Multiverse as an ensemble of stable and unstable Universes

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Abstract

Estimates of the Higgs and top quark masses, $m_H \simeq 125.10 \pm 0.14$ [GeV] and $m_t \simeq 172.76 \pm 0.30$ [GeV] based on the experimental results place the Standard Model in the region of the metastable vacuum. A consequence of the metastability of the Higgs vacuum is that it should induce the decay of the electroweak vacuum in the early Universe with catastrophic consequences. It may happen that certain universes were lucky enough to survive the time of canonical decay, that is the exponential decay, and live longer. This means that it is reasonable to analyze conditions allowing for that. We analyze properties of an ensemble of Universes with unstable vacua considered as an ensemble of unstable systems from the point of view of the quantum theory of unstable states. We found some symmetry relations for quantities characterizing the metastable state. We also found a relation linking the decay rate, $\Gamma$ of the metastable vacuum state with the Hubble parameter $H(t)$, which may help to explain why a universe with an unstable vacuum that lives longer then the canonical decay times need not decay.

Key words: Unstable (false) vacuum, Quantum decay process, Cosmological constant problem

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1
1 Introduction

In cosmology a discussion of a problem of the false vacuum and the possibility of its decay began from the papers by Coleman and his colleagues [1, 2, 3]. Krauss in [4] analyzing properties of the false vacuum as a quantum unstable (quasi–stationary) state $|M\rangle$ drawn an attention to the problem that there may exist universes in which the lowest energy state is the false vacuum state and such that they can survive up to the times much later than times $t$ when the canonical exponential decay law holds (see also [5]). The study of cosmological models with unstable vacua has became particularly important in the context of the discovery of the Higgs boson and of finding its mass $m_H$ [6, 7] to be $125.1 \pm 0.14$ [GeV] and the top quark mass $m_t \simeq 172.76 \pm 0.30$ [GeV] [8]. It is because the Standard Model calculations performed for the Higgs particle suggest that the electroweak vacuum is unstable if the mass of the Higgs particle is around 125 — 126 GeV (see eg. [9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20], which means that our Universe may be the Universe with unstable vacuum. For this reason various mechanisms slowing the vacuum decay down or even stopping it, have been discussed in many papers (see e. g. [21, 22] and and also [23, 24, 25, 26] and references therein).

In this paper we analyze multiverse made up of ensembles of stable and unstable universes. The property of the universe "to be unstable" or "to be stable" is determined by properties of the vacuum state: If it is a false vacuum then it is unstable and decays into the true vacuum state and thus this universe decays too. The decay of the false vacuum is a quantum decay process and in this paper we will use this fact as an assumption. Any quantum decay process, whether it is a decay of a particle or a decay of an excited level in an atom, or of the decay of the metastable false vacuum, no matter how, (e. g. via the quantum tunneling through a potential barrier), must exhibit all the general properties resulting form the quantum theory of unstable states. Therefore, in our opinion, the quantum theory of unstable states seems to be an appropriate tool for the general analysis of the decay process of the false vacuum state and can help to understand and explain the various subtleties and properties of this process. The vacuum decay plays an extremely important role in cosmology. It cannot be ruled out that without the decay of a metastable vacuum it will be impossible to explain some issues, as stated in [20] at the end of Sec. 6, where one can find the following sentence: Assuming that the present acceleration of the universe is due to a small cosmological constant, and accepting the conjecture that quantum gravity is
ill-defined in a de Sitter space, we argue that vacuum decay is a necessary way out for the universe. Now, suppose, following the idea of Krauss and Dent [4], that certain universes were lucky enough to survive the times of the canonical decay and they are still alive. (The canonical decay times means times when the decay law (the survival probability) has an exponential form to very good approximation). This idea can be applied to our Universe if we assume that its current age is longer than the canonical decay times of the false vacuum state. It is worth noting here that there are cosmological models under study in which the lifetime of a false vacuum is very short, and even significantly shorter than the duration of the inflationary phase (see e.g. [27, 28]). The important question is what conditions should be satisfied in order that in such and similar cases some universes could survive up to times much later than the canonical decay times and how long they are able to survive. Here we attempt to clarify this issue considering unstable universes as an ensemble of unstable quantum particles and analyzing their behavior at very late times. The tools we use for this purpose are the general properties of the quantum decay law, the decay rate, $\Gamma$, and the energy of the system in a metastable state in the region of very long times. From the general principles of the quantum theory it results that the decay rate depends on time, $\Gamma = \Gamma(t)$, and $\Gamma(t) \to 0$ as $t \to \infty$ whereas at canonical decay times $\Gamma(t) \simeq \Gamma^0 = \text{const.}$ to very good approximation. In my opinion, these properties of the decay rate $\Gamma(t)$ may cause a universe whose vacuum is a false vacuum to survive longer than the lifetime of this false vacuum.

The paper is organized as follows: In Sec. 2 one can find a quantum description of the decay process and parameters characterizing this process. In Sec. 3 a simplified toy model of the combined process of the expansion of a universe with unstable vacuum and of the quantum decay process of the unstable vacuum state is analyzed. Sec. 4 contains analysis of the long time properties of the survival amplitude and connected with these properties a behavior of the decay rate as a function of time $t$. Properties of the energy of the metastable vacuum state as a function of time $t$ and the related properties of the density of the vacuum energy are considered in Sec. 5. Sec. 6 contains a discussion and conclusions.
2 Preliminaries: Quantum description of the decay process

From experiments it is known that for some unstable systems decay process is relatively fast or very fast while for others it is slow or very slow. The rate of this process is characterized by parameter called the "lifetime" or the "decay rate". In decay experiments one has an ensemble of unstable physical systems in a certain area, which is surrounded by counters that detect decay products. The counting rate, i.e. the number of decay per second \( \frac{\delta N(t)}{\delta t} \), is proportional to the number of unstable particles \( N(t) \) in a given volume at instant \( t \). The proportionality coefficient,

\[
\lambda = \frac{\frac{\delta N(t)}{\delta t}}{N(t)},
\]

is connected with the average lifetime (or simply lifetime) of the unstable objects considered (see, e.g. [29] for a discussion). Indeed, if \( N(t) \) is very large, then the ratio of \( N(t) \) by the initial number \( N_0 \) of such object at the initial instant \( t_0^{\text{init}} \), \( N_0 = N(t_0^{\text{init}}) \), in this area is the probability, \( \mathcal{P}(t) \), of finding an unstable object undecayed in this area at a given instant of time \( t \) (i.e., the survival probability \( \mathcal{P}(t) \)). There is \( \mathcal{P}(t) \simeq N(t)/N_0 \) and \( \lim_{t \to \infty} N(t) = 0 \) so \( \mathcal{P}(\infty) = 0 \) and \( \mathcal{P}(t_0^{\text{init}}) = 1 \). The number of decays \( \delta N(t) \) per unit of time \( \delta t \) equals: \( \delta N(t) \simeq N(t) - N(t + \delta t) \). There is \( N(t) > N(t + \delta t) \) in the case of decay processes and thus

\[
\lim_{\delta t \to 0} \frac{\delta N(t)}{\delta t} = -\frac{dN(t)}{dt}.
\]

