Sparse Matrix Code Dependence Analysis Simplification at Compile Time

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Analyzing array-based computations to determine data dependences is useful for many applications including automatic parallelization, race detection, computation and communication overlap, verification, and shape analysis. For sparse matrix codes, array data dependence analysis is made more difficult by the use of index arrays that make it possible to store only the nonzero entries of the matrix (e.g., in A[B[i]], B is an index array). Here, dependence analysis is often stymied by such indirect array accesses due to the values of the index array not being available at compile time. Consequently, many dependences cannot be proven unsatisfiable or determined until runtime. Nonetheless, index arrays in sparse matrix codes often have properties such as monotonicity of index array elements that can be exploited to reduce the amount of runtime analysis needed. In this paper, we contribute a formulation of array data dependence analysis that includes encoding index array properties as universally quantified constraints. This makes it possible to leverage existing SMT solvers to determine whether such dependences are unsatisfiable and significantly reduces the number of dependences that require runtime analysis in a set of eight sparse matrix kernels. Another contribution is an algorithm for simplifying the remaining satisfiable data dependences by discovering equalities and/or subset relationships. These simplifications are essential to make a runtime-inspection-based approach feasible.

Additional Key Words and Phrases: Dependence Analysis, Sparse Matrices, Inspector Simplification, Decision Procedure, SMT

1 INTRODUCTION

Data dependence analysis answers questions about which memory accesses in which loop iterations access the same memory location thus creating a partial ordering (or dependence) between loop iterations. Determining this information enables iteration space slicing [Pugh and Rosser 1997], provides input to race detection, makes automatic parallelization and associated optimizations such as tiling or communication/computation overlap possible, and enables more precise data-flow analysis, or abstract interpretation. A data dependence exists between two array accesses that (1) access the same array element with at least one access being a write and (2) that access occurs within the loop bounds for each of the accesses’ statement(s). These conditions for a data dependence has been posed as a constraint-based decision problem [Banerjee et al. 1993], a data-flow analysis with polyhedral set information [Feautrier 1991], and linear memory access descriptors [Paek et al. 2002]. However, such approaches require a runtime component when analyzing codes with indirect
memory accesses (e.g., A[B[i]], B being an index array) such as those that occur in sparse matrix codes. In this paper, we present an approach to improve the precision of compile-time dependence analysis for sparse matrix codes and simplification techniques for decreasing the complexity of any remaining runtime checks.

Sparse matrix computations occur in many codes, such as graph analysis, partial differential equation solvers, and molecular dynamics solvers. Sparse matrices save on storage and computation by only storing the nonzero values in a matrix. Figure 1 illustrates one example of how the sparse matrix vector multiplication ($y = Ax$) can be written to use a common sparse matrix format called compressed sparse row (CSR). In CSR, the nonzeros are organized by row, and for each nonzero, the column for that nonzero is explicitly stored. Since the percentage of nonzeros in a matrix can be less than 1%, sparse matrix formats are critical for many computations to fit into memory and execute in a reasonable amount of time. Unfortunately, the advantage sparse matrices have on saving storage compared to dense matrices comes with the cost of complicating program analysis. Compile-time only approaches resort to conservative approximation [Barthou et al. 1997; Paek et al. 2002]. Some approaches use runtime dependence tests to complement compile time analysis and obtain more precise answers [Oancea and Rauchwerger 2012; Pugh and Wonnacott 1995; Rus et al. 2003]. Runtime dependence information is also used to detect task-graph or wavefront parallelism that arises due to the sparsity of dependences [Rauchwerger et al. 1995a; Saltz et al. 1991; Streit et al. 2015].

The data dependence analysis approach presented here is constraint-based. Some constraint-based data dependence analysis approaches [Pugh and Wonnacott 1994, 1995, 1998; Venkat et al. 2016] represent the index arrays in sparse matrix computations as uninterpreted functions in the data dependence relations. For example, the loop bounds for the $k$ loop in Figure 1 can be represented as $rowptr(i) \leq k < rowptr(i+1)$. Previous work by others generates simplified constraints at compile time that can then be checked at runtime with the goal of finding fully parallel loops [Pugh and Wonnacott 1995]. We build on the previous work of [Venkat et al. 2016]. In that work, dependences in a couple of sparse matrix codes were determined unsatisfiable manually, simplified using equalities found through a partial ordering of uninterpreted function terms, or approximated by removing enough constraints to ensure a reasonable runtime analysis complexity. In this paper, we automate the determination of unsatisfactory data dependences, find equalities using the Integer Set Library (ISL) [Verdoolaege 2010], have developed ways to detect
data dependence subsets for simplifying runtime analysis, and perform an evaluation with eight popular sparse kernels.

**In this paper, we have two main goals:** (1) prove as many data dependences as possible to be unsatisfiable, thus reducing the number of dependences that require runtime tests; and (2) simplify the satisfiable data dependences so that a runtime inspector for them has complexity less than or equal to the original code. Figure 2 shows an overview of our approach. We use the ISL library [Verdoolaege 2018] like an SMT solver to determine which data dependences are unsatisfiable. Next, we manipulate any remaining dependence relations using IEGenLib [Strout et al. 2016] and ISL libraries to discover equalities that lead to simplification.

Fortunately, much is known about the index arrays that represent sparse matrices as well as the assumptions made by the numerical algorithms that operate on those matrices. For example, in the CSR representation shown in Figure 1, the `rowptr` index array is monotonically strictly increasing.

In Section 3.1 we explain how such information can be used to add more inequality and equality constraints to a dependence relation. The new added constraints in some cases cause conflicts, and hence we can detect that those relations are unsatisfiable.

The dependences that cannot be shown as unsatisfiable at compile time, still require runtime tests. For those dependences, the goal is to simplify the constraints and reduce the overhead of any runtime test. Sometimes index array properties can be useful for reducing the complexity of runtime inspector by finding equalities that are implied by the dependence constraints in combination with assertions about the index arrays. The equalities such as \( i = \text{col}(m') \) can help us remove a loop level from the inspector, \( i \) in the example. Since some equality constraint would allow us deduce value of an iterator in the inspector from another one, e.g., we can deduce \( i \) values from \( m' \) values using \( i = \text{col}(m') \). Another simplification involves determining when data dependence relations for a code are involved in a subset relationship. When this occurs, runtime analysis need only occur for the superset.

This paper makes the following contributions:

1. An automated approach for the determination of unsatisfiable dependences in sparse codes.
2. An implementation of an instantiation-based decision procedure that discovers equality relationships in satisfiable dependences.
3. An approach that discovers subset relationships in satisfiable dependences thus reducing run-time analysis complexity further.
4. A description of common properties of index arrays arising in sparse matrix computations, expressed as universally quantified constraints.
5. Evaluation of the utility of these properties for determining unsatisfiability or simplifying dependences from a suite of real-world sparse matrix codes.
// forward solve assuming a lower triangular matrix.

for (i=0; i<N; i++) {
    tmp = f[i];
    for (j=0; j<i; j++) {
        S1: tmp -= A[i][j]*u[j];
    }
    S2: u[i] = tmp / A[i][i];
}

Fig. 3. Forward solve for a dense matrix.

2 BACKGROUND: DATA DEPENDENCE ANALYSIS

Data dependence analysis of a loop nest is a common code analysis that is used in different applications, such as automatic parallelization [Brandes 1988] and data race detection [Atzeni et al. 2016a]. This section reviews the data dependence analysis process and how that process differs when analyzing sparse matrix codes. Then, we review some of the applications of data dependence analysis including an example of its use for finding wavefront parallelism in sparse codes.

2.1 Data Dependence Analysis

A data dependence occurs between two iterations of a loop when both of the iterations access the same memory location and at least one of the accesses is a write. Data dependence constraints are of the following form:

\[
\text{Dep : } (\exists \vec{I}, \vec{I}') (\vec{I} < \vec{I}' \land F(\vec{I}) = G(\vec{I}') \land \text{Bounds}(\vec{I}) \land \text{Bounds}(\vec{I}'))
\]

where \(\vec{I}\) and \(\vec{I}'\) are iteration vector instances from the same loop nest, \(F\) and \(G\) are array index expressions to the same array with at least one of the accesses being a write, and \(\text{Bounds}(\vec{I})\) expands to the loop nest bounds for the \(\vec{I}\) iteration vectors. In this paper, the term dependence relation is used interchangeably with dependence constraints by viewing them as a relation between \(\vec{I}\) and \(\vec{I}'\).

