizes $r_+ \gg v^{-1}$, in which case the discrete symmetry is well approximated through an effective description in terms of light states with $\mathbb{Z}_N$-preserving interactions.

As anticipated in section 2.1, the properties of a BH carrying discrete charge are therefore much like those of its uncharged counterpart, up to exponentially small corrections. However, the semiclassical expansion will break down as the BH gets small. If the separation of scales between the vector mass and the scale of spontaneous symmetry breaking is not too large, i.e. if the coupling $gN$ is not much smaller than 1, then the semiclassical analysis breaks down as we approach the thick-string limit, and the size of the BH becomes $r_+ \sim v^{-1}$. In this regime, the effect of discrete hair on such small BHs is potentially large. In particular, the existence of an unsuppressed electric field would allow the BH to lose its hair by the same mechanism through which the Reissner-Nördstrom solution sheds its charge, i.e. through a combination of Schwinger pair production of particle and anti-particle pairs, and asymmetric Hawking evaporation. Demanding that a BH of size $v^{-1}$, and therefore mass $M \sim M_{Pl}^2/v$, is kinematically allowed to lose $\sim N$ units of discrete hair requires

$$m \cdot v \lesssim \frac{M_{Pl}^2}{N},$$

which coincides with Eq.(1.3) after identifying $\Lambda \sim v$.

We expect Eq.(2.5) to also approximately hold when the discrete group is $\mathbb{Z}_2^N$, instead of $\mathbb{Z}_N$. In this case, the appropriate version of Eq.(2.2) now reads

$$\log \left( \frac{Z(\beta, Q)}{Z(\beta)} \right)^{-1} \sim N \left[ 1 - \cos (\pi Q) \right] e^{-\Delta S_v},$$

and the electric field of Eq.(2.3) would now include $N$ different components, corresponding to the individual $\mathbb{Z}_2$ factors. Despite the $N$-dependent enhancement present in this case, the expression for $\Delta S_v$ is the same as in the $\mathbb{Z}_N$ case, assuming all $\mathbb{Z}_2$ factors are UV-completed into an Abelian Higgs model at the same scale $v$. Thus, up to corrections scaling as $\log N$, we obtain the same parametric bound as in Eq.(2.5).

We emphasize that verifying the expectation that small BHs are capable of losing their hair as a result of cosmic string loop nucleation would require a dedicated calculation well beyond the regime in which existing results are applicable [40, 44, 45]. Our discussion, however, raises the more conservative possibility that $\Lambda$, understood as a proxy for the scale at which physics responsible for discrete hair loss must appear, may simply correspond to the scale of UV completion into an Abelian Higgs model, as opposed to being intrinsically related to physics of the gravitational sector.
3 Towards a Weak Gravity Conjecture for discrete gauge symmetries

We now explore the extent to which the above bounds on discrete gauge symmetries may be further understood by applying the electric form of the WGC to Abelian Higgs models and their dual descriptions. In section 3.1 we review how a discrete gauge theory realised in terms of a spontaneously broken $U(1)$ gauge group admits a dual description in terms of two $U(1)$ factors coupled through a topological term. In 3.2 we show how applying the electric form of the WGC in the dual picture leads to constraints consistent with those of Eq.(1.3) and (1.4), so long as $\Lambda$ is identified with the scale of spontaneous symmetry breaking.

3.1 Dual description of Higgsing

An equivalent description of a $Z_N$ gauge theory at low energies can be written in terms of two $U(1)$ gauge groups coupled through a topological term [6, 40, 41]. We refer to the two Abelian factors as $U(1)_A$ and $U(1)_B$, with the associated gauge potentials being a 1-form $A$ and a 2-form $B$. The gauge couplings corresponding to the $U(1)_A$ and $U(1)_B$ factors are $g$ and $f$, and have mass dimensions 0 and 1 respectively.

