ERDŐS DISTINCT DISTANCES IN HYPERBOLIC SURFACES

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ABSTRACT. In this paper, we study the number of distinct distances among any
N points in hyperbolic surfaces. For Y from a large class of hyperbolic surfaces,
we show that any N points in Y determines \( \geq c(Y)N/\log N \) distinct distances
where \( c(Y) \) is some constant depending only on \( Y \). In particular, for \( Y \) being
modular surface or standard regular of genus \( g \geq 2 \), we evaluate \( c(Y) \) explicitly.
We also derive new sum-product type estimates.

1. Introduction

1.1. Distinct distances problem in hyperbolic surfaces. In 1946, Erdős [6]
posed the distinct distances problem which asks for the least number of distinct
distances among any \( N \) points in the Euclidean plane, and conjectured that it is
\( \sim N/\sqrt{\log N} \). Guth-Katz [10] obtained the nearly optimal bound \( \geq N/\log N \) (we
use the notation \( f \geq g \) to mean that there is an absolute constant \( C > 0 \) with
\( f \geq Cg \)). Erdős also considered the higher dimensional generalization of the problem
in \( \mathbb{R}^d \) \( (d \geq 3) \) and conjectured the lower bound \( \geq N^{2/d} \). For \( d \geq 3 \), Solymosi-Vu [29]
obtained the lower bound \( \geq N^{(2d-2)/(d+2)} \) by an induction on the dimension with
the best known lower bound in the plane at that time as the base case. Combining
the Guth-Katz bound with the induction of Solymosi-Vu, one may get better lower
bounds for higher dimensional Euclidean spaces. For example when \( d = 3 \), it gives
the lower bound \( \geq N^{3/5-\epsilon} \) for any \( \epsilon > 0 \), see Sheffer [27] for details. There is also
a continuous analogue of the problem in geometric measure theory, the Falconer’s
conjecture, asking about the lower bound of Hausdorff dimension of the sets in \( \mathbb{R}^d \) for
which the difference set has positive Lebesgue measure. Interested readers may check
[7], [9], [14] etc. In addition to the Euclidean space, Erdős-Falconer type problems
have also been studied in vector spaces over finite fields and in the complex plane,
see e.g. Bourgain-Katz-Tao [3], Iosevich-Rudnev [15], Hart-Iosevich-Koh-Rudnev
[12], and Sheffer-Zahl [28] etc.

In the present paper, we establish lower bounds of distinct distances problem for
a large class of hyperbolic surfaces.

Theorem 1.1. Assume \( Y \) is the modular surface or a surface whose fundamental
group is co-compact as a Fuchsian group. A set of \( N \) points in \( Y \) determines \( \geq c(Y)N/\log N \) distinct distances for some constant \( c(Y) > 0 \) depending only on \( Y \).
In order to deal with various hyperbolic surfaces, we propose the concept of \textit{geodesic-covering number} (see Section 3.1) of a hyperbolic surface, which itself can be of independent interest. The finiteness of the geodesic-covering number implies such type of lower bound in the above theorem for distinct distances problem.

In particular for $Y_g$ being standard regular of genus $g \geq 2$, whose fundamental domain in the upper half plane $\mathbb{H}^2$ can be chosen as a standard regular $4g$-gon, we are able to estimate $c(Y_g)$ explicitly and get the following theorem.

\textbf{Theorem 1.2.} For $Y_g$ being standard regular of genus $g \geq 2$, the lower bound of distinct distances among any $N$ points in $Y_g$ is $\geq c\left(\frac{N}{\log N + \log g}\right)$ for some absolute constant $c > 0$.

To evaluate $c(Y_g)$ explicitly for standard regular $Y_g$, we rely on hyperbolic trigonometry and connect it with the hyperbolic circle problem. As a natural analogue of the Gauss circle problem in $\mathbb{H}^2$, the hyperbolic circle problem asks for the asymptotics of $\#\{\gamma \in \Gamma : d_{\mathbb{H}^2}(z_0, \gamma \cdot z_0) \leq Q\}$ for discrete subgroups $\Gamma \leq PSL_2(\mathbb{R})$ and $Q > 0$. This problem and related generalizations have been widely studied by various mathematicians including Delsarte [4], Huber [13], Selberg [26], Margulis [21], Patterson [24], Iwaniec [16], Phillips-Rudnick [25], Boca-Zaharescu [2], Kontorovich [18] etc.

In the case of the hyperbolic plane, Rudnev-Selig [23] demonstrated a proof of the lower bound $\gtrsim N/\log N$ based on a general idea utilizing the Klein quadric in Plücker coordinates motivated by a blog of Tao\textsuperscript{1}. In Section 2, we provide an approach for the hyperbolic plane in a more Guth-Katz ethnic language by working concretely with isometries of the upper half plane $\mathbb{H}^2$.

More generally, we also derive a lower bound for the number of distinct distances between points of any two finite sets $P_1$ and $P_2$ in $\mathbb{H}^2$, which implies new sum-product type estimates.

\textbf{Theorem 1.3.} Let $P_1, P_2 \subset \mathbb{H}^2$ be any finite sets. Then we have

$$\left|\{d_{\mathbb{H}^2}(p_1, p_2) : p_1 \in P_1, p_2 \in P_2\}\right| \gtrsim \frac{|P_1|^2|P_2|^2}{|P_1 \cup P_2|^3\log |P_1 \cup P_2|}.$$ 

\textbf{Remark 1.} When $P_1$ and $P_2$ are roughly the same size, this lower bound is sharp up to a factor of log.

For any finite sets $A, B \subset \mathbb{R}, C, D \subset \mathbb{R}\setminus\{0\}$, define $P_1 = \{a + i|c| : a \in A, c \in C\}, P_2 = \{b + i|d| : b \in B, d \in D\}$ and $P'_2 = \{-b + i|d| : b \in B, d \in D\}$. Note that explicitly we have the hyperbolic distance formula

$$2 \cosh(d_{\mathbb{H}^2}(x_1 + iy_1, x_2 + iy_2)) = \frac{(x_1 - x_2)^2 + y_1^2 + y_2^2}{y_1 y_2}$$

and $\left|\{|x| : x \in E\}\right| \geq \frac{1}{2}|E|$ for any finite set $E \subset \mathbb{R}$, by applying Theorem 1.3 to $P_1, P_2$ and $P_1, P'_2$, we get

\textsuperscript{1}Lines in the Euclidean group SE(2), http://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/
Corollary 1.4. Let $A, B \subset \mathbb{R}$, $C, D \subset \mathbb{R} \setminus \{0\}$ be finite sets. Then we have
\[
\left\{ \frac{(a-b)^2 + c^2 + d^2}{cd} : a \in A, b \in B, c \in C, d \in D \right\} \gtrsim \frac{|A|^2|B|^2|C|^2|D|^2}{\Delta^3 \log(\Delta)},
\]
and
\[
\left\{ \frac{(a+b)^2 + c^2 + d^2}{cd} : a \in A, b \in B, c \in C, d \in D \right\} \gtrsim \frac{|A|^2|B|^2|C|^2|D|^2}{\Delta^3 \log(\Delta)},
\]
where $\Delta = |A||C| + |B||D|$. By adding or subtracting 2 on the elements in the above sets, the factor $c^2 + d^2$ can be replaced by $(c+d)^2$ or $(c-d)^2$.

