Topologically massive higher spin gauge theories

Sergei M. Kuzenko and Michael Ponds

Department of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia

Email: sergei.kuzenko@uwa.edu.au, michael.ponds@research.uwa.edu.au

Abstract

We elaborate on conformal higher-spin gauge theory in three-dimensional (3D) curved space. For any integer $n > 2$, we introduce a conformal spin-$\frac{n}{2}$ gauge field $h^{(n)} = h_{\alpha_1...\alpha_n}$ (with $n$ spinor indices) of dimension $(2 - n/2)$ and argue that it possesses a Weyl primary descendant $C^{(n)}$ of dimension $(1 + n/2)$, which is divergenceless and gauge invariant in every conformally flat space. Primary fields $C^{(3)}$ and $C^{(4)}$ coincide with the linearised Cottino and Cotton tensors, respectively. The properties of $C^{(n)}$ imply that the Chern-Simons-type action $S_{CS}^{(n)} \propto \int d^3x \, e h_{\alpha(n)} C^{\alpha(n)}$ is Weyl and gauge invariant in any conformally flat space. The actions $S_{CS}^{(n)}$, which for $n > 4$ are higher-spin extensions of linearised conformal gravity, are used to construct gauge-invariant formulations for massive higher-spin fields in Minkowski and anti-de Sitter (AdS) space. As is known, the irreducible unitary massive spin-$\frac{n}{2}$ representations of the 3D Poincaré and AdS groups (for $n > 1$) are realised in terms of symmetric rank-$n$ spinor fields obeying first-order differential equations. We demonstrate that these equations follow from the equations of motion in certain higher-derivative gauge theories. If $n = 2s$, our massive action is $S_{\text{massive}}^{(2s)} = \lambda S_{CS}^{(2s)} + \mu^{2s-3} S_{F}^{(2s)}$, where $S_{F}^{(2s)}$ denotes the Fronsdal-type action for a massless spin-$s$ field in AdS$_3$. If $n = 2s+1$, our massive action is $S_{\text{massive}}^{(2s+1)} = \lambda S_{CS}^{(2s+1)} + \mu^{2s-1} S_{FF}^{(2s+1)}$, where $S_{FF}^{(2s+1)}$ denotes the Fang-Fronsdal-type action for a massless spin-$(s+1/2)$ field in AdS$_3$. Finally we develop $\mathcal{N} = 1$ supersymmetric extensions of the above results.
# Contents

1 Introduction

2 On-shell massive (super)fields
   2.1 Massive fields .................................................. 5
   2.2 Massive superfields ............................................. 6

3 Conformal higher-spin fields
   3.1 Conformal gravity .............................................. 8
   3.2 Conformal gauge fields ....................................... 9
   3.3 Higher-spin Cotton tensor in Minkowski space ............. 12
   3.4 Higher-spin Cotton tensor in conformally flat spaces .... 13

4 Massive higher-spin actions in maximally symmetric spaces 15
   4.1 Massive higher-spin actions in AdS space ............... 15
   4.2 Massive higher-spin actions in Minkowski space: The fermionic case .... 17
   4.3 Massive higher-spin actions in Minkowski space: The bosonic case .... 20

5 Conformal higher-spin gauge superfields 22
   5.1 Conformal supergravity .................................... 23
   5.2 Conformal gauge superfields ................................. 24
   5.3 Higher-spin super-Cotton tensor in Minkowski superspace .... 26
   5.4 Higher-spin super-Cotton tensor in conformally flat superspaces .... 27
   5.5 Massive supersymmetric higher-spin theories in AdS superspace .... 28
      5.5.1 First-order massless actions ............................ 28
      5.5.2 Second-order massless actions .......................... 29
   5.6 From AdS superspace to AdS space ........................... 30
1 Introduction

A unique feature of three spacetime dimensions (3D) is the existence of topologically massive Yang-Mills and gravity theories. These theories are obtained by augmenting the usual Yang-Mills action or the gravitational action by a gauge-invariant topological mass term. Such a mass term coincides with a Chern-Simons functional in the Yang-Mills case \cite{1, 2, 3, 4, 5} and with a Lorentz Chern-Simons term in the case of gravity \cite{4, 5}. The Lorentz Chern-Simons term is required to make the gravitational field possess nontrivial dynamics, for the pure gravity action propagates no local degrees of freedom. The Lorentz Chern-Simons term can be interpreted as the action for 3D conformal gravity \cite{6, 7}.\footnote{The usual Einstein-Hilbert action for 3D gravity with a cosmological term can also be interpreted as the Chern-Simons action for the anti-de Sitter group \cite{8, 9}.}

Topologically massive gravity possesses supersymmetric extensions. In particular, $\mathcal{N} = 1$ topologically massive supergravity was constructed in \cite{10, 11}. The off-shell formulations for...
\( \mathcal{N} \)-extended topologically massive supergravity theories were presented in \[12, 13\] for \( \mathcal{N} = 2 \), in \[14\] for \( \mathcal{N} = 3 \), and in \[14, 15\] for the \( \mathcal{N} = 4 \) case. In all of these theories, the action functional is a sum of two terms, one of which is the action for pure \( \mathcal{N} \)-extended supergravity (Poincaré or anti-de Sitter) and the other is the action for \( \mathcal{N} \)-extended conformal supergravity. The off-shell actions for \( \mathcal{N} \)-extended supergravity theories in three dimensions were given in \[16\] for \( \mathcal{N} = 1 \), \[17, 18\] for \( \mathcal{N} = 2 \) and \[17\] for the cases \( \mathcal{N} = 3, 4 \). The off-shell actions for \( \mathcal{N} \)-extended conformal supergravity were given in \[6\] for \( \mathcal{N} = 1 \), \[19\] for \( \mathcal{N} = 2 \), \[20\] for \( \mathcal{N} = 3, 4, 5 \), and in \[21, 22\] for the \( \mathcal{N} = 6 \) case. Refs. \[20, 22\] made use of the off-shell formulation for \( \mathcal{N} \)-extended conformal supergravity proposed in \[23\]. The on-shell formulation for \( \mathcal{N} \)-extended conformal supergravity with \( \mathcal{N} > 2 \) was given in \[24\]. On-shell approaches to \( \mathcal{N} \)-extended topologically massive supergravity theories with \( 4 \leq \mathcal{N} \leq 8 \) were presented in \[25, 26, 27, 28, 29\].

Topologically massive \( \mathcal{N} = 1 \) supergravity, with or without a cosmological term, may be linearised about a maximally supersymmetric solution. The resulting linearised actions for the gravitino and the gravitational field contain higher derivatives. However, the genuine massive states prove to obey first-order differential equations. This paper is devoted to the description of higher-spin extensions of the linearised actions for topologically massive gravity and \( \mathcal{N} = 1 \) supergravity. In particular, for every (half-)integer spin \( n/2 \), where \( n = 5, 6, \ldots \), we present a gauge-invariant higher-derivative action in Minkowski space that propagates a single massive state of helicity \( +n/2 \) or \( -n/2 \) on the mass shell. The action is of the form

\[
S_{\text{massive}} = S_{\text{massless}} + S_{\text{CS}}. \tag{1.1}
\]

Here \( S_{\text{massless}} \) denotes the 3D massless spin-\( \frac{n}{2} \) gauge action of the Fronsdal-Fang type \[30, 31\], with no propagating degrees of freedom. The second term in the right-hand side of (1.1) is a conformal spin-\( \frac{n}{2} \) gauge action \[32, 33\] described by a Lagrangian of the schematic form

\[
\mathcal{L}_{\text{CS}} \propto \varphi(n) \partial^{n-1} \varphi(n),
\]

where \( \varphi(n) \) stands for the conformal spin-\( \frac{n}{2} \) field. We show that \( S_{\text{massive}} \) propagates a single massive state described by the equations (2.1). We also present AdS extensions of the actions introduced, as well as their \( \mathcal{N} = 1 \) supersymmetric generalisations.

In the case of Minkowski space, our actions (1.1) are in fact contained, at the component level, in the massive supersymmetric higher-spin models proposed in \[34, 35\]. However, the analysis in \[34, 35\] was carried out mostly in terms of superfields so that the component actions were not studied. All the massive higher-spin gauge models in AdS, which are presented in this paper, are new.

This paper is organised as follows. In section 2 we review field realisations of the irreducible massive spin-\( \frac{n}{2} \) representations \( n = 2, 3, \ldots \) of the 3D Poincaré and AdS groups. We also review the structure of on-shell massive higher-spin superfields for both 3D \( \mathcal{N} = 1 \) Poincaré
and AdS supersymmetry. In section 3 we introduce, for any integer \( n \geq 2 \), a conformal spin-\( \frac{n}{2} \) gauge field \( h(n) = h_{\alpha_1...\alpha_n} = h(\alpha_1...\alpha_n) \) and argue that it possesses a Weyl primary descendant \( \mathcal{C}(n) \) of dimension \( (1 + \frac{n}{2}) \) with the following properties: (i) is of the schematic form \( \nabla^{n-1} h(n) \); (ii) \( \mathcal{C}(n) \) is divergenceless and gauge invariant in an arbitrary conformally flat space. These descendants \( \mathcal{C}(n) \) are constructed in any conformally flat space. Making use of the primary fields \( \mathcal{C}(n) \), we propose Chern-Simons-type actions \( S_{CS}^{(n)} \propto \int d^3x \epsilon^{\alpha_1...(n)} \mathcal{C}_\alpha(n) \) which are Weyl and gauge invariant in any conformally flat space, and which are higher-spin extensions of the linearised action for 3D conformal gravity. These conformal higher-spin actions are then used to construct massive higher-spin gauge theories in AdS, described by the actions (4.5a) and (4.5b). In section 4 we study the dynamics of the flat-space counterparts to the gauge theories (4.5a) and (4.5b).

Sections 5 and 6 are devoted to supersymmetric extensions of the results presented in sections 3 and 4. In section 5 we introduce conformal higher-spin gauge superfields \( \mathcal{H}_\alpha(n) \) in curved \( \mathcal{N} = 1 \) superspace. These conformal gauge superfields are argued to possess primary descendants \( \mathcal{W}_\alpha(n) \) of dimension \( (1 + \frac{n}{2}) \) that are locally supersymmetric extensions of the linearised higher-spin super-Cotton tensors [33, 35]. For any conformally flat superspace background, the primary superfields \( \mathcal{W}_\alpha(n) \) are explicitly constructed, and are shown to be gauge invariant and conserved. Making use of \( \mathcal{H}_\alpha(n) \) and \( \mathcal{W}_\alpha(n) \), we construct a higher-spin extension of the action for linearised \( \mathcal{N} = 1 \) conformal gravity, \( S_{SCS}^{(n)}[\mathcal{H}(n)] \), which is given by eq. (5.21). We employ \( S_{SCS}^{(n)}[\mathcal{H}(n)] \) to construct massive higher-spin gauge actions in \( \mathcal{N} = 1 \) AdS superspace, given by eqs. (5.39a) and (5.39b). Section 6 describes the component structure of the supersymmetric higher-spin theories introduced in section 5, with the analysis being restricted to the flat-superspace case. Concluding comments and discussion are given in section 7. The main body of the paper is accompanied by three appendices. Appendix A describes our notation and conventions. Appendix B reviews the Tyutin-Vasiliev action [36]. Appendix C provides two realisations for the higher-spin Cotton tensor \( C_{\alpha(n)} \) as a descendant of gauge-invariant field strengths corresponding to two different higher-spin massless models[2].

2 On-shell massive (super)fields

In this section we review the structure of irreducible massive higher-spin (super)fields in Minkowski space and in anti-de Sitter (AdS) space.

[2] A similar result in the \( \mathcal{N} = 2 \) supersymmetric case was given in [34].
2.1 Massive fields

We first recall the definition of on-shell massive fields in Minkowski space. Given a positive integer \( n > 1 \), a massive field, \( \phi_{\alpha_1\cdots\alpha_n} = \bar{\phi}_{\alpha_1\cdots\alpha_n} = \phi_{(\alpha_1\cdots\alpha_n)} \), is a real symmetric rank-\( n \) spinor field which obeys the differential conditions \[36\] (see also \[37\])

\[
\partial^{\beta\gamma} \phi_{\beta\gamma\alpha_1\cdots\alpha_{n-2}} = 0 \tag{2.1a}
\]
\[
\partial^\beta (\alpha_1 \phi_{\alpha_2\cdots\alpha_n})_{\beta} = m\sigma \phi_{\alpha_1\cdots\alpha_n} \, , \quad \sigma = \pm 1 \, , \tag{2.1b}
\]

with \( m \) being the mass of the field. In the spinor case, \( n = 1 \), eq. \((2.1a)\) is absent, and the massive field is defined to obey the Dirac equation \((2.1b)\). It is easy to see that \((2.1a)\) and \((2.1b)\) imply the mass-shell equation

\[
(\Box - m^2) \phi_{\alpha_1\cdots\alpha_n} = 0 \, . \tag{2.2}
\]

In the spinor case, \( n = 1 \), eq. \((2.2)\) follows from the Dirac equation \((2.1b)\). The helicity of \( \phi_{\alpha(n)} \) is \( \lambda = n/2 \sigma \), and the spin of \( \phi_{\alpha(n)} \) is \( n/2 \).

It should be remarked that the system of equations \((2.1a)\) and \((2.2)\) is equivalent to the 3D version of the Fierz-Pauli field equations \[38\]. The general solution to \((2.1a)\) and \((2.2)\) is a superposition of two massive states of helicity \( +\frac{n}{2} \) and \( -\frac{n}{2} \), respectively. Twenty years ago, Tyutin and Vasiliev \[36\] proposed Lagrangian formulations for massive higher-spin fields that lead to the equations \((2.1a)\) and \((2.1b)\) on the mass shell. Their actions did not possess gauge invariance. In the present paper, we propose gauge-invariant formulations for massive higher-spin fields in Minkowski space that lead to the equations \((2.1a)\) and \((2.1b)\) on-shell.

In the case of AdS space, massive fields are defined by the following equations \[39, 40\] (see also \[41\])

\[
\nabla^{\beta\gamma} \phi_{\beta\gamma\alpha_1\cdots\alpha_{n-2}} = 0 \, , \tag{2.3a}
\]
\[
\nabla^\beta (\alpha_1 \phi_{\alpha_2\cdots\alpha_n})_{\beta} = \mu \phi_{\alpha_1\cdots\alpha_n} \, , \tag{2.3b}
\]

for some real mass parameter \( \mu \). Equation \((2.3b)\) implies that

\[
(\nabla^a \nabla_a + 2(n+2)S^2 - \mu^2) \phi_{\alpha(n)} = 0 \, , \tag{2.4}
\]

where the parameter \( S \) is related to the AdS curvature via eq. \((3.45a)\). Equation \((2.4)\) can be rewritten in terms of the quadratic Casimir operator of the 3D AdS group \( \text{SO}(2,2) \),

\[
\mathcal{Q} := \nabla^a \nabla_a - 2S^2 M^{\alpha\delta} M_{\gamma\delta} \, , \quad [\mathcal{Q}, \nabla_a] = 0 \, , \tag{2.5}
\]
with $M_{\gamma\delta}$ the Lorentz generators, see Appendix A.

Equations (2.3a) and (2.4) constitute the 3D AdS counterpart to the Fierz-Pauli field equations. They describe a reducible representation of the AdS isometry group. Gauge-invariant Lagrangian formulations for massive higher-spin fields in AdS, which lead to the equations (2.3a) and (2.4) on the mass shell, were developed in \[42, 43, 44, 45\], including $\mathcal{N} = 1$ supersymmetric extensions obtained by combining the bosonic and fermionic actions (on-shell supersymmetry). The formulations given in \[42, 43, 44, 45\] are based on Zinoviev’s gauge-invariant approach \[46\] to describe massive higher-spin fields. In the present paper, we propose different gauge-invariant formulations for massive higher-spin fields in AdS that lead to the equations (2.3a) and (2.3b) on-shell.

2.2 Massive superfields

For $n > 0$, a massive superfield $T_{\alpha(n)}$ is defined to be a real symmetric rank-$n$ spinor, $T_{\alpha_1...\alpha_n} = \bar{T}_{\alpha_1...\alpha_n} = T_{(\alpha_1...\alpha_n)}$, which obeys the differential conditions \[47\] (see also \[35\])

\[
D^\beta T_{\beta\alpha_1...\alpha_{n-1}} = 0 \implies \partial^{\beta\gamma} T_{\beta\gamma\alpha_1...\alpha_{n-2}} = 0 , \tag{2.6a}
\]

\[
-\frac{i}{2} D^2 T_{\alpha_1...\alpha_n} = m\sigma T_{\alpha_1...\alpha_n} , \quad \sigma = \pm 1 . \tag{2.6b}
\]

Here $D^2 = D^\alpha D_\alpha$, and $D_\alpha$ is the spinor covariant derivative of $\mathcal{N} = 1$ Minkowski superspace. It follows from (2.6a) that

\[
-\frac{i}{2} D^2 T_{\alpha_1...\alpha_n} = \partial^{\beta} (T_{\alpha_1 T_{\alpha_2...}\alpha_n})_\beta , \tag{2.7}
\]

and thus $T_{\alpha(n)}$ is an on-shell superfield,

\[
\partial^{\beta} (T_{\alpha_1 T_{\alpha_2...}\alpha_n})_\beta = m\sigma T_{\alpha_1...\alpha_n} , \quad \sigma = \pm 1 . \tag{2.8}
\]

It follows from (2.6b) that

\[
(\Box - m^2) T_{\alpha(n)} = 0 . \tag{2.9}
\]

For the superhelicity of $T_{\alpha(n)}$ we obtain

\[
\kappa = \frac{1}{2} \left( n + \frac{1}{2} \right) \sigma . \tag{2.10}
\]

\[\text{3The equations (2.6a) and (2.9) are the $\mathcal{N} = 1$ supersymmetric extension of the Fierz-Pauli equations.}\]
We define the superspin of $T_{\alpha(n)}$ to be $n/2$. The massive supermultiplet $T_{\alpha(n)}$ contains two ordinary massive fields of the type (2.1), which are

$$\phi_{\alpha_1...\alpha_n} := T_{\alpha_1...\alpha_n}|_{\theta=0}, \quad \phi_{\alpha_1...\alpha_{n+1}} := i^{n+1}D_{(\alpha_1}T_{\alpha_2...\alpha_{n+1})}|_{\theta=0}. \quad (2.11)$$

Their helicity values are $\frac{n}{2}\sigma$ and $\frac{n+1}{2}\sigma$, respectively.

The off-shell gauge-invariant formulations for massive higher-spin $\mathcal{N} = 1$ supermultiplets in Minkowski superspace, which lead to the equations (2.6a) and (2.6b) on the mass shell, were constructed in [35].

In the case of $\mathcal{N} = 1$ AdS supersymmetry, on-shell massive superfields are described by the equations [47]

$$\mathcal{D}^\beta T_{\alpha_1...\alpha_{n-1}\beta} = 0, \quad (2.12a)$$
$$-\frac{i}{2}D^2 T_{\alpha_1...\alpha_n} = \mu T_{\alpha_1...\alpha_n}, \quad (2.12b)$$

with $\mu$ a real mass parameter and $D^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha$. Here $\mathcal{D}_\alpha = (D_a, D_\alpha)$ are the covariant derivatives of the $\mathcal{N} = 1$ AdS superspace, see section 5 for the details. It can be shown that

$$-\frac{1}{4}D^2D^2 = \mathcal{D}^\alpha \mathcal{D}_\alpha - 2iS\mathcal{D}^2 + 2SD^{\alpha\beta}M_{\alpha\beta} - 2S^2M^{\alpha\beta}M_{\alpha\beta}. \quad (2.13)$$

This differential operator, which is the square of the operator in the left-hand side of (2.12b), can be expressed via the quadratic Casimir operator of the 3D $\mathcal{N} = 1$ AdS supergroup,

$$Q = -\frac{1}{4}D^2D^2 + iSD^2, \quad [Q, \mathcal{D}_\alpha] = 0. \quad (2.14)$$

It is worth pointing out that the left-hand side of (2.12b) can be rewritten as

$$-\frac{i}{2}D^2 T_{\alpha_1...\alpha_n} = \mathcal{D}_{(\alpha_1} \beta T_{\alpha_2...\alpha_n)\beta} + (n + 2)ST_{\alpha_1...\alpha_n}, \quad (2.15)$$

where we have made use of (2.12a).

