$k$-Servers with a Smile: 
Online Algorithms via Projections

Niv Buchbinder∗  Anupam Gupta†  Marco Molinaro‡  Joseph (Seffi) Naor§

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Abstract

We consider the $k$-server problem on trees and HSTs. We give an algorithm based on Bregman projections. This algorithm has a competitive ratios that match some of the recent results given by Bubeck et al. (STOC 2018), whose algorithm was based on mirror-descent-based continuous dynamics prescribed via a differential inclusion.

1 Introduction

The $k$-server problem is one of the cornerstones of online algorithms and competitive analysis. It captures many other classic online problems (like paging) that maintain “feasible” stages while satisfying a sequence of requests arriving online. Given a metric space $(X, d)$ on $n = |X|$ points, the input is a sequence of requests $r_1, r_2, \ldots, r_T, \ldots$, where each request $r_t$ is a point in the metric space. The algorithm maintains a set $A_t \subseteq X$ of $k$ points in the metric, which gives the locations of the $k$ servers. We require that $r_t \in A_t$ for all $t$, i.e., at each time-step there is a server at the requested point. The cost of the algorithm is the sum of the (earthmover) distances between the consecutive states of the algorithm; i.e., the total distance traveled by the $k$ servers while occupying locations $A_t$ at time $t$.

The problem has a long rich history; we list some relevant events and refer to [BBMN11, BCL+17] for more references. Manasse et al. [MMS90] introduced it and conjectured a (deterministic) $k$-competitive algorithm for all metrics; there is a deterministic lower bound of $k$ even for the uniform metric (which is equivalent to the paging problem). This conjecture technically still remains open, though the $(2k−1)$-competitive algorithm of Koutsoupias and Papadimitriou [KP95] settled it in spirit. The focus then shifted to the randomized $k$-server conjecture: can we get an $O(\log k)$-competitive randomized algorithm for general metrics? Such a result would be tight, since there is a lower bound of $\Omega(\log k)$, again coming from the paging problem. The first non-trivial improvement over the deterministic case was a polylog($k, n$)-competitive randomized algorithm which is due to Bansal et al. [BBMN11]. Very recently, Bubeck et al. [BCL+17] achieved a breakthrough,
introducing several new ideas to give an algorithm that is $O(D \log k)$-competitive on depth-$D$ trees, and $O(\log^2 k)$-competitive on HSTs. Moreover, they gave a dynamic tree-embedding result to show an $O(\log^3 k \log \Delta)$-competitiveness result for general metrics. Subsequently, Lee [Lee17] employed further new ideas to remove the dependence on $\Delta$ and achieve an $O(\log^6 k)$-competitive randomized algorithm for general metrics. This is the first polylog $k$-competitive algorithm for general metrics.

The [BCL+17] paper defined a differential inclusion, whose (unique) solution gives a fractional solution to an LP relaxation for the problem. Our first result is a different algorithm (albeit directly inspired by theirs) with the same asymptotic guarantees.

We first give the result for trees with small hop-diameter:

**Theorem 1.1** (Low-Depth Trees). There is a deterministic algorithm that outputs a fractional solution for the $k$-server problem when the metric $(X,d)$ is a tree metric, with competitive ratio $O(D \log k)$, where $D$ is the hop-diameter of the tree.

Recall that the hop-diameter of a tree is the maximum number of edges on any simple path in the tree. A useful sub-class of trees are $\tau$-HSTs; these have a designated root, and consecutive edge-lengths decrease by a factor of $\tau > 1$ along any root-leaf path. (We suppress the $\tau$ and just refer to HSTs when the precise value of $\tau$ is not important.) It is easy to transform any $\tau$-HST into one that has depth $O(\log n)$, while changing distances by a factor of at most $\frac{2\tau}{\tau-1}$. Moreover, on such trees it is possible to randomly round fractional solutions to integer ones using ideas from [BBMN11, BCL+17]. This implies the following result:

**Corollary 1.2.** There is a randomized algorithm that is $O(\min\{D, \log n\} \log k)$-competitive for the $k$-server problem when the metric space $(X,d)$ is induced by an HST. Again, $D$ is the hop-diameter of the tree.

Finally, we can improve this guarantee to get an $O(\log^2 k)$ guarantee:

**Theorem 1.3** (HSTs). There is a randomized algorithm that is $O(\log^2 k)$-competitive for the $k$-server problem when the metric space $(X,d)$ is induced by a $\tau$-HST for $\tau \leq 1/10$.

**Bicriteria Problems.** Our algorithm naturally extends to the $(h,k)$-server problem, where the algorithm has $k$ servers, but its cost is compared to the cost of the best solution with only $h \leq k$ servers. For the weighted star metric, this problem admits $\frac{k}{k-h+1}$-competitive deterministic [ST85, You94] and $O(\log \frac{k}{k-h+1})$-competitive randomized algorithms [BBN10]; these guarantees approach 1 as $k/h \to \infty$. For more general tree metrics, such strong guarantees are not possible. Bansal et al. [BEJK17] showed a lower bound of 2.41 on the competitiveness of deterministic algorithms for the $(h,k)$-server problem on depth-2 HSTs, even when $h \ll k$. They also gave a deterministic algorithm for depth-$D$ trees with competitive ratio $D(1-1/(1+\varepsilon)^{(1/D-1)})^{D+1}$ where $k/h = (1+\varepsilon)$. E.g., for $\varepsilon \in (0,1]$, this factor is about $D(2D/\varepsilon)^{D+1}$; contrast this with the deterministic $(1/\varepsilon)$-competitiveness for paging.

A small change to the algorithm and analysis from Theorems 1.1 and 1.3 gives the following:
Theorem 1.4. There is a deterministic algorithm that outputs a fractional solution for the \((h, k)\)-server problem when the metric \((X, d)\) is a tree metric, with competitive ratio \(O(D \log(1/\epsilon))\), where \(k/h = 1 + \epsilon\). For the case of HSTs, we get a bound of \(O(\min\{D, \log k\} \log(1/\epsilon))\), and can round it to get a randomized algorithm with the same asymptotic competitive ratio.

We point out that the algorithms of [BCL+17] also extend to the \((h, k)\)-server setting and give a competitive factor of \(O(\min\{D, \log k\} \log(1/\epsilon))\) [Bub18].

Techniques. The algorithm is easy to state. We use the elegant linear programming relaxation of the \(k\)-server problem given by [BCL+17], which defines a feasible polytope \(P\) amenable to online computation. Let \(x_t\) denote the “anti-server” solution at time \(t\). The points that serve the request at node \(r_t\) at time \(t\) are those in some subspace \(P \cap \{x : x_{r_t} = 0\}\). Now, given the previous solution \(x_{t-1}\), getting a solution \(x_t\) for time \(t\) is easy: we project \(x_{t-1}\) onto this subspace \(P \cap \{x : x_{r_t} = 0\}\).

The projection is not a Euclidean projection, but is with respect to a “natural” distance function in the context of trees—the distance corresponding to the (negative) multiscale entropy function \(D\). Formally,

\[
x_t \leftarrow \arg \min_{x \in P \cap \{x : x_{r_t} = 0\}} D(x \| x_{t-1}^t).
\]

This amounts to solving a convex program\(^2\). Note that this projection-based algorithm makes a sequence of discrete jumps, one for each time step, and hence differs from the mirror-descent approach of [BCL+17] which takes infinitesimal steps and keeps using Bregman projections to get back into \(P\), until feasibility is achieved.

The analysis also draws significantly on [BCL+17], with some differences because of the discrete steps. The proof of competitiveness is via a potential-function argument. This potential is (more or less) the distance between the optimal solution and that of the algorithm (measured according to the distance function used for the projection):

\[
\Phi_t := D(OPT_t \| ALG_t).
\]

However, since we project at each time using a distance that is a Bregman divergence, we use the “reverse-Pythagorean” property of such distance functions to relate the distance between \(x_{t-1}^t\) and \(x_t\) to the drop in potential, and thereby show

\[
D(ALG_t \| ALG_{t-1}) + (\Phi_t - \Phi_{t-1}) \leq \alpha \cdot \Delta OPT.
\]

(To obtain this inequality we also need to relate the change in \(OPT\) to the change in potential, which we do by choosing divergences \(D(\cdot \| \cdot)\) that are “Lipschitz” in the first argument.) The technical work is then to relate the actual movement cost \(||ALG_t - ALG_{t-1}||\) to this Bregman distance \(D(ALG_t \| ALG_{t-1})\). These proofs are short for set cover and weighted paging (which we give here for completeness and intuition), and longer for \(k\)-server. We also show that our algorithm maintains many of the side invariants that hold for the continuous process of [BCL+17].

We emphasize that Bregman projections are not new in the context of online algorithms: they

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\(^1\)The indicator \(x_{tu} = 0\) means that there is a server at node \(u\) and \(x_{tu} = 1\) indicates otherwise.

\(^2\)We are glossing over an important detail—we will need to project on the affine slice \(P \cap \{x : x_{r_t} = \delta\}\) for some positive small \(\delta\) to ensure the gradients are Lipschitz. See §3 for the full story.
have been explicitly used by, e.g., [BCN14, BCL+17], and also implicitly underlie many online primal-dual algorithms for packing and covering problems, even though the algorithms may not be explicitly described in this language. Moreover, the projection method is not a panacea: there are problems for which this approach does not seem to give us the desired fine-grained control over the solutions; for other problems like MTS we need to incorporate service costs. Yet, we hope this perspective will be useful in other contexts.

Finally, the authors of [BCL+17] inform us that their algorithm can also be discretized, by repeatedly taking $\varepsilon = \varepsilon(n,k)$-sized steps, and then (Bregman) projecting back onto the polytope of feasible points [Lee18]. Indeed, since the proof of [BCL+17, Theorem 5.6] proceeds by finding a discrete sequence of feasible points and then taking limits, this proof can be used to show that such a discretization process works.

**Roadmap.** As a warm-up we give a rephrasing of the primal-dual algorithm for the (unweighted) set cover problem in terms of projections. This provides the basic ingredients of the analysis: how the KKT conditions allow us to prove useful properties of the projected points, which in turn are used in the analysis based on the reverse-Pythagorean property of Bregman projections. We then build on these ideas to give algorithms for the $(h,k)$-weighted paging problem in §A, the $(h,k)$-server problem on trees in §3, and for HSTs in §4. We defer the paging example to an appendix to get to $k$-server earlier, but the non-expert reader may want to read the paging example first to gain some more familiarity with the ideas.

We emphasize that the length of some of our proofs comes from spelling out all the details. E.g., readers familiar with basics of convex optimization can easily compress the proofs for set cover and paging to a page each; other proofs also can be considerably shortened.

1.1 Related Work

The fact that many primal-dual algorithms could be viewed as mirror descent was observed by Buchbinder, Chen, and Naor [BCN14], who used this viewpoint to give approximation algorithms for set cover with service costs, which is a simultaneous extension of set cover and metrical task systems (MTS) on a weighted star. Many authors have used convex programs to analyze and solve online problems, e.g., Devanur and Jain [DJ12], Anand et al. [AGK12], Gupta et al. [GKP12], Devanur and Huang [DH14], Kim and Huang [HK15], Im et al. [IKM18], and others have developed primal-dual and dual-fitting techniques using convex programs. However, the ideas and techniques used there are different from the ones in this paper.

1.2 Notation and Preliminaries

We merely give some definitions that we need in this paper; for details about convexity and convex optimization, see Rockafellar [Roc70], Hiriart-Urruty and Lemarechal [HUL01], or Boyd and Vanderberghe [BV04].

Given a convex function $h : \mathbb{R}^n \rightarrow \mathbb{R}$, the Bregman divergence associated with $h$ is given by

$$D_h(p \parallel q) := h(p) - h(q) - \langle \nabla h(q), p - q \rangle.$$

In words, this is the amount by which the linear approximation of convex function $h$ at point $q$
underestimates the function value at $p$. We get an underestimate because $h$ is convex. Hence the divergence is non-negative for any $p, q$, and is zero when $p = q$. (If the function $h$ is strictly convex, then the converse also holds, i.e., $D_h(p \parallel q) = 0 \implies p = q$.) The Bregman projection of a point $q$ onto a convex body $P$ is simply $p = \arg \min_{x \in P} D_h(x \parallel q)$.

One commonly used Bregman divergence between non-negative vectors is the (unnormalized) Kullback-Liebler divergence:

$$D(p \parallel q) = \sum_i \left( p_i \log \frac{p_i}{q_i} - p_i + q_i \right)$$

which arises from the negative entropy function $h(p) = \sum_i p_i \log p_i$. If we consider $p$ and $q$ in the probability simplex (or indeed, if $p$ and $q$ have the same $\ell_1$ norm), then the linear terms fall away and $D(p \parallel q)$ becomes the well-known normalized KL-divergence (or relative entropy function) $\sum_i p_i \log \frac{p_i}{q_i}$. Now, projecting a point $x$ onto the probability simplex using KL-divergence, simply scales up each coordinate by the same factor—this is the multiplicative-weight update rule! We will use variants of KL-divergence extensively.

We need the reverse-Pythagorean property of Bregman divergences. Given a convex body $K$, with point $y \in K$, and the Bregman projection $x' := \min_{z \in K} D(z \parallel x)$ for some point $x$,

$$D(y \parallel x) \geq D(y \parallel x') + D(x' \parallel x).$$

The name comes from the illuminating (visual) proof of this fact for the squared Euclidean distance Bregman divergence $D(p \parallel q) := \frac{1}{2} \|p - q\|_2^2$.

