Global smooth solutions of the 3D Hall-magnetohydrodynamic equations with large data

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Abstract: In this paper, we establish the global existence to the three-dimensional incompressible Hall-MHD equations for a class of large initial data, whose \(L^\infty\) norms can be arbitrarily large. In addition, we give an example to show that such a large initial value does exist. Our idea is splitting the generalized heat equations from Hall-MHD system to generate a small quantity for large time \(t\).

Keywords: Hall-MHD; Global existence; Large initial data.

MSC (2010): 35Q35; 76D03; 86A10

1 Introduction

This paper studies the Cauchy problem for the following 3D incompressible, resistive Hall-MHD equations

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \mu (\Lambda)^\alpha u + \nabla p &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b - \nu \Delta b + \nabla \times ((\nabla \times b) \times b) &= b \cdot \nabla u, \\
\text{div} u &= \text{div} b = 0, \\
(u, b)|_{t=0} &= (u_0, b_0),
\end{aligned}
\]

where \(u = (u_1(t, x), u_2(t, x), u_3(t, x))\) and \(b = (b_1(t, x), b_2(t, x), b_3(t, x))\) denote the velocity field and magnetic field, respectively, \(p \in \mathbb{R}\) is the scalar pressure. \(\mu\) is the viscosity and \(\nu\) is the magnetic diffusivity. \(\Lambda = (-\Delta)^{\frac{\gamma}{2}}\) is the Zygmund operator and the fractional power operator \(\Lambda^\gamma\) with \(0 < \gamma < 1\) is defined by Fourier multiplier with symbol \(|\xi|^{\gamma}\) (see e.g. \([10]\)), namely,

\[
\Lambda^\gamma u(x) = \mathcal{F}^{-1}[|\xi|^{\gamma}\mathcal{F}u(\xi)].
\]

Comparing with the standard MHD system, the Hall term \(\nabla \times ((\nabla \times b) \times b)\) is included due to Ohm’s law, which is believed to be a key issue for understanding magnetic reconnection in the case of large magnetic shear and describes many physical phenomena such as magnetic reconnection
in space plasmas [8], star formation [2, 23], neutron stars [21] and also geo-dynamo [18]. For the physical background of the magnetic reconnection and the Hall-MHD, we refer the readers to [8, 11, 22] and references therein. When \( \alpha = 1 \) and the Hall term is neglected, (1.1) reduces to the standard MHD equation which has been extensively studied, and there are a lot of excellent work, see [13, 16, 17, 20] and references therein.

For the standard Hall-MHD equations, Acheritogaray et al. [1] obtained the global weak solutions in the periodic setting by using the Galerkin approximation. Chae, Degond and Liu [3] proved the global existence of weak solutions as well as the local well-posedness of smooth solutions in the whole space. Meanwhile, they showed that if \( \|u_0\|_{H^m} + \|b_0\|_{H^m}(m > \frac{5}{3}) \) is small enough, the local smooth solution is global in time. Chae and Lee [4] improved their results under weaker smallness assumptions on the initial data. Laterly, the conditions on the initial data were once more refined by Wan and Zhou [28]. Recently, Wan, Zhou [29] removed the restriction on \( \epsilon \) in [28], established the global existence of strong solution for standard Hall-MHD equations with the Fujita-Kato type initial data. Global small solutions for generalized Hall-MHD equations, we refer the readers to [31, 19, 27]. The mathematical studies on (1.1) have motivated a relatively large number of research papers concerning the local well-posedness [6, 26], regularity criterion [9, 7, 30], asymptotic behavior [5, 24, 25, 27] and we can refer the readers to the reference therein.

