A NOTE ON THE CORE OF STEINBERG ALGEBRAS

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Abstract. In this short note we show that, for an ample Hausdorff groupoid \( G \) and the Steinberg algebra \( A_R(G) \) with coefficients in the commutative ring \( R \), the centraliser of subalgebra \( A_R(G^{(0)}) \) of unit space of \( G \), is \( A_R(\text{Iso}(G)) \), the algebra of the interior of the isotropy. This will unify several results in the literature and the corresponding results for Leavitt path algebras follow.

Let \( E \) be a graph and \( L_R(E) \) the Leavitt path algebra associated to \( E \) with coefficients in the commutative ring \( R \). The \( R \)-subalgebra of \( L_R(E) \) generated by all the monomials \( aa^* \), where \( a \) is a path in \( E \), is called the \textit{diagonal subalgebra} and denoted by \( D(E) \). The \textit{commutative core} subalgebra of \( L_R(E) \) is \( R \)-subalgebra generated by all the monomials \( aba^* \), where \( a \) is a path and \( b \) is a cycle with no exit and denoted by \( M(E) \). The commutative core subalgebra of a Leavitt path algebras where considered by Gil Cantoa and Nasr-Isfahani [9]. They prove that \( M(E) \) is a maximal commutative subalgebra of \( L_R(E) \) which coincides with the centraliser of \( D(E) \). They also raised an open question as to which \( R \) and \( E \), the Leavitt path algebra \( L_R(E) \) has the property that \( Z(L_R(E)) = M_R(E) \), where \( Z(L_R(E)) \) is the centre of \( L_R(E) \). Their results are the algebraic analogue of an earlier work by Nagy and Reznikoff who introduced the \( M(E) \) in the setting of graph \( C^* \)-algebras and proved that it is a masa [10]. A key tool in the above papers is the conditional expectation which was used in [10] and then developed in [9] in the algebraic setting. The diagonal subalgebra \( D(E) \) and consequently the core subalgebra \( M(E) \) play vital roles in the structure and classification theory of graph algebras [6].

In this note we consider the \textit{core} in the setting of Steinberg algebras. We show that the core is indeed the centraliser of the diagonal subalgebra of Steinberg algebra. Despite the fact that these algebras are much more general than graph algebras, the proofs we present here are very short. Our approach does not require the use of conditional expectation and therefore it is direct and short. Specialising to Leavitt path algebras we obtain the result of [9] and answer the open question raised there.

1. Ample groupoids and Steinberg algebras

A groupoid is a small category in which every morphism is invertible. For a groupoid \( G \) and \( g \in G \), denote \( s(g) := g^{-1}g \) and \( r(g) := gg^{-1} \). The pair \((g_1, g_2)\) is composable if and only if \( r(g_2) = s(g_1) \). The set \( G^{(0)} := s(G) = r(G) \) is called the \textit{unit space} of \( G \). The \textit{isotropy group} at a unit \( u \) of \( G \) is the group \( \text{Iso}(u) = \{ g \in G \mid s(g) = r(g) = u \} \). Let \( \text{Iso}(G) = \bigsqcup_{u \in G^{(0)}} \text{Iso}(u) \). For \( U, V \subseteq G \), we define
\[
UV = \{ g_1g_2 \mid g_1 \in U, g_2 \in V \text{ and } r(g_2) = s(g_1) \}.
\]

