Three Dimensional Corners: 
A Box Norm Proof

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Abstract

For any discrete additive abelian group \((G, +)\), we define a \(d\)-dimensional corner to be the \(d + 1\) points in \(G^d\) given by

\[
g, g + h e_r, \quad 1 \leq r \leq d, \quad h \in G - \{0\},
\]

\[
e_r = (0, \ldots, 1, \ldots, 0), \quad 1 \leq r \leq d.
\]

The Ramsey numbers of interest are \(R(G, d)\), the maximum cardinality of a subset \(A \subset G^d\) which does not contain a \(d\)-dimensional corner. We give a new proof of a special case of the Theorem of Furstenberg and Katznelson \([3]\) that in dimension \(d = 3\), for the group \(G\) a finite field of characteristic 5,

\[
R(\mathbb{F}_5^n, 3) = o(|\mathbb{F}_5^n|^3), \quad n \to \infty.
\]

Our proof, specialized to one dimension, would reduce to Gowers’ proof \([4]\) of four term arithmetic progressions in dense subsets of the integers. (Also see \([7]\).) Nevertheless, there are significant difficulties to overcome, and as a result this proof does not yield new quantitative bounds.
Contents

1 Introduction 3

2 Overview of the Proof 5

3 Principal Lemmata 12

4 Box Norms 17

5 Linear Forms for the Analysis of Box Norms 23

6 Linear Forms for the Analysis of Corners 35

7 Proof of the von Neumann Lemma 38

8 The Paley-Zygmund Inequality for the Box Norm and the set $T$ 47
   8.1 One-Dimensional Obstructions 49
   8.2 Two-Dimensional Obstructions 50
   8.3 Three-Dimensional Obstructions 56

9 Proof of Uniformizing Lemma 60
   9.1 Martingales 61
   9.2 Partitions 63
   9.3 Useful Propositions 64
   9.4 The $U(3)$ Norm 66
1 Introduction

For any discrete abelian group \((G, +)\), we define a \emph{d-dimensional corner} to be the \(d + 1\) points in \(G^d\) given by

\[
g, g + h(1, 0, 0, \ldots, 0), g + h(0, 1, 0, \ldots, 0), \ldots, g + h(0, 0, 0, \ldots, 1), \quad h \in G \setminus \{0\}.
\]

The Ramsey numbers of interest are \(R(G, d)\), the maximum cardinality of a subset \(A \subset G^d\) which does not contain a \(d\)-dimensional corner.

The principal result in the subject is the Theorem of Furstenberg and Katznelson [3], a generalization of the Szemerédi Theorem [22] to arbitrary dimension.

**1.1 Furstenberg-Katznelson Theorem.** We have the estimate below, for any dimension \(d\).

\[
R(Z_N, d) = o(N^d), \quad N \to \infty.
\]

Our principal result of this result is a new proof of this Theorem, in dimension \(d = 3\), for a finite field.

**1.2 Main Theorem.** We have this estimate, where \(N = 5^n = |F_5^n|\),

\[
R(F_5^n, 3) = o(N^3), \quad n \to \infty.
\]

The quantitative bound we provide is of Ackerman type, and accordingly we do not attempt to specify it. In the two dimensional case, there is a much better quantitative bound, doubly logarithmic in nature, due to Shkredov [18, 19].
1.3 Shkredov’s Two Dimensional Theorem. There is a $0 < c < 1$ for which we have the estimate below in the two dimensional case.

$$R(\mathbb{Z}_N, 2) \lesssim \frac{N^2}{(\log \log N)^c}, \quad N \to \infty.$$ 

In the simpler case of the finite field, one can get a better estimate, in that the constant $c$ can be specified. See [15], also [9]. Indeed it would appear that any improvement in the constant below would require new ideas.

1.4 Theorem. In the finite field setting, we have the estimate below in the two dimensional case. Set $N = p^n$ for prime $p$.

$$R(\mathbb{F}_p^n, 2) \lesssim N^2 \frac{\log \log \log N}{\log \log N}, \quad N \to \infty.$$ 

Our methods of proof are those of arithmetic combinatorics, which in most instances give better quantitative bounds. However in this proof, our bounds are of Ackerman type. It took some time for a purely combinatorial proof of the Furstenberg-Katznelson proof to be found [5,6,16] and the commentary in [20]. Thus, our proof using the Gowers norms [20], and the double recursion argument of Shkredov [18], might have some independent interest.

The Theorem we discuss is the first ‘hard’ case, as it corresponds to four-term arithmetic progressions [4,21]. The ‘hardness’ is expressed in terms of the very weak information that we get from the Box Norm, an issue we go into in more depth in the next section, see also §8. The rigorous results on Box Norm are Lemma 8.2 below, and a more sophisticated variant Lemma 8.3.

A central question in the subject of Ergodic Theory concerns the identification of the characteristic factors for multi-linear ergodic averages, especially in the sense of Host and Kra [12,14]. In the case of commuting transformations, the only complete information about these factors is in the case of two commuting transformations, a result of Conze and Lesigne [2], also [14]. Incorporating their results in to a proof of Shkredov’s Theorem is of substantial interest. Our ignorance of these factors is also a hindrance in the result of Bergelson, Leibman and Lesigne [1]. Perhaps this approach can shed some light on this question.

