Cosmological evolutions of $F(R)$ nonlinear massive gravity

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Abstract

Recently a new extended nonlinear massive gravity model has been proposed which includes the $F(R)$ modifications to dRGT model. We follow the $F(R)$ nonlinear massive gravity and study its implications on cosmological evolutions. We derive the critical points of the cosmic system and study the corresponding kinetics by performing the phase-plane analysis.

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I. INTRODUCTION

The search for a consistent covariant modification of General Relativity in which graviton is allowed to acquire a mass has been initiated since Fierz and Pauli (FP) proposed a quadratic Lagrangian which describes a massive spin-2 field [1]. The Lagrangian is ghost-free [2] but cannot recover Einstein gravity in the limit of vanishing graviton mass, due to the coupling between the longitude mode of the graviton and the trace of the energy momentum tensor [3, 4]. Nonlinear terms were introduced and the troublesome mode could be suppressed at macroscopic length scales via the Vainshtein mechanism [2]. However the same nonlinear terms are responsible for the existence of the Boulware-Deser (BD) ghost [6–10] which would make the theory unstable. Until recent, a scheme was invented by de Rham, Gabadadze and Tolley (dRGT) so that only the suitable nonlinear terms enter the theory thus eliminating the BD ghost once for all [11–16] (see [17, 18] for review), implications of the dRGT model has been studied in [19–60].

Some extended nonlinear massive gravity theories were introduced shortly after. In mass-varying massive gravity theory [61], the graviton mass is promoted to vary with a dynamical scalar field, the cosmological evolutions of the model have been studied in [62–67]. Especially in [35], bounce and cyclic cosmology has been built, which might have interesting implications [68–71] and help to confronting the massive gravity with observation, e.g. explaining CMB anomalies [72–74]. In quasi-dilaton theory [75], a dynamical scalar field is also present but it is non-trivial coupling with the massive graviton instead, see also [76, 77]. The extended theories have more theoretical freedom and thus allow for more desirable cosmological solutions [78].

Recently a new extended theory was proposed which introduce the \( F(R) \) modifications to the dRGT model [79]. The theory contains modification of GR not only in IR regimes like all the other massive gravity theories but also in UV regimes. The theory is free of the DB ghost as proven in [79], and inherit the theoretical advantages of \( F(R) \) paradigm [80]. Later it is also claimed to be free of ghosts instabilities at perturbative level which was found in dRGT model [81]. The theory allows a huge class of interesting cosmological behaviors at early and late times, and is promising in fitting the current observations. Ghost-free \( F(R) \) bigravity was proposed in [82] which would give variety of cosmic acceleration models [83]. A different model of ghost-free massive \( F(R) \) gravity was develop in [84].
In the paper we analyze the $F(R)$ nonlinear massive gravity model and study its implications on cosmic evolution by performing a phase-space and stability analysis. The dynamical analysis of the regular $F(R)$ gravity has been carried out in some papers \[85, 87\], for general analyses see \[88, 89\]. $F(G)$ model has been analyzed in \[90\], see also \[91\]. However in the present model the dynamic of the system becomes more complicated, not only the dimensionless variables have more complex evolution, but some of the variables are not completely independent, and their relations yield extra constraints on the system which make the system more difficult to study. In the previous paper \[85\] a technique was developed which enables one to analyze the dynamic of the system without a specific model of $F(R)$, we will see that this technique is partially valid in the present work. The stability of fixed points has a strong correlation with the specific model of $F(R)$, so we will analyze two models after giving a general discussion of solution space of the system.

The present paper is organized as follows. In Section II, we briefly review the $F(R)$ nonlinear massive gravity model and its cosmological equations of motion. Then we construct the dynamics of $F(R)$ nonlinear massive gravity and develop a method to deal with a system with extra constraints in Section III. We provide the solutions of the cosmic system described by this theory by performing detailed phase-space and stability analyses of two $F(R)$ models and summarize the results in Section IV. Finally, we conclude with a discussion in Section V.

II. COSMOLOGY OF $F(R)$ NONLINEAR MASSIVE GRAVITY

To begin with, we briefly review the $F(R)$ nonlinear massive gravity model constructed in \[79\]. This model imposes a UV sector modification of the dRGT model with the scalar curvature $R$ replaced by an arbitrary function of it. Therefore, the complete action can be expressed as

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{|g|} \left[ F(R) + 2m_g^2 U_M \right],$$

where $M_p$ is the Planck mass, $g$ is the physical metric and $m_g$ is the graviton mass. Graviton potential is given by $U_M = U_2 + \alpha_3 U_2 + \alpha_4 U_2$, where $\alpha_3, \alpha_4$ are dimensionless parameters, and

$$U_2 = K^{\mu}_{[\mu} K^\nu_{\nu]}, \quad U_3 = K^{\mu}_{[\mu} K^\nu_{\nu} K^\rho_{\rho]}, \quad U_4 = K^{\mu}_{[\mu} K^\nu_{\nu} K^\rho_{\rho} K^\sigma_{\sigma]},$$

where $K_{\mu\nu}$ is the extrinsic curvature.
where $\mathcal{K} \equiv \mathcal{I} - \sqrt{g^{-1}f}$, and $f$ denote the fiducial metric. We begin with a Minkowski fiducial metric

\[ f_{AB} = \eta_{AB}, \tag{3} \]

and an open FRW physical metric

\[ ds^2 = -N^2 dt^2 + a^2(t) \gamma^K_{ij} dx^i dx^j, \tag{4} \]

where

\[ \gamma^K_{ij} dx^i dx^j = \delta_{ij} dx^i dx^j - \frac{a_0^2 (\delta_{ij} x^i x^j)^2}{1 + a_0^2 \delta_{ij} x^i x^j} \]

and $a_0 = \sqrt{|\mathcal{K}|}$, $a_0$ is associated with the spatial curvature. Variation of the action with respect to $b$, $N$ and $a$ gives three equations

\[ (\dot{a} - a_0) Y_1 = 0, \tag{5} \]

\[ 3M_p^2 F_{,R} \left( H^2 - \frac{a_0^2}{a^2} \right) = \rho_m + \rho_{IR} + \rho_{UV}, \tag{6} \]

\[ M_p^2 F_{,R} \left( -2H^2 + \frac{a_0^2}{a^2} \right) = p_m + p_{IR} + p_{UV}, \tag{7} \]

