Every orientation of a 4-chromatic graph has a non-bipartite acyclic subgraph

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Abstract

Let \( f(n) \) denote the smallest integer such that every directed graph with chromatic number larger than \( f(n) \) contains an acyclic subgraph with chromatic number larger than \( n \). The problem of bounding this function was introduced by Addario-Berry et al., who noted that \( f(n) \leq n^2 \). The only improvement over this bound was obtained by Nassar and Yuster, who proved that \( f(2) = 3 \) using a deep theorem of Thomassen. Yuster asked if this result can be proved using elementary methods. In this short note we provide such a proof.

Mathematics Subject Classifications: 05C15, 05C20

1 Introduction

The relation between the chromatic number of a graph and properties of orientations of its edges have long been investigated. For the sake of brevity, we refer the reader to [4] for a general survey on this topic, and to the discussions in [3, 6], which are more closely related to our investigation here.

We consider the following problem introduced by Addario-Berry, Havet, Sales, Reed and Thomassé; given an integer \( n \), what is the smallest integer \( f(n) \) so that if \( G \) has chromatic number more than \( f(n) \) then in every orientation of \( G \)'s edges, one can find an acyclic subgraph of chromatic number more than \( n \). The best known general upper bound for this function is \( f(n) \leq n^2 \). This follows from taking any oriented version of \( G \), splitting it into two acyclic subgraphs, denoted \( G_1, G_2 \), and applying the well known

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The fact that the chromatic number of $G$ is at most the product of the chromatic numbers of $G_1, G_2$. The only known improvement over this general bound was obtained by Nassar and Yuster [6] who proved that $f(2) = 3$, by establishing the following.

**Theorem 1** (Nassar–Yuster [6]). Suppose $G$ is a graph of chromatic number 4. Then every orientation of its edges contains an acyclic odd cycle.

The proof in [6] relied on a deep theorem of Thomassen [7], which confirmed a conjecture of Toft [8]. Yuster [9] asked if one can prove Theorem 1 using elementary methods. In this short paper we give such a proof. The main idea is to take advantage of properties of 4-critical graphs.

## 2 An elementary proof of Theorem 1

We may and will assume that $G$ is 4-critical, that is, that the removal of every edge of $G$ reduces its chromatic number. This will allow us to use important properties of 4-critical graphs. We proceed by induction on $|V(G)|$, with the base case being $K_4$. It is easy to see that every orientation of $K_4$ contains an acyclic $K_3$ (in fact, two) so the base case holds. We now proceed with the induction step. We consider separately the case where $G$ is 3-connected (in which case we will not need induction) and the case where it has a separating pair of vertices.

Assume first that $G$ is 3-connected, and let $C$ be a shortest odd cycle in $G$. Since $C$ must be induced and $G$ has chromatic number 4, there must be a vertex $v \notin C$. Since $G$ is assumed to be 3-connected, there are 3 vertex disjoint paths connecting $v$ to $C$. Let $P, Q, R$ denote these paths, and $p, q, r$ denote their meeting points with $C$, see Figure 1a. If $C$ is acyclic we are done, so suppose wlog that $C$ is oriented as in Figure 1a. Clearly not all three paths $P \cup Q$, $P \cup R$ and $Q \cup R$ can be directed paths, as they all intersect internally in the vertex $v$. Assume wlog that $P \cup Q$ is not directed. Then, since $|C|$ is odd, one of the cycles $P \cup Q \cup P_{pq}$ or $P \cup Q \cup P_{rp} \cup P_{qr}$ is an acyclic odd cycle.

![Figure 1: The two cases considered in the proof](image-url)
Suppose now that $G$ is not 3-connected, that is, it has a pair of vertices $u, v$ whose removal breaks it into at least two (non-empty) connected components. In what follows, if $(u, v) \notin E(G)$ then we use $G + (u, v)$ to denote the graph obtained by adding the edge $(u, v)$ to $G$. We use $G/\{u, v\}$ to denote the graph obtained from $G$ by contracting $u, v$, that is, the graph obtained by replacing $u, v$ with a new vertex and connecting it to all the vertices that were connected to either $u$ or $v$. We will need the following well known result of Dirac [2], see also Problem 9.22 in [5] for a short proof.

Lemma 2 (Dirac [2]). Let $k \geq 4$ be an integer, let $G$ be a $k$-critical graph, and let $u, v \in V(G)$ be such that $G \setminus \{u, v\}$ (the graph obtained from $G$ by deleting the vertices $u, v$) is disconnected. Then:

1. $u \neq v$, that is, $G$ is 2-connected;
2. $(u, v) \notin E(G)$;
3. $G \setminus \{u, v\}$ has exactly two components;
4. There are unique proper induced subgraphs $G_1, G_2$ of $G$ such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u, v\}$, and the graphs $G_1 \setminus \{u, v\}$ and $G_2 \setminus \{u, v\}$ are the two components of $G \setminus \{u, v\}$. Also, $u, v$ have no common neighbor in $G_2$, and $G_1 + (u, v)$ and $G_2/\{u, v\}$ are $k$-critical.

By induction and Lemma 2, the graph $G_2/\{u, v\}$ has an acyclic odd cycle $C_2$. If $C_2$ does not contain the vertex $w$ that resulted from contracting $\{u, v\}$, it is also a cycle in $G$ and we are done. Also, if the two neighbors of $w$ on $C_2$ are both neighbors of $v$ or both neighbors of $u$, then we can again conclude that $C_2$ is also an acyclic odd cycle in $G$. So assume one neighbor of $w$ is a neighbor of $v$ and one is a neighbor of $u$. Then we may infer that in $G$ we have a path $P_2$ connecting $u$ and $v$, so that $|P|$ is even and $P$ is not directed from $u$ to $v$ or from $v$ to $u$. See Figure 1b.

By induction and Lemma 2, the graph $G_1 + (u, v)$ has an acyclic odd cycle $C_1$ (no matter how we orient the edge $(u, v)$). If $C_1$ does not use the edge $(u, v)$, it is also an acyclic odd cycle in $G$ and we are done. Suppose then that it does, implying that $G$ contains a path $P_1$ connecting $u$ to $v$ with $|P_1|$ odd. Then item (4) in Lemma 2 guarantees that $|P_1 \cup P_2| = |P_1| + |P_2| - 2$ so $P_1 \cup P_2$ is an odd cycle. The assertion at the end of the previous paragraph guarantees that it is acyclic. This completes the proof of Theorem 1.

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\footnote{We will only apply this operation when $u, v$ are not connected and have no common neighbor, so this operation will not create loops or parallel edges.}
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