The Moduli Space of $\mathbb{CP}^1$ Stringy Cosmic Lumps

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ABSTRACT

We examine the low-energy dynamics of $\mathbb{CP}^1$ lumps coupled to gravity, taking into account the gravitational back-reaction of the spacetime geometry. We show that the single lump moduli space is equipped with a three-dimensional metric, and we derive stability bounds on the scalar coupling constant. We also derive an expression for the multi-lump moduli space metric.
1. Introduction

One particularly useful method of probing the low energy physics of objects arising in M-theory is the moduli space approximation. This has been particularly apparent in the determining of the low energy dynamics of black holes of $\mathcal{N} = 2$ supergravity theories in four and five dimensions which are associated with intersecting brane configurations in ten and eleven dimensions compactified on Calabi-Yau manifolds [1, 2, 3, 4, 5]. Moreover, these geometries exhibit a sufficiently high degree of symmetry that the associated sigma models typically admit supersymmetric extensions. On quantizing these supersymmetric sigma models, interesting results such as the existence of bound states in a near horizon limit have been obtained [6].

In addition to black holes, there are other interesting supergravity solutions which are obtained from branes. In particular, we shall concentrate on a class of stringy cosmic string solutions of four dimensional supergravity theories [7, 8]. Such solutions may be obtained in various different ways. For example, one may wrap an M5-brane on a supersymmetric co-associative 4-cycle of a 7-manifold of $G_2$ holonomy. Compactifying along the $G_2$ manifold, one obtains string solutions of $\mathcal{N} = 1$, $D = 4$ supergravity theory [9]. Alternatively, stringy cosmic string solutions of a four-dimensional $\mathcal{N} = 2$ theory may be obtained from compactifying heterotic string theory on $K_3 \times T^2$ [10]. Compactifying further in the direction parallel to the string, one obtains lump solutions of a 3-dimensional theory.

In this paper we shall examine the low energy dynamics of lump solutions of 3-dimensional gravity coupled to a complex scalar $\mathbb{CP}^1$ sigma model obtained from the compactification of $\mathcal{N} = 2$, $D = 4$ supergravity, in which the scalar is a holomorphic rational function. A considerable amount is known about the moduli space geometries of such sigma models in various fixed backgrounds [11, 12, 13]. One feature is that some of the moduli are frozen due to divergences in the integrals involved in the computations. It has been argued in [14] that coupling the sigma model to gravity provides a natural way of fixing an appropriate background
geometry, and that this geometry effectively un-freezes some of the moduli. A probe computation was used to derive a six-dimensional metric on the moduli space of a single lump. Although such computations provide useful information about the low energy dynamics of objects in supergravity theories, they do not generically take into account the back reaction of the geometries to the motion of the bodies involved, and so many aspects of the dynamics are neglected. This may be readily observed in the case of low energy black hole dynamics.

The plan of this paper is as follows. In section 2 we describe some aspects of the static solution. In section 3 we present the moduli space perturbation computation including the back-reaction of the 2+1 dimensional spacetime, and show how the single lump moduli space is described by a 3-dimensional conical geometry. In section 4 we present some conclusions.

2. The Static Solution

The action which we shall consider is that of a non-linear sigma model coupled to gravity in 2+1 dimensions. It is given by

$$S = \int d^3x \sqrt{|g|} \left( R - 4\lambda(1 + |W|^2)^{-2}\partial_M W \partial^M \bar{W} \right)$$  \hspace{1cm} (1)

where $\lambda > 0$ is a real constant, $g$ is the metric, and $W$ is a complex scalar. The Einstein equations are given by

$$G_{MN} + 2\lambda(1 + |W|^2)^{-2}(\partial_S W \partial^S \bar{W} g_{MN} - 2\partial_{(M} W \partial_{N)} \bar{W}) = 0$$  \hspace{1cm} (2)

