Inequalities and Separations among Assisted Capacities of Quantum Channels

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We exhibit discrete memoryless quantum channels whose quantum capacity assisted by two-way classical communication, $Q_2$, exceeds their unassisted one-shot Holevo capacity $C_H$. These channels may be thought of as having a data input and output, along with a control input that partly influences, and a control output that partly reveals, which of a set of unitary evolutions the data undergoes en route from input to output. The channel is designed so that the data’s evolution can be exactly inferred by a classically coordinated processing of 1) the control output, and 2) a reference system entangled with the control input, but not from either of these resources alone. Thus a two-way classical side channel allows the otherwise noisy evolution of the data to be corrected, greatly increasing the capacity. The same family of channels provides examples where the classical capacity assisted by classical feedback, $C_B$, and the quantum capacity assisted by classical feedback $Q_B$, both exceed $C_H$. A related channel, whose data input undergoes dephasing before interacting with the control input, has a classical capacity $C = C_H$ strictly less than its $C_2$, the classical capacity assisted by independent classical communication.

1. The echo effect and retrocorrectable channels

Before introducing the class of channels that we will be using to prove capacity separations, we consider the underlying phenomenon in a simpler setting, that of a classic Bell-inequality experiment. In a typical such experiment, each member of a pair of polarization-entangled photons enters a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization. The output of the analyzer is two photons entering a separate analyzer which chooses randomly, and independently of the other analyzer, one of two nonorthogonal bases in which to measure the photon’s polarization.
a system entangled with the channel input (in this case the other photon of the entangled pair), but could not have been accurately inferred had the input instead been a known unentangled pure state. The fact that supplying an input photon of known polarization is worse than supplying one with an entangled partner to be measured later is an essential manifestation of entanglement and the violations of Bell’s inequality it gives rise to.

In our applications the externally echoed quantity, instead of being emitted as an output, is used internally within the channel to control the processing of another input. In more detail, we consider channels whose input and output spaces are each conceptually factorized into a control part and a data part. The channel performs a stochastic mapping of the data input variable onto a corresponding output variable, in a way that is partly influenced by the control input, and is partly revealed by the control output. The goal is to design the channel so that the stochastic mapping of the data variable can be corrected and made noiseless with the help of measurements on the control output and on a reference system entangled with the control input, but not by either resource alone. Two-way communication allows the measurements at the sending and receiving ends to be coordinated and exploited, thereby increasing the capacity above what could be achieved by any noninteractive protocol. Of course it is necessary to be sure that the control input and output do not, by their mere presence, increase the Holevo capacity so much as to neutralize the gains achieved by using them to correct the data variable’s stochastic evolution.

Specifically we consider a family of channels, for integers \( c, d \geq 2 \), which we call standard retrocorrectable channels and denote \( \mathcal{R}_{c,d} \). Their internal operation is depicted in Figure 1.

2. Holevo capacity and coherent information

Owing to the retrocorrectable channels’ high symmetry, their one-shot Holevo capacity can be calculated assuming a uniform distribution over the data input and an arbitrary fixed value of the control input. The control outputs do not contribute to the Holevo capacity because they are uncorrelated with the input. In the \( c = d = 2 \) (qubit) case

\[
C_H = 1 + (\pi^2/18 - 5/6)/\ln 2 = 0.5888
\]  

(1)

If we allow \( c \) to increase slightly superlinearly with \( d \) (e.g., as \( c \approx d \log^3 d \)), the randomization of the data becomes more efficient for larger \( d \), making \( C_H \) tend to zero as \( d \to \infty \). This follows the fact that under these conditions, 1) all but an asymptotically vanishing fraction of the probability distribution of outcomes \( j \) is contributed by outcomes of probability less than \( 1/(d \log^2 d) \), and 2) except for these improbable outcomes, the application of an unknown unitary from a known set of random unitaries on \( \mathcal{H}_d \) of cardinality \( d \log^2 d \) constitutes an asymptotically randomizing quantum channel in the sense of Hayden et al.

The one-shot coherent information is similarly maximized by choosing a uniform ensemble for the data variable and a fixed value for the control variable. Although the one-shot coherent information may be only a lower bound on the unassisted quantum capacity for this channel, the latter is in any case bounded above by the unassisted classical capacity \( C \), and so, if \( C_H \) is additive, by \( C_H \). The maximal one-shot coherent information is 0.4262 for \( c=d=2 \) and approaches zero for \( d \to \infty \), if, as before, \( c \) is allowed to increase slightly superlinearly with \( d \).

