Cosmic Axion Bose-Einstein Condensation.

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Abstract

QCD axions are a well-motivated candidate for cold dark matter. Cold axions are produced in the early universe by vacuum realignment, axion string decay and axion domain wall decay. We show that cold axions thermalize via their gravitational self-interactions, and form a Bose-Einstein condensate. As a result, axion dark matter behaves differently from the other proposed forms of dark matter. The differences are observable.

1 QCD Axions

The theory of strong interactions, called "quantum chromodynamics" or QCD for short, has in its Lagrangian density a "θ-term" [1] [2] [3] [4]

\[ \mathcal{L}_\theta = \frac{\theta g_s^2}{32\pi^2} \tilde{G}^{a\mu\nu} \tilde{G}_a^{\mu\nu} \] (1)

where \( \theta \) is an angle between 0 and 2\( \pi \), \( g_s \) is the coupling constant for strong interactions, and \( \tilde{G}^{a\mu\nu} \) is the gluon field tensor. The θ-term is a 4-divergence and therefore has no effects in perturbation theory. However, it can be shown to have non-perturbative effects, and these are important at low energies/long distances. Since \( \mathcal{L}_\theta \) is P and CP odd, QCD violates those discrete symmetries when \( \theta \neq 0 \). The strong interactions are observed to be P and CP
symmetric, and therefore $\theta$ must be small. The experimental upper bound on the electric dipole moment of the neutron implies $\theta \lesssim 0.7 \times 10^{-11}$ \cite{5,6}. In the Standard Model of particle physics there is no reason for $\theta$ to be small; it is expected to be of order one. That $\theta$ is less than $10^{-11}$ is a puzzle, referred to as the strong CP problem.

Peccei and Quinn proposed \cite{7,8} solving the strong CP problem by introducing a global $U(1)_{PQ}$ symmetry which is spontaneously broken. When some conditions are met, the parameter $\theta$ is promoted to a dynamical field $\frac{\phi(x)}{f_a}$, where $f_a$ is the energy scale at which $U(1)_{PQ}$ is spontaneously broken, and $\phi(x)$ the associated Nambu-Goldstone boson field. The theory now depends on the expectation value of $\phi(x)$. The latter minimizes the QCD effective potential. It can be shown that the minima of the QCD effective potential occur where $\theta = 0$ \cite{9}. The strong CP problem is thus solved if there is a Peccei-Quinn symmetry.

Axions are the quanta of the field $\phi(x)$ \cite{10,11}. Axions acquire mass due to the non-perturbative effects that make QCD depend on $\theta$. The axion mass is given by

$$ m_a \simeq 10^{-6} \text{eV} \left( \frac{10^{12} \text{GeV}}{f_a} \right) \quad (2) $$

when the temperature is zero.

## 2 Production of cold axions

The equation of motion for $\phi(x)$ is

$$ D_\mu D^\mu \phi(x) + V'_a(\phi(x)) = 0 \quad (3) $$

where $V'_a$ is the derivative of the effective potential with respect to the axion field and $D_\mu$ is the covariant derivative with respect to space-time coordinates. The effective potential may
be written

\[ V_a = m_a(t)^2 f_a^2 \left[ 1 - \cos \left( \frac{\phi(x)}{f_a} \right) \right]. \]  

(4)

The axion mass is temperature- and hence time-dependent. It reaches its zero-temperature value, Eq. (2), at temperatures well below 1 GeV. At temperatures much larger than 1 GeV, \( m_a \) is practically zero. The axion field starts to oscillate\[12, 13, 14\] at a time \( t_1 \) after the Big Bang given by

\[ m(t_1) \cdot t_1 = 1. \]  

(5)

Throughout we use units in which \( \hbar = c = 1 \). \( t_1 \) is approximately \( 2 \times 10^{-7} s \left( f_a 10^{12} \text{ GeV} \right)^{1/3} \). The temperature of the primordial plasma at that time is \( T_1 \approx 1 \text{ GeV} \left( 10^{12} \text{ GeV} \right)^{1/6} \). The \( \phi(x) \) oscillations describe a population of axions called “of vacuum realignment”. Their momenta are of order \( t_1^{-1} \) at time \( t_1 \), and are red-shifted by the expansion of the universe after \( t_1 \):

\[ \delta p(t) \sim \frac{1}{t_1} \frac{R(t_1)}{R(t)} \]  

(6)

where \( R(t) \) is the scale factor. As a result the axions are non-relativistic soon after \( t_1 \), and today they are extremely cold. The fact that they are naturally abundant, weakly coupled and very cold, and that they solve the strong CP problem as well, makes axions an attractive candidate for the dark matter of the universe.

