A class of non-convex polytopes that admit no orthonormal basis of exponentials

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Abstract

A conjecture of Fuglede states that a bounded measurable set \( \Omega \subset \mathbb{R}^d \), of measure 1, can tile \( \mathbb{R}^d \) by translations if and only if the Hilbert space \( L^2(\Omega) \) has an orthonormal basis consisting of exponentials \( e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle \). If \( \Omega \) has the latter property it is called spectral. Let \( \Omega \) be a polytope in \( \mathbb{R}^d \) with the following property: there is a direction \( \xi \in S^{d-1} \) such that, of all the polytope faces perpendicular to \( \xi \), the total area of the faces pointing in the positive \( \xi \) direction is more than the total area of the faces pointing in the negative \( \xi \) direction. It is almost obvious that such a polytope \( \Omega \) cannot tile space by translation. We prove in this paper that such a domain is also not spectral, which agrees with Fuglede’s conjecture.

Let \( \Omega \) be a measurable subset of \( \mathbb{R}^d \), which we take for convenience to be of measure 1. Let also \( \Lambda \) be a discrete subset of \( \mathbb{R}^d \). We write

\[
E_\Lambda = \{ e_\lambda : \lambda \in \Lambda \} \subset L^2(\Omega).
\]

The inner product and norm on \( L^2(\Omega) \) are

\[
\langle f, g \rangle_\Omega = \int_{\Omega} f \overline{g}, \quad \| f \|_\Omega^2 = \int_{\Omega} |f|^2.
\]

**Definition 1** The pair \( (\Omega, \Lambda) \) is called a spectral pair if \( E_\Lambda \) is an orthonormal basis for \( L^2(\Omega) \). A set \( \Omega \) will be called spectral if there is \( \Lambda \subset \mathbb{R}^d \) such that \( (\Omega, \Lambda) \) is a spectral pair. The set \( \Lambda \) is then called a spectrum of \( \Omega \).

**Example:** If \( Q_d = (-1/2, 1/2)^d \) is the cube of unit volume in \( \mathbb{R}^d \) then \( (Q_d, \mathbb{Z}^d) \) is a spectral pair (d-dimensional Fourier series).

We write \( B_R(x) = \{ y \in \mathbb{R}^d : |x - y| < R \} \).

**Definition 2** (Density) The discrete set \( \Lambda \subset \mathbb{R}^d \) has density \( \rho \), and we write \( \rho = \text{dens} \Lambda \), if we have

\[
\rho = \lim_{R \to \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|},
\]

uniformly for all \( x \in \mathbb{R}^d \).

We define translational tiling for complex-valued functions below.

**Definition 3** Let \( f : \mathbb{R}^d \to \mathbb{C} \) be measurable and \( \Lambda \subset \mathbb{R}^d \) be a discrete set. We say that \( f \) tiles with \( \Lambda \) at level \( w \in \mathbb{C} \), and sometimes write “\( f + \Lambda = w \mathbb{R}^d \)”, if

\[
\sum_{\lambda \in \Lambda} f(x - \lambda) = w, \quad \text{for almost every (Lebesgue) } x \in \mathbb{R}^d,
\]
with the sum above converging absolutely a.e. If $\Omega \subset \mathbb{R}^d$ is measurable we say that $\Omega + \Lambda$ is a tiling when $1_\Omega + \Lambda = w\mathbb{R}^d$, for some $w$. If $w$ is not mentioned it is understood to be equal to 1.

**Remark 1**
If $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, and $f + \Lambda = w\mathbb{R}^d$, then the set $\Lambda$ has density

$$\text{dens } \Lambda = \frac{w}{\int f}.$$}

The following conjecture is still unresolved in all dimensions and in both directions.

**Conjecture:** (Fuglede [F74]) If $\Omega \subset \mathbb{R}^d$ is bounded and has Lebesgue measure 1 then $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if there exists $\Lambda \subset \mathbb{R}^d$ such that $\Omega + \Lambda = \mathbb{R}^d$ is a tiling.

Fuglede’s conjecture has been confirmed in several cases.

1. Fuglede [F74] shows that if $\Omega$ tiles with $\Lambda$ being a lattice then it is spectral with the dual lattice $\Lambda^*$ being a spectrum. Conversely, if $\Omega$ has a lattice $\Lambda$ as a spectrum then it tiles by the dual lattice $\Lambda^*$.

2. If $\Omega$ is a convex non-symmetric domain (bounded, open set) then, as the first author of the present paper has proved [K00], it cannot be spectral. It has long been known that convex domains which tile by translation must be symmetric.

