Gulon-pair-Creation Production Model of Strong Interaction Vertices

Bing-Dong Wan$^1$ and Cong-Feng Qiao$^{1,2}$

$^1$ School of Physics, University of Chinese Academy of Science, Yuquan Road 19A, Beijing 10049
$^2$ CAS Center for Excellence in Particle Physics, Beijing 10049, China

By studying the $\eta_c$ decays exclusively to double glueballs, we introduce a model to mimic phenomenologically the gluon-pair-vacuum interaction vertices, namely the $0^{++}$ model. Based on this model, we study glueball production in $\eta_c$ decay, explicitly $\eta_c \to f_0(1500)\eta(1405)$. Among them $f_0(1500)$ is well-known scalar glueball candidate and $\eta(1405)$ is thought of a candidate for pseudoscalar glueball. We discuss the possibility of finding these light glueballs in their production via the $0^{++}$ model. We also discuss the heavier glueball production in $\eta_b$ decays, which might be detectable in the LHCb and Belle-II experiments.

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I. INTRODUCTION

According to the theory of strong interaction, the Quantum Chromodynamics (QCD) [1], gluons have self-interaction, which suggests in some sense the existence of glueball. The search for glueballs has experienced a long history, however the existence evidence is still vague. Being short of reliable glueball production and decay mechanisms makes the corresponding investigation rather difficult. Another hurdle hinder- ing the glueball searching lies in the fact that usually glue balls are around 1 GeV, which is lower than the results from quenched lattice QCD.

In this paper, we discuss the glueball production in $\eta_c$ decay by introducing a model for the gluon-pair-vacuum interaction vertices, namely the $0^{++}$ model, as shown in the the Fig.(1). We assume the gluon pair is created homogeneously in space with equal probability. Comparing to the $J^P_0$ model [27–35], which models the quark-antiquark pair creation in the vacuum, we formulate an explicit vacuum gluon-pair-transition matrix and estimate the strength of the gluon-pair creation. Employing the $0^{++}$ model, we then investigate the $\eta_c$ decay to scalar and pseudoscalar glueballs. Based on our knowledge about glueballs, we choose $f_0(1500)$ and $\eta(1405)$ as scalar and pseudoscalar glueball candidates respectively. The possibility of finding $\eta_c \to f_0(1500)\eta(1405)$ is discussed.

The rest of the paper is arranged as follows. After the introduction, we construct a model for gluon-pair-vacuum interaction vertices in Sec.II. In Sec.III, we evaluate the process $\eta_c \to f_0(1500)\eta(1405)$. Last section is remained for summary and outlooks.

![FIG. 1: Schematic diagram for glueball production in $\eta_c$ decay in $0^{++}$ model.](image-url)
II. CONSTRUCTION OF THE 0++ MODEL

In quantum fields theory, the physical vacuum is the ground state of energy and a large number of particle fields fluctuate in it. Therefore, there are certain probabilities of quark pairs and gluon pairs with vacuum quantum numbers being induced from the vacuum by energy fluctuation. It is reasonable to conjecture that gluon pairs would be created with equal amplitude in space, just like the quark-antiquark pairs do in $3P_0$ model. Created from the vacuum, the gluon pairs hence possess the quantum numbers $J^P = 0^+$, and so the $0^+$ model is named.

Following we investigate glueball production in $\eta_c$ decays by means of the $0^+$ model, as shown in the Fig.(1). The transition amplitude of $\eta_c$ exclusively decay to double glueballs for instance can be formulated as

$$\langle G_1 G_2 | T | \eta_c \rangle = g_s^2\gamma_\eta \langle G_1 G_2 | \bar{q}_1 \gamma_m \gamma_\rho A^2_m | \eta_c \rangle \times \delta_{\delta_c \delta_{\bar{c}}} A^c_A \delta_{\delta_{\bar{c}} A} A^\rho \langle \eta_c \rangle .$$

(1)

Here, $G_1$ and $G_2$ represent glueballs, $g_s$ is the strong coupling constant, $\gamma_\eta$ is a constant with energy dimension representing the strength of gluon pair creation from the vacuum which can be extracted by fitting to the experimental data, $\delta_{\delta_c \delta_{\bar{c}}}$ is known as the Minkowski metric, $q_i$ is the quark field with color index, $A^\rho$ is for the gluon field with Lorentz superscript and color subscript, $\rho$ is the Gell-Mann matrix of SU$(3)$ group. The $\delta_{\delta_c \delta_{\bar{c}}} A^c_A \delta_{\delta_{\bar{c}} A}$ term is employed to produce the gluon pair from the vacuum.

