Riesz summability on boundary lines of holomorphic functions of finite order generated by Dirichlet series

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Abstract
A particular consequence of the famous Carleson–Hunt theorem is that the Taylor series expansion of a bounded holomorphic function on the open unit disk converges almost everywhere on the boundary, whereas on single points the convergence may fail. In contrast, Bayart, Konyagin, and Queffélec constructed an example of an ordinary Dirichlet series \( \sum a_n n^{-s} \), which on the open right half-plane \( \text{Re} > 0 \) converges pointwise to a bounded holomorphic function—but diverges at each point of the imaginary line, although its limit function extends continuously to the closed right half plane. Inspired by a result of M. Riesz, we study the boundary behavior of holomorphic functions \( f \) on the right half-plane which for some \( \ell \geq 0 \) satisfy the growth condition \( |f(s)| = O((1 + |s|)^{\ell}) \) and are generated by some Riesz germ, i.e., there is a frequency \( \lambda = (\lambda_n) \) and a \( \lambda \)-Dirichlet series \( \sum a_n e^{-\lambda_n s} \) such that on some open subset of \( \text{Re} > 0 \) and for some \( m \geq 0 \) the function \( f \) coincides with the pointwise limit (as \( x \to \infty \)) of so-called \( (\lambda, m) \)-Riesz means \( \sum_{\lambda_n < x} a_n e^{-\lambda_n s}(1 - \frac{\lambda_n}{x})^m, \ x > 0 \). Our main results present criteria for pointwise and uniform Riesz summability of such functions on the boundary line \( \text{Re} = 0 \), which include conditions that are motivated by classics like the Dini-test or the principle of localization.

Keywords General Dirichlet series · Finite order · Riesz summability · Almost everywhere convergence · Hardy spaces

Mathematics Subject Classification Primary 43A17; Secondary 30B50 · 43A50

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Dedicated to our brave and much-loved Matea, who will always occupy a substantial place in our hearts

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1 Introduction

A \( \lambda \)-Dirichlet series is a series of the form \( D = \sum a_n(D) e^{-\lambda_n s} \), where \( (a_n(D)) \) is a sequence of complex coefficients (called Dirichlet coefficients), \( \lambda = (\lambda_n) \) a strictly increasing, non-negative real sequence (called frequency), and \( s \) a complex variable. A fundamental property states that, whenever \( D \) converges at some complex number \( s = \sigma + i \tau \), it converges on the open half plane \( \text{Re} > \sigma \), where its limit defines a holomorphic function.

To recall two prominent examples, observe that the choice \( \lambda = (\log n) \) leads to ordinary Dirichlet series \( \sum a_n n^{-s} \), whereas the choice \( \lambda = (n) \) after the substitution \( z = e^{-s} \) generates power series \( \sum a_n z^n \) in one variable.

Given a frequency \( \lambda = (\lambda_n) \) and a holomorphic function \( f : [\text{Re} > 0] \to \mathbb{C} \), we call a \( \lambda \)-Dirichlet series \( D = \sum a_n(D) e^{-\lambda_n s} \) a \( \lambda \)-Riesz germ of \( f \), whenever on some open subset \( U \subset [\text{Re} > 0] \) and for some \( m \geq 0 \)

\[
f(s) = \lim_{x \to \infty} R^\lambda_{x,m}(D)(s), \quad s \in U,
\]

where the \((\lambda, m)\)-Riesz means of \( D \) in \( s \in \mathbb{C} \) are defined by

\[
R^\lambda_{x,m}(D)(s) = \sum_{\lambda_n < x} a_n(D) e^{-\lambda_n s} \left(1 - \frac{\lambda_n}{x}\right)^m, \quad x > 0.
\]

In [10, Corollary 2.15] it is proved that \( \lambda \)-Riesz germs \( D \) of \( f \), whenever they exist, are unique, and as a consequence we in this case may assign to every such \( f \) the unique sequence

\[
(a_n(f))_n = (a_n(D))_n,
\]
which we call the ‘sequence of Bohr coefficients of \( f \)’. 

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Consequently, given a holomorphic function \( f : [\text{Re} > 0] \to \mathbb{C} \) generated by the \( \lambda \)-Riesz germ \( D \), we may define the \( x \)th Riesz mean of order \( k \geq 0 \) of \( f \) in \( s \in \mathbb{C} \) by
\[
R^{\lambda,k}_x(f)(s) = \sum_{\lambda_n < x} a_n(f)e^{-\lambda ns} \left(1 - \frac{\lambda_n}{x}\right)^k,
\]
and a natural question then is to which extend these Riesz means ‘reproduce’ the function itself.

Within this setting a more precise formulation of the above questions reads as follows: Given a frequency \( \lambda \) and a holomorphic function \( f : [\text{Re} > 0] \to \mathbb{C} \) generated by the \( \lambda \)-Riesz germ \( D \),

- is there any \( k \geq 0 \) such that \( f \) on \([\text{Re} > 0]\) is pointwise \( (\lambda, k) \)-Riesz summable on \([\text{Re} > 0]\),
- and if yes, to which extend does this approximation transfer to the boundary \([\text{Re} = 0]\)?

In the rest of this introduction we want to indicate that the first part of this question is fairly well understood, and why we hence are going to concentrate on the second part.

### 1.1 Classics

Let us illustrate all this, recalling some classics for the power series case \( \lambda = (n) \). Each function \( f \) from the Hardy space \( H^\infty(D) \) of all bounded and holomorphic functions on the open complex unit ball \( D \) determines its (formal) Taylor series
\[
P(z) = \sum \frac{\partial^n f(0)}{n!} z^n, \quad z \in \mathbb{C}
\]
(i.e. the \( (n) \)-Dirichlet series \( D = \sum a_n(D)e^{-ns} \) with \( a_n(D) = \frac{\partial^n f(0)}{n!} \) after the substitution \( z = e^{-s} \)), and moreover, the function \( f \) is represented by its Taylor series in the sense that for every \( z \in D \)
\[
f(z) = \sum_{n=0}^{\infty} \frac{\partial^n f(0)}{n!} z^n.
\]

Since by Fatou’s theorem the radial limits of \( f \), i.e.
\[
f^*(t) = \lim_{r \to 1} f(re^{it}),
\]
exist for almost all \( t \in [0, 2\pi[, \) one may ask, if \( P \) converges almost everywhere on the boundary \( \mathbb{T} \) and if in this case its pointwise limit coincides with \( f^* \). A consequence of the famous Carleson–Hunt theorem indeed shows that for almost every \( t \in [0, 2\pi[ \)
\[
f^*(t) = \sum_{n=0}^{\infty} \frac{\partial^n f(0)}{n!} e^{-int}.
\]

On the other hand, it is well-known that there exists a function \( f \in H^\infty(D) \), that is uniformly continuous on \D, although its Taylor series diverges at certain points of the boundary \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \).

Several criteria for pointwise convergence of the Taylor series from (1) on the boundary are known. Having in mind that \( H^\infty(D) \) via the mapping \( f \mapsto f^* \) is isometrically isomorphic to \( H^\infty(\mathbb{T}) \) and that this mapping preserves Taylor coefficients \( (\frac{\partial^n f(0)}{n!}) \) and Fourier coefficients \( (\hat{f^*}(n)) \), i.e.
\[
H^\infty(D) = H^\infty(\mathbb{T}),
\]
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the classical Dini test (see e.g. [17, p. 53]) states that for \( f \in H_\infty(D) \) and \( z_0 \in \mathbb{T} \)

\[
f^* (z_0) = \sum_{n=0}^{\infty} \frac{\partial^n f(0)}{n!} z_0^n,
\]

whenever

\[
\int_{\mathbb{T}} \left| \frac{f^*(z) - f^*(z_0)}{z - z_0} \right| \, dz < \infty.
\]

Moreover, if \( I \subset \mathbb{T} \) is an open set such that \( f^*(z) = 0 \) for all \( z \in I \), then

\[
\sum_{n=0}^{\infty} \frac{\partial^n f(0)}{n!} z^n = 0, \quad z \in I,
\]

a fact known as the principle of localization (see [17, p. 54]).

1.2 Ordinary Dirichlet series—a counter example

The situation changes dramatically, if we jump from the frequency \( \lambda = (n) \) to the frequency \( \lambda = (\log n) \). Recall that

\[
D_\infty = D_\infty((\log n))
\]

denotes the linear space of all ordinary Dirichlet series \( D = \sum a_n n^{-s} \) which converge on some half-plane \( \text{Re} > \sigma_0 \) and have a limit function \( f \) on this half-plane extending to a bounded holomorphic function on all of \( \text{Re} > 0 \).

A fundamental theorem of Bohr from [3] (see e.g. [6, Theorem 1.5]) shows that every \( D \in D_\infty \) in fact converges uniformly on all half-planes \( \text{Re} > \epsilon, \epsilon > 0 \). This fact has many non-trivial consequences—among others that \( D_\infty \) endowed with the supremum norm \( \|D\|_\infty = \sup_{\text{Re} s > 0} |f(s)| \) is a Banach space. For all needed information on ordinary Dirichlet series see the monographs [6] and [19].

It may come as a surprise that in contrast to the case \( \lambda = (n) \), Bayart, Konyagin, and Queffélec for the case \( \lambda = (\log n) \) in their article [2] prove the existence of a Dirichlet series \( D \in D_\infty \), that diverges at every point on the imaginary line \( \text{Re} = 0 \) although its limit function \( f \) extends continuously to the closed half plane \( \text{Re} \geq 0 \).

Let us explain that this result may be interpreted as a result in infinite dimensional holomorphy as well as a result in harmonic analysis. Indeed, denote by \( H_\infty(B_{c_0}) \) the Banach space of all holomorphic (Fréchet differentiable) functions \( g : B_{c_0} \to X \) endowed with the sup norm, where \( B_{c_0} \) denotes the open unit ball of the Banach space \( c_0 \) of all complex null sequences. Then there is a unique isometric linear bijection

\[
D_\infty = H_\infty(B_{c_0}), \quad D \mapsto g,
\]

which preserves Dirichlet coefficients \( (a_n(D)) \) and monomial coefficients \( (\frac{\partial^\alpha g(0)}{\alpha!}) \) in the sense that \( a_n(D) = \frac{\partial^\alpha g(0)}{\alpha!} \), whenever \( n = p^\alpha \); here \( \alpha = (\alpha_n) \) stands for a finite multi index with entries from \( \mathbb{N}_0 \) (we write \( \alpha \in \mathbb{N}_0^{(\mathbb{N})} \)) and \( p = (p_n) \) for the sequence of primes (see [13] and [6, Theorem 3.8] for details).
If $D$ and $g$ are associated to each other according to (5), then by [6, Theorem 3.8] for every $s = u + it \in \{\Re > 0\}$
\[
g(p^{-s}) = \sum_{n=1}^{\infty} a_n(D)n^{-s} = \lim_{x \to \infty} \sum_{p^\alpha < x} \frac{\partial^\alpha g(0)}{\alpha!} \frac{1}{p^{\alpha u}} \frac{1}{p^{\alpha t}}.
\]
(6)

But for $u = 0$ this is in general not true—the Bayart–Konyagin–Queffélec example shows the existence of a function $g \in H_\infty(B_{c_0})$, such that none of the limits
\[
\lim_{x \to \infty} \sum_{p^\alpha < x} \frac{\partial^\alpha g(0)}{\alpha!} \frac{1}{p^{\alpha t}}, \quad t \in \mathbb{R},
\]
exists.

For a reformulation of (7) in terms of harmonic analysis recall that the countable product $T^\infty$ of the torus $T = \{z \in \mathbb{C} : |z| = 1\}$ forms a compact abelian group, where the Haar measure is given by the countable product of the normalized Lebesgue measure. The Hardy space $H_\infty(T^\infty)$ is the closed subspace of all $f \in L_\infty(T^\infty)$ such that the Fourier transforms $\hat{f} : \overline{T^\infty} = \mathbb{Z}^{(N)} \to \mathbb{C}$ have their supports in $\mathbb{N}_0^{(N)}$. Then (as an analog of (2)) there is an isometric isomorphism
\[
H_\infty(B_{c_0}) = H_\infty(T^\infty), \quad g \mapsto f
\]
(8)

preserving monomial coefficients ($\frac{\partial^\alpha g(0)}{\alpha!}$) and Fourier coefficients ($\hat{f}(\alpha)$) (see e.g. [6, Theorem 5.1]). Reformulating (7), we see that there is a function $f \in H_\infty(T^\infty)$ such that none of the limits
\[
\lim_{x \to \infty} \sum_{p^\alpha < x} \hat{f}(\alpha) \frac{1}{p^{\alpha t}}, \quad t \in \mathbb{R}
\]
exists.

Hedenmalm and Saaksman in [14] proved a Carleson type convergence theorem for $L_2(T^\infty)$, i.e. the Fourier series of every function in $L_2(T^\infty)$ converges almost everywhere. A particular consequence of this deep fact is that there is an alternative theorem which sometimes may compensate for the loss caused by the Bayart–Konyagin–Queffélec example.

To explain this, recall that all characters $\chi : \mathbb{N} \to \mathbb{T}$, so all completely multiplicative mappings from $\mathbb{N}$ into $\mathbb{T}$, in a natural way form a compact abelian group (identifying them with $T^\infty$). Then we know from [14, Theorem 1.4] (see also [9, Theorem 2.1]), that for every Dirichlet series $D = \sum a_n n^{-s}$ with $(a_n) \in \ell_2$ (so in particular for Dirichlet series in $D_\infty$, see e.g. [7, Corollary 4.11])
\[
\sum a_n \chi(n)n^{-it}
\]
(10)
converges for almost all characters $\chi : \mathbb{N} \to \mathbb{T}$ and almost all $t \in \mathbb{R}$. This is a considerable improvement of an earlier result of Helson from [15] (see also [16, Theorem 9]).

Supplementing all this, we are going to show (see Corollary 5.4) that the limit function $f$ of any ordinary Dirichlet series $D = \sum a_n n^{-s} \in D_\infty$ extends almost everywhere to the imaginary line $i\mathbb{R}$, where it is almost everywhere $((\log n), k)$-Riesz-summable at any order $k > 0$, i.e. for almost all $t \in \mathbb{R}$ and all $k > 0$ the $((\log n), k)$-Riesz limit
\[
\lim_{x \to \infty} \sum_{\log n < x} a_n \frac{1}{n^{it}} \left(1 - \frac{\log n}{x}\right)^k
\]
(11)
exists. Moreover, it will turn out that under certain further analytic assumptions on the limit function \( f \) of \( D \) on \([\text{Re} > 0]\) this convergence improves considerably (Sect. 7).

### 1.3 General Dirichlet series and uniform almost periodicity

The results on ordinary Dirichlet series which we just indicated, will be performed within a setting of general Dirichlet series—a far more challenging task.

Fixing a frequency \( \lambda \), an extensive ‘modern’ study of \( \lambda \)-Dirichlet series \( \sum a_n e^{-\lambda n s} \) has been started in the recent articles \([1, 5–9, 21, 22]\), and one of the major concepts is the introduction of so-called Hardy spaces \( \mathcal{H}_p(\lambda) \) of \( \lambda \)-Dirichlet series (in Sect. 9 we repeat the definition).

The particular case \( p = \infty \) is of special interest, since then \( \mathcal{H}_\infty(\lambda) \) may be described in terms of holomorphic functions on the right half-plane, and in fact our purposes in this article demand only this case.

Therefore, recall that \( \mathcal{H}_\infty^\lambda(\text{Re} > 0) \) (as defined in \([8]\)) denotes the linear space of all holomorphic and bounded functions \( f : [\text{Re} > 0] \to \mathbb{C} \), which are almost periodic on all vertical lines \([\text{Re} = \sigma] \) (or equivalently, some line \([\text{Re} = \sigma] \)) and have Bohr coefficients

\[
a_n(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (\sigma + it)e^{-(\sigma + it)x}dt, \quad x \in \mathbb{R}
\]  

supported in \( \{\lambda_n : n \in \mathbb{N}\} \). Note that here the limits (12) are independent of the choice of \( \sigma > 0 \). See e.g. \([19, \text{Section 1.5.2.2}]\) for the definition of almost periodic functions on \( \mathbb{R} \), in particular \([19, \text{Theorem 1.5.5}]\) for a couple of important equivalent reformulations of its definition.

