WEIGHTED UNIPOLAR HARDY INEQUALITY WITH OPTIMAL CONSTANT

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Abstract. We state the following weighted Hardy inequality
\[ c_\mu \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu + K \int_{\mathbb{R}^N} \varphi^2 \, d\mu \quad \forall \varphi \in H_\mu^1 \]
in the context of the study of the Kolmogorov operators
\[ Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \]
with \( \mu \) probability density in \( \mathbb{R}^N \), and of the related evolution problems. We prove the optimality of the constant \( c_\mu \) and state existence and nonexistence results following the Cabré-Martel’s approach [7] and using results stated in [14, 8].

1. Introduction

This paper fits in the more general framework of the study of Kolmogorov operators on smooth functions
\[ Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \]
with \( \mu \) probability density on \( \mathbb{R}^N \), and of the related evolution problem
\[ (P) \quad \left\{ \begin{array}{l}
\partial_t u(x, t) = Lu(x, t) + V(x)u(x, t), \quad x \in \mathbb{R}^N, t > 0, \\
u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu.
\end{array} \right. \]
The operator \( L \) in \( (P) \) is perturbed by the singular potential \( V(x) = \frac{c}{|x|^2} \) and \( L^2_\mu := L(\mathbb{R}^N, d\mu) \), with \( d\mu(x) = \mu(x) \, dx \).

The interest in inverse square potentials of type \( V \sim \frac{c}{|x|^2} \) relies in the criticality: they have the same homogeneity as the Laplacian and do not belong to the Kato’s class, then they cannot be regarded as a lower order perturbation term. Furthermore interest in singular potentials is due to the applications to many fields, for example in many physical contexts as molecular physics [15], quantum cosmology (see e.g. [3]) and combustion models [13].

A remarkable result stated in 1984 by P. Baras and J. A. Goldstein in [3] show that in the case \( V(x) = \frac{c}{|x|^2} \) the evolution problem \( (P) \) with \( L = \Delta \) admits a unique positive solution if \( c \leq c_o = \left( \frac{N-2}{2} \right)^2 \) and no positive solutions exist if \( c > c_o \). When it exists, the solution is exponentially bounded, on the contrary, if \( c > c_o \), there is the so called instantaneous blowup phenomenon.

An analogous result has been obtained in 1999 by X. Cabré and Y. Martel [7] for more general potentials \( 0 \leq V \in L^1_{loc}(\mathbb{R}^N) \) with a different approach.

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To state the existence and nonexistence results we use the relation between the weak solution of \((P)\) and the \textit{bottom of the spectrum} of the operator \(- (L + V)\)

\[
\lambda_1 (L + V) := \inf_{\varphi \in H^1_\mu (\{ 0 \})} \left( \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu - \int_{\mathbb{R}^N} V \varphi^2 d\mu}{\int_{\mathbb{R}^N} \varphi^2 d\mu} \right)
\]

with \(H^1_\mu\) suitable weighted Sobolev space.

When \(\mu = 1\) Cabré and Martel showed that the boundedness of \(\lambda_1 (\Delta + V)\), \(0 \leq V \in L^1_{loc} (\mathbb{R}^N)\), is a necessary and sufficient condition for the existence of positive exponentially bounded in time solutions to the associated initial value problem. Later in [14, 8] analogous results have been extended to Kolmogorov operators. For Ornstein-Uhlenbeck type operators \(Lu = \Delta u - \sum_{i=1}^n A(x - a_i) \cdot \nabla u, a_i \in \mathbb{R}^N, i = 1, \ldots, n\), perturbed by multipolar inverse square potentials a weight multipolar Hardy inequality and related existence and nonexistence results were stated in [9].

In [8] the authors state a similar inequality in the weighted case using a different approach based on improved Hardy inequalities. This requires other conditions on \(\mu\) and we use is different. Furthermore we present some examples of function \(\mu\) which satisfy some general conditions

\[
c_\mu \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K \int_{\mathbb{R}^N} \varphi^2 d\mu, \quad \forall \varphi \in H^1_\mu, \quad K > 0,
\]

and prove the optimality of the constant \(c_\mu\).

