KATO SQUARE ROOT PROBLEM WITH UNBOUNDED LEADING COEFFICIENTS

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Abstract. We prove the Kato conjecture for elliptic operators, \( L = -\nabla \cdot ((A + D)\nabla) \), with \( A \) a complex measurable bounded coercive matrix and \( D \) a measurable real-valued skew-symmetric matrix in \( BMO(\mathbb{R}^n) \); i.e., the domain of \( \sqrt{L} \) is the Sobolev space \( H^1(\mathbb{R}^n) \) in any dimension, with the estimate \( \|\sqrt{L}f\|_2 \lesssim \|\nabla f\|_2 \).

1. Introduction

Let \( A = (a_{ij}) \) be an \( n \times n \) matrix of complex, \( L^\infty \) coefficients defined on \( \mathbb{R}^n \) and satisfying the ellipticity or accretivity condition

\[
\lambda |\xi|^2 \leq \Re\langle A\xi, \xi \rangle \equiv \Re \sum_{i,j} a_{ij}(x) \xi_j \overline{\xi_i}, \quad \|A\|_\infty \leq \lambda^{-1},
\]

for \( \xi \in \mathbb{C}^n \) and for some \( 0 < \lambda \leq 1 \). We consider a divergence form operator

\[
Lu \equiv -\nabla \cdot (A(x)\nabla u).
\]

The accretivity condition enables one to define a square root \( \sqrt{L} \) and a fundamental issue was “solve the square root problem”, i.e. to establish the estimate

\[
\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \leq N\|\nabla f\|_{L^2(\mathbb{R}^n)},
\]

with \( N \) depending on \( n \) and \( \lambda \). The latter estimate is connected with the question of the analyticity of the mapping \( A \to \sqrt{L} \), which in turn has applications to the perturbation theory for certain classes of hyperbolic equations. We remark that (1.3) is equivalent to the opposite inequality for the square root of the adjoint operator \( L^* \).

In [21, 22] Kato conjectured that an abstract version of (1.3) might hold, for “regularly accretive operators”. A counter-example to this abstract conjecture was obtained by McIntosh [26], who then reformulated the conjecture in the following form, bearing in mind that Kato’s interest in the problem had been motivated by the special case of elliptic differential operators:

The estimate (1.3) holds for \( L \) defined as in (1.2), for any \( L^\infty \), \( n \times n \) matrix \( A \) with complex entries, for which (1.1) holds.
To establish the validity of this conjecture became known as the Kato Problem or square root problem. In 1982 it was solved in one dimension \( \text{[9]} \), where it is essentially equivalent to the problem of proving the \( L^2 \) boundedness of the Cauchy integral operator on Lipschitz curves \( \text{[23]} \).

For \( n > 1 \), a restricted version of the conjecture, also essentially posed by Kato in \( \text{[22]} \), was proved by P. Auscher, S. Hofmann, J.L. Lewis and P. Tchamitchian in \( \text{[4]} \). The restricted version treated the case that \( A \) is close in the \( L^\infty \) norm to a real symmetric matrix of bounded measurable coefficients. It is this version that yields the perturbation results for hyperbolic equations alluded to above \( \text{[27]} \).

Prior to the latter result, the conjecture was proved in higher dimensions when \( \| A - I \|_{L^\infty(\mathbb{R}^n)} \leq \epsilon(n) \; \text{[3, 13, 14, 20]} \). \( \text{[12]} \) gave a different proof using the \( T(1) \) theorem. Sharper bounds for the constant \( \epsilon(n) \) on the order of \( n^{-1/2} \) were obtained in \( \text{[20]} \). In \( \text{[6]} \) it was proved when \( \| A - I \|_{\text{BMO}(\mathbb{R}^n)} \) is small.

Later, the validity of the conjecture was established when the heat kernel of the operator \( L \) satisfies the “Gaussian” property, first in 2 dimensions \( \text{[17]} \), and then in all dimensions \( \text{[18]} \); i.e., let \( G(x, y, t) \) denote the kernel of the operator \( e^{-tL} \), we say that \( L \) satisfies the Gaussian property, if there are positive constants \( 0 < \alpha \leq 1 \) and \( N \) such that

\[
(i) \quad |G(x, y, t)| \leq N t^{-\frac{n}{2}} e^{-|x-y|^2/4Nt} \\
(ii) \quad |G(x + h, y, t) - G(x, y, t)| + |G(x, y + h, t) - G(x, y, t)| \\
\quad \leq N \left( |h|/\sqrt{t} \right) t^{-\frac{n}{2}} e^{-|x-y|^2/4Nt},
\]

where the latter holds when \( t > 0 \) and either, \( |h| \leq t \), or \( |h| \leq |x - y|/2 \).

The Gaussian property holds when \( A \) is real-valued by results of Aronson \( \text{[2]} \) and in some cases for complex \( A \): in two dimensions from \( \text{[5]} \) and for perturbations of real operators \( \text{[3]} \). Hence, \( \text{[17], 18} \) solve the conjecture in the former two dimensional cases or the latter \( n \)-dimensional cases.

Finally, the conjecture was solved for general complex, bounded and coercive matrices \( A \) satisfying \( \text{[1.1]} \) in \( \text{[7]} \).

The purpose of this note is to show that minor modifications of the reasoning in \( \text{[7]} \) also yield the following extension. Let

\( H^1(\mathbb{R}^n) := \{ f \in L^2(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n) \} \)

denote the usual Sobolev space, and \( \dot{H}^1(\mathbb{R}^n) \) its homogeneous version; i.e., \( \dot{H}^1 \) is the closure of \( C_0^\infty(\mathbb{R}^n) \) with respect to the seminorm \( \| f \|_{\dot{H}^1(\mathbb{R}^n)} := \| \nabla f \|_{L^2(\mathbb{R}^n)} \).

**Theorem 1.** For any operator

\[
L = -\nabla \cdot ((A(x) + D(x))\nabla) 
\]

with \( A \) a bounded complex-valued coercive matrix satisfying \( \text{[1.1]} \) and \( D \) a real-valued skew-symmetric matrix in \( \mathbb{R}^n \) with entries in \( \text{BMO}(\mathbb{R}^n) \) satisfying \( \text{[1.5]} \), the domain of \( \sqrt{L} \) contains \( H^1(\mathbb{R}^n) \) and \( \text{[1.3]} \) holds over \( \dot{H}^1(\mathbb{R}^n) \).

We remark in passing that the Gaussian property has been shown to hold with \( \alpha = \alpha(\lambda, n) \) and \( N = N(\lambda, n) \), when

\[
Lu \equiv -\nabla \cdot ((A(x) + D(x))\nabla u),
\]
with $A$ a real-valued, bounded symmetric and coercive matrix satisfying (1.1) and $D = (d_{ij}(x))$ a real-valued skew-symmetric $BMO(\mathbb{R}^n)$ matrix \cite{31, 28} with
\begin{equation}
\|D\|_{BMO(\mathbb{R}^n)} \leq \lambda^{-1}.
\end{equation}
The arguments of \cite{18} could be modified to treat this restricted case. On the other hand, as in \cite{7}, we do not require the Gaussian property in the proof of Theorem 1 in the present paper.