where $\dot{a} = a'$ and $H = \frac{\dot{a}}{a}$. In the above expressions IR (massive gravity) and UV ($F(R)$ sector) effective contributions are defined as follow

\[ \rho_{IR} = m_g^2 M_p^2 (\mathcal{B} - 1)(Y_1 + Y_2), \tag{8} \]

\[ p_{IR} = -m_g^2 M_p^2 (\mathcal{B} - 1)Y_2 - m_g^2 M_p^2 (\dot{b} - 1)Y_1, \tag{9} \]

\[ \rho_{UV} = M_p^2 \left[ \frac{RF_{,R} - F}{2} - 3H \dot{R} F_{,RR} \right], \tag{10} \]

\[ p_{UV} = M_p^2 \left[ \ddot{R}^2 F_{,RRR} + 2H \dot{R} F_{,RR} + \dot{R} F_{,RR} + \frac{F - RF_{,R}}{2} \right], \tag{11} \]

where the polynomials $Y_{1,2}$ are given by $Y_1 = (3 - 2\mathcal{B}) + \alpha_3 (3 - \mathcal{B})(1 - \mathcal{B}) + \alpha_4 (1 - \mathcal{B})^2$ and $Y_2 = (3 - \mathcal{B}) + \alpha_3 (1 - \mathcal{B})$ with $\mathcal{B} = \frac{a b}{a}$. Similar to all massive gravity scenarios, the nontrivial solutions of Eq. (5) correspond to the case of $Y_1 = 0$ and yield

\[ \mathcal{B}_\pm = \frac{1 + 2\alpha_3 - 2\alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4}. \tag{12} \]

This relation can be fulfilled by choosing $b(t) \propto a(t)$, and therefore it yields $\rho_{IR} = -p_{IR}$ to be constant.
III. DYNAMICAL FRAMEWORK

In this section we perform a detailed phase-space analysis of cosmic evolutions describing the $F(R)$ nonlinear massive gravity model following the method developed in [90, 92–103] (see also [104] for a recent analysis in the frame of generalized Galileon cosmology). First we give a brief review of autonomous system and transform the dynamical system into the autonomous form, then discuss a problem in the phase-space analysis and try to give a possible solution.

A. Autonomous system and its evolution

For a general dynamical system, one group of suitable auxiliary variables can be chosen so that the corresponding equations of motion will be first-order differential equations. The group of auxiliary variables can be written as a vector $\vec{x}$, its equation of motion is

$$\frac{d\vec{x}}{dt} = \vec{f}(x).$$

(13)

The system is said to be autonomous if $\vec{f}(x)$ do not contain explicit time-dependent terms. We want to find out which state will the system be, eventually. If the system is stabilized at one particular state, the speed of the variables must equal to 0, assuming the number of variables is $n$, this condition corresponds to:

$$\begin{cases}
  f_1(x_1, x_2, \cdots, x_n) = 0 \\
  \vdots \\
  f_n(x_1, x_2, \cdots, x_n) = 0.
\end{cases}$$

(14)

The solutions to these equations are called fixed points, they are the candidates for stable states. In order to find out whether a solution is stable or not, we need to analyze the perturbation around it. By taking the perturbation of the system we get

$$\frac{d\delta\vec{x}}{dt} = A \cdot \delta\vec{x},$$

(15)

where $A$ is a matrix with element $A_{ij} = \frac{\partial f_i}{\partial x_j}(\vec{x}_0)$, $\vec{x}_0$ is the fixed point under study. We can view the equation above as the equation of motion of the perturbation. Assuming eigenvalues of $A$ are $(\mu_1 \cdots \mu_n)$, and the corresponding eigenvectors are $(\vec{v}_1 \cdots \vec{v}_n)$, if the perturbation starts as $\vec{v}_m$, then its evolution takes the form of $\vec{v}_m \exp[\mu_m t]$. Because the
eigenvectors of \( A \) are linearly independent in most cases, any perturbation can be written as \( \delta \vec{x}(t = 0) = \sum_{i=1}^{n} \alpha_i \vec{\nu}_i \), where \( \alpha_i \) is arbitrary coefficient, and because Eq. 15 is linear, the evolution of the perturbation follows the equation

\[
\delta \vec{x}(t) = \sum_{i=1}^{n} \alpha_i \vec{\nu}_i e^{\mu_i t}.
\]  

(16)

If the perturbation around a fixed point becomes smaller over time and approaches 0 eventually, that is \( \lim_{t \to +\infty} \delta \vec{x} = 0 \), we call this fixed point asymptotic stable. One can see that this requirement is satisfied when the real parts of all the eigenvalues are negative. In cosmology, we mainly concern the stable fixed points, for they contain information about late-time evolution of the universe.

B. Dynamics of \( F(R) \) nonlinear massive gravity

We begin to discuss the dynamic system of the \( F(R) \) nonlinear massive gravity in detail. First we define 6 dimensionless variables

\[
x_1 = \frac{\rho_{1R}}{3M_p^2 F,RR H^2}, x_2 = \frac{R}{6H^2}, x_3 = -\frac{F}{6F,RR H^2}, x_4 = -\frac{\dot{R} F,RR}{F,RR H},
\]

\[
x_5 = \frac{a^2_0}{a^2 H^2}, \Omega_m = \frac{\rho_m}{3M_p^2 F,RR H^2}.
\]

Eq. 6 can be reduced to

\[
\Omega_m + x_1 + x_2 + x_3 + x_4 + x_5 = 1.
\]

(17)

Using Eq. 6 and Eq. 7, we can write the equations of motion of the variables as follow

\[
\frac{d}{dN} x_1 = x_1 (x_4 - 2 (x_2 + x_5) + 4),
\]

(18)

\[
\frac{d}{dN} x_2 = x_2 (4 - 2 (x_2 + x_5)) - \frac{x_2 x_4}{m},
\]

(19)

\[
\frac{d}{dN} x_3 = x_3 (x_4 - 2 (x_2 + x_5) + 4) + \frac{x_2 x_4}{m},
\]

(20)

\[
\frac{d}{dN} x_4 = x_4 (x_4 - x_2 - x_5) + x_5 - x_2 - 3 (x_1 + x_3) + 3 \omega \Omega_m - 1,
\]

(21)

\[
\frac{d}{dN} x_5 = 2 x_5 (1 - (x_2 + x_5)) ,
\]

(22)

\[
\frac{d}{dN} \Omega_m = \Omega_m (x_4 - 2 (x_2 + x_5) + 1 - 3 \omega),
\]

(23)
where $N$ stands for $\ln a$, and $m = \frac{F_{,RR}R}{F_{,R}}$, it's a parameter that depends on the form of $F(R)$. We can define another parameter $r = \frac{x_2}{x_3} = -\frac{F_{,R}}{F}$, once we have the exact form of $F(R)$, we can derive $R$ from $r$ and then substituting it into $m$, in the end the parameter $m$ will be a function of $x_2$, $x_3$ and the dynamical system will become autonomous.

