Extremes of Gaussian random fields with regularly varying dependence structure

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Abstract Let \( X(t), t \in \mathcal{T} \) be a centered Gaussian random field with variance function \( \sigma^2(\cdot) \) that attains its maximum at the unique point \( t_0 \in \mathcal{T} \), and let \( M(\mathcal{T}) = \sup_{t \in \mathcal{T}} X(t) \). For \( \mathcal{T} \) a compact subset of \( \mathbb{R} \), the current literature explains the asymptotic tail behaviour of \( M(\mathcal{T}) \) under some regularity conditions including that \( 1 - \sigma(t) \) has a polynomial decrease to 0 as \( t \to t_0 \). In this contribution we consider more general case that \( 1 - \sigma(t) \) is regularly varying at \( t_0 \). We extend our analysis to Gaussian random fields defined on some compact set \( \mathcal{T} \subset \mathbb{R}^2 \), deriving the exact tail asymptotics of \( M(\mathcal{T}) \) for the class of Gaussian random fields with variance and correlation functions being regularly varying at \( t_0 \). A crucial novel element is the analysis of families of Gaussian random fields that do not possess locally additive dependence structures, which leads to qualitatively new types of asymptotics.

Keywords Non-stationary Gaussian processes · Gaussian random fields · Extremes · Fractional Brownian motion · Regular variation · Uniform approximation

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1 Introduction

Let $X(t), t \geq 0$ be a centered stationary Gaussian processes with continuous trajectories, unit variance and correlation function $r(\cdot)$ satisfying Pickands’s condition

$$1 - r(t) \sim a|t|^\alpha, \quad t \downarrow 0, \quad \text{and} \quad r(t) < 1, \forall t \neq 0,$$

with $a > 0, \alpha \in (0, 2]$. In our notation $\sim$ means asymptotic equivalence when the argument tends to 0 or $\infty$.

In the seminal contribution (Pickands 1969) it is shown that under (1), for any $T$ positive

$$\mathbb{P}\left( \sup_{t \in [0,T]} X(t) > u \right) \sim T^{\mathcal{H}_\alpha}(\sqrt{au})^{2/\alpha} \mathbb{P}(X(0) > u), \quad u \to \infty,$$

where the Pickands constant $\mathcal{H}_\alpha$ is defined by

$$\mathcal{H}_\alpha = \lim_{S \to \infty} S^{-1} \mathcal{H}_\alpha[0,S] \in (0, \infty) \quad \text{with} \quad \mathcal{H}_\alpha[S_1, S_2] = \mathbb{E}\left\{ \sup_{t \in [S_1, S_2]} e^{\sqrt{2B_\alpha(t) - \frac{1}{2}t^2}} \right\}, \quad S_1 < S_2,$$

with $B_\alpha(t), t \geq 0$ a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0, 1]$, see Pickands (1969); Piterbarg (1972, 1996); Michna (1999); Burnecki and Michna (2002); Dębicki (2002); Dieker (2005); Dębicki and Kosiński (2014); Dieker and Yakir (2014); Dębicki et al. (2014); Piterbarg (2015); Dębicki et al. (2015); Dieker and Mikosch (2015); Dębicki et al. (2016); Arendarczyk (2016) for various properties of $\mathcal{H}_\alpha$ and related constants.

The asymptotics in Eq. 2 is extended in various directions, including $\alpha(t)$-locally-stationary Gaussian processes (see Dębicki and Kisowski (2008)), and general non-stationary Gaussian processes and random fields, see e.g., Piterbarg (2015). A particularly important place in this theory is taken by the result of Piterbarg and Prisjažnuk (1978), where the exact tail asymptotics of $\sup_{t \in [0,T]} X(t)$ is derived in the case that the variance function $\sigma^2$ of a centered Gaussian process $X$ has a unique point of maximum in $[0, T]$, say $t_0$. For simplicity assume that $t_0 \in (0, T)$ and $\sigma(t_0) = 1$. Similarly to the stationary case, in Piterbarg and Prisjažnuk (1978) it is assumed that the correlation function $r(s, t) = \text{Corr}(X(s), X(t))$ satisfies for some $a > 0, \alpha \in (0, 2]$

$$1 - r(s, t) \sim a|t - s|^\alpha, \quad s, t \to t_0,$$

whereas the behaviour of the variance function around the unique maximizer $t_0$ satisfies

$$1 - \sigma(t) \sim b|t - t_0|^\beta, \quad t \to t_0$$

for some $b, \beta > 0$. Assume further that for $C > 0, \nu \in (0, 2]$ the following Hölder continuity condition

$$\mathbb{E}\left\{ (X(t) - X(s))^2 \right\} \leq C|t - s|^\nu, \quad \forall s, t \in [t_0 - \theta, t_0 + \theta]$$

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