MORSE SUBSETS OF CAT(0) SPACES ARE STRONGLY
CONTRACTING

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Abstract. We prove that Morse subsets of CAT(0) spaces are strongly contracting. This generalizes and simplifies a result of Sultan, who proved it for Morse quasi-geodesics. Our proof goes through the recurrence characterization of Morse subsets.

In this note we give a short proof of the following technical result:

Proposition. If \( Z \) is a closed, \( \rho \)-recurrent subset of a CAT(0) space then \( Z \) is \( 12\rho(21) \)-strongly contracting.

This is the final piece of the following theorem, which says that a number of properties that are equivalent to quasi-convexity in hyperbolic spaces are also equivalent to one another in CAT(0) spaces:

Theorem. Let \( X \) be a geodesic metric space. Let \( Z \) be a closed, unbounded subset of \( X \). For \( x \in X \), let \( \pi_Z(x) := \{ z \in Z \mid d(x, z) = d(x, Z) \} \). The following are equivalent:

- \( Z \) is Morse: There is a function \( \mu \): \([1, \infty) \times [0, \infty) \to [0, \infty) \) defined by \( \mu(L, A) := \sup_{(L, A)} \sup_{w \in \gamma} d(w, Z) \), where the first supremum is taken over \( (L, A) \)-quasi-geodesic segments \( \gamma \) with both endpoints on \( Z \).

- \( Z \) is contracting: There is a function \( \sigma \): \([0, \infty) \to [0, \infty) \) defined by \( \sigma(r) := \sup_{d(x, y) \leq d(x, Z) \leq r} \) \( \frac{\text{diam} \pi_Z(x) \cup \pi_Z(y)}{} \) that satisfies \( \lim_{r \to \infty} \frac{\sigma(r)}{r} = 0 \).

- \( Z \) is recurrent: There is a function \( \rho \): \([1, \infty) \to [0, \infty) \) defined by \( \rho(q) := \sup_{\Delta(\gamma) \leq q} \inf_{w \in \gamma} d(w, Z) \), where the first supremum is taken over rectifiable segments \( \gamma \) with endpoints \( z, z' \in Z \) such that \( \Delta(\gamma) := \frac{\text{len}(\gamma)}{d(z, z')} \leq q \) and \( Z' = Z \) with the open balls of radius \( d(z, z')/3 \) about \( z \) and \( z' \) removed.

If \( X \) is hyperbolic or CAT(0) then these conditions are equivalent to:

- \( Z \) is strongly contracting: \( Z \) is contracting and the contraction gauge \( \sigma \) is a bounded function.

We refer the reader to [2] for background on hyperbolic and CAT(0) spaces.

Corollary. Morse subsets of CAT(0) spaces are strongly contracting.

The corollary confirms a conjecture of Russell, Spriano, and Tran [6] and generalizes results of Sultan [7], who proved that Morse quasi-geodesics in CAT(0) spaces are strongly contracting, and Genevois [5], who proved that Morse subsets of CAT(0) cube complexes are strongly contracting.

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The condition that $Z$ is closed is inessential. It guarantees that the empty set is not in the image of $\pi_Z$. This hypothesis can be avoided by defining $\pi_Z(x) := \{ z \in Z \mid d(x,z) \leq d(x,Z) + 1 \}$. Extra bookkeeping is then required to compute an explicit contraction bound in the proof of the proposition.

The four properties are trivially satisfied for bounded sets, with the possible exception that recurrence can fail if some $Z'$ is empty. For example, a two point set is not recurrent, but its contraction gauge is bounded by its diameter.

Proof of the theorem. The contraction condition was introduced in [1], where it was shown to be equivalent to the Morse condition. The recurrence condition was used to characterize Morse quasi-geodesics in [4], and this characterization can be extended to arbitrary subsets, as in [3, Theorem 2.2]. Strong contraction obviously implies contraction. It is easy to see that all of these properties are equivalent to quasi-convexity in hyperbolic spaces. The proposition supplies the remaining implication.

Proof of the proposition. Define $D := \rho(21)$. Supposing the contraction gauge $\sigma$ of $Z$ is not bounded by $12D$, we derive a contradiction. Failure of the contraction bound means there exist points $x, y \in X$ such that $d(x, y) \leq d(x, Z)$ and such that $diam(\pi_Z(x) \cup \pi_Z(y)) > 12D$. We may assume $d(x, Z) \geq d(y, Z)$, because otherwise $d(x, y) \leq d(y, Z')$ and we can swap the roles of $x$ and $y$. Choose $x' \in \pi_Z(x)$ and $y' \in \pi_Z(y)$ such that $P := d(x', y') > 12D$. Let $Z'$ denote the set $Z$ with the open balls of radius $P/3$ about $x'$ and $y'$ removed.