and the scalar equation obtained by varying $W$ is

$$\partial_M \left( \sqrt{|g|}(1 + |W|^2)^{-2} \partial^M \bar{W} \right) + 2\sqrt{|g|} |W|(1 + |W|^2)^{-3} \partial_M W \partial^M \bar{W} = 0 ,$$  \hspace{1cm} (3)

and taking the complex conjugate of this one obtains the $\bar{W}$ scalar equation.
From now on, we shall adopt co-ordinates $x^M = [t, z, \bar{z}]$, where $z$, $\bar{z}$ are complex co-ordinates, and $t$ is real. The solution which corresponds to $N \mathbb{CP}^1$ lumps is given by taking

$$W = \frac{P}{Q}$$

(4)

where $P = P(z)$ and $Q = Q(z)$ are degree $N$ polynomials in $z$ with no common roots. The spacetime metric is given by

$$ds^2 = -dt^2 + \Omega^2 dzd\bar{z}$$

(5)

where

$$\Omega = (|P|^2 + |Q|^2)^{-\lambda}.$$  

(6)

3. Moduli Metric Computation

In order to perturb the $N$-lump system we allow the coefficients of $P$ and $Q$ to depend on $t$. Such a solution automatically satisfies the scalar equations up to first order in the velocities. Hence, for a sigma model which is not coupled to gravity, it is sufficient to simply substitute this solution back into the action and obtain the terms second order in the time derivatives, which in turn defines the metric on the moduli space. However, when one couples the sigma model to gravity the Einstein equations must also be considered. It is straightforward to see that the $tz$ and $t\bar{z}$ components are not satisfied to first order. In order to rectify this it is necessary to perturb the $tz$ and $t\bar{z}$ components of the metric according to

$$\delta g_{tz} = p_z$$
$$\delta g_{t\bar{z}} = p_{\bar{z}}$$

(7)

where $p_z$ and $p_{\bar{z}}$ are first order in the velocities and are to be determined by solving
the Einstein equations. Specifically, \( p \) must satisfy
\[
\partial_z [\Omega^{-2}(\partial_z p - \partial_z \bar{p}) + \lambda(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-1}(\overline{\mathcal{P}}\partial_t \mathcal{P} - \mathcal{P} \partial_t \overline{\mathcal{P}} + \overline{\mathcal{Q}}\partial_t \mathcal{Q} - \mathcal{Q} \partial_t \overline{\mathcal{Q}})] = 0 ,
\]
(8)
together with the complex conjugate of this expression. This is solved by taking
\[
\Omega^{-2}(\partial_z p - \partial_z \bar{p}) + \lambda(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-1}(\overline{\mathcal{P}}\partial_t \mathcal{P} - \mathcal{P} \partial_t \overline{\mathcal{P}} + \overline{\mathcal{Q}}\partial_t \mathcal{Q} - \mathcal{Q} \partial_t \overline{\mathcal{Q}}) = i\mathcal{L}
\]
(9)
where \( \mathcal{L} \) is real, independent of \( z \) and \( \bar{z} \) and is first order in the velocities. Defining
\[
\mathcal{D} \equiv \mathcal{L}(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-2\lambda + i\lambda(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-2\lambda - 1}(\overline{\mathcal{P}}\partial_t \mathcal{P} - \mathcal{P} \partial_t \overline{\mathcal{P}} + \overline{\mathcal{Q}}\partial_t \mathcal{Q} - \mathcal{Q} \partial_t \overline{\mathcal{Q}}) ,
\]
(10)
we note that (9) is solved by finding \( V = V(z, \bar{z}) \in \mathbb{C} \) such that
\[
\partial_z V = \frac{i}{2} \mathcal{D} ,
\]
(11)
and \( p \) is given by
\[
p_z = \bar{V} + \partial_z \mathcal{T}
p_{\bar{z}} = V + \partial_{\bar{z}} \mathcal{T} ,
\]
(12)
where \( \mathcal{T} = \mathcal{T}(z, \bar{z}) \in \mathbb{R} \) is arbitrary. It is then straightforward to compute the metric on the moduli space. In particular, substituting (7) into (1) one obtains the following second order terms
\[
S^{(2)} = \int dt \, dz \, d\bar{z} \left[ \mathcal{F} + \lambda(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-2\lambda - 2}((|\mathcal{Q}|^2 - \lambda|\mathcal{P}|^2)\partial_t \mathcal{P}\partial_t \overline{\mathcal{P}}
+ (|\mathcal{P}|^2 - \lambda|\mathcal{Q}|^2)\partial_t \mathcal{Q}\partial_t \overline{\mathcal{Q}} - (1 + \lambda)(\mathcal{Q}\partial_t \overline{\mathcal{Q}}\partial_t \mathcal{P} + \mathcal{P}\partial_t \overline{\mathcal{P}}\partial_t \mathcal{Q})
- \Omega^{-2}(\partial_z p_{\bar{z}} - \partial_{\bar{z}} \mathcal{P}_z + \lambda(|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-2\lambda - 1}(\overline{\mathcal{P}}\partial_t \mathcal{P} - \mathcal{P} \partial_t \overline{\mathcal{P}} + \overline{\mathcal{Q}}\partial_t \mathcal{Q} - \mathcal{Q} \partial_t \overline{\mathcal{Q}}))\right]^2 ,
\]
(13)
where \( \mathcal{F} \) integrates to give surface terms only;
\[
\mathcal{F} = -4\partial_z \partial_{\bar{z}}((|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{2\lambda}p_z p_{\bar{z}}) + 4\partial_z ((|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{2\lambda}p_z \partial_z p_{\bar{z}}
+ 4\partial_{\bar{z}}((|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{2\lambda}p_\bar{z} \partial_\bar{z} p_z + 4\lambda \partial \partial_z ((|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-1}p_z (\mathcal{P}\partial_t \mathcal{P} + \overline{\mathcal{Q}}\partial_t \mathcal{Q})
+ 4\lambda \partial \partial_{\bar{z}} ((|\mathcal{P}|^2 + |\mathcal{Q}|^2)^{-1}p_{\bar{z}} (\mathcal{P}\partial_t \mathcal{P} + \overline{\mathcal{Q}}\partial_t \mathcal{Q} )) .
\]
(14)
In particular, we shall only consider solutions in which the contribution from \( \mathcal{F} \)
vanishes. Then substituting (9) into (13) we obtain the second order effective action