**Remark 2.** This result has the flexibility that all the four sets involved are arbitrary and the lower bounds only depend on the sizes of these sets. In particular, if $|A|, |B|, |C|, |D|$ are all about the size $\asymp N$, the above lower bounds become $\gtrsim \frac{N^2}{\log N}$.

A variation of distinct distances problem has been previously used by Roche-Newton and Rudnev [22] to study sum-product type estimates. See also Jones [19] for estimates of other sum-product types using incidence geometry. Very recently, Sheffer-Zahl [28] derived sum-product type estimates for complex numbers.

At the end of this subsection, we illustrate a bit more on the proof of Theorem 1.1, which is a consequence of Theorem 3.2, and Propositions 3.1 and 3.4. For any surface $Y$ with universal cover $\mathbb{H}^2$, its fundamental group is isomorphic to a Fuchsian group $\Gamma_Y \leq \text{PSL}_2(\mathbb{R})$. Note that $\Gamma_Y$ acts on $\mathbb{H}^2$ by Möbius transformation, we have $Y \simeq \Gamma_Y \setminus \mathbb{H}^2$ endowed the hyperbolic metric from $\mathbb{H}^2$. For any points $p, q \in Y$, we pick two representatives (still denoted by $p, q$) in a fundamental domain $F$ of $\Gamma_Y$. Then $d_Y(p, q) = \min_{\gamma \in \Gamma_Y} d_{\mathbb{H}^2}(p, \gamma \cdot q)$. We want to find a subset $\Gamma_0 \subset \Gamma_Y$ such that for any $p, q \in Y$, we have $d_Y(p, q) = d_{\mathbb{H}^2}(p, \gamma \cdot q)$ for some $\gamma \in \Gamma_0$. We call the patched region $\cup_{\gamma \in \Gamma_0} \gamma(F)$ a geodesic cover of $Y$ and call the smallest $|\Gamma_0|$, denoted by $K_Y$ (or $K_{\Gamma_Y}$), the geometric covering number of $Y$ (or $\Gamma_Y$).

In Section 3 we show that co-compact Fuchsian groups have finite geodesic-covering number. If a Fuchsian group $\Gamma$ is co-compact, its fundamental domain is a closed region without ideal points as vertices. This is equivalent to that $\Gamma \setminus \mathbb{H}^2$ has finite hyperbolic area and $\Gamma$ contains no parabolic elements, see Corollary 4.2.7 of [17]. Especially closed hyperbolic surfaces of genus $g \geq 2$ belong to this case. Moreover, Proposition 3.3 establishes the estimate $K_{Y_g} \lesssim g^3$ for $Y_g$ being standard regular of genus $g \geq 2$. For groups which are not co-compact, we show by explicit analysis that the modular group has finite geodesic-covering number. More specifically, Proposition 3.4 establishes the estimate $K_{\text{PSL}_2(\mathbb{Z})} \leq 10$.

Now given any $N$ points $P \subset Y$, if $K_Y < \infty$ we duplicate the points to be $\tilde{P} = \cup_{\gamma \in \Gamma_0} \gamma(P) \subset \mathbb{H}^2$ on a geodesic cover $\cup_{\gamma \in \Gamma_0} \gamma(F)$ of $Y$ with $|\Gamma_0| = K_Y$. By definition, the distances among points of $P$ in $Y$ all belong to the distances among points of $\tilde{P}$ in $\mathbb{H}^2$. However, we are not allowed to apply the lower bound for the hyperbolic plane to points of $\tilde{P}$ directly, since we have more number of points now
and the inequality actually goes to wrong direction. Instead, we resort to counting of distance quadruples (see Theorem 2.4) of $\tilde{P} \subset \mathbb{H}^2$ to establish Theorem 1.1. See Theorem 3.2 for details.

Remark 3. For completeness we include the case of flat tori, i.e. $g = 1$. We may similarly define $K_\Gamma$ for any discrete subgroup $\Gamma$ of the rigid motion group of $\mathbb{R}^2$. For flat tori which correspond to $\Gamma \simeq \mathbb{Z}^2$, we immediately see that $K_\Gamma < \infty$. Thus by the result of Guth-Katz [10], the number of distinct distances among $N$ points on any flat torus is $\gtrsim N/\log N$.

Remark 4. There is also an analogue of the unit distance problem in hyperbolic surfaces. Running through the arguments of Section 7.6 of [8] using estimates of crossing numbers, one may generalize the Spencer-Szemerédi-Trotter bound to $\mathbb{H}^2$, i.e. the number of pairs with unit (or equal) distance among any $N$ points in $\mathbb{H}^2$ is $\lesssim N^{4/3}$. For any set of $N$ points in a hyperbolic surface $Y$ with $K_Y$ finite, we lift it to a set of $K_Y N$ points on a geodesic cover of $Y$. By Spencer-Szemerédi-Trotter one may bound the number of unit (or equal) distances among any $N$ points on $Y$ by $\lesssim (K_Y N)^{4/3}$. In particular for standard regular surfaces $Y_g$ of genus $g \geq 2$, by Proposition 3.3, the upper bound becomes $\lesssim g^4 N^{4/3}$.

1.2. Sharpness of Theorem 1.2 and conjectures on geodesic-covering number. In order to analyze the sharpness of Theorem 1.2, we connect it with the equilateral dimension of hyperbolic surfaces. The equilateral dimension of a metric space is defined to be the maximal number of points with pairwise equal distance. For the simplest example the equilateral dimension of the Euclidean space $\mathbb{E}^d$ is always $d + 1$. The equilateral dimensions of various spaces have been studied by Alon-Milman [1], Guy [11], Koolen [20] etc. We are not aware of any non-trivial bound of equilateral dimension on hyperbolic surfaces in literature. We observe that our results can be applied to the equilateral dimension problem on hyperbolic surfaces. And in converse, the results for equilateral dimensions could also help us to analyze the sharpness of Theorem 1.2.

We claim that Theorem 1.2 implies equilateral dimension of standard regular surfaces $Y_g$ of genus $g$ is $\lesssim g^{9/2 + \epsilon}$. Suppose to the contrary for infinitely many $g$, the surface $Y_g$ has equilateral dimension $\geq C g^{9/2 + \epsilon}$ for some constant $C > 0$. Then for each such $g$ there exists a set of $M_g = C g^{9/2 + \epsilon}$ points in $Y_g$ with pairwise equal distance. Hence its number of distinct distances is 1. On the other hand, by Theorem 1.2, the number of distinct distances for any set of $M_g$ points is $\gtrsim \frac{M_g}{g^9 \log(g M_g)} \gtrsim g^\epsilon$ which would approach infinity as $g \to \infty$. Contradiction.

