Discrete dynamical systems associated with root systems of indefinite type

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Abstract

A geometric characterization of the equation found by Hietarinta and Viallet, which satisfies the singularity confinement criterion but which exhibits chaotic behavior, is presented. It is shown that this equation can be lifted to an automorphism of a certain rational surface and can therefore be considered to be a realization of a Cremona isometry on the Picard group of the surface. It is also shown that the group of Cremona isometries is isomorphic to an extended Weyl group of indefinite type. A method to construct the mappings associated with some root systems of indefinite type is also presented.

1 Introduction

The singularity confinement method has been proposed by Grammaticos et al. [6] as a criterion for the integrability of (finite or infinite dimensional) discrete dynamical systems. The singularity confinement method demands that when singularities appear due to particular initial values such singularities should disappear after a finite number of iteration steps, in which case the information on the initial values ought to be recovered (hence the dynamical system has to be invertible).

However “counter examples” were found by Hietarinta and Viallet [8]. These mappings satisfy the singularity confinement criterion but the orbits of their solutions exhibit chaotic behavior. The authors of [8] introduced the notion of algebraic entropy in order to test the degree of complexity of successive iterations. The algebraic entropy is defined as

\[ s := \lim_{n \to \infty} \frac{\log(d_n)}{n} \]

where \( d_n \) is the degree of the \( n \)th iterate. This notion is linked to Arnold’s complexity since the degree of a mapping gives the intersection number of the image of a line and a hyperplane. While the degree grows exponentially for a generic mapping, it was shown that it only grows polynomially for a large class of integrable mappings [1, 2, 3, 4, 5].

The discrete Painlevé equations have been extensively studied [9, 14]. Recently it was shown by Sakai [15] that all of (from the point of view of symmetries) these are obtained by studying rational surfaces in connection with the extended affine Weyl groups.

Surfaces obtained by successive blow-ups [7] of \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \) have been studied by several authors by means of connections between the Weyl groups and the groups of Cremona isometries on the Picard group of the surfaces [1, 2, 3, 4]. Here, the Picard group of a rational surface \( X \) is the group of isomorphism classes of invertible sheaves on \( X \) and it is isomorphic to the group of linear equivalent classes of divisors on \( X \). A Cremona isometry is an isomorphism of the Picard group such that a) it preserves the intersection number of any pair of divisors, b) it preserves the canonical divisor \( K_X \) and c) it leaves the
set of effective classes of divisors invariant. In the case where 9 points (in the case of \( \mathbb{P}^2 \), 8 points in the case of \( \mathbb{P}^1 \times \mathbb{P}^1 \)) are blown up, if the points are in general position the group of Cremona isometries becomes isomorphic with an extension of the Weyl group of type \( E_8^{(1)} \). In case the 9 points are not in general position, the classification of connections between the group of Cremona isometries and the extended affine Weyl groups was first studied by Looijenga \([11]\) and more generally by Sakai. Birational (bi-meromorphic) mappings on \( \mathbb{P}^2 \) (or \( \mathbb{P}^1 \times \mathbb{P}^1 \)) are obtained by interchanging the procedure of blow downs. Discrete Painlevé equations are recovered as the birational mappings corresponding to the translations of affine Weyl groups. It was shown that for every Painlevé equation in Sakai’s list the order of the \( n \)th iterate is at most \( O(n^2) \) \([17]\).

Our aim in this paper is to characterize some birational mappings which satisfy the singularity confinement criterion but exhibit chaotic behavior from the point of view of the theory of rational surfaces. Considering one such mapping and the space of its initial values, we obtain a rational surface associated with some root system of hyperbolic type. Conversely, we recover the mapping from the surface and consequently obtain the extension of the mapping to its non-autonomous version. It is important to remark that this method also allows the construction of other mappings starting from suitable rational surfaces. We also show some other examples of such constructions.

In Section 2, we start from one of the mappings found by Hietarinta and Viallet (we call it the HV eq. in this paper) and construct the space such that the mappings is lifted to an automorphism, i.e. bi-holomorphic mapping, of the surface. The mapping \( \varphi' \) is called a mapping lifted from the mapping \( \varphi \) if \( \varphi' \) coincides with \( \varphi \) on any point where \( \varphi \) is defined. For this purpose we compactify the original space of initial values, \( \mathbb{C}^2 \), to \( \mathbb{P}^1 \times \mathbb{P}^1 \) and blow-up 14 times.

In Section 3, we study the symmetry of the space of initial values. We show that the group of all the Cremona isometries of the Picard group of the surface is isomorphic to an extended Weyl group of hyperbolic type. As a corollary, we prove that there does not exist any Cremona isometry whose action on the Picard group commutes with the action of the HV eq. except the action itself.

In Section 4, we show a method to recover the HV eq. from the surface as an element of the extended Weyl group. Each element of the extended Weyl group which acts on the Picard group as a Cremona isometry, is realized as a Cremona transformation (i.e. a birational mapping) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) by interchanging the blow down structure. Here, a blow down structure is the sequence designating the procedure of blow downs. As a result of this we obtain the non-autonomous version of the equation.

In Section 5, we discuss the construction of other mappings from certain rational surfaces and show some examples which are associated with root systems of indefinite type.

## 2 Construction of the space of initial values by blow-ups

We consider the dynamical system written by the birational (bi-meromorphic) mapping

\[
\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \\
\begin{pmatrix} x_n \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ -x_n + y_n + a/y_n^2 \end{pmatrix}
\]  

(1)

where \( a \in \mathbb{C} \) is a nonzero constant. This equation was found by Hietarinta and Viallet \([8]\) and we call it the HV eq.. To test the singularity confinement, let us assume \( x_0 \neq 0 \) and
In this way, blowing up at $(\frac{1}{x}, y)$ denote the mapping $\phi$ of the mapping $X_2$ isomorphism from $Y_2$. Let the coordinates of $X$ indeterminacy.

A sequence of algebraic varieties $X_i$ is (or $X_i$ themselves are) called the space of initial values for the sequence of rational mapping $\varphi_i$, if each $\varphi_i$ is lifted to an isomorphism from $X_i$ to $X_{i+1}$, for all $i$.

Our aim in this section is to construct the surface $X$ by blow-ups $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\varphi$ is lifted to an automorphism of $X$, where the mapping $\varphi'$ is called a mapping lifted from the mapping $\varphi$ if $\varphi'$ coincides with $\varphi$ on any point where $\varphi$ is defined.

### 2.1 Regular mapping from $Y_1$ to $\mathbb{P}^1 \times \mathbb{P}^1$

Let the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ be $(x, y), (x, 1/y), (1/x, y)$ and $(1/x, 1/y)$ and let $x = \infty$ denote $1/x = 0$. We denote the HV eq. as

$$\varphi : (x, y) \mapsto (\overline{x}, \overline{y}) = (y, -x + y + a/y^2) \quad (2)$$

where $(\overline{x}, \overline{y})$ means the image of $(x, y)$ by the mapping. This mapping has two indeterminate points: $(x, y) = (\infty, 0), (\infty, \infty)$. By blowing up at these points we can ease the indeterminacy.

We denote blowing up at $(x, y) = (x_0, y_0) \in \mathbb{C}^2$:

$$\mu_{(x_0, y_0)} : \{(x, y) : x, y \in \mathbb{C}\} \rightarrow \{(x - x_0, y - y_0; \zeta_1 : \zeta_2) \mid x, y, \zeta_1, \zeta_2 \in \mathbb{C}\}$$

by

$$(x, y) \mapsto (x - x_0, (y - y_0)/(x - x_0)) \cup ((x - x_0)/(y - y_0), y - y_0). \quad (3)$$

In this way, blowing up at $(x, y) = (x_0, y_0)$ gives meaning to $(x - x_0)/(y - y_0)$ at this point.
First we blow-up at \((x, y) = (\infty, 0)\), \((1/x, y) \leftrightarrow (1/xy, y) \cup (1/xy, y)\) and denote the obtained surface by \(Y_0\). Then \(\varphi\) is lifted to a rational mapping from \(Y_0\) to \(\mathbb{P}^1 \times \mathbb{P}^1\). For example, in the new coordinates \(\varphi\) is expressed as

\[
(u_1, v_1) := (1/x, xy) \mapsto (\overline{x}, \overline{y}) = (u_1v_1, (-u_1v_1^2 + u_1^2v_1^2 + a)/(u_1^2v_1^2))
\]

\[
(u_2, v_2) := (1/xy, y) \mapsto (\overline{x}, \overline{y}) = (v_2, (-v_2 + u_2v_2^2 + au_2)/(u_2v_2^2))
\]

This maps the exceptional curve at \((x, y) = (\infty, 0)\), i.e. \(u_1 = 0\) and \(v_2 = 0\), almost to \((\overline{x}, \overline{y}) = (\infty, 0)\) but has an indeterminate point on the exceptional curve: \((u_2, v_2) = (0, 0)\). Hence we have to blow-up again at this point. In general it is known that, if there is a rational mapping \(X \to X'\) where \(X\) and \(X'\) are smooth projective algebraic varieties, the procedure of blowing up can be completed in a finite number of steps, after which one obtains a smooth projective algebraic variety \(Y\) such that the rational mapping is lifted to a regular mapping from \(Y\) to \(X'\) (theorem of the elimination of indeterminacy \(\text{(a)}\)).

