On BiHom-analogue of generalized Lie algebras

Shuangjian Guo\textsuperscript{1}, Xiaohui Zhang\textsuperscript{2}, Shengxiang Wang\textsuperscript{3}\textsuperscript{*}

1. School of Mathematics and Statistics, Guizhou University of Finance and Economics
   Guiyang 550025, P. R. of China
2. School of Mathematical Sciences, Qufu Normal University
   Qufu 273165, P. R. of China
3. School of Mathematics and Finance, Chuzhou University
   Chuzhou 239000, P. R. of China

ABSTRACT

In this paper, we introduce the definition of generalized BiHom-Lie algebras and generalized BiHom-Lie admissible algebras in the category $\mathcal{H} \mathcal{M}$ of left comodules for any coquasitriangular Hopf algebra $(\mathcal{H}, \sigma)$. Also, we provide some constructions of generalized BiHom-Lie algebras and generalized BiHom-Lie admissible algebras. Finally, we study the structure of the BiHom-associative algebras and the BiHom-Lie algebras in $\mathcal{H} \mathcal{M}$ and describe the BiHom-Lie ideal structures of the BiHom-associative algebras.

Key words: BiHom-associative algebra; BiHom-Lie admissible algebra; comodules category.

2010 Mathematics Subject Classification: 16T05; 17B75; 17B99.

Introduction

Hom-algebras were firstly studied by Hartwig, Larsson and Silvestrov \cite{8}, where they introduced the structure of Hom-Lie algebras in the context of the deformations of Witt and Virasoro algebras. Later, motivated by the new examples arising as applications of the general quasi-deformation construction and the desire to be able to treat within the same framework of the super and color Lie algebras, Larsson and Silvestrov \cite{9} extended the notion of Hom-Lie algebras to quasi-Hom Lie algebras and quasi-Lie algebras, such that the classical definition of this algebraic structure is "deformed" by means of this

\textsuperscript{*}Corresponding author(Shengxiang Wang): wangshengxiang@chzu.edu.cn
endomorphism. The theory of Hom-type algebraic structures has seen an enormous growth in recent years.

An elementary but important property of Lie algebras is that each associative algebra gives rise to a Lie algebra via the commutator bracket. Makhlouf and Silvestrov ([11], [12], [13]) generalized the associativity to twisted associativity and naturally proposed the notion of Hom-associative algebras. Furthermore they obtained that a Hom-associative algebra gives rise to a Hom-Lie algebra via the commutator bracket. Caenepeel and Goyvaerts [4] studied the Hom-Hopf algebras from a categorical viewpoint. In nowadays mathematics, much of the research on certain algebraic object is to study its representation theory. The representation theory of an algebraic object reveals some of its profound structures hidden underneath. A good example is that the structure of a complex semisimple Lie algebra is much revealed via its representation theory. Sheng [14] studied the representation theory of Hom-Lie algebras. Later on, Wang [16] studied the representation theory of Hom-Lie algebras in Yetter-Drinfeld categories.

A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms \( \alpha, \beta \). This class of algebras was introduced from a categorical approach in [7] as an extension of the class of Hom-algebras. More applications of BiHom-Lie algebras, BiHom-algebras, BiHom-Lie superalgebras and BiHom-Lie colour algebras can be found in ([1], [5], [10], [17], [19]). What forms in the BiHom-case are Wang’s results in [16]? In this paper, we give a positive answer to the question.

This article is organized as follows. In Section 2, we introduce the notion of BiHom-Lie algebras in \( HM \) and show that a BiHom-associative algebra in \( HM \) gives rise to a BiHom-Lie algebra in \( HM \) by the natural bracket product (see Theorem 2.3). In Section 3, we obtain a more generalized algebra class called generalized BiHom-Lie admissible algebras, and explore some other general class of algebras: generalized \( G \)-BiHom-associative algebras, where \( G \) is any subgroup of the symmetric group \( S_3 \), using which we classify all the generalized BiHom-Lie admissible algebras. In Section 4, we study the structure of the BiHom-associative algebras and the BiHom-Lie algebras in \( HM \) (see Theorem 4.3). In Section 5, we determine the the BiHom-Lie ideal structures of the BiHom-associative algebras (see Theorem 5.11).

1 Preliminaries

In this section we recall some basic definitions and results related to our paper. Throughout the paper, all algebraic systems are supposed to be over a field \( k \). The reader is referred to [17] as general references about BiHom-associative algebras. If \( C \) is a coalgebra, we use the Sweedler-type notation [15] for the comultiplication: \( \Delta(c) = c_1 \otimes c_2 \), for all \( c \in C \).
Recall that if $H$ is a bialgebra and $M$ is a left $H$-comodule with coaction
\[
\rho : M \rightarrow H \otimes M, \quad m \mapsto m_{(-1)} \otimes m_0, \quad \forall m \in M,
\]
the coassociativity of the coaction means $(\Delta \otimes id) \circ \rho = (id \otimes \rho) \circ \rho$.

In this paper we consider objects in the category of left $H$-comodules $\mathcal{H}M$. In particular, a left $H$-comodule algebra $A$ is an algebra in the category $\mathcal{H}M$, this means that multiplication in $A$ is an $H$-comodule map:
\[
\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_0b_0, \quad \rho(1_A) = 1_H \otimes 1_A, \quad \forall a, b \in A. \quad (1.1)
\]

1.1. Coquasitriangular structure A pair $(H, \sigma)$ is called a coquasitriangular Hopf algebra if $H$ is a Hopf algebra and $\sigma : H \otimes H \rightarrow k$ is a $k$-bilinear form satisfying:

\[
\begin{align*}
(a) & \quad \sigma(h_1, g_1)g_2h_2 = h_1g_1\sigma(h_2, g_2), \\
(b) & \quad \sigma \text{ is convolution invertible in } \text{Hom}_k(H \otimes H, k), \\
(c) & \quad \sigma(h, gl) = \sigma(h_1, g)\sigma(h_2, l), \\
(d) & \quad \sigma(hg, l) = \sigma(g, l_1)\sigma(h, l_2),
\end{align*}
\]

for any $h, g, l \in H$. If, in addition, $\sigma$ is symmetric, that is
\[
(e) \quad \sigma(h_1, g_1)\sigma(g_2, h_2) = \varepsilon(g)\varepsilon(h),
\]
then $(H, \sigma)$ is called cotriangular.

In the category $\mathcal{H}M$, the braiding $\tau : M \otimes N \rightarrow N \otimes M$ is given by
\[
\tau(m \otimes n) = \sigma(m_{(-1)}, n_{(-1)})n_0 \otimes m_0,
\]
for all $m \in \mathcal{H}M$ and $n \in \mathcal{H}M$.

