Estimation of the parameters of multivariate stable distributions

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ABSTRACT

In this paper, we first discuss some of the well-known methods available in the literature for the estimation of the parameters of a univariate/multivariate stable distribution. Based on the available methods, a new hybrid method is proposed for the estimation of the parameters of a univariate stable distribution. The proposed method is further used for the estimation of the parameters of a strictly multivariate stable distribution. The efficiency, accuracy and simplicity of the new method is shown through Monte-Carlo simulation. Finally, we apply the proposed method to the univariate and bivariate financial data.

1. Introduction

In finance, economics, statistical physics, and various other engineering fields, we often encounter datasets where the “fitted” distribution deviates from the normal distribution and exhibits excess skewness, kurtosis and heavy tails. To address this concern, Lévy (1924), in his study on Generalized Central Limit Theorem, introduced a rich class of distributions known as the stable distributions. Each univariate distribution, in this class, is characterized by four parameters, namely $\alpha$, $\beta$, $\sigma$, and $\delta$, which, respectively, denote the index of stability, skewness, scale and shift of the distribution. Their respective ranges are given by $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$ and $\delta \in \mathbb{R}$.

On the other hand, a $d$-dimensional stable random vector is determined by $\alpha \in (0, 2]$, the shift vector $\delta \in \mathbb{R}^d$ and the spectral measure $\mathbf{C}$ (a finite Borel measure) on $S^d = \{s : ||s||_2 = 1\}$ which denotes a unit sphere in $\mathbb{R}^d$.

Next, it is natural to fit these distributions on the datasets showing excess skewness, kurtosis and heavy tails, and this brings us to the problem of estimation of the above mentioned parameters. We now begin our discussion with the introduction to some well-known methods that efficiently estimate the parameters of the univariate stable distribution. This will be followed by the discussion of methods available for the multivariate case.

Fama and Roll (1971) estimate the parameters of symmetric stable distribution (i.e., $\beta = 0$) using quantile method which is later generalized and improved by McCulloch (1986) to incorporate the skewed case (i.e., $\beta \neq 0$). Press (1972) estimates the parameters using method of moments, while Koutrouvelis (1980) and Kogon and Williams (1998) estimate the parameters using the characteristic function and regression. DuMouchel (1973) proposed the maximum likelihood method which is further studied by Mittnik et al. (1999) and Nolan (2001). Though, these techniques are beneficial in modeling heavy-tailed data (see Nolan 2001, 2003, 2014; Wang et al. 2015;...
Kateregga, Mataramvura, and Taylor (2017) simulation studies reveal certain limitations for each of these methods (see Adler, Feldman, and Taqqu 1998; Borak, Hardle, and Weron 2005). McCulloch’s quantile method (McCulloch 1986) is computationally faster than the regression-based estimation by Koutrouvelis (1980) and Kogon and Williams (1998), but fails to provide an estimate whenever \( \alpha < 0.6 \). The estimates obtained via the method of moments (see Press 1972) are of poor quality and are not recommended for more than preliminary estimation. Koutrouvelis regression-based method is iterative in nature and requires the use of look-up tables which makes the estimation of the parameters quite complex. Thus, Kogon and Williams (1998) simplify and eliminate the need of numerous iterations and the use of look-up tables thereby making the method considerably faster and better in comparison to the method of Koutrouvelis especially near \( \alpha = 1 \) and \( \beta \neq 0 \). However, the method gives slightly worse estimates for very small \( \alpha \). Finally, the maximum likelihood method (DuMouchel 1973; Mittnik et al. 1999; Nolan 2001) seems to give the most accurate estimates, however, it involves numerical complexities. For this purpose, Nolan (1997) introduced a program called STABLE that includes a fast pre-computed spline approximation to stable densities for \( \alpha / \gamma > 0 \), for quick maximum likelihood estimation of stable parameters.

In comparison to the univariate case, not much is known about the estimation of the parameters of the multivariate stable distribution. However, some of the estimation methods specifically focusing on the estimation of the spectral measure are by Rachev and Xin (1993), Cheng and Rachev (1995), Nolan, Panorska, and McCulloch (2001), and Mohammadi, Mohammadpour, and Ogata (2015). Their methods are based on the use of characteristic functions. Pivato and Seco (2003), used spherical harmonic analysis while Teimouri, Rezakhah, and Mohammadpour (2017) make use of the U-statistic proposed by Fan (2006). Ogata (2013) proposed the use of generalized empirical likelihood (GEL) method where they constructed the estimating function by empirical and theoretical characteristic function.

In this paper, for the univariate case, we propose a new hybrid method of estimation. In terms of the accuracy and root mean square error of the estimates, the new method outperforms the Kogon-Williams and McCulloch’s quantile-based method. On the other hand, in terms, of computational efficiency, the method outperforms the iterative Koutrouvelis regression method and the maximum likelihood method (Nolan 2001), however, is computationally slow in comparison to the maximum likelihood estimation done using the STABLE program. Further, motivated by Nolan, Panorska, and McCulloch (2001), we use our proposed hybrid method which jointly estimates all the parameters of a strictly multivariate stable distribution and outperforms the methods of Mohammadi, Mohammadpour, and Ogata (2015) and Teimouri, Rezakhah, and Mohammadpour (2017), both, in terms, of computational efficiency and accuracy of the estimators. The term “hybrid” is used to reflect the combination and modification in the methods of Koutrouvelis and Kogon-Williams. The efficiency, accuracy and simplicity of this new technique are shown through simulation results.