Using the inequality $1 + x \leq e^x$ with $x = \log(b/a)$ gives what we call the “poor-man’s Pinsker” inequality: for all $a, b \geq 0$,

$$a - b \leq a \log \frac{a}{b}. \quad \text{(PMP)}$$

## 2 The Unweighted Set Cover Problem

As a warm-up, we solve and analyze (unweighted) set cover in the projection perspective. This is to show the main steps: the derived algorithm is essentially that from Alon et al. [AAA+09], and the projection viewpoint was already noted in, e.g., [BCN14]. The steps in subsequent sections are similar to those here, only more involved.

We are given $n$ sets, and “set covering” constraints $\langle a_t, x \rangle \geq 1$ arrive online, where each request vector $a_t \in \{0, 1\}^n$. This defines the set covering polytope:

$$P_t := \{ x \geq 0 \mid \langle a_s, x \rangle \geq 1 \forall s \leq t \}.$$

The goal is to maintain a fractional solution $x^t \in P_t$ that is monotone (i.e., $x^t \geq x^{t-1}$) and is an approximately good solution to the linear program:

$$\min_{x \in P_t} \langle 1, x \rangle.$$
We compare ourselves to the optimal integer solution \( y^t \in \{0, 1\}^n \). We start with the initial solution \( x_i^0 = \delta \), where \( \delta = \frac{1}{n} \); hence we start off with the fractional solution buying one set in total. As long as the request sequence contains at least one request, this extra (fractional) set does not affect the competitive ratio except by at most a factor of 2.

2.1 The Projection Algorithm

Project the old point \( x^{t-1} \) onto the new body \( P_t \) using the Bregman divergence:

\[
D(x \parallel x') := \sum_i \left( x_i \log \frac{x_i}{x'_i} - x_i + x'_i \right)
\]

I.e., set \( x^t := \arg\min_{x \in P_t} D(x \parallel x^{t-1}) \). Since all entries \( a_{ti} \geq 0 \) (i.e., the polytope is a covering polytope), we claim this projection is equivalent to just projecting onto the convex set \( Q_t := \mathbb{R}^n \cap \{ \langle a_t, x \rangle \geq 1 \} \). To see this, define \( z := \arg\min_{z \in Q_t} D(z \parallel x^{t-1}) \). We show monotonicity below: that \( z_i \geq x_i^{t-1} \) for all \( i \), and hence \( z \in P_{t-1} \). Since \( P_{t-1} \cap Q_t = P_t \), this shows an equivalence between the two projections, and hence \( x^t = z \).

To show monotonicity, consider the point \( z \): it is the solution to the convex program

\[
\min \sum_i \left( z_i \log \frac{z_i}{x_i^{t-1}} - z_i + x_i^{t-1} \right), \quad \sum_i a_{ti} z_i \geq 1
\]

The KKT optimality conditions imply:

\[
\log \frac{z_i}{x_i^{t-1}} = \lambda_t a_{ti}, \quad (2.1)
\]

where the Lagrange multiplier \( \lambda_t \geq 0 \) satisfies the complementary slackness condition \( \lambda_t (\sum_i a_{ti} z_i - 1) = 0 \). (Since the constraints are affine and only the objective function is convex, strong duality holds as long as the problem is feasible. Hence we do not have problems with duality gaps, and so can assume the existence of optimal duals/KKT certificates/multipliers for the problems in this paper.)

Rewriting (2.1), \( z_i = x_i^{t-1} e^{\lambda_t a_{ti}} \geq x_i^{t-1} \) because the exponent is non-negative. This proves monotonicity, and the preceding argument then gives \( x^t = z \). Moreover, defining \( P_t^\delta = P_t \cap [\delta, 1]^n \), we get that \( x^t \in P_t^\delta \) for all \( t \).

2.2 Analysis

We use a potential function to measure the “distance” from the optimal solution to the algorithm’s solution. Define the function

\[
\Phi(x \parallel x') = \sum_i x_i \log \frac{x_i}{x'_i}.
\]
Observe that $\Phi(x \parallel x') = D(x \parallel x') + \langle 1, x - x' \rangle$. Define the potential after serving the $t^{th}$ request to be:

$$\Phi(y^t \parallel x^t) := \sum_i y^t_i \log \frac{y^t_i}{x^t_i} = \sum_{i: y^t_i = 1} \log \frac{1}{x^t_i}.$$  

Here $y^t$ is the optimal integer solution for $P_t$. Since the fractional solution $x^t \in P^0_t$, each term of this potential is non-negative, and at most $\log 1/\delta$. (As an aside, we could set $\Phi(\cdot \parallel \cdot) = D(\cdot \parallel \cdot)$ with tiny changes, but we find the current view cleaner.)

**When OPT moves:** We can ensure that OPT only changes entries from 0 to 1. In this case the increase in potential is at most $\log 1/\delta$, so we get

$$\Phi(y^t \parallel x^{t-1}) - \Phi(y^{t-1} \parallel x^{t-1}) \leq (\log 1/\delta) \cdot \Delta OPT. \quad (\star)$$

**When ALG moves:** Since we used a Bregman divergence to project a point $x^{t-1}$ down to $x^t \in P_t$ (and also because the optimal point $y^t \in P_t$), we can use the reverse-Pythagorean property of these projections to claim

$$D(y^t \parallel x^{t-1}) \geq D(y^t \parallel x^t) + D(x^t \parallel x^{t-1}).$$

Substituting the definition of $D(\cdot \parallel \cdot)$ and canceling linear terms from both sides, we get

$$\Phi(y^t \parallel x^{t-1}) \geq \Phi(y^t \parallel x^t) + \Phi(x^t \parallel x^{t-1}),$$

or equivalently,

$$\underbrace{\Phi(x^t \parallel x^{t-1}) + (\Phi(y^t \parallel x^t) - \Phi(y^t \parallel x^{t-1}))}_{\text{"shadow" cost}} \leq 0. \quad (**)$$

Since all coordinates of $x$ are non-decreasing, the actual cost for step $t$ is

$$\|x^t - x^{t-1}\|_1 \leq \sum_{i} (x^t_i - x^{t-1}_i) \leq \sum_{i} x^t_i \log \frac{x^t_i}{x^{t-1}_i} = \Phi(x^t \parallel x^{t-1}). \quad (***)$$

Summing the starred equations shows that in serving any request,

$$(\text{cost for ALG}) + \Delta \Phi \leq (\log 1/\delta) \cdot (\text{cost for OPT}).$$

Since $\Phi_0 = 0$ and $\Phi_t \geq 0$, we get the following theorem.

**Theorem 2.1** (Set Cover). For the online (unweighted) set cover problem, the projection algorithm maintains a fractional solution with at most $(\log n) \cdot OPT + 1$ sets, where OPT is the optimal integer solution.

Observe that increasing the $\delta$ term improves the multiplicative guarantee but worsens the additive term. By a guess-and-double approach, we can get an $O(\log (n/\delta) \cdot OPT)$-approximation. We chose the analysis for the unweighted case for its simplicity. To extend to the case, e.g., where sets have costs, we need to consider a slightly different Bregman divergence. We defer this discussion to a later version of the paper.
3 The \((h, k)\)-Server Problem on Trees

We now consider the \((h, k)\)-server problem on trees. For readers wishing to gain more familiarity with projection-based algorithms, and analyses using KKT conditions, we recommend §A for an analysis of a projection-based weighted paging algorithm. However, reading §A is optional for the experts, since the present section is self-contained.

Given a tree \(T\), the root \(r\) is at depth 0; the depth for a node \(u\) is the number of edges on the \(r-u\) path. We assume all the leaves are at depth \(D\); this is for convenience, and incurs no significant loss of generality. The vertices at depth \(d\) are denoted by \(V_d\), and hence \(V_0 = \{r\}\), whereas \(V_D\) is the set of all leaves. Let \(w_u\) be the weight of node \(u\); we define the distance from \(u\) to \(v\) as the total sum of weights of vertices on the path from \(u\) to \(v\) (this is equivalent to defining suitable edge weights). Let \(n = |V_D|\) be the number of leaves of the tree, and we associate the leaves with the set \([n]\). Without loss of generality we assume that requests only appear at the leaves.

Similarly to the paging problem, the goal in the weighted \((h, k)\)-server problem is to respond at each time \(t\) to requests \(r_t \in [n]\) by producing a vector \(z_t \in \{0, 1\}^n\), with \(\|z_t\|_1 = k\), such that \(z_{rt} = 1\) for all \(t\). The objective is to minimize the total weighted change:

\[
d(z^t, z^{t-1}) := \sum_{u \in V(T)} w_u |z^t(T_u) - z^{t-1}(T_u)|. \tag{3.2}
\]

Here \(T_u\) denotes the set of vertices in the subtree rooted at \(u\), and \(z^t(T_u)\) denotes the number of servers in this subtree as defined by \(z^t\). We compare our performance to that of the optimal solution that uses only \(h\) servers.

3.1 Atoms and the Anti-Server Polytope

Following the paging setting, we consider an “anti-server” polytope, where an \(x_u\)-value of 0 indicates that there is a server at \(u\), and 1 otherwise. In particular, we use the anti-server polytope proposed by Bubeck et al. [BCL+17], which has many nice features. It will be crucial to define variables for both leaves and internal nodes; in fact, each internal node may have several variables corresponding to it, as we now explain.

Let \(L_u\) be the set of leaves in the tree \(T_u\). Each vertex \(u \in V(T)\) has associated variables \(x_{u,j}\) for \(j \in \{1, 2, \ldots, |L_u|\}\). We refer to pairs \((u, j)\) as atoms. Let \(\chi_u := \{(v, \ell) \mid v \text{ is a child of } u, \ell \in [|L_v|]\}\) be the atoms corresponding to the children of \(u\), and let \(\chi_u\) be the children of \(u\) in the tree \(T\). Since the leaves are all at the same level, the total number of atoms (and hence the number of variables) at each level of the tree is exactly \(n\), the number of leaves. Moreover, each leaf \(u\) has only a single atom \((u, 1)\) and hence a single variable \(x_{u,1}\). Let \(N = n(D + 1)\) be the total number of atoms.

The anti-server polytope proposed by Bubeck et al. [BCL+17] is the following:

\[
P := \left\{ x \in [0, 1]^N \left| \begin{array}{l}
x_{r,j} \geq 1_{(j \geq h)} \\
\sum_{j \leq |S|} x_{u,j} \leq \sum_{(v, \ell) \in S} x_{v,\ell} \\
\forall u, S \subseteq \chi_u
\end{array} \right. \right\}
\]
Remember that $P$ represents solutions in the anti-server world as follows: to encode an integer solution $B^t$, set $y_{\ell,1} = 0$ for leaves $\ell$ containing servers and $y_{\ell,1} = 1$ otherwise; set $y_{u,j} = 0$ if the subtree under $u$ contains at least $j$ servers, and 1 otherwise. It is easy to check that the constraints are satisfied by this integral solution.

The reader may find it convenient to think of an “ideal” fractional solution for $P$ as follows: the “ideal” setting for a $y_{u,}$- vector at an internal node $u$ is to take the vectors of its children, concatenate them, and then sort the entries of this concatenated vector in non-decreasing order. While such a sortedness condition is not required, and may not even hold, it may be useful for intuition about the LP.

### 3.3.1 Translating between Servers and Anti-Servers

Define $\delta := \frac{k-h+1/2}{k+1/2}$, as for paging. We define the shifted polytope $P_\delta := P \cap \{ x \mid x_{(u,1)} \geq \delta \ \forall u \in L_r \}$ to be the subset of points such that all leaf atoms have value at least $\delta > 0$. The algorithm maintains a fractional solution $x^t \in P_\delta$, with $x^t_{(u,1)} = \delta$ and $\|x^t\| = n - h$. We can define a norm on vectors in $\mathbb{R}^N$ as follows:

$$
\|x\|_{\ell_1(T)} := \sum_u w_u \sum_{j \in [\|x_u\|]} |x_{u,j}|. \tag{3.3}
$$

Note: the norm (3.3) is defined for vectors in $\mathbb{R}^N$ that assign values to all atoms in the tree, whereas the distance (3.2) is defined for vectors in $\mathbb{R}^n$ that assign values only to the leaves. However, the two distances can be related to each other using Lemma B.5.

Define the vector of servers $z^t \in [0, 1]^n$ by setting $z^t_u := \frac{1 - x^t_{(u,1)}}{1 - \delta}$ for each leaf $u$. This gives a fractional vector with $\|z^t\| = k + 1/2$; Lemma B.5 shows that

$$
d(z^t, z^{t-1}) \leq \frac{1}{1-\delta} \cdot \|x^t - x^{t-1}\|_{\ell_1(T)}.
$$

Finally, for hierarchically well-separated trees (HSTs) we can use [BCL+17, Lemma 3.4] and [BBMN11, §5.2] to round the fractional solutions with $k + 1/2$ servers to integer solutions with just $k$ servers, so that

$$
\mathbb{E}[d(z^t, z^{t-1})] \leq O(d(z^t, z^{t-1})).
$$

The main result of this section is an algorithm to maintain a fractional point $x^t$ as follows.

**Theorem 3.1** (Main Theorem: $(h,k)$-server). Given a tree of depth $D$, there exists an algorithm that maintains fractional solutions $x^t \in P_\delta$ with $x^t_{(u,1)} = \delta$ and $\|x^t\| = n - h$, such that for any integer feasible solutions $y^t$, we have

$$
\|x^t - x^{t-1}\|_{\ell_1(T)} \leq O(D \log(1 + 1/\delta)) \cdot \|y^t - y^{t-1}\|_{\ell_1(T)} + C' = O(D \log(1 + 1/\delta)) \cdot d(y^t, y^{t-1}) + C'.
$$

for an additive term $C'$ that depends only on $T, k, \delta$ but not on the input sequence.