A \textit{topological groupoid} is a groupoid endowed with a topology under which the inverse map is continuous, and such that composition is continuous with respect to the relative topology on \( G^{(2)} := \{(g_1, g_2) \in G \times G : s(g_1) = r(g_2)\} \) inherited from \( G \times G \). An \textit{étale} groupoid is a topological groupoid \( G \) such that the domain map \( s : G \to G^{(0)} \) is a local homeomorphism. In this case, the range map \( r \) is also a local homeomorphism. Throughout this article, we assume that the étale groupoids are locally compact Hausdorff. An \textit{open bisection} of \( G \) is an open subset \( U \subseteq G \) such that \( s|_U \) and \( r|_U \) are homeomorphisms onto an open subset of \( G^{(0)} \). If \( G \) is an étale groupoid, then there is a base for the topology on \( G \) consisting of open bisections with compact closure. As demonstrated in [13], if \( G^{(0)} \) is totally disconnected and \( G \) is étale, then there is a basis for the topology on \( G \) consisting of compact open bisections. We say that an étale groupoid \( G \) is \textit{ample} if there is a basis consisting of compact open bisections for its topology. For an ample groupoid \( G \) and a commutative ring \( R \) with the discrete topology, the \( R \)-value maps on \( G \) which are locally constant and have compact support constitute an algebra called \textit{Steinberg algebra} and denoted by \( A_R(G) \). One can show that [7, §4]
\[
A_R(G) = \left\{ \sum_{B \in F} r_B 1_B \mid F: \text{mutually disjoint finite collection of compact open bisections} \right\},
\]
addition and scalar multiplication of functions are pointwise, multiplications on the generators are \( 1_B 1_D = 1_{BD} \).

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Throughout this note we work with the Steinberg algebras associated to the amenable groupoid $G$, its unit space $G^0$ and its the interior of isotropy $Iso(G)^o$. One can consider $A_R(G^0)$ and $A_R(Iso(G)^o)$ as subalgebras of $A_R(G)$. For a basis of the theory of Steinberg algebras we refer the reader to [7, 13].

**Definition 1.1.** Let $G$ be an ample Hausdorff groupoid and $A_R(G)$ the Steinberg algebra associated to $G$. We call the subalgebra $A_R(Iso(G)^o)$ the core algebra of $A_R(G)$.

We need the following notation: For a subset $X$ of a ring $A$, its centraliser (or commutant) is defined as $C_A(X) := \{a \in A \mid xa = ax, \text{ for all } x \in X\}$. For a subring $B \subseteq A$, we have $C_A(B) = B$ if and only if $B$ is a maximal commutative subring of $A$.

**Theorem 1.2.** Let $G$ be an ample Hausdorff groupoid and $A_R(G)$ the Steinberg algebra associated to $G$. Then

$$C_{A_R(G)}(A(G^0)) = A_R(Iso(G)^o).$$

**Proof.** Let $V \subseteq Iso(G)^o$ be a compact open bisection and $U \subseteq G^0$ a compact open set. We first observe that $UV = VU$. If $x \in UV$, then $x = uv$, where $u \in U \subseteq G^0$ and $v \in V$. Since $s(v) = r(v) = u$, it follows that $x = uv = vu \in VU$. The converse follows from symmetry. Now for any element $x \in A_R(Iso(G)^o)$ and $y \in A(G^0)$ write $x = \sum s_i 1_{V_i}$ and $y = \sum j v_i 1_{U_i}$, where $V_i \subseteq Iso(G)^o$ and $U_i \subseteq G^0$ are compact open bisections. Since $V_i$ and $U_j$ commute, we have $xy = yx$. This shows that $A_R(Iso(G)^o) \subseteq C_{A_R(G)}(A(G^0))$.

Now suppose $f \in A_R(G)$ commutes with elements of the diagonal algebra $A(G^0)$. Write $f = \sum_{i=1}^n r_i 1_{V_i}$, where $0 \neq r_i \in R$ and $V_i \subseteq G$ are disjoint compact open bisections. We show that $V_i \subseteq Iso(G)$, for all $i$. Suppose that there is $x \in V_i$ such that $s(x) \neq r(x)$. Since $G^0$ is Hausdorff, there is a compact open set $U \subseteq G^0$ such that $s(x) \in U$ but $r(x) \notin U$. It follows that $x \in V_k U$ but $x \notin UV_k$. Since $1_U f = f 1_U$, it follows that

$$\left( \sum_{i=1}^n r_i 1_{V_i} \right)(x) = \left( \sum_{i=1}^n r_i 1_{V_i U} \right)(x).$$

