ERDÉLYI-KOBER INTEGRALS ON THE CONE OF POSITIVE
DEFINITE MATRICES AND RADON TRANSFORMS ON
GRASSMANN MANIFOLDS

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Abstract. We introduce bi-parametric fractional integrals of the Erdélyi-
Kober type that generalize known Gårding-Gindikin constructions associated
to the cone of positive definite matrices. It is proved that the Radon trans-
form, which maps a zonal function on the Grassmann manifold $G_{n,m}$ of $m$-
dimensional linear subspaces of $\mathbb{R}^n$ into a function on the similar manifold
$G_{n,k}, 1 \leq m < k \leq n - 1$, is represented as analytic continuation of the cor-
responding Erdélyi-Kober integral. This result shows that different Grinberg-
Rubin’s formulas for such transforms [GR] have, in fact, a common structure.

1. Introduction

Radon transforms of different kinds have a long history and numerous applica-
tions; see [Eh], [GGG], [GGV], [H1], [Ru2], and references therein. In the present
paper, we focus on important connection between Radon transforms on Grass-
mann manifolds and higher rank fractional integrals. Let $G_{n,m}$ and $G_{n,k}$ be a pair
of Grassmann manifolds of $m$-dimensional and $k$-dimensional linear subspaces of
$\mathbb{R}^n$, respectively; $1 \leq m < k \leq n - 1$. We use the notation $\tau_m$ and $\tau_k$ for the
respective elements of these Grassmannians. The Radon transform of a function $f$
on $G_{n,m}$ is a function $Rf$ on $G_{n,k}$ defined by

$$ (Rf)(\tau_k) = \int_{\{\tau_m : \tau_m \subset \tau_k\}} f(\tau_m) d\tau_m, $$

where $\tau_k \in G_{n,k}$ and $d\tau_m$ is the relevant probability measure. For $m = 1$, a function
$f$ on $G_{n,1}$ can be identified with an even function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.
In this case, $Rf$ represents the totally geodesic transform that assigns to a function $f$
on the unit sphere $S^{n-1}$ its integrals over the set of $(k - 1)$-dimensional totally
geodesic submanifolds of $S^{n-1}$. Different aspects of the Radon transform (1.1) were
investigated by Gel’fand and collaborators [GGR], [GGS], Grinberg [Gr], Grinberg
and Rubin [GR], Kakehi [K1], [K2], Petrov [P1], Zhang [Zh1], and others.

There is a remarkable connection between the Radon transform (1.1) and the
following Gårding-Gindikin fractional integrals associated to the cone $P_\ell$ of positive

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define symmetric $\ell \times \ell$ matrices:

\begin{equation}
(I^\alpha f)(s) = \frac{1}{\Gamma_\ell(\alpha)} \int_0^s f(r) \det(s - r)^{\alpha - (\ell + 1)/2} dr,
\end{equation}

\begin{equation}
(I^\alpha f)(s) = \frac{1}{\Gamma_\ell(\alpha)} \int_s^{I_\ell} f(r) \det(r - s)^{\alpha - (\ell + 1)/2} dr.
\end{equation}

Here $I_\ell$ is the identity $\ell \times \ell$ matrix,

\begin{align*}
\int_0^s = \int_{\mathcal{P}_\ell \cap (s - \mathcal{P}_\ell)}, & \quad \int_s^{I_\ell} = \int_{(s + \mathcal{P}_\ell) \cap (I_\ell - \mathcal{P}_\ell)}, & \quad s \in \mathcal{P}_\ell;
\end{align*}

$\Gamma_\ell(\alpha)$ is the Siegel gamma function \[2.2\]; see [Ga], [Gl], [OR1] for more details. For sufficiently good $f$, the integrals $I^\alpha f$ converge absolutely if $\Re \alpha > (\ell - 1)/2$, and extend to all $\alpha \in \mathbb{C}$ as entire functions of $\alpha$.

The following result from [GR] is of our main concern. A function $f$ on $G_{n,m}$ is canonically identified with right $O(m)$-invariant function on the Stiefel manifold $V_{n,m}$ of $n \times m$ real matrices $v$ satisfying $v'v = I_m$, $v'$ being the transpose of $v$.

Abusing notation, we write $f(\tau_m) = f(v)$. Fix an integer $\ell$ so that $1 \leq \ell \leq k - m$ and suppose that $f$ is $\ell$-zonal, i.e., $f(v) = f_0(r)$, where $r = \sigma'_r v' \sigma_r$, $\sigma_r = \begin{bmatrix} 0 & I_\ell \\ I_\ell & \end{bmatrix} \in V_{n,\ell}$.

It is proved in [GR] Theorem 4.5, that for $m \geq \ell$, $\mathcal{R} f$ is represented by the Gårding-Gindikin integral associated to $\mathcal{P}_\ell$, namely,

\begin{equation}
(\mathcal{R} f)(\tau_k) = c_1 \det(s)^{\ell - k/2}(I_+^{(k - m)/2}) f_0(s), \quad s = \sigma'_r \mathcal{P}_\ell \sigma_r,
\end{equation}

\[\tilde{f}_0(r) = \det(r)^{(m - \ell - 1)/2} f_0(r), \quad r \in \mathcal{P}_\ell, \quad c_1 = \frac{\Gamma_\ell(k/2)}{\Gamma_\ell(m/2)},\]

$\mathcal{P}_\ell \sigma_r$ denotes the orthogonal projection on $\tau_k$. In the case $m < \ell$, when $\text{rank}(r) < \ell$, the following formula was obtained in [GR] Theorem 4.5:

\begin{equation}
(\mathcal{R} f)(\tau_k) = c_2 \int_0^1 \det(I_m - r)^{\gamma} \det(r)^{\delta} dr \int_{V_{\ell,m}} f_0(s^{1/2} u r u' s^{1/2}) du,
\end{equation}

$\gamma = (\ell - m - 1)/2$, $\delta = (k - m - \ell - 1)/2$, $c_2 = 2^{-m} \pi^{\ell m/2} \Gamma_\ell(k/2) / \Gamma_\ell((k - \ell)/2)$.

Our aim is to show that right hand sides of (1.4) and (1.5) can be regarded as different forms of the same fractional integral, which is introduced below. The latter resembles well known Erdélyi-Kober operators in fractional calculus [SKM].