Eq. (17) must hold at anytime, differentiating it respect to $N$ gives

$$\frac{d}{dN}(x_1 + x_2 + x_3 + x_4 + x_5 + \Omega_m) = 0. \quad (24)$$

Adding Eq. (18) through Eq. (23) we get

$$\frac{d}{dN}(x_1 + x_2 + x_3 + x_4 + x_5 + \Omega_m)$$

$$= (1 - 2(x_2 + 2x_5) + x_4)(\Omega_m + x_1 + x_2 + x_3 + x_4 + x_5 - 1). \quad (25)$$

One can see from Eq. (25) that the constraint equation Eq. (17) is not automatic satisfied, we must impose it when we solve the system, and after doing so we can eliminate one variable, we eliminate $\Omega_m$ for convenience and do not consider Eq. (23).

From the definitions of $x_1$, $x_2$ and $x_3$, we can express $R$ and $H$ in terms of any two variables and substitute them into the third one, which allows us to eliminate one variable directly, thus reducing the dimension of the dynamical system by 1. However for some complicated expression of $F(R)$, it is not always possible to give the resolved expression of $R$ and $H$, and in some cases eliminating one variable would actually make the dynamics of system more complicated because the complex relations between $x_1$, $x_2$ and $x_3$ will be involved in the equations of motion. If we do not consider the relations between the variables, we could get false result, unnecessary fixed points may appear, and the analysis from the whole phase space may be unreliable. We will discuss this issue in the next section.

C. System with hidden constraint

We now consider the issue that one must compute the fixed points and study the stability of these fixed points with redundant degrees of freedom in the system. $x_1$, $x_2$ and $x_3$ are not entirely independent, and the system is restricted to a low-dimensional surfaces. The surface is uniquely determined, since $x_1$, $x_2$ and $x_3$ are all functions of $R$ and $H$, $R$ and $H$ are the natural coordinates of the surface, we will call this surface $h(x_1, x_2, x_3) = 0$. 

7
\( x_1, x_2 \) and \( x_3 \) do not contain explicit time-dependent terms, neither does the surface \( h = 0 \), thus all the orbits of the system are guaranteed to stay on the surface and the speed of the system is tangent to the surface at any time, the condition is characterized by the expression

\[
\sum_n \frac{\partial h}{\partial x_n} \frac{dx_n}{dN} = 0,
\]

which implies that \( h = c \) is an integral, where \( c \) is a constant and \( n \) is the number of variables. But the actual system only stays on the surface of \( h = 0 \). In the previous section, we have another constraint equation

\[
\Omega_m + x_1 + x_2 + x_3 + x_4 + x_5 = 1.
\]

It is low-dimensional plane with normal vector \((1, 1, 1, 1, 1, 1)\), and Eq.\((25)\) prove that it is an integral. Usually finding the integral of a dynamic system is a difficult task, but we know the system under study must have integrals, because the precise forms of the variables are given. It seems straightforward that one should include equation \( h = 0 \) when solving the system for fixed points and analyzing the perturbations of the system around these fixed points, reflecting this fact is that all the fixed points must stay on the constraint surface and the perturbation of the system belongs to the tangent plane of the given point,

\[
\sum_n \frac{\partial h}{\partial x_n} \delta x_n = 0.
\]

A natural question is that will the perturbation of the system stay in the same tangent plane as it evolves. One can see that this condition is not always satisfied by taking the time derivative of the equation above,

\[
\sum_n \left( \frac{\partial h}{\partial x_n} \frac{d\delta x_n}{dN} + \sum_m \frac{\partial^2 h}{\partial x_m \partial x_n} \frac{dx_m}{dN} \delta x_n \right) = 0.
\]

If \( h \) is a linear function, then \( \frac{\partial^2 h}{\partial x_m \partial x_n} \) equals 0, equation above becomes \( \sum_n \frac{\partial h}{\partial x_n} \frac{d\delta x_n}{dN} = 0 \), which means that at any point the speed of the perturbation always stays in the tangent plane, so does the perturbation. The constraint surface we encounter before is linear function. But for general surfaces \( \frac{\partial^2 h}{\partial x_m \partial x_n} \neq 0 \), in this case the speed of the perturbation will no longer stay in the tangent plane because \( \sum_n \frac{\partial h}{\partial x_n} \frac{d\delta x_n}{dN} \neq 0 \), neither will the perturbation. But we will see that this fact won’t cause trouble because we only consider the perturbations around fixed
points, and at the fixed point we have \( \frac{dx_m}{dN} = 0 \), thus \( \sum_{n} \frac{\partial h}{\partial x_n} \frac{dk_{x_n}}{dN} = 0 \) still holds, this result could simplify the calculation. We could assume that the perturbations start on the tangent plane and studying their evolutions without concerning their unwanted behaviors.

The perturbation around any given fixed point could be written as

\[
\delta \vec{x} = \sum_{i=1}^{n-1} \beta_i \vec{e}_i = \sum_{j=1}^{n} \alpha_j \vec{\nu}_j,
\]

where \( \vec{e}_i \) is the base vector of the tangent plane, \( \vec{\nu}_j \) is the eigenvector of the system, because the eigenvectors of a certain point are linearly independent, \( \vec{e}_i \) and \( \vec{n} \) (the normal vector of the surface) can be written as the linear combinations of these eigenvectors, then the perturbation can be written as combination of \( \vec{\nu}_j \), \( \alpha_j \) is the coefficient. When a fixed point is stable viewing from the whole phase space, the stable subspace is \( n \) dimensional, tangent plane must belong to it, and the fixed point is stable because the perturbation will approach 0 as time passing by regardless of the actual value of the coefficient \( \beta_i \). If the stable subspace is \( n - 1 \) dimensional, there is a chance that the tangent plane belongs to it and the point is still stable.