Note that $UV_1, \ldots, UV_n$ are disjoint and so are $V_k U, \ldots, V_n U$. Thus (1.2) reduces to $r_k = 0$ which is a contradiction. So $s(x) = r(x)$. This implies that $V_i \subseteq Iso(G)$ and consequently $f \in A_R(Iso(G)^o)$. □

For the next result we assume that the interior of the isotropy $Iso(G)^o$, is abelian (that is, it is a bundle of abelian groups). A large class of topological groupoids have abelian isotropy, such as Deaconu-Renault groupoids [11] (and in particular graph groupoids [2]).

**Corollary 1.3.** Let $G$ be an ample Hausdorff groupoid and $A_R(G)$ the Steinberg algebra associated to $G$. Suppose $Iso(G)^o$ is abelian. Then we have the followings.

(i) The algebra $A_R(Iso(G)^o)$ is a maximal commutative subalgebra of $A_R(G)$.

(ii) The Steinberg algebra $A_R(G)$ is commutative if and only if $Z(A_R(G)) = C_{A_R(G)}(A(G^0))$.

**Proof.** (1) Since $Iso(G)^o$ is abelian, the subalgebra $A_R(Iso(G)^o)$ is commutative. Now the results follows from Theorem 1.2.

(2) Suppose $Z(A_R(G)) = C_{A_R(G)}(A(G^0))$. Let $V$ be a compact open bisection of $G$. Suppose that there is $x \in V$ such that $s(x) \neq r(x)$. Since $G^0$ is Hausdorff, there is a compact open set $U \subseteq G^0$ such that $s(x) \in U$ but $r(x) \notin U$. It follows that $x \in UV$ but $x \notin UV$. By assumption, $1_U \in A_R(G^0) \subseteq Z(A_R(G))$ and thus $1_U 1_V = 1_V 1_U = 1_V 1_U = 1_V$, i.e., $UV = VU$ which is a contradiction. Thus $V \subseteq Iso(G)$. So $A_R(G) = A(G^0)$, and therefore $A_R(G)$ is commutative. The converse is clear. □

If $G$ is effective, then $G^0 = Iso(G)^o$, therefore by Corollary 1.3(i), $A_R(G^0)$ is a maximal commutative subalgebra of $A_R(G)$. Since topologically principal is a stronger condition than effectiveness, we obtain [2, Lemma 2.1] from Corollary 1.3.

2. Commutative core of Leavitt path algebras

Let $E = (E^0, E^1, s, r)$ be a directed graph. We refer the reader to [1] for the basics on the theory of Leavitt path algebras. We denote by $E^{\infty}$ the set of infinite paths in $E$. Set $X := E^{\infty} \cup \{ \mu \in E^* \mid r(\mu) \text{ is not a regular vertex} \}$. Let $G_E := \{ (\alpha x, |\alpha| = |\beta|, x) \mid \alpha, \beta \in E^*, x \in X, r(\alpha) = r(\beta) = s(x) \}$.

We view each $(x, k, y) \in G_E$ as a morphism with range $x$ and source $y$. The formulas $(x, k, y)(y, l, z) = (x, k + l, z)$ and $(x, k, y)^{-1} = (y, -k, x)$ define composition and inverse maps on $G_E$ making it a groupoid with $G_E^0 = \{(x, 0, x) \mid x \in X \}$ which we identify with the set $X$. Next, we describe a topology on $G_E$. For $\mu \in E^*$ define

$$Z(\mu) = \{ \mu x \mid x \in X, r(\mu) = s(x) \} \subseteq X.$$
For $\mu \in E^*$ and a finite $F \subseteq s^{-1}(r(\mu))$, define

$$Z(\mu \setminus F) = Z(\mu) \setminus \bigcup_{\alpha \in F} Z(\mu\alpha).$$

The sets $Z(\mu \setminus F)$ constitute a basis of compact open sets for a locally compact Hausdorff topology on $X = \mathcal{G}_E^{(0)}$ (see [14, Theorem 2.1]). For $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, and for a finite $F \subseteq E^*$ such that $r(\mu) = s(\alpha)$ for $\alpha \in F$, we define

$$Z((\mu, \nu) \setminus F) = \{ (\mu x, |\mu| - |\nu|, \nu x) \mid x \in X, r(\mu) = s(x) \},$$

and then

$$Z((\mu, \nu) \setminus F) = Z((\mu, \nu) \setminus F) \setminus \bigcup_{\alpha \in F} Z((\mu, \nu)\alpha).$$

