VOLUME RIGIDITY OF PRINCIPAL CIRCLE BUNDLES OVER THE COMPLEX PROJECTIVE SPACE

PAUL W.Y. LEE

Abstract. In this paper, we prove that principal circle bundles over the complex projective space equipped with the standard Sasakian structures are volume rigid among all $K$-contact manifolds satisfying positivity conditions of tensors involving the Tanaka-Webster curvature.

1. Introduction

One of the main consequences of the classical Bishop-Gromov inequality states that a Riemannian manifold with the Ricci curvature bounded below by a positive constant has its volume controlled by the corresponding model which is the sphere of constant curvature. Moreover, equality of the volumes holds only if the manifold is isometric to the model. This result is also the starting point of the theory of almost rigidity (see [10] and reference therein).

Recently, there is a surge of interest in extending various comparison type results to the sub-Riemannian setting (see [1, 2, 3, 4, 18, 16, 17, 5]). However, extension of the above rigidity result to the sub-Riemannian setting seems to be missing. One of the purposes of this paper is to provide such a result for $K$-contact manifolds.

Recall that a contact metric manifold is a contact manifold $M$ equipped with a Riemannian metric $(\cdot, \cdot)$ and $(1, 1)$-tensor $J$ (which is a complex structure defined on the contact distribution) satisfying some compatibility conditions (see Section 2 for more details). It is $K$-contact if the Reeb vector field $\xi$ is Killing.

An example of a $K$-contact metric manifold is given by the Hopf fibration which is a principal bundle $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$. There is a $K$-contact structure which make $S^{2n+1}$ a Sasakian manifold. Moreover, the quotient $\mathbb{C}P^n$ inherits from this Sasakian structure a Kähler structure which a constant multiple of the standard one given by the Fubini-Study metric. We denote the total space and the base of the

Date: October 21, 2018.
Hopf fibration equipped with this structure by $\mathcal{P}(k,1)$ and $\mathcal{B}(k,1)$, respectively. Here $k$ is the number given by $\langle Rm(JX, X)X, JX \rangle = k^2$, where $Rm$ is the Riemann curvature tensor of $\mathcal{B}(k,1)$ and $X$ is any unit tangent vector. Finally, let $\mathcal{P}(k,m) = \mathcal{P}(k,1)/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts on $\mathcal{P}(k,1)$ by a discrete subgroup of $S^1$ of order $m$. These are all the model spaces of our volume rigidity results. (See Section 3 for more details.)

In order to state the main result, we also need the Tanaka-Webster curvature $Rm$ which is the curvature of a connection defined in the CR case in [28] and more general contact case in [29]. The tensor $N$ is defined by $N(X) = P(\nabla_X J)X$, where $P$ is the orthogonal projection onto the contact distribution and $\nabla$ is the Levi-Civita connection.

**Theorem 1.1.** Assume that the above assumptions hold and that the following curvature conditions are satisfied:

1. $\langle Rm(JX, X)X, JX \rangle - |N(X)|^2 \geq k^2$ for all unit tangent vectors $X$,
2. $\text{tr}_{(X,JX)}(Y \mapsto \langle Rm(Y, X)X, Y \rangle) - |N(X)|^2 \geq \frac{(2n-2)k^2}{4}$ for all unit tangent vectors $X$.

Then

$$m := \frac{\text{len}(\bar{C})}{\text{len}(C)} \leq \frac{\text{vol}(\mathcal{P}(k,1))}{\text{vol}(M)},$$

where $C$ and $\bar{C}$ are closed orbits of the Reeb fields on $M$ and $\mathcal{P}(k,1)$, respectively. Moreover, if equality holds, then $m$ is a positive integer and $M$ is isometrically contactomorphic to $\mathcal{P}(k,m)$.

Here isometrically contactomorphic means there is an isometry between the two spaces which is also a contactomorphism. Remark that the curvature conditions appeared earlier in a related work of the author [18].

We also remark that the existence of a closed orbit in a compact contact manifold is a long standing open problem called the Weinstein conjecture [30] (see [6], Chapter 2 for a brief discussion and numerous references). In the case of $K$-contact manifolds, there are in fact multiple closed orbits (see [26] and references therein).

A symplectic analogue of the above result follows immediately by considering the Boothby-Wang fibrations [7]. First, we recall that a contact manifold is regular if the flow of the corresponding Reeb vector field $\xi$ is regular. On a compact manifold, it means that each orbit of $\xi$ is closed. Assume that the contact manifold is regular. A result in [7] shows that one can multiply a positive function $f$ to the Reeb field $\xi$ such that $M$ is the total space of a principal circle bundle $\pi$:
M \to N$, called a Boothby-Wang fibration, with action defined by the flow of $f\xi$. Moreover, the base space $N$ of this bundle is an integral symplectic manifold. If $\omega$ is the symplectic form on $N$, then $\pi^*\omega$ is the exterior derivative of the new contact form $\eta$. Conversely, a result \cite{15} shows that any integral symplectic manifold is the base of a Boothby-Wang fibration. If $J_N$ defines an almost complex structure with a compatible Riemannian metric $\langle \cdot, \cdot \rangle^N$, then the lifts of these structures to $M$ together with $\eta$ define a $K$-contact manifold (see \cite{6} and references therein for further details).

Let $M$ be an integral symplectic manifold equipped with an almost complex structure $J$ and a compatible Riemannian metric $\langle \cdot, \cdot \rangle$. The following result is a consequence of the above discussion and Theorem \ref{1.1} (the same notations are used for both the symplectic and the contact case though it will be clear from the context which case we are in).

**Theorem 1.2.** Assume that the following curvature conditions hold:

1. $\langle Rm(JX,X)X,JX \rangle - |\mathcal{N}(X)|^2 \geq k^2$ for all unit tangent vectors $X$,

2. $tr_{\{X,JX\}}(Y \mapsto \langle Rm(Y,X)X,Y \rangle - |\mathcal{N}(X)|^2 \geq \frac{(2n-2)k^2}{4}$ for all unit tangent vectors $X$,

where $\mathcal{N}$ is the tensor defined by $\mathcal{N}(X) = (\nabla_X J)X$.

Then the volume of $M$ is bounded above by that of $B(k)$. Moreover, equality holds only if $M$ is isometrically symplectic to $B(k)$.

Note that comparison results for Kähler manifolds are well-studied (see \cite{20} and references therein) but not in the symplectic case.

The organization of the paper is as follows. In section 2, we recall various notions on contact metric manifolds and the corresponding sub-Riemannian geodesic flows needed in this paper. In section 3, we give a brief discussion on the model spaces, circle bundles over $\mathbb{C}P^n$. Section 4 is devoted to the proof of some Myers’ type maximal diameter theorems. In Section 5, we prove a few comparison type results for a closed Reeb orbit in the spirit of \cite{13}. The equality case of these estimates is discussed in Section 6. Finally, we finish the proof of Theorem 1.1 in Section 7. The appendix summarizes several formulas relating the structures defined on the contact metric manifolds which are needed in this paper.

2. Contact Manifolds and their Sub-Riemannian Geodesics

In this section, we recall several notions about contact manifolds and the sub-Riemannian geodesics which are needed for this paper (see \cite{6}
and references therein for more details about Riemannian geometry of contact manifolds and see [21] for sub-Riemannian geometry). Let \( M \) be a \( 2n + 1 \) dimensional manifold equipped with a contact form \( \eta \) (i.e. \( d\eta \) is non-degenerate on \( \ker \eta \)). Let \( \xi \) be the corresponding Reeb field \( \xi \) defined by the conditions \( \eta(\xi) = 1 \) and \( d\eta(\xi, \cdot) = 0 \). A smoothly varying inner product defined on \( \ker \eta \) is called a Carnot-Caratheodory or sub-Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( \ker \eta \). This can be extended to a Riemannian metric, denoted by the same symbol \( \langle \cdot, \cdot \rangle \), by the conditions \( \langle \xi, X \rangle = 0 \) and \( |\xi| = 1 \) for all \( X \) in \( \ker \eta \). A \((1,1)\)-tensor \( J \) together with \( \xi, \eta \), and the Riemannian metric \( \langle \cdot, \cdot \rangle \) is an contact metric structure if

\[
J\xi = 0, \quad J^2(X_1) = -X_1, \quad \langle JX_1, JX_2 \rangle, \quad \text{and} \quad d\eta(X_1, X_2) = \langle X_1, JX_2 \rangle \text{ for all } X_1 \text{ and } X_2 \text{ in } \ker \eta.
\]

Finally, let \( h \) be the \((1,1)\)-tensor defined by \( h = L_\xi J \). A contact metric manifold is \( K \)-contact if \( \xi \) is an isometry and this is equivalent to \( h = 0 \). It is a CR manifold if

\[
\nabla_u J(v) = \frac{1}{2} \langle u + hu, v \rangle \xi - \frac{1}{2} \langle \xi, v \rangle (u + hu).
\]

A \( K \)-contact CR manifold is Sasakian.

By the Chow-Rashevskii Theorem [11, 24], any two points can be connected by a horizontal curve (i.e. a curve tangent to \( \ker \eta \)). The length of the shortest horizontal curve connecting two points \( x_0 \) and \( x_1 \) in \( M \) is called the Carnot-Caratheodory or sub-Riemannian distance between \( x_0 \) and \( x_1 \). It is denoted by \( d_{CC}(x_0, x_1) \). The function \( g(\cdot) = d_{CC}(x_0, \cdot) \) is locally semi-concave outside the diagonal \( \downarrow \) and so it is twice differentiable Lebesgue almost everywhere [12, 9]. Moreover, since there is no abnormal minimizer (see [21] for more detail), the function \( g \) is \( C^\infty \) along a sub-Riemannian geodesic except at the endpoints.