We recall that a function $\beta : \mathbb{R}^n \to \mathbb{C}$ is in $BMO(\mathbb{R}^n)$ or has bounded mean oscillation \cite{19}, when it is is locally integrable and
\begin{equation}
\|\beta\|_{BMO(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \int_{Q} |\beta - m_Q(\beta)| \, dx < +\infty,
\end{equation}
where $Q$ ranges over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axis and
\begin{equation}
m_Q(\beta) = \int_{Q} \beta \, dx.
\end{equation}
If one defines other norms by
\begin{equation}
\|\beta\|_{BMO(\mathbb{R}^n)_p} = \sup_{Q \subset \mathbb{R}^n} \left( \int_{Q} |\beta - m_Q(\beta)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 < p < \infty,
\end{equation}
the John-Nirenberg inequality \cite{19} implies that all the $BMO(\mathbb{R}^n)_p$-norms are equivalent, when $1 \leq p < \infty$. Finally, $BMO(\mathbb{R}^n)$ is the dual of $H^1_{at}(\mathbb{R}^n)$, the real Hardy space in $\mathbb{R}^n$, where $f$ in $L^1(\mathbb{R}^n)$ is in $H^1_{at}(\mathbb{R}^n)$, when
\begin{equation}
\sup_{\epsilon > 0} |\theta_\epsilon * f|
\end{equation}
is in $L^1(\mathbb{R}^n)$, where $\theta_\epsilon = \epsilon^{-n} \theta(x/\epsilon)$ and $\theta$ is any smooth non-negative compactly supported mollifier with integral equal to 1 \cite{15}. In particular, when $\beta$ is in $BMO(\mathbb{R}^n)$ and $f$ is in $H^1_{at}(\mathbb{R}^n)$, the principal value of the integral of $\beta f$ is well defined \cite{24} and
\begin{equation}
\left| \int_{\mathbb{R}^n} \beta f \, dx \right| \leq N(n) \|\beta\|_{BMO(\mathbb{R}^n)} \|f\|_{H^1_{at}(\mathbb{R}^n)},
\end{equation}
with
\begin{equation}
\|f\|_{H^1_{at}(\mathbb{R}^n)} = \|\sup_{\epsilon > 0} |\theta_\epsilon * f|\|_{L^1(\mathbb{R}^n)}.
\end{equation}

Following \cite{28}, when
\begin{equation}
L = -\nabla \cdot (A \nabla) - b \cdot \nabla
\end{equation}
with $A$ a complex-valued bounded matrix verifying (1.1) and $b$ a real-valued divergence-free vector field with
\begin{equation}
\sup_{x \in \mathbb{R}^n, r > 0} r \int_{B_r(x)} |b| \, dx < \infty,
\end{equation}
the matrix
\begin{equation}
D = \Delta^{-1}(\nabla b - \nabla b^\top)
\end{equation}
is skew-symmetric and real-valued matrix with
\begin{equation}
\|D\|_{BMO(\mathbb{R}^n)} \leq N(n) \sup_{x \in \mathbb{R}^n, r > 0} r \int_{B_r(x)} |b| \, dx
\end{equation}
and $L$ can be written in the form $\{1.4\}$. Thus, according with Theorem $\{1.1\}$ the domain of the square root of the accretive operator $\{1.6\}$ contains $H^1(\mathbb{R}^n)$, when $A$ is as above, $b$ is real-valued and $\{1.7\}$ holds.

The proof of Theorem $\{1.1\}$ requires simple modifications to the original reasonings in $\{7\}$ but most importantly the following two compensated compactness-type results.

**Proposition 1.** When $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ are in $H^1(\mathbb{R}^n)$ and $i, j \in \{1, \ldots, n\}$, the functions $\partial_i f \partial_j g - \partial_j f \partial_i g$ and $f \partial_i f$ are in $H^1(\mathbb{R}^n)$ at $(\mathbb{R}^n)$ and there is $N = N(n)$ such that

\[(1.8) \quad \|\partial_i f \partial_j g - \partial_j f \partial_i g\|_{H^1(\mathbb{R}^n)} \leq N \|\nabla f\|_{L^2(\mathbb{R}^n)} \|\nabla g\|_{L^2(\mathbb{R}^n)}\]

and

\[(1.9) \quad \|f \partial_i f\|_{H^1(\mathbb{R}^n)} \leq N \|f\|_{L^2(\mathbb{R}^n)} \|\nabla f\|_{L^2(\mathbb{R}^n)}.
\]

The reader can find the proofs of $\{1.8\}$ and $\{1.9\}$ in $\{10\}$ and $\{31\}$ respectively.

In the next section we explain the minor modifications one must make to the reasonings in the proof of the Conjecture $1.4$ in $\{7\}$, to derive its extension in Theorem $\{1.1\}$. Throughout the next pages $N$ denotes a constant which depends at most on $\lambda$ and $n$, $B$, an open ball in $\mathbb{R}^n$ of radius $r > 0$, $Q$ a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axis, $x_Q$ its center and $\delta(Q)$ its side length.

2. **Proof of Theorem $\{1.1\}$**

Setting

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g}(x) \, dx,$$

for $f, g \in L^2(\mathbb{R}^n)$, it follows from Hölder’s inequality, $\{1.1\}$, $\{1.5\}$, the identity

\[(2.1) \quad \int_{\mathbb{R}^n} D(x) \nabla u \cdot \nabla v \, dx = \frac{1}{2} \int_{\mathbb{R}^n} d_{ij}(x) (\partial_i u \partial_j v - \partial_j u \partial_i v) \, dx,
\]

