UNIFORM ESTIMATES OF RESOLVENTS IN HOMOGENIZATION THEORY OF
ELLiptIC SYSTEMS

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Abstract. In this paper, we study the estimates of resolvents
\[ R(\lambda, L_\varepsilon) = (L_\varepsilon - \lambda I)^{-1} \]
where
\[ L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) \]
is a family of second elliptic operators with symmetric, periodic and oscillating coefficients defined on
a bounded domain \( \Omega \) with \( \varepsilon > 0 \). For \( 1 < p < \infty \), we will establish uniform \( L^p \to L^p \), \( W^{-1,p} \to L^p \) and \( W^{-1,p} \to W^{1,p}_0 \) estimates by using the real variable method. Meanwhile, we use Green
functions for operators \( L_\varepsilon - \lambda I \) to study the asymptotic behavior of \( R(\lambda, L_\varepsilon) \) and obtain convergence
estimates in \( L^p \to L^p \), \( L^p \to W^{1,p}_0 \) norm.

Keywords: Homogenization; Uniform estimates; Resolvents; Convergence.

1. Introduction and main results

The main purpose of this paper is to study estimates of resolvents for a family of elliptic operators with
rapidly oscillating and symmetric coefficients. Precisely speaking, we consider the resolvents of operators
\[ L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) = -\partial_{x_i}\left\{ a_{ij}^{i\beta}(x/\varepsilon) \partial_{x_j}\right\}, \quad \varepsilon > 0, \]
where \( 1 \leq i,j \leq d \) and \( 1 \leq \alpha,\beta \leq m \). Here, \( d \geq 2 \) denotes the dimension of Euclidean space and \( m \geq 1 \) is the number of equations in the system. The summation convention for repeated indices is used
throughout the paper. In discussion, we will always assume that the measurable matrix-valued functions
\[ A(y) = (a_{ij}^{i\alpha}(y)) : \mathbb{R}^d \to \mathbb{R}^{m \times d^2} \] satisfy the symmetry condition
\[ a_{ij}^{i\beta}(y) = a_{ij}^{j\alpha}(y) \]
the uniform ellipticity condition
\[ \mu|\xi|^2 \leq a_{ij}^{i\beta}(y)\xi^i_\alpha \xi^j_\beta \leq \mu^{-1}|\xi|^2 \]
for any \( y \in \mathbb{R}^d \) and \( \xi = (\xi^i_\alpha) \in \mathbb{R}^{m \times d} \),
where \( \mu > 0 \) is a positive constant and the periodicity condition
\[ A(y + z) = A(y) \]
for any \( y \in \mathbb{R}^d \) and \( z \in \mathbb{Z}^d \).

To ensure \( L^p \), \( W^{1,p} \) and Lipschitz estimates of operators \( L_\varepsilon \), in some situations, we need more smoothness
conditions on the coefficients matrix \( A \), i.e. the Hölder regularity of \( A \),
\[ |A(x) - A(y)| \leq \tau|x - y|^\nu \]
for any \( x, y \in \mathbb{R}^d \),
where \( \tau > 0, \nu \in (0,1) \) and the VMO condition (vanishing mean oscillation condition),
\[ \sup_{x \in \mathbb{R}^d, 0 < \rho < 1} \left| \int_{B(x,\rho)} A(y) - \int_{B(x,\rho)} A \right| dy \leq \omega(t) \]
where \( \omega(t) \) is a continuous nondecreasing function. To make the notation simpler, we denote that \( A \in \text{VMO(\mathbb{R}^d)} \) if \( A \) satisfies the VMO condition (1.6).
Assume that $A(y) = (a^{\alpha\beta}_{ij}(y))$, the coefficient matrix of $L_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla)$, satisfies (1.3) and (1.4). Let $\chi_{ij}^\beta(y) = (\chi_{i}^\alpha(y))$ denote the matrix of correctors for $L_1 = -\text{div}(A(x)\nabla)$ in $\mathbb{R}^d$, where $\chi_{ij}^\beta(y) = (\chi_{ij}^{1\beta}(y),...,\chi_{ij}^{m\beta}(y)) \in H^1_{\text{per}}(Y;\mathbb{R}^m)$ is defined by the following cell problem

\[
\begin{aligned}
-\text{div}(A(x)\nabla \chi_{ij}^\beta) &= \text{div}(A(x)\nabla P_{ij}^\beta) \text{ in } [0,1)^d, \\
\chi_{ij}^\beta &= \text{ periodic with respect to } \mathbb{Z}^d \text{ and } \int_Y \chi_{ij}^\beta \, dy = 0,
\end{aligned}
\]

(1.7)

where $1 \leq j \leq d$, $1 \leq \beta \leq m$, $Y = [0,1)^d \cong \mathbb{R}^d/\mathbb{Z}^d$ and $P_{ij}^\beta = (P_{ij}^\alpha(x)) = (x_j\delta^{\alpha\beta})$ with $\delta^{\alpha\beta} = 1$ if $\alpha = \beta$, $\delta^{\alpha\beta} = 0$ otherwise. The homogenized operator is defined by $L_0 = -\text{div}(\hat{A}\nabla)$, where the coefficients $\hat{A} = (\hat{a}_{ij}^\alpha)$ are given by

\[
\hat{a}_{ij}^\alpha = \int_Y \left[ a^{\alpha\beta}_{ij}(y) + a^{\alpha\gamma}_{ik}(y) \frac{\partial}{\partial y_k} \chi_{ij}^{\beta}(y) \right] \, dy.
\]

(1.8)

It is easy to show that if $A$ satisfies (1.2), then $\hat{A}$ is also symmetric, that is

\[
\hat{a}_{ij}^\beta = \hat{a}_{ji}^\beta \text{ for any } 1 \leq i,j \leq d \text{ and } 1 \leq \alpha, \beta \leq m.
\]

Moreover, if $A$ satisfies (1.3), then there exists $\mu_1 > 0$ depending only on $\mu$, such that

\[
\mu_1|\xi|^2 \leq \hat{a}_{ij}^\alpha \xi_i \xi_j \leq \mu_1^{-1}|\xi|^2 \text{ for any } \xi = (\xi_\alpha) \in \mathbb{R}^{m\times d}.
\]

(1.10)

For the sake of simplicity, we set min($\mu, \mu_1$) as new $\mu$ in the rest of the paper. Let

\[
b_{ij}^\beta(y) = a_{ij}^{\alpha\beta}(y) - a_{ik}^{\alpha\gamma}(y) \frac{\partial}{\partial y_k} \chi_{ij}^{\beta}(y),
\]

(1.11)

where $1 \leq i,j \leq d$ and $1 \leq \alpha, \beta \leq m$. It is easy to see that $\int_Y b_{ij}^{\alpha\beta}(y) \, dy = 0$ by (1.8). Then in view of (1.7) and (1.8), there exists the flux corrector $(F_{kij}^{\alpha\beta}(y))_{1 \leq \alpha, \beta \leq m} \in H^1_{\text{per}}(Y;\mathbb{R}^m)$ such that

\[
b_{ij}^\beta(y) = \frac{\partial}{\partial y_k} \left( F_{kij}^{\alpha\beta}(y) \right) \text{ and } F_{kij}^{\alpha\beta}(y) = -F_{ikj}^{\alpha\beta}(y)
\]

(1.12)

for any $1 \leq i,j,k \leq d$ and $1 \leq \alpha, \beta \leq m$. Moreover, $(F_{kij}^{\alpha\beta}(y)) \in L^\infty(Y)$ if $(\chi_{ij}^{\beta}(y))$ is Hölder continuous. For details about the proof of this fact, one can refer to Chapter 2 of [21].

Let $A$ satisfy (1.2), (1.3) and (1.4). For $F \in L^2(\Omega;\mathbb{C}^m)$ and $\varepsilon \geq 0$, we can define a linear operator $T_\varepsilon : L^2(\Omega;\mathbb{C}^m) \to H^1_0(\Omega;\mathbb{C}^m) \subset L^2(\Omega;\mathbb{C}^m)$ by $T_\varepsilon(F) = u_\varepsilon \in H^1_0(\Omega;\mathbb{C}^m)$ such that $L_\varepsilon(u_\varepsilon) = F$ in $\Omega$ and $u_\varepsilon = 0$ on $\partial\Omega$. Using standard arguments in [14], operators $T_\varepsilon$ with $\varepsilon \geq 0$ are positive and self-adjoint. Therefore, applying the spectrum theory, it is natural to consider properties of resolvents $R(\lambda, L_\varepsilon) = (L_\varepsilon - \lambda I)^{-1}$ with $\lambda \in \mathbb{C}\setminus(0,\infty)$. To simplify notations, we will use $R(\lambda, L)$ to denote the resolvent of the elliptic operator $L$ in the rest of the paper. In order to better characterize resolvents of operators, for $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}\setminus(0,\infty)$, we define a constant $c(\lambda, \theta)$ by

\[
c(\lambda, \theta) = \begin{cases} 1 & \text{ if } \theta \in [\pi/2,3\pi/2] \text{ or } \lambda = 0 \\ \sin \theta^{-1} & \text{ if } \theta \in (0,\pi/2) \cup (3\pi/2,2\pi) \text{ and } |\lambda| > 0. \end{cases}
\]

(1.13)

To present our results more conveniently and study the estimates of resolvents, we need to introduce the matrix of Dirichlet correctors $\Phi_\varepsilon(x) = (\Phi_{\varepsilon,j}^\beta(x))_{1 \leq j \leq d, 1 \leq \beta \leq m}$ in $\Omega$, defined by

\[
L_\varepsilon(\Phi_{\varepsilon,j}^\beta(x)) = 0 \text{ in } \Omega \quad \text{and} \quad \Phi_{\varepsilon,j}^\beta(x) = P_{ij}^\beta(x) \text{ on } \partial\Omega.
\]

(1.14)

The Dirichlet correctors were first introduced in [2] to study the uniform Lipschitz estimates of homogenization problems. It is known that if $A$ satisfies (1.3), (1.4), (1.5) and $\Omega$ is a bounded $C^{1,\eta}$ $\eta \in (0,1)$
domain in $\mathbb{R}^d$ with $d \geq 2$, then
\begin{equation}
\|\Phi_{\varepsilon,j} - P_{j}^{\beta}\|_{L^\infty(\Omega)} \leq C\varepsilon \quad \text{and} \quad \|\nabla \Phi_{\varepsilon}\|_{L^\infty(\Omega)} \leq C, \tag{1.15}
\end{equation}
where $C$ depends only on $\mu, d, m, \tau, \nu, \eta$ and $\Omega$. The following are the main results of this paper. For the sake of simplicity, we will denote $\text{diam}(\Omega)$, the diameter of the bounded domain $\Omega$ in $\mathbb{R}^d$ with $d \geq 2$ by $R_0$ throughout this paper.

**Theorem 1.1** ($L^2$ and $H^1_0$ convergence of resolvents). Suppose that $d \geq 2$, $A$ satisfies (1.2), (1.3), (1.4) and (1.5). Let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}\setminus(0, \infty)$. For $\varepsilon \geq 0$ and $F \in L^2(\Omega; \mathbb{C}^m)$, let $u_{\varepsilon,\lambda} \in H^1_0(\Omega; \mathbb{C}^m)$ be the unique solution of the Dirichlet problem $(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F$ in $\Omega$ and $u_{\varepsilon,\lambda} = 0$ on $\partial \Omega$. Then
\begin{equation}
\|u_{\varepsilon,\lambda} - u_{0,\lambda}\|_{L^2(\Omega)} \leq C\varepsilon^2(\lambda, \theta)(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|F\|_{L^2(\Omega)}, \tag{1.16}
\end{equation}
\begin{equation}
\|u_{\varepsilon,\lambda} - u_{0,\lambda} - (\Phi_{\varepsilon,j} - P_{j}^{\beta})\partial_{x_j} u_{0,\lambda}\|_{H^1_0(\Omega)} \leq C\varepsilon^2(\lambda, \theta)\|F\|_{L^2(\Omega)}, \tag{1.17}
\end{equation}
where $C$ depends only on $\mu, d, m, \tau, \nu, \Omega$ and $\Phi_{\varepsilon}$ is given by (1.14). In operator forms,
\begin{equation}
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0)\|_{L^2(\Omega) \to L^2(\Omega)} \leq C\varepsilon^2(\lambda, \theta)(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}, \tag{1.18}
\end{equation}
\begin{equation}
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0) - K_\varepsilon(\lambda)\|_{L^2(\Omega) \to H^1_0(\Omega)} \leq C\varepsilon^2(\lambda, \theta), \tag{1.19}
\end{equation}
where $K_\varepsilon(\lambda)$ are the operator correctors given by the formula
\begin{equation}
K_\varepsilon(\lambda) = \{K_\varepsilon^{\alpha}(\lambda)\}_{1 \leq \alpha \leq m} = \{(\Phi_{\varepsilon,j}^{\alpha \beta}(x) - P_{j}^{\alpha \beta}(x))\partial_{x_j} R(\lambda, \mathcal{L}_0)^{\beta}\}^{1 \leq \alpha \leq m}. \tag{1.20}
\end{equation}

Theorem 1.1 is actually a quantitative result of the homogenization theory for the operator $\mathcal{L}_\varepsilon - \lambda I$. In fact, if $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$ and $A$ satisfies (1.3), (1.4), it can be shown that $u_{\varepsilon,\lambda} \to u_{0,\lambda}$ weakly in $H^1_0(\Omega; \mathbb{C}^m)$ and strongly in $L^2(\Omega; \mathbb{C}^m)$. For more details about the homogenization problems of elliptic systems, one can refer to [4] and [21].

In [22], assuming that $\Omega$ is a bounded and $C^{1,1}$ domain, the convergence rates of such resolvents are established, that is, for $\lambda \in \mathbb{C}\setminus[0, \infty)$ and $0 \leq \varepsilon < 1$,
\begin{equation}
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0)\|_{L^2(\Omega) \to L^2(\Omega)} \leq C\varepsilon^2(\lambda, \theta)(\varepsilon|\lambda|)^{-\frac{1}{2}} + \varepsilon^2, \tag{1.21}
\end{equation}
\begin{equation}
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0) - K_0(\varepsilon, \lambda)\|_{L^2(\Omega) \to H^1(\Omega)} \leq C\varepsilon^2(\lambda, \theta)\varepsilon^{\frac{1}{2}}, \tag{1.22}
\end{equation}
where $K_0(\varepsilon, \lambda)$ is some corrector of the homogenization problem and $C$ depends only on $\mu, d, m$ and $\Omega$. These estimates are obtained by applying the results when $\Omega = \mathbb{R}^d$ in [5], [6] and some extension theorems. By using such approximation estimates for $R(\lambda, \mathcal{L}_\varepsilon)$, [16] and [17] gave the convergence rates for homogenization problems of parabolic and hyperbolic systems in $L^2$ and $H^1_0$ space. What is new for Theorem 1.1 is that we use different operator correctors and obtain a more brief proof under higher regularity assumptions of $A$. The main method is developed in the proof of Theorem 2.4 in [14], which is used to deal with the case that $\lambda = 0$. In Theorem 1.5 of [26], the author generalized convergence results in [14] to elliptic operators with lower order terms. Such results are somewhat similar to this paper, but this does not mean that the conclusions of this paper can be trivially covered. To some extent, Theorem 1.1 is a generalization of Theorem 2.4 in [14] and Theorem 1.5 in [26]. The difference between this and Theorem 1.5 in [26] is that the constants on the right hand side of (1.16) do not depend on the module of $\lambda$, i.e. $|\lambda|$. Moreover, we remark that these estimates in Theorem 1.1 can also be applied to evolution systems and obtain similar results given in [12], [16] and [17].
Given a sectorial domain

$$\Sigma_{\theta_0} = \{ \lambda = |\lambda|e^{i\theta} \in \mathbb{C} : |\lambda| > 0, |\arg \theta| > \pi - \theta_0 \}, \quad (1.23)$$

where $\theta_0 \in (0, \frac{\pi}{2})$, we can obtain the following estimates.

**Theorem 1.2 (L^p and W^{1,p}_0 estimates of resolvents).** Suppose that $\varepsilon \geq 0$ and $d \geq 2$. Let $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Assume that $A$ satisfies (1.2), (1.3), (1.4) and (1.6). Then for any $1 < p < \infty$, $F \in L^p(\Omega; \mathbb{C}^m)$ and $f \in L^p(\Omega; \mathbb{C}^{m \times d})$, there exists a unique $u_{\varepsilon, \lambda} \in W^{1,p}_0(\Omega; \mathbb{C}^m)$ such that $(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f)$ in $\Omega$, $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$ and satisfies the uniform estimates

$$\|u_{\varepsilon, \lambda}\|_{L^p(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-1}\|F\|_{L^p(\Omega)} + C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \quad (1.24)$$

$$\|\nabla u_{\varepsilon, \lambda}\|_{L^p(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|F\|_{L^p(\Omega)} + C_{p, \theta_0}\|f\|_{L^p(\Omega)}, \quad (1.25)$$

where $C_{p, \theta_0}$ depends only on $\mu, \sigma, m, \omega(t), p, \theta_0$ and $\Omega$. In operator forms,

$$\|R(\lambda, \mathcal{L}_\varepsilon)||_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-1}, \quad (1.26)$$

$$\|R(\lambda, \mathcal{L}_\varepsilon)||_{W^{-1,p}(\Omega) \rightarrow L^p(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}, \quad (1.27)$$

$$\|R(\lambda, \mathcal{L}_\varepsilon)||_{L^p(\Omega) \rightarrow W^{1,p}_0(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}, \quad (1.28)$$

$$\|R(\lambda, \mathcal{L}_\varepsilon)||_{W^{-1,p}(\Omega) \rightarrow W^{1,p}_0(\Omega)} \leq C_{p, \theta_0}, \quad (1.29)$$

where $W^{-1,p}(\Omega; \mathbb{C}^m) \triangleq (W^{1,p'}(\Omega; \mathbb{C}^m))^*$ with $p' = \frac{p}{p-1}$ being the conjugate number of $p$.

The $L^p \rightarrow L^p$ estimates of resolvents for elliptic operators are widely studied and have abundant materials. We will list some results for non-homogenization problems. For $m = 1$ and BMO coefficients, see [13]; for constant coefficients and Dirichlet boundary conditions, see [18]; for constant coefficients and Neumann boundary conditions, see [25]; and for variable coefficients and Lipschitz domains, see [24]. It is noteworthy that in [24], the authors derived the $L^p \rightarrow L^p$ estimate of resolvents $R(\lambda, \mathcal{L}_1)$ without any regularity assumptions on $A$ with $p$ being closed to 2 when $\Omega$ is a bounded Lipschitz domain. These results can be used for homogenization problems. In this point of view, Theorem 1.2 gives stronger results under more assumptions of $A$ and $\Omega$. We see that the estimates established in Theorem 1.2 are sharp in view of the estimates for $p = 2$, which will be given later.

The main tool for the proof of Theorem 1.2 is real variable method, which was original used in [7] to deal with $W^{1,p}$ estimates of elliptic equations. We remark that for the operator $\mathcal{L}_\varepsilon - \lambda I$, the use of real variable method is slightly different from the case for $\mathcal{L}_\varepsilon$.

Besides the uniform $W^{-1,p}, L^p \rightarrow L^p, W^{1,p}_0$ estimates of resolvents, one may also concern about the convergence rates in $L^p \rightarrow L^p$ or $L^p \rightarrow W^{1,p}_0$ norm with $1 < p < \infty$. For these topics, we have the following theorems.

**Theorem 1.3 (L^p convergence rates of resolvents).** Suppose that $d \geq 2$, $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $A$ satisfies (1.2), (1.3), (1.4), (1.5). Let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. For $\varepsilon \geq 0$ and $F \in L^p(\Omega; \mathbb{C}^m)$, let $u_{\varepsilon, \lambda} \in W^{1,p}_0(\Omega; \mathbb{C}^m)$ be the unique solution of the Dirichlet problem $(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$. Then for any $1 < p < \infty$,

$$\|u_{\varepsilon, \lambda} - u_{0, \lambda}\|_{L^p(\Omega)} \leq C_{p, \theta_0}(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|F\|_{L^p(\Omega)}, \quad (1.30)$$
where $C_{p,\theta_0}$ depends only on $\mu, d, m, \tau, \nu, p$ and $\Omega$. In operator forms,
\[
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0)\|_{L^p(\Omega) \to L^p(\Omega)} \leq C_{p,\theta_0} \varepsilon (R_0^{-2} + |\lambda|)^{-\frac{1}{2}}.
\]  
(1.31)

**Theorem 1.4** ($L^p \rightarrow W_0^{1,p}$ convergence rates of resolvents). Suppose that $d \geq 2, \lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $A$ satisfies (1.2), (1.3), (1.4), (1.5). Let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. For $\varepsilon \geq 0$ and $F \in L^p(\Omega; \mathbb{C}^m)$, let $u_{\varepsilon,\lambda} \in W_0^{1,p}(\Omega; \mathbb{C}^m)$ be the unique solution of the Dirichlet problem ($\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F$ in $\Omega$ and $u_{\varepsilon,\lambda} = 0$ on $\partial \Omega$. Then for any $1 < p < \infty$,
\[
\|u_{\varepsilon,\lambda} - u_{0,\lambda} - (\Phi_{\varepsilon,j} - P_j^\lambda) \partial u_{0,\lambda} / \partial x_j\|_{W_0^{1,p}(\Omega)} \leq C_{p,\theta_0} \varepsilon \left\{\ln[\varepsilon^{-1} R_0 + 2]\right\}^{\frac{d}{2} - \frac{2}{p}} \|F\|_{L^p(\Omega)},
\]  
(1.32)
where $C_{p,\theta_0}$ depends only on $\mu, d, m, \tau, \nu, p$ and $\Omega$. In operator forms,
\[
\|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0) - K_\varepsilon(\lambda)\|_{L^p(\Omega) \to W_0^{1,p}(\Omega)} \leq C_{p,\theta_0} \varepsilon \left\{\ln[\varepsilon^{-1} R_0 + 2]\right\}^{\frac{d}{2} - \frac{2}{p}},
\]  
(1.33)
where $K_\varepsilon(\lambda)$ is defined by (1.20).

For the operator $\mathcal{L}_\varepsilon$, the estimates of $L^p$ and $W_0^{1,p}$ convergence rates are established in [15]. These estimates are established by obtaining the convergence rates of Green functions of operators $\mathcal{L}_\varepsilon$. The existences and pointwise estimates of Green functions for operators $\mathcal{L}_\varepsilon$ are well-known and one can refer to [11] and [23] for details. However, for operators $\mathcal{L}_\varepsilon - \lambda I$ with $\lambda \in \Sigma_{\theta_0}$ and $\theta_0 \in (0, \frac{\pi}{2})$, the constructions and estimates of Green functions are still unknown. We will construct the Green functions for operators $\mathcal{L}_\varepsilon - \lambda I$ in this paper and establish the uniform estimates of them. We point out that the constructions of Green functions for operators $\mathcal{L}_\varepsilon - \lambda I$ with $d \geq 3$ are similar to which of the Green functions of elliptic operators with lower order terms in [26]. There are some differences between these two. The first is that the problems considered in this paper are about complex valued functions. This makes calculations a little more complicated. The second difference is that the impact of the parameter $\lambda$ should be calculated explicitly. On the other hand, for $d = 2$, we need to employ arguments in [8], [9] and [23]. The constructions of Green functions for $d \geq 3$ and $d = 2$ are rather different and to deal with the case $d = 2$, we need some properties of BMO space and Hardy space. Moreover, proofs of uniform estimates on two dimensional Green functions are much more difficult than that of the case $d \geq 3$. These estimates are new and are the most important innovations of this paper. What is essential is that all the uniform regularity estimates of Green functions in this paper are scaling invariant (see [27] for such estimates of fundamental solutions related to generalized elliptic operators with lower order terms).

After constructing Green functions for the operator $\mathcal{L}_\varepsilon - \lambda I$, we will use almost the same methods in [15] to prove Theorem 1.3-1.4. In view of the convergence rates with $p = 2$, we can also see that estimates (1.31) and (1.33) are sharp.

The rest of this paper is organized as follows. In Section 2, we will give some basic ingredients in the proof, including the $L^2$ regularity theory, Caccioppoli’s inequality and some properties about BMO, Hardy space. In Section 3, we will establish the $W^{1,p}$, Hölder and Lipschitz estimates for the operator $\mathcal{L}_\varepsilon - \lambda I$. In Section 4, we will first prove Theorem 1.2 by using real variable methods. Then we will construct Green functions $G_{\varepsilon,\lambda}(x, y)$ for operators $\mathcal{L}_\varepsilon - \lambda I$ and obtain the regularity estimates for them. In Section 5, firstly, we will use standard methods given in [14] to prove Theorem 1.1. Next, we will calculate the convergence rates of Green functions for ($\mathcal{L}_\varepsilon - \lambda I$). Using these tools, we can prove Theorem 1.3-1.4.
2. Preliminaries

2.1. Energy estimates and Caccioppoli’s inequality. A notable observation gives that for any \( \xi = \xi^{(1)} + i\xi^{(2)} \in \mathbb{C}^{m \times d} \), where \( \xi^{(1)}, \xi^{(2)} \in \mathbb{R}^{m \times d} \),

\[
 a_{kj}^{\alpha\beta}(y)\xi_{\alpha}^{(1)}\xi_{\beta}^{(2)} = a_{kj}^{\alpha\beta}(y)\left((\xi_{k}^{(1)})^{\alpha} + i\xi_{k}^{(2)}\alpha\right)\left((\xi_{j}^{(1)})^{\beta} - i\xi_{j}^{(2)}\beta\right) = a_{kj}^{\alpha\beta}(y)\xi_{k}^{(1)}\xi_{j}^{(1)}\beta + a_{kj}^{\alpha\beta}(y)\xi_{k}^{(2)}\xi_{j}^{(2)}\beta.
\]

This, together with (2.7), implies that for any \( y \in \mathbb{R}^{d} \) and \( \xi = (\xi_{\alpha}^{\alpha}) \in \mathbb{C}^{m \times d} \).

\[
2\mu|\xi|^{2} \leq a_{ij}^{\alpha\beta}(y)\xi_{\alpha}^{i}\xi_{\beta}^{j} \leq 2\mu^{-1}|\xi|^{2} \text{ for any } y \in \mathbb{R}^{d} \text{ and } \xi = (\xi_{\alpha}^{\alpha}) \in \mathbb{C}^{m \times d}.
\]

(2.1)

Similarly, in view of (1.9) and (1.10), we can also infer that

\[
2\mu|\xi|^{2} \leq \tilde{a}_{ij}^{\alpha\beta}\xi_{\alpha}^{i}\xi_{\beta}^{j} \leq 2\mu^{-1}|\xi|^{2} \text{ for any } y \in \mathbb{R}^{d} \text{ and } \xi = (\xi_{\alpha}^{\alpha}) \in \mathbb{C}^{m \times d}.
\]

(2.2)

Moreover, for \( d \geq 2 \), let \( \Omega \) be a bounded domain in \( \mathbb{R}^{d} \), \( \varepsilon \geq 0 \), \( \lambda = |\lambda|e^{i\theta} \in \mathbb{C}\setminus(0, \infty) \) and \( A \) satisfy (1.2), (1.3). Define a bilinear form \( B_{\varepsilon, \lambda, \Omega}[\cdot, \cdot] : H^{1}_{0}(\Omega; \mathbb{C}^{m}) \times H^{1}_{0}(\Omega; \mathbb{C}^{m}) \to \mathbb{C} \) by

\[
B_{\varepsilon, \lambda, \Omega}[u, v] = \int_{\Omega} A_{\varepsilon}(x)\nabla u(x)\overline{\nabla v(x)}dx - \lambda \int_{\Omega} u(x)\overline{v(x)}dx,
\]

(2.3)

where \( A_{\varepsilon}(x) = A(x/\varepsilon) \) for \( \varepsilon > 0 \) and \( A_{0}(x) = \tilde{A} \). For \( \lambda \in \mathbb{C}\setminus[0, \infty) \) and \( u \in H^{1}_{0}(\Omega; \mathbb{C}^{m}) \),

\[
B_{\varepsilon, \lambda, \Omega}[u, u] = \int_{\Omega} A_{\varepsilon}(x)\nabla u(x)\overline{\nabla u(x)}dx - \lambda \int_{\Omega} |u(x)|^{2}dx.
\]

(2.4)

If \( \text{Re}(\lambda) \geq 0 \), we can take imaginary parts of both sides of (2.4) and obtain that

\[
|B_{\varepsilon, \lambda, \Omega}[u, u]| \geq |\text{Im}(\lambda)|\int_{\Omega} |u(x)|^{2}dx = \|u\|_{L^{2}(\Omega)}^{2} \leq c(\lambda, \theta)|\lambda|^{-1}|B_{\varepsilon, \lambda, \Omega}[u, u]|.
\]

(2.5)

If \( \text{Re}(\lambda) < 0 \), we can take real parts of both sides of (2.4) and get that

\[
|B_{\varepsilon, \lambda, \Omega}[u, u]| \geq |\text{Re}(\lambda)|\int_{\Omega} |u(x)|^{2}dx.
\]

(2.6)

Adding (2.6) by (2.5), it can be easily shown that

\[
\|u\|_{L^{2}(\Omega)}^{2} \leq Cc(\lambda, \theta)|\lambda|^{-1}|B_{\varepsilon, \lambda, \Omega}[u, u]| \text{ for any } \lambda \in \mathbb{C}\setminus[0, \infty).
\]

(2.7)

This, together with (2.1), (2.2) and (2.4), implies that

\[
\mu\|\nabla u\|_{L^{2}(\Omega)}^{2} \leq |B_{\varepsilon, \lambda, \Omega}[u, u]| + |\lambda|\|u\|_{L^{2}(\Omega)}^{2} \leq 2c(\lambda, \theta)|B_{\varepsilon, \lambda, \Omega}[u, u]|.
\]

(2.8)

If \( \lambda = 0 \), it is obvious that (2.8) is still true. By using Poincaré’s inequality, then

\[
\|u\|_{L^{2}(\Omega)}^{2} \leq CR^{2}_{\Omega}\|\nabla u\|_{L^{2}(\Omega)}^{2} \leq Cc(\lambda, \theta)R^{2}_{\Omega}|B_{\varepsilon, \lambda, \Omega}[u, u]|,
\]

which, together with (2.7), implies that for any \( \lambda \in \mathbb{C}\setminus(0, \infty) \),

\[
\|u\|_{L^{2}(\Omega)}^{2} \leq Cc(\lambda, \theta)(R^{2}_{\Omega} + |\lambda|^{-1})|B_{\varepsilon, \lambda, \Omega}[u, u]|.
\]

