FRACTIONAL GAGLIARDO-NIRENBERG INTERPOLATION INEQUALITY AND BOUNDED MEAN OSCILLATION

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Abstract. We prove Gagliardo–Nirenberg interpolation inequalities estimating the Sobolev semi-norm in terms of the bounded mean oscillation semi-norm and of a Sobolev semi-norm, with some of the Sobolev semi-norms having fractional order.

1. Introduction

The homogeneous Gagliardo–Nirenberg interpolation inequality for Sobolev space states that if $d \in \mathbb{N} \setminus \{0\}$ and if $0 \leq s_0 < s < s_1$, $1 \leq p, p_0, p_1 \leq \infty$ and $0 < \theta < 1$ fulfill the condition
\[
(s, \frac{1}{p}) = (1 - \theta)(s_0, \frac{1}{p_0}) + \theta(s_1, \frac{1}{p_1}),
\]
then, for every function $f \in \dot{W}^{s_0, p_0}(\mathbb{R}^d) \cap \dot{W}^{s_1, p_1}(\mathbb{R}^d)$, one has $f \in \dot{W}^{s, p}(\mathbb{R}^d)$, and
\[
\|f\|_{\dot{W}^{s, p}(\mathbb{R}^d)} \leq C\|f\|_{\dot{W}^{s_0, p_0}(\mathbb{R}^d)}^{1-\theta}\|f\|_{\dot{W}^{s_1, p_1}(\mathbb{R}^d)}^\theta,
\]
unless $s_1$ is an integer, $p_1 = 1$ and $s_1 - s_0 \leq 1 - \frac{1}{p_0}$.

When $s = 0$, we use the convention that $\dot{W}^{0, p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$, and when $s \in \mathbb{N} \setminus \{0\}$ is a positive integer, $\dot{W}^{s, p}(\mathbb{R}^d)$ is the classical integer-order homogeneous Sobolev space of $s$ times weakly differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $D^s f \in L^p(\mathbb{R}^d)$ and
\[
\|f\|_{\dot{W}^{s, p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |D^s f|^p \right)^{\frac{1}{p}}.
\]

For $s_0, s_1, s \in \mathbb{N}$ the inequality (1.2) was proved by Gagliardo [15] and Nirenberg [26] (see also [14]).

When $s \not\in \mathbb{N}$, the homogeneous fractional Sobolev–Slobodeckiĭ space $\dot{W}^{s, p}(\mathbb{R}^d)$ can be defined as the set of measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ which are $k$ times weakly differentiable with a finite Gagliardo semi-norm:
\[
\|f\|_{\dot{W}^{s, p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D^k f(y) - D^k f(x)|^p}{|y-x|^{d + \sigma p}} \, dy \, dx\right)^{\frac{1}{p}} < \infty,
\]

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with \( k \in \mathbb{N} \), \( \sigma \in (0,1) \) and \( s = k + \sigma \); the characterisation of the range in which the Gagliardo–Nirenberg interpolation inequality (1.2) holds was performed in a series of works \([11,9,11]\) up to the final complete settlement by Brezis and Mironescu \([5]\).

We focus on the endpoint case where \( s_0 = 0 \) and \( p_0 = \infty \). In this case, the inequality (1.2) becomes

\[
\|f\|_{W^{s,p}(\mathbb{R}^d)} \leq C \|f\|_{L^{s+1,p_1}(\mathbb{R}^d)}^{p_1-p}\|f\|_{W^{s+1,p_1}(\mathbb{R}^d)}^{p_1},
\]

and holds under the assumption that \( sp = s_1p_1 \) and either \( s_1 \neq 1 \) or \( p_1 > 1 \). It is natural to ask whether the inequality (1.5) can be strengthened by replacing the uniform norm \( \|\cdot\|_{L^{s+1,p_1}(\mathbb{R}^d)} \) by John and Nirenberg’s bounded mean oscillation (BMO) semi-norm \( \|\cdot\|_{\text{BMO}(\mathbb{R}^d)} \), which plays an important role in harmonic analysis, calculus of variations and partial differential equations \([18]\), that is, whether we have the inequality

\[
\|f\|_{W^{s,p}(\mathbb{R}^d)} \leq C \|f\|_{\text{BMO}(\mathbb{R}^d)}^{p_1-p}\|f\|_{W^{s_1,p_1}(\mathbb{R}^d)}^{p_1},
\]

where the bounded mean oscillation semi-norm \( \|\cdot\|_{\text{BMO}(\mathbb{R}^d)} \) is defined for any measurable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) as

\[
\|f\|_{\text{BMO}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(z)| \, dy \, dz.
\]

