Structure and Stability of Steady State Bifurcation in a Cannibalism Model with Cross-Diffusion

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Received 10 February 2020; Revised 10 June 2020; Accepted 25 June 2020; Published 21 July 2020

Academic Editor: Gen Q. Xu

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This paper deals with spatial patterns of a predator-prey crossdiffusion model with cannibalism. By applying the asymptotic analysis and Rabinowitz bifurcation theorem, we consider the local structure of steady state to the model and determine an explicit formula of the nonconstant steady state. Furthermore, the criteria of the stability/instability for the steady state with small amplitude are established.

1. Introduction

Multicomponent system is widely existed in nature and engineering applications, for instance, in combustion, chemical reactors, tumor growth, gas mixtures, and animal crowds. On the diffusive level, these systems can be described by crossdiffusion equations taking into account multicomponent diffusion and reaction [1]. Specifically, crossdiffusion is a phenomenon in which the concentration gradient of one species induces a flux of the other species. The possibility of crossdiffusion terms in multicomponent systems was proposed by Onsager and Fuoss [2], while Baldwin et al. [3] undertook the experimental verification of the existence of crossdiffusion and also observed that the crossdiffusion coefficients can be quite significant. Since then, various crossdiffusion mathematical models have been suggested to interpret and predict many interesting features of natural multicomponent dynamics [4–10].

For example, Darcy’s law implies that the velocity is proportional to the negative pressure gradient, and the pressure is defined by a state equation imposed by the volume extension of the mixture. Druet and Jungel [9] considered the convective transport in a multicomponent isothermal compressible fluid subject to the mass continuity equations. According to the idea that decomposes the system into a porous-medium-type equation for the volume extension and transport equations for the modified number fractions, they proved the global-in-time existence of classical and weak solutions in a bounded domain with no-penetration boundary conditions. Jungel and Ptashnyk [10] considered crossdiffusion systems defined in a heterogeneous medium, where the heterogeneity is reflected in spatially periodic diffusion coefficients or by the perforated domain. By combining two-scale convergence and the boundedness-by-entropy method, they proved two-scale homogenization limits of parabolic crossdiffusion systems in a heterogeneous medium with no-flux boundary conditions. Practical applications of such approach would investigate the problem of reducing a heterogeneous material to a homogeneous, specifically, predicting the response of refractory concrete to high-temperature exposure in steel furnaces. In nature, the impact of heat and mass transfer processes become particularly evident at high temperatures, where the increased pressure in pores, large temperature gradients, and temperature-induced creep may lead to catastrophic service failures [11]. Benes et al. [12] discussed a nonlinear numerical scheme arising from the implicit time discretization of the Bazant–Thongthai model (with crossdiffusion) for hygrothermal behavior of concrete at high temperatures. By theoretical analysis and numerical simulation, they found that the model reproduces well the rapid increase of pore pressure in wet concrete due to
extreme heating. In particular, there are three different zones
which can be predicted: one corresponds to elements failed
due to spalling damage, the other one indicates the part of
the structure in which the local strength is sufficiently high to
sustain the pore pressure, but its stability is lost due the
explosive spalling of the former region, and the last one
shows the portion of the cross-section is still capable of
transmitting stresses due to mechanical loading, which is
thereby responsible for the structural safety during fire. This
phenomenon is particularly meaningful for the safety as-
essment of concrete structures prone to thermally induced
spalling.

In fact, the rapid growth of the field of system biology
has further contributed to interest in reaction-diffusion
systems. Nowadays, crossdiffusion terms, which are in-
spired by the model, proposed by Shigesada et al. [13] in
1979 when they have studied the segregation of competing
species, also have attracted wide attentions used in reac-
tion-diffusion equations encountered in models from
mathematical biology [14–21]. As we all know, the growth
of biological population depends not only on time but also
on spatial distribution. Spatial species interaction includes
the self-diffusion and crossdiffusion. In this paper, we
investigate a predator-prey system with both cannibalism
and crossdiffusion in the form

\[
\begin{align*}
&u_t - d_1 \Delta u = -u + v + auw + uw, & x \in \Omega, t > 0, \\
v_t - \Delta \left( d_2 v + \frac{d_3 v}{\varepsilon + u^2} \right) = bu - gv - uv, & x \in \Omega, t > 0, \\
w_t - d_3 \Delta w = sw - rw^2 - eww, & x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega, t > 0, \\
u(x, 0) \geq 0, v(x, 0) \geq 0, w(x, 0) \geq 0, & x \in \Omega,
\end{align*}
\]

(1)

where \(u(t), v(t), \) and \(w(t)\) are the biomass of an adult
predator, a juvenile predator, and prey at time \(t\).
\(a, b, g, s, r, \varepsilon, \) and \(\varepsilon\) are positive constants. \(a\) denotes
the cannibalism rate. The positive constants \(d_1, d_2, \) and \(d_3\)
are diffusion coefficients, and \(d_4\) is the crossdiffusion coefficient.
The initial values \(u(x, 0), v(x, 0), \) and \(w(x, 0)\) are nonneg-
ative smooth functions which are not identically zero.
\(\Omega \subseteq \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega,\)
and \(v\) is the outward unit normal vector of the boundary \(\partial \Omega.\)
The homogeneous Neumann boundary condition indicates
that this system is self-contained with zero population flux
across the boundary. For more details on the backgrounds of
crossdiffusion models, one can see [21, 22].

The corresponding kinetic system of (1) was introduced
by Magnusson in [23]. In this ODE system, Magnusson
found that cannibalism, which is used as a means of population
control to prevent them from overbreeding [21–31], has a
destabilizing effect. When the large prey carries capacity, the
juvenile mortality rate is high and adult recruitment rate is
low, and stability of the equilibrium will be lost through
Hopf bifurcation, the appearance of which is caused by the
increasing level of cannibalism. And sustained oscillations
can set in for sufficiently high levels of cannibalism. After,
Kaewmanee and Tang [32] optimized the results of Mag-
nusson, they obtained that stability of the equilibrium is lost
due to the increase of the cannibalism attack rate past a
bifurcation point that depends on other parameters. Later,
the stability, topological properties, and types of bifurcations
of the ODE system have been studied more explicitly by
Marilk and Pribilova [33]. The authors proved that both
subcritical and supercritical bifurcations may occur with the
rates of two factors (the cannibalization and the benefit from
cannibalism) as bifurcation parameters.