3. Assisted Capacities

By using the echo effect to retrospectively correct the otherwise noisy evolution of the data qubit, we can show that for appropriate choices of \( c \) and \( d \), the retrocorrectable channel’s channel’s assisted capacities, \( C_B, Q_2, C_2, Q_E \) and \( C_E \) all can be made to exceed the one-shot

FIG. 1: The standard retrocorrectable channel \( \mathcal{R}_{c,d} \) has a data input living in a \( d \)-dimensional Hilbert space, and a corresponding output; a control input living in a \( c \)-dimensional Hilbert space; and a classical control output living in a large Hilbert space. The channel internally selects a random basis \( B \), for \( \mathcal{H}_c \) and a set of \( c \) random unitaries \( \{U\} = U_1...U_c \) on \( \mathcal{H}_d \). The channel measures the control input in the basis \( B \), yielding result \( j \in \{1...c\} \), and according to that result applies one of the unitaries \( U_j \) to the data input, which is then emitted as the data output. The channel also emits a classical control output consisting of the random basis \( B \) and the set of random unitaries \( \{U\} = U_1...U_c \). It does not, however, emit the measurement result \( j \) but keeps it hidden, or, in another way of speaking, discards it into the channel’s inaccessible environment.
Holevo capacity $C_H$. Indeed, by allowing $c$ to increase slightly superlinearly with $d$ as in the previous section, all the assisted capacities can be made to increase linearly with $\log d$, while $C_H$ tends to zero.

Figures 2–5 show respectively how the channel $R_{2,2}$ can be used

- to transmit a faithful qubit in the presence of two-way communication;
- to generate a faithful ebit in the presence of classical back communication; and
- to transmit a faithful qubit without back communication, but consuming an ebit previously shared between sender and receiver.

The essential trick, shown in Figure 2, is to feed the control input half of a maximally entangled pair, whose other half she later measures in the basis $B^T$, after Bob has told her $B$ through a classical back channel. This measurement, yields, via the “echo effect,” the same outcome $j$ as occurred earlier within the channel. Alice tells Bob $j$ through a forward classical channel, after which he can undo the unitary transformation $U_j$ that the channel performed, restoring the data output qubit to the same state as it had initially.

Continuing to use $S$ to represent the standard retrocorrectable channel $R_{2,2}$ of Figure 1, we have the following reducibilities.

\begin{align}
1 \text{ ebit} & \leq S + \text{back communication} \\
1 \text{ qubit} & \leq S + 1 \text{ ebit} \\
2 \text{ cbit} & \leq 3 S + \text{back communication}
\end{align}

These constructions may be extended to variable $c$ and $d$, and, in the limit of large $d$ with $c$ increasing slightly
superlinearly with \(d\), we have the following:

\[
\lim_{d \to \infty} C_H = 0 \tag{4}
\]
\[
\lim_{d \to \infty} Q_2 = \log d \tag{5}
\]
\[
\lim_{d \to \infty} Q_B \geq \frac{1}{2} \log d \tag{6}
\]
\[
\lim_{d \to \infty} C_B \geq \frac{2}{3} \log d \tag{7}
\]

The last two expressions are lower bounds, because it is possible (though we have no evidence for it) that a higher \(Q_B\) or \(C_B\) might be achieved by some other protocol than ours.

4. Other similar channels

Other simpler channels with nontrivial echo effects can be constructed. For example, consider a simplified (2,2) channel that measures its control qubit in a random one of two fixed conjugate bases and, according to result, either does or doesn’t depolarize its data qubit input before emitting it as the data output. The control output is then a single classical bit indicating the measurement basis.

While the limited control information does not allow the data variable’s evolution to be corrected completely, it does allow depolarization events to be converted to less costly erasures, thereby creating a separation between \(Q_2\) and \(C_H\). The maximum \(Q_2\) of \(\cos^2 \frac{\pi}{8} = 0.85355\) is achieved by setting the control qubit midway between the \(|0\rangle\) eigenstates of the two conjugate bases, while the maximum \(C_H\) of \(\frac{1}{2} + \frac{1}{2}(1 - h_2(\frac{1}{2})) = 0.59436\) is obtained by setting it equal to the \(|0\rangle\) eigenstate of one of them. This channel is the simplest example of what might be called a partially retrocorrectable channel.