In this paper we propose off-shell gauge-invariant formulations for massive higher-spin supermultiplets in $\mathcal{N} = 1$ AdS superspace that lead to the equations (2.12a) and (2.12b) on-shell.

\footnote{It is of interest to compare (2.14) with the quadratic Casimir operator of the 4D $\mathcal{N} = 1$ AdS supergroup (given by eq. (29) in [48]), which plays an important role in the quantisation [48] of the massless higher-spin supermultiplets [49] in AdS4.}
3 Conformal higher-spin fields

The concept of conformal higher-spin field theory was introduced by Fradkin and Tseytlin in four dimensions \[51\]. (Super)conformal higher-spin field theories in three dimensions were discussed in \[32, 52\]. In this section, our starting points will be (i) the description of conformal higher-spin gauge fields in Minkowski space given in \[32, 33\]; and (ii) the approach advocated in \[53\].

3.1 Conformal gravity

The gravitational field may be described in terms of the torsion-free covariant derivatives

\[
\nabla_a = e_a + \omega_a = e^m_a \partial_m + \frac{1}{2} \omega^b_a M_{bc}, \quad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}^{\quad cd} M_{cd}. \tag{3.1}
\]

Here \( M_{bc} = -M_{cb} \) denotes the Lorentz generators, \( e^m_a \) the inverse vielbein, \( e^m_a e^b_m = \delta^b_a \), and \( \omega^a_{\ bc} \) the torsion-free Lorentz connection. Finally, \( R_{ab}^{\quad cd} \) is the Riemann curvature tensor. In three dimensions, \( R_{ab}^{\quad cd} \) is determined by the Ricci tensor \( R_{ab} := \eta^{cd} R_{cadb} = R_{ba} \) and the scalar curvature \( R = \eta^{ab} R_{ab} \).

The Weyl tensor is identically zero in three dimensions, which means

\[
R_{abcd} = \eta_{ac} R_{bd} - \eta_{ad} R_{bc} - \eta_{bc} R_{ad} + \eta_{bd} R_{ac} - \frac{1}{2} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}) R. \tag{3.2}
\]

The role of the Weyl tensor is played by the Cotton tensor \( W_{abc} = -W_{bac} \), which is defined in terms of the 3D Schouten tensor \( P_{ab} = R_{ab} - \frac{1}{3} \eta_{ab} R \) as follows

\[
W_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}. \tag{3.3}
\]

Spacetime is conformally flat if and only if the Cotton tensor vanishes \[50\] (see \[23\] for a modern proof). The algebraic properties of the Cotton tensor are

\[
W_{abc} + W_{bca} + W_{cab} = 0, \quad W_{ab}^\ b = 0. \tag{3.4}
\]

They imply that \( W_{ab} := \frac{1}{2} \varepsilon_{acd} W^{cd}_{\ b} \) is symmetric and traceless,

\[
W_{ba} = W_{ab}, \quad W^a_a = 0. \tag{3.5}
\]

It is also divergenceless,

\[
\nabla^a W_{ab} = 0, \tag{3.6}
\]
as a consequence of the Bianchi identity \( \nabla^b R_{ab} = \frac{1}{2} \nabla_a R \).

The condition of vanishing torsion is invariant under Weyl (local scale) transformations of the form

\[
\nabla_a \to \nabla'_a = e^\sigma (\nabla_a + \nabla^b \sigma \bar{M}_{ba}) ,
\]

with the parameter \( \sigma(x) \) being completely arbitrary. In the infinitesimal case, the Weyl transformation laws of \( R_{ab} \) and \( R \) are

\[
\delta_{\sigma} R_{ab} = 2\sigma R_{ab} + \nabla_a \nabla_b \sigma + \eta_{ab} \Box \sigma , \quad \delta_{\sigma} R = 2\sigma R + 4\Box \sigma , \tag{3.7}
\]

where \( \Box = \nabla^c \nabla_c \). The Cotton tensor is a Weyl primary field of weight +3,

\[
\delta_{\sigma} W_{ab} = 3\sigma W_{ab} . \tag{3.8}
\]

In what follows, we often convert every vector index into a pair of spinor ones using the well-known correspondence: a three-vector \( V_a \) can equivalently be realised as a symmetric spinor \( V_{\alpha \beta} = V_{\beta \alpha} \). The relationship between \( V_a \) and \( V_{\alpha \beta} \) is as follows:

\[
V_{\alpha \beta} := (\gamma^a)_{\alpha \beta} V_a = V_{\beta \alpha} , \quad V_a = -\frac{1}{2} (\gamma_a)^{\alpha \beta} V_{\alpha \beta} . \tag{3.9}
\]

Associated with the traceless part of the Ricci tensor, \( R_{ab} - \frac{1}{3} \eta_{ab} R \), and the Cotton tensor, \( W_{ab} \), are the following completely symmetric rank-4 spinors:

\[
R_{\alpha \beta \gamma \delta} := (\gamma^a)_{\alpha \beta} (\gamma^b)_{\gamma \delta} (R_{ab} - \frac{1}{3} \eta_{ab} R) = R_{(\alpha \beta \gamma \delta)} , \tag{3.10}
\]

\[
W_{\alpha \beta \gamma \delta} := (\gamma^a)_{\alpha \beta} (\gamma^b)_{\gamma \delta} W_{ab} = W_{(\alpha \beta \gamma \delta)} = \nabla^\rho (\gamma^a R_{\beta \gamma \delta})_\rho . \tag{3.11}
\]

The Weyl transformation of \( R_{\alpha \beta \gamma \delta} \) is

\[
\delta_{\sigma} R_{\alpha \beta \gamma \delta} = 2\sigma R_{\alpha \beta \gamma \delta} + \nabla_{(\alpha \beta} \nabla_{\gamma \delta)} \sigma . \tag{3.12}
\]

### 3.2 Conformal gauge fields

A real tensor field \( h_{(n)} := h_{\alpha_1 ... \alpha_n} = h_{(\alpha_1 ... \alpha_n)} \) is said to be a conformal spin-\( \frac{n}{2} \) gauge field if

(i) it is Weyl primary of some weight \( d_n \),

\[
\delta_{\sigma} h_{(n)} = d_n \sigma h_{(n)} ; \tag{3.13}
\]

and (ii) it is defined modulo gauge transformations of the form

\[
\delta_{\zeta} h_{(n)} = \nabla_{(\alpha_1 \alpha_2} \zeta_{\alpha_3 ... \alpha_n)} , \tag{3.14}
\]
with the real gauge parameter $\zeta_{\alpha(n-2)}$ being also Weyl primary. These conditions uniquely fix the Weyl weight of $h_{\alpha(n)}$ to be

$$d_n = 2 - \frac{n}{2}.$$  \hfill (3.16)

Starting with $h_{\alpha(n)}$ one can construct its descendant, $c_{\alpha(n)}$, defined uniquely, modulo a normalisation, by the following the properties:

1. $c_{\alpha(n)}$ is of the form $Ah_{\alpha(n)}$, where $A$ is a linear differential operator involving the covariant derivatives, the curvature tensors $R_{\alpha(4)}$ and $R$ and their covariant derivatives.

2. $c_{\alpha(n)}$ is Weyl primary of weight $(1 + n/2)$,

$$\delta_\sigma c_{\alpha(n)} = \left(1 + \frac{n}{2}\right)\sigma c_{\alpha(n)} .$$  \hfill (3.17)

3. The gauge variation of $c_{\alpha(n)}$ vanishes if the spacetime is conformally flat,

$$\delta_\zeta c_{\alpha(n)} = O(W_{(4)}) ,$$  \hfill (3.18)

where $W_{(4)}$ is the Cotton tensor.

4. $c_{\alpha(n)}$ is divergenceless if the spacetime is conformally flat,

$$\nabla_\beta c_{\alpha(n)} = O(W_{(4)}) .$$  \hfill (3.19)

Here and in (3.18), $O(W_{(4)})$ stands for contributions containing the Cotton tensor and its covariant derivatives.

We now consider several examples. Given a conformal spin-1 gauge $h_{\alpha\beta} = h_{\beta\alpha}$,

$$\delta_\sigma h_{\alpha\beta} = \sigma h_{\alpha\beta} ,$$  \hfill (3.20)

the required Weyl primary descendant is $c_{\alpha\beta} = \nabla^\gamma (h_{\beta\gamma})$ and coincides with the gauge-invariant field strength, $c_{ab} = \nabla_a h_b - \nabla_b h_a$, of the one-form $h_a$. This implies that $c_{\alpha(2)}$ is conserved,

$$\nabla^\beta \nabla_\gamma c_{\beta\gamma} = 0 .$$  \hfill (3.21)

Next consider a conformal spin-$\frac{3}{2}$ gauge field $h_{\alpha(3)}$ (i.e. conformal gravitino),

$$\delta_\sigma h_{\alpha(3)} = \frac{1}{2} \sigma h_{\alpha(3)} .$$  \hfill (3.22)
The required Weyl primary descendant is
\[ C_{\alpha}^{(3)} = \frac{3}{4} \nabla_{\alpha_1} \nabla_{\alpha_2} \beta_1 h_{\alpha_3\beta_1\beta_2} + \frac{1}{4} \Box h_{\alpha(3)} + \frac{3}{4} R_{\beta_1\beta_2(\alpha_1\alpha_2} h_{\alpha_3)} \beta_1 \beta_2 - \frac{1}{16} R h_{\alpha(3)} \]  
\[ (3.23) \]

Its gauge transformation is
\[ \delta_\z C_{\alpha}^{(3)} = -\frac{1}{2} W_{(3)\beta} \z^\beta . \]
\[ (3.24) \]

Computing its divergence gives
\[ \nabla^\beta C_{\beta\gamma\alpha} = -\frac{1}{2} W_{\alpha(3)\beta} h^{\beta(3)} . \]
\[ (3.25) \]

Our last example is a conformal spin-2 gauge field \( h_{\alpha(4)} \) (i.e. conformal graviton),
\[ \delta_\sigma h_{\alpha(4)} = 0 . \]
\[ (3.26) \]

The required Weyl primary descendant of \( h_{\alpha(4)} \) is
\[ C_{\alpha}^{(4)} = \frac{1}{2} \nabla_{\alpha_1} \nabla_{\alpha_2} \beta_1 h_{\alpha_3\beta_1\beta_2} + \frac{1}{2} \Box \nabla_{\alpha_1} h_{\alpha_2\alpha_3\alpha_4}) \beta_1 + \left( \nabla_{\alpha_1} \beta_1 R_{\alpha_2\alpha_3} \beta_2 \beta_3 \right) h_{\alpha_4}) \beta(3) \]
\[ + \frac{1}{12} \left( \nabla_{\alpha_1} \beta_1 R \right) h_{\alpha_2\alpha_3\alpha_4}) \beta_1 - \frac{1}{12} R \nabla_{\alpha_1} h_{\alpha_2\alpha_3\alpha_4}) \beta_1 + 2 R_{\beta_1\beta_2(\alpha_1\alpha_2} \nabla_{\alpha_3} h_{\alpha_4)} \beta(3) \]
\[ - \frac{3}{4} R_{\beta_1} (\delta_{\alpha_1\alpha_2} \nabla \beta_2 h_{\alpha_3\alpha_4}) \beta(2) . \]
\[ (3.27) \]

Its gauge transformation is
\[ \delta_\z C_{\alpha}^{(4)} = \left( \nabla^\gamma W_{\gamma(\alpha_1\alpha_2\alpha_3} \z_{\alpha_4)} \right) \delta + \frac{1}{2} \left( \nabla_{\alpha_1\alpha_2} W_{\alpha_3\alpha_4}) \beta(2) \right) \z_{\gamma(2)} - W_{\gamma_1(\alpha_1\alpha_2\alpha_3} \nabla_{\alpha_4)} \gamma_{\gamma_2} \]
\[ + \frac{11}{12} W_{\alpha(4)} \nabla\beta(2) \z_{\beta(2)} + \frac{1}{2} W_{\gamma(\alpha_1\alpha_2} \nabla_{\alpha_3} \nabla_{\gamma_4)} \z_{\gamma_4} ) \beta . \]
\[ (3.28) \]

The divergence of \( C_{\alpha}^{(4)} \) may be shown to be
\[ \nabla^\beta C_{\beta\gamma\alpha(2)} = -\frac{1}{2} \left( \nabla_{\gamma(\alpha_1} W^{\gamma(\beta(3) h_{\alpha_2))\beta(3)} + \frac{5}{12} \left( \nabla_{\gamma(\alpha_1} W^{\beta(4)} h_{\beta(4)} + W_{\alpha(2)}^{(2)} \nabla(2) \z_{\beta(2)} - \frac{1}{12} W_{\gamma(\alpha_1} \beta(2) \nabla_{\gamma_2} h_{\beta(2)} \gamma(2) \right) \right) \]
\[ + \frac{3}{2} W_{\gamma(\alpha_1} \beta(2) \nabla_{\gamma_2} h_{\alpha_2\beta(2)} - \frac{1}{12} W_{\gamma(\alpha_1} \beta(2) \nabla_{\gamma_2} h_{\beta(2)} \right) . \]
\[ (3.29) \]

Suppose that the spacetime under consideration is conformally flat,
\[ W_{\alpha(4)} = 0 . \]
\[ (3.30) \]

Then the tensor \( C_{\alpha(n)} \) is gauge invariant and conserved,
\[ \delta_\z C_{\alpha(n)} = 0 , \]
\[ (3.31a) \]
\[ \nabla^{\beta \gamma} \mathcal{C}_{\beta \gamma \alpha(n-2)} = 0. \tag{3.31b} \]

These properties and the Weyl transformation law (3.17) tell us that the action
\[ S^{(n)}_{\text{CS}}[h_{\alpha(n)}] = \frac{i^n}{2^{[n/2]+1}} \int d^3x \, e^{\alpha(n)} \mathcal{C}_{\alpha(n)} \, , \quad e^{-1} = \det(e^m_a) \tag{3.32} \]
is gauge and Weyl invariant,
\[ \delta_\zeta S^{(n)}_{\text{CS}}[h_{\alpha(n)}] = 0 \, , \quad \delta_\sigma S^{(n)}_{\text{CS}}[h_{\alpha(n)}] = 0 \, . \tag{3.33} \]

Here \([x]\) denotes the floor function; it coincides with the integer part of a real number \(x \geq 0\).

The above action is actually Weyl invariant in an arbitrary curved space. Condition (3.30) is required to guarantee the gauge invariance of \(S^{(n)}_{\text{CS}}[h_{\alpha(n)}]\) for \(n > 2\).

It follows from the Weyl transformation law (3.17) that \(\nabla^{\beta \gamma} \mathcal{C}_{\beta \gamma \alpha(n-2)}\) is a primary field,
\[ \delta_\sigma \left( \nabla^{\beta \gamma} \mathcal{C}_{\beta \gamma \alpha(n-2)} \right) = \left( 2 + \frac{n}{2} \right) \sigma \nabla^{\beta \gamma} \mathcal{C}_{\beta \gamma \alpha(n-2)} \, . \tag{3.34} \]

This property means that the conservation equation (3.31b) is Weyl invariant.

### 3.3 Higher-spin Cotton tensor in Minkowski space

In Minkowski space, the linearised higher-spin Cotton tensor \(C_{\alpha(n)}(h)\), with \(n \geq 2\), is given by the expression [33]
\[ C_{\alpha(n)}(h) := \frac{1}{2^{n-1}} \sum_{j=0}^{[n/2]} \left( \frac{n}{2j+1} \right) \Box^j \partial_{\alpha_1} \beta_1 \ldots \partial_{\alpha_{n-2j-1}} \beta_{n-2j-1} h_{\alpha_{n-2j} \ldots \alpha_n} \beta_1 \ldots \beta_{n-2j-1} \, . \tag{3.35} \]

It is a descendant of the conformal field \(h_{\alpha(n)}\) defined modulo gauge transformations of the form
\[ \delta h_{\alpha(n)} = \partial_{(\alpha_1 \alpha_2} \zeta_{\alpha_3 \ldots \alpha_n)} \, . \tag{3.36} \]

The field strength is invariant under these gauge transformations,
\[ \delta_\zeta C_{\alpha(n)} = 0 \, , \tag{3.37} \]
and obeys the Bianchi identity
\[ \partial^{\gamma} C_{\beta \gamma \alpha_1 \ldots \alpha_{n-2}} = 0 \, . \tag{3.38} \]

The higher-spin Chern-Simons action
\[ S^{(n)}_{\text{CS}}[h_{(n)}] = \frac{i^n}{2^{[n/2]+1}} \int d^3x \, h^{\alpha(n)} C_{\alpha(n)}(h) \tag{3.39} \]
is conformal and invariant under (3.36).

In the case of even rank, \( n = 2s \), with \( s = 1, 2, \ldots \), the field strength (3.35) can be shown to coincide with the bosonic higher-spin Cotton tensor given originally by Pope and Townsend [32]. It reduces to the linearised Cotton tensor for \( n = 4 \), and to the Maxwell field strength for \( n = 2 \). The fermionic case, \( n = 2s + 1 \), with \( s = 2, \ldots \), was not considered in [32]. It was presented for the first time in [33].

It should be pointed out that the conformal spin-3 case, \( n = 6 \), was studied for the first time in [54]. The spin-3/2 case, \( n = 3 \), was considered in [55]. The field strength \( C_{\alpha(3)} \) is the linearised version of the Cotton vector spinor [10, 56].

The normalisation of \( C_{\alpha(n)}(h) \) defined by (3.35) can be explained as follows. The gauge freedom (3.36) allows us to impose a gauge condition

\[
\partial^{\beta\gamma} h_{\beta\gamma\alpha(n-2)} = 0 .
\] (3.40)

Under this gauge condition, the field strength (3.35), with \( s = 1, 2, \ldots \), takes the form

\[
C_{\alpha(2s)} = \Box^{s-1} \partial^\beta (\alpha_1 h_{\alpha_2 \ldots \alpha_{2s}}) = \Box^{s-1} \partial^\beta \alpha_1 h_{\alpha_2 \ldots \alpha_{2s}} , \quad (3.41a)
\]

\[
C_{\alpha(2s+1)} = \Box^s h_{\alpha(2s+1)} , \quad (3.41b)
\]

as a consequence of the identity

\[
\sum_{j=0}^{[n/2]} \binom{n}{2j+1} = 2^{n-1} .
\] (3.42)

The field strength (3.35) proves to be the general solution to the conservation equation (3.38). This result has recently been proved in [57] in the bosonic case, \( n = 2s \), and the proof given is quite nontrivial (see also [58]). An alternative proof, which is valid for arbitrary integer \( n > 1 \) and is based on supersymmetry considerations, was given in [33].