There should be no essential difficulty in rewriting this proof to treat the estimate $R(\mathbb{Z}_N, 3) = o(N^3)$. We have adopted the finite field setting just as a matter of convenience,
making the arguments of §9 technically a little easier (though admittedly there is little gain in simplicity by this choice.) It appears to be an interesting question, requiring additional insight, to extend this argument to higher dimensions.

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2 Overview of the Proof

There is a substantial jump in difficulty of the proof in passing from the two dimensional case to the three case. The three dimensional case, projected back to one dimension, gives a result about four term arithmetic progressions, explaining part of this difficulty. Accordingly, we begin with a description of the two dimensional case.

In two dimensions, there are three important coordinate directions: \(e_1 = (1, 0)\), \(e_2 = (0, 1)\), and \(e_3 = e_1 + e_2\), associated with the endpoints of the corners.

We exploit these three choices of coordinate directions by this mechanism. Consider three functions \(\lambda_j : \mathbb{Z}_N^3 \to \mathbb{Z}_N^2\) given by

\[
\lambda_j(x_1, x_2, x_3) = \sum_{k : k \neq j} x_k e_k
\]

The point of these definitions is that \(\lambda_j\) is not a function of \(x_j\).

For a given set \(A \subset \mathbb{Z}_N^2\), the expected number of corners in \(A\) is

\[
E_{x_1, x_2, x_3 \in \mathbb{Z}_N} A(x_1, x_2)A(x_1 + x_3, x_2)A(x_1, x_2 + x_3)
= E_{x_1, x_2, x_3 \in \mathbb{Z}_N} A(x_1, x_2)A(x_3 - x_2, x_2)A(x_1, x_3 - x_1)
= E_{x_1, x_2, x_3 \in \mathbb{Z}_N} \prod_{j=1}^3 A \circ \lambda_j(x_1, x_2, x_3).
\]

Each of the three functions is a function of just two of the three variables \(x_1, x_2, x_3\).
There is a specific mechanism to address expectations of such products: the Gowers Box norms. Define one of these norms on a function \(g\) of \(x_1, x_2\) as follows.

\[
\|g\|_{\Box \{1,2\}} = \left[ \mathbb{E}_{x_1, x_2, x'_2 \in \mathbb{Z}_N} g(x_1, x_2) g(x_1, x'_2) g(x'_1, x_2) g(x'_1, x'_2) \right]^{1/4} 
\]

which is the cross-correlation of \(g\) at the four points of an average rectangle selected from \(\mathbb{Z}_N \times \mathbb{Z}_N\). Write \(\delta = \mathbb{P}(A)\), and \(f = A - \delta\), which is, following Gower’s terminology, the balanced function of \(A\). We then expand one of the \(A\)’s in the expectation above as \(A = \delta + f\),

\[
\mathbb{E}_{x_1, x_2, x_3 \in \mathbb{Z}_N} \prod_{j=1}^3 A \circ \lambda_j(x_1, x_2, x_3) = C_1 + C_2
\]

\[
C_1 = \delta \mathbb{E}_{x_1, x_2, x_3 \in \mathbb{Z}_N} \prod_{j=1}^2 A \circ \lambda_j(x_1, x_2, x_3)
\]

\[
C_2 = \mathbb{E}_{x_1, x_2, x_3 \in \mathbb{Z}_N} f \circ \lambda_3 \prod_{j=1}^2 A \circ \lambda_j(x_1, x_2, x_3)
\]

For the first of these terms, one can check directly that

\[
C_1 \geq \delta \mathbb{E}_{x_1} |\mathbb{E}_{x_2} A(x_1, x_2)|^2 \geq \delta^3.
\]

For sets \(A\) with the number of corners approximately equal to the number of corners that one would naively expect, this should be the dominant term. On the other hand, it is the import and power of the Gowers Box Norms that we have the inequality

\[
|C_0| \leq \|f\|_{\Box \{1,2\}}
\]

Thus, if this last quantity is less than, say, \(\frac{1}{2} \delta^3\), the \(A\) has at least one-half of the expected number of corners.

