We show how to perform error correction of single qubit dephasing by encoding a single qubit into a minimum of three. This may be performed in a manner closely analogous to classical error correction schemes. Further, the resulting quantum error correction schemes are trivially generalized to the minimal encoding of arbitrarily many qubits so as to allow for multiqubit dephasing correction under the sole condition that the environment acts independently on each qubit.

89.70.+c, 89.80.+h, 02.70.-c, 03.65.-w

The ability to store and manipulate quantum information for long periods is at the heart of many exciting new applications such as quantum computation [1], quantum communication across noisy channels [2], quantum cryptographic schemes [3] and networks [4] as well as possible attacks on simpler quantum cryptosystems [5]. The hope of realising such storage has been spurred by the theoretical construction of quantum error correction codes and circuits [6]. The original scheme corrected for arbitrary one-qubit errors in a single qubit encoded within an error-correcting coded state requiring 9 qubits [6]. (A qubit is the information encoded in a two-state quantum system [11].) This code was soon reduced to requiring 7 qubits [8] and finally the minimum code requiring only 5 qubits [9]. The importance of using the minimal resources in constructing quantum error correcting codes is based on our current difficulty in performing operations on even two qubits [10]. In this context, a scheme which could perform quantum error correction with even fewer qubits could have important consequences. This paper shows that 1-qubit dephasing can be corrected with a minimum of 3 qubits encoding a single qubit of quantum information. A similar scheme for correcting dephasing was presented by Steane [14]. However, it uses 3 auxiliary qubits for encoding which is a total of 4 qubits. Further, Steane’s scheme uses external detection and manipulation conditioned the measurement results for decoding. By contrast, the schemes discussed here work without the necessity of external detection.

To date, dephasing is the primary anticipated cause of failure of a quantum computation [11] and of quantum information storage in general. In fact, it is the rule that dephasing time for a quantum system is no longer, and usually a lot shorter, than the population decay time. With a suitable design, quantum computers might even approach the performance of classical computers for their insensitivity to random bit-flips. Indeed, classical computers virtually run without error correction [12] except over comparatively noisy networks.

We demonstrate here a recipe for converting classical coding schemes, which protect against random bit-flips, to quantum schemes for protecting against dephasing. In the simplest case this leads to a 3-qubit scheme of pure-quantum error correction. More generally, it automatically provides us with the minimal encoding and decoding circuits to store arbitrary numbers of qubits and to correct against multibit dephasing. The generalization requires an environment that acts independently on each qubit (and through these interactions causes decoherence). Since between computational steps the qubits are decoupled one from another, it is expected that this is not too limiting a restriction on the applicability of the results presented here. In any case, the economy of the schemes make them attractive for implementation in at least the first generation of quantum memories and computers.

The schemes studied here rely on the following geometric representation: Single qubit dephasing is generated by random rotations about the z-axis of the Bloch sphere. Such single particle dephasings may be individually converted into random bit-flips by a π/2 rotation about the y-axis of the Bloch sphere of each particle, see Fig. 1. Random bit flips alone, however, can be optimally corrected by classical coding schemes [12]. Thus, by taking a classical error correction circuit and reinterpreting it as
acting on qubits, we may use it to correct independent qubit dephasing. In effect we are translating qubit dephasing into qubit flipping for which the error-correction circuit can correct.

\[ |\psi\rangle \rightarrow U |\psi\rangle \]

FIG. 2. Classical error correction circuit. Here each classical bit \( \psi \) is recorded as a redundant triple \( \psi \psi \psi \). The shaded region represents the possible introduction of a single randomly flipped bit. After this error the decoding circuit successfully restores the original value of \( \psi \) in the upper line.

FIG. 3. Quantum 1-bit dephasing correction. Here \( \hat{U} \) is the rotation \( \exp(-i\pi/4) \).

The minimal requirements to correct one-bit dephasing of an encoded qubit can be studied by arguments similar to those developed in Ref. [9]. There the environment could produce ‘rotations’ in the Bloch sphere about any axis, whereas here we assume the environment is free only to produce rotations about a single axis. Such arguments show that the minimal dephasing correction scheme requires 3 qubits, which has just been achieved in Fig. 3. More directly, however, the circuit is optimal because the classical version of it was.

Generalizing the scheme shown in Fig. 3 is also trivial. Taking a classical coding scheme, we perform single bit rotations to convert the single bit dephasings into bit flips. Examples of such generalizations are easily constructed as shown in Fig. 4 for a scheme protecting one qubit against two-bit dephasing. Again, it is easy to check that this quantum scheme is minimal for what it achieves. The main disadvantage of these generalizations is not immediately obvious from our discussion of dephasing or quantum error correction: The scheme is designed to correct dephasing which was incurred independently on each qubit (by independent interactions with the environment). If, however, a conditional dephasing occurs on one qubit based on the state of a second qubit then our scheme fails. Such errors are not converted into bit-flip errors by our strategy and so the ‘classical’ schemes employed are inadequate.

Two final issues require addressing in all quantum error correction schemes. First, the circuitry of quantum error correction schemes are typically derived and work when ‘expensive’ gates are employed — gates having no extraneous signs. Under purely unitary evolution the presence of extraneous signs is unimportant if one follows the simple rules of reversible programming.
FIG. 4. Quantum 2-bit dephasing correction of the qubit $|\psi\rangle$ requiring a minimum of 5 qubits. The decoding requires a new gate which is a bit flip $\oplus$ conditioned on the majority of the qubits passing through the ‘majority’ box being in state $|1\rangle$. Such multibit dephasing generalizations require that the environment dephases each qubit independently.

However, in quantum error correction, where part of the evolution is non-unitary, these rules are inadequate. It appears that all quantum error correction circuits constructed so far require expensive gates.

Second, error correction is, in some sense, static; it aims at ‘refrigerating’ particular degrees of freedom of interest to us and is ideal for providing us with stable quantum memories. But how do we combine this static nature with the dynamics necessary for computation? This is an unsolved problem. Computational (dynamical) steps are indistinguishable from many errors. Therefore, in order to compute we must make the computer susceptible to errors, by suspending error correction. One proposed solution [4] is to apply error correction to the overall state of the computer between computational steps. In this way the correction occurs only while the computer is static. Unfortunately, this also means that all errors during the actual computation remain uncorrected and accumulate with every step taken. This application yields no advantage over an error correction free computation.

An alternate approach is to apply error correction to those bits unused in the current computational step. Naturally, this alternative fails to protect against errors in bits that are involved in the computation; in principle, these could be as few as two bits. Yet another approach is to attempt to design computational steps mapping states protected by error correction code directly to each other. In this way we could ‘freeze’ the state into the correct result of the computation using ‘dynamic error correction.’ The last approach appears to be complicated. Clearly, more work is required.

In conclusion, we have shown how to perform error correction of single-qubit dephasing by encoding a single qubit into the minimum of three. This economy in qubits makes this scheme likely to be one of the first to be implemented. The scheme described is simple to construct, understand and generalize. Generalizations to multiqubit dephasing corrections, however, require that the environment acts independently on each qubit.

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