The paper is organized as follows. In Sec. 2, we discuss some well-known facts related to multivariate stable distributions. In Sec. 3, a new hybrid method (univariate case) is proposed which efficiently estimates \( \alpha, \beta, \sigma \) and \( \delta \). The estimates found are then used to obtain the estimates of the parameters of a strictly multivariate stable distribution. The estimate of the spectral measure \( \Gamma \) is obtained using the empirical characteristic function method for the multivariate case. In Sec. 4, the new method is compared with some of the well-known methods. Finally, using financial data, the efficiency of the new method, both for the univariate and multivariate case, is demonstrated in Sec. 5.

**2. Preliminaries and notations**

In general, for the univariate stable distributions, closed forms for densities are not available, except for a few well-known distributions viz. normal \( (\alpha = 2, \beta = 0) \), Cauchy \( (\alpha = 1, \beta = 0) \) and
Lévy (α = 1/2, β = 1). However, closed form representation for the characteristic function of a univariate/multivariate stable distribution is available. We first define the multivariate stable random vector \( \mathbf{X} \in \mathbb{R}^d \) and the characteristic function representation of the distribution of \( \mathbf{X} \). For details, see Samorodnitsky and Taqqu (1994).

**Definition 1.** A random vector \( \mathbf{X}=(X_1, X_2, X_3, \ldots, X_d) \) is said to be a stable random vector in \( \mathbb{R}^d \) if: \( \forall A > 0 \) and \( B > 0 \) \( \exists C > 0 \) and \( \mathbf{D} \in \mathbb{R}^d \) such that:

\[
AX^{(1)} + BX^{(2)} \overset{d}{=} CX + \mathbf{D}
\]

where \( X^{(1)} \) and \( X^{(2)} \) are independent copies of \( \mathbf{X} \) and \( C = (A^2 + B^2)^{1/\alpha} \).

**Definition 2.** A random vector \( \mathbf{X} \in \mathbb{R}^d \) is stable if for any \( n \geq 2 \), there is an \( \alpha \in (0,2] \) and a vector \( \mathbf{D}_n \) such that

\[
X^{(1)} + X^{(2)} + \cdots + X^{(n)} \overset{d}{=} n^{1/\alpha} \mathbf{X} + \mathbf{D}_n
\]

where \( X^{(1)}, X^{(2)}, \ldots, X^{(n)} \) are independent copies of \( \mathbf{X} \).

The vector \( \mathbf{X} \) is strictly stable when \( \mathbf{D} = 0 \) \( \forall A > 0 \) and \( B > 0 \) in (1) and \( \mathbf{D}_n = 0 \) \( \forall n \geq 2 \) in (2). The vector \( \mathbf{X} \) is symmetric stable if it is stable and satisfies the relation:

\[
P\{ \mathbf{X} \in A \} \overset{d}{=} P\{ -\mathbf{X} \in A \}
\]

for any Borel set \( A \) of \( \mathbb{R}^d \). The index \( \alpha \) in (2) is called the index of stability of the vector \( \mathbf{X} \) which represents the tail thickness of the distribution.

We need the following notations to define the characteristic function representation. Let \( \phi(t) = \mathbb{E}(e^{i<\mathbf{X}, t>}) \) denote the characteristic function of \( \mathbf{X} \), where \( i \) is the unit imaginary number and \( t \in \mathbb{R}^d \). Also, let \( \Im(\cdot) \) and \( \Re(\cdot) \) respectively denote the imaginary and real part of the argument and sign(\( \cdot \)) denote the sign function. The standard parametrization of the characteristic function of a stable random vector \( \mathbf{X} \) when \( \alpha \in (0,2] \) is as follows

\[
\phi(t) = \mathbb{E}(e^{i<\mathbf{X}, t>}) = e^{-\delta(t)}, \; t \in \mathbb{R}^d
\]

where \( < \cdot, \cdot > \) denotes the dot product between the two vectors and

\[
I(t) = \int_{S^d} \psi_\alpha(\langle t, s \rangle) \Gamma(ds) + i < \delta, t >
\]

where

\[
\psi_\alpha(u) = \begin{cases} 
|u|^\alpha(1-i \text{sign}(u) \tan \frac{\pi \alpha}{2}), & \alpha \neq 1, \\
|u|(1 + i \text{sign}(u) \ln |u|), & \alpha = 1.
\end{cases}
\]

**Remark 1.** In the multivariate case, \( \beta = 1 \) and \( \sigma = 1 \). See (STABLE for \( R \)).

**Remark 2.** The standard parametrization is discontinuous at \( \alpha = 1 \), since \( |\tan \frac{\pi \alpha}{2}| \to \infty \) as \( \alpha \to 1 \). As a result, \( \Gamma \) and \( \delta \) are poorly estimated whenever \( \alpha \to 1 \). To overcome this problem, one can use the multivariate version of parametrization given by Zolotarev (1989) also termed as the continuous parametrization defined in Nolan (1998) where

\[
\psi_\alpha(u) = \begin{cases} 
|u|^\alpha(1+i \text{sign}(u) \tan \frac{\pi \alpha}{2}(|u|^{1-\alpha}-1)), & \alpha \neq 1, \\
|u|(1 + i \frac{2}{\pi} \text{sign}(u) \ln |u|), & \alpha = 1.
\end{cases}
\]
Remark 3. For \(d = 1\), the univariate stable random variable \(X\) is described by four parameters \((\alpha, \beta, \sigma, \delta)\). The two main characteristic function representations for random variable \(X\) are given by

\[
\phi(t) = \begin{cases} 
\exp \left\{ -(\alpha |t|)^\alpha \left[ 1 + i\beta \text{sign}(t) \tan \left( \frac{\pi \alpha}{2} \right) \right] + i\delta_1 t \right\}, & \alpha \neq 1, \\
\exp \left\{ -\sigma |t| \left[ 1 + i\beta \frac{2}{\pi} \text{sign}(t) \ln |t| \right] + i\delta_1 t \right\}, & \alpha = 1;
\end{cases}
\]

where \(\delta = \begin{cases} 
\delta_1 + \beta \sigma \tan \frac{\pi \alpha}{2}, & \alpha \neq 1 \\
\delta_1 + \beta \frac{2}{\pi} \sigma \ln \sigma, & \alpha = 1.
\end{cases}\)

