Theoretical investigation of the quantum noise in ghost imaging

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Abstract

Ghost imaging is a method to nonlocally image an object by transmitting pairs of entangled photons through the object and a reference optical system respectively. We present a theoretical analysis of the quantum noise in this imaging technique. The dependence of the noise on the properties of the apertures in the imaging system are discussed and demonstrated with a numerical example. For a given source, the resolution and the signal-to-noise ratio cannot be improved at the same time.

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In recent years, there has been an increasing interest in the topic of ghost imaging. Ghost imaging is a kind of correlated imaging and relies on the quantum entanglement of the photon pairs created by spontaneous parametric down conversion (SPDC). The photons of a pair are spatially separated. One of them propagates through a known (reference) optical imaging system, while the other travels through an unknown (test) optical imaging system in which an object is placed. By measuring the coincidence rate of these photon pairs at the reference and test detectors, one can obtain the image of the object as a function of the transverse position of the reference photon. The theory of ghost imaging was firstly studied by Klyshko. Then the experiments were demonstrated in mid-1990s. A systematic theory was given by the Boston group in [3, 4]. Gatti et al. generalized the theory to deal with the case in which the number of entangled photons are large [7].

Resolution and noise are two of the most important factors to characterize an imaging system. Resolution of a ghost imaging system has been investigated in some papers. In references [4] and [5], the authors have analyzed the dependence of imaging resolution on some physical parameters. However, the noise in ghost imaging attracts little attention. In this paper, we give a theoretical investigation of the quantum noise in ghost imaging. Based on the general theory given in [3, 4], we derive a formula to calculate the variance of the coincidence rate. Especially, our main interest is to study the dependence of the quantum noise on the impulse respond function of both reference and test imaging systems. For simplicity, the possible fluctuations from the detectors are not considered. We find that, when the imaging system has a large cutoff frequency (corresponding to good resolution), the variance of the coincidence rate may be very large and leads to bad signal-to-noise ratio (SNR).

Let’s consider the setup of a ghost imaging system given in Fig 1. The source $S$ produces pairs of entangled photons. These photon pairs are transmitted through a reference optical system and a test optical system which contains the object to be imaged. These two optical systems are characterized by their impulse response function $h_r(x, x_r)$ and $h_t(x, x_t)$ respectively. Two detectors $D_1$ and $D_2$ record the intensity distribution of the test and reference photons. The coincidence rate of photon pairs at these two detectors ($G^{(2)}(x_r, x_t)$)
FIG. 1: A setup of entangled ghost imaging. A pump field incidents on a nonlinear crystal $S$. The source $S$ emits pairs of entangled photons. One of the photons transmits through the test system $h_t(x,t)$ which contains an unknown object, and the other photon transits the reference system with a known $h_r(x',r)$. Two detectors $D_t$ and $D_r$ record the intensity distribution. The coincidence rate $G^{(2)}(x_r,x_t)$ is measured to give an image of the object.

is proportional to the fourth-order correlation function

$$G^{(2)}(x_r,x_t) = \left| \int dxdx' \varphi(x,x') h_t(x,t) h_r(x,r,x') \right|^2,$$

(1)

where $\varphi(x,x')$ represents the wave function of entangled photons.

For simplicity, we define an operator

$$\hat{S}(x_t,x_r) = \hat{E}^{-}_t(x_t) \hat{E}^{-}_r(x_r) \hat{E}^{+}_t(x_t) \hat{E}^{+}_r(x_r),$$

(2)

where $\hat{E}^{-}_t(x_t)$, $\hat{E}^{-}_r(x_r)$, $\hat{E}^{+}_t(x_t)$, $\hat{E}^{+}_r(x_r)$ are operators for the negative and positive frequency portions of the test and reference photons at the positions $x_t$ and $x_r$. As shown in Ref [3], the probability of coincidence of photons at the positions $x_t$ and $x_r$ is the expected value of operator $\hat{S}$

$$G^{(2)}(x_r,x_t) = \langle \Psi | \hat{S}(x_t,x_r) | \Psi \rangle,$$

(3)

The photon pairs can be described by the pure two-photon state as

$$|\Psi\rangle = \int dxdx' \varphi(x,x') \hat{a}^{\dagger}_t(x) \hat{a}^{\dagger}_r(x') |0,0\rangle,$$

(4)

where $|0,0\rangle$ is the vacuum state, $a$ and $a^\dagger$ are creation operators for the test and reference photons at position $x$ and $x'$. The relations between $\hat{E}^{\pm}_t(x_t)$, $\hat{E}^{\pm}_r(x_r)$ and the operators at
\[\hat{E}_t^+(x_t) = \int dx_t h_t(x_t, x) \hat{a}_t(x), \quad (5)\]

\[\hat{E}_r^+(x_r) = \int dx_t h_r(x_r, x') \hat{a}_r(x'). \quad (6)\]

Eq. (5) is derived by directly substituting Eqs. (4,5,6) into Eq. (3).

To study the quantum noise in ghost imaging, we need to calculate the variance of the operator \(\hat{S}\). The quantum fluctuation of the coincidence rate \(\Delta G^{(2)}\) is obtained from

\[\Delta G^{(2)}(x_r, x_t) = \sqrt{\langle \Psi | [\hat{S}(x_t, x_r)]^2 | \Psi \rangle - [G^{(2)}(x_r, x_t)]^2}. \quad (7)\]

By direct calculations, we can obtain a mathematic formula for \(\langle \Psi | [\hat{S}(x_t, x_r)]^2 | \Psi \rangle\),

\[\langle \Psi | [\hat{S}(x_t, x_r)]^2 | \Psi \rangle = G^{(2)}(x_r, x_t) \times \int dx |h_t(x_t, x)|^2 \times \int dx' |h_r(x_r, x')|^2. \quad (8)\]

Eq. (7) and Eq. (8) are the main equations in this paper.