There is however, the alternative that \(\|f\|_{\Box \{1,2\}} \geq \frac{1}{2} \delta^3\), which point brings us to an unfortunate fact concerning these Box Norms: The definition in (2.2) makes perfect sense on the product of arbitrary probability spaces. Accordingly, the consequence of the Box Norm being large can only have a probabilistic consequence. In the two dimensional case, it is this: There is are subsets \(R_1, R_2 \subset \mathbb{Z}_N\) so that \(A\) correlates with the product set \(R_1 \times R_2\), namely \(\mathbb{P}(A \mid R_1 \times R_2) \geq \delta + \frac{1}{3} \delta^{12}\), and the product set \(R_1 \times R_2\) is non-trivial, in that we have the estimates \(\mathbb{P}(R_1), \mathbb{P}(R_2) \geq c \delta^{12}\), for appropriate constant \(c\). There is however no additional structure on the sets \(R_1\) and \(R_2\).
The natural path, originating in Roth’s proof [17] for three term arithmetic progressions, is to iterate this alternative. We can only hope to achieve an increment in density of \( A \) by an amount of \( \delta^{12} \) a finite number of times. But without an additional insight, the iteration cannot go forward as the use of the Gowers Box Norms requires at least a little arithmetic information through the use of the change of variables. Shkredov [18] found a solution to this problem by introducing a secondary iteration, the result of which is that one finds further subsets \( R'_1 \subset R_1 \) and \( R'_2 \subset R_2 \) which satisfy three conditions. First, we maintain the property that \( A \) has a higher density on \( R'_1 \times R'_2 \), namely \( \mathbb{P}(A \mid R'_1 \times R'_2) \geq \delta + \frac{1}{8} \delta^{12} \). Second, the sets \( R'_1 \) and \( R'_2 \) are non-trivial, in that they have a lower bound on their probabilities. Third, \( R'_1 \) and \( R'_2 \) have arithmetic properties, in that their one-dimensional Box Norms are small. Specifically, \( R_1, R_2 \) are subsets of a subspace \( H \leq \mathbb{F}_2^n \), where there is a lower bound on the dimension of \( H \), and the norms

\[
\|R_j(x_1 + x_2) - \mathbb{P}(R_j \mid H)H(x_1 + x_2)\|_{\Box^{[1,2]} H \times H}, \quad j = 1, 2
\]

are small. The first two conditions are certainly required. It is the third property that permits the iteration to continue, as a subtle refinement of the inequality (2.3) is available.

There is one additional feature of this discussion that we should bring forward, as it plays a decisive role in the three-dimensional case. Namely, the discussion above placed a distinguished role on the standard basis \((e_1, e_2)\), whereas the formulation of the question makes sense any any choice of basis from the three vectors \(\{e_1, e_2, e_3\}\). One can phrase a ‘coordinate-free’ version of Shkredov’s argument, which is the viewpoint of [15]. This is the viewpoint we adopt in the three-dimensional case.

We turn to the three dimensional case. We again have the the standard basis \(e_j\), for \( j = 1, 2, 3 \) in \( \mathbb{Z}_N^3 \). The fourth relevant basis element is \( e_4 = \sum_{j=1}^3 e_j \) associated to the endpoints of the corner. The analogs of the functions \( \lambda_j \) in (2.1) are now four distinct functions from \( \mathbb{Z}_N^4 \rightarrow \mathbb{Z}_N^3 \) given by

\[
\lambda_j(x_1, x_2, x_3, x_4) = \sum_{k: k \neq j} x_k e_k .
\]

The point to exploit is that \( \lambda_j \) is not a function of \( x_j \).

For a given set \( A \subset \mathbb{Z}_N^3 \), the average number of corners in \( A \) is given by

\[
\mathbb{E}_{x_1, x_2, x_3, x_4 \in \mathbb{Z}_N} A(x_1, x_2, x_3) \prod_{j=1}^3 A((x_1, x_2, x_3) + x_4 e_j) = \mathbb{E}_{x_1, x_2, x_3, x_4 \in \mathbb{Z}_N} \prod_{j=1}^4 A \circ \lambda_j(x_1, x_2, x_3, x_4) .
\]
This is a four-linear term, which each of the four terms being dependent upon just three variables.

Again, there is a Gowers Box Norm that is relevant. This norm, of a function \( g(x_1, x_2, x_3) \) has a definition that can be given recursively as

\[
\|g(x_1, x_2, x_3)\|_{\Box[1,2,3]}^8 = \|\mathbb{E}_{x_3 \in \mathbb{Z}_N} g(x_1, x_2, x_3)\|_{\Box[1,2]}^4
\]

It has a similar interpretation as the average cross-correlation of \( g \) at the eight corners of a randomly chosen box in \( \mathbb{Z}_N^3 \). To exploit the norm, we make the same expansion of \( A \).

Setting \( \delta = \mathbb{P}(A | \mathbb{Z}_N^3) \), write \( A = \delta + f \). Use this expansion just on \( A \circ \lambda_4 \) above, so that we can write

\[
\mathbb{E}_{x_1, x_2, x_3, x_4 \in \mathbb{Z}_N} A \circ \lambda_j = C_1 + C_0
\]

\[
C_1 = \delta \mathbb{E}_{x_1, x_2, x_3, x_4 \in \mathbb{Z}_N} A \circ \lambda_j
\]

\[
C_0 = \mathbb{E}_{x_1, x_2, x_3, x_4 \in \mathbb{Z}_N} f \circ \lambda_4 \prod_{j=1}^3 A \circ \lambda_j.
\]

The Box Norm is introduced because it controls the second term.

(2.4) \[ |C_0| \leq \|f\|_{\Box[1,2,3]} \cdot \]

Thus, if the Box Norm is sufficiently small, \( C_0 \) should be negligible. Turning to the term \( C_1 \), typically we would expect \( C_1 \) to be of the order of \( \delta^4 \), but we do not have any simple recourse to establishing such a bound. Indeed, \( C_1 \) is an instance of the two-dimensional question, as \( C_1 \) is \( \delta \) times the average number of two-dimensional corners in \( A \), with the two-dimensional corners located on hyperplanes of the form \((x_1, x_2, x_3) \cdot e_4 = c\), for some \( c \).

