ON THE LEHMER CONJECTURE AND COUNTING IN FINITE FIELDS

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ABSTRACT. We give a reformulation of the Lehmer conjecture about algebraic integers in terms of a simple counting problem modulo \( p \).

Let \( S_d \) be the set of all polynomials of degree at most \( d \) with coefficients in \( \{0, 1\} \). Recall that if \( \alpha \) is an algebraic number, with minimal polynomial \( \pi_\alpha(X) = a_0 \prod_{i=1}^n (X - \alpha_i) \in \mathbb{Z}[X] \), then the Mahler measure of \( \alpha \) is the quantity:

\[
M(\alpha) = |a_0| \prod_{1}^{n} \max\{1, |\alpha_i|\}.
\]

In [BV16] we have shown that the Mahler measure is related to the growth rate of the cardinality of the set

\[
S_d(\alpha) := \{ P(\alpha); P \in S_d \}
\]

as follows:

**Theorem 1** ([BV16 Thm 5, Lem 16]). There is a numerical constant \( c > 0 \) such that for every algebraic number \( \alpha \),

\[
\min\{2, M(\alpha)\}^c \leq \lim_{d \to +\infty} \frac{|S_d(\alpha)|^{1/d}}{d} \leq \min\{2, M(\alpha)\}.
\]

One can take \( c = 0.44 \). The limit above always exists by sub-multiplicativity of \( d \mapsto |S_d(\alpha)| \). The celebrated Lehmer conjecture posits the existence of a numerical constant \( c_0 > 0 \) such that every algebraic number \( \alpha \), which is not a root of unity must have

\[
M(\alpha) > 1 + c_0.
\]

In particular the following reformulation of the Lehmer conjecture follows immediately from Theorem 1:

**Corollary 2.** The following are equivalent

1. there exists \( c_1 > 0 \) such that

\[
\lim_{d \to +\infty} \frac{|S_d(\alpha)|^{1/d}}{d} > 1 + c_1
\]

for every complex number \( \alpha \), which is not a root of unity,
2. the Lehmer conjecture is true.

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Note that if $\alpha$ is transcendental, or simply not a root of a polynomial of the form $P - Q$, $P, Q$ in $S_d$ for some $d$, then $|S_d(\alpha)| = 2^{d+1}$ for all $d$. Note further that if $\alpha$ is a root of unity, then $|S_d(\alpha)|$ grows at most polynomially in $d$.

The set $S_d(\alpha)$ is the support of the random process $\sum_0^d \epsilon_i \alpha^i$ where the $\epsilon_i$ are Bernoulli random variables equal to 0 or 1 with probability $\frac{1}{2}$. The proof of the lower bound in Theorem 1 consists in establishing a lower bound on the entropy of the above process.

The purpose of this short note is to give a mod version of the above equivalence, showing that Lehmer’s conjecture is equivalent to an easy to formulate assertion about finite fields.

We first propose the following reformulation. We write $\log$ for the base 2 logarithm. Given $C \geq 1$, we say that a prime $p$ is $C$-wild if there is $x \in \mathbb{F}_p^\times$ of multiplicative order at least $(\log p)^2$ such that not every element of $\mathbb{F}_p$ is a sum of at most $(\log p)^C$ elements from the geometric progression $H := \{x, x^2, \ldots, x^{[C \log p]}\}$, where $[y]$ denotes the integer part of $y$. (Here we allow choosing the elements of $H$ multiple times.)

**Theorem 3.** The following are equivalent.

(a) The Lehmer conjecture is true.

(b) For some $C > 1$, almost no prime is $C$-wild, namely as $X \rightarrow +\infty$ $|\{p \leq X \text{ and } p \text{ is } C\text{-wild}\}| = o(|\{p \leq X\}|)$.

The implication (b) $\Rightarrow$ (a) is the easier one. Inasmuch this result can hardly been seen as making any progress towards a positive answer to the Lehmer conjecture.

The main point of this note is to show the converse implication. A crucial ingredient in the proof is the lower bound for $|S_d(\alpha)|$ in Theorem 1.

As it turns out the method of proof actually yields a more precise result: the validity of Lehmer’s conjecture implies a stronger statement than (b), and vice versa a weaker version of (b) already implies Lehmer’s conjecture. We spell this out in Theorem 4 below and to this aim we first introduce some notation.

