A Consistent Estimator for Skewness of Partial Sums of Dependent Data

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Abstract

We introduce an estimation method for the scaled skewness coefficient of the sample mean of short and long memory linear processes. This method can be extended to estimate higher moments such as curtosis coefficient of the sample mean. Also a general result on computing all asymptotic moments of partial sums is obtained, allowing in particular a much easier derivation of some existing central limit theorems for linear processes. The introduced skewness estimator provides a tool to empirically examine the error of the central limit theorem for long and short memory linear processes. We also show that, for both short and long memory linear processes, the skewness coefficient of the sample mean converges to zero at the same rate as in the i.i.d. case.

Keywords : Linear processes, Long memory, Short memory, Stationarity, Skewness

1 Introduction

Skewness is a characteristic of a distribution which is often used to measure its departure from symmetry. The skewness of the marginal distribution of a set of data is a different quantity from that of an estimator which is built as an aggregation of the data. In particular, the skewness of \( \overline{X} \), which is the sample mean of \( n \), \( n \geq 1 \) i.i.d. observations with a finite third moment, is of the same order as \( 1/\sqrt{n} \) times the skewness of the marginal distribution. In other words, the marginal skewness of i.i.d. is the same as \( \sqrt{n} \) times the skewness of \( \overline{X} \). As a result, in the i.i.d. case, having an empirical estimator for the marginal skewness means having an estimator for the skewness of the sample mean. In the case of stationary and dependent observations, the relation between the skewness of the sample mean and that of the marginal distribution of the data is no longer as straightforward as that of the i.i.d. data. For stationary dependent data, unlike in the i.i.d. case, estimating the skewness of the marginal distribution and estimating that of the sample mean do not amount to the same task. Although, the marginal skewness of the stationary data has been investigated and contributions have been made to the area (cf. e.g. Bai and Ng (2005), Grigoletto and Lisi (2009) and references therein) less so has been done for their sample mean. In this paper, a consistent empirical estimator for the third cumulant of partial sum of \( n \) stationary linear processes is introduced. This estimator is then combined with some existing consistent estimators for the variance of partial sums of linear processes to obtain an estimator for the skewness of partial sums of linear processes. The results in this
paper also show that the rate at which the skewness of partial sums of long and short memory linear processes vanishes is, counter-intuitively for long memory, $\sqrt{n}$ and that long memory effect tends to make the sampling distribution more and more quickly symmetrical. The rest of the paper is organized as follows: in section 2 we establish and discuss (the impact of) a general result (Theorem 1) on asymptotically computing all higher order moments of partial sums of linear processes. This result is interesting on its own as, given its short proof, it opens the door to reproofing some existing central limit theorems without using the machinery of stochastic integrals or cumulant-based methods to prove asymptotic normality. In section 2 we also give a second important result which is the skewness estimation (Theorem 2) where we essentially develop a method to estimate the rightly normalized third moment of partial sum. This method can be extended to estimate all higher moments. In section 4 we discuss some Monte Carlo studies to illustrate the skewness estimation. Section 5 is devoted to proofs.

2 Skewness of partial sums of linear processes

For throughout use in this paper we let $X_1, \ldots, X_n$, be a stationary sample of size $n$, $n \geq 1$, with the linear representation

$$X_t = \mu + \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}, \quad t \geq 1,$$

where $\mu \in \mathbb{R}$, $a_i$ are squarely summable and $\varepsilon_i, \ i \geq 0$, are i.i.d. white noise with variance $\sigma^2$, where $0 < \sigma^2 < \infty$. Furthermore, we assume that $\varepsilon_1$ has a finite third moment. The validity of some of the results in this paper will require that $\varepsilon_1$ has a finite sixth moment.

The linear process (1), when $\sum_{i=0}^{\infty} |a_i| < \infty$ is said to be of short memory. For long memory linear processes, we adopt the conventional definition in which $X_t$ is called long memory if, as $i \to \infty$, and for some constant $c(d) > 0$ (depending on $d$), $a_i \sim c(d)i^{d-1}$, with $0 < d < 1/2$, where, here and also throughout this paper, $\ell_n \sim \lambda_n$ stands for the asymptotic equivalence $\ell_n = \lambda_n(1 + o(1))$.

The parameter $d$ is the memory parameter whose large values (i.e., closer to $1/2$) indicate a stronger dependence between the observations on the linearly structured temporal series $X_t$ as in (1).

To unify our notation we denote a short memory linear process $X_t$, as in (1), with $d = 0$.

We note that linear processes of the form (1) englobe the well known fractionally autoregressive integrated moving average FARIMA($p, d, q$) processes of the form

$$\phi(B)(1 - B)^d(X_t - \mu) = \theta(B)\varepsilon_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(b) = 1 + \theta_1 B + \cdots + \theta_q B^q$ are respectively the $p$ order and $q$ order autoregressive and moving average polynomials, and $B$ is the backshift operator, namely $BX_t = X_{t-1}$. From corollary 3.1 of Kokoszka and Taqqu (1995), such processes have an explicit linear representation (1) with

$$a_i \sim \frac{\theta(1)}{\phi(1)} i^{d-1}, \quad 0 < d < 1/2, \quad \text{as } i \to \infty,$$
and when $d = 0$, i.e. just ARMA($p, q$), the coefficients $a_i$ decrease exponentially fast towards zero. In all what follows and without restriction of generality, we assume that $\mu = 0$. Let $\mathbb{E}(e_1^2) = \eta$. Also, let $S_n = X_1 + \cdots + X_n$. Before presenting the skewness estimation, we first give a general result on asymptotically computing all higher order moments of partial sums of linear processes according to whether the higher order is even or odd.

