It is quite well known for some time that string inspired axionic terms of the form \( \nu(\phi) \tilde{R}R \), known also as Chern-Simons terms, do not affect the scalar perturbations and the background evolution for a flat Friedman-Robertson-Walker Universe. In this paper we study and quantify the implications of the presence of the above term in the context of vacuum \( f(R) \). Particularly, we assume that axionic dark matter is present during inflation, and we examine in a quantitative way the effects of axionic Chern-Simons terms on the tensor perturbations. The axion field is quantified in terms of a canonical scalar field, with broken Peccei-Quinn symmetry. The model perfectly describing axions as potential dark matter candidates is based on the so-called misalignment mechanism, in which case the axion is frozen near its non-zero vacuum expectation value during early times with \( H \gg m_a \). In effect, the inflationary era is mainly controlled by the \( f(R) \) gravity and the Chern-Simons term. As we demonstrate, the Chern-Simons term may achieve to make a non-viable \( f(R) \) gravity theory to be phenomenologically viable, due to the fact that the tensor-to-scalar ratio is significantly reduced, and the same applies to the spectral index of the tensor perturbations \( n_T \). Also by studying the Starobinsky model in the presence of the Chern-Simons term, we demonstrate that it is possible to further reduce the amount of primordial gravitational radiation. The issues of having parity violating gravitational waves, also the graceful exit from inflation due to axion oscillations and finally the unification of dark energy-inflation and axion dark matter in the same \( f(R) \) gravity-axion dark matter model, are also briefly discussed.

I. INTRODUCTION

Cosmology is currently at a stage in which many observational data exist that need to be appropriately explained theoretically. Three are the most intriguing mysteries in cosmology, the dark matter nature, the dark energy issue for the late-time Universe, and the primordial era. These mysteries are currently constrained by observations, but for the moment we are at the stage of speculating and fitting models that may appropriately describe these in a consistent way. With regard to the primordial era, it is currently accepted that an inflationary era preceded the radiation and matter domination eras, and many theoretical proposals have appeared after the pioneer papers \cite{1,2} that can describe this accelerating era. The inflationary era and the dark energy epoch may be successfully described by modified gravity \cite{4,5}, and actually a unified description can be provided in terms of a modified gravity, see for example \cite{6} for an early \( f(R) \) gravity model unifying dark energy and inflation.

Dark matter was introduced in order to describe theoretically the galactic rotation curves, and its nature is mysterious. Although modified gravity may explain to some extent certain aspects of dark matter, after the GW170817 event \cite{11}, these theories were seriously questioned \cite{12}. Thus the only consistent for the moment theoretical proposal is that dark matter is some weakly interacting massive particle. In fact, one fascinating event that strongly supports the particle nature of dark matter is collision of galaxies observed in the Bullet Cluster, which indicates that there is an invisible gravitating component in the collision. In the literature there exist many candidate particles that may describe dark matter, see for example \cite{13}, but still no direct detection of dark matter occurred. One promising candidate is the axion \cite{14,15}, which occurs in various theoretical physics contexts, even in string theory, see also \cite{16} for some related works on axion dark matter. The axion is a low-mass particle which is the goldstone boson of the spontaneously broken Peccei-Quinn symmetry, in the context of ordinary field theory and also in string theory. In string theory the axions are massless when the extra dimensional theory is compactified in four dimensions, and
non-perturbative effects can provide mass to axions. The most interesting model for axion dark matter is the so-called misalignment axion model, in which initially the axion has a small and constant mass during and before the inflationary era, and starts to oscillate during the radiation domination era. In this paper we shall be interested in investigating the effects of the existence of axions in an $f(R)$ gravity inflationary theory. We shall assume that the axion is described by a canonical scalar field at early times and also that the axion field is a misalignment axion and it thus has a non-zero vacuum expectation value. Also we shall assume that terms coming from string corrections are present in the theory, of the form $\nu(\phi)\hat{R}R$. These are well motivated from string theory, due to the fact that string theory is the only consistent UV description of all known theories describing the four fundamental interactions. It is thus possible that some terms may survive during the inflationary era, and these can have some effect on the dynamics of inflation. Terms of the form $\nu(\phi)\hat{R}R$ are dubbed Chern-Simons (CS) terms, however the term $\nu(\phi)\hat{R}R$ is simply the Chern-Pontryagin density which actually is related to a three dimensional Chern-Simons term via the exterior derivative $\nu(\phi)\hat{R}R = d(\text{Chern}\,−\,	ext{Simons})$. An important stream of papers on Chern-Simons gravity can be found in Refs. [17–30] and references therein. The effect of the CS term on theories of the form $f(R, \phi)$ were thoroughly studied by Hwang and Noh some time ago [31] and as they proved, the CS term affects the tensor perturbations and leaves intact the scalar perturbations and the background evolution. In this paper we are interested to quantify the effect of the presence of an axion dark matter, with a CS term present, on vacuum $f(R)$ gravity theories. Our aim is two-fold: Firstly we choose some $f(R)$ gravity which is known to provide a non-viable phenomenology, and try to investigate whether the axion dark matter with CS term can make the theory viable. Secondly we shall choose the most successful model of $f(R)$ gravity up-to-date, namely the Starobinsky model [2], and we shall investigate quantitatively the effect of CS corrected axion dark matter on the vacuum $f(R)$ gravity. As we shall demonstrate, the effect of the CS term is to reduce the value of tensor-to-scalar ratio and of the spectral index of the primordial tensor perturbations, and we provide a thorough quantitative analysis of how these reductions can be produced. The resulting picture of the $f(R)$-dark matter model we introduce is quite interesting theoretically, since the presence of early-time dark matter affects the dynamics of inflation, and also has a non-trivial effect on the propagation models of gravitational waves, discriminating the two different polarizations.

