Weil-Petersson translation distance and volumes of mapping tori

Jeffrey F. Brock *

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Abstract

Given a closed hyperbolic 3-manifold $T_\psi$ that fibers over the circle with monodromy $\psi: S \to S$, the monodromy $\psi$ determines an isometry of Teichmüller space with its Weil-Petersson metric whose translation distance $\|\psi\|_{\text{WP}}$ is positive. We show there is a constant $K \geq 1$ depending only on the topology of $S$ so that the volume of $T_\psi$ satisfies $\|\psi\|_{\text{WP}}/K \leq \text{vol}(T_\psi) \leq K\|\psi\|_{\text{WP}}$.

1 Introduction

In this paper we generalize our study of the relation of Weil-Petersson distance to volumes of hyperbolic 3-manifolds to the context of those hyperbolic 3-manifolds that fiber over $S^1$.

Let $S$ be a closed surface of genus at least 2, and let $\text{Mod}(S)$ denote its mapping class group, or the group of isotopy classes of orientation preserving self-homeomorphisms of $S$. Let $\psi$ be a pseudo-Anosov element of $\text{mod}(S)$, i.e. an element that preserves no isotopy class of simple closed curves on the surface other than the trivial one.

Elements of $\text{Mod}(S)$ act by isometries on the Teichmüller space $\text{Teich}(S)$ with the Weil-Petersson metric. Letting $d_{\text{WP}}(.,.)$ denote Weil-Petersson distance, let

$$\|\psi\|_{\text{WP}} = \inf_{X \in \text{Teich}(S)} d_{\text{WP}}(X, \psi(X))$$

denote the Weil-Petersson translation distance of $\psi$ acting as an isometry of $\text{Teich}(S)$.

Choosing a particular representative $\hat{\psi}: S \to S$ of $\psi$, we may form the mapping torus $T_\psi = S \times [0, 1]/(x, 1) \sim (\hat{\psi}(x), 0)$.

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Thurston exhibited a hyperbolic structure on $T_\psi$ for any pseudo-Anosov $\psi$; the \textit{hyperbolic volume} $\text{vol}(T_\psi)$ is then naturally associated to the mapping class $\psi$. The question of whether the assignment $\psi \rightarrow \text{vol}(T_\psi)$ is a previously encountered isotopy invariant of a homeomorphism is a natural one.

This paper relates the hyperbolic volume of $T_\psi$ to the Weil-Petersson translation distance for $\psi$.

\textbf{Theorem 1.1} Given $S$ there is a constant $K > 1$ and so that if $T_\psi$ is a pseudo-Anosov mapping torus, we have

$$\frac{1}{K} \|\psi\|_{\text{WP}} \leq \text{vol}(T_\psi) \leq K \|\psi\|_{\text{WP}}.$$

The proof of the main theorem mirrors the proof of a similar result for \textit{quasi-Fuchsian} hyperbolic 3-manifolds [Br]: if $Q(X,Y) = \mathbb{H}^3/\Gamma(X,Y)$ is the hyperbolic quotient of $\mathbb{H}^3$ by the Bers \textit{simultaneous uniformization} of the pair $(X,Y) \in \text{Teich}(S) \times \text{Teich}(S)$, then we have the following theorem.

\textbf{Theorem 1.2 (Main Theorem of [Br])} Given the surface $S$, there are constants $K_1 > 1$ and $K_2 > 0$ so that

$$\frac{d_{\text{WP}}(X,Y)}{K_1} - K_2 \leq \text{vol}(\text{core}(Q(X,Y))) \leq K_1 d_{\text{WP}}(X,Y) + K_2.$$

Here, $\text{core}(Q(X,Y))$ is the \textit{convex core} of $Q(X,Y)$: the smallest convex subset of $Q(X,Y)$ carrying its fundamental group.