**Plan of the paper and main results.** Section 2 contains preliminaries. In Section 3, we introduce the following weighted versions of the Gårding-Gindikin integrals:

\begin{equation}
(J^\alpha f)(s) = \frac{\det(s)^{d - \alpha - \beta}}{\Gamma_\ell(\beta)} (\mathcal{R}_+^\alpha g_\beta)(s),
\end{equation}

\begin{equation}
g_\beta(r) = \det(r)^{\beta - d} f(r), \quad d = (\ell + 1)/2.
\end{equation}

For $m = 1$, $J^\alpha f$ coincide up to $1/\Gamma_\ell(\beta)$ with the classical Erdélyi-Kober fractional integrals; see [SKM]. The newly introduced normalizing factor $1/\Gamma_\ell(\beta)$ is needed for
analytic continuation of $J^{\alpha,\beta}_\pm f$ in the $\beta$-variable. We call (1.6) *fractional integrals of the Erdélyi-Kober type.*

If $f$ is good enough, then integrals $J^{\alpha,\beta}_\pm f$ converge absolutely for $Re \alpha, Re \beta > d - 1$, and extend as entire functions of $\alpha$ and $\beta$. We obtain explicit representations of $J^{\alpha,m/2}_\pm f$, $m \in \mathbb{N}$, provided that $Re \alpha > d - 1$, see (3.4), (3.5). This allows us to define $J^{\alpha,\beta}_\pm f$ for $Re \alpha > d - 1$ and $\beta$ belonging to the Wallach-like set $\mathbb{W}$

$$(1.7) \quad \mathbb{W}^\beta = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{d-1}{2} \right\} \cup \left\{ \beta : Re \beta > \frac{d-1}{2} \right\}$$

see Definitions 3.4 and 3.6.

In Section 4, we establish connection between the Radon transform of $\ell$-zonal functions and integrals (1.6).

**Theorem 1.1.** Let $f$ be an integrable $\ell$-zonal function on $V_{n,m}$, that is, $f(v) \equiv f_0(r)$, $r = \sigma_i^j v^j \sigma_i$. If $1 \leq \ell \leq k - m$, then

$$(1.8) \quad (\mathcal{R}f)(\tau_k) = \Gamma_{\ell}(k/2) (J^{k-m}_\ell \tau f_0)(s),$$

where $\tau_k \in G_{n,k}$, $s = \sigma_i^j \Pr \tau_s \sigma_i$.

Formula (1.8) obviously coincides with (1.4) in the case $m \geq \ell$. As we shall see below (Remark 4.9), it also includes (1.5) in the case $m < \ell$. An analogue of Theorem 1.1 holds for the dual Radon transform, see Theorem 4.10.

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## 2. Preliminaries

### 2.1. Notation.

Let $\mathcal{W}_{n,m} \sim \mathbb{R}^{nm}$ be the space of $n \times m$ real matrices $x = (x_{i,j})$ with the volume element $dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$. In the following, $x'$ denotes the transpose of $x$, $I_m$ is the identity $m \times m$ matrix, and 0 stands for zero entries. Given a square matrix $a$, we denote by $|a| = \det(a)$ the determinant of $a$, $\text{tr}(a)$ stands for the trace of $a$. We use standard notations $O(n)$ and $SO(n)$ for the orthogonal group and the special orthogonal group of $\mathbb{R}^n$ with the normalized invariant measure of total mass 1.

Let $S_\ell \sim \mathbb{R}^{\ell(\ell+1)/2}$ be the space of $\ell \times \ell$ real symmetric matrices $s = (s_{i,j})$ with the volume element $ds = \prod_{i<j} ds_{i,j}$. We denote by $\mathcal{P}_\ell$ the cone of positive definite matrices in $S_\ell$; $\overline{\mathcal{P}_\ell}$ is the closure of $\mathcal{P}_\ell$, that is the set of all positive semi-definite $\ell \times \ell$ matrices. For $r \in \mathcal{P}_\ell$ ($r \in \overline{\mathcal{P}_\ell}$), we write $r > 0$ ($r \geq 0$). Given $a, b \in S_\ell$, the inequality $a > b$ means $a - b \in \mathcal{P}_\ell$, the symbol $\int_a^b f(s)ds$ denotes the integral over the set $(a + \mathcal{P}_\ell) \cap (b - \mathcal{P}_\ell)$. The group $G = GL(\ell, \mathbb{R})$ of real non-singular $\ell \times \ell$ matrices $g$ acts transitively on $\mathcal{P}_\ell$ by the rule $r \rightarrow grg'$.

The corresponding $G$-invariant measure on $\mathcal{P}_\ell$ is $[\mathbb{I}]$ p. 18]

$$(2.1) \quad d_s r = |r|^{-d} dr, \quad |r| = \det(r), \quad d = (\ell + 1)/2.$$  

The Siegel gamma function of $\mathcal{P}_\ell$ is defined by

$$(2.2) \quad \Gamma_{\ell}(\alpha) = \int_{\mathcal{P}_\ell} \exp(-\text{tr}(r)) |r|^{\alpha} d_s r = \pi^{\ell(\ell+1)/4} \prod_{j=0}^{\ell-1} \Gamma(\alpha - j/2),$$
The relevant beta function has the form

$$B_\ell(\alpha, \beta) = \int_0^\ell |r|^{-d} |r\ell - r\beta|^{-d} dr = \frac{\Gamma_\ell(\alpha) \Gamma_\ell(\beta)}{\Gamma_\ell(\alpha + \beta)}.$$  

(2.3)

These integrals converge absolutely if and only if Re $\alpha$, Re $\beta > d - 1$. For $1 \leq k < \ell$, $k \in \mathbb{N}$, the equality (2.2) yields

$$\frac{\Gamma_\ell(\alpha)}{\Gamma_\ell(\alpha + k/2)} = \frac{\Gamma_k(\alpha + (k - \ell)/2)}{\Gamma_k(\alpha + k/2)}.$$  

(2.4)

For $r = (r_{i,j}) \in \mathcal{P}_\ell$, the differential operators acting in the $r$-variable are defined by

$$D_+ \equiv D_{+, r} = \det \left( \eta_{i,j} \frac{\partial}{\partial r_{i,j}} \right), \quad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 1/2 & \text{if } i \neq j, \end{cases}$$  

(2.5)

and

$$D_- \equiv D_{-, r} = (-1)^\ell D_{+, r}.$$  

(2.6)

In the following, all function spaces on a subspace $S$ of $\mathcal{S}$ are identified with the corresponding spaces on a subspace of $\mathbb{R}^{(\ell+1)/2}$. For instance, $\mathcal{S}(S_\ell)$ denotes the Schwartz space of infinitely differentiable rapidly decreasing functions; $\mathcal{D}(S)$ is the space of functions $f \in C^\infty(S)$ with supp$f \subset S$.