This paper is organized as follows: In section II we shall discuss in some detail the CS modified axion dark matter $f(R)$ gravity model, focusing also on the essential properties of the axion misalignment model. In section III we shall investigate the implications of the CS axion dark matter on the power-law $f(R)$ gravity which is known to provide a non-viable inflationary phenomenology in vacuum. Particularly we shall investigate when the resulting model can be compatible with the Planck [32] and BICEP2/Keck-Array [33] observational data. In addition, we examine the effects of the CS modified axion dark matter on the Starobinsky model, and finally the conclusions follow in the end of the paper.

Before we start off, until it is stated differently, in this paper we shall assume that the geometric background is a flat Friedmann-Robertson-Walker (FRW), with metric,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,$$

(1)

with $a(t)$ being the scale factor.

II. CHERN-SIMONS MODIFIED AXION DARK MATTER $f(R)$ GRAVITY

The model we propose composes from a vacuum $f(R)$ gravity in the presence of an axion scalar field, with a CS term present too. The gravitational action is,

$$S = \int d^4x\sqrt{g} \left[ \frac{1}{2\kappa^2}f(R) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) + \frac{1}{8} \nu(\phi)\hat{R}R \right],$$

(2)

where $R\hat{R} = \epsilon^{abcd} R_{ab}^{ef} R_{cdef}$, $\kappa^2 = \frac{1}{8G}$, with $G$ Newton’s gravitational constant, and $\epsilon^{abcd}$ is the totally antisymmetric Levi-Civita tensor. Basically the Chern-Pontryagin density is an analogue of the term $*F_{\mu\nu}F^{\mu\nu}$ built from the curvature $F_{\mu\nu}$ on a principal bundle with connection components $A_\mu$, but we will call it Chern-Simons term in order to comply with the literature.

By varying the action (2) with respect to the metric, and by using the FRW metric of Eq. (1), we obtain the following equations of motion,

$$3H^2 F = \kappa^2 \frac{1}{2} \dot{\phi}^2 + \frac{RF - f + 2V\kappa^2}{2} - 3H \dot{F},$$

(3)
\[-3FH^2 + 2\dot{H}F = \kappa^2 \frac{1}{2} \dot{\phi}^2 - \frac{RF - f + 2V}{2} + \ddot{F} + 2HF,\]

while the variation with respect to the scalar field yields,

\[\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,\]  

where \(V'(\phi) = \frac{\partial V}{\partial \phi}\) and \(F = \frac{\partial f}{\partial \phi}\). As it can be seen from the Eqs. (3) and (4), the CS term does not affect the background equations, as was also demonstrated by Hwang and Noh [31], see also [32]. In addition, it was shown in [31, 32] that the scalar perturbations remain intact by the presence of the CS-term. The reason for this is that it is not possible to form a scalar \(T_{00}\) energy momentum tensor nor vector \(T_{0a}\) or symmetric tensor \(T_{a\beta}\), which contains \(e_{abcd}\) and scalar derivatives only [32]. Therefore, the only effect of the CS term is on tensor perturbations and let us present here the slow-roll indices of the \(f(R)\)-scalar theory. We adopt the notation of [31], and the slow-roll indices are,

\[\epsilon_1 = \frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\ddot{F}}{2HF}, \quad \epsilon_4 = \frac{\dot{E}}{2HE}, \quad \epsilon_6 = \frac{\dot{Q}_t}{2HQ_t},\]

where \(E\) and \(Q_t\) are defined as follows,

\[E = \frac{F}{\dot{\phi}^2\kappa^2} \left( \dot{\phi}^2 + \frac{3\dot{F}^2}{2F\kappa^2} \right), \quad Q_t = \frac{F}{\kappa^2} + 2\lambda_1\nu k/a.\]  

The parameter \(\lambda_t\) in Eq. (6) characterizes the polarization of the primordial gravity waves with wavenumber \(k\) and takes values \(\lambda_L = -1\) and \(\lambda_R = 1\) for left ad right handed polarization states, while \(a\) is simply the scale factor. As it was shown in Ref. [31], the spectral index of the primordial scalar curvature perturbations is,

\[n_s = 2\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4,\]  

while the spectral index of the tensor perturbations is,

\[n_T = 2\epsilon_1 - \epsilon_6.\]  

