EXPLICIT HOLOMORPHIC STRUCTURES FOR EMBEDDINGS
OF CLOSED 3-MANIFOLDS INTO $\mathbb{C}^3$

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Abstract. Expanding on my former work along with the more recent work
of Kasuya and Takase, we demonstrate that for a given link $L \subset M$ which is
null-homologous in $H_1(M)$ and for any smooth oriented 2-plane field $\eta$ over
$L$ there exists a smooth embedding $F : M \rightarrow \mathbb{C}^3$ so that the set of complex
tangents to the embedding is exactly $L$ and at each $x \in L$ the holomorphic
tangent space is exactly $\eta_x$. Furthermore, we demonstrate how the analyticity
of a complex tangent, as given by the Bishop invariant, may be determined
exactly from the angle formed between the holomorphic complex line and the
the curve of complex tangents.

I. Introduction

Complex tangents to an embedding $M^k \rightarrow \mathbb{C}^n$ are points $x \in M$ so that the
tangent space to $M$ at $x$ (considered as a subspace of the tangent space of $\mathbb{C}^n$)
contains a complex line. If $k > n$, all points of $M$ are necessarily complex tangent,
by virtue of the dimensions. If $k \leq n$, some (or all) points of $M$ may have strictly
real tangent space. If all points of $M$ are real, we say that the embedding is totally
real. In general, the dimension of the maximal complex tangent space of $M$ at
$x$ is called the dimension of the complex tangent. An embedding is called CR
(Cauchy-Riemann) if the complex dimension of all points of $M$ are the same. In
[4], M. Gromov used the h-principle to demonstrate that the only spheres admitting
totally real embeddings $S^n \rightarrow \mathbb{C}^n$ are $S^1$ and $S^3$.

In his paper [3], F. Forstneric showed that every closed oriented 3-manifold can
be embedded totally real into $\mathbb{C}^3$. In my paper [2], I demonstrate a lack of such
flexibility in higher dimensions, particularly we show that the 5-sphere $S^5$ cannot
be CR-embedded into $\mathbb{C}^4$. We also show analogous results for different dimensions.

In this work, we will focus on embeddings of closed 3-manifolds $M \rightarrow \mathbb{C}^3$. In this
situation, all complex tangents are necessarily of dimension 1 and generically arise
along a link. In our paper [1], we demonstrated that for every knot (or link) type
in $S^3$ there exists an embedding $S^3 \rightarrow \mathbb{C}^3$ which assumes complex tangents exactly
along a link of the prescribed type (with one point being degenerate). In 2018,
Kasuya and Takase proved that given any smooth link $L$ in a closed 3-manifold
$M$ with trivial fundamental class $[L] \in H_1(M)$, there exists a smooth embedding
$M \rightarrow \mathbb{C}^3$ whose complex tangents form precisely the given link $L$ and are all non-
degenerate (see [5]).

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We will here generalize the result of Kasuya and Takase to assert that given any smooth orientable 2-plane field over such a link \( L \), there exists an embedding whose complex tangents have holomorphic tangent spaces being precisely the specified 2-plane field. In effect, we show that any potential configuration of complex structure to an embedding of a 3-manifold can actually arise to some embedding; the only restriction is that the set of complex tangents must be null-homologous.

II. Known Results

The result of Kasuya-Takase in [5] demonstrates that given a closed 3-manifold \( M \) and a null-homologous link \( L \subset M \), we may construct a smooth embedding \( M \hookrightarrow \mathbb{C}^3 \) which is complex tangent precisely along \( L \). This is in contrast with my earlier result in [1], where I show that any topological type of link may arise as the set of complex tangents to an embedding \( S^3 \hookrightarrow \mathbb{C}^3 \), with the caveat that the embedding is only \( C^n \) \((n < \infty)\) and must have one complex degenerate point.

We now state the result of Kasuya-Takase formally:

**Theorem 1.** ([5]): Let \( M \) be a closed orientable 3-manifold and \( L \subset M \) a closed 1-dimensional submanifold. Then there exists a smooth embedding \( M \hookrightarrow \mathbb{C}^3 \) complex tangent exactly along \( L \) if and only if \( [L] \in H_1(M) \) is trivial.

