Localized numerical impulses solutions in diffuse neural networks modeled by the complex fractional Ginzburg-Landau equation

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Abstract

We investigate a network of diffusively Hindmarsh-Rose neurons with long-range synaptic coupling. By means of a specific perturbation technique, we show by using the Lienard form of the model that it can be governed by a complex fractional Ginzburg-Landau (CFGL) equation where analytical as well as numerical nonlinear wave solutions can be obtained. We propose the semi implicit Riesz fractional finite-difference scheme to solve efficiently the obtained CFGL equation. From numerical simulations, it is found that the fractional solutions for the nerve impulse are well-localized impulses whose shape and stability depend on the value of the long-range parameter.

Keywords: Hindmarsh-Rose neurons model, long-range coupling, complex

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1. Introduction

Recent investigations of diffusively neural networks have alerted us to the significant role played by oscillations, spatial structures and waves in the dynamics of excitable systems [1, 2, 3, 4, 5]. This is because the conditions under which cortical waves occur are very primordial in the understanding of the normal processing of sensory stimuli as well as more pathological forms of behavior such as epileptic seizures, migraines, Parkinson diseases and others [6, 7]. Many nontrivial examples of dynamical systems have been provided by phenomenological and neurophysiological models developed to reproduce the activities of neural oscillators. The Hindmarsh-Rose (HR) model [8], a simplification version of Hodgkin-Huxley model [9] and a generalization of the Fitzhugh equations [10], represents a paradigmatic example of these systems.

The HR neural model is one of the most recent studied in the neuroscience which increasingly becomes an active area of research. It aims to study the spiking-bursting behavior of the membrane potential observed in the single neuron experiments. The model is governed by a set of three nonlinear ordinary differential equations given by

\[
\begin{align*}
\dot{u} &= v - au + bu^2 - w + I \\
\dot{v} &= c - du^2 - ev \\
\dot{w} &= r[s(u - u_0) - w],
\end{align*}
\]

(1)
where $u$ is the membrane potential (nerve impulse), $v$ is the spiking variable which takes into account the measure of the rate at which transport of sodium and potassium ions is made through fast ion channels, and $w$ is the bursting variable which takes into account the rate at which the transport of other ions ($Cl^-$ and proteins anions) made through slow ions channels. The values of the parameters are $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $r = 0.008$, $s = 4.0$, $e = 1.0$, $u_0 = -1.60$, and $I$ is the stimulation current, which is also the bifurcation parameter determining the qualitative behavior of the neuron. The parameter $r$ is the ratio of fast/slow time scales, $s$ is the recovery variable, $u_0$ is the equilibrium coordinate when $I = 0$ and $w = 0$.

In fact, a biological neural network is a series of interconnected neurons whose activation defines a recognizable linear pathway. In the multicellular state, cells interact with each other directly or indirectly. Neurons interact or communicate among themselves through synapses to form complex network structures that perform specialized functions. In a neural network, the coupling between neurons may occur via the electrical or chemical synapses \[11\]. In the former case, the coupling occurs through gap junctions and its strength depends linearly on the difference between the membrane potential. The neurons must be very close to each other in order to realize this synaptic connection. In that sense, neurons only make electrical connections with their nearest neighbors. In the chemical case, one neuron releases neurotransmitter molecules into a small space (the synaptic cleft) that is adjacent to another neuron. Then, the synapse is mediated by neurotransmitters and the connection occurs between the dendrites and the axons, therefore this type of connection allows long-range synaptic interactions \[12\]. Hence, the
dynamics of an individual neuron may be influenced by the interaction or coupling with other neurons. Such interactions are used in living systems to coordinate and control many biological functions. 

The search of confined wave packet is one of the major challenges in nonlinear neural dynamics. In this context, many studies have been carried out that indicate the presence of nonlinear waves such as solitons in the neural systems. Solitons are localized solutions of a widespread class of weakly nonlinear dispersive partial differential equations. The soliton model in neuroscience is a recent developed model that attempts to explain how signals are conducted within neurons. This model proposes that the signals travel along the cell’s membrane in the form of pulse solitons. As such the model presents a direct challenge to the widely accepted Hodgkin-Huxley model, which proposes that signals travel as action potentials. Up to the present, in diffuse neural models the soliton model has been exclusively addressed in the case of the nearest neighbors coupling. The behavior of most biological systems has memory or aftereffects and therefore, the modeling of these systems by fractional-order differential equations has more advantages than classical mathematical modeling using integer-order in which such effects are neglected. According to some recent studies, the non-locality originating from the long-range coupling or interaction results in the dynamic equations with space derivatives of fractional order (see, e.g., and references therein). In the present paper, we show that the HR neural model can be governed in the infrared limit by a complex fractional Ginzburg-Landau (CFGL) equation when the long-range coupling are taken into account. To solve efficiently our CFGL equation, we follow closely.
in space discretization and propose the semi implicit Riesz fractional finite-difference scheme where only one linear system is solved by time iteration. The obtained numerical solutions of the nerve impulse reveal localized short impulse properties.

