MULTI-ATOMS AND MONOTONICITY OF GENERALIZED KOSTKA POLYNOMIALS

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Abstract. The Poincaré polynomials of isotypic components of the graded $GL(n)$-modules given by twists by line bundles of coordinate rings of closures of conjugacy classes of nilpotent matrices, are $q$-analogues of the Littlewood-Richardson coefficients giving multiplicities in a tensor product of irreducible $GL(n)$-modules indexed by rectangular partitions. These polynomials are the $q$-enumeration of a set of Young tableaux (called LR tableaux) with a generalized charge statistic. These polynomials satisfy a monotonicity property which extends that of the Kostka-Foulkes. In this paper the monotonicity property is realized combinatorially by a directed system of embeddings of graded posets whose objects are sets of LR tableaux. The final object in this directed system is a weakened dual of the cyclage poset of Lascoux and Schützenberger on the set of column strict tableaux of a fixed content. For certain sequences of rectangles, the image of the embedding of a set of LR tableaux into column-strict tableaux, is shown to be a set of catabolizable tableaux, proving a conjecture of the author and J. Weyman.

1. Introduction

Let $X_\mu \subset gl(n, \mathbb{C})$ be the Zariski closure of the conjugacy class of the nilpotent $n \times n$ Jordan matrix with block sizes given by the parts of the transpose $\mu^t$ of the partition $\mu$ of $n$. $X_\mu$ is a subvariety of the affine space $gl(n, \mathbb{C})$, so there is a canonical surjection of coordinate rings $\mathbb{C}[x_{ij} : 1 \leq i, j \leq n] = C[gl(n, \mathbb{C})] \to \mathbb{C}[X_\mu]$. Moreover, $X_\mu$ is closed under multiplication by scalar matrices, so $\mathbb{C}[X_\mu]$ is not only filtered but graded. The variety $X_\mu$, being the closure of a conjugacy class, is stable under conjugation, inducing a rational graded action of $GL(n, \mathbb{C})$ on $\mathbb{C}[X_\mu]$.

These adjoint orbit closures satisfy $X_\mu \subset X_\nu$ if $\mu \geq \nu$, inducing surjections $\mathbb{C}[X_\nu] \to \mathbb{C}[X_\mu]$ of graded $GL(n)$-modules, given by restriction of functions. For the dominant integral weight $\tau$ of $GL(n)$, let $\tilde{K}_{\tau, \mu}(q)$ denote the Poincaré polynomial of the $\tau$-th isotypic component of $\mathbb{C}[X_\mu]$. It follows immediately that the Poincaré polynomials satisfy the monotonicity property

$$K_{\tau, \mu}(q) \leq K_{\tau, \nu}(q) \quad \text{if } \mu \geq \nu$$

(coefficientwise), for every $\tau$. The (normalized) Kostka-Foulkes polynomial $\tilde{K}_{\lambda, \mu}(q)$ occurs as the Poincaré polynomial of an isotypic component of $\mathbb{C}[X_\mu]$ \cite{24}. More precisely, let $\{\omega_i : 1 \leq i \leq n-1\}$ be the fundamental weights for $sl_n$. If $\tau = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \omega_i$ for a partition $\lambda$ of $n$, then $\tilde{K}_{\tau, \mu}(q) = \tilde{K}_{\lambda', \mu}(q)$ where $\lambda'$ is the transpose of the partition $\lambda$ \cite{24}. In this case (1.1) specializes (after transposing $\lambda$) to

$$\tilde{K}_{\lambda, \mu}(q) \leq \tilde{K}_{\lambda', \mu}(q) \quad \text{if } \mu \geq \nu$$

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for every partition \( \lambda \) of \( n \).

In \([9]\) \([12]\) \([13]\) \([14]\) Lascoux and Schützenberger provided a beautiful combinatorial explanation for (1.2), giving the following results and constructions.

1. \( \tilde{K}_{\lambda, \mu}(q) \) is the \( q \)-enumeration of the set \( \text{CST}(\lambda, \mu) \) of column-strict tableaux of shape \( \lambda \) and content \( \mu \) by the map cocharge : \( \text{CST}(\lambda, \mu) \rightarrow \mathbb{N} \).

2. The set \( \text{CST}(\mu) = \bigcup_{\lambda} \text{CST}(\lambda, \mu) \) has the structure of a poset (called the cyclage poset) that is graded by cocharge.

3. If \( \mu \succeq \nu \) then there is an embedding of graded posets

\[ \theta^\mu_\nu : \text{CST}(\mu) \rightarrow \text{CST}(\nu) \]

that preserves the shapes of tableaux.

4. A map from standard tableaux to partitions is defined (called catabolism type in \([23]\) ), and it is asserted that the image of the embedding \( \theta^{(1^n)}_\mu \) is the set of standard tableaux whose catabolism type dominates \( \mu \).

The coordinate rings \( \mathbb{C}[X_\mu] \) are examples of a larger family of \( \mathbb{C}[X_\mu] \)-modules that afford a compatible graded \( GL(n) \)-module structure. Consider the category of finitely generated graded \( \mathbb{C}[x_{ij}] \)-modules supported in \( X_\mu \) that afford a compatible graded action of \( GL(n) \). In \([4]\) a set of generators is given for the graded Grothendieck group of this category. Among these modules are a distinguished subfamily of modules \( M_R \) that may be indexed by sequences \( R \) of rectangular partitions; these are twists of the coordinate ring \( \mathbb{C}[X_\mu] \) by line bundles. The Poincaré polynomial \( K_{\lambda, R}(q) \) of the \( \lambda \)-th isotypic components of the module \( M_R \) is a \( q \)-analogue of the Littlewood-Richardson (LR) coefficient given by the multiplicity of an irreducible \( GL(n) \)-module in the tensor product of irreducibles indexed by the rectangular partitions of \( R \). In \([23]\) it is observed that there are natural surjections between certain of these modules, yielding monotonicity results that generalize (1.1).

In \([18]\) generalizations of 1 and 2 were proven for the Poincaré polynomials \( K_{\lambda, R}(q) \). In this paper a generalization of 3 conjectured in \([8]\) is proven using the generalized cyclage poset of \([18]\), yielding a combinatorial proof of the monotonicity property for \( K_{\lambda, R}(q) \). In the case that the sequence of rectangles \( R \) is linearly ordered by containment, the generalization of 4 conjectured in \([23]\) is proven, obtaining a new formula for \( K_{\lambda, R}(q) \) as the \( q \)-enumeration of \( R \)-catabolizable tableaux of shape \( \lambda \) by the charge map of Lascoux and Schützenberger.

A proof is given for the conjecture \([8]\) that relates the polynomials \( K_{\lambda, R}(q) \) and \( K_{\lambda^t, R^t}(q) \), where \( \lambda^t \) is the transpose of the partition \( \lambda \) and \( R^t \) is the sequence of transposes of the rectangles in \( R \). Using this it is shown precisely how our results generalize 3 and 4 above.

All undefined notation and definitions are taken from \([18]\).

While finishing this paper, the author was informed of the work of Schilling andWarnaar \([22]\), which has considerable overlap with this paper and \([18]\) \([19]\).

2. Monotonicity via embeddings

2.1. The Poincaré polynomials. Let \( R = (R_1, R_2, \ldots, R_t) \) be a sequence of rectangular partitions and \( \lambda \) a partition. There is a polynomial \( K_{\lambda, R}(q) \) with integer coefficients such that \( K_{\lambda, R}(1) \) is the multiplicity of the \( GL(n) \)-irreducible \( V_\lambda \) of highest weight \( \lambda \) in the tensor product \( V_{R_1} \otimes V_{R_2} \otimes \cdots \otimes V_{R_t} \) \([23]\). There is a set
Write $\xi R$ be the partition whose parts are the heights of the rectangles in the structure of a graded poset with order called the Section 3.3]) and grading function given by a generalized charge map called charge $K$ Embeddings.

2.3. constructions of the embeddings defined in [8], which were conjectured to have such The inequality. Let $R$ be a sequence of rectangles. For $k \geq 1$ let $\xi^k(R)$ be the partition whose parts are the heights of the rectangles in $R$ of width $k$. Write $\xi(R) = (\xi^1(R), \xi^2(R), \ldots)$ Observe that $R$ and $R'$ are reorderings of each other if and only if $\xi(R) = \xi(R')$. If $\xi = (\xi^1, \xi^2, \ldots)$ and $\phi = (\phi^1, \phi^2, \ldots)$ are two sequences of partitions, write $\xi \succeq \phi$ for the partial order given by the direct product of dominance partial orders, that is, if $\xi^i \succeq \phi^i$ for all $i$.

Write $R \succeq R'$ if $\xi(R) \succeq \xi(R')$. This is a pseudo order; it is the direct product of dominance partial orders on the sequences of rectangles modulo reordering.

Theorem 2. [23] Let $R$ and $R'$ be dominant sequences of rectangles such that $R \succeq R'$. Then

$$K_{\lambda;R}(q) \leq K_{\lambda;R'}(q)$$

coefficientwise.

This was proved by algebro-geometric methods. There is a graded $GL(n)$-module $M_R$ whose $\lambda$-th isotypic component has Poincaré polynomial given by $K_{\lambda;R}(q)$.

Theorem 2 follows immediately from the existence of canonical surjections of graded $GL(n)$-modules $M_{R'} \to M_R$.

In particular if $R$ and $R'$ are dominant and $\xi(R) = \xi(R')$ then $K_{\lambda;R}(q) = K_{\lambda;R'}(q)$.

2.3. Embeddings. It is shown in [8] that the set $\text{LRT}(R) = \cup_{\lambda} \text{LRT}(\lambda; R)$ has the structure of a graded poset with order called the $\tilde{R}$-cycclage (defined in Section 3.3]) and grading function given by a generalized charge map called charge $K$ Embeddings.

Let us recall the constructions of the embeddings defined in [8], which were conjectured to have such properties.

The pseudo order $R \succeq R'$ is generated by relations of two forms:

(E1) $R \succeq R'$ where $R_i = R_i'$ for $i > 2$, $R_1 = (k^a)$, $R_2 = (k^b)$, $R_1' = (k^{a-1})$, $R_2' = (k^{b+1})$ for some $a - 1 \geq b + 1$ and $k$.

(E2) $R \succeq \tau_p R$ where $\tau_p R$ is obtained from $R$ by exchanging the $p$-th and $(p + 1)$-st rectangles.

For the relation (E2), it is known that the bijection $\tau_p : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R')$ [8 Section 2.5] is an isomorphism of posets that preserves the grading [8 Theorem 20]. Write $\theta_{\pi}^p : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; \tau_p R)$ for this isomorphism.