The number of axions produced depends on various circumstances, in particular whether inflation occurred before or after the phase transition in which \( U(1)_{\text{PQ}} \) is spontaneously broken, hereafter called the PQ phase transition. For a review, see ref. [15]. If inflation occurs after, it homogenizes the axion field within the observable universe. The initial value of the axion field may then be accidentally close to the CP conserving value in which case the cold axion population from vacuum realignment is suppressed. If inflation occurs before the PQ phase transition, there is always a vacuum realignment contribution (because the
axion field has random unrelated values in different QCD horizons) and there are additional contributions from axion string decay and axion domain wall decay. The number density of cold axions is

\[ n(t) \simeq \frac{4 \times 10^{47}}{\text{cm}^3} X \left( \frac{f_a}{10^{12} \text{GeV}} \right)^{5/3} \left( \frac{R(t_1)}{R(t)} \right)^3 \]  

(7)

where \( X \) is a fudge factor. If inflation occurs before the PQ phase transition, \( X \) is of order 2 or 20 depending on whose estimate of the string decay contribution one believes. If inflation occurs after the PQ phase transition, \( X \) is of order \( \frac{1}{2} \sin^2 \alpha_1 \), where \( \alpha_1 = \phi(t_1)/f_a \) is the initial misalignment angle.

Cold axions are effectively stable because their lifetime is vastly longer than the age of the universe. The number of axions is effectively conserved. The phase-space density of cold axions implied by Eqs. (6) and (7) is \[ N \sim n \left( \frac{2\pi}{3} \frac{m \delta v}{(m \delta v)^3} \right)^{\frac{8}{3}} \sim 10^{61} X \left( \frac{f_a}{10^{12} \text{ GeV}} \right)^{8/9} \]  

(8)

\( N \) is the average occupation number of those axion states that are occupied. Because their phase-space density is huge and their number is conserved, cold axions may form a Bose-Einstein condensate (BEC). The remaining necessary and sufficient condition for the axions to form a BEC is that they thermalize. Assuming thermal equilibrium, the critical temperature is \[ T_c(t) = \left( \frac{\pi^2 n(t)}{\zeta(3)} \right)^{1/3} \simeq 300 \text{ GeV} \left( \frac{f_a}{10^{12} \text{ GeV}} \right)^{5/9} \left( \frac{R(t_1)}{R(t)} \right) \]  

(9)

The critical temperature is enormous because the cosmic axion density is so very high. The formula given in Eq. (9) differs from the one for atoms because, in thermal equilibrium, most of the non-condensate axions would be relativistic.

The question is whether the axions thermalize. This is not at all obvious since axions
are extremely weakly coupled. Note that for Bose-Einstein condensation to occur, it is not necessary that full thermal equilibrium be reached. It is sufficient that the rate of condensation into the lowest energy available state be larger than the inverse age of the universe. Whether this happens is the issue which we address next.

3 Axion-axion interactions

Axions interact by $\lambda \phi^4$ self-interactions and by gravitational self-interactions. In this section we discuss these two processes in detail and calculate the corresponding relaxation rates [17]. Let us introduce a cubic box of volume $V = L^3$, with periodic boundary conditions at the surface. The axion field and its canonically conjugate field $\pi(\vec{x}, t)$ may be written as

$$\phi(\vec{x}, t) = \sum_{\vec{n}} (a_{\vec{n}}(t)\Phi_{\vec{n}}(\vec{x}) + a_{\vec{n}}^\dagger(t)\Phi_{\vec{n}}^*(\vec{x}))$$ (10)

$$\pi(\vec{x}, t) = \sum_{\vec{n}} (-i\omega_{\vec{n}})(a_{\vec{n}}(t)\Phi_{\vec{n}}(\vec{x}) - a_{\vec{n}}^\dagger(t)\Phi_{\vec{n}}^*(\vec{x}))$$ (11)