For example, consider the dense matrix implementation for forward solve in Figure 3. Forward solve solves for the vector \(\bar{u}\) in the equation \(A\bar{u} = \bar{f}\) assuming that the matrix is lower triangular (i.e., nonzeros are only on the diagonals or below as shown in the example in Figure 1). The dense forward solve code has the following dependences for the outermost loop \(i\):

- A loop-carried dependence due to the scalar \(\text{tmp}\) variable. However, since \(\text{tmp}\) is written before being read in each iteration of the \(i\) loop, it is privatizable, which means each processor in a potential parallelization of the \(i\) loop can be given its own private copy of \(\text{tmp}\).

- A loop-carried dependence between the write \(u[i]\) in Statement S2 and the read \(u[j]\) in Statement S1 with constraints

\[
(\exists i, j, i', j') (i < i' \land i = j' \land 0 \leq i, i' < N \land 0 \leq j < i \land 0 \leq j' < i').
\]

The iterators \(i'\) and \(j'\) are different instances of \(i\) and \(j\). This dependence due to the writes and reads to array \(u\) is satisfiable because the computation for any row \(i'\) depends on all previous rows \(i < i'\). This means that the outer loop in dense forward solve is fully ordered due to data dependences and therefore not parallelizable.

2.2 Sparse Codes and Runtime Parallelism

For sparse codes, after compile time dependence analysis, some remaining dependences may involve index arrays as subscript expressions. The data dependence constraints can use uninterpreted
// Forward solve assuming a lower triangular matrix.
for(i=0; i<N; i++) {
    tmp = f[i];
    for(k=rowptr[i]; k<rowptr[i+1]-1; k++) {
        S1: tmp -= val[k]*u[col[k]];
    }
    S2:u[i] = tmp / val[rowptr[i+1]-1];
}

functions to represent the index arrays at compile time. Because the values of the index arrays
are unknown until run time, proving such dependences are unsatisfiable may require runtime
dependence testing. However, even when dependences arise at runtime, it still may be possible
to implement a sparse parallelization called wavefront parallelization. Identifying wavefront par-
allelizable loops combines compile time and runtime analyses. The compiler generates inspector
code to find the data dependence graph at runtime.

We now consider the sparse forward solve with Compressed Sparse Row CSR format in Figure 4.
We are interested in detecting loop-carried dependences of the outermost loop. There are two pairs
of accesses on array u in S1 and S2 that can potentially cause loop-carried dependences: u[col[k]]
(read), u[i] (write); and u[i] (write), u[i] (write). The constraints for the two dependence tests
are shown in the following.

Dependences for the write/write u[i] in S2:

\begin{align}
(1) \quad & i = i' \land i < i' \land 0 \leq i < N \land 0 \leq i' < N \\
& \land rowptr(i) \leq k < rowptr(i+1) \land rowptr(i') \leq k' < rowptr(k' + 1)
\end{align}

\begin{align}
(2) \quad & i = i' \land i' < i \land 0 \leq i < N \land 0 \leq i' < N \\
& \land rowptr(i) \leq k < rowptr(i+1) \land rowptr(i') \leq k' < rowptr(k' + 1)
\end{align}

Dependences for read u[col[k]] and write u[i] in S1, and S2:

\begin{align}
(3) \quad & i = col(k') \land i < i' \land 0 \leq i < N \land 0 \leq i' < N \\
& \land rowptr(i') \leq k' < rowptr(i' + 1)
\end{align}

\begin{align}
(4) \quad & i = col(k') \land i' < i \land 0 \leq i < N \land 0 \leq i' < N \\
& \land rowptr(i') \leq k' < rowptr(i' + 1)
\end{align}

These dependences can be tested at runtime when concrete interpretations for the index arrays
(contents of arrays rowptr and col) are available. The runtime dependence analyzers, called
inspectors [Saltz et al. 1991], may be generated from the dependence constraints [Venkat et al. 2016].

Suppose the matrix in the forward solve code in Figure 4 had the nonzero pattern as in Figure 1. The runtime check would create the dependence graph for this example based on the four
dependences above as shown in Figure 5. Once the dependence graph is constructed a breadth-first
traversal of the dependence graph can derive sets of iterations that may be safely scheduled in
parallel without a dependence violations, with each level set being called a wavefront as shown in
Figure 5.
2.3 Applications of the Sparse Data Dependence Analysis

Besides wavefront parallelism, there are many other uses for sparse data dependence analysis. Any application of sparse data dependence analysis would benefit from a reduction in the number of data dependences that need to be inspected at runtime and from any complexity reduction of data dependences that do require runtime inspection. Here we summarize some of those applications.

**Race detection:** Dynamic race detection is an essential prerequisite to the parallelization of existing sequential codes. While the front-end static analysis methods employed in these checkers [Atzeni et al. 2016b] can often suppress race checks on provably race-free loops, they fail to do so when presented with non-affine access patterns that occur in sparse matrix codes. In addition to significantly increasing runtimes, the shadow memory cells employed by dynamic race checkers also increases memory pressure, often by a factor of four. The techniques presented in this paper can help suppress race checks when we can prove the independence of loop iterations.

**Dynamic program slicing:** Pugh and Rosser introduced the concept of iteration space slicing where program slicing is done on a loop iteration basis using Presburger representations [Pugh and Rosser 1997]. Similar dynamic approaches for tiling across loops in sparse codes were presented by various groups [Douglas et al. 2000; Strout et al. 2004]. All of these techniques would require runtime data dependence analysis, thus disproving dependences or reducing the complexity of inspecting dependences at runtime would be applicable.

**High-level synthesis:** Optimizations in high-level synthesis (HLS) uses runtime dependence checks. In HLS, it is important to pipeline the innermost loops to get efficient hardware. Alle et al. have proposed using runtime dependence checks to dynamically check if an iteration is in conflict with those currently in the pipeline, and add delays only when necessary [Alle et al. 2013].

**Distributed memory parallelization:** Another possible application of our work can be found in the work by [Ravishankar et al. 2015]. The authors produce distributed parallel code that uses MPI for loops where there might be indirect loop bounds, and/or array accesses. The read and write sets/data elements of each process are computed via an inspector where indirect accesses are involved to determine if each process is reading/writing data that is owned by other processes. Basumallik and Eigenmann use run-time inspection of data dependences to determine how to reorder a loop to perform computation and communication overlap [Basumallik and Eigenmann 2006].

3 AUTOMATING (UN)SATISFIABILITY ANALYSIS FOR SPARSE DATA DEPENDENCES

For any application of data dependence analysis for sparse codes, the best outcome is to determine that a potential data dependence is unsatisfiable. Any dependence that is unsatisfiable does not for runtime analysis. Previous work used domain-specific knowledge about the index arrays used to represent sparse matrices to guide manual determination of unsatisfactory data dependences [Venkat et al. 2016]. In this paper, we show how to automate this process by specifying the

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**Fig. 5.** Dependence graph for forward solve for sparse matrix in Figure 1.
domain-specific knowledge as universally quantified constraints on uninterpreted functions and then using instantiation methods similar to those used by SMT solvers to produce more constraints that can cause conflicts.

3.1 Detecting Unsatisfiable Dependences Using Domain Information

As an example of how domain information can be used to show dependences are unsatisfiable consider the following constraints from a dependence relation:

\[ i' < i \land k = m' \land 0 \leq i, i' < n \land rowptr(i) \leq k < rowptr(i+1) \land rowptr(i'-1) \leq m' < rowptr(i'). \]

Some relevant domain information is that the \( rowptr \) index array is strictly monotonically increasing:

\[ (\forall x_1, x_2)(x_1 < x_2 \implies rowptr(x_2) < rowptr(x_2)). \]

Since the dependence relation in question has the constraints \( i' < i \). Then, using the above strict monotonicity information would result in adding \( rowptr(i') < rowptr(i) \). But, considering the constraint, \( k = k' \), \( rowptr(i) \leq k \), and \( m' < rowptr(i') \) we know that, \( rowptr(i) < rowptr(i') \). This leads to a conflict,

\[ rowptr(i) < rowptr(i') \land rowptr(i') < rowptr(i). \]

This conflict would indicate the dependence in question was unsatisfiable and therefore does not require any runtime analysis.