2.2. Stiefel manifolds. For $n \geq m$, let $V_{n,m} = \{ v \in \mathbb{M}_{n,m} : v^t v = I_m \}$ be the Stiefel manifold of orthonormal $m$-frames in $\mathbb{R}^n$. The group $O(n)$ acts transitively on $V_{n,m}$ by the left matrix multiplication. This is also true for $SO(n)$ if $n > m$.

We fix an invariant measure $dv$ on $V_{n,m}$ normalized by

$$\sigma_{n,m} \equiv \int_{V_{n,m}} dv = \frac{2^m m!^{m/2}}{\Gamma_m(n/2)},$$  

(2.7)

in [Mu, p. 70], and denote $d_* v = \sigma_{n,m}^{-1} dv$. The following statement can be found, e.g., in [Herz], [Mu], [FK].

Lemma 2.1. (polar decomposition) Let $x \in \mathbb{M}_{n,m}$, $n \geq m$. If rank$(x) = m$, then

$$x = vv^t 1/2, \quad v \in V_{n,m}, \quad r = x'x \in \mathcal{P}_m,$$

and $dx = 2^{-m} |v|^{(n-m-1)/2} dr dv$.

Lemma 2.2. (bi-Stiefel decomposition). Let $k$, $m$, and $n$ be positive integers satisfying $1 \leq k, m \leq n - 1$, $k + m \leq n$. Almost all matrices $v \in V_{n,m}$ can be represented in the form

$$v = \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix}, \quad a \in \mathbb{M}_{k,m}, \quad u \in V_{n-k,m},$$  

(2.9)

so that

$$\int_{V_{n,m}} f(v) dv = \int_{0 < a'a < I_m} \int_{V_{n-k,m}} d\mu(a) \int_{V_{n,m}} f \left( \begin{bmatrix} a \\ u(I_m - a'a)^{1/2} \end{bmatrix} \right) du,$$

(2.10)

$$d\mu(a) = |I_m - a'a|^{\delta} da, \quad \delta = (n - k - m - 1)/2.$$  

For $m=1$, this is a well known bispherical decomposition [VK, pp. 12, 22]. In the higher rank case, this statement is due to [Herz, p. 495] for $k = m$ and to Grinberg and Rubin [GR] for all $k + m \leq n$, see also [OlR], [Zh1].
2.3. The Laplace transform. Let \( z = \sigma + i\omega, \sigma \in \mathcal{P}_\ell, \omega \in \mathcal{S}_\ell, \) be a complex symmetric matrix. Suppose that \( f \) is a locally integrable function on \( \mathcal{S}_\ell \) satisfying \( f(r) = 0 \) if \( r \notin \mathcal{P}_\ell \), and \( \exp(-\text{tr}(\sigma r))f(r) \in L^1(\mathcal{S}_\ell) \) for some \( \sigma_0 \in \mathcal{P}_\ell \). The integral
\[
(Lf)(z) = \int_{\mathcal{P}_\ell} \exp(-\text{tr}(zr))f(r)dr
\]
is called the Laplace transform of \( f \). This integral is absolutely convergent in the (generalized) half-plane \( \text{Re} \ z > \sigma_0 \). Let
\[
(Fg)(\omega) = \int_{\mathcal{S}_\ell} \exp(\text{tr}(i\omega s))g(s)ds, \quad \omega \in \mathcal{S}_\ell,
\]
be the Fourier transform of a function \( g \) on \( \mathcal{S}_\ell \). Then \( (Lf)(z) = (Fg_\sigma)(-\omega) \), where \( g_\sigma(r) = \exp(-\text{tr}(\sigma r))f(r) \in L^1(\mathcal{S}_\ell) \) for \( \sigma > \sigma_0 \). Thus, all properties of the Laplace transform are obtained from the general Fourier transform theory for Euclidean spaces [Herz], [Vl, p. 126]. The following uniqueness result for the Laplace transform follows from injectivity of the Fourier transform of tempered distributions.

**Lemma 2.3.** If \( f_1(r) \) and \( f_2(r) \) satisfy \( \exp(-\text{tr}(\sigma_0 r))f_j(r) \in L^1(\mathcal{P}_\ell), \quad j = 1, 2, \) for some \( \sigma_0 \in \mathcal{P}_\ell \), and \( (Lf_1)(z) = (Lf_2)(z) \) whenever \( \text{Re} \ z > \sigma_0 \), then \( f_1(r) = f_2(r) \) almost everywhere on \( \mathcal{S}_\ell \).

2.4. Gårding-Gindikin distributions. Let \( f \) belong to the Schwartz space \( S(\mathcal{S}_\ell) \). The Gårding-Gindikin distribution associated to the cone \( \mathcal{P}_\ell \) is defined by
\[
\mathcal{G}_\alpha(f) = \frac{1}{\Gamma(\ell)} \int_{\mathcal{P}_\ell} f(r)r^{\alpha-d}dr, \quad d = (\ell + 1)/2.
\]
The integral \( 2.13 \) converges absolutely for \( \text{Re} \alpha > d - 1 \) and admits analytic continuation as an entire function of \( \alpha \) so that \( \mathcal{G}_0(f) = f(0) \). The integrals of half-integral order have the form
\[
\mathcal{G}_{m/2}(f) = \pi^{-\ell m/2} \int_{\mathfrak{M}_{m,\ell}} f(\omega)\omega d\omega, \quad m = 1, 2, \ldots,
\]
see [FK] pp. 132–134, [OR2].

