Comparing Formulations of Generalized Quantum Mechanics for Reparametrization-Invariant Systems

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Abstract

A class of decoherence schemes is described for implementing the principles of generalized quantum theory in reparametrization-invariant ‘hyperbolic’ models such as minisuperspace quantum cosmology. The connection with sum-over-histories constructions is exhibited and the physical equivalence or inequivalence of different such schemes is analyzed. The discussion focuses on comparing constructions based on the Klein-Gordon product with those based on the induced (a.k.a. Rieffel, Refined Algebraic, Group Averaging, or Spectral Analysis) inner product. It is shown that the Klein-Gordon and induced products can be simply related for the models of interest. This fact is then used to establish isomorphisms between certain decoherence schemes based on these products.

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I. INTRODUCTION

Generalized quantum mechanics is a comprehensive framework for quantum theories of closed systems. This framework includes the usual quantum generalizations that may be necessary for quantum theories of dynamical spacetime geometry. Indeed, the framework permits many such generalizations corresponding to different ways in which its principles can be implemented.

Time reparametrization invariance is a characteristic feature, both classically and quantum mechanically, of dynamical theories of spacetime such as general relativity. This paper is concerned with generalized quantum theories that incorporate this invariance for model systems with a single reparametrization invariance. The relativistic world line or the homogeneous minisuperspace models in quantum cosmology are simple examples.

A quantum theory of a reparametrization invariant system must deal with the constraint associated with the invariance. In Dirac quantization the physical states are selected from an extended space of states by the condition that they satisfy an operator form of the classical constraint. In this paper we introduce a class of generalized quantum theories that incorporate the constraints associated with reparametrization invariance in a natural way. They will be called “product space decoherence schemes”. We also show how earlier sum-over-histories generalized quantum theories for systems with constraints can be formulated as members of this class. We show how different members of the class can be equivalent in the sense of leading to the same physical predictions.

The objective of a generalized quantum theory is the prediction of the probabilities of the individual members of sets of alternative coarse-grained histories of a closed system given the boundary conditions that define initial and final conditions for the system. In the case of a model cosmology, these histories might be alternative histories of the four-dimensional geometry of the model universe. Only sets of histories that have negligible interference between their members can be consistently assigned probabilities. A central element in a generalized quantum theory is therefore the decoherence functional $D(\alpha', \alpha)$ that incorporates the system’s boundary conditions and measures the interference between pairs of histories $(\alpha', \alpha)$ in a coarse-grained set. A set of histories is said to (medium) decohere when the “off-diagonal” elements of $D$ are negligible. The probabilities $p(\alpha)$ of a decoherent set of histories are the diagonal elements of $D$. These definitions are summarized by

$$D(\alpha', \alpha) = \delta_{\alpha' \alpha} p(\alpha) .$$

(1.1)

Generalized quantum theory can be defined axiomatically and various specific constructions of decoherence functionals consistent with these axioms have been given. The most general representation is that of Isham, Linden, and Schreckenberg. We will refer to a particular algorithm for constructing decoherence functionals as a ‘decoherence scheme’.

In this paper we consider a specific class of ‘product space’ decoherence schemes for model systems with a single reparametrization invariance. In these schemes, initial and final boundary conditions are represented by physical states that satisfy the constraints in an extended space of states. Reparametrization invariant histories are represented by ‘class operators’ that are annihilated by the constraints. Decoherence functionals are constructed by combining these elements using a product operation $\circ$. The natural product on the extended space of states is typically not a candidate for $\circ$, because the product between two
physical states diverges in that product. Rather, a special construction is required for the inner product between physical states, and this paper is concerned with the relationship between the decoherence schemes defined with different choices for $\circ$. In particular, we investigate the relationship between decoherence schemes defined through the Klein-Gordon product $\circ_{KG}$ employed, for example, in the sum-over-histories constructions of [2,8,9], and those defined through a product which been discussed under a variety of names that include ‘spectral analysis’ [10], ‘group averaging’ [11], ‘refined algebraic quantization’ [12,13] and the ‘Rieffel induced’ [14] inner product. We shall refer to it here as simply the ‘induced product’ $\circ_I$. The definition of this product will be reviewed in Section III. The induced product is a promising route to a rigorous definition of the inner product between physical states in non-perturbative canonical quantum gravity (see, for example [12,15–18] for results concerning this inner product). The relationship between the canonical and sum-over-histories approaches to the quantum mechanics of spacetime is of great interest. Relating the decoherence schemes based on the natural products associated with the two approaches is a step in the direction of clarifying that connection.

This relationship between the Klein-Gordon and induced products in particular cases raises the possibility that the decoherence schemes based on the two products could also be related or could even be even isomorphic. An important consideration, however, is that decoherence schemes involve more structures than just this product. A precise definition of a decoherence scheme and of an isomorphism between decoherence schemes is needed to give meaning to this question; it will be provided in Section II.

We begin in Section II with various preliminaries and definitions: a description of the class of reparametrization-invariant models to be considered (the asymptotically free hyperbolic or AFH models), a full definition of a decoherence scheme, a description of the product space schemes (the schemes on which we will focus), and a discussion of how the sum-over-histories schemes of [4] can be written as product space decoherence schemes. Section III contains a description of the product $\circ_I$ and shows that it is related to the Klein-Gordon product for many asymptotically free hyperbolic models. Section IV constructs isomorphisms between decoherence schemes based on the two products. Section VII contains a summary and some brief conclusions.

II. PRELIMINARIES

This section contains various background material for our main discussion. In particular, we introduce the notion of a decoherence scheme, describe the class of (asymptotically free hyperbolic [AFH]) models to be considered, describe the class of ‘product space’ decoherence schemes, and finally show that the product space schemes described above are sufficiently general to include the sum-over-histories constructions of [4]. This sets the stage for further discussion of the products in Section III and the construction of isomorphisms in Section IV.
A. A Definition of Decoherence Scheme

We shall be concerned in this work with comparing different ways of constructing decoherence functionals for theories with a single reparametrization invariance. We aim not merely to compare the decoherence functionals that might be appropriate for different systems, but rather to compare the different algorithms or processes or methods of constructing decoherence functionals for this broad class of theories, somewhat in the way that it is possible to compare different “methods of quantization”\(^1\). In particular we shall be interested in the question of when two different algorithms for the construction of decoherence functionals give physically equivalent results.

A precise but still broad notion of an algorithm for the construction of decoherence functionals is provided by the idea of a *decoherence scheme* which we now introduce. We begin with the specification of a vector space \(V_N\) of ‘states’ together with a space of ‘class operators’ \(C\) acting on \(V\) which represent the possible individual histories of system. The space \(C\) contains a preferred unit operator \(U\) representing the identity alternative, and of course zero representing the empty alternative. ‘Exhaustive’ sets of alternative histories \(\Pi\) (for ”partition”) are certain subsets of \(C\) satisfying in particular the condition \(\sum_{C \in \Pi} C = U\) under some appropriate addition operation. A *decoherence scheme* over \((V_N, C)\) is then just the assignment of a decoherence functional \(D(\phi, \psi, \Pi)\) to each triple \((\phi, \psi, \Pi)\) where \(\psi \in V_N\) is the ‘initial condition,’ \(\phi \in V_N\) is the ‘final condition,’ and \(\Pi \subset C\) is an exhaustive set of alternative histories. (Such sets \(\Pi\) will also be called ‘partitions of the space of histories’ or just ‘partitions’ for short.)