Accordingly, the tensor-to-scalar ratio for the action (2) is equal to,

\[r = 16|\epsilon_1 - \epsilon_3| \sum_{l=L,R} \frac{1}{2\lambda_l\nu^2k^2} \left( 1 + 2\lambda_l\frac{\nu^2k^2}{aF} \right)^{-1}.\]

Basically, the effect of the CS term of the axion field can be seen only on the slow-roll parameter \(\epsilon_6\), which affects the spectral index of the tensor perturbations, and on the tensor-to-scalar ratio, as was expected, since the background equations and the slow-roll indices being involved in the scalar perturbations are not affected. Thus only the tensor perturbations related phenomenology of \(f(R)\) gravity theories is affected by the presence of the CS term. The aim of this paper is quantify the impact of the CS term on some characteristic models of \(f(R)\) gravity. Prior getting into that, let us briefly discuss the essential phenomenology of axion dark matter theories. A recent review on axion cosmology is given in Ref. [14] and we shall adopt the notation and conventions of this review. The axion field is described by a canonical scalar field with potential \(V(\phi)\), and with the additional assumption that a string theory inspired CS term is present in the action. The most promising model of axions that can produce some of the dark matter present today in our Universe, is the so-called misalignment model. In the context of misalignment model, the original unbroken Peccei-Quinn \(U(1)\) symmetry is broken and the axion field has a non-zero vacuum expectation value, which is strongly model dependent, but quite large, and actually of the order \(\mathcal{O}(10^{19})\) GeV or larger. The axion field is at the broken phase during inflation and the mass of the axion, which we denote \(m_a\) is almost constant, and remains constant until the radiation domination era. Also during the inflationary era, when \(H \gg m_a\), the axion has a potential,

\[V(\phi(t)) \approx \frac{1}{2} m_a^2 \phi^2(t),\]

and the value of the scalar field is actually the constant non-zero expectation value of the axion. During the inflationary era, the field is overdamped, and it is practically frozen at its vacuum expectation value. For all cosmic times during this era, we have practically the following conditions holding true for the dynamics of the axion,

\[\dot{\phi}(t_i) = \delta \ll 1, \quad \phi(t_i) = f_0 \theta_a,\]
where \( f_a \) is the axion decay constant, and \( \theta_a \) is the initial misalignment angle. Thus during inflation, the axion effective equation of state is approximately \( w_{\text{eff}} = -1 \) and it basically acts as a cosmological constant term in the cosmological equations, since the potential is constant. Hence, the energy density of the axion at early times and before it starts oscillations, is solely determined by the axion mass and by the initial field displacement \( \phi_i \). Practically the axion field is very slowly varying, so that \( \dot{\phi} = \delta \ll 1 \) and also \( \dot{\phi} = \zeta \ll 1 \) during the whole inflationary era, until it starts to oscillate during the radiation domination era. We shall assume that low-scale inflation occurs, so by taking into account the compatibility with the Planck and BICEP2/Keck-Array data, the inflationary scale is of the order \( H_I = \mathcal{O}(10^{15}) \text{GeV} \) in the low-scale inflation scenario. Also for phenomenological reasons having to do with overproduction of axion dark matter via isocurvature perturbations backreaction and also by taking the allowed axion energy density into account, the most convenient choices for \( f_a \) and \( \theta_a \) that fit the data is \( n = 4, \theta_a \sim \mathcal{O}(1) \). \( f_a \sim \mathcal{O}(10^{11}) \text{GeV}, \theta_a \sim \mathcal{O}(1) \). (12)

Also the mass of the axion field is assumed to be, \[ m_a \sim \mathcal{O}(10^{-12}) \text{eV}, \]

which is the maximum allowed value in the context of the broken Peccei-Quinn symmetry scenario. From the form of the equations of motion \( [6] \) it is obvious that if the \( f(R) \) gravity is simply the Einstein-Hilbert gravity \( f(R) = R \), the contribution of the action in the cosmological evolution is significant, and actually it simply contributes a cosmological constant \( \sim V(\phi) \) in the cosmological equations. However, if an \( f(R) \) gravity is chosen that diverges faster for large curvatures, then the contribution of the axion can be reduced. In all cases, the CS term \( \nu RR \) affects only tensor perturbations and not the background evolution. In the following sections we shall quantify our study by choosing specific \( f(R) \) gravity models and we examine in detail their inflationary phenomenology.

III. EFFECTS OF CHERN-SIMONS MODIFIED AXION DARK MATTER ON POWER-LAW \( f(R) \) AND STAROBINSKY GRAVITY

Let us first consider an \( f(R) \) gravity with problematic phenomenology, which is, \[ f(R) = R + \beta R^n, \] (14)

with \( n \) being chosen in the interval \( n = \left[ \frac{\log \sqrt{2}}{2}, 2 \right] \) in order to have accelerating expansion. For the moment the approach we shall adopt for the power-law model, will not cover the Starobinsky model case, which corresponds to \( n = 2 \), because the Starobinsky model can be treated in a more accurate way in order to find the exact Hubble rate realized by the model in the slow-roll approximation. The differences between the power-law model for \( n \neq 2 \) and the Starobinsky model are clarified in the Appendix.