The rest of the paper is organized as follows. In Section 2, we present the neural model and derive its Lienard form. In Section 3, by means of the perturbation technique, we derive the CFGL equation. In Section 4, we present our efficient semi implicit Riesz fractional finite-difference scheme for the CFGL equation and numerical results. Our work is summarized in Section 5.

2. The Hindmarsh-Rose coupled model with the long-range coupling

In this work, we consider the HR neural network as a system of $N$ neural oscillators in which the configuration of couplings is assumed to be long-range. In this case, each unit of HR neural model is coupled to any other. We also assume that the coupling between the neural oscillators is nonlocal. Then, the model can be reformulated by means of the following nonlinear ordinary differential equations:

\[
\begin{align*}
\dot{u}_n &= v_n - au_n^3 + bu_n^2 - w_n + I + \sum_{m=1,m\neq n}^{N} K_\alpha(n-m)(u_n - u_m) \\
\dot{v}_n &= c - du_n^2 - ev_n \\
\dot{w}_n &= r[s(u_n - u_0) - w_n],
\end{align*}
\]

(2)
with $1 \leq n \leq N$. The nonlocal coupling is given by the following power function

$$K_\alpha(n - m) = \frac{K}{|n - m|^{\alpha+1}}. \quad (3)$$

The absolute difference $|m - n|$ measures the distance between neural oscillators $n$ and $m$. From the biological point of view, $|m - n|$ means the synaptic cleft length. The parameter $K$ is the synapse strength and $\alpha$ is the long-range coupling parameter, it physically describes a level of collective coupling of neural oscillators. The nonlocal coupling Eq. (3) was developed for the first time in [18] to study thermodynamics and phase transitions, and recently in [15, 19] to study complex systems with long-range interactions.

As we are interested by nonlinear waves in the network, we therefore rewrite the system of Eq. (2) in the wave form. We achieve this by differentiating the first equation in this system and substituting $\dot{v}_n$ into the obtained second-order ODE. Then, we rewrite suitably Eq. (2) in a Lienard form, that is a second-order differential equation with a small damping term, such that

$$\begin{align*}
\ddot{v}_n + \Omega_0^2 v_n + (\eta_0 + \eta_1 v_n + \eta_2 v_n^2)\dot{v}_n + \lambda_1 v_n^2 + \frac{\eta_3}{3} v^3_n + \lambda_3 w_n + I_0 \\
= \sum_{m=1, m \neq n}^{N} \frac{1}{|n-m|^{\alpha+1}} [c_0(\psi_n - \psi_m) + c_1(\dot{\psi}_n - \dot{\psi}_m)] \quad (4)
\end{align*}$$

where the parameter $\Omega_0, \eta_0, \eta_1, \eta_2, \lambda_3, \lambda_3, I_0, c_0$ and $c_1$ are constant parameters related to those of Eq. (1). A set of coupled nonlinear ODEs Eq. (4) is now similar to those that generally describe the dynamics of atomic chain. In general, the solutions of Eq. (4) can be obtaining using perturbation techniques. In that sense, we introduce the following variables $u_n = \varepsilon \psi_n$.
\[ w_n = \varepsilon \beta_n, \quad \text{where } \varepsilon << 1 \text{ is a smallness parameter. By keeping in the development the first two nonlinear terms, the governing equations then become} \]

\[
\begin{align*}
\ddot{\psi}_n + \Omega_0^2 \psi_n + \varepsilon (\varepsilon \eta_0 + \eta_1 \psi_n + \varepsilon \eta_2 \psi_n^2) \dot{\psi}_n + \varepsilon \lambda_1 \psi_n^2 + \varepsilon^2 \frac{\eta_0}{3} \psi_n^3 + \varepsilon^2 \lambda_3 \beta_n &= \sum_{m=1,m \neq n}^{N} \frac{1}{|n-m|^\alpha + 1} \left[ c_0 (\psi_n - \psi_m) + \varepsilon^2 c_1 (\dot{\psi}_n - \dot{\psi}_m) \right] \\
\dot{\beta}_n + r \beta_n - \Omega_0^2 \psi_n &= 0.
\end{align*}
\]

(5)

While writing Eq. (5), the coupling parameter \( \lambda_3 \) of the membrane potential with the bursting variable has been perturbed of order \( \varepsilon^2 \), taking into account the fact that the variation of the bursting variable is slower than the one of the membrane potential. In addition, as we are interested by an analysis in a weakly dissipative medium, we have assumed the parameters \( \eta_0 \) and \( c_1 \) to be perturbed at the order \( \varepsilon^2 \).