Recently, Fu and Yang [34] considered the corre-
spanding pure diffusion system of (1) \((d_4 = 0)\) and
proved that the positive equilibrium in this system has the same
stability properties when it is regarded as equilibrium of the
ODE system. It is shown that the decisive factor of desta-
bilization for semilinear reaction-diffusion system is still
cannibalism. Therefore, they further discussed the following
strongly coupled crossdiffusion system (1).

Obviously, if \(r \neq 0\) and

\[
\begin{align*}
br + gs - gr & > 0, \\
s + eg + r(1 - ab) & > 0, \\
s(1 - ab) + (e - g)b & > 0,
\end{align*}
\]

(2)

then (1) has the unique positive equilibrium point
\(\bar{u} = (\bar{u}, \bar{v}, \bar{w}), \) where
\(\bar{u} = ((\alpha + \sqrt{\beta^2 + 4e\beta})/2e),
\bar{v} = (\bar{u}/(g + \bar{u})), \) and \(\bar{w} = ((s - \bar{e}u)/r), \) and
\(\alpha = abr + s - r - eg, \beta = br + gs - gr.\) By the linearization analysis, Fu
and Yang proved that positive equilibrium can undergo stability
switch from stable in the ODE system and semilinear system to
unstable in the crossdiffusion system if \(d_4 \) is suffi-
ciently large. This means that the decisive factor of destabiliza-
tion for the positive equilibrium point in (1) is the cross-
diffusion rate, and cannibalism is an auxiliary destabilis-
ing force. Besides, by using the Leray–Schauder degree
type, they also obtained the existence of nonconstant
positive steady states. For completeness, we introduce this
existence result, which can be proved by similar argu-
ments as Theorem 5 in [21].

**Theorem 1** (see [34]). Let \(a, b, g, s, r, \varepsilon, d_1, d_2, \) and \(d_3\) be
fixed positive constants such that \(0 < e < \varepsilon d_4, ag < 1, r > s,\)
and \((A1)\) hold. If \(\bar{u} \in (\mu_{-1}, \mu_{+1})\) \((n \geq 2)\) and the sum
\(\sum_{\nu=1}^{n} \dim E(\mu_{\nu})\) is odd, then there exists a positive constant \(d_4^*\)
such that the steady state problem corresponding to (1) has at
least one nonconstant positive solution for \(d_4 \geq d^*_4\). Here, let \(\mu_n\) and \(\mu_{n+1}\) be the eigenvalues of the operator \(-\Delta\) on \(\Omega\) with the homogeneous Neumann boundary condition.

Our main purpose in this paper is to describe in detail the local structure of the nonconstant steady states and discuss the stability and instability of bifurcation steady states.

The rest of the paper is organized as follows. In Section 2, the local bifurcation analysis is performed to examine the structure of bifurcating steady states. Furthermore, the stability and instability of bifurcation steady states with small amplitude will be given in Section 3; meanwhile, the paper ends with a brief discussion.

2. Local Structure and Formula of Steady State Bifurcation

The existence of nonconstant steady state of system (1) is established under the condition that \(\mu \in (\mu_n, \mu_{n+1})\) \((n \geq 2)\) and the sum \(\sum_{i=2}^{n} \dim E(\mu_i)\) is odd in [34]. In this section, using the asymptotic analysis and bifurcation theory similar to those in [35–38], we choose \(d_4\) as a bifurcation parameter and fix the rest of the parameter to explore the local structure of nonconstant steady states of (1) bifurcating from the constant steady state \(\bar{u}\) in one dimension.

Before proceeding, we present some properties about the negative Laplace operator. Let \(0 = \mu_1 < \mu_2 < \mu_3 < \cdots\) be the eigenvalues of the operator \(-\Delta\) on \(\Omega\) with the homogeneous Neumann boundary condition, and let \(E(\mu_i)\) be the eigenspace corresponding to \(\mu_i\) in \(H^1(\Omega)\). Let \(X\) be the closure of \([C^1(\Omega)]^3\) in \([H^1(\Omega)]^3\), \(\{\phi_{ij}; j = 1, 2, \ldots, \dim E(\mu_i)\}\) be an orthonormal basis of \(E(\mu_i)\), and \(X_{ij} = \{c\phi_{ij}; c \in \mathbb{R}\}^3\). Then,

\[
X = \bigoplus_{i=1}^{\infty} X_i, \\
X_i = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}.
\]

In particular, for \(\Omega = (0, 1)\), it is well known that the problem

\[
\begin{align*}
-w''(x) &= \mu w(x), \quad x \in (0, 1), \\
w'(x) &= 0, \quad x = 0, 1,
\end{align*}
\]

has a sequence of simple eigenvalues

\[
\mu_j = \left(\frac{\pi j}{L}\right)^2, \quad j = 0, 1, 2, \ldots,
\]

whose corresponding eigenfunctions are given by

\[
\omega_j(x) = \begin{cases} 
1, & j = 0, \\
\cos\left(\frac{\pi j x}{L}\right), & j > 0.
\end{cases}
\]

This set of eigenfunctions is an orthogonal basis in \(L^2(0, 1)\). For later use, we now define a Banach space \(X\) by

\[
X = \{(u, v, w); u, v, w \in C^2([0, 1]), u' = v' = w' = 0 \text{ at } x = 0, 1\},
\]

equipped with usual \(C^2\) norm, and a Hilbert space \(Y\)

\[
Y = L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1),
\]

with the inner product

\[
\langle (\rho_1, \rho_2, \rho_3), (u, v, w) \rangle = \langle (\rho_1, \rho_2, \rho_3), (\mu_1, \mu_2, \mu_3) \rangle \quad (\mu_1 \neq \mu_2 \neq \mu_3),
\]

for \(\rho_i = (u_i, v_i, w_i), i = 1, 2, 3\).