$\{1.8\}$ and the fact that

$$\Re D(x) \xi \cdot \bar{\xi} = 0, \text{ when } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{C}^n,$$

that the sesquilinear form

$$\mathcal{B}(u, v) = \int_{\mathbb{R}^n} (A(x) + D(x)) \nabla u \cdot \nabla v \, dx,$$

associated to the unbounded operator $L$ in $\{1.4\}$, with domain

$$\mathcal{D}(L) = \{u \in H^1(\mathbb{R}^n) : Lu \in L^2(\mathbb{R}^n)\}$$

and by the relation

$$\langle Lu, v \rangle = \mathcal{B}(u, v), \text{ when } u \in \mathcal{D}(L) \text{ and } v \in H^1(\mathbb{R}^n),$$

is bounded and coercive on $H^1(\mathbb{R}^n)$ with

$$|\mathcal{B}(u, v)| \leq N \|\nabla u\|_{L^2(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)}$$

and

$$\Re \mathcal{B}(u, u) \geq \lambda \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \text{ when } u \in H^1(\mathbb{R}^n),$$

when the matrices $A$ and $D$ satisfy the conditions in Theorem $\{1.1\}$. $L$ is an accretive unbounded operator,

$$\Re \langle Lu, u \rangle = \Re \mathcal{B}(u, u) \geq 0, \text{ when } u \in \mathcal{D}(L);$$
\[ L \text{ is also m-accretive [21, p. 279] and the operators} \]
\[ (2.2) \quad (1 + t^2 L)^{-1}, \quad t\nabla (1 + t^2 L)^{-1}, \quad (1 + t^2 L)^{-1} t \nabla \cdot \quad \text{and} \quad t^2 \nabla (1 + t^2 L)^{-1} \nabla \cdot \]
are uniformly \( L^2(\mathbb{R}^n) \)-bounded with bounds depending \( n \) and \( \lambda \) for all \( t > 0 \), where
\[ u = (1 + t^2 L)^{-1} f \quad \text{and} \quad w = (1 + t^2 L)^{-1} t \nabla \cdot f \]
are the unique Lax-Milgram weak solutions in \( H^1(\mathbb{R}^n) \) satisfying respectively
\[ (2.3) \quad \int_{\mathbb{R}^n} w \nabla \cdot (A + D) \nabla u = -\int_{\mathbb{R}^n} f \nabla \pi \]
and
\[ (2.4) \quad \int_{\mathbb{R}^n} u \nabla \pi \] for all \( v \) in \( C_0^\infty(\mathbb{R}^n) \), when \( f \) and \( \pi \) are in \( L^2(\mathbb{R}^n) \). Similar bounds hold when \( L \) is replaced above by the adjoint of \( L \)
\[ L^* = -\nabla \cdot ((A^* - D) \nabla), \]
where \( A^* \) denotes the transpose conjugate matrix of \( A \).

Following T. Kato [21, p. 281], \( L \) has a unique m-accretive square root \( \sqrt{L} \) given by
\[ \sqrt{L} f = \frac{1}{\pi} \int_0^{+\infty} \lambda^{-\frac{1}{2}} (\lambda + L)^{-1} Lf \, d\lambda, \quad \text{when} \ f \in \mathcal{D}(L). \]
The identities
\[ (\lambda + L)^{-k-1} = -\frac{1}{k} \frac{d}{d\lambda} (\lambda + L)^{-k}, \quad \lambda > 0, \]
for \( k = 1, 2 \), integration by parts and the change of variables \( \lambda = 1/t^2 \), show that
\[ \sqrt{L} f = \frac{2}{\pi} \int_0^{+\infty} (1 + t^2 L)^{-3} t^3 L^2 f \frac{dt}{t}, \quad \text{when} \ f \in \mathcal{D}(L^2) = (1 + L)^{-1} \mathcal{D}(L) \]
and as in [21], we use the latter resolution formula for \( \sqrt{L} \) to prove Theorem 1.

As is [21], Theorem 1 follows once (1.3) is derived for \( f \) in a dense subspace of \( H^1(\mathbb{R}^n) \), as \( \mathcal{D}(L^2) \) (here, \( L^2 = L \circ L \)); because then (2.7), (2.12) and the closedness of \( \sqrt{L} \) as an unbounded operator over \( L^2(\mathbb{R}^n) \), show that \( H^1(\mathbb{R}^n) \) is contained the domain of \( \sqrt{L} \) and (1.3) holds for \( f \) in \( H^1(\mathbb{R}^n) \). Finally, \( H^1(\mathbb{R}^n) \) is dense in \( H^1(\mathbb{R}^n) \) and \( \sqrt{L} \) can be uniquely extended by density to \( H^1(\mathbb{R}^n) \).

We have
\[ |\langle \sqrt{L} f, g \rangle| = \left| \int_0^{+\infty} \langle (1 + t^2 L)^{-1} t Lf, (1 + t^2 L^*)^{-2} t^2 L^* g \rangle \frac{dt}{t} \right| \]
\[ \leq \left( \int_0^{+\infty} \| (1 + t^2 L)^{-1} t Lf \|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right)^{\frac{1}{2}} \]
\[ \times \left( \int_0^{+\infty} \| (1 + t^2 L^*)^{-2} t^2 L^* g \|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right)^{\frac{1}{2}} \]
and
\[ (2.5) \quad \left( \int_0^{+\infty} \| (1 + t^2 L^*)^{-2} t^2 L^* g \|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right)^{\frac{1}{2}} \leq N \| g \|_{L^2(\mathbb{R}^n)}. \]
To verify the later inequality, define
\[ S_t = (1 + t^2L)^{-2} t^2L = (1 + t^2L)^{-1} - (1 + t^2L)^{-2}. \]

By duality
\[
\int_0^{+\infty} \| (1 + t^2L^*)^{-2} t^2L^*g \|_{L^2(\mathbb{R}^n)} dt = (\int_0^{+\infty} S_t S_t^* g dt, g)
\]
\[
\leq \| S_t S_t^* g \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}
\]
and because the operators \(S_t\) are uniformly bounded in \(\mathcal{B}(L^2(\mathbb{R}^n))\) and \(\| S_t S_t^* \|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq N \min \{t/s, s/t\}\), when \(s, t > 0\), Cotlar’s Lemma for integrals [11] [29] and (2.6) imply that
\[
\| \int_0^{+\infty} S_t S_t^* g dt \|_{L^2(\mathbb{R}^n)} \leq N \| g \|_{L^2(\mathbb{R}^n)},
\]
which gives (2.5). Thus,
\[
\| \sqrt{t} f \|_{L^2(\mathbb{R}^n)} \leq N \left( \int_0^{+\infty} \| (1 + t^2L)^{-1} tL f \|_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}},
\]

provided that (2.6) holds. Write then for \(t, s > 0\),
\[
\langle S_t^* S_s f, g \rangle = \langle (1 + s^2L)^{-2} s^2 Lf, (1 + t^2L)^{-2} t^2 Lg \rangle
\]
\[
= (s^2L (1 + s^2L)^{-2} f, (1 + t^2L)^{-1} g - (1 + t^2L)^{-2} g)
\]
\[
= -\frac{s}{t} \langle (A + D) s \nabla (1 + s^2L)^{-2} f, t \nabla (1 + t^2L)^{-1} g - t \nabla \cdot (1 + t^2L)^{-2} g \rangle
\]
\[
= -\frac{s}{t} \langle s \nabla (1 + s^2L)^{-1} f - s \nabla \cdot (1 + s^2L)^{-2} f, (A + D) t \nabla (1 + t^2L)^{-2} g \rangle
\]

and use Hölder’s inequality, (2.1), (2.2), and (1.8) to derive (2.6), from the previous identities.