Combining this theorem with the above chain of inequalities gives us a randomized algorithm for the $(h,k)$-server problem with a competitive ratio of $O(D \log(1 + 1/\delta)) = O(D \log \frac{2k-h+1}{k-h+1/2})$, at least when $h \in \Omega(k)$. Note that for $h = k$ the algorithm is $O(D \log k)$-competitive, and for $h = k/2$ the algorithm is $O(D)$-competitive. In the rest of this section, we present the proof of Theorem 3.1.
3.2 The Projection Algorithm

When a request arrives at leaf \( r_t \) at time \( t \), we consider the polytope

\[ P_t := P \cap \{ x_{r_t,1} \leq \delta \}. \]

Given the previous solution \( x^{t-1} \in P_{t-1} \cap P_\delta \), the new point \( x^t \) is the projection of \( x^{t-1} \) onto \( P_t \), with respect to the Bregman divergence for the (shifted) multilevel entropy function:

\[ D(x \parallel x') := \sum_u w_u \sum_j \left( \tilde{x}_{u,j} \log \frac{\tilde{x}_{u,j}}{\tilde{x}_{u,j}^{t-1}} - \tilde{x}_{u,j} + \tilde{x}_{u,j}^{t-1} \right). \]

(Here the tildes over the variables denote an additive shift by \( \delta \), so that \( \tilde{x} = (x + \delta) \).) In other words, we set \( x^t := \arg \min_{x \in P_t} D(x \parallel x^{t-1}) \). Since \( P_t \neq P_\delta \), we need to show that \( x^t \) indeed lies in \( P_\delta \); this appears in Claim 3.3.

3.2.1 The Optimality Conditions

The projection problem above can be written as:

\[
\begin{align*}
\min \ & \sum_{u \neq r_t} w_u \sum_j \left( \tilde{x}_{u,j} \log \frac{\tilde{x}_{u,j}}{\tilde{x}_{u,j}^{t-1}} - \tilde{x}_{u,j} + \tilde{x}_{u,j}^{t-1} \right) \\
\text{s.t.} \ & x_{r,j} \geq 1, \quad (j > h) \quad \forall j \\
\sum_{j \leq |S|} x_{u,j} \leq \sum_{(v, \ell) \in S} x_{v,\ell} \quad \forall \text{ non-leaves } u, \forall S \subseteq \chi_u \\
x_{r_t,1} \leq \delta
\end{align*}
\]

Each constraint of the form (3.5) is uniquely specified by some set \( S \) of atoms that share a common parent, which we denote by \( p(S) \) — in other words, \( S \subseteq \chi_{p(S)} \).

The KKT optimality conditions show that for all \( u \notin \{r, r_t\} \),

\[
w_u \log \frac{\tilde{x}_{u,j}^t}{\tilde{x}_{u,j}^{t-1}} = \sum_{S \subseteq \chi_{p(u)} : (u,j) \in S} \lambda_S - \sum_{T \subseteq \chi_u : |T|} \lambda_T \quad \text{(KKT2a)}
\]

where all the Lagrange multipliers are non-negative. While we omit the superscripts \( t \) for the variables \( \lambda, a_{u,j}, b_{u,j} \), we emphasize that all these terms are different for each time \( t \). Note that the leaves \( u \) have no \( b_{u,j} \) terms, only the (positive) \( a_{u,j} \) terms. Hence all non-\( r_t \) leaves can only increase in value. (We sometimes refer to \( b_{u,j} \) for a leaf \( u \), in which case imagine \( b_{u,j} = 0 \).)

For the root atoms, the KKT conditions have the dual variable \( \lambda_{r,j} \geq 0 \) corresponding to constraint (3.4), but they do not have any \( a_{r,j} \) terms:

\[
w_r \log \frac{\tilde{x}_{r,j}^t}{\tilde{x}_{r,j}^{t-1}} = \lambda_{r,j} - \sum_{T \subseteq \chi_{r,j} : |T|} \lambda_T . \quad \text{(KKT2b)}
\]
For the demanded vertex $r_t$, which has a single atom $(r_t, 1)$:

$$w_{r_t} \log \frac{\tilde{x}_{r_t,1}}{\bar{x}_{r_t,1}} = \sum_{S \subseteq \chi_{p(r_t)}(r_t,1) \in S} \lambda_S - \gamma_t \quad (\text{KKT2c})$$

where $\gamma_t \geq 0$ corresponds to constraint (3.6). Again, we assume $b_{r_t,1} = 0$. In the rest of the paper, we define

$$A^t_{u,j} := \tilde{x}_{u,j} a^t_{u,j} \quad \text{and} \quad B^t_{u,j} := \tilde{x}_{u,j} b^t_{u,j} \quad (3.7)$$

Finally, complementary slackness implies:

$$\left( \lambda_S > 0 \iff \sum_{j \leq |S|} x^t_{p(S),j} = \sum_{(v,\ell) \in S} x^t_{v,\ell} \right) \iff \left( \lambda_S \sum_{j \leq |S|} x^t_{p(S),j} = \lambda_S x^t(S) \right). \quad (\text{CS2})$$

where $p(S)$ is the “parent” node for the set of nodes in $S$. The other two variables give us:

$$\lambda_{r,j} \cdot (x^t_{r,j} - 1_{(j > h)}) = 0, \quad \gamma_t x^t_{r_t,1} = \gamma_t \delta. \quad (3.8)$$

$$\gamma_t x^t_{r_t,1} = \gamma_t \delta. \quad (3.9)$$

### 3.3 Properties of the Projected Point

We prove some useful properties for the new optimal solution $x^t$. These properties are satisfied by $x^0$ by construction, and we inductively assume that they hold for $x^{t-1}$, in order to prove them for $x^t$. We defer the proofs until later; these are very similar to those in [BCL+17].

**Claim 3.2 (Root is Tight).** For the root vertex $r$, $x^t_{r,1} = 1_{(j > h)}$.

**Claim 3.3 (Box Constraints).** For all $u, j$, $0 \leq x^t_{u,j} \leq 1$. Moreover, $x^t_{r_t,1} = \delta$. Finally, $x^t_{u,1} \geq \delta$ for leaves $u$, and hence $x^t \in P_\delta$.

**Lemma 3.4 (Flow).** For each internal node $u$, $\sum_j x^t_{u,j} = \sum_{(v,\ell) \in \chi_u} x^t_{v,\ell}$. This implies that for any depth $d$,

$$\sum_{u \in V_d} \sum_j x^t_{u,j} = n - h.$$ 

So the difference between $x^{t-1}$ and $x^t$ can be viewed as a flow from $r_t$ to the other leaves in $T$.

The following lemma, using complementary slackness, is crucial to relate the dual values across levels.

**Lemma 3.5 (Relating Consecutive Levels).** For any node $u$ in the tree

$$\sum_j B^t_{u,j} = \sum_{(v,\ell) \in \chi_u} A^t_{v,\ell} = \sum_{T \subseteq \chi_u} \lambda_T \tilde{x}^t(T).$$
3.4 The Potential Function

If $y^t \in \{0, 1\}^N$ is the optimal solution, and $x^t \in [0, 1]^N$ is our solution, the potential is defined as

$$\Phi(y^t \parallel x^t) = \sum_u w_u \sum_j y^t_{u,j} \log \frac{y^t_{u,j}}{x^t_{u,j}}$$

Observe that each term in the inner sum lies in the range $[-\delta \log(1 + 1/\delta), (1 + \delta) \log(1 + 1/\delta)]$.

3.4.1 When OPT Moves: Upper Bounding the Potential Gain

Suppose OPT moves from $y^{t-1}$ to $y^t$. Changing a coordinate $y_{u,j}$ from 0 to 1 causes OPT to pay $w_u$ for such an increase—recall that it only pays for increases, and not decreases. Moreover, the increase in potential is

$$w_u((1 + \delta) \log(1 + \delta) - \delta \log \delta - \log \tilde{x}^t_{u,j} - \log \tilde{x}^{t-1}_{u,j}) \leq w_u(1 + \delta) \log(1 + 1/\delta).$$

Moreover, changing $y_{u,j}$ from 1 to 0 only decreases the potential. Hence, we get

**Lemma 3.6.** $\Phi(y^t \parallel x^{t-1}) - \Phi(y^{t-1} \parallel x^{t-1}) \leq (1 + \delta) \log(1 + 1/\delta) \cdot \Delta OPT$.

An aside: to see why the variables in the Bregman divergence, and hence the potential, are shifted by $\delta$, observe that we do not ensure that all $x_{u,j}$ variables are at least $\delta$—only the leaves are at least $\delta$. However, to prove the above lemma, we need to control the potential change and make the gradients Lipschitz even at the non-leaf nodes, so the terms are shifted explicitly by $\delta$.

3.4.2 When ALG Moves: Lower Bounding the Potential Drop

Next, the algorithm changes its solution from $x^{t-1}$ to $x^t$, and the rest of the argument will be to bound the amortized cost. Since we use a Bregman divergence to project a point $x^{t-1}$ down to $x^t \in P_t$ (and also because $y^t \in P_t$), the reverse-Pythagorean property implies:

$$D(y^t \parallel x^{t-1}) \geq D(y^t \parallel x^t) + D(x^t \parallel x^{t-1})$$

$$\Rightarrow \left(\Phi(x^t \parallel x^{t-1}) + (\Phi(y^t \parallel x^t) - \Phi(y^t \parallel x^{t-1}))\right) \leq 0. \quad (3.10)$$

We now bound the algorithm’s movement cost by some small factor times this “shadow” cost $\Phi(x^t \parallel x^{t-1})$. Substituting the expressions for $w_u \log \frac{\tilde{x}^t_{u,j}}{\tilde{x}^{t-1}_{u,j}}$ from (KKT2a-KKT2c) into the definition of $\Phi(\cdot \parallel \cdot)$, and observing that $\gamma_t \tilde{x}^t_{t+1,j} = \delta \gamma_t$, we get that the “shadow cost” is:

$$\Phi(x^t \parallel x^{t-1}) = \sum_u w_u \sum_j \tilde{x}^t_{u,j} \log \frac{\tilde{x}^t_{u,j}}{\tilde{x}^{t-1}_{u,j}}$$

$$= \sum_u \sum_j \tilde{x}^t_{u,j} (a_{u,j} - b_{u,j}) - 2\delta \gamma_t = \sum_u \sum_j (A^t_{u,j} - B^t_{u,j}) - 2\delta \gamma_t \quad (3.11)$$

Recall that the leaf nodes (i.e., nodes at depth $D$) do not have any $b_{u,j}$ terms in (KKT2a).
Lemma 3.5 now allows us to cancel the \( B_{u,j}^t \) with the \( A_{u,t}^t \) terms on the next level, giving:

\[
\Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \geq \Phi(x^t \parallel x^{t-1}) = \sum_j A_{r,j}^t - 2\delta\gamma_t. \tag{3.12}
\]

It now suffices to bound the movement cost of the algorithm by some constant factor times the expression in (3.12). We will not manage to do that; instead we give another lower bound on the drop in potential. For some \( x \), let \( W(x) := \sum_u \sum_j w_u x_{u,j} \). Observe that \( D(y \parallel x) = \Phi(y \parallel x) - W(y) + W(x) \), and hence the \( W(\cdot) \) function captures the difference between \( D(\cdot \parallel \cdot) \) and \( \Phi(\cdot \parallel \cdot) \).

**Lemma 3.7 (Second Lower Bound).** \( D(y^t \parallel x^{t-1}) - D(y^t \parallel x^t) \geq \delta\gamma_t \). Thus,

\[
\Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \geq \delta\gamma_t + W(x^t) - W(x^{t-1}).
\]

We defer the proof to §B.2. Averaging the expression in Lemma 3.7 with (3.12) gives us

\[
\Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \geq \frac{1}{3} \sum_j A_{r,j}^t + \frac{2}{3}(W(x^t) - W(x^{t-1})). \tag{3.13}
\]

The linear terms \( W(x^t) - W(x^{t-1}) \) will telescope over time, and hence the interesting term is the summation \( \sum_j A_{r,j}^t \). In the next section, we relate the movement cost of the algorithm to this summation, which will complete the argument.

### 3.5 Bounding the Movement by the Shadow Cost: Shallow Trees

We now bound the movement cost \( \sum_t \sum_u w_u \sum_j (x_{u,j}^t - x_{u,j}^{t-1})^+ \) for the entire sequence.\(^3\) First, let us record a simple observation.

**Lemma 3.8.** Suppose we have values \( y, y' \geq 0 \) such that \( c \log \frac{y}{y'} \leq (a - b) \) with some \( a, b \geq 0 \) and \( c > 0 \). Then

\[
c \cdot (y - y')^+ \leq \tilde{y} \cdot a.
\]

**Proof.** If \( y \leq y' \) then \( c \cdot (y - y')^+ = 0 \leq \tilde{y} \cdot a \). Else, when \( y > y' \),

\[
c \cdot (y - y')^+ = c \cdot (y - y') = c \cdot \left(\tilde{y} - \tilde{y}' \right) (PMP) \leq c \cdot \tilde{y} \log \frac{\tilde{y}}{y'} \leq \tilde{y}(a - b) \leq \tilde{y} \cdot a. \quad \square
\]

Using (KKT2a-KKT2c) in conjunction with Lemma 3.8 (where we set \( a = a_{u,j}^t, b = b_{u,j}^t, c = w_u \)),

\[
\sum_u w_u \sum_j (x_{u,j}^t - x_{u,j}^{t-1})^+ \leq \sum_u \sum_j A_{u,j}^t. \tag{3.14}
\]

We do not get the \( \gamma_t \) term, because this corresponds to the requested node \( r_t \); since the \( x_{r_t}^t \) value decreases, the corresponding \( (x_{r_t,1}^t - x_{r_t,1}^{t-1})^+ \) term is in fact zeroed out.