The solution of Eq. (1) in the case \( \delta t \to 0 \) takes the following form

\[
\frac{N(t)}{N_0} = e^{-\lambda t}.
\]

So, in this case there is \( \mathcal{P}(t) \simeq \exp[-\lambda t] \) and the density of the probability of the decay at time \( t \) during the time interval \( t + dt \), \( \rho_{\mathcal{P}}(t) \), equals \( \rho_{\mathcal{P}}(t) = -\frac{d\mathcal{P}(t)}{dt} \equiv \lambda \exp[-\lambda t] \equiv \lambda \mathcal{P}(t) \). Taking for simplicity \( t_0^{\text{init}} = 0 \) it is easy to verify that \( \int_0^\infty \rho_{\mathcal{P}}(t) \, dt = 1 \) as it should be. Using \( \rho_{\mathcal{P}}(t) \) and keeping for a moment \( t_0^{\text{init}} = 0 \) one can find the average lifetime,

\[
\tau = \langle t \rangle = \int_0^\infty t \rho_{\mathcal{P}}(t) \, dt \equiv \frac{1}{\lambda}.
\]
Thus, in general

$$\lambda \equiv \frac{1}{\tau} = - \frac{d\mathcal{P}(t)}{dt} \equiv \frac{\Gamma}{\hbar},$$

(5)

where $\Gamma$ is the decay rate.

Within the quantum theory, similarly as in the case of the classical physics, the number of unstable particles $N(t)$, which at the time $t$ can be found in the area considered, is equal to the product of the probability, $\mathcal{P}(t)$, of finding an unstable object undecayed in this area at a given instant of time $t$, (i.e. of the survival probability $\mathcal{P}(t)$), and the initial number $N_0$ of such objects:

$$N(t) = \mathcal{P}(t) N_0.$$  

(6)

where the survival probability $\mathcal{P}(t)$ (or the decay law) is defined as follows:

$$\mathcal{P}(t) = |A(t)|^2.$$  

(7)

and

$$A(t) = \langle M|M(t) \rangle,$$  

(8)

is the survival amplitude, $|M\rangle$ is the unstable (metastable) state under considerations, $|M\rangle \in \mathcal{H}$ (where $\mathcal{H}$ is the Hilbert space of states of the considered system), and $|M(t)\rangle$ is the solution of the Schrödinger equation

$$i \hbar \frac{\partial}{\partial t}|M(t)\rangle = \mathcal{H}|M(t)\rangle,$$  

(9)

for the initial condition $|M(t_{0}^{\text{init}})\rangle = |M\rangle$. Here $\mathcal{H}$ is the total self-adjoint Hamiltonian for the system under consideration and $t_{0}^{\text{init}}$ is the initial instant. The vector $|M(t)\rangle = \exp \left[ - \frac{i}{\hbar} (t - t_{0}^{\text{init}}) \mathcal{H} \right] |M\rangle \equiv |M(t - t_{0}^{\text{init}})\rangle$ is the solution of Eq. (9).

It is easy to find that

$$A(t) \equiv A(t - t_{0}^{\text{init}}) = \langle M| \exp \left[ - \frac{i}{\hbar} (t - t_{0}^{\text{init}}) \mathcal{H} \right] |M\rangle \equiv A^*[t - (t_{0}^{\text{init}})].$$  

(10)

So, there are some symmetries of quantities characterizing the decaying state. The first one is given by Eq. (10). The second one is a direct consequence of Eq (10). Namely using (10) one finds that there is the following symmetry,

$$\mathcal{P}(t) \equiv A(t) A^*(t) = A(t) A(-t) = \mathcal{P}(-t).$$  

(11)
From (5), (7) we obtain that,

\[
\frac{\Gamma}{\hbar} \equiv \frac{\Gamma_M(t)}{\hbar} = -\left( \frac{1}{A(t)} \frac{\partial A(t)}{\partial t} + \frac{1}{A^*(t)} \frac{\partial A^*(t)}{\partial t} \right). \tag{12}
\]

Using (10) and (12) another symmetry is easy to find. This time for \( \Gamma_M(t) \). There is

\[
\Gamma_M(t) = -\Gamma_M(-t). \tag{13}
\]

If to define the following quantity [36]:

\[
h_M(t) = \frac{i\hbar}{A(t)} \frac{\partial A(t)}{\partial t} \tag{14}
\]

then the relation (12) means that simply

\[
\Gamma_M(t) = -2\Im [h_M(t)], \tag{15}
\]

where \( \Im [z] \) denotes the imaginary parts of \( z \) (similarly, \( \Re [z] \) is the real part of \( z \)).

Note that one can find also the symmetry for \( h_M(t) \) that results directly from Eq (10) and from the definition (14) of \( h_M(t) \). There is

\[
h_M^*(t) = h_M(-t). \tag{16}
\]

From basic principles of the quantum theory it follows that the amplitude \( A(t) \), and thus the decay law \( \mathcal{P}(t) \) of the unstable state \( |M\rangle \), can be completely determined by the density of the energy distribution \( \omega(E) \) for the system in this state [30, 31]

\[
A(t) = \int_{\operatorname{Spec}(\mathcal{H})} \omega(E) e^{-\frac{i}{\hbar} E (t - t^{\text{init}}_0)} dE \equiv A(t - t^{\text{init}}_0). \tag{17}
\]

where \( \omega(E) \geq 0 \).

In [32] assuming that the spectrum of \( \mathcal{H} \) must be bounded from below, \( \operatorname{Spec}(\mathcal{H}) \overset{\text{def}}{=} \sigma(\mathcal{H}) = [E_{\text{min}}, \infty) \) and \( E_{\text{min}} > -\infty \), and using the Paley–Wiener Theorem [33] it was proved that in the case of unstable states there must be

\[
|A(t)| \geq B e^{-b t^q} \quad \text{for} \quad |t| \to \infty, \tag{18}
\]

(where \( B > 0, b > 0 \) and \( 0 < q < 1 \)). This means that the decay law \( \mathcal{P}(t) \) of unstable states decaying in the vacuum can not be described by an
exponential function of time $t$ if time $t$ is suitably long, $t \to \infty$, and that for these lengths of time $P(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of $t$. The analysis of the models of the decay processes shows that $P(t) \simeq \exp[-\Gamma_0^M t/\hbar]$, to a very high accuracy from $t$ suitably later than the initial instant $t_{0\text{init}}$ up to $t \gg \tau_M$, (where $\tau_M = \hbar/\Gamma_0^M$ is the life–time of the state $\ket{M}$, $\Gamma_0^M$ is the decay width of the unstable state $\ket{M}$ calculated within the one pole approximation [34]), and smaller than $t = T_1$, where $T_1$ denotes the time $t$ at which the non–exponential deviations of $A(t)$ begin to dominate.

