Back-of-the-envelope swaptions in a very parsimonious multicurve interest rate model

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Abstract

We propose an elementary model to price European physical delivery swaptions in multicurve setting with a simple exact closed formula. The proposed model is very parsimonious: it is a three-parameter multicurve extension of the two-parameter [Hull and White (1990)] model. The model allows also to obtain simple formulas for all other plain vanilla Interest Rate derivatives. Calibration issues are discussed in detail.

Keywords: Multicurve interest rates, parsimonious modeling, calibration cascade.

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1 Introduction

The financial crisis of 2007 has had a significant impact also on Interest Rate (hereinafter IR) modeling perspective. On the one hand, multicurve dynamics have been observed in main inter-bank markets (e.g. EUR and USD), on the other volumes on exotic derivatives have considerably decreased and liquidity has significantly declined even on plain vanilla instruments. While on the first issue there exist nowadays excellent textbooks (see, e.g. [Henrard 2014, Grbac and Runggaldier 2015]), the main consequence of the second issue, i.e. the need of very parsimonious models, has been largely forgotten in current financial literature where the additional complexity of today financial markets is often faced with parameter-rich models. In this paper the focus is on the two relevant issues of parsimony and calibration.

First, the parsimony feature is crucial: in today (less liquid) markets one often needs to handle models with very few parameters both from a calibration and from a risk management perspective. In this paper we focus on a three-parameter multicurve extension of the well known two-parameters Hull and White (1990) model. This choice is very parsimonious: one of the most parsimonious Multicurve HJM model in the existing literature is the one introduced by Moreni and Pallavicini (2014) that, in the simplest WG2++ case, requires ten free parameters. Another one has been recently proposed by Grbac et al. (2016), that in the simplest model parametrization involves at least seven parameters.

Second, the model should allow for a calibration cascade, the methodology followed by practitioners, that consists in calibrating first IR curves via bootstrap techniques and then volatility parameters. This cascade is crucial and the reason is related again to liquidity. Instruments used in bootstrap, as FRAs, Short-Term-Interest-Rate (STIR) futures and swaps, are several order of magnitude more liquid than the corresponding options on these instruments.

The proposed model, besides the calibration of the initial discount and pseudo-discount curves, allows to price with exact and simple closed formulas all plain vanilla IR options: caps/floors, STIR options and European swaptions. While caps/floors and STIR options can be priced with straightforward modifications of solutions already present in the literature (see, e.g. Henrard 2010, Baviera and Cassaro 2015), in this paper we focus on pricing European physical delivery swaption derivatives (hereinafter swaptions). We also show in a detailed example the calibration cascade, where the volatility parameters are calibrated via swaptions.

The remainder of the paper is organized as follows. In Section 2, we recall the characteristics of a swaption derivative contract in a general multicurve setting. In Section 3 we introduce the Multicurve HJM framework and the parsimonious model within this framework; we also prove model swaption closed formula. In section 4 we show in detail model calibration. Section 5 concludes.
2 Interest Rate Swaptions in a multicurve setting

Multicurve setting for interest rates can be found in the two textbooks of Henrard (2014) and Grbac and Runggaldier (2015). In this section we briefly recall interest rate notation and some key relations, with a focus on swaption pricing in a multicurve setting.

Let \((\Omega, \mathcal{F}, \mathbb{P})\), with \(\{\mathcal{F}_t : t_0 \leq t \leq T^*\}\), be a complete filtered probability space satisfying the usual hypothesis, where \(t_0\) is the value date and \(T^*\) a finite time horizon for all market activities. Let us define \(B(t, T)\) the discount curve with \(t_0 \leq t < T < T^*\) and \(D(t, T)\), the stochastic discount, s.t.

\[
B(t, T) = \mathbb{E}[D(t, T)|\mathcal{F}_t].
\]

(1)

The quantity \(B(t, T)\) is often called also risk-free zero-coupon bond. For example, market standard in the Euro interbank market is to consider as discount curve the EONIA curve (also called OIS curve). As in standard single curve models, forward discount \(B(t; T, T + \Delta)\) is equal to the ratio \(B(t, T + \Delta)/B(t, T)\). A consequence of (1) is that \(B(t; T, T + \Delta)\) is a martingale in the \(T\)-forward measure\(^1\).

As in Henrard (2014), also a pseudo-discount curve is considered. The following relation holds for Libor rates \(L(T, T + \Delta)\) and the corresponding forward rates \(L(t; T, T + \Delta)\) in \(t\)

\[
B(t, T + \Delta) L(t; T, T + \Delta) := \mathbb{E}[D(t, T + \Delta) L(T, T + \Delta)|\mathcal{F}_t],
\]

(2)

where the lag \(\Delta\) is the one that characterizes the pseudo-discount curve; e.g. 6-months in the Euribor 6m case.

The (forward) pseudo-discounts are defined as

\[
\hat{B}(t; T, T + \Delta) := \frac{1}{1 + \delta(T, T + \Delta) L(t; T, T + \Delta)}
\]

(3)

with \(\delta(T, T + \Delta)\) the year-fraction between the two calculation dates for a Libor rate and the spread is defined as

\[
\beta(t; T, T + \Delta) := \frac{B(t; T, T + \Delta)}{B(t; T, T + \Delta)}.
\]

From equation (2) one gets

\[
B(t, T) \beta(t; T, T + \Delta) = \mathbb{E}[D(t, T) \beta(T, T + \Delta)|\mathcal{F}_t]
\]

(4)

i.e. \(\beta(t; T, T + \Delta)\) is a martingale in the \(T\)-forward measure. This is the unique property that process \(\beta(t; T, T + \Delta)\) has to satisfy.