The next goal is to show that the operator
\[
\mathcal{J}_t = (1 + t^2L)^{-1} t^2L = I - (1 + t^2L)^{-1}
\]
has Gaffney bounds and a well defined action over \(L^\infty(\mathbb{R}^n)\) and the space of Lipschitz functions over \(\mathbb{R}^n\). To show it we prove first the following Lemma.

**Lemma 1.** There are \(\theta = \theta(\lambda, n)\) and \(N\) such that the inequalities
\[
\| e^{x\xi/t} (1 + t^2L)^{-1} f \|_{L^2(\mathbb{R}^n)} + \| e^{x\xi/t} t \nabla (1 + t^2L)^{-1} f \|_{L^2(\mathbb{R}^n)} \leq N \| e^{x\xi/t} f \|_{L^2(\mathbb{R}^n)},
\]
\[
\| e^{x\xi/t} (1 + t^2L)^{-1} t \nabla \cdot f \|_{L^2(\mathbb{R}^n)} + \| e^{x\xi/t} t^2 \nabla (1 + t^2L)^{-1} \nabla \cdot f \|_{L^2(\mathbb{R}^n)} \leq N \| e^{x\xi/t} f \|_{L^2(\mathbb{R}^n)},
\]
hold when \(\xi\) is in \(\mathbb{R}^n\) and \(|\xi| \leq \theta\).

**Proof.** We first prove the Lemma when the domain of \(L\) is replaced by
\[
\mathcal{D}(L) = \{ f \in H^1_0(\Omega) : Lf \in L^2(\Omega) \},
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $L^2(\mathbb{R}^n)$ is replaced by $L^2(\Omega)$ in Lemma 1. In that case, when $f$ is in $L^2(\Omega)$, $u = (1 + t^2 L)^{-1} f$ is the unique Lax-Milgram weak solution in $H^1_0(\Omega)$, which satisfies

\begin{equation}
(2.9) \quad \int_\Omega u + t^2 (A + D) \nabla u \cdot \nabla v \ dx = \int_\Omega f v \ dx,
\end{equation}

for all $v$ in $H^1_0(\Omega)$. Take then $v = e^{2x \xi/t} u$ in (2.9) to find that

\begin{equation}
\int_\Omega e^{2x \xi/t} \left[ |u|^2 + t^2 (A + D) \nabla u \cdot (\nabla u + 2(\xi/t) u) \right] \ dx = \int_\Omega e^{2x \xi/t} f u \ dx.
\end{equation}

Taking real parts, we get

\begin{align*}
&\|e^{x \xi/t} u\|_{L^2(\Omega)}^2 + \lambda \|e^{x \xi/t} t \nabla u\|_{L^2(\Omega)}^2 \leq \|e^{x \xi/t} f\|_{L^2(\Omega)} \|e^{x \xi/t} u\|_{L^2(\Omega)} \\
&+ \lambda^{-1} |\xi| \|e^{x \xi/t} t \nabla u\|_{L^2(\Omega)} \|e^{x \xi/t} u\|_{L^2(\Omega)} - t \int_\Omega \nabla \left( |e^{x \xi/t} u|^2 \right) \cdot \xi \ dx.
\end{align*}

According with (1.9) and because $e^{x \xi/t} u$ is in $H^1_0(\Omega) \subset H^1(\mathbb{R}^n)$, the absolute value of the last integral above is bounded by

\begin{equation}
\lambda^{-1} |\xi| \|e^{x \xi/t} u\|_{L^2(\Omega)} \|e^{x \xi/t} u\|_{L^2(\Omega)} + |\xi| \|e^{x \xi/t} u\|_{L^2(\Omega)}^2
\end{equation}

and the inequality

\begin{equation}
(2.10) \quad \|e^{x \xi/t} u\|_{L^2(\Omega)}^2 + \|e^{x \xi/t} t \nabla u\|_{L^2(\Omega)}^2 \leq N \|e^{x \xi/t} f\|_{L^2(\Omega)},
\end{equation}

follows, when $|\xi| \leq \theta, \xi$ is in $R^n$ and $\theta$ is sufficiently small.

For $f$ in $C^\infty_c(\mathbb{R}^n)$ with the support of $f$ contained in $B_R$, let $u_R$ denote the Lax-Milgram weak solution to (2.9), when $\Omega = B_R$. Because $D \nabla \varphi$ is in $L^2(\mathbb{R}^n)$, when $\varphi$ is in $C^\infty_c(\mathbb{R}^n)$ and the bounds that we have for $u_R$ and $\nabla u_R$ are independent of $R \geq 1$, we can derive that $u_R$ converges to $u$ in $L^2_{loc}(\mathbb{R}^n)$ and $\nabla u_R$ converges weakly to $\nabla u$ in $L^2_{loc}(\mathbb{R}^n)$, where now $u = (1 + t^2 L)^{-1} f$ is as in Lemma 1. The first part of the Lemma now follows from (2.10) and the local weak convergences of $e^{x \xi/t} u_R$ and $e^{x \xi/t} \nabla u_R$ to $e^{x \xi/t} u$ and $e^{x \xi/t} \nabla u$, when $R$ tends to infinity.

The second part of the Lemma follows after replacing (2.10) by (2.14) and taking $v = e^{2x \xi/t} w$.

\begin{lemma}
There is $N$ such that the following inequalities hold for all cubes $Q$ in $\mathbb{R}^n$ with side length $\delta(Q)$ and $t > 0$

\begin{align*}
&\| (1 + t^2 L)^{-1} f \|_{L^2(Q)} + \| t \nabla (1 + t^2 L)^{-1} f \|_{L^2(Q)} \leq Ne^{-2^k \delta(Q)/Nt} \| f \|_{L^2(\mathbb{R}^n)}, \\
&\| (1 + t^2 L)^{-1} t \nabla \cdot f \|_{L^2(Q)} + \| t^2 \nabla (1 + t^2 L)^{-1} \nabla \cdot f \|_{L^2(Q)} \leq Ne^{-2^k \delta(Q)/Nt} \| f \|_{L^2(\mathbb{R}^n)},
\end{align*}

when the supports of $f$ and $f$ are contained in $2^{k+1} Q \setminus 2^k Q$ and $k \geq 1$.
\end{lemma}

\begin{proof}
Without loss of generality we may assume that the cube $Q$ is centered at the origin and $2^k Q = \{ x \in \mathbb{R}^n : \| x \|_\infty \leq 2^{k-1} \delta(Q) \}$, when $k \geq 1$. Assume then that $f$ is supported in $2^{k+1} Q \setminus 2^k Q$ and write $\mathbb{R}^n \setminus \{ 0 \}$ as the union of the sets, $\bigcup_{i=1}^n A_i \cup B_i$, where

$A_i = \{ x \in \mathbb{R}^n : \| x \|_\infty = x_i \}$ and $B_i = \{ x \in \mathbb{R}^n : \| x \|_\infty = -x_i \}$, $i = 1, \ldots, n$.