Here we obtain the surface \(Y_1\) defined by the following sequence of blow-ups. (For simplicity we take only one coordinate of \((\text{b})\).

\[
(x, y) \xleftarrow{(\infty, 0)}_{\mu_1} (1/xy, y) \xleftarrow{(0, 0)}_{\mu_2} (1/xy, xy^2)
\]

\[
\phantom{(x, y)} \xleftarrow{(0,a)}_{\mu_3} (1/xy, xy(xy^2 - a)) \xleftarrow{(0,0)}_{\mu_4} (1/xy, x^2y^2(xy^2 - a))
\]

\[
(x, y) \xleftarrow{(\infty, \infty)}_{\mu_9} (1/x, y) \xleftarrow{(0,1)}_{\mu_10} (1/x, x(\frac{y}{x} - 1))
\]

where \(\mu_i\) denotes the \(i\)th blow-up. Of course the above sequence is not unique since there is freedom in choosing the coordinates.

### 2.2 Automorphism of \(X\)

We have obtained a mapping from \(Y_1\) to \(\mathbb{P}^1 \times \mathbb{P}^1\) which is lifted from \(\varphi\). But our aim is to construct a rational surface \(X\) such that \(\varphi\) is lifted to an automorphism of \(X\).

First we construct the rational surface \(Y_2\) such that \(\varphi\) is lifted to a regular mapping from \(Y_2\) to \(Y_1\). For this purpose it is sufficient to eliminate the indeterminacy of mapping from \(Y_1\) to \(Y_1\). Consequently we obtain \(Y_2\) defined by the following sequence of blow-ups.

\[
\left(\frac{1}{x}, x(\frac{y}{x} - 1)\right) \xleftarrow{(0,0)}_{\mu_{11}} \left(\frac{1}{x^2(x/y - 1)}, x(\frac{y}{x} - 1)\right)
\]

\[
\phantom{\left(\frac{1}{x}, x(\frac{y}{x} - 1)\right)} \xleftarrow{(0,0)}_{\mu_{12}} \left(\frac{1}{x^2(x/y - 1)}, x^3(\frac{y}{x} - 1)^2\right)
\]

\[
\phantom{\left(\frac{1}{x}, x(\frac{y}{x} - 1)\right)} \xleftarrow{(0,a)}_{\mu_{13}} \left(\frac{1}{x^2(x/y - 1)}, x^2(\frac{y}{x} - 1)(x^3(\frac{y}{x} - 1)^2 - a)\right)
\]

\[
\phantom{\left(\frac{1}{x}, x(\frac{y}{x} - 1)\right)} \xleftarrow{(0,0)}_{\mu_{14}} \left(\frac{1}{x^2(x/y - 1)}, x^4(\frac{y}{x} - 1)^2(x^3(\frac{y}{x} - 1)^2 - a)\right)
\]

Next eliminating the indeterminacy of mapping from \(Y_2\) to \(Y_2\), we obtain \(Y_3\) defined by the following sequence of blow-ups.

\[
(x, y) \at \xleftarrow{(0, \infty)}_{\mu_5} \left(x, \frac{1}{xy}\right) \xleftarrow{(0,0)}_{\mu_6} \left(x^2y, \frac{1}{xy}\right)
\]

\[
\phantom{(x, y) \at \xleftarrow{(0, \infty)}_{\mu_5}} \xleftarrow{(a,0)}_{\mu_7} \left(xy(x^2y - a), \frac{1}{xy}\right) \xleftarrow{(0,0)}_{\mu_8} \left(x^2y^2(x^2y - a), \frac{1}{xy}\right)
\]
It can be shown that the mapping from \( Y_3 \) to \( Y_3 \) which is lifted from \( \varphi \) does not have any indeterminate points.

To show this, let us define the total and proper transforms.

**Definition.** Let \( S \) be the set of zero points of \( \bigwedge_{i \in I} f_i(u, v) = 0 \), where \((u, v) \in \mathbb{C}^2\) and the \( \{f_i\}_{i \in I} \) is a finite set of polynomials, and let \( U_1 : (u_1, v_1) \) and \( U_2 : (u_2, v_2) \) the new coordinates of blow-up at the point \((u, v) = (a, b)\), i.e. \((u_1, v_1) = (u - a, (v - b)/(u - a)), (u_2, v_2) = ((u - a)/(v - b), v - b)\). The total transform of \( S \) is

\[
\{(u_1, v_1) \in U_1; f_i(u_1 + a, u_1v_1 + b) = 0\} \cup \{(u_2, v_2) \in U_2; f_i(u_2v_2 + a, v_2 + b) = 0\}
\]

and the proper transform of \( S \) is

\[
\{(u_1, v_1) \in U_1; f_i(u_1 + a, u_1v_1 + b) = 0\} \cup \{(u_2, v_2) \in U_2; f_i(u_2v_2 + a, v_2 + b) = 0\}
\]

where \( m \) or \( n \) is the maximum integer simplifying the respective equations for \( u_1 \) or \( v_2 \) respectively.

For example, by blowing up at \((u, v) = (0, 0)\), the total transform of \( u = 0 \) is \( \{(u_1, v_1) \in U_1; u_1 = 0\} \cup \{(u_2, v_2) \in U_2; u_2v_2 = 0\} \) and its proper transform is \( \{(u_1, v_1) \in U_1; u_1 = 0\} = \emptyset \cup \{(u_2, v_2) \in U_2; u_2 = 0\} \).

We denote the total transform of the point of the \( i \)th blow-up by \( E_i \) and denote the proper transform of the exceptional curves of the \( i \)th blow-up by

\[
D_0, D_1, D_2, C_0 = E_4, D_3, D_4, D_5, C_1 = E_8, D_6, D_{12}, D_7, D_8, D_9, C_2 = E_{14}.
\]

Moreover we denote the proper transforms of \( x = 0, x = \infty, y = 0, y = \infty \) as

\[
C_3, D_{10}, C_4, D_{11}
\]

(See Fig[1].)

The proper transforms of \( C_4, D_0, D_1, D_2 \) and \( C_0 \) are written as

- \( C_4 : (x, y) = (x, 0) \)
- \( D_0 : (u_1, v_1) = (u_1, 0) \)
- \( D_1 : (u_2, v_2) = (0, v_2) \)
- \( D_2 : (u_3, v_3) = (0, v_3) \)
- \( C_0 : (u_4, v_4) = (0, v_4) \)

where \( u_i \) and \( v_i \) are the new coordinate of the \( i \)th blow-up (more precisely, these express the total transforms of curves and we have to write each curve by using the coordinates of the last blow-up but this makes the notation rather complicated) and therefore the relations

\[
x = 1/(u_1v_1), \quad y = v_1,
\]

\[
u_1 = u_2, \quad v_1 = u_2v_2,
u_2 = u_3, \quad v_2 = u_3v_3 + a,
u_3 = u_4, \quad v_3 = u_4v_4
\]
Figure 1:

hold. The proper transforms of $C_1, D_5, D_4, D_3$ and $C_3$ are written as

$\begin{align*}
C_1 & : (u_8, v_8) = (u_8, 0) \\
D_5 & : (u_7, v_7) = (u_7, 0) \\
D_4 & : (u_6, v_6) = (u_6, 0) \\
D_3 & : (u_5, v_5) = (0, v_5) \\
C_3 & : (x, 1/y) = (0, 1/y)
\end{align*}$

and the relations

$\begin{align*}
u_8 &= u_7/v_7, & v_8 &= v_7, \\
u_7 &= (u_6 - a)/v_6, & v_7 &= v_6, \\
u_6 &= u_5/v_5, & v_6 &= v_5, \\
u_5 &= x, & v_5 &= 1/(xy)
\end{align*}$

hold.

Using these relations one can calculate the images of the curves. For example, in the case of $C_4$: From the above relations and (2) we can calculate $(\overline{u}_8, \overline{v}_8)$ using initial values corresponding to $C_1$ as

$$(\overline{u}_8, \overline{v}_8)_{(x,y)=(x,0)} = \left. \left( -x + y \right) \left( a + y^2(-x+y) \right)^2, \frac{y}{a - xy^2 + y^3} \right|_{(x,y)=(x,0)} = (-a^2x, 0)$$

This then implies that the image of $C_4 \ (= \overline{C}_4)$ is $C_1$. 
Analogously, from the equation
\[
\begin{align*}
(u_7, v_7)|_{(u_1, v_1) = (u_1, 0)} & = \left(\frac{-1 + u_1 v_1^2}{u_1^2}, \frac{u_1 v_1}{au_1 - v_1 + u_1 v_1^3}\right)|_{(u_1, v_1) = (u_1, 0)} \\
& = \left(-\frac{a}{u_1}, 0\right)
\end{align*}
\]
which implies \( D_0 = D_5 \). In this way we can show that
\[
(D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}, D_{12}, C_0, C_1, C_2, C_4) \rightarrow (D_5, D_4, D_3, D_7, D_8, D_9, D_6, D_0, D_1, D_2, D_{12}, D_{10}, C_3, C_2, C_0, C_1).
\]
\( (6) \)

It is obvious that this mapping has an inverse (the mapping lifted from \( \varphi^{-1} \)). Hence we obtain the following theorem.

**THEOREM 1** The HV eq. (2) can be lifted to an automorphism of \( X(= Y_3) \).

### 3 The Picard group and symmetry

#### 3.1 Action on the Picard group

We denote the (linear equivalent classes of) total transform of \( x = \text{constant} \) (or \( y = \text{constant} \)) on \( X \) by \( H_0 \) (or \( H_1 \) respectively) and the (linear equivalent classes of) total transform of the point of the \( i \)th blow-up by \( E_i \). From [7] we know that the Picard group of \( X \), \( \text{Pic}(X) \), is
\[
\text{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_{14}
\]
and that the canonical divisor of \( X \), \( K_X \), is
\[
K_X = -2H_0 - 2H_1 + E_1 + \cdots + E_{14}
\]
It is also known that the intersection form, i.e. the intersection numbers of pairs of base elements, is
\[
H_i \cdot H_j = 1 - \delta_{i,j}, \quad E_k \cdot E_l = -\delta_{k,l}, \quad H_i \cdot E_k = 0
\]
where \( \delta_{i,j} \) is 1 if \( i = j \) and 0 if \( i \neq j \), and the intersection numbers of any pairs of divisors are given by their linear combinations.