Let $A$ be an algebra in $\mathcal{H}M$, $A$ is called $H$-commutative if
\[
\sigma(a_{(-1)}, b_{(-1)})b_0a_0 = ab, \quad \forall a, b \in A. \quad (1.2)
\]

1.2. BiHom-associative algebra Recall from [7] that a BiHom-associative algebra is a 4-tuple $(A, m, \alpha, \beta)$ consisting of a linear space $A$, a bilinear map $m : A \otimes A \rightarrow A$ and homomorphisms $\alpha, \beta : A \rightarrow A$ such that, for all $a, b, c \in A$,
\[
\alpha \circ \beta = \beta \circ \alpha, \quad \alpha(a)(bc) = (ab)\beta(c). \quad (1.3)
\]

In particular, if $\alpha(ab) = \alpha(a)\alpha(b)$ and $\beta(ab) = \beta(a)\beta(b)$, we call $A$ a multiplicative BiHom-associative algebra. If there exists an element $1_A \in A$ such that $1_Aa = \beta(a)$ and $a1_A = \alpha(a)$ for all $a \in A$, we call $A$ a unital BiHom-associative algebra.
1.3. BiHom-Lie algebra  Recall from [7] that a BiHom-Lie algebra is a 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\) consisting of a linear space \(L\), a bilinear map \([\cdot, \cdot] : L \otimes L \to L\) and homomorphisms \(\alpha, \beta : L \to L\) satisfying:

\[
\alpha \circ \beta = \beta \circ \alpha, \\
[\beta(l), \alpha(l')] = -[\beta(l'), \alpha(l)], \quad (\text{Skew-symmetry}), \\
\bigotimes_{l, l', l''} \left[ \beta^2(l), \left[ \beta(l'), \alpha(l'') \right] \right] = 0, \quad (\text{BiHom-Jacobi identity}),
\]

for any \(l, l', l'' \in L\), where \(\bigotimes\) denotes the summation over the cyclic permutation on \(l, l', l''\).

2 Generalized BiHom-Lie algebras

In this section we introduce the concept of BiHom-Lie algebras in the category \(^H\mathcal{M}\) and provide a construction of BiHom-Lie algebras in \(^H\mathcal{M}\) through BiHom-associative algebras in \(^H\mathcal{M}\).

**Definition 2.1.** Let \((H, \sigma)\) be a coquasitriangular Hopf algebra. A BiHom-Lie algebra in the category \(^H\mathcal{M}\) (called a generalized BiHom-Lie algebra) is a 4-tuple \((L, [\cdot, \cdot], \alpha, \beta)\), where \(L\) is an object in \(^H\mathcal{M}\), \(\alpha, \beta : L \to L\) are homomorphisms in \(^H\mathcal{M}\) and \([\cdot, \cdot] : L \otimes L \to L\) is a morphism in \(^H\mathcal{M}\) satisfying:

\[
\alpha \circ \beta = \beta \circ \alpha, \quad (2.1) \\
\alpha([l, l']) = [\alpha(l), \alpha(l')], \quad \beta([l, l']) = [\beta(l), \beta(l')], \quad (2.2) \\
[\beta(l), \alpha(l')] = -\sigma(l_{(-1)}, l'_{(-1)})[\beta(l'), \alpha(l_0)], \quad (2.3) \\
\{l \otimes l' \otimes l''\} + \{(1 \otimes \tau)(\tau \otimes 1)(l \otimes l' \otimes l'')\} + \{(1 \otimes \tau)(\tau \otimes 1)(l \otimes l' \otimes l'')\} = 0, \quad (2.4)
\]

for any \(l, l', l'' \in L\), where \(\{l \otimes l' \otimes l''\}\) denotes \([\beta^2(l), [\beta(l'), \alpha(l'')]]\) and \(\tau\) the braiding for \(L\).

**Example 2.2.** (1) If \(\beta = \alpha\) in Definition 2.1, then the generalized BiHom-Lie algebra \(L\) is just the generalized Hom-Lie algebra in Wang [16].

(2) If \(H = kG\) in Definition 2.1, then the generalized BiHom-Lie algebra \(L\) is just the BiHom-Lie colour algebra in Abdaoui [1].

**Theorem 2.3.** Let \((H, \sigma)\) be a cotriangular Hopf algebra and \((A, \alpha, \beta)\) a BiHom-associative algebra in \(^H\mathcal{M}\) (called a generalized BiHom-associative algebra) with two bijective homomorphisms \(\alpha\) and \(\beta\). Then the 4-tuple \((A, [\cdot, \cdot], \alpha, \beta)\) is a generalized BiHom-Lie algebra, where the bracket product \([\cdot, \cdot] : A \otimes A \to A\) is defined by

\[
[a, b] = ab - \sigma(a_{(-1)}, b_{(-1)})(\alpha^{-1}(b_0))(\alpha(b^{-1}(a_0)), \quad (2.5)
\]

for all \(a, b \in A\).
Proof Similar to \[1\].

**Example 2.4.** Let \( \{x_1, x_2\} \) be a basis of a 2-dimensional linear space \( A \). The following multiplication \( m \) and linear maps \( \alpha, \beta \) on \( A \) define a BiHom-associative algebra (\[7\]) :

\[
m(x_1, x_1) = x_1, m(x_1, x_2) = bx_2,
\]
\[
m(x_2, x_1) = -x_2, m(x_2, x_2) = 0,
\]
\[
\alpha(x_1) = x_1, \alpha(x_2) = -x_2,
\]
\[
\beta(x_1) = x_1, \beta(x_2) = bx_2,
\]

where \( b \) is a parameter in \( k \).

Let \( G \) be the cyclic group of order 2 generated by \( g \). The group algebra \( H = kG \) is a Hopf algebra in the usual way. We can define a \( k \)-bilinear form on \( H \) by \( \sigma(g, g) = -1, \ \sigma(e, g) = 1, \ \sigma(g, e) = 1, \ \sigma(e, e) = 1 \), where \( e \) is the unit of the group \( G \). It is easy to check that \( H \) is a cotriangular Hopf algebra. Define the left-\( H \)-comodule structure of \( A \) by

\[
\rho(x_1) = e \otimes x_1, \rho(x_2) = g \otimes x_2.
\]

It is not hard to check that \( A \) is a generalized BiHom-associative algebra.

The braiding \( \tau \) is given by \( \tau(x_2 \otimes x_2) = -x_2 \otimes x_2 \) and \( \tau(x_i \otimes x_j) = x_j \otimes x_i \) for other cases. Then according to Theorem 2.3, we obtain a generalized BiHom-Lie algebra with the bracket product \([,]\) satisfying the following non-vanishing relation

\[
[x_1, x_2] = 2bx_2, \quad [x_2, x_1] = -x_2 + bx_1.
\]

**Example 2.5.** Let \( A \) be the three dimensional Heisenberg Lie algebra, which consists of the strictly upper-triangular complex \( 3 \times 3 \) matrices. It has a standard linear basis

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Let \( G \) be the cyclic group of order 2 generated by \( g \). The group algebra \( H = kG \) is a Hopf algebra in the usual way, we can define a \( k \)-bilinear form on \( H \) by \( \sigma(g, g) = -1, \ \sigma(e, g) = 1, \ \sigma(g, e) = 1, \ \sigma(e, e) = 1 \), it is easy to check that \( H \) is a cotriangular Hopf algebra. Define the left-\( H \)-comodule structure of \( A \) by

\[
\rho(x_1) = g \otimes x_1, \rho(x_2) = g \otimes x_2, \rho(x_3) = e \otimes x_3,
\]

where \( e \) is the unit of the group \( G \). It is not hard to check that \( A \) is an object in \( H_M \).
The braiding $\tau$ is given by

$$\tau(x_1 \otimes x_1) = -x_1 \otimes x_1, \quad \tau(x_1 \otimes x_2) = -x_2 \otimes x_1, \quad \tau(x_1 \otimes x_3) = x_3 \otimes x_1,$$

$$\tau(x_2 \otimes x_1) = -x_1 \otimes x_2, \quad \tau(x_2 \otimes x_2) = -x_2 \otimes x_2, \quad \tau(x_2 \otimes x_3) = x_3 \otimes x_2,$$

$$\tau(x_3 \otimes x_1) = x_1 \otimes x_3, \quad \tau(x_3 \otimes x_2) = x_2 \otimes x_3, \quad \tau(x_3 \otimes x_3) = x_3 \otimes x_3.$$

Define a bracket product $[,]$ on $A$ by

$$[x_i, x_3] = [x_3, x_i] = [x_1, x_1] = [x_2, x_2] = 0,$$

$$[x_1, x_2] = [x_2, x_1] = x_3, \quad i = 1, 2, 3.$$

One may verify that $(A, [,])$ is a generalized Lie algebra.