### 3.4 Higher-spin Cotton tensor in conformally flat spaces

Consider a conformally flat spacetime \( \mathcal{M}^3 \). Its covariant derivatives \( \nabla_a \) are related to the flat-space ones by

\[
\nabla_a = e^\sigma (\partial_a + \partial^b \sigma M_{ba}) ,
\] (3.43)

for some scale factor \( \sigma \). The linearised higher-spin Cotton tensor \( \mathcal{C}_{\alpha(n)} \) in \( \mathcal{M}^3 \) is related to the flat-space one, eq. (3.35), by the rule

\[
\mathcal{C}_{\alpha(n)} = e^{(1+\frac{n}{2})\sigma} C_{\alpha(n)} .
\] (3.44)
In general, it is a difficult technical problem to express \( c_{\alpha(n)} \) in terms of the covariant derivatives \( \nabla_a \) and the gauge prepotential \( h_{\alpha(n)} = e^{(2-n/2)\sigma} h_{\alpha(n)} \). As an example, let us consider the case of AdS space, whose geometry is described by covariant derivatives satisfying the algebra

\[
[\nabla_a, \nabla_b] = -4S^2 M_{ab}
\]

or in two-component notation

\[
[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = 4S^2 \left( \varepsilon_{\gamma(\alpha} M_{\beta)\delta} + \varepsilon_{\delta(\alpha} M_{\beta)\gamma} \right),
\]

Here the parameter \( S \) is related to the AdS scalar curvature as \( R = -24S^2 \). The Cotton tensor \( (3.44) \) for the cases \( n = 3, 4, 5 \) and 6 proves to be

\[
c_{\alpha(3)} = \frac{1}{2^3} \left( 3\nabla_{(\alpha} \nabla_{\beta)} h_{\alpha\beta} + \mathcal{Q} h_{\alpha(3)} - 9S^2 h_{\alpha(3)} \right),
\]

\[
c_{\alpha(4)} = \frac{1}{2^4} \left( 4\nabla_{(\alpha} \nabla_{\beta} \nabla_{\gamma} h_{\alpha\beta\gamma} + 4\mathcal{Q} \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta} - 80S^2 \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} \right),
\]

\[
c_{\alpha(5)} = \frac{1}{2^5} \left( 5\nabla_{(\alpha} \nabla_{\beta} \nabla_{\gamma} \nabla_{\delta} h_{\alpha\beta\gamma\delta} + 10\mathcal{Q} \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} + 2\mathcal{Q} h_{\alpha(5)} \right) - 330S^2 \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} - 82S^2 \mathcal{Q} h_{\alpha(5)} + 1425S^4 h_{\alpha(5)}
\]

\[
c_{\alpha(6)} = \frac{1}{2^6} \left( 6\nabla_{(\alpha} \nabla_{\beta} \nabla_{\gamma} \nabla_{\delta} \nabla_{\epsilon} h_{\alpha\beta\gamma\delta\epsilon} + 20\mathcal{Q} \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} + 36\mathcal{Q} h_{\alpha(6)} \right) + 6S^2 \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} - 960S^2 \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} h_{\alpha(6)} + 704S^2 \mathcal{Q} \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta} + 18432S^4 \mathcal{Q} \nabla_{(\alpha} h_{\beta\gamma\delta)\beta\gamma\delta\beta\delta}
\]

where \( \mathcal{Q} \) is the quadratic Casimir of the 3D AdS group, \( \text{SO}(2, 2) \), given by eq. (2.5). Each of the tensors \( c_{\alpha(n)} \) given above can be written as \( c_{\alpha(n)}(h_{\alpha(n)}) = \mathcal{A} h_{\alpha(n)} \), where the linear differential operator \( \mathcal{A} \) is symmetric in the sense that

\[
\int d^3x \epsilon^\alpha(\eta) \mathcal{A} h_{\alpha(n)} = \int d^3x \epsilon^\alpha(\eta) \mathcal{A} g_{\alpha(n)}
\]

for arbitrary prepotentials \( g_{\alpha(n)} \) and \( h_{\alpha(n)} \). This means that it suffices to prove one of the two properties in \((3.31)\), and then the second property follows.
4 Massive higher-spin actions in maximally symmetric spaces

The conformal higher-spin actions in conformally flat spaces, eq. (3.32), are formulated in terms of the gauge fields $h_{\alpha(n)}$. The same gauge field can be used to construct massless Fronsdal-Fang-type actions [30, 31, 59, 60] in maximally symmetric spaces. Such actions however, will involve not only $h_{\alpha(n)}$ but also some compensators.

Here we describe these massless higher-spin gauge actions in AdS and then use them to construct gauge-invariant models for massive higher-spin fields.

4.1 Massive higher-spin actions in AdS space

There are two types of the higher-spin massless actions, first-order and second-order ones. Given an integer $n \geq 4$, the first-order model is described by real fields $h_{\alpha(n)}$, $\eta_{\alpha(n-2)}$ and $\eta_{\alpha(n-4)}$ which are defined modulo gauge transformations of the form

$$
\delta_\xi h_{\alpha(n)} = \nabla_{(\alpha_1 \alpha_2 \xi_{\alpha_3 \ldots \alpha_n})},
$$

$$
\delta_\xi \eta_{\alpha(n-2)} = \frac{1}{n} \nabla^\beta (\alpha_1 \xi_{\alpha_2 \ldots \alpha_{n-2})}^\beta + S\eta_{\alpha(n-2)},
$$

$$
\delta_\xi \eta_{\alpha(n-4)} = \nabla^{\beta(2)} \xi_{\alpha(n-4)\beta(2)}.
$$

The Fang-Fronsdal-type gauge-invariant action, $S_{FF}^{(n)} = S_{FF}^{(n)}[h_{(n)}, \eta_{(n-2)}, \eta_{(n-4)}]$, is

$$
S_{FF}^{(n)} = \frac{i^n}{2^{[n/2]}} \int d^3 x \left\{ h_{(n-1)\gamma} \nabla_{\gamma} \delta h_{\alpha(n-1)} + 2(n-2) \eta_{(n-2)\gamma} \nabla^{\beta(2)} h_{\alpha(n-2)\beta(2)}
\right.
$$

$$
+ 4(n-2) \eta_{(n-3)\gamma} \nabla_{\gamma} \delta \eta_{\alpha(n-3)} + 2 \frac{n(n-3)}{(n-1)} \eta_{(n-4)\gamma} \nabla^{\beta(2)} \eta_{\alpha(n-4)\beta(2)}
$$

$$
- \frac{(n-3)(n-4)}{(n-1)(n-2)} \eta_{(n-5)\gamma} \nabla_{\gamma} \delta \eta_{\alpha(n-5)} + (n-2) S h_{\alpha(n)} \eta_{\alpha(n)}
$$

$$
- 4n(n-2) S \eta_{(n-2)\gamma} \eta_{\alpha(n-2)} - \frac{n(n-3)}{(n-1)} S h_{\alpha(n-4)} \eta_{\alpha(n-4)} \right\}.
$$

Here $[n/2]$ stands for the ceiling function, which is equal to $s$ for $n = 2s$ and $s+1$ for $n = 2s+1$, with $s \geq 0$ an integer.

Given an integer $n \geq 4$, the second-order model is described by real fields $h_{\alpha(n)}$ and $\eta_{\alpha(n-4)}$ defined modulo gauge transformations of the form

$$
\delta_\xi h_{\alpha(n)} = \nabla_{(\alpha_1 \alpha_2 \xi_{\alpha_3 \ldots \alpha_n})},
$$

$$
\delta_\xi \eta_{\alpha(n-4)} = \nabla^{\beta(2)} \xi_{\alpha(n-4)\beta(2)}.
$$
\[ \delta \xi \eta_{\alpha(n-4)} = \frac{n-2}{n-1} \nabla^{\beta(2)} \xi_{\alpha(n-4)\beta(2)} . \] (4.3b)

The Fronsdal-type gauge-invariant action, \( S_F^{(n)} = S_F^{(n)}[\partial_\alpha \eta, \eta_{(n-4)}] \), is

\[ S_F^{(n)} = \frac{i^n}{2^{[n/2]+1}} \int d^3x \left\{ \eta^{\alpha(n)} \square \eta_{\alpha(n)} - \frac{n}{4} \nabla_\gamma \gamma(2) \eta^{\gamma(2)\alpha(n-2)} \nabla_\beta \beta(2) \eta_{\alpha(n-2)\beta(2)} ight. \\
- \frac{n-3}{2} \eta^{\alpha(n-4)} \nabla^{(2)} \nabla^{(2)\gamma} \eta_{\alpha(n-4)\beta(2)\gamma(2)} - n(n-6) \delta^4 \eta^{\alpha(n)} \eta_{\alpha(n)} \\
- \frac{(n-3)}{n} \left[ 2 \eta^{\alpha(n-4)} \square \eta_{\alpha(n-4)} - 2(n^2 - 2n + 4) \delta^4 \eta^{\alpha(n-4)} \eta_{\alpha(n-4)} \\
+ \frac{(n-4)(n-5)}{4(n-2)} \nabla_\gamma \gamma(2) \eta^{\gamma(2)\alpha(n-6)} \nabla^{\beta(2)} \eta_{\beta(2)\alpha(n-6)} \right] \right\} . \tag{4.4}

Our action (4.2) is a unique gauge-invariant extension to AdS space of the flat-space action given by Tyutin and Vasiliev [36], see Appendix B for a review. When \( n \) is odd, \( n = 2s+1 \), (4.2) is the unique gauge-invariant 3D counterpart to the Fang-Fronsdal action in AdS$_4$ [60]. When \( n \) is even, \( n = 2s \), our action (4.4) is the unique gauge-invariant 3D counterpart to the Fronsdal action in AdS$_4$ [59]. The Fronsdal action [59] can also be generalised to \( d \)-dimensional AdS backgrounds [68, 69]. Such an action in AdS$_d$ is formulated in terms of a symmetric double-traceless field and it is fixed by the condition of gauge invariance.

Each of the gauge-invariant actions (3.32), (4.2) and (4.4) proves to describe no propagating degrees of freedom. We claim that the following models

\[ S_{\text{massive}}^{(2s+1)} = \lambda S_{CS}^{(2s+1)}[\partial_\alpha \eta_{(2s+1)}] + \mu^{2^{s-1}} S_{FF}^{(2s+1)}[\eta_{(2s+1)}, \eta_{(2s-1)}, \eta_{(2s-3)}] \tag{4.5a} \]

\[ S_{\text{massive}}^{(2s)} = \lambda S_{CS}^{(2s)}[\partial_\alpha \eta_{(2s)}] + \mu^{2^{s-3}} S_{FF}^{(2s)}[\eta_{(2s)}, \eta_{(2s-4)}] \tag{4.5b} \]

describe irreducible massive fields in AdS. Here the parameter \( \lambda \) is dimensionless, while \( \mu \) has dimension of mass. Since we do not have a closed form expression for \( C_{\alpha(n)} \) in AdS$_3$, for arbitrary \( n \), our analysis below will be restricted to the case of Minkowski space, \( \mathbb{M}^3 \).

5It is worth pointing out that the Fang-Fronsdal action for a massless spin-(\( s + \frac{1}{2} \)) field [31] is also described in terms of a triplet of fermionic gauge fields, \( \Psi_{\alpha(s+1)\alpha(s)} \), \( \Psi_{\alpha(s-1)\alpha(s)} \) and their conjugates, if one makes use of the two-component spinor notation, see section 6.9 of [61]. More generally, there exist bosonic and fermionic higher-spin triplet models in higher dimensions [62, 63, 64, 65, 66]. On-shell supersymmetric formulations for the generalised triplets in diverse dimensions have recently been given in [67].

6The dynamical equations for massless higher-spin fields in AdS$_d$ were studied by Metsaev [70, 71, 72, 73]. For alternative descriptions of massless higher-spin dynamics in AdS$_d$, see [74, 75].
4.2 Massive higher-spin actions in Minkowski space: The fermionic case

In this section we study the dynamics of the flat-space counterparts to the gauge theories \((4.5a)\) and \((4.5b)\). In fact, the resulting flat-space actions are contained at the component level in the massive supersymmetric higher-spin models proposed in \([34, 35]\). However, the analysis in \([34, 35]\) was carried out mostly in terms of superfields so that the component actions were not studied in detail.

We first analyse the flat-space limit of the fermionic model \((4.5a)\). It is described by the action

\[
S_{\text{massive}}^{(2s+1)}[h^{(2s+1)}] + \mu^{2s-1}S_{\text{FF}}^{(2s+1)}[h^{(2s+1)}, y^{(2s-1)}, y^{(2s-3)}],
\]

where the massless sector is

\[
S_{\text{FF}}^{(2s+1)} = \frac{i}{2} \left( -\frac{1}{2} \right)^s \int d^3 x \left\{ h^{\alpha(2s)} \gamma^\delta \partial_\gamma \delta_{\alpha(2s)} + 2(2s-1)y^{\alpha(2s-1)} \partial^{(2)} h_{\alpha(2s-1)\beta(2)} \right.
\]
\[
+ 4(2s-1)y^{\alpha(2s-2)} \gamma^\delta y_{\alpha(2s-2)} + \frac{2}{s} (2s+1)(s-1)y^{\alpha(2s-3)} \partial^{(2)} y_{\alpha(2s-3)\beta(2)}
\]
\[
\left. - \frac{(s-1)(2s-3)}{s(2s-1)} y^{\alpha(2s-4)} \gamma^\delta y_{\delta(2s-4)} \right\}.
\]

The action \((4.6)\) is invariant under the following gauge transformations:

\[
\delta_\zeta h_{\alpha(2s+1)} = \partial_{(\alpha_1\alpha_2...\alpha_{2s+1})} \zeta,
\]
\[
\delta_\zeta y_{\alpha(2s-1)} = \frac{1}{2s+1} \partial^{(2)}_{(\alpha_1\alpha_2...\alpha_{2s-1})} \zeta,
\]
\[
\delta_\zeta y_{\alpha(2s-3)} = \partial^{(2)}_{(2)} \zeta_{(2s-3)\beta(2)}.
\]

The equations of motion corresponding to the model \((4.6)\) are

\[
0 = \mu^{2s-1} \left( \partial^{(2)}_{(2s-1)} h_{\alpha(2s-1)\beta(2)} + \frac{(s-1)(2s+1)}{s(2s-1)} \partial_{(\alpha_1\alpha_2...\alpha_{2s-1})} y_{\alpha(2s-1)} \right),
\]
\[
0 = \partial^{(2)} h_{\alpha(2s-1)\beta(2)} + 4 \partial^{(2)}_{(\alpha_1 y_{\alpha_2...\alpha_{2s-1})} y_{\alpha(2s-1)} \beta(2)} - \frac{(s-1)(2s+1)}{s(2s-1)} \partial_{(\alpha_1 y_{\alpha_2...\alpha_{2s-1})} y_{\alpha(2s-1)} \beta(2)}
\]
\[
0 = (2s-1) \partial^{(2)} y_{\alpha(2s-3)\beta(2)} - \frac{2s-3}{2s+1} \partial^{(2)}_{(\alpha_1 y_{\alpha_2...\alpha_{2s-1})} y_{\alpha(2s-3)\beta(2)}}.
\]

We now demonstrate that the model \((4.6)\) indeed describes an irreducible massive spin-\((s+\frac{1}{2})\) field on the equations of motion. The gauge transformation \((4.8c)\) tells us that \(y_{\alpha(2s-3)}\) can be completely gauged away, that is, we are able to impose the gauge condition

\[
y_{\alpha(2s-3)} = 0.
\]
Then, the residual gauge freedom is described by $\zeta_{\alpha(2s-1)}$ constrained by

$$
\partial^{\beta(2)}\zeta_{\alpha(2s-3)\beta(2)} = 0 \quad \implies \quad \partial^{\beta}_{(\alpha_1}\zeta_{\alpha_2\ldots\alpha_{2s-1}\beta)} = \partial^{\beta}_{\alpha_1}\zeta_{\alpha_2\ldots\alpha_{2s-1}\beta} .
$$

(4.11)

In the gauge (4.10), the equation of motion (4.9c) becomes the condition for $y_{\alpha(2n-1)}$ to be divergenceless,

$$
\partial^{\beta(2)}y_{\alpha(2s-3)\beta(2)} = 0 \quad \implies \quad \partial^{\beta}_{(\alpha_1}y_{\alpha_2\ldots\alpha_{2s-1}\beta)} = \partial^{\beta}_{\alpha_1}y_{\alpha_2\ldots\alpha_{2s-1}\beta} .
$$

(4.12)

Due to (4.12), the gauge transformation (4.8b) becomes

$$
\delta\zeta y_{\alpha(2s-1)} = \frac{1}{2s+1} \partial^{\beta}_{\alpha_1}\zeta_{\alpha_2\ldots\alpha_{2s+1}\beta} .
$$

(4.13)

Since $y_{\alpha(2s-1)}$ and $\zeta_{\alpha(2s-1)}$ have the same functional type, we are able to completely gauge away the $y_{\alpha(2s-1)}$ field,

$$
y_{\alpha(2s-1)} = 0 .
$$

(4.14)

In accordance with (4.13) and (4.14), the residual gauge freedom is described by the parameter $\zeta_{\alpha(2s-1)}$ constrained by

$$
\partial^{\beta}_{\alpha_1}\zeta_{\alpha_2\ldots\alpha_{2s-1}\beta} = 0 \quad \implies \quad \square\zeta_{\alpha(2s-1)} = 0 .
$$

(4.15)

In the gauge (4.14), the equation of motion (4.9b) tells us that $h_{\alpha(2s+1)}$ is divergenceless,

$$
\partial^{\beta(2)}h_{\alpha(2s-1)\beta(2)} = 0 \quad \implies \quad \partial^{\beta}_{(\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta)} = \partial^{\beta}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta} .
$$

(4.16)

So far the above analysis has been identical to that given in Appendix B of [34] for the massless model (4.7).

Due to (4.16), the Cotton tensor (3.35) reduces to the expression (3.41b). In the gauge (4.14), the equation of motion (4.9a) becomes

$$
\mu^{2s-1}\partial^{\beta}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta} + \lambda \square^s h_{\alpha(2s+1)} = 0 .
$$

(4.17)

This equation has two types of solutions, massless and massive ones,

$$
\partial^{\beta}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta} = 0 \quad \implies \quad \square h_{\alpha(2s+1)} = 0 ;
$$

(4.18a)

$$
\mu^{2s-1}h_{\alpha(2s+1)} + \lambda \square^{s-1}\partial^{\beta}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta} = 0 .
$$

(4.18b)

We point out that $\Psi_{\alpha_1\ldots\alpha_{2s+1}} := \partial^{\beta}_{\alpha_1}h_{\alpha_2\ldots\alpha_{2s+1}\beta}$ is completely symmetric and divergenceless, $\Psi_{\alpha_1\ldots\alpha_{2s+1}} = \Psi_{(\alpha_1\ldots\alpha_{2s+1})}$ and $\partial^{\beta\gamma}\Psi_{\beta\alpha_1\ldots\alpha_{2s-1}} = 0$.
Let us show that the massless solution (4.18a) is a pure gauge degree of freedom. Since both the gauge field \( h_{\alpha(2s+1)} \) and the residual gauge parameter \( \zeta_{\alpha(2s-1)} \) are on-shell massless, it is useful to switch to momentum space, by replacing \( h_{\alpha(2s+1)}(x) \rightarrow h_{\alpha(2s+1)}(p) \) and \( \zeta_{\alpha(2s-1)}(x) \rightarrow \zeta_{\alpha(2s-1)}(p) \), where the three-momentum \( p^a \) is light-like, \( p^{\alpha\beta}p_{\alpha\beta} = 0 \). For a given three-momentum, we can choose a frame in which the only non-zero component of \( p\) is \( p^{22} = p_{11} \). Then, the conditions \( p^{2(\alpha_1,1)}h_{\alpha_2...\alpha_{2s+1}\beta}(p) = 0 \) and \( p^{2(\alpha_1,1)}\zeta_{\alpha_2...\alpha_{2s-1}\beta}(p) = 0 \) are equivalent to

\[
h_{\alpha(2s)}(p) = 0, \quad \zeta_{\alpha(2s-2)}(p) = 0. \tag{4.19}
\]

Thus the only non-zero components of \( h_{\alpha(2s+1)}(p) \) and \( \zeta_{\alpha(2s-1)}(p) \) are \( h_{1...1}(p) \) and \( \zeta_{1...1}(p) \). The residual gauge freedom, \( \delta h_{1...}(p) \propto p_{11}\zeta_{1...1} \), allows us to gauge away the field \( h_{\alpha(2s+1)} \) completely.