(2.9)

where \( C \) depends only on \( \mu, d, m \) and \( \Omega \). Another important fact is that if \( A \) satisfies (1.2), the adjoint operator of \( L_{\varepsilon} - \lambda I \) is \( L_{\varepsilon} - \overline{\lambda}I \). Indeed,

\[
\langle (L_{\varepsilon} - \overline{\lambda}I)(u), v \rangle_{H^{-1}(\Omega) \times H^{1}_{0}(\Omega)} = B_{\varepsilon, \lambda, \Omega}[u, v] = \overline{B_{\varepsilon, \lambda, \Omega}[v, u]} = \langle u, (L_{\varepsilon} - \overline{\lambda}I)(v) \rangle_{H^{1}_{0}(\Omega) \times H^{-1}(\Omega)}.
\]

In view of (2.8) and well-known Lax-Milgram theorem, it is easy to show the following theorem for existence of solutions corresponding to operators \( L_{\varepsilon} - \lambda I \).
Theorem 2.1. Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$ with $d \geq 2$. Assume that $A$ satisfies (1.2) and (1.3). Then for any $\varepsilon \geq 0$, $F \in H^{-1}(\Omega; \mathbb{C}^m)$, there is a unique weak solution $u_{\varepsilon, \lambda} \in H^1_0(\Omega; \mathbb{C}^m)$ satisfying the Dirichlet problem $(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$. That is, for any $\varphi \in H^1_0(\Omega; \mathbb{C}^m)$,

$$B_{\varepsilon, \lambda, \Omega}[u_{\varepsilon, \lambda}, \varphi] = (F, \varphi)_{H^{-1}(\Omega) \times H^1_0(\Omega)}.$$ 

Lemma 2.2. Suppose that $d \geq 2$ and $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}\backslash(0, \infty)$. Let $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Assume that $A$ satisfies (1.2) and (1.3). Then for any $\varepsilon \geq 0$, $F \in L^2(\Omega; \mathbb{C}^m)$ and $f \in L^2(\Omega; \mathbb{C}^{m \times d})$, there exists a unique weak solution $u_{\varepsilon, \lambda} \in H^1_0(\Omega; \mathbb{C}^m)$ of the Dirichlet problem $(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f)$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$, satisfying the uniform energy estimates

$$\|u_{\varepsilon, \lambda}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta) \left\{ (R_0^{-2} + |\lambda|)^{-1}\|F\|_{L^2(\Omega)} + (R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|f\|_{L^2(\Omega)} \right\},$$

$$\|\nabla u_{\varepsilon, \lambda}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta) \left\{ (R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right\},$$

where $C$ depends only on $\mu, d, m$ and $\Omega$. Moreover, if $\Omega$ is a bounded $C^{1,1}$ domain, $\varepsilon = 0$ and $f \equiv 0$, then

$$\|\nabla^2 u_{0, \lambda}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)\|F\|_{L^2(\Omega)}.$$

In operator forms,

$$\|\mathcal{R}(\lambda, \mathcal{L}_\varepsilon)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)(R_0^{-2} + |\lambda|)^{-1},$$

$$\max \left\{ \|\mathcal{R}(\lambda, \mathcal{L}_\varepsilon)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)}, \|\mathcal{R}(\lambda, \mathcal{L}_\varepsilon)\|_{L^2(\Omega) \rightarrow H^1_0(\Omega)} \right\} \leq C(\varepsilon, \lambda, \theta)(R_0^{-2} + |\lambda|)^{-\frac{1}{2}},$$

$$\max \left\{ \mathcal{R}(\lambda, \mathcal{L}) \right\}_{L^2(\Omega) \rightarrow H^2(\Omega)}, \|\mathcal{R}(\lambda, \mathcal{L}_\varepsilon)\|_{H^{-1}(\Omega) \rightarrow H^1_0(\Omega)} \leq C(\varepsilon, \lambda, \theta).$$

Proof. The existence is ensured by Theorem 2.1, we only need to show (2.10) and (2.11). Firstly, we can choose $u_{\varepsilon, \lambda}^{(1)}$ and $u_{\varepsilon, \lambda}^{(2)}$ in $H^1_0(\Omega; \mathbb{C}^m)$ such that

$$\begin{align*}
(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}^{(1)}) &= F \quad \text{in} \quad \Omega, \\
(u_{\varepsilon, \lambda}^{(1)}) &= 0 \quad \text{on} \quad \partial \Omega, \\
(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}^{(2)}) &= \text{div}(f) \quad \text{in} \quad \Omega, \\
(u_{\varepsilon, \lambda}^{(2)}) &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}$$

The next thing is to estimate $u_{\varepsilon, \lambda}^{(1)}$ and $u_{\varepsilon, \lambda}^{(2)}$ respectively. For $u_{\varepsilon, \lambda}^{(1)}$, we need to prove that

$$\|\nabla u_{\varepsilon, \lambda}^{(1)}\|_{L^2(\Omega)} + (R_0^{-2} + |\lambda|)^{\frac{1}{2}}\|u_{\varepsilon, \lambda}^{(1)}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)(R_0^{-2} + |\lambda|)^{-\frac{1}{2}}\|F\|_{L^2(\Omega)}.$$  

Choosing $u_{\varepsilon, \lambda}^{(1)}$ as the test function, it can be obtained that

$$B_{\varepsilon, \lambda, \Omega}[u_{\varepsilon, \lambda}^{(1)}, u_{\varepsilon, \lambda}^{(1)}] = \int_{\Omega} F(x)u_{\varepsilon, \lambda}^{(1)}(x)dx = (F, u_{\varepsilon, \lambda}^{(1)})_{L^2(\Omega) \times L^2(\Omega)}.$$  

In view of (2.8) and (2.9), we have

$$\|u_{\varepsilon, \lambda}^{(1)}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)R_0^{-2}\|F\|_{L^2(\Omega)}.$$  

$$\mu\|\nabla u_{\varepsilon, \lambda}^{(1)}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)\|u_{\varepsilon, \lambda}^{(1)}\|_{L^2(\Omega)}\|F\|_{L^2(\Omega)},$$

which complete the proof of (2.15). For $u_{\varepsilon, \lambda}^{(2)}$, it suffices to show that

$$\|\nabla u_{\varepsilon, \lambda}^{(2)}\|_{L^2(\Omega)} + (R_0^{-2} + |\lambda|)^{\frac{1}{2}}\|u_{\varepsilon, \lambda}^{(2)}\|_{L^2(\Omega)} \leq C(\varepsilon, \lambda, \theta)\|F\|_{L^2(\Omega)}.$$  

Choosing $u_{\varepsilon, \lambda}^{(2)}$ as the test function, then

$$B_{\varepsilon, \lambda, \Omega}[u_{\varepsilon, \lambda}^{(2)}, u_{\varepsilon, \lambda}^{(2)}] = -\int_{\Omega} f(x)\nabla u_{\varepsilon, \lambda}^{(2)}(x)dx = -(f, \nabla u_{\varepsilon, \lambda}^{(2)})_{L^2(\Omega) \times L^2(\Omega)}.$$  

(2.18)
Again, by applying (2.8) and (2.9), it is easy to get that
\[ \| u_{\varepsilon, \lambda}^{(2)} \|_{L^2(\Omega)}^2 \leq C \varepsilon (\lambda, \theta) (R_0^{-2} + |\lambda|)^{-1} \| \nabla u_{\varepsilon, \lambda}^{(2)} \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}, \]
\[ \mu \| \nabla u_{\varepsilon, \lambda}^{(2)} \|_{L^2(\Omega)}^2 \leq C \varepsilon (\lambda, \theta) \| \nabla u_{\varepsilon, \lambda}^{(2)} \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}, \]
which implies (2.17). At last, for the proof of (2.11), since \( \Omega \) is \( C^{1,1} \), one can deduce that
\[ \| \nabla^2 u_{0, \lambda} \|_{L^2(\Omega)} \leq C \left\{ \| f \|_{L^2(\Omega)} + \| u_{0, \lambda} \|_{L^2(\Omega)} \right\} \leq C \varepsilon (\lambda, \theta) \| f \|_{L^2(\Omega)}, \]
where, for the first inequality, we have used standard \( H^2 \) estimates for elliptic systems with constant coefficients and for the second inequality, we have used (2.10).

\[ \square \]

**Remark 2.3.** For \( d \geq 3 \) and \( \lambda \in \mathbb{C} \setminus (0, \infty) \), using Hölder’s inequality and Sobolev embedding theorem \( H^1_0(\Omega) \subset L^{2^*}(\Omega) \), we have
\[ |(F, u_{\varepsilon, \lambda}^{(1)})(L^2(\Omega) \times L^2(\Omega))| \leq \| F \|_{L^{2^*}(\Omega)} \| u_{\varepsilon, \lambda}^{(1)} \|_{L^{2^*}(\Omega)} \leq \| F \|_{L^{2^*}(\Omega)} \| \nabla u_{\varepsilon, \lambda} \|_{L^2(\Omega)}. \]

By using the same arguments in the proof of Theorem 2.2, it is easy to find that for any \( \varepsilon \geq 0 \),
\[ \| \nabla u_{\varepsilon, \lambda} \|_{L^2(\Omega)} + (R_0^{-2} + |\lambda|)^{\frac{1}{2}} \| u_{\varepsilon, \lambda} \|_{L^2(\Omega)} \leq C \varepsilon (\lambda, \theta) \left\{ \| F \|_{L^{2^*}(\Omega)} + \| f \|_{L^2(\Omega)} \right\}, \tag{2.19} \]
where \( C \) depends only on \( \mu, d, m, \) and \( \Omega \). Similarly, if \( d = 2 \), in view of the Sobolev embedding \( H^1_0(\Omega) \subset L^q(\Omega) \) with \( q' = \frac{q}{q-1} > 1 \), it can be easily seen that for any \( q > 1 \),
\[ \| \nabla u_{\varepsilon, \lambda} \|_{L^2(\Omega)} + (R_0^{-2} + |\lambda|)^{\frac{1}{2}} \| u_{\varepsilon, \lambda} \|_{L^2(\Omega)} \leq C \varepsilon (\lambda, \theta) \left\{ R_0^{2(1-\frac{1}{q'})} \| F \|_{L^q(\Omega)} + \| f \|_{L^2(\Omega)} \right\}, \tag{2.20} \]
where \( C \) depends only on \( \mu, m, q, \) and \( \Omega \).

Next, we will establish the Caccioppoli’s inequality for the operator \( \mathcal{L}_\varepsilon - \lambda I \). In some point of view, Caccioppoli’s inequality can be seen as the localization of (2.19). In this paper, we need to obtain Caccioppoli’s inequality that is scaling invariant. This means that \( \lambda \) cannot be regarded as a constant and its influence on the constants of this inequality should be calculated explicitly. The scaling invariant Caccioppoli’s inequality plays a vital role in the proofs of other scaling invariant inequalities. The idea comes from [18, 27] and we combine the methods in both of them to complete the proof for the sake of completeness. To simplify the notations, we can define
\[ \Omega(x_0, R) = \Omega \cap B(x_0, R) \quad \text{and} \quad \Delta(x_0, R) = \partial\Omega \cap B(x_0, R) \quad \tag{2.21} \]
for any \( 0 < R < R_0 \) and \( x_0 \in \Omega \). Sometimes we will use \( \Omega_r \) and \( \Delta_r \) to denote \( \Omega(x_0, r) \) and \( \Delta(x_0, r) \) if no confusion would be made.

**Lemma 2.4** (Caccioppoli’s inequality). Let \( \varepsilon \geq 0, \ d \geq 2, \ \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \) and \( \Omega \) be a bounded \( C^1 \) domain in \( \mathbb{R}^d \). Assume that \( A \) satisfies (1.2) and (1.3). Suppose that \( 0 < R < R_0, \ x_0 \in \Omega, \ f \in L^2(\Omega(x_0, 2R); \mathbb{C}^m) \) and \( F \in L^q(\Omega(x_0, 2R); \mathbb{C}^m) \), where \( q = \frac{d+2}{d-2} \) if \( d \geq 3 \) and \( q > 1 \) if \( d = 2 \). If \( \Delta(x_0, 2R) \neq \emptyset \), assume that \( u_{\varepsilon, \lambda} \in H^1(\Omega(x_0, 2R), \mathbb{C}^m) \) is the weak solution of the boundary problem
\[ (\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f) \text{ in } \Omega(x_0, 2R) \quad \text{and} \quad u_{\varepsilon, \lambda} = 0 \text{ on } \Delta(x_0, 2R). \]
If \( \Delta(x_0, 2R) = \emptyset \), assume that \( u_{\varepsilon, \lambda} \in H^1(B(x_0, 2R); \mathbb{C}^m) \) is the weak solution of the interior problem
\[ (\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f) \text{ in } B(x_0, 2R). \]
Then for any \( k \in \mathbb{N}_+ \),
\[
\left( \int_{\Omega(x_0,R)} |u_{\varepsilon,\lambda}|^2 \right)^{\frac{1}{2}} \leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^k} \left( \int_{\Omega(x_0,2R)} |u_{\varepsilon,\lambda}|^2 \right)^{\frac{1}{2}} \\
+ \frac{C_{k,\theta_0} R}{(1 + |\lambda| R^2)^k} \left\{ R \left( \int_{\Omega(x_0,2R)} |F|^q \right)^{\frac{1}{q}} + \left( \int_{\Omega(x_0,2R)} |f|^2 \right)^{\frac{1}{2}} \right\},
\]
where \( C_{k,\theta_0} \) depends only on \( \mu, d, m, q, k, \theta_0, \Omega \).

**Proof.** We will only prove the case for \( d \geq 3 \), since the case \( d = 2 \) follows from almost the same methods by substituting the Sobolev embedding theorem \( H_0^1 \subset L^{\frac{2d}{d-2}} \) with \( H_0^1 \subset L^{d'} \), \( 1 < q < \infty \). Firstly, we can assume that \( R = 1 \) and \( x_0 = 0 \) and for general case, (2.22) and (2.23) follows by rescaling and translation. For \( \lambda = 0 \), the results are obvious by standard Caccioppoli’s inequality for the operator \( \mathcal{L}_\varepsilon \). Then we can assume that \( \lambda \neq 0 \). Choose \( \varphi \in C_0^\infty (B(0,2); \mathbb{R}) \) such that \( \varphi \equiv 1 \) for \( x \in B(0,1) \), \( \varphi \equiv 0 \) for \( x \in B(0,\frac{3}{4})^c \), \( |\nabla \varphi| \leq C \) and \( 0 \leq \varphi \leq 1 \). We can set \( \psi = \varphi^2 u_{\varepsilon,\lambda} \) as the test function. By applying the definition of weak solutions, we have
\[
\int_{\Omega_2} A_\varepsilon \nabla \psi \nabla (\varphi^2 u_{\varepsilon,\lambda}) - \lambda \int_{\Omega_2} \varphi^2 |u_{\varepsilon,\lambda}|^2 = \int_{\Omega_2} F \varphi^2 u_{\varepsilon,\lambda} - \int_{\Omega_2} \int \nabla (\varphi^2 u_{\varepsilon,\lambda}),
\]
where \( A_\varepsilon = A(x/\varepsilon) \) if \( \varepsilon > 0 \) and \( A_0 = \tilde{A} \). In the following calculations, we use \( C_{\theta_0} \) to denote a constant depending on \( \theta_0 \) and \( C_{k,\theta_0} \) a constant depending on \( \theta_0 \) and \( k \). If \( \text{Re} \lambda \geq 0 \), we can take the imaginary parts of both sides of (2.24), then
\[
|\text{Im} \lambda| \int_{\Omega_2} \varphi^2 |u_{\varepsilon,\lambda}|^2 \leq C \left\{ \int_{\Omega_2} \varphi |\nabla u_{\varepsilon,\lambda}| |\nabla \varphi||u_{\varepsilon,\lambda}| + \int_{\Omega_2} \varphi^2 |F| |u_{\varepsilon,\lambda}| + \int_{\Omega_2} \varphi |f| (|\nabla \varphi||u_{\varepsilon,\lambda}| + |\varphi| |\nabla u_{\varepsilon,\lambda}|) \right\}.
\]
Noticing that \( |\lambda| = c(\lambda, \theta) |\text{Im} \lambda| \), it actually means that
\[
\int_{\Omega_2} \varphi^2 |u_{\varepsilon,\lambda}|^2 \leq \frac{C_{\theta_0}}{|\lambda|} \left\{ \int_{\Omega_2} |\nabla \varphi|^2 |u_{\varepsilon,\lambda}|^2 + \delta \int_{\Omega_2} \varphi^2 |\nabla u_{\varepsilon,\lambda}|^2 \Delta + \int_{\Omega_2} \varphi F |\varphi u_{\varepsilon,\lambda}| + \int_{\Omega_2} \varphi^2 |f|^2 \right\},
\]
where we have used Cauchy’s inequality and the following inequality
\[
ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for any } \delta > 0 \text{ and } a, b > 0,
\]
with \( \delta > 0 \) being sufficiently small. By using Hölder’s inequality, it is easy to see that
\[
\int_{\Omega_2} \varphi^2 |u_{\varepsilon,\lambda}|^2 \leq \frac{C_{\theta_0}}{|\lambda|} \left\{ \int_{\Omega_2} |\nabla \varphi|^2 |u_{\varepsilon,\lambda}|^2 + \delta \int_{\Omega_2} \varphi^2 |\nabla u_{\varepsilon,\lambda}|^2 \\
+ \left( \int_{\Omega_2} |\varphi F| \frac{4\delta}{4\delta - 2} \right)^{\frac{4\delta - 2}{4\delta}} \left( \int_{\Omega_2} |\varphi u_{\varepsilon,\lambda}| \frac{4\delta}{4\delta - 2} \right)^{\frac{4\delta - 2}{4\delta}} + \int_{\Omega_2} \varphi^2 |f|^2 \right\}.
\]
If \( \Re \lambda < 0 \), we can take real parts of both sides of (2.24) and conclude that
\[
|\Re \lambda| \int_{\Omega_2} \varphi^2 |u_{e,\lambda}|^2 
\leq C \left\{ \int_{\Omega_2} \varphi |\nabla u_{e,\lambda}| |\nabla \varphi||u_{e,\lambda}| + \int_{\Omega_2} \varphi^2 |F||u_{e,\lambda}| + \int_{\Omega_2} \varphi |f| (|\nabla \varphi||u_{e,\lambda}| + |\nabla u_{e,\lambda}|) \right\}. \tag{2.28}
\]
By adding (2.25) to (2.28), we can also derive (2.27) for the case \( \Re \lambda < 0 \). Then (2.27) is true for any \( \lambda \in \Sigma_{\theta_0} \). Owing to \( H_0^1(\Omega_2) \subset L^{\frac{2d}{d-2}}(\Omega_2) \) and (2.26), it can be seen that
\[
\left( \int_{\Omega_2} |\varphi F|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}} \left( \int_{\Omega_2} |\varphi u_{e,\lambda}|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}} 
\leq C \left( \int_{\Omega_2} |\varphi F|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{2d}} + \delta \left( \int_{\Omega_2} (|\nabla \varphi u_{e,\lambda}| + |\nabla u_{e,\lambda}|)^2 \right)^{\frac{d+2}{d}}.
\]
This, together with (2.27) and the properties of \( \varphi \), leads to
\[
\left( \int_{\Omega_2} (|u_{e,\lambda}|^2)^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}} \leq C_{\theta_0} \left\{ \left( \int_{\Omega_2} |u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} + \left( \int_{\Omega_2} |F|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}} + \left( \int_{\Omega_2} |f|^2 \right)^{\frac{d}{d+2}} \right\}. \tag{2.29}
\]
Combining (2.24) and (2.29), we have
\[
\left( \int_{\Omega_1} |\nabla u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} \leq C_{\theta_0} \left\{ \left( \int_{\Omega_2} |u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} + \left( \int_{\Omega_2} |F|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}} + \left( \int_{\Omega_2} |f|^2 \right)^{\frac{d}{d+2}} \right\}. \tag{2.30}
\]
In view of the obvious fact that \( \|u_{e,\lambda}\|_{L^2(\Omega_1)} \leq C \|u_{e,\lambda}\|_{L^2(\Omega_2)} \), we can deduce from (2.29) that for any \( \lambda \in \Sigma_{\theta_0} \),
\[
\left( \int_{\Omega_1} |u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} \leq \frac{C_{\theta_0}}{(1 + |\lambda|)^{\frac{d}{2}} \left\{ \left( \int_{\Omega_2} |u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} + \left( \int_{\Omega_2} |F|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}} + \left( \int_{\Omega_2} |f|^2 \right)^{\frac{d}{d+2}} \right\}}.
\]
This, together with (2.30), gives the proof by repeating the procedure for \( 2k \) times. \( \square \)

**Remark 2.5.** Choose \( \varphi \in C_0^\infty(B(0,2R); \mathbb{R}) \) such that \( \varphi \equiv 1 \) for \( x \in B(x_0, R) \), \( \varphi \equiv 0 \) for \( x \in B(x_0, \frac{R}{2})^c \), \( |\nabla \varphi| \leq C/R \) and \( 0 \leq \varphi \leq 1 \). According to (2.25), (2.26) and (2.28), it can be easily seen that, if \( \lambda \neq 0 \), \( f \equiv 0 \) and \( F \equiv 0 \), then
\[
\int_{\Omega(x_0,2R)} \varphi^2 |u_{e,\lambda}|^2 \leq \frac{C_{\theta_0}}{|\lambda|} \int_{\Omega(x_0,2R)} |\nabla u_{e,\lambda}| |\nabla \varphi||u_{e,\lambda}| 
\leq \frac{C_{\theta_0}}{|\lambda|^2} \int_{\Omega(x_0,2R)} |\nabla \varphi|^2 |\nabla u_{e,\lambda}|^2 + \frac{1}{2} \int_{\Omega(x_0,2R)} \varphi^2 |u_{e,\lambda}|^2,
\]
which, together with the properties of \( \varphi \), implies that
\[
\left( \int_{\Omega(x_0,2R)} |u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} \leq \frac{C_{\theta_0}}{|\lambda|} \left( \int_{\Omega(x_0,2R)} |\nabla \varphi|^2 |\nabla u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}} \leq \frac{C_{\theta_0}}{|\lambda| R} \left( \int_{\Omega(x_0,2R)} |\nabla u_{e,\lambda}|^2 \right)^{\frac{d}{d+2}}. \tag{2.31}
\]

### 2.2. BMO and Hardy space.
To deal with the Green functions with \( d = 2 \), we will need some basic knowledge on real analysis, mainly about BMO space and Hardy space.

**Definition 2.6.** (BMO space and atom functions). Let \( d \geq 2 \). Assume that \( x_0 \in \mathbb{R}^d \), \( r > 0 \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^d \). Denote \( \Omega(x_0, r) = \Omega \cap B(x_0, r) \) as before. BMO(\( \Omega; \mathbb{C}^m \)) is a space containing all measurable, \( \mathbb{C}^m \)-valued functions such that
\[
\|u\|_{\text{BMO}(\Omega)} = \sup \left\{ \int_{\Omega(x_0,r)} |u - u_{x_0,r}| : x_0 \in \Omega, r > 0 \right\} < \infty, \tag{2.32}
\]
where
\[
u_{x_0, r} := \begin{cases} 
0 & \text{if } r \geq \delta(x_0) = \text{dist}(x_0, \partial \Omega), \\
\int_{\Omega(x_0, r)} u & \text{if } r < \delta(x_0) = \text{dist}(x_0, \partial \Omega).
\end{cases}
\] (2.33)

We call a bounded measurable function \(a(x)\) as an atom function in \(\Omega\) if \(\text{supp}(a) \subset \Omega(x_0, r)\) with \(x_0 \in \overline{\Omega}\), \(0 < r < R_0\) and
\[
\|a\|_{L^\infty(\Omega)} \leq \frac{1}{|\Omega(x_0, r)|}; \quad a_{x_0, r} = 0.
\]

**Definition 2.7** (Hardy space). Let \(\Omega\) be a \(C^1\) domain in \(\mathbb{R}^d\) with \(d \geq 2\). A function \(f\) is an element in the Hardy space \(H^1(\Omega; \mathbb{C}^m)\), if there exist a sequence of atoms \(\{a_i\}_{i=1}^\infty\) and a sequence of complex numbers \(\{\eta_i\}_{i=1}^\infty \subset l^1(\mathbb{C})\), such that \(f(x) = \sum_{i=1}^\infty \eta_i a_i(x)\). We define the norm in this space by
\[
\|f\|_{H^1(\Omega)} = \inf \left\{ \sum_{i=1}^\infty |\eta_i| : f(x) = \sum_{i=1}^\infty \eta_i a_i(x) \right\}.
\]
We notice the expression
\[
\sup \left\{ \int_{\Omega} a(y)u(y)dy : a = a(x) \text{ is an atom in } \Omega \right\},
\]
gives the equivalent norm of \(\text{BMO}(\Omega)\). This is because that \(\text{BMO}(\Omega)\) is the dual space of \(H^1\).

**3. Uniform regularity estimates**

It is well-known that for the homogenization problem with real vector values, the \(W^{1,p}\) estimates for \(u_\varepsilon\) are established (see [2] and [20]). By simple observations, we can obtain similar results for the problem with complex vector values.

**Lemma 3.1** \((W^{1,p})\) estimates for the operator \(L_\varepsilon\). For \(\varepsilon \geq 0\) and \(d \geq 2\), let \(2 < p < \infty\) and \(\Omega\) be a bounded \(C^1\) domain in \(\mathbb{R}^d\). Suppose that \(A\) satisfies (1.2), (1.3), (1.4) and (1.6). Assume that \(f = (f^a) \in L^p(\Omega; \mathbb{C}^{m \times d})\) and \(F \in L^q(\Omega; \mathbb{C}^m)\) with \(q = \frac{pd}{p+d}\). Then the weak solution of \(L_\varepsilon(u_\varepsilon) = F + \text{div}(f)\) in \(\Omega\) and \(u_\varepsilon = 0\) on \(\partial \Omega\) satisfies the uniform estimate
\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \left\{ \|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \right\},
\] (3.1)

where \(C\) depends only on \(\mu, d, m, p, q, \omega(t)\) and \(\Omega\).

**Proof.** For the case that \(f \in L^p(\Omega; \mathbb{R}^{m \times d})\) and \(F \in L^q(\Omega; \mathbb{R}^m)\), the proof is trivial. For \(f \in L^p(\Omega; \mathbb{C}^{m \times d})\) and \(F \in L^q(\Omega; \mathbb{C}^m)\), we can write that \(f = g + ih\) and \(F = H + iG\) with \(g, h \in L^p(\Omega; \mathbb{R}^{m \times d})\), \(G, H \in L^q(\Omega; \mathbb{R}^m)\) and use the \(W^{1,p}\) estimates for the case of real functions. \(\square\)

**Theorem 3.2** (Localization of \(W^{1,p}\) estimates for the operator \(L_\varepsilon - \lambda I\)). For \(\varepsilon \geq 0\) and \(d \geq 2\), let \(\lambda \in \Sigma_{\theta_0} \cup \{0\}\) with \(\theta_0 \in (0, \frac{\pi}{2})\), \(\Omega\) be a bounded \(C^1\) domain in \(\mathbb{R}^d\) and \(2 < p < \infty\). Suppose that \(A\) satisfies (1.2), (1.3), (1.4) and (1.6). Assume that \(x_0 \in \Omega\), \(0 < R < R_0\), \(f \in L^p(\Omega(x_0, 2R) ; \mathbb{C}^{m \times d})\), \(F \in L^q(\Omega(x_0, 2R) ; \mathbb{C}^m)\) and \(q = \frac{pd}{p+d}\). If \(\Delta(x_0, 2R) \neq \emptyset\), assume that \(u_{\varepsilon, \lambda} \in H^1(\Omega(x_0, 2R) ; \mathbb{C}^m)\) is the weak solution of the boundary problem
\[
(L_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f) \text{ in } \Omega(x_0, 2R) \quad \text{and} \quad u_{\varepsilon, \lambda} = 0 \text{ on } \Delta(x_0, 2R).
\]

If \(\Delta(x_0, 2R) = \emptyset\), assume that \(u_{\varepsilon, \lambda} \in H^1(B(x_0, 2R) ; \mathbb{C}^m)\) is the weak solution of the interior problem
\[
(L_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F + \text{div}(f) \text{ in } B(x_0, 2R).
\]
Then there exists \( n \in \mathbb{N}_+ \), a constant integer depending only on \( d \), such that for any \( k \in \mathbb{N}_+ \),

\[
\left( \frac{\int_{\Omega(x_0, R)} |\nabla u_{\varepsilon, \lambda}|^p}{\int_{\Omega(x_0, R)} |u_{\varepsilon, \lambda}|^2} \right)^{\frac{1}{p}} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k R} \left( \frac{\int_{\Omega(x_0, 2R)} |u_{\varepsilon, \lambda}|^2}{\int_{\Omega(x_0, 2R)} |u_{\varepsilon, \lambda}|^2} \right)^{\frac{1}{2}} + C_{k, \theta_0} (1 + |\lambda| R^2)^n \left\{ R^2 \left( \frac{\int_{\Omega(x_0, 2R)} |F|^q}{\int_{\Omega(x_0, 2R)} |F|^p} \right)^{\frac{1}{q}} + R \left( \frac{\int_{\Omega(x_0, 2R)} |f|^p}{\int_{\Omega(x_0, 2R)} |f|^p} \right)^{\frac{1}{p}} \right\},
\]

(3.2)

\[
\left( \frac{\int_{\Omega(x_0, R)} |u_{\varepsilon, \lambda}|^p}{\int_{\Omega(x_0, R)} |u_{\varepsilon, \lambda}|^2} \right)^{\frac{1}{p}} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \left( \frac{\int_{\Omega(x_0, 2R)} |u_{\varepsilon, \lambda}|^2}{\int_{\Omega(x_0, 2R)} |u_{\varepsilon, \lambda}|^2} \right)^{\frac{1}{2}} + C_{k, \theta_0} (1 + |\lambda| R^2)^n \left\{ R^2 \left( \frac{\int_{\Omega(x_0, 2R)} |F|^q}{\int_{\Omega(x_0, 2R)} |F|^p} \right)^{\frac{1}{q}} + R \left( \frac{\int_{\Omega(x_0, 2R)} |f|^p}{\int_{\Omega(x_0, 2R)} |f|^p} \right)^{\frac{1}{p}} \right\},
\]

(3.3)

where \( C_{k, \theta_0} \) depends only on \( \mu, d, m, k, \theta_0, p, \omega(t) \) and \( \Omega \).

