CURVATURE ESTIMATES FOR SUBMANIFOLDS WITH
PRESCRIBED GAUSS IMAGE AND MEAN CURVATURE

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Abstract. We study that the $n$–graphs defining by smooth map $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, m \geq 2$, in $\mathbb{R}^{m+n}$ of the prescribed mean curvature and the Gauss image. We derive the interior curvature estimates

$$\sup_{D_R(x)} |B|^2 \leq \frac{C R^2}{R^2}$$

under the dimension limitations and the Gauss image restrictions. If there is no dimension limitation we obtain

$$\sup_{D_R(x)} |B|^2 \leq C R^{-a} \sup_{D_{2R}(x)} (2 - \Delta_f)^{-\left(\frac{s}{2} + \frac{1}{2}\right)}, \quad s = \min(m, n)$$

with $a < 1$ under the condition

$$\Delta_f = \left[ \det \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2.$$ 

If the image under the Gauss map is contained in a geodesic ball of the radius $\sqrt{\frac{2}{\pi}}$ in $G_{n,m}$ we also derive corresponding estimates.

1. Introduction

There are many beautiful results on minimal hypersurfaces and we have a fairly profound understanding of the issue of minimal hypersurfaces in many aspects, the issue of minimal submanifolds of higher codimension seems more complicated. Lawson-Osserman’s paper revealed several important different phenomena in higher codimension [7].

For higher codimensional Bernstein problem Hildebrandt-Jost-Widman [4] gave us the following result.

Theorem 1.1. Let $z^\alpha = f^\alpha(x), \alpha = 1, \cdots, m, x = (x^1, \cdots, x^n) \in \mathbb{R}^n$ be the $C^2$ solution to the system of minimal surface equations. Let there exist $\beta$, where

$$\beta \leq \cos^{-s} \left( \frac{\pi}{2\sqrt{s} K} \right), \quad K = \begin{cases} 1 & \text{if } s = 1 \\ 2 & \text{if } s \geq 2 \end{cases}, \quad s = \min(m, n)$$

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such that for any $x \in \mathbb{R}^n$,

$$\Delta f = \left[ \det \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < \beta,$$

then $f^1, \cdots, f^m$ are affine linear functions on $\mathbb{R}^n$, whose graph is an affine $n$–plane in $\mathbb{R}^{m+n}$.

The theorem not only generalized Moser’s result [8] to higher codimension, but also introduced the Gauss image assumption. The geometric meaning of the conditions in the above theorem is that the image under the Gauss map lies in a closed subset of an open geodesic ball of the radius $\sqrt{2\pi}$ in the Grassmannian manifold $G_{n,m}$. It would be remind that the Gauss maps play an important role in minimal surface theory. For general surfaces in $\mathbb{R}^3$ the Gauss map and the mean curvature determine a simply connected surface completely [6].

Later, in the author’s joint work with J. Jost [5], the above theorem has been improved that the number in the theorem is 2, instead of $\cos^{-s} \left( \frac{\pi}{2\sqrt{sK}} \right)$, which is independent of the dimension and the codimension. The key point is to find larger geodesic convex set $B_{IJX} \subset G_{n,m}$ which contains the geodesic ball of radius $\sqrt{2\pi}$.

In author’s previous work, Schoen-Simons-Yau type curvature estimates [10] and Ecker-Huisken type curvature estimates [1] have been generalized to the flat normal bundle situation [14] [12]. Using some techniques in [2], Fröhlich-Winklmann [3] derived interior curvature estimates for the flat normal bundle case and generalized our results in [12].

Recently, we studied complete minimal submanifolds with codimension $m \geq 2$ and curved normal bundle, but with the convex Gauss image. Thus, we can construct auxiliary functions, which enable us to carry out both Schoen-Simons-Yau type curvature estimates and Ecker-Huisken type curvature estimates in [16] and [17]. From the estimates several geometrical conclusions follow, including the following Bernstein type theorems.

**Theorem 1.2.** Let $M$ be a complete minimal $n$-dimensional submanifold in $\mathbb{R}^{n+m}$ with $n \leq 6$ and $m \geq 2$. If the Gauss image of $M$ is contained in an open geodesic ball of $G_{n,m}$ centered at $P_0$ and of radius $\sqrt{2\pi}$, then $M$ has to be an affine linear subspace.

**Theorem 1.3.** Let $M = (x, f(x))$ be an $n$-dimensional entire minimal graph given by $m$ functions $f^\alpha(x^1, \cdots, x^n)$ with $n \leq 4$ and $m \geq 2$. If

$$\Delta f = \left[ \det \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2,$$

then $f^\alpha$ has to be affine linear functions representing an affine $n$-plane.
Theorem 1.4. Let $M$ be a complete minimal $n$-dimensional submanifold in $\mathbb{R}^{n+m}$. If the Gauss image of $M$ is contained in an open geodesic ball of $G_{n,m}$ centered at $P_0$ and of radius $\sqrt{\frac{2}{4}}\pi$, and $(\sqrt{\frac{2}{4}}\pi - \rho \circ \gamma)^{-1}$ has growth

\begin{equation}
(\sqrt{\frac{2}{4}}\pi - \rho \circ \gamma)^{-1} = o(R),
\end{equation}

where $\rho$ denotes the distance on $G_{n,m}$ from $P_0$ and $R$ is the Euclidean distance from any point in $M$. Then $M$ has to be an affine linear subspace.

Theorem 1.5. Let $M = (x, f(x))$ be an $n$-dimensional entire minimal graph given by $m$ functions $f^\alpha(x^1, \cdots, x^n)$ with $m \geq 2$. If

\begin{equation}
\Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2,
\end{equation}

and

\begin{equation}
(2 - \Delta_f)^{-1} = o(R^\frac{1}{2}),
\end{equation}

where $R^2 = |x|^2 + |f|^2$. Then $f^\alpha$ has to be affine linear functions and hence $M$ has to be an affine linear subspace.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^m$ be a smooth map whose graph is an $n$-submanifold $S$ in $\mathbb{R}^{m+n}$. We always assume that $m \geq 2$ in this paper. On $S$ there is extrinsic distance $r$, restriction to $S$ of Euclidean distance from $x \in S$. Denote the closed ball of radius $R$ and centered at $x \in S$ by $B_R(x) \subset \mathbb{R}^{m+n}$. Its restriction to $S$ is denoted by

$$ D_R(x) = B_R(x) \cap S. $$

We also have the mean curvature $H$ and the Gauss image restriction $\gamma : M \to V \subset G_{n,m}$, where $V$ will be given in §3. We will give interior estimates for the squared norm of the second fundamental form $B$ of $S$ in $\mathbb{R}^{m+n}$ in terms of those geometric data. Since a complete submanifold in Euclidean space with our Gauss image restrictions has to be a graph. Those results can be viewed as generalizations of the above Theorems 1.2-1.5.