3.2 Universally Quantified Assertions about Index Arrays

Even if a formula that includes uninterpreted function calls is satisfiable in its original form, additional constraints about the uninterpreted functions may make it unsatisfiable. This has been exploited abundantly in program verification community to obtain more precise results [Bradley et al. 2006; Ge and de Moura 2009; Habermehl et al. 2008]. A common approach to express such additional constraints is to formulate them as universally quantified assertions. For instance, [Bradley et al. 2006] use following to indicate that array \( a \) is sorted within a certain domain:

\[ (\forall x_1, x_2)(i < x_1 \leq x_2 < j \implies a(x_1) \leq a(x_2)). \]

There are several methods that SMT solvers use to reason about quantified formulas, the most common one being quantifier instantiation [Bradley et al. 2006; Ge and de Moura 2009; Löding et al. 2017; Moura and Bjørner 2007; Reynolds et al. 2015, 2014]. In quantifier instantiation, instances of universally quantified assertions, where the universally quantified variables are replaced with ground terms, are added to the original formula. Any of the added constraints might contradict constraint(s) in the formula that would show the original formula is unsatisfiable. For the general case of quantified first order logic, there is no complete instantiations procedure. That means the instantiation can go on forever not exactly knowing whether the formula is satisfiable or unsatisfiable. In some limited cases, the quantified assertions can be completely replaced by a set of quantifier instances to construct an equisatisfiable quantifier-free formula [Bradley et al. 2006; Ge and de Moura 2009].

Combining the constraints from dependences with arbitrary universally quantified assertions would create a first order logic theory that in general is undecidable. Undecidability would imply that we cannot implement an algorithm for deciding the formulas that would always terminate. Numerous works such as [Bradley et al. 2006; Ge and de Moura 2009; Habermehl et al. 2008] present different decidable fragments of first order logic. The approach that these works use to make decidable fragments is to put restrictions what type of universally quantified assertions can be used. The restriction are usually on on syntax of the allowed assertions [Bradley et al. 2006;
Habermehl et al. 2008], and sometimes specific properties that a specific instantiation procedure for assertions must have [Ge and de Moura 2009]. We perform a terminating instantiation that is sound but incomplete. In other words, the dependences we determine unsatisfiable are in fact unsatisfiable, but we may characterize some unsatisfiable constraints as may satisfiable.

### 3.3 Domain Information about Index Arrays

We represent domain information about index arrays as universally quantified assertions. In this section, we illustrate some assertions relevant to numerical benchmarks and relate the corresponding assertions to the existing theory fragments. Table 2 in Section 7 lists all the assertions we use in the evaluation. Below are some example properties.

For the forward solve with compressed sparse row (CSR) code in Figure 4, we know the following:

- **Monotonic index arrays**: The row index array values increase monotonically. This property of index arrays can be expressed with an assertion about the uninterpreted function symbol that represents the index array. For instance, in the example the rowptr() function is monotonically increasing. If we assume that all the sparse matrix rows have at least one nonzero, then rowptr() is strictly monotonically increasing. This assertion can be encoded as follows:

  \[(\forall x_1, x_2) (x_1 < x_2 \iff \text{rowptr}(x_1) < \text{rowptr}(x_2)).\]

- **Lower Triangular Matrix**: The forward solve algorithm shown in Figure 4 operates on lower triangular matrices. For the CSR format that leads to the following domain-specific assertion:

  \[(\forall x_1, x_2) (x_1 + c_1 \leq x_2 = \Rightarrow f(x_1) + c_2 \leq g(x_2))\]

  This indicates that nonzeros of rows before row \(i\) have columns less than \(i\).

The domain information in Table 2 in Section 7 can be represented with following general forms:

1. \[(\forall x_1, x_2) (x_1 + c_1 \leq x_2 = \Rightarrow f(x_1) + c_2 \leq f(x_2))\]
2. \[(\forall x_1, x_2) (x_1 + c_1 \leq x_2 = \Rightarrow f(x_1) + c_2 \leq g(x_2))\]
3. \[(\forall x_1, x_2) (x_1 + c_1 \leq f(x_2) = \Rightarrow g(x_1) + c_2 \leq x_2))\]

Where \(c_1\) and \(c_2\) can be 0 or 1. The first and second assertions fit the decidable LIA fragment that is presented by [Habermehl et al. 2008]. However, to the best of our knowledge the third assertion form does not fit any previously presented decidable fragment, and its decidability remains open.

Modern SMT solvers are equipped with heuristic-based quantifier instantiations to reason about quantified formulas. Existing techniques for quantifier instantiation can construct the set of instantiations for deciding some of our assertions, e.g., non-strict monotonicity, but not for all of them. For unsatisfiable formulas with universal quantifiers where the solver only needs a small set of relevant instances to find contradicting constraints, the existing heuristics can work well. For all our examples, both Z3 and CVC4 were able to identify all unsatisfiable dependences. The solvers also time out for satisfiable ones given a small timeout (5 seconds). This is as expected, since specific instances of universally quantified formulas usually do not help in proving that the quantified formula is satisfiable.

Nonetheless, we cannot just use a conventional SMT solver like Z3 in our context. The key reason is that we are not just interested in satisfiability of the dependence constraints. If unsatisfiability cannot be proven statically, runtime checks will be generated. It is important for these runtime checks to be as fast as possible, and hence we are also interested in using the assertions to decrease the cost of runtime checks. For example, additional equalities means the data dependence inspector iteration space has lower dimensionality, thus reducing the algorithmic complexity of runtime
checks. We illustrate the complexity reduction through instantiation of assertions with two examples in Section 4.

### 3.4 Detecting Unsatisfiable Sparse Dependences

Instantiation-based quantifier elimination is a natural choice for our context, since we seek to either prove unsatisfiability or find additional constraints that simplifies runtime checks. Unfortunately, our assertions are not fully covered by decidable fragments [Bradley et al. 2006; Ge and de Moura 2009] where equisatisfiable quantifier-free formulas can always be obtained. Nonetheless, using inspiration from the decidable fragments [Bradley et al. 2006; Ge and de Moura 2009] we have a procedure that detects all unsatisfiable examples from our benchmark suite that represent a wide range of numerical analysis codes.

Note that we can show our general assertions (1), (2), and (3), presented in Section 3.3 as:

\[
\forall \vec{x}, \varphi_I(\vec{x}) \implies \varphi_V(\vec{x})
\]

Where \(\vec{x}\) denotes vector of quantified variables, \(\varphi_I(\vec{x})\) denotes antecedent of the assertion, and \(\varphi_V(\vec{x})\) denotes consequent of the assertion. Then the following definitions define our procedure to instantiate quantified variables, and potentially use a SMT to detect their unsatisfiability.

**Definition 1 (E)** We define E to be the set of expressions used as arguments to all uninterpreted function calls in the original set of constraints. We use this set to instantiate quantified assertions.

**Definition 2 (UNSAT)\(\psi\)**

1. The inference rules for turning the universally quantified predicates into quantifier-free predicates is as follows:

\[
\forall \vec{x}, \varphi_I(\vec{x}) \implies \varphi_V(\vec{x}) \quad \text{for all } \vec{x} \in E^n
\]

where \(E^n\) is the set of vectors of size \(n = |\vec{x}|\) produced as Cartesian product of \(E\).

2. Solve the quantifier-free formula \(\psi\) output of step with an SMT solver that decide union of quantifier-free theories of uninterpreted functions with equality and Presburger Arithmetics.

**Correctness:** Although the above procedure is incomplete, we do have soundness. This means if a dependence is determined unsatisfiable, it in fact is not a dependence. However, if a dependence is determined satisfiable at compile time, it could be that at runtime the actual values of index arrays lead to the dependence not being satisfiable. Since our procedure is conservatively correct, it is sound.

To show that the decidability procedure \(UNSAT_\psi\) is sound, we need to show that if the original formula \(\psi\) is satisfiable, then so is the unquantified formula \(\psi'\),

\[
\psi \in SAT \implies \psi' \in SAT.
\]

This is equivalent to

\[
\psi' \notin SAT \implies \psi \notin SAT.
\]

Since universal quantification is being replaced with specific expression instantiations to create \(\psi', \psi'\) is a potentially weaker set of constraints than \(\psi\). This means that \(\psi'\) is a conservative approximation of \(\psi\). As such, if \(\psi'\) is not satisfiable, then \(\psi\) is not satisfiable.

### 4 SIMPLIFYING THE DEPENDENCES UTILIZING EQUALITIES

The finite instantiation proposed in Section 3.4 can prove many of the dependence relations to be unsatisfiable. However, some of the relations remain satisfiable, thus requiring runtime checks. It is
for(int colNo = 0; colNo < n; ++colNo) {
    std::fill_n(f,n,0); //Zero initialization
    f[r[nzNo]] = values[nzNo];
    for(int i = prunePtr[colNo], sw=0; i < prunePtr[colNo + 1]; ++i){
        for (int l = lC[pruneSet[i]], bool sw=false;; l < lC[pruneSet[i] + 1]; ++l){
            if (lR[l] == colNo && !sw) {
                tmp = lValues[l];
                sw=true;
            }
            if(sw){
                f[lR[l]] -= lValues[l] * tmp;
            }
        }
        if (f[colNo] <= 0) return false; //The matrix is not SPD
    }
    lValues[lC[colNo]] = sqrt(f[colNo]);
    for(int j = lC[colNo] + 1; j < lC[colNo + 1]; ++j)
        Values[j] = f[lR[j]] / sqrt(f[colNo]);
}

Fig. 6. Static Left Cholesky code, which is a modified version of Left Cholesky code [Cheshmi et al. 2017].

then important to minimize the runtime cost by simplifying the dependence relations as much as possible. In this section, we discuss one of such simplifications utilizing additional equalities after finite instantiations.