The example of standard quantum mechanics may help to make these ideas concrete. There the space \(V_N\) is the usual Hilbert space of normalizable states. Partitions \(\Pi\) are formed from sequences of orthogonal projections taken from exhaustive sets, and the space \(C\) is the space of all the strings of projections made up from these sets. The unit alternative is the identity operator. It should be clear from this example that the space \(C\) has a considerable amount of structure arising from the conditions which determine just what is an admissible history and the requirements for sets of alternative histories. The preferred unit operator \(U\) is the most general example of such structure. Another is the closure of \(C\) under addition if the logical ‘or’ operation holds for histories. In that case \(C\) has the structure of an Abelian semi-group.

In the above definition, \(V_N\) is a space of ‘pure’ initial and final conditions (in the language of \(\mathbb{2}\)). Decoherence functionals for ‘mixed states’ can be defined in terms of those for the pure states. Indeed, such mixed initial and final conditions are necessary for realistic cosmological models where the final boundary condition is likely to be indifference with respect to final state (See, e.g. \(\mathbb{4}\)). If \(I = (\{\psi_a\}, \pi^I_a)\) is a set of initial states together with their probabilities \(\pi^I_a\) and \(F = (\{\phi_b\}, \pi^F_b)\) is the corresponding set for a final condition, then the decoherence functional for such mixed initial and final states is just

\(^1\) The Isham-Linden-Schreckenberg \(\mathbb{7}\) theorem gives a standard representation by which to compare any two decoherence functionals. However, here we aim at comparing different algorithms for constructing decoherence functionals in the specific class of reparametrization invariant theories. For this purpose the more restrictive notion of decoherence scheme introduced below is useful.
\[ D(F, I, \Pi) = \mathcal{N} \sum_{ab} \pi_a^f \pi_b^i D(\phi_a, \psi_b, \Pi), \] (2.1)

the normalizing constant \( \mathcal{N} \) being chosen so \( D = 1 \) when \( \Pi \) contains only the unit alternative.

A general notion of equivalence between decoherence schemes may be introduced as follows: Decoherence schemes over \( (\mathcal{V}_{N,1}, \mathcal{C}_1) \) and over \( (\mathcal{V}_{N,2}, \mathcal{C}_2) \) will be called isomorphic when there exist isomorphisms \( \sigma : \mathcal{V}_{N,1} \rightarrow \mathcal{V}_{N,2} \) and \( \chi : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) such that \( D_1(\psi, \phi, \Pi) = D_2(\sigma(\psi), \sigma(\phi), \chi(\Pi)) \) where \( \chi(\Pi) = \{ \chi(C) : C \in \Pi \} \) and such that \( \chi \) places the partitions of \( U_1 \) and \( U_2 \) in one-to-one correspondence. When two decoherence schemes are isomorphic, they encode the same structure; all of their predictions, both for decoherence and probabilities, are the same in the sense that any result derived for states \( \psi, \phi \) and a partition \( \Pi \) in the first scheme can also be derived for some states \( \psi', \phi' \) and some partition \( \Pi' \) in the second scheme. The two schemes are then physically equivalent.

The simplest example of an isomorphism between decoherence schemes is provided in usual quantum mechanics by a unitary transformation of the operators representing the histories and the states representing the initial and final states [20]. This operation clearly leaves the decoherence functional unchanged and is an isomorphism in the sense described above.

The structures of the set \( \mathcal{C} \) are in general altered under an isomorphism of the kind described. For example, the unit operator \( U \) will in general be mapped to a new operator serving the same purpose. It is sometimes desirable to fix these structures and, as a result, to restrict the possible isomorphisms. For example for reparametrization invariant systems it will prove useful to examine isomorphisms within the class of product space decoherence schemes (to be defined below) where the physical states and class operators satisfy fixed constraints. In that case the mapping must preserve this property.

B. The Asymptotically Free Hyperbolic Models

A reasonably general class of models with a single reparametrization invariance involves a phase space spanned by \( n \) coördinates \( x^A \) and their conjugate momenta \( p_A \), with a (classical) constraint of the form

\[
H_{cl} = G^{AB} p_A p_B + V(x^A)
\]

where \( G^{AB} \) is a metric with signature \((-,-,+,-,\cdots,+\)) on the configuration space \( \mathcal{Q} = \mathbb{R}^n \). The coordinates \( x^A \) are assumed to form a global chart on \( \mathcal{Q} \). We take \( (\mathcal{Q}, G) \) to be time orientable and \( G \) to be asymptotically flat in the distant past. We frequently denote these coördinates by \( x \), writing for example \( V(x) \). We denote the timelike coördinate \( x^0 \) by \( t \) and the \( n-1 \) spacelike coördinates collectively by \( \vec{x} \).

An obvious example is a single relativistic world line. Then \( \mathcal{Q} \) is four-dimensional space-time, the \( x^A \) are four spacetime coördinates, \( G^{AB} \) is the Minkowski metric, and \( V(x^A) = m^2 \) where \( m \) is the particle’s rest mass. It should be emphasized, however, that in general cases \( \mathcal{Q} \) is not a spacetime even though the metric \( G \) defines a causal structure on \( \mathcal{Q} \) (so that we may use the terms ‘past’ and ‘future’ in this context). The dependence of \( V \) on \( x^0 \) means that a typical classical history may wander back and forth in ‘time’ in complete disregard for this causal structure. This is quite common in the case of cosmological models, where the ‘timelike’ coordinate usually describes the size or ‘scale factor’ of the cosmology. Thus,
any classical solution that first expands and then recollapses moves first ‘forwards’ and then ‘backwards’ in so-called ‘time.’

A more general class of examples is the set of homogeneous minisuperspace models in quantum cosmology \[21–23\]. Their spacetime geometries are given by metrics of the form

\[ ds^2 = -dt^2 + e^{2\alpha(t)}(e^{2\beta_{ij}}\omega^i\omega^j). \tag{2.3} \]

Here \(\alpha\) is a function while \(\beta_{ij}\) is a \(3 \times 3\) traceless matrix — both depending only on \(t\). The \(\omega^i\) are three one-forms whose commutation relations are the Lie algebra of a group expressing spatial homogeneity. In the case of Bianchi IX models with a homogeneous scalar field \(\phi(t)\), we may take

\[ x^A = (\alpha, \beta^+, \beta^-, \phi) \tag{2.4} \]

where \(\beta^\pm\) are two of the principal values of \(\beta_{ij}\). The metric in (2.2) is given by

\[ G^{AB} = \text{diag}(-1, 1, 1, 1) \tag{2.5} \]

and the potential \(V(x^A)\) by

\[ V(x^A) = e^{4\alpha}[V_\beta(\beta^+, \beta^-) - 1] + e^{6\alpha}[V_\phi(\phi) + \Lambda] \tag{2.6} \]

where \(\Lambda\) is the cosmological constant and the functions \(V_\beta\) and \(V_\phi\) can be found in \[24\]. All of these models have the feature that \(V(x)\) becomes constant as \(t \equiv x^0\) tends to \(-\infty\) and the metric becomes asymptotically flat in this region:

\[ V(x^A) \to \text{const.}, x^0 \to -\infty. \tag{2.7} \]

We shall in fact assume a slightly stronger properties in what follows. We will be interested in models for which the metric is asymptotically flat at past null and timelike infinity and for which the potential becomes a nonnegative constant in this region. While not all of the above cosmological models fall into this class (in particular, the Bianchi IX potential diverges on past null infinity), models such as Bianchi II and Bianchi V do \[23\] fulfill our requirements. Such models will be called ‘asymptotically free hyperbolic models’ or AFH models for short.