Then we state an existence and nonexistence Theorem following the Cabré-Martel’s approach using the results stated in [14, 8] when the function \(\mu\) belongs to \(C^{1,\lambda}_{loc} (\mathbb{R}^N)\) or belongs to \(C^{1,\lambda}_{loc} (\mathbb{R}^N \setminus \{ 0 \})\) respectively, for some \(\lambda \in (0, 1)\).

The proof of the weighted Hardy inequality is different from the others in literature. In [8] the authors state a similar inequality in the weighted case using a different approach based on improved Hardy inequalities. This requires other conditions on \(\mu\). The our technique allows us to achieve the best constant (cf. [8, Theorem 1.3]). To state the optimality of the constant in the estimate we need further assumptions on \(\mu\) as usually it is done. The choice of the function \(\varphi\) plays a fundamental role in the proof. The technique we use is close to the one used in [8] but the function \(\varphi\) we use is different. Furthermore we present some examples of function \(\mu\) which satisfy hypotheses of the main Theorems.

2. \textbf{Weighted unipolar Hardy inequalities}

Let \(\mu\) a weight function in \(\mathbb{R}^N\). We define the weighted Sobolev space \(H^1_\mu = H^1 (\mathbb{R}^N, \mu (x) dx)\) as the space of functions in \(L^2_\mu := L^2 (\mathbb{R}^N, \mu (x) dx)\) whose weak derivatives belong to \((L^2_\mu)^N\).
As first step we consider the following conditions on \( \mu \) which we need to state a preliminary weighted Hardy inequality.

\( \text{H}_1 \) \( \mu \geq 0, \mu \in L^1_{\text{loc}}(\mathbb{R}^N); \)
\( \text{H}_2 \) \( \nabla \mu \in L^1_{\text{loc}}(\mathbb{R}^N); \)
\( \text{H}_3 \) there exist constants \( k_1, k_2 \in \mathbb{R}, k_2 > 2 - N, \) such that if 
\[ f_\varepsilon = (\varepsilon + |x|^2)^{\alpha}, \quad \alpha < 0, \quad \varepsilon > 0, \]
it holds
\[ \frac{\nabla f_\varepsilon}{f_\varepsilon} \cdot \nabla \mu = \frac{\alpha x}{\varepsilon + |x|^2} \cdot \nabla \mu \leq \left( k_1 + \frac{k_2 \alpha}{\varepsilon + |x|^2} \right) \mu \]
for any \( \varepsilon > 0. \)

The condition \( \text{H}_3 \) contains the requirement that the scalar product \( \alpha x \cdot \nabla \mu \mu \) is bounded in \( B_R, R > 0, \) while \( \alpha x \varepsilon + |x|^2 \cdot \nabla \mu \) is bounded in \( \mathbb{R}^N \setminus B_R, \) where \( B_R \) is a ball of radius \( R \) centered in zero.

The reason we use the function \( f_\varepsilon \) will be clear in the proof of the weighted Hardy inequality \( \text{(wHi)} \) which we will state below.

The idea to introduce the functions \( f_\varepsilon \) is due to [11] but our proof is different.

Finally we observe that we need the condition \( k_2 > 2 - N \) to apply Fatou’s lemma in the proof of Theorem 2.1.

**Theorem 2.1.** Under conditions \( \text{H}_1\)–\( \text{H}_3 \) there exists a positive constant \( c \) such that
\[ c \int_{\mathbb{R}^N} \varphi^2 \frac{1}{|x|^2} d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu, \] (2.1)
for any function \( \varphi \in C_0^\infty(\mathbb{R}^N), \) where \( c \in (0, c_o(N+k_2)] \) with \( c_o(N+k_2) = (\frac{N+k_2-2}{2})^2. \)

**Proof.** As first step we start from the integral of the square of the gradient of the function \( \varphi. \) Then we introduce \( \psi = \frac{\varphi}{f_\varepsilon}, \) with \( f_\varepsilon \) defined in \( \text{H}_3 \), and integrate by parts taking in mind \( \text{H}_1 \) and \( \text{H}_2 \).