The estimate (1.6) was proved indeed when \( s = 1, p = 4, s_1 = 2 \) and \( p_1 = 2 \) via a Littlewood–Paley decomposition by Meyer and Rivière \([24]\) theorem 1.4, and for \( s, s_1 \in \mathbb{N} \) via the duality between \( \text{BMO}(\mathbb{R}^d) \) and the real Hardy space \( \mathcal{H}^1(\mathbb{R}^d) \) by Strezelecki \([28]\); a direct proof was been given recently by Miyazaki \([25]\) (in the limiting case \( s_0 = s_1 = 0 \), see \([8,21]\) theorem 2.2)); when \( s_1 < 1 \), the estimate (1.6) has been proved by Brezis and Mironescu through a Littlewood–Paley decomposition \([6\text{, lemma 15.7}]\) (see also \([2,20]\) for similar estimates in Riesz potential spaces).

The main result (theorem 1) of the present work is the estimate (1.6) when \( s_1 = 1 \) and \( 0 < s < 1 \), with a proof which is quite elementary: the main analytical tool is the classical maximal function theorem. We also show how the same ideas can be used to give a direct proof of (1.6) when \( s_1 < 1 \), depending only on the definitions of the Gagliardo and bounded mean oscillation semi-norms (theorem 7). Finally, we show how a last interpolation result (theorem 10) allows one to obtain the full range of interpolation between \( \text{BMO}(\mathbb{R}^d) \) and higher-order fractional Sobolev–Slobodeckii spaces \( W^{s,p}(\mathbb{R}^d) \) with \( s \in (1,\infty) \).

Our proofs can be considered as fractional counterparts of Miyazaki’s direct proof in the integer-order case \([25]\). We also refer to Dao’s recent work \([12]\) for an alternative approach via negative-order Besov spaces to the results in the present paper.

## 2. Interpolation between first-order Sobolev semi-norm and mean oscillation

We prove the following interpolation inequality between the first-order Sobolev semi-norm and the mean oscillation seminorm into fractional Sobolev spaces.

\[
\|f\|_{W^{s,p}(\mathbb{R}^d)} \leq C \|f\|_{\text{BMO}(\mathbb{R}^d)}^{p_1-p}\|f\|_{W^{s_1,p_1}(\mathbb{R}^d)}^{p_1},
\]
Theorem 1. For every $d \in \mathbb{N} \setminus \{0\}$ and every $p \in (1, \infty)$, there exists a constant $C(p) > 0$ such that for every $s \in (1/p, 1)$, every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\varkappa(\Omega) < \infty$ and every function $f \in W^{1,sp}(\Omega) \cap \text{BMO}(\Omega)$, one has $f \in W^{s,p}(\Omega)$ and

\[
\|f\|_{W^{s,p}(\Omega)} \leq C(p)\varkappa(\Omega)^{sp} \left(\frac{sp}{sp - 1} \right)^{s(p - 1)} \|f\|_{\text{BMO}(\Omega)} \int_{\Omega} |Df|^p.
\]

We define here for a domain $\Omega \subseteq \mathbb{R}^d$, the bounded mean oscillation semi-norm of a measurable function $f : \Omega \to \mathbb{R}$ as

\[
\|f\|_{\text{BMO}(\Omega)} := \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |f(y) - f(z)| \, dy \, dz,
\]

and the geometric quantity

\[
\varkappa(\Omega) := \sup \left\{ \frac{\mathcal{L}^d(B_r(x))}{\mathcal{L}^d(\Omega \cap B_r(x))} \mid x \in \Omega \text{ and } r \in (0, \text{diam}(\Omega)) \right\}.
\]

For the latter quantity, one has for example

\[
\varkappa(\mathbb{R}^d) = 1
\]

and

\[
\varkappa((\mathbb{R}^d)^+) = 2.
\]

If the set $\Omega$ is convex and bounded, we have $\Omega \subseteq B_{\text{diam}(\Omega)}(x)$ and $t\Omega + (1-t)x \subseteq \Omega \cap B_r(x)$, with $t := r/\text{diam}(\Omega)$, so that

\[
\mathcal{L}^d(\Omega \cap B_r(x)) \geq t^d \mathcal{L}^d(\Omega) = \frac{\mathcal{L}^d(\Omega)}{\text{diam}(\Omega)^d} r^d,
\]

and thus

\[
\varkappa(\Omega) \leq \frac{\mathcal{L}^d(B_1)}{\mathcal{L}^d(\Omega)} \text{diam}(\Omega)^d.
\]

The quantity $\varkappa(\Omega)$ can be infinite for some unbounded convex sets such as $\Omega = (0,1) \times \mathbb{R}^{d-1}$ and $\Omega = \{(x', x_d) \in \mathbb{R}^d \mid x_d \geq |x'|^2\}$.

