Existence of a Variational Principle for PDEs with Symmetries and Current Conservation

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June 27, 2019

Abstract

We prove that under certain assumptions a partial differential equation can be derived from a variational principle. It is well-known from Noether’s theorem that symmetries of a variational functional lead to conservation laws of the corresponding Euler-Lagrange equation. We reverse this statement and prove that a differential equation which satisfies sufficiently many symmetries and corresponding conservation laws leads to a variational functional whose Euler-Lagrange equation is the given differential equation.

1 Introduction

In this paper we prove that a differential equation which satisfies certain kinds of symmetries and conservation laws can be written as an Euler-Lagrange equation, that is, it can be derived from a variational principle. Before we make this statement more precise we provide some motivation and explain possible applications.

The variational principle is very important in theoretical physics and it seems that all the differential equations which describe the laws of nature on a fundamental level can be derived from a variational principle, as for
example, Einstein’s field equations and the equations in the standard model. Feynman’s path integral formalism also relies on it which can be used to quantize a classical system. Since Lagrange formulated this principle in 1788 \[19\], it is hard to imagine doing theoretical physics without it.

Because the variational principle has been used very successfully in physics, it is desirable to have some explanation why it works so well, and to understand whether we can state it as a kind of superordinate axiom in physics, or whether we can prove it as a consequence of other conditions, like symmetries and conservation laws. Symmetries, like Lorentz-, Poincaré-, or certain kinds of gauge invariance can be possibly assumed for any fundamental differential equation in physics. For the conservation laws, we may assume energy-, mass-, momentum-, angular momentum-, or charge conservation, among others.

The above question, whether a differential equation which satisfies certain symmetries and conservation laws is necessarily derivable from a variational principle, was first formulated by Floris Takens in 1977. He also proved three different theorems, in which he answered this question affirmatively \[24\]. Others have generalized his proofs in a series of papers which was initiated by I. M. Anderson and J. Pohjanpelto in 1994. Scalar PDEs of second order are considered in \[4\], system of PDEs (SPDEs) of first order in \[23\], SPDEs of second order in \[6, 20\], SPDEs of third order in \[3\], and SPDEs of arbitrary order, but only polynomial expressions, are investigated in \[5\].

In the above mentioned articles there are usually additional assumptions in their statements, beside the assumptions of order, scalar PDEs, SPDEs, and polynomial structure. In \[23, 6, 20, 5\] very strong restrictions for the symmetries and conservation laws are made. For the symmetries, very often some kind of translation-, and gauge invariance is assumed. Since there is a connection between symmetries and conservation laws (similar as in Noether’s theorem), the conservation law assumptions in these theorems are also of a very special type. For example, gauge invariance leads to the condition that the differential equation must be divergence-free. In this article we generalize the case of second order SPDEs in the way that we do not restrict to very special symmetries, especially as it is done for second order SPDEs in \[6, 20\]. Symmetries are describe by a set of vector fields, and it can be shown that such a set is a Lie algebra \[14, \text{p.177}\]. Our main theorem delivers a proof for any non-abelian, or abelian Lie algebra of symmetries which spans a certain tangent space pointwise. It generalizes and unifies previously know results by Takens, Anderson, Poljanpelto, Manno, and Vitolo of a similar nature. A more detailed list of our generalizations can be found below after we have
introduce some more notation.

A differential equation \( f(x,u(x),Du(x),...) = 0 \) contains independent coordinates \( x \) and dependent coordinates \( u \). The concept of independent- and dependent coordinates is realized by a fiber bundle \( \pi : E \to M \), where \( x \) are local coordinates on the base manifold \( M \) and \( u \) are local coordinates of the fibers of \( E \). We let \( n = \dim M \) and \( n + m = \dim E \). Derivatives \( Du, D^2u \) and so on are then local coordinates on certain fibers of the jet bundle \( J^kE \). Symmetries are \( \pi \)-projectable vector fields on \( E \) which are lifted to vector fields on \( J^kE \) and this lift is called prolongation.

Instead of investigating the differential equation \( f = 0 \) directly, we consider a weak formulation of the differential equation, that is, \( \int f \varphi dx = 0 \), where \( \varphi \) are test functions with certain properties, that is, they correspond to vertical vector fields on \( E \), as we will see below. The rough formulation of our main theorem is the following (the precise formulation can be found in Theorem 4.1):

**Theorem 1.1** Let \( \pi : E \to M \) be a fiber bundle with base dimension \( n \) and fiber dimension \( m \), where \( n, m \in \mathbb{N} \) are arbitrary, and \( M \) is oriented. Furthermore, let

\[
K_f(u; \varphi) = \int_M f_{\alpha}(x,u(x),Du(x),D^2u(x))\varphi(x)dx = 0
\]

be the weak formulation of a second order system of PDEs

\[
f_{\alpha}(x,u(x),Du(x),D^2u(x)) = 0, \quad \alpha = 1,2,...,m.
\]

Assume:

1. \( K_f \) is invariant under a set of symmetry vector fields which span the tangent space \( T_pE \) at each \( p \in E \) (the invariance of \( K_f \) is explained in (15)).

2. Each symmetry vector field generates a corresponding conservation law of the form of a divergence expression (so-called continuity equation, see Definition 3.2).
Then \( f \) must be variational, that is, there exists a functional

\[
I(u) = \int_M L(x,u(x),Du(x),D^2u(x))dx
\]

with Lagrange function \( L \), such that \( K_f(u; \varphi) = \delta I(u; \varphi) \), where \( \delta I \) is the first variation of \( I \).

Later, the weak formulation \( K_f \) will be replaced by the so-called source form \( \Delta = f_\alpha du^\alpha \wedge dx \), see (13), and we also consider the problem only locally, that is, for sufficiently small subsets \( U \subset E \) and corresponding subset of \( J^k E \) (written as \( (\pi^{k,0})^{-1}U \)). We state our main theorem precisely in Section 4, but here we highlight in which ways it generalizes previously known results for second order system of PDEs:

- We do not make strong symmetry assumptions, like translation-, and gauge invariance in \([6, 20]\). We only assume that the symmetry vector fields span the tangent space \( T_p E \) pointwise, see (32).

- As already mentioned, there is a connection between symmetries and conservation laws. Therefore, with the span-condition (32) we do not make strong conservation law assumptions, like divergence-free in \([6, 20]\). The divergence-free-condition forces a certain polynomial structure of the differential equations in the second order coordinates at the very beginning and, for example, the Monge-Ampère equation is excluded. In our theorem the Monge-Ampère equation is not excluded.

- We do not need to assume that the set of symmetry vector fields, or certain subalgebras, are an abelian Lie algebra which is the case for translation invariance in \([6, 20]\). Our proof works for non-abelian, or abelian Lie algebras of any type which satisfy the mentioned span-condition (32).

- We do not need to assume that \( n = m \), that is, the dimension of the base manifold \( M \) is equal to the dimension of the fibers of \( E \) which is 1.

\[1\] While in many classical cases the Lagrangian can be chosen to depend on \((x,u,Du)\) only, there are cases in which second derivatives \( D^2u \) are needed, for example the Monge-Ampère equation \( u_{xx}u_{yy} - u_{xy}u_{xy} = 0 \).
assumed in [6]. There are also restrictions for the fibers of $E$ in [20].

Our theorem works for any $n,m$ without restrictions on the fibers of $E$. For example, the vanishing of $H_{ab,i}^{D}$ in [6] is derived when explicitly using $n = m$.

In a sense, our result can be seen as the most general theorem for arbitrary system of PDEs of second order. Weakening our symmetry assumptions is only possible when restricting to first order systems which are free of topological obstructions, see [23], or having other restrictions, like equations which are polynomial in their arguments $u$ and derivatives $Du, D^2u$ and so on. Furthermore, we introduce a new inductive method to prove our main theorem. In our opinion, this method is much easier to handle and understand than, for example, the so-called $d$-fold method used in [5, Lemma 2.3] and [20, p.12] to solve similar problems. One reason for this is that the inductive method reduces a big problem to several very simple problems and the steps are easy to check. On the other hand, the $d$-fold method solves the mentioned problems in one relatively complicated step which is also hard to check. Of course, both methods can be useful in different situations and should be developed further.

Notice that the assumptions of symmetries and conservation laws in Theorem 1.1 are in some sense very natural, when we recall Noether’s theorem [21]. Theorem 1.1 can be seen as a reverse of Noether’s theorem. Actually, there are several so-called Noether’s theorems which should be distinguished here. Let $f(x,u(x),Du(x),...)=0$ be a variational differential equation with corresponding functional $I$. Roughly speaking, Noether’s theorems can be formulated as

\[ \text{‘}f \text{ is variational'} \& \text{ ‘}K_f \text{ is invariant'} \Rightarrow \text{ ‘}f \text{ satisfies conservation laws'}, \]

\[ \text{‘}f \text{ is variational'} \& \text{ ‘}f \text{ satisfies conservation laws'} \Rightarrow \text{ ‘}K_f \text{ is invariant’}. \]

Beside the both formulations of Noether’s theorems, we usually also distinguish the cases whether the symmetries are given by a finite dimensional group of transformations (also called Noether’s first theorems), or if they are given by an infinite dimensional group of transformations (also called Noether’s second theorems). Theorem 1.1 is to prove

\[ \text{‘}K_f \text{ is invariant'} \& \text{ ‘}f \text{ satisfies conservation laws'} \Rightarrow \text{ ‘}f \text{ is variational’}. \]
The paper is organized as follows. In Section 2, we present a simple instructive example which should help the reader to understand the problem in more detail, but without introducing much notation. In Section 3, we introduce our notation which is the notation of jet bundles, and we precisely formulate what we mean by a differential equation, symmetry, conservation law, and what it means to have a variational formulation. The main theorem and proof can be found in Section 4. Some of the longer calculations of Section 4 are carried out in Section 5 and 6. Finally, in Section 7 we provide a critical discussion of our result, explain the main open problems, and discuss some applications in physics.

Acknowledgment This article is a result of my Ph.D. thesis [8], written at the Carl von Ossietzky University Oldenburg, and I want to thank my supervisor Prof. Dr. Daniel Grieser very much for supporting this project with helpful advice and comments about this paper. In particular, I want to thank Daniel Grieser for giving me a lot of freedom in my research and supporting my interests. I also want to thank Prof. Dr. Ian M. Anderson for fruitful discussions about Takens’ problem during my stay at Utah State University and for a very pleasant time.

2 A simple instructive example

In this example we consider the differential equation $f = 0$ itself, instead of the weak formulation $\int f \varphi dx = 0$. This is because we then need to introduce less notation, and for translations $f$ and $K_f$ satisfy the same symmetry and both formulations are in some sense equivalent. Let $f(x,u(x),u_x(x),u_{xx}(x)) = 0$ be a scalar differential equation for the unknown function $u : \mathbb{R} \rightarrow \mathbb{R}$, where $f = u + u_{xx}$. As usual, $u_x(x) = \frac{du(x)}{dx}$ and $u_{xx}(x) = \frac{d^2u(x)}{dx^2}$. This equation is translation invariant in $x$-direction, that is, $f(x+s,u,u_x,u_{xx}) = f(x,u,u_x,u_{xx})$ for all $s \in \mathbb{R}$. The infinitesimal generator of this symmetry, written as

$$\frac{d}{ds} f(x+s,u,u_x,u_{xx})|_{s=0} = \frac{\partial}{\partial x} f(x,u,u_x,u_{xx}) = 0 \text{ for all } (x,u,u_x,u_{xx}),$$

is the vector field $\frac{\partial}{\partial x}$. When we consider $x$ as the time variable, and $u$ as the (one-dimensional) position of a particle at the time $x$, then the equation
\[ f = u + u_{xx} = 0 \] satisfies energy conservation. More precisely, when we multiply \( f \) by \( u_x \), then we get

\[ u_x f = u_x (u + u_{xx}) = \frac{d}{dx} \left( \frac{1}{2} u_x^2 + \frac{1}{2} u^2 \right) = 0 \quad \text{on solutions, where } f = 0 \]

\[ \Rightarrow \quad \frac{1}{2} (u_x^2 + u^2) = \text{constant} \quad \text{on solutions, where } f = 0. \quad (1) \]

The term \( \frac{1}{2} (u_x^2 + u^2) \) can be interpreted as the kinetic plus the potential energy of the particle and it delivers a conservation law, called energy conservation.

As we already mentioned above, the structure of the theorem we prove can be written as

'\( f \) (or \( K_f \)) is invariant' \& '\( f \) satisfies conservation laws' \( \Rightarrow \) '\( f \) is variational'.

The fact that \( f = u + u_{xx} \) is variational is easy to check in this case. For example, let us consider the Lagrangian \( L = \frac{1}{2} u^2 - \frac{1}{2} u_x^2 \), and we get

\[ \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u_x} = u + u_{xx} = f. \]

It is well-known, from Noether’s first theorem, that time translation invariance leads to energy conservation, and this correlation can be observed in our example here. Actually, so far, in this example we did not have to prove anything, but now we may conjecture the following (the conjecture is actually true, as we show in the Appendix):

Any scalar differential equation \( f(x,u,u_x,u_{xx}) = 0 \) of second order, where \( u : \mathbb{R} \to \mathbb{R} \), which is \( \frac{\partial}{\partial x} \)-invariant, and which has a corresponding conservation law of the form \( u_x f = \frac{d}{dx} E \), where \( E \) is a no more specified (energy) function, can be written as the Euler-Lagrange equation for some Lagrangian \( L \). (Let us also assume that the functions \( f \) and \( E \) are smooth in \( x,u,u_x \) and so on.)

The proof does not require a lot of non-standard notation in this scalar ODE case and one can easily check the proof in the Appendix first, to get an understanding of the problem.

### 3 Preliminaries

Let \( x = (x^i)_{i=1,2,...,n} \) denote the independent coordinates, and \( u = (u^\beta)_{\beta=1,2,...,m} \) the dependent coordinates which are used to describe a \( k \)-th order system of
PDEs $f(x,u(x),Du(x),...,D^ku(x)) = 0$, where we use the short notation

$$D^l u(x) := \left( \frac{\partial^l u^\alpha(x)}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_l}} \right)_{i_1,j_2,...,i_l=1,2,...,n}, \quad 1 \leq l \leq k,$$

for the set of all possible $l$-th order partial derivatives of $u(x)$. When we want to consider PDEs on manifolds, or when we want to make our descriptions independent of the choice of coordinates, then it is reasonable to use the language of fiber- and jet bundles.

**The jet bundle:** Let $\pi : E \to M$ be a fiber bundle with local coordinates $(x,u)$ on $E$ and adapted local coordinates $x$ on $M$, that is, $u$ describes the coordinates of the fibers. We have $\dim E = n + m$ and $\dim M = n$. Local charts on $E$ are written as $(U,\varphi)$, where $U \subset E$ is open, and we have $\varphi : U \to \mathbb{R}^{n+m}$. Furthermore we write $(\bar{U},\varphi^0)$ for local charts on $M$, where $\bar{U} \subset M$ is open, and we have $\varphi^0 : \bar{U} \to \mathbb{R}^n$. Adapted local coordinates means that $\bar{U} = \pi U$ and that $\varphi, \varphi^0$ satisfy $\varphi^0(\pi p) = \pi^\varphi(p)$ for all $p \in U$, where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the canonical projection.

(Local) sections of $\pi$ are written as $\sigma : \Omega \to E$, where $\Omega \subset M$, and $\pi \sigma$ is the identity map on $\Omega$. We have $\varphi \circ \sigma \circ (\varphi^0)^{-1}(x) = (x,s(x))$ on $\varphi^0(\Omega \cap \bar{U})$, and in local coordinates the section $\sigma$ can be identified with the function $s : \mathbb{R}^n \to \mathbb{R}^m$. In fiber bundle notation, a differential equation is written as $f(x,s(x),Ds(x),...,D^k s(x)) = 0$, where $f$ is a function defined on the so-called jet bundle, and solutions are certain sections of $\pi$.

