GENERALIZED RELAXATION OF STRING AVERAGING OPERATORS BASED ON STRICTLY RELAXED CUTTER OPERATORS

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ABSTRACT. We present convergence analysis of a generalized relaxation of string averaging operators which is based on strictly relaxed cutter operators on a general Hilbert space. In this paper, the string averaging operator is assembled by averaging of strings’ endpoints and each string consists of composition of finitely many strictly relaxed cutter operators. We also consider projected version of the generalized relaxation of string averaging operator. To evaluate the study, we recall a wide class of iterative methods for solving linear equations (inequalities) and use the subgradient projection method for solving nonlinear convex feasibility problems.

1. INTRODUCTION

In this paper we consider a fixed point iteration method for solving convex feasibility problems which are used in different areas of mathematics and physical sciences. A convex feasibility problem consists in finding a point in the intersection of closed convex sets \( \{C_\ell\}_{\ell=1}^N \). Using string averaging method [18], which is particularly suitable for parallel computing and therefore have the ability to handle huge-size problems, may accelerate the fixed point iteration method. The output of the string averaging process is an operator, called string averaging operator, which is used in the fixed point iteration method. In this paper, the string averaging operator is made by averaging of finitely many operators which are composition of finitely many strictly relaxed cutter operators. The string averaging process is studied in many research works as [7, 8, 10, 13, 23, 25, 26] which are based on projection operators. Recently in [32, 46] and [22], a fixed point iteration method is analyzed based on strict paracontraction operators, strictly quasi-nonexpansive operators and cutter operators respectively. Moreover, the string averaging scheme has been extended in [23, 43] and [49]. The cutter operators are introduced and investigated in [5] and studied in several research works as [6, 13] and references therein.

To accelerate a fixed point iteration algorithm, one may use relaxation parameters or its generalized version which is called generalized relaxation.

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The generalized relaxation strategy is recently studied for composition of cutter operators in [13] and in [47] without considering string averaging process. On the other hand, various extrapolation schemes applied to a pure convex combination can be found in literature; see, for example, [1, 4, 14, 28, 29, 30, 39, 48] and [11, Chapter 4, Chapter 5]. A recent work [46] analyzes a fixed point iteration method based on generalized relaxation of the string averaging operator which is based on strictly quasi-nonexpansive operators. Note that the set of strictly quasi-nonexpansive operators involves all cutter operators. The analysis in [46] indicates that the generalized relaxation of cutter operators is inherently able to make more acceleration comparing with [13], see section 3.1 for more details.

We study a fixed point iteration method based on generalized relaxation of string averaging operator using strictly relaxed cutter operators. The class of relaxed cutter operators contains relaxed projection operators [27], relaxed subgradient projections [21, 44], relaxed firmly nonexpansive operators [9], the resolvents of a maximal monotone operators [6, 35], contraction operators [3, 11], averaged operators [2] and strongly quasi-nonexpansive operators [3].

One may ask: Is there any connection between the set of strictly relaxed cutter operators and the set of strictly quasi-nonexpansive operators? The answer is yes. Actually any strictly relaxed cutter operator is strictly quasi-nonexpansive operator, see Remark 2.5. On the other hand, the convergence analysis of a fixed point iteration method based on the generalized relaxation of the string averaging operator has been shown in [46] where each string consists of composition of finitely many strictly quasi-nonexpansive operators. Therefore, the next question is: What are the advantages of using strictly relaxed cutter operators instead of strictly quasi-nonexpansive operators? In [46], the relaxation parameters are chosen from (0, 1) whereas our relaxation parameters lie in (0, 2). Also, we will show that using strictly relaxed cutter operators gives faster reduction in error. The next advantage of these operators is related to demi-closedness property. In the string averaging process we have finitely many operators, say \( \{U_t\}_{t=1}^E \), which are composition of some finitely many other operators, i.e., \( U_t = T_{m_t} \cdots T_1 \). Also, there is an averaging process on \( U_t \) which makes the string averaging operator, say \( T \). We consider a fixed point iteration method based on the generalized relaxation of \( T \). To get convergence result for the fixed point iteration method, it is assumed in [46] that \( T - Id \) is demi-closed at zero or alternatively \( \{U_t - Id\}_{t=1}^E \) are demi-closed at zero. In addition to demi-closedness of \( T - Id \) or \( \{U_t - Id\}_{t=1}^E \), using strictly relaxed cutter operators allows us to consider, similar to [13], the demi-closedness of all operators \( T_t - Id \).

The paper is organized as follows. In section 2 we recall some definitions and properties of relaxed cutter operators. We reintroduce string averaging process and give its convergence analysis based on strictly relaxed cutter operators in section 3. A short discussion on error reduction and choosing
generalized relaxation of string averaging operators 3

In section 3.1, we present the projected version of generalized relaxation of string averaging operator with convergence proof. At the end, the capability of the main result is examined in section 4 by employing the subgradient projection method.

2. Preliminaries and Notations

Throughout this section, we consider $T : H \to H$ with nonempty fixed point set, i.e., $\text{Fix}T \neq \emptyset$ where $H$ is a Hilbert space and $Id$ denotes the identity operator on $H$. The following definitions, see [11], will be useful in our future analysis.

**Definition 2.1.** An operator $T$ is quasi-nonexpansive (QNE) if

$$\|T(x) - z\| \leq \|x - z\|$$

for all $x \in H$ and $z \in \text{Fix}T$. Also, one may use the term strictly quasi-nonexpansive (sQNE) by replacing strict inequality in (2.1), i.e., $\|T(x) - z\| < \|x - z\|$ for all $x \in H \setminus \text{Fix}T$ and $z \in \text{Fix}T$. Moreover, a continuous sQNE operator is called paracontracting operator, see [39].

Another useful class of operators is the class of cutter operators, namely, an operator $T : H \to H$ with nonempty fixed point set is called cutter if

$$\langle x - T(x), z - T(x) \rangle \leq 0$$

for all $x \in H$ and $z \in \text{Fix}T$. Using [11, Remark 2.1.31], the operator $T$ is a cutter if and only if

$$\langle T(x) - x, z - x \rangle \geq \|T(x) - x\|^2$$

for all $x \in H$ and $z \in \text{Fix}T$.

**Definition 2.2.** Let $T : H \to H$ and $\alpha \in [0, 2]$. The following operator

$$T_\alpha := (1 - \alpha)Id + \alpha T$$

is called an $\alpha$-relaxation or, shortly, relaxation of the operator $T$. If $\alpha \in (0, 2)$, then $T_\alpha$ is called a strictly (or strict) relaxation of $T$.

Based on [11] Remark 2.1.31, an $\alpha$-relaxed cutter operator is defined as follows.

**Definition 2.3.** Let $T : H \to H$ has a fixed point. Then the operator $T$ is an $\alpha$-relaxed cutter, or, shortly, relaxed cutter where $\alpha \in [0, 2]$, if

$$\langle T_\alpha(x) - x, z - x \rangle = \alpha \langle T(x) - x, z - x \rangle \geq \|T(x) - x\|^2$$

for all $x \in H$ and $z \in \text{Fix}T$. If $\alpha \in (0, 2)$, then $T_\alpha$ is called a strictly relaxed cutter operator of $T$.

