Massive spin 2 propagators on de Sitter space

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Abstract
We compute the Pauli-Jordan, Hadamard and Feynman propagators for the massive metrical perturbations on de Sitter space. They are expressed both in terms of mode sums and in invariant forms.

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I Introduction

The apparently simplest model of space-time beyond Minkowski space is de Sitter space. Its high (maximal) degree of symmetry makes it the typical framework to investigate quantum field theory outside flat space.

One of the essential ingredients of quantum field theory are the various Green functions of the fields. For the scalar field on de Sitter space, the first work on the subject is, in our knowledge, the paper of Géhéniau and Schomblond [1]. These authors have used the harmonicity property of de Sitter space, i.e. the possibility to solve Klein-Gordon equation by a function depending only on the geodesic distance, to obtain the expression of the Pauli-Jordan propagator $\Delta(x, y)$. Soon after, Cahen, Géhéniau, Günther and Schomblond [2] have obtained the expression of the analogous Green function $S(x, y)$ for the spinorial field. Their method consisted essentially to compute $S(x, y)$ first with $y$ fixed at the origin of a coordinate patch, and then, to generalize the expression so obtained for an arbitrary couple of points by using parallel transport. Later, Schomblond and one of the authors of this work (Ph.S) have obtained a Fock space description of the scalar [3], spinorial and vectorial [4] propagators by computing them as mode sums. The main result in [3] was that by imposing invariance conditions (with respect to the isometries of the space) and fixing the behaviour at short distances of the propagator, a uniqueness theorem holds. All ambiguities about the definition of particles, expressed by arbitrary Bogoljubov’s transformations, are resolved. At the same time, Candelas and Raine [5] obtained a similar result using the harmonicity properties of the space to write a Schwinger rep-
representation of the Feynman propagator depending only on the geodesic distance and satisfying a regularity boundary condition (imposed on the kernel of the Schwinger representation).

Let us emphasize that, for the massless scalar field ($\Box \Phi = 0$), it is impossible to obtain a fully de Sitter invariant vacuum state. This result was first noticed by Spindel \cite{6} and independently rediscovered in 1982 by Vilenkin and Ford \cite{7}. It has been discussed in details by Allen and Folacci \cite{8, 9}. Allen has also obtained explicitly de Sitter invariant representations for the Green functions of the spinorial, vectorial and gravitational fields \cite{10, 11, 12}, under the assumption a priori that these Green functions could be expressed in terms of products of functions of the geodesic distance with maximally symmetric bispinors or bitensors.

A drawback of this geometrical construction of the Green functions is that we have no information about the existence of an underlying Fock space, i.e. a vacuum state such that expectations values of fields products with respect to it give the corresponding Green functions. Actually, for the gravitational field, the situation is the same as for the massless spin 0 field \cite{1}: there exists de Sitter invariant Green functions, but no corresponding vacuum state. Representations of these Green functions have been obtained by Allen and Turyn \cite{12, 13} and by Antoniadis and Mottola \cite{14}. Both results, which differ only by gauge choice, are obtained as analytic continuation of Green functions built on the Euclidean 4-sphere. A direct evaluation of the gravitational propagator as mode sums in physical space (de Sitter space) has been done by Tsamis and Woodard \cite{15}. Their construction leads to a result analogous to the one already obtained for the massless, minimally coupled scalar field: there is no de
Sitter invariant vacuum state for the massless spin 2 field.

In this note, we shall perform the calculation of the propagator for the “massive spin 2 field”. More precisely, we consider the gravitational perturbation field equations introduced by Lichnerowicz [16], which corresponds to a mixture of spin 0 and spin 2 fields. In section 2, we summarize the field equations and remind the mode sums representations of the various propagators. In the third section, we specialize the field equations on (3+1) de Sitter space and solve them explicitly on a half de Sitter space. We then obtain the propagators by summing modes. In section 4, we establish coordinate-free and manifestly $O(4,1)$ invariant representations of the propagator. In section 5, we discuss quickly the analytic continuation of the modes and propagators on the full de Sitter space.

II Metric perturbation on Einstein space

Following Lichnerowicz [16], we write the equations of motion for “massive metric perturbations” on an Einstein background space ($R_{\alpha\beta} = \Lambda g_{\alpha\beta}$) as

$$ \delta R_{\alpha\beta}(h) = \mu h_{\alpha\beta} $$

(1)

where $\delta R_{\alpha\beta}(h)$ denotes the terms linear in the components of the tensor $h$ in the expansion of the Ricci tensor evaluated on the metric $g' = g + h$. The factor $\mu$ on the right hand side of eq.(1) is related to a mass term as $M^2 = 2(\Lambda - \mu)$. This definition of mass is meaningful because $M^2 = 0$ corresponds to pure gravity. Hereafter, we shall denote by $\frac{m}{R^2} = -2\mu$ the eigenvalues of the quadratic Casimir operator $I_1$ considered
by Börner and Dürr [17] who have used the notations \( m^2 = -2\mu \) and \( m_0 = i\nu \) (see eq.(47)).

From the Bianchi identities, written for the metric \( g' \), we deduce that solutions of eq.(1) satisfy automatically the de Donder conditions:

\[
(\mu - \Lambda) \nabla_{\alpha} h^{\alpha\beta} = 0 .
\]  

(2)

In these equations, as in the rest of this paper, we have introduced the Einsteinian conjugate tensor \( h_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h_{\mu\mu} \), and the covariant derivatives refer to the Levi-Civita connection built on the metric \( g \) which is used to lower and raise the indices. Consequently, eq.(1) is equivalent to the system constituted by eq.(2) and

\[
\Box h_{\alpha\beta} + 2R_{\alpha\sigma\beta\rho} h^{\sigma\rho} - 2(\Lambda - \mu)h_{\alpha\beta} = 0 .
\]  

(3)

A particular solution of eqs (2,3) is given by:

\[
h_{\alpha\beta} = \nabla_{\alpha} \nabla_{\beta} \phi + (\Lambda - \mu) g_{\alpha\beta} \phi
\]  

(4)

when \( \phi \) satisfies the scalar field equation :

\[
\Box \phi + 2\mu \phi = 0 .
\]  

(5)

This scalar field is proportional to the trace:

\[
h_{\mu\mu} = (4\Lambda - 6\mu) \phi ,
\]  

(6)

and describes the spin 0 content of \( h \), which is a mixture of spins 0 and 2.

Equation (1) can be derived from the Lagrangian :
\( \mathcal{L} = \frac{1}{2} \nabla^\alpha h^\beta \nabla^\mu h^{\nu\rho} (g_{\alpha\mu}g_{\beta\nu}g_{\gamma\rho} - g_{\alpha\rho}g_{\beta\mu}g_{\gamma\nu} - g_{\mu\beta}g_{\nu\alpha}g_{\rho\gamma} \\
+ g_{\rho\nu}g_{\beta\alpha}g_{\gamma\lambda} - g_{\alpha\nu}g_{\beta\rho}g_{\gamma\mu}) - \frac{\mu}{2} h^{\alpha\beta} h^{\mu
u} (g_{\alpha\mu}g_{\beta\nu} \\
+ g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu}) \equiv \frac{1}{2} \nabla^\alpha h^\beta \nabla^\mu h^{\nu\rho} Q_{\alpha\beta\gamma,\mu\nu\rho} - \frac{\mu}{2} h^{\alpha\beta} h^{\mu\nu} P_{\alpha\beta,\mu\nu} \)

which is unique, up to trivial transformations (rescaling and addition of divergences).