3. Equation of motion: The derivation of the complex fractional Ginzburg-Landau equation

The non-locality features of the medium often impose the necessity of using non-traditional tools. In that follows, we first assume the following solutions for Eq. (5):

\[
\begin{align*}
\psi_n &= B_n^{(1)} e^{i \theta_n} + B_n^{(2)*} e^{-i \theta_n} + \varepsilon [C_n^{(1)} e^{2i \theta_n} + D_n^{(1)*} e^{-2i \theta_n}] + 0(\varepsilon^2) \\
\beta_n &= B_n^{(2)} e^{i \theta_n} + B_n^{(2)*} e^{-i \theta_n} + \varepsilon [C_n^{(2)} e^{2i \theta_n} + D_n^{(2)*} e^{-2i \theta_n}] + 0(\varepsilon^2),
\end{align*}
\]

(6)

with \( \theta_n = -kn - \Omega t \), where \( k \) is the normal mode wave vector and \( \Omega \) is the angular velocity of the wave. The variable \( t \) is rescaled as \( t \rightarrow \varepsilon^2 t \).
In the following, we replace the solutions \((6)\) and their derivatives in the new membrane potential equation of motion given by the first equation of Eq. \((5)\). We then group terms in the same power of \(\varepsilon\), which leads us to a system of equations. Each of these equations will correspond to each approximation for specific harmonics. To reach this goal, we consider the infinite network of neural oscillators \((N \to \infty)\) and introduce the following functions

\[
\begin{align*}
    f^j(k,t) &= \sum_{n=-\infty}^{\infty} e^{-ink} B_n^{(j)}(t) \\
    g^j(k,t) &= \sum_{n=-\infty}^{\infty} e^{-ink} C_n^{(j)}(t) \\
    h^j(k,t) &= \sum_{n=-\infty}^{\infty} e^{-2ink} D_n^{(j)}(t) \\
    \tilde{J}_\alpha(k) &= \sum_{n=-\infty}^{\infty} e^{-ink} \frac{1}{|n|^{\alpha+1}},
\end{align*}
\]

with \(j = 1, 2, \) and

\[
\tilde{J}_\alpha(0) = 2 \sum_{n=1}^{\infty} \frac{1}{|n|^\alpha} = 2\zeta(\alpha + 1),
\]

where \(\zeta(\alpha)\) is the Riemann zeta function.

In the long-wave limit, we may adopt \(f^j(k,t)\), \(g^j(k,t)\) and \(h^j(k,t)\) as a \(k^{th}\) Fourier component of continuous function \(B^j(x,t)\), \(C^j(x,t)\) and \(D^j(x,t)\), respectively such that \(B_n^j(t) \to B^j(x,t)\), \(C_n^j(t) \to C^j(x,t)\) and \(D_n^j(t) \to D^j(x,t)\). The functions are related each other by the Fourier transforms such that
\[
\left\{
\begin{align*}
B^j(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} f^j(k,t) dk \\
C^j(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} g^j(k,t) dk \\
D^j(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} h^j(k,t) dk 
\end{align*}
\] (8)

After some development and simplifications, terms without exponential dependence give

\[
C^1 = -\frac{2\lambda_1}{\Omega_0^2} |B^1|^2
\]

At the order \(\varepsilon^1\) after the annihilation of terms in \(e^{\pm i\theta}\), we obtain the relation

\[
\Omega^2 = \Omega_0^2 + c_0 a_\alpha |k|^\alpha,
\] (9)

which determines the dispersion relation of linear waves of the system. As displayed in Fig. 1, the corresponding linear spectrum is reduced when \(\alpha\) increases. To obtain Eq. (9), we have used the infrared approximation [15, 20]

\[
[\tilde{J}_\alpha(0) - \tilde{J}_\alpha(k)] \approx a_\alpha |k|^\alpha, \quad (0 < \alpha < 2, \alpha \neq 1)
\] (10)

where \(a_\alpha = 2\Gamma(-\alpha) \cos(\pi \alpha/2)\).

Terms with \(e^{2i\theta}\) give the relation

\[
D^1 = \frac{\lambda_1 - i\Omega \eta_1}{\Omega_0^2} (B^1)^2
\]

For Eq. (4), the terms with \(e^{i\theta}\) give

\[
B^2 = \frac{\Omega_0^2 (r + i\Omega) B^1}{r^2 + \Omega^2}
\]
Figure 1: The dispersion relation of the nerve impulse according to three different values of $\alpha$, namely $\alpha = 1.6$, $\alpha = 1.7$ and $\alpha = 1.8$. $\Omega_0^2 = 0.032$ and $c_0 = 0.001$. It appears that the long-rang parameter $\alpha$ affects the dispersion area.