This suggests to us that we will need to use a two-dimensional Box Norm on the hyperplanes just described. Namely, and this is an essential point, control of the Box Norm in (2.4) is not sufficient to control the number of corners in \( A \). Control of one more Box Norm, in a second set of coordinates, is required. This situation can be avoided in the two-dimensional case.
We adopt a method that places the four coordinate vectors \( \{ e_j \mid 1 \leq j \leq 4 \} \) on equal footing. For each choice of subset \( I \subset \{ 1, 2, 3, 4 \} \), we have a Box Norm corresponding to the basis for \( \mathbb{Z}_N \) given by \( \{ e_j \mid j \in I \} \). A sufficient condition for \( A \) to have a corner is that

\[
\max_{I \subset [1,2,3,4]} \| f \|_{\Box I} < 2^{-8} \delta^4.
\]

These norms are distinct, namely that one can have \( \| f \|_{\Box \{ 1,2,3 \}} \) very small, while \( \| f \|_{\Box \{ 1,2,4 \}} \) is much larger, a situation that does not arise in the one-dimensional case, as all of these norms turn out to be the same after a change of variables.

Turning to the alternative, suppose that we have \( \| f \|_{\Box \{ 1,2,3 \}} > 2^{-8} \delta^4 \). Again, the Box Norm admits a formulation on the three-fold product of probability spaces. Accordingly we can only have a probabilistic consequence of the Box Norm being large, and it is a dramatically weaker statement than in the two-dimensional case. It is this: Associate \( \mathbb{Z}_N^3 \) to \( \mathbb{Z}_N^{1,2,3} \), with the superscripts signifying the coordinates. For \( J \subset \{ 1, 2, 3 \} \) of cardinality 2, associate \( \mathbb{Z}_N^J \) to the corresponding face of \( \mathbb{Z}_N^{1,2,3} \). For each such \( J \), there is a subset \( R_J \subset \mathbb{Z}_N^J \). Consider the fibers that lie above this set, denoted by

\[
\overline{R}_J = \left\{ (x_1, x_2, x_3) \in \mathbb{Z}_N^{1,2,3} \mid \{(x_1, x_2, x_3) \cdot e_j \mid j \in J\} \in R_J \right\}.
\]

Then, the conclusions are two fold. First, \( A \) has a higher density in \( \prod_{\substack{J \subset [1,2,3]\,|\,|J|=2}} \overline{R}_J \), and second the latter set is non-trivial, in that it admits a lower bound on its probability. Namely, the conclusions are

\[
P(A \mid \prod_{\substack{J \subset [1,2,3]\,|\,|J|=2}} \overline{R}_J) \geq \delta + c \delta^C, \tag{2.5}
\]

\[
P\left( \prod_{\substack{J \subset [1,2,3]\,|\,|J|=2}} \overline{R}_J \right) \geq c \delta^C. \tag{2.6}
\]

Here \( 0 < c, C \) are absolute constants. Note that both conclusions are substantive. There is no a priori reason that the set in (2.6) should admit this lower bound in its probability. The other conclusion (2.5) gives a correlation with a set, unfortunately, this set has substantially less structure than in the two-dimensional case.

Another essential complication arises from the fact that one must consider the 6 sets \( R_J \), for \( J \subset \{ 1, 2, 3, 4 \}, J \) consisting of two elements. If we consider the three-fold intersection
$\Pi_{J \subset \{1,2,3\}} \overline{R}_J$, one can see that it is well-behaved with respect to corners if the individual
sets $R_J$ are well-behaved with respect to two-dimensional Box Norms, and their one-
dimensional projections are well-behaved with respect to the $U(3)$ norm.

But, there is no reason that the 3-dimensional set formed from the 6-fold intersection
$\Pi_{J \subset \{1,2,3,4\}} \overline{R}_J$ should be well-behaved with respect to any Box Norm. To overcome this
difficulty, we introduce an auxiliary set $T \subset \overline{R}_J$ for all $J$. This set is required to be uniform
with respect to all four three-dimensional Box Norms, but the Box Norm is taken relative
to the sets $R_J$.

We are left with the following task: Find the appropriate ‘uniformity’ conditions on the
sets $R_J$ and the set $T$ so that these conditions are met. First, we can obtain a variant of the
inequality (2.4), namely if the set $A$ is uniform in the ‘Box Norms adapted to $T$’ then $A$ has
a corner. Second, assuming that $A$ is not uniform with respect to a ‘Box Norms adapted to
$T$,’ then we can find suitable variants of (2.5) and (2.6).

This must be done in a manner that is consistent with the choice of any of the four
possible coordinate systems from $\{e_1, e_2, e_3, e_4\}$.

The remainder of the paper is organized as follows.

• § 3 presents the most important definitions and three Lemmas which combine to
prove our main result, Theorem 1.2. These three Lemmas set out, in broad terms the
iteration scheme of Shkredov [18], but the formulation of the definitions is hardly
clear.

  - A critical definition is that of a corner-system, Definition 3.1. Such a system
    consists of the set $A$, in which we seek a corner, and a number of auxiliary sets,
    such as the sets $R_J$ mentioned above. If the auxiliary sets are ‘suitably uniform’
    the the corner-system is called admissible, see Definition 3.4.