From it we deduce the expression of a conserved current, a sesquilinear form on the space of complex solutions of eq.(I):

\( J_\alpha(h, k) = i(h^* \beta \gamma \nabla^\mu k^{\nu\rho} - k^* \beta \gamma \nabla^\mu h^{\nu\rho}) Q_{\alpha\beta\gamma,\mu\nu\rho} \)

and, by integration on a Cauchy surface \( \Sigma \), a symplectic structure:

\( h*k = \int_\Sigma d\sigma^\alpha J_\alpha(h, k) = -(k*h)^* = -(k^* h^*) \).

If \( \{^A h\} \) is a complete set of positive frequency modes, labelled by the index \( A \), and satisfying the relations

\[ ^A h * ^B h = \delta_{AB} \]
\[ ^A h^* * ^B h^* = -\delta_{AB} \]
\[ ^A h^* * ^B h = 0 = ^A h * ^B h^* \]

we may obtain as mode sums the usual Green functions of the quantum field \( \hat{h} \) associated to the classical field \( h \). The Pauli-Jordan propagator \( \Delta(x, y) \), defined by the commutator \([\hat{h}(x), \hat{h}(y)]\), is given by:

\( \Delta(x, y) = -i \sum_A ^A h(x)^A h^*(y) - ^A h^*(x)^A h(y) \).
while the symmetric (often called Hadamard) propagator $\Delta^1(x, y)$, defined as the vacuum expectation value of the anti-commutator $\langle \{\hat{h}(x), \hat{h}(y)\} \rangle$, is:

$$\Delta^1(x, y) = \sum_A A h(x)^A h^*(y) + A^* h(x)^A h(y) , \quad (12)$$

and the Feynman propagator $\Delta_F(x, y)$:

$$\Delta_F(x, y) = \frac{1}{2} [\Delta^1(x, y) + i\epsilon(x, y)\Delta(x, y)] , \quad (13)$$

where $\epsilon(x, y) = \pm 1$ according as the point $x$ is in future or the past of $y$.

### III Field equations on de Sitter space

The four dimensional de Sitter space $H^4$ can be seen as the homogeneous coset space $O(4,1)/O(3,1)$, i.e as the sphere of equation:

$$\eta_{AB} X^A X^B = R^2 \quad (A, B = 0, ..., 4) \quad (14)$$

imbedded in a five dimensional (flat) Minkowski space $M^5$. Using the parametrization:

$$\lambda = \frac{R^2}{X^4 - X^0} , \quad x^i = \frac{R X^i}{X^4 - X^0} \quad (i = 1, 2, 3) \quad (15)$$

the metric induced on $H^4$ reads as

$$g = \frac{R^2}{\lambda^2} \left(-d\lambda^2 + \sum_i (dx^i)^2 \right) . \quad (16)$$
De Sitter space-time is a space of constant curvature: 
\[ R_{\alpha\beta\gamma\delta} = \frac{\Lambda}{3} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \]
the relation between the cosmological constant \( \Lambda \) and the radius \( R \) of the space being:
\[ \Lambda = \frac{3}{R^2}. \]

On de Sitter space eq.(3) becomes equivalent to

\[ \Box \tilde{h}_{\alpha\beta} + \left( \frac{8}{3} \Lambda - 2\mu \right) \tilde{h}_{\alpha\beta} - \frac{2}{3} \Lambda g_{\alpha\beta} \tilde{h}_{\tau\tau} = 0. \]  
\hspace{1cm} (17)

Hereafter we shall in a first step restrict ourselves to the chart \( \lambda > 0 \), which covers only one half of the full de Sitter space. It is the domain corresponding to the causal past of a physical observer (region \( \mathcal{O} \) on fig.1).

To solve the system of equations (17) we have found useful to use the rescaled quantities introduced in [15]:

\[ k_{\mu\nu} = \frac{\lambda^2}{R^2} \tilde{h}_{\mu\nu} \]  
\hspace{1cm} (18)

and to pass to the Fourier transformed variables :

\[ K_{\mu\nu}(\lambda, \vec{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3p \ e^{-i\vec{p} \cdot \vec{x}} k_{\mu\nu}(\lambda, \vec{x}). \]  
\hspace{1cm} (19)

where \( \vec{p}, \vec{x} = \sum_i p^i x^i \).
Moreover it is natural to express these Fourier components in a local frame adapted to the vector $\vec{p}$. To this aim, we introduce the four vectors $u_{(\alpha)}(\vec{p})$ whose components with respect to the natural coordinate frame $(\partial_\lambda, \partial_{x^i})$ are:

\begin{align}
    u_{(0)}^\alpha &= (1, 0, 0, 0) \\
    u_{(3)}^\alpha &= (0, p^i/p) , \quad p = \sqrt{\sum_i (p^i)^2} \\
    u_{(1,2)}^\alpha &= (0, \varepsilon_{(1,2)}^i), \quad \text{with} \quad \sum_i p^i \varepsilon_{(1,2)}^i = 0 .
\end{align}

They satisfy the orthogonality relations

\begin{equation}
    g_{\mu\nu} u^{\mu}_{(\alpha)}(\vec{p}) u^{\nu}_{(\beta)}(\vec{p}) = \frac{R^2}{\lambda^2} \eta_{\alpha\beta} .
\end{equation}

We shall also make use of the projector on the spacelike directions orthogonal to $\vec{p}$:

\begin{align}
    \perp^\nu_{\mu} &= \delta^\nu_{\mu} + \frac{\lambda^2}{R^2} \left( u_{(0)}^\mu u_{(0)}^{\nu} - u_{(3)}^\mu u_{(3)}^{\nu} \right) \\
    &= \frac{\lambda^2}{R^2} \left( u_{(1)}^\mu u_{(1)}^{\nu} + u_{(2)}^\mu u_{(2)}^{\nu} \right) .
\end{align}

Once expressed in this frame, the components of (19) split into longitudinal and transverse parts:

\begin{equation}
    K^L = u_{(3)}^\alpha K_{0\alpha} = \frac{p^i}{p} K_{0i} ,
\end{equation}
\[ K^\perp_\alpha = (0, K^\perp_i) = (0, \perp_i^j K_0^j) \]  \hspace{1cm} (24)

\[ K^{LL} = u^\alpha_{(3)} u^\beta_{(3)} K_{\alpha\beta} \]  \hspace{1cm} (25)

\[ K^\perp\perp_\alpha = (0, K^\perp\perp_i) = u^\mu_{(3)} \perp^\nu_\alpha K_{\mu\nu} \]  \hspace{1cm} (26)

\[ K^\perp\perp_{\alpha\beta} = \perp_{\alpha\beta}^\mu K_{\mu\nu} \]  \hspace{1cm} (27)

\[ K^\perp\perp_{0\alpha} = K^\perp\perp_{\alpha\bar{0}} = 0 \]  \hspace{1cm} (28)

