Lévy flights in random environments

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Abstract

We consider Lévy flights characterized by the step index $f$ in a quenched random force field. By means of a dynamic renormalization group analysis we find that the dynamic exponent $z$ for $f < 2$ locks onto $f$, independent of dimension and independent of the presence of weak quenched disorder. The critical dimension, however, depends on the step index $f$ for $f < 2$ and is given by $d_c = 2f - 2$. For $d < d_c$ the disorder is relevant, corresponding to a non trivial fixed point for the force correlation function.

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There is a current interest in the dynamics of fluctuating manifolds in quenched random environments [1]. This fundamental issue in modern condensed matter physics is encountered in problems as diverse as vortex motion in high temperature superconductors, moving interfaces in porous media, and random field magnets and spin glasses.

The simplest case is that of a random walker in a random environment, corresponding to a zero dimensional fluctuating manifold. This problem has been treated extensively in the literature \[2, 3, 4\] and many results are known.

In the case of ordinary Brownian motion, characterized by a finite mean square step, in a pure environment without disorder, the central limit theorem \[5\] implies that the statistics of the walk is given by a Gaussian distribution with a mean square deviation proportional to the number of steps or, equivalently, the elapsed time, i.e., the mean square displacement

$$\langle r^2(t) \rangle \propto D t^{2/z},$$

where the dynamic exponent \(z = 2\) for Brownian walk and \(D\) is the diffusion coefficient.

There are, however, many interesting processes in nature which are characterized by an anomalous diffusion with \(z \neq 2\) due to the statistical properties of the environments \[3, 4\]. Examples are found in chaotic systems \[6\], turbulence \[7, 8\], flow in fractal geometries \[9\], and Lévy flights \[10, 11\], which generally lead to enhanced diffusion or superdiffusion with \(z < 2\). We note that the ballistic case corresponds to \(z = 1\). The other case of subdiffusion or dispersive behaviour with \(z > 2\) is encountered in various constrained systems like doped crystals, glasses or fractals \[12, 13, 14, 15, 16\].

Irrespective of the spatial dimension \(d\), ordinary Brownian motion traces out a manifold of fractal dimension \(d_F = 2\) \[17\]. In the presence of a quenched disordered force field in \(d\) dimensions the Brownian walk is unaffected for \(d > d_F\), i.e., for \(d\) larger than the critical dimension \(d_c = d_F\) the walk is transparent and the dynamic exponent \(z\) locks onto the value 2 for the pure case. Below the critical dimension \(d_c = 2\) the long time characteristics of the walk is changed to subdiffusive behaviour with \(z > 2\) \[18, 19, 20\]. In \(d = 1\), \(\langle r^2(t) \rangle \propto [\log t]^4\), independent of the strength of the quenched disorder \[21\].

Lévy flights constitute an interesting generalization of ordinary Brownian walks.
Here the step size is drawn from a Lévy distribution characterized by the step index $f$. The Lévy distribution has a long range algebraic tail corresponding to large but infrequent steps, so-called rare events. This step distribution has the interesting property that the central limit theorem does not hold in its usual form. For $f > 2$ the second moment or mean square deviation of the step distribution is finite, the central limit theorem holds and the dynamic exponent $z$ for the Lévy walk locks onto 2, corresponding to ordinary diffusive behaviour; however, for $f < 2$ the mean square step deviation diverges, the rare large step events prevail and determine the long time behaviour, and the the dynamic exponent $z$ depends on the microscopic step index $f$ according to $z = f$ ($f < 2$), indicating anomalous enhanced diffusion, that is superdiffusion [11, 22, 23]. The ‘built in’ superdiffusive characteristics of Lévy flights have been used to model a variety of physical processes such as self diffusion in micelle systems [24], and transport in heterogeneous rocks [22].