The effective lagrangian contains kinetic terms for both gauge fields, as well as a $B \wedge F$ coupling of the form [6]

$$
\mathcal{L} \supset -\frac{1}{12f^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{N}{8\pi} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}F_{\rho\sigma},
$$

where $F \equiv dA$, $H \equiv dB$, and we have used a normalization where the gauge coupling sits in front of the kinetic term. In this description, the order of the discrete symmetry corresponds to the integer $N$ in front of the $B \wedge F$ coupling. The lagrangian may contain further interaction terms, not included in Eq.(3.1), involving charged dynamical objects, with electric charge under $U(1)_A$ and $U(1)_B$ quantized in units of $g$ and $f$ respectively.

Objects carrying electric $U(1)_A$ charge correspond to point particles, whereas those charged under $U(1)_B$ are strings. Wilson loop operators describing the dynamics of charged objects are given by

$$
W_A(q, C) = \exp \left( iq \oint_C A \right), \quad \text{and} \quad W_B(k, \Sigma) = \exp \left( ik \oint_{\Sigma} B \right),
$$

for a particle with world-line $C$ carrying $q$ units of $U(1)_A$ charge, and a string with world-sheet $\Sigma$ carrying $k$ units of $U(1)_B$ charge, respectively.

Although the operators in Eq.(3.2) can be defined for any $q, k \in \mathbb{Z}$, only operators defined modulo $N$ label different observables. In physical terms, the only long-range interaction available to measure $U(1)_A$ and $U(1)_B$ charge is Aharonov-Bohm scattering of particles and strings. The Aharonov-Bohm phase that a string carrying $k$ units of electric $U(1)_B$ charge picks up when looped around a particle carrying $U(1)_A$ charge $q$ is given by

$$
\exp \left( \frac{2\pi q k}{N} \right). \quad (3.3)
$$
Eq. (3.3) makes it manifest that observables in this theory are labelled in $\mathbb{Z}_N$, with electric $U(1)_A$ and $U(1)_B$ charge only defined modulo $N$. Indeed, a gauge invariant order parameter that would distinguish the $\mathbb{Z}_N$ phase of the theory from the completely broken one (i.e. $N = 1$) necessarily involves the two types of operators in Eq. (3.2), with $C$ intersecting $\Sigma$ once [46], essentially capturing the operation described above.

The theory of Eq. (3.1) is dual to a more familiar realization of a $\mathbb{Z}_N$ gauge symmetry: an Abelian Higgs model in which the charge of the condensate is $N$ times the corresponding charge quantum. To make the relation between the two theories manifest it is convenient to rewrite the last term in Eq. (3.1) in terms of $H$ (after integration by parts), and introduce an extra degree of freedom $\varphi$ acting as a Lagrange multiplier that would enforce the exactness of $H$ dynamically. After rescaling the gauge fields so that the corresponding kinetic terms are canonically normalized, the lagrangian is given by

$$L \supset -\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{Ngf}{12\pi} \varepsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho} A_{\sigma} - \frac{1}{6} \varphi \varepsilon^{\mu\nu\rho\sigma} \partial_{\mu} H_{\nu\rho\sigma}. \tag{3.4}$$

We may now integrate out $H$ from Eq. (3.4) by using the corresponding equations of motion, which read

$$H_{\mu\nu\rho} = -\varepsilon^{\mu\nu\rho\sigma} \left( \partial_{\sigma} \varphi - \frac{Ngf}{2\pi} A_{\sigma} \right). \tag{3.5}$$

Finally, plugging Eq. (3.5) back into (3.4), one finds

$$L \supset -\frac{1}{2} (\partial_{\mu} \varphi - N g v A_{\mu})^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{3.6}$$

where we have defined $v \equiv f/2\pi$.