\(^3\)Note that it suffices to bound the increase in coordinates, since the total movement is at most twice this amount, plus an additive constant that depends only on the instance and is independent of the request sequence.
Since the lower bound on OPT is just in terms of the $A_{r,j}^t$ terms, we want to argue that all the non-root terms in \((3.14)\) are bounded by the terms corresponding to $r$. That is almost what we now show (modulo a certain additive term that telescopes over time). For brevity, given the $x_{u,j}^t$ values, define
\[
x_{u}^t := \sum_j x_{u,j}^t.
\] *(3.15)*

**Lemma 3.9.** For each non-leaf node $u$, we have
\[
\sum_{(v,\ell) \in \mathcal{X}_u} A_{v,\ell}^t \leq \sum_j A_{u,j}^t - w_u(x_{u,j}^t - x_{u,j}^{t-1}).
\] *(3.16)*

**Proof.** We focus on a node $u$ that is neither a leaf nor the root; the root case is very similar. Non-negativity of Bregman divergences (or equivalently, \((PMP)\)) and $w_u \geq 0$ imply
\[
w_u \left( x_{u,j}^t \log \frac{x_{u,j}^t}{x_{u,j}^{t-1}} - \tilde{x}_{u,j}^t + \tilde{x}_{u,j}^{t-1} \right) \geq 0.
\]
Now applying \((KKT2a)\), using definition \((3.15)\) and cancelling the additive $\delta$ terms gives
\[
(A_{u,j}^t - B_{u,j}^t) - w_u(x_{u,j}^t - x_{u,j}^{t-1}) \geq 0.
\]
Finally, summing up over all $j$, and using Lemma 3.5 to replace $\sum_j B_{u,j}^t$ by $\sum_{(v,\ell) \in \mathcal{X}_u} A_{v,\ell}^t$ completes the proof. The proofs for the root $r$ is similar, using \((KKT2b)\) instead. For the root $r$, observe that the total mass does not change, so the term $w_r(x_r^t - x_r^{t-1}) = 0$. \hfill \Box

Now we can multiply \((3.16)\) by $(D - d)$ for vertices $u \in V_d$ with $d = 0, 1, 2, \ldots, D - 1$, sum these up, and add $\sum_j A_{r,j}^t$ to both sides to get
\[
\sum_{u,j} A_{u,j}^t \leq (D + 1) \sum_j A_{r,j}^t - \sum_{d=1}^{D-1} \sum_{u \in V_d} (D - d) \cdot w_u(x_{u}^t - x_{u}^{t-1}).
\]
Summing up over all times $t$ and using \((3.14)\) gives us
\[
ALG \leq (D + 1) \sum_{t=1}^{T} \sum_j A_{r,j}^t - \sum_{d=1}^{D-1} \sum_{u \in V_d} (D - d) \cdot w_u(x_{u}^T - x_{u}^{0})
\]
\[
\leq (D + 1) \sum_{t=1}^{T} \sum_j A_{r,j}^t + D(W(x^0) - W(x^T)).
\]
Combining with \((3.13)\), this implies
\[
ALG \leq 3(D + 1) \sum_t \left( \Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \right) + O(D)[W(x^0) - W(x^T)].
\]
Since $|x_u^T - x_u^0| \leq h$ for each node $u$, where $h$ is the number of servers that the optimal algorithm
has in the \((h, k)\) server problem, \(W(x^T) - W(x^0) \leq h \sum_u w_u\). Now using Lemma 3.6 to bound the change in potential due to \(\text{OPT}\), we get

\[
\text{ALG} \leq O(D \log(1 + 1/\delta)) \cdot \text{OPT} + 3(D + 1) \left( \Phi(y^0 \parallel x^0) - \Phi(y^T \parallel x^T) \right) + O(Dh) \sum_u w_u.
\]

This proves Theorem 3.1 with the additive term \(C' = O(Dh \sum_u w_u + D \cdot (\Phi(y^0 \parallel x^0) - \Phi(y^T \parallel x^T))\).

### 3.5.1 An \(O(\log n \log k)\)-competitive algorithm for HSTs

Let us focus on the \(k\)-server problem; the extensions to \((h, k)\)-server are immediate. For the \(k\)-server problem on HSTs, Theorem 3.1 implies an \(O(\log \Delta \log k)\)-competitive algorithm, by setting \(\delta = \frac{1}{2k+1}\), and using the fact that the depth of any HST is \(O(\log \Delta)\). Here \(\Delta\) is the aspect ratio of the tree, the ratio of the largest to smallest distance in the tree.

To get the improved result of \(O(\log n \log k)\), we simply use the fact that for any HST, there is another tree with depth \(O(\log n)\) which changes distances by at most a constant factor (see, [BBMN11, Theorem 5.1] for a formal statement). The basic idea for obtaining this tree is simple: for each vertex \(u\), if it contains a child \(v\) such that \(|L_v| \geq |L_u|/2\), i.e., the number of leaves under \(v\) is at least half the number under \(u\), then we contract the edge \((u, v)\), and make the new node have weight equal to the parent’s weight. This ensures that traversing each edge reduces the number of leaves by a factor of 2 and hence gives a tree with depth \(O(\log n)\); moreover it does not change distances by more than a constant factor. This implies the following:

**Theorem 3.10.** There is an \(O(\log n \log k)\)-competitive randomized algorithm for the \(k\)-server problem on HSTs.

In the next section, we improve this bound to get \(O(\log^2 k)\)-competitiveness.

### 4 An \(O(\log^2 k)\) Bound for \(k\)-server

We now give the proof of Theorem 1.3. The proof here is somewhat longer and more involved than in [BCL+17]—while it can conceivably be shortened, we currently believe that some of the complexity is due to the algorithm being defined as a sequence of discrete jumps, rather than via a continuous trajectory. That being said, the high-level idea of the proof is simple and modular (and parallels that in [BCL+17]).

Recall that for a leaf \(u\) we defined \(z_t^u := \frac{1 - x_t^u}{1 - \delta}\). We extend this definition for an internal node \(u\) as \(z_t^u := \sum_{v \in \text{leaves}(T_u)} z_t^v\). Define \(\|v\|_u^+ := \sum_u w_u v_u^+\). The main result of this section is the following:

**Theorem 4.1.** Let \(T\) be a \(\tau\)-HST, where \(0 < \tau \leq 1/10\). Then there exists a potential function \(\Psi\) such that for each time \(t\),

\[
\|x^t - x^{t-1}\|_u^+ \leq O(\log(k/\delta)) \cdot \sum_j A_{x,j}^t + \Psi(z^t) - \Psi(z^{t-1}).
\]

Moreover, \(\Psi(z^T) - \Psi(z^0) = O(\sum_u \frac{w_u}{1 - \delta}(k + \log \frac{k}{\delta}))\).
The above left hand side is the algorithm’s (positive) movement. Combining with (3.13) and Lemma 3.6, the same arguments give

\[ ALG \leq O(\log k/\delta \log 1/\delta) \cdot OPT + \Psi(z^T) - \Psi(z^0) + O(\log k/\delta)(\Phi(y^0 \parallel x^0) - \Phi(y^T \parallel x^T) + O(h \sum_u w_u). \]

Setting \(\delta = 1/k\) gives the \(O(\log^2 k)\)-competitiveness, and hence the proof of Theorem 1.3.

4.1 Proof of Theorem 4.1

The proof contains many ingredients in common with that of [BCL+17], but the projection-based approach means we need some further ideas (such as the potential \(\Psi_2\) below). As in their work, we show the result in two steps. We first define values \(\alpha_u^t\) for each vertex. Let \(q_u^t := \alpha_p^t - \alpha_u^t \geq 0\) for each non-root vertex, and \(q_r^t := 0\). For a given vector \(q\), define \(||\cdot||_q^+ := \sum_u q_u w_u v_u^+\).

**Lemma 4.2.** There exists choices of \(\alpha_u^t\) and a potential function \(\Psi_1\) such that for each time \(t\),

\[ ||x^t - x^{t-1}\parallel_{q^+}^+ \leq O(\log(k/\delta)) \cdot \sum_j A_{r,j}^t + \Psi_1(z^t) - \Psi_1(z^{t-1}). \]

Observe that this lemma bounds \(||\cdot||_{q^+}^+\) rather than \(||\cdot||_w\), so we relate these two norms next:

**Lemma 4.3.** For the choice of \(\alpha_u^t\) from Lemma 4.2, there exist universal constants \(c, c'\) and another potential function \(\Psi_2\) such that for each time \(t\),

\[ ||x^t - x^{t-1}\parallel_w^+ \leq c \cdot \left( c' \cdot ||x^t - x^{t-1}\parallel_{q^+}^+ - 2[\Psi_2(z^t) - \Psi_2(z^{t-1})] \right). \]

Combining these two results and setting \(\Psi(z) = c \cdot c' \cdot \Psi_1(z) - 2c \cdot \Psi_2(z)\) immediately gives Theorem 4.1. The proofs of these two lemmas appear in the following sections.

4.1.1 Proof of Lemma 4.2

Recall \(x_u^t := \sum_j x_{u,j}^t\). Following the proof of (3.14), using (KKT2a-KKT2c) in conjunction with Lemma 3.8,

\[ \sum_u w_u q_u^t (x_{u,j}^t - x_{u,j}^{t-1})^+ \leq \sum_u w_u q_u^t \sum_j (x_{u,j}^t - x_{u,j}^{t-1})^+ \leq \sum_u q_u^t \sum_j A_{u,j}^t. \quad (4.17) \]

The non-negativity of Bregman divergences and the fact that \(\alpha_u^t \geq 0\) implies

\[ 0 \leq \sum_j \left( \frac{\tilde{x}_{u,j}^t}{x_{u,j}^{t-1}} - 1 \right) \log \left( \frac{\tilde{x}_{u,j}^t}{x_{u,j}^{t-1}} + 1 \right) \]

\[ \leq \sum_j (A_{u,j}^t - B_{u,j}^t) + \alpha_u^t w_u \sum_j (x_{u,j}^{t-1} - x_{u,j}^t) \]

\[ = \alpha_u^t \sum_j A_{u,j}^t - \alpha_u^t \sum_{(v,t) \in \chi_u} A_{v,t}^t + \alpha_u^t w_u (x_u^{t-1} - x_u^t). \quad (by\ (KKT2a)-(KKT2c)) \]

(by Lemma 3.5)
Summing over all $u$, using $x_u^t - x_u^{t-1} = \frac{z_u^{t-1} - z_u^t}{1-\delta}$, and rearranging,

$$\sum_{u \neq r} \sum_j A_{u,j}(\alpha_{p(u)}^t - \alpha_u^t) \leq \sum_j \alpha_j^t A_{r,j}^t + \frac{1}{1-\delta} \sum_u \alpha_u^t w_u(z_u^t - z_u^{t-1}).$$  \hfill (4.18)

Since $q_u^t = \alpha_{p(u)}^t - \alpha_u^t$, the left hand side of (4.18) equals the right hand side of (4.17). Therefore, to prove Lemma 4.2 we need to:

(i) choose $\alpha_u^t$ so that $\alpha_u^t \leq O(\log k/\delta)$ and $q_u^t \geq 0$; and

(ii) choose a potential $\Psi_1(\cdot)$ such that $\frac{1}{1-\delta} \sum u \alpha_u^t w_u(z_u^t - z_u^{t-1}) \leq \Psi_1(z^t) - \Psi_1(z^{t-1})$.

(Of course, we want this choice of $\alpha_u^t$ to allow us to prove Lemma 4.3 as well.)

To this end, we define

$$\alpha_u^t := \frac{1}{z_u^{t-1} - z_u^t} \int_{z_u^t}^{z_u^{t-1}} \ln \left(1 + \frac{z}{\delta}\right) dz.$$  \hfill (4.19)

Since $0 \leq z_u^t \leq k$, the value $\alpha_u^t \in [0, \ln(1 + \frac{\delta}{k})]$. Moreover, $\alpha_{p(u)}^t \geq \alpha_u^t$, since $\alpha_u^t$ is the average value of an increasing function between two endpoints, and the corresponding endpoints in $\alpha_{p(u)}^t$ are at least those in $\alpha_u^t$ (since $z_u^t \geq z_u^{t-1}$ and $\frac{z_u^t}{z_u^{t-1}} \geq z_u^t$). Thus, the first condition above is satisfied. Satisfying the potential condition is easy. Define

$$\Psi_1(z) = \sum_u \frac{w_u}{1 - \delta} \int_{z_u^t}^{z_u} \ln \left(1 + \frac{z}{\delta}\right) dz.$$  

Since

$$\alpha_u^t(z_u^t - z_u^{t-1}) = \int_{z_u^{t-1}}^{z_u^t} \ln \left(1 + \frac{z}{\delta}\right) dz,$$  

we immediately get

$$\Psi_1(z^{t-1}) + \sum_u \frac{w_u}{1 - \delta} \alpha_u^t(z_u^t - z_u^{t-1}) = \Psi_1(z^t),$$  

as we want. Putting these together proves Lemma 4.2.

4.1.2 Proof of Lemma 4.3

We now turn our attention to Lemma 4.3 and show the choice of $\alpha^t$ from (4.19) suffices. We consider a $\tau$-HST, for $\tau \leq 1/10$. We consider the potential function $\Psi_2(z) := \sum_u w_u(z_u - (2/3)z_{p(u)})^+$, and want to show

$$\frac{1}{c} \|z^{t-1} - z^t\|_c^+ \leq c' \cdot \|z^{t-1} - z^t\|_{q^+}^+ - 2 \left[\Psi_2(z^t) - \Psi_2(z^{t-1})\right].$$  \hfill (4.20)

Observation 4.4. For any $x, y$, $\langle x + y\rangle^+ \leq x^+ + y^+$.