2.5. The Gårding-Gindikin fractional integrals. Let \( Q = \{ r \in \mathcal{P}_\ell : 0 < r < I_\ell \} \) be the “unit interval” in \( \mathcal{P}_\ell \). For \( f \in L^1(Q) \) and \( \text{Re} \alpha > d - 1 \), the Gårding-Gindikin integrals are defined by
\[
(I^\alpha_+ f)(s) = \frac{1}{\Gamma(\ell)} \int_0^s f(r)s-r)^{\alpha-d}dr,
\]
\[
(I^\alpha_- f)(s) = \frac{1}{\Gamma(\ell)} \int_s^1 f(r)r-s)^{\alpha-d}dr,
\]
where \( s \in Q \). Both integrals are absolutely convergent. For \( f \in \mathcal{D}(Q) \) and \( \text{Re} \alpha \leq d - 1 \), the analytic continuation of the integrals \( 2.15 \) and \( 2.16 \) can be defined by
\[
(I^\alpha_\pm f)(s) = (I^\alpha_\pm D^j_\pm f)(s) \quad \text{if} \quad d - 1 - j < \text{Re} \alpha \leq d - j; \quad j = 1, 2, \ldots,
\]
Lemma 2.4. For \( f \in \mathcal{D}(Q) \) and \( \alpha \in \mathbb{C} \),
\[
(I_\alpha f)(s) = (I_\alpha^\alpha g)(I_\ell - s), \quad g(r) = f(I_\ell - r).
\]

Proof. Since the integrals \((I_\alpha^\alpha f)(s)\) and \((I_\alpha^\alpha f)(s)\) are entire functions of \( \alpha \), it suffices to prove (2.18) for \( \Re \alpha > d - 1 \). This can be easily done by changing variables \( r \to I_\ell - r \).

\[\square\]

Theorem 2.5. If \( f \in \mathcal{D}(Q) \) and all \( m \in \mathbb{N} \),
\[
(I^{m/2}_+ f)(s) = \pi^{-\ell m/2} \int_{\{\omega \in M_{m,I}: \omega' \omega < s\}} f(s - \omega') d\omega,
\]
\[
(I^{m/2}_- f)(s) = \pi^{-\ell m/2} \int_{\{\omega \in M_{m,I}: \omega' \omega < I_\ell - s\}} f(s + \omega') d\omega.
\]

Moreover,
\[
(I^0_\pm f)(s) = f(s).
\]

Proof. Formulas (2.19) and \((I^0_\pm f)(s) = f(s)\) are verified in \([OR2]\) for arbitrarily \( s \in \mathcal{P}_\ell \); (2.20) and the second equality in (2.21) follow from (2.18) and the corresponding properties of the integral \( I_\alpha^\alpha f \).

\[\square\]

Theorem 2.6. If \( f \in L^1(Q) \) and \( m \in \mathbb{N} \), then \((I^{m/2}_\pm f)(s)\) converge absolutely for almost all \( s \in Q \).

Proof. It was shown in \([OR2]\) that for nonnegative \( f \) and every \( a \in \mathcal{P}_\ell \),
\[
\int_0^a (I^{m/2}_+ f)(s) ds \leq \frac{\Gamma(\ell)(d)}{\Gamma(\ell/2 + d)} \left|a\right|^{m/2} \int_0^a f(r) dr.
\]
This proves that \((I^{m/2}_\pm f)(s)\) is absolutely convergent for almost all \( s \in Q \) provided that \( f \in L^1(Q) \). The statement for the right-sided integral is a consequence of (2.18).

\[\square\]

According to Theorems 2.5 and 2.6, the Gårding-Gindikin fractional integrals can be defined for arbitrary integrable functions \( f \) and \( \alpha \) belonging to the Wallach set
\[
W^\alpha_\ell = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{\ell - 1}{2}\right\} \cup \left\{\alpha \mid \Re \alpha > \frac{\ell - 1}{2}\right\}.
\]

3. The generalized Erdélyi-Kober fractional integrals

Let \( f \) be a function defined on the “unit interval” \( \bar{Q} = \{r \in \mathcal{P}_\ell : 0 \leq r \leq I_\ell\} \) in \( \mathcal{P}_\ell \). For \( \Re \alpha, \Re \beta > d - 1 \), we introduce the generalized Erdélyi-Kober fractional
integrals

\[
(J^\alpha_{+}f)(s) = \frac{|s|^{d-\alpha-\beta}}{\Gamma(\ell(\alpha))\Gamma(\ell(\beta))} \int_{0}^{s} f(r)|r|^{\beta-d}|s-r|^\alpha dr,
\]

(3.1)

\[
(J^\alpha_{-}f)(s) = \frac{|s|^{d-\alpha-\beta}}{\Gamma(\ell(\alpha))\Gamma(\ell(\beta))} \int_{s}^{1} f(r)|r|^{\beta-d}|r-s|^\alpha dr,
\]

(3.2)

where \( s \in Q, d = (\ell + 1)/2 \). Since

\[
(J^\alpha_{\pm}f)(s) = \frac{|s|^{d-\alpha-\beta}}{\Gamma(\ell(\beta))} (I^n_{\pm}g_\beta)(s), \quad g_\beta(r) = |r|^{\beta-d}f(r),
\]

(3.3)

these integrals converge absolutely for almost all \( s \in Q \) whenever \( g_\beta \in L^1(Q) \). Suppose that for fixed \( \beta \) satisfying \( \Re \beta > d-1 \), the function \( g_\beta \) is infinitely differentiable and supported in \( Q \). Then the integrals \( J^\alpha_{\pm}f \) admit analytic continuation as entire functions of \( \alpha \) so that \( (J^\alpha_{\pm}f)(s) = f(s)/\Gamma(\ell(\beta)) \). Our next goal is to show that integrals (3.1), (3.2) extend analytically as entire functions of \( \beta \) and obtain an explicit form of \( J^\alpha_{\pm}f \) for \( m \in \mathbb{N} \) provided that \( \Re \alpha > d-1 \).

