RESTRICTION OF CHARACTERS AND PRODUCTS OF CHARACTERS

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ABSTRACT. Let $G$ be a finite $p$-group, for some prime $p$, and $\psi, \theta \in \text{Irr}(G)$ be irreducible complex characters of $G$. It has been proved that if, in addition, $\psi, \theta$ are faithful characters, then the product $\psi \theta$ is a multiple of an irreducible or it is the nontrivial linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters of $G$. We show that if we do not require the characters to be faithful, then given any integer $k > 0$, we can always find a $p$-group $G$ and irreducible characters $\Psi$ and $\Theta$ such that $\Psi \Theta$ is the nontrivial combination of exactly $k$ distinct irreducible characters. We do this by translating examples of decompositions of restrictions of characters into decompositions of products of characters.

1. Introduction

Let $G$ be a finite group. Denote by $\text{Irr}(G)$ the set of irreducible complex characters of $G$. Let $\psi$ and $\theta$ be characters in $\text{Irr}(G)$. Since a product of characters is a character, the product $\psi \theta$, where $\psi \theta(g) = \psi(g) \theta(g)$ for all $g \in G$, is a character. Then the decomposition of the character $\psi \theta$ into its distinct irreducible constituents $\phi_1, \phi_2, \ldots, \phi_k \in \text{Irr}(G)$ has the form

$$\psi \theta = \sum_{i=1}^{k} n_i \phi_i$$

where $k > 0$ and $n_i = (\psi \theta, \phi_i) > 0$ is the multiplicity of $\phi_i$ in $\psi \theta$ for each $i = 1, \ldots, k$. Let $\eta(\psi \theta) = k$ be the number of distinct irreducible constituents of the character $\psi \theta$.

Given any subgroup $H$ of $G$, we denote by $\chi_H$ the restriction of the character $\chi$ of $G$ to $H$.

Let $G$ be a finite $p$-group, where $p > 2$ is a prime number. Given any two faithful characters $\psi, \theta \in \text{Irr}(G)$, in Theorem A of [1] it is proved that the product $\psi \theta$ is either a multiple of an irreducible, i.e $\eta(\psi \theta) = 1$, or $\psi \theta$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible constituents, i.e. $\eta(\psi \theta) \geq \frac{p+1}{2}$. Thus there is not $5$-group $G$ with faithful characters $\psi, \theta \in \text{Irr}(G)$ such that $\psi \theta$ has exactly two distinct irreducible constituents. But can we find a $5$-group $P$ with characters $\psi, \theta \in \text{Irr}(P)$ such that $\psi \theta$ has exactly two distinct irreducible constituents? The answer is yes. Moreover
Theorem A. Fix a prime number $p$. Let $k > 0$ and $n_i > 0$, $i = 1, \ldots, k - 1$, be integers. Choose $r > 0$ such that $p^r > \sum_{i=1}^{k-1} n_i$. Let $n_k = p^r - \sum_{i=1}^{k-1} n_i$. Then there exists a finite $p$-group $G$ with characters $\Psi, \Theta \in \text{Irr}(G)$ such that

$$\Psi \Theta = \sum_{i=1}^{k} n_i \Phi_i,$$

where $\Phi_1, \Phi_2, \ldots, \Phi_k$ are distinct irreducible characters in $\text{Irr}(G)$ and $n_i = (\Psi \Theta, \Phi_i)$. Thus $\eta(\Psi \Theta) = k$.

The main key for the previous result is the following

Theorem B. Let $P$ be a finite $p$-group, $Q < P$ be a subgroup of $P$ and $\psi \in \text{Irr}(G)$. Assume that

(i) $\psi_Q = \sum_{i=1}^{k} n_i \phi_i$,

where $\phi_1, \phi_2, \ldots, \phi_k$ are distinct irreducible characters in $\text{Irr}(Q)$ and $n_i = (\psi_Q, \phi_i) > 0$ for each $i = 1, 2, \ldots, k$. Then there exists a $p$-group $G$ with characters $\Psi, \Theta \in \text{Irr}(G)$ such that

(ii) $\Psi \Theta = \sum_{i=1}^{k} n_i \Phi_i$,

where $\Phi_1, \Phi_2, \ldots, \Phi_k$ are distinct irreducible characters in $\text{Irr}(G)$ and $n_i = (\Psi \Theta, \Phi_i) > 0$ for each $i = 1, 2, \ldots, k$.