The function \( g \) satisfies the equation

\[
|\nabla_H g(x)| = 1
\]

for each \( x \) where \( g \) is differentiable. The vector field \( \nabla_H g \) is the horizontal gradient of \( g \) which is defined as the orthogonal projection of the gradient vector field \( \nabla g \) onto the distribution \( \ker \eta \). Here the gradient is taken with respect to the Riemannian metric defined above. Moreover, if \( \gamma \) is a geodesic which starts from \( x \) and ends at \( x_0 \), then

\[
\dot{\gamma}(t) = -\nabla_H g(\gamma(t)).
\]

A computation using this relation gives
Lemma 2.1. A sub-Riemannian geodesic $\gamma$ satisfies the following system of equations:

$$\frac{D^2}{dt^2} \gamma(t) = a(t) J \dot{\gamma}(t) - \frac{h(\dot{\gamma}(t), J \dot{\gamma}(t))}{2} \xi(t),$$

$$\dot{a}(t) = \frac{1}{2} \langle h(\dot{\gamma}(t)), J \dot{\gamma}(t) \rangle,$$

where $\frac{D}{dt}$ denotes the covariant derivative of the Riemannian metric defined above.

Proof. By differentiating (2.1) and applying Proposition 8.2,

$$0 = \langle \nabla^2 g(\nabla g), X \rangle - \langle \nabla g, \xi \rangle \langle \nabla^2 g(\xi), X \rangle - \langle \nabla g, \xi \rangle \langle \nabla g, \nabla \xi \rangle$$

By differentiating (2.2) and applying Proposition 8.2,

$$\frac{D^2}{dt^2} \gamma = -\nabla^2 g(\dot{\gamma}(t)) + \langle \nabla^2 g(\dot{\gamma}(t)), \xi \rangle_{\gamma(t)} \xi(\gamma(t))$$

The first assertion follows with $a(t) = -\langle \nabla g, \xi \rangle_{\gamma(t)}$. The second one follows from

$$\frac{d}{dt} \langle \nabla g, \xi \rangle_{\gamma(t)} = \langle \nabla^2 g(\dot{\gamma}(t)), \xi \rangle_{\gamma(t)} + \langle \nabla g, \nabla \dot{\gamma}(t) \rangle_{\gamma(t)} = \frac{1}{2} \langle \dot{\gamma}(t), Jh \dot{\gamma}(t) \rangle.$$  

Next, we define a family of orthonormal frames along a sub-Riemannian geodesic.

Lemma 2.2. There is a family of orthonormal frames $v(t) = (v_0(t), v_1(t), v_2(t), ..., v_{2n}(t))^T$ defined along the geodesic $t \mapsto \gamma(t)$ which span the orthogonal complements of $\dot{\gamma}(t)$ such that $v_0(t) = \xi(\gamma(t))$, $v_1(t) = \dot{\gamma}(t)$, $v_2(t) = J \dot{\gamma}(t)$, and $\dot{v}(t) = W(t)v(t)$, where
(1) \( a(t) = -\langle \nabla g, \xi \rangle \gamma(t) \),

(2) \( H_{ij}(t) = \langle h(v_i(t), v_j(t)) \rangle \),

(3) \( N_i(t) = \langle (\nabla_{v_i(t)} J)v_1(t), v_i(t) \rangle \),

(4) \( W(t) = \begin{pmatrix} W_1(t) & W_2(t)U(t)^T \\ -U(t)W_2(t)^T & O \end{pmatrix} \),

(5) \( U(t) \) is a family of \((2n-2) \times (2n-2)\) orthogonal matrices,

(6) \( W_1(t) = \begin{pmatrix} 0 & \frac{H_{12}(t)}{2} & -\frac{1+H_{11}(t)}{2} \\ -\frac{H_{12}(t)}{2} & 0 & a(t) \\ \frac{1+H_{11}(t)}{2} & -a(t) & 0 \end{pmatrix} \),

(7) \( W_2(t) = \begin{pmatrix} H_{32}(t) & \ldots & H_{2n,2}(t) \\ 0 & \ldots & 0 \\ N_3(t) & \ldots & N_{2n}(t) \end{pmatrix} \).

Proof. A computation shows that

\[
\begin{align*}
\dot{v}_0(t) &= \nabla \gamma(t) \xi = -\frac{1}{2} (J + Jh) \dot{\gamma}(t) = -\frac{1}{2} (v_2(t) - hv_2(t)), \\
\dot{v}_1(t) &= a(t)v_2(t) - \frac{H_{12}(t)}{2} v_0(t), \\
\dot{v}_2(t) &= (\nabla \gamma(t) J)v_1(t) + J\dot{v}_1(t) = (\nabla_{v_i(t)} J)v_1(t) - a(t)v_1(t).
\end{align*}
\]

Note that \( N_0(t) = \frac{1}{2}(1 + H_{11}(t)) \) and \( N_1 = N_2 = 0 \) by Proposition 8.3.

Let \( \bar{v}_3(t), \ldots, \bar{v}_{2n}(t) \) be a family of bases for the orthogonal complement of \( \{v_0(t), v_1(t), v_2(t)\} \). Let \( A(t) \) be the family of matrices defined by \( A_{ij}(t) = \langle \bar{v}_i(t), \bar{v}_j(t) \rangle \), where \( i, j = 3, \ldots, 2n+1 \). Let \( U(t) \) be the family of orthogonal matrices defined by \( U(0) = I \) and \( \dot{U}(t) = -U(t)A(t) \).

Finally, let \( v_i(t) = \sum_{j=3}^{2n+1} U_{ij}(t) \bar{v}_j(t) \).

\[
\begin{pmatrix} \dot{v}_3(t) \\ \vdots \\ \dot{v}_{2n+1}(t) \end{pmatrix} = (\dot{U}(t) + U(t)A(t)) \begin{pmatrix} \bar{v}_3(t) \\ \vdots \\ \bar{v}_{2n+1}(t) \end{pmatrix} - U(t)W_2(t)^T \begin{pmatrix} v_0(t) \\ v_1(t) \\ v_2(t) \end{pmatrix}
\]

\[
= -U(t)W_2(t)^T \begin{pmatrix} v_0(t) \\ v_1(t) \\ v_2(t) \end{pmatrix}
\]

The assertion follows. \( \square \)

Using the above frames, we can show that the Hessian of \( g \) satisfies a matrix Riccati equation.

Lemma 2.3. Let

(1) \( S_{ij}(t) = -\langle \nabla^2 g(v_i(t)), v_j(t) \rangle \),

(2) \( H_{ij}(t) = \langle h(v_i(t)), v_j(t) \rangle \),

(3) \( J_{ij}(t) = \langle J_{v_i(t)}, v_j(t) \rangle \).
(4) \( S_1(t) = S(t) + \frac{a(t)}{2} H(t)J(t) \),
(5) \( R_{ij}(t) = \langle \text{Rm}(v_i(t), \dot{\gamma}(t))\dot{\gamma}(t), v_j(t) \rangle \),
(6) \( K_{1,ij}(t) = \langle (\nabla_{v_i(t)} J)v_i(t) + (\nabla_{v_j(t)} J)v_j(t), v_1(t) \rangle \),
(7) \( C_i = \begin{pmatrix} \bar{C}_i & 0 \\ 0 & 0_{2n-2} \end{pmatrix} \) with \( i = 1, 2 \),
(8) \( \bar{C}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \),
(9) \( \bar{C}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \),
(10) \( \bar{C}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

Then, for all \( t \) in the open interval \((0, 1)\),

\[
\dot{S}_1(t) - \left( W(t) - \frac{a(t)}{2} J(t) + \frac{1}{2}(I + H(t))C_2 \right) S_1(t) \\
- S_1(t) \left( W(t) - \frac{a(t)}{2} J(t) + \frac{1}{2}(I + H(t))C_2 \right)^T + S_1(t)C_3S_1(t) \\
= -R(t) + \frac{H_{12}(t)}{4} H(t)J(t) + \frac{1}{4}(I + H(t))C_1(I + H(t)) - \frac{(a(t))^2}{4} C_3 - \frac{a(t)}{2} K_1(t).
\]

Proof. By differentiating (2.1) twice,

\[
0 = \nabla^3 g(\nabla_H g, Y, X) + (\text{Rm}(Y, \nabla_H g)\nabla g, X) + \nabla^2 g(\nabla^2 g(Y), X) \\
- \nabla^2 g(\xi, Y)\nabla^2 g(\xi, X) - \langle \nabla g, \nabla Y \xi \rangle \nabla^2 g(\xi, X) \\
- \langle \nabla g, \nabla Y \xi \rangle \nabla^2 g(\nabla Y X) - \frac{\langle \nabla^2 g(Y), \xi \rangle}{2} \langle J\nabla g + hJ\nabla g, X \rangle \\
- \frac{\langle \nabla g, \nabla Y \xi \rangle}{2} \langle J\nabla g + hJ\nabla g, X \rangle \\
- \frac{\langle \nabla g, \xi \rangle}{2} \langle J\nabla^2 g(Y) + hJ\nabla^2 g(Y), X \rangle \\
- \frac{\langle \nabla g, \xi \rangle}{2} \langle \nabla Y (J + hJ)\nabla g, X \rangle.
\]
By setting $Y = v_i(t)$, $X = v_j(t)$, and $S_{ij}(t) = -\nabla^2 g(v_i(t), v_j(t))$,

$$0 = -\nabla^3 g(v_i(t), v_i(t), v_j(t)) + R_{ij}(t)$$

$$+ a(t) \langle Rm(v_i(t), v_1(t))v_0(t), v_j(t) \rangle + \sum_{k \neq 0} S_{ik}(t)S_{kj}(t)$$

$$- \frac{1}{2}(\delta_{i2} + H_{i2}(t))S_{0j}(t) + \frac{a(t)}{2} \sum_k \left( J_{ik}(t) + \sum_l H_{il}(t)J_{lk}(t) \right) S_{kj}(t)$$

$$- \frac{S_{0i}(t)}{2}(\delta_{2j} + H_{2j}(t)) - \frac{(\delta_{i2} + H_{i2}(t))(\delta_{j2} + H_{j2}(t))}{4}$$

$$- \frac{a(t)}{2} \sum_k S_{ik}(t) \left( J_{kj}(t) + \sum_l J_{kl}(t)H_{lj}(t) \right)$$

$$+ \frac{a(t)^2}{2} \langle (\mathcal{J} + h\mathcal{J})\nabla_{v_i(t)}\xi, v_j(t) \rangle$$

$$- \frac{a(t)}{2} \langle \nabla_{v_i(t)}(\mathcal{J} + h\mathcal{J})v_1(t), v_j(t) \rangle$$