**Theorem 1.** Assume that $X_t$ satisfies (1) with $0 \leq d < 1/2$, and that $\mathbb{E}[\tau_1^{k}]$ is finite. Let

$$m(d) = \begin{cases} 
\sum_{i=0}^{\infty} a_i, & \text{if } d = 0, \\
\frac{c(d)}{d}, & \text{if } 0 < d < 1/2.
\end{cases}$$

Then as $n \to \infty$,

$$n^{-p(1+2d)}\mathbb{E}\left[S_n^k\right] \to (m(d))^k \sigma^k \left(\frac{k!}{2^p(p!)}\right) \left(\frac{1}{1+2d} + \int_0^{\infty} \left((1+x)^d - x^d\right)^2 dx\right)^p$$

if $k = 2p$, and

$$n^{-1/2(1+2d)}\mathbb{E}\left[S_n^k\right] \to (m(d))^k \sigma^k \left(\frac{3 + 2\ell}{3}\right) \left(\frac{(2\ell)!}{2^p(p!)}\right) \times \left(\frac{1}{1+3d} + \int_0^{\infty} \left((1+x)^d - x^d\right)^3 dx\right) \left(\frac{1}{1+2d} + \int_0^{\infty} \left((1+x)^d - x^d\right)^2 dx\right)$$

if $k = 3 + 2\ell$, $\ell \geq 0$.

**Remark 1.** The previous theorem provides another (much easier method of moments) proof of some Central limit theorems for linear processes (see for example Davydov 1970) when all moments exist. Actually, from one side, it is well known that if $Z = \mathcal{N}(0, 1)$ then for all integer $p \geq 0$,

$$\mathbb{E}(Z^{2p}) = \frac{(2p)!}{2^p(p!)}, \quad \text{and } \mathbb{E}(Z^{2p+1}) = 0,$$

and from another side, it is straightforward from the theorem above to see that

$$\mathbb{E}\left[\left(\frac{S_n}{\sqrt{\text{Var}(S_n)}}\right)^k\right] \to \begin{cases} 
\frac{(2p)!}{2^p(p!)}, & \text{if } k = 2p, \\
0, & \text{if } k = 2p + 1.
\end{cases}$$

We also note that the previous theorem goes deeper than showing that odd moments of normalized partial sums converge to zero and gives an explicit rate of $\sqrt{n}$ of this convergence regardless of the type of the memory whether it be short or long.

**Remark 2.** We note that for all FARIMA processes, the coefficient $m(d)$ in (2) is continuous in $d$ as $d \to 0$. Actually, and as mentioned prior to the previous theorem, for FARIMA($p, d, q$), (and noting that $\Gamma(d) \sim 1/d$ as $d \to 0$), $c(d) = \theta(1)/\phi(1)\Gamma(d) \sim (\theta(1))/(\phi(1)d$ so that
$m(d) \to m(0)$ as $d \to 0$, since FARIMA($p, 0, q$) is just ARMA($p, q$), for which the coefficients $a_i$ in (1) satisfy

$$\sum_{k=0}^{\infty} a_k = \frac{\theta(1)}{\phi(1)},$$

since

$$\sum_{k=0}^{\infty} a_k z^k = \frac{\theta(z)}{\phi(z)}, \quad \text{for all } |z| \leq 1.$$  

We now define a measure of skewness for the partial sum of $X_1, \cdots, X_n$, i.e., the first $n \geq 1$ observations of the linear process $X_t$, as follows:

$$\beta_n := \frac{E(S_n^3)}{[\text{Var}(S_n)]^{3/2}}. \quad (3)$$

Note that $\beta_n = 0$ for all $n$ for linear processes $X_t$, as in (1), with symmetric innovations, as we will have in this case $E(\varepsilon^3_1) = 0$, but in general, due to the CLT, we will always have $\beta_n \to 0$ as $n \to \infty$.

The rate at which the skewness $\beta_n$ of the partial sums of a linear process vanishes is given in the following corollary which is an immediate consequence of Theorem 1 by evaluating the second and the third moments of $S_n$.