For relations of the form (E1), let us define an embedding $\theta_{\pi}^0 : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R')$. Suppose first that $R = (R_1, R_2)$. There is a unique embedding $\iota_{k,n_1,n_2} : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R')$ since $0 \leq |\text{LRT}(\lambda; R)| \leq |\text{LRT}(\lambda; R')| \leq 1$; these tableaux
are described explicitly in [18, Prop. 33]. Moreover in this two-rectangle case
\[ \text{charge}_R(\iota_{k,m,n}(T)) = \text{charge}_R(T) \]

since \( \iota_{k,m,n} \) preserves shape and both \( R \) and \( R' \) have the same maximum column size \( k \).

**Example 3.** Let \( n = 6, k = 3, \eta_1 = 4, \eta_2 = 2, \) and \( \lambda = (54321) \). The tableaux \( T \in \text{LRT}(\lambda; R) \) and \( T' \in \text{LRT}(\lambda; R') \) are given below.

\[
T = \begin{array}{c}
1 & 1 & 1 & 5 & 5 \\
2 & 2 & 2 & 6 \\
4 & 4 & 4 \\
5 & 6 \\
6 & 6 
\end{array}
\quad T' = \begin{array}{c}
1 & 1 & 1 & 4 & 4 \\
2 & 2 & 2 & 5 \\
4 & 5 & 6 \\
5 & 6 \\
6 & 6 
\end{array}
\]

In general, let \( R = ((k^{\eta_1}), (k^{\eta_2}), \tilde{R}), R' = ((k^{\eta_1 - 1}), (k^{\eta_2 + 1}), \tilde{R}) \) and \( B = [\eta_1 + \eta_2] \). Define \( \theta_{R}^{R'} : \text{LRT}(R) \to \text{LRT}(R') \) by:
\[
\theta_{R}^{R'}(T)|_{B} = \iota_{k,m,n}(T|_{B}) \\
\theta_{R}^{R'}(T)|_{[n]-B} = T|_{[n]-B}
\]

where \( T|_{B} \) denotes the restriction of the tableau \( T \) to the subalphabet \( B \). To extend these functions to words, recall the definition of the set \( W(R) \) of \( R \)-LR words [18, Section 2.3]. Let
\[
\theta_{R}^{R'} : W(R) \to W(R') \\
P(\theta_{R}^{R'}(w)) = \theta_{R}^{R'}(P(w)) \\
Q(\theta_{R}^{R'}(w)) = Q(w)
\]

where \( P \) is the Schensted \( P \) tableau and \( Q \) is the standard row insertion recording tableau.

The maps \( \theta_{R}^{R'} \) and \( \tau_{R} \) shall be called elementary embeddings.

Now let \( R \triangleright R' \) be arbitrary. Let \( R = R^i \triangleright R^j \triangleright \ldots \triangleright R^m = R' \) such that for all \( i \), \( R^i \triangleright R^{i+1} \) is a relation of the form \( (E1) \) or \( (E2) \). Define an embedding \( \theta_{R}^{R'} : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R') \) by the composition of elementary embeddings
\[
\theta_{R}^{R'} = \theta_{R^m}^{R^{m-1}} \circ \ldots \circ \theta_{R^3}^{R^2} \circ \theta_{R^2}^{R^1}.
\]

**Theorem 4.**
1. Let \( R \triangleright R' \). Then the map \( \theta_{R}^{R'} \) is independent of the chain in the pseudo-order from \( R \) to \( R' \).
2. If \( R \triangleright R' \triangleright R'' \) then \( \theta_{R}^{R''} = \theta_{R'}^{R''} \circ \theta_{R}^{R'} \).
3. For \( R \triangleright R' \), \( \theta_{R}^{R'} \) is an injective map \( \text{LRT}(R) \to \text{LRT}(R') \) that preserves the shape of tableaux and gives an isomorphism of the \( R \)-cocyclage poset with the full subposet induced by its image under \( \theta_{R}^{R'} \).
4. For \( R \triangleright R' \),
\[ \text{charge}_R \circ \theta_{R}^{R'} = \text{charge}_{R'} . \]

Theorem 4 is proven over the next several sections.
2.4. Embeddings and R-cocyclage. It is shown that the elementary embedding associated to a pseudo-order covering relation of the form (E1), preserves the partial orders on LR tableaux. The notation is taken from [8, Section 2.2].

Proposition 5. Let \( R \supset R' \) be as in (E1). Then the elementary embedding \( \theta = \iota_{k,n_1,n_2} : \text{LRT}(R) \to \text{LRT}(R') \) is an isomorphism of \( \text{LRT}(R) \) under \( R \)-cocyclage, with the full subposet given by its image inside \( \text{LRT}(R') \) under \( R'-\)cocyclage.

Proof. It must be shown that for all \( S, T \in \text{LRT}(R), S <_R T \) is a covering relation in the \( R \)-cocyclage poset if and only if \( \theta(S) <_{R'} \theta(T) \) is a covering relation in the \( R' \)-cocyclage poset. Write \( S' = \theta(S) \) and \( T' = \theta(T) \).

First it is shown that the forward direction suffices. Let \( S' <_{R'} T' \) be an \( R' \)-cocyclage covering relation with starting cell \( s' \). Since \( T \) has the same shape as \( T' \), it admits an \( R \)-cocyclage covering relation \( S'' <_R T \) that starts at the same cell \( s \). By the forward direction, \( \theta(S'') <_{R'} T' \) is a covering relation that starts at \( s' \). This implies that \( \theta(S'') = S' \) since both are \( R' \)-cocyclages of \( T' \) starting at the cell \( s' \). But \( S' = \theta(S) \) and \( \theta \) is injective, so \( S'' = S \) and the covering relation \( S' <_{R'} T' \) is the image under \( \theta \) of a covering relation \( S <_R T \).

Let \( ux \in W(R) \) such that \( T = P(ux) = P(\chi_R(ux)) = P(w^R_0 x(w^R_0 u)) \) (where \( \chi_R \) and \( w^R_0 \) are defined in [8, Section 3.2]), and the difference of the shapes of \( T \) and \( U = P(u) \) is a cell \( s \) that lies strictly east of the \( a \)-th column, where \( a \) is the maximum number of columns among the rectangles in \( R \). Let \( A_i \) and \( A'_i \) be the subintervals corresponding to the rectangles \( R_i \) and \( R'_i \) [8, Section 2.2]. By [8, Theorem 21 (C2), Prop. 23], \( x \in A_i \) for some \( i > 1 \).

It will be shown that \( \chi_{R'}(\theta(ux)) = \theta(\chi_R(ux)) \). Write \( R = (R_1, R_2, \widehat{R}) \) and \( R' = (R_1', R_2', \widehat{R}) \). By definition \( w^R_0 \) acts on \( A_j \) by \( w^A_j \) for \( 1 \leq j \leq 2 \) and by \( w^R_0 \) on \([n] - B \). Similarly \( w^R_i \) acts on \( A'_j \) by \( w^{A'}_j \) for \( 1 \leq j \leq 2 \) and by \( w^R_i \) on \([n] - B \).

Write \( \theta(ux) = vy \) where \( y \) is a letter.

Suppose that \( i > 2 \), in which case \( y = x \). Now \( \theta \) affects only the letters of \( B \), and \( \chi_R \) only changes the letters outside of \( A_i \) by a right shift by one position [8, Remark 36]. It follows that \( \chi_{R'}(\theta(ux)) = \theta(\chi_R(ux)) \).

Otherwise \( x \in A_2 \). In passing from \( ux \) to either \( \chi_{R'}(\theta(ux)) \) or \( \theta(\chi_R(ux)) \), the letters in \([n] - B \) are merely shifted to the right by one position. By restricting to \( B \) it may be assumed that \( R = (R_1, R_2) \). By [8, Prop. 38] the shape of \( S = P(\chi_R(ux)) \) is obtained from the shape of \( T \) by removing the cell \( s \) and adjoining a cell \( s' \) that is uniquely determined by the shape of \( T \) and \( s \). Now \( P(\theta(ux)) = \theta(P(ux)) = \theta(T) \) has the same shape as \( T \). The difference of the shapes of \( P(u'x') \) and \( P(u') \) is the cell of the row insertion recording tableau \( Q(u'x') \) that contains the maximum letter. But \( Q(\theta(ux)) = Q(ux) \) so this cell is \( s \). By [8, Prop. 38] the shape of \( P(\chi_{R'}(u'x')) \) is obtained from that of \( P(u'x') \) by removing \( s \) and adjoining the same cell \( s' \) as above. The tableaux \( P(\chi_{R'}(\theta(ux))) \) and \( P(\theta(\chi_R(ux))) \) are both in \( \text{LRT}(R') \) and have the same shape and \( R' \) has only two rectangles, so the tableaux coincide. Moreover, since \( \theta \) doesn’t change \( Q \) tableaux and both \( \chi_R \) and \( \chi_{R'} \) induce cocyclages starting at the same corner cell, it follows that the \( Q \) tableaux of the words \( \chi_{R'}(\theta(ux)) \) and \( \theta(\chi_R(ux)) \) coincide, hence the words themselves do also by the bijectivity of the RS correspondence.

This shows that \( \chi_{R'}(\theta(ux)) = \theta(\chi_R(ux)) \). Now \( P(u') \) has the same shape as \( P(u) \) (since \( Q(ux) = Q(u'x') \) and \( T \) has the same shape as \( T' \). It follows that \( S' = P(\chi_{R'}(\theta(ux))) \) and that \( S' <_{R'} T' \) is a covering relation under \( R' \)-cocyclage.
Proposition 6. Suppose $R \triangleright R'$ such that $R$ and $R'$ are both dominant. Say $R_j$ has $\mu_j$ columns for all $j$. Let $\tilde{R}$ and $\tilde{R}'$ be obtained from $R$ and $R'$ by removing all rectangles that have $\mu_1$ columns, and $B$ the alphabet corresponding to $\tilde{R}$ (and also $\tilde{R}'$). Then $\tilde{R} \triangleright \tilde{R}'$ and for all $R$-cocyclage minimal tableaux $T \in \text{LRT}(R)$,

$$\theta_R^{R'}(T)|_B = \theta_{R'}^{R}(T)|_B.$$  

Proof. Observe that $T \in \text{LRT}(R)$ is $R$-cocyclage minimal if and only if it has exactly $\mu_1$ columns. Let $M$ be the sum of the numbers of rows of rectangles in $R$ (or $R'$) having exactly $\mu_1$ columns. Let $B = [M + 1, n]$. Then by the definition of LR tableau, $T|_{[M]} = Y := \text{key}((\mu_1^M))$ (see [8] Section 5.1) for the definition of key). Let $S = \theta_R^{R'}(T)$. Since $S$ has the same shape as $T$ it follows that $S$ is $R'$-cocyclage minimal. For the same reasons $S|_{[M]} = Y$. By definition of LR tableau $T|_B \in \text{LRT}(\tilde{R})$ and $S|_B \in \text{LRT}(\tilde{R}')$.

