The time-development of energy spectrum of a quantized vortex ring

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Abstract. The present work is devoted to the study of time dependence of the energy spectrum that is associated with quantized vortex ring movement in the superfluid helium at different temperatures. Temperature changes from 1.3 K to 1.9 K. The full Biot-Savart equation is used to calculate velocity fields. The energy spectrum is calculated by using the method of structure functions. It is established that the structure function amplitude is associated with energy dissipation, while the energy spectrum is independent of temperature and time. The relationship between the energy dissipation rate and the vortex ring length is also discussed.

1. Introduction
Superfluid or quantum turbulence attracts significant attention from the theoretical and experimental points of view [1]. Generally, superfluid turbulence is represented by a tangle of quantized vortex lines, loops and rings that originate in the superfluid component of helium. Of particular interest in this area are the energy spectra of various vortex configurations and the mechanisms of their formation. Additional interest is associated with active attempts to explain some properties of classical turbulence in terms of quantum turbulence [1]. However, in the general case, the problem is extremely complex. An additional factor complicating the task is the time dependence of the energy spectra that takes place during the development of vortex tangles [2]. A reasonable approach to solving the general problem is to analyze the time dependence of the energy spectrum, created by the simplest element of the vortex tangle, that is, by the vortex ring. The vortex ring is a relatively well theoretically studied object; see, for example, [3–5]. Moreover, the dynamics of the vortex ring and the substantially deformed vortex loop are in many respects similar [6]. Thus, the main goal of this work is to study the time dependence of the energy spectrum created by a single smooth vortex ring in superfluid helium. The velocity field related to vortex ring is studied by using the vortex filament method and the Biot-Savart equation. The spectral characteristics of the velocity field are determined according to the method of structure functions.

2. Methodology for calculating the energy spectrum

The energy spectrum $E(k)$ describes the distribution of energy in the space of wave numbers $k$. To find it, the method of structure functions is used, namely, the longitudinal structure velocity function of the second order is calculated [7–9]:

$$C_2(l) = \left\langle \delta v^2(l) \right\rangle,$$
Here we use the following notation: \( \mathbf{r} \) is the radius vector that determines the position of a fluid element in space, and \( \mathbf{v} \) is the velocity of this element. Angle brackets mean averaging over the entire ensemble of points for which the velocity was determined, i.e. over all possible pairs of points separated by a distance \( l \). Using Fourier transform it can be shown that the dependence \( C_\alpha(l) \sim l^\alpha \) in the coordinate space is equivalent to the dependence of \( E(k) \sim k^{-\alpha-1} \) in the wavenumber space. To find the second-order structure function, it is necessary to calculate the velocity field created by the vortex ring. The dynamics of a vortex ring is calculated on the basis of the vortex filament method using the full Biot-Savart equation; for more details, see, for example, [10]. Next, the velocity field corresponding to the found configuration of the vortex ring is calculated (see, for example, Figure 1). For this, the full Biot-Savart equation is used:

\[
\mathbf{V}_s(\mathbf{r}) = \frac{\kappa}{4\pi L} \frac{(\mathbf{s} - \mathbf{r}) \times d\mathbf{s}}{|\mathbf{s} - \mathbf{r}|},
\]

where \( \mathbf{s} \) is the radius vector of the points on the vortex ring, and \( \kappa \) is the quantum of circulation. Integration is carried out along the entire length of the vortex ring \( L \).

**Figure 1.** An example of the velocity field generated by single smooth vortex ring.

A specific calculation of the velocity field values was carried out at fifteen thousand points (in order to obtain acceptable accuracy), selected randomly from the region occupied by the vortex configuration. The region for determining the velocity field was chosen in the form of a cube. The cube edge was equal to 0.005 cm; for comparison, the initial radius of the vortex ring was equal to 0.001 cm. It should also be noted that the vortex loop moved in space, therefore the center of the volume in which the velocity field was calculated superposed on the "center of mass" of the vortex ring, i.e. the region in which the velocity field was calculated moved with the vortex ring. Thus, the velocity field was found, that was used to calculate the second-order structure function and the corresponding energy spectrum.