Then,

\begin{equation}
(1 + t^2 L)^{-1} f = \sum_{i=1}^n (1 + t^2 L)^{-1} f(x_{A_i}) + (1 + t^2 L)^{-1} f(x_{B_i})
\end{equation}

\end{proof}
and we show that the first inequality in the Lemma holds for each of these parts of \((1 + t^2L)^{-1} f\). To get it for \((1 + t^2L)^{-1} (fF_{A1})\), apply Lemma 1 to \(fF_{A1}\) with \(\xi = -\theta e_1\), \(e_1 = (1, 0, \ldots, 0)\) and observe that \(e^{-\theta e_1/t} \geq e^{-\theta^2 R^2/(2t)}\) inside \(Q\) and 
\[e^{-\theta e_1/t} \leq e^{-\theta e^2 R^2/(2t)}\] inside \(A_1 \cap 2^k Q \setminus 2^k Q\).

The other inequalities in Lemma 2 follow in the same way from Lemma 1. \(\square\)

For \(f\) in \(L^\infty(\mathbb{R}^n)\), define
\[
(1 + t^2L)^{-1} f = \lim_{R \to +\infty} (1 + t^2L)^{-1} (fF_{B_R(x_0)})
\]
where \(x_0\) is any point in \(\mathbb{R}^n\) and the limit is taken in the \(L^2_{loc}(\mathbb{R}^n)\)-sense. The limit is well defined due to the Gaffney bounds in Lemma 2 for \(x_1\) is any other point in \(\mathbb{R}^n\), the symmetric difference between \(B_R(x_0)\) and \(B_R(x_1)\) is contained in an annulus \(B_{2R} \setminus B_{\frac{R}{2}}\) for \(R\) sufficiently large and
\[
\| (1 + t^2L)^{-1} (fF_{B_R(x_0)}) - (1 + t^2L)^{-1} (fF_{B_R(x_1)}) \|_{L^2(B_{\frac{R}{2}})} \leq N R^\frac{n}{2} e^{-R/Nt} \|f\|_{L^\infty(\mathbb{R}^n)}.
\]

Also, after writing
\[
\chi_{B_{2R} \setminus B_{R_1}} = \sum_{i=0}^{l} \chi_{B_{2^i R_1} \setminus B_{2^{i+1} R_1}},
\]
when \(2R < R_1 < 2^{l+1} R_1 < R_2 \leq 2^{l+2} R_1\), it follows from Lemma 2 that
\[
\| (1 + t^2L)^{-1} (fF_{B_{R_2}}) - (1 + t^2L)^{-1} (fF_{B_{R_1}}) \|_{L^2(B_{R_1})} \leq N t^{l+1} R_1^{-1} \|f\|_{L^\infty(\mathbb{R}^n)},
\]
which shows that \((1 + t^2L)^{-1} (fF_{B_R})\) is a Cauchy sequence in \(L^2_{loc}(\mathbb{R}^n)\), when \(f\) is in \(L^\infty(\mathbb{R}^n)\). Also, the Gaffney control that we have in Lemma 2 over the operator
\[
t \nabla (1 + t^2L)^{-1}
\]
shows with similar reasonings that for \(f\) in \(L^\infty(\mathbb{R}^n)\), \(u = (1 + t^2L)^{-1} f\) is a weak \(H^1_{loc}(\mathbb{R}^n)\) solution over \(\mathbb{R}^n\) to \(u + t^2L u = f\).

In particular, \((1 + t^2L)^{-1} 1 = 1\) and \(\nabla (1 + t^2L)^{-1} 1 = 0\) in the above sense, because if \(\eta_R(x) = \eta(x/R)\), with \(\eta\) in \(C^\infty_0(\mathbb{R}^n)\), \(\eta = 1\) in \(B_1\) and \(\eta = 0\) outside \(B_2\), \(u_R = (1 + t^2L)^{-1} (\eta_R)\) satisfies
\[
uu_R - \nabla(u_R - \eta_R) = -t^2L \eta_R.
\]
At the same time, the skew-symmetry of \(D\) implies that in the sense of distributions
\[-t^2L \eta_R = t^2 \nabla \cdot (\mathbf{A} + D - m_{B_{2R}}(D)) \nabla \eta_R), \quad m_{B_{2R}}(D) = \int_{B_{2R}} D \, dx.
\]
Then, the second inequality in Lemma 2 gives
\[
\|u_R - 1\|_{L^2(B_{\frac{R}{2}})} + \|t \nabla u_R\|_{L^2(B_{\frac{R}{2}})} \\
\leq N t e^{-R/Nt} \|\mathbf{A} \nabla \eta_R + (D - m_{B_{2R}}(D)) \nabla \eta_R\|_{L^2(B_{2R})} \\
\leq N t R^{\frac{n}{2}} e^{-R/Nt},
\]
which tends to zero as \(R\) tends to \(+\infty\). The latter shows that the \(L^2(\mathbb{R}^n)\)-uniformly bounded operators \(T_t\) defined by (2.8) verify Gaffney bounds, map \(L^\infty(\mathbb{R}^n)\) into \(L^2_{loc}(\mathbb{R}^n)\) and \(T_t(1) = 0\), for \(t > 0\).
For a Lipschitz function $f$ in $\mathbb{R}^n$, define in a similar manner
\[
\mathcal{J}_t(f) = \lim_{R \to +\infty} \mathcal{J}_t \left( (f - f(x_0)) \chi_{B_R(x_1)} \right),
\]
where $x_0$ and $x_1$ are any points in $\mathbb{R}^n$. The limit is measured in the $L^2_{loc}(\mathbb{R}^n)$-sense and the definition is again independent of the choices of $x_0$ and $x_1$. Clearly, for $f$ Lipschitz, $\mathcal{J}_t(f)$ is a weak $H^1_{loc}(\mathbb{R}^n)$ solution over $\mathbb{R}^n$ to
\[
\mathcal{J}_t(f) + t^2 L \mathcal{J}_t(f) = t^2 L f.
\]
This follows from the Gaffney bounds verified by the operators $\mathcal{J}_t$ and the following Lemma.