**Remark.** Let \( X \) be a rational surface. It is known that \( \text{Pic}(X) \), the group of isomorphism classes of invertible sheaves of \( X \), is isomorphic to the following groups.

i) The group of linear equivalent classes of divisors on \( X \).

ii) The group of numerically equivalent classes of divisors on \( X \), where divisors \( D \) and \( D' \) on \( X \) are numerically equivalent if and only if for any divisors \( D'' \) on \( X \), \( D \cdot D'' = D' \cdot D'' \) holds.

Hence we identify them in this paper.

The (linear equivalent classes of) prime divisors in [4], [5] as elements of \( \text{Pic}(X) \) are described as
\[
C_0 = E_4, \quad C_1 = E_8, \quad C_2 = E_{14}, \quad C_3 = H_0 - E_5, \quad C_4 = H_1 - E_1 (\text{\(-1 \) curve})
\]
$$D_0 = E_1 - E_2, \quad D_1 = E_2 - E_3, \quad D_2 = E_3 - E_4,$$
$$D_3 = E_5 - E_6, \quad D_4 = E_6 - E_7, \quad D_5 = E_7 - E_8,$$
$$D_6 = E_9 - E_{10}, \quad D_7 = E_{11} - E_{12}, \quad D_8 = E_{12} - E_{13}, \quad D_9 = E_{13} - E_{14} \quad (-2 \text{ curve})$$
$$D_{10} = H_0 - E_1 - E_2 - E_9, \quad D_{11} = H_1 - E_5 - E_6 - E_9,$$
$$D_{12} = E_{10} - E_{11} - E_{12} \quad (-3 \text{ curve})$$

where by $n$ curve we mean a curve whose self-intersection number is $n$. See Fig. 2.

The anti-canonical divisor $-K_X$ can be reduced uniquely (see appendix A) to prime divisors as

$$D_0 + 2D_1 + D_2 + D_3 + 2D_4 + D_5$$
$$+3D_6 + D_7 + 2D_8 + D_9 + 2D_{10} + 2D_{11} + 2D_{12}$$

and the connection of $D_i$ are expressed by the following diagram.

Figure 2:
The HV eq. (1) acts on curves as (6). Hence the HV eq. acts on Pic(X) as

\[
\begin{pmatrix}
H_0 \\
H_1, E_1, E_2 \\
E_3, E_4, E_5, E_6 \\
E_7, E_8, E_9, E_{10} \\
E_{11}, E_{12}, E_{13}, E_{14}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10} \\
H_0, H_0 - E_8, H_0 - E_7 \\
H_0 - E_6, H_0 - E_5, E_{11}, E_{12} \\
E_{13}, E_{14}, H_0 - E_{10}, H_0 - E_9 \\
E_1, E_2, E_3, E_4
\end{pmatrix}
\]

(this table means $\overline{H_0} = 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10}$, $\overline{H_1} = H_0, \overline{E_1} = H_0 - E_8$ and so on) and their linear combinations.

**Remark.** Let $\theta$ be an isomorphism from the rational surface $X$ to the rational surface $X'$. Let $D$ be a divisor and $[D]$ its class. The class of $\theta(D)$ coincides with the class $\theta([D]) \in \text{Pic}(X')$ and the action of $\theta$ on Pic(X) ($\cong \text{Pic}(X')$) is linear.

### 3.2 Cremona isometries and the root system

**Definition.** An automorphism $s$ of Pic(X) is called a *Cremona isometry* if the following three properties are satisfied:

a) $s$ preserves the intersection form in Pic(X);
b) $s$ leaves $K_X$ fixed;
c) $s$ leaves the semigroup of effective classes of divisors invariant.

In general, if a birational mapping can be lifted to an isomorphism from $X$ to $X'$ by blow-ups, its action on the resulting Picard group is always a Cremona isometry. We will show that the group of Cremona isometries is an extended Weyl group of hyperbolic type. In the next section we will show these Cremona isometries can be realized as Cremona transformations, i.e. birational mappings, on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma 1** Let $s$ be a Cremona isometry, then
c') $s$ is an automorphism of the diagram (8).

**Proof.** First we show that for any $i \in \{0, 1, \cdots, 12\}$ there exists $j \in \{0, \cdots, 12\}$ such that $D_i = s(D_j)$. Notice that $-K_X$ can be uniquely reduced to prime divisors in the form $-K_X = \sum_i m_iD_i$ (see (8)) and the condition b). We have $s(-K_X) = -K_X = \sum_i m_i s(D_i)$, where all $s(D_i)$ are effective divisors due to the condition c) (and moreover $D_i \cdot D_i = s(D_i) \cdot s(D_i)$). By the uniqueness of decomposition of $-K_X$, we have that for any $i$ there exists $j$ such that $D_i = s(D_j)$ and $m_i = m_j$. According to this fact and the condition a) we have the lemma. (Another proof is shown in (11, 15))

Let us define $< D_i >$ and $< D_i >^\perp$ as

\[
< D_i > = \sum_{i=0}^{12} \mathbb{Z}D_i
\]

and

\[
< D_i >^\perp = \{ \alpha \in \text{Pic}(X); \alpha \cdot D_i = 0 \text{ for } i = 0, 1, \cdots, 12 \}.
\]

**Lemma 2** The Cremona isometry $s$ leaves $< D_i >^\perp$ invariant.
Proof. Let $\alpha \in < D_i >^\perp$. By the condition $c'$ we have that for any $i \in \{0, \ldots, 12\}$ there exists $j \in \{0, \ldots, 12\}$ such that $s(\alpha) \cdot D_i = s(\alpha) \cdot s(D_j) = \alpha \cdot D_j = 0$. It implies $s(\alpha) \in < D_i >^\perp$.

In this case $< D_i >^\perp$ can be written as

$$< D_i >^\perp = < \alpha_i > := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$$

where

$$\begin{align*}
\alpha_1 &= 2H_1 - E_1 - E_2 - E_3 - E_4, \\
\alpha_2 &= 2H_0 - E_5 - E_6 - E_7 - E_8, \\
\alpha_3 &= 2H_0 + 2H_1 - 2E_9 - 2E_{10} - E_{11} - E_{12} - E_{13} - E_{14}.
\end{align*}$$

We consider $< \alpha_i >$ with the intersection form to be a root lattice with a symmetric bilinear form. Let us define the transformation $w_i (i = 1, 2, 3)$ on $< \alpha_i >$ as

$$w_i (\alpha) = \alpha - 2 \frac{\alpha_i \cdot \alpha}{\alpha_i \cdot \alpha_i} \alpha_i$$

for $\alpha \in < \alpha_i >$. The transformation $w_i (\alpha_j)$ has the form $w_i (\alpha_j) = \alpha_j - c_{ij} \alpha_i$, where $c_{ij} = 2(\alpha_i \cdot \alpha_j)/(\alpha_i \cdot \alpha_i)$, and the matrix $c_{ij}$ becomes a generalized Cartan matrix. Here, the generalized Cartan matrix and its Dynkin diagram are of the hyperbolic type $H_{71}^{(5)}$ as follows:

$$\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

and

$$\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{array}$$

Hence the group $W$ generated by the actions $w_1, w_2, w_3$, is a Weyl group of hyperbolic type. The extended (including the full automorphism group of the Dynkin diagram) Weyl group, $\bar{W}$, is generated by

$$\{w_1, w_2, w_3, \sigma_{12}, \sigma_{13}\}$$

and the fundamental relations:

$$w_i^2 = \sigma_{ij}^2 = 1, \quad (\sigma_{12}\sigma_{13})^3 = 1,$$

$$\begin{align*}
\sigma_{12}w_1 &= w_2\sigma_{12}, & \sigma_{12}w_2 &= w_1\sigma_{12}, & \sigma_{12}w_3 &= w_3\sigma_{12} \\
\sigma_{13}w_1 &= w_3\sigma_{13}, & \sigma_{13}w_2 &= w_2\sigma_{13}, & \sigma_{13}w_3 &= w_1\sigma_{13}
\end{align*}$$

(15)

where the action of $\sigma_{12}$ or $\sigma_{13}$ on $< \alpha_i >$ is defined by the exchange of indices of $\alpha_i$; the action of $\sigma_{ij}$ and $w_k$ on $\alpha_i$ can be summarized as follows:

$$\begin{array}{c|c|c|c|c|c|c}
\sigma_{12} & \sigma_{13} & w_1 & w_2 & w_3 \\
\hline
\alpha_1 & \alpha_2 & \alpha_3 & -\alpha_1 & \alpha_1 + 2\alpha_2 & \alpha_1 + 2\alpha_3 \\
\alpha_2 & \alpha_1 & \alpha_2 & -\alpha_2 & \alpha_2 + 2\alpha_3 \\
\alpha_3 & \alpha_1 & \alpha_3 & -\alpha_3 & \alpha_3 + 2\alpha_2 \\
\end{array}$$

Moreover, by the following property we have the fact that the group of Cremona isometries is included in $\pm \bar{W}$.
PROPOSITION 1 ([10] §5.10) If the generalized Cartan Matrix $c_{ij}$ is a symmetric matrix of finite, affine, or hyperbolic type, then the group of all automorphisms of $<\alpha_i>$ preserving the bilinear form is $\pm\tilde{W}$.