The model \( [4] \) leads to non-zero tensor spectral index and also one does not have simultaneous compatibility of the spectral index of primordial scalar perturbations and of the tensor-to-scalar ratio with the observational data. In this section we shall investigate the phenomenological implications of the CS term and of the presence of the dark matter axion field in the theory. Firstly, by recalling that during the inflationary era \( \dot{\phi} \) is constant and \( \dot{\phi} = \delta \ll 1 \), and also that the scalar field is almost constant, see the conditions \( [11] \), the first equation of motion in Eq. \( [3] \) becomes, \[ 3H^2n\beta R^{n-1} = \frac{\kappa^2 \delta^2}{2} + \frac{\beta(n-1)R^{n-1}}{2} - 3n(n-1)\beta H R^{n-2}\dot{\delta} + \frac{\kappa^2}{2} m_a^2 f_a^2 \theta_a^2, \] (15)

so by using the fact that at \( R = 12H^2 + 6\dot{H} \), we get at leading order, \[ 3H^2n\beta \simeq 6\beta(n-1)H^2 - 6n\beta(n-1)\dot{H} + 3\beta(n-1)\dot{H} + \frac{\kappa^2}{2(12H^2)^{n-1}} \delta^2 + \frac{\kappa^2}{2(12H^2)^{n-1}} m_a^2 f_a^2 \theta_a^2. \] (16)

The last two terms in Eq. \( [18] \) are much smaller than the rest of the terms, therefore these can be neglected. In order to have a concrete idea on how small are the last two terms in Eq. \( [18] \), let us use the numerical values for the axion field and for the low-scale inflation we presented in the previous section. The first term is obviously suppressed due to the presence of \( \kappa^2 \delta^2 \) in the numerator and \( (12H^2)^{n-1} \) in the denominator, so let us focus on the last term in Eq. \( [18] \). Recall that \( n = \left[ \frac{\log \sqrt{2}}{2}, 2 \right] \), so let us use the values for \( m_a, f_a \) and \( \theta_a \) appearing in Eqs. \( [12] \) and \( [13] \),
and also we assume that the Hubble rate during inflation is \( H_f = O(10^{10}) \text{GeV} \). By using these values and also since \( \kappa^2 = 4.10 \times 10^{-28} \text{eV} \), we get that the last term is of the order,

\[
\frac{\kappa^2}{2(12H^2)^{n-1}m_a^2f_a^2g_a^2} = O(10^{-70}/\beta) \text{eV},
\]

while the first term in Eq. \( \text{(18)} \) is of the order \( O(10^{38}) \text{eV} \). Hence, due to the fact that in most polynomial inflationary \( f(R) \) gravities, \( \beta \ll 1 \), it is obvious that the last two terms are subleading for the axion dark matter \( f(R) \) gravity model, so these can be safely neglected. Therefore, the differential equation that determines the Hubble rate is,

\[
3H^2n\beta \simeq 6\beta(n-1)H^2 - 6n\beta(n-1)\dot{H} + 3\beta(n-1)\dot{H},
\]

which can be solved to yield,

\[
H(t) = \frac{-2n^2 + 3n - 1}{(n-2)t}.
\]

Using the Hubble rate \( \text{(19)} \), the slow-roll indices \( \epsilon_i, i = 1,\ldots,4 \) can be easily calculated for the \( f(R) \) gravity \( \text{(14)} \), at the horizon crossing time instance \( t_k \) during inflation, and these are,

\[
\epsilon_1 = \frac{n - 2}{1 - 3n + 2n^2}, \quad \epsilon_2 \simeq 0, \quad \epsilon_3 = (n - 1)\epsilon_1, \quad \epsilon_4 = \frac{n - 2}{n - 1},
\]

where we used the fact that \( \phi(t_k) = \delta \ll 1 \) and \( \tilde{\phi}(t_k) = \zeta \ll 1 \), so \( \epsilon_2 \simeq 0 \). The slow-roll indices \( \text{(20)} \) have the same functional form as the vacuum \( f(R) \) gravity, however the slow-roll index \( \epsilon_6 \) contains the effects of the CS axion term, and now we shall find its explicit functional form. From Eqs. \( \text{(5)} \) and \( \text{(4)} \), we have,

\[
\epsilon_6 = \frac{1}{2} \sum_{l=L,R} \left( \frac{\dot{F}}{2HF} + \frac{\lambda_l\dot{x}k^2}{2FH} \right) \left( \frac{1}{1 + \frac{\Lambda xk^2}{F}} \right),
\]

where \( x = \frac{2\kappa}{a}, \quad F \simeq n\beta R^{n-1} \). The slow-roll parameter \( \epsilon_6 \) can be rewritten,

\[
\epsilon_6 = \frac{1}{2} \sum_{l=L,R} \left( \epsilon_3 + \frac{\lambda_l\dot{x}k^2}{2FH} \right) \left( \frac{1}{1 + \frac{\Lambda xk^2}{F}} \right).
\]

