Multiquark-Oriented QCD Sum Rules

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Abstract. We propose to increase the factual reliability of descriptions of exotic multiquark hadrons utilizing the approach to bound states of strongly interacting constituents known as QCD sum rules, by allowing exclusively all contributions that potentially bear some relevance for multiquark states to enter the correlation functions that form the main ingredient of this framework. The route to this goal is illustrated for the (presumably least involved) special case of tetraquark states.

1 Goal: Adaptation of QCD Sum-Rule Approach to Multiquark States

Evaluation of correlation functions of operators interpolating one’s hadrons of interest at both phenomenological (hadron) and fundamental (QCD) levels by application of operator product expansion [1] and Borel transformations, and assuming a complete cancellation of hadron and perturbative-QCD contributions above optimized [2–4] effective thresholds, provides analytic relationships, denoted as QCD sum rules [5], between, on the one hand, observable properties of hadrons and, on the other hand, basic parameters of QCD. In this context QCD accounts for both perturbative contributions, given by integrals over spectral densities, and nonperturbative power corrections (that involve vacuum condensates multiplied by powers of Borel variables).

The colour-singlet bound states governed by QCD encompass not just quark–antiquark mesons and three-quark baryons, labelled conventional hadrons, but also multiquark hadrons: tetraquarks, pentaquarks, hexaquarks, etc. For attempts to take hold of such exotic hadrons by QCD sum rules, a systematic scrutiny [6–10] points to the necessity of adapting the traditional wisdom to the peculiarities of multiquarks: Retaining just contributions potentially capable of providing information on multiquark features by getting rid of all contributions not related to multiquarks yields the novel sort of multiquark-adequate QCD sum rules; for some tetraquark categories [11], all of the latter involve, at least, two gluon exchanges of appropriate topology.

2 Proof of Assertion by Brief Sketch of Underlying Line of Argument

We illustrate our considerations for the simplest type of multiquark hadrons, the tetraquark $T$, a meson constituted of two antiquarks $\bar{q}_a, \bar{q}_c$ and two quarks $q_b, q_d$ (of masses $m_a, m_b, m_c, m_d$):

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\[ T = [\bar{q}_a q_b \bar{q}_c q_d] , \quad a, b, c, d \in \{a, d, s, c, b\} . \]

Leaving aside the spin degrees of freedom, by application of a Fierz transformation [12] every tetraquark-interpolating operator may be proven [13] to be equivalent to a linear combination of just two products \( \theta \) of colour-singlet quark–antiquark bilinear currents \( f_{\theta\lambda}(x) \equiv \bar{q}_a(x) q_b(x) \):

\[
\theta_{\mu\nu\lambda}(x) \equiv j_{\mu\nu}(x) j_{\rho\sigma}(x) , \quad \theta_{\mu\lambda\sigma}(x) \equiv j_{\mu\lambda}(x) j_{\rho\sigma}(x) .
\]

Accordingly, we regard it preferable to exploit (potential) manifestations of tetraquarks by means of intermediate-state poles in the scattering amplitudes of two conventional mesons (of momenta \( p_1, p_2 \)) to two conventional mesons, by scrutiny of vacuum averages of time-ordered products of four quark–antiquark bilinear currents \( f^{(i)} \), i.e., of four-point correlation functions

\[
\left\langle T\left( j(y) j(y') f^i(x) j^i(x') \right) \right\rangle .
\]

Configuration-space contractions of the latter yield all required types of correlation functions:

- two-point correlation functions of two operators \( \theta \), interpolating the tetraquark investigated,

\[
\left\langle T\left( \theta(y) \theta^i(x) \right) \right\rangle = \lim_{x' \to x} \left\langle T\left( j(y) j(y') f^i(x) j^i(x') \right) \right\rangle ;
\]

- three-point correlators of one current \( \theta \) and two conventional-meson interpolating currents \( j \)

\[
\left\langle T\left( j(y) j(y') \theta^i(x) \right) \right\rangle = \lim_{x' \to x} \left\langle T\left( j(y) j(y') f^i(x) j^i(x') \right) \right\rangle .
\]

