Revisiting the relation between the binding energy of finite nuclei and the equation of state of infinite nuclear matter

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The energy density is calculated in coordinate space for \( ^{12}\text{C} \), \( ^{40}\text{Ca} \), \( ^{48}\text{Ca} \), and \( ^{208}\text{Pb} \) using a dispersive optical model (DOM) constrained by all relevant data including the energy of the ground state. For \( ^{12}\text{C} \), the energy density is also calculated using the variational Monte-Carlo method employing the Argonne/Urbana two and three-body interactions. The nuclear interior minimally contributes to the total binding energy due to the \( 4\pi r^2 \) phase space factor. Thus, the volume contribution to the energy in the interior is not well constrained. The DOM energy densities are in reasonable agreement with \textit{ab initio} self-consistent Green’s function calculations of infinite nuclear matter restricted to treat only short-range and tensor correlations. These results call into question the degree to which the equation of state for nuclear matter is constrained by the empirical mass formula. In particular, the results in this letter indicate that the binding energy of saturated nuclear matter does not require the canonical value of 16 MeV per particle but only about 13-14 MeV when the interior of \( ^{208}\text{Pb} \) is considered.

The investigation of the binding energy of atomic nuclei dates back to the origins of nuclear physics [1]. The well-known empirical mass formula developed by Bethe and Weizsäcker [2, 3] accurately describes the global aspects of nuclear binding for most of the nuclear chart. Its success is largely due to the saturating nature of the constituent nucleons in nuclei. The evidence for nuclear saturation came from measurements of the root-mean-squared (rms) charge radius of nuclei which revealed that the volume of a given nucleus scales linearly with the squared (rms) charge radius of nuclei which revealed that the volume of a given nucleus scales linearly with the density in the interior of nuclei saturates at a value around \( \rho_0 \approx 0.16 \) fm [4, 5]. In order to understand the mechanism behind nuclear saturation, infinite nuclear matter (NM) is an ideal system that is often studied [6–8]. Depending on the method and realistic nucleon-nucleon (NN) interaction used, the calculated value of \( \rho_0 \) in NM can stray from the experimental value as discussed \textit{e.g.} in Ref. [9]. In addition to the density at saturation, the associated binding energy, \( E_0 \), plays a vital role in the nuclear equation of state (EOS), which is relevant for astrophysical research on supernovae and neutron stars [10–12].

While the value of \( \rho_0 \) is determined experimentally, \( E_0 \) is determined empirically from an extrapolation of the empirical mass formula [4, 13, 14]

\[
BE(A, Z) = a_Y A - a_S A^{2/3} + a_C Z (Z - 1) A^{-1/3} - \frac{1}{2} a_A (A - 2Z)^2 A^{-1} + \delta,
\]

where \( a_Y \), \( a_S \), \( a_C \), \( a_A \), and \( \delta \) are parameters fit to nuclear masses [1]. Because the only link between Eq. (1) and NM is the volume term, the canonical value of the saturation energy is assumed to be \( E_0/A = -a_Y \approx -16 \text{ MeV} \) [4, 13]. However, this involves a serious extrapolation that neglects proper consideration of long-range correlations in both finite and infinite systems [6, 15–17].

In this letter, the connection between the empirical mass formula and the value of \( E_0 \) is investigated through energy densities calculated using a nonlocal dispersive optical-model (DOM) which constrains a complex self-energy using both scattering and bound-state data [18, 19]. The DOM was originally developed by Mahaux and Sartor [18], employing local real and imaginary potentials connected through dispersion relations. However, only with the introduction of nonlocality can realistic self-energies be obtained [19, 20]. The Dyson equation then determines the single-particle propagator, or Green’s function, \( G_{\nu}(r, r'; E) \) from which bound-state and scattering observables are obtained. Results of DOM fits of \( ^{12}\text{C} \), \( ^{40}\text{Ca} \), \( ^{48}\text{Ca} \), and \( ^{208}\text{Pb} \) are considered here.

In these fits, the self-energy at negative energies was not only constrained by quasihole energies, particle numbers, and charge densities, but also by the total binding energy of each nucleus. The energy density can then be defined such that its volume integral is the total binding energy. These energy densities can be used to relate the energy of these nuclei to self-consistent Green’s function (SCGF) calculations in NM [9, 21]. In light of the present analysis coupled with the NM results from Ref. [9, 21], the validity of the canonical value for \( E_0 \) is re-examined. This is done by comparing three different methods of obtaining the value of \( E_0 \): using the canonical value of \( a_Y \), calculating the minimum energy in infinite nuclear matter, and calculating the energy in the interior of finite nuclei.