**Corollary 1.** Let $X_1, \cdots, X_n$ be the first $n$ terms of the linear process (1) with $0 \leq d < 1/2$ fixed. If $\eta$ is finite then, as $n \to \infty$,

$$\sqrt{n} \beta_n \to k(d) := \frac{\eta}{\sigma^3} \frac{1}{1+3d} + \int_0^{\infty} ((1 + x)^d - x^d)^3 dx \left( \frac{1}{1+2d} + \int_0^{\infty} ((1 + x)^d - x^d)^2 dx \right)^{3/2}. \quad (4)$$

**Remark 3.** From the previous corollary one can see that the skewness of sums of short and long memory linear processes asymptotically vanishes at the same convergence rate $\sqrt{n}$ as that of sums of i.i.d. data. The effect of the short or long range dependence appears only in terms of $d$ in the limiting constant $k(d)$ in (4) which, surprisingly enough, (in absolute value) is decreasing in $d \in [0, 1/2]$, with $k(0) = \eta/\sigma^3$ and $k(.5) = 0$. While short memory effect (such as in ARMA models) tends to asymptotically disappear, long memory effect tends to make normalized partial sums more and more quickly symmetrical. This is mainly due to the fact that, at fixed $n$ and as $d \to .5$, the variance of the partial sum becomes infinite, while its third moment remains bounded. Of course these findings have to be considered with some caution as we still do not know how fast (as function of $n$ and $d$) convergence (4) is taking place.

### 3 A consistent estimator for skewness of partial sums of linear processes

To construct a consistent estimator for $\beta_n$, one approach is to construct separate consistent estimators for $E(S_n)^3$ and $\text{Var}(S_n)$. In this section, we estimate only the former as a consistent
estimator for the variance of the partial sums of short and long memory linear process is obtained in Abadir et al (2009). More precisely, Abadir et al used the well known fact that

\[ n^{-1-2d} \text{Var} \left( \sum_{k=1}^{n} X_k \right) = n^{-2d} \left( \gamma(0) + 2 \sum_{h=1}^{\frac{n}{h}} \left( 1 - \frac{h}{n} \right) \gamma(h) \right) \to \nu(d) \]

where \( \nu(d) > 0 \) is the limiting variance and \( \gamma(h) = \text{Cov}(X_1, X_{1+h}) \), to construct the long run variance estimator from the sample covariance function

\[ \hat{\gamma}(h) = \frac{1}{n} \sum_{j=1}^{n-h} (X_j - \bar{X})(X_{j+h} - \bar{X}), \]

in the form of

\[ q_0^{-2d} \left( \tilde{\gamma}_0 + 2 \sum_{h=1}^{q_0} \left( 1 - \frac{h}{q_0} \right) \tilde{\gamma}_h \right) \overset{p}{\to} \nu(d), \]

where \( q_0 \to \infty \) and \( q_0 = o(n) \).

To estimate the limiting normalized third moment of the partial sum (for \( 0 \leq d < 1/2 \))

\[ S_3(d) := n^{-1-3d} \mathbb{E}(S_n^3) \to \frac{n}{\sigma^3} n(d) \left( \frac{1}{1+3d} + \int_0^\infty \left( (1+x)^d - x^d \right)^3 dx \right), \quad (5) \]

we define the estimator

\[ \hat{S}_3(d) := q_1^{-3d} \tilde{\Delta}(0) + 3q_2^{-3d} \sum_{h=1}^{q_2} \left( 1 - \frac{h}{q_2} \right) \tilde{\Delta}(h) + 6q_3^{-3d} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3-h} \left( 1 - \frac{h + h'}{q_3} \right) \tilde{\Delta}(h, h'), \quad (6) \]

where, as \( n \to \infty \), \( q_i \to \infty \), for \( i = 1, 2, 3 \), with \( q_2 = o(n) \), \( q_3 = o(n^{1/2}) \), and \( \tilde{\Delta}(h) \) and \( \tilde{\Delta}(h, h') \) are third order sample covariances defined as

\[ \tilde{\Delta}(h) := \frac{1}{n} \sum_{j=1}^{n-h} ((X_j - \bar{X})^2(X_{j+h} - \bar{X}) + (X_j - \bar{X})(X_{j+h} - \bar{X})^2), \]

\[ \tilde{\Delta}(h, h') := \frac{1}{n} \sum_{j=1}^{n-h-h'} (X_j - \bar{X})(X_{j+h} - \bar{X})(X_{j+h+h'} - \bar{X}). \]

The idea of having such estimator stems from the fact that we can break \( S_3(d) \) into similar third order theoretical covariances (according to the cases where all indices are equal, or two are equal, or all are different)

\[ S_3(d) = n^{-3d} \left( \Delta(0) + 3 \sum_{h=1}^{\frac{n}{h}} \left( 1 - \frac{h}{n} \right) \Delta(h) + 6 \sum_{h=1}^{\frac{n}{h}} \sum_{h'=1}^{\frac{n}{h}} \left( 1 - \frac{h + h'}{n} \right) \Delta(h, h') \right), \quad (7) \]

where

\[ \Delta(h) = \mathbb{E}(X_1X_1^2) + \mathbb{E}(X_1^2X_1), \quad \Delta(h, h') = \mathbb{E}(X_1X_{1+h}X_{1+h+h'}). \]
In practice we would use $S_3(d)$ where $d$ is a consistent estimator for $d$. The bandwidth numerical sequence $q_n = q$ is such that, as $n \to \infty$, $q \to \infty$ and $q^2 = o(n)$.

The following theorem is the second main result of this paper which shows that $S_3(d)$ is a consistent estimator for $S_3(d)$, the normalized third moment of partial sum of linear processes.

