On the Topological Tverberg Theorem

Diploma Thesis
Submitted by
Torsten Schöneborn

Supervised by Prof. Dr. Günter M. Ziegler
Coreferee Priv.-Doz. Dr. Michael Joswig

Institut für Mathematik, Fakultät II,
Technische Universität Berlin

Berlin, 13th November 2018
Helge Tverberg proved in 1966 that for every linear map from the $\Delta^{(d+1)(q-1)}$ into $\mathbb{R}^d$ there is a set of $q$ disjoint faces of this simplex such that their images intersect in a point [Tve66].

It is conjectured that such a set of disjoint faces exists for every continuous map $\Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ as well, but no complete proof of such a “Topological Tverberg Theorem” is known yet. Up to now, it has only been proven that the conjecture holds in the case that $q$ is a prime power [Vol96]. A proof of the Topological Tverberg Theorem for arbitrary $q$ is considered as one of the biggest challenges in topological combinatorics.

Furthermore, it is still unclear if the $q$ disjoint faces in the Tverberg Theorem can be uniquely determined by the mapping or if there are several so called Tverberg partitions. Gerard Sierksma conjectured that for every linear map from the $\Delta^{(d+1)(q-1)}$ into $\mathbb{R}^d$ there are at least $(q-1)!^d$ different Tverberg partitions. He has found an example that attains exactly that number of partitions, but a comprehensive proof is not known yet. A lower bound is proven only for the case that $q$ is prime [VˇZ93], but this bound is far below the bound conjectured by Sierksma.

In Chapter 1, I briefly describe the origin of the Topological Tverberg Theorem and summarize the current status of research. In doing so, all fundamental definitions and theorems used in this thesis are introduced.

A linear map from the simplex is determined by its behavior on the vertices of the simplex. In order to find the Tverberg partitions of a map, it is therefore enough to know where the vertices of the simplex are mapped to.

A continuous map on the other hand is not completely determined by its behavior on a subskeleton. In this thesis, I prove that for the Topological Tverberg Theorem it suffices to know the mapping on the $(d-1)$-skeleton. The restriction to this subskeleton is a considerable simplification, since the dimension of the simplex is much larger than $d-1$ if $q$ is big. We introduce a “Winding Number Conjecture”:

For every continuous map from the $(d-1)$-skeleton of the $\Delta^{(d+1)(q-1)}$ into $\mathbb{R}^d$, we can choose $q$ disjoint, at most $d$-dimensional faces of the simplex $\Delta^{(d+1)(q-1)}$ together with a point in $\mathbb{R}^d$, such that for every face either the point is in the
image of the face or the image of the boundary of the face “winds around” the point.

In Chapter 2, I prove that the Winding Number Conjecture and the Topological Tverberg Theorem are equivalent. This is the main theorem of this thesis. The proof is structured in two parts:

First we point out, that we can restrict the search for Tverberg partitions to the $d$-skeleton. Therefore, we introduce the $d$-Skeleton Conjecture and prove its equivalence to the Topological Tverberg Theorem. In a second step, we deal with the equivalence of the $d$-Skeleton Conjecture and the Winding Number Conjecture. We prove this for the higher dimensional cases ($d \geq 3$) first. Afterwards, we show that the $d$-dimensional case of the Winding Number Conjecture follows from the $d+1$-dimensional case.

At the end of the second chapter, we see that Sierksma’s conjecture about the number of Tverberg partitions transfers to the Winding Number Conjecture.

The Winding Number Conjecture is particularly intuitive in the case $d = 2$, since it deals with maps from the complete graph $K_{3(q-1)+1}$ into the plane. It claims that in every image of this graph either $q - 1$ triangles wind around one vertex or $q - 2$ triangles wind around the intersection of two edges, where the triangles, edges and vertices are disjoint.

In Chapter 3, we examine which graphs have this property, especially which subgraphs of $K_{3(q-1)+1}$. The most interesting result of this chapter is the following: If $q$ is prime, then the graph $K_{3(q-1)+1}$ has this property even after deleting a maximal matching. For the case $q = 3$, this even constitutes the minimal subgraph of $K_7$ having this property.

I would like to thank Prof. Ziegler for the supervision of this thesis and his many stimulating suggestions. Furthermore, I would like to thank Stephan Hell and Arnold Wassmer for the motivating cooperation and their continuing interest in this thesis.
Zusammenfassung

Helge Tverberg bewies 1966, dass es zu jeder linearen Abbildung des \((d + 1)(q - 1))\)-dimensionalen Simplex \(\Delta^{(d+1)(q-1)}\) in den \(\mathbb{R}^d\) eine Menge von \(q\) disjunkten Seiten dieses Simplex gibt, deren Bilder sich in einem Punkt schneiden [Tve66]. Man vermutet, dass es auch zu jeder stetigen Abbildung \(\Delta^{(d+1)(q-1)} \rightarrow \mathbb{R}^d\) eine Menge disjunkter Seiten mit dieser Eigenschaft gibt, allerdings ist bis heute noch kein vollständiger Beweis für ein solches „Topologisches Tverberg-Theorem“ gefunden worden. Bislang konnte lediglich gezeigt werden, dass die Vermutung zutrifft, falls \(q\) eine Primzahlpotenz ist [Vol96]. Die Gültigkeit des Topologischen Tverberg-Theorems für beliebige \(q\) gilt als eine der größten Herausforderungen der topologischen Kombinatorik.

Ebenfalls unklar ist bislang, ob die \(q\) disjunkten Seiten in Tverbergs Theorem eindeutig festgelegt sein können, oder ob es in Abhängigkeit von \(d\) und \(q\) für jede Abbildung mehrere solcher sogenannten Tverberg-Partitionen gibt. Gerard Sierksma vermutet, dass es für jede lineare Abbildung des \((d + 1)(q - 1))\)-dimensionalen Simplex in den \(\mathbb{R}^d\) mindestens \((q - 1)!^d\) verschiedene Tverberg-Partitionen gibt. Er hat ein Beispiel gefunden, das genau diese Anzahl an Partitionen hat; ein allgemeiner Beweis steht aber noch aus. Nur für den Fall, dass \(q\) eine Primzahl ist, kennt man bislang eine untere Schranke [VĚZ93]. Diese liegt jedoch weit unter der von Sierksma vermuteten.

In Kapitel 1 beschreibe ich kurz die Entstehung des Topologischen Tverberg-Theorems und fasse den aktuellen Stand der Forschung zusammen. Dabei werden alle grundlegenden Definitionen und Theoreme aufgeführt, die in dieser Arbeit benutzt werden.

Eine lineare Abbildung des Simplex ist bestimmt durch die Abbildung der Ecken des Simplex. Um die Tverberg-Partitionen einer Abbildung zu finden, genügt es daher zu wissen, wohin die Ecken des Simplex abgebildet werden.

Im Gegensatz dazu ist eine stetige Abbildung nicht durch ihr Verhalten auf einem Teilskelett vollständig festgelegt. In dieser Arbeit zeige ich, dass es für das Topologische Tverberg-Theorem dennoch genügt, die Abbildung auf dem \((d-1)\)-Skelett zu kennen. Durch die Beschränkung auf das \((d-1)\)-Skelett ergibt sich eine Vereinfachung, da die Dimension des Simplex für große \(q\) sehr
viel größer als $d - 1$ ist. Wir stellen die „Windungszahlvermutung“ auf:

Für jede stetige Abbildung vom $(d - 1)$-Skelett des $((d + 1)(q - 1))$-Simplex in den $\mathbb{R}^d$ können wir $q$ disjunkte, höchstens $d$-dimensionale Seiten des Simplex $\Delta^{(d+1)(q-1)}$ und einen Punkt des $\mathbb{R}^d$ auswählen, so dass für jede Seite entweder ihr Bild den Punkt selbst trifft oder aber das Bild ihres Randes den Punkt “umläuft”.

In Kapitel 2 beweise ich, dass die Windungszahlvermutung und das Topologische Tverberg-Theorem äquivalent sind. Dies ist das Hauptresultat dieser Arbeit. Der Beweis gliedert sich in zwei Teile:

Zuerst machen wir uns klar, dass man sich bei der Suche nach Tverb erg-Partitionen auf das $d$-Skelett beschränken kann. Dazu stellen wir eine $d$-Skelett-Vermutung auf und zeigen ihre Äquivalenz zum Topologischen Tverberg-Theorem. Im zweiten Schritt behandeln wir die Äquivalenz von $d$-Skelett-Vermutung und Windungszahl-Vermutung. Diese beweisen wir zuerst für die höherdimensionalen Fälle ($d \geq 3$). Danach zeigen wir, dass aus dem $d$-dimensionalen Fall der Windungszahl-Vermutung der $(d - 1)$-dimensionale folgt.

Am Schluss des zweiten Kapitels sehen wir, dass sich Sierkmas Vermutung über die Anzahl der Tverberg-Partitionen auf die Windungszahlvermutung übertragen lässt.

Besonders anschaulich ist die Windungszahlvermutung im Fall $d = 2$, da sie sich hier mit Abbildungen des vollständigen Graphen $K_{3(q-1)+1}$ in die Ebene beschäftigt. Sie behauptet, dass in jedem Bild dieses Graphen entweder eine Ecke von $q - 1$ Dreiecken umlaufen wird oder aber der Schnittpunkt zweier Kanten von $q - 2$ Dreiecken umlaufen wird, wobei die Dreiecke, Kanten und Ecken paarweise disjunkt sind. In Kapitel 3 untersuchen wir, welche Graphen (insbesondere welche Teilgraphen des $K_{3(q-1)+1}$) diese Eigenschaft haben. Das interessanteste Resultat dieses Kapitels ist das folgende: Der Graph $K_{3(q-1)+1}$ hat sogar abzüglich eines maximalen Matchings diese Eigenschaft, falls $q$ eine Primzahl ist. Für den Fall $q = 3$ ist damit sogar der minimale Teilgraph von $K_7$ mit dieser Eigenschaft gefunden.

Ich danke Herrn Prof. Ziegler für das interessante Thema, viele Anregungen und insbesondere für die immer geöffnete Tür. Arnold Wassmer und Stephan Hell bin ich dankbar für die motivierende Zusammenarbeit und das stete Interesse an dieser Arbeit. Euch beiden und ganz besonders Henryk Gerlach danke ich für das Korrekturlesen - ohne Euch hätte diese Arbeit (noch) mehr Fehler. Ein besonderer Dank geht an meine liebe Freundin Anna für das viele Verständnis, wenn ich mal wieder in Gedanken war. Meinen Eltern möchte ich danken für die Unterstützung während des gesamten Studiums, dessen Abschluss diese Arbeit darstellt.
Contents

1 Introduction ................................................................. 1
   1.1 The Tverberg Theorem ........................................ 1
   1.2 The Topological Tverberg Theorem ........................ 2
   1.3 How many Tverberg partitions are there? ................. 7

2 The Winding Number Conjecture ..................................... 9
   2.1 Step 1: Reduction to the $d$-skeleton ...................... 12
      2.1.1 Maps in general position ............................ 13
      2.1.2 Tverberg partitions in the $d$-skeleton ............ 15
      2.1.3 The connection between Tverberg partitions in the full simplex and in the $d$-skeleton ................................. 17
   2.2 Step 2: Reduction to the $(d-1)$-skeleton .................. 18
      2.2.1 Reduction to piecewise linear maps in general position ............................ 19
      2.2.2 The case $d > 3$ ...................................... 20
      2.2.3 The case $d = 2$ of the Winding Number Conjecture .................................. 25
      2.2.4 The connection between Tverberg partitions and winding partitions 27
   2.3 The number of winding partitions and Tverberg partitions 28

3 $Q$-winding Graphs .................................................... 32
   3.1 1-winding graphs ............................................... 32
   3.2 2-winding graphs and $\Delta - Y$-operations .................. 32
   3.3 3-winding graphs and $q$-winding subgraphs of complete graphs 35

Bibliography ........................................................................ 39
Chapter 1

Introduction

In this chapter, we first discuss three different versions of the classical Tverberg Theorem. Then, we introduce the Topological Tverberg Theorem (which is really a conjecture) and discuss bounds for the number of Tverberg partitions.

1.1 The Tverberg Theorem

The historical starting point of the topic of this thesis is the following theorem from linear geometry.

**Theorem 1.1.1 (Tverberg Theorem).** Let \( d \) and \( q \) be positive integers. No matter how \( (d+1)(q-1)+1 \) points are chosen in \( \mathbb{R}^d \), it is always possible to partition them into \( q \) disjoint sets such that the convex hulls of these sets intersect, i.e., such that they have a point in common.

The first proof was delivered by Helge Tverberg [Tve66]. Today, several different ways of proving it are known. Tverberg himself offered another proof in [Tve81].