\[
S_{(2)} = \int dt \, dz \, d\bar{z} \left[ -4\lambda^2 (|P|^2 + |Q|^2)^{-2\lambda - 1}(|\partial_t P|^2 + |\partial_t Q|^2) \\
+ 2\lambda(1 + 2\lambda)(|P|^2 + |Q|^2)^{-2\lambda - 2}|P\partial_t Q - Q\partial_t P|^2 + (|P|^2 + |Q|^2)^{-2\lambda}L^2 \right].
\]

(15)

Here \( L \) is to be determined by the requirement that \( p \) is smooth, bounded and \( F \) gives no contribution to the moduli space metric. It is possible to determine \( p \) and \( L \) explicitly in the simplest case of a single lump, where the polynomials \( P \) and \( Q \) are of degree 1.

### 3.1. Moduli Space of a \( \mathbb{C}P^1 \) Lump

It is straightforward to use the reasoning set out in the previous section to compute the metric on the moduli space of a single \( \mathbb{C}P^1 \) lump. In this case \( P \) and \( Q \) are degree 1 polynomials, and it is most convenient to write them as

\[
P = \sin \frac{\theta}{2} e^{-\frac{i}{2}(\psi - \phi)}(z - \gamma) + \beta \cos \frac{\theta}{2} e^{\frac{i}{2}(\psi + \phi)}
\]

\[
Q = \cos \frac{\theta}{2} e^{-\frac{i}{2}(\psi + \phi)}(z - \gamma) - \beta \sin \frac{\theta}{2} e^{\frac{i}{2}(\psi - \phi)},
\]

(16)

where \( \gamma \in \mathbb{C} \), \( \beta \in \mathbb{R}^+ \), and \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi < 4\pi \). Then

\[
|P|^2 + |Q|^2 = \beta^2 + |z - \gamma|^2,
\]