Let $\lambda_1, \lambda_2$ be two nonzero scalars in $k$. Consider the maps $\alpha, \beta : A \to A$ defined on the basis element by

$$\alpha(x_1) = \lambda_1 x_1, \quad \alpha(x_2) = \lambda_2 x_2, \quad \alpha(x_3) = \lambda_1 \lambda_2 x_3,$$

$$\beta(x_1) = \lambda_1 x_1, \quad \beta(x_2) = \lambda_2 x_2, \quad \beta(x_3) = \lambda_1 \lambda_2 x_3.$$

It is straightforward to check that $\alpha, \beta$ defines a Lie algebra endomorphism in $H\mathcal{M}$. We obtain a generalized BiHom-Lie algebra $(A, [\cdot, \cdot], \alpha, \beta)$, whose bracket product satisfies the following non-vanishing relation

$$[x_1, x_2]' = \lambda_1 \lambda_2 x_3, \quad [x_2, x_1]' = \lambda_1 \lambda_2 x_3.$$

3 Generalized BiHom-Lie admissible algebras

In this section, we aim to extend the notions and results about Hom-Lie admissible algebras and BiHom-Lie colour admissible algebras to a more generalized case: BiHom-Lie admissible algebras in $H\mathcal{M}$.

**Definition 3.1.** Let $(H, \sigma)$ be a coquasitriangular Hopf algebra and $(A, \mu, \alpha, \beta)$ a generalized BiHom-algebra with two bijective homomorphisms $\alpha$ and $\beta$. Then $(A, \mu, \alpha, \beta)$ is called a generalized BiHom-Lie admissible algebra if the bracket product

$$[x, y] = \mu(x \otimes y) - \sigma(x_{(-1)}, y_{(-1)}) \mu(\alpha^{-1}(y_0) \otimes \alpha \beta^{-1}(x_0)),$$

satisfies the $H$-BiHom-Jacobi identity Eq. (2.4) for all elements $x, y \in A$.

**Proposition 3.2.** Let $(H, \sigma)$ be a cotriangular Hopf algebra and $(L, [\cdot, \cdot], \alpha, \beta)$ a generalized BiHom-Lie algebra with two bijective homomorphisms $\alpha$ and $\beta$. Then $(A, [\cdot, \cdot], \alpha, \beta)$ is a generalized BiHom-Lie admissible algebra.

**Proof** Define a new bracket product $[,]'$ by

$$[x, y]' = [x, y] - \sigma(x_{(-1)}, y_{(-1)})[\alpha^{-1}(y_0), \alpha \beta^{-1}(x_0)], \quad x, y \in L.$$

6
A BiHom-associator as \( L \),

Since \((\cdot), \mu, \alpha, \beta\) of \(L\) be a generalized BiHom-associative algebra and define

\[
\rho(x, y) = \rho(x, y) - \sigma(x_{(-1)}, y_{(-1)})[\alpha^{-1}\beta(y_0), \alpha\beta^{-1}(x_0)],
\]

\[
(1 \otimes [\cdot], \cdot)\rho(x \otimes y) = (1 \otimes [\cdot])(x_{(-1)}y_{(-1)} \otimes x_0 \otimes y_0),
\]

\[
x_{(-1)}y_{(-1)} \otimes [x_0, y_0] - \sigma(x_{(-1)}y_{(-1)}x_{(-1)}y_{(-1)})[\alpha^{-1}\beta(y_0), \alpha\beta^{-1}(x_0)]
\]

\[
x_{(-1)}y_{(-1)} \otimes [x_0, y_0] - \sigma(x_{(-1)}y_{(-1)}x_{(-1)}y_{(-1)})[\alpha^{-1}\beta(y_0), \alpha\beta^{-1}(x_0)]
\]

Hence \([\cdot, \cdot]’\) is a morphism in \(H\), as desired.

One may check that \(H\)-BiHom-skew-symmetry holds straightforwardly. To verify the \(H\)-BiHom-Jacobi identity Eq. (2.4), we note \(x \otimes y \otimes z = [\beta^2(x), [\beta(y), \alpha(z)]’]’\) and calculate

\[
\{x \otimes y \otimes z\}' = \{\beta^2(x), [\beta(y), \alpha(z)]’\}'
\]

\[
= \{\beta^2(x), [\beta(y), \alpha(z)]’\}'
\]

\[
= \{\beta^2(x), [\beta(y), \alpha(z)]’ - \sigma(y_{(-1)}, z_{(-1)})[\alpha^{-1}(\beta^2(y_0)), \beta^{-1}(\alpha^2(y_0))]\}
\]

\[
= \{\beta^2(x), [\beta(y), \alpha(z)]’ - \sigma(y_{(-1)}, z_{(-1)})[\beta^2(x), [\alpha^{-1}(\beta^2(y_0)), \beta^{-1}(\alpha^2(y_0))]\}
\]

\[
= \{\beta^2(x), [\beta(y), \alpha(z)]’ - \sigma(x_{(-1)}y_{(-1)}z_{(-1)}), [\alpha^{-1}(\beta(y_0), \alpha(z_0)), \beta^{-1}\alpha(\beta^2(x))] - [\beta^2(x), \sigma(y_{(-1)}, z_{(-1)})[\alpha^{-1}(\beta^2(y_0)), \beta^{-1}(\alpha^2(y_0))] + \sigma(y_{(-1)}, z_{(-1)})
\]

\[
\sigma(x_{(-1)}, z_{(-1)}y_{(-1)}[\alpha^{-1}\beta\alpha^{-1}(\beta^2(y_0)), \beta^{-1}(\alpha^2(y_0)))], \alpha(\beta(x))
\]

\[
= 2[\beta^2(x), [\beta(y), \alpha(z)]’ - 2\sigma(x_{(-1)}, y_{(-1)}z_{(-1)})[\alpha^{-1}\beta^2(y_0), \beta(\beta(z_0))], \alpha(\beta(x))
\]

\[
= 4[\beta^2(x), [\beta(y), \alpha(z)]’ = 4\{x \otimes y \otimes z\}.
\]

By similar calculations, we have

\[
\{(\tau \otimes 1)(1 \otimes \tau)(x \otimes y \otimes z)\}' = 4\{(\tau \otimes 1)(1 \otimes \tau)(x \otimes y \otimes z)\},
\]

\[
\{(1 \otimes \tau)(\tau \otimes 1)(x \otimes y \otimes z)\}' = 4\{(1 \otimes \tau)(\tau \otimes 1)(x \otimes y \otimes z)\}.
\]

Since \((L, [\cdot], \alpha, \beta)\) is a generalized BiHom-Lie algebra, we have

\[
\{x \otimes y \otimes z\}' + \{(\tau \otimes 1)(1 \otimes \tau)(x \otimes y \otimes z)\}' + \{(1 \otimes \tau)(\tau \otimes 1)(x \otimes y \otimes z)\}' = 0,
\]

as required, and this completes the proof.