Theorem 1.6. Let $S \subset \mathbb{R}^{m+n}$, $m \geq 2$ be a graph given by $f^1, \cdots, f^m$. Suppose $D_{2R}(x) \subset \subset S$. If one of the following conditions is satisfied

1. $2 \leq n \leq 6$ and the image under the Gauss map is contained in an open geodesic ball of radius $\sqrt{\frac{2}{4}}\pi$;

2. $2 \leq n \leq 5$ and

\begin{equation}
\Delta_f = \left[ \det \left( \delta_{ij} + \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2.
\end{equation}
Then,\[
\sup_{D_R(x)} |B|^2 \leq \frac{C}{R^2}
\]
with the constant $C$ depending on $n, m, R \sup_{D_{2R}(x)} |H|, R^2 \sup_{D_{2R}(x)} (|\nabla H| + \lambda_1)$ and \( R^3 \sup_{D_{2R}(x)} \lambda_2 \).

**Theorem 1.7.** Let $S \subset \mathbb{R}^{m+n}, m \geq 2$ be a graph given by $f^1, \ldots, f^m$. Suppose $D_{2R}(x) \subset S$. If the image under the Gauss map is contained in an open geodesic ball of radius $\sqrt{2} \frac{\pi}{4}$, then

\[
(1.4) \quad \sup_{D_R(x)} |B|^2 \leq CR^{-a} \sup_{D_{2R}(x)} \left( \frac{\sqrt{2}}{4} \pi - \rho \circ \gamma \right)^{-2}.
\]

If

\[
\Delta_f = \left[ \det \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{\frac{1}{2}} < 2,
\]

then

\[
(1.5) \quad \sup_{D_R(x)} |B|^2 \leq CR^{-a} \sup_{D_{2R}(x)} (2 - \Delta_f)^{-(\frac{3s}{2} + \frac{s}{2})}, \quad s = \min(m, n).
\]

The above constant $C$ depends on $n, m, R \sup_{D_{2R}(x)} |H|, R^2 \sup_{D_{2R}(x)} (|\nabla H| + \lambda_1)$ and \( R^3 \sup_{D_{2R}(x)} \lambda_2 \) and the constant $a < 1$.

**Remark 1.1.** In the case of parallel mean curvature we can use Lemma 5.1 with $g = 0$ and the estimates in Theorem 1.7 could be improved as

\[
\sup_{D_R(x)} |B|^2 \leq CR^{-a} \sup_{D_{2R}(x)} \left( \frac{\sqrt{2}}{4} \pi - \rho \circ \gamma \right)^{-2}
\]

or

\[
\sup_{D_R(x)} |B|^2 \leq CR^{-2} \sup_{D_{2R}(x)} (2 - \Delta_f)^{-\frac{3}{2}}
\]

respectively, where the constant $C$ depends on $n, m, R \sup_{D_{2R}(x)} |H|$.

**Remark 1.2.** $\lambda_1$ and $\lambda_2$ in Theorem 1.6 and Theorem 1.7 are determined by mean curvature and Gauss image assumption. The precise definitions are given by (3.9) and (2.5), respectively.

The paper will be arranged as follows. In §2 notations and basic formulas will be given, especially, the Bochner-Simons type inequality will be derived for general submanifolds in $\mathbb{R}^{m+n}$ with $m \geq 2$. In §3 we describe the two kinds Gauss image restrictions which enable us to define auxiliary functions. Those are important in $L^p-$curvature estimates which is given in §4. We will give Schoen-Simons-Yau type and Ecker-Huisken type estimates in our general setting. In the final section we prove our main results. It is done by using $L^p-$curvature estimates in §4 and mean value inequality of Fröhlich-Winklman in [3].
2. A Bochner-Simons type inequality

Let $M \to \mathbb{R}^{m+n}$ be an $n-$submanifold in $(m+n)-$dimensional Euclidean space with the second fundamental form $B$ which can be viewed as a cross-section of the vector bundle $\text{Hom}(\otimes^2 TM, NM)$ over $M$, where $TM$ and $NM$ denote the tangent bundle and the normal bundle along $M$, respectively. A connection on $\text{Hom}(\otimes^2 TM, NM)$ is induced from those of $TM$ and $NM$ naturally. We consider the situation of higher codimension $m \geq 2$ in this paper.

Taking the trace of $B$ gives the mean curvature vector $H$ of $M$ in $\mathbb{R}^{m+n}$, a cross-section of the normal bundle.

To have the curvature estimates we need the Simons version of the Bochner type formula for the squared norm of the second fundamental form. It is done in [11] for minimal submanifolds in an arbitrary ambient Riemannian manifold. Now, for any submanifold in Euclidean space, by the same calculation as in the paper [11] we have the following formula:

\[
(\nabla^2 B)_{XY} = \nabla_X \nabla_Y H + \langle B_{Xe_i}, H \rangle B_{Ye_i} - \langle B_{XY}, B_{e_ie_j} \rangle B_{e_ie_j} + 2 \langle B_{Xe_i}, B_{Ye_i} \rangle B_{e_ie_j} - \langle B_{XY}, B_{e_ie_j} \rangle B_{Xe_i} + \langle B_{XY}, B_{e_ie_j} \rangle B_{Ye_j},
\]

where $\nabla^2$ stands for the trace Laplacian operator, \{e_i\} is a local tangential orthonormal frame field of $M$. Here and in the sequel we use the summation convention.

Then, we have (see [15] for details)

**Proposition 2.1.**

\[
\Delta |B|^2 \geq 2 |\nabla B|^2 + 2 \langle \nabla_i \nabla_j H, B_{ij} \rangle + 2 \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle - 3 |B|^4,
\]

where $\nabla_i$ denotes $\nabla_{e_i}$ and $B_{ij} = B_{e_ie_j}$.

In (2.2), the terms involving the mean curvature can be estimated as follows.

\[
| \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle | \leq |H| |B|^3 \leq \varepsilon |B|^4 + \frac{1}{\varepsilon'} |H|^2 |B|^2, \quad \varepsilon' > 0
\]

or

\[
| \langle B_{ij}, H \rangle \langle B_{ik}, B_{jk} \rangle | \leq \sqrt{n} |B|^4.
\]

Define

\[
\lambda_2 = \begin{cases} -\left( \frac{\langle \nabla_i \nabla_j H, B_{ij} \rangle}{|B|^2} \right)^-, & \text{if } |B| > 0, \\ 0, & \text{if } |B| = 0,
\end{cases}
\]

where $(\cdots)^-$ denotes the negative part of the quantity. Obviously,

\[
\lambda_2 \leq |\nabla \nabla H|.
\]
In order to use the formula (2.2) we also need to estimate $|\nabla B|^2$ in terms of $|\nabla|B||^2$. Schoen-Simon-Yau [10] did such an estimate for codimension $m = 1$. The following lemma is for any prescribed mean curvature $H$ and any codimension. In the case of $H = 0$, the following estimates also improve our previous estimates in [16].