(17)

and

\[
P\partial_t P - P\partial_t \bar{P} + Q\partial_t Q - Q\partial_t \bar{Q} = i\xi(\beta^2 - |z - \gamma|^2) + \bar{\chi}(z - \gamma) - \chi(\bar{z} - \bar{\gamma}),
\]

(18)

where

\[
\xi = \dot{\psi} + \cos \theta \dot{\phi},
\]

(19)
\[ \chi = \beta e^{i\psi}(\dot{\theta} - i \sin \theta \dot{\phi}) + \dot{\gamma} . \]  \hspace{1cm} (20)

Then (9) is solved by taking
\[ V = i \frac{2}{(\bar{z} - \gamma)} (\beta^2 + |z - \gamma|^2)^{-2\lambda} \left[ \xi \beta^2 + \frac{\mathcal{L} + \xi \lambda}{1 - 2\lambda} (\beta^2 + |z - \gamma|^2) \right] \]
\[ - \frac{\chi}{4} (\beta^2 + |z - \gamma|^2)^{-2\lambda} - \frac{\bar{\chi}}{4(1 - 2\lambda)(\bar{z} - \gamma)^2} (\beta^2 + |z - \gamma|^2)^{-2\lambda} (\beta^2 + 2\lambda |z - \gamma|^2) \]
\[ T = \frac{1}{4(2\lambda - 1)} (\beta^2 + |z - \gamma|^2)^{-2\lambda + 1} \left( \frac{\chi}{z - \gamma} + \frac{\bar{\chi}}{\bar{z} - \gamma} \right), \]  \hspace{1cm} (21)

so that
\[ p_z = -\frac{\bar{\chi}}{2} (\beta^2 + |z - \gamma|^2)^{-2\lambda} - \frac{i}{2(\bar{z} - \gamma)} \left[ \xi \beta^2 (\beta^2 + |z - \gamma|^2)^{-2\lambda} \right. \]
\[ + (1 - 2\lambda)^{-1} (\mathcal{L} + \xi \lambda) (\beta^2 + |z - \gamma|^2)^{-2\lambda + 1} \]
\[ = -\frac{\chi}{2} (\beta^2 + |z - \gamma|^2)^{-2\lambda} + \frac{i}{2(z - \gamma)} \left[ \xi \beta^2 (\beta^2 + |z - \gamma|^2)^{-2\lambda} \right. \]
\[ + (1 - 2\lambda)^{-1} (\mathcal{L} + \xi \lambda) (\beta^2 + |z - \gamma|^2)^{-2\lambda + 1} \]  \hspace{1cm} (22)

In order to ensure that \( p \) remains bounded at \( z = \gamma \) we set \( \mathcal{L} = (\lambda - 1)\xi \) so that
\[ p_z = \frac{1}{2} (\beta^2 + |z - \gamma|^2)^{-2\lambda} \left( -\bar{\chi} + i\xi (\bar{z} - \gamma) \right) \]
\[ p_{\bar{z}} = \frac{1}{2} (\beta^2 + |z - \gamma|^2)^{-2\lambda} \left( -\chi - i\xi (z - \gamma) \right). \]  \hspace{1cm} (23)

We also impose \( \lambda > \frac{1}{2} \) which ensures that the surface terms \( \mathcal{F} \) give no contribution to the moduli space metric. In this case the volume of space takes the finite value \( \frac{\pi \beta^2}{2\lambda - 1} \), and the boundary at \( |z| = \infty \) is a conical singularity of deficit angle \( 4\pi (1 - \lambda) \). So, on substituting this value of \( \mathcal{L} \) into (15) and performing the spatial integration we obtain
\[ S_{(2)} = \pi \int dt (1 - 2\lambda) \beta^{-4\lambda} \dot{\beta}^2 + \frac{(1 - \lambda)}{2(2\lambda - 1)} \beta^{2 - 4\lambda} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2). \]  \hspace{1cm} (24)