In general, in the case of quasi–stationary (metastable) states it is convenient to express $A(t)$ in the following form

$$A(t) = A_c(t) + A_{lt}(t), \quad (for \ t \gg \tau_M),$$

(19)

where $A_c(t)$ is the exponential part of $A(t)$, that is $A_c(t) = N \exp[-\frac{i}{\hbar}(t - t_{0\text{init}})(E_M^0 - \frac{1}{2} \Gamma_M^0)]$, ($N$ is the normalization constant, $E_M^0$ is the energy of the system in the unstable state $\ket{M}$ calculated within the one pole approximation), and $A_{lt}(t)$ is the late time non–exponential part of $A(t)$. For times $t \sim \tau_M$: $|A_c(t)| \gg |A_{lt}(t)|$. Using (19) one finds that

$$P(t) = |A(t)|^2 = |A_c(t)|^2 + 2\Re [A_c(t) A_{lt}^*(t)] + |A_{lt}(t)|^2. \quad (20)$$

The solution, $t$, of the equation

$$|A_c(t)|^2 = 2\Re [A_c(t) A_{lt}^*(t)], \quad (21)$$

(let us denote it as $t = T_1$) is usually considered as an approximate, conventional end of the canonical phase of a decay process, where the survival probability has an exponential form: For $t < T_1$ there is $P(t) \simeq |A(t)|^2 = \exp[-\frac{i}{\hbar} \Gamma_0^M t]$ to a very good approximation. Solving the following equation

$$2\Re [A_c(t) A_{lt}^*(t)] = |A_{lt}(t)|^2, \quad (22)$$

one finds the time $t = T_2$. The time $T_2$ is the time from which the late time phase of the decay process begins: For $t > T_2$ the survival probability has a form of powers of $(1/t)$. The transition phase of a decay process is the epoch when time $t$ is passing the time–interval $(T_1, T_2)$. At this point, a mention should also be made of the so-called "cross–over time" used by some author (see, e. g. [35]). The crossover time, denoted above as $T$, is the time when
contributions to the survival probability $\mathcal{P}(t)$ of its exponential (canonical) and late time non-exponential parts are the same:

$$|A_c(t)|^2 = |A_{\text{lt}}(t)|^2,$$

(23)

and $T$ is the solution of this equation. There is $T_1 < T < T_2$.

At this point, it should be noted that the consideration of asymptotic late time properties of the amplitude $A(t)$ and the quantities defined within the use of $A(t)$ is justified by experimental results. Namely, in an experiment described in the Rothe paper [35], the experimental evidence of deviations from the exponential decay law at long times, much later than the crossover time $T$, was reported.

From relations (7), (12), (15) it is seen that the amplitude $A(t)$ contains information about the decay law $\mathcal{P}(t)$ of the state $|M\rangle$ and about the decay rate $\Gamma(t)$. It was also shown that using (14) the information about the energy $E_M(t)$ of the system in the unstable state considered can also be extracted from the survival amplitude $A(t)$: The energy of the system in the unstable state $|M\rangle$ (the instantaneous energy), $E_M(t)$, is equal to the real part of the effective hamiltonian $h_M(t)$ (see eg. [36]),

$$E_M(t) = \Re [h_M(t)].$$

(24)

and in general we have,

$$h_M(t) = E_M(t) - \frac{i}{2} \Gamma_M(t).$$

(25)

There is the following symmetry for $E_M(t)$ completing the symmetry relation (13), which results directly from Eqs (14), (10):

$$E_M(t) = E_M(-t).$$

(26)

Now let us focus an attention on the survival amplitude $A(t)$ and on the survival probability $\mathcal{P}(t)$ given by (8) and (7) and on the description of the decay of a metastable false vacuum. $\mathcal{P}(t)$ is the probability to find the system at time $t$ in the metastable state $|M(t_0^{\text{init}})\rangle \equiv |M\rangle$ prepared in the initial instant $t_0^{\text{init}} > 0$. If there were a suitable large number $N_0$ of identical unstable objects at the initial instant $t_0^{\text{init}}$ then according to (6) one should detect $N(t) = \mathcal{P}(t) N_0 < N_0$ of them at $t > t_0^{\text{init}}$. There is no such a simple correspondence of $\mathcal{P}(t)$ with the results of measurements when one is able to
prepare only one particle (or a few particles) at $t_{\text{init}}^0$. On the other hand if
one is able to prepare at $t_{\text{init}}^0$ a system containing only one unstable object
and to make a large number $N_0$ of indistinguishable copies of this system
then the problem reduces to the previous one: $N(t) = P(t) N_0$ copies of the
system will contain this unstable object undecayed at $t > t_{\text{init}}^0$. When there
are no $N_0 \gg 1$ copies of the system at $t = t_{\text{init}}^0$ but one has to deal with
only one particle system then one can never be sure whether one will detect
this particle undecayed at $t > t_{\text{init}}^0$ or not. This same concerns a universe
with the metastable (false) vacuum: One can expect that an ensemble of $N_0$
universes with unstable vacua will behave analogously as a system containing
$N_0$ unstable objects. So, let $|M\rangle \equiv |0^M\rangle$ be the metastable (false) vacuum
state of a universe considered and $|0^M\rangle \neq |0^{\text{true}}\rangle$, (where $|0^{\text{true}}\rangle$ is the true
ground state describing the state in which the energy of the system under
considerations has the absolute minimum). Let us assume that this universe
was created at the instant $t = t_{\text{init}}^0 > 0$ and the volume occupied by this
universe at $t = t_{\text{init}}^0$ was $V_{\text{init}}^0 = V(t)|_{t=t_{\text{init}}^0}$. Thus in fact one should take
into account that there is $|0^M\rangle \equiv |0^M; V_{\text{init}}^0\rangle$, where $|0^M; V_{\text{init}}^0\rangle$ is the vacuum
state of the universe of the volume $V_{\text{init}}^0$. It is convenient to choose the
normalization condition for $|0^M; V_{\text{init}}^0\rangle$ in the following form,

$$
\langle V_{\text{init}}^0; M | 0^M; V_{\text{init}}^0 \rangle = 1. \quad (27)
$$

In this case an analysis of the survival probability $P(t)$ can not give a con-
clusive answer whether the universe of the volume $V_{\text{init}}^0$ will still exist in the
state $|0^M; V_{\text{init}}^0\rangle$ at instant $t > t_{\text{init}}^0$ or not. The problem becomes much more
complicated if to take into account that in addition to the pure quantum
tunneling process leading to decay of the false vacuum state [1, 2, 3] there
exists another completely different process forcing the universe of the vol-
ume $V_{\text{init}}^0$ to expand. This effect was considered in [4], where Krauss and
Dent analyzing a false vacuum decay pointed out that in eternal inflation,
even though regions of false vacua by assumption should decay exponentially,
gravitational effects force the space region of the volume $V_{\text{init}}^0$ that has not
decayed yet to grow exponentially fast. This effect causes that many false
vacuum regions or many universes forming a multiverse can survive up to the
times much later than times when the exponential decay law holds. What
is more, the particle physics can provide us with hints suggesting what may
happen in such or similar cases: A free neutron is unstable and decays but
the neutron inside a nucleus is subjected to other additional interactions and
does not decay. These processes both can be described using the survival amplitude \( |A(t)|^2 \to 0 \) as \( t \to \infty \). This property is not the case of the neutron inside the nucleus. In general, when an unstable particle is subjected to different interactions described by suitable commuting Hamiltonians, then it may happen that the decay process can be slowed or even stopped. Similarly, as it was shown in \( \textbf{3} \), the gravitation may stop the decay of the false vacuum. So when analyzing the stability of the false vacuum state by means of the survival amplitude \( A(t) \) the correct conclusion can not be drawn if to use only the Hamiltonian \( \mathcal{H} \) describing the ”pure” decay through quantum tunneling. One can expect that the correct result can be obtained if to replace this \( \mathcal{H} \) in (8), (10) by the sum \( \mathcal{H} + \mathcal{H}_V \), where \( \mathcal{H}_V \) describes more or less accurately the expansion process of the volume \( V_{\text{init}} \).