**Theorem 2.** If $\mathbb{E}(\varepsilon_1^6) < \infty$ then, as $n, q_i \to \infty$ in such a way that $q_i^2 = o(n)$, for $i = 1, 2, 3$ used in the definition (6), we have

$$S_3(d) - S_3(d) = o_P(1).$$

A direct consequence of Theorem 2 and Abadir et al yields a consistent estimator for the scaled skewness $\sqrt{n}\beta_n$, where $\beta_n$ is as in (3). This consistency is stated in the following corollary.

**Corollary 2.** Assume that $\mathbb{E}(\varepsilon_1^6) < \infty$. As $n, q_0, q_i \to \infty$ in such a way that $q_i^2 = o(n)$, for $i = 1, 2, 3$, and $q_0 = o(n)$, we have

$$\hat{k}(d) \to k(d),$$

where $k(d)$ is given in (4) and

$$\hat{k}(d) := \frac{S_3}{\left(q_0^{-2d}(\bar{\gamma}_0 + 2 \sum_{h=1}^{d-1} (1 - h/q_0)\bar{\gamma}_h)\right)^{3/2}}. \quad (8)$$

### 4 Numerical Illustrations

The results in Tables 1 and 2 illustrate the performance of the empirical estimator $\hat{k}(d)$ above, in estimating $k(d)$. In Tables 1 and 2, for each sample size, 2000 replications of an ARMA(1,1) process $X_t = \phi X_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t$ were produced. In Tables 3 and 4 the same number of replications were produced from a fractionally integrated process $X_t = (1 - B)^{-d}\varepsilon_t$, where $d$ is the memory parameter, $B$ is the back shift operator. In both cases, the innovations $\varepsilon_t$ are i.i.d. exponentially distributed with mean 1. In all the tables, the following empirical mean square error is computed:

$$\overline{MSE}(\hat{k}(d)) = \frac{\sum_{b=1}^{2000} (\hat{k}_b(d) - k(d))^2}{2000},$$

where $\hat{k}_b(d)$ is computed based on the $b$th replication of the linear process, as indicated in each table.

| Table 1: $X_t = 0.5X_{t-1} + 0.5\varepsilon_{t-1} + \varepsilon_t$ | Table 2: $X_t = -0.5X_{t-1} - 0.5\varepsilon_{t-1} + \varepsilon_t$ |
|---|---|
| $n$ | $\overline{MSE}(\hat{k}(d))$ | $n$ | $\overline{MSE}(\hat{k}(d))$ |
| 200 | 1.075 | 200 | 1.800 |
| 1000 | 0.575 | 1000 | 0.586 |
| 5000 | 0.298 | 5000 | 0.171 |
Table 3: ARIMA(0,0.2,0)

| n    | MSE(\hat{k}(d)) |
|------|-----------------|
| 200  | 0.374           |
| 1000 | 0.113           |
| 5000 | 0.048           |

Table 4: ARIMA(0,0.4,0)

| n    | MSE(\hat{k}(d)) |
|------|-----------------|
| 200  | 0.121           |
| 1000 | 0.027           |
| 5000 | 0.023           |

In the preceding tables, the limiting values of the normalized skewness coefficient for the two ARIMA(1,1) and ARIMA(0,0.2,0) and ARIMA(0,0.4,0), when innovations \( \epsilon_t \) are exponentially distributed (i.e. with skewness coefficient 2), are respectively 2 and 1.7 and 0.675, as we immediately see from (4) that \( k(0) = 2 \) and we can numerically compute \( k(0.2) = 1.7 \) and \( k(0.4) = 0.675 \). It is worth reiterating that for all short memory linear processes, the limiting normalized skewness coefficient \( k(0) \) is equal to the skewness of the innovations \( \epsilon_t \).

**Remark 4.** It was observed empirically that for long memory processes with all memory parameters \( 0 < d < 1/2 \), the choice \( q_1 = q_2 = [n^{0.2}], q_3 = [n^{0.1}], q_4 = [n^{0.5-d}] \), in (6) and (8) yields relatively good estimates for \( k(d) \). For short memory processes, the choice \( q_2 = q_3 = q_4 = [n^{0.33}] \) is deemed to be a good choice. The preceding choices were implemented in the numerical illustrations of Tables 1-4.

5 **Proofs**

5.1 **Proof of Theorem 1**

Assume that \( 0 < d < 1/2 \). With the assumption that \( \mu = 0 \) and writing (1) as

\[ X_t = \sum_{j=-\infty}^{t} a_{t-j} \epsilon_j \]

we get

\[
\mathbb{E} \left[ \left( \sum_{t=1}^{n} X_t \right)^k \right] = \mathbb{E} \left[ \left( \sum_{j=-\infty}^{n} \left( \sum_{t=\max(j,1)}^{n} a_{t-j} \right) \epsilon_j \right)^k \right] \\
= \sum_{j_1=\infty}^{n} \cdots \sum_{j_k=\infty}^{n} \left( \sum_{t=\max(j_1,1)}^{n} a_{t-j_1} \right) \cdots \left( \sum_{t=\max(j_k,1)}^{n} a_{t-j_k} \right) \mathbb{E} (\epsilon_{j_1} \cdots \epsilon_{j_k}) . \tag{9}
\]