Collecting all the terms depending on $e^{i\theta}$ in Eq. (5) at the second order of the perturbation, we obtain the following equation

\[-2i\Omega \frac{\partial B^1}{\partial t} = i\Omega \eta_0 B^1 + (i\Omega \eta_1 - 2\lambda_1)(B^1 C^1 + B^{1*} D^1) + (i\Omega - 1)\eta_2 |B^1|^2 B^1 - \lambda_3 \frac{\Omega_0^2 (r + i\Omega)}{r^2 + \Omega^2} B^1 - iC_1 \Omega a_\alpha |k|^\alpha B^1.\]

(11)

Rewriting this equation taking into account the connection between the Riesz fractional derivative and its Fourier transform [21]

\[|k|^\alpha = -\frac{\partial^\alpha}{\partial |x|^\alpha},\]  

(12)

we obtain
\[ \frac{\partial B^1}{\partial t} = \gamma B^1 + P_r \frac{\partial^\alpha B^1}{\partial |x|^\alpha} - Q|B^1|^2 B^1, \]  
(13)

where the coefficients \( \gamma \), \( P_r \) and \( Q \) are given by
\[ \gamma = \gamma_r + i \gamma_i, \quad P_r = c_1 a_\alpha / 2 \quad \text{and} \quad Q = Q_r + i Q_i. \]

The coefficients \( \gamma_r \) and \( \gamma_i \) are the real and imaginary parts of the dissipation coefficient, for the nonlinearity coefficient the same terminology is used. The coefficients \( Q_r, Q_i, \gamma_r \) and \( \gamma_i \) are given by
\[ \gamma_r = \frac{\lambda_3 \Omega_0^2 - \eta_0}{2(r^2 + \Omega^2)} - \frac{\eta_0}{2}, \quad \gamma_i = -\frac{r \lambda_3 \Omega_0^2}{2 \Omega (r^2 + \Omega^2)}; \]
\[ Q_r = \frac{\eta_2}{2} + \frac{\eta_1 \lambda_1}{\Omega_0^2}, \quad Q_i = \frac{1}{w} \left( \frac{\eta_2}{2} - \frac{\Omega^2 \eta_1^2}{2 \Omega_0^2} - \frac{\lambda_1^2}{\Omega_0^2} \right). \]

Equation (13) which is a new general theoretical framework derived in our neural network is the well known complex fractional Ginzburg-Landau equation. In this work, we confirm once more the fact that using the Fourier transforms and infrared limit, the long-range term interaction leads under special conditions to the fractional dynamics \[15, 20, 21\]. The fractional Ginzburg-Landau equation has been initially proposed by Weitzner and Zaslavsky \[22\] to describe the dynamical processes in a medium with fractal dispersion. In \[23\] the fractional Ginzburg-Landau equation is derived from the variational Euler Lagrange equation for fractal media. In the present work, the motion of modulated waves are proven to be described by the complex fractional Ginzburg-Landau equation (CFGL) equation. Thus, the infrared limit of an infinite network of neural oscillators with the long-range coupling can be described by equations with the fractional Riesz coordinate derivative of order \( \alpha < 2 \). To the best of our knowledge, this is the first research work that attempts to describe the dynamical behavior of neural
networks with equation of fractional order. The result suggests that neurons can participate in a collective processing of long-scale information, a relevant part of which is shared over all neurons. The brain may actively work effectively using the spatial dimension for information processing but not only in time domain [24, 25, 26].

The diffusively HR neural network with nearest neighbors coupling admit plane wave and spatial localized solutions [4]. In the present work, the CFGL equation (13) admits plane wave solution in the form [15]

\[ B_1(x,t) = \sqrt{\frac{\gamma_r - P_r|k|^\alpha}{Q_r}} e^{i(kx - \omega_\alpha(k)t + \theta_0)}. \] (14)

where \( \omega_\alpha(k) = \frac{i}{Q_i}(Q_i\gamma_r - Q_r\gamma_i - Q_iP_r|k|^\alpha) \). The parameter \( k \) is a fixed wave number and \( \theta_0 \) is a constant parameter.

The plane wave solution Eq. (14) is stable if the parameters \( \gamma_r, \gamma_i, Q_i \) and \( P_r \) satisfy the condition

\[ 0 < \gamma_r - P_r|k|^\alpha < \frac{\gamma_i}{Q_i} < 3(\gamma_r - P_r|k|^\alpha). \] (15)

This condition defines the region of parameters for plane waves where the synchronization exists [15].

4. Localized numerical wave solutions

In the previous section, we have demonstrated that the HR neural network can be elegantly described by the CFGL equation which admits plane wave solutions. In general, analytical and closed solutions of fractional equations cannot be obtained. In that case, numerical techniques are used to identify the solution behavior of such fractional equations. The interest in
investigating and controlling the propagation of waves in neural tissue has been increasingly growing during the last decades. In this section, we use the semi-implicit Riesz fractional finite-difference scheme to find numerically localized wave solutions for the CFGL equation (13).