Let \((A, \mu, \alpha, \beta)\) be a generalized BiHom-associative algebra and define

\[
[x, y] = xy - \sigma(x_{(-1)}, y_{(-1)})(\alpha^{-1}\beta(y_0)) (\alpha\beta^{-1}(x_0)), \quad \forall x, y \in A.
\]

A BiHom-associator \(a_{\alpha, \beta}(x, y, z)\) of \(\mu\) is defined by

\[
a_{\alpha, \beta}(x, y, z) = \alpha(x)(y\beta) - (xy)\beta(z), \quad \forall x, y, z \in A.
\]
Then \((A, \mu, \alpha, \beta)\) a generalized BiHom-associative algebra if and only if \(as_{\alpha, \beta}(x, y, z) = 0\).

Now let us introduce the notation:

\[
S(x, y, z) = as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z)) + \{(\tau \otimes 1)(1 \otimes \tau)as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z))\}
+ \{(1 \otimes \tau)(\tau \otimes 1)as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z))\}.
\]

Then we have the following properties.

**Lemma 3.3.** Let \((A, \mu, \alpha, \beta)\) be a generalized BiHom-associative algebra and \([,]\) the associated supercommutator. Then

\[
S(x, y, z) = [\beta^2(x), \beta(y)\alpha(z)] + \{(\tau \otimes 1)(1 \otimes \tau)[\beta^2(x), \beta(y)\alpha(z)]\}
+ \{(1 \otimes \tau)(\tau \otimes 1)[\beta^2(x), \beta(y)\alpha(z)]\}.
\]

**Proof** For any \(x, y, z \in A\), we have

\[
\begin{align*}
[\beta^2(x), \beta(y)\alpha(z)] &+ \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})[\alpha^{-1}\beta^2(y_0), \beta(z_0)\alpha^2(x_0)] \\
&+ \sigma(x_{(-1)}y_{(-1)}, z_{(-1)})[\alpha^{-1}\beta^2(z_0), \alpha\beta(x_0)\alpha(y_0)] \\
&= \beta^2(x)(\beta(y)\alpha(z)) - \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})\alpha^{-1}\beta(\beta(y_0)\alpha(z_0))\alpha\beta(x_0) \\
&+ \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})\alpha^{-1}\beta^2(y_0)(\beta(z_0)\alpha^2(x_0)) \\
&- \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})\beta(y_0)z_{(-1)}z_{(-1)}\alpha^{-1}\beta(z_0)\alpha^2(x_00))\beta(y_00) \\
&+ \sigma(x_{(-1)}y_{(-1)}, z_{(-1)})\alpha^{-1}\beta^2(z_0)(\alpha\beta(x_0)\alpha(y_0)) - \sigma(x_{(-1)}y_{(-1)}, z_{(-1)}) \\
&\sigma(z_0_{(-1)}, x_0_{(-1)}y_0_{(-1)})\alpha^{-1}\beta(\alpha\beta(x_00)\alpha(y_00))\beta(z_00) \\
&= \beta^2(x)(\beta(y)\alpha(z)) - \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})\alpha^{-1}\beta(\beta(y_0)\alpha(z_0))\alpha\beta(x_0) \\
&+ \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})\alpha^{-1}\beta^2(y_0)(\beta(z_0)\alpha^2(x_0)) \\
&- \sigma(x_{(-1)}, y_{(-1)}z_{(-1)})y_{(-1)}z_{(-1)}\alpha^{-1}\beta(z_0)\alpha^2(x_00))\beta(y_00) \\
&+ \sigma(x_{(-1)}y_{(-1)}, z_{(-1)})\alpha^{-1}\beta^2(z_0)(\alpha\beta(x_0)\alpha(y_0)) - \alpha^{-1}\beta(\alpha\beta(x)\alpha(y))\beta(z) \\
&= as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z)) + \{(\tau \otimes 1)(1 \otimes \tau)as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z))\} \\
&+ \{(1 \otimes \tau)(\tau \otimes 1)as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha\beta(z))\} \\
&= S(x, y, z),
\end{align*}
\]

as required, and this finishes the proof. \(\square\)

**Proposition 3.4.** Let \((H, \sigma)\) be a cotriangular Hopf algebra and \((A, \mu, \alpha, \beta)\) a generalized BiHom-associative algebra with two bijective homomorphisms \(\alpha\) and \(\beta\). Then \((A, \alpha, \beta)\) is a generalized BiHom-Lie admissible algebra if and only if it satisfies

\[
S(x, y, z) = \sigma(y_{(-1)}, z_{(-1)})S(x, z_0, y_0).
\]
Proof For any $x, y, z \in A$, we have

$$S(x, y, z) - \sigma(y_{-1}, z_{-1}) S(x, y_{0}, y_{0})$$

$$= [\beta^2(x), \beta(y) \alpha(z)] + \{(\tau \otimes 1)(1 \otimes \tau)[\beta^2(x), \beta(y) \alpha(\tau)]\}$$

$$+ \{(1 \otimes \tau)(\tau \otimes 1)[\beta^2(x), \beta(y) \alpha(\tau)]\} - \sigma(y_{-1}, z_{-1})[\beta^2(x), \beta(y_0) \alpha(z_0)]$$

$$+ \sigma(y_{-1}, z_{-1})\{(\tau \otimes 1)(1 \otimes \tau)[\beta^2(x), \beta(y_0) \alpha(z_0)]\}$$

$$+ \sigma(y_{-1}, z_{-1})\{(1 \otimes \tau)(\tau \otimes 1)[\beta^2(x), \beta(y_0) \alpha(z_0)]\}$$

$$= [\beta^2(x), \beta(y) \alpha(z) - \sigma(y_{-1}, z_{-1}) \beta(z) \alpha(y)]$$

$$+ \{(\tau \otimes 1)(1 \otimes \tau)[\beta^2(x), \beta(y) \alpha(z) - \sigma(y_{-1}, z_{-1}) \beta(z) \alpha(y)]\}$$

$$+ \{(1 \otimes \tau)(\tau \otimes 1)[\beta^2(x), \beta(y) \alpha(z) - \sigma(y_{-1}, z_{-1}) \beta(z) \alpha(y)]\}$$

$$= [\beta^2(x), [\beta(y), \alpha(z)] + \{(\tau \otimes 1)(1 \otimes \tau)[\beta^2(x), [\beta(y), \alpha(z)]\}$$

$$+ \{(1 \otimes \tau)(\tau \otimes 1)[\beta^2(x), [\beta(y), \alpha(z)]\} = 0,$$

as desired, and this ends the proof. \qed

In the following, we will provide a classification of generalized BiHom-Lie admissible algebras (i.e., BiHom-Lie admissible algebras in $H\mathcal{M}$) using the symmetric group $S_3$, whereas it was classified in [2][11][18] for Hom-Lie admissible algebras, Hom-Lie admissible superalgebras and Hom-Lie color admissible algebras, respectively.