Let $\alpha \geq 0$ and assume that $T : H \to H$ has a fixed point. We say that $T$ is $\alpha$-strongly quasi-nonexpansive ($\alpha$-SQNE), if

$$\|T(x) - z\|^2 \leq \|x - z\|^2 - \alpha \|T(x) - x\|^2$$
for all $x \in H$ and $z \in FixT$. Also, the operator $T$ satisfying (2.6) with $\alpha > 0$ is called strongly quasi-nonexpansive (SQNE) operator.

Following theorem presents a relationship between strictly relaxed cutter and SQNE operators.

**Theorem 2.4.** [11, Theorem 2.1.39 and Corollary 2.1.40] Assume that $T : H \rightarrow H$ has a fixed point and let $\lambda \in (0, 2]$. Then $T$ is a $\lambda$-relaxed cutter if and only if $T$ is \( \frac{2-\lambda}{\lambda} \)-SQNE, i.e.,

\[
\|T_\lambda(x) - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda} \|T_\lambda(x) - x\|^2
\]

for all $x \in H$ and all $z \in FixT$.

The following remark gives a relationship between sQNE and strictly relaxed cutter operators.

**Remark 2.5.** [11, Remark 2.1.44.] Assume that $T : H \rightarrow H$ has a fixed point. If $T$ is SQNE, then $T$ is sQNE. Therefore, all properties of sQNE operators are also valid for SQNE and strictly relaxed cutter operators.

A very useful property of $\alpha$-relaxed cutter and sQNE operators is their closedness respect to convex combination and composition of the operators. Furthermore, any cutter operator is 1-relaxed cutter, compare [11, Remark 2.1.31] and Definition 2.3. However, the class of cutter operators is not necessarily closed with respect to composition of operators. The closedness property of sQNE operators is presented in [11, Theorem 2.1.26]. Also, the class of strictly relaxed cutter operators has the closedness property, see following theorem.

**Theorem 2.6.** [11, Theorem 2.1.48 and Theorem 2.1.50] Let $L_{\alpha_i} : X \rightarrow X$ be an $\alpha_i$-relaxed cutter, where $\alpha_i \in (0, 2)$ and $i \in I = \{1, 2, \ldots, m\}$. Let \( \bigcap_{i \in I} FixL_{\alpha_i} \neq \emptyset \) and $P_m := L_{\alpha_m}L_{\alpha_{m-1}} \ldots L_{\alpha_1}$. Then the operator $P_m$ is a $\gamma_m$-relaxed cutter, with

\[
\gamma_m = \frac{2}{(\sum_{i=1}^{m} \frac{\alpha_i}{2-\alpha_i})^{-1} + 1}.
\]

Furthermore, the operator $Q_m = \sum_{i \in I} \omega_i L_{\alpha_i}$ is a $\mu$-relaxed cutter where $\mu = \sum_{i \in I} \omega_i \alpha_i$, $\sum_{i \in I} \omega_i = 1$ and $\omega_i \geq 0$. Moreover, $FixP_m = FixQ_m = \bigcap_{i \in I} FixL_{\alpha_i}$.

We next reintroduce, see also [11][12][13][14][46], the generalized relaxation of an operator, which allows to accelerate locally a fixed point iteration method.

**Definition 2.7.** Let $T : H \rightarrow H$ and $\sigma : H \rightarrow (0, \infty)$ be a step size function. The generalized relaxation of $T$ is defined by

\[
T_{\sigma, \lambda}(x) = x + \lambda\sigma(x)(T(x) - x)
\]

where $\lambda$ is a relaxation parameter in $[0, 2]$. 

If $\lambda \sigma(x) \geq 1$ for all $x \in H$, then the operator $T_{\sigma, \lambda}$ is called an extrapolation of $T$. For $\sigma(x) = 1$ we get the relaxed version of $T$, namely, $T_{1, \lambda} := T_{\lambda}$. Furthermore, it is clear that $T_{\sigma, \lambda}(x) = x + \lambda(T_{\sigma}(x) - x)$ where $T_{\sigma} = T_{\sigma, 1}$ and $\text{Fix } T_{\sigma, \lambda} = \text{Fix } T$ for any $\lambda \neq 0$.

**Definition 2.8.** An operator $T : H \to H$ is demi-closed at 0 if for any weakly converging sequence $x^k \rightharpoonup y \in H$ with $T(x^k) \to 0$ we have $T(y) = 0$.

**Remark 2.9.** [11, p. 108] It is well known that the operator $T - Id$ is demi-closed at 0 where $T : H \to H$ is a nonexpansive operator.

**Remark 2.10.** Assume that $\{T_i\}_{i=1}^m$ are strictly relaxed cutter operators such that $\{T_i - Id\}_{i=1}^m$ are demi-closed at 0 and $\bigcap_{i=1}^m \text{Fix } T_i \neq \emptyset$. Based on [16, Theorem 4.1, Theorem 4.2], [49, Lemma 3.4] and Theorem 2.4, $V - Id$ is demi-closed at 0 while assuming that $V$ is defined either by a composition $V = T_m \ldots T_1$ or by a convex combination $V = \sum_{i=1}^m \omega_i T_i$.

### 3. Main result

In this section we reintroduce the string averaging process which is based on the class of strictly relaxed cutter operators. We next consider a fixed point iteration method based on the generalized relaxation of string averaging operator and present its convergence analysis. We give the projected version of generalized relaxation of string averaging operator with convergence analysis. We also compare the error reduction of our algorithm with [13] and [46] and give a short discussion on choosing relaxation parameters of strictly relaxed cutter operators.

We first give a short review of research works on the string averaging algorithm. The string averaging algorithmic scheme is first proposed in [18]. Their analysis was based on the projection operators, whereas the algorithm is defined for any operators, for solving consistent convex feasibility problems. Studying the algorithm in a more general setting is considered by [7]. The inconsistent case is analyzed by [23] and they proposed a general algorithmic scheme for string averaging method without any convergence analysis. A special case of the algorithm is studied under summable perturbation in [10, 34]. A dynamic version of the algorithm is presented in [24]. In [40] the string averaging method is compared with other methods for sparse linear systems. Other applications of the string averaging scheme, such as constrained minimization and variational inequalities can be found, for example, in [25] and [15], respectively.

Recently, a perturbation resilience iterative method with an infinite pool of operators is studied in [43] which answers some open problems mentioned by [18] whereas these problems are partially answered by [29]. Also the proposed general algorithmic scheme of [23, Algorithm 3.3], which was presented without any convergence analysis, is extended with a convergence proof in [43]. Another general form of the string averaging scheme appeared in [49].
All the above mentioned research works are based on projection operators. In \[22\], the string averaging algorithm is studied for cutter operators and the sparseness of the operators is used in averaging process. In \[32, 33\], the string averaging method is used for finding common fixed point problem of strict paracontraction operators.

We next reintroduce the string averaging algorithm as follows.

**Definition 3.1.** The string \( I_t = (i_{t1}, i_{t2}, \ldots, i_{tm_t}) \) is an ordered subset of \( I = \{1, 2, \ldots, m\} \) such that \( \bigcup_{t=1}^{E} I_t = I \). Define
\[
U_t = T_{i_{tm_t}} \ldots T_{i_{t2}} T_{i_{t1}}, \quad t = 1, 2, \ldots, E
\]
\[
T = \sum_{t=1}^{E} \omega_t U_t
\]
where \( \omega_t > 0 \) and \( \sum_{t=1}^{E} \omega_t = 1 \). Here \( T_{i \in I} \) are operators on a Hilbert space \( H \).