At \(x^0 \to +\infty\) a variety of behaviors of \(V(x^A)\) is possible. When the metric \(G\) is asymptotically flat and the potential \(V\) approaches a constant as \(x^0 \to +\infty\), we will say that the model is also asymptotically free in the far future.

C. The Product Space Schemes

We consider systems with a single reparametrization invariance such as the AFH models described above. Recall that, as a consequence of this invariance, the classical canonical coordinates and momenta are subject to a constraint

\[ H_{cl}(p_i, x^i) = 0. \tag{2.8} \]
For example, for the relativistic particle $H_{cl} = p^2 + m^2$ where $p$ is the particle’s four-momentum.

Construction of a product space scheme begins by introducing a linear space of states $\mathcal{V}$ of smooth ‘wave functions’ $\psi(x^i)$ of the canonical coordinates on the configuration space $\mathcal{Q}$. $\mathcal{V}$ should be equipped with an involution $\ast$, which is usually complex conjugation of the wave functions $\psi(x)$. Initial and final conditions are represented by density operators in $\mathcal{V}$ which satisfy a quantum version of the constraints. Thus, for a pure state $\psi$, we require

$$H\psi(x^i) = 0,$$  \hspace{1cm} (2.9)

where $H$ is an operator version of the classical constraint $H_{cl}$. We assume that the constraint is real, in the sense that $\psi$ satisfies (2.8) if and only if $\psi^\ast$ satisfies (2.8). Individual coarse-grained histories (class operators) are described by elements of $\mathcal{V} \otimes \mathcal{V}$ and are often represented by functions (called ‘matrix elements’) $C(x''^i, x'^i)$ on $\mathcal{Q} \times \mathcal{Q}$. We assume that these satisfy the constraints as well, separately in each argument. Thus

$$(H \otimes \mathbb{1})C = (\mathbb{1} \otimes H)C = 0.$$  \hspace{1cm} (2.10)

Any such operator $C$ may be chosen to be the unit alternative $U$ for the scheme.

The last element of the construction is a bilinear product operation $\circ$. However, this product is not defined on the entire space $\mathcal{V} \times \mathcal{V}$. Instead, it is only defined on $\mathcal{V}_N \times \mathcal{V}_N$ for some subspace $\mathcal{V}_N \subset \mathcal{V}$ of solutions to the constraints. We require that $\mathcal{V}_N$ is preserved under the involution $\ast$. The subspace $\mathcal{V}_N$ will be called the space of $\circ$-normalizable states; only $\mathcal{V}_N$ will provide initial and final conditions for the decoherence functionals. We require the product to take values in the complex numbers and to satisfy $(\phi \circ \psi)^\ast = (\psi^\ast \circ \phi^\ast)$ where $\ast$ denotes either complex conjugation or the involution on $\mathcal{V}$.

In a product space scheme, the decoherence functionals are constructed in terms of the above objects as follows. For a ‘pure’ initial state $\phi$, a pure final state $\psi$, and a partition $\Pi$, the decoherence functional $D(\phi, \psi, \Pi)$ is given by

$$D(\phi, \psi, \Pi)_{\alpha', \alpha} = \mathcal{N}(\phi^\ast \circ C_{\alpha'} \circ \psi) (\phi^\ast \circ C_{\alpha} \circ \psi)^\ast.$$  \hspace{1cm} (2.11a)

The quantity $\mathcal{N}$ that ensures that the decoherence functional is normalized is given by

$$\mathcal{N}^{-1} = |(\phi \circ U \circ \psi)|^2.$$  \hspace{1cm} (2.11b)

For the typical quantum cosmology case, $\mathcal{V}$ is a set of functions on the configuration space $\mathcal{Q}$, but the inner product that makes (a large subspace of) $\mathcal{V}$ into the Hilbert space $L^2(\mathcal{Q})$ is not a natural choice for $\circ$ because none of the nontrivial solutions to the constraints (2.8) are contained in this Hilbert space. For instance, in the case of the relativistic particle, (2.8) is the massive wave equation and its solutions are not square integrable over the whole of four-dimensional space. At least two different choices for $\circ$ have been proposed for a generalized quantum theory of the relativistic world line. The first is the standard Klein-Gordon inner product $\circ_{KG}$ defined on a hypersurface $\sigma$ of spacetime.

$$\psi \circ_{KG} \phi = i \int_\sigma d\Sigma^\mu \bar{\psi} \tilde{\nabla}_\mu \phi.$$  \hspace{1cm} (2.12)
The product $\sigma_{KG}$ is independent of $\sigma$ because $\psi$ and $\phi$ satisfy the constraint. The second product has been referred to by many names that include ‘spectral analysis [10]’, ‘group averaging [11]’, ‘refined algebraic quantization [12,13]’ and the ‘Rieffel induced [14]’ inner product. We shall refer to it here as simply the ‘induced product’ $\sigma_I$. Very roughly, the induced product is defined by

$$\int_Q d\psi^* \phi = \delta(0) (\psi^* \sigma_I \phi). \quad (2.13)$$

A more precise definition will be given in Section III below, but for full details the reader should refer to [13] or [12,14]. Note that while the product described in these references is actually a Hermitian inner product, we have inserted an extra application of $\ast$ in the definition (2.13) in order that $\sigma_I$ above is a complex bilinear product in accordance with the structure presented above.

D. Sum-Over-Histories

We now show how previous work on sum-over-histories constructions of generalized quantum theories for systems with a single reparametrization can be incorporated in the framework described in this paper. By generalized sum-over-histories quantum mechanics we shall refer to the framework described in [4]. Class operators are then constructed through sums over paths of $\exp(iS)$ where $S$ is the action for the reparametrization invariant system. A canonical action for the systems with a single reparametrization invariance discussed in Subsection B can be defined by introducing a multiplier $N(\lambda)$ to enforce the constraint and writing

$$S[N(\lambda), p_A(\lambda), x^A(\lambda)] = \int_0^1 d\lambda [p_A \dot{x}^A - NH] \quad (2.14)$$

where a dot denotes a derivative with respect to the parameter $\lambda$. This action is invariant under reparametrizations generated by $\lambda \rightarrow f(\lambda)$.

A sum-over-histories generalized quantum theory begins by positing a set of paths as the unique set of fine-grained histories. In the present case these may be taken to be the configuration space paths $(x^A(\lambda), N(\lambda))$ or, slightly more generally, phase space paths $(p_A(\lambda), x^A(\lambda), N(\lambda))$. Sets of coarse-grained histories are partitions of these paths into reparametrization-invariant classes $\{c_\alpha\}, \alpha = 1, 2, \cdots$. Each class $c_\alpha$ is an individual coarse-grained history.