Together with the sup norm, taken on the right half-plane, \( \mathcal{H}_\infty^\lambda(\text{Re} > 0) \) forms a Banach space, and by \([8, \text{Theorem 2.16}]\) there is a coefficient preserving isometric linear bijection identifying the Hardy space \( \mathcal{H}_\infty(\lambda) \) of \( \lambda \)-Dirichlet series and \( \mathcal{H}_\infty^\lambda(\text{Re} > 0) \),

\[
\mathcal{H}_\infty(\lambda) = \mathcal{H}_\infty^\lambda(\text{Re} > 0).
\]  

Moreover, by \([10, \text{Corollary 3.11}]\) we know that for every bounded and holomorphic function \( f : [\text{Re} > 0] \to \mathbb{C} \) we have

\[
f \in \mathcal{H}_\infty^\lambda(\text{Re} > 0) \text{ if and only if } f \text{ has a } \lambda \text{-Riesz germ.}
\]  

Let us consider an important subspace of \( \mathcal{H}_\infty^\lambda(\text{Re} > 0) \). By \( \mathcal{D}_\infty(\lambda) \) we denote the space of all Dirichlet series \( \sum a_n(D) e^{-\lambda_n s} \) which converge on \([\text{Re} > 0] \) and have a bounded limit function \( f : [\text{Re} > 0] \to \mathbb{C} \). Then \( \mathcal{D}_\infty(\lambda) \) is a normed space if we endow it with the supremum norm \( \|f\| = \sup_{\text{Re}s > 0} |f(s)| \), and by Helson \([16, \text{Theorem 8}]\) (see also \([8, \text{Corollary 2.17}]\)) we may interpret it as an isometric subspace of \( \mathcal{H}_\infty^\lambda(\text{Re} > 0) \), where the Dirichlet and Bohr coefficients are preserved. We say that a frequency \( \lambda \) satisfies Bohr’s theorem whenever every \( \lambda \)-Dirichlet series \( D = \sum a_n(D) e^{-\lambda_n s} \), which converges on some half-plane and has a limit function extending to a bounded, holomorphic function to \([\text{Re} > 0] \), in fact converges uniformly on all half-planes \([\text{Re} > \varepsilon], \varepsilon > 0 \); in other terms,

\[
f(s) = \lim_{x \to \infty} R_x^{\lambda,0} f(s) = \lim_{x \to \infty} \sum_{\lambda_n < x} a_n(f) e^{-\lambda_n s}
\]  

uniformly on \([\text{Re} > \varepsilon] \) for all \( \varepsilon > 0 \). As indicated in the preceding section the frequency \( \lambda = (\log n) \) satisfies Bohr’s theorem.
A delicate question, which came up in [9], then is whether we have
\[ \mathcal{D}_\infty(\lambda) = H_\infty^\lambda[\text{Re} > 0], \] (15)
i.e., each \( f \in H_\infty^\lambda[\text{Re} > 0] \) is represented by its Dirichlet series in the sense that \( f(s) = \sum_{n=1}^\infty a_n e^{-\lambda_n s} \) for all \( s \in [\text{Re} > 0] \).

A positive answer is provided by the so-called equivalence theorem from [9, Theorem 5.1] stating that (15) holds if and only if \( \lambda \) satisfies Bohr’s theorem if and only if \( \mathcal{D}_\infty(\lambda) \) is a Banach space if and only if for every \( \sigma > 0 \) there is a constant \( C > 0 \) such that for all complex sequences \( (a_n) \) and \( M \in \mathbb{N} \)
\[
\sup_{N \leq M} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n e^{-it\lambda_n} \right| \leq C e^{\sigma M} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^M a_n e^{-it\lambda_n} \right|,
\]
(for the third equivalence see [5, Theorem 4.12]). Counterexamples of frequencies \( \lambda \), failing the equality in (15), are then provided by [22, Theorem 5.2], and concrete sufficient conditions on \( \lambda \) were given by Bohr [4], Landau [18], and more recently Bayart [1]. These criteria in particular prove that
\[ \mathcal{D}_\infty((n)) = H_\infty^{(n)}[\text{Re} > 0] \text{ and } \mathcal{D}_\infty((\log n)) = H_\infty^{(\log n)}[\text{Re} > 0]. \] (16)

More generally than what we announced for the ordinary case in (11), we are going to show that every \( f \in H_\infty^\lambda[\text{Re} > 0] \) extends almost everywhere to the imaginary axis, where it for all \( k > 0 \) is \((\lambda, k)\)-Riesz summable almost everywhere, i.e. for almost all \( t \in \mathbb{R} \) the horizontal limits
\[ f^*(it) = \lim_{\varepsilon \to 0} f(\varepsilon + i\tau) \]
exist, and for all \( k > 0 \)
\[ f^*(it) = \lim_{x \to \infty} R_x^{\lambda,k}(f)(it) = \sum_{\lambda_n < x} a_n(f)e^{-it\lambda_n} \left( 1 - \frac{\lambda_n}{x} \right)^k \] (17)
(see Corollary 5.3). In particular, (17) holds true for limit functions of Dirichlet series from \( \mathcal{D}_\infty(\lambda) \), since (as mentioned before) \( \mathcal{D}_\infty(\lambda) \subseteq H_\infty^\lambda[\text{Re} > 0] \). Moreover, the convergence in (17) improves considerably whenever \( f \) fulfills certain additional assumptions.

What happens, if we consider unbounded holomorphic functions on \([\text{Re} > 0]\)?

### 1.4 Two theorems of M. Riesz

Indeed, already M. Riesz in his article [20] from 1909 gave a positive answer, stating a sufficient condition for a wider class of holomorphic functions on \([\text{Re} > 0]\) which are not necessarily bounded. We recall his two beautiful results from [12, Theorem 41, 42], which (here reformulated using our notions) in fact were the starting point of our research.

**Theorem 1.1** Let \( f : [\text{Re} > 0] \to \mathbb{C} \) be holomorphic with a \( \lambda \)-Riesz germ. Assume that there is \( \ell \geq 0 \) such that
\[ \forall \varepsilon > 0 \exists C(\varepsilon) > 0 : |f(s)| \leq C(\varepsilon)|s|^\ell, \ s \in [\text{Re} > \varepsilon]. \] (18)
Then for every \( k > \ell \) and \( s \in [\text{Re} > 0] \)
\[ f(s) = \lim_{x \to \infty} R_x^{\lambda,k}(f)(s). \]
Theorem 1.2 Let \( f : [\text{Re} > 0] \to \mathbb{C} \) be holomorphic with a \( \lambda \)-Riesz germ and \( k > \ell \geq 0 \). Assume that \( f \) extends continuously to \([\text{Re} \geq 0]\) with the exception of finitely many poles \( p_1, \ldots, p_m \) on \([\text{Re} = 0]\) of order \(< k + 1\). If there exist \( C, \tau_0 > 0 \) such that for all \( s = \sigma + i \tau \in [\text{Re} > 0] \) with \(|\tau| \geq \tau_0\)
\[
|f(s)| \leq C|s|^{\ell},
\]
then for every \( i \tau \notin \{ p_1, \ldots, p_m \} \) we have
\[
f(i \tau) = \lim_{x \to \infty} R^\lambda_x, k(f)(i \tau).
\]
Moreover, on every compact interval \( I \subset [\text{Re} = 0]\)\{p_1, \ldots, p_m\} the convergence is uniform.

We illustrate the last theorem with a well-known example (see e.g. [12]). Take the zeta-Dirichlet series \( \sum n^{-s} \), that converges absolutely on \([\text{Re} > 1]\), and consider its analytic continuation \( \zeta \), namely the zeta function, on \([\text{Re} > 0]\) given by the formula
\[
(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}. \quad (19)
\]
Recall that \( \zeta \) has a simple pole at \( s = 1 \), and satisfies the estimate
\[
|\zeta(1 + it)| \leq C \log(|t|), \quad |t| \geq 1. \quad (20)
\]
After an obvious translation, Theorem 1.2 is applicable for every \( \ell > 0 \), and so as a consequence for every \( k > 0 \) and \( \tau \in \mathbb{R}\{0\} \)
\[
\zeta(1 + i \tau) = \lim_{x \to \infty} \sum_{\log(n) < x} n^{-(1+i\tau)} \left( 1 - \frac{\log(n)}{x} \right)^{k}, \quad (21)
\]
with uniform convergence on every compact interval \( I \subset \mathbb{R}\{0\} \).

1.5 New results in a new setting

Motivated by these two theorems of Riesz we in [10] define, given a frequency \( \lambda \) and \( \ell \geq 0 \), the space
\[
H^\lambda_{\infty, \ell}[\text{Re} > 0],
\]
collecting all holomorphic functions \( f : [\text{Re} > 0] \to \mathbb{C} \), which are generated by a \( \lambda \)-Riesz germ and satisfy the growth condition
\[
\|f\|_{\infty, \ell} = \sup_{\text{Re}s > 0} \left| \frac{f(s)}{(1 + s)^{\ell}} \right| < \infty. \quad (22)
\]
That this in fact leads to a Banach space \( (H^\lambda_{\infty, \ell}[\text{Re} > 0]), \| \cdot \|_{\infty, \ell} \) is a non-trivial fact proved in [10, Theorem 3.16]. The case \( \ell = 0 \) is of special interest, since then
\[
H^\lambda_{\infty}[\text{Re} > 0] = H^\lambda_{\infty, 0}[\text{Re} > 0] \quad \text{isometrically;} \quad (23)
\]
this was already remarked in (14) within a slightly different context.

In [10] we performed a sort of structure theory of these Banach spaces—mainly based on a considerable extension of Theorem 1.1. In fact, most of the results we derive there, are
consequences of the following approximation theorem from [10, Theorem 3.7] for functions in $H^\lambda_{\infty, \ell}[\text{Re} > 0]$ in terms of their Riesz means.

**Theorem 1.3** Let $k > \ell \geq 0$ and $f \in H^\lambda_{\infty, \ell}[\text{Re} > 0]$. Then for every $u > 0$

$$\lim_{x \to \infty} R^{\lambda, k}_x (f)(u + \cdot) = f(u + \cdot) \quad \text{in} \quad H^\lambda_{\infty, \ell}[\text{Re} > 0].$$

In particular, for every $s \in [\text{Re} > 0]$

$$f(s) = \lim_{x \to \infty} R^{\lambda, k}_x (f)(s).$$

The central question we intend to study in this article then is, to which extent functions in $H^\lambda_{\infty, \ell}[\text{Re} > 0]$ are Riesz summable on the imaginary line.

Let us sketch the main results we establish by fixing a frequency $\lambda$. We in Remark 5.2 observe that, if the $(\lambda, k)$-Riesz limit of a Dirichlet series $D$ exists at some point $i \tau_0 \in i\mathbb{R}$, and so the limit function $g$ of $D$ defines a holomorphic function on $[\text{Re} > 0]$, then necessarily

$$\lim_{x \to \infty} R^{\lambda, k}_x (D)(i \tau_0) = \lim_{u \to 0} g(u + i \tau_0).$$

Hence, one of the main properties of functions $f \in H^\lambda_{\infty, \ell}[\text{Re} > 0]$ we take advantages of, is given by the fact that the horizontal limits

$$f^*(it) = \lim_{u \to 0} f(u + it) \quad (24)$$

exist for almost every $t \in \mathbb{R}$ and define a measurable function (see Proposition 2.1 and Corollary 2.2). The proof needs Fatou’s famous theorem on boundary limits (within Stolz regions) of bounded holomorphic function on the disc $\mathbb{D}$. For these horizontal limits we then in Theorem 5.1 prove that for all $k > \ell \geq 0$ and for almost all $\tau \in \mathbb{R}$

$$f^*(i \tau) = \lim_{x \to \infty} R^{\lambda, k}_x (f)(i \tau).$$

In Sect. 10.1 (see also Corollary 3.2) we provide an internal characterization of the closed subspace of $L^\infty(\mathbb{R})$ of all horizontal limits $f^*$ generated by functions $f \in H^\lambda_{\infty, \ell}[\text{Re} > 0]$. The main tool for all this is a far-reaching Perron-type formula for such horizontal limits (Theorem 4.3). Elaborating these almost everywhere results, we show in Theorem 7.1 that, if $f^*$ is continuous on some open interval $I \subset [\text{Re} = 0]$, then for all $i \tau \in I$

$$\lim_{x \to \infty} R^{\lambda, k}_x (f)(i \tau) = f^*(i \tau) \quad (25)$$

with uniform convergence on every compact subinterval $J \subset I$. Even more, the sequence $(R^{\lambda, k}_x (f))_{x > 0}$ converges uniformly on all ‘flattened cones’, which include all rectangles of the form $[0, \sigma] + J, \sigma > 0$. In Theorem 6.1 we show a principle of localization: Assuming that $g$ is another function in $H^\lambda_{\infty, \ell}[\text{Re} > 0]$ such that $f^* = g^*$ on some open interval $I \subset [\text{Re} = 0]$, we prove that for all $i \tau \in I$

$$\lim_{x \to \infty} R^{\lambda, k}_x (f)(i \tau) \text{ exists if and only if } \lim_{x \to \infty} R^{\lambda, k}_x (g)(i \tau) \text{ exists},$$

and in this case

$$\lim_{x \to \infty} R^{\lambda, k}_x (f)(i \tau) = \lim_{x \to \infty} R^{\lambda, k}_x (g)(i \tau).$$

We finish with a Dini test in Theorem 8.1: If for $\tau \in \mathbb{R}$ there is $\delta > 0$ such that

$$\int_{-\delta}^{\delta} \frac{|f^*(iy + \tau) - f^*(i\tau)|}{|y|^{1+k+\ell}} dy < \infty,$$
then
\[ \lim_{x \to \infty} R^x_\lambda(f)(i\tau) = f^*(i\tau). \]

We note that Theorem 1.2 in [12] is stated without proof. Since the original article [20] of M. Riesz from 1909 is not easily accessible, we at the end of our article (Sect. 10.2) provide a full (more modern) proof of Theorem 1.2.

Eventually, we comment how Theorem 1.2 and the main contributions of this article are related to each other. First, observe that a function \( f \in H^\lambda_{\infty,\ell}[\Re > 0] \), which can be continuously extended to all of \([\Re \geq 0]\) with the exception of a finite number of points on the boundary line \([\Re = 0]\), never can have poles at these points (see again (22)). So for example the result from (21), being a consequence of Theorem 1.2, can not be derived from (25) (Theorem 7.1). On the other hand, we hope to convince our reader that focusing on functions from \( H^\lambda_{\infty,\ell}[\Re > 0] \), leads to far more knowledge which can not be reached under the restrictions assumed in Theorem 1.2.

### 2 Horizontal limits

As mentioned in the introduction for \( f \in H^\lambda_{\infty,\ell}[\Re > 0] \) we define the measurable function

\[ f^*: i\mathbb{R} \to \mathbb{C}, \quad f^*(it) = \begin{cases} \lim_{\epsilon \to 0} f(\epsilon + it) & \text{if the limit exists} \\ 0 & \text{else,} \end{cases} \]

and call it the horizontal limit function of \( f \). The purpose of this section is to ensure that this definition indeed is reasonable (see Corollary 2.2). This fact is based on the following seemingly well-known consequence of Fatou’s theorem on non-radial limits of holomorphic functions on the open unit disc \( \mathbb{D} \) (see e.g. [6, Lemma 11.22]):

Given a bounded and holomorphic function \( f: [\Re > 0] \to \mathbb{C} \), for almost every \( t \in \mathbb{R} \) the horizontal limit

\[ f^*(it) = \lim_{\epsilon \to 0} f(\epsilon + it) \]

exists. In fact, we in the following need a variant of this result, namely the following improvement.