The sets $Z((\mu, \nu) \setminus F)$ constitute a basis of compact open bisections for a topology under which $G_E$ is an ample groupoid. We have an isomorphism the map

$$\pi_E : L_R(E) \longrightarrow A_R(\mathcal{G}_E)$$

$$v \mapsto 1_{Z(v)},$$

$$e \mapsto 1_{Z(e, r(e))},$$

$$e^* \mapsto 1_{Z(r(e), e)},$$

where $v \in E^0$ and $e \in E^1$. Recall that the diagonal subalgebra $D(E)$ is generated by monomials $aa^*$, where $a$ is a path in $E$, whereas the commutative core subalgebra $M_R(E)$ is generated the monomials $aba^*$, where $a$ is a path and $b$ is a cycle with no exit. Observe that the isomorphism $\pi_E$ restricts to isomorphisms

$$D_R(E) \longrightarrow A_R(\mathcal{G}_E^{(0)})$$

$$M_R(E) \longrightarrow A_R(\text{Iso}(\mathcal{G}))^\circ.$$ (2.1)

We can recover the results of [9] and answer the open question raised there [9, p. 245].

**Corollary 2.1.** Let $E$ be an arbitrary graph and $L_R(E)$ the Leavitt path algebra associated to $E$. Then

(i) The centraliser of the diagonal algebra $D_R(E)$ is the core algebra $M_R(E)$.

(ii) The core algebra $M_R(E)$ is the maximal commutative subalgebra of $L_R(E)$.

(iii) If $Z(L_R(E)) = M_R(E)$ then $L_R(E)$ is either $R$ or $R[x, x^{-1}]$, i.e., the graph $E$ is either a single vertex or a vertex with a loop.

**Proof.** (i) and (ii) immediately follow from Theorem 1.2 and Corollary 1.3 by considering the graph groupoid $\mathcal{G}_E$ (see 2.1).

(iii) By Corollary 1.3, $L_R(E)$ is commutative. Thus the graph $E$ is either a single vertex or a vertex with a loop. $\square$

In a similar manner we can determine the centraliser of the diagonal algebra of Kumjian-Pask algebras as they can also be described as a Deaconu-Renault groupoid algebras. Let $\Lambda$ be a row-finite $k$-graph without sources and KP$_k(\Lambda)$ the Kumjian–Pask algebra of $\Lambda$. We refer the reader to [3] for the basics on the theory Kumjian-Pask algebras. Following [3], $\Lambda^\infty$ denotes the set of all degree-preserving functor $x : \Omega_k \rightarrow \Lambda$. Here $\Omega_k$ is the $k$-graph defined as a set by

$$\Omega_k = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n \}$$

with $d(m, n) = n - m$, $\Omega_k^0 = \mathbb{N}^k$, $r(m, n) = m$, $s(m, n) = n$ and $(m, n)(n, p) = (m, p)$.

The diagonal algebra $D_R(\Lambda)$ is generated by $\lambda^*\lambda$ where $\lambda \in \Lambda$. We define the core subalgebra

$$M_R(\Lambda) := \text{span}_R \{ \lambda\mu^* \mid \lambda x = \mu x, \text{ for any } x \in \Lambda^\infty \}.$$ 

By appealing to Theorem 1.2 and Corollary 1.3, similar to the case of Leavitt path algebras, we have.