After performing the sum over the polarization states, and by taking into account that \( \lambda_L = -1 \) and \( \lambda_R = 1 \), Eq. \( \text{(22)} \) can be rewritten as follows,

\[
\epsilon_6 = \frac{\epsilon_3}{2} \left( \frac{1}{1 - \frac{\epsilon_3}{\sqrt{2}}} + \frac{1}{1 + \frac{\epsilon_3}{\sqrt{2}}} \right) + \frac{1}{2} \sum_{l=L,R} \lambda_l\kappa^2\dot{x} \left( \frac{1}{1 + \frac{\Lambda xk^2}{F}} \right).
\]

In addition, the tensor-to-scalar ratio for the model at hand is equal to,

\[
r = 16|\epsilon_4 - \epsilon_3| \frac{1}{2} \left( \frac{1}{1 - \frac{\epsilon_3}{\sqrt{2}}} + \frac{1}{1 + \frac{\epsilon_3}{\sqrt{2}}} \right).
\]

At this point we can obtain some phenomenological results for the CS axion corrected power-law \( f(R) \) gravity model. In order to have a spectral index of scalar primordial perturbations with value \( n_s = 0.965 \), we easily find from Eq. \( \text{(7)} \), we must have \( n = 1.817 \). So the simple power-law \( f(R) \) gravity model without the CS axion corrections yields a tensor-to-scalar ratio \( r_v = 0.24 \), which is excluded from both Planck \( \text{[22]} \) and BICEP2/Keck-Array data \( \text{[34]} \). The CS corrected tensor-to-scalar ratio can be compatible with the observations, due to the presence of the \( x/F \) terms. Particularly, suppose that we need to achieve a value compatible with the BICEP2/Keck-Array data, so assume that \( r = 0.06 \). This can be achieved if \( \kappa^2 = 4.37 \), and in addition, if \( \kappa^2 = 24.93 \), then \( r = 0.01 \). Thus the phenomenology of the power-law \( f(R) \) gravity model \( \text{(14)} \) becomes refined by the presence of the CS corrected axion dark matter.

Now let us turn our focus on the tensor spectral index \( n_T \), which in the vacuum \( f(R) \) gravity case can be found from Eqs. \( \text{(8)} \) and \( \text{(23)} \) (for \( x = 0 \)) that it is equal to,

\[
|n_T| = 0.03.
\]
which is non-zero and thus the theory is problematic. However as we shall show, the tensor spectral index can be significantly smaller in the context of CS corrected axion $f(R)$ gravity case. This can be seen from Eq. (22), due to the presence of the $x$-dependent terms. The result is strongly model dependent, so let us choose the function $\nu(\phi)$ to be,

$$\nu(\phi) = \Lambda e^{k\phi}$$  \hspace{1cm} (26)

where $\Lambda$ a free parameter, and $\dot{\nu}, \ddot{\nu}$ are equal to,

$$\dot{\nu} \simeq \Lambda k e^{k\phi} \delta, \quad \ddot{\nu} \simeq \Lambda k^2 \delta^2 e^{k\phi}.$$  \hspace{1cm} (27)

There is no specific reason for choosing the $\nu(\phi)$ function in Eq. (26), apart from the fact that the derivatives of $\nu(\phi)$ with respect to the cosmic time contain the same exponential term $\sim e^{k\phi}$ so a direct comparison between the resulting terms may lead to a conclusion on which is dominant at leading order. One can choose the function $\nu(\phi)$ freely for the moment, because there is no direct proof of the existence of the axion, or the presence of the string axion coupling. However, the observation of axions and also the presence of non-equivalent polarizations in the primordial gravitational waves may provide sufficient data to find the functional form of the function $\nu(\phi)$. Thus for the moment we use the exponential form only for simplicity.

Having Eq. (27) at hand we can evaluate $\epsilon_6$ at the horizon crossing time instance $t = t_k$, when $k = Ha$, and $\phi_k = \theta_a f_a$, so we have,

$$\kappa^2 x \frac{dx}{F} \simeq \frac{\delta \kappa^2 A \beta^3 - 2n \Lambda^3 e^{k\phi} \theta_a \kappa}{\beta n},$$  \hspace{1cm} (28)

and in addition,

$$\kappa^2 \frac{\dot{x}}{2FH} = \frac{\delta^2 \kappa^3 A H \delta^{1-n} e^{k\phi} \theta_a \kappa}{\beta n} - \frac{c_a \delta \kappa^3 A H \delta^{1-n} e^{k\phi} \theta_a \kappa}{\beta n},$$  \hspace{1cm} (29)

where $c_a$ is equal to,

$$c_a = \frac{n - 2}{-2n^2 + 3n - 1}.$$  \hspace{1cm} (30)