4.1. The Semi-implicit Riesz fractional finite-difference scheme

Let us reconsider our CFGL equation in a mathematical setting

\[
\begin{align*}
\frac{\partial B^1}{\partial t} &= P_r \frac{\partial^\alpha B^1}{\partial |x|^\alpha} + (\gamma_r + i\gamma_i)B^1 - (Q_r + iQ_i)|B^1|^2B^1, \\
B^1(x, 0) &= g(x), \quad x \in (0, b), \\
B^1(0, t) &= B^1(b, t) = 0, \quad t \in [0, T],
\end{align*}
\]

where homogeneous periodical boundary conditions are used without loss of generality. Note that the function \( g \) is the initial solution, \( T > 0 \) is the final time and \( \frac{\partial^\alpha}{\partial |x|^\alpha} \) is space Riesz fractional derivative of order \( \alpha \) given for \( 0 < \alpha < 2 \), \( \alpha \neq 1 \) by

\[
\frac{\partial^\alpha}{\partial |x|^\alpha} = -c_\alpha \left( -\infty D^\alpha_x + x D^\alpha_{+\infty} \right),
\]

where the coefficient

\[
\begin{align*}
c_\alpha &= \frac{1}{2 \cos(\alpha \pi/2)} \\
-\infty D^\alpha_x B^1 &= \left( \frac{d}{dx} \right)^m \left[ \infty I_x^{m-\alpha} B^1(x, t) \right] \\
x D^\alpha_{+\infty} B^1 &= (-1)^m \left( \frac{d}{dx} \right)^m \left[ x I_{+\infty}^{m-\alpha} B^1(x, t) \right],
\end{align*}
\]

with \( m \in \mathbb{N} \) such that \( m - 1 < \alpha \leq m \). The terms \( -\infty D^\alpha_x \) and \( x D^\alpha_{+\infty} \) are respectively the left and the right side Riemann-Liouville fractional derivatives. The left and right side Weyl fractional integrals used in Eq. (18) are
defined by
\[ I^{m-\alpha}_x B^1(x, t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \zeta)^{\alpha-1} B^1(\zeta, t) d\zeta \]
\[ I^{m-\alpha}_{+\infty} B^1(x, t) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} (x - \zeta)^{\alpha-1} B^1(\zeta, t) d\zeta \] (19)

By setting \( B^1 = U + iV \), where \( U \) and \( V \) are respectively the real and imaginary parts of \( B^1 \), Eq. (16) is equivalent to the following coupled system
\[
\begin{align*}
\frac{\partial U}{\partial t} &= P_r \frac{\partial^\alpha U}{\partial |x|^\alpha} + \gamma_r U - \gamma_i V - (Q_r U - Q_i V) (U^2 + V^2) \\
\frac{\partial V}{\partial t} &= P_r \frac{\partial^\alpha V}{\partial |x|^\alpha} + \gamma_i U + \gamma_r V - (Q_i U + Q_r V) (U^2 + V^2)
\end{align*}
\] (20)

Let us use the following identification \( B^1 \equiv (U, V)^T \). By setting
\[
A = \begin{pmatrix}
P_r \frac{\partial^\alpha}{\partial |x|^\alpha} & -\gamma_i \\
\gamma_i & P_r \frac{\partial^\alpha}{\partial |x|^\alpha}
\end{pmatrix}
\] (21)
\[
F(B^1) = \begin{pmatrix}
-(Q_r U - Q_i V) (U^2 + V^2) \\
-(Q_i U + Q_r V) (U^2 + V^2)
\end{pmatrix}
\] (22)

the coupled system Eq. (20) becomes
\[
\begin{align*}
\frac{\partial B^1}{\partial t} &= AB^1 + F(B^1) \\
B^1(x, 0) &= g(x), \quad x \in (0, b) \\
B^1(0, t) &= B^1(b, t) = (0, 0)^T.
\end{align*}
\] (23)

For space discretization, we used the weighted Riesz fractional finite-difference approximation as presented in [17]. We divide the interval \((0, b)\) into \( M \) sub-interval with the step \( h = b/M \). In order to perform the space discretization
with our homogeneous boundary conditions, the function $B_1$ should be extended to the whole $\mathbb{R}$ (see [17]) as

$$B_1^*(x, t) \equiv (U^*, V^*)^T = \begin{cases} 
B_1(x, t), & x \in (0, b) \\
(0, 0)^T, & x \in (-\infty, 0) \cup (0, +\infty).
\end{cases} \tag{24}$$

Using Eq. (24), each component of the function $B_1$ can be discretized by the centered finite difference as follows for $0 < \alpha < 2, \alpha \neq 1$

$$\frac{\partial^\alpha B_1}{\partial |x|^\alpha} = \begin{pmatrix}
-\frac{1}{h^\alpha} \sum_{-\infty}^{+\infty} w_k^\alpha U^*(x-kh,t) + O(h^2) \\
-\frac{1}{h^\alpha} \sum_{-\infty}^{+\infty} w_k^\alpha V^*(x-kh,t) + O(h^2)
\end{pmatrix}. \tag{25}$$