A general kind of partition of the fine-grained paths is obtained by classifying paths by the values of a reparametrization-invariant functional, $F[p, x, N]$. Consider for example the class of paths $c_\alpha$ for which $F$ lies in a range $\Delta_\alpha$. A corresponding set of ‘matrix elements’ would be constructed as follows

$$\langle x'' | | \tilde{C}_\alpha || x' \rangle = \int_{(x', x'')} \delta p \delta x \delta N e_\alpha[F] \Delta_\Phi \delta[\Phi] \exp(iS[p, x, N]). \quad (2.15)$$

Here $e_\alpha$ is the characteristic function for the interval $\Delta_\alpha$, $\Phi = 0$ is a parametrization fixing condition, and $\Delta_\Phi$ is the associated determinant. The paths $x(\lambda)$ go from the endpoint $x'$...
to the endpoint \( x'' \). We will not need the further details of the measure or of the conditions on the paths in \( p \) and \( N \); they can be found in [2].

The sum-over-histories construction of the decoherence functional parallels (2.11) and utilizes the Klein-Gordon product. Initial and final spacelike surfaces \( \sigma' \) and \( \sigma'' \) are selected on which the Klein-Gordon products in (2.11) are evaluated. The analog of the unit operator is supplied by the sum over all paths in (2.15).

However, there is an important difference between the sum-over-histories construction just adumbrated and the product space decoherence schemes described in Subsection C. The matrix elements defined by (2.11) do not satisfy the constraints over the whole of configuration space (as in (2.10) for the most general and interesting choices of the integration. They cannot be taken to be the matrix elements of a class operator in a product space decoherence scheme. A simple argument for understanding this follows.

We may express the characteristic function in (2.15) as a Fourier transform, creating an exponent containing the effective action

\[
S[p, x, N] + \mu F[p, x, N]
\]

(2.16)

where \( \mu \) is the parameter of the Fourier transform. The formal manipulations which argue that invariantly constructed path integrals satisfy the constraints (e.g. [25]) would, in this case, lead to the conclusion that the \( \langle x''|\tilde{C}_\alpha||x'\rangle \) satisfy an operator version of the equation

\[
\frac{\delta S}{\delta N} + \mu \frac{\delta F}{\delta N} = 0
\]

This is indeed the constraint when \( F \) is a reparametrization-invariant functional of \( x^A \) and \( p_A \) alone — the kind of "observable" usually considered in canonical quantization. But it is not the case for the more general reparametrization invariant functionals that can depend on \( N(\lambda) \). As a result, there is some subtlety in using (2.15) to define a 'class operator' of the product space schemes of Section IIC.

It is now straightforward to show that generalized sum-over-histories quantum mechanics can be written as a product space decoherence scheme. By solving the constraint equation using \( \langle x''|\tilde{C}_\alpha||x'\rangle \) on \( (\sigma'', \sigma') \) and its normal derivative as initial data on these surfaces, it should be possible to construct a function \( C(x'', x') \) on \( Q \times Q \) which satisfies the constraints everywhere and whose value and normal derivative coincide with those of \( \langle x''|\tilde{C}_\alpha||x'\rangle \) on the surfaces \( (\sigma', \sigma'') \). The class operators thus defined will depend on the specification of these surfaces. The product space decoherence scheme constructed from the resulting \( C(x'', x') \) and the Klein-Gordon product will then coincide with that obtained by the sum-over-histories prescription.

### III. THE INDUCED AND KLEIN-GORDON PRODUCTS

We now review the formulation of the induced product and show that it can be related to the Klein-Gordon product for many of the AFH models described in Section II. This relationship is a generalization of the observation of [13] that the induced product corresponds to a kind of ‘absolute value’ of the Klein-Gordon product when the metric \( G \) is flat and the potential vanishes. This simple relationship will be of great use in comparing the Klein-Gordon and induced product based decoherence schemes in Section IV below. Many readers may find the relationship between the induced and Klein-Gordon inner products to be their best source of intuition when dealing with the induced product. As a result, some
readers may wish to glance quickly at Section IIIB before studying the review of the induced product in IIIA.

A. Review of the induced Product

We now briefly review the induced product. This discussion is intended to provide only a passing familiarity with the scheme to allow the unfamiliar reader to follow certain calculations and to have a general understanding of the results. For this reason, we consider only the most straightforward application of the techniques of [12–14] to the asymptotically free hyperbolic models of Section IIA; more general and more detailed treatments can be found in [11,12,14–16,26] and especially [13].

The induced product is motivated by attempts to construct a physical inner product for Dirac-style canonical quantization [27] of constrained systems. Following Dirac, we will be interested only in solutions \( \psi \) of the constraint equation \( H\psi = 0 \). Such solutions give the so-called ‘physical states,’ and it is on these states that the induced product will be defined.

We consider an AFH model as in section IIB and associate with our system the ‘auxiliary’ Hilbert space \( \mathcal{H}_{aux} = L^2(M, \sqrt{-G} d^n x) \); this space is called auxiliary because the final physical states will not live in \( \mathcal{H}_{aux} \) — they will not be normalizable in its inner product. However, we will use this space to ‘induce’ an inner product on a space \( \mathcal{H}_{phys} \) of physical states. We are interested in the case where \( H \) is self-adjoint and has purely continuous spectrum on \( \mathcal{H}_{phys} \); this occurs whenever \((Q,G)\) is asymptotically flat and \( V \) decays sufficiently fast in a sufficiently large region near infinity. The inner product of two states \( |\phi\rangle \) and \( |\psi\rangle \) in \( \mathcal{H}_{aux} \) will be denoted by \( \langle \phi || \psi \rangle \).

In this situation, and under a certain further technical assumption concerning the operator \( H \), the physical Hilbert space is not difficult to construct. Note that what we would really like is to ‘project’ \( \mathcal{H}_{aux} \) onto the (generalized) states which are zero-eigenvalue eigenvectors of \( H \). Of course, since none of these states are normalizable, this will not be a projection in the technical sense. Instead, it will correspond to an object which we will call \( \delta(H) \), a Dirac delta ‘function.’ Given the above mentioned assumption on \( H \) (see [13]), the object \( \delta(H) \) can be shown to exist and to be uniquely defined. Technically speaking however, it exists not as an operator in the Hilbert space \( \mathcal{H}_{aux} \), but as a map from a dense subspace \( S \) of \( \mathcal{H}_{aux} \) to the (for our purposes, topological) dual \( S' \) of \( S \). The space \( S \) may typically be thought of as a Schwarz space; that is, as the space of smooth rapidly decreasing functions (‘test functions’) on the configuration space. In this case, \( S' \) is the usual space of tempered distributions. Not surprisingly, this is reminiscent of the study of generalized eigenfunctions through Gel’fand’s spectral theory [28] and \( S \subset \mathcal{H}_{aux} \subset S' \) forms a rigged Hilbert triple.

The key point is as follows: While generalized eigenstates of \( H \) do not lie in \( \mathcal{H}_{aux} \), they can be related to normalizable states through the action of the ‘operator’ \( \delta(H) \). That is, generalized eigenstates \( |\psi_{phys}\rangle \) of \( H \) with eigenvalue 0, can always be expressed in the form \( \delta(H)|\psi_0\rangle \), where \( |\psi_0\rangle \) is a normalizable state in \( S \subset \mathcal{H}_{aux} \). This choice of \( |\psi_0\rangle \) is of course not unique and, in fact, we associate with the physical state \( |\psi_{phys}\rangle \) the entire equivalence class of normalizable states \( \psi \in S \) satisfying

\[
\delta(H)|\psi\rangle = |\psi_{phys}\rangle.
\] (3.1)
Each equivalence class of normalizable states will form a single state of the physical Hilbert space.