However, from another approach one may show that the equilateral dimension of $Y_g$ is actually $\lesssim g$. Suppose there are $N_g$ points in $Y_g$ with pairwise equal distance $r > 0$. Choosing a fundamental domain $F$ of $Y_g$, we draw a circle of radius $r$ in $\mathbb{H}^2$ centered at one representative of the $N_g$ points, say $p_0$. By definition, each point has a representative lying on the circle with distance at least $r$ from each other. We order these representatives by $p_i, i = 1, \ldots, N_g - 1$. For adjacent $p_i, p_j$, let $\alpha_{ij}$ be the smaller positive angle between geodesics connecting $p_0, p_i$ and $p_0, p_j$. By hyperbolic
trigonometry, since \( d_{H^2}(p_i, p_j) \geq r \),
\[
\sin(\frac{\alpha_{ij}}{2}) = \frac{\sinh(\frac{d_{H^2}(p_i, p_j)}{2})}{\sinh(r)} \geq \frac{\sinh(r/2)}{2 \cosh(r/2)}.
\]

In the proof of Proposition 3.3, we get the upper bound \( \cosh r \lesssim g^2 \), hence \( \alpha_{ij} \gtrsim \frac{1}{g} \). This shows that \( N_g \lesssim g \) and hence the equilateral dimension of \( Y_g \) is \( \lesssim g \).

By the above analysis on equilateral dimensions, we see that the lower bound for the number of distinct distances among any \( N \) points should be better than trivial in the range \( g \lesssim N \lesssim g^{9+\epsilon} \). Therefore the factor of \( g \) in Theorem 1.2 is not sharp.

One possible approach to improve Theorem 1.2 is trying to get a better bound for geodesic-covering number \( K_{Y_g} \). One may modify the definition of geodesic cover a little bit, to choose a set \( \Gamma_1 \subset \Gamma_{Y} \) for a surface \( Y \) such that for any \( p, q \in Y \),
\[
(1) \quad d_Y(p, q) = \min_{\gamma_1, \gamma_2 \in \Gamma_1} d_{H^2}(\gamma_1 \cdot p, \gamma_2 \cdot q) = \min_{\gamma_1, \gamma_2 \in \Gamma_1} d_{H^2}(p, \gamma_1^{-1} \gamma_2 \cdot q),
\]
for \( p, q \) treated as representatives in some fundamental domain of \( \Gamma_{Y} \). Then \( \Gamma_0 = \Gamma_1^{-1} \Gamma_0 \) is a geodesic cover in the original definition and \( |\Gamma_1| \) may be estimated as \( \sim |\Gamma_0|^{1/2} \) in many cases. However this definition appears not as convenient for computation. We also observe that for rectangle tori, the four fundamental polygons around a vertex patched together gives a geodesic cover in the sense above. Thus we are tempted to conjecture that the fundamental polygons around one vertex may also work for the hyperbolic case.

**Conjecture 1.** For standard regular surfaces \( Y_g \) of genus \( g \geq 2 \), the geodesic-covering number \( K_{Y_g} \) is \( \lesssim g \).

In addition, since for any finite index subgroup \( \Gamma' \) of a Fuchsian group \( \Gamma \), its fundamental domain is the union of finitely many fundamental domains of \( \Gamma \). If \( K_{\Gamma} \) is finite, one may expect that \( K_{\Gamma'} \) is also finite. We further make the conjecture.

**Conjecture 2.** For any subgroup \( \Gamma \leq PSL_2(\mathbb{Z}) \) of finite index, its geodesic-covering number is finite.

There are more examples of Fuchsian groups with finite geodesic-covering number. For example, the translation group
\[
\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}
\]
has the strip \( \{ x + iy : 0 < x \leq 1, y > 0 \} \) as fundamental domain and its geodesic-covering number is \( \leq 3 \). However it is an infinite index subgroup of \( PSL_2(\mathbb{Z}) \). Other simple examples include finite subgroups of \( PSL_2(\mathbb{R}) \). We further make the following more general conjecture.

**Conjecture 3.** For any discrete subgroup \( \Gamma \leq PSL_2(\mathbb{R}) \) whose fundamental domain has finitely many sides (geometrically finite), its geodesic-covering number is finite.
Notation. Throughout this paper we use the notation \( f \gtrsim g \) to mean that there is an absolute constant \( C > 0 \) such that \( f \geq Cg \), and we use \( f' \lesssim g' \) to mean that \( |f'| \leq C'g' \) for some absolute constant \( C' > 0 \). We use \( f \asymp g \) to mean that \( f \lesssim g \) and also \( f \gtrsim g \).

2. Distinct distances in the hyperbolic plane

In this section, we use Elekes-Sharir framework to reduce the counting of distinct distances to incidence problem of lines in the real projective space \( \mathbb{P}^3 \). To overcome the difficulty of linearizing projective lines in \( \mathbb{P}^3 \), we turn the incidence of lines in \( \mathbb{P}^3 \) into that of lines in \( \mathbb{R}^3 \) by certain conjugation. Then fulfilling the requirements for our lines in \( \mathbb{R}^3 \) as of Proposition 2.8 of Guth-Katz \[10\] amounts to a more concrete proof of the lower bound \( \gtrsim N/\log N \) of distinct distances among \( N \) points in \( \mathbb{H}^2 \).

2.1. Framework. We modify the Elekes-Sharir framework in \[10\] and apply it to the hyperbolic case. Let \( \mathbb{H}^2 \) be the hyperbolic plane and \( G = \text{PSL}_2(\mathbb{R}) \) be its isometry group which acts on \( \mathbb{H}^2 \) by M"obius transformation:

\[
\gamma \cdot z = \frac{az + b}{cz + d}, \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}), \quad z \in \mathbb{H}^2.
\]

Let \( P \subset \mathbb{H}^2 \) be a set of \( N \) points and define the set of distance quadruples

\[
Q(P) := \{(p_1, p_2; p_3, p_4) \in P^4 : d(p_1, p_2) = d(p_3, p_4) \neq 0\},
\]

where \( d(\cdot, \cdot) \) denotes the hyperbolic metric. Denote the distance set by

\[
d(P) := \{d(p_1, p_2) : p_1, p_2 \in P\}.
\]

Then we have a close relation between \( d(P) \) and \( Q(P) \) as follows. Suppose \( d(P) = \{d_i : 1 \leq i \leq m\} \) and \( n_i \) is the number of pairs of points in \( P \) with distance \( d_i \). So \( |Q(P)| = \sum_{i=1}^{m} n_i^2 \). Since \( \sum_{i=1}^{m} n_i = 2\binom{N}{2} = N^2 - N \), by Cauchy-Schwarz inequality we get

\[
(N^2 - N)^2 = \left(\sum_{i=1}^{m} n_i\right)^2 \leq \left(\sum_{i=1}^{m} n_i^2\right)m = |Q(P)||d(P)|.
\]

Rearranging the inequality gives

\[
|d(P)| \geq \frac{N^4 - 2N^3}{|Q(P)|}.
\]

Moreover any quadruple \((p_1, p_2, p_3, p_4) \in Q(P)\) uniquely determines an isometry \( g \in G \) such that \( g(p_1) = p_3, g(p_2) = p_4 \). Suppose \( p_1 = x + iy, p_3 = x' + iy' \in \mathbb{H}^2 \) \((y, y' > 0)\) and there is some \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \) such that

\[
A \cdot (x + iy) = \frac{a(x + iy) + b}{c(x + iy) + d} = x' + iy',
\]
for $i = \sqrt{-1}$. Rearranging terms we get
\[ ax + b + iay = cxx' + dx' - cyy' + i(cx'y + dy' + cx'y), \]
or equivalently the system of linear equations
\[
\begin{aligned}
xa + b + (yy' - xx')c - x'd &= 0 \\
ya - (xy' + x'y)c - y'd &= 0.
\end{aligned}
\]
Its solution set in $\mathbb{R}^4$ is the intersection of two distinct hyperplanes, which turns out to be a two dimensional plane passing through the origin. If moreover $A \cdot p_2 = p_1$, the point $(a, b, c, d)$ also lies in another distinct two dimensional plane since $p_1 \neq p_2, p_3 \neq p_4$. Hence $(a, b, c, d)$ lies in their intersection at a line in $\mathbb{R}^4$ which projects to a point in $[a : b : c : d] \in \mathbb{P}^3$.