The $k$-th order jet bundle of $E$ is written as $J^kE$ and it has (adapted) local coordinates

$$\left( x^i, u_0^{\alpha_0}, u_{i_1}^{\alpha_1}, \ldots, u_{i_1i_2...i_k}^{\alpha_k} \right)_{i,j_1,...,i_j=1,2,...,n}$$

where the indices $i_1i_2...i_l$ in the coordinates $u_{i_1i_2...i_l}^{\alpha}$ are ordered as $1 \leq i_1 \leq i_2 \leq \ldots \leq i_l \leq n$ for every $1 \leq l \leq k$. These local coordinates are in one to one correspondence with a $k$-th order equivalence class $[\sigma]^k(q)$ of local sections of $\pi$ at a point $q \in M$, where $\varphi^0(q) = x$, such that all local sections in this equivalence class satisfy

$$u^\alpha = s^\alpha(x) \big|_{\bar{x}=x}, \quad \alpha = 1,2,...,m,$$

$$u_{i_1i_2...i_l}^{\alpha} = \frac{\partial^l s^\alpha(x)}{\partial x^{i_1} \cdots \partial x^{i_l}} \big|_{\bar{x}=x}, \quad \alpha = 1,2,...,m, \quad 1 \leq i_1 \leq i_2 \leq \ldots \leq i_l \leq n, \quad (2)$$
where we define $J$. We make the identification $u$ also comes out naturally by (2), since partial derivatives commute. Even the calculations below much easier. Therefore, we call the number $l = 0$ for higher order coordinates on $J^k E$. Then a point in $J^k E$ is given by the coordinates $(x,u,\partial u,\ldots,\partial^k u)$. We call $(x,u,\partial u,\ldots,\partial^k u)$ the $k$-th order jet coordinates and a single coordinate $u_{i_1 i_2 \ldots i_l}$ runs from 1 to $n$ and there is no ordering of $i_1 \ldots i_l$ in the Einstein summation.

Let us introduce the short notation $\partial^l u := (u_{i_1 i_2 \ldots i_l})_{i_1 i_2 \ldots i_l = 1,2,\ldots,n}$, $1 \leq l \leq k$ for higher order coordinates on $J^k E$. Then a point in $J^k E$ is given by the coordinates $(x,u,\partial u,\ldots,\partial^k u)$. We call $(x,u,\partial u,\ldots,\partial^k u)$ the $k$-th order jet coordinates and a single coordinate $u_{i_1 i_2 \ldots i_l}$ runs from 1 to $n$ and there is no ordering of $i_1 \ldots i_l$ in the Einstein summation.

The jet bundle has different types of projections:

\[
\pi^k : J^k E \to M,
\]

\[
\pi^{k,0} : J^k E \to E,
\]

\[
\pi^{k,l} : J^k E \to J^l E, \quad 0 \leq l \leq k,
\]

where we define $J^0 E = E$ and $\pi^{k,k}$ is the identity map on $J^k E$. These projections are an immediate consequence of the definition of $J^k E$, since we can always map a point $[\sigma]_k(q) \in J^k E$ to a point $\pi^{k,l}[\sigma]_k(q) = [\sigma]_l(q) \in J^l E$, $l \leq k$, when ignoring the equivalence of higher order derivatives of sections $s$ in the equivalence class $[\sigma]_k(q)$, see (2).

If $g \in C^\infty(J^k E)$ then we call the number

\[
l = \min_{h \in C^\infty(J^k E),\ g = (\pi^{k,l}h)^*h} \{ r \}, \tag{3}
\]

the order of $g$. For example, $u_{i}^{\beta} u_{kl}$ is a second order function.

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\footnote{For example, later we will introduce the so-called total derivative operator $D_i$ and it satisfies $D_i u_{i_1 i_2 \ldots i_l}^\alpha = u_{i_1 i_2 \ldots i_l}^\alpha$ when we use the identification and we do not have to care in which position we have to write the index $i$. Another reason is that the identification also comes out naturally by (2), since partial derivatives commute.}
Prolongation of sections: As already mentioned, points in $J^k E$ are given as equivalence class of sections of $\pi$. There is a very special set of points in $J^k E$, namely the set which consists of a so-called prolonged section. Let $\sigma: \Omega \to E, \Omega \subset M$ be a (local) section of $\pi$. Then we define $\text{pr}^k \sigma: \Omega \to J^k E$ as the (local) section of $\pi^k$, which maps $q \mapsto [\sigma]_k(q)$ for every $q \in \Omega$. This map is called **prolongation of $\sigma$**, see [17, p.30]. Prolongation lifts a section of $\pi$ to a section of $\pi^k$. Notice that every point $[\sigma]_k(q) \in J^k E$ can also be written as $\text{pr}^k \sigma(q)$ for a certain section $\sigma$ of $\pi$ and a certain $q \in M$. Sometimes it is appropriate to write $\text{pr}^k \sigma(q)$ and sometimes $[\sigma]_k(q)$.

Vector fields: Let $\mathfrak{X}(J^k E)$ denote the set of vector fields on $J^k E$. In local coordinates, a vector field $V \in \mathfrak{X}(J^k E)$ is written as

$$V = V^i \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial u^\alpha} + V^\alpha_{i_1 i_2 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_k}} + \ldots + V^\alpha_{i_1 i_2 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_k}}.$$

Notice that we may assume that $V^\alpha_{i_1 \ldots i_l} = V^\alpha_{\tau(i_1) \ldots \tau(i_l)}$ for every permutation $\tau$, since by our previous definition we have $\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_l}} = \frac{\partial}{\partial u^\alpha_{\tau(i_1) \ldots \tau(i_l)}}$, and we may assume that the coefficients $V^\alpha_{i_1 i_2 \ldots i_l}$ are completely symmetrized in the indices $i_1 i_2 \ldots i_l$. There are $\frac{n!}{l_1! l_2! \ldots l_n!}$ possibilities how to rearrange the numbers $i_1, i_2, \ldots, i_l$, where $l_r$ is the number of occurrences of the index $r$ in the multi-index $i_1 i_2 \ldots i_l$ and we have $1 \leq r \leq n$. Therefore, let us also define the weighted partial derivatives

$$\partial_i := \frac{\partial}{\partial x^i}, \quad \partial_\alpha := \frac{\partial}{\partial u^\alpha}, \quad \partial^{i_1 i_2 \ldots i_l}_\alpha := \frac{l_1! l_2! \ldots l_n!}{l!} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_l}}, \quad 1 \leq l \leq k,$$

see [23].

On fiber bundles, there are certain vector fields with additional structure, given by the projection map $\pi$. We call a vector field $V \in \mathfrak{X}(E)$ **$\pi$-projectable** if $\pi_* V$ exists and if it is a vector field on $M$. Moreover, we call a vector field $V \in \mathfrak{X}(E)$ **$\pi$-vertical** if $\pi_* V = 0$. In local coordinates, these

3When using concrete labels, like $V^1$, then we write $V^1_x$, or $V^1_u$ to indicate if it belongs to a coefficient $V^i$, or $V^\alpha$.

4Again, when we have concrete labels, like $\partial_1$, then we write $\partial_{x_1}$ or $\partial_{u_1}$ to indicate if it belongs to $\partial_i$ or $\partial_\alpha$.
vector fields are written as
\[ V = V^i(x) \partial_i + V^\alpha(x,u) \partial_\alpha, \quad (\pi\text{-projectable vector field on } E), \]
\[ V = V^\alpha(x,u) \partial_\alpha, \quad (\pi\text{-vertical vector field on } E). \]

Notice that the flow of a \(\pi\)-projectable vector field transforms sections of \(\pi\) to sections of \(\pi\) and for this reason they will be very important later on. Since \(J^kE\) has different types of projections \(\pi^k, \pi^{k,l}\), we also define \(\pi^k, \pi^{k,l}\text{-projectable}\) and \(\pi^k, \pi^{k,l}\text{-vertical}\) vector fields in the same way.

There is another kind of vector field we want to mention here and this is the so-called total derivative. The total derivative on \(J^kE\), written with respect to some local coordinates on \(J^{k+1}E\), is defined as
\[ D_i = D_i(k) := \partial_i + u^\alpha_i \partial_\alpha + u^\alpha_{i_1} \partial_\alpha^{i_1} + ... + u^\alpha_{i_1i_2...i_k} \partial_\alpha^{i_1i_2...i_k}, \quad (4) \]
and it is a kind of vector field, however it is not vector field in the usual sense, since the last coefficients \(u^\alpha_{i_1i_2...i_k}\) are not defined in \(J^kE\), rather in \(J^{k+1}E\). A coordinate invariant definition is the following one: Let \(V = V^i \partial_i\) be a vector field on \(M\) and \(\phi_t^0\) its flow on \(M\). Furthermore, let \([\sigma]_{k+1}(q)\) be a point in \(J^{k+1}E\). Then we assign a tangent vector \(\text{tot}^kV\) at the point \(\pi^{k+1,k}[\sigma]_{k+1}(q) = [\sigma]_k(q) \in J^kE\) as follows: We take any local section \(\sigma : \Omega \to E\) in the equivalence class \([\sigma]_{k+1}(q)\) and we compute the tangent vector
\[ \text{tot}^kV|_{[\sigma]_k(q)} := \frac{d}{dt} \text{pr}^k \sigma(\phi_t^0(q))|_{t=0} \]
at the point \([\sigma]_k(q)\) which always works for sufficiently small \(t\), such that we have \(\phi_t^0(q) \in \Omega\). The tangent vector \(\text{tot}^kV|_{[\sigma]_k(q)}\) depends on the equivalence class \([\sigma]_{k+1}(q)\). When we consider the union of all these tangent vectors at all points \([\sigma]_k(q) \in J^kE\) then we get a kind of vector field on \(J^kE\). However, this is not a vector field in the usual sense as a map \(J^kE \to TJ^kE\), it is rather a map \(J^{k+1}E \to TJ^kE\) (sometimes called a vector field along the map \(\pi^{k+1,k}\), or a generalized vector field). In local coordinates, we get \(\text{tot}^kV = V^i D_i\), where we again find the total derivative on \(J^kE\). We call \(\text{tot}^kV\) the total vector field of \(V\) on \(J^kE\). More generally, for every \(\pi^k\)-projectable vector field \(W\) on \(J^kE\) we can assign a unique total vector field \(\text{tot}^kW\) by the above construction, where \(V = \pi^kW\). This construction is also called horizontalization of \(W\), see [14] p.17. For later purposes it is important to notice here that when we apply a total vector field to functions we get a map \(\text{tot}^kV : C^\infty(J^kE) \to C^\infty(J^{k+1}E)\), that is, the order is increased by one. This also means that when we write several total derivatives repeated, like \(D_iD_j\), we actually mean \(D_i(k+1)D_j(k)\). Fur further details also see [11][15].

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Prolongation of flows and vector fields: As we saw above, a section of \( \pi \) can be lifted to a section of \( \pi^k \) and we called this lift prolongation of the section. In a similar way we can lift a \( \pi \)-projectable vector field on \( E \) to a vector field on \( J^kE \). Let \( V \) be a \( \pi \)-projectable vector field on \( E \) and \( \phi_t \) its flow. Moreover, let \( \pi_*V \) be the corresponding vector field on \( M \) with flow \( \phi_t^0 \). It can be shown that \( \phi_t \circ \sigma \circ (\phi_t^0)^{-1} \) is a section of \( \pi \) (basically by showing that \( \pi(\phi_t \circ \sigma \circ (\phi_t^0)^{-1})(q) = q \) for all \( q \in \Omega \subset M \)). Now we can prolong this section and we get a section of \( \pi^k \). Let us define \( \text{pr}^k \phi_t \) as the prolongation of the flow \( \phi_t \) through

\[
\text{pr}^k \phi_t(\text{pr}^k \sigma(q)) := \text{pr}^k[\phi_t \circ \sigma \circ (\phi_t^0)^{-1}] (\phi_t^0(q))
\]

for every point \( \text{pr}^k \sigma(q) \in J^kE \). It is a map \( \text{pr}^k \phi_t : J^kE \to J^kE \) and it is also an isomorphism, see [17, p.32]. The section \( \phi_t \circ \sigma \circ (\phi_t^0)^{-1} \) depends on the parameter \( t \), and when we prolong it and take the derivative with respect to \( t \)

\[
\text{pr}^k V(\text{pr}^k \sigma(q)) := \frac{d}{dt} \text{pr}^k[\phi_t \circ \sigma \circ (\phi_t^0)^{-1}] (\phi_t^0(q))|_{t=0},
\]

we get a vector field \( \text{pr}^k V \) on \( J^kE \) at \( t = 0 \), depending on the point \( \text{pr}^k \sigma(q) \in J^kE \) which is called the prolongation of the vector field \( V \). Such vector fields are used to describe symmetries of differential equations, or to describe perturbations of integral functionals in the calculus of variations. In local coordinates, a prolonged vector field can be written as

\[
\text{pr}^k V = V^i \partial_i + V^\alpha \partial_\alpha + \xi_1^\alpha \partial_1^i + \xi_2^\alpha \partial_2^i + \ldots + \xi_l^\alpha \partial_{l+1}^i + \ldots + \xi_k^\alpha \partial_{k+1}^i,
\]

where the \( \xi \)-coefficients are defined recursively as

\[
\xi_1^\alpha = D_i V^\alpha - u_i^\alpha D_i V^i,
\]

\[
\xi_{i_1i_2 \ldots i_l}^\alpha = D_i \xi_{i_1i_2 \ldots i_l}^\alpha - u_i^\alpha \xi_{i_1i_2 \ldots i_{l-1}}^\alpha - u_{i_1i_2 \ldots i_{l-1}}^\alpha D_i V^i, \quad 2 \leq l \leq k,
\]

see [17] p.32 and [14] p.26. It also turns out, that on \( J^{k+1}E \), the prolongation of \( V \) can be written equivalently as

\[
\text{pr}^{k+1} V = V^i D_i + V_{ev}^\alpha \partial_\alpha + (D_{i_1} V_{ev}^\alpha) \partial_1^i + \ldots + (D_{i_1} D_{i_2} \ldots D_{i_k} V_{ev}^\alpha) \partial_k^i,
\]

where \( (V_{ev}^\alpha) := (V^\alpha - u_i^\alpha V^i) \), and \( V_{ev} = V_{ev}^\alpha \partial_\alpha \) is called evolutionary vector field of \( V \), see [14] p.198. The right hand side in (6) is not defined in \( J^kE \),
rather in \( J^{k+1}E \), the same as with the total derivative operator. If there is no danger of confusion we either write (5), or (6) for pr\( ^k \)\( V \) and we do not explicitly notice one or the other. The formal prolongation of \( V \) is defined as

\[
\text{pr}^k V_{ev} := V_{ev}^\alpha \partial_\alpha + (D_i V_{ev})^\alpha \partial_i^\alpha + \ldots + (D_{i_1} D_{i_2} \ldots D_{i_k} V_{ev})^\alpha \partial_{i_1 \ldots i_k}^\alpha.
\]

Then (6) can be written as pr\( ^k \)\( V \) = tot\( ^k \)\( V \) + pr\( ^k \)\( V_{ev} \). Roughly speaking, pr\( ^k \)\( V \) decomposes into a total-, and a \( \pi \)-vertical part, but the decomposition cannot be done in \( J^k E \), rather in \( J^{k+1} E \).

When we think of \( \pi \)-projectable vector fields (or their prolongations) as symmetries and Lie algebras, the following might be helpful to know: If \( V,W \) are \( \pi \)-projectable vector fields on \( E \) and if they form a Lie algebra, then pr\( ^k \)\( V \), pr\( ^k \)\( W \) also form a Lie algebra and we have \([\text{pr}^k V,\text{pr}^k W] = \text{pr}^k [V,W] \), where \([\ldots]\) is the commutator, see [14, p.29].

**Zero-functionals and the Lagrange form:** In the calculus of variations we investigate integral functionals of the form

\[
I_\sigma = \int_\Omega (\text{pr}^k \sigma)^* (L dx) = \int_\Omega L(x,s(x),D_s(x),\ldots,D^k s(x)) dx,
\]

where \( L = L(x,u,\ldots,\partial^k u) \) is the Lagrange function, the closure \( \bar{\Omega} \subset M \) is a compact set, and \( \Omega \) is oriented. We use the short notation \( dx := dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \). We also call (7) a **zero-functional**. The so-called **Lagrange form**

\[
\lambda := L dx
\]

is a main ingredient in the integral functional (7), and for many calculations it is sufficient to consider \( \lambda \) instead of (7). For example, when we want to prove Noether’s theorem, see Theorem 3.3. For an intrinsic definition of \( \lambda \) and further information see [14] in §4. Let \( \phi_t \) be the flow of a \( \pi \)-projectable vector field \( V \in \mathfrak{X}(E) \) and \( \phi_t^0 \) the flow of \( \pi_* V \). The transformation of (7) under \( \phi_t \) is defined as

\[
\phi_t I_\sigma := \int_{\phi_t^0 \Omega} \text{pr}^k (\phi_t \circ \sigma \circ \phi_{-t}^0)^* \lambda = \int_{\Omega} (\text{pr}^k \sigma)^* [(\text{pr}^k \phi_t)^* \lambda],
\]

see [17, p.42] and [14, p.111] (we may assume that the transformation is defined for at least sufficiently small \( t \)). In the following, \( \mathcal{L}_W \) denotes the Lie
derivative with respect to the vector field $W$. The expression (8) allows us to define a formal Lie derivative for integral functionals as

$$L_{\text{pr}V} I_\sigma := \frac{d}{dt} \int_{\phi_0^\ast \Omega} \text{pr}^k (\phi_t \circ \sigma \circ \phi_0^{-1})^\ast \lambda |_{t=0} = \int_{\Omega} (\text{pr}^k \sigma)^\ast (L_{\text{pr}V} \lambda),$$  \tag{9}$$

where $L_{\text{pr}V} \lambda$ is the Lie derivative in the usual sense, applied to the differential form $\lambda$, see [17, p.42] and [14, p.111]. The formal Lie derivative in (9) can be used for two things:

1. When $V \in \mathfrak{X}(E)$ is a $\pi$-vertical vector field, such that $\text{supp} \ V \subset \pi^{-1} \Omega$, then (9) describes the first variation of $I$ in direction of $V$. Support in $\pi^{-1} \Omega \subset E$ implies that $V$ must vanish at the boundary $\pi^{-1} \partial \Omega \subset E$ and this means that we can do partial integration without getting boundary terms which will lead to the Euler-Lagrange equation.