Defining \( r = \beta^{1 - 2\lambda} \), we obtain the metric on the moduli space (up to an overall
constant conformal factor of $\frac{\pi}{1-2\lambda}$ as

$$ds^2 = dr^2 + \frac{1}{2}(\lambda - 1)r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$  \hfill (25)

We observe that for $\lambda = 3$ we have $ds^2 = ds^2(\mathbb{E}^3)$, and for $\lambda = 1$ the metric is degenerate. For $\lambda \neq 1$, one may solve the geodesic equations explicitly; in particular, if $J^2 = r^4(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2)$ denotes the total conserved angular momentum, then $\beta$ is given by

$$\beta = \frac{1}{\sqrt{\alpha^{-1}\left[\frac{1}{2}(\lambda - 1)J^2 + \alpha^2(t + \delta)^2\right]^\frac{1}{\lambda - 1}}} \hfill (26)$$

where $\alpha \in \mathbb{R}^+$ and $\delta \in \mathbb{R}$ are constants of integration. Hence, if $\frac{1}{2} < \lambda < 1$ then the moduli space approximation breaks down at some finite time, at which the spatial volume becomes infinite and this is a naked curvature singularity. However, if $\lambda > 1$ then the moduli space approximation is generically valid for all time and the volume of space is bounded above and tends to zero as $t \to \pm \infty$. In particular, we recall that (1) may be obtained from a four-dimensional action via an appropriate compactification, and it has been shown in [15] that in order for the scalar sigma model portion of this action to describe a Kähler-Hodge manifold, one must have $\lambda \in \mathbb{N}$. So, for $\lambda \neq 1$, it follows that the moduli metrics corresponding to single lump solutions of supersymmetric theories have a conical geometry, except for $\lambda = 3$ when $ds^2 = ds^2(\mathbb{E}^3)$.

### 3.2. Multi-Lump Problem

It is also possible to use the reasoning presented here to compute the metric on the moduli space of multi-lump solutions. In fact, it is not even necessary to compute $p$ explicitly to determine this. To see this we note that (9) may be written in a form notation as

$$dp = 2D dx \wedge dy \hfill (27)$$

where $z = z + iy$, $\bar{z} = x - iy$ and $x, y$ are real co-ordinates. Then for smooth $p$ such that $|p| \sim |z|^{-c}$ for $c > 1$ as $|z| \to \infty$, applying Stokes' theorem to (27) one
obtains the constraint
\[ \int \mathcal{D}(x, y) \, dxdy = 0 \] (28)
which fixes \( \mathcal{L} \) uniquely as
\[ \mathcal{L} = -i\lambda \left[ \int dxdy \left( |P|^2 + |Q|^2 \right)^{-2\lambda-1} \left( \bar{P} \partial_t P - P \partial_t \bar{P} + \bar{Q} \partial_t Q - Q \partial_t \bar{Q} \right) \right] \left( \int dxdy \left( |P|^2 + |Q|^2 \right)^{-2\lambda} \right)^{-1}. \] (29)