**Corollary 2.2.** Let $\Lambda$ be a row-finite $k$-graph with no sources and KP$_k(\Lambda)$ the Kumjian–Pask algebra associated to $\Lambda$. Then

(i) The centraliser of the diagonal algebra $D_R(\Lambda)$ is the core algebra $M_R(\Lambda)$.

(ii) The core algebra $M_R(\Lambda)$ is the maximal commutative subalgebra of KP$_k(\Lambda)$.

(iii) If $Z(KP_R(\Lambda)) = M_R(\Lambda)$ then KP$_R(\Lambda)$ is $R[x^\pm 1, \ldots, x_k^\pm 1]$.

**Remark 2.3.** In the setting of groupoid $C^*$-algebras, the general version of Theorem 1.2 yet to be established. In this setting, with an extra assumption that $\text{Iso}(\mathcal{G})^\circ$ is abelian, the $C^*$-version of Theorem 1.2 was established in [5] (see [5, Corollary 5.3, 5.4]), and also [4, Theorem 4.3]). The latter Theorem was used to give the $C^*$-version of Corollary 2.2 (i) and (ii) (see [4, Corollary 4.6]). We finish the note by remarking that Exel, Clark and Pardo [8] have established the algebraic version of uniqueness theorem of [4, Theorem 3.1]. Namely, let $\mathcal{G}$ be a second-countable, ample, Hausdorff groupoid, $R$ a unital commutative ring, and let $\pi : A_R(\mathcal{G}) \rightarrow A$ be a homomorphism of rings. Then $\pi$ is injective if and only if the restriction of $\pi$ on the core subalgebra $A(\text{Iso}(\mathcal{G})^\circ)$ is injective.
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References

[1] G. Abrams, P. Ara, M. Siles Molina, Leavitt path algebras, Lecture Notes in Mathematics 2191 Springer, 2017.
[2] P. Ara, J. Bosa, R. Hazrat, A. Sims, Reconstruction of graded groupoids from graded Steinberg algebras, Forum Math. 29(5) (2017), 1023–1037.
[3] G. Aranda Pino, J. Clark, A. an Huef, I. Raeburn, Kumjian–Pask algebras of higher-rank graphs, Trans. Amer. Math. Soc. 365 (2013), no. 7, 3613–3641.
[4] J. Brown, G. Nagy, S. Reznikoff, A. Sims, D. Williams, Cartan subalgebras in C∗-algebras of Hausdorff étale groupoids, Integral Equations Operator Theory 85 (2016), no. 1, 109–126.
[5] T.M. Carlsen, E. Ruiz, A. Sims, M. Tomforde, Reconstruction of groupoids and C∗-rigidity of dynamical systems, arXiv:1711.01052.
[6] T.M. Carlsen, ∗-isomorphism of Leavitt path algebras over Z, Adv. in Math. 324 (2018) 326–335.
[7] L.O. Clark, R. Hazrat, Étale groupoid and Steinberg algebras, a concise introduction, arXiv:1901.01612.
[8] L.O. Clark, R. Exel, E. Pardo, A generalised uniqueness theorem and the graded ideal structure of Steinberg algebras, Forum Mathematicum, 30 (2018), no. 3, 533–552.
[9] C. Gil Cantoa, A. Nasr-Isfahani The commutative core of a Leavitt path algebra, J. of Algebra 511 (2018) 227–248.
[10] G. Nagy, S. Reznikoff, Abelian core of graph algebras, J. Lndon Math. Soc. 85 (2012) 889–908.
[11] J. Renault, Cuntz-like algebras, Operator theoretical methods. (Timioara, 1998), Theta Found (2000) 371–386.
[12] J. Renault, A groupoid approach to C∗-algebras, Lecture Notes in Mathematics, 793 Springer, Berlin, 1980.
[13] B. Steinberg, A groupoid approach to discrete inverse semigroup algebras, Adv. Math. 223 (2010), 689–727.
[14] S.B.G. Webster, The path space of a directed graph, Proc. Amer. Math. Soc. 142 (2014), 213–225.

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