This gives a map $E : Q(P) \to G$. Define for any $p, q \in \mathbb{H}^2$
\[ S_pq := \{g \in G : g(p) = q\}, \]
which are one dimensional curves in $G$. Similar with Lemma 2.4 and 2.6 in [10], we have
(i) if $|P \cap gP| = k$, then $|E^{-1}(g)| = 2\binom{k}{2}$;
(ii) and $|P \cap gP| = k$ if and only if $g$ lies in at least $k$ of the curves $\{S_{pq}\}_{p, q \in P}$.

Thus we derive that
\[
|Q(P)| = \sum_{k=2}^{N} 2\binom{k}{2} |\{g : |P \cap gP| = k\}| = \sum_{k=2}^{N} (2k - 2)|G_k(P)|,
\]
where $G_k(P) \subset G$ consists of $g \in G$ with $|P \cap gP| \geq k$. Henceforth we focus on estimating $|G_k(P)|$ for $k = 2$ and $k \geq 3$ as in Sections 3 and 4 of [10].

2.2. Incidence of projective lines in $\mathbb{P}^3$. For any $g \in G$, we have $d(gp, gq) = d(p, q)$ so that shifting $P$ to $gP$ does not affect counting of distinct distances. Now for a quadruple $(p_1, p_2, p_3, p_4) \in Q(P)$, suppose $E((p_1, p_2, p_3, p_4)) = h$ i.e. $hp_1 = p_3, hp_3 = p_4$, after shifting we get
\[ E((gp_1, gp_2, gp_3, gp_4)) = ghg^{-1}. \]

In the matrix form of $G = \text{PSL}_2(\mathbb{R})$, we manage to reshape the distance quadruples as follows.

**Proposition 2.1.** For any finite set of points $P \subset \mathbb{H}^2$, there exists isometry $g \in \text{PSL}_2(\mathbb{R})$ such that all matrices in $E(Q(gP))$ have non-vanishing upper-left corners.

**Proof.** We use translations $T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in \mathbb{R}$. For any $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})$ we calculate that
\[
T_x h T_x^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cx & -cx^2 + (d - a)x + b \\ c & d - cx \end{pmatrix}.
\]
Suppose $E(Q(P))$ consists of $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \text{PSL}_2(\mathbb{R}), 1 \leq i \leq K$. Note that $a_i$ and $c_i$ can not be both zero, we choose nonzero $x$ such that $a_i + c_i x \neq 0$ for all $i = 1, \ldots, K$. For such $x$ we have $E(Q(T_x P)) = T_x E(Q(P)) T_x^{-1}$ consisting of matrices with non-vanishing upper-left corners. \qed

Remark 5. For any finite set of points in upper half plane, we may also dilate points by hyperbolic isometries so that they all have sufficiently large absolute values. Note that a Möbius transformation $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \cdot z = \frac{b}{cz + d}$ basically inverses absolute value of $z$, so that it can not map points with large absolute values to points with large absolute values. Thus after dilation, Möbius transformations with vanishing upper left corners do not occur as isometries in consideration.

Hence without loss of generality, we assume $p_x, p_y, q_x, q_y \gg 0$, i.e. far away in the first quadrant. Moreover, we have the following observation through (4). First, each $S_{pq}$ is a projective line in $\mathbb{P}^3 \supset G = \text{PSL}_2(\mathbb{R})$. We use the natural manifold atlas $\mathbb{P}^3 = \mathbb{R}^3_1 \cup \mathbb{R}^3_2 \cup \mathbb{R}^3_3 \cup \mathbb{R}^3_4$ with $\mathbb{R}^3_1 = \{[1 : b : c : d] \mid b, c, d \in \mathbb{R}\} \simeq \mathbb{R}^3$ and $\mathbb{R}^3_i \simeq \mathbb{R}^3, i = 2, 3, 4$ similarly defined with $i$-th entry equal to 1 in the projective coordinate. Analogously we use the partition $G = \bigcup_{i=1}^{4} G_i$, $G_i = \text{PSL}_2(\mathbb{R}) \cap \mathbb{R}^3_i$.

In particular, $G_1$ consists of matrices with non-vanishing upper left corners. Then the restriction $S_{pq} \cap G_i$ becomes a real line in $\mathbb{R}^3_i$, and moreover by Proposition 2.1, there exists $g \in G$ such that $G_k(gP) \subset G_1$ for each $k \geq 2$. Abusing notations, we always denote by $L_{pq}$ the real line $S_{(gp)(qg)} \cap \mathbb{R}^3_1$ in the manifold atlas of $\mathbb{P}^3$. The incidences among curves $S_{pq}$ are now equivalent to that of lines $L_{pq}$ in $\mathbb{R}^3 (\mathbb{R}^3_1)$. Explicitly $L_{pq}$ has the following linear parametrization.

Proposition 2.2. For any $p = p_x + ip_y, q = q_x + iq_y \in \mathbb{H}^2$, the line $L_{pq}$ can be parametrized as

\[ t \begin{pmatrix} p_y(q_x^2 + q_y^2) \\ p_x(q_x^2 + q_y^2) \\ p_x(q_x^2 + q_y^2) \end{pmatrix} - \frac{q_y(p_x^2 + p_y^2) + p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y} \begin{pmatrix} p_x + q_x \\ p_y + q_y \\ 0 \end{pmatrix}, \]

for $t \in \mathbb{R}$.

Proof. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p = q$ with $a = 1$ and $t = d + 1$ as parameter, we get from (4)

\[ b = -\frac{q_y(p_x^2 + p_y^2) + p_y(q_x^2 + q_y^2)}{p_x q_y + q_x p_y} + p_y(q_x^2 + q_y^2) t, \quad c = \frac{p_y + q_y}{p_x q_y + q_x p_y} - \frac{q_y}{p_x q_y + q_x p_y} t, \]

which gives us the parametrization of points $(b, c, t) \in L_{pq}$. \qed
Remark 6. There are other parametrizations of $L_{pq}$, say for $b = t$ as the parameter. Here the roles of $p$ and $q$ are symmetric in that the intersection of $L_{pq}$ and $L_{qp}$ is on the plane $t = 0$.

Note that there are non-linear terms in our parametrization which is not a problem in Guth-Katz [10], we have to consider different families of lines that rule surfaces and the vector fields on reguli to get the following.

Proposition 2.3. For any set of $N$ points $P \subset \mathbb{H}^2_0 := \{x + iy : x, y > 0\}$ and $\mathcal{L} = \{L_{pq} : p, q \in P\}$, no more than $N$ lines of $\mathcal{L}$ lie in a common plane and no more than $O(N)$ lines of $\mathcal{L}$ lie in a common regulus.