  - A ‘generalized von Neumann Lemma,’ to use the phrase of Ben Green and
    Terrance Tao [8]. Lemma 3.13 states that if the corner-system is admissible, and
    $A$ is suitably ‘uniform’ in a non-obvious sense (and $A$ is not too small, a weak
    condition) then $A$ has a corner.

  - An ‘increment Lemma,’ Lemma 3.16. This Lemma tells us that in the event
    that the hypothesis of of Lemma 3.13 fails, we can find a new corner-system,
    which is non-trivial, in which $A$ has a larger density. It is this step that provides
termination in our iteration, as the density of a set can never exceed one. The non-triviality comes from suitable lower bounds on the probabilities associated to the sets in the corner-system. This Lemma, probabilistic in nature does not provide for an admissible corner-system.

- A ‘Uniformizing Lemma,’ Lemma 3.17, in which a non-admissible corner-system is made admissible, permitting the recursion to continue.

These three Lemmas are combined, in a known way see §10, to prove the Main Theorem.

- §4 sets out notation for the Box Norms which are essential for the entire paper, in particular the Gowers-Cauchy-Schwartz Inequality 4.2. These considerations have to be set out in some generality, as the later arguments will encounter a variety of Box Norms, and multi-linear forms consisting of up to 56 functions. Most, but not all, of this section is standard, but worked out in a setting in which the underlying sets have relatively large probabilities.

- §5 applies the results on the Box Norm to some classes of linear forms which arise in the context of the three-dimensional Box Norm. These results have proofs which are appropriate refinements of the proof of the Gowers-Cauchy-Schwartz Inequality, taking into account the fact that the underlying sets we are interested have very small probabilities. This section introduces a notion of uniformity with respect to linear forms of a bounded complexity, Definition 5.2. An important component of the argument, is that the sets we consider only have a uniformity in the sense of Definition 5.2 of a bounded complexity. Also in this section, and particularly important, is the First Proposition on Conservation of Densities, Proposition 5.11, and its corollary Lemma 5.14.

- §6 is a reprise of the previous section. In principle, we could have written the one section to encompass both this section and §5 but felt that this might make the paper harder to read. This section contains the Second Proposition on Conservation of Densities, Proposition 6.4. Both of these sections are central to the remainder of the argument.

- §7 will prove the first of the three Lemmas, Lemma 3.13 by a subtle reworking of a standard Box Norm inequality. In its simplest form, this argument was found by Shkredov [8], but has a more refined elaboration in the current context.

- §8 presents a Lemma we refer to as a ‘Paley-Zygmund inequality for the Box Norm,’ see Lemma 8.2. Namely, assuming that the Box Norm is big, deduce, e.g., the
conclusions (2.5) and (2.6) above. This Lemma is presented in the simplest context in the two dimensional setting. We then present the same Lemma as above, but in the ‘weighted context.’ That is, in a context where the underlying spaces is not just a tensor product space. See Lemma 8.3. Both of these Lemmas are stated in some generality, as the more general formulation is required in §9. The main result of this section, Lemma 8.3 requires a careful elaboration of the proof in the ‘unweighted’ case.

• §9 we address the fact that the data provided to us from Lemma 8.2 and Lemma 8.3 does not have any uniformity properties. This is remedied by selecting a variety of partitions of the underlying space, with most of the ‘atoms’ of the partitions are sufficiently uniform. It is in this section that the Ackerman function will arise. The main Lemma is Lemma 3.17

• The three Lemmas of §3 are combined to prove our main Theorem in §10.

3 Principal Lemmata

Our proof is recursive, with each step in the recursion identifying a new subspace $H \leq \mathbb{F}_5^n$ in which we work. $H$ is of course a copy of $\mathbb{F}_5^n$, just with a smaller value of $n$. We maintain a lower bound on the dimension of $H$.

$H \times H \times H$ has the standard basis elements $e_1, e_2, e_3$. We also use the basis element $e_4 = e_1 + e_2 + e_3$, which is the element associated with the ‘endpoints’ of the corner. A corner has an equivalent description in terms of any three elements of the four basis elements $\{e_i \mid 1 \leq i \leq 4\}$.

Below, we will work with sets $S_i$, $1 \leq i \leq 4$. They can be viewed as subsets of $H \times H \times H$, as follows:

$$\overline{S_i} = \{x \in H \times H \times H \mid x \cdot e_i \in S_i\}, \quad 1 \leq i \leq 4.$$ 

Thus, the fibers over $\overline{S_i}$ are copies of $H \times H$. 

12
Likewise we will work with sets $R_{i,j} \subset S_i \times S_j$. They can be viewed as subsets of $H \times H \times H$ by setting

$$
\overline{R}_{j,k} = \{ x \in H \times H \times H \mid (x \cdot e_j, x \cdot e_k) \in R_{j,k} \}, \quad 1 \leq i < j \leq 4.
$$

Thus, the fibers of $\overline{R}_{j,k}$ are copies of $H$.

### 3.1 Definition.

By an *corner-system* we mean the data

(3.2) \[ \mathcal{A} = \{ H , S_i , R_{i,j} , T , A \mid 1 \leq i , j \leq 4 \} \]

where these conditions are met.