The challenge of theorem 1.1 is primarily to relate the techniques involved in the proof of theorem 1.2 to the setting of 3-manifolds fibering over $S^1$. Here is a quick sketch of the argument:

- The cover $M_\psi$ of $T_\psi$ corresponding to the inclusion $\iota: S \rightarrow T_\psi$ of the fiber is an infinite volume doubly degenerate manifold homeomorphic to $S \times \mathbb{R}$ and invariant by a translational isometry $\Psi: M_\psi \rightarrow M_\psi$, for which $\Psi_\ast = \psi_\ast$.

- We control the volume of a fundamental domain $D$ for the action of $\Psi$ from below by estimating up to bounded error a lower bound for the number of bounded length closed geodesics in $D$. Since $D$ is not convex, there is potential spill-over of volume into other translates of $D$; this is rectified by a limiting argument.
We control the volume of $T_\psi$ from above by building a model manifold $N_\Delta \cong T_\psi$ built out of 3-dimensional tetrahedra, and a degree-1 homotopy equivalence $f: N_\Delta \to T_\psi$ that is simplicial (the lift $\tilde{f}: \tilde{N}_\Delta \to \mathbb{H}^3$ sends each tetrahedron to the convex hull of the image of its vertices).

A priori bounds on the volume of a tetrahedron in $\mathbb{H}^3$ give an estimate on the total volume of the image, and by a spinning trick (as in [Br]) we may homotope $f$ to force all but $k\|\psi\|_{\text{WP}}$ to have arbitrarily small volume.

The constructions in each case are directly those of [Br], to which we refer often in the interest of brevity. The paper will introduce vocabulary in section 2 and prove the main theorem in section 3.

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2 Vocabulary

We define our terms. Throughout, $S$ will be a closed surface of genus at least 2.

Teichmüller space. The Teichmüller space $\text{Teich}(S)$ is the set of marked hyperbolic structures on $S$, namely pairs $(f,X)$ of a hyperbolic surface $X$ and a homeomorphism $f: S \to X$, up to the equivalence relation $(f,X) \sim (g,Y)$ if there is an isometry $\phi: X \to Y$ so that $\phi \circ f \simeq g$. The Teichmüller space inherits a topology from the marked bi-Lipschitz distance:

$$d_{\text{bL}}((f,X),(g,Y)) = \inf_{\phi \simeq g \circ f^{-1}} \log(L(\phi))$$

where

$$L(\phi) = \sup_{x \in X, v \in T_x(X)} |D\phi(v)||v|$$

denotes the best bi-Lipschitz constant for $\phi$.

Considering $X = \mathbb{H}/\Gamma$ as a Riemann surface, let $Q(X)$ denote the vector space of holomorphic quadratic differentials $\phi(z)dz^2$ on $X$. The space $Q(X)$ is naturally the holomorphic cotangent space to $\text{Teich}(S)$. The Weil-Petersson Hermitian metric on $\text{Teich}(S)$ is induced by the inner product on $Q(X)$

$$\langle \phi, \psi \rangle_{\text{WP}} = \int_X \frac{\overline{\phi \psi}}{\rho^2} d\bar{z}dz$$
(where $\rho(z)|dz|$ is the Poincaré metric on $X$) via the duality $(\mu, \phi) = \int_X \mu \phi$ where $\mu$, a Beltrami differential, is a tangent vector to Teich$(S)$. Here, we consider the Riemannian part $g_{WP}$ of the Weil-Petersson metric, and its associated distance function $d_{WP}(.,.)$.

We will often refer to a hyperbolic Riemann surface $X$, assuming an implicit marking $(f: S \to X)$. Hyperbolic 3-manifolds. Moving up a dimension, we denote by $V(S) = \text{Hom}(\pi_1(S), \text{Isom}^+(\mathbb{H}^3))/\text{conjugation}$ the representation variety for $\pi_1(S)$. Each equivalence class $[\rho]$ lying in the subset $AH(S) \subset V(S)$ consisting of representations that are discrete and faithful determines a complete hyperbolic 3-manifold $M = \mathbb{H}^3/\rho(\pi_1(S))$. Since hyperbolic 3-manifolds are $K(\pi,1)$s, points in $AH(S)$ are in bijection with the set of all marked hyperbolic 3-manifolds $(f: S \to M)$, where $f$ is a homotopy equivalence, modulo the equivalence relation $(f: S \to M) \sim (g: S \to N)$ if there is an isometry $\phi: M \to N$ so that $\phi \circ f \simeq g$.