\[
\begin{align*}
\int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu &= \int_{\mathbb{R}^N} |\nabla (\psi f_\varepsilon)|^2 d\mu \\
&= \int_{\mathbb{R}^N} |\nabla \psi f_\varepsilon + \nabla f_\varepsilon \psi|^2 d\mu \\
&= \int_{\mathbb{R}^N} |\nabla \psi|^2 f_\varepsilon^2 d\mu + \int_{\mathbb{R}^N} \psi^2 |\nabla f_\varepsilon|^2 d\mu + 2 \int_{\mathbb{R}^N} f_\varepsilon \psi \nabla \psi \cdot \nabla f_\varepsilon d\mu \tag{2.2} \\
&= \int_{\mathbb{R}^N} |\nabla \psi|^2 f_\varepsilon^2 d\mu + \int_{\mathbb{R}^N} \psi^2 |\nabla f_\varepsilon|^2 d\mu \\
&- \int_{\mathbb{R}^N} \psi^2 |\nabla f_\varepsilon|^2 d\mu - \int_{\mathbb{R}^N} f_\varepsilon^2 \psi^2 \frac{\Delta f_\varepsilon}{f_\varepsilon} d\mu - \int_{\mathbb{R}^N} f_\varepsilon^2 \psi^2 \nabla f_\varepsilon \cdot \nabla \mu dx.
\end{align*}
\]

Observing that
\[ \Delta f_\varepsilon = \frac{\alpha(N-2+\alpha)|x|^2 + \alpha \varepsilon N}{(\varepsilon + |x|^2)^{\frac{2-N}{2}}} \]
and using hypothesis $H_3$ we deduce that
\[
\int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu \geq - \int_{\mathbb{R}^N} \frac{\Delta f_\varepsilon}{f_\varepsilon} \varphi^2 d\mu - \int_{\mathbb{R}^N} \nabla f_\varepsilon \cdot \nabla \mu \varphi^2 d\varepsilon
\]
\[
\geq - \left[ \alpha(N-2) + \alpha^2 \right] \int_{\mathbb{R}^N} \frac{|x|^2}{\varepsilon + |x|^2} \varphi^2 d\mu - \varepsilon \alpha N \int_{\mathbb{R}^N} \frac{\varphi^2}{\varepsilon + |x|^2} d\mu
\]
\[
- k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu - k_2 \alpha \int_{\mathbb{R}^N} \frac{\varphi^2}{\varepsilon + |x|^2} d\mu
\]
\[
= [-\alpha(N-2+k_2) - \alpha^2] \int_{\mathbb{R}^N} \frac{|x|^2}{\varepsilon + |x|^2} \varphi^2 d\mu
\]
\[
- \varepsilon \alpha (N+k_2) \int_{\mathbb{R}^N} \frac{\varphi^2}{\varepsilon + |x|^2} d\mu - k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu.
\]

The constant $-\alpha(N-2+k_2) - \alpha^2$ is greater than zero since $\alpha < 0$ and $k_2 > 2 - N$, so by Fatou’s lemma we state the following estimate letting $\varepsilon \to 0$
\[
\int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu \geq c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu,
\]
with $c = -\alpha(N-2+k_2) - \alpha^2$. Finally we observe that
\[
\max_\alpha [-\alpha(N+k_2-2) - \alpha^2] = \left( \frac{N+k_2-2}{2} \right)^2 := c_o(N+k_2),
\]
attained for $\alpha_o = -\frac{N+k_2-2}{2}$. \hfill \Box

We observe that in the case $\mu = 1$ we obtain the classical Hardy inequality.

**Remark 2.2.** In an alternative way we can define $f_\varepsilon$ in $H_3$ setting $\alpha = \alpha_o$ and get the estimate (2.1) with $c = c_o(N+k_2)$. Although the result it goes in the same direction, in the proof we point out that $c_o(N+k_2)$ is the maximum value of the constant $c$.