We observe that the constraint \( \lambda > \frac{1}{2N} \) is sufficient to ensure the appropriate asymptotic behaviour of \( p \) as \( |z| \to \infty \). This constraint is satisfied if \( \lambda \in \mathbb{N} \), which corresponds to supersymmetric theories which are of most interest. We shall only consider such \( \lambda \). Even with this simplification, evaluating the integrals explicitly for generic \( N \geq 2 \) and \( \lambda \in \mathbb{N} \) is difficult. It is however instructive to examine the two-body problem. In particular, suppose we consider the symmetric configuration given by taking
\[ P = \sin \left( \frac{\theta}{2} \right) e^{-\frac{i}{2}(\psi-\phi)} H + \beta \cos \left( \frac{\theta}{2} \right) e^{\frac{i}{2}(\psi+\phi)} \]
\[ Q = \cos \left( \frac{\theta}{2} \right) e^{-\frac{i}{2}(\psi+\phi)} H - \beta \sin \left( \frac{\theta}{2} \right) e^{\frac{i}{2}(\psi-\phi)} , \] (30)
where \( \beta \in \mathbb{R}^+ \), \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \), \( 0 \leq \psi < 4\pi \) and
\[ H = z^2 - \gamma \] (31)
for \( \gamma \in \mathbb{C} \). The lumps are centred around \( z = \pm \sqrt{\gamma} \). Then
\[ |P|^2 + |Q|^2 = \beta^2 + |H|^2 , \] (32)
and
\[ \bar{P} \partial_t P - P \partial_t \bar{P} + \bar{Q} \partial_t Q - Q \partial_t \bar{Q} = i\xi (\beta^2 - |H|^2) + \bar{\chi} H - \chi \bar{H} , \] (33)
where
\[ \xi = \dot{\psi} + \cos \theta \dot{\phi} , \] (34)
and
\[ \chi = \beta e^{i\psi}(\dot{\theta} - i \sin \theta \dot{\phi}) + \dot{\gamma} \quad (35) \]

It is convenient to define for \( c \geq 1 \),
\[ \Upsilon_c = \int dx dy (\beta^2 + |H|^2)^{-c} \quad (36) \]

With these conventions, \( \mathcal{L} \) is fixed as
\[ \mathcal{L} = \lambda \xi \left( \frac{2 \beta^2 \Upsilon_{2\lambda+1}}{\Upsilon_{2\lambda}} - 1 \right) \quad (37) \]

and it follows that
\[
S_{(2)} = \lambda \int dt \left[ \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \left( \frac{1}{2} \Upsilon_{2\lambda} - (1 + 2\lambda)\beta^2 \Upsilon_{2\lambda+1} + (1 + 2\lambda)\beta^4 \Upsilon_{2\lambda+2} \right) \\
+ 2\dot{\beta}^2 \left( \Upsilon_{2\lambda+1} - (1 + 2\lambda)\beta^2 \Upsilon_{2\lambda+2} \right) + (\dot{\gamma} \bar{x} + \dot{x} \bar{\gamma})(-2\lambda \Upsilon_{2\lambda+1} + (1 + 2\lambda)\beta^2 \Upsilon_{2\lambda+2}) \\
+ 2\beta^2 \xi^2 \left( \Upsilon_{2\lambda+1} - (1 + 2\lambda)\beta^2 \Upsilon_{2\lambda+2} + 2\lambda \beta^2 \frac{\Upsilon_{2\lambda+1}^2}{\Upsilon_{2\lambda}} \right) \right] . \quad (38)
\]

We observe that the double integral in (36) may be rewritten for \( c \in \mathbb{N} \) as
\[ \Upsilon_c = \frac{\pi^2 (2c - 2)!}{2^{2c-1}((c-1)!)^2} \beta^{1-2c} \, _2F_1 \left( \frac{1}{2}, c - \frac{1}{2}; 1; -\frac{|\gamma|^2}{\beta^2} \right) , \quad (39) \]

where \(_2F_1\) denotes the analytic continuation of the hypergeometric function defined on \( \mathbb{C}/[1, \infty) \) with the branch cut along the positive real axis.
4. Conclusions

We have shown that the second order effective action which determines the low energy dynamics of a single $\mathbb{CP}^1$ lump is simplified considerably when one includes the back reaction of the spacetime geometry. In particular, for stable solutions, we require that $\lambda > 1$. In the case of $\lambda = 3$ it is clear from the reasoning in [16] that the moduli space metric $ds^2(E^3)$ admits a supersymmetric extension to give a $\mathcal{N} = 4$ supersymmetric sigma model, even though there are only three moduli involved in (25). For $\lambda = 3$, this result is consistent with the correspondence between moduli metrics of supersymmetric gravity and brane world-volume theories and supersymmetric sigma models. For $\lambda \neq 3$, the moduli metric is not of the form given in [16], and the correspondence is less clear.