4.1 Discovering New Equality Constraints and Their Usefulness

Sometimes index array properties can help reduce the complexity of runtime inspectors through introducing equalities to the dependence’s constraints. The new equalities are discoverable after instantiating the universally quantified assertions and combining those with other inequality and equality relationships. For instance, consider the following set of constraints; it is a satisfiable dependence that needs a runtime inspector with complexity of $O(n^2)$ to traverse the space of values for $i$ and $i'$:

\[(i \leq i') \land (f(i') \leq f(i)) \land (0 \leq i, i' < n).\]

And assume we also know following universally quantified rule about the uninterpreted function $f$ (strict monotonicity):

\[(\forall x_1, x_2, x_1 < x_2) \implies (f(x_1) < f(x_2)).\]

With any universally quantified implication, if the left side of the implication is true, then the right side must be true to satisfy the assertion (i.e., $p \implies q$). It is also the case that the contrapositive is true (i.e., $\neg q \implies \neg p$). For this example, the negation of the right-hand side of the implication is $f(x_2) \leq f(x_1)$, which matches one of the constraints in the dependence. Thus the negation of the left-hand side must be true and therefore $x_2 \leq x_1$. With $x_1$ matching $i$ and $x_2$ matching $i'$, we find $i' \leq i$. Thus an equality has been found:

\[(i \leq i' \land i' \leq i) \implies i = i'.\]

Using this equality we can iterate over either $i$ or $i'$ in the inspector and calculate the other by taking advantage the equality. The runtime inspection would have complexity of only $O(n)$. 

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4.2 Finding Equalities Example: Left Cholesky

For a more realistic example from one of the benchmarks used in the evaluation, consider a maybe satisfiable dependence from our Static Left Cholesky shown in Figure 6. Following dependence is coming from a read in S1 (lValues[i]), and a write in S2 (lValues[j]):

\[
\{[\text{colNo}] \Rightarrow [\text{colNo}'] : \exists j, i', l' (j = l') \land (\text{colNo} < \text{colNo}')
\land (0 \leq \text{colNo} < n) \land (0 \leq \text{colNo}' < n) \land (\text{lcolptr(pruneSet}(i')) \leq l' < \text{lcolptr(pruneSet}(i' + 1))
\land (\text{prunePtr}(\text{colNo}') \leq i' < \text{prunePtr}(\text{colNo}' + 1)) \land (\text{lcolptr}(\text{colNo}) < j < \text{lcolptr}(\text{colNo} + 1))\}
\]

An inspector for this dependence is shown in Figure 7a. We do not need loops for j and l’ in the inspector, because they can be projected out. Note, index array prunePtr points to nonzeros in the sparse matrix, ranging from 0 to number of nonzeros, nnz, and n denotes the number of column. The two loops, colNop and ip, combined are traversing all the nonzero values and hence have a combined complexity of nnz, followed by the colNo loop traversing over columns, n. Consequently, the complexity of this inspector is \(n(nnz)\).

The equality \(\text{colNo} = \text{pruneSet}(i')\) is found using the additional knowledge that lcolptr is strictly monotonically increasing as demonstrated in the following.

We have the following constraints in the original dependence:

\[
\text{lcolptr(pruneSet}(i')) \leq l' < \text{lcolptr(pruneSet}(i' + 1))
\land j = lp \land \text{lcolptr(colNo)} < j < \text{lcolptr(colNo + 1)},
\]

which gives the following through transitivity:

\[
\text{lcolptr(pruneSet}(i')) < \text{lcolptr(colNo + 1)}
\land \text{lcolptr(colNo)} < \text{lcolptr(pruneSet}(i' + 1)).
\]

We have the following assertion:

\[
(\forall x_1, x_2)(\text{lcolptr}(x_1) < \text{lcolptr}(x_2) \implies x_1 < x_2)
\]

where two instances \(x_1 = \text{pruneSet}(i')\), \(x_2 = \text{colNo} + 1\) and \(x_1 = \text{colNo}, x_2 = \text{pruneSet}(i') + 1\) give new constraints:

\[
\text{pruneSet}(i') < \text{colNo} + 1 \land \text{colNo} < \text{pruneSet}(i') + 1
\implies \text{pruneSet}(i') \leq \text{colNo} \land \text{colNo} \leq \text{pruneSet}(i')
\implies \text{colNo} = \text{pruneSet}(i')
\]

The optimized inspector based on new discoveries is shown in Figure 7b. We do not need loop for \(\text{colNo}\), since we can get its values from \(\text{pruneSet}(i')\) based on \(\text{colNo} = \text{pruneSet}(i')\). This simplified inspector has a complexity of \((nnz)\), which is significantly better than the original \(n(nnz)\).

5 SIMPLIFYING THE DEPENDENCES UTILIZING SUPERSET RELATIONSHIP

Another way to deal with data dependence relations that cause complex runtime analysis is to remove it from consideration by determining it is a subset of a less expensive relation. Consider two dependence relations R1 and R2, and two iterations of the outermost loop i and i’. If we can show that for all i and i’ that are dependent according to R2, the same pairs of i and i’ are also dependent according to R1, then it is sufficient to only test R1. We say that R1 is a superset of R2, written \(R1 \supseteq R2\), in such cases, and remove R2 from runtime check. Note that in the above definition, R1 may have more pairs of outermost iterators that are dependent than R2.
for(colNop = 0; colNop < n; colNop++)
for(ip = prunePtr(colNop);
    ip < prunePtr(colNop+1); ip++) {
    for(colNo=0; colNo<n; colNo++) {
        if(lcolptr(colNo) < lcolptr(colNo+1) && ...
            // Add a flow dependence between
            colNo and ColNop
        }
    }
}

(a) Inspector with the original dependence constraints.

for(colNop = 0; colNop < n; colNop++)
for(ip = prunePtr(colNop);
    ip < prunePtr(colNop+1); ip++) {
    colNo = pruneSet(ip);
        if(lcolptr(colNo) < lcolptr(colNo+1) && ...
            // Add a flow dependence between colNo
            and ColNop
        }
    }

(b) Inspector with an additional equality: colNo = pruneSet(i').

Fig. 7. Inspector pseudo-code for dependence constraints in Section 4.2, before and after utilizing index array properties to add new equalities. We obtain the equality colNo = pruneSet(i') using the properties as described in Section 4.2. Notice how this equality is used to remove loop iterating over i in Line 3.

for (i = 0; i < n; i++) {
S1: val[colPtr[i]] = sqrt(val[colPtr[i]]);
for (m = colPtr[i] + 1; m < colPtr[i+1]; m++)
S2: val[m] = val[m] / val[colPtr[i]];
for (m = colPtr[i] + 1; m < colPtr[i+1]; m++)
    for (k = colPtr[rowIdx[m]] ; k < colPtr[rowIdx[m]+1]; k++)
        for ( l = m; l < colPtr[i+1] ; l++)
            if (rowIdx[l] == rowIdx[k] && rowIdx[l+1] <= rowIdx[k])
                S3: val[k] -= val[m]* val[l];
}

Fig. 8. Incomplete Cholesky code from SparseLib++ [Pozo et al. 1996]. Some variable names have been changed. The arrays col and row are to represent common colPtr, and rowIdx in CSC format.

Taking advantage of this redundancy can result in lower complexity runtime analysis. As an example, consider the Incomplete Cholesky code shown in Figure 8. In section, we refer to an array access \(A\) at statement \(S\) as \(A@S\) for brevity. One of the dependence tests is between the write \(val[k]@S3\) and the read \(val[m]@S3\). This test is redundant with the test between the write \(val[k]@S3\) and the read \(val[m]@S2\). This is because an iteration of the i loop that make the read from \(val[m]\) in \(S3\) is guaranteed to access the same memory location while executing the loop surrounding \(S2\). Thus, the more expensive check between accesses in \(S3\) can be removed.

In this section, we describe our approach to identify redundant dependence relations. The key challenge is to determine superset relations between two dependence tests involving uninterpreted functions. We present two approaches that cover important cases, and discuss possible extensions.