Now we suppose that $H_4$)
\[\mu \geq 0, \sqrt{\mu} \in H^1_{\text{loc}}(\mathbb{R}^N);\]
\[H_5) \mu^{-1} \in L^1_{\text{loc}}(\mathbb{R}^N).\]

Let us observe that in the hypotheses $H_4) - H_5)$ the space $C_c^\infty(\mathbb{R}^N)$ is dense in $H^1_\mu$ and $H^1_\mu$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Sobolev norm
\[
\| \cdot \|_{H^1_\mu}^2 := \| \cdot \|^2_{L^2_\mu} + \| \nabla \cdot \|^2_{L^2_\mu}
\]
(see [20]). For some interesting papers on density of smooth functions in weighted Sobolev spaces and related questions we refer, for example, to [17] [12] [18] [10] [21] [4] [6].

So we can deduce the following result from Theorem 2.1 by density argument.

**Theorem 2.3.** Under conditions $H_2) - H_5)$ there exists a positive constant $c$ such that
\[
c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 d\mu,
\]
for any function $\varphi \in H^1_\mu(\mathbb{R}^N)$, where $c \in (0, c_o(N+k_2)]$ with $c_o(N+k_2) = \left( \frac{N+k_2-2}{2} \right)^2$. 

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Remark 2.4. We remark that our proof of \( wHi \) is different from the others in literature, in particular it is different from the one stated in [8] for the weighted case. Furthermore some hypotheses are different. We achieve the best constant \( c_0(N + k_2) \) (cf. [8, Theorem 1.3]) without the further requirements which allow the authors to use improved Hardy inequalities (see [1, 19]) to get the result.

We present some examples of functions \( \mu \) which satisfy the hypotheses of Theorem 2.3.

We remark that, in the hypotheses \( \mu = \mu(|x|) \in C^1 \) for \( |x| \in [r_0, +\infty[, r_0 \geq 0 \), a class of weight functions \( \mu \) which satisfies \( H_3 \) is the following

\[
\mu(x) \geq Ce^{-\frac{k_1}{|x|^2} |x|^{k_2 - \frac{k_1}{2}}}, \quad \text{for } |x| \geq r_0,
\]

where \( C \) is a constant depending on \( \mu(r_0) \) and on \( r_0 \).

In the case of radial functions, \( \mu(x) = \mu(|x|) \), if we set \( |x| = \rho \), the condition \( H_3 \) states that \( \mu \) satisfies the following inequality

\[
\frac{\alpha \rho}{\varepsilon + \rho^2} \mu'(|\rho|) \leq \left( k_1 + \frac{k_2 \alpha}{\varepsilon + \rho^2} \right) \mu(\rho),
\]

which implies

\[
\mu'(\rho) \geq a(\rho) \mu(\rho)
\]

with

\[
a(\rho) = \left[ \frac{k_1}{\alpha} \left( \frac{\varepsilon + \rho^2}{\rho} \right) + \frac{k_2}{\rho} \right].
\]

Integrating in \([r_0, r] \) we get

\[
\mu(r) \geq \mu(r_0) e^{\int_{r_0}^{r} a(s) ds} = \mu(r_0) \left( \frac{r}{r_0} \right)^{k_2 - \frac{k_1}{|\alpha|}} \rho^{\frac{k_2 - k_1}{|\alpha|}} e^{-\frac{k_1}{2|\alpha|} (r^2 - r_0^2)} \quad \text{for } r \geq r_0,
\]

from which

\[
\mu(r) \geq \frac{\mu(r_0)}{k_2 - \frac{k_1}{|\alpha|}} \rho^{\frac{k_2 - k_1}{|\alpha|}} e^{-\frac{k_1}{2|\alpha|} (r^2 - r_0^2)} \quad \text{for } r \geq r_0.
\]

Example 2.5. Another class of weight functions satisfying \( H_3 \), when \( k_1 = k_2 = 0 \), consists of the bounded increasing functions, as, for example, \( \cos e^{-|x|^2} \). Such a function verifies the requirements in the Theorem 2.3.

In the following example we consider a wide class of functions which contains the Gaussian measure and polynomial type measures. A class of functions which behaves as \( \frac{1}{|x|^n} \) when \( |x| \) goes to zero.