3 A simplified toy model

Astrophysical observations lead to the conclusions that our Universe is expanding in time. In \( \textbf{4} \) an observation was made that in inflationary processes, even if some space regions of false (unstable) vacua decay exponentially, gravitational effects force space in a region that did not have time to decay, to grow exponentially fast (see also \( \textbf{5} \)). So, in general the expansion process affect the process of decay of the universes (domains) with the false vacua. The problem is how to describe this expansion so that variations in time of the volume \( V(t) \) occupied by the Universe had the form of Schrödinger Equation (9) or a similar form with a suitable effective hamiltonian \( \mathcal{H}_V \). The volume \( V(t) \) is an increasing function of time \( t \) in the present epoch, so its evolution is non–unitary and \( \mathcal{H}_V \) can not be hermitian. The non–unitary evolution operator solving the Schrödinger–like equation with this \( \mathcal{H}_V \) and acting on the initial state \( |0^M; V_{\text{init}}^0 \rangle \) should transform this state into the vector \( |\psi(t)\rangle = |0^\text{false}(t); V(t)\rangle \equiv \alpha \ [V(t)]^{1/2} \exp \left[-\frac{i}{\hbar}(t - t_{\text{init}}^0)\mathcal{H}\right]|0^\text{false}; V_{\text{init}}^0\rangle \), where \( \alpha \) is a complex or real number. The simplest \( \mathcal{H}_V \), which seems to be sufficient for the simplified qualitative analysis of the problem, may be
chosen as follows,

\[ \mathcal{H}_V \equiv \mathcal{H}_V(t) = (E_V + i \hbar \frac{d}{dt} \ln [a^{3/2}(t)]) \mathbb{I} \quad (28) \]

\[ = (E_V + i \hbar \frac{3}{2} H(t)) \mathbb{I}, \quad (29) \]

where \( a(t) = R(t)/R_0 \) is the scale factor, \( R(t) \) is the proper distance at epoch \( t \), \( R_0 = R(t_0) \) is the distance at the reference time \( t_0 \), (it can be also interpreted as the radius of the Universe now) and here \( t_0 \) denotes the present epoch (see, e. g. [37]), \( H(t) = \frac{\ddot{a}(t)}{a(t)} \) is the Hubble parameter, \( \dot{a}(t) = \frac{d}{dt}a(t) \) (in the general case \( \dot{f}(t) \equiv \frac{df(t)}{dt} \)), \( \mathbb{I} \) is the unit operator, \( \mathcal{H}_V \) is the non–hermitian effective Hamiltonian, \( E_V \) is a real parameter having dimension of the energy. The scale factor \( a(t) \) is a solution of Einstein’s equations, which with the Robertson—Walker metric in the standard form of Friedmann equations [37, 38], look as follows: The first one,

\[ \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k c^2}{R_0^2 a^2(t)} = \frac{8\pi G_N}{3} \rho + \frac{\Lambda c^2}{3}, \quad (30) \]

and the second one,

\[ \frac{\ddot{a}(t)}{a(t)} = - \frac{4\pi G_N}{3} \left( \frac{3\rho}{c^2} + \rho \right) + \frac{\Lambda c^2}{3}. \quad (31) \]

where the parameter \( \Lambda \) is known as the cosmological constant, \( \rho \) and \( p \) are mass density and pressure respectively, \( k \) denote the curvature signature, The pressure \( p \) and the density \( \rho \) are related to each other through the equation of state, \( p = w \rho c^2 \), where \( w \) is constant [37]. There is \( w = 0 \) for a dust (for a matter dominated era), \( w = 1/3 \) for a radiation and \( w = -1 \) for a vacuum energy.

The volume \( V(t) \) equals:

\[ V(t) = \frac{4}{3}\pi [R(t)]^3 \equiv \frac{4}{3}\pi [a(t)R_0]^3, \]

and similarly, \( V_0^{\text{init}} = \frac{4}{3}\pi [a(t_0^{\text{init}})R_0]^3 \). Therefore

\[ V(t) = \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^3 V_0^{\text{init}}. \quad (32) \]

We are looking for the solutions of the Schrödinger equation with the Hamiltonian \( \mathcal{H} + \mathcal{H}_V \) and a matrix element of the form \( \langle V_0^{\text{init}}, \text{false} | \psi(t) \rangle \) with \( |\psi(t_0^{\text{init}})\rangle = |0^M, V_0^{\text{init}}\rangle \). So we need solutions of the following equation

\[ i\hbar \frac{d}{dt} |\psi(t)\rangle = (\mathcal{H} + \mathcal{H}_V) |\psi(t)\rangle, \quad (33) \]
with the initial condition $|\psi(t_0^{\text{init}})\rangle = |0^M; V_0^{\text{init}}\rangle$. Here $\mathcal{H}$ is a hermitian operator (Hamiltonian) responsible for the decay of the false vacuum state $|0^M; V_0^{\text{init}}\rangle$ and $[\mathcal{H}, \mathcal{H}_V] = 0$. Now, let $|\psi(t)\rangle$ be of the form

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H}} |M(t)\rangle,$$

and

$$|M(t_0^{\text{init}})\rangle = |0^M; V_0^{\text{init}}\rangle. \quad (35)$$

Inserting (34) into (33) one obtains

$$\mathcal{H} e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H}} |M(t)\rangle + e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H} - \frac{i}{\hbar} \mathcal{H}_V} |M(t)\rangle = \mathcal{H} e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H}} |M(t)\rangle + e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H} - \frac{i}{\hbar} \mathcal{H}_V} |M(t)\rangle, \quad (36)$$

This means that our problem reduces to finding a solution of the following equation

$$i\hbar \frac{d}{dt} |M(t)\rangle = (E_V + i\hbar \frac{d}{dt} \ln [a^{3/2}(t)])|M(t)\rangle. \quad (37)$$

Putting

$$|M(t)\rangle = f(t) |M(t_0^{\text{init}})\rangle \equiv f(t) |0^M; V_0^{\text{init}}\rangle, \quad (38)$$

where $f(t)$ is a real or complex scalar function and $f(t_0^{\text{init}}) = 1$, we can rewrite Eq. (37) as follows

$$i\hbar \frac{df(t)}{dt} |0^M; V_0^{\text{init}}\rangle = (E_V + i\hbar \frac{d}{dt} \ln [a^{3/2}(t)]) f(t) |0^M; V_0^{\text{init}}\rangle. \quad (39)$$