Nolan (2014) recommends (6) for numerical computations and statistical analysis, as the characteristic function is jointly continuous in all parameters, while (7) can be used in the study of theoretical properties of the stable distribution.

Remark 4. Let \(X\) be a univariate stable random variable with characteristic function as defined in (6) and (7).

1. If \(\alpha \neq 1\), then \(X\) is strictly stable if and only if
   \[
   \delta = \delta_1 + \beta \sigma \tan \frac{\pi \alpha}{2} = 0
   \]
2. If \(\alpha = 1\), then \(X\) is strictly stable if and only if \(\beta = 0\).

Remark 5. Let \(X\) be a multivariate stable random vector with the characteristic function representation defined above. We say that \(X\) is strictly stable random vector if \(\delta = 0\).

3. Estimation of the parameters of the stable random vector

Our main goal is to efficiently and accurately estimate the parameters \((\alpha, \Gamma, \delta)\), given \(X^{(1)}, X^{(2)}, ..., X^{(n)}\) as independent copies of \(X\), a stable random vector. The parameters are known as \(\alpha\), the characteristic exponent, \(\Gamma\), the spectral measure and \(\delta\), the shift vector.

3.1. Estimation of \(\alpha\) and \(\delta\)

For the estimation of \(\alpha\) and the shift vector \(\delta\), Nolan, Panorska, and McCulloch (2001) suggested using some univariate method to estimate the one-dimensional parameters \((\hat{\alpha}_j, \hat{\beta}_j, \hat{\sigma}_j, \hat{\delta}_j), j = 1, 2, ..., d\) for each of the coordinates of the \(d\)-dimensional dataset. In Nolan, Panorska, and McCulloch (2001), the vector \(\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, ..., \hat{\delta}_d)\) is used as an estimate of the shift vector and \(\hat{\alpha} = (\sum_{j=1}^d \hat{\alpha}_j)/d\) is used as an estimate of the joint index of stability \(\alpha\).

We now discuss our proposed hybrid method (univariate case), useful in obtaining the estimates of \(\alpha, \beta, \sigma\) and \(\delta\). The estimates obtained are then used to estimate \(\alpha\) and the shift vector \(\delta\) for the multivariate case as discussed above.
3.1.1. Proposed hybrid Method-Univariate case

Step 1.

a. Given a sample of iid observations \(x_1, x_2, \ldots, x_n\), we obtain the preliminary estimates of \(r\) and \(d\), say \(\hat{r}_0\) and \(\hat{d}_0\) respectively, using the maximum likelihood estimation method suggested by Nolan (2001).

b. Next, we normalize the sample data with the initial estimates of scale (\(\hat{r}_0\)) and shift parameter (\(\hat{d}_0\)) using the transformation

\[x'_j = \frac{x_j - \hat{d}_0}{\hat{r}_0}, j = 1, 2, \ldots, n.\]

The above normalization is necessary, in order to remove the dependence of the estimators on \(r\) and \(d\) as originally suggested by Paulson, Holcomb, and Leitch (1975) and for the optimal selection of the sample characteristic arguments which is the part of modified Koutrouvelis-Kogon-Williams empirical characteristic function based method to obtain the parameter estimates as discussed in Step 2 and Step 3.

Step 2. Compute the sample characteristic function

\[\hat{\phi}(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itx'_j}\]

of the normalized sample \(x'_1, x'_2, \ldots, x'_n\). From (6), observe that, for \(\alpha \neq 1\)

\[\ln (-\ln |\phi(t)|^2) = \ln (2\sigma^2) + \alpha \ln |t|\]

(8)

Using (8), obtain the estimates of \(\alpha\) and \(\sigma\) for the normalized sample data using least squares regression in the model

\[y_k = \mu + x a_k + \epsilon_k, \quad k = 1, 2, \ldots, K,\]

(9)

where \(y_k = \ln (-\ln |\hat{\phi}(t_k)|^2), \mu = \ln (2\sigma^2), a_k = \ln |t_k|, \epsilon_k\) denotes the error term and \(K = (|f(x, n)|), t_k = \frac{2k}{25}\) are points obtained using (10). Let \(\hat{a}_1\) and \(\hat{\sigma}_1\) denote the regression estimates of \(\alpha\) and \(\sigma\) respectively.