2. Symmetries of $I$ are defined as $\pi$-projectable vector fields $V \in \mathfrak{X}(E)$, such that (9) vanishes for all sections $\sigma$ of $\pi$.

**One-functionals and the source form:** Let $R \in \mathbb{N}$ and

$$(\text{pr}^k \sigma)^\ast f_r = f_r(x,s(x),Ds(x),...,D^k s(x)) = 0, \quad r = 1,2,...,R \tag{10}$$

be a system of PDEs, where $f_r$ are functions defined on $J^k E$, and the section $\sigma$ is a solution of the differential equation. We can multiply (10) by test functions $\varphi_r = \varphi_r(x)$ which are defined on $\Omega \subset M$, the closure $\bar{\Omega}$ is compact, and we can integrate over $\Omega$, such that we get

$$\tilde{K}_\sigma(\varphi) = \int_{\Omega} \sum_{r=1}^{R} (\text{pr}^k \sigma)^\ast (f_r \varphi_r) dx = 0, \quad \text{(for all test functions)} \tag{11}$$

which is known as a **weak formulation of** (10). As usual, test functions are $C^\infty(M)$-functions with support in $\Omega$. When $R = m$, then we can define a very special weak formulation, namely

$$\tilde{K}_\sigma(V) = \int_{\Omega} (\text{pr}^k \sigma)^\ast (f_\alpha V^\alpha) dx = 0, \tag{12}$$
where the test functions $\varphi_r$ can be identified with $\pi$-vertical vector fields $V = V^\alpha \partial_\alpha \in \mathfrak{X}(E)$, and they have the additional property that their support is in $\pi^{-1}\Omega \subset E$, such that $(\text{pr}^k \sigma)^*V^\alpha$ is a test function with support $\Omega$. The Einstein summation in (12) shows that $(f_\alpha)$ now must have a very specific transformation property under local coordinate changes which is not necessarily the case for any equation of the form (10). For example, let $L = L(x,u,\partial u)$ be a first order Lagrangian, then the first variation (9) leads to such a particular weak formulation, where $f_\alpha = (\partial_\alpha - D_i \partial_\alpha^i)L$ is the so-called Euler-Lagrange expression which satisfies this transformation property. We can also work with (12) when $(f_\alpha)$ cannot be written as an Euler-Lagrange expression, but the transformation property for $(f_\alpha)$ must be the same as for Euler-Lagrange equations. More precisely, it is given as $f_\alpha \rightarrow f_\alpha \frac{\partial u^\alpha}{\partial y^r} \det \frac{\partial x}{\partial y}$, where $\frac{\partial x}{\partial y}$ is a short notation for the Jacobian matrix and the $\det \frac{\partial x}{\partial y}$-term comes from the differential $dx$ in (12). As we have seen above, that the main ingredient of zero-functionals is the Lagrange form $\lambda$, we now want to find in a similar way the main ingredient of the weak formulation (12). It turns out that this is the so-called source form

$$\Delta := f_\alpha du^\alpha \wedge dx,$$

which satisfies the above mentioned transformation property. The source form is a $(n+1)$-form on $J^kE$. For an intrinsic definition and further details see [24] and [25, p.560]. With the help of the source form, we define the one-functional

$$K_\sigma(\text{pr}^k V) := \int_{\Omega} (\text{pr}^k \sigma)^*((\text{pr}^k V) \cdot \Delta),$$

for every $\pi$-projectable $V \in \mathfrak{X}(E)$. Then (12) can be written equivalently as $K_\sigma(\text{pr}^k V) = 0$ for all $\pi$-vertical $V \in \mathfrak{X}(E)$ with $\text{supp} V \subset \pi^{-1}\Omega$. We now want to define how to transform one-functionals under the flow of $\pi$-projectable vector fields. This can be used, for example, to define what it means that a one-functional is invariant under certain transformations, see Theorem 1.1. Let $K$ be a one-functional. We define the transformation of

---

5Because $R = m$, and because of the specific transformation of $(f_\alpha)$, we cannot assign a source form to every differential equation of the form (10) and the assignment is also not unique. For example, let $R = 2$ and assume $(f_r)$ is given, then we can assign $\Delta = f_1 du^1 \wedge dx + f_2 du^2 \wedge dx$, or $\Delta = f_2 du^1 \wedge dx + f_1 du^2 \wedge dx$. Also see the discussion in Section 7.
$K$ under $\phi_t$ as

$$(\phi_t K)_\sigma(pr^k V) := \int_\Omega (pr^k \sigma)^* \{ (pr^k V) \lrcorner ((pr^k \phi_t)^* \Delta) \}$$

for every section $\sigma$ and for every prolonged vector field $pr^k V$. Moreover, we define the formal Lie derivative of $K$ with respect to $pr^k W$ as

$$(L_{pr^k W} K)_\sigma(pr^k V) := \frac{d}{dt}(\phi_t K)_\sigma(pr^k V)|_{t=0} = \int_\Omega (pr^k \sigma)^* [pr^k V \lrcorner (L_{pr^k W} \Delta)],$$

(15)

where $W \in \mathfrak{X}(E)$ is $\pi$-projectable and $\phi_t$ the flow of $W$. We say that $K$ is invariant under the $\pi$-projectable vector field $W \in \mathfrak{X}(E)$ if (15) vanishes for all sections $\sigma$ and for all $pr^k V$. This is equivalent to $L_{pr^k W} \Delta = 0$ (which is a version of the Noether-Bessel-Hagen equation, see [14, p.177]).

**Integrability conditions:** Let us consider a one-functional $K$, see (14). We say that $K$ is variational, if there exists a zero-functional $I$, such that

$$(L_{pr^k V} I)_\sigma = K_\sigma(pr^k V)$$

(16)

for every section $\sigma$, and for every $\pi$-vertical $V \in \mathfrak{X}(E)$ with $\text{supp} V \subset \pi^{-1}\Omega$. This is analogous how we define an exact one-form $\omega$, where a zero-form $\eta$ must exist, such that the equation

$$(L_v \eta)_p = (v \lrcorner \omega)_p$$

(17)

is satisfied for all vector fields $v$ and for all points $p$. The only difference is that (16) only holds for certain vector fields with the additional assumptions ‘$\pi$-vertical’ and ‘support in $\pi^{-1}\Omega \subset E$’, whereas (17) holds for all vector fields, and that the points in the calculus of variations are described by an infinite dimensional space of sections, whereas the points $p$ in (17) are described by a finite dimensional space. Notice that (16) is a global definition. By partial integration, the definition in (16) can be transferred to source forms and to the coefficients $(f_\alpha)$, where we get the following definition:

**Definition 3.1** Let $\Delta = f_\alpha du^\alpha \wedge dx$ be a source form on $J^k E$. We say that $\Delta$ is locally variational, if for each $p \in E$ there exists a neighborhood $U \subset E$, 

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a Lagrange form $\lambda = Ldx$ on $(\pi^{k,0})^{-1}U \subset J^k E$, and local coordinates on $(\pi^{k,0})^{-1}U \subset J^k E$, such that we can write

$$f_\alpha = (\partial_\alpha - D_i \partial^{i}_\alpha + \ldots + (-1)^k D_i D_j \ldots D_k \partial^{i_1i_2\ldots i_k}_\alpha) L$$

(18)
on $(\pi^{k,0})^{-1}U$. If there exists a Lagrange form $\lambda$ on $J^k E$, such that we can write $\Delta$ as (18) in every local coordinate system, then we say $\Delta$ is globally variational.

Notice that (18) is independent of the choice of local coordinates. For a coordinate independent definition also see [26].

Now we want to derive conditions for $(f_\alpha)$ under which they can be written as (18) for some $L$. We start the discussion in a simple case. Let us assume that we have a source form $\Delta = f_\alpha du^\alpha \wedge dx$ of 0-th order, that is, where $(f_\alpha)$ only depend on $(x,u)$. Then it is easy to check that $\Delta$ is locally variational if there exists a Lagrange form $\lambda = Ldx$ on $E$, such that locally $\Delta = d\lambda$.

A necessary and sufficient condition for a 0-th order source form $\Delta$ to be locally variational is that $d\Delta = 0$, that is, $\Delta$ must be closed and this leads to the integrability conditions $\partial_\beta f_\alpha(x,u) - \partial_\alpha f_\beta(x,u) = 0$. This is well-known by the De-Rham theory. For higher order source forms, that is, where $(f_\alpha)$ depend on $(x,u,\partial u,\ldots,\partial^k u)$, the necessary and sufficient conditions for $\Delta$ to be locally variational are the so-called Helmholtz conditions. When $\Delta$ is of second order, then they are given by the equations

$$\begin{align*}
\partial_\beta f_\alpha - \partial_\alpha f_\beta + D_i \partial^{i}_\alpha f_\beta - D_i D_j \partial^{ij}_\alpha f_\beta &= 0, \\
\partial_\beta f_\alpha + \partial^{i}_\alpha f_\beta - 2D_j \partial^{ij}_\alpha f_\beta &= 0, \\
\partial^{ij}_\beta f_\alpha - \partial^{ij}_\alpha f_\beta &= 0.
\end{align*}$$

(19)

These conditions are known for a very long time and there is quite a lot of literature about them [11, 17, 14, 7]. Beside the Helmholtz conditions, we also define the so-called Helmholtz expressions (for second order $\Delta$)

$$\begin{pmatrix}
H_{\alpha\beta}(\Delta) \\
H'_{\alpha\beta}(\Delta) \\
H''_{\alpha\beta}(\Delta)
\end{pmatrix} :=
\begin{pmatrix}
\partial_\beta f_\alpha - \partial_\alpha f_\beta + D_i \partial^{i}_\alpha f_\beta - D_i D_j \partial^{ij}_\alpha f_\beta \\
\partial^{i}_\beta f_\alpha + \partial^{i}_\alpha f_\beta - 2D_j \partial^{ij}_\alpha f_\beta \\
\partial^{ij}_\beta f_\alpha - \partial^{ij}_\alpha f_\beta
\end{pmatrix}.$$

(20)

Notice that the expressions in (20) are also defined when $(f_\alpha)$ is not variational. The Helmholtz expressions, as well as the Helmholtz conditions, can
be extended to arbitrary order, for example, see [14], [4].

Since the the Helmholtz conditions (and expressions) will be very important later on, we give a brief overview how they can be derived and what the main idea is (the idea is pretty much the same as one derives integrability conditions for exact differential forms in the De-Rham theory). According to (9), we get that when \( \sigma \) is a zero-functional, then
\[
L_{pr^k \mathcal{V}}I_\sigma = K_\sigma(pr^k \mathcal{V})
\]
is one-functional for every \( \pi \)-vertical vector field \( \mathcal{V} \in \mathfrak{X}(E) \) with \( \text{supp } \mathcal{V} \subset \pi^{-1}\Omega \) (this is the first variation of \( I \)) and in (15) we defined how to apply the formal Lie derivative to one-functionals. By direct computation it then follows that the equation
\[
L_{pr^k \mathcal{W}}L_{pr^k \mathcal{V}}I_\sigma - L_{pr^k \mathcal{V}}L_{pr^k \mathcal{W}}I_\sigma - L_{pr^k [\mathcal{W}, \mathcal{V}]}I_\sigma = 0,
\]
must be satisfied for every \( \pi \)-vertical vector fields \( \mathcal{V}, \mathcal{W} \in \mathfrak{X}(E) \) with \( \text{supp } \mathcal{V}, \mathcal{W} \subset \pi^{-1}\Omega \) and \([\mathcal{W}, \mathcal{V}]\) is the commutator of \( \mathcal{V}, \mathcal{W} \). Every zero-functional \( I \) satisfies (21), but not every one-functional \( K \) satisfies the analogous equation
\[
(L_{pr^k \mathcal{W}}K)_\sigma(pr^k \mathcal{V}) - (L_{pr^k \mathcal{V}}K)_\sigma(pr^k \mathcal{W}) - K_\sigma(pr^k [\mathcal{W}, \mathcal{V}]) = 0,
\]
for all \( \pi \)-vertical \( \mathcal{V}, \mathcal{W} \in \mathfrak{X}(E) \) with \( \text{supp } \mathcal{V}, \mathcal{W} \subset \pi^{-1}\Omega \) (since this is only the case when \( K_\sigma(pr^k \mathcal{V}) = L_{pr^k \mathcal{V}}I_\sigma \), at least locally). Therefore, the condition (22) delivers a necessary condition for a source form to be variational (it can be shown that it also delivers a locally sufficient condition). With the help of (22), one then deduces the Helmholtz conditions with the following procedure: First, (22) can be written as
\[
\int_{\Omega} pr^k \sigma^*[\left(V^\alpha L_{pr^k \mathcal{W}}f_\alpha - W^\alpha L_{pr^k \mathcal{V}}f_\alpha\right)dx] = 0
\]
and the integrand in (23) can be written as
\[
V^\alpha L_{pr^k \mathcal{W}}f_\alpha - W^\alpha L_{pr^k \mathcal{V}}f_\alpha = V^\alpha [W^\beta \partial_\beta + (D_i W^\beta) \partial_i^\beta + (D_i D_j W^\beta) \partial^{ij}_\beta] f_\alpha - \\
- W^\alpha [V^\beta \partial_\beta + (D_i V^\beta) \partial_i^\beta + (D_i D_j V^\beta) \partial^{ij}_\beta] f_\alpha.
\]
Second, the main technique is then to do partial integration with the expression (24) in a systematic way, namely we shift all total derivatives of \( D_i V^\beta \) and \( D_i D_j V^\beta \) to other terms plus a divergence term (in a similar way as one does partial integration in the first variational formula to derive the Euler-Lagrange equations, where one shifts all total derivatives away of the test
functions). It turns out that we get
\[ V^\alpha \mathcal{L}_{\mu^k \nu^l} f^\alpha - W^\alpha \mathcal{L}_{\mu^k \nu^l} f^\alpha = 
= V^\alpha W^\beta H_{\alpha \beta} + V^\alpha (D_i W^\beta) H_{i \alpha \beta} + V^\alpha (D_j D_i W^\beta) H_{ij \alpha \beta} + 
+ D_i [V^\beta D_j (W^\alpha \partial_{ij} f^\alpha)] - W^\alpha V^\beta \partial_{ij} f^\alpha - W^\alpha (D_j V^\beta) \partial_{ij} f^\alpha. \]  
(25)

Equation (25) can be seen as the definition of the Helmholtz expressions. Now we evaluate the integral in (23). By Gauss’s theorem, the divergence expression
\[ D_i [V^\beta D_j (W^\alpha \partial_{ij} f^\alpha)] - W^\alpha V^\beta \partial_{ij} f^\alpha - W^\alpha (D_j V^\beta) \partial_{ij} f^\alpha \]
vanishes when integrated, since \( V, W \) have support in \( \pi^{-1} \Omega \). The remaining three terms in (25) include the Helmholtz expressions, and the coefficients \( V^\alpha W^\beta, V^\alpha D_i W^\beta \) and \( V^\alpha D_i D_j W^\beta \). It can be shown that these three coefficients can be chosen independently and therefore the Helmholtz expressions must vanish under the assumption that (22) holds. To understand this independence, let us consider
\[ (\text{pr}^k \sigma)^*(V^\alpha W^\beta) = \varphi^\alpha(x) \psi^\beta(x), \]
\[ (\text{pr}^k \sigma)^*(V^\alpha D_i W^\beta) = \varphi^\alpha(x) \psi^\beta_i(x), \]
\[ (\text{pr}^k \sigma)^*(V^\alpha D_i D_j W^\beta) = \varphi^\alpha(x) \psi^\beta_{ij}(x), \]
where \( \varphi^\alpha(x) \) and \( \psi^\beta(x) \) can be seen as test functions and derivatives are written as \( \psi^\beta_i = \frac{\partial \psi^\beta}{\partial x^i} \) and \( \psi^\beta_{ij} = \frac{\partial^2 \psi^\beta}{\partial x^i \partial x^j} \). Since these test functions can be chosen arbitrarily, we can choose \( \text{supp} \varphi^\alpha \subset \text{supp} \psi^\beta \) and \( \psi^\beta(x) \equiv 1 \) in the support of \( \varphi^\alpha \), that is, derivatives \( \psi^\beta_i \) and \( \psi^\beta_{ij} \) vanish in the support of \( \varphi^\alpha \). A pull-back by a section \( \sigma \) of the expression (25) then leads to \( \varphi^\alpha(x) (\text{pr}^k \sigma)^*H_{\alpha \beta} \) and this expression must vanish for all test functions \( \varphi^\alpha \) with \( \text{supp} \varphi^\alpha \subset \text{supp} \psi^\beta \) which finally leads to \( H_{i \alpha \beta} = 0 \). A similar discussion then can be done to show that \( H_{ij \alpha \beta} = 0 \) and \( H_{ij \alpha \beta} = 0 \).

