Spinless Duffin-Kemmer-Petiau Oscillator in a Galilean Non-commutative Phase Space

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Received: 12 January 2012 / Accepted: 7 March 2012 / Published online: 23 March 2012 © Springer Science+Business Media, LLC 2012

Abstract We examine Galilei-invariant linear wave equations in a non-commutative phase space. Specifically, we establish and solve the Galilean covariant Duffin-Kemmer-Petiau equation for spin-0 fields in a harmonic oscillator potential. We obtain these wave equations with a Galilean covariant approach, based on a $(4 + 1)$-dimensional manifold with light-cone coordinates followed by a reduction to a $(3 + 1)$-dimensional spacetime. We find the exact wave functions and their energy levels, and we examine the effects of non-commutativity.

Keywords Galilean covariance · Non-commutative phase space · Duffin-Kemmer-Petiau equations

1 Introduction

In this paper, we exploit a higher-dimensional formulation of Galilean covariance to study the non-relativistic Duffin-Kemmer-Petiau (DKP) oscillator for a spin-zero field in a non-commutative phase space; that is, where both coordinates and momenta are non-commuting. The DKP wave equation, which is of first order, can be seen as a counterpart of the Dirac
equation for spin-zero and spin-one fields. Its form is similar to the Dirac equation with
the gamma matrices replaced by matrices which satisfy the so-called DKP algebra [1–5].
The fact that the DKP equation has not received much attention in the literature might be
explained by the equivalence between the Klein-Gordon equation and the DKP equation,
and the more complex algebraic structure of the latter [6, 7]. Over the years, that equivalence
has been challenged; some of these claims have allegedly been put to rest in Ref. [8]. The
relativistic DKP oscillator is discussed, for instance, in Ref. [9, 10].

As far as we know, the first paper on the idea that configuration-space coordinates do not
commute was published by Snyder in 1947 [11, 12]. According to Ref. [13–16], the idea
first came to Heisenberg in the late 1930s as a possible cure for short-distance singularities.
Heisenberg mentioned his idea to Peierls, who relayed it to Pauli, who in turn mentioned it
to Oppenheimer, who asked his student H Snyder to develop this idea. The recent interest in
non-commutative quantum mechanics was motivated by studies of the low-energy effective
theory of D-branes in the background of a Neveu-Schwarz B-field in a non-commutative
space [17–20]. Among recent applications, let us mention the quantum Hall effect on non-
commutative spaces [21–24], the Landau problem on the non-commutative plane [25–28],
planar quantum systems with central potentials [29, 30], and studies of the relativistic DKP
oscillator in a non-commutative space [31–35]. Papers investigating Galilei-invariant sys-
tems with non-commutative geometry are in Refs. [36–41].

Our main interest in the present problem stems from the connection between non-
commutative coordinates and discrete space-time, following the original paper by Sny-
der [11, 12]. We expect that a Galilean version should be of interest in condensed matter
physics for the study of non-relativistic lattice models. Particle physics and condensed mat-
ter physics share many tools of quantum field theory, for instance: gauge invariance, spon-
taneous symmetry breaking, Goldstone bosons, and so on. The Galilean covariance with a
metric in an extended manifold is but one further unifying feature. It consists in enforcing
Lorentz-like covariance (ubiquitous in high-energy physics) in a (4 + 1)-dimensional man-
ifold in such a way that the resulting theory is Galilean invariant (encountered in condensed
matter physics and low-energy physics). Note that in this paper, a (4 + 1) manifold refers to
a (3, 1) space-time augmented by 1 space-like coordinate.

A Galilean covariant theory is obtained by the addition of an extra coordinate, \( s \) or \( x^5 \),
embedded in a (4 + 1) Minkowski manifold [42–44]. This extended manifold consists of
five-vectors with coordinates

\[
x^\mu = (x^1, x^2, x^3, x^4, x^5) = (r, t, s),
\]

which transform under Galilean boosts as

\[
\begin{align*}
r' &= r - vt, \\
t' &= t, \\
s' &= s - r \cdot v + \frac{1}{2} v^2 t.
\end{align*}
\]

This transformation leaves invariant the scalar product

\[
(r, t, s) \cdot (r', t', s') \equiv r \cdot r' - ts' - t's,
\]

defined by the following metric,

\[
g^{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}.
\]
Hereafter we shall refer to this as the *Galilean metric*, even though this is equivalent to the Lorentz metric in \((4 + 1)\) space-time. The term “Galilean” describes the procedure which consists in projecting down to four space-time dimensions, thereby obtaining a Galilean theory. We note that the extra coordinate, \(s\), appears to be related to the quasi-invariance of the free particle Lagrangian under Galilean transformations, since it transforms like the phase of the quantum wavefunction that ensures the invariance of the Schrödinger equation under Galilean transformations [42–44]. If we consider “energy-mass eigenstates” \(\Psi\) that satisfy
\[
i\hbar \partial_4 \Psi = E \Psi \quad \text{and, in an analogous manner,} \quad i\hbar \partial_5 \Psi = m \Psi,\]
then we obtain
\[
p_\mu = -i\hbar \partial_\mu = (p, -E, -m), \quad (2)
\]
so that \(p^4 = -p^5 = m\) is the mass, and \(p^5 = -p^4 = E\) is the energy. Thus, it suggests that \(x^5\) could be seen as being conjugate to \(m\), similarly to time-energy conjugation relation. (The consequences of this interpretation—including a “mass-x^5 uncertainty principle”—remain to be explored.)

The relativistic analogue of the present work is described in Ref. [31], and we shall compare our results with it. Let us consider the usual position and momentum operators, \(r_i\) and \(p_i\), which satisfy the canonical commutations relations:
\[
[r_i, r_j] = 0, \quad [p_i, p_j] = 0, \quad [r_i, p_j] = i\hbar \delta_{ij}.
\]
Following Ref. [31], we consider a non-commutative space described by the operators \(\hat{r}_i\) and \(\hat{p}_i\):
\[
\hat{r}_i = r_i - \frac{\Theta_{ij}}{2\hbar} p_j = r_i + \frac{(\Theta \times p)_i}{2\hbar}, \\
\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = p_i - \frac{(\Omega \times r)_i}{2\hbar}.
\]
(3)
They satisfy the following commutation relations:
\[
[\hat{r}_i, \hat{r}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\Omega_{ij}, \quad [\hat{r}_i, \hat{p}_j] = i\hbar \Delta_{ij}, \quad (5)
\]
with \(\Theta_{ij} = \epsilon_{ijk} \Theta_k, \Omega_{ij} = \epsilon_{ijk} \Omega_k\), where \(\Theta_i\) and \(\Omega_i\) \((i = 1, 2, 3)\) are real parameters. As mentioned in Ref. [32] (see also Ref. [20, 45]), the bounds on the non-commutativity parameters are currently given by
\[
\Theta < 4 \times 10^{-40} \text{ m}^2, \quad \Omega < 1.76 \times 10^{-61} \text{ kg}^2 \text{ m}^2 / \text{s}^2.
\]
The matrix \(\Delta_{ij}\) is given by
\[
\Delta_{ij} = \left(1 + \frac{\Theta \cdot \Omega}{4\hbar^2}\right) \delta_{ij} - \frac{\Omega \Theta_j}{4\hbar^2}.
\]
From the experimental bounds on \(\Theta\) and \(\Omega\), we see that the second term in the parenthesis is less than \(10^{-33}\).