3. When $\Omega$ is a smooth convex domain it is clear that it admits no translational tilings. Iosevich, Katz and Tao [IKT] have shown that it is also not spectral.

4. There has also been significant progress in dimension 1 (the conjecture is still open there as well) by Laba [La, Lb]. For example, the conjecture has been proved in dimension 1 if the domain $\Omega$ is the union of two intervals.

In this paper we describe a wide class of, generally non-convex, polytopes for which Fuglede’s conjecture holds.

**Theorem 1** Suppose $\Omega$ is a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that

$$\sum_i \sigma^*(\Omega_i) \neq 0.$$}

The finite sum is extended over all faces $\Omega_i$ of $\Omega$ which are orthogonal to $\xi$ and $\sigma^*(\Omega_i) = \pm \sigma(\Omega_i)$, where $\sigma(\Omega_i)$ is the surface measure of $\Omega_i$ and the $\pm$ sign depends upon whether the outward unit normal vector to $\Omega_i$ is in the same or opposite direction with $\xi$.

Then $\Omega$ is not spectral.

Such polytopes cannot tile space by translation for the following, intuitively clear, reason. In any conceivable such tiling the set of positive-looking faces perpendicular to $\xi$ must be countered by an equal area of negatively-looking $\xi$-faces, which is impossible because there is more (say) area of the former than the latter.
It has been observed in recent work on this problem (see e.g. [K00]) that a domain (of volume 1) is spectral with spectrum $\Lambda$ if and only if $|\hat{\chi}_\Omega|^2 + \Lambda$ is a tiling of Euclidean space at level 1. By Remark 1 this implies that $\Lambda$ has density 1.

By the orthogonality of $e_\lambda$ and $e_\mu$ for any two different $\lambda$ and $\mu$ in $\Lambda$, it follows that
\[
\hat{\chi}_\Omega(\lambda - \mu) = 0. \tag{2}
\]
It is only this property, and the fact that any spectrum of $\Omega$ must have density 1, that are used in the proof.

**Proof of Theorem 1.**

The quantities $P, Q, N, \ell$ and $K$, which are introduced in the proof below, will depend only on the domain $\Omega$. (The letter $K$ will denote several different constants.)

Suppose that $\Lambda$ is a spectrum of $\Omega$. Define the Fourier transform of $\chi_\Omega$ as
\[
\hat{\chi}_\Omega(\eta) = \int_\Omega e^{-2\pi i (x, \eta)} \, dx.
\]
By an easy application of the divergence theorem we get
\[
\hat{\chi}_\Omega(\eta) = -\frac{1}{i|\eta|} \int_{\partial \Omega} e^{-2\pi i (x, \eta)} \left( \frac{\eta}{|\eta|}, \nu(x) \right) \, d\sigma(x), \quad \eta \neq 0,
\]
where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\partial \Omega$ at $x \in \partial \Omega$ and $d\sigma$ is the surface measure on $\partial \Omega$.

From the last formula we easily see that for some $K \geq 1$
\[
|\nabla \hat{\chi}_\Omega(\eta)| \leq \frac{K}{|\eta|}, \quad |\eta| \geq 1. \tag{3}
\]
Without loss of generality we assume that $\xi = (0, \ldots, 0, 1)$. Hence
\[
\hat{\chi}_\Omega(t\xi) = -\frac{1}{it} \int_{\partial \Omega} e^{-2\pi itx, \nu_d(x)} \, d\sigma(x).
\]
Now it is easy to see that each face of the polytope other than any of the $\Omega_i$s contributes $O(t^{-2})$ to $\hat{\chi}_\Omega(t\xi)$ as $t \to \infty$. Therefore
\[
\left| \hat{\chi}_\Omega(t\xi) + \frac{1}{it} \sum_i e^{-2\pi i \lambda_i t} \sigma^* (\Omega_i) \right| \leq \frac{K}{t^2}, \quad t \geq 1, \tag{4}
\]
where $\lambda_i$ is the value of $x_d$ for $x = (x_1, \ldots, x_d) \in \Omega_i$.

Now define
\[
f(t) = \sum_i \sigma^*(\Omega_i) e^{-2\pi i \lambda_i t}, \quad t \in \mathbb{R}.
\]
f is a finite trigonometric sum and has the following properties:

(i) $f$ is an almost-periodic function.

(ii) $f(0) \neq 0$ by assumption. Without loss of generality assume $f(0) = 1$. 
(iii) \( |f'(t)| \leq K \), for every \( t \in \mathbb{R} \).