In this paper, we assume that all \( T_{i \in I} \) of Definition \[3.1\] are strictly relaxed cutter operators on \( H \) and \( \bigcap_{i \in I} \text{Fix} T_i \neq \emptyset \). It should be mentioned that the averaging process \( (3.1) \) is a special case of \[18\].

**Remark 3.2.** Note that all \( \{U_t\}_{t=1}^{E} \) and consequently the operator \( T \) belong to the class of strictly relaxed cutter operators where \( T_{i \in I} \) are strictly relaxed cutter operators, see Theorem \[2.6\].

We now consider the following fixed point iteration algorithm which is based on generalized relaxation of \( T \).

**Algorithm 3.3.**
\[\text{Initialization: } x^0 \in H \text{ is arbitrary.}\]
**Iterative Step:** Given \( x^k \), compute
\[
x^{k+1} = T_{\sigma, \lambda_k}(x^k).
\]

To simplifying the notation of Definition \[3.1\], we denote \( i^t_{\ell} \) by \( \ell \) for \( \ell = 1, 2, \ldots, m_t \). Analogues with \[13\], we consider the following notations
\[
S_0 = \text{Id}, \quad S_i = T_i \ldots T_1 \text{ for } i = 1, \ldots, m_t
\]
which leads to \( U_t = S_{m_t} \). Furthermore, let
\[
\begin{aligned}
u_0 &= x, & u^i &= T_i u^{i-1}, & y^i &= u^i - u^{i-1} \\
\end{aligned}
\]
for \( i = 1, \ldots, m_t \) and \( t = 1, \ldots, E \). Using \[3.2\] and \[3.3\] we have
\[
\begin{aligned}
u^i &= S_i x, & \sum_{j=i}^{m_t} y^j &= U_t(x) - S_{i-1} x \text{ for } i = 1, 2, \ldots, m_t \\
\end{aligned}
\]
and particularly \( \sum_{j=1}^{m_t} y^j = U_t(x) - x \).
Lemma 3.4. Let $T_i \in \mathcal{I} : H \to H$ be $\alpha_i$-relaxed cutter operators such that $\bigcap_{i=1}^m \text{Fix}T_i \neq \emptyset$ and $\alpha_i \in (0, 2)$. For any $z \in \bigcap_{i=1}^m \text{Fix}T_i$ we have

$\langle z - x, T(x) - x \rangle \geq \sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( \frac{1}{\alpha_i} y^i \Big| \sum_{j=i+1}^{m_t} y^j \right)$

$\geq \sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \|y^i\|^2$

$\geq \sum_{t=1}^{E} \omega_t \left( \frac{1}{\alpha_t} - \frac{1}{2} \right) \sum_{i=1}^{m_t} \|y^i\|^2$

$\geq \left( \frac{1}{\gamma} - \frac{1}{2} \right) \sum_{t=1}^{E} \frac{\omega_t}{m_t} \|U_t(x) - x\|^2$

$\geq \frac{1}{\bar{m}_t} \|T(x) - x\|^2$

where $\bar{\alpha}_t := \max_{1 \leq i \leq m_t} \alpha_i$, $\gamma := \max_{1 \leq t \leq E} \bar{\alpha}_t$ and $\bar{m}_t := \max_{1 \leq t \leq E} m_t$.

Proof. We first assume $E = 1$ which leads to the case $m_t = m$. The inequality (3.5) is directly followed by Definition 2.3 where $m = 1$. Now let $m = 2$. 
Since $T_i$ is an $\alpha_i$-relaxed cutter operator and using (3.3), we have

\[
\langle z - x, T_2T_1(x) - x \rangle = \langle z - x, T_2T_1(x) - T_1(x) + T_1(x) - x \rangle \\
= \langle z - T_1(x), T_2T_1(x) - T_1(x) \rangle + \langle z - x, T_1(x) - x \rangle \\
+ \sum_{i=1}^{\infty} \frac{1}{\alpha_i} \|T_2T_1(x) - T_1(x)\|^2 + \frac{1}{\alpha_1} \|T_1(x) - x\|^2 \\
+ \langle T_1(x) - x, T_2T_1(x) - T_1(x) \rangle \\
= \frac{1}{\alpha_2} \|y^2\|^2 + \frac{1}{\alpha_1} \|y^1\|^2 + \langle y^1, y^2 \rangle \\
= \sum_{i=1}^{2} \left( \frac{1}{\alpha_i} y^i + \sum_{j=i+1}^{2} y^j, y^i \right) \\
= \frac{1}{\alpha_2} \|y^2\|^2 + \frac{1}{\alpha_1} \|y^1\|^2 \\
+ \frac{1}{2} \left( \|y^1\|^2 + \|y^2\|^2 \right) \\
= \left( \frac{1}{\alpha_2} - \frac{1}{2} \right) \|y^2\|^2 + \left( \frac{1}{\alpha_1} - \frac{1}{2} \right) \|y^1\|^2 \\
+ \frac{1}{2} \|y^1 + y^2\|^2 \\
\geq \left( \frac{1}{\alpha_2} - \frac{1}{2} \right) \|y^2\|^2 + \left( \frac{1}{\alpha_1} - \frac{1}{2} \right) \|y^1\|^2 \\
\geq \left( \frac{1}{\alpha} - \frac{1}{2} \right) \left( \|y^1\|^2 + \|y^2\|^2 \right)
\]

(using convexity of $\|.,\\|\|^2$) \[\geq \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{2} \right) \|y^1 + y^2\|^2\]

where $\sum_{j=\ell}^{k} y^j = 0$ for $k < \ell$ and $\bar{\alpha} = \max_{1 \leq i \leq 2} \alpha_i$. Now suppose that the inequality (3.5) holds for $m = t$ (induction hypothesis). Similar to [13], Lemma 7], define $V_1 = I_d$, $V_i = T_iT_{i-1} \cdots T_2$ for $i = 2, 3, \ldots, t + 1$ and put $v^1 = h$, where $h$ is an arbitrary element of $H$, $v^i = T_i v^{i-1}$ and $z^i = v^i - v^{i-1}$, $i = 2, 3, \ldots, t + 1$. If we set $h = T_i x$, then $S_i x = V_i h$, $u^i = v^i$ and $y^i = z^i$ for $i = 2, 3, \ldots, t + 1$. Using the induction hypothesis we have

\[
(3.8) \quad \langle V_{t+1} h - h, z - h \rangle \geq \sum_{i=2}^{t+1} \left( \frac{1}{\alpha_i} z^i + \sum_{j=i+1}^{t+1} z^j, z^i \right)
\]
for all \( h \in H \) and \( z \in \bigcap_{i=2}^{t+1} \text{Fix}T_i \). If \( h = T_1x \) then for \( x \in H \) and \( z \in \bigcap_{i=1}^{t+1} \text{Fix}T_i \) we obtain that