**Theorem 3.1.** Let \( f \) be an infinitely differentiable function supported in \( Q \). The integrals \( J^\alpha_{\pm}f \) are entire functions of \( \alpha \) and \( \beta \). For all \( m \in \mathbb{N} \), and \( \Re \alpha > d-1 \),

\[
(J^\alpha_{\pm,m/2}f)(s) = \frac{\pi^{-\ell m/2}|s|^{d-\alpha-\beta}}{\Gamma(\ell(\alpha))} \int_{\{\omega \in \mathcal{M}_{\ell,m}: \omega^\prime < s\}} |s - \omega^\prime |^{\alpha-d} f(\omega')d\omega,
\]

(3.4)

\[
(J^\alpha_{-m/2}f)(s) = \frac{\pi^{-\ell m/2}|s|^{d-\alpha-\beta}}{\Gamma(\ell(\alpha))} \int_{\{\omega \in \mathcal{M}_{\ell,m}: \omega^\prime < 1-s\}} |s + \omega^\prime |^{\alpha-d} f(\omega')d\omega.
\]

(3.5)

**Proof.** We apply the Laplace transform technique to prove (3.4). One can assume that \( f \) is a function on \( S_\ell \) which is identically zero outside \( Q \), so that \( f \in S(S_\ell) \). Denote

\[
(J^{\alpha,\beta}f)(s) = |s|^\alpha |\beta-d| (J^\alpha_{+}f)(s), \quad s \in \mathcal{P}_\ell
\]

(3.6)

It is known (see, e.g., [OR2]) that for \( f \in D(\mathcal{P}_\ell), s \in \mathcal{P}_\ell \), and \( \alpha \in \mathbb{C} \),

\[
(LL^\alpha_{\ell}f)(z) = \det(z)^{-\alpha}(Lf)(z), \quad \Re z > 0.
\]

(3.7)

Here, \( \det(z)^{-\alpha} = \exp(-\alpha \log \det(z)) \), where the branch of \( \log \det(z) \) is chosen so that \( \det(z) = |\sigma| \) for real \( z = \sigma \in \mathcal{P}_m \). Hence, for \( \Re \beta > d-1 \) and \( \Re z > 0 \),

\[
(LL^\alpha_{\ell})(z)^{-\beta-d} = \frac{\det(z)^{-\alpha}}{\Gamma(\ell(\beta))} (Lg_\beta)(z) = \frac{\det(z)^{-\alpha}}{\Gamma(\ell(\beta))} \int_{\mathcal{P}_\ell} |r|^{\beta-d} \exp(-\tr(rz))f(r)dr.
\]

(3.7)

Denote \( F_z(r) = \exp(-\tr(zr))f(r) \). Then \( (LL^\alpha_{\ell}f)(z) = \det(z)^{-\alpha}G_\beta(F_z) \), where \( G_\beta(F_z) \) is the Gårding-Gindikin distribution [2,4], which is entire function of \( \beta \). By [2,14], for \( m = 1, 2, \ldots \), we obtain

\[
(LL^\alpha_{m,\ell}m/2(f)(z) = \pi^{-\ell m/2} \det(z)^{-\alpha} \int_{\mathcal{M}_{m,\ell}} \exp(-\tr(z\omega'))f(\omega')d\omega.
\]

(3.8)
On the other hand, the Laplace transform of the integral

\[ I(s) = \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int_{\omega \in \mathcal{M}_{m,\ell} : \omega' < s} |s - \omega'\omega|^{\alpha-d} f(\omega')d\omega \]

can be evaluated as follows

\[ (LI)(z) = \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int_{\mathcal{M}_{m,\ell}} f(\omega')d\omega \int_{\omega'\omega}^{\infty} \exp(-\text{tr}(zs))|s - \omega'\omega|^{\alpha-d}ds \]

\[ = \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int_{\mathcal{M}_{m,\ell}} \exp(-\text{tr}(z\omega')f(\omega')d\omega \int_{\mathcal{P}_{\ell}} \exp(-\text{tr}(zs))|s|^{\alpha-d}ds. \]

By the the well known formula

\[ \int_{\mathcal{P}_{\ell}} \exp(-\text{tr}(zr))|r|^{\alpha-d}dr = \Gamma(\alpha)\det(z)^{-\alpha}, \quad \text{Re } \alpha > d - 1, \]

see, e.g., [Herz, p. 479], we obtain

\[ (LI)(z) = \pi^{-\ell m/2}\det(z)^{-\alpha} \int_{\mathcal{M}_{m,\ell}} \exp(-\text{tr}(z\omega')f(\omega')d\omega. \]

According to the uniqueness property of the Laplace transform, it follows that

\[ (\tilde{J}^{\alpha, m/2}_+ f)(s) = \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int_{\omega \in \mathcal{M}_{m,\ell} : \omega'\omega < I_{\ell}} |s - \omega'\omega|^{\alpha-d} f(\omega')d\omega. \]

This proves (3.4). The statement for the right-sided integral is a consequence of (3.3), (2.18), and (3.4).

\[ \square \]

**Theorem 3.2.** Let a function \( f \) on \( \bar{Q} \) satisfy the inequality

\[ (3.9) \quad \int_{\mathcal{M}_{m,\ell}} |f(\omega')d\omega |d\omega < \infty, \quad m \in \mathbb{N}. \]

Then the integrals \((\tilde{J}^{\alpha, m/2}_+ f)(s)\) converge absolutely for \( \text{Re } \alpha > d - 1 \) and almost all \( s \in Q \).

**Proof.** It suffices to show that the integrals \( \int_{0}^{1} (\tilde{J}^{\alpha, m/2}_+ f)(s)ds \) are finite for nonnegative \( f \), where \( \tilde{J}^{\alpha, m/2}_+ f \) are defined by (3.6). For the left-sided integral, changing
the order of integration yields
\[
\int_0^{I_\ell} (J_+^{\alpha, m/2} f)(s) \, ds = \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int f(\omega') \, d\omega \int_0^{I_\ell} |s - \omega'|^{\alpha-d} \, ds
\]
\[
= \frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int f(\omega') \, d\omega \int_0^{I_\ell} |s - \omega'|^{\alpha-d} \, ds
\]
\[
= \frac{\pi^{-\ell m/2}}{\Gamma(\alpha + d)} \int |s - \omega'|^{\alpha-d} f(\omega') \, d\omega
\]
as required. Here, we used the substitution \( r = a^{1/2} s a^{1/2}, \ dr = |a|^{d} ds \ [M] \ pp. \ 57–59], to evaluate the integral
\[
\int_0^{a} |r|^{\alpha-d} \, dr = |a|^{\alpha} \int_0^{I_\ell} |s|^{\alpha-d} \, ds = B_\ell(\alpha, d)|a|^{\alpha}, \quad a = I_\ell - \omega'.
\]
The proof for \( J_-^{\alpha, m/2} f \) is the same.

\[ \square \]

**Remark 3.3.** For \( m \geq \ell \), condition \([3.9]\) is equivalent to \(|r|^{m/2-d} f(r) \in L^1(Q)\).

Theorem 3.2 allows us to define integrals \( J_+^{\alpha, \beta} f \) for Re\( \alpha > d - 1 \) and all \( \beta \) belonging to the Wallach set \([1.7]\) provided that \(|r|^{\beta-d} f(r) \in L^1(Q)\) when Re\( \beta > d - 1 \), and \( f \) satisfies \([3.9]\) when \( \beta = m/2, m = 1, 2, \ldots, \ell - 1 \).