However, there are few results of global existence for Hall-MHD equations with general initial data without smallness conditions. It is also worth to mention that when \( \alpha = 1 \), \( b = 0 \), the system (1.1) is reduced to the Navier-Stokes equations. Lei, Lin and Zhou [12] constructed a family of finite energy smooth large solutions to the Navier-Stokes equations with the initial data close to a Beltrami flow. Li, Yang and Yu [14] established a class of global large solution to the 2D MHD equations with damping terms whose initial energy can be arbitrarily large. But the Hall term heightens the level of nonlinearity of the standard MHD system from a second-order semilinear to a second-order quasilinear level, significantly making its qualitative analysis more difficult. Until very recently, Zhang[32] obtained global large smooth solutions in the sense that the initial data can be arbitrarily large in \( H^3(\mathbb{R}^3) \). Li et al. [15] constructed a class of large solution with spectrally supported \( u_0 \) and \( b_0 = -\nabla \times u_0 \). Different from the constructed large initial data in [32, 15], for some class of large initial data whose \( L^\infty \) norms can be arbitrarily large, by splitting the generalized heat equations from system (1.1) to generate a small quantity, the solution of the system (1.1) with \( \mu, \nu > 0 \) evolve into a global solution.

Our main result is stated as follows.

**Theorem 1.1** Let \( 0 \leq \alpha \leq 1 \) and \( U_0 \) be a smooth function satisfying \( \text{div} U_0 = 0 \), \( \nabla \times U_0 = \Lambda U_0 \) and

\[
\text{supp} \hat{U}_0(\xi) \subset \mathcal{C} \triangleq \left\{ \xi \in \mathbb{R}^3 \mid 1 - \varepsilon \leq |\xi| \leq 1 + \varepsilon \right\}, \quad 0 < \varepsilon < \frac{2 - \sqrt{2}}{2}. \tag{1.2}
\]

Assume that the initial data fulfills \( u_0 = v_0 + U_0 \) and \( b_0 = c_0 + U_0 \), then there exists a sufficiently small positive constant \( \delta \), and a universal constant \( C \) such that if

\[
\left( ||v_0||_{H^3}^2 + ||c_0||_{H^3}^2 + \varepsilon ||U_0||_{L^2}^2 (1 + ||\hat{U}_0||_{L^1}) \right) \left( C(||1 + ||\hat{U}_0||_{L^1}||\hat{U}_0||_{L^1} + \varepsilon ||U_0||_{L^2}^2) \right) \leq \delta,
\]

then the system (1.1) has a unique global solution.

**Remark 1.1** Assume that \( \hat{a}_0 : \mathbb{R}^3 \to [0, 1] \) be a radial, non-negative, smooth function which is supported in \( \mathcal{C} \) and \( \hat{a}_0 \equiv 1 \) for \( 1 - \frac{1}{2} \varepsilon \leq |\xi| \leq 1 + \frac{1}{2} \varepsilon \).
Notice that
\[ a_0(x) = \int_{\mathbb{R}^3} \cos(x \cdot \xi) \hat{a}_0(\xi) d\xi \quad \text{and} \quad \Lambda^{-1} a_0(x) = \int_{\mathbb{R}^3} \cos(x \cdot \xi) \frac{\hat{a}_0(\xi)}{|\xi|} d\xi, \]
then we have \( a_0, \Lambda^{-1} a_0 \in \mathbb{R} \) and also let \( U_0 = V_0 + \Lambda^{-1} \nabla \times V_0 \) with
\[ V_0 = \varepsilon^{-1} \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}} \nabla \times \begin{pmatrix} a_0 \\ 0 \\ 0 \end{pmatrix} = \varepsilon^{-1} \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}} \begin{pmatrix} 0 \\ \partial_3 a_0 \\ -\partial_2 a_0 \end{pmatrix}. \]
Here, we can verify that \( \text{div} U_0 = 0 \) and \( \nabla \times U_0 = \Lambda U_0. \)
Moreover, we also have
\[ \hat{U}_0 = \varepsilon^{-1} \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}} \hat{\begin{pmatrix} c_2^2 + \xi_3^2 \\ -\xi_1 \xi_2 + i \xi_3 |\xi| \\ -\xi_1 \xi_3 - i \xi_2 |\xi| \end{pmatrix}} \frac{\hat{a}_0(\xi)}{|\xi|}. \]
Then, direct calculations show that
\[ ||\hat{U}_0||_{L^1} \approx \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}} \quad \text{and} \quad ||U_0||_{L^2} \approx \varepsilon^{-\frac{3}{2}} \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}}. \]
Thus, the left side of (1.3) becomes
\[ C \varepsilon^{\frac{3}{2}} \log \log \frac{1}{\varepsilon} \exp \left( C \log \log \frac{1}{\varepsilon} \right). \]
Therefore, choosing \( \varepsilon \) small enough, we deduce that the system (1.1) has a global solution. Notice that \( U_0^1 = -\varepsilon^{-1} \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}} (\partial_2^2 + \partial_3^2) \Lambda^{-1} a_0 \) and \( \hat{U}_0^1 \geq 0 \), we can deduce that
\[ ||U_0^1||_{L^\infty} \approx ||\hat{U}_0^1||_{L^1} \gtrsim \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}}. \]
Let \( v_0 = c_0 = 0 \), then we have \( ||u_0||_{L^\infty} = ||b_0||_{L^\infty} \gtrsim \left( \log \log \frac{1}{\varepsilon} \right)^{\frac{3}{2}}. \)