Inserting the completeness relation $\sum_G \langle G | G \rangle = 2E_G$ into Eq.(1), we get

$$\langle G_1 G_2 | T | \eta_c \rangle = \frac{1}{2E_G} \sum_G g_s^2 \gamma_\eta \langle G_1 G_2 | \bar{q}_1 \gamma_m \gamma_\rho A^2_m | G \rangle \times \delta_{\delta_c \delta_{\bar{c}}} A^c_A \delta_{\delta_{\bar{c}} A} A^\rho \langle \eta_c \rangle \times \delta_{\delta_c \delta_{\bar{c}}}$$

(2)

where $|G\rangle$ is the shorthand notation for two gluons $g_1$ and $g_2$, emitted from $\eta_c$ and the phase space integration is hidden in $|G\rangle$ state, as shown in the following Eq.(6). $T_1$ is the transition operator for $G g_3 g_4 \rightarrow G_1 G_2$, where $g_3 g_4$ represent the gluon pair created from the vacuum and $T_2$ is the transition operator for $\eta_c \rightarrow g_1 g_2$.

![Fig. 2: The Feynman Diagram for $T_1$ and $T_2$.](image)

The transition matrix $T_1$ can be decomposed as follows (see Fig.(2) for illustration):

$$T_1 = I_1 \otimes I_2 \otimes T_{vac} ,$$

(3)

where $T_{vac}$ is the vacuum-gluon pair transition amplitude, $I_i$ are identity matrices indicating the propagations of $g_1$ and $g_2$. Considering $g_3$ and $g_4$ are created in vacuum and gluon spin equals 1, the spin third-components of two gluons thus have three different combinations, that is $|m_1, m_2\rangle$ may take $|1, -1\rangle, |0, 0\rangle,$ or $|-1, 1\rangle$. The total spin state of the vacuum produced gluon pair, $|S, M_G\rangle$, is a singlet, and can be formulated as

$$\chi^{34}_{0,0} = \frac{1}{\sqrt{3}} (|1, -1\rangle m_1 m_2 - |0, 0\rangle m_1 m_2 + |-1, 1\rangle m_1 m_2).$$

(4)

The $T_{vac}$ can be expressed as

$$T_{vac} = \gamma \int d^3k_1 d^3k_4 \delta^3(k_3 + k_4) Y_{00} \left(\frac{k_3 - k_4}{2}\right) \times \chi^{34}_{0,0} \delta_{\delta_c \delta_{\bar{c}}} A^c_A \delta_{\delta_{\bar{c}} A} A^\rho \langle \eta_c \rangle .$$

(5)

Here, $k_3$ and $k_4$ represent 3-momenta of gluons $g_3$ and $g_4$, $A^c_A$ and $A^\rho \langle \eta_c \rangle$ represent 3-momenta of gluons with color indices, and $Y_{00} (\theta, \phi)$ is the 0th solid harmonic polynomial that gives the momentum-space distribution of the produced gluon pair.

The state $|G\rangle$ obviously possesses the same quantum numbers of $|\eta_c\rangle$, i.e. $J^P = 0^+$, thus can be expressed as

$$|G\rangle = \sqrt{2E_G} \int d^3k_1 d^3k_2 \delta^3 (K_G - k_1 - k_2) \times \sum_{M_{G_1}, M_{G_2}} \langle L_G M_{G_1}, S_{G_1} G_{S_1}, J_{G_1} M_{J_1} | G \rangle \times \psi_{\eta_c, L_G M_{G_1}} (k_1, k_2) \chi^{12}_{S_{G_1} M_{J_1}} \delta_{\delta_c \delta_{\bar{c}}} | g_1 g_2 \rangle ,$$

(6)

where $k_1$ and $k_2$ represent 3-momenta of gluons $g_1$ and $g_2$, $\psi_{\eta_c, L_G M_{G_1}} (k_1, k_2)$ is the spatial wave function with $n, L, S, J$ the principal quantum number, orbital angular momentum, total spin and the total angular momentum of $|G\rangle$, respectively. $\chi^{12}$ is the corresponding spin state, and can be expressed as $|S_{G_1} M_{J_1}\rangle$. $\langle L_G M_{G_1}, S_{G_1} G_{S_1}, J_{G_1} M_{J_1} | G \rangle$ represents the Clebsch-Gordan coefficient and for $|G\rangle$ state it reads $\langle m_1; 1 - m_0|00\rangle$. Various $K$ with different subscripts represent the corresponding 3-momenta. The normalization conditions write

$$\langle G (K_G) | G (K_G') \rangle = 2E_G \delta^3 (K_G - K_G') ,$$

(7)

$$\langle g^c_A (k_i) | g^b_A (k_j) \rangle = \delta_{ij} \delta^{ab} \delta^3 (k_i - k_j) ,$$

(8)

$$\int d^4k_1 d^4k_2 \delta^3 (K_G - k_1 - k_2) \psi_{G_1} (k_1, k_2) \psi_{G_2} (k_1, k_2) = \delta_{G G} .$$

(9)

Similarly we may have expressions for $G_1$ and $G_2$ states.