Our first tool to prove theorem 1 is an estimate by the maximal function of the average distance of values on a ball to a fixed value; this formula is related to the Lusin–Lipschitz inequality [11] lemma II.1; [3] p. 404; [17] (3.3); [21] lemma 2.

Lemma 2. If the set $\Omega \subseteq \mathbb{R}^d$ is open and convex and if $f \in W^{1,1}_{\text{loc}}(\Omega)$, then for every $r \in (0, \text{diam}(\Omega))$ and almost every $x \in \Omega$,

\[
\int_{\Omega \cap B_r(x)} |f(z) - f(x)| \, dz \leq \varkappa(\Omega) r \mathcal{M}|Df|(x).
\]

Here $\mathcal{M}g : \mathbb{R}^d \to [0, +\infty]$ denotes the classical Hardy–Littlewood maximal function of the function $g : \Omega \to \mathbb{R}$, defined for each $x \in \mathbb{R}^d$ by

\[
\mathcal{M}g(x) := \sup_{r > 0} \frac{1}{\mathcal{L}^d(B_r(x))} \int_{\Omega \cap B_r(x)} |g|.
\]
Proof of lemma 2. Since $\Omega$ is convex and $f \in \dot{W}^{1,1}(\Omega)$, for almost every $x \in \Omega$ and every $r \in (0, \infty)$, we have

\begin{equation}
\int_{\Omega \cap B_r(x)} |f(z) - f(x)| \, dz \leq \int_{\Omega \cap B_r(x)} \int_0^1 |Df((1-t)x + tz)| \, dt \, dz.
\end{equation}

By convexity of the set $\Omega$, for every $z \in \Omega \cap B_r(x)$ and $t \in [0,1]$ we have $(1-t)x + tz \in \Omega \cap B_{tr}(x)$. We deduce from (2.9) and (2.10) through the change of variable $y = (1-t)x + tz$ that

\begin{equation}
\int_{\Omega \cap B_r(x)} |f(z) - f(x)| \, dz \leq \int_0^1 \int_{\Omega \cap B_r(x)} \frac{|Df(y)[y-x]|}{t^{d+1}} \, dy \, dt
\end{equation}

in view of the definition (2.2) of the maximal function, and the conclusion (2.7) then follows from the definition of the geometric quantity $\varkappa(\Omega)$ in (2.3). \qed

Our second tool to prove theorem 1 is the following property of averages of functions of bounded mean oscillation (see [7, §3]).

Lemma 3. If the set $\Omega \subseteq \mathbb{R}^d$ is open and convex, if $f \in \text{BMO}(\Omega)$ and if $r_0 < r_1$, then

\begin{equation}
\int_{\Omega \cap B_{r_0}(x)} \int_{\Omega \cap B_{r_1}(x)} |f(y) - f(z)| \, dy \, dz \leq e(1 + d \ln(r_1/r_0)) \|f\|_{\text{BMO}(\Omega)}.
\end{equation}

In (2.11), $e$ denotes Euler’s number.

The proof of lemma 3 will use the following triangle inequality for averages

Lemma 4. Let $\Omega \subseteq \mathbb{R}^d$. If the function $f : \Omega \to \mathbb{R}$ is measurable, and the sets $A, B, C \subseteq \mathbb{R}^d$ are measurable and have positive measure, then

\[ \int_A \int_B |f(y) - f(x)| \, dy \, dx \leq \int_A \int_C |f(z) - f(x)| \, dz \, dx + \int_B \int_C |f(y) - f(z)| \, dy \, dz. \]