Eq. (3.6) is precisely the effective lagrangian describing the spontaneous breaking of a continuous Abelian gauge theory through a Higgs field with vev $v$, after integrating out the radial mode. The energy scale defined by the gauge coupling of the $U(1)_B$ factor corresponds in the Abelian Higgs model to the scale of spontaneous symmetry breaking. In the dual picture, strings carrying $k$ units of electric $U(1)_B$ charge correspond in the Higgs description to cosmic strings threaded by $k$ units of magnetic flux.

Our discussion makes it manifest that electric $U(1)_A$ charge matches on to charge under the spontaneously broken $U(1)$ in the Abelian Higgs model. This is the type of charge we have referred to as discrete hair, which BHs can carry [34, 40]. It is then natural to wonder whether BHs may also carry electric $U(1)_B$ hair. Indeed, non-trivial solutions to the Einstein-Maxwell-Higgs equations exist that correspond to a BH threaded by an infinitely long flux-tube [47, 48], the $U(1)_B$ charge of the configuration corresponding to the units of flux confined inside the string. Although there is a sense in which such charge may be regarded as BH hair [48], it is clearly different from the type of hair we have so far been discussing. In particular, it is unclear how arguments similar to those of [37–39]

5 Strictly speaking, the far infrared regime of a discrete gauge symmetry admits a universal description as given by the last term in Eq. (3.1), corresponding to a purely topological theory [6]. We note that such a theory admits UV completions different than an Abelian Higgs model, although we will not consider any alternative possibilities in this work.
could be made. We will not discuss this type of BHs further in this work, but note that a more detailed consideration of their properties might provide valuable insight.

If the gauge group is $\mathbb{Z}_N^2$, instead of $\mathbb{Z}_N$, the equivalent version of Eq.(3.1) is given by

$$\mathcal{L} \supset -\sum_{j=1}^{N} \left( \frac{1}{12f^2} H_{j\mu
u\rho} H_{j}^{\mu\nu\rho} + \frac{1}{4g^2} F_{j\mu\nu} F_{j}^{\mu\nu} + \frac{1}{4\pi} \epsilon_{\mu\nu\rho\sigma} B_{j\mu\nu} F_{j\rho\sigma} \right),$$  

(3.7)

where, for simplicity, we have assumed that the gauge couplings of the $N U(1)_A$ and $U(1)_B$ factors are all equal to $g$ and $f$ respectively. Eq.(3.7) is dual to $N$ copies of a $U(1)$ gauge symmetry spontaneously broken down to $\mathbb{Z}_2$ by Higgs fields with vev $v = f/2\pi$, and charge twice the electric charge quantum.

### 3.2 Weak Gravity Conjecture

The value of rewriting the effective description of a discrete gauge theory in terms of the dual picture described in section 3.1 lies on the fact that electric $\mathbb{Z}_N$ charge and magnetic flux quanta are put on the same footing: in the dual description, they correspond to electric charge of the $U(1)_A$ and $U(1)_B$ factors, respectively. Although $U(1)_A$ and $U(1)_B$ charge is only defined modulo $N$, and only their product can be measured at long distances, the dual description suggests that if there is a version of the WGC that applies for theories with discrete gauge groups, it might involve both $U(1)$ factors.

We can gain some insight into how a version of the WGC for theories with $\mathbb{Z}_N$ gauge group may ultimately arise by applying the electric form of the conjecture in the dual picture to both $U(1)_A$ and $U(1)_B$.\(^6\) Doing so translates into the requirement that there be particles with mass $m$ charged under $U(1)_A$, and strings with tension $T_s$ charged under $U(1)_B$ satisfying

$$m \lesssim g M_{Pl}, \quad \text{and} \quad T_s \lesssim f M_{Pl}.$$  

(3.8)

We may further interpret Eq.(3.8) in the context of an spontaneously broken $U(1)$ gauge theory, where $f \sim v$, and $T_s \sim v^2$. Moreover, for an Abelian Higgs UV completion to remain valid up to arbitrarily high scales, the theory must remain weakly coupled, and so $g \lesssim 1/N$. In this context, Eq.(3.8) therefore implies

$$m \lesssim \frac{M_{Pl}}{N}, \quad \text{and} \quad v \lesssim M_{Pl}.$$  

(3.9)