Equipped with the gluon-to-gluball transition operator $T_1$ and expressions for initial and final states, we can now calculate the transition matrix element.
\[
(G_1 G_2 | T_1 | G) = \gamma_s \sqrt{8E_G E_G} \sum_{(M_{1G}, M_{1G}, M_{1G})} \langle M_{1G}, M_{1G}, M_{1G} | (M_{1G}, M_{1G}, M_{1G}) \rangle \\
\times \langle L_G M_{1G} S_{G1} M_{S1} | j G | M_{1G}, M_{1G}, M_{1G} \rangle \langle L_G M_{1G} S_{G1} | j G | M_{1G}, M_{1G}, M_{1G} \rangle \\
\times \langle \chi_{13}^{13} \chi_{24}^{24} \chi_{12}^{12} \chi_{24}^{24} \chi_{00}^{00} \rangle \langle M_{1G}, M_{1G}, M_{1G} \rangle \delta_{\alpha \beta \gamma \delta} \delta_{\alpha \beta \gamma \delta} \text{color-octet}.
\]

Here the momentum space integral \(I_{M_{1G}, M_{1G}, M_{1G}}(K)\) is

\[
I_{M_{1G}, M_{1G}, M_{1G}}(K) = \int d^3 k_1 d^3 k_2 d^3 k_3 d^3 k_4 \delta^3(k_1 + k_2) \delta^3(k_3 + k_4) \delta^3(k G_1 - k_1 - k_3) \delta^3(k G_2 - k_2 - k_4) \\
\times \psi_{\eta, \eta_0, \eta_0}^* \psi_{\eta, \eta_0, \eta_0} (k_1, k_3) \psi_{\eta, \eta_0, \eta_0}^* (k_2, k_4) \psi_{\eta, \eta_0, \eta_0} (k_1, k_2) \delta_{\alpha \beta \gamma \delta} \delta_{\alpha \beta \gamma \delta} \text{color-octet}.
\]

The table shows that the \(G_1 G_2 | T_1 | G\) state wave functions are conjectured to be in harmonic oscillator (HO) form

\[
\psi_{nLM}(k) = \mathcal{N}_n \exp \left( -\frac{k^2}{2} \right) Y_{LM}(k) P(k^2),
\]

where \(k\) is the relative momentum between two gluons inside the states, \(\mathcal{N}_n\) is the normalization coefficient and \(P(k^2)\) is a polynomial of \(k^2\) [31]. \(\langle \chi_{13}^{13} \chi_{24}^{24} \chi_{12}^{12} \chi_{24}^{24} \chi_{00}^{00} \rangle\) which denotes the spin coupling can be expressed by Wigner’s 9j symbol [29]

\[
\langle \chi_{S1}^{S1} \chi_{S2}^{S2} \chi_{S3}^{S3} \rangle = (-1)^{S_1} \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G - 1 \rangle \langle S G + 1 \rangle \langle 2S_G + 1 \rangle \\
+ \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G - 1 \rangle \langle 2S_G - 1 \rangle \langle 2S_G + 1 \rangle \\
+ \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G - 1 \rangle \langle 2S_G + 1 \rangle \langle 2S_G - 1 \rangle \\
+ \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G + 1 \rangle \langle 2S_G - 1 \rangle \langle 2S_G + 1 \rangle \\
+ \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G + 1 \rangle \langle 2S_G + 1 \rangle \langle 2S_G - 1 \rangle \\
+ \delta_{S_1}^{S_2} \delta_{S_2}^{S_3} \delta_{S_3}^{S_1} \langle 2S_G - 1 \rangle \langle 2S_G - 1 \rangle \langle 2S_G - 1 \rangle.
\]

Here, \(s_1\) is spin of the gluon \(g_i\) with \(i = 1, 2, 3, 4\), and \(\sum_{S,M} \langle S M \rangle \langle S M \rangle = \text{completeness relation}.

The helicity amplitude \(M^{M_{1G}, M_{1G}, M_{1G}}\) may be read off from

\[
\langle G_1 G_2 | T_1 | G \rangle = \delta^3(K G_1 + K G_2 - K G_2) M^{M_{1G}, M_{1G}, M_{1G}} \cdot
\]

by which the decay width for process \(\eta_e \rightarrow G_1 G_2\) can be readily calculated through [31]:

\[
\Gamma = \frac{\alpha^2}{2 \pi} \frac{M^2}{M_{\eta_e}} \sum_{JL} |M^{JL}|^2.
\]