Remark. If $s$ is a Cremona isometry, then $-s$ can not satisfy the condition c).

Next we consider uniqueness for the extension of action of $\tilde{W}$ to the action on Pic($X$).

LEMMA 3 Let $s$ and $s'$ be Cremona isometries such that the action of $s$ is identical to the action of $s'$ on $<\alpha_i>$, then $s = s'$ as Cremona isometries, i.e. $s$ is identical to $s'$ as an automorphism of Pic($X$).

Proof. Let $s$ and $s'$ be Cremona isometries such that the actions of $s$ is identical to the action of $s'$ on $<\alpha_i>$. The actions of $s'\circ s^{-1}$ on $<\alpha_i>$ is the identity.

We investigate where the exceptional divisor $E_4$ is moved by the action of $s'\circ s^{-1}$. In $\{D_i ; i = 0, \cdots, 12\}$, only $D_2$ has an intersection with $E_4$. By the condition c'), $s'\circ s^{-1}(D_2)$ is $D_0$, $D_2$, $D_3$, $D_5$, $D_7$ or $D_9$ and only $s'\circ s^{-1}(D_2)$ has an intersection with $s'\circ s^{-1}(E_4)$ in $\{s'\circ s^{-1}(D_i) ; i = 0, \cdots, 12\}$.

i) Assume $s'\circ s^{-1}(D_2) = D_2$. $s'\circ s^{-1}(E_4)$ has an intersection only with $D_2$ in $\{D_i ; i = 0, \cdots, 12\}$ (this condition on the coefficients of basis of Pic($X$) can be considered to be a system of linear equations of order 13). Then we have the general solution with three integers $z_1, z_2, z_3$:

$$s'\circ s^{-1}(E_4) = E_4 + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3.$$ 

Multiplying this equation by $s'\circ s^{-1}(\alpha_1) = \alpha_1$, we have the system of linear equations:

$$\begin{cases}
1 - 4z_1 + 4z_2 + 4z_3 = 1 \\
0 + 4z_1 - 4z_2 + 4z_3 = 0 \\
0 + 4z_1 + 4z_2 - 4z_3 = 0
\end{cases}$$

It implies $s'\circ s^{-1}(E_4) = E_4$.

ii) Assume $s'\circ s^{-1}(D_2) = D_0$. We have $s'\circ s^{-1}(E_4) = (H_1 - E_1) + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3$.

Multiplying this equation by $s'\circ s^{-1}(\alpha_1) = \alpha_1$, one has that this equation does not have integer solutions.

iii) The other cases. $s'\circ s^{-1}(D_2) = D_3, D_5, D_7$ or $D_9$ implies $s'\circ s^{-1}(E_4) = L + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3$, where $L = H_0 - E_5, E_8, H_0 + H_1 - E_9 - E_{10} - E_{11}$ or $E_{14}$ respectively. This implies that this equation does not have integer solutions.

According to i),ii) and iii) $s'\circ s^{-1}(E_4) = E_4$ and $s'\circ s^{-1}(D_2) = D_2$.

Analogously we have $s'\circ s^{-1}(H_1 - E_1) = H_1 - E_1$ and $s'\circ s^{-1}(D_0) = D_0$ and so on. Due to this fact and the condition c'), $s'\circ s^{-1}$ must be the identity as an Cremona isometry. This implies the lemma.

Next we consider the extension of actions of elements of $\tilde{W}$ on $<\alpha_i>$ to the actions on Pic($X$). Let us define $\alpha_{i,j}$ ($i = 1, 2, 3$; $j = 1, 2$) as $\alpha_{1,1} = H_1 - E_1 - E_4$, $\alpha_{1,2} = H_1 - E_2 - E_3$, $\alpha_{2,1} = H_0 - E_5 - E_7$, $\alpha_{2,2} = H_0 - E_6 - E_7$, $\alpha_{3,1} = H_0 + H_1 - E_9 - E_{10} - E_{11} - E_{14}$, $\alpha_{3,2} = H_0 + H_1 - E_9 - E_{10} - E_{12} - E_{13}$ and define the action of $\alpha_{i,j}$ ($i = 1, 2, 3$; $j = 1, 2$) on $\alpha \in <\alpha_i>$ as $w_{i,j}(\alpha) := \alpha - 2\frac{\alpha_{i,j} \cdot \alpha}{\alpha_{i,j} \cdot \alpha_{i,j}} \alpha_{i,j}$. 

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It is easy to see that \( w_1(\alpha) = w_{i,1} \circ w_{i,2}(\alpha) = w_{i,2} \circ w_{i,1}(\alpha) \). We define the action of \( w_i \) on \( D \in \text{Pic}(X) \) as \( w_i(D) := w_{i,1} \circ w_{i,2}(D) = w_{i,2} \circ w_{i,1}(D) \). These actions are explicitly written as follows (See Fig. 2). (For brevity we did not write the invariant elements under each action.)

\[
\begin{align*}
\sigma_{12} : \begin{pmatrix} H_0, & H_1, & E_1, & E_2, & E_3 \end{pmatrix} & \rightarrow \begin{pmatrix} H_0, & H_1, & E_0, & E_5, & E_6, & E_7 \end{pmatrix} \\
\sigma_{13} : \begin{pmatrix} E_3, & E_4, & E_9, & E_{10} \end{pmatrix} & \rightarrow \begin{pmatrix} E_3, & E_4, & E_9, & E_{10} \end{pmatrix}
\end{align*}
\]

(16)

We define the action of \( \sigma_{12} \) and \( \sigma_{13} \) on \( \text{Pic}(X) \) as follows.

\[
\begin{align*}
\sigma_{12} : \begin{pmatrix} H_0, & H_1, & E_1, & E_2, & E_3 \end{pmatrix} & \rightarrow \begin{pmatrix} H_0, & H_1, & E_0, & E_5, & E_6, & E_7 \end{pmatrix} \\
\sigma_{13} : \begin{pmatrix} E_3, & E_4, & E_9, & E_{10} \end{pmatrix} & \rightarrow \begin{pmatrix} E_3, & E_4, & E_9, & E_{10} \end{pmatrix}
\end{align*}
\]

(17)

By direct calculation, it is easy to check that each \( w_i \) (or \( \sigma_{1i} \)) expressed by (15) (or (17) resp.) acts on \( < \alpha_i > \) as (11) (or as the exchanges of indices of \( \alpha_i \) resp.) and that they satisfy the fundamental relations (13) (the later property is of course guaranteed by the uniqueness of extension of \( \tilde{W} \)). Moreover it is also easy to check that the actions of all elements of \( \tilde{W} \) on \( \text{Pic}(X) \) satisfy the conditions a), b) and c').

**THEOREM 2** The group of Cremona isometries of \( X \) is isomorphic to \( \tilde{W} \), where \( \tilde{W} \) is generated by \( \{ w_1, w_2, w_3, \sigma_{12}, \sigma_{13} \} \) and the fundamental relations (13). The actions of elements of \( \tilde{W} \) on \( \text{Pic}(X) \) are given by (16) and (17) and their composition.

To show this theorem, it is enough to show that (16) and (17) satisfy the condition c). For this purpose, it is enough to realize them as isomorphisms from \( X \) to \( X' \), where \( X \) and \( X' \) have the same semigroup of classes of effective divisors. We show this fact in the next subsection.

From (16) and (17) it is straightforward to show that the action of the HV eq. on \( \text{Pic}(X) \) is identical to the action of \( w_2 \circ \sigma_{13} \circ \sigma_{12} \).

**COROLLARY 1** There does not exist any Cremona isometry of \( X \) whose action on \( \text{Pic}(X) \) commutes with the action of the HV eq. except \( (w_2 \circ \sigma_{13} \circ \sigma_{12})^m \), where \( m \in \mathbb{Z} \).

**Proof.** In this proof we denote \( \sigma_{12} \) or \( \sigma_{13} \) by \( \sigma_2 \) or \( \sigma_3 \) respectively and omit the symbol of composition \( \circ \). Each element of \( W \) can be uniquely written in the form

\[
w_{i_1} w_{i_2} \cdots w_{i_n} s
\]

where all indices of \( w \) (or \( \sigma \)) are considered in \( \text{Mod} \ 3 \) (or \( 2 \) resp.) and \( i_j \neq i_{j+1} \) and \( s \in \{ 1, \sigma_j, \sigma_j \sigma_{j+1}, \sigma_{j+1} \sigma_j \} \). Assume \( g = w_{i_1} w_{i_2} \cdots w_{i_n} s \) commutes with \( w_2 \sigma_3 \sigma_2 \).

i) The case of \( s = 1 \). According to the relation

\[
w_2 \sigma_3 \sigma_2 w_{i_1} w_{i_2} \cdots w_{i_n} = w_{i_1} w_{i_2} \cdots w_{i_n} w_2 \sigma_3 \sigma_2,
\]

we have the relation

\[
w_{i_1} w_{i_2} \cdots w_{i_n} = w_{i_1} w_{i_2} \cdots w_{i_n} w_2 \sigma_3 \sigma_2.
\]
It implies \( i_1 \equiv 2, i_2 \equiv 3, \ldots, i_n \equiv n + 1, 2 \equiv n + 2 \) and therefore there exists the integer \( m \) such that \( n = 3m \). On the other hand \((w_2\sigma_3\sigma_2)^{3m} = w_2w_3 \cdots w_{n+1}\). It implies \( g = (w_2\sigma_3\sigma_2)^{3m}\).