In the present letter we consider Lévy flights in the presence of a quenched random force field and examine the interplay between the ‘built in’ superdiffusive behaviour of the Lévy flights and the pinning effect of the random environment generally leading to subdiffusive behaviour. Generalizing the discussion in refs.[18, 20, 25] we find that in the case of enhanced diffusion for $f < 2$ the dynamic exponent $z = f$, independent of the presence of weak disorder. On the other hand, we can still identify a critical dimension $d_c = 2f - 2$, depending on the step index $f$ for $f < 2$. Below $d_c$ the quenched disorder becomes relevant as indicated by the emergence of a non trivial fixed point in the renormalization group analysis. Here we give a brief outline of our arguments while details will be published elsewhere.

It is convenient to discuss Lévy flights in terms of a Langevin equation with ‘power law’ noise [24]. In an arbitrary drift force field $F(r)$, representing the quenched disordered environment, the equation takes the form

$$\frac{dr(t)}{dt} = F(r(t)) + \eta(t).$$

(2)

Here $\eta$ is the instantly correlated power law white noise with distribution

$$p(\eta)d^d\eta \propto \eta^{-1-f}d\eta.$$  

(3)

We assume an isotropic distribution characterized by the step index $f$. In order to ensure normalizability we have introduced a lower cutoff $\eta \sim a$ of the order of a microscopic length $a$ and chosen $f > 0$. For $f > 2$ the second moment, $\langle \eta^2 \rangle = \int p(\eta)\eta^2d^d\eta$, is finite and a characteristic step size is given by the root mean square deviation $\sqrt{\langle \eta^2 \rangle}$. For $1 < f < 2$ the second moment diverges but the mean step,
\(\langle \eta \rangle\), is finite. In the interval \(0 < f < 1\) the first moment diverges and even a mean step size is not defined.

The power law noise \(\eta\), describing the consecutive Lévy steps, drives the position \(r\) of the walker. For the quenched force field \(F(r)\) we assume a Gaussian distribution,

\[
p(F) \propto \exp \left( -\frac{1}{2} \int d^d r d^d r' F^\alpha(r) \Delta^{\alpha \beta}(r-r')^{-1} F^\beta(r') \right)
\]

Here \(\Delta^{\alpha \beta}(r-r')\) is the force correlation function expressing the range and vector nature of \(F(r)\).

In the absence of the force field, i.e., \(F(r) = 0\), the distribution for the position of the walker is easily inferred. From the definition \(P(r,t) = \langle \delta(r - r(t)) \rangle\), the solution of Eq. (2), and averaging according to Eq. (3), we find the scaling form \([11, 17]\),

\[
P(r,t) = \int \frac{d^d k}{(2\pi)^d} \exp \left( i k r - D k^\mu / | t | \right) \propto | t |^{-\frac{d}{\mu}} G(r/ | t |^{\frac{1}{\mu}}).
\]

\(D\) is a diffusion coefficient setting the time scale. This procedure is equivalent to applying the central limit theorem to a sum of random variables with the Lévy distribution in Eq. (3) \([3, 17]\).

The scaling exponent \(\mu\) depends on the step index \(f\), characterizing the Lévy distribution. For \(f > 2\), i.e., the case of a finite mean square step, \(\mu\) locks onto the value 2 and the scaling function \(G\) takes the Gaussian form for ordinary Brownian walk, \(G(x) = \exp(-x^2)\). This is a consequence of the central limit theorem which here leads to universal behaviour. For \(f < 2\) the scaling exponent \(\mu = f\) and the scaling function \(G\) can only be given explicitly in terms of known functions for \(\mu = 1\), the ballistic case, where we find the Cauchy distribution \(F(x) = (1 + x^2)^{-(d+1)/2}\). It is, however, easy to show that \(G \rightarrow \text{const.}\) for \(x \rightarrow 0\) and \(G \rightarrow 0\) for \(x \rightarrow \infty\). From the distribution in Eq. (6) we deduce the scaling form for the mean square displacement of the walker in time \(t\):

\[
\langle r^2(t) \rangle = \int P(r,t)r^2 d^dr \propto t^\frac{\mu}{2}
\]

and following from Eq. (1) the dynamic exponent \(z = \mu\). We also note that the fractal dimension of a Lévy flight is \(d_F = \mu\) \([17]\).
In the presence of the quenched force field given by Eqs. (4) and (5) it is convenient to recast the problem given by the Langevin equation (2) in terms of the associated Fokker-Planck equation,

\[
\frac{\partial P(r, t)}{\partial t} = -\nabla(F(r)P(r, t)) + D\nabla^\mu P(r, t). \tag{8}
\]