Hereinafter, as market standard, all discounts and OIS derivatives refer to the discount curve, while forward forward Libor rates are always related to the corresponding pseudo-discount curve via (3).

2.1 Swaption

A swaption is a contract on the right to enter, at option’s expiry date \(t_\alpha\), in a payer/receiver swap with a strike rate \(K\) established when the contract is written.

The underlying swap at expiry date \(t_\alpha\) is composed by a floating and a fixed leg; typically payments do not occur with the same frequency in the two legs (and they can have also different daycount) and this fact complicates the notation. Flows end at swap maturity date \(t_\omega\). We indicate floating

\(^1\)The \(T\)-forward measure is defined as the probability measure s.t. \(B(t, T) \mathbb{E}^{(T)}[\bullet|\mathcal{F}_t] = \mathbb{E}[D(t, T) \bullet|\mathcal{F}_t]\) (see, e.g. Musiela and Rutkowski 2006).
A receiver swaption is the expected value at value date of the discounted payoff where we have also rewritten the expectation in the $t$ and the remaining sum of $B$
The two following two properties hold

\[
\begin{aligned}
B_{\alpha j}(t) & := B(t; t_\alpha, t_j) \\
B_{\alpha' \iota}(t) & := B(t; t'_{\alpha'}, t'_\iota) \\
\beta(t) & := \beta(t; t'_{\alpha'}, t'_{\iota+1}) \\
\delta(t) & := \delta(t') \delta(t'_{\iota+1}) \\
\delta_j & := \delta_j K \quad \text{for } j = \alpha + 1, \ldots, \omega - 1 \quad \text{and} \quad 1 + \delta_{\omega} K \quad \text{for } j = \omega
\end{aligned}
\]

Let us introduce the following shorthands

\[
\begin{aligned}
\text{i)} & \quad N_{\alpha\omega}(t) := \sum_{i=\alpha'}^{\omega'-1} D(t, t'_{i+1}) \delta_i \left[L(t_i, t'_{i+1}) \mid \mathcal{F}_{t_i}\right] = 1 - B(t, t_\omega) + \sum_{i=\alpha'}^{\omega'-1} B(t, t'_i) [\beta_i(t) - 1] , \\
\text{ii) } & \quad R_{\alpha\omega}(t) \quad \text{receiver swaption payoff)}
\end{aligned}
\]

A swap rate forward start in $t_\alpha$ and valued in $t \in [t_0, t_\alpha]$, $S_{\alpha\omega}(t)$, is obtained equating in $t$ the Net-Present-Value of the floating leg and of the fixed leg

\[
S_{\alpha\omega}(t) = \frac{N_{\alpha\omega}(t)}{BPV_{\alpha\omega}(t)}
\]

and the numerator equal to the expected value in $t$ of swap’s floating leg flows

\[
N_{\alpha\omega}(t) := \mathbb{E} \left[ \sum_{i=\alpha'}^{\omega'-1} D(t, t'_{i+1}) \delta_i \left[L(t_i, t'_{i+1}) \mid \mathcal{F}_{t_i}\right] \right] = 1 - B(t, t_\omega) + \sum_{i=\alpha'}^{\omega'-1} B(t, t'_i) [\beta_i(t) - 1] ,
\]

where the last equality is obtained using relations (1) and (4). Let us observe that the sum of floating leg flows is composed by two parts: the term $[1 - B(t, t_\omega)]$, equal to the single curve case, and the remaining sum of $B(t, t'_i) [\beta_i(t) - 1]$ that corresponds to the spread correction present in the multicurve setting.

Receiver swaption payoff at expiry date is

\[
R_{\alpha\omega}(t_\alpha) := BPV_{\alpha\omega}(t_\alpha) [K - S_{\alpha\omega}(t_\alpha)]^+ = [K BPV_{\alpha\omega}(t_\alpha) - N_{\alpha\omega}(t_\alpha)]^+ .
\]

A receiver swaption is the expected value at value date of the discounted payoff

\[
\begin{aligned}
R_{\alpha\omega}(t_0) := & \quad \mathbb{E} \{ D(t_0, t_\alpha) R_{\alpha\omega}(t_\alpha) \mid \mathcal{F}_{t_0} \} = B(t_0, t_\alpha) \mathbb{E}^{(a)} \{ R_{\alpha\omega}(t_\alpha) \mid \mathcal{F}_{t_0} \}
\end{aligned}
\]

where we have also rewritten the expectation in the $t_\alpha$-forward measure.

**Lemma 1** The two following two properties hold

i) $N_{\alpha\omega}(t)$ and $BPV_{\alpha\omega}(t)$ are martingale processes in the $t_\alpha$-forward measure for $t \in [t_0, t_\alpha]$;

ii) Receiver swaption payoff (7) reads

\[
\begin{aligned}
R_{\alpha\omega}(t_\alpha) & = \left[ B(t_\alpha, t_\omega) + K BPV_{\alpha\omega}(t_\alpha) + \sum_{i=\alpha'}^{\omega'-1} B(t_\alpha, t'_i) [1 - \beta_i(t_\alpha)] - 1 \right]^+ = \\
& \left[ \sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_\alpha) + \sum_{i=\alpha'+1}^{\omega'-1} B_{\alpha' i}(t_\alpha) - \sum_{i=\alpha'}^{\omega'-1} \beta_i(t_\alpha) B_{\alpha' i}(t_\alpha) \right]^+
\end{aligned}
\]
Proof. Straightforward given the definitions of discount and pseudo-discount curves ♠

This lemma has some relevant consequences. On the one hand, property i) allows generalizing the Swap Market Model approach in [Jamshidian 1997] to swaptions in the multicurve case, hence it allows obtaining market swaption formulas choosing properly the volatility structure. One can get the Black, Bachelier or Shifted-Black market formula (see, e.g. [Brigo and Mercurio 2007]) where flows are discounted with the discount curve and forward Libor rates are related to pseudo-discounts via (3), as considered in market formulas. Moreover, property i) implies also that put-call parity holds also for swaptions in a multicurve setting.

