Stability analysis and numerical simulation of a composite laminated piezoelectric rectangular plate system

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Abstract. This paper investigates the stability behavior in three-degree-freedom composite laminated piezoelectric rectangular plate system based on the Routh-Hurwitz criterion and the normal form theory. Firstly, the stability behavior of the composite laminated piezoelectric rectangular plate is determined by applying the Routh-Hurwitz criterion. Then, the normal forms of the composite laminated piezoelectric rectangular plate systems for the case of one non-semi-simple double zero and two pairs of pure imaginary eigenvalues are obtained by using the normal form theory. Finally, numerical simulations are carried out to illustrate the main results.

1. Introduction

The general idea of the normal form theory, which was introduced by Poincaré [1], is to simplify the ordinary differential equations by a series of near identity nonlinear transformations. The method of normal forms turns out to be a powerful tool in study of the complex dynamical behavior of nonlinear systems near the equilibrium point.

The computation of the classical normal forms has been studied extensively in [2-4]. Note that the normal form is not unique and it can be further reduced. Further reduction of the classical normal forms is necessary. Jing and Yuan [5] derived a modular algorithm for computing the generalized Hermite normal form of matrices over principal ideal domain $\mathbb{Z}[x]$. Moreover, a powerful device for obtaining the normal forms of nonlinear systems is obtained in reference [6, 7], and the new method is applied to study the bifurcation behavior of nonlinear systems. By using the adjoint operator method, Chen et al. [8] extended the method that is used to compute the normal form for four-dimensional nonlinear systems to the six-dimensional nonlinear systems.

The well-known Hurwitz-Routh criterion [9] is a mathematical criterion in the control theory. It can be used to assess the stability of nonlinear dynamical systems by determining whether the roots of characteristic polynomials of the linear system have negative real parts. Al-Azzawi [10] investigated the bifurcation and stability behavior of a pan chaotic system using the Routh-Hurwitz method. Tan et al. [11] also analyzed the stability boundary of the piezoelectric self-elastic absorber system using the Routh-Hurwitz conditions.

Rectangular laminated composite piezoelectric plates are small structures, which have received a great attention and are increasingly used in aerospace and space structures. A large number of mechanical models related to the piezoelectric rectangular plate have been presented. However, there are few studies on complex dynamical analysis for the high-dimensional piezoelectric rectangular plate system. Li et al. [12] investigated the active control of random vibration for laminated composite
plates using piezoelectric fiber reinforced composites. Zhu and Fu [13] investigated the nonlinear
dynamic responses and fatigue damage evolution of the piezoelectric laminated plates by using the
finite difference method. Zhang et al. [14, 15] studied the nonlinear oscillations of a simply supported
symmetric cross-ply composite laminated piezoelectric plate subjected to the transverse, in-plane and
the excitation excitations. Tanzadeh and Amoushahi [16] discussed the buckling and free vibration of
the laminated piezoelectric composite laminated plates by using the finite strip method.

The next section presents the averaged equations of laminated composite piezoelectric rectangular
plate. The stability of the mechanical system and its normal form are discussed in Section 3.
Conclusion is drawn in Section 4.

2. Averaged equations of the problem

Figure 1 shows a laminated composite piezoelectric rectangular plate. As we can see from Figure 1
that the edge thickness, length and width are $h, a$ and $b$, respectively. The in-plane excitations along
$x$ direction at $y = 0$ and $y$ direction at $x = 0$ are given by $q_0 + q_1 \cos \Omega t$ and $q_i + q_2 \cos \Omega t$
respectively. The transverse excitation is written as $q \cos \Omega t$.

![Figure 1. The mechanical model of composite laminated piezoelectric rectangular plate.](image)

Using the Hamilton principle and Galerkin method, the nonlinear governing equations of motion of the
laminated composite piezoelectric rectangular plate in the dimensionless form can be expressed as follows [15, 17].