When the integrability conditions (22) are satisfied then, by Poincaré lemma, we can construct \( I \) (at least locally) as
\[ I_{\phi_1 \sigma} = \int_0^1 K_{\phi_1 \sigma} \left( \frac{d}{dt} \text{pr}^k \phi_i \right) dt = \int_0^1 \left( \int_{\Omega} f^\alpha(x, s_t(x), D s_t(x), \ldots, D^k s_t(x)) \frac{ds^\alpha(x)}{dt} dx \right) dt, \]  
(26)
where $\phi_t$ is the flow of a $\pi$-vertical vector field $V \in \mathfrak{X}(E)$ such that $\text{supp } V \subset \pi^{-1}\Omega$, and $(x,s_t(x),Ds_t(x),...,Ds_t(x))$ describes the local coordinates in $J^kE$ of the prolonged section $\phi_t\sigma$ at time $t$. Formula (26) is the same as one constructs the zero-form $\eta$ in (17), where we get

$$
\eta = \int_0^1 \omega_{\phi_t} \left( \frac{d\phi_t}{dt} \right) dt,
$$

and the map $\phi_t$ connects two points $\phi_0 = p_0, \phi_1 = p_1$ in the manifold, where $\omega$ is defined. The functional $\tilde{I}_\sigma$ in (26) is defined for any section $\tilde{\sigma}$ of $\pi$, since when we choose $\sigma$ such that $\sigma|_{\partial \Omega}$ is any function at the boundary we want, then we can always find a flow $\phi_t$, with the above properties, such that $\tilde{\sigma} = \phi_1 \circ \sigma$ and $\phi_0 \circ \sigma = \sigma$.[6]

**Helmholtz dependencies:** As already mentioned, the Helmholtz conditions (19) guarantee that a second order source form is locally variational, but it turns out that these conditions are not independent and we would actually need less of them to guarantee that a source form is locally variational. The main reason for this is that the partial integration technique for two vector fields to get from (24) to (25) does not necessarily deliver independent Helmholtz expressions (in contrast to the partial integration which is done in the first variational formula, where only one vector field is involved). The Helmholtz expressions (20) are dependent and we get the following relations

$$
H_{\alpha\beta} + H_{\beta\alpha} - D_i H_{i\alpha\beta} + D_i D_j H_{ij}^{\alpha\beta} = 0,
$$

$$
H_{i\alpha\beta} - H_{i\beta\alpha} - 2D_j H_{ij}^{\alpha\beta} = 0,
$$

$$
H_{ij}^{\alpha\beta} + H_{ij}^{\beta\alpha} = 0
$$

(27)

which are always satisfied, whether $(f_\alpha)$ is variational or not (in contrast to the Helmholtz conditions which are only satisfied when $(f_\alpha)$ is variational). We call these relations the **Helmholtz dependencies**. They can be proven by a straightforward calculation, when substituting the definition (20) into (27). Using these relations is crucial in our main proof later, since they reduce drastically the number of unknowns. These relations can be found in

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6Notice that $\text{supp } V \subset \pi^{-1}\Omega$ implies $\frac{d}{dt}\text{pr}^k(\phi_t \circ \sigma)|_{\pi^{-1}\partial \Omega} = 0$ and we can do partial integration without getting any boundary terms. And, for $\pi$-vertical vector fields we get $\pi_* V = 0$ and the flow $\phi_t^0$ of $\pi_* V$ on $M$ is the identity map. The prolonged transformation of a section $\sigma$ under $\phi_t$ then simplifies to $\text{pr}^k(\phi_t \circ \sigma \circ (\phi_t^0)^{-1})(\phi_t^0) = \text{pr}^k(\phi_t \circ \sigma)$. 

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and [3, p.8], and they are already used in [24], but not explicitly noticed.

Notice that there are also integrability conditions for functions $G_{\alpha\beta}, G^{i}_{\alpha\beta}, G^{ij}_{\alpha\beta}$ under which they can be written as (20) for some functions $(f_{\alpha})$. These conditions have to be distinguished from the Helmholtz dependencies which are not these integrability conditions. See [18, p.86], where these integrability conditions can be found for $n = 1$ and third order (they are pretty complicated in general).

**Symmetries:** We define a symmetry of $\Delta$ as a $\pi$-projectable vector field $V \in \mathfrak{X}(E)$ such that $\mathcal{L}_{\text{pr}^k V} \Delta = 0$ for all points in $J^k E$ and $\mathcal{L}$ denotes again the Lie-derivative. Notice that this definition is in general not equivalent to $\mathcal{L}_{\text{pr}^k V} f_{\alpha} = 0$, $\alpha = 1, 2, \ldots, m$. In any case, we need this definition when we want to solve Takens’ problem. Roughly speaking, we require that the weak formulation of a differential equation satisfies certain symmetries and not the differential equation itself. And, the symmetry does not only hold for solutions of the differential equation, it holds for all points in $J^k E$.

**Conservation laws and continuity equations:** In the example in Section 2, we used a very vague concept of conservation law and in this paragraph we will give a precise definition. First of all, it is not obvious how to use the intuitive concept of conservation law in physics to get a meaningful mathematical definition which covers all these physical ‘conservation laws’. Moreover, ODEs and PDEs can behave quite differently in this regard and, in our opinion, we should distinguish these cases carefully and not give certain equations the same name conservation laws. In the case of ODEs, a conservation law is usually considered as a first integral of the differential equation, see the example in Section 2. For PDEs, a conservation law is usually considered as a divergence expression, or in physics known as a continuity equation. A well-known example of a continuity equation in physics is $\frac{\partial \rho}{\partial t} + \nabla j = 0$, where $\rho$ is the charge-, and $j$ the current density in Maxwell’s equations. In relativistic notation, this can also be written as $\partial_{\mu} J^{\mu} = 0$, where $(J^{\mu}) = (\rho, j)$ is the so-called four-current in physics. We will just call it current density, see below. Another example of a continuity equation

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\footnote{In our notation, $\partial_{\mu}$ would be the total derivative operator $D_{\mu}$.}
is $\rho = |\Psi|^2$ and $j = \frac{\hbar}{2m}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi)$ in Schrödinger’s equation. These examples now motivate what we will define as a current density and a continuity equation. For further information we refer to [4] and [22].

We define a **current density** $J$ as an $(n - 1)$-form on $J^kE$, written in local coordinates as

$$J = \sum_{i=1}^{n} (-1)^{i+1} J^i dx^1 \wedge ... \wedge \hat{dx}^i \wedge ... \wedge dx^n,$$

where the hat $\hat{dx}^i$ denotes omission, and $J^i$ are functions on $J^kE$, see [4, p.199]. The intrinsic property of (28) is that $J$ is a so-called horizontal $(n - 1)$-form on $J^kE$ which means that there are no $du^\alpha, du^\alpha_i, ..., du^\alpha_{i_1...i_k}$-forms involved, see [17, p.33] and [14, p.36]. For any horizontal $(n - 1)$-form $\omega \in \Omega^{n-1}(J^kE)$, written in local coordinates as $\omega = \omega_{i_1...i_{n-1}} dx^{i_1} \wedge ... \wedge dx^{i_{n-1}}$, we define the **total divergence operator** $E_n$ in local coordinates as

$$E_n \omega := (D_i d\omega_{i_1...i_{n-1}}) dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_{n-1}},$$

where $D_i$ is the total derivative operator in (4). The fact that $E_n$ is coordinate invariant can be seen when writing $E_n$ as $E_n = h d$, where $h$ is the so-called horizontalization operator and $d$ is the exterior derivative, see [14, p.289], [23, p.33] and [14, p.54]. For any current density $J$ we get $E_n J = (D_i J^i) dx$, that is, where the $dx^i$-forms are ordered in an increasing way, and $D_i J^i$ delivers a divergence expression which is coordinate invariantly defined by these forms.

**Definition 3.2** We say that a source form $\Delta$ satisfies a **local continuity equation**, if for each $p \in E$ there exists a neighborhood $U \subset E$, a current density $J$ on $(\pi^{k,0})^{-1} U \subset J^kE$, and a corresponding $\pi^k$-vertical vector field $Q$ on $(\pi^{k,0})^{-1} U \subset J^kE$, such that the equation $E_n J = Q \cdot \Delta$ is satisfied on $(\pi^{k,0})^{-1} U \subset J^kE$. If there exists a current density $J$ on $J^kE$, and a corresponding $\pi^k$-vertical vector field $Q \in \mathfrak{X}(J^kE)$, such that $E_n J = Q \cdot \Delta$ is satisfied on $J^kE$, then we say that $\Delta$ satisfies a **global continuity equation**.

The vector field $Q = Q^\alpha \partial_\alpha + ... + Q_{i_1...i_k}^{\alpha_1...i_k} \partial_{\alpha_1...i_k}$ is also called characteristic vector field for the continuity equation and $(Q^\alpha) = (Q^\alpha(x,u,\partial u,...,\partial^k u))$ is called the **characteristic of the continuity equation**, see [4, p.199] and [22, p.270]. Notice that the equation $E_n J = Q \cdot \Delta$, or in local coordinates

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$D_i J^i = Q^a f_a$, can be understood as follows: If we are on solutions of the differential equation $f_\alpha = 0$, $\alpha = 1, 2, ..., m$, then we get $D_i J^i = 0$ on solutions of the differential equation and $D_i J^i = 0$ is interpreted as a continuity equation which leads to some kind of conserved quantity, like charge conservation in Maxwell’s- or Schrödinger’s equations.

The Euler-Lagrange-, and Helmholtz operator: Let us define the Euler-Lagrange operator $E_\alpha$ in local coordinates as

$$E_\alpha := \partial_\alpha - D_{i_1} \partial_{i_1}^\alpha + ... + (-1)^k D_{i_1} D_{i_2} ... \partial_{i_k}^{i_1 i_2 ... i_k}, \quad \alpha = 1, 2, ..., m,$$

which acts on Lagrange functions $L = L(x,u,...,\partial^k u)$ as $E_\alpha L$. To define the operator in a coordinate invariant way, we define the operator $E_{n+1}$, acting on Lagrange forms $\lambda = L dx$ as

$$E_{n+1}(\lambda) := (E_\alpha L)du^\alpha \wedge dx.$$

Let us also define the Helmholtz operator $H_{\alpha \beta}^\gamma$ (for second order source forms) in local coordinates as

$$H_{\alpha \beta}^\gamma := \left(\begin{array}{c}
H_{\alpha \beta}^\gamma \\
H_{\alpha \beta}^{i \gamma} \\
H_{\alpha \beta}^{ij}
\end{array}\right) = \left(\begin{array}{c}
\partial_\beta \delta_\alpha^\gamma - \partial_\alpha \delta_\beta^\gamma + D_{i_1} \partial_{i_1}^{\alpha} \delta_\beta^\gamma - D_{i_1} D_{i_2} \partial^{i_1 i_2 \alpha} \delta_\beta^\gamma \\
\partial_\beta \delta_\alpha^\gamma + \partial_{\alpha} \delta_\beta^\gamma - 2D_{i_1} \partial^{i_1} \delta_\beta^\gamma \\
\partial_\beta \delta_\alpha^\gamma - \partial_{\alpha} \delta_\beta^\gamma
\end{array}\right),$$

where $\alpha, \beta, \gamma = 1, 2, ..., m$, which acts on functions $(f_\gamma)$, $\gamma = 1, 2, ..., m$ as $H_{\alpha \beta}^\gamma f_\gamma$, such that we get the Helmholtz expressions in (20) ($\delta_\alpha^\gamma$ denotes the Kronecker-delta). Notice that the operator can be defined for higher order $(f_\gamma)$ when considering the Helmholtz expressions for higher order and they can be derived with the method we have explained above. Again, to define the operator in a coordinate invariant way, we define the operator $E_{n+2}$ acting on (second order) source forms $\Delta = f_a du^a \wedge dx$ as

$$E_{n+2}(\Delta) := [H_{\alpha \beta}^\gamma f_\gamma]du^\alpha + (H_{\alpha \beta}^{i \gamma} f_\gamma)du_\gamma + (H_{\alpha \beta}^{ij} f_\gamma)du_\gamma^i du_\gamma^j \wedge du^\beta \wedge dx.$$

The fact that $E_{n+1}$ and $E_{n+2}$ are coordinate invariant can be shown when writing $E_{n+1} = Id$, acting on certain kinds of differential forms, where $I$ is the so-called interior Euler-Lagrange operator and $d$ is the exterior derivative. The same holds for $E_{n+2}$. See [26] and [16, p.7] for the finite jet bundle $J^k E$, and [1] for the infinite jet bundle $J^\infty E$. The operator $I$ is basically needed to do partial integration, as we explained, for example, in (25) when we derived the Helmholtz expressions.

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Locally exact sequence: It is clear that $\mathcal{H}^\gamma_{\alpha\beta} \mathcal{E}_\gamma L = 0$ for every function $L$, since variational expressions $f_\gamma = \mathcal{E}_\gamma L$ satisfy the Helmholtz conditions, see \[19\]. As an operator identity we therefore get $\mathcal{H}^\gamma_{\alpha\beta} \mathcal{E}_\gamma = 0$, or $E_{n+2} \circ E_{n+1} = 0$. It can also be shown that $\mathcal{E}_\alpha D_i = 0$, or $E_{n+2} \circ E_n = 0$. This is because any Lagrangian of the form $L = D_i J^i$, where $(J^i)$ are some functions on $J^k E$, leads to a zero-functional of the form

$$I_\sigma = \int_{\Omega} (\text{pr}_k \sigma)^* (D_i J^i) dx \bigg|_{\partial \Omega} = \int_{\partial \Omega} (\text{pr}_k \sigma)^* (J^i n_i dS) = I_\sigma |_{\partial \Omega},$$

where, by Gauss’s theorem, $I_\sigma$ does only depend on the boundary of $\Omega$ and therefore perturbations of $I_\sigma |_{\partial \Omega}$, which are described by the Euler-Lagrange operator $\mathcal{E}_\alpha$ and $\pi$-vertical vector fields which have support in $\pi^{-1}\Omega$, lead to the trivial Euler-Lagrange equation $\mathcal{E}_\alpha (D_i J^i) = 0$ for any functions $(J^i)$ $(n_i$ are the components of the unit normal vector of the surface $\partial \Omega)$. Moreover, the operators $E_n$, $E_{n+1}$ and $E_{n+2}$ lead to a locally exact sequence of the form

$$... \rightarrow \{J\} \xrightarrow{E_n} \{\lambda\} \xrightarrow{E_{n+1}} \{\Delta\} \xrightarrow{E_{n+2}} ... \quad (29)$$

where $\{J\}$ denotes the set of current densities, $\{\lambda\}$ the set of Lagrange forms, and $\{\Delta\}$ the set of source forms. Notice that the operators $E_n$, $E_{n+1}$ and $E_{n+2}$ can be extended and they can be applied to any differential form on $J^k E$, see \[14\]. There are actually different ways how to realize this sequence, or sequences similar to them. For example, Krupka uses a finite order sequence with quotient spaces \[13, 14\], and Anderson uses an infinite order sequence which is constructed with the help of vertical and horizontal exterior derivatives \[1\].

Cartan’s formula and its generalizations: As it is well-known, by Cartan’s formula, the Lie derivative decomposes into two terms $\mathcal{L}_V = dV \lrcorner + V \lrcorner d$.

8To prove $\mathcal{E}_\alpha D_i = 0$ (and also to prove $\mathcal{H}^\gamma_{\alpha\beta} \mathcal{E}_\gamma = 0$) it is also possible to commute $D_i$ with the partial derivatives $\partial_{i_1...i_l}^{\alpha}$ in the operator $\mathcal{E}_\alpha$ (or in $\mathcal{H}^\gamma_{\alpha\beta}$) and use the commutation relation $[\partial_{i_1...i_l}^{\alpha}, D_i] = \delta_{i_1...i_l}^{\beta} \delta_{\beta}^{\alpha} \mathcal{E}_{i_1...i_l}^{\beta}$. However, we usually prefer to prove these identities with the methods explained above, since commuting these operators can get very complicated, at least to show that $\mathcal{H}^\gamma_{\alpha\beta} \mathcal{E}_\gamma = 0$.