All that is left now is to ‘induce’ the physical inner product from the auxiliary Hilbert space. Naively, the inner product of two physical states $\phi_{\text{phys}}$ and $\psi_{\text{phys}}$ may be written $\langle \phi||\delta(H)\delta(H)||\psi \rangle$, where $|\phi\rangle$ and $|\psi\rangle$ are normalizable states in the appropriate equivalence classes. This inner product is clearly divergent, as it contains $[\delta(H)]^2$. The resolution is simply to ‘renormalize’ this inner product by defining the physical (induced) product $\circ_I$ to be

$$\phi^*_{\text{phys}} \circ_I \psi_{\text{phys}} = \langle \phi||\delta(H)||\psi \rangle.$$ (3.2)

Note that (3.2) does not depend on which particular states $|\phi\rangle$, $|\psi\rangle \in S$ were chosen to represent the physical states $|\phi_{\text{phys}}\rangle$ and $|\psi_{\text{phys}}\rangle$. This construction parallels the case of purely discrete spectrum as, if $P_H$ were a projection onto normalizable zero-eigenvalue eigenstates of $H$, we would have $[P_H]^2 = P_H$. Although $\delta(H)$ is not strictly speaking an operator, taking $|\phi\rangle$ and $|\psi\rangle$ to lie in $S$ makes the above inner product well defined. As a result, $\circ_I$ is a bilinear product with the reality properties required to build a product space decoherence scheme.

**B. Relating the Induced and Klein-Gordon Products**

We now show that, for a large class of AFH models, the Klein-Gordon and induced products are connected by a simple relation. Indeed, linearity leads us to expect that any two bilinear product operations $\circ'$ and $\circ''$ on the same space $V$ should be related by

$$\phi \circ'' \psi = \phi \circ' A\psi$$ (3.3)

for some linear operator $A$, up to possible subtleties concerning the domain. In the remainder of this section we show that there is indeed such a relation between the Klein-Gordon and induced products for AFH systems and we exhibit the corresponding operator $A$.

The connection is most easily found by introducing the usual apparatus of $\delta$-function normalized states, for example, the eigenfunctions $\psi_\lambda$ of the constraints such that

$$H\psi_\lambda = \lambda\psi_\lambda,$$ (3.4)

normalized so that

$$\langle \psi_\lambda||\psi_{\lambda'} \rangle = \delta(\lambda - \lambda').$$ (3.5)

It follows from (3.3) that if $\psi_\lambda$ is a continuous one parameter family of eigenfunctions approaching $\psi_{\text{phys}}$ as $\lambda \rightarrow 0$, then

$$\langle \psi_\lambda||\phi_{\text{phys}} \rangle = \delta(\lambda) \left( \psi^*_{\text{phys}} \circ_I \phi_{\text{phys}} \right)$$ (3.6)

This is what was meant by (2.13). But, for our systems, the left hand side of (3.6) may be evaluated in terms of the Klein–Gordon product on a spacelike slice $\sigma$ yielding the connection.
between $\sigma_{KG}$ and the induced product. We will always restrict to the case $\lambda \leq 0$ so that the Klein-Gordon product is in fact independent of $\sigma$, so long as $\sigma$ ends only at spatial infinity.

There are a number of particular cases according to how the potential behaves at large positive times. In considering these, it is useful to keep in mind that, with the sign conventions of (2.2), it is $-V$ that would function like an effective potential as far as one dimensional motion in $t = x^0$ is concerned. The simplest case is when there is a single asymptotic free region at $t \to -\infty$ and $V(x) \to -\infty$ sufficiently fast at $t \to +\infty$ so that all generalized eigenstates of $H$ must vanish there. Effectively, there is a repulsive barrier for propagation to large, positive, time. We begin with this situation.

The explicit form of the inner product on the left hand side of (3.6) is

$$\langle \psi_\lambda || \phi_{phys} \rangle = \int d^n x \psi_\lambda^*(x) \phi_{phys}(x).$$

(3.7)

The key point is that this integral must yield the $\delta$-function form on the right hand side of (3.6) with no finite additions. Any finite range of the integral in $t$ will not contribute to the singular $\delta$-function. Neither will the asymptotic region at $t \to +\infty$ since we have assumed that the wave functions vanish there. Only the asymptotic region $t \to -\infty$ contributes to the coefficient of the $\delta$-function. There, $\psi_\lambda$ may be expanded in terms of a complete set of positive and negative frequency solutions of $\nabla^2 \psi = \lambda \psi$ in the form

$$\psi_\lambda(x) = \int d^{n-1} p \left[ a^{(+)\lambda}_{\vec{p}} f^{(+)}_{\lambda \vec{p}}(x) + a^{(-)\lambda}_{\vec{p}} f^{(-)}_{\lambda \vec{p}}(x) \right]$$

(3.8)

for some coefficients $a^{(+)\lambda}_{\vec{p}}$, $a^{(-)\lambda}_{\vec{p}}$. Explicitly, e.g.

$$f^{(+)}_{\lambda \vec{p}}(x) = \left[ (2\pi)^3 (2\omega_{\lambda \vec{p}}) \right]^{-\frac{1}{2}} \exp \left[ i \left( -\omega_{\lambda \vec{p}} t + \vec{p} \cdot \vec{x} \right) \right]$$

(3.9)

for the $3 + 1$ case, where

$$\omega_{\lambda \vec{p}} = \left( \vec{p}^2 - \lambda \right)^{\frac{1}{2}}.$$  

(3.10)

Since $\lambda \leq 0$ and, in this region, the metric is flat and the potential is a nonnegative constant, the separation of positive and negative frequency states is well defined. This provides a definition of the ‘positive and negative frequency parts’ $\phi^{(+)\lambda}$, $\phi^{(-)\lambda}$ of $\phi_\lambda$. There is a similar expression for $\phi_{phys}(x)$.

The integral (3.7) in the asymptotic region $t \to -\infty$ may be evaluated to yield the coefficient of $\delta(\lambda)$ in (3.6) (and hence the induced product) in terms of the constants $a^{(+)\lambda}_{\vec{p}}$, $a^{(-)\lambda}_{\vec{p}}$ and the similar constants for $\phi_{phys}$. Not surprisingly, since only an asymptotic regime of $t$ is involved, this expression can also be evaluated in terms of the Klein-Gordon product on a surface of constant $t$. The following relation between the induced and Klein-Gordon products emerges when the potential exhibits a single asymptotically free region

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2In order for a generalized eigenstate to lie in $S'$ as required, it must be a tempered distribution. That is, it may increase as $x^0 \to +\infty$, but not in an exponential manner.
\[
(\psi_{\text{phys}} \circ_I \phi_{\text{phys}}) = \pi \left[ \left( \psi_{\text{phys}}^{(+)} \circ_{KG} \phi_{\text{phys}}^{(+)} \right) - \left( \psi_{\text{phys}}^{(-)} \circ_{KG} \phi_{\text{phys}}^{(-)} \right) \right].
\] (3.11)

It is not necessary to indicate on which surface the Klein-Gordon product is evaluated because, being between solutions of the constraints for \( \lambda \leq 0 \) and \( V \geq 0 \) at infinity, it is independent of the surface (so long as this surface ends only at spatial infinity). The important point is that the decomposition of \( \phi_{\text{phys}} \) and \( \psi_{\text{phys}} \) into their positive and negative frequency parts occurs at the asymptotic surface \( t \to -\infty \). In this respect induced product methods are similar to the approach advocated by Wald in [29].