Proof. We have successively, in view of the triangle inequality,

\[ \int_A \int_B |f(y) - f(x)| \, dy \, dx = \int_A \int_B \int_C |f(y) - f(x)| \, dz \, dy \, dx \]

\[ \leq \int_A \int_B |f(z) - f(x)| + |f(y) - f(z)| \, dz \, dy \, dx \]

\[ = \int_A \int_C |f(z) - f(x)| \, dz \, dx + \int_B \int_C |f(y) - f(z)| \, dy \, dz. \] \qed
Proof of lemma. We first note that since $r_1 > r_0$, we have in view of (2.2)
\[
\int_{\Omega \cap B_{r_0}(x)} \int_{\Omega \cap B_{r_1}(x)} |f(y) - f(z)| \, dy\, dz
\leq \frac{\mathcal{L}^d(\Omega \cap B_{r_1}(x))}{\mathcal{L}^d(\Omega \cap B_{r_0}(x))} \int_{\Omega \cap B_{r_1}(x)} \int_{\Omega \cap B_{r_1}(x)} |f(y) - f(z)| \, dy\, dz
\leq \left( \frac{r_1}{r_0} \right)^d \|f\|_{BMO(\Omega)},
\]
since by convexity $r_0/r_1(\Omega \cap B_{r_1}(x)) \subseteq \Omega \cap B_{r_0}(x)$ and thus $\mathcal{L}^d(\Omega \cap B_{r_1}(x))/r_1^d \leq \mathcal{L}^d(\Omega \cap B_{r_0}(x))/r_0^d$. Applying $k \in \mathbb{N} \setminus \{0\}$ times the inequality (2.12), we get thanks to the triangle inequality for mean oscillation of lemma
\[
\int_{\Omega \cap B_{r_0}(x)} \int_{\Omega \cap B_{r_1}(x)} |f(y) - f(z)| \, dy\, dz
\leq \sum_{j=0}^{k-1} \int_{\Omega \cap B_{r_0(r_1/r_0)^j/k}(x)} \int_{\Omega \cap B_{r_0(r_1/r_0)^j/k}(x)} |f(y) - f(z)| \, dy\, dz
\leq k \left( \frac{r_1}{r_0} \right)^{d/k} \|f\|_{BMO(\Omega)}.
\]
Taking $k \in \mathbb{N} \setminus \{0\}$ such that $k - 1 < d \ln(r_1/r_0) \leq k$, we obtain the conclusion (2.11).

Our last tool to prove theorem is the following integral identity.

Lemma 5. For every $p \in (1, \infty)$ and $\alpha \in (0, \infty)$, one has
\[
\int_1^\infty \frac{(\ln r)^p}{r^{1+\alpha}} \, dr = \frac{\Gamma(p+1)}{\alpha^{p+1}}.
\]
Proof. One performs the change of variable $r = \exp(t/\alpha)$ in the left-hand side integral and uses the classical integral definition of the Gamma function. 

We now proceed to the proof of theorem.

Proof of theorem. For every $x, y \in \Omega$, we have by the triangle inequality and the domain monotonicity of the integral
\[
|f(y) - f(x)| \leq \int_{\Omega \cap B_{|y-x|/2}(\frac{d+p}{d})} |f(y) - f| + \int_{\Omega \cap B_{|y-x|/2}(\frac{d+p}{d})} |f - f(x)|
\leq 2^d \int_{\Omega \cap B_{|y-x|}(y)} |f(y) - f| + 2^d \int_{\Omega \cap B_{|y-x|}(x)} |f - f(x)|,
\]
since by convexity $\Omega \cap B_{|y-x|/2(\frac{d+p}{d})} \subseteq \frac{1}{2}(B_{|y-x|}(x) \cap \Omega) + \frac{d}{2}$. It follows thus from (2.11) by integration and by symmetry that
\[
\iint_{\Omega \times \Omega} \frac{|f(y) - f(x)|^p}{|y-x|^{d+sp}} \, dy\, dx \leq C_1 \int_{\Omega \times \Omega} \left( \int_{\Omega \cap B_{|y-x|}(x)} |f - f(x)| \right)^p \frac{dy\, dx}{|y-x|^{d+sp}}
\leq C_2 \int_0^{\text{diam} \Omega} \int_0^{\text{diam} \Omega} \left( \int_{\Omega \cap B_{r}(x)} |f - f(x)| \right)^p \frac{dr\, dx}{r^{1+sp}}.
\]
If \( \rho \in (0, \text{diam}(\Omega)) \), we first have by lemma [2] for almost every \( x \in \Omega \),

\[
\int_0^\rho \left( \int_{\Omega \cap B_r(x)} |f - f(x)|^p \right)^{1\sp p} \, dr \leq (\varkappa(\Omega) \mathcal{M}|Df|(x))^{p} \int_0^\rho r^{(1-s)p-1} \, dr \\
= \frac{\rho^{(1-s)p} (\varkappa(\Omega) \mathcal{M}|Df|(x))^p}{(1-s)p}.
\]