Notations: Let \( \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3 \) be a multi-index and \( D^\beta = \partial^{||\beta||} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3} \) with \( ||\beta|| = \beta_1 + \beta_2 + \beta_3 \). For the sake of simplicity, \( a \lesssim b \) means that \( a \leq C b \) for some "harmless" positive constant \( C \) which may vary from line to line. \([A, B]\) stands for the commutator operator \( AB - BA \), where \( A \) and \( B \) are any pair of operators on some Banach space \( X \). We also use the notation \( ||f_1, \cdots, f_n||_X \triangleq ||f_1||_X + \cdots + ||f_n||_X \).

2 Reformulation of the System
Setting \( U = e^{-\mu t} U_0 \) and \( B = e^{-\nu t} U_0 \), we know that \((U, B)\) solve the following system
\[
\begin{aligned}
\partial_t U + \mu(-\Delta)^\alpha U &= \mu((-\Delta)^\alpha - I) U := F, \\
\partial_t B - \nu \Delta B &= \nu(-\Delta - I) B := G, \\
\text{div} U &= \text{div} B = 0, \\
(U, B)|_{t=0} &= (U_0, U_0).
\end{aligned}
\] (2.1)
Noticing the fact that $B \cdot \nabla U - U \cdot \nabla B = 0$ and denoting $v = u - U$ and $c = b - B$, we can reformulate the system (1.1) and (2.1) equivalently as

$$
\begin{aligned}
\partial_t v + v \cdot \nabla v - c \cdot \nabla c - \nabla \left( p + \frac{|U|^2 - |B|^2}{2} \right) &= f + f_1 - F, \\
\partial_t c + v \cdot \nabla c - c \cdot \nabla v - \Delta c + \nabla \times ((\nabla \times c) \cdot c) &= g + g_1 + g_2 - G,
\end{aligned}
$$

(2.2)

where

$$
\begin{aligned}
f &= B \cdot \nabla B - U \cdot \nabla U - \nabla \left( \frac{|B|^2 - |U|^2}{2} \right) = (\Lambda B - B) \times B - (\Lambda U - U) \times U, \\
f_1 &= B \cdot \nabla c + c \cdot \nabla B - U \cdot \nabla v - v \cdot \nabla U, \\
g &= -\nabla \times ((\nabla \times B) \times B) = -\nabla \times ((\Lambda B - B) \times B), \\
g_1 &= B \cdot \nabla v + c \cdot \nabla U - U \cdot \nabla c - v \cdot \nabla B, \\
g_2 &= -\nabla \times ((\nabla \times c) \times B) - \nabla \times ((\nabla \times B) \times c).
\end{aligned}
$$

3 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Before proceeding on, we present some estimates which will be used in the proof of Theorem 1.1.