ii) The case of \( s = \sigma_3\sigma_2 \) or \( \sigma_2\sigma_3 \). Similar to the case i), \( n \) must be \( n = 3m + 1 \) or \( n = 3m + 2 \) respectively and \( g \) becomes \((w_2\sigma_3\sigma_2)^n\).

iii) The case of \( s = \sigma_j \). Suppose \( j = 2 \). According to the relation

\[
w_2\sigma_3\sigma_2w_1w_{i_2} \cdots w_{i_n}\sigma_2 = w_1w_{i_2} \cdots w_{i_n}\sigma_2w_2\sigma_3\sigma_2,
\]

we have the relation

\[
w_2w_{i_1+1}w_{i_2+1} \cdots w_{i_n+1}\sigma_3 = w_{i_1}w_{i_2} \cdots w_{i_n}w_1\sigma_2\sigma_3\sigma_2.
\]

It implies \( \sigma_3 = \sigma_2\sigma_3\sigma_2 \) but this is a contradiction. Similarly \( s = \sigma_3 \) is impossible.

v) The case \( s = \sigma_3\sigma_3 \). Suppose \( j = 2 \). Similar to the case iii), we have the relation

\[
w_2w_{i_1+1}w_{i_2+1} \cdots w_{i_n+1}\sigma_2 = w_{i_1}w_{i_2} \cdots w_{i_n}w_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2.
\]

It implies \( \sigma_2 = \sigma_2\sigma_3\sigma_2\sigma_3\sigma_2 \) and hence \( 1 = \sigma_3\sigma_2\sigma_3\sigma_2 \) which leads to a contradiction. Similarly it can be shown that \( s = \sigma_3\sigma_2\sigma_3 \) is impossible.

\[\square\]

**Conclusion of the section.**

We have shown in this section that a) the “Dynkin diagram” of the irreducible components of the anti-canonical divisor \(-K_X\) does not correspond to a generalized Cartan matrix, b) the group of Cremona isometries is isomorphic to an extended Weyl group of hyperbolic type and c) there does not exist a Cremona isometry which is commutative with the action of the HV eq. except itself, while for the discrete Pailevé equations a) affine type, b) affine type and c) there does exist such a Cremona isometries except for the trivial exceptions (the type of root system is \(A_1^{(1)}\) etc.).

### 4 The inverse problem

A birational mapping is called a Cremona transformation. One method for obtaining a Cremona transformation such that its action on \(\text{Pic}(X)\) is a Cremona isometry is to interchange the blow down structures, i.e. to interchange the procedure of blow downs. Following this method, we construct the Cremona transformations on \(\mathbb{P}^1 \times \mathbb{P}^1\) which yield the extended Weyl group (14), (15) and thereby recover the HV eq. from its action on \(\text{Pic}(X)\).

An element of \(\tilde{W}\) is an automorphism of \(\text{Pic}(X)\) but does not have to be an automorphism of \(X\) itself, i.e. the blow-up points can be changed with a transformation satisfying the condition a), b) and c) in Section 3.2. In order to do this, one has to consider not only autonomous but also non-autonomous mappings.

By \(a_0, a_1, a_2, a_3, a_4, a_5, a_6\) or \(a_7\) we denote the point of the 10, 3, 4, 7, 8, 11, 13, 14-th blow-up or the corresponding coordinates and we call them “the parameters”. In short these points can be expressed as follows:

\[
\left( \frac{1}{xy}, xy^2 \right) = (0, a_1), \quad \left( \frac{1}{xy}, xy(xy^2 - a_1) \right) = (0, a_2), \quad \left( x^2y, \frac{1}{xy} \right) = (a_3, 0), \quad \left( xy(x^2y - a_3), \frac{1}{xy} \right) = (a_4, 0),
\]

\[
\left( \frac{1}{xy}, xy^2 \right) = (0, a_1), \quad \left( \frac{1}{xy}, xy(xy^2 - a_1) \right) = (0, a_2), \quad \left( x^2y, \frac{1}{xy} \right) = (a_3, 0), \quad \left( xy(x^2y - a_3), \frac{1}{xy} \right) = (a_4, 0),
\]
\[
\left( \frac{1}{x}, \frac{x}{y} \right) = (0, a_0), \quad \text{where we normalize } a_0 \text{ to be } a_0 = 1,
\]
\[
\left( \frac{1}{x}, x(\frac{x}{y} - 1) \right) = (0, a_5), \quad \left( \frac{1}{x(\frac{x}{y} - 1) - a_3}, x(\frac{x}{y} - 1) - a_5 \right)^2 = (0, a_6),
\]
\[
\left( \frac{1}{x(\frac{x}{y} - 1) - a_3}, x(\frac{x}{y} - 1) - a_5 \right) \left\{ x(\frac{x}{y} - 1) - a_5 \right\}^2 - a_6 \right) = (0, a_7),
\]
where \( a_i \in \mathbb{C} \) and \( a_1, a_3, a_6 \) are nonzero.

The point of the 2, 6, 12-th blow-up is determined by intersection numbers. Moreover the point of the 1, 5, 9, 10, 11-th blow-up can be fixed by acting with a suitable automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \), i.e. a Möbius transformation of each coordinate combination with an exchange of the coordinates. We call this operation “normalization”. It can also be seen that we can normalize \( a_5 = 1 \) except the case \( a_5 = 0 \).

In this section we consider a realization of the generating elements of \( \tilde{W} \) as Cremona transformations which can be lifted to isomorphisms from \( \mathbb{X} \) to \( \mathbb{X} \), where \( \mathbb{X} \) is the same rational surface as \( X \) except for a difference in parameters.

First we realize the action of \( w_2 \) as a Cremona transformation on \( \mathbb{P}^1 \times \mathbb{P}^1 \)

### 4.1 The calculation of interchanging the blow down structure

In the following we shall present a scheme in which the blow down structure is changed. This method is based on the following fact:

By \( F_n \) we denote the \( n \)-th Hirzebruch surface with the coordinate system

\[
(O, \Delta) \cup (O, \frac{1}{\Delta}) \cup (\frac{1}{O}, O^n \Delta) \cup (\frac{1}{O}, \frac{1}{O^n \Delta}).
\]

Blowing up the \( n \)-th Hirzebruch surface at the point \((1/O, O^n \Delta) = (0, 0)\) and blowing down along the line \( 1/O = 1/(O^{n+1} \Delta) = 0 \), we obtain the \( n + 1 \)-th Hirzebruch surface as follows:

\[
\begin{array}{cccc}
\text{up} & (O, \Delta) \cup (O, \frac{1}{\Delta}) & \cup (\frac{1}{O}, O^n \Delta) & \cup (\frac{1}{O}, \frac{1}{O^n \Delta}) \\
\text{down} & (O, \Delta) \cup (O, \frac{1}{\Delta}) & \cup (\frac{1}{O}, O^n \Delta) & \cup (\frac{1}{O}, \frac{1}{O^n \Delta})
\end{array}
\]

On the other hand, blowing up the \( n \)-th Hirzebruch surface at the point \((1/O, 1/(O^n \Delta)) = (0, 0)\) and blowing down along the line \( 1/O = O^{n-1} \Delta = 0 \), we obtain the \( n - 1 \)-th Hirzebruch surface as follows:

\[
\begin{array}{cccc}
\text{up} & (O, \Delta) \cup (O, \frac{1}{\Delta}) & \cup (\frac{1}{O}, O^n \Delta) & \cup (\frac{1}{O}, \frac{1}{O^n \Delta}) \\
\text{down} & (O, \Delta) \cup (O, \frac{1}{\Delta}) & \cup (\frac{1}{O}, O^n \Delta) & \cup (\frac{1}{O}, \frac{1}{O^{n-1} \Delta})
\end{array}
\]

The Next figure shows the order of the blow-ups and the blow downs to obtain the Cremona transformation corresponding to \( w_2 \). (This table has to be read from the left to the right.)
where double lines mean the lines which are blown down in the next steps.

The calculation of changing the blow down structure is as follows.

\[
(x, y) \cup (x, \frac{1}{y}) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) = \mathbb{P}^1 \times \mathbb{P}^1
\]

\[
\begin{align*}
&\leftarrow (x, y) \cup (x, \frac{1}{xy}) \cup (xy, \frac{1}{y}) \cup (\frac{1}{x}, \frac{1}{y}) \\
&\rightarrow (x, xy) \cup (x, \frac{1}{xy}) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) = F_1
\end{align*}
\]

\[
\begin{align*}
&\leftarrow (x, xy) \cup (x, \frac{1}{x^2y}) \cup (x^2y, \frac{1}{x}) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) \\
&\rightarrow (x, x^2y) \cup (x, \frac{1}{x^2y}) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) = F_2
\end{align*}
\]

\[
\begin{align*}
&\sim (x, x^2y - a_3) \cup (x, \frac{1}{x^2y - a_3}) \cup (\frac{1}{x}, \frac{x^2 - a_3}{x}) \cup (\frac{1}{x}, \frac{x^2}{x}) = F_2 \\
&\leftarrow (x, x^2y - a_3) \cup (\frac{x}{x^2y - a_3}, x^2y - a_3) \cup (x, \frac{1}{x^2y - a_3}) \cup (\frac{1}{x}, \frac{x^2 - a_3}{x}) \cup (\frac{1}{x}, \frac{x^2}{x}) = F_2
\end{align*}
\]