Conversely, the components of (19) with respect to the natural frame read

\[ K_{0i} = u^i_{(3)} K^L + K^\perp_i \]  \hspace{1cm} (29)

\[ K_{ij} = u^i_{(3)} u^j_{(3)} K^{LL} + u^i_{(3)} K^\perp\perp_j + u^j_{(3)} K^\perp\perp_i + K^\perp\perp_{ij} \]  \hspace{1cm} (30)

In terms of these variables eq.(2) splits into:

\[ \dot{K}_{00} - i \rho K^L - \frac{4}{\lambda} K_{00} - \frac{1}{\lambda} K = 0 \]  \hspace{1cm} (31)

\[ \dot{K}^L - i \rho K^{LL} - \frac{4}{\lambda} K^L = 0 \]  \hspace{1cm} (32)

\[ \dot{K}^\perp_i - i \rho K^\perp\perp_i - \frac{4}{\lambda} K^\perp_i = 0 \]  \hspace{1cm} (33)

where we have denoted by dots derivatives with respect to \( \lambda \) and by \( K \) the trace of \( h_{\mu\nu} \):

\[ K = -K_{00} + \sum_i K_{ii} = h^\mu_\mu = -h^\mu_\mu \]  \hspace{1cm} (34)
Eqs (17) written with $m = -2\mu R^2$ become:

\[
\ddot{K}_{00} - \frac{6}{\lambda} \dot{K}_{00} + \left( p^2 + \frac{16 + m}{\lambda^2} \right) K_{00} + \frac{4}{\lambda^2} K = 0 \quad (35)
\]

\[
\ddot{K}^L - \frac{4}{\lambda} \dot{K}^L - \frac{2i}{\lambda} p K_{00} + \left( p^2 + \frac{10 + m}{\lambda^2} \right) K^L = 0 \quad (36)
\]

\[
\ddot{K}^\perp_j - \frac{4}{\lambda} K^\perp_j + \left( p^2 + \frac{10 + m}{\lambda^2} \right) K^\perp_j = 0 \quad (37)
\]

\[
\ddot{K}^{LL} - \frac{2}{\lambda} \dot{K}^{LL} + \left( p^2 + \frac{6 + m}{\lambda^2} \right) K^{LL} - \frac{4i}{\lambda} p K^L - \frac{2}{\lambda^2} (K + K_{00}) = 0 \quad (38)
\]

\[
\ddot{K}^{\perp \perp}_j - \frac{2}{\lambda} \dot{K}^{\perp \perp}_j + \left( p^2 + \frac{6 + m}{\lambda^2} \right) K^{\perp \perp}_j - \frac{2}{\lambda^2} \perp_j (K_{00} + K) = 0 \quad (39)
\]

Summing the appropriate equations (35, 38, 40) we recover with the help of eqs (31, 32, 33) the trace equation (5) written in terms of its Fourier transformed variable:

\[
\ddot{K} - \frac{2}{\lambda} \dot{K} + \left( p^2 + \frac{m}{\lambda^2} \right) K = 0 \quad . \quad (41)
\]

The general solution of this equation is given by a combination of Hankel functions (see [18]):

\[
K(\lambda, \vec{p}) = (\lambda \vec{p})^{3/2} \left[ a(\vec{p}) \mathcal{H}^{(1)}_{\nu_0}(\lambda \vec{p}) + b(\vec{p}) \mathcal{H}^{(2)}_{\nu_0}(\lambda \vec{p}) \right] \quad (42)
\]

with
\[ \nu_0 = i \sqrt{m - \frac{9}{4}} \quad (43) \]

and

\[ \mathcal{H}_\nu^{(1)}(\lambda p) = e^{i\nu \frac{\pi}{2}} \mathcal{H}_\nu^{(1)}(\lambda p) = [\mathcal{H}_\nu^{(2)}(\lambda p)]^* \quad (44) \]

where \( \mathcal{H}_\nu^{(1)}, \mathcal{H}_\nu^{(2)} \) are the usual Hankel functions defined in [18].

From eqs (4, 5) and eq.(35) we obtain (assuming \( m + 4 \neq -n(n+1) \), \( n \in \mathbb{Z} \), see comment after eq.(66)):

\[ K_{00} = Q - \frac{\lambda^2}{3(m+4)} \left( \frac{3}{\lambda} \dot{K} - (p^2 - \frac{3}{\lambda^2})K \right) \quad (45) \]

where \( Q \) is the general solution of the homogeneous part \( (K = 0) \) of eq.(35):

\[ Q = (\lambda p)^{7/2} \left[ c(\vec{p}) \mathcal{H}_\nu^{(1)}(\lambda p) + d(\vec{p}) \mathcal{H}_\nu^{(2)}(\lambda p) \right] \quad (46) \]

and here

\[ \nu = i \sqrt{m + \frac{15}{4}} \quad . \quad (47) \]
From these solutions and eq. (31) we obtain algebraically the $K^L$ component:

$$K^L = \frac{-i}{p} \left[ \dot{Q} - \frac{4}{\lambda} Q + \frac{\lambda^2 p^2}{3(m+4)} \left( \dot{K} + \frac{1}{\lambda} K \right) \right].$$  \hspace{1cm} (48)

In the same way eq. (32) gives immediately

$$K^{LL} = \frac{1}{p^2} \left[ \frac{2}{\lambda} \dot{Q} + (p^2 + \frac{m-4}{\lambda^2}) Q \right] - \frac{\lambda^2}{3(m+4)} \left[ \frac{1}{\lambda} \dot{K} - (p^2 + \frac{m+3}{\lambda^2}) K \right].$$  \hspace{1cm} (49)

The equation (37) is decoupled from the others. Its general solution is:

$$K^\perp_j = (\lambda p)^{5/2} \left[ c_j(\vec{p}) H^{(1)}_\nu(\lambda p) + d_j(\vec{p}) H^{(2)}_\nu(\lambda p) \right].$$  \hspace{1cm} (50)

where $c_j(\vec{p})$ and $d_j(\vec{p})$ are 3-vectors orthogonal to $\vec{p}$ and $\nu$ is again given by eq. (47).

This leads directly, thanks to eq. (33), to

$$K^{L\perp}_j = -\frac{i}{p} \left( \dot{K}^\perp_j - \frac{4}{\lambda} K^\perp_j \right).$$  \hspace{1cm} (51)

Finally it remains to solve eq. (40) for $K^{\perp\perp}_{jl}$. The general solution of the homogenous part is still given by a combination of Hankel functions of the same index $\nu$ (see eq. (47)).
\[ Q_{jl}^{\perp\perp} = (\lambda p)^{3/2} \left[ c_{jl}(\vec{p}) \mathcal{H}_{\nu}^{(1)}(\lambda p) + d_{jl}(\vec{p}) \mathcal{H}_{\nu}^{(2)}(\lambda p) \right] , \] (52)

while a particular solution can be expressed in terms of \( Q \) and \( K \), leading to the general solution:

\[
K_{jl}^{\perp\perp} = Q_{jl}^{\perp\perp} - \perp_{jl} \left[ \frac{1}{3(m + 4)} (3 + m) \dot{K} - \left( 2 \lambda \dot{Q} + (m - 4) \dot{Q} \right) \right] \] (53)

The integration constants \( c_{jl} \) (and similarly \( d_{jl} \)) can be expressed with the help of the projector (22) as:

\[
c_{jl}(\vec{p}) = \sum_{m,n} (\perp_{jm} \perp_{ln} - \frac{1}{2} \perp_{jl} \perp_{mn}) \mathcal{E}_{mn}(\vec{p}) \] (54)

where \( \mathcal{E}_{mn}(\vec{p}) \) is arbitrary. This form insures that \( c_{jl} \) is transverse to \( \vec{p} \) and traceless in order to verify eq.(30).