Here the first term on the right hand side of Eq. (8) is the usual drift term due to the motion of the walker in the force field, the second term follows from Eq. (6) and the assumption of independent contributions to the probability current. The ‘fractional’ gradient operator \(\nabla^\mu\) is the Fourier transform of \(-k^\mu\) and is a spatially non local integral operator reflecting the long range Lévy steps; for \(\mu = 2\) it reduces to the usual Laplace operator describing ordinary diffusion.

The Fokker-Planck equation in Eq. (8) together with the distribution in Eqs. (4) and (5) defines the problem of a random Lévy walker in a quenched random force field. The ‘microscopic’ Lévy steps occurring on a fast time scale represented by the noise term in the Langevin equation (2) has now been absorbed and replaced by the anomalous diffusion term in the Fokker-Planck equation (8). The remaining randomness due to the quenched force field is assumed static compared to the time scale of \(r(t)\).

There are a variety of techniques available in order to treat the random Fokker-Planck equation (8). Applying the Martin-Siggia-Rose formalism in functional form \[27, 28, 29, 30\] and using either the replica method \[4\] or an explicit causal time dependence \[18, 29, 32\], one can average over the quenched force field and construct an effective field theory. A more direct method, which we shall adhere to in the present discussion, amounts to an expansion of the Fokker-Planck equation (8) in powers of the force field and an average over products of \(F(r)\) according to the distribution in Eqs. (4) and (5) \[25\].

In order to deduce the scaling properties of the force averaged distribution \(\langle P(r, t) \rangle_F\) and the mean square displacement \(\langle r^2(t) \rangle_F\) we carry out a renormalization group analysis of Eq. (8) following the momentum shell integration method propounded in refs. \[33, 34, 25\] and also drawing on the results in ref. \[18\].

Defining the Fourier transform

\[
P(k, \omega) = \int d^d r dt \exp(i\omega t - ikr)P(r, t)\theta(t), \tag{9}
\]
where $\theta(t)$ is the step function, we obtain the Fokker-Planck equation

$$(-i\omega + Dk^\mu)P(k, \omega) = P_o(k) - i k \int \frac{d^dp}{(2\pi)^d} F(k-p)P(p, \omega). \tag{10}$$

Here the force field is averaged according to Eqs. (4) and (5), or

$$\langle F^\alpha(k)F^\beta(p) \rangle_F = \Delta^\alpha\beta(k)(2\pi)^d\delta(k+p) \tag{11}$$

for all pairwise force contractions (Wick’s theorem for the Gaussian distribution); $P_o(k) = P(t = 0)$ is the initial distribution. For simplicity we consider the case of isotropic zero range force correlations, i.e., $\Delta^\alpha\beta(k) = \Delta\delta^\alpha\beta$; the general case has been discussed in refs.[4, 25, 35]. We introduce a microscopic UV cut off and assume $0 < k, p < 1$. Iterating Eq. (11), identifying self energy and force correlation corrections to second order in $\Delta$, and performing a momentum shell integration, i.e., averaging over the force in the shell $e^{-\ell} < k, p < 1$, we obtain the ‘corrected’ Fokker-Planck equation

$$(-i\omega + Dk^\mu + \delta Dk^2)P_k(k, \omega) = P_o(k) - i k \int \frac{d^dp}{(2\pi)^d} F(k-p)P(p, \omega). \tag{12}$$

and the force correlation function

$$\langle F^\alpha(k)F^\beta(p) \rangle_F = (\Delta + \delta\Delta)\delta^\alpha\beta(2\pi)^d\delta(k+p)$$