Let $S_3$ be the symmetric group generated by $\varphi_1 = (12), \varphi_2 = (23)$ and $A = (V, \mu, \alpha, \beta)$ a generalized BiHom-algebra. Suppose that $S_3$ acts on $V^{\times 3}$ in the usual way, i.e., $\varphi(x_1, x_2, x_3) = (x_{\varphi(1)}, x_{\varphi(2)}, x_{\varphi(3)})$. Let $(-1)^{\epsilon(\varphi)}$ denotes the the signature of $\varphi \in S_3$, we have the following Lemma.

**Lemma 3.5.** A generalized BiHom-algebra $A = (V, \mu, \alpha, \beta)$ is a generalized BiHom-Lie admissible algebra if and only if the following condition holds:

$$\sum_{\varphi \in S_3} (-1)^{\epsilon(\varphi)} a_{\alpha, \beta} \circ (\alpha^{-1} \beta^2 \otimes \beta \otimes \alpha) \circ \varphi(x_1, x_2, x_3) = 0,$$

for any elements $x_1, x_2, x_3 \in V$.

**Proof** It is sufficient to verify the $H$-BiHom-Jacobi identity Eq. (2.4). By Lemma 3.4, we have

$$\bigotimes_{x_1, x_2, x_3} \sigma(x_{\varphi(1)}, x_{\varphi(1)})[\beta^2(x_{\varphi(1)}), [\beta(x_{\varphi(1)}), \alpha(x_{\varphi(1)})]]$$

$$= \sum_{\varphi \in S_3} (-1)^{\epsilon(\varphi)} a_{\alpha, \beta} \circ (\alpha^{-1} \beta^2 \otimes \beta \otimes \alpha) \circ \varphi(x_1, x_2, x_3) = 0,$$

as desired. \qed

Let $G$ be a subgroup of $S_3$, any generalized BiHom-algebra $(V, \mu, \alpha, \beta)$ is said to be $G$-BiHom-associative if the following equation holds:

$$\sum_{\varphi \in G} (-1)^{\epsilon(\varphi)} a_{\alpha, \beta} \circ (\alpha^{-1} \beta^2 \otimes \beta \otimes \alpha) \circ \varphi(x_1, x_2, x_3) = 0,$$
for any elements \(x_1, x_2, x_3 \in V\).

**Proposition 3.6.** Let \(G\) be a subgroup of the symmetric group \(S_3\). Then any \(G\)-generalized BiHom-associative algebra \((V, \mu, \alpha, \beta)\) is generalized BiHom-Lie admissable.

**Proof** The \(H\)-BiHom-symmetry Eq. \((2.3)\) follows straightaway from the definition above. Assume that \(G\) is a subgroup of \(S_3\), then \(S_3\) can be written as the disjoint union of the left cosets of \(G\). Say \(S_3 = \bigcup_{\varphi \in I} I\), with \(I \subseteq S_3\), and for any \(\varphi, \varphi' \in I\), Therefore, we have

\[
\sum_{\varphi \in S_3} (-1)^{\varepsilon(\varphi)} \alpha_{\alpha, \beta} \circ (\alpha^{-1} \beta^2 \otimes \beta \otimes \alpha) \circ \varphi(x_1, x_2, x_3)
\]

\[
= \sum_{\tau \in I} \sum_{\varphi \in \tau G} (-1)^{\varepsilon(\varphi)} \alpha_{\alpha, \beta} \circ (\alpha^{-1} \beta^2 \otimes \beta \otimes \alpha) \circ \varphi(x_1, x_2, x_3) = 0,
\]

for any elements \(x_1, x_2, x_3 \in V\). By Lemma 3.5, \((V, \mu, \alpha, \beta)\) is a generalized BiHom-Lie admissable algebra. The proof is completed. \(\square\)

Now we provide a classification of the generalized BiHom-Lie admissable algebras via generalized \(G\)-BiHom-associative algebras. The subgroups of \(S_3\) are

\[
G_1 = \{id\}, \quad G_2 = \{id, \varphi_1\}, \quad G_3 = \{id, \varphi_2\},
\]

\[
G_4 = \{id, \varphi_2 \varphi_1 \varphi_2 = (13)\}, \quad G_5 = A_3, \quad G_6 = S_3,
\]

where \(A_3\) is the alternating subgroup of \(S_3\). Then by Lemma 3.5, we obtain the following types of generalized BiHom-Lie admissable algebras.

1. The generalized \(G_1\)-BiHom-associative algebras are the usual generalized BiHom-associative algebras.

2. The generalized \(G_2\)-BiHom-associative algebras satisfy the condition:

\[
\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha \beta(z))
\]

\[
= \sigma(x_{-1}, y_{-1}) \{\mu(\alpha \beta(y_0), \mu(\alpha^{-1} \beta^2(x_0), \alpha(z))) - \mu(\beta(y_0), \alpha^{-1} \beta^2(x_0)), \alpha \beta(z))\}.
\]

3. The generalized \(G_3\)-BiHom-associative algebras satisfy the condition:

\[
\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha \beta(z))
\]

\[
= \sigma(y_{-1}, z_{-1}) \{\mu(\beta^2(x), \mu(\alpha(z_0), \alpha(y_0))) - \mu(\alpha^{-1} \beta^2(x), \alpha(z_0)), \beta^2(y_0))\}.
\]

4. The generalized \(G_4\)-BiHom-associative algebras satisfy the condition:

\[
\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\mu(\alpha^{-1} \beta^2(x), \beta(y)), \alpha \beta(z))
\]

\[
= \sigma(y_{-1}, z_{-1}) \sigma(x_{-1}, z_0) \sigma(x_0_{-1}, y_{0-1}) \{\mu(\alpha(z_0), \beta(y_0), \alpha^{-1} \beta^2(x_0))) - \mu(\mu(\alpha(z_0), \beta(y_0)), \alpha^{-1} \beta^3(x_0))\}.
\]
The generalized $G_5$-BiHom-associative algebras satisfy the condition:

\[
\mu(\beta^2(x), \mu(\beta(y), \alpha(z))) - \mu(\alpha^{-1}\beta^2(x), \beta(y)), \alpha\beta(z)) \\
- \sigma(z_{(-1)}y_{(-1)}, x_{(-1)})\{\mu(\alpha\beta(y_0), \mu(\alpha(z_0), \alpha^{-1}\beta^2(x_0))) - \mu(\beta(y_0), \alpha(z_0)), \alpha^{-1}\beta^3(x_0))\}
\]

(6) The generalized $G_6$-BiHom-associative algebras are the generalized BiHom-Lie admissible algebras.

4 Generalized BiHom-associative algebras which are sums of generalized sub-BiHom-associative algebras

In this section we generalize some results of Wang [16] to BiHom-associative algebras in $H\mathcal{M}$.

Lemma 4.1. Let $(H, \sigma)$ be a coquasitriangular Hopf algebra and $(A, \alpha, \beta)$ a BiHom-associative algebra in $H\mathcal{M}$ with sub-BiHom-associative algebras $X$ and $Y$ in $H\mathcal{M}$ which are $H$-commutative such that $A = X + Y$. Then

\[
a_{(-1)} \otimes y_{(-1)} \otimes (a_0y_0)^X_{(-1)} \otimes (a_0y_0)^X_0 \\
+ a_{(-1)} \otimes y_{(-1)} \otimes (a_0y_0)^Y_{(-1)} \otimes (a_0y_0)^Y_0 \\
= a_{(-1)} \otimes y_{(-1)} \otimes a_0(-1)y_0(-1) \otimes (a_0y_0y_0)^X \\
+ a_{(-1)} \otimes y_{(-1)} \otimes a_0(-1)y_0(-1) \otimes (a_0y_0y_0)^Y \tag{4.1}
\]

for all $a \in X$ and $b \in Y$, where $a_0y_0 = (a_0y_0)^X + (a_0y_0)^Y \in X + Y$.