**Lemma 3.** Let $f$ be a Lipschitz function in $\mathbb{R}^n$ and $Q$ be a cube in $\mathbb{R}^n$ with $0 < t \leq \delta(Q)$. Then,
\[
||\mathcal{J}_t(f)||_{L^2(Q)} \leq N |t| |Q|^{1/2} \|\nabla f\|_{L^\infty(\mathbb{R}^n)}
\]
and
\[
||\nabla \mathcal{J}_t(f)||_{L^2(Q)} \leq N |Q|^{1/2} \|\nabla f\|_{L^\infty(\mathbb{R}^n)}.
\]

**Proof.** Let $x_Q$ denote the center of the cube $Q$. Write
\[
\mathcal{J}_t(f) = \lim_{R \to +\infty} \mathcal{J}_t \left( (f - f(x_Q)) \chi_{B_R(x_Q)} \right)
= \mathcal{J}_t \left( (f - f(x_Q)) \eta_0 \right) + \sum_{k=0}^{+\infty} \mathcal{J}_t \left( (f - f(x_Q)) (\eta_{k+1} - \eta_k) \right),
\]
where $\eta \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 in $2Q - x_Q$, 0 outside $3Q - x_Q$ and $\eta_k(x) = \eta(x - x_Q/2^k)$, $k \geq 0$. Then, the Gaffney bounds in Lemma 2 show that
\[
(2.11) \quad ||\mathcal{J}_t((f - f(x_Q)) (\eta_{k+1} - \eta_k))||_{L^2(Q)} + ||t \nabla \mathcal{J}_t ((f - f(x_Q)) (\eta_{k+1} - \eta_k))||_{L^2(Q)}
\leq N 2^{-k} \|\nabla f\|_{L^\infty(\mathbb{R}^n)} t |Q|^{1/2},
\]
when $k \geq 0$. Next, $u = \mathcal{J}_t ((f - f(x_Q)) \eta_0)$ is a weak $H^1(\mathbb{R}^n)$ solution to
\[
u + t^2 L u = -t^2 \nabla \cdot (A + D) \nabla ((f - f(x_Q)) \eta_0)
\]
and recalling that the distribution
\[
\nabla \cdot (D \nabla ((f - f(x_Q)) \eta_0))
\]
is the same as
\[
\nabla \cdot ((D - m_Q(D)) \nabla ((f - f(x_Q)) \eta_0)),
\]
we find that
\[
u = - (1 + t^2 L)^{-1} t^2 \nabla \cdot [(A + (D - m_Q(D)) \nabla ((f - f(x_Q)) \eta_0)].
\]
Then, the uniform boundedness of the last two operators in (2.2) give
\[
||\nu||_{L^2(\mathbb{R}^n)} + ||t \nabla \nu||_{L^2(\mathbb{R}^n)} \leq N |t| |Q|^{1/2} (1 + \|D\|_{BMO}) \|\nabla ((f - f(x_Q)) \eta_0)\|_{L^\infty(\mathbb{R}^n)}
\leq N |t| |Q|^{1/2} (1 + \|D\|_{BMO}) \|\nabla f\|_{L^\infty(\mathbb{R}^n)},
\]
and the Lemma follows after adding up (2.11) and the last inequality.

Next, we recall the following result in [1] Lemma 3.9.

**Lemma 4.** Let $\{\mathcal{J}_t : t > 0\}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$ satisfying for some $N > 0$
(1) \( \sup_{t>0} \| \mathcal{T}_t \|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq N. \)
(2) \( \mathcal{T}_t \) verifies Gaffney bounds; i.e. when \( Q \) is a cube in \( \mathbb{R}^n \) and \( k \geq 1 \)
\[ \| \mathcal{T}_t (f \chi_{2^{k+1}Q \setminus 2^k Q}) \|_{L^2(Q)} \leq N e^{-2^{k+1} \delta(Q)/N t} \| f \chi_{2^k Q \setminus 2^{k+1} Q} \|_{L^2(\mathbb{R}^n)}, \]
(3) \( \mathcal{T}_t(1) \equiv 0 \) in \( L^2_{loc}(\mathbb{R}^n) \).

Then,
\[ \left( \int_{\mathbb{R}^n} \left( \frac{1}{t^2} \mathcal{T}_t(f)^2 \right) \frac{dx dt}{t} \right)^{\frac{1}{2}} \leq N \left( 1 + \frac{1}{t^2} \mathcal{T}_t(\Phi) \right) \| \nabla f \|_{L^2(\mathbb{R}^n)}, \]
for all \( f \) in \( H^1(\mathbb{R}^n) \), where
\[ \| \frac{1}{t^2} \mathcal{T}_t(\Phi) \|_C = \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{RQ} \frac{1}{t^2} \mathcal{T}_t(\Phi)^2 \frac{dx dt}{t} \right)^{\frac{1}{2}}, \]
\( \Phi \) is the identity map of \( \mathbb{R}^n \) and \( RQ \) the Carleson box \( Q \times (0, \delta(Q)) \).

Hence, as in [7], Lemma 4 implies that
\[ \left( \int_0^{+\infty} \| (1 + t^2 L)^{-1} t L f \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq N \| \nabla f \|_{L^2(\mathbb{R}^n)} \]
after one shows with \( \mathcal{T}_t \) as in (2.8), that the measure
\[ \left| \frac{1}{t^2} \mathcal{T}_t(\Phi) \right|_{L^2(\mathbb{R}^n)} \]
is a Carleson measure with
\[ \| \frac{1}{t^2} \mathcal{T}_t(\Phi) \|_C \leq N \]
and (1.3) for \( f \) in \( \mathcal{D}(L^2) \) follows from (2.7) and (2.12).

To obtain (2.13), it suffices to adapt the construction of [18] to verify a variant of the \( T(b) \) theorem for square roots [6]: for a fixed cube \( Q \) in \( \mathbb{R}^n \), \( 0 < \epsilon < 1 \) and \( \xi \) a unit vector in \( \mathbb{C}^n \), define the scalar-valued function
\[ f_{Q, \epsilon, \xi} = \Phi_Q \cdot \xi - \mathcal{T}_t (\Phi_Q \cdot \xi), \]
with \( \Phi_Q(x) = x - x_Q \) and \( t = \epsilon \delta(Q) \). Then, if follows from Lemma 3 with \( Q \) replaced by \( 10Q \), \( t = \epsilon \delta(Q) \) and \( f = \Phi_Q \cdot \xi \) that
\[ \left( \int_{10Q} |f_{Q, \epsilon, \xi} - \Phi_Q \cdot \xi|^2 \, dx \right)^{\frac{1}{2}} \leq N \epsilon \delta(Q), \]
\[ \left( \int_{10Q} |\nabla f_{Q, \epsilon, \xi} - \xi|^2 \, dx \right)^{\frac{1}{2}} \leq N. \]
Also \( f_{Q, \epsilon, \xi} \) is a weak \( H^1_{loc}(\mathbb{R}^n) \) solution to \( f_{Q, \epsilon, \xi} + t^2 L f_{Q, \epsilon, \xi} = \Phi_Q \cdot \xi \) over \( \mathbb{R}^n \), with \( t = \epsilon \delta(Q) \) and
\[ \left( \int_{10Q} |L f_{Q, \epsilon, \xi}|^2 \, dx \right)^{\frac{1}{2}} \leq N / (\epsilon \delta(Q)). \]