To be complete we have still to check that the longitudinal components (48, 49, 51), obtained from the divergence equations, are really solutions of the second order equations (36, 38 and 39). They are!

Now we may write a complete set of modes, solutions of eqs (217) as:

\[
\mathcal{h}_{\mu\nu}(\lambda, \vec{x}, \vec{p}) = \frac{e^{i\vec{p}.\vec{x}}}{(2\pi)^{3/2}} \frac{R^2}{\lambda^2} K_{\mu\nu}(\lambda, \vec{p}) .
\] (55)
However instead of considering their natural components it is more useful to split the modes according to their spin contents. Indeed, such a decomposition leads automatically to orthogonal modes and it will just remain to normalize them. So we shall write the general complex solution of the field equations as

$$\hat{h}_{\mu\nu}(\lambda, \vec{x}) = \sum_{I} \int d^3p [a_I(\vec{p}) \bar{h}_I^{\mu\nu}(\lambda, \vec{x}, \vec{p}) + b_I^+(\vec{p}) \bar{h}_I^{\mu\nu}(\lambda, \vec{x}, \vec{p})]$$  (56)

where the index $I$ runs over six values corresponding to the spin 0 and spin 2 content of the field (See appendix for the explicit form of the modes).

If we assume that the amplitudes $a_I(\vec{p}), b_I(\vec{p})$ are operators obeying usual canonical commutation relations:

$$[a_I(\vec{p}), a_{I'}^\dagger(\vec{p}')] = \delta_{II'} \delta(\vec{p} - \vec{p}')$$
$$[b_I(\vec{p}), b_{I'}^\dagger(\vec{p}')] = \delta_{II'} \delta(\vec{p} - \vec{p}')$$
$$[a_I(\vec{p}), a_{I'}(\vec{p}')] = [b_I(\vec{p}), b_{I'}(\vec{p}')] = [a_I(\vec{p}), b_{I'}(\vec{p}')] = ... = 0$$  (57)

the modes $\bar{h}_I^{\mu\nu}$ have to be normalized to $\delta^3(\vec{p} - \vec{p}')$.

Anticipating on the discussion of section V, we impose now that all modes depends only on $H^{(2)}$ functions, with their various "$d" coefficients equal to 1. This choice is compatible with the commutation relations (57). It results from the requirements that 1\textsuperscript{o}) the Green functions are de Sitter invariant 2\textsuperscript{o}) they have the same short
distance behaviour as in flat space. The coefficients of normalization of the modes with respect to the scalar product (9) are displayed in the appendix. Inserting these modes in the general expression of the Green functions (13) we obtain

\[ \Delta_{00,00'}(x; y) = \frac{R^4}{6(m+4)(m+\frac{15}{2})} \left( \frac{3}{\lambda} \partial_\lambda - \vec{\nabla}_x \cdot \vec{\nabla}_y + \frac{3}{\lambda x^2} \right) \Delta_0(p) \]

\[ + \frac{2R^4}{3(m+6)(m+4)} \left( \vec{\nabla}_x \cdot \vec{\nabla}_y \right)^2 \Delta_\nu(p) \]

\[ \Delta_{00,00'}(x; y) = \frac{R^4}{6(m+4)(m+\frac{15}{2})} \partial_\lambda \left( \frac{3}{\lambda} \partial_\lambda - \vec{\nabla}_x \cdot \vec{\nabla}_y + \frac{3}{\lambda x^2} \right) \Delta_0(p) \]

\[ - \frac{2R^4}{3(m+6)(m+4)} \partial_\lambda \left( \vec{\nabla}_x \cdot \vec{\nabla}_y \right)(\partial_\lambda - \frac{2}{\lambda}) \Delta_\nu(p) \] (58)

\[ \Delta_{00,00'}(x; y) = \frac{R^4}{6(m+4)(m+\frac{15}{2})} \partial_\lambda \left( \frac{3}{\lambda} \partial_\lambda - \vec{\nabla}_x \cdot \vec{\nabla}_y + \frac{3}{\lambda x^2} \right) \Delta_0(p) \]

\[ + \frac{2R^4}{6(m+4)(m+\frac{15}{2})} \partial_\lambda \partial_\nu \left( \frac{3}{\lambda} \partial_\lambda - \vec{\nabla}_x \cdot \vec{\nabla}_y + \frac{3}{\lambda x^2} \right) \Delta_0(p) \]

\[ - \frac{2R^4}{3(m+6)(m+4)} \partial_\lambda \partial_\nu \left( \vec{\nabla}_x \cdot \vec{\nabla}_y \right)(\partial_\lambda + \frac{2}{\lambda}) \Delta_\nu(p) \] (59)

\[ \Delta_{00,00'}(x; y) = \frac{R^4}{6(m+4)(m+\frac{15}{2})} \partial_\lambda \partial_\nu \left( \partial_\lambda + \frac{1}{\lambda} \right)(\partial_\nu + \frac{1}{\lambda}) \Delta_0(p) \]

\[ + \frac{2R^4}{3(m+6)(m+4)} \partial_\lambda \partial_\nu \left( \partial_\lambda - \frac{2}{\lambda} \right)(\partial_\nu - \frac{2}{\lambda}) \Delta_\nu(p) \]

\[ + \frac{R^4}{2(m+6)} \eta_{j'j}(\vec{\nabla}_x \cdot \vec{\nabla}_y) \frac{1}{\lambda x^2} \Delta_\nu(p) \]

\[ - \frac{R^4}{2(m+6)} \partial_\lambda \partial_\nu \left( \frac{1}{\lambda} \right) \Delta_\nu(p) \] (60)

\[ \Delta_{00,00'}(x; y) = \frac{R^4}{6(m+4)(m+\frac{15}{2})} \eta_{j'k'}(\partial_\lambda + \frac{1}{\lambda})(\partial_{\nu} - \frac{3}{\lambda x^2}) \partial_i \Delta_0(p) \]

\[ + \frac{R^4}{6(m+4)(m+\frac{15}{2})} \partial_\lambda \partial_{j'} \partial_{k'} \partial_\lambda \partial_\nu \left( \partial_\lambda + \frac{1}{\lambda} \right) \Delta_0(p) \]

\[ - \frac{2R^4}{3(m+6)(m+4)} \eta_{j'k'} \partial_\lambda (\partial_\lambda - \frac{2}{\lambda})(\partial_{\nu} - \frac{m}{2\lambda x^2}) \Delta_\nu(p) \]

\[ - \frac{R^4}{2(m+6)} \eta_{j'k'} \partial_\lambda \left( \frac{1}{\lambda} \right) \Delta_\nu(p) \] (61)

\[ - \frac{R^4}{2(m+6)} \eta_{j'k'} \partial_\lambda \left( \frac{1}{\lambda} \right) \Delta_\nu(p) \]