for $0 < k, p < e^{-\ell}$. For small values of the scale parameter $\ell$ the corrections $\delta D$ and $\delta\Delta$ arising from the momentum shell integrations are proportional to $\ell$. From the diagrammatic contributions to $\delta D$ and $\delta\Delta$ given in refs. [18, 25], involving three propagators $(-i\omega + Dk^\mu)^{-1}$ and two force contractions $\Delta$ for $\delta D$ and two propagators and two force contractions for $\delta\Delta$, all evaluated in the static limit $\omega = 0$ and on the shell $k = 1$, we obtain

$$\delta D = -A\frac{\Delta^2}{D^3}\ell \quad \delta\Delta = -B\frac{\Delta^2}{D^2}\ell, \tag{14}$$

where $A$ and $B$ are geometric factors associated with the area of a d-dimensional unit sphere. In order to derive the ‘renormalized’ Fokker-Planck equation we introduce scaled quantities $k' = ke^{\ell}$, $p' = pe^{\ell}$, $\omega' = \omega e^{\alpha(\ell)}$, $P'(k', \omega') = P(k, \omega)e^{-\alpha(\ell)}$, and $F'(k') = F(k)e^{-(d+1)\ell+\alpha(\ell)}$ such that $0 < k', p' < 1$ and find

$$(-i\omega' + Dk'^\mu e^{-\mu\ell+\alpha(\ell)} + \delta Dk'^2 e^{-2\ell+\alpha(\ell)})P'(k', \omega') = P_o(k') - i k' \int \frac{d^dp'}{(2\pi)^d} F'(k'-p')P'(p', \omega') \tag{15}$$
\[ \langle F^{\alpha}(k')F^{\beta}(p') \rangle_F = (\Delta + \delta \Delta)e^{-(d+2)\ell+2\alpha(\ell)}\delta^{\alpha\beta}(2\pi)^d\delta^d(k' + p'). \]  

(16)

From the renormalized Fokker-Planck equation and force correlation function we read off the renormalization group equations for \( D \) and \( \Delta \),

\[ D' = De^{\mu\ell+\alpha(\ell)} \]  

(17)

\[ \Delta' = (\Delta + \delta \Delta)e^{-(d+2)\ell+2\alpha(\ell)} \]  

(18)

or in differential form setting \( \alpha(\ell) = \int^{\ell}_0 z(\ell')d\ell' \) and introducing Eq. (14)

\[ \frac{dD(\ell)}{d\ell} = (z(\ell) - \mu)D(\ell) \]  

(19)

\[ \frac{d\Delta(\ell)}{d\ell} = (2z(\ell) - d - 2)\Delta(\ell) - B\frac{\Delta(\ell)^2}{D(\ell)^2}. \]  

(20)

Before we discuss the renormalization group equations for the scale dependent diffusion coefficient \( D(\ell) \) and force correlation \( \Delta(\ell) \) we notice immediately that for Lévy flights with \( \mu = f < 2 \) there is no correction to the anomalous term \( Dk^\mu \) in the Fokker Planck equation. The perturbative contribution comes with a leading power \( k^2 \) which is subdominant compared with the Lévy term \( k^\mu \). In the case of ordinary Brownian motion for \( \mu = 2 \) there is a correction and we recover the results in refs. [18].

Proceeding with the discussion of Eqs. (19) and (20) we fix as usual the dynamic exponent \( z(\ell) = \mu \) such that the diffusion coefficient stays constant under renormalization, i.e., \( D(\ell) = D \). The equation for \( \Delta(\ell) \) then takes the form

\[ \frac{d\Delta(\ell)}{d\ell} = (2\mu - d - 2)\Delta(\ell) - B\frac{\Delta(\ell)^2}{D^2}. \]  

(21)