For the sake of simplicity, we investigate the structure of nonconstant positive steady state of (1) in one-dimensional interval \(\Omega = (0, 1)\), i.e., consider the associated elliptic problem:

\[
\begin{align*}
-d_1 u'' &= -u + v + au + w, \quad x \in (0, 1), \\
\left(-d_2 \frac{dv}{dx}\right)' &= bu - gv - uv, \quad x \in (0, 1), \\
-d_3 w'' &= sw - rw^2 - ew, \quad x \in (0, 1), \\
u' &= v' = w' = 0, \quad x = 0, 1.
\end{align*}
\]

Define the map \(F: (0, \infty) \times X \to Y\) as

\[
F(d_4, \bar{u}) = \begin{pmatrix} \Delta\left(\frac{d_4 v}{\varepsilon + u^2}\right) + bu - gv - uv \\
\Delta \frac{d_4 \omega}{\varepsilon + \bar{u}^2} + sw - rw^2 - ew \end{pmatrix}.
\]

Then, \(u = (u, v, w)\) is a solution of (10), equivalent to it is a zero-point of the map \(F\). Clearly,

\[
F(d_4, \bar{u}) = 0, \quad d_4 > 0.
\]

Notice that

\[
f(u, v) = \frac{v}{\varepsilon + u^2}
\]

\[
= f(u, v) = f(u, \bar{u}) + [f_u(u, \bar{u})](u - \bar{u})
\]

\[
+ f_v(u, \bar{u})(v - \bar{u}) + \cdots
\]

\[
= \frac{\bar{u}}{\varepsilon + \bar{u}^2} - \frac{2\bar{u}^2}{\varepsilon + \bar{u}^2}(u - \bar{u}) + \frac{1}{\varepsilon + \bar{u}^2}(v - \bar{u}) + \cdots.
\]

Assume

\[
d_4 = d_4^0 + \sum_{k=1}^{\infty} \varepsilon^k d_4^k.
\]

For nonconstant solution \(\bar{u}_i = (u_i, v_i, w_i)\) of (10) bifurcating from \(\bar{u}\) with small amplitude, let
\[
\begin{align*}
\mathcal{L}_3(v, u, w) &= (d_2 + d_4 h(\overline{u})) v'' - \left( g + \overline{u} + \frac{d_4^0 h'(\overline{u}) v}{d_1} (1 + a\overline{u}) \right) v \\
&\quad + \left( b - \overline{v} + \frac{d_4^0 h'(\overline{u}) v^2}{d_1 \overline{u}} \right) u - \frac{d_4^2 h'(\overline{u}) \overline{v} v}{d_1} w,
\end{align*}
\]

where \(0 < \tau \ll 1\). Substituting (14) and (15) into (10) and equating the \(O(\tau)\) and \(O(\tau^2)\) terms, respectively, we derive two systems

\[
\begin{align*}
\begin{cases}
    d_2 u_1'' + (-1 + a\overline{v} + \overline{w}) u_1 + (1 + a\overline{u}) v_1 + \overline{u} w_1 = 0, & x \in (0, l), \\
    \mathcal{L}_3(v_1, u_1, w_1) = 0, & x \in (0, l), \\
    d_2 w_1'' - e \overline{w} u_1 - r \overline{w} w_1 = 0, & x \in (0, l), \\
    u_1' = v_1' = w_1' = 0, & x = 0, l,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    d_2 u_2'' + (-1 + a\overline{v} + \overline{w}) u_2 + (1 + a\overline{u}) v_2 + \overline{u} w_2 = F_1, & x \in (0, l), \\
    \mathcal{L}_3(v_2, u_2, w_2) = F_2, & x \in (0, l), \\
    d_2 w_2'' - e \overline{w} u_2 - r \overline{w} w_2 = F_3, & x \in (0, l), \\
    u_2' = v_2' = w_2' = 0, & x = 0, l,
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
F_1 &= -au_1 v_1 - u_1 w_1, \\
F_3 &= ru_2^2 + eu_1 w_1, \\
h(\overline{u}) &= \frac{1}{\overline{u} + \varepsilon}, \\
h'(\overline{u}) &= \frac{-2\overline{u}}{(\overline{u} + \varepsilon)^2}, \\
h''(\overline{u}) &= \frac{2(3\overline{u}^2 - \varepsilon)}{(\overline{u} + \varepsilon)^3}, \\
F_2 &= -d_4^0 \left[ (h'(\overline{u}) u_1 v_1)'' + (h''(\overline{u}) \overline{v} u_1 u_1)' \right] - d_4^1 \left[ h(\overline{u}) v_1'' + h'(\overline{u}) \overline{v} u_1'' \right] + u_1 v_1.
\end{align*}
\]
It is easy to get nonzero solutions for (unique up to a constant multiple for any given positive integer \( j \), and this constant can be absorbed into \( \tau \) in (15)) for (17) as

\[
\begin{cases}
u_1 = e_1(j)\omega_j, \\
e_1(j) = \frac{1}{1 + a\bar{u}} \left( \frac{\bar{v}}{u} + \frac{e\bar{u}\bar{w}}{r\bar{w} + d_3\mu_j} + d_4\mu_j \right) > 0, \\
\omega_1 = h_1(j)\omega_j, \\
h_1(j) = -\frac{e\bar{w}}{r\bar{w} + d_3\mu_j} < 0,
\end{cases}
\]

as long as \( d_4^0 \) is given by

\[
d_4^0 = \frac{(g + \bar{u} + d_3\mu_j)e_1 - (b - \bar{v})}{-h(\bar{u})\mu_j \varepsilon_j - h'(\bar{u})\bar{w}\mu_j} \quad j = 1, 2, \ldots
\]

(21)

It is clearly known that \( d_4^0 > 0 \) if

\[
\frac{r\bar{w} + d_3\mu_j}{e_1} < \frac{2\bar{u}v}{e + \bar{u}^2}.
\]

(22)

Here, in order to get the uniqueness of solution, we need assume that \( d_4^{0j} \neq d_4^{0k} \) for any integer \( j \neq k \).

Set

\[
d_{4\min} = \min_{j \in \mathbb{Z}} d_{4j}^0 = \min_{j \in \mathbb{Z}} \left\{ \frac{(g + \bar{u} + d_3\mu_j)e_1 - (b - \bar{v})}{-h(\bar{u})\mu_j \varepsilon_j - h'(\bar{u})\bar{w}\mu_j} \right\} = d_4^{0j_0}
\]

(23)

for a positive integer \( j_0 \). Since \( d_4 \) is regarded as the bifurcation parameter, \( d_{4j}, j = 1, 2, \ldots \) is called the possible bifurcation location for the formation of new patterns [38]. This means that the first bifurcation occurs when the parameter \( d_4 \) crosses the bifurcation value \( d_{4\min} \). If the bifurcation is stable, it will be the pattern \((u_1, v_1, w_1)\) with formulae given in (15) and (20).