The modification of the upper bound on $m$ in going from Eq.(3.8) to (3.9) is better justified under the assumption that the perturbativity bound on $g$ is roughly saturated. This is the regime of interest in our discussion of the effect of discrete hair on BHs in section 2.2, and we assume it holds in the following. In particular, from Eq.(3.9) it follows that

$$m \cdot v \lesssim \frac{M_{Pl}^2}{N}.$$  

\(^6\)Of course, in the dual picture with gauge potentials $A$ and $B$, we can further dualize $A$ to another 1-form that may be interpreted as a matter field Higgsing the $U(1)_B$ symmetry down to $\mathbb{Z}_N$ [6]. In this case one may question the utility of applying the WGC to $U(1)_A$ and $U(1)_B$, insofar as there is always another duality frame in which one is a matter field responsible for Higgsing the other. There are, however, a variety of arguments in favor of applying the WGC to this scenario (see e.g. [49]), and we proceed apace.
which coincides with the result of Eq. (2.5).

The independent upper bound on $m$ in Eq. (3.9) is more restrictive than the result in Eq. (2.5), which allows $m$ to be as high as $M_{Pl}/\sqrt{N}$ so long as the theory is UV-completed into an Abelian Higgs model at roughly the same scale. Indeed, the arguments of section 2 only justify the weaker version Eq. (2.5). A more sophisticated treatment might lead to Eq. (2.5) as the true version of the WGC that applies to theories with $Z_N$ gauge group. Meanwhile, we regard it as encouraging that Eq. (2.5) can be obtained from this approach. (Perhaps an argument could be made that the WGC bound should only be applied to the product of $m$ and $T_s$, as opposed to individually as in Eq. (3.8), based on the observation that only the product of $U(1)_A$ and $U(1)_B$ charge can be measured asymptotically.) Likewise, while the bound on $v$ from Eq. (3.9) is equivalent to simply requiring that the Higgs vev lie below the gravitational cutoff, it is reassuring that the same conclusion emerges directly from applying the WGC to $U(1)_B$.

Discussing the WGC in the context of the dual theory also clarifies some of the potential differences between $Z_N$ and $Z_N^2$ theories. In the latter case, applying the conjecture to a theory with $N$ copies of the $U(1)_A$ and $U(1)_B$ factors leads to

$$m \lesssim \frac{gM_{Pl}}{\sqrt{N}} , \quad \text{and} \quad T_s \lesssim \frac{fM_{Pl}}{\sqrt{N}} , \quad (3.10)$$

where the factors of $\sqrt{N}$ correspond to the modification of the WGC when applied to theories with multiple gauge groups [22], and we have made the simplifying assumption that all unit-charge particles and strings have the same mass $m$, and tension $T_s$ respectively. In an Abelian Higgs model, $f \sim v$, and $T_s \sim v^2$ as before, but now perturbativity only requires $g \lesssim 1/2 \sim 1$. Eq. (3.10) therefore implies independent upper bounds on both $m$ and $v$, of the form

$$m, v \lesssim \frac{M_{Pl}}{\sqrt{N}} , \quad (3.11)$$

which coincides with Eq. (1.4) so long as $\Lambda$ is identified with the Higgs vev. This result further fits in well with the expectation that the gravitational cutoff is lowered down to $M_{Pl}/\sqrt{N}$ for a $Z_N^2$ gauge theory. In this context, Eq. (3.11) just corresponds to the reasonable statement that both the mass of the unit-charge particle, and the scale of spontaneous symmetry breaking, must be below the cutoff scale.