Here, \(M^{JL} = \frac{M^{JL}}{2E_{\eta_e}}\). \(M_2\) is the amplitude of process \(\eta_e \rightarrow g g\), and \(M^{JL}\) is the partial wave amplitude which can be obtained by converting the helicity amplitude \(M^{M_{1G}, M_{1G}, M_{1G}}\) through the Jacob-Wick formula [36]

\[
M^{JL} = \frac{\sqrt{2 J + 1}}{2 J + 1} \sum_{M_{1G}, M_{G}} \langle L0 J M_{1G} | J G M_{G} \rangle \\
\times \langle J G, M_{1G}, J G, M_{1G}, J G, M_{1G} | M_{1G}, M_{1G}, M_{1G} \rangle \delta_{\alpha \beta \gamma \delta} \delta_{\alpha \beta \gamma \delta} \text{color-octet}.
\]

with \(J = J_{G_1} + J_{G_2}\) and \(L = J_{G_2} - J_{G_1}\).

III. GLEEBALL PRODUCTION IN \(\eta_e\) DECAYS VIA THE \(0^{++}\) MODEL

In this section, we estimate the scalar and the pseudoscalar glueballs production in \(\eta_e\) decays via the \(0^{++}\) model, taking \(f_0(1500)\) and \(\eta(1405)\) as the corresponding candidates, named \(G_1\) and \(G_2\) respectively. The quantum numbers of the states involve in the process are presented in Table I, where \(|G\rangle\) and \(|\eta_e\rangle\) are the same in quantum number.

| \(J^{PC}\) | \(L\) | \(E\) | \(S\) | \(M\) |
|---|---|---|---|---|
| \(0^{++}\) | 1 | 0 | 0 | 0 |
| \(f_0(1500)\) | 0 | 0 | 0 | 0 |
| \(\eta(1405)\) | 0 | 1 | 0 | 0 |

TABLE I: Quantum numbers of \(\eta_e\), \(f_0(1500)\) and \(\eta(1405)\). The values of \(M_0\) and \(M_2\) can be \(-1, 0\) and \(1\).

A. The calculation of \(T_1\)

In Eq.(10), the color contraction gives a number 8, and for scalar glueballs, the spin and orbital angular momentum coupling leads the C-G coefficient to be \((00; 0000) = 1\). Therefore, Eq.(10) in this situation turns to

\[
\langle G_1 G_2 | T_1 | G \rangle = \sum_{M_0, M_2} \gamma_s \sqrt{8E_G E_G} \times \langle M_{1G} | 1 - M_{2G} | 00 \rangle \\
\times \langle 1 - M_{2G} | 1 - M_{2G} | 00 \rangle \\
\times \langle \chi_{13}^{13} \chi_{24}^{24} \chi_{12}^{12} \chi_{24}^{24} \chi_{00}^{00} \rangle \langle M_0, M_0 | \gamma_0 | M_0, M_0 \rangle (K) \cdot
\]

The spin coupling \(\langle \chi_{13}^{13} \chi_{24}^{24} \chi_{12}^{12} \chi_{24}^{24} \chi_{00}^{00} \rangle\) in Eq.(13) is in Wigner’s 9j symbol, which is a representation of 4-particle
spin coupling and can be expanded as series of 2-particle spin couplings represented by Wigner’s 3j symbol [29]. The Wigner’s 3j and 9j symbols read
\[
\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}} \langle j_1j_2m_1m_2|j, -m \rangle \tag{18}
\]
and
\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{pmatrix} = \sum_m \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ m_7 & m_8 & m_9 \end{pmatrix} \tag{19}
\]
respectively. Applying to Eq.(13), the spin coupling term then reduces to
\[
\langle \chi_{00}^{13} \chi_{11}^{12} | M_1 \rangle = 3 \sum_{S\,M_S} \langle 00; 1 - M_2 | S \rangle S \rangle \times \langle S \, M_S | 1 - M_0; 00 \rangle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & S \end{pmatrix} \tag{20}
\]
In above equation, evidently \langle 00; 1 - M_2 | S \rangle S \rangle and \langle S \, M_S | 1 - M_0; 00 \rangle will be nonzero only when \( S = 1 \), which means \( M_S \) can be any of 1, 0 or -1. Thus all possible \( S \, M_S \) are \( |1, 1\rangle, |1, 0\rangle \) and \( |1, 1\rangle \). In addition, \langle 00; 1 - M_2 | S \rangle S \rangle and \langle S \, M_S | 1 - M_0; 00 \rangle will be zero unless \( M_0 = M_2 = -M_S \).