Let $\delta, \epsilon > 0$. We will say that a prime number $p$ is $\delta$-bad if there is a non-zero residue class $x$ modulo $p$ with multiplicative order at least $\log p \log \log \log p$ such that $|S_d(x)| \leq \frac{1}{2} p$ for all $d \leq \frac{1}{\delta} \log p$ and all $K \leq 2^{\epsilon d}$.

We can now state our second reformulation:

**Theorem 4.** The following are equivalent.

(a) The Lehmer conjecture is true.

(b) There is $\delta > 0$ such that for all $\epsilon > 0$ as $X \rightarrow +\infty$ $|\{p \leq X \text{ and } p \text{ is } (\delta, \epsilon)\text{-very bad}\}| = o(|\{p \leq X\}|)$. 

For $K \in \mathbb{N}$, we let $S^K_d$ be the family of polynomials of degree at most $d$ with integer coefficients in the interval $[-K, K]$. Clearly

$$S_d \subset S^K_d \subset S_d \pm \ldots \pm S_d \subset S^{2K}_d,$$

where there are $2K$ summands in the sunset.

We will say that $p$ is $(\delta, \epsilon)$-very bad if there is a non-zero residue class $x$ modulo $p$ with multiplicative order at least $\sqrt{\frac{p}{\log p}}$ such that $|S^K_d(x)| \leq \frac{1}{2} p$ for all $d \leq \frac{1}{\delta} \log p$ and all $K \leq 2^{\epsilon d}$.

We can now state our second reformulation:
(c) For every $\varepsilon > 0$ there is $\delta > 0$ such that as $X \to +\infty$,

\[
\left\{ p \leq X; p \text{ is } \delta\text{-bad} \right\} \ll \varepsilon \cdot X^\varepsilon.
\]

We have used the Vinogradov notation $\ll$ to mean that the inequality holds up to a multiplicative constant depending only on $\varepsilon$.

It is clear that (c) implies (b) in the above theorem, because for $\delta, \varepsilon \in (0, \frac{1}{2})$ every large enough $(\delta, \varepsilon)$-very bad prime must be $\delta$-bad.

The relevance of the Lehmer conjecture to the study of the family of polynomials $S_d$ is well-known and appears for example in Konyagin’s work [K92, K99]. As we mentioned above, the growth of $|S_d(\alpha)|$ is intimately related to the behavior of the random process $\sum_0^d \epsilon_i \alpha^i$ where the $\epsilon_i$ are Bernoulli random variables equal to 0 or 1 with probability $\frac{1}{2}$. These processes have been studied by Chung, Diaconis and Graham [CDG87] and Hildebrand [Hil90]. Very recently they have been used to study irreducibility of random polynomials in [BV18].

In the remainder of this paper we prove the above theorems. We first handle Theorem 4 and then deduce Theorem 3.

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1. Proof that $(b)$ implies $(a)$ in Theorem 1

We recall the following fairly well-known lemma (see Lemma 16 in [BV16] and §8 in [B07]).

**Lemma 5.** Let $\alpha$ be an algebraic number. Then there is a constant $C_\alpha > 0$ such that for every $d, K \geq 1$,

\[
|S_d^K(\alpha)| \leq C_\alpha (Kd)^{C_\alpha} M(\alpha)^d.
\]

If Lehmer’s conjecture fails, then for each $\delta > 0$ there is an algebraic unit $\alpha$ such that $M(\alpha) < 1 + \delta^2/2$. By the lemma above we conclude that if we choose $\epsilon$ in the interval $(0, C_\alpha^{-1} \delta^2/4)$, then for all large enough $d$ we have:

\[
|S_d^{2^d}(\alpha)| \leq 2^{\delta^2 d}.
\]

Let $F$ be the Galois closure of $\mathbb{Q}(\alpha)$. Let $p$ be a prime number which splits completely in $F$ and let $p$ be a prime ideal in $F$ above $p$. Denoting by $\bar{\alpha}$ the residue class of $\alpha$ modulo $p$ we must have:

\[
|S_d^{2^d}(\bar{\alpha})| \leq 2^{\delta^2 d},
\]

for all $d \in \mathbb{N}$. In particular if $d \leq \frac{1}{\delta} \log p$, we get

\[
|S_d^{2^d}(\bar{\alpha})| \leq p^\delta.
\]

In particular if the multiplicative order of $\bar{\alpha}$ is at least $\sqrt{\frac{p}{\log p}}$, then $p$ is $(\delta, \varepsilon)$-very bad.