In addition, we have computed the metric on the moduli space for a symmetric configuration of two lumps. The metric cannot be written in a simple closed form. It would be interesting to determine if there is a relationship between this metric and the class of $SO(3) \times SO(3)$ invariant Kähler metrics discussed in [17] as it is clear that including the gravitational back-reaction alters the form of the metrics. Furthermore, the methods used here may be easily extended to probe the moduli dynamics of a symmetric configuration of $N$ lumps by making the replacement $z^2 - \gamma \rightarrow z^N - \gamma$ in (31). It may also be possible to determine some of the properties of more general non-symmetric $N$-lump moduli space geometries for particular values of the coupling $\lambda$ from the integral expressions presented here, using techniques similar to those used to analyse black hole moduli space geometries.

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REFERENCES

1. G. W. Gibbons and P. J. Ruback, *The Motion of extreme Reissner-Nordström black holes in the low velocity limit*, Phys. Rev. Lett. **57** (1986) 1492.

2. R. C. Ferrell and D. M. Eardley, *Slow motion scattering and coalescence of maximally charged black holes*, Phys. Rev. Lett. **59** (1987) 1617.

3. J. Michelson and A. Strominger, *Superconformal Multi-Black Hole Quantum Mechanics*, JHEP 9909 (1999) 005; [hep-th/9908044](http://arxiv.org/abs/hep-th/9908044).

4. J. Gutowski and G. Papadopoulos, *The Dynamics of Very Special Black Holes*, Phys. Lett. **B472** (2000) 45; [hep-th/9910022](http://arxiv.org/abs/hep-th/9910022).

5. J. Gutowski and G. Papadopoulos, *Moduli Spaces for Four- and Five-Dimensional Black Holes*, Phys. Rev. **D62** (2000) 064023; [hep-th/0002242](http://arxiv.org/abs/hep-th/0002242).

6. R. Britto-Pacumio, A. Strominger and A. Volovich, *Two-Black-Hole Bound States*, JHEP 0005 (2000) 020; [hep-th/0004017](http://arxiv.org/abs/hep-th/0004017).

7. A. Comtet and G. W. Gibbons, *Bogomolny Bounds for Cosmic Strings*, Nucl. Phys. **B299**: 719 (1988).

8. B. R. Greene, A. Shapere, C. Vafa and S-T Yau, *Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds*, Nucl. Phys. **B337**:1 (1990).

9. J. Gutowski and G. Papadopoulos, *Moduli Spaces and Brane Solitons for M-Theory Compactifications on Holonomy G2 Manifolds*; [hep-th/0104105](http://arxiv.org/abs/hep-th/0104105).

10. A. Kehagias, *N=2 Heterotic Stringy Cosmic Strings*; [hep-th/9611110](http://arxiv.org/abs/hep-th/9611110).

11. R. S. Ward, *Slowly Moving Lumps in the CP1 Model in (2+1) dimensions*, Phys. Lett. **B158** (1985) 424.

12. J. M. Speight, *Lump Dynamics in the CP1 model on the torus*, Commun. Math. Phys. **194** (1998) 513; [hep-th/9707101](http://arxiv.org/abs/hep-th/9707101).

13. J. M. Speight, *Low Energy Dynamics of a CP1 lump on the sphere*, Phys. Rev. **D49** (1994) 6914; [hep-th/9712089](http://arxiv.org/abs/hep-th/9712089).
14. J. M. Speight and I. A. B. Strachan, *Gravity Thaws the frozen Moduli of the \( \mathbb{CP}^1 \) lump*, Phys. Lett. **B457** (1999) 12; hep-th/9903264.

15. E. Witten and J. Bagger, *Quantization of Newton’s Constant in Certain Supergravity Theories*, Phys. Lett. **B115** (1982) 202.

16. A. Maloney, M. Spradlin and A. Strominger, *Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions*; hep-th/9911001.

17. J. M. Speight, *The \( L^2 \) Geometry of spaces of harmonic maps \( S^2 \rightarrow S^2 \) and \( \mathbb{RP}^2 \rightarrow \mathbb{RP}^2 \)*; math.DG/0102038.