**Proof.** By rescaling and translation, it can be assumed that \( R = 1 \) and \( x_0 = 0 \). We can choose \( \varphi \in C_0^\infty(B(0, 2); \mathbb{R}) \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in \( B(0, 1) \), \( \varphi \equiv 0 \) in \( B(0, \frac{3}{2}) \) and \( |\nabla \varphi| \leq C \). Then by setting \( A_\varepsilon \) as \( A(x/\varepsilon) \) if \( \varepsilon > 0 \) and \( \tilde{A} \) if \( \varepsilon = 0 \), we have

\[
L_\varepsilon (\varphi u_{\varepsilon, \lambda}) = F_\varepsilon + \text{div}(f_\varepsilon) \quad \text{in} \quad \Omega_2 \quad \text{and} \quad \varphi u_{\varepsilon, \lambda} = 0 \quad \text{on} \quad \partial(\Omega_2),
\]

where \( F_\varepsilon, f_\varepsilon \) are defined by

\[
F_\varepsilon = \lambda \varphi u_{\varepsilon, \lambda} + \varphi F - f \nabla \varphi - A_\varepsilon \nabla u_{\varepsilon, \lambda} \nabla \varphi \quad \text{and} \quad f_\varepsilon = \varphi f - A_\varepsilon \nabla \varphi u_{\varepsilon, \lambda}.
\]

Then owing to (3.1) and Hölder’s inequality, it follows that

\[
\| \nabla (\varphi u_{\varepsilon, \lambda}) \|_{L^p(\Omega_3/2)} \leq C \left\{ \| F_\varepsilon \|_{L^q(\Omega_3/2)} + \| f_\varepsilon \|_{L^p(\Omega_3/2)} \right\}
\]

\[
\leq C \left\{ |\lambda| \| u_{\varepsilon, \lambda} \|_{L^q(\Omega_3/2)} + \| F \|_{L^q(\Omega_3/2)} + \| u_{\varepsilon, \lambda} \|_{L^q(\Omega_3/2)} + \| \nabla u_{\varepsilon, \lambda} \|_{L^q(\Omega_3/2)} + \| f \|_{L^p(\Omega_3/2)} \right\},
\]

where \( q = \frac{pd}{p+d} \). By using Hölder’s inequality again, this implies that

\[
\| \nabla u_{\varepsilon, \lambda} \|_{L^p(\Omega)} \leq C \left\{ (1 + |\lambda|) \| u_{\varepsilon, \lambda} \|_{L^q(\Omega_3/2)} + \| F \|_{L^q(\Omega_3/2)} + \| \nabla u_{\varepsilon, \lambda} \|_{L^q(\Omega_3/2)} + \| f \|_{L^p(\Omega_3/2)} \right\}
\]

\[
\leq C \left\{ (1 + |\lambda|) \left( \| \nabla u_{\varepsilon, \lambda} \|_{L^q(\Omega)} + \| u_{\varepsilon, \lambda} \|_{L^2(\Omega)} \right) + \| F \|_{L^q(\Omega)} + \| f \|_{L^p(\Omega)} \right\}.
\]

Here, we have used the inequality that for any \( 2 < p < \infty \) and \( u \in W^{1,q}(\Omega; \mathbb{C}^m) \) with \( q = \frac{pd}{p+d} \),

\[
\| u \|_{L^p(\Omega)} \leq C \| u \|_{W^{1,q}(\Omega)} \leq C \left\{ \| \nabla u \|_{L^q(\Omega)} + \| u \|_{L^2(\Omega)} \right\},
\]

(3.4)

where \( C \) depends only on \( p, d, m, \Omega \). One can refer to [1] for details about (3.4). By iterating for finite times (depending only on \( d \)), we can get \( n = n(d) \in \mathbb{N}_+ \), such that

\[
\| \nabla u_{\varepsilon, \lambda} \|_{L^p(\Omega_2)} \leq C (1 + |\lambda|)^n \| u_{\varepsilon, \lambda} \|_{L^2(\Omega_2)} + C (1 + |\lambda|)^{n-1} \left\{ \| F \|_{L^q(\Omega_2)} + \| f \|_{L^p(\Omega_2)} \right\}.
\]

In view of (2.22), this yields that for any \( k \in \mathbb{N}_+ \),

\[
\| \nabla u_{\varepsilon, \lambda} \|_{L^p(\Omega_2)} \leq C_{k, \theta_0} (1 + |\lambda|)^{n+k} \| u_{\varepsilon, \lambda} \|_{L^2(\Omega_2)} + C_{k, \theta_0} (1 + |\lambda|)^n \left\{ \| F \|_{L^q(\Omega_2)} + \| f \|_{L^p(\Omega_2)} \right\}
\]

\[
\leq C_{k, \theta_0} (1 + |\lambda|)^{-k} \| u_{\varepsilon, \lambda} \|_{L^2(\Omega_2)} + C_{k, \theta_0} (1 + |\lambda|)^n \left\{ \| F \|_{L^q(\Omega_2)} + \| f \|_{L^p(\Omega_2)} \right\}.
\]

This inequality implies (3.2). On the other hand, (3.3) follows from Poincaré’s inequality and the same arguments of localization. \( \square \)
By using Sobolev embedding theorem and convex arguments (see [19]), we can use $W^{1,p}$ estimates to obtain the Hölder and $L^\infty$ estimates as follows. The proofs are trivial and for more details, one can refer to [26].

**Corollary 3.3** (Localization of Hölder and $L^\infty$ estimates for the operator $\mathcal{L}_\varepsilon - \lambda I$). For $\varepsilon \geq 0$ and $d \geq 2$, let $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Suppose that $A$ satisfies (1.2), (1.3), (1.4) and (1.6). Assume that $x_0 \in \Omega$, $0 < R < R_0$, $f \in L^p(\Omega(x_0,2R); \mathbb{C}^{m})$, $F \in L^q(\Omega(x_0,2R); \mathbb{C}^m)$ and $q = \frac{pd}{p+d}$. If $\Delta(x_0,2R) \neq \emptyset$, assume that $u_{\varepsilon,\lambda} \in H^1(\Omega(x_0,2R); \mathbb{C}^m)$ is the weak solution of the boundary problem

$$(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F + \text{div}(f) \text{ in } \Omega(x_0,2R) \quad \text{and} \quad u_{\varepsilon,\lambda} = 0 \text{ on } \Delta(x_0,2R).$$

If $\Delta(x_0,2R) = \emptyset$, assume that $u_{\varepsilon,\lambda} \in H^1(B(x_0,2R); \mathbb{C}^m)$ is the weak solution of the interior problem

$$(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F + \text{div}(f) \text{ in } B(x_0,2R).$$

Then there exists $n \in \mathbb{N}_+$, a constant integer depending only on $d$, such that for $\gamma = 1 - \frac{d}{p}$, any $k \in \mathbb{N}_+$ and $0 < s < \infty$,

$$\begin{aligned}
[u_{\varepsilon,\lambda}]_{C^{0,\gamma}(\Omega(x_0,R))} &\leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^s R^\gamma} \left( \int_{\Omega(x_0,2R)} |u_{\varepsilon,\lambda}|^2 \right)^{\frac{1}{2}} \\
&\quad + \frac{C_{k,\theta_0}(1 + |\lambda| R^2)^s R}{R^\gamma} \left\{ R \left( \int_{\Omega(x_0,2R)} |f|^q \right)^{\frac{1}{q}} + \left( \int_{\Omega(x_0,2R)} |f|^p \right)^{\frac{1}{p}} \right\}, \\
\|u_{\varepsilon,\lambda}\|_{L^\infty(\Omega(x_0,R))} &\leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^s} \left( \int_{\Omega(x_0,2R)} |u_{\varepsilon,\lambda}|^2 \right)^{\frac{1}{2}} \\
&\quad + \frac{C_{k,\theta_0}(1 + |\lambda| R^2)^n}{R^\gamma} \left\{ R^2 \left( \int_{\Omega(x_0,2R)} |f|^q \right)^{\frac{1}{q}} + R \left( \int_{\Omega(x_0,2R)} |f|^p \right)^{\frac{1}{p}} \right\},
\end{aligned}$$

where $C_{k,\theta_0}$ depends only on $\mu$, $d$, $m$, $k$, $\theta_0$, $p$, $s$, $\omega(t)$ and $\Omega$.

**Theorem 3.4** (Localization of Lipschitz estimates for the operator $\mathcal{L}_\varepsilon$). For $\varepsilon \geq 0$ and $d \geq 2$, let $\Omega$ be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ with $0 < \eta < 1$. Suppose that $A$ satisfies (1.3), (1.4) and (1.5). Assume that $x_0 \in \Omega$, $0 < R < R_0$ and $F \in L^p(\Omega(x_0,2R); \mathbb{C}^m)$ with $p > d$. If $\Delta(x_0,2R) \neq \emptyset$, assume that $u_\varepsilon \in H^1(\Omega(x_0,2R); \mathbb{C}^m)$ is the weak solution of the boundary problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \text{ in } \Omega(x_0,2R) \quad \text{and} \quad u_\varepsilon = 0 \text{ on } \Delta(x_0,2R).$$

If $\Delta(x_0,2R) = \emptyset$, assume that $u_\varepsilon \in H^1(B(x_0,2R); \mathbb{C}^m)$ is the weak solution of the interior problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \text{ in } B(x_0,2R).$$

then

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^\infty(\Omega(x_0,R))} &\leq \frac{C}{R} \left\{ \left( \int_{\Omega(x_0,2R)} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + R^2 \left( \int_{\Omega(x_0,2R)} |f|^p \right)^{\frac{1}{p}} \right\},
\end{aligned}$$

where $C$ depends only on $\mu$, $d$, $m$, $\tau$, $\nu$, $\eta$ and $\Omega$.

**Proof.** See [3] or Theorem 3.1.1 and Theorem 4.5.1 of [21].
Theorem 3.5 (Localization of Lipschitz estimates for the operator $\mathcal{L}_\varepsilon - \lambda I$). For $\varepsilon \geq 0$ and $d \geq 2$, let $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\Omega$ be a bounded $C^{1,\gamma}$ domain in $\mathbb{R}^d$ with $0 < \gamma < 1$. Suppose that $A$ satisfies (1.2), (1.3), (1.4) and (1.5). Assume that $x_0 \in \Omega$, $0 < R < R_0$ and $F \in L^p(\Omega(x_0, 2R); \mathbb{C}^m)$ with $p > d$. If $\Delta(x_0, 2R) \neq \emptyset$, assume that $u_{\varepsilon, \lambda} \in H^1(\Omega(x_0, 2R); \mathbb{C}^m)$ is the weak solution of the boundary problem

$$(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F \text{ in } \Omega(x_0, 2R) \quad \text{and} \quad u_{\varepsilon, \lambda} = 0 \text{ on } \Delta(x_0, 2R).$$

If $\Delta(x_0, 2R) = \emptyset$, assume that $u_{\varepsilon, \lambda} \in H^1(B(x_0, 2R); \mathbb{C}^m)$ is the weak solution of the interior problem

$$(\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F \text{ in } B(x_0, 2R).$$

Then there exists $n \in \mathbb{N}_+$, a constant integer depending only on $d$, such that for any $k \in \mathbb{N}_+$,

$$\|\nabla u_{\varepsilon, \lambda}\|_{L^\infty(\Omega(x_0, R))} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^{k-1}} \left( \int_{\Omega(x_0, 2R)} |u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + C_{k, \theta_0} (1 + |\lambda| R^2)^n \left( \int_{\Omega(x_0, 2R)} |F|^p \right)^{\frac{1}{p}},$$

(3.8)

where $C_{k, \theta_0}$ depends only on $\mu, d, k, \theta_0, \tau, \nu, \eta$ and $\Omega$.

Proof. In view of (3.4), we can infer that

$$\|\nabla u_{\varepsilon, \lambda}\|_{L^\infty(\Omega_R)} \leq \frac{C}{R} \left\{ \left( \int_{\Omega_{3/2R}} |u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + R^2 \left( \int_{\Omega_{3/2R}} |F|^p \right)^{\frac{1}{p}} + |\lambda| R^2 \left( \int_{\Omega_{3/2R}} |F|^p \right)^{\frac{1}{p}} \right\}.$$

By applying (3.3) to $u_{\varepsilon, \lambda}$ for index $p$, this yields that for any $k \in \mathbb{N}_+$,

$$\|\nabla u_{\varepsilon, \lambda}\|_{L^\infty(\Omega_R)} \leq \frac{C}{R} \left( \int_{\Omega_{3/2R}} |u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + CR \left( \int_{\Omega_{3/2R}} |F|^p \right)^{\frac{1}{p}} + \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^{k-1}} \left\{ \left( \int_{\Omega_{3/2R}} |u_{\varepsilon, \lambda}|^p \right)^{\frac{1}{p}} + (1 + |\lambda| R^2)^n R^2 \left( \int_{\Omega_{3/2R}} |F|^p \right)^{\frac{1}{p}} \right\}.$$

This, together with (2.22), implies (3.8).

4. Green functions of operators

4.1. Proof of Theorem 1.2 and relevant estimates. To begin with, we will use the well-known real variable method to prove Theorem 1.2. Unlike what had been done in the proof of $W^{1,p}$ estimates for $\mathcal{L}_\varepsilon$, the operators $\mathcal{L}_\varepsilon - \lambda I$ do not have the homogeneous property, that is, if $\lambda \neq 0$ and $0 \neq c \in \mathbb{C}^m$ is a constant vector, $(\mathcal{L}_\varepsilon - \lambda I)(c) \neq 0$. For this reason, we need to make some adjustments for the original method employed on $\mathcal{L}_\varepsilon$. For simplicity, we use $B = B(x, r)$ to denote a ball in $\mathbb{R}^d$ ($d \geq 2$) and $tB = B(x, tr)$ ($t \in \mathbb{R}_+$) to denote balls with center $x$ and radius $tr$ if no confusion would be caused.

Theorem 4.1 (Real variable method). Let $q > 2$ and $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$. Let $F \in L^2(\Omega; \mathbb{C}^m)$ and $f \in L^p(\Omega; \mathbb{C}^{m \times d})$ for some $2 < p < q$. Suppose that for each ball $B$ with the property that $|B| \leq c_0 |\Omega|$ and either $4B \subset \Omega$ or $B$ is centered on $\partial \Omega$, there exist two measurable functions
where \( W \) and \( \delta \) are constants.

Proof. It is proved for the case that \( 0 \leq \eta < \eta_0 \), then \( F \in L^p(\Omega; \mathbb{C}^m) \) and

\[
\left( \int_{\Omega} |F|^p \right)^{\frac{1}{p}} \leq C \left( \int_{\Omega} |F|^2 \right)^{\frac{1}{2}} + \left( \int_{\Omega} |f|^p \right)^{\frac{1}{p}},
\]

where \( C \) depends at most on \( N_1, N_2, c_0, p, q \) and the Lipschitz character of \( \Omega \).

**Proof of Theorem 1.2.** Firstly, we assume that \( \lambda \neq 0 \), since the case \( \lambda = 0 \) is trivial in view of the results for \( W^{1,p} \) estimates of \( \mathcal{L}_e \). If \( p = 2 \), the results are given by Lemma 2.2. For \( 2 < p < \infty \), choose \( q = p + 1 \).

Consider functions

\[
H = |u_{\varepsilon,\lambda}|, \quad h = (R_0^2 + |\lambda|)^{-1} |F| + (R_0^2 + |\lambda|)^{-\frac{1}{2}} |f|, \quad G = |\nabla u_{\varepsilon,\lambda}|
\]

and

\[
g = (R_0^2 + |\lambda|)^{-\frac{1}{2}} |F| + |f|.
\]

Now we will apply Theorem 4.1 to complete the proof. For each ball \( B \) with the property that \( |B| \leq \frac{1}{100} |\Omega| \) and \( 4B \subset \Omega \), we write \( u_{\varepsilon,\lambda} = u_{\varepsilon,\lambda,1} + u_{\varepsilon,\lambda,2} \) in \( 2B \), where \( u_{\varepsilon,\lambda,1} \in H^1(4B; \mathbb{C}^m) \) is the weak solution of \( \mathcal{L}_e(u_{\varepsilon,\lambda,1} - \lambda u_{\varepsilon,\lambda,1} = F + \text{div}(f) \) in \( 4B \) and \( u_{\varepsilon,\lambda,1} = 0 \) on \( \partial(4B) \). Let

\[
H_B = |u_{\varepsilon,\lambda,1}|, \quad R_B = |u_{\varepsilon,\lambda,2}|, \quad G_B = |\nabla u_{\varepsilon,\lambda,1}| \text{ and } T_B = |\nabla u_{\varepsilon,\lambda,2}|.
\]

Obviously, \( |H| \leq H_B + R_B \) and \( |G| \leq G_B + T_B \) in \( 2B \). It follows from Lemma 2.2 that

\[
\left( \int_{4B} |H_B|^2 \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0}}{r^{-2} + |\lambda|} \left( \int_{4B} |F|^2 \right)^{\frac{1}{2}} + \frac{C_{\theta_0}}{(r^{-2} + |\lambda|)^{\frac{1}{2}}} \left( \int_{4B} |f|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \int_{4B} |h|^2 \right)^{\frac{1}{2}}
\]

and

\[
\left( \int_{4B} |G_B|^2 \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0}}{(r^{-2} + |\lambda|)^{\frac{1}{2}}} \left( \int_{4B} |F|^2 \right)^{\frac{1}{2}} + C_{\theta_0} \left( \int_{4B} |f|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \int_{4B} |g|^2 \right)^{\frac{1}{2}},
\]

where \( r \) is the radius of \( B \). Moreover, we note that \( u_{\varepsilon,\lambda,2} \in H^1(4B; \mathbb{C}^m) \) and \( \mathcal{L}_e(u_{\varepsilon,\lambda,2} - \lambda u_{\varepsilon,\lambda,2} = 0 \) in \( 4B \). Owing to (3.3), (4.4) and Lemma 2.2, we can obtain that

\[
\left( \int_{2B} |R_B|^p \right)^{\frac{1}{p}} \leq C_{\theta_0} \left( \int_{4B} |u_{\varepsilon,\lambda,2}|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \int_{4B} |u_{\varepsilon,\lambda,1}|^2 \right)^{\frac{1}{2}} + \left( \int_{4B} |u_{\varepsilon,\lambda,2}|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \int_{4B} |H|^2 \right)^{\frac{1}{2}} + \left( \int_{4B} |h|^2 \right)^{\frac{1}{2}}.
\]
To estimate $T_B$, we first note that

$$( \mathcal{L}_\varepsilon - \lambda I ) \left( u_{\varepsilon, \lambda, 2} - \frac{1}{3B} \int_{3B} u_{\varepsilon, \lambda, 2} \right) = \lambda \left( \frac{1}{3B} \int_{3B} u_{\varepsilon, \lambda, 2} \right) \text{ in } 3B.$$

Then it is easy to get that

$$\left( \frac{1}{2B} |T_B|^q \right)^{\frac{1}{q}} \leq \frac{C_{\theta_0}}{r} \left( \frac{1}{3B} \int_{3B} |u_{\varepsilon, \lambda, 2} - \frac{1}{3B} \int_{3B} u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}} + C_{\theta_0} (1 + |\lambda|^2)^n |\lambda| r \left( \frac{1}{3B} \int_{3B} |u_{\varepsilon, \lambda, 2}| \right)$$

$$\leq C_{\theta_0} \left\{ \left( \frac{1}{3B} \int_{3B} |\nabla u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}} + |\lambda| r \left( \int_{\frac{1}{2}B} |u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}} \right\} \leq C_{\theta_0} \left( \frac{1}{4B} \int_{4B} |\nabla u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}}$$

$$\leq C_{\theta_0} \left\{ \left( \frac{1}{4B} \int_{4B} |\nabla u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{4B} \int_{4B} |\nabla u_{\varepsilon, \lambda, 2}|^2 \right)^{\frac{1}{2}} \right\},$$

where for the second inequality, we have used (2.22) with $k = n$, Poincaré’s inequality, Hölder’s inequality and for the third inequality, we have used (2.31).

For ball $B = B(x, r)$ such that it is centered at $\partial \Omega$ with $0 < r < \frac{1}{10}R_0$, write $u_{\varepsilon, \lambda} = u_{\varepsilon, \lambda, 3} + u_{\varepsilon, \lambda, 4}$ in $4B \cap \Omega$, where $u_{\varepsilon, \lambda, 3} \in \mathcal{H}_0^1(4B; \mathbb{C}^n)$ is the weak solution of $\mathcal{L}_\varepsilon(\varepsilon, \lambda) = F + \text{div}(f)$ in $4B \cap \Omega$ and $u_{\varepsilon, \lambda, 3} = 0$ on $\partial(4B \cap \Omega)$. Let

$$H_B = |u_{\varepsilon, \lambda, 3}|, \quad R_B = |u_{\varepsilon, \lambda, 4}|, \quad G_B = |\nabla u_{\varepsilon, \lambda, 3}| \text{ and } T_B = |\nabla u_{\varepsilon, \lambda, 4}|.$$

By using almost the same arguments, we can obtain $|H| \leq H_B + R_B, \quad |G| \leq G_B + T_B$ in $2B \cap \Omega$,

$$\left( \frac{1}{2B \cap \Omega} \int_{2B \cap \Omega} |R_B|^q \right)^{\frac{1}{q}} \leq C_{\theta_0} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |H|^2 \right)^{\frac{1}{2}} + C_{\theta_0} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |h|^2 \right)^{\frac{1}{2}},$$

$$\left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |G|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |h|^2 \right)^{\frac{1}{2}} \text{ and } \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |H|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |h|^2 \right)^{\frac{1}{2}}.$$

Moreover, since $u_{\varepsilon, \lambda, 4} = u_{\varepsilon, \lambda} - u_{\varepsilon, \lambda, 3} = 0$ on $\partial \Omega \cap 4B$, then in view of Poincaré’s inequality and (3.2), we have

$$\left( \frac{1}{2B \cap \Omega} \int_{2B \cap \Omega} |T_B|^q \right)^{\frac{1}{q}} \leq \frac{C_{\theta_0}}{r} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |u_{\varepsilon, \lambda, 4}|^2 \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |\nabla u_{\varepsilon, \lambda, 4}|^2 \right)^{\frac{1}{2}}$$

$$\leq C_{\theta_0} \left\{ \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |\nabla u_{\varepsilon, \lambda, 3}|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |\nabla u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} \right\}$$

$$\leq C_{\theta_0} \left\{ \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |G|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{4B \cap \Omega} \int_{4B \cap \Omega} |g|^2 \right)^{\frac{1}{2}} \right\}.$$

Then by using Theorem 4.1, we have, for any $2 < p < \infty$,

$$\left( \frac{1}{\Omega} \int_{\Omega} |u_{\varepsilon, \lambda}|^p \right)^{\frac{1}{p}} \leq C_{\theta_0} \left\{ \left( \frac{1}{\Omega} \int_{\Omega} |u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{\Omega} \int_{\Omega} |h|^p \right)^{\frac{1}{p}} \right\},$$

$$\left( \frac{1}{\Omega} \int_{\Omega} |\nabla u_{\varepsilon, \lambda}|^p \right)^{\frac{1}{p}} \leq C_{\theta_0} \left\{ \left( \frac{1}{\Omega} \int_{\Omega} |\nabla u_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{\Omega} \int_{\Omega} |g|^p \right)^{\frac{1}{p}} \right\}.$$

In view of Lemma 2.2, definitions of $g, h$ and Hölder’s inequality, we can complete the proof for the case $2 < p < \infty$. For $1 < p < 2$, the results follow form duality arguments. Using the same definitions of $u_{\varepsilon, \lambda}^{(1)}$
and \( u_{ε,λ}^{(2)} \) in the proof of Lemma 2.2, we only need to show that for any \( 1 < p < 2 \),

\[
\| \nabla u_{ε,λ}^{(1)} \|_{L^p(Ω)} + (R_0^{-2} + |λ|)^{\frac{p}{2}} \| u_{ε,λ}^{(1)} \|_{L^p(Ω)} \leq C_0 \| R_0^{-2} + |λ| \|^{-\frac{p}{2}} \| F \|_{L^p(Ω)},
\]

\[
\| \nabla u_{ε,λ}^{(2)} \|_{L^p(Ω)} + (R_0^{-2} + |λ|)^{\frac{p}{2}} \| u_{ε,λ}^{(2)} \|_{L^p(Ω)} \leq C_0 \| f \|_{L^p(Ω)}.
\]

For \( F_1 \in L^{p'}(Ω; C^m) \) and \( f_1 \in L^{p'}(Ω; C^{m \times d}) \) with \( p' = \frac{p}{p-1} \) being the conjugate number of \( p \), let \( v_{ε,λ}^{(1)} \) and \( v_{ε,λ}^{(2)} \) be solutions of Dirichlet problems:

\[
\begin{align*}
(\mathcal{L}_ε - \lambda I)(v_{ε,λ}^{(1)}) &= F_1 \text{ in } Ω, \quad \text{and} \quad (\mathcal{L}_ε - \lambda I)(v_{ε,λ}^{(2)}) = \text{div}(f_1) \text{ in } Ω, \\
v_{ε,λ}^{(1)}(Ω) &= 0 \text{ on } ∂Ω, \quad \text{and} \quad v_{ε,λ}^{(2)}(Ω) = 0 \text{ on } ∂Ω.
\end{align*}
\]

Then it follows from direct calculations and the definition of \( B_{ε,λ,Ω}[\cdot, \cdot] \) that

\[
\begin{align*}
\int Ω F_1 u_{ε,λ}^{(1)} dx &= B_{ε,λ,Ω}[v_{ε,λ}^{(1)}, u_{ε,λ}^{(1)}] = B_{ε,λ,Ω}[u_{ε,λ}^{(1)}, v_{ε,λ}^{(1)}] = \int Ω F^{(1)} v_{ε,λ}^{(1)} dx, \\
- \int Ω f_1 \nabla u_{ε,λ}^{(1)} dx &= B_{ε,λ,Ω}[v_{ε,λ}^{(2)}, u_{ε,λ}^{(1)}] = B_{ε,λ,Ω}[u_{ε,λ}^{(1)}, v_{ε,λ}^{(2)}] = \int Ω \nabla F^{(2)} v_{ε,λ}^{(2)} dx, \\
\int Ω F_1 u_{ε,λ}^{(2)} dx &= B_{ε,λ,Ω}[v_{ε,λ}^{(1)}, u_{ε,λ}^{(2)}] = B_{ε,λ,Ω}[u_{ε,λ}^{(2)}, v_{ε,λ}^{(1)}] = - \int Ω \nabla F^{(1)} v_{ε,λ}^{(1)} dx, \\
- \int Ω f_1 \nabla u_{ε,λ}^{(2)} dx &= B_{ε,λ,Ω}[v_{ε,λ}^{(2)}, u_{ε,λ}^{(2)}] = B_{ε,λ,Ω}[u_{ε,λ}^{(2)}, v_{ε,λ}^{(2)}] = - \int Ω \nabla F^{(2)} v_{ε,λ}^{(2)} dx.
\end{align*}
\]

These, together with the results for \( 2 < p < ∞ \), imply that

\[
\begin{align*}
\left| \int Ω F_1 u_{ε,λ}^{(1)} dx \right| &\leq \| F \|_{L^p(Ω)} \| v_{ε,λ}^{(1)} \|_{L^{p'}(Ω)} \leq C_0 (R_0^{-2} + |λ|)^{-1} \| F \|_{L^p(Ω)} \| F_1 \|_{L^{p'}(Ω)}, \\
\left| \int Ω f_1 \nabla u_{ε,λ}^{(1)} dx \right| &\leq \| F \|_{L^p(Ω)} \| v_{ε,λ}^{(2)} \|_{L^{p'}(Ω)} \leq C_0 (R_0^{-2} + |λ|)^{-\frac{p}{2}} \| F \|_{L^p(Ω)} \| f_1 \|_{L^{p'}(Ω)}, \\
\left| \int Ω F_1 u_{ε,λ}^{(2)} dx \right| &\leq \| f \|_{L^p(Ω)} \| \nabla v_{ε,λ}^{(1)} \|_{L^{p'}(Ω)} \leq C_0 (R_0^{-2} + |λ|)^{-\frac{p}{2}} \| f \|_{L^p(Ω)} \| f_1 \|_{L^{p'}(Ω)}, \\
\left| \int Ω f_1 \nabla u_{ε,λ}^{(2)} dx \right| &\leq \| f \|_{L^p(Ω)} \| \nabla v_{ε,λ}^{(2)} \|_{L^{p'}(Ω)} \leq C_0 \| f \|_{L^p(Ω)} \| f_1 \|_{L^{p'}(Ω)},
\end{align*}
\]

which give the proof. □

**Corollary 4.2.** Assume that \( Ω \) is a bounded \( C^{1,1} \) domain in \( \mathbb{R}^d \) with \( d \geq 2 \) and \( λ \in Σ_{θ_0} \cup \{0\} \) with \( θ_0 \in (0, \frac{d}{2}) \). If \( u_{0,λ} \in W^{2,p}(Ω; C^m) \) is the unique weak solution for the Dirichlet problem

\[
(\mathcal{L}_0 - λ I)(u_{0,λ}) = F \text{ in } Ω \quad \text{and} \quad u_{0,λ} = 0 \text{ on } ∂Ω
\]

with \( 1 < p < ∞ \) and \( F \in L^p(Ω; C^m) \). Then

\[
\| \nabla^2 u_{0,λ} \|_{L^p(Ω)} \leq C_{θ_0} \| F \|_{L^p(Ω)},
\]

where \( C_{θ_0} \) depends only on \( μ, d, m, p, θ_0 \) and \( Ω \). In operator forms,

\[
\| R(λ, \mathcal{L}_0) \|_{L^p(Ω) \to W^{2,p}(Ω)} \leq C_{θ_0}.
\]

**Proof.** The results follow from the \( W^{2,p} \) estimates for \( \mathcal{L}_0 \), i.e.

\[
\| \nabla u_{0,λ} \|_{L^p(Ω)} \leq C \{ |λ| \| u_{0,λ} \|_{L^p(Ω)} + \| F \|_{L^p(Ω)} \}
\]

and \( \| u_{0,λ} \|_{L^p(Ω)} \leq C_{θ_0} (R_0^{-2} + |λ|)^{-1} \| F \|_{L^p(Ω)} \), which is given by Theorem 1.2. □
Lemma 4.3. Assume that $\Omega$ is $C^{1,1}$ domain in $\mathbb{R}^d$ with $d \geq 2$, $x_0 \in \Omega$ and $0 < R < R_0$. If $\Delta(x_0, 2R) \neq \emptyset$, assume that $u_{0, \lambda} \in H^2(\Omega(x_0, 2R); \mathbb{C}^m)$ is the weak solution of the boundary problem

$$(L_0 - \lambda I)(u_{0, \lambda}) = 0 \text{ in } \Omega(x_0, 2R) \quad \text{and} \quad u_{0, \lambda} = 0 \text{ on } \Delta(x_0, 2R),$$

if $\Delta(x_0, 2R) = \emptyset$, assume that $u_{0, \lambda} \in H^2(B(x_0, 2R); \mathbb{C}^m)$ is the weak solution of the interior problem

$$(L_0 - \lambda I)(u_{0, \lambda}) = 0 \text{ in } B(x_0, 2R),$$

with $x_0 \in \Omega$, $0 < R < R_0$ and $1 < p < \infty$. Then there exists $n \in \mathbb{N}_+$, a constant depending only on $d$, such that for any $k \in \mathbb{N}_+$,

$$ \left( \int_{\Omega(x_0, R)} |\nabla^2 u_{0, \lambda}|^p \right)^{\frac{1}{p}} \leq C_{k, \theta_0} \left( \frac{1 + |\lambda| R^2)^k R^2}{\int_{\Omega(x_0, 2R)} |u_{0, \lambda}|^2} \right)^{\frac{1}{p}}, \quad (4.8) $$

If we further assume that $\Omega$ is a bounded $C^{2,1}$ domain and $\rho \in (0, 1)$, then

$$ \left\| \nabla^2 u_{0, \lambda} \right\|_{L^\infty(\Omega(x_0, R))} \leq C_{k, \theta_0} \left( \frac{1 + |\lambda| R^2)^k R^2}{\int_{\Omega(x_0, 2R)} |u_{0, \lambda}|^2} \right)^{\frac{1}{p}}, \quad (4.9) $$

$$ \left[ \nabla^2 u_{0, \lambda} \right]_{L^{\rho,p}(\Omega(x_0, R))} \leq C_{k, \theta_0} \left( \frac{1 + |\lambda| R^2)^k R^2}{\int_{\Omega(x_0, 2R)} |u_{0, \lambda}|^2} \right)^{\frac{1}{p}}, \quad (4.10) $$

where $C$ depends on $\mu, d, m, k, \theta_0, \rho, \lambda$ and $\Omega$.