Since $B_1^*(x, t) = (0, 0)^T$ for $x \in (-\infty, 0) \cup (0, +\infty)$, we therefore have

$$\frac{\partial^\alpha B_1}{\partial |x|^\alpha} = \begin{pmatrix}
-\frac{1}{h^\alpha} \sum_{-(b-x)/h}^{(x-0)/h} w_k^\alpha U(x-kh,t) + O(h^2) \\
-\frac{1}{h^\alpha} \sum_{-(b-x)/h}^{(x-0)/h} w_k^\alpha V(x-kh,t) + O(h^2)
\end{pmatrix}. \tag{26}$$

where

$$w_k^\alpha = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\alpha/2 - k + 1) \Gamma(\alpha/2 + k + 1)}. \tag{27}$$

Denote by $U_i^h(t)$ and $V_i^h(t)$ the approximated values of $U(x_i, t)$ and $V(x_i, t)$ respectively, the central finite difference approximation is therefore given by

$$\frac{\partial^\alpha B_1(x_i, t)}{\partial |x|^\alpha} \approx \begin{pmatrix}
-\frac{1}{h^\alpha} \sum_{i-M+i} w_k^\alpha U_{i-k}^h(t) \\
-\frac{1}{h^\alpha} \sum_{i-M+i} w_k^\alpha V_{i-k}^h(t)
\end{pmatrix}. \tag{28}$$

By setting $U_h = (U_i^h)_{1 \leq i \leq M}$, $V_h = (V_i^h)_{1 \leq i \leq M}$ and $B_1^h = (U_h, V_h)^T$, the semi-discrete version of Eq. (23) after space discretization is given by

$$\begin{align*}
\frac{dB_1^h}{dt} &= A_h B_1^h + F(B_1^h) \\
B_1^h(0) &= (g(x_i))_{1 \leq i \leq M}.
\end{align*} \tag{29}$$
where

\[ A_h = \begin{pmatrix} -P + \gamma I & -\gamma I \\ \gamma I & -P + \gamma I \end{pmatrix} \]  

(30)

\[ P = (p_{i,j})_{1 \leq i,j \leq M-1}, \quad p_{i,j} = \frac{P r}{h^\alpha w_{i-j}^\alpha}. \]  

(31)

Note that \( I \) is the \((M-1) \times (M-1)\) identity matrix. Please also note that the matrix \( A_h \) is more than 50 % full.

Let \( N \) being the time subdivision, we use the constant time step \( \tau = T/N \). In order to fully discretize Eq. (23), let \( B_{h,n}^1 \) being our approximated solution of \( B^1(n\tau) = (B^1(x_i, n\tau))_{1 \leq i \leq M-1} \). From Eq. (29), the \( \theta \)-Euler Riesz fractional finite-difference scheme to approximate Eq. (23) is given by

\[ B_{h,n}^1 + \frac{\tau}{\theta} \left( A_h B_{h,n}^1 + F(B_{h,n}^1) \right) + (1 - \theta) \left( A_h B_{h,n+1}^1 + F(B_{h,n+1}^1) \right) = 0 \leq \theta \leq 1. \]  

(32)

For \( \theta = 1 \), the scheme is an explicit Riesz fractional finite-difference scheme, while for \( \theta = 0 \), the scheme is a fully implicit Riesz fractional finite-difference scheme. The high order accuracy in time is obtained for \( \theta = \frac{1}{2} \), which corresponds to the Crank-Nicholson Riesz fractional finite-difference approximation scheme. Note the for \( \theta = 0 \), the corresponding explicit scheme is only stable for very small time step \( \tau \). For \( \theta \neq 1 \), the scheme is more stable, but the fact that the matrix \( A_h \) is more than 50 % full makes the Newton iterations less efficient.

To solve the efficiency drawback of the implicit schemes, we propose in this work the semi-implicit Riesz fractional finite-difference scheme where the linear part of Eq. (29) is approximated implicitly and the nonlinear part explicitly. Following ([27, 28, 29]), the corresponding scheme is given by

\[ \frac{B_{h,n+1}^1 - B_{h,n}^1}{\tau} = A_h B_{h,n+1}^1 + F(B_{h,n}^1). \]  

(33)
Obviously the semi-implicit scheme given at Eq. (33) is very efficient than the implicit schemes given in Eq. (32) for \( \theta \neq 1 \), as only one linear system is solved per time iteration.

4.2. Numerical simulations

All numerical simulations in this work will be performed using the semi-implicit Riesz fractional finite-difference given at Eq. (33). We choose the initial condition of \( B^1(x, t) \) in the form of a solitary wave solution \([30, 31]\)

\[
B^1(x, 0) = B_0 \left[ \frac{e^{-kx} + \cos(2\mu k x)e^{-kx}}{2\cosh(2kx) + \cos(2\mu k x)} - \frac{\sin(2\mu k x)e^{-kx}}{2\cosh(2kx) + \cos(2\mu k x)} \right], \tag{34}
\]

where \( \mu = \beta + \sqrt{2 + \beta^2} \) and \( \beta = \frac{3Q_i}{2Q_r} \).