Given \( M = M_S \) the Wigner’s 9j symbol can then be calculated as follows:
\[
\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \sum_m \begin{pmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ m_3 & m_4 & -M \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ M & 0 & -M \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ m_1 & m_2 & -M \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ m_3 & m_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & M & -M \end{pmatrix}
\]
\[
= \frac{1}{9} \langle 1m_1; 1m_3|00 \rangle \langle 1m_2; 1m_4|1M \rangle \times \langle 1M; 00|1M \rangle \langle 1m_1; 1m_3|00; 1M \rangle \tag{21}
\]
Every term in above equation can be evaluated by normal C-G coefficient. That is:
\[
\langle 1m_1; 1m_3|00 \rangle = \sqrt{\frac{1}{3}} (\delta_{m_1,1}\delta_{m_3,-1} - \delta_{m_1,0}\delta_{m_3,0} + \delta_{m_1,-1}\delta_{m_3,1}) \tag{22}
\]
\[
\langle 1M; 00|1M \rangle = \sqrt{\frac{2}{2}} \tag{23}
\]
\[
\langle 1m_1; 1m_3|00; 1M \rangle = \sqrt{\frac{1}{3}} (\delta_{m_1,1}\delta_{m_3,-1} - \delta_{m_1,0}\delta_{m_3,0} + \delta_{m_1,-1}\delta_{m_3,1}) \tag{24}
\]
\[
\langle 1m_2; 1m_4|00 \rangle = \sqrt{\frac{1}{3}} (\delta_{m_2,1}\delta_{m_4,-1} - \delta_{m_2,0}\delta_{m_4,0} + \delta_{m_2,-1}\delta_{m_4,1}) \tag{25}
\]
\[
\langle 1m_2; 1m_4|1M \rangle = \sqrt{\frac{2}{2}} \tag{26}
\]
\[
\langle 1m_1; 1m_3|1M \rangle = \frac{\sqrt{3}}{3} \times \delta_{m_1,1}\delta_{m_3,-1} - \frac{\sqrt{3}}{3} \times \delta_{m_1,0}\delta_{m_3,0} + \frac{\sqrt{3}}{3} \times \delta_{m_1,-1}\delta_{m_3,1} \tag{27}
\]
\[
\langle 1m_1; 1m_3|1M \rangle = \frac{\sqrt{3}}{3} \times \delta_{m_1,1}\delta_{m_3,-1} - \frac{\sqrt{3}}{3} \times \delta_{m_1,0}\delta_{m_3,0} + \frac{\sqrt{3}}{3} \times \delta_{m_1,-1}\delta_{m_3,1} \tag{29}
\]
\[
\langle 1m_1; 1m_3|1M \rangle = \frac{\sqrt{3}}{3} \times \delta_{m_1,1}\delta_{m_3,-1} - \frac{\sqrt{3}}{3} \times \delta_{m_1,0}\delta_{m_3,0} + \frac{\sqrt{3}}{3} \times \delta_{m_1,-1}\delta_{m_3,1} \tag{31}
\]
After inserting above ingredients into Eq.(20), we get the corresponding spin couplings:
\[
\langle \chi_{00}^{13} \chi_{11}^{12} | M_1 \rangle = \frac{1}{72} \times \delta_{m_1,1}\delta_{m_3,-1} - \frac{1}{72} \times \delta_{m_1,0}\delta_{m_3,0} + \frac{1}{72} \times \delta_{m_1,-1}\delta_{m_3,1} \tag{32}
\]
which equals \( -\frac{1}{72} \) for \( m_1 = -1, m_2 = 0, m_3 = 1, m_4 = -1 \) or \( m_1 = 0, m_2 = -1, m_3 = 1, m_4 = -1 \) and 0 for all other cases.
\[
\langle \chi_{00}^{13} \chi_{11}^{12} | M_1 \rangle = \frac{1}{72} \times \delta_{m_1,1}\delta_{m_3,-1} - \frac{1}{72} \times \delta_{m_1,0}\delta_{m_3,0} + \frac{1}{72} \times \delta_{m_1,-1}\delta_{m_3,1} \tag{33}
\]
which equals \( -\frac{1}{72} \) for \( m_1 = -1, m_2 = 1, m_3 = 1, m_4 = -1 \) or \( m_1 = 1, m_2 = -1, m_3 = -1, m_4 = 1 \) and 0 for all other cases.
Substituting above spin couplings into Eq.(17), \(T_1\) will be reduced to

\[
\langle G_1 G_2 | T_1 | G \rangle = -\frac{1}{\sqrt{8}} \sqrt{\frac{8 E_G E_{G_1} E_{G_2}}{\pi}} \times \left( \langle 11, 1 | \psi \rangle \langle 11, 1 | \psi \rangle I_{0,0,0}(K) \right) \\
+ \langle 10, 1 | \psi \rangle \langle 10, 1 | \psi \rangle I_{0,0,0}(K) \\
+ \langle 1, 1 | \psi \rangle \langle 1, 1 | \psi \rangle I_{0,0,0}(K) \\
= -\frac{\gamma_8}{27} \sqrt{\frac{8 E_G E_{G_1} E_{G_2}}{\pi}} \\
\times \left( I_{0,0,0}(K) + I_{0,0,0}(K) + I_{0,0,0}(K) \right). \tag{35} \]