\[ \ddot{w}_1 + \epsilon \mu w_1 + \omega^2 w_1 + \epsilon \left( a_1 \cos \Omega t + a_1 \cos \Omega t + a_1 \cos \Omega t \right) w_1 + \epsilon \alpha_1 w_1^3 w_2 + \epsilon \alpha_1 w_1^5 w_3 \\
+ \epsilon \alpha_1 w_1^3 w_2 + \epsilon \alpha_1 w_1^5 w_3 + \epsilon \alpha_1 w_1^5 w_4 + \epsilon \alpha_1 w_1^3 w_5 + \epsilon \alpha_1 w_1^5 w_6 + \epsilon \alpha_1 w_1^5 w_7 + \epsilon \alpha_1 w_1^5 w_8 \]
\[ + \epsilon \alpha_1 w_1^3 w_2 + \epsilon \alpha_1 w_1^5 w_3 + \epsilon \alpha_1 w_1^5 w_4 + \epsilon \alpha_1 w_1^3 w_5 + \epsilon \alpha_1 w_1^5 w_6 + \epsilon \alpha_1 w_1^5 w_7 + \epsilon \alpha_1 w_1^5 w_8, \]  
\[ (1a) \]

\[ \ddot{w}_2 + \epsilon \mu w_2 + \omega^2 w_2 + \epsilon \left( b_1 \cos \Omega t + b_1 \cos \Omega t + b_1 \cos \Omega t \right) w_2 + \epsilon b_1 w_2^3 w_3 + \epsilon b_1 w_2^5 w_5 \\
+ \epsilon b_1 w_2^3 w_3 + \epsilon b_1 w_2^5 w_5 + \epsilon b_1 w_2^3 w_4 + \epsilon b_1 w_2^5 w_6 + \epsilon b_1 w_2^3 w_7 + \epsilon b_1 w_2^5 w_8 \\
+ \epsilon b_1 w_2^3 w_3 + \epsilon b_1 w_2^5 w_5 + \epsilon b_1 w_2^3 w_4 + \epsilon b_1 w_2^5 w_6 + \epsilon b_1 w_2^3 w_7 + \epsilon b_1 w_2^5 w_8 \]
\[ + \epsilon b_1 w_2^3 w_3 + \epsilon b_1 w_2^5 w_5 + \epsilon b_1 w_2^3 w_4 + \epsilon b_1 w_2^5 w_6 + \epsilon b_1 w_2^3 w_7 + \epsilon b_1 w_2^5 w_8 = \epsilon f_1 \cos \Omega t, \]  
\[ (1b) \]

\[ \ddot{w}_3 + \epsilon \mu w_3 + \omega^2 w_3 + \epsilon \left( d_1 \cos \Omega t + d_1 \cos \Omega t + d_1 \cos \Omega t \right) w_3 + \epsilon d_1 w_3^3 w_4 + \epsilon d_1 w_3^5 w_6 \\
+ \epsilon d_1 w_3^3 w_4 + \epsilon d_1 w_3^5 w_6 + \epsilon d_1 w_3^3 w_5 + \epsilon d_1 w_3^5 w_7 + \epsilon d_1 w_3^5 w_8 + \epsilon d_1 w_3^5 w_9 \\
+ \epsilon d_1 w_3^3 w_4 + \epsilon d_1 w_3^5 w_6 + \epsilon d_1 w_3^3 w_5 + \epsilon d_1 w_3^5 w_7 + \epsilon d_1 w_3^5 w_8 + \epsilon d_1 w_3^5 w_9 \]
\[ + \epsilon d_1 w_3^3 w_4 + \epsilon d_1 w_3^5 w_6 + \epsilon d_1 w_3^3 w_5 + \epsilon d_1 w_3^5 w_7 + \epsilon d_1 w_3^5 w_8 + \epsilon d_1 w_3^5 w_9 = \epsilon f_2 \cos \Omega t. \]  
\[ (1c) \]

With the method of multiple scales, Yao et al. transformed the three-degree-of-freedom (3DoF)
system to the following six-dimensional averaged equations:

\[ \dot{x}_1 = -\frac{1}{2} \mu x_1 - \frac{1}{2} \sigma x_2 - \frac{1}{4} \left( a_1 + a_3 + a_4 \right) x_2 + a_1 x_2 \left( x_2^2 + x_2^4 \right) + a_3 x_2 \left( x_3^2 + x_3^4 \right) \\
+ \frac{3}{2} a_1 x_2 \left( x_2^2 + x_2^4 \right) + \frac{1}{2} a_3 \left( -x_1 x_2 x_4 + x_2 x_4 x_6 + x_1 x_3 x_5 + x_2 x_3 x_5 \right), \]  
\[ (2a) \]
\[\dot{x}_2 = -\frac{1}{2} \mu_1 x_2 + \frac{1}{2} \sigma_1 x_1 - \frac{1}{4} (a_1 + a_2 + a_3) x_1 - a_1 x_1 \left( x_1^2 + x_2^2 \right) - a_2 x_1 \left( x_1^2 + x_6^2 \right) - \frac{3}{2} a_3 x_1 \left( x_1^2 + x_2^2 \right) - \frac{1}{4} f_1 - \frac{1}{2} a_{14} \left( x_2 x_3 x_6 + x_1 x_4 x_6 - x_2 x_5 x_5 + x_1 x_3 x_3 \right), \tag{2b}\]

