Research Article

Wave Breaking for the Rotational Camassa–Holm Equation on the Circle

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1. Introduction

Recently, a rotational Camassa–Holm equation was proposed in [1–4], which reads as

\[ m_t + u m_x + 2u_x m + cu_x - \frac{\beta_0}{\beta} u_{xxx} + \frac{\omega_1}{\alpha} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x = 0, \]

where \( \beta \) is a parameter related to the Coriolis effect. In the whole space, Gui et al. [3] established the local well-posedness in \( H^s(\mathbb{R}) \), \( s > 3/2 \). Sufficient conditions to guarantee wave breaking phenomena were also studied in [3]. Tu et al. [4] established the global existence and uniqueness of the energy conservative weak solutions. In [5], they found some explicit solutions by elliptic integrals. They also classified the members of the equation describing pseudo-spherical surfaces. Generic regularity of conservative solutions was investigated in [6]. Moon [7] studied the question of nonexistence of periodic peaked traveling wave solution for rotational Camassa–Holm equation.

If \( \Omega = 0 \), it follows \( \omega_1 = \omega_2 = 0 \), and then system (1) reduces to the Dullin–Gottwald–Holm equation [8]. Some mathematical studies can be found in [9–12].

Another highly related model is the well-known Camassa–Holm equation [13]

\[ m_t + u m_x + 2u_x m = 0, m = u - u_{xx}, \]

with

\[ c = \sqrt{1 + \Omega^2} - \Omega, \]
\[ \alpha = \frac{c^2}{1 + c^2}, \]
\[ \beta_0 = \frac{c(c^4 + 6c^2 - 1)}{6(c^2 + 1)^2}, \]
\[ \beta = \frac{3c^4 + 8c^2 - 1}{6(c^2 + 1)^2}, \]
\[ \omega_1 = \frac{3c(c^2 - 1)(c^2 - 2)}{2(c^2 + 1)^3}, \]
\[ \omega_2 = \frac{(c^2 - 1)^2(c^2 - 2)(8c^2 - 1)}{2(c^2 + 1)^5}, \]

where \( \Omega \) is a parameter related to the Coriolis effect. In the whole space, Gui et al. [3] established the local well-posedness in \( H^s(\mathbb{R}) \), \( s > 3/2 \). Sufficient conditions to guarantee wave breaking phenomena were also studied in [3]. Tu et al. [4] established the global existence and uniqueness of the energy conservative weak solutions. In [5], they found some explicit solutions by elliptic integrals. They also classified the members of the equation describing pseudo-spherical surfaces. Generic regularity of conservative solutions was investigated in [6]. Moon [7] studied the question of nonexistence of periodic peaked traveling wave solution for rotational Camassa–Holm equation.

If \( \Omega = 0 \), it follows \( \omega_1 = \omega_2 = 0 \), and then system (1) reduces to the Dullin–Gottwald–Holm equation [8]. Some mathematical studies can be found in [9–12].

Another highly related model is the well-known Camassa–Holm equation [13] \( m_t + u m_x + 2u_x m = 0, m = u - u_{xx} \). This equation has a physical background with shallow water propagation. The Camassa–Holm equation [14,15] has infinitely many conservation laws. In [16,17], they established the local well-posedness. Wave breaking phenomena were widely studied in [16–19]. McKean [20] (see also [21] for a simple proof) established a necessary and sufficient condition on the initial datum \( u_0 \), which depends on the shape of \( y = u - u_{xx} \). In [22], the orbital stability of the peakons was proved. In [23, 24], they studied persistence properties and unique continuation of solutions. The long-time behavior for the support of
momentum density of the Camassa–Holm equation was discussed in [25]. Mathematical studies for the related models can been found in [26–28].

For the convenience of research, in this paper, we consider the rotational Camassa–Holm equation as the following form on the circle:

\[ m_t + um_x + 2u_x u_x = au_x + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, \quad (3) \]

where \( m = u - u_{xx} \) and \( \alpha, \beta, \gamma, \Gamma \) are real constants. \( x \in \mathbb{S} \), where \( \mathbb{S} \) denotes the unit circle, i.e., \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \).