By \( \Delta^N \) we denote the \( N \)-dimensional simplex, by \( \Delta^N_k \) its \( k \)-skeleton. We will normally not distinguish between a simplicial complex and its realization, unless it could cause confusion. We can express the Tverberg Theorem in terms of a linear map:

**Theorem 1.1.2 (Tverberg Theorem (Equivalent version I)).** For every linear map

\[
f : \Delta^{(d+1)(q-1)} \rightarrow \mathbb{R}^d
\]

there are \( q \) disjoint faces of \( \Delta^{(d+1)(q-1)} \) such that their images have a point in common.

To see that this is equivalent to the original formulation of Tverberg’s theorem, observe that the convex hull of \( n \) points in \( \mathbb{R}^d \) is precisely the image of the linear map \( \Delta^{n-1} \rightarrow \mathbb{R}^d \) that maps the \( n \) vertices of \( \Delta^{n-1} \) to these \( n \) points.
**Definition 1.1.3.** Let $k$ be a nonnegative integer and let $f$ be a (not necessarily linear) map $f : \Delta^{(d+1)(q-1)}_k \to \mathbb{R}^d$ and $S$ a set of $q$ disjoint faces $\sigma$ of $\Delta^{(d+1)(q-1)}_k$. We call $S$ a **Tverberg partition** for the map $f$ if the images of the faces in $S$ have a point in common, that is, if

$$\bigcap_{\sigma \in S} f(\sigma) \neq \emptyset.$$ 

Every point in this nonempty intersection is called a **Tverberg point**. There might be vertices of $\Delta^{(d+1)(q-1)}_k$ that are not contained in any face of $S$, although this can happen only in degenerated cases.

Using this definition, we can formulate Tverberg’s theorem even simpler:

**Theorem 1.1.4 (Tverberg Theorem (Equivalent version II)).** For every linear map

$$f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$$

there is a Tverberg partition.

Two questions arise now.

- Does this theorem hold for a wider class of maps, for example continuous ones, as well?
- How many Tverberg partitions are there at least?

We will deal with these questions in the following subsections.

### 1.2 The Topological Tverberg Theorem

The following conjecture is a generalization of Tverberg’s theorem to arbitrary continuous maps. It is misleadingly referred to as the “Topological Tverberg Theorem”, although up to now no complete proof of this conjecture is known.

**Conjecture 1.2.1 (“Topological Tverberg Theorem”).** For every continuous map

$$f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$$

there is a Tverberg partition.

**Remark 1.2.2.** If we want to restrict the parameters $d$ and $q$ in this conjecture, we for example talk about “the case $d = 4$ of the Topological Tverberg Theorem” or “the case $q = 6$ of the Topological Tverberg Theorem”.

The Topological Tverberg Theorem was proven in three cases:

- The case $d = 1$ is equivalent to the mean value theorem for continuous functions $f : \mathbb{R} \to \mathbb{R}$. 
Table 1.1: The checkmarks indicate the proven cases of the Topological Tverberg Theorem.

- The Topological Tverberg Theorem for higher dimensions $d$ was first proven for prime $q$ by Bárány, Shlosman and Szücs [BSS81] using deleted products. A proof using deleted joins and the $\mathbb{Z}_q$-index is given in [Mat03].

- Özaydin proved the more general case of $q$ a prime power 1987 in a still unpublished manuscript. Later, Aleksei Volovikov gave an alternative proof [Vol96]. An elaborate version of Sarkaria’s proof using characteristic classes [Sar00] can be found in de Longueville [dL01].

All other cases still remain open; the smallest open case is therefore $d = 2$, $q = 6$ (see also Table 1.1). This case deals with maps from the 15-dimensional simplex to $\mathbb{R}^2$. In Matoušek’s opinion, “the validity of the Topological Tverberg Theorem for arbitrary (nonprime) $q$ is one of the most challenging problems in this field [topological combinatorics]” [Mat03, p.154]. It is known that lower dimensional cases follow from higher dimensional ones:

**Proposition 1.2.3** (de Longueville [dL01]). If the Topological Tverberg Theorem holds for $q$ and $d$, then it also holds for $q$ and $d - 1$.

I will now give an outline of the proof of the Topological Tverberg Theorem in the case that $q$ is a prime.

**Theorem 1.2.4** (Bárány, Shlosman and Szücs [BSS81]). The Topological Tverberg Theorem is valid if $q$ is a prime.

We follow the proof presented in [Mat03]. The central definition is the deleted join, which we will use as a configuration space. We need two definitions – the first one for simplicial complexes, the second one for topological spaces.
**Definition 1.2.5.** Let $q$ be a positive integer. For $q$ sets $A_1, \ldots, A_q$, let $A_1 \cup A_2 \cup \ldots \cup A_q$ be the set

$$(A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup \ldots \cup (A_q \times \{q\}).$$

Let $\Delta_1, \Delta_2, \ldots, \Delta_q$ be a simplicial complexes with vertex sets $V_1, V_2, \ldots, V_q$. The **join** $\Delta_1 \ast \Delta_2 \ast \ldots \ast \Delta_q$ has vertex set

$$V_1 \cup V_2 \cup \ldots \cup V_q$$

and face set

$$\{F_1 \cup F_2 \cup \ldots \cup F_q \mid F_i \text{ is a face of } \Delta_i \text{ for all } i\}.$$ 

Let $t_1, \ldots, t_q$ be nonnegative real numbers with $\sum_{i=1}^q t_i = 1$. For all $i$, let $p_i$ be a point in a realization of $F_i$. We define the following notation for points in the realization of the face $F_1 \cup F_2 \cup \ldots \cup F_q$:

$$(t_1p_1 + t_2p_2 + \ldots + tqp_q) := \sum_{v \in F_1} t_1 \cdot t(v \times \{1\}) + \ldots + \sum_{v \in F_q} t_q \cdot t(v \times \{q\})$$

where $t_{i,v} \geq 0$ such that $p_i = \sum_{v \in F_i} t_{i,v} \cdot v$ and $\sum_{v \in F_i} t_{i,v} = 1$ for all $i$.

Let $\Delta$ be a simplicial complex with vertex set $V$. The **$q$-fold join** $\Delta^{*q}$ of $\Delta$ is

$$\Delta^{*q} := \Delta \ast \Delta \ast \ldots \ast \Delta\text{, } q \text{ times.}$$

The **$q$-fold pairwise deleted join** $\Delta^{*q}_{\Delta(2)}$ is the subcomplex of $\Delta^{*q}$ with the face set

$$\{F_1 \cup F_2 \cup \ldots \cup F_q \mid \text{the } F_i \text{ are pairwise disjoint faces of } \Delta\}.$$ 

**Definition 1.2.6.** Let $X_1, X_2, \ldots, X_q$ be topological spaces. The **join** $X_1 \ast X_2 \ast \ldots \ast X_q$ is the topological space

$$X_1 \ast X_2 \ast \ldots \ast X_q := X_1 \times X_2 \times \ldots \times X_q \times Y/ \approx$$

where $Y$ is the convex hull of the standard unit vectors in $\mathbb{R}^q$, that is, the set

$$\{(t_1, \ldots, t_q) \in \mathbb{R}^q \mid \sum_{i=1}^q t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\},$$

and $\approx$ is given by

$$(p_1, \ldots, p_q, (t_1, \ldots, t_q)) \approx (p'_1, \ldots, p'_q, (t'_1, \ldots, t'_q))$$

if and only if $(t_i = t'_i) \text{ and } (t_i \neq 0 \Rightarrow p_i = p'_i)$ for all $i$.

We write

$$(t_1p_1 + t_2p_2 + \ldots + tqp_q) := (p_1, \ldots, p_q, (t_1, \ldots, t_q)).$$
Let $X$ be a topological space. The $q$-fold join $X^*q$ of $X$ is the topological space

$$X \ast X \ast \ldots \ast X.$$ 

The $q$-fold $q$-wise deleted join $X^*_\Delta$ is the following subspace of $X^*q$:

$$X^*_\Delta := X^*q \setminus \{(p, \ldots, p, \left(\frac{1}{q}, \ldots, \frac{1}{q}\right)) \mid p \in X\}$$

$$= X^*q \setminus \{\left(\frac{1}{q}p \oplus \ldots \oplus \frac{1}{q}p\right) \mid p \in X\}$$

**Definition 1.2.7.** Let $f : \Delta \to X$ be a continuous map from the realization of a simplicial complex $\Delta$ to a topological space $X$. We define the $q$-fold join $f^*q$ of $f$ to be the following map.

$$f^*q : \Delta^*q \to X^*q$$

$$f^*q(t_1p_1 \oplus t_2p_2 \oplus \ldots \oplus t_qp_q) := (t_1f(p_1) \oplus t_2f(p_2) \oplus \ldots \oplus t_qf(p_q))$$

The definitions of deleted joins are tailored for our purposes: Let us assume that a counterexample $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ exists. First, we take its $q$-fold join

$$f^*q : (\Delta^{(d+1)(q-1)})^*q \to (\mathbb{R}^d)^*q$$

and restrict it to the deleted join

$$f^*q|_{(\Delta^{(d+1)(q-1)})_{(\Delta^*q)}}^*q : (\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)} \to (\mathbb{R}^d)^*q.$$ 

Observe that every face of $(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)}$ represents $q$ (possibly empty) disjoint faces of $\Delta^{(d+1)(q-1)}$. Because we assumed $f$ to be a counterexample, these faces do not form a Tverberg partition. Thus no point $(t_1p \oplus \ldots \oplus t_qp)$ of $(\mathbb{R}^d)^*q$ is in the image $f^*q|_{(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)}}^*q$, where $p$ is in $\mathbb{R}^d$ and the $t_i$ are positive real numbers. In particular no point $(\frac{1}{q}p \oplus \ldots \oplus \frac{1}{q}p)$ is attained. Therefore we can reduce the co-domain to $(\mathbb{R}^d)^*q_{(\Delta^*q)}$ and obtain a map

$$f^*q|_{(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)}}^*q : (\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)} \to (\mathbb{R}^d)^*q_{(\Delta^*q)}.$$ 

If we show that there is no such map $f^*q$, then we can immediately deny the existence of a counterexample for the Topological Tverberg Theorem. Of course, a map $(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)} \to (\mathbb{R}^d)^*q_{(\Delta^*q)}$ exists (for example the constant map), but $f^*q$ has an important property: It is $\mathbb{Z}_q$-equivariant. $\mathbb{Z}_q$ operates on the $q$-fold join $A^*q$ of a simplicial complex or topological space $A$ by

$$\mu.(t_1p_1 \oplus t_2p_2 \oplus \ldots \oplus t_qp_q) := (t_2p_2 \oplus t_3p_3 \oplus \ldots \oplus t_qp_q \oplus t_1p_1)$$

where $\mu$ is a generator of $\mathbb{Z}_q$. Note that $\mathbb{Z}_q$ operates in the same way on deleted joins. Furthermore, the $q$-fold join of a map is $\mathbb{Z}_q$-equivariant under this operation. As we have seen, the question whether an arbitrary map $(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)} \to (\mathbb{R}^d)^*q_{(\Delta^*q)}$ exists is not fruitful; but does a $\mathbb{Z}_q$-equivariant map $(\Delta^{(d+1)(q-1)})^*q_{(\Delta^*q)} \to (\mathbb{R}^d)^*q_{(\Delta^*q)}$ exist?

The answer is “no”, if $q$ is a prime. We prove this using index theory.
**Definition 1.2.8.** Let $X$ be a topological space or a simplicial complex with a $\mathbb{Z}_q$-operation. We define the $\mathbb{Z}_q$-index of $X$ with respect to this operation as

$$\text{ind}_{\mathbb{Z}_q}(X) := \min\{n \mid \text{there is a } \mathbb{Z}_q\text{-equivariant map } X \to (\Delta^n)^{\ast_q}_{\Delta(2)}\}$$

Here, we regard $(\Delta^n)^{\ast_q}_{\Delta(2)}$ equipped with the $\mathbb{Z}_q$-operation described above.

**Lemma 1.2.9.** Let $X$ and $Y$ be spaces with $\mathbb{Z}_q$-operation. There is no $\mathbb{Z}_q$-invariant map $X \to Y$ if $\text{ind}_{\mathbb{Z}_q}(X)$ is greater than $\text{ind}_{\mathbb{Z}_q}(Y)$.