Their method of proof involves using stable maps into \( \mathbb{R}^2 \) and immersion lifts. In particular, they first demonstrate that given any such null-homologous link \( L \subset M \), there exists a liftable stable map \( f : M \rightarrow \mathbb{R}^2 \) which is singular exactly along the link \( L \). Writing the immersion lift as \((f, g) : M \rightarrow \mathbb{R}^4\), we may assume that the immersion lift is an embedding in a tubular neighborhood of the link (perturbing \( g \) if necessary).

Then writing \( f = (f_1, f_2) \) and \( g = (f_3, f_4) \), the following smooth immersion is constructed: \( G = (f_1, f_2, f_3, f_4, f_1, -f_2) : M \rightarrow \mathbb{C}^3 \) whose complex tangents they show forms precisely the link \( L \).

With the fact that the immersion is an embedding in a tubular neighborhood of \( L \) and using the relative form of the \( h \)-principle, the existence of a smooth embedding \( M \hookrightarrow \mathbb{C}^3 \) complex tangent exactly along \( L \) is assured, and the theorem is proven.

III. Specifying Complex Configurations

Let \( M \) be a closed 3-manifold and let \( L \subset M \) be a link, null-homologous in \( M \). By the work of Kasuya and Takase, there exists a smooth embedding \( F : M \hookrightarrow \mathbb{C}^3 \) so that the complex tangents of the embedding form exactly \( L \). Let \( \tau_x \subset T_x M \) denote the holomorphic tangent space at \( x \in L \) to this embedding. We wish to show that for any oriented 2-plane field \( \eta \in \Gamma(Gr_2(TM|_L)) \) (section of the Grassmann
that the matricies \( m \) will have columns being the component derivatives (columns of \( \mu \) using the component functions and the derivative will be the Jacobian which itself

is spanned (over complex numbers!) by \( \mu \) at each point). Note that this adds six equation conditions, one for each of the component functions individually and forming \( \mu \) satisfies these conditions. Denote these spaces by:

\[ S_x = \{ m \in GL_6(\mathbb{R}) | m \cdot \tau_x = \eta_x \} \subset GL_6(\mathbb{R}) \]

Recall then the Whitney Extension Theorem from real analysis which states that for a compact set \( K \subset \mathbb{R}^n \) and vector field along \( K \), any function \( f : K \to \mathbb{R} \) can be extended to a function \( F : \mathbb{R}^n \to \mathbb{R} \) so that \( F|_K = f \) and the derivative of \( F \) at the points of \( K \) form exactly the prescribed vector field. Note that the conditions for applying the Whitney Extension Theorem with \( K = \partial L \) are satisfied trivially as \( \dim(L) = 1 \) (see [6]). Except that since \( L \equiv S^1 \) (or a link of such) is closed, the appropriate consistency condition must satisfied. We will explicitly describe this condition for the function \( f = F|_L \):

\[ \oint_L f(s) \cdot T \, ds = 0, \] that is the integral of the derivatives in the direction of the link (component) about the link must be 0 (T is the unit tangent to the link at each point). Note that this adds six equation conditions, one for each of the components in the ambient space \( \mathbb{R}^6 \).

A generalization of the Whitney’s Theorem could be developed a given function \( f : K \to \mathbb{R}^n \) and prescribed derivatives at each \( x \in K \) (now \( (n \times n) \)-matrices \( m_x \)) there exists an extension \( F : \mathbb{R}^n \to \mathbb{R}^n \) so that: \( F|_K = f \) and \( df|_x = m_x \). This is seen by considering each of the component functions individually and forming \( F \) using the component functions and the derivative will be the Jacobian which itself will have columns being the component derivatives (columns of \( m_x \)).