The backbone (at time $t$) is the path from the root $r$ to the requested node $r_t$. By the definition of $z_u^t$ for internal nodes $u$, we can visualize the difference between $z_u^{t-1}$ and $z_u^t$ as a flow over $T$, where a total $1 - z_{rt}^{t-1}$ amount of flow is sent from the non-$r_t$ leaves to $r_t$. For each leaf $u \neq r_t$, there is a
flow path $P_u$, and $z_u^{t-1} - z_u^t$ flow is sent along this path, reducing the $z$-value at $u$ and increasing it at $r_t$; the $z$ values at internal nodes are obtained by summing over all leaves in their subtree. To track the difference between solutions $z^{t-1}$ and $z^t$, we introduce an intermediary solution $z'$ obtained by sending just the flows from leaves lying within a “$z^{t-1}$-light” subtree in the backbone.

More precisely, let $a$ be the highest node on the backbone, where

$$z^{t-1}(T_a) \leq \frac{1}{10}. \tag{4.21}$$

If $a = r_t$, or no such node exists, simply define $z' = z^{t-1}$. Else, consider the flow defined above, and let $z'$ be obtained by applying to $z^{t-1}$ all the flows on paths contained within $T_a$. Since all these flows stay within the subtree $T_a$, the value of its root $a$ remains unchanged—i.e., $z'_a = z_a^{t-1}$.

It is technically simpler to track the change from the intermediate solution $z'$ to $z^t$, instead of tracking it from $z^{t-1}$ to $z'$. By the next lemma, the $\|\cdot\|_w^+$ movement from $z^{t-1}$ to $z^t$ can be bounded in terms of the movement starting from the intermediate solution $z'$.

**Lemma 4.5.** There is a constant $c$ such that

$$\|z^{t-1} - z^t\|_w^+ \leq c \cdot (\|z' - z^t\|_w^+ - 2 [\Psi_2(z') - \Psi_2(z^{t-1})]). \tag{4.22}$$

**Proof.** If $z' = z^{t-1}$ the claim is vacuous, so assume the backbone node $a$ exists. By Observation 4.4,

$$\|z^{t-1} - z^t\|_w^+ \leq \|z^{t-1} - z'\|_w^+ + \|z' - z^t\|_w^+. \tag{4.23}$$

We first claim that $\|z' - z^t\|_w^+ \geq (9/10)w_a$, i.e., the remaining flow is large after we apply the flow paths contained in $T_a$. To see this, notice $z_{r_t}^t \leq z_a' = z_a^{t-1} \leq \frac{1}{10}$, and by feasibility of $z^t$ we have $z_{r_t}^t = 1$. Thus, to move from solution $z'$ to $z^t$ at least $9/10$ units of flow need to be sent into $r_t$. Moreover, since all this flow comes from leaves outside $T_a$, each unit of this flow pays at least $w_a$ when it “enters the backbone”.

Next, we claim $\|z^{t-1} - z'\|_w^+ \leq (2/90)w_a$. Using Observation 4.4,

$$\|z^{t-1} - z'\|_w^+ = \sum_{u \in T_a \setminus a} w_u(z_u^{t-1} - z_u')^+ \leq \sum_{u \in T_a \setminus a} w_u(z_u^{t-1} + z_u')^+ \leq \sum_{u \in T_a \setminus a} w_u(z_u^{t-1} + z_u') = w_a \sum_{\ell \geq 1} \tau^\ell \left[ \sum_{u \text{level } \ell \text{ of } T_a} z_u^{t-1} + \sum_{u \text{level } \ell \text{ of } T_a} z_u' \right] = w_a \sum_{\ell \geq 1} \tau^\ell (z_a^{t-1} + z_a') \leq (2/90)w_a, \tag{4.24}$$

where the last inequality uses the fact that we have a $\tau$-HST with $\tau \leq 1/10$, and that by the definition of $a$ we have $z_a^{t-1} = z_a' \leq 1/10$. By the previous paragraph, we now get $\|z^{t-1} - z^t\|_w^+ \leq \frac{9}{80}\|z' - z^t\|_w^+$. Finally, we claim $\Psi_2(z') - \Psi_2(z^{t-1}) \leq (4/90)w_a$ (which is $\leq \frac{4}{80}\|z' - z^t\|_w^+$): the subadditivity of $(\cdot)^+$
implies subadditivity of $\Psi_2$, hence (let $y := Z' - Z_t^{-1}$)
\[
\Psi_2(Z') - \Psi_2(Z_t^{-1}) \leq \Psi_2(Z' - Z_t^{-1}) = \sum_{u \in T_a \setminus a} w_u (y_u - (2/3)y_{p(u)})^+
\]
\[
\leq \sum_{u \in T_a \setminus a} w_u \left( y_u^+ + (2/3)y_{p(u)}^+ \right) \leq 2 \sum_{u \in T_a \setminus a} w_u y_u^+,
\]
where the last inequality uses the fact $y_a = 0$. Again by Observation 4.4 $y_u^+ \leq Z_u' + Z_u^{t-1}$, so part of inequality (4.23) gives $\sum_{u \in T_a \setminus a} w_u y_u^+ \leq (2/90) w_a$. This proves the claim. Moreover,
\[
RHS(4.21) = c \cdot (\|Z' - Z_t\|_w^+ - 2 \left[ \Psi_2(Z') - \Psi_2(Z_t^{-1}) \right]) \geq c \cdot (73/81)\|Z' - Z_t\|_w^+.
\]
By (4.22) and the consequence of (4.23), $LHS(4.21) = \|Z_t^{-1} - Z_t\|_w^+ \leq (1 + 2/81)\|Z' - Z_t\|_w^+$. Now we have $LHS \leq RHS$ for $c \geq 73/81$, concluding the proof. \hfill \Box

Next we track the changes from the intermediate solution $Z'$ to $Z_t$. Since the value of $q^t$ is changing during this process, we need some notation to track it carefully. Given two solutions $Z$ and $Z'$, we define
\[
q(Z', Z) := \alpha(Z'_{p(u)}, Z_{p(u)}) - \alpha(Z_u', Z_u),
\]
where
\[
\alpha(Z', Z) := \frac{1}{Z' - Z} \int_{Z}^{Z'} \ln(1 + x/\delta)dx.
\]
Notice that $q_u^t = q(Z_t^{-1}, Z_t^t)$. The main technical part of this section will be to prove the following.

**Lemma 4.6.** The following holds:
\[
\|Z' - Z_t^t\|_w^+ \leq c \cdot \|Z' - Z_t\|_{q(Z', Z)}^+ - 2 \left[ \Psi_2(Z_t^t) - \Psi_2(Z') \right].
\] (4.24)

Before we prove Lemma 4.6, let us prove the main result using the above lemmas.

**Proof of Lemma 4.3.** We prove inequality (4.20). Putting Lemmas 4.5 and 4.6 together we get
\[
\frac{1}{c} \|Z_t^{-1} - Z_t^t\|_w^+ \leq c' \cdot \|Z' - Z_t\|_{q(Z', Z)}^+ - 2 \left[ \Psi_2(Z_t^t) - \Psi_2(Z') \right].
\]
This is almost what we wanted to prove, except that we have $\|Z' - Z_t\|_{q(Z', Z)}^+$ instead of $\|Z_t^{-1} - Z_t\|_{q(Z_t^{-1}, Z_t^t)}^+$ on the RHS. But this is easily handled. First we change the $q$ and claim that $\|Z' - Z_t\|_{q(Z', Z)}^+ = \|Z' - Z_t\|_{q(Z_t^{-1}, Z_t^t)}^+$. Indeed, $(Z_u - Z_u^t)^+ > 0$ only for nodes $u \not\in T_a$; since $Z_u = Z_u^{t-1}$ for those nodes, we immediately get
\[
q(Z', Z_t^t) = \alpha(Z'_{p(u)}, Z_{p(u)}) - \alpha(Z_u^t, Z_u) = \alpha(Z_u^{t-1}, Z_{p(u)}) - \alpha(Z_u^{t-1}, Z_u^t) = q(Z_u^{t-1}, Z_t^t).
\]
Finally, using the fact that we send flows from other leaves to the requested node $r_t$, the solutions
satisfy $z_u^t \leq z'_u \leq z_u^{t-1}$ for all nodes outside the backbone, we have
\[
\|z' - z\|^+_w = \sum_{u \notin \text{backbone}} w_u \cdot (z_u^{t-1}, z'_u) \cdot (z'_u - z_u^{t})^+ \\
\leq \sum_{u \notin \text{backbone}} w_u \cdot (z_u^{t-1}, z'_u) \cdot (z_u^{t-1} - z_u^{t})^+ = \|z_u^{t-1} - z'_u\|^+_w.
\]

Putting these together concludes the proof. \qed

4.1.3 Proof of Lemma 4.6

In this section we give the proof of Lemma 4.6. Given a path $P$ from any leaf in the tree to $r_t$, we denote by $\text{top}(P)$ its topmost vertex, which is always on the backbone. The proof takes the residual flow $z' - z$, decomposes it as “small” flows on paths from leaves outside $T_a$ to the request node $r_t$, discharging them iteratively. We order these paths $P$ so that their vertex $\text{top}(P)$ becomes higher on the backbone over time, in order to control the change in value of the $q$’s. (Observe that by pushing $\varepsilon$ flow over a path $P = u \sim \text{top}(P) \sim r_t$ for a leaf $u \neq r_t$ causes the $z$-value of all but the last node in the subpath $u \sim \text{top}(P)$ to drop by $\varepsilon$, and the $z$-value of all but the first node in the subpath $\text{top}(P) \sim r_t$ to increase by $\varepsilon$.)

Since several flow paths may have the same $\text{top}(\cdot)$ node, and the $q$ function changes with each flow discharge, we handle this carefully. Consider a sequence of flow paths that all have $\text{top}(\cdot) = \ell$ that transform $z \rightarrow \ldots \rightarrow z' \rightarrow z'' \rightarrow \ldots \rightarrow z'''$, and we are presently concerned with a particular flow path $P$ that takes us from $z'$ to $z''$. We say that an edge $(u, v)$ is heavy for $z$ if $z_u > (2/3)z_{p(u)}$, i.e., it contributes to the potential $\Psi_2$. (If we have equality $z_u = (2/3)z_{p(u)}$, we may or may not consider this edge heavy).

Lemma 4.7 (One Path). Consider LP solutions $z, z', z'', z'''$ and a backbone node $\ell$ (with a child $v'$ on the backbone, and some child $v$ outside the backbone) such that:

(i) $z_\ell \geq \frac{1}{17}$,
(ii) $z_\ell = z'_\ell = z''_\ell = z'''_\ell$ and $z'''_\ell \leq z'_\ell \leq z''_\ell \leq z_v$,
(iii) $z'''$ is obtained from $z'$ by pushing $\varepsilon$ flow over a leaf-to-$r_t$ path $P$ that has $\text{top}(P) = \ell$ and that passes through $v$ and $v'$, and
(iv) the heavy edges in $z'$ and $z''$ are the same, where edges with $z_u = (2/3)z_{p(u)}$ may be considered heavy or not, as needed.

Then
\[
\|z' - z''\|^+_w \leq c \cdot \|z' - z''\|^+_w - 2 \left[ \Psi_2(z'') - \Psi_2(z') \right]. \tag{4.25}
\]

Proof. Observe that the left hand side of (4.25) is at most $w_v \varepsilon (1 + \tau + \tau^2 + \ldots) = \frac{w_v \varepsilon}{1 - \tau}$. To bound the right hand side from below, we consider two cases, based on which of the top two edges in $P$ may be heavy.

Case 1: $z'_v \geq \frac{2}{3} z'_\ell$. Hence the edge $(\ell, v)$ is heavy. Since the first term on the right hand side is non-negative, it is at least $-2 \left[ \Psi_2(z'') - \Psi_2(z') \right] = 2 \sum_{(v,u)} [\Psi_2(z'')_u - \Psi_2(z')_u]$, where $\Psi_2(y)_u = w_u(y_u - (2/3)y_{p(u)})^+$. Let $P_+$ and $P_-$ be the subpaths of $P$ from $r_t$ to $\text{top}(P)$, and down from there, such that $z$ increases on the former and decreases on the latter, as we push flow.
Now \(-|\Psi_2(\bar{z}'')_v - \Psi_2(\bar{z}')_v| = w_v \varepsilon\); this potential change for vertices \(u\) on \(P_+\) (apart from vertex \(v'\)) is \(-w_u \varepsilon / 3\), and this change for vertices whose parents lie on \(P_-\) is \(-2w_u \varepsilon / 3\). Since all these vertices \(u\) are at a greater depth than \(v\) is, using the HST property means that the total change in potential is at least \(2w_u \varepsilon (1 - \tau - \tau^2 - \ldots) = 2w_u \varepsilon \frac{1}{1 - \tau^2} \). Since \(\tau \leq 1/10\), the right hand side is at least \(w_u \varepsilon / 1 - \tau\) and hence the left hand side in this case.