**Proof.** Assume an equivariant map $f : X \to Y$ exists. By definition, there is also an equivariant map $g : Y \to (\Delta^{\text{ind}_{\mathbb{Z}_q}(Y)})^{\ast_q}_{\Delta(2)}$. By combining these, we obtain an equivariant map

$$g \circ f : X \to (\Delta^{\text{ind}_{\mathbb{Z}_q}(Y)})^{\ast_q}_{\Delta(2)}.$$

Again by definition we conclude that $\text{ind}_{\mathbb{Z}_q}(X) \leq \text{ind}_{\mathbb{Z}_q}(Y)$, which contradicts the conditions of the lemma. \hfill $\square$

**Lemma 1.2.10** (see Matoušek [Mat03, Sections 6.2 and 6.3]).

- Let $X$ and $Y$ be spaces with $\mathbb{Z}_q$-operation. $\mathbb{Z}_q$ operates on $X \ast Y$ by operating on both factors simultaneously and we have

$$\text{ind}_{\mathbb{Z}_q}(X \ast Y) \leq \text{ind}_{\mathbb{Z}_q}(X) + \text{ind}_{\mathbb{Z}_q}(Y) + 1.$$

- Let $\mathbb{Z}_q$ operate on $S_1$ by a rotation of angle $\frac{2\pi}{q}$. Then we have

$$\text{ind}_{\mathbb{Z}_q}(S^1) = 1.$$

- The index of a pairwise deleted join of a simplex is

$$\text{ind}_{\mathbb{Z}_q}((\Delta^n)^{\ast_q}_{\Delta(2)}) = n.$$

- If $q$ is a prime, then the index of $(\mathbb{R}^d)^{\ast_q}_{\Delta}$ is

$$\text{ind}_{\mathbb{Z}_q}((\mathbb{R}^d)^{\ast_q}_{\Delta}) = (d + 1)(q - 1) - 1.$$

**Proof of Theorem 1.2.4.** If a counterexample existed, then there would be an equivariant map $f^{\ast_q}_{\Delta} : (\Delta^{(d+1)(q-1)})^{\ast_q}_{\Delta(2)} \to (\mathbb{R}^d)^{\ast_q}_{\Delta}$. But there is no such equivariant map, since

$$\text{ind}_{\mathbb{Z}_q}((\Delta^{(d+1)(q-1)})^{\ast_q}_{\Delta(2)}) = (d + 1)(q - 1) > (d + 1)(q - 1) - 1 = \text{ind}_{\mathbb{Z}_q}((\mathbb{R}^d)^{\ast_q}_{\Delta}).$$

\hfill $\square$
1.3 How many Tverberg partitions are there?

Sierksma conjectured that for every linear map \( f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d \) there are at least \( ((q-1)!)^d \) Tverberg partitions. This number is attained for the configuration of \( d + 1 \) tight clusters, with \( q - 1 \) points each, placed at the vertices of a simplex, and one point in the middle.

For \( d = 1 \), the mean value theorem implies Sierksma’s conjecture. In almost all other cases, Sierksma’s conjecture is still unresolved at the time of writing (see Table 1.2). Nevertheless, for special values of \( q \), a lower bound is known:

**Theorem 1.3.1 (Vučić and Živaljević [VZ93].)** If \( q \) is a prime, then there are at least

\[
\frac{1}{(q-1)!} \cdot \left(\frac{q}{2}\right)^{(d+1)(q-1)/2}
\]

Tverberg partitions for every continuous map \( f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d \).

A nice proof can also be found in Matoušek [Mat03, Theorem 6.6.1]. For arbitrary \( q \) but linear \( f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d \), the best known lower bound is 1, given by the classical Tverberg Theorem. Furthermore, no non-topological method is known to yield a good lower bound.
1.3 How many Tverberg partitions are there?

| $d \setminus q$ | 1     | 2     | 3     | 4     | 5     | 6     |
|-----------------|-------|-------|-------|-------|-------|-------|
| 1               | 1 (1) | 1 (1) | 2 (2) | 6 (6) | 24 (24) | 120 (120) |
| 2               | 1 (1) | 1 (1) | 4 (2) | 36 (1) | 576 (11) | 14400 (0) |
| 3               | 1 (1) | 1 (1) | 8 (3) | 216 (1) | 13824 (64) | 1728000 (0) |
| 4               | 1 (1) | 1 (1) | 16 (4) | 1296 (1) | 331776 (398) | 2.07·10$^{10}$ (0) |
| 5               | 1 (1) | 1 (1) | 32 (6) | 7776 (1) | 7962624 (2484) | 2.48·10$^{10}$ (0) |
| 6               | 1 (1) | 1 (1) | 64 (9) | 46656 (1) | 1.91·10$^{12}$ (15523) | 2.98·10$^{12}$ (0) |
| 7               | 1 (1) | 1 (1) | 128 (13) | 279936 (1) | 4.58·10$^{10}$ (97013) | 3.58·10$^{14}$ (0) |
| 8               | 1 (1) | 1 (1) | 256 (20) | 1679616 (1) | 2.96·10$^{12}$ (606330) | 4.29·10$^{16}$ (0) |
| 9               | 1 (1) | 1 (1) | 512 (29) | 10077696 (1) | 3.46·10$^{12}$ (3789562) | 5.15·10$^{18}$ (0) |
| 10              | 1 (1) | 1 (1) | 1024 (44) | 60466176 (1) | 6.34·10$^{13}$ (23684758) | 6.19·10$^{20}$ (0) |
| 11              | 1 (1) | 1 (1) | 2048 (65) | 362797056 (1) | 1.52·10$^{15}$ (148029737) | 7.43·10$^{22}$ (0) |

Table 1.2: The number of Tverberg partitions conjectured by Sierksma. The number in brackets shows the currently highest proven lower bound for continuous maps.
Chapter 2

The Winding Number Conjecture

We saw in the introduction that there are two equivalent versions of the Tverberg Theorem for linear maps: Theorem 1.1.2, that deals with maps $f$ from the entire $((d + 1)(q - 1))$-dimensional Simplex to $\mathbb{R}^d$, and the original version (Theorem 1.1.1), that uses only the image of the vertices of the simplex, that is, it talks about $f|_{\Delta^{(d+1)(q-1)}}$. In this chapter, we will establish conjectures that are equivalent to the Topological Tverberg Theorem, but talk only about $f|_{\Delta^{(d+1)(q-1)}}$ for some integer $k$ less than $(d + 1)(q - 1)$.

In fact, we will see two such conjectures: The $d$-Skeleton Conjecture $(k = d)$ claims that every continuous map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ has a Tverberg partition in the $d$-skeleton. We will derive the equivalence to the Topological Tverberg Theorem by reducing the problem to maps in “general position”.

The Winding Number Conjecture $(k = d - 1)$ claims that for every continuous map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$, the boundary of many simplices of $\Delta^{(d+1)(q-1)}$ wind about a point contained in the image of some of the faces. We also consider bounds on the number of Tverberg partitions respectively winding partitions.

We start our discussion with the Winding Number Conjecture, since it is more powerful, and introduce the $d$-Skeleton Conjecture later as a link between the Winding Number Conjecture and the Topological Tverberg Theorem. We need the following definition.

**Definition 2.0.2.** First, let us assume $d \geq 2$. Let $f : S^{d-1} \to \mathbb{R}^d$ be a continuous map from the $(d - 1)$-dimensional sphere to $\mathbb{R}^d$, and let $p$ be a point in $\mathbb{R}^d$. We choose an isomorphism $I : \pi_{d-1}(\mathbb{R}^d \setminus \{p\}) \to \mathbb{Z}$. If $f$ does not attain $p$, that is, if $p \notin f(S^{d-1})$, then we define the **winding number of $f$ with respect to $p$** as

$$W(f, p) := I([f]) \in \mathbb{Z}.$$
The sign of $W(f, p)$ depends on the choice of $I$, but the expression

$$W(f, x) = 0$$

is independent of this choice. Let $\partial \Delta^d := \Delta^d_{d-1}$. For maps $f : \partial \Delta^d \to \mathbb{R}^d$, we define $W(f, p)$ via $h : S^{d-1} \xrightarrow{\sim} \Delta^d_{d-1} = \partial \Delta^d$ to be $W(f, x) := W(f \circ h, x)$.

Now if $d = 1$, then we do not have such an isomorphism $I$. In this thesis, we are not interested in the exact value of $W(f, p)$, but only in whether it is zero. Therefore it is sufficient for our purposes to define $W(f, p)$ to be zero if the two points $f(S^0)$ lie in the same component of $\mathbb{R}\{p\}$. Otherwise we say that $W(f, p) \neq 0$.

**Winding Number Conjecture.** For all positive integers $d$ and $k$ and every continuous map $f : \Delta_{d-1}^{d+1}(q-1) \to \mathbb{R}^d$ there are $q$ disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta_{d-1}^{d+1}(q-1)$ and a point $p \in \mathbb{R}^d$ such that for every $i$, one of the following holds:

- $\dim(\sigma_i) \leq d - 1$ and $p \in f(\sigma_i)$
- $\dim(\sigma_i) = d$ and $(p \in f(\partial \sigma_i) \text{ or } W(f|_{\partial \sigma_i}, p) \neq 0)$

Such a set $S = \{\sigma_1, \ldots, \sigma_q\}$ will be called a **winding partition**; $p$ will be called a **winding point**.

We intentionally included the case “$p \in f(\partial \sigma_i)$” in the previous conjecture. See also Remark 2.0.5.

**Example 2.0.3.** Let us look at a concrete example of a continuous map $\Delta_{d-1}^{d+1}(q-1) \to \mathbb{R}^d$. In the case $d = 2$, this is really a drawing of $K_{3(q-1)+1}$, the complete graph with $3(q - 1) + 1 = 3q - 2$ vertices. If the drawing is in “general position” (in a way made precise in the next section), then the Winding Number Conjecture says that in the drawing of $K_{3q-2}$ either $q - 1$ (possibly distorted) triangles wind around one vertex, or $q - 2$ triangles wind around the intersection of two edges, with the triangles, edges and the vertex being pairwise disjoint.

The following way to draw $K_n$ is called “the alternating linear model” and was proposed in [Saa64]. Draw the $n$ vertices on a line and number them from left to right. Draw the edges $[i, i + 1]$ on this line and the edges $[i, i + k], k \geq 2$ on one side of the line (e.g. above) if $i$ is odd and on the other side (e.g. below) if $i$ is even. Figure 2.1 illustrates the situation for $K_7$ and $K_{10}$.

For this drawing the Winding Number Conjecture is satisfied: The vertex with number $2q - 1$ is a winding point. For example the $q - 1$ disjoint triangles $(1, 2, 3q - 2), (3, 4, 3q - 3), \ldots, (2q - 3, 2q - 2, 3q - q)$ wind around it.

This is not surprising, because the classical Tverberg Theorem guarantees that every rectilinear drawing of $K_{3q-2}$ satisfies the Winding Number Conjecture, and there is a rectilinear drawing for the alternating linear model.
Figure 2.1: The alternating linear model of $K_7$ and $K_{10}$. The thick black lines in the drawing of $K_7$ form a winding partition.

We can give an alternative description of the term “winding partition”:

**Definition 2.0.4.** For any map $f : \partial \Delta^d \to \mathbb{R}^d$, we define

$$W_{\neq 0}(f) := f(\partial \Delta^d) \cup \{x \in \mathbb{R}^d \setminus f(\partial \Delta^d) \mid W(f, x) \neq 0\}.$$  

**Remark 2.0.5.** Later on, it will be advantageous that $W_{\neq 0}(f)$ is a closed set containing $f(\partial \Delta^d)$ even in degenerated cases where $W_{\neq 0}(f)$ might be empty. This is why we had to add $f(\partial \Delta^d)$ to the definition of $W_{\neq 0}(f)$ and include “$p \in f(\partial \sigma_i)$” in our first formulation of the Winding Number Conjecture.

**Lemma 2.0.6.** A set $S = \{\sigma_1, \ldots, \sigma_q\}$ of $q$ disjoint faces of $\Delta^{(d+1)(q-1)}_d$ is a winding partition for $f : \Delta^{(d+1)(q-1)}_{d-1} \to \mathbb{R}^d$ if and only if

$$\bigcap_{\dim(\sigma_i) < d} f(\sigma_i) \cap \bigcap_{\dim(\sigma_i) = d} W_{\neq 0}(f|_{\partial \sigma_i}) \neq \emptyset.$$  

**Winding Number Conjecture (Equivalent version).** For every continuous map $f : \Delta^{(d+1)(q-1)}_{d-1} \to \mathbb{R}^d$ there are $q$ disjoint faces $\sigma_1, \ldots, \sigma_q$ of $\Delta^{(d+1)(q-1)}_d$ such that

$$\bigcap_{\dim(\sigma_i) < d} f(\sigma_i) \cap \bigcap_{\dim(\sigma_i) = d} W_{\neq 0}(f|_{\partial \sigma_i}) \neq \emptyset.$$  

This conjecture can be proved easily if $d = 1$ (see Proposition 2.3.1). The rest of this chapter covers the proof of the following theorem.