By induction it may be assumed that there is a decreasing sequence in the pseudo order from $R$ to $R'$ that has at most one step of the form (E1).

Suppose first that there is no step of the form (E1). Without loss of generality it may be assumed that $\theta_R^{R'} = \tau_p$, which by the dominance assumption must exchange rectangles with the same number of columns.

If $\tau_p$ exchanges rectangles with $\mu_1$ columns then it acts by the identity on $R$-cocyclage minimal tableaux and (2.2) holds trivially. If $\tau_p$ exchanges rectangles with strictly less than $\mu_1$ columns, then by induction,

$$\theta_R^{R'}(T)|_B = \tau_p(T)|_B = \tau_p(T|_B) = \theta_{R'}^{R}(T|_B)$$

and again (2.2) holds.

So it may be assumed that there is a step of the form (E1). Let $k$ be as in (E1). If $k = \mu_1$ then $\theta_R^{R'}$ is the identity on $R$-cocyclage minimal tableaux and again (2.2) holds trivially.

So assume that $k < \mu_1$. A decreasing chain in the pseudo order from $R$ to $R'$ can be given by switching the rectangles $(k^{\mu_1})$ and $(k^{\mu_2})$ to the front, then applying the map $\iota_{k,\eta_1,\eta_2}$, then switching back to obtain the sequence $R'$. By [8] Remark 39], these rectangle-switching bijections are easily computed directly and satisfy (2.2).

2.5. Embeddings and $\text{charge}_R$.

Proposition 7. Let $R \triangleright R'$. Then

$$\text{charge}_R(\theta_R^{R'}(T)) = \text{charge}_R(T)$$

for all $T \in \text{LRT}(R)$.

Proof. It is enough to assume that $\theta_R^{R'}$ has either the form $\iota_{k,\eta_1,\eta_2}$ or $\tau_p$ and to show that the left hand side of (2.3) satisfies the axioms (C1) through (C4) that characterize $\text{charge}_R$ [8, Theorem 21]. This is accomplished using Propositions [9] and [8, Theorem 20], and induction.

2.6. Proof of Theorem 4.

Proof. Consider an arbitrary chain in the pseudo-order from $R$ to $R'$ and consider the composition $\theta_R^{R'}$ of elementary embeddings defined by this chain. Then for this
map \( \theta_R^{R'} \), 3 holds, in light of Proposition \([3]\) and \([8]\). Theorem 20]. 4 also holds by Proposition \([3]\).

2 follows from 1, for to compute \( \theta_R^{R''} \) one may take a chain in the pseudo-order from \( R \) to \( R' \) and then from \( R' \) to \( R'' \).

The proof of 1 proceeds by induction on the number of rectangles in \( R \) and on charge_{\gamma(R)}. Consider two different chains from \( R \) to \( R' \) in the pseudo-order and call the corresponding embeddings \( \theta_1 \) and \( \theta_2 \). Let \( T \in \text{LRT}(R) \). Suppose first that \( T \) is not \( R \)-cocyclage-minimal. Then it admits an \( R \)-cocyclage covering relation \( S <_R T \). By induction on charge(\( R \)), \( \theta_1(S) = \theta_2(S) \). But by 3, \( \theta_1(T) = \theta_2(T) \). Otherwise suppose \( T \) is \( R \)-cocyclage-minimal. By applying some bijections \( \tau_p \) to \( T \) and then to both \( \theta_1(T) \) and \( \theta_2(T) \), it may be assumed that both \( R \) and \( R' \) are dominant. Let \( a \) be the maximal number of columns among the rectangles \( R_j \). The proof of Proposition \([3]\) shows that one may write \( T = \hat{T}Y \) and that \( \theta_i(T) = \hat{\theta_i}(\hat{T})Y \) where \( \hat{\theta_i}(\hat{T}) \) computes the map \( \theta_R^{R'}(\hat{T}) \). But by induction on the number of rectangles in \( R \), \( \theta_1(\hat{T}) = \theta_2(\hat{T}) \), so \( \theta_1(T) = \theta_2(T) \).

\[ \square \]

3. Embedding of LR tableaux into column-strict tableaux

Given a sequence of rectangles \( R \), let \text{rows}(\( R \)) be the sequence of one row rectangles of lengths given by the weight \( \gamma(\( R \)) = (\mu_1^1, \mu_2^2, \ldots) \in \mathbb{Z}^n \), that is, \text{rows}(\( R \)) is obtained by splitting each of the rectangles \( R_j \) in \( R \) into its constituent rows. Clearly \( R \succeq \text{rows}(\( R \)) \) and \( \text{rows}(\( R \)) \) is minimal in the pseudo-order. Since \( \text{rows}(\( R \)) \) consists of one row rectangles, \( \text{LRT}(\text{rows}(\( R \))) \) is the set of column-strict tableaux of content \( \gamma(\( R \)) \) and partition shape, with a weak version of the cocyclage poset structure of Lascoux and Schützenberger and graded by the usual charge statistic. But the embedding \( \theta_R := \theta_R^{\text{rows}(\( R \))} \) preserves grading and the shape of tableaux by Theorems \([3]\) and \([3]\). For certain \( R \) it is shown that the image of \( \theta_R \) is given by the \( R \)-catabolizable tableaux of \([3]\). Thus \( K_{\lambda;R}(q) \) has another formula, as the \( q \)-enumeration of the \( R \)-catabolizable tableaux of shape \( \lambda \) by the charge statistic.

The map \( \theta_R \) is a generalization of the standardization map in \([3]\) which is a cyclage- and cocharge-preserving embedding of the column-strict tableaux into standard tableaux; a precise statement of this is given in Remark \([3]\).

3.1. Catabolizable tableaux. Given a (possibly skew) column-strict tableau \( S \) and index \( r \) (resp. index \( c \)), let \( H_r(S) = P(S_nS_n) \) (resp. \( V_r(S) = P(S_nS_w) \)), where \( S_n \) and \( S_s \) (resp. \( S_c \) and \( S_w \)) are the north and south (resp. east and west) subtableaux obtained by slicing \( S \) horizontally (resp. vertically) between its \( r \)-th and \( (r+1) \)-st rows (resp. \( c \)-th and \((c+1) \)-st columns).

Let \( S \) be a column-strict tableau of partition shape in the alphabet \( [n] \), \( R = (R_1, R_2, \ldots) \) such that \( R_j \) is the rectangular partition with \( \eta_j \) rows and \( \mu_j \) columns, and \( \hat{R} = (R_2, R_3, \ldots) \). Suppose \( S|_{A_1} = Y_1 \). In this case the \( R_1 \)-catabolism of \( S \) (the \( R_1 \)-column-catabolism) is defined to be the tableau \( \text{cat}_{R_1}(S) = H_n(S - Y_1) \) (resp. \( \text{colc}_{R_1}(S) = V_n(S - Y_1) \)).

Define \( \theta(R) \) (resp. \( \theta(R) \)-column-catabolizability) by the following rules.

1. The empty tableau is the unique (resp. column) catabolizable tableau for the empty sequence of rectangles.
2. Otherwise, $S$ is $R$-catabolizable (resp. $R$-column-catabolizable) if and only if $S|A_1 = Y_1$ and $\text{cat}_{R_1}(S)$ (resp. $\text{colcat}_{R_1}(S)$) is $\hat{R}$-catabolizable (resp. $\hat{R}$-column-catabolizable) in the alphabet $[\eta_1 + 1, n]$.

Observe that if $S$ is $R$-catabolizable then $\text{content}(S) = \gamma(R)$, and that if $R$ is dominant and consists of single rows, then $S$ is $R$-catabolizable if and only if $S$ is column-strict.

**Example 8.** Let $R = ((33), (33), (222))$ so that $\mu = (332)$, $\eta = (223)$, $n = 7$, $A_1 = [1, 2], A_2 = [3, 4], A_3 = [5, 7]$, and

$$Y_1 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \end{array} \quad Y_2 = \begin{array}{ccc} 3 & 3 & 3 \\ 4 & 4 & 4 \\ \end{array} \quad Y_3 = \begin{array}{ccc} 5 & 5 \\ 6 & 6 & 7 \\ \end{array}$$

The following computation shows that the tableau $S$ is both catabolizable and column-catabolizable.

$$S = \begin{array}{cccc} 1 & 1 & 1 & 3 & 4 & 5 \\ 2 & 2 & 2 & 4 & 5 & 6 \\ 3 & 3 & 6 \\ 4 & 7 & 7 \\ \end{array}$$

$$S_eS_w = \begin{array}{cccc} \times & \times & \times & 3 & 3 & 6 \\ \times & \times & \times & 4 & 7 & 7 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ \end{array}$$

$$\text{cat}_{R_1}(S) = P(S_eS_w) = \begin{array}{cccc} 3 & 3 & 3 & 6 & 7 \\ 4 & 4 & 4 & 7 \\ 5 & 5 & \end{array}$$

The following result solves a special case of [23, Conjecture 26].

**Theorem 9.** Suppose $R$ is dominant and $R^t$ is dominant, that is, $R_j \supseteq R_{j+1}$ for all $j$. Then the image of $\theta_R$ is the set of $R$-catabolizable tableaux.

The rest of the section follows [23]. Two weak versions of the cocyclage poset on column-strict tableaux [14] are now defined. Let $u$ be a word and $x$ a letter, $S = P(ux), U = P(u), T = P(xu)$, and $s$ (resp. $s'$) the cell that is the difference of the shapes of $S$ (resp. $T$) and $U$. Contrary to [23], define $T <_{(\text{r}, c)} S$ (resp. $T <_{(\text{c}, c)} S$) if the cell $s = (i, j)$ (resp. $s' = (i', j')$) satisfies $j > c$ (resp. $i' > r$).

Given a sequence of rectangles $R$, let $R_j$ have $\eta_j$ rows and $\mu_j$ columns, $a = \max_j \mu_j$ and $r > 0$. Then the set $\text{CST}(\gamma(R))$ is a graded poset under the order $<_{(\text{r}, r)}$ and also under $<_{(\text{c}, c)}$. This follows from [14, Lemme 2.13] and variants. Let $b = \max_j \eta_j$. Only the case $r = b$ is used here.