**Proposition 2.1** Let \( f: [\Re > 0] \to \mathbb{C} \) be bounded and holomorphic. Then there is a null set \( E \) in \( i\mathbb{R} \) such that for all \( t \in i\mathbb{R}\setminus E \) and all \( y \in \mathbb{R} \) we have that

\[ f^*(it) = \lim_{\epsilon \to 0} f(\epsilon + iy + it) \]

exists.

In order to verify this result we have to take a deeper look into the proof of [6, Lemma 11.22], which basically relies on a classical improvement of Fatou’s theorem showing that bounded holomorphic functions on \( \mathbb{D} \) not only have radial limits almost everywhere—but even boundary limits almost everywhere within so-called Stolz regions.

To do this, let \( \varphi \) be the Cayley transformation, i.e.

\[ \varphi: \overline{\mathbb{D}} \setminus \{1\} \to [\Re \geq 0], \quad \varphi(z) = \frac{1 + z}{1 - z}, \]
with its inverse
\[ \varphi^{-1}: [\text{Re} \geq 0] \to \overline{\mathbb{D}} \setminus \{1\}, \quad \varphi^{-1}(s) = \frac{s - 1}{s + 1}. \]

Fix some bounded and holomorphic function \( f : [\text{Re} > 0] \to \mathbb{C} \), and define for every \( \alpha > 1 \)
the set
\[
N(\alpha) = \left\{ w \in \mathbb{T} : \lim_{z \to w} f(\varphi(z)) \text{ does not exist} \right\},
\]
where
\[
S(\alpha, w) = \{ z \in \mathbb{D} : |z - w| \leq \alpha(1 - |z|) \}
\]
is the so-called Stolz region with respect to \( w \) and \( \alpha \). Then by the mentioned variant of Fatou’s
theorem (see e.g. [6, Section 23]) we know that \( N(\alpha) \) for every \( \alpha > 1 \) is a null set in \( \mathbb{T} \).
Moreover, we have that
\[
N(\alpha) \subset N(\beta) \text{ for every choice of } 1 < \alpha < \beta, \tag{26}
\]
since \( S(\alpha, w) \subset S(\beta, w) \).

**Proof of Proposition 2.1** We first show that for every \( y \in \mathbb{R} \), every \( \alpha > |1 + iy| \), and every \( t \in \mathbb{R} \)
\[
\exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : \varphi^{-1}(\varepsilon + iy \varepsilon + it) \in S(\alpha, \varphi^{-1}(it)). \tag{27}
\]
Indeed, for every \( \varepsilon > 0 \)
\[
\frac{|\varphi^{-1}(\varepsilon + iy \varepsilon + it) - \varphi^{-1}(it)|}{1 - |\varphi^{-1}(\varepsilon + iy \varepsilon + it)|} = \frac{|\varepsilon + iy \varepsilon + it - 1|}{|\varepsilon + iy \varepsilon + it + 1| - |\varepsilon + iy \varepsilon + it - 1|} =: A_\varepsilon(t, y),
\]
and we claim that \( \lim_{\varepsilon \to 0} A_\varepsilon(t, y) = |1 + iy| \). Calculating the numerator
\[
|(\varepsilon + iy \varepsilon + it - 1)(it + 1) - (it - 1)(\varepsilon + iy \varepsilon + it + 1)| = |2it + 2(\varepsilon + iy \varepsilon + it)| = 2|\varepsilon + iy \varepsilon|,
\]
gives
\[
A_\varepsilon(t, y) = \frac{2|\varepsilon + iy \varepsilon|}{|it + 1|} \frac{1}{|\varepsilon + iy \varepsilon + it + 1| - |\varepsilon + iy \varepsilon + it - 1|}.
\]
We extend the fraction and obtain
\[
A_\varepsilon(t, y) = \frac{|1 + iy|}{2|1 + it|} (|\varepsilon + iy \varepsilon + it + 1| + |\varepsilon + iy \varepsilon + it - 1|),
\]
since after multiplication
\[
(|\varepsilon + iy \varepsilon + it + 1| - |\varepsilon + iy \varepsilon + it - 1|) \cdot (|\varepsilon + iy \varepsilon + it + 1| + |\varepsilon + iy \varepsilon + it - 1|)
= |\varepsilon + iy \varepsilon + it + 1|^2 - |\varepsilon + iy \varepsilon + it - 1|^2
= (1 + \varepsilon)^2 - (1 - \varepsilon)^2 = 2 + 2\varepsilon^2 = 4\varepsilon.
\]
Consequently, $A_\varepsilon(t, y)$ tends to $|1 + iy|$ as $\varepsilon \to 0$, and this completes the proof of (27). Now define for every $y \in \mathbb{R}$

$$\alpha_y := |1 + iy| + 1,$$

as well as

$$E(y) := \varphi(N(\alpha_y)) \subset i\mathbb{R} \quad \text{and} \quad E := \bigcup_{y \in \mathbb{R}} E(y) \subset i\mathbb{R}.$$

Note that by (26) we have

$$E(y_1) \subset E(y_2) \text{ for every choice of } y_1 < y_2,$$

and we claim that this fact shows that $E$ is a null set in $i\mathbb{R}$. Indeed, for every $y \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $|y| < n$, and hence by (28)

$$E = \bigcup_{y \in \mathbb{R}} E(y) \subset \bigcup_{n \in \mathbb{N}} E(n).$$

But since the latter set is a countable union of null sets, the claim follows. Now for every $it \in i\mathbb{R} \setminus E$ we for all $y \in \mathbb{R}$ have that

$$\lim_{\varepsilon \to 0} g(\varphi(z)) \text{ exists.}$$

By (27) we know that for all $it \in i\mathbb{R}$ and all $y \in \mathbb{R}$

$$\varphi^{-1}(\varepsilon + i\varepsilon y + it) \in S(\alpha_y, \varphi^{-1}(it)),$$

whenever $\varepsilon$ is small enough. Consequently, we deduce from (29) and the fact that by continuity

$$\lim_{\varepsilon \to 0} \varphi^{-1}(\varepsilon + iy + it) = \varphi^{-1}(it),$$

that for every $it \in i\mathbb{R} \setminus E$ and every $y \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} f(\varepsilon + iy + it) \text{ exists .}$$

On the other hand, again by (27), for all $t \in i\mathbb{R}$ and all $y \in \mathbb{R}$

$$\varphi^{-1}(\varepsilon + it) \in S(\alpha_0, \varphi^{-1}(it)) \subset S(\alpha_y, \varphi^{-1}(it)),$$

so that another application of (29) assures that for every $it \in i\mathbb{R} \setminus E$ and every $y \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} f(\varepsilon + it) = \lim_{\varepsilon \to 0} f(\varepsilon + iy + it).$$

This finishes the proof. 

Proposition 2.1 easily transfers to functions in $H^\lambda_{\infty, \ell}[\text{Re} > 0]$, which are not necessarily bounded.

**Corollary 2.2** Let $\ell \geq 0$ and $f \in H^\lambda_{\infty, \ell}[\text{Re} > 0]$. Then for almost every $t \in \mathbb{R}$ the horizontal limit

$$f^*(it) = \lim_{\varepsilon \to 0} f(\varepsilon + it).$$
exists. More generally, there is a null set \( E \) in \( i \mathbb{R} \) such that for all \( t \in \mathbb{R} \setminus E \) and all \( y \in \mathbb{R} \) we have that
\[
f^*(it) = \lim_{\varepsilon \to 0} f(\varepsilon + i\varepsilon y + it).
\]

**Proof** The argument is immediate—apply Proposition 2.1 to the bounded and holomorphic function \( g(s) = f(s)(1 + s)^{-\ell}, \ s \in [\text{Re} > 0] \).

### 3 Convolution with weighted horizontal limits

For functions \( f \in H^\infty_{\infty, \ell}[\text{Re} > 0] \) the following convolution formula in Theorem 3.1 is crucial for the forthcoming sections. In fact, this result is the central tool to establish a Perron-type representation of Riesz means in terms of the horizontal limit functions \( f^* \) of \( f \) (Theorem 4.3). We recall that for \( u > 0 \) the classical Poisson kernel \( P_u : \mathbb{R} \to \mathbb{R} \) is given by
\[
P_u(t) = \frac{1}{\pi} \frac{u}{u^2 + t^2}, \quad t \in \mathbb{R},
\]
and satisfies \( \|P_u\|_{L^1(\mathbb{R})} = 1 \) with Fourier transform
\[
\hat{P}_u(x) = e^{-u|x|} \text{ for all } x \in \mathbb{R}.
\]

**Theorem 3.1** Let \( \ell \geq 0 \) and \( f \in H^\infty_{\infty, \ell}[\text{Re} > 0] \). Then for every \( u + i\tau \in [\text{Re} > 0] \)
\[
\left[ \frac{f^*(i\cdot)}{(1 + i\cdot)^\ell} * P_u \right](\tau) = \frac{f(u + i\tau)}{(1 + u + i\tau)^\ell}.
\]

**Proof** We show that for all \( \varepsilon, u > 0 \) and all \( \tau \in \mathbb{R} \)
\[
\frac{f(u + \varepsilon + i\tau)}{(1 + u + i\tau)^\ell} = \left[ \frac{f(\varepsilon + i\cdot)}{(1 + i\cdot)^\ell} * P_u \right](\tau).
\]
If then \( \varepsilon \to 0 \), the conclusion follows by continuity, the dominated convergence theorem and the observation that \( \frac{f^*(i\cdot)}{(1 + i\cdot)^\ell} \in L^\infty(\mathbb{R}) \) with
\[
\left\| \frac{f^*(i\cdot)}{(1 + i\cdot)^\ell} \right\|_{\infty} \leq \|f\|_{\infty, \ell},
\]
where the latter is valid, since for fixed and admissible \( t \in \mathbb{R} \)
\[
\left| \frac{f^*(it)}{(1 + it)^\ell} \right| = \lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon + it)}{(1 + it)^\ell} \right| \leq \|f\|_{\infty, \ell} \lim_{\varepsilon \to 0} \frac{|1 + \varepsilon + it|^\ell}{|1 + it|^\ell} = \|f\|_{\infty, \ell}.
\]

Note first, that looking at Theorem 1.3, it suffices to check (33) only for \( f(s) = e^{-sx} \) with \( x \geq 0 \). To do so, we recall that for all \( \ell > 0 \) and \( s \in [\text{Re} > 0] \)
\[
\frac{\Gamma(\ell)}{s^\ell} = \mathcal{L}(t^{\ell-1})(s) = \int_0^\infty e^{-st} t^{\ell-1} dt,
\]
where $\mathcal{L}$ denotes the Laplace transform. Together with (31) we obtain for all $\ell, \varepsilon, u > 0$ and all $\tau \in \mathbb{R}$

$$\Gamma(\ell) \frac{f(u + \varepsilon + i \tau)}{(1 + u + i \tau)^\ell} = \Gamma(\ell) e^{-(u+\varepsilon+i\tau)t}$$

$$\frac{1}{(1 + u + i \tau)^\ell} = e^{-(u+\varepsilon+i\tau)x} \mathcal{L}(t^{\ell-1})(1 + u + i \tau)$$

$$= \int_0^\infty e^{-(u+\varepsilon+i\tau)x} e^{-(1+u+i\tau)t} t^{\ell-1} dt$$

$$= e^{-(\varepsilon+i\tau)x} \int_0^\infty e^{-u(t+x)} e^{-(1+i\tau)t} t^{\ell-1} dt$$

$$= e^{-(\varepsilon+i\tau)x} \int_\mathbb{R} P_u(y) e^{iy(t+x)} e^{-(1+i\tau)t} t^{\ell-1} dt$$

$$= e^{-(\varepsilon+i\tau)x} \int_\mathbb{R} P_u(y) e^{iyx} \mathcal{L}(t^{\ell-1})(1 + i(\tau - y)) dy$$

$$= e^{-(\varepsilon+i\tau)x} \int_\mathbb{R} P_u(y) e^{iyx} \frac{\Gamma(\ell)}{(1 + i(\tau - y))^{\ell}} dy$$

$$= \Gamma(\ell) \int_\mathbb{R} P_u(y) e^{-(\varepsilon+i(\tau - y)x)} (1 + i(\tau - y))^{\ell} dy = \Gamma(\ell) \left[ \frac{f(\varepsilon + i \cdot)}{(1 + i \cdot)^\ell} \right] P_u(\tau),$$

which finishes the argument for $\ell > 0$ dividing both sides by $\Gamma(\ell)$. For $\ell = 0$ the same argument works since then $\mathcal{L}(t^{-1}) = 1$. \hfill \Box

**Corollary 3.2** Let $\ell \geq 0$. Then the mapping

$$T : H_{\infty, \ell}^{\lambda}([\text{Re} > 0]) \leftrightarrow L_{\infty}(\mathbb{R}), \quad f \mapsto \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell}$$

defines an isometric embedding.

**Proof.** As by (34) and Theorem 3.1 we already know that $T$ is a continuous, injective contraction, it remains to show that $T$ is isometric. Indeed, applying [10, Theorem 3.7], Eq. (32) and the fact that the convolution of $L_{\infty}(\mathbb{R})$ and $L_1(\mathbb{R})$ functions is continuous, we obtain

$$\|f\|_{\infty, \ell} = \lim_{u \to 0} \sup_{\tau \in \mathbb{R}} \left| \frac{f(u + i \tau)}{(1 + u + i \tau)^\ell} \right| = \lim_{u \to 0} \sup_{\tau \in \mathbb{R}} \left| \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell} * P_u \right| \left(\tau\right)$$

$$= \lim_{u \to 0} \left\| \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell} * P_u \right\|_{L_{\infty}(\mathbb{R})} \leq \sup_{u > 0} \left\| \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell} * P_u \right\|_{L_{\infty}(\mathbb{R})}$$

$$\leq \sup_{u > 0} \left\| \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell} \right\|_{L_{\infty}(\mathbb{R})} \|P_u\|_{L_1(\mathbb{R})} = \left\| \frac{f^*(i \cdot)}{(1 + i \cdot)^\ell} \right\|_{L_{\infty}(\mathbb{R})}. \hfill \Box$$

We refer to Sect. 10.1 from our appendix, where we for the sake of completeness in Theorem 10.1 give an internal description of the range of the operator in Corollary 3.2. For the particular case $\ell = 0$ we in Theorem 10.4 show that a function $g \in L_{\infty}(\mathbb{R})$ belongs to the image of $T$ if and only if the continuous function $g * P_u$ is almost periodic for every $u > 0$ and the Bohr coefficients

$$a_x(g * P_u) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T (g * P_u)(t)e^{-ixt} dt, \quad x \in \mathbb{R},$$

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vanish, whenever \( x \notin \{ \lambda_n : n \in \mathbb{N} \} \).

## 4 Perron’s formula in terms of horizontal limits

Given a frequency \( \lambda \), some \( \ell \geq 0 \) and \( f \in H_{\infty, \ell}^{\lambda} [\text{Re} > 0] \), the aim of this section is to prove an integral formula for the Riesz means \( R_{x,k}^{\lambda}(f) \) in terms of the horizontal limit function \( f^* \), whenever \( k > \ell \). Later we are going to see that this integral description incorporates most of the information we need for the understanding of Riesz summation on the imaginary line.

The following Perron-type formula is an indispensable tool from [10, Theorem 3.5], which in fact up to some point rules the structure theory of the scale of Banach spaces \( H_{\infty, \ell}^{\lambda} [\text{Re} > 0] \), \( \ell \geq 0 \).