\[
\begin{align*}
&\rightarrow (x, x^2y - a_3) \cup (x, \frac{1}{x^2y - a_3}) \cup (\frac{1}{x}, \frac{x^2 - a_3}{x}) \cup (\frac{1}{x}, \frac{x^2}{x^2y - a_3}) = F_1 \\
&\sim (x, \frac{x^2y - a_3}{a_3x}) \cup (x, \frac{a_3x}{a_3(x^2y - a_3) - a_4x}) \ldots = F_1
\end{align*}
\]
\[ \begin{align*}
\left( x, \frac{a_3(x^2y - a_3) - a_4x}{a_3x^2} \right) \cup \left( \frac{a_3x^2}{a_3(x^2y - a_3) - a_4x} \right) \\
\cup \left( x, \frac{a_3(x^2y - a_3) - a_4x}{a_3x} \right) \cup \cdots \\
\left( x, \frac{a_3(x^2y - a_3) - a_4x}{a_3} \right) \cup \left( \frac{a_3x^2}{a_3(x^2y - a_3) - a_4x} \right) \cup \cdots = \mathbb{P}^1 \times \mathbb{P}^1
\end{align*} \]

where \( \cdots \) is
\[ \left( \frac{1}{x}, \frac{a_3(x^2y - a_3) - a_4x}{a_3x^2} \right) \cup \left( \frac{1}{x}, \frac{a_3x^2}{a_3(x^2y - a_3) - a_4x} \right) \]

and \( \sim \) means an automorphism of the Hirzebruch surface and is determined as the point of blow-up in (13) is moved to the origin.

Writing
\[ w'_2 : (x, y) \mapsto (x, y - \frac{a_3}{x^2} - \frac{a_4}{a_3x}) \]
we obtain \( w_2 = t \circ w'_2 \) where \( t \) is an automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \). By taking a suitable \( t \), we can normalize \( w_2 \) to get the required result.

### 4.2 Normalization and the action on the space of parameters

First we determine the automorphism for normalization \( t \). By (14), \( w_2 \) does not move the points \((x, y) = (\infty, 0), (\infty, \infty) \). According to the fact: \( w'_2 : (\infty, 0) \mapsto (\infty, 0), (\infty, \infty) \mapsto (\infty, \infty) \), \( t \) is reduced to the mapping \( t : (x, y) \mapsto (c_1x + c_2, c_3y) \), where \( c_1, c_2, c_3 \in \mathbb{C} \) are nonzero constants.

Similarly, \( w_2 \) moves the point \( a_3 \) to the proper transform of the point of the 6-th blow-up. We denote this fact as
\[ \left( \frac{u_6, v_6}{u_6, v_6} \right) \mid_{(u_6, v_6) = (a_3, 0)} = (0, 0), \]

where \((u_n, v_n)\) is the coordinate of the \( n \)-th blow-up. On the other hand,
\[ \left. \frac{(u_6, v_6)}{(u_6, v_6) = (a_3, 0)} \right|_{(u_6, v_6) = (a_3, 0)} = \left( \frac{x^2y}{x^2}, 1 \right) \]
\[ = \left( x^2(y - \frac{a_3}{x^2} - \frac{a_4}{a_3x}), \frac{a_3x}{a_3(x^2y - a_3) - a_4x} \right) \mid_{(u_6, v_6) = (a_3, 0)} = \left( -a_3 + u_6 - \frac{a_4u_6v_6}{a_3} - \frac{a_3u_6v_6}{a_3^2 - a_3u_6 + a_4u_6v_6} \right) \mid_{(u_6, v_6) = (a_3, 0)} = (0, 0) \]

holds. Hence \( t \) does not move the point \((x, y) = (0, \infty)\) and therefore \( c_2 = 0 \).

Similarly, since \( w_2 \) does not move the points of the 10-th and the 11-th blow-ups, we have \( c_1 = c_3 = 1 \) (moreover we can normalize \( a_5 \) to be \( a_5 = 1 \) or \( a_5 = 0 \) by taking a suitable value of \( c_1 = c_3 \)). Hence \( t \) has to be the identity.

Next we calculate how the parameters \( a_1, a_2, a_3, a_4, a_6, a_7 \) are changed by the action of \( w_2 \). Notice that \( w_2 \) is an isomorphism from \( X \) to \( X' \), where \( w_2 \) satisfy the conditions a)\,b) and c) in Section 3.2. and therefore \( X \) and \( X' \) have the same sequence of blow-ups except their parameters. By \( \overline{a_i} \) we denote the parameter of the \( i \)-th blow-up of \( X' \).
Since the action of \( w_2 \) moves the points of blow-ups as follows

\[
\begin{align*}
(u_2, v_2) &= (0, a_1) \quad \mapsto \quad (w_2, v_2) = (0, a_1) \\
(u_3, v_3) &= (0, a_2) \quad \mapsto \quad (w_3, v_3) = (0, a_2) \\
(u_6, v_6) &= (0, 0) \quad \mapsto \quad (w_6, v_6) = (a_3, 0) \\
(xy, 1/y) &= (0, 0) \quad \mapsto \quad (w_7, v_7) = (a_4, 0) \\
(u_{12}, v_{12}) &= (0, a_6) \quad \mapsto \quad (w_{12}, v_{12}) = (0, a_6) \\
(u_{13}, v_{13}) &= (0, a_7) \quad \mapsto \quad (w_{13}, v_{13}) = (0, a_7),
\end{align*}
\]

\( \overline{w_1} \) can be calculated. For example \( \overline{w_1} \) is calculated as follows

\[
(0, \overline{a_1}) = \left( \frac{1}{xy} \bar{xy} \bar{xy}^2 - \overline{a_1} \right)_{(u_2, v_2) = (0, a_1)}
= \left( \frac{1}{xy} \bar{xy} \bar{xy}^2 - \overline{a_1} \right)_{(u_3, v_3) = (0, a_2)}
= (0, a_2 - \frac{a_1 a_4}{a_3}),
\]

\[
(\overline{a_3}, 0) = \left( \bar{xy} \bar{xy}^2 - \overline{a_3} \right)_{(u_6, v_6) = (0, 0)}
= (-a_3, 0),
\]

\[
(\overline{a_4}, 0) = \left( \bar{xy} \bar{xy}^2 - \overline{a_3} \right)_{(xy, 1/y) = (0, 0)}
= (a_4, 0).
\]

Consequently \( w_2 \) changes the parameters \( a_i \) as

\[
\begin{align*}
\overline{a_1} &= a_1, \quad \overline{a_2} = a_2 - 2a_1 a_4/a_3, \quad \overline{a_3} = -a_3, \quad \overline{a_4} = a_4, \\
\overline{a_5} &= a_5, \quad \overline{a_6} = a_6, \quad \overline{a_7} = a_7 + 2a_4 a_6/a_3.
\end{align*}
\]

We write the action of \( w_2 \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) and the space of parameters together as

\[
w_2 : \quad (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
\quad \mapsto \quad (\bar{x}, \bar{y}; \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7})
= \left( x, y - \frac{a_3}{a_3 x}, \frac{a_4}{a_3 x}; a_1, a_2 - \frac{2a_1 a_4}{a_3}, -a_3, a_4, a_5, a_6, a_7 + \frac{2a_4 a_6}{a_3} \right).
\]

Here, in the calculation of the next iteration step we have to use \( \overline{a_3} = -a_3 \) instead of \( a_3 \) and so on.

As was remarked before the mapping \( w_2 \) is of order 2 as an element of an extended Weyl group and can be lifted to an isomorphism from \( X \) to \( \overline{X} \).
4.3 The actions of other elements

Next we calculate the action of $\sigma_{13}$ from $X$ to $\overline{X}$.

The following figure shows the order of the blow-ups and the blow downs.

Its calculation is as follows.

\[
(x, y) \cup (\frac{1}{y}, x) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) = \mathbb{P}^1 \times \mathbb{P}^1
\]

\[
\leftarrow (x, y) \cup (\frac{1}{y}, x) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y})
\]

\[
\rightarrow (x, y) \cup (\frac{1}{y}, x) \cup (\frac{1}{x}, y) \cup (\frac{1}{x}, \frac{1}{y}) = F_1
\]

\[
\sim (x, y - x) \cup (x, \frac{1}{y - x}) \cup (\frac{1}{x}, y - x) \cup (\frac{1}{x}, \frac{1}{y - x}) = F_1
\]

\[
\leftarrow (x, y - x) \cup (x, \frac{1}{y - x}) \cup (\frac{1}{x}, y - x) \cup (\frac{1}{x}, \frac{1}{y - x})
\]

\[
\rightarrow (x, y - x) \cup (x, \frac{1}{y - x}) \cup (\frac{1}{x}, y - x) \cup (\frac{1}{x}, \frac{1}{y - x}) = \mathbb{P}^1 \times \mathbb{P}^1
\]

Similar to the case of $w_2$, we have the action of $\sigma_{13}$ on $X$ and the space of parameters as follows

\[
\sigma_{13} : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)
\]

\[
\mapsto (x, x - y - a_5 ; a_6, a_7 - 2a_2a_6, - a_3, a_4, a_5, a_1, a_2 + 2a_1a_5^2) .
\]

Similarly the action of $\sigma_{12}$ on $X$ and the space of parameters is

\[
\sigma_{12} : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)
\]

\[
\mapsto (-y, - x ; - a_3, - a_4, - a_1, - a_2, a_5, - a_6, a_7 - 4a_2^2a_6) .
\]