The paper is organized as follows. In Section 2, we introduce some useful lemmas. The main result and its proof will be shown in Section 3.

2. Preliminaries

Let \( \Lambda = (1 - \partial_x^2)^{1/2} \); then, the operator \( \Lambda^{-2} \) can be expressed by its associated Green’s function as

\[ u = \Lambda^{-2} m(x, t) = G \ast m(x, t), \]

\[ G(x) = \frac{\cosh (x - [x] - (1/2))}{2 \sinh (1/2)}. \quad (4) \]

Then, (3) can be rewritten as

\[ u_t + (u + \Gamma)u_x + \partial_x G \ast \left( u^2 + \frac{u_x^2}{2} \right) = \partial_x G \ast \left( \alpha + \Gamma \right) u + \frac{\beta}{3} u^3 + \frac{\gamma}{3} u^4 \quad (5) \]

By applying Kato’s semigroup theory [29] and similar arguments in [3], we can also have the local well-posedness on the circle.

Theorem 1 (see [3]). Given \( u(x, t = 0) = u_0 \in H^s(\mathbb{S}) \), \( s > 3/2 \), then there exist a maximal \( T = T(u_0) > 0 \) and a unique solution \( u \) to the rotational Camassa–Holm equation (3) such that

\[ u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{S})) \cap C^1([0, T); H^{s - 1}(\mathbb{S})). \quad (6) \]

Then, the precise blow-up scenario will be shown as follows.

Theorem 2 (see [3]). Assume that \( u_0 \in H^2(\mathbb{S}) \) and let \( T \) be the maximal existence time of the solution \( u(x, t) \) to equation (3) with the initial data \( u_0(x) \). Then, the corresponding solution of the rotational Camassa–Holm equation (3) blows up in finite time if and only if

\[ \lim_{t \to T} \inf_{x \in \mathbb{S}} [u_x(x, t)] = -\infty. \quad (7) \]

Then, we introduce some useful inequality in the circle.

Lemma 1 (see [30]). For all \( f \in H^1(\mathbb{S}) \), the following inequality holds:

\[ \max_{x \in [0, 1]} f^2(x) \leq C_0 \|f\|^2_{H^1(\mathbb{S})}, \quad (8) \]

where \( C_0 = e^{1/2} + e^{-1/2}/2 (e^{1/2} - e^{-1/2}) \).

Lemma 2 (see [30]). Let \( f \in H^1(\mathbb{S}) \). If \( \int \mathbb{S} f dx = 0 \), then the following inequalities hold:

\[ \left\| f \right\|^2_{L^2(\mathbb{S})} \leq \frac{1}{12} \left\| f \right\|^2_{H^1(\mathbb{S})}, \]

\[ \int \mathbb{S} f^2(x) dx \leq \frac{1}{12} \int \mathbb{S} f^2(x) dx, \quad (9) \]

\[ \int \mathbb{S} f^2(x) dx \leq \frac{1}{12} \left( \int \mathbb{S} f^2(x) dx \right)^2. \]

3. Main Results

In this section, we firstly establish a sufficient condition to guarantee the blow up of the solution to the rotational Camassa–Holm equation (3). We give the particle trajectory as

\[ \begin{cases} \begin{align*} &q_t(x, t) = u(q(x, t), t) + \Gamma, \quad 0 < t < T, x \in \mathbb{S}, \\ &q(x, 0) = x, \quad x \in \mathbb{S}, \end{align*} \end{cases} \]

where \( T \) is the lifespan of the solution. Taking derivative (10) with respect to \( x \), we obtain

\[ \frac{dq_t(x, t)}{dx} = q_x q_t(x, t), \quad t \in (0, T). \quad (11) \]

Therefore,

\[ \begin{cases} \begin{align*} &q_x = \exp \left\{ \int_{0}^{t} u_t(x, s) ds \right\}, \quad 0 < t < T, x \in \mathbb{S}, \\ &q_x(x, 0) = 1, \quad x \in \mathbb{S}, \end{align*} \end{cases} \]

which is always positive before the blow-up time.