The reasonings in [7] Lemma 5.4 show that given functions \( f_{Q, \epsilon, \xi} \) in \( H^1_{loc}(\mathbb{R}^n) \) verifying (2.15) and (2.16) for some \( N > 0 \), there is \( 0 < \epsilon \leq 1 \), \( \epsilon = \epsilon(N, n) \) and a
finite set $W$ of unit vectors in $\mathbb{C}^n$, whose cardinality depends only on $\epsilon$ and $n$, such that the inequality

$$
\|\Psi\|_{C} \leq N \sum_{\xi \in W} \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{R_Q} |\chi_{Q}(\nabla \hat{\Phi}_{Q,\xi})| \cdot |S_t^Q(\nabla f_{Q,\xi})|^2 \frac{dxdt}{t} \right)^{\frac{1}{2}},
$$

holds for all measurable functions $\Psi : \mathbb{R}^{n+1} \rightarrow \mathbb{C}^n$ in $L^2_{\text{loc}}(\mathbb{R}^{n+1})$, where for each $Q$ cube in $\mathbb{R}^n$, $S_t^Q$ denotes the dyadic averaging operator associated to the dyadic mesh generated by $Q$, i.e.

$$
S_t^Q(h(x)) = \int_{Q'} h(y) \, dy,
$$

for $x$ in the dyadic cube $Q'$ with $\frac{1}{2} \delta(Q') < t \leq \delta(Q')$. In fact, the proof of (2.18) in [7, Lemma 5.4] uses the compactness of the unit sphere in $\mathbb{C}^n$, properties of the distance function in $\mathbb{C}^n$, Hölder’s inequality, the boundedness of the Hardy-Littlewood maximal function in $L^2(\mathbb{R}^n)$, a suitable stopping time argument independent of $\Psi$ and the interpolation inequality in [7, Lemma 5.15]. Thus, its proof is independent of the choice of $\Psi$ and (2.18) holds with $\Psi = \frac{1}{7} T_t(\hat{\Phi})$, when $L$ is as in Theorem 11 and for the choice of functions $f_{Q,\xi}$ defined in (2.14).

Then, (2.18) follows from (2.18) with $\Psi = \frac{1}{7} T_t(\hat{\Phi})$ and Lemma 5 below, which adapts [7, Lemma 5.5] to the more general hypothesis on the coefficients matrix of $L$ in Theorem 11.

**Lemma 5.** Let $\epsilon = \epsilon(N, n)$ be the choice of $\epsilon$ in (2.18). Then, there is $N > 0$ such that

$$
\left( \frac{1}{|Q|} \int_{R_Q} |\nabla \hat{\Phi}_{Q}(\chi f)| \cdot |S_t^Q(\nabla f_{Q,\xi})|^2 \frac{dxdt}{t} \right)^{\frac{1}{2}} \leq N,
$$

for all cubes $Q$ in $\mathbb{R}^n$ and $\xi$ a unit vector in $\mathbb{C}^n$.

**Proof.** Fix $Q$, $\xi$ in $\mathbb{C}^n$ with $|\xi| = 1$ and make $\epsilon = \epsilon(N, n)$. Let $\chi$ be in $C_0^\infty(4Q)$ with $\chi = 1$ in $2Q$, $\chi = 0$ outside $3Q$ and

$$
\|\chi\|_\infty + \delta(Q)\|\nabla \chi\|_\infty \leq N.
$$

To simplify the notation set $f = f_{Q,\xi}$ and $S_t = S_t^Q$. Then,

$$
\|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} = \|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)}
$$

because $\nabla(\chi f) = \nabla f$ over $2Q$ and $S_t(\nabla f)$ only reads information about $\nabla f$ inside $Q$ to calculate its values at points $(x, t)$ in $R_Q$. Next, let $P_t$ denote the convolution with an even smooth mollifier, $\theta_t(x) = t^{-\alpha} \theta(x/t)$, $\theta$ with integral 1 and supported in $B_1$. We have

$$
\begin{align*}
\|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} & \leq \|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} \\
& \leq \|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} \\
& \leq \|\frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} + \frac{1}{t} \nabla \hat{\Phi}_{Q}(\chi f)\|_{L^2(R_Q, dxdt/t)} \\
& = I + II + III.
\end{align*}
$$

Then, $I$ and $II$ in the right hand side above are handled exactly as its analogues in [7, Lemma 5.5]. In particular, the only information about $f$ that one needs to
bound $I$ and $II$ by $N\sqrt{|Q|}$ is that (2.15) and (2.16) imply the bound
\begin{equation}
(\int_{5Q} |\nabla (\chi f)|^2 \, dx)^{\frac{1}{2}} \leq N
\end{equation}
and it suffices to apply the same harmonic analysis techniques, which allow to handle the operators $\frac{1}{t} \mathcal{I}_t(\Phi) \cdot (S_t^Q - P_t^2)$ and $\frac{1}{t} \left( (\mathcal{I}_t(\Phi) \cdot \nabla P_t^2 - \mathcal{I}_t) \right)$ in [7, Lemma 5.5]. In particular, for $I$ use that $S_t$ is a projection operator; i.e. $S_t^2 = S_t$,
\begin{equation}
\| \frac{1}{t} \mathcal{I}_t(\Phi) \cdot (S_t - P_t^2) \nabla (\chi f) \|_{L^2(\mathbb{R}^{n+1}, dx dt / t)}
= \| \frac{1}{t} \mathcal{I}_t(\Phi) \cdot (S_t + P_t)(S_t - P_t) \nabla (\chi f) \|_{L^2(\mathbb{R}^{n+1}, dx dt / t)}
\end{equation}
that $\frac{1}{t} \mathcal{I}_t(\Phi) \cdot (S_t + P_t)$ is a bounded operator on $L^2(\mathbb{R}^{n})$ because the pointwise bounds of the kernel of $S_t$ and $P_t$ and duality show that
\begin{equation}
\| \frac{1}{t} \mathcal{I}_t(\Phi)(S_t + P_t) \|_{L^2(\mathbb{R}^{n})} \leq N \| P_N (|\frac{1}{t} \mathcal{I}_t(\Phi)|^2 P_N t) \|_{B(\mathbb{R}^{n})},
\end{equation}
while the first inequality in Lemma 5 implies that
\begin{equation}
\left( \int_{B_{2Nt}(x)} |\frac{1}{t} \mathcal{I}_t(\Phi)|^2 dx \right)^{\frac{1}{2}} \leq N,
\end{equation}
which shows that the kernel of $P_N (|\frac{1}{t} \mathcal{I}_t(\Phi)|^2 P_N t)$ is bounded by $N t^{-n} \chi_{|x-y| \leq 4Nt}$. Finally, the proof of the inequality
\begin{equation}
\left( \int_{\mathbb{R}^{n+1}} |(S_t - P_t)(h)|^{2 \frac{2 dx dt}{t}} \right) \leq N \| h \|_{L^2(\mathbb{R}^{n})}, \text{ for } h \in L^2(\mathbb{R}^{n}),
\end{equation}
is explained in [20] or [6] pp. 168 and 172-173]. The bound for $II$ follows from (2.10) and Lemma 4 applied to the family of operators $\mathcal{I}_t(\Phi) \cdot \nabla P_t^2 - \mathcal{I}_t$, which are uniformly bounded in $L^2(\mathbb{R}^{n})$, verify Gaffney bounds and map 1 and $\Phi$ to zero.