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\[
\Delta_{ij,k} \ln \frac{R^4}{2 (m + 6)} \eta_{k}\partial_j \left( \frac{1}{\lambda} \partial_{\lambda'} - \frac{3}{\lambda} \Delta_{\nu} (p) \right) \]
\[
- \frac{2R^4}{3(m + 6)(m + 4)} \left( \frac{3}{2} + \frac{3}{4\nu} (m + 3) \right) \partial_i \partial_j \partial_k \frac{1}{\lambda} \Delta_{\nu+1} (p) \]
\[
- \frac{2R^4}{3(m + 6)(m + 4)} \left( \frac{3}{2} - \frac{3}{4\nu} (m + 3) \right) \partial_k \partial_j \partial_k \frac{1}{\lambda} \Delta_{\nu-1} (p) \]
\[
+ \frac{2R^4}{3(m + 6)(m + 4)} \partial_i \partial_j \partial_k \left( \partial_{\lambda} - \frac{2}{\lambda} \right) \Delta_{\nu} (p) \]

\[
\Delta_{ij,k\nu'} (x, y) = \eta_{j\nu'} \eta_{k\nu'} \left\{ \begin{array}{c}
\frac{R^4}{6(m + 4)(m + \frac{15}{2})} (\lambda \partial_{\lambda} - 3 - m) (\lambda' \partial_{\lambda'} - 3 - m) \Delta_{\nu_0} (p) \\
- \frac{2R^4}{3(m + 6)(m + 4)} \left( \frac{\partial_{\lambda}}{\lambda} + \frac{m}{2\lambda^2} \right) \Delta_{\nu} (p) \\
- \frac{R^4}{2(\lambda' \lambda^2) \Delta_{\nu} (p)} \\
+ \eta_{j\nu} \partial_{k\nu} \left\{ \frac{R^4}{6(m + 4)(m + \frac{15}{2})} (\lambda \partial_{\lambda} - 3 - m) \Delta_{\nu_0} (p) \right. \\
+ \frac{2R^4}{3(m + 6)(m + 4)} \left( \frac{\partial_{\lambda}}{\lambda} + \frac{m}{2\lambda^2} \right) \Delta_{\nu} (p) \\
- \frac{1}{\lambda \lambda'} 2R^4 \\
\frac{24m + 90 + 36\nu - 12m\nu}{16\nu^2} \Delta_{\nu-1} (p) \\
- \frac{1}{\lambda \lambda'} \frac{24m + 90 + 36\nu + 12m\nu}{16\nu^2} \Delta_{\nu+1} (p) \\
\left. \right\} \\
+ \eta_{k\nu'} \partial_{j\nu} \left\{ \frac{R^4}{6(m + 4)(m + \frac{15}{2})} (\lambda' \partial_{\lambda'} - 3 - m) \Delta_{\nu_0} (p) \right. \\
+ \frac{2R^4}{3(m + 6)(m + 4)} \left( \frac{\partial_{\lambda'}}{\lambda'} + \frac{m}{2\lambda'^2} \right) \Delta_{\nu} (p) \\
- \frac{1}{\lambda' \lambda} 2R^4 \\
\frac{24m + 90 - 36\nu - 12m\nu}{16\nu^2} \Delta_{\nu-1} (p) \\
- \frac{1}{\lambda' \lambda} \frac{24m + 90 + 36\nu + 12m\nu}{16\nu^2} \Delta_{\nu+1} (p) \\
\left. \right\} \\
- (\eta_{ik} \partial_{j\nu'} + \eta_{ik} \partial_{j\nu} + \eta_{jk} \partial_{i\nu'} + \eta_{jk} \partial_{i\nu}) \\
\frac{R^4}{2 (m + 6) \lambda \lambda'} \frac{12\nu + 8m + 30}{16\nu^2} \Delta_{\nu-1} (p) + \frac{-12\nu + 8m + 30}{16\nu^2} \Delta_{\nu+1} (p) \left( p \right) \right. \\
+ (\eta_{ik} \eta_{jk'} + \eta_{ik} \eta_{jk'}) \frac{R^4}{2(\lambda '\lambda^2)} \Delta_{\nu} (p) \\
+ \partial_i \partial_j \partial_{k\nu} \frac{2R^4}{3(m + 6)(m + 4) \Delta_{\nu} (p) \left( p \right) \right. \}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
m + 6 + 4(m + \frac{1}{2}) \Delta_{\nu_0}(p) + \frac{4 + m}{19 + 4m} \Delta_{\nu}(p) \\
\frac{1}{4(\nu - 1)^2} \left\{ \frac{-3(4 + m)(6 + m)}{60 + 16m} - 3(m + 4)(\frac{1}{2} + \frac{3}{4\nu})^2 - \frac{9(m + 3 + \nu)}{60 + 16m} \right\} \Delta_{\nu - 2}(p) \\
+ \frac{1}{4(\nu - 1)^2} \left\{ \frac{3(4 + m)(6 + m)}{60 + 16m} - 3(m + 4)(-\frac{1}{2} + \frac{3}{4\nu})^2 - \frac{9(m + 3 - \nu)}{60 + 16m} \right\} \Delta_{\nu + 2}(p)
\end{array} \right. \\
\end{align*}
\]

where

\[
\Delta_{\nu_0}(p) = \frac{\pi}{4R^2} \left( \frac{\lambda \lambda'}{2\pi} \right)^{3/2} \int d^3k e^{i\vec{k} \cdot (\vec{x} - \vec{y})} (\mathcal{H}_{\nu_0}^{(1)}(\lambda k)\mathcal{H}_{\nu_0}^{(2)}(\lambda' k) - \mathcal{H}_{\nu_0}^{(2)}(\lambda k)\mathcal{H}_{\nu_0}^{(1)}(\lambda' k))
\]

\[
= \frac{-1}{8\pi R^2 \cos(\nu_0 \pi)} (\lambda - \lambda') \Im \left[ \sum_{1} \left( \nu_0 \frac{3}{2} + \nu_0, \frac{3}{2} + 1 + p \right) \right]
\]  

(64)

and a similar representation for \( \Delta_{\mu_\nu,\rho\sigma}^{(1)}(x, y) \) and \( \Delta_{\mu_\nu,\rho\sigma}^{(2)}(x, y) \) where \( \Delta_{\nu_0} \) is replaced respectively by:

\[
\Delta_{\nu_0}^{(1)}(p) = \frac{\pi}{4R^2} \left( \frac{\lambda \lambda'}{2\pi} \right)^{3/2} \int d^3k e^{i\vec{k} \cdot (\vec{x} - \vec{y})} (\mathcal{H}_{\nu_0}^{(1)}(\lambda k)\mathcal{H}_{\nu_0}^{(2)}(\lambda' k) + \mathcal{H}_{\nu_0}^{(2)}(\lambda k)\mathcal{H}_{\nu_0}^{(1)}(\lambda' k))
\]

\[
= \frac{1}{8\pi R^2 \cos(\nu_0 \pi)} \Re \left[ \sum_{1} \left( \nu_0 \frac{3}{2} + \nu_0, \frac{3}{2} + 1 + p \right) \right]
\]  

(65)

and

\[
\Delta_{\nu_0}^{(2)}(p) = \frac{1}{16\pi R^2 \cos(\nu_0 \pi)} \sum_{1} \left( \nu_0 \frac{3}{2} + \nu_0, \frac{3}{2} + 1 + p \right) - i\epsilon)
\]