9To construct $J$, see the total homotopy operator in \[22\] p.364 in equation (5.112). To construct $\lambda$, we can use the so-called Vainberg-Tonti Lagrangian, see \[17\] p.56, \[14\] p.136 and also see formula \[26\].
There is a similar formula with the operators \( E_n, E_{n+1} \) and \( E_{n+2} \) which is stated in the following two theorems:

**Theorem 3.3** Let \( \lambda \) be a first order Lagrange form, and \( V \in \mathfrak{X}(E) \) be a \( \pi \)-projectable vector field. Then we get the decomposition

\[
\mathcal{L}_{pr^k V} \lambda = E_n (pr^k V \cdot \Lambda) + (pr^k V_e v) \cdot E_{n+1} (\lambda) = D_i (V^i L + V^a \partial_a^i L) dx + (V^a \xi_a^i L) dx.
\]

where

\[
\Lambda = L dx + \sum_{i=1}^{n} (\partial_a^i L) dx^1 \wedge \ldots \wedge dx^{i-1} \wedge (du^a - u_j^a dx^j) \wedge dx^{i+1} \wedge \ldots \wedge dx^n
\]

is the so-called Poincaré-Cartan form, see [17, 14, 12].

For possible higher order version on the Poincaré-Cartan form see [10].

**Theorem 3.4** Let \( \Delta \) be a second order source form, and \( V \in \mathfrak{X}(E) \) be a \( \pi \)-projectable vector field. Then we get the decomposition

\[
\mathcal{L}_{pr^k V} \Delta = E_{n+1} (pr^k V_e v) \cdot \Delta) + (pr^k V_e v) \cdot E_{n+2} (\Delta) = \{ \xi_a^i (V^a \xi_b^j \gamma_a^j) + [V^a \xi_d^b \xi_a^e \delta^i_j + (D_i V^a {\xi_a}^b) \xi_d^j + (D_j D_i V^a {\xi_a}^b) \xi_d^j] \} du^a \wedge dx.
\]

Theorem 3.3 describes the first variational formula and the classical Noether’s theorem. Theorem 3.4 can be found in [11, p.202] and it also holds for higher order source forms and it is used in Takens’ problem. It also delivers an implicit formulation of Noether’s theorem.

**The formulation of Takens’ problem:** Now let us explain how we use all the objects and definitions we have made above which was first discovered by Takens [24]. Let us assume that:

- \( \Delta \) satisfies the symmetry \( \mathcal{L}_{pr^k V} \Delta = 0 \) for every point on \( J^k E \).
• For every symmetry \( V \in \mathfrak{X}(E) \) we get that \( \Delta \) satisfies a corresponding global continuity equation of the form \( E_n J = (\text{pr}_k V_{ev}) . \Delta \), where the characteristic of the conservation law is \( Q^\alpha = V_{ev}^\alpha \). Due to the locally exact sequence, it also works with a local continuity equation such that \( E_{n+1}(\text{pr}_k V_{ev} . \Delta) = 0 \). \(^{10}\)

• \( \Delta \) is of second order (in what we investigate in this paper).

Then Theorem 3.4 forces the equation

\[
V_{ev}^\beta H_{\alpha \beta} + (D_i V_{ev}^\beta) H_{\alpha \beta}^i + (D_j D_i V_{ev}^\beta) H_{\alpha \beta}^{ij} = 0, \tag{30}
\]

since the \( E_{n+1} \)-term vanishes, because \( E_{n+1} \) annihilates \( E_n \)-expressions. We call \( \text{(30)} \) the equation of continuities and symmetries (ECS). If we can conclude from \( \text{(30)} \) that \( H_{\alpha \beta}, H_{\alpha \beta}^i, H_{\alpha \beta}^{ij} = 0 \), then, due to the locally exact sequence, \( \Delta \) must be locally variational.

In the next section, we have different kinds of symmetry vector fields \( V_{\mathcal{A}} \), where \( \mathcal{A} = 1, 2, \ldots, R \) label these symmetries. For every symmetry vector field we get an equation \( \text{(30)} \). For sufficiently many symmetries, such that the matrix

\[
(V_{ev, \mathcal{A}}, D_i V_{ev, \mathcal{A}}, D_j D_i V_{ev, \mathcal{A}})_{l, i, j, i, \mathcal{A} = 1, 2, \ldots, m; \mathcal{A} = 1, 2, \ldots, R} \tag{31}
\]

is invertible on \( J^k E \) (as a quadratic matrix), the linear equation \( \text{(30)} \) immediately forces that \( H_{\alpha \beta}, H_{\alpha \beta}^i, H_{\alpha \beta}^{ij} = 0 \) (notice that \( l, i, j, \alpha, \beta, \gamma \) label the columns of the matrix \( \text{(31)} \) and \( \mathcal{A} \) the rows). However, \( \text{(31)} \) does in general not have full rank. One reason for this is that the entries of the matrix \( \text{(31)} \) are always related by each other through total derivatives \( D_i \) and \( D_j D_i \) and this means that the columns are in some sense dependent. And, in our main theorem later, the integer \( R \) is much smaller then \( m + nm + \frac{n(n+1)}{2} m \) which also means that the matrix \( \text{(31)} \) cannot have full rank and a much deeper investigation of \( \text{(30)} \) is necessary which makes it a really hard problem.

\(^{10}\)One should be aware that when the symmetry vector field \( V \) generates a conservation law, then the corresponding characteristic \( Q^\alpha = V^\alpha - u^i V_i^\alpha \) must have a very special form and this is actually a strong restriction.
4 The main result

In this section we formulate and prove our main theorem. To specify our assumptions, let us define the space $\mathcal{V}$ of symmetry vector fields on $E$ as

$$\mathcal{V} := \{ V \in \mathfrak{X}(E) : V \text{ is } \pi\text{-projectable and } \mathcal{L}_{pr^k V} \Delta = 0 \text{ on } J^k E \}. $$

Moreover, let us define the condition

$$\text{span}_R \{ V_p : V \in \mathcal{V} \} = T_p E \text{ for all } p \in E. \quad (32)$$

The main result of this paper is the following:

**Theorem 4.1** Let $\pi : E \to M$ be a fiber bundle with base dimension $n$ and fiber dimension $m$, where $n, m \in \mathbb{N}$ are arbitrary. Furthermore, let $\Delta$ be a second order source form on $J^2 E$. Assume:

1. The set $\mathcal{V}$ of symmetries of $\Delta$ satisfies (32).
2. For each $V \in \mathcal{V}$ we have a corresponding local continuity equation, that is, $E_{n+1} (pr^k V_{ev} \Delta) = 0$, where $V_{ev}$ is the evolutionary vector field of $V$.

Then $\Delta$ must be locally variational.

The proof of Theorem 4.1 will be mainly done in local coordinates and we have to prepare two propositions and some notation first, before we go to the main part of the proof. Roughly speaking, the set $\mathcal{V}$ of symmetry vector fields can be very large and complicated and the first propositions shows that near a point $p \in E$ it is sufficient to consider a finite set of symmetry vector fields. Moreover, the second proposition shows that for such a finite set of symmetry vector fields it is possible to simplify equation (30), that is, we can invert a part of the matrix (31).

**Proposition 4.2** Let $\mathcal{W} \subset \mathfrak{X}(E)$ be a set of $\pi$-projectable vector fields on $E$ such that $\text{span}_R \{ V_p : V \in \mathcal{W} \} = T_p E$ for each $p \in E$. Then for every $p_0 \in E$ there exists a small neighborhood $U_{p_0} \subset E$ of $p_0$ such that we can choose $n+m$ vector fields $\{ V_1, V_2, \ldots, V_{n+m} \} \subset \mathcal{W}$ such that $\text{span}_R \{ V_{1,p}, V_{2,p}, \ldots, V_{n+m,p} \} = T_p E$ for all $p \in U_{p_0}$. 

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The proof is straightforward. In the following, we always label finitely many symmetry vector fields \( V_{iA} \) by \( A \) and since they should span \( T_pE \) we always have \( A = 1, 2, \ldots, n+m \). Therefore, \( A \) also labels the columns in the matrix \( B := (V_{iA}^i, V_{iA}^\alpha)_{iA} \) and \( i; \alpha \) label the rows.

**Proposition 4.3** Let \( V_{iA} \) be \( \pi \)-projectable vector fields on \( E \) and \( A = 1, 2, \ldots, n+m \). If \( \text{span}_\mathbb{R}\{V_{pA}, \, A = 1, 2, \ldots, n+m\} = T_pE \) for each \( p \in U \subset E \), then there exists a \((n+m) \times (n+m)\)-matrix \( C = C(x,u) \) on \( U \subset E \), such that the \((n+m) \times (n+m)\)-matrix \( B \) satisfies \( C \cdot B = \text{Id} \), where \( \text{Id} \) is the identity matrix. That is, \( C \) is the inverse matrix of \( B \). Moreover, there exists a row \( c_A = c_A(x,u) \) of the matrix \( C \) such that either \( c_A^i V_{iA}^i = \delta_{ij} \), \( c_A^\alpha V_{iA}^\alpha = 0 \), or such that \( c_A^i V_{iA}^j = 0 \), \( c_A^\alpha V_{iA}^\alpha = \delta^{\alpha\beta} \), where \( \delta_{ij}, \delta^{\alpha\beta} \) are Kronecker deltas.

The proof follows directly by definition of \( \text{span}_\mathbb{R}\{V_{pA}, \, A = 1, 2, \ldots, n+m\} = T_pE \) for each \( p \in U \subset E \).

**The notation of order:** Varying terms in equations independently from others, and thereby showing that they must vanish, will be one of our main techniques to prove Theorem 4.1. With varying terms independently we mean that the coordinates \( x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1i_2\ldots i_k} \) can be varied independently if an equation must hold on \( J^kE \) (for example, the equation \( L_{pi} \Delta = 0 \) on \( J^kE \)). Usually we start our discussion with the highest order coordinate \( u^\alpha_{i_1i_2\ldots i_k} \) and all the lower order coordinates are not relevant in this moment. Therefore, it is reasonable to have a notation for such lower order-, or in other words non-important terms, and we will write \( O(k-1) \) which stands for a function of order \( k-1 \), see (3).

Total derivatives increase the order by one when applied to functions. More precisely, the order is increased affine linear in the highest order coordinates and we get

\[
D_i g = \partial_i g + u^\alpha_i \partial_\alpha g + \ldots + u^\alpha_{i_1i_2\ldots i_k} \partial^{i_1i_2\ldots i_k}_{\alpha} g = O(k) + u^\alpha_{i_1i_2\ldots i_k} O(k) \tag{33}
\]

for every function \( g \in C^\infty(J^kE) \). Since \( J^{k+1}E \) is an affine linear bundle over \( J^kE \), this notation is invariant under coordinate transformations. We also want to introduce the notation \( g = O_1(k) \) if \( g \) is affine linear in the \( k \)-th order coordinates, or in other words a polynomial of degree one in \( u^\alpha_{i_1i_2\ldots i_k} \).
In general, we write \( g = O_P(k) \) if \( g \) is a polynomial of degree \( P \) in the \( k \)-th order coordinates. However, notice that later \( O_\mathcal{A}(k) \) does not indicate a polynomial of degree \( \mathcal{A} \), it rather labels the different kinds of symmetries. Sometimes we will write a few indices on the expression \( O(k) \), for example, \( O_{ij}(k) \) or \( O_{\mathcal{A},ij}(k) \), and always when we use the indices \( \mathcal{A},\alpha,\beta,i,j \) then we do not describe polynomial degree (this will also be clear from the context). Also notice that the definition of objects \( O_P(k) \) is invariant under local coordinate transformations. The \( O_P(k) \)-notation satisfies the properties

\[
O_{P_1}(k)O_{P_2}(k) = O_{P_1+P_2}(k), \quad \text{for all } k \geq 1,
\]
\[
O_{P_1}(k)O_{P_2}(l) = O_{P_1}(k), \quad \text{for all } k > l \geq 0,
\]
\[
D_j O_P(k) = O_1(k + 1).
\]

Let us now explain where we use this notation. The leading orders and the polynomial degree of local coordinates in the columns of the matrix (31) will be crucial in the proof of Theorem 4.1. Recall that the assumptions of Theorem 4.1 are that the symmetry vector fields are \( \pi \)-projectable, and therefore we get

\[
V_{ev}^\beta = V^\beta - u_i^\beta V^i,
\]
\[
D_k V_{ev}^\beta = D_k (V^\beta - u_i^\beta V^i) = O^\beta_k(1) - u_i^\beta V^i,
\]
\[
D_l D_k V_{ev}^\beta = D_l (O^\beta_k(1) - u_i^\beta V^i) = O^\beta_{lk}(2) - u_i^\beta V^i,
\]
where \( V^i = V^i(x) \) are functions only depending on \( x \).

Proof of Theorem 4.1. Let us consider an arbitrary point \( p_0 \in E \). According to Proposition 4.2 we can find a small neighborhood \( U_{p_0} \subset E \) of \( p_0 \), such that \( n + m \) symmetry vector fields \( \{ V_{\mathcal{A}}, \mathcal{A} = 1, 2, \ldots, n + m \} \subset V \) span \( T_pE \) for each \( p \in U_{p_0} \), and according to symmetry and continuity equation assumptions, we get equation (30), that is,

\[
(V_{\mathcal{A}}^\beta - u_i^\beta V_{\mathcal{A}}^i) H_{\alpha \beta} + (O_{i,\mathcal{A}}^\beta(1) - u_i^\beta V_{\mathcal{A}}^i) H_{\alpha \beta}^k + (O_{j,\mathcal{A}}^\beta(2) - u_i^\beta V_{\mathcal{A}}^i) H_{\alpha \beta}^{lk} = 0
\]

(34)

for every \( \alpha = 1, 2, \ldots, m \) and every \( \mathcal{A} = 1, 2, \ldots, n + m \). We will show that equation (34) forces that the Helmholtz conditions must be satisfied in \( (\pi^{k,0})^{-1}U_{p_0} \subset J^kE \) and since we can show this for every open subset \( U_{p_0} \subset E \) the Helmholtz conditions must be satisfied everywhere on \( (\pi^{k,0})^{-1}E = J^kE \). Then, due to
the locally exactness of the variational sequence \( \text{(29)} \) we know that \( \Delta \) must be locally variational.

Now we will discuss the ECS \( \text{(34)} \) in more detail, and we divide the proof into seven steps. The main results in every step will be written in a box. The results in these boxes will then be needed in the next steps. Things which are not written in boxes are basically the proofs of what is written in the boxes.