On the other hand,

$$\frac{d}{dt} \nabla^2 g_\gamma(t)(v_i(t), v_j(t)) = \nabla^3 g_\gamma(t)(\dot{\gamma}(t), v_i(t), v_j(t))$$

$$+ \sum_k W_{ik}(t)\nabla^2 g_\gamma(t)(v_k(t), v_j(t)) + \sum_k W_{jk}(t)\nabla^2 g_\gamma(t)(v_1(t), v_k(t)).$$

Therefore,

$$- \dot{S}_{ij}(t) + \sum_k W_{ik}(t)S_{kj}(t) + \sum_k W_{jk}(t)S_{ki}(t)$$

$$= R_{ij}(t) + a(t) \langle Rm(v_i(t), v_1(t))v_0(t), v_j(t) \rangle + \sum_{k \neq 0} S_{ik}(t)S_{kj}(t)$$

$$- \frac{1}{2}(\delta_{i2} + H_{i2}(t))S_{0j}(t) + \frac{a(t)}{2} \sum_k \left( J_{ik}(t) + \sum_l H_{il}(t)J_{lk}(t) \right) S_{kj}(t)$$

$$- \frac{S_{0i}(t)}{2}(\delta_{2j} + H_{2j}(t)) - \frac{(\delta_{i2} + H_{i2}(t))(\delta_{j2} + H_{j2}(t))}{4}$$

$$- \frac{a(t)}{2} \sum_k S_{ik}(t) \left( J_{kj}(t) + \sum_l J_{kl}(t)H_{lj}(t) \right)$$

$$+ \frac{a(t)^2}{2} \langle (\mathcal{J} + h\mathcal{J})\nabla_{v_i(t)}\xi, v_j(t) \rangle$$

$$- \frac{a(t)}{2} \langle \nabla_{v_i(t)}(\mathcal{J} + h\mathcal{J})v_1(t), v_j(t) \rangle$$
By Proposition 8.1 and 8.3

\[-\frac{1}{2} \langle v_1(t), \nabla_{v_1(t)}(\mathcal{J} + \mathcal{J}h)v_j(t) \rangle - \langle \text{Rm}(v_1(t), v_1(t))v_0(t), v_j(t) \rangle \]

\[= -\frac{1}{2} \langle (\nabla_{v_j(t)}\mathcal{J})(v_i(t)) + (\nabla_{v_i(t)}\mathcal{J})v_j(t), v_1(t) \rangle - \frac{1}{2} \langle \nabla_{v_i(t)}(\mathcal{J}h)v_i(t), v_j(t) \rangle . \]

It follows that

\[-\dot{S}_{ij}(t) + \sum_k W_{ik}(t)S_{kj}(t) + \sum_k W_{jk}(t)S_{ki}(t) \]

\[= R_{ij}(t) + \frac{a(t)}{2} \langle \nabla_{v_i(t)}\mathcal{J}(v_j(t)), v_1(t) \rangle + \frac{a(t)}{2} \langle \nabla_{v_j(t)}\mathcal{J}(v_i(t)), v_1(t) \rangle \]

\[+ \frac{a(t)}{2} \langle \nabla_{v_i(t)}(\mathcal{J}h)(v_i(t)), v_j(t) \rangle + \sum_{k \neq 0} S_{ik}(t)S_{kj}(t) \]

\[-\frac{1}{2}(\delta_{j2} + H_{i2}(t))S_{0j}(t) + \frac{a(t)}{2} \sum_k \left( J_{ik}(t) + \sum_l H_{il}(t)J_{lk}(t) \right) S_{kj}(t) \]

\[-\frac{S_{0k}(t)}{2}(\delta_{j2} + H_{j2}(t)) - \frac{(\delta_{i2} + H_{i2}(t))(\delta_{j2} + H_{j2}(t))}{4} \]

\[-\frac{a(t)}{2} \sum_k S_{ik}(t) \left( J_{kj}(t) + \sum_l J_{kl}(t)H_{lj}(t) \right) \]

\[-\frac{a(t)^2}{4} \langle (\mathcal{J} + h\mathcal{J})(\mathcal{J} + \mathcal{J}h)v_i(t), v_j(t) \rangle . \]

In other words,

\[-\dot{S}(t) + W(t)S(t) + S(t)W(t)^T \]

\[= R(t) + \frac{a(t)}{2} K_1(t) \]

\[+ \frac{a(t)}{2} \left( \frac{d}{dt}(H(t)J(t)) - W(t)H(t)J(t) - H(t)J(t)W(t)^T \right) \]

\[+ S(t)C_3S(t) - \frac{1}{2}(I + H(t))C_2S(t) + \frac{a(t)}{2}(J(t) + H(t)J(t))S(t) \]

\[-\frac{1}{2} S(t)C_2^T(I + H(t)) - \frac{1}{4}(I + H(t))C_1(I + H(t)) \]

\[-\frac{a(t)}{2} S(t)(J(t) + J(t)H(t)) + \frac{a(t)^2}{4}(C_3 + 2H(t) + H(t)^2) . \]
By rewriting this in terms of $S_1(t)$ and using $C_2H(t) = 0$,

$$0 = \dot{S}_1(t) + R(t) + \frac{a(t)}{2}K_1(t) - \frac{H_{12}(t)}{4}H(t)J(t) + S_1(t)C_3S_1(t)$$

$$- S_1(t) \left( W(t) - \frac{a(t)}{2}J(t) + \frac{1}{2}(I + H(t))C_2 \right)^T$$

$$- \left( W(t) - \frac{a(t)}{2}J(t) + \frac{1}{2}(I + H(t))C_2 \right)S_1(t)$$

$$- \frac{1}{4}(I + H(t))C_1(I + H(t)) + \frac{a(t)^2}{4}C_3$$

as claimed. \(\square\)

3. ON PRINCIPAL CIRCLE BUNDLES OVER \(\mathbb{C}P^n\)

In this section, we will give a brief discussion on the model space, circle bundles over the complex projective space (see \([6, 14, 21]\) for further details).

The Hopf fibration is a principal circle bundle \(S^1 \to S^{2n+1} \to \mathbb{C}P^n\). We consider the total space \(S^{2n+1}\) as the subset of the complex vector space \(\mathbb{C}^{n+1}\)

$$S^{2n+1} = \{z \in \mathbb{C}^{n+1}||z|^2 = 4\}.$$

The circle action is defined by \(\theta \mapsto e^{-i\theta/2}z\). Its infinitesimal generator \(-iz/2\) defines the Reeb vector field of the contact structure and its orbits have length \(4\pi\) (assuming that the Reeb field has length 1).

The Riemannian metric \(\langle \cdot, \cdot \rangle\), the contact form \(\eta\), and the tensor \(J\) are defined by \(\langle v, w \rangle = \text{Re} \left( \sum_{i=1}^{n+1} \bar{v}_i w_i \right)\), \(\eta(w) = \langle -iz/2, w \rangle\), and \(Jv = iv\), respectively, where \(v = (v_1, ..., v_{n+1})\).

Points in \(S^{2n+1}\) are of the form

$$(2 \cos \theta z_0, 2 \sin \theta z')$$

where \(|z_0| = |z'| = 1\). The points \((2 \cos \theta z_0, 0, ..., 0)\) and \((0, 2z')\) are joined by the unit speed (Riemannian or sub-Riemannian) geodesic \(t \mapsto (2 \cos(t/2)z_0, 2 \sin(t/2)z')\) of length \(\pi\).

The standard Euclidean structure on \(\mathbb{C}^{n+1}\) induces a Riemannian metric on \(S^{2n+1}\) with constant sectional curvature \(1/4\). It induces a metric on the quotient \(\mathbb{C}P^n\) such that the projection map is a Riemannian submersion. By the formula in \([23]\), the curvature on \(\mathbb{C}P^n\) is given by

$$\langle Rm(Y,X)X,Y \rangle = \frac{1}{4} + \frac{3}{4} \langle X, JY \rangle^2$$

for all unit tangent vectors \(X\) and \(Y\) such that \(\langle X, Y \rangle = 0\). So this Riemannian structure on \(\mathbb{C}P^n\) is a constant multiple of the Fubini-Study metric.
If we multiply this Riemannian metric on \( CP^n \) and the contact form \( \eta \) on \( S^{2n+1} \) by a constant \( 1/k^2 \), then the curvature of the new Riemannian metric will satisfy
\[
\langle Rm(Y, X)X, Y \rangle = \frac{k^2}{4} + \frac{3k^2}{4} \langle X, JY \rangle^2
\]
for all unit tangent vectors \( X \) and \( Y \) which satisfies \( \langle X, Y \rangle = 0 \) with respect to the new Riemannian metric. In order to make everything compatible, one also need to multiply the old Reeb field by \( k^2 \) to get the new field. The length of an orbit of the new Reeb field becomes \( \frac{4\pi}{k^2} \). The complex projective space equipped with this Kähler structure is denoted by \( B(k, 1) \). The total space of the Hopf fibration together with contact metric structure induced from \( B(k, 1) \) is denoted by \( \mathcal{P}(k, 1) \). Finally, the quotient of \( \mathcal{P}(k, 1) \) by the discrete subgroup of \( S^1 \) of order \( m \) is denoted by \( \mathcal{P}(k, m) \).