The binding energy of a nucleus can be expressed as the expectation value of the Hamiltonian using the full \( A \)-body wave function, \( E_0^A = \langle \Psi_0^A | \hat{H} | \Psi_0^A \rangle \). The energy
density, $\mathcal{E}_A(r)$, of a nucleus can then be defined such that

$$E_0^A = \int d^3 r \mathcal{E}_A(r) = 4\pi \int_0^\infty dr r^2 \mathcal{E}_A(r).$$  \hfill (2)

The energy of the ground state can be recast into the Migdal-Galitski sum rule [22] for both proton and neutron contributions with $E_0^A = E_N^0 + E_N^0$ [6]. Since the DOM is calculated in a coordinate-space basis of Lagrange functions [23], $\mathcal{E}_A(r)$ can be calculated using

$$\mathcal{E}_A(r) = \frac{1}{2} \int_0^{r_p} \left[ E_{S_h}(r, r; E) + \int_0^{r_p} dr' r'^2 \left( r' | \hat{T} | r' \right) S_h(r, r'; E) \right] dE.$$  \hfill (3)

The second term in Eq. (3) represents the kinetic energy density (see supplement [24]), while the first term corresponds to a combination of the kinetic and potential densities [6].

It is important to note that this derivation assumes there are no three-body terms in the nuclear interaction [25]. While it is known that there is a three-body force [26], the arguments below do not change in any essential way by the assumption that Eq. (3) is exact. Variational Monte Carlo (VMC) leading to exact Green’s function Monte Carlo results (GFMC) [27] require only a modest three-body contribution to the binding energy. With chiral interactions [28], the three-body force is important to generate NM saturation.

With Eq. (2), the binding energy of nuclei are also included in DOM fits with an accuracy of about 1.5% and shown for $^{12}$C, $^{40}$Ca, $^{48}$Ca, and $^{208}$Pb in Table I. The details of these DOM fits can be found in Refs. [24, 29–31] respectively. The agreement with experiment in Table I along with the reproduction of experimental charge densities indicates that the hole spectral densities are well constrained.

The energy density of $^{40}$Ca weighted by the volume element $4\pi r^2$ and its separation in kinetic and potential energy density are shown in Fig. 1. The weighting is chosen to emphasize the parts of the energy density that contribute to the integral in Eq. (2). The nucleon point-density is shown in addition to the energy densities in Fig. 1 to demonstrate that the radial dependence of the energy density is very similar.

Self-consistent Green’s function (SCGF) calculations in NM from Ref. [21] are represented by points in Fig. 1. Each different symbol corresponds to a different NN interaction in the SCGF calculation, where the triangles correspond to the charge-dependent Bonn (CD-Bonn) interaction [32], the circles correspond to the Argonne $v_{18}$ (AV18) interaction [33], and the squares correspond to the Idaho next-to-next-to-next-to-leading order (N3LO) chiral interaction [34]. The calculation in NM is for specific values of the nuclear density which are mapped to radii using the DOM matter density. These results cannot be directly compared to the energy density in finite nuclei because there is no Coulomb force included in NM. Since there are an equal number of protons and neutrons in $^{40}$Ca, isospin symmetry implies that their distributions would be the same if the Coulomb force were ignored. Thus, using twice the neutron energy density in $^{40}$Ca is an effective way of removing the influence of the Coulomb force. This is how the lines in Fig. 1 are generated. The agreement with the NM calculations is striking provided that only short-range (SRC) and tensor correlations are included as suggested in Ref. [15]. This implies that the interior of $^{40}$Ca exhibits NM-like properties.