### 5.1 Trivial Superset Relations

Given a polyhedral dependence relation, it is easy to characterize the pairs of loop iterations that are dependent. All the indices that do not correspond to the loop iterators in question can be projected out to obtain the set of dependent iterations. These sets can be compared to determine if a dependence test is subsumed by another. In principle, this is what we do to check if a dependence relation is redundant with another. However, dependence relations from sparse codes may have variables passed as parameters to uninterpreted functions. Such variables cannot be projected out.
Thus, we employ an approach based on similarities in the constraint systems. The trivial case is when the dependence relation \( R_1 \) is expressed with a subset of constraints in another relation \( R_2 \). If this is the case, then \( R_1 \) can be said to be superset equal to \( R_2 \).

We illustrate this approach with the earlier example from Incomplete Cholesky. We take two dependence relations, \( R_1 \) between \( \text{val}[k]@S3 \) and \( \text{val}[m]@S2 \), and \( R_2 \) between \( \text{val}[k]@S3 \) and \( \text{val}[m]@S3 \). The relations—omitting the obviously common constraints for \( \text{val}[k]@S3 \)—are:

\[
R_1 = \{ [i, m, k, l] \rightarrow [i', m'] : k = m' \land i < i' \land 0 \leq i' < n \land \text{col}(i') + 1 \leq m' < \text{col}(i' + 1) \}
\]

\[
R_2 = \{ [i, m, k, l] \rightarrow [i', m', k', l'] : k = m' \land i < i' \land 0 \leq i' < n \land \text{col}(i') + 1 \leq m' < \text{col}(i' + 1) \\
\land \text{col}(\text{row}(m')) \leq k' < \text{col}(\text{row}(m) + 1) \land m' \leq l' \land \text{col}(i + 1) \\
\land \text{row}(l') = \text{row}(k') \land \text{row}(l' + 1) \leq \text{row}(k') \}
\]

Since \( R_1 \) is expressed with a subset of constraints in \( R_2 \), we may conclude that \( R_1 \supseteq R_2 \).

### 5.2 Superset Relation due to Overlapped Accesses

The trivial check is sufficient for many pairs of relations. However, some relations require a more involved process. We use a different dependence relation from Incomplete Cholesky (Figure 8) as an example of such cases. We consider the dependence relation \( R_3 \) between \( \text{val}[k]@S3 \) and \( \text{val}[1]@S3 \) that is redundant with \( R_1 \). This can be intuitively observed from the code structure: the set of memory locations that may be accessed by the read of \( \text{val}[1] \) when \( l = m \), i.e., the first iteration of the \( l \) loop, is exactly the same as the reads by \( \text{val}[m]@S2 \). This guarantees that even if the guard on S3 always evaluated to true, the dependence between iterations of the \( l \) loop would be redundant with that imposed by S2.

The constraints for \( R_3 \) (omitting those for \( \text{val}[k]@S3 \)) are as follows:

\[
R_3 = \{ [i, m, k, l] \rightarrow [i', m', k', l'] : k = l' \land i < i' \land 0 \leq i' < n \land \text{col}(i') + 1 \leq m' < \text{col}(i' + 1) \\
\land \text{col}(\text{row}(m')) \leq k' < \text{col}(\text{row}(m) + 1) \land m' \leq l' \land \text{col}(i + 1) \\
\land \text{row}(l') = \text{row}(k') \land \text{row}(l' + 1) \leq \text{row}(k') \}
\]

(1) We first identify that \( k = m' \) in \( R_1 \) is not a constraint in \( R_3 \).
(2) The equality \( k = m' \) has a similar (one side of the equality is the same) equation \( k = l' \) in \( R_3 \).
(3) The bounds on \( m' \) and \( l' \) are collected from the respective constraints.
(4) Because the bound on \( m' \) subsumes that of \( l' \), and since \( k = m' \) was the only constraint that was not in \( R_3 \), we may conclude that \( R_1 \supseteq R_3 \).

It is important to note that the bounds on \( l' \)—the set of values accessed in the subset relation—can be conservative, i.e., may accesses, but the bounds on \( m' \)—the set of values accessed in the superset relation—must be exact. If both bounds represent may accesses, then the superset relation does not hold. This is important for situations as illustrated in the example above, where statements have data-dependent guards.

### 5.3 Limitations and Extensions

Although the approach presented above was able to cover all the important cases we encountered, it is by no means complete. The difficulty of manipulating integer sets with uninterpreted function symbols have led us to work directly on the constraints. This may cause our approach to miss some superset relations, since the same relation can be expressed in many different ways. Adding some form of normalization to the constraint system will help us avoid such pitfalls.
The overlapped iterator approach to finding a superset in Section 5.2 was developed specifically for the problematic data dependence relation R3. Future work includes developing a more general simplification approach based on this overlapping iterator concept.

In terms of scaling, there is potentially a problem of selecting the pairs of dependence relations to test for redundancy. We currently try all possible candidate pairs, which does not pose a problem since a large number of dependence relations are filtered out through unsatisfiability test described in Section 3.4. However, selecting promising pairs to limit the number of tests would be an useful extension.

6 IMPLEMENTATION

The data dependence analysis and simplification have been automated except for the superset simplification. This section summarizes the software packages the implementation relies on, discusses some important optimization to make our implementation scalable, and compares the ISL-based implementation with that of an SMT solver.

6.1 Software Description

The artifact for reproducing the results presented in this article can be found at the following public github repository: https://github.com/CompOpt4Apps/Artifact-datadepsimplify-arXiv-July2018

We use three software packages to automate applying methods described in this paper: IEGenLib library [Strout et al. 2016], ISL library [Verdoolaege 2018], and CHiLL compiler framework [chi 2018]. CHiLL is a source-to-source compiler framework for composing and applying high level loop transformations to improve the performance of nested loop written in C. We utilize CHiLL to extracted the dependence relations from the benchmarks. ISL is a library for manipulating integer sets and relations that only contain affine constraints. It can act as a constraint solver by testing the emptiness of integer sets. It is also equipped with a function for detecting equalities in sets and relations. ISL does not support uninterpreted functions, and thus cannot directly represent the dependence constraints in sparse matrix code. IEGenLib is a set manipulation library that can manipulate integer sets/relations that contain uninterpreted function symbols. The IEGenLib library utilizes ISL for some of its fundamental functionalities. We have implemented detecting unsatisfiable dependences and finding the equalities utilizing the IEGenLib and ISL libraries.

The following briefly describes how the automation works. First, we extract the dependence relations utilizing CHiLL, and store them in IEGenLib. The user defined index array properties are also stored in IEGenLib. Next, the instantiation procedure is carried out in IEGenLib. Then inside IEGenLib, the uninterpreted functions are removed by replacing each call with a fresh variable, and adding constraints that encode functional consistency [Kroening and Strichman 2016, Chapter 4]. Next, ISL can be utilized by IEGenLib to find the empty sets, i.e, unsatisfiable relations. Additionally, equality detection is available as one of many operations supported by ISL. The finite instantiations described in Section 3.4 are intersections of the assertions with the dependence relation.

6.2 Optimization

A straightforward approach to implementing the procedure in Section 3.4 would be to take the quantifier-free formula resulting from instantiation, replace the uninterpreted functions, and directly pass it to ISL. However, this approach does not scale to large numbers of instantiations. An instantiated assertion is encoded as a union of two constraints ($\neg p \lor q$). Given $n$ instantiations, this approach introduces $2^n$ disjunctions to the original relation, although many of the clauses may be empty. In some of our dependence relations, the value of $n$ may exceed 1000, resulting in a prohibitively high number of disjunctions. We have observed that having more than 100 instantiations causes ISL to start having scalability problems.
We apply an optimization to avoid introducing disjunctions when possible. Given a set of instantiations, the optimization adds the instantiations to the dependence relation in two phases. The first phase only instantiates those that do not introduce disjunctions to the dependence relation. During this phase, we check if the antecedent is already part of the dependence constraint, and thus is always true. If this is the case, then \( q \) can be directly added to the dependence relation. We also perform the same for \( \neg q \implies \neg p \) and add \( \neg p \) to the dependence relation if \( \neg q \) is always true. The second phase adds the remaining instantiations that introduce disjunctions. This optimization helps reducing the cost of dependence testing in two ways: (1) if the relation is unsatisfiable after the first phase, disjunctions are completely avoided; and (2) the second phase only instantiates the remainder, reducing the number of disjunctions.

If the dependence relation remains non-empty after the second phase, then the relation is checked at runtime. All equalities in a relation is made explicit before inspector code generation with ISL so that the code generator can take advantage of the equalities to simplify the generated code.