### 4. Myers’ Type Maximal Diameter Theorems for Symplectic and K-contact Manifolds

A contact metric manifold is K-contact if the Reeb field \( \xi \) is Killing. In this section, we assume that the contact manifold is K-contact and give the proof of the following Myers’ type maximal diameter theorem. This guarantees that all manifolds which satisfy the conditions in Theorem 4.1 are compact.

**Theorem 4.1.** Let \( N : \ker\eta \rightarrow \ker\eta \) be the bilinear form defined by \( N(X) = P(\nabla_X J)X \), where \( P : TM \rightarrow \ker\eta \) is the orthogonal projection.

1. Assume that
\[
\langle Rm(JX, X)X, JX \rangle - |N(X)|^2 \geq \frac{k_1^2}{4}
\]
for all unit tangent vectors \( X \). Then the diameter of \( M \) with respect to the metric \( d_{CC} \) is bounded above by \( \frac{2n}{k_1} \).

2. Assume that
\[
\text{tr}_{\{X, JX \}^\perp} (Y \mapsto \langle Rm(Y, X)X, Y \rangle) - |N(X)|^2 \geq \frac{k_2^2}{4}
\]
for all unit tangent vector \( X \), where \( \text{tr}_{\{X, JX \}^\perp} \) denotes the trace of the bilinear form defined on \( \{X, JX \}^\perp \). Then the diameter of \( M \) with respect to the metric \( d_{CC} \) is bounded above by \( \frac{\sqrt{2n-2n}}{k_2} \).

Note that (4.1) becomes a lower bound on a CR analogue of holomorphic sectional curvature \( \langle Rm(JX, X)X, JX \rangle \) if the manifold is Sasakian.

Next, we state a result which is a symplectic analogue of Theorem 4.1. In the following theorem, \( M \) is a manifold of dimension \( 2n \)
equipped with a symplectic structure $\omega$, an almost complex structure $J$, and a compatible Riemannian metric $\langle \cdot, \cdot \rangle$. In particular, $\langle X, JY \rangle = \omega(X, Y)$.

**Theorem 4.2.** Let $N : TM \to TM$ be the bilinear form defined by $N(X) = (\nabla_X J)X$.

1. Assume that

$$\langle Rm(JX, X)X, JX \rangle - |N(X)|^2 \geq k_1^2$$

for all unit tangent vectors $X$. Then the diameter of $M$ with respect to the Riemannian metric $d$ is bounded above by $\pi k_1$.

2. Assume that

$$\text{tr}_{\{X, JX\}^\perp} (Y \mapsto \langle Rm(Y, X)X, Y \rangle) - |N(X)|^2 \geq k_2^2$$

for all unit tangent vector $X$, where $\text{tr}_{\{X, JX\}^\perp}$ denotes the trace of the bilinear form defined on $\{X, JX\}^\perp$. Then the diameter of $M$ with respect to the Riemannian metric $d$ is bounded above by $\sqrt{2n - 2\pi k_2}$.

Note that when the manifold $M$ is Kähler, both conditions (4.3) and (4.4) are satisfied if the bisectional curvature is bounded below by a positive constant (see [27] for a closely related result).

**Proof of Theorem 4.1.** Since the manifold is $K$-contact, the tensor $h$ vanishes. It also follows that $a(t)$ is independent of time. Using the same notations as in the previous section, we have

$$\dot{S}_1(t) + L'(t) + S_1(t)C_3S_1(t) - S_1(t)W'(t)^T - W'(t)S_1(t) = 0,$$

where

$$W'(t) = W(t) - \frac{a}{2} J(t) + \frac{1}{2} C_2$$

and

$$L'(t) = R(t) + \frac{a}{2} K_1(t) - \frac{1}{4} C_1 + \frac{a^2}{4} C_3.$$

Let $S_1(t) = \begin{pmatrix} S_{1,0}(t) & S_{1,1}(t) \\ S_{1,1}(t)^T & S_{1,2}(t) \end{pmatrix}$, $W'(t) = \begin{pmatrix} W'_0 & W'_1 \\ -W'_1 & W'_2 \end{pmatrix}$, $L'(t) = \begin{pmatrix} L'_0 & L'_1 \\ L'_1^T & L'_2 \end{pmatrix}$, $K_1(t) = \begin{pmatrix} K_{1,0}(t) & K_{1,1}(t) \\ K_{1,1}(t)^T & K_{1,2}(t) \end{pmatrix}$, and $J(t) = \begin{pmatrix} J_0 & 0 \\ 0 & J_2(t) \end{pmatrix}$, where $S_{1,0}(t)$, $W'_0(t)$, $L'_0(t)$, $K_{1,0}(t)$, and $J_0$ are $3 \times 3$ blocks.
A computation using Theorem 2.3, Proposition 8.1, and Proposition 8.3 shows that

\[
J_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, \\
K_{1,0} = \begin{pmatrix}
0 & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
W'_0 = \begin{pmatrix}
0 & 0 & -\frac{1}{2} \\
0 & 0 & \frac{a}{2} \\
1 & -\frac{a}{2} & 0
\end{pmatrix}, \\
W'_1(t) = \begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
N_3(t) & \ldots & N_{2n}(t)
\end{pmatrix} U(t)^T, \\
W'_2(t) = -\frac{a}{2} J_2(t).
\]

The block \(S_{1,0}(t)\) satisfies

\[
0 = \dot{S}_{1,0}(t) - W'_0 S_{1,0}(t) - W'_1(t) S_{1,1}(t)^T - S_{1,0}(t)^T W'_0 T \\
- S_{1,1}(t) W'_1(t)^T + S_{1,0}(t) \tilde{C}_3 S_{1,0}(t) + S_{1,1}(t) S_{1,1}(t)^T + L'_0(t) \\
\geq \dot{S}_{1,0}(t) - W'_0 S_{1,0}(t) - S_{1,0}(t)^T W'_0 T \\
+ S_{1,0}(t) \tilde{C}_3 S_{1,0}(t) + L'_0(t) - W'_1(t) W'_1(t)^T,
\]

where \(L'_0(t) = \begin{pmatrix}
\frac{1}{4} & -\frac{a}{4} & 0 \\
-\frac{a}{4} & \frac{a^2}{4} & 0 \\
0 & 0 & \bar{R}_{22}(t) - 1 + \frac{a^2}{4}
\end{pmatrix}\),

\[
W_1(t) W_1(t)^T = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & |N(t)|^2
\end{pmatrix}, \text{ and}
\]

\[|N(t)|^2 = N_3(t)^2 + \ldots + N_{2n}(t)^2.\]

Here \(\text{Rm}\) is the Tanaka-Webster curvature and

\[
\bar{R}_{22}(t) = \langle \text{Rm}(v_2(t), v_1(t)) v_1(t), v_2(t) \rangle.
\]

Under the assumptions of the theorem, we have

\[\bar{R}_{22}(t) - |N(t)|^2 \geq k_1^2.\]

Solutions of the comparison equation

\[
0 = \dot{S}_0(t) - W'_0 S_0(t) - S_0(t)^T W'_0 T + \bar{S}_0(t) \tilde{C}_3 \bar{S}_0(t) + \bar{L}'_0
\]
can be computed explicitly using the method in [19]. Here

\[
\tilde{L}_0'(t) = \begin{pmatrix}
\frac{1}{4} & -\frac{a}{4} & 0 \\
-\frac{a}{4} & \frac{a^2}{4} & 0 \\
0 & 0 & k_1^2 - 1 + \frac{a^2}{4}
\end{pmatrix}.
\]

A solution is given by

\[
\bar{S}_0(t) = \begin{pmatrix}
\frac{2a^2 - 2a^2 \cos(c_1 t) + k_1^2 c_1 t \sin(c_1 t)}{t} \\
\frac{2c_1^2 - 2 + 2(1 - c_1^2) \cos(c_1 t) + c_1 t \sin(c_1 t)}{2a} \\
s(t)
\end{pmatrix},
\]

where \( s(t) = 2 - 2 \cos(c_1 t) - c_1 t \sin(c_1 t) \) and \( c_1 = \sqrt{k_1^2 + a^2} \).

As \( t \to 0 \), it grows like

\[
\begin{pmatrix}
\frac{12}{a^2} \\
\frac{a}{6} \\
\frac{a}{2}
\end{pmatrix},
\]

so \( \lim_{t \to 0} \bar{S}_0(t) \to \infty \) as \( t \to 0 \).

By the comparison theorem [25] of matrix Riccati equations, \( \bar{S}_0(t) \geq S_{1,0}(t) \). Note that \( S_{1,0}(0) \) is the matrix defined by \( -\nabla^2 g(\gamma(t_0)) \) which is well-defined for all small positive \( t_0 \) (recall that \( g \) is \( C^\infty \) along \( \gamma \) except at the end-points). This approach does not require any short time asymptotic. It follows that the geodesic is no longer minimizing if \( t > \frac{2\pi}{k_1} \). Therefore, the diameter of \( M \) is less than or equal to \( \frac{2\pi}{k_1} \).