Proof. By applying $C^{2,\rho}$ estimates for the constant elliptic system (see Theorem 5.23 in [10]), (2.22), (2.23), (4.6) and Theorem 1.2. \qed

Remark 4.4. Under conditions that $A$ satisfies (1.2), (1.3), (1.4), (1.6), $\varepsilon \geq 0$ and $\Omega$ is a bounded $C^1$ domain in $\mathbb{R}^d$ with $d = 2$, if $u_{\varepsilon, \lambda}$ is the unique solution for the Dirichlet problem $(L_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = F$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$, then in view of (3.1) and Theorem 1.2, we have

$$ \left\| \nabla u_{\varepsilon, \lambda} \right\|_{L^p(\Omega)} \leq C \left\{ \left\| u_{\varepsilon, \lambda} \right\|_{L^{\frac{2p}{p-1}}(\Omega)} + \left\| F \right\|_{L^{\frac{2p}{p-1}}(\Omega)} \right\} \leq C_{\theta_0} \left\| F \right\|_{L^{\frac{2p}{p-1}}(\Omega)}, \quad (4.11) $$

for $2 < p < \infty$. If $u_{\varepsilon, \lambda}$ is the unique solution for the Dirichlet problem $(L_\varepsilon - \lambda I)(u_{\varepsilon, \lambda}) = \text{div}(f)$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$, it is not hard to see that

$$ \left\| u_{\varepsilon, \lambda} \right\|_{L^{\frac{2p}{p-1}}(\Omega)} \leq C_{\theta_0} \left\| f \right\|_{L^{\frac{2p}{p-1}}(\Omega)}, \quad \text{for any } 2 < p < \infty, \quad (4.12) $$

Indeed, for any $F \in C^\infty_0(\Omega; \mathbb{C}^m)$, we can choose $w_{\varepsilon, \lambda}$ such that $(L_\varepsilon - \overline{\lambda} I)(w_{\varepsilon, \lambda}) = F$ in $\Omega$ and $w_{\varepsilon, \lambda} = 0$ on $\partial \Omega$. Then it can be easily seen by duality that

$$ \int_{\Omega} F w_{\varepsilon, \lambda} \, dx = B_{\varepsilon, \lambda, \Omega}[w_{\varepsilon, \lambda}, v_{\varepsilon, \lambda}] = B_{\varepsilon, \lambda, \Omega}[v_{\varepsilon, \lambda}, w_{\varepsilon, \lambda}] = - \int_{\Omega} \overline{L} w_{\varepsilon, \lambda} \, dx $$

In view of Hölder’s inequality and (4.11), it can be got that

$$ \left\| \int_{\Omega} F w_{\varepsilon, \lambda} \, dx \right\| \leq \left\| F \right\|_{L^{\frac{p'}{p}}(\Omega)} \left\| \nabla w_{\varepsilon, \lambda} \right\|_{L^p(\Omega)} \leq C_{\theta_0} \left\| F \right\|_{L^{\frac{2p}{p-1}}(\Omega)} \left\| F \right\|_{L^{\frac{2p}{p-1}}(\Omega)}, $$

which directly implies (4.12). Moreover, for $F \in \tilde{W}^{-1,p'}(\Omega)$, where $\tilde{W}^{-1,p'}(\Omega)$ denotes the dual space for the homogeneous Sobolev space $\tilde{W}_0^{-1,p'}(\Omega)$ and $p'$ is the conjugate number of $p$, we have

$$ \left\| u_{\varepsilon, \lambda} \right\|_{L^{\frac{2p}{p-1}}(\Omega)} \leq C_{\theta_0} \left\| F \right\|_{\tilde{W}^{-1,p'}(\Omega)}. \quad (4.13) $$
This conclusion is also proved by using the duality arguments. Actually, for all $g \in C_0^1(\Omega; \mathbb{C}^m)$ we can choose $v_{\varepsilon, \lambda}$ such that $(L_{\varepsilon} - \lambda I)(v_{\varepsilon, \lambda}) = g$ in $\Omega$ and $v_{\varepsilon, \lambda} = 0$ on $\partial \Omega$. Thus,

$$\int_{\Omega} g u_{\varepsilon, \lambda} \, dx = B_{\varepsilon, \lambda, \Omega}[v_{\varepsilon, \lambda}, u_{\varepsilon, \lambda}] = B_{\varepsilon, \lambda, \Omega}[u_{\varepsilon, \lambda}, v_{\varepsilon, \lambda}] = \langle F, v_{\varepsilon, \lambda} \rangle_{W^{-1, \sigma'}(\Omega) \times W_0^{1, \sigma'}(\Omega)}.$$

By using Hölder’s inequality and (4.11), this shows that

$$\left| \int_{\Omega} g u_{\varepsilon, \lambda} \, dx \right| = \left| \langle F, v_{\varepsilon, \lambda} \rangle_{W^{-1, \sigma'}(\Omega) \times W_0^{1, \sigma'}(\Omega)} \right| \leq \| F \|_{W^{-1, \sigma'}(\Omega)} \| v_{\varepsilon, \lambda} \|_{W_0^{1, \sigma'}(\Omega)} \leq C_{\theta_0} \| F \|_{W^{-1, \sigma'}(\Omega)} \| g \|_{L^{\sigma/2}(\Omega)},$$

which gives the proof of (4.13).

**Lemma 4.5.** Assume that $A$ satisfies (1.2), (1.3), (1.4), (1.6), $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\Omega$ is a $C^1$ bounded domain in $\mathbb{R}^2$. Let $a = a(x)$ be an atom function in $\Omega$. If $u_{\varepsilon, \lambda}$ is the unique weak solution for the Dirichlet problem $(L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = a$ in $\Omega$ and $u_{\varepsilon, \lambda} = 0$ on $\partial \Omega$, then there exists a constant $C_{\theta_0}$ depending only on $\mu, \omega(t), m, \theta_0$ and $\Omega$ such that

$$\| u_{\varepsilon, \lambda} \|_{L^\infty(\Omega)} \leq C_{\theta_0}. \tag{4.14}$$

**Proof.** For atom function $a = a(\cdot)$, assume that

$$\text{supp}(a) \subset \Omega(x_0, \rho) \quad \text{and} \quad \| a \|_{L^\infty(\Omega)} \leq \frac{1}{|\Omega(x_0, \rho)|},$$

with $x_0 \in \Omega$ and $0 < \rho < R_0$. Fix $z \in \Omega$, we can choose $2 < p < \infty$. Then

$$|u(z)| \leq |u(z) - u_{\varepsilon, \rho}| + |u_{\varepsilon, \rho}| \leq C \rho^{1 - \frac{2}{p}} \| a \|_{L^p(\Omega(z, \rho))} + |u_{\varepsilon, \rho}| \leq C \rho^{1 - \frac{2}{p}} \| \nabla u_{\varepsilon, \lambda} \|_{L^p(\Omega)} + \rho^\frac{2}{p} - 1 \| u_{\varepsilon, \lambda} \|_{L^{\frac{2p}{p-2}}(\Omega)},$$

due to Morrey’s inequality and Hölder’s inequality. Using (4.11) and (4.13), we have

$$\| \nabla u_{\varepsilon, \lambda} \|_{L^p(\Omega)} \leq C_{\theta_0} \| a \|_{L^{p(\Omega; \mathbb{R}^m)}} \leq C_{\theta_0} \rho^\frac{2}{p} - 1 \quad \text{and} \quad \| u_{\varepsilon, \lambda} \|_{L^{\frac{2p}{p-2}}(\Omega)} \leq C_{\theta_0} \| a \|_{W^{-1, \sigma'}(\Omega)}.$$

We claim that $\| a \|_{W^{-1, \sigma'}(\Omega)} \leq \rho^{1 - \frac{2}{p}}$. It is because that for all $v \in W_0^{1, p}(\Omega; \mathbb{R}^m)$,

$$\left| \int_{\Omega} a^\alpha(y) v^\alpha(y) \, dy \right| \leq \left| \int_{\Omega} a^\alpha(y)(v^\alpha(y) - v_{x_0, \rho}^\alpha) \, dy \right| \leq \| a \|_{L^1(\Omega)} \| v - v_{x_0, \rho} \|_{L^\infty(\Omega(x_0, \rho))} \leq C \rho^{1 - \frac{2}{p}} \| \nabla v \|_{L^p(\Omega)},$$

where we have used Morrey’s theory again for the last inequality. This completes the proof. \qed

**Remark 4.6.** The key point of (4.14) is that $C_{\theta_0}$ does not depend on the module of $\lambda$. Such estimates are extremely important in the constructions of Green functions with $d = 2$. If $C_{\theta_0}$ depends on the module of $\lambda$, we will not be able to obtain related estimates similar to the case that $d \geq 3$.

4.2. Green functions with dimension no less than three.

**Theorem 4.7** (Green functions of $L_{\varepsilon} - \lambda I$ with $d \geq 3$). For $\varepsilon \geq 0$ and $d \geq 3$, let $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Suppose that $A$ satisfies (1.2), (1.3), (1.4) and (1.6). Then there exists a unique Green function, $G_{\varepsilon, \lambda} = (G_{\varepsilon, \lambda}^{\alpha\beta}(\cdot, \cdot)) : \Omega \times \Omega \to \mathbb{C}^{m^2} \cup \{\infty\}$ with $1 \leq \alpha, \beta \leq m$, \ldots
such that \( G_{\varepsilon,\lambda}(\cdot, y) \in H^1(\Omega \setminus B(y, r); \mathbb{C}^{m^2}) \cap W_0^{1,p}(\Omega; \mathbb{C}^{m^2}) \) for any \( s \in [1, \frac{d}{d-2}] \), \( y \in \Omega \) and \( 0 < r < R_0 \). \( G_{\varepsilon,\lambda} \) satisfies

\[
B_{\varepsilon,\lambda,\Omega}[G_{\varepsilon,\lambda}^\gamma(\cdot, y), \phi(\cdot)] = \phi^\gamma(y),
\]

for any \( 1 \leq \gamma \leq m \), \( \phi \in W_0^{1,p}(\Omega; \mathbb{C}^m) \) with \( p > d \). Particularly, if \( F \in L^q(\Omega; \mathbb{C}^m) \) with \( q > d/2 \),

\[
u_{\varepsilon,\lambda}(x) = \int_\Omega G_{\varepsilon,\lambda}(x, y) F(y) \, dy,
\]

satisfies the Dirichlet problem \((\mathcal{L}_\varepsilon - \lambda I)(\nu_{\varepsilon,\lambda}) = F\) in \( \Omega \) and \( u_{\varepsilon,\lambda} = 0 \) on \( \partial \Omega \). Moreover, let \( G_{\varepsilon,\lambda}(x, y) \) be the Green function of the operator \( \mathcal{L}_\varepsilon - \lambda I \), then

\[
G_{\varepsilon,\lambda}(x, y) = \left[ G_{\varepsilon,\lambda}(x, y) \right]^T,
\]

which means that \( G_{\varepsilon,\lambda}^{\alpha\beta}(x, y) = G_{\varepsilon,\lambda}^{\beta\alpha}(x, y) \) for any \( 1 \leq \alpha, \beta \leq m \), \( x, y \in \Omega \) and \( x \neq y \). For any \( \sigma_1, \sigma_2 \in (0, 1) \) and \( k \in \mathbb{N}_+ \), the following estimates

\[
|G_{\varepsilon,\lambda}(x, y)| \leq \frac{C_{k,\theta_0}}{(1 + |x - y|^{2k})|x - y|^{d - 2}} \min \left\{ 1, \frac{[\delta(x)]^{\sigma_1}}{|x - y|^{\sigma_1}}, \frac{[\delta(y)]^{\sigma_2}}{|x - y|^{\sigma_2}} \right\}
\]

hold for any \( x, y \in \Omega \) and \( x \neq y \), where \( \delta(x) = \text{dist}(x, \partial \Omega) \) denotes the distance from \( x \) to the boundary of \( \Omega \) and \( C_{k,\theta_0} \) depends only on \( \mu, d, m, k, \theta_0, \omega(t), \sigma_1, \sigma_2 \) and \( \Omega \).

**Proof.** For \( \lambda = 0 \), the results follow from constructions of Green functions of standard elliptic operators \( \mathcal{L}_\varepsilon \). Then we only need to consider the case that \( \lambda \neq 0 \). First of all, let \( I(u) = \int_{\Omega(x, \rho)} u^\gamma(\cdot) \), then \( I \in H^{-1}(\Omega; \mathbb{C}^m) \). Precisely speaking, for any \( u \in H_0^1(\Omega; \mathbb{C}^m) \), we can easily get by using Sobolev embedding theorem \( L^\frac{2d}{d-2}(\Omega) \subset H_0^1(\Omega) \) that

\[
|I(u)| \leq C|\Omega(x, \rho)|^{-\frac{d-2}{2}} \left\| u \right\|_{L^\frac{2d}{d-2}(\Omega)} \leq C|\Omega(x, \rho)|^{-\frac{d-2}{2}} \left\| u \right\|_{H_0^1(\Omega)} \leq C\rho^{-\frac{d-2}{2}} \left\| u \right\|_{H_0^1(\Omega)}.
\]

Consider the approximating Green function \( G_{\rho,\varepsilon,\lambda}(\cdot, y) = (G_{\rho,\varepsilon,\lambda}^{\alpha\beta}(\cdot, y)) \in H_0^1(\Omega; \mathbb{C}^{m^2}) \) with \( \rho > 0 \) and \( 1 \leq \alpha, \beta \leq m \), such that

\[
B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y), u(\cdot)] = \int_{\Omega(x, \rho)} u^\gamma(\cdot) \quad \text{for any } u \in H_0^1(\Omega; \mathbb{C}^m),
\]

where \( 1 \leq \gamma \leq m \). We see that the existence of \( G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \) is a direct consequence of Theorem 2.1. Choose \( G_{\rho,\varepsilon,\lambda}^{\alpha\beta}(\cdot, y) \) \((1 \leq \beta \leq m)\) itself as the test function, we have

\[
B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y), G_{\rho,\varepsilon,\lambda}^{\beta\alpha}(\cdot, y)] = I(G_{\rho,\varepsilon,\lambda}^{\gamma\beta}(\cdot, y)).
\]

In view of (2.7), (2.8) and (4.19), it is not hard to see that

\[
\left\| G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}|\lambda|^{-\frac{d}{d-2}} \left\| \nabla G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)},
\]

\[
\left\| \nabla G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}\rho^{-\frac{d-2}{2}} \left\| \nabla G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)},
\]

where \( \nabla_i \) \((i = 1, 2)\) denote the derivatives for the first or second variable of Green functions and we will use these notations throughout this paper. Then

\[
\left\| \nabla G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}\rho^{-\frac{d-2}{2}} \text{ and } \left\| G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}|\lambda|^{-\frac{d}{2}}\rho^{-\frac{d-2}{2}}.
\]

On the other hand, using Poincaré’s inequality, we can deduce that

\[
\left\| G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0} \left\| \nabla G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}\rho^{-\frac{d-2}{2}}.
\]

This, combined with (4.22) and the case of \( \lambda = 0 \), gives the following estimate

\[
\left\| G_{\rho,\varepsilon,\lambda}^\gamma(\cdot, y) \right\|_{L^2(\Omega)} \leq C_{\theta_0}(R_0^{-2} + |\lambda|)^{-\frac{d}{2}}\rho^{-\frac{d-2}{2}},
\]
for any $\lambda \in \Sigma_{\theta_0} \cup \{0\}$ and $1 \leq \gamma \leq m$. Let $F \in C^0_0(\Omega; \mathbb{C}^m)$, consider the Dirichlet problem $(\mathcal{L}_\epsilon - \mathcal{M})(u_{\epsilon, \lambda}) = F$ in $\Omega$ and $u_{\epsilon, \lambda} = 0$ on $\partial \Omega$. If we choose $G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y)$ as the test function, it follows from simple calculations that

$$
\int_{\Omega} G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y) F(\cdot) = B_{\epsilon, \lambda, \Omega}[u_{\epsilon, \lambda}(\cdot), G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y)] = B_{\epsilon, \lambda, \Omega}[G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y), u_{\epsilon, \lambda}(\cdot)] = \int_{\Omega(\rho, \rho)} u_{\epsilon, \lambda}(\cdot).
$$

Suppose that $\text{supp} F \subset B(y, R) \subset \Omega$, by using (3.6), it can be got that

$$
\left| \int_{\Omega} G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y) F(\cdot) \right| \leq \|u_{\epsilon, \lambda}\|_{L^\infty(\Omega(\rho, \rho))} \leq \|u_{\epsilon, \lambda}\|_{L^\infty(B(y, \frac{4}{3}))}
$$

\begin{equation}
\leq C_{\theta_0} \left( \int_{B(y, \frac{4}{3})} |u_{\epsilon, \lambda}|^2 \right)^{\frac{q}{2}} + (1 + |\lambda| R^2)^n R^2 \left( \int_{B(y, \frac{4}{3})} |F|^q \right)^{\frac{1}{q}}
\end{equation}

for any $\rho < \frac{4}{3}$ and $q > \frac{d}{2}$. In view of (4.24), we need to estimate $\left( \int_{B(y, \frac{4}{3})} |u_{\epsilon, \lambda}|^2 \right)^{\frac{q}{2}}$. Owing to (2.19), we can obtain without difficulty that

$$
\|\nabla u_{\epsilon, \lambda}\|_{L^2(\Omega)} \leq C_{\theta_0} \|F\|_{L^{\frac{dq}{d+q}}(\Omega)} \quad \text{and} \quad \|u_{\epsilon, \lambda}\|_{L^2(\Omega)} \leq C_{\theta_0}(R_0^2 + |\lambda|)^{\frac{q}{2}} \|F\|_{L^{\frac{dq}{d+q}}(\Omega)}.
$$

These, together with Sobolev embedding theorem that $H^1_0(\Omega) \subset L^{\frac{dq}{d+q}}(\Omega)$, imply that

$$
\left( \int_{B(y, \frac{4}{3})} |u_{\epsilon, \lambda}|^2 \right)^{\frac{q}{2}} \leq \left( \int_{B(y, \frac{4}{3})} |u_{\epsilon, \lambda}|^{\frac{2d}{d+q}} \right)^{\frac{2q}{2d}} \leq CR^{1-\frac{d}{q}} \left( \int_{\Omega} |u_{\epsilon, \lambda}|^{\frac{2d}{d+q}} \right)^{\frac{2q}{2d}}
$$

$$
\leq CR^{1-\frac{d}{q}} \left( \int_{\Omega} \|\nabla u_{\epsilon, \lambda}\|^2 \right)^{\frac{q}{2}} \leq CR^{1-\frac{d}{q}} \left( \int_{\Omega} |F|^\frac{dq}{d+q} \right)^{\frac{2q}{d+q}} \leq C_{\theta_0} R^2 \left( \int_{B(y, R)} |F|^\frac{dq}{d+q} \right)^{\frac{2q}{d+q}}.
$$

Since $d \geq 3$, we have $q > \frac{d}{2} > \frac{2d}{d+2}$. Then by using Hölder’s inequality and (4.25), it is not hard to see that

$$
\left( \int_{B(y, \frac{4}{3})} |u_{\epsilon, \lambda}|^2 \right)^{\frac{q}{2}} \leq C_{\theta_0} R^2 \left( \int_{B(y, R)} |F|^\frac{dq}{d+q} \right)^{\frac{2q}{d+q}} \leq C_{\theta_0} R^2 \left( \int_{B(y, R)} |F|^q \right)^{\frac{q}{q}}.
$$

This, together with (4.24), gives that

$$
\left| \int_{\Omega} G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y) F(\cdot) \right| \leq C_{\theta_0}(1 + |\lambda| R^2)^n R^2 \left( \int_{B(y, R)} |F|^q \right)^{\frac{q}{q}}.
$$

By using duality arguments, we can get that

$$
\left( \int_{B(y, R)} |G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y)|^q \right)^{\frac{1}{q}} \leq C_{\theta_0} \left( 1 + |\lambda| R^2 \right)^n R^{2-d},
$$

for any $\rho < \frac{4}{3}$, $R \leq \delta(x)$ and $s \in (1, \frac{d}{d+2})$. For $x, y \in \Omega$ such that $x \neq y$, set $r = |x - y|$. If $r \leq \frac{1}{4} \delta(y)$, choosing $\rho \in (1, \frac{d}{d+2})$, we have $(\mathcal{L}_\epsilon - \mathcal{M})(G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y)) = 0$ in $B(x, \frac{r}{2})$ for any $\rho < \frac{1}{4} r$. Then by using (3.6) and (4.28), it is easy to show that for any $s \in (1, \frac{d}{d+2})$

$$
|G^\gamma_{\rho, \epsilon, \lambda}(x, y)| \leq \frac{C_{\theta_0}}{(1 + |\lambda| r^2)^n} \left( \int_{B(x, \frac{r}{2})} |G^\gamma_{\rho, \epsilon, \lambda}(\cdot, y)|^s \right)^{\frac{1}{s}} \leq \frac{C_{\theta_0}}{(1 + |\lambda| r^2)^n} \frac{r^{2-d}}{|x - y|^{d-2}}.
$$
where we have chosen \( k = n \in \mathbb{N}_+ \). Assume that \( R < \frac{1}{4}\delta(y) \), then for any \( \rho < \frac{R}{4} \), we have \((L_{\varepsilon} - \lambda I)(G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)) = 0 \) in \( \Omega \setminus B(x, R) \). Choose \( \varphi \in C_0^1(\Omega; \mathbb{R}) \), such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in \( \Omega \setminus B(y, 2R) \), \( \varphi \equiv 0 \) in \( B(y, R) \) and \( |\nabla \varphi| \leq \frac{C_5}{R} \). Set \( u(z) = \varphi^2 G_{\rho,\varepsilon,\lambda}^\gamma(z,y) \) as the test function, then it is not hard to obtain that
\[
\omega^2 A_{\varepsilon}(\cdot)\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y) - \lambda \omega^2 |G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2 = - \int_\Omega 2\varphi A_{\varepsilon}(\cdot)\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\nabla \varphi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y),
\]
(4.30)
where \( A_{\varepsilon}(x) = A(x/\varepsilon) \) if \( \varepsilon > 0 \) and \( A_0(x) = \hat{A} \). For \( \Re \lambda \geq 0 \), we can take the imaginary parts of both sides of (4.30) and get that
\[
\|\varphi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|^2_{L^2(\Omega)} \leq \frac{C_5}{|\lambda|} \int_\Omega |\nabla \varphi(\cdot)||G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)||\varphi(\cdot)||\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|
\leq \frac{C_5}{|\lambda|} \left\{ |\nabla \varphi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2_{L^2(\Omega)} + \delta |\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2_{L^2(\Omega)} \right\},
\]
(4.31)
where \( \delta > 0 \) is sufficiently small. For \( \Re \lambda < 0 \), we can take the real parts of both sides of (4.30) and obtain that (4.31) is also true. Then (4.31) is true for any \( \lambda \in \Sigma_{\theta_0} \). Owing to (4.30), (4.31) and properties of \( \varphi \), it is obvious that
\[
\|\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|^2_{L^2(\Omega \setminus B(y, 2R))} \leq C_{\theta_0} \|\nabla \varphi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|^2_{L^2(\Omega)}
\leq \frac{C_{\theta_0}}{R^2} \|G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|^2_{L^2(\Omega \setminus B(y, 2R))}
\leq \frac{C_{\theta_0}}{R^2} \int_{B(y, 2R) \setminus B(y, R)} \frac{1}{|z - y|^{2d - 4}}dz \leq C_{\theta_0} R^{2 - d}.
\]
(4.32)
On the other hand, if \( \rho > \frac{R}{4} \), according to (4.22), it can be obtained that
\[
\int_{\Omega \setminus B(y, 2R)} |\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2 \leq C_{\theta_0} R^{2 - d}.
\]
(4.33)
(4.32), together with (4.33), yields that
\[
\int_{\Omega \setminus B(y, 2R)} |\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2 \leq C_{\theta_0} R^{2 - d},
\]
(4.34)
for any \( \rho > 0 \) and \( R < \frac{1}{4}\delta(y) \). Let \( \psi \in C_0^1(\Omega; \mathbb{R}) \) such that \( 0 \leq \psi \leq 1 \), \( \psi \equiv 0 \) in \( B(y, 2R) \), \( \psi \equiv 1 \) in \( \Omega \setminus B(y, R) \) and \( |\nabla \psi| \leq \frac{C_5}{R} \). With the help of the definition of \( \psi \) and Sobolev embedding theorem, we can deduce that for any \( \rho < \frac{R}{4} \) and \( R < \frac{1}{4}\delta(y) \),
\[
\|\psi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|_{L^\infty(\Omega)} \leq C \|\nabla \psi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|_{L^2(\Omega)}
\leq C \|\nabla \psi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|_{L^2(\Omega)} + C \|\psi G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)\|_{L^2(\Omega)} \leq C_{\theta_0} R^{2 - d},
\]
(4.35)
where for the third inequality, we have used (4.29) and (4.34). Combining (4.35) and properties of \( \psi \), it can be easily inferred that
\[
\int_{\Omega \setminus B(y, \frac{R}{2})} |G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|_{L^2(\Omega)} \leq C_{\theta_0} R^{2 - d} \text{ for any } \rho < \frac{R}{4} \text{ and } R < \frac{1}{4}\delta(y).
\]
For the case that \( \rho \geq \frac{R}{4} \), due to (4.22), one can find that
\[
\int_{\Omega \setminus B(y, \frac{R}{2})} |G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|_{L^2(\Omega)} \leq \left( \int_\Omega |\nabla_1 G_{\rho,\varepsilon,\lambda}^\gamma(\cdot,y)|^2 \right)^{\frac{1}{p}} \leq C_{\theta_0} R^{2 - d}.
\]
We now address ourselves to the uniform estimates of \(G_{\gamma,\varepsilon,\lambda}^{\gamma}(:,y)\) and \(\nabla_1G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y)\) with respect to parameter \(\rho > 0\). In the case of \(t > (\frac{\varepsilon}{\lambda})^{2-d}\), we can choose \(R = t^{-\frac{d}{2-d}} < \frac{4}{\varepsilon}\delta(y)\) and obtain that for any \(\rho > 0\),
\[
\left\{ x \in \Omega : \left| \nabla_1G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right| > t \right\} \leq CR^d + t^{-2} \int_{\Omega \setminus B(y,2R)} \left| \nabla_1G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right|^2 \leq C R^d + C_{b0} t^{-2} R^{2-d} \leq C_{b0} t^{-\frac{d}{2-d}}. \tag{4.36}
\]
When \(t > (\frac{\varepsilon}{\lambda})^{2-d}\), similarly, we can choose \(R = t^{-\frac{d}{2-d}}\) and obtain that for any \(\rho > 0\),
\[
\left\{ x \in \Omega : \left| G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right| > t \right\} \leq CR^d + t^{-\frac{d}{2-d}} \int_{\Omega \setminus B(y,R)} \left| G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right|^2 \leq C R^d + C_{b0} t^{-\frac{d}{2-d}} R^{-d} \leq C_{b0} t^{-\frac{d}{2-d}}. \tag{4.37}
\]
Then in view of (4.36) and (4.37), it can be shown by simple calculations that
\[
\int_{\Omega} \left| G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right|^s \leq C [\delta(y)]^{s(2-d)} + C_{b0} \int_{(\delta(y)/4)^{2-d}}^t t^{s-1} t^{-\frac{d}{2-d}} dt \leq C_{b0} \left\{ [\delta(y)]^{s(2-d)} + [\delta(y)]^{s(2-d)} \right\},
\]
for any \(s \in [1,\frac{d}{1-d})\) and
\[
\int_{\Omega} \left| \nabla G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right|^s \leq C_{b0} \left\{ [\delta(y)]^{s(1-d)} + [\delta(y)]^{s(1-d)} \right\},
\]
for any \(s \in [1,\frac{d}{1-d})\). From the uniform estimates above, it follows that there exists a subsequence of \(\{G_{\rho_n,\varepsilon,\lambda}^{\gamma}(:,y)\}_{n=1}^{\infty}\) and \(G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y)\) such that for any \(s \in (1,\frac{d}{1-d})\) and \(1 \leq \gamma \leq m\),
\[
G_{\rho_n,\varepsilon,\lambda}^{\gamma}(:,y) \rightharpoonup G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \text{ weakly in } W_0^{1,s}(\Omega;\mathbb{C}^m) \text{ as } n \to \infty. \tag{4.38}
\]
Hence, we have, for any \(\phi \in W_0^{1}(\Omega;\mathbb{C}^m)\) with \(p > d\)
\[
B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y),\phi(\cdot)] = \lim_{n \to \infty} B_{\varepsilon,\lambda,\Omega}[G_{\rho_n,\varepsilon,\lambda}^{\gamma}(:,y),\phi(\cdot)] = \lim_{n \to \infty} \int_{\Omega(y,p\varepsilon)} \phi^\gamma(\cdot) = \phi^\gamma(y),
\]
where we have used the definition of the approximating Green matrix. We note that there exists a weak solution \(u_{\varepsilon,\lambda} \in W_0^{1,p}(\Omega;\mathbb{C}^m)\) satisfying \((L_{\varepsilon} - \bar{M})(u_{\varepsilon,\lambda}) = F\) in \(\Omega\) and \(u_{\varepsilon,\lambda} = 0\) on \(\partial\Omega\) for any \(F \in L^2(\Omega;\mathbb{C}^m)\) with \(p > d\left(\frac{4d}{3d-2} < \frac{2}{\gamma}\right)\). Thus we can obtain that
\[
u_{\varepsilon,\lambda}(y) = B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y),\phi(\cdot)] = \frac{B_{\varepsilon,\lambda,\Omega}[u_{\varepsilon,\lambda}(\cdot),G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y)]}{\int_{\Omega} G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y)F(\cdot)}. \tag{4.39}
\]
We now verify the uniqueness. If \(\tilde{G}_{\varepsilon,\lambda}^{\gamma}(:,y)\) is another Green matrix, we can also derive by representation formula that \(\tilde{u}_{\varepsilon,\lambda}(y) = \int_{\Omega} \tilde{G}_{\varepsilon,\lambda}^{\gamma}(:,y)F(\cdot)\). It follows from the uniqueness of the weak solution that
\[
\int_{\Omega} \left[ G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) - G_{\rho,\varepsilon,\lambda}^{\gamma}(:,y) \right]F = 0 \text{ for any } F \in L^2(\Omega;\mathbb{C}^m). \tag{4.40}
\]
Then \(\tilde{G}_{\varepsilon,\lambda}^{\gamma}(:,y) = G_{\varepsilon,\lambda}^{\gamma}(:,y)\) a.e. in \(\Omega\) due to the arbitrariness of \(F\). Next, let \(G_{\tau,\varepsilon,\lambda}^{\gamma}(:,x)\) denote the approximating for Green functions of the operator \(L_{\varepsilon} - \bar{M}\), which satisfy
\[
B_{\varepsilon,\lambda,\Omega}[G_{\tau,\varepsilon,\lambda}^{\gamma}(:,x),u(\cdot)] = \int_{\Omega(x,\tau)} u(\cdot) \text{ for any } u \in H_0^1(\Omega;\mathbb{C}^m). \tag{4.41}
\]
By the same argument, we can derive the existence and uniqueness of $G_{ε,λ}^{τ}(\cdot, x)$. Thus for any $τ, ρ > 0$ and $1 ≤ γ, ξ ≤ m$, it is obvious to see that

$$
\int_{Ω(τ, ρ)} G_{τ, ε, λ} G_{ρ, ε, λ} = B_{ε, λ, τ}(G_{τ, ε, λ}, G_{ρ, ε, λ}) = \int_{Ω(τ, ρ)} G_{ε, λ}(\cdot, x),
$$

Note that $(L_{ε} - \lambda I)(G_{ε, λ}^{τ}(\cdot, x)) = 0$ in $Ω \setminus B(x, τ)$ and $(L_{ε} - \lambda I)(G_{ρ, ε, λ}^{\rho, ε, λ}(\cdot, y)) = 0$ in $Ω \setminus B(y, ρ)$. In view of the $W^{1,p}$ estimates, $G_{ε, λ}^{τ}(\cdot, x)$ and $G_{ρ, ε, λ}^{\rho, ε, λ}(\cdot, y)$ are locally Hölder continuous. Therefore, by taking $τ, ρ → ∞$, we have $G_{ε, λ}^{τ}(x, y) = G_{ε, λ}^{τ}(y, x)$, which implies that $G_{ε, λ}(x, y) = (G_{ε, λ}(x, y))^{T}$ for any $x, y \in Ω$ such that $x ≠ y$. This gives the proof of (4.17).