**Proposition 10.** Let $R$ be dominant, $S$ an $R$-column-catabolizable tableau and $T <_{(\text{r}, r)} S$. Then $T$ is $R$-column catabolizable.
Proof. Observe that in the above notation, $S$, $T$, and $U$ all contain $Y_1$. So define $S_e$ and $S_w$ as in the definition of $\text{colcat}_{R_1}(S)$ and similarly for $T$ and $U$. In particular $S_e = P(U_e x)$ and $S_w = U_w$. By assumption $S$ is $R$-column-catabolizable, so $P(S_e S_w) = \text{colcat}_{R_1}(S)$ is $\hat{R}$-column-catabolizable where $\hat{R} = (R_2, R_3, \ldots)$. By definition it suffices to show that $\text{colcat}_{R_1}(T) = P(T_e T_w)$ is $\hat{R}$-column-catabolizable.

Suppose that during the column insertion of $x$ into $U$, that a letter $y$ is bumped from the $a$-th column of $U$ to the $(a + 1)$-st. This can only happen if $U$ has $a$ columns. Then $P(x U_w) = P(T_w y)$ where the difference of the shapes of $P(T_w y)$ and $T_w$ is a cell in the last column, which is the $(a + 1)$-st since $U_w$ and hence $T_w$ has $a$ columns. Finally $P(y U_e) = T_e$. Then

$$\text{colcat}_{R_1}(T) = P(T_e T_w) = P(y U_e T_w)$$

and

$$P(U_e T_w y) = P(U_e x U_w) = P(S_e S_w) = \text{colcat}_{R_1}(S).$$

Since the row insertion of $y$ into $T_w$ ends at the cell $(a + 1, 1)$, by [20] Lemma 22 the row insertion of $y$ into $P(U_e T_w)$ ends strictly to the east of the $a$-th column. It follows that

$$\text{colcat}_{R_1}(T) = P(U_e T_w y) <_{(>,a)} P(y U_e T_w) = \text{colcat}_{R_1}(S).$$

By induction $\text{colcat}_{R_1}(T)$ is $\hat{R}$-column-catabolizable.

Otherwise suppose that no such $y$ exists; then $S_e = P(U_e x)$, $S_w = U_w$, $P(x U_w) = T_w$, $T_e = U_e$, and

$$\text{colcat}_{R_1}(T) = P(T_e T_w) = P(U_e T_w)$$

$$= P(U_e x U_w) = P(S_e S_w) = \text{colcat}_{R_1}(S).$$

In an analogous manner one proves the following.

**Proposition 11.** Suppose $R^t$ is dominant, $T$ is $R$-catabolizable and $T <_{(>,b)} S$. Then so is $S$.

The next result relates the catabolism and column-catabolism operators.

**Proposition 12.** Suppose $R$ is dominant, $R^t$ is dominant, and $S$ contains $Y_1$. Then

1. $\text{colcat}_{R_1}(S) \leq_{(>,a)} \text{cat}_{R_1}(S)$.
2. $\text{colcat}_{R_1}(S) \leq_{(>,b)} \text{cat}_{R_1}(S)$.

**Proof.** Slice $S$ into four parts, cutting both horizontally as in the definition of $\text{cat}_{R_1}(S)$ and vertically as in that of $\text{colcat}_{R_1}(S)$. Denote the northeast, southeast, and southwest tableaux by $S_{ne}$, $S_{se}$, and $S_{sw}$. Then $\text{colcat}_{R_1}(S) = P(S_{se} S_{ne} S_{sw})$ and $\text{cat}_{R_1}(S) = P(S_{ne} S_{sw} S_{se})$.

Now the row insertion of $S_{se}$ into $S_{sw}$ creates all cells strictly east of the $a$-th column. It follows that the same happens for the row insertion of $S_{se}$ into $P(S_{sw} S_{se})$ and that a sequence of $<_{(>,a)}$-cocyclages lead from $\text{cat}_{R_1}(S)$ to $\text{colcat}_{R_1}(S)$. This proves 1. 2 is proven similarly.

For special sequences of rectangles, the two kinds of $R$-catabolizability are equivalent.
Proposition 13. Let \( R \) be dominant and \( R^t \) dominant. Then \( S \) is \( R \)-catabolizable if and only if \( S \) is \( R \)-column-catabolizable.

Proof. It may be assumed that \( S \) contains \( Y_1 \). Under this assumption each of the following implies the next, using induction and Propositions 10 and 12.

1. \( S \) is \( R \)-catabolizable.
2. \( \text{cat}_{R_1}(S) \) is \( \tilde{R} \)-catabolizable.
3. \( \text{cat}_{R_1}(S) \) is \( \tilde{R} \)-column-catabolizable.
4. \( \text{colcat}_{R_1}(S) \) is \( \tilde{R} \)-column-catabolizable.
5. \( S \) is \( R \)-column-catabolizable.

To reverse the implications, replace 3 by “colcat\( R_1 \)(\( S \)) is \( \tilde{R} \)-catabolizable” and Proposition 11 by 11. Then each statement implies the previous one.

3.2. Strong \( R \)-cocyclage. Our proof of Theorem 4 requires the strongest version of the cocyclage poset. Suppose \( ux \in W(R) \) with \( x \) a letter. Write \( P(\chi_R(ux)) < P(ux) \) if, for every element \( \tau \) in the symmetric group that permutes \( R \), the last letter of \( \tau(ux) \) is not in the first subalphabet of \( \tau L R \); call the resulting partial order the strong \( R \)-cocyclage. By [18, Prop. 23] this holds if and only if charge\( R(\chi_R(ux)) \) = charge\( R(\chi_R(ux)) - 1 \). The strong \( R \)-cocyclage poset is graded by charge\( R \) and is obtained from the \( R \)-cocyclage poset by adding some relations.

Theorem 14. Suppose some rearrangement of \( R \) is decreasing with respect to containment and \( R \triangleright R' \). Then \( \theta = \theta^{R'}_R \) is an embedding of the strong \( R \)-cocyclage poset with the full subposet of its image in the strong \( R' \)-cocyclage poset. Moreover, if \( w \in W(R) \) and \( P(\chi_R(w)) < P(w) \) then

\[
\theta(\chi_R(w)) = \chi_{R'}(\theta(w))
\]

In general it is difficult to check the strong \( R \)-cocyclage condition; one must examine the entire orbit of a word. However, if the rectangles of \( R \) are linearly ordered by containment, then looking at one orbit element suffices.

Lemma 15. Suppose \( R_j \supseteq R_{j+1} \) for all \( j \) and \( ux \in W(R) \) with \( x \) a letter. Then \( P(\chi_R(ux)) < P(ux) \) if and only if \( x \) is not in the first subalphabet \( A_1 \) for \( R \).

Proof. The forward direction holds by definition. Consider the two rectangle case \( R = (R_1, R_2) \). Write \( vy = \tau_1(ux) \) and \( A_j \) and \( A'_j \) the subalphabets for \( R \) and \( R' = (R_2, R_1) \). Since \( R_1 \supseteq R_2 \), it follows that \( x \in A_1 \) if and only if \( y \in A'_1 \) by [18, Prop. 38].

Now let \( R = (R_1, \ldots, R_t) \). Since \( \tau_p \) for \( p > 1 \) does not change letters in the first subalphabet, it is enough to check the last letter of \( \tau(ux) \) for \( \tau \) a minimal coset representative in \( S_1 \times S_{t-1} \setminus S_t \), that is, \( \tau \) has the form \( \tau = \tau_1 \tau_2 \ldots \tau_r \). Now \( \tau R \) starts with \( (R_r, R_1, \ldots) \). By restriction to these first two subalphabets and the two rectangle case above, it follows that the last letter of \( \tau(ux) \) is not in the first subalphabet. But this holds for all \( r \), so \( P(\chi_R(ux)) < P(ux) \).

Proof of Theorem 4:

Proof. It follows immediately from the definitions that if two sequences of rectangles are rearrangements of each other, then their strong cocyclage posets are isomorphic. It is clear that since \( R \) has a rearrangement that is decreasing with respect to containment, so must \( R' \). So it may be assumed without loss of generality that \( R \)
and $R'$ are related as in (E1). Write $A_j$ and $A'_j$ for the subalphabets for $R_j$ and $R'_j$ respectively. Let $R = (R_1, \ldots, R_t)$.

Let $w = ux$ and $vy = \theta(ux)$ with $x$ and $y$ letters. By restriction to the first two subalphabets and the explicit description of the two-rectangle LR tableaux in [18, Prop. 33, 38], it follows that $x \in A_1$ if and only if $y \in A'_1$. By Lemma [17] this establishes Theorem [14] except for the equation (3.1).

Assume $P(\chi_R(ux)) < P(ux)$ so that $x \notin A_1$ and $y \notin A'_1$. Consider the two words in (E1). Suppose first that $x \in A_j$ for $j > 2$. By definition, $\theta$ only changes letters in the first two subalphabets, so $y = x$. By [18, Remark 39] $\chi_R$ only affect the alphabets other than $A_j$ by shifting letters to the right by one position. In this case it is clear that (3.1) holds. Since $x \notin A_1$ the only other case is $x \in A_2$ and hence $y \in A'_2$. Again by [18, Remark 39] the letters in $A_j$ for $j > 2$ are merely shifted to the right by $\chi_R$, so by restriction to $A_1 \cup A_2$ it may be assumed that $R = (R_1, R_2)$. In this case [18, Prop. 38] establishes (3.1). \hfill $\Box$

3.3. Proof of Theorem [3].

**Lemma 16.** Let $R$ be a dominant sequence of rectangles, $R = (R_1, \hat{R})$, and $u \in W(\hat{R})$ in the alphabet $[\eta_1 + 1, n]$. Then $\theta_R(uY_1) = \theta_{\hat{R}}(u)Y_1$ where $\theta_{\hat{R}}$ is understood to have image in the alphabet $[\eta_1 + 1, n]$ and the tableau $Y_1$ is identified with its row-reading word.

**Proof.** Observe that $uY_1 \in W(R)$ since $u \in W(\hat{R})$, so that $\theta_R(uY_1)$ makes sense.

It suffices to show that the two words $\theta_R(uY_1)$ and $\theta_{\hat{R}}(u)Y_1$ have the same $P$ tableaux and same $Q$ tableaux.

Let $M$ be the length of $u$ and $N$ the length of $uY_1$. Then by the definition of recording tableau and $\theta$,

$$Q(\theta_R(u)Y_1)|_{[M]} = Q(\theta_R(u)) = Q(u) = Q(uY_1)|_{[M]} = Q(\theta_R(uY_1))|_{[M]}.$$ 

Observe that the letters of $Y_1$ are smaller than those in $u$, so $P(uY_1)$ is obtained from $P(u)$ by pushing each of the first $\mu_1$ columns down by $\eta_1$ cells and placing $Y_1$ in the vacated positions. The same is true for $P(\theta_R(u))$ and $P(\theta_{\hat{R}}(u)Y_1)$. Moreover it is clear that for all $1 \leq j \leq \mu_1$, the $j$-th column of $Q(uY_1)|_{[M+1, N]}$ is equal to the $j$-th column of $Q(Y_1) + M$. This shows the equality of recording tableaux

$$Q(\theta_R(uY_1)) = Q(uY_1) = Q(\theta_{\hat{R}}(u)Y_1).$$

For equality of $P$-tableaux, let $U = P(u)$ have shape $\lambda$, say.