Our purpose is to apply the \((4 + 1)\)-dimensional Galilean covariant formalism to define the non-relativistic non-commutative DKP oscillator for spinless fields. In Sect. 2, we begin by outlining the commutative version of the Galilean covariant DKP equation. Then we write its non-commutative version and solve it. In both commutative and non-commutative cases, we can use projection operators, developed for the Galilean covariant DKP equation in Ref. [46].
2 Galilean DKP Oscillator in a Commutative Space

We begin this section by reviewing the Galilean DKP formulation in the commutative phase space. In Sect. 2.1, we recall from Refs. [47–49] the spinless field representation. In Sect. 2.2, we apply the projection operators of the Galilean DKP fields and focus on the spin-zero field [46]. We shall establish and discuss solutions of the DKP equations for the non-commutative Galilean covariant oscillator in Sect. 3.

The Lagrangian density for the Galilean covariant free DKP field \( \Psi \) in \((4 + 1)\) dimensions is given by

\[
L = \frac{1}{2} \overline{\Psi} \beta^\mu \partial_\mu \Psi - \frac{1}{2} \partial_\mu \overline{\Psi} \beta^\mu \Psi - k \overline{\Psi} \Psi, \quad \mu = 1, \ldots, 5. \tag{6}
\]

The adjoint of the spinor field \( \Psi \) is denoted \( \Psi^\dagger \). It is defined by \( \overline{\Psi} = \Psi^\dagger \eta \) where

\[
\eta = (\beta^4 + \beta^5)^2 + 1. \tag{7}
\]

In Eq. (6), \( k \) is a constant, and \( \beta^\mu \) are matrices that satisfy the DKP algebra [1–5, 50]

\[
\beta^\mu \beta^\rho + \beta^\rho \beta^\mu = g^{\mu\nu} \beta^\nu + g^{\rho\nu} \beta^\mu,
\]

with the metric \( g_{\mu\nu} \) given by Eq. (1). The Lagrangian in Eq. (6) leads to the Galilean DKP wave equation and its adjoint:

\[
(\beta^\mu \partial_\mu + k) \Psi = 0, \quad \overline{\Psi} (\beta^\mu \partial_\mu - k) = 0. \tag{8}
\]

With appropriate representations of the \( \beta \)-matrices, these equations describe spinless and spin-one fields (see detail in Refs. [47–49]). The \( \beta \)-matrices are given by representations of the Lie algebra so(5,1); this is analogous to the representations of so(4,1) in a 4-dimensional space-time. For the Galilean DKP wave equations, the relevant representations are six-dimensional for spinless fields (in Sect. 2.1), and 15-dimensional for spin-one fields. We will examine the spin-one field with an oscillator in a separate paper.

2.1 DKP-Oscillator Wave Equation

In Ref. [49], we utilized the following 6-by-6 representation for the spin-zero DKP field:

\[
\begin{align*}
\beta^1 &= e_{1.6} + e_{6.1}, \\
\beta^2 &= e_{2.6} + e_{6.2}, \\
\beta^3 &= e_{3.6} + e_{6.3}, \\
\beta^4 &= e_{4.6} - e_{6.5}, \\
\beta^5 &= e_{5.6} - e_{6.4}.
\end{align*}
\]

The notation \( e_{jk} \) is a shorthand for square matrices whose only non-zero entry is \( jk \); that is,

\[
(e_{jk})_{mn} = \delta_{jm} \delta_{kn}.
\]

The spin-zero oscillator is described by substituting these matrices into Eq. (8), acting of the 6-vector \( \Psi = (\psi_1, \ldots, \psi_6)^t \), where \( t \) denotes transpose. The momentum representation of Eq. (8) is

\[
(\beta^\mu p_\mu - \ii k) \Psi = 0,
\]
into which we insert the non-minimal coupling,

$$ p \rightarrow p + im\omega r. $$

(9)

The explicit form becomes

$$ \left[ \beta \cdot (p + im\omega r) + \beta^4 p_4 + \beta^5 p_5 - ik \right] \Psi = 0, $$

which leads to the equations

$$ -ik\psi_j + (p_j - im\omega r_j)\psi_6 = 0, \quad j = 1, 2, 3, $$

$$ -ik\psi_4 + p_4\psi_6 = 0, $$

$$ -ik\psi_5 + p_5\psi_6 = 0. $$

(10)

If we proceed as in Refs. [47–49]), we obtain

$$ E\psi_6 = \left( \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{3}{8}h\omega \right)\psi_6. $$

(11)

This result was obtained in Ref. [49] with the 5-dimensional Galilean covariant formalism, and through a low-velocity limit process from the relativistic DKP equation, in Ref. [51].

2.2 DKP Projectors

Given a general representation of the DKP matrices $\beta^\mu$, the selection of the scalar or vector sector can be done through projection operators [46]. The spinless sector can be selected by the operator $P$:

$$ P = -\frac{1}{2}(\beta^4 + \beta^5)^2(\beta^1)^2(\beta^2)^2(\beta^3)^2, $$

which satisfies the properties

$$ P^2 = P, $$

$$ P^\mu = P\beta^\mu, $$

$$ P^\mu \beta^\nu = P\beta^{\mu\nu}, \quad P^i\eta = P^i, \quad P\eta = -P. $$

(12)

This operator allows us to write Eq. (8) as

$$ (\beta^\mu \partial_\mu + k)(P\Psi) = 0, $$

where $P\Psi$ transforms like a scalar under Galilean boosts. Note that $P^\mu \Psi$ transforms like a pseudo-vector [46].