Simple closed curves. Let $S$ denote the set of isotopy classes of essential simple closed curves on $S$. Given two curves $\alpha$ and $\beta$ in $S$ their geometric intersection number $i(\alpha, \beta)$ is obtained by counting the minimal number of points of intersection over all representatives of $\alpha$ and $\beta$ on $S$.

A pants decomposition $P$ of $S$ is a maximal collection of distinct elements of $S$ with pairwise disjoint representatives on $S$. Two pants decompositions $P$ and $P'$ are related by an elementary move if $P'$ can be obtained from $P$ by removing a curve $\alpha \in P$ and replacing it with a curve $\beta \neq \alpha$ so that $i(\alpha, \beta)$ is minimal among all choices for $\beta$ (see figure [1]). Denote by $C_P(S)$ the graph whose vertices are pants decompositions of $S$ and whose edges join pants decompositions differing by an elementary move. Hatcher and Thurston introduced the complex $C_P(S)$ in [HT], where they prove $C_P(S)$ is connected (see also [HLS]). It therefore carries a natural path metric, obtained by assigning each edge length 1. We will be interested in the distance between vertices: we let $d_P(P, P')$ to be the length of the minimal length path in $C_P(S)$ joining $P$ to $P'$.

In [Br] we proved the following theorem.

**Theorem 2.1** The graph $C_P(S)$ is coarsely quasi-isometric to Teich$(S)$ with its Weil-Petersson metric.

(See [Br, Thm 1.1]). A quasi-isometry $Q: C_P(S) \to \text{Teich}(S)$ is obtained by taking $Q$ to be any mapping for which $Q(P) = X_P$ lies in the set

$$V(P) = \{X \in \text{Teich}(S) \mid \ell_X(\alpha) < L \text{ for each } \alpha \in P\}$$
where $L$ is Bers’ constant, namely, the minimal constant for which each hyperbolic surface $X \in \text{Teich}(S)$ admits a pants decomposition $P$ all of whose representatives have length less than $L$ on $X$ (see [Bus]).

**Simplicial hyperbolic surfaces.** Let $\text{Sing}_k(S)$ denote the marked singular hyperbolic structures on $S$ with at most $k$ cone singularities, namely surfaces that are hyperbolic away from at most $k$ cone points, each with cone angle at least $2\pi$, equipped with homeomorphisms $g: S \to Z$, up to marking preserving isometry. As with $\text{Teich}(S)$, $\text{Sing}_k(S)$ is topologized via the marked bi-Lipschitz distance, defined analogously.

Let $M$ be a complete hyperbolic 3-manifold. A simplicial hyperbolic surface is a path-isometric map $h: Z \to M$ of a singular hyperbolic surface into $M$ so that $h$ is totally geodesic on the faces of a geodesic triangulation $T$ of $Z$, in the sense of [Hat] (cf. [Br, Sec. 4]). We say $g$ is adapted to $T$.

Given $(f: S \to M)$ in $\text{AH}(S)$, we denote by $\text{SH}_k(S)$ the simplicial hyperbolic surfaces homotopic to $f$, in other words, those simplicial hyperbolic surfaces $h: Z \to M$, so that if $g: S \to Z$ is the marking on $Z$, then $h \circ g$ is homotopic to $f$.

As in [Br], the following result of Canary (see [Can, Sec. 5]) will be instrumental in what follows.