Finally, we will prove (4.18). Let $r = |x - y|$ and $F ∈ C_{0}^{∞}(Ω(x, \frac{r}{2}); C^{m})$. Assume that $u_{ε, λ}$ is the solution of $(L_{ε} - \lambda I)(u_{ε, λ}) = F$ in $Ω$ and $u_{ε, λ} = 0$ on $∂Ω$. Then $u_{ε, λ}(y) = \int_{Ω} F(\cdot) G_{ε, λ}(\cdot, y)$. Since $(L_{ε} - \lambda I)(u_{ε, λ}) = 0$ in $Ω \setminus (Ω(x, \frac{r}{2})$ and $Ω(y, \frac{r}{2}) ⊂ Ω \setminus (Ω(x, \frac{r}{2})$, it follows from (2.19) and (3.6) that for any $k ∈ N_{+},$

$$
|u_{ε, λ}(y)| ≤ \frac{C_{k, θ_{0}}}{(1 + |λ| r^{2})^{k}} \left( \int_{Ω(\frac{r}{2})} |u_{ε, λ}(y)|^{2} \right)^{\frac{1}{2}} ≤ \frac{C_{k, θ_{0}}}{(1 + |λ| r^{2})^{k}} \left| \int_{Ω(\frac{r}{2})} |d_{(y)} u_{ε, λ}(y)|^{2} \right|^{\frac{1}{2}} \left[ \int_{Ω(\frac{r}{2})} |d_{(y)} u_{ε, λ}(y)|^{2} \right].
$$

In view of duality arguments, we can infer that for any $k ∈ N_{+},$

$$
\left( \int_{Ω(\frac{r}{2})} |G_{ε, λ}(\cdot, y)|^{2} \right)^{\frac{1}{2}} ≤ \frac{C_{k, θ_{0}}}{(1 + |λ| r^{2})^{k}} r^{2 - d}. \tag{4.40}
$$

Note that $(L_{ε} - \lambda I)(G_{ε, λ}^{τ}(\cdot, y)) = 0$ in $Ω \setminus B(y, r)$ for any $r > 0$. So in the case of $\frac{r}{2} ≤ δ(x)$, it follows from (3.6) and (4.40) that for any $k ∈ N_{+},$

$$
|G_{ε, λ}(x, y)| ≤ C_{θ_{0}} \left( \int_{Ω(\frac{r}{2})} |G_{ε, λ}(\cdot, y)|^{2} \right)^{\frac{1}{2}} ≤ \frac{C_{k, θ_{0}}}{(1 + |λ| r^{2})^{k}} \frac{1}{|x - y|^{d - 2}}.
$$

When $\frac{r}{2} > δ(x)$, on the other hand, in view of (3.5) and (4.40), for any $σ_{1} ∈ (0, 1)$ and $k ∈ N_{+},$ we have

$$
|G_{ε, λ}(x, y)| = |G_{ε, λ}(x, y) - G_{ε, λ}(x, y)| ≤ |G_{ε, λ}(\cdot, y)| + |G_{ε, λ}(\cdot, y)| C_{σ_{1}} \left( \int_{Ω(x, \frac{r}{2})} |x - y|^{d - 2 + σ_{1}} \right)^{\frac{1}{d - 2 + σ_{1}}}
$$

$$
≤ C_{θ_{0}} \left( \frac{|x - y|}{r} \right)^{σ_{1}} \left( \int_{Ω(\frac{r}{2})} |G_{ε, λ}(\cdot, y)|^{2} \right)^{\frac{1}{2}} ≤ \frac{C_{k, θ_{0}}}{(1 + |λ| r^{2})^{k}} \frac{|δ(x)|^{σ_{1}}}{|x - y|^{d - 2 + σ_{1}}},
$$

where $x ∈ ∂Ω$ is the point such that $|x - y| = δ(x)$. It is easy to complete the proof of (4.18) by considering the same estimates for $G_{ε, λ}(x, y)$ and using property (4.17).

4.3. Two dimensional Green functions.

Theorem 4.8 (Green functions of $L_{ε} - λ I$ with $d = 2$). Suppose that $A$ satisfies (1.2), (1.3), (1.4), (1.6), $ε ≥ 0$ and $Ω$ is a bounded $C^{1}$ domain in $R^{2}$. If $λ ∈ Σ_{θ_{0}} \cup \{0\}$ with $θ_{0} ∈ (0, \frac{r}{2})$, then there exists a unique
Green function \( G_{\varepsilon,\lambda} = (C_{\varepsilon,\lambda}^{\alpha\beta}(\cdot, \cdot)) : \Omega \times \Omega \to \mathbb{C}^{m^2} \cup \{\infty\} \) with \( 1 \leq \alpha, \beta \leq m \), such that
\[
G_{\varepsilon,\lambda}(\cdot, y) \in \text{BMO}(\Omega; \mathbb{C}^{m^2}) \quad \text{i.e.} \quad \|G_{\varepsilon,\lambda}(\cdot, y)\|_{\text{BMO}(\Omega)} \leq C_{\theta_0} \text{ uniformly for } y \in \Omega. \quad (4.41)
\]
Moreover, for all \( u_{\varepsilon,\lambda} \) being the weak solution for the Dirichlet problem \((\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F \) in \( \Omega \) and \( u_{\varepsilon,\lambda} = 0 \) on \( \partial \Omega \), where \( F \in L^p(\Omega; \mathbb{C}^m) \), \((p > 1)\), we have
\[
C_{\varepsilon,\lambda}(x) = \int \Omega G_{\varepsilon,\lambda}(x, y) F(y) dy. \quad (4.42)
\]
Furthermore, for Green functions \( G_{\varepsilon,\lambda}(x, y) \) corresponding to the operators of \( \mathcal{L}_\varepsilon - \lambda I \), we have \( G_{\varepsilon,\lambda}(x, y) = \frac{G_{\varepsilon,\lambda}(y, x)}{4} \). For all \( \sigma_1, \sigma_2, \sigma \in (0, 1) \) and \( k \in \mathbb{N}_+ \), Green functions satisfy pointwise estimates
\[
|G_{\varepsilon,\lambda}(x, y)| \leq C_{k,\theta_0} \left( 1 + |\lambda| |x - y|^2 \right)^{\frac{\sigma}{2}} \text{ for any } x, y \in \Omega, \quad (4.43)
\]
\[
|G_{\varepsilon,\lambda}(x, y)| \leq C_{k,\theta_0} \left( 1 + |\lambda| |x - y|^2 \right)^{\frac{\sigma_1}{2} + \theta} \text{ for any } x, y \in \Omega, \quad (4.44)
\]
\[
|G_{\varepsilon,\lambda}(x, y)| \leq \frac{C_{k,\theta_0} [\delta(x)]^{\sigma_1}}{1 + |\lambda| |x - y|^2 |x - y|^{\sigma_1}} \text{ if } \delta(x) < \frac{1}{4} |x - y|, \quad (4.45)
\]
\[
|G_{\varepsilon,\lambda}(x, y)| \leq \frac{C_{k,\theta_0} [\delta(y)]^{\sigma_2}}{1 + |\lambda| |x - y|^2 |x - y|^{\sigma_2}} \text{ if } \delta(y) < \frac{1}{4} |x - y|, \quad (4.46)
\]
\[
|G_{\varepsilon,\lambda}(x, y)| \leq \frac{C_{k,\theta_0} [\delta(x)]^{\sigma_1} [\delta(y)]^{\sigma_2}}{1 + |\lambda| |x - y|^2 |x - y|^{\sigma_1 + \sigma_2}} \text{ if } \min \{ \delta(x), \delta(y) \} < \frac{1}{4} |x - y|, \quad (4.47)
\]
where \( \delta(x) = \text{dist}(x, \partial \Omega) \) denotes the distance from \( x \) to the boundary of \( \Omega \), \( x \neq y \) and \( C_{k,\theta_0} \) are constants depending only on \( \sigma_1, \sigma_2, \sigma, \mu, \omega(t), k, \theta_0, m \) and \( \Omega \).

**Proof.** For any \( y \in \Omega \) and \( \rho > 0 \), in view of Theorem 2.1, there exists a matrix-valued function \( G_{\rho,\varepsilon,\lambda}(\cdot, y) = (G_{\rho,\varepsilon,\lambda}^{\alpha\beta}(\cdot, y)) : \Omega \to \mathbb{C}^{m^2} \), such that for any \( 1 \leq \gamma \leq m \), \( G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma(\cdot, y) \in H_0^1(\Omega; \mathbb{C}^m) \) satisfies
\[
B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma(\cdot, y), u(\cdot)] = \int_{\Omega(y, \rho)} u(\cdot) \text{ for any } u \in H_0^1(\Omega; \mathbb{C}^m). \]

For any atom function in \( \Omega \) denoted by \( a = a(\cdot) \), s.t.
\[
\text{supp}(a) \subset \Omega(y, \rho) \text{ and } \|a\|_{L^\infty(\Omega)} \leq \frac{1}{|\Omega(y, \rho)|},
\]
we can get \( v_{\varepsilon,\lambda} \in H_0^1(\Omega; \mathbb{R}^m) \) such that \((\mathcal{L}_\varepsilon - \lambda I)(v_{\varepsilon,\lambda}) = a \) in \( \Omega \) and \( v_{\varepsilon,\lambda} = 0 \) on \( \partial \Omega \). Then
\[
\int_{\Omega(y, \rho)} v_{\varepsilon,\lambda}(\cdot) = B_{\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma, v_{\varepsilon,\lambda}(\cdot)] = B_{\varepsilon,\varepsilon,\lambda,\Omega}[G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma, v_{\varepsilon,\lambda}(\cdot)] = \int_{\Omega} G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma(\cdot, y) a(\cdot, y).
\]
This, together with (4.14), implies the following estimate
\[
\left| \int_{\Omega} G_{\rho,\varepsilon,\lambda}^{\gamma}\gamma(\cdot, y) a(\cdot, y) \right| \leq \int_{\Omega(y, \rho)} v_{\varepsilon,\lambda}(\cdot) \leq \|v_{\varepsilon,\lambda}\|_{L^\infty(\Omega(y, \rho))} \leq \|v_{\varepsilon,\lambda}\|_{L^\infty(\Omega)} \leq C_{\theta_0},
\]
where \( C_{\theta_0} \) depends only on \( \mu, \omega(t), \theta_0, m \) and \( \Omega \). According to the fact that \( \mathcal{H}^1 \) is the dual space of BMO space, we can derive that \( G_{\rho,\varepsilon,\lambda}(\cdot, y) \) has a uniform boundedness \( C_{\theta_0} \) in BMO space, where \( C_{\theta_0} \) depends only on \( \mu, \omega(t), m, \theta_0 \) and \( \Omega \). In view of Banach-Alaoglu theorem, we have, for all \( y \in \Omega \), there exists a sequence \( \{\rho_j\} \) such that \( \rho_j \to 0 \) when \( j \to \infty \) and functions \( G_{\rho_j,\varepsilon,\lambda}^{\alpha\beta}(\cdot, y) \in \text{BMO}(\Omega) \) such that \( G_{\rho_j,\varepsilon,\lambda}^{\alpha\beta}(\cdot, y) \) converge to \( G_{\varepsilon,\lambda}^{\alpha\beta}(\cdot, y) \) in the space \( \text{BMO}(\Omega) \) with the sense of the weak*-topology. For
Let $F \in L^q(\Omega; \mathbb{R}^m)$ where $q > 1$, we can choose $1 < q_1 < q$, $p = \frac{2q}{2 - q_1} > 2$ and $u_{\epsilon, \lambda} \in W^{1,p}_0(\Omega; \mathbb{R}^m)$, such that $(L_z - \lambda I)(u_{\epsilon, \lambda}) = F$ and $u_{\epsilon, \lambda} = 0$ on $\partial \Omega$. Then it can be obtained that
\[
\int_{\Omega(y, \rho)} u_{\epsilon, \lambda}^\gamma(\cdot) = B_{\epsilon, \lambda, \Omega} \left[ G_{\epsilon, \lambda}^\gamma(\cdot, y), u_{\epsilon, \lambda}(\cdot) \right] = B_{\epsilon, \lambda, \Omega} \left[ u_{\epsilon, \lambda}(\cdot), G_{\rho, \epsilon, \lambda}^\gamma(\cdot, y) \right] = \int_{\Omega} G_{\rho, \epsilon, \lambda}^\gamma(\cdot, y) F^\alpha(\cdot).
\]
(4.48)
In view of Remark 4.4, Poincaré’s inequality and Hölder’s inequality, we have
\[\|u_{\epsilon, \lambda}\|_{L^p(\Omega)} \leq C\|\nabla u_{\epsilon, \lambda}\|_{L^q(\Omega)} \leq C_{\theta_0} \|F\|_{L^{q_1}(\Omega)} \leq C_{\theta_0} \|F\|_{L^q(\Omega)} ;\]
where $C_{\theta_0}$ is a constant depending only on $\mu, \omega(t), \theta_0, m, p, q$ and $\Omega$. Using Sobolev embedding theorem, we see that $u_{\epsilon, \lambda}$ is continuous. Letting $\rho_j \to 0$, the left hand side of (4.48) converges to $u_{\epsilon, \lambda}(y)$. On the other hand, according to the embedding theorem $L^p \subset H^1$, one can obtain that
\[
\frac{\partial}{\partial \rho_j} G_{\epsilon, \lambda}^\gamma(\cdot, y) \rightarrow \frac{\partial}{\partial \rho_j} G_{\epsilon, \lambda}^\gamma(\cdot, y) \text{ in } H^1(\Omega).
\]
Since the approximation formula, uniqueness of Green functions and duality property follow from almost the same arguments for the case $d \geq 3$, here we will not repeat the proofs of them.

Finally, we will prove the pointwise estimates of Green functions. Namely, we will prove (4.43)-(4.47).

Letting $x_0, y_0 \in \Omega$ with $x_0 \neq y_0$ and assuming that $\delta(x_0) < \frac{1}{2}|x_0 - y_0| = \frac{1}{2}r$, we have $\Omega(x_0, \frac{1}{2}r) \subset \Omega \setminus \{y_0\}$.

According to the definition of $G_{\epsilon, \lambda}^\gamma(\cdot, y_0)$, we have $(L_z - \lambda I)(G_{\epsilon, \lambda}^\gamma(\cdot, y_0)) = 0$ in $\Omega(x_0, \frac{1}{2}r)$ and $G_{\epsilon, \lambda}^\gamma(\cdot, y_0)$ is continuous on $\partial \Omega \cap B(x_0, \frac{1}{2}r)$. Then by (3.6), we can obtain that for any $k \in \mathbb{N}_+$,
\[\left| G_{\epsilon, \lambda}(x_0, y_0) \right| \leq C \left| G_{\epsilon, \lambda}(\cdot, y_0) \right|_{L^\infty(\Omega(x_0, \frac{1}{2}r))} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}} \int_{\Omega(x_0, \frac{1}{2}r)} |G_{\epsilon, \lambda}(\cdot, y_0)|.
\]
According to the definition of $H^1(\Omega)$ and the observation that $G_{\epsilon, \lambda}(\cdot, y_0)_{x_0, \frac{1}{2}r} = 0$ by (2.33), it is easy to see that for any $k \in \mathbb{N}_+$,
\[
\frac{1}{k} \left| G_{\epsilon, \lambda}(x_0, y_0) \right| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}} \int_{\Omega(x_0, \frac{1}{2}r)} |G_{\epsilon, \lambda}(z, y_0) - G_{\epsilon, \lambda}(\cdot, y_0)_{x_0, \frac{1}{2}r}| dz \\
\leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}} \left| G_{\epsilon, \lambda}(\cdot, y_0) \right|_{H^1(\Omega(x_0, \frac{1}{2}r))} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}} \left| G_{\epsilon, \lambda}(\cdot, y_0) \right|_{H^1(\Omega(x_0, \frac{1}{2}r))}.
\]
where $C_{k, \theta_0}$ depends only on $\mu, \omega(t), k, \theta_0, m$ and $\Omega$. With the help of estimates above, we can find that if $\delta(x) < \frac{1}{2}|x - y|$, then
\[
\left| G_{\epsilon, \lambda}(x, y) \right| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda|)^{\frac{k}{2}}}.
\]
(4.50)
Assume that $\delta(x_0) < \frac{1}{2}|x_0 - y_0| = \frac{1}{2}r$ and $z_0 \in \partial \Omega$ is chosen such that $|x_0 - z_0| = \delta(x_0)$, we note that $(L_z - \lambda I)(G_{\epsilon, \lambda}^\gamma(\cdot, y_0)) = 0$ in $\Omega(x_0, \frac{1}{2}r)$ and $G_{\epsilon, \lambda}^\gamma(\cdot, y_0)$ is continuous on $\partial \Omega \cap B(z_0, \frac{1}{2}r)$. In view of the localized boundary Hölder estimates (3.5), then for all $\sigma_1 \in (0, 1)$, there is
\[
\left| G_{\epsilon, \lambda}(x_0, y_0) \right| = \left| G_{\epsilon, \lambda}(x_0, y_0) - G_{\epsilon, \lambda}(z_0, y_0) \right| \\
\leq |x_0 - z_0|^{\sigma_1} \left| G_{\epsilon, \lambda}(\cdot, y_0) \right|_{C^{\sigma_1, \sigma_1}(\Omega(z_0, \frac{1}{2}r))} \\
\leq C_{\theta_0} \left( \frac{\delta(x_0)}{r} \right)^{\sigma_1} \int_{\Omega(z_0, \frac{1}{2}r)} \left| G_{\epsilon, \lambda}(\cdot, y_0) \right| \frac{1}{r^\frac{\sigma_1}{2}}.
\]
(4.51)
Simple observations give that for all $x \in \Omega(z_0, \frac{1}{2}r)$, we have $\delta(x) < \frac{1}{2}r$. Then due to (4.50) and (4.51),
\[
\left| G_{\epsilon, \lambda}(x_0, y_0) \right| \leq \frac{C_{k, \theta_0}\delta(x_0)^{\sigma_1}}{(1 + |\lambda|)^{\frac{k}{2}}} |x_0 - y_0|^{\sigma_1}.
\]
This proves (4.45). On the other hand, (4.46) can be obtained naturally by considering the same estimates for Green functions of $L_{x} - M_{y}$. In addition, we can see from the above proof that $\frac{1}{4}$ is not an essential constant in (4.45)-(4.46). That is, (4.45)-(4.46) are also true if we change $\frac{1}{4}$ to any constants $C_{0}$ such that $0 < C_{0} < \frac{1}{4}$ by using almost the same arguments. Next, let us prove (4.47). We can assume that $\delta(x_{0}) < \frac{1}{4}|x_{0} - y_{0}|$ and $\delta(y_{0}) < \frac{1}{4}|x_{0} - y_{0}|$. Otherwise, (4.47) follows from (4.45)-(4.46) directly. By using almost the same methods, we have, for any $1 \leq \gamma \leq m$,

$$
|G_{x,\lambda}^{\gamma}(x_{0}, y_{0})| \leq C_{b_{0}} \left((\frac{\delta(x_{0})}{r})^{\sigma_{1}} \left(\int_{\Omega(x_{0}, \frac{7r}{16})} |G_{x,\lambda}^{\gamma}(\cdot, y_{0})|^{2} dy\right)^{\frac{1}{2}}\right).
$$

(4.52)

For all $y \in B(z_{0}, \frac{7r}{16}) \cap \Omega$, it is easy to find by triangular inequality that $|y - y_{0}| \geq |x_{0} - y_{0}| - |x_{0} - y| \geq \frac{3}{4}r$. Meanwhile, it is not hard to obtain that

$$
|G_{x,\lambda}^{\gamma}(y_{0})| \leq \frac{C_{b_{0}} \|\delta(y_{0})\|^{\sigma_{2}}}{(1 + |\lambda||z - y_{0}|^{2})^{k}|z - y_{0}|^{\sigma_{2}}}
$$

for any $k \in \mathbb{N}_{+}$ and $z \in \Omega \left(z_{0}, \frac{7r}{16}\right)$. This, together with (4.52), gives the proof of (4.47). Indeed, for any $k \in \mathbb{N}_{+}$ and $\sigma_{1}, \sigma_{2} \in (0, 1)$,

$$
|G_{x,\lambda}^{\gamma}(x_{0}, y_{0})| \leq C_{b_{0}} \left((\frac{\delta(x_{0})}{r})^{\sigma_{1}} \left(\int_{B(z_{0}, \frac{7r}{16}) \cap \Omega} |\delta(y_{0})|^{2\sigma_{2}} \left((1 + |\lambda||y - y_{0}|^{2})^{k}|y - y_{0}|^{2\sigma_{2}}\right) dy\right)^{\frac{1}{2}}\right).
$$

At last, if $\delta(x_{0}) \geq \frac{1}{4}|x_{0} - y_{0}|$ and $\delta(y_{0}) \geq \frac{1}{4}|x_{0} - y_{0}|$, we choose $F \in C_{0}^{\infty}(\Omega_{(x_{0}, \frac{7}{4})}; \mathbb{C}^{m})$. Obviously, here, $\Omega_{(x_{0}, \frac{7}{4})} = B(x_{0}, \frac{7}{4})$. Let $w_{x,\lambda} \in H_{0}^{1}(\Omega; \mathbb{C}^{m})$ satisfy $(L_{x} - M_{y})(w_{x,\lambda}) = F$ in $\Omega$ and $w_{x,\lambda} = 0$ on $\partial\Omega$. In view of the representation theorem, we have that $w_{x,\lambda}(y) = \int_{\Omega} G_{x,\lambda}(\cdot, y) F(\cdot)$. Since $F \equiv 0 \in \Omega \setminus \Omega_{(x_{0}, \frac{7}{4})}$, then $(L_{x} - M_{y})(w_{x,\lambda}) = 0$ in $\Omega \setminus \Omega_{(x_{0}, \frac{7}{4})}$. For all $p > 2$ and $1 < q < 2$, by using (3.6) and Sobolev embedding theorem $H_{0}^{1}(\Omega) \subset L^{p}(\Omega)$ with $p > 1$, it is not hard to get that

$$
|w_{x,\lambda}(y)| \leq C_{b_{0}} \left(\int_{\Omega(y_{0}, \frac{7}{4})} |w_{x,\lambda}|^{2} \right)^{\frac{1}{2}} \leq C_{b_{0}} \left(\int_{\Omega(y_{0}, \frac{7}{4})} |w_{x,\lambda}|^{p} \right)^{\frac{1}{2}} = C_{b_{0}} R_{0}^{\frac{2}{p} - \frac{2}{q}} \left(\int_{\Omega} |\nabla w_{x,\lambda}|^{2} \right)^{\frac{1}{2}}
$$

\leq C_{b_{0}} R_{0}^{\frac{2}{p} + 2 - \frac{2}{q} - \frac{2}{p} \frac{q}{p}} \left(\int_{\Omega} |F|^{q} \right)^{\frac{1}{2}} \leq C_{b_{0}} R_{0}^{\frac{2}{p} + 2 - \frac{2}{q} - \frac{2}{p} \frac{q}{p} - \frac{2}{q} \frac{q}{p}} \left(\int_{\Omega(x_{0}, \frac{7}{4})} |F|^{2} \right)^{\frac{1}{2}}.
$$

where for the third inequality, we have used (2.20). This implies that

$$
\int_{\Omega(x_{0}, \frac{7}{4})} G_{x,\lambda}(\cdot, y) F(\cdot) \leq C_{b_{0}} R_{0}^{\frac{2}{p} + 2 - \frac{2}{q} - \frac{2}{p} \frac{q}{p} - \frac{2}{q} \frac{q}{p}} \left(\int_{\Omega(x_{0}, \frac{7}{4})} |F|^{2} \right)^{\frac{1}{2}}.
$$

(4.53)

Owing to duality arguments, (4.53) implies that for all $p > 2$ and $1 < q < 2$

$$
\left(\int_{\Omega(x_{0}, \frac{7}{4})} |G_{x,\lambda}(\cdot, y_{0})|^{2} \right)^{\frac{1}{2}} \leq C_{b_{0}} \left(\frac{R_{0}}{r}\right)^{\frac{2}{p} - \frac{2}{q} + 2}.
$$

(4.54)

For any $\sigma \in (0, 1)$, we can choose special $p, q$ such that $- \frac{2}{p} + \frac{2}{q} - 2 = - \sigma$. Then (4.54), together with (3.6), gives the proof of (4.43). Of course, there is still a certain distance between this and (4.44), so we need to make more precise estimates.

For $x_{0}, y_{0} \in \Omega$, letting $r_{1} = \frac{1}{4}|x_{0} - y_{0}|$, it can be seen that if $\delta(x_{0}) < \frac{1}{4}|x_{0} - y_{0}|$, then by (4.50), we have $|G_{x,\lambda}(x_{0}, y_{0})| \leq C_{b_{0}}$. If $\delta(x_{0}) \geq \frac{1}{4}|x_{0} - y_{0}|$, we need to consider the sequence of subsets of $\Omega$ denoted
as $\Omega_j = \Omega(x_0, 2^j r_1)$ with $j = 0, 1, ..., N$ such that $2^N r_1 \geq \delta(x_0)$ and $2^{N-1} r_1 < \delta(x_0)$. Obviously, we can bound $N$ by the inequality

$$N \leq C \left\{ 1 + \ln \left( \frac{R_0}{|x_0 - y_0|} \right) \right\}.$$  \hspace{1cm} (4.55)

According to the fact that $G_{\varepsilon, \lambda}(\cdot, y_0) \in BMO(\Omega)$, we have, if $1 \leq j \leq N - 2$, then

$$|G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, 2^j r_1} - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, 2^{j+1} r_1}| = \left| \int_{\Omega(x_0, 2^j r_1)} G_{\varepsilon, \lambda}(\cdot, y_0) - \int_{\Omega(x_0, 2^{j+1} r_1)} G_{\varepsilon, \lambda}(\cdot, y_0) \right| \hspace{1cm} (4.56)

\leq C \int_{\Omega(x_0, 2^{j+1} r_1)} |G_{\varepsilon, \lambda}(x, y_0) - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, 2^{j+1} r_1}| dx \leq C \|G_{\varepsilon, \lambda}(\cdot, y_0)\|_{BMO(\Omega)} \leq C_{\theta_0}.