Instead of Eq. (9), we can consider general non-minimal couplings, that allow us to describe interactions between scalar bosons and an external vector potential $C(r)$:

$$ p \rightarrow p + C\eta. $$

From this coupling, if we consider the action of the operator $P$ on the DKP equation as in Eq. (8), and $p_\mu$ as in Eq. (2) and Refs. [47–49], we obtain the wave equation

$$ EP\Psi = \frac{1}{2m}(p^2 - C^2 - i\nabla \cdot C)P\Psi. $$

Clearly, the oscillator described in Sect. 2.1 corresponds to the special case

$$ C = im\omega r. $$

(13)
This leads to the following equation [46]:
\[
E(P/\Psi) = \left( \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2 + \frac{3}{2} \hbar \omega \right) (P/\Psi),
\]
in agreement with Eq. (11).

In Sect. 3.3, we shall need the counterpart of Eq. (12),
\[
\mu P = \beta \mu P, \tag{14}
\]
such that the wave equations for \( \Psi \) and \( \overline{\Psi} \) lead to
\[
P^{\mu} \Psi = - \frac{1}{\hbar k} \partial^{\mu} P \Psi \tag{15}
\]
and
\[
\overline{\Psi}^{\mu} P = \frac{1}{\hbar k} \partial^{\mu} \overline{\Psi} P. \tag{16}
\]
We shall use these relations, as well as
\[
\beta^{\mu} = \mu P + P^{\mu}, \tag{17}
\]
when we normalize the DKP wave functions.

3 DKP Oscillator in a Non-commutative Space

In this section, we turn to the DKP wave equation in a non-commutative phase space. We formulate these equation by substituting into the DKP equation (8) the non-commutative coordinates and momenta, \( \hat{r}_j \) and \( \hat{p}_j \), given by Eqs. (3) and (4). In Sect. 3.1, we consider a general DKP wave equation and utilize the projector approach to obtain the spin-zero equation. We determine the energy spectrum in Sect. 3.2 via the separation of variables, and describe the normalized wave functions in Sect. 3.3.

3.1 DKP Wave Equation in a Non-commutative Space

The DKP equation with a non-minimal coupling \( C \), in a non-commutative space, is written as
\[
\left( \beta^{\mu} \pi_\mu - i \hbar k \right) \Psi = 0, \tag{18}
\]
where \( \pi_\mu = (\hat{p} + C \eta, p_4, p_5) \) with \( C = C(\hat{r}) \). If we apply the operators \( P \) and \( P^{\mu} \) to each term in Eq. (18), we obtain
\[
i \hbar k P^j \Psi = \left( \hat{p}^j - C^j \right) P \Psi,
i \hbar k P^4 \Psi = -m P \Psi,
i \hbar k P^5 \Psi = -E P \Psi,
i \hbar k P \Psi = \left( (\hat{p}_i + C_i) P^i + E P^4 + m P^5 \right) \Psi,
\]
so that Eq. (18) becomes
\[
E P \Psi = \frac{1}{2m} \left( \hat{p}^2 - C^2 + [\hat{p}_i, C_i] \right) P \Psi. \tag{19}
\]
This is the wave equation for the scalar field \( P/\Psi_1 \) in a non-commutative space with a general non-minimal coupling. In other words, if we have the functional dependence for the vector potential \( C(\hat{r}) \) in a non-commutative space, then it is possible to write down the complete wave equation that describes the interaction.

For instance, the free field corresponds to \( C = 0 \). Then we can recast Eq. (19) as

\[
E P/\Psi_1 = \frac{1}{2m} \left( \frac{\hbar^2}{2m} p^2 - \frac{1}{\hbar} \Omega \cdot L + \frac{1}{4\hbar^2} (r \times \Omega)^2 + \hbar^2 k^2 \right) P/\Psi_1.
\]

This equation can be interpreted as a non-relativistic free particle in a commutative space with spin-orbit coupling in the presence of a constant magnetic field, given in terms of the non-commutative parameter vector \( \Omega \).

Now let us couple the scalar field to the three-dimensional harmonic oscillator in a non-commutative space. From Eq. (19) with the potential given in Eq. (13), we find that Eq. (19) reduces to

\[
E P/\Psi_1 = \frac{1}{2m} \left[ p^2 + m^2 \omega^2 r^2 - 3m\hbar\omega - \frac{1}{\hbar} \left( \Omega + m^2 \omega^2 \Theta \right) \cdot L \right.
\]

\[
+ \frac{1}{4\hbar^2} \left( (r \times \Omega)^2 + m^2 \omega^2 (p \times \Theta)^2 \right) - \frac{m\omega}{2\hbar} \Theta \cdot \Omega + \hbar^2 k^2 \left. \right] P/\Psi_1. \tag{20}
\]

Let us denote the field simply by \( \psi \equiv P/\Psi \). From now on, we choose the non-commutativity vectors to point in the \( z \)-direction,

\[
\Theta = (0, 0, \Theta), \quad \Omega = (0, 0, \Omega).
\]

3.2 Energy Spectrum

Hereafter, we substitute the previous expressions into the explicit representation utilized to obtain Eq. (10), and reduce these equations into a single equation for \( \psi_6 \). Equivalently, we can use Eq. (20) and substitute the values of \( \Theta \) and \( \Omega \). With cylindrical coordinates \((\rho, \phi, z)\), we obtain

\[
E \psi = \left[ -\left( \frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8\hbar^2} \right) \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + \left( \frac{1}{2} \frac{m\omega^2}{8m\hbar^2} \right) \rho^2 \right] \psi
\]

\[
+ \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m\omega^2 z^2 - \frac{3}{2} \hbar \omega \right] \psi
\]

\[
- \left[ \frac{1}{2m\hbar} (\Omega + m^2 \omega^2 \Theta) L_3 + \frac{\omega}{4\hbar} \Theta \Omega - \frac{\hbar^2 k^2}{2m} \right] \psi.
\]

We perform the separation of variables as follows:

\[
\psi(\rho, \phi, z) = \chi(\rho) \Phi(\phi) \Xi(z). \tag{21}
\]

The function \( \Phi(\phi) \) is given by

\[
\Phi(\phi) = \exp(i|m_l|\phi), \tag{22}
\]

with \( m_l \) given by

\[
L_3 \psi = m_l \hbar \psi.
\]