Next we have by the triangle inequality, by lemma [2] again and by lemma [3] for every \( r \in (\rho, \text{diam}(\Omega)) \),

\[
\int_{\Omega \cap B_r(x)} |f - f(x)| \leq \int_{\Omega \cap B_{\rho}(x)} |f - f(x)| + \int_{\Omega \cap B_{\rho}(x)} |f(y) - f(z)| \, dy \, dz \\
\leq (\rho \varkappa(\Omega) \mathcal{M}|Df|(x) + e(1 + d \ln(\rho / \rho))(\|f\|_{BMO(\Omega)}),
\]

and hence, integrating (2.17), we get

\[
\int_0^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{\rho}(x)} |f - f(x)| \right)^p \frac{dr}{r^{1+s p}} \\
\leq C_3 \left( \int_0^{\infty} \rho^p \mathcal{M}|Df|(x)^p \, dr \right. \\
\left. + \int_0^{\infty} \varkappa(\Omega)^p \|f\|_{BMO(\Omega)}^p \left(1 + d \ln(\rho / \rho)\right)^p \, dr \right) \\
\leq C_4 \left( \frac{\rho^{(1-s)p} \varkappa(\Omega)^p \mathcal{M}|Df|(x)^p}{s} + \frac{\Gamma(p + 1) \|f\|_{BMO(\Omega)}^p}{(sp)^{p+1} \rho^p} \right),
\]

in view of lemma [3]. Putting (2.16) and (2.18) together, we get, since \( sp > 1 \),

\[
\int_0^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{\rho}(x)} |f - f(x)| \right)^p \frac{dr}{r^{1+s p}} \\
\leq C_5 \left( \frac{\rho^{(1-s)p} \varkappa(\Omega)^p \mathcal{M}|Df|(x)^p}{1-s} + \frac{\|f\|_{BMO(\Omega)}^p}{\rho^{sp}} \right).
\]

If \( \|f\|_{BMO(\Omega)} \leq \text{diam}(\Omega) \varkappa(\Omega) \mathcal{M}|Df|(x) \), taking \( \rho := \|f\|_{BMO(\Omega)}/(\varkappa(\Omega) \mathcal{M}|Df|(x)) \) in (2.19), we obtain

\[
\int_0^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{\rho}(x)} |f - f(x)| \right)^p \frac{dr}{r^{1+s p}} \leq \frac{C_6}{1-s} (\varkappa(\Omega) \mathcal{M}|Df|(x))^{sp} \|f\|_{BMO(\Omega)}^{(1-s)p},
\]

otherwise we take \( \rho := \text{diam}(\Omega) \leq \|f\|_{BMO(\Omega)}/(\varkappa(\Omega) \mathcal{M}|Df|(x)) \) in (2.16) and also obtain (2.20). Integrating the inequality (2.20), we reach the conclusion (2.1) by the quantitative version of the classical maximal function theorem in \( L^p(\mathbb{R}^d) \) since \( sp > 1 \) (see for example [27, theorem 1.1]). \( \square \)

We conclude this section by pointing out that theorem [4] admits a localised version in terms of Fefferman and Stein’s sharp maximal function \( f^\sharp : \Omega \to [0, \infty] \) which is defined
for every $x \in \Omega$ (see [13, (4.1)]) as

$$f^*(x) := \sup_{r>0} \int_{\Omega \cap B_r(x)} \int_{\Omega \cap B_r(x)} |f(y) - f(z)| \, dy \, dz;$$

noting that the proof of lemma 3 yields in fact the estimate

$$\int_{\Omega \cap B_{r_0}(x)} \int_{\Omega \cap B_{r_1}(x)} |f(y) - f(z)| \, dy \, dz \leq e(1 + d\ln(r_1/r_0)) f^*(x)$$

and following then the proof of theorem 1, we reach the following local counterpart of

$$\text{(2.20)}$$

Proposition 6. For every $d \in \mathbb{N} \setminus \{0\}$ and for every $p \in (1, \infty)$, there exists a constant $C > 0$ such that for every $s \in (1/p, 1)$, for every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\nabla(\Omega) < \infty$, for every function $f \in W^{1,1}_{\text{loc}}(\Omega)$ and for almost every $x \in \Omega$, we have

$$\int_0^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_r(x)} |f - f(x)| \right)^p \frac{dr}{r^{1+sp}} \leq \frac{C}{1-s} (\nabla(\Omega))^{(1-s)p} (\nabla(\Omega) \mathcal{M}|Df|(x))^{sp}. \tag{2.23}$$