With the choice of $N$, we can get that

$$\frac{1}{\Omega_{N-1}} \int_{\Omega_N} |G_{\varepsilon, \lambda}(\cdot, y_0)| \leq C \int_{\Omega_N} |G_{\varepsilon, \lambda}(\cdot, y_0)| \leq C \|G_{\varepsilon, \lambda}(\cdot, y_0)\|_{BMO(\Omega)} \leq C_{\theta_0}. \hspace{1cm} (4.57)$$

In the view of (3.6), if $p > 2$ and $q = \frac{2p}{p+2}$, then for any $k \in \mathbb{N}_+$,

$$|G_{\varepsilon, \lambda}(x, y_0) - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| r_1^2)^k} \int_{B(x_0, r_1)} |G_{\varepsilon, \lambda}(x, y_0) - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| dx + C_{k, \theta_0} (1 + |\lambda| r_1^2)^n |\lambda| r_1^2 |G_{\varepsilon, \lambda}(\cdot, y_0)|_{x_0, r_1} \hspace{1cm} (4.58)

\leq \frac{C_{k, \theta_0}}{(1 + |\lambda| r_1^2)^k} + C_{k, \theta_0} (1 + |\lambda| r_1^2)^n |\lambda| r_1^2 |G_{\varepsilon, \lambda}(\cdot, y_0)|_{x_0, r_1},$$

where the second inequality in (4.58) is derived from (4.56) and the fact that

$$\frac{1}{B(x_0, r_1)} \int_{B(x_0, r_1)} |G_{\varepsilon, \lambda}(x, y_0) - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| dx \leq \|G_{\varepsilon, \lambda}(\cdot, y_0)\|_{BMO(\Omega)} \leq C_{\theta_0}.$$  \hspace{1cm}

In view of (4.56), (4.57) and (4.58), it is not hard to see that

$$|G_{\varepsilon, \lambda}(x, y_0)| \leq C \sum_{j=1}^{N-2} \left\{ \int_{\Omega_j} G_{\varepsilon, \lambda}(\cdot, y_0) - \int_{\Omega_{j+1}} G_{\varepsilon, \lambda}(\cdot, y_0) \right\} + \int_{\Omega_{N-1}} |G_{\varepsilon, \lambda}(\cdot, y_0)| \leq C_{\theta_0} \left( 1 + \ln \left( \frac{R_0}{|x_0 - y_0|} \right) \right) + C |G_{\varepsilon, \lambda}(x, y_0) - G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| \hspace{1cm} (4.59)

\leq C_{\theta_0} \left( 1 + \ln \left( \frac{R_0}{|x_0 - y_0|} \right) \right) + C_{\theta_0} (1 + |\lambda| r_1^2)^n |\lambda| r_1^2 |G_{\varepsilon, \lambda}(\cdot, y_0)|_{x_0, r_1},$$

where we denote $\int_{\Omega_{N-1}} G_{\varepsilon, \lambda}(\cdot, y_0) = G_{\varepsilon, \lambda}(x_0, y_0)$ and $C_{\theta_0}$ depends only on $\mu, \omega(t), \theta_0, m$ and $\Omega$. Setting $\sigma \in (0, 1)$ and using (4.43), then for any $k \in \mathbb{N}_+$,

$$|G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| = \int_{\Omega_0} G_{\varepsilon, \lambda}(\cdot, y_0) \leq \int_{\Omega_0} |G_{\varepsilon, \lambda}(\cdot, y_0)| \leq \int_{B(x_0, r_1)} \frac{C_{k, \theta_0} R_0^k}{(1 + |\lambda| |x - y_0|^2)^k |x - y_0|^\sigma} dx \leq \frac{C_{k, \theta_0} R_0^k}{(1 + |\lambda| |x_0 - y_0|^2)^k |x_0 - y_0|^\sigma}.$$  \hspace{1cm}

Choosing $k \in \mathbb{N}_+$ such that $k > n$, we have

$$C_{\theta_0} (1 + |\lambda| r_1^2)^n |\lambda| r_1^2 |G_{\varepsilon, \lambda}(\cdot, y_0)_{x_0, r_1}| \leq C_{\theta_0} |\lambda|^{\frac{n}{2}} R_0^\sigma.$$

This, together with (4.59), completes the proof. \hfill$$\square$$
4.4. More estimates of Green functions.

**Theorem 4.9.** For \( \varepsilon \geq 0 \) and \( d \geq 2 \), \( \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \), let \( \Omega \) be a bounded \( C^{1,\eta} \) \( (0 < \eta < 1) \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). Then for any \( k \in \mathbb{N}_+ \), the Green functions of \( (L_\varepsilon - \lambda I) \) satisfy the uniform pointwise estimates

\[
|\nabla_1 G_{\varepsilon,\lambda}(x,y)| + |\nabla_2 G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d-2}}, \tag{4.60}
\]

\[
|\nabla_1 G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}\delta(y)}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d-1}}, \tag{4.61}
\]

\[
|\nabla_2 G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}\delta(x)}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d}}, \tag{4.62}
\]

\[
|\nabla_1 \nabla_2 G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d}}, \tag{4.63}
\]

for \( x, y \in \Omega \) and \( x \neq y \), where \( C_{k,\theta_0} \) depends only on \( \mu, d, m, \tau, \nu, k, \theta_0, \eta \) and \( \Omega \).

**Remark 4.10.** The Lipschitz estimates of Green functions for operators \( L_\varepsilon - \lambda I \) given above are sharp. These are of great significance in this paper. Further proofs rely heavily on these estimates. In fact, we point out that after proving these estimates, the proofs of theorems on convergence rates for Green functions are standard and the reason why the standard proofs can be carried out later is that we calculated the impact of \( \lambda \) in Lipschitz estimates to the best.

**Remark 4.11.** The proof of Theorem 4.9 is quite different from which of Lipschitz estimates of Green functions for the operator \( L_\varepsilon \), especially when \( d = 2 \). Recall that in [21], the case \( d = 2 \) was discussed by using the fact that \( L_\varepsilon(c) = 0 \) for any constant vector \( c \in \mathbb{C}^m \). However operator \( L_\varepsilon - \lambda I \) do not have such property. In this point of view, we need to modify the proofs in [21] and use some technical calculations here.

**Proof of Theorem 4.9.** Firstly, we consider the case \( d \geq 3 \). Under the condition that \( A \) is Hölder continuous, we can improve the estimates (4.18) to the case that \( \sigma_1 = \sigma_2 = 1 \). Setting \( R = |x-y| \) and applying (4.18), it is apparent that if \( \frac{R}{\varepsilon} \leq \delta(x) \), then

\[
|G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d-2}}, \tag{4.64}
\]

for any \( k \in \mathbb{N}_+ \). If \( \frac{R}{\varepsilon} > \delta(x) \), we can choose \( \overline{x} \in \partial \Omega \) such that \( |x - \overline{x}| = \delta(x) \). Then it follows from (3.8) that for any \( k \in \mathbb{N}_+ \),

\[
|G_{\varepsilon,\lambda}(x,y)| = |G_{\varepsilon,\lambda}(x,y) - G_{\varepsilon,\lambda}(\overline{x},y)| \leq |x - \overline{x}| \|
abla_1 G_{\varepsilon,\lambda}(\cdot,y)\|_{L^\infty(\Omega(x,\frac{\varepsilon}{4}))}
\]

\[
\leq \frac{C_{k,\theta_0}\delta(x)}{1 + |\lambda|R^2} \frac{1}{|x-y|^{d-1}}\left( \int_{\Omega(x,\frac{\varepsilon}{4})} G_{\varepsilon,\lambda}(\cdot,y)^2 \right)^{\frac{1}{2}}. \tag{4.40}
\]

This, together with (4.40), implies that for any \( k \in \mathbb{N}_+ \),

\[
|G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}\delta(x)}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d-1}}. \tag{4.64}
\]

By applying the same arguments on \( G_{\varepsilon,\lambda}(x,y) \) and using (4.17), we have

\[
|G_{\varepsilon,\lambda}(x,y)| \leq \frac{C_{k,\theta_0}}{1 + |\lambda||x-y|^2} \frac{1}{|x-y|^{d-2}} \min \left\{ 1, \frac{\delta(x)}{|x-y|} \right\}. \tag{4.64}
\]
Noticing \((\mathcal{L}_\varepsilon - \lambda I)(G_{\varepsilon,\theta}(\cdot, y)) = 0\) for any \(1 \leq \gamma \leq m\) in \(\Omega \setminus \{y\}\), it follows from (3.8) and (4.40) that

\[
|\nabla_1 G_{\varepsilon,\theta}(x, y)| \leq \frac{C_{\theta_0}}{R} \left( \frac{\int_{\Omega(x, R)} |G_{\varepsilon,\theta}(\cdot, y)|^2}{(1 + |\lambda||x - y|^2)^k} \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0}}{(1 + |\lambda||x - y|^2)^k} R^{d-1}
\]

(4.65)

for any \(k \in \mathbb{N}_+\). Again, in view of (4.17), we can see that (4.60) is true. Similarly, if we use the estimate

\[
|G_{\varepsilon,\theta}(x, y)| \leq \frac{C_{\theta_0} \delta(y)}{(1 + |\lambda||x - y|^2)^k}
\]

for the second inequality of (4.65), it is easy to show that

\[
|\nabla_1 G_{\varepsilon,\theta}(x, y)| \leq \frac{C_{\theta_0} \delta(y)}{(1 + |\lambda||x - y|^2)^k} |x - y|^d.
\]

(4.66)

which gives the proof of (4.61). Similarly, (4.62) can be derived by using (4.17) and (4.66). Now, we only need to show (4.63). Obviously, for any \(1 \leq \gamma \leq m\),

\[(\mathcal{L}_\varepsilon - \lambda I)(\nabla_2 G_{\varepsilon,\theta}(\cdot, y)) = 0 \text{ in } \Omega \setminus \{y\}.
\]

For any \(k_1, k_2 \in \mathbb{N}_+, \ k = k_1 + k_2\), using (4.60), we have

\[
|\nabla_1 \nabla_2 G_{\varepsilon,\theta}(x, y)| \leq \frac{C_{k_1 \theta_0}}{(1 + |\lambda||x|^2)^{k_2} R} \left( \frac{\int_{\Omega(x, R)} |\nabla_2 G_{\varepsilon,\theta}(\cdot, y)|^2}{(1 + |\lambda||x - y|^2)^k} \right)^{\frac{1}{2}} \leq \frac{C_{k_1 \theta_0}}{(1 + |\lambda||x - y|^2)^{k_1} R^{d-1}} \leq \frac{C_{k_1 \theta_0}}{(1 + |\lambda||x - y|^2)^{k_1}} |x - y|^d.
\]

which completes the proof for the case \(d \geq 3\). Next, we consider the case \(d = 2\). Set \(r = |x_0 - y_0|\) with \(x_0 \neq y_0\) and \(x_0, y_0 \in \Omega\). For \(F \in C_0^\infty(\Omega(x_0, \frac{r}{2}); \mathbb{C}^m)\), let \(v_{\varepsilon,\lambda} \) satisfy \((\mathcal{L}_\varepsilon - \lambda I)(v_{\varepsilon,\lambda}) = F\) in \(\Omega\) and \(v_{\varepsilon,\lambda} = 0\) on \(\partial \Omega\). Then \(v_{\varepsilon,\lambda}(y) = \int_\Omega G_{\varepsilon,\lambda}(\cdot, y) F(\cdot)\) due to representation formula (4.42). Since \(F \equiv 0\) in \(\Omega \setminus \Omega(x_0, \frac{r}{2})\), we have, \((\mathcal{L}_\varepsilon - \lambda I)(v_{\varepsilon,\lambda}) = 0 \text{ in } \Omega \setminus \Omega(x_0, \frac{r}{2})\). Owing to (3.6), (4.11), Poincaré’s inequality and Hölder’s inequality, we can get, for any \(p > 2\),

\[
|v_{\varepsilon,\lambda}(y_0)| \leq C_{\theta_0} \left( \frac{\int_{\Omega(y_0, \frac{r}{2})} |v_{\varepsilon,\lambda}|^2}{(1 + |\lambda||y_0|^2)^{k_1}} \right)^{\frac{1}{2}} \leq C_{\theta_0} \left( \frac{\int_{\Omega(y_0, \frac{r}{2})} |v_{\varepsilon,\lambda}|^p}{(1 + |\lambda||y_0|^2)^{k_1} R^{d-1}} \right)^{\frac{1}{p}} = C_{\theta_0} \delta(y_0)r^{-\frac{2}{p}} \left( \frac{\int_{\Omega} |\nabla v_{\varepsilon,\lambda}|^p}{(1 + |\lambda||y_0|^2)^{k_1} R^{d-1}} \right)^{\frac{1}{p}} \leq C_{\theta_0} \delta(y_0) \|F\|_{L^2(\Omega(x_0, \frac{r}{2}))}
\]

By duality arguments, this implies that

\[
\left( \frac{\int_{\Omega(x_0, \frac{r}{2})} |G_{\varepsilon,\lambda}(\cdot, y_0)|^2}{(1 + |\lambda||x_0 - y_0|^2)^k} \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0} \delta(y_0)}{|x_0 - y_0|} \leq \frac{C_{\theta_0} \delta(y_0)}{|x_0 - y_0|^{\frac{1}{2}}},
\]

(4.67)

According to the Lipschitz estimates (3.8) and (4.67), then for any \(k \in \mathbb{N}_+\),

\[
|\nabla_1 G_{\varepsilon,\lambda}(x_0, y_0)| \leq \frac{C_{k_1 \theta_0}}{(1 + |\lambda||x_0 - y_0|^2)^{k_2} R} \left( \frac{\int_{\Omega(x_0, \frac{r}{2})} |G_{\varepsilon,\lambda}(\cdot, y_0)|^2}{(1 + |\lambda||x_0 - y_0|^2)^k} \right)^{\frac{1}{2}} \leq \frac{C_{k_1 \theta_0} \delta(y_0)}{(1 + |\lambda||x_0 - y_0|^2)^{k_1} |x_0 - y_0|^2}.
\]

Then (4.61) is proved and (4.62) follows directly by considering same estimates of \(G_{\varepsilon,\lambda}(x, y)\). Moreover, for \(v_{\varepsilon,\lambda}\), we have \((\mathcal{L}_\varepsilon - \lambda I)(v_{\varepsilon,\lambda} - (v_{\varepsilon,\lambda})_{y_0, \frac{r}{2}}) = \chi(v_{\varepsilon,\lambda})_{y_0, \frac{r}{2}}\) in \(\Omega\). By using (2.10), (3.8), (4.11), Hölder’s
inequality and Poincaré’s inequality, it can be obtained that

$$|\nabla v_{\varepsilon, \lambda}(y_0)| \leq \frac{C_{\theta_0}}{r} \left( \int_{\Omega(y_0, \frac{r}{4})} |v_{\varepsilon, \lambda} - (v_{\varepsilon, \lambda})_{y_0, \frac{r}{4}}|^2 \right)^{\frac{1}{2}} + C_{\theta_0} (1 + |\lambda|^2)^n |\lambda| r (v_{\varepsilon, \lambda})_{y_0, \frac{r}{4}}$$

$$\leq C_{\theta_0} \left( \int_{\Omega(y_0, \frac{r}{4})} |\nabla v_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} + C_{\theta_0} (1 + |\lambda|^2)^n |\lambda| r \left( \int_{\Omega(y_0, \frac{r}{4})} |v_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}}$$

$$= C_{\theta_0} r^{-\frac{2}{p}} \left( \int_{\Omega} |\nabla v_{\varepsilon, \lambda}|^p \right)^{\frac{1}{p}} + C_{\theta_0} (1 + |\lambda|^2)^n |F|_{L^p(\Omega(x_0, \frac{r}{4}))}$$

$$\leq C_{\theta_0} r^{-\frac{2}{p}} |F|_{L^2(\Omega(x_0, \frac{r}{4}))}^p + C_{\theta_0} (1 + |\lambda|^2)^n |F|_{L^2(\Omega(x_0, \frac{r}{4}))}$$

$$\leq C_{\theta_0} (1 + |\lambda|^2)^n |F|_{L^2(\Omega(x_0, \frac{r}{4}))},$$

when $2 < p < \infty$. In view of the definition of $v_{\varepsilon, \lambda}$, we can get that

$$\left| \int_{\Omega(x_0, \frac{r}{4})} \nabla_2 G_{\varepsilon, \lambda}(\cdot, y_0) F(\cdot) \right| \leq C_{\theta_0} (1 + |\lambda|^2)^n |F|_{L^2(\Omega(x_0, \frac{r}{4}))}. \quad (4.68)$$

Again, by duality arguments, it can be inferred that

$$\left( \int_{\Omega(x_0, \frac{r}{4})} |\nabla_2 G_{\varepsilon, \lambda}(\cdot, y_0)|^2 \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0} (1 + |\lambda|^2)^n}{r}. \quad (4.69)$$

Similarly, with the help of Lipschitz estimates (3.8) and (4.69), for any $k \in \mathbb{N}_+$,

$$|\nabla_1 \nabla_2 G_{\varepsilon, \lambda}(x_0, y_0)| \leq \frac{C_k}{(1 + |\lambda|^2)^{k+n} r} \left( \int_{\Omega(x_0, \frac{r}{4})} |\nabla_2 G_{\varepsilon, \lambda}(z, y_0)|^2 dz \right)^{\frac{1}{2}}$$

$$\leq \frac{C_k}{(1 + |\lambda|^2)^{k+n} r} \leq \frac{C_k}{(1 + |\lambda|^2)^{k+n} r^2} \leq C_{k, \theta_0} \left( \frac{1}{1 + |\lambda||x_0 - y_0|^2} \right).$$

This completes the proof of (4.63). Then we only need to show (4.60). To begin with, we note that if $|y - z| < \frac{1}{2}|x - y|$, then

$$|\nabla_1 G_{\varepsilon, \lambda}(x, y) - G_{\varepsilon, \lambda}(x, z)| \leq \|\nabla_1 \nabla_2 G_{\varepsilon, \lambda}(x, \cdot)|_{L^\infty(\Omega(y, |y - y_0|))}|y - z|$$

$$\leq \frac{C_k}{(1 + |\lambda||x - y|^2)^k |x - y|} \leq \frac{C_k}{(1 + |\lambda||x - y|^2)^k |x - y|}. \quad (4.70)$$

for any $k \in \mathbb{N}_+$. If $\frac{1}{4}|x_0 - y_0| \leq \delta(y_0)$, we get that

$$|x - x_0| \leq \frac{1}{2}|x - y_0| \text{ for any } x \in B \left(x_0, \frac{1}{4}|x_0 - y_0| \right). \quad (4.71)$$

At first, there exists a point $\overline{y} \in \partial \Omega$ (see Figure 1), such that

$$\frac{x_0 - \overline{y}}{|x_0 - y_0|} \text{ and } (x_0 - \overline{y}) \cdot (x_0 - y_0) > 0.$$
Moreover, in view of the definition of \( \{ y_j \}_{j=1}^N \), one can obtain that
\[
|x_0 - y_j| = \frac{5}{4}|x_0 - y_{j-1}| = \ldots = \left( \frac{5}{4} \right)^j |x_0 - y_0|, \quad j = 1, \ldots, N. \tag{4.72}
\]

By applying (4.70), we can obtain that for any \( k \in \mathbb{N}_+ \),
\[
|\nabla_1 (G_{\varepsilon, \lambda}(x_0, y_0) - G_{\varepsilon, \lambda}(x_0, y_1))| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x_0 - y_0|^2)^k |x_0 - y_0|}. \tag{4.73}
\]
Similarly, since \( |y_{j+1} - y_j| = \frac{1}{2}|x_0 - y_j| < \frac{1}{2}|x_0 - y_j| \), we have
\[
|\nabla_1 (G_{\varepsilon, \lambda}(x_0, y_j) - G_{\varepsilon, \lambda}(x_0, y_{j+1}))| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x_0 - y_j|^2)^k |x_0 - y_j|}, \tag{4.74}
\]
for any \( j = 1, \ldots, N - 1 \). Finally, owing to the simple observation that
\[
\frac{1}{4}|x_0 - y_N| > |y_N - x| \geq \delta(y_N),
\]
it can be inferred that
\[
|\nabla_1 G_{\varepsilon, \lambda}(x_0, y_N)| \leq \frac{C_{k, \theta_0} \delta(y_N)}{(1 + |\lambda||x_0 - y_N|^2)^k |x_0 - y_N|^2} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x_0 - y_N|^2)^k |x_0 - y_N|}, \tag{4.75}
\]
where we have used (4.61). From (4.73)-(4.75), we can obtain that
\[
|\nabla_1 G_{\varepsilon, \lambda}(x_0, y_0)| \leq \sum_{j=0}^{N} \left( \frac{1}{4} \right)^j \frac{C_{k, \theta_0}}{(1 + |\lambda| \left( \frac{1}{2} \right)^j |x_0 - y_0|^2)^k |x_0 - y_0|} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x_0 - y_0|^2)^k |x_0 - y_0|},
\]
for any \( k \in \mathbb{N}_+ \) and \( \frac{1}{2} |x_0 - y_0| \leq \delta(y_0) \). This, together with (4.61) and (4.17), gives (4.60). \( \square \)

**Remark 4.12.** Under the same assumptions of Theorem 4.8, if we further assume that \( A \) satisfies (1.5), we can improve (4.44) to the sharp estimate. That is, for any \( k \in \mathbb{N}_+ \),
\[
|G_{\varepsilon, \lambda}(x, y)| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x - y|^2)^k \left( 1 + \ln \left( \frac{R_0}{|x - y|} \right) \right)}, \tag{4.76}
\]
where \( C_{k, \theta_0} \) depends only on \( \mu, m, \tau, \nu, k, \theta_0, \eta \) and \( \Omega \). The proof is similar to Theorem 4.9. Like what we have done in (4.70), one can find that if \( |y - z| < \frac{1}{2}|x - y| \), then
\[
|G_{\varepsilon, \lambda}(x, y) - G_{\varepsilon, \lambda}(x, z)| \leq \| \nabla_2 G_{\varepsilon, \lambda}(x, \cdot) \|_{L^\infty(\Omega(y, \frac{R_0}{|x - y|}))} |y - z| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x - y|^2)^k |x - y|} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x - y|^2)^k}, \tag{4.77}
\]
where we have used (4.60). Then using almost the same arguments, we can obtain that
\[
|G_{\varepsilon, \lambda}(x_0, y_j) - G_{\varepsilon, \lambda}(x_0, y_{j+1})| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda||x_0 - y_0|^2)^k}, \tag{4.78}
\]
Lemma 4.13. For \( \varepsilon \geq 0 \) and \( d \geq 2 \), let \( \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \) and \( \Omega \) be a bounded \( C^1 \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.6). Then for any \( x \in \Omega \), \( \sigma \in (0, 1) \) and \( k \in \mathbb{N}_+ \),

\[
\int_{\Omega} |G_{\varepsilon, \lambda}(x, y)| dy \leq C_{\theta_0}(R_0^{-2} + |\lambda|)^{-1} \cdot \begin{cases} 
1 & \text{if } d \geq 3, \\
(1 + |\lambda|R_0^2)^\sigma & \text{if } d = 2,
\end{cases}
\]

which completes the proof.

**Proof.** By using (4.18) with \( k = 2 \), it is easy to infer that

\[
\int_{\Omega} |G_{\varepsilon, \lambda}(x, y)| dy \leq \int_{B(x, R_0)} \frac{C_{\theta_0}}{(1 + |\lambda||x - y|^2)^2} \frac{1}{|x - y|^{d-2}} dy = \int_0^{R_0} \frac{C_{\theta_0} \rho^d}{(1 + |\lambda|\rho^2)^2} d\rho = C_{\theta_0} \frac{1}{|\lambda|(1 + |\lambda|\rho^2)} \left| \frac{\rho}{|\lambda|} \right| \leq C_{\theta_0},
\]

(4.83)

On the other hand, it can be easily seen that

\[
\int_{\Omega} |\nabla G_{\varepsilon, \lambda}(x, y)| dy \leq \int_{B(x, R_0)} \frac{C_{\theta_0}}{(1 + |\lambda||x - y|^2)^2} \frac{1}{|x - y|^{d-2}} dy \leq \int_0^{R_0} C_{\theta_0} \rho d\rho \leq C_{\theta_0} R_0^2.
\]

This, together with (4.83), implies (4.80) with \( d \geq 3 \). For \( d = 2 \), owing to (4.43), we can get that, for any \( \sigma \in (0, 1) \),

\[
\int_{\Omega} |G_{\varepsilon, \lambda}(x, y)| dy \leq \sum_{j=0}^{\infty} \int_{\Omega(x_2^{-j}R_0) \setminus \Omega(x_2^{-j-1}R_0)} \frac{C_{\theta_0} R_0^2}{(1 + |\lambda||x - y|^2)^2 |x - y|^\sigma} dy \\
\leq \sum_{j=0}^{\infty} \frac{C_{\theta_0}(2^{-j}R_0)^2 R_0^2}{(1 + |\lambda|(2^{-j}R_0)^2)^{1-\sigma}(2^{-j}R_0)^\sigma} \leq C_{\theta_0} (1 + |\lambda| R_0^2)^{\sigma} (R_0^{-2} + |\lambda|)^{-1}.
\]

For \( R > 0 \), consider the annulus \( \Omega(x, 2R) \setminus \Omega(x, R) \). One can use small \( d \) dimensional balls, whose radius are \( \frac{R}{2} \) and centers are on \( \partial B(x, \frac{R}{2}) \cap \Omega \), to cover the annulus \( \Omega(x, 2R) \setminus \Omega(x, R) \). We denote these small balls as \( \{B(x_i, \frac{R}{2})\}_{i=1}^N \). Obviously, \( N \leq C \frac{\pi(2R)^d}{\pi(\frac{R}{2})^d} \leq C \), where \( C \) is a constant, independent of \( R \).
Then by using Hölder’s inequality, we have
\[
\int_{\Omega(x, 2R) \setminus \Omega(x, R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)| dy \leq C \left( \int_{\Omega(x, 2R) \setminus \Omega(x, R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} R^{d/2}.
\]
and
\[
\leq C \sum_{i=1}^{N} \left( \int_{\Omega(x_i, \frac{R}{2} R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} R^{d/2}.
\]
In view of Caccioppoli’s inequality (2.23) and (4.18), if \( d \geq 3 \), then for any \( k \in \mathbb{N}_+ \),
\[
\left( \int_{\Omega(x, \frac{\varepsilon}{2} R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \left( \int_{\Omega(x, \frac{\varepsilon}{2} R)} |G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} \leq \frac{C_{k, \theta_0} R^{d/2}}{(1 + |\lambda| R^2)^k} \left( \frac{1}{(1 + |\lambda| R^2)^{d-1}} \right).
\]
This implies that
\[
\int_{\Omega(x, 2R) \setminus \Omega(x, R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)| dy \leq \frac{C_{k, \theta_0} R}{(1 + |\lambda| R^2)^k}.
\]
Thus
\[
\int_{\Omega} |\nabla^2 G_{\varepsilon, \lambda}(x, y)| dy \leq \sum_{i=0}^{\infty} \int_{\Omega(x, 2^{-i} R_0) \setminus \Omega(x, 2^{-i-1} R_0)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)| dy \leq \sum_{i=0}^{\infty} \frac{C_{\theta_0} (2^{-i} R_0)}{(1 + |\lambda| (2^{-i} R_0)^2)^{d-1}} 
\]
\[
\approx \int_{B(0, R_0)} \frac{1}{(1 + |\lambda| R^2)^{d-1}} \frac{1}{|x|^{d-1}} |x| dy \leq C (R_0^{-2} + |\lambda|)^{-\frac{1}{2}},
\]
which gives the proof of (4.81) with \( d \geq 3 \). If \( d = 2 \), to show (4.81), we have
\[
\int_{\Omega(x, 2R) \setminus \Omega(x, R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)| dy \leq \left( \int_{\Omega(x, 2R) \setminus \Omega(x, R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} \leq C \sum_{i=1}^{N} \left( \int_{\Omega(x_i, \frac{\varepsilon}{2} R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} R.
\]
Choose \( f \in C^0_0(\Omega(x_i, \frac{\varepsilon}{2} R) \setminus \Omega(x_i, R); \mathbb{C}^{m \times d}) \), \( w_{\varepsilon, \lambda} \) such that \( (\mathcal{L}_{\varepsilon} - \lambda I)(w_{\varepsilon, \lambda}) = \text{div}(f) \) in \( \Omega \) and \( w_{\varepsilon, \lambda} = 0 \) in \( \partial \Omega \). Then \( w_{\varepsilon, \lambda}(z) = \int_{\Omega} \nabla^2 G_{\varepsilon, \lambda}(z, \cdot) f(\cdot) \) due to (4.42). In view of (3.6), (4.12), Hölder’s inequality, the fact that \( (\mathcal{L}_{\varepsilon} - \lambda I)(w_{\varepsilon, \lambda}) = 0 \) in \( \Omega \setminus \Omega(x_i, \frac{\varepsilon}{2} R) \) and \( \Omega(x_i, \frac{1}{20} R) \subset \Omega(x_i, \frac{\varepsilon}{2} R) \), it is not hard to deduce that for any \( 2 < p < \infty \) and \( k \in \mathbb{N}_+ \),
\[
|w_{\varepsilon, \lambda}(x)| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \left( \int_{\Omega(x, \frac{\varepsilon}{2} R)} |w_{\varepsilon, \lambda}|^2 \right)^{1/2} \leq \frac{C_{k, \theta_0} R^{-1+\frac{1}{p}}}{(1 + |\lambda| R^2)^k} \|w_{\varepsilon, \lambda}\|_{L^{2p}(\Omega)},
\]
\[
\leq \frac{C_{k, \theta_0} R^{-1+\frac{1}{p}}}{(1 + |\lambda| R^2)^k} \|f\|_{L^{2p}(\Omega)},
\]
Then by using the representation of \( w_{\varepsilon, \lambda} \) given above, we have
\[
\left| \int_{\Omega(x_i, \frac{\varepsilon}{2} R)} \nabla^2 G_{\varepsilon, \lambda}(x, y) f(x, y) dy \right| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \|f\|_{L^2(\Omega(x_i, \frac{\varepsilon}{2} R))}.
\]
By duality arguments, we can infer that
\[
\left( \int_{\Omega(x_i, \frac{\varepsilon}{2} R)} |\nabla^2 G_{\varepsilon, \lambda}(x, y)|^2 dy \right)^{1/2} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^{k}}.
\]
With the help of (4.84), it can be seen that for any \( k \in \mathbb{N}_+ \),
\[
\int_{\Omega(x,2R) \setminus \Omega(x,R)} |\nabla_2 G_{\varepsilon,\lambda}(x,y)| \, dy \leq \frac{C_{k,\theta_0} R}{(1 + |\lambda|R^2)^k}.
\] (4.85)

Then according to the calculations with \( d \geq 3 \),
\[
\int_{\Omega} |\nabla_2 G_{\varepsilon,\lambda}(x,y)| \, dy \leq \sum_{i=0}^\infty \int_{\Omega(x,2^{-i}R_0) \setminus \Omega(x,2^{-i-1}R_0)} |\nabla_2 G_{\varepsilon,\lambda}(x,y)| \, dy
\]
\[
\leq \sum_{i=0}^\infty \frac{C_{\theta_0} 2^{-i} R_0}{(1 + |\lambda|(2^{-i} R_0)^2)^{\frac{d}{2}}} \leq C_{\theta_0} (R_0^{-2} + |\lambda|)^{-\frac{d}{2}}.
\]

For the proof of (4.82), we simply note that by (4.60) with \( k = 2 \),
\[
\int_{\Omega} |\nabla_1 G_{\varepsilon,\lambda}(x,y)| \, dy \leq \int_{\Omega} \frac{C_{\theta_0}}{(1 + |\lambda||x - y|^2)|x - y|^{d-1}} \, dy \leq C_{\theta_0} (R_0^{-2} + |\lambda|)^{-\frac{d}{2}}.
\]
Then we can complete the proof. \( \square \)

5. Convergence estimates of resolvents

5.1. Convergence of Green functions. Now we turn to estimate \( |G_{\varepsilon,\lambda}(x,y) - G_{0,\lambda}(x,y)| \), where \( G_{\varepsilon,\lambda} \) and \( G_{0,\lambda} \) are Green functions for operators \( \mathcal{L}_\varepsilon - \lambda I \) and \( \mathcal{L}_0 - \lambda I \) respectively. Such estimates are essential in the proof of Theorem 1.3 and 1.4.