**Lemma 2.1.** For any real number $\varepsilon > 0$

\[(2.6)\quad |\nabla B|^2 \geq \left(1 + \frac{2}{n + \varepsilon}\right)|\nabla|B||^2 - C(n, \varepsilon)|\nabla H|^2,\]

where

\[C(n, \varepsilon) = \frac{2(n - 1 + \varepsilon)}{\varepsilon(n + \varepsilon)}.\]

If $M$ has parallel mean curvature, then

\[(2.7)\quad |\nabla B|^2 \geq \left(1 + \frac{2}{n}\right)|\nabla|B||^2.\]

**Proof.** It is sufficient for us to prove the inequality at the points where $|B|^2 \neq 0$. Choose a local orthonormal tangent frame field $\{e_1, \cdots, e_n\}$ and a local orthonormal normal frame field $\{\nu_1, \cdots, \nu_m\}$ of $M$ near the considered point $x$. Denote the shape operator $A^\alpha = A^\nu_\alpha$. Then obviously $|B|^2 = \sum_\alpha |A^\alpha|^2$ and

\[\nabla|B|^2 = \sum_\alpha \nabla|A^\alpha|^2.\]

Let

\[A^\alpha e_i = h_{aij}e_j, \quad h_{aij} = h_{aji}.\]

By triangle inequality

\[|\nabla|B||^2 = \left|\sum_\alpha \nabla|A^\alpha|^2\right| \leq \sum_\alpha |\nabla|A^\alpha|^2|.\]

Therefore,

\[(2.8)\quad |\nabla|B||^2 = \frac{|\nabla|B|^2|^2}{4|B|^2} \leq \frac{(\sum_\alpha |\nabla|A^\alpha|^2|^2)}{4\sum_\alpha |A^\alpha|^2}.\]

Since $|B|^2 \neq 0$, we can assume $|A^\alpha|^2 > 0$ for each $\alpha$ without loss of generality. Let $1 \leq \gamma \leq m$ such that

\[\frac{|\nabla|A^\gamma|^2|^2}{|A^\gamma|^2} = \max_\alpha \left\{\frac{|\nabla|A^\alpha|^2|^2}{|A^\alpha|^2}\right\} < +\infty,\]

then from (2.8),

\[(2.9)\quad |\nabla|B||^2 \leq \frac{|\nabla|A^\gamma|^2|^2}{4|A^\gamma|^2}.\]
$|A^\gamma|^2$ and $\nabla |A^\gamma|^2$ is independent of the choice of $\{e_1, \cdots, e_n\}$, then without loss of generality we can assume $h_{\gamma ij} = 0$ whenever $i \neq j$. Then

$$|
abla |A^\gamma|^2|^2 = 4 \sum_k \left( \sum_i h_{\gamma ii} h_{\gamma ik} \right)^2$$

$$\leq \left( \sum_i h_{\gamma ii}^2 \right) \left( \sum_{i,k} h_{\gamma iik}^2 \right) = 4 |A^\gamma|^2 \sum_{i,k} h_{\gamma iik}^2$$

and from (2.9)

$$|
abla B|^2 \leq \sum_{i,k} h_{\gamma iik}^2.$$

Since

$$\sum_i h_{\gamma iiii}^2 = \sum_i (|\nabla e_i H^\gamma| - \sum_{j \neq i} h_{\gamma jjj})^2$$

$$= |\nabla H^\gamma|^2 + \sum_i \left( \sum_{j \neq i} h_{\gamma jjj} \right)^2 - 2 |\nabla H^\gamma| \sum_i \sum_{j \neq i} h_{\gamma jjj}$$

$$\leq |\nabla H^\gamma|^2 + (n - 1) \sum_{j \neq i} h_{\gamma jjj}^2 + \frac{n - 1}{\varepsilon} |\nabla H^\gamma|^2 + \frac{\varepsilon}{n - 1} \sum_{j \neq i} h_{\gamma jjj}^2$$

$$\leq \left( 1 + \frac{n - 1}{\varepsilon} \right) |\nabla H^\gamma|^2 + (n - 1 + \varepsilon) \sum_{j \neq i} h_{\gamma jjj}^2,$$

we obtain

$$|
abla B|^2 \leq \sum_{i,k} h_{\gamma iik}^2 = \sum_{i \neq k} h_{\gamma iik}^2 + \sum_i h_{\gamma iiii}^2$$

(2.10)

$$\leq \left( 1 + \frac{n - 1}{\varepsilon} \right) |\nabla H^\gamma|^2 + (n + \varepsilon) \sum_{j \neq i} h_{\gamma jjj}^2.$$

On the other hand, a direct calculation shows

$$|
abla |B|^2|^2 = \sum_{k} \left( \sum_{\alpha,i,j} h_{\alpha i j} h_{\alpha i j k} e_k \right)^2 = 4 \sum_{\alpha,\beta,i,j,s,t,k} h_{\alpha i j} h_{\alpha i j k} h_{\beta s t} h_{\beta s t k},$$

$$|
abla B|^2 - |
abla |B|^2|^2 = |\nabla B|^2 - \frac{|
abla |B|^2|^2}{4 |B|^2}$$

$$= \sum_{\alpha,i,j,k} h_{\alpha i j k}^2 - \frac{\sum_{\alpha,\beta,i,j,s,t,k} h_{\alpha i j k} h_{\alpha i j k} h_{\beta s t} h_{\beta s t k}}{\sum_{\beta,s,t} h_{\beta s t}^2}$$

$$= \frac{\sum_{\alpha,\beta,i,j,s,t,k} (h_{\alpha i j k} h_{\beta s t} - h_{\beta s t k} h_{\alpha i j})^2}{2 |B|^2}$$

$$\geq \frac{\sum_{\beta,i \neq j,s,t,k} h_{\gamma i j k}^2 h_{\beta s t}^2 + \sum_{\alpha,s \neq t,i,j,k} h_{\gamma s t k}^2 h_{\alpha i j}^2}{2 |B|^2}.$$
\[ = \sum_{i \neq j, k} h_{\gamma i j k}^2 \geq \sum_{i \neq j, k} (h_{\gamma i k i}^2 + h_{\gamma j k k}^2) \]

(2.11)

\[ = 2 \sum_{i \neq k} h_{\gamma i k i}^2. \]

Noting (2.10) and (2.11), we arrive at (2.6).