The action of $w_1$ or $w_3$ is determined by the relation $w_1 = \sigma_{12} \circ w_2 \circ \sigma_{12}$ or $w_3 = \sigma_{13} \circ w_1 \circ \sigma_{13}$ respectively as follows

\[
w_1 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)
\]

\[
\mapsto \left( x - \frac{a_1}{y^2} - \frac{a_2}{a_1y}, y ; - a_1, a_2, a_3, a_4 - \frac{2a_3a_4}{a_1}, a_5, a_6, a_7 + \frac{2a_2a_6}{a_1} \right) .
\]
\[ w_3 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \]
\[ \mapsto \left( x - \frac{a_6}{(x - y - a_5)^2} - \frac{a_7 - 2a_5^2a_6}{a_6(x - y - a_5)}, y - \frac{a_6}{(x - y - a_5)^2} - \frac{a_7 - 2a_5^2a_6}{a_6(x - y - a_5)} \right); \]
\[ a_1, a_2 + \frac{2a_1(a_7 - 2a_5^2a_6)}{a_6}, a_3, a_4 + \frac{2a_3(a_7 - 2a_5^2a_6)}{a_6}, a_5, -a_6, a_7 - 4a_5^2a_6). \]

### 4.4 The non-autonomous HV equation

The composition \( w_2 \circ \sigma_{13} \circ \sigma_{12} \) is reduced to

\[
\begin{align*}
w_2 \circ \sigma_{13} \circ \sigma_{12} : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
\mapsto (-y, x - y - a_5 - \frac{a_1}{y^2} - \frac{a_2}{a_1y}; -a_6, a_7 - 2a_5^2a_6 - \frac{2a_2a_6}{a_1}, -a_1, -a_2, a_5, -a_3, -a_4 - 2a_3a_5^2 + \frac{2a_2a_3}{a_1})
\end{align*}
\]

where \( a_5 \) can be normalized to be \( a_5 = 0 \) or 1.

Of course this mapping satisfies the singularity confinement criterion by construction and in the case of \( a_2 = a_4 = a_5 = a_7 = 0 \) and \( a_1 = a_3 = a_6 = a \) it coincides with the HV eq.\( \text{(1)} \) except their signs. The difference between them comes from the assumption \( a_5 = a_5 \). Assuming \( a_5 = -a_5 \) under the actions of \( w_2, \sigma_{13} \) and \( \sigma_{12} \), we have \(-w_2, -\sigma_{13} \) and \(-\sigma_{12} \) as new \( w_2, \sigma_{13} \) and \( \sigma_{12} \) and therefore \( \text{(19)} \) becomes as follows

\[
\begin{align*}
w_2 \circ \sigma_{13} \circ \sigma_{12} : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
\mapsto (y, -x + y + a_5 + \frac{a_1}{y^2} + \frac{a_2}{a_1y}; a_6, -a_7 + 2a_5^2a_6 + \frac{2a_2a_6}{a_1}, a_1, a_2, -a_5, a_3, a_4 + 2a_3a_5^2 - \frac{2a_2a_3}{a_1}).
\end{align*}
\]

Actually in the case of \( a_2 = a_4 = a_5 = a_7 = 0 \) and \( a_1 = a_3 = a_6 = a \) it coincides with the HV eq.\( \text{(13)} \).

### 5 Some other examples

We present some examples of rational mappings which satisfy the singularity confinement criterion and some of which have positive algebraic entropy.

#### 5.1 Example 1

Let \( w_1, w_2, w_3 \) and \( a_1 \) be as in § 4. First we consider the mapping \( w_1 \circ w_2 \)

\[
\begin{align*}
w_1 \circ w_2 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
\mapsto \left( x - \frac{a_1}{y^2} - \frac{a_2}{a_1y}, y - \frac{a_3}{(x - a_1y)^2} - \frac{a_4 - 2a_2a_3/a_1}{a_3(x - a_1y)^2 - a_2/(a_1y)} \right); \\
-a_1, a_2 + \frac{2(2a_1a_4 - 2a_2a_3)}{a_3}, -a_3, -a_4 + \frac{2a_2a_3}{a_1}, a_5, a_6, a_7 + \frac{2a_2a_6}{a_1} + 2(a_4 - 2a_2a_3)/a_3)
\end{align*}
\]

This mapping has the following properties.

1) This mapping satisfies the singularity confinement criterion.
2) The order of the \( n \)th iterate of mapping is \( O(n^2) \) (easily seen from the action on the
root lattice $\langle \alpha_i \rangle$ [17].

3) The actions on the parameters $a_5, a_6, a_7$ can be ignored.

This mapping is nothing but one of the discrete Painlevé equations, since the surface obtained by blowing down the curves $E_9, E_{10}, \cdots, E_{14}$ in $X$ is also the space of initial values and the type of Dynkin diagram corresponding to the irreducible components of anti-canonical divisor is $D^{(1)}_7$ with the symmetry $A^{(1)}_1$. Actually the irreducible components of anti-canonical divisor are

$$ E_1 - E_2, E_2 - E_3, E_3 - E_4, E_5 - E_6, E_6 - E_7, E_7 - E_8,$$

$$ H_0 - E_1 - E_2, H_1 - E_5 - E_6$$

and the root basis of orthogonal lattice is

$$\alpha_1 = 2H_1 - E_1 - E_2 - E_3 - E_4,$$

$$\alpha_2 = 2H_0 - E_5 - E_6 - E_7 - E_8.$$

Next we consider the mapping $w_3 \circ w_2 \circ w_1$. This mapping is almost identical to the nonautonomous HV eq. after 3 steps. Actually the latter becomes $w_2 \circ w_3 \circ w_1$.

At last we consider the mapping $w_2 \circ w_3 \circ w_2 \circ w_1$. This mapping satisfies the singularity confinement criterion and its algebraic entropy is $17 + 12\sqrt{2}$.

### 5.2 Example 2

We consider the following diagram as irreducible components of the anti-canonical divisor.

This diagram is realized by the sequence of blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ as follows

$$(x, y) \xrightarrow{(\infty, \infty)} \frac{1}{x y} \xrightarrow{\mu_1} \frac{1}{x y^2} \xrightarrow{\mu_2} \frac{1}{x y^3} \xrightarrow{\mu_3} \frac{1}{x y^4} \xrightarrow{\mu_4} \frac{1}{x y^5} \xrightarrow{\mu_5} \frac{1}{x y^6}$$

and

$$(x, y) \xrightarrow{\mu_7} \frac{1}{x} \xrightarrow{\mu_8} \frac{1}{x^2} \xrightarrow{\mu_9} \frac{1}{x^3} \xrightarrow{\mu_{10}} \frac{1}{x^4} \xrightarrow{\mu_{11}} \frac{1}{x^5} \xrightarrow{\mu_{12}} \frac{1}{x^6}$$

and

$$(x, y) \xrightarrow{\mu_{13}} \frac{1}{x} \xrightarrow{\mu_{14}} \frac{1}{x^2} \xrightarrow{\mu_{15}} \frac{1}{x^3} \xrightarrow{\mu_{16}} \frac{1}{x^4} \xrightarrow{\mu_{17}} \frac{1}{x^5} \xrightarrow{\mu_{18}} \frac{1}{x^6} \xrightarrow{\mu_{19}} \frac{1}{x^7}$$
where we denote \( z := x(\frac{x}{y} - 1) - a_7 \),

\[
\frac{1}{x^2} \cdot z = \frac{1}{x^2} \cdot x \cdot z^3 = \frac{1}{x^2} \cdot xz^2(xz^3 - a_8) \cdot \frac{1}{x^2} \cdot xz^2(xz^3 - a_8) - a_9
\]

\[
\frac{1}{x^2} \cdot xz^2(xz^3 - a_8) - a_9
\]

Similar to the case of the HV eq. (§ 4), we obtain

\[ w_2 : (x; y; a_1, a_2, \cdots, a_{10}) \]

\[
\mapsto (x, y - \frac{a_4}{x^3} \cdot \frac{a_5}{a_4x^2} - \frac{a_6}{a_4} - \frac{a_2^2}{a_4^2} \cdot \frac{1}{x} ; a_1, a_2, a_3 + \frac{3a^2_7a_9}{a_4^2}, -a_4, a_5, -a_6, a_7, a_8, a_9, a_{10} - \frac{3a^2_7a_9}{a_4^2} + \frac{a_6a_9^2}{a_4^2})
\]

\[ \sigma_{13} : (x, y; a_1, a_2, \cdots, a_{10}) \]

\[
\mapsto (x, y - a_7; a_8, a_9, a_{10} - 3a^2_7a_9, -a_4, a_5, -a_6, a_7, a_8, a_9, a_{10} + 6a_7^2a_9^2)
\]

\[ \sigma_{12} : (x, y; a_1, a_2, \cdots, a_{10}) \]

\[
\mapsto (-y, -x; a_4, -a_5, a_6, a_1, -a_2, a_3, a_7, -a_8, a_9, -a_{10} + 6a_7^2a_9^2)
\]

and finally

\[
w_2 \circ \sigma_{13} \circ \sigma_{12} : (x; y; a_1, a_2, \cdots, a_{10}) \mapsto
\]

\[
(-y, -y + x - a_7 - \frac{a_1}{y^2} - \frac{a_2}{a_1y^2} + \frac{a_3}{a_1^2} + \frac{a_2^2}{a_1} \cdot \frac{1}{y}; -a_8, a_9, -a_{10} + 3a^2_7a_9^2 - \frac{3a^2_7a_9}{a_1^2} + \frac{3a^2_7a_9}{a_1^2}, a_1, -a_2, a_3, a_7, a_8, -a_5, a_6 + 3a^2_7a_9^2 - \frac{3a^2_7a_9}{a_1^2} - \frac{3a^2_7a_9}{a_1^2})
\]

In the case \( a_i = 0 \) for \( i = 2, 3, 5, 6, 9, 10 \), this mapping reduces to

\[
w_2 \circ \sigma_{13} \circ \sigma_{12} : (x, y; a_1, a_4, a_7, a_8)
\]

\[
\mapsto (-y, -y - a_7 + \frac{a_1}{y^2}; -a_8, a_1, a_7, a_4).
\]

We present some basic properties of this mapping.