After dividing each term of Eq. (21) by \( \chi(\rho) \Phi(\phi) \Xi(z) \), it becomes
\[ E = -\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\chi}{d\rho} \right) + \left( \frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8} \right) \frac{m_i^2}{\rho^2} + \left( \frac{\hbar^2}{2m} + \frac{\Omega^2}{8m\hbar^2} \right) \rho^2 - \frac{m\omega^2\Theta^2}{8} \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\chi}{d\rho} \right) \frac{1}{\chi} \]

\[ -\frac{\hbar^2}{2m} \frac{d^2 \Xi}{d\rho^2} + \frac{1}{2} m\omega^2 \rho^2 \]

\[ -\frac{3}{2} \hbar \omega - \frac{m_i}{2m} (\Omega + m^2\omega^2\Theta) - \frac{\omega}{4\hbar} \Theta \Omega + \frac{\hbar^2}{2m} \theta^2 \]

Note that the terms of the first two lines on the right-hand side of Eq. (23) depend on \( \rho \) only; we set their sum equal to the constant \( E_\rho \). The third line depends on \( z \) only; we set it equal to the constant \( E_{nz} \). The remaining terms (\( E \) from the left-hand side, and the fourth line of Eq. (23)) are independent of the coordinates. Thus each set of terms is equal to a constant, and when we separate the variables, the third line of Eq. (23) gives

\[ \frac{\hbar^2}{2m} \frac{d^2 \Xi}{d\rho^2} + \left( E_{nz} - \frac{1}{2} m\omega^2 \rho^2 \right) \Xi(z) = 0, \]

and the first two lines of Eq. (23) lead to

\[ \left( \frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8} \right) \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\chi}{d\rho} \right) + \left( E_\rho - \left( \frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8} \right) \frac{m_i^2}{\rho^2} \right) \chi(\rho) = 0. \]

The constants \( E_{nz} \) and \( E_\rho \) are related to the fourth line of Eq. (23) as follows:

\[ E_{nz} + E_\rho = E + \frac{3}{2} \hbar \omega + \frac{m_i}{2m} (\Omega + m^2\omega^2\Theta) + \frac{\omega}{4\hbar} \Theta \Omega - \frac{\hbar^2}{2m} \theta^2. \]

Of course, Eq. (24) is the one-dimensional Schrödinger equation for the simple harmonic oscillator, whose solution is (for instance, see Chap. 5 of Ref. [52])

\[ \Xi(z) = 2^{-n_z/2}(n_z!)^{-1/2} \left( \frac{m\omega}{\hbar^2} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} z^2 \right) H_{n_z} \left( \sqrt{\frac{m\omega}{\hbar}} z \right), \]

where \( H_{n_z} \) denotes the Hermite polynomial of degree \( n_z \), with the corresponding energy eigenvalue given by

\[ E_{nz} = \left( n_z + \frac{1}{2} \right) \hbar \omega. \]

Let us return to the radial, or \( \rho \)-dependent, part of Eq. (25) by first rewriting it as

\[ \left[ \frac{\hbar^2}{2M} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m_i^2}{\rho^2} \right) + E_\rho - \frac{1}{2} M \omega^2_\Theta \rho^2 \right] \chi(\rho) = 0, \]

where

\[ M = \frac{4m\hbar^2}{4\hbar^2 + m^2\omega^2\Theta^2}, \]

\[ \omega_\Theta = \frac{1}{4m\hbar^2} \sqrt{(4m^2\hbar^2\omega^2 + \Omega^2)(4\hbar^2 + m^2\omega^2\Theta^2)}. \]

We notice that \( M \) becomes equal to \( m \) as the non-commutativity parameter \( \Theta \) approaches zero.
If we change the variable from $\rho$ to 
\[ y = \frac{M\omega_{\Theta,\Omega}}{2\hbar} \rho^2, \] (31)
then Eq. (29) can be cast into the form 
\[ \left( y \frac{d^2}{dy^2} + \frac{d}{dy} - \frac{m_i^2}{4y} - y + \beta \right) \chi(y) = 0, \] (32)
where 
\[ \beta = \frac{E_\rho}{\hbar \omega_{\Theta,\Omega}}. \]

This equation is the same as in the relativistic DKP equation (see Eq. (22) in Ref. [31]).

Let us introduce the function $\phi(y)$, given by 
\[ \chi(y) = e^{-y\sqrt{|m_i|}/2} \phi(y). \] (33)
If we substitute this into Eq. (32), we obtain the following differential equation for $\phi(y)$:
\[ \left[ y \frac{d^2}{dy^2} + (\gamma - 2y) \frac{d}{dy} + \beta - \gamma \right] \phi(y) = 0, \]
where $\gamma \equiv |m_i| + 1$. By taking $w \equiv 2y$ and $-2\alpha \equiv \beta - \gamma$, we finally obtain
\[ w \frac{d^2\phi}{dw^2} + (\gamma - w) \frac{d\phi}{dw} - \alpha \phi = 0. \]

This is Kummer’s differential equation, whose solution is given by the confluent hypergeometric function (see Sect. 13.1.1 in Ref. [53]), so that
\[ \phi(w) = N \left[ \, _1 F_1 (\alpha; \gamma; w) \, \right], \] (34)
where $N$ is a normalization constant, and
\[ _1 F_1 (\alpha; \gamma; w) = 1 + \frac{\alpha w}{\gamma} + \frac{(\alpha_2 w^2)}{(\gamma_2)2!} + \cdots + \frac{(\alpha_n w^n)}{(\gamma_n)n!} + \cdots, \]
with the Pochhammer symbol defined as
\[ (a)_n \equiv a(a + 1)(a + 2) \cdots (a + n - 1), \quad (a)_0 \equiv 1. \] (35)

From the boundary condition, $w \to \infty$ (which follows from $\rho \to \infty$), which implies $\phi(w) \to 0$ (so that $\psi \to 0$), we obtain
\[ \alpha = \frac{1}{2} \left( |m_i| + 1 - \frac{E_\rho}{\hbar \omega_{\Theta,\Omega}} \right) = -n_\rho, \quad n_\rho = 0, 1, 2, \ldots \]
so that
\[ E_\rho = (2n_\rho + |m_i| + 1)\hbar \omega_{\Theta,\Omega}. \] (36)

To summarize, the energy eigenvalue, $E_{n_\rho m_i n_z}$, of the DKP oscillator is obtained by substituting Eqs. (28) and (36) into Eq. (26) and solving for $E$. If we absorb $k$ within the energy, we find that
\[ E_{n_\rho m_i n_z} = (n_z - 1)\hbar \omega + (2n_\rho + |m_i| + 1)\hbar \omega_{\Theta,\Omega} - \frac{m_i}{2m} \left( \Omega + m^2 \omega^2 \Theta \right) - \frac{\omega^4}{4\hbar} \Theta \Omega. \] (37)
where $\omega_{\Theta,\Omega}$ is given in Eq. (30). The resulting energy spectrum is non-degenerate.
3.3 Normalized Wave Functions