Thus, it remains to analyse the general solution of the equation \( (4.18a) \), which implies

\[
\left( \Box^{2s-1} - (m^2)^{2s-1} \right) h_{\alpha(2s+1)} = 0, \quad m := \left| \frac{\mu}{\lambda^{1/(2s-1)}} \right|. \tag{4.20}
\]

This equation in momentum space yields

\[
\left( 1 - \left( -\frac{p^2}{m^2} \right)^{2s-1} \right) h_{\alpha(2s+1)}(p) = 0. \tag{4.21}
\]

Since the polynomial equation \( z^{2s-1} - 1 = 0 \) has only one real root, \( z = 1 \), the only real solution to \( (4.21) \) is \( p^2 = -m^2 \), from which it follows that \( h_{\alpha(2s+1)} \) satisfies the ordinary Klein-Gordon equation,

\[
\left( \Box - m^2 \right) h_{\alpha(2s+1)} = 0. \tag{4.22}
\]

Applying \( (4.22) \) to \( (4.17) \) reveals that \( h_{\alpha(2s+1)} \) satisfies the equation of motion corresponding to a massive spin \(( s + \frac{1}{2})\)-field with mass \( m \) and helicity \( \sigma(s + \frac{1}{2}) \),

\[
\partial^{\alpha_1}h_{\alpha_2...\alpha_{2s+1}\beta} = \sigma mh_{\alpha(2s+1)} \quad \text{or} \quad \sigma := -\text{sign}(\mu\lambda). \tag{4.23}
\]

Finally, it is worth reminding the proof of the fact that equation \( (4.23) \) describes a single propagating degree of freedom. The field \( h_{\alpha(2s+1)} \) is on-shell with momentum satisfying \( p^2 = -m^2 \), we can therefore transform equation \( (4.23) \) into momentum space and boost into the rest frame where \( p^a = (m, 0, 0) \) \( \Rightarrow \) \( p^{11} = p^{22} = 0, \ p^{12} = -p^{21} = m \),

\[
\imath h_{\alpha(2s)}(p) - \sigma h_{\alpha(2s)}(p) = 0. \tag{4.24}
\]

Due to the symmetry of the field \( h_{\alpha(2s+1)} \), equation \( (4.24) \) states that there is only a single degree of freedom. Taking the independent field component to be \( h_{11...1}(p) \) allows us to express all other components in terms of it.

19
Along with the fermionic model (4.6), which corresponds to $n = 2s + 1$, we could consider a bosonic one described by the action

$$S_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)}[h_{(2s)}] + \mu^{2s-2}S_{F}^{(2s)}[h_{(2s)}, y_{(2s-2)}, y_{(2s-4)}],$$

which corresponds to $n = 2s$. Most of the above analysis would remain valid in this case as well. However, in place of eq. (4.21) we would have

$$(1 - \left(-\frac{p^2}{m^2}\right)^{2s-2}) h_{\alpha(2s)}(p) = 0. \quad (4.26)$$

This equation has both physical ($p^2 = -m^2$) and tachyonic ($p^2 = m^2$) solutions. Therefore, the model (4.25) is unphysical. This may be interpreted as a consequence of the spin-statistics theorem.

### 4.3 Massive higher-spin actions in Minkowski space: The bosonic case

Our next goal is to analyse the flat-space limit of the bosonic model (4.5b). It is described by the action

$$S_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)}[h_{(2s)}] + \mu^{2s-3}S_{F}^{(2s)}[h_{(2s)}, y_{(2s-4)}],$$

where the second term is

$$S_{F}^{(2s)} = \frac{1}{2} \left(\frac{-1}{2}\right)^{s} \int d^{3}x \left\{ h_{\alpha(2s)} \Box h_{\alpha(2s)} - \frac{s}{2} \partial_{\gamma(2)} h_{\gamma(2)\alpha(2s-2)} \partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} - \frac{(2s-3)}{2s} \left[ sy_{\alpha(2s-4)} \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} + 2y^{\alpha(2s-4)} \Box y_{\alpha(2s-4)} \right] + \frac{(s-2)(2s-5)}{4(s-1)} \partial_{\gamma(2)} y^{\gamma(2)\alpha(2s-6)} \partial^{\beta(2)} y_{\beta(2)\alpha(2s-6)} \right\}. \quad (4.28)$$

The action (4.27) is invariant under the gauge transformations

$$\delta_{\zeta} h_{\alpha(2s)} = \partial_{(\alpha_{1} \cdots \alpha_{2s})} \zeta_{\alpha_{3} \cdots \alpha_{2s}}, \quad (4.29a)$$

$$\delta_{\zeta} y_{\alpha(2s-4)} = \frac{2s-2}{2s-1} \partial^{\beta(2)} \zeta_{\alpha(2s-4)\beta(2)}. \quad (4.29b)$$

The equations of motion corresponding to (4.27) are

$$0 = \mu^{2s-3} (\Box h_{\alpha(2s)} + \frac{1}{2} s \partial^{\beta(2)} \partial_{(\alpha_{1} \cdots \alpha_{2s})} h_{\alpha_{3} \cdots \alpha_{2s})\beta(2)} +$$
\[ -\frac{1}{4} (2s - 3) \partial^{(\alpha_1 \alpha_2} \partial_{\alpha_3 \alpha_4} y_{\alpha_5 \ldots \alpha_{2s})} + \lambda C_{\alpha(2s)} \]  
\[ 0 = \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} + \frac{4}{s} \Box y_{\alpha(2s-4)} + \]  
\[ - \frac{(s - 2)(2s - 5)}{2s(s - 1)} \partial^{\beta(2)} \partial_{(\alpha_1 \alpha_2 y_{\alpha_3 \ldots \alpha_{2s-4})\beta(2).} \]  
\[ (4.30a) \]

We will now show that on-shell, the model \( S^{(2s)}_{\text{massive}} \) describes a massive spin-s field which propagates a single degree of freedom. As follows from the gauge transformation (4.29b), it is possible to completely gauge away \( y_{\alpha(2s-4)} \),

\[ y_{\alpha(2s-4)} = 0. \]  
\[ (4.31) \]

Then, the residual gauge freedom is described by a parameter \( \zeta_{\alpha(2s-2)} \) constrained by

\[ \partial^{(2)} \zeta_{\alpha(2s-4)\beta(2)} = 0 \implies \partial^{(a_1 \zeta_{a_2 \ldots a_{2s+1}})\beta} = \partial^{a_1 \zeta_{a_2 \ldots a_{2s+1}}}. \]  
\[ (4.32) \]

In the gauge (4.31), the equation of motion (4.30b) becomes

\[ \partial^{\gamma(2)} \partial^{\beta(2)} h_{\alpha(2s-4)\beta(2)\gamma(2)} = 0. \]  
\[ (4.33) \]

According to (4.29a), the divergence of \( h_{\alpha(2s)} \) transforms as

\[ \delta \left( \partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} \right) = \partial^{a_1 a_2} \partial_{(a_1 \alpha_2 \zeta_{a_3 \ldots a_{2s-2}})\beta(2)} = -\frac{2}{s} \Box \zeta_{\alpha(2s-2)} \]  
\[ (4.34) \]

where we have made use of (4.32). Since \( \zeta_{\alpha(2s-2)} \) and \( \partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} \) have the same functional type, it is possible to completely gauge away the divergence of \( h_{\alpha(2s)} \),

\[ \partial^{\beta(2)} h_{\alpha(2s-2)\beta(2)} = 0 \implies \partial^{\beta(a_1 h_{a_2 \ldots a_{2s}})\beta} = \partial^{a_1 h_{a_2 \ldots a_{2s}}}. \]  
\[ (4.35) \]

Under the gauge conditions imposed, there still remains some residual gauge freedom described by a gauge parameter constrained by (4.32) and \( \Box \zeta_{\alpha(2s-2)} = 0 \). So far the above analysis has been identical to that given in Appendix B of [34] for the massless model (4.28).

As a consequence of (4.35), the Cotton tensor (3.35) reduces to the simple form (3.41a). Making use of the gauge conditions (4.31) and (4.35) in conjunction with eq. (3.41a), the equation of motion (4.30a) becomes

\[ \left( \mu^{2s-3} \partial^{a_1} + \lambda \Box^{s-2} \partial^{a_1} \right) \Box h_{a_2 \ldots a_{2s}} = 0. \]  
\[ (4.36) \]

This equation has two types of solutions, massless and massive ones,

\[ \Box h_{\alpha(2s)} = 0; \]  
\[ (4.37a) \]
\[ \mu^{2s-3} h_{\alpha(2s)} + \lambda s^{-2} \partial^\beta \alpha_1 h_{\alpha_2...\alpha_{2s}\beta} = 0 \, . \quad (4.37b) \]

Let us show that the massless solution (4.37a) is a pure gauge degree of freedom. Since both the gauge field \( h_{\alpha(2s)} \) and the gauge parameter \( \zeta_{\alpha(2s-2)} \) are on-shell massless, it is useful to switch to momentum space by replacing \( h_{\alpha(2s)}(x) \rightarrow h_{\alpha(2s)}(p) \) and \( \zeta_{\alpha(2s-2)}(x) \rightarrow \zeta_{\alpha(2s-2)}(p) \), where the three-momentum \( p^\alpha \) is light-like, \( p^\alpha p_\alpha = 0 \). As in the fermionic case studied in the previous subsection, we can choose a frame in which the only non-zero component of \( p^\alpha = (p_{11}, p_{12} = p_{21}, p_{22}) \) is \( p_{22} = p_{11} \). In this frame, the equations (4.32) and (4.35) are equivalent to

\[ h_{\alpha(2s-2)22}(p) = 0 \, , \quad \zeta_{\alpha(2s-4)22}(p) = 0 \, . \quad (4.38) \]

These conditions tell us that the only non-zero components in this frame are \( h_{1...1}(p) \), \( h_{1...12}(p) \) and \( \zeta_{1...1}(p) \), \( \zeta_{1...12}(p) \). However, the gauge transformation (4.29a) is equivalent to \( \delta h_{1...1}(p) \propto \zeta_{1...1}(p) \) and \( \delta h_{1...12}(p) \propto \zeta_{1...12}(p) \), allowing us to completely gauge away the \( h_{\alpha(2s)} \) field.

Let us turn to the other equation (4.37b), which implies

\[ \left( \Box^{2s-3} - (m^2)^{2s-3} \right) h_{\alpha(2s)}(p) = 0 \, , \quad m := \left\vert \frac{\mu}{\lambda_{1/(2s-3)}} \right\vert . \quad (4.39) \]

Here the mass parameter has the same form as in the fermionic case, eq. (4.20). Transforming eq. (4.39) to momentum space gives

\[ \left( 1 - \left( \frac{-p^2}{m^2} \right)^{2s-3} \right) h_{\alpha(2s)}(p) = 0 \, . \quad (4.40) \]

This equation has a unique real solution, which is \( p^2 = -m^2 \), in complete analogy with the fermionic case considered in the previous subsection. It follows that \( h_{\alpha(2s)} \) satisfies the Klein-Gordon equation,

\[ (\Box - m^2) h_{\alpha(2s)} = 0 \, . \quad (4.41) \]

As a consequence, the equation of motion (4.37b) leads to

\[ \partial^\beta \alpha_1 h_{\alpha_2...\alpha_{2s}\beta} = \sigma m h_{\alpha(2s)} \, , \quad \sigma := -\text{sign}(\mu \lambda) . \quad (4.42) \]

Therefore \( h_{\alpha(2s)} \) is an irreducible on-shell massive field with mass \( m \) and helicity \( \lambda = \sigma s \). Equation (4.42) implies that \( h_{\alpha(2s)} \) describes a single propagating degree of freedom.

5 Conformal higher-spin gauge superfields

Conformal higher-spin gauge superfields in \( \mathcal{N} = 1 \) Minkowski superspace were introduced in [33, 35], as a by-product of the \( \mathcal{N} = 2 \) approach of [34]. In this section we start by generalising this concept to the case of \( \mathcal{N} = 1 \) supergravity, building on the ideas advocated in [53].
### 5.1 Conformal supergravity

Consider a curved $\mathcal{N} = 1$ superspace, $\mathcal{M}^{3|2}$, parametrised by local real coordinates $z^M = (x^m, \theta^\mu)$, with $m = 0, 1, 2$ and $\mu = 1, 2$, of which $x^m$ are bosonic and $\theta^\mu$ fermionic. We introduce a basis of one-forms $E^A = (E^a, E^\alpha)$ and its dual basis $E_A = (E_a, E_\alpha)$, defined by the supervielbein and its inverse, respectively. The superspace structure group is $\text{SL}(2, \mathbb{R})$, the double cover of the connected Lorentz group $\text{SO}_0(2,1)$. The covariant derivatives have the form:

$$ D_A = (D_a, D_\alpha) = E_A + \Omega_A , $$

where

$$ \Omega_A = \frac{1}{2} \Omega_A{}^{bc} M_{bc} = -\Omega_A{}^b M_b = \frac{1}{2} \Omega_A{}^{\beta\gamma} M_{\beta\gamma} $$

is the Lorentz connection.

The covariant derivatives are characterised by graded commutation relations

$$ \{D_A, D_B\} = T_{AB} C_D D_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} , $$

where $T_{AB} C$ and $R_{AB}{}^{cd}$ are the torsion and curvature tensors, respectively. To describe supergravity, the covariant derivatives have to obey certain torsion constraints [16] such that the algebra (5.4) takes the form [17]

$$ \{D_\alpha, D_\beta\} = 2i D_{\alpha\beta} - 4i S M_{\alpha\beta} , $$

$$ [D_a, D_\beta] = (\gamma_a)_{\beta\gamma} \left[ S D_\gamma + i C_{\gamma\delta\mu} M^{\delta\mu} \right] - \frac{2}{3} \left[ D_\beta S \delta^c_a - 2 \varepsilon_{ab} (\gamma^b)_{\beta\gamma} D_\gamma S \right] M_c , $$

$$ [D_a, D_b] = \varepsilon_{abc} \left\{ \left[ \frac{1}{2} (\gamma^c)_{\alpha\beta} C^{\alpha\beta\gamma} - \frac{2i}{3} (\gamma^c)_{\beta\gamma} D_\beta S \right] D_\gamma 

+ \left[ \frac{1}{2} (\gamma^c)_{\alpha\beta} (\gamma^d)_{\gamma\delta} D_\alpha C_{\beta\gamma\delta} + \left( \frac{2i}{3} D^2 S + 4 S^2 \right) \eta^{cd} \right] M_d \right\} . $$

Here the scalar $S$ and the symmetric spinor $C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}$ are real. The dimension-2 Bianchi identities imply that

$$ D_\alpha C_{\beta\gamma\delta} = D_\alpha (C_{\beta\gamma\delta}) + \varepsilon_{\alpha(\beta} D_{\gamma)\delta} S \quad \implies \quad D^\gamma C_{\alpha\beta\gamma} = \frac{4}{3} D_{\alpha\beta} S . $$

We use the notation $D^2 := D^\alpha D_\alpha$. 

23
The algebra of covariant derivatives is invariant under the following super-Weyl transformations \[76, 77, 78\]

\[
\delta_\sigma D_\alpha = \frac{1}{2} \sigma D_\alpha + D_\beta \sigma M_{\alpha\beta} ,
\]
\[
\delta_\sigma D_a = \sigma D_a + \frac{i}{2} (\gamma_a)^{\gamma\delta} D_\gamma \sigma D_\delta + \varepsilon_{abc} D^b \sigma M^c ,
\]

with the parameter \(\sigma\) being a real unconstrained superfield, provided the torsion super fields transform as

\[
\delta_\sigma S = \sigma S - \frac{i}{4} D^2 \sigma , \quad \delta_\sigma C_{\alpha\beta\gamma} = \frac{3}{2} \sigma C_{\alpha\beta\gamma} - \frac{i}{2} D_{(\alpha\beta} D_{\gamma)} \sigma .
\]

The \(\mathcal{N} = 1\) supersymmetric extension of the Cotton tensor [3.3] was constructed in [79]. It is given by the expression

\[
\mathcal{W}_{\alpha\beta\gamma} = \left(\frac{i}{2} D^2 + 4 S\right) C_{\alpha\beta\gamma} + i D_{(\alpha\beta} D_{\gamma)} S .
\]

The super-Weyl transformation of \(\mathcal{W}_{\alpha\beta\gamma}\) proves to be

\[
\delta_\sigma \mathcal{W}_{\alpha\beta\gamma} = \frac{5}{2} \sigma \mathcal{W}_{\alpha\beta\gamma} .
\]

It can be shown [23] that the curved superspace is conformally flat if and only if \(\mathcal{W}_{\alpha\beta\gamma} = 0\).

### 5.2 Conformal gauge superfields

A real tensor superfield \(\mathfrak{H}_{\alpha(n)}\) is said to be a conformal gauge supermultiplet if (i) it is super-Weyl primary of dimension \((1 - n/2)\),

\[
\delta_\sigma \mathfrak{H}_{\alpha(n)} = \left(1 - \frac{n}{2}\right) \sigma \mathfrak{H}_{\alpha(n)} ;
\]

and (ii) it is defined modulo gauge transformations of the form

\[
\delta_\lambda \mathfrak{H}_{\alpha(n)} = i^n D_{(\alpha_1} \lambda_{\alpha_2...\alpha_n)} ,
\]

with the gauge parameter \(\lambda_{\alpha(n-1)}\) being real but otherwise unconstrained. The super-Weyl weight of \(\mathfrak{H}_{\alpha(n)}\), given by \((1 - n/2)\), is uniquely fixed by requiring \(\lambda_{\alpha(n-1)}\) and \(\delta_\lambda \mathfrak{H}_{\alpha(n)}\) to be super-Weyl primary.

Starting with \(\mathfrak{H}_{\alpha(n)}\) one can construct its descendant, \(\mathfrak{M}_{\alpha(n)}(\mathfrak{H})\), defined uniquely, modulo a normalisation, by the following the properties:
1. $\mathcal{W}_{\alpha(n)}$ is of the form $\mathcal{A}_{\alpha(n)}$, where $\mathcal{A}$ is a linear differential operator involving $D_A$, the
torsion tensors $\mathcal{C}_{\alpha\beta\gamma}$ and $\mathcal{S}$ and their covariant derivatives.

2. $\mathcal{W}_{\alpha(n)}$ is super-Weyl primary of weight $(1 + n/2)$,
$$\delta_\sigma \mathcal{W}_{\alpha(n)} = (1 + \frac{n}{2}) \sigma \mathcal{W}_{\alpha(n)} \ .$$
(5.13)

3. The gauge variation of $\mathcal{W}_{\alpha(n)}$ vanishes if the superspace is conformally flat,
$$\delta_\lambda \mathcal{W}_{\alpha(n)} = O(\mathcal{W}_{(3)}) \ ,$$
(5.14)
where $\mathcal{W}_{(3)}$ is the super-Cotton tensor [5.9].

4. $\mathcal{W}_{\alpha(n)}$ is divergenceless if the superspace is conformally flat,
$$D_\beta \mathcal{W}_{\beta\alpha(n-1)} = O(\mathcal{W}_{(3)}) \ .$$
(5.15)

As a simple example, we consider a U(1) vector multiplet coupled to supergravity, which
corresponds to the $n = 1$ case. This multiplet is described by a real spinor prepotential
$\mathcal{H}_\alpha$ which is super-Weyl primary of weight $1/2$ and is defined modulo gauge transformations
$\delta_\lambda \mathcal{H}_\alpha = i D_\alpha \lambda$, where the gauge parameter $\lambda$ is an unconstrained real superfield. The required
super-Weyl primary descendant of weight $3/2$ is given by
$$\mathcal{W}_\alpha = - \frac{i}{2} D^\beta D_\alpha \mathcal{H}_\beta - 2 S \mathcal{H}_\alpha$$
(5.16)
and proves to be gauge invariant,
$$\delta_\zeta \mathcal{W}_\alpha = 0 \ .$$
(5.17)

The field strength obeys the Bianchi identity
$$D^\alpha \mathcal{W}_\alpha = 0 \ .$$
(5.18)
For $n > 1$ the right-hand sides of (5.14) and (5.15) are non-vanishing.

Suppose that our background curved superspace $\mathcal{M}^{3|2}$ is conformally flat,
$$\mathcal{W}_{\alpha(3)} = 0 \ .$$
(5.19)
Then the tensor superfield $\mathcal{W}_{\alpha(n)}$ is gauge invariant and conserved,

\begin{align}
\delta_{\lambda} \mathcal{W}_{\alpha(n)} &= 0 , \\
D_{\beta} \mathcal{W}_{\beta\alpha(n-1)} &= 0 .
\end{align}

These properties and the super-Weyl transformation laws \([5.11]\) and \([5.13]\) imply that the action

\[S_{SCS}^{(n)}[\mathcal{H}(n)] = -\frac{i^n}{2|n/2|+1} \int d^3z \ E \mathcal{H}^{(n)} \mathcal{W}_{\alpha(n)}(\mathcal{H}) , \quad E^{-1} = \text{Ber}(E_A^M)\]

is gauge and super-Weyl invariant,

\[\delta_{\lambda} S_{SCS}^{(n)}[\mathcal{H}(n)] = 0 , \quad \delta_{\sigma} S_{SCS}^{(n)}[\mathcal{H}(n)] = 0 .\]

We now turn to constructing the linearised higher-spin super-Cotton tensors $\mathcal{W}_{\alpha(n)}$ on such a conformally flat superspace.