\[
\langle S_{t+1}x - x, z - x \rangle = \langle V_{t+1}h - x, z - x \rangle = \langle V_{t+1}h - h, z - x \rangle + \langle V_{t+1}h - h, h - x \rangle + \frac{1}{\alpha_1} \| y^1 \|^2 \\
\geq \langle V_{t+1}h - h, z - h \rangle + \langle V_{t+1}h - h, h - x \rangle + \frac{1}{\alpha_1} \| y^1 \|^2 \\
= \sum_{i=2}^{t+1} \left( \frac{1}{\alpha_i} z^i + \sum_{j=i+1}^{t+1} z^i, z^i \right) + \left( \sum_{i=2}^{t+1} z^i, h - x \right) + \frac{1}{\alpha_1} \| y^1 \|^2 \\
= \sum_{i=2}^{t+1} \left( \frac{1}{\alpha_i} y^i + \sum_{j=i+1}^{t+1} y^j, y^j \right) + \left( \sum_{i=2}^{t+1} y^i, y^1 \right) + \frac{1}{\alpha_1} \| y^1 \|^2 \\
= \sum_{i=1}^{t+1} \left( \frac{1}{\alpha_i} y^i + \sum_{j=i+1}^{t+1} y^j, y^j \right) \\
\geq \sum_{i=2}^{t+1} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 + \sum_{i=1}^{t+1} \frac{1}{2} \| y^i \|^2 \\
= \sum_{i=2}^{t+1} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 + \sum_{i=1}^{t+1} \frac{1}{2} \| y^i \|^2 \\
= \sum_{i=1}^{t+1} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 \\
\geq \frac{1}{m} \left( \frac{1}{\alpha} - \frac{1}{2} \right) \| \sum_{i=1}^{m} y^i \|^2.
\]

Therefore, the inequality (3.5) is true for \( m = t + 1 \) which completes the induction. The inequality (3.6) can be obtained by

\[
\sum_{i=1}^{m} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 + \sum_{i=1}^{m} \frac{1}{2} \| y^i \|^2 \\
= \sum_{i=1}^{m} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 + \sum_{i=1}^{m} \frac{1}{2} \| y^i \|^2 \\
\geq \frac{1}{m} \left( \frac{1}{\alpha} - \frac{1}{2} \right) \| \sum_{i=1}^{m} y^i \|^2.
\]
Now we complete the proof by considering the case \( E \geq 1 \). Using Definition 3.1, we have

\[
\langle z - x, T(x) - x \rangle = \sum_{t=1}^{E} \omega_t \langle z - x, U_t(x) - x \rangle
\]  

(3.12)  

(\text{using (3.9)}) \geq \sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( \frac{1}{\alpha_t} y^i + \sum_{j=i+1}^{m_t} y^j \right) \]  

(3.13)  

(\text{using (3.10), (3.11)}) \geq \sum_{t=1}^{E} \omega_t \frac{1}{\alpha_t} \sum_{i=1}^{m_t} \| y^i \|^2  

(3.14)  

\geq \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \sum_{t=1}^{E} \omega_t \| U_t(x) - x \|^2  

(3.15)  

(\text{using convexity of } \| . \| ^2) \geq \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \sum_{t=1}^{E} \omega_t \| U_t(x) - x \|^2  

(3.16)  

\geq \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \| T(x) - x \|^2  

(3.17)  

which completes the proof. \( \square \)

We next show that the generalized relaxation operator \( T_{\sigma, \lambda} \) is a \( \lambda \)-relaxed cutter operator under a condition on \( \sigma(x) \).

**Theorem 3.5.** Let \( T_{\sigma, \lambda} \) be a generalized relaxation of \( T = \sum_{t=1}^{E} \omega_t U_t \) and \( \bigcap_{i \in I} Fix T_i \neq \emptyset \). Then \( T_{\sigma, \lambda} \) is a \( \lambda \)-relaxed cutter operator if

\[
\frac{\sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( U_t(x) - S_i(x) + \frac{1}{\alpha_t} \zeta_i(x), \zeta_i(x) \right)}{\| T(x) - x \|^2} > 0 < \sigma(x) < \frac{\sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( U_t(x) - S_i(x) + \frac{1}{\alpha_t} \zeta_i(x), \zeta_i(x) \right)}{\| T(x) - x \|^2}
\]  

(3.17)  

where \( x \in H \setminus Fix T \) and \( \zeta_i(x) = S_i(x) - S_{i-1}(x) \). Furthermore, the step size function

\[
\sigma_{\text{max}}(x) = \begin{cases} 
\frac{\sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( U_t(x) - S_i(x) + \frac{1}{\alpha_t} \zeta_i(x), \zeta_i(x) \right)}{\| T(x) - x \|^2}, & x \in H \setminus Fix T \\
1, & x \in Fix T 
\end{cases}
\]  

(3.18)  

is bounded below by \( \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \).
Proof. For $z \in \text{Fix} \ T_{\sigma, \lambda}$ and $x \in H \setminus \text{Fix} \ T_{\sigma, \lambda}$ one gets

$$\langle z - x, T_{\sigma, \lambda}(x) - x \rangle = \langle z - x, \lambda \sigma(x)(T(x) - x) \rangle = \lambda \sigma(x) \langle z - x, T(x) - x \rangle$$

(3.19) (using (3.15))

$$\geq \lambda \sigma(x) \sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( \frac{1}{\alpha_i} y_i + \sum_{j=i+1}^{m_t} y_j \right)$$

(3.20)

$$= \lambda \sigma(x) \sum_{t=1}^{E} \omega_t \sum_{i=1}^{m_t} \left( U_t(x) - S_i(x) + \frac{1}{\alpha_i} \zeta_i(x), \zeta_i(x) \right)$$

(3.21) (using (3.17))

$$\geq \frac{\lambda^2 \sigma^2(x)}{\lambda} \| T(x) - x \|^2$$

(3.22)

$$= \frac{1}{\lambda} \| T_{\sigma, \lambda}(x) - x \|^2.$$

So $T_{\sigma, \lambda}$ is a $\lambda$-relaxed cutter operator. Using (3.13), (3.16), (3.19) and (3.20), the lower bound of $\sigma_{\max}$ is

$$\sigma_{\max}(x) \geq \sum_{t=1}^{E} \omega_t \left( \frac{1}{\alpha_t} - \frac{1}{2} \right) \sum_{i=1}^{m_t} \| y_i \|^2$$

(3.23)

$$\geq \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \| T(x) - x \|^2$$

(3.24)

$$\geq \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right)$$

(3.25)

where $x \in H \setminus \text{Fix} T$. It completes the proof. □

Corollary 3.6. $T_{\sigma, \lambda} - \text{Id}$ is demi-closed at 0 assuming that $T - \text{Id}$ is demi-closed at 0. This happens, for example, when $T_i - \text{Id}$ is demi-closed at 0 for $i = 1, \ldots, m$, or $U_t - \text{Id}$ is demi-closed at 0 for $t = 1, \ldots, E$.