In order to bound $III$, the presence of the $BMO(\mathbb{R}^{n})$ matrix $D$, obliges to use some additional information about the gradient of $f = f_{Q, \xi}$. In particular, local higher integrability; i.e. there is $p = p(\lambda, n) > 0$ independent of $\delta(Q)$ and $\xi$ such that
\begin{equation}
(\int_{5Q} |\nabla f_{Q, \xi}|^p \, dx)^{\frac{1}{p}} \leq N.
\end{equation}
Once the later is known, the skew-symmetry of $D$ implies that as a distribution
\begin{equation}
L(\chi f) = \chi Lf - \nabla \cdot (f (A + D - m_{4Q}(D)) \nabla \chi) - (A + D - m_{4Q}(D)) \nabla f \cdot \nabla \chi
\end{equation}
and
\begin{equation}
\frac{1}{t} \mathcal{I}_t(\chi f) = (1 + t^2 L)^{-1} t L(\chi f)
= t (1 + t^2 L)^{-1} (\chi Lf)
- (1 + t^2 L)^{-1} t \nabla \cdot (f (A + D - m_{4Q}(D)) \nabla \chi) + (A + D - m_{4Q}(D)) \nabla f \cdot \nabla \chi
\end{equation}
Then, (2.22) and (2.17) give
\begin{equation}
\| t (1 + t^2 L)^{-1} (\chi Lf) \|_{L^2(Q)} \leq N t \delta(Q)^{-1} \sqrt{|Q|},
\end{equation}
while the Gaffney bounds in Lemma 2.14, 2.15, 2.20, 2.16, Hölder’s inequality and the Poincaré-Sobolev inequality over 4Q, imply that for 0 < t ≤ δ(Q)

\[
\| (1 + t^2L)^{-1} t \nabla \cdot (f(A + D - m_{4Q}(D)) \nabla \chi) \|_{L^2(Q)} + \| (1 + t^2L)^{-1} t [(A + D - m_{4Q}(D)) \nabla f \cdot \nabla \chi] \|_{L^2(Q)} \\
\leq e^{-\delta(Q)/Nt}\delta(Q)^{-1} \left[ \| (A + D - m_{4Q}(D)) (f - m_{4Q}(f)) \|_{L^2(4Q)} + |m_{4Q}(f)| \sqrt{|Q|} \right] \\
+ e^{-\delta(Q)/Nt}t\delta(Q)^{-1} \| (A + D - m_{4Q}(D)) \nabla f \|_{L^2(4Q)} \leq Ne^{-\delta(Q)/2Nt}\sqrt{|Q|}.
\]

Finally, (2.21), (2.22) and (2.23) show that III is also bounded by N√|Q|, which proves Lemma 5.

It only remains to show that (2.20) holds; but this follows from (2.15) and (2.16) and standard higher integrability methods [16, 30, 28] because f = f_{Q,ξ} is a weak $H^1_{loc}(\mathbb{R}^n)$ solution to $f + t^2Lf = \Phi_Q \cdot \xi$ over $\mathbb{R}^n$, with $t = \epsilon \delta(Q)$. We include the details for the reader’s convenience:

When $B_{2r}$ is any ball, multiply the equation

\[-t^2 \nabla \cdot ((A + D) \nabla f) = \Phi_Q \cdot \xi - f\]

by $(\overline{f} - m_{B_{2r}}(\overline{f})) \eta^2$, with $\eta = 1$ over $B_r$, $\eta \in C_0^\infty(B_{2r})$. It yields

\[
\int t^2 A \nabla f \cdot \nabla \eta^2 + 2t^2 (A + D) \nabla f \cdot \nabla \eta (\overline{f} - m_{B_{2r}}(\overline{f})) \eta \, dx \\
= \int (\Phi_Q \cdot \xi - f) (\overline{f} - m_{B_{2r}}(\overline{f})) \eta^2 \, dx.
\]

Taking real parts, dividing by $t^2$ and using the cancellations provided by the skew-symmetry of the matrix $m_{B_{2r}}(D)$, one gets

\[
\int |\nabla f|^2 \eta^2 \, dx \leq r^{-2} \int_{B_{2r}} (1 + |D - m_{B_{2r}}(D)|^2) |f - m_{B_{2r}}(f)|^2 \, dx \\
+ \left( \int_{B_{2r}} |(\Phi_Q \cdot \xi - f) t^{-2} |\frac{2\eta}{n} \right)^{\frac{n+2}{n}}.
\]

Next, by Hölder’s inequality and a Sobolev-Poincaré inequality

\[
\int_{B_{2r}} |D - m_{B_{2r}}(D)|^2 |f - m_{B_{2r}}(f)|^2 \, dx \\
\leq \left( \int_{B_{2r}} |D - m_{B_{2r}}(D)|^{2n} \, dx \right)^{\frac{1}{n}} \left( \int_{B_{2r}} |f - m_{B_{2r}}(f)|^{\frac{2n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \\
\leq Nr \left( \int_{B_{2r}} |\nabla f|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{n}}.
\]
Hence, recalling that $t = \varepsilon \delta(Q)$, one gets

\begin{equation}
\left( \int_{B_{2r}} |\nabla f|^2 \, dx \right)^{\frac{1}{2}} \leq N \left( \int_{B_{2r}} |\nabla f|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}} + N \left( \int_{B_{2r}} |(\Phi_Q \cdot \xi - f) \delta(Q)|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}},
\end{equation}

when $B_{2r}$ is any ball contained in $10Q$. From (30) and (2.24), there is some $p = p(\lambda, n) > 2$ such that

\begin{equation}
\left( \int_{5Q} |\nabla f|^p \, dx \right)^{\frac{1}{p}} \leq N \left( \int_{10Q} |\nabla f|^2 \, dx \right)^{\frac{1}{2}} + N \delta(Q)^{-1} \left( \int_{10Q} |\Phi_Q \cdot \xi - f|^p \, dx \right)^{\frac{1}{p}},
\end{equation}

Finally, we may assume that $2 < p < \frac{2n}{n-2}$ and the interpolation of (2.15) and (2.16) shows that the second term in the right hand of (2.25) is bounded by $N$, while (2.16) implies that the same holds with the first term.

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