Proof. We consider the families $L_q := \{L_{pq}\}_{p \in \mathbb{H}^2_0}$ of lines targeting at $q$. First, for any $p' \neq p$, the line $L_{p'q}$ does not intersect $L_{pq}$. Note that $L_{pq} \subset S_{pq}$, suppose $L_{pq} \cap L_{p'q} \neq \emptyset$ there would be some $g \in G$ such that $gp' = gp = q$, a contradiction. Moreover by (6), the directions of $L_{pq}$ and $L_{p'q}$ are different:

$$\left(\frac{p_y(q_z^2 + q_y^2)}{p_xq_y + q_zp_y}, -\frac{q_y}{p_xq_y + q_zp_y}, 1\right) = (\xi_1, \xi_2, 1)$$

has a unique solution for fixed $q$ and $\xi_1, \xi_2$. Note that $\xi_1, \xi_2$ cannot be zero since $p_x, p_y, q_x, q_y > 0$. Indeed, equivalently we have

$$\begin{pmatrix}
-\xi_1 q_y & q_z^2 + q_y^2 - q_x \xi_1 \\
\xi_2 q_y & q_z^2 \\
q_y
\end{pmatrix}
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix} = \begin{pmatrix}
0 \\
-q_y
\end{pmatrix},
$$

whose associate matrix has determinant $-(q_z^2 + q_y^2)\xi_2 q_y \neq 0$. Hence lines of $L_q$ are pairwise skew and no two of its lines lie in a common plane. Therefore any plane intersects each $L_q$ at most one line and intersects $\mathcal{L}$ at most $N$ lines.

To prove the second part, we construct a vector field $V = (V_1, V_2, V_3)$ on $\mathbb{R}^3$ tangent to lines of $L_q$ for any fixed $q = q_x + iq_y \in \mathbb{H}^2_{+0}$. By (4) we locate $p$ such that $L_{pq}$ passing through any given $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as follows ($a = 1, x_1 = b, x_2 = c, x_3 = d$):

$$\begin{cases}
p_x + x_1 + (p_y q_y - p_x q_x) x_2 - q_x x_3 = 0 \\
p_y - (p_x q_y + p_z p_y) x_2 - q_y x_3 = 0,
\end{cases}$$

or equivalently,

$$\begin{cases}
(1 - q_x x_2) p_x + (q_y x_2) p_y = q_x x_3 - x_1 \\
(-q_y x_2) p_x + (1 - q_x x_2) p_y = q_y x_3,
\end{cases}$$

which has solution

$$\begin{pmatrix}
p_x \\
p_y
\end{pmatrix} = \frac{1}{(1 - q_x x_2)^2 + q_y^2 x_2^2} \begin{pmatrix}
q_x x_1 x_2 - (q_z^2 + q_y^2) x_2 x_3 - x_1 + q_x x_3 \\
-q_y x_1 x_2 + q_y x_3
\end{pmatrix}.$$

By (6), we set the direction of $L_{pq}$ as

$$((q_z^2 + q_y^2)p_y, -q_y, q_y p_x + q_x p_y) = \frac{1}{(1 - q_x x_2)^2 + q_y^2 x_2^2} (V_1, V_2, V_3),$$

where $((V_1, V_2, V_3) \in \mathbb{R}^3$ such that $V_3 > 0$ and $V_3 \neq 0$. This completes the proof.
where
\begin{align*}
V_1 &= -q_y(q_x^2 + q_y^2)(x_1x_2 - x_3), \\
V_2 &= -q_y[(1 - q_xx_2)^2 + q_y^2x_2^2], \\
V_3 &= -q_y(q_x^2 + q_y^2)x_2x_3 - q_yx_1 + 2q_xq_yx_3,
\end{align*}
Let \( V = (V_1, V_2, V_3) \) then \( V \) has degree 2. Note that \( p \in \mathbb{H}^2_{>0} \), the vector field is defined over the open subset
\[ U_q := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : q_x^2x_2 - (q_x^2 + q_y^2)x_2x_3 - x_1 + q_yx_3 > 0, -q_yx_1 + q_yx_3 > 0 \}, \]
and we always consider the pieces of reguli restricted in \( U_q \).

Now suppose a line \( L_{pq} \) lies in a regulus \( R \) defined by a degree 2 irreducible polynomial \( f \) in \( \mathbb{R}^3 \). Then at any point \( x \in L_{pq} \) we have the Taylor expansion
\[ f(x + tV(x)) = f(x) + \nabla(f)^T \cdot V(x)t + \frac{1}{2}V^T H(f)Vt^2, \]
where \( \nabla(f) \) is the gradient of \( f \) and \( H(f) \) is the Hessian matrix of \( f \).

By Bezout’s lemma (Lemma 3.1 of [10]), if more than 9 lines of \( L_q \) are contained in \( R \), \( f \) would have a common factor with both \( \nabla(f) \cdot V \) and \( V^T H(f)V \), which have degree 3 and 4. By irreducibility, \( f \) must be the common factor so that \( f \) vanishes on each line of \( L_q \) with direction \( V(x) \) for any \( x \in R \) by the Taylor expansion above, i.e. \( L_q \) is a ruling of \( R \). Since a regulus has only two rulings, \( R \) can only contain at most 8 lines from \( N - 2 \) families \( L_q \) which are not rulings of \( R \) and \( 2N \) lines of \( L_{q_1}, L_{q_2} \) if they are rulings of \( R \). Totally there are at most \( \leq 2N + 8(N - 1) = 10N - 8 \) lines of \( L \) lying in \( R \). \( \square \)

Now we already reduce the problem to incidence geometry in the Euclidean space. Applying ruled surface theory and polynomial partition in the work of Guth-Katz [10], we get the following lower bound for distinct distances problem in the hyperbolic plane.

**Theorem 2.4.** For \( P \subset \mathbb{H}^2 \) any set of \( N \) points and \( \mathcal{L} = \{ L_{pq} \mid p, q \in P \} \), let \( G_k \) be the set of points where at least \( k \) lines of \( \mathcal{L} \) meet. Then
\[ |G_k| \lesssim N^3 k^{-2}. \]
Consequently by (5) \( |Q(P)| \lesssim N^3 \log N \) and by (3), we have \( |d(p)| \gtrsim N/ \log N \).

It has the same strength as the result of Guth-Katz for the Euclidean plane.

2.3. Distinct distances between two sets. This subsection contributes to the proof of Theorem 1.3.

Let \( P_1, P_2 \subset \mathbb{H}^2 \) be finite sets and define
\[ d(P_1, P_2) := \{ d_{\mathbb{H}^2}(p_1, p_2) : p_1 \in P_1, p_2 \in P_2 \}, \]
and
\[ Q(P_1, P_2) := \{ (p_1, p_2; q_1, q_2) : p_1, q_1 \in P_1, p_2, q_2 \in P_2, d_{\mathbb{H}^2}(p_1, p_2) = d_{\mathbb{H}^2}(q_1, q_2) \neq 0 \}. \]
Suppose $d(P_1, P_2) = \{d_1, \ldots, d_m\}$ and $n_i$ is the number of pairs $(p_1, p_2)$ for $p_1 \in P_1, p_2 \in P_2$ with $d(p_1, p_2) = d_i$. We see that $|P_1||P_2| - |P_1 \cap P_2| = \sum_{i=1}^{m} n_i$ and $|Q(P_1, P_2)| = \sum_{i=1}^{m} n_i^2$. Then by Cauchy-Schwarz inequality we get

$$|P_1|^2|P_2|^2 \lesssim (|P_1||P_2| - |P_1 \cap P_2|)^2 \leq m \sum_{i=1}^{m} n_i^2 = m|Q(P_1, P_2)|.$$  

Using the estimate from Theorem 2.4, we have

$$|Q(P_1, P_2)| \leq |Q(P_1 \cup P_2)| \lesssim |P_1 \cup P_2|^3 \log |P_1 \cup P_2|,$$

and consequently

$$|d(P_1, P_2)| \gtrsim \frac{|P_1|^2|P_2|^2}{|P_1 \cup P_2|^3 \log |P_1 \cup P_2|}.$$  

This finishes the proof of Theorem 1.3.