The proof of Corollary 3.6 follows immediately by Remark 2.10 and the estimate established in Theorem 3.5, that is, we have

$$\| T_{\sigma, \lambda}(x) - x \| = \sigma(x) \lambda \| T(x) - x \| \geq \lambda \frac{1}{m_t} \left( \frac{1}{\gamma} - \frac{1}{2} \right) \| T(x) - x \|.$$

Theorem 3.7. Let $\sigma = \sigma_{\max}$ be step size function, $\bigcap_{i \in I} \text{Fix} T_i \neq \emptyset$ and $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. The sequence generated by Algorithm 3.3 converges weakly to a point in $\text{Fix} T$, if one of the following conditions is satisfied

(i) $T - \text{Id}$ is demi-closed at 0, or
(ii) $U_t - \text{Id}$ are demi-closed at 0, for all $t = 1, 2, \ldots, E$, or
(iii) $T_i - \text{Id}$ are demi-closed at 0, for all $i = 1, 2, \ldots, m$. 
Proof. For $z \in Fix T, x^k \in H \setminus Fix T$ one gets
\begin{align*}
e_{k+1}^2 &= \| T_{\sigma, \lambda_k}(x^k) - z \|^2 \\
&= \| x^k + \lambda_k \sigma(x^k)(T(x^k) - x^k) - z \|^2 \\
&= \| x^k + \lambda_k(T_{\sigma}(x^k) - x^k) - z \|^2 \\
(3.26) &= e_k^2 + \xi_k + \lambda^2_k \| T_{\sigma}(x^k) - x^k \|^2
\end{align*}
where $\xi_k = 2\lambda_k \langle x^k - z, T_{\sigma}(x^k) - x^k \rangle$ and $e_k^2 = \| x^k - z \|^2$. Using Theorem 3.5 we obtain that
\begin{equation}
\left\langle x^k - z, T_{\sigma}(x^k) - x^k \right\rangle \leq -\| T_{\sigma}(x^k) - x^k \|^2
\end{equation}
and consequently
\begin{equation}
\xi_k \leq -2\lambda_k \| T_{\sigma}(x^k) - x^k \|^2.
\end{equation}
Using (3.27) and Theorem 3.5 we conclude
\begin{align*}
e_{k+1}^2 &\leq \| x^k - z \|^2 - 2\lambda_k \| T_{\sigma}(x^k) - x^k \|^2 + \lambda^2_k \| T_{\sigma}(x^k) - x^k \|^2 \\
(3.28) &= e_k^2 - \lambda_k(2 - \lambda_k)\sigma^2(x^k)\| T(x^k) - x^k \|^2 \\
(3.29) &\leq e_k^2 - \frac{\lambda_k(2 - \lambda_k)}{\bar{m}^2} \left( \frac{1}{\gamma} - \frac{1}{2} \right) ^2 \| T(x^k) - x^k \|^2.
\end{align*}
Therefore, we get that $\{e_k\}$ decreases and consequently $\{x^k\}$ is bounded and
\begin{equation}
\| T(x^k) - x^k \| \to 0.
\end{equation}
Using Fejér monotonicity of $\{x^k\}$ and [3. Theorem 2.16 (ii)], it suffices to show that every weak cluster point $x^*$ of $\{x^k\}$ lies in $Fix T$. To this end we assume that $\{x^{n_k}\}$ is a subsequence of $\{x^k\}$ which converges weakly to some point $x^* \in H$. In view of Remark 2.10 the operator $T - Id$ is demi-closed at 0 if any of conditions (i), (ii) and (iii) is satisfied. Thus the fact (3.30) and demi-closeness of $T - Id$ at 0 implies that $x^* \in Fix T$, which proves that $\{x^k\}$ converges weakly to some point in $Fix T$. \hfill \square

Remark 3.8. The following example shows that distinguishing between (i), (ii) and (iii) in Theorem 3.7 is necessary. Let $U(x) = T_2 T_1(x)$ where
\begin{align*}
T_1(x) &= \begin{cases} 1 + \frac{1}{2} \left( \frac{1}{n} + \frac{1}{n+1} \right), & x = 1 + \frac{1}{n} \\
\frac{1}{2} x, & \text{otherwise} \end{cases} \\
\text{and} \\
T_2(x) &= \begin{cases} 1 + \frac{\sqrt{2}}{\gamma} \left( \frac{1}{n} + \frac{1}{n+1} \right), & x = 1 + \frac{\sqrt{2}}{n} \\
\frac{1}{2} x, & \text{otherwise}. \end{cases}
\end{align*}

It is easy to check that $T_1$ and $T_2$ are strictly relaxed cutter operators and consequently $U$ is a strictly relaxed cutter operator. Furthermore, we have $Fix U = Fix T_1 \cap Fix T_2 = \{0\}$. Assuming $x^n = 1 + \frac{1}{n}$, we get that...
lim_{n \to \infty} \|T_1(x^n) - x^n\| = 0. Since lim_{n \to \infty} x^n = 1 \notin Fix T_1, we conclude that $T_1 - Id$ is not demi-closed at 0. Similarly, considering $x^n = 1 + \frac{\sqrt{2}}{n}$ gives that $T_2 - Id$ is not demi-closed at 0. Therefore, assuming only the case $(iii)$ in Theorem 3.7 does not guaranty the convergence results. We next verify that $U - Id$ is demi-closed at 0 which shows the case $(ii)$ is satisfied. We have

$$U(x) - x = \begin{cases} \frac{1}{2} - \frac{3}{4n} + \frac{1}{4(n+1)}, & x = 1 + \frac{1}{n} \\ -1 - \frac{3\sqrt{2}}{4n} + \frac{\sqrt{2}}{2(n+1)}, & x = 2 + \frac{2\sqrt{2}}{n} \\ -\frac{3}{4}, & \text{otherwise} \end{cases}$$

and consequently $\lim_{n \to \infty} \|U(x^n) - x^n\| = 0$ if and only if $\lim_{n \to \infty} x^n = 0$. Therefore, we get that $0 \in Fix U$ and $U - Id$ is demi-closed at 0.

### 3.1. Error reduction.

We now compare results of [13, theorem 9], [11, Theorem 4.9.1] and [46, Theorem 14] with Theorem 3.7. Since the result of [13, theorem 9] is based on one string, we here assume $E = 1$, see Definition 3.1. Let $e_k = \|x^k - z\|$ for $k \geq 0$ and $z \in Fix T$. Using (3.29), we obtain

$$e_k^2 - e_{k+1}^2 \geq \frac{\lambda_k(2 - \lambda_k)}{m^2} \left( \frac{1}{\gamma} - \frac{1}{2} \right)^2 \|T(x^k) - x^k\|^2$$

which shows how much is big the difference of successive squared errors. Indeed bigger right hand side in (3.31) gives faster decay in error. Therefore, the minimum reduction of error, assuming $\lambda_k = 1$ in (3.31), is

$$\left( \frac{1}{\gamma} - \frac{1}{2} \right)^2 \|T(x^k) - x^k\|^2$$

whereas this value in [13, theorem 9] is

$$\frac{1}{4m^2} \|T(x^k) - x^k\|^2.$$

Therefore, using $\alpha$-relaxed cutter operators, with proper $\alpha$, leads to have faster decay in error than using 1-relaxed cutter operators.

Setting $\{\alpha_i\}_{i=1}^m = 1$ in Algorithm 3.3 i.e. assuming all operators $T_i \in I$ are 1-relaxed cutter, leads to $\gamma = \max_{1 \leq i \leq m} \alpha_i = 1$. Therefore, we obtain that both lower bounds (3.32) and (3.33) are equal.

**Remark 3.9.** For simplicity assume that $\{\alpha_i\}_{i=1}^m = \alpha$. Since $\|T(x^k) - x^k\|^2$ converges to zero, we conclude that the quantity (3.32) takes a large value if $\alpha$ is a small number. Furthermore, using small value for $\alpha$ diminishes the effect of each operator $T_i$ of Definition 3.1. Therefore, we are not able to select proper relaxation parameters only based on (3.32).