For points $a, b \in X$, let $[a, b] : [0, 1] \to X$ denote the geodesic segment from $a$ to $b$, parameterized proportional to arc length. Concatenation is denoted `$+$'.

\begin{align*}
(*) & \quad \text{If } d(w, Z') \leq D \text{ for some } w \in X \text{ then } w \notin [x', x] + [x, y] + [y, y'].
\end{align*}

To see this, first suppose $w \in [x', x]$. Then $x' \in \pi_Z(w)$, so $P/3 \leq d(x', Z') \leq d(x', w) + d(w, Z') = d(w, Z) + d(w, Z') \leq 2d(w, Z') \leq 2D$, which is a contradiction, since $P > 12D$. Similarly, $w \notin [y', y]$. If $w \in [x, y]$ then:

$$d(x, w) + d(w, y) = d(x, y) \leq d(x, Z) \leq d(x, w) + D$$

Thus, $d(w, y) \leq D$, which implies:

$$P/3 \leq d(y', Z') \leq d(y', y) + d(y, Z') \leq 2d(y, Z') \leq 2d(y, w) + d(w, Z') \leq 4D$$

Again, this contradicts the hypothesis that $P > 12D$, so $(*)$ is verified.

Case 1, $d(x, x') \leq 6P$: Define $\gamma := [x', x] + [x, y] + [y, y']$. Then $\text{len}(\gamma) \leq 18P < 21P$, so recurrence says there is a point $w \in \gamma$ such that $d(w, Z') \leq D$. By $(*)$, this is impossible.
Case 2, \(d(x, x') > 6P \) and \(d(y, y') \leq 4P\): Let \(a := [x', x]\left(\frac{6P}{d(x, y)}\right)\) and \(b := [y, x]\left(\frac{6P}{d(x, y)}\right)\), so that \(d(a, x') = \frac{6P}{d(x, y)} \cdot d(x, x') = 6P\) and \(d(b, y) = \frac{6P}{d(x, y)} \cdot d(x, y) \leq 6P\). Since \(d(x', y) \leq 5P\), the CAT(0) condition implies \(d(a, b) < 5P\). Define \(\gamma := [x', a] + [a, b] + [b, y] + [y, y']\). Since \(\text{len}(\gamma) \leq 6P + 5P + 6P + 4P = 21P\), recurrence says there is a point \(w \in \gamma\) with \(d(w, Z') \leq D\). By \((\ast)\), \(w \in [a, b]\), but this is impossible because \(d(a, b, Z) \geq d(a, Z) - d(a, b) > 6P - 5P = P > D\).

Case 3, \(d(x, x') > 6P\) and \(d(y, y') > 4P\): Let \(a := [x', x]\left(\frac{4P}{d(x, y)}\right)\) and let \(c := [y', y]\left(\frac{4P}{d(x, x')}\right)\). Then \(d(x, a) = 4P\) and:

\[
4P \leq d(y', c) = \frac{4P}{d(x, x')} \cdot d(x, x') \leq \frac{4P}{d(x, x')} \cdot (d(x, x') + P) \leq \frac{14}{3} P
\]

Let \(b\) be the point of \([y', y]\) at distance \(4P\) from \(y'\), and let \(e\) be the point of \([y', x]\) at distance \(4P\) from \(y'\), so \(d(c, e) \leq \frac{2}{3} P\). The CAT(0) condition implies that \(d(a, c) < P\) and, since \(d(x, y) \leq d(x, y')\), that \(d(e, b) \leq 4\sqrt{2} P\).

Define \(\gamma := [x', a] + [a, c] + [c, e] + [e, b] + [b, y']\). Then \(\text{len}(\gamma) < 4P + P + \frac{2}{3} P + 4\sqrt{2} P + 4P < 21P\), so recurrence demands a point \(w \in \gamma\) with \(d(w, Z') \leq D\). By \((\ast)\), \(w \notin [a', a] + [b, y']\). We cannot have \(w \in [a, c] + [c, e]\) because \(d([a, c] + [c, e], Z) \geq d(a, c) + d(c, e) \geq 4P - P - \frac{2}{3} P > D\). Thus, \(w \in [e, b]\), so \(d(e, b) = d(e, w) + d(w, b)\). However, \(d(w, b) \geq d(b, Z) - d(w, Z) \geq 4P - D > \frac{11}{12} P\). By the same reasoning, \(\frac{11}{12} P < d(a, w)\), but \(d(a, w) \leq P + \frac{2}{3} P + d(e, w)\), so \(d(e, w) > \frac{11}{12} P\).

This gives us a contradiction:

\[
6P < \frac{74}{12} P < P < d(e, w) + d(w, b) = d(e, b) \leq 4\sqrt{2} P < 6P
\]

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