Finally, we have

**Proposition 2.2.** In the case of \( H \neq 0 \)

\[ \Delta |B|^2 \geq 2 \left( 1 + \frac{2}{n + \varepsilon} \right) |\nabla |B||^2 - (3 + 2 \varepsilon')|B|^4 \]

\[ - 2\lambda_2 |B| - 2C(n, \varepsilon)|\nabla H|^2 - \frac{2}{\varepsilon}|H|^2|B|^2 \]

with \( \varepsilon \) and \( \varepsilon' \) and

(2.13)

\[ \Delta |B|^2 + (3 + 2\sqrt{n})|B|^4 \geq - 2\lambda_2 |B|. \]

In the minimal case

(2.14)

\[ \Delta |B|^2 \geq 2 \left( 1 + \frac{2}{n} \right) |\nabla |B||^2 - 3|B|^4 \]

3. Grassmannian Manifolds and Gauss Maps

Let \( \mathbb{R}^{n+m} \) be an \((n + m)\)-dimensional Euclidean space. All oriented \( n \)-subspaces constitute the Grassmannian manifolds \( G_{n,m} \), which is an irreducible symmetric space of compact type.

In the Grassmanian manifolds \( G_{n,m} \) the sectional curvature of the canonical metric varies in \([0, 2]\). The radius of the largest convex ball is \( \frac{\sqrt{2}}{4\pi} \).

We consider the two cases.

1. On an open geodesic ball \( B_{\frac{\sqrt{2}}{4\pi}}(P_0) \subset G_{n,m} \) of radius \( \frac{\sqrt{2}}{4\pi} \) and centered at \( P_0 \).

Let

\[ h = \cos(\sqrt{2}\rho) \]

be a positive function on \( B_{\frac{\sqrt{2}}{4\pi}}(P_0) \), where \( \rho \) is the distance function from \( P_0 \) in \( G_{n,m} \). Then, the Hessian comparison theorem gives

(3.1)

\[ \text{Hess}(h) = h'\text{Hess}(\rho) + h''d\rho \otimes d\rho \leq -2h \ g \]

with the metric tensor \( g \) on \( G_{n,m} \).

Let

\[ h_1 = \sec^2(\sqrt{2}\rho), \]
where $\rho$ is the distance function from $P_0$ in $G_{n,m}$. We then have

$$\text{Hess}(h_1) = h'_1\text{Hess}(\rho) + h''_1d\rho \otimes d\rho$$

$$\geq 4h_1 \ g + \frac{3}{2}h_1^{-1}dh_1 \otimes dh_1$$

2. For $P_0 \in G_{n,m}$, which is expressed by a unit $n-$vector $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$. For any $P \in G_{n,m}$, expressed by an $n-$vector $e_1 \wedge \cdots \wedge e_n$, we define an important function on $G_{n,m}$

$$w \overset{\text{def.}}{=} \langle P, P_0 \rangle = \langle e_1 \wedge \cdots \wedge e_n, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rangle = \det W,$$

where $W = (\langle e_i, \varepsilon_j \rangle)$. The Jordan angles between $P$ and $P_0$ are defined by

$$\theta_\alpha = \arccos(\lambda_\alpha),$$

where $\lambda_\alpha \geq 0$ and $\lambda_\alpha^2$ are the eigenvalues of the symmetric matrix $W^TW$. The distance between $P_0$ and $P$ is

$$d(P_0, P) = \sqrt{\sum \theta_\alpha^2}.$$

Denote

$$U = \{P \in G_{n,m} : w(P) > 0\}.$$

On $U$ we can define

$$v = w^{-1} = \prod_{\alpha} \sec \theta_\alpha.$$

Define

$$B_{JX}(P_0) = \{P \in U : \text{sum of any two Jordan angles between } P \text{ and } P_0 < \frac{\pi}{2}\}.$$

This is a geodesic convex set, larger than the geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ and centered at $P_0$. This was found in a previous work of Jost-Xin [5]. For any real number $a$ let $V_a = \{P \in G_{n,m}, \ v(P) < a\}$. From ([5], Theorem 3.2) we know that

$$V_2 \subset B_{JX} \quad \text{and} \quad \overline{V}_2 \cap \overline{B}_{JX} \neq \emptyset.$$

**Theorem 3.1.** [17]

$v$ is a convex function on $B_{JX}(P_0) \subset U \subset G_{n,m}$, and

$$\text{Hess}(v) \geq v(2-v)g + \left(\frac{v-1}{pv^2} + \frac{p+1}{pv}\right)dv \otimes dv$$

on $\nabla_2$, where $g$ is the metric tensor on $G_{n,m}$ and $p = \min(n, m)$.

Let

$$h = v^{-k}(2-v)^k$$

(3.3)
define a positive function on \( V_2 \), where \( k = \frac{3}{4} + \frac{1}{2s} \) and \( s = \min(m, n) \). From (3.2) we have (see (4.4) in [17])

\[
(3.4) \quad \text{Hess}(h) \leq -\left( \frac{3}{2} + \frac{1}{s} \right) h \ g,
\]

where \( g \) is the metric tensor on \( G_{n,m} \).

Let

\[
(3.5) \quad h_1 = h^{-2},
\]

then

\[
(3.6) \quad \text{Hess}(h_1) = h'_1 \text{Hess}(h) + h''_1 dh \otimes dh
\]

\[
\geq \left( 3 + \frac{2}{s} \right) h_1 g + \frac{3}{2} h_1^{-1} dh_1 \otimes dh_1,
\]

where \( g \) is the metric tensor on \( G_{n,m} \).

In each of the above cases we have a positive functions \( h \) and \( h_1 \) defined on an open subset on \( V \subset G_{m,n} \) satisfying

\[
(3.7) \quad \text{Hess}(h) \leq -\lambda h g,
\]

where \( \lambda = 2 \) in the first case and \( \lambda > \frac{3}{2} \) in the second case and

\[
(3.8) \quad \text{Hess}(h_1) \geq \mu h_1 g + \frac{3}{2} h_1^{-1} dh_1 \otimes dh_1,
\]

where \( \mu = 4 \) in the first case and \( \mu > 3 \) in the second case.

For \( n \)-dimensional submanifold \( M \) in \( \mathbb{R}^{n+m} \). The Gauss map \( \gamma : M \to G_{n,m} \) is defined by

\[
\gamma(x) = T_x M \in G_{n,m}
\]

via the parallel translation in \( \mathbb{R}^{n+m} \) for arbitrary \( x \in M \). If \( m = 1 \), the image of the Gauss map is the unit sphere. This is just the hypersurface situation. Otherwise, the image of the Gauss map is a Grassmannian manifold.

The energy density of the Gauss map (see [13] Chap.3, §3.1) is

\[
e(\gamma) = \frac{1}{2} \langle \gamma_\ast e_i, \gamma_\ast e_i \rangle = \frac{1}{2} |B|^2.
\]

We assume that the image of \( M \) under the Gauss map is contained in \( V \subset G_{m,n} \).