### 5.3 Higher-spin super-Cotton tensor in Minkowski superspace

In Minkowski superspace, $M^{3|2}$, the higher-spin super-Cotton tensor \([33, 35]\) is

\[W_{\alpha_1...\alpha_n} = \left(-\frac{i}{2}\right)^n D_{\beta_1} D_{\alpha_1} ... D_{\beta_n} D_{\alpha_n} H_{\beta_1...\beta_n} = W_{(\alpha_1...\alpha_n)} ,\]

with $D_A = (\partial_a, D_{\alpha})$ being the flat-superspace covariant derivatives. This tensor is invariant under the gauge transformation

\[\delta H_{\alpha_1...\alpha_n} = i^n D_{(\alpha_1} \zeta_{\alpha_2...\alpha_n)} ,\]

and obeys the conservation identity

\[D^\beta W_{\beta\alpha_1...\alpha_{n-1}} = 0 .\]

The fact that $W_{\alpha_1...\alpha_n}$ defined by \((5.23)\) is completely symmetric, is a corollary of the identities

\[D^\alpha D_\beta D_\alpha = 0 \quad \Rightarrow \quad [D_\alpha D_\beta, D_\gamma D_\delta] = 0 .\]

The normalisation in \((5.23)\) is explained as follows. The gauge freeform \((5.24)\) allows us to impose a gauge condition

\[D_\beta H_{\beta\alpha(n-1)} = 0 ,\]

The super-Weyl transformation of the superspace integration measure is $\delta_{\sigma} E = -2\sigma E$. 

26
under which the expression for the super-Cotton tensor simplifies,

\[ D^\beta H_{\beta_1...\alpha_{n-1}} = 0 \implies W_{\alpha(n)} = \partial_{\alpha_1}^{\beta_1} \ldots \partial_{\alpha_n}^{\beta_n} H_{\beta_1...\beta_n} . \]  

(5.28a)

This result can be fine-tuned to

\[ W_{\alpha(2s)} = \Box^s H_{\alpha(2s)} , \]  

(5.28b)

\[ W_{\alpha(2s+1)} = \Box^s \partial^{\beta_1}(\alpha_1 H_{\alpha_2...\alpha_{2s+1}}) = \Box^s \partial^{\beta_1} H_{\alpha_2...\alpha_{2s+1}} , \]  

(5.28c)

where \( s > 0 \) is an integer.

For completeness, we also give another representation for the higher-spin super-Cotton tensor derived in [33, 35]:

\[ W_{\alpha_1...\alpha_n} := \frac{1}{2^n} \sum_{j=0}^{[n/2]} \left\{ \binom{n}{2j} \square^j \partial_{\alpha_1}^{\beta_1} \ldots \partial_{\alpha_{n-2j}}^{\beta_{n-2j}} H_{\alpha_{n-2j+1}...\alpha_n} \beta_1...\beta_{n-2j} \right. 
\]

\[ - \frac{i}{2} \left( \binom{n}{2j+1} D^2 \square^j \partial_{\alpha_1}^{\beta_1} \ldots \partial_{\alpha_{n-2j+1}}^{\beta_{n-2j+1}} H_{\alpha_{n-2j}...\alpha_n} \beta_1...\beta_{n-2j+1} \right) \} . \]  

(5.29)

The following higher-spin action [33, 35]

\[ S_{SCS}^{(n)}[H_{(n)}] = \frac{i^n}{2^{[n/2]+1}} \int d^{3|2}z H^{(n)} W_{\alpha(n)}(H) \]  

(5.30)

is \( \mathcal{N} = 1 \) superconformal. It is clearly invariant under the gauge transformations (5.24).

### 5.4 Higher-spin super-Cotton tensor in conformally flat superspaces

Consider a conformally flat spacetime \( \mathcal{M}^{3|2} \). Its covariant derivatives \( \mathcal{D}_A \) are related to the flat-space ones by

\[ \mathcal{D}_a = e^{\frac{i}{2} \sigma} \left( D_a + D^\beta \sigma M_{a\beta} \right) , \]  

(5.31)

\[ \mathcal{D}_a = e^\sigma \left( \partial_a + \frac{i}{2} (\gamma_a)^{\alpha\beta} \mathcal{D}_a \sigma D_\beta + \partial^\beta \sigma M_{ba} - \frac{i}{8} (\gamma_a)^{\alpha\beta} (D^\gamma \sigma) D_\gamma \sigma M_{a\beta} \right) , \]  

(5.32)

for some scale factor \( \sigma \). In accordance with (5.13), the higher-spin super-Cotton tensor \( \mathfrak{W}_{\alpha(n)} \) in \( \mathcal{M}^{3|2} \) is related to the flat-space one, eq. (5.23) or equivalently (5.29), by the rule

\[ \mathfrak{W}_{\alpha(n)} = e^{(1+\frac{1}{2})\sigma} W_{\alpha(n)} . \]  

(5.33)

In general, it is a difficult technical problem to express \( \mathfrak{W}_{\alpha(n)} \) in terms of the covariant derivatives \( \mathcal{D}_A \) and the gauge prepotential \( \mathfrak{f}_{\alpha(n)} = e^{(1-n/2)\sigma} H_{\alpha(n)} \). As an example, we only give
expressions for the supersymmetric photino $\mathbb{W}_\alpha$ and Cottino $\mathbb{W}_{\alpha(2)}$ tensors in AdS superspace. The geometry of AdS$^{3|2}$ is encoded in the following algebra of covariant derivatives:

\[
\begin{align*}
\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} &= 2i\mathcal{D}_{\alpha\beta} - 4iS\mathcal{M}_{\alpha\beta}, \\
[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma] &= -2S\varepsilon_\gamma(\alpha\mathcal{D}_\beta), \\
[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma]\delta] &= 4S^2(\varepsilon_\gamma(\alpha\mathcal{M}_\beta)\delta + \varepsilon_\delta(\alpha\mathcal{M}_\beta)\gamma),
\end{align*}
\]

with $S$ being a non-zero real parameter. The tensors $\mathbb{W}_\alpha$ and $\mathbb{W}_{\alpha(2)}$ are expressed in terms of the operator

\[
\Delta^\beta_\alpha := -\frac{i}{2}\mathcal{D}^\beta\mathcal{D}_\alpha - 2S\delta^\beta_\alpha,
\]

with the properties

\[
\begin{align*}
[\Delta^\beta_{\alpha_1}, \Delta^\beta_{\alpha_2}] &= \varepsilon_{\alpha_1\alpha_2}S(D^\beta_{\alpha_1\alpha_2} - 2SM^\beta_{\alpha_1\alpha_2}) - \varepsilon_{\beta_1\beta_2}S(\mathcal{D}_{\alpha_1\alpha_2} - 2SM_{\alpha_1\alpha_2}).
\end{align*}
\]

These properties follow from the identity

\[
D^\beta\mathcal{D}_\alpha D^\beta = 4iSD_\alpha \quad \implies \quad D^\alpha \Delta^\beta_\alpha = 0.
\]

The expressions for $\mathbb{W}_\alpha$ and $\mathbb{W}_{\alpha(2)}$ are:

\[
\begin{align*}
\mathbb{W}_\alpha &= \Delta^\beta_\alpha \mathcal{H}_\beta, \\
\mathbb{W}_{\alpha_1\alpha_2} &= \Delta^\beta_{\alpha_1} \Delta^\beta_{\alpha_2} \mathcal{H}_{\beta_1\beta_2} - 2S\Delta^\beta_{\alpha_1} \mathcal{H}_{\alpha_2}\beta.
\end{align*}
\]

5.5 Massive supersymmetric higher-spin theories in AdS superspace

Massive supersymmetric higher-spin actions in AdS involve different massless sectors depending on the value of superspin.

\[
\begin{align*}
S^{(2s)}_{\text{massive}} &= \lambda S^{(2s)}_{\text{SCS}}[\mathcal{H}^{(2s)}_{(2s)}] + \mu^{2s-1}S^{(2s)}_{\text{FO}}[\mathcal{H}^{(2s)}_{(2s)}, \mathcal{Z}^{(2s-2)}], \\
S_{\text{massive}}^{(2s+1)} &= \lambda S^{(2s+1)}_{\text{SCS}}[\mathcal{H}^{(2s+1)}_{(2s+1)}] + \mu^{2s-1}S^{(2s+1)}_{\text{SO}}[\mathcal{H}^{(2s+1)}_{(2s+1)}, \mathcal{X}^{(2s-2)}].
\end{align*}
\]

5.5.1 First-order massless actions

We introduce a gauge theory described by a reducible gauge superfield $\mathcal{H}_{\beta,\alpha_1...\alpha_{n-1}} = \mathcal{H}_{\beta,(\alpha_1...\alpha_{n-1})}$. This superfield is defined modulo gauge transformations of the form

\[
\delta \mathcal{H}_{\beta,\alpha_1...\alpha_{n-1}} = i^n\mathcal{D}_{\beta}^{\alpha_1...\alpha_{n-1}}.
\]
A supersymmetric gauge-invariant action of lowest order in derivatives is
\[
S^{(n)}_{\text{FO}} = \frac{i^{n+1}}{2^{[(n+1)/2]}} \int d^{3/2} z E \mathcal{H}^{\gamma,\alpha_1...\alpha_{n-1}} \left( D^\gamma D_{\beta} - 4iS\delta^\gamma_{\beta} \right) \mathcal{H}_{\gamma,\alpha_1...\alpha_{n-1}} .
\] (5.41)

The gauge invariance of $S^{(n)}_{\text{FO}}$ follows from the identity (5.37). Our action (5.41) is a higher-spin AdS extension of the model for the massless gravitino multiplet \((n = 2)\) in Minkowski superspace proposed by Siegel [1] (see also [16]).

The gauge superfield $\mathcal{H}^{\beta,\alpha_1...\alpha_{n-1}}$ can be decomposed into irreducible SL(2, $\mathbb{R}$) superfields
\[
\mathcal{H}^{\beta,\alpha_1...\alpha_{n-1}} = \mathcal{S}^{\beta,\alpha_1...\alpha_{n-1}} + \sum_{k=1}^{n-1} \varepsilon^{\beta\alpha_k} \mathcal{Y}^{\alpha_1...\hat{\alpha_k}...\alpha_{n-1}},
\] (5.42)
where $\mathcal{S}^{\alpha(n)}$ and $\mathcal{Y}^{\alpha(n-2)}$ are completely symmetric tensor superfields. Then the gauge transformation (5.40) turns into
\[
\begin{align*}
\delta \mathcal{S}^{\alpha(n)} &= i^n D_{(\alpha_1...\alpha_{n-2})}, \\
\delta \mathcal{Y}^{\alpha(n-2)} &= \frac{i^n}{n} D^\beta \lambda^{\beta,\alpha_1...\alpha_{n-2}}.
\end{align*}
\] (5.43a, 5.43b)

The supersymmetric gauge-invariant action takes the form
\[
S^{(n)}_{\text{FO}} = \frac{i^{n+1}}{2^{[(n+1)/2]}} \int d^{3/2} z E \left\{ \mathcal{S}^{\beta,\alpha(n-1)} D^\gamma D_{\beta} \mathcal{S}^{\gamma,\alpha(n-1)} + 2i(\alpha - 1)\mathcal{Y}^{\alpha(n-2)} D^\gamma \mathcal{S}^{\gamma,\alpha(n-2)} \\
+ (\alpha - 1) \left( \mathcal{Y}^{\alpha(n-2)} D^2 \mathcal{Y}^{\alpha(n-2)} + (-1)^{\alpha} (n - 2) D_{\beta} \mathcal{Y}^{\beta,\alpha(n-3)} D^\gamma \mathcal{Y}^{\gamma,\alpha(n-3)} \right) \\
- 4Si \left( \mathcal{S}^{\alpha(n)} \mathcal{S}^{\alpha(n)} + (n - 1) \mathcal{Y}^{\alpha(n-2)} \mathcal{Y}^{\alpha(n-2)} \right) \right\} .
\] (5.44)

When $n$ is even, $n = 2s$, this action is the unique gauge-invariant AdS extension of the massless integer superspin action of [35].

### 5.5.2 Second-order massless actions

The massless half-integer superspin action in AdS is
\[
S^{(2s+1)}_{\text{SO}} = \left( -\frac{1}{2} \right)^s \int d^{3/2} z E \left\{ -\frac{i}{2} \mathcal{S}^{\alpha(2s+1)} \mathcal{S}^{\alpha(2s+1)} - \frac{i}{8} D_{\beta} \mathcal{S}^{\beta,\alpha(2s)} D^2 D^\gamma \mathcal{S}^{\gamma,\alpha(2s)} \\
+ \frac{i}{4} s D_{\beta} \mathcal{S}^{\beta,\alpha(2s-1)} D^\gamma \mathcal{S}^{\gamma,\lambda(2s-1)} - \frac{1}{2} (2s - 1) \mathcal{X}^{\alpha(2s-2)} D^\gamma D^\delta \mathcal{S}^{\delta,\gamma,\alpha(2s-2)} \\
+ \frac{1}{2}(2s - 1) \left[ \mathcal{X}^{\alpha(2s-2)} D^2 \mathcal{X}^{\alpha(2s-2)} - \frac{s - 1}{s} D_{\beta} \mathcal{X}^{\beta,\alpha(2s-3)} D^\gamma \mathcal{X}^{\gamma,\alpha(2s-3)} \right] \\
+ is S\mathcal{S}^{\beta,\alpha(2s)} D^\gamma \mathcal{S}^{\gamma,\alpha(2s)} + \frac{1}{2} (s + 1) S\mathcal{S}^{\alpha(2s+1)} D^2 \mathcal{S}^{\alpha(2s+1)} \right\} .
\] (5.45)
\begin{align*}
+is(2s-3)S^2\mathcal{H}^{\alpha(2s+1)}\mathcal{H}_{\alpha(2s+1)} + \frac{(2s-1)(s^2-3s-2)}{s}S\mathcal{H}^{\alpha(2s-2)}X_{\alpha(2s-2)} \right),
\end{align*}

where \( Q \) is the quadratic Casimir operator \( \{2,14\} \). One can express \( Q \) in the form

\begin{align*}
Q &= D^a D_a - i SD^2 + 2SD\alpha\beta M_{\alpha\beta} - 2S^2M^{\alpha\beta}M_{\alpha\beta}.
\end{align*}

The action (5.45) is invariant under gauge transformations

\begin{align*}
\delta \mathcal{H}_{\alpha(2s+1)} &= iD_{(\alpha_1\alpha_2...\alpha_{2s+1})}, \\
\delta X_{\alpha(2s-2)} &= \frac{s}{2s+1}D^{\beta\gamma}\lambda_{\beta\gamma\alpha_1...\alpha_{2s-2}}.
\end{align*}

The action (5.45) is the unique gauge-invariant AdS extension of the massless half-integer superspin action of [35].

### 5.6 From AdS superspace to AdS space

To conclude this section, we briefly discuss the key aspects of component reduction for supersymmetric field theories formulated in AdS superspace, AdS\(^3\|2\). In general, the action functional of such a theory is given by

\begin{align*}
S = \int d^{3\|2}z E \mathcal{L},
\end{align*}

where the Lagrangian \( \mathcal{L} \) is a scalar superfield. In accordance with the general formalism described in section 6.4 of [61], the isometry transformations of AdS\(^3\|2\) are generated by the Killing vector fields \( \xi^A E_A \) which are defined to obey the master equation [80]

\begin{align*}
[\xi + \frac{1}{2}\Lambda^{bc}M_{bc}, D_A] = 0, \quad \xi := \xi^B D_B = \xi^b D_b + \xi^\beta D_\beta,
\end{align*}

for some Lorentz superfield parameter \( \Lambda^{bc} = -\Lambda^{cb} \). An infinitesimal isometry transformation acts on a tensor superfield \( T \) as

\begin{align*}
\delta \xi T = (\xi + \frac{1}{2}\Lambda^{bc}M_{bc})T.
\end{align*}

The action (5.48) is invariant under the isometry group of AdS\(^3\|2\).

As shown in [80], the parameters in (5.49) obey the following Killing equations:

\begin{align*}
D_\alpha \xi_\beta &= \frac{1}{2}\Lambda_{\alpha\beta} + S\xi_{\alpha\beta} = D_\beta \xi_\alpha, \\
D_\beta \xi^{\beta\alpha} + 6i\xi^\alpha &= 0, \quad D_\beta \Lambda^{\beta\alpha} + 12Si\xi^\alpha = 0,
\end{align*}

As shown in [80], the parameters in (5.49) obey the following Killing equations:
\[ \mathcal{D}_{(\alpha\xi\beta\gamma)} = 0, \quad \mathcal{D}_{(\alpha\Lambda\beta\gamma)} = 0, \quad (5.51c) \]

which imply

\[ \begin{align*}
\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a &= 0, \\
\mathcal{D}^2\xi_a - 12iS\xi_a &= 0, \\
\mathcal{D}_{\alpha\beta}\xi^\beta + 2S\xi_a &= 0.
\end{align*} \quad (5.52a, b, c) \]

Equation (5.52a) tells us that \( \xi_a \) is a Killing vector, while (5.52c) means that \( \xi_a \) is a Killing spinor. The component form of the action (5.48) is computed using the formula (see also [79])

\[ S = \frac{1}{4} \int d^3x \ e \left( i\mathcal{D}^2 + 8S \right) \mathcal{L} \, . \quad (5.53) \]

Here and in what follows, the \( \theta \)-independent component \( T|_{\theta=0} \) of a superfield \( T(x, \theta) \) will simply be denoted \( T| \). To complete the formalism of component reduction, we only need the following relation

\[ (\mathcal{D}_a T)| = \nabla_a T|, \quad (5.54) \]

where \( \nabla_a \) is the standard torsion-free covariant derivative of AdS space. Making use of the AdS transformation law \( \delta \xi \mathcal{L} = \xi \mathcal{L} \) in conjunction with the identities (5.51) and (5.52), one may check that the action (5.53) is invariant under arbitrary isometry transformations of the AdS superspace.

6 Supersymmetric higher-spin actions in components

In this section we will describe the component structure of the supersymmetric higher-spin theories introduced in the previous section. Our analysis will be restricted to the flat-superspace case. As in [35], the integration measure for \( \mathcal{N} = 1 \) Minkowski superspace is defined as follows:

\[ \int d^{3|2}z \ L = \frac{i}{4} \int d^3x \ D^2 L \big|_{\theta=0} . \quad (6.1) \]

6.1 Superconformal higher-spin action

We start by reducing the superconformal higher-spin action (5.30) to components. The gauge freedom (5.24) can be used to impose a Wess-Zumino gauge

\[ H_{\alpha_1...\alpha_n} = 0, \quad D^3 H_{\beta\alpha_1...\alpha_{n-1}} = 0. \quad (6.2) \]

This definition implies that \( \int d^{3|2}z \ V = \int d^3x \ F \), for any scalar superfield \( V(x, \theta) = \cdots + i\theta^2 F(x) \).
In this gauge, there remain two independent component fields

\[ h_{\alpha_1...\alpha_{n+1}} := i^{n+1} D_{( \alpha_1 H_{\alpha_2...\alpha_{n+1}} )} , \quad h_{\alpha_1...\alpha_n} := -\frac{i}{4} D^2 H_{\alpha_1...\alpha_n} \] . \tag{6.3} 

Due to the conservation equation \((5.25)\), the higher-spin super-Cotton tensor \((5.29)\) also has two independent components, which we define as

\[ C_{\alpha_1...\alpha_n} := W_{\alpha_1...\alpha_n} \] ,  
\[ C_{\alpha_1...\alpha_{n+1}} := i^{n+1} D_{( \alpha_1 W_{\alpha_2...\alpha_{n+1}} )} \] . \tag{6.4} 

The field strengths \(C_{\alpha(n)}\) and \(C_{\alpha(n+1)}\) are given in terms of the gauge potentials \(h_{\alpha(n)}\) and \(h_{\alpha(n+1)}\), respectively, according to eq. \((3.35)\). To prove this statement for \(C_{\alpha(n+1)}\), one has to use the identity

\[ \binom{n}{2j} + \binom{n}{2j+1} = \binom{n+1}{2j+1} \] . \tag{6.5} 

Reducing the action \((5.30)\) to components gives

\[ S_{\text{SCS}}^{(n)}[H_{(n)}] = S_{\text{CS}}^{(n)}[h_{(n)}] + S_{\text{CS}}^{(n+1)}[h_{(n+1)}] , \tag{6.6} \]

where the conformal higher-spin action \(S_{\text{CS}}^{(n)}[h_{(n)}]\) is defined by eq. \((3.39)\).