**Theorem 2.2 (Canary)** Let $M \in \text{AH}(S)$ have no accidental parabolics, and let $(g_1, Z_1)$ and $(g_2, Z_2)$ lie in $\text{SH}_1(M)$ where $(g_1, Z_1)$ and $(g_2, Z_2)$ are adapted to $\alpha$ and $\beta$. Then there is a continuous family $(g_t: Z_t \to M) \subset \text{SH}_2(M)$, $t \in [1, 2]$. 

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**Figure 1. Elementary moves on pants decompositions.**
3 Proof of the main theorem

Given $\psi \in \text{Mod}(S)$, let

$$T_\psi = S \times I / (x, 0) \sim (\psi(x), 1)$$

be the mapping torus for $\psi$. By a theorem of Thurston ([Th], see also [Mc], [Otal]), $T_\psi$ admits a complete hyperbolic structure if and only if $\psi$ is pseudo-Anosov: no power of $\psi$ fixes any non-peripheral isotopy class of essential simple closed curves on $S$.

To prove theorem 1.1 we relate the volume of the mapping torus $T_\psi$ to the translation distance of $\psi$ on the pants complex $C_P(S)$.

**Proof:** (of theorem 1.1) Let $M_\psi$ be the cover of $T_\psi$ corresponding to the inclusion of the fiber $\iota: S \to T_\psi$; $\iota$ naturally lifts to a marking ($\tilde{\iota}: S \to M_\psi$) of $M_\psi$, so $M_\psi$ is naturally an element of $AH(S)$.

We will apply the techniques of [Br] directly to larger and larger subsets of the manifold $M_\psi$.

**Bounding volume from below.** Consider any simplicial hyperbolic surface $(h: Z \to M_\psi) \in SH_1(M_\psi)$. Let $(h_\psi: Z_\psi \to M_\psi) \in SH_1(M_\psi)$ be the simplicial hyperbolic surface obtained by post-composing the image with the generating covering transformation $\Psi: M_\psi \to M_\psi$ acting on $\pi_1(M_\psi)$ by $\psi\ast$.

The hyperbolic surface $Z^h$ in the same conformal class as $Z$ differs from the hyperbolic surface $Z^h_\psi$ by the action of $\psi$ on Teich($S$). If $Z^h$ lies in $V(P)$, then, the surface $Z^h_\psi$ lies in $V(\psi(P))$.

By an argument using theorem 2.2 in an exactly analogous manner to its use in [Br, Sec. 4], there is a sequence

$$g = \{P = P_0, P_1, \ldots, P_n = \psi(P)\}$$

and a continuous family $(h_t: Z_t \to M_\psi) \subset SH_k(M_\psi)$, $k = 2$, $0 \leq t \leq n$, of simplicial hyperbolic surfaces joining $Z$ and $Z_\psi$ so that the following holds:

- For each $i$, the simplicial hyperbolic surface $h_i: Z_i \to M_\psi$ satisfies $\ell_{Z_i}(\alpha) < L$ for each $\alpha \in P_i$.

- Successive pants decompositions have distance $d_{P_i}(P_i, P_{i+1}) < D$ where $D$ depends only on $S$.

Let $S_g \subset S$ denote the set of curves

$$S_g = \{\alpha \in S \mid \alpha \in P_t, P_t \in g\}.$$
Then we phrase the following lemma of [Br] (lemma 4.2) to suit our context:

**Lemma 3.1** There is a constant $K$ depending only on $S$ so that

$$d_P(P, \psi(P)) \leq K|S_g|$$

where $|S_g|$ denotes the number of elements in $S_g$.

Consider the image of the homotopy $H: [0, n] \times S \to M_\psi$ for which $H(t, .) = h_t \circ g_t: S \to M_\psi$ and $g_t: S \to Z_t$ is the implicit marking on $Z_t$.