In the following, we find the formula of \( d_4^1 \). The adjoint system of the homogeneous system associated with (18) is

\[
\begin{cases}
d_4 \bar{u}''_2 + (-1 + a\bar{v} + \bar{w})\bar{w}_2 + \left( b - \bar{v} + \frac{d_4^0h'(\bar{u})\bar{v}^2}{d_4\bar{u}} \right) \bar{v}_2 - e\bar{w}\bar{w}_2 = 0, & x \in (0, l), \\
\left( d_2 + d_4^0h(\bar{u}) \right) \bar{v}_2'' + (1 + a\bar{u})\bar{w}_2 - \left( g + \bar{u} + \frac{d_4^0h'(\bar{u})\bar{v}(1 + a\bar{u})}{d_4} \right) \bar{v}_2 = 0, & x \in (0, l), \\
d_3 \bar{w}''_2 + \bar{u}\bar{w}_2 - \frac{d_4^0h'(\bar{u})\bar{v}\bar{w}}{d_4} \bar{v}_2 - r\bar{w}\bar{w}_2 = 0, & x \in (0, l), \\
\bar{u}_2' = \bar{v}_2' = \bar{w}_2' = 0, & x = 0, l,
\end{cases}
\]

(24)

which has one solution

\[
\begin{cases}
\bar{u}_2 = e_2(j)\omega_j, \\
e_2(j) = \frac{1}{1 + a\bar{u}} \left( d_2\mu_j + d_4^0h(\bar{u})\mu_j + g + \bar{u} + \frac{d_4^0h'(\bar{u})\bar{v}(1 + a\bar{u})}{d_4} \right), \\
\bar{v}_2 = \omega_j, \\
\bar{w}_2 = h_2(j)\omega_j, \\
h_2(j) = \frac{\bar{u}}{r\bar{w} + d_3\mu_j} \left( 1 + a\bar{u} \right) \left( d_2\mu_j + d_4^0h(\bar{u})\mu_j + g + \bar{u} \right) > 0,
\end{cases}
\]

(25)
where \( \omega_j \) is defined as in (6) for \( j = 1, 2, \ldots \). By the solvable condition for (18), that is, the vectors \((\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)\) and \((\tilde{v}_2, \tilde{v}_3, \tilde{w}_3)\) should be orthogonal in \( L^2(0,1) \), we have the solvability equation for \( d_x^1 \) as follows:

\[
\int_0^1 (\tilde{v}_2 \tilde{F}_1 + \tilde{v}_2 \tilde{F}_2 + \tilde{w}_3 \tilde{F}_3) \, dx = 0. \tag{26}
\]

A direct computation gives

\[
d_x^1 = d_x^j = 0, \quad \text{for all } j = 1, 2, \ldots. \tag{27}
\]

When \( d_x^1 = 0 \), for each \( j, F_2 \) in (19) can be simplified to

\[
F_2 = \frac{\epsilon_1}{2} + \left( 2d_x^0 h' (\tilde{\mu}) \epsilon_1 \mu_j + d_x^0 h'' (\tilde{\mu}) \tilde{\nu} \mu_j + \frac{\epsilon_1}{2} \right) \cos (2\sqrt{\mu_j} \chi). \tag{28}
\]

From this, a particular solution \((u_2, v_2, w_2)\) can be found as

\[
\begin{align*}
   u_2 &= a_1(j) + a_2(j) \cos (2\sqrt{\mu_j} \chi), \\
   v_2 &= a_3(j) + a_4(j) \cos (2\sqrt{\mu_j} \chi), \\
   w_2 &= a_5(j) + a_6(j) \cos (2\sqrt{\mu_j} \chi),
\end{align*}
\]

where

\[
\begin{align*}
A &= -1 + a\tilde{v} + \tilde{w}, \\
B &= 1 + a\tilde{\nu}, \\
C &= d_2 + d_x^i h (\tilde{\nu}), \\
D &= b - \tilde{v} + \frac{d_x^0 h' (\tilde{\mu}) \tilde{\nu}^2}{d_x^1 \tilde{\mu}}, \\
E &= g + \tilde{\nu} + \frac{d_x^0 h' (\tilde{\mu}) \tilde{\nu} (1 + a\tilde{\nu})}{d_x^1}, \\
F &= \frac{d_x^0 h' (\tilde{\mu}) \tilde{\nu} \tilde{\nu}^2}{d_x^1}, \\
G &= 2d_x^0 h' (\tilde{\mu}) \epsilon_1 \mu_j + d_x^0 h'' (\tilde{\mu}) \tilde{\nu} \mu_j + \frac{\epsilon_1}{2}, \\
N_1 &= (\tilde{\nu} E - BF) r h_1^2 + (\epsilon\tilde{\nu} E - \epsilon BF - \tilde{\nu} E) h_1 + \epsilon_1 \tilde{\nu} r (B - AE), \\
M_1 &= 2\tilde{\nu} [(AE + BD)r + (BF - \tilde{\nu} E)e], \\
M_2 &= 64d_1 d_2 C \mu_j^2 + 16(\tilde{\nu} d_1 C - d_2 AC + d_1 d_2 E) \mu_j^2 \\
&\quad + 4(-d_2 BD - d_2 AE - \tilde{\nu} AC + \tilde{\nu} d_1 E + \epsilon\tilde{\nu} C) \mu_j \\
&\quad - r \tilde{\nu} (AE + BD) + (\tilde{\nu} E - FB)e\tilde{w}, \\
N_2 &= 16d_1 C (ae_1 + h_1) \mu_j^2 + 4\left[ -r h_1^2 C + (\epsilon\tilde{\nu} C + \tilde{\nu} E) h_1 + \epsilon_1 (\tilde{\nu} C + d_1 E) - 2d_1 GB \mu_j + (BF - \tilde{\nu} E) r h_1^2 \\
&\quad + (\epsilon BF - \epsilon\tilde{\nu} E + \tilde{\nu} E) h_1 + (ae_1 E - 2GB) \tilde{\nu} \tilde{w},
\end{align*}
\]