Suppose first that $\lambda_1 > \mu_1$. Let $U' <_{\hat{R}} U$ be the $\hat{R}$-cocyclage on $U$ starting at the cell $s$ at the bottom of the rightmost column $\lambda_1$. Since the letters of $Y_1$ are smaller than those in $U$, it follows that $P(U'Y_1) <_{\hat{R}} P(UY_1)$. We have

$$P(\theta_R(UY_1)) = \theta_R(P(UY_1))$$

$$>_{\text{rows}(\hat{R})} \theta_R(P(U'Y_1))$$

$$= P(\theta_R(U'Y_1))$$

$$= P(\theta_{\hat{R}}(U')Y_1).$$

by the definition of $\theta_R$, Theorem [3], and and induction.

On the other hand, $\theta_{\hat{R}}(U') <_{\text{rows}(\hat{R})} \theta_R(U)$, by Theorem [3] and [18, Prop. 15]. Applying the above argument to add $\eta_1$ single rows ($\mu_1$) to the front of the sequence rows($\hat{R}$), it follows that $P(\theta_{\hat{R}}(U')Y_1) <_{\text{rows}(\hat{R})} P(\theta_R(U)Y_1)$. 


There are two rows\((R)-\)cocyclages to \(P(\theta_{R}(U')Y_{1})\) from either \(P(\theta_{R}(UY_{1}))\) or \(P(\theta_{R}(U)Y_{1})\) and both are induced by a cell in the last column on tableaux of the same shape. It follows that \(P(\theta_{R}(UY_{1})) = P(\theta_{R}(U)Y_{1})\).

The other case is that \(\lambda_{1} \leq \mu_{1}\). Consider the following way to compute \(\theta_{R}\):

\[
(R_{1}, R_{2}, \ldots) \triangleright ((\mu_{1})^{m}, R_{2}, R_{3}, \ldots) \\
\triangleright (R_{2}, R_{3}, \ldots, (\mu_{1})^{m}) \\
\triangleright \text{(rows}(\hat{R}), (\mu_{1})^{m}) \\
\triangleright \text{rows}(R).
\]

Let us describe the corresponding embeddings of sets of LR tableaux. It is not hard to show that the first embedding, which chops the first rectangle \(R_{1} = (\mu_{1})^{m}\) into its constituent rows, is the identity on the appropriate LR tableaux. The second and fourth maps are compositions of rectangle-switching bijections. The third map is equal to \(\theta_{R}\) when restricted to the alphabet \([n - m]\), and the identity on \([n - m + 1, n]\). Moreover, the two rectangle-switching steps always exchange subtableaux vertically since the number of columns in the tableau \(P(UY_{1})\) is equal to the number of columns in \(Y_{1}\) [Remark 39].

This given, tracing the image of the tableau \(P(UY_{1})\) through this map produces the tableaux \(P(UY_{1}), P((Y_{1} + (n - m))(U - m)), P((Y_{1} + (n - m))(\theta_{R}(U) - m)),\) and finally \(P(\theta_{R}(U)Y_{1})\). By the previous computation and the definition of \(\theta\) we have

\[
P(\theta_{R}(uY_{1})) = \theta_{R}(P(uY_{1})) = \theta_{R}(P(UY_{1})) \\
= P(\theta_{R}(UY_{1})) = P(\theta_{R}(U)Y_{1}) \\
= P(\theta_{R}(P(u))Y_{1}) = P(\theta_{R}(u))Y_{1} \\
= P(\theta_{R}(u)Y_{1}).
\]

Proof of Theorem [13].

Proof. The proof proceeds by induction on the number of rectangles in \(R\). Let \(S\) be a column-strict tableau of shape \(\lambda\) and content \(\gamma(\lambda)\).

For the forward direction, suppose \(S = \theta_{R}(T)\) for some \(T \in \text{LRT}(R)\). Define \(T'_{c} = w_{0}^{R}T_{c}\) and \(T'_{w} = w_{0}^{R}T_{w}\). Then \(P(T'_{c}T'_{w}Y_{1}) <_{R} T\). One may show that inducing corresponding cocyclages on \(S\) yields \(P(S_{c}S_{w}Y_{1}) <_{\{>,<\}} S\). By Theorem [13] \(\theta_{R}(T'_{c}T'_{w}Y_{1}) = S_{c}S_{w}Y_{1}\). By Lemma [13] this implies \(\theta_{R}(T'_{c}T'_{w}) = S_{c}S_{w}\), so that \(\text{colcat}(S) = P(S_{c}S_{w})\) is in the image of \(\theta_{R}\). By induction \(\text{colcat}(S)\) is \(\hat{R}\)-column-catabolizable, so that \(S\) is column-catabolizable by definition.

Conversely suppose \(S\) is \(R\)-column-catabolizable. Then \(S|_{A_{1}} = Y_{1}\) and \(\hat{S} = \text{colcat}_{R_{1}}(S) = P(S_{c}S_{w})\) is \(\hat{R}\)-column-catabolizable. By induction there is a tableau \(\hat{T} \in \text{LRT}(\hat{R})\) such that \(\theta_{R}(\hat{T}) = \hat{S}\). Since \(\theta_{R}\) preserves recording tableaux, there are tableaux \(T'_{c}\) and \(T'_{w}\) of the same shapes as \(S_{c}\) and \(S_{w}\) respectively, such that \(P(T'_{c}T'_{w}) = \hat{T}\) and \(\theta_{R}(T'_{c}T'_{w}) = S_{c}S_{w}\). By Lemma [13] \(\theta_{R}(T'_{c}T'_{w}Y_{1}) = S_{c}S_{w}Y_{1}\). Write \(T_{c} = w_{0}^{R}T'_{c}\) and \(T_{w} = w_{0}^{R}T'_{w}\). Let \(j\) be the number of letters in \(T'_{c}\). Observe that one has a sequence of covering relations in the strong \(R\)-cocyclage that gives

\[
P(\chi_{R}(T_{w}Y_{1}T_{c})) < P(T_{w}Y_{1}T_{c}).
\]
By Theorem 14, \[ \chi^j_{\text{rows}(R)}(\theta_R(T_wY_1T_c)) = \theta_R(\chi^j_{\text{rows}(R)}(T_wY_1T_c)) = \theta_R(T_wY_1T_c) = S_eS_wY_1. \]

Now \( \chi_{\text{rows}(R)} \) is merely right circular rotation since \( w_0^{\text{rows}(R)} \) is the identity. Then
\[
P(\theta_R(T_wY_1T_c)) = P(S_wY_1S_e) = S
\]
which shows that \( S \) is in the image of \( \theta \). \( \Box \)

3.4. Multi-atoms and catabolism multi-type. In this section the notion of an atom \( \bullet \) is generalized.

Let \( \gamma \) be a partition of length \( n \). Consider the collection of (dominant) sequences of rectangles \( R \) such that \( \gamma(R) = \gamma \). For each such \( R \) there is an embedding \( \theta_R : \text{LRT}(R) \to \text{CST}(\gamma) \). If \( R \cong R' \) then \( \gamma(R) = \gamma(R') \) and the image of \( \theta_R' \) contains the image of \( \theta_R \), by Theorem 4. Define the multi-atom \( \text{matom}(R) \) to be the set of \( S \in \text{CST}(\gamma) \) such that \( S \) is in the image of \( \theta_R \) but not in the image of \( \theta_R' \) where \( R \cong R' \) and \( R' \not\cong R \).

We give a somewhat awkward characterization of the image of \( \theta_R \) in CST(\gamma(R)) for arbitrary \( R \). This condition, which requires the calculation of several images of a given word of content \( \gamma \) under the automorphisms of conjugation, shows that the multi-atoms for \( R \) such that \( \gamma = \gamma(R) \), induce a set partition of CST(\gamma). The multi-atoms \( \text{matom}(R) \) where \( R \) is linearly ordered by containment, have a nicer characterization. Given a tableau \( S \) of content \( \gamma \), we construct a sequence of partitions \( \text{ctype}(S) = (\xi^1(S),\xi^2(S),\ldots) \) called its catabolism multi-type.

Conjecture 17. Let \( \gamma \) be a partition with at most \( n \) parts, \( S \in \text{CST}(\gamma) \), and \( R \) such that \( \gamma(R) = \gamma \). Then \( S \in \text{matom}(R) \) if and only if \( \text{ctype}(S) = \xi(R) \). In particular,
\[
K_{\lambda;R}(q) = \sum_{S} q^{\text{charge}(S)}
\]
where \( S \) runs over the column-strict tableaux of shape \( \lambda \), content \( \gamma \), such that \( \text{ctype}(S) \cong \xi(R) \).

Theorem 18. Conjecture 17 holds for \( R \) such that \( R_j \cong R_{j+1} \) for all \( j \).

3.5. Image characterization. Let \( \gamma \in \mathbb{Z}^n \) and \( R \) be related by \( \gamma = \gamma(R) \). For a given positive integer \( k \), let \( R^k \) be the subsequence of \( R \) consisting of the rectangles that have exactly \( k \) columns.

Proposition 19. Let \( S \in \text{CST}(\gamma(R)) \). The following are equivalent:

1. \( S \in \text{Im} \theta_R \).
2. For all \( k \geq 1 \) there is a permutation \( \sigma \in S_n \) such that \( \sigma \) sends the union of the subalphabets of the rectangles in \( R^k \), to an initial interval \( I \subseteq |n| \) and \( (\sigma S)|_I \in \text{Im} \theta_R \) where \( \sigma \) acts by an automorphism of conjugation.

Proof. Let \( S \in \text{CST}(\gamma(R)) \). Suppose that \( S \) has strictly more than \( a = \max_j \mu_j \) columns, so that there is a tableau \( T \) such that \( T <_{(>,a)} S \). By Theorem 4 and induction on charge \( R \), it is enough to show that \( S \) satisfies 2 if and only if \( T \) does. Write \( S = P(Ux) \) with \( x \) a letter, \( U \) a tableau such that the difference of the shapes of \( S \) and \( U \) is a cell \( s \) that lies in a column strictly east of the \( a \)-th, and \( T = P(xU) \). Fix \( k, \sigma, \) and \( I \) as in the hypothesis. Write \( \sigma(Ux) = U'x' \). Then \( P(x'U') <_{(>,a)} P(U'x') \), with the cocyclage starting at the same cell \( s \). If \( x' \in I \),
then since $I$ is an initial interval, the cell $s'$ that comprises the difference of the shapes of $P(U'x')|_I$ and $U'|_I$, is weakly east of $s$, so that

$$
(\sigma(T))|_I = \sigma(P(xU)|_I) = P(\sigma(xU)|_I) = P((x'U')|_I) = P((U'|_I)x') = P(U'|_I)|_I = \sigma(S)|_I.
$$

This case is completed by applying Theorem 1 for $\theta_R^k$.