Example 2.6. We consider the following weight functions

\[
\mu(x) = \frac{1}{|x|^\gamma} e^{-\delta |x|^m}, \quad \delta \geq 0, \quad \gamma < N - 2
\]

and state for which values of \( \gamma \) and \( m \) the functions in (2.5) are “good” functions to get \( wHi \).

The weight \( \mu \) satisfies \( H_2 \), \( H_4 \) and \( H_5 \) if \( \gamma > -N \). The condition \( H_3 \)

\[
\frac{\alpha (-\gamma - \delta m x^m)}{\varepsilon + |x|^2} \leq k_1 + \frac{\alpha k_2}{\varepsilon + |x|^2}
\]
is fulfilled if
\[- (\alpha \gamma + \alpha k_2 + k_1 \varepsilon) - \alpha \delta m |x|^m - k_1 |x|^2 \leq 0. \tag{2.6}\]
In the case \( \delta = 0 \) we only need to require that \( \gamma \leq -k_2 - \frac{k_1}{\alpha} \varepsilon \) and we are able to get the Caffarelli-Nirenberg inequality
\[
\left( \frac{N - 2 - \gamma}{2} \right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} |x|^{-\gamma} \, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 |x|^{-\gamma} \, dx \quad \forall \varphi \in H^1_{\mu}.
\]
While if \( \gamma = 0 \) the inequality (2.6) holds if \( m \leq 2 \) and \( k_1 \) is large enough.

In general to get (2.6) we need the following conditions on parameters and on the constant \( k_1 \)

i) \( \gamma \in (-N, -k_2], \delta = 0, k_1 = 0, \)

ii) \( \gamma \in (-N, -k_2], k_1 \geq -2 \alpha \delta, m = 2, \)

iii) \( \gamma \in (-N, -k_2), k_1 \geq \tilde{k}_1, m < 2, \)

where \( \tilde{k}_1 = \frac{\tilde{\gamma}(1 - \frac{1}{\alpha})}{\alpha(\gamma + k_2)} \), to get the inequality (2.3).

**Example 2.7.** The function \( \mu(x) = [\log(1 + |x|)]^{-\gamma}, \) for \( \gamma \) as in i), behaves as \( \frac{1}{|x|^\gamma} \) when \( |x| \) goes to 0. So can state the weighted Hardy inequality (2.3) with \( k_1 = 0 \) as in the previous case.

3. Optimality of the constant in \( wHi \)

To state the optimality of the constant \( c_o(N + k_2) \) in the estimate (2.3) we need further assumptions on \( \mu \) as usually it is done. We remark that in the proof of optimality the choice of the function \( \varphi \) plays a fundamental role. The technique we use is close to the one used in [8] but the function \( \varphi \) we use is different.

We suppose

\( H_6 \) there exists \( \sup_{\delta \in \mathbb{R}} \left\{ \frac{1}{|x|^\delta} \in L^1_{\text{loc}}(\mathbb{R}^N, d\mu) \right\} := N_0 \) and \( k_2 = N_0 - N. \)

We observe that the condition \( H_6 \) is necessary for the technique used in the proof of the optimality. In [8] there is a similar condition to get the optimality of the constant when \( c_\mu = \left( \frac{N_0 - 2}{2} \right)^2 \). In our case the requirement \( H_6 \) involves the constant \( k_2 \).

For example the functions \( \mu \) such that
\[
\lim_{x \to 0} \frac{\mu}{|x|^{k_2}} = l, \quad l > 0,
\]
verify \( H_6 \).

The result below states the optimality of the constant \( c_o(N + k_2) \) in the Hardy inequality.

**Theorem 3.1.** In the hypotheses of Theorem 2.3 and if \( H_6 \) holds, for \( c > c_o(N + k_2) = \left( \frac{N + k_2 - 2}{2} \right)^2 \) the inequality (2.3) doesn't hold for any \( \varphi \in H^1_{\mu}. \)
Proof. Let $\theta \in C^\infty_c(\mathbb{R}^N)$ a cut-off function, $0 \leq \theta \leq 1$, $\theta = 1$ in $B_1$ and $\theta = 0$ in $B_2^c$. We introduce the function
\[
\varphi_\varepsilon = \begin{cases} 
(\varepsilon + |x|)^{\eta} & \text{if } |x| \in [0, 1], \\
(\varepsilon + |x|)^{\eta} \theta(x) & \text{if } |x| \in [1, 2], \\
0 & \text{if } |x| \in [2, +\infty[,
\end{cases}
\]
where $\varepsilon > 0$ and the exponent $\eta$ is such that
\[
\max \left\{ -\sqrt{c}, -\frac{N + k_2}{2} \right\} < \eta < \min \left\{ -\frac{N + k_2 - 2}{2}, 0 \right\}.
\]
The function $\varphi_\varepsilon$ belongs to $H^1_0$ for any $\varepsilon > 0$.