Lemma 3.1 For small enough $\varepsilon$, under the assumptions of Theorem 1.1, the following estimates hold

$$
||F||_{H^3} + ||G||_{H^3} \leq C \max\{\mu, \nu\} e^{-\min\{\mu, \nu\} t} \varepsilon ||U_0||_{L^2}
$$

(3.1)

and

$$
||f||_{H^3} + ||g||_{H^3} \leq C e^{-\min\{\mu, \nu\} t} \varepsilon ||U_0||_{L^2} ||\hat{U}_0||_{L^1}.
$$

(3.2)

Proof of Lemma 3.1 For the term $f$, we can show that

$$
||F||_{H^3}^2 = \mu^2 e^{-2\mu t} \int_{1-\varepsilon \leq |\xi| \leq 1+\varepsilon} (1 + |\xi|^2)^3 ||\xi||^{2\alpha} - 1^2 |\hat{U}_0|^2 d\xi \leq C \mu^2 e^{-2\mu t} \varepsilon^2 \alpha^2 ||U_0||_{L^2}^2.
$$

Similar argument as the term $F$, we also have

$$
||G||_{H^3}^2 \leq C \nu^2 e^{-2\nu t} \varepsilon^2 ||U_0||_{L^2}^2.
$$

Using the classical Kato-Ponce product estimates and the fact the Fourier transform of a distribute belonging to $L^1$ lies in $L^\infty$, after a simple calculation, we obtain

$$
||f||_{H^3} \lesssim ||U||_{L^\infty} ||(\Lambda - I) U||_{H^3} + ||U||_{H^3} ||(\Lambda - I) U||_{L^\infty}
+ ||B||_{L^\infty} ||(\Lambda - I) B||_{H^3} + ||B||_{H^3} ||(\Lambda - I) B||_{L^\infty}
\leq C e^{-\min\{\mu, \nu\} t} \varepsilon ||U_0||_{L^2} ||\hat{U}_0||_{L^1},
$$
Thus, we complete the proof of Lemma 3.1. □

**Proof of Theorem 1.1** Applying $D^\beta$ on (2.2) \textsuperscript{1} and (2.2) \textsuperscript{2} respectively and taking the scalar product of them with $D^\beta v$ and $D^\beta c$ respectively, adding them together and then summing the result over $|\beta| \leq 3$, we get

\[
\frac{1}{2} \frac{d}{dt} (||v||^2_{H^3} + ||c||^2_{H^3}) + ||\Lambda^\alpha v||^2_{H^3} + ||\nabla c||^2_{H^3} \leq \sum_{i=1}^{11} I_i, \tag{3.3}
\]

where

\[
I_1 = - \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} [D^\beta, v] \nabla v \cdot D^\beta v dx - \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} [D^\beta, v] \nabla c \cdot D^\beta c dx,
\]

\[
I_2 = \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} [D^\beta, c] \nabla c \cdot D^\beta v dx + \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} [D^\beta, c] \nabla v \cdot D^\beta c dx,
\]

\[
I_3 = \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta ((\nabla \times c) \times c) \cdot D^\beta (\nabla \times c) dx,
\]

\[
I_4 = - \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (U \cdot \nabla v) \cdot D^\beta v dx - \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (U \cdot \nabla c) \cdot D^\beta c dx,
\]

\[
I_5 = \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (B \cdot \nabla c) \cdot D^\beta v dx + \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (B \cdot \nabla v) \cdot D^\beta c dx,
\]

\[
I_6 = \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (c \cdot \nabla B) \cdot D^\beta v dx - \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (v \cdot \nabla B) \cdot D^\beta c dx,
\]

\[
I_7 = \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (c \cdot \nabla U) \cdot D^\beta c dx - \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (v \cdot \nabla U) \cdot D^\beta v dx,
\]

\[
I_8 = \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta ((\nabla \times c) \times B) \cdot D^\beta (\nabla \times c) dx,
\]

\[
I_9 = \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta ((\nabla \times B) \times c) \cdot D^\beta (\nabla \times c) dx,
\]

\[
I_{10} = \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (f - F) \cdot D^\beta v dx + \sum_{0\leq|\beta|\leq 3} \int_{\mathbb{R}^3} D^\beta (g - G) \cdot D^\beta c dx.
\]