inside the box, where

$$\Phi_{\vec{n}}(\vec{x}) = \frac{e^{i\vec{p}_{\vec{n}} \cdot \vec{x}}}{\sqrt{2\omega_{\vec{n}}V}},$$ (12)

$$\vec{p}_{\vec{n}} = 2\pi \vec{n}/L, \ \vec{n} = (n_1, n_2, n_3)$$ where $n_1, n_2, n_3$ are integers, and $\omega_{\vec{n}} = \sqrt{p_{\vec{n}}^2 + m^2}$. The creation and annihilation operators satisfy canonical equal-time commutation relations

$$[a_{\vec{n}}(t), a_{\vec{n}'}^\dagger(t)] = \delta_{\vec{n}, \vec{n}'}, \quad [a_{\vec{n}}(t), a_{\vec{n}'}(t)] = 0.$$ (13)

The Hamiltonian, including $\lambda \phi^4$ self-interactions, is

$$H = \sum_{\vec{n}} \omega_{\vec{n}} a_{\vec{n}}^\dagger a_{\vec{n}} + \sum_{\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4} \frac{1}{4} \Lambda_{\vec{s}_{\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4}} a_{\vec{n}_1} a_{\vec{n}_2} a_{\vec{n}_3} a_{\vec{n}_4}$$ (14)
where

$$\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4} = \frac{-\lambda}{4m^2V} \delta_{\vec{n}_1+\vec{n}_2,\vec{n}_3+\vec{n}_4}. \quad \text{(15)}$$

The Kronecker-delta ensures 3-momentum conservation. When deriving Eq. \((14)\), axion number violating terms such as \(aaa, a^\dagger a^\dagger a^\dagger a\), \(a^\dagger a^\dagger a^\dagger a\) are neglected. Indeed, in lowest order they allow only processes that are forbidden by energy-momentum conservation. In higher orders they do lead to axion number violating processes but only on times scales that are vastly longer than the age of the universe.

The gravitational self-interactions of the axion fluid are described by Newtonian gravity since we only consider interactions on sub-horizon scales. The interaction Hamiltonian is

$$H_g = -\frac{G}{2} \int d^3x \, d^3x' \frac{\rho(\vec{x},t)\rho(\vec{x}',t)}{||\vec{x} - \vec{x}'||} \quad \text{(16)}$$

where \(\rho(\vec{x},t) = \frac{1}{4}(\pi^2+m^2\phi^2)\) is the axion energy density. In terms of creation and annihilation operators \([17]\)

$$H_g = \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3,\vec{n}_4} \frac{1}{4} \Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4} a_{\vec{n}_1}^\dagger a_{\vec{n}_2}^\dagger a_{\vec{n}_3} a_{\vec{n}_4} \quad \text{(17)}$$

where

$$\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4} = -\frac{4\pi Gm^2}{V} \delta_{\vec{n}_1+\vec{n}_2,\vec{n}_3+\vec{n}_4} \left( \frac{1}{|\vec{p}_{\vec{n}_1} - \vec{p}_{\vec{n}_3}|^2} + \frac{1}{|\vec{p}_{\vec{n}_1} - \vec{p}_{\vec{n}_4}|^2} \right). \quad \text{(18)}$$

\(H_g\) must be added to the RHS of Eq. \((14)\). In summary, we have found that the axion fluid is described by a set of coupled quantum harmonic oscillators. We now estimate the resulting relaxation rates. There are two different regimes of relaxation depending on the relative values of the relaxation rate \(\Gamma\) and the energy dispersion \(\delta\omega\). The condition \(\Gamma << \delta\omega\) defines the “particle kinetic regime”, whereas \(\Gamma >> \delta\omega\) defines the “condensed regime”. Most physical systems relax in the particle kinetic regime. Axions on the other hand relax in the condensed regime.
3.1 Particle kinetic regime