For \( d \) greater that the critical dimension \( d_c = 2\mu - 2 \) Eq. (21) has the trivial fixed point \( \Delta^* = 0 \) and the quenched disorder is irrelevant. For \( d < d_c \) we obtain the non trivial fixed point \( \Delta^* = (d_c - d)D^2/B \) and the quenched disorder becomes relevant. We also note that unlike the case of Brownian motion the critical dimension \( d_c = 2\mu - 2 \) is less than the fractal dimension \( d_F = \mu \).
In order to derive the scaling properties of the force averaged distribution 
\( \langle P(k, \omega) \rangle_F \) and the means square displacement \( \langle \langle r^2(t) \rangle \rangle_F \) we use the methods discussed in refs. [23, 33, 34]. From the derivation of the renormalization group equations we infer the scaling relation

\[
\langle P(k, \omega, \Delta) \rangle_F = e^{\alpha(\ell)} \langle P(k\ell, \omega e^{\alpha(\ell)}, \Delta(\ell)) \rangle_F.
\]

(22)

In the vicinity of either the trivial fixed point \( \Delta^* = 0 \) for \( d > d_c = 2\mu - 2 \) or the fixed point \( \Delta^* = (d_c - d)D^2/B \) for \( d < d_c \) we have, setting \( \alpha(\ell) \propto \mu \ell \),

\[
\langle P(k, \omega, \Delta) \rangle_F = e^{\mu \ell} \langle P(k\ell, \omega e^{-\mu \ell}, \Delta^*) \rangle_F.
\]

(23)

Choosing \( k\ell \sim 1 \) we obtain the scaling form

\[
\langle P(k, \omega, \Delta) \rangle_F = k^{-\mu} L(k/\omega^{1/\mu}),
\]

(24)

where \( L \) is a scaling function. From Eq. (24) follows directly

\[
\langle \langle r^2(t) \rangle \rangle_F \propto t^{2/\mu} = t^{2/z}.
\]

(25)

Since the scaling analysis in the Lévy case is quite similar to the Brownian case discussed in refs. [18, 25, 35] we shall not repeat this analysis here but simply discuss and summarize our results below.

For Lévy flights in random quenched environment we have demonstrated that the dynamic exponent \( z \) locks onto the scaling index \( \mu \), depending on the Lévy step index \( f \), independent of the presence of weak quenched disorder. The long range superdiffusive behaviour characteristic of Lévy flights enables the walker to escape the inhomogeneous pinning environment and the long time behaviour is the same as in the pure case. This result is in contrast to the case of a Brownian walk where the dynamic exponent \( z > 2 \) below \( d = 2 \), corresponding to subdiffusive behaviour. We have also identified a critical dimension \( d_c = 2\mu - 2 \), depending on the scaling exponent \( \mu \). Below the critical dimension the weak disorder becomes relevant as shown by the emergence of a non trivial fixed point.

Bouchaud [31] has given a heuristic argument yielding the critical dimension \( d_c = \mu \) in the Lévy case. This result is at variance with the critical dimension \( d_c = 2\mu - 2 \) given here based on a renormalization group analysis. It would be of interest to construe a qualitative heuristic argument for the critical dimension \( d_c = 2\mu - 2 \) given here and the insensitivity of the dynamic exponent \( z = \mu \) to the weak quenched disorder.
Below we have in Figure 1 plotted the scaling exponent $\mu$ and the dynamic exponent $z$ as function of the Lévy step index $f$. In Figure 2 we have plotted the critical dimension $d_c$ as a function of $\mu$.

![Graph of $\mu, z$ (z = $\mu$)](image)

Fig.1. Plot of the scaling index $\mu$ and the dynamic exponent $z$ as functions of the Lévy step index $f$. For $f > 2$ we have normal Brownian diffusion and we obtain corrections to $z$ below $d = 2$; for $0 < f < 2$ we have anomalous Lévy superdiffusion.
Fig. 2. Plot of the critical dimension $d_c$ as a function of the scaling index $\mu$. For $\mu = 2$ we have the Brownian case $d_c = 2$; for $1 < \mu < 2$ $d_c$ depends linearly on $\mu$. Note that $d_c = 0$ in the ballistic case for $\mu = 1$.

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