We show that in (9), in the case \( k = 2p \), only the indices \( j_1, \ldots, j_k \) that are equal two by two will be the leading terms, and in case \( k = 3 + 2p \), only the cases corresponding two three of them are equal and the rest are equal to by two will lead. Consider first the case \( k = 2p \). The two by two equal indices give

\[
\left( \frac{k!}{2^p(p!)^2} \right) \sigma^k \left[ \left( \sum_{j=-\infty}^{n} \left( \sum_{t=\max(j,1)}^{n} a_{t-j} \right)^2 \right)^p - B_n \right] ,
\]

\[7\]
Hence in both cases of short and long memory, and when \( k \) and \( \alpha_1, \ldots, \alpha_s \geq 1 \). The coefficient \( k! / (2^p (p!)) \) corresponds to the number of two-by-two configurations, which is the number of pairings of \( k = 2p \) individuals. Now we have (distinguishing between \( j \geq 1 \) and \( j \leq 0 \), as \( n \to \infty \),

\[
\sigma^k \left( \sum_{j=-\infty}^{n} \left( \sum_{t=\max(j,1)}^{n} a_{t-j} \right)^2 \right)^p
\]

\[
= \sigma^k \left( \sum_{j=1}^{n} \left( \sum_{t=j}^{n} a_{t-j} \right)^2 + \sum_{j=0}^{\infty} \left( \sum_{t=1}^{n} a_{t+j} \right)^2 \right)^p
\]

\[
= \sigma^k \left( \sum_{j=1}^{n} \left( \sum_{i=0}^{j} a_i \right)^2 + \sum_{j=1}^{\infty} \left( \sum_{t=1}^{n} a_{t+j} \right)^2 \right)^p
\]

\[
\sim \sigma^k \left( \frac{c(d)}{d} \right)^k \left( \frac{1}{2d+1} n^{2d+1} + \sum_{j=1}^{\infty} \left( \frac{c(d)}{d} (n+j)^d - (j+1)^d \right)^2 \right)^p
\]

\[
= \sigma^k \left( \frac{c(d)}{d} \right)^k n^{(2d+1)p} \left( \frac{1}{2d+1} + \int_{0}^{\infty} \left( (x+1)^d - x^d \right)^2 dx \right)^p
\]

For the case of short memory \( (d = 0) \), where the coefficients \( a_i \) are summable, we have

\[
\sum_{j=1}^{\infty} \left( \sum_{t=1}^{n} a_{t+j} \right)^2 = \sum_{t=1}^{n} \sum_{s=1}^{\infty} \sum_{j=1}^{n} a_{t+j} a_{s+j} = \sum_{t=1}^{n} \sum_{j=1}^{\infty} a_{t+j} \sum_{s=1}^{n} a_{s+j} = \sum_{t=1}^{n} o(1) = o(n),
\]

and

\[
\sum_{j=1}^{n} \left( \sum_{i=0}^{j} a_i \right)^2 \sim n \left( \sum_{i=0}^{\infty} a_i \right)^2.
\]

Hence in both cases of short and long memory, and when \( k = 2p \), we get that

\[
n^{-p-2dp} \left( \sum_{j=-\infty}^{n} \left( \sum_{t=\max(j,1)}^{n} a_{t-j} \right)^2 \right)^p
\]

\[
\to (m(d))^k \left( \frac{1}{2d+1} + \int_{0}^{\infty} \left( (x+1)^d - x^d \right)^2 dx \right)^p.
\]

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Similar calculations show that
\[
B_n \sim \sum_{s=1}^{p-1} (m(d))^k n^{2sd+s} \sum_{\alpha_1 + \cdots + \alpha_s = p} \prod_{j=1}^{s} \left( \frac{1}{2d\alpha_j + 1} + \int_0^\infty (x + 1)^d - x^d \right)^{2\alpha_j} = o(n^{2pd+p}).
\]