On the other hand, property ii) clarifies that a complete specification of the model for swaption pricing requires only the dynamics for the forward discount and spread curves as specified in the next section.

3 A Multicurve Gaussian HJM model with closed form swaption solution

A Multicurve HJM model (hereinafter MHJM) is specified providing initial conditions for the discount curve \( B(t_0, t) \) and the spread curve \( \beta(t_0; T, T + \Delta) \), and indicating their dynamics. Discount and spread curves’ dynamics in the MHJM framework we consider in this paper are

\[
\begin{align*}
\text{d}B(t; t_\alpha, t_i) &= -B(t; t_\alpha, t_i) \left[ \sigma(t, t_i) - \sigma(t, t_\alpha) \right] \cdot \left[ dW_t + \rho \sigma(t, t_\alpha) \, dt \right] \quad t \in [t_0, t_\alpha] \\
\text{d}\beta(t; t_i, t_{i+1}) &= \beta(t; t_i, t_{i+1}) \left[ \eta(t, t_{i+1}) - \eta(t, t_i) \right] \cdot \left[ dW_t + \rho \sigma(t, t_i) \, dt \right] \quad t \in [t_0, t_i]
\end{align*}
\]

(9)

where \( \sigma(t, T) \) and \( \eta(t, T) \) are d-dimensional vectors of adapted processes (in particular in the Gaussian case they are deterministic functions of time) with \( \sigma(t, t) = \eta(t, t) = 0, x \cdot y \) is the canonical scalar product between \( x, y \in \mathbb{R}^d \), and \( W \) is a d-dimensional Brownian motion with instantaneous covariance \( \rho = (\rho_{i,j=1,...,d}) \)

\[
dW_{i,t} dW_{j,t} = \rho_{i,j} \, dt .
\]

Model (9) is the most natural extension of the single-curve [Heath et al. 1992] model. The first equation in (9) corresponds to the usual HJM model for the discount curve (see, e.g. [Musiela and Rutkowski 2006]). The second equation in (9) is a very general continuous process satisfying condition (4) for the spread.

We do not impose any other additional condition for curves’ dynamics as the independence hypothesis in [Henrard 2014] or the orthogonality condition in [Baviera and Cassaro 2015].

Change of measures are standard in this framework, because they are a straightforward generalization of single curve modeling approaches (see, e.g. [Musiela and Rutkowski 2006]). The process

\[
dW_{i}^{(i)} := dW_i + \rho \sigma(t, t_i) \, dt
\]

is a d-dimensional Brownian motion in the \( t_i \)-forward measure. It is immediate to prove that, given dynamics (9), \( B(t; t_\alpha, t) \) is martingale in the \( t_\alpha \)-forward measure and \( \beta(t; t_i, t_{i+1}) \) is martingale in the \( t_i \)-forward measure.

Remark 1. Given equations (9), the dynamics for the pseudo-discounts (3) in the \( t_i \)-forward measure is

\[
d\hat{B}(t; t_i, t_{i+1}) = -\hat{B}(t; t_i, t_{i+1}) \left[ \sigma_i(t) + \eta_i(t) \right] \cdot \left[ dW_{i}^{(i)} - \rho \eta_i(t) \, dt \right] \quad t \in [t_0, t_i]
\]
where $\sigma_i(t) := \sigma(t, t_{i+1}) - \sigma(t, t_i)$ and $\eta_i(t) := \eta(t, t_{i+1}) - \eta(t, t_i)$. The pseudo-discount has a volatility which is the sum of discount volatility $\sigma_i(t)$ and of spread volatility $\eta_i(t)$.

In this paper we consider an elementary 1-dimensional Gaussian model within MHJM framework \cite{9}. Volatilities for the discount curve $\sigma(t, T)$ and for the spread curve $\eta(t, T)$ are modeled as

$$
\begin{align*}
\sigma(t, T) &= (1 - \gamma) \ v(t, T) \\
\eta(t, T) &= \gamma \ v(t, T)
\end{align*}
$$

with $v(t, T) := \begin{cases} 
\sigma 1 - e^{-a(T-t)} & a \in \mathbb{R}^+ \setminus \{0\} \\
\sigma (T - t) & a = 0
\end{cases}$ \ (10)

with $a, \sigma \in \mathbb{R}^+$ and $\gamma \in [0, 1]$, the three model parameters.

This model is the most parsimonious (non-trivial) extension of Hull and White (1990) to multicurve dynamics, for this reason we call it Multicurve Hull White (hereinafter MHW) model. For all parameters choices volatility $v(t, T)$ is strictly positive.

The selection of this model originates from two facts related to the IR derivatives available for calibration. On the one hand, in the calibration cascade, “linear” IR derivatives (i.e depos, FRAs, STIR futures and swaps) are used for discount and pseudo-discount initial curve bootstrap, while the other parameters are calibrated on IR options. In the market, liquid IR option are STIR options, caps/floors and swaptions; unfortunately options on OIS are not liquid in the market place (see, e.g. Moreni and Pallavicini 2014 and references therein).

On the other hand, in liquid IR options, the key driver is the pseudo-discount curve $\hat{B}(t, T)$ via a Libor rate or a swap rate, where the latter can be seen as combinations of Libor rates (see, e.g. eq.(1.28) in Grbac and Runggaldier 2015). Hence, when IR curves move, the main driver is pseudo-discount curve, directly related to option underlyings; the discount curve appears only in weights or discount factors, and swaption sensitivities w.r.t. the discount curve are less than the corresponding sensitivities w.r.t. the pseudo-discount curve.