The interaction with the best agreement with the DOM energy density in Fig. 1 is AV18. It is interesting that, unlike the other two interactions, AV18 correctly reproduces the nuclear saturation density, $\rho \approx$...
the SCGF calculation reported in Ref. [9], but saturates at about $-11.5$ MeV. This is in disagreement with the canonical value which comes from the empirical mass formula. However, it is clear from Fig. 1 that the interior of the nucleus does not determine the binding energy and its contribution to Eq. (1) is not well determined. Therefore, with the interpretation that NM is representative of the core of finite nuclei, there is no strong constraint that the binding energy of NM has to be $a_V$. This implies that the AV18 interaction produces consistent results for not only the density at saturation, but also the energy $[9]$ at least for $^{40}$Ca. Furthermore, AV18 + Urbana-IX $[35]$ (3-body interaction) was used to derive the APR EOS of nuclear matter $[12]$. It is widely used in calculations of neutron star structure, all of which are consistent with current observations of neutron stars, including the recent neutron star merger event $[12,36]$. The APR EOS correctly predicts the value of $\rho_0$ but with a minimum energy of $E_0 = -12.6$ MeV. While the value of this minimum energy has been seen as a defect of the APR EOS, its success in describing nuclear systems further supports a saturation energy different from $a_V$. We note here that quantum Monte Carlo studies of drops of atomic helium, both bosonic $^4\text{He}$ $[37]$ and fermionic $^3\text{He}$ $[38]$ using the HFDHE2 atom-atom interaction $[39]$, are able to extract a reasonable volume binding energy from finite drops in a liquid drop mass formula, but only by including additional terms beyond the standard volume and surface terms of Eq. (1). Fitting the drop energies with only volume and surface terms predicts a volume binding energy of $-1.42$ K while adding a curvature term $\propto A^{1/3}$ generates a much better fit, with a volume term of $-2.09$ K, much closer to the infinite liquid result of $-2.36$ K and the experimental value of $-2.47$ K.

The fact that the binding energy density traces the nucleon density in Fig. 1 is not surprising when considering the decomposition of the binding energy using full $A$-body wave functions,

$$E_0^A = \langle \Psi_0^A | \hat{H} | \Psi_0^A \rangle = E_0^A \langle \Psi_0^A | \Psi_0^A \rangle$$

$$= E_0^A \int d^3r \left[ \int d^3r_1 \ldots d^3r_A \left| \Psi_0^A (r_1, r_2, \ldots, r_A) \right|^2 \right]. \quad (4)$$

where the complete set $\{ | r_1, r_2, \ldots, r_A \rangle \}$ has been inserted and all other quantum numbers are suppressed for clarity. Noting that the bracketed term in Eq. (4) is the one-body density distribution $\rho(r)$, the binding energy can be written as

$$E_0^A = \frac{E_0^A}{A} \int d^3r \rho_A(r) \Rightarrow E_A(r) = \left( \frac{E_0^A}{A} \right) \rho_A(r). \quad (5)$$

The exact result in Eq. (5) reveals that the energy density is simply the nucleon density scaled by the binding energy. While Eq. (5) is exact, it cannot be used as a replacement for Eq. (3) because there is no guarantee that the DOM propagator is equal to the exact propagator, which would be built from the exact $A$-body ground-state wave function $[6]$. This is demonstrated in Fig. 2, which shows the energy density in $^{40}$Ca calculated using both Eq. (3) and Eq. (5). The general agreement of the curves in Fig. 2 is quantified by the similarity of the rms radii of the displayed energy and scaled nucleon density of 3.477 and 3.480 fm, respectively. This reveals that the DOM description of the density is close to exact. It is not surprising that there are deviations, since the DOM fit constrains the density which is only an indirect way of constraining the full $A$-body wave function. In principle, Eq. (5) is a good test to determine how close a given many-body approximation is to solving for the actual eigenfunction of the Hamiltonian or the accuracy of the Hamiltonian if the method is exact.

A method that is well-suited to calculate the energy-density using Eq. (4) is VMC. A VMC calculation of the

FIG. 2. The energy density of Eq. (3) (solid line) compared to the scaled nucleon density of Eq. (5) (dashed line) in $^{40}$Ca.

FIG. 3. Results of a VMC calculation of $^{12}$C with $\rho$ representing the density, $\mathcal{E}$ the energy density, $\mathcal{T}$ kinetic energy density, $\mathcal{V}$ the two-body potential energy density, and $\mathcal{U}$ the three-body potential energy density, respectively.
$^{12}$C binding-energy density is shown in Fig. 3 generating a total kinetic energy of 395 MeV, a two-body potential energy of -449 MeV, a three-body potential energy of -10.5 MeV, and a total energy of -64.8 MeV. In this calculation, the AV18 + Urbana-X [40] interactions were employed to generate the ground-state wave function. It is reasonable to expect that no differences in the shapes will occur in a GFMC treatment which generates a total energy of -93.3 MeV [26] compared to the experimental value of -92.16 MeV. The shape of the VMC energy density is very similar to the shape of the DOM energy density, and also traces the particle density. Included in Fig. 3 is the contribution of the three-body interaction to the energy density. Comparing the two- and three-body potential density clarifies that the latter contributes modestly to the total energy density and does not alter its shape. This implies that ignoring the three-body interaction by using Eqs. (2) and (3) in the DOM analysis does not alter the shape of the binding-energy density. Hence, the conclusions of this letter are robust.