**Definition 3.4.** The left-sided Erdélyi-Kober type fractional integral is defined by

\[
(J_+^{\alpha, \beta} f)(s) = \begin{cases} 
\frac{|s|^{d-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s f(r)|r|^{\beta-d} |s - r|^{\alpha-d} \, dr & \text{if Re} \, \beta > d - 1, \\
\frac{\pi^{-\ell m/2}}{\Gamma(\alpha)} \int_{\{\omega \in \mathcal{M}_{m, \ell}: \omega' < s\}} |s - \omega'|^{\alpha-d} f(\omega') \, d\omega & \text{if } \beta = m/2.
\end{cases}
\]

**Remark 3.5.** For \( 0 < m < \ell \), the second integral in Definition \([3.4]\) can be rewritten as follows:

\[
(3.10) \quad (J_+^{\alpha, m/2} f)(s) = c \int_0^{I_\ell} |q|^{(\ell - m - 1)/2} |I_m - q|^{\alpha-d} \, dq \int_{V_{\ell, m}} f(s^{1/2} uq' s^{1/2}) \, du,
\]
\[
c = 2^{-m} \pi^{-\ell m/2} / \Gamma(\alpha).
\]
Definition 3.6. The right-sided Erdélyi-Kober type fractional integral is defined as follows

\[
(J^{\alpha,\beta}_f)(s) = \begin{cases} 
\frac{|s|^{d-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{s} f(r)|r|^{\beta-d}|r-s|^{\alpha-d}dr & \text{if } \Re \beta > d - 1, \\
\frac{\pi^{-m/2}|s|^{d-\alpha-\beta}}{\Gamma(\alpha)} \int_{\omega \in \mathcal{M}_m : \omega |w| < |s|} |s + \omega |^{\alpha-d}f(\omega) d\omega & \text{if } \beta = m/2.
\end{cases}
\]

4. The Radon Transform on Grassmannians

4.1. Definitions. Let \(1 < m < k \leq n - 1\). Given a pair of Grassmann manifolds \(G_{n,m}\) and \(G_{n,k}\), the Radon transform of a function \(f(\tau_m)\) on \(G_{n,m}\) is defined by (1.1). The corresponding dual transform of a function \(\varphi(\tau_k)\) on \(G_{n,k}\) is

\[
(\mathcal{R}^*\varphi)(\tau_m) = \int_{\{\tau_k : \tau_m \subset \tau_k\}} \varphi(\tau_k) d\tau_k, \quad \tau_m \in G_{n,m}.
\]

Let \(\tau_m = \{x \in \mathbb{R}^n : \zeta'x = 0, \zeta \in V_{n,n-m}\}\), \(\tau_k = \{x \in \mathbb{R}^n : \zeta'x = 0, \zeta \in V_{n,n-k}\}\). Denote by \(\nu = \zeta^\perp\) the orthogonal complement of \(\zeta\). To give precise meanings to integrals (1.1), (4.1), we use the following parameterizations:

\[
\tau_m \equiv \tau_m(v), \quad v \in V_{n,m}; \quad \tau_k \equiv \tau_k(\xi), \quad \xi \in V_{n,n-k}.
\]

The functions \(f(\tau_m)\) on \(G_{n,m}\) and \(\varphi(\tau_k)\) on \(G_{n,k}\) are identified with the right-invariant functions on the Stiefel manifolds \(V_{n,m}\) and \(V_{n,n-k}\), respectively. Condition \(\tau_m \subset \tau_k\) is equivalent to \(\zeta'v = 0\). The set of all \(v\) satisfying the last equation is represented as \(v = g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix}, \omega \in V_{k,m}\), where \(g_\xi\) is an arbitrary rotation with the property

\[
g_\xi \xi_0 = \xi, \quad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.
\]

This observation leads to the following.

Definition 4.1. The Radon transform of a right-invariant function \(f(v)\) on \(V_{n,m}\) is defined by the formula

\[
(\mathcal{R}f)(\xi) = \int_{V_{n,m}} f \left( g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega, \quad \xi \in V_{n,n-k}.
\]

Remark 4.2. Definition (4.3) of the Radon transform differs from that in [GR] by the parametrization of the plane \(\tau_k\). Specifically, in [GR], \((\mathcal{R}f)(\tau_k) \equiv (\mathcal{R}f)(\zeta^\perp)\).

Let \(\gamma_v\) be an arbitrary rotation with the property

\[
\gamma_v v_0 = v, \quad v_0 = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in V_{n,m}.
\]

The set of all \(\xi\) that obey \(\zeta'v = 0\) has the form \(\xi = \gamma_u \begin{bmatrix} u \\ 0 \end{bmatrix}, \quad u \in V_{n-m,n-k}^\perp\).
Definition 4.3. The dual Radon transform of a right-invariant function $\varphi(\xi)$ on $V_{n,n-k}$ is defined by

$$\langle R^* \varphi \rangle (v) = \int_{V_{n,m,n-k}} \varphi\left(\gamma_v \begin{bmatrix} u \\ 0 \end{bmatrix}\right) d_u, \quad v \in V_{n,m}. \tag{4.5}$$

The expressions (4.3) and (4.5) are independent of the choice of rotations $g_\xi : \xi_0 \to \xi$, and $\gamma_v : v_0 \to v$, respectively.

Lemma 4.4. The duality relation

$$\int_{V_{n,m}} f(v) \langle R^* \varphi \rangle (v) d_v = \int_{V_{n,n-k}} (Rf)(\xi) \varphi(\xi) d_\xi \tag{4.6}$$

is valid provided that either side of this equality is finite for $f$ and $\varphi$ replaced by $|f|$ and $|\varphi|$, respectively.