\[
\begin{align*}
a_1(j) &= \frac{N_1}{M_1}, \\
a_2(j) &= \frac{N_1}{2M_2}.
\end{align*}
\]
\[ a_3(j) = \frac{1}{B} \left( A - \frac{\bar{u}e}{r} \right) a_1 - \frac{\bar{u}(r\bar{h}_1^2 + e\bar{h}_1)}{2r\bar{w}} + \frac{ae_1 + h_1}{2}, \]

\[ a_4 = \frac{1}{B} \left( 4d_1\mu_j - A + \frac{e\bar{u}\bar{w}}{4d_1\mu_j + r\bar{w}} \right) a_2 + \frac{(r\bar{h}_1^2 + e\bar{h}_1)\bar{u}}{2(4d_1\mu_j + r\bar{w})} - \frac{ae_1 + h_1}{2}, \]

\[ a_5 = -\frac{e}{r}a_1 - \frac{r\bar{h}_1^2 + e\bar{h}_1}{2r\bar{w}}, \]

\[ a_6 = -\frac{e\bar{u}a_2}{4d_3\mu_j + r\bar{w}} - \frac{r\bar{h}_1^2 + e\bar{h}_1}{2(4d_1\mu_j + r\bar{w})}. \]  

(30)

Since \( d_1^2 = 0 \), it is necessary for further analysis to obtain \( d_2^2 \). Following the process of getting (17) and (18), further computation up to the order \( O(\tau^3) \) and equating the \( O(\tau^3) \) term yields

\[
\begin{align*}
\left\{ 
& d_1u''_3 + (-1 + a\bar{v} + \bar{w})u_3 + (1 + a\bar{u})v_3 + \bar{u}w_3 = F_4, \quad x \in (0, l), \\
& \mathcal{X}(v_1, u_3, w_3) = F_5, \quad x \in (0, l), \\
& d_1w''_3 - e\bar{u}u_3 - r\bar{u}w_3 = F_6, \quad x \in (0, l), \\
& u''_1 = v''_1 = w''_1 = 0, \quad x = 0, l,
\end{align*}
\]

(31)

where

\[ F_4 = -au_1v_2 - av_1u_2 - u_1w_2 - w_1u_2, \]

\[ F_6 = 2rw_1w_2 + eu_1w_2 + eu_2w_1, \]

\[
\begin{align*}
\bar{h}''(\bar{u}) &= -\frac{24(\bar{u}^3 - \bar{u}e)}{(\bar{e} + \bar{u}^2)^3}, \\
F_5 &= -d_4^0 \left\{ \bar{h}''(\bar{u})(u_1v_2 + u_2v_1)'' + \bar{h}''(\bar{u}) \left( (\bar{v}u_1u_2)'' + \left( \frac{1}{2}u_1^2v_1'' + u_1v_1u_1'' \right) \right) + \frac{1}{2} \bar{h}''(\bar{u})\bar{v}(u_1''u_1') \right\} \\
&\quad - d_4^2(\bar{h}(\bar{u})v_1'' + \bar{h}''(\bar{u})\bar{v}u_1'' + u_1v_2 + u_2v_1).
\end{align*}
\]

(32)

The solvability condition can be simplified as

\[ \int_0^l \left( F_4e_2 \cos \left( \frac{\pi jx}{l} \right) + F_5 \cos \left( \frac{\pi jx}{l} \right) + F_6h_2 \cos \left( \frac{\pi jx}{l} \right) \right) dx = 0. \]

(33)

\[
\begin{align*}
d_4^{2j} &= \frac{-d_4^0}{4(\bar{h}(\bar{u})e_1 + \bar{h}''(\bar{u})\bar{u})} \left\{ 2\bar{h}''(\bar{u}) \left[ 2e_1a_1(j) + e_1a_2(j) + 2a_3(j) + a_4(j) \right] \\
&\quad + \bar{h}''(\bar{u}) \left[ 4a_1(j) + 2a_2(j) + 3e_1 \right] + 2h''(\bar{u})\bar{u} \right\} \\
&\quad + \frac{1}{2(\bar{h}(\bar{u})e_1 + \bar{h}''(\bar{u})\bar{u})\mu_j} \left\{ e_2(2ae_1 + h_1)(2a_4(j) + a_2(j)) + a(2a_3 + a_4) \right\}
\end{align*}
\]
According to the above computation and bifurcation theorem [38], the local bifurcation of (10) is given as follows.

**Theorem 2.** If for any positive integer \( j \neq k, \ d_4^{(j)} \neq d_4^{(k)}, \) then \( d_4^{(j)} \) is a bifurcation value of the equation \( F(d_4, u) = 0 \) with respect to the curve \( (d_4, \bar{u}), d_4 > 0. \) Moreover, there is a one-parameter family of nontrivial solutions \( \Gamma(\tau) = (d_4(\tau), u^*(\tau), v^*(\tau), w^*(\tau)) \) of (10) for \( |\tau| \) sufficiently small, where \( d_4(\tau), u^*(\tau), v^*(\tau), \) and \( w^*(\tau) \) are continuous functions such that \( d_4^{(i)} = d_4^{(j)} \) and

\[
\begin{align*}
    u^*(\tau) &= \bar{u} + \tau u_1 + \tau^2 u_2 + o(\tau^2), \\
    v^*(\tau) &= \bar{v} + \tau v_1 + \tau^2 v_2 + o(\tau^2), \\
    w^*(\tau) &= \bar{w} + \tau w_1 + \tau^2 w_2 + o(\tau^2).
\end{align*}
\]

The zero-point set of \( F(d_4, u) \) constitutes two curves \( (d_4, \bar{u}), \Gamma(\tau) \) in a neighborhood of the bifurcation point \( (d_4^{(j)}, \bar{u}). \)