When $\Gamma \ll \delta \omega$, the rate of change of the occupation numbers $N_i$ ($i = 1, 2, \ldots, M$) of $M$ coupled oscillators is given by

$$\langle \dot{N}_i \rangle = \sum_{i,j,k=1}^{M} \frac{1}{2} |\Lambda_{ij}^{kl}|^2 [N_i N_j (N_i + 1) - (N_i + 1)(N_j + 1)N_i N_k] 2\pi \delta (\Omega_{ij}^{kl}) + \mathcal{O}(\Lambda^3)$$

(19)

where $\Omega_{ij}^{kl} = \omega_k + \omega_l - \omega_i - \omega_j$, and the $\Lambda_{ij}^{kl}$ are the relevant couplings, such as are given in Eqs. (15) and (18) for axions. If we substitute the couplings due to $\lambda \phi^4$ interactions, Eq. (15), and replace the sums over modes by integrals over momenta, we obtain [18, 17]

$$\langle \dot{N}_1 \rangle = \frac{1}{2\omega_1} \int \frac{d^3 p_2}{(2\pi)^3 2\omega_2} \frac{d^3 p_3}{(2\pi)^3 2\omega_3} \frac{d^3 p_4}{(2\pi)^3 2\omega_4} \lambda^2 (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4)$$

$$\times \frac{1}{2} [(N_1 + 1)(N_2 + 1)N_3 N_4 - N_1 N_2 (N_3 + 1)(N_4 + 1)] ,$$

(20)

where $N_1 \equiv N_{\vec{p}_1}$ and so forth. When the states are not highly occupied ($N \lesssim 1$), Eq. (20) implies the standard formula for the relaxation rate

$$\Gamma \sim \frac{\dot{N}}{N} \sim n\sigma \delta v$$

(21)

where $\sigma = \lambda^2 / 64\pi m^2$ is the scattering cross-section due to $\lambda \phi^4$ interactions, $n$ is the particle density and $\delta v$ is the velocity dispersion. On the other hand, when the states are highly occupied ($N \gg 1$), Eq. (20) implies

$$\Gamma \sim n\sigma \delta v \mathcal{N}.$$  

(22)

The relaxation rate is enhanced by the degeneracy factor, which is huge ($\mathcal{N} \sim 10^{61}$) in the axion case. The process of Bose-Einstein condensation occurs as a result of scatterings
\[ a(\vec{p}_1) + a(\vec{p}_2) \leftrightarrow a(\vec{p}_3) + a(\vec{p}_4) \] in which \( N_1, N_2 \) and \( N_3 \) are of order the large degeneracy factor \( \mathcal{N} \) whereas \( N_4 << \mathcal{N} \). Eq. (20) implies that, as a result of such scatterings, the occupation number of the lowest available energy state grows exponentially with the rate given in Eq. (22) [18, 16, 19].

In contrast to \( \lambda \phi^4 \) interactions, gravitational interactions are long-range. The cross-section for gravitational scattering is infinite due to the contribution from very small angle (forward) scattering. But forward scattering does not contribute to relaxation, whereas scattering through large angles does contribute. (The issue does not arise in the case of \( \lambda \phi^4 \) interactions, for which there is no peak in the differential cross-section for forward scattering and scattering is generically through large angles.) The upshot is that Eqs. (21) and (22) are still valid for estimating the relaxation rate by gravitational interactions in the particle kinetic regime provided one uses for \( \sigma \) the cross-section for large angle scattering. That cross-section is finite and equals

\[
\sigma_g \sim \frac{4G^2m^2}{(\delta v)^4}
\]  

in order of magnitude.

### 3.2 Condensed regime

When \( \Gamma >> \delta \omega \), one cannot use Eq. (19) because the derivation of that equation involves an averaging over time that is valid only when \( \Gamma << \delta \omega \). Instead we will use the equations

\[
i\dot{a}_i(t) = \omega_i a_i(t) + \sum_{i,j,k=1}^{M} \frac{1}{2} \Lambda_{kl}^{ij} a_k^\dagger a_i a_j
\]  