To compare Theorem 3.7 with extrapolated simultaneous cutter operator which is analyzed in [11, Theorem 4.9.1], we assume $E = m$ and $m_t = 1$ for
\( t = 1, \ldots, E \) in Definition 3.1. In this case, we have
\[
\epsilon_k^2 - \epsilon_{k+1}^2 \geq \lambda_k (2 - \lambda_k) \left( \frac{1}{\gamma} - \frac{1}{2} \right)^2 \| T(x^k) - x^k \|^2.
\]
Assuming \( \lambda_k = 1 \), the difference of successive squared errors is
\[
\left( \frac{1}{\gamma} - \frac{1}{2} \right)^2 \| T(x^k) - x^k \|^2,
\]
whereas this value in [11, Theorem 4.9.1] is estimated by
\[
\| T(x^k) - x^k \|^2.
\]
Therefore, choosing proper \( \alpha \)-relaxed cutter operators leads to have faster reduction of error than 1-relaxed cutter operators.

Comparing Theorem 3.7 with results in [46, Theorem 14] it should be mentioned that the analysis in [46] are based on sQNE operators and the relaxation parameters \( \lambda_k \) lie in \([\varepsilon, 1 - \varepsilon]\). Furthermore, their convergence analysis is valid if at least \( T - Id \) or \( \{ U_t - Id \}_{t=1}^E \) is demi-closed at zero whereas in Theorem 3.7 the demi-closedness of \( \{ T_i - Id \}_{i=1}^m \) gives the convergence result too. As it is seen in Theorem 3.5 the step-size function depends on \( \| y^i \| \) and \( \| U_t(x) - x \| \) for \( i = 1, \ldots, m_t \) and \( t = 1, \ldots, E \). It allows us to use demi-closeness property of \( T_i - Id \) and \( U_t - Id \) in Theorem 3.7. However, the step-size function in [46] is defined as
\[
\sigma_{max}(x) = \sum_{t=1}^E \omega_t \frac{\| U_t(x) - x \|^2}{\| T(x) - x \|^2}.
\]
It only depends on the \( \| U_t(x) - x \| \) for \( t = 1, \ldots, E \). Since assuming the demi-closeness of \( T_i - Id \) does not give any convergence result, this property is not assumed in [46, Theorem 14]. Therefore, to compare Theorem 3.7 with the results in [46] in the equal conditions, we do not consider demi-closeness of \( T_i - Id \) for \( i = 1, \ldots, m \). In this case, Theorem 2.6 and Lemma 3.4 lead to the following results
\[
\langle z - x, T(x) - x \rangle = \sum_{t=1}^E \omega_t \langle z - x, U_t(x) - x \rangle \\
\geq \sum_{t=1}^E \frac{\omega_t}{\gamma_{m_t}} \| U_t(x) - x \|^2 \\
\geq \sum_{t=1}^E \frac{\omega_t}{\gamma_{m_t}} \| U_t(x) - x \|^2 \\
\geq \frac{1}{\gamma} \| T(x) - x \|^2,
\]
where \( \bar{\gamma} = \max_{t=1,\ldots,E} \gamma_m \). Therefore the step-size function in Theorem 3.5 can be rewritten as
\[
\sigma_{\max} = \frac{\sum_{t=1}^{E} \frac{\omega_t}{\gamma_m} \| U_t(x) - x \|^2}{\| T(x) - x \|^2}
\]
which is bounded below by \( \frac{1}{\bar{\gamma}} \). If we use step-size defined in (3.37) in Theorem 3.7, we have
\[
e^2_k - e^2_{k+1} \geq \lambda_k (2 - \lambda_k) \frac{1}{\bar{\gamma}^2} \| T(x^k) - x^k \|^2.
\]
Assuming \( \lambda_k = 1 \), the difference of successive squared errors is
\[
\frac{1}{\bar{\gamma}^2} \| T(x^k) - x^k \|^2,
\]
whereas this value in [46, Theorem 14] is estimated
\[
\frac{1}{4} \| T(x^k) - x^k \|^2.
\]
Since \( \bar{\gamma} < 2 \), we conclude that for every relaxed cutter operator we have faster reduction of error comparing with [46].

At the end, consider again \( E = 1 \) and \( m_t = m \) in Definition 3.1 and let \( T_{i} \in I \) are 1-relaxed cutter operators. If we assume that \( T_i - Id \) are demi-closed at 0 then the minimum reduction of error for generalized relaxation of cutter operators is derived in (3.33) whereas this value without assuming the demi-closeness property of \( T_i - Id \) is estimated in (3.39) and (3.40) for strictly relaxed cutter and \( SQNE \) operators, respectively. Therefore the generalized relaxation of cutter operators was inherently able to have faster reduction of error.

3.2. Constraints. In this section we consider the projected version of Algorithm 3.3 for a general Hilbert space and for finite dimensional Euclidean space, i.e., \( \mathbb{R}^n \).

Let \( \Omega \) be a closed convex subset of \( H \) and \( FixT_{\sigma,\lambda} \cap \Omega \neq \emptyset \). Since \( \sigma \) is far from zero, see Theorem 3.5, we conclude that \( FixT_{\sigma,\lambda} = FixT \). Therefore we assume \( FixT \cap \Omega \neq \emptyset \). We now consider projected version of Algorithm 3.3 with constant relaxation parameter \( \lambda_k = \lambda \) as follows.

Algorithm 3.10.

Initialization: \( x^0 \in H \) is arbitrary.
Iterative Step: Given \( x^k \), compute
\[
x^{k+1} = P_\Omega T_{\sigma,\lambda}(x^k)
\]
where \( P_\Omega \) denotes the orthogonal projection onto \( \Omega \).

Using Theorem 3.5 and Theorem 2.6 we conclude that \( P_\Omega T_{\sigma,\lambda} \) is an \( \eta \)-relaxed cutter operator. Based on Theorem 2.4 \( P_\Omega T_{\sigma,\lambda} \) is an \( \frac{2-\eta}{\eta} \)-SQNE.
On the other hand, based on [11, Theorem 3.4.3] any SQNE operator is asymptotically regular, namely,

\[
\lim_{k \to \infty} \|P_{\Omega} T_{\sigma,\lambda} x^k - x^k\| = 0.
\]

If we assume that \(P_{\Omega} T_{\sigma,\lambda} - \text{Id}\) is demi-closed at 0 then, using the similar arguments to those in the proof of Theorem 3.7, in particular, by [3, Theorem 2.16 (ii)] and [11, Theorem 2.1.26], the sequence \(\{x^k\}\) generated by Algorithm 3.10 converges weakly to a point \(x^* \in \text{Fix} P_{\Omega} T_{\sigma,\lambda} = \Omega \cap \text{Fix} T\).

By applying Remark 2.10 and in view of Remark 2.9, it is not difficult to see that \(P_{\Omega} T_{\sigma,\lambda} - \text{Id}\) is demi-closed at 0 if either \(T - \text{Id}, U_t - \text{Id}\), \(t = 1, 2, \ldots, E\) or \(T_i - \text{Id}, i = 1, \ldots, m\) is demi-closed at 0. We summarize the above results as follows.