To clarify the relationship between the solution \( (u_j, v_j, w_j) \) and its bifurcation location \( d_4^{(j)}, \bar{u}). \) we may relabel \( (u_j, v_j, w_j) \) as \( (u_{sj}, v_{sj}, w_{sj}) \), i.e.,

\[
\begin{cases}
    d_4\varphi'' + (-1 + a v_{sj} + w_{sj})\varphi + (1 + a u_{sj})\xi + u_{sj}\eta = \lambda\varphi, & x \in (0, l), \\
    (d_4 + d_4 h(u_{sj}))(\varphi'' + \mathcal{H}\varphi'' + \mathcal{H}\xi' + \mathcal{H}\varphi' + \mathcal{B}\xi + \mathcal{H}\varphi = \lambda\xi, & x \in (0, l), \\
    d_3\eta'' - c w_{sj}\varphi + (s - 2r w_{sj} - c u_{sj})\eta = \lambda\eta, & x \in (0, l), \\
    \varphi' = \xi' = \eta' = 0, & x = 0, l,
\end{cases}
\]

where

\[
\begin{align*}
    P &= d_4 h(u_{sj})v_{sj}, \\
    \mathcal{H} &= 2d_4 h'(u_{sj})u_{sj}, \\
    \mathcal{H} &= 2d_4(h'(u_{sj})v_{sj} + h''(u_{sj})v_{sj}u_{sj}''), \\
    \mathcal{B} &= d_4(h'(u_{sj})w_{sj} + h''(u_{sj})u_{sj}'') - u_{sj} - g, \\
    \mathcal{H} &= d_4(h'(u_{sj})v_{sj}'' + h''(u_{sj})v_{sj}u_{sj}'' + h''(u_{sj})u_{sj}''v_{sj} + 2h''(u_{sj})u_{sj}'v_{sj}) + b - v_{sj}.
\end{align*}
\]

Setting

\[
\begin{align*}
    u_{sj} &= \bar{u} + \tau u_1 + \tau^2 u_2 + \cdots + \tau^k u_k + \cdots, \\
    v_{sj} &= \bar{v} + \tau v_1 + \tau^2 v_2 + \cdots + \tau^k v_k + \cdots, \\
    w_{sj} &= \bar{w} + \tau w_1 + \tau^2 w_2 + \cdots + \tau^k w_k + \cdots.
\end{align*}
\]

As specified in [27], \((\bar{u}, \bar{v}, \bar{w})\) is called the base term of the pattern, and the pattern shape and its amplitude are primarily determined by the leading term \((u_1, v_1, w_1)\) when \( \tau \) is small.

### 3. Stability of Steady State Bifurcation

This section is devoted to study the stability of the pattern solution \( (u_{sj}, v_{sj}, w_{sj}) \) bifurcated from \( (d_4^{(j)}, \bar{u}) \) by analyzing the sign of the principal eigenvalue.

For (10), set

\[
\begin{align*}
    u &= u_{sj} + \phi e^{\lambda t}, \\
    v &= v_{sj} + \xi e^{\lambda t}, \\
    w &= w_{sj} + \eta e^{\lambda t}.
\end{align*}
\]

Then,

\[
\begin{align*}
    d_4\varphi'' + (-1 + a v_{sj} + w_{sj})\varphi + (1 + a u_{sj})\xi + u_{sj}\eta &= \lambda\varphi, & x \in (0, l), \\
    (d_4 + d_4 h(u_{sj}))(\varphi'' + \mathcal{H}\varphi'' + \mathcal{H}\xi' + \mathcal{H}\varphi' + \mathcal{B}\xi + \mathcal{H}\varphi = \lambda\xi, & x \in (0, l), \\
    d_3\eta'' - c w_{sj}\varphi + (s - 2r w_{sj} - c u_{sj})\eta &= \lambda\eta, & x \in (0, l), \\
    \varphi' = \xi' = \eta' = 0, & x = 0, l,
\end{align*}
\]
\[
\begin{aligned}
\{ \varphi &= \varphi_0 + \tau \varphi_1 + \tau^2 \varphi_2 + \cdots, \\
\xi &= \xi_0 + \tau \xi_1 + \cdots, \\
\eta &= \eta_0 + \tau \eta_1 + \cdots, \\
\lambda &= \lambda_0 + \tau \lambda_1 + \cdots, \\
\}
\end{aligned}
\quad (40)
\]

and substituting them into (38), we can obtain a system by equating the \(O(1)\) terms

\[
\begin{aligned}
u_{ij} &= \bar{u} + \tau u_1 + \tau^2 u_2 + \cdots, \\
\nu_{ij} &= \bar{v} + \tau v_1 + \tau^2 v_2 + \cdots, \\
\nu_{ij} &= \bar{w} + \tau w_1 + \tau^2 w_2 + \cdots, \\
d_{ij}^t &= d_{ij}^t + \tau d_{ij}^t + \tau^2 d_{ij}^t + \cdots, \\
\end{aligned}
\quad (41)
\]

\[
\begin{aligned}
d_1 \varphi_0'' + (-1 + a\bar{v} + \bar{u}) \varphi_0 + (1 + a\bar{u}) \xi_0 + \bar{u} \eta_0 = \lambda_0 \varphi_0, & \quad x \in (0, l), \\
\mathcal{L}_3 (\xi_0, \varphi_0, \eta_0) = \lambda_0 \xi_0 - \lambda_0 \frac{d_{ij}^t h'(\bar{u})\bar{v}}{d_{ij}^t} \varphi_0, & \quad x \in (0, l), \\
d_3 \eta''_0 - c\bar{w} \varphi_0 - r \bar{w} \eta_0 = \lambda_0 \eta_0, & \quad x \in (0, l), \\
\varphi'_0 = \xi'_0 = \eta'_0 = 0, & \quad x = 0, l,
\end{aligned}
\quad (42)
\]

where

\[
\mathcal{L}_3 (\xi, \varphi, \eta) = \left( d_2 \xi'' + d_{ij}^t h'(\bar{u}) \xi' \right) - \left( g + \bar{u} + \frac{d_{ij}^t h'(\bar{u})\bar{v}(1 + a\bar{u})}{d_{ij}^t} \right) \xi \\
+ \left( b - \bar{v} + \frac{d_{ij}^t h'(\bar{u})\bar{v}}{d_{ij}^t} \right) \varphi - \frac{d_{ij}^t h'(\bar{u})\bar{v}}{d_{ij}^t} \eta. 
\]

\quad (43)