Step 1 (transform the ECS)\footnote{This step was developed during my stay at Utah State University when working together with Ian M. Anderson.} Using Proposition \( \text{4.3} \) we can take linear combinations

\[
c^\alpha (V_{\alpha}^\beta - u_{k}^\beta V_{\alpha}^i) H_{a}^{\alpha \beta} + c^\beta (O_{\alpha,k}^\beta (1) - u_{k}^\beta V_{\alpha}^i) H_{a}^{k} + \]

\[
+ c^\alpha (O_{\alpha,k}^\beta (2) - u_{k}^\beta V_{\alpha}^i) H_{a}^{kl} = 0
\]

such that we get

\[\begin{align*}
\text{i) } & 0 = -u_{j}^\beta H_{a}^{\alpha \beta} + (O_{k}^\beta (1) - u_{j}^\beta) H_{a}^{k} + (O_{k}^\beta (2) - u_{j}^\beta) H_{a}^{kl}, \\
\text{ii) } & 0 = H_{\alpha \gamma} + O_{\gamma,k}^\beta (1) H_{a}^{\alpha \beta} + O_{\gamma,k}^\beta (2) H_{a}^{kl}.
\end{align*}\]

To derive equation i), we use \( c^\alpha V_{\alpha}^i = \delta^i{}^j \) and \( c^\beta V_{\alpha}^i = 0 \), and to derive equation ii), we use \( c^\alpha V_{\alpha}^i = 0 \) and \( c^\beta V_{\alpha}^i = \delta^\beta{}^\gamma \). Then we take another linear combination of i) and ii), namely

\[
\begin{align*}
-u_{j}^\beta H_{a}^{\alpha \beta} + (O_{k}^\beta (1) - u_{j}^\beta) H_{a}^{k} + (O_{k}^\beta (2) - u_{j}^\beta) H_{a}^{kl} & \\
+ u_{j}^\delta H_{\alpha \gamma} + O_{\gamma,k}^\beta (1) H_{a}^{\alpha \beta} + O_{\gamma,k}^\beta (2) H_{a}^{kl} & = 0
\end{align*}
\]

to derive the transformed ECS

\[
\begin{array}{|l|}
\hline
\text{I) } & 0 = (O_{j,k}^\beta (1) - u_{j}^\beta) H_{a}^{k} + (O_{j,k}^\beta (2) - u_{j}^\beta) H_{a}^{kl}, & j = 1,2,\ldots,n, \quad \alpha = 1,2,\ldots,m, \\
\hline
\text{II) } & 0 = H_{\alpha \gamma} + O_{\gamma,k}^\beta (1) H_{a}^{\alpha \beta} + O_{\gamma,k}^\beta (2) H_{a}^{kl}, & \alpha,\gamma = 1,2,\ldots,m,
\end{array}
\]

where we eliminated \( H_{a}^{\alpha \beta} \) in I), furthermore II) is the same as ii). From now on we simply call (35) equation I) and (36) equation II) and we should remember these equations since they are used several times. Equations I) and
II) are together \( m^2 + nm \) equations, but \( H_{\alpha\beta}, H^i_{\alpha\beta}, H^{ij}_{\alpha\beta} \) are together \( m^2 + nm^2 + \frac{n(n+1)m(m-1)}{2} \) unknowns, which can be seen by definition \((20)\). Therefore, for large \( n,m \) we have a highly under-determined system and only for \( m = 1 \) we can immediately determine the solution. To determine to solution of I) and II) for \( m > 1 \) we have to investigate the deeper structure of \( H_{\alpha\beta}, H^i_{\alpha\beta}, H^{ij}_{\alpha\beta} \) and their relations. Since \((f_{\alpha})\) are of second order, the polynomial structure of \( H_{\alpha\beta}, H^i_{\alpha\beta}, H^{ij}_{\alpha\beta} \) in fourth and third order coordinates is given as

\[
H_{\alpha\beta} = \partial_\beta f_\alpha - \partial_\alpha f_\beta + D_k \partial^k_\alpha f_\beta - D_k D_l \partial^{kl}_\alpha f_\beta = O_1(3) - u^\delta_r u^\gamma_{ij} \partial^r_\delta \partial^i_\gamma \partial^j_\alpha f_\beta - u^\gamma_{ijkl} \partial^i_\gamma \partial^j_\alpha f_\beta,
\]

\[
H^i_{\alpha\beta} = \partial^i_\beta f_\alpha + \partial^i_\alpha f_\beta - 2D_k \partial^i_\alpha f_\beta = O_1(3),
\]

\[
H^{ij}_{\alpha\beta} = \partial^{ij}_\beta f_\alpha - \partial^{ij}_\alpha f_\beta = O(2).\]

In Step 2, we now investigate terms which depend on the fourth order coordinates of degree one and terms which depend on third order coordinates of degree two. These terms only occur in equation II) and they must vanish separately. They are generated by the double total derivatives \( D_k D_l \) in \( H_{\alpha\beta} \) as we can see in \((37)\). Later on we will also discuss terms which are generated by single total derivatives \( D_k \) and after that the remaining terms of second order coordinates. In other words, we discuss successively the leading orders and the leading polynomial degree in equation I) and II).

Step 2 (fourth order of degree one and third order of degree two in II)):

As we already mentioned in Step 1, the fourth order terms of degree one

\[
u^\gamma_{ijkl} \partial^i_\gamma \partial^j_\alpha \partial^{kl}_\beta f_\beta = 0 \iff \partial^i_\gamma \partial^{ij}_\alpha f_\beta = 0
\]

and the third order terms of degree two

\[
u^\delta_{krs} u^\gamma_{ij} \partial^r_\delta \partial^i_\gamma \partial^{kl}_\alpha f_\beta = 0 \iff \partial^i_\gamma \partial^{ij}_\alpha f_\beta = 0
\]

must vanish separately in equation II). The brackets \( (\ldots) \) mean symmetrization in the indices \( ijk,l, rsk, \) and \( lij \). It is possible to solve the system of differential equations \((38)\) and \((39)\) and we can determine the most general solution. Surprisingly, the set of solutions can be described by a finite dimensional vector space which is usually not the case for systems of PDEs (so-called overdetermined system). The solution \( f_\beta \) satisfies the following...
properties:

\[ f_\beta \text{ must be a polynomial of degree } \leq n \text{ in the second order coordinates and the coefficients of the polynomial are first order functions.} \]  

(40)

There are further restrictions on the first order coefficients of \( f_\beta \) and we could also describe this structure in detail, but the additional structure of these coefficients is not needed to complete the proof and we will not bother the reader at this point (the underlying structure is that \( f_\beta \) is a sum of so-called Hyperjacobians of second order)\(^\text{12}\) The proof was found by Anderson and Duchamp [2, pp.786].

Step 3 (third order of degree one in I)): With the help of equation I) and \((40)\), we show that

\[ H^i_{\alpha\beta} = O(2), \]
\[ H^{ij}_{\alpha\beta} = 0. \]  

(41)

It needs a lot of work to deduce \((41)\), and therefore we will prove it separately in Section 5 \(^\text{13}\).

Step 4 (third order of degree one in II)): This step is quite simple. We use \((41)\) and we plug these expressions into equation II), which then can be written as

\[ \text{II) } \quad H_{\alpha\gamma} + O^\beta_{\gamma,k}(1)H^k_{\alpha\beta} + O^\beta_{\gamma,kl}(2)H^{kl}_{\alpha\beta} = 0, \]

\[ = O(2), \text{ see } (41) \quad = 0, \text{ see } (41) \]

and therefore we get

\[ H_{\alpha\gamma} = O(2). \]  

(42)

\(^{12}\)For \( n = 1 \), equation \((38)\) reduces to \( \partial_{\gamma}^{11} \partial_{\alpha}^{11} f_\beta = 0 \) and an easy integration leads to \( f_\beta = A_\beta + B_\beta u_1 \), where \( A_\beta, B_\beta \) are first order functions. Therefore, \( f_\beta \) is obviously a polynomial of degree one in second order coordinates. Equation \((39)\) is automatically satisfied, but this is no longer true for \( n > 1 \) and integrating these differential equations gets much more complicated.

\(^{13}\)This step can maybe also be proven with a modification of the so called \( d \)-fold operator, used in the proof of Theorem 1.1 in [6, p.379] or in the proof of Theorem 1 in [20, p.12]. The same is maybe true for Step 6.
Step 5 (we consider the Helmholtz dependencies (27)): This step is again very simple. We use (41) and (42) and we plug them into the Helmholtz dependencies (27), which delivers

\[ H_{\alpha \beta} + H_{\beta \alpha} = D_k H_{\alpha \beta}^k - D_k D_l H_{\alpha \beta}^{kl}, \]

and this shows that \( D_k H_{\alpha \beta}^k = O(2) \). Now let us consider the equation \( D_k H_{\alpha \beta}^k = O(2) \) in more detail, where we again use (41) to write

\[ O(2) = D_k H_{\alpha \beta}^k = \partial_k H_{\alpha \beta}^k + u_\gamma^k \partial_\gamma H_{\alpha \beta}^k + u_\gamma^k \partial_\gamma H_{\alpha \beta}^k, \quad (43) \]

From (43) we get that \( \partial_{(ij)} H_{\alpha \beta}^k = 0 \), where \((ijk)\) means symmetrization in \(ijk\), since third order terms must vanish separately. When we set \(i = j = k\) then symmetrization in \( \partial_{(ij)} H_{\alpha \beta}^k = 0 \) reduces to a single term and we get

\[ \partial_{ij} H_{\alpha \beta}^i = 0. \quad (44) \]

Even if we could use the stronger condition \( \partial_{(ij)} H_{\alpha \beta}^k = 0 \), instead of (44), it turns out that (44) is actually sufficient to complete the proof.

Step 6 (second order in I)): With the help of equation I), (40), (41) and (44) we show that

\[ H_{\alpha \beta}^i = 0. \quad (45) \]

This step is quite difficult and we therefore prove it separately in Section 6\[superscript{14}\].

Step 7 (second order in II)): With the help of equation II), (41) and (45) we get that

\[ H_{\alpha \beta} = 0, \]

and therefore all Helmholtz conditions are satisfied. \(\square\)

It remains to prove Step 3 and Step 6 in the following sections.

\[superscript{14}\] Notice that this step can maybe also be proven with a different method, when using the so-called \(d\)-fold operator in [6, p.379] or [20, p.12].
5 Proof of Step 3

We briefly explain the notation in this section before we start with the proof. From Step 2 in Section 4 we know that \( (f_\alpha) \) must be polynomials of degree \( \leq n \) in second order coordinates, see (40). Therefore, we can write

\[
f_\alpha = A_\alpha + A^{ij}_{\alpha|\gamma} u^\gamma_{ij} + \ldots + A^{ij_1\ldots i_n j_n}_{\alpha|\gamma_1 \ldots \gamma_n} u^\gamma_{i_1 i_2} \ldots u^\gamma_{i_n j_n}, \tag{46}
\]

where the coefficients \( A^{ij_{1\ldots i_l}}_{\alpha|\gamma_1 \ldots \gamma_l} \) are first order functions. Now recall the Helmholtz expression \( H^{ij}_{\alpha\beta} = \partial^{ij}_{\beta} f_\alpha - \partial^{ij}_{\alpha} f_\beta \). When \( f_\alpha \) and \( f_\beta \) have the form (46), then \( H^{ij}_{\alpha\beta} \) is of degree \( \leq n-1 \) in the second order coordinates and we get

\[
\partial^{i_1 j_1}_{\gamma_1} \partial^{i_2 j_2}_{\gamma_2} \ldots \partial^{i_n j_n}_{\gamma_n} H^{ij}_{\alpha\beta} = 0. \tag{47}
\]

To determine \( H^{i}_{\alpha\beta} \) in more detail, let us introduce the following short notation: We write

\[
f_\alpha = A^0_\alpha + A^1_\alpha u(2) + A^2_\alpha u(2) u(2) + \ldots + A^n_\alpha u(2) u(2) \ldots u(2),
\]

where the coefficients \( A^l_\alpha \) are first order functions, and \( u^\gamma_{ij} \) will be identified with \( u(2) \), that is, we suppress some of the indices in comparison with (46).

With this short notation we do a symbolic calculation of the following form (the main focus is on the last expression \( 2D_k \partial^{jk}_{\alpha} f_\beta \))

\[
H^{i}_{\alpha\beta} = \partial^{i}_{\beta} f_\alpha + \partial^{i}_{\alpha} f_\beta - 2D_k \partial^{jk}_{\alpha} f_\beta =
= O_n(2) - 2D_k \partial^{jk}_{\alpha} (A^0_{\beta} + A^1_{\beta} u(2) + A^2_{\beta} u(2) u(2) + \ldots + A^n_{\beta} u(2) u(2) \ldots u(2)) =
= O_n(2) + D_k (B^{0,k}_{\alpha\beta} + B^{1,k}_{\alpha\beta} u(2) + B^{2,k}_{\alpha\beta} u(2) u(2) + \ldots + B^{n-1,k}_{\alpha\beta} u(2) u(2) \ldots u(2)) =
= O_n(2) + C^0_{\alpha\beta} u(3) + C^1_{\alpha\beta} u(2) u(3) + \ldots + C^{n-2}_{\alpha\beta} u(2) u(2) \ldots u(2) u(3), \tag{48}
\]

where \( B^{l,k}_{\alpha\beta} \) and \( C^l_{\alpha\beta} \) are again first order functions and \( u^\gamma_{ijk} \) is identified with \( u(3) \). In exact notation, we can write

\[
\partial^{i_1 j_1}_{\gamma_1} \partial^{i_2 j_2}_{\gamma_2} \ldots \partial^{i_n j_n}_{\gamma_n} H^{ij}_{\alpha\beta} = O_1(2) = O(2). \tag{49}
\]
Next, we want to apply second order partial derivatives $\partial_{\gamma_1}^{i_1} \partial_{\gamma_2}^{j_2} \ldots \partial_{\gamma_r}^{i_r}$, $1 \leq l \leq n$, to equation (I) and use the conditions (47) and (49) to derive further restrictions for $H_{\alpha \beta}^i$ and $H_{\alpha \beta}^{ij}$. But, we do not apply all kinds of second order partial derivatives $\partial_{\gamma_1}^{i_1} \partial_{\gamma_2}^{j_2} \ldots \partial_{\gamma_r}^{i_r}$, we only apply those which can be written as

\begin{align*}
a) \quad (\partial_{\gamma}^{JJ})^r &:= \underbrace{\partial_{\gamma_1}^{JJ} \partial_{\gamma_2}^{JJ} \ldots \partial_{\gamma_r}^{JJ}}_r, \quad 1 \leq r \leq n \\
b) \quad \partial_{\gamma}^{JJ}(\partial_{\gamma}^{JJ})^{r-1} &:= \underbrace{\partial_{\gamma}^{JJ} \partial_{\gamma_1}^{JJ} \ldots \partial_{\gamma_{r-1}}^{JJ}}_{(r-1)-times}, \quad 1 \leq r \leq n-1 \\
c) \quad \partial_{\gamma}^{JJ}(\partial_{\gamma}^{JJ})^{r-1} &:= \underbrace{\partial_{\gamma}^{JJ} \partial_{\gamma_1}^{JJ} \ldots \partial_{\gamma_{r-1}}^{JJ}}_{(r-1)-times}, \quad 1 \leq r \leq n-1
\end{align*}

where $J$ is a placeholder for (different) numbers in $\{1,2,\ldots,n\}$ such that $J \neq j$. Here is the reason why we choose these operators: Recall equation (I), which is

\begin{equation}
I) \quad (O_{j k}^{\beta}(1) - u_{j k}^\beta) H_{\alpha \beta}^k + (O_{j k l}^{\beta}(2) - u_{j k l}^\beta) H_{\alpha \beta}^{kl} = 0, \tag{50}
\end{equation}

for $j = 1,2,\ldots,n$ and $\alpha = 1,2,\ldots,m$. The operators in a), b), c) are defined in such a way that they commute with $u_{j k}$ in (50), except this does not hold for $\partial_{\gamma}^{JJ}$ and $\partial_{\gamma}^{JJ}$. Notice that there is summation over $k$, $l$, and $\beta$ in (50), but there is no summation over the indice $j$ and this is crucial. Again, the $J$’s can take all possible values $J \in \{1,2,\ldots,n\}$ such that $J \neq j$ and this holds for every single indice $J$ individually, that is, the (different) $J$’s can have different values. For example, for a) we can equivalently write

\begin{equation}
(\partial_{\gamma}^{JJ})^r = \partial_{\gamma_1}^{J_1} \partial_{\gamma_2}^{J_2} \ldots \partial_{\gamma_{r-1}}^{J_{r-1}} \partial_{\gamma_r}^{J_r}, \quad \text{for } J_s \in \{1,2,\ldots,n\} \text{ and } J_s \neq j, \tag{51}
\end{equation}

and a similar notation holds for $b)$ and $c)$. To indicate that $J$ is in general not a fixed number, we will also write $J$’s $\in \{1,2,\ldots,n\}$ instead of $J \in \{1,2,\ldots,n\}$. We also use the notation $(\gamma)^r = \gamma_1 \ldots \gamma_r$, that is, multi-index notation in $(\gamma)^r$ is assumed. Notice that (51) actually defines a set of operators for certain $J$’s $\in \{1,2,\ldots,n\}$, $J$’s $\neq j$ and there are many combination of such $J$’s which lead to an operator $(\partial_{\gamma}^{JJ})^r$. Therefore, let us now clarify which kinds of combinations we are interested in and what the further restrictions on the $J$’s are.

Below, we consider the set of operators $(\partial_{\gamma}^{JJ})^r$, $J$’s $\in \{1,2,\ldots,n\}$, $j \neq
$J$, but we will also have further restrictions (similar for $\partial_\delta^J(\partial_\nu^J)^{r-1}$ and $\partial_\delta^J(\partial_\nu^J)^{r-1}$). For example, $J$'s $\in \{2,3,\ldots,n\}$ and $J$'s $\neq j$. In general, we denote $S_r \subset \{1,2,\ldots,n\}$ as a subset of $\{1,2,\ldots,n\}$ with $r$ elements (we also define $S_0 := \emptyset$ as the empty set). When we have further restrictions, we simply write $J$'s $\in S_r$, and we do not always add $J$'s $\neq j$, since this is always be assumed. Notice that $j$ is allowed to be in the set $S_r$, but it can also be excluded. And last, when we write $J$'s $\in S_r$, then there is only one possibility to construct a set $S_n$ with $n$ elements, namely $S_n = \{1,2,\ldots,n\}$. However, when, for example, we write $J$'s $\in S_{n-1}$, then there are $n$ possibilities to construct a set $S_{n-1}$ with $n - 1$ numbers and we also need a notation for all such possibilities. Therefore, let us define $\mathcal{I}_{n-1}$ as the set which consists of all subsets of $\{1,2,\ldots,n\}$ with $n - 1$ elements, that is,

$$\mathcal{I}_{n-1} := \{\{2,3,\ldots,n\}, \{1,3,4,\ldots,n\}, \{1,2,4,5,\ldots,n\}, \ldots, \{1,2,\ldots,n-1\}\}$$

and a similar definition holds for all the other sets $\mathcal{I}_l$, $0 \leq l \leq n$, which consist of subsets of $\{1,2,\ldots,n\}$ with $l$ elements ($\mathcal{I}_0 = \emptyset$). This notation allows us to write a large amount of equations in a structured and compact form, for example, see (52), (53), (54).