1. $H$ is a subspace of $F_5^n$.
2. $S_i \subset H$, $1 \leq i \leq 4$.
3. $R_{j,k} \subset S_j \times S_k$, $1 \leq j < k \leq 4$.
4. $T \subset \overline{R}_{j,k}$, $1 \leq j < k \leq 4$.
5. $A \subset T$.

By a *T-system* we mean the data

(3.3) \[ \mathcal{T} = \{ H , S_i , R_{i,j} , T \mid 1 \leq i , j \leq 4 \} \]

which is the same as a corner system, except that the set $A$ is not listed, and so condition (5) above is not needed.

For such systems we use the notations

$$
T_\ell := \bigcap_{1 \leq j < k \leq 4 \atop j,k \neq \ell} \overline{R}_{j,k}, \quad 1 \leq \ell \leq 4,
$$

$$
\delta_j := \mathbb{P}(S_j \mid H), \quad \delta_{j,k} := \mathbb{P}(R_{j,k} \mid S_j \times S_k), \quad 1 \leq j < k \leq 4,
$$

$$
\delta_{T \mid \ell} := \mathbb{P}(T \mid T_\ell), \quad 1 \leq \ell \leq 4.
$$
The sets $T_\ell$ play an essential role in this proof for the following reason. They are built up from lower dimensional objects in a natural way, and presuming that the lower dimensional objects are themselves well behaved with respect to box norms, then the $T_\ell$ is as well. The same conclusion does not seem to hold for the 6-fold intersection $\cap_{1 \leq i < j \leq k} R_{j,k}$. That in turn lead us to the introduction of the auxiliary set $T \subset R_{j,k}$. Working on this indeterminant set $T$ leads to most of the complications of this paper.

We use the notation $R_{j,k} \subset S_j \times S_k$ rather than the (more natural) $S_{j,k}$, as we will use the notation $S_{j,k} := S_j \times S_k$ in association with a number of Box Norms throughout the paper.

3.4 Definition. Let $C_{\text{admiss}} \geq 64$ be a fixed large constant, and $0 < \kappa_{\text{admiss}} < 1$ be a fixed small constant. Given $0 < \epsilon < 1$, and $T$-system $\mathcal{T}$ as in (3.3), we say that $\mathcal{T}$ is $\epsilon$-admissible iff

$$\|T - \delta_{T_\ell}|_{\square[i | i \neq \ell]}\| \leq \kappa_{\text{admiss}} \epsilon C_{\text{admiss}} \cdot P(T | T_\ell)^{C_{\text{admiss}}}, \quad 1 \leq \ell \leq 4,$$

$$\|R_{i,j} - \delta_{S_{ij}}(S_j \times S_k)\| \leq \kappa_{\text{admiss}} \epsilon C_{\text{admiss}} P(T | H \times H \times H)^{C_{\text{admiss}}}, \quad 1 \leq i < j \leq 4,$$

$$\|S_i - \delta_i\|_{U(3)} \leq \kappa_{\text{admiss}} \epsilon C_{\text{admiss}} P(T | H \times H \times H)^{C_{\text{admiss}}}, \quad 1 \leq i \leq 4.$$

All conditions require uniformity of the objects in terms of the density of $T$ in that object. But the condition in (3.5) can not be strengthened in any way, and it is the condition that turns out to be the most subtle. In particular, it will turn out that we can compute the expression $\|T_\ell|_{\square[i | i \neq \ell]}$ in (3.5), but it is also the case that $T_\ell$ is not uniform with respect to the norm $\square[i | i \neq \ell]$.

The norms in (3.5) and (3.6) are detailed in Definition 4.1 and (3.10), but also given explicitly in the next definition.

3.8 Definition. Let $X$, $Y$ and $Z$ be finite sets. For any function $f : X \rightarrow \mathbb{R}$, we use the notation for expectation, namely

$$\mathbb{E}_{x \in X} f(x) = |X|^{-1} \sum_{x \in X} f(x).$$

Corresponding notation for probability $P(A)$, conditional probabilities, and conditional expectations, and conditional variance are also used.

For a function $f : X \times Y \rightarrow \mathbb{R}$, define

$$\|f\|_{4}^{4}(X \times Y) := \mathbb{E}_{x,x' \in X \atop y,y' \in Y} f(x, y) f(x', y') f(x, y') f(x', y).$$
Note that the right hand side is the average of the cross-correlation of \( f \) over all combinatorial rectangles in \( X \times Y \).

For a function \( f : X \times Y \times Z \rightarrow \mathbb{R} \), define
\[
\| f \|_{U(3)} := \mathbb{E}_{x,z' \in Z} \| f(\cdot, \cdot, z) f(\cdot, \cdot, z') \|_{D}^{4} \times \mathbb{E}_{y,y' \in Y} f(x, y, z)f(x', y, z) f(x, y', z)f(x', y', z') f(x, y', z')f(x', y', z')
\]
This has a similar interpretation as the norm in (3.9). In (3.5), we use the notation
\[
(3.10) \quad \| g \|_{\square \{ i \mid i \neq \ell \}} := \| g \|_{\square \{ i \mid i \neq \ell \} (H \times H \times H)}.
\]
This notation is consistent with (3.10) below.