Now, we give some discussions on the quantum noise in ghost imaging. Firstly, when the coincidence rate \(G^{(2)}\) is increasing, the noise \(\Delta G^{(2)}\) is also increasing, but the signal-to-noise ratio (SNR) will be enhanced,

\[\text{SNR} = \frac{G^{(2)}(x_r, x_t)}{\Delta G^{(2)}(x_r, x_t)} = \frac{1}{\sqrt{\int dx|h_t(x_t, x)|^2 \int dx'|h_r(x_r, x')|^2} - 1}. \quad (9)\]

Further, the noise also depends on the two impulse response functions. Since \(\langle \Psi | [\hat{S}(x_t, x_r)]^2 | \Psi \rangle\) is proportional to the integral of \(|h_t|^2\) and \(|h_r|^2\), one needs to carefully design the optical imaging systems to control the noise.

An optical imaging system may contain different optical elements, such as lens and apertures. Generally, all apertures can be projected through an effective exit aperture characterized by its pupil function \(p(x_a)\). As discussed in Fourier optics, for a coherent imaging system, the relation between the image \(u_i(x_i)\) and the object \(u_o(x_o)\) is

\[u(x_i) = \int dx h(x_i - x_o) u_o(M x_o), \quad (10)\]

where \(M\) is the magnification of the system. The impulse response function is given by

\[h(x) = \mathcal{F}[p(\eta x)], \quad (11)\]
where $F[f(x)]$ means the Fourier transformation of function $f(x)$ and $\eta$ is a constant dependent on the geometric parameters. In many cases, these impulse response functions are shift-invariant. Then the integral $\int dx_0 |h(x_i, x_o)|^2$ is determined by the cutoff frequency of the optical transfer function. Since the cutoff frequency is proportional to the size of the aperture, this integral is proportional to the areas of the exit aperture. Thus, the noise will be small if the exit aperture is small. However, small aperture means small cutoff frequency, so the image resolution will be decreased. On the other hand, using a large aperture can increase the image resolution, but the noise is also increased. So good resolution and small noise can not be realized at the same time.

In practical experiments, thousands of photon pairs are used to get an image. Suppose there are $N$ entangled photon pairs which are generated independently, then the fluctuation of the averaged coincidence rate will be $\sqrt{N}$ times smaller and the SNR will be enhanced by a factor $\sqrt{N}$.

We give a numerical example to see the dependence on the apertures. The test imaging system consists of an object characterized by a transmission function $t(x)$, a lens, and a detector $(D_t)$. The lens is located at a focal distance $f$ from the object and from $D_t$. If the size of the lens is much larger than the object, $h_t$ has the form

$$h_t(x_t, x) = -\frac{i}{\lambda f} t(x) \exp \left(-\frac{2\pi i}{\lambda f} x_t x\right),$$

(12)

where $\lambda$ is the wavelength. In the reference imaging system, a lens is placed at a distance $2f$ both from the source and the detector $D_r$. Then $h_r$ has the form

$$h_r(x_r, x') = \frac{1}{4\lambda^2 f^2} P \left( \frac{x_r + x'}{2\lambda f} \right) \exp \left( \frac{i\pi}{2\lambda f} (x_r^2 + x'^2) \right),$$

(13)

where $P(u)$ is the Fourier transformation of the pupil function of the lens $p(x)$. Such a kind of ghost imaging system can image the object in the reference detector \cite{4, 8}.

A double slit is used in our calculation as the object, with the width of the two slits $w = 0.05$mm and the distance between them $d = 1$mm. The size of the lens is $D_f = 100$mm, $\lambda = 650$nm and $N = 10000$ are used in our calculation. The precise formula of $\varphi(x, x')$ may be very complicated, as given in \cite{3},

$$\varphi(x, x') \propto \int dy E_p(y) \zeta(x - y, x' - y),$$

(14)

where $E_p(x)$ is the pump field and $\zeta(x, x')$ is a phase-matching function depends on the
crystal parameters. Here we use a simplified formula

$$\varphi(x, x') = C \exp\left(-\frac{x^2 + x'^2}{a^2}\right) \exp\left(-\frac{(x - x')^2}{b^2}\right),$$

where parameter $a = 2\text{mm}$ decides the size of the source, $b = 0.05\text{mm}$ determines the degree of the entanglement, $C$ is a normalized constant.

First, for a large aperture $D = 10\text{mm}$, in Fig (2), the dashed line shows the conditional coincidence rate ($G^{(2)}(x_r, 0)$) and the solid line is the quantum fluctuation of the conditional coincidence rate ($\Delta G^{(2)}(x_r, 0)$). These two curves are normalized with the maximum value of $G^{(2)}(x_r, 0)$. The resolution is not bad and the two slits are very clear. Also the noise is not very large, the SNR is about 4. So the image quality is good in ghost imaging for given parameters.

As we have discussed before, the aperture of the imaging system can affect the image quality. In Fig (3), we use a small aperture lens with $D = 2\text{mm}$. Small lens make the resolution degrade. But, as shown in Fig (3), the amplitude of the noise is decreased about two times compared with Fig (2). To obtain a good image, we need to balance the requirements on resolution and noise in a ghost imaging system.

In conclusion, the quantum noise in entangled ghost imaging has been investigated theoretically in this paper. We have presented the mathematical formula to calculate the quantum fluctuation of the coincidence rate suitable for various imaging configurations. Apertures in the imaging system affect the imaging quality significantly. Using small aper-
FIG. 3: The same as in Fig [2], but $D = 2$mm.

tures will decrease the noise but also degrade the resolution. It is impossible to improve both the resolution and SNR at the same time by designing the imaging system only.

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