6.3 Contrasting SMT with ISL

SMT solvers are specialized for solving satisfiability problems expressed as a combination of background theories. ISL is a library for manipulating integer sets, and is specialized for the theory of Presburger arithmetic over integers.

The finite instantiation in Section 3.4 is well-suited for SMT solvers. In fact, SMT solvers are equipped with their own instantiation algorithms that also work well for our dependence relations. However, SMT solvers do not provide any equality relationships they might derive while answering the satisfiability question. Although it is possible to use SMT solvers to test if an equation is true for a set of constraints, we cannot search for an equation given the constraints (unless all candidates are enumerated—but there are infinite candidates in general).

For our implementation, the choice was between adding finite instantiation to ISL or adding equality detection to SMT solvers. We have chosen the former option as it seemed simpler to do, and also because we are more familiar with ISL.

7 EVALUATION OF UNSATISFIABILITY AND SIMPLIFICATION APPROACHES

In this section, we study the impact of our approach of utilizing domain information about index arrays on the data dependence analysis of eight sparse kernels. Our approach may help data dependence analysis in three ways: (1) The runtime check can be completely removed if the dependences are proven unsatisfiable; (2) Deriving equalities from instantiated universally quantified assertions about domain information can simplify dependences and reduce respected runtime check complexity; and (3) Reducing all maybe satisfiable relations of a given code to a set of dependence relations that encompass all potential dependences. We do this by finding relations that are superset equal of other relations. This can discard even more dependence relations that potentially might need expensive runtime checks.

We first describe the suite of numerical kernels that we have compiled to evaluate our approach. Then we evaluate the impact of each step in our approach, from the relevance of the index property assertions to the simplification using superset relations. Finally, we report the complexity of inspectors with and without our proposed simplifications.

7.1 Numerical Algorithms in Benchmark

We have included some of the most popular sparse kernels in a benchmark suite: (1) The Cholesky factorization, Incomplete LU0 and Incomplete Cholesky0, and the sparse triangular solver, which are commonly used in direct solvers and as preconditioners in iterative solvers; (2) sparse matrix vector multiplication, and Gauss-Seidel methods, which are often used in iterative solvers. Table 1
Table 1. The benchmark suite used in this paper. The suite includes the fundamental blocks in several applications. The suite is also selected to cover both static index arrays, such as Gauss-Seidel, and dynamic index arrays, such as Left Cholesky. The modification column shows the type of modification applied to the original code.

| Algorithm name          | Format  | Library source       | Mod. |
|-------------------------|---------|----------------------|------|
| Gauss-Seidel solver     | CSR     | Intel MKL [Wang et al. 2014] | None |
| Gauss-Seidel solver     | BCSR    | Intel MKL [Wang et al. 2014] | None |
| Incomplete LU           | CSR     | Intel MKL [Wang et al. 2014] | None |
| Incomplete Cholesky     | CSC and CSR | SparseLib++ [Pozo et al. 1996] | None |
| Forward solve           | CSC     | Sympiler [Cheshmi et al. 2017] | None |
| Forward solve           | CSR     | [Vuduc et al. 2002] | None |
| Sparse MV Multiply      | CSR     | Common                | None |
| Static Left Cholesky    | CSC     | Sympiler [Cheshmi et al. 2017] | Pa + Rb |

a Privatization of temporary arrays  
b Removal of dynamic index array updates

summarizes the benchmarks indicating which library each algorithm came from and how the benchmark compares with the implementations in existing libraries.

We modified one of the benchmarks, left Cholesky, to make temporary arrays privatizable and to remove dynamic index array updates so that the compiler can analyze the sparse code.

**Left Cholesky:** This code has following changes compared to a more common implementation in CSparse [Davis 2006]: (i) **Privatization of temporary arrays:** We analyzed dependences between reads and writes to temporary arrays to detect privatizable arrays. This can be challenging for a compiler to do with sparse codes since accesses to these arrays are irregular. We set the values of these arrays to zero at the beginning of each loop so a compiler could identify them as privatizable. (ii) **Removal of dynamic index array updates:** Previous data dependence analysis work focuses on cases where index arrays are not updated. However, in some numerical codes, updating index arrays is a common optimization. We refer to this as **dynamic index array updates**, and it usually occurs when the nonzero structure of an output matrix is modified in the sparse code during the computation. This would make dependence analysis very complicated for the compiler. We removed dynamic index arrays by partially decoupling symbolic analysis from the numerical code in these benchmarks. **Symbolic analysis** here refers to terminology used in the numerical computing community. Symbolic analysis uses the nonzero pattern of the matrix to determine computation patterns and the sparsity of the resulting data. To remove dynamic index array updates, we decouple symbolic analysis from the code similar to the approach used by [Cheshmi et al. 2017].

**Performance Impact:** The changes made to Left Cholesky do not have a noticeable effect on the code performance. Based on our experiments using five matrices\(^1\) from the Florida Sparse Matrix Collection [Davis and Hu 2011] the performance cost of these modifications is on average less than 10% than the original code.

### 7.2 Relevance of Index Array Properties

We have extracted the constraints to test for dependences that are carried by the outermost loop for the sparse matrix codes in Table 1. A total of 124 data dependences relations were collected from the benchmarks. Of those 124, only 83 of them were unique, the repetition coming from accesses

\(^1\)Problem1, rdb450l, wang2, ex29, Chebyshev2
Table 2. Categorization of index array properties in our evaluation of their utility in detecting unsatisfiability.

| Array property | Formulation with examples from Left Cholesky code | What codes found in |
|----------------|---------------------------------------------------|---------------------|
| Monotonocity   | \((x_1 < x_2 \Leftrightarrow lcolptr(x_1) < lcolptr(x_2))\). | All                 |
| Correlated Monotonicity | \((x_1 = x_2 \Rightarrow rowPtr(x_1) \leq diagPtr(x_2)). (x_1 < x_2 \Rightarrow diagPtr(x_2) < rowPtr(x_1)).\) | Incomplete LU0, Forward Solves |
| Triangular Matrix | \((lcolptr(x_1) < x_2 \Rightarrow x_1 < lrow(x_2)). (x_1 < prunePtr(x_2) \Rightarrow pruneSet(x_1) < x_2)).\) | Cholesky’s, Forward Solves |

with same access indices in the same statements, or other situations. Table 2 summarizes the index array assertions relevant to the benchmarks.

We are not claiming to have found all the array properties that exist either in our example suite nor in general. Also, we only consider dependence relations for outermost loops, however, dependence relations can be extracted for other loop levels in a loop nest and can be used for vectorization and in other applications of dependence analysis.

![Fig. 9. Reduction in the number of different inspectors’ complexities after adding array properties individually and in combination. Please note, \(nnz\) is number of non-zeros, and \(n\) is number of columns or rows in a matrix. The array properties discussed in the paper can help us detect 45 relations as unsatisfiable out of 71 baseline relations. Note, the number of unsatisfiable relations detected with combination of information is not the accumulation of all others. Sometimes combination of information together helps detect unsatisfiability.](image)

7.3 Detecting Unsatisfiability

In this section, we show the impact of using index array properties to detect unsatisfiability for the relations collected from dependences from our benchmark suite. To not conflate the impact of the index array properties that we are evaluating with what traditional methods are capable of, we first apply functional consistency in the theory of Presburger arithmetic combined with uninterpreted functions [Shostak 1979]. This detects 12 dependences as unsatisfiable. Nevertheless, we must note that, all of the 12 dependences have inconsistencies in their affine parts and functional consistency does not help detect any more unsatisfiable relations; like the first two dependences from the Forward Solve CSR example in Section 2.2. After detecting 12 out of 83 dependences as
unsatisfiable we are left with 71 dependences to use in our evaluation. We call these 71 dependences that are satisfiable just by looking at their affine constraints our baseline.

Figure 9 categorizes the complexity of an inspector for each dependence into 7 different classes in total. In this figure, \( nnz \) is number of non-zeros, and for simplicity \( n \) denotes the number of rows or columns of the input matrix. The black bar, "baseline", in each class shows the baseline number of relations with that complexity in our suite. The bars show how many dependences would remain after we instantiate certain index array properties. The last bar in each class, the red bar, shows the effect of adding all the information in combination.

The main observations from analyzing Figure 9 are as follows: (1) Combining the array properties and non-domain information has the biggest impact and helps detect significantly more unsatisfiable dependences than any single property. Combining all the index array properties helped us detect 45 out of 71 relations as unsatisfiable, with 26 remaining as maybe satisfiable. (2) Monotonicity has the highest impact on detecting unsatisfiable relations when array properties are applied independently. (3) The Triangular Matrix property helped detect 3 relations when applied independently and 11 more in combination with Monotonicity (not obvious in the figure). This property helped us detect unsatisfiability in cases where Monotonicity was completely handicapped; see the \( nnz \) and \( nnz \times n \) classes in Figure 9.