Next, we look at the equation satisfied by \( S_{1,2}(t) \).

\[
0 = \dot{S}_{1,2}(t) + L_2'(t) + S_{1,1}(t)T\tilde{C}_3 S_{1,1}(t) + S_{1,2}(t)^2 + W_1'(t)^TS_{1,1}(t)
- W_2'(t)^TS_{1,2}(t) + S_{1,1}(t)^TW_1'(t) - S_{1,2}(t)^TW_2'(t)
= \dot{S}_{1,2}(t) + L_2'(t) + (S_{1,1}(t) + W_1'(t))^T\tilde{C}_3(S_{1,1}(t) + W_1'(t)) + S_{1,2}(t)^2
- W_1'(t)^TW_1'(t) - W_2'(t)^TS_{1,2}(t) - S_{1,2}(t)^TW_2'(t)
\geq \dot{S}_{1,2}(t) + S_{1,2}(t)^2 + L_2'(t) - W_1'(t)^TW_1'(t) + \frac{a}{2}J_2^TS_{1,2}(t) + \frac{a}{2}S_{1,2}(t)^TJ_2.
\]

Let \( s(t) = \text{tr}(S_{1,2}(t)) \). After taking the trace, we obtain

\[
0 \geq \dot{s}(t) + \frac{1}{2n - 2}s(t)^2 + \text{tr}(L_2'(t) - W_1'(t)^TW_1'(t))
\geq \dot{s}(t) + \frac{1}{2n - 2}s(t)^2 + k_2^2.
\]

Let \( \bar{s}(t) \) be

\[
\bar{s}(t) = \sqrt{2n - 2}k_2 \cot\left(\frac{k_2 t}{\sqrt{2n - 2}}\right).
\]
It is a solution of the comparison equation \( \dot{s}(t) + \frac{1}{2n-2}s(t)^2 + k_2^2 \). By the same comparison principle as above, \( s(t) \leq \bar{s}(t) \). It follows that the diameter is bounded by \( \frac{\sqrt{2n-2}}{k_2} \) in this case.

**Proof of Theorem 4.2**. The proof is similar to and simpler than the one for Theorem 4.1. We give a very brief sketch.

Let \( \gamma : [0,d(x,x_0)] \to M \) be a geodesic which starts from \( x \) and ends at \( x_0 \). Let us fix an orthonormal frame \( \{v_1(t), \ldots, v_{2n}(t)\} \) defined along \( \gamma \) such that \( v_1(t) = \dot{\gamma}(t), v_2(t) = J\dot{\gamma}(t) \), and \( v_i(t) \in \text{span}\{v_1(t), v_2(t)\} \) for all \( i = 3, \ldots, 2n \).

Let \( v(t) = (v_1(t) \ldots v_{2n}(t))^T \). A computation shows that
\[
\dot{v}(t) = \begin{pmatrix} 0 & A_1(t) \\ -A_1(t)^T & 0 \end{pmatrix} v(t)
\]

where \( A_1(t) = \begin{pmatrix} 0 & \ldots & 0 \\ N_3(t) & \ldots & N_{2n}(t) \end{pmatrix} \) and \( N_i(t) = \langle (\nabla_{\dot{\gamma}(t)} J) \dot{\gamma}(t), v_i(t) \rangle \).

The curve \( \gamma \) satisfies \( \dot{\gamma}(t) = -\nabla g(\dot{\gamma}(t)) \), where \( g(x) = d(x_0, x) \). The function \( g \) satisfies \( |\nabla g| = 1 \) and a computation shows that \( S_{ij}(t) = -\nabla^2 g(\dot{\gamma}(t)) (v_i(t), v_j(t)) \) satisfies
\[
\dot{S}(t) + S(t)^2 - A(t)S(t) - S(t)A(t)^T + R(t) = 0.
\]

Let \( S(t) = \begin{pmatrix} S_0(t) & S_1(t) \\ S_1(t)^T & S_2(t) \end{pmatrix} \) and \( R(t) = \begin{pmatrix} R_0(t) & R_1(t) \\ R_1(t)^T & R_2(t) \end{pmatrix} \), where \( S_0(t) \) and \( R_0(t) \) are \( 2 \times 2 \) blocks. The blocks \( S_0(t) \) and \( S_2(t) \) satisfy
\[
0 = \dot{S}_0(t) + S_0(t)^2 + S_1(t)S_1(t)^T - A_1(t)S_1(t)^T - S_1(t)A_1(t)^T + R_0(t)
\]

and
\[
0 \geq \dot{s}(t)^2 + \frac{1}{2n-2}s(t)^2 + \text{tr}(-A_1(t)^T A_1(t) + R_2(t)).
\]

where \( s(t) = \text{tr}(S_2(t)) \).

The results follow as in the proof of Theorem 4.1.

\[\square\]

5. **Comparison Theorems for the Closed Reeb Orbit**

In this section, we prove two comparison type theorems in the spirit of [13] for the closed Reeb orbit. First, a Myers’ type result.

**Theorem 5.1.** Let \( C \) be a closed orbit of \( \xi \) and let \( \gamma : [0, T] \to M \) be a unit speed minimizing geodesic which starts from a point \( x \) and ends at a point in \( C \) which is closest to \( x \).
(1) Assume that (4.1) holds. Then $T \leq \frac{\pi}{k_1}$.

(2) Assume that (4.2) holds. Then $T \leq \frac{\sqrt{2n-2k_1}}{2k_2}$.

The second one is a volume growth estimate.

**Theorem 5.2.** Assume that conditions (4.1) and (4.2) hold with $k_2 = \sqrt{2n-2k_1}$. Let $C$ be a closed orbit of $\xi$. Let $V(C, T)$ be the neighborhood of $C$ of radius $T$

$$V(C, T) := \text{vol}\{x \in M|d_{CC}(C, x) < T\}.$$

Let $\bar{V}(T)$ be the corresponding volume in the model $\mathcal{P}(k_1, 1)$. Then $\frac{V(C, T)}{\bar{V}(T)}$ is non-increasing in $T$.

**Proof of Theorem 5.1.** Let $\psi : U := \{(x, v) \in \ker \eta_C| |v| = 1\} \rightarrow M$ be the map defined as the solution of the initial value problem:

$$\frac{D^2}{dt^2} \psi(x, tv) = 0, \quad \psi(x, 0) = x, \quad \frac{D}{dt} \psi(x, tv)|_{t=0} = v.$$

i.e. the restriction of the exponential map to the distribution (in this case the exponential map can be the Riemannian or the sub-Riemannian one).

For each $(x, v)$ in $U$, let $e_1(0) = \xi(x)$ and $e_2(0) = \mathcal{J}v$. Let $e_i(t)$ be defined as above along $\psi(x, tv) =: \gamma(t)$ such that $\dot{e}_i(t)$ is contained in $\text{span}\{e_1(t), e_2(t)\}$, where $i = 3, \ldots, 2n$. Let $E(t) = (e_1(t) \ldots e_{2n}(t))^T$ and let $A(t)$ be a family of matrices defined by $\dot{E}(t) = A(t)E(t)$. A computation shows that

$$\dot{e}_1(t) = \nabla_\gamma \xi = -\frac{1}{2} e_2(t),$$

$$\dot{e}_2(t) = (\nabla_\gamma \mathcal{J}) \dot{\gamma}(t) = \frac{1}{2} e_0(t) + \sum_{i \geq 3} N_i(t)e_i(t),$$

where $N_i(t) = \langle (\nabla_{e_i(t)} \mathcal{J}) e_1(t), e_i(t) \rangle$.

Hence, $A(t) = \begin{pmatrix} A_0(t) & A_1(t) \\ -A_1(t)^T & 0 \end{pmatrix}$, where $A_0(t) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, $A_1(t) = \begin{pmatrix} 0 \\ N' \end{pmatrix}$, and $N' = \begin{pmatrix} N_3 & \ldots & N_{2n} \end{pmatrix}$.

For $i \neq 1$, let $\sigma_i$ be the curve defined by $\sigma_i(0) = tv$ and $\dot{\sigma}_i(t) = \dot{e}_i(t)$, where $\dot{e}_i(t)$ is the vector induced by $e_i(0)$. For $i = 1$, let $\sigma_1(s) = d\Phi_s(v)$. Let $d\psi_{(x, tv)}(\bar{E}(t)) = B(t)E(t)$. 


When $t = 0$, $B(0) = I$. Since $\Phi_s$ is an isometry, $\Phi_s(\gamma)$ is also a geodesic if $\gamma$ is. It follows that $\psi(d\Phi_s(tv)) = \Phi_s(\psi(tv))$. Therefore,

$$\sum_i B_{1i}(t)e_i(t) = d\psi(x, tv)(\bar{e}_1(t)) = \frac{D}{ds}\psi(d\Phi_s(tv))\bigg|_{s=0} = \xi(\psi(tv))$$

So $B_{1i}(t) = \delta_{1i}$.

Let $t \mapsto \varphi_t(x)$ be the geodesic connecting from $x$ to the point in closed orbit $C$ which is closest to $x$. It satisfies $\dot{\varphi}_t(x) = \nabla H g(\varphi_t(x)) = \nabla g(\varphi_t(x))$, where $g(x) = -d(C, x)$. Note that the curve $\psi(x, tv)$ coincides with $\varphi_{T-t}(\psi(x, Tv))$ if $t \in [0, T] \mapsto \psi(x, tv)$ is minimizing.

For $i \neq 1$,

$$\frac{D}{dt}d\psi(x, tv)(t\bar{e}_i(t)) = -T\nabla^2 f(d\varphi_{T-t}(d\psi(x, Tv)(\bar{e}_i(0))))$$

and

$$\sum_j (B_{ij}(t)e_j(t) + t\dot{B}_{ij}(t)e_j(t) + t \sum_k B_{ij}(t)A_{jk}(t)e_k(t))$$

By setting $t = T$, it follows that

$$-D(T)^{-1}\dot{D}(T) - A(T) = TF(0)$$

is symmetric, where $D(t) = \begin{pmatrix} B_1(t) \\ tB_2(t) \\ \vdots \\ tB_n(t) \end{pmatrix}$. 