Thus, we have the function \( \tilde{h} = h \circ \gamma \) and \( \tilde{h}_1 = h_1 \circ \gamma \) defined on \( M \). We denote \( h \) for \( \tilde{h} \) and \( h_1 \) for \( \tilde{h}_1 \) in the sequel for simplicity. Define

\[
(3.9) \quad (dh(\tau(\gamma)))^+ = \lambda_1 h,
\]

where \( \tau(\gamma) \) is the tension field of the Gauss map, which is zero when \( M \) has parallel mean curvature by Ruh-Vilms theorem [9].
From (3.7) and the composition formula we have
\[ \Delta h \leq -\lambda |B|^2 h + dh(\tau(\gamma)) \]
(3.10)
\[ \leq -\lambda |B|^2 h + \lambda_1 h. \]
We then obtain
\[ \lambda |B|^2 \phi^2 \leq -h^{-1}\phi^2 \Delta h + \lambda_1 \phi^2 \]
and
\[ \lambda \int_M |B|^2 \phi^2 * 1 \leq - \int_M h^{-1}\phi^2 \Delta h * 1 + \int_M \lambda_1 \phi^2 * 1. \]
Since
\[ -\int_M h^{-1}\phi^2 \Delta h * 1 = -\int_M \nabla(h^{-1}\phi^2 \nabla h) * 1 + \int_M \nabla(h^{-1}\phi^2) \nabla h * 1 \]
\[ = \int_M (\nabla h^{-1})(\nabla h) \phi^2 * 1 + \int_M h^{-1} \nabla h \nabla \phi^2 * 1 \]
(3.11)
\[ = -\int_M h^{-2} |\nabla h|^2 \phi^2 * 1 + 2 \int_M h^{-1} \phi \nabla h \cdot \nabla \phi * 1 \]
\[ \leq -\int_M h^{-2} |\nabla h|^2 \phi^2 * 1 + \int_M h^{-2} \phi^2 |\nabla h|^2 * 1 + \int_M |\nabla \phi|^2 * 1 \]
\[ = \int_M |\nabla \phi|^2 * 1, \]
we obtain
\[ \lambda \int_M |B|^2 \phi^2 \leq \int_M |\nabla \phi|^2 * 1 + \int_M \lambda_1 \phi^2 * 1 \]
(3.12)
for arbitrary function \( \phi \) with compact support \( D \subset M \).

Define
\[ \mu_1 = -h_1^{-1}(dh_1(\tau(\gamma)))^- \]
Then from (3.8) and the composition formula we have
\[ \Delta h_1 \geq \mu h_1 |B|^2 + \frac{3}{2} h_1^{-1} |\nabla h_1|^2 - \mu_1 h_1, \]
(3.13)
where \( \mu > 3 \).

In the graphic situation the \( v \)-function on \( G_{n,m} \) composed with the Gauss map \( \gamma \) is just
\[ \Delta f = \left[ \text{det} \left( \delta_{ij} + \sum_{\alpha} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} \right) \right]^{-\frac{3}{2}}, \]
which is the volume element of our graph. It follows that \( \text{vol}(D_R(x)) \leq 2R^\alpha \) for the second case. As for the first case
\[ \Delta f \leq \left( \sec \left( \frac{\sqrt{2}}{4\pi} \right) \right)^s \]
with \( s = \min(m, n) \). So in each cases we have

\[
\text{vol}(D_R(x)) \leq C \ R^n
\]

with the constant \( C \) depending on \( n \) and \( m \).

4. \( L^p \)-Curvature estimates

Replacing \( \phi \) by \( |B|^{1+q}\phi \) in (3.12) gives

\[
\int_M |B|^{4+2q}\phi^2 * 1 \leq \lambda^{-1} \int_M |\nabla(|B|^{1+q}\phi)|^2 * 1 + \lambda^{-1} \int_M \lambda_1 |B|^{2+2q}\phi^2
\]

\[
= \lambda^{-1} (1 + q)^2 \int_M |B|^{2q} |\nabla|B|^2 \phi^2 * 1 + \lambda^{-1} \int_M |B|^{2+2q}(|\nabla\phi|^2 + \lambda_1 \phi^2) * 1
\]

\[
+ 2\lambda^{-1}(1+q) \int_M |B|^{1+2q} \nabla|B| \cdot \phi \nabla\phi * 1.
\]

(4.1)

Using Bochner type formula (2.12), which is equivalent to

\[
\frac{2}{n+\varepsilon} |\nabla|B|^2|^2 \leq |B|\Delta|B| + \frac{(3+2\varepsilon')}{2} |B|^4
\]

\[
+ \lambda_2 |B| + \frac{C(n,\varepsilon)}{2\varepsilon'} |\nabla H|^2 + \frac{C(n)}{2\varepsilon'} |H|^2 |B|^2.
\]

(4.2)

Multiplying \( |B|^{2q}\phi^2 \) with both sides of (4.2) and integrating by parts, we have

\[
\frac{2}{n+\varepsilon} \int_M |B|^{2q} |\nabla|B||^2 \phi^2 * 1
\]

\[
\leq -(1 + 2q) \int_M |B|^{2q} |\nabla|B||^2 \phi^2 * 1
\]

\[
- 2 \int_M |B|^{1+2q} \nabla|B| \cdot \phi \nabla\phi * 1 + \frac{3 + \varepsilon'}{2} \int_M |B|^{4+2q}\phi^2 * 1
\]

\[
+ \int_M \lambda_2 |B|^{1+2q}\phi^2 * 1 + \frac{C(n)}{2\varepsilon'} \int_M |H|^2 |B|^{2+2q}\phi^2 * 1
\]

\[
+ \frac{C(n,\varepsilon)}{2} \int_M |\nabla H|^2 |B|^{2q}\phi^2 * 1.
\]

(4.3)
By multiplying $\frac{3 + \varepsilon'}{2}$ with both sides of (4.1) and then adding up both sides of it and (4.3), we have

\[
\left(\frac{2}{n + \varepsilon} + 1 + 2q - \frac{3 + \varepsilon'}{2}\lambda^{-1}(1 + q)^2\right) \int_M |B|^{2q}|\nabla |B||^2 \phi^2 * 1 \\
\leq \frac{3 + \varepsilon'}{2}\lambda^{-1} \int_M |B|^{2+2q}|(\nabla \phi|^2 + \lambda_1 \phi^2) * 1 \\
+ \frac{C(n, \varepsilon)}{2} \int_M |\nabla \phi|^2|B|^{2q} \phi^2 * 1. \tag{4.4}
\]

By using Young's inequality and letting $\varepsilon = \varepsilon'$, (4.4) becomes

\[
\left(\frac{2}{n + \varepsilon} + 1 + 2q - \frac{3 + \varepsilon}{2}\lambda^{-1}(1 + q)^2 - \varepsilon\right) \int_M |B|^{2q}|\nabla |B||^2 \phi^2 * 1 \\
\leq C_1(\varepsilon, \lambda, q, n) \int_M |B|^{2+2q}|(\nabla \phi|^2 + \lambda_1 \phi^2 + |H|^2 \phi^2) * 1 \\
+ C_1(\varepsilon, \lambda, q, n) \int_M \lambda_2 |B|^{1+2q} \phi^2 \\
+ C_1(\varepsilon, \lambda, q, n) \int_M |\nabla H|^2|B|^{2q} \phi^2. \tag{4.5}
\]