Next, we need to formulate an exact and easily applicable criterion [14, 15] that allows for unequivocal identification of all contributions impacting any correlation function under study:

**Tetraquark-phile** [16, 17] contributions depend in a nonpolynomial form on the Mandelstam variable \( s \equiv (p_1 + p_2)^2 \) and – as can be established by fulfilment of the Landau equations [18] – enable true intermediate four-quark states, by exhibiting branch cuts that start at branch points

\[
\hat{s} = (m_a + m_b + m_c + m_d)^2 .
\]

As an example, we focus to the subset of flavour-exotic tetraquarks, which encompass two quarks and two antiquarks carrying four unequal quark flavours, generically labelled \( a, b, c, d \).

For such choice of quark-flavour contents, in the course of analysis it proves advantageous to discriminate two categories of four-point correlation functions, differing in the distributions of the four quark flavours among the pairs of interpolating currents \( f^{(i)} \) in initial or final states:

- **flavour-retaining** correlation functions [6, 8, 9], identified by identical flavour distributions,

\[
\left\langle T\left( j_{\mu\nu}(y) j_{\rho\sigma}(y') f^i_{\mu\nu}(x) j^i_{\rho\sigma}(x') \right) \right\rangle , \quad \left\langle T\left( j_{\mu\nu}(y) j_{\rho\sigma}(y') f^i_{\mu\nu}(x) j^i_{\rho\sigma}(x') \right) \right\rangle ;
\]

- **flavour-rearranging** correlation functions [6–9], identified by unequal flavour distributions,

\[
\left\langle T\left( j_{\mu\nu}(y) j_{\rho\sigma}(y') f^i_{\mu\nu}(x) j^i_{\rho\sigma}(x') \right) \right\rangle .
\]

Armed with our rigorous criterion [14, 15] above, for every two- or three-point correlation function it is then a straightforward task to identify the tetraquark-phile [16, 17] contributions:

**Flavour-retaining ones** [6, 8–10] with no or at most a single gluon exchange [Fig. 1(a, b)] are spatially separable and vanish at all or involve \( s \) in merely polynomial form, implying that just ones with two or more relevant gluon exchanges do affect tetraquarks [Fig. 1(c) and Fig. 2(a)].

**Flavour-reordering ones** [6–10] are not spatially separable and will support tetraquarks only if resorting to two or more gluon exchanges, as revealed by the Landau equations [Fig. 2(b, c)].
Flavour-preserving correlation functions of two tetraquark-interpolating operators $\theta$, resulting from double configuration-space contractions of a pair of operators $j$ interpolating conventional mesons: generic contributions of lowest (a), next-to-lowest (b) and next-to-next-to-lowest (c) perturbative orders.

Figure 1.

Exemplary tetraquark-phile contributions of lowest perturbative order to (a) flavour-retaining correlation functions of one tetraquark-interpolating current $\theta$ and two quark–antiquark bilinear currents $j$ and to flavour-reordering correlation functions of either (b) a pair of tetraquark-interpolating operators $\theta$ or (c) one tetraquark-interpolating operator $\theta$ and two ordinary-meson interpolating currents $j$, inferred by single (a,c) or double (b) configuration-space contraction of two meson-interpolating currents $j$, resp.

Figure 2.
3 Outcome: QCD Sum Rules Tailored to Peculiarities of Tetraquarks

Upon feeding the above concepts or insights into the traditional framework of QCD sum rules, it becomes feasible to disentangle, in any deduced relation, the unavoidable intertwining of expressions genuinely related to tetraquarks and those involving exclusively ordinary mesons:

- Every flavour-preserving relation (Fig. 3) consists of two kinds of QCD sum rules [6, 8–10]; the first of these provides two unconnected QCD sum rules for conventional mesons (Fig. 4) and the second one the required tetraquark-adequate QCD sum rule (Fig. 5) [6]. Refraining from one contraction of interpolating currents \( j^{(\dagger)} \) disentangles the three-point analogue [6].