For $i \neq 1$, we have

\[
0 = \left. \frac{D}{ds} \frac{D}{dt} \psi(x, t \sigma_i(s)) \right|_{s=0} = \frac{D}{dt} \frac{D}{dt} d\psi_{(x, tv)}(t \bar{e}_i(0)) + Rm(d\psi_{(x, tv)}(t \bar{e}_i(0)), \dot{\gamma}) \dot{\gamma}
\]

\[
= \sum_j \frac{D}{dt} \left( \dot{D}_{ij}(t)e_j(t) + D_{ij}(t) \sum_k A_{jk}(t)e_k(t) \right) + \sum_{k,l} D_{ik}(t)R_{kl}(t)e_l(t)
\]

\[
= \sum_j \left( \dot{D}_{ij}(t)e_j(t) + 2 \dot{D}_{ij}(t) \sum_k A_{jk}(t)e_k(t) \right) + D_{ij}(t) \sum_k (\dot{A}_{jk}(t) + A_{jk} \dot{A}_{lk}) e_k(t) + \sum_{k,l} D_{ik}(t)R_{kl}(t)e_l(t).
\]

For $i = 1$, let $\Phi_s$ be the flow of $\xi$.

\[
0 = \left. \frac{D}{ds} \frac{D}{dt} \psi(d\Phi_s(tv)) \right|_{s=0} = \frac{D}{dt} \frac{D}{dt} d\psi_{(x, tv)}(\bar{e}_0(t)) + Rm(d\psi_{(x, tv)}(\bar{e}_0(t)), \dot{\gamma}) \dot{\gamma}
\]

\[
= \sum_j \frac{D}{dt} (\dot{D}_{0j}(t)e_j(t) + D_{0j}(t) \sum_k A_{jk}(t)e_k(t)) + \sum_{k,l} D_{0k}(t)R_{kl}(t)e_l(t)
\]

\[
= \sum_j \left( \dot{D}_{0j}(t)e_j(t) + 2 \dot{D}_{0j}(t) \sum_k A_{jk}(t)e_k(t) \right) + D_{0j}(t) \sum_k (\dot{A}_{jk}(t) + A_{jk} \dot{A}_{lk}) e_k(t) + \sum_{k,l} D_{0k}(t)R_{kl}(t)e_l(t).
\]

By combining the above two equations, we obtain

\[
D(t)^{-1} \ddot{D}(t) + 2D(t)^{-1} \dot{D}(t)A(t) + \dot{A}(t) + A(t)^2 + R(t) = 0.
\]

Let $T(t) = D(t)^{-1} \dot{D}(t)$ and $S(t) = T(t) + A(t)$. The matrix $S(t)$ satisfies

\[
(5.1) \quad \dot{S}(t) + S(t)^2 + S(t)A(t) + A(t)^T S(t) + R(t) = 0.
\]
Let $S(t) = \begin{pmatrix} S_0(t) & S_1(t) \\ S_1(t)^T & S_2(t) \end{pmatrix}$, where $S_0(t)$ is a $2 \times 2$ block.

$$0 = \dot{S}_0(t) + S_0(t)^2 + S_1(t)S_1(t)^T + S_0(t)A_0(t) - S_1(t)A_1(t)^T + A_0(t)^T S_0(t) - A_1(t)S_1(t)^T + R_0(t)$$

$$= \dot{S}_0(t) + S_0(t)^2 + (S_1(t) - A_1(t))(S_1(t) - A_1(t))^T + S_0(t)A_0(t)$$

$$+ A_0(t)^T S_0(t) - A_1(t)A_1(t)^T + R_0(t). \tag{5.2}$$

Let us split $S_0(t)$ further $S_0(t) = \begin{pmatrix} S_{0,0}(t) & S_{0,1}(t) \\ S_{0,1}(t) & S_{0,2}(t) \end{pmatrix}$. A computation shows that $S_{0,0} = 0$, $S_{0,1} = -1/2$, and $A_{0,1} = -1/2$.

The function $S_{0,2}(t)$ satisfies

$$0 \geq \dot{S}_{0,2}(t) + S_{0,2}(t)^2 + S_{0,2}(t)^2$$

$$+ 2S_{0,1}(t)A_{0,1}(t) - |N'(t)|^2 + R_{22}(t)$$

$$= \dot{S}_{0,2}(t) + S_{0,2}(t)^2 - |N'(t)|^2 + R_{22}(t)$$

$$\geq \dot{S}_{0,2}(t) + S_{0,2}(t)^2 + k_1^2. \tag{5.3}$$

A solution of the comparison equation $\dot{S}_{0,2}(t) + S_{0,2}(t)^2 + k_1^2 = 0$ with the condition $\lim_{t \to 0} \dot{S}_{0,2}(t) = \infty$ is given by $\dot{S}_{0,2}(t) = k_1 \cot(k_1 t)$.

By the comparison principle of Riccati equations [25], $S_{0,2}(t) \leq \bar{S}_{0,2}(t)$. Since $g$ is $C^\infty$ along minimizing geodesics except at the endpoints, it follows that $T \leq \frac{\pi}{k_1}$.

Similarly, the block $S_2(t)$ satisfies

$$0 = \dot{S}_2(t) + (S_1(t) + A_1(t))^T(S_1(t) + A_1(t))$$

$$+ S_2(t)^2 - A_1(t)^T A_1(t) + R_2(t)$$

$$\geq \dot{S}_2(t) + S_2(t)^2 - A_1(t)^T A_1(t) + R_2(t). \tag{5.4}$$

Let $s(t) = \text{tr}(S_2)$. It follows that

$$0 \geq \dot{s}(t) + \frac{1}{2n-2} s(t)^2 + k_2^2. \tag{5.5}$$

A solution to the comparison equation $\dot{s} + \frac{1}{2n-2} s^2 + k_2^2 = 0$ which satisfies the condition $\lim_{t \to 0} \bar{s}(t) = \infty$ is given by $\bar{s}(t) = \sqrt{2n-2k_2 \cot(k_2 t/\sqrt{2n-2})}$. 

VOLUME RIGIDITY OF $S^1 \to \mathcal{P} \to \mathbb{CP}^n$
Once again, by the comparison principle, we have $\text{tr}(S(t)) \leq \text{tr}(\bar{S}_0(t)) + s(t) = \text{tr}(\bar{S}(t))$. Therefore, $T \leq \frac{\sqrt{2n-3n}}{k_2}$.

**Proof of Theorem 5.2.** Let $U = \{(x, v) \in \ker \eta | x \in C \text{ and } |v| = 1\}$. Below, we denote the geometric object in the case of the model $\mathbb{CP}^n$ by adding a bar above the symbol. For instance, $\bar{U}$ denotes the set $U$ in the case of the model.

Let $r : U \rightarrow \mathbb{R}$ be the first time for which the curve $t \mapsto \psi(x, tv)$ is no longer minimizing.

$$V(C, T) := \int_{\psi((x, tv) | (x, v) \in U \text{ and } t \in [0, \min\{T, r(x, v)\}])} \text{vol}$$

$$= \int_U \int_0^{\min\{T, r(x, v)\}} \det(B(t)) dt$$

$$= \int_U \int_0^T \det(B(t)) dt.$$  

Here we extend $\det(B(t))$ by zero when $t > r(x, v)$.

By using the fact that $B_{1i}(t) = \delta_{1i}$ and $\text{tr}(S) \leq \text{tr}(\bar{S}_0) + s = \text{tr}(\bar{S})$,

$$\frac{d}{dt} \det(B(t)) = \frac{\det(B(t)) \det(\bar{B}(t))(\text{tr}(S) - \text{tr}(\bar{S}))}{\det(\bar{B}(t))^2} \leq 0,$$

where $\bar{B}(t)$ is $B(t)$ in the case when $M$ is the complex projective space $\mathbb{CP}^n$.

It follows that $\frac{\det(B(t))}{\det(\bar{B}(t))}$ is non-increasing. This also holds when $t > r(x, v)$. The average with respect to $\det(\bar{B}(t)) dt$ is also non-increasing. It follows that

$$T \mapsto \frac{\text{vol}(\bar{U}) \int_U \int_0^T \det(B(t)) dt}{\text{vol}(U) \int_U \int_0^T \det(\bar{B}(t)) dt} = \frac{\text{vol}(U) V(C, T)}{\text{vol}(U) \bar{V}(T)}$$

is also non-increasing. \hfill \Box

### 6. The Equality case

By Theorem 5.2,

$$\frac{V(C, \frac{\pi}{k_1})}{\text{len}(C)} \leq \frac{\bar{V}(\frac{\pi}{k_1})}{\text{len}(\bar{C})},$$

where $\text{len}(C)$ and $\text{len}(\bar{C})$ are the lengths of closed orbits $C$ and $\bar{C}$ in $M$ and the model $\mathcal{P}(k_1, 1)$, respectively. This section is devoted to the proof of the following key lemma which deals with the case when the above inequality becomes an equality.
Lemma 6.1. Let $T = \frac{\pi}{k_1}$. Assume that $m := \frac{\bar{V}(T)}{\bar{V}(C)} = \frac{\text{len}(\bar{C})}{\text{len}(C)}$. Let $X$ be the set of points in $M$ which are of distance $\frac{\pi}{k_1}$ away from $C$. Then

1. $m$ is a positive integer,
2. $X$ is a totally geodesic submanifold of $M$,
3. the tangent bundle $TX$ of $X$ is invariant under the tensor $J$,
4. the Reeb field is tangent to $X$,
5. the exponential map is a $m$-fold covering from the set $P_x := \{(x, v) \in \text{ker} \eta \mid |v| = \pi/k_1\}$ to $X$ for each $x$ in $C$,
6. the $(2n - 1)$-dimensional volume of $X$ is equal to that of the corresponding submanifold $\bar{X}$ in the model $P(k_1, m)$.