The total wave function \( \psi(\rho, \phi, z) \), given by Eq. (21) (with \( \chi(\rho) \) obtained in Eqs. (33), (31) and (34), \( \Phi(\phi) \) given in Eq. (22), and \( \Xi(z) \) obtained in Eq. (27)), can be expressed as follows:

\[
\psi(\rho, \phi, z) = \mathcal{N} \rho^{|m|} e^{i|m|} e^{-m^2 \frac{\omega}{2M} \rho^2} \frac{M \omega_\Theta \Omega}{\hbar} e^{-\frac{M \omega_\Theta \Omega}{\hbar} \rho^2} \left( -n \rho; |m| + 1; \frac{M \omega_\Theta \Omega}{\hbar} \rho^2 \right) H_{\Xi}(\frac{m \omega}{\hbar} z),
\]

where \( \mathcal{N} \) is given by

\[
\mathcal{N} = \frac{1}{(2^n n!)^{1/2}} \left( \frac{m \omega}{h \pi} \right)^{1/4} \left( \frac{M \omega_\Theta \Omega}{2 \hbar} \right)^{|m|/2}.
\]

Our normalization follows from the fourth component, \( j^4 \), of the conserved current \( j_\mu = i \hbar k^2 m / \psi \beta_\mu / \psi \), so that we have

\[
i \hbar k \int_0^\infty \int_0^\infty \psi(4P + P^4) \psi \rho \rho d\rho d\phi = 1.
\]

If we use \( \beta^4 = 4P + P^4 \) from Eq. (17), the previous equation becomes

\[
i \hbar k \int_0^\infty \int_0^\infty \psi(4P + P^4) \psi \rho \rho d\rho d\phi = 1,
\]

so that when we substitute Eqs. (15) and (16), as well as Eq. (2), in the previous equation, we obtain

\[
i \hbar k \int_0^\infty \psi(4P + P^4) \psi \rho \rho d\rho d\phi = 1.
\]

Note that the Hermite function, which describes the oscillating motion in \( z \), is already properly normalized. Likewise, the exponential in \( \phi \) is already normalized. After integrating over \( \phi \) and \( \rho \), we find

\[
(2\pi)^{2^{-|m|}} N^2 \int_0^\infty \left( \frac{M \omega_\Theta \Omega}{\hbar} \rho^2 \right)^{|m|} e^{-\frac{M \omega_\Theta \Omega}{\hbar} \rho^2} \left( \frac{M \omega_\Theta \Omega}{\hbar} \rho^2 \right)^{|m|/2} \rho d\rho = 1.
\]

(The factor \( 2\pi \) follows from the integration over \( \phi \).

Let us define \( x = \frac{M \omega_\Theta \Omega}{\hbar} \rho^2 \), so that \( \rho d\rho = \frac{\hbar}{M \omega_\Theta \Omega} dx \). Then we find

\[
\int_0^\infty \Gamma(a) \Gamma(b) \Gamma(|m| + i + j + 1) = 1.
\]

The sums are from the Kummer functions and \( (a)_n \) is given in Eq. (35). Next, we utilize the integral \( \int_0^\infty y^{a-1} e^{-y} dy = \Gamma(a) \), we have

\[
\frac{N^2 \hbar}{2^{|m|} M \omega_\Theta \Omega} \sum_{i,j=0}^\infty \frac{(a)_i (a)_j}{(b)_i (b)_j i! j!} \Gamma(|m| + i + j + 1) = 1.
\]

This result can be written in the form

\[
\frac{N^2 \hbar \Gamma(|m| + 1)}{2^{|m|} M \omega_\Theta \Omega} \sum_{i,j=0}^\infty \frac{(a)_i (a)_j}{(b)_i (b)_j i! j!} = 1,
\]

(38)
as well as
\[
\frac{N^2 \hbar \Gamma(|m_l| + 1)}{2^{|m_l|} M \bar{\omega}_{\bar{\theta}, \Omega}} F_2[|m_l| + 1, a, a; b, b; 1, 1] = 1,
\]
where we have used the following expression for the Appell hypergeometric series:
\[
F_2[a, b, b'; c, c'; x, y] = \sum_{n,m=0}^{\infty} \frac{(a)^m(b)^m(b')^n x^m x^n}{m! n!}.
\]

On the other hand, the result in Eq. (38) can be rewritten in another way by redefining the index as \(i + j = n\); this leads to
\[
\frac{N^2 \hbar}{2^{|m_l|} M \bar{\omega}_{\bar{\theta}, \Omega}} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}!(n-i)!} = 1,
\]
which agrees with the coefficient obtained by Yang et al. [31]. Then, we can express the constant \(N\) in two forms: first, with Eq. (39),
\[
N^2 = \frac{2^{|m_l|} M \bar{\omega}_{\bar{\theta}, \Omega}}{\hbar \Gamma(|m_l| + 1)} F_2[|m_l| + 1, a, a; b, b; 1, 1],
\]
or by using Eq. (40),
\[
N^2 = \frac{2^{|m_l|} M \bar{\omega}_{\bar{\theta}, \Omega}}{\hbar} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(|m_l|+n)! a_i(a)_{n-i}}{(b)_i(b)_{n-i}!(n-i)!}.
\]

Then \(\bar{N}\) is given by
\[
\bar{N} = \left[ \frac{1}{\sqrt{\pi} \hbar^{3/2} k_{\text{ext}} |m_l| + 1} \left( \frac{M \bar{\omega}_{\bar{\theta}, \Omega}}{\hbar} \right)^{|m_l| + \frac{1}{2}} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(|m_l|+n)! a_i(a)_{n-i}}{(b)_i(b)_{n-i}!(n-i)!} \right]^{1/2}.
\]

Now let us return to the complete spinor \(\Psi\), given by Eq. (18),
\[
\Psi = \frac{1}{i \hbar \beta^\mu \pi_\mu} \Psi.
\]