- By investing a little bit more efforts, every flavour-rearranging relation may be unscrambled [6–10] and systematically reorganized as two relationships, one that does not [Fig. 6(a)] and one that does [Fig. 6(b)] involve two-meson s-channel cuts and potentially tetraquark poles.

\[
\text{Figure 3. Decomposing a generic flavour-retaining relationship into a pair of conventional-meson QCD sum rules (top row) and a QCD sum rule that possibly involves, among others, tetraquarks (bottom row).}
\]

\[
\text{Figure 4. Generic QCD sum rule for ordinary mesons (blue dashed lines) following traditional folklore.}
\]

\[
\text{Figure 5. Tetraquark-adequate QCD sum rule, possibly involving tetraquarks (blue dashed double line).}
\]

In addition to the perturbative contributions to the QCD sum rules aimed at, various power corrections have to be considered, too. Their inclusion is more tricky but accomplishable [19].

Ultimately, incorporating the aforementioned findings should offer a reliable extraction of the fundamental properties of tetraquarks [their masses \( M \), decay constants \( f_{abcd} \equiv \langle 0|\theta_{abcd}|T \rangle \) and \( f_{adcb} \equiv \langle 0|\theta_{adcb}|T \rangle \), and momentum-space amplitudes \( A(T \rightarrow j_{ab} j_{cd}) \) and \( A(T \rightarrow j_{ad} j_{cb}) \)] from tetraquark-adequate QCD sum rules for all two- and three-point correlation functions of...
the fundamental properties of tetraquarks [their masses corrections have to be considered, too. Their inclusion is more tricky but accomplishable [19].

Every flavour-preserving relation (Fig. 3) consists of two kinds of QCD sum rules [6, 8–10]; expressions genuinely related to tetraquarks and those involving exclusively ordinary mesons: it becomes feasible to disentangle, in any deduced relation, the unavoidable intertwinement of

![Diagram](Image)

**Figure 6.** Decomposing each *flavour-rearranging* relation into (a) one without four-quark or two-meson s-channel cut and (b) a *tetraquark-adequate* QCD sum rule supporting tetraquarks (blue horizontal bar).

interest involving flavour-preserving or -reordering tetraquark-phile spectral densities $\rho_{p, x}$ and $\Delta_{p, x}$ and effective thresholds $s_{\text{eff}}(\tau)$ depending on Borel parameters $\tau$, symbolically of the form

\[
(f_{\pi\pi\pi\pi})^2 \exp(-M^2 \tau) \\
= \int_{\hat{\mathcal{H}}} ds \exp(-s \tau) \rho_{p}(s) + \text{Borel-transformed power corrections},
\]

\[
f_{\pi\pi\pi\pi} A(T \rightarrow j_{\pi\pi\pi\pi}) \exp(-M^2 \tau) \\
= \int_{\hat{\mathcal{H}}} ds \exp(-s \tau) \Delta_{p}(s) + \text{Borel-transformed power corrections},
\]

\[
f_{\pi\pi\pi\pi} f_{\pi\pi\pi\pi} \exp(-M^2 \tau) \\
= \int_{\hat{\mathcal{H}}} ds \exp(-s \tau) \rho_{x}(s) + \text{Borel-transformed power corrections},
\]

\[
f_{\pi\pi\pi\pi} A(T \rightarrow j_{\pi\pi\pi\pi}) \exp(-M^2 \tau) \\
= \int_{\hat{\mathcal{H}}} ds \exp(-s \tau) \Delta_{x}(s) + \text{Borel-transformed power corrections}.
\]

Last but not least, the *cluster reducibility* [20] of any multiquark (its group-theory-implied potential decomposition into a cluster of ordinary hadrons) requires to regard a multiquark, on an equal footing, as tightly bound compact and loosely bound molecular-type hadron [21, 22].

**Acknowledgements.** D. M. and H. S. express sincere gratitude for support by joint CNRS/RFBR Grant PRC Russia/19-52-15022, D. M. for support by the Austrian Science Fund (FWF), Project P29028-N27, H. S. for support by EU research and innovation program Horizon 2020 under Grant Agreement 824093.

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