For the non-trivial situation, that is \(M_0 = M_2 = -M\), the momentum integral \(I_{M_0, M_1, M_2}(K)\) in Eq.(35) now reduces to

\[
I_{M_0, M_1, M_2}(K) = \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \delta^3(k_1 + k_2) \\
\times \delta^3(k_3 + k_4) \delta^3(k_{G_1} - k_1 - k_3) \\
\times \delta^3(k_{G_2} - k_2 - k_4) \psi_{n_0, 00}^*(m, k_1, k_3) \\
\times \psi_{n_1, 1M}^*(m, k_2, k_4) \psi_{n_1, 1M}(m, k_1, k_2) \\
\times \gamma_8 \left( \frac{k_1 - k_2}{2} \right). \tag{36} \]

Provided the ground state dominates in our calculation, we may simply take the principal quantum numbers \(n_0, n_1\) and \(n_2\) to be 1. The wave function \(\psi\) then turns to

\[
\psi_{100}(k) = \frac{1}{\sqrt{3} \pi^{3/2}} R^{3/2} \exp \left( -\frac{R^2 k_1^2}{2} \right), \tag{37} \]

\[
\psi_{11M}(k) = i \sqrt{\frac{2}{3} \pi^{3/2}} R^{3/2} c_1 \exp \left( -\frac{R^2 k_1^2}{2} \right), \tag{38} \]

where \(k = (k_x + i k_y)/\sqrt{2}\) and \(0 = k_z\), are the spherical components of vector \(k\).

Simplifying Eq.(36), we now have

\[
I_{M_0, M_1, M_2}(K) = \psi_{100} (k_1) \psi_{100} (k_1) \int d^3k_1 \\
\times \psi_{11M}^*(k_2, k_3) \psi_{11M} (k_1, k_2) \\
\times \gamma_8 \left( \frac{k_1 - k_2}{2} \right). \tag{39} \]

Here, in the \(\eta_\ell\) center of mass frame, \(K_G = K_{\eta_\ell} = 0\) and \(K_{G_1} = -K_{G_2} = K\). Applying which to Eqs.(37) and (38), the spatial wave functions now write

\[
\psi_{100}^* (k_1) = \frac{R^{3/2}}{\pi^{1/4}} \exp \left( -\frac{R^2 (2k_1 - K)^2}{8} \right), \tag{40} \]

\[
\psi_{11M} (k_1, k_2) = -i \frac{R^{3/2}}{\sqrt{2} \pi^{3/4}} (2k_1 - K) \exp \left( -\frac{R^2 (2k_1 - K)^2}{8} \right). \tag{41} \]

\[
\psi_{11M}^* (k_1, k_2) = \frac{R^{3/2}}{\pi^{1/4}} \exp \left( -\frac{R^2 (2k_1 - K)^2}{8} \right), \tag{42} \]

\[
\eta_{00} = \frac{1}{\sqrt{4\pi}}. \tag{43} \]

where \(R_0, R_1\) and \(R_2\) are the most probable radii of \(\eta_\ell, f_0(1500)\) and \(\eta(1405)\), respectively. After performing the integration, one may notice that the states \(M = 1\) and \(M = -1\) give null contribution, i.e. \(I_{1,0,1} = I_{-1,0,-1} = 0\, and\)

\[
I_{0,0,0} = -\frac{\gamma_8}{8 R^{3/2} (R_1^2 + R_2^2)} 6 \frac{\gamma_8}{\pi^{1/4}} E_G E_{G_1} E_{G_2} \times \exp \left( -\frac{K^2 R_1^2 + R_2^2}{8 (R_1^2 + R_2^2)} \right) \times \left( R_1^2 + R_2^2 \left[ K^4 (R_1^2 + R_2^2)^2 - 96 \right] + 12 R_1^2 \left[ K^2 (R_1^2 + R_2^2)^2 - 4 \right] - 12 (R_1^2 + R_2^2)^2 \left[ K^2 (R_1^2 + R_2^2) + 4 \right] \right). \tag{44} \]

Given \(\delta^3(K_G - K_{G_1} - K_{G_2})I = I_{0,0,0}\) and considering of Eqs.(14), (35) and (44), we have

\[
\langle G_1 G_2 | T_1 | G \rangle = \frac{\gamma_8}{27} \sqrt{\frac{8 E_G E_{G_1} E_{G_2}}{\pi}} \times \exp \left( -\frac{K^2 R_1^2 + R_2^2}{8 (R_1^2 + R_2^2)} \right) \times \left( R_1^2 + R_2^2 \left[ K^4 (R_1^2 + R_2^2)^2 - 96 \right] + 12 R_1^2 \left[ K^2 (R_1^2 + R_2^2)^2 - 4 \right] - 12 (R_1^2 + R_2^2)^2 \left[ K^2 (R_1^2 + R_2^2) + 4 \right] \right). \tag{45} \]

from which \(M_{10, 0, 0} \cdot M_{0, 0, 0} \cdot M_{0, 0, 0}\) can be extracted, i.e.