We may replace $|P_1 \cup P_2|$ by $\max\{|P_1|, |P_2|\}$ in the above inequality. If $|P_1|^2 \leq |P_2|$, the inequality gives a trivial lower bound.

3. Distinct distances in hyperbolic surfaces

In this section we establish lower bounds for the number of distinct distances on hyperbolic surfaces. Especially we consider closed surfaces of genus $g \geq 2$ and the modular surface which are endowed with hyperbolic metric from $\mathbb{H}^2$.

We first introduce the concept of geodesic-covering number for discrete subgroups of $\text{PSL}_2(\mathbb{R})$ then use its estimates to deal with the distinct distances problem in closed hyperbolic surfaces and the modular surface.

3.1. Geodesic-covering number. In general, let $\Gamma \leq G = \text{PSL}_2(\mathbb{R})$ be a Fuchsian group, which acts on $\mathbb{H}^2$ discontinuously. The discrete subgroup $\Gamma$ is of first kind if it has finite co-volume, i.e. a fundamental domain of $\Gamma \backslash \mathbb{H}^2$ has finite hyperbolic volume. In particular, surface groups and the modular group $\text{PSL}_2(\mathbb{Z})$ are all Fuchsian groups of first kind.

Generally for any discrete subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$, let $Y$ be the hyperbolic surface associated with $\Gamma$ and $F$ be the fundamental domain of $Y$, we propose the question of finding a subset $\Gamma_0 \subset \Gamma$ such that

$$d_Y(p, q) = \min_{\gamma \in \Gamma_0} d_{\gamma^*}(p, \gamma(q)), \ \forall p, q \in Y.$$  

We call the patched region of fundamental domains $U = \bigcup_{\gamma \in \Gamma_0} \gamma(F)$ a geodesic cover of $Y$. We say $U$ is minimal if the cardinality of $\Gamma_0$ attains the minimal and $U \supset F$ (roughly $1 \in \Gamma_0$) and we denote by $K_\Gamma$ the smallest $|\Gamma_0|$. We call it the geodesic-covering number of $\Gamma$.

We expect the geodesic-covering number is finite for many discrete subgroups of $\text{PSL}_2(\mathbb{R})$.

Proposition 3.1. For any co-compact discrete subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$, its geodesic-covering number $K_\Gamma$ is finite.
Proof. If \( \Gamma \) is co-compact, we may choose a closed fundamental domain \( F \subset \mathbb{H}^2 \) without ideal points as vertices. Then its diameter
\[
\text{diam}(F) := \sup_{x,y \in F} \min_{\gamma \in \Gamma} d_{\mathbb{H}^2}(x, \gamma(y))
\]
is finite. Let \( U \subset \mathbb{H}^2 \) be
\[
U := \{ z \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(z, F) \leq \text{diam}(F) \}
\]
and \( \tilde{U} \supset U \) be
\[
\tilde{U} := \bigcup \{ \gamma(F) \mid \gamma \in \Gamma, \gamma(F) \cap U \neq \emptyset \}.
\]
We claim that the shortest geodesic connecting \( x \) and \( y \) in \( \mathbb{H}^2 \) lies in \( \tilde{U} \). Otherwise, \( d_Y(p,q) = d_{\mathbb{H}^2}(x, \gamma(y)) \) for some \( \gamma \in \Gamma \) with \( \gamma(F) \cap U = \emptyset \), contradicting with \( d_{\mathbb{H}^2}(x, \gamma(y)) > \text{diam}(F) \geq d_Y(p,q) \).

Now for each fundamental domain \( F' \subset \tilde{U} \), we choose a \( \gamma' \in \Gamma \) such that \( F' = \gamma'(F) \). The set \( \Gamma_0 \) consisting of these isometries satisfies (8). The number of fundamental domains \( F' \subset \tilde{U} \) is finite and so \( K_\Gamma \leq |\Gamma_0| < \infty \). \( \square \)

Now we connect the geodesic-covering number with distinct distances problem on any hyperbolic surface \( Y \) with corresponding fundamental group \( \Gamma \subset G \).

**Theorem 3.2.** Assume \( Y \) is a hyperbolic surface with fundamental group \( \Gamma \) and \( K_\Gamma \) is finite. Then a set of \( N \) points on \( Y \) determines
\[
\gtrsim \frac{N}{K_\Gamma^3 \log(K_\Gamma N)}
\]
distinct distances.

Proof. For any set \( P \) of \( N \) points on \( Y \), we choose a minimal geodesic cover \( \Gamma_0 \subset \Gamma \) with \( |\Gamma_0| = K_\Gamma \) such that
\[
d_Y(P) := \{ d_Y(p,q) : p, q \in Y \} \subset d_{\mathbb{H}^2}(\cup_{\gamma \in \Gamma_0} \gamma(P)).
\]
Then
\[
Q_Y(P) := \{ (p_1, p_2; p_3, p_4) \in P^4 : d_Y(p_1, p_2) = d_Y(p_3, p_4) \neq 0 \}
\subset Q(\cup_{\gamma \in \Gamma_0} \gamma(P)),
\]
where \( Q(P) \) is defined in (2). Since \( |\cup_{\gamma \in \Gamma_0} \gamma(P)| \leq K_\Gamma |P| = K_\Gamma N \), by Theorem 2.4 we get
\[
|Q_Y(P)| \leq |Q(\cup_{\gamma \in \Gamma_0} \gamma(P))| \lesssim (K_\Gamma N)^3 \log(K_\Gamma N).
\]
Similar to (3), by Cauchy-Schwarz inequality, we have
\[
|d_Y(P)| \geq \frac{N^4 - 2N^3}{|Q_Y(P)|} \gtrsim \frac{N}{K_\Gamma^3 \log(K_\Gamma N)}.
\]
We get the desired lower bound. \( \square \)

In the next two subsections, we give precise estimates for geodesic-covering numbers of closed hyperbolic surfaces and the modular surface.
3.2. Closed hyperbolic surfaces of genus \( g \geq 2 \). In this subsection we deal with surface groups. Here a surface group \( \Gamma_g \) is the fundamental group of a closed hyperbolic surface \( Y_g \) of genus \( g \geq 2 \), with the following presentation

\[
\Gamma_g := \langle a_i, b_i : 1 \leq i \leq g \rangle,
\]

in which \( a_i, b_i \in G \) satisfy \([a_1, b_1] \cdots [a_g, b_g] = 1\). Here \([a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}\) is the commutator. Topologically, the generators \( a_i, b_i \) represent the homotopy classes of closed geodesics on \( Y_g \) and the unique relator is derived from the condition of gluing sides of a \( 4g \)-gon in \( \mathbb{H}^2 \) as a fundamental domain of \( Y_g \). Note that for non-isometric closed surfaces of fixed genus \( g \), the subgroups \( \Gamma_g \) could be different. The moduli space of isometry classes of surfaces of genus \( g \) is characterized by the Teichmüller space \( T(Y_g) \simeq \mathbb{R}^{6g-6} \).