The nuclear energy density can be further explored in $^{48}$Ca and $^{208}$Pb. The agreement between Eq. (5) and Eq. (3) in $^{48}$Ca and $^{208}$Pb is comparable to that of $^{40}$Ca. The case of $^{208}$Pb is particularly interesting because the interior is more extended than in $^{40}$Ca and $^{48}$Ca. This implies that finite-size (surface) effects are reduced in this region of $^{208}$Pb, making it an ideal system to compare with NM. Using isospin symmetry to remove the effect of the Coulomb interaction on the energy density of $^{40}$Ca is not valid in $^{208}$Pb, since $N > Z$. While removing the Coulomb energy density from $\mathcal{E}(r)$ would provide a NM-like energy density, the Coulomb potential is still reflected in the matter density of $^{208}$Pb. So, one way to compare with the NM calculations from Ref. [21] is to completely remove the Coulomb potential from the DOM self-energy. To preserve the proton number, the proton Fermi energy must therefore be shifted such that it remains between the particle-hole gap of the protons. The resulting Coulomb-less matter density exactly confirms the expected 0.16 fm$^{-3}$ in the interior of $^{208}$Pb. The matter density can also be extrapolated from the proton distribution by scaling with $A/Z$ to account for the asymmetry [6], generating a central value of 0.18 fm$^{-3}$.

The energy in the interior can be approximately calculated from the energy density using Eq. (5), $E_A(r) \approx \mathcal{E}_A(r) \left(\frac{A}{\rho_A(r)}\right)$. This approximation should be valid for small values of $r$, where the nuclear density is relatively constant and saturated. The binding energy with Coulomb removed as a function of $r$ in $^{208}$Pb is shown in Fig. 4. The ambiguity to determine the Coulomb-less interior density is reflected in the wide band shown in Fig. 4. The thin band represents the interpolation of SCGF calculations from Ref. [21] using AV18 at densities corresponding to 0.08, 0.12, and 0.16 fm$^{-3}$ obtained in the same way. These NM results require an additional 2-3 MeV per particle attraction to reproduce the DOM result, which is not inconsistent with the trend obtained for the required contribution of the three-body interaction to accurately describe the energies of light nuclei with GFMC [26]. The contribution of the symmetry energy per nucleon of $^{208}$Pb is $E_{sym} = 1.04$ MeV, leading to the expectation of the interior energy of $^{208}$Pb to be $E^{sym}_{208} = -15.0$ MeV based on the empirical mass formula (see dashed line in Fig. 4). This analysis therefore suggests that the energy in the interior (and hence the saturation energy) is less bound than what is expected from the empirical mass formula.

A comparison of the DOM energy as a function of radius for $^{12}$C, $^{40}$Ca, $^{48}$Ca, and $^{208}$Pb is shown in Fig. 5, where the Coulomb contribution has been removed from

![FIG. 4. Binding energy as a function of radius in $^{208}$Pb. The thick blue band covers the range of energies of $^{208}$Pb calculated using the DOM matter density (top) and the use of the DOM proton density scaled by $^{208}/82$ (bottom), both with Coulomb removed. The narrow band is similarly obtained from the SCGF calculations for the AV18 [21] (see text). The dashed line is the expected energy from the empirical mass formula.](image)

![FIG. 5. Binding energy as a function of radius in $^{12}$C (dashed line), $^{40}$Ca (solid line), $^{48}$Ca (dot-dot-dashed), and $^{208}$Pb (dot-dashed). The latter reflects the middle of the band in Fig. 4. The canonical 16 MeV/A binding is also shown.](image)
each nucleus. The energies in the core of each nucleus are all within a few MeV of each other, all of which are significantly less bound than 16 MeV per particle. Furthermore, Fig. 1 clearly shows that the interior of the nucleus does not significantly contribute to the total binding energy. The interpretation that the interior of the nucleus is a close approximation to NM leads to the inevitable conclusion that the saturation energy of symmetric NM is less than the canonical value of 16 MeV per particle. It has been noted in the past [15–17] that long-range correlations in finite nuclei and nuclear matter are not commensurate, implying an uncertainty in the extrapolation from Eq. (1) to NM. One way to address this is to make use of the fact that the energy density tracks the matter density, as shown in Fig. 2 and Eq. (5), and construct an alternate nuclear mass formula which depends on the density distribution and asymmetry rather than $A$ and $Z$.