Proposition 6 is stronger than theorem 1 in the sense that the integration of the estimates (2.23) yields (2.1). Proposition 6 is a counterpart of the interpolation involving maximal and sharp maximal function of derivatives [22, (4)], which generalised a priori estimates in terms of maximal functions [19, 24, theorem 1]; proposition 6 generalises the corresponding result for integer-order Sobolev spaces [25, remark 2.2].

3. Interpolation between first-order Sobolev semi-norm and mean oscillation

We explain how the tools of the previous section can be used to prove the fractional BMO Gagliardo–Nirenberg interpolation inequality as presented by Brezis and Mironescu’s [6, lemma 15.7].

Theorem 7. For every $d \in \mathbb{N} \setminus \{0\}$, every $s, s_1 \in (0, 1)$ and every $p, p_1 \in (1, +\infty)$ satisfying $s < s_1$ and $s_1 p_1 = sp$, there exists a constant $C > 0$ such that for every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\nabla(\Omega) < \infty$ and for every function $f \in W^{s_1,p_1}(\Omega) \cap \text{BMO}(\Omega)$, one has $f \in W^{s,p}(\Omega)$ and

$$\int_{\Omega \times \Omega} \frac{|f(y) - f(x)|^p}{|y - x|^{d+sp}} \, dy \, dx \leq C \|f\|_{\text{BMO}(\Omega)}^{p-p_1} \nabla(\Omega)^{sp_1} \int_{\Omega \times \Omega} \frac{|f(y) - f(x)|^p}{|y - x|^{d+s_1p_1}} \, dy \, dx. \tag{3.1}$$

The proof of theorem 7 will follow essentially the proof of theorem 1, the main difference being the replacement of lemma 2 by its easier fractional counterpart.

Lemma 8. For every $p \in (1, \infty)$, there exists a constant $C > 0$ such that if the set $\Omega \subseteq \mathbb{R}^d$ is open and convex, if $s \in (0, 1)$ and if $f : \Omega \to \mathbb{R}$ is measurable, then for every $r \in (0, \text{diam}(\Omega))$ and every $x \in \Omega$,

$$\int_{\Omega \cap B_r(x)} |f - f(x)| \leq C \nabla(\Omega) r^s \left( \int_{\Omega} \frac{|f(y) - f(x)|^p}{|y - x|^{d+sp}} \, dy \right)^{\frac{1}{p}}. \tag{3.2}$$
Proof. By Hölder’s inequality we have for every \( r \in (0, \text{diam}(\Omega)) \) and for every \( x \in \Omega \),
\[
\int_{\Omega \cap B_r(x)} |f - f(x)| \leq \left( \int_{\Omega} \frac{|f(y) - f(x)|^{p}}{|y - x|^{d + sp}} \, dy \right)^{\frac{1}{p}} \left( \int_{B_r(x)} |y - x|^{\frac{d + sp}{p-1}} \, dy \right)^{1-\frac{1}{p}}.
\]
Noting that
\[
\int_{B_r(x)} |y - x|^{\frac{d + sp}{p-1}} \, dy = C_{7} \frac{p - 1}{d + sp} (r^{s} \mathcal{L}^{d}(B_r(x)))^{\frac{p-1}{p}} \leq C_{8} (r^{s} \mathcal{L}^{d}(B_r(x)))^{\frac{p-1}{p}},
\]
we reach the conclusion \( \text{(3.2)} \) thanks to the definition of the geometric quantity \( \mathcal{K}(\Omega) \) in \( \text{(2.18)} \).