A solution, $f(t)$, of this equation is

$$f(t) = N_f e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})E_V} e^{\int_{t_0^{\text{init}}}^{t} \frac{d}{dx} \ln [a^{3/2}(x)] dx} f(t_0^{\text{init}}) = N_f e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})E_V} \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^{3/2}, \quad (40)$$

where $N_f$ is a normalization factor. Now inserting this $f(t)$ into (38) and then using (34) we obtain the solution, $|\psi(t)\rangle$, of Eq. (33),

$$|\psi(t)\rangle = N_f e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})E_V} \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^{3/2} e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})\mathcal{H}} |0^M; V_0^{\text{init}}\rangle. \quad (41)$$
Thus
\[
\langle V_0^{\text{init}}, M|\psi(t) \rangle \equiv N_f e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})E_V} \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^{3/2} \times \\
\times \langle V_0^{\text{init}}, M|e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})H}|0^M; V_0^{\text{init}} \rangle,
\]
and
\[
\Pi(t) \overset{\text{def}}{=} |\langle V_0^{\text{init}}, M|\psi(t) \rangle|^2 \equiv N_f^2 \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^3 \mathcal{P}_0(t).
\]

Here
\[
\mathcal{P}_0(t) \overset{\text{def}}{=} |\langle V_0^{\text{init}}, M|e^{-\frac{i}{\hbar}(t - t_0^{\text{init}})H}|0^M, V_0^{\text{init}} \rangle|^2,
\]
is the survival probability of the system in the initial false vacuum state $|0^M; V_0^{\text{init}} \rangle$ assuming that volume $V_0^{\text{init}}$ occupied by this system remains unchanged. The function $\Pi(t)$ describes the combined effect of the processes of a decay and an expansion of the initially created universes.

There is $V(t) = \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^3 V_0^{\text{init}}$ but the use of normalization factor, $N_f$, allows us to write volume $V(t)$ as $V(t) \equiv \left[ \frac{a(t)}{a(t_0^{\text{init}})} \right]^3$. So
\[
\Pi(t) \equiv N_f^2 V(t) \mathcal{P}_0(t),
\]
and
\[
\Pi(t) N_0 \equiv N_f^2 V(t) \mathcal{P}_0(t),
\]
where $N_0$ is the number of universes of volume $V_0^{\text{init}}$ created at the initial instant $t_0^{\text{init}}$ with the vacua described by $|0^M; V_0^{\text{init}} \rangle$ and $V(t)$ is volume occupied by all these universes at the instant $t > t_0^{\text{init}}$ and it corresponds with $N(t)$ in (6). Relations (45), (46) describe in our simplified toy model the combined effect of the processes of a decay and of an expansion of the initially created universes of volumes $V_0^{\text{init}}$. In the case when the decay process is the dominant process then $\Pi(t)$ appearing in (46) is a decreasing function of time $t$ and it tends to zero as $t \to \infty$. If the expansion process prevails over the decay process or these processes both are in balance then $\Pi(t)$ is non–decreasing function of $t$. In such a case
\[
\frac{d}{dt}\Pi(t) \geq 0,
\]
that is,
\[
\frac{d}{dt}[\Pi(t)N_0] = N_0^2 \left( \frac{\dot{V}(t)}{V(t)} + \frac{\dot{P}_0(t)}{P_0(t)} \right) V(t) P_0(t)
\]
\[
= N_0^2 \left( 3 \frac{\dot{a}(t)}{a(t)} - \frac{\Gamma_M(t)}{\hbar} \right) V(t) P_0(t)
\]
\[
= N_0^2 \left( 3 H(t) - \frac{\Gamma_M(t)}{\hbar} \right) V(t) P_0(t) \geq 0. \quad (48)
\]

So, if there exists such time, say \( t = T_L > 0 \), that for all \( t \geq T_L \) the relation,
\[
d_{H,\Gamma} \overset{\text{def}}{=} 3 \frac{H(t)}{\hbar} - \frac{\Gamma_M(t)}{\hbar} \geq 0, \quad (49)
\]
is satisfied then the function \( \Pi(t) \) is a non-decreasing function of time \( t \) (it increases or is constant in time). This means that in such a case the decay process of the volumes \( V(t) = \left[ \frac{a(t)}{a(\text{init})} \right]^3 V_0^{\text{init}} \) should be stopped. Therefore if some universes had the luck to survive until time \( T_L \) such that for all \( t \geq T_L \) the relation \( (49) \) is fulfilled then later, when \( t > T_L \), these universes should be found undecayed.

### 4 Late time properties of the decay rate \( \Gamma_M(t) \) and related quantities

As it was mentioned in Sec. 2, the experimental evidence of deviations from the exponential decay law at long times, much later than the crossover time \( T \), was reported in \([35]\). This result gives rise to problem which is important for our considerations: If (and how) deviations from the exponential decay law at long times affect the decay rate of the unstable state and the energy of the system in this state.

From the condition \((18)\) for the amplitude \( A(t) \) and from \((7)\) it results that at the long time region the lowest bound for the survival probability \( \mathcal{P}(t) \) has the form
\[
\mathcal{P}(t) \sim B^2 e^{-2b|t|} \quad \text{for} \quad |t| \to \infty. \quad (50)
\]
This and the relation \((5)\) lead to the conclusion that (see \([36]\))
\[
\Gamma_M(t) \sim 2\hbar b t^{t-1} \quad \text{for} \quad |t| \to \infty, \quad (51)
\]

14
and thus \( \Gamma_M(t) \to 0 \) as \( t \to \infty \) because \( q < 1 \). A more accurate estimation of \( \Gamma_M(t) \) can be found using the amplitude \( A(t) \) instead of the condition (18) for the modulus \(|A(t)|\) of \( A(t) \).

So let us assume that we know the amplitude \( A(t) \). Equivalently it is sufficient to know the energy distribution \( \omega(E) \) of the system in the unstable state considered: In such a case \( A(t) \) can be calculated using (17). Then starting with the \( A(t) \) and using the expression (14) one can calculate the effective Hamiltonian \( h_M(t) \) in a general case for every \( t \). Thus, one can find the instantaneous energy, \( E_M(t) \), and the instantaneous decay rate, \( \Gamma_M(t) \), of the system in the metastable state \(|M\rangle\) for canonical decay times, when \( t \sim \tau_M < T_1 \), for transition times \( t \in (T_1, T_2) \) and for asymptotically late times \( t > T_2 \) (for details see: [39, 40, 41]).