In our setting, let \( K = \partial L \) and let \( f \) be the identity on \( L \). We can then construct a smooth function \( E : \mathbb{C}^3 \to \mathbb{C}^3 \) so that \( dE_x = m_x \), where \( m_x \in S_x \) for each \( x \in L \). Since the respective 2-plane fields vary smoothly in \( x \) by construction, it is clear that the matricies \( m_x \) will also vary smoothly in \( x \).
We will further need the condition that the matrices $m_x$ commute with the complex structure $J$ for each $x \in L$. Note the complex structure $J$ takes the form of a 6x6-matrix over $\mathbb{R}^6(\cong \mathbb{C}^3)$ as follows:

$$J = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Let $x \in L$ represent a complex tangent as in our previous construction. We have a 24-dimensional space of matrices $S_x \subset GL_6(\mathbb{R})$ which satisfy the conditions as we gave before. We now want to further impose the condition that $m_x$ commutes with the complex structure $J$. Namely we have 18 more real equations on $m_x$ given by the entries of the matrix equation:

$$J \cdot m_x = m_x \cdot J$$

More precisely we will have that:

$$\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

$$\times$$

$$\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{pmatrix}$$

$$= \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{pmatrix} \times
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

We can directly see that the only equations that will need to be satisfied are determined by the skew-symmetric behavior: $a_{ij} = -a_{ji}$ for all $i \neq j, 1 \leq i, j \leq 6$. This will give 15 equations resulting from the choice of two from the six rows, i.e. $C_2^6$ combinations of two indices from 6. In addition, we will have three more equations resulting from the necessary equal diagonal elements for each of the "complex pairs":

$$a_{11} = a_{22}, a_{33} = a_{44}, a_{55} = a_{66}$$

to give a total of 18 equations on the 24-dimensional space $S_x$ for each complex tangent $x \in \eta$. Let us now write (without loss) $S_x$ to be the 6 dimensional space of matrices which satisfy all the earlier conditions, and also commute with the complex structure $J$.

We will now construct a new embedding $\tilde{F} = E \circ F : M \to \mathbb{C}^3$ which will be a composition of smooth functions and hence itself smooth:
Claim: The sets \( m_x(T_x M \cap J(T_x M)) = T_x(\tilde{F}(M)) \cap J(T_x(\tilde{F}(M))) \) for every \( x \in L \).

First, we note that since \( d\tilde{F}|_x = m_x \cdot dF_x \), we have \( T_x(\tilde{F}(M)) = m_x T_x M \) for each \( x \in L \). By dimensionality arguments, it will suffice to show:

\[ m_x(T_x M \cap J(T_x M)) \subset m_x T_x M \cap J(m_x T_x M). \]

Let \( \varphi \in T_x M \cap J(T_x M) = \tau \). Clearly, \( \varphi \in T_x M \). So it remains to show that \( m_x \varphi \in J(m_x T_x M) \).

Since \( m_x \in S_x \), it will commute with the complex structure \( J \) by constructio, and hence \( J(m_x \cdot T_x(M)) = m_x \cdot J(T_x M) \) we see that \( m_x(T_x M \cdot J(T_x M)) \subset m_x T_x M \cap J(m_x T_x M) \) and by linear algebra dimensionality reasons, the sets must be equal, and the claim is proven.

As a direct consequence of this claim, it is clear that the link \( L \) will be complex tangent for any such embedding \( \tilde{F} : M \hookrightarrow \mathbb{C}^3 \) and the holomorphic tangent space at \( x \in L \) is \( \eta_x \).

It remains for us to prove that we can find a family \( \{ m_x \in S_x \} \), \( x \in L \) so that the resulting embedding \( \tilde{F} \) will not have any new complex tangents off of \( L \). Note that it will suffice to prove this is true for a tubular neighborhood of \( N \subset M \) of \( L \) and then proceed using Gromov’s h-principle for extensions.

Consider the set: \( S = \cup S_x \subset GL_6(\mathbb{R}) \) over each \( x \in L \). This forms a fiber bundle over \( L \) with fibers \( S_x \) over each \( x \in L \). Consider further the set of sections to this bundle \( \Gamma = \{ s : L \to S \} \). By our work above, for each section \( s \) there exists a smooth automorphism \( E : \mathbb{C}^3 \to \mathbb{C}^3 \) so that the function \( \tilde{f}_s = E_s \circ f \) has complex tangents along \( L \) with holomorphic tangent space exactly the desired complex line field \( \eta \). Also, let \( N \) be a tubular neighborhood of \( L \subset M \).

Consider now the (collection of) Gauss map: \( \mathcal{G} : N \times \Gamma \to Gr_{3,6} \) that sends \((x, s)\) to the tangent space of the embedding \( \tilde{f}_s \) at \( x \). We wish to show that \( \mathcal{G} \) is transverse to \( \mathbb{W} = \{ P \subset Gr_{3,6} | P \cap JP \cong \mathbb{C} \} \), the set of planes which contain a complex line.