Proof Straightforward. \hfill $\square$

Lemma 4.2. Let $(H, \sigma)$ be a coquasitriangular Hopf algebra and $(A, \alpha, \beta)$ a BiHom-associative algebra in $H\mathcal{M}$ with sub-BiHom-associative algebras $X$ and $Y$ in $H\mathcal{M}$ which are $H$-commutative such that $A = X + Y$. Then

\[
\sigma(a_{(-1)}, x_{(-1)})\sigma(b_{(-1)}, y_{(-1)})(x_0a_0)(y_0b_0) \\
= \sigma(a_{(-1)}, x_{(-1)}b_{(-1)})\epsilon(y_{(-1)})x_0b_0(a_0y_0)^X \\
+ \sigma(x_{(-1)}b_{(-1)}, a_{(-1)})\epsilon(a_{(-1)})(a_0y_0)^Y(x_0b_0), \tag{4.2}
\]

for all $a, b \in X$ and $x, y \in Y$, where $a_0y_0 = (a_0y_0)^X + (a_0y_0)^Y \in X + Y$.\hfill $\square$
**Theorem 4.3.** Let \((H, \sigma)\) be a cotriangular Hopf algebra and \((A, \alpha, \beta)\) a BiHom-associative algebra in \(H\mathcal{M}\) with sub-BiHom-associative algebras \(X\) and \(Y\) in \(H\mathcal{M}\) which are \(H\)-commutative such that \(A = X + Y\). Then \(A\) satisfies the identity \([A, A][A, A] = 0\).

**Proof** The proof of this item follows the similar steps as in Wang [16].

**Example 4.4.** We apply Theorem 4.3 to the generalized BiHom-Lie admissible algebra in Example 2.5. Let \(\{x_1, x_2, x_3\}\) be a basis of a 3-dimensional linear space \(A\) with sub-BiHom-associative algebras \(X = \{x_1\}\) and \(Y = \{x_2, x_3\}\). It is easy to check that \(A\) satisfies the following identity \([A, A][A, A] = 0\).

5 On the Generalized BiHom-Lie ideals structures of generalized BiHom-associative algebras

In this section, we consider some \(H\)-analogous of classical concepts of ring theories and Lie theories as follows.

Let \((A, \alpha, \beta)\) be a generalized BiHom-associative algebra. An \(H\)-BiHom-ideal \(U\) of \(A\) is an ideal such that \(\alpha(U) \subseteq U, \beta(U) \subseteq U\) and \((AU)A = A(UA) \subseteq U\).

Let \((L, \alpha, \beta)\) be a generalized BiHom-Lie algebra. An \(H\)-BiHom-Lie ideal \(U\) of \(L\) is a Lie ideal such that \(\alpha(U) \subseteq U, \beta(U) \subseteq U\) and \([U, L] \subseteq U\). Define the center of \(L\) to be \(Z_H(L) = \{l \in L | [l, L]_H = 0\}\).
Lemma 5.3. Assume that $(L, \alpha, \beta)$ is called prime if the product of any two non-zero $H$-BiHom-ideals of $L$ is non-zero. It is called semiprime if it has no non-zero nilpotent $H$-BiHom-ideals, and is called simple if it has no nontrivial $H$-BiHom-ideals.

Lemma 5.1. (1) If $(A, \alpha, \beta)$ is a generalized BiHom-associative algebra, then

$$x_{(-1)} \otimes x_0 a = \rho(xa_0)(S^{-1}(a_{(-1)}) \otimes 1), x, a \in A.$$  

(2) If $A$ is a generalized BiHom-Lie algebra, then

$$x_{(-1)} \otimes [x_0, a] = \rho([x, a_0])(S^{-1}(a_{(-1)}) \otimes 1), x, a \in A.$$  

Proof Straightforward from Lemma 4.1 in Wang [16].

Lemma 5.2. Let $(A, \alpha, \beta)$ be a generalized BiHom-associative algebra. Then

1. $[\alpha \beta(a), bc] = [\beta(a), b \beta(c)] + \sigma(a_{(-1)}, b_{(-1)}) \beta(b_0)[\alpha(a_0), c],$

2. $[ab, \alpha \beta(c)] = \alpha(a)[b, \alpha(c)] + \sigma(b_{(-1)}, c_{(-1)})[a, \beta(c_0)]\alpha(b_0),$

for all $a, b, c \in L.$

Proof We only prove (1), and similarly for (2). For any $a, b, c \in L,$

$$[\beta(a), b \beta(c)] + \sigma(a_{(-1)}, b_{(-1)})\beta(b_0)[\alpha(a_0), c]$$

$$= (\beta(a)b)\beta(c) - \sigma(a_{(-1)}, b_{(-1)})\alpha^{-1}\beta(b_0)\alpha^{-1}\beta(a_0)\beta(c) + \sigma(a_{(-1)}, b_{(-1)})\beta(b_0)(\alpha(a_0)c)$$

$$- \sigma(a_{(-1)}, b_{(-1)})\alpha^{-1}\beta(c_0)\alpha^{-1}\beta(c_0)\beta(a_0))$$

$$= \alpha\beta(a)(bc) - \sigma(a_{(-1)}, b_{(-1)})\beta(b_0)(\alpha(a_0)c) + \sigma(a_{(-1)}, b_{(-1)})\beta(b_0)(\alpha(a_0)c)$$

$$- \sigma(a_{(-1)}, b_{(-1)})\beta(b_0)(\alpha^{-1}\beta(c_0)\alpha^{-1}\beta(c_0)\beta(a_0))$$

$$= \alpha\beta(a)(bc) - \sigma(a_{(-1)}, b_{(-1)})\alpha^{-1}\beta((bc_0)\alpha^{-1}\beta(a_0))$$

$$= [\alpha \beta(a), bc].$$

Lemma 5.3. Assume that $(L, \alpha, \beta)$ is a semiprime unital generalized BiHom-associative algebra, and let $U$ be both an $H$-BiHom-Lie ideal and a BiHom-associative subalgebra of $L$. If $[U, U] \neq 0,$ then there exists a non-zero $H$-BiHom-ideal of $L.$