Generally speaking, if \( \alpha_j \) equals 0, the corresponding eigenvalue do not effect the stability of the fixed point. This happens when the eigenvector is normal to the surface or the other eigenvectors belong to the tangent plane. After we find out which eigenvalue is responsible for the stability of the fixed point, we could give the parameter range for the point to be stable.

IV. PHASE-SPACE ANALYSIS AND RESULTS

A special method has been developed for analyzing \( F(R) \) model which enables one to obtain a general understanding of the system without specific form of \( F(R) \).\(^8\) Instead of solving the specific form of \( m(r) \), one could solve the system for fixed points assuming \( m \) is another unknown variable. Some fixed points do not contain \( m \) and are assumed to be independent of form of \( F(R) \), the rest fixed points contain \( m \) and can give a new relation between \( m \) and \( r \) because by definition \( r \) is the ratio of \( x_2 \) and \( x_3 \), both of which contain \( m \). Together with the function of \( m(r) \) imply by the \( F(R) \), one can find the exact value of \( r \) and \( m \), and the fixed points are determined. One can even carry out the stability analysis without knowing the form of \( m(r) \) and the value of \( r \), although in \(^8\), the authors claim
that the stability analysis in this way can be troublesome, and because the form of $F(R)$ strongly influences the stability of these fixed points in the $F(R)$ nonlinear massive gravity, we will not carry out the stability analysis in this way, instead we will study two models and give detailed analysis of each fixed point.

The system may have less fixed points due to the constraints of the system, but all the possible fixed points can be found, after that finding the fixed points existed in a certain $F(R)$ model is relatively easy, we just keep the fixed points that satisfy the constraint. All the possible fixed points are listed

\begin{align*}
A & : (5 - x_3, 0, x_3, -4, 0) \\
B & : (2 - x_3, 0, x_3, -2, 1) \\
C & : (-1 - x_3, 2, x_3, 0, 0) \\
D & : (0, 0, 0, 0, 1) \\
E & : (0, 0, 0, 1, 0) \\
F & : (0, 0, 0, -1 + 3\omega, 0) \\
G & : (0, 0, 0, 1 + 3\omega, 1) \\
H & : (0, -1 - 3\omega, 2 + 6\omega, -3 (1 + \omega), 0) \quad m \rightarrow \frac{-1}{2} \\
I & : \left(0, -\frac{3}{2} x_3 (1 + \omega), x_3, 1 + 3\omega, \frac{1}{2} (2 + 3 x_3 (1 + \omega))\right) \quad m \rightarrow \frac{1}{2} (1 + 3\omega) \\
J & : \left(0, 2 m (1 + m), -2 m, 2 m, 1 - 2 m - 2 m^2\right) \\
K & : \left(0, \frac{4 m^2 + 3 m - 1}{m (1 + 2 m)}, \frac{1 - 4 m}{m + 2 m^2}, \frac{2 - 2 m}{1 + 2 m}, 0\right) \\
L & : \left(0, \frac{1 + 4 m - 3 \omega}{2 + 2 m}, \frac{3 \omega - 1 - 4 m}{2 (1 + m)^2}, \frac{3 m (1 + \omega)}{1 + m}, 0\right).
\end{align*}

From points I, J, K, L we can derive the relation

\begin{equation}
m (r) = -1 - r. \tag{31}
\end{equation}

For each $F(R)$ model one or more values of $r$ can be determined, and the exact form of fixed points can be obtained. Notice that A, B, C, I are lines of equilibria instead of fixed points, we will not be bothered by this fact because when considering the system with a constraint surface, the lines of equilibria will intersect with it and the points of intersection are the fixed points of the system.
| Lable | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\Omega_m$ |
|-------|-------|-------|-------|-------|-------|-----------|
| $A_1$ | 5     | 0     | 0     | -4    | 0     | 0         |
| $B_1$ | 2     | 0     | 0     | -2    | 1     | 0         |
| $C_1$ | -1 + \frac{2}{n} | 2 | -\frac{2}{n} | 0 | 0 | 0 |
| $D_1$ | 0     | 0     | 0     | 0     | 1     | 0         |
| $E_1$ | 0     | 0     | 0     | 1     | 0     | 0         |
| $F_1$ | 0     | 0     | 0     | -1 + 3\omega | 0 | 2 - 3\omega |
| $G_1$ | 0     | 0     | 0     | 1 + 3\omega | 1 | -3\omega - 1 |
| $J_1$ | 0     | 2n(n - 1) | 2(1 - n) | 2(n - 1) | 1 + 2n(1 - n) | 0 |
| $K_1$ | 0     | $\frac{n(4n-5)}{(2n-1)(n-1)}$ | $\frac{(5-4n)}{(2n-1)(n-1)}$ | $\frac{2(n-2)}{1-2n}$ | 0 | 0 |
| $L_1$ | 0     | $\frac{4n-3\omega-3}{2n}$ | $\frac{3\omega-4n+3}{2n^2}$ | $\frac{3(n-1)(\omega+1)}{n}$ | $\frac{n(9\omega+13-2n(3\omega+4))-3(\omega+1)}{2n^2}$ | 0 |

TABLE I: The fixed points in $F(R)$ nonlinear massive gravity with $F(R) = R^n$.

The stability of the fixed points is related to the constraint surface and cannot be analyzed without the precise form of $F(R)$, we will consider two $F(R)$ models and analyze the stability of each fixed point.

A. $R^n$ model

Let us consider the Lagrangian $F(R) = R^n$. Corresponding constraint surface can be written as

$$nx_3 + x_2 = 0.$$  \hspace{1cm} (32)

First we check if the surface is an integral, after some calculation we get

$$\vec{n} \cdot \frac{d\vec{x}}{dN} = (nx_3 + x_2)(2x_2 + 4 - 2(x_5 + x_4)), \hspace{1cm} (33)$$

so the surface is an integral, the normal vector of the surface is $\vec{n} = (0, 1, n, 0, 0)$, $m$ takes the constant value of $n - 1$. We summarize the fixed points of this autonomous system and their stability in Table I and Table II, respectively.