Since we now have clarified our notation, we can start with the main part of the proof of Step 3. The proof is based on a sort of induction. We will write Step 3.0 for the $k$-th step in the induction.

Step 3.0 (Start of the proof of Step 3): The starting point of the induction are equations (47) and (49). To get the induction going, we actually only need the weaker conditions written in the following boxes

\begin{align*}
\text{a)} & \quad (\partial_\gamma^J)^n H_{\alpha\beta}^{kl} = 0, \quad J \text{'s} \in S_n, \quad S_n \in \mathcal{I}_n, \quad j = 1,2,\ldots,n, \quad (52) \\
\text{b)} & \quad (\partial_\gamma^J)^n-1 H_{\alpha\beta}^j = O(2), \quad J \text{'s} \in S_{n-1}, \quad S_{n-1} \in \mathcal{I}_{n-1}, \quad j = 1,2,\ldots,n, \quad (53) \\
\text{c)} & \quad (\partial_\gamma^J)^n-1 H_{\alpha\beta}^j = O(2), \quad J \text{'s} \in S_{n-1}, \quad S_{n-1} \in \mathcal{I}_{n-1}, \quad j = 1,2,\ldots,n. \quad (54)
\end{align*}

By convention, (52), (53) and (54) hold for all $k,l = 1,2,\ldots,n$, for all $\alpha,\beta = 1,2,\ldots,m$ and for all multi-indices $\gamma$, but we do not explicitly notice it because of lack of space (also in the formulas below). Also notice that the condition $J$'s $\in S_n$, $S_n \in \mathcal{I}_n$ and the condition $J$'s $\in S_{n-1}$, $S_{n-1} \in \mathcal{I}_{n-1}$ are equivalent in this case, since $J$ cannot take $n$ different values, because we always assume $J \neq j$ for some $j$. We choose this notation here to keep the same structure in the induction later on.
Step 3.1: We apply the operators $(\partial_{\gamma}^{IJ})^{n-1}$, $J's \in S_{n-1}$, $S_{n-1} \in \mathcal{S}_{n-1}$ to equation (50), and since we can commute them with $O(1)$, $u^{\beta}_{jk}$ and $u^{\beta}_{jkl}$, we get

\begin{align*}
\text{a) } 0 &= (O(1) - u^{\beta}_{jk}) (\partial_{\gamma}^{IJ})^{n-1} H_{\alpha\beta}^{k} + (\partial_{\gamma}^{IJ})^{n-1} [O(2) H_{\alpha\beta}^{kl}] - u^{\beta}_{jkl} (\partial_{\gamma}^{IJ})^{n-1} H_{\alpha\beta}^{kl}.
\end{align*}

The term $A$ is $O(2)$, see (53) and (54). The term $B$ is also $O(2)$, since $H_{\alpha\beta}^{kl}$ is $O(2)$. Moreover, the term $C$ is $O(2)$, and since it is the only term which contains a third order coordinate $u^{\beta}_{jkl}$, which can be varied independently of all other coordinates on $J^k E$, and $H_{\alpha\beta}^{kl}$ is symmetric in $k,l$, we get

Result a): $(\partial_{\gamma}^{IJ})^{n-1} H_{\alpha\beta}^{kl} = 0$, $J's \in S_{n-1}$, $S_{n-1} \in \mathcal{S}_{n-1}$, $j = 1,2,...,n$, which is the result of Step 3.1 a).

Then we apply the operators $\partial_{\delta}^{IJ} (\partial_{\gamma}^{IJ})^{n-2}$, $J's \in S_{n-2}$, $S_{n-2} \in \mathcal{S}_{n-2}$ to equation (50) and we get

\begin{align*}
\text{b) } 0 &= O(1) \partial_{\delta}^{IJ} (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{k} - \partial_{\delta}^{IJ} [u^{\beta}_{jk} (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{k}] + \\
&+ \partial_{\delta}^{IJ} (\partial_{\gamma}^{IJ})^{n-2} [O(2) H_{\alpha\beta}^{kl}] - u^{\beta}_{jkl} \partial_{\delta}^{IJ} (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{kl} = \\
= O(2) - (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{J} - u^{\beta}_{jk} \partial_{\delta}^{IJ} (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{k} = O(2) - (\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{J},
\end{align*}

(55)

The term $A$ and $D$ is $O(2)$, see (53) and (54). The term $B$ is also $O(2)$ since $H_{\alpha\beta}^{kl}$ is $O(2)$. The term $C$ vanishes, since when we choose the $J's$ in a set $S_{n-2}$, then the set $j \cup S_{n-2}$ is a set $S_{n-1}$ and we can apply the result of Step 3.1 a). Therefore, we get

Result b): $(\partial_{\gamma}^{IJ})^{n-2} H_{\alpha\beta}^{J} = O(2)$, $J's \in S_{n-2}$, $S_{n-2} \in \mathcal{S}_{n-2}$, $j = 1,2,...,n$, which is the result of Step 3.1 b).

Next, we apply the operators $\partial_{\delta}^{ij} (\partial_{\gamma}^{IJ})^{n-2}$, $J's \in S_{n-2}$, $S_{n-2} \in \mathcal{S}_{n-2}$ to

\footnote{For example, for $n = 2$ we get $\partial_{\gamma}^{11} H_{\alpha\beta}^{kl} = 0$ and $\partial_{\gamma}^{22} H_{\alpha\beta}^{kl} = 0$, but we do not get $\partial_{\gamma}^{12} H_{\alpha\beta}^{kl} = 0$, since we can only choose from a set $S_{n-1}$ of $n - 1 = 1$ numbers.}

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equation (50) and we get

c) \[ 0 \equiv O(1) \delta^j_i (\partial^J_J)^{n-2} H^k_{\alpha \beta} - \partial^j_i [u^\beta_j (\partial^J_J)^{n-2} H^k_{\alpha \beta}] + \]
\[ + \partial^j_i (\partial^J_J)^{n-2} [O(2) H^k_{\alpha \beta}] - u^\beta_j \partial^j_i (\partial^J_J)^{n-2} H^k_{\alpha \beta} = \]
\[ = O(2) - (\partial^J_J)^{n-2} H^j_{\alpha \delta} - u^\beta_j \partial^j_i (\partial^J_J)^{n-2} H^k_{\alpha \beta} = O(2) - (\partial^J_J)^{n-2} H^j_{\alpha \delta}, \]

where the terms \( A,B,C,D \) have the same properties as already explained in b), see (55). Therefore, we get

Result c): \((\partial^J_J)^{n-2} H^j_{\alpha \delta} = O(2), \) \( J's \in S_{n-2}, \) \( S_{n-2} \in \mathcal{I}_{n-2}, \) \( j = 1,2,\ldots,n, \)

which is the result of Step 3.1 c).

Together, the results in a), b) and c) deliver

\[
\begin{align*}
(\partial^J_J)^{n-1} H^k_{\alpha \beta} & = 0, & J's \in S_{n-1}, & S_{n-1} \in \mathcal{I}_{n-1}, & j = 1,2,\ldots,n, \quad (56) \\
(\partial^J_J)^{n-2} H^k_{\alpha \delta} & = O(2), & J's \in S_{n-2}, & S_{n-2} \in \mathcal{I}_{n-2}, & j = 1,2,\ldots,n. \quad (57)
\end{align*}
\]

Notice that we always have to prove a) first, since it is needed to prove b) and c).

**Step 3.2:** Now we are ready to do this inductively, that is, we can repeat exactly the same argument in every step, and we always use the result in the box from the previous step to derive the result in the actual step. We briefly write down Step 3.2 once again.

In Step 3.2 a), we apply the operators \((\partial^J_J)^{n-2}, \) \( J's \in S_{n-2}, \) \( S_{n-2} \in \mathcal{I}_{n-2}\) to equation (50), and with the result from the previous step (see (56) and (57)) we derive \((\partial^J_J)^{n-2} H^k_{\alpha \beta} = 0, \) \( J's \in S_{n-2}, S_{n-2} \in \mathcal{I}_{n-2}, \) \( j = 1,2,\ldots,n. \)

Then in Step 3.2 b), we apply \( \partial^i_j (\partial^J_J)^{n-3}, \) \( J's \in S_{n-3}, S_{n-3} \in \mathcal{I}_{n-3}\) to equation (50) which leads to \((\partial^J_J)^{n-3} H^j_{\alpha \delta} = O(2), \) \( J's \in S_{n-3}, S_{n-3} \in \mathcal{I}_{n-3}, \) \( j = 1,2,\ldots,n. \) Again, if we choose the \( J's \) in \( S_{n-3} \) then \( j \cup S_{n-3} \) can be considered as a set \( S_{n-2} \) and we can apply (57) and the result in Step 3.2 a).

In Step 3.2 c), we apply \( \partial^j_i (\partial^J_J)^{n-3}, \) \( J's \in S_{n-3}, S_{n-3} \in \mathcal{I}_{n-3}\) to equation (50) and this leads to \((\partial^J_J)^{n-3} H^j_{\alpha \delta} = O(2), \) \( J's \in S_{n-3}, S_{n-3} \in \mathcal{I}_{n-3}, \)

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Together, with the results a), b), c) we get

$$
(\partial^JJ)^{n-2}H_{\alpha\beta}^{kl} = 0, \quad J's \in S_{n-2}, \quad S_{n-2} \in \mathcal{I}_{n-2}, \quad j = 1,2,...,n,
$$

$$
(\partial^JJ)^{n-3}H_{\alpha\delta}^k = O(2), \quad J's \in S_{n-3}, \quad S_{n-3} \in \mathcal{I}_{n-3}, \quad j = 1,2,...,n.
$$

We repeat these arguments

until we get Step 3. (n - 2).

**Step 3. (n - 2):** In this step we get

$$
(\partial^JJ)^{n-(n-2)}H_{\alpha\beta}^{kl} = 0, \quad J's \in S_2, \quad S_2 \in \mathcal{I}_2, \quad j = 1,...,n,
$$

$$
(\partial^JJ)^{n-(n-1)}H_{\alpha\delta}^k = \partial^JJ H_{\alpha\delta}^k = O(2), \quad J's \in S_1, \quad S_1 \in \mathcal{I}_1, \quad j = 1,...,n.
$$

(58)

For example, for $n = 3$ we get $\partial_{11}^{11}H_{\alpha\delta}^k = 0$, $\partial_{22}^{22}H_{\alpha\delta}^k = 0$ and $\partial_{33}^{23}H_{\alpha\delta}^k = 0$. But we do not get one of the mixed partial derivatives $\partial_{12}^{12}H_{\alpha\delta}^k = 0$, $\partial_{13}^{13}H_{\alpha\delta}^k = 0$, or $\partial_{33}^{23}H_{\alpha\delta}^k = 0$. This will be crucial in the next step, since applying $\partial^JJ$ does not work any longer.

**Step 3. (n - 1):** The second last step is different in comparison with all the previous steps, since applying the $\partial^JJ$-operators does not work any longer, as already mentioned. Formally, in this step, with a), b) and c), we would get

$$
(\partial^JJ)^{n-(n-1)}H_{\alpha\beta}^{kl} = 0, \quad J's \in S_1, \quad S_1 \in \mathcal{I}_1, \quad j = 1,2,...,n,
$$

$$
(\partial^JJ)^{n-n}H_{\alpha\delta}^k = H_{\alpha\delta}^k = O(2), \quad J's \in S_0 = \emptyset, \quad j = 1,2,...,n.
$$

(59)

and since we cannot choose $J's \in \emptyset$, there must be something wrong. However, part a) still works, where we get

$$
a) \quad 0 = (O(1) - u_{jkl}^\beta) \underbrace{\partial^JJ H_{\alpha\beta}^{kl}}_{=O(2), \text{see (58)}} + \underbrace{\partial^JJ[O(2)H_{\alpha\beta}^{kl}]}_{=O(2)} - u_{jkl}^\beta \partial^JJ H_{\alpha\beta}^{kl},
$$

where $J's \in S_1, \quad S_1 \in \mathcal{I}_1$, that is, both of the $JJ$ must be the same now. Therefore, we get

**Result a):** $\partial^JJ H_{\alpha\beta}^{kl} = 0, \quad J's \in S_1, \quad S_1 \in \mathcal{I}_1.$

(60)
Because of (58), and because of (60), Step 3. \((n-1)\) \(b\) does not work any longer, where we would formally get

\[
0 = O(1) \left( \partial^j J H^k_{\alpha\beta} - \partial^j J \left[ u^\beta_{jk} H^k_{\alpha\beta} \right] + \partial^j J \left[ O(2) H^{kl}_{\alpha\beta} \right] - u^\beta_{jkl} \partial^j J H^{kl}_{\alpha\beta} \right),
\]

but we always assume \(j \neq J\).

Surprisingly, part \(c\) still works, where we get

\[
0 = O(1) \left( \partial^j J H^k_{\alpha\beta} - \partial^j J \left[ u^\beta_{jk} H^k_{\alpha\beta} \right] + \partial^j J \left[ O(2) H^{kl}_{\alpha\beta} \right] - u^\beta_{jkl} \partial^j J H^{kl}_{\alpha\beta} \right) = O(2) - H^j_{\alpha\delta} - u^\beta_{jk} \partial^j J H^k_{\alpha\beta} = O(2),
\]

and there does not occur any \(J\). Therefore, we get \(H^j_{\alpha\delta} = O(2)\).

Together with the results in \(a\), \(b\) and \(c\) we get

\[
(\partial^J J)^{n-(n-1)} H^{kl}_{\alpha\beta} = \partial^J J H^{kl}_{\alpha\beta} = 0, \quad J's \in S_1, \quad S_1 \in \mathcal{S}_1, \quad j = 1,2,...,n, \quad (formally J's \in \emptyset), \quad j = 1,2,...,n.
\]

In the last step, we now also show that \(H^{kl}_{\alpha\beta} = 0\).

**Step 3.** Formally, we would apply \((\partial^J J)^{n-n} = 1\) to equation (50) and deduce

\[
a) \quad 0 = (O(1) - u^\beta_{jk}) H^k_{\alpha\beta} + O(2) H^{kl}_{\alpha\beta} - u^\beta_{jkl} H^{kl}_{\alpha\beta},
\]

and since \(H^{kl}_{\alpha\beta}\) is \(O(2)\) and symmetric in \(k,l\), and third order terms must vanish separately, we get \(H^{kl}_{\alpha\beta} = 0\), which is the result of Step 3.\(n\) \(a\). Equation \(b\) and \(c\) do not provide new information in Step 3.\(n\).

Now (61) and the result of Step 3.\(n\) \(a\) provide

\[
H^{kl}_{\alpha\beta} = 0, \quad H^k_{\alpha\delta} = O(2),
\]

and the proof of Step 3 is complete.

\(\square\)
6 Proof of Step 6

The proof will be again a kind of induction and we will write Step 6.\(k\) for the \(k\)-th step in the induction. Equation I) can be written as

\[
1) \quad (O^\beta_{jk}(1) - u^\beta_{jk}) H^k_{\alpha\beta} = 0, \quad j = 1, 2, ..., n, \quad \alpha = 1, 2, ..., m, \tag{62}
\]

when we plug in the result (41) into (35). In the following, a partial derivative operator \(\partial^j_i\), where \(i = j\), is called a **derivative with same pairs**.

Let us consider the differential operator

\[
\partial^1_{\gamma_1} \wedge j_{j_1} \wedge j_{j_2} \wedge j_{j_r} \ldots \partial^n_{\gamma_n},
\]

where \(\wedge j_{j_1}, \wedge j_{j_2}, \ldots, \wedge j_{j_r}\) means that the the derivatives \(\partial_{\gamma_{j_1}}, \partial_{\gamma_{j_2}}, \ldots, \partial_{\gamma_{j_r}}\) are omitted in \(\partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^n_{\gamma_n}\). For example,

\[
\partial^1_{\gamma_1} \ldots \wedge j_{j} \ldots \partial^n_{\gamma_n} = \partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^{(j-1)(j-1)}_{\gamma_{j-1}} \partial^{(j+1)(j+1)}_{\gamma_{j+1}} \ldots \partial^n_{\gamma_n}.
\]

**Step 6.0 (Start of the proof of Step 6):** From (44) we know that

\[
\partial^j_i H^j_{\alpha\beta} = 0
\]

for all derivatives with same pairs. Notice that there is no summation over \(j\) in (63). Let us consider the expression \(\partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^n_{\gamma_n} H^j_{\alpha\beta}\). In the derivatives \(\partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^n_{\gamma_n}\), with same pairs, we can always find a \(\partial^j_i\)-derivative and \(\partial^j_i H^j_{\alpha\beta} = 0\). Therefore, we get

\[
\partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^n_{\gamma_n} H^j_{\alpha\beta} = 0. \tag{64}
\]

Let us now use (63) and (64) as the starting point in the induction.