Theorem 3. Assume that \( u_0 \in H^2(\mathbb{S}) \) and there exists \( x_0 \in \mathbb{S} \) such that

\[ u_{0x}(x_0) < -\sqrt{2C^*}, \quad (13) \]

where \( C^* = (1/2)C_0\|u_0\|_{H^1} + 2 \|u_0\|_{H^1} + (2\beta/3)\|u_0\|_{H^1}^2 + (\gamma/4)\|u_0\|_{H^1}^4 \); then, the corresponding solution \( u(x, t) \) to equation blows up at a finite time \( T \) bounded by

\[ T \leq \frac{1}{-(1/2)u_{0x}(x_0) + (C^*/u_{0x}(x_0))}. \quad (14) \]

Proof. Let \( \mathcal{H}(u) = (\alpha + \Gamma)u + (\beta/3)u^3 + (\gamma/4)u^4 \). Differentiating (5) with respect to \( x \) yields

\[ u_{xt} + (u + \Gamma)u_{xx} + u_x^2 + \partial_x^2 G \ast \left( u^2 + \frac{u_x^2}{2} \right) = \partial_x^2 G \ast \mathcal{H}. \quad (15) \]

Then, we have \( u_{xt} \) at the point \( (q(x, t), t) \) as
\[ u_{xt}(q(x,t),t) = \left( u^2 - \frac{u_x^2}{2} - G \ast \left( u^2 + \frac{u_x^2}{2} \right) - \mathcal{H}(u) + G \ast \mathcal{H} \right) \cdot (q(x,t),t). \]  

Without loss of generality, we choose \( 0 \leq q(x,t) \leq 1 \), and we have

\[ G \ast \left( u^2 + \frac{u_x^2}{2} \right)(q(x,t),t) = \frac{1}{2 \sinh(1/2)} \int_0^{q(x,t)} e^{-\eta\left(u^2 + \frac{u_x^2}{2}\right)}(\eta,t) d\eta \]

\[ + \frac{1}{2 \sinh(1/2)} \int_{q(x,t)}^{1} e^{\eta\left(u^2 + \frac{u_x^2}{2}\right)}(\eta,t) d\eta. \]

Note that

\[ \int_0^{q(x,t)} e^{-\eta\left(u^2 + \frac{u_x^2}{2}\right)}(\eta,t) d\eta \geq -2 \int_0^{q(x,t)} e^{-\eta u u_x}(\eta,t) d\eta = -e^{-\eta u^2}(\eta,t) \bigg|_{0}^{q(x,t)} - \int_0^{q(x,t)} e^{-\eta u^2}(\eta,t) d\eta. \]

We have

\[ \int_0^{q(x,t)} e^{-\eta\left(u^2 + \frac{1}{2} u_x^2\right)} q(x,t) d\eta \geq - \frac{1}{2} e^{-\eta u^2}(\eta,t) \bigg|_{0}^{q(x,t)}. \]

Similar argument yields that

\[ \int_0^{q(x,t)} e^{-\eta\left(u^2 + \frac{1}{2} u_x^2\right)}(\eta,t) d\eta \geq \frac{1}{2} \int_0^{q(x,t)} e^{-\eta u^2}(\eta,t) d\eta \]

\[ \int_0^{1} e^{-\eta\left(u^2 + \frac{1}{2} u_x^2\right)}(\eta,t) d\eta \geq \frac{1}{2} \int_0^{q(x,t)} e^{-\eta u^2}(\eta,t) d\eta \]

\[ \int_0^{1} e^{-\eta\left(u^2 + \frac{1}{2} u_x^2\right)}(\eta,t) d\eta \geq \frac{1}{2} \int_0^{q(x,t)} e^{-\eta u^2}(\eta,t) d\eta. \]

Then, we have

\[ G \ast \left( u^2 + \frac{u_x^2}{2} \right)(q(x,t),t) \geq \frac{1}{2} u^2 (q(x,t),t). \]

Combining (21) into (16), we obtain

\[ u_{xt}(q(x,t),t) \leq \left( u^2 - \frac{u_x^2}{2} - \mathcal{H}(u) + G \ast \mathcal{H} \right)(q(x,t),t). \]