Now for the indices that are not two by two equal, and similarly to \( B_n \), they will yield quantities of the form
\[
\mathbb{E}(\epsilon_1^{\beta_1}) \cdots \mathbb{E}(\epsilon_1^{\beta_s}) n^{kd+s} (m(d))^k \prod_{j=1}^{s} \left( \frac{1}{d\beta_j + 1} + \int_0^\infty (x + 1)^d - x^d \right)^{\beta_j}
\]
where \( s = 1, \ldots, p - 1 \), and \( \beta_j \geq 2 \) and \( \beta_1 + \cdots + \beta_s = k \). Clearly each quantity above is \( o(n^{2pd+p}) \). This completes the first part of the theorem (i.e. when \( k \) is even). Now we consider \( k = 3 + 2\ell \), with \( \ell \geq 0 \), we proceed in a similar way as above for the \( 2\ell \) part as we must have three indices (among \( j_1, \ldots, j_k \)) equal and the remaining \( 2p \) indices must be two by two equal in order to build the leading term. I.e. when \( k = 3 + 2\ell \), we have \( \alpha(k) = \binom{k}{3} (2p)! / (2^p (p!) \) ways of selecting three indices (to be equal) and paring the remaining \( 2p \), so that \( \beta \) is asymptotically equivalent to
\[
\alpha(k) \sum_{j_0 = -\infty}^{n} \sum_{j_1 = -\infty}^{n} \cdots \sum_{j_\ell = -\infty}^{n} \left( \sum_{t = \max(j_0,1)}^{n} a_{t-j_0} \right)^3 \left( \sum_{t = \max(j_1,1)}^{n} a_{t-j_1} \right)^2 \cdots \left( \sum_{t = \max(j_\ell,1)}^{n} a_{t-j_\ell} \right)^2 \mathbb{E}(\epsilon_1^3) \mathbb{E}(\epsilon_1^2)^\ell
\]
\[
\sim \alpha(k) \eta \sigma^{k-3} (m(d))^k n^{3d+1} n^{2d+\ell} \left( \frac{1}{3d+1} + \int_0^\infty (x + 1)^d - x^d \right)^{3} dx \left( \frac{1}{2d+1} + \int_0^\infty (x + 1)^d - x^d \right)^{2} dx \right)^\ell
\]
\[
= \alpha(k) \eta \sigma^{k-3} (m(d))^k n^{kd+\frac{k-3}{2}},
\]
which completes the proof of the second part of the theorem.

**5.2 Proof of Theorem**

We first introduce the following notations.
\[
\hat{\Delta}(h) = \frac{1}{n} \sum_{j=1}^{n-h} [(X_j - \mu)^2(X_{j+h} - \mu) + (X_j - \mu)(X_{j+h} - \mu)^2],
\]
and
\[
\hat{\Delta}(h, h') = \frac{1}{n} \sum_{j=1}^{n-h} (X_j - \mu)(X_{j+h} - \mu)(X_{j+h+h'} - \mu).
\]

Let
\[
\hat{S}_3 = q_1^{-3d} \Delta(0) + 3q_2^{-3d} \sum_{h=1}^{q_2} \left( 1 - \frac{h}{q_2} \right) \Delta(h) + 6q_3^{-3d} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \left( 1 - \frac{h + h'}{q_3} \right) \Delta(h, h').
\]

If \( \mathbb{E}(\epsilon_1^\theta) < \infty \) then, as \( n \to \infty \), we have
\[
\mathbb{S}_3 - \hat{S}_3 = o_P(1).
\]
Equation (10) follows from the law of large numbers for the sample mean of linear processes.

In view of (10), the conclusion of this theorem is equivalent to

$$\hat{S}_3 - S_3 = o_P(1).$$ (11)

To prove (11), we assume that $\mu = 0$ and, to make notations lighter, we will drop the coefficients 
$(1 - h/n), (1 - h/q_i), (1 - (h + h')/n), (1 - (h + h')/q_i)$ and replace all of them by their limit 
(as $n, q_i \to \infty$) 1. Also, from the proof of Theorem 1, we have for $0 < d < 1/2$,

$$\Delta(h) = \eta \left( \sum_{i=0}^{\infty} a_i^2 a_{i+h} + \sum_{i=0}^{\infty} a_i a_{i+h} \right) \sim E(X_1^2 X_{1+h}).$$

By virtue of the preceding asymptotic equivalence, we take

$$\hat{\Delta}(h) = E(X_1^2 X_{1+h}),$$

and accordingly we will also consider that

$$\hat{\Delta}(h) = \frac{1}{n} \sum_{j=1}^{n-h} X_j^2 X_{j+h}.$$ (12)

Clearly $\hat{\Delta}(0) \to \Delta(0)$, so, we just need to deal with the two remaining terms in $\hat{S}_3$. That is we

need to show that

$$q_2^{-d} \left( \sum_{h=1}^{q_2} \hat{\Delta}(h) - \sum_{h=1}^{q_2} \Delta(h) \right) = o_P(1)$$ (13)

and that

$$q_3^{-3d} \left( \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3-h} \hat{\Delta}(h, h') - \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3-h} \Delta(h, h') \right) = o_P(1).$$ (14)

To prove (13), without loss of generality and just to avoid the use of absolute value at different
places, we will assume that $\Delta(h) \geq 0$ and $a_j \geq 0$ and $a_j$ are nonincreasing. Then, in view of
(12), we have

$$q_2^{-d} \left| \sum_{h=1}^{q_2} \mathbb{E} \left( \hat{\Delta}(h) - \Delta(h) \right) \right| = q_2^{-d} \sum_{h=1}^{q_2} \frac{h}{n} \Delta(h) \leq \frac{q_2}{n} q_2^{-d} \sum_{h=1}^{q_2} \Delta(h) = O(q_2/n).$$

Consequently,

$$q_2^{-2d} \mathbb{E} \left( \sum_{h=1}^{q_2} \left( \hat{\Delta}(h) - \Delta(h) \right)^2 \right) \sim q_2^{-2d} \text{Var} \left( \sum_{h=1}^{q_2} \hat{\Delta}(h) \right) \sim q_2^{-2d} \mathbb{E} \left( \left( \sum_{h=1}^{q_2} \hat{\Delta}(h) \right)^2 \right) - q_2^{-2d} \left( \sum_{h=1}^{q_2} \Delta(h) \right)^2.$$ (15)
We now consider the configurations can be treated in the same way.