These facts lead to the conclusion that is much more difficult to calibrate volatility parameters specific to the discount curve. Thus Remark 1 plays a crucial role when selecting the most parsimonious model within framework \cite{9}: $\hat{B}(t, T)$ dynamics has volatility equal to $v(t, T)$ in MHW model \cite{10}. A parsimonious choice should associated a fraction $1 - \gamma$ of volatility $v(t, T)$ to the discount curve and the remaining fraction $\gamma$ to the spread dynamics; in fact, as previously discussed, options on OIS are not liquid enough and then a separate calibration of $\sigma(t, T)$ and $\eta(t, T)$ in a generic MHJM is not feasible in practice.

Moreover, MHW model \cite{10} allows pricing IR options in an elementary way. STIR options and caps/floors Black-like formulas can be obtained via a straightforward generalization of the solutions in \cite{Henrard2010} and \cite{BavieraCassaro2015}. In this section we show that it is possible to price also swaptions via a simple closed formula. To the best of our knowledge, MHW model \cite{10} is the first Multicurve HJM where all plain vanilla derivatives can be written with simple exact closed formulas that are extensions of Black (1976) formulas.

The remaining part of this section is divided as follows. We first show in Lemma 2 how to write, within MHW model \cite{10}, each element in receiver swaption payoff \cite{8} as a simple function of one single Gaussian r.v. $\xi$. Then, (technical) Lemma 3 shows that swaption payoff can be rewritten as a function of $\xi$ and this function presents interesting properties. Finally in Proposition 1 we prove the key result of this section: the exact closed formula for swaptions according to model \cite{10}. 


It is useful to introduce the following shorthands

\[
\begin{align*}
    v_{\alpha'_t} &:= v(t_\alpha, t'_t) & t = \alpha', \ldots, \omega' \\
    s_{\alpha'_t} &:= (1 - \nu) v_{\alpha'_t} & t = \alpha', \ldots, \omega \\
    \nu_{\alpha'_t} &:= s_{\alpha'_t} - (\eta(t_\alpha, t'_{t+1}) - \eta(t_\alpha, t'_t)) & t = \alpha', \ldots, \omega' - 1.
\end{align*}
\]

**Remark 2.** Volatilities \( \{v_{\alpha'_t}\}_{i=\alpha'+1}^{\omega} \) are always positive and are strictly increasing with \( t \). The quantities \( \{\nu_{\alpha'_t}\}_{i=\alpha'+1}^{\omega'} \) change sign depending on the value of \( \gamma \). In fact

\[
\nu_{\alpha'_t} = v_{\alpha'_t} - \gamma v_{\alpha'_t+1} = v_{\alpha'_t+1} (\tilde{\gamma}_t - \gamma)
\]

with \( \tilde{\gamma}_t := \nu_{\alpha'_t}/\nu_{\alpha'_t+1} \in (0, 1) \). Then, when \( \gamma = 0 \) all \( \{\nu_{\alpha'_t}\}_{i=\alpha'+1}^{\omega'-1} \) are positive and \( \nu_{\alpha'_t} \) is negative, while for larger values of \( \gamma \) some \( \nu_{\alpha'_t} \) become negative. For \( \gamma \) equal or close to 1 all \( \{\nu_{\alpha'_t}\}_{i=\alpha'}^{\omega'-1} \) are negative. Due to these possible negative values, \( \{\nu_{\alpha'_t}\}_t \) are not volatilities; we call them *extended* volatilities.

**Lemma 2** Discount and spread curves in \( t_\alpha \) can be written, according to the MHW model \((10)\) in the \( t_\alpha \)-forward measure, as

\[
\begin{align*}
    B_{\alpha'_t}(t_\alpha) &= B_{\alpha'_t}(t_0) \exp \left\{ -s_{\alpha'_t} \xi - s_{\alpha'_t}^2 \zeta^2 / 2 \right\} & t = \alpha' + 1, \ldots, \omega' \\
    \beta(t_\alpha) B_{\alpha'_t}(t_\alpha) &= \beta(t_0) B_{\alpha'_t}(t_0) \exp \left\{ -\nu_{\alpha'_t} \xi - \nu_{\alpha'_t}^2 \zeta^2 / 2 \right\} & t = \alpha', \ldots, \omega' - 1
\end{align*}
\]

where

\[
\xi := \int_{t_0}^{t_\alpha} dW^{(a)}_u e^{-a(t_\alpha - u)}
\]

(12)

a zero mean Gaussian r.v. whose variance is

\[
\zeta^2 := \begin{cases} 
    \frac{1 - e^{-2a(t_\alpha - t_0)}}{2a} & a \in \mathbb{R}^+ \setminus \{0\} \\
    \frac{t_\alpha - t_0}{a} & a = 0.
\end{cases}
\]

**Proof.** A straightforward application of Itô calculus, given dynamics \((9)\) and deterministic volatilities \((10)\)  

A consequence of previous lemma is that receiver swaption payoff \((8)\) in the \( t_\alpha \)-forward measure can be written as a function of a unique r.v. \( \xi \) as

\[
R_{\alpha \omega}(t_\alpha) := [f(\xi)]^+.
\]

(13)

In the following lemma we show that \( f(\xi) \) is equal to a finite sum of exponential functions of \( \xi \), i.e.

\[
f(\xi) = \sum_i w_i e^{\lambda_i \xi} \quad \text{with} \quad w_i, \lambda_i \in \mathbb{R}
\]

where some \( w_i < 0 \) and some \( \lambda_i \geq 0 \). Hence, the swaption looks like a non-trivial spread option, with a number of terms equal to \( \omega - \alpha - 1 + 2(\omega' - \alpha') \).