It is obvious that the sign of \(\lambda_0\) determines the stability of the stationary solution \((u_{ij}, v_{ij}, w_{ij})\). To solve the eigenvalue problem (42), we can use \(-\mu_k (\varphi_0, \xi_0, \eta_0)\) to replace \((\varphi''_0, \xi''_0, \eta''_0)\) for some integer \(k \geq 0\). As to the existence of nonzero solution \((\varphi_0, \xi_0, \eta_0)\), we have

\[
\lambda_0^3 + k_1 (k) \lambda_0^2 + k_2 (k) \lambda_0 + k_3 (k) = 0, 
\]

\quad (44)

where

\[
\begin{aligned}
k_1 (k) &= \left( d_1 + d_2 + d_3 + d_{ij}^t h'(\bar{u}) \right) \mu_k + r \bar{w} + g + \frac{\bar{v}}{\bar{u}} > 0, \\
k_2 (k) &= \left( (d_1 + d_3) d_{ij}^t h'(\bar{u}) + d_1 d_2 + d_1 d_3 + d_2 d_3 \right) \mu_k^2 \\
&\quad + \left( (g + \bar{u} + r \bar{w}) d_1 + \left( \frac{\bar{v}}{\bar{u}} + r \bar{w} \right) d_2 + \left( \frac{\bar{v}}{\bar{u}} + g + \bar{u} \right) d_3 \right) \mu_k \\
&\quad + \left( \frac{\bar{v}}{\bar{u}} + r \bar{w} \right) h'(\bar{u}) (1 + a \bar{u} \bar{v} h'(\bar{u})) d_{ij}^t \mu_k \\
&\quad + (g + \bar{u}) r \bar{w} + \bar{u} + \frac{\bar{v}}{\bar{u}} + (e - 1) \bar{u} \bar{w} - b (1 + a \bar{u}) \\
&\quad + \frac{(r \bar{w} + g) \bar{v}}{\bar{u}} - \frac{d_{ij}^t h'(\bar{u})}{d_{ij}^t} \left( \left( \frac{\bar{v}}{\bar{u}} + a \bar{u} \bar{v} \right)^2 + \bar{u} \bar{v} (\bar{v} - 1) (1 + a \bar{u}) \right),
\end{aligned}
\]
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for the stability of $k$ and the steady state the principle wave mode. This means that the first bifurcation $(u_{sj}, v_{sj}, w_{sj})$ for some $k$ and $d_3$. It follows from the Routh–Hurwitz stability criteria that $d_3 > 0$ for all $k$. Hence there exist some eigenvalues with positive real parts if one of the conditions above fails for some $k$ in $\mathbb{N}^*$. We set $d_{4 \text{min}} = d_{4 \text{min}}^j$ for a positive integer $j_0$ which is called the principle wave mode. This means that the first bifurcation will occur when the parameter $d_4$ crosses the bifurcation value $d_{4 \text{min}}$. Hence, $d_{4 \text{j}}^j > d_{4 \text{min}}$ if $j \neq j_0$ and there exists an integer $k = j_0$ such that $k_3 < 0$. As such, equation (44) at least has a root with positive real part. Therefore, a necessary condition for the stability of $(u_{sj}, v_{sj}, w_{sj})$ is stated as below.

**Theorem 3** (stability criterion). If $j \neq j_0$, then $d_{4 \text{j}}^j > d_{4 \text{min}}$ and the steady state $(u_{sj}, v_{sj}, w_{sj})$ in (15) is unstable. In other words, if $(u_{sj}, v_{sj}, w_{sj})$ is stable, then $j = j_0$.

\[ k_3(k) = \left( d_1 d_2 d_3 h(\bar{u}) + d_1 d_2 d_3 \right) \mu_0^2 + \left( d_1 d_2 d_3 h(\bar{u}) + \bar{v}(1 + a\bar{u})d_3 h'(\bar{u}) \right) d_4^j + r \bar{u} d_1 + (g + \bar{u}) d_1 + \left( d_1 d_2 d_3 \right) \mu_0^2 + \left( d_1 d_2 d_3 \right) \mu_0^2 \]

and $d_{4 \text{j}}^j$ is given by (21).

Now, we determine the stability of the steady state $(u_{sj}, v_{sj}, w_{sj})$. When $j = j_0$, we have $d_{4 \text{j}}^j = d_{4 \text{min}}$. Obviously, the principal eigenvalue for (42) is $\lambda_0 = 0$ with eigenvector:

\[ \begin{pmatrix} \varphi_0, \xi_0, \eta_0 \end{pmatrix} = \begin{pmatrix} \omega_{j_0} \frac{1}{1 + a\bar{u}} \left( \frac{\bar{v}}{\bar{u}} + \frac{e\bar{u} \bar{v}}{r \bar{u} + d_3 \mu_0} + d_4 \mu_0 \right) \omega_{j_0}, e\bar{u} \frac{\bar{v}}{r \bar{u} + d_3 \mu_0} \omega_{j_0} \end{pmatrix} \]

In order to obtain the stability of $(u_{sj}, v_{sj}, w_{sj})$, we need to evaluate $\lambda_1$ and $\lambda_2$. Again substituting (40) and (41) into (38) and equating the $O(\tau)$ terms, we have

\[ \begin{align*}
   d_1 \varphi'' + (-1 + a\bar{v} + \bar{u}) \varphi_1 + (1 + a\bar{u}) \xi_1 + \bar{u} \eta_1 &= \lambda_1 \varphi_0 + G_1, \quad x \in (0, l), \\
   \mathcal{L}_3^j(\xi_0, \varphi_0, \eta_0) &= \lambda_1 \xi_0 - \lambda_1 \bar{d}_4^j h'(\bar{u}) \varphi_0 + G_2, \quad x \in (0, l), \\
   d_3 \eta'' - e\bar{u} \varphi_1 - r \bar{u} \eta_1 &= \lambda_1 \eta_0 + G_3, \quad x \in (0, l), \\
   \varphi_1 = \xi_1 = \eta_1 &= 0, \quad x = 0, l,
\end{align*} \]
where

\[ G_1 = -(an_1 + w_1)\varphi_0 - au_1\xi_0 - u_1\eta_0, \]
\[ G_2 = -d_4^{ij} h' (\bar{u}) \nu_1 \varphi_0 + h'' (\bar{u}) \nu_1 \varphi_0 + h' (\bar{u}) u_1 \xi_0 \]
\[ + v_1 \varphi_0 + u_1 \xi_0, \]
\[ G_3 = ew_1 \varphi_0 + (2w_1 + eu_1)\eta_0. \]