Basically, the term $\kappa^2 \frac{\dot{x}}{2FH}$ is subleading in Eq. (23), as we now demonstrate. Indeed, by taking the values for the free parameters as in the previous section, we have,

$$\kappa^2 \frac{dx}{F} = \frac{4.165985637444943 \times 10^{-80} \delta \Lambda}{\beta}.$$  \hspace{1cm} (31)

So in order to have $r = 0.01$, we must have $\kappa^2 \frac{dx}{F} = 24.93$, thus from Eq. (31) the parameters $\beta, \Lambda$ and $\delta$ must satisfy,

$$\frac{\beta}{\delta \Lambda} \simeq 1.73 \times 10^{-81},$$  \hspace{1cm} (32)

thus we have approximately,

$$\kappa^2 \frac{\dot{x}}{x F} \simeq -1.07436 \times 10^{-19} + 4.9281 \times 10^{-46} \delta,$$  \hspace{1cm} (33)

and since $\delta \ll 1$, the second term in Eq. (23) is significantly suppressed, so we have approximately,

$$\epsilon_6 \simeq \frac{\epsilon_3}{2} \left( \frac{1}{1 - \kappa^2 x} + \frac{1}{1 + \kappa^2 x} \right).$$  \hspace{1cm} (34)

This is the reason we chose the exponential function $\nu(\phi)$, in order to have a clear picture of which terms are dominant, due to the presence of the same term $\sim e^{k\phi}$ in the derivatives of $\nu(\phi)$ with respect to the cosmic time.

Therefore, for $\frac{\dot{x}}{x F} \simeq 24.93$, we have $|n_T| \simeq 0.000116871$, which is $10^{-3}$ times smaller in comparison to the vacuum $f(R)$ gravity result of Eq. (25). Thus it is clear that the CS axion dark matter $f(R)$ gravity results to better phenomenological results, in comparison to the vacuum $f(R)$ gravity case. Of course, the results are model dependent
and considerate fine-tuning is required in order to obtain refined phenomenological results, but the general outcome is phenomenologically more appealing in comparison to the axion free $f(R)$ gravity, at least for the polynomial $f(R)$ gravity of Eq. (14). Also we need to mention that the presence of $F$ in the denominator of the term $\kappa^2 x^2 / 2 F H$ requires less fine tuning in order to achieve a desirable value for $\kappa^2 x^2 / 2 F H$ in comparison to the Einstein-Hilbert considerations of Ref. [32], in which case $F = 1$. The fine-tuning is unavoidable for the moment in order to see how the theory can be fit to the present observational data. Indeed, we have two sources of uncertainty in the present outcome, firstly the $f(R)$ gravity model, and secondly the axion coupling $\nu(\phi)$. The purpose of this example is to show the new possibilities that arise in $f(R)$ gravity phenomenology by the presence of a CS axion coupling, and the result is that even non-viable vacuum $f(R)$ gravity models may become compatible with the observations. In the future, the observational data will possibly indicate if some model of $f(R)$ gravity is indeed the correct description for inflation, and of course if the axion exists. In this case, one may severely constrain the free parameters of $f(R)$ gravity, in the case at hand $\beta$, and also find hints for the presence (or non presence) of the axion CS coupling. Hence, these fine-tunings will be less severe, and of course more physically motivated.

In the context of CS axion dark matter corrected $f(R)$ gravity, it is also possible to make a viable $f(R)$ gravity to have even smaller primordial gravitational radiation. In view of this aspect, we now discuss the case of Starobinsky inflation [2]. In this case, the $f(R)$ gravity is of the form,

$$f(R) = R + \frac{1}{36H_i} R^2,$$

and the Friedman equation in the presence of the misalignment axion is,

$$\dot{H} - \frac{\dot{H}^2}{2H} + 3H_i H = -3H \dot{H}.$$  

(36)

The above result is due to the fact that the kinetic term of the axion scalar field and of the corresponding scalar potential are significantly suppressed during the inflationary era, and the axion is “frozen” in its vacuum expectation value. In order to have a concrete idea on how much suppressed are these terms, the potential term over $F$ is in this case,

$$\frac{\kappa^2}{2(12H^2)} m_a^2 \dot{\phi}_a^2 = \mathcal{O}(10^{-39} / \beta) eV,$$

(37)

where we took into account that the inflationary scale $H_I$ in this case is $H_I = \mathcal{O}(10^{13})$ GeV and we used the previous conventions for the axion field.