**Case 2:** \(\bar{z}'_v \leq \frac{2}{5} \bar{z}'_v\). In this case the edge \((\ell, v)\) is not heavy, but \((\ell, v')\) may be heavy. Hence the expression \(-[\Psi_2(\bar{z}')_v - \bar{z}']_u\) is at least \(-w_u \varepsilon\) for \(v'\). Again the same calculations as in the previous case imply \(-2[\Psi_2(\bar{z}'') - \Psi_2(\bar{z}')] \geq -2 \frac{w_v}{\tau} \). Hence, for the right hand side to be larger than the left hand side, we need to take the first term on the right hand side into account. This is \(c \cdot \|\bar{z}' - \bar{z}''\|_{q(\bar{z}, \bar{z}'')}^+ \geq c \cdot w_v \cdot q(\bar{z}, \bar{z}'') \cdot (\bar{z}' - \bar{z}'')^+ = c \cdot w_v \cdot q(\bar{z}, \bar{z}'') \cdot \varepsilon\). For this to be at least \(\frac{2}{5} w_v \varepsilon\), it suffices to show \(q(\bar{z}, \bar{z}'')\) is at least a constant, since we can then set \(c\) large enough.

To simplify the notation, let \(f(x) = \ln(1 + x/\delta)\). Using property (ii), \(\bar{z}_\ell = \bar{z}''\) and so we have

\[
q(\bar{z}, \bar{z}'')_v = \alpha(\bar{z}_\ell, \bar{z}''_\ell) - \alpha(\bar{z}_v, \bar{z}''_v) = f(\bar{z}_\ell) - \alpha(\bar{z}_v, \bar{z}''_v),
\]

and we now want to upper-bound the last term. Using again property (ii), \(\bar{z}''_v \leq \bar{z}_v \leq \bar{z}''_v \leq \bar{z}_v\), so

\[
\alpha(\bar{z}_v, \bar{z}''_v) = \alpha(\bar{z}''_v, \bar{z}_v) \leq \alpha(\bar{z}_v, \bar{z}_v).
\]

Moreover, in the current case \(2, \bar{z}_v' \leq \frac{2}{5} \bar{z}_v'\) and \(\bar{z}_v \leq \bar{z}_\ell\), so \(\alpha(\bar{z}_v, \bar{z}_\ell) \leq \alpha(\bar{z}_v, \bar{z}_v)\). Now letting \(X\) be a random variable uniformly distributed in \([1/3 \bar{z}_\ell, \bar{z}_\ell]\) (with median \(\text{med}(X) = \frac{1}{3} \bar{z}_\ell\)) we have

\[
\alpha((2/3) \bar{z}_\ell, \bar{z}_\ell) = \mathbb{E} f(X) = \frac{1}{2} \mathbb{E} [f(X) \mid X \leq \text{med}(X)] + \frac{1}{2} \mathbb{E} [f(X) \mid X \geq \text{med}(X)]
\]

\[
\leq \frac{1}{2} f(\text{med}(X)) + \frac{1}{2} f(\bar{z}_\ell) = \frac{1}{2} f((5/6) \bar{z}_\ell) + \frac{1}{2} f(\bar{z}_\ell).
\]

Putting everything together and applying it to (4.26) we get

\[
q(\bar{z}, \bar{z}'')_v \geq f(\bar{z}_\ell) - (1/2 f((5/6) \bar{z}_\ell) + 1/2 f(\bar{z}_\ell)) = 1/2 \left( f(\bar{z}_\ell) - f((5/6) \bar{z}_\ell) \right) = 1/2 \left( \ln \left( \frac{1 + \bar{z}_\ell/\delta}{1 + (5/6) \bar{z}_\ell/\delta} \right) \right).
\]

This final expression is at least a constant, because \(\bar{z}_\ell \geq 1/10\) by property (i). This concludes the proof.

Using Lemma 4.7 repeatedly, we can understand the process of going from \(\bar{z} \rightarrow \ldots \rightarrow \bar{z}'\) via pushing flow on multiple paths, as long as all these paths share the same top node.

**Lemma 4.8 (All Paths with top(\(P\)) = \(\ell\)). Consider LP solutions \(\bar{z}, \bar{z}'\) and a node \(\ell\) on the backbone such that: (i) \(\bar{z}_\ell \geq 1/\ell\), and (ii) \(\bar{z}'\) is obtained from \(\bar{z}\) by pushing flow along multiple paths with top(\(\cdot\)) = \(\ell\). Then

\[
\|\bar{z} - \bar{z}'\|_w^+ \leq c \cdot \|\bar{z} - \bar{z}'\|_{q(\bar{z}, \bar{z}'')}^+ - 2 \left[ \Psi_2(\bar{z}') - \Psi_2(\bar{z}) \right].
\]

**Proof.** Consider a sequence of solutions \(\bar{z} = \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_m = \bar{z}'\), where \(\bar{z}_{i+1}\) is obtained from \(\bar{z}_i\) by pushing flow over a leaf-to-\(\tau\_i\) path \(P_i\) with \(\text{top}(P_i) = \ell\); we construct this sequence of flows such that the set of heavy edges with respect to each consecutive pair \(z_i\) and \(z_{i+1}\) is the same, for all \(i\).

Applying Lemma 4.7 with \(\bar{z} = \bar{z}_0, \bar{z}' = \bar{z}_i, \bar{z}'' = \bar{z}_{i+1}\), and \(\bar{z}''' = \bar{z}_m\), we get

\[
\|\bar{z}_i - \bar{z}_{i+1}\|_w^+ \leq c \cdot \|\bar{z}_i - \bar{z}_{i+1}\|_{q(\bar{z}_0, \bar{z}_m)}^+ - 2 \left[ \Psi_2(\bar{z}_{i+1}) - \Psi_2(\bar{z}_i) \right].
\]
Summing this up over all indices \( i \), and observing that \( \sum_i \| z_i - z_{i+1} \|_w^+ = \| z_0 - z_m \|_w^+ \) (since for each vertex \( u \), every flow either increases its value or decreases it), concludes the proof. \( \square \)

Now we can finally give the proof of Lemma 4.6.

**Proof of Lemma 4.6.** Recall the node \( a \) was defined to be the highest node on the backbone with \( z^{i-1} \)-value at most \( 1/10 \). Let \( a, v_1, v_2, \ldots, v_m \) be the vertices on the backbone starting from \( a \), going from bottom to top. In order to change the solution \( z' \) into \( z' \), push the remaining flow on paths “from bottom to top”, i.e., those with \( \text{top}(\cdot) = v_1 \) first, then those with \( \text{top}(\cdot) = v_2 \) next, etc., to get a sequence of solutions \( z' = z_0 \to z_1 \to \ldots \to z_m = z' \).

Since all these remaining flows have \( \text{top}(\cdot) \) higher than node \( a \), we have \( z_{\text{top}(\cdot)} \geq 1/10 \) at all times during this process, so Lemma 4.8 applied to each choice of \( \text{top}(\cdot) \) gives

\[
\sum_i \| z_i - z_{i+1} \|_w^+ \leq c \sum_i \| z_i - z_{i+1} \|_{q(z_i, z_{i+1})}^+ - 2 [\Psi_2(z_m) - \Psi_2(z_0)]. \tag{4.27}
\]

Again each intermediate step causes the \( z \)-value on the backbone to rise, and off the backbone to fall: this monotonicity means the left hand side equals \( \| z_0 - z_m \|_w^+ \).

Now we claim that

\[
\| z_i - z_{i+1} \|_{q(z_i, z_{i+1})}^+ \leq \| z_i - z_{i+1} \|_{q(z_0, z_m)}^+.
\]

Since \( (z_i)_u \geq (z_{i+1})_u \) only in the subforest \( T_{v_i} \setminus \{v_i\} \cup T_{v_{i-1}} \), it suffices to show that \( q(z_i, z_{i+1})_u \leq q(z_0, z_m)_u \) for nodes \( u \) in this set. In fact, the flows that convert \( z_i \) to \( z_{i+1} \) are the only ones that change the \( z \)-value of these nodes, so we have \( (z_0)_u = (z_i)_u \) and \( (z_{i+1})_u = (z_m)_u \). Now since the \( q \)-value at node \( u \) depends on the \( z \)-values at the node and its parent, we have equality \( q(z_i, z_{i+1})_u = q(z_0, z_m)_u \) for all nodes in \( T_{v_i} \setminus \{v_i\} \cup T_{v_{i-1}} \), except perhaps when \( u \) is a child of \( v_i \). For \( u \) being a child of \( v_i \) (and not on the backbone), we have

\[
q(z_i, z_{i+1})_u = \alpha((z_i)_v, (z_{i+1})_v) - \alpha((z_i)_0, (z_{i+1})_0) \\
= \alpha((z_i)_v, (z_{i+1})_v) - \alpha((z_0)_v, (z_m)_v) \\
= \alpha((z_0)_v, (z_{i+1})_v) - \alpha((z_0)_v, (z_m)_v) \quad \text{(since \( (z_0)_v = (z_i)_v \))} \\
\leq \alpha((z_0)_v, (z_m)_v) - \alpha((z_0)_v, (z_m)_v) = q(z_0, z_m)_u,
\]

where the inequality uses the fact that \( (z_m)_v \geq (z_{i+1})_v \geq (z_0)_v \). This proves the claim.

Thus, the right hand side of (4.27) is at most

\[
c \cdot \sum_i \| z_i - z_{i+1} \|_{q(z_0, z_m)}^+ - 2 [\Psi_2(z_m) - \Psi_2(z_0)].
\]

Again, since there is no cancellation in the flows, the sum equals \( \| z_0 - z_m \|_{q(z_0, z_m)}^+ \). This concludes the proof. \( \square \)

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A  The Weighted \((h, k)\)-Paging Problem

In this section we consider the weighted \((h, k)\)-paging problem, where we maintain a fractional solution with \(k\) pages, and compare to an optimal solution that maintains \(h \leq k\) pages. The goal is to respond to requests \(r_t \in [n]\) at each time \(t\) by producing a vector \(z_t \in \{0, 1\}^n\) with \(\|z_t\|_1 = k\), such that \(z_{rt} = 1\) for all \(t\). Define the \(\ell_1\) norm for \(x \in \mathbb{R}^n\) by

\[
\|x\|_{\ell_1(w)} := \sum_i w_i |x_i|.
\]

The objective is to minimize the total weighted movement cost:

\[
\sum_t \|z_t - z_{t-1}\|_{\ell_1(w)}.
\]

This problem is equivalent to the \((h, k)\)-server problem on a weighted star metric.

A.1  Polytopes and Solutions

As is common for fractional paging, consider the anti-paging polytope

\[
P := \{ x \in [0, 1]^n \mid \sum_i x_i \geq n - h \}.
\]

For \(\delta := \frac{k-h+1/2}{k+1/2}\), define the shifted polytope

\[
P_\delta := \{ x \in [\delta, 1]^n \mid \sum_i x_i \geq n - h \}.
\]

We maintain the following invariant:

Invariant A.1. The algorithm’s solutions are fractional vectors \(x_t \in P_\delta\) with \(\|x_t\|_1 = n - h\), and the optimal solutions are Boolean vectors \(y_t \in P \cap \{0, 1\}^n\), again with \(\|y_t\|_1 = n - h\). Moreover, for \(t \geq 1\), \(x_{rt} = \delta\) and \(y_{rt} = 0\).

At the beginning, if the optimal servers are at some set \(B_0 \subseteq [n]\) with \(|B_0| = h\), define \(y^0_i = 1_{i \notin B_0}\). If the algorithm’s servers are initially at \(A^0\) with \(|A^0| = k\), define \(x^0_i = \delta\) for all \(i \in A^0\), and \(x^0_i = \frac{n-h-\delta k}{n-k}\) for \(i \notin A^0\). It is easy to verify that \(x^0, y^0\) satisfy the invariant.

Interpreting the Boolean vector \(y^t\) is simple: the adversary has servers exactly at the \(h\) locations \(i \in [n]\) where \(y^t_i = 0\), i.e., its paging solution is given by \(1 - y^t\). On the other hand, converting the algorithm’s fractional solution \(x^t \in P_\delta\) from the shifted anti-paging polytope to a fractional paging solution \(z^t\) requires handling this shift: define \(z^t_i := \frac{1-x^t_i}{1-\delta}\) for each \(i \in [n]\). This new fractional solution \(z^t\) has \(z^t_{rt} = 1\) since \(x^t_{rt} = \delta\), and it uses

\[
\|z^t\|_1 = \frac{n - \|x^t\|_1}{1-\delta} = \frac{n - (n-h)}{h/(k+1/2)} = k + 1/2
\]

servers. It satisfies \(\|z^t - z^{t-1}\|_{\ell_1(w)} = \|x^t - x^{t-1}\|_{\ell_1(w)} \cdot 1/(1-\delta)\).
We finally use the following result that combines [BCL+17, Lemma 3.4] with [BBMN11, §5.2] to round fractional solutions $z^t$ to integer ones:

**Theorem A.2** (Rounding Theorem). There exists an absolute constant $C > 1$ and an efficient randomized algorithm that takes a sequence of fractional solutions $z^t = \frac{1-x^t}{h}$ to the weighted paging problem, each with $\|z^t\|_1 = k + 1/2$ pages and with $z^t_{r_t} = 1$, and rounds them to integer solutions $\hat{z}^t$ each with $\|\hat{z}^t\| = k$ pages and $\hat{z}^t_{r_t} = 1$, so that the expected movement cost is

$$\mathbb{E}\left[\|\hat{z}^t - \hat{z}^{t-1}\|_{\ell_1(w)}\right] \leq C \|z^t - z^{t-1}\|_{\ell_1(w)} \leq \frac{C}{1 - \delta} \|x^t - x^{t-1}\|_{\ell_1(w)}.$$

Henceforth, we only consider the problem of maintaining the fractional solution $x^t \in P_\delta$. Our main theorem for computing fractional solutions for weighted paging is the following:

**Theorem A.3** (Main Theorem: Weighted $(h,k)$-Paging). There is an algorithm that maintains a sequence of fractional solutions $x^t \in P_\delta$ with $x^t_{r_t} = \delta$ and $\|x^t\|_1 = n - h$, such that

$$\|x^t - x^{t-1}\|_{\ell_1(w)} \leq (\log 1/\delta) \cdot \|y^t - y^{t-1}\|_{\ell_1(w)} + C',$$

for any sequence of feasible solutions $y^t$ where $y^t \in P \cap \{0,1\}^n$ and $y^t_{r_t} = 0$. Here $C'$ is a constant that depends on the weights $w_i$ and the values of $k$ and $h$, but is independent of the request sequence.