A direct calculation gives

\[ |G \ast \mathcal{H}(q(x,t),t)| = \frac{1}{2 \sinh(1/2)} \int_0^{q(x,t)} e^{\eta\left(u^2 - \frac{1}{2} u_x^2\right)}(\eta,t) d\eta + \frac{1}{2 \sinh(1/2)} \int_{q(x,t)}^{1} e^{\eta\left(u^2 + \frac{1}{2} u_x^2\right)}(\eta,t) d\eta \]

\[ \leq \max_{x \in \mathbb{R}} |\mathcal{H}| \left( \frac{1}{2 \sinh(1/2)} \int_0^{q(x,t)} e^{\eta\left(u^2 - \frac{1}{2} u_x^2\right)}(\eta,t) d\eta + \frac{1}{2 \sinh(1/2)} \int_{q(x,t)}^{1} e^{\eta\left(u^2 + \frac{1}{2} u_x^2\right)}(\eta,t) d\eta \right) \]

\[ = \max_{x \in \mathbb{R}} |\mathcal{H}|. \]
By Lemma 1, we have
\[ \frac{u^2}{2} \leq \frac{1}{2} C_0 \| u_0 \|_{H^1}^2. \] (24)

By (22) and the definition of $C^*$, we have
\[ u_{xt} (q(x,t), t) \leq - \frac{u^2}{2} (q(x,t), t) + C^*. \] (27)

This is a Riccati type inequality. By the fundamental ODE methods, the proof is completed by the initial condition.

**Theorem 4.** Assume that $u_0 \in H^1(\mathbb{S})$, $\int_S u_0 \, dx = 0$, and there exists $x_0 \in \mathbb{S}$ such that
\[ u_{0x}(x_0) < - \sqrt{2C^*}, \] (28)
where $C^* = (1/24) \| u_0 \|_{H^1}^2 + (2 (|a| + |\Gamma|)/\sqrt{12}) \| u_0 \|_{H^1} + (|\beta|/18 \sqrt{12}) \| u_0 \|_{H^2} + (|\gamma|/576) \| u_0 \|_{H^3}$; then, the corresponding solution $u(x,t)$ to equation blows up at a finite time $T$ bounded by
\[ T \leq \frac{1}{-(1/2)u_{0x}(x_0) + (C^*/u_{0x}(x_0))}. \] (29)

**Proof.** Recall that
\[ u_{xt} (q(x,t), t) \leq \left( \frac{u^2}{2} - \frac{u^2}{2} - \mathcal{H}(u) + G \mathcal{H} \right) (q(x,t), t). \] (30)

With the initial condition $\int_S u_0 \, dx = 0$, by Lemma 2, we have
\[ \frac{u^2}{2} \leq \frac{1}{24} \| u_0 \|_{H^1}^2. \] (31)

For $\gamma \geq 0$, we have
\[ \mathcal{H}(u) \leq - (\alpha + \Gamma)u - \frac{\beta}{3} u^3 \leq (|a| + |\Gamma|) \sqrt{C_0} \| u_0 \|_{H^1} + \frac{|\beta|}{3} C_0^{3/2} \| u_0 \|_{H^1}^3 + \frac{|\gamma|}{4} C_0 \| u_0 \|_{H^1}^4. \] (25)

For $\gamma < 0$, we have
\[ \mathcal{H}(u) \leq (|a| + |\Gamma|) \sqrt{C_0} \| u_0 \|_{H^1} + \frac{|\beta|}{3} C_0^{3/2} \| u_0 \|_{H^1}^3, \] (26)

By (22) and the definition of $C^*$, we have
\[ u_{xt} (q(x,t), t) \leq - \frac{u^2}{2} (q(x,t), t) + C^*. \] (27)

This is a Riccati type inequality. By the fundamental ODE methods, the proof is completed by the initial condition.

Combining the above arguments into (30), we have
\[ u_{xt} (q(x,t), t) \leq - \frac{u^2}{2} (q(x,t), t) + C^*, \] (34)
where $C^*$ is defined in Theorem 4. The proof of Theorem 4 is completed by the fundamental ODE methods.

**Data Availability**

No data were used to support this study.
Conflicts of Interest
The author declares that there are no conflicts of interest.

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