3.1.1.1. Derivation of \(f(x, n)\). For finding the optimal value \(K\), we make use of Koutrouvelis look-up Table 1 and obtain a function corresponding to the values given under \(x\) and each sample size \((n = 200, 800\) and \(1600)\) via the method of least squares regression that best fits the dataset given in Table 1. The function that best fits the given dataset is

\[f(x, n) = \begin{cases} 
24.36x^{-1.47}, & 1 < n \leq 200, \\
20.58x^{-1.43}, & 201 < n \leq 800, \\
122.9x^4 - 648.2x^3 + 1245x^2 - 1040x + 335.2, & 801 < n \leq 1600.
\end{cases}\]

(10)

The \(R^2\) values for the two power curves when \(1 < n \leq 200\) and \(201 < n \leq 800\) are 0.949 and 0.963 respectively, while the \(R^2\) value is 0.996 when \(801 < n \leq 1600\). The function is then evaluated at \(\hat{a}_0\), the preliminary estimate of \(x\) obtained using maximum likelihood method suggested by Nolan (2001). For \(n > 1600\), we recommend the division of sample size into equal parts, so
that each part has the sample size smaller than 1600 and apply the proposed method to each sub-sample. The final estimate of each parameter is the average of the estimates obtained from each sub-sample.

**Step 3.** From (6), observe that for \( \alpha \neq 1 \),

\[
\mathcal{Z}(\ln \phi(t)) = \delta_1 t + \beta \sigma t (|\sigma t|^{2\alpha - 1} - 1) \tan \frac{\pi \alpha}{2}.
\]

Using (11), obtain the estimates of \( \beta \) and \( \delta_1 \) for the normalized sample data using least squares regression in the model

\[
z_k = \delta_1 t_k + \beta v_k + \eta_k, \quad k = 1, 2, ..., K,
\]

where \( z_k = \mathcal{Z}(\ln \hat{\phi}(t_k)) \), \( v_k = \hat{\sigma}_1 t_k (|\hat{\sigma}_1 t_k|^{2\alpha - 1} - 1) \tan \frac{\pi \alpha}{2} \), \( \eta_k \) denotes the error term and \( K, t_k \) are the points used in the estimation of \( \alpha \) and \( \sigma \) for the normalized sample as discussed in Step 2. Let \( \hat{\beta}_1 \) and \( \hat{\delta}_1 \) denote the regression estimates of \( \beta \) and \( \delta_1 \) respectively.

**Step 4.** Compute the final estimates of the sample data as \( \hat{z} = \hat{z}_1 = \hat{\beta}_1, \quad \hat{\alpha} = \hat{\alpha}_0 \hat{\sigma}_1, \quad \hat{\delta} = \hat{\sigma}_0 \hat{\delta}_1 + \hat{\delta}_0 \)

**3.2. Estimation of \( \Gamma \)**

The estimation of the spectral measure is vital in the modeling of stochastic processes. For example, in portfolio optimization, the dependence structure between the individual stocks is studied and analyzed through the spectral measure estimation. More applications can be seen in Tsakalides and Nikias (1995).

Nolan, Panorska, and McCulloch (2001) suggested two methods namely, empirical characteristic function and the projection method for the estimation of the spectral measure. In our proposed method we make use of the empirical characteristic method to get the estimate of \( \Gamma \). Observe that if \( X \) is a stable random vector, then \( X - \delta \) is strictly stable random vector.

**3.2.1. Empirical characteristic function method for strictly stable random vectors**

Given an iid sample \( X^{(1)}, X^{(2)}, ..., X^{(n)} \) of stable random vectors with the spectral measure \( \Gamma \), let \( \hat{\phi}_n(t) \) and \( \hat{I}_n(t) \) be the empirical counterparts of \( \phi \) and \( I \) respectively defined as

\[
\hat{\phi}_n(t) = (1/n) \sum_{i=1}^{n} e^{-<t, X^{(i)}>}, \quad \hat{I}_n(t) = -\ln \hat{\phi}_n(t)
\]

For the estimation of the spectral measure \( \Gamma \), Nolan, Panorska, and McCulloch (2001) considered a discrete approximation to the exact spectral measure (see Byczkowski, Nolan, and Rajput 1993) of the form

\[
\Gamma^s = \sum_{l=1}^{L} \gamma_l \delta_{s_l}
\]

where \( \gamma_l = \Gamma(A_l) \), \( l = 1, ..., L \) are the weights at point \( s_l \in S^d \), a unit sphere and \( \delta_{s_l} \) is a point mass at \( s_l \). The patches that partition the sphere \( S^d \), with some “center” \( s_l \) are represented by \( A_l \).

Thus, the characteristic function \( \phi(t) \) is transformed to \( \hat{\phi}(t) = e^{-\sum_{l=1}^{L} \psi_l(<t, s_l>) \gamma_l} \).

Next, for given frequencies \( t_1, ..., t_L \in \mathbb{R}^d \), define an \( L \times L \) matrix \( \psi \) whose \((k, l)\)-th element is \( \psi_k(<t_k, s_l>) \). Finally obtain the expression \( I = \psi \gamma \), where \( \gamma = (\gamma_1, ..., \gamma_L)' \). Replacing \( I \) by \( \hat{I} = (\hat{I}(t_1), ..., \hat{I}(t_L)) \) and choosing \( t_1, t_2, ..., t_L \) in such a way that \( \psi^{-1} \) exists, we obtain the discretized estimator \( \hat{\gamma} = \psi \hat{x}^{-1} \hat{I} \) of the spectral measure \( \Gamma \).

For a general spectral measure \( \Gamma \) (not discrete and/or the location of the point masses are unknown) consider the discrete approximation defined above in (13).
When \(d = 2\), we take \(t_i = s_i = (\cos(2\pi(l - 1)/L), \sin(2\pi(l - 1)/L)) \in \mathbb{S}^d\), and arcs \(A_l = (2\pi(l - (3/2))/L, 2\pi(l - (1/2))/L), l = 1, ..., L\). In order to eliminate the problem of imaginary weights \(\gamma_p\), the properties of the trigonometric moments of the spectral measure and a symmetric grid are used.