Otherwise $x' \notin I$. Then $\sigma(T)|_I = \sigma(S)|_I$, and the equivalence of 2 for $k$, $T$ and $S$ is obvious in this case.

Otherwise it may be assumed that $S$ has exactly $a$ columns (and also that $R$ is dominant). If $k = a$, then $\sigma$ is the identity and $S|_I$, being a canonical rectangular tableau with $a$ columns, is automatically in the image of any map $\theta_R^k$. If $k < a$, in light of [18, Remark 39], it is not hard to show that $\sigma(S)|_I$ is also obtained by removing the rectangles in $R^k$ by restriction to the complement of an initial interval, followed by an automorphism of conjugation that moves the union of alphabets of $R^k$ to an initial interval, then restricts to that initial interval. The proof is completed by Lemma 16 and induction on the number of rectangles in $R$. \hfill \square

3.6. Nilpotent orbit case. Suppose that $R$ is a sequence of partitions, all of which have $k$ columns. Let $S$ have content $(k^n) = \gamma(R)$. We associate to $S$ a partition $\xi^k$ as follows. Consider the sequence of tableaux $S$, $V_k(S)$, $V_k^2(S)$, etc. If $V_k^j(S)$ has more than $k$ columns, then passing from $V_k^j(S)$ to $V_k^{j+1}(S)$ is a composition of $(,>k)$-cocyclages and the charge drops. So the sequence stabilizes. Let $y_k(S)$ be the positive integer $j$ such that $S|_{[j]} = \text{key}((k^j))$. Define $\xi^k_1 = y_k(S)$ and $\xi^k_{j+1} = y_k(V_k^j(S)) - y_k(V_k^{j-1}(S))$.

**Lemma 20.** For $S$ of content $(k^n)$, $\xi^k(S)$ is a partition.\hfill \square

*Proof.* By restriction to the initial subinterval $[1, i]$ where $i$ is maximal such that $\gamma_1 = \gamma_i$ and induction, it may be assumed that $\gamma = (k^n)$. Let $S$ be column-strict of content $(k^n)$ and partition shape. Write $y = y_k$, $\xi^k = \xi$ and $m = y(S)$. Let $Y$ be the column-strict tableau of shape $(k^m)$ whose $i$-th row consists of $k$ copies of the letter $i$, for all $1 \leq i \leq m$. Let $S_e$ and $S_w$ be as in the definition of $\text{cat}(k^m)(S)$ and write $\tilde{S} = \text{cat}(k^m)(S)$. By induction $\xi(\tilde{S})$ is a partition of the number $n - m$. Since $P(\tilde{SY}) = V_k(S)$, it follows that the $j$-th part of $\xi(\tilde{S})$ is the $(j-1)$-st part of $\xi(S)$. This given, it is enough to show that the second part $m'$ of $\xi(S)$ does not exceed the first part $m$. For this, without loss of generality, by restriction to the subalphabet $[m + m']$, it may be assumed that $\xi(S) = (m, m')$. This means that $P(S_e, S_w) = Y'$ where $Y'$ is the tableau of shape $(k^m)$, whose $i$-th column consists of $k$ copies of the letter $m + i$ for all $1 \leq i \leq m'$. By [18] Prop. 32, the tableau $S_w$ must be lattice in the alphabet $[m + 1, m + m']$ and the shape of $S_e$ must be the 180-degree rotation of the skew shape given by the complement of the shape of $S_w$ in the rectangular shape $(k^m)$. By the maximality of $m$, $S_w$ cannot contain all $k$ of the letters $m + 1$, so this means that $S_e$ has exactly $m'$ rows. On the other hand, since $S$ has partition shape, $S_e$ has at most $m$ rows. Therefore $m \geq m'$. \hfill \square

3.7. Catabolism multi-type. Let $\gamma$ be a partition. $\text{ctype}(S) = (\xi^1(S), \xi^2(S), \ldots)$ is defined as follows. Set $\xi^k(S) = 0$ if $k > \gamma_1$. Next, let $k = \gamma_1$. Define $\xi^k(S)$ as
before, ignoring the restriction on content. Recall that for $j$ large, $V^j_k(S)$ stabilizes. Let $\hat{S}$ be the tableau obtained by restricting this stable tableau to the numbers strictly greater than $m_k$, where $m_k$ is the number of occurrences of the part $k = \gamma_1$ in $\gamma$.

Inductively define

$$\xi^j(S) = \xi^j(\hat{S})$$

for the tableau $\hat{S}$ of content $(0^{m_k}, \gamma_{m_k+1}, \gamma_{m_k+2}, \ldots, \gamma_n)$.

**Example 21.** Using the tableau $S$ in the previous example, $\text{ctype}(S)$ is computed below.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 3 & 4 & 5 \\
2 & 2 & 2 & 4 & 5 & 6 \\
3 & 3 & 6 \\
4 & 7 & 7 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 1 & 1 & 6 & 7 \\
2 & 2 & 2 & 7 \\
3 & 3 & 3 \\
4 & 4 & 4 \\
5 & 5 \\
6 & 7 & 7 \\
\end{array}
\]

\[
\begin{array}{cccc}
V_3(S) = & 4 & 4 & 4 \\
V_3^2(S) = & 4 & 4 & 4. \\
\end{array}
\]

We have $\text{ctype}(S) = (((), (3), (2, 2), ()), \ldots)$. In this case $\text{ctype}(S) = \xi(R)$.

**Proposition 22.** Let $R$ be such that $R_j \supseteq R_{j+1}$ for all $j$ and $S \in \text{CST}(\gamma(R))$. Then $\text{ctype}(S) \supseteq \xi(R)$ if and only if $S$ is $R$-column-catabolizable.

**Proof.** Let $\gamma = \gamma(R)$. By definition $|\xi^i(S)| = |\xi^i(R)|$ for all $i$. Let $k$ be maximal such that $\xi^k(S)$ is nonempty. Write $M := \xi^k(S)_1$ and $m := \xi^k(R)_1$ so that $R_1 = (k^m)$.

Write $R' = (R_2, R_3, \ldots)$ and $\gamma' = (0^m, \gamma_{m+1}, \ldots, \gamma_n)$.

Note that $M \geq m$ since $\text{ctype}(S) \supseteq \xi(R)$ and $M$ and $m$ are the first parts of $\xi^k(S)$ and $\xi^k(R)$ respectively. On the other hand, if $S$ is $R$-column-catabolizable then $S|_{[m]} = Y_1$ and hence $M \geq m$. So without loss of generality it may be assumed that $M \geq m$ and $S|_{[m]} = Y_1$.

Divide $S$ into $Y_1$, $S_c$ and $S_w$ as in the definition of $R$-column-catabolizability. Then $S' = \text{colcat}_{R_1}(S) = P(S_cS_w)$ and $V_k(S) = P(S_cS_wY_1)$.

Because of the assumptions that $M \geq m$ and $S|_{[m]} = Y_1$, it is enough to show that the following are equivalent:

1. $\text{ctype}(S) \supseteq \xi(R)$.
2. $\text{ctype}(S') \supseteq \xi(R')$.
3. $S'$ is $R'$-column-catabolizable.
4. $S$ is $R$-column-catabolizable.

The equivalences $2 \iff 3$ and $3 \iff 4$ hold by induction and definition respectively.
1 \Leftrightarrow 2: Note that
\[ P(V^p_k(S')Y_1) = V^p_k(P(S'Y_1)) = V^{p+1}_k(S) \]
for all \( p \geq 0 \) since row insertion of \( Y_1 \) and \( V_k \) commute on all column-strict tableaux of content \( \gamma' \). It follows from the definitions that \( \xi^j(S') = \xi^j(S) \) for \( j < k \), that the first part of \( \xi^k(S') \) is \( M - m + \xi^k(S)_2 \), and that the \( j \)-th part is \( \xi^k(S)_{j+1} \) for all \( j > 1 \). Since \( M \geq m \) it follows that \( 1 \Leftrightarrow 2 \).

3.8. Image characterization revisited. Proposition 22 and Theorem 3 show that if \( R_j \supseteq R_{j+1} \) for all \( j \), then \( S \in \text{Im} \, \theta_R \) if and only if \( \text{ctype}(S) \geq \xi(R) \).

For general \( R \), in the notation of Proposition 23 one may reduce the test for \( S \in \text{Im} \, \theta_R \) to that of \( (\sigma(S))_j \in \text{Im} \, \theta_R \). But the sequence of rectangles \( R^k \) has a rearrangement that is linearly ordered by containment, since all of the rectangles in \( R^k \) have \( k \) columns. Thus the catabolism multi-type can be used for each \( k \).

4. Transposition of poset structure

The transposition maps are necessary to describe the relationship between the embeddings \( \theta \) and the cyclage-preserving embeddings \( \text{CST}(\mu) \to \text{CST}(1^n) \) given in 1.

4.1. \( R \)-cyclage. The set \( \text{LRT}(R) \) has another structure as a graded poset which shall be called the \( R \)-cyclage. Its grading function, called cocharge \( R \), is complementary to charge \( R \). But it is not the dual of the \( R \)-cocyclage poset, but rather the “transpose” of the \( R^e \)-cocyclage poset, where \( R^e \) is the sequence of rectangles given by the transposes of the rectangles in \( R \).

Let \( b = \max \epsilon_i, \eta_i \) and \( w = xu \in W(R) \) with \( x \) a letter. Denote by \( P(ux) <_R P(xu) \) for a covering relation for the \( R \)-cyclage poset; this relation holds if the cell given by the difference of the shapes of the tableaux \( P(xu) \) and \( P(u) \) is in a row strictly south of the \( b \)-th.

Let us define a function cocharge \( R : W(R) \to \mathbb{N} \) in the same way as charge \( R \) except the function \( d_{R_1,R_2} : W((R_1,R_2)) \to \mathbb{N} \) is replaced by \( \overline{d}_{R_1,R_2} : W((R_1,R_2)) \to \mathbb{N} \), where \( \overline{d}_{R_1,R_2}(w) \) is the number of cells of the shape of \( P(w) \) that are strictly below the \( \max(\eta_1, \eta_2) \)-th row. Note that \( d_{R_1,R_2}(w) + \overline{d}_{R_1,R_2}(w) = |R_1 \cap R_2| \).

Theorem 23. 1. \( \text{LRT}(R) \) is a graded poset under the \( R \)-cyclage relation with grading function cocharge \( R \).