(66)

The occurrence of the factor \( \sec(\nu \pi) \) in eqs(64,65) implies that the special values of \( \nu = n + \frac{1}{2} \), i.e. \( m + 4 = n(n + 1) \) need a special analysis. These values of the index of the modes correspond to eigenvalues of the Laplace-Beltrami operator on the 4-sphere \( S^4 \), i.e. situations where the analytic continuation of the propagators built on \( S^4 \) onto de Sitter space fails [14]. In these cases it doesn’t exist de Sitter invariant states; the invariant propagators describe expectations values of the field with respect to a density matrix.
IV Invariant representation of the Green functions

The \((\lambda, \vec{x})\) coordinates cover only one half of de Sitter space corresponding to the causal past of a physical observer (region \(\mathcal{O}\) on fig.1). This domain is bounded by a future event horizon \((X^0 = X^4\) in eq.\((13)\)) and is invariant under a seven parameter subgroup of the full de Sitter group \(O(4, 1)\). This group is isomorphic to \(R^+ \times E(3)\), as it is obvious from the writing \((16)\) of the metric. It consists of the \(E(3)\) euclidean motions preserving \(\sum_i (dx^i)^2\) in eq.\((16)\) and the dilatations \(\lambda \mapsto k\lambda, \vec{x} \mapsto k\vec{x}\).

On this domain we may express the Green functions in terms of obviously geometrically invariant quantities. Let us consider two points \((P, Q)\) belonging to \(\mathcal{O}\) and denote by \(X^A\) and \(Y^A\) \((A = 0, ..., 4)\) their coordinates in the embedding \(M^5\) space. The tangent vectors at the ends of the unique geodesic in \(\mathcal{O}\) joining them are:

\[
T_P^A = \frac{Y_A - pX_A}{R|p^2 - 1|^{1/2}}, \quad T_Q^A = \frac{-X_A + pY_A}{R|p^2 - 1|^{1/2}},
\]

where:

\[
p = \frac{\eta_{AB}X^AY^B}{R^2} \equiv \frac{XX^A - pY^A}{R^2} = \frac{\lambda_P^2 + \lambda_Q^2 - (\vec{x}_P - \vec{x}_Q)^2}{2\lambda_P\lambda_Q}.
\]

The components \(V_Q^A\) of the vector \(V_P^A\) parallely transported from \(P\) to \(Q\) are

\[
V_Q^A = V_P^A - \frac{(T_P.v)}{T_P.T_P} T_P + \frac{V.T_Q}{T_Q.T_Q} T_Q
\]

and \(V_Q^A = V_P^A\) when the geodesic is a null one \((T_P.T_P = 0 = T_Q.T_Q)\).

We deduce immediately from this the expression the \(M^5\) components of the tensor of parallel transport from \(P\) to \(Q\) :

\[
\Theta_{B}^{A'} = \delta_{B}^{A'} - \frac{X^{A'}X_B + Y^{A'}Y_B + X^{A'}X_B - pY^{A'}Y_B}{R^2(p + 1)}.
\]

18
In \((\lambda, \vec{x})\) coordinate, with \(\vec{r} = \vec{x}_Q - \vec{x}_P\), the components of these objects read:

\[
T_\alpha^P = \left( \frac{\lambda_Q^2 - \lambda_P^2 - r^2}{2\lambda_Q} \frac{\lambda_P}{\lambda_Q} \frac{r^i}{r^i} \right) \frac{1}{R^2 \sqrt{|p^2 - 1|^{1/2}}} \\
T_{\alpha'}^Q = \left( -\frac{\lambda_P^2 - \lambda_Q^2 - r^2}{2\lambda_P} \frac{\lambda_Q}{\lambda_P} \frac{r^i}{r^i} \right) \frac{1}{R^2 \sqrt{|p^2 - 1|^{1/2}}} \\
\]

(71)

and

\[
\Theta_{\alpha\beta}^{\mu\nu'} = \left( \frac{\lambda_Q}{\lambda_P} \frac{(\lambda_Q + \lambda_P)^2 + r^2}{(\lambda_Q + \lambda_P)^2 - r^2} \frac{r^i}{r^i} \frac{\lambda_Q + \lambda_P}{\lambda_P (p+1)} \frac{\delta_{\mu\nu'}}{\lambda_P (p+1)} + \frac{r^i r^j'}{\lambda_P (p+1)} \right) .
\]

(72)

Following Allen [10] we have to consider the five invariant bitensors defined by:

\[
O_{1}^{\alpha\beta,\mu\nu'} = g^{\alpha\beta} g^{\mu\nu'} , \\
O_{2}^{\alpha\beta,\mu\nu'} = T_P T_P T_Q T_{\nu'} , \\
O_{3}^{\alpha\beta,\mu\nu'} = (\Theta^{\alpha\mu'} \Theta^{\beta\nu'} + \Theta^{\alpha\mu'} \Theta^{\beta\nu'}) , \\
O_{4}^{\alpha\beta,\mu\nu'} = (g^{\alpha\beta} T_Q T_{\nu'} T_P T_P g^{\mu\nu'}) , \\
O_{5}^{\alpha\beta,\mu\nu'} = 4T_P^{(\alpha} \Theta^{\beta)\mu'} T_{\nu')} .
\]

(73)

Using the previous expressions of the components of \(T_P, T_Q\) and \(\Theta\) and expliciting the action of the derivatives in eqs (58) to (63) we obtain by identification invariant expressions of the Green functions. The details of the calculations are very tedious and we don’t reproduce them here. To illustrate the method, we shall consider only the case of the massive vector field. It has been demonstrated in [9] that the \((\lambda, \lambda')\) component (in the coordinates system \((13)\) ) of the Feynman propagator, defined by the equation \((g_{\alpha\beta} \Box - R_{\alpha\beta} - M^2 g_{\alpha\beta}) \Delta^{F\beta\gamma}_{\lambda\mu}(x, y) = 0\) is given by:

\[
\Delta^{F\lambda}_{\lambda'}(x, y) = -\frac{R^2}{\lambda^2 M^2} \frac{1}{M^2} (\vec{\nabla}_x \cdot \vec{\nabla}_y) \left[ \frac{(\lambda\lambda')^2}{R^2} \Delta^{F}_{\sigma}(p) \right] .
\]

(74)

with:

\[
\sigma = i \sqrt{M^2 R^2 - \frac{1}{4}} .
\]

(75)
The others components are given by similar expressions (differential operators acting
on invariant functions of \(p\), see [6]). On the other hand, the most general maxi-
mally symmetric invariant bitensor with the same index structure as the vectorial
propagator reads as the combination:

\[
\Theta_\alpha^\alpha F(p) + T_\alpha^\alpha T\gamma T\alpha^\gamma G(p)
\] (76)