With the expressions (14) and (17), this spinor can be written as
\[
\Psi = \frac{1}{i \hbar \beta^\mu \pi_\mu} \left( \mu P + P^\mu \right) \pi_\mu \Psi,
\]
as well as
\[
\Psi = \frac{1}{i \hbar k} \left[ i P (\hat{\beta}_i - C_i) + P^i (\hat{\beta}_i + C_i) + \left( \hat{\beta} \right)^4 \psi + P^4 \psi + \left( \hat{\beta} \right)^5 \psi + P^5 \psi \right] \psi,
\]
where the operator \(\hat{\beta}_i\) and \(C_i\) are written in terms of cylindrical coordinates. This expression shows us that all we need is to obtain the wave function \(P \Psi\), so that all the other components
of $\Psi$ are obtained by the derivatives with respect to the coordinates. Also, if we use the $6 \times 6$ representation presented at the beginning of Sect. 2.1, we can express the spinor as follows,

$$
\Psi = \frac{1}{i\hbar} \begin{pmatrix}
\hat{p}_1 - C_1 \\
\hat{p}_2 - C_2 \\
\hat{p}_3 - C_3 \\
p_4 \\
p_5 \\
1
\end{pmatrix} P \Psi.
$$

Next, if we apply

$$
\hat{p}_1 - C_1 = -i\hbar \partial_x + \frac{\Omega y}{2\hbar} - im\omega \left( x + i\hbar \frac{\Theta \partial_y}{2\hbar} \right)
$$

$$
= -i\hbar \left( \cos \phi \partial_\rho - \frac{\sin \phi}{\rho} \partial_\phi \right) - \frac{\Omega}{2\hbar} \rho \sin \phi
$$

$$
- im\omega \left( \rho \cos \phi + i\hbar \frac{\Theta}{2\hbar} \left( \sin \phi \partial_\rho + \frac{\cos \phi}{\rho} \partial_\phi \right) \right),
$$

to $P \Psi = \psi$, we find

$$
(\hat{p}_1 - C_1) \psi = -i\hbar \left( \cos \phi + i m\omega \frac{\Theta}{2\hbar} \sin \phi \right) \partial_\rho \psi
$$

$$
+ \left( -\frac{\hbar |m_1|}{\rho} \sin \phi + i\hbar \frac{|m_1|}{\rho} \cos \phi + \frac{\Omega}{2\hbar} \rho \sin \phi - im\omega \rho \cos \phi \right) \psi.
$$

If we perform the same operation for $\hat{p}_2 - C_2$, we find

$$
\hat{p}_2 - C_2 = -i\hbar \partial_y - \frac{\Omega x}{2\hbar} - im\omega \left( y - i\hbar \frac{\Theta \partial_x}{2\hbar} \right)
$$

$$
= -i\hbar \left( \sin \phi \partial_\rho + \frac{\cos \phi}{\rho} \partial_\phi \right) - \frac{\Omega}{2\hbar} \rho \cos \phi
$$

$$
- im\omega \left( \rho \sin \phi - i\hbar \frac{\Theta}{2\hbar} \left( \cos \phi \partial_\rho - \frac{\sin \phi}{\rho} \partial_\phi \right) \right),
$$

which, when applied to $P \Psi = \psi$, gives

$$
(\hat{p}_2 - C_2) \psi = -i\hbar \left( \sin \phi - i m\omega \frac{\Theta}{2\hbar} \cos \phi \right) \partial_\rho \psi
$$

$$
+ \left( \frac{\hbar |m_1|}{\rho} \cos \phi + i\hbar \frac{|m_1|}{\rho} \sin \phi + \frac{\Omega}{2\hbar} \rho \sin \phi - im\omega \rho \cos \phi \right) \psi.
$$

Note that

$$
\partial_\rho \psi = \overline{N} \rho |m_1| e^{i|m_1|\phi} \left( |m_1| + \frac{M \Theta}{\hbar} \rho ^2 \right) e^{- \frac{m\omega \rho^2}{2\hbar^2} - \frac{M |m_1| \rho^2}{2\hbar^2} - \frac{M \rho^2}{2\hbar^2} - \frac{\rho^2}{\hbar^2}}
$$

$$
\times _1 F_1 \left( -n_\rho; |m_1| + 1; \frac{M \Theta}{\hbar} \rho ^2 \right) H_{\rho z} \left( \frac{m\omega}{\hbar} z \right)
$$

$$
+ 2 \frac{M \Theta}{\hbar} \rho \overline{N} \rho |m_1| e^{i|m_1|\phi} e^{- \frac{m\omega \rho^2}{2\hbar^2} - \frac{M |m_1| \rho^2}{2\hbar^2} - \frac{M \rho^2}{2\hbar^2} - \frac{\rho^2}{\hbar^2}}
$$