Proof. The equality $\frac{V(C, T)}{\text{len}(C)} = \frac{\bar{V}(T)}{\bar{V}(C)}$ implies that all inequalities including (5.2), (5.3), (5.4), and (5.5) become equality and the coefficients equal to that of the model case. First, we have $r(x, v) = \frac{\pi}{k_1}$. It follows from (5.2) that $S_1(t) = A_1(t)$, from (5.3) that $N' = 0$ and $\bar{R}_{22}(t) = k_1^2$ and $S_{0,2}(t) = k_1 \cot(k_1 t)$, from (5.4) that $S_1(t) = -A_1(t) = 0$, and from (5.5) that $S_2(t) = c \cot(ct) I$, where $c = \frac{k_2}{\sqrt{2n-2}} = \frac{k_1}{2}$.

The above arguments show that $S(t)$ and $A(t)$ coincides with the corresponding quantities in the model case. By substituting this into (5.1), it follows that the same holds for $R(t)$. Since $B(t)$ satisfies the same equations and initial condition as the corresponding quantity in the model case, the two quantities coincide as well. In summary, we obtain the followings:

$$S(t) = \begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & k_1 \cot(k_1 t) & 0 \\ 0 & 0 & c \cot(ct) I \end{pmatrix},$$

$$A(t) = \begin{pmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 1 & -\cos(k_1 t) & 0 \\ \frac{1-\cos(k_1 t)}{k_1 t} & \frac{\sin(k_1 t)}{k_1 t} & 0 \\ 0 & 0 & \frac{\sin(ct)}{ct} I \end{pmatrix}. $$

Therefore,

$$D(T) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{k_1} & 0 & 0 \\ 0 & 0 & \frac{2}{k_1} I \end{pmatrix}. $$
\[ (6.3) \quad \dot{D}(T) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

It follows that the derivative of the exponential map restricted to \( P_x \) has full rank and so it is an immersion. The second row of \( D(T) \) in (6.2) shows that the derivative of the exponential map sends \( e_1(0) \) to the Reeb field \( \xi \). Therefore, \( \xi \) is tangent to the submanifold \( X \). Since the second column of \( D(T) \) vanishes and the vector field \( e_2(T) \) is orthogonal to \( X \). The third and the fourth assertions follow.

Let \( (\Phi_t(x), d\Phi_t(v)) \) be another vector in \( P_{\Phi_t(x)} \). Let \( \gamma \) be the geodesic connecting \( x \) to \( X \). Since \( \Phi_t \) is distance preserving and \( \xi \) is tangent to \( X \), \( \Phi_t \circ \gamma \) has the same length as \( \gamma \) and it connects \( \Phi_t(x) \) to \( X \). It follows that the image of \( P_{\Phi_t(x)} \) under the exponential map is contained in that of \( P_x \). By symmetry, they are the same.

Let \( v(t) = d\Phi_t(v) \). Since \( \nabla_\xi J = 0 \). It follows that \( Jv(t) = d\Phi_t(Jv) \). Indeed,

\[ \frac{d}{dt} v(t) = \frac{d}{ds} \xi(\Phi_t(\gamma(s))) \bigg|_{s=0} = -\frac{1}{2} J(d\Phi_t(v)), \]

where \( \dot{\gamma}(0) = v \).

It follows that

\[ \frac{d}{dt} (Jv(t)) = \frac{1}{2} d\Phi_t(v) = \frac{d}{dt} d\Phi_t(Jv) \]

and the claim follows from this and that two vector fields coincide at \( t = 0 \).

Let \( w(t) = \frac{\pi}{k_1} (\cos(ct)d\Phi_t(v) + \sin(ct)Jd\Phi_t(v)) \). A computation shows that

\[ \frac{d}{dt} w(t) \bigg|_{t=t_0} = \frac{d}{dt} \left( \frac{\pi}{k_1} (\cos(ct_0)d\Phi_t(v) + \sin(ct_0)Jd\Phi_t(v)) \right) \]

\[ + \frac{c\pi}{k_1} (-\sin(ct_0)d\Phi_{t_0}(v) + \cos(ct_0)Jd\Phi_{t_0}(v)) \]

The first term on the right equals to \( \bar{e}_1(0) \) and the second one equals to \( c\bar{e}_2(0) \) defined at \( w(t_0) \). This discussion together with (6.2) gives

\[ \frac{d}{dt} \exp (w(t)) \bigg|_{t=t_0} = d\exp_{w(t)} \left( \frac{d}{dt} w(t) \right) \bigg|_{t=t_0} = 0. \]

if \( c = \frac{k_1^2}{2} \).
This shows that $\exp(w(t))$ is a point in $X$ independent of time $t$. Moreover, by (6.3),
\[
\frac{D}{ds} \frac{d}{dt} \exp(t w(s)) \bigg|_{t=T, s=0} = \frac{D}{dt} \frac{d}{ds} \exp(t w(s)) \bigg|_{t=T, s=0} = -ce_2(0).
\]
Therefore, the tangent vectors $\frac{d}{dt} \exp(t w(s)) \bigg|_{t=T}$ is rotating at speed $c$ in a circle in the normal bundle of $X$ at $\exp(w(t))$. Since the speed of rotation is $c$, it takes time $\frac{4\pi}{k_1^2} = \text{len}(\bar{C})$ to rotate around once.

On the other hand, since
\[
\frac{d}{dt} \exp(t w(0)) \bigg|_{t=T} = \frac{d}{dt} \exp(t (4\pi/k_1^2)) \bigg|_{t=T},
\]
$w(4\pi/k_1^2) = w(0)$ by the uniqueness of the geodesic. It follows that $m$ is the number of time the curve $s \in [0, 4\pi/k_1^2] \mapsto \Phi_s(x)$ hits the point $x$ and so it is a positive integer proving the first statement. By continuity, this integer $m$ is the same for each $v$ in the normal bundle. Therefore, the exponential map restricted to $P_x$ is a $m$-fold covering of the submanifold $X$.

Since the volume of $X$ can be written in terms of $B(T)$ and $m$, it equals to the corresponding volume in the model space $P(k_1, m)$. This is the last assertion.

Finally, the fact that $X$ is totally geodesic follows from (6.3). Indeed, from the above discussion, any normal vector of $X$ is of the form $\frac{d}{dt} \exp(tv) \bigg|_{t=T}$, where $v$ is any unit vector in the normal bundle of $C$. The shape operator is given by
\[
\frac{D}{ds} \frac{d}{dt} \exp(t v(s)) \bigg|_{t=T, s=0} = \frac{D}{dt} \frac{d}{ds} \exp(t v(s)) \bigg|_{t=T, s=0}.
\]
The above quantity is essentially given by the lower right diagonal block of $\bar{D}(T)$ which vanishes. So the shape operator is zero everywhere and $X$ is totally geodesic. 

\[\square\]

7. Proof of Theorem 1.1

By Lemma 6.1, the submanifold $X$ is totally geodesic submanifold of $M$, the Reeb field $\xi$ is tangent to $X$, and the tangent bundle $TX$ is invariant under $J$. It follows that all the geometric structures restrict to $X$ and give it a $K$-contact manifold structure of dimensions two lower than that of $M$. Moreover, since $X$ is totally geodesic, the curvature and hence the conditions in Theorem 1.1 is preserved. Therefore, by
induction $X$ is isometrically contactomorphic to the model $P(k_1, m)$. The exponential map restricted to the subset 
\[ \{ (x, v) \in \ker \eta \| v \| < \pi / k_1 \} \]
of the normal bundle of $C$ defines a diffeomorphism onto $M - X$. This map together with the corresponding exponential map of the model space $P(k_1, m)$ defines a diffeomorphism $\Psi_1$ from $M - X$ to $P(k_1, m) - \bar{X}$. By (6.1), the matrix $B(t)$, which is the matrix representation of the derivative of the exponential map with respect to the orthonormal moving frames, agrees with the corresponding one in the model. Therefore, $\Psi_1$ is an isometry. The first row of $B(t)$ shows that $\Psi_1$ sends the Reeb field on $M$ to the Reeb field on the model, so it is a contactomorphism.

Finally, by induction (see below for the argument in the three dimensional case), both submanifolds $X$ and $\bar{X}$ are isometrically contactomorphic to $P(k_1, m)$ of dimension $2n - 1$. It follows that the exponential maps restricted to the normal bundles of $X$ and $\bar{X}$ defines an isometric contactomorphism $\Psi_2$ from $M - C$ to $P(k_1, m) - \bar{C}$. The analysis in the proof of Lemma 6.1 shows that the $\Psi_1$ and $\Psi_2$ paste together to form a map from $M$ to $P(k_1, m)$.

It remains to consider the three dimensional case. In this case $X$ and $\bar{X}$ are closed Reeb orbits with the length of the later one equals to $m$-times of that of the former one. The same argument used above works. This finishes the proof.

8. Appendix

In this section, we recall several known formulas which are needed for this paper. The proof of them can be found in [6] though the formulas here have slightly different constants since there are differences in the notations.

Let us first recall that $M$ is a contact metric manifold. It means that $M$ is equipped with tensors $J, \langle \cdot, \cdot \rangle, \eta, \xi$ such that $\eta(\xi) = 1, d\eta(\xi, \cdot) = 0, \langle \xi, X_1 \rangle = 0, \langle JX_1, JX_2 \rangle = \langle X_1, X_2 \rangle$, and $\langle X_1, JX_2 \rangle = d\eta(X_1, X_2)$ for all tangent vectors $X_1$ and $X_2$ in $\ker \eta$.

Recall that the Nijenhuis tensor is defined by

\[ [J, J](u, v) = J^2[u, v] + [J u, J v] - J [J u, v] - J [u, J v]. \]

**Proposition 8.1.** The followings hold for all $u, v, w$ in $TM$.

1. $\nabla_\xi \xi = 0$,
2. $[J, J](v, w) = (\nabla_{Jv} J)w - (\nabla_{Jw} J)v + J(\nabla_w J)v - J(\nabla_v J)w$,
3. $\langle (\nabla_u J)w, v \rangle + \langle (\nabla_v J)u, w \rangle + \langle (\nabla_w J)v, u \rangle = 0$, 


The followings hold.