Next we check if the perturbation around fixed point stays on the tangent plane, we compute $\vec{n} \cdot \frac{d\vec{x}}{dN}$ and find out that for Point $A_1, B_1$ and $C_1$, it automatically equals to 0 suggesting that those points have 0 eigenvalue, for the rest of the points it equals to 0 if we
Point \( \omega_{\text{eff}} \) Stable when
\begin{align*}
A_1 & : \quad \frac{1}{3} \quad \text{not stable} \\
B_1 & : \quad -\frac{1}{3} \quad 0 < n < 1, \, \omega > -1 \\
C_1 & : \quad -1 \quad 1 < n < 2, \, \omega > -1 \\
D_1 & : \quad -\frac{1}{3} \quad \text{not stable} \\
E_1 & : \quad \frac{1}{3} \quad \text{not stable} \\
F_1 & : \quad \frac{1}{3} \quad \text{not stable} \\
G_1 & : \quad -\frac{1}{3} \quad n < 0, \, \omega < \frac{1}{3}(2n - 3) \text{ or } 0 < n < 1, \, \omega < -1 \\
J_1 & : \quad -\frac{1}{3} \quad \frac{1}{2} - \frac{\sqrt{3}}{2} < n < 0, \, \omega > \frac{1}{3}(2n - 3) \\
K_1 & : \quad \frac{n(7 - 6n) + 1}{6n^2 - 9n + 3} \quad n < \frac{1}{2} \left(1 - \sqrt{3}\right) \text{ or } \frac{1}{2} < n < 1 \text{ or } n > 2, \, \omega > -\frac{8n^2 + 13n - 3}{6n^2 - 9n + 3} \\
L_1 & : \quad \frac{\omega + 1}{n} - 1 \quad \text{stable}
\end{align*}

TABLE II: The stability of the fixed points in \( F(R) \) nonlinear massive gravity with \( F(R) = R^n \).

require that the perturbation satisfy \( n\delta x_3 + \delta x_2 = 0 \), which means the perturbation starts on the tangent plane. One can see that points \( H, I \) in the general case corresponding to \( L_1, F_1 \), we will analyze the stability of each point in detail.

Point \( A_1 \): The eigenvalues of the linearised system are

\[ 0, -5, 2, \frac{4n}{n - 1}, -3(\omega + 1), \]

corresponding eigenvectors are

\[ \{-1, 0, 1, 0, 0\}, \{-1, 0, 0, 1, 0\}, \left\{\frac{-15}{7}, 0, 0, \frac{8}{7}, 1 \right\}, \]

\[ \left\{\frac{5(1 - 3n)}{6n}, \frac{9n - 5}{6(n - 1)}, \frac{5 - 9n}{6n(n - 1)}, 1, 0 \right\}, \left\{\frac{-5}{3(\omega + 1)}, 0, 0, 1, 0 \right\}. \]

The eigenvectors of \(-5, 2, \frac{4n}{n - 1}, -3(\omega + 1)\) are normal to \( \vec{n} \), so they belong to the tangent plane. If we analyze the system with one variable eliminated we would get the same eigenvalues. The eigenvector of \( 0 \) is not normal to the tangent plane, it is tangent to Line \( A \), it won't effect the stability of the point. The point is not stable.

Point \( B_1 \): The eigenvalues of the linearised system are

\[ 0, -2, -2, \frac{2n}{n - 1}, -3(\omega + 1), \]
corresponding eigenvectors are

\[
\{-1, 0, 1, 0, 0\}, \{-1, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0\}, \\
\left\{ \frac{(n(3n - 2) + 1)}{n(2n - 1)}, \frac{2n - 1}{1 - n}, \frac{2n - 1}{n(n - 1)}, \frac{n + 1}{1 - 2n}, 1 \right\}, \left\{ -\frac{2}{3(\omega + 1)}, 0, 0, 1, 0 \right\}.
\]

The eigenvectors of \(-2, \frac{2n}{n-1}, -3(\omega + 1)\) belong to the tangent plane, so they must be the eigenvalues of the system with one variable eliminated. The eigenvectors seem insufficient to determine the base vectors of the tangent plane, so we compute the eigenvalues of the system with one variable eliminated and get eigenvalues of \(-2, -2, \frac{2n}{n-1}, -3(\omega + 1)\). The eigenvector of 0 is not normal to the tangent plane, but it won’t effect the stability of the point. The point is stable when

\[0 < n < 1, \ \omega > -1.\]

The effective equation of state is \(\omega_{\text{eff}} = -\frac{1}{3}\).

Point \(C_1\): The eigenvalues of the linearised system are

\[0, -2, \frac{\sqrt{(n - 1)(25n - 41)}}{2(1 - n)} - \frac{3}{2}, \frac{\sqrt{(n - 1)(25n - 41)}}{2(n - 1)} - \frac{3}{2}, -3(\omega + 1),\]

and eigenvectors are

\[
\{-1, 0, 1, 0, 0\}, \left\{ \frac{1}{n}, -2, \frac{2}{n}, 0, 1 \right\}, \\
\left\{ \frac{n - 5 + \sqrt{(n - 1)(25n - 41)}}{4n}, \frac{\sqrt{(n - 1)(25n - 41)}}{4(1 - n)} - \frac{5}{4}, \frac{\sqrt{(n - 1)(25n - 41)}}{4n(n - 1)} + \frac{5}{4n}, 1, 0 \right\}, \\
\left\{ \frac{n - 5 - \sqrt{(n - 1)(25n - 41)}}{4n}, \frac{\sqrt{(n - 1)(25n - 41)}}{4(1 - n)} - \frac{5}{4}, \frac{\sqrt{(n - 1)(25n - 41)}}{4n(n - 1)} + \frac{5}{4n}, 1, 0 \right\}, \\
\left\{ \frac{(n - 2)(3\omega(n - 1) - n - 3)}{3n(n - 1)(\omega + 1)(3\omega - 1)}, \frac{2}{(n - 1)(3\omega - 1) - \frac{2}{n(n - 1)(3\omega - 1)}}, 1, 0 \right\}.
\]

Same as Point \(A_1, B_1\), eigenvectors of the none-zero eigenvalues belong to the tangent plane therefore these eigenvalues are responsible for the stability of the fixed point. Point \(C_1\) is stable when

\[1 < n < 2, \ \omega > -1. \quad (34)\]

The effective equation of state is \(\omega_{\text{eff}} = -1\).
Point $D_1$: The eigenvalues of the linearised system are

$$-2, 2, 2, 2, -1 - 3\omega,$$

corresponding eigenvectors are

$$\{0, 0, 0, -1, 1\}, \{1, -2, 0, 0, 1\}, \{-1, 0, 0, 1, 0\}, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}.$$ (35)

The eigenvectors of $-2, -1-3\omega$ and one of 2 belong to the tangent plane, and the eigenspace of 2 intersects with the tangent plane therefor provides the system with another eigenvalue. The point is not stable.