### 7.4 Simplifying Inspector Complexity Utilizing Equalities

As stated in the previous section, instantiating index array properties results in 45 out of 71 dependence relations being detected as unsatisfiable. At this point, without any further simplification, to perform a partial parallelism transformation, inspectors are needed for the remaining 26 dependences. One question we can ask about those 26 inspector is whether their complexity is even reasonable. We consider a runtime dependence analysis complexity reasonable, if it is bound by the complexity of the original computation. The computations would certainly do much more operations compared to the analysis as numerical algorithms usually call these computations several time for the same sparse matrix nonzero structure. Thus runtime data dependence analysis is reasonable if it is the same complexity as the original computation. Nonetheless, for numerical algorithms, it is common to aim for a runtime data dependence analysis that is of \( O(nnz) \), where \( nnz \) is the number of nonezeros in the input.

By instantiating index array properties with expressions from the data dependences, it is also possible to derive equalities between some of the iterators in the dependence. These new useful equalities can be used to eliminate extra loops in the runtime inspector. Table 3 shows that the additional equalities increases the number of dependence relations with reasonable complexities (\( \leq \) kernel). For instance, the Left Cholesky code have 4 high complexity dependence relation left. As illustrated in Section 4.2, the additional equalities can be used to reduce the complexity of all those relations. Finding equalities also help reduce the complexity of 4 dependences for Incomplete Cholesky\(^0\) and 2 dependences of Incomplete LU\(^0\) to become reasonable.

We should also mention that in addition to these 10 complexity reductions, the complexity of another 4 dependence relations were reduced. However, the complexity after simplification is still higher than the kernel, and hence these simplifications are not visible in Table 3.

### 7.5 Impact of Utilizing Superset Relationship

The superset relations we identify uncovers dependence relations that are redundant. We can discard the dependence relations that are found to be subsets of another and only generate runtime inspectors for remaining relations. As shown in Table 3, this results in fewer dependence relations to be checked at runtime. Most notably, the number of runtime checks were reduced from 4 to 2 for Left Cholesky, and both of those dependences are less complex than the original algorithm.
Table 3. Effect of simplifications based additional equalities (Section 4) and redundancy elimination (Section 5) on the remaining 26 maybe satisfiable dependences for each code in the benchmark. The Total columns show the number of dependence relations that needs to be checked at runtime. The ≤ kernel columns show the number of such tests that have the same or lower complexity than the kernel. Equality Impact is the numbers after using additional equalities, reducing the number of high complexity checks. Superset Impact is the composed effect of using superset relations after adding equalities, reducing the total number of checks.

| Kernel name             | Remaining satisfiables | Equality Impact | Superset Impact |
|-------------------------|------------------------|-----------------|-----------------|
|                         | ≤ kernel | Total | ≤ kernel | Total | ≤ kernel | Total |
| Gauss-Seidel CSR        | 2        | 2     | 2        | 2     | 2        | 2     |
| Gauss-Seidel BCSR       | 4        | 4     | 4        | 4     | 2        | 2     |
| Incomplete LU           | 0        | 4     | 2        | 4     | 2        | 4     |
| Incomplete Cholesky     | 1        | 9     | 5        | 9     | 2        | 2     |
| Forward solve CSR       | 1        | 1     | 1        | 1     | 1        | 1     |
| Forward solve CSC       | 2        | 2     | 2        | 2     | 1        | 1     |
| Sparse MV Mul.          | 0        | 0     | 0        | 0     | 0        | 0     |
| Left Cholesky           | 0        | 4     | 4        | 4     | 2        | 2     |

As discussed in Section 5, the superset relation may reveal that a relation is redundant being subset of another relation with lower complexity. The Incomplete Cholesky kernel were left with 4 expensive relations even after adding equalities. As you can see in Table 3, these relations are removed from runtime checks by identifying the superset relations. For Incomplete Cholesky kernel, we have found 2 relations with less than original algorithm complexity to be superset of all the dependences that we need to have a runtime check for. The composed effect of our proposed technique reduces the inspector cost to 2 or fewer inexpensive tests for all of our kernels, except for the Incomplete LU.

7.6 Putting It All Together

We have presented a series of techniques to simplify dependence relations with the main motivation being automatic generation of efficient inspector code. Our approach aims to simplify the dependence relations starting from array properties that can be succinctly specified by the experts. We show that the array properties can be used to automatically disprove a large number of potential dependences, as well as reduce the complexity of remaining dependences. Combined with a method for detecting redundancies in dependence tests, we are able to generate efficient inspectors.

Table 4 summarizes the impact of our proposed approach on inspector complexity. It is interesting to note that Incomplete LU0 is the only kernel left with expensive inspector (more complex than kernel). This case is discussed further in Section 7.7.

7.7 Discussion: Limitations

Table 4 demonstrates that our method significantly reduces both the number of runtime checks and their complexity. Nonetheless, our approach is not free of limitations, which are discussed in this section.

Two of the original kernels include dynamic index arrays and temporary arrays that require privatization. As discussed in Section 7.1, these kernels can be preprocessed such that it can be accepted by our compiler. This preprocessing is currently done manually.

Using the associativity of reductions is important for Forward Solve CSC and Incomplete Cholesky0. We do not automate the reduction detection in this paper, as it is a complex task.
Table 4. The impact of our simplifications on inspector complexity. The baseline inspector complexity is when all possible dependences are tested at runtime, without using any of the simplifications proposed in this paper. The simplified inspector complexity reports the final cost of inspection generated by our approach. The overall complexity of inspectors decreases considerably. The complexity of the kernels are included for comparison; \( k \) and \( K \) denote constant factors, with \( K \) signaling a bigger number.

| Kernel name                | Inspector complexity                      | Simplified inspector | Kernel complexity |
|----------------------------|------------------------------------------|----------------------|-------------------|
| Gauss-Seidel CSR           | \((n) + 2(nnz)\)                        | \(2(nnz)\)          | \(k(nnz)\)        |
| Gauss-Seidel BCSR          | \(4(n) + 4(nnz)\)                      | \(2(nnz)\)          | \(k(nnz)\)        |
| Incomplete LU CSR          | \(4(nnz \times (nnz/n)) + \left(n^2\right) + (2n \times nnz) + 2(nnz^2) + 2(nnz^2 \times (nnz/n)^2) + 2(nnz^2 \times (nnz/n)^3)\) | \(2(nnz \times (nnz/n)^2) + 2(nnz \times (nnz/n)^4)\) | \(K(nnz \times (nnz/n)^2)\) |
| Incomplete Cholesky CSR    | \(10(n^2) + 8(nnz^2) + 6(nnz^2 \times (nnz/n)) + 4(nnz^2 \times (nnz/n)^3)\) | \((nnz \times (nnz/n)) + (nnz \times (nnz/n)^2)\) | \(K(nnz \times (nnz/n)^2)\) |
| Forward solve CSC          | \(3(n) + 4(nnz)\)                      | \(nnz\)              | \(k(nnz)\)        |
| Forward solve CSR          | \((n) + 2(nnz)\)                       | \(nnz\)              | \(k(nnz)\)        |
| Sparse MV Mul. CSR         | \(3(n)\)                               | \(0\)                | \(k(nnz \times (nnz/n))\) |
| Left Cholesky CSC          | \(8(n \times nnz) + 4(n^2)\)           | \(2(nnz)\)          | \(K(nnz \times (nnz/n))\) |

on its own. It is common for compilers and programming models, such as openMP, to provide pragma interfaces for programmers to signal which update should be considered as a reduction. We have followed the same approach.

Incomplete LU0 has two dependence relations that has higher complexity than the kernel, even with domain information. Related work by [Venkat et al. 2016] presents approximation techniques that reduce the inspector complexity for these high complexity relation to \(nnz \times (nnz/n)\). Such approximation can potentially result in loss of some parallelism. Nevertheless, [Venkat et al. 2016] show that the approximation of dependences does not significantly affect the performance of the partial parallelism for this code. We have not used approximations in our work, but it would be interesting to see how the two approaches can be combined.

8 RELATED WORK

Array data dependence analysis has been used for a variety of applications, including automatic parallelization [Paek et al. 2002], locality optimization [Wolfe 1989], communication generation, program slicing [Pugh and Rosser 1997], detecting race conditions [Zheng et al. 2015a], and high-level synthesis [Alle et al. 2013]. For sparse matrix codes, this analysis is made more difficult due to indirection through index arrays, such that the source and sink of dependences cannot be resolved until their values are available at runtime. For these and other situations where dependences arise that cannot be resolved until runtime, a number of techniques for compile time and runtime dependence analysis have been developed.