$$
\times _1 F_1 \left( 1 - n_\rho; |m_1| + 2; \frac{M \Theta}{\hbar} \rho ^2 \right) H_{\rho z} \left( \frac{m\omega}{\hbar} z \right).$$
Therefore, we can write
\[
(\hat{p}_1 - C_1)\psi = G_{11} F_1 \left( -n_\rho; |m_l| + 1; \frac{M\Omega_1 \rho^2}{\hbar} \right) \\
+ G_{12} F_1 \left( 1 - n_\rho; |m_l| + 2; \frac{M\Omega_1 \rho^2}{\hbar} \right) H_n(z) \left( \sqrt{\frac{m\omega}{\hbar} z} \right),
\]
where
\[
G_{11} = \mathcal{N} \left[ -i\hbar \left( \cos \phi + \frac{i m\omega \Theta}{2\hbar} \sin \phi \right) \left( |m_l| + \frac{M\Omega_1 \rho^2}{\hbar} \right) \rho^{-1} \\
+ \left( -\frac{\hbar |m_l|}{\rho} \sin \phi + \frac{i\hbar |m_l|}{\rho} \cos \phi + \frac{\Omega}{2\hbar} \rho \sin \phi - im\omega \rho \cos \phi \right) \right] \Lambda H_n(z) \left( \sqrt{\frac{m\omega}{\hbar} z} \right),
\]
and
\[
G_{12} = -2i\mathcal{N} M\Omega_1 \left( \cos \phi + \frac{i m\omega \Theta}{2\hbar} \sin \phi \right) \rho \Lambda H_n(z) \left( \sqrt{\frac{m\omega}{\hbar} z} \right).
\]
The symbol \( \Lambda \) is a short-hand for
\[
\Lambda = \rho |m_l| e^{i|m_l|\phi} e^{-\frac{m\omega}{2\hbar} z^2 - \frac{M\Omega_1 \rho^2}{2\hbar}}.
\]
For \( \hat{p}_2 - C_2 \), we obtain
\[
(\hat{p}_2 - C_2)\psi = G_{21} F_1 \left( -n_\rho; |m_l| + 1; \frac{M\Omega_1 \rho^2}{\hbar} \right) \\
+ G_{22} F_1 \left( 1 - n_\rho; |m_l| + 2; \frac{M\Omega_1 \rho^2}{\hbar} \right),
\]
where
\[
G_{21} = \mathcal{N} \left[ -i\hbar \left( \sin \phi - \frac{im\omega \Theta}{2\hbar} \cos \phi \right) \left( |m_l| + \frac{M\Omega_1 \rho^2}{\hbar} \right) \rho^{-1} \\
+ \left( \frac{\hbar |m_l|}{\rho} \cos \phi + \frac{i\hbar |m_l|}{\rho} \sin \phi + \frac{\Omega}{2\hbar} \rho \sin \phi - im\omega \rho \cos \phi \right) \right] \Lambda H_n(z) \left( \sqrt{\frac{m\omega}{\hbar} z} \right),
\]
and
\[
G_{22} = -2i\mathcal{N} M\Omega_1 \left( \sin \phi - \frac{i m\omega \Theta}{2\hbar} \cos \phi \right) \Lambda H_n(z) \left( \sqrt{\frac{m\omega}{\hbar} z} \right).
\]
If we proceed similarly for \( \hat{p}_3 - C_3 = p_3 - im\omega z \), we find
\[
(p_3 - im\omega z)\psi = (-i\hbar \partial_z - im\omega z)\psi = -i\hbar \partial_z \psi \\
= G_{31} F_1 \left( -n_\rho; |m_l| + 1; \frac{M\Omega_1 \rho^2}{\hbar} \right),
\]
where
\[
G_{3} = -2i\sqrt{\frac{m\omega}{\hbar}} \mathcal{N} \Lambda_{n_z-1} \left( \sqrt{\frac{m\omega}{\hbar} z} \right).
\]
Therefore, we can rewrite the spinor \( \Psi \) as
\[ i\hbar \psi = \begin{pmatrix} G_{11} \\ G_{21} \\ G_{3} \\ E \\ m \end{pmatrix} \begin{pmatrix} -n_\rho \\ |m| + 1 \\ \frac{M\bar{\omega}_{\Theta,\Omega}}{\hbar} \rho^2 \end{pmatrix} F_1 \left(-n_\rho; |m| + 1; \frac{M\bar{\omega}_{\Theta,\Omega}}{\hbar} \rho^2\right) + \begin{pmatrix} G_{12} \\ G_{22} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 - n_\rho \\ |m| + 2; \frac{M\bar{\omega}_{\Theta,\Omega}}{\hbar} \rho^2 \end{pmatrix} F_1 \left(1 - n_\rho; |m| + 2; \frac{M\bar{\omega}_{\Theta,\Omega}}{\hbar} \rho^2\right). \]

### 4 Concluding Remarks

We have obtained and solved the Galilean DKP wave equation for spin-zero fields in the oscillator potential for a non-commutative (both for coordinates and momenta) space. We obtained the equation by a Lorentz-like approach called ‘Galilean covariance’ where we begin with manifestly covariant equations in a \((4+1)\)-dimensional manifold using light-cone coordinates, and then reduce to the Newtonian 4-dimensional space-time. We have determined the exact wave functions and the corresponding energy levels.

In order to discuss the effects of non-commutativity, notice that Eq. (30) leads to

\[
\bar{\omega}_{\Theta=0,\Omega=0} = \omega, \\
\bar{\omega}_{\Theta=0,\Omega} = \frac{1}{2m\hbar} \sqrt{4m^2\hbar^2\omega^2 + \Omega^2}, \\
\bar{\omega}_{\Theta,\Omega=0} = \frac{\omega}{2\hbar} \sqrt{4\hbar^2 + m^2\omega^2\Theta^2}.
\]

If we take \(\Omega = 0\) and \(\Theta = 0\) in Eq. (37), then the energy eigenvalues are given by

\[ E = (2n_\rho + |m| + n_z)\hbar \omega, \quad (\Omega = 0, \Theta = 0). \]

If we take only \(\Theta = 0\) in Eq. (37), this renders the momenta commuting among themselves while keeping the coordinates mutually non-commuting, and the energy eigenvalues become

\[ E = (n_z - 1)\hbar \omega + (2n_\rho + |m| + 1)\hbar \bar{\omega}_{\Theta=0,\Omega} - \frac{m \Omega}{2m}, \quad (\Theta = 0). \]

Instead, if we take only \(\Omega = 0\) in Eq. (37), so that we have commuting coordinates and non-commuting momenta in Eq. (5), then the energy is given by

\[ E = (n_z - 1)\hbar \omega + (2n_\rho + |m| + 1)\hbar \bar{\omega}_{\Theta,\Omega=0} - \frac{1}{2} m \omega^2 \Theta, \quad (\Omega = 0). \]

We are currently extending the present work in two directions: to the non-commutative Galilean covariant Dirac oscillator (or ‘Lévy-Leblond oscillator’) and the non-commutative spin-one Galilean DKP oscillator. The commutative version of the Galilean Dirac-like equation was examined by Lévy-Leblond in Ref. [54]; its Galilean covariant version is discussed in Ref. [55, 56]. The relativistic Dirac oscillator in a non-commutative phase space has been investigated in Ref. [57]. Finally, it should be interesting to consider the analogy between the oscillator in a non-commutative space and a constant magnetic field in a commutative space, especially since there exist two Galilean limits (so-called ‘electric’ and ‘magnetic’) of electromagnetism (see [58] and Santos et al. [55, 56]).
Acknowledgements

We acknowledge partial support by the Natural Science and Engineering Research Council (NSERC) of Canada (MdeM) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) of Brazil (ESS). We are grateful to the referee for useful comments.