It is easily seen that every direction \( v \in \mathbb{R}^6 \) can arise in the image of the differential \( dE^*_{x,v} = m_x \) for some \( s \in \Gamma \), for any given \( x \in L \). Hence, we see that the differential \( d\mathcal{G} \) is onto and \( \mathcal{G} \) is necessarily transverse to \( \mathbb{W} \).

By parametric transversality theorem, for almost any \( s \in \Gamma \), the map \( \tilde{f}_s \) will be generic, in that it is either totally real or assumes complex tangents along a link. As \( \tilde{f} \) assumes complex tangents along \( L \), we can take \( N \) to be "sufficiently thin" so that no other knots will be obtained in the transverse intersection. As the intersection of the Gauss map with \( \mathbb{W} \) is transverse for such a generic \( s \), the complex tangents are all non-degenerate and will form exactly the link \( L \). We may then use the totally real h-principle for extensions to construct a smooth embedding \( M \hookrightarrow \mathbb{C}^3 \) which is (non-degenerate) complex tangent along \( L \), and totally real off
of $L$, with holomorphic tangent spaces being the arbitrary 2-plane field $\eta$ along $L$, and our theorem is proven.

**QED**

We believe that this result demonstrates the full (topological) flexibility of real embeddings into complex Euclidean space in the dimension 3. It may also be interesting to now ask if the same kind of flexibility will extend to the structure of the holomorphic hulls for real 3-dimensional submanifolds of complex space.

**IV. A Topological Formulation for Analyticity**

The Bishop Invariant assigns to each (non-degenerate) one-dimensional complex tangent $m \in M$ a number $\gamma_m \in [0, \infty]$. Errett Bishop formulated the invariant in the 1950’s (see [1]) for any embedding of a closed real $n$-manifold $M \hookrightarrow \mathbb{C}^n$ using a certain normal form for $M$ at a one-dimensional complex tangent $m$. This invariant gives rise to the concept of the **analyticity** of the complex tangent, and how the manifold may be considered to be (locally) elliptic, hyperbolic, or parabolic near the given complex tangent. It also has implications regarding the holomorphic hull of $M$. The definition is given via Bishop’s normal form, which in our case in dimension 3 is given in a local holomorphic coordinate system $\{z_1, z_2, z_3\}$ as follows:

$$
M : R = (r_1, r_2, r_3) = 0
$$

$$
R = \bar{R}
$$

$$
dr_1 \wedge dr_2 \wedge dr_3 \neq 0
$$

$$
\partial r_1 \wedge \partial r_2 \wedge \partial r_3 = Bdz_1 \wedge dz_2 \wedge dz_3
$$

It is direct to see that $B$ is a "volume form" type function and is given by:

$$
B = \frac{\partial(r_1, r_2, r_3)}{\partial(z_1, z_2, z_3)} = \frac{1}{2} \frac{\partial F}{\partial z_1} + O(2)
$$

It is then clear that the set of points $\eta = \{ B = 0, R = 0 \}$ form exactly the complex tangents of the embedding. For $m \in \eta$, and $H_m = T_m M \cap JT_m M$ the holomorphic tangent space, we can write:

$$
\mathbb{C} \otimes H_m = H'_m \oplus H''_m, H''_m = \bar{H}'_m.
$$

We then need to choose a vector field $X$ which spans $H'_m$. In our situation the $z = z_1$ coordinate represents exactly the holomorphic line $H'_m$, and we may choose $X = \frac{\partial}{\partial z}.$

The Bishop Invariant in this situation may then be defined as follows:

$$
\gamma_m = \frac{1}{2} \left| \frac{\partial F}{\partial z} \right| (m). \text{ We say that } m \in \eta \text{ is degenerate if } B_z = B_{\bar{z}} = 0 \text{ at } m \text{ and to be non-degenerate otherwise. From [7], we find that if } m \in \eta \text{ is non-degenerate then } \eta \text{ has a tangent line at } m.
$$
The holomorphic tangent space $H_m$ at a non-degenerate complex tangent $m$ is given exactly by the complex variable $z = z_1$. The tangent space $T_m(M)$ may then be written as the direct sum of the plane $H_m$ and a real line, which we take to form a new real axis which we may designate as the $x$-axis, and now the tangent space $T_m(M) = \{(z, x)| z \in \mathbb{C}, x \in \mathbb{R}\}$.