Proof. By (4.5), the left-hand side of (4.6) equals

$$l.h.s. = \int_{V_{n,m}} f(v) d_v \int_{V_{n,m,n-k}} \varphi\left(\gamma_v \begin{bmatrix} u \\ 0 \end{bmatrix}\right) d_u$$

$$= \int_{SO(n-m)} d_\gamma \int_{SO(n)} f(\beta v_0) \varphi\left(\beta \gamma_v \begin{bmatrix} \gamma \\ 0 \\ I_m \end{bmatrix} \tilde{\xi}_0 \right) d_\beta,$$

where $\tilde{\xi}_0 = \begin{bmatrix} I_{n-k} \\ 0 \end{bmatrix} \in V_{n,n-k}$. The change of variables $\beta \gamma_v \begin{bmatrix} \gamma \\ 0 \\ I_m \end{bmatrix} \to \beta$ yields

$$l.h.s. = \int_{SO(n-m)} d_\gamma \int_{SO(n)} \varphi(\beta \tilde{\xi}_0) f\left(\beta \begin{bmatrix} \gamma' \\ 0 \\ I_m \end{bmatrix} \gamma'_v v_0 \right) d\beta$$

$$= \int_{SO(n)} \varphi(\beta \tilde{\xi}_0) f(\beta v_0) d\beta.$$

Similarly, according to (4.3), the right-hand side of (4.6) is evaluated as follows

$$r.h.s. = \int_{V_{n,n-k}} \varphi(\xi) d_\xi \int_{V_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) d_\omega$$

$$= \int_{SO(n)} \varphi(\beta \tilde{\xi}_0) f(\beta \tilde{v}_0) d\beta = \int_{SO(n)} \varphi(\beta \tilde{\xi}_0) f(\beta v_0) d\beta,$$

where $\tilde{v}_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m}$. This proves the duality relation. \qed

Theorem 4.5. The Radon transform $Rf$ and the dual Radon transform $R^* \varphi$ are well defined almost everywhere on $V_{n,n-k}$ and $V_{n,m}$, respectively, for any integrable functions $f$ and $\varphi$. 
Proof. It follows from (4.6) that

\[
\begin{align*}
(4.7) \quad \int_{V_{n,m-k}} (Rf)(\xi) d_\xi &= \int_{V_{n,m}} f(v) d_v, \\
(4.8) \quad \int_{V_{n,m}} (R^*\varphi)(v) d_v &= \int_{V_{n,n-k}} \varphi(\xi) d_\xi.
\end{align*}
\]

Hence, the integrals \((Rf)(\xi), (R^*\varphi)(v)\) exist for almost all \(\xi\) and \(v\), respectively. \(\square\)

4.2. Radon transform and its dual of invariant functions. The notion of \(\ell\)-zonal function on the Stiefel manifold was introduced in [GR].

Definition 4.6. Let \(1 \leq \ell \leq n-1\), \(\rho \in O(n-\ell)\), \(g_\rho = \begin{bmatrix} \rho & 0 \\ 0 & I_\ell \end{bmatrix} \in O(n)\). A function \(f(v)\) on \(V_{n,m}\) is called \(\ell\)-zonal if \(f(g_\rho v) = f(v)\) for all \(\rho \in O(n-\ell)\).

Lemma 4.7. [GR] p. 798 Let \(m+\ell \leq n\), \(m \geq \ell\). A function \(f(v)\) on \(V_{n,m}\) is \(\ell\)-zonal and \(O(m)\) right-invariant (simultaneously) if and only if there is a function \(f_0\) on \(P_\ell\) such that \(f(v) \overset{\text{a.e.}}{=} f_0(r)\), \(r = \sigma_\ell vv'\sigma_\ell\), \(\sigma_\ell = \begin{bmatrix} 0 & I_\ell \\ I_\ell & 0 \end{bmatrix} \in V_{n,\ell}\). Here, \(P_\ell\sigma_\ell\) is the orthogonal projection of \(\sigma_\ell\) onto \(v\).

The following theorem establishes connection between the Radon transform of a function \(f(v)\) of the form \(f(v) \equiv f_0(r)\), \(r = \sigma_\ell vv'\sigma_\ell\), and the Erdélyi-Kober type fractional integrals associated to the cone \(P_\ell\). Note that for \(m \geq \ell\), the function \(f_0\) is defined on \(P_\ell\), and the integral \(J_+^{(k-m)/2, m/2} f_0\) exists in the usual sense, cf. (3.1). For \(m < \ell\), \(f_0\) is a function on the boundary \(\partial P_\ell\), and \(J_+^{(k-m)/2, m/2} f_0\) is understood in the sense of analytic continuation [GR].

Theorem 4.8. Let \(f(v)\) be an integrable function on \(V_{n,m}\). Suppose that \(f(v)\) has the form \(f(v) \equiv f_0(r)\), where \(r = \sigma_\ell vv'\sigma_\ell \in \overline{P_\ell}\), and denote \(s = I_\ell - \sigma_\ell \xi_\ell'\sigma_\ell = \sigma_\ell' P_{\xi_\ell} \sigma_\ell\). If \(1 \leq \ell \leq k-m\), then

\[
(4.9) \quad (Rf)(\xi) = \Gamma_{\ell}(k/2)(J_+^{(k-m)/2, m/2} f_0)(s),
\]

where \(J_+^{(k-m)/2, m/2}\) is the the Erdélyi-Kober type operator.