Now for the standard regular surfaces, we estimate their geodesic-covering numbers concretely as follows.

**Proposition 3.3.** For the surface of genus \( g \) with standard regular fundamental \( 4g \)-gon of inner angle \( \frac{\pi}{2g} \), we have \( K_\Gamma \lesssim g^3 \).

**Proof.** Let \( \Gamma \subset G \) be the corresponding surface group. For a standard regular geodesic \( 4g \)-gon \( F \subset \mathbb{H}^2 \) centered at \( i \) (denote by \( O \)) serving as a fundamental domain of \( Y_g \), we estimate its diameter as follows:

**Figure 1.** Distance between \( P \) and \( Q \)

First we determine a bound for \( \text{diam}(Y_g) \). For any \( P, Q \in F \) (see Figure 1), choose two vertices \( A \) and \( B \) of \( F \) that are closest to \( P \) and \( Q \) correspondingly. Since there exists \( \gamma \in \Gamma \) such that \( \gamma(A) = B \), we have \( (d = d_{\mathbb{H}^2}) \) by triangle inequality

\[
d_{Y_g}(P, Q) \leq d(O, Q) + d(O, P), \quad d_{Y_g}(P, Q) \leq d(P, A) + d(Q, B),
\]
hence
\[ 2d_{Y_g}(P, Q) \leq d(O, P) + d(P, A) + d(Q, Q) + d(Q, B). \]

We claim that \( d(O, P) + d(P, A) \leq d(O, D) + d(A, D) \). Indeed, if we extend the geodesic between \( O \) and \( P \) to \( E \), by triangle inequality we have
\[
d(O, P) + d(P, A) \leq d(O, D) + d(D, E) + d(P, E) + d(E, A),
\]
so that
\[
d(O, P) + d(P, A) = d(O, E) - d(P, E) + d(P, A) \leq d(O, D) + d(D, E) - d(P, E) + d(E, A) = d(O, D) + d(D, E) + d(E, A) = d(O, D) + d(A, D).
\]

Since \( F \) is regular and \( P, Q \) are arbitrary, we have
\[ \text{diam}(Y_g) \leq d(O, D) + d(A, D). \]

Note that \( (O, D, A) \) forms a right triangle and \( \angle AOD = \angle OAD = \frac{\pi}{4g} =: \beta \), hyperbolic trigonometry gives
\[ \cosh(d(O, D)) = \cosh(d(D, A)) = \cot \beta, \]
thus
\[ \cosh(\text{diam}(Y_g)) \leq \cosh(d(O, D) + d(D, A)) = \cosh(2d(O, D)) = 2 \cot^2(\beta) - 1. \]

Also, \( \cosh(d(O, A)) = \cot^2(\beta). \)

By the construction of geodesic cover \( \tilde{U} \) in the proof of Proposition 3.1, we only need to choose \( \gamma \in \Gamma \) with \( d(\gamma(i), i) \leq d(O, A) + \text{diam}(Y_g) \). Since for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}), \)
\[ 2 \cosh(d(\gamma(i), i)) = ||\gamma||^2 = a^2 + b^2 + c^2 + d^2, \]
we get
\[ K_\Gamma \leq \# \{ \gamma \in \Gamma : ||\gamma||^2 \leq 2 \cosh(d(O, A) + \text{diam}(Y_g)) \}. \]

By the sum of arguments formula,
\begin{equation}
(9)
\cosh(d(O, A) + \text{diam}(Y_g)) = \cosh(d(O, A)) \cosh(\text{diam}(Y_g)) + \sinh(d(O, A)) \sinh(\text{diam}(Y_g))
= \cot^2(\beta)(2 \cot^2(\beta) - 1) + \sqrt{\cot^4(\beta) - 1} \sqrt{(2 \cot^2(\beta) - 1)^2 - 1}
\lesssim g^2 \cdot g^2 + g^2 \cdot g^2 \lesssim g^4.
\end{equation}

By the result of counting hyperbolic lattices inside a circle (see [2] or [18]), we have asymptotically
\[ \# \{ \gamma \in \Gamma : ||\gamma|| \leq R \} \sim \frac{\pi}{\text{Area}(\Gamma \setminus \mathbb{H}^2)} R^2. \]

This is the so-called hyperbolic circle problem. Combining with the fact that \( \text{Area}(\Gamma \setminus \mathbb{H}^2) = (4g - 2)\pi - 2\pi = 4(g - 1)\pi \), we get
\[ K_\Gamma \lesssim \frac{g^4}{g} = g^3. \]

We finish the proof. \( \square \)
**Remark 7.** Note that \( \text{diam}(Y_g) \geq d_{Y_g}(O,A) = d_{\mathbb{H}^2}(O,A) \) in Figure 1, we have \( \cosh(\text{diam}(Y_g)) \geq \cosh(d(O,A)) = \cot^2(\beta) \geq g^2 \).

Thus by (9) the upper bound \( g^3 \) is optimal up to a constant in our method.

3.3. **Modular surface.** The key to prove Proposition 3.1 is that co-compact Fuchsian groups have fundamental domains of finite diameter. For other Fuchsian groups \( \Gamma \) whose fundamental domain is not of finite diameter, the geodesic-covering number \( K_\Gamma \) may still exist. In particular for the modular group \( \text{PSL}_2(\mathbb{Z}) \) we have the following result.

**Proposition 3.4.** For modular surface, we have \( K_{\text{PSL}_2(\mathbb{Z})} \leq 10 \) and the number of distinct distances among \( N \) points on the modular surface \( X \) is \( \gtrsim N/\log N \).

**Proof.** Let \( F \) be the standard fundamental domain

\[
F := \{ z \in \mathbb{H}^2 \mid -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > 1 \}.
\]

For any \( z_1 = x_1 + y_1i, z_2 = x_2 + y_2i \in F \), it is immediately to verify the Möbius transformation

\[
z_j = \begin{pmatrix} \sqrt{y_j} & x_j \\ \sqrt{y_j} & 0 \\ 0 & \sqrt{y_j} \end{pmatrix} \cdot i = \gamma_j(i), \ j = 1, 2.
\]

Then for each \( \gamma \in \text{SL}_2(\mathbb{Z}) \) we get

\[
2 \cosh(d_{\mathbb{H}^2}(z_1, \gamma(z_2))) = 2 \cosh(d_{\mathbb{H}^2}(i, \gamma^{-1}\gamma_2(i))) = \|\gamma^{-1}\gamma_2\|^2.
\]

Note that \( \cosh(x) \) is monotonic for \( x > 0 \), we see that

\[
d_X(z_1, z_2) = \arccosh(\min_{\gamma \in \text{SL}_2(\mathbb{Z})} \|\gamma^{-1}\gamma_2\|/2).
\]