**Step 6.1:** We consider equation (62), we apply the operator \(\partial^1_{\gamma_1} \partial^2_{\gamma_2} \ldots \partial^n_{\gamma_n}\) and we get (there is no summation over \(j\))

\[
O^\beta_{jk}(1) \underbrace{\partial^1_{\gamma_1} \ldots \partial^n_{\gamma_n} H^k_{\alpha\beta}}_{=0, \text{see Step 6.0, (63)}} - \underbrace{\partial^1_{\gamma_1} \ldots \wedge j_{j} \ldots \partial^n_{\gamma_n} H^j_{\alpha\beta}}_{=0, \text{see Step 6.0, (64)}} - u^\beta_{jk} \underbrace{\partial^1_{\gamma_1} \ldots \partial^n_{\gamma_n} H^k_{\alpha\beta}}_{=0, \text{see Step 6.0, (63)}} = 0.
\]

41
which leads to $\partial_{\gamma_1}^{11} \wedge^{j} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0$, where the $\partial_{\gamma_j}^{ij}$-derivative is omitted. For example, for $n = 2$, we get $\partial_{\gamma_1}^{22} H_{\alpha \beta}^{1} = 0$ and $\partial_{\gamma_1}^{11} H_{\alpha \beta}^{2} = 0$, but we do not (yet) get $\partial_{\gamma_1}^{11} H_{\alpha \beta}^{1} = 0$ and $\partial_{\gamma_1}^{22} H_{\alpha \beta}^{2} = 0$. But because of (63), we also know that $\partial_{\gamma_1}^{11} \wedge^{k} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0$, $k \neq j$, that is, where the $\partial_{\gamma_j}^{ij}$-derivative is included, which together provides

$$\partial_{\gamma_1}^{11} \wedge^{k} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0 \quad \text{for all } j, k = 1, 2, ..., n.$$  (65)

Notice: In the result (65) in Step 6.1 there is one derivative omitted, in the result in Step 6.2 there will be two derivatives omitted and so on.

Step 6.2: Now we can do this inductively, that is, we can repeat exactly the same argument from the previous step. Let us formulate this once again. We consider equation (62) and we apply the differential operator $\partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn}$, since we want to use the result (65) from the previous step to derive further conditions. Thus, we consider the equation

$$I) \quad \partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn} \left[ (O_{\alpha \beta}^{\gamma_k}(1) - u_{\alpha \beta}^{\gamma_k} ) H_{\alpha \beta}^{k} \right] = 0,$$  (66)

where $l = 1, 2, ..., n$. If $j = l$ in (66), then we can commute all derivatives with $O_{\alpha \beta}^{\gamma_k}(1)$ and $u_{\alpha \beta}^{\gamma_k}$ and we do not get any new information. Therefore, let $j \neq l$, that is, the $\partial_{\gamma_j}^{ij}$-derivative is included in $\partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn}$. Then we get

$$O(1) \left[ \partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn} H_{\alpha \beta}^{k} - \partial_{\gamma_1}^{11} \wedge^{jj} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} - \partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} \right]$$

$$= 0, \text{ see Step 6.1, (65)}$$

$$- u_{\alpha \beta}^{\gamma_k} \partial_{\gamma_1}^{11} \wedge^{ll} ... \partial_{\gamma_n}^{mn} H_{\alpha \beta}^{k} = 0 \quad (67)$$

which leads to $\partial_{\gamma_1}^{11} \wedge^{jj} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0$, $j \neq l$, that is, the derivatives $\partial_{\gamma_j}^{ij}$, $\partial_{\gamma_j}^{il}$ are omitted. Again, because of (63), we can also write $\partial_{\gamma_1}^{11} \wedge^{rr} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0$, $r \neq l$, and $r, l \neq j$, that is, the $\partial_{\gamma_j}^{ij}$-derivative is included in $\partial_{\gamma_1}^{11} \wedge^{rr} ... \partial_{\gamma_n}^{mn}$. Together, we get

$$\partial_{\gamma_1}^{11} \wedge^{rr} ... \partial_{\gamma_n}^{mn} H_{\alpha \gamma_j}^{j} = 0, \quad \text{for all } j, r, l = 1, 2, ..., n, \quad r \neq l.$$  (68)

Notice that in the sum over $k$ in (67) there exists a $k$ such that $k = l$ and therefore it is not possible to apply (63) and we definitely need the result from Step 6.1 in (65).
Step 6.3: When we do the procedure we get
\[ \partial_{\gamma_1}^{11} \ldots \Lambda^{kk} \ldots \Lambda^{rr} \ldots \Lambda^{ll} \ldots \partial_{\gamma_n}^{nn} H_{\alpha\gamma_j}^j = 0, \quad \text{for all } k, r, l, j = 1, 2, \ldots, n, \]
\[ k, r, l \text{ are different.} \]

We repeat exactly the same argument from the 6.k-th in the 6.(k+1)-th Step:
\[ \vdots \]
until we get Step 6.(n − 1).

Step 6.(n − 1): We do the same calculation as before and we get
\[ \partial_{\gamma_1}^{ll} H_{\alpha\gamma_j}^j = 0, \quad \text{for all } j, l = 1, 2, \ldots, n, \]  
(69)
and this is a generalization of (63), since the equation now also holds for \( j \neq l \).

Step 6.n: In the last step, we apply the differential operator \( \partial_{\gamma_j}^{jj} \) to equation (62), which leads to
\[ O(1) \begin{array}{c} \partial_{\gamma_j}^{jj} H_{\alpha\beta}^k = 0, \\
= 0, \text{ see (69)} \end{array} \quad \begin{array}{c} -H_{\alpha\gamma_j}^j - u_{jk}^{\beta} \partial_{\gamma_j}^{jj} H_{\alpha\beta}^k = 0, \\
= 0, \text{ see (69)} \end{array} \]
and therefore we get
\[ H_{\alpha\gamma_j}^j = 0, \]
which completes the proof of Step 6. \( \square \)

7 Open problems and conclusion

Concerning Takens’ question, there are several open problems in terms of applications and pure mathematics.

When we want to apply Theorem 4.1 in the context of physics, as it is motivated in the introduction, we have at least to check two things: First, we have to check whether physically interesting symmetries satisfy the span-condition (32). Since the set of symmetries we may assume in physics is
usually large this condition should be satisfied in a lot of interesting cases. Second, we have to check if the corresponding conservation laws have physical relevance. In both cases, a coordinate invariant definition is in general difficult without further assumptions. For example, energy conservation is usually an equation of the form \( u^\alpha f_\alpha = D_t J^i \), where \( (f_\alpha) \) describes a second order differential equation, the characteristic is \( (Q^\alpha) = (u^\alpha) \), and \( (J^i) \) is the current density, also see (1). An open question is, what is the coordinate invariant definition of energy conservation, and does an equation of the form \( u^\alpha f_\alpha = D_t J^i \) also make sense for higher order source forms, like fourth order. Also the conservation laws in [6, 20] are only given in local coordinates and the additional assumption \( M = \mathbb{R}^n \) is made. In the context of applications, we should also find an explanation why the symmetries \( V \) and continuity equations with characteristics \( Q \) are connected in the very special form \( Q^\alpha = V_{\nu}^\alpha \), see Definition 3.2.

Also a very big open question is to find an explanation why a differential equation, given by functions \( f_r, r = 1, 2, ..., R \), should allow for a source form formulation \( \Delta = f_\alpha du^\alpha \wedge dx \), where \( R = m \), and where we force a very specific transformation property of \( f \), as we explained in Section 3. These are, in a sense, the additional and hidden assumptions in Theorem 4.1 (beside the assumptions of symmetries and conservation laws). This also means that we do not consider the symmetries of the function \( f \), rather the symmetries of the source form, that is, the weak formulation of the differential equation.

Another quite difficult problem which is also important in the pure mathematical context is the definition of a variational differential equation in general. When we already assigned a source form to some function \( f \) then there is a clear answer if the source form is variational or not, see Definition 3.1. But when we only consider the differential equation without an assignment to a source form, then there are many equivalent reformulations of the differential equation. Some of them will be variational and some will not. For example, Maxwell’s equations are not variational when formulated with the fields \( E \) and \( B \) as \( \nabla E = \rho, \nabla B = 0, \nabla \times E = -\partial_t B, \nabla \times B = j + \partial_t E \), but they are variational when formulated with the vector potential \( A_\mu \) as \( \partial_\mu F^{\mu\nu} = j^\nu \), where \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Therefore, we would need an explanation why we should consider the one or the other formulation of the differential equation. This problem is also connected to the so-called variational multiplier method which transforms equations equivalently to other differential equations, see [9, 7]. However, the variational multiplier transformation does not cover all equivalent reformulations of differential equations, especially order reduction.
or order increase methods are not included, as explained for Maxwell’s equations.

The question of Takens is very interesting, as motivated in the introduction, and we should probably try to find a reformulation, such that we can avoid the above mentioned problems, or at least some of them. In our opinion, this is the most crucial problem at the time. For example, a reasonable question could be: Is a source form which satisfies certain symmetries and corresponding continuity equations (or some kinds of conservation laws in general) always equivalent to a variational one and can we always assign a source form to any differential expressions \( f_r, r = 1, 2, ..., R \) in a certain way, such that we get a meaningful weak formulation, for example, by order reduction or order increase methods.

8 Appendix

**Proof of the conjecture in Section 2.** We write \( u^{(n)}(x) = \frac{d^n u(x)}{dx^n} \). The so-called total derivative operator \( \frac{d}{dx} \) is the operator \( \frac{d}{dx} := \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + ... \), see (4) for further details. When we have a conservation law of the form \( u_x f = \frac{d}{dx} E \), and we may assume \( E = E(x,u,u_x,...,u_{(k)}) \) depends on coordinates up to order \( k \geq 2 \), then the equation

\[
 u_x f = \frac{\partial E}{\partial x} + u_x \frac{\partial E}{\partial u} + u_{xx} \frac{\partial E}{\partial u_x} + u_{xxx} \frac{\partial E}{\partial u_{xx}} + ... + u_{(k+1)} \frac{\partial E}{\partial u_{(k)}} \tag{70}
\]

leads to the following cascade of conditions:

- The term \( u_x f \) on the left hand side in (70) only depends on the coordinates \( (x,u,u_x,u_{xx}) \), since we assume that \( f \) is of second order. On the right hand side in (70), the term \( u_{(k+1)} \frac{\partial E}{\partial u_{(k)}} \) must vanish, since otherwise this would be the only term where we have a \( u_{(k+1)} \)-coordinate, which can be varied independently of the remaining coordinates \( (x,u,u_x,...,u_{(k)}) \). This forces that \( \frac{\partial E}{\partial u_{(k)}} = 0 \) for all values \( (x,u,u_x,...,u_{(k)}) \). This means that \( E = E(x,u,u_x,...,u_{(k-1)}) \).

- We repeat these arguments until we get \( E = E(x,u,u_x) \).
Now let us again consider equation (70), which now reduces to
\[ u_x f(x,u,u_x,u_{xx}) = \frac{\partial E(x,u,u_x)}{\partial x} + u_x \frac{\partial E(x,u,u_x)}{\partial u} + u_{xx} \frac{\partial E(x,u,u_x)}{\partial u_x}. \] (71)

Equation (71) shows that \( f(x,u,u_x,u_{xx}) \) must be affine linear in \( u_{xx} \), and therefore we can write \( f = A + u_{xx} B \) for some functions \( A = A(x,u,u_x) \) and \( B = B(x,u,u_x) \). More precisely, we get \( A = \frac{1}{u_x} \left( \frac{\partial E}{\partial x} + u_x \frac{\partial E}{\partial u} \right) \) and \( B = \frac{1}{u_x} \frac{\partial E}{\partial u_x} \) whenever \( u_x \neq 0 \).

When \( f \) is \( \frac{\partial}{\partial x} \)-invariant, then \( f = f(u,u_x,u_{xx}) \), and this also means that \( A = A(u,u_x) \) and \( B = B(u,u_x) \). Therefore, we get \( f = A(u,u_x) + u_{xx} B(u,u_x) \). Plugging in \( f = A + u_{xx} B \) into (71), and sorting all terms with respect to the \( u_{xx} \)-coordinate, and terms which do not involve a \( u_{xx} \)-coordinate, delivers the two equations
\[ u_x A(u,u_x) = \frac{\partial E(x,u,u_x)}{\partial x} + u_x \frac{\partial E(x,u,u_x)}{\partial u}, \] (72)
\[ u_x B(u,u_x) = \frac{\partial E(x,u,u_x)}{\partial u_x}. \] (73)

We apply \( \frac{\partial}{\partial u_x} \) to equation (72), and \( \frac{\partial}{\partial x}, \frac{\partial}{\partial u} \) to equation (73), to deduce
\[ A + u_x \frac{\partial A}{\partial u_x} = \frac{\partial^2 E}{\partial x \partial u_x} + \frac{\partial E}{\partial u} + u_x \frac{\partial^2 E}{\partial u \partial u_x}, \] (74)
\[ 0 = \frac{\partial^2 E}{\partial x \partial u_x}, \] (75)
\[ u_x \frac{\partial B}{\partial u} = \frac{\partial^2 E}{\partial u \partial u_x}. \] (76)

Plugging in (75) and (76) into (74) delivers
\[ A + u_x \frac{\partial A}{\partial u_x} = \frac{\partial E}{\partial u} + u_x^2 \frac{\partial B}{\partial u}. \] (77)

Multiplying equation (77) by \( u_x \), and using (72), delivers
\[ u_x^2 \frac{\partial A}{\partial u_x} = -\frac{\partial E}{\partial x} + u_x^3 \frac{\partial B}{\partial u}. \] (78)

Applying \( \frac{\partial}{\partial x} \) to (77) delivers
\[ 0 = \frac{\partial^2 E}{\partial x \partial u}, \] (79)
since \(A\) and \(B\) do not explicitly depend on \(x\). Now equation (75) and (79) show that \(\frac{\partial E}{\partial x}\) does not depend on \((u,u_x)\), that is, \(\frac{\partial E}{\partial x} = \frac{\partial E(x)}{\partial x}\). We use this condition to rewrite equation (78) as

\[
u^2 \frac{\partial A(u,u_x)}{\partial u_x} - u_x \frac{\partial B(u,u_x)}{\partial u} = - \frac{\partial E(x)}{\partial x}.\]

(80)

When we let \(u_x \to 0\) in equation (80), and when we assume that \(\frac{\partial A}{\partial u} - u_x \frac{\partial B}{\partial u}\) is a smooth function in \(u_x\), then we get \(\frac{\partial E(x)}{\partial x} = 0\) for all \(x\). When \(\frac{\partial E}{\partial x} = 0\) and when we divide (71) by \(u_x\), then we get

\[f(u,u_x,u_{xx}) = \frac{\partial E(u,u_x)}{\partial u} + u_{xx} \frac{\partial E(u,u_x)}{\partial u_x} = \left( \frac{\partial}{\partial u} + \frac{u_{xx}}{u_x} \frac{\partial}{\partial u_x} \right) E,\]

(81)

where we know by assumption that \(B = \frac{1}{u_x} \frac{\partial E}{\partial u}\) is a smooth function (there exists a unique smooth continuation for \(u_x \to 0\)). We can always write \(E\) as \(E = L - u_x \frac{\partial L}{\partial u_x}\) for some function \(L = L(u,u_x)\), since this differential equation can always be solved for a suitable \(L\). Formally, the solution is \(L = c(u)u_x - u_x \int \frac{E(u^2)}{u_x^2} du_x\), where \(c(u)\) is an arbitrary function depending on \(u\) only. That there exists a one time differentiable function \(L\) in the whole range of definition is left to the reader. For example, the equation \(u_x = L - u_x \frac{\partial L}{\partial u_x}\) has a singular solution \(L = -u_x \ln u_x\), where \(L\) is not one time differentiable (the resonance case). Notice that \(u_x c(u) = \frac{d}{dx} C(u)\) is a trivial Lagrangian, but we can not ignore it, since it is needed to construct a one time differentiable \(L\). Now we can write (81) as

\[f = \left( \frac{\partial}{\partial u} + \frac{u_{xx}}{u_x} \frac{\partial}{\partial u_x} \right) \left( L - u_x \frac{\partial L}{\partial u_x} \right) = \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u_x},\]

which proves that \(f\) is variational. When we consider \(\frac{dE}{dx} = \frac{dL}{dx} - u_{xx} \frac{\partial L}{\partial u_x} - u_x \frac{d}{dx} \frac{\partial L}{\partial u_x}\) and when we divide through \(u_x\), then we also get that \(\frac{d}{dx} \frac{\partial L}{\partial u_x}\) must be continuous.

□

Notice, when we assume that \(f\) is \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial u}\) invariant, then we would not need to make the smoothness assumption of the function \(f\) (see Theorem 31).
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