Combining Theorems A.2 and A.3, and using our choice of $\delta$, the competitive factor of our randomized algorithm is $O(\log \frac{k + 1/2}{h - k/2})$ for $h = \Omega(k)$. Note that for $h = k$ the algorithm is $O(\log k)$-competitive, and for $h = k/2$ the algorithm is $O(1)$-competitive. In the rest of this section, we prove Theorem A.3.

### A.2 The Projection Algorithm

Given a request at location $r_t$, define the body $P_t = P \cap \{x_{r_t} \leq \delta\}$. Project the old point $x^{t-1} \in P_\delta$ onto the new body $P_t$ using a weighted form of the unnormalized KL divergence:

$$D(x \| x') := \sum_i w_i \left( x_i \log \frac{x_i}{x'_i} - x_i + x'_i \right)$$

I.e., set $x^t := \arg\min_{x \in P_t} D(x \| x^{t-1})$. That’s the entire algorithm.

Since $P_t$ is not contained within $P_\delta$, we must prove that the new point $x^t$ lies in the polytope $P_\delta$.

**Lemma A.4.** For each $t$, the solution $x^t$ satisfies Invariant A.1.

**Proof.** We assume that $x^{t-1}$ satisfies the invariant, and then prove it for $x^t$. As the base case, the invariant holds for $x^0$. Denote $x := x^{t-1}$ and $x' := x^t$, to avoid visual clutter. The projection operation that defines $x'$ can be written as follows:

$$\min \sum_i w_i \left( x_i \log \frac{x_i}{x'_i} - x_i + x'_i \right) \sum_i x_i \geq n - h$$

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The KKT optimality conditions give us dual multipliers $\lambda, \gamma_i \geq 0$ that for all $i$,

$$w_i \log \frac{x_i'}{x_i} = \lambda - \gamma_i.$$  \hspace{1cm} (A.28)

Moreover, by complementary slackness, $\lambda > 0$ implies $\sum_i x_i' = n - h$, and $\gamma_i > 0$ implies $x_i' = 1$ for $i \neq r_t$, and that $x_i' = \delta$ for $i = r_t$.

If $x$ already belongs to $P_t$ then the projection returns $x' = x$, in which case the proof trivially follows. Hence assume that $x_{r_t} > \delta$. It now follows that:

(a) $x_{r_t}' = \delta$. Indeed, observe that $x_{r_t}' \leq \delta < x_{r_t}$, and by (A.28) this decrease can only come about if $\gamma_{r_t} > 0$. Now complementary slackness for $\gamma_{r_t}$ implies $x_{r_t}' = \delta$.

(b) All other coordinates rise, i.e., $x_i' \geq x_i$ for all $i \neq r_t$. Indeed, if $\gamma_i > 0$, we have $x_i' = 1 \geq x_i$, else $x_i' = x_i e^\lambda \geq x_i$ since $\lambda \geq 0$. In particular, if we start off with $x_i \geq \delta$, the final solution also satisfies $x_i' \geq \delta$.

(c) $\|x'\|_1 = n - h$. Indeed, inductively assume $\|x\|_1 = n - h$. Then either $x_{r_t} = \delta$ already, and hence there is no change, so $x' = x$. Else setting $x_{r_t}' \leftarrow \delta$ means we must increase some coordinates, so $\lambda > 0$ and hence $\|x'\|_1 = n - h$.

Since $x_{r_t}' = \delta$ from Lemma A.4, let us give an equivalent view of our algorithm. Define the auxiliary Bregman divergence

$$D(x \parallel x')^{wo} := \sum_{i \neq r_t} w_i \left( x_i \log \frac{x_i}{x_i'} - x_i + x_i' \right).$$ \hspace{1cm} (A.29)

Our algorithm is equivalent to first setting $x_{r_t}' = \delta$, and then projecting the rest of the coordinates of $x^{t-1}$ onto $P_t$ by $\min_{x \in P_t} D(x \parallel x^{t-1})^{wo}$.

A.3 Bounding the Movement Cost

**The Potential.** We use a KL-divergence-type potential to measure the “distance” from the optimal solution $y^t \in P$ to $x^t$:

$$\Phi(y^t \parallel x^t) := \sum_i w_i y_i^t \log \frac{y_i^t}{x_i^t} = \sum_{i:y_i^t=1} w_i \log \frac{1}{x_i^t}.$$ \hspace{1cm} (A.30)

Observe that $\Phi \geq 0$ as long as $y \in \{0,1\}^n$ and $x \in (0,1]^n$. Again, $D(y^t \parallel x^t)$ could be used as the potential, but removing the linear terms makes the arguments cleaner. Recall that this potential has already been used in a potential function proof by Bansal et al. [BBN10]; it arises naturally given our algorithm.

**When OPT moves:** Say OPT pays for moving pages out of the cache, i.e., for increasing $y$ from 0 to 1, it pays $w_i$. In this case the potential increases by at most $w_i \log(1/\delta)$, because the
A sequence of numbers $x_i^{t-1} \geq \delta$. This gives us:

$$\Delta \Phi^{OPT}_t := \Phi(y^t \| x^{t-1}) - \Phi(y^{t-1} \| x^{t-1}) \leq \log(1/\delta) \cdot \sum_i w_i (y^t_i - y^{t-1}_i)^+. \quad (A.31)$$

**When ALG moves:** Recall the auxiliary Bregman divergence $(A.29)$. The equivalent view of the algorithm (discussed above) shows

$$x^t \in \arg \min_{x \in P_t} D(x \| x^{t-1})_{\delta o}.$$  

Hence $x^t$ is a projection of $x^{t-1}$ onto $P_t$ with respect to a Bregman divergence $D(\| \|)^{\delta o}$. Since the optimal solution $y^t$ lies in $P_t = P \cap \{x_{r_t} \leq \delta\}$, the reverse-Pythagorean property gives

$$D(y^t \| x^{t-1})_{\delta o} \geq D(y^t \| x^t)_{\delta o}^+ + D(x^t \| x^{t-1})_{\delta o}^+$$

$$\Rightarrow \Phi(x^t \| x^{t-1})_{\delta o}^+ + (\Phi(y^t \| x^t) - \Phi(y^t \| x^{t-1})) \leq 0. \quad (A.32)$$

Here, the “without-$r_t$” potential is defined much as you would expect: $\Phi(x \| x')_{\delta o} := \sum_{i \neq r_t} w_i x_i \log \frac{x_i}{x'_i}$. The second line in $(A.32)$ follows from the first by using the definition $(A.29)$ and canceling linear terms on both sides. Moreover, $\Phi(y^t \| \cdot)_{\delta o} = \Phi(y^t \| \cdot)$ since $y^t_{r_t} = 0$.

Since all coordinates except for $r_t$ increase, we get

$$\sum_i w_i (x^t_i - x^{t-1}_i)^+ = \sum_{i \neq r_t} w_i (x^t_i - x^{t-1}_i) \leq \sum_{i \neq r_t} w_i x^t_i \log \frac{x^t_i}{x^{t-1}_i} = \Phi(x^t \| x^{t-1})_{\delta o}. \quad (A.33)$$

**A.3.1 Wrapping Up**

Combining $(A.31)$, $(A.32)$ and $(A.33)$ gives us

$$\sum_i w_i (x^t_i - x^{t-1}_i)^+ + (\Phi(y^t \| x^t) - \Phi(y^{t-1} \| x^{t-1})) \leq (\log 1/\delta) \cdot \sum_i w_i (y^t_i - y^{t-1}_i)^+.$$

Summing up over all times $t$, and using the property that the final potential is non-negative, we have

$$\sum_t \sum_i w_i (x^t_i - x^{t-1}_i)^+ \leq (\log 1/\delta) \cdot \sum_t \sum_i w_i (y^t_i - y^{t-1}_i)^+ + \Phi(y^0 \| x^0). \quad (A.34)$$

We would like to translate the cost in terms of the weighted $\ell_1$ metric. For this, observe that for any sequence of numbers $p_0, p_1, \ldots, p_T \in [0, M]$, we have

$$\sum_t (p_t - p_{t-1})^+ \leq \sum_t |p_t - p_{t-1}| \leq 2 \sum_t (p_t - p_{t-1})^+ + M.$$
Applying this to each term in the summations from (A.34), we get
\[
\sum_t \|x_t^i - x_{t-1}^i\|_{\ell_1(w)} \leq 2 \log(1/\delta) \cdot \sum_t \|y_t^i - y_{t-1}^i\|_{\ell_1(w)} + O\left(\sum_i w_i\right) + \Phi(y^0 \parallel x^0).
\]
Observe that the last term is at most
\[
\sum_{i \in A \setminus B} w_i \log(1/\delta) \leq \sum_{i \in [n]} w_i \log(1/\delta),
\]
and hence setting \(C' = O(\sum_{i \in [n]} w_i \log(1/\delta))\) completes the proof of Theorem A.3. \qed

An aside: while the above proof for paging proceeded via the \(D(\parallel \cdot \parallel_{wo})\) divergence, it could have instead followed the arguments in §3.4.2 using Lemma 3.7, which would give very similar results.

B Proofs from Section 3

B.1 Omitted Proofs from Section 3.3

We now give proofs of properties we claimed in §3.3, as well as some supporting claims.

Claim 3.2 (Root is Tight). For the root vertex \(r\), \(x_{r,j}^t = 1_{(j > h)}\).

Proof. Constraint (3.4) yields \(x_{r,j}^t \geq 1_{(j > h)}\), so it remains to prove this is an equality. The previous solution \(x_{t-1}^j\) satisfies the equality (by induction on \(t\)), and (KKT2b) implies that for \(x_{r,j}^t > x_{r,j}^{t-1}\) we must have \(\lambda_{r,j} > 0\). But then complementary slackness (3.8) implies \(x_{r,j}^t = 1_{(j > h)}\). \qed

Claim 3.3 (Box Constraints). For all \(u, j\), \(0 \leq x_{u,j}^t \leq 1\). Moreover, \(x_{r,t,1}^t = \delta\). Finally, \(x_{u,1}^t \geq \delta\) for leaves \(u\), and hence \(x^t \in P_h\).

Proof. The lower bound is by induction on the levels of the tree. For the base case, the root satisfies \(x_{r,j}^t \geq 0\). For \(x_{u,j}^t\) with \(u\) at depth \(d\), its parent \(p(u)\) satisfies \(x_{p(u),1}^t \geq 0\) by the induction hypothesis. Now constraint (3.5) for \(S = \{(u, j)\}\) completes the inductive step.

For the upper bound, suppose it does not hold. Then, choose the highest node \(u\) in the tree for which some \(x_{u,j}^t > 1\). Since \(x_{u,j}^t \leq 1\) (by induction), this coordinate of \(x^t\) has increased. Hence \(a_{u,j} > 0\), which means by (KKT2a) there exists some \(S \subseteq \chi_{p(u)}\) such that \((u, j) \in S\) for which \(\lambda_S > 0\). Now (CS2) implies that \(x^t(S) = \sum_{\ell \leq |S|} x^t_{p(u), \ell}\). Thus
\[
x^t(S - \{(u, j)\}) = x^t(S) - x_{u,j}^t < \sum_{\ell \leq |S|} x_{p(u), \ell} - 1 \leq \sum_{\ell \leq |S - \{(u, j)\}|} x_{p(u), \ell}.
\]
The strict inequality uses that \(x_{u,j}^t > 1\), and the last inequality uses the fact that \(x_{p(u), \ell} \leq 1\) by our choice of \(u\). But this violates constraint (3.5) of the convex program, yielding a contradiction.

Secondly, \(x_{r,t,1}^t \geq \delta\) (by induction \(x_{t-1}^t \in P_\delta\)), so if \(x_{r,t,1}^t < \delta\) then its value has decreased. Then by (KKT2c) we must have \(\gamma_t > 0\), which would imply \(x_{r,t,1}^t = \delta\) by complementary slackness (3.9), a contradiction.

For the last claim, for all non-\(r_t\) leaves, (KKT2a) has no \(b_{u,j}\) terms. Thus \(x_{u,j}^t \geq x_{u,j}^{t-1} \geq \delta\). \qed
Lemma B.1 (Monotonicity). \( x_{u,j}^t \leq x_{u,j+1}^t \).

Proof. Again assume that \( x_{u,j}^{t-1} \leq x_{u,j+1}^{t-1} \). In (KKT2a) observe that \( b_{u,j} \geq b_{u,j+1} \) because it sums up over more non-negative terms; this means \( x_{u,j}^t \) cannot be larger than \( x_{u,j+1}^t \) because of these terms. So we focus on the \( a_{u,j} \) terms in (KKT2a).