Point $E_1$: The eigenvalues of the linearised system are

$$2, 5, 5, \frac{5 - 4n}{1 - n}, 2 - 3\omega,$$

corresponding eigenvectors are

$$\{0, 0, 0, -1, 1\}, \{-1, 0, 0, 1, 0\}, \{-1, 0, 1, 0, 0\},$$

$$\left\{0, -\frac{n}{n - 1}, \frac{1}{n - 1}, 1, 0\right\}, \{0, 0, 0, 1, 0\}.$$ (35)

The eigenvectors of $2, \frac{5 - 4n}{1 - n}, 2 - 3\omega$ and one of 5 belong to the tangent plane, therefor effect the stability of the point. The point is not stable.

Point $F_1$: The eigenvalues of the linearised system are

$$2, \frac{4n - 3(\omega + 1)}{n - 1}, 3(\omega + 1), 3(\omega + 1), 3\omega - 2,$$

corresponding eigenvectors are

$$\left\{0, 0, 0, \frac{6}{3\omega - 4} + 2, 1\right\}, \left\{0, \frac{n(3n(\omega - 2) + 5)}{3(n - 1)(\omega(2n - 1) - 1)} - \frac{3n(\omega - 2) + 5}{3(n - 1)(\omega(2n - 1) - 1)}, 1, 0\right\},$$

$$\left\{-\frac{5}{3(\omega + 1)}, 0, 0, 1, 0\right\}, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}.$$ (35)

The eigenvectors of $2, \frac{4n - 3(\omega + 1)}{n - 1}, 3\omega - 2$ and one of $3(\omega + 1)$ belong to the tangent plane, therefor effect the stability of the point. The point is not stable.

Point $G_1$: The eigenvalues of the linearised system are

$$-2, \frac{2n - 3(\omega + 1)}{n - 1}, 3(\omega + 1), 3(\omega + 1), 3\omega + 1,$$
corresponding eigenvectors are
\[
\begin{cases}
0, 0, 0, 
\begin{cases}
0, 
\frac{2\omega}{\omega + 1}, 1
\end{cases}
, \\
0, 
\frac{5 - 4n + 3\omega}{2(n-1)}, 
\frac{5 - 4n + 3\omega}{2n(1-n)}, 
- \frac{(2n - 3(\omega + 1))(n(6\omega + 4) - 3\omega - 5)}{2n(n(3\omega - 1) + 2)}, 1
\end{cases}, \\
- \frac{2}{3(\omega + 1)}, 0, 0, 1, 0
\end{cases}
, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}.
\]
The eigenvectors of \(-2, \frac{2n - 3(\omega + 1)}{n - 1}, 3\omega + 1\) and one of \(3(\omega + 1)\) belong to the tangent plane, therefor effect the stability of the point. The point is stable when
\[
n < 0, \ \omega < \frac{1}{3}(2n - 3) \ \text{or} \ 0 < n < 1, \ \omega < -1.
\]
The effective equation of state is \(\omega_{eff} = -\frac{1}{3}\).

Point \(J_1\): The eigenvalues of the linearised system are
\[
2n, 2n, n - \sqrt{3n(3n - 4)} - 2, n + \sqrt{3n(3n - 4)} - 2, 2n - 3(\omega + 1),
\]
corresponding eigenvectors are
\[
\begin{cases}
\frac{1}{2n(n-1) - 1}, \frac{n}{2n(n-1) - 1} - 1, 0, \frac{n + 1}{1 - 2n(n-1)}, 1
\end{cases}, \{-1, 0, 1, 0, 0\}, \\
\begin{cases}
0, 
\frac{n(5 - 4n) - \sqrt{3n(3n - 4)}}{2(2n(n-1) - 1)}, 
- \frac{n(5 - 4n) - \sqrt{3n(3n - 4)}}{2n(2n(n-1) - 1)}, 
\frac{n(7 - 5n) + (n - 1)\sqrt{3n(3n - 4)}}{2n(2n(n-1) - 1)}, 1
\end{cases}, \\
\begin{cases}
0, 
\frac{n(5 - 4n) + \sqrt{3n(3n - 4)}}{2(2n(n-1) - 1)}, 
- \frac{n(5 - 4n) + \sqrt{3n(3n - 4)}}{2n(2n(n-1) - 1)}, 
\frac{(1 - n)\sqrt{3n(3n - 4)} - n(5n - 7)}{2n(2n(n-1) - 1)}, 1
\end{cases}, \\
\begin{cases}
0, 
\frac{2n(2n - 3) + 3\omega + 1}{2(1 - 2n(n-1))}, 
\frac{2n(2n - 3) + 3\omega + 1}{2n(1 - 2n(n-1))}, 
\frac{(2n - 3\omega - 1)(2n - 3(\omega + 1))}{4n(2n(n-1) - 1)}, 1
\end{cases}.
\]
The eigenvectors of \(n - \sqrt{3n(3n - 4)} - 2, n + \sqrt{3n(3n - 4)} - 2, 2n - 3(\omega + 1)\) belong to the tangent plane, and the eigenspace of \(2n\) intersects with tangent plane, therefor they all effect the stability of the point. The point is stable when
\[
\frac{1}{2} - \frac{\sqrt{3}}{2} < n < 0, \ \omega > \frac{1}{3}(2n - 3).
\]
The effective equation of state is \(\omega_{eff} = -\frac{1}{3}\).