The integral representation (17) of \( A(t) \) means that \( A(t) \) is the Fourier transform of the energy distribution function \( \omega(E) \). Using this fact we can find asymptotic form of \( A(t) \) for \( t \to \infty \), that is \( A_{lt}(t) \) (see [40] for details): As it has been shown in [40], if to assume that \( \lim_{E \to E_{min}} \omega(E) \equiv \omega_0 > 0 \) and \( \omega(E < E_{min}) = 0 \) and derivatives \( \omega^{(k)}(E) \), \( (k = 0, 1, 2, \ldots, n) \), are continuous in \([E_{min}, \infty)\), (that is if for \( E > E_{min} \) all \( \omega^{(k)}(E) \) are continuous and all the limits \( \lim_{E \to E_{min}} \omega^{(k)}(E) \) exist), and also all these \( \omega^{(k)}(E) \) are absolutely integrable functions, then

\[
A(t) \equiv A(t - t_0^{init}) \sim \frac{-i\hbar}{t - t_0^{init}} e^{-\frac{i}{\hbar}E_{min}(t - t_0^{init})} \times \\
\times \sum_{k=0}^{n-1} (-1)^k \left( \frac{i\hbar}{t - t_0^{init}} \right)^k \omega_0^{(k)} = A_{lt}(t), \quad (52)
\]

where \( \omega_0^{(k)} \equiv \lim_{E \to E_{min}} \omega^{(k)}(E) \) (see [40, 41]).

Bearing in mind the purpose of our considerations, which is to look from the point of view of the quantum theory of unstable states at the fate of the universe at times \( t \) very distant from the moment of its creation, \( t_0^{init} \), we assume that \( t > T_2 \gg t_0^{init} \). As a result we can write that \( (t - t_0^{init}) \approx t \) and we will use this conclusion in our late time asymptotic formulae for \( t \to \infty \) considered in this paper.

In the case of a universal more general form of \( \omega(E) \), when

\[
\omega(E) = (E - E_{min})^\lambda \eta(E) \in L_1(-\infty, \infty), \quad (53)
\]
where $0 < \lambda < 1$, and it is assumed that $\eta(E_{\min}) > 0$, $\eta(E < E_{\min}) = 0$ and derivatives $\eta^{(k)}(E)$, $(k = 0, 1, \ldots, n)$, exist and they are continuous in $[E_{\min}, \infty)$, and limits $\lim_{E \to E_{\min}+} \eta^{(k)}(E)$ exist, $\lim_{E \to \infty} (E - E_{\min})^{\lambda} \eta^{(k)}(E) = 0$ for all above mentioned $k$, there is,

$$
A(t) \sim (-1)^{-1} e^{-i E_{\min} t} \left[ \left( -\frac{i \hbar}{t} \right)^{\lambda+1} \Gamma(\lambda + 1) \eta_0 + \lambda \left( -\frac{i \hbar}{t} \right)^{\lambda+2} \Gamma(\lambda + 2) \eta_0^{(1)} + \ldots \right] = A_{lt}(t),
$$

as it has been shown in [40]. Here $\Gamma(z)$ is the Euler’s Gamma Function.

Starting from the asymptotic expression (54) for $A(t)$ and using (14) after some algebra one finds that in general for times $t > T_2$ (see [40])

$$
h_{\text{M}}(t)_{t \to \infty} \simeq E_{\min} + (-\frac{i \hbar}{t}) c_1 + (-\frac{i \hbar}{t})^2 c_2 + \ldots,
$$

where $c_i = c_i^*$, $i = 1, 2, \ldots$, (coefficients $c_i$ are determined by $\omega(E)$).

This last relation means that (see [42]),

$$
\Gamma_{\text{M}}(t) \simeq 2 c_1 \frac{\hbar}{t} - 2 c_3 \frac{\hbar^3}{t^3} \ldots, \text{ (for } t > T_2),
$$

and similarly

$$
E_{\text{M}}(t) \simeq E_{\min} - c_2 \frac{\hbar^2}{t^2} + c_4 \frac{\hbar^4}{t^4} + \ldots, \text{ (for } t > T_2),
$$

These properties take place for all unstable states which survived up to times $t > T_2$. From (57) it follows that $\lim_{t \to \infty} E_{\text{M}}(t) = E_{\min}$.

Note that the symmetry relations (13), (16) oraz (26) also hold for the asymptotic expansions (55), (56) and (57).

For the most general form (53) of the density $\omega(E)$ (i.e. for $A(t)$ having the asymptotic form given by (54)) we have (see [42] and references herein).

$$
c_1 = \lambda + 1 > 0, \quad c_2 = (\lambda + 1) \frac{\eta^{(1)}(E_{\min})}{\eta(E_{\min})} > 0.
$$

As an example let us consider a typical form of $\omega(E)$. Namely, properties of metastable systems are described in many papers with sufficient accuracy.
using $\omega(E)$ having the form of the Breit–Wigner energy distribution function, $\omega_{BW}(E)$,

$$\omega(E) \equiv \omega_{BW}(E) = \frac{N}{2\pi} \Theta(E - E_{\text{min}}) \frac{\Gamma_M^0}{(E - E_M^0)^2 + (\Gamma_M^0/2)^2}. \quad (59)$$

There is

$$c_1 = 1, \quad c_2 = -2 \frac{E_0 - E_{\text{min}}}{(\Gamma_M^0)^2 (\beta^2 + \frac{1}{4})}, \quad (60)$$

for $\omega(E) = \omega_{BW}(E)$, (see [43] for details). Here $\beta = \frac{E_0 - E_{\text{min}}}{\Gamma_M^0}$. In general, the sign of $c_2$ depends on the model considered (that is on the form of $\omega(E)$) and, contrary to the case of $\omega(E) = \omega_{BW}(E)$, there is $c_2 > 0$ for a wide class of $\omega(E) \neq \omega_{BW}(E)$.

The typical form of the survival probability $P(t) = |A(t)|^2$ at transition times is shown below in panel A of Fig (1) and Fig (2). The behavior of $\Gamma_M(t)$ at canonical decay times $t < T_1$, at transition times $t \in (T_1, T_2)$ and asymptotically late times $t > T_2$ is shown in panel B of Figs (1) and (2). These results are the direct, mathematical consequence (by (14) and (24)) of properties of the amplitude $A(t)$ at these time regions. It is seen from these Figures that at times $t < T_1$, $\Gamma_M(t) \simeq \Gamma_M^0$ to a very high accuracy, then rapid and large fluctuations of $\Gamma_M(t)$ occur at the transitions time region $t \in (T_1, T_2)$, and for very late times, $t > T_2$, $\Gamma_M(t) \to 0$ as $t \to \infty$ according to the result (56).

There is a widespread belief that the quantum theory accurately depicts reality. This belief is based on the facts that predictions of the quantum theory were confirmed experimentally to a very high accuracy. So it should be expected with the probability close to a certainty that the experimental confirmation of the presence of late time deviations from the exponential decay [35] means that the late time properties of $\Gamma_M(t)$ and $E_M(t)$ described in Eqs (56) and (57) and effects shown in panel (B) of Figs (1) and (2) should take place too, and should manifest itself under suitable conditions. Results presented in Figs (1), (2) were obtained for the Breit–Wigner energy distribution function (59) assuming for simplicity that $\beta = 10$. 

17
5 Instantaneous energy \( E_M(t) \) and the vacuum energy density at late times

From the point of view of the purpose of the paper specified in the Introduction and the results presented in Sec. 3 the most important is the knowledge of the late time asymptotic properties of the decay rate, \( \Gamma_M(t) \). Nevertheless, for the sake of completeness and for the convenience of readers, this Section will briefly discuss the late time asymptotic properties of the energy \( E(t) \) of an unstable system, which can be applied to the analysis of the evolution of a universe having metastable vacuum.