**Theorem 2.0.7.** The Winding Number Conjecture is equivalent to the Topological Tverberg Theorem.
The line of argument of the proof is illustrated in Figure 2.2.

Remark 2.0.8. The basic idea of the proof are the following two speculations.

- Let $F : \Delta^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ be a continuous map. Every winding partition of $F|_{\Delta_{d-1}^{(d+1)(q-1)}}$ is a Tverberg partition of $F$.

- Let $f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ be a continuous map. Then $f$ can be extended to a continuous map $F : \Delta^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ such that every Tverberg partition of $F$ is a winding partition of $f$.

The first statement turns out to be true, but the second one needs some adjustment, as we will see in the course of the proof. We will come back to the above speculations in Theorem 2.2.14.

2.1 Step 1: Reduction to the $d$-skeleton

First, we show that the Topological Tverberg Theorem guarantees the existence of a Tverberg partition in the $d$-skeleton of $\Delta^{(d+1)(q-1)}$. 
Step 1: Reduction to the $d$-skeleton

**Figure 2.3**: Images $f(\Delta)$ of linear maps $f : \Delta \to \mathbb{R}^2$ in general position.

**$d$-Skeleton Conjecture.** Every continuous map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ has a Tverberg partition.

**Proposition 2.1.1.** The $d$-Skeleton Conjecture is equivalent to the Topological Tverberg Theorem.

It is obvious that the $d$-Skeleton Conjecture implies the Topological Tverberg Theorem. The converse is harder. Its proof is the aim of this subsection. We divide the proof into the Lemmas 2.1.7 and 2.1.9.

### 2.1.1 Maps in general position

For the first lemma, we need the following definition.

**Definition 2.1.2.** Let $\Delta$ be a simplicial complex. A map $f : \Delta \to \mathbb{R}^d$ is **linear** if it is linear on every face of $\Delta$. Such an linear map $f$ is in **general position** if for every set of disjoint faces $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ of $\Delta$ the inequality

$$\text{codim}\left(\bigcap_{i=1}^{k} f(\sigma_i)\right) \geq \sum_{i=1}^{k} \text{codim}(f(\sigma_i))$$

holds, where $\text{codim}(\sigma) := d - \dim(\sigma)$ if $\sigma \subset \mathbb{R}^d$. We use the convention $\dim(\emptyset) = -\infty$ and thus set $\text{codim}(\emptyset) = \infty$. The last equation includes the case $\bigcap_{i=1}^{k} f(\sigma_i) = \emptyset$. Thus, in that case the condition above holds independently of the right hand side because we have $\text{codim}\left(\bigcap_{i=1}^{k} f(\sigma_i)\right) = \infty$.

We need to restrict ourselves to piecewise linear maps to exclude “wild” maps.

**Definition 2.1.3.** Let $\Delta$ be a simplicial complex. A map $f : \Delta \to \mathbb{R}^d$ is **piecewise linear** if there is a subdivision $s : \Delta' \to \Delta$ such that the composition $f \circ s : \Delta' \to \mathbb{R}^d$ is a linear map. Furthermore, we call $f$ in **general position** if we can choose the subdivision $s$ such that the linear map $f \circ s$ is in general position.
2.1 Step 1: Reduction to the $d$-skeleton

Figure 2.4: Images $f(\Delta)$ of linear maps $f : \Delta \to \mathbb{R}^2$ not in general position. In the last picture, the complex $\Delta$ consists of two lines.

Figure 2.5: Images $f(\Delta)$ of piecewise linear maps $f : \Delta \to \mathbb{R}^2$. In the first three pictures, $\Delta$ consists of two lines, in the last picture $\Delta$ consists of a triangle and a line. The two pictures on the left are in general position, the two on the right are not.

Whether $f \circ s$ is in general position depends on the subdivision $s$. For example, the map $f$ depicted on the very left in Figure 2.5 combined with the second barycentric subdivision gives a linear map not in general position, although $f$ itself is in general position.

The definition of general position made here may seem overly restrictive for the purpose of this section, but we need it in Proposition 2.2.8.

The key point of maps in general position is the following lemma.

**Lemma 2.1.4.** Let $\Delta$ be a simplicial complex and $f : \Delta \to \mathbb{R}^d$ a piecewise linear map in general position. If $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a set of disjoint faces of $\Delta$, then we have

$$\text{codim}(\bigcap_{i=1}^k f(\sigma_i)) \geq \sum_{i=1}^k \max(0, (d - \dim(\sigma_i))).$$

**Remark 2.1.5.** $\bigcap_{i=1}^k f(\sigma_i)$ might have parts of different dimension. We use the convention

$$\dim(A \cup B) := \max(\dim(A), \dim(B))$$

and thus

$$\text{codim}(A \cup B) := \min(\text{codim}(A), \text{codim}(B))$$
2.1 Step 1: Reduction to the $d$-skeleton

Proof. Let $s : \Delta' \to \Delta$ be a subdivision such that $f \circ s$ is a linear map in general position.

$$\text{codim} \left( \bigcap_{i=1}^{q} f(\sigma_i) \right) = \min_{\tilde{\sigma}_i \subset \sigma_i} \text{codim} \left( \bigcap_{i=1}^{q} f(s(\tilde{\sigma}_i)) \right)$$

$$\geq \min_{\tilde{\sigma}_i \subset \sigma_i} \sum_{i=1}^{q} \text{codim}(f \circ s(\tilde{\sigma}_i))$$

$$= \sum_{i=1}^{q} \min_{\tilde{\sigma}_i \subset \sigma_i} (d - \text{dim}(f \circ s(\tilde{\sigma}_i)))$$

$$= \sum_{i=1}^{q} (d - \max_{\tilde{\sigma}_i \subset \sigma_i} \text{dim}(f \circ s(\tilde{\sigma}_i)))$$

$$\geq \sum_{i=1}^{q} (d - \min(d, \text{dim} \sigma_i))$$

$$= \sum_{i=1}^{k} \max(0, (d - \text{dim} \sigma_i)).$$

We need an approximation lemma to tackle continuous maps.

Lemma 2.1.6 (Piecewise Linear Approximation Lemma). Let $\Delta$ be a simplicial complex with a subcomplex $\Delta_0 \subset \Delta$ and let $\varepsilon > 0$. Furthermore, let $f : \Delta \to \mathbb{R}^d$ be a continuous map that is piecewise linear on $\Delta_0 \subset \Delta$. Then there is a piecewise linear map $\tilde{f} : \Delta \to \mathbb{R}^d$ that equals $f$ on $\Delta_0$ and approximates it on the rest of $\Delta$, that is, $\tilde{f}|_{\Delta_0} = f|_{\Delta_0}$ and

$$\|\tilde{f} - f\|_{\infty} = \max \{ |\tilde{f}(x) - f(x)| : x \in \Delta \} < \varepsilon.$$

Proof. Let $s : \Delta' \to \Delta_0$ be a subdivision such that $f|_{\Delta_0} \circ s$ is a linear map. We can extend $s$ to a subdivision $S : \Delta' \to \Delta$ of all of $\Delta$. Since $\Delta$ is compact, there is an iterated barycentric subdivision $S : \tilde{\Delta} \to \Delta'$ of $\Delta'$ with the following property: If $p \in \tilde{\Delta}$ is contained in the face $\{v_1, \ldots, v_k\}$ of $\tilde{\Delta}$, then $\|f(p) - f(v_i)\| < \varepsilon$. Let $\tilde{f} : \tilde{\Delta} \to \mathbb{R}^d$ be the linear map that is given on the vertices $v$ of $\Delta$ by $\tilde{f}(v) := f(S(S(v)))$. Therefore $\tilde{f}$ is piecewise linear on $\Delta$, equals $f$ on $\Delta_0$ (because already $f \circ S$ is linear on this subcomplex) and approximates $f$ on the rest of $\Delta$. \qed

2.1.2 Tverberg partitions in the $d$-skeleton

Using the Approximation Lemma and the properties of piecewise linear maps, we can now prove that if the Topological Tverberg Theorem holds, then the $d$-Skeleton Conjecture holds as well.
Lemma 2.1.7. Every Tverberg partition of any piecewise linear map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ in general position contains only faces of dimension at most $d$.

Corollary 2.1.8. If the Topological Tverberg Theorem is true, then the $d$-Skeleton Conjecture holds for all piecewise linear maps in general position.

Lemma 2.1.9. For every continuous map $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ there is an $\varepsilon_f > 0$ such that the following holds: If $\tilde{f} : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ is a continuous map with $\|\tilde{f} - f\|_{\infty} < \varepsilon_f$, then every Tverberg partition of $\tilde{f}$ is also a Tverberg partition of $f$.

This lemma states that by distorting $f$ by less than $\varepsilon_f$, we do not create new Tverberg partitions.

Corollary 2.1.10. If the $d$-Skeleton Conjecture holds for all piecewise linear maps in general position, then it is true in general (i.e., for all continuous maps).

By these two corollaries, the Topological Tverberg Theorem implies the $d$-Skeleton Conjecture.

Proof of Lemma 2.1.7. Let $f$ be in general position that has an arbitrary Tverberg partition $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$.

$$
\begin{align*}
    d & \geq (1) \quad \text{codim}(\bigcap_{i=1}^{q} f(\sigma_i)) \\
    & \geq (2) \sum_{i=1}^{q} \max(0, (d - \dim \sigma_i)) \\
    & \geq (\ast) \sum_{i=1}^{q} (d - \dim \sigma_i) \\
    & = qd - \left( \sum_{i=1}^{q} ((\text{number of vertices of } \sigma_i) - 1) \right) \\
    & \geq qd - (\text{(number of vertices of } \Delta^{(d+1)(q-1)} - q) \\
    & = qd - ((d + 1)(q - 1) + 1 - q) \\
    & = d.
\end{align*}
$$

(1): This holds because $\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$ is a Tverberg partition and thus $\bigcap_{i=1}^{q} f(\sigma_i) \neq \emptyset$.

(2): This holds because $f$ is in general position.

In (\ast), equality holds only if $d - \dim(\sigma_i) \geq 0$, or equivalently if $\dim(\sigma_i) \leq d$ for all $i$, which is what we had to prove. \qed
2.1 Step 1: Reduction to the $d$-skeleton

**Proof of Corollary 2.1.8.** Let us assume we are given a piecewise linear map $f : \Delta_{d}^{(d+1)(q-1)} \to \mathbb{R}^d$ in general position. First, we extend this map piecewise linearly to a map $\tilde{F} : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$. This is possible by taking a continuous extension, which exists because $\mathbb{R}^d$ is contractible, and then taking a piecewise linear approximation without changing the map on $\Delta_{d}^{(d+1)(q-1)}$. Now we can obtain a piecewise linear map $F : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ in general position by perturbing $\tilde{F}$, again leaving it unchanged on $\Delta_{d}^{(d+1)(q-1)}$.

By the Topological Tverberg Theorem, there is a Tverberg partition for $F$, which must lie in the $d$-skeleton by Lemma 2.1.7 and thus be a Tverberg partition of $f$ as well. \qed

**Proof of Lemma 2.1.9.** Let $\tilde{f} : \Delta_{d}^{(d+1)(q-1)} \to \mathbb{R}^d$ be a map that satisfies $\|\tilde{f} - f\|_\infty < \varepsilon$. We want to show that if $\varepsilon$ is sufficiently small, then we can be sure that every Tverberg partition of $\tilde{f}$ is a Tverberg partition of $f$.

If $S$ is a Tverberg partition of $\tilde{f}$, then we obtain

$$\emptyset \neq \bigcap_{\sigma \in S} \tilde{f}(\sigma) \subseteq \bigcap_{\sigma \in S} \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, f(\sigma)) \leq \varepsilon\}.$$ 

The last expression denotes a compact set that gets smaller when $\varepsilon$ decreases. If it is empty for $\varepsilon = 0$, then it must therefore already be empty for a sufficiently small $\varepsilon$, that is, there is an $\varepsilon_S > 0$ such that

$$\bigcap_{\sigma \in S} f(\sigma) = \emptyset \Rightarrow \bigcap_{\sigma \in S} \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, f(\sigma)) \leq \varepsilon_S\} = \emptyset.$$ 

If we choose

$$\varepsilon := \varepsilon_f := \min \{\varepsilon_S \mid S \text{ a set of } q \text{ disjoint faces of } \Delta^{(d+1)(q-1)}\},$$

then we can be sure that every Tverberg partition of $\tilde{f}$ is also a Tverberg partition of $f$. \qed

**Proof of Corollary 2.1.10.** For all $\varepsilon > 0$ and continuous maps $f : \Delta_{d}^{(d+1)(q-1)} \to \mathbb{R}^d$, we can obtain a piecewise linear map $\tilde{f}$ in general position by distorting $f$ by less than $\varepsilon$. This distortion can, for example, be carried out via a piecewise linear approximation to obtain a piecewise linear map followed by a small adjustment to get this map into general position.