The \( U(3) \) norm used in (3.7) has a definition that is similar to the Box Norms, but has an additive component.

3.11 Definition. For \( f : H \rightarrow \mathbb{R} \), we define
\[
\| f \|_{U(3)} := \| f(x + y + z) \|_{D}^{4} H \times H \times H.
\]
In these definitions, observe

- A \( \delta \) represents a ‘density,’ and this will most frequently be a relative density. Thus, \( \delta_{i,j} \) is the density of \( R_{i,j} \) in \( S_{i} \times S_{j} \). In some of these notations, this relative density is indicated explicitly, as in the definition for \( \delta_{T | \ell} \).

- Likewise, the Box Norms in (3.5) and (3.6) are relative Box Norms. In (3.6), this relative norm is indicated in the notation. But, in (3.5) this is indicated by the division by \( \| T_{\ell} \|_{\square \{ i \mid i \neq \ell \}} \).

- Notice that the uniformity conditions (3.5)—(3.6) are phrased relative to the the ‘higher dimensional objects in question.’ Thus, the uniformity condition on \( T \) in (3.5) is phrased in terms of the densities of \( T \) in \( T_{\ell} \).

- The previous point, not anticipated by the two-dimensional version of this Theorem, is important to the proof of our critical Lemma 3.17 below. And it complicates the proof of Lemma 3.13.
• It is possible that the degree of uniformity require on \( S_i \) in (3.7) and \( R_{i,j} \) in (3.6) is too high. For instance, one could imagine that (3.7) should be replaced by

\[
\| S_i - \delta_i \|_{U(3)} \leq \kappa \epsilon \text{Adm} \mathbb{P}(T \mid S_i) \text{Adm}, \quad 1 \leq i \leq 4.
\]

As it turns out, the conditions (3.7) and (3.6) are available to us by this proof, and so we use them. The distinction between (3.12) and (3.7) could be important in extensions of this argument to higher dimensions.

The three Lemmas are very much as in [15, 18], though with more complicated statements in the current setting. The first Lemma asserts that for admissible corner-systems, if dimension is not too small, and the Box Norms \( \| A - \delta A \|_{T, \| i | \neq \ell} \) are sufficiently small, uniformly in \( \ell \) then \( A \) has a corner.

3.13 The von Neumann Lemma. Suppose that we are given an corner-system \( \mathcal{A} \) as in (3.2). Set \( \delta_{A \mid T} = \mathbb{P}(A \mid T) \), and assume that \( \mathcal{A} \) is \( \delta_{A \mid T} \)-admissible. The following two conditions are then sufficient for \( A \) to have a corner.

\[
\delta_{A \mid T} \cdot \prod_{j=1}^{4} \delta_j \cdot \prod_{1 \leq j < k \leq 4} \delta_{jk} \cdot \prod_{\ell=1}^{4} \delta_{T \mid \ell} \cdot \mathbb{H}^4 > 4|A|,
\]

\[
\max_{1 \leq \ell \leq 4} \frac{\| A - \delta_{A \mid T} \|_{\| i | \neq \ell} \| T \|_{\| i | \neq \ell}}{\| T \|_{\| i | \neq \ell}} \leq \kappa \delta_{A \mid T}.
\]

The condition (3.14) is the condition, typical to the subject, that the ‘average number of corners’ in \( A \) exceed the number of ‘trivial corners’ in \( A \). The second condition (3.15) is the all important uniformity condition. The second Lemma is the alternative if (3.15) does not hold.

3.16 Density Increment Lemma. There is an absolute constant \( \kappa \) for which the following holds. Suppose that the corner-system in (3.2) is \( \delta_{A \mid T} \)-admissible, and that (3.15) does not hold. Then, there are sets

\[
S'_i \subset S_i, \quad R'_{i,j} \subset R_{i,j}, \quad T' \subset T' = \prod_{1 \leq i, j \leq 4} S'_{i,j}
\]

These sets satisfy the estimates \( \mathbb{P}(T' \mid T) \geq \delta_{A \mid T}^{1/\kappa} \) and \( \mathbb{P}(A \mid T') \geq \delta_{A \mid T} + \delta_{A \mid T}^{1/\kappa} \).

It is the last estimate that provides a termination for our algorithm in §10. The previous Lemma, which is probabilistic in nature, does not supply us with admissible data. This is rectified in the next Lemma.
3.17 Uniformizing Lemma. There are functions
\[ \Psi_{\text{dim}}, \Psi_T : [0, 1]^3 \to \mathbb{N} \]
for which the following holds for all \( 0 < v < \delta < 1 \). Let \( \mathcal{A} \) be a corner-system as in (3.2). Assume that \( \mathbb{P}(A \mid T) \geq \delta + v \). There is a new corner-system
\[ \mathcal{A}' = \{ H', S'_i, R'_{i,j}, T', A' \mid 1 \leq i, j \leq 4 \} \]
so that for some \( x \in H \), \( A' \subset A + x \), and similarly for \( T' \subset T + x \). More importantly, we have:
\[
\begin{align*}
(3.18) & \quad \dim(H') \geq \dim(H) - \Psi_{\text{dim}}(v, \delta) \\
(3.19) & \quad \mathbb{P}(A' \mid T') \geq \delta + \frac{v}{4} \\
(3.20) & \quad \mathcal{A}' \text{ is } \delta\text{-admissible,} \\
(3.21) & \quad \mathbb{P}(T' \mid H' \times H' \times H') \geq \Psi_T(\delta, v, \mathbb{P}(T \mid H \times H \times H)).
\end{align*}
\]

We remark that in (3.18), if the dimension of \( H \) is too small, then \( \mathcal{A}' \) will be trivial in that \( T' \) consists of only one point. These Lemmas are combined in a standard way to prove our Main Theorem. The details are in \( \S 10 \).