For the first term of the right hand side of (15) we write

\[
q_2^{-d} \sum_{h=1}^{q_2} \Delta(h) = q_2^{-d} \sum_{h=1}^{q_2} \mathbb{E} \left[ \left( \sum_{i=0}^{\infty} a_i \varepsilon_{1-h} \right)^2 \right] = \eta q_2^{-d} \sum_{h=1}^{q_2} \sum_{j=0}^{\infty} a_j^2 a_{k+h} \to \eta c_1(d).
\]

For the first term of the right hand side of (15) we write

\[
q_2^{-2d} \mathbb{E} \left( \sum_{h=1}^{q_2} \Delta(h) \right)^2 = q_2^{-2d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \mathbb{E} (\Delta(h) \Delta(h')) = \frac{q_2^{-2d}}{n^2} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{i=1}^{n-h} \sum_{j=1}^{n-h'} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{f=0}^{\infty} \sum_{f'=0}^{\infty} a_s a_t a_f a_{s'} a_{t'} a_{f'} \mathbb{E} \left[ \varepsilon_{i-s} \varepsilon_{i-t} \varepsilon_{i+h-f} \varepsilon_{j-s'} \varepsilon_{j-t'} \varepsilon_{j+h'-f'} \right] .
\]

We now show that only the configuration corresponding to the t hree by three equality of indices \( i - s = i - t = i + h - f \neq j - s' = j - t' = j + h' - f' \) will contribute to the limit as \( n \) (and therefore \( q_2 \)) tends to infinity, in the equation (16). This configuration corresponds to the following quantity, with \( \eta = \mathbb{E}(\varepsilon_1^3) \), the subtracted term corresponds to taking off the diagonal elements with \( t' = t + k \), where \( k = j - i \):

\[
\eta^2 q_2^{-2d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \left( \sum_{t=0}^{\infty} a_t^2 a_{t+h} \right) \left( \sum_{t=0}^{\infty} a_t^2 a_{t+h'} \right) - \frac{\eta^2 q_2^{-2d}}{n} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{t=0}^{\infty} \sum_{k=-n}^{n} a_t^2 a_{t+h} a_{t+k} a_{t+h'}.
\]

The preceding term is nonnegative and bounded above by

\[
\frac{q_2^{-2d}}{n} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \Delta(h) \Delta(h') = O(1/n).
\]

As a result, with the configuration \( i - s = i - t = i + h - f \neq j - s' = j - t' = j + h' - f' \), the term on the right hand side of the last equation in (16), as \( n, q_2 \to \infty \) such that \( q_2 = o(n^{1/2}) \), converges to the following nonzero limit:

\[
\lim_{{q_2 \to \infty}} q_2^{-2d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \Delta(h) \Delta(h') = \lim_{{q_2 \to \infty}} \left( q_2^{-d} \sum_{h=1}^{q_2} \Delta(h) \right)^2 .
\]

All other configurations of the indices of the summations on the right hand side of (16), result in asymptotically negligible terms, as \( n, q_2 \to \infty \). In what follows we show the negligibility of selected configurations of the indices, noting that the remaining cases of each configuration can be treated in the same way.

We now consider the configurations \( i - s = i - t = j - s' \neq i + h - f = j - t' = j + h' - f' \), which is equivalent to \( s = s' = t + k \), with \( k = j - i \), \( f = t' + h + k \), \( f' = t' + h' \), and \( i - s = i - t = i + h - f \neq j - s' = j - t' = j + h' - f' \). Defining \( M = (\eta^2 + \mathbb{E}(\varepsilon_1^3)) \), we can
bound the sum resulting from these two configurations by
\[
\frac{M q_2^{-2d}}{n} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{t=0}^{q_2} \sum_{t'=0}^{q_2} a_t^2 a_t+k a_t' a_t'+h-k a_t'+h'
\]
\[
\leq 2 \frac{M q_2^{-2d}}{n} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{t=0}^{q_2} \sum_{t'=0}^{q_2} \Delta(k) \gamma(h') \sim \epsilon q_2^{-2d} q_2 d^2 = o(n^{-1/2}) \rightarrow 0.
\]

Consider now the configuration in (16) where two indices are equal and another four are equal.

An example of such configuration is when \(i-s = i-t \neq i+h-f = j-s' = j-t' = j+h' - f'\) so that \(s = t\) and \(s' = t', f' = t' + h'\) and \(f = t' + h + k\) where \(k = i - j\). This configuration of the indices results in the following quantity
\[
\mathbb{E}(\varepsilon h h') \Delta(u, u')
\]
which is \(O(q_2^2/n)\). This completes the proof of (13).