**Theorem 4.1** Let \( f \in H_{\infty, \ell}^{\lambda} [\text{Re} > 0] \) and \( k > \ell \geq 0 \). Then for all \( s_0 \in [\text{Re} \geq 0] \), \( x > 0 \) and \( c > 0 \)

\[
R_{x,k}^{\lambda}(f)(s_0) = \frac{\Gamma(1 + k)}{2\pi i} x^{-k} \int_{c-i\infty}^{c+i\infty} \frac{f(s + s_0)}{s^{1+k}} e^{sx} ds.
\]

We start modifying this formula. First, observe that regarding summation on the imaginary line by translation, it suffices to handle the case \( s_0 = 0 \). Then, the choice \( c = x^{-1} \) leads to

\[
R_{x,k}^{\lambda}(f)(0) = \frac{\Gamma(1 + k)}{2\pi i} x^{-k} \int_{x^{-1}-i\infty}^{x^{-1}+i\infty} \frac{f(s)}{s^{1+k}} e^{sx} ds.
\]

\[
= \frac{\Gamma(1 + k)}{2\pi} x^{-k} \int_{-\infty}^{\infty} \frac{f(x^{-1} + it)}{(x^{-1} + it)^{1+k}} e^{ix(x^{-1}+it)} dt.
\]

We fix this observation.

**Remark 4.2** Let \( k > \ell \geq 0 \) and \( f \in H_{\infty, \ell}^{\lambda} [\text{Re} > 0] \). Then for all \( x > 0 \)

\[
R_{x,k}^{\lambda}(f)(0) = \int_{-\infty}^{\infty} f(x^{-1} + it)e^{ixt} K_x^k(t) dt,
\]

where

\[
K_x^k(t) = \frac{\Gamma(1 + k)}{2\pi} \frac{x}{(1 + ixt)^{1+k}}, \quad t \in \mathbb{R}.
\]

Observe now that the functions \((K_x^k)_{x > 0}\) are generated by the kernel

\[
K(x) = \frac{\Gamma(1 + k)}{2\pi} \frac{1}{(1 + iy)^{1+k}}, \quad y \in \mathbb{R}
\]

in the sense that \( K_x^k(t) = xK^k(xt) \) for all \( x > 0 \) and \( t \in \mathbb{R} \). Moreover, note that, provided \( \lambda_1 = 0 \), the function \( f = 1 \) belongs to \( H_{\infty, \ell}^{\lambda} [\text{Re} > 0] \) with \( R_{x,k}^{\lambda}(f)(0) = 1 \) for all \( x > 0 \). Hence, by Remark 4.2 \((x = 1)\) we see that

\[
\int_{\mathbb{R}} e^{iy} K^k(y) dy = 1.
\]
With the aid of Theorem 3.1 we now continue the modification of the Perron-type formula from Remark 4.2. The following result is the main contribution of this subsection—Perron’s formula in terms of horizontal limits.

**Theorem 4.3** Let \( f \in H^\lambda_{\infty, \ell} [\text{Re} > 0] \) and \( k > \ell \geq 0 \). Then

\[
R^{\lambda, k}_x(f)(0) = \int_{\mathbb{R}} \frac{f^*(iy)}{(1 + iy)^{\ell}} R^{k, \ell}(x, y) dy, \quad x > 0,
\]

where

\[
R^{k, \ell}(x, y) = \int_{\mathbb{R}} P_{x^{-1}}(t - y)e^{i\ell t} (1 + x^{-1} + it)^{\ell} K^k_{\ell}(t) dt, \quad y \in \mathbb{R}.
\]

**Proof.** By Remark 4.2 and the convolution formula from Theorem 3.1 we have

\[
R^{\lambda, k}_x(f)(0) = \int_{\mathbb{R}} f(x^{-1} + it)e^{i\ell t} K^k_{\ell}(t) dt
\]

\[
= \int_{\mathbb{R}} \frac{f(x^{-1} + it)}{1 + x^{-1} + it} e^{i\ell t} (1 + x^{-1} + it)^{\ell} K^k_{\ell}(t) dt
\]

\[
= \int_{\mathbb{R}} \left( \frac{f^*(i\cdot)}{1 + i\cdot} * P_{x^{-1}}(\cdot) \right)(t) e^{i\ell t} (1 + x^{-1} + it)^{\ell} K^k_{\ell}(t) dt
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f^*(iy)}{(1 + iy)^{\ell}} P_{x^{-1}}(t - y) e^{i\ell t} (1 + x^{-1} + it)^{\ell} K^k_{\ell}(t) dy dt
\]

\[
= \int_{\mathbb{R}} \frac{f^*(iy)}{(1 + iy)^{\ell}} \int_{\mathbb{R}} P_{x^{-1}}(t - y) e^{i\ell t} (1 + x^{-1} + it)^{\ell} K^k_{\ell}(t) dt dy
\]

\[
= \int_{\mathbb{R}} \frac{f^*(iy)}{(1 + iy)^{\ell}} R^{k, \ell}(x, y) dy.
\]

In Lemma 6.3 we are going to show how to control the \( L_1(\mathbb{R}) \)-norm of the functions \( R^{k, \ell}(x, \cdot) \), which will be essential in all our applications of the preceding formula.

## 5 Almost everywhere convergence

The completion of the preceding preparations have paved the way for the first of our four main contributions. As before, the sequence \( \lambda = (\lambda_n) \) always denotes an arbitrary frequency.

**Theorem 5.1** Let \( f \in H^\lambda_{\infty, \ell} [\text{Re} > 0] \) and \( k > \ell \geq 0 \). Then the Dirichlet series \( \sum a_n(f)e^{-\lambda_n \cdot} \) is \((\lambda, k)\)-Riesz summable almost everywhere on the imaginary line, and for almost all \( \tau \in \mathbb{R} \)

\[
\lim_{x \to \infty} R^{\lambda, k}_x(f)(i\tau) = f^*(i\tau).
\]

Observe that this result for \( k = \ell = 0 \) fails in the ordinary case \( \lambda = (\log n) \). This follows from the Bayart–Konyagin–Queffélec example from Sect. 1.2 and the equality in (23).

Before we come to the proof of Theorem 5.1 we remark a sort of converse of this result: If \( \sum a_n(f)e^{-\lambda_n \cdot} \) is \((\lambda, k)\)-Riesz summable at some point of the boundary line, then the horizontal limit of \( f \) exists and equals the \((\lambda, k)\)-Riesz sum of \( f \) at this point. For the special case \( \lambda = (n) \) and \( k = 0 \) this (after the standard reformulation) is nothing else than Abel’s classical convergence theorem for power series.
**Remark 5.2** Let $k \geq 0$ and $f : \{\text{Re} > 0\} \to \mathbb{C}$ a holomorphic function with a $\lambda$-Riesz germ $k \geq 0$. Assume that $f$ is $(\lambda, k)$-Riesz summable at $0$ with limit $A$, i.e. $\lim_{x \to \infty} R_x^{\lambda, k}(f)(0) = A$ exists. Then

$$\lim_{\sigma \to 0} f(\sigma) = A.$$ 

**Proof** We assume (without loss of generality) that $A = 0$. Note first that by assumption the $\lambda$-Riesz germ of $f$ converges on $\{\text{Re} > 0\}$, and that its limit function coincides with $f$. Then we know from [10, Theorem 2.9] that for each $\sigma > 0$

$$f(\sigma) = \frac{1}{\Gamma(k + 1)} \sigma^{1+k} \int_0^\infty S_t^{\lambda, k}(f)(0)e^{-\sigma t}dt,$$

where for $t > 0$

$$S_t^{\lambda, k}(f)(0) = t^k R_t^{\lambda, k}(f)(0) = \sum_{\lambda_n < t} a_n(f)(t - \lambda_n)^k$$

denotes the summatory function in $0$. Now fix some $\varepsilon > 0$, and choose $\tau_0 > 0$ such that for all $t > \tau_0$

$$|S_t^{\lambda, k}(f)(0)| \leq \varepsilon t^k.$$

Consequently, for each $\sigma > 0$

$$|f(\sigma)| \leq \frac{1}{\Gamma(k + 1)} \sigma^{1+k} \int_0^{\tau_0} S_t^{\lambda, k}(f)(0)e^{-\sigma t}dt + \frac{1}{\Gamma(k + 1)} \sigma^{1+k} \varepsilon \int_{\tau_0}^\infty t^k e^{-\sigma t}dt \leq \frac{1}{\Gamma(k + 1)} \sigma^{1+k} \int_0^{\tau_0} S_t^{\lambda, k}(f)(0)dt + \varepsilon.$$

Since the first term tends to 0 whenever $\sigma$ tends to 0, we obviously obtain the conclusion. 

**Proof of Theorem 5.1** Applying the substitution $y = tx$ in Remark 4.2, we obtain for all $\tau > 0$ and all $x > 0$

$$R_x^{\lambda, k}(f)(i\tau) = \int_{-\infty}^\infty f(x^{-1} + iyx^{-1} + i\tau)e^{iy}K^k(y)dy.$$ 

Since by Corollary 2.2 for all $y \in \mathbb{R}$ and almost all $\tau > 0$ we have

$$\lim_{x \to \infty} f(x^{-1} + iyx^{-1} + i\tau) = f^*(i\tau),$$

the dominated convergence theorem and the use of (36) imply

$$\lim_{x \to \infty} R_x^{\lambda, k}(f)(i\tau) = f^*(i\tau) \int_{-\infty}^\infty e^{iy}K^k(y)dy = f^*(i\tau).$$
Indeed, fixing $\tau > 1$ and $x > 1$, we for $y \in \mathbb{R}$ have
\[
|f(x^{-1} + iy^{-1} + i\tau)e^{iy} - 1/(1 + iy)^{1+k}| 
\leq \|f\|_\infty,\ell \frac{|1 + (x^{-1} + iy^{-1} + i\tau)|^\ell}{|1 + iy|^{1+k}} 
\leq \|f\|_\infty,\ell 2^{\max(0,\ell-1)} \left( \frac{|1 + iy^{-1}|^\ell}{|1 + iy|^{1+k}} + \frac{|x^{-1} + i\tau|^\ell}{|1 + iy|^{1+k}} \right) 
\leq \|f\|_\infty,\ell 2^{\max(0,\ell-1)} \left( \frac{1}{|1 + iy|^{1+k-\ell}} + \frac{|1 + i\tau|^\ell}{|1 + iy|^{1+k}} \right).
\]

From (23) we immediately deduce the case $\ell = 0$, which is of special interest.

**Corollary 5.3** Let $f \in H^\lambda_\infty[\Re > 0]$ and $k > 0$. Then for almost every $\tau \in \mathbb{R}$
\[
\lim_{x \to \infty} R^{\lambda,k}_x(f)(i\tau) = f^*(i\tau).
\]
In particular, every Dirichlet series $D \in \mathcal{D}_\infty(\lambda)$ is almost everywhere $(\lambda, k)$-Riesz summable on the imaginary line.

In view of the Bayart–Konyagin–Queffélec counterexample (see Sect. 1.2), it seems worthwhile to mention the special case $\lambda = (\log n)$ separately.

**Corollary 5.4** Let $f \in H^{(\log n)}_\infty[\Re > 0]$ and $k > 0$. Then for almost every $\tau \in \mathbb{R}$
\[
\lim_{x \to \infty} R^{(\log n),k}_x(f)(i\tau) = f^*(i\tau).
\]
In particular, if $D = \sum a_n n^{-s} \in \mathcal{D}_\infty$ is the Dirichlet series associated to $f$ (see again (16)), then for almost all $\tau \in \mathbb{R}$
\[
\lim_{x \to \infty} \sum_{\log n < x} a_n \frac{1}{n^{i\tau}} \left( 1 - \frac{\log n}{x} \right)^k = f^*(i\tau).
\]

### 6 A principle of localization

The second main result (after Theorem 5.1) may be seen as a principle of localization—compare with what we recalled in (4) for the one variable case.

**Theorem 6.1** Let $k > \ell \geq 0$ and $f, g \in H^\lambda_\infty,\ell[\Re = 0]$. Assume that $f^* = g^*$ on some open interval $I \subset [\Re = 0]$. Then, given $i\tau \in I$, the limit $\lim_{x \to \infty} R^{\lambda,k}_x(f)(i\tau)$ exists if and only if $\lim_{x \to \infty} R^{\lambda,k}_x(g)(i\tau)$ exists, and in this case
\[
\lim_{x \to \infty} R^{\lambda,k}_x(f)(i\tau) = \lim_{x \to \infty} R^{\lambda,k}_x(g)(i\tau).
\]

The proof of this principle is given at the end of this section, and it turns out to be a simple consequence of the following independently interesting result. Recall the definition of the kernel functions $R^{k,\ell}(x, \cdot), x > 0$ from Theorem 4.3.
Theorem 6.2 Let \( f \in H^2_{-\infty, \ell}[\Re u > 0] \) and \( k > \ell \geq 0 \). Then for every \( \delta > 0 \)

\[
\lim_{x \to \infty} \int_{|y| \geq \delta} \frac{f^*(iy)}{(1 + iy)^\ell} R^{k,\ell}(x, y) dy = 0.
\]

The proof of Theorem 6.2 requires to control the norm of \( R^{k,\ell}(x, \cdot) \), which is provided by the following lemma.

Lemma 6.3 Let \( k > \ell \geq 0 \). Then there is a constant \( C(k, \ell) > 0 \) such that for each \( x > 1 \) and every \( y \in \mathbb{R} \)

\[
|R^{k,\ell}(x, y)| \leq C(k, \ell) \begin{cases} \frac{x}{|1 + ixy|^{1+k}} , & k < 1, \\ \frac{x}{|1 + ixy|^2} + \frac{x}{|1 + ixy|^{1+k-\ell}}, & k \geq 1 \text{ and } k - \ell \leq 1, \\ \frac{x}{|1 + ixy|^2}, & k \geq 1 \text{ and } k - \ell > 1. \end{cases}
\]

Moreover, for every \( \delta > 0 \)

\[
\lim_{x \to \infty} \int_{|y| \geq \delta} |R^{k,\ell}(x, y)| dy = 0. \quad (37)
\]

Let us first deduce Theorem 6.2 from Lemma 6.3.

Proof of Theorem 6.2 Corollary 3.2 and (37) imply

\[
\lim_{x \to \infty} \int_{|y| \geq \delta} \frac{f^*(iy)}{(1 + iy)^\ell} R^{k,\ell}(x, y) dy \leq \|f\|_{\infty, \ell} \lim_{x \to \infty} \int_{|y| \geq \delta} |R^{k,\ell}(x, y)| dy = 0,
\]

the conclusion. \( \square \)

Proof of Lemma 6.3 Recall (from Theorem 4.3 and (35)) that for \( y \in \mathbb{R} \) and \( x > 0 \)

\[
R^{k,\ell}(x, y) = \frac{\Gamma(1 + k)e^{\pi i}}{2\pi} \int_{\mathbb{R}} P_{x^{-1}}(t - y) e^{itx} \frac{x(1 + x^{-1} + it)\ell}{(1 + ixt)^{1+k}} dt,
\]

Then

\[
|R^{k,\ell}(x, y)| \leq \frac{\Gamma(1 + k)e}{2\pi} \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{1 + x^{-1} + it\ell}{|1 + ixt|^{1+k}} dt,
\]

and hence for \( x > 1 \) (implying \( |x^{-1} + it| \leq |1 + ixt| \) for every \( t \in \mathbb{R} \)) we get

\[
\int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x(1 + x^{-1} + it)^\ell}{(1 + ixt)^{1+k}} dt \\
\leq C(\ell) \left( \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k}} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x|x^{-1} + it\ell}{|1 + ixt|^{1+k}} dt \right) \\
\leq C(\ell) \left( \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k}} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k-\ell}} dt \right),
\]

where \( C(\ell) = 2^{\max(0, \ell-1)} \). So it remains to control the last two integrals. We already know from [8, Lemma 3.4] (with \( u = 0 \) and \( v = x^{-1} \)) that for all \( 0 < \alpha \leq 1 \)

\[
\int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|x^{-1} + iy|^{1+\alpha}} dt \leq \frac{2}{|x^{-1} + iy|^{1+\alpha}},
\]

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Now assume first that $k < 1$, which implies $k - \ell \leq 1$. Then
\[
\int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k}} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k-\ell}} dt
\]
\[
= x^{-k} \int_{\mathbb{R}} \frac{P_{x^{-1}}(t - y)}{|x^{-1} + it|^{1+k}} dt + x^{-k+\ell} \int_{\mathbb{R}} \frac{P_{x^{-1}}(t - y)}{|x^{-1} + it|^{1+k-\ell}} dt
\]
\[
\leq x^{-k} \int_{\mathbb{R}} \frac{2}{|x^{-1} + iy|^{1+k} + x^{-k+\ell}} \frac{x}{|x^{-1} + iy|^{1+k-\ell}} dt
\]
\[
= 2 \frac{x}{|1 + ixy|^{1+k} + 2} \frac{x}{|1 + ixy|^{1+k-\ell}} \leq 4 \frac{x}{|1 + ixy|^{1+k-\ell}},
\]
which proves the first claim. Assume second that $k \geq 1$ and $k - \ell \leq 1$. Then $|1 + ixt|^2 \leq |1 + ixt|^{1+k}$ for $t \in \mathbb{R}$, and consequently
\[
\int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k}} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k-\ell}} dt
\]
\[
\leq \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^2} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k-\ell}} dt
\]
\[
= x^{2-k} \int_{\mathbb{R}} \frac{P_{x^{-1}}(t - y)}{|x^{-1} + it|^{2}} dt + x^{-k+\ell} \int_{\mathbb{R}} \frac{P_{x^{-1}}(t - y)}{|x^{-1} + it|^{1+k-\ell}} dt
\]
\[
\leq x^{-2} \frac{2}{|x^{-1} + iy|^2} \frac{x}{|x^{-1} + iy|^{1+k-\ell}}
\]
\[
= 2 \frac{x}{|1 + ixy|^2} + \frac{2x}{|1 + ixy|^{1+k-\ell}}.
\]
Similarly, we handle the case $k \geq 1$ and $k - \ell > 1$ getting
\[
\int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k}} dt + \int_{\mathbb{R}} P_{x^{-1}}(t - y) \frac{x}{|1 + ixt|^{1+k-\ell}} dt \leq \frac{4x}{|1 + ixy|^2}.
\]
The ‘moreover part’ follows by substitution. Since all three cases follow the same lines, we only consider the case $k < 1$. Then
\[
\int_{|y| > \delta} |R^{k,\ell}(x, y)| dy \leq C(k, \ell) \int_{|y| > \delta} \frac{x}{|1 + ixy|^{1+k-\ell}} dy
\]
\[
= C(k, \ell) \int_{|y| > \delta} \frac{1}{|1 + it|^{1+k-\ell}} dt,
\]
which tends to zero as $x \to \infty$. This completes the proof.