in order of magnitude.
which follow directly from the Hamiltonian, Eq. (14). It is convenient to define
\[ c_l(t) \equiv \alpha_l(t)e^{i\omega_l t} \]
in terms of which Eq. (24) becomes
\[ \dot{c}_l(t) = -i \sum_{i,j,k=1}^{M} \frac{1}{2} \Lambda_{kl}^{ij} c_k^\dagger c_i c_j e^{i\Omega_{ij}^{kl} t} \]  
(25)
where \( \Omega_{ij}^{kl} \equiv \omega_k + \omega_j - \omega_i - \omega_l \), as before. Further, because the occupation numbers of the occupied states are huge, we write \( c_l \) as a sum of a classical part \( C_l \) and a quantum part \( d_l \)
\[ c_l(t) = C_l(t) + d_l(t) \]  
(26)
The \( C_l \) are c-number functions of order \( \sqrt{N_l} \) describing the bulk of the axion fluid. They satisfy the equations of motion
\[ \dot{C}_l(t) = -i \sum_{i,j,k=1}^{M} \frac{1}{2} \Lambda_{kl}^{ij} C_k^* C_i C_j e^{i\Omega_{ij}^{kl} t} \]  
(27)
The \( d_l \) and \( d_l^\dagger \) are annihilation and creation operators satisfying canonical commutation relations. Quantum statistics plays the essential role in determining the outcome of relaxation to be the Bose-Einstein distribution. However, we may use classical physics to estimate the rate of relaxation. The relaxation rate is the inverse time scale over which \( C_l(t) \) changes by an amount of order \( C_l(t) \).
The sum in Eq. (27) is dominated by those states that are highly occupied. Let \( K \) be the number of such states. Using the fact that in the condensed regime \( \Omega_{ij}^{kl} t << 1 \), we may rewrite Eq. (27) as
\[ \dot{C}_l(t) \sim -i \sum_{i,j,k=1}^{K} \frac{1}{2} \Lambda_{kl}^{ij} C_k^* C_l C_j \]  
(28)
If we substitute Eq. (15) for $\lambda \phi^4$ interactions, we get

$$\dot{C}_{\vec{p}_1}(t) \sim i \frac{\lambda}{4m^2V} \sum_{\vec{p}_2, \vec{p}_3} \frac{1}{2} C_{\vec{p}_2}^* C_{\vec{p}_3} C_{\vec{p}_4}$$

(29)

where $\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$ and the sum is restricted to the highly occupied states. The sum is similar to a random walk with each step of order $\sim N^{3/2}$ and the number of steps of order $K^2$. Hence

$$\dot{C}_{\vec{p}} \sim \frac{\lambda}{4m^2V} K N^{3/2} \sim \frac{\lambda}{4m^2V} N N^{1/2}$$

(30)

where we used $K \sim N/N$. Since $C_t \sim \sqrt{N}$, the relaxation rate due to $\lambda \phi^4$ interactions in the condensed regime is [16] [17]

$$\Gamma_\lambda \sim \frac{1}{4} n \lambda m^{-2}$$

(31)

where $n = N/V$ is the number density of the particles in highly occupied states. Likewise the relaxation rate for gravitational scattering is found to be

$$\Gamma_g \sim 4\pi G n m^2 \ell^2$$

(32)

where $\ell = 1/\delta p$ is the correlation length of the particles.

The expressions estimating the relaxation rates in the condensed regime, Eqs. (31) and (32), are very different from the expression, Eq. (22), in the particle kinetic regime. In particular, in the condensed regime, the relaxation rate is first order in the coupling, whereas it is second order in the particle kinetic regime. But the expressions are compatible. At the boundary between the two regimes, where $\delta \omega \sim \Gamma$, the two estimates agree. At that boundary, up to factors of order 2 or so,

$$\delta v N \sim \delta v \frac{n}{(\delta p)^3} \sim \frac{n}{m^2 \delta \omega} \sim \frac{n}{m^2 \Gamma}$$

(33)
Substituting this into Eq. (22) yields Eq. (31). Similarly for the relaxation rate due to gravitational self-interactions.

4 Axion BEC

For a system of particles to form a BEC, four conditions must be satisfied:

1. the particles must be identical bosons,
2. their number must be conserved,
3. they must be degenerate, i.e. the average occupation number $N$ of the states that they occupy should be order 1 or larger,
4. they must thermalize.