In the gauge \((6.2)\), the residual gauge freedom is characterised by the conditions

\[ D_{( \alpha_1 \lambda_{\alpha_2...\alpha_n} )} = 0 , \quad D^2 \lambda_{\alpha_1...\alpha_{n-1}} = -2i \frac{n-1}{n+1} \partial_{( \alpha_1 \lambda_{\alpha_2...\alpha_{n-1}} )} \beta \] . \tag{6.7} 

At the component level, the remaining independent gauge transformations are generated by \(\zeta_{\alpha(n-1)} \propto \lambda_{\alpha(n-1)}\) and \(\zeta_{\alpha(n-2)} \propto i^n D^2 \lambda_{\beta \alpha(n-2)}\).

### 6.2 Massless first-order model

We now turn to working out the component structure of the first-order model \((5.44)\) in the flat-superspace limit. In Minkowski superspace, the action can be written in the form

\[ S_{\text{FO}}^{(n)} = \frac{i^{n+1}}{2^{|n+1|/2}} \int d^{|n+1|/2} z \left\{ i H^{\beta_{\alpha_1...\alpha_{n-1}} d_{\beta} H_{\gamma_{\alpha_1...\alpha_{n-1}}} + \frac{1}{2} H_{\alpha_1...\alpha_n} D^2 H_{\alpha_1...\alpha_n} \right. \]
\[ + 2i (n-1) Y^{\alpha_1...\alpha_{n-2}} \partial_{\beta} H_{\beta \gamma_{\alpha_1...\alpha_{n-2}}} + (n-1) Y^{\alpha_1...\alpha_{n-2}} D^2 Y_{\alpha_1...\alpha_{n-2}} \]
\[ \left. + (-1)^n (n-1)(n-2) D_{\beta} Y^{\beta_{\alpha_1...\alpha_{n-3}} D^2 Y_{\alpha_1...\alpha_{n-3}} \right\} . \tag{6.8} \]

It is invariant under the gauge transformations

\[ \delta H_{\alpha_1 \alpha_2...\alpha_n} = i^n D_{( \alpha_1 \lambda_{\alpha_2...\alpha_n} )} , \tag{6.9a} \]
\[ \delta Y_{\alpha_1...\alpha_{n-2}} = \frac{i^n}{n} D^\beta \lambda_{\alpha_1...\alpha_{n-2}} \tag{6.9b} \]

with the gauge parameter \( \lambda_{\alpha(n-1)} \) being a real unconstrained superfield. When \( n \) is even, \( n = 2s \), the action (6.8) describes the massless integer superspin model of [35].

The gauge freedom allows us to choose a Wess-Zumino gauge

\[ H_{\alpha_1...\alpha_n} = 0 , \quad D^\beta H_{\beta\alpha_1...\alpha_{n-1}} = 0 , \quad Y_{\alpha_1...\alpha_{n-2}} = 0 . \tag{6.10} \]

Then, the residual gauge freedom is characterised by the conditions

\[ D_{\alpha_1} \lambda_{\alpha_2...\alpha_n} = 0 , \quad D^2 \lambda_{\alpha_1...\alpha_{n-1}} = -2i \frac{n-1}{n+1} \partial^\beta (\alpha_1 \lambda_{\alpha_2...\alpha_{n-1}}) \beta \tag{6.11} \]

These conditions imply that there remains only one independent gauge parameter at the component level. We define it as

\[ \zeta_{\alpha_1...\alpha_{n-1}}(x) := (-1)^{n+1} \lambda_{\alpha_1...\alpha_{n-1}} \tag{6.12} \]

We define the component fields as

\[ h_{\alpha_1...\alpha_{n+1}} := i^{n+1} D_{(\alpha_1 H_{\alpha_2...\alpha_{n+1}})} \], \tag{6.13a} \]
\[ h_{\alpha_1...\alpha_n} := -\frac{i}{4} D^2 H_{\alpha_1...\alpha_n} \], \tag{6.13b} \]
\[ y_{\alpha_1...\alpha_{n-1}} := \frac{i^{n+1}}{2n} D_{(\alpha_1 Y_{\alpha_2...\alpha_{n-1}})} \], \tag{6.13c} \]
\[ Z_{\alpha_1...\alpha_{n-2}} := \frac{i}{4} D^2 Y_{\alpha_1...\alpha_{n-2}} \]. \tag{6.13d} \]

Their gauge transformation laws are

\[ \delta h_{\alpha_1...\alpha_{n+1}} = \partial (\alpha_1 \alpha_2 \zeta_{\alpha_3...\alpha_{n+1}}) \], \tag{6.14a} \]
\[ \delta y_{\alpha_1...\alpha_{n-1}} = \frac{1}{n+1} \partial^\beta (\alpha_1 \zeta_{\alpha_2...\alpha_{n-1}}) \beta \], \tag{6.14b} \]
\[ \delta y_{\alpha_1...\alpha_{n-3}} = \partial^\beta \gamma \zeta_{\beta\gamma\alpha_1...\alpha_{n-3}} \], \tag{6.14c} \]
\[ \delta h_{\alpha_1...\alpha_n} = 0 \], \tag{6.14d} \]
\[ \delta Z_{\alpha_1...\alpha_{n-2}} = 0 \]. \tag{6.14e} \]

Direct calculations of the component action give

\[ S^{(n)}_{\text{FO}} = \frac{i^n}{2^{[n/2]}} \int d^3x \left\{ h_{\alpha_1...\alpha_n} h_{\alpha_1...\alpha_n} + Z_{\alpha_1...\alpha_{n-2}} Z_{\alpha_1...\alpha_{n-2}} \right\} \]
\[ + \frac{i^{n+1}}{2^{[(n+1)/2]}} \int d^3x \left\{ h_{\beta\alpha_1...\alpha_n} \partial^\gamma h_{\gamma\alpha_1...\alpha_n} + 2(n-1) y_{\alpha_1...\alpha_{n-1}} \partial^\beta h_{\beta\gamma\alpha_1...\alpha_{n-1}} \right\} \]
\[ +4(n - 1)g^{\beta\alpha_1...\alpha_{n-2}}\partial_\gamma y_{\alpha_1...\alpha_{n-2}} \]  
\[ + \frac{2(n - 2)(n + 1)}{n}g^{\alpha_1...\alpha_{n-3}}\partial^{\gamma\delta}y_{\gamma\alpha_1...\alpha_{n-3}} \]  
\[ - \frac{(n - 2)(n - 3)}{n(n - 1)}g^{\alpha_1...\alpha_{n-4}}\partial_\gamma h_{\gamma\alpha_1...\alpha_{n-4}} \] .  
\[(6.15)\]

The fields \( h_{\alpha(n)} \) and \( Z_{\alpha(n-2)} \) appear in the action without derivatives. This action can be rewritten in the form

\[
S^{(n)}_{\text{FO}} = \frac{i^n}{2^{n/2}} \int d^3x \left\{ h_{\alpha(n)}^{(n)} h_{\alpha(n)} + Z_{\alpha(n-2)}^{(n-2)} Z_{\alpha(n-2)} \right\} + S^{(n+1)}_{\text{FF}}[h_{(n+1), y(n-1), y(n-3)}] , \quad (6.16)
\]

where \( S^{(n+1)}_{\text{FF}} \) is the flat-space version of (4.2), eg. (13.5), with \( n \) replaced by \( (n + 1) \).

### 6.3 Massive integer superspin action

We are now prepared to read off the component form of a massive integer superspin action that is obtained from (5.39a) in the flat-superspace limit,

\[
S^{(2s)}_{\text{massive}} = \lambda S^{(2s)}_{\text{CS}}[H_{(2s)}] + \mu^{2s-1} S^{(2s)}_{\text{FO}}[H_{(2s)}, Y_{(2s-2)}] . \quad (6.17)
\]

Choosing \( n = 2s \) in the component actions (6.6) and (6.16) gives

\[
S^{(2s)}_{\text{massive}} = \lambda S^{(2s)}_{\text{CS}}[h_{(2s)}] + \frac{1}{2} \left( - \frac{1}{2} \right)^s \mu^{2s-1} \int d^3x h^{\alpha(2s)} h_{\alpha(2s)}
\]
\[ + \lambda S^{(2s+1)}_{\text{CS}}[h_{(2s+1)}] + \mu^{2s-1} S^{(2s+1)}_{\text{FF}}[h_{(2s+1), y(2s-1), y(2s-3)}]
\]
\[ + \frac{1}{2} \left( - \frac{1}{2} \right)^s \mu^{2s-1} \int d^3x Z^{\alpha(2s-2)} Z_{\alpha(2s-2)} \] .  
\[(6.18)\]

It is seen that the \( Z_{\alpha(2s-2)} \) field appears only in the third line of (6.18) and without derivatives, and thus \( Z_{\alpha(2s-2)} \) is an auxiliary field. Next, the expression in the second line of (6.18) constitutes the massive gauge-invariant spin-\((s + \frac{1}{2})\) action (4.6). The two terms in the first line of (6.18) involve the \( h_{\alpha(2s)} \) field. Unlike \( S^{(2s)}_{\text{CS}}[h_{(2s)}] \), the second mass-like term is not gauge invariant. However, the action

\[
\tilde{S}^{(2s)}_{\text{massive}} = \lambda S^{(2s)}_{\text{CS}}[h_{(2s)}] + \frac{1}{2} \left( - \frac{1}{2} \right)^s \mu^{2s-1} \int d^3x h^{\alpha(2s)} h_{\alpha(2s)}
\]
\[(6.19)\]

does describe a massive spin-\( s \) field on-shell. Indeed, the equation of motion is

\[
\lambda C_{\alpha(2s)} + \mu^{2s-1} h_{\alpha(2s)} = 0 . \quad (6.20)
\]

Since \( C_{\alpha(2s)} \) is divergenceless, eq. (3.38), the equation of motion implies that \( h_{\alpha(2s)} \) is divergenceless, eq. (4.35). As a consequence, \( C_{\alpha(2s)} \) takes the simple form given by (3.41a), and the above equation of motion turns into (compare with eq. (4.36))

\[
\lambda \Box^{s-1} \partial^\alpha_{\alpha_1} h_{\alpha_2...\alpha_{2s}} + \mu^{2s-1} h_{\alpha(2s)} = 0 , \quad (6.21)
\]

34
which implies
\[
\left( \Box^{2s-1} - (m^2)^{2s-1} \right) h_{\alpha(2s)} = 0 , \quad m := \left| \frac{\mu}{\lambda^{1/(2s-1)}} \right| ,
\]
and should be compared with (1.39). Since the polynomial equation \( z^{2s-1} - 1 = 0 \) has only one real root, \( z = 1 \), we conclude that (6.22) leads to the Klein-Gordon equation \(4.22\). As a result, the higher-derivative equation (6.21) reduces to the first-order one, eq. (4.23).

The above component analysis clearly demonstrates that the model (6.17) describes a single massive supermultiplet subject to the equations (2.6a) and (2.6b) with \( n = 2s \) on the mass shell. The superfield proof was provided in [35].

### 6.4 Massless second-order model

Finally we consider the massless half-integer superspin model describe by the action [35]
\[
S^{(2s+1)}_{SO} = \left( -\frac{1}{2} \right)^s \int d^{3|2}z \left\{ -\frac{i}{2} H^\alpha({2s+1}) \Box H_{\alpha(2s+1)} - \frac{i}{8} D_\beta H^{\beta\alpha(2s)} D^2 D^\gamma H_{\gamma\alpha(2s)} \\
+ \frac{i}{4} s \partial_\beta H^{\beta\gamma\alpha(2s-1)} \partial^\lambda H_{\lambda\alpha(2s-1)} - \frac{1}{2} (2s - 1) X^\alpha(2s-2) \partial^\beta H_{\beta\gamma\delta\alpha(2s-2)} \\
+ \frac{i}{2} (2s - 1) \left[ X^\alpha(2s-2) D^2 X_{\alpha(2s-2)} - \frac{s}{s} D_\beta X^{\beta\alpha(2s-3)} D^\gamma X_{\gamma\alpha(2s-3)} \right] \right\} .
\]
(6.23)

It is invariant under the following gauge transformations
\[
\delta H_{\alpha(2s+1)} = i D_{(\alpha_1} \lambda_{\alpha_2...\alpha_{2s+1})} , \quad \delta X_{\alpha(2s-2)} = \frac{s}{2s + 1} \partial^\beta \lambda_{\beta\gamma_1...\gamma_{2s-2}} .
\]
(6.24a)
(6.24b)

The gauge freedom allows us to choose a Wess-Zumino gauge of the form
\[
H_{\alpha(2s+1)} = 0 , \quad D^\beta H_{\beta\alpha(2s)} = 0 .
\]
(6.25)

To preserve these conditions, the residual gauge symmetry has to be constrained by
\[
D_{(\alpha_1} \lambda_{\alpha_2...\alpha_{2s+1})} = 0 , \quad D^2 \lambda_{\alpha(2s)} = -\frac{2is}{s + 1} \partial^\beta_{(\alpha_1} \lambda_{\alpha_2...\alpha_{2s})\beta)} .
\]
(6.26)

Under the gauge conditions imposed, the independent component fields of \( H_{\alpha(2s+1)} \) can be chosen as
\[
h_{\alpha(2s+2)} := -D_{(\alpha_1 H_{\alpha_2...\alpha_{2s+2})} , \quad h_{\alpha(2s+1)} := \frac{i}{4} D^2 H_{\alpha(2s+1)} .
\]
(6.27)
The remaining independent component parameters of $\lambda_{\alpha(2s)}$ can be chosen as

$$
\zeta_{\alpha(2s)} := \lambda_{\alpha(2s)}|, \quad \xi_{\alpha(2s-1)} := -i\frac{s}{2s + 1}D^\beta\lambda_{\beta\alpha(2s-1)}.
$$

(6.28)

The gauge transformation laws of $h_{\alpha(2s+2)}$ and $h_{\alpha(2s+1)}$ can be shown to be

$$
\delta \zeta h_{\alpha(2s+2)} = \partial(\alpha_1\alpha_2\xi_{3...\alpha_{2s+2}}),
$$

(6.29a)

$$
\delta \xi h_{\alpha(2s+1)} = \partial(\alpha_1\alpha_2\xi_{3...\alpha_{2s+1}}).
$$

(6.29b)

We now define the component fields of $X_{\alpha(2s-2)}$ as follows:

$$
y_{\alpha(2s-2)} := 2X_{\alpha(2s-2)}|,
$$

(6.30a)

$$
y_{\alpha(2s-1)} := -\frac{i}{2}D_{(\alpha_1}X_{\alpha_2...\alpha_{2s-1})|, \quad y_{\alpha(2s-3)} := -iD^\beta X_{\beta\alpha(2s-3)}|,
$$

(6.30b)

$$
F_{\alpha(2s-2)} := \frac{i}{4}X_{\alpha(2s-2)}|.
$$

(6.30c)

The gauge transformation laws of $y_{\alpha(2s-2)}$, $y_{\alpha(2s-1)}$ and $y_{\alpha(2s-3)}$ are as follows:

$$
\delta \zeta y_{\alpha(2s-2)} = \frac{2s}{2s + 1}\partial^{\gamma\gamma}\zeta_{\gamma\alpha(2s-2)},
$$

(6.31a)

$$
\delta \xi y_{\alpha(2s-1)} = \frac{1}{2s + 1}\partial^{\beta}(\alpha_1\xi_{\alpha_2...\alpha_{2s-1}})\beta,
$$

(6.31b)

$$
\delta \xi y_{\alpha(2s-3)} = \partial^{\gamma\gamma}\xi_{\gamma\alpha(2s-3)}.
$$

(6.31c)

In principle, we do not need to derive the gauge transformation of $F_{\alpha(2s-2)}$ since this field turns out to be auxiliary.

The bosonic transformation laws (6.29a) and (6.31a) correspond to the massless spin-$(s + 1)$ action $S^{(2s+2)}_F$ defined by eq. (4.28). The fermionic transformation laws (6.29b), (6.31b) and (6.31c) correspond to the massless spin-$(s + \frac{1}{2})$ action $S^{(2s+1)}_{FF}$ defined by eq. (4.7).

The component action follows from (6.23) by making use of the reduction rule (6.31). Direct calculations lead to the following bosonic Lagrangian:

$$
2(-2)^{s+1}L_{bos} = h^{\alpha(2s+2)}\Box h_{\alpha(2s+2)} - \frac{1}{2}(s + 1)\partial(\gamma(2)h^{(2)\alpha(2s)}\partial(2)h_{\alpha(2s)}\beta(2) - \frac{1}{2}(2s - 1)y^{\alpha(2s-2)}\partial(2)\gamma(2)h_{\alpha(2s-2)}\beta(2)\gamma(2) - \frac{(s + 1)(2s - 1)}{2s}y^{\alpha(2s-4)}\Box y_{\alpha(2s-4)} - 4s(2s - 1)[(s + 1)F^{\alpha(2s-2)}F_{\alpha(2s-2)} - \frac{s}{2s}F^{\alpha(2s-2)}\partial(\alpha_1y_{\alpha_2...\alpha_{2s-2}})\beta].
$$

(6.32)

Eliminating the auxiliary field $F_{\alpha(2s-2)}$ leads to

$$
2(-2)^{s+1}L_{bos} = h^{\alpha(2s+2)}\Box h_{\alpha(2s+2)} - \frac{1}{2}(s + 1)\partial(\gamma(2)h^{(2)\alpha(2s)}\partial(2)h_{\alpha(2s)}\beta(2)
$$

36
\[-\frac{1}{2}(2s - 1)\left[ y^{\alpha(2s-2)} \partial^{\beta(2)} \partial^{\gamma(2)} h_{\alpha(2s-2)\beta(2)\gamma(2)} + \frac{2}{s+1} y^{\alpha(2s-2)} \Box y_{\alpha(2s-2)} \right. \\
+ \left. \frac{(s - 1)(2s - 3)}{4(s + 1)} \partial_{\gamma(2)} y^{\gamma(2)\alpha(2s-4)} \partial^{\beta(2)} y^{\beta(2)\alpha(2s-4)} \right]. \tag{6.33} \]

This Lagrangian corresponds to the massless spin-(s + 1) action \( S_F^{(2s+2)} \) obtained from \([4,28]\) by replacement \( s \to s + 1 \). The fermionic sector of the component action proves to coincide with the massless spin-(s + \(\frac{1}{2}\)) action, \( S_{\text{FF}}^{(2s+1)}[h_{(2s+1)}, y_{(2s-3)}] \).

### 6.5 Massive half-integer superspin action

We now have all ingredients at our disposal to read off the component form of the massive half-integer superspin action that is obtained from \((5.39b)\) in the flat-superspace limit,

\[
S_{\text{massive}}^{(2s+1)} = \lambda S_{\text{SCS}}^{(2s+1)}[H_{(2s+1)}] + \mu^{2s-1} S_{\text{SO}}^{(2s+1)}[H_{(2s+1)}, X_{(2s-2)}] \\
\approx \lambda S_{\text{CS}}^{(2s+2)}[h_{(2s+2)}] + \mu^{2s-1} S_{\text{FF}}^{(2s+2)}[h_{(2s+2)}, y_{(2s-2)}] \\
+ \lambda S_{\text{CS}}^{(n+1)}[H_{(2s+1)}] + \mu^{2s-1} S_{\text{FF}}^{(2s+1)}[h_{(2s+1)}, y_{(2s-1)}, y_{(2s-3)}]. \tag{6.34} \]

Here the symbol ‘\(\approx\)’ indicates that the auxiliary field has been eliminated.