For this choice of $\eta$ we obtain $\eta^2 < c$, $|x|^{2^\eta} \in L^1_{\text{loc}}(\mathbb{R}^N, d\mu)$ and $|x|^{2^{\eta-2}} \notin L^1_{\text{loc}}(\mathbb{R}^N, d\mu)$. Let us assume that $c > c_\varepsilon(N + k_2)$. Our aim is to prove that the bottom of the spectrum of the operator $-(L + V)$
\[
\lambda_1 = \inf_{\varphi \in H^1_0 \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu - \int_{\mathbb{R}^N} \frac{c}{|x|^2} \varphi^2 d\mu}{\int_{\mathbb{R}^N} \varphi^2 d\mu} \right).
\] (3.1)
is $-\infty$. For this purpose we estimate at first the numerator in (3.1),
\[
\begin{align*}
\int_{\mathbb{R}^N} \left( |\nabla \varphi_\varepsilon|^2 - \frac{c}{|x|^2} \varphi_\varepsilon^2 \right) d\mu &= \\
= \int_{B_1} \left[ |\nabla (\varepsilon + |x|)^{\eta}|^2 - \frac{c}{|x|^2} (\varepsilon + |x|)^{2^\eta} \right] d\mu \\
&+ \int_{B_1^c} \left[ |\nabla (\varepsilon + |x|)^{\eta} \theta|^2 - \frac{c}{|x|^2} (\varepsilon + |x|)^{2\eta} \theta^2 \right] d\mu \\
= \int_{B_1} \left[ \eta^2 (\varepsilon + |x|)^{2^\eta - 2} - \frac{c}{|x|^2} (\varepsilon + |x|)^{2^\eta} \right] d\mu \\
&+ \eta^2 \int_{B_1^c} (\varepsilon + |x|)^{2^{\eta-2}} \theta^2 d\mu + \int_{B_1^c} (\varepsilon + |x|)^{2^\eta} |\nabla \theta|^2 d\mu \\
&+ 2\eta \int_{B_1^c} (\varepsilon + |x|)^{2^\eta - 2} \theta^2 + \int_{B_1^c} (\varepsilon + |x|)^{2^\eta} |\nabla \theta|^2 d\mu
\end{align*}
\] (3.2)
where $C_1 = (2\eta^2 + 2 \|\nabla \theta\|_\infty) \int_{B_1^c} d\mu$.

Furthermore
\[
\int_{\mathbb{R}^N} \varphi_\varepsilon^2 d\mu \geq \int_{B_2 \setminus B_1} (\varepsilon + |x|)^{2^\eta \theta^2} d\mu = C_{2,\varepsilon}.
\] (3.3)
Put together (3.2) and (3.3) from (3.1) we get
\[ \lambda_1 \leq \frac{\int_{B_1}(\varepsilon + |x|)^{2\eta} \left( \frac{\eta^2}{(\varepsilon + |x|)^2} - \frac{c}{|x|^2} \right) d\mu + C_1}{C_{2\varepsilon}}. \]

Letting \( \varepsilon \to 0 \) in the numerator above, taking in mind that \( |x|^{2\eta} \in L^1_{loc}(\mathbb{R}^N, d\mu) \) and Fatou’s lemma, we obtain
\[ \lim_{\varepsilon \to 0} \int_{B_1}(\varepsilon + |x|)^{2\eta} \left( \frac{\eta^2}{(\varepsilon + |x|)^2} - \frac{c}{|x|^2} \right) d\mu \leq - (c - \eta^2) \int_{B_1} |x|^{2\eta - 2} d\mu = -\infty \]
and, then, \( \lambda_1 = -\infty. \) □