Proof Since $[U, U] \neq 0,$ there exists $x, y \in U$ such that $[x, y] \neq 0,$ and $[lx, \alpha(y)] \in U$ for all $l \in L.$ By Lemma 5.2(2), $\alpha(l)[x, \alpha(y)] = [lx, \alpha(y)] - \sum \sigma(\alpha(x_{(-1)}), y_{(-1)})[l, \beta(y_0)]\alpha(x_0),$ 

thus $[l, \beta(y_0)] \in U.$ Because $U$ is an BiHom-associative subalgebra of $L,$ then $[l, \beta(y_0)]\alpha(x_0) \in U,$ therefore $\alpha(l)[x, \alpha(y)] \in U$ for all $l, m \in L.$ It follows that $I = (L[x, y]L = L([x, y]L) \in U.$ In fact, take $I = \{ \sum a_i \alpha_{m_i} \beta^m_i ([x, y])b_i | a_i, b_i \in L, m_i, n_i \in Z \}.$ Then by Eq. (2.3),
one has

\[
\sum_i (a_i \alpha^{n_i} \beta^{m_i}([x, y]))b_i = \sum_i [a_i \alpha^{n_i} \beta^{m_i}([x, y]), b_i] + \sum_i \sum \sigma(a_{i(-1)}x_{(-1)}y_{(-1)}, b_{i(-1)})\alpha^{-1} \beta(b_{i0})\alpha\beta^{-1}(a_{i0}\alpha^{n_i} \beta^{m_i}([x_0, y_0])) \\
= \sum_i [\alpha^{n_i} \beta^{m_i}(\alpha^{-n_i} \beta^{-m_i}(a_i)[x, y]), b_i] + \sum_i \sum \sigma(a_{i(-1)}x_{(-1)}y_{(-1)}, b_{i(-1)})\alpha^{n_i+1}((\alpha^{-n_i-1}(\alpha^{-1} \beta(b_{i0})\alpha\beta^{-1}(a_{i0})))\alpha^{n_i+1} \beta^{m_i-1}[x_0, y_0]) \\
= \sum_i [\alpha^{n_i} \beta^{m_i}(1_A(\alpha^{-n_i-1} \beta^{-m_i-1}(a_i)[x, y])1_A), b_i] + \sum_i \sum \sigma(a_{i(-1)}x_{(-1)}y_{(-1)}, b_{i(-1)})\alpha^{n_i+1} \beta^{-m_i-1}(((\alpha^{-n_i-1} \beta^{-m_i-1}(1_A \alpha^{-1}(b_{i0})a_{i0})))[x_0, y_0])1_A \\
\in U + U \subseteq U,
\]

since \(\alpha(U) \subseteq U, \beta(U) \subseteq U\). Moreover, \(I \neq 0\), for otherwise \([x, y] \in L\) will generate a nilpotent \(H\)-BiHom-ideal of \(L\).

**Theorem 5.4.** Assume that \((L, \alpha, \beta)\) is an prime unital generalized BiHom-associative algebra, and \(U\) be a \(H\)-BiHom-Lie ideal of \(L\) such that \([U, U] \neq 0\). Then there exists a \(H\)-BiHom-ideal \(I\) of \(L\) such that \(0 \neq [I, L] \subseteq U\).

**Proof** Define \(N_L(U) = \{x \in L | [x, L] \subseteq U\}\). Note that \(U \subseteq N_L(U)\). Since \(x_{(-1)} \otimes [x_0, l] = \rho([x, l_0])(S^{-1}(l_{(-1)}) \otimes 1) \in H \otimes U\), we have \([N_L(U), L] \subseteq U \subseteq N_L(H)\) and also \(N_L(U)\) is a \(H\)-BiHom-Lie ideal. It is easy to see that \([N_L(U), N_L(U)] \supseteq [U, U] \neq 0\). Applying Lemma 5.3 to \(N_L(U)\), we may find a non-zero \(H\)-BiHom-ideal \(I\) of \(L\) such that \(I \subseteq N_L(U)\), i.e., \([I, L] \subseteq U\).

Now we prove \([I, L] \neq 0\). If not, choose \(x \in I\) and \(l, m \in L\), then by Lemma 5.2(2), \(\alpha(x)[l, \alpha(m)] = [x, \alpha(m)] - \sum \sigma(l_{(-1)}, m_{(-1)})[x, \beta(m_0)]\alpha(l_0)\), since \(1_A(x\alpha(l)) \in I\), it is not hard to check that \([x[l, m] = 0, [L, L] \subseteq Ann_L(I)\). It is not hard to show that \(Ann_L(I)\) is a \(H\)-BiHom-ideal. In fact, let \(z \in Ann_L(I)\) and \(z \in I\). By Lemma 5.1(1), \(x_{(-1)} \otimes x_0z = \rho(x_0z)(S^{-1}(z_{(-1)}) \otimes 1) = 0\). This implies that \(Ann_L(I)\) is an ideal. \(Ann_L(I)\) is clearly a \(H\)-BiHom-ideal. It follows that \(Ann_L(I)\) is a \(H\)-BiHom-ideal. The simplicity of \(L\) gives \([L, L] = 0\), a contradiction. \(\square\)

**Corollary 5.5.** Let \((L, \alpha, \beta)\) be a simple unital generalized BiHom-associative algebra. If \(U\) is a \(H\)-BiHom-Lie ideal with \([U, U] \neq 0\), then \([L, L] \subseteq U\).

As usual, we define a sequence of \(H\)-BiHom-ideals (the derived series) by

\[
L^0 = L, L^{(1)} = [L, L], L^{(2)} = [L^{(1)}, L^{(1)}], \ldots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}].
\]
Lemma 5.6. Let $U$ be a $H$-BiHom-ideal of $A$ and $<U>$ defined as above. Then $<U>$ is a $H$-BiHom-Lie ideal of $A$, $<[U], A] \subseteq U$.

Proof Straightforward. □

Lemma 5.7. Let $(L, \alpha, \beta)$ be an $H$-simple unital generalized BiHom-associative algebra. Then

(1) If $L^{(2)} \neq 0$, then $L = <L^{(1)}>$. \\
(2) If $L^{(3)} \neq 0$, then $L^{(2)} = L^{(1)}$.

Proof (1) Let $S = \langle L^{(1)} \rangle$. By Lemma 5.6(1), $S$ is a $H$-BiHom-Lie ideal of $L$. By hypothesis, $[S, S] \supseteq L^{(2)} \neq 0$. Obviously, $\alpha(S) \subseteq S$. Thus by Lemma 5.3, $S$ contains a non-zero $H$-BiHom-ideal of $L$. So $S = L$ since $L$ is $H$-simple.

(2) Let $V = L^{(2)}$. The BiHom-Jacobi identity implies that $V$ is a $H$-BiHom-Lie ideal of $L^{(3)} = [V, V] \neq 0$. Also it is easy to see that $\alpha(V) \subseteq V$. By the Theorem 4.4, there is a $H$-BiHom-ideal $I$ of $L$. Thus $V \supseteq [L, L]$. Clearly, $V \subseteq [L, L]$. This finishes the proof. □

Lemma 5.8. Let $V$ be an $H$-BiHom-Lie ideal of $[L, L]$ and $T(V) = \{l \in L|[l, L] \subseteq V\}$. Then $T(V)$ is an $H$-BiHom-associative subalgebra of $L$.

Proof Straightforward. □

Lemma 5.9. With the notations as above, then $[V, V] \subseteq T(V)$ and $[[T(V), T(V)], L] \subseteq T(V)$.

Proof Let $x, y \in V$ and $l \in L$. By the $H$-BiHom-Jacobi identity, we have

$$[eta^2(l), [eta(x), \alpha(y)]]$$

$$= -\sigma(l_{(-1)}x_{(-1)}, y_{(-1)})[eta^2(y_0), [eta(l_0), \alpha(x_0)]]$$

$$-\sigma(l_{(-1)}x_{(-1)}l_{(0(-1)}y_{(-1)})[eta^2(x_0), [eta(y_0), \alpha(l_0)]]$$.