**Theorem 5.1** (Convergence of Green functions I). For \( \varepsilon \geq 0 \), \( d \geq 2 \), \( \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in \left(0, \frac{\pi}{2}\right) \), let \( \Omega \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). Then for any \( k \in \mathbb{N}_+ \) and \( x, y \in \Omega \) with \( x \neq y \),
\[
|G_{\varepsilon,\lambda}(x,y) - G_{0,\lambda}(x,y)| \leq \frac{C_{k,\theta_0} \varepsilon}{(1 + |\lambda||x - y|^2)^{k}|x - y|^{d-1}},
\] (5.1)

where \( C_{k,\theta_0} \) depends only on \( \mu, d, m, \nu, \tau, k, \theta_0 \) and \( \Omega \).

For continuous function \( u \in \Omega \), the nontangential maximal function is defined by
\[
(u)^*(y) = \sup \{ |u(x)| : x \in \Omega \text{ and } |x - y| < C_0 \delta(x) \}
\] (5.2)
for \( y \in \partial \Omega \), where \( C_0 = C_0(\Omega) > 1 \) is sufficiently large depending on \( \Omega \).

**Theorem 5.2** (Nontangential-maximal-function estimates). For \( \varepsilon \geq 0 \) and \( d \geq 2 \), \( \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in \left(0, \frac{\pi}{2}\right) \), let \( \Omega \) be a bounded \( C^{1,1-\eta} \) domain in \( \mathbb{R}^d \) with \( 0 < \eta < 1 \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4), (1.5) and \( 1 < p \leq \infty \). For \( g \in L^p(\partial \Omega; \mathbb{C}^m) \), let \( u_{\varepsilon,\lambda} \) be the unique solution to the Dirichlet problem \( (\mathcal{L}_\varepsilon - \lambda I)u_{\varepsilon,\lambda} = 0 \) in \( \Omega \) and \( u_{\varepsilon,\lambda} = g \) on \( \partial \Omega \) with the property \( (u_{\varepsilon,\lambda})^* \in L^p(\partial \Omega) \). Then
\[
\|(u_{\varepsilon,\lambda})^*\|_{L^p(\partial \Omega)} \leq C_{\theta_0} \|g\|_{L^p(\partial \Omega)},
\] (5.3)

where \( C_{\theta_0} \) depends only on \( \mu, d, m, \nu, \tau, \theta_0, \eta \) and \( \Omega \). In the case that \( p = \infty \), (5.3) implies,
\[
\|u_{\varepsilon,\lambda}\|_{L^\infty(\Omega)} \leq C_{\theta_0} \|g\|_{L^\infty(\partial \Omega)}.
\] (5.4)

**Proof.** By using the definition of Green functions, we can represent \( u_{\varepsilon,\lambda} \) by the formula
\[
u_{\varepsilon,\lambda}(x) = \int_{\partial \Omega} P_{\varepsilon,\lambda}(x,y)g(y)\, d\sigma(y),
\]
where the Poisson kernel \( P_{\varepsilon,\lambda}(x,y) = \left(P_{\varepsilon,\lambda}^\gamma(x,y)\right) \) for \( \mathcal{L}_\varepsilon - \lambda I \) in \( \Omega \) is given by
\[
P_{\varepsilon,\lambda}^\gamma(x,y) = -n_i(y)a_{ij}^\gamma(y/\varepsilon) \frac{\partial}{\partial y_j}\{G_{\varepsilon,\lambda}(x,y)\}
\]
for \( x \in \Omega, \ y \in \partial \Omega \) and \( n(x) = (n_1(x), n_2(x), \ldots, n_d(x)) \) denotes the outward unit normal to \( \partial \Omega \). In view of the estimate (4.62), we have, for any \( k \in \mathbb{N}_+ \),
\[
|P_{\varepsilon, \lambda}(x, y)| \leq \frac{C_{k, \theta_0} \delta(x)}{(1 + |\lambda||x - y|^2)^k |x - y|^d} \leq \frac{C_{\theta_0} \delta(x)}{|x - y|^d}.
\]
Hence, we can obtain that
\[
|u_{\varepsilon, \lambda}(x)| \leq \int_{\partial \Omega} \frac{C_{\theta_0} \delta(x)}{|x - y|^d} |g(y)| dS(y) \text{ for any } x \in \Omega.
\]

The rest of the proof is standard, which can be obtained in Theorem 4.6.5 of [21].

\[\square\]

**Lemma 5.3.** For \( \varepsilon \geq 0, \ d \geq 2, \ \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \), let \( \Omega \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). If \( \Delta(x_0, 3R) \neq \emptyset \), assume that \( u_{\varepsilon, \lambda} \in H^1(\Omega(x_0, 3R); \mathbb{C}^m) \) is a solution of the boundary problem
\[
(L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = 0 \text{ in } \Omega(x_0, 3R) \text{ and } u_{\varepsilon, \lambda} = f \text{ on } \Delta(x_0, 3R),
\]
with \( \|f\|_{L^\infty(\Omega)} < \infty \). If \( \Delta(x_0, 3R) = \emptyset \), assume that \( u_{\varepsilon, \lambda} \in H^1(B(x_0, 3R); \mathbb{C}^m) \) is the weak solution of the interior problem
\[
(L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = 0 \text{ in } B(x_0, 3R).
\]

Then for any \( k \in \mathbb{N}_+ \),
\[
\|u_{\varepsilon, \lambda}\|_{L^\infty(\Omega(x_0, R))} \leq C_{k, \theta_0} \|f\|_{L^\infty(\Delta(x_0, 3R))} + \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega(x_0, 3R)} |u_{\varepsilon, \lambda}|, \tag{5.5}
\]
where \( C_{k, \theta_0} \) depends only on \( \mu, d, m, \theta_0, k, \nu, \tau \) and \( \Omega \).

**Proof.** We only need to show the case that \( \Delta(x_0, 3R) \neq \emptyset \) since the other follows directly from (3.6). If \( f \equiv 0 \), the estimate is also a consequence of (3.6). To treat the general case, let \( v_{\varepsilon, \lambda} \) be the solution to \( (L_{\varepsilon} - \lambda I)(v_{\varepsilon, \lambda}) = 0 \) in \( \bar{\Omega} \) with the Dirichlet condition \( v_{\varepsilon, \lambda} = f \) on \( \partial \Omega \cap \partial \Omega \) and \( v_{\varepsilon, \lambda} = 0 \) on \( \partial \Omega \setminus \partial \Omega \), where \( \bar{\Omega} \) is a \( C^{1,1} \) domain such that \( \Omega(x_0, 2R) \subset \bar{\Omega} \subset \Omega(x_0, 3R) \). By the maximum principle (5.4), we have,
\[
\|v_{\varepsilon, \lambda}\|_{L^\infty(\bar{\Omega})} \leq C_{\theta_0} \|v_{\varepsilon, \lambda}\|_{L^\infty(\partial \bar{\Omega})} \leq C_{\theta_0} \|f\|_{L^\infty(\Delta(x_0, 3R))}.
\]
This, together with the fact that for any \( k \in \mathbb{N}_+ \),
\[
\|u_{\varepsilon, \lambda} - v_{\varepsilon, \lambda}\|_{L^\infty(\Omega)} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda} - v_{\varepsilon, \lambda}| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda}| + C_{k, \theta_0} \|f\|_{L^\infty(\Delta(x_0, 3R))},
\]
gives the results.

\[\square\]

**Lemma 5.4.** For \( \varepsilon \geq 0 \) and \( d \geq 2, \ \lambda \in \Sigma_{\theta_0} \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \), let \( \Omega \) be a bounded \( C^{1,1} \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). Let \( u_{\varepsilon, \lambda} \in H^1(\Omega(x_0, 4R); \mathbb{C}^m) \) and \( u_{0, \lambda} \in W^{2,p}(\Omega(x_0, 4R); \mathbb{C}^m) \) for some \( d < p < \infty \). If \( \Delta(x_0, 4R) \neq \emptyset \), assume that \( (L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = (L_0 - \lambda I)(u_{0, \lambda}) \) in \( \Omega(x_0, 4R) \) and \( u_{\varepsilon, \lambda} = u_{0, \lambda} \) on \( \Delta(x_0, 4R) \), and if \( \Delta(x_0, 4R) = \emptyset \), assume that \( (L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = (L_0 - \lambda I)(u_{0, \lambda}) \) in \( B(x_0, 4R) \).
Then for any $k \in \mathbb{N}_+$,
\[
\|u_{\varepsilon, \lambda} - u_{0, \lambda}\|_{L^\infty(\Omega_R)} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{3R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| + C_{k, \theta_0} \varepsilon \left\{ R^{1 - \frac{1}{p}} \|\nabla u_{0, \lambda}\|_{L^p(\Omega_{3R})} + (1 + |\lambda| R^2) \|\nabla u_{0, \lambda}\|_{L^\infty(\Omega_{3R})} \right\},
\]  
(5.6)
where $C_{k, \theta_0}$ depends on $\mu, d, m, \nu, \tau, p, k, \theta_0$ and $\Omega$.

**Proof.** Firstly, we consider the case that $\Delta(x_0, 3R) \neq 0$. Choose a domain $\widetilde{\Omega}$, which is $C^{1, 1}$ such that $\Omega(x_0, 3R) \subset \widetilde{\Omega} \subset \Omega(x_0, 4R)$. Consider
\[
w_{\varepsilon, \lambda}(x) = u_{\varepsilon, \lambda}(x) - u_{0, \lambda}(x) - \varepsilon \chi_j^\beta(x/\varepsilon) \frac{\partial u_{0, \lambda}^\beta}{\partial x_j} = w_{\varepsilon, \lambda}^{(1)}(x) + w_{\varepsilon, \lambda}^{(2)}(x) \quad \text{in} \quad \widetilde{\Omega},
\]
where $w_{\varepsilon, \lambda}^{(1)}$ and $w_{\varepsilon, \lambda}^{(2)}$ are weak solutions for the Drichlet problems
\[
(\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon, \lambda}^{(1)}) = (\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon, \lambda}) \quad \text{in} \quad \widetilde{\Omega} \quad \text{and} \quad w_{\varepsilon, \lambda}^{(1)} \in H_0^1(\widetilde{\Omega}; \mathbb{C}^m),
\]
\[
(\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon, \lambda}^{(2)}) = 0 \quad \text{in} \quad \widetilde{\Omega} \quad \text{and} \quad w_{\varepsilon, \lambda}^{(2)} = w_{\varepsilon, \lambda} \quad \text{on} \quad \partial \widetilde{\Omega}.
\]
(5.7)
(5.8)
Since $w_{\varepsilon, \lambda}^{(2)} = w_{\varepsilon, \lambda} = -\varepsilon \chi_j^\beta(x/\varepsilon) \nabla u_{0, \lambda}$ on $\Delta(x_0, 3R)$ and $\|\chi\|_{L^\infty(\Omega)} \leq C$, it follows from (5.5) that for any $k \in \mathbb{N}_+$,
\[
w_{\varepsilon, \lambda}^{(2)} \|_{L^\infty(\Omega_R)} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{3R}} |w_{\varepsilon, \lambda}^{(2)}| + \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{3R}} |w_{\varepsilon, \lambda}^{(1)}| + \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{3R}} |w_{\varepsilon, \lambda}| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \left\{ \int_{\Omega_{3R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| + \|w_{\varepsilon, \lambda}^{(1)}\|_{L^\infty(\Omega_{3R})} + \varepsilon \|\nabla u_{0, \lambda}\|_{L^\infty(\Omega_{3R})} \right\}.
\]
By using definitions of $u_{\varepsilon, \lambda}$, $w_{\varepsilon, \lambda}^{(1)}$ and $w_{\varepsilon, \lambda}^{(2)}$, this gives
\[
\|u_{\varepsilon, \lambda} - u_{0, \lambda}\|_{L^\infty(\Omega_R)} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k} \int_{\Omega_{3R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| + C_{k, \theta_0} \left\{ \|w_{\varepsilon, \lambda}^{(1)}\|_{L^\infty(\Omega_{3R})} + \varepsilon \|\nabla u_{0, \lambda}\|_{L^\infty(\Omega_{3R})} \right\}.
\]
(5.9)
To estimate $w_{\varepsilon, \lambda}^{(1)}$ on $\Omega(x_0, 3R)$, we use the representation formula to obtain that
\[
w_{\varepsilon, \lambda}^{(1)}(x) = \int_{\widetilde{\Omega}} \tilde{G}_{\varepsilon, \lambda}(x, y)(\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon, \lambda})dy,
\]
where $\tilde{G}_{\varepsilon, \lambda}(x, y)$ denotes the matrix of the Green function for $\mathcal{L}_\varepsilon - \lambda I$ in $\widetilde{\Omega}$. Note that
\[
\alpha = -\varepsilon \frac{\partial}{\partial x_i} \left\{ r^{\alpha \beta}_{ij}(x/\varepsilon) \frac{\partial^2 u_{0, \lambda}^\beta}{\partial x_j \partial x_k} + \varepsilon \chi_j^\beta(x/\varepsilon) \frac{\partial u_{0, \lambda}^\beta}{\partial x_j} \right\} + \varepsilon \frac{\partial}{\partial x_i} \left\{ a^{\alpha \beta}_{ij}(x/\varepsilon) \chi_k^\beta(x/\varepsilon) \frac{\partial^2 u_{0, \lambda}^\beta}{\partial x_j \partial x_k} \right\}.
\]
(5.10)
Then by using the representation formula of Green functions (4.16) and (4.42),
\[
w_{\varepsilon, \lambda}^{(1)}(x) = \varepsilon \int_{\Omega} \frac{\partial}{\partial y_i} \left( \tilde{G}_{\varepsilon, \lambda}(x, y) \right) \left[ r^{\alpha \beta}_{ij}(y/\varepsilon) - a^{\alpha \beta}_{ij}(y/\varepsilon) \chi_k^\beta(y/\varepsilon) \right] \frac{\partial^2 u_{0, \lambda}^\beta}{\partial y_j \partial y_k} dy + \varepsilon \chi_j^\beta \int_{\Omega} \tilde{G}_{\varepsilon, \lambda}(x, y) \chi_k^\beta(y/\varepsilon) \frac{\partial u_{0, \lambda}^\beta}{\partial y_j} dy.
\]
Since \( \|F_{ij}\|_{L^\infty(\Omega)}, \|\chi_j\|_{L^\infty(\Omega)} \leq C \) and \( p > d \), we have
\[
\varepsilon \lambda \int_{\Omega} \tilde{G}_{\varepsilon,\lambda}(x, y) \chi_j^{\alpha\beta}(y/\varepsilon) \frac{\partial u_{0,\lambda}^\beta}{\partial y_j} dy \leq C \theta_0 \varepsilon |\lambda| \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})} \int_{\tilde{\Omega}} |\tilde{G}_{\varepsilon,\lambda}(x, y)| dy
\]
\[
\leq C \theta_0 \varepsilon |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})},
\]
where we have used (4.80) for the second inequality. Then
\[
|w_{\varepsilon,\lambda}^{(1)}(x)| \leq C \varepsilon \delta \int_{\Omega} |\nabla^2 \tilde{G}_{\varepsilon,\lambda}(x, y)||\nabla^2 u_{0,\lambda}(y)| dy + C \theta_0 \varepsilon |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})}
\]
\[
\leq C \varepsilon \|\nabla^2 u_{0,\lambda}\|_{L^p(\Omega_{4R})} \left( \int_{\tilde{\Omega}} |\nabla^2 \tilde{G}_{\varepsilon,\lambda}(x, y)|^p' dy \right)^{\frac{1}{p}} + C \theta_0 \varepsilon |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})}.
\]
In view of (4.60) with \( k = 0 \), we have \( \left( \int_{\Omega} |\nabla^2 \tilde{G}_{\varepsilon,\lambda}(x, y)|^p' dy \right)^{\frac{1}{p'}} \leq C \theta_0 R^{1 - \frac{d}{p}} \) and then
\[
|w_{\varepsilon,\lambda}^{(1)}(x)| \leq C \theta_0 \varepsilon \left\{ R^{1 - \frac{d}{p}} \|\nabla^2 u_{0,\lambda}\|_{L^p(\Omega_{4R})} + |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})} \right\}.
\]
This, together with (5.9), gives the proof for the case that \( \Delta(x_0, 3R) \neq \emptyset \). If \( \Delta(x_0, 3R) = \emptyset \), we can choose \( w_{\varepsilon,\lambda}^{(1)} \) and \( w_{\varepsilon,\lambda}^{(2)} \) by
\[
(\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon,\lambda}^{(1)}) = (\mathcal{L}_\varepsilon - \lambda I)(\hat{w}_{\varepsilon,\lambda}) \text{ in } B(x_0, 3R) \quad \text{and} \quad w_{\varepsilon,\lambda}^{(1)} \in H_0^1(B(x_0, 3R); \mathbb{C}^m),
\]
\[
(\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon,\lambda}^{(2)}) = 0 \text{ in } B(x_0, 3R) \quad \text{and} \quad w_{\varepsilon,\lambda}^{(2)} = \hat{w}_{\varepsilon,\lambda} \text{ on } \partial B(x_0, 3R).
\]
By using (3.6), it can be seen that for any \( k \in \mathbb{N}_+ \),
\[
\|w_{\varepsilon,\lambda}^{(2)}\|_{L^\infty(B_R)} \leq \frac{C k \theta_0}{(1 + |\lambda| R^2)^k} \left\{ \int_{B_{3R}} |w_{\varepsilon,\lambda}^{(1)}| + \int_{B_{3R}} |w_{\varepsilon,\lambda}| \right\}
\]
\[
\leq \frac{C k \theta_0}{(1 + |\lambda| R^2)^k} \left\{ \int_{B_{3R}} |u_{\varepsilon,\lambda} - u_{0,\lambda}| + \|w_{\varepsilon,\lambda}^{(1)}\|_{L^\infty(B_{3R})} + \varepsilon \|\nabla u_{0,\lambda}\|_{L^\infty(B_{3R})} \right\}.
\]
Then (5.9) is still true. Choosing \( \tilde{\Omega} = B(x_0, 3R) \) and using almost the same arguments for the case that \( \Delta(x_0, 3R) \neq \emptyset \), we can complete the proof. \( \square \)

**Proof of Theorem 5.1.** Fix \( x_0, y_0 \in \Omega \) and \( R = \frac{|x_0 - y_0|}{16} > 0 \). For \( F \in C_0^\infty(\Omega(y_0, R); \mathbb{C}^m) \), let
\[
u_{\varepsilon,\lambda}(x) = \int_{\Omega} G_{\varepsilon,\lambda}(x, y) F(y) dy \quad \text{and} \quad u_{0,\lambda}(x) = \int_{\Omega} G_{0,\lambda}(x, y) \overline{F(y)} dy.
\]
Then \( (\mathcal{L}_\varepsilon - \lambda I)(\nu_{\varepsilon,\lambda}) = (\mathcal{L}_\varepsilon - \lambda I)(u_{0,\lambda}) = F \) in \( \Omega \) and \( u_{\varepsilon,\lambda} = u_{0,\lambda} = 0 \) on \( \partial \Omega \). For any \( x \in \Omega \) and \( p > d \), using (4.60) with \( k = 0 \), it follows by Hölder’s inequality that
\[
|\nabla u_{0,\lambda}(x)| \leq \left| \int_{\Omega} \nabla_1 G_{0,\lambda}(x, y) \overline{F(y)} dy \right| \leq C \|F\|_{L^p(\Omega(y_0, R))} \left( \int_{\Omega(y_0, R)} |\nabla_1 G_{0,\lambda}(x, y)|^p' dy \right)^{\frac{1}{p'}}
\]
\[
\leq C \theta_0 \left( \int_{\Omega(y_0, R)} \frac{1}{|x - y|^{(d-1)p}} dy \right)^{\frac{1}{p'}} \|F\|_{L^p(\Omega(y_0, R))} \leq C \theta_0 R^{1 - \frac{d}{p}} \|F\|_{L^p(\Omega(y_0, R))}.
\]
To this end, we can obtain the \( L^\infty \) estimates of \( \nabla u_{0,\lambda} \), that is,
\[
\|\nabla u_{0,\lambda}\|_{L^\infty(\Omega)} \leq C \theta_0 R^{1 - \frac{d}{p}} \|F\|_{L^p(\Omega(y_0, R))}.
\]
To estimate \( \|u_{\varepsilon,\lambda} - u_{0,\lambda}\|_{L^\infty(\Omega(y_0, R))} \), set \( w_{\varepsilon,\lambda} \) by
\[
w_{\varepsilon,\lambda}(x) = u_{\varepsilon,\lambda}(x) - u_{0,\lambda}(x) - \varepsilon \chi_j^\beta \frac{\partial u_{0,\lambda}^\beta}{\partial x_j} = v_{\varepsilon,\lambda}(x) + z_{\varepsilon,\lambda}(x),
\]
where we have used (4.80) for the second inequality. Then
\[
|w_{\varepsilon,\lambda}^{(1)}(x)| \leq C \varepsilon \delta \int_{\Omega} |\nabla^2 \tilde{G}_{\varepsilon,\lambda}(x, y)||\nabla^2 u_{0,\lambda}(y)| dy + C \theta_0 \varepsilon |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})}
\]
\[
\leq C \varepsilon \|\nabla^2 u_{0,\lambda}\|_{L^p(\Omega_{4R})} \left( \int_{\tilde{\Omega}} |\nabla^2 \tilde{G}_{\varepsilon,\lambda}(x, y)|^p' dy \right)^{\frac{1}{p}} + C \theta_0 \varepsilon |\lambda| R^2 \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_{4R})}.
\]
where \( v_{\varepsilon, \lambda} \in H_0^1(\Omega; \mathbb{C}^m) \) and \( (\mathcal{L}_\varepsilon - \lambda I)(v_{\varepsilon, \lambda}) = (\mathcal{L}_\varepsilon - \lambda I)(w_{\varepsilon, \lambda}) \) in \( \Omega \). By using (2.10), (4.6) and (5.10) with \( p = 2 \), we can obtain
\[
\| \nabla v_{\varepsilon, \lambda} \|_{L^2(\Omega)} \leq C_{\theta_0} \varepsilon \| \nabla^2 u_{0,\lambda} \|_{L^2(\Omega)} + C_{\theta_0} \varepsilon \| \lambda \|_\infty \| \nabla u_{0,\lambda} \|_{L^2(\Omega)} \\
\leq C_{\theta_0} \varepsilon \| F \|_{L^2(\Omega)} \leq C_{\theta_0} \varepsilon \| F \|_{L^2(\Omega(y_0,R))},
\]

By Hölder’s and Sobolev’s inequalities, this implies that if \( d \geq 3 \), we have
\[
\| v_{\varepsilon, \lambda} \|_{L^2(\Omega(y_0,R))} \leq CR\| v_{\varepsilon, \lambda} \|_{L^\frac{2d}{d-2}(\Omega(y_0,R))} \leq CR\| \nabla v_{\varepsilon, \lambda} \|_{L^2(\Omega)} \\
\leq C_{\theta_0} \varepsilon \| F \|_{L^2(\Omega(y_0,R))} \leq C_{\theta_0} \varepsilon R^{1+\frac{4}{d}-\frac{2}{q}} \| F \|_{L^p(\Omega(y_0,R))}, \tag{5.12}
\]

where \( p > d \). If \( d = 2 \), one can use the following estimate
\[
\| u_{\varepsilon, \lambda} - u_{0,\lambda} \|_{L^2(\Omega(x_0,R))} \leq C\| F \|_{L^2(\Omega(y_0,R))},
\]
in place of (5.12). Indeed, for any \( 2 < q < \infty \), by using (5.10), Hölder’s inequality and Sobolev embedding theorem that \( W_0^{1,\frac{2d}{d-2}}(\Omega) \subset L^q(\Omega) \), it can be got that
\[
\| v_{\varepsilon, \lambda} \|_{L^2(\Omega(y_0,R))} \leq CR\| v_{\varepsilon, \lambda} \|_{L^q(\Omega(y_0,R))} \leq CR\| \nabla v_{\varepsilon, \lambda} \|_{L^\frac{2d}{d-2}(\Omega(y_0,R))} \leq CR\| \nabla v_{\varepsilon, \lambda} \|_{L^\frac{2d}{d-2}(\Omega)}/2
\]

where for the forth and fifth inequalities, we have used Theorem 1.2 and (4.6). Since \( (\mathcal{L}_\varepsilon - \lambda I)(z_{\varepsilon, \lambda}) = 0 \) in \( \Omega \) and \( z_{\varepsilon, \lambda} = w_{\varepsilon, \lambda} \) on \( \partial\Omega \), by maximum principle (5.4),
\[
\| z_{\varepsilon, \lambda} \|_{L^\infty(\Omega)} \leq C_{\theta_0} \| z_{\varepsilon, \lambda} \|_{L^\infty(\partial\Omega)} \leq C_{\theta_0} \varepsilon \| \nabla u_{0,\lambda} \|_{L^\infty(\partial\Omega)}.
\]

In view of (5.11), we obtain
\[
\| u_{\varepsilon, \lambda} - u_{0,\lambda} \|_{L^2(\Omega(x_0,R))} \leq \| u_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + C\varepsilon R^{\frac{4}{d}} \| \nabla u_{0,\lambda} \|_{L^\infty(\Omega)} \\
\leq \| v_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + \| z_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + C\varepsilon R^{\frac{4}{d}} \| \nabla u_{0,\lambda} \|_{L^\infty(\Omega)} \\
\leq \| v_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + C_{\theta_0} \varepsilon R^{\frac{4}{d}} \| \nabla u_{0,\lambda} \|_{L^\infty(\Omega)} \\
\leq C_{\theta_0} \varepsilon R^{1+\frac{4}{d}-\frac{2}{q}} \| F \|_{L^p(\Omega(y_0,R))},
\]

where \( p > d \) and \( d \geq 3 \). Similarly, if \( d = 2 \), we have, for \( p > 2 \),
\[
\| u_{\varepsilon, \lambda} - u_{0,\lambda} \|_{L^2(\Omega(x_0,R))} \leq \| v_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + \| z_{\varepsilon, \lambda} \|_{L^2(\Omega(x_0,R))} + C\varepsilon R \| \nabla u_{0,\lambda} \|_{L^\infty(\Omega)} \\
\leq C_{\theta_0} \varepsilon R^{2+\frac{4}{d}-\frac{2}{q}} \| F \|_{L^p(\Omega(y_0,R))}.
\]

This, together with Lemma 5.4 and (4.6), gives
\[
\| u_{\varepsilon, \lambda}(x_0) - u_{0,\lambda}(x_0) \| \leq C_{\theta_0} \varepsilon R^{1+\frac{4}{d}-\frac{2}{q}} \| F \|_{L^p(\Omega(y_0,R))}.
\]

Then it follows by duality arguments that
\[
\left( \int_{\Omega(y_0,R)} \left| G_{\varepsilon, \lambda}(x_0, y) - G_{0,\lambda}(x_0, y) \right|^p \, dy \right)^{\frac{1}{p}} \leq C_{\theta_0} \varepsilon R^{1-\frac{2}{q}} \text{ for any } p > d. \tag{5.14}
\]
Finally, since \( (L_\varepsilon - \lambda I)(G_{\varepsilon,\lambda}^\gamma(x_0, \cdot)) = (L_0 - \lambda I)(G_{0,\lambda}^\gamma(x_0, \cdot)) = 0 \) in \( \Omega(y_0, R) \) for any \( 1 \leq \gamma \leq m \), we may invoke Lemma 5.4 again to conclude that for any \( k \in \mathbb{N}_+ \),

\[
|G_{\varepsilon,\lambda}(x_0, y_0) - G_{0,\lambda}(x_0, y_0)| \leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^k R^d} \int_{\Omega(y_0, R)} |G_{\varepsilon,\lambda}(x, y) - G_{0,\lambda}(x, y)| dy + C_{k,\theta_0} \varepsilon R^{1-\frac{d}{2}} \|\nabla^2 G_{0,\lambda}(x_0, \cdot)\|_{L^p(\Omega(y_0, R))} + C_{k,\theta_0} \varepsilon (1 + |\lambda| R^2) \|\nabla^2 G_{0,\lambda}(x_0, \cdot)\|_{L^\infty(\Omega(y_0, R))},
\]

(5.15)

Firstly for the first term of (5.15), in view of (5.14), we can obtain that it is bounded by

\[
C_{k,\theta_0} \varepsilon (1 + |\lambda| R^2)^{-k} R^{1-d}.
\]

(5.16)

For the third term of (5.15), using (4.60), it is also bounded by (5.16). For the second term, if \( d \geq 3 \), to obtain the same boundedness, we can use the \( W^{2,p} \) estimates for \( L_0 - \lambda I \), (4.8), that is,

\[
\left( \frac{1}{\Omega(y_0, R)} \int_{\Omega(y_0, R)} |\nabla^2 G_{0,\lambda}(x, y)|^p dy \right)^{\frac{1}{p}} \leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^k R^d} \left( \frac{1}{\Omega(y_0, 2R)} \int_{\Omega(y_0, 2R)} |G_{0,\lambda}(x, y)|^2 dy \right)^{\frac{1}{2}}.
\]

(5.17)

This, together with (4.18), completes the proof for the case that \( d \geq 3 \). If \( d = 2 \), we divide the proof into two cases. If \( \Delta(y_0, 3R) \neq \emptyset \), then it follows directly from (4.50) and (5.17). If \( \Delta(y_0, 3R) = \emptyset \), we can obtain the same result by using the interior Lipschitz estimate for the function \( \nabla^2 G_{0,\lambda}(x_0, y) \),

\[
|\nabla^2 G_{0,\lambda}(x_0, y)| \leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^k R} \left( \frac{1}{B(y_0, 2R)} \int_{B(y_0, 2R)} |\nabla^2 G_{0,\lambda}(x, y)|^2 dy \right)^{\frac{1}{2}}
\]

(5.18)

and (4.60).

\[ \square \]

**Theorem 5.5** (Convergence of Green functions II). For \( \varepsilon > 0 \) and \( d \geq 2 \), \( \lambda \in \Sigma_\theta_0 \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \), let \( \Omega \) be a bounded \( C^{2,1} \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). Then for any \( k \in \mathbb{N}_+ \), \( 1 \leq \alpha \leq m \) and \( x, y \in \Omega \) with \( x \neq y \),

\[
|\frac{\partial}{\partial x_i} \{ G_{\varepsilon,\lambda}^\alpha(x, y) \} - \frac{\partial}{\partial x_i} \{ G_{0,\lambda}^\alpha(x, y) \}| \leq \frac{C_{k,\theta_0} \varepsilon \ln(\varepsilon^{-1} |x - y| + 2)}{1 + |\lambda| |x - y|^2} \|x - y\|^2.
\]

(5.19)

where \( C_{k,\theta_0} \) depends only on \( \mu, d, m, \nu, \tau, k, \theta_0 \) and \( \Omega \).