When \(L = 2m\), let the grid be given by \(t_i = s_i = (\cos(2\pi(l - 1)/L), \sin(2\pi(l - 1)/L))\). Observe that, if \(I_l = I_{l+m}\) and the entries of the \(\psi\) satisfy \(\psi_{k,l} = \psi(k, t_l) = \psi(2k-l, \pi/L)\) for \(k = 1, ..., L\). Thus, for \(l = 1, ..., m, R_l = (I_l + I_{l+m})/2\) and \(\exists_l = -(I_l - I_{l+m})/2\). Define the real vector \(c = (R_{1l}, R_{2l}, ..., R_{ml}, \exists_{1l}, \exists_{2l}, ..., \exists_{ml})^t\) and the real \(L \times L\) matrix \(A = a_{k,l}\) by

\[
\begin{align*}
   a_{k,l} = \begin{cases} 
   R_{k,l} & k = 1, ..., m \\
   \exists_{k,l} & k = m + 1, ..., L
   \end{cases}
\end{align*}
\]

then

\[
c = A\gamma
\]

In order to avoid the chance of getting complex or negative values for some of the weights, we use the nnls() (Mullen and van Stokkum 2012) library in R that solves the minimization problem

\[
\text{Minimize } \|c - A\gamma\|_2 \text{ subject to } \gamma \geq 0. \tag{14}
\]

When \(L = 2m + 1\), again let \(t_i = s_i = (\cos(2\pi(l - 1)/L), \sin(2\pi(l - 1)/L))\). In this case, we obtain the discretized estimator \(\hat{\gamma}\) of the spectral measure \(\Gamma\) as \(\hat{\gamma} = \left|\psi^{-1}R(I)\right|\). Thus,
4. Simulation and comparative analysis

4.1. Performance analysis of the proposed hybrid method—univariate case

In this section, we specifically compare the estimation accuracy of three methods namely, Nolan’s Maximum Likelihood method (ML-STABLE), McCulloch’s quantile method (MC) and Kogon-Williams regression method (KW) with that of our proposed Hybrid method through Monte Carlo simulation. In addition, we also check the computational efficiency of each of the methods introduced above. Each method is then applied to a data whose distribution is stable. The data is generated by the method of Chambers, Mallows, and Stuck (1976). The three methods mentioned above are available in the “STABLE” package (RobustAnalysisInc 2010) introduced by Nolan for R and all related simulations are carried out using the package.

For a selected set of values of the parameters \( \alpha, \beta, \sigma, \delta \) and the sample size \( n \), a simulation is run where 500 replicates of iid stable random variables each of length \( n \) are generated. For each replicate, we then obtain the estimates of the parameters \( \alpha, \beta, \sigma \) and \( \delta \) by implementing various estimation techniques. The quantitative evaluation of the performance of the parameter estimators is given in Tables 2–6.

### 4.1.1. Estimation of \( \alpha, \beta, \sigma \) and \( \delta \)

For estimating \( \alpha \), the parameters \( \beta, \sigma, \delta \) are fixed to 0, 1, 0 respectively, for sample size \( n = 100, 700, 1500, 3200 \) and 10000. The parameter \( \alpha \) is allowed to vary from 0.4 to 1.9. From Table 2, it is quite evident that the RMSE values of the Hybrid method, when estimating \( \alpha \), is significantly lower than the KW and MC method and closer to the ML-STABLE method for each sample size thereby depicting the stability of our method. We also observe that, in comparison to the other methods, the RMSE values of the MC method increases significantly as \( \alpha \to 2 \).

Next, we estimate \( \beta = 0.1 \), for different values of sample size \( n \) and \( \alpha \) with the parameters \( \sigma \) and \( \delta \) fixed to 1, 0 respectively. From Table 3, we observe that the Hybrid method outperforms...
Table 4. Estimation of \( \beta \) for various \( \beta \) and \((\alpha = 0.5, \sigma = 1, \delta = 0)\). The observations denote the mean (RMSE) of 500 stable replicates.

| Methods     | \( n = 100 \)          | \( n = 700 \)          | \( n = 1500 \)         | \( n = 3200 \)         | \( n = 10000 \)        |
|-------------|-------------------------|------------------------|------------------------|------------------------|------------------------|
| Hybrid      | -0.043 (0.1763)         | -0.009 (0.0717)        | -0.009 (0.0426)        | -0.0033 (0.0340)       | -0.0007 (0.0153)       |
| ML-STABLE 0 | -0.017 (0.1302)         | -0.005 (0.0505)        | -0.004 (0.0327)        | 0.0012 (0.0161)        | 0.0005 (0.0058)        |
| MC          | -0.088 (0.3083)         | -0.021 (0.170)         | -0.007 (0.751)         | 7.44 \times 10^{-6}    | -0.0020 (0.0322)       |
| KW          | 0.026 (0.3216)          | 0.013 (0.1060)         | 0.009 (0.0714)         | 0.0065 (0.0470)        | 0.0031 (0.0309)        |
| Hybrid      | 0.178 (0.1649)          | 0.205 (0.6651)         | 0.204 (0.470)          | 0.203 (0.374)          | 0.204 (0.161)          |
| ML-STABLE 0.2 | 0.194 (0.1298)         | 0.195 (0.0438)         | 0.197 (0.0292)         | 0.199 (0.0197)         | 0.197 (0.0104)         |
| MC          | 0.094 (0.2878)          | 0.176 (0.1162)         | 0.194 (0.0715)         | 0.201 (0.0537)         | 0.199 (0.0288)         |
| KW          | 0.259 (0.3197)          | 0.200 (0.1122)         | 0.201 (0.0655)         | 0.206 (0.0495)         | 0.204 (0.0283)         |
| Hybrid      | 0.364 (0.1890)          | 0.411 (0.0662)         | 0.409 (0.0478)         | 0.394 (0.0502)         | 0.407 (0.0180)         |
| ML-STABLE 0.4 | 0.398 (0.1207)         | 0.396 (0.0431)         | 0.398 (0.0298)         | 0.400 (0.0197)         | 0.397 (0.0106)         |
| MC          | 0.290 (0.2924)          | 0.387 (0.0980)         | 0.402 (0.0567)         | 0.405 (0.0432)         | 0.404 (0.0232)         |
| KW          | 0.461 (0.3353)          | 0.408 (0.1142)         | 0.408 (0.0810)         | 0.404 (0.0530)         | 0.401 (0.0329)         |