By computation,

\[
\gamma^{-1}\gamma_2 = \begin{pmatrix} \sqrt{y_2} / y_1 (a - x_1c) & x_2a + b - x_1x_2c - x_1d \\ \sqrt{y_1y_2} & \sqrt{y_1} / y_2 (x_2c + d) \end{pmatrix},
\]

whence

\[
\|\gamma^{-1}\gamma_2\|^2 = \frac{y_2}{y_1} (a - x_1c)^2 + \frac{1}{y_1y_2} (x_2a + b - x_1x_2c - x_1d)^2 + y_1y_2c^2 + \frac{y_1}{y_2} (x_2c + d)^2.
\]

If \( c = 0 \), then \( \gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \) and

\[
\|\gamma^{-1}\gamma_2\|^2 = \frac{y_2}{y_1} + \frac{1}{y_1y_2} (x_2 - x_1 \pm b)^2 + \frac{y_1}{y_2}.
\]
Note that \(-\frac{1}{2} < x_1, x_2 < \frac{1}{2}\), the module \(\|\gamma_1^{-1} \gamma_2\|^2\) attains minimum at \(|b| \leq 1\) whose value is \(< \frac{y_2}{y_1} + \frac{1}{4y_1 y_2} + \frac{y_1}{y_2} := U(0)\).

If \(c \neq 0\), we have \(\frac{a}{c} \pm \frac{1}{c^2} = \frac{b}{c}\) and

\[
\|\gamma_1^{-1} \gamma_2\|^2 = c^2 \left[ \frac{y_2}{y_1} \left( \frac{a}{c} - x_1 \right)^2 + \frac{1}{y_1 y_2} \left( \frac{a}{c} x_2 + \frac{b}{c} - x_1 x_2 - \frac{d}{c} \right)^2 + y_1 y_2 + \frac{y_1}{y_2} \left( \frac{d}{c} + x_2 \right)^2 \right]
= c^2 \left[ \frac{y_2}{y_1} \left( \frac{a}{c} - x_1 \right)^2 + \frac{1}{y_1 y_2} \left( \frac{1}{c} \left( \frac{d}{c} + x_2 \right) - \frac{1}{c^2} \right)^2 + y_1 y_2 + \frac{y_1}{y_2} \left( \frac{d}{c} + x_2 \right)^2 \right]
\geq c^2 y_1 y_2.
\]

Comparing it with \(U(0)\) and note that \(-\frac{1}{2} < x_2, x_1 < \frac{1}{2}\) we get

\[
\|\gamma_1^{-1} \gamma_2\|^2 - U(0) \geq c^2 y_1 y_2 - \frac{y_2}{y_1} - \frac{1}{4y_1 y_2} - \frac{y_1}{y_2}
= \frac{c^2 y_1^2 y_2^2 - y_1 y_2 - 1}{y_1 y_2}
= \left(\frac{|y_1 y_2| - \frac{1}{|c|}}{|y_1 y_2|}\right) \left(\frac{|y_1 y_2| - \frac{1}{|c|}}{-\frac{1}{|c|}}\right) - \frac{1}{|c|^2} - \frac{1}{4}.
\]

For \(|c| \geq 2\) we have

\[
\|\gamma_1^{-1} \gamma_2\|^2 - U(0) \geq \frac{\left(\frac{2y_1^2 - 1}{2}\right) \left(\frac{2y_2^2 - 1}{2}\right) - 1}{y_1 y_2}
= \frac{\left(\frac{3}{2} - \frac{1}{2}\right)^2 - \frac{1}{2}}{y_1 y_2} > 0
\]
since \(y_2 > \sqrt{3}/2, j = 1, 2\). Thus in order to choose for \(\Gamma_0\) as in (8), we only need \(\gamma \in \text{PSL}_2(\mathbb{Z})\) with \(|c| \leq 1\).

For \(|c| = 1\), we have \(ad \pm b = 1\) (so that \(a\) and \(d\) can be chosen arbitrarily). We claim that in this case, (10) attains minimum when \(|a| \leq 1, |d| \leq 1\). By choosing \(\gamma \in \text{SL}_2(\mathbb{Z})/\{\pm 1\}\) we may assume \(c = 1\). Let \(t_1 = a - x_1\) and \(t_2 = d + x_2\), then (10) becomes

\[
\|\gamma_1^{-1} \gamma_2\|^2 = \frac{y_2}{y_1} t_1^2 + \frac{1}{y_1 y_2} (t_1 t_2 - 1)^2 + \frac{y_1}{y_2} t_2^2 + y_1 y_2.
\]

(11)
We prove the claim by refuting the contradictory cases: (i) if $c = 1, |a| \geq 2, |d| \geq 2$, note that $|x_1| \leq 1/2, |x_2| \leq 1/2$, then $|t_1| \geq 3/2, |t_2| \geq 3/2$ and (11) becomes

$$\|\gamma^{-1}_1 \gamma_2\|^2 \geq \frac{y_2}{y_1} \cdot \frac{9}{4} + \frac{1}{y_1 y_2} \cdot \frac{25}{16} + \frac{y_1}{y_2} \cdot \frac{9}{4} + y_1 y_2 > U(0);$$

(ii) if $c = 1, |a| \leq 1, |d| \geq 2$, then $|t_2| \geq 3/2$ and we take the difference ($t_1$ or $a$ fixed)

$$\min_{c=1,|d|\geq 2} \|\gamma^{-1}_1 \gamma_2\|^2 - \min_{c=1,|d|\leq 1} \|\gamma^{-1}_1 \gamma_2\|^2$$

$$\geq \frac{1}{y_1 y_2} \left( |t_1| \cdot \left( \frac{3}{2} - 1 \right)^2 + \frac{y_1}{y_2} \cdot \frac{9}{4} - \min_{|d|\leq 1} \left\{ \frac{1}{y_1 y_2} (|t_1||d + x_2| - 1)^2 + \frac{y_1}{y_2} (d + x_2)^2 \right\} \right)$$

$$\geq \min_{|d|\leq 1} \left\{ \frac{1}{y_1 y_2} \left[ 2t_1^2 + (2|d + x_2| - 3)|t_1| + 2y_1^2 \right] \right\}$$

$$\geq \frac{1}{y_1 y_2} \left[ 2 \left( |t_1| - \frac{3}{4} \right)^2 - \frac{9}{8} + 2 \cdot \frac{3}{4} \right] > 0,$$

noting that $|y_1| \geq \sqrt{3}/2$; (iii) for $c = 1, |a| \geq 2, |d| \leq 1$, the above difference (for $t_2$ fixed) stays positive if symmetrically the roles of $t_1, d$ are replaced by $t_2, a$. Thus the claim is proved.

In conclusion, we may choose $\Gamma_0 \subset \text{PSL}_2(\mathbb{Z})$ consisting of

$$1, \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. $$

Thus $K_{\text{PSL}_2(\mathbb{Z})} \leq 10$. Then by Theorem 3.2 we get the desired lower bound for distinct distances on modular surface. \qed

Here the geodesic cover $\cup_{\gamma \in \Gamma_0} \gamma(\mathbb{F})$ is $F$ together with the nine neighbouring fundamental domains on $\mathbb{H}^2$. Actually we may only choose the geodesic cover in the sense of (1) as

$$\Gamma_1 = \left\{ 1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

since $\Gamma_1^{-1} \Gamma_1 = \Gamma_0$.

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