Proof of theorem 7. We begin as in the proof of theorem 1. Instead of \( \text{(2.16)} \), we have
\[
\int_{0}^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{r}(x)} |f - f(x)| \right)^{\frac{p}{r^1 + sp}} \, dr \leq C_{9} \mathcal{K}(\Omega)^{\frac{p}{sp}} \left( \int_{\Omega} \frac{|f(y) - f(x)|^{p}}{|y - x|^{d + s_1 p_1}} \, dy \right)^{\frac{p}{p_1}} \int_{0}^{\text{diam}(\Omega)} r^{(s_1 - s)p - 1} \, dr.
\]
Next instead of \( \text{(2.18)} \), we have
\[
\int_{0}^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{r}(x)} |f - f(x)| \right)^{\frac{p}{r^1 + sp}} \, dr \leq C_{10} \frac{\mathcal{K}(\Omega)^{p(s_1 - s)p}}{sp} \left( \int_{\Omega} \frac{|f(y) - f(x)|^{p}}{|y - x|^{d + s_1 p_1}} \, dy \right)^{\frac{p}{p_1}} + \frac{\|f\|_{\text{BMO}(\Omega)}}{(sp)^{p+1} \mathcal{K}(\Omega)}.
\]
Taking \( \varrho \in (0, \text{diam}(\Omega)) \) such that
\[
\|f\|_{\text{BMO}(\Omega)}^{p} = \varrho^{s_1 p} \mathcal{K}(\Omega)^{p} \left( \int_{\Omega} \frac{|f(y) - f(x)|^{p}}{|y - x|^{d + s_1 p_1}} \, dy \right)^{\frac{p}{p_1}}
\]
if possible, and otherwise taking \( \varrho := \text{diam}(\Omega) \), we obtain, since \( s_1 p_1 = sp \), by \( \text{(3.5)} \), \( \text{(3.6)} \) and \( \text{(3.7)} \)
\[
\int_{0}^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_{r}(x)} |f - f(x)| \right)^{\frac{p}{r^1 + sp}} \, dr \leq C_{12} \|f\|_{\text{BMO}(\Omega)}^{p} \mathcal{K}(\Omega)^{p_1} \int_{\Omega} \frac{|f(y) - f(x)|^{p}}{|y - x|^{d + s_1 p_1}} \, dy.
\]
We conclude by integration of \( \text{(3.8)} \). □

As previously, we point out that the estimate \( \text{(3.8)} \) admits a localised version, which is the fractional counterpart of proposition 6.
convex set $\Omega$ one has $f \in \mathcal{E}$ satisfying $s < s_1$ for every $x \in \Omega$,

$$
\int_0^{\text{diam}(\Omega)} \left( \int_{\Omega \cap B_r(x)} |f - f(x)|^p \right)^{1/p} \frac{dr}{r^{1+sp}} \leq C \left( f^2(x) \right)^{p-p_1} \chi(\Omega)^{p_1} \int_{\Omega} \frac{|f(y) - f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \, dy.
$$

The estimate (3.11) can be seen as a consequence of the integration of (3.9).

4. Higher-order fractional spaces estimates

The last ingredient to obtain the full scale of Gagliardo–Nirenberg interpolation inequalities between fractional Sobolev–Slobodeckii spaces and the bounded mean oscillation space is the following estimate.

**Theorem 10.** For every $d \in \mathbb{N} \setminus \{0\}$, every $k_1 \in \mathbb{N} \setminus \{0\}$, every $\sigma_1 \in (0,1)$ and every $p, p_1 \in (1,\infty)$ satisfying

$$
k_1 p = (k_1 + \sigma_1)p_1,
$$

there exists a constant $C > 0$ such that for every function $f \in \dot{W}^{k_1,\sigma_1,p_1}(\mathbb{R}^d) \cap \text{BMO}(\mathbb{R}^d)$, one has $f \in \dot{W}^{k_1,p}(\mathbb{R}^d)$ and

$$
\int_{\mathbb{R}^d} |D^{k_1} f|^p \leq C \| f \|_{\dot{W}^{k_1,\sigma_1,p_1}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x-y|^{d_1 + \sigma_1 p_1}} \, dx \, dy.
$$

As a consequence of theorem 10, we have that $f \in \dot{W}^{k_1+p}(\mathbb{R}^d)$ whenever $k \in \mathbb{N}$, $\sigma \in [0,1)$ and $p \in (1,\infty)$ satisfy $k + \sigma < k_1 + \sigma_1$ and $(k + \sigma)p = (k_1 + \sigma_1)p_1$. Indeed for $\sigma = 0$ and $k = k_1$, this follows from theorem 10 and then for $k \in \{1, \ldots, k_1 - 1\}$ by the Gagliardo–Nirenberg interpolation inequality for integer-order Sobolev space [25, 28]; for $0 < \sigma < 1$ and $k = 0$ one then uses theorem 1 whereas for $0 < \sigma < 1$ and $k \in \mathbb{N} \setminus \{0\}$ one uses the classical fractional Gagliardo–Nirenberg interpolation inequality [6].