KW and MC method, in terms of, the mean and RMSE values for values of \( \alpha \in (0, 1) \) and for different values of \( n \). However, the ML-STABLE method seems to outperform all the remaining methods in terms of the mean and RMSE values. We also varied \( \beta = 0, 0.2, 0.4 \), while holding \( \alpha = 0.5 \) with the values shown in Table 4. We considered the positive values of \( \beta \) since the performance is identical for the corresponding negative values. We observe that the proposed Hybrid method performs significantly better than the KW and MC method.

For the estimation of \( \sigma = 1 \) and \( \delta = 0 \), we consider various values of \( \alpha \) and \( n \). We fix \( \beta \) and \( \delta \), both to zero, when estimating \( \sigma \). On the other hand, for the estimation of \( \delta \), we fix \( \sigma = 1 \) and \( \beta = 0 \). From Tables 5 and 6, we clearly see that the Hybrid method outperforms the KW and MC method and is almost at par with ML-STABLE method in terms of the RMSE values for each sample size \( n \).

4.1.2. Computational burden and accuracy

Table 7 showcases the time taken (in seconds) by the estimation methods to estimate the parameters \( \alpha, \beta, \sigma \) and \( \delta \) for different values of \( n \). We observe that, in terms of computational efficiency,
the Hybrid method outperforms the usual Maximum Likelihood (ML) method and the Iterative Koutrouvelis (Kout) method, both available in the “StableEstim” (StableEstim) package in R. On the other hand, in terms of the accuracy of the estimates, Tables 6–8 showcase the effectiveness of the Hybrid method in comparison to KW, MC and ML-STABLE method, all computationally efficient than the proposed method.
Table 8. Estimation of $\Gamma$ for $n = 1000$, $\alpha = 1.3$, $\delta = (0, 0)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (1/4, 1/4, 1/4, 1/4)$.

| Values | SU       | RMSE | TM       | RMSE | MH       | RMSE |
|--------|----------|------|----------|------|----------|------|
| $\alpha$ | 1.305    | 0.0331 | 1.304    | 0.0378 | 1.310    | 0.0697 |
| $\gamma_1$ | 0.248    | 0.0163 | 0.250    | 0.0144 | 0.251    | 0.0144 |
| $\gamma_2$ | 0.251    | 0.0155 | 0.252    | 0.0148 | 0.252    | 0.0149 |
| $\gamma_3$ | 0.250    | 0.0143 | 0.248    | 0.0163 | 0.248    | 0.0169 |
| $\gamma_4$ | 0.252    | 0.0149 | 0.251    | 0.0156 | 0.251    | 0.0161 |

4.2. Performance analysis of the proposed hybrid method in the estimation of $\alpha$, $\Gamma$ and $\delta$—bivariate case

The bivariate strictly stable data is simulated using the Modarres and Nolan (1994) procedure and the R package “alphastable” (Teimouri, Mohammadpour, and Nadarajah 2019) is used to perform all the simulations. As suggested by Nolan et al. (Nolan, Panorska, and McCulloch 2001), we use the proposed Hybrid method to estimate the one-dimensional parameters $(\hat{\alpha}_j, \hat{\beta}_j, \hat{\sigma}_j, \hat{\delta}_j)$, $j = 1, 2$ for each of the coordinates of the 2-dimensional dataset. Thus, the estimate of $\alpha$ is obtained as $\hat{\alpha} = (\sum_{j=1}^{2} \hat{\alpha}_j)/2$ and the estimate of shift vector $\delta = (\delta_1, \delta_2)$ is $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2)$.

For the estimation of $\Gamma$, we use $\hat{\alpha}$ obtained using the Hybrid method in the ECF method as discussed in Subsec. 3.2. Our proposed method, namely the Sathe Upadhye (SU) method, useful in obtaining the discretized estimator $\hat{\gamma}$ of the spectral measure $\Gamma$, is compared with the method proposed by Mohammadi et al. (MH) (Mohammadi, Mohammadpour, and Ogata 2015) and Teimouri et al. (TM) (Teimouri, Rezakakh, and Mohammadpour 2017) in terms of the mean, RMSE (root mean squared error) and computational efficiency. For a selected set of values of the parameters $\alpha$, $\gamma$, $\delta = (0, 0)$ and the sample size $n$, a simulation is run where 100 replicates of bivariate strictly stable random vector each of length $n$ are generated. We choose different values of $\alpha$ and $n$ with the shift vector $\delta = (0, 0)$ fixed and consider four different cases for discrete spectral measure $\gamma$ with $l = 4$ and $l = 8$ point masses as follows