2. A tableau in \( \text{LRT}(R) \) is \( R \)-cyclage minimal if and only if it has exactly \( b \) rows.

3. \( \tau_p \) is an isomorphism from \( \text{LRT}(R) \) under \( R \)-cyclage to \( \text{LRT}(\tau_p R) \) under \( (\tau_p R) \)-cyclage.

The proof is similar to those of 18, Theorems 19, 20. The statistics charge \( R \) and cocharge \( R \) are complementary in the following sense. Let \( r_{i,j}(R) \) be the number of indices \( k \) such that the rectangle \( R_k \) contains the cell \( (i,j) \). Write
\[ n(R) = \sum_{(i,j)} \left( \frac{r_{i,j}(R)}{2} \right). \]

In the Kostka case where \( \eta_i = 1 \) for all \( i \), \( n(R) = n(\mu) = \sum_i (i-1)\mu_i \) as in [14, I.1.5], and the maps charge \( R \) and cocharge \( R \) are the charge and cocharge statistics of Lascoux and Schützenberger.

Proposition 24. For every word \( w \in W(R) \), charge \( R(w) + \text{cocharge} \( R \)(w) = n(R). \)
Proof. Let \( R = (R_1, \ldots, R_t) \). Let us write \( \chi(P) \) to give the value 1 if \( P \) is true and 0 otherwise. Starting with the definition of charge in [18, Section 3.2], we have

\[
\begin{align*}
\text{charge}_R(w) + \text{cocharge}_R(w) &= \frac{1}{t!} \sum_{\sigma \in S_t} \sum_{i=1}^{t-1} (t-i)(d_{i,\sigma}(\sigma w) + \tilde{d}_{i,\sigma}(\sigma w)) \\
&= \frac{1}{t!} \sum_{\sigma \in S_t} \sum_{i=1}^{t-1} (t-i)\sigma(R)_i \cap \sigma(R)_{i+1} \\
&= \frac{1}{t!} \sum_{1 \leq k < l \leq t} |R_k \cap R_l| \sum_{i=1}^{t-1} (t-i) \sum_{\sigma \in S_t} \delta_{k,\sigma(i)} \delta_{l,\sigma(i+1)} \\
&= \frac{1}{t!} \sum_{1 \leq k < l \leq t} |R_k \cap R_l| \sum_{i=1}^{t-1} (t-i)(t-2)! = \frac{1}{2} \sum_{1 \leq k < l \leq t} |R_k \cap R_l| \\
&= \frac{1}{2} \sum_{1 \leq k < l \leq t} \sum_{(i,j)} \chi((i,j) \in R_k) \chi((i,j) \in R_l) = \frac{1}{2} \sum_{(i,j)} r_{i,j}(R)(r_{i,j}(R) - 1) \\
&= n(R).
\end{align*}
\]

4.2. Transpose bijection. Let \( R \) be a sequence of rectangles. Recall the definition of the subintervals \( A_i \) from [18, Section 2.2]. Let \( R_t \) denote the sequence of transposes of the rectangles of \( R \) and \( A_t^i \) the subintervals for \( R_t \).

Define the map \( \text{tr}_R : W(R) \to W(R^t) \) by \( w \mapsto w' \) where \( w' \) is obtained from \( w \) by replacing the \( c \)-th copy (from the left) of the \( r \)-th largest letter of \( A_i \) by the \( c \)-th largest letter in \( A_t^i \) for all \( i, c, \) and \( r \), and then reversing the resulting word. To see that \( \text{tr}_R(w) \in W(R^t) \), by restriction to the interval \( A_t^i \) it is enough to check it for the case \( R = (R_1) \), in which case it follows from the fact that \( w \in W((R_1)) \) if and only if \( w \) is lattice of content \( R_1 \).

Using the same words as above, one has a bijection \( \text{tr}_R : \text{LRT}(R) \to \text{LRT}(R^t) \) that transposes the shapes of tableaux. This bijection on LR tableaux appears in [8].

Example 25. Let \( R \) and \( \lambda \) be as in the previous example. We have \( R^t = ((222), (222), (33)) \), \( A_t^1 = [1,3] \), \( A_t^2 = [4,6] \) and \( A_t^3 = [7,8] \), with tableaux

\[
\begin{align*}
Y_t^1 &= 1 \ 1 \\
&\ 2 \ 2 \\
3 &\ 3
\end{align*} \quad
\begin{align*}
Y_t^2 &= 4 \ 4 \\
&\ 5 \ 5 \\
6 &\ 6
\end{align*} \quad
\begin{align*}
Y_t^3 &= 7 \ 7 \\
&\ 8 \ 8 \\
8 &\ 8
\end{align*}
\]

A tableau \( T \in \text{LRT}(\lambda; R) \) and \( \text{tr}_R(T) \in \text{LRT}(\lambda^t; R^t) \) are given below.

\[
\begin{align*}
T &= \begin{bmatrix}
1 & 1 & 1 & 3 & 3 & 5 \\
2 & 2 & 2 & 4 & 5 & 6
\end{bmatrix} \\
T^t &= \begin{bmatrix}
1 & 1 & 4 & 4 \\
2 & 2 & 5 & 7 \\
3 & 3 & 7 & 8 \\
4 & 7 & 7 & 8 \\
6 & 7 & 8 & 8
\end{bmatrix}
\end{align*}
\]
Theorem 26. The map $\text{tr}_R : \text{LRT}(R) \to \text{LRT}(R')$ is a bijection that transposes the shapes of tableaux and gives a poset isomorphism from $R$-cocycage to $R'$-cycage that sends $\text{charge}_R$ to $\text{cocharge}_{R'}$.

Remark 27. Suppose in Theorem 26 the $R$-cocycage is replaced by the strong $R$-cocycage and $R'$-cycage by an obvious definition of strong $R'$-cycage. The resulting theorem follows immediately from Propositions 31 and 32.

Before proving Theorem 26, note that it and Proposition 24 solve the following conjecture of [8].

Theorem 28. For $R$ dominant and $R'$ a dominant rearrangement of $R'$,

$$K_{\lambda',\mu'}(q) = q^{n(R)} K_{\lambda,\mu}(q^{-1})$$

Remark 29. In the Kostka case (that is, $R_i = (\mu_i)$ for all $i$), we have $n(R) = n(\mu) := \sum_i (i-1)\mu_i$, and Theorem 28 yields

$$K_{\lambda',\mu'}(q) = q^{n(\mu)} K_{\lambda,\mu}(q^{-1}) =: \bar{K}_{\lambda,\mu}(q)$$

where the left hand side is the Poincaré polynomial and the right hand side is the normalized Kostka-Foulkes polynomial. The cycage standardization map in [3] is a cocharge-preserving map from column-strict tableaux of content $\mu$ (where $\mu$ is a partition of $n$) to standard tableaux (those of content $(1^n)$), precisely the map $\text{tr}_{((1^n))} \circ \theta_{R'} \circ \text{tr}_R$. Note that $\text{tr}_{((1^n))}$ is just the obvious transposition of standard tableaux that have $n$ letters. If $T$ is the input tableau and $S$ the output tableau, then one has $\text{charge}(S) = n((1^n)) - (n(\mu) - \text{charge}(T))$ by the properties of $\text{tr}$ and $\theta$. But this means precisely that $\text{cocharge}(T) = \text{cocharge}(S)$.

Proposition 30. Let $w \in W(R)$.

(T1) $\text{tr}_{R'} \circ \text{tr}_R$ is the identity on $W(R)$ and LRT($R$).

(T2) $w$ is the row-reading word of a column-strict tableau of (possibly skew) shape $D$ if and only if $\text{tr}_R(w)$ is the column-reading word of a column-strict tableau of shape $D'$. 

(T3) $\text{tr}_R$ preserves Knuth equivalence.

(T4) $P(\text{tr}_R(w)) = \text{tr}_R(P(w))$. 

(T5) $Q(\text{tr}_R(w)) = \text{ev}(Q(w))$ where $\text{ev}$ means evacuation.

Proof. (T1) follows easily from the definitions. Let $w \in W(R)$ so that $\text{tr}_R(w) \in W(R')$. By Remark 33, $\text{rev}((\text{cstd}(w))) = \text{std}(\text{tr}_R(w))$ holds and this equality characterizes $\text{tr}_R(w)$.

By Lemma 36 (see the appendix) the maps cstd and std preserve the property of being the row-reading word of a column-strict tableau of shape $D$, send Knuth equivalence to Knuth equivalence, and preserve $Q$ tableaux.

Let $v$ be a standard word. It is obvious that $v$ is the row-reading word of a standard tableau of shape $D$, if and only if $\text{rev}(v)$ is the column-reading word of a standard tableau of shape $D'$. This proves (T2). It is well-known that $Q(\text{rev}(v)) = \text{ev}(Q(v))$, proving (T5). Reversal of standard words preserves Knuth equivalence, proving (T3) and (T4).

Proposition 31. For all $w \in W(R)$,

$$\text{tr}_R(\chi_R(w)) = \chi_R^{-1}(\text{tr}_R(w)).$$
Proof. Let \( A_j \) be the subalphabet for the transposed rectangle \( R_j' \). Now \( \chi_R \) cyclicly rotates the positions of the letters of \( A_j \) in \( w \) and \( \text{tr}_R \) sends the letters of \( A_j \) to letters of \( A_j' \) occurring in positions that are complementary with respect to the length of the word \( w \). This given, it is clear that the letters of \( A_j' \) occur in the same set of positions in both words in \([18]\). Therefore it suffices to show that the restrictions of each of these words to \( A_j' \) coincide for all \( j \). Suppose the last letter of \( w \) is in \( A_j \). Then the last letter of \( \text{tr}_R(w) \) is in \( A_j' \). By the definition of \( \chi_R \) and \( \text{tr}_R \), the restrictions of the two words to \( A_j' \) agree for all \( i \neq j \).

Thus it may be assumed that \( R = (R_1) \). The \( P \)-tableaux of both sides are equal since there is only one \( R \)-LR tableau for \( R = (R_1) \). For the standard row-insertion recording tableaux, recall from \([18]\) the promotion operator \( \text{pr}_1 \), evacuation \( \text{ev} \), and shape transposition \( (\cdot)^t \). Then

\[
Q(\text{tr}_R(\chi_R(w))) = \text{ev}(Q(\chi_R(w)))^t = \text{ev}(\text{pr}_1(Q(w)))^t = \text{pr}_1^{-1}(\text{ev}(Q(w))^t) = Q(\chi_{R_1}^{-1}(\text{tr}_R(w))).
\]

These equalities of rectangular standard tableaux hold by (T5), \([18]\). Proof of Proposition 15], the fact that \( \text{ev} \text{pr}_1 = \text{pr}_1^{-1} \text{ev} \) on standard rectangular tableaux, and the commutation of \( \text{ev} \) and \( \text{pr}_1 \) with transpose. \( \square \)

The transpose map is also compatible with rectangle-switching.