4 Naturalness in the Swampland

The aim behind the Swampland Program is to identify consistency conditions that effective field theories need to satisfy in order to be compatible with a further UV completion into a theory of quantum gravity [50]. Much of the intuition behind Swampland criteria, most of which remain in conjectural form, stems from patterns observed in known string theory constructions, with infrared arguments based on BH physics, when available, providing extra support (a list of current Swampland conjectures includes [1, 50–56]). The WGC is perhaps the best-known condition in the list of Swampland criteria, and potential implications of the conjecture have been studied in a variety of circumstances. For example,
variations of the conjecture can lead to significant constraints on models of inflation that require super-Planckian field ranges [57, 58], rule out models that include parametrically light St"uckelberg massive photons [49], or even explain the apparent fine-tuning of the weak scale through arguments linking it to the size of the cosmological constant [59, 60]. Although these statements crucially depend on additional assumptions, the most obvious being the specific form of the conjecture to be applied in the low energy regime of effective theories [8, 61–63], such efforts are valuable to the extent that they link our intuition about quantum gravity to experimental observation.

Indeed, the form of Eq.(1.1), which sets an upper bound on the mass of a state charged under an Abelian gauge symmetry, raises the question of whether the conjecture could provide an explanation of the smallness of the weak scale compared to $M_{Pl}$. Although not directly applicable to the SM Higgs, as the corresponding symmetry would be broken, it could still be applied to a different state whose mass arises from the Higgs vev, so that Eq.(1.1) would translate into an indirect constraint on $v_{SM}$. The simplest version of this idea, proposed in [22], is realised if the $U(1)_{B-L}$ symmetry of the SM is gauged, and the constraint Eq.(1.1) applied to one of the neutrinos, i.e. $m_\nu \lesssim g_{B-L} M_{Pl}$. Since $m_\nu \lesssim 0.1$ eV this would require $g_{B-L} \lesssim 10^{-28}$, which is consistent with the current experimental upper bound of $10^{-24}$ [64, 65]. If neutrino masses arise through a Yukawa coupling to the SM Higgs, then the weak scale would be indirectly constrained by the WGC.

The above argument employing the electric WGC is, however, undone by its magnetic counterpart. Specifically, the magnetic version of the conjecture, which applies to unbroken $U(1)$ gauge theories, demands some form of UV completion at the same scale at which the electrically charged super-extremal state is present. Obvious ways of addressing this issue, such as embedding the $U(1)_{B-L}$ into an $SU(2)$ gauge group at the scale in question, are clearly in conflict with experimental observation.

This difficulty disappears if $U(1)_{B-L}$ is broken down to a gauge $Z_N$ subgroup. 7 Now, the scale at which particles carrying $Z_N$ charge appear may be parametrically below the scale of UV completion into an Abelian Higgs model. If $U(1)_{B-L}$ is spontaneously broken at a scale $v \sim M_{Pl}$, then electrically charged states must be present below $\sim M_{Pl}/N$. As applied to neutrinos, this leads to

$$m_\nu \lesssim \frac{M_{Pl}}{N} \, ,$$

(4.1)

which with $N \gtrsim 10^{28}$ requires $m_\nu \lesssim 0.1$ eV. If the neutrino mass arises through a tiny Yukawa coupling with the SM Higgs, $m_\nu \sim y_\nu v_{SM}$, then Eq.(4.1) implies

$$v_{SM} \lesssim \frac{M_{Pl}}{y_\nu N} \, ,$$

(4.2)

and the correct value of the weak scale is obtained for $y_\nu \sim 10^{-12}$.