The relation between the induced and Klein-Gordon products may be expressed more compactly by introducing the operator

\[
\Omega^{(-)} = \pi \text{sign} \left( p^{(-)}_A \right)
\] (3.12)

where \( p^{(-)}_A \) is the momentum in the asymptotic region \( t \to -\infty \). Then we have

\[
\psi_{\text{phys}} \circ_I \phi_{\text{phys}} = \psi_{\text{phys}} \circ_{KG} A \phi_{\text{phys}}
\] (3.13)

with \( A = \Omega^{(-)} \). It is important to note that when \( \Omega^{(-)} \) is interpreted as an operator from states normalizable in the induced product to states normalizable in the Klein-Gordon product, it is not surjective (even on a dense subspace). In fact, any solution \( \psi \) that is normalizable in the induced product decreases rapidly at \( x^0 \to +\infty \) so that its Klein-Gordon norm vanishes. The Klein-Gordon norm of \( \Omega^{(-)} \psi \) then vanishes as well. As a result, the above relation can be inverted to give the Klein-Gordon product in terms of the induced product only for a special set of states despite the fact that an inverse for \( \pi \text{sign} \left( p^{(-)}_0 \right) \) exists on the space of Klein-Gordon normalizable states.

Other asymptotic forms of the potential give the relation (3.13) between the induced and Klein-Gordon products but with differing operators \( A \). For example, for systems that are asymptotically free in both past and future (as with the free relativistic world line), a form of (3.3) holds with

\[
A = \Omega^{(+)} + \Omega^{(-)}
\] (3.14)

\[
= \pi \left[ \text{sign} \left( p^{(+)}_0 \right) + \text{sign} \left( p^{(-)}_0 \right) \right] \equiv \Omega.
\] (3.15)

In this case the relation (3.13) is invertible. Thus we may also write

\[
\psi_{\text{phys}} \circ_{KG} \phi_{\text{phys}} = \psi_{\text{phys}} \circ_I \Omega^{-1} \phi_{\text{phys}}.
\] (3.16)

The expression (3.13) can be used to write the inner product in terms of the ‘Bogoliubov coefficients’ associated with the constraint equation \( H \psi = 0 \). For example, suppose that two solutions \( \psi \) and \( \psi' \) are purely positive frequency in the far past, but that \( \psi = \alpha + \beta \) and \( \psi' = \alpha' + \beta' \) where \( \alpha, \alpha' \) are purely positive frequency in the far future and \( \beta, \beta' \) are purely negative frequency in the far future. Then we have

\[
\psi^* \circ_I \psi' = 2\pi \alpha^* \circ_{KG} \alpha'.
\] (3.17)

Another interesting case is provided by a potential of the form (2.4) with \( \Lambda > 0 \). The potential \( V \) then approaches \( -\infty \) at \( t \to +\infty \), becoming infinitely attractive for motion in \( t \).
Indeed, it is sufficiently attractive that the constraint $H$ is not automatically self-adjoint but permits various self-adjoint extensions. These are clearly discussed in [30]. The various self-adjoint extensions are equivalent to inserting an impenetrable “wall” at some large value of $t$. As far as the relation between the induced and Klein-Gordon products this case is therefore the same as $V \rightarrow -\infty$. Thus we have (3.13) with $A = \Omega(-)$.

We expect that our asymptotically free assumption may be relaxed somewhat and that (3.11) or (3.16) will continue to hold. In support of this conjecture, note that when the $t$-dynamics is separable (so that it decouples from the other ($\vec{x}$) coordinates) and the Klein-Gordon norm is conserved, these results can be derived without imposing additional assumptions on the dynamics of the $\vec{x}$ coordinates. As a result, we would not be surprised to find that (3.11) holds also for complicated models such as Bianchi IX cosmologies.

**IV. KLEIN-GORDON AND INDUCED PRODUCT SPACE SCHEMES**

The connections between the Klein-Gordon and induced products uncovered in the previous Section naturally give rise to the question of whether there are corresponding relationships between the product space decoherence schemes which employ them. We shall investigate such relationships in this Section, especially to see when two different decoherence schemes yield equivalent physical predictions. We will consider specifically the case of AFH models with two asymptotically free regions and the case of a single asymptotically free region in the past and a sufficiently repulsive potential in the future. When the spaces of states are appropriately chosen and the other structures are chosen in the usual way, product space decoherence schemes based on the Klein-Gordon and induced products are isomorphic in the sense of section IIA. However, this always involves choosing a space of states which is smaller than what one would naively use for one of the schemes. That is not necessarily a disadvantage in quantum cosmology where one deals with fixed initial and final conditions prescribed by fundamental laws.

Let us begin with a schematic description of a notion of equivalence between product space decoherence schemes which is a restriction of the notion of isomorphism between decoherence schemes described in Section IIA. For the moment we ignore issues such as invertibility of operators, domains, and so on. We will then state below how the spaces of states may be chosen so that our schematic argument does in fact define an isomorphism of decoherence schemes. Below, the involution $\ast$ is complex conjugation of functions on the configuration space $Q$.

The results of Section IIIB tell us that a change of product (from Klein-Gordon to induced) can always be compensated by a vector space isomorphism $A$ on the space of states. That is, for a fixed class operator $C$:

$$\phi^* \circ_{KG} C \circ_{KG} \psi = (A\phi)^* \circ_I C \circ_I (A\psi) \quad (4.1)$$

where we have used the fact that the operator $A$ from Section IIIB is always real under $\ast$ and self-adjoint with respect to $\circ_I$. Let us take the partitions $\Pi$ to be defined only by the condition that $\sum_{\alpha} C_{\alpha} = U$. Then the constraint $H$ and the the unit alternatives $U_{KG}$ and $U_I$ (to be used with the Klein-Gordon and induced schemes respectively) provide the only remaining general structures in a product space decoherence scheme. The constraint $H$ will
be preserved if $H$ commutes with $A$. If the operators $\hat{U}_I = U_I \circ I$ and $\hat{U}_{KG} = U_{KG} \circ I$ are invertible, then the pair

$$\sigma = A, \quad \chi = \hat{U}_I \hat{U}_{KG}^{-1} \otimes I$$

is an isomorphism from the Klein-Gordon decoherence scheme to the induced scheme that preserves $H$.