Proof. By (4.3),

\[
(4.10) \quad (Rf)(\xi) = \int_{V_{k,m}} f_0(y_\omega y_\omega') d_\omega, \quad y_\omega = \sigma_\ell' g_\xi \begin{bmatrix} \omega \\ 0 \end{bmatrix}.
\]

Denote

\[
a = g_\xi^{-1} \sigma_\ell = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad a_1 \in \mathfrak{M}_{k,\ell}, \quad a_2 \in \mathfrak{M}_{n-k,\ell},
\]

so that \(y_\omega = a_1' \omega\). By Lemma 2.1, \(a_1 = us^{1/2}\), where \(u \in V_{k,\ell}\) and

\[
s = a_1' a_1 = \sigma_\ell' g_\xi \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix} g_\xi^{-1} \sigma_\ell = \sigma_\ell' P_{\xi_\ell} \sigma_\ell.
\]
The change of variables $\phi$ has the form

\[
\sigma \quad \text{where} \quad \sigma
\]

Hence, (4.11) \( \Gamma \) according to (2.10), we obtain

which yields

\[
c = \frac{\pi^{\ell/2}\sigma_{k-\ell, m}}{\Gamma_{\ell}(k_m/2)} = \frac{\Gamma_{\ell}(k_m/2)\Gamma_{m}(k_{\ell}/2)}{\Gamma_{m}(k_{\ell}/2)}.
\]

By (2.11),

(4.11)

\[
\frac{\Gamma_{m}(k_{\ell}/2)}{\Gamma_{m}(k_{\ell}/2)} = \frac{\Gamma_{\ell}(k_m/2)\Gamma_{m}(k_{\ell}/2)}{\Gamma_{m}(k_{\ell}/2)}
\]

which yields $c = \Gamma_{\ell}(k/2)$.  

\[\]  

Remark 4.9. It follows from (3.10) and (4.11) that for $m = 1, 2, \ldots, \ell - 1$, formula (4.9) coincides with (4.3).

Let us consider the dual Radon transform.

Theorem 4.10. Let $\varphi(\xi)$ be an integrable function on $V_{n,n-k}$. Suppose that $\varphi(\xi)$ has the form $\varphi(\xi) = \varphi_0(s)$, where $s = \sigma_{\ell}^{\ell} \xi^{\ell} \sigma_{\ell} \in \mathbb{R}^{\ell}$, and denote $r = I_{\ell} - \sigma_{\ell}^{\ell} \nu' \sigma_{\ell} = \sigma_{\ell}^{\ell} \mathcal{P}_{\nu'} \sigma_{\ell}$. If $1 \leq \ell \leq \min\{k - m, n - m\}$, then

(4.12)

\[
(R^{\ast} \varphi)(v) = \Gamma_{\ell}((n - m)/2)(J_{\ell}^{\kappa_{m} - n - k} \varphi_0)(r).
\]
Proof. By (4.5),

\[(4.13) \quad (\mathcal{R}^* \varphi)(v) = \int_{V_{n-m,n-k}} \varphi_0(z_n z_n^*) d_u, \quad z_u = \sigma'_{\gamma_v} \begin{bmatrix} u \\ 0 \end{bmatrix}.\]

Denote

\[a = \gamma_w^{-1} \sigma_{\ell} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad a_1 \in \mathcal{M}_{n-m,\ell}, \quad a_2 \in \mathcal{M}_{m,\ell},\]

so that \(z_u = a'_1 u\). By Lemma 2.1, \(a_1 = w r^{1/2}, w \in V_{n-m,\ell}\), and

\[r = a'_1 a_1 = \sigma'_{\gamma_v} \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} \gamma_w^{-1} \sigma_{\ell} = \sigma'_{\gamma_w} \mathcal{P}_{r^{1/2}} \sigma_{\ell}.\]

Therefore,

\[(\mathcal{R}^* \varphi)(v) = \int_{V_{n-m,n-k}} \varphi_0(a'_1 u a_1) d_u \]

\[= \int_{V_{n-m,n-k}} \varphi_0(r^{1/2} w u u'r^{1/2}) d_u \]

\[= \int_{V_{n-m,n-k}} \varphi_0(r^{1/2} w_0 u u'r^{1/2}) d_u, \quad w_0 = \begin{bmatrix} I_{\ell} \\ 0 \end{bmatrix} \in V_{n-m,\ell}.\]

Applying the bi-Stiefel decomposition

\[u = \begin{bmatrix} b \\ u_1(I_{n-k} - b/b^{1/2}) \end{bmatrix}, \quad b \in \mathcal{M}_{\ell,n-k}, \quad u_1 \in V_{n-m-\ell,n-k},\]

we obtain

\[(\mathcal{R}^* \varphi)(v) = c \int_{\{b \in \mathcal{M}_{\ell,n-k} : 0 < b < r_{n-k}\}} |I_{n-k} - b/b^{(k-m-\ell-1)/2} \varphi_0(r^{1/2} b b^{1/2})| db \]

\[= c \int_{\{y \in \mathcal{M}_{n-k,\ell} : 0 < y < r_{\ell}\}} |I_{\ell} - y' y^{(k-m-\ell-1)/2} \varphi_0(r^{1/2} y y^{1/2})| dy,\]

where \(c = \sigma_{n-m-\ell,n-k}/\sigma_{n-m,n-k}\). The change of variables \(z = y r^{1/2}, dz = |r|^{(n-k)/2} dy\), gives

\[(\mathcal{R}^* \varphi)(v) = c |r|^{(k-n)/2} \int_{\{z \in \mathcal{M}_{n-k,\ell} : 0 < z' z < r\}} |I_{\ell} - z^{-1/2} z' z r^{-1/2} |(k-m-\ell-1)/2 \varphi_0(z' z)| dz \]

\[= c |r|^{(m-n+\ell+1)/2} \int_{\{z \in \mathcal{M}_{n-k,\ell} : 0 < z' z < r\}} |r - z' z|^{(k-m-\ell-1)/2} \varphi_0(z' z) dz.\]

According to (3.4), we obtain

\[(\mathcal{R}^* \varphi)(v) = c_1 \left( J_{+} \frac{k-n}{m-n} \varphi(0) \right)(r),\]
where
\[ c_1 = \frac{\pi ^{k-\frac{n}{2}} \sigma_{n-m-k-n-k} \Gamma_k \left( \frac{k-m}{2} \right)}{\sigma_{n-m-k-n-k} \Gamma_k \left( \frac{n-m}{2} \right)} = \frac{\Gamma_k \left( \frac{k-m}{2} \right) \Gamma_k \left( \frac{n-m}{2} \right)}{\Gamma_k \left( \frac{n-m-k}{2} \right)} \]

By (2.4),
\[ \frac{\Gamma_k \left( \frac{n-m-k}{2} \right)}{\Gamma_k \left( \frac{n-m}{2} \right)} = \frac{\Gamma_k \left( \frac{k-m}{2} \right)}{\Gamma_k \left( \frac{n-m}{2} \right)}, \]
and therefore,
\[ c = \Gamma_k \left( \left( \frac{n-m}{2} \right) \right). \]

□

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