Applying the argument of lemma 4.1 of [Br], there are constants $c_1 > 1$ and $c_2 > 0$ depending only on $S$ so that the $\epsilon$ neighborhood of the image $N_\epsilon(H([0, n] \times S))$ has volume

$$\text{vol}(N_\epsilon(H([0, n] \times S))) > \frac{|S_g|}{c_1} - c_2 \geq \frac{d_P(P, \psi(P))}{c_1 K} - c_2.$$

The image $H([0, n] \times S)$ is a compact subset of $M_\psi$; choose an embedding $h: S \to M_\psi$ homotopic to $\iota$ so that $h(S)$ does not intersect the $\epsilon$-neighborhood $N_\epsilon(H([0, n] \times S))$ of the image $H([0, n] \times S)$. Let $\Psi: M_\psi \to M_\psi$ be the isometric covering transformation so that $\iota \circ \psi \simeq \Psi \circ \iota$.

Let $n_0$ be an integer so that $\Psi^{n_0} \circ h(S)$ is also disjoint from $N_\epsilon(H([0, n] \times S))$ and so that $N_\epsilon(H([0, n] \times S))$ lies in the compact subset $K$ of $M_\psi$ bounded by $h(S)$ and $\Psi^{n_0} \circ h(S)$. Note that

$$\text{vol}(K_{n_0}) = n_0 \text{vol}(T_\psi)$$

since the region bounded by each $\Psi^i \circ h(S)$ and $\Psi^{i+1} \circ h(S)$ is a fundamental domain for the action on $\Psi$ on $M_\psi$.

It follows that if

$$C_j = \bigcup_{i=0}^{j} \Psi^i(N_\epsilon(H([0, n] \times S)))$$

then we have

$$(n_0 + j)\text{vol}(T_\psi) \geq \text{vol}(C_j) \geq \frac{j d_P(P, \psi(P))}{c_1 K} - jc_2.$$
Taking the limit as \( j \) tends to infinity, we have

\[
\text{vol}(T_\psi) \geq \frac{d_P(P, \psi(P))}{c_1 K} - c_2.
\]

Applying theorem 2.1, there are constants \( d_1 > 1 \) and \( d_2 > 0 \) so that

\[
\text{vol}(T_\psi) \geq \frac{\|\psi\|_{WP}}{d_1} - d_2.
\]

But given \( S \), there is an \( \epsilon_S \) so that

\[
\min\{\text{vol}(T_\psi) \mid \psi \in \text{Mod}(S) \text{ is pseudo-Anosov}\}
\]

is greater than \( \epsilon_S \), so there is a thus a \( K_0 > 1 \) so that

\[
\text{vol}(T_\psi) \geq \frac{\|\psi\|_{WP}}{K_0},
\]

proving one direction of the theorem.

Bounding volume from above. To bound \( \text{vol}(T_\psi) \) from above, we adapt our construction of a simplicial model 3-manifold for a quasi-Fuchsian 3-manifold in [Br, Sec. 5] to build a triangulated 3-manifold \( N \cong T_\psi \), and a degree one homotopy equivalence \( f: N \to T_\psi \) that is simplicial: the lift \( \tilde{f}: \tilde{N} \to \mathbb{H}^3 \) maps each simplex \( \Delta \subset N \) to the convex hull of the images of its vertices.

Consider the surface \( Z^h \) in \( V(P) \), and let \( T \) be a standard triangulation suited to \( P \), in the sense of [Br, Defn. 5.3]. Let \( (g_0: Z_0 \to M_\psi) \in SH_k(M_\psi) \) be a simplicial hyperbolic surface adapted to \( T \) that realizes each \( \alpha \in P \). As in [Br] we may choose \( (g_0: Z_0 \to M_\psi) \) so that the vertices of \( T \) map to pairs of antipodal vertices on each geodesic \( \alpha^* \); in other words the two vertices that lie in \( \alpha^* \) separate \( \alpha^* \) into two segments of the same length.
Let \((g_1: Z_1 \to M_\psi)\) be the simplicial hyperbolic surface
\[
g_1 = \Psi \circ g_0 \circ \psi^{-1}.
\]

Since \(g_1 \circ \psi\) is homotopic to \(\Psi \circ g_0\), the simplicial hyperbolic surface \((g_1: Z_1 \to M_\psi)\) lies in \(\mathcal{SH}_k(M_\psi)\). Moreover, \(g_1\) realizes the pants decomposition \(\psi(P)\) and is adapted to \(\psi(T)\).