The Picard group of the space of initial values is

\[ \text{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_20 \]

and the canonical divisor of \( X \) is

\[ K_X = -2H_0 - 2H_1 + E_1 + \cdots + E_{20} \]

The irreducible components of the anti-canonical divisor are

\[ E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4 - E_5, E_5 - E_6, \]

\[ E_6 - E_7, E_7 - E_8, E_8 - E_9, E_9 - E_{10}, E_{11} - E_{12}, \]

\[ E_{12} - E_{13}, E_{13} - E_{14}, E_{14} - E_{15}, E_{15} - E_{16}, E_{16} - E_{17}, \cdots, E_{19} - E_{20}, \]

\[ H_0 - E_1 - E_2 - E_3 - E_{13}, H_1 - E_5 - E_6 - E_7 - E_{13}, E_{14} - E_{15} - E_{16} - E_{17} \]
and the root basis is
\[
\begin{align*}
\alpha_1 &= 3H_1 - E_1 - E_2 - E_3 - E_4 - E_5 - E_6, \\
\alpha_2 &= 3H_0 - E_7 - E_8 - E_9 - E_{10} - E_{11} - E_{12}, \\
\alpha_3 &= 3H_0 + 3H_1 - 3E_{13} - 3E_{14} - E_{15} - E_{16} - E_{17} - E_{18} - E_{19} - E_{20}.
\end{align*}
\]

The Cartan matrix \(2(\alpha_i \cdot \alpha_j)/(\alpha_i \cdot \alpha_i)\) is
\[
\begin{bmatrix}
2 & -3 & -3 \\
-3 & 2 & -3 \\
-3 & -3 & 2
\end{bmatrix}.
\tag{20}
\]

This Cartan matrix is not finite, affine nor hyperbolic type.

The algebraic entropy is \(2 + \sqrt{3}\).

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### A Uniqueness of the decomposition of the anti-canonical divisor

**Theorem 3** Let \(X\) be the space of initial values of the HV eq. obtained in Section 2. The anti-canonical class of divisors \(-K_X\) can be reduced uniquely to prime divisors as
\[
D_0 + 2D_1 + D_2 + 2D_3 + 2D_4 + D_5 \\
+ 3D_6 + 2D_7 + 2D_8 + D_9 + 2D_{10} + 2D_{11} + 2D_{12}
\]

**Proof.** Suppose that \(-K_X\) is reduced to prime divisors as \(-K_X = \sum_{i=1}^{m} f_i F_i\), where \(F_i\) is a prime divisor and \(f_i\) is a positive integer. \(F_i\) has the form
\[
h_0^{(i)} H_0 + h_1^{(i)} H_1 - \sum_{j=1}^{14} e_j^{(i)} E_j,
\]
where \(h_j\) and \(e_j\) are nonnegative integers and \(h_j \leq 2\) and moreover \(h_0\) or \(h_1\) is strictly positive (remember that \(-K_X = 2H_0 + 2H_1 - \sum_{j=1}^{14} E_j\)), or in the case the curve is included in the total transform of a blow-up point:
\[
E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4, E_5 - E_6, E_6 - E_7, E_7 - E_8, E_8, \\
E_9 - E_{10}, E_{10} - E_{11}, E_{11} - E_{12}, E_{12} - E_{13}, E_{13} - E_{14}, E_{14}.
\]

Let us first suppose that there does not exist \(F_i\) such that \(F_i = E_{13} - E_{14}\). Then there exists a \(F_i\) such that \(F_i\) has the form as \(h_0 H_0 + h_1 H_1 - \sum_{j=1}^{14} e_j E_j\) where \(e_{14}\) is strictly positive. This means that \(F_i\) has an intersection point with \(E_{14}\) and therefore this divisor passes the point of the 14-th blow-up point before the blow-up. Namely,
\[
\left(1 \over x^2(x/y - 1), x^2(y/x - 1)(x^3(y/x - 1)^2 - a)\right) = (0,0)
\]
as \(x, y \to \infty\). Denoting \(u = 1/x, v = 1/y\), we have \(u^3/(v-u) = 0\) and \((v-u)(v^2 - 2v + 1 - au^5)/u^8 = 0\) as \(u, v \to 0\). This implies \(v^2 - 2v + 1 - au^5 = o(u^8)\) and hence we have \(h_0 \geq 5\), which is a contradiction.
Similarly there does not exist a divisor which has the form \( h_0 H_0 + h_1 H_1 - \sum_{j=1}^{13} e_j E_j \) where \( e_{12} \) or \( e_{13} \) is strictly positive and \( h_j \leq 2 \).

Now, we can consider the case where there does not exist a \( F_i \) such that \( F_i = E_{14} \). Then according to the above result, there should exist integers \( j, k \) such that \( F_j = E_{13} - E_{14} \), \( F_k = E_{12} - E_{13} \) and \( f_j = 1, f_k = 2 \) (in the absence of a prime divisor \( E_{14} \), the form of \(-K_X\) forces \( f_j = 1 \) and subsequently \( f_k = 2 \) in order to have the correct \( E_{13} \) dependence). Considering the new coefficient of \( E_{12} \), we have that the sum of the coefficients of \( E_{11} - E_{12} \) and \( E_{10} - E_{11} - E_{12} \) has to be 3.

Then we have the following possibilities:

i) the coefficient of \( E_{11} - E_{12} \) is 0;
ii) the coefficient of \( E_{11} - E_{12} \) is 1;
iii) the coefficient of \( E_{11} - E_{12} \) is 2;
iv) the coefficient of \( E_{11} - E_{12} \) is 3.

In the case i), ii), iii) or iv) the coefficient of \( E_{11} \) is \(-3, -1, 1 \) or 3 respectively. Hence i) is impossible. To pass the point of the 11-th blow-up, a divisor whose class has the form \( h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j \) must satisfy the equation \( u = 0 \) and \((v-u)/u^2 = 0\). This implies \( h_0 \geq 2, h_1 \geq 1 \) and \( e_{11} = 1 \) and therefore iii) and iv) are impossible. Consequently the coefficient of \( E_{11} - E_{12} \) has to be 1 and the coefficient of \( E_{10} - E_{11} - E_{12} \) has to be 2.

Along the same lines, considering the coefficient of \( E_{10} \), we have the following possibilities:

a) the coefficient of \( E_9 - E_{10} \) is 0, in which case the coefficient of \( E_9, E_{10} \) is 0, 2;

b) the coefficient of \( E_9 - E_{10} \) is 1, in which case the coefficient of \( E_9, E_{10} \) is 1, 1;

c) the coefficient of \( E_9 - E_{10} \) is 2, in which case the coefficient of \( E_9, E_{10} \) is 2, 0;

d) the coefficient of \( E_9 - E_{10} \) is 3, in which case the coefficient of \( E_9, E_{10} \) is 3, -1.

To pass the point of the 10-th blow-up, a divisor whose class has the form \( h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j \) must satisfy the equation \( u = 0 \) and \( v/u = 1 \). It implies two possibilities: i) \( h_0 \geq 1, h_1 \geq 1, e_9 = e_{10} = 1 \) and \( e_2, e_3, e_4, e_6, e_7, e_8 = 0 \) (since if \( e_2 \) would be strictly positive, the intersection of the divisor \( h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j \) and \( x = \infty \) (= \( H_0 \) is greater than or equal to 3, which is impossible) or ii) \( h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j = 2H_0 + 2H_1 - 2E_9 - 2E_{10} \). Therefore a) is impossible. In the case b) the coefficient of \( E_9 \) becomes \(-1 \) (or \( 1, 3 \) in the cases c), d) respectively) and the coefficients of \( H_0 \) and \( H_1 \) become equal to 2 (or become greater than or equal to 1, become 0 in the cases c), d) respectively).

In the case b) the coefficients of \( E_1, E_2, \cdots, E_8 \) is 0 and hence the remaining contribution \(-E_1 - \cdots - E_8 \) must be a sum of the terms:

\[
E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4 - E_5, E_5 - E_6, E_6 - E_7, E_7 - E_8, E_8
\]  

(21)

with nonnegative integer coefficients. Obviously this is impossible. In the case c) the remaining contribution has the form \( b_0' H_0 + b_1' H_1 - b_1 E_1 - \cdots - b_8 E_8 - 2E_9 \), where \( b_0, b_1 \) = 0 or 1, \( b_2, b_3, b_4, b_6, b_7, b_8 = 1 \) and \( b_1, b_2 = 0 \) or 1 and hence it must be a sum of \( H_0 - E_1 - E_2 - E_3, H_1 - E_5 - E_6 - E_9 \) and \( E_0 \) with nonnegative integer coefficients. By straight forward discussion it can be seen that this case is also impossible.

The coefficient of \( E_9 - E_{10} \) therefore has to be equal to 3 and hence the coefficient of \( E_9 \) has to be 3 as well, from which point on it can be easily seen that the remaining contributions to \(-K_X\) are uniquely determined.

Similarly, starting from the supposition that there does exist a \( F_i \) such that \( F_i = E_{14} \), similar proof can be given.\( \square \)
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