Finally, we come back to the proof of Theorem 6.1.

**Proof of Theorem 6.1** Translating, if necessary, we may assume that $0 \in I$ and $\tau = 0$. We choose some $\delta > 0$ such that $i[-\delta, \delta] \subset I$. Then by the Perron-type formula from Theorem 4.3 we have
\[
R_{x}^{\lambda, k}(f)(0) = \int_{|y| \geq \delta} \frac{f^*(iy)}{(1 + iy)^{\ell}} R^{k,\ell}(x, y) dy + \int_{|y| \leq \delta} \frac{f^*(iy)}{(1 + iy)^{\ell}} R^{k,\ell}(x, y) dy. \quad (38)
\]
Since $f^* = g^*$ on $i[-\delta, \delta] \subset I$, we then observe that the claim is an immediate consequence of Theorem 6.2. \qed
7 Uniform convergence

We come to our third main result, which among others recovers Theorem 1.2 for functions in \( H^\lambda_{\infty,\ell}[\Re > 0], \ell \geq 0 \).

**Theorem 7.1** Let \( k > \ell \geq 0 \) and \( f \in H^\lambda_{\infty,\ell}[\Re > 0] \). If \( f^* \) is continuous on some open interval \( I \subset [\Re = 0] \), then for all \( i\tau \in I \)

\[
\lim_{x \to \infty} R^\lambda_{\ell,x}(f)(i\tau) = f^*(i\tau),
\]

with uniform convergence on every compact sub interval \( J \subset I \). Moreover, in this case \( (R^\lambda_{\ell,x}(f))_{x > 0} \) converges uniformly on each ‘flattened cone’

\[
K(\gamma, J) = \{ z \in [\Re > 0] : z = iy + w \text{ with } iy \in J \text{ and } \arg(w) < \gamma \}, \ 0 < \gamma < \frac{\pi}{2},
\]

and for each \( z = iy + w \in K(\gamma, J) \)

\[
f(z) = \lim_{x \to \infty} R^\lambda_{\ell,x}(f)(z) = \frac{w^{k+1}}{\Gamma(1 + k)} \int_0^\infty t^k R^\lambda_{\ell,x}(f)(iy)e^{-wt}dt.
\]

**Remark 7.2** Indeed, a closer look at the proof of the ‘moreover-part’ shows, that we in fact prove the following: Given a Dirichlet series \( D = \sum a_n e^{-\lambda_n z} \), an interval \( J \subset [\Re = 0] \), and \( 0 < \gamma < \frac{\pi}{2} \), then \( D \) is uniformly \((\lambda, k)\)-Riesz summable on \( K(\gamma, J) \), provided \( D \) is uniformly \((\lambda, k)\)-Riesz summable on \( J \). Knowing this, the ‘moreover-part’ of Theorem 7.1 is an immediate consequence of the first part.

Before we prove Theorem 7.1 we illustrate this result with two simple examples.

The first example shows that the case \( k = 1 \) and \( \lambda = (n) \) in fact is covered by Fejer’s theorem applied to functions in \( H_{\infty}(\mathbb{T}) \cap C(\mathbb{T}) \). In fact, given \( g \in C(\mathbb{T}) \), Fejer’s theorem shows that uniformly for \( z \in \mathbb{T} \)

\[
g(z) = \lim_{x \to \infty} \sum_{n < x} \hat{g}(n) z^n \left( 1 - \frac{n}{x} \right) = \lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^x \sum_{k=1}^n a_k z^k.
\]

(39)

For functions from \( H_{\infty}(\mathbb{T}) \cap C(\mathbb{T}) \) this may be deduced from Theorem 7.1. Indeed, for each \( g \in H_{\infty}(\mathbb{T}) \cap C(\mathbb{T}) \) there is a function \( f \in H_{\infty}(\mathbb{D}) \) such that \( f^* = g \) on \( \mathbb{T} \) and \( \partial_n f(0)/n! = \hat{g}(n) \) for all \( n \in \mathbb{N}_0 \). But then as \( g \) is continuous on \( \mathbb{T} \) the outcome of Theorem 7.1 with \( k = 1 \) and \( \lambda = (n) \) precisely is (39), since the substitution \( z = e^{-s} \) leads to a coefficient preserving isometry from \( H_{\infty}(\mathbb{D}) \) onto \( H_{\infty,0}^{(n)}[\Re > 0] \).

To see another example, denote by \( \xi : \mathbb{C}\backslash\{1\} \to \mathbb{C} \) the zeta-function, which is holomorphic with a simple pole in \( s = 1 \), and which on \( [\Re > 1] \) is the pointwise limit of the zeta-Dirichlet series \( \sum n^{-s} \). Moreover, consider the entire function

\[
\eta : \mathbb{C} \to \mathbb{C}, \quad \eta(s) = (1 - 2^{1-s})\xi(s),
\]

which on \( [\Re > 0] \) is nothing else then the pointwise limit of the \( \eta \)-Dirichlet series \( \sum (-1)^{n+1} n^{-s} \). As remarked in [10, Section 3.1]

\[
\ell > \frac{1}{2} \quad \Rightarrow \quad \eta \in H_{\infty,\ell}^{(\log n)}[\Re > 0] \quad \Rightarrow \quad \ell \geq \frac{1}{2},
\]

and in particular, \( \eta \notin \mathcal{H}_{\infty}((\log n)) \). Hence, Theorem 7.1 implies that for \( k > 1/2 \)

\[
\eta(it) = \lim_{x \to \infty} \sum_{\log(n) < x} (-1)^n n^{-it} \left( 1 - \frac{\log(n)}{x} \right)^k
\]

(40)
uniformly on every closed interval $I \subset [\text{Re} = 0]$. In fact, a stronger result is true: The abscissa $\sigma_{\hat{c}, k}^\lambda(\eta)$ of $(\lambda, k)$-Riesz summability of the eta-Dirichlet series equals $-k$ (see e.g. [12, VI, 8.1]).

**Proof of Theorem 7.1** Without loss of generality we all over the proof assume that $\lambda_1 = 0$.

For the proof of the first part it (after translation) suffices to check that $f^*$ is $(\lambda, k)$-Riesz summable in $\tau = 0$, assuming without loss of generality that $0 \in I$. Moreover, we assume that $f^*(0) = 0$, since, if this is not the case, we may consider $f - f^*(0) \in H_{\infty, \ell}^\lambda([\text{Re} > 0]$ instead of $f^*$. As before we distinguish the two cases $k < 1$ and $k \geq 1$, and start with the first one.

Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|f^*(iy)| \leq \varepsilon$ for all $|y| \leq \delta$, which is possible using the continuity of $f^*$ at the origin. Then by Lemma 6.3 and substitution

$$\left| \int_{|y| \leq \delta} f^*(iy) \frac{R_{k, \ell}(x, y)}{(1 + iy)^\ell} dy \right| \leq \int_{|y| \leq \delta} |f^*(iy)| R_{k, \ell}(x, y)|dy$$

$$
\leq \varepsilon C(k, \ell) \int_{|y| \leq \delta} \frac{x}{|1 + iy|^1 + k - \ell} dy \leq \varepsilon C(k, \ell) \int_{\mathbb{R}} \frac{1}{|1 + it|^1 + k - \ell} dt.
$$

Hence, splitting like in (38) and using Theorem 6.2, we finally get that

$$\lim_{x \to \infty} R_{x, k}^\lambda(f)^{0} = 0 = f^*(0).$$

Since the second case $k \geq 1$ follows the same lines, the first part of Theorem 7.1 is accomplished.

For the proof of the second part assume that $J \subset I$ is a compact sub interval. Let $\delta_0 > 0$ be such that $J_0 = J \pm i\delta_0 \subset I$, and note that $f^*$ is uniformly continuous on $J_0$. Fix $\varepsilon > 0$ and let $\delta_0 > \delta > 0$ such that $|f^*(iy + \tau) - f^*(\tau)| \leq \varepsilon$ for all $|y| \leq \delta$ and $i\tau \in J$. Looking at (38), using Theorem 3.1, Corollary 3.2, and again Lemma 6.3 as before, for all $\varepsilon > 0$ and $i\tau \in J$

$$|R_{x, k}^\lambda(f)(i\tau) - f^*(i\tau)|$$

$$\leq \left| \int_{|y| \geq \delta} f^*(i(y + \tau)) \frac{R_{k, \ell}(x, y)}{(1 + iy)^\ell} dy \right| + \left| \int_{|y| \leq \delta} f^*(i(y + \tau)) \frac{R_{k, \ell}(x, y)}{(1 + iy)^\ell} dy \right|$$

$$\leq \| f \|_{\infty, \ell} \int_{|y| \geq \delta} \frac{|(1 + i(y + \tau))|^{\ell} \left| R_{k, \ell}(x, y) \right| dy + \varepsilon \int_{|y| \leq \delta} \left| R_{k, \ell}(x, y) \right| dy$$

$$\leq \| f \|_{\infty, \ell} C(\ell, \delta, J) \int_{|y| \geq \delta} \left| R_{k, \ell}(x, y) \right| dy + \varepsilon C(k, \ell),$$

which then by (37) (from Lemma 6.3) ensures that $(R_{x, k}^\lambda(f)(\cdot))_{x > 0}$ converges uniformly to $f^*$ on $J$.

To verify the ‘moreover part’ is slightly more involved. Choose $m \in \mathbb{N}_0$ such that $m < k \leq m + 1$. Fixing $iy \in J$, we define the $\lambda$-Dirichlet series

$$D^\lambda = -f^*(iy) + \sum a_n(f) e^{-\lambda_n s}$$

(recall that $\lambda_1 = 0$) and observe that for all $x > 0$ and $s \in \mathbb{C}$

$$R_{x, k}^\lambda(D^\lambda)(s) = -f^*(iy) + R_{x, k}^\lambda(f)(s). \quad (41)$$

Applying [10, Lemma 4.11] to $D$ (or more precisely to the horizontal translation $D^\lambda_{iy}$ of $D^\lambda$ about $iy$), we see that there is a constant $L = L(m, k)$ such that for all $z = \sigma + i\tau \in [\text{Re} > 0]$

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of the form $z = iy + w \in K(J, \gamma)$ (so $iy \in J$ and $\arg(w) < \gamma$) we have

$$
\left| \frac{x^k}{\Gamma(m + 2)} \int_0^x t^{m+1} R_{t}^{\lambda, m+1} (D^y)(iy)w^{m+2} \left(1 - \frac{t}{x}\right)^k e^{-wt} dt - x^k R_x^{\lambda,k} (D^y)(u) \right|
$$

$$
= \left| \frac{x^k}{\Gamma(m + 2)} \int_0^x t^{m+1} R_{t}^{\lambda, m+1} (D^y)(0)w^{m+2} \left(1 - \frac{t}{x}\right)^k e^{-wt} dt - x^k R_x^{\lambda,k} (D^y)(w) \right|
$$

$$
\leq L(m, k) \sum_{j=1}^{m+1} |\sec(\gamma)|^j \int_0^x |t^k R_t^{\lambda,k} (D^y)(iy)| t^{-j} (x - t)^{j-1} dt
$$

$$
+ e^{-\sigma x} |x^k R_x^{\lambda,k} (D^y)(iy)|.
$$

From [10, Lemma 4.12] we know that uniformly in $y$ and $w$ (as above)

$$
\lim_{x \to \infty} \int_0^x t^{m+1} R_{t}^{\lambda, m+1} (D^y)(iy)w^{m+2} \left(1 - \frac{t}{x}\right)^k e^{-wt} dt
$$

$$
= \int_0^\infty t^{m+1} R_{t}^{\lambda, m+1} (D^y)(iy)w^{m+2} e^{-wt} dt,
$$

and by [10, Lemma 4.5]

$$
\int_0^\infty t^{m+1} R_{t}^{\lambda, m+1} (D^y)(iy)w^{m+2} e^{-wt} dt = \frac{\Gamma(m + 2)}{\Gamma(k + 1)} \int_0^\infty t^k R_t^{\lambda,k} (D^y)(iy)w^{k+1} e^{-wt} dt,
$$

so together

$$
\lim_{x \to \infty} \frac{1}{\Gamma(m + 2)} \int_0^x t^{m+1} R_{t}^{\lambda, m+1} (D^y)(iy)w^{m+2} \left(1 - \frac{t}{x}\right)^k e^{-wt} dt
$$

$$
= \frac{1}{\Gamma(k + 1)} \int_0^\infty t^k R_t^{\lambda,k} (D^y)(iy)w^{k+1} e^{-wt} dt.
$$

Moreover, following the proof of [10, Theorem 2.9] and using (41), we have that

$$
\lim_{x \to \infty} x^{-k} \sum_{j=1}^{m+1} |\sec(\gamma)|^j \int_0^x |t^k R_t^{\lambda,k} (D^y)(iy)| t^{-j} (x - t)^{j-1} dt
$$

$$
\leq \lim_{x \to \infty} x^{-k} \sum_{j=1}^{m+1} |\sec(\gamma)|^j \int_0^x \sup_{iy \in J} |R_t^{\lambda,k} (f)(iy) - f^*(iy)| t^{-j} (x - t)^{j-1} dt = 0,
$$

still uniformly in $y$. All together we obtain that uniformly in $y$ and $w$ (so uniformly in $z$)

$$
\lim_{x \to \infty} -f^*(iy) + R_x^{\lambda,k} (f)(z)
$$

$$
= \frac{w^{k+1}}{\Gamma(1 + k)} \int_0^\infty t^k (-f^*(iy) + R_t^{\lambda,k} (f)(iy)) e^{-wt} dt
$$

$$
= -f^*(iy) \frac{w^{k+1}}{\Gamma(1 + k)} \int_0^\infty t^k e^{-wt} dt + \frac{w^{k+1}}{\Gamma(1 + k)} \int_0^\infty t^k R_t^{\lambda,k} (f)(iy) e^{-wt} dt
$$

$$
= -f^*(iy) + \frac{w^{k+1}}{\Gamma(1 + k)} \int_0^\infty t^k R_t^{\lambda,k} (f)(iy) e^{-wt} dt.
$$
and finally
\[ \lim_{x \to \infty} R_{x}^{\lambda, k}(f)(z) = \frac{w^{k+1}}{\Gamma(1+k)} \int_{0}^{\infty} t^{k} R_{t}^{\lambda, k}(f)(iy)e^{-wt}dt, \]
which completes the argument. \qed

8 A Dini test

Finally, we come to the last main contribution announced in the introduction—a Dini test for functions in \( H^{\lambda, \ell}_{\infty}[\Re > 0] \).