In Lemma \[3\] we prove that, even if the function \( f \), for some parameters choices, is not a decreasing function of \( \xi \), however there exists a unique value \( \xi^* \) s.t. \( f(\xi^*) = 0 \), i.e. the equality \( S_{\alpha \omega}(t_\alpha) = K \) is satisfied for this unique value.
Lemma 3 According to MHW model \([10]\), the function \(f(\xi)\) in swaption payoff is equal to

\[
f(\xi) = \sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_0) e^{-\varsigma_{\alpha j} \xi - \varsigma^2_{\alpha j}/2} \quad (a)
\]

\[
+ \sum_{i=\alpha'+1}^{\omega'-1} B_{\alpha' i}(t_0) e^{-\varsigma_{\alpha' i} \xi - \varsigma^2_{\alpha' i}/2} \quad (b)
\]

\[
- \sum_{i=\alpha'}^{\omega'-1} \beta_i(t_0) B_{\alpha' i}(t_0) e^{-\nu_{\alpha' i} \xi - \nu^2_{\alpha' i}/2} \quad (c)
\]

and \(\exists! \xi^*\) s.t. \(f(\xi^*) = 0\) for \(a, \sigma \in \mathbb{R}^+\) and \(\gamma \in [0, 1]\). Moreover, the function \(f\) is greater than zero for \(\xi < \xi^*\).

Proof. See Appendix A ♣

We have proven that, even if function \(f\) is not monotonic in its argument, there is a unique solution for equation \(f(\xi) = 0\). This fact grants the possibility to extend to MHW the approach of [Jamshidian (1989)]. In the following proposition we prove that a closed form solution holds for a receiver swaption for model \([10]\).

Proposition 1 A receiver swaption, according to MHW model \([10]\), can be computed with the closed formula

\[
\mathcal{R}^{\text{MHW}}_{\alpha \omega}(t_0) = B(t_0, t_\alpha) \left\{ \sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_0) N \left( \frac{\xi^*}{\zeta} + \zeta c_{\alpha j} \right) + \sum_{i=\alpha'+1}^{\omega'-1} B_{\alpha' i}(t_0) N \left( \frac{\xi^*}{\zeta} + \zeta c_{\alpha' i} \right) - \sum_{i=\alpha'}^{\omega'-1} \beta_i(t_0) B_{\alpha' i}(t_0) N \left( \frac{\xi^*}{\zeta} + \zeta \nu_{\alpha' i} \right) \right\}
\]

(14)

where \(N(\bullet)\) is the standard normal CDF and \(\xi^*\) is the unique solution of \(f(\xi) = 0\).

Proof. See Appendix A ♣

Let us comment above proposition, which is the most relevant analytical result of this paper. It generalizes the celebrated result of [Jamshidian (1989)] to this Multicurve HJM model. The main difference is that also negative addends appear in the receiver swaption \(\mathcal{R}^{\text{MHW}}_{\alpha \omega}(t_0)\) and there are extended volatilities instead of standard volatilities. It is straightforward to prove that, mutatis mutandis, a similar solution holds for a payer swaption.

4 Model calibration

In this section we show in detail model calibration of market parameters in the Euro market considering European ATM swaptions vs Euribor 6m with the end-of-day market conditions of September 10, 2010 (value date).

As discussed in the introduction, the calibration cascade is divided in two steps. First, we bootstrap the discount and the pseudo-discount curves from 6m-Depo, three FRAs \((1 \times 7, 2 \times 8\) and \(3 \times 9)\) and swaps (both OIS and vs Euribor 6m). Then, we calibrate the three MHW parameters \(p := (a, \sigma, \gamma)\) with European ATM swaptions vs Euribor 6m on the 10y-diagonal (i.e. considering the \(M = 9\) ATM swaptions 1y9y, 2y8y, \ldots, 9y1y).
|     | OIS rate (%) | swap rate vs 6m (%) |
|-----|--------------|---------------------|
| 1w  | -0.132       | -                   |
| 2w  | -0.132       | -                   |
| 1m  | -0.132       | -                   |
| 2m  | -0.133       | -                   |
| 3m  | -0.136       | -                   |
| 6m  | -0.139       | -                   |
| 1y  | -0.147       | 0.044               |
| 2y  | -0.135       | 0.080               |
| 3y  | -0.083       | 0.154               |
| 4y  | 0.008        | 0.259               |
| 5y  | 0.122        | 0.377               |
| 6y  | 0.254        | 0.512               |
| 7y  | 0.392        | 0.652               |
| 8y  | 0.529        | 0.786               |
| 9y  | 0.655        | 0.909               |
| 10y | 0.766        | 1.016               |
| 11y | 0.866        | 1.109               |
| 12y | 0.957        | 1.195               |
| 15y | 1.160        | 1.383               |

Table 1: OIS rates and swap rates vs Euribor 6m in percentages: end-of-day mid quotes (annual 30/360 day-count convention for swaps vs 6m, Act/360 day-count for OIS) on 10 September 2015.