As shown in [Br], there is a triangulated model 3-manifold \(N \cong S \times I\) built out of 3-dimensional tetrahedra, together with a simplicial mapping \(g: N \to M_\psi\) with the following properties:

1. There is a constant \(k\) so that all but \(kd_P(P, \psi(P))\) of the tetrahedra in \(N\) have the property that one edge maps by \(g\) to a geodesic \(\alpha^*\), where \(\alpha \in S_g\).

2. If \(\partial^- N = S \times \{0\}\) and \(\partial^+ N = S \times \{1\}\), then \(g|_{\partial^- N}\) factors through the simplicial hyperbolic surface \((g_0: Z_0 \to M_\psi)\), and \(g|_{\partial^+ N}\) factors through the simplicial hyperbolic surface \((g_1: Z_1 \to M_\psi)\).

After adjusting \(g|_{\partial^+ N}\) by precomposition with an isotopy, we may glue \(\partial^- N\) to \(\partial^+ N\) by a homeomorphism \(h \simeq \psi\) to obtain a manifold \(N_\psi \cong T_\psi\), and a homotopy equivalence \(f_\psi: N_\psi \to T_\psi\). By a theorem of Stallings [St], \(f_\psi\) is homotopic to a homeomorphism, and is thus surjective since \(T_\psi\) is closed.

There is a constant \(V_3\) so that the volume of each tetrahedron in \(H^3\) is less than \(V_3\), so the volume of the image of \(g\) is less than \(V_3\) times the number of tetrahedra in \(N\). Arguing as in [Br, Sec. 5], we may spin the mapping \(f_\psi\) by pulling the vertices \(p_\alpha\) and \(f_\alpha\) around the geodesic \(\alpha^*\), keeping the map simplicial. In the process, the volume of the image of each tetrahedron \(\Delta\) in \(N\) for which \(g\) sends an edge of \(\Delta\) to \(\alpha^*\) for some \(\alpha \in S_g\), can be made arbitrarily small by spinning the mapping sufficiently far (see [Br, Lem. 5.10]).

We can, then, for each \(\epsilon\) spin the mapping \(f_\psi\) to a mapping \(f_\psi^\epsilon\) so that the image of \(f_\psi^\epsilon\) has volume less than
\[
kV_3d_P(P, \psi(P)) + \epsilon.
\]
It follows from theorem 2.1 that there are constants \(e_1 > 1\) and \(e_2 > 0\) so that
\[
\text{vol}(T_\psi) \leq e_1 d_{WP}(Z, \psi(Z)) + e_2.
\]
But there is a constant \(C(Z)\) depending on the surface \(Z\), so that
\[
d_{WP}(Z, \psi^n(Z)) \leq C(Z) + \|\psi^n\|_{WP} = C(Z) + n\|\psi\|_{WP}.
\]
(The constant $C(Z)$ measures twice the distance to the min-set for the action of the isometry $\psi$ of the Weil-Petersson completion $\text{Teich}(S)$).

Since $\text{vol}(T_{\psi^n}) = n\text{vol}(T_{\psi})$, we have

$$n\text{vol}(T_{\psi}) \leq e_1(C(Z) + n\|\psi\|_{\text{WP}}) + e_2.$$  

Taking limits of both sides as $n \to \infty$, we have

$$\text{vol}(T_{\psi}) \leq e_1\|\psi\|_{\text{WP}}.$$  

Setting $K = \max\{K_0, e_1\}$ proves the theorem.  $\square$

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**Math Department, University of Chicago, 5734 S. University Ave., Chicago, IL 60637**

**Email:** brock@math.uchicago.edu