On the other hand, the Frobenius-Chebotarev density theorem tells us that there is a positive proportion of primes $p$ which split completely in $F$. Therefore in order to complete the proof, it is enough to show that there is only a vanishing proportion of primes $p$ for which $\bar{\alpha}$ has multiplicative order at most $\sqrt{\frac{p}{\log p}}$, namely:
Lemma 6. Let $F\mid\mathbb{Q}$ be a finite Galois number field. Let $\mathcal{O}_F$ be its ring of integers and let $\alpha \in \mathcal{O}_F \setminus \{0\}$. Let $\mathcal{P}_\alpha$ be the set of primes $p \in \mathbb{N}$ such that there is a prime ideal $\mathfrak{p}$ in $F$ above $p$ such that $|\mathcal{O}_F/\mathfrak{p}| = p$ and $\alpha \mod \mathfrak{p}$ is non-zero and of multiplicative order in $\mathbb{F}_p^\times$ at most $\sqrt{\frac{p}{\log p}}$. Then as $X \to +\infty$,

$$|\{p \leq X; p \in \mathcal{P}_\alpha\}| = o(|\{p \leq X\}|).$$

Proof. Note that if $\alpha^n - 1 \in \mathfrak{p}$, then $p = N_{F\mid\mathbb{Q}}(\mathfrak{p})$ divides $N_{F\mid\mathbb{Q}}(\alpha^n - 1)$. In particular

$$\prod_{p \in \mathcal{P}_\alpha: \sqrt{X} \leq p \leq X} p \text{ divides } \prod_{n \leq X/\log X} N_{F\mid\mathbb{Q}}(\alpha^n - 1).$$

On the other hand:

$$|N_{F\mid\mathbb{Q}}(\alpha^n - 1)| \leq \prod_{\sigma \in \text{Gal}(F\mid\mathbb{Q})} (1 + |\sigma(\alpha)|^n) \leq \prod_{\sigma \in \text{Gal}(F\mid\mathbb{Q})} 2 \max\{1, |\sigma(\alpha)|\}^n \leq 2^{[F:Q]} M(\alpha)^n$$

We consider this inequality for $n = 1, \ldots, (X/\log X)^{1/2}$ and get

$$|\mathcal{P}_\alpha \cap [\sqrt{X}, X]| \log X \ll \left(\frac{X}{\log X}\right)^{1/2}[F:Q] + \frac{X}{\log X} \log M(\alpha).$$

The conclusion follows immediately given that $|\{p \leq X\}| \gg X/\log X$. \hfill \Box

Remark 7. Clearly we can replace $(p/\log p)^{1/2}$ by $p^{1/2}e(p)$ for any function $e(p)$ tending to 0 as $p \to +\infty$ with the same proof. Erdős and Murty improve this even further to $pe(p)$ in [LM99] in the special case $F = \mathbb{Q}$ assuming the Riemann hypothesis for certain Dedekind zeta functions.

2. Proof that (a) implies (c) in Theorem 4

The proof of this implication uses the harder part (the lower bound) in Theorem 4

Let $\mathcal{P}_d$ be the set of polynomials with degree at most $d$ and coefficients in $\{-1,0,1\}$. Note that $\mathcal{P}_d = \mathcal{S}_d - \mathcal{S}_d$. Let $\mathcal{I}_d$ be the set of monic $\mathbb{Q}$-irreducible polynomials dividing at least one non-zero polynomial in $\mathcal{P}_d$. We will say that a prime $p$ is $d$-exceptional if it divides the resultant $Res(D_1, D_2)$ of some pair of distinct irreducible polynomials $D_1, D_2$ in $\mathcal{I}_d$.

Recall that given a field $F$ every field element $\alpha \in F$ determines a partition $\pi_{\alpha,d}$ of $\mathcal{S}_d$ made of the preimages of the map $P \mapsto P(\alpha)$ from $\mathcal{S}_d$ to $F$. Clearly $|\mathcal{S}_d(\alpha)|$ is the number of parts of $\pi_{\alpha,d}$.

Claim 1. If $p$ is not $d$-exceptional, then for every $\alpha \in \mathbb{F}_p$ there is at most one $D \in \mathcal{I}_d$ such that $D(\alpha) = 0$ and moreover $\pi_{\alpha,d} = \pi_{\beta,d}$ if $\beta \in \mathbb{Q}$ is a root of $D$.