So, the vectorial invariant Feynman propagator \(\Delta^{F\beta}_{\gamma\gamma'}\) can be written as:

\[
\Delta^{F\beta}_{\gamma\gamma'} = \Theta_\beta^\gamma \alpha(p) + T_\beta^\gamma T\gamma T\gamma' \beta(p)
\] (77)

for some functions \(\alpha(p), \beta(p)\). Equation (74) can be reexpressed as a function of \(p\)
and \(\xi = \frac{\lambda}{R^2}\):

\[
\Delta^F_{\lambda\lambda'}(x, y) = -\frac{1}{M^2} \left[ \frac{\lambda^2}{R} \left[ \frac{3}{\lambda\lambda'} \frac{d}{dp} - \frac{r}{\lambda\lambda'} \frac{d^2}{dp^2} \right] \Delta_\sigma^F(p) \right.
\]

\[
= -\frac{1}{R^2 M^2} \left[ 3\xi \frac{d}{dp} - \xi(2p - \xi - \xi^{-1}) \frac{d^2}{dp^2} \right] \Delta_\sigma^F(p)
\] (78)

thanks to eq.(68). Comparing this expression with the \((\lambda, \lambda')\) component of eq.(77), in
which the terms are grouped together according to their powers of \(\xi\), we may identify
the coefficients \(\alpha(p)\) and \(\beta(p)\) of the decomposition:

\[
\alpha(p) = \frac{1}{m^2 R^2} \left[ 3p \frac{d}{dp} + (p^2 - 1) \frac{d^2}{dp^2} \right] \Delta_\sigma^F(p) , \quad \beta(p) = \frac{1}{m^2 R^2} \left[ 3(1 - p) \frac{d}{dp} + (1 - p^2) \frac{d^2}{dp^2} \right] \Delta_\sigma^F(p) \] (79)

(80)

and possibly use the other components of the propagator to check the results. They
are in agreement with those given in [10].

A similar calculation involving only two types of components: \(\Delta^{00,0'1'}\) and \(\Delta^{00,0'2'}\)
allows to determine the invariant form of the propagator:

\[
\Delta^{F\mu\nu,\rho\sigma'}(p) = \alpha(p)O^{\mu\nu,\rho\sigma'}_1 + \beta(p)O^{\mu\nu,\rho\sigma'}_2 + \gamma(p)O^{\mu\nu,\rho\sigma'}_3 + \delta(p)O^{\mu\nu,\rho\sigma'}_4 + \epsilon(p)O^{\mu\nu,\rho\sigma'}_5
\] (81)
with:

\[ \alpha(p) = \frac{2}{3(m+4)(m+6)} \left\{ -(m+6) \left[ (p^2-1)(m+8)+2 \right] \Delta_{\nu}^F - \frac{p \left[ 2(m+6)(p^2-1)+8 \right] \Delta_{\nu}^{F'} }{2(p^2-1)} \right. \\
\left. + \frac{[(p^2-1)(m^2+5m+9)-m] \Delta_{\nu_0}^F + [p(p^2-1)(2m+3)-4p] \Delta_{\nu_0}^{F'} }{p^2-1} \right\} \] (82)

\[ \beta(p) = \frac{2}{3(m+4)(m+6)} \cdot \left\{ -(m+6) \left[ (m+8)(p^2-1)-20p-28 \right] \Delta_{\nu}^F + \frac{[(p^2-1)(2p(m+6)+10(m+2))-112p-80] \Delta_{\nu}^{F'} }{p^2-1} \right. \\
\left. + \frac{[(p^2-1)(m^2-16m)-2m(10p+14)] \Delta_{\nu_0}^F + [(p^2-1)(8mp-48p+4m-64)-80-112p] \Delta_{\nu_0}^{F'} }{p^2-1} \right\} \] (83)

\[ \gamma(p) = \frac{2}{3(m+4)(m+6)} \left\{ \frac{(m+6) \left[ 3(p^2-1)(m+8)-4 \right] \Delta_{\nu}^F + [(p^2-1)6p(m+6)-16p] \Delta_{\nu}^{F'} }{4(p^2-1)} \right. \\
\left. - \frac{m \Delta_{\nu_0}^F + 4p \Delta_{\nu_0}^{F'} }{p^2-1} \right\} \] (84)

\[ \delta(p) = \frac{2 \varepsilon(p^2-1)}{3(m+4)(m+6)} \left\{ \frac{-(m+6) \left[ (p^2-1)(m+8)+12 \right] \Delta_{\nu}^F - [(p^2-1)(m+6)2p+48p] \Delta_{\nu}^{F'} }{2(p^2-1)} \right. \\
\left. + \frac{m \left[ (p^2-1)(m-1)-6 \right] \Delta_{\nu_0}^F + p \left[ (p^2-1)5m-24 \right] \Delta_{\nu_0}^{F'} }{p^2-1} \right\} \] (85)

\[ \epsilon(p) = \frac{2 \varepsilon(p^2-1)}{3(m+4)(m+6)} \cdot \left\{ \frac{(m+6) \left[ 3(p^2-1)(m+8)-20p-4 \right] \Delta_{\nu}^F + [(p^2-1)(6p(m+6)+10(m+2))-16(p+5)] \Delta_{\nu}^{F'} }{4(p^2-1)} \right. \\
\left. + \frac{-m(1+5p) \Delta_{\nu_0}^F + [(p^2-1)(m-16)-4(p+5)] \Delta_{\nu_0}^{F'} }{p^2-1} \right\} \] (86)

where \( \Delta_{\nu}^{F'} = \frac{d}{dp} \Delta_{\nu}^F \)
\section{Scalar Green function revisited}

In ref \cite{3} we have shown that the Green’s functions of the scalar field equation on the domain $\mathcal{O}$:

\begin{equation}
(\Box - M^2)\varphi \equiv \left( \frac{\lambda^2}{R^2} (-\partial^2 + \vec{\nabla}^2) + \frac{2\lambda}{R} \partial_\lambda - M^2 \right) \varphi = 0 \tag{87}
\end{equation}

can be written as a superpositions of modes expressed in terms of Hankel functions:

\begin{align*}
  u_{\vec{p}}(\lambda) &= \sqrt{\frac{\pi}{2R}} \lambda^{3/2} \left[ c(\vec{p}) \mathcal{H}^{(1)}_{\nu_0}(\lambda p) + d(\vec{p}) \mathcal{H}^{(2)}_{\nu_0}(\lambda p) \right] e^{i\vec{p}.\vec{x}} \\
  \nu_0 &= i \sqrt{m^2 R^2 - \frac{9}{4}} \tag{88}
\end{align*}

with $|d(\vec{p})|^2 - |c(\vec{p})|^2 = 1$, and $c(\vec{p})d(-\vec{p}) - c(-\vec{p})d(\vec{p}) = 0$, the last conditions resulting from the normalisation condition $u_{\vec{p}} * u_{\vec{p}'} = \delta^3(\vec{p} - \vec{p}')$. These conditions are not sufficient to fix the vacuum (the positive frequency modes). If we impose the vacuum to be invariant with respect to the 7-parameter isometry group of $\mathcal{O}$, extra (necessary) conditions appear. The coefficients $c(\vec{p})$ and $d(\vec{p})$ have to be constant. The resulting Green functions still depend on three parameters. Definite values of these parameters are obtained by imposing that the short distance singularities of the Feynman propagator are the same as in flat space:

\begin{equation}
\lim_{\sigma \to 1} \sigma^2 \Delta^F(\sigma) = -\frac{1}{2\pi^2}, \quad \text{where} \quad p = \cosh\left(\frac{\sigma}{R}\right) \tag{89}
\end{equation}