**Proposition 32.** For all \( w \in W(R) \),

\[(4.2) \quad \text{tr}_{pr}(\tau_p(w)) = \tau_p(\text{tr}_R(w)).\]

**Proposition 33.** Let \( B = A_p \cup A_{p+1} \). From the definitions, it is easy to check that at positions in the two words in \([18]\) that contain a letter not in \( B \), the two words coincide. Thus it may be assumed that \( R = (R_1, R_2) \) and \( p = 1 \). The \( P \) tableaux of the words are equal,

\[
P(\text{tr}_{pr}(\tau_p(w))) = \tau_p(\text{tr}_R(P(w))) = \tau_p(\text{tr}_R(P(w))) = P(\tau_p(\text{tr}_R(w))),
\]

by (T4), \([18]\) Theorem 9 (A3)], and the fact that \( V_{R_1} \otimes V_{R_2} \) is multiplicity-free. The \( Q \) tableaux also coincide:

\[
Q(\text{tr}_{pr}(\tau_p(w))) = \text{ev}(Q(\tau_p(w)))^t = \text{ev}(Q(w))^t = Q(\text{tr}_R(w)) = Q(\tau_p(\text{tr}_R(w))),
\]

by (T5) and \([18]\) Theorem 9 (A4)].

Proof of Theorem 24:

Proof. Let \( S <_R T \) be an \( R \)-cocyclage covering relation in \( \text{LRT}(R) \). Let \( s \) be the corner cell of the shape \( \lambda \) of \( T \), \( x \) be the letter and \( U \) the column-strict tableau of shape \( \lambda - s \), such that \( T = P(Ux) \) and \( S = P(\chi_R(Ux)) = P((w_0^R)x(w_0^R)U) \).

Write \( S' = \text{tr}_R(S), T' = \text{tr}_R(T) \), and \( x'u = \text{tr}_R(Ux) \) where \( x' \) is a letter. It must be shown that \( S' <_R T' \). We have

\[
T' = \text{tr}_R(T) = \text{tr}_R(P(Ux)) = P(\text{tr}_R(Ux)) = P(x'u)
\]

\[
S' = \text{tr}_R(S) = \text{tr}_R(P(\chi_R(Ux))) = P(\text{tr}_R(\chi_R(Ux))) = P(\chi_{R_1}^{-1}(\text{tr}_R(Ux)))
\]

Therefore, \( S' <_R T' \).
by (T4) and Proposition 31. By the definition of R-cocyclage, it is enough to show
that \( P(u) \) and \( U \) have transpose shape. Now \( Ux \) fits the skew shape \( D \otimes (1) \) where
\( D \) is the shape of the tableau \( U \). By (T2) \( \text{tr}_R(Ux) = x'U \) fits the skew shape
\( (D \otimes (1))^t = (1) \otimes D^t \). In fact by definition \( u \) is the column-reading word of a
column-strict tableau \( U' \) of shape \( D^t \). Thus \( P(u) = U' \) and \( U \) have transpose
shape, and \( \text{tr}_R \) sends the R-cocyclage to \( R^t \)-cyclage.

To show that the transpose map sends \( \text{charge}_R \) to \( \text{cocharge}_R \), by the above poset
isomorphism it is enough to show that this holds for R-cocyclage minimal tableaux.
In light of Proposition 32 it may be assumed that \( R \) is dominant.

Let \( Y_1 \) and \( Y_1^t \) denote the first Yamanouchi tableaux for \( R \) and \( R^t \) respectively,
\( R = (R_2, R_3, \ldots) \) and \( R^t = (R_2^t, R_3^t, \ldots) \). Let \( T \in \text{LRT}(\lambda; R) \) be R-cocyclage
minimal and let \( T \) consist of \( Y_1 \) sitting atop \( \hat{T} \in \text{LRT}(R) \). Since \( \text{tr}_R \) is shape-
transposing it follows that \( \text{tr}_R(T) \) is \( R^t \)-cyclage minimal and therefore consists of
\( Y_1^t \) sitting to the left of a tableau \( \hat{T}' \in \text{LRT}(\hat{R}^t) \). By the definition of \( \text{tr}_R \) it is easy
to see that \( \hat{T}' = \text{tr}_R(\hat{T}) \). Then by induction

\[
\text{charge}_R(T) = \text{charge}_R(\hat{T}) = \text{cocharge}_R(\text{tr}_R(\hat{T}))
= \text{cocharge}_R(\text{tr}_R(T) - Y_1^t) = \text{cocharge}_R(\text{tr}_R(T)).
\]

\[\square\]

5. Appendix

The purpose of this section is to describe a generalization of Schensted’s stan-
dardization map that has enough flexibility for use with the bijection \( \text{tr}_R \) on R-LR
words and tableaux. First we recall the original standardization map of Schensted
\[21\].

**Proposition 34.** Let \( \text{std} \) be the map (Schensted’s standardization) from words of
length \( N \) to words of content \((1^N)\) such that \( \text{std}(w)_i < \text{std}(w)_j \) if \( w_i < w_j \), or
\( w_i = w_j \) and \( i < j \). The following are equivalent for \( \alpha = (\alpha_1, \alpha_2, \ldots)\).

1. There is a (necessarily unique) word \( w \) of content \( \alpha \) such that \( v = \text{std}(w) \).
2. For every value \( i \) such that \( i + 1 \) precedes \( i \) in \( v \), \( i \) must have the form \( i = \alpha_1 + \cdots + \alpha_r \) for some \( r \).

Let \( A = A_1 + A_2 + \cdots + A_t \) be a segmentation of an alphabet \( A \), that is, a set
partition of \( A \) into subintervals \( A_i \) such that for all \( i < j \) then \( x < y \) for all \( x \in A_i \) and \( y \in A_j \).

Let \( R = (R_1, R_2, \ldots, R_t) \) be a sequence of partitions and \( Y = (Y_1, \ldots, Y_t) \) a
sequence of column-strict tableaux such that \( Y_i \) has shape \( R_i \) and letters in \( A_i \)
for all \( i \). Denote by \( W(R, Y) \) the set of words \( w \) in the alphabet \( A \) such that
\( P(w|_{A_i}) = Y_i \) for all \( i \). The set of R-LR words is a special case of this construction.

Let \( B = B_1 + B_2 + \cdots + B_t \) be a segmentation of the alphabet \( B \) and \( Z = (Z_1, \ldots, Z_t) \) such that \( Z_i \) is column-strict of shape \( R_i \) in the alphabet \( B_i \) for all \( i \). Due to the bijectivity of the RS correspondence, there is a unique bijection
\( \text{std}_Z : W(R, Y) \to W(R, Z) \) such that:

1. For all \( i \), the positions of the letters of \( B_i \) in \( \text{std}_Z(w) \) are equal to those of \( A_i \)
in \( w \).
2. \( P|_{B_i}(w) = Z_i \) for all \( i \).
3. \( Q|_{B_i}(w) = Q|_{A_i}(w) \) for all \( i \).
For a partition $\lambda$ of $N$, define the rowwise tableau of shape $\lambda$ to be the standard tableau of shape $\lambda$ in which the first row is comprised of the first $\lambda_1$ letters in the interval $[N]$, the second row is comprised of the next $\lambda_2$ letters in $[N]$, etc. Define the columnwise tableau of shape $\lambda$ in the same way except that rows are replaced by columns and $\lambda_i$ by $\lambda'_i$.

Let $R = (R_1, \ldots, R_t)$ be a sequence of partitions, $B = [N]$ where $N = \sum_i |R_i|$, and $B = B_1 + B_2 + \cdots + B_t$ the segmentation of $B$ such that $|B_i| = |R_i|$. For all $i$, let $Z_i$ be the rowwise standard tableau of shape $R_i$ in the alphabet $B_i$. Then the map $\text{std}^Z_i$ coincides with Schensted’s standardization map $\text{std}$ (see Lemma 36). If $Z_i$ is taken to be the columnwise standard tableau of shape $R_i$ in the alphabet $B_i$, then write $\text{cstd}$ for the map $\text{std}^Z_i$.

**Remark 35.** Now let us require that each of the $R_i$ be rectangles. Let $w \in W(R)$, so that $\text{tr}_R(w) \in W(R')$. Then by construction, $\text{rev}(\text{cstd}(w)) = \text{std}(\text{tr}_R(w))$. Moreover, this condition determines $\text{tr}_R(w)$ uniquely in light of Proposition 34. It should be mentioned that the map $\text{tr}_R$ can be defined for arbitrary sequences of partitions, or more generally, for skew shapes, using Zelevinsky’s definition of a picture [25]. Even more generally, in [17] there is a transpose construction that generalizes all of the above.

In the present case it is possible to use a direct relabeling, as opposed to the Schensted constructions that must be used in [17].

Here are the main properties of the generalized standardization maps. A non-trivial construction (a crystal lowering operator) is used in the proof.

**Lemma 36.** In the above notation,

1. $w \in W(R,Y)$ is the row-reading word of a column-strict tableau of the (skew) shape $D$ if and only if $\text{std}^Z_i(w) \in W(R,Y)$ is. In this case, if $w = \text{word}(T)$ where $T$ is column-strict of shape $D$, then write $\text{std}^Z_i(T)$ for the column-strict tableau of shape $D$ whose word is $\text{std}^Z_i(w)$.
2. If $v \sim_K w$ where $v, w \in W(R,Y)$ then $\text{std}^Z_i(v) \sim_K \text{std}^Z_i(w)$ in $W(R,Z)$.
3. $P(\text{std}^Z_i(w)) = \text{std}^Z_i P(w)$.
4. $Q(\text{std}^Z_i(w)) = Q(w)$.

**Proof.** Suppose one of the tableaux $Y_i$ is not the unique column-strict tableau of shape and content $R_i$ in the alphabet $A_i$. Then $Y_i$ admits a crystal lowering operator $e_r$, say, where $r \in A_i$. Then $e_r$ induces the bijection $\text{std}^D_r$ from $W(R,Y)$ to $W(R,Y')$ where $Y'_j = Y_j$ for $j \neq i$ and $Y'_i = e_r Y_i$. But it is well-known that $e_r$ satisfies the above properties. By applying this process to alter $Y$ and $Z$, it may be assumed that for all $i$, $Y_i$ and $Z_i$ are the unique column-strict tableaux of shape and content $R_i$ in the alphabets $A_i$ and $B_i$ respectively. In this case $\text{std}^Z_i$ is a trivial relabeling and the above properties are obviously satisfied.

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