The previous result allows us to translate all examples of decompositions of restrictions of characters into decompositions of products of characters. Since the product $\psi \theta$ of two characters $\psi, \theta$ of a group $G$ can be regarded as the restriction of the character $\psi \times \theta$ of the direct product group $G \times G$ to the diagonal subgroup $D(G) = \{(g, g) | g \in G\}$, which is another copy of $G$ inside $G \times G$, the converse also holds.

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2. Proofs

Lemma 2.1. Given any finite $p$-group $P$ and any subgroup $Q$ of $P$, there exists an elementary abelian $p$-group $E$ and $\lambda \in \text{Lin}(E)$ such that $P$ acts on $E$ as automorphisms and $Q$ is the stabilizer of $\lambda$ in $P$.

Proof. Let $S$ be a finite set on which $P$ acts transitively with $Q$ as the stabilizer of one point $s \in S$. Then extend the action of $P$ on $S$ to one of $P$ as automorphisms of the elementary abelian $p$-group $E = \langle S \rangle$ with $S$ as a basis. Finally, take $\lambda$ to be any linear character on $E$ sending $s$ to some primitive $p$-th root of unity, and every $s' \in S - \{s\}$ to 1. □

Notation. We will denote by $1_G$ the trivial character of $G$. We will also denote by $\text{Lin}(G)$ the set of linear characters of $G$. Also we denote by $1 \in G$ the identity element of $G$.

Proof of Theorem B. Let $E$ be an elementary abelian $p$-group and $\lambda \in \text{Lin}(E)$ such that $P$ acts on $E$ as automorphisms of $E$ and the stabilizer $P_\lambda$ of $\lambda$ in $P$ is $Q$. We define $G$ to be the semi-direct product $P \ltimes E$. We identify both $P$ and $E$ with their images in $G$. So

$$G = P E = P \ltimes E.$$
We define \( \Psi \) to be the inflation of \( \psi \in \text{Irr}(P) \) to an irreducible character of \( G \) with the values
\[
\Psi(\sigma \tau) = \psi(\sigma)
\]
for any \( \sigma \in P \) and \( \tau \in E \).

Because \( Q \) fixes \( \lambda \), the factor group \( QE/\text{Ker}(\lambda) \) is naturally isomorphic to \( Q \times (E/\text{Ker}(\lambda)) \). It follows that there is a unique character \( \phi_i \lambda \in \text{Irr}(QE) \), for each \( i = 1, 2, \ldots, k \), such that
\[
(\phi_i \lambda)(\sigma \tau) = \phi_i(\sigma)\lambda(\tau)
\]
for any \( \sigma \in Q \) and \( \tau \in E \). The characters \( \phi_i \lambda \), for \( i = 1, 2, \ldots, k \), are clearly distinct and lie over \( \lambda \). Because \( QE = P \lambda E \) is the stabilizer \( G \lambda \) of \( \lambda \) in \( G \), Clifford theory tells us that the characters
\[
\Phi_i = (\phi_i \lambda)^G
\]
induced by these \( \phi_i \lambda \) are distinct and lie in \( \text{Irr}(G) \).

The trivial character \( 1_Q \) on \( Q \) also defines an irreducible character \( 1_Q \lambda \) of \( QE \), with the values
\[
(1_Q \lambda)(\sigma \tau) = \lambda(\tau)
\]
for any \( \sigma \in Q \) and \( \tau \in E \). This also induces an irreducible character
\[
(1.3) \quad \Theta = (1_Q \lambda)^G \in \text{Irr}(G).
\]

We now prove that
\[
\Psi \Theta = \sum_{i=1}^{k} n_i \Phi_i.
\]

To see this, we start with the formula
\[
\Psi \Theta = \Psi(1_Q \lambda)^G = (\Psi_{QE}(1_Q \lambda))^G,
\]
which holds because \( \Theta \) is induced in \( (1.3) \) from the character \( 1_Q \lambda \) of \( QE \). Since \( \Psi \) is inflated \( (2.2) \) from \( \psi \in \text{Irr}(P) \), its restriction \( \Psi_{QE} \) is inflated from the character \( \psi_Q \) of \( Q \simeq QE/E \). It follows that \( \Psi_{QE}(1_Q \lambda) \) is precisely the product character \( \psi_Q \lambda \). In view of (i) this product character is
\[
\Psi_{QE}(1_Q \lambda) = \psi_Q \lambda = \sum_{i=1}^{k} n_i \phi_i \lambda.
\]