Then one obtains:

\begin{align*}
  c &= 0, \quad d = 1 \tag{90}
\end{align*}

because the phase of $d$ becomes irrelevant. The Feynman propagator for $M \neq 0$, is given by:

\begin{equation}
\Delta^F(x, y) = \frac{1}{16\pi R^2} \frac{(M^2 R^2 - 2)}{\cos \nu_0 \pi} F \left( \frac{3}{2} + \nu_0, \frac{3}{2} - \nu_0; 2; \frac{1 + p}{2} - i\epsilon \right). \tag{91}
\end{equation}
while for $M = 0$ one obtains \[3\]:

\[
\Delta^F(x, y) = \frac{1}{4\pi^2 R^2} \left[ \frac{1}{1 - p} - \ln \left| \frac{\lambda \lambda'}{R^2} (p - 1) \right| + c^e \right] \\
- \frac{i}{4\pi R^2} \epsilon(\lambda - \lambda') (p - 1 + \theta(p - 1))
\]

which is only $E(3)$ invariant. We plan to discuss the physical significance of this choice of modes in a forthcoming publication \[19\].

Up to now, all the expressions of the modes that we have considered were defined only on $\mathcal{O}$. If we extend the definition of $\lambda$ and $\vec{x}$ by eq. (15) on the full de Sitter space (excepted on the horizon $H$, i.e. the 3-surface $X_4 = X_0$) we may analytically continue the modes by considering the behaviour of a wave packet near $H$. Typically, such wave packet behaves as:

\[
\varphi(\lambda, \vec{x}) = \int \sqrt{\frac{\pi}{2R}} |\lambda|^{\frac{3}{2}} e^{i\vec{p} \cdot \vec{x}} H_\nu^2(\lambda p) \left( \frac{e^{i\vec{p} \cdot \vec{x}}}{(2\pi)^{\frac{3}{2}}} \right) f(\vec{p}) d^3p
\]

The Kirchoff (Poisson) formula giving the solution of the massless scalar wave equation in flat space insures that this expression remains finite on the horizon, despite the presence of the divergent factor $|\lambda|$. The continuation of the modes across the horizon is obtained by looking in the region $\lambda < 0$ which combination of Hankel functions have an asymptotic expansion that matches with the one used in eq. (93). This leads us immediately to an expression of the modes valid on the full de Sitter space:

\[
u_p(\lambda) = \frac{\sqrt{\pi}}{2R} |\lambda|^{3/2} [\theta(\lambda) H_\nu^{(2)}((\lambda - i\epsilon)p) - i\theta(-\lambda) H_\nu^{(1)}(-(\lambda - i\epsilon)p)] e^{i\vec{p} \cdot \vec{x}}
\]

Inserting this expressions of modes in integrals like those considered in eqs \[64, 65\], we conclude that the expression (11) is valid on whole de Sitter space, with $p$ still given by eq. (68) whatever are the signs of $\lambda$ and $\lambda'$.

Note that the region $\mathcal{O} = H^4 \setminus \mathcal{O}$ is isometric to $\mathcal{O}$ but with time running in the opposite way. This is in accord with the fact that it is precisely the Hankel function $H^{(1)}(|\lambda p|)$ which is coupled to $H^{(2)}(|\lambda p|)$, these two functions being of opposite frequencies on both $\mathcal{O}$ and $\mathcal{O}$. Finally, note also that by the continuation $R \mapsto iR$ we obtain invariant expressions of Green functions on anti-de Sitter space.
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Appendix

We have collected in this appendix the explicit expressions of the modes we have used to obtain the propagators. They are expressed for each $\vec{p}$ in the base $\{\vec{e}_0 = \partial_0, \vec{e}_1, \vec{e}_2, \vec{e}_3 = \frac{\vec{p}}{p}\}$. They read as:

\[ \bar{h}^S_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
O_1 & 0 & 0 & O_3 \\
0 & O_7 & 0 & 0 \\
0 & 0 & O_7 & 0 \\
O_3 & 0 & 0 & O_5 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m-4}}(\lambda p) \] (A.1)

\[ \bar{h}^{TT}_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
1 & 0 & 0 & O_2 \\
0 & O_6 & 0 & 0 \\
0 & 0 & O_6 & 0 \\
O_2 & 0 & 0 & O_4 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m+4}}(\lambda p) \] (A.2)

\[ \bar{h}^{\perp 1}_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & O_2 \\
0 & 0 & 0 & 0 \\
0 & O_2 & 0 & 0 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m+4}}(\lambda p) \] (A.3)

\[ \bar{h}^{\perp 2}_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & O_2 \\
0 & 0 & O_2 & 0 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m+4}}(\lambda p) \] (A.4)

\[ \bar{h}^{\perp \perp 1}_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m+4}}(\lambda p) \] (A.5)

\[ \bar{h}^{\perp \perp 2}_{\mu\nu}(\vec{p}, \lambda, \vec{x}) = \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right) \frac{1}{(\lambda p)^2} \sqrt{p} e^{i p \cdot x} \hat{\mathcal{H}}^{(2)}(\lambda p) \frac{1}{i \sqrt{m+4}}(\lambda p) \] (A.6)

where:

\[ O_1 = (3\lambda \partial_\lambda - (\lambda p)^2 + 3) \] (A.7)

\[ O_2 = \frac{-i}{p} (\partial_\lambda - \frac{4}{\lambda}) \] (A.8)

\[ O_3 = ip\lambda^2(\partial_\lambda + \frac{1}{\lambda}) \] (A.9)

\[ O_4 = \frac{2}{(\lambda p)^2} \partial_\lambda + 1 + \frac{m-4}{(\lambda p)^2} \] (A.10)

\[ O_5 = (\lambda \partial_\lambda - ((p\lambda)^2 + m + 3)) \] (A.11)
\[ O_6 = \left( -\frac{1}{p^2\lambda} \partial_\lambda + \frac{4-m}{2(p\lambda)^2} \right) \]  
(A.12) 

\[ O_7 = (\lambda \partial_\lambda - 3 - m) \]  
(A.13)

These modes are orthogonal for the scalar product (9). In order that they satisfy the orthonormalization condition (10), their coefficients must be chosen, up to a phase, as:

\[ N_S = \frac{1}{R} \sqrt{\frac{208\pi^2(m+4)(m+\frac{15}{2})}{48\pi}} \]  
(A.14)

\[ N_{TT} = \frac{1}{R} \sqrt{\frac{48\pi^2(m+6)(m+4)}{48\pi^2}} \]  
(A.15)

\[ N_\perp = \frac{1}{R} \sqrt{\frac{64\pi^2(m+6)}{64\pi^2}} \]  
(A.16)

\[ N_{\perp\perp} = \frac{1}{R} \sqrt{64\pi^2} \]  
(A.17)
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Figure Caption

Penrose diagram of de Sitter space [20]. Region $\mathcal{O} (\lambda > 0)$ corresponds to the causal past of observers $\vec{x} = cst, \quad \lambda > 0$. Their common future event horizon $\mathcal{H}$ is the boundary of the two coordinate patches $(\lambda > 0, \vec{x})$ and $(\lambda < 0, \vec{x})$. Dashed lines represent $\vec{x} = cst$ world lines.