As it was mentioned in Sec. 2 in [4] the idea was formulated that in the case of the metastable vacuum states some space regions or universes can survive up to times much later than times when the exponential decay law holds. In the mentioned paper by Krauss and Dent the attention was focused on the possible behavior of the unstable false vacuum at very late times, where deviations from the exponential decay law become to be dominant. In [44] it was concluded that such an effect has to change the energy, \( E_M^0 \) of the system in the false (metastable) vacuum state at these times \( t \) so that at very long times \( E_M^0 \) is replaced by \( E_M(t) \) and at these times the typical form of \( E_M(t) \) looks as it results from the formula (57).

The typical behavior of the energy \( E_M(t) \) at canonical decay times \( t < T_1 \), at transition times \( t \in (T_1, T_2) \), (or \( t \sim T \)), and asymptotically late times \( t > T_2 \), is shown in panels (C) of Figs (1), (2) (see also [42, 45]) where the function

\[
\kappa(t) = \frac{E_M(t) - E_{\text{min}}}{E_M^0 - E_{\text{min}}} \quad (61)
\]

is presented. The red dashed line in these Figures denotes the value

\[
\kappa(t) = 1, \quad (t < T_1), \quad (62)
\]

that is \( E_M(t) = E_M^0 \). Note that there is \( E_M^0 > E_{\text{min}} \). From these Figures it is seen that for \( t < T_1 \) we have \( E_M(t) = E_M^0 \) whereas for \( t > T_2 \) there is

\[
E_M(t) - E_{\text{min}} \simeq \pm c_2 \frac{\hbar^2}{t^2}, \quad (t > T_2). \quad (63)
\]
Figure 1: Typical form of the decay curve (panel (A)), the decay rate (panel (B)) and instantaneous energy (panel (C)) of an unstable state as a function of time. Axes: In all panels $x = t/\tau_M$ (The time $t$ is measured as a multiple of the lifetime $\tau_M$); Panel (A) — $y = \mathcal{P}(t)$ (the logarithmic scale) — the survival probability; Panel (B) — $y = \Gamma_M(t)/\Gamma^0_M$; Panel (C) — $\kappa(t)$ (The instantaneous energy in relation to the energy measured at canonical decay times). The horizontal dashed line $y = 1$ represents in Panel (B) the value of $\Gamma_M(t)/\Gamma^0_M = 1$ whereas in Panel (C) it represents $\kappa(t) = 1$
Figure 2: Enlarged part of Fig 1 showing the behavior the survival probability $\mathcal{P}(t)$, decay rate $\Gamma_M(t)$ and $E(t)$ at the of the transition times region. Axes: $y = \Gamma_M(t)/\Gamma_M^0$, $x = t/\tau_M$. The horizontal dashed line $y = 1$ represents in Panel (B) the value of $\Gamma_M(t) \equiv \Gamma_M^0$ whereas in Panel (C) $\kappa(t) = 1$

When one considers a meta–stable (unstable or decaying) vacuum state,
If one prefers to consider $\Lambda(t)$ instead of $\rho_M(t)$ then one obtains,

$$\Lambda(t) - \Lambda_{\text{bare}} = (\Lambda_0 - \Lambda_{\text{bare}}) \kappa(t),$$

(68)

Analogous relations (with the same $\kappa(t)$) take place for $\Lambda(t) = \frac{8\pi G}{c^2} \rho_M(t)$.

The important property of $\kappa(t)$ is a presence of rapid fluctuations of the high amplitude for times $t \sim T$, i.e., for $t \in (T_1, T_2)$. This means that in the case of a decaying (unstable) vacuum analogous fluctuations of the energy density $\rho_M(t)$ and $\Lambda(t)$ should take place for $t \in (T_1, T_2)$. So if our Universe is the Universe with the unstable vacuum as the mass of Higgs boson suggests then in agreement with ideas expressed in [4] we can conclude that the lifetime of the false vacuum may be shorter by at least a few or even much more orders than the age of our Universe. This means that our Universe may place itself at the region of times described by the form of $\kappa(t)$ and $\Gamma_M(t)$ for $t > T_2$.

If one prefers to consider $\Lambda(t)$ instead of $\rho_M(t)$ then one obtains,
or,
\[ \Lambda(t) = \Lambda_{\text{bare}} + (\Lambda_0 - \Lambda_{\text{bare}}) \kappa(t), \] (69)
where \( \Lambda_0 = \frac{8\pi G}{c^2} \rho_M^0 \) and \( \Lambda_{\text{bare}} = \frac{8\pi G}{c^2} \rho_{\text{bare}} \).

One may expect that \( \Lambda_0 \) equals to the cosmological constant calculated within quantum field theory [46]. From (69) it is seen that for \( t < T_1 \),
\[ \Lambda_M(t) \simeq \Lambda_0, \quad \text{for} \quad (t < T_1), \] (70)
because \( \kappa(t < T_1) \simeq 1 \). Now if to assume that \( \Lambda_0 \) corresponds to the value of the cosmological constant \( \Lambda \) calculated within the quantum field theory, than one should expect that [46]
\[ \frac{\Lambda_0}{\Lambda_{\text{bare}}} \geq 10^{120}, \] (71)
(see [46]) which allows one to write down Eq. (69) as follows
\[ \Lambda_M(t) \simeq \Lambda_{\text{bare}} + \Lambda_0 \kappa(t). \] (72)

Note that for \( t > T_2 \) there should be (see (67))
\[ \Lambda_0 \kappa(t) \simeq \frac{8\pi G}{c^2} d_2^2 \frac{\hbar^2}{t^2}, \quad \text{for} \quad (t > T_2), \] (73)
that is
\[ \Lambda_M(t) \simeq \Lambda_{\text{bare}} \pm \frac{b_2}{t^2}, \quad \text{for} \quad (t > T_2), \] (74)
where \( b_2 = \frac{8\pi G}{c^2} \hbar^2 d_2 \) and the sign of \( b_2 \) is determined by the sign of \( d_2 \).

Note that a parametrization following from the quantum theoretical treatment of metastable vacuum states can explain why the cosmologies with the time–dependent cosmological constant \( \Lambda(t) \) are worth considering and may help to explain the cosmological constant problem [47, 48]. The time dependence of \( \Lambda \) of the type \( \Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2} \) was assumed eg. in [49] but there was no any explanation what physics suggests such a choice of the form of \( \Lambda \). Earlier analogous form of \( \Lambda \) was obtained in [50], where the invariance under scale transformations of the generalized Einstein equations was studied. Such a time dependence of \( \Lambda \) was postulated also in [51] as the result of the analysis of the large numbers hypothesis. The cosmological model with time dependent \( \Lambda \) of the above postulated form was studied also in [52] and in much more recent papers.
The nice feature and maybe even the advantage of the formalism presented in Section 4 and in this Section is that in the case of the universe with metastable (false) vacuum if one realizes that the decay of this unstable vacuum state is the quantum decay process then it emerges automatically that there have to exist the true ground state of the system that is the true (or bare) vacuum with the minimal energy, $E_{\text{min}} > -\infty$, of the system corresponding to this vacuum, and equivalently, $\rho_{\text{bare}} = E_{\text{min}}/V$, or $\Lambda_{\text{bare}}$. What is more, in this case the $\Lambda \equiv \Lambda(t)$ having the form described by equations (72) — (74) emerges quite naturally. In such a case the function $\kappa(t)$ given by the relation (61) describes time dependence for all times $t$ of the energy density $\rho_M(t)$ (or the cosmological ”constant” $\Lambda_M(t)$) and its general form is presented in panels (C) in Figs (1) and (2). Note that results presented in Sections 4 and 5 are rigorous.