(4) $2\langle (\nabla_u \mathcal{J})v, w \rangle = \langle [\mathcal{J}, \mathcal{J}](v, w), \mathcal{J}u \rangle + d\eta(\mathcal{J}v, u)\eta(w) - d\eta(\mathcal{J}w, u)\eta(v)$.
(5) $\nabla_\xi \mathcal{J} = 0$,

where $\nabla$ denotes the Levi-Civita connection.

**Proposition 8.2.** Assume that $M$ is of dimension $2n+1$. Let $h = L_{\xi} \mathcal{J}$. The followings hold for all $u, v, w$ in $TM$.

(1) $\mathcal{L}_{Ju} \eta(v) = \mathcal{L}_{Jv} \eta(u)$,
(2) $h\xi = 0$,
(3) $\langle hu, v \rangle = \langle hv, u \rangle$,
(4) $\nabla_u \xi = -\frac{1}{2}(Ju + Jhu)$,
(5) $Ju + hJ = 0$,
(6) $tr(h) = 0$,
(7) $\mathcal{J}Rm(\xi, u)\xi = \frac{1}{2}(\nabla_\xi h)u - \frac{1}{4}Ju + \frac{1}{4}h^2 Ju$,
(8) $Rm(\xi, w)\xi - J Rm(\xi, Ju)\xi = \frac{1}{2} J^2 u + \frac{1}{2} h^2 u$,
(9) $Rc(\xi, \xi) = \frac{n}{2} - \frac{1}{4} tr(h^2)$.

**Proposition 8.3.** The followings hold.

(1) $2\langle (\nabla_v \mathcal{J})w, u \rangle + 2\langle (\nabla_{Jv} \mathcal{J})w, Ju \rangle$
   $= \eta(u) \langle v, w + hv \rangle - 2\eta(w) \langle u, v \rangle + \eta(u)\eta(v)\eta(w)$,
(2) $\langle (\nabla_v \mathcal{J})v, \xi \rangle = \frac{1}{2} \langle v, v + hv \rangle - \langle \xi, v \rangle^2$,
(3) $Rm(v, u)\xi = -\frac{1}{2}(\nabla_v \mathcal{J})(u) - \frac{1}{2}\nabla_v (Ju)(u) + \frac{1}{2}(\nabla_u \mathcal{J})(v) + \frac{1}{2}\nabla_u (Ju)v$,
(4) $\langle Rm(\xi, w)v, u \rangle - \langle Rm(\xi, w)Ju, Ju \rangle$
   $+ \langle Rm(\xi, Ju)Jw, v \rangle + \langle Rm(\xi, \mathcal{J}w)v, Ju \rangle$
   $= \frac{1}{2} \eta(u) \langle v, w + hw \rangle - \frac{1}{2} \eta(v) \langle u, w + hw \rangle + \langle (\nabla_h Ju)u, v \rangle$.

Let $\nabla$ be the generalized Tanaka connection defined by

$$\nabla_u v = \nabla_v u + \frac{1}{2} \eta(u)Jv - \eta(v)\nabla_u \xi + \langle \nabla_\xi, v \rangle \xi.$$  

Let $T$ and $\overline{Rm}$ be the torsion and the curvature of $\nabla$, respectively.

**Proposition 8.4.** The followings hold for all $u, v$ in $TM$ and all $X, Y, Z$ in $\ker \eta$.

(1) $T(u, v) = \frac{1}{2} \eta(v)Ju - \frac{1}{2} \eta(u)Jh + g(u, Jv)\xi$,
(2) $T(\xi, Jv) = -JT(\xi, v)$,
(3) $T(X, Y) = g(X, JY)\xi = d\eta(X, Y)\xi$,
(4) $\overline{Rm}(u, v)\xi = 0$,
(5) $\overline{Rm}(X, Y)Z = P\overline{Rm}(X, Y)Z + \frac{1}{2} g(JY + JhY, Z)(JX + JhX) - \frac{1}{2} g(JX + JhX, Z)(JY + JhY) + \frac{1}{2} g(X, JY)JZ$,
(6) $\overline{Rm}(X, \xi)Z = P\overline{Rm}(X, \xi)Z + \frac{1}{2} P(\nabla_X \mathcal{J})Z$.

**Proposition 8.5.** The followings hold for all $Y$ in $\ker \eta$.

(1) $\overline{Rc}(\xi, Y) = Rc(\xi, Y) = -\frac{1}{2} \langle \text{div} h, JY \rangle$, 
\[
(2) \quad \overline{\text{Rc}}(Y, Y) = \text{Rc}(Y, Y) - \langle \text{Rm}(\xi, Y)Y, \xi \rangle - \frac{1}{4} \langle hY, hY \rangle + \frac{3}{4} \langle Y, \xi \rangle, \\
(3) \quad \overline{\text{Rc}}(Y, Z) = \sum_i \langle \text{Rm}(X_i, Y)Z, X_i \rangle + \frac{3}{4} \langle Y, Z \rangle - \frac{1}{4} \langle hY, hZ \rangle. 
\]

A contact manifold is CR if
\[ J[X, Y] - J[JX, Y] - [JX, Y] - [X, JY] = 0 \]
for all horizontal vector fields \(X\) and \(Y\). A computation using the first and the second propositions gives

**Proposition 8.6.** A contact metric manifold is CR if and only if
\[
\nabla_u J(v) = \frac{1}{2} (u + hu, v) \xi - \frac{1}{2} \langle \xi, v \rangle (u + hu).
\]

**References**

[1] A. Agrachev, P.W.Y. Lee: Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds. Math. Ann. 360 (2013), no. 1-2, 209-253.

[2] A. Agrachev, P.W.Y. Lee: Bishop and Laplacian comparison theorems on three-dimensional contact sub-Riemannian manifolds with symmetry. J. Geom. Anal. 25 (2015), no. 1, 512-535.

[3] F. Baudoin, N. Garofalo: Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries. J. Eur. Math. Soc. (JEMS) 19 (2017), no. 1, 151-219.

[4] F. Baudoin, M. Bonnefont, N. Garofalo: A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincaré inequality. Math. Ann. 358 (2014), no. 3-4, 833-860.

[5] F. Baudoin, E. Grong, K. Kuwada, A. Thalmaier: Sub-Laplacian comparison theorems on totally geodesic Riemannian foliation. [arXiv:1706.08489](https://arxiv.org/abs/1706.08489)

[6] D.E. Blair: Riemannian geometry of contact and symplectic manifolds. Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.

[7] W.M. Boothby, H.C. Wang: On contact manifolds. Ann. of Math. (2) 68 1958 721-734.

[8] P. Cannarsa, L. Rifford: Semiconcavity results for optimal control problems admitting no singular minimizing controls. Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 4, 773-802.

[9] P. Cannarsa, C. Sinestrari: Semiconcave functions, Hamilton-Jacobi equations, and optimal control. Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston, Inc., Boston, MA, 2004.

[10] J. Cheeger, T.H. Colding: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math. (2) 144 (1996), no. 1, 189-237.

[11] W.L. Chow: Über systeme van linearen partielen differentialgleichungen erster ordnung. Math. Annalen 117:98-105.

[12] L.C. Evans, R.F. Gariepy: Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[13] E. Heintze, H. Karcher: A general comparison theorem with applications to volume estimates for submanifolds. Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 451-470.
[14] H. Karcher: Riemannian comparison contructions. Global differential geometry, 170-222, MAA Stud. Math., 27, Math. Assoc. America, Washington, DC, 1989.

[15] S. Kobayashi: Principal fibre bundles with the 1-dimensional toroidal group. Tôhoku Math. J. (2) 8 (1956), 29-45.

[16] P.W.Y. Lee: On measure contraction property without Ricci curvature lower bound. Potential Anal. 44 (2016), no. 1, 27-41.

[17] P.W.Y. Lee: Ricci curvature lower bounds on Sasakian manifolds. [arXiv:1511.09381]

[18] P.W.Y. Lee, C. Li, I. Zelenko: Ricci curvature type lower bounds for sub-Riemannian structures on Sasakian manifolds. Discrete Contin. Dyn. Syst. 36 (2016), no. 1, 303-321.

[19] J.J. Levin: On the matrix Riccati equation. Proc. Amer. Math. Soc. 10 (1959) 519-524.

[20] P. Li, J.Wang: Comparison theorem for Kähler manifolds and positivity of spectrum. J. Diff. Geom. 69 (2005) 43-74.

[21] R. Montgomery: A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[22] S.B. Myers: Riemannian manifolds with positive mean curvature. Duke Math. J. 8 (1941), 401-404.

[23] B. O’Neil: The fundamental equations of a submersion. Michigan Math. J. 13 (1966), 459-469.

[24] P.K. Rashevskii: About connecting two points of complete nonholonomic space by admissible curve. Uch. Zapiski ped. inst. Libnextra (2):83-94.

[25] H.L. Royden: Comparison theorems for the matrix Riccati equation. Comm. Pure Appl. Math. 41 (1988), no. 5, 739-746.

[26] P. Rukimbira: Topology and closed characteristics of K-contact manifolds. Bull. Belg. Math. Soc. 349-356.

[27] L.F. Tam, C.Yu: Some comparison theorems for Kähler manifolds. Manuscripta Math. 137, 483-495 (2012).

[28] N. Tanaka: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. Japan J. Math. 2, 131-190.

[29] S. Tanno: Variational problems on contact Riemannian manifolds. Trans. Amer. Math. Soc. 314 349-379.

[30] A. Weinstein: Periodic orbits for convex hamiltonian systems, Ann. of Math., 108, 507-518.

E-mail address: wylee@math.cuhk.edu.hk

Room 216, Lady Shaw Building, The Chinese University of Hong Kong, Shatin, Hong Kong