The explicit structure of the component action \((6.34)\) clearly demonstrates that the model

\[
S_{\text{massive}}^{(2s+1)} = \lambda S_{\text{SCS}}^{(2s+1)}[H_{(2s+1)}] + \mu^{2s-1} S_{\text{SO}}^{(2s+1)}[H_{(2s+1)}, X_{(2s-2)}] \tag{6.35} \]

describes a single massive supermultiplet subject to the equations \((2.6a)\) and \((2.6b)\) with \( n = 2s + 1 \) on the mass shell. The superfield proof was provided in \([35]\).

### 7 Concluding comments

All massive higher-spin theories in Minkowski space, which have been presented in this paper, were extracted from off-shell supersymmetric field theories. As shown in section 6, all the theories studied in section 4 are contained at the component level in the \( \mathcal{N} = 1 \) supersymmetric massive higher-spin theories proposed in \([35]\). The latter models were obtained from the \( \mathcal{N} = 2 \) supersymmetric massive higher-spin theories of \([34]\) by carrying out the \( \mathcal{N} = 2 \to \mathcal{N} = 1 \) superspace reduction. Furthermore, the off-shell structure of the massless 3D \( \mathcal{N} = 2 \) supersymmetric higher-spin actions of \([34]\), which constitute one of the two sectors of the \( \mathcal{N} = 2 \) massive actions, were designed following the pattern of the gauge off-shell formulations for massless 4D \( \mathcal{N} = 1 \) higher-spin supermultiplets developed in the early 1990s \([81, 82]\).
Our supersymmetric massive higher-spin theories, which are formulated in AdS$^{3|2}$ superspace and are described by the actions (5.39a) and (5.39b), contain two different models for a massive integer-spin field in AdS at the component level. One of them is the gauge-invariant model (4.5a). The second model is described by the action

$$\tilde{S}_{\text{massive}}^{(2s)} = \lambda S_{\text{CS}}^{(2s)} \mathcal{h}_{(2s)} + \frac{1}{2} \left( - \frac{1}{2} \right)^s \mu^{2s-1} \int d^3x ~ e \mathcal{h}^{a(2s)} \mathcal{h}_{a(2s)} ,$$

(7.1)

which does not possess gauge invariance and which is the AdS uplift of the model (6.19). The action (7.1) leads to the equation of motion

$$\lambda \mathcal{C}_{(2s)} + \mu^{2s-1} \mathcal{h}_{a(2s)} = 0 \implies \nabla^{\beta\gamma} \mathcal{h}_{\beta\gamma a(2s-2)} = 0 .$$

(7.2)

The action (7.1) can be turned into a gauge-invariant one by making use of the St"uckelberg trick. An interesting feature of the model (7.1) is that it is well-defined in an arbitrary conformally flat space.

The models (4.5b) and (7.1) are higher-spin analogues of the two well-known equivalent models for a massive vector field (see [83, 84] and references therein) with Lagrangians

$$\mathcal{L}_T = -\frac{1}{4} F_{ab} F^{ab} + \frac{m}{4} \varepsilon^{abc} V_a F_{bc} , \quad F_{ab} = \partial_a V_b - \partial_b V_a ,$$

(7.3a)

$$\mathcal{L}_{SD} = \frac{1}{2} f_a f_a - \frac{1}{2m} \varepsilon^{abc} f_a \partial_b f_c .$$

(7.3b)

New duality transformations were introduced in [33] for theories formulated in terms of the linearised higher-spin Cotton tensors $C_{\alpha(n)}$ and their $\mathcal{N} = 1$ supersymmetric counterparts $W_{\alpha(n)}$. These duality transformations can readily be generalised to arbitrary conformally flat backgrounds, with $C_{\alpha(n)}$ and $W_{\alpha(n)}$ replaced with $\mathcal{C}_{\alpha(n)}$ and $\mathcal{W}_{\alpha(n)}$, respectively.

In the present paper, we have been unable to obtain closed-form expressions for $\mathcal{C}_{\alpha(n)}$ and $\mathcal{W}_{\alpha(n)}$ in terms of the covariant derivatives of AdS (super)space for arbitrary $n$. These are interesting open problems.

The field strengths $\mathcal{C}_{\alpha(n)}$ and $\mathcal{W}_{\alpha(n)}$ are the higher-spin extensions of the linearised Cotton and super-Cotton tensors, respectively. The actions (3.32) and (5.21) are the higher-spin extensions of the linearised actions for conformal gravity and supergravity, respectively. An intriguing question is: Do nonlinear higher-spin extensions exist? Within the approach initiated in [85, 86], Linander and Nilsson [87] constructed the full nonlinear spin-3 Cotton equation coupled to spin-2. They made use of the frame field description and the Chern-Simons formulation for 3D (super)conformal field theory due to Fradkin and Linetsky [52]. The construction of the nonlinear spin-3 Cotton tensor [87] requires an elimination of certain auxiliary fields,
a procedure that becomes extremely difficult for \( s > 3 \). However, so far this is unexplored territory. There exist nonlinear formulations for the massless spin-3 theory \[88, 89\], and the generalisation from \( s = 3 \) to \( s > 3 \) is shown in \[89\] to be trivial within the formulation developed. These results indicate that it is possible to construct a nonlinear topologically massive higher-spin field theory. The fundamental results by Prokushkin and Vasiliev \[90, 91\] should be essential of course. Any attempt to construct a supersymmetric interacting higher-spin theory should inevitably be an extension of the conformal superspace approach \[23, 20\].

**Acknowledgements:**

SMK is grateful to Mirian Tsulaia for collaboration at the initial stages of the project, as well as for pointing out important references. The work of SMK is supported in part by the Australian Research Council, project No. DP160103633. The work of MP is supported by the Hackett Postgraduate Scholarship UWA, under the Australian Government Research Training Program.

**A Notation and conventions**

We follow the notation and conventions adopted in \[17\]. In particular, the Minkowski metric is \( \eta_{ab} = \text{diag}(-1, 1, 1) \). The spinor indices are raised and lowered using the \( \text{SL}(2, \mathbb{R}) \) invariant tensors

\[
\varepsilon_{\alpha \beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha \beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha \gamma} \varepsilon_{\gamma \beta} = \delta_{\beta}^{\alpha}
\]

by the standard rule:

\[
\psi^{\alpha} = \varepsilon_{\alpha \beta} \psi_{\beta}, \quad \psi_{\alpha} = \varepsilon^{\alpha \beta} \psi_{\beta}.
\]

We make use of real gamma-matrices, \( \gamma_{a} := (\gamma_{a})_{\alpha}^{\beta} \), which obey the algebra

\[
\gamma_{a} \gamma_{b} = \eta_{ab} \mathbb{1} + \varepsilon_{abc} \gamma^{c},
\]

where the Levi-Civita tensor is normalised as \( \varepsilon^{012} = -\varepsilon_{012} = 1 \). The completeness relation for the gamma-matrices reads

\[
(\gamma_{a})_{\alpha \beta} (\gamma_{a})^{\rho \sigma} = -\left( \delta_{\rho \beta}^{\sigma} \delta_{\alpha}^{\delta} + \delta_{\rho \alpha}^{\delta} \delta_{\beta}^{\sigma} \right).
\]

Here the symmetric matrices \( (\gamma_{a})_{\alpha \beta} \) and \( (\gamma_{a})_{\alpha \beta} \) are obtained from \( \gamma_{a} = (\gamma_{a})_{\alpha}^{\beta} \) by the rules \[A.2\]. Some useful relations involving \( \gamma \)-matrices are

\[
\varepsilon_{abc} (\gamma^{b})_{\alpha \beta} (\gamma^{c})_{\gamma \delta} = \varepsilon_{\gamma (\gamma_{a})_{\beta} \delta} + \varepsilon_{\delta (\gamma_{a})_{\beta} \gamma},
\]

(A.5a)
\[\text{tr}[\gamma_a \gamma_b \gamma_c \gamma_d] = 2\eta_{ab}\eta_{cd} - 2\eta_{ac}\eta_{db} + 2\eta_{ad}\eta_{bc} \]. (A.5b)

Given a three-vector \(x_a\), it can be equivalently described by a symmetric second-rank spinor \(x_{\alpha\beta}\) defined as
\[
x_{\alpha\beta} := (\gamma^a)_{\alpha\beta}x_a = x_{\beta\alpha}, \quad x_a = \frac{1}{2} (\gamma^a)^{\alpha\beta}x_{\alpha\beta}.
\] (A.6)

In the 3D case, an antisymmetric tensor \(F_{ab} = -F_{ba}\) is Hodge-dual to a three-vector \(F_a\), specifically
\[
F_a = \frac{1}{2} \varepsilon_{abc} F^bc, \quad F_{ab} = -\varepsilon_{abc} F^c.
\] (A.7)

Then, the symmetric spinor \(F_{\alpha\beta} = F_{\beta\alpha}\), which is associated with \(F_a\), can equivalently be defined in terms of \(F_{ab}\):
\[
F_{\alpha\beta} := (\gamma^a)_{\alpha\beta} F_a = \frac{1}{2} (\gamma^a)^{\alpha\beta} \varepsilon_{abc} F^bc.
\] (A.8)

These three algebraic objects, \(F_a\), \(F_{ab}\) and \(F_{\alpha\beta}\), are in one-to-one correspondence to each other, \(F_a \leftrightarrow F_{ab} \leftrightarrow F_{\alpha\beta}\). The corresponding inner products are related to each other as follows:
\[
-F^a G_a = \frac{1}{2} F^{ab} G_{ab} = \frac{1}{2} F_{\alpha\beta} G_{\alpha\beta}.
\] (A.9)

The Lorentz generators with two vector indices \((M_{ab} = -M_{ba})\), one vector index \((M_a)\) and two spinor indices \((M_{\alpha\beta} = M_{\beta\alpha})\) are related to each other by the rules: \(M_a = \frac{1}{2} \varepsilon_{abc} M^bc\) and \(M_{\alpha\beta} = (\gamma^a)^{\alpha\beta} M_a\). These generators act on a vector \(V_c\) and a spinor \(\Psi_\gamma\) as follows:
\[
M_{ab} V_c = 2\eta_{[a} V_{b]} , \quad M_{\alpha\beta} \Psi_\gamma = \varepsilon_{\gamma(\alpha} \Psi_{\beta)}.
\] (A.10)

## B First-order higher-spin model

In this appendix we review the first-order higher-spin model in Minkowski space used by Tyutin and Vasiliev [36] in their formulation for massive higher-spin fields. It is realised in terms of a reducible field \(h_{b,\alpha_1...\alpha_{n-2}} = h_{b,\{\alpha_1...\alpha_{n-2}\}}\) which is defined modulo gauge transformations of the form
\[
\delta h_{b,\alpha_1...\alpha_{n-2}} = \partial_b \xi_{\alpha_1...\alpha_{n-2}} , \quad \xi_{\alpha_1...\alpha_{n-2}} = \xi_{(\alpha_1...\alpha_{n-2})}.
\] (B.1)

The structure of this transformation implies that the following action
\[
S_{FF}^{(n)} = -\frac{i^n}{2^{[n/2]}} \int d^3x \varepsilon^{bcd} h_{b,\alpha_1...\alpha_{n-2}} \partial_c h_{d,\alpha_1...\alpha_{n-2}}
\] (B.2)
is gauge invariant.

The field $h_{\beta\gamma,\alpha_1...\alpha_{n-2}} := (\gamma^b)_{\beta\gamma} h_{b,\alpha_1...\alpha_{n-2}}$ contains three irreducible $\text{SL}(2,\mathbb{R})$ fields that we define as follows:

\begin{align}
  h_{\alpha_1...\alpha_n} &:= h_{(\alpha_1\alpha_2,\alpha_3...\alpha_n)} , \\
  y_{\alpha_1...\alpha_{n-2}} &:= \frac{1}{n} h^\beta_{(\alpha_1,\alpha_2...\alpha_{n-2})} , \\
  y_{\alpha_1...\alpha_{n-4}} &:= h^\beta\gamma,\beta\gamma\alpha_1...\alpha_{n-4} .
\end{align}

In accordance with (B.1), the gauge transformation laws of these fields are

\begin{align}
  \delta h_{\alpha_1...\alpha_n} &= \partial_{(\alpha_1\alpha_2}\xi_{\alpha_3...\alpha_n)} , \\
  \delta y_{\alpha_1...\alpha_{n-2}} &= \frac{1}{n} \partial^\beta_{(\alpha_1\xi_{\alpha_2...\alpha_{n-2}})} , \\
  \delta y_{\alpha_1...\alpha_{n-4}} &= \partial^\beta\gamma\xi_{\beta\gamma\alpha_1...\alpha_{n-4}} .
\end{align}

The action (B.2) turns into

\begin{equation}
  S^{(n)}_{\text{FF}} = \frac{i}{2^n[n/2]+1} \int d^3x \left\{ h^\beta\gamma,\alpha_1...\alpha_{n-1} \partial_{\beta\gamma} h_{\alpha_1...\alpha_{n-1}} + 2(n-2) y^\beta_{\alpha_1...\alpha_{n-2}} \partial^\beta\gamma h_{\beta\gamma\alpha_1...\alpha_{n-2}} \\
  + 4(n-3) y^\alpha_{\beta_1...\alpha_{n-3}} \partial_{\beta\gamma} h_{\gamma\alpha_1...\alpha_{n-3}} + 2 \frac{n(n-3)}{n-1} y^\alpha_{\alpha_1...\alpha_{n-4}} \partial^\beta\gamma h_{\beta\gamma\alpha_1...\alpha_{n-4}} \\
  - \frac{(n-3)(n-4)}{(n-1)(n-2)} y^\beta_{\alpha_1...\alpha_{n-3}} \partial_{\beta\gamma} h_{\gamma\alpha_1...\alpha_{n-3}} \right\} .
\end{equation}

This is the flat-space limit of the first-order action (4.2). When $n$ is odd, $n = 2s + 1$, the functional $S^{(2s+1)}_{\text{FF}}$ coincides with plain $4D \rightarrow 3D$ dimensional reduction of the Fang-Fronsdal action [31].

C Higher-spin Cotton tensor as a descendent of gauge-invariant field strengths

The Cotton tensor is defined in terms of the Ricci tensor according to (3.3). The latter determines the equations of motion corresponding to the Einstein-Hilbert action. In this appendix we show that analogous properties hold for the linearised higher-spin Cotton tensor defined by eq. (3.35).
C.1 The first-order case

We begin by demonstrating that the higher-spin Cotton tensor (3.35) is a descendant of gauge-invariant field strengths which determine the equations of motion in the first-order model (B.5). Associated with the dynamical variables $h_{\alpha(n)}, y_{\alpha(n-2)}$ and $y_{\alpha(n-4)}$ are the following gauge-invariant field strengths:

$$F_{\alpha(n)} := \partial_{(\alpha_1} h_{\alpha_2 \ldots \alpha_n)\beta} - (n - 2) \partial_{(\alpha_1} y_{\alpha_2 \ldots \alpha_n)} \beta) , \quad (C.1a)$$

$$G_{\alpha(n-2)} := \partial^{\beta(2)} h_{\alpha(n-2)\beta(2)} + 4 \partial_{(\alpha_1} y_{\alpha_2 \ldots \alpha_{n-2})\beta} - \frac{n(n - 3)}{(n - 1)(n - 2)} \partial_{(\alpha_1} y_{\alpha_2 \ldots \alpha_{n-2})\beta} , \quad (C.1b)$$

$$H_{\alpha(n-4)} := (n - 2) \partial^{\beta(2)} y_{\alpha(n-4)\beta(2)} - \frac{n - 4}{n} \partial_{(\alpha_1} y_{\alpha_2 \ldots \alpha_{n-4})\beta} . \quad (C.1c)$$

The equations of motion corresponding to (B.5) are the conditions that these field strengths vanish. Furthermore, the gauge symmetry implies that $F_{\alpha(n)}, G_{\alpha(n-2)}$ and $H_{\alpha(n-4)}$ are related to each other via the Noether identity

$$0 = \partial^{\beta(2)} F_{\alpha(n-2)\beta(2)} - \frac{n - 2}{n} \partial_{(\alpha_1} G_{\alpha_2 \ldots \alpha_{n-2})\beta} + \frac{n(n - 3)}{(n - 1)(n - 2)} \partial_{(\alpha_1} H_{\alpha_2 \ldots \alpha_{n-2})} . \quad (C.2)$$

We claim that the Cotton tensor $C_{\alpha(n)}(h)$ may be expressed as $C_{\alpha(n)} = (A_1 F)_{\alpha(n)} + (A_2 G)_{\alpha(n)} + (A_3 H)_{\alpha(n)}$, for some linear differential operators $A_i$ of order $n - 2$. A suitable ansatz for such an expression is

$$C_{\alpha(n)} = \sum_{j=0}^{[\frac{n}{2}]-1} a_j \Box^j \partial_{(\alpha_1} \beta_1 \ldots \partial_{\alpha_{n-2j-2}}^{\beta_1} \ldots \partial_{\alpha_{n-2j-2}}^{\beta_{n-2j-2}} F_{\alpha_{n-2j-1} \ldots \alpha_{n-2j-2})\beta_1 \ldots \beta_{n-2j-2}}$$

$$+ \sum_{k=0}^{[\frac{n}{2}]-2} b_k \Box^k \partial_{(\alpha_1} \beta_1 \ldots \partial_{\alpha_{n-2k-3}}^{\beta_1} \ldots \partial_{\alpha_{n-2k-3}}^{\beta_{n-2k-3}} G_{\alpha_{n-2k-2} \ldots \alpha_{n-2k-1})\beta_1 \ldots \beta_{n-2k-3}}$$

$$+ \sum_{l=0}^{[\frac{n}{2}]-2} c_l \Box^l \partial_{(\alpha_1} \beta_1 \ldots \partial_{\alpha_{n-2l-4}}^{\beta_1} \ldots \partial_{\alpha_{n-2l-4}}^{\beta_{n-2l-4}} H_{\alpha_{n-2l-3} \ldots \alpha_{n-2l-2})\beta_1 \ldots \beta_{n-2l-4}}$$

for some coefficients $a_j, b_k$ and $c_l$. It may be shown that the values of these coefficients are not unique and that there are $[\frac{n}{2}] - 1$ free parameters. For example, when $n = 5$ one may show that the general solution is

$$\begin{pmatrix}
  a_0 \\
  a_1 \\
  b_0 \\
  b_1 \\
  c_0
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{2} + \frac{18}{5} c_0 \\
  \frac{1}{2} - \frac{18}{5} c_0 \\
  \frac{9}{80} - \frac{36}{25} c_0 \\
  \frac{3}{80} - \frac{18}{25} c_0 \\
  c_0
\end{pmatrix}.$$
We may use this freedom to completely eliminate the $\lfloor \frac{n}{2} \rfloor - 1$ coefficients $c_l$ so that only the field strengths $F_{\alpha(n)}$ and $G_{\alpha(n-2)}$ appear in (C.3). This fixes the solution uniquely to

$$a_j = \frac{1}{2^{n-2} (2j+1)} \left( \frac{n-2}{2j} \right)$$

for $0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, \hspace{1cm} (C.4a)

$$b_k = \frac{1}{2^{n-1} n (2k+1)} \left( \frac{n-3}{2k} \right)$$

for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$, \hspace{1cm} (C.4b)

$$c_l = 0$$

for $0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$. \hspace{1cm} (C.4c)

The fact that there are $\lfloor \frac{n}{2} \rfloor - 1$ free parameters may be understood as a consequence of the Noether identity (C.2). To see this, observe that, in principle, we may use (C.2) to replace all occurrences of $H_{\alpha(n-4)}$ with $F_{\alpha(n)}$ and $G_{\alpha(n-2)}$ in the ansatz (C.3). There will then be only two sets of independent coefficients, say $\tilde{a}_j$ and $\tilde{b}_k$, whose unique values coincide with those of (C.4a) and (C.4b).