The operators $A$ that relate the products have already been considered in Section IIIB. Since these operators involve only the signs of asymptotic momenta, they commute with the constraint $H$ and solutions of the constraint are mapped into solutions of the constraint by $\sigma$. It remains to discuss the unit alternatives $U_{KG}$ and $U_I$ on whose invertibility the isomorphism depends. We take as our model for Klein-Gordon schemes the sum-over-histories schemes of [2], in which the unit alternative $U_{KG}$ is based on the Feynman propagator $\Delta_F$, which is in turn defined by the path integral of the form (2.15) with a sum over positive lapse and all paths. As remarked in Section IIC, the result is not a class operator of the kind needed for a product space scheme as $\Delta_F$ does not fully satisfy the constraints. (Indeed, $\Delta_F$ is well known to be a Green’s function for the constraint $H$.) However, following Section IIC, we can translate this unit operator into the language of product space schemes. The result is that this unit alternative becomes just the ‘positive frequency function’ $U_{KG} = G^{(+)\otimes}$, the bi-solution to the constraints that is positive frequency in $y$ when $y$ is far to the future and is negative frequency in $x$ when $x$ is far to the past. The appendix shows that, for the case of two asymptotically free regions, the matrix elements of $G^{(+\otimes \otimes)}$ are just

$$G^{(+\otimes \otimes)}(x, y) = \langle x||\Pi^{+\otimes \otimes}_{-\infty} \delta(H) \Omega||y \rangle$$

where $\Pi_{-\infty}^{(+\otimes \otimes)} (\Pi^{+\otimes \otimes}_{-\infty})$ again denotes the projection onto solutions of the constraint that are purely positive frequency in the far future (past). Now $\Omega$ is invertible (in fact, its inverse is bounded on the space of induced normalizable states) but, due to the presence of the projections in (4.3), the invertibility of $G^{(+\otimes \otimes)}$ will depend on choosing the proper space of states.

In the case where only the past is asymptotically free (and where the potential is sufficiently repulsive in the future) the appendix shows that,

$$G^{(+\otimes \otimes)}(x, y) = -2\pi \langle x||\Pi^{+\otimes \otimes}_{-\infty} \delta(H) \Omega||y \rangle.$$  (4.4)

Again, any invertibility issues will rest on the proper choice of state space.

On the other hand, in schemes based on the induced product, it is natural to choose the unit alternative to be represented by the matrix elements $U_I(x'', x') = \langle x''||\delta(H) \Omega||x' \rangle$. This is because $\delta(H) \circ I \psi$ is just $\psi$ itself. The constructions of \cite{13,15–17} may be thought of as providing a decoherence scheme of this type. Since $\delta(H) \circ I$ is just the unit operator on $\circ I$-normalizable states it is in general invertible.

We now provide the appropriate details to turn our schematic argument above into an actual isomorphism. We begin with the Klein-Gordon scheme for the case of two asymptotically flat regions. Because of the projections inherent in $G^{(+\otimes \otimes)}$, we must restrict ourselves to, say, states which are positive frequency in the far future (else the normalization coefficient $N$ (2.11b) would diverge). In particular, let us take the space $V_{N}^{(KG)}$ on which the Klein-Gordon product is to be defined to consist of those smooth functions on $Q$ which are purely positive
frequency in the far future and are normalizable in $H_I$. Note that $\Omega^{-1} : V_N^{(KG)} \rightarrow \Omega^{-1}V_N^{(KG)}$ (defined by (3.10)) is injective. As a result, if we define the induced product on the space $V_N^{(I)} \equiv \Omega^{-1}V_N^{(KG)}$, the invertibility of the vector space map $(\Omega^{-1})$ is assured. Our schematic argument will then provide an isomorphism so long as $\hat{U}^{-1}_{KG}$ can be defined as a map from $V_N^{(I)} = \Omega^{-1}V_N^{(KG)}$ to the space of $\sigma_I$ normalizable states. Now, for $\phi \in V_N^{(KG)}$, we have

$$\Omega^{-1} \hat{U}_{KG}(\Omega^{-1}\phi) = \Omega^{-1}\phi$$

(4.5)

since $\phi$ is positive frequency in the far future. As a result, we may take $\hat{U}^{-1}_{KG} = \Omega^{-1}$ and define an isomorphism of decoherence schemes through (4.2). Note, however, that the space of states chosen for the induced scheme is considerably smaller than the Hilbert space used in, for example, [15].

For the case of a single asymptotically flat region (and such that the potential is sufficiently repulsive in the far future), it is convenient to consider the induced product scheme first. We will take the space $V_N^{(I)}$ on which $\sigma_I$ is defined to consist of those smooth solutions to the constraints with finite norm in $H_I$. The Klein-Gordon scheme will use $V = \Omega^{(-)}V_N^{(I)}$. In this case, the required isomorphism is just $(\sigma = \Omega^{(-)}, \chi = 1 \otimes 1)$. To show this, we simply note that if $\psi = \Omega^{(-)}\alpha$, $\phi = \Omega^{(-)}\beta$ for $\alpha, \beta \in V_N^{(I)}$ then

$$\psi^* \circ_{KG} U_{KG} \circ_{KG} \phi = -\pi \alpha^* \circ_I \beta$$

(4.6)

since $(\alpha^{(+)})^* \circ_{KG} \beta^{(+)} = -(\alpha^{(-)})^* \circ_{KG} \beta^{(-)}$, where $\alpha^{(\pm)}, \beta^{(\pm)}$ are the positive and negative frequency parts of $\alpha, \beta$ (in the distant past). The constant $-\pi$ is irrelevant in computing the decoherence functional and again we see that the relation between the products has led directly to an isomorphism of decoherence schemes. However, this time it is the space of states for the Klein-Gordon scheme which has to be artificially restricted.

V. CONCLUSION

Generalized quantum theory is a comprehensive framework for quantum theories of closed systems. As a result, its principles can be implemented in many ways that are different from the way they are implemented in the usual quantum theory. Classical physics, for example, can be considered as a generalized quantum theory [1]. This generality may be needed to deal with dynamical quantum spacetime geometry.

However, as the present paper illustrates, there is less freedom in the construction of generalized quantum theories than might naively be supposed. Two decoherence schemes may be equivalent in the sense that they yield identical decoherence functionals for corresponding sets of alternative histories. The predictions for the probabilities of decoherent sets of alternative histories are then isomorphic. This occurs when the elements representing histories, boundary conditions, and auxiliary structures can be mapped into one another preserving all of the associated decoherence functionals.

In this paper we have compared certain decoherence schemes appropriate for the quantum mechanics of systems, such as the relativistic world line and homogeneous minisuperspace models, which have a single reparametrization invariance. Each scheme involves a bilinear product $\circ$, but there will often be an isomorphism between decoherence schemes constructed
from two different products, or perhaps between appropriate restrictions of such schemes. We have seen that schemes based on the Klein-Gordon and induced products are equivalent (when appropriately restricted) in models with either one or two asymptotically free regions. Note that, due to the restriction of the state space, it is natural to restrict the set of class operators (or observables) as well.

Equivalences of the kind described here will be useful in narrowing the choice of decoherence functionals for reparametrization-invariant systems and, in addition, they may also allow the utilization of different techniques for calculation and approximation corresponding to the different but equivalent ways in which the decoherence functional can be expressed.

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**APPENDIX A: THE CLASS OPERATOR \( U = G^{(+)} \)**

In this appendix, we will derive a number of useful expressions for the class operator \( G^{(+)}(x'', x') \) in various AFH models. Recall that this class operator is the bi-solution of the constraint constructed from the Feynman propagator \( \Delta_F(x, y) \) for \( x \) near past infinity and \( y \) near future infinity.

We begin by considering models with two asymptotically free regions, one to the past and one to the future. For such models, the spaces of states that are normalizable in the induced and Klein-Gordon products are essentially the same. In addition, for such cases our system resembles a scattering experiment; the states may be thought of as free in both the distant past and the far future. Here, it is important to keep in mind that ‘past’ and ‘future’ as we use the words have no direct physical meaning in terms of the scattering experiment, but merely label regions of the configuration space. The most direct analogy is to, for example, a one dimensional quantum mechanical scattering problem in which the states are free both on the far left and on the far right.