\[
M_{10, 0, 0} = \frac{\gamma_8}{27} I \sqrt{\frac{8 E_G E_{G_1} E_{G_2}}{\pi}}. \tag{46} \]

The most probable radius \(R\) of the HO wave function can be estimated through \(R = 1/\alpha\), where \(\alpha = \sqrt{\mu/\bar{n}}\). Here, \(\mu\) denotes the reduced mass, \(\omega\) is the angular frequency of harmonic oscillator which is given by \(E_m = (2n + L + 3/2)\hbar\omega\), with \(E_m\) being the gluonball inner energy, \(n\) the radial quantum number, and \(L\) the orbital angular momentum. Note, the inner energy equals to the glueball mass. As discussed in Refs.[37, 38], here we also take the effective mass of the constituent quark to be 0.6 GeV, which yields \(\mu = 0.3\) GeV for glueballs. In our calculation, the masses of \(\eta_\ell, f_0(1500)\) and \(\eta(1405)\) are obtained from PDG [39], i.e. \(M_{\eta_\ell} = 2.98\) GeV, \(M_0 = 1.50\) GeV and \(M_2 = 1.41\) GeV. In \(\eta_\ell\) center-of-mass system, the momenta of \(f_0(1500)\) and \(\eta(1405)\) are fixed to be 0.32 GeV, and hence their total energies are known, as given in Table II.

Taking into account above discussion and input values, we can readily get \(I = 0.41 GeV^{-3/2}\) and \(M_{I00} = 0.11\gamma_8\), and in case of \((LO/IM_0 | J_G M_{J_G}) = \langle L_0 J_0 | 000 | 000000 \rangle = (LO/IM_0 | J_G M_{J_G}) = \langle L_0 J_0 | 000000 | 1, M_{I00} = M_{I00} = 0.11\gamma_8\).
TABLE II: The energy, mass, $\omega$, $\alpha$ and the most probable radii of $\eta_c$, $f_0(1500)$ and $\eta(1405)$.

| $E$(GeV) | $E_\omega$(GeV) | $\omega$(GeV) | $\alpha$(GeV) | $R$(GeV)$^{-1}$ |
|----------|-----------------|---------------|---------------|-----------------|
| $G_1$    | 2.98            | 2.98          | 0.66          | 0.45            | 2.24            |
| $G_2$    | 1.53            | 1.50          | 0.43          | 0.36            | 2.79            |
| $G_3$    | 1.45            | 1.41          | 0.31          | 0.31            | 3.26            |

B. The calculation of $T_2$

The calculation of the process $\eta_c \to gg$ is quite straightforward. In leading order of perturbative QCD, there are only two Feynman diagrams, as shown in Fig.3. Their decay amplitudes read:

\[ iM_1^{\text{gravab}} \bar{e}_p(k_1)e_q(k_2) = (ig_s)^2\bar{\eta}(p_2)\gamma^\mu p_1 \frac{i}{p_1 - k_1 - m_\eta} \gamma^\mu p_1 \times u(p_1)\bar{e}_p(k_1)e_q(k_2); \]

\[ iM_2^{\text{gravab}} \bar{e}_p(k_1)e_q(k_2) = (ig_s)^2\bar{\eta}(p_2)\gamma^\mu p_1 \frac{i}{p_1 - k_2 - m_\eta} \gamma^\mu p_1 \times u(p_1)\bar{e}_p(k_1)e_q(k_2); \]

where $u$ and $\bar{v}$ are Dirac spinors of charm quarks, $\bar{e}_p$ denotes gluon polarization vector, and $g_s$ is the QCD coupling constant. For quark pair to form a pseudoscalar quarkonium, in normal routine one can realize it by performing the following projection [40]:

\[ u(p)\bar{u}(-p) \to \frac{i\gamma_5R_{\eta_c}(0)}{2\sqrt{2\pi}} \otimes \left( \frac{1}{\sqrt{N_c}} \right); \]

where in $\eta_c$ center-of-mass system, $p_1 = p_2 = p$, $m_c$ is the charm quark mass, $R(0)_{\eta_c}$ is the $\eta_c$ radial wave function at the origin, which in our calculation takes $|R(0)_{\eta_c}|^2 = 0.527 \pm 0.013$ GeV$^3$ [40]. Considering of above discussion, the $\eta_c \to gg$ matrix element squared can be readily obtained:

\[ |M_2|^2 = \frac{4g_s^2|\bar{\eta}(p_2)|^2}{3\pi m_c}. \]

C. Estimation of $\gamma_s$

We estimate the strength of glue-pair-vacuum coupling, in analogous to the $^3P_0$ model, where the strength of quark pair creation in vacuum is represented by $\gamma$. According to the argument of Ref. [33], $\gamma = \frac{g}{2\alpha}$, where $g$ is a constant with energy dimension, $m$ is the created quark mass. To avoid constructing new model to mimic the non-perturbative process of gluon pair production in vacuum, we simply infer the $\gamma_s$ by comparing the relative strength of $^3\bar{q}q \to gg$ to $q\bar{q} \to q\bar{q}$ processes, as shown in Fig.4. The $\gamma_s^2/g^2$ is conjectured to be at the same order of the relative rate of those two processes.