By (i), for every \( \epsilon > 0 \) there exists an \( \ell > 0 \) such that every interval of \( \mathbb{R} \) of length \( \ell \) contains a translation number \( \tau \) of \( f \) belonging to \( \epsilon \): 

\[
\sup_{t} |f(t + \tau) - f(t)| \leq \epsilon \tag{5}
\]

(see [B32]).

Fix \( \epsilon > 0 \) to be determined later (\( \epsilon = 1/6 \) will do) and the corresponding \( \ell \). Fix the partition of \( \mathbb{R} \) in consecutive intervals of length \( \ell \), one of them being \([0, \ell] \). Divide each of these \( \ell \)-intervals into \( N \) consecutive equal intervals of length \( \ell/N \), where 

\[
N > \frac{6K\ell\sqrt{d-1}}{\epsilon}.
\]

In each \( \ell \)-interval there is at least one \( \frac{\ell}{N} \)-interval containing a number \( \tau \) satisfying (3). For example, in \([0, \ell] \) we may take \( \tau = 0 \) and the corresponding \( \frac{\ell}{N} \)-interval to be \([0, \ell/N] \).

Define the set \( L \) to be the union of all these \( \frac{\ell}{N} \)-intervals in \( \mathbb{R} \). Then \( L \) is a copy of \( L \) on the \( x_d \)-axis. Construct \( M \) by translating copies of the cube \([0, \ell/N] \times \mathbb{R}^d \) along the \( x_d \)-axis so that they have their \( x_d \)-edges on the \( \frac{\ell}{N} \)-intervals of \( L \).

The point now is that there can be no two \( \lambda \)s of \( \Lambda \) in the same translate of \( M \), at distance \( D > \frac{2K}{\epsilon} \) from each other. Suppose, on the contrary, that 

\[
\lambda_1, \lambda_2 \in \Lambda, \quad |\lambda_1 - \lambda_2| \geq D, \quad \lambda_1, \lambda_2 \in M + \eta.
\]

Then \( \lambda_1 = t_1 \xi + \eta_1, \lambda_2 = t_2 \xi + \eta_1 + \eta_2 \), for some \( t_1, t_2 \in L, \eta_1, \eta_2 \in \mathbb{R}^d \) with 

\[
|\eta_1|, |\eta_2| < \frac{\ell}{N}\sqrt{d} - 1 < \frac{\epsilon}{6K}.
\]

Hence, \( \lambda_1 - \lambda_2 = (t_1 - t_2) \xi + \eta_1 - \eta_2 \) and an application of the mean value theorem together with (3) and (4) gives 

\[
|\hat{\chi}_\Omega((t_1 - t_2) \xi)| \leq \frac{3K}{|t_1 - t_2|} |\eta_1 - \eta_2|.
\]

From (4) we get 

\[
|f(t_1 - t_2)| \leq 3K|\eta_1 - \eta_2| + \frac{K}{|t_1 - t_2|} < 2\epsilon.
\]

Now, since \( t_1, t_2 \in L \), there exist \( \tau_1, \tau_2 \) satisfying (3) so that 

\[
|\tau_1 - t_1|, |\tau_2 - t_2| < \frac{\ell}{N}
\]

and hence (by (iii)) 

\[
|f(\tau_1 - \tau_2) - f(\tau_1 - t_2)|, |f(\tau_1 - t_2) - f(t_1 - t_2)| < K\frac{\ell}{N} < \epsilon.
\]

Therefore 

\[
2\epsilon > |f(t_1 - t_2)| \\
\geq |f(0)| - |f(0) - f(-\tau_2)| - |f(-\tau_2) - f(\tau_1 - \tau_2)| \\
- |f(\tau_1 - \tau_2) - f(\tau_1 - t_2)| - |f(\tau_1 - t_2) - f(t_1 - t_2)| \\
\geq 1 - \epsilon - \epsilon - \epsilon - \epsilon.
\]
It suffices to take $\epsilon = 1/6$ for a contradiction.

Therefore, as the distance between any two $\lambda$s is bounded below by the modulus of the zero of $\hat{\chi}_\Omega$ that is nearest to the origin, there exists a natural number $P$ so that every translate of $M$ contains at most $P$ elements of $\Lambda$ and, hence, there exists a natural number $Q$ (we may take $Q = 2NP$) so that every translate of 

$$\mathbb{R}\xi + [0, \frac{\ell}{N}]^d$$

contains at most $Q$ elements of $\Lambda$.

It follows that $\Lambda$ cannot have positive density, a contradiction as any spectrum of $\Omega$ (which has volume 1) must have density equal to 1.

\[\square\]

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