Proof. If $p$ is not $d$-exceptional, then the reductions mod $p$ of the polynomials in $\mathcal{I}_d$ are pairwise relatively prime. In particular they cannot have a common root in $\mathbb{F}_p$. So each $\alpha \in \mathbb{F}_p$ can be the root of at most one $D \in \mathcal{I}_d$. If there is such a $D$, it is unique, and for any $P, Q \in \mathcal{S}_d$ we have the equivalences: $P(\alpha) = Q(\alpha)$ if and only if $D$ divides $P - Q$ and if and only if $P(\beta) = Q(\beta)$, where $\beta$ is chosen to be a complex root of $D$. This ends the proof. \hfill \Box
Note that if \( \pi_{\alpha,d} = \pi_{\beta,d} \), then \( |S_n(\alpha)| = |S_n(\beta)| \) for all \( n \leq d \). Similarly for any \( n \leq d \) we have \( \alpha^n = 1 \) in \( \mathbb{F}_p \) if and only if \( \beta^n = 1 \) in \( \overline{\mathbb{Q}} \). Here we used that the \( n \)’th cyclotomic polynomial is in \( I_d \) for \( n \leq d \).

**Claim 2.** For \( d \) large enough, there are at most \( 10^d \) \( d \)-exceptional primes.

**Proof.** If \( D_1, D_2 \) are two polynomials in \( I_d \), their resultant can be bounded above by Hadamard’s inequality:

\[
|\text{Res}(D_1, D_2)| \leq \|D_1\|_2^{\deg D_2} \|D_2\|_2^{\deg D_1},
\]

where \( \|P\|_2 \) denotes the \( \ell^2 \)-norm of the coefficients of a polynomial \( P \). Recall that for every \( P \in \mathbb{C}[X] \) we have \( \|P\|_1 \leq 2^{\deg P} M(P) \). This is easily seen by expressing the coefficients of \( P \) as sums of products of roots of \( P \). Recall also that \( M(P) \leq \|P\|_1 \) as can be seen using the Jensen formula for \( M(P) \). However if \( D \in I_d \), then \( D \) divides some \( P \in \mathcal{P}_d \) and hence \( M(D) \leq M(P) \leq \|P\|_1 \leq d + 1 \), and thus

\[
\|D\|_2 \leq \|D\|_1 \leq 2d(d + 1).
\]

It follows that

\[
|\text{Res}(D_1, D_2)| \leq 2^{2d^2} (d + 1)^2d \leq 2^{4d^2}.
\]

The number of distinct prime factors of \( |\text{Res}(D_1, D_2)| \) is thus at most \( 4d^2 \) when \( d \) is large enough. Since there are at most \( d \) irreducible factors of \( P \) for any \( P \in \mathcal{P}_d \), \( |\mathcal{I}_d| \leq d3^{d+1} \). Hence there can only be at most \( 4d^2|\mathcal{I}_d|^2 \leq 4d^49^{d+1} \) \( d \)-exceptional primes.

We are now ready to conclude the proof of the implication \((c) \Rightarrow (a)\) in Theorem 1. We assume that Lehmer’s conjecture holds inasmuch as there is \( \delta_0 > 0 \) such that \( M(\beta) > e^{\delta_0} \) for every algebraic number \( \beta \), which is not a root of unity. Let \( \delta > 0 \) and \( X \geq 1 \). Assume that \( \delta < (c\delta_0/2)^2 \), where \( c \) is the constant from Theorem 1.

Let \( d \) be the integer part of \( \sqrt{\delta} \log X \).

**Claim 3.** Let \( p \in [\sqrt{X}, X] \) be a prime number. If \( p \) is \( \delta \)-bad, then \( p \) is \( d \)-exceptional.

**Proof.** If \( p \) is \( \delta \)-bad, there is \( \alpha \in \mathbb{F}_p \setminus \{0\} \) of multiplicative order at least \( \log p \log \log p \) such that \( |S_n(\alpha)| \leq p^\delta \) for all \( n \leq \log p \). In particular

\[
|S_{\frac{1}{2} \log X}(\alpha)| \leq X^\delta.
\]

On the other hand if \( p \) is not \( d \)-exceptional, by Claim 1 above, there is \( \beta \in \overline{\mathbb{Q}} \) of degree at most \( d \) such that \( |S_n(\alpha)| = |S_n(\beta)| \) for all \( n \leq d \). In particular if say \( \delta < \frac{1}{2} \), we have \( d \leq \frac{1}{2} \log X \) and thus

\[
|S_d(\beta)| \leq X^\delta \leq e^{\sqrt{\delta}(d+1)} \leq e^{2\sqrt{\delta}d}.
\]