Point \(K_1\): The eigenvalues of the linearised system are
\[
\begin{cases}
\frac{1}{n - 1} - 4, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}, 
- \frac{2n(n - 2)}{(n - 1)(2n - 1)}
\end{cases}, \\
\frac{2n - 4}{(1 - 2n)(n - 1)} - 2, 
- \frac{1 + n}{(n - 1)(1 - 2n)} - 3\omega - 4,
\]

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corresponding eigenvectors are
\[
\begin{align*}
\{0, n(8n^2 - 22n + 15), 2n - 1, 4(n - 2)(n - 1)\}, \\
\{0, (2n(2n - 5) + 7)(1 - n), 2n(2n - 5) + 7, 1\}, \\
\{0, -\frac{n}{n - 1}, \frac{1}{n - 1}, 1, 0\}, & \quad \left\{\frac{2(8 - 3n)n - 11}{6(n - 1)^2}, \frac{5 - 4n}{6(n - 1)^2}, 0, 1, 0\right\}, \\
\{0, \frac{n(4n - 5)}{3(n - 1)^2((2n - 1)\omega - 1)}, \frac{5 - 4n}{3(n - 1)^2((2n - 1)\omega - 1)}, 1, 0\}.
\end{align*}
\]

None of the eigenvectors are parallel to the normal vector there for all the eigenvalues effect the stability of the point, however when \(n\) equals \(\frac{5}{4}\) or \(\frac{3}{2}\), eigenvector of \(-\frac{n}{n - 1} - 4\) may be parallel to \(\vec{n}\), but in that case the point is not stable. The point is stable when \(n < \frac{1}{2} \left(1 - \sqrt{3}\right)\), or \(\frac{1}{2} < n < 1\), or \(n > 2\) and
\[
\omega > \frac{-8n^2 + 13n - 3}{6n^2 - 9n + 3}.
\]

The effective equation of state is \(\omega_{eff} = \frac{n(7 - 6n) + 1}{6n^2 - 9n + 3}\).

Point \(L_1\): The eigenvalues of the linearised system are
\[
3(\omega + 1), 3(\omega + 1), \frac{-2n + 3\omega + 3}{n}, \\
-\frac{\sqrt{n - 1}}{4n^3(3\omega + 8)^2 + \cdots + 3n((2n - 3)\omega - 1) + 3\omega + 3}, \\
\frac{\sqrt{n - 1}}{4n^3(3\omega + 8)^2 + \cdots + 3n((2n - 3)\omega - 1) + 3\omega + 3}.
\]

Both the specific forms of corresponding eigenvectors and the parameter range for the point to be stable are complicated, we do not give their exact forms, the point can be stable. The effective equation of state is \(\omega_{eff} = \frac{1 +\omega}{n^2} - 1\), the point is a solution depending on matter fluid with \(\Omega_m = \frac{n(9\omega + 13 - 2n(3\omega + 4)) - 3(\omega + 1)}{2n^2}\).

B. \(ln(R)\) model

Let us discuss now the case of Lagrangian \(F(R) = ln(R)\). Corresponding constraint surface can be written as
\[
x_1 - \gamma x_2 = 0 \left(\gamma = \frac{2\rho_m}{M_p^2}\right).
\]
The normal vector is \((1, -\gamma, 0, 0, 0)\). First we check if the surface is an integral, after some calculation we get
\[
\vec{n} \cdot \frac{d\vec{x}}{dN} = (x_1 - \gamma x_2)(-2(x_2 + x_5) + x_4 + 4),
\]
so the surface is an integral. \(m\) takes the constant value of \(-1\). We summarize the fixed points of this autonomous system and their stability analysis in Table III.

Next we check if the perturbation stays on the tangent plane at fixed point. We compute \(\vec{n} \cdot \frac{d\delta\vec{x}}{dN}\) and find out that for Point \(A_2, B_2\) and \(C_2\), it automatically equals to 0 suggesting that those points have 0 eigenvalue, for the rest of the points it equals to 0 if we require that the perturbation satisfy \(\delta x_1 - \gamma \delta x_2 = 0\), which means the perturbation starts on the tangent plane. We now analyze each fixed point in detail.

Point \(A_2\): The eigenvalues of the linearised system are
\[-5, 2, 0, 0, -3(\omega + 1),\]
corresponding eigenvectors are
\[
\{0, 0, -1, 1, 0\}, \left\{0, 0, -\frac{15}{7}, \frac{8}{7}, 1\right\}, \left\{-\frac{11}{6}, \frac{5}{6}, 0, 1, 0\right\}, \\
\{-1, 0, 1, 0, 0\}, \left\{0, 0, -\frac{5}{3(\omega + 1)}, 1, 0\right\}.
\]
The eigenvectors of \(-5, 2, -3(\omega + 1)\) are normal to \(\vec{n}\), so they belong to the tangent plane, therefore \(-5, 2, -3(\omega + 1)\) are eigenvalues of the system with the redundant variable eliminated. The eigenspace of 0 intersects with tangent plane, so 0 is also the eigenvalues of

| \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(x_5\) | \(\Omega_m\) | \(\omega_{eff}\) | Stable when |
|------|------|------|------|------|------|------|----------|
| \(A_2\) | 0 | 0 | 5 | -4 | 0 | 0 | \(\frac{1}{3}\) |
| \(B_2\) | 0 | 0 | 2 | -2 | 1 | 0 | \(-\frac{1}{3}\) |
| \(C_2\) | \(2\gamma\) | 2 | \(-2\gamma - 1\) | 0 | 0 | 0 | \(-1\) Not stable |
| \(D_2\) | 0 | 0 | 0 | 0 | 1 | 0 | \(-\frac{1}{3}\) Not stable |
| \(E_2\) | 0 | 0 | 0 | 1 | 0 | 0 | \(\frac{1}{3}\) Not stable |
| \(F_2\) | 0 | 0 | 0 | \(-1 + 3\omega\) | 0 | 2 | \(-3\omega\) | \(\frac{1}{3}\) Not stable |
| \(G_2\) | 0 | 0 | 0 | 1 | \(3\omega\) | 1 | \(-3\omega - 1\) | \(-\frac{1}{3}\) \(\omega < -1\) |

**TABLE III**: The fixed points in F(R) nonlinear massive gravity with \(F(R) = \ln R\)
the system with the redundant variable eliminated. The stability of this doubly degenerate equilibrium can be analysed with more advance technique, and is beyond the scope of this paper.

Point $B_2$: The eigenvalues of the linearised system are

$$-2, -2, 0, 0, -3(\omega + 1),$$

Corresponding eigenvectors are

$\{0, 0, -1, 1, 0\}, \{0, 0, 0, 0, 0\}, \{-1, -1, 0, 1, 1\},$

$\{-1, 0, 1, 0, 0\}, \left\{0, 0, -\frac{2}{3(\omega + 1)}, 1, 0\right\}.$

Therefore $-2, -3(\omega + 1), 0$ are eigenvalues of the system with the redundant variable eliminated. Point $B_2$ is also doubly degenerate equilibrium same as Point $A_2$, we will not discuss its stability.