### 3. Results and discussion

It should be mentioned at once that the calculations at a temperature near absolute zero will not be considered in detail, although they did take place. The fact is that at this temperature the momentum,
angular momentum and energy of the vortex ring are preserved, and this should lead to the invariance of the energy spectrum. This hypothesis was confirmed by numerical simulation, i.e. we obtained a spectrum that does not change with time and has the form $E(k) \sim k^{-3/2}$. The main calculations were carried out at nonzero temperatures, namely: 1.3 K, 1.6 K, and 1.9 K. However, the results obtained at different temperatures turned out to be qualitatively similar. The differences in the results for different temperatures consisted in the compression rate of the vortex rings, as well as in the rate of change of the amplitudes of the structure functions; therefore, without loss of generality, we give below the results only for a temperature of 1.9 K. Spectral characteristics of velocity fields associated with a vortex ring, which is compressed by the friction force, were calculated according to the algorithm given in the previous section. The change in the length of the vortex ring with time is shown in Figure 2; here, round hollow markers denote the lengths of the vortex rings (and corresponding times) at which the velocity fields’ spectra were calculated. As can be seen from the figure, the spectra were calculated for seven different lengths of the vortex ring.

![Figure 2](image)

**Figure 2.** The dependence of the length of the single vortex ring on time at temperature of 1.9 K. The round hollow markers indicate the length values at which the velocity field spectrum was calculated.

Figure 3 shows the structure functions calculated for the corresponding seven vortex ring lengths. With a decrease in the length of the vortex ring, the amplitude of the structure functions also decreases, that occurs due to energy dissipation during the interaction of the vortex ring with the normal component of superfluid helium. However, the type of the spectrum is almost unchanged, as it is evidenced by the approximating curves for the structure functions. In Figure 3, these curves are shown by solid lines. They are all proportional to $l^{1/2}$ that corresponds to the spectrum $E(k) \sim k^{-3/2}$.

![Figure 3](image)

**Figure 3.** Dotted lines represent the dependence of the structure functions on $l$ for different times. The solid lines correspond to functions $-l^{1/2}$. The horizontal dashed line corresponds to function $-l^0$. 
Another way of representing power dependences is to construct them in logarithmic coordinates. So the data corresponding to Figure 3 are presented in Figure 4. It should be noted that the linear slope for the six configurations well illustrates the obtained power-law dependences. However, for the seventh configuration corresponding to the vortex ring before its collapse, a region close to horizontal appears in the structure function. In figures 3 and 4 it is marked with a dashed line. The horizontal section of the structure function corresponds to the spectrum $E(k) \sim k^{-1}$ characteristic of a straight, smooth vortex line. This circumstance seems natural, given the small size of the vortex loop in comparison with the linear size of the computational domain of the energy spectrum.

**Figure 4.** The dependences are similar to those shown in Figure 3, but they are plotted in logarithmic coordinates.

We consider the amplitude of the structure function in more detail. In [11] Kolmogorov obtained (for homogeneous isotropic turbulence in an incompressible fluid) a relation between the second-order structure function and the energy dissipation rate $\varepsilon$, namely $C_\nu(l) = \varepsilon^{2/3} l^{1/3}$. In the case under consideration $C_\nu(l) = \alpha l^{1/2}$, where $\alpha$ is the amplitude of the structure function at a particular moment in time. From the approximating curves the dependence of the amplitude of the second-order structure function on time was calculated. In Figure 5, the time dependence $c \cdot \alpha$ is superimposed on the time dependence of the vortex ring length $L$. Here, $c$ is an empirically selected coefficient equal to 0.135. In other words, $L = 0.135 \cdot \alpha$. Since, like normal turbulence, the amplitude of the structure function must be related to the rate of energy dissipation, there is an intention to establish a specific relationship between the length of the vortex line and the rate of energy dissipation. However, numerical calculations do not allow establishing the dimension of the found proportionality coefficient. In this regard, the correspondence of the time dependences of the vortex ring length and the amplitude of the structural function only hint at a functional connection $L = f(\varepsilon)$, but do not allow establishing its specific form.
Figure 5. The dependence of the length of the single vortex ring on time at temperature of 1.9 K. The round solid markers indicate the value of function $c \cdot \alpha$.

Conclusions
Thus, in current research, the time dependence of the energy spectrum of the velocity field associated with a smooth vortex ring has been analyzed. It is shown that the type of the spectrum remains almost unchanged up to the moment of the collapse of the vortex loop and has the form $E(k) \sim k^{-3/2}$. The change in the structure function amplitude is associated with energy dissipation. In turn, the energy dissipation for a smooth single vortex ring is due to the only possible mechanism, and in this case, it is friction between the vortex ring and the normal component of superfluid helium. In addition, the energy dissipation rate is shown to functionally relate to the length of the vortex line.

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