Next, we need to estimate the above terms one by one.

According to the commutate estimate (See [17]),

\[
\sum_{|\alpha|\leq m} ||[D^\alpha, g] f||_{L^2} \leq C(||f||_{H^{m-1}} ||\nabla g||_{L^\infty} + ||f||_{L^\infty} ||g||_{H^m}), \tag{3.4}
\]
we obtain

\[
I_1 \leq \sum_{0<|\beta|\leq 3} ||[D^\beta, v] \nabla v||_{L^2} ||\nabla v||_{H^2} + \sum_{0<|\alpha|\leq 3} ||[D^\beta, v] \nabla c||_{L^2} ||\nabla c||_{H^2}
\]

\[
\leq C ||\nabla v||_{L^\infty} ||v||_{H^3} ||\nabla v||_{H^2} + C ||v||_{H^3} ||\nabla c||_{H^2}^2
\]

\[
\leq C ||v||_{H^3} \left( ||\nabla v||_{H^2}^2 + ||\nabla c||_{H^2}^2 \right) \leq C ||v||_{H^3} \left( ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^2}^2 \right), \quad (3.5)
\]

\[
I_2 \leq \sum_{0<|\alpha|\leq 3} ||[D^\alpha, c] \nabla c||_{L^2} ||\nabla v||_{H^2} + \sum_{0<|\alpha|\leq 3} ||[D^\alpha, c] \nabla v||_{L^2} ||\nabla c||_{H^2}
\]

\[
\leq C ||\nabla c||_{H^2} ||\nabla v||_{H^2} ||\nabla c||_{H^3}
\]

\[
\leq C ||c||_{H^3} \left( ||\nabla v||_{H^2}^2 + ||\nabla c||_{H^2}^2 \right) \leq C ||c||_{H^3} \left( ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^2}^2 \right). \quad (3.6)
\]

Using the cancelation equality \([D^\beta (\nabla \times c) \times c] \cdot D^\beta (\nabla \times c) = 0\) and (3.4), we get

\[
I_3 = - \sum_{0<|\beta|\leq 3} \int_{\mathbb{R}^3} [D^\beta, c \times ](\nabla \times c) \cdot D^\beta (\nabla \times c) dx
\]

\[
\leq C ||\nabla c||_{L^\infty} ||c||_{H^3} ||\nabla c||_{H^3}
\]

\[
\leq C ||c||_{H^3} ||\nabla c||_{H^2}^2 + \frac{1}{8} ||\nabla c||_{H^2}^2. \quad (3.7)
\]

Invoking the following calculus inequality which is just a consequence of Leibniz’s formula,

\[
\sum_{|\beta|\leq 3} ||D^\beta, g||_{L^2} \leq C \left( ||\nabla g||_{L^\infty} + ||\nabla^3 g||_{L^\infty} \right) ||f||_{H^2},
\]

we obtain

\[
I_4 \leq \sum_{0<|\beta|\leq 3} ||[D^\beta, \cdot \nabla v||_{L^2} ||\nabla v||_{H^2} + \sum_{0<|\beta|\leq 3} ||[D^\beta, \cdot \nabla c||_{L^2} ||\nabla c||_{H^2}
\]

\[
\leq C \left( ||\nabla U||_{L^\infty} + ||\nabla^3 U||_{L^\infty} \right) \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) \leq C ||U||_{L^\infty} \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right), \quad (3.8)
\]