It is well known that at the tree level

\[ |\bar{M}(q\bar{q} \to q\bar{q})|^2 = \frac{4}{9} \left( \frac{s^2 + u^2}{t^2 + s^2} - \frac{2u^2}{3st} \right); \]

\[ |\bar{M}(q\bar{q} \to gg)|^2 = \frac{32}{27} \left( \frac{9(t^2 + u^2)}{4s^2} + \frac{t + u}{t} \right), \]

and considering the relationship between Mandelstam variables, we can get

\[ \gamma_s^2/g^2 \approx \frac{\sigma(q\bar{q} \to gg)}{\sigma(q\bar{q} \to q\bar{q})} = 0.0288, \]

where the interaction energy is set to be $2m_u$. Since in $^3P_0$ model $\gamma = 3 \times 6.9$ [31], we then have $\gamma_s^2 \approx 0.0288 \times \gamma^2 = 0.0288 \times (2m_u)^2 = 0.239^{+0.146}_{-0.079}$ GeV$^2$, where $m_u$ is the u-quark mass, equals $2.2^{+0.6}_{-0.4}$ MeV [39].

D. The glueball production rates in $0^{++}$ model

From the Eq.(2) we know $M_{UL}^I = \frac{M_{UL}^I M_{UL}^0}{\omega m_c}$, in which $|M_2|^2 = \frac{4g_s^2|\bar{\eta}(p_2)|^2}{3\pi m_c}$ and $M_{UL}^I$ has only one nonzero matrix element, the $M_{10}^0 = \frac{1}{2\pi} \sqrt{8E_{\gamma}E_{G_{\gamma}}}$. Substituting them into Eq.(15), we readily get the decay width of $\eta_c \to f_0(1500)\eta(1405)$ process:

\[ \Gamma = \pi^2 \frac{|K|}{M_{UL}^0} \sum_{JL} |M_{UL}^I|^2 = \pi^2 \frac{|K|}{M_{UL}^0} |M_1^{(0)}|^2 |M_2|^2 \]

\[ = \frac{8\pi^2 g_s^2 |\bar{\eta}(p_2)|^2 |K|E_{\gamma}E_{G_{\gamma}}}{3\pi m_c M_{UL}^0} = 50.73^{+35.47}_{-18.37} \text{ keV}. \]

In above calculation, charm quark mass $m_c = (1.27 \pm 0.03)$ GeV, strong coupling constant $\alpha_s = 0.25$, and all other inputs are given in preceding sections. From PDG [39] we know that the $\eta_c$ total decay width

\[ \Gamma = 50.73^{+35.47}_{-18.37} \text{ keV}. \]
is \( \Gamma_{\text{total}} = 31.8 \pm 0.8 \) MeV, and hence the branching ratio of
\( \eta_c \to f_0(1500)\eta(1405) \) process is about

\[
Br_{\eta_c \to f_0(1500)\eta(1405)} = \frac{\Gamma_{\eta_c \to f_0(1500)\eta(1405)}}{\Gamma_{\text{total}}} = 1.80^{+0.98}_{-0.81} \times 10^{-3}. \quad (55)
\]

Similar as \( \eta_c \), its upsilon family partner \( \eta_u \) may also exclusively decay to scalar and pseudoscalar glueballs, which can be evaluated by \( 0^{++} \) model. From lattice QCD calculation [2–5, 24], there are scalar and pseudoscalar glueball candidates with masses 1.75 GeV and 2.39 GeV, respectively. With the same procedure performed in above for \( \eta_c \), we can readily get \( \Gamma_{\eta_c \to G^0^+ G^0^-} = 0.19^{+0.12}_{-0.07} \) MeV and \( Br_{\eta_c \to G^0^+ G^0^-} = 1.90^{+0.27}_{-0.16} \times 10^{-2} \).

**IV. SUMMARY**

In this paper, we discuss the glueball exclusive production in pseudoscalar quarkonium exclusive decay to glueballs. Refinement of the \( 0^{++} \) model still needs a lot of tedious works. Due to the importance of glueball physics, various mechanisms of glueball decay and production are deserved to explore.

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