In this appendix, we will make use of the space \( \mathcal{H}_{aux} = L^2(\mathbb{R}^n) \) and adapt our notation accordingly. The Feynman propagator can be defined as the operator

\[
\Delta_F = -i \int_0^\infty dN \exp(iNH) = \frac{1}{H + i\epsilon}
\]  

on this space. It is convenient to write the matrix elements of \( \Delta_F \) in terms of a complete set of eigenfunctions \( \psi^+_k \) of \( H \). Here we choose \( k = (k_0, \vec{k}) \) to label the asymptotic momenta in the far future; this is indicated by the superscript \(+\) on the wave functions. The eigenfunctions satisfy

\[
H\psi^+_k = (-k_0^2 + \vec{k}^2 + (m^+)^2)\psi^+_k = (k^2 + (m^+)^2)\psi^+_k_{k_0, \vec{k}}
\]  

(A2)
where \((m^+)^2\) is the asymptotic value of the potential in the far future. We will take these states to be normalized so as to have the inner products

\[
\langle \psi_{p_0,\mathbf{k}}^+ | \psi_{p_0,\mathbf{k}}^+ \rangle = \delta^n(k - p).
\]  

(A3)

As a result, the Feynman propagator takes the familiar form

\[
\langle x||\Delta_F||y \rangle = \int d^n k \, \frac{\psi^{+*}(x) \psi^{+*}(y)}{k^2 + (m^+)^2 + i\epsilon}.
\]  

(A4)

We wish to compute the Klein-Gordon product (on the right) of this expression with a solution \(\psi(y)\) to the constraint equation. This product can be evaluated on any hypersurface \(\Sigma\) lying to the future of the point \(x\) and having its boundary at spatial infinity. Let us therefore take this surface to lie in the far future so that \(\psi_{k}(y)\) becomes just an oscillatory exponential with wave vector \(k\). For \(y \gg |x|\), we may perform the \(k_0\) integral in (A4) by the usual method of closing the contour in the upper half plane. The contour encloses only the pole at positive frequency, so that we obtain

\[
\langle x||\Delta_F||y \rangle = 2\pi i \int d^{n-1} \mathbf{k} \, \frac{\psi^{+*}(x) \psi^{+*}(y)}{2\omega(k)} \Omega_{\omega(k)}(k, k). 
\]  

(A5)

where \(\omega(k) = \sqrt{k^2 + (m^+)^2}\). It is then clear that the Klein-Gordon product of \(\Delta_F\) and \(\psi\) yields just \(i\) times the part of \(\psi\) which is positive frequency in the far future. Denoting the associated projection by \(\Pi^+_{\pm \infty}\), we have

\[
G^{(+)} \circ KG \psi = -\Pi^+_{\pm \infty} \psi, 
\]  

(A6)

where the Klein-Gordon product is taken on a surface in the far future of the point at which both sides are evaluated.

Similarly, \(\psi \circ KG G^{(+)} = -(\Pi^+_{\pm \infty} \psi)^*\) with the product evaluated at \(x^0 \to -\infty\). Taking the Klein-Gordon product on both sides of \(\Delta_F\) (as \(x^0 \to \pm \infty\) respectively) and using the fact that the Klein-Gordon and induced products are related by a factor of \(\Omega\) allows one to derive the relation \(\Pi^+_{\pm \infty} \delta(H) \Omega = \Omega \delta(H) \Pi^+_{\pm \infty}\). The relation can also be checked by more direct means. Note that the matrix elements of the class operator \(U = G^{(+)}\) may therefore be written

\[
U(x, y) = \langle x||\Pi^+_{\pm \infty} \delta(H) \Omega||y \rangle. 
\]  

(A7)

We now turn to the case of a single asymptotic region as in Section IIA; we will take the far past to be asymptotically free. We proceed much as before, keeping in mind that we are only interested in \(\Delta_F(x, y)\) for large negative \(x\) and large positive \(y\). The matrix elements of \(\Delta_F\) can again be written in terms of the eigenfunctions of \(H\) on \(\mathcal{H}_{aux}\). Of course, because states that are purely positive or negative frequency in the far past grow exponentially in the far future, any solution to the constraint which is normalizable in the induced product contains both positive and negative frequency parts in the far past. We may, however, consider a basis of states \(\psi_{k_0, \mathbf{k}}^+\) for \(k_0 > 0\) with asymptotic momentum \(\mathbf{k}\) in the spacelike
directions and which satisfy $P_0^2 \psi_k = k_0^2 \psi_k$, where $P_0$ is the asymptotic momentum (in the far past) in the $x^0$ direction. We may then express the Feynman propagator as

$$\langle x||\Delta_F||y \rangle = \int_{k_0>0} d^n k \frac{\psi_k(x)\psi^*_k(y)}{k^2 + (m^-)^2 + i\epsilon}$$

$$= \int_{k_0} d^n k \frac{\phi_k(x)\psi^*_k(y)}{k^2 + (m^-)^2 + i\epsilon} \tag{A8}$$

where $(m^-)^2$ is the asymptotic value of the potential and $\phi_k(y)$ for $k_0 > 0$ ($k_0 < 0$) is the solution of the constraint that matches the positive (negative) frequency part of $\psi|_{k_0}$. For $x \ll -|y|$, we can again close the $k_0$ contour in the upper half plane. This time, we find

$$\langle x||\Delta_F||y \rangle = 2\pi i \int d^{n-1} k \frac{\phi_{\omega(k),\vec{k}}(x)\psi^*_{\omega(k),\vec{k}}(y)}{2\omega(k)} \tag{A9}$$

where $\omega(k) = \sqrt{k^2 + (m^-)^2}$.

Suppose now that $\phi$ is some $\omega_I$-normalizable state. Due to the form of (A9), we do not expect $\phi \omega_I G^{(+)}$ to have any well-defined meaning. However, taking the product with the state $\phi$ on the right yields

$$G^{(+)} \omega_I \phi = -2\pi \Pi^{+\infty} \phi \tag{A10}$$

where $\Pi^{+\infty}$ again denotes the `positive frequency projection' defined at $t = -\infty$, although $\Pi^{+\infty} \phi$ is not normalizable with respect to the induced product.

It is also interesting to consider the Klein-Gordon product of (A9) with a $\omega_I$ normalizable state. Now a solution $\psi_k$ is orthogonal in the Klein-Gordon product to all $\phi_p$ unless $\vec{p} = \vec{k}$ and $p_0 = \pm k_0$. As a result, taking the Klein-Gordon product of a state $\psi$ (that is normalizable in the induced product) on the left with $G^{(+)}$ yields just $\psi \omega_{KG} G^{(+)} = -\sqrt{2} \psi$. On the other hand, taking the Klein-Gordon inner product with a state $\phi$ (which is normalizable in the induced product) on the right yields $G^{(+)} \omega_{KG} \phi = 0$. This may be seen from the fact that the inner product may be taken on any hypersurface which ends only at spatial infinity. In particular, we may take this hypersurface to lie in the far future. There, however, both $\phi$ and $\psi_{\omega(k),\vec{k}}$ vanish.
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