If

\[
\lambda > \frac{3}{2}(1 - \frac{2}{n}), \tag{4.6}
\]

then

\[
\frac{2}{n} + 1 + 2q - \frac{3}{2}\lambda^{-1}(1 + q)^2 > 0
\]

whenever

\[
q \in \left[0, -1 + \frac{2}{3} \lambda + \frac{1}{3} \sqrt{4\lambda^2 - 6(1 - \frac{2}{n}) \lambda}\right]. \tag{4.7}
\]

Thus we can choose $\varepsilon$ sufficiently small, such that

\[
\int_M |B|^{2q}|\nabla |B||^2 \phi^2 * 1 \leq C_2 \int_M |B|^{2+2q}|(\nabla \phi|^2 + \lambda_1 \phi^2 + |H|^2 \phi^2) * 1 \\
+ C_2 \int_M \lambda_2 |B|^{1+2q} \phi^2 * 1 + C_2 \int_M |\nabla H|^2|B|^{2q} \phi^2 \tag{4.8}
\]

where $C_2$ only depends on $n$, $\lambda$ and $q$. 
Combining with (4.1) and (4.8), we can derive
\[
\int_{\mathcal{M}} |B|^{4+2q}\phi^2 * 1 \leq C_3(n, \lambda, q) \int_{\mathcal{M}} |B|^{2+2q}(|\nabla \phi|^2 + \lambda_1 \phi^2 + |H|^2 \phi^2) * 1 \\
+ C_3 \int_{\mathcal{M}} \lambda_2 |B|^{1+2q}\phi^2 * 1 + C_3 \int_{\mathcal{M}} |\nabla H|^2 |B|^{2q}\phi^2
\]
(4.9)
by using Young’s inequality again.

Replacing \(\phi\) by \(\phi^{q+2}\) in (4.8) yields
\[
\int_{\mathcal{M}} |B|^{4+2q}\phi^{4+2q} * 1 \leq C \int_{\mathcal{M}} |B|^{2+2q}\phi^{2+2q}(|\nabla \phi|^2 + \lambda_1 \phi^2 + |H|^2 \phi^2) * 1 \\
+ C \int_{\mathcal{M}} |B|^{1+2q}\phi^{1+2q}\lambda_2 \phi^3 * 1 + C \int_{\mathcal{M}} |B|^{2q}\phi^{2q} \nabla |H|^2 \phi^4 * 1,
\]
(4.10)
in what follows \(C\) may be different in different expressions which depending on \(n\), \(\lambda\) and \(q\).

By using Young’s inequality, namely for any positive real number \(\alpha, a, b, s, t\) with \(\frac{1}{s} + \frac{1}{t} = 1\)
\[
\frac{\alpha^s a^s}{s} + \frac{\alpha^{-t} b^t}{t} \geq ab,
\]
we have
\[
C |B|^{2+2q}\phi^{2+2q} |\nabla \phi|^2 \leq \varepsilon |B|^{4+2q}\phi^{4+2q} + \alpha_0 |\nabla \phi|^{4+2q},
\]
\[
C |B|^{2+2q}|H|^2 \leq \varepsilon |B|^{4+2q} + \alpha_1 |H|^{4+2q},
\]
\[
C |B|^{2+2q}\lambda_1 \leq \varepsilon |B|^{4+2q} + \alpha_2 \lambda_1^{2+q},
\]
\[
C |B|^{2q}|\nabla H|^2 \leq \varepsilon |B|^{4+2q} + \alpha_3 |\nabla H|^2 + \lambda_2
\]
Finally, (4.10) becomes
\[
\int_{\mathcal{M}} |B|^{4+2q}\phi^{4+2q} * 1 \\
\leq C \int_{\mathcal{M}} |\nabla \phi|^{4+2q} * 1 \\
+ C \int_{\mathcal{M}} \left(|H|^{1+2q} + |\nabla H|^{2+2q} + \lambda_1^{2+2q} + \lambda_2^{4+2q}\right) \phi^{4+2q} * 1.
\]
(4.11)

Let \(r\) be a function on \(\mathcal{M}\) with \(|\nabla r| \leq 1\). For any \(R \in [0, R_0]\), where \(R_0 = \sup_{\mathcal{M}} r\), suppose
\[
M_R = \{x \in \mathcal{M}, \ r \leq R\}
\]
is compact.

(4.7) and (4.11) enable us to prove the following results.
Theorem 4.1. Let $M$ be an $n$-dimensional submanifolds of $\mathbb{R}^{n+m}$ with mean curvature $H$. If the Gauss image of $M_{2R}$ is contained in an open geodesic ball of radius $\frac{\sqrt{2}}{4}\pi$ in $G_{n,m}$, then we have the $L^p$-estimate

$$
\|B\|_{L^p(M_{2R})} \leq C R^{-1} \text{Vol}(M_{2R})^{\frac{1}{p}}
$$

for

$$
p \in \left[ 4, 4 + \frac{2}{3} + \frac{4}{3} \sqrt{1 + \frac{6}{n}} \right],
$$

where $C$ is depending on $n, R \sup_{M_{2R}} |H|, R^2 \sup_{M_{2R}} (|\nabla H| + \lambda_1)$ and $R^3 \sup_{M_{2R}} \lambda_2$.

Proof. Take $\phi \in C^\infty(M_R)$ to be the standard cut-off function such that $\phi \equiv 1$ in $M_R$ and $|\nabla \phi| \leq CR^{-1}$; then (4.11) yields

$$\int_{M_R} |B|^p \leq C R^{-p} \text{Vol}(M_{2R}),$$

where $p = 4 + 2q$. Thus the conclusion immediately follows from (4.11). \hfill \Box

Theorem 4.2. Let $M$ be an $n$-dimensional submanifolds of $\mathbb{R}^{n+m}$ with the mean curvature $H$. If the Gauss image of $M_{2R}$ is contained in $\{ P \in U \subset G_{n,m} : v(P) < 2 \}$, then we have the estimate

$$
\|B\|_{L^p(M_{2R})} \leq C R^{-1} \text{Vol}(M_{2R})^{\frac{1}{p}}
$$

for

$$
p \in \left[ 4, 4 + \sqrt{\frac{8}{n}} \right],
$$

where $C$ is depending on $n, R \sup_{M_{2R}} |H|, R^2 \sup_{M_{2R}} (|\nabla H| + \lambda_1)$ and $R^3 \sup_{M_{2R}} \lambda_2$.