We restrict the distortion and adjustment to $\varepsilon_f$, that is, we make sure that $\|\tilde{f} - f\| < \varepsilon_f$. By assumption, $\tilde{f}$ has a Tverberg partition. This is also a Tverberg partition of $f$ (Lemma 2.1.9). \qed

2.1.3 The connection between Tverberg partitions in the full simplex and in the $d$-skeleton

In the previous section, we proved the equivalence of the Topological Tverberg Theorem and the $d$-Skeleton Conjecture. The arguments that we saw establish the following stronger result as well.
Proposition 2.1.11. Let $F : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map. Every Tverberg partition of $F|_{\Delta^{(d+1)(q-1)}_d}$ is also a Tverberg partition of $F$.

Let $f : \Delta^{(d+1)(q-1)}_d \to \mathbb{R}^d$ be a continuous map. We can extend a slightly distorted version of $f$ to a continuous map $F : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ such that every Tverberg partition of $F$ is also a Tverberg partition of $f$.

Proof. The first part is obvious. For the second part, first approximate $f$ by a piecewise linear map $\bar{f} : \Delta^{(d+1)(q-1)}_d \to \mathbb{R}^d$ in general position that is sufficiently close to $f$ (Lemma 2.1.9) and extend $\bar{f}$ to $F$ in the way described in the proof of Corollary 2.1.8.

Corollary 2.1.12. The $d$-Skeleton Conjecture is valid if $d = 1$ and if $q$ is a prime power.

Therefore Table 1 also applies to the $d$-Skeleton Conjecture.

2.2 Step 2: Reduction to the $(d-1)$-skeleton

Now we proceed to prove the equivalence of the Winding Number Conjecture and the $d$-Skeleton Conjecture.

Proposition 2.2.1. The Winding Number Conjecture implies the $d$-Skeleton Conjecture.

Proof. Let $f : \Delta^{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map and $\sigma_1, \ldots, \sigma_q$ be a winding partition for $f|_{\Delta^{(d+1)(q-1)}_d}$ with winding point $P \in \mathbb{R}^d$. This winding partition is also a Tverberg partition for $f$:

- If $\dim(\sigma_i) \leq d - 1$, we have $P \in f|_{\Delta^{(d+1)(q-1)}_{d-1}}(\sigma_i) = f(\sigma_i)$.
- If $\dim(\sigma_i) = d$, then $W(f|_{\partial \sigma_i}, P) \neq 0$, hence $P \in f(\sigma_i)$.

The proof of the converse is harder. For this we want to show that any map

$$f : \Delta^{(d+1)(q-1)}_{d-1} \to \mathbb{R}^d$$

can be extended to a map

$$F : \Delta^{(d+1)(q-1)}_d \to \mathbb{R}^d$$

such that every Tverberg partition of $F$ is a winding partition of $f$. This would be easy to do if for each $d$-dimensional face $\sigma \subset \Delta^{(d+1)(q-1)}$, we could satisfy $F(\sigma) \subset W_{\neq 0}(f|_{\partial \sigma})$. Unfortunately, this is not always possible because of the following proposition:
Proposition 2.2.2. Not every continuous map \( f : S^{d-1} \to \mathbb{R}^d \) is nullhomotopic within \( W_{\neq 0}(f) \).

We look at two examples. Let \( \mathbb{B}^d \) be the \( d \)-dimensional ball with boundary \( S^{d-1} \).

Example 2.2.3. The first counterexample is a map \( f : S^1 \to \mathbb{R}^2 \) illustrated by Figure 2.6. The topological space \( W_{\neq 0}(f) \) is homotopy equivalent to the wedge of two spheres \( S^1 \). The fundamental group \( \pi_1(W_{\neq 0}(f)) \) is therefore equal to \( \pi_1(S^1 \vee S^1) = \mathbb{Z} \ast \mathbb{Z} \), the free product of \( \mathbb{Z} \) with itself. The element \( [f] \in \pi_1(W_{\neq 0}(f)) \) can be written as the nonzero term \( aba^{-1}b^{-1} \) if we choose generators \( a, b \) of \( \mathbb{Z} \ast \mathbb{Z} \) suitably.

If we extend \( f \) to \( \mathbb{B}^2 \), then its image covers at least one of the two “holes” in \( W_{\neq 0}(f) \) entirely, which is a 2-dimensional set. There is no one-dimensional subset \( V \subset \mathbb{R}^2 \) such that \( f \) is contractible in \( W_{\neq 0}(f) \cup V \).

The suspension \( Sf : S^2 \to \mathbb{R}^3 \) is not a counterexample. We have

\[
W_{\neq 0}(Sf) = SW_{\neq 0}(f) = S(S^1 \vee S^1) = S^2 \vee S^2
\]

again, but this time the homotopy group \( \pi_2(S^2 \vee S^2) \) is not a free product but a free sum \( \mathbb{Z} \oplus \mathbb{Z} \), therefore we calculate \( [f] = aba^{-1}b^{-1} = aa^{-1}bb^{-1} = 0 \) in \( \pi_2(W_{\neq 0}(Sf)) \).

Example 2.2.4. For \( d \geq 4 \), the homotopy group \( \pi_{d-1}(S^{d-2}) \) is nontrivial (see Hatcher [Hat02 Sections 4.1 and 4.2]). For example, the Hopf map \( S^3 \to S^2 \) is not nullhomotopic. Choose such a map \( f : S^{d-1} \to S^{d-2} \) that is not nullhomotopic. Let \( i : S^{d-2} \to \mathbb{R}^d \) be an embedding of the sphere in a \((d-1)\)-dimensional linear subspace of \( \mathbb{R}^d \). Then \( W_{\neq 0}(i \circ f) = i(f(S^{d-2})) \), hence \( i \circ f \) can not be contracted in \( W_{\neq 0}(i \circ f) \).

An important difference between this example and the previous one is that here, \( i \circ f \) can be contracted within the \((d-1)\)-dimensional subspace that contains \( i(S^{d-2}) \). No \( d \)-dimensional set outside of \( W_{\neq 0}(i \circ f) \) is necessary for a contraction.

Because of these problems, we have to take a more technical route.

2.2.1 Reduction to piecewise linear maps in general position

We need an approximation lemma similar to 2.1.9

Lemma 2.2.5. For every continuous map \( f : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d \) there is an \( \varepsilon_f > 0 \) such that the following holds: If \( \tilde{f} : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d \) is a continuous map with \( \| \tilde{f} - f \|_\infty < \varepsilon_f \), then every winding partition of \( \tilde{f} \) is also a winding partition of \( f \).

Corollary 2.2.6. If the Winding Number Conjecture holds for piecewise linear maps in general position, then it also holds for all continuous maps.
2.2 Step 2: Reduction to the \((d-1)\)-skeleton

Proof of Lemma 2.2.5. Let \(\tilde{f} : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d\) be a map that satisfies \(\|\tilde{f} - f\|_{\infty} < \varepsilon\). We want to show that if \(\varepsilon\) is sufficiently small, then we can be sure that every winding partition of \(\tilde{f}\) is a winding partition of \(f\).

If \(S\) is a winding partition of \(\tilde{f}\), we obtain

\[
\emptyset \neq \left( \bigcap_{\sigma \in S, \dim(\sigma) < d} \tilde{f}(\sigma) \right) \cap \left( \bigcap_{\sigma \in S, \dim(\sigma) = d} W_{\neq 0}(\tilde{f}|_{\partial\sigma}) \right)
\]
\[
\subseteq \left( \bigcap_{\sigma \in S, \dim(\sigma) < d} \{ x \in \mathbb{R}^d \mid \text{dist}(x, f(\sigma)) \leq \varepsilon \} \right)
\]
\[
\cap \left( \bigcap_{\sigma \in S, \dim(\sigma) = d} \{ x \in \mathbb{R}^d \mid \text{dist}(x, W_{\neq 0}(f|_{\partial\sigma})) \leq \varepsilon \} \right).
\]

The last expression denotes a compact set that gets smaller when \(\varepsilon\) decreases. If it is empty for \(\varepsilon = 0\), then it is empty for a sufficiently small \(\varepsilon_S\). If we choose

\[
\varepsilon_f := \min \{ \varepsilon_S \mid S \text{ a set of disjoint faces of } \Delta_d^{(d+1)(q-1)} \},
\]
then we can be sure that every winding partition of \(\tilde{f}\) is also a winding partition of \(f\).

Proof of Corollary 2.2.6. Identical to the proof of Corollary 2.1.10.

2.2.2 The case \(d \geq 3\)

Definition 2.2.7. A triangulation of \(\mathbb{R}^d\) is a simplicial complex \(\Delta\) with a fixed linear map \(\|\Delta\| \xrightarrow{\cong} \mathbb{R}^d\). We do not distinguish between a face of the triangulation and the corresponding set in \(\mathbb{R}^d\).
Let $\Delta_1, \Delta_2, \ldots, \Delta_\ell$ be triangulations of $\mathbb{R}^d$. They are in general position with respect to each other if for every subset $S \subset \{1, \ldots, \ell\}$ and faces $\sigma_i$ of $\Delta_i$, we have

$$\text{codim}(\bigcap_{i \in S} \sigma_i) \geq \sum_{i \in S} \text{codim}(\sigma_i).$$

**Proposition 2.2.8.** Let $k \geq 3$. If the $d$-Skeleton Conjecture is true for $d = k$, then the Winding Number Conjecture holds for $d = k$.

**Proof.** By Corollary 2.2.6, we can restrict ourselves to piecewise linear maps. Let $f : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d$ be a piecewise linear map in general position. We divide the proof in three steps:

1. Choose a triangulation $\Delta_\sigma$ of $\mathbb{R}^d$ for every face $\sigma \subset \Delta_{d}^{(d+1)(q-1)}$.

2. Extend $f : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d$ to a continuous map $F : \Delta_{d}^{(d+1)(q-1)} \to \mathbb{R}^d$ “compatible” to the $\Delta_\sigma$.

3. Show that every Tverberg partition of $F$ is a winding partition of $f$. By the $d$-Skeleton Conjecture, $F$ has a Tverberg partition, that thus is a winding partition for $f$.

**Step 1:** For every face $\sigma \subset \Delta_{d}^{(d+1)(q-1)}$ choose a triangulation $\Delta_\sigma$ of $\mathbb{R}^d$ such that

- for $\dim(\sigma) \leq d - 1$, the set $f(\sigma)$ is a subset of the $\dim(\sigma)$-skeleton of $\Delta_\sigma$ and
- for $\dim(\sigma) = d$, the set $f(\partial\sigma)$ is a subset of the $(d - 1)$-skeleton of $\Delta_\sigma$.

Choose the $\Delta_\sigma$ such that if $\sigma_1, \ldots, \sigma_\ell$ are disjoint faces of $\Delta_{d}^{(d+1)(q-1)}$, then $\Delta_{\sigma_1}, \ldots, \Delta_{\sigma_\ell}$ are in general position with respect to each other. This is possible because $f$ is in general position. (Here we need the restrictive definition of “general position”!) Furthermore, choose them such that for every $\Delta_\sigma$ there is a map $b_\sigma : \mathbb{S}^{d-1} \to \Delta_\sigma(= \mathbb{R}^d)$ with the following properties:

- The map $b_\sigma$ is a simplicial embedding with respect to a suitably chosen triangulation of $\mathbb{S}^{d-1}$.

Step 2: Now, we extend $f$ to a $d$-face $\sigma \subset \Delta_{d}^{(d+1)(q-1)}$. Let $\sigma_1, \ldots, \sigma_k$ be the $d$-faces of $B_\sigma \cap (\mathbb{R}^d \setminus W_{\neq 0}(f|_{\partial\sigma}))$ with respect to the triangulation $\Delta_\sigma$. For every $i$ in $\{1, \ldots, k\}$, choose a point $x_i$ in $\sigma_i$. Let

$$v_i : B_\sigma \setminus \{x_1, \ldots, x_k\} \to \mathbb{R}^d \setminus \{x_i\}$$
be the inclusion and let \( I_i : \pi_{d-1}(\mathbb{R}^d \setminus \{x_i\}) \xrightarrow{\cong} \mathbb{Z} \) be the isomorphism used for the definition of the winding number \( W(\cdot, x_i) \). We have the following commutative diagram:

\[
\pi_{d-1}(B_\sigma \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k)) \xrightarrow{r_1}_* \oplus_{i=1}^k \pi_{d-1}(\mathbb{R}^d \setminus \{x_i\}) \xleftarrow{(I_1, \ldots, I_k)} \oplus_{i=1}^k \pi_{d-1}(\mathbb{R}^d \setminus \{x_i\}) \xrightarrow{(j_1)_* \oplus \ldots \oplus (j_k)_*} \pi_{d-1}(B_\sigma \setminus \{x_1, \ldots, x_k\}) \xrightarrow{(\tau_1)_*} \oplus_{i=1}^k \pi_{d-1}(S^{d-1}) \xrightarrow{\text{proj}_i} \pi_{d-1}(S^{d-1})
\]

(1): By blowing up the points \( x_i \) until they fill up all of \( \hat{\sigma}_i \), we obtain a deformation retraction \( r_1 \) from \( B_\sigma \setminus \{x_1, \ldots, x_k\} \) to \( B_\sigma \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k) \) (see Figure 2.7).