Consider now the normal bundle $S$ of $\eta \subset M$ and let $S^*$ denote its conormal bundle. It is direct to see that $S^*$ will be spanned by the coframe $dB$ along $\eta$.

Consider then the composite function:

$$\varphi : H \hookrightarrow T(M)|_\eta \to (T(M)|_\eta)/T(\eta) \equiv S.$$ 

In reference to Webster in his work in [7], we say that $m \in \eta$ is elliptic if $\varphi_m$ is orientation-reversing. This will be true if and only if $\gamma_m < \frac{1}{2}$.

We may also say $m$ is hyperbolic if $\varphi_m$ is orientation-preserving. This will be true if and only if $\gamma_m > \frac{1}{2}$.

We may say $m$ is parabolic if $\varphi_m$ is singular of rank 1. This is equivalent to the situation where $\gamma_m = \frac{1}{2}$.

It is then clear that $m$ is parabolic precisely when $T_m(\eta) \subset H_m$. It is also evident that if the tangent line $T(\eta)$ is oriented to be "above" the holomorphic space $H_m$ in agreement with the orientation of the positive normal of $H_m$ then $m$ will be a hyperbolic complex tangent. Analogously, if $T(\eta)$ is oriented "below" and in disagreement with the orientation of $H_m$, then $m$ will be an elliptic complex tangent.

If $T(\eta) \perp H_m$, we say that $m$ is an exceptional complex tangent. In accordance with the definition of the Bishop invariant, the condition that $\gamma = 0$ will be the extreme case for ellipticity and will satisfy that $T(\eta) = (H_m)^\perp$ with opposite orientation to that induced by $H_m$. Analogously, $\gamma = \infty$ will correspond to the extreme hyperbolic case $T(\eta) = (H_m)^\perp$ in agreement with orientation induced by $H_m$.

From our previous work in Section 3, we are able to construct an embedding with the set of complex tangents along any curve $\gamma \subset M$ with the holomorphic planes along any 2-plane field $\xi \subset Gr_2(M)|_\eta$ and as such we can control the flow of the tangents of $\eta$ along the planes $\xi$ and by the above we can form the embedding to have any "Bishop structure" along the complex tangents as we so choose.

We may then consider the angle $\theta_m$ formed by the unit normal $\rho_m$ to $H_m \subset T_m(M)$ (using the natural orientation of induced by the complex structure) and $t_m \in T_m(N)$ representing the unit positive tangent to the curve of complex tangents $\eta$ at $m$. We then summarize our work above in the classification of the complex tangents of the embedding in terms the angles $\theta_m = \angle(t_m, \rho_m)$ as follows:
Theorem 3. ([5]): Let $M$ be a closed orientable 3-manifold and $L \subset M$ a closed 1-dimensional submanifold, and let $\eta$ be a smooth oriented 2-plane field over $L \subset M$. Then there exists a smooth embedding $M \hookrightarrow \mathbb{C}^3$ non-degenerate complex tangent exactly along $L$ with holomorphic tangent space $\eta_x$ for each $x \in L$. Furthermore, for each $m \in L$ the analyticity of $m$ may be (pre-) determined by the angle $\theta_m = \angle(t_m, \rho_m)$ between the complex line and the curve of complex tangents as follows:

1. If $\theta_m \in [0, \pi/2)$ then $m$ is a hyperbolic complex tangent.
2. If $\theta_m \in (\pi/2, \pi]$ then $m$ is an elliptic complex tangent.
3. If $\theta_m = \pi/2$ then $m$ is a parabolic complex tangent.

If $\theta_m \in \{0, \pi\}$ then $m$ is an exceptional complex tangent, of elliptic type if $\theta_m = \pi$ and of hyperbolic type if $\theta_m = 0$.

We also note that it could be of interest to find a new formula for the Bishop invariant in terms of the angle $\theta_m$ amongst the set of complex tangents $m \in N$. We will leave this work for another paper as it would need an analytic sophistication that we do not make use of in our current work, which has been in the spirit of complex differential topology in nature.

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