\[\dot{x}_3 = -\frac{1}{2} \mu_3 x_3 - \frac{1}{2} \sigma_3 x_3 + \frac{1}{2} b_3 x_3 \left( x_3^2 + x_2^2 \right) + \frac{1}{2} b_{10} x_3 \left( x_3^2 + x_6^2 \right) + \frac{3}{4} b_{12} x_3 \left( x_3^2 + x_4^2 \right), \tag{2c}\]

\[\dot{x}_4 = \frac{1}{4} \sigma_2 x_4 - \frac{1}{2} \mu_4 x_4 - \frac{1}{2} b_4 x_3 \left( x_3^2 + x_2^2 \right) - \frac{1}{8} f_2 - \frac{1}{4} b_6 \left( x_2^2 x_3 - x_3^2 x_2 - 2 x_1 x_2 x_6 \right) - \frac{1}{2} b_{10} x_3 \left( x_3^2 + x_2^2 \right) - \frac{3}{4} b_{12} x_3 \left( x_3^2 + x_4^2 \right), \tag{2d}\]

\[\dot{x}_5 = -\frac{1}{2} \mu_5 x_5 - \frac{1}{2} \sigma_5 x_5 + \frac{1}{4} d_6 \left( x_6^2 + x_2^2 \right) + \frac{1}{2} d_{10} \left( x_6^2 x_4 - x_2^2 x_4 - 2 x_1 x_2 x_6 \right) + \frac{1}{4} d_{12} x_6 \left( x_6^2 + x_4^2 \right) + \frac{3}{8} d_{11} x_6 \left( x_6^2 + x_6^2 \right), \tag{2e}\]

\[\dot{x}_6 = \frac{1}{8} \sigma_6 x_6 - \frac{1}{2} \mu_6 x_6 - \frac{1}{4} d_6 \left( x_6^2 + x_2^2 \right) - \frac{1}{8} f_3 + \frac{1}{4} d_{10} \left( x_6^2 x_3 - x_2^2 x_3 - 2 x_1 x_2 x_4 \right) - \frac{1}{4} d_{12} x_6 \left( x_6^2 + x_6^2 \right) - \frac{3}{8} d_{11} x_6 \left( x_6^2 + x_6^2 \right). \tag{2f}\]

3. The stability analysis of the mechanical model

It is known that system (2) has a zero solution \((x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 0, 0)\) at which the Jacobian matrix can be obtained as

\[
J = \begin{bmatrix}
-\frac{1}{2} \mu_1 & -\frac{1}{2} \sigma_1 & -1/4 & 0 & 0 & 0 \\
1/2 \sigma_1 & 1/4 & -1/2 \mu_1 & 0 & 0 & 0 \\
0 & 0 & -1/2 \mu_2 & -1/4 \sigma_2 & 0 & 0 \\
0 & 0 & 1/4 \sigma_2 & -1/2 \mu_2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2 \mu_3 & 1/8 \sigma_3 \\
0 & 0 & 0 & 0 & 1/8 \sigma_3 & -1/2 \mu_3 
\end{bmatrix}, \tag{3}\]

where \(\alpha = \alpha_1 + \alpha_2 + \alpha_3\).

The characteristic polynomial corresponding to the zero solution is

\[
f(\lambda) = \lambda^6 + c_1 \lambda^5 + c_2 \lambda^4 + c_3 \lambda^3 + c_4 \lambda^2 + c_5 \lambda + c_6,
\tag{4}\]

where

\[
c_1 = \mu_1 + \mu_2 + \mu_3,
\]

\[
c_2 = \mu_1 (\mu_2 + \mu_3) + \mu_2 \mu_3 - \frac{\alpha^2}{16} + \frac{\sigma_1^2}{16} + \frac{\sigma_2^2}{16} + \frac{\sigma_3^2}{64} + \frac{1}{4} \left( \mu_1^2 + \mu_2^2 + \mu_3^2 \right),
\]