Point $C_2$: The eigenvalues of the linearised system are

$$-\frac{1}{2} \left(3 + \sqrt{41}\right), -2, \frac{1}{2} \left(\sqrt{41} - 3\right), 0, -3(\omega + 1),$$

Corresponding eigenvectors are

$\left\{-\frac{1}{4} \left(5 + \sqrt{41}\right) \gamma, \frac{4}{5 - \sqrt{41}}, \frac{1}{4} \left(\left(5 + \sqrt{41}\right) \gamma + \sqrt{41} + 1\right), 1, 0\right\},$

$\{-2 \gamma, -2, 2 \gamma + 1, 0, 1\}, \left\{\frac{4 \gamma}{5 + \sqrt{41}}, \frac{1}{4} \left(\sqrt{41} - 5\right), \frac{1}{4} \left(\left(5 - \sqrt{41}\right) \gamma - \sqrt{41} + 1\right), 1, 0\right\},$

$\{-1, 0, 1, 0, 0\}, \left\{\frac{2 \gamma}{1 - 3 \omega}, \frac{2}{1 - 3 \omega}, \frac{6 \gamma (\omega + 1) + 9 \omega + 1}{9 \omega^2 + 6 \omega - 3}, 1, 0\right\}.$

Here the eigenvector of 0 is tangent to Line C, we are able to analyze the stability of this point. None of the eigenvectors are parallel to $\vec{n}$, so they all are responsible for the stability of the fixed point(eigenvector of $\frac{1}{2} \left(\sqrt{41} - 3\right)$ may be parallel to $\vec{n}$, but it requires $\gamma$ to be imaginary). Because $\frac{1}{2} \left(\sqrt{41} - 3\right)$ is positive, the point is not stable.

Point $D_2$: The eigenvalues of the linearised system are

$$-2, 2, 2, 2, -3 \omega - 1$$

Corresponding eigenvectors are

$\{0, 0, 0, -1, 1\}, \{-1, -2, 0, 0, 1\}, \{-1, 0, 0, 1, 0\},$

$\{-1, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}.$
Eigenvectors of $-2, -3\omega - 1$ are on the tangent plane. The eigenspace of $2$ intersects with tangent plane, so the system with the redundant variable eliminated have the eigenvalues of $2, 2, -2, -3\omega - 1$. The point is not stable.

Point $E_2$: The eigenvalues of the linearised system are

$$5, 5, 5, 2, 2 - 3\omega,$$

corresponding eigenvectors are

$$\{-1, 0, 0, 1, 0\}, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, -1, 1\}, \{0, 0, 0, 1, 0\}.$$ 

Eigenvectors of $2, 2 - 3\omega$ are on the tangent plane. The eigenspace of $5$ intersects with tangent plane, so the system with the redundant variable eliminated have the eigenvalues of $5, 5, 2, 2 - 3\omega$ (the eigenspace of $5$ seems insufficient to determine a base vector of the tangent plane, so we compute the eigenvalues of the system with one variable eliminated and get the desired eigenvalues). The point is not stable.

Point $F_2$: The eigenvalues of the linearised system are

$$2, 3(\omega + 1), 3(\omega + 1), 3(\omega + 1), 3\omega - 2,$$

corresponding eigenvectors are

$$\left\{0, 0, 0, \frac{6}{3\omega - 4} + 2, 1\right\}, \left\{-\frac{5}{3(\omega + 1)}, 0, 0, 1, 0\right\}, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 1, 0\}.$$ 

eigenvectors of $2, 3\omega - 2$ are on the tangent plane. The eigenspace of $3(\omega + 1)$ intersects with tangent plane, so the system with the redundant variable eliminated have the eigenvalues of $2, 3\omega - 2, 3(\omega + 1), 3(\omega + 1)$ (the same situation as Point $E_2$). The point is not stable.

Point $G_2$: The eigenvalues of the linearised system are

$$-2, 3(\omega + 1), 3(\omega + 1), 3(\omega + 1), 3\omega + 1$$

Corresponding eigenvectors are

$$\left\{0, 0, 0, \frac{2\omega}{\omega + 1}, 1\right\}, \left\{-\frac{2}{3(\omega + 1)}, 0, 0, 1, 0\right\}, \{-1, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0\}.$$
Eigenvectors of $-2, 3\omega + 1$ are on the tangent plane. The eigenspace of $3(\omega + 1)$ intersects with tangent plane, so the system with the redundant variable eliminated have the eigenvalues of $3(\omega + 1), 3(\omega + 1), -2, 3\omega + 1$ (the same situation as Point $E_2$). The point is stable when $\omega < -1$, the effective equation of state is $\omega_{\text{eff}} = -\frac{1}{3}$.

V. CONCLUSIONS

In this paper we have studied the dynamical behavior of the $F(R)$ nonlinear massive gravity by recasting the field equations into a 6 dimensional autonomous system. However after reducing the dimension of the system by one via the constraint equation, we could see that the system still has a redundant variable which gives rise to a hidden constraint equation. The hidden constraint equation which depends on the model of $F(R)$ would change the behavior of the perturbation greatly, and ignoring it may result in false conclusion in the stability analysis of the fixed points. We study the stability of the fixed points by analyzing the relations between the eigenvector of a certain eigenvalue and the tangent plane of the constraint surface at the fixed point. If the eigenvector is normal to the tangent plane, the corresponding eigenvalue do not effect the stability of the fixed point, otherwise the eigenvalue should be considered. We analyze the system in this way instead of eliminating one variable because it would result in complicated relations between variables thus making the dynamic of the system utterly complicated and hard to analyze.

Notice that some lines of equilibria have emerged instead of fixed points, this situation happens when the line of equilibria has a 0 eigenvalue whose eigenvector is tangent to the line of equilibria. The lines of equilibria would intersect with the constraint surface and the points of intersection are considered as the fixed points of the system, and we could carry out the analysis as discussed above.

We consider two specific models of $F(R)$ which are $R^n$ and $\ln(R)$. The models are relatively simple and the specific forms of constraint surfaces are easy to obtain, but the same process can be carried out with a more complicated model of $F(R)$. Both models present a few stable points which may have interesting cosmological implication. However $F(R)$ nonlinear massive gravity possesses plentiful phenomenological properties due to its features inherited both from nonlinear massive gravity and $F(R)$ gravity. More study is needed to fully understand the cosmological behavior of this model.
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