Notice that even though Eq.(4.2) is bounding the observed value of the weak scale by parameters $N$ and $y_\nu$ that are technically natural, the theory would still be fine-tuned from an effective field theory perspective, since the cutoff scale of the theory, $M_{Pl}$, is well

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7 This possibility was first suggested in [39], but as we will discuss momentarily, the different conjectural bounds on $m$ and $\Lambda$ in [39] lead to different conclusions.
above $v_{SM}$. This is therefore not so much a solution to the hierarchy problem, but rather an explanation of why nature appears finely-tuned: those versions of the theory that are natural would fall into the Swampland of theories that do not satisfy the WGC as applied to discrete gauge symmetries. Our suggestion therefore lacks an experimental ‘smoking-gun’ signature that often models of naturalness provide. However, we note that the scenario proposed here would be falsified if (neutrinoless) double-beta decay was experimentally observed, as a Majorana mass for the neutrinos would break $U(1)_{B-L}$ down to a $Z_2$ factor, therefore invalidating our argument.

A potentially more satisfying approach would involve lowering the gravitational cutoff, therefore not only constraining the experimental value of the weak scale but also making the theory natural, in the traditional sense of lowering the scale of radiative corrections to the Higgs mass parameter. This can be achieved if the theory contains $N$ copies of a $Z_2$ symmetry, with $N \sim 10^{32}$, as first noted in [37]. In the context of a single discrete symmetry such an approach was first suggested in [39], as applied to a $Z_N$ $B-L$ gauge group using the the weaker bound $m_\nu \lesssim M_{Pl}/\sqrt{N}$ advocated therein. In this case, fixing the value of the weak scale via a bound on the neutrino mass would require $N \gtrsim 10^{56}$, with Eq.(1.3) further implying that the theory requires a UV completion at roughly the same scale. This suggestion has several difficulties. If the discrete symmetry is realised through an Abelian Higgs model, then $\Lambda \sim v$, and a scalar excitation carrying $U(1)_{B-L}$ charge would have to be present at roughly the scale of the neutrino mass. This is in conflict with experiment, and, as discussed in section 2, such a UV completion does not require a gravitational cutoff at scales parametrically below $M_{Pl}$. Even if one insisted on a UV completion in which $\Lambda$ corresponds to the scale at which gravitational effects become important, as advocated in [39], then a gravitational cutoff would be at the scale of the neutrino mass, obviously in contradiction with experimental observations.

Finally, we acknowledge that $Z_N$ gauge symmetries with parametrically large $N$ may be hard to realise while maintaining an arbitrarily high cutoff scale. Attempts at finding controlled string constructions with parametrically large charges have not been successful [66]. However, to the best of our knowledge, no consistency requirements of a theory of quantum gravity forbids a scenario with a Higgs field carrying charge $N \gg 1$.

5 Conclusions

Arguments based on the absence of BH remnants stabilized by discrete charge lead to bounds on discrete gauge symmetries [37–39] that are reminiscent of the Weak Gravity Conjecture. In this work we have pursued deeper connections between the two sets of conjectures in the context of Abelian Higgs models that realize discrete gauge symmetries in the infrared. We have shown that the bounds of [37–39] are consistent with expectations based on the effect of discrete hair on BHs in theories with spontaneously broken symmetries. Moreover, we have shown that parametrically similar bounds can be obtained by applying the WGC to an alternative description of the Abelian Higgs model in terms of two $U(1)$ factors coupled through a topological term. This highlights the sense in which
conjectured bounds on discrete gauge symmetries may be thought of as residuals of the WGC applied to spontaneously broken continuous gauge groups.

Applying the WGC to a dual description of the Abelian Higgs model suggests a novel way in which a $\mathbb{Z}_N B - L$ gauge symmetry could help explain the apparent fine-tuning of the weak scale, if neutrinos satisfy the version of the WGC applicable to discrete gauge symmetries. Although this requires an extremely large value of $N$ that may be hard to realise in specific string constructions, it provides another example of how Swampland conjectures may impact our understanding of physics at low scales.

Of course, discrete gauge symmetries need not originate from spontaneously broken continuous gauge symmetries, and extending the discussion presented here to other UV completions is bound to shed further light on the relationship between various conjectured bounds on continuous and discrete gauge symmetries.

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