\[
c_3 = \mu_1 \left( \frac{\sigma_2^2}{16} + \frac{\mu_2^2}{4} + \mu_2 \mu_3 + \frac{\mu_3^2}{4} \right) + \left( \mu_2 + \mu_3 \right) \left( \frac{\alpha^2}{16} + \frac{\sigma_2^2}{16} + \frac{\mu_3^2}{4} \right) + \mu_3 \left( \frac{\sigma_2^2}{16} + \frac{\mu_3^2}{4} \right) + \mu_3 \left( \frac{\sigma_3^2}{64} + \frac{\mu_3^2}{4} \right),
\]
investigated by using the fourth-order Runge-Kutta algorithm. We use $\mu_1$, $\mu_2$ and $\mu_3$ as the parameters and find from figure 2 that the trajectory starting from an initial point $(x_1, x_2, x_3, x_4, x_5, x_6) = (0.0003, -0.0008, 0.00002, 0.0003, 0.001, 0.0002)$ converges to the origin.
Figure 2. Trajectory projection converged to the region at $(\mu_1, \mu_2, \mu_3) = (0.1, 0.2, 0.1)$.

4. Conclusions
The stability of the composite laminated piezoelectric rectangular plate under in-plane and transverse excitations is studied in this paper. By simplifying the averaged equations of the system at the equilibrium and analyzing the associated characteristic equation, we find that the initial equilibrium point is stable when $d_i$ exceeds zero. Besides, the stability phenomenon at the equilibrium is numerically demonstrated using the fourth-order Runge-Kutta method.

References
[1] Poincaré H 1879 *Sur les propriétés des fonctions définies par des equations aux différences partielles*thèse inaugural (Paris: Gauthier-Villars)
[2] Wiggins S 2004 *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (New York: Springer)
[3] Kuznetsov Y A 2010 *Elements of Applied Bifurcation Theory* (New York: Springer)
[4] Baider A and Sanders J 1992 Further reduction of the Takens-Bogdanov normal form J. Differ. Equ. 99 205-224.
[5] Jing R J and Yuan C M 2017 A modular algorithm to compute the generalized hermite normal form for $\mathbb{Z}[x]$-lattices J. Symb. Comput. 81 97-118.
[6] Chen G and Della Dora J 2000 Further reduction of normal forms for dynamical systems J. Differ. Equ. 166 1-25.
[7] Chen S P and Qian Y H 2014 Normal form for high-dimensional nonlinear system and its application to a viscoelastic moving belt Abstr. Appl. Anal. 2014 879564.
[8] Zhang W, Chen Y and Cao D X 2006 Computation of normal forms for eight-dimensional nonlinear dynamical system and application to a viscoelastic moving belt Int. J. Nonlin. Sci. Num. 7 35-58.
[9] Brauer F and Nohel J A 1989 The Qualitative Theory of Ordznary Differential Equations: An Introduction (New York: Dover)
[10] AL-Azzawi S F 2012 Stability and bifurcation of pan chaotic system by using Routh-Hurwitz and Gardan methods Appl. Math. Comput. 219 1144-52.
[11] Tan T, Yan Z M, Zou Y J and Zhang W M 2019 Optimal dual-functional design for a piezoelectric autoparametric vibration absorber Mech. Syst. Signal Proc. 123 513-32.
[12] Li J Q, Ma Z R, Wang Z H and Narita Y 2016 Random vibration control of laminated composite plates with piezoelectric fiber reinforced composites Acta Mech. Solida Sin. 29 316-27.
[13] Zhu F H and Fu Y M 2008 Nonlinear dynamic response and fatigue damage evolution for piezoelectric laminated plates with matrix cracks Theor. Appl. Fract. Mech. 49 291-304.
[14] Zhang W, Gao M J, Yao M H, and Yao Z G 2009 Higher-dimensional chaotic dynamics of a composite laminated piezoelectric rectangular plate Sci. China Phys. Mech. 52 1989-2000.
[15] Zhang W, Yao Z G, and Yao M H 2009 Periodic and chaotic dynamics of composite laminated piezoelectric rectangular plate with one-to-two internal resonance Sci. China Ser. E 52 731-42.
[16] Tanzadeh H and Amoushahi H 2019 Buckling and free vibration analysis of piezoelectric laminated composite plates using various plate deformation theories Eur. J. Mech. A-Solids 74 pp 242-56.
[17] Yao Z 2009 Studies on complicated nonlinear dynamics and control of composite laminated piezoelectric structures (Beijing: Beijing University of Technology)
[18] Ogata K 1970 Modern Control Engineering (New York : Prentice-Hall)