Since $\alpha(V) \subseteq V, \beta(V) \subseteq V$ and $V$ is an $H$-BiHom-Lie ideal of $[L, L]$, we have $[\beta^2(y_0), [\beta(l_0), \alpha(x_0)]] \in V$, therefore $\sum \sigma(l_{(-1)}x_{(-1)}, y_{(-1)})[eta^2(y_0), [\beta(l_0), \alpha(x_0)]] \in V$. Also we have $\sigma(l_{(-1)}x_{(-1)}l_{(0(-1)}y_{(-1)})[eta^2(x_0), [\beta(y_0), \alpha(l_0)]] \in V$, so $[V, V] \subseteq T(V)$.

To show $[[T(V), T(V)], L] \subseteq T(V)$, we take $x, y \in T(V)$ and $l, l' \in L$ and calculate

$$[[[x, y], l], l']$$

$$= -\sum \sigma(l_{(-1)}x_{(-1)}y_{(-1)}l_{(-1)}l'_{(-1)})[\alpha^{-1}\beta(l'_0), \alpha\beta^{-1}[[x_0, y_0], l_0]]$$

$$= \sum \sigma(l''_{(-1)}x_{0(-1)}y_{0(-1)}l''_{(0(-1)}l''_{0(-1)}y_{0(-1)})[eta(l_{00}), [\alpha^{-1}l''_{00}, \alpha^2\beta^{-2}[x_{00}, y_{00}]]]$$

$$+ \sum \sigma(l''_{(-1)}x_{0(-1)}y_{0(-1)}l''_{0(-1)}y_{0(-1)})[\alpha[x_{00}, y_{00}], [l_{00}, \beta^{-1}l'_{00}]]$$.
where \( l''_0 = \sum \sigma(x_{(-1)}y_{(-1)}l'_{(-1)}, l''_{(-1)})l'_0 \). It is easy to see that

\[
\sum \sigma(l''_{(-1)}, x_{0(-1)}y_{0(-1)})\sigma(l''_{0(-1)}, l_{0(-1)}l''_{00(-1)})[\alpha[x_{00}, y_{00}], [l_{00}, \beta^{-1}(l''_{00})]] \in V
\]

since \([x_0, y_0] \in T(V)\). Therefore \([l''_{(-1)}x_{0(-1)}y_{0(-1)} : \beta(l_0), [\alpha^{-1}(l''_0), \alpha^2\beta^{-2}[x_{00}, y_{00}]]] \in V\), as desired.

\[ \square \]

**Remark 5.10.** Similar to the conclusion of Lemma 5.9, we have \([V, T(V)] \subseteq T(V)\), \([L, T(V)] \subseteq T(V)\), and \(L([T(V), T(V)], [T(V), T(V)]L) \subseteq V + T(V)\).

**Theorem 5.11.** Let \( L \) be a simple unital generalized BiHom-associative algebra and \( V \subseteq [L, L] \) be a \( H \)-BiHom-Lie ideal of \([L, L]\) such that \( V \neq [L, L] \). Then \( V \) is a solvable generalized BiHom-Lie subalgebra of \([L, L]\). Moreover \([V, V] \) is nilpotent.

**Proof** Firstly if \( L^{(3)} = 0 \), it is easy to see that \( V^{(3)} = 0 \), and so \( V \) is a solvable Hom-Lie subalgebra of \([L, L]\). And if \( L^{(3)} \neq 0 \), suppose \([L, L] \subseteq T(V)\), then \([L, [L, L]] \subseteq [L, T(V)] \subseteq V\). But then \( L^{(2)} = [[L, L], [L, L]] \subseteq [T(V), T(V)] \subseteq V\). By Lemma 5.7(2), \( L^{(2)} = [L, L]\), and so \([L, L] \subseteq V\), which contradicts the hypothesis \( V \neq [L, L] \). Thus we may assume that \([l, m] \in T(V)\) for some \( l, m \in L\). By Lemma 5.9 and Remark 7.10, we have \([V + T(V), V + T(V)] \subseteq [V, V] + [V, T(V)] + [T(V), T(V)] \subseteq T(V)\), Hence \( V + T(V) \neq L\). Also by Remark 5.10, \( L([T(V), T(V)], [T(V), T(V)]L) \subseteq V + T(V)\). This contradicts \( L\) being \( H \)-simple, unless \(([T(V), T(V)], [T(V), T(V)]) = 0\), i.e., \( T(V)^{(2)} = 0\). Also, since \([V, V] \subseteq T(V)\), it follows that \( V^{(3)} = 0\). Thus \( V \) is a solvable \( H \)-BiHom-Lie subalgebra of \([L, L]\). Moreover \([V, V] \) is nilpotent.

**ACKNOWLEDGEMENT**

The paper is supported by the NSF of China (Nos. 11761017 and 11801304) and the Anhui Provincial Natural Science Foundation (Nos. 1908085MA03 and 1808085MA14).

**REFERENCES**

[1] K. Abdaoui, A. Ben Hassine and A. Makhlouf, *BiHom-Lie colour algebras structures*, arxiv: 1706.02188v1(2017).

[2] F. Ammar and A. Makhlouf, *Hom-Lie superalgerbas and Hom-Lie asmissible superalgebras*, J. Algebra, 324(2010), 1513-1528.

[3] F. Ammar, A. Makhlouf and N. Saadaoui, *Cohomology of Hom-Lie superalgebras and \( q \)-deformed Witt superalgebra*, Czech. Math. J., 63(2013), 721-761.

[4] S. Caenepeel, and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra, 39(2011), 2216-2240.
[5] Y. Cheng and H. Qi, *Representations of BiHom-Lie algebras*, arXiv: 1610.04302v1(2016).

[6] A. Gohr, *On hom-algebras with surjective twisting*, J. Algebra, 324(2010), 1483-1491.

[7] G. Graziani, A. Makhlouf, C. Menini and F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, Symmetry Integrability and Geometry: Methods and Applications, 11(2015), 1-34.

[8] J. T. Hartwig, D. Larsson and S. D. Silvestrov, *Deformations of Lie algebras using \(\sigma\)-derivations*, J. Algebra, 295(2006), 314-361.

[9] D. Larsson, S. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra, 288(2005), 321-344.

[10] J. Li, L. Chen, Y. Cheng. *Representations of Bihom-Lie superalgebras*, Linear Multilinear A., 67(2), 299-326 (2019).

[11] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl., 2(2008), 51-64.

[12] A. Makhlouf and S. Silvestrov, *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, J. Gen. Lie Theory in Mathematics, Physics and beyond. Springer-Verlag, Berlin, 2009, pp. 189-206.

[13] A. Makhlouf and S. Silvestrov, *Hom-algebras and Hom-coalgebras*, J. Algebra Appl., 9(2010), 553-589.

[14] Y. Sheng, *Representations of Hom-Lie algebras*, Algebr. Represent. Theory, 15(2012), 1081-1098.

[15] M. E. Sweedler, *Hopf Algebras*, New York: Benjamin, 1969.

[16] S. Wang and S. Wang, *Hom-Lie algebras in Yetter-Drinfeld categories*, Comm. Algebra, 42(2014), 4540-4561.

[17] S. Wang and S. Guo, *BiHom-Lie superalgebra structures*, arXiv:1610.02290v1 (2016).

[18] L. Yuan, *Hom-Lie color algebra structures*, Comm. Algebra, 40(2012), 575-592.

[19] J. Zhang, L. Chen, C. Zhang, *On split regular BiHom-Lie superalgebras*, J. Geom. Phys., 128, 38-47 (2018).