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Revisiting the relation between the binding energy of finite nuclei and the equation of state of infinite nuclear matter

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KINETIC-ENERGY DENSITY

As stated in the main text, the energy density is calculated using the DOM spectral functions using the following expression,

\[ \mathcal{E}_A(r) = \frac{1}{2} \int_0^\infty \left[ E S_h(r, r; E) - \int_0^\infty dr' r'^2 \langle r | \hat{T} | r' \rangle S_h(r, r'; E) \right] dE. \quad (1) \]

The second term in Eq. (1) represents the kinetic energy density,

\[ T(r) = \int_0^\infty dE \int_0^\infty dr_1 r_1^2 \langle r_1 | \hat{T} | r \rangle S_h(r_1, r; E), \]

where the volume integral of \( T(r) \) is the total kinetic energy of the nucleus. The kinetic-energy operator in coordinate space,

\[ \langle r | \hat{T} | r' \rangle = \delta^3(r - r') \frac{-\hbar^2 \nabla_r^2}{2\mu} \]

is used to calculate \( T(r) \), resulting in the following expression:

\[ r^2 T(r) = -\hbar^2 \left[ \frac{d^2}{dr^2} - \frac{\ell(\ell + 1)}{r^2} \right] |rn(r, r')|_r, \]

where \( n(r) \) is the one-body density matrix defined as

\[ n(r, r') = \int_0^\infty dE S_h(r, r'; E). \]

PARAMETRIZATION OF THE POTENTIALS

The parametrization and fit of \(^{12}\)C is presented in this section. The parametrizations of \(^{40}\)Ca, \(^{48}\)Ca, and \(^{208}\)Pb can be found in Refs. [1–3], respectively. Table I displays the parameters for the \(^{12}\)C self-energy.

RESULTS

The results of the \(^{12}\)C fit are presented in this section. The constraint of the number of particles was incorporated to include contributions from \( \ell = 0 \) to 10. Such a range of \( \ell \)-values generates a sensible convergence with \( \ell \) when short-range correlations are included as in Ref. [4]. We obtain 5.8 protons from all \( \ell = 0 \) to 10 partial wave terms including \( j = \ell \pm \frac{1}{2} \) and 5.9 for neutrons. The corresponding binding energy can be found in the main text. If in future higher \( \ell \)-values are included, we expect a slight but not essential change in the fitted parameters.

We found the DOM self-energy by minimizing the \( \chi^2 \) using experimental data in the form of elastic-scattering cross sections, total and reaction cross sections, charge density, and particle number. The results of this fit led to the curves shown in this supplementary material.

The resulting elastic-scattering cross sections are shown in Fig. 1, the proton analyzing powers are shown in Fig. 2, the proton reaction cross section is shown in Fig. 3, and the neutron total cross section is shown in Fig. 4. The charge density is shown in Fig. 5.

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TABLE I. Fitted parameter values for proton and neutron potentials in $^{12}$C. This table also lists the number of the equation that defines each individual parameter.

| Parameter | Value |
|-----------|-------|
| $V_{HF}^{n}$ [MeV] | 90.8 |
| $r_{HF}^{n}$ [fm] | 0.952 |
| $a_{HF}^{n}$ [fm] | 0.417 |
| $\beta_{n}^{vol}$ [fm] | 0.908 |
| $\beta_{n}^{vol}$ [fm] | 0.738 |
| $\rho_{n}^{SB}$ [fm] | 0.911 |
| $\rho_{n}^{SB}$ [fm] | 1.01 |
| $V_{so}^{n}$ [MeV] | 26.6 |
| $\alpha_{so}^{n}$ [fm] | 0.540 |
| $\beta_{s}^{vol}$ [fm] | 0.755 |
| $A_{so}^{n}$ [MeV] | 1.23 |
| $B_{so}^{n}$ [MeV] | -1.62 |
| $\alpha$ | 66.4 |