(2): There is another deformation retraction \( r_2 \) from \( B_\sigma \setminus \{x_1, \ldots, x_k\} \) to a subset \( S \) of \( \mathbb{R}^d \) that is homeomorphic to the wedge of spheres \( \bigvee_{i=1}^k S^{d-1} \) (see Figure 2.7 again). Let \( j : \bigvee_{i=1}^k S^{d-1} \rightarrow S \) be a homeomorphism. Then let \( j_i : S^{d-1} \rightarrow S \subset \mathbb{R}^d \) be the maps such that \( j = j_1 \vee \ldots \vee j_k \).

(3): Let \( \tilde{j}_i \) be the inclusion \( S^{d-1} \hookrightarrow \bigvee_{i=1}^k S^{d-1} \) of the \( i \)th summand of the wedge sum. Furthermore, let \( \text{proj}_i \) be the projection

\[
\text{proj}_i : \bigoplus_{i=1}^k \pi_{d-1}(S^{d-1}) \rightarrow \pi_{d-1}(S^{d-1})
\]

on the \( i \)th summand. For \( d \geq 3 \), the group homomorphism

\[
\tilde{j} := (\tilde{j}_1)_* \circ \text{proj}_1 + \ldots + (\tilde{j}_k)_* \circ \text{proj}_k : \bigoplus_{i=1}^k \pi_{d-1}(S^{d-1}) \rightarrow \pi_{d-1}(\bigvee_{i=1}^k S^{d-1})
\]

is an isomorphism, see Hatcher [Hat02, Example 4.26]. For \( d = 2 \), this is false. See Remark 2.2.10.

(4): Each \( (j_i)_* \) is an isomorphism, since each \( j_i \) is a homotopy equivalence.

(5): This holds because each \( I_i \) is an isomorphism.
The diagram is commutative. For every continuous map
\[ c : S^{d-1} \rightarrow B_{\sigma} \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k), \]
the equivalence class \([c]\) in \(\pi_{d-1}(B_{\sigma} \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k))\) is mapped by these isomorphisms to \((W(c, x_1), \ldots, W(c, x_k)) \in \bigoplus_{i=1}^k \mathbb{Z}\). The equivalence class \([f|_{\partial \sigma}]\) is therefore mapped to \((W(f, x_1), \ldots, W(f, x_k)) = (0, \ldots, 0)\). Hence \([f|_{\partial \sigma}] = 0\) already in \(\pi_{d-1}(B_{\sigma} \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k))\).

In other words, \(f|_{\partial \sigma}\) is contractible in \(B_{\sigma} \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k)\). This means we can extend \(f\) continuously to \(\tilde{f} : \Delta_{(d+1)(q-1)} \cup \sigma \rightarrow \mathbb{R}^d\) such that \(\tilde{f}(\sigma) \subset B_{\sigma} \setminus (\hat{\sigma}_1 \cup \ldots \cup \hat{\sigma}_k)\), that is, such that \(\tilde{f}(\sigma)\) lies within \(W \neq 0\) and the \((d - 1)\)-skeleton of the complement of \(W \neq 0\). By applying this argument to all \(d\)-faces of \(\Delta_{(d+1)(q-1)}\), we obtain a continuous map \(F : \Delta_{(d+1)(q-1)} \rightarrow \mathbb{R}^d\).

Step 3: We prove that every Tverberg partition of \(F\) is a winding partition of \(f\). Let \(P \in \mathbb{R}^d\) be a Tverberg point and \(\sigma_1, \ldots, \sigma_q \subset \Delta_{(d+1)(q-1)}\) a Tverberg partition for \(F\). We now show that \(\sigma_1, \ldots, \sigma_q\) is also a winding partition for \(f\) with winding point \(P\):

- \(\dim(\sigma_j) \leq d - 1\): In that case we immediately have \(P \in F(\sigma_j) = f(\sigma_j)\).
- \(\dim(\sigma_j) = d\): Suppose \(W(f|_{\partial \sigma_j}, P) = 0\). For \(1 \leq i \leq q\), let \(\tilde{\sigma}_i\) be the face of \(\Delta_{\sigma_i}\) that contains \(P\) in its interior, i.e., the minimal face containing \(P\). We have

\[
\begin{align*}
d & \geq \text{codim} \left( \bigcap_{i=1}^q \tilde{\sigma}_i \right) \\
& \geq \sum_{i=1}^q \max(0, (d - \dim(\tilde{\sigma}_i))) \\
& = \sum_{i=1}^q (d - \dim(\tilde{\sigma}_i)) \\
& \geq q(d - \dim(\sigma_i)) \\
& = qd - ((d + 1)(q - 1) + 1 - q) \\
& = d.
\end{align*}
\]

(1): This holds because \(\bigcap_{i=1}^k \tilde{\sigma}_i\) contains \(P\) and therefore is not empty. 
(2): This holds because the \(\Delta_{\sigma_i}\) are in general position.
2.2 Step 2: Reduction to the \((d-1)\)-skeleton

Figure 2.7: The two retractions of \( B_\sigma \setminus \{x_1, \ldots, x_k\} \): The dots are the points \( \{x_1, \ldots, x_k\} \), the thick line is \( b_\sigma(\mathbb{S}^{d-1}) \) and the thin lines are the \((d-1)\)-skeleton of \( \Delta_\sigma \). The left hand side shows the retraction \( r_1 \) to \( B_\sigma \setminus (\mathring{\sigma}_1 \cup \ldots \cup \mathring{\sigma}_k) \), while the right hand side shows the retraction \( r_2 \) to \( \bigvee_{i=1}^k \mathbb{S}^{d-1} \), divided into two retractions \( r_{2a} \) and \( r_{2b} \).
The inequality (*) is an equality if and only if \( \dim(\tilde{\sigma}_i) = \dim(\sigma_i) \) for all \( i \) and in particular for \( i = j \). Hence \( \dim(\tilde{\sigma}_j) = \dim(\sigma_j) = d \). Outside of \( W_{\neq 0}(f|_{\partial \sigma}) \), the image \( F(\sigma_i) \) lies entirely in the \((d-1)\)-skeleton of \( \Delta_{\sigma_i} \); therefore \( P \) must lie in \( W_{\neq 0}(f|_{\partial \sigma}) \).

\( \square \)

Remark 2.2.9. We used the \( d \)-Skeleton Conjecture for continuous maps. This is necessary because we can not bring a piecewise linear approximation of \( F \) into general position such that \( F(\sigma) \subseteq B_\sigma \setminus (\tilde{\sigma}_1 \cup \ldots \cup \tilde{\sigma}_k) \).

Remark 2.2.10. The problem with the case \( d = 2 \) is that \( \pi_1 \) is not abelian. Instead of

\[
\pi_{d-1}(\mathbb{R}^d \setminus \{x_1, \ldots, x_k\}) \cong \bigvee_{i=1}^k \mathbb{S}^{d-1} \cong \bigoplus_{i=1}^k \pi_{d-1}(\mathbb{S}^{d-1}) \cong \mathbb{Z}^k,
\]

which holds for \( d \geq 3 \), we have

\[
\pi_1(\mathbb{R}^2 \setminus \{x_1, \ldots, x_k\}) \cong \bigvee_{i=1}^k \mathbb{S}^1 \cong \bigoplus_{i=1}^k \pi_1(\mathbb{S}^1)
\]

where \( F_k \) is the free group on \( k \) generators. In particular, \( f|_{\partial \sigma} \) need not be contractible in \( \mathbb{R}^2 \setminus \{x_1, \ldots, x_k\} \), see Figure 2.6.

\subsection*{2.2.3 The case \( d = 2 \) of the Winding Number Conjecture}

We will not show that the cases \( d = 2 \) of the Winding Number Conjecture and the \( d \)-Skeleton Conjecture are equivalent. Instead, we take a different route:

**Proposition 2.2.11.** If the Winding Number Conjecture holds for \( d + 1 \), then it also holds for \( d \).

**Corollary 2.2.12.** The case \( d = 3 \) of the \( d \)-Skeleton Conjecture implies the case \( d = 2 \) of the Winding Number Conjecture.

**Corollary 2.2.13.** The \( d \)-Skeleton Conjecture implies the Winding Number Conjecture.

**Proof of Proposition 2.2.11.** The idea of this proof is based on the proof of Proposition 1.2.3 presented in [dL01].

From any continuous map \( f : \Delta_{d-1}^{(d+1)(q-1)} \to \mathbb{R}^d \), we construct a continuous map \( F : \Delta_{d}^{(d+2)(q-1)} \to \mathbb{R}^{d+1} \). Regard \( \mathbb{R}^d \) as the set of all points in \( \mathbb{R}^{d+1} \) that have last coordinate zero. Furthermore, regard \( \Delta_{d-1}^{(d+1)(q-1)} \) as a face of \( \Delta_{d}^{(d+2)(q-1)} \). We denote the vertices of \( \Delta_{d-1}^{(d+1)(q-1)} \) with \( e_0, e_1, \ldots, e_{(d+1)(q-1)} \).
2.2 Step 2: Reduction to the \((d-1)\)-skeleton

The plane \(\mathbb{R}^2\) contains the image of \(\Delta_1^4 = K_4\), the three points above the plane are \(Q_1, Q_2, Q_3\) and the point below is \(Q\).

and the \(q - 1\) additional vertices of \(\Delta^{(d+2)(q-1)}_d\) with \(P_1, P_2, \ldots, P_{q-1}\). Now choose \(q\) points \(Q, Q_1, Q_2, \ldots, Q_{q-1}\) in \(\mathbb{R}^{d+1}\), such that \(Q\) is below \(\mathbb{R}^d\) (i.e., in \(\mathbb{R}^d \times \mathbb{R}^-\)) and \(Q_1, \ldots, Q_{q-1}\) are above \(\mathbb{R}^d\) (i.e., in \(\mathbb{R}^d \times \mathbb{R}^+\)). The points \(Q_i\) need not be linearly independent.

Define \(F|_{\Delta^{(d+1)(q-1)}_d} := f\) and \(F(P_i) := Q_i\). Extend this to all faces of \(\Delta^{(d+2)(q-1)}_d\) containing at least one of the \(P_i\) by taking cones over \(F|_{\Delta^{(d+1)(q-1)}_{d-1}}\) with the \(Q_i\) as their tips. More precisely, for all nonnegative numbers \(t_i, s_i\) satisfying \(\sum_{i=0}^{(d+1)(q-1)} t_i + \sum_{i=1}^{q-1} s_i = 1\), define

\[
F \left( \sum_{i=0}^{(d+1)(q-1)} t_i e_i + \sum_{i=1}^{q-1} s_i P_i \right) := \left( \sum_{i=0}^{(d+1)(q-1)} t_i \right) f \left( \sum_{i=0}^{(d+1)(q-1)} t_i e_i \right) + \sum_{i=1}^{q-1} s_i Q_i.
\]

If \(\sum_{i=0}^{(d+1)(q-1)} t_i = 0\), then the first summand on the right hand side has to be omitted. Extend this further to all \(d\)-faces containing none of the \(P_i\) (these are the \(d\)-faces of \(\Delta^{(d+1)(q-1)}_{d+1}\)) by using their barycentres to take the cone over \(F|_{\Delta^{(d+1)(q-1)}_{d-1}}\) with \(Q\) as its tip (see Figure 2.8).