1. $\alpha = 1.3$, $n = 1000$, $l = 4$, $\gamma = (1/4, 1/4, 1/4, 1/4)^T$
2. $\alpha = 1.5$, $n = 1200$, $l = 4$, $\gamma = (1/2, 0, 1/2, 0)^T$
3. $\alpha = 1.5$, $n = 1000$, $l = 8$, $\gamma = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)^T$
4. $\alpha = 1.2$, $n = 1500$, $l = 8$, $\gamma = (0.1, 0.2, 0.3, 0.4, 0.1, 0.2, 0.3, 0.4)^T$

In Table 8, for $\alpha$, $\gamma_3$ and $\gamma_4$, SU performs significantly better than TM and MH in terms of mean and RMSE values. In Table 9, for $\alpha$, $\gamma_1$, $\gamma_2$ and $\gamma_3$, SU performs better than TM. On the other hand, for $\alpha$, $\gamma_1$ and $\gamma_3$, SU outperforms MH in terms of the mean and RMSE values.

In Table 10, for $\alpha$, $\gamma_3$, $\gamma_4$, $\gamma_7$ and $\gamma_8$, SU performs significantly better than TM and MH in terms of mean and RMSE values. We observe that, overall, MH method gives less accurate estimates with high RMSE values in comparison to SU and TM. Finally, in Table 11, for $\gamma_2$, $\gamma_3$, $\gamma_4$, $\gamma_6$, $\gamma_7$ and $\gamma_8$, SU works better than TM and MH, with a significant difference in the RMSE values. The MH method gives less accurate estimates with high RMSE values in comparison to TM and SU.

4.2.1. Computational burden and accuracy

To check the computational efficiency of the methods, we generated iid realizations from bivariate stable random vector of size $n = 4000$, $\alpha = 1.2$, $\gamma = (0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25, 0.25)$ and $\delta = (0, 0)$. Table 12, showcases the time taken (in sec) and the estimates of the parameters obtained using SU, TM and MH methods. We observe that the SU method is computationally efficient than the TM method. Though, the MH method is the most efficient computationally, it
gives less accurate estimates and high RMSE values in comparison to the TM and PM method, as observed from Tables 8–11.

5. Applications to financial data

In this section, we apply our above introduced method to real life data. We illustrate two examples based on univariate and strictly bivariate stable distribution.
5.1. **International Business Machines Corporation (IBM)**

The data has been obtained from Yahoo Finance for the datasets dealing with the prices of stock for our empirical analysis. The first dataset that we have considered is, International Business Machines Corporation (IBM) from New York Stock Exchange for the period January 19, 2012–March 19, 2018 comprising of 1550 daily returns value of the adjusted closing price. The price and return of IBM adjusted closing price are depicted in Figure 1.

In order to ensure that the given dataset can be modeled by a heavy tailed distribution specifically a non-Gaussian stable distribution, some plots and normality tests have been carried out. The \( p \)-values obtained via Anderson-Darling and Shapiro-Wilk test were far lesser than 0.05 thereby disabling us to accept the null hypothesis on the support of normality. From the QQ-normal plot in Figure 2, it is again evident that the data is not normally distributed as the points in the plot do not lie on a straight diagonal line. Also the plot of the empirical cumulative distribution function on a log-log scale, in Figure 2, shows that the dataset cannot even be modeled by a power law distribution which is another type of heavy-tailed distribution.

Next, we check which model whether Gaussian or non-Gaussian stable model, namely, Hybrid, KW, MC and Maximum Likelihood (ML-STABLE) gives a better fit to the observed returns of IBM. The estimates obtained from each model are given in Table 13. The performance of each of the models is graphically assessed through the density plots as seen in Figure 3. The theoretical assessment is done via the Kolmogorov-Smirnov (K-S) goodness of fit test. The K-S statistic measures the distance between the empirical cumulative distribution function (ECDF) of the sample data and the cumulative distribution function (CDF) of the reference distribution. It is defined as

\[
D = \sup_{x \in \mathbb{R}} ||F(x; \alpha, \beta, \sigma, \delta) - \hat{F}(x; \alpha, \beta, \sigma, \delta)||
\]

where sup is the supremum, \( F \) and \( \hat{F} \) denote the ECDF and CDF computed from the estimated probability density function. The \( D \) values obtained using different models are given Table 14.

From Figure 4, we observe that the density of the normal distribution is too low near the middle, high in the midrange and quite low on the tails. While on the other hand, it is interesting to observe how all the stable models including the Hybrid model approximate the data well over almost the whole range. A Table 14 showcases the values of \( D \) obtained using the K-S goodness of fit test.

5.2. **Tata Consultancy Services Limited (TCS.NS) and National Thermal Power Corporation Limited (NTPC.NS)**

The daily returns of 1475 adjusted closing prices for the two components of Nifty, namely NTPC.NS and TCS.NS is obtained from Yahoo Finance for the time period, January 3, 2011 to
Table 13. stable fit to 1550 IBM daily returns of the adjusted closing price obtained using different models.

| Method       | $\alpha$ | $\beta$ | $\sigma$ | $\delta$ |
|--------------|----------|---------|----------|----------|
| stable fit   | 1.6011   | 0.0508  | 0.0061   | 0.00007  |
| Hybrid       | 1.7518   | 0.02207 | 0.0063   | 0.0001   |
| ML-STABLE    | 1.7506   | 0.4819  | 0.0064   | 0.0001   |

Figure 2. QQ-plot and CCDF plot.