**Theorem 8.1** Let \( k > \ell \geq 0 \) and \( f \in H^{\lambda, \ell}_{\infty}[\Re > 0] \). If for \( \tau \in \mathbb{R} \) there is \( \delta > 0 \) such that
\[ \int_{-\delta}^{\delta} \frac{|f^*(iy+\tau) - f^*(i\tau)|}{|y|^{1+k-\ell}}dy < \infty, \]
then
\[ \lim_{x \to \infty} R_{x}^{\lambda, k}(f)(i\tau) = f^*(i\tau). \]

**Proof** As argued in the proof of Theorem 7.1 we may assume that \( \tau = 0 \) and \( f^*(0) = 0 \) (provided that w.l.o.g. \( \lambda_1 = 0 \)). According to the splitting from (38) and Theorem 6.2, the claim follows once we show that
\[ \lim_{x \to \infty} \int_{|y| \leq \delta} f^*(iy) \frac{R^{k, \ell}(x, y)}{(1+iy)^{\ell}}dy = 0. \]
Indeed, by Lemma 6.3, provided \( k < 1 \),
\[ \left| \int_{-\delta}^{\delta} f^*(iy) \frac{R^{k}(x, y)}{(1+iy)^{\ell}}dy \right| \leq C(k, \ell) \int_{-\delta}^{\delta} \frac{|f^*(iy)|^x}{|1+iy|^{1+k-\ell}}dy \]
\[ = C(k, \ell)x^{-(k-\ell)} \int_{-\delta}^{\delta} \frac{|f^*(iy)|}{|x-1|+i|y|^{1+k-\ell}}dy \leq C(k, \ell)x^{-(k-\ell)} \int_{-\delta}^{\delta} \frac{|f^*(iy)|}{|y|^{1+k-\ell}}dy, \]
which by assumption vanishes as \( x \to \infty \). The case \( k \geq 1 \) follows the same lines using Lemma 6.3 accordingly. \qed

9 A link to Carleson’s theorem

Recall from (10) that for every Dirichlet series \( D = \sum a_{n}n^{-s} \) with \( (a_{n}) \in \ell_{2} \) (so in particular, if \( D \in \mathcal{D}_{\infty} \)) the so-called vertical limits
\[ \sum a_{n} \chi(n)n^{-it} \quad (42) \]
converge for almost all characters \( \chi : \mathbb{N} \to \mathbb{T} \) and almost all \( t \in \mathbb{R} \). In short, for such Dirichlet series almost all vertical limits of \( D \) converge almost everywhere on the boundary line.

In the introduction we indicate that this result in fact is a consequence of a Carleson-type convergence theorem for functions in \( L^{2}(\mathbb{T}^{\infty}) \). In [9] we proved a Carleson-type theorem for \( \lambda \)-Dirichlet series which belong to the Hardy spaces \( \mathcal{H}_{p}(\lambda), 1 < p \leq \infty \). Inspired by

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(42), this result has consequences for the boundary behavior of almost all vertical limits of \( \lambda \)-Dirichlet series—in particular if these series belong to the Banach space (see again (13) and (23))

\[
\mathcal{H}_\infty(\lambda) = H^\lambda_{\infty,0}[Re > 0] = H^\lambda_{\infty}[Re > 0].
\]

The aim of this subsection is to compare this output with what we now know about the boundary behaviour of functions in \( H^\lambda_{\infty}[Re > 0] \). We start with a brief introduction to all relevant notions.

**\( \lambda \)-Dirichlet groups.** Let \( G \) be a compact abelian group and \( \beta : (\mathbb{R}, +) \to G \) a homomorphism of groups. Then the pair \( (G, \beta) \) is called Dirichlet group, if \( \beta \) is continuous and has dense range. In this case the dual map \( \hat{\beta} : \hat{G} \to \mathbb{R} \) is injective, where we identify \( \mathbb{R} = (\mathbb{R}, +) \) (note that we do not assume \( \beta \) to be injective). Consequently, the characters \( e^{-ix} : \mathbb{R} \to \mathbb{T}, x \in \hat{\beta}(\hat{G}) \), are precisely those which define a unique \( h_x \in \hat{G} \) such that \( h_x \circ \beta = e^{-ix} \). In particular, we have that

\[
\hat{G} = \{ h_x \mid x \in \hat{\beta}(\hat{G}) \}.
\]

Now, given a frequency \( \lambda \), we call a Dirichlet group \( (G, \beta) \) a \( \lambda \)-Dirichlet group whenever \( \lambda \subset \hat{\beta}(\hat{G}) \), or equivalently whenever for every \( e^{-i\lambda u} \in \hat{\beta}(\hat{G}) \) there is (a unique) \( h_{\lambda u} \in \hat{G} \) with \( h_{\lambda u} \circ \beta = e^{-i\lambda u} \).

For every \( u > 0 \) the Poisson kernel \( P_u \) defines a measure \( p_u \) on \( G \), which we call the Poisson measure on \( G \) (the push forward measure of \( P_u dt \) under \( \beta \)). We have \( \| p_u \| = \| P_u \|_{L^1(\mathbb{R})} = 1 \) and

\[
\widehat{p_u}(h_x) = \widehat{p_u}(x) = e^{-u|x|} \text{ for all } x \in \hat{\beta}(\hat{G}).
\]

Finally, recall from [7, Lemma 3.11] that, given a measurable function \( g : G \to \mathbb{C} \), for almost all \( \omega \in G \) there are measurable functions \( g_\omega : \mathbb{R} \to \mathbb{C} \) such that \( g_\omega = g(\omega \beta(.)) \) almost everywhere on \( \mathbb{R} \), and if \( g \in L_1(G) \), then all these \( g_\omega \) are locally integrable. \( g_\omega(t) = g(\omega \beta(.)) \) almost everywhere on \( \mathbb{R} \).

**Hardy spaces on \( \lambda \)-Dirichlet groups.** Given a \( \lambda \)-Dirichlet group \( (G, \beta) \) and \( 1 \leq p \leq \infty \), by \( H^\lambda_p(G) \) we denote the Hardy space of all functions \( g \in L_p(G) \) (recall that being a compact abelian group, \( G \) allows a unique normalized Haar measure) having a Fourier transform \( \hat{g} \) supported on \( \{ h_{\lambda n} : n \in \mathbb{N} \} \subset \hat{G} \). Being a closed subspace of \( L_p(G) \), this clearly defines a Banach space. A fundamental fact from [7, Theorem 3.20] is that the definition of \( H^\lambda_p(G) \) in the following sense is independent of the chosen \( \lambda \)-Dirichlet group \( (G, \beta) \): If \( (G_1, \beta_1) \) and \( (G_2, \beta_2) \) are two \( \lambda \)-Dirichlet groups, then there is a Fourier coefficient preserving, isometric and linear bijection identifying \( H^\lambda_p(G_1) \) and \( H^\lambda_p(G_1) \), i.e.

\[
H^\lambda_p(G_1) = H^\lambda_p(G_2).
\]

By \( \mathcal{B}(g) = \sum \hat{g}(h_{\lambda n})e^{-\lambda n s} \) every \( g \in H^\lambda_p(G) \) naturally generates a \( \lambda \)-Dirichlet series, and the Hardy space \( \mathcal{H}_p(\lambda) \) of \( \lambda \)-Dirichlet series is then defined to be the Banach space of all such Dirichlet series, i.e.

\[
\mathcal{H}_p(\lambda) = \{ D = \sum \hat{g}(h_{\lambda n})e^{-\lambda n s} : g \in H^\lambda_p(G) \},
\]

together with the norm \( \| D \|_p = \| g \|_p \), whenever \( D = B(g) \).

The Carleson type theorem from [9, Theorem 2.1] proves that, given \( D = \sum a_n e^{-\lambda n s} \in \mathcal{H}_p(\lambda) \) and a \( \lambda \)-Dirichlet group \( (G, \beta) \), for almost every \( \omega \in G \) the Dirichlet series
\[ D^{\omega} = \sum a_n h_{\lambda_n}(\omega)e^{-\lambda_n s} \] (a so-called vertical limit of \( D \)) converges almost everywhere on the boundary line \([\text{Re} = 0]\), provided \( p > 1 \).

**Examples.** Note that for every \( \lambda \) there exists a \( \lambda \)-Dirichlet group \((G, \beta)\) (which is not unique). To see a very first example, take the Bohr compactification \( D \) together with the mapping

\[ \beta_{\mathbb{R}} : \mathbb{R} \to \mathbb{T}, \quad t \mapsto \left[ x \mapsto e^{itx} \right]. \] (43)

Then \( \beta_{\mathbb{R}} \) is continuous and has dense range, and so the pair \((\mathbb{R}, \beta_{\mathbb{R}})\) forms a \( \lambda \)-Dirichlet group for all \( \lambda \)'s. We refer to [7] for more ‘universal examples’ of \( \lambda \)-Dirichlet groups. Looking at the frequency \( \lambda = (n) = (0, 1, 2, \ldots) \), the group \( G = \mathbb{T} \) together with

\[ \beta_{\mathbb{T}} : \mathbb{R} \to \mathbb{T}, \quad \beta_{\mathbb{T}}(t) = e^{-it}, \]

forms a \( \lambda \)-Dirichlet group, and the so-called Kronecker flow

\[ \beta_{\mathbb{T}^\infty} : \mathbb{R} \to \mathbb{T}^\infty, \quad t \mapsto p^{-it} = (2^{-it}, 3^{-it}, 5^{-it}, \ldots), \]

turns the infinite dimensional torus \( \mathbb{T}^\infty \) into a \( \lambda \)-Dirichlet group for \( \lambda = (\log n) \). We note that, identifying \( \hat{\mathbb{T}} = \mathbb{Z} \) and \( \mathbb{T}^\infty = \mathbb{Z}^{(\mathbb{N})} \) (all finite sequences of integers), in the first case \( h_n(z) = z^n \) for \( z \in \mathbb{T}, n \in \mathbb{Z} \), and in the second case \( h_n(z) = z^\alpha \) for \( z \in \mathbb{T}^\infty, \alpha \in \mathbb{Z}^{(\mathbb{N})} \).

There is a useful reformulation of the Dirichlet group \((\mathbb{T}^\infty, \beta_{\mathbb{T}^\infty})\), which is already presented in the introduction. Denote by \( \Xi \) the set of all characters \( \chi : \mathbb{N} \to \mathbb{T} \), i.e. \( \chi \) is completely multiplicative in the sense that \( \chi(nm) = \chi(n)\chi(m) \) for all \( n, m \). So every character is uniquely determined by its values on the primes. If we on \( \Xi \) consider pointwise multiplication, then

\[ \iota : \Xi \to \mathbb{T}^\infty, \quad \chi \mapsto (\chi(p_n)), \]

is a group isomorphism which turns \( \Xi \) into a compact abelian group. The Haar measure \( d\chi \) on \( \Xi \) is the push forward of the normalized Lebesgue measure \( dz \) on \( \mathbb{T}^\infty \) through \( \iota^{-1} \). Hence also \( \Xi \) together with

\[ \beta_{\Xi} : \mathbb{R} \to \Xi, \quad t \mapsto [p_k \mapsto p_k^{-it}], \]

forms a \((\log n)\)-Dirichlet group. Note that \( \mathbb{Z}^{(\mathbb{N})} = \widehat{\Xi}, \alpha \mapsto \varphi, \) where \( \varphi(\chi) = \chi(\varphi)^\alpha \) for \( \chi \in \Xi \).

**Applying Carleson’s theorem.** Fix some \( f \in H^\lambda_{\infty}[\text{Re} > 0] \) and \( \omega \in G \), where \((G, \beta)\) is a \( \lambda \)-Dirichlet group. From [8, Theorem 2.16] (see also again (13)) we know that there is an isometric and coefficient preserving identity

\[ H^\lambda_{\infty}[\text{Re} > 0] = H^\lambda_{\infty}(G), \quad f \mapsto g. \] (44)

Hence, we deduce from [7, Proposition 4.3] that there is a unique function

\[ f^{\omega} \in H^\lambda_{\infty}[\text{Re} > 0] \]

such that \( a_n(f^{\omega}) = a_n(f)h_{\lambda_n}(\omega) \) for all \( n \) and \( ||f^{\omega}||_\infty = ||f||_\infty \). We call the function \( f^{\omega} \) vertical limit of \( f \) with respect to \( \omega \) (and refer to [7, Proposition 4.6] which motivates this name). Then by Theorem 1.3 for each \( k > 0 \) and \( s \in [\text{Re} > 0] \) the limit

\[ f^{\omega}(s) = \lim_{x \to \infty} \sum_{\lambda_n \leq s} a_n(f)h_{\lambda_n}(\omega) \left(1 - \frac{\lambda_n}{x}\right)^k e^{-\lambda_n s} \]

exist, i.e. \( f^{\omega} \) is \((\lambda, k)\)-Riesz summable on \([\text{Re} > 0]\) for every \( k > 0 \).
This is in general not true for $k = 0$ and all $\omega \in G$ (look at $\omega = e$, the unit in $G$, and some $\lambda$ not satisfying Bohr’s theorem) and not true for $k = 0$ and all $s \in \{\text{Re } = 0\}$ (look at $\omega = e$, $\lambda = (\log n)$, and the Bayart–Konyagin–Queffélec example).

But an application of Carleson’s theorem in the form given in [9, Theorem 2.2] shows that for each $f \in H^\lambda_{\infty}[\text{Re } > 0]$ the vertical limits $f^\omega$ for almost all $\omega \in G$ are $(\lambda, 0)$-Riesz-summable almost everywhere on the imaginary axis. Moreover, as we show now, if $g \in H^\lambda_{\infty}(G)$ is the function uniquely assigned to $f$ in the sense of (44), then for almost all $\omega \in G$ the horizontal limit $(f^\omega)^*$ of the vertical limit $f^\omega$ equals almost everywhere on $\mathbb{R}$ the ‘restriction’ $g_\omega(\tau) = g(\omega \beta(\tau))$, $\tau \in \mathbb{R}$.