Hence
\[
\Psi \Theta = \left( \sum_{i=1}^{k} n_i \phi_i \lambda \right)^G = \sum_{i=1}^{k} n_i (\phi_i \lambda)^G = \sum_{i=1}^{k} n_i \Phi_i.
\]

Thus (ii) holds. \( \square \)
Example 2.4. Fix a prime number $p$. Let $k > 0$ and $n_i > 0$, $i = 1, \ldots, k - 1$, be integers. Choose integers $r, t > 0$ such that $p^r > \sum_{i=1}^{k-1} n_i$ and $p^t \geq k$. Let $Z_{p^n}$ be a cyclic group of order $p^n$. Let $N = Z_{p^r} \times Z_{p^r} \times \cdots \times Z_{p^r}$ be the direct product of $p^r$ copies of $Z_{p^n}$. Thus $|N| = (p^r)^{p^r}$. Observe that $Z_{p^n}$ acts on $N$ by permuting the entries of $N$. Set $P = Z_{p^r} \times N = Z_{p^n}N$ and thus $P = Z_{p^r} \wr Z_{p^n}$ is the wreath product of $Z_{p^n}$ by $Z_{p^r}$.

Fix a generator $c$ in $Z_{p^r}$ and $\alpha \in \text{Lin}(Z_{p^r})$ such that $\alpha(c)$ is a primitive $p^t$-th root of unity. Set $\lambda = (\alpha, 1_{Z_{p^r}}, \ldots, 1_{Z_{p^r}}) \in \text{Lin}(N)$. We can check that the stabilizer $P_{\lambda}$ of $\lambda$ in $N$, and so $\psi = \lambda^p \in \text{Irr}(P)$ is a character of degree $p^r$. Observe also that $\psi_N = \sum_{i=1}^{p^r} \lambda_i$, where $\lambda_i = (1_{Z_{p^r}}, \ldots, 1_{Z_{p^r}}, \alpha, 1_{Z_{p^r}}, \ldots, 1_{Z_{p^r}}) \in \text{Lin}(N)$ is the character with $\alpha$ in the $i$-th position, for $i = 1, \ldots, p^r$.

Fix

$$q = (c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_3, \ldots, c_3, \ldots, c_{k-1}, \ldots, c_{k-1}, 1, 1, \ldots, 1) \in N,$$

where the first $n_1$-entries are $c$, the $(n_1 + 1)$-th entry to the $(n_1 + n_2)$-th is $c^2$ and so for.

Let $Q$ be the subgroup of $N$ generated by $q$. Observe that $Q$ is a cyclic subgroup of $N$ of order $p^t$. Let $\delta \in \text{Lin}(Q)$ such that $\delta(q) = \alpha(c)$. Since $p^t \geq k$ and $\alpha(c)$ is a primitive $p^t$-root of unity, then for $1 \leq i, j \leq k - 1$ we have that $\delta^i \neq \delta^j$ if $i \neq j$. Observe that $\lambda_i(q) = \alpha(c^i) = \delta(q)$ for $1 \leq l \leq n_1$ and so $n_1 = (\psi_Q, \delta)$. For $l > n_1$, we can check that $\lambda_l(q) = \alpha(c^l) = \alpha^j(c) = \delta^j(q)$ if and only if $\sum_{i=1}^{j-1} n_i < l \leq \sum_{i=1}^{j} n_i$ and so $n_j = (\psi_Q, \delta^j)$. Observe that if $l > \sum_{i=1}^{k-1} n_i$, then $\lambda_l(q) = \alpha(1) = 1$ and so $n_k = p^r - \sum_{i=1}^{k-1} n_i = (\psi_Q, 1_Q)$. Thus $\psi_Q = \sum_{i=1}^{k-1} n_i \delta^i + n_k 1_Q$.

Observe that Theorem A follows then by the previous example and Theorem B.

References

[1] E. Adan-Bante, Products of characters and finite $p$-groups, J. Algebra 277 (1), 236-255.