As mentioned in the introduction to this Section, its aim is to inform readers about the late–time properties of the energy density $\rho_M(t)$ in the false vacuum state and how they can affect on the behavior of $\rho_M(t)$ and $\Lambda_M(t)$ at late times (see Eqs (67), and (72), (73)). We do not present here a more detailed analysis of a possible cosmological consequences of these properties because detailed discussion and analysis of the consequences of the late time properties of the density of the vacuum energy $\rho_M(t)$ and $\Lambda(t)$ briefly described in this Section readers can find in [45, 46, 53, 54, 55, 56].

6 Discussion and conclusions

The problem how the process of an expansion of a universe and its decay process affect together on this universe is analyzed in Sec. 3. The possible result of these combined processes is characterized by the condition (49). The obvious next step in the considerations in Sc. 3 and 4 is to apply the results obtained in them to the analysis of the possible future fate of the universe with an unstable vacuum. In the case of very late times assuming that the lifetime of the metastable false vacuum is shorter by at least a few or even much more orders than the age of our Universe it can be done by inserting into (49), eg. the present value of the Hubble expansion rate $H(t) = H(t_0) = H_0$ and the late time asymptotic form of the decay rate $\Gamma_M(t)$ given by relations (56) and (58),

$$\Gamma_M(t) \simeq 2(\lambda + 1) \frac{\hbar}{t} \text{ (for } t > T_2),$$

(75)
where the coefficient $c_3$ in (56) is neglected, and assuming that $t = t_0$, (where $t_0$ is the age of the our Universe). The only problem is to choose the appropriate value of $\lambda$ in (58). If to choose $\lambda$ appearing in the case of the decays into two particles, that is, $\lambda = 1/2$ (see, eg. [57]), then inserting the present values of $H_0$ and $t_0$ [8] into (49) one obtains that

$$d_{H,\Gamma_M} = 3H_0 - \frac{\Gamma_M(t_0)}{\hbar} \simeq 3 \left[ H_0 - \frac{1}{t_0} \right],$$

where $H_0 = H(t_0)$ is the present–day value of the Hubble parameter [8], which gives

$$d_{H,\Gamma_M} \simeq -0.001065 \text{ [Gyr]}^{-1} \simeq -3.3764 \times 10^{-19} \text{ [s]}^{-1} < 0. \quad (77)$$

This result suggests that in the case $\lambda = 1/2$ the Universe may decay at late times, but such a conclusion can not be considered to be decisive and final. First, taking into account the neglected term $c_3\frac{\hbar^3}{4\pi}$ in (75) can result in changing the sing of $d_{H,\Gamma_M}$. The second, there is no certainty that the choice $\lambda = 1/2$ is the correct choice. In fact, it is not known what value of $\lambda$ is correct for decays of the unstable vacuum states and this problem requires further studies. So, we need some bounds for the values of $\lambda$ that lead to the nonnegative $d_{H,\Gamma_M}$. The solution of the equation

$$d_{H,\Gamma} \overset{\text{def}}{=} 3H(t) - \frac{2(\lambda + 1)}{t} = 0, \quad (78)$$

which follows from (49) and (75) is

$$\lambda \simeq 0.426546. \quad (79)$$

This solution is obtained for the same values of $H(t) = H_0$ and $t = t_0$ which were used to find the result (77). The result (79) means that there should be

$$d_{H,\Gamma} \geq 0 \text{ for } 0 \leq \lambda \leq 0.426546 \quad (80)$$

and

$$d_{H,\Gamma} < 0 \text{ for } \lambda > 0.426546, \quad (81)$$

within the considered late time approximation (75) for $\Gamma(t)$. Thus, if the energy distribution $\omega(E)$ for the universe in the metastable vacuum state is given by the relation (53) with such $\lambda$ that $0 \leq \lambda \leq 0.426546$, then such a
universe should rather not decay. This conclusion show how important is to find \( \omega(E) \) and thus \( \lambda \) for the metastable vacuum state of the universe. To complete this discussion let us note that the Breit–Wigner energy distribution function (59) corresponds to the case \( \lambda = 0 \). This means that in the considered case of the late times, when the late time approximations for \( \Lambda(t) \) and \( \Gamma_M(t) \) hold, the use of the Breit–Wigner form of \( \omega(E) \) to characterize the false vacuum state can give our Universe stability. Unfortunately, it is not certain currently whether such \( \omega(E) \) correctly characterizes the energy distribution density in the false vacuum state. Among other things, for this reason, it is necessary to study properties of the metastable false vacuum state and the corresponding \( \omega(E) \). As it is seen from the results presented in Sec. 4, the coefficients \( c_1, c_2, \ldots \), in late time asymptotic expansions of \( \Gamma(t) \) and \( E_M(t) \) depend on the form of \( \omega(E) \) (see Eq. (60)). Therefore simply the knowledge of the correct \( \omega(E) \) is necessary when one wants to find the proper form, values and sign of the coefficient \( c_2 \) appearing in relations (57) and (63) and then \( d_2 \) in (67), and also \( c_3 \) in (56), but above all, knowing the correct \( \omega(E) \) we will be able to answer the question of whether the hypothesis mentioned in Sec. 1 and formulated by Krasus and Dent in [4] is realized in our Universe.

One may ask what do the results presented in this paper really mean? Suppose that our Universe was created in the metastable false vacuum state and the lifetime of this vacuum is much shorter than the time \( T_2 \) defined in Sec. 2 by Eq. (22) and that this \( T_2 \) is much shorter than the age of the Universe. Then in our epoch its survival probability, \( \mathcal{P}(t_0) \), is negligibly small: One can even say that it is zero to very high accuracy. The methods used in this paper and the quantum theory of unstable states do not give an answer for the question when this Universe should decay but they can explain why this Universe still exists and whether will it exist longer? In the light ideas presented in, e.g. [20] and in other papers mentioned in Sec. 1, such an information seems to be very important.

Note that formalism and results described in Sections 2, 4 and 5 are rigorous. The approach described in Sections 2 and 5 was applied in [45, 46, 53, 54, 55, 56], where cosmological models with \( \Lambda(t) = \Lambda_{\text{bare}} \pm \frac{\alpha^2}{t^2} \) were studied. From the results presented therein and in this paper, in the light of the LHC result concerning the mass of the Higgs boson [8] and its cosmological consequences, the conclusion follows that further studies of this approach are necessary.
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