**Theorem 3.11.** Let \(\sigma = \sigma_{\text{max}}\) be step size function, \(\bigcap_{i \in I} \text{Fix} T_i \neq \emptyset\) and \(\lambda \in [\varepsilon, 2-\varepsilon]\) for an arbitrary constant \(\varepsilon \in (0, 1)\). Assume that \(\Omega \cap \text{Fix} T \neq \emptyset\). The sequence generated by Algorithm 3.10 converges weakly to a point in \(\text{Fix} T \cap \Omega\), if one of the following conditions is satisfied

(i) \(T - \text{Id}\) is demi-closed at 0, or
(ii) \(U_t - \text{Id}\) are demi-closed at 0, for all \(t = 1, 2, \ldots, E\), or
(iii) \(T_i - \text{Id}\) are demi-closed at 0, for all \(i = 1, 2, \ldots, m\).

We now consider the case \(H = \mathbb{R}^n\). We assume that every operator \(T_i \in I\), is continuous and strictly relaxed cutter. In this case we don’t need to explicitly assume that \(T_i - \text{Id}, U_t - \text{Id}\) or \(T - \text{Id}\) is demi-closed at zero. Indeed, the continuity of \(T_i \in I\) implies that all of the above-mentioned operators are demi-closed at zero. Thus, we have the following corollary.

**Corollary 3.12.** Let \(\sigma = \sigma_{\text{max}}\) be step size function, \(\bigcap_{i \in I} \text{Fix} T_i \neq \emptyset\) and \(\lambda \in [\varepsilon, 2-\varepsilon]\) for an arbitrary constant \(\varepsilon \in (0, 1)\). Assume that \(\Omega \cap \text{Fix} T \neq \emptyset\) and \(\{T_i\}_{i \in I}\) are continuous and strictly relaxed cutter operators. The generated sequence of Algorithm 3.10 where \(H = \mathbb{R}^n\) converges to a point in \(\text{Fix} T \cap \Omega\).

**Remark 3.13.** The same convergence analysis as Corollary 3.12 is still true for Algorithm 3.3 where \(\{T_i\}_{i \in I}\) are continuous and strictly relaxed cutter with \(\text{Fix} T \neq \emptyset\).

4. **Applications**

In this section we reintroduce two state-of-the-art iteration methods which are based on strictly relaxed cutter operators. First we begin with block iterative methods which are used for solving linear systems of equations (inequalities) and later we employ subgradient projections for solving non-linear convex feasibility problems. In all numerical tests, the case \(\alpha_t = 1\) with various \(\lambda_k\) where the number of string is one, i.e. \(E = 1\), means that the only cutter operator, see [13], is used. The rest of pairs in all tables are our results for various \(\alpha_t\) and \(\lambda_k\). Also, for simplicity, we assume \(\alpha_t = \alpha\) and...
λ_k = λ in all tests where t = 1, . . . , p and k ≥ 0. All the numerical results are performed with Intel(R) Xenon(R) E5440 CPU 2.83 GHz, 8GB RAM, and the codes are written in Matlab R2013a.

4.1. Block iterative methods. Let A ∈ ℝ^{m×n} and b ∈ ℝ^m are given. We assume the consistent linear system of equations

\[ Ax = b. \]

Let A and b be partitioned into p row-blocks \( \{ A_t \}_{t=1}^p \) and \( \{ b^t \}_{t=1}^p \), respectively. We now consider the following operators

\[ T_t(x) = x + \frac{\alpha_t}{\rho(A_t^T M_t A_t)} A_t^T M_t (b^t - A_t x) \]

for \( t = 1, \ldots, p \)

where \( \{ \alpha_t \}_{t=1}^p \) and \( \{ M_t \}_{t=1}^p \) stand for relaxation parameters and symmetric positive definite weight matrices respectively. Also \( \rho(B) \) denotes the spectral radius of \( B \). If \( E = 1 \) and \( \sigma = 1 \) then Algorithm 3.3 with operators (4.2), is called fully simultaneous method and fully sequential method where \( p = 1 \) and \( p = m \), respectively. With \( M_t \) equal to the identity we get the classical Landweber method [42]. Other choices give rise to, e.g., Cimmino’s method [27], the CAV method [20], and, with a componentwise scaling, the DROP algorithm [19].

If \( 0 < \varepsilon \leq \alpha_t \leq 2 - \varepsilon \) for \( t = 1, \ldots, p \), then [45, lemmas 3 and 4] gives that the operator \( T_t \) of (1.2) is not only strictly relaxed cutter but also is nonexpansive. Note that the operator \( T_t \) is an \( \alpha_t \)-relaxed cutter. Since \( \{ T_t \}_{t=1}^p \) are nonexpansive, we conclude that \( \{ T_t - Id \}_{t=1}^p \) are demi-closed at zero, see Remark 2.9. Since composition and convex combination of nonexpansive operators are nonexpansive, we get that operators \( U_t - Id \) and \( T - Id \) are demi-closed at zero. Therefore all conditions of Theorem 3.7 are satisfied. However, based on Corollary 3.12 and Remark 3.13 we do not need to verify the demi-closedness property of \( T_t \).

**Remark 4.1.** It should be noted that a wide range of iterative methods, for solving linear systems of equations, is based on strictly relaxed cutter operators, see [45, lemma 4] and [17, 19, 20, 34, 36, 37, 41]. Furthermore, such iterations appear in many applications which one can find, for example, in signal processing, system theory, computed tomography, proton computerized tomography and other areas.

Using (3.10) the step size function (3.18) can be written as

\[
\sigma_{max} = \frac{\sum_{t=1}^E \omega_t \left( \sum_{i=1}^{m_t} \left( \frac{1}{\alpha_i} - \frac{1}{2} \right) \| y^i \|^2 + \frac{1}{2} \| \sum_{i=1}^{m_t} y^i \|^2 \right)}{\| T(x) - x \|^2}.
\]
where \( x \in H \setminus FixT \). Therefore, using (4.2), we have

\[
\sigma_{\text{max}} = \sum_{i=1}^{E} \omega_i \left( \sum_{i=1}^{m} \frac{\alpha_i(2-\alpha_i)}{\rho(A_i^T M_i(b_i - A_iu^{i-1}))^2} \left\| A_i^T M_i(b_i - A_iu^{i-1}) \right\|^2 + \left\| U_i(x) - x \right\|^2 \right) \frac{1}{2} \sum_{i=1}^{E} \omega_i \left\| U_i(x) - x \right\|^2
\]

where \( u^i \) is defined in (3.3).

**Remark 4.2.** As a special case of (4.2), we assume \( p = m \) and \( M_i = \frac{1}{\|a^i\|^2} \) for \( t = 1, \ldots, m \) where \( a^i \) and \( b^t \) show the \( t \)-row of \( A \) and \( b \) respectively. Therefore we obtain \( T_t \) as below

\[
T_t(x) = x + \alpha_t \frac{b^t - \langle a^t, x \rangle}{\|a^t\|^2} a^t
\]

which is the orthogonal projection, where \( \alpha_t = 1 \), of \( x \) onto hyperplane \( \{ x \in \mathbb{R}^n \mid \langle a^t, x \rangle = b^t \} \) for \( t = 1, 2, \ldots, m \). Therefore, Algorithms 3.3 and 3.10 are the accelerated version of full sequential Kaczmarz’s method and its projected version, respectively.