The discount curve is bootstrapped from OIS quoted rates with the same methodology described in Baviera and Cassaro (2015). Their quotes at value date are reported in Table 1 (with market conventions, i.e. annual payments and Act/360 day-count); in the same table we report also the swap rates (annual fixed leg with 30/360 day-count). In Table 2 we show the relevant FRA rates and the Euribor 6m fixing on the same value date (both with Act/360 day-count). All market data are provided by Bloomberg. Convexity adjustments for FRAs, present in the MHW model, are neglected because they do not impact the nodes relevant for the diagonal swaptions co-terminal 10y considered in this calibration and they are very small in any case. In figure 1 we show the discount and pseudo-discount curves obtained via the bootstrapping technique.

|     | rate (%) |
|-----|----------|
| Euribor 6m | 0.038    |
| FRA 1 × 7   | 0.038    |
| FRA 2 × 8   | 0.041    |
| FRA 3 × 9   | 0.043    |

Table 2: Euribor 6m fixing rate and FRA in percentages (day-count Act/360). FRA rates are end-of-day mid quotes at value date.

We show the swaption ATM volatilities in basis points (bps) in Table 3; the swaption market prices are obtained according to the standard normal market model; a model choice that allows for negative interest rates.
Figure 1: Discount OIS curve (in red) and pseudo-discount Euribor-6m curve (in blue) on September 10, 2010, starting from the settlement date and up to a 12y time horizon.

| expiry | tenor | volatility (bps) |
|--------|-------|------------------|
| 1y     | 9y    | 64.70            |
| 2y     | 8y    | 66.78            |
| 3y     | 7y    | 68.53            |
| 4y     | 6y    | 70.91            |
| 5y     | 5y    | 72.36            |
| 6y     | 4y    | 73.07            |
| 7y     | 3y    | 73.21            |
| 8y     | 2y    | 73.51            |
| 9y     | 1y    | 73.45            |

Table 3: Normal volatilities for ATM diagonal swaptions co-terminal 10y in bps on 10 September 2015.

We minimize the square distance between swaption model and market prices

$$\text{Err}^2(p) = \sum_{i=1}^{M} [\mathcal{N}_i^{\text{MW}}(p;t_0) - \mathcal{N}_i^{\text{MKT}}(t_0)]^2$$

where market ATM swaption pricing formula according to the multicurve normal model is reported in Appendix B.

We obtain the parameter estimations minimizing the $\text{Err}$ function w.r.t. $a, \gamma$ and $\sigma := \sigma/a$; the solution is stable for a large class of starting points. As estimations we obtain $a = 13.31\%$, $\sigma = 1.27\%$ and $\gamma = 0.06\%$. The difference between model and market swaption prices are shown in figure 2: calibration results look good despite the parsimony of the proposed model.

It is interesting to observe that the dependence of the $\text{Err}$ function w.r.t. $\gamma$ is less pronounced compared to the one w.r.t. $a$ and $\sigma$; even if the minimum values for the $\text{Err}$ function are achieved for very low values of $\gamma$, however, differences in terms of mean squared error are very small increasing, even significantly, $\gamma$: another evidence that the most relevant dynamics for swaption
valuation is the one related to the pseudo-discount curve, where the corresponding volatility does not depend on $\gamma$ parameter.

5 Conclusions

Is it possible to consider a parsimonious multicurve IR model without assuming constant spreads? In this paper we introduce a three parameter generalization of the two parameters Hull and White (1990) model, where the additional parameter $\gamma$ lies in the interval $[0, 1]$. The limiting cases correspond to some models already known in the literature: the case with $\gamma = 0$ corresponds to the S0 hypothesis in Henrard (2010), where the spread curve is constant over time, while $\gamma = 1$ corresponds to the S1 assumption in Baviera and Cassaro (2015).

We have proven that the model allows a very simple closed formula for European physical delivery swaptions (14) with a formula, very similar to the one of Jamshidian (1989), with the presence of extended volatilities, that can assume negative values. Model calibration is immediate: we have shown in detail how to implement the calibration cascade on the September 10, 2010 end-of-day market conditions.

The proposed model allows also Black-like formulas for the other liquid IR options (caps/floors and STIR options) and simple analytical convexity adjustments for FRAs and STIR futures; furthermore numerical techniques similar to the HW model can be applied.

This very parsimonious model is justified by the good calibration properties on ATM swaption prices and by the observation that the pseudo-discount dynamics is the relevant one in the valuation of liquid IR options. Furthermore a very parsimonious model, as the proposed MHW model (10), can be the choice of election in challenging tasks where the multicurve IR dynamics is just one of the modeling elements: two significant examples are the pricing and the risk management of illiquid corporate bonds, and the XVA valuations including all contracts between two counterparts within a netting set at bank level.
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Appendix A

Proof of Lemma 3. Function $f(\xi)$ is obtained from direct substitution of swaption payoff components (11) in Receiver payoff (8). $f(\xi)$ is a sum of exponentials $\exp(\lambda_i \xi)$ multiplied by some coefficients $\omega_i$, where both $\lambda_i, \omega_i \in \mathbb{R}$. Function $f(\xi)$ is composed by different parts: positive addends with negative exponentials (terms $a$ and $b$) and a negative term with a positive exponential (first addend in $c$ for $\iota = \alpha'$), which becomes a negative constant for $\gamma = 0$. The remaining coefficients in $(c)$ are always negative and they can be divided into three parts; one with negative exponentials ($\nu_{\alpha t} > 0$), another one with positive exponentials ($\nu_{\alpha t} < 0$) and a third part constant when at least one $\nu_{\alpha t}$ is equal to 0.

Let us study $f(\xi)$ as a function of $\xi \in \mathbb{R}$. It is a very regular function ($C^\infty$), a finite sum of exponentials and constants. We divide the addends of function $f$ in two parts. In the first one $f_+(\xi)$ we consider the sum of all positive addends (i.e. terms $a$ and $b$) and in the second one $f_-\left(\xi\right)$ the sum of all negative addends (i.e. term $c$) in absolute value, i.e.

$$f(\xi) =: f_+(\xi) - f_-\left(\xi\right)$$

where both $f_+\left(\xi\right)$ and $f_-\left(\xi\right)$ are positive functions of their argument: $f_+\left(\xi\right)$ is the sum of negative exponentials while $f_-\left(\xi\right)$ can be the sum of both positive, negative exponentials and a constant (only for a finite set of values for $\gamma$, for the values of $\gamma$ equal to one of the $\{\tilde{\gamma}_t\}_{t=\alpha'+1,\ldots,\alpha'}$).