We now prove (14). Noting that
\[
q_2^{3-6d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \frac{1}{n} \left| \mathbb{E}(\hat{\Delta}(h, h')) - \Delta(h, h') \right| = O\left(\frac{q_2^2}{n^2}\right) \rightarrow 0
\]
it will suffice to prove that
\[
q_2^{3-6d} \mathbb{E}\left(\left(\sum_{h=1}^{n} \sum_{h'=1}^{q_2} \hat{\Delta}(h, h')\right)^2\right) - q_2^{3-6d} \left(\sum_{h=1}^{q_2} \sum_{h=1}^{q_2} \Delta(h, h')\right)^2 \rightarrow 0. \tag{17}
\]

We have
\[
\mathbb{E}(\hat{\Delta}(h, h') \hat{\Delta}(u, u'))
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n-h-h'-n-u-u'} \sum_{j=1}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{s'=0}^{\infty} \sum_{t'=0}^{\infty} \sum_{a_t a_{t'} a_{s} a_{s'} a_{t'} a_{t'} \epsilon_i \epsilon_i + h+h' + f \epsilon_j \epsilon_j + u+u'} \mathbb{E}\left[\varepsilon_i \varepsilon_i + h+h' + f \epsilon_j \epsilon_j + u+u'\right]. \tag{18}
\]

We show that the three by three configuration corresponding to \(i-t = i+h-s = i+h+h'-f \neq j-t' = j+u-s' = j+u+u'-f'\) will coincide in the limit with
\[
q_2^{3-6d} \left(\sum_{h=1}^{q_2} \sum_{h=1}^{q_2} \Delta(h, h')\right)^2.
\]

First, summing over \(h, h', u, u', i, j, k\), this configuration of (18) results in the following quantity (subtracting the diagonal elements with \(k = j - i\))
\[
\eta^2 q_2^{3-6d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{u=1}^{q_2} \sum_{u'=1}^{q_2} \sum_{t=0}^{\infty} \sum_{t'=0}^{\infty} a_t a_{t+h} a_{t+h} a_{t'+u} + u a_{t'+u} + u + u' a_{t'+u} + u + u' \]
\[
- \eta^2 q_2^{3-6d} \sum_{h=1}^{q_2} \sum_{h'=1}^{q_2} \sum_{u=1}^{q_2} \sum_{u'=1}^{q_2} \sum_{t=0}^{\infty} \sum_{t'=0}^{\infty} a_t a_{t+h} a_{t+h} a_{t+k} a_{t+k} a_{t+k} a_{t+k} a_{t+k} a_{t+k} a_{t+k}. \tag{19}
\]
The second term in (19) is nonnegative and bounded by
\[
\frac{q_3^{-6d}}{n} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u=1}^{q_3} \sum_{u'=1}^{q_3} \Delta(h, h') \Delta(u, u') = O(1/n) \to 0.
\]

The first term in (19) equals
\[
\left( q_3^{-3d} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \Delta(h, h') \right)^2.
\]

Therefore, in order to complete the proof of (17), it suffices to show that all the other configurations of (18) result in quantities converging to zero. Here again we will show this for selected cases of each remaining configuration. In the three by three configuration, when we take \(i - t = i + h = s = j - t'\) and \(i + h + h' = j + u + s' = j + h + h'\), (note that this also includes the all-indices-are-equal configuration), with \(k = j - i\), we see that, with \(M = E(e_1^2) + \eta^2\). Such a configuration results in a quantity bounded by
\[
\sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u=1}^{q_3} \sum_{u'=1}^{q_3} Mq_3^{6d} \frac{n}{n} \sum_{k=-n}^{n} \sum_{u=0}^{\infty} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_t a_t + h a_t + k a_f + k + u - (h + h') a_f + k + u + u' - (h + h')
\]
\[
= Mq_3^{-6d} \frac{n}{n} \sum_{u=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u'=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_t a_t + h a_t + k a_f + k + u - (h + h') a_f + k + u + u' - (h + h')
\]
\[
+ Mq_3^{-6d} \frac{n}{n} \sum_{u=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u'=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_t a_t + h a_t + k a_f + k + u - (h + h') a_f + k + u + u' - (h + h')
\]
\[
\leq c Mq_3^{-6d} \frac{n}{n} \sum_{u=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u'=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} n^d (h') (h'') = O \left( \frac{q_3^{-6d}}{n} \frac{2 + 4d}{n^d} \right) = O \left( \frac{q_3^2}{n} \right)^{1-d} \to 0.
\]

As to showing the negligibility of the two by four configuration of (18) we consider, for example, the case when the first two indices equal and the remaining four are equal. For this, we get (with \(k = j - i\)) a term bounded by
\[
\frac{E(e_1^2) E(e_1^4) q_3^{-6d}}{n} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=-n}^{n} \sum_{k=0}^{\infty} a_t a_t + h a_t + k a_f + a_f + u a_f + u + u'
\]
\[
\leq 2 \frac{E(e_1^2) E(e_1^4) q_3^{-6d}}{n} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_t a_t + h a_t + k a_f + a_f + u a_f + u + u'
\]
\[
= 2 \frac{E(e_1^2) E(e_1^4) q_3^{-6d}}{n} \sum_{h=1}^{q_3} \sum_{h'=1}^{q_3} \sum_{u=1}^{q_3} \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_k (h) (h') = O \left( \frac{q_3^2 q_3^{2d} q_3^{3d}}{q_3^{4d} n} \right) = O \left( \frac{q_3}{n} \right)^{1-d} \to 0.
\]

This completes the proof of (17) and that of (14).

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