**Lemma 5.6.** For \( \varepsilon > 0 \) and \( d \geq 2 \), \( \lambda \in \Sigma_\theta_0 \cup \{0\} \) with \( \theta_0 \in (0, \frac{\pi}{2}) \), let \( \Omega \) be a bounded \( C^{2,1} \) domain in \( \mathbb{R}^d \). Suppose that \( A \) satisfies (1.2), (1.3), (1.4) and (1.5). Assume that \( u_{\varepsilon,\lambda} \in H^1(\Omega(x_0, 4R); \mathbb{C}^m) \) and \( u_{0,\lambda} \in C^{2,p}(\Omega(x_0, 4R); \mathbb{C}^m) \) for some \( 0 < \rho < 1 \). If \( \Delta(x_0, 4R) \neq \emptyset \), assume that

\[
(L_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = (L_0 - \lambda I)(u_{0,\lambda}) \quad \text{in} \quad \Omega(x_0, 4R) \quad \text{and} \quad u_{\varepsilon,\lambda} = u_{0,\lambda} \quad \text{on} \quad \Delta(x_0, 4R).
\]

If \( \Delta(x_0, 4R) = \emptyset \), assume that

\[
(L_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = (L_0 - \lambda I)(u_{0,\lambda}) \quad \text{in} \quad B(x_0, 4R).
\]

Then for any \( 0 < \varepsilon < r \), \( 1 \leq \alpha \leq m \) and \( k \in \mathbb{N}_+ \),

\[
\| \frac{\partial u_{\varepsilon,\lambda}^\alpha}{\partial x_i} - \frac{\partial u_{0,\lambda}^\alpha}{\partial x_i} \|_{L^\infty(\Omega_R)} \leq \frac{C_{k,\theta_0}}{(1 + |\lambda| R^2)^k R} \int_{\Omega_R} |u_{\varepsilon,\lambda} - u_{0,\lambda}| dy + C_{k,\theta_0} \varepsilon \{ (1 + |\lambda| R^2) R^{-1} \|\nabla u_{0,\lambda}\|_{L^\infty(\Omega_R)} + R^d \|\nabla^2 u_{0,\lambda}\|_{C^{0,p}(\Omega_R)} \}.
\]

(5.20)

where \( C_{k,\theta_0} \) depends on \( \mu, d, m, k, \theta_0, \nu, \tau, p, \rho \) and \( \Omega \).
Proof. We only prove the case that \( \Delta(x_0, 3R) \neq \emptyset \) and the other is similar in view of the proof of Lemma 5.4. We start by choosing a \( C^{2,1} \) domain \( \overline{\Omega} \) such that \( \Omega(x_0, 3R) \subset \overline{\Omega} \subset \Omega(x_0, 4R) \). Let

\[
w_{\varepsilon, \lambda}(x) = u_{\varepsilon, \lambda}(x) - u_{0, \lambda}(x) - [\Phi_{\varepsilon, j}^\beta(x) - P_j^\beta(x)] \frac{\partial u_{0, \lambda}^\beta}{\partial x_j}.
\]

Simple calculations imply that

\begin{equation}
\begin{aligned}
\{(L_\varepsilon - \lambda I)(w_{\varepsilon, \lambda})\}^\alpha &= -\varepsilon \frac{\partial}{\partial x_i} \left\{ F_{\varepsilon, \gamma}^\alpha(x/\varepsilon) \frac{\partial^2 u_{0, \lambda}^\gamma}{\partial x_j \partial x_k} \right\} \\
&\quad + a_{ij}^\alpha(x/\varepsilon) \frac{\partial}{\partial x_j} \left[ \Phi_{\varepsilon, k}^\beta(x) - x_k \delta^\beta_\gamma - \varepsilon \chi^\beta_\lambda(x/\varepsilon) \right] \frac{\partial^2 u_{0, \lambda}^\gamma}{\partial x_j \partial x_k} \\
&\quad + \lambda \frac{\partial}{\partial x_i} \left\{ a_{ij}^\alpha(x/\varepsilon) \left[ \Phi_{\varepsilon, k}^\beta(x) - x_k \delta^\beta_\gamma \right] \frac{\partial^2 u_{0, \lambda}^\gamma}{\partial x_j \partial x_k} \right\} \\
&\quad + \lambda \chi^\alpha_\lambda(x) \frac{\partial}{\partial x_i} \left[ \Phi_{\varepsilon, k}^\beta(x) \right] \frac{\partial u_{0, \lambda}^\beta}{\partial x_k}.
\end{aligned}
\end{equation}

(5.21)

Note that \( w_{\varepsilon, \lambda} = 0 \) on \( \Delta(x_0, 4R) \). Write \( w_{\varepsilon, \lambda} = v_{\varepsilon, \lambda} + z_{\varepsilon, \lambda} \) in \( \overline{\Omega} \), where \( v_{\varepsilon, \lambda} \in H^1_0(\overline{\Omega}; \mathbb{C}^m) \) and \( (L_\varepsilon - \lambda I)(z_{\varepsilon, \lambda}) = 0 \) in \( \overline{\Omega} \). Since \( (L_\varepsilon - \lambda I)(z_{\varepsilon, \lambda}) = 0 \) in \( \overline{\Omega} \) and \( z_{\varepsilon, \lambda} = w_{\varepsilon, \lambda} = 0 \) on \( \Delta(x_0, 3R) \), it follows from the boundary Lipschitz estimate (3.6) and (3.8) that for any \( k \in \mathbb{N}_+ \),

\[
\| \nabla z_{\varepsilon, \lambda} \|_{L^\infty(\Omega_R)} \leq \frac{C_{\theta_0}}{R} \left( \int_{\Omega_{3/2R}} |z_{\varepsilon, \lambda}|^2 \right)^{\frac{1}{2}} \leq \frac{C_{\theta_0}}{R} \| z_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{3/2R})}
\]

\[
\begin{aligned}
&\leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^2 R} \int_{\Omega_{2R}} |z_{\varepsilon, \lambda}| \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^2 R} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda}| + C_{k, \theta_0} \| v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} \\
&\leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^2 R} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| + C_{k, \theta_0} \{ \| v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} + \varepsilon \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{2R})} \},
\end{aligned}
\]

where we have used (1.15). This implies that

\[
\| \nabla w_{\varepsilon, \lambda} \|_{L^\infty(\Omega_R)} \leq \| \nabla v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_R)} + \| \nabla z_{\varepsilon, \lambda} \|_{L^\infty(\Omega_R)}
\]

\[
\begin{aligned}
&\leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^2 R} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| + C_{k, \theta_0} \left\{ R \| \nabla v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} + \varepsilon \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{2R})} \right\},
\end{aligned}
\]

where we have used \( \| v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} \leq CR \| \nabla v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} \) since \( v_{\varepsilon, \lambda} = 0 \) on \( \partial \overline{\Omega} \). Owing to

\[
\frac{\partial u_{\varepsilon, \lambda}^\alpha}{\partial x_i} = \frac{\partial u_{0, \lambda}^\beta}{\partial x_i} + \frac{\partial \Phi_{\varepsilon, j}^\beta}{\partial x_i} \frac{\partial u_{0, \lambda}^\beta}{\partial x_j} - [\Phi_{\varepsilon, j}^\beta(x) - x_j \delta^\beta_\gamma] \frac{\partial^2 u_{0, \lambda}^\gamma}{\partial x_i \partial x_j},
\]

we can obtain that for any \( 1 \leq \alpha \leq m \) and \( k \in \mathbb{N}_+ \),

\[
\| \frac{\partial u_{\varepsilon, \lambda}^\alpha}{\partial x_i} - \frac{\partial \Phi_{\varepsilon, j}^\beta}{\partial x_i} \frac{\partial u_{0, \lambda}^\beta}{\partial x_j} \|_{L^\infty(\Omega_R)} \leq \| \nabla w_{\varepsilon, \lambda} \|_{L^\infty(\Omega_R)} + \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_R)}
\]

\[
\begin{aligned}
&\leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^2 R} \int_{\Omega_{2R}} |u_{\varepsilon, \lambda} - u_{0, \lambda}| \\
&\quad + C_{k, \theta_0} \left\{ R^{-1} \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{2R})} + \| \nabla v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{2R})} + \varepsilon \| \nabla^2 u_{0, \lambda} \|_{L^\infty(\Omega_{2R})} \right\}.
\end{aligned}
\]

(5.22)

It remains to estimate \( \nabla v_{\varepsilon, \lambda} \) in \( \Omega_{2R} \). To this end, we use the representation formula

\[
v_{\varepsilon, \lambda}(x) = \int_{\Omega} \bar{G}_{\varepsilon, \lambda}(x, y)(L_\varepsilon - \lambda I)(w_{\varepsilon, \lambda})(y) dy,
\]
where $G_{ε,λ}(x, y)$ is the Green function for $L_ε - λI$ in the $C^{2,1}$ domain $\tilde{Ω}$. Let
\[
f_ε^α(x) = -εF_{jik}^\alpha(x/ε) \frac{∂^2 u_{0,λ}^j}{∂x_j∂x_k} + a_{ij}^\alpha(x/ε) [Φ_ε^{βγ}(x) - x_κδ^{βγ}] \frac{∂^2 u_{0,λ}^j}{∂x_j∂x_k}.
\]

In view of (5.21), we can obtain
\[
v_{ε,λ}(x) = - \int_{Ω} \frac{∂}{∂y_i} \{ G_{ε,λ}(x, y) \} [f_i(y) - f_i(x)] dy
\]
\[
+ \int_{Ω} G_{ε,λ}(x, y) a_{ij}(y/ε) \frac{∂}{∂y_i} [Φ_{ε,k}(y) - P_k(y) - εχ_k(y/ε)] \frac{∂^2 u_{0,λ}}{∂y_i∂y_k} dy
\]
\[
+ Χ \int_{Ω} G_{ε,λ}(x, y) [Φ_{ε,k}(y) - P_k(y)] \frac{∂u_{0,λ}}{∂y_k} dy.
\]

It follows that
\[
|∇v_{ε,λ}(x)| ≤ \int_{Ω} |∇_1 ∇_2 G_{ε,λ}(x, y)||f(y) - f(x)| dy
\]
\[
+ C||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} \int_{Ω} |∇_1 G_{ε,λ}(x, y)||∇[Φ_ε(y) - P(y) - εχ(y/ε)]| dy
\]
\[
+ C|λ||∇u_{0,λ}||_{L^∞(Ω_ε)} \int_{Ω} |G_{ε,λ}(x, y)||Φ_ε(y) - P(y)| dy.
\]

In view of the Lipschitz estimates of Green functions, (4.63), we have
\[
|∇_1 ∇_2 G_{ε,λ}(x, y)| \leq \frac{C_{k,θ_0}}{(1 + |λ||x - y|^2)|x - y|^d} \text{ for any } k \in N_+.
\]

Meanwhile, simple calculations and (1.15) give the $L^∞$ and Hölder estimates of $f$, that is
\[
||f||_{L^∞(Ω_ε)} \leq Cε||∇^2 u_{0,λ}||_{L^∞(Ω_ε)},
\]
\[
|f(x) - f(y)| ≤ C|x - y|^p \{ ε^{1-p}||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} + ε||∇^2 u_{0,λ}||_{C^{0,ρ}(Ω_ε)} \}.
\]

Choosing $k = 0$ in (5.24), it can be obtained that
\[
\int_{Ω} |∇_1 ∇_2 G_{ε,λ}(x, y)||f(y) - f(x)| dy
\]
\[
≤ C_{θ_0} ε||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} \int_{Ω \setminus B(x, ε)} \frac{dy}{|x - y|^d}
\]
\[
+ C_{θ_0} \{ ε^{1-p}||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} + ε||∇^2 u_{0,λ}||_{C^{0,ρ}(Ω_ε)} \} \int_{Ω \cap B(x, ε)} \frac{dy}{|x - y|^{d-ρ}}
\]
\[
≤ C_{θ_0} \{ ε ln[ε^{-1} R + 2]||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} + Cε^{1+ρ}||∇^2 u_{0,λ}||_{C^{0,ρ}(Ω_ε)} \}.
\]

Finally, using the estimates
\[
|∇_1 G_{ε,λ}(x, y)| \leq \frac{C_{k,θ_0}}{(1 + |λ||x - y|^2)|x - y|^{d-1}} \min \left\{ 1, \frac{\text{dist}(y, ∂Ω)}{|x - y|} \right\} \text{ for any } k \in N_+,
\]
as well as the observation that for any $1 ≤ j ≤ d$ and $1 ≤ β ≤ m$,
\[
|∇ \{ Φ_ε^{βγ}(x) - P_j^β(x) - εχ_j(x/ε) \}| ≤ C \min \left\{ 1, ε \text{dist}(x, ∂Ω)^{-1} \right\},
\]
we can bound the second term in the right hand side of (5.23) by
\[
C_{θ_0} ||∇^2 u_{0,λ}||_{L^∞(Ω_ε)} \left\{ ε \int_{Ω \setminus B(x, ε)} \frac{dy}{|x - y|^d} + \int_{Ω \cap B(x, ε)} \frac{dy}{|x - y|^{d-1}} \right\}
\]
\[
≤ C_{θ_0} ε ln[ε^{-1} R + 2]||∇^2 u_{0,λ}||_{L^∞(Ω_ε)}.
\]
For the third term of the right hand side of (5.23), by rescaling and (4.80), we can obtain that this term is bounded by

\[ C_{\theta_0} \varepsilon (1 + |\lambda| R^2) R^{-1} \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{4R})}. \]

As a result, we have proved that

\[
\| \nabla v_{\varepsilon, \lambda} \|_{L^\infty(\Omega_{3R})} \leq C_{\theta_0} \varepsilon (1 + |\lambda| R^2) R^{-1} \| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{3R})} + C_{\theta_0} \varepsilon |\varepsilon^{-1} R + 2 \| \nabla^2 u_{0, \lambda} \|_{L^\infty(\Omega_{3R})} + \varepsilon R^\rho \| \nabla^2 u_{0, \lambda} \|_{C^{0, \rho}(\Omega_{4R})}.\]

This, together with (5.22), completes the proof. □

**Proof of Theorem 5.5.** Fix \( x_0, y_0 \in \Omega \) and \( R = \frac{|x_0 - y_0|}{16} \). We may assume that \( 0 < \varepsilon < r \), since the case \( \varepsilon \geq r \) is trivial and follows directly from the size estimates of \( |v_1 G_{\varepsilon, \lambda}(x, y)|, |v_2 G_{\varepsilon, \lambda}(x, y)| \) (see (4.60)) and (1.15). For any \( 1 \leq \gamma \leq m \), let \( u_{\varepsilon, \lambda}(x) = G^\gamma_{\varepsilon, \lambda}(x, y_0) \) and \( u_{0, \lambda}(x) = G^\gamma_{0, \lambda}(x, y_0) \). Observe that \((L_{\varepsilon} - \lambda I)(u_{\varepsilon, \lambda}) = (L_0 - \lambda I)(u_{0, \lambda}) = 0 \) in \( \Omega(x_0, 4R) \) and \( u_{\varepsilon, \lambda} = u_{0, \lambda} = 0 \) on \( \Delta(x_0, 4R) \) (if \( \Delta(x_0, 4R) \neq \emptyset \)). By Theorem 5.1, it can be obtained,

\[
\| u_{\varepsilon, \lambda} - u_{0, \lambda} \|_{L^\infty(\Omega(x_0, 4R))} \leq C_{\theta_0} \varepsilon R^{1 - d}.
\]

Also, since \( \Omega \) is \( C^{2,1} \), we have, for any \( k \in \mathbb{N}_+ \),

\[
\| \nabla u_{0, \lambda} \|_{L^\infty(\Omega_{4R})} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k R^d}, \quad \| \nabla^2 u_{0, \lambda} \|_{L^\infty(\Omega_{4R})} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k R^{d + \rho}},
\]

and \( \| \nabla^2 u_{0, \lambda} \|_{C^{0, \rho}(\Omega_{3R})} \leq \frac{C_{k, \theta_0}}{(1 + |\lambda| R^2)^k R^{d + \rho}} \).

Here, we have used (4.9), (4.10), (4.60) and arguments in the proof of Theorem 5.1. Hence, in view of (5.20), we can complete the proof. □

**5.2. Proof of Theorem 1.1, Theorem 1.3 and 1.4.**

**Proof of Theorem 1.1.** For \( u_{\varepsilon, \lambda} = R(\lambda, L_\varepsilon)F \) and \( u_{0, \lambda} = R(\lambda, L_0)F \), let

\[
w_{\varepsilon, \lambda}(x) = u_{\varepsilon, \lambda}(x) - u_{0, \lambda}(x) - \left[ \Phi^\beta_{\varepsilon, j}(x) - P^\beta_j(x) \right] \frac{\partial u_{0, \lambda}}{\partial x_j}.
\]

Using the equality (5.21) and choosing \( w_{\varepsilon, \lambda} \) as the test function, we can obtain that

\[
B_{\varepsilon, \lambda, \Omega}[w_{\varepsilon, \lambda}, w_{\varepsilon, \lambda}] = \int_{\Omega} A(x/\varepsilon) \nabla w_{\varepsilon, \lambda} \nabla w_{\varepsilon, \lambda} dx - \lambda \int_{\Omega} |w_{\varepsilon, \lambda}|^2 dx = J_{\varepsilon, \lambda, \Omega}[u_{0, \lambda}, w_{\varepsilon, \lambda}], \quad \varepsilon > 0, \tag{5.25}
\]

where the bilinear form \( J_{\varepsilon, \lambda, \Omega}[\cdot, \cdot] : H^1_0(\Omega; \mathbb{C}^m) \times H^1_0(\Omega; \mathbb{C}^m) \to \mathbb{C} \) is defined by

\[
J_{\varepsilon, \lambda, \Omega}[u, v] = -\int_{\Omega} \varepsilon F^\gamma_{ijk}(x/\varepsilon) \frac{\partial^2 u^\gamma}{\partial x_j \partial x_k} \frac{\partial^2 v^\gamma}{\partial x_i \partial x_k} dx + \lambda \int_{\Omega} \left[ \Phi^\beta_{\varepsilon, k}(x) - x_k \delta^\beta \gamma \right] \frac{\partial^2 u^\beta}{\partial x_k \partial x_i} \frac{\partial^2 v^\gamma}{\partial x_i \partial x_k} dx

- \int_{\Omega} a^\alpha_{ij}(x/\varepsilon) \frac{\partial^2 u^\alpha}{\partial x_j \partial x_i} \frac{\partial^2 v^\beta}{\partial x_k \partial x_k} dx

+ \int_{\Omega} a^\beta_{ij}(x/\varepsilon) \frac{\partial}{\partial x_k} \left[ \Phi^\gamma_{\varepsilon, k}(x) - x_k \delta^\beta \gamma \right] \frac{\partial^2 u^\gamma}{\partial x_k \partial x_i} \frac{\partial^2 v^\beta}{\partial x_i \partial x_k} dx. \tag{5.26}
\]
Taking \( u = u_{0,\lambda} \) and \( v = v_{\epsilon,\lambda} \) and using (1.15), it can be inferred that
\[
|J_{\epsilon,\lambda,\Omega}|w_{\epsilon,\lambda}|w_{\epsilon,\lambda}| \leq C \varepsilon \int_{\Omega} |\nabla^2 u_{0,\lambda}| |\nabla w_{\epsilon,\lambda}| dx + C \varepsilon |\lambda| \int_{\Omega} |\nabla u_{0,\lambda}| |w_{\epsilon,\lambda}| dx \\
+ C \int_{\Omega} |\nabla (\Phi_{\epsilon}(x) - P(x) - \varepsilon \chi(\varepsilon))| |\nabla^2 u_{0,\lambda}| |w_{\epsilon,\lambda}| dx \\
\leq C \varepsilon |\lambda||\nabla u_{0,\lambda}| |L^2(\Omega)| |w_{\epsilon,\lambda}| |L^2(\Omega)| + C \varepsilon |\nabla^2 u_{0,\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)| \\
+ C |\nabla (\Phi_{\epsilon}(x) - P(x) - \varepsilon \chi(\varepsilon))| |w_{\epsilon,\lambda}| |L^2(\Omega)| |\nabla^2 u_{0,\lambda}| |L^2(\Omega)|.
\]

To estimate \( |J_{\epsilon,\lambda,\Omega}|u_{0,\lambda}, w_{\epsilon,\lambda}|w_{\epsilon,\lambda}| \), we first claim that for any \( 1 \leq j \leq d \) and \( 1 \leq \beta \leq m, \)
\[
|\nabla (\Phi_{\epsilon,\lambda}(\cdot) - P(\cdot) - \varepsilon \chi(\varepsilon))| |w_{\epsilon,\lambda}| |L^2(\Omega)| \leq C \varepsilon |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)|. \tag{5.27}
\]

To see (5.27), we fix \( 1 \leq \beta_0 \leq m, 1 \leq j_0 \leq d \) and let
\[
h_{x}(x) = \Phi_{\epsilon,j_0}(x) - P_{\beta_0}(x) - \varepsilon \chi(\varepsilon), \text{ where } x \in \Omega.
\]
Note that \( h_{x} \in H^{1}(\Omega; \mathbb{C}^{m}) \cap L^{\infty}(\Omega; \mathbb{C}^{m}) \) and \( L_{\beta_0}(h_{x}) = 0 \) in \( \Omega \). It follows that
\[
\mu \int_{\Omega} |\nabla h_{x}|^2 |w_{\epsilon,\lambda}|^2 dx \leq \int_{\Omega} A^{\beta_0}_{ij}(x/\varepsilon) \frac{\partial h_{x}^{\beta_0}}{\partial x_j} \frac{\partial h_{x}^{\beta_0}}{\partial x_i} |w_{\epsilon,\lambda}|^2 dx \\
= - \int_{\Omega} \frac{\partial A^{\beta_0}_{ij}(x/\varepsilon)}{\partial x_j} \frac{\partial h_{x}^{\beta_0}}{\partial x_i} |w_{\epsilon,\lambda}|^2 dx - \int_{\Omega} \frac{\partial A^{\beta_0}_{ij}(x/\varepsilon)}{\partial x_i} \frac{\partial h_{x}^{\beta_0}}{\partial x_j} |w_{\epsilon,\lambda}|^2 dx,
\]
where we have used integration by parts. Hence
\[
\int_{\Omega} |\nabla h_{x}|^2 |w_{\epsilon,\lambda}|^2 dx \leq C \int_{\Omega} |h_{x}| |\nabla h_{x}| |w_{\epsilon,\lambda}| |w_{\epsilon,\lambda}| dx,
\]
where \( C \) depends on \( d, m \) and \( \mu \). This directly implies the claim by noticing that \( ||h_{x}||_{L^{\infty}(\Omega)} \leq C \varepsilon \) and using the inequality
\[
\int_{\Omega} |\nabla h_{x}|^2 |w_{\epsilon,\lambda}|^2 dx \leq \frac{1}{2} \int_{\Omega} |\nabla h_{x}|^2 |w_{\epsilon,\lambda}|^2 dx + C \int_{\Omega} |h_{x}|^2 |\nabla w_{\epsilon,\lambda}|^2 dx,
\]
where we have used the (2.26). Then the claim implies that
\[
|J_{\epsilon,\lambda,\Omega}|u_{0,\lambda} |w_{\epsilon,\lambda}| \leq C \varepsilon |\lambda||\nabla u_{0,\lambda}| |L^2(\Omega)| |w_{\epsilon,\lambda}| |L^2(\Omega)| + C \varepsilon |\nabla^2 u_{0,\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)|. \tag{5.28}
\]

In view of (2.9), (2.26) and (2.25), it can be easily shown that
\[
|w_{\epsilon,\lambda}|_{L^2(\Omega)} \leq \frac{C \varepsilon |\lambda|}{R^2 + |\lambda|} |B_{\epsilon,\lambda,\Omega}| |w_{\epsilon,\lambda}| \leq \frac{C \varepsilon |\lambda|}{R^2 + |\lambda|} |J_{\epsilon,\lambda,\Omega}|u_{0,\lambda} |w_{\epsilon,\lambda}|
\leq \frac{C \varepsilon |\lambda|}{R^2 + |\lambda|} \left\{ |\lambda||\nabla u_{\lambda}| |L^2(\Omega)| |w_{\epsilon,\lambda}| |L^2(\Omega)| + ||\nabla^2 u_{\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)| \right\}
\leq C \varepsilon^2 c^2(\lambda, \theta)|\nabla u_{0,\lambda}| |L^2(\Omega)| + \frac{1}{2} |w_{\epsilon,\lambda}| |L^2(\Omega)| + C \varepsilon |\lambda| |\nabla^2 u_{0,\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)|.
\]

Then it is not hard to obtain the \( L^2 \) estimate of \( w_{\epsilon,\lambda} \), that is,
\[
|w_{\epsilon,\lambda}| |L^2(\Omega)| \leq C \varepsilon^2 c^2(\lambda, \theta)|\nabla u_{0,\lambda}| |L^2(\Omega)| + C \varepsilon \frac{c(\lambda, \theta)}{R^2 + |\lambda|} |\nabla^2 u_{0,\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)|. \tag{5.29}
\]

Similarly, owing to (2.8), it can be obtained without difficulty that
\[
|\nabla w_{\epsilon,\lambda}| |L^2(\Omega)| \leq C \varepsilon |\lambda| |B_{\epsilon,\lambda,\Omega}| |w_{\epsilon,\lambda}| \leq C \varepsilon |\lambda| |J_{\epsilon,\lambda,\Omega}|u_{0,\lambda} |w_{\epsilon,\lambda}|
\leq C \varepsilon |\lambda| \left\{ |\lambda||\nabla u_{\lambda}| |L^2(\Omega)| |w_{\epsilon,\lambda}| |L^2(\Omega)| + ||\nabla^2 u_{\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)| \right\}
\leq C \varepsilon^2 c^2(\lambda, \theta)|\nabla u_{0,\lambda}| |L^2(\Omega)| + |\lambda||w_{\epsilon,\lambda}| |L^2(\Omega)| + C \varepsilon |\lambda| |\nabla^2 u_{0,\lambda}| |L^2(\Omega)| |\nabla w_{\epsilon,\lambda}| |L^2(\Omega)|.
This, together with (2.26) and (5.29), gives that
\[ \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)} \leq C\varepsilon^2 c^2(\lambda, \theta) |\lambda| \|\nabla u_{0,\lambda}\|_{L^2(\Omega)}^2 + C\varepsilon c(\lambda, \theta) \|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega)} \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)} \]
\[ \leq C\varepsilon^2 c^2(\lambda, \theta) |\lambda| \|\nabla u_{0,\lambda}\|_{L^2(\Omega)}^2 + C\varepsilon^2 c^2(\lambda, \theta) \|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega)} \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)} + \frac{1}{2} \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)}^2. \]
In view of (2.10) and (2.11), we can estimate \( \|\nabla u_{0,\lambda}\|_{L^2(\Omega)} \) and \( \|\nabla^2 u_{0,\lambda}\|_{L^2(\Omega)} \). Then
\[ \|\nabla w_{\varepsilon,\lambda}\|_{L^2(\Omega)} \leq C\varepsilon c^2(\lambda, \theta) \|F\|_{L^2(\Omega)}, \]
which completes the proof of (1.17). This, together with (5.29), shows (1.16). \( \square \)

**Proof of Theorem 1.3.** In view of the representation formula (4.16) and (5.1), it can be seen that
\[ \|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0)\|_{L^\infty(\Omega) \to L^\infty(\Omega)} \leq C_{\theta_0} \varepsilon (R_0^{-2} + |\lambda|)^{-\frac{2}{p}}, \]
\[ \|R(\lambda, \mathcal{L}_\varepsilon) - R(\lambda, \mathcal{L}_0)\|_{L^1(\Omega) \to L^1(\Omega)} \leq C_{\theta_0} \varepsilon (R_0^{-2} + |\lambda|)^{-\frac{2}{p}}. \]
These, together with (1.18) and the M. Riesz interpolation theorem, give (1.31). \( \square \)

**Proof of Theorem 1.4.** It follows directly from (5.19) and the arguments in Theorem 6.5.2 in [21]. We give the proof here for the sake of completeness. Let \( u_{\varepsilon,\lambda}, u_{0,\lambda} \in H^1_\varepsilon(\Omega; \mathbb{C}^m) \) such that \( (\mathcal{L}_\varepsilon - \lambda I)(u_{\varepsilon,\lambda}) = F \) and \( (\mathcal{L}_0 - \lambda I)(u_{0,\lambda}) = F \) with \( F \in L^p(\Omega; \mathbb{C}^m) \). By using (1.15) and (4.6), we only need to show that for any \( 1 \leq p \leq \infty \) and \( 1 \leq \alpha \leq m, \)
\[ \left\| \frac{\partial u_{\varepsilon,\lambda}^{\alpha}}{\partial x_i} - \frac{\partial \Phi_{\varepsilon,j}^{\alpha}}{\partial x_i} \right\|_{L^p(\Omega)} \leq C_{\theta_0} \varepsilon \{\ln[\varepsilon^{-1} R_0 + 2]\}^{4 \frac{1}{2} - \frac{2}{p}} \|F\|_{L^p(\Omega)}. \]
In view of Theorem 5.5, we can obtain that
\[ \left\| \frac{\partial u_{\varepsilon,\lambda}^{\alpha}}{\partial x_i} - \frac{\partial \Phi_{\varepsilon,j}^{\alpha}}{\partial x_i} \right\| \leq C_{\theta_0} \int_\Omega K_\varepsilon(x, y) |F(y)| dy, \]
where the kernel \( K_\varepsilon \) is defined by
\[ K_\varepsilon(x, y) = \begin{cases} \varepsilon |x - y|^{-d} \ln[\varepsilon^{-1}|x - y| + 2] & \text{if } |x - y| \geq \varepsilon, \\ |x - y|^{-d} & \text{if } |x - y| < \varepsilon. \end{cases} \]
Direct computations imply that
\[ \sup_{x \in \Omega} \int_\Omega K_\varepsilon(x, y) dy + \sup_{y \in \Omega} \int_\Omega K_\varepsilon(x, y) dy \leq C_{\theta_0} \{\ln[\varepsilon^{-1} R_0 + 2]\}^2. \quad (5.30) \]
This gives (1.32) for cases \( p = 1 \) and \( \infty \). Thus, by the M. Riesz interpolation theorem, the result is a direct consequence of the case \( p = 2 \), which is given by (1.17). \( \square \)

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