**Proof of theorem 10** Fixing a function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \eta = 1$ and supp $\eta \subseteq B_1$, we have for every $x \in \mathbb{R}^d$ and every $\varrho \in (0,\infty)$,

$$
D^{k_1} f(x) = \frac{1}{\varrho^d} \int_{\mathbb{R}^d} \eta \left( \frac{x-y}{\varrho} \right) \left( D^{k_1} f(x) - D^{k_1} f(y) \right) \, dy + \frac{1}{\varrho^d} \int_{\mathbb{R}^d} \eta \left( \frac{x-y}{\varrho} \right) D^{k_1} f(y) \, dy.
$$

We estimate the first term in the right-hand side of (4.3) by Hölder's inequality

$$
\left| \frac{1}{\varrho^d} \int_{\mathbb{R}^d} \eta \left( \frac{x-y}{\varrho} \right) \left( D^{k_1} f(x) - D^{k_1} f(y) \right) \, dy \right| \leq C_3 \varrho^d \left( \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x-y|^{d_1 + \sigma_1 p_1}} \, dx \right)^{\frac{1}{p_1}} \left( \int_{B_\varrho(x)} \frac{1}{|x-y|^{\frac{d_1 + \sigma_1 p_1}{p_1 - 1}}} \, dx \right)^{1 - \frac{1}{p_1}}.
$$

$$
\leq C_4 \varrho^{\sigma_1} \left( \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x-y|^{d_1 + \sigma_1 p_1}} \, dx \right)^{\frac{1}{p_1}}.
$$
For the second-term in the right-hand side of (4.3), for every $x \in \mathbb{R}^d$, we have by weak differentiability,

\begin{equation}
\left( 4.5 \right) \frac{1}{d + k_1} \int_{\mathbb{R}^d} \eta \left( \frac{x - y}{\bar{e}} \right) D^{k_1} f(y) \, dy = \frac{1}{d + k_1} \int_{\mathbb{R}^d} D^{k_1} \eta \left( \frac{x - y}{\bar{e}} \right) f(y) \, dy
\end{equation}

and thus by (4.5) and by definition of bounded mean oscillation (1.7), we have

\begin{equation}
\left( 4.6 \right) \left| \frac{1}{d + k_1} \int_{\mathbb{R}^d} \eta \left( \frac{x - y}{\bar{e}} \right) D^{k_1} f(y) \, dy \right| \leq \frac{C_{15}}{d + k_1 \| f \|_{\text{BMO}(\mathbb{R}^d)}}.
\end{equation}

Choosing $\bar{e} \in (0, \infty)$ such that

\begin{equation}
\left( 4.7 \right) \bar{e}^{k_1 + \sigma_1} \left( \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x - y|^{d + \sigma_1 p_1}} \, dy \right)^{\frac{1}{p_1}} = \| f \|_{\text{BMO}(\mathbb{R}^d)},
\end{equation}

we get from (4.3), (4.4) and (4.6), for every $x \in \mathbb{R}^d$,

\begin{equation}
\left( 4.8 \right) |D^{k_1} f(x)| \leq C_{16} \| f \|_{\text{BMO}(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x - y|^{d + \sigma_1 p_1}} \, dy \right)^{\frac{k_1}{(k_1 + \sigma_1) p_1}},
\end{equation}

and thus in view of the condition (4.1), the estimate (4.2) follows by integration. \hfill \square

**Proposition 11.** For every $d \in \mathbb{N} \setminus \{0\}$, every $k_1 \in \mathbb{N} \setminus \{0\}$, every $\sigma_1 \in (0, 1)$ and every $p_1 \in (1, \infty)$, there exists a constant $C > 0$ such that for every function $f \in W^{k_1, 1}_{\text{loc}}(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$,

\begin{equation}
\left( 4.9 \right) |D^{k_1} f(x)| \leq C \left( f^2(x) \right)^{\frac{k_1}{2(k_1 + \sigma_1)}} \left( \int_{\mathbb{R}^d} \frac{|D^{k_1} f(x) - D^{k_1} f(y)|^{p_1}}{|x - y|^{d + \sigma_1 p_1}} \, dy \right)^{\frac{k_1}{(k_1 + \sigma_1) p_1}}.
\end{equation}

As previously, the integration of (4.9) yields (4.2).

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