When the four conditions are satisfied, a macroscopically large fraction of the particles go to the lowest energy available state. It may be useful to clarify the notion of lowest energy available state \[20\]. Thermalization involves interactions. By lowest energy available state we mean the lowest energy state that can be reached by the thermalizing interactions. In general the system has states of yet lower energy. For example, and at the risk of stating the obvious, when a beaker of superfluid $^4$He is sitting on a table, the condensed atoms are in their lowest energy available state. This is not their absolute lowest energy state since the energy of the condensed atoms can be lowered by placing the beaker on the floor. In the case of atoms, it is relatively clear what state the atoms condense into when BEC occurs. The case of axions is more confusing because the thermalizing interactions, both gravity and the $\lambda\phi^4$ self-interactions, are attractive and therefore cause the system to be unstable. When the system is unstable, the restriction to the lowest energy available state is especially crucial.

We saw in the first two sections that, for cold dark matter axions, the first three conditions for BEC are manifestly satisfied. In this section we show that the fourth condition is satisfied
as well \[16\] [17]. Cold axions will thermalize if their relaxation time $\tau$ is shorter than the age $t$ of the universe, or equivalently if their relaxation rate $\Gamma \equiv 1/\tau$ is greater than the Hubble expansion rate $H \sim 1/t$.

The cold axion energy dispersion is

$$\delta \omega(t) \simeq \frac{(\delta p(t))^2}{2m(t)}.$$  \hspace{1cm} (34)

In view of Eqs. (31) and (30), $\delta \omega(t) \sim 1/t$. If axions thermalize at time $t_1$, we have $\Gamma(t_1) > 1/t_1$ and therefore the thermalization is in the condensed regime or at the border between the particle kinetic and condensed regimes. After time $t_1$, $\delta \omega(t) < 1/t$ since $m(t)$ increases sharply for a period after $t_1$ whereas $(\delta p(t))^2 \propto R(t)^{-2} \propto 1/t$, since $R(t) \propto \sqrt{t}$ in the radiation dominated era. So after $t_1$, axions can only thermalize in the condensed regime.

To see whether the axions thermalize by $\lambda \phi^4$ self-interactions at time $t_1$, we may use either Eq. (31) or (22). Both estimates yield $\Gamma_\lambda(t_1) \sim H(t_1)$ indicating that the axions thermalize at time $t_1$ by $\lambda \phi^4$ self-interactions but only barely so. After $t_1$ we must use Eq. (31). It informs us that $\Gamma_\lambda(t)/H(t) \propto R(t)^{-3}t \propto t^{-\frac{3}{2}}$, i.e. that even if axions thermalize at time $t_1$ they stop doing so shortly thereafter. Nothing much changes as a result of this brief epoch of thermalization since in either case, whether it occurs or not, the correlation length $\ell(t) \equiv 1/\delta p(t) \sim t_1 R(t)/R(t_1)$.

To see whether the axions thermalize by gravitational self-interactions we use Eq. (32). It implies

$$\frac{\Gamma_g(t)}{H(t)} \sim 8\pi G n m^2 \ell^2 t \sim 5 \cdot 10^{-7} \frac{R(t_1)}{R(t)} \frac{t}{t_1} X \left( \frac{f_a}{10^{12}\text{GeV}} \right)^{\frac{3}{2}}$$ \hspace{1cm} (35)

once the axion mass has reached its zero temperature value, shortly after $t_1$. Gravitational self-interactions are too slow to cause thermalization of cold axions near the QCD phase transition but, because $\Gamma_g/H \propto R^{-1}(t)t \propto R(t)$, they do cause the cold axions to thermalize later on. The RHS of Eq. (35) reaches one at a time $t_{\text{BEC}}$ when the photon temperature is
of order

$$T_{\text{BEC}} \sim 500 \text{ eV} \times \left( \frac{f_a}{10^{12} \text{ GeV}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (36)

The axions thermalize then and form a BEC as a result of their gravitational self-interactions. The whole idea may seem far-fetched because we are used to think that gravitational interactions among particles are negligible. The axion case is special, however, because almost all particles are in a small number of states with very long de Broglie wavelength, and gravity is long range.

