Title: "Self-inverse Sheffer sequences and Riordan involutions.”

Authors: Ana Luzón* and Manuel A. Morón**

Address: *Departamento de Matemática Aplicada a los Recursos Naturales. E.T.S. I. Montes. Universidad Politécnica de Madrid. 28040-Madrid, SPAIN.

**Departamento de Geometria y Topologia. Facultad de Matematicas. Universidad Complutense de Madrid. 28040- Madrid, SPAIN.

E-mail: *anamaria.luzon@upm.es

**mamoron@mat.ucm.es

All correspondence should be sent to:

Ana Maria Luzón Cordero

e-mail: anamaria.luzon@upm.es

Address:

Departamento de Matemática Aplicada a los Recursos Naturales.

E.T.S.I. Montes.

Universidad Politécnica de Madrid.

28040-Madrid, SPAIN

Phone number: 34 913366399

Fax number: 34 915439557

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SELF-INVERSE SHEFFER SEQUENCES AND RIORDAN INVOLUTIONS

ANA LUZÓN* AND MANUEL A. MORÓN**

*Departamento de Matemática Aplicada a los Recursos Naturales. E.T. Superior de Ingenieros de Montes. Universidad Politécnica de Madrid. 28040-Madrid, SPAIN.

anamaria.luzon@upm.es

**Departamento de Geometría y Topología. Facultad de Matemáticas. Universidad Complutense de Madrid. 28040- Madrid, SPAIN.

mamoron@mat.ucm.es

Abstract. In this short note we focus on self-inverse Sheffer sequences and involutions in the Riordan group. We translate the results of Brown and Kuczma on self-inverse sequences of Sheffer polynomials to describe all involutions in the Riordan group.

Keywords: Riordan group, involution, self-inverse Sheffer sequence.

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Very recently, first in [7] later in [16], see also [11], it has been established a very close relation between the Sheffer group and the Riordan group, see [13], [15]. In fact, there is a natural isomorphism between both groups. This means that the group properties can
be translated from one group to the other equivalently. One of those properties is just the structure of their finite subgroups.

In this short note we focus on self-inverse Sheffer sequences and involutions in the Riordan group. They determine the corresponding subgroups of order two. In fact, we translate the results of Brown and Kuczma, [2], on self-inverse sequences of Sheffer polynomials to describe all involutions in the Riordan group. Although the translation is almost automatic we think that it is still interesting to point it out, because of the relations to some problems about involutions in the Riordan group posed in [14] that motivated the paper [3] and that has been recently solved see [4], [5], [6]. Also in pages 2264-2265 of [10] we have some related results.

In some sense we want to point out that some aspects of Shapiro’s problem was solved, even before it was posed, if we reinterpret this in terms of Sheffer sequences.

In this paper $\mathbb{K}$ always represents a field of characteristic zero and $\mathbb{N}$ is the set of natural numbers including 0. The notation used herein for Riordan arrays is that introduced in [9], see also [10].

Up to the inconvenience that produce the fact that we call, following [11], a generalized Appell sequence associated to Hadamard invertible series $h(x) = \sum_{n \geq 0} h_n x^n$ just to the sequence obtained by multiplying by $h_n$ the $n$-term of the polynomial sequence named by the same way in [2], we have the following obvious result. The notation used below is just that used in [11], the operation group $\sharp_h$ is the umbral composition as it is also described in [7] for the particular case $h(x) = e^x$, and $\star$ represents the Hadamard product of series.
Proposition 1. Let $N$ be a natural number. Suppose a polynomial sequence of Riordan type $(p_n(x))_{n \in \mathbb{N}}$ and $h(x) = \sum_{n \geq 0} h_n x^n$ be a series with $h_n \neq 0 \ \forall n \in \mathbb{N}$. Consider the Hadamard $h$-weighted sequence $(p_n^h(x)) = (p_n(x) \ast h(x))$. Then, the $N$-fold umbral composition, by means of $\sharp_h$, of the sequence $(p_n^h(x))$ is the neutral element in $(\mathcal{R}_h, \sharp_h)$ if and only if $D^N = I$ in the Riordan group where $D = (d_{n,k})$ is the Riordan matrix given by $p_n(x) = \sum_{k=0}^{n} d_{n,k} x^k$.

Remark 2. Note that the sequence $e_n(x) = h_n x^n$ is the neutral element in $(\mathcal{R}_h, \sharp_h)$.