The Winding Number Conjecture for \(d + 1\) applied to \(F\) gives us a winding point \(P\) in \(\mathbb{R}^{d+1}\) with a winding partition consisting of \(q\) disjoint faces \(\sigma_1, \ldots, \sigma_q\) of \(\Delta^{(d+2)(q-1)}_{d+1}\). If \(P\) were above \(\mathbb{R}^d\), then all of the \(F(\sigma_i)\) would have to be at least partially above \(\mathbb{R}^d\), therefore all of the \(\sigma_i\) would have to contain at least one of the \(P_i\). But this can not be, since the \(\sigma_i\) are disjoint,
and there are only \( q - 1 \) points \( P_i \). If \( P \) were below \( \mathbb{R}^d \), then all of the \( F(\sigma_i) \) would have to be at least partially below \( \mathbb{R}^d \), hence all of the \( \sigma_i \) would have to contain \( d + 1 \) of the vertices of \( \Delta_{d+1}^{(d+1)(q-1)} \). This cannot be either, since the \( \sigma_i \) are disjoint and there are only \( (d + 1)(q - 1) + 1 < (d + 1)q \) vertices of \( \Delta_{d+1}^{(d+1)(q-1)} \). Therefore \( P \) has to be in \( \mathbb{R}^d \). Define \( \tilde{\sigma}_i := \sigma_i \cap \Delta_{d+1}^{(d+1)(q-1)} \). Then \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_q \) are \( q \) disjoint faces that form a winding partition for \( f \). To see this, we differentiate three cases.

- \( \text{dim}(\sigma_i) \leq d - 1 \): \( P \) is in \( f(\tilde{\sigma}_i) \), since
  \[
P \in F(\sigma_i) \cap \mathbb{R}^d = F(\sigma_i \cap \Delta_{d-1}^{(d+1)(q-1)}) = F(\tilde{\sigma}_i) = f(\tilde{\sigma}_i)
  \]

- \( \text{dim}(\sigma_i) = d \): \( P \) is in \( f(\partial \tilde{\sigma}_i) \), since
  \[
P \in F(\sigma_i) \cap \mathbb{R}^d = F(\sigma_i \cap \Delta_{d-1}^{(d+1)(q-1)}) = F(\partial \tilde{\sigma}_i) = f(\partial \tilde{\sigma}_i)
  \]

- \( \text{dim}(\sigma_i) = d + 1 \): W.l.o.g. we may assume that \( P \) is not in \( F(\partial \sigma_i) \). We know that \( P \) lies in \( W_{\neq 0}(F|_{\sigma_i}) \cap \mathbb{R}^d \), therefore \( F(\partial \sigma_i) \) must contain points both above and below \( \mathbb{R}^d \). Thus \( \sigma_i \) contains exactly one of the \( P_j \), \( \sigma_i \) is \( d \)-dimensional and we get
  \[
P \in W_{\neq 0}(F|_{\sigma_i}) = \{ x \in \mathbb{R}^{d+1} \mid W(F|_{\partial \sigma_i}, x) \neq 0 \} \cap \mathbb{R}^d
  = \{ x \in \mathbb{R}^d \mid W(f|_{\partial \tilde{\sigma}_i}, x) \neq 0 \}
  = W_{\neq 0}(f|_{\partial \tilde{\sigma}_i})
  \]

\[\Box\]

2.2.4 The connection between Tverberg partitions and winding partitions

Again, we have proved a stronger statement than just the equivalence of the \( d \)-Skeleton Conjecture and the Winding Number Conjecture.

**Theorem 2.2.14.** Let \( F : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \) (or even \( F : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \)) be a continuous map. Every winding partition of \( F|_{\Delta_{d-1}^{(d+1)(q-1)}} \) is a Tverberg partition of \( F \).

Let \( f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \) be a continuous map. If \( d \geq 3 \), then we can extend a slightly distorted version of \( f \) to a continuous map \( F : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \) (and even to \( F : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \)) such that every Tverberg partition of \( F \) is a winding partition of \( f \).

If \( d = 2 \), then we can extend a slightly distorted version of \( f \) to a continuous map \( F : \Delta_{3}^{(d+1)(q-1)} \rightarrow \mathbb{R}^3 \) (and even to \( F : \Delta_{3}^{(d+1)(q-1)} \rightarrow \mathbb{R}^3 \)) such that for every Tverberg partition \( \{ \sigma_1, \ldots, \sigma_q \} \) of \( F \), the set \( \{ \sigma_1 \cap \Delta_{2}^{(d+1)(q-1)}, \ldots, \sigma_q \cap \Delta_{2}^{(d+1)(q-1)} \} \) is a winding partition of \( f \).
2.3 The number of winding partitions and Tverberg partitions

Proof. The first part follows by the proof of Proposition 2.2.1.

For the second part, we first approximate \( f \) by a piecewise linear map \( \bar{f} : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \) in general position without adding any new winding partitions (possible by Lemma 2.2.5). We then extend \( \bar{f} \) to the map \( F : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \) constructed in the proof of Proposition 2.2.8. From that proof we know that every Tverberg partition of \( F \) is a winding partition of \( \bar{f} \) and thus of \( f \).

For the third part, extend \( f \) to a continuous map \( \tilde{f} : \Delta_4^{2(q-1)} \) by the suspension described in the proof of Proposition 2.2.11 and proceed with \( \tilde{f} \) like we did in the second part.

This theorem is the correct formulation of the two statements we speculated about in Remark 2.0.8.

Corollary 2.2.15. The Winding Number Conjecture is valid if \( d = 1 \) and if \( q \) is a prime power.

Therefore Table 11 applies to the Winding Number Conjecture, too.

2.3 The number of winding partitions and Tverberg partitions

Proposition 2.3.1 (The case \( d=1 \)). For every continuous mapping \( f : \Delta_0^{2(q-1)} \rightarrow \mathbb{R} \), there are at least \( (q-1)! \) winding partitions. For every continuous map \( f : \Delta_1^{2(q-1)} \rightarrow \mathbb{R} \) respectively \( f : \Delta_2^{2(q-1)} \rightarrow \mathbb{R} \), there are at least \( (q-1)! \) Tverberg partitions.

Proof. \( \Delta_0^{2(q-1)} \) is a set of \( 2(q-1) + 1 = 2q-1 \) vertices. \( f(\Delta_0^{2(q-1)}) \) is a set of \( 2(q-1) + 1 \) real numbers (counted with multiplicity). Denote the points of \( \Delta_0^{2(q-1)} \), ordered by their function value, by \( P_1, \ldots, P_{q-1}, M, Y_1, \ldots, Y_{q-1} \). A partition of these points into \( q \) sets is a winding partition for \( f \) if one of the sets is \( \{M\} \) and all the other sets contain exactly one of the \( P_i \) and one of the \( Q_j \). There are \( (q-1)! \) such partitions. The statement about the number of Tverberg partitions follows directly, because every winding partition of \( f|_{\Delta_0^{2(q-1)}} \) is a Tverberg partition of \( f \).

Proposition 2.3.2 (The case \( d \geq 3 \)). If \( d \geq 3 \), then the following three numbers are equal.

- The minimal number of Tverberg partitions for a continuous map \( f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \).
- The minimal number of Tverberg partitions for a continuous map \( f : \Delta_{d}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d \).
• The minimal number of winding partitions for a continuous map $f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$.

If $d = 2$, then at least the first two of these numbers are equal.

Proof. Proposition 2.1.11 and Theorem 2.2.14.

If Sierksma’s conjecture on the minimal number of Tverberg partitions is correct, then the equivalence established in the previous proposition carries over to the case $d = 2$:

Theorem 2.3.3. The following three statements are equivalent:

1. Sierksma’s conjecture: For all positive integers $d$ and $q$ and every continuous map $f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions.

2. For every continuous map $f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ there are at least $((q-1)!)^d$ Tverberg partitions.

3. For every continuous map $f : \Delta_{d-1}^{(d+1)(q-1)} \rightarrow \mathbb{R}^d$ there are at least $((q-1)!)^d$ winding partitions.

Proof. Again by Proposition 2.1.11 and Theorem 2.2.14 we know that Statements 1 and 2 are equivalent and that Statement 3 implies Statement 2, which in turn guarantees Statement 3 if $d \neq 2$.

We now prove that the case $d = 3$ of Statement 3 implies the case $d = 2$. By Lemma 2.2.5 it is sufficient to examine piecewise linear maps $f : \Delta_1^{3(q-1)} \rightarrow \mathbb{R}^2$ in general position. Regard $f$ as a map $\Delta_1^{3(q-1)} \rightarrow \mathbb{R}^3$ in the way we did in the proof of Proposition 2.2.11. For each pair $e_1, e_2$ of 1-dimensional faces of $\Delta_1^{3(q-1)}$, define one of them to be the “upper” and the other one to be the “lower” one of the pair. Now alter $f$ in the following way: For each intersection $P \in f(e_1) \cap f(e_2)$ of the images of two lines, change $f$ slightly so that the image of the “upper” line runs above the image of the “lower” line at $P$, i.e., has a bigger last coordinate (see Figure 2.9). We call this new map $\tilde{f} : \Delta_1^{3(q-1)} \rightarrow \mathbb{R}^3$.

We continue similar to the proof of Proposition 2.2.11 and choose points $Q_1, \ldots, Q_{q-1}$ high above $\mathbb{R}^2$ and a point $Q$ far below $\mathbb{R}^2$ and extend $\tilde{f}$ to a map $F : \Delta_2^{4(q-1)} \rightarrow \mathbb{R}^3$ by taking cones using the $Q_i$ and $Q$. Let $\{\sigma_1, \ldots, \sigma_q\}$ be a winding partition for $F$ and denote $\bar{\sigma}_i := \sigma_i \cap \Delta_1^{3(q-1)}$. By the argument given in that proof, $\{\bar{\sigma}_1, \ldots, \bar{\sigma}_q\}$ is a winding partition for $f$. Since $f$ is in general position, there are two possibilities for the $\bar{\sigma}_i$.

• All but one of the $\bar{\sigma}_i$ are 2-dimensional. The one that is not 2-dimensional, say $\bar{\sigma}_1$, is therefore 0-dimensional. Since $\{\sigma_1, \ldots, \sigma_q\}$ is a winding partition for our constructed $F$, the faces $\sigma_2, \ldots, \sigma_q$ have to be 3-dimensional and the face $\sigma_1$ has to be 0-dimensional. Therefore each
of the faces $\sigma_2, \ldots, \sigma_q$ contains exactly one of the vertices $P_i$. Hence the winding partition $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_q\}$ of $f$ corresponds to $(q - 1)!$ winding partitions of $F$.

- All but two of the $\tilde{\sigma}_i$ are 2-dimensional. The ones that are not 2-dimensional, say $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, are therefore 1-dimensional. W.l.o.g. let $\tilde{\sigma}_1$ be the “upper” one of the two. Since $\{\sigma_1, \ldots, \sigma_q\}$ is a winding partition for $F$, the faces $\sigma_3, \ldots, \sigma_q$ have to be 3-dimensional, the face $\sigma_2$ has to be 2-dimensional and the face $\sigma_1$ has to be 1-dimensional. Hence the winding partition $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_q\}$ of $f$ corresponds to $(q - 1)!$ winding partitions of $F$.

In any case, a winding partition of $f$ corresponds to $(q - 1)!$ winding partitions of $F$. Since there are at least $((q - 1)!)^3$ winding partitions of $F$, there have to be at least $((q - 1)!)^2$ winding partitions of $f$, which is the bound we wanted to obtain.

We now know that the proved and conjectured lower bounds for the number of Tverberg partitions given in Table 1.2 also apply to the number of winding partitions – except for the proved bound in the case $d = 2$.

**Proposition 2.3.4 (The case $d=2$).** Let $q$ be a prime. There are at least

$$\frac{1}{((q - 1)!)^2} \cdot \left(\frac{q}{2}\right)^{2(q-1)}$$

winding partitions for every map $f : \Delta_1^{3(q-1)} \to \mathbb{R}^2$.

**Proof.** In the case $d = 3$, there are at least $b := \frac{1}{(q-1)!} \cdot \left(\frac{q}{2}\right)^{2(q-1)}$ Tverberg partitions (Theorems 1.3.1 and 2.3.2) and thus the same number of winding partitions. By the proof of the previous theorem, $\frac{b}{(q-1)!}$ is a bound for the number of winding partitions for $d = 2$. \qed

The proved and conjectured bounds for $d = 2$ are compared in Table 2.1

**Example 2.3.5.** For the alternating linear model of $K_n$ described in Example 2.0.3, there are $((q - 1)!)^2$ winding partitions, exactly the bound conjectured in the previous Theorem.
Table 2.1: A comparison of bounds for the number of Tverberg partitions and winding partitions in the case $d = 2$. The numbers are rounded up.
Chapter 3

Q-winding Graphs

The Winding Number Conjecture for \( d = 2 \) claims that complete graphs have a certain property. We now discuss which other graphs have this property, too.

**Definition 3.0.6.** We call a graph \( G \) **\( q \)-winding** if for every map \( f : G \to \mathbb{R}^2 \) there are \( q \) disjoint paths or cycles \( P_1, \ldots, P_q \) in \( G \) with

\[
\left( \bigcap_{P_i \text{ is a path}} f(P_i) \right) \cap \left( \bigcap_{P_i \text{ is a cycle}} W_{\neq 0}(f|_{P_i}) \right) \neq \emptyset.
\]

In accordance with the definition in the previous section, we call \( P_1, \ldots, P_q \) a \( q \)-winding partition for \( f \).