We now study the $L^p$–curvature estimates in terms of $h_1$. From (2.12) and (3.13) we compute

$$
\Delta(|B|^{2p} h_1^q) \geq (\mu q - 3p - 2\varepsilon' p)|B|^{2p+2} h_1^q
$$

$$
+ 2p \left( 2p - 1 + \frac{2}{n + \varepsilon} \right) |B|^{2p-2} h_1^q |\nabla |B||^2
$$

$$
+ q \left( q + \frac{1}{2} \right) |B|^{2p} h_1^{q-2} |\nabla h_1|^2 + 4pq |B|^{2p-1} |\nabla h_1| \cdot h_1^{q-1} \nabla h_1
$$

$$
- \frac{2p}{\varepsilon'} |H|^2 |B|^{2p-1} h_1^q + q\mu |B|^{2p} h_1^q
$$

$$
- 2p \lambda_2 |B|^{2p-1} h_1^q - 2p C(n, \varepsilon) |B|^{2p-2} h_1^q |\nabla H|^2.
$$

Using Young’s inequality, when $p \geq \frac{1}{2} - \frac{1}{n} + (1 - \frac{2}{n}) q$, we obtain

$$
\Delta(|B|^{2p} h_1^q) \geq \left( \mu q - 3p - 2\varepsilon' p \right) |B|^{2p+2} h_1^q
$$

$$
- \frac{2p}{\varepsilon'} |H|^2 |B|^{2p-1} h_1^q + 2pC(n, \varepsilon) |B|^{2p-2} h_1^q |\nabla H|^2.
$$
In particular, we have when $p \geq n - 1$

\begin{equation}
\Delta (|B|^{p-1}h_1^p) \geq \frac{3}{2} |B|^{p+1}h_1^p - \frac{p-1}{\varepsilon'} |H|^2 |B|^{p-1}h_1^p - \frac{p}{2} \mu_1 |B|^{p-1}h_1^p
\end{equation}

\begin{equation}
-(p-1) \lambda_2 |B|^{p-2}h_1^p - (p-1)C(n, \varepsilon)|B|^{p-3}h_1^p |\nabla H|^2.
\end{equation}

Multiplying $|B|^{p-1}h_1^p \eta^{2p}$, integrating by parts and using Young’s inequality lead to

\begin{equation}
\int_M |B|^{2p}h_1^p \eta^{2p} \leq \frac{2}{3} b^2 \int_M |B|^{2p-2}h_1^p \eta^{2p-2} |\nabla \eta|^2 \ast 1
\end{equation}

\begin{equation}
+ \frac{2(p-1)}{3\varepsilon'} \int_M |H|^2 |B|^{2p-2}h_1^p \eta^{2p} \ast 1 + \frac{p}{3} \int_M \mu_1 |B|^{2p-2}h_1^p \eta^{2p} \ast 1
\end{equation}

\begin{equation}
+ \frac{2}{3} (p-1) \int_M \lambda_2 |B|^{2p-3}h_1^p \eta^{2p} \ast 1
\end{equation}

\begin{equation}
+ \frac{2}{3} (p-1)C(n, \varepsilon) \int_M |B|^{2p-4}h_1^p |\nabla H|^2 \eta^{2p} \ast 1,
\end{equation}

where $\eta$ is a smooth function with compact support. By using Young’s inequality again, we obtain

\begin{equation}
\int_M |B|^{2p}h_1^p \eta^{2p} \ast 1 \leq C \int_M h_1^p |\nabla \eta|^{2p} \ast 1
\end{equation}

\begin{equation}
+ C \int_M \left( |H|^{2p} + \mu_1^p + \lambda_2^{2p} + |\nabla H|^p \right) h_1^p \eta^{2p} \ast 1,
\end{equation}

where $C$ is a constant depending on $p$ and $n$. Take $\eta \in C_\infty^\infty(D_{2R}(x))$ to be the standard cut-off function such that $\eta \equiv 1$ in $D_R$ and $|\nabla \eta| \leq CR^{-1}$; then from (4.18) we have the following estimate.

**Theorem 4.3.** Let $M$ be an $n$-dimensional minimal submanifold of $\mathbb{R}^{n+m}$. If there exists a positive function $h_1$ on $M$ satisfying (3.13), then

\begin{equation}
\int_M |B|^{2p}h_1^p \eta^{2p} \ast 1 \leq CR^{n-2p} \sup_{D_{2R}} h_1^p
\end{equation}

where $C$ depends on $p, n, m, R \sup_{D_{2R}} h_1^p |H|, R^2 \sup_{D_{2R}} (\mu_1 + |\nabla H|), R^3 \sup_{D_{2R}} \lambda_2$.

### 5. Proof of the main results

From $L^p$-estimates to pointwise estimates we need the following mean value inequality in [3]:

**Lemma 5.1.** Let $S$ be an $n$-graph in $\mathbb{R}^{m+n}$. Suppose that $u$ is a nonnegative solution of

\begin{equation}
\Delta u + Qu \geq g \quad \text{on} \ S
\end{equation}
where $Q \in L^{q'}(S)$ and $g \in L^{q'}(S)$ with $q', p' > n$. If $D_{2R}(x) \subset S$, then we have the estimates

\begin{equation}
\sup_{D_{2R}(x)} u \leq C \left(R^{-\frac{n}{2}}||u||_{L^2(D_{2R}(x))} + k(R)\right),
\end{equation}

where

\begin{equation}
k(R) = R^{2\left(1-\frac{n}{p'}\right)}||g||_{L^{p'}\left(D_{2R}(x)\right)},
\end{equation}

the constants $C$ depending on $n$, $p'$, $q'$, $R^{2\left(1-\frac{n}{p'}\right)}||Q||_{L^{q'}(D_{2R}(x))}$, $R\sup_{D_{2R}(x)}|H|$ and $R^{-n}\text{Vol}(D_{2R}(x))$.

**Proof of Theorem 1.6**

Now, noting (2.13) we use Lemma 5.1 with $u = |B|^2$, $Q = (3 + 2\sqrt{n})|B|^2$, $g = -2\lambda_2|B|$ and $p' = 2q' = 2p$.

Using Theorem 4.1 and Theorem 4.2 have

\begin{equation}
||Q||_{L^{q'}\left(D_{2R}(x)\right)} = (3 + 2\sqrt{n}) \left(\int_{D_{2R}(x)}|B|^p * 1\right)^{\frac{1}{p}} \leq C R^{-2}\text{Vol}(D_{2R}(x))\frac{2}{p},
\end{equation}

\begin{equation}R^{2\left(1-\frac{n}{p'}\right)}||Q||_{L^{q'}\left(D_{2R}(x)\right)} \leq C R^{-\frac{2n}{p'}}\text{Vol}(D_{2R}(x))\frac{2}{p},
\end{equation}

\begin{equation}||g||_{L^{p'}\left(D_{2R}(x)\right)} = ||g||_{L^p(D_{2R}(x))} \leq C R^{-3} \left(\int_{D_{2R}(x)}|B|^p * 1\right)^{\frac{1}{p}} \leq C R^{-4}\text{Vol}(D_{2R}(x))\frac{1}{p},
\end{equation}

\begin{equation}k(R) = R^{2\left(1-\frac{n}{p'}\right)}||g||_{L^p(D_{2R}(x))} \leq C R^{-2} R^{-\frac{n}{p'}}\text{Vol}(D_{2R}(x))\frac{1}{p},
\end{equation}

\begin{equation}||u||_{L^2(D_{2R}(x))} = \left(\int_{D_{2R}(x)}|B|^4 * 1\right)^{\frac{1}{2}} \leq CR^{-2}\text{Vol}(D_{2R}(x))^{\frac{1}{2}},
\end{equation}

\begin{equation}R^{-\frac{n}{2}}||u||_{L^2(D_{2R}(x))} \leq C R^{-2} R^{-\frac{n}{2}}\text{Vol}(D_{2R}(x))^{\frac{1}{2}},
\end{equation}