We now translate the result in [2] on Self-inverse Sheffer sequences into Riordan involutions. In particular, using the results of Section 3 in [2] we can give a procedure to compute all the elements $T(f \mid g)$ of the Riordan group such that $T^2(f \mid g) = T(1 \mid 1) = I$. First if we impose $g = 1$, then $T^2(f \mid g) = T(1 \mid 1)$ if and only if $f = 1$ or $f = -1$. To construct the remaining cases we proceed in the following way:

Choose any series $\phi = \sum_{n \geq 0} \phi_n x^n$ with $\phi_0 \neq 0$. Consider the Riordan matrix $T(1 \mid \phi)$. As a consequence of the results in [8] we get that $T(1 \mid A) = T^{-1}(1 \mid \phi)$ where $A$ is the so called, [13], [12], the $A$-sequence of $T(1 \mid \phi)$. It is clear that $\frac{x}{A}\left(\frac{x}{\phi}\right) = \frac{x}{\phi}\left(\frac{x}{A}\right) = x$, that is $\frac{x}{A}$ is the inverse, for the composition of the series $\frac{x}{\phi}$. Following [2] we take

(1) \[ g = \frac{x}{\frac{x}{\phi}} \]

Choose now any odd power series $u = \sum_{k \geq 1} u_{2k-1} x^{2k-1}$, finally take

(2) \[ f = \frac{\pm x}{\frac{x}{A}\left(\frac{x}{\phi}\right)} e^u(\frac{x}{\phi}) \]

Consequently we have
Proposition 3. Any Riordan involution different from the identity \( I = T(1 \mid 1) \) and \(-I = T(-1 \mid 1)\) is of the form \( T(f \mid g) \) for \( f \) and \( g \) satisfying (2) and (1) respectively.

We want to point out that the pair of series \((f, g)\) above is far from being univocally determined by \( \phi \) and \( u \). For example

Proposition 4. Let \( \phi = \sum_{n \geq 0} \phi_n x^n \) be a series with \( \phi_0 \neq 0 \). Suppose that \( \frac{x}{A} \) is the compositional inverse of \( \frac{x}{\phi} \) (equivalently \( A \) is the A-sequence of \( T(1 \mid \phi) \)). Suppose also that \( g = \frac{x}{A} \left( -\frac{x}{\phi} \right) \). Then \( g = -1 \) if and only if \( \phi(x) = \phi(-x) \) (i.e. \( \phi \) is even).

Proof. Note that \( \phi(x) = \phi(-x) \) if and only if the series \( \frac{x}{\phi} \) is odd.

If \( \phi \) is even then \( \frac{x}{\phi} \) is odd and since \( \frac{x}{A} \) is the compositional inverse of an odd power series then \( \frac{x}{A} \) is odd itself. Consequently \( g = -1 \). On the other hand if \( g = -1 \) then \( \frac{x}{A} \left( -\frac{x}{\phi} \right) = -x \). Composing by the right by \( \frac{x}{A} \) we have \( \frac{x}{A}(-x) = \frac{-x}{A(-x)} = -\frac{x}{A(x)} \). This implies that \( \frac{x}{\phi} \) is odd and the \( \phi \) is even. \( \square \)

In [10] we proved that for any \( \alpha \neq 0 \), the Riordan matrices \( T(\pm 1 \mid \alpha x - 1) \) are involutions, see page 2265 in [10]. Now we are going to recover this result using the construction above.

In fact we will get a more general class of Riordan involutions:

Let \( \alpha \neq 0 \) and take

\[
\phi(x) = \frac{\alpha x}{\log(1 - \alpha x)}
\]

consequently

\[
A(x) = \frac{\alpha x}{1 - e^{\alpha x}}
\]
This implies that

\[ g = \frac{x}{\alpha \left( \frac{-x}{\alpha} \right)} = \alpha x - 1 \]

We know that if

\[ f = \pm(\alpha x - 1)e^{u(\frac{1}{\alpha} \log(1-\alpha x))} \]

then \( T(f \mid g) \) is an involution when \( u \) is an odd series. In particular if \( u(x) = -\alpha x \) then we obtain that \( T(\pm1 \mid \alpha x - 1) \) is a Riordan involution.

From this point of view the fact that Pascal triangle is a pseudo-involution, \([3]\), is equivalent to the fact that the classical Laguerre polynomials are self-inverse see \([2]\).

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