The solvability condition for equation (47) gives

\[ \int_0^l (\lambda_1 \phi_0 + G_1) \overline{\pi}_2 \, dx + \int_0^l \left[ \lambda_1 \xi_0 - \lambda_1 \frac{d_4^{ij} h' (\bar{u}) \nu}{d_1} \varphi_0 + G_2 \right] \overline{\pi}_2 \, dx \]
\[ + \int_0^l (\lambda_1 \eta_0 + G_3) \overline{\omega}_2 \, dx = 0, \]

where \((\overline{\pi}_2, \overline{\pi}_2, \overline{\omega}_2)\) is given in (24) with \(j = j_0\). Therefore,

\[ \lambda_1 = \frac{\int_0^l (G_1 \overline{\pi}_2 + G_2 \overline{\pi}_2 + G_3 \overline{\omega}_2) \, dx}{\int_0^l (\phi_0 \overline{\pi}_2 + (\xi_0 - (d_4^{ij} h' (\bar{u}) \nu) / d_1) \varphi_0) \overline{\pi}_2 + \eta_0 \overline{\omega}_2) \, dx}. \]

A simple calculation gives

\[ \int_0^l (G_1 \overline{\pi}_2 + G_2 \overline{\pi}_2 + G_3 \overline{\omega}_2) \, dx = 0. \]  

(52)

Thus, \(\lambda_1 = 0\). For this reason, we need further to compute \(\lambda_2\). Since \(G_2\) can be expressed as

\[ G_2 = c_1 + \left(4d_4^{ij} h' (\bar{u}) \nu_1 \mu_j + 2d_4^{ij} h'' (\bar{u}) \nu_1 \mu_j + c_1 \right) \cos (2\sqrt{\mu} x), \]

and we can find a particular solution \((\varphi_1, \xi_1, \eta_1)\) of (51) as

\[ \varphi_1 = \overline{\pi}_1 + \overline{\pi}_2 + 2\sqrt{\mu} x, \]
\[ \xi_1 = \overline{\pi}_3 + \overline{\pi}_4 + 2\sqrt{\mu} x, \]
\[ \eta_1 = \overline{\pi}_5 + \overline{\pi}_6 + 2\sqrt{\mu} x, \]

where \(\overline{\pi}_i = 2\alpha_i (j_0), \quad i = 1, \ldots, 6, \)

with \(\alpha_i (j_0), i = 1, \ldots, 6, \) as defined in (30). A further similar computation by equating the \(O(\tau^2)\) term gives the following system:

\[ \begin{cases} 
    d_1 \varphi'' + (-1 + a\bar{v} + \bar{u}) \varphi_2 + (1 + a\bar{u}) \xi_2 + \bar{u} \eta_2 = \lambda_2 \varphi_0 + G_4, & x \in (0, l), \\
    -d_0 \xi' + \frac{d_0 h' (\bar{u}) \nu}{d_1} \varphi_0 + G_5, & x \in (0, l), \\
    d_2 \eta'' - e\bar{u} \varphi_2 - r\varphi_2 + \lambda_2 \eta_0 + G_6, & x \in (0, l), \\
    \varphi_2 = \xi_2 = \eta_2 = 0, & x = 0, l,
\end{cases} \]

where


\[ G_d = -(av_1 + w_1)u_0 - (av_1 + w_1)v_1 - au_1u_0 - au_1v_0 - u_1u_0 - u_1v_0, \]

\[ G_d = -d_4h'((u))\left[\left(\frac{1}{2}d_4^h\xi_0 + u_1u_0\right)\right] ' + h''((u))\left\{\left(\frac{1}{2}d_4^h\xi_0 + u_1u_0\right)\right\} ' - d_4^h\left[h((u))e_0 + h'((u))v_0\right] + u_1\xi_0 + u_1\xi_0 + v_1 + v_2. \]

\[ G_d = csw_0 + csw_1 + (r2csw_2 + csw_0)\eta_0 + (r2csw_1 + csw_0)\eta_1. \]

Similarly, by applying the solvability condition of (56), we have

\[ \lambda_2 = -\int_0^l \left(G_d\frac{d_2}{\xi_0} + G_d\eta_2 + G_d\eta_2\right) \frac{d\xi_0}{d_4^h} \]

\[ A \text{ tedious computation leads to} \]

\[ \int_0^l \left(G_d\xi_0 + G_d\eta_2 + G_d\eta_2\right) \frac{d\xi_0}{d_4^h} = -2\mu \lambda, \]

with

\[ \lambda = [h((u))e_0 + h'((u))v_0]d_4^h. \]

\( \lambda_2 < 0 \) and the small-amplitude steady state \((u_{ij}, v_{ij}, w_{ij})\) is stable.

The following example can illustrate the result of Theorem 4.

**Example 1.** For (1), in the interval \( l = [0, 10n] \), set \( a = 1, b = 0.5, g = 0.6, s = 1.5, r = 1.6, \varphi = 1, d_1 = 0.3, d_2 = 0.2, d_3 = 0.2, \) and \( d = 0.01, \) and choose \( d_4 \) as the variable parameter. It is not hard to verify that the equilibrium point \( u = (0.91, 0.3, 0.37), \) \( d_4^h = 6.02 (j_0 = 9), \) and \( d_4^h = 0.9, 1.19 < 1 < 1.9. \) From Theorem 4, \((u_{ij}, v_{ij}, w_{ij})\) is stable.

In conclusion, we study a predator-prey crossdiffusion model with cannibalism and discuss the effects of cross-diffusion on steady-state solutions of (1). We obtain the local structure of steady state bifurcation from the homogeneous steady states \((u_0, v_0, w_0)\) in one dimension by treating the crossdiffusion coefficient \( d_4 \) as a bifurcation parameter. Moreover, we establish the stability criteria and find a selection mechanism of the principal eigenvalue. More specifically, if the stationary pattern is stable, then its principal wave mode \( j \) is a positive integer \( j_0 \) at which \( d_4^h \) is minimized. And all the rest bifurcations are unstable if \( j \neq j_0 \). In addition, the stable patterns can be derived if \( j = j_0 \) and \( \lambda_2 < 0 \).

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Acknowledgments**

This research was supported by the National Natural Science Foundation of China (nos. 11761063 and 11661051).

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