However according to Theorem 1 \( \max\{2, M(\beta)\}^\varepsilon \leq |S_d(\beta)|^{1/d} \) (this holds for all \( d \) by sub-multiplicativity of \( d \mapsto |S_d(\beta)| \)). Hence

\[
M(\beta) \leq e^{\frac{1}{2} \sqrt{\delta}}.
\]

Since we assume the validity of the Lehmer conjecture, it follows that \( \beta \) must be a root of unity. However \( \beta \) is a root of a polynomial \( P \) in \( \mathcal{P}_d \), hence has degree at most \( d \) over \( \mathbb{Q} \). Its minimal polynomial must be a cyclotomic polynomial \( \Phi_n \).
dividing $P$ hence in $I_d$. But $p$ is not $d$-exceptional, so by Claim 1 we know that $\Phi_m(\alpha) = 0$.

It is well known that there is an absolute numerical constant $C > 0$ such that if $x$ is a root of unity in $\mathbb{C}$ and has degree at most $d$ as an algebraic number over $\mathbb{Q}$, then the order of $x$ is at most $Cd \log \log d$. Indeed, the degree of a root of unity of order $n$ is given by the Euler totient function, which satisfies the lower estimate: $\phi(n) \gg n/\log \log n$ (see [HW79 Thm 328]).

It follows that $\alpha$ has multiplicative order at most $m \leq C_0d \log \log d$. As this is less than $\log p \log \log \log p$ if $X$ is large enough (and $\delta_0$ chosen small enough), a contradiction.

Now combining Claims 2 and 3 we get:

$$\{|p \in \sqrt{X}, X; p \text{ is } \delta\text{-bad}| \leq 10^d \leq X^{3\sqrt{\delta}},$$

for all $X \geq 1$, hence

$$|\{p \leq X; p \text{ is } \delta\text{-bad}| \leq X^\delta + \sum_{k \geq 0; 2^k \leq 1/\delta} X^{3\sqrt{\delta}/2^k} \ll_\delta X^{3\sqrt{\delta}}.$$

This concludes the proof of Theorem 4.

3. Proof of Theorem 3

The proof that (b) implies (a) is immediate from the corresponding implication in Theorem 4. Indeed given $\epsilon \in (0, \frac{1}{2})$ and $C > 1$ every large enough $(\frac{1}{2}, \epsilon)$-very bad prime will be $C$-wild.

In the converse direction, in view of Theorem 4 it is enough to show that assertion (c) of Theorem 4 implies (b) of Theorem 3. Let $\epsilon = \frac{1}{2}$ and let $\delta$ be given by assertion (c). It is enough to find $C = C(\delta) > 1$ such that every $C$-wild prime is $\delta$-bad. We will show the contrapositive. So assume $p$ is not $\delta$-bad. Then for every $x \in F_p^\infty$ with multiplicative order at least $(\log p)^2$ we have

$$|\mathcal{S}_d(x)| \geq p^\delta$$

for some $d \leq \log p$. Now recall the sum-product theorem over $F_p$:

**Theorem 8** (sum-product theorem, see 2.58 and 4.10 in [TV10]). There is $\epsilon > 0$ such that for all primes $p$ and all subsets $A \subset F_p$ at least one of three possibilities can occur: either $AA + AA + AA = F_p$, or $|A + A| > |A|^{1+\epsilon}$, or $|AA| > |A|^{1+\epsilon}$.

Note further that $\mathcal{S}_d^K(x) + \mathcal{S}_d^K(x) \subset \mathcal{S}_d^{2K}(x)$, while $\mathcal{S}_d^K(x), \mathcal{S}_d^K(x) \subset \mathcal{S}_{2d}^{dK^2}(x)$. In particular, starting from $\mathcal{S}_d(x)$ and applying at most $n$ times either a sumset or a product set, we obtain a subset of $\mathcal{S}_{2^n d}(x)$.

Applying the sum-product theorem, we see that after some $n = n(\delta)$ steps $\mathcal{S}_{2^n [\log p]^{dK^2}}(x)$ must be all of $F_p$. Setting $C = 2^{n+1}$, we conclude (by 4) that $p$ is not $C$-wild as desired. This concludes the proof of Theorem 3.

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