The parameter values used are \([4]\): \( \Omega_0^2 = 0.032 \), \( \lambda_1 = 0.01, \lambda_3 = 0.023 \), \( \eta_0 = 0.1, \eta_1 = 0.001, \eta_2 = 0.15, r = 0.008 \), \( B_0 = 0.5, c_0 = 0.001 \) and \( c_1 = 0.001 \).

Figure 2 displays the spatio-temporal evolution on the initial solution for \( \alpha = 1.8 \). We have assumed the dissipation term to be purely real \((\gamma_i = 0)\), that is we neglect the motion of ions and protein anions across slow ions channels. We observe in this figure that the solution is well localized nonlinear excitation in space and time, it has the shape of a short pulse and propagates without any change of its profile. It is clear from there that as time evolves, the form of the pulse does not change; it is structurally stable.

Figure 3 displays spatial profiles of the amplitude of the solution for four distinct values of the parameters \( \alpha \), namely \( \alpha = 1.5, \alpha = 1.8, \alpha = 1.9 \) and \( \alpha = 1.99 \). We observe in the graphs of Figure 3 that the solution is
well localized in space with the shape of a short pulse. However, as $\alpha$ increases, profiles of the solution reflect more and more short impulses with well localized shapes, consistently with the reduction in the space course. On Figure 3(d), the pulse shape of the solution is clearly apparent at positive spaces for sufficiently large values of the parameter $\alpha$ in contrast to Figure 3(a), Figure 3(b) and Figure 3(c), where the shape of the pulse is not more pronounced. Remarkably, the pulse profiles in Figure 3 are in qualitative agreement with the typical results reported in electrodynamics theory in both myelinated and myelin-free nerve fiber contexts \[3\].

Figure 4 displays the effect of the imaginary term of the dissipation in the solution. In figure 4a, we have chosen $\alpha = 1.8$ and $\gamma_i \neq 0$. We observe that the imaginary term of the dissipation can not act on the dynamics of the pulse. The amplitude of the pulse is not altered. The same behavior is observed in Figure 4(b) where we have chosen $\alpha = 1.99$.

Figure 2: Profile of the numerical solution $|B^1(x,t)|^2$ according to time and space for $\alpha = 1.8$ and $\gamma_i = 0$. Parameter values are given in the text.
Figure 3: Spatial profile of the numerical solution $|B^1(x,t)|^2$ of Eq. (11) for $t=0.2$ with initial nonlinear localized solution, according to 4 values of $\alpha$. (a) $\alpha = 1.5$, (b) $\alpha = 1.8$, (c) $\alpha = 1.9$, (d) $\alpha = 1.99$.

Figure 4: The effect of the imaginary part of the dissipation coefficient on $|B^1(x,t)|^2$. (a) $\alpha = 1.8$, $\gamma_i \neq 0$. (b) $\alpha = 1.99$, $t=0.5$. 

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5. Conclusion

The goal of this paper was to study the nonlinear dynamics of diffusively Hindmarsh-Rose neural network with long-range coupling. Performing a perturbation technique, we have shown that the dynamics of modulated waves in our neural network can be elegantly described by the complex fractional Ginzburg-Landau equation. The result has confirmed the fact that the brain may actively work also effectively use the spatial dimension for information processing. In general, exact analytical solutions of fractional nonlinear equations cannot be obtained. We have proposed the semi implicit Riesz fractional finite-difference scheme to solve efficiently the complex fractional Ginzburg-Landau equation. It has been revealed that the numerical solutions for the nerve impulse are well-localized short impulses whose shape and stability depend on the value of the long-range parameter. The fractional properties observed in our neural network may be advantageous in excitable systems for crucial intuitions into spatio-temporal dynamics, synchronization and chaos. The approaches used in this work give also the opportunity to familiarize with improved fractional analytical and numerical methods which can be used to study other physical systems with long-range interactions.

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References

[1] S. Raghavachari and J.A. Glazier. Waves in Diffusively Coupled Bursting Cells. *Phys. Rev. Lett.* **82**, 2991 (1999).

[2] V.A. Makarov, V.I. Nekorkin, M.G. Velarde. Spiking Behavior in a Noise-Driven System Combining Oscillatory and Excitatory Properties. *Phys. Rev. Lett.* **86**, 431 (2001).

[3] A.M. Dikande and G.A. Bartholomew. Localized short impulses in a nerve model with self-excitable membrane. *Phys. Rev. E*, **80**, 041904 (2009).

[4] F.M. Moukam Kakmeni, E.M. Inack, E.M. Yamakou. Localized nonlinear excitations in diffusive Hindmarsh-Rose neural networks. *Phys. Rev. E* **89**, 052919 (2014).

[5] A. Destexhe, A. Babloyantz, T.J. Sejnowski. Ionic mechanisms for intrinsic slow oscillations in thalamic relay neurons. *Biophys. J.* **65**, 1538 (1993).