Systems dominated by gravitational self-interactions are inherently unstable. In this regard the axion BEC differs from the BECs that occur in superfluid $^4$He and dilute gases. The axion fluid is subject to the Jeans gravitational instability and this is so whether the axion fluid is a BEC or not [16]. The Jeans instability causes density perturbations to grow at a rate of order the Hubble rate $H(t)$, i.e. on a time scale of order the age of the universe at the moment under consideration. Each mode of the axion fluid is Jeans unstable. We showed however that, after $t_{\text{BEC}}$, the thermalization rate is faster than the Hubble rate. The rate at which quanta of the axion field jump between modes is faster than the rate at which the Jeans instability develops. So the modes are essentially frozen on the time scale over which the axions thermalize.

Finally, we comment on a misapprehension that appears in the literature. The axions do not condense in the lowest momentum mode $\vec{p} = 0$. Condensation into the $\vec{p} = 0$ state would mean that the fluid becomes homogeneous and at rest. Of course this is not what happens in the axion case since the axion fluid is Jeans unstable. Despite a common misconception, it is not a rule of BEC that the particles condense into the $\vec{p} = 0$ state. The rule instead is that they condense into the lowest energy available state, as defined earlier. Only in empty space, and only if the total linear momentum and the total angular momentum of the particles are zero, is the lowest energy state a state of zero momentum. It should be obvious that the
particles do not condense in the $\vec{p} = 0$ state if they are moving or rotating. Nonetheless, Bose-Einstein condensation occurs.

5 Observational implications

For a long time, it was thought that axions and the other proposed forms of cold dark matter behave in the same way on astronomical scales and are therefore indistinguishable by observation. Axion BEC changed that. On time scales longer than their thermalization time scale $\tau$, axions almost all go to the lowest energy state available to them. The other dark matter candidates, such as weakly interacting massive particles (WIMPs) and sterile neutrinos, do not do this. It was shown in Ref. [16] that, on all scales of observational interest, density perturbations in axion BEC behave in exactly the same way as those in ordinary cold dark matter provided the density perturbations are within the horizon and in the linear regime. On the other hand, when density perturbations enter the horizon, or in second order of perturbation theory, axions generally behave differently from ordinary cold dark matter because the axions rethermalize so that the state most axions are in tracks the lowest energy available state.

A distinction between axions and the other forms of cold dark matter arises in second order of perturbation theory, in the context of the tidal torquing of galactic halos. Tidal torquing is the mechanism by which galaxies acquire angular momentum. Before they fall onto a galactic halo, the axions thermalize sufficiently fast that the axions that are about to fall into a particular galactic gravitational potential well go to their lowest energy available state consistent with the total angular momentum they acquired from nearby protogalaxies through tidal torquing [20]. That state is a state of net overall rotation, more precisely a state of rigid rotation on the turnaround sphere. In contrast, ordinary cold dark matter falls into a galactic gravitational potential well with an irrotational velocity field [21]. The inner
caustics are different in the two cases. In the case of net overall rotation, the inner caustics are rings [22] whose cross-section is a section of the elliptic umbilic $D_{-4}$ catastrophe [23], called caustic rings for short. If the velocity field of the infalling particles is irrotational, the inner caustics have a ‘tent-like’ structure which is described in detail in ref. [21] and which is quite distinct from caustic rings. Evidence was found for caustic rings. A summary of the evidence is given in ref. [24]. Furthermore, it was shown in ref. [25] that the assumption that the dark matter is axions explains not only the existence of caustic rings but also their detailed properties, in particular the pattern of caustic ring radii and their overall size.

Vortices appear in the axion BEC as it is spun up by tidal torquing. The vortices in the axion BEC are attractive, unlike those in superfluid $^4$He and dilute gases. Hence a large fraction of the vortices in the axion BEC join into a single big vortex along the rotation axis of the galaxy [20]. Baryons and ordinary cold dark matter particles that may be present, such as WIMPs and/or sterile neutrinos, are entrained by the axion BEC and acquire the same velocity distribution. The resulting baryonic angular momentum distribution gives a good qualitative fit [20] to the angular momentum distributions observed in dwarf galaxies [26]. This resolves a long-standing problem with ordinary cold dark matter called the ”galactic angular momentum problem” [27, 28]. A minimum fraction of cold dark matter must be axions to explain the data. That fraction is of order 35% [20].

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