4. Kolmogorov Operators and Existence and Nonexistence Results

In the standard setting one considers \( \mu \in C^{1,\lambda}_{loc}(\mathbb{R}^N) \) for some \( \lambda \in (0, 1) \) and \( \mu > 0 \) for any \( x \in \mathbb{R}^N. \)

We consider Kolmogorov operators
\[
Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u, \quad (4.1)
\]
on smooth functions, where the probability density \( \mu \) in the drift term is not necessarily \((1, \alpha)-\) Hölderian in the whole space but belongs to \( C^{1,\lambda}_{loc}(\mathbb{R}^N \setminus \{0\}). \)

These operators arise from the bilinear form integrating by parts
\[
a_{\mu}(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v d\mu = - \int_{\mathbb{R}^N} (Lu)v d\mu.
\]

The purpose is to get existence and nonexistence results for weak solutions to the the initial value problem on \( L^2_\mu \) corresponding to the operator \( L \) perturbed by an inverse square potential
\[
(P) \quad \left\{ \begin{array}{l}
\partial_t u(x, t) = Lu(x, t) + V(x)u(x, t), \quad x \in \mathbb{R}^N, t > 0, \\
u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu,
\end{array} \right.
\]
where \( V(x) = \frac{c}{|x|^2}, \) with \( c > 0. \)

We say that \( u \) is a weak solution to \( (P) \) if, for each \( T, R > 0, \) we have
\[ u \in C([0, T], L^2_\mu), \quad V u \in L^1(B_R \times (0, T), d\mu dt) \]
and
\[ \int_0^T \int_{\mathbb{R}^N} u(-\partial_t \phi - L\phi) d\mu dt - \int_{\mathbb{R}^N} u_0\phi(\cdot, 0) d\mu = \int_0^T \int_{\mathbb{R}^N} Vu\phi d\mu dt \]
for all \( \phi \in W^{2,1}_{\text{loc}}(\mathbb{R}^N \times [0, T]) \) having compact support with \( \phi(\cdot, T) = 0, \) where \( B_R \) denotes the open ball of \( \mathbb{R}^N \) of radius \( R \) centered at \( 0. \) For any \( \Omega \subset \mathbb{R}^N, \)
\( W^{2,1}_{\text{loc}}(\Omega \times (0, T)) \) is the parabolic Sobolev space of the functions \( u \in L^2(\Omega \times (0, T)) \) having weak space derivatives \( D^\alpha_x u \in L^2(\Omega \times (0, T)) \) for \( |\alpha| \leq 2 \) and weak time derivative \( \partial_t u \in L^2(\Omega \times (0, T)) \) equipped with the norm
\[
\|u\|_{W^{2,1}_{\text{loc}}(\Omega \times (0, T))} := \left( \|u\|^2_{L^2(\Omega \times (0, T))} + \|\partial_t u\|^2_{L^2(\Omega \times (0, T))} + \sum_{1 \leq |\alpha| \leq 2} \|D^\alpha u\|^2_{L^2(\Omega \times (0, T))} \right)^{\frac{1}{2}}.
\]
Let us assume that the function $\mu$ is a probability density on $\mathbb{R}^N$, $\mu > 0$. In the hypothesis

$H_7$) $\mu \in C_{loc}^{1,\lambda}(\mathbb{R}^N)$, $\lambda \in (0, 1)$

it is known that the operator $L$ with domain

$$D_{max}(L) = \{ u \in C_b(\mathbb{R}^N) \cap W_{loc}^{2,p}(\mathbb{R}^N) \text{ for all } 1 < p < \infty, Lu \in C_b(\mathbb{R}^N) \}$$

is the weak generator of a not necessarily $C_0$-semigroup in $C_b(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} Lu \, d\mu = 0$ for any $u \in C_c^\infty(\mathbb{R}^N)$, then $d\mu = \mu(x)\,dx$ is the invariant measure for this semigroup in $C_b(\mathbb{R}^N)$. So we can extend it to a positivity preserving and analytic $C_0$-semigroup \{T(t)\}_{t \geq 0}$ on $L^2_{\mu}$, whose generator is still denoted by $L$ (see [16]).