\[
I_5 \leq \sum_{0<|\beta|\leq 3} ||[D^\beta, B \cdot \nabla c||_{L^2} ||\nabla v||_{H^2} + \sum_{0<|\beta|\leq 3} ||[D^\beta, B \cdot \nabla v||_{L^2} ||\nabla c||_{H^2}
\]

\[
\leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^3 B||_{L^\infty} \right) \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) \leq C ||B||_{L^\infty} \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right), \quad (3.9)
\]

\[
I_8 \leq \sum_{0<|\beta|\leq 3} ||[D^\beta, B \times ](\nabla \times c)||_{L^2} ||\nabla c||_{H^3}
\]

\[
\leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^3 B||_{L^\infty} \right) ||\nabla c||_{H^2} ||\nabla c||_{H^3}
\]

\[
\leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^3 B||_{L^\infty} \right)^2 ||c||_{H^3} + \frac{1}{8} ||\nabla c||_{H^3}^2 \leq C \left( ||B||_{L^\infty} ||c||_{H^3} + \frac{1}{8} ||\nabla c||_{H^3}^2. \quad (3.10)
\]
By Leibniz’s formula and Hölder’s inequality, one has
\[ I_6 \leq [c \cdot \nabla B]_{H^3} ||v||_{H^3} + ||v \cdot \nabla B||_{H^3} ||c||_{H^3} \]
\[ \leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^4 B||_{L^\infty} \right) (||v||_{H^3}^2 + ||c||_{H^3}^2) \leq C ||B||_{L^\infty} (||v||_{H^3}^2 + ||c||_{H^3}^2), \]
\[ (3.11) \]
\[ I_7 \leq [c \cdot \nabla U]_{H^3} ||c||_{H^3} + ||v \cdot \nabla U||_{H^3} ||v||_{H^3} \]
\[ \leq C \left( ||\nabla U||_{L^\infty} + ||\nabla^4 U||_{L^\infty} \right) (||v||_{H^3}^2 + ||c||_{H^3}^2) \leq C ||U||_{L^\infty} (||v||_{H^3}^2 + ||c||_{H^3}^2), \]
\[ (3.12) \]
\[ I_9 \leq ||(\nabla \times B) \times c||_{H^3} ||\nabla c||_{H^3} \]
\[ \leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^4 B||_{L^\infty} \right) ||c||_{H^3} ||\nabla c||_{H^3} \]
\[ \leq C \left( ||\nabla B||_{L^\infty} + ||\nabla^4 B||_{L^\infty} \right)^2 ||c||_{H^3}^2 + \frac{1}{8} ||\nabla c||_{H^3}^2 \leq C ||B||_{L^\infty}^2 ||c||_{H^3}^2 + \frac{1}{8} ||\nabla c||_{H^3}^2. \]
\[ (3.13) \]
Using Hölder’s inequality and Young inequality, we deduce
\[ I_{10} \leq C ||f, g, F, G||_{H^3} (||v||_{H^3}^2 + ||c||_{H^3}^2) \]
\[ \leq C ||f, g, F, G||_{H^3} + C ||f, g, F, G||_{H^3} (||v||_{H^3}^2 + ||c||_{H^3}^2). \]
\[ (3.14) \]
Taking all the estimates (3.5)–(3.14) into (3.3), we obtain
\[ \frac{d}{dt} \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) + ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^3}^2 \]
\[ \lesssim \left( ||c||_{H^3} + ||c||_{H^3}^2 \right) \left( ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^3}^2 \right) \]
\[ + \left( ||B, U||_{L^\infty} + ||B||_{L^\infty}^2 \right) (||v||_{H^3}^2 + ||c||_{H^3}^2) + ||f, g, F, G||_{H^3}^2 \]
\[ \lesssim \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right)^\frac{3}{2} + ||c||_{H^3}^2 \left( ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^3}^2 \right) \]