**4.2. Subgradient projection.** The subgradient method uses the subgradient calculations instead of orthogonal projections onto the individual sets for solving convex feasibility problem. We will examine the subgradient projection operator in Algorithms 3.3 and 3.10. As it is mentioned before, one can use Algorithm 3.10 based on Theorem 3.11.

Let \( i \in J = \{1, 2, \ldots, M\} \), the index set, and \( g_i : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) be convex functions. We consider finding a solution \( x^* \in D \) (assuming its existence) of the following system of convex inequalities

\[
g_i(x) \leq 0, \quad \text{for } i \in J.
\]

Let \( g_i^+(x) = \max \{0, g_i(x)\} \), and denote the solution set of (4.6) by \( S = \{ x \mid g_i(x) \leq 0, \ i \in J \} \). Thus \( g_i^+(x) \) is a convex function and

\[
S = \{ x \mid g_i^+(x) = 0, \ i \in J \}.
\]

Let \( \ell_i(x) \) and \( \partial g_i^+(x) \) denote subgradient and set of all subgradients of \( g_i \) at \( x \), respectively. Here a vector \( t \in \mathbb{R}^n \) is called subgradient of a convex function \( g \) at a point \( y \in \mathbb{R}^n \) if \( \langle t, x - y \rangle \leq g(x) - g(y) \) for every \( x \in \mathbb{R}^n \). It is known that the subgradient of a convex function always exist. We consider the following operators which are used in cyclic subgradient projection method, see [21],

\[
T_t(x) = x - \alpha_t \frac{g_i^+(x)}{\left\| \ell_i(x) \right\|^2} \ell_i(x)
\]

where \( 0 < \varepsilon \leq \alpha_t \leq 2 - \varepsilon \). Clearly \( T_t \) is an \( \alpha_t \)-relaxed cutter. Based on analysis of [13] section 4], see also [11] theorem 4.2.7], \( T_t - Id \) is demiclosed at 0 under a mild condition and this condition holds for any finite dimensional spaces. Therefore the last condition of Theorem 3.7 is satisfied.
Similar to (4.4), we obtain the following step size function for the operator (4.8)

$$\sigma_{\text{max}} = \sum_{t=1}^{E} \omega_t \left( \sum_{i=1}^{m_t} \alpha_i (2 - \alpha_i) \left( \frac{g_i^+(u_i^{t-1})}{\|G_i(u_i^{t-1})\|} \right)^2 + \|U_t(x) - x\|^2 \right)^{1/2} \sum_{t=1}^{E} \omega_t U_t(x) - x.$$  (4.9)

**Remark 4.3.** Assuming $E = 1$ and $\alpha_i = 1$ for $i = 1, \cdots, m$ give the special case of Algorithm 3.3. If we use $T_t$ of (4.8) and the step size function (4.9) then Algorithm 3.3 leads to accelerated scheme of cyclic subgradient projections method which was proposed in [13] and [44].

**Table 1.** The results of 10 quadratic examples using different values of $\alpha_t$ and $\lambda_k$ where $E = 1$. The first and the second parts of a pair indicate the average of iteration numbers and the computational times (per second), respectively.

| $\lambda_k$ | $\alpha$ | 0.001 | 0.5 | 1 | 1.5 | 2 - 0.001 |
|------------|----------|-------|-----|---|-----|-----------|
| 0.001      | (*)      | (57, 45, 65) | (56, 44, 76) | (60, 47, 81) | (74, 58, 98) | (84, 66, 80) |
| 0.5        | (28, 22, 81) | (36, 28, 39) | (42, 34, 94) | (40, 32, 10) | (44, 36, 16) |           |
| 1          | (14, 11, 57) | (25, 26, 10) | (17, 13, 18) | (14, 11, 39) | (15, 12, 22) |           |
| 1.5        | (19, 15, 47) | (25, 26, 08) | (22, 17, 71) | (19, 15, 29) | (15, 12, 17) |           |
| 2 - 0.001  |           |       |     |   |     |           |

We next examine 10 nonlinear systems of inequalities with 500 variables which are produced randomly. Each nonlinear system consists of 100 convex functions. We now explain how one of them is produced. After generating the matrices $G_i \in \mathbb{R}^{500 \times 500}$ and the vectors $c_i \in \mathbb{R}^{500}$ for $i = 1, \cdots, 100$, we consider the following convex functions

$$g_i(x) = x^T G_i^T G_i x + c_i^T x + d_i, \quad i = 1, \cdots, 100$$  (4.10)

and calculate $d_i$ such that $g_i(y) \leq 0$ where $y = (1, \cdots, 1)^T$. Therefore the solution set $S = \{x|g_i(x) \leq 0, \ i = 1, \cdots, 100\}$ has at least one point. The components of $G_i$ and $c_i$ lie in the interval $[-1, 1]$. We generate randomly a starting point, which its components lie in $[-10, 10]$ for all 10 problems. Also we stop the iteration when $g_i^+(x^k) \leq 10^{-6}$ for all $i = 1, \cdots, 100$ or $\|T x^k - x^k\| \leq 10^{-16}$ or the number of iterations exceeds $k = 1000$. Also, we use equal weights for all numerical tests.

In Table 1 we consider one string, i.e., $E = 1$. Also, the first and the second parts of a pair indicate the average of iteration numbers and the computational times (per second), respectively. The sign “*” means no feasible point is achieved within our mentioned criteria. Based on Table 1 the best result of [13], i.e., using $\alpha = 1$ with different $\lambda$, is obtained by choosing $\alpha = 1$ and $\lambda = 1$. However, based on our analysis which allows us to use $\alpha \in (0, 2)$, by setting $\alpha = 1.5$ and $\lambda = 1.5$ we reduce both, the number of iterations and the computational time. Therefore, based on Table 1 one may select a proper value for $\alpha$ to reduce the number of iterations.
To see the effect of using generalized relaxation technique (grt), we consider various number of strings, i.e. $E = 2, 4, 5, 10, 20$, and assume $\alpha = \lambda = 1$. As it is seen in Table 2 using grt reduces the number of iterations and consequently the computational times. On the other hand, the only increasing of the number of strings does not guarantee to reduce the number of iterations, see the last line of Table 2. Comparing the results of Table 2 emphasizes that using grt together with many strings is able to reduce notably the number of iterations.

Table 2. The results of 10 quadratic examples using different sizes of strings where $\alpha = \lambda = 1$. The first and the second parts of a pair indicate the average of iteration numbers and the computational times (per second), respectively.

| $E$ | grt | | | | |
|---|---|---|---|---|---|
| 2 | (36,22.36) | (31,14.50) | (36,15.65) | (28,10.77) | (28,9.68) |
| 4 | (85,52.47) | (170,79.41) | (212,92.01) | (430,160.91) | (868,298.74) |

5. Conclusion

In this paper we studied the generalized relaxation of string averaging operator which is based on strictly relaxed cutter operators. We showed that this operator is strictly relaxed cutter operator by restricting the step size function. We analyzed a fixed point iteration method with its projected version, based on this operator. The capability of the method was examined by employing state-of-the-art iterative methods. Our numerical tests showed that one may choose relaxation parameters $\alpha \neq 1$ to reduce the number of iterations comparing with the case $\alpha = 1$. The numerical results emphasize that using generalized relaxation technique together with many strings is able to reduce remarkably the number of iterations.

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