[6] Z.P. Kilpatrick and P.C. Bressloff. Stability of bumps in piecewise smooth neural fields with nonlinear adaptation. *Physica D* **239**, 547 (2010).

[7] D.D. Clarke and L. Sokoloff. Circulation and Energy Metabolism of the Brain. In: G.J. Siegel *et al.* (Eds.) *Basic Neurochemistry*: 6th edition,
Molecular, Cellular and Medical Aspects, Lippincott-Raven, Philadelphia, 1999.

[8] J.L. Hindmarsh and R.M. Rose. A model of neuronal bursting using three coupled first order differential equations. *Proc. R. Soc. London, Ser. B* 221, 87-102 (1984).

[9] A.L. Hodgkin and A.F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. *J. Physiol. (London)* 117, 500-544 (1952).

[10] R. Fitzhugh. Impulses and Physiological States in Theoretical Models of Nerve Membrane. *Biophys. J.* 1, 445-466 (1961).

[11] E.R. Kandel, J.M. Schwartz, T.M. Jessel. Principles of neural Science. 4th edn. McGraw Hill, 2000.

[12] T. Periera, M.S. Baptista, J. Kurths. Multi-time-scale synchronization and information processing in bursting neuron networks. *Eur. Phys. J. Special Topics* 146, 155-168 (2007).

[13] M. Freeman and J.B. Gurdon. Regulatory principles of developmental signaling. Regulatory principles of developmental signaling. *Annu. Rev. Cell Dev. Biol.* 18, 512 (2002);

[14] T. Heimburg and A.D. Jackson. On soliton propagation in biomembranes and nerves. *Proc. Natl. Acad. Sci. U.S.A.* 102 (2), 9790 (2005) role of anesthetics. *Biophys. Rev. Lett.* 2, (2007) 5778;
[15] V.E. Tarasov and G.M. Zaslavsky. Fractional dynamics of systems with long-range interaction. *Commun. Nonlinear Sci. Numer. Simul.* 11, 885-898 (2006).

[16] F.A. Rihan, Numerical modeling of fractional-Order biological systems. *Abstract and Applied Analysis*, Vol. 2013, Article ID 816803.

[17] S. Shen, F. Liu, V.Anh, I. Turner, and J. Chen. A novel numerical approximation for the space fractional advection-dispersion equation. *IMA journal of Applied Mathematics*, 79, 431-444 (2014).

[18] F.J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet. *Commun. Math. Phys.* 12, 91-107 (1969).

[19] A. Mvogo, G.H. Ben-Bolie, T.C. Kofane. Coupled fractional nonlinear differential equations and exact Jacobian elliptic solutions for exciton phonon dynamics. *Phys. Lett. A* 378, 2509 (2014).

[20] N. Laskin and G.M. Zaslavsky. Nonlinear fractional dynamics on a lattice with longrange interactions. *Physica A* 368, 3814 (2006).

[21] S.G. Samko, A.A. Kilbas, O.I. Marichev. Fractional Integrals and Derivatives Theory and Applications. *Gordon and Breach, New York*, 1993.

[22] H. Weitzner and G. M. Zaslavsky. Some applications of fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.* 8, 273 (2003).

[23] V.E. Tarasov and G.M. Zaslavsky. Fractional Ginzburg-Landau equation for fractal media. *Physica A*, 354, 249261 (2005).
[24] C.D. Negro, C.-F. Hsiao, S. Chandler, A. Garfinkel. Evidence for a novel bursting mechanism in rodent trigeminal neurons. *BioPhys. J.* **75**, 174-182 (1998).

[25] C.M. Pedroarena *et al.*. Oscillatory membrane potential activity in the soma of a primary afferent neuron. *J. Neurophysiology* **82**(1465) (1999).

[26] A.K. Al Azad and P. Ashwin. Within-burst synchrony changes for coupled elliptic bursters. *SIAM J. Appl. Dyn. Syst.* **9**, 261 (2010).

[27] A. Tambue, Efficient Numerical schemes for Porous Media Flow. *PhD thesis, Department of Mathematics, Heriot–Watt University*, 2010.

[28] A. Tambue, G. J. Lord, S. Geiger. An exponential integrator for advection-dominated reactive transport in heterogeneous porous media. *Journal of Computational Physics*, **229**(10), 3957-3969 (2010).

[29] A. Tambue, S. Geiger and G. J. Lord. Exponential Time integrators for 3D Reservoir Simulation. *In proceedings of the 12th European Conference on the Mathematics of Oil Recovery, Oxford, UK*, 2010.

[30] K. Nozaki and N. Bekki. Exact Solutions of the Generalized Ginzburg-Landau Equation. *J. Phys. Soc. Jpn.* **53**, 1581-1582 (1984).

[31] N. R. Pereira and L. Stenflo. Nonlinear Schrödinger equation including growth and damping. *Phys. Fluids* **20**, 1733-1734 (1977).