The channels considered so far are not entanglement-breaking. To obtain an echo effect in an entanglement-breaking channel, we modify the the standard (2,2) retrocorrectable channel \(S = R_{2,2}\) by unconditionally dephasing its data input in the computational basis before applying the conditional unitary. Being entanglement-breaking, this channel can have no quantum capacity without entanglement assistance \((Q = Q_B = Q_2 = 0)\), but it does have a \(C_2\) capacity of 1, strictly greater than its Holevo capacity of 0.5888. Since it is entanglement-breaking, its \(C_B\) capacity must also equal the Holevo capacity, by the argument of Bowen and Nagarajan. Therefore this channel definitely (without any assumptions about additivity of \(C_B\)) violates the second part of the equality \(C = C_B = C_2\) obeyed by all classical DMCs. (Fig. 5) showing a double hierarchy with classical capacities \(C \leq C_B \leq C_2 \leq C_E\) on one side and corresponding quantum capacities \(Q \leq Q_B \leq Q_2 \leq Q_E\) on the other. Each quantum capacity is upper bounded by its corresponding classical capacity, and in every case but \(Q_E\) vs \(C_E\), the inequality can be saturated. Here \(C_2\) denotes a channel’s classical capacity when assisted by an arbitrary classical two-way side conversation, subject only to the limitation that, taken as a whole, the side conversation be independent of the message being transmitted through the main protocol. With this restriction it is easy to show that \(C_2 = C\) for any classical DMC, whereas without it the side conversation would become a short circuit making \(C_2\) trivially infinite. It has not been customary to impose a similar independence restriction in the definition of \(Q_2\), where no short circuit problem exists. But in fact, without loss of generality, the classical side conversation in \(Q_2\) can also be required to be independent of the (quantum) message being transmitted through the main protocol, because if it were not independent, it could be used as a means of eavesdropping on the quantum message without disturbing it. From another viewpoint, \(C_2\) represents a channel’s private classical capacity when assisted by two-way public communication, the adversary being given access to the side conversation but not the channel environment (cf \(S\)).

In passing, we note that attempting to define capacities such as \(Q_E\) which would allow unlimited amounts of both shared entanglement and bidirectional classical communication, leads to a more serious short circuit problem, because independence does not prevent the assistive resources from being used for teleportation, making the capacity infinite.

Returning to Fig. 5, the general goal is to determine, for every pair of capacities, whether they are related by

- a strict inequality, as in the case of \(Q_E < C_E\), with the inequality being saturated only trivially when both sides vanish;
- a saturable inequality as in \(Q \leq C\); or
- an incomparability as between \(Q_E\) and \(C\), in which, depending on the channel, either side may be greater.

The former two relations are indicated by a solid line in the ladder diagram, with the greater quantity being higher. Incomparability is indicated by a dashed line.

For each of the solid lines in Fig. 5, we need to find a proof of the general inequality and examples showing both equality and separation. Referring to the notes \(a\) through \(h\) in the diagram,

\(a\): The general inequality \(\leq\) can be shown by an argument involving monotonicity of the conditional mutual information (cf \(\mathcal{I}\)). Equality is witnessed by the classical bit channel (i.e., a 100% dephasing qubit channel),
for which it is easy to show that both capacities equal 1. Separation is witnessed by the noiseless qubit channel where we can show that $C_2 = 1$ but by superdense coding $C_E = 2$.

**b:** The general inequality follows from the fact that any protocol that achieves $C_B$ can be modified to decouple the back communication from the message (cf [2]). Equality is witnessed by the classical bit channel, inequality by the dephased retrocorrectable channel of the previous section, for which $C_2 = 1$ but $C_B = C = C_H < 1$.

**c:** The general inequality is obvious. Equality holds for the classical bit channel, separation for the standard retrocorrectable channel (assuming $C = C_H$).

**d:** The general inequality is obvious. Equality holds for the qubit channel, separation for appropriate high dimensional retrocorrectable channels.

**e:** The general inequality is obvious. Equality holds for the qubit channel. We suspect but do not know how to prove that $Q_B < Q_2$ for channels such as $R_{2,2}$. This separation would be implied by incomparability $k$ which we also don’t know how to prove.

**f:** The general inequality can be proved by a monotonicity argument similar to note $a$ above. Equality is witnessed by the qubit channel, separation by channels such as the strongly depolarizing channel, for which $Q_2$ is zero but $C$ and hence $Q_E$ are positive.

**g:** These capacities are related by a constant factor of 2, making inequality strict unless both capacities vanish.

**h:** The general inequalities are obvious, from the fact that a qubit channel can simulate a bit channel. Separations are witnessed by the classical bit channel.

Turning now to the incomparabilities in Fig. 5,

**i:** $C_2 < Q_E$ may be witnessed by the 2/3 depolarizing channel. This channel is known to have $Q_E > C_H$, and because it is unital, $C = C_H$. If we can show that $C_2 = C_H$ for this channel then we have a separation. The other separation $C_2 > Q_E$ is witnessed by the classical bit channel.

**j:** $C < Q_E$ is witnessed by $R_{2,2}$ retrocorrectable channel (assuming additivity of $C_H$). $C > Q_E$ is witnessed by classical bit channel.

**k:** $C_B > Q_2$ is witnessed by classical channel for which $Q_2 = 0$. We conjecture that the retrocorrectable channel $R_{2,2}$, with $Q_2 = 1$ and $C_B \geq 1/2$ witnesses the inequality in the other direction, but we do not have a nontrivial upper bound on $C_B$ for this channel.

**l:** $C < Q_B$ holds for high dimensional retrocorrectable channels, assuming additivity of $C_H$. In the other direction, $C > Q_B$ is witnessed by the classical bit channel, for which $Q_2 = Q_B = 0$.

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