The differential equation (38) can easily be solved, and it yields the quasi-de Sitter evolution,

$$H(t) = H_0 - H_it,$$

(38)

and due to the fact that the slow-roll indices in the Starobinsky inflation case satisfy $\epsilon_i \ll 1$, the spectral index of the primordial scalar perturbations and the tensor-to-scalar ratio in the presence of the axion field are at leading order,

$$n_s = 1 - \frac{2}{N}, \quad r \approx \frac{r_T^v}{2} \left( \frac{1}{|1 - \frac{\kappa^2 x^2}{2 F H}|} + \frac{1}{|1 + \frac{\kappa^2 x^2}{2 F H}|} \right),$$

(39)

where $r_T^v = 48\epsilon_i^2$ is the vacuum $f(R)$ gravity tensor-to-scalar ratio. The presence of the term $\sim \kappa^2 x^2 / F$ can further reduce the value of the tensor-to-scalar ratio below the vacuum $f(R)$ gravity value which is $r_T^v = 0.0033$ which is obtained for $N = 60$ e-foldings. For example if $\frac{\kappa^2 x^2}{F} = \mathcal{O}(3 \times 10^2)$, the tensor-to-scalar ratio is $r = \mathcal{O}(10^{-5})$, while for large values, for example $\frac{\kappa^2 x^2}{F} = \mathcal{O}(3 \times 10^8)$, the tensor-to-scalar ratio is $r = \mathcal{O}(10^{-11})$. Finally let us discuss the effect of the CS axion coupling on the tensor spectral index $n_T$. In the vacuum Starobinsky model, this is exactly equal to zero, however in the presence of the CS axion coupling term, the tensor spectral index is not equal to zero anymore, due to a non-trivial $\epsilon_i$ slow-roll index. In this case, a similar analysis as in the previous case indicates that $n_T \sim 0$, when $\frac{\kappa^2 x^2}{F} \gg 1$, so this means that the Starobinsky inflation model in the presence of the CS axion dark matter has the same spectral index as the vacuum $f(R)$ gravity model, but many orders reduced tensor-to-scalar ratio, and thus it produces smaller amounts of inflationary gravitational radiation.

Before closing this section, let us briefly discuss an interesting perspective of CS axion dark matter $f(R)$ gravity models. In the context of the misalignment axion dark matter, during the inflationary era the axion field is frozen at its vacuum expectation value, and thus affects inflation via the tensor perturbations quantified by the CS coupling.
function $\eta(\phi)$. Thus $f(R)$ gravity drives inflation, at least the background evolution and also controls the primordial scalar perturbations, however the axion affects the tensor perturbations, reducing the amount of primordial gravity waves. As the Hubble rate drops, and specifically when $H \sim m_a$, the axion field starts to oscillate, and thus this could be viewed as a natural graceful exit mechanism and also some reheating type. When the Hubble rate satisfies $H \ll m_a$, the curvature is small and thus the axion field starts to dominate the cosmological evolution. The WKB approximation in the context of standard Einstein-Hilbert gravity yields a solution $\phi(t) \sim \cos(m_a t + \theta)$, so if indeed the $f(R)$ gravity does not control the cosmological evolution during this era, this WKB extracted solution will be indeed a solution to the scalar equation of motion. This yields an energy density for the axion $\rho \sim a^{-3}$ and also an averaged axion effective equation of state parameter $\langle w_{eff} \rangle \sim 0$, for $t \gg 1/m_a$, independently of the background being radiation or matter dominated \cite{14}. Thus if one finds a model of $f(R)$ gravity which dominates at early times, like the Starobinsky model, and remains subdominant at intermediate evolutionary stages of the Universe, while it dominates at late times again, then the CS axion dark matter $f(R)$ gravity could potentially provide an appealing unified description of early and late-time acceleration eras, with the intermediate eras. Here we just sketched a qualitative picture however, but this issue is quite interesting for future development.

IV. DISCUSSION AND CONCLUDING REMARKS

The possibility that the axion could be the main constituent of dark matter is stimulating, and the detection of the axion is one of the main goals in several observational and experimental proposals \cite{35,37}. Actually the misalignment models which are based on small mass axions, could provide a major candidate for dark matter, and in the very appealing experimental proposals of Refs. \cite{35,37}, the search for low mass axions is the main aim. Also axions can have indirect effects to neutron stars via their interaction with the thermal photons near the core of neutron stars, and several observational proposals exist \cite{38}, see also \cite{39,40}, that could actually verify the existence of axions in the near future. Actually axions could interact with photons \cite{41} and electric fields in a plasma \cite{42,43} and therefore neutron stars could be a virtual future laboratory for seeking axion induced effects. In the literature there exist several theoretical proposals discussing the possibility of detecting axions, see for example \cite{44}. Thus if the axion is one or the main components of dark matter, this could be revealed in the near future.