First, let us observe that a positive addend is leading for small $\xi$. This fact is a consequence of the following inequalities that hold $\forall \iota = \alpha' + 1,\ldots,\omega'$

$$v_{\alpha'\iota - 1} < v_{\alpha'\iota}, \quad \nu_{\alpha'\iota} \leq (1 - \gamma) v_{\alpha'\iota}$$

where the equality holds only for $\gamma = 0$, immediate consequences of volatility definitions (10). For all values of $\gamma$ the leading term of $f(\xi)$ for small $\xi$ is

$$c_{\omega} B_{\alpha \omega}(t_0) e^{-(1-\gamma) v_{\alpha \omega} \xi} + \ldots$$

because, due to inequalities (15), $-(1 - \gamma) v_{\alpha \omega}$ is the lowest exponent coefficient that multiplies $\xi$ among the exponentials in $f(\xi)$; i.e. there exists always a $\hat{\xi}$ s.t. $\forall \xi < \hat{\xi}$ $f_+(\xi) > f_-\left(\xi\right)$.

Then, let us define $\hat{\tilde{\gamma}} := \max_i \tilde{\gamma}_i$ and let us distinguish three cases depending on $\gamma$ value:

1. When $\hat{\tilde{\gamma}} \leq \gamma \leq 1$, $f_-\left(\xi\right)$, due to Remark 2, is a positive linear combination of positive exponentials (and a positive constant when $\gamma = \hat{\tilde{\gamma}}$). Also this case admits one unique intersection with $f_+\left(\xi\right)$, which is a sum of negative exponentials for $\gamma < 1$, as mentioned above, while is a constant for $\gamma = 1$.

2. When $0 < \gamma < \hat{\tilde{\gamma}}$, $f_-\left(\xi\right)$ is a u-shaped positive function since it is a positive linear combination of positive and negative exponentials (and a constant for some values of $\gamma$). Moreover $f_+\left(\xi\right)$ and $f_-\left(\xi\right)$ present one unique intersection, because $f_+\left(\xi\right)$ goes to $+\infty$ for $\xi \to -\infty$ faster than $f_-\left(\xi\right)$ and to 0 for $\xi \to +\infty$. 


The ATM formula simplifies to

\[ f(\xi) = \sum_{j=\alpha+1}^\omega c_j B_{\alpha j}(t_0) e^{-\nu_{\alpha j} \xi - v_{\alpha j}^2 \xi^2 / 2} \]

all addends are negative exponentials and constants, and then the limit for \( \xi \to +\infty \) is equal to \(-\beta_{\alpha'}(t_0)\). Moreover, due to inequalities (15), \(-v_{\alpha' \alpha' + 1}\) (always lower than zero) is the largest exponent coefficient that multiplies \( \xi \) among the exponentials in \( f(\xi) \), the leading term for large \( \xi \) is

\[ -(\beta_{\alpha' + 1}(t_0) - 1) B_{\alpha' \alpha' + 1}(t_0) e^{-v_{\alpha' \alpha' + 1} \xi - v_{\alpha' \alpha' + 1}^2 \xi^2 / 2} \]

hence \( f(\xi) \) tends to \(-\beta_{\alpha'}(t_0) < 0\) from below for \( \xi \to \infty \). With similar arguments applied to the first derivative of \( f(\xi) \), one can show that the function has one minimum. Summarizing, for \( \gamma = 0 \) the function \( f(\xi) \) is a decreasing function up to its minimum \( \xi_{\text{min}} \) (reaching a value lower than \(-\beta_{\alpha'}(t_0) < 0\) and then it gradually goes to \(-\beta_{\alpha'}(t_0)\) from below for \( \xi > \xi_{\text{min}} \). Also in this case the function \( f(\xi) \) presents a unique intersection with zero.

We have then proven that, for all parameters choices, there exists a unique value \( \xi^* \) s.t \( f(\xi^*) = 0 \). The proof is complete once we observe that, for \( \xi < \xi^* \), the function \( f(\xi) \) is larger than zero in the three cases described above  

**Proof of Proposition 1.** Due to Lemma 3, swaption receiver is equivalent to

\[ \mathcal{R}_{\alpha \omega}(t_0)/B(t_0, t_\alpha) = \mathbb{E} \{ f(\xi) \}^+ = \mathbb{E} \{ f(\xi) \} \mathbb{1}_{\xi \leq \xi^*} \]

\[ = \sum_{j=\alpha+1}^\omega c_j \mathbb{E} \left\{ \left[ B_{\alpha j}(t_0) e^{-\omega_{\alpha j} \xi - v_{\alpha j}^2 \xi^2 / 2} \right] \mathbb{1}_{\xi \leq \xi^*} \right\} + \sum_{t=\alpha' + 1}^{\omega' - 1} \mathbb{E} \left\{ \left[ B_{t \alpha}(t_0) e^{-\omega_{t \alpha} \xi - v_{t \alpha}^2 \xi^2 / 2} \right] \mathbb{1}_{\xi \leq \xi^*} \right\} \]