References

1. Petiau, G.: Académie Royale de Belgique. Classe des Sciences, Mémoires, Collection (8) 16(2) (1936)
2. Duffin, R.J.: Phys. Rev. 54, 1114 (1938)
3. Kemmer, N.: Proc. R. Soc. A, Math. Phys. Eng. Sci. 166, 127 (1938)
4. Kemmer, N.: Proc. R. Soc. A, Math. Phys. Eng. Sci. 173, 91 (1939)
5. Fischbach, E., Louck, J.D., Nieto, M.M., Scott, C.K.: J. Math. Phys. 15, 60 (1974) (and references therein)
6. Fainberg, V.Ya., Pimentel, B.M.: Phys. Lett. A 271, 16 (2000)
7. Pimentel, B.M., Fainberg, V.Ya.: Theor. Math. Phys. 124, 1234 (2000)
8. Lunardi, J.T., Pimentel, B.M., Teixeira, R.G., Valverde, J.S.: Phys. Lett. A 268, 165 (2000)
9. Bednar, M., Ndimubandi, N., Nikitin, A.G.: Can. J. Phys. 75, 283 (1997)
10. Kulikov, D.A., Tutik, R.S., Yaroshenko, A.P.: Mod. Phys. Lett. A 20, 43 (2005)
11. Snyder, H.S.: Phys. Rev. 71, 38 (1947)
12. Dimakis, A., Müller-Hoissen, F., Striker, T.: J. Phys. A, Math. Gen. 26, 1927 (1993)
13. von Meyenn, K. (ed.): Letter from Heisenberg to Peierls (1930). In: Wolfgang Pauli, Scientific Correspondence, vol. 2, p. 15. Springer, Berlin (1985)
14. von Meyenn, K. (ed.): Letter from Pauli to Oppenheimer (1946). In: Wolfgang Pauli, Scientific Correspondence, vol. 3, p. 380. Springer, Berlin (1993)
15. von Meyenn, K. (ed.): Letter from Pauli to Bohr (1947). In: Wolfgang Pauli, Scientific Correspondence, vol. 2, p. 414. Springer, Berlin (1985)
16. Jackiw, R.: Nucl. Phys. B, Proc. Suppl. 127, 53 (2004)
17. Connes, A., Douglas, M.R., Schwartz, A.: J. High Energy Phys. 02, 003 (1998)
18. Douglas, M.R., Hull, C.: J. High Energy Phys. 02, 008 (1998)
19. Seiberg, N., Witten, E.: J. High Energy Phys. 09, 032 (1999)
20. Bertolami, O., Rosa, J.G., de Araújo, C.M.L., Castorina, P., Zappalà, D.: Phys. Rev. D 72, 025010 (2005)
21. Dayi, O.F., Jellal, A.: J. Math. Phys. 43, 4592 (2002) [Erratum 45, 827 (2004)]
22. Basu, B., Ghosh, S.: Phys. Lett. A 346, 133 (2005)
23. Carey, A.L., Hannabuss, K., Mathai, V.: J. Geom. Symmetry Phys. 6, 16 (2006)
24. Harms, B., Micu, O.: J. Phys. A 40, 10337 (2007)
25. Gamboa, J., Loewe, M., Mendez, F., Rojas, J.C.: Mod. Phys. Lett. A 16, 2075 (2001)
26. Dayi, O.F., Kelleyane, L.T.: Mod. Phys. Lett. A 17, 2002 (2003)
27. Horváthy, P.A.: Ann. Phys. 299, 128 (2002)
28. Alvarez, P.D., Gomis, J., Kamimura, K., Plyushchay, M.S.: Phys. Lett. B 906 (2008)
29. Gamboa, J., Loewe, M., Rojas, J.C.: Phys. Rev. D 64, 067901 (2001)
30. Bellucci, S., Yeranyan, A.: Phys. Lett. B 609, 418 (2005)
31. Yang, Z.H., Long, C.Y., Qin, S.J., Long, Z.W.: Int. J. Theor. Phys. 49, 644 (2010)
32. Guo, G., Long, C., Yang, Z., Qin, S.: Can. J. Phys. 87, 989 (2009)
33. Falek, M., Merad, M.: Commun. Theor. Phys. 50, 587 (2008)
34. Falek, M., Merad, M.: J. Math. Phys. 50, 023508 (2009)
35. Falek, M., Merad, M.: J. Math. Phys. 51, 033516 (2010)
36. Lukierski, J., Stichel, P.C., Zakrzewski, W.: Ann. Phys. 260, 224249 (1997)
37. Duval, C., Horváthy, P.A.: J. Phys. A, Math. Gen. 34, 10097 (2001)
38. Horváthy, P.A., Martina, L., Stichel, P.C.: Phys. Lett. B 564, 149 (2003)
39. Horváthy, P.A., Martina, L., Stichel, P.C.: Nucl. Phys. B 673, 301 (2003)
40. Horváthy, P.A.: Talk given at the Int. Conf. Noncommutative Geometry and Quantum Physics, Kolkata (2006). Available at arXiv:hep-th/0602133
41. Horváthy, P.A., Martina, L., Stichel, P.C.: SIGMA 6, 060 (2010)
42. Takahashi, Y.: Fortschr. Phys. 36, 63 (1988)
43. Takahashi, Y.: Fortschr. Phys. 36, 83 (1988)
44. Omote, M., Kamefuchi, S., Takahashi, Y., Ohnuki, Y.: Fortschr. Phys. 37, 933 (1989)
45. Carroll, S.M., Harvey, J.A., Kostelecký, V.A., Lane, C.D., Okamoto, T.: Phys. Rev. Lett. 87, 141601 (2001)
46. Santos, E.S., Abreu, L.M.: J. Phys. A, Math. Theor. 41, 075407 (2008)
47. de Montigny, M., Khanna, F.C., Santana, A.E., Santos, E.S., Vianna, J.D.M.: J. Phys. A, Math. Gen. 33, L273 (2000)
48. Abreu, L.M., Ferreira, F.J.S., Santos, E.S.: Braz. J. Phys. 40, 235 (2010)
49. de Montigny, M., Khanna, F.C., Santana, A.E., Santos, E.S.: J. Phys. A, Math. Gen. 34, 8901 (2001)
50. Corson, E.M.: Introduction to Tensors, Spinors, and Relativistic Wave Equations. Blackie, London (1955)
51. Nedjadi, Y., Barrett, R.C.: J. Phys. G, Nucl. Part. Phys. 19, 87 (1993)
52. Merzbacher, E.: Quantum Mechanics, 3rd edn. Wiley, New York (1998)
53. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1972)
54. Lévy-Leblond, J.M.: Commun. Math. Phys. 6, 286 (1967)
55. Santos, E.S., de Montigny, M., Khanna, F.C., Santana, A.E.: J. Phys. A, Math. Gen. 37, 9771 (2004)
56. Santos, E.S., de Melo, G.R.: Int. J. Theor. Phys. 50, 332 (2011)
57. Cai, S., Jing, T., Guo, G., Zhang, R.: Int. J. Theor. Phys. 49, 1699 (2010)
58. Le Bellac, M., Lévy-Leblond, J.M.: Nuovo Cimento B 14, 217 (1973)