One of the issues we would like now to briefly discuss is the implications of the CS axion term $\nu(\phi) \tilde{R} R$ on the propagation of gravitational waves. Particularly, it is known for quite some time \cite{31} that the presence of the CS term discriminates the two different polarizations of the primordial gravity waves. In the literature this is a well studied possibility \cite{45,46}, an effect which is known as parity violating gravity waves, and this non-equivalence in the propagation of gravity waves could have an observable effect on the Cosmic Microwave Background \cite{47}, which could be captured in the next generation of experiments. In this paper we also demonstrated that the presence of the CS term can reduce significantly the tensor-to-scalar ratio of $f(R)$ gravity theories. We specified our analysis by using two concrete examples, a power-law $f(R) = R + \beta R^n$ gravity (apart of realistic $n = 2$ case which corresponds to Starobinsky inflation) and also the $R^n$ gravity. In the case of power-law gravity ($n < 2$), the vacuum theory was unable to provide a phenomenologically viable theory, however the presence of CS corrected axion dark matter can modify the resulting theory, suppressing the tensor perturbations and the corresponding tensor-to-scalar ratio. In the context of $f(R)$ gravity, the tensor-to-scalar is suppressed due to the presence of the term $\kappa^2 \beta \nu \tilde{R}/\rho$. In the case of Einstein-Hilbert gravity, the term $F = \frac{\partial f}{\partial R}$ is equal to one, however in the case of $f(R)$ gravity, it is proportional to powers or functions of the curvature, hence the suppression of the term $\kappa^2 \beta \nu \tilde{R}/\rho$ caused by the $\kappa^2$ in the Einstein-Hilbert case, is not an issue anymore in the context of $f(R)$ gravity. Finally, with regard to the Starobinsky case in the context of CS axion dark matter gravity, we demonstrated in a quantitative way that the resulting inflationary theory could have significantly smaller tensor-to-scalar ratio in comparison to the vacuum theory. Thus, the CS axion dark matter could extend the viability of already viable $f(R)$ gravity cosmological models, in a way that the scalar perturbations and the background evolution are unaffected, but only the gravitational radiation is affected. Finally, we should briefly note that the axion sinusoidal oscillations when $H \sim m_a$ could actually trigger the graceful exit from inflation even in the context of polynomial $f(R)$ gravity, which can be problematic in the vacuum theory, and also can contribute to the $f(R)$ gravity reheating mechanism. We hope to address this issue in more detail in a future work more focused on this issue, but this study should require a numerical approach.

Appendix: Power-law and Starobinsky $f(R)$ Gravity Models in the slow-roll Approximation

In this section we shall demonstrate the differences in deriving the cosmological evolution stemming from the polynomial $f(R)$ gravity model of Eq. \cite{13} for $n \neq 2$ and for the Starobinsky model \cite{35}. As we now show, the
Starobinsky model leads to less complicated and more accurate results, due to the fact that \( n = 2 \). We shall consider the vacuum case, since the scalar field contribution will be neglected eventually. The first Friedman equation for the vacuum \( f(R) \) gravity is,

\[
3H^2F = \frac{RF - f}{2} - 3H\dot{F}.
\]  

(40)

Let us first consider the polynomial \( f(R) \) gravity of Eq. (14), in which case, by assuming that \( F \sim n\beta R^{n-1} \) the Friedman equation (40) becomes,

\[
3H^2n\beta R^{n-1} = \frac{\beta(n-1)R^{n-1}}{2} - 3n(n-1)\beta HR^{n-2}\dot{R},
\]  

(41)

so by using the fact that at \( R = 12H^2 + 6\dot{H} \), and by further taking the simplification \( R \sim 12H^2 \) and \( \dot{R} \sim 24H\ddot{H} \), the Friedman equation (41) becomes approximately at leading order,

\[
3H^2n\beta \simeq 6\beta(n-1)H^2 - 6n\beta(n-1)\dot{H} + 3\beta(n-1)\dot{H} \frac{1}{2(12H^2)^{n-1}}\delta^2.
\]  

(42)

So by simplifying the above we get,

\[
3H^2n\beta \simeq 6\beta(n-1)H^2 - 6n\beta(n-1)\dot{H} + 3\beta(n-1)\dot{H},
\]  

(43)

which can be solved to yield,

\[
H(t) = \frac{-2n^2 + 3n - 1}{(n-2)t}.
\]  

(44)

Let us now consider the Starobinsky model case,

\[
f(R) = R + \frac{1}{36H^2}R^2,
\]  

(45)

so by substituting \( R = 12H^2 + 6\dot{H} \) and \( \dot{R} = 24H\ddot{H} + 6\dot{H} \) and \( F = 1 + \frac{R}{18H^2} \) in the Friedman equation (40), we obtain exactly the following differential equation,

\[
\ddot{H} - \frac{\dot{H}^2}{2H} + 3H\dot{H} = -3H\dot{H}.
\]  

(46)

The first two terms can be disregarded during the slow-roll era, so the resulting differential equation is,

\[
3H_\epsilon\dot{H} = -3H\dot{H},
\]  

(47)

which when solved yields the quasi-de Sitter evolution of Eq. (38). So basically in the Starobinsky model, there is no fractional power of \( n \) complicating things, and we take \( R = 12H^2 + 6\dot{H} \) and \( \dot{R} = 24H\ddot{H} + 6\dot{H} \). The only simplification assumed is the slow-roll condition in Eq. (46). In the power-law case we had to do three stages of simplifications in order to extract an analytic result, so it is a more complicated case.

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