### C.2 The second-order case

We now consider the flat-space version of the second-order model (4.4). It is described by the real fields $h_{\alpha(n)}$ and $h_{\alpha(n-4)}$. Associated with these two fields are the following gauge-invariant field strengths:

$$F_{\alpha(n)} = \Box h_{\alpha(n)} + \frac{n}{4} \partial^{\beta(2)} \partial_{(\alpha_1 \alpha_2} h_{\alpha_3...\alpha_{n})\beta(2)} - \frac{n-3}{4} \partial_{(\alpha_1 \alpha_2} \partial_{\alpha_3 \alpha_4 y_{\alpha_5...\alpha_{n})} ;$$

(C.5a)

$$G_{\alpha(n-4)} = \partial^{\beta(2)} \partial^{\beta(2)} h_{\alpha(n-4)\beta(4)} + \frac{8}{n} \Box y_{\alpha(n-4)} - \frac{(n-4)(n-5)}{n(n-2)} \partial^{\beta(2)} \partial_{(\alpha_1 \alpha_2 y_{\alpha_3...\alpha_{n-4}})\beta(2)} .$$

(C.5b)

The equations of motion for the model are $F_{\alpha(n)} = 0$ and $G_{\alpha(n-4)} = 0$. The two field strengths are related by the Noether identity

$$\partial^{\beta(2)} F_{\alpha(n-2)\beta(2)} = \frac{(n-3)(n-2)}{4(n-1)} \partial_{(\alpha_1 \alpha_2} G_{\alpha_3...\alpha_{n-2})} .$$

(C.6)

We claim that the Cotton tensor $C_{\alpha(n)}(h)$ may be written as $C_{\alpha(n)} = (A_1 F)_{\alpha(n)} + (A_2 G)_{\alpha(n)}$ where the $A_i$ are linear differential operators of order $n - 3$. A suitable ansatz for such an expression is

$$C_{\alpha(n)} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 2} a_j \Box^j \partial_{(\alpha_1} \partial^{(\beta_1} \cdots \partial_{\alpha_{n-2j-3}}^{(a_{n-2j-3}} F_{\alpha_{n-2j-2}...\alpha_{n})_{\beta_1...\beta_{n-2j-3}}}$$

(C.7)
\[ \sum_{k=0}^{\left\lceil \frac{n}{2} \right\rceil - 3} b_k \partial_{(\alpha_1} \beta_1 \cdots \partial_{\alpha_{n-2k-5}} \beta_{n-2k-5}^\rangle \partial_{\alpha_{n-2k-5}} \partial_{\alpha_{n-2k-4}} \partial_{\alpha_{n-2k-3}} \partial_{\alpha_{n-2k-2}} G_{\alpha_{n-2k} \cdots \alpha_n}) \beta_1 \cdots \beta_{n-2k-5} \] 

for some coefficients \( a_j \) and \( b_k \). It may be shown that the choice of these coefficients is not unique, and that there are \( \left\lceil \frac{n}{2} \right\rceil - 2 \) free parameters. For example, when \( n = 6 \) one may show that the general solution is

\[
\begin{pmatrix}
a_0 \\
a_1 \\ b_0
\end{pmatrix} = \begin{pmatrix}
\frac{5}{8} - \frac{10}{3} b_0 \\
\frac{3}{8} + \frac{10}{3} b_0 \\
b_0
\end{pmatrix}.
\]

We can use this freedom to completely eliminate the \( \left\lceil \frac{n}{2} \right\rceil - 2 \) coefficients \( b_k \) so that only the top field strength, \( F_{\alpha(n)} \), appears in (C.7). This gives the unique solution

\[
a_j = (j+1) \left(\frac{n-3}{2j+3}\right) \frac{n(n-1)}{3 \cdot 2^{n-2}} \quad \text{for} \quad 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 , \quad (C.8a)
\]

\[
b_k = 0 \quad \text{for} \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 3 . \quad (C.8b)
\]

References

[1] W. Siegel, “Unextended superfields in extended supersymmetry,” Nucl. Phys. B 156, 135 (1979).
[2] R. Jackiw and S. Templeton, “How super-renormalizable interactions cure their infrared divergences,” Phys. Rev. D 23, 2291 (1981).
[3] J. F. Schonfeld, “A mass term for three-dimensional gauge fields,” Nucl. Phys. B 185, 157 (1981).
[4] S. Deser, R. Jackiw and S. Templeton, “Three-dimensional massive gauge theories,” Phys. Rev. Lett. 48, 975 (1982).
[5] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” Annals Phys. 140, 372 (1982) [Erratum-ibid. 185, 406 (1988)].
[6] P. van Nieuwenhuizen, “D = 3 conformal supergravity and Chern-Simons terms,” Phys. Rev. D 32, 872 (1985).
[7] J. H. Horne and E. Witten, “Conformal gravity in three dimensions as a gauge theory,” Phys. Rev. Lett. 62, 501 (1989).
[8] A. Achúcarro and P. K. Townsend, “A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories,” Phys. Lett. B 180, 89 (1986).
[9] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system,” Nucl. Phys. B 311, 46 (1988).
[10] S. Deser and J. H. Kay, “Topologically massive supergravity,” Phys. Lett. B 120, 97 (1983).
[11] S. Deser, “Cosmological topological supergravity,” in Quantum Theory Of Gravity, S. M. Christensen (Ed.), Adam Hilger, Bristol, 1984, pp. 374-381.
[12] N. S. Deger, A. Kaya, H. Samtleben and E. Sezgin, “Supersymmetric warped AdS in extended topologically massive supergravity,” Nucl. Phys. B 884, 106 (2014) [arXiv:1311.4583 [hep-th]].
[13] S. M. Kuzenko, U. Lindström, M. Roček, I. Sachs and G. Tartaglino-Mazzucchelli, “Three-dimensional N=2 supergravity theories: From superspace to components,” Phys. Rev. D 89, 085028 (2014) [arXiv:1312.4267 [hep-th]].
[14] S. M. Kuzenko and J. Novak, “Supergravity-matter actions in three dimensions and Chern-Simons terms,” JHEP 1405, 093 (2014) [arXiv:1401.2307 [hep-th]].
[15] S. M. Kuzenko, J. Novak and I. Sachs, “Minimal $\mathcal{N} = 4$ topologically massive supergravity,” JHEP 1703, 109 (2017) [arXiv:1610.09895 [hep-th]].
[16] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, Superspace, or One Thousand and One Lessons in Supersymmetry, Benjamin/Cummings (Reading, MA), 1983, hep-th/0108200.
[17] S. M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, “Off-shell supergravity-matter couplings in three dimensions,” JHEP 1103, 120 (2011) [arXiv:1101.4013 [hep-th]].
[18] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, “Three-dimensional N=2 (AdS) supergravity and associated supercurrents,” JHEP 1112, 052 (2011) [arXiv:1109.0496 [hep-th]].
[19] M. Roček and P. van Nieuwenhuizen, “N ≥ 2 supersymmetric Chern-Simons terms as d = 3 extended conformal supergravity,” Class. Quant. Grav. 3, 43 (1986).
[20] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Conformal supergravity in three dimensions: Off-shell actions,” JHEP 1310, 072 (2013) [arXiv:1305.3132 [hep-th]].
[21] M. Nishimura and Y. Tanii, “N=6 conformal supergravity in three dimensions,” JHEP 1310, 123 (2013) [arXiv:1306.1205 [hep-th]].
[22] S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “N=6 superconformal gravity in three dimensions from superspace,” JHEP 1401, 121 (2014) [arXiv:1308.5552 [hep-th]].
[23] D. Butter, S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Conformal supergravity in three dimensions: New off-shell formulation,” JHEP 1309, 072 (2013) [arXiv:1305.3132 [hep-th]].
[24] U. Lindström and M. Roček, “Superconformal gravity in three dimensions as a gauge theory,” Phys. Rev. Lett. 62, 2905 (1989).
[25] X. Chu and B. E. W. Nilsson, “Three-dimensional topologically gauged N=6 ABJM type theories,” JHEP 1006 (2010) 057 [arXiv:0906.1655 [hep-th]].
[26] U. Gran, J. Greitz, P. S. Howe and B. E. W. Nilsson, “Topologically gauged superconformal Chern-Simons matter theories,” JHEP 1212 (2012) 046 [arXiv:1204.2521 [hep-th]].
[27] B. E. W. Nilsson, “Critical solutions of topologically gauged N = 8 CFTs in three dimensions,” JHEP 1404 (2014) 107 [arXiv:1304.2270 [hep-th]].
[28] F. Lauf and I. Sachs, “On topologically massive gravity with extended supersymmetry,” Phys. Rev. D 94, 065028 (2016) [arXiv:1605.00103 [hep-th]].

[29] F. Lauf and I. Sachs, “Complete superspace classification of three-dimensional Chern-Simons-matter theories coupled to supergravity,” JHEP 1802, 154 (2018) [arXiv:1709.01461 [hep-th]].

[30] C. Fronsdal, “Massless fields with integer spin,” Phys. Rev. D 18, 3624 (1978).

[31] J. Fang and C. Fronsdal, “Massless fields with half-integral spin,” Phys. Rev. D 18, 3630 (1978).

[32] C. N. Pope and P. K. Townsend, “Conformal higher spin in (2+1) dimensions,” Phys. Lett. B 225, 245 (1989).

[33] S. M. Kuzenko, “Higher spin super-Cotton tensors and generalisations of the linear-chiral duality in three dimensions,” Phys. Lett. B 763, 308 (2016) [arXiv:1606.08624 [hep-th]].

[34] S. M. Kuzenko and D. X. Ogbum, “Off-shell higher spin N=2 supermultiplets in three dimensions,” Phys. Rev. D 94, no. 10, 106010 (2016) [arXiv:1603.04668 [hep-th]].

[35] S. M. Kuzenko and M. Tsulaia, “Off-shell massive N=1 supermultiplets in three dimensions,” Nucl. Phys. B 914, 160 (2017) [arXiv:1609.06010 [hep-th]].

[36] I. V. Tyutin and M. A. Vasiliev, “Lagrangian formulation of irreducible massive fields of arbitrary spin in (2+1) dimensions,” Teor. Mat. Fiz. 113N1, 45 (1997) [Theor. Math. Phys. 113, 1244 (1997)] [hep-th/9704132].

[37] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “On higher derivatives in 3D gravity and higher spin gauge theories,” Annals Phys. 325, 1118 (2010) [arXiv:0911.3061 [hep-th]].

[38] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A 173, 211 (1939).

[39] S. Deger, A. Kaya, E. Sezgin and P. Sundell, “Spectrum of D = 6, N=4b supergravity on AdS in three-dimensions x S**3,” Nucl. Phys. B 536, 110 (1998) [hep-th/9804166].

[40] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, “On critical massive (super)gravity in adS3,” J. Phys. Conf. Ser. 314, 012009 (2011) [arXiv:1011.1153 [hep-th]].

[41] N. Boulanger, D. Ponomarev, E. Sezgin and P. Sundell, “New unfolded higher spin systems in AdS3,” Class. Quant. Grav. 32, no. 15, 155002 (2015) [arXiv:1412.8209 [hep-th]].

[42] I. L. Buchbinder, T. V. Snegirev and Y. M. Zinoviev, “Gauge invariant Lagrangian formulation of massive higher spin fields in (A)dS3 space,” Phys. Lett. B 716, 243 (2012) [arXiv:1207.1215 [hep-th]].

[43] I. L. Buchbinder, T. V. Snegirev and Y. M. Zinoviev, “Frame-like gauge invariant Lagrangian formulation of massive fermionic higher spin fields in AdS3 space,” Phys. Lett. B 738, 258 (2014) [arXiv:1407.3918 [hep-th]].

[44] I. L. Buchbinder, T. V. Snegirev and Y. M. Zinoviev, “Lagrangian formulation of the massive higher spin supermultiplets in three dimensional space-time,” JHEP 1510, 148 (2015) [arXiv:1508.02829 [hep-th]].

[45] I. L. Buchbinder, T. V. Snegirev and Y. M. Zinoviev, “Lagrangian description of massive higher spin supermultiplets in AdS3 space,” JHEP 1708, 021 (2017) [arXiv:1705.06103 [hep-th]].
[46] Y. M. Zinoviev, “On massive high spin particles in AdS,” hep-th/0108192.

[47] S. M. Kuzenko, J. Novak and G. Tartaglino-Mazzucchelli, “Higher derivative couplings and massive supergravity in three dimensions,” JHEP 1509, 081 (2015) arXiv:1506.09063 [hep-th].

[48] I. L. Buchbinder, S. M. Kuzenko and A. G. Sibiryakov, “Quantization of higher spin superfields in the anti-De Sitter superspace,” Phys. Lett. B 352, 29 (1995) hep-th/9502148.

[49] S. M. Kuzenko and A. G. Sibiryakov, “Free massless higher-superspin superfields on the anti-de Sitter superspace” Phys. Atom. Nucl. 57, 1257 (1994) [Yad. Fiz. 57, 1326 (1994)] arXiv:1112.4612 [hep-th].

[50] L. P. Eisenhart, Riemannian Geometry, Princeton University Press, Princeton, 1926.

[51] E. S. Fradkin and A. A. Tseytlin, “Conformal supergravity,” Phys. Rept. 119, 233 (1985).

[52] E. S. Fradkin and V. Y. Linetsky, “A superconformal theory of massless higher spin fields in $D = (2 + 1)$,” Annals Phys. 198, 293 (1990).

[53] S. M. Kuzenko, R. Manvelyan and S. Theisen, “Off-shell superconformal higher spin multiplets in four dimensions,” JHEP 1707, 034 (2017) arXiv:1701.00682 [hep-th].

[54] T. Damour and S. Deser, “‘Geometry’ of spin 3 gauge theories,” Ann. Inst. H. Poincare Phys. Theor. 47, 277 (1987).

[55] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin and P. K. Townsend, “Massive 3D supergravity,” Class. Quant. Grav. 27, 025010 (2010) arXiv:0907.4658 [hep-th].

[56] G. W. Gibbons, C. N. Pope and E. Sezgin, “The general supersymmetric solution of topologically massive supergravity,” Class. Quant. Grav. 25, 205005 (2008) arXiv:0807.2613 [hep-th].

[57] M. Henneaux, S. Hörtner and A. Leonard, “Higher spin conformal geometry in three dimensions and prepotentials for higher spin gauge fields,” JHEP 1601, 073 (2016) arXiv:1511.07389 [hep-th].

[58] T. Basile, R. Bonezzi and N. Boulanger, “The Schouten tensor as a connection in the unfolding of 3D conformal higher-spin fields,” JHEP 1704, 054 (2017) arXiv:1701.08645 [hep-th].

[59] C. Fronsdal, “Singletons and massless, integral-spin fields on de Sitter space,” Phys. Rev. D 20, 848 (1979).

[60] J. Fang and C. Fronsdal, “Massless, half-integer-spin fields in de Sitter space,” Phys. Rev. D 22, 1361 (1980).

[61] I. L. Buchbinder and S. M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity, Or a Walk Through Superspace, IOP, Bristol, 1995 (Revised Edition 1998).

[62] D. Francia and A. Sagnotti, “On the geometry of higher spin gauge fields,” Class. Quant. Grav. 20, S473 (2003) hep-th/0212185.

[63] A. Sagnotti and M. Tsulaia, “On higher spins and the tensionless limit of string theory,” Nucl. Phys. B 682, 83 (2004) hep-th/0311257.

[64] A. Fotopoulos, K. L. Panigrahi and M. Tsulaia, “Lagrangian formulation of higher spin theories on AdS space,” Phys. Rev. D 74, 085029 (2006) hep-th/0607248.

[65] D. P. Sorokin and M. A. Vasiliev, “Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation,” Nucl. Phys. B 809, 110 (2009) arXiv:0807.0206 [hep-th].
[66] A. Agugliaro, F. Azzurli and D. Sorokin, “Fermionic higher-spin triplets in AdS,” Nucl. Phys. B 907, 633 (2016) [arXiv:1603.02251 [hep-th]].

[67] D. Sorokin and M. Tsulaia, “Supersymmetric reducible higher-spin multiplets in various dimensions,” Nucl. Phys. B 929, 216 (2018) [arXiv:1801.04615 [hep-th]].

[68] A. Y. Segal, “A Generating formulation for free higher spin massless fields,” hep-th/0103028.

[69] I. L. Buchbinder, A. Pashnev and M. Tsulaia, “Lagrangian formulation of the massless higher integer spin fields in the AdS background,” Phys. Lett. B 523, 338 (2001) [hep-th/0109067].

[70] R. R. Metsaev, “Lowest eigenvalues of the energy operator for totally (anti)symmetric massless fields of the n-dimensional anti-de Sitter group,” Class. Quant. Grav. 11 (1994) L141.

[71] R. R. Metsaev, “Free totally (anti)symmetric massless fermionic fields in d-dimensional anti-de Sitter space,” Class. Quant. Grav. 14, L115 (1997) [hep-th/9707066].

[72] R. R. Metsaev, “Fermionic fields in the d-dimensional anti-de Sitter space-time,” Phys. Lett. B 419, 49 (1998) [hep-th/9802097].

[73] R. R. Metsaev, “Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space,” Lect. Notes Phys. 524, 331 (1999) [hep-th/9810231].

[74] A. Campoleoni and D. Francia, “Maxwell-like Lagrangians for higher spins,” JHEP 1303, 168 (2013) [arXiv:1206.5877 [hep-th]].

[75] D. Francia, G. L. Monaco and K. Mkrtchyan, “Cubic interactions of Maxwell-like higher spins,” JHEP 1704, 068 (2017) [arXiv:1611.00292 [hep-th]].

[76] B. M. Zupnik and D. G. Pak, “Superfield formulation of the simplest three-dimensional gauge theories and conformal supergravities,” Theor. Math. Phys. 77 (1988) 1070 [Teor. Mat. Fiz. 77 (1988) 97].

[77] B. M. Zupnik and D. G. Pak, “Differential and integral forms in supergauge theories and supergravity,” Class. Quant. Grav. 6, 723 (1989).

[78] U. Lindstr¨om and M. Roˇcek, “A super-Weyl-invariant spinning membrane,” Phys. Lett. B 218, 207 (1989).

[79] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, “Conformal supergravities as Chern-Simons theories revisited,” JHEP 1303, 113 (2013) [arXiv:1212.6852 [hep-th]].

[80] S. M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, “Three-dimensional (p,q) AdS superspaces and matter couplings,” JHEP 1208, 024 (2012) [arXiv:1205.4622 [hep-th]].

[81] S. M. Kuzenko, V. V. Postnikov and A. G. Sibiryakov, “Massless gauge superfields of higher half-integer superspins,” JETP Lett. 57, 534 (1993) [Pisma Zh. Eksp. Teor. Fiz. 57, 521 (1993)].

[82] S. M. Kuzenko and A. G. Sibiryakov, “Massless gauge superfields of higher integer superspins,” JETP Lett. 57, 539 (1993) [Pisma Zh. Eksp. Teor. Fiz. 57, 526 (1993)].

[83] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen, “Self-duality in odd dimensions,” Phys. Lett. 136B, 38 (1984) Addendum: [Phys. Lett. 137B, 443 (1984)].

[84] S. Deser and R. Jackiw, “Self-duality of topologically massive gauge theories,” Phys. Lett. 139B, 371 (1984).
[85] B. E. W. Nilsson, “Towards an exact frame formulation of conformal higher spins in three dimensions,” JHEP 1509, 078 (2015) [arXiv:1312.5883 [hep-th]].

[86] B. E. W. Nilsson, “On the conformal higher spin unfolded equation for a three-dimensional self-interacting scalar field,” JHEP 1608, 142 (2016) [arXiv:1506.03328 [hep-th]].

[87] H. Linander and B. E. W. Nilsson, “The non-linear coupled spin 2 - spin 3 Cotton equation in three dimensions,” JHEP 1607, 024 (2016) [arXiv:1602.01682 [hep-th]].

[88] M. Henneaux and S. J. Rey, “Nonlinear $W_{\infty}$ as asymptotic symmetry of three-dimensional higher spin AdS gravity,” JHEP 1012, 007 (2010) [arXiv:1008.4579 [hep-th]].

[89] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” JHEP 1011, 007 (2010) [arXiv:1008.4744 [hep-th]].

[90] S. F. Prokushkin and M. A. Vasiliev, “Higher spin gauge interactions for massive matter fields in 3-D AdS space-time,” Nucl. Phys. B 545, 385 (1999) [hep-th/9806236].

[91] S. F. Prokushkin and M. A. Vasiliev, “Cohomology of arbitrary spin currents in AdS(3),” Theor. Math. Phys. 123, 415 (2000) [Teor. Mat. Fiz. 123, 3 (2000)] [hep-th/9907020].