Hence,

\begin{equation}\sup_{D_{2R}(x)} u \leq C R^{-2-\frac{n}{4}}\text{Vol}(D_{2R}(x))^{\frac{1}{2}}.
\end{equation}

In §3 we have shown the volume growth under our Gauss image assumption. We then finish the proof Theorem 1.6.
Proof of Theorem 1.7

From (4.15) we also have (in case of $p \geq n - 1$)
\begin{equation}
\Delta(|B|^{2p} h_1^p) \geq -\frac{2p}{\varepsilon'} |H|^2 |B|^{2p} h_1^p - p \mu_1 |B|^{2p} h_1^p - 2p \lambda_2 |B|^{2p} - 2p C(n, \varepsilon) |B|^{2p} \nabla H^2.
\end{equation}

By Young’s inequality
\begin{equation}
|B|^{2p-1} h_1^p \lambda_2 = |B|^{2p-1} h_1^p \frac{2p-1}{p} \lambda_2 \frac{2p-1}{p} h_1^p \frac{2p-1}{p} \lambda_2 \frac{2p-1}{p} h_1^p
\end{equation}
\begin{equation}
\leq C_1 |B|^{2p} h_1^p \lambda_2^2 + C_2 h_1^p \lambda_2 \frac{2p+1}{3},
\end{equation}
\begin{align*}
|B|^{2p-2} h_1^p \nabla H^2 &= |B|^{2p-2} h_1^p \nabla H |h_1| \nabla H| \\
&\leq C_3 |B|^{2p} h_1^p \nabla H |h_1|^{\frac{p}{p-1}} + C_4 h_1^p |\nabla H|^p.
\end{align*}

Then (5.6) becomes
\begin{equation}
\Delta(|B|^{2p} h_1^p) + C(n, p) (|H|^2 + \mu_1 + \lambda_2^2 + |\nabla H|^\frac{p}{p-1}) |B|^{2p} h_1^p
\end{equation}
\begin{equation}
\geq -C(n, p) (\lambda_2 \frac{2(p+1)}{3} h_1^p + |\nabla H|^p h_1^p).
\end{equation}

We use Lemma 5.1 with
\begin{align*}
u &= |B|^{2p} h_1^p, \quad Q = C(n, p) (|H|^2 + \mu_1 + \lambda_2^2 + |\nabla H|^\frac{p}{p-1}), \\
g &= -C(n, p) (\lambda_2 \frac{2(p+1)}{3} h_1^p + |\nabla H|^p h_1^p) \quad \text{and} \quad p' = 2q' = 2p > 2n.
\end{align*}

Since
\begin{equation}
\left( \int_{D_2 R(x)} |H|^{p} \overset{1}{\ast} 1 \right)^{\frac{p}{p'}} \leq CR^{\frac{2n}{p} - 2}, \quad \left( \int_{D_2 R(x)} \mu_1 \overset{1}{\ast} 1 \right)^{\frac{p}{p'}} \leq CR^{\frac{2n}{p} - 2}, \\
\left( \int_{D_2 R(x)} \lambda_2 \overset{1}{\ast} 1 \right)^{\frac{p}{p'}} \leq CR^{\frac{2n}{p} - 2}, \quad \left( \int_{D_2 R(x)} |\nabla H|^{p} \overset{1}{\ast} 1 \right)^{\frac{p}{p'}} \leq CR^{\frac{2n}{p} - 2},
\end{equation}
we obtain
\begin{equation}
R^{2(1 - \frac{p}{p'})} ||Q||_{L^p(D_2 R(x))} \leq C,
\end{equation}
where the constant $C$ depending on $n, m, p, R \sup_{D_2 R(x)} |H|, R^2 \sup_{D_2 R(x)} (|\nabla H| + \mu_1)$ and $R^3 \sup_{D_2 R(x)} \lambda_2$. We also have
\begin{equation}
\left( \int_{D_2 R(x)} \left( \lambda_2 \frac{2(p+1)}{3} h_1^p \right) \overset{1}{\ast} 1 \right)^{\frac{p}{p'}} \leq CR^{\frac{2n}{p} - p - 1} \sup_{D_2 R(x)} h_1^p
\end{equation}
\begin{equation}
R^{2(1 - \frac{p}{p'})} ||\lambda_2 \frac{2(p+1)}{3} h_1^p||_{L^p(D_2 R(x))} \leq CR^{p + 1} \sup_{D_2 R(x)} h_1^p
\end{equation}
and

\[(5.11) \quad \left( \int_{D_{2R}(x)} (|\nabla H|^p h_1^p)^p * 1 \right)^{\frac{1}{p}} \leq C R^{-\frac{p}{2} - 2p} \sup_{D_{2R}(x)} h_1^p,\]

\[(5.12) \quad R^{2(1 - \frac{p}{2})} \|||\nabla H|^p h_1^p||_{L^p(D_{2R}(x))} \leq CR^{-2p + 2} \sup_{D_{2R}(x)} h_1^p.\]

It follows that

\[k(R) \leq CR^{-p+1} \sup_{D_{2R}(x)} h_1^p.\]

From Theorem 4.3 we have

\[\left( \int_{D_{2r}(x)} |B|^{4p} h_1^{2p} * 1 \right)^{\frac{1}{2}} \leq CR^{-\frac{p}{2} - 2p} \sup_{D_{2R}(x)} h_1^p\]

and

\[R^{-\frac{p}{2}} \|||B|^{2p} h_1^p||_{L^2(D_{2R}(x))} \leq R^{-2p} \sup_{D_{2R}(x)} h_1^p.\]

In the case of Gauss image is contained in a geodesic ball of radius \(\sqrt{\frac{2}{4}}\)

\[h_1 = \sec^2(\sqrt{2} \rho).\]

It is easily seen that

\[\sec(\sqrt{2} \rho) \leq C \left( \frac{\sqrt{2}}{4} - \rho \right)^{-1}\]

for the constant \(C > 0\).

In the case of \(\Delta_f < 2\)

\[h_1 = \left( \frac{v}{2 - v} \right)^{-\left(\frac{3}{2} + \frac{1}{v}\right)} \leq C(2 - v)^{-\left(\frac{3}{2} + \frac{1}{v}\right)}\]

with the constant \(C > 0\). We then finish the proof of Theorem 1.7.

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