The case \( d = 2 \) of the Winding Number Conjecture claims that \( K_{3q-2} \) is \( q \)-winding.

**Proposition 3.0.7.** If \( q \) is a prime power, then \( K_{3q-2} \) is \( q \)-winding.

We now take a closer look at 1-, 2- and 3-winding graphs.

### 3.1 1-winding graphs

Every non-empty graph is 1-winding.

### 3.2 2-winding graphs and \( \Delta \)-\( Y \)-operations

Here are two examples of 2-winding graphs:
3.2 2-winding graphs and $\Delta$-$Y$-operations

Proposition 3.2.1. $K_4$ and $K_{2,3}$ are 2-winding.

Before we can give a proof, we need to introduce $\Delta$-$Y$-operations. We discuss their effect on $q$-winding graphs in general before we return to the proof of the proposition above.

Definition 3.2.2. A $\Delta$-$Y$-operation deletes the three edges of a triangle and adds a 3-valent vertex with edges going from that vertex to the three vertices of the triangle. A $Y$-$\Delta$-operation is the reverse of a $\Delta$-$Y$-operation.

Figure 3.1: An example of a $\Delta$-$Y$-operation.

Lemma 3.2.3. Let $G$ and $G'$ be two graphs. If there exists a continuous map $f : G \to G'$ that maps disjoint paths to disjoint paths, then $G'$ is $q$-winding if $G$ is $q$-winding.

Proof. Let $g : G' \to \mathbb{R}^2$ be a drawing of $G'$. Then $g \circ f : G \to \mathbb{R}^2$ is a drawing of $G$. Since $G$ is $q$-winding, there are $q$ disjoint paths in $G$ that form a $q$-winding partition for $g \circ f$. These paths are mapped under $f$ to $q$ disjoint pathes in $G'$ that form a $q$-winding partition for $g$. Since $g$ was arbitrary, $G'$ is $q$-winding. 

$\Box$
Proposition 3.2.4. A graph obtained from a Δ-Y-operation on a q-winding graph is again q-winding. A graph obtained from a Y-Δ-operation on a q-winding graph need not be q-winding.

Proof. Assume that $G'$ is obtained from $G$ by a Δ-Y-operation, more precisely by deleting the edges $v_1v_2, v_2v_3$ and $v_1v_3$ and adding the vertex $v$ together with the edges $vv_1, vv_2$ and $vv_3$. Define $f : G \to G'$ as the identity on all vertices of $G$ and all edges of $G$ except the three deleted ones. For these, define

\[
\begin{align*}
  f(v_1v_2) &:= v_1vv_2, \\
  f(v_2v_3) &:= v_2vv_3, \\
  f(v_1v_3) &:= v_1vv_3.
\end{align*}
\]

The function $f$ maps disjoint paths to disjoint paths. $G'$ is thus $q$-winding if $G$ is $q$-winding.

Figure 3.2 illustrates that performing $Y$-Δ-operations may destroy the property of being $q$-winding.

![Figure 3.2: A $Y$-$\Delta$-operation that transforms this 2-winding graph into a graph that is not 2-winding.](image)

We return to the discussion of 2-winding graphs.

Proof of Proposition 3.2.1. The Winding Number Conjecture holds for $q = 2$, hence $K_4$ is 2-winding. The graph $K_{2,3}$ can be obtained from $K_4$ by a Δ-Y-operation and hence is 2-winding as well.

Theorem 3.2.5. A graph is 2-winding if and only if it contains $K_4$ or $K_{2,3}$ as a minor.

Proof. Every graph that has a $q$-winding minor is itself $q$-winding. Therefore every graph containing $K_4$ or $K_{2,3}$ as a minor is 2-winding.
On the other hand, if a graph does not contain one of these two graphs as a minor, then it is outerplanar, that is, it has a planar drawing with all vertices lying on the exterior region (Chartrand and Harary [CH67]). In such a drawing no two edges intersect (the drawing is planar!) and no cycle winds around a vertex. Hence the graph is not 2-winding. □

3.3 3-winding graphs and \( q \)-winding subgraphs of complete graphs

We prove two general results about \( q \)-winding subgraphs of \( K_{3q-2} \) and obtain the minimal 3-winding subgraph of \( K_7 \).

**Theorem 3.3.1.** Let \( p \geq 3 \) be a prime and \( M \) a maximal matching in \( K_{3p-2} \). Then \( K_{3p-2} - M \) is \( p \)-winding.

**Proof.** Let \( N := 4(p - 1) \) and let \( f : K_{3p-2} \to \mathbb{R}^2 \) be a drawing of \( K_{3p-2} \). We divide the proof in three steps.

1. We describe a \( \mathbb{Z}_p \)-invariant subcomplex \( L \) of \( (\Delta N)^* \Delta(2) \).

2. We show that \( \text{ind}_{\mathbb{Z}_p}(L) \geq N > N - 1 = \text{ind}_{\mathbb{Z}_p}((\mathbb{R}^3)^* \Delta) \). By Lemma 1.2.9 on index theory, \( L \) cannot be mapped to \( (\mathbb{R}^3)^* \Delta \mathbb{Z}_p \)-equivariantly.

3. We extend the drawing \( f \) to a map \( F : \Delta N \to \mathbb{R}^3 \) and examine the Tverberg partitions of \( F \) and winding partitions of \( f \) that correspond to \( L \).

**Step 1:** The maximal simplices of \( (\Delta N)^* \Delta(2) \) correspond to the edges of a complete \( (N + 1) \)-partite hypergraph with \( p \) vertices in each shore. In Figure 3.3, the \( N + 1 \) rows represent the \( N + 1 \) shores. We extend the matching \( M \) of \( K_{3p-2} \) to a maximal matching the vertices of \( \Delta N \) and group the rows into pairs accordingly. One row remains single. For each pair of rows we now choose a \( \mathbb{Z}_p \)-invariant cycle in the complete bipartite graph generated by these two shores, such that the cycles contain no vertical lines. The maximal simplices of \( L \) shall be the maximal simplices of \( (\Delta N)^* \Delta(2) \) whose corresponding edge in the hypergraph contains an edge of each cycle (see Figure 3.3). Through this, \( L \) is completely determined and \( \mathbb{Z}_p \)-invariant in \( (\Delta N)^* \Delta(2) \).

**Step 2:** \( L \) can be interpreted as the join of its \( N/2 \) circles and the remaining row of \( p \) points:

\[
L \cong (S^1)^{N/2} * D_p.
\]

By Lemma 1.2.10 we obtain

\[
\text{ind}_{\mathbb{Z}_p}(L) = \text{ind}_{\mathbb{Z}_p}((S^1)^{N/2} * D_p) \\
\geq \frac{N}{2} \text{ind}_{\mathbb{Z}_p}(S^1) + \frac{N}{2} + \text{ind}_{\mathbb{Z}_p}(D_p)
\]
3.3 3-winding graphs and $q$-winding subgraphs of complete graphs

Figure 3.3: This figure illustrates the correspondence between $(\Delta^N)_\Delta^{p}\Delta(2)$ and the complete $(N + 1)$-partite hypergraph with $p$ vertices in each shore. The left hand side shows the case $p = 3$ and $N = 8$: The rows represent the 9 shores of 3 vertices each. For each pair of rows a cycle is drawn. The thick line corresponds to a maximal face of $L$, the partition of the vertices of $(\Delta^8)_\Delta^3\Delta(2)$ represented by this face is drawn in black below the dots. The right hand side of the figure shows the hypergraph corresponding to $(\Delta^{13})_\Delta^4\Delta(2)$. 
3.3 3-winding graphs and $q$-winding subgraphs of complete graphs

The identity $N - 1 = \text{ind}_{\mathbb{Z}_p}((\mathbb{R}^3)_\Delta^p)$ was also stated in Lemma 1.2.10.

**Step 3:** By Theorem 2.2.14, we can extend a slightly distorted version of $f$ to a continuous map $F : \Delta^{(q-1)} \rightarrow \mathbb{R}^3$, such that for every Tverberg partition $\{\sigma_1, \ldots, \sigma_q\}$ of $F$, the set $\{\sigma_1 \cap \Delta^{3(q-1)}, \ldots, \sigma_q \cap \Delta^{3(q-1)}\}$ is a winding partition for $f$.

The maximal simplices of $L$ correspond to sets of disjoint faces of $\Delta^N$. For every continuous map $\Delta^N \rightarrow \mathbb{R}^3$, at least one of these sets is a Tverberg partition, because $L$ can not be mapped $\mathbb{Z}_p$-equivariantly to $(\Delta^N)^\ast_{\Delta(2)}$.

Since we chose the cycles in Step 1 such that they contain no vertical lines, we can be sure that in every such Tverberg partition $\{\sigma_1, \ldots, \sigma_q\}$ of $F$ the two vertices that form a pair do not belong to the same face. This also holds for the corresponding winding partition $\{\sigma_1 \cap \Delta^{3(q-1)}, \ldots, \sigma_q \cap \Delta^{3(q-1)}\}$ of $f$. Therefore we can delete the edges in $K_{3p-2}$ connecting the two vertices of a pair (that is, we can delete the maximal matching $M$) and still have a $p$-winding graph.

**Remark 3.3.2.** The complex $L$ was used before to obtain a lower bound for the number of Tverberg partitions (Theorem 1.3.1, see Matoušek [Mat03, Theorem 6.6.1]).

**Proposition 3.3.3.** Let $N$ be $q - 1$ edges of $K_{3q-2}$ meeting in one vertex. Then $K_{3q-2} - N$ is not $q$-winding.

**Proof.** All we need to do is to find a drawing of $K_{3q-2} - N$ without a $q$-winding partition. We can use the alternating linear model of $K_n$ described in Example 2.0.3. All we have to do is to order the vertices such that the meeting vertex is at the right end of the drawing and the other vertices of $N$ have the numbers $1, 3, 5, \ldots, 2q - 5, 2q - 3$. The edges of $N$ are then in the upper half. Figure 3.4 illustrates the situation for $K_7$ and $K_{10}$.

**Corollary 3.3.4.** The unique minimal 3-winding minor of $K_7$ is $K_7 - M$, where $M$ is a maximal matching.

**Proof.** $K_7 - M$ is a 3-winding minor of $K_7$ (Theorem 3.3.1). It is minimal, because all edges not in $M$ are adjacent to an edge in $M$ and thus must not be deleted (Proposition 3.3.3).

If on the other hand $K$ is 3-winding minor of $K_7$, then only a matching can be deleted (again by Proposition 3.3.3). For $K$ to be minimal, this matching must be maximal. 

**Proposition 3.3.5.** Not every 3-winding graph has $K_7$ minus a maximal matching as a minor.
Proof. Let $M$ be a maximal matching in $K_7$. Execute a $\Delta$-$Y$-operation on an $K_7 - M$; the resulting graph is 3-winding, but does not have $K_7 - M$ as a minor. \qed
Bibliography

[BSS81] Imre Bárány, Senya B. Shlosman, and András Szücs. On a topological generalization of a theorem of Tverberg. *J. London Math. Soc. (2)*, 23(1):158–164, 1981.

[CH67] Gary Chartrand and Frank Harary. Planar permutation graphs. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 3:433–438, 1967.

[dL01] Mark de Longueville. Notes on the topological Tverberg theorem. *Discrete Math.*, 241(1-3):207–233, 2001. Selected papers in honor of Helge Tverberg.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Mat03] Jiří Matoušek. *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*. Universitext. Springer-Verlag, Berlin, 2003.

[Saa64] Thomas L. Saaty. The minimum number of intersections in complete graphs. *Proc. Nat. Acad. Sci. U.S.A.*, 52:688–690, 1964.

[Sar00] Karanbir S. Sarkaria. Tverberg partitions and Borsuk-Ulam theorems. *Pacific J. Math.*, 196(1):231–241, 2000.

[Tve66] Helge Tverberg. A generalization of Radon’s theorem. *J. London Math. Soc.*, 41:123–128, 1966.

[Tve81] Helge Tverberg. A generalization of Radon’s theorem. II. *Bull. Austral. Math. Soc.*, 24(3):321–325, 1981.

[Vol96] Aleksei Yu. Volovikov. On a topological generalization of Tverberg’s theorem. *Mat. Zametki*, 59(3):454–456, 1996. Translation in Math. Notes 59 (1996), no. 3–4, 324-325.

[VŽ93] Aleksandar Vučić and Rade T. Živaljević. Note on a conjecture of Sierksma. *Discrete Comput. Geom.*, 9(4):339–349, 1993.