8.1 User-Provided Assertions

[McKinley 1991] exploit user assertions about index arrays to increase the precision of dependence testing. The assertions certify common properties of index arrays, e.g., an index array can be a permutation array, monotonically increasing, and monotonically decreasing. [Lin and Padua 2000]
present a compile time analysis for determining index array properties, such as monotonicity. They use the analysis results for parallelization of sparse matrix computations.

Our approach also uses these assertions, but in addition we use more domain-specific assertions and provide a way to automate the general use of such assertions. In this paper, the idea of applying constraint instantiation of universally quantified constraints as is done in SMT solvers to find unsatisfactory dependences is novel and the assertions about index arrays we use are more general.

8.2 Proving Index Arrays Satisfy the Assertions

In this work, we assume that the assertions provided by the programmer is correct. It is useful to verify the user-provided assertions by analyzing the code that constructs the sparse matrix data structures. There is a large body of work in abstract interpretation that address this problem.

The major challenge in verifying the assertion about programs that manipulate arrays is the trade-off between scalability and precision. When there is a large number of updates to an array, keeping track of individual elements do not scale, but approximating the whole array as a single summary significantly degrades the precision. Many techniques to verify/infer important properties about array contents from programs have been developed, e.g., [Cousot et al. 2011; Gopan et al. 2005; Halbwachs and Péron 2008].

In the work by [Henzinger et al. 2010], the authors present an approach for inferring shape invariants for matrices. While their work does not deal with sparse matrices and index arrays, it may help generate domain-specific assertions that we could employ to show that the data dependences are unsatisfiable.

The main subject of our work - dependence tests - does not involve array updates, since all the index arrays, which alter the control-flow and indexing of the data arrays, are not updated. This makes the verification of the assertions a closely related but orthogonal problem, which we do not address in this paper.

8.3 More General Quantifier Elimination Techniques

The area of SMT-solving is advancing at a significant pace; the webpage for SMT-COMP provides a list of virtually all actively developed solvers, and how they fared in each theory category. As these solvers are moving into a variety of domains, quantifier instantiation and elimination has become a topic of central interest. Some of the recent work in this area are: E-matching [Moura and Bjørner 2007], Model-Based [Ge and de Moura 2009], Conflict-Based [Reynolds et al. 2014], and Counter-Example Guided [Reynolds et al. 2015].

These efforts make it clear that quantifier instantiation is challenging, and is an area of active development. SMT solvers often rely on heuristic-driven instantiations to show unsat for difficult problems. In this context, our work can be viewed as heuristic instantiation where the heuristic is inspired by decidable fragments of the array theory.

Dependence constraints with universally quantified assertions are related to the first order theory fragments described by [Bradley et al. 2006] as undecidable extensions to their array theory fragment. However, [Löding et al. 2017] claim that the proofs for undecidability of extension theories by [Bradley et al. 2006] are incorrect, and declare their decidability status as an open problem. Regardless of whether the theory fragment that encompasses our dependence constraints is decidable or not following is true: if we soundly prove that a relation is unsatisfiable just with compile time information, the unsatisfiability applies in general, and having runtime information would not change anything. However, if a dependence detected to be satisfiable just with compile

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2http://smtcomp.sourceforge.net/2017/
time information, we need to have runtime tests to see if it is actually satisfiable given runtime information, and even if it is, run time tests would determine for what values the dependence holds.

8.4 Dependence Analysis for Full Parallelization

Some compilation techniques have been developed to extend the dependence analysis to sparse, or non-affine programs [Benabderrahmane et al. 2010]. These techniques extend to non-affine programs of various forms: while loops, polynomial expressions, function calls, data-dependent conditions, and indirection arrays. The outcome of such analysis is often an approximation, which is quite pessimistic for sparse computations involving indirection arrays. The focus of our work is not to identify (approximated) dependences, but to reduce the cost of runtime dependence checks by disproving potential dependences as much as possible at compile-time.

The work by [Pugh and Wonnacott 1998] also formulate the problem in the theory of Presburger sets with uninterpreted functions. However, they only allow affine expressions of unquantified variables as indexing expressions to the function symbols, excluding some of the examples in this paper. They propose an analysis to identify conditions for a dependence to exist through the use of gist operator that simplifies the constraint system given its context. The result of this analysis may involve uninterpreted functions, and can be used to query the programmer for their domain knowledge. This is an interesting direction of interaction that complements our work.

Several runtime approaches focus on identifying loops we denoted fully parallel whose iterations are independent and can safely execute in parallel [Barthou et al. 1997; Moon and Hall 1999; Pugh and Wonnacott 1998] or speculatively execute in parallel while testing safety [Rauchwerger and Padua 1999].

8.5 Dependence Analysis for Wavefront Parallelization

For sparse codes, even when loops carry dependences, the dependences themselves may be sparse, and it may be possible to execute some iterations of the loop in parallel (previously denoted partially parallel). The parallelism is captured in a task graph, and typically executed as a parallel wavefront. A number of prior works write specialized code to derive this task graph as part of their application [Bell and Garland 2009; Park et al. 2014a,b; Rauchwerger et al. 1995a; Saltz et al. 1991; Zhuang et al. 2009] or with kernel-specific code generators [Byun et al. 2012]. For example, Saltz and Rothbergers worked on manual parallelization of sparse triangular solver codes in the 1990s [Rothberg and Gupta 1992; Saltz 1990]. There is also more recent work on optimizing sparse triangular solver NVIDIA GPUs and Intel’s multi-core CPUs [Rennich et al. 2016; Wang et al. 2014]. Even though these manual optimizations have been successful at achieving high performance in some cases, significant programmer effort has to be invested for each of these codes and automating these parallelization strategies can significantly reduce this effort.

Other approaches automate the generation of inspectors that find task-graph, wavefront or partial parallelism. [Rauchwerger et al. 1995b] and others [Huang et al. 2013] have developed efficient and parallel inspectors that maintain lists of iterations that read and write each memory location. By increasing the number of dependences found unsatisfiable, the approach presented in this paper reduces the number of memory accesses that would need to be tracked. For satisfiable dependences, there is a tradeoff between inspecting iteration space dependences versus maintaining data for each memory access. That choice could be made at runtime. There are also other approaches for automatic generation of inspectors that have looked at simplifying the inspector by finding equalities, using approximation, parallelizing the inspector, and applying point-to-point synchronization to the executor [Venkat et al. 2016].
8.6 Algorithm-Specific Data Dependence Analysis

An algorithm-specific approach to represent data dependences and optimize memory usage of sparse factorization algorithms such as Cholesky [Pothen and Toledo 2004] uses an elimination tree, but to the best of our knowledge, this structure is not derived automatically from source code. When factorizing a column of a sparse matrix, in addition to nonzero elements of the input matrix new nonzero elements, called fill-in, might be created. Since the sparse matrices are compressed for efficiency, the additional fills during factorization make memory allocation ahead of factorization difficult. The elimination tree is used to predict the sparsity pattern of the L factor ahead of factorization so the size of the factor can be computed [Coleman et al. 1986] or predicted [Gilbert 1994; Gilbert and Ng 1993], and captures a potential parallel schedule of the tasks. Prior work has investigated the applicability of the elimination tree for dependence analysis for parallel implementation [George et al. 1989; Gilbert and Schreiber 1992; Hénon et al. 2002; Hogg et al. 2010; Karypis and Kumar 1995; Pothen and Sun 1993; Rennich et al. 2016; Schenk and Gärtner 2002; Zheng et al. 2015b]. Some techniques such as [George et al. 1989; Hénon et al. 2002; Pothen and Sun 1993] use the elimination tree for static scheduling while others use it for runtime scheduling.

9 CONCLUSION

In this paper, we present an automated approach for showing sparse code data dependences are unsatisfiable or if not reducing the complexity for later runtime analysis. Refuting a data dependence brings benefits to many areas of sparse matrix code analysis, including verification and loop optimizations such as parallelization, pipelining, or tiling by completely eliminating the high runtime costs of deploying runtime dependence checking. Additionally, when a dependence remains satisfiable, our approach of performing constraint instantiation within the context of the Integer Set Library (ISL) enables equalities and subset relationships to be derived that simplify the runtime complexity of inspectors for a case study with wavefront parallelism. Parallelization of these sparse numerical methods is an active research area today, but one where most current approaches require manual parallelization. It is also worth noting that without inspector complexity reduction, most inspectors would timeout, thus underscoring the pivotal role of the work in this paper in enabling parallelization and optimization of sparse codes. Our results are established over 71 dependences extracted from 8 sparse numerical methods.

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