\[ + \left( ||B, U||_{L^\infty} + ||B||_{L^\infty}^2 \right) (||v||_{H^3}^2 + ||c||_{H^3}^2) + ||f, g, F, G||_{H^3}. \]
\[ (3.15) \]
The Hausdorff-Young inequality together with the estimates for \( f, g, F, G \) in Lemma 3.1 yields
\[ \frac{d}{dt} \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) + ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^3}^2 \]
\[ \leq C \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right)^\frac{3}{2} + ||c||_{H^3}^2 \left( ||\Lambda^\alpha v||_{H^3}^2 + ||\nabla c||_{H^3}^2 \right) \]
\[ + C e^{-\min\{\mu, \nu\} t} \left( 1 + ||\hat{U}_0||_{L^1} \right) \left( ||\hat{U}_0||_{L^1} + \varepsilon ||U_0||_{L^2} \right) \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) \]
\[ + C e^{-\min\{\mu, \nu\} t} \varepsilon ||U_0||_{L^2} \left( 1 + ||\hat{U}_0||_{L^1} \right). \]
\[ (3.16) \]
Now, we define
\[ \Gamma \triangleq \sup \left\{ t \in [0, T^*) : \sup_{\tau \in [0, t]} \left( ||v(\tau)||_{H^3}^2 + ||c(\tau)||_{H^3}^2 \right) \leq \eta \right\}, \]
where \( \eta \) is a small enough positive constant which will be determined later on.
Assume that \( \Gamma < T^* \). For all \( t \in [0, \Gamma] \), we obtain from (3.16) that
\[ \frac{d}{dt} \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) \leq C e^{-\min\{\mu, \nu\} t} \left( 1 + ||\hat{U}_0||_{L^1} \right) \left( ||\hat{U}_0||_{L^1} + \varepsilon ||U_0||_{L^2} \right) \left( ||v||_{H^3}^2 + ||c||_{H^3}^2 \right) \]
\[ + C e^{-\min\{\mu, \nu\} t} \varepsilon ||U_0||_{L^2} \left( 1 + ||\hat{U}_0||_{L^1} \right), \]
which follows from the assumption that
\[
||v||_{H^3}^2 + ||c||_{H^3}^2 \leq C \left( ||v_0||_{H^3}^2 + ||c_0||_{H^3}^2 + \epsilon ||U_0||_{L^2} (1 + ||\dot{U}_0||_{L^1}) \right) 
\exp \left( C (||1 + ||\dot{U}_0||_{L^1}||\dot{U}_0||_{L^1} + \epsilon ||U_0||_{L^2}) \right) \leq C \delta.
\]
Choosing \( \eta = 2C \delta \), thus we can get
\[
\sup_{\tau \in [0,t]} \left( ||v(\tau)||_{H^3}^2 + ||c(\tau)||_{H^3}^2 \right) \leq \frac{\eta}{2} \quad \text{for} \quad t \leq \Gamma.
\]
So if \( \Gamma < T^* \), due to the continuity of the solutions, we can obtain there exists \( 0 < \epsilon \ll 1 \) such that
\[
\sup_{\tau \in [0,t]} \left( ||v(\tau)||_{H^3}^2 + ||c(\tau)||_{H^3}^2 \right) \leq \frac{\eta}{2} \quad \text{for} \quad t \geq \Gamma + \epsilon < T^*,
\]
which is contradiction with the definition of \( \Gamma \).
Thus, we can conclude \( \Gamma = T^* \) and
\[
\sup_{\tau \in [0,t]} \left( ||v(\tau)||_{H^3}^2 + ||c(\tau)||_{H^3}^2 \right) \leq C < \infty \quad \text{for all} \quad t \in (0, T^*),
\]
which implies that \( T^* = +\infty \). This completes the proof of Theorem 1.1.

Acknowledgments

J. Li is supported by the National Natural Science Foundation of China (Grant No.11801090).

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