Figure 3. Density plots of IBM using different estimation methods.
Table 14. K-S goodness of fit test to 1550 IBM daily returns of the adjusted closing price obtained using different models.

| Method      | D  |
|-------------|----|
| Hybrid      | 0.0347 |
| MC          | 0.0137 |
| KW          | 0.0221 |
| ML-STABLE   | 0.0180 |
| Gaussian    | 0.4842 |

Figure 4. Density plots of IBM.

December 31, 2016. Figure 5 shows the scatter and the contour plots of their returns respectively and reveals that the data is heavily skewed downwards and several points are away from the origin. Also from the contour plot it is clear that the distribution of the data is neither normal nor elliptical stable. Thus, multivariate stable distributions are more appropriate than multivariate normal or elliptical distributions. The theoretical assessment of whether the data is multivariate normal is done using the MVN package in R and the test used is Henze-Zirkler test. The test reveals that the data is not multivariate normal. The details of this test on the bivariate data are shown in Table 15.

We now model the pair $X = (TCS.NS, NTPC.NS)$ using a bivariate stable distribution with $L = 12$ points of masses for spectral measure given by $t_l = s_l = (\cos(2\pi(l - 1)/L), \sin(2\pi(l - 1)/L)) \in S^2, l = 1, 2, \ldots, L$. The estimates of the location vector $\hat{\delta}$ obtained using our proposed Hybrid method are $\hat{\delta} = (0.0003077732, -0.0001562171)$ while $\hat{\alpha} = 1.85591$, obtained after taking the mean of $\hat{\alpha}_1 = 1.841935$ and $\hat{\alpha}_2 = 1.87007$. To fit a strictly stable distribution transform $X$ to $X - \hat{\delta}$. The results of the estimates of the spectral measure after fitting a strictly stable
distribution to the shifted data is shown in Table 16 and is compared with other well-known methods such as Mohammadi, Mohammadpour, and Ogata (2015) and Teimouri, Rezakhah, and Mohammadpour (2017). We observe that the values of the estimated masses obtained via our method are very much closer to the other two methods.

6. Concluding remarks

To conclude, we make the following observations in relation to our proposed method.

1. Our proposed hybrid method (univariate) is non-iterative in nature as no further improvement in the RMSE’s of the estimators resulted after the first iteration. Also, as in the case of Kogon-Williams method, the regressions are performed only once using ordinary least squares as opposed to the numerous iterations for Koutrouvelis’ method where regressions are performed using generalized least squares.

2. Koutrouvelis makes use of look-up tables in order to perform regressions while Kogon-Williams eliminated the need to use look-up tables for estimation. Our proposed hybrid method is simple and shows that if we retain the look-up table and modify it in order to obtain the regression estimates using ordinary least squares, we obtain accurate estimates with low RMSE.

3. In terms of the computational efficiency, our proposed hybrid method works better than the iterative Koutrouvelis method and the usual maximum likelihood method, see Table 7.

4. For the multivariate case, we make use of our proposed hybrid method to obtain the estimators \( \hat{\alpha} \) and the shift vector \( \hat{\delta} \). The discretized estimator \( \hat{\gamma} \) of the spectral measure \( \Gamma \) is

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**Table 15.** Results obtained using the mvn package in R.

| Test                | Statistic | p-value | Normality | Mean  | Standard Deviation | Median | Skew | Kurtosis |
|---------------------|-----------|---------|-----------|-------|--------------------|--------|------|----------|
| Henz-Zirkler        | 7.2338    | 0       | NO        | 0.0074| 0.01616             | 0.0035 | 0.14523 | 3.9688   |
| Shapiro-Wilk (TCS.NS) | 0.9681 | < 0.001 | NO        | 0.0012| 0.01692             | 0      | —19950 | 4.2812   |
| Shapiro-Wilk (NTPC.NS) | 0.9676 | < 0.001 | NO        | 0.0012| 0.01692             | 0      | —19950 | 4.2812   |

**Table 16.** Estimates of the spectral measure obtained using different method for a strictly stable bivariate data.

| Methods | \( \gamma_1 \) | \( \gamma_2 \) | \( \gamma_3 \) | \( \gamma_4 \) | \( \gamma_5 \) | \( \gamma_6 \) | \( \gamma_7 \) | \( \gamma_8 \) | \( \gamma_9 \) | \( \gamma_{10} \) | \( \gamma_{11} \) | \( \gamma_{12} \) |
|---------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| MH      | 1.8349          | 0               | 0               | 0               | 0               | 0               | 0.00017         | 0               | 0.00019         | 0.00002         | 0               | 0               |
| TM      | 1.6802          | 0               | 0               | 0               | 0               | 0               | 0.00020         | 0               | 0.00025         | 0               | 0               | 0               |
| SU      | 1.8559          | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0.00020         | 0               | 0.00012         | 0.000005        |
obtained via the empirical characteristic method suggested by Nolan, Panorska, and McCulloch (2001) with slight modifications. Thus, in terms of computational efficiency and accuracy, the new method outperforms the method of Mohammadi, Mohammadpour, and Ogata (2015) and Teimouri, Rezakhah, and Mohammadpou (2017), see Table 12.

5. Finally, we give two applications of our proposed method using financial data, where, the distribution of the datasets considered is stable. Though the maximum likelihood method of estimation is said to give the most accurate estimate, however, it is computationally, the slowest when applied to the two financial data. For the bivariate data, we observe that the values of the estimated masses obtained via our method are very close to the values obtained through the method of Mohammadi, Mohammadpour, and Ogata (2015) and Teimouri, Rezakhah, and Mohammadpur (2017).

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