\[ - \sum_{t=\alpha'}^{\omega' - 1} \mathbb{E} \left\{ \left[ \beta_t(t_0) B_{t \alpha}(t_0) e^{-\omega_{t \alpha} \xi - v_{t \alpha}^2 \xi^2 / 2} \right] \mathbb{1}_{\xi \leq \xi^*} \right\} \]

and then, after straightforward computations, one proves the proposition  

**Appendix B**

In this appendix we report the Normal-Black formula for a receiver swaption:

\[ \mathcal{R}_{\alpha \omega}(t_0) = B(t_0, t_\alpha) BPV_{\omega \omega}(t_0) \left\{ \left[ K - S_{\omega \omega}(t_0) \right] N(-d) + \sigma_{\omega \omega} \sqrt{t_\alpha - t_0} \phi(d) \right\} \]

where \( N(\bullet) \) is the standard normal CDF, \( \phi(\bullet) \) the standard normal density function and \( \sigma_{\omega \omega} \) the corresponding implied normal volatility

\[ d := \frac{S_{\omega \omega}(t_0) - K}{\sigma_{\omega \omega} \sqrt{t_\alpha - t_0}} . \]

The ATM formula simplifies to

\[ \mathcal{R}_{\omega \omega}(t_0) = B(t_0, t_\alpha) BPV_{\omega \omega}(t_0) \sigma_{\omega \omega} \sqrt{\frac{t_\alpha - t_0}{2\pi}} . \]
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## Notation and shorthands

| Symbol | Description |
|--------|-------------|
| $a, \sigma, \gamma$ | Multicurve Hull and White (10) parameters; $a, \sigma \in \mathbb{R}^+$ and $\gamma \in [0, 1]$ |
| $B(t, T)$ | discount curve, zero-coupon bond in $t$ with maturity $T$ |
| $B(t; T, T + \Delta)$ | forward discount in $t$ between $T$ and $T + \Delta$, $t \leq T < T + \Delta$ |
| $\hat{B}(t; T, T + \Delta)$ | forward pseudo-discount in $t$ between $T$ and $T + \Delta$, $t \leq T < T + \Delta$ |
| $\beta(t; T, T + \Delta)$ | forward spread in $t$ between $T$ and $T + \Delta$, $t \leq T < T + \Delta$ |
| $\beta(t, T)$ | spread curve in $t$ with maturity $T$ |
| $\delta(t, t_{j+1})$ | year-fraction between two payment dates in swap’s fixed leg |
| $\delta(t', t'_{i+1})$ | year-fraction between two payment dates in swap’s floating leg |
| $\Delta$ | the lag that characterizes the pseudo-discounts, e.g. 6-months for Eur6m |
| $K$ | strike rate |
| $N(\bullet)$ | the standard normal CDF |
| $\rho$ | correlation matrix in $\mathbb{R}^{d \times d}$ s.t. $dW_{i,t} dW_{j,t} = \rho_{ij} dt$ |
| $\sigma(t, T)$ | HJM discount volatility in $\mathbb{R}^d$ between $t$ and $T$ |
| $\eta(t, T)$ | HJM spread volatility in $\mathbb{R}^d$ between $t$ and $T$ |
| $\mathcal{R}_{\alpha\omega}(t_\alpha)$ | receiver swaption payoff at expiry |
| $\mathcal{R}_{\alpha\omega}(t_0)$ | receiver swaption price at value date |
| $t_0$ | value date |
| $t_\alpha$ | swaption expiry date |
| $t_\omega$ | underlying swap maturity date |
| $t := \{t_j\}_j$ | underlying swap fixed leg payment dates, $j = \alpha + 1, \ldots, \omega$ |
| $t' := \{t'_i\}_i$ | underlying swap floating leg payment dates, $i = \alpha' + 1, \ldots, \omega'$ |
| $W_t$ | vector of correlated Brownian motions in $\mathbb{R}^d$ s.t. $dW_{i,t} dW_{j,t} = \rho_{ij} dt$ |
| $x \cdot y$ | canonical scalar product in $\mathbb{R}^d$ |
| $x^2$ | scalar product $x \cdot \rho x$ with $x \in \mathbb{R}^d$ and $\rho$ correlation matrix |
| $\xi$ | Gaussian r.v. defined in (12) with zero mean and variance $\zeta^2$ |
| $\xi^*$ | the unique solution of $f(\xi) = 0$; $f(\xi)$ defined in (13) |
| $\zeta$ | standard deviation of the Gaussian r.v. $\xi$ |
Shorthands

\[ B_{\alpha j}(t) : B(t; t_\alpha, t_j) \]
\[ B_{\alpha' \iota}(t) : B(t; t'_{\alpha'}, t'_{\iota}) \]
\[ \beta_{\iota}(t) : \beta(t; t'_{\iota}, t'_{\iota+1}) \]
\[ \delta'_{\iota} : \delta(t'_{\iota}, t'_{\iota+1}) \]
\[ \delta_j : \delta(t_j, t_{j+1}) \]
\[ c_j : \delta_j K \quad \text{for} \quad j = \alpha + 1, \ldots, \omega - 1 \quad \text{and} \quad 1 + \delta_\omega K \quad \text{for} \quad j = \omega \]
\[ v_{\alpha' \iota} : v(t_\alpha, t'_\iota) \]
\[ s_{\alpha' \iota} : (1 - \gamma) v_{\alpha' \iota} \]
\[ v_{\alpha' \iota} : s_{\alpha' \iota} - (\eta(t_\alpha, t'_{\iota+1}) - \eta(t_\alpha, t'_\iota)) \]
\[ \text{IR : Interest Rate} \]
\[ \text{MHW : Multicurve Hull White model (10)} \]
\[ \text{r.v. : random variable} \]
\[ \text{s.t. : such that} \]
\[ \text{w.r.t. : with respect to} \]