When the assumptions on $\mu$ allow degeneracy at one point, we require the following conditions to get $L$ generates a semigroup:

$H_8$) $\mu \in C_{loc}^{1,\lambda}(\mathbb{R}^N \setminus \{0\})$, $\lambda \in (0, 1)$, $\mu \in H^1_{loc}(\mathbb{R}^N)$, $\sum_{x \in K} \mu(x) > 0$ for any compact set $K \subset \mathbb{R}^N$.

So by [2, Corollary 3.7]), we have that the closure of \{(L, C^\infty_c(\mathbb{R}^N))\} on $L^2_{\mu}$ generates a strongly continuous and analytic Markov semigroup \{T(t)\}_{t \geq 0} on $L^2_{\mu}$.

We observe that the function $e^{-\delta|x|^m}$ in Example 2.6 fully satisfies the condition $H_8$ while $\cos e^{-|x|^2}$ in Example 2.5 is $(1, \alpha)$-Hölderian in $\mathbb{R}^N$.

For weight functions $\mu$ satisfying assumption $H_7$ or $H_8$ there are some interesting properties regarding the semigroup \{T(t)\}_{t \geq 0} generated by the operator $L$. These properties listed in the Proposition below are well known under hypothesis $H_7$ (see [16]) and have been proved in [8] if $\mu$ satisfies $H_8$.

**Proposition 4.1.** Assume that $\mu$ satisfies $H_7$ or $H_8$. Then the following assertions hold:

(i) $D(L) \subset H^1_{\mu}$.

(ii) For every $f \in D(L)$, $g \in H^1_{\mu}$ we have

$$\int Lfg \, d\mu = -\int \nabla f \cdot \nabla g \, d\mu.$$ 

(iii) $T(t)L^2_{\mu} \subset D(L)$ for all $t > 0$.

The following Theorem stated in [14] for functions $\mu$ satisfying condition $H_7$, was proved in [8] for functions $\mu$ under condition $H_8$.

**Theorem 4.2.** Let $0 \leq V(x) \leq \frac{1}{|x|^2}$. Assume that the weight function $\mu$ satisfies $H_4$, $H_5$ and $H_8$. Then the following assertions hold:

(i) If $\lambda_1(L+V) > -\infty$, then there exists a positive weak solution $u \in C([0, \infty), L^2_{\mu})$ of (P) satisfying

$$\|u(t)\|_{L^2_{\mu}} \leq Me^{\omega t}\|u_0\|_{L^2_{\mu}}, \quad t \geq 0 \quad (4.2)$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

(ii) If $\lambda_1(L+V) = -\infty$, then for any $0 \leq u_0 \in L^2_{\mu} \setminus \{0\}$, there is no positive weak solution of (P) satisfying (4.2).
To get existence and nonexistence of solutions to \((P)\) we put together the weighted Hardy inequality \((2.3)\), Theorem 3.1 and Theorem 4.2. So we can state the following result.

**Theorem 4.3.** Assume that the weight function \(\mu\) satisfies hypotheses \(H_2)\)–\(H_6\) and \(H_8\). The following assertions hold:

(i) If \(0 \leq c \leq c_0(N + k_2) = \left(\frac{N + k_2 - 2}{2}\right)^2\), then there exists a positive weak solution \(u \in C([0, \infty), L^2_\mu)\) of \((P)\) satisfying
\[
\|u(t)\|_{L^2_\mu} \leq Me^{\omega t}\|u_0\|_{L^2_\mu}, \quad t \geq 0
\] (4.3)
for some constants \(M \geq 1\), \(\omega \in \mathbb{R}\), and any \(u_0 \in L^2_\mu\).

(ii) If \(c > c_0(N + k_2)\), then for any \(0 \leq u_0 \in L^2_\mu\), \(u_0 \neq 0\), there is no positive weak solution of \((P)\) with \(V(x) = \frac{c}{|x|^2}\) satisfying (4.3).

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