Spin-orbit

| Parameter | Value |
|-----------|-------|
| $a_{s}^{vol}$ [fm] | 0.536 |
| $r_{s}^{vol}$ [fm] | 1.27 |
| $\rho_{s}^{vol}$ [fm] | 0.340 |
| $\alpha_{s}^{vol}$ [fm] | 0.256 |
| $\beta_{s}^{vol}$ [fm] | 1.02 |
| $A_{s}^{vol}$ [MeV] | 1.08 |
| $B_{s}^{vol}$ [MeV] | 6.51 |
| $E_{s}^{vol}$ [MeV] | 25.3 |
| $A_{so}^{vol}$ [MeV] | 2.31 |
| $B_{so}^{vol}$ [MeV] | 16.9 |
| $E_{so}^{vol}$ [MeV] | 8.97 |
| $E_{s}^{so}$ [MeV] | 1.61 |
| $E_{so}^{vol}$ [MeV] | 23.4 |
| $E_{so}^{vol}$ [MeV] | 67.5 |
| $\alpha$ | 0.189 |

Volume imaginary

| Parameter | Value |
|-----------|-------|
| $a_{v}^{vol}$ [fm] | 0.943 |
| $r_{v}^{vol}$ [fm] | 1.40 |
| $\rho_{v}^{vol}$ [fm] | 3.38 |
| $\alpha_{v}^{vol}$ [fm] | 0.316 |
| $\beta_{v}^{vol}$ [fm] | 0.631 |
| $A_{v}^{vol}$ [MeV] | 1.72 |
| $B_{v}^{vol}$ [MeV] | 13.0 |
| $C_{v}^{vol}$ [MeV] | 26.3 |
| $A_{v}^{so}$ [MeV] | 198 |
| $B_{v}^{so}$ [MeV] | 199 |
| $C_{v}^{so}$ [MeV] | 199 |
| $A_{v}^{so}$ [MeV] | 28.0 |
| $B_{v}^{so}$ [MeV] | 23.11 |
| $C_{v}^{so}$ [MeV] | 20.0 |
| $\alpha$ | 94.9 |

Surface imaginary

| Parameter | Value |
|-----------|-------|
| $a_{s}^{sur}$ [fm] | 0.189 |
| $r_{s}^{sur}$ [fm] | 0.493 |
| $\rho_{s}^{sur}$ [fm] | 1.40 |
| $\alpha_{s}^{sur}$ [fm] | 3.38 |
| $\beta_{s}^{sur}$ [fm] | 0.316 |
| $\beta_{s}^{sur}$ [fm] | 0.631 |
| $A_{s}^{sur}$ [MeV] | 1.72 |
| $B_{s}^{sur}$ [MeV] | 13.0 |
| $C_{s}^{sur}$ [MeV] | 26.3 |
| $A_{s}^{so}$ [MeV] | 198 |
| $B_{s}^{so}$ [MeV] | 199 |
| $C_{s}^{so}$ [MeV] | 199 |
| $A_{s}^{so}$ [MeV] | 28.0 |
| $B_{s}^{so}$ [MeV] | 23.11 |
| $C_{s}^{so}$ [MeV] | 20.0 |
| $\alpha$ | 94.9 |

FIG. 1. Calculated and experimental proton elastic-scattering angular distributions of the differential cross section $d\sigma/d\Omega$. The data at each energy is offset by factors of ten to help visualize all of the data at once. Refs. [5–23] contain the proton experimental data. Refs. [24–30] contain the neutron experimental data.

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FIG. 2. Proton analyzing power generated from the DOM self-energy. Refs. [11, 14, 19, 31–34] contain the proton experimental data. Refs. [25, 35, 36] contain the neutron experimental data.

FIG. 3. Proton reaction cross section generated from the DOM self-energy. The experimental data can be found in Refs. [37–39].

FIG. 4. Neutron total cross section (solid line) an reaction cross section (dashed line) generated from the DOM self-energy. The total cross section data can be found in Ref. [40]. The reaction cross section data can be found in Ref. [41].

FIG. 5. Experimental and fitted $^{12}$C charge density. The solid black line is calculated using the DOM self-energy and folding with the proton charge distribution while the experimental band represents the 1% error associated with the extracted charge density from elastic electron scattering experiments using the sum of Gaussians parametrization [42, 43].

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