**Theorem 9.1** Let $f \in H^\lambda_{\infty}[\text{Re } > 0]$. Then for every $\lambda$-Dirichlet group $(G, \beta)$, almost every $\tau \in \mathbb{R}$ and almost every $\omega \in G$

$$\lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n}(\omega) e^{-i\lambda_n \tau} = (f^\omega)^*(i\tau).$$

Moreover, if $g \in H^\lambda_{\infty}(G)$ is the unique function such that $a_n(f) = \widehat{g}(h_{\lambda_n})$ for all $n$, then for almost all $\tau \in \mathbb{R}$ and almost all $\omega \in G$

$$g_{\omega}(\tau) = (f^\omega)^*(i\tau).$$

**Proof** Let $g \in H^\lambda_{\infty}(G)$ be the unique function such that $a_n(f) = \widehat{g}(h_{\lambda_n})$ for all $n$. Then $g \in H^\lambda_{\infty}(G)$, and by a variant of Carleson’s convergence theorem from [9, Theorem 2.2] we know that

$$g = \lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n} \quad \text{almost everywhere on } G.$$

Consequently, for almost all $\omega \in G$ by [8, Lemma 1.4] the limit

$$g_{\omega}(\tau) = \lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n}(\omega) e^{-i\lambda_n \tau}$$

exists for almost everywhere $\tau \in \mathbb{R}$. But by Corollary 2.2 (used in the first equation of the following calculation) and [8, Proposition 2.4] (used in the fourth step to change limits), for almost all $\omega \in G$ and for almost all $\tau \in \mathbb{R}$ we have

$$(f^\omega)^*(i\tau) = \lim_{\varepsilon \to 0} f^\omega(\varepsilon + i\tau)$$

$$= \lim_{\varepsilon \to 0} \lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n}(\omega) \left(1 - \frac{\lambda_n}{x}\right)^k e^{-\varepsilon \lambda_n} e^{-i\lambda_n \tau}$$

$$= \lim_{\varepsilon \to 0} \lim_{x \to \infty} R^\lambda_x(g \ast p_\varepsilon)(\omega \beta(\tau))$$

$$= \lim_{\varepsilon \to 0} \lim_{x \to \infty} R^\lambda_x(g \ast p_\varepsilon)(\omega \beta(\tau))$$

$$= \lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n}(\omega) \left(1 - \frac{\lambda_n}{x}\right)^k e^{-i\lambda_n \tau}$$

$$= \lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) h_{\lambda_n}(\omega) e^{-i\lambda_n \tau} = g_{\omega}(\tau),$$

where the penultimate equation follows from the fact that a $(\lambda, \ell)$-Riesz summable series is $(\lambda, k)$-Riesz summable for each $0 \leq \ell \leq k$ with the same limit (see e.g [12, Theorem 16, p. 29]). This completes the argument. \(\square\)
We again believe that the ordinary case is of independent interest.

**Corollary 9.2** Let \( f \in H_\infty^{(\log n)}(\Re > 0) \). Then for almost every \( \tau \in \mathbb{R} \) and almost every \( \chi \in \mathbb{Z} \) we have

\[
\lim_{x \to \infty} \sum_{\lambda_n \leq x} a_n(f) \chi(n) n^{-i \tau} = (f^\wedge)(i \tau).
\]

Moreover, if \( g \in H_\infty^{(\log n)}(\mathbb{Z}) \) is the function associated to \( f \), i.e. \( \widehat{g}(\alpha) = a_n(f) \) for \( n = p^\alpha \) and \( \widehat{g}(\alpha) = 0 \) else, then for almost all \( \tau \in \mathbb{R} \) and almost all \( \chi \in \mathbb{Z} \)

\[
(f^\wedge)(i \tau) = g(n \mapsto \chi(n) n^{-i \tau}).
\]

We finally illustrate Theorem 9.1 looking at bounded, holomorphic functions on the infinite dimensional polydisc \( B_{c,0}. \) Take some \( f \in H_\infty^{(\log n)}(B_{c,0}) \). Then \( \mathbb{T}_\infty \) may be interpreted as the ‘distinguished boundary’ of \( B_{c,0} \), and we may ask to which extent \( f \) has boundary values.

We deduce as a consequence of Theorem 5.3 (together with (5), (6), and (16)) that, given \( k > 0 \), for almost every \( t \in \mathbb{R} \)

\[
\lim_{\varepsilon \to 0} f(p - (\varepsilon + it)) = \lim_{x \to \infty} \sum_{p^\alpha < x} \frac{\partial^\alpha f(0)}{\alpha!} \left( 1 - \frac{\log p^\alpha}{x} \right)^k \frac{1}{p^\alpha t}.
\]

What can we in this case conclude from Theorem 9.1? To see this, let \( g \in H_\infty^{(\log n)}(\mathbb{T}_\infty) \) be associated to \( f \) in the sense that \( \widehat{g}(\alpha) = \partial^\alpha f(0) / \alpha! \) for \( \alpha \in \mathbb{N}_0 \), and \( \widehat{g}(\alpha) = 0 \) else. Then by Theorem 9.1 for almost every \( z \in \mathbb{T}_\infty \) and almost all \( t \in \mathbb{R} \)

\[
g(z p^{-i t}) = \lim_{\varepsilon \to 0} f(z (p - (\varepsilon + it))) = \lim_{x \to \infty} \sum_{p^\alpha < x} \frac{\partial^\alpha f(0)}{\alpha!} \frac{1}{p^\alpha t}.
\]

### 10 Appendix

#### 10.1 Elaborating Corollary 3.2

Recall from Corollary 3.2 that the mapping

\[
T : H_\infty^{\lambda,\ell}[\Re > 0] \leftrightarrow L_\infty(\mathbb{R}), \quad f \mapsto \frac{f^\wedge(i \cdot)}{(1 + i \cdot)\ell}
\]

defines an isometric embedding. Starting with the following definition, we in Theorem 10.1 below prove an internal description of the range of \( T \):

Given a frequency \( \lambda \) and \( \ell \geq 0 \),

\[
H_\infty^{\lambda,\ell}(\mathbb{R})
\]

denotes the subspace of all \( g \in L_\infty(\mathbb{R}) \) for which there are a \( \lambda \)-Dirichlet series \( D = \sum a_n e^{-\lambda_n s} \) and \( m > 0 \) such that for every \( u > 0 \)

\[
\lim_{x \to \infty} \sup_{t \in \mathbb{R}} \frac{|(1 + u + it)^\ell[g \ast P_u](t)}{(1 + it)^\ell} - R^\frac{\lambda, m}{\ell}(D)(u + it) | = 0.
\]

For \( \ell = 0 \) we write

\[
H_\infty^{\lambda,0}(\mathbb{R}) = H_\infty^\lambda(\mathbb{R}).
\]
The following result is an elaboration of Corollary 3.2.

**Theorem 10.1** Let \( \ell \geq 0 \) and \( \lambda \) an arbitrary frequency. Then the mapping

\[
\Psi : H^\lambda_{\infty, \ell}(\mathbb{R}) \to H^\lambda_{\infty, \ell}[Re > 0], \quad \Psi(g)(u + it) = (1 + u + it)^{\ell}[g * P_u](t)
\]

defines a bijective isometry with inverse

\[
\Psi^{-1} : H^\lambda_{\infty, \ell}[Re > 0] \to H^\lambda_{\infty, \ell}(\mathbb{R}), \quad \Psi^{-1}(f)(t) = \frac{f^*(it)}{(1 + it)^{\ell}}.
\]

where \( f^* \) denotes the horizontal limit of \( f \).

**Proof** Fix \( g \in H^\lambda_{\infty, \ell}(\mathbb{R}) \), and let \( D = \sum a_n e^{-\lambda_n s} \) be some Dirichlet series for which (45) holds. Consider the function

\[
f(u + it) = (1 + u + it)^{\ell}[g * P_u](t) : [Re > 0] \to \mathbb{C}.
\]

We claim that \( f \) is holomorphic and that \( D \) is the \( \lambda \)-Riesz germ of \( f \). Indeed, condition (45) implies that for every \( s \in [Re > 0] \)

\[
f(s) = \lim_{x \to \infty} R^\lambda_{\infty, m}(D)(s).
\]

Then the sequence \( (R^\lambda_{\infty, m}(D))_x \) converges to \( f \) uniformly on all compact subsets of \([Re > 0]\) (see e.g. [10, Theorem 2.9]), and so \( f \) is holomorphic on \([Re > 0]\) and \( D \) is the \( \lambda \)-Riesz germ of \( f \). Moreover,

\[
\|f\|_{H^\lambda_{\infty, \ell}[Re > 0]} = \sup_{u > 0} \sup_{t \in \mathbb{R}} \left| \frac{f(u + it)}{(1 + u + it)^{\ell}} \right| = \sup_{u > 0} \|g * P_u\|_{L^\infty(\mathbb{R})} \leq \|g\|_{\infty},
\]

which eventually shows that \( f \in H^\lambda_{\infty, \ell}[Re > 0] \). Let now \( f \in H^\lambda_{\infty, \ell}[Re > 0] \). Then by Corollary 3.2 the function \( g = \frac{f^*(it)}{(1 + it)^{\ell}} \) belongs to \( L^\infty(\mathbb{R}) \) with \( \|g\|_{\infty} = \|f\|_{\infty, \ell} \). It remains to show that \( g \in H^\lambda_{\infty, \ell}(\mathbb{R}) \). To do this, let \( D \) be the \( \lambda \)-Riesz germ of \( f \), and note that by Theorem 3.1 for every \( u + it \in [Re > 0] \)

\[
(1 + u + it)^{\ell}[g * P_u](t) = f(u + it).
\]

Then, by Theorem 1.3, [10, Theorem 3.17] and continuity (of the function \( f(u + \cdot) \) on \([Re \geq 0]\)), we for every \( u > 0 \) and \( k > \ell \) obtain that

\[
0 = \lim_{x \to \infty} \sup_{s \in [Re > 0]} \left| \frac{f(u + s)}{(1 + s)^{\ell}} - \frac{R^\lambda_{\infty, k}(D)(u + s)}{(1 + s)^{\ell}} \right| = \lim_{x \to \infty} \lim_{v \to 0} \sup_{t \in \mathbb{R}} \left| \frac{f(u + v + it)}{(1 + v + it)^{\ell}} - \frac{R^\lambda_{\infty, k}(D)(u + v + it)}{(1 + v + it)^{\ell}} \right|
\]

\[
\geq \lim_{x \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{f(u + it)}{(1 + it)^{\ell}} - \frac{R^\lambda_{\infty, k}(D)(u + it)}{(1 + it)^{\ell}} \right| = \lim_{x \to \infty} \sup_{t \in \mathbb{R}} \left| \frac{(1 + u + it)^{\ell}[g * P_u](t) - R^\lambda_{\infty, k}(D)(u + it)}{(1 + it)^{\ell}} \right|
\]

which proves that \( g \in H^\lambda_{\infty, \ell}(\mathbb{R}) \). Finally, we show that almost everywhere on \( \mathbb{R} \) we have

\[
\Psi(g)^*(t) = (1 + it)^{\ell}g(t),
\]
and so \( \Psi^{-1}(\Psi(g)) = g \). Obviously (48) is equivalent to the fact that for almost everywhere on \( \mathbb{R} \)

\[
\lim_{u \to 0} [P_u \ast g](t) = g(t).
\]

(49)

But since \( g \in L_\infty(\mathbb{R}) \), this is standard, and we only for the sake of completeness add an argument. Indeed, define \( \alpha = g \chi_{[-N,N]} \) and \( \beta = g - g \chi_{[-N,N]} \). Then \( \beta \) vanishes on \([-N,N]\], and so we almost everywhere on \([-N,N]\] have

\[
\lim_{u \to 0} [P_u \ast \beta](t) = \beta(t) = 0
\]

(see e.g. [11, Theorem 1.2.19, p. 27]). Additionally, since \( \alpha \in L_1(\mathbb{R}) \), almost everywhere on \([-N,N]\]

\[
\lim_{u \to 0} [P_u \ast \alpha](t) = \alpha(t) = g(t).
\]

(see e.g. Theorem [11, Theorem 2.1.14., p. 94] and also Example 2.1.15, p.95). Together, almost everywhere on \([-N,N]\]

\[
\lim_{u \to 0} [P_u \ast g](t) = \lim_{u \to 0} [P_u \ast \alpha](t) + [P_u \ast \beta](t) = g(t).
\]

Now collecting countably many zero sets, proves (48), which finishes the proof.

The case \( \ell = 0 \) is of special interest. In view of (23) we may reformulate Theorem 10.1 as follows.

**Corollary 10.2** Let \( \lambda \) be a frequency. Then the mapping

\[
\Psi : H^\lambda_\infty(\mathbb{R}) \to H^\lambda_\infty([Re > 0]), \quad \Psi(g)(u + it) = [g \ast P_u](t)
\]

is a bijective isometry with inverse

\[
\Psi^{-1} : H^\lambda_\infty([Re > 0]) \to H^\lambda_\infty(\mathbb{R}), \quad \Psi(f)(t) = f^*(it).
\]

Recall from Sect. 1.3 that the Banach space \( H^\lambda_\infty([Re > 0]) \) is defined in terms of almost periodicity. In the remaining part of this subsection we show how to give a similar description for functions in \( H^\lambda_\infty(\mathbb{R}) \).

By \( AP(\mathbb{R}) \) we denote the Banach space of all almost periodic functions (equipped with the sup-norm). If \( g \in AP(\mathbb{R}) \), then the uniquely assigned Bohr coefficients of \( g \) are given by

\[
a_x(g) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t)e^{-ixt} dt, \quad x \in \mathbb{R};
\]

for functions in \( H^\lambda_\infty([Re > 0]) \) compare this with (12). Given a frequency \( \lambda = (\lambda_n) \), by

\[
AP^\lambda(\mathbb{R})
\]

we denote all functions \( g \in AP(\mathbb{R}) \) for which the Bohr coefficients are supported on \( \lambda \), i.e. the coefficient \( a_x(g) \) vanishes, whenever \( x \notin \{\lambda_n : n \in \mathbb{N}\} \). We need a simple observation.

**Lemma 10.3** Let \( g \in AP^\lambda(\mathbb{R}) \). Then \( g \ast P_u \in AP^\lambda(\mathbb{R}) \) for every \( u > 0 \).
Proof Recall that all trigonometric polynomials are dense in $AP(\mathbb{R})$ (see [19, Theorem 1.5.5]). So, if $(p_n)$ is a sequence of trigonometric polynomials such that $\lim_{n \to \infty} p_n = g$ uniformly on $\mathbb{R}$, then also $\lim_{n \to \infty} p_n * P_u = g * P_u$ uniformly on $\mathbb{R}$, and so $g * P_u \in AP(\mathbb{R})$ for every $u > 0$. Checking on trigonometric polynomials and using density, shows

$$a_x(g * P_u) = e^{-ux}a_x(g), \ x \in \mathbb{R},$$

which completes the argument. \qed

The following result characterizes functions in $H^\lambda_{\infty}(\mathbb{R})$ in terms of almost periodicity.

**Theorem 10.4** Let $\lambda$ be a frequency and $g \in L^\infty(\mathbb{R})$. Then $g \in H^\lambda_{\infty}(\mathbb{R})$ if and only if $g * P_u \in AP^\lambda(\mathbb{R})$ for every $u > 0$. Moreover, in this case

$$\|g\|_\infty = \sup_{u > 0} \|g * P_u\|_\infty,$$

and there is a $\lambda$-Dirichlet series $D$ such that for all $k > 0$ and all $u > 0$

$$\lim_{x \to \infty} \sup_{t \in \mathbb{R}} \left| [g * P_u](t) - R^\lambda_{\mathbb{R}}(D)(u + it) \right| = 0.$$

We prepare the proof with the following lemma.

**Lemma 10.5** The mapping

$$\Phi: AP^\lambda(\mathbb{R}) \leftrightarrow H^\lambda_{\infty}[\text{Re} > 0], \ g \mapsto f,$$

where

$$f(u + it) := [g * P_u](it) \text{ for all } u + it \in [\text{Re} > 0],$$

defines an isometric and coefficient preserving embedding.

Proof Recall first that a function on $\mathbb{R}$ is almost periodic if and only if it has a unique extension $\tilde{g}$ to the Bohr compactification $\mathbb{R}$ (see again [19, Theorem 1.5.5]). This in particular implies that the mapping

$$I: AP^\lambda(\mathbb{R}) \to C(\mathbb{R}), \ g \mapsto \tilde{g}$$

is an isometric embedding (here it is used that $\beta_{\mathbb{R}}$ from (43) has dense range). Moreover, $I$ preserves Bohr- and Fourier coefficients, i.e. for all $x \in \mathbb{R}$

$$a_x(g) = \hat{\tilde{g}}(h_x)$$

(see e.g. [7, Proposition 3.10]). So the range of $I$ is in fact contained in $H^\lambda_{\infty}(\mathbb{R})$. On the other hand, we know from (44) that there is an isometric, coefficient preserving bijection

$$J: H^\lambda_{\infty}(\mathbb{R}) \to H^\lambda_{\infty}[\text{Re} > 0], \ g \mapsto f.$$

Combining, we get an isometric, coefficient preserving embedding

$$J \circ I: AP^\lambda(\mathbb{R}) \to H^\lambda_{\infty}[\text{Re} > 0], \ g \mapsto f,$$

and then it remains to show that

$$[J \circ I](g)(u + it) = [g * P_u](it), \ u + it \in [\text{Re} > 0].$$
Indeed, fix some $u > 0$. Then for all $n$

$$a_{\lambda_n}(g) = a_{\lambda_n}([J \circ I]g) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} ([J \circ I]g)(u + it)e^{-(u+it)\lambda_n}dt,$$

(see again (12)), and hence by Lemma 10.3, (51) and (50)

$$a_{\lambda_n}(g * P_u) = a_{\lambda_n}(g)e^{-u\lambda_n} = a_{\lambda_n}(([J \circ I]g)(u + i \cdot)).$$

Since almost periodic functions are uniquely determined by their Bohr coefficients, (53) is proved.