Suppose \( x_{u,j}^t > x_{u,j+1}^t \), then it must be that \( a_{u,j} > a_{u,j+1} \). So there is some \( \lambda_S > 0 \) with \( (u,j) \in S \), yet \( (u,j+1) \notin S \). Consider \( S' \) in (Flow) \( S' = (S - \{(u,j)\}) \cup \{(u,j+1)\} \). By constraint (3.5) for the set \( S' \),

\[
\sum_{(v,\ell) \in S'} x_{v,\ell}^t \geq \sum_{i \leq |S'|} x_{p(v),i}^t = \sum_{i \leq |S|} x_{p(u),i}^t = \sum_{(v,\ell) \in S} x_{v,\ell}^t,
\]

where the last equality holds because \( \lambda_S > 0 \). This implies \( x_{u,j+1}^t \geq x_{u,j}^t \), hence a contradiction. \( \square \)

We say a set \( S \) is tight if \( \sum_{j \leq |S|} x_{p(S),j}^t = x^t(S) := \sum_{(v,\ell) \in S} x_{v,\ell}^t \). Let \( \mathcal{C}_u \) denote the collection of tight sets in \( \chi_u \), and \( \mathcal{C} := \bigcup_u \mathcal{C}_u \) be all the tight sets.

Lemma B.2 (Uncrossing). For \( S_1, S_2 \subseteq \chi_u \), \( S_1, S_2 \in \mathcal{C} \implies S_1 \cup S_2 \in \mathcal{C} \). I.e., the union of tight sets with a common parent gives a tight set.

Proof. Let \( M := \max_{(v,j) \in S_1 \cup S_2} x_{v,j}^t \) be the largest value in the union (say it is in \( S_1 \)) and let \( (v^*, j^*) \in S_1 \) achieving this maximum value. Consider any \( (v,j) \in S_2 \setminus S_1 \); since the set \( S_1 \) is tight and \( S' := (S_1 \cup \{(v,j)\}) \setminus \{(v^*, j^*)\} \) satisfies the constraint (3.5), we infer that \( x_{v,j}^t = M \). From feasibility for the set \( S_1 \setminus \{(v^*, j^*)\} \), we know that \( x_{u,|S_1|}^t \geq M \) so using monotonicity of Lemma B.1,

\[
\sum_{i \leq |S_1 \cup S_2|} x_{u,i}^t \geq x^t(S_1) + |S_2 \setminus S_1| \cdot M = x^t(S_1 \cup S_2).
\]

The converse direction (inequality) follows from feasibility, and hence \( S_1 \cup S_2 \) is also tight. \( \square \)

Lemma 3.4 (Flow). For each internal node \( u \), \( \sum_j x_{u,j}^t = \sum_{(v,\ell) \in \chi_u} x_{v,\ell}^t \). This implies that for any depth \( d \),

\[
\sum_{u \in V_d} \sum_j x_{u,j}^t = n - h.
\]

So the difference between \( x^{t-1} \) and \( x^t \) can be viewed as a flow from \( r_1 \) to the other leaves in \( T \).

Proof. We assume these properties hold for \( x^{t-1} \), and show them for \( x^t \). For each vertex \( u \), let \( S_u \subseteq \chi_u \) be the largest tight set with respect to \( x^t \) in \( \mathcal{C}_u \). By definition of tight sets, for each depth \( d \),

\[
\sum_{u \in V_d} \sum_{j \leq |S_u|} x_{u,j}^t = \sum_{u \in V_d} x^t(S_u). \tag{B.35}
\]

We claim that

\[
\sum_{u \in V_d} \sum_{j \leq |S_u|} x_{u,j}^{t-1} \geq \sum_{(v,\ell) \in V_{d+1} \setminus \bigcup_{u \in V_d} S_u} x_{v,\ell}^t. \tag{B.36}
\]

By the constraints (3.5), we have \( \sum_{u \in V_d} \sum_{j \leq |S_u|} x_{u,j}^{t-1} \leq \sum_{u \in V_d} x^{t-1}(S_u) \). Since we inductively assumed the \( x^{t-1} \)-mass at each level of the tree was the same, collecting the terms not appearing
in the above inequality gives us
\[ \sum_{u \in V_d} \sum_{j > |S_u|} x_{u,j}^{t-1} \geq \sum_{(v,\ell) \in V_{d+1} \setminus \{u \in V_d S_u\}} x_{v,\ell}^{t-1}. \] (B.37)

Now, for any \( x_{u,j}^t \) on the left hand side of (B.36), every set \( T \subseteq \chi_u \) with \( j \leq |T| \) is not tight, and has \( \lambda_T = 0 \). This means \( b_{u,j} = 0 \) in (KKT2a) and hence \( x_{u,j}^t \) is increasing, i.e., \( x_{u,j}^t \geq x_{u,j}^{t-1} \).

Moreover, each term on the right side of (B.36) has \( a_{v,\ell} = 0 \) and is decreasing, i.e., \( x_{u,j}^t \leq x_{u,j}^{t-1} \) : from the maximality of \( S_u \) and Lemma B.2 all sets \( T \) containing \((v,\ell)\) are non-tight and hence have \( \lambda_T = 0 \). Using these inequalities in (B.37) gives us (B.36). And together with (B.35) gives us \( \sum_{u \in V_d} \sum_j x_{u,j}^t \geq \sum_{u \in V_{d+1}} \sum_j x_{u,j}^t \). From (3.5) we have the converse direction (inequalities)
\[ \sum_j x_{u,j}^t \leq \sum_{(v,\ell) \in \chi_u} x_{v,\ell}^t \] for each node \( u \in V_d \), so each such inequality must be tight.

Since each leaf \( u \neq r_t \) has \( a_{u,1} \) terms but no \( b_{u,1} \) terms in (KKT2a), \( x_{u,1}^t \geq x_{u,1}^{t-1} \). Since we just proved that the \( x^t \)-value at each node equals the \( x^t \)-value at its children, there is a “flow” of \( x^t \)-measure from \( r_t \) to all the other leaves. In particular, for each node not on the \( r_t \)-path, \( \sum_j x_{u,j}^t \geq \sum_j x_{u,j}^{t-1} \).

**Lemma 3.5 (Relating Consecutive Levels).** For any node \( u \) in the tree
\[ \sum_j B_{u,j}^t = \sum_{(v,\ell) \in \chi_u} A_{v,\ell}^t = \sum_{T \subseteq \chi_u} \lambda_T \bar{x}^t(T). \]

**Proof.** Expanding the first summation gives
\[ \sum_j B_{u,j}^t = \sum_j \bar{x}_{u,j}^t \sum_{T \subseteq \chi_u : |T| \geq j} \lambda_T = \sum_{T \subseteq \chi_u} \lambda_T \sum_{j \leq |T|} \bar{x}_{p(T),j}^t. \]

By complementary slackness (CS2), we have \( \lambda_T \sum_{j \leq |T|} x_{p(T),j}^t = \lambda_T \sum_{j \in T} x_{u,j}^t = \lambda_T x^t(T) \). However, since there are \( |T| \) terms on both sides, adding \( \lambda_T |T| \delta \) to both sides gives us the desired inequality for the shifted variables \( \bar{x} \). Now summing up over all \( T \) gives
\[ \sum_{T \subseteq \chi_u} \lambda_T \sum_{j \leq |T|} \bar{x}_{p(T),j}^t = \sum_{T \subseteq \chi_u} \lambda_T \bar{x}^t(T). \] (B.38)

And the second summation in the statement of the lemma is
\[ \sum_{(v,\ell) \in \chi_u} A_{v,\ell}^t = \sum_{(v,\ell) \in \chi_u} \bar{x}_{v,\ell}^t \sum_{S \subseteq \chi_u : (v,\ell) \in S} \lambda_S = \sum_{S \subseteq \chi_u} \lambda_S \sum_{(v,\ell) \in S} \bar{x}_{v,\ell}^t = \sum_{S \subseteq \chi_u} \lambda_S \bar{x}^t(S). \] (B.39)

The two expressions are equal, hence the claim. \( \square \)

**B.2 Proof of Lemma 3.7**

Recall the statement of Lemma 3.7, where \( W(x) := \sum_u w_u \sum_j x_{u,j} \).

**Lemma 3.7 (Second Lower Bound).** \( D(y^t \parallel x^{t-1}) - D(y^t \parallel x^t) \geq \delta \gamma_t. \) Thus,
\[ \Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \geq \delta \gamma_t + W(x^t) - W(x^{t-1}). \]
If \((\lambda, \gamma_t)\) are the optimal dual variables for the projection problem (3.4)-(3.6), we can get an equivalent characterization of the optimal solution \(x^t\) as follows:

**Claim B.3.** Vector \(x^t\) is an optimal solution for the optimization problem

\[
\min_{x \in P} D(x \parallel x^{t-1}) + \gamma_t x_{r_t}. \tag{B.40}
\]

**Proof.** Recall that we defined \(x^t\) to be the minimizer of just the first term, subject to the constraints that \(x \in P\) and \(x \leq \delta\). If we now Lagrangify the second constraint, and use the fact that \(\gamma_t\) is an optimal Lagrange multiplier, we get the equivalent problem \(\min_{x \in P} D(x \parallel x^{t-1}) + \gamma_t x_{r_t}\). □

Observe that \(D(x \parallel x')\) is the Bregman divergence corresponding to the strongly convex function

\[
h(x) := \sum_u w_u \sum_j \tilde{x}_{uj} \log \tilde{x}_{uj}.
\]

Now, using the definition of Bregman divergences and some simple algebra, we get that

\[
D(y^t \parallel x^{t-1}) - D(y^t \parallel x^t) = D(x^t \parallel x^{t-1}) - \langle \nabla h(x^{t-1}) - \nabla h(x^t), x^t - y^t \rangle \\
\geq \langle \nabla h(x^{t-1}) - \nabla h(x^t), x^t - y^t \rangle + W(x^t) - W(x^{t-1}). \tag{B.41}
\]

where the inequality uses non-negativity of Bregman divergences. Manipulating the linear terms,

\[
\Phi(y^t \parallel x^{t-1}) - \Phi(y^t \parallel x^t) \geq \langle \nabla h(x^{t-1}) - \nabla h(x^t), x^t - y^t \rangle + W(x^t) - W(x^{t-1}). \tag{B.42}
\]

To prove Lemma 3.7, we need to bound the inner product term from below. For a point \(u \in P\), define the normal cone at \(u\) to be \(N_P(u) := \{d \mid \langle d, v - u \rangle \leq 0 \forall v \in P\}\).

**Claim B.4.** Let \(e_{r_t} \in \{0, 1\}^N\) be the vector that has a 1 in the coordinate corresponding to leaf \(r_t\), and 0s otherwise. Then

\[
\nabla h(x^{t-1}) - \nabla h(x^t) = \gamma_t e_{r_t} + d
\]

where \(d\) belongs to the normal cone \(N_P(x^t)\).

**Proof.** Since \(x^t\) solves the optimization problem (B.40), the first-order optimality criteria implies that the gradient of the objective function, when evaluated at \(x^t\), belongs to the negative normal cone \(-N_P(x^t)\). Recall that the gradient at any point \(x\) is

\[
\nabla D(x \parallel x^{t-1}) + \gamma_t e_{r_t} = \nabla h(x) - \nabla h(x^{t-1}) + \gamma_t e_{r_t},
\]

so \(\nabla h(x^t) - \nabla h(x^{t-1}) + \gamma_t e_{r_t} = -d\), where \(d \in N_P(x^t)\). Rearranging completes the proof. □

Substituting this expression into (B.42) implies that the inner product term is

\[
\langle \gamma_t e_{r_t} + d, x^t - y^t \rangle \geq \langle \gamma_t e_{r_t}, x^t - y^t \rangle = \gamma_t (x^t_{r_t} - y^t_{r_t}) = \gamma_t \delta. \tag{B.43}
\]

The inequality uses the definition of the normal cone \(N_P(x^t)\) and that \(y^t \in P\). The last equality uses \(x^t_{r_t} = \delta\) and \(y^t_{r_t} = 0\). This proves Lemma 3.7.
B.3 Miscellaneous Lemmas

**Lemma B.5.** Consider vectors \( \mathbf{x}, \mathbf{x}' \in P \), and let \( \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n \) be their respective restrictions to the leaf atoms. Then

\[
d(\mathbf{x}, \mathbf{x}') \leq \| \mathbf{x} - \mathbf{x}' \|_{\ell_t(T)}.
\]

Moreover, if \( \mathbf{x}, \mathbf{x}' \) are integer vectors, we get equality above.

**Proof.** From the flow conservation Lemma 3.4, for each node \( u \) we have \( \sum_j x_{u,j} = \sum_{v \in \text{leaves}(T_u)} x_{v,1} = \sum_{v \in \text{leaves}(T_u)} x_{v,1} \), and the same holds for \( \mathbf{x}' \) and \( \mathbf{x}' \). Therefore

\[
d(\mathbf{x}, \mathbf{x}') = \sum_u w_u \left| \sum_{v \in \text{leaves}(T_u)} x_{v,1} - \sum_{v \in \text{leaves}(T_u)} x'_{v,1} \right|
= \sum_u w_u \left| \sum_j (x_{u,j} - x'_{u,j}) \right|
\leq \sum_u w_u \sum_j |x_{u,j} - x'_{u,j}| = \| \mathbf{x} - \mathbf{x}' \|_{\ell_t(T)},
\]

concluding the first part of the proof.

For the second part, when \( \mathbf{x} \) and \( \mathbf{x}' \) are integral \( | \sum_j (x_{u,j} - x'_{u,j}) | \) equals \( | \# \text{1's in } (x_{u,j})_j - \# \text{1's in } (x'_{u,j})_j | \). Moreover, by the monotonicity Lemma B.1, \( \sum_j |x_{u,j} - x'_{u,j}| \) equal the same quantity. Thus, inequality (B.44) holds at equality and hence \( d(\mathbf{x}, \mathbf{x}') = \| \mathbf{x} - \mathbf{x}' \|_{\ell_t(T)} \). This concludes the proof. \( \square \)