**Proof of Theorem 10.4** If $g \in H_{\infty}^{\lambda}(\mathbb{R})$, then $g * P_u$ by definition is the uniform limit of $\lambda$-Dirichlet polynomials, and so clearly $g * P_u \in AP^{\lambda}(\mathbb{R})$ for every $u > 0$.

Conversely, assume that $g \in L_{\infty}(\mathbb{R})$ such that $g * P_u \in AP^{\lambda}(\mathbb{R})$ for every $u > 0$. We define the bounded function $f(u + it) := (g * P_u)(t), u + it \in [\text{Re } 0]$, and claim that $f \in H_{\infty}^{\lambda}[\text{Re } 0]$ with $\Psi^{-1}f = g$, which finishes the proof by Corollary 10.2. To do this, we apply Lemma 10.5 to $h_n = g * P_{\frac{1}{n}}, n \in \mathbb{N}$, and obtain that the function $f_n$ defined by

$$f_n(u + it) = ((g * P_{\frac{1}{n}}) * P_{\frac{1}{n}})(t) = ((g * P_{\frac{1}{n^2}})(t), u + it \in [\text{Re } 0]$$

belongs to $H_{\infty}^{\lambda}[\text{Re } 0]$. Since for every $s \in [\text{Re } 0 + \frac{1}{n}]$ and $n \in \mathbb{N}$

$$f(s) = f\left(s - \frac{1}{n} + \frac{1}{n}\right) = f_n\left(s - \frac{1}{n}\right),$$

we see that $f$ is holomorphic on $[\text{Re } 0]$. Hence, being bounded and almost periodic on all vertical lines $[\text{Re } \sigma]$, the function $f$ belongs to $H_{\infty}^{\lambda}[\text{Re } 0]$. Moreover, the argument for (49) implies that for almost every $t \in \mathbb{R}$

$$(\Psi^{-1}f)(t) = f^t(it) = \lim_{u \to 0} f(u + it) = \lim_{u \to 0} (g * P_u)(t) = g(t),$$

which finishes the proof of the first claim. Finally, note that the first statement of the ‘moreover part’ then is also evident, whereas the second follows from Theorem 1.3 looking at the $\lambda$-Riesz germ $D$ of $f$.

In view of Theorem 9.1 and Corollary 10.2 we add the following observation.

**Corollary 10.6** Let $(G, \beta)$ be a $\lambda$-Dirichlet group and $g \in H_{\infty}^{\lambda}(G)$. Then $g_\omega \in H_{\infty}^{\lambda}(\mathbb{R})$ for almost every $\omega \in G$.

**Proof** For almost every $\omega \in G$ we know from [7, Lemma 3.11] that $g_\omega \in L_{\infty}(\mathbb{R})$, and moreover by Theorem 9.1, that $g_\omega$ is the horizontal limit of a function $f_\omega \in H_{\infty}^{\lambda}[\text{Re } 0]$. Hence by Corollary 10.2, we obtain $g_\omega \in H_{\infty}^{\lambda}(\mathbb{R})$ for almost every $\omega \in G$.

We complete this section by another observation, which states that in the ordinary case $H_{\infty}^{(\log n)}(\mathbb{R})$ may be describe in terms of Cesàro limits.

**Corollary 10.7** A function $g \in L_{\infty}(\mathbb{R})$ belongs to $H_{\infty}^{(\log n)}(\mathbb{R})$ if only if there is an ordinary Dirichlet series $D = \sum a_n n^{-s}$ such that for every $u > 0$ the Cesàro means of $D$ converges uniformly on $[\text{Re } u]$ to $g * P_u$, i.e. for every $u > 0$

$$\lim_{x \to \infty} \sup_{t \in \mathbb{R}} \left| g * P_u(t) - \frac{1}{x} \sum_{y \leq x} \sum_{n \leq y} a_n n^{-(u+it)} \right| = 0.$$
Proof: Observe that by Theorem 10.4 condition (54) immediately implies that \( g \in H^k_{\infty}(\mathbb{R}) \). Conversely, if \( g \in H^k_{\infty}(\mathbb{R}) \), then Theorem 10.4 implies that there is an ordinary Dirichlet series \( D \) such that the \((n, k)\)-Riesz means of \( D \) of any order \( k > 0 \) on \( \text{Re} = u \) converges uniformly to \( g \ast P_u \) for every \( u > 0 \). Now using [12, Theorems 17 and 30] (see also [10, Theorem 2.7, (ii)] and [10, Theorem 2.8]), and repeating the vector-valued argument of [10, Corollary 2.17] for fixed \( u > 0 \), we obtain that the \((n, k)\)-Riesz means of \( D \) of any order \( k > 0 \) converge uniformly on \( \text{Re} > 2u \) with limit \( g \ast P_{2u} \) for every \( u > 0 \). Since that the \((n, k)\)-Riesz means of \( D \) are precisely the Cesàro means of \( D \), the proof is complete. \( \square \)

10.2 A proof of Riesz’ Theorem 1.2

A crucial ingredient of our proof of Theorem 1.2 is given by the Perron-type formula from [10, Theorem 2.13]:

For \( k \geq 0 \) let \( D = \sum a_n e^{-\lambda_n s} \) be a somewhere \((\lambda, k)\)-Riesz summable \( \lambda \)-Dirichlet series and \( f : [\text{Re} > \sigma_c^{\lambda, k}(D)] \to \mathbb{C} \) its limit function, where \( \sigma_c^{\lambda, k}(D) \) stands for the abscissa of \((\lambda, k)\)-Riesz summability of \( D \). Then

\[
R_x^{\lambda, k}(f)(0) = \frac{\Gamma(1 + k)}{2\pi i} \frac{x^{-k}}{c - i\infty} \int_{c - i\infty}^{c + i\infty} \frac{f(s)}{s^{1+k}} e^{xs} ds, \quad c > 0. \tag{55}
\]

Given \( s \in \mathbb{C} \) and \( r > 0 \), the euclidean ball in \( \mathbb{C} \) of radius \( r \) and center \( s \) is denoted by \( B_r(s) \).

Proof of Theorem 1.2: After translation we may assume that the poles of \( f \) are given by \( 0 < |p_1| \leq |p_2| \leq \ldots \leq |p_N| \) with orders \( m_j = m(p_j) < 1 + k \). We claim that for every \( 0 < 2\delta < |p_1| \) and \( I = [-i\delta, i\delta] \)

\[
\lim_{x \to \infty} \sup_{i \tau \in I} |R_x^{\lambda, k}(f)(i\tau) - f(i\tau)| = 0. \tag{56}
\]

Indeed, provided that this claim is established, take an arbitrary compact interval

\[
I = i[a, b] \subset [\text{Re} = 0] \setminus \{p_1, \ldots, p_N\}.
\]

Then for every \( i \tau \in I \) the translation \( f_{i\tau}(s) = f(s + i\tau) \) of \( f \) about \( i\tau \) is uniformly \((\lambda, k)\)-Riesz summable on \([i\delta(\tau), i\delta(\tau)]\) for some \( \delta(\tau) > 0 \), and so \( f \) is uniformly \((\lambda, k)\)-Riesz summable on \([-i\delta(\tau) + i\tau, i\delta(\tau) + i\tau]\). Since

\[
I \subset \bigcup_{\tau \in I} [-i\delta(\tau) + i\tau, i\delta(\tau) + i\tau],
\]

by compactness there are finitely many \( \tau_1, \ldots, \tau_K \in I \) such that

\[
I \subset \bigcup_{j=1}^{K} [-i\delta(\tau_j) + i\tau_j, i\delta(\tau_j) + i\tau_j],
\]

and consequently \( f \) is uniformly \((\lambda, k)\)-Riesz summable on \( I \).

Let us start the proof of (56), fixing some \( 0 < 2\delta < |p_1| \). The choice \( c = x^{-1} \) in (55) leads to

\[
R_x^{\lambda, k}(f)(i\tau) = \frac{\Gamma(1 + k)e^{-|\tau|}}{2\pi i} \int_{-\infty}^{\infty} f(x^{-1} + i(t + \tau)) e^{itx} \frac{x}{(1 + itx)^{1+k}} dt.
\]
The idea now is to split this integral with respect to a disjoint union of sub intervals of \( \mathbb{R} \). To do so, choose some \( \varepsilon > 0 \) such that
\[
\bigcap_{j=1}^{N} [p_j - 2\varepsilon, p_j + 2\varepsilon] = \emptyset
\] (57)
and consider the disjoint decomposition
\[
\mathbb{R} = S \cup [\mathbb{R} \setminus S], \text{ where } S = ] - \delta, \delta[ \cup \bigcup_{j=1}^{N} [p_j - \varepsilon, p_j + \varepsilon[.
\]

Observe that \( \mathbb{R} \setminus S \) is the union of finitely many disjoint intervals \( J \) formed by the connected components of \( \mathbb{R} \setminus S \). Now we show first that
\[
\lim_{x \to \infty} \sup_{\tau \in I} \left| \int_{-\delta}^{\delta} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + ixt)^{1+k}} dt - f(i\tau) \right| = 0, \quad (58)
\]
then second that for all \( 1 \leq j \leq N \)
\[
\lim_{x \to \infty} \sup_{\tau \in I} \left| \int_{p_j - \varepsilon}^{p_j + \varepsilon} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + ixt)^{1+k}} dt \right| = 0, \quad (59)
\]
and finally that for all connected components \( J \) of \( \mathbb{R} \setminus S \)
\[
\lim_{x \to \infty} \sup_{\tau \in I} \left| \int_{J} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + ixt)^{1+k}} dt \right| = 0. \quad (60)
\]

Note that the proof is complete, whenever these three claims are provided.

**Proof of (58):** By substitution for every \( i\tau \in I \) and \( x > 1 \)
\[
\int_{-\delta}^{\delta} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + ixt)^{1+k}} dt = \int_{-\delta}^{\delta} f(x^{-1} + iy^{-1} + \tau))e^{iy} \frac{1}{(1 + iy)^{1+k}} dy \]
\[
= \int_{R} \chi_{[\delta, \delta]}(y)f(x^{-1} + iy^{-1} + \tau))e^{iy} \frac{1}{(1 + iy)^{1+k}} dy.
\]
Moreover,
\[
|\chi_{[\delta, \delta]}(y)f(x^{-1} + iy^{-1} + \tau))e^{iy}(1 + iy)^{-(1+k)}| \leq \sup_{|y| \leq 1 + 2\delta} |f(s)||1 + iy|^{-(1+k)},
\]
since \( |x^{-1} + iy^{-1} + \tau| \leq 1 + 2\delta \) for all \( y \in x[-\delta, \delta] \). Additionally, since \( f \) is uniformly continuous on \( [0, 1] \times 2\delta \), we for every \( y \in \mathbb{R} \) have uniformly for \( i\tau \in I \)
\[
\lim_{x \to \infty} \chi_{[\delta, \delta]}(y)f(x^{-1} + iy^{-1} + \tau))e^{iy}(1 + iy)^{-(1+k)} = f(i\tau)e^{iy}|1 + iy|^{-(1+k)}.
\]
Hence the (uniform) dominated convergence theorem together with (36) shows that uniformly for \( i\tau \in I \)
\[
\lim_{x \to \infty} \int_{-\delta}^{\delta} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + ixt)^{1+k}} dt = f(i\tau) \int_{-\infty}^{\infty} e^{iy} \frac{1}{(1 + iy)^{1+k}} dy = f(i\tau).
\]
Proof of (59): Fix $1 \leq j \leq N$ and let $x > 1$. Since by assumption and (57)

$$\sup_{s \in B_r(i p_j) \cap (\Re > 0)} |(s - i p_j)^{m_j} f(s)| = C_j < \infty,$$

we for every $i \tau \in I$ conclude that

$$\left| \int_{p_j - \varepsilon}^{p_j + \varepsilon} f(x^{-1} + i(t + \tau)) e^{i \tau t} \frac{x}{(1 + i xt)^{1+k}} dt \right| \leq C_j \int_{p_j - \varepsilon}^{p_j + \varepsilon} \frac{1}{|x^{-1} + i(t + p_j)|^{m_j}} \frac{x}{|1 + i xt|^{1+k}} dt \leq C_j \int_{-\varepsilon}^{\varepsilon} \frac{1}{|x^{-1} + i(y + \tau)|^{m_j}} \frac{1}{|1 + i(x + y)|^{1+k}} dy \leq C_j x^{-k} p_j^{-(1+k)} \int_{-\varepsilon}^{\varepsilon} \frac{1}{|x^{-1} + i(y + \tau)|^{m_j}} dy,$$

where, since $x > 1$, for the last estimate we use that $p_j \leq |1 + i(x + p_j)|$. Moreover,

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{|x^{-1} + i(y + \tau)|^{m_j}} dy = \chi_{m_j - 1} \int_{-\varepsilon}^{\varepsilon} \frac{x}{|1 + i xy|^{m_j}} dy \leq \chi_{m_j - 1} \int_{-\varepsilon}^{\varepsilon} \frac{1}{|1 + i r|^{m_j}} dr.$$

If $m_j \geq 2$, then

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{|1 + i r|^{m_j}} dr < \infty$$

and if $m_j = 1$, there is $C = C(I, \varepsilon) > 0$ such that for all $x, \tau$

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{|1 + i r|} dr \leq \int_{-\varepsilon}^{\varepsilon} \frac{1}{|1 + i r|} dr = 2 \int_{0}^{C x} \frac{1}{|1 + i r|} dr \leq 2 \left( 1 + \int_{1}^{C x} \frac{1}{|1 + i r|} dr \right) \leq 2 \int_{1}^{C} \frac{1}{r} dr = 2 + \ln(Cx).$$

Hence all in all we obtain some $D = D(I, m_j, \varepsilon) > 0$ such that for all $x, \tau$

$$\sup_{\tau \in I} \left| \int_{p_j - \varepsilon}^{p_j + \varepsilon} f(x^{-1} + i(t + \tau)) e^{i \tau t} \frac{x}{(1 + i xt)^{1+k}} dt \right| \leq D p_j^{-(1+k)} \ln(x) x^{m_j - (1+k)},$$

which vanishes as $x \to \infty$, since $m_j < k + 1$.

Proof of (60): Note first that each of the finitely many connected components $J$ of $\mathbb{R} \setminus S$ is an interval, and that all of them except two are bounded. Fix such interval $J = [a, b]$. Then, using in the bounded case the continuity of $f$ on $[0, 1] + J$ and in the unbounded case moreover the assumption made on the growth of $f$, we for all $i \tau \in I$ and $i \tau \in J$ have

$$|f(x^{-1} + i(t + \tau))| \leq C(J) |x^{-1} + i(t + \tau)|^\ell \leq C(J, I) |1 + itx|^\ell.$$
Consequently

\[
\left| \int_{J} f(x^{-1} + i(t + \tau))e^{itx} \frac{x}{(1 + itx)^{1+k+\epsilon}} \, dt \right| \leq C(J, I) \int_{a}^{b} \frac{x}{|1 + itx|^{1+k-\epsilon}} \, dt
\]

\[
= C(I, J) \int_{ax}^{bx} \frac{1}{|1 + iy|^{1+k-\epsilon}} \, dy,
\]

which vanishes as \( x \to \infty \).

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