4 Box Norms

It will be helpful to recall the Gowers uniformity or Box Norms in a more general form. In this we follow the the presentation in the appendices of [11], with most, but not all, Lemmas similar in statement to that reference. The notion of a Box Norm is critical to all the principal arguments of this paper; accordingly, we have pulled these general results together into their own section.

4.1 Definition of Gowers Box Norms. Let \( \{ X_u \}_{u \in U} \) be a finite non-empty collection of finite non-empty sets indexed by \( u \in U \). For any \( V \subseteq U \) write \( X_V := \prod_{v \in V} X_v \) for the Cartesian product. For a complex-valued function \( f_U : X_U \to \mathbb{C} \), we define the Gowers Box Norm (or just Box Norm) \( \| f_U \|_{\square_U(X_U)} \in \mathbb{R}^+ \) to be
\[
\| f_U \|_{\square_U(X_U)}^2 := \mathbb{E}_{x_U \in X_U} \prod_{\omega_U \in \{0, 1\}^U} C^{\omega_U} f_U(x_U^{\omega_U})
\]
where \( C : z \mapsto \overline{z} \) is complex conjugation, and for any \( x_U^0 = (x_u^0)_{u \in U} \) and \( x_U^1 = (x_u^1)_{u \in U} \) in \( X_U \) and \( \omega_U = (\omega_u)_{u \in U} \) in \( \{0, 1\}^U \), we write \( x_U^{\omega_U} := (x_u^{\omega_u})_{u \in U} \) and \( |\omega_U| := \sum_{u \in U} \omega_u \). In the special case that \( U \) is empty, forcing \( f_U \) to be a constant, we have \( \| f_U \|_{\square_U(X_U)} := |f_U| \).
Above, we use the notation $A^B$ for the class of maps from $B$ into $A$, which notation will be used throughout the paper. If $U = \{ u \}$, then $\| f_U \|_{L^1(X_U)} = |E_{X_u} f|$. In particular this is non-negative, and can be zero. Note that if $A \subset X_U$, $\|A\|_{L^1(X_U)}$ is the average number of ‘boxes’ in $A$. Thus, $\|A - \mathbb{P}(A \mid X_U)\|_{L^1(X_U)}$ measures the degree to which $A$ behaves as expected, in regards to the number of boxes it contains. It is also easy to verify that if $A$ is a randomly selected subset of $X_U$, then $\|A - \mathbb{P}(A \mid X_U)\|_{L^1(X_U)}$ is small. A similar point is essential to this section: Sets which are small with respect to this semi-norm behave in a manner similar to randomly selected subsets. A set $A$ for which $\|A - \mathbb{P}(A \mid X_U)\|_{L^1(X_U)}$ is small we will call uniform.

The Box Norms arise through the following inequality, proved by inductive application of the Cauchy-Schwartz inequality. For this Lemma, see [11, Lemma B.2].

**4.2 Gowers-Cauchy-Schwartz Inequality.** Let $U$ be non-empty, and $\{X_u\}_{u \in U}$ be a finite collection of finite non-empty sets. For every $\omega_U \in \{0, 1\}^U$ let $f_{\omega_U} : X_U \to \mathbb{C}$ be a function. Then

\[
\left| \mathbb{E}_{x_U, x_U \in X_U} \prod_{\omega_U \in \{0, 1\}^U} C^{\omega_U} f_{\omega_U}(\chi^\omega) \right| \leq \prod_{\omega_U \in \{0, 1\}^U} \| f_{\omega_U} \|_{L^1(X_U)}.
\]

From this, it follows that one has the Gowers Triangle Inequality.

\[
\|f_U + g_U\|_{L^1(X_U)} \leq \|f_U\|_{L^1(X_U)} + \|g_U\|_{L^1(X_U)}
\]

Indeed, raise both sides of the equation above to the power of $2^{|U|}$ and use (4.3).

We will also refer to this corollary to the Gowers-Cauchy-Schwartz inequality.

**4.4 Corollary.** Let $\{X_u\}_{u \in U}$ be a finite collection of finite non-empty sets. For $V \subset U$, let $f_V : X_V \to \{z \in \mathbb{C} \mid |z| \leq 1\}$. Then,

\[
\left| \mathbb{E}_{x_U \in X_U} \prod_{V \subset U} f_V(x_V) \right| \leq \|f_U\|_{L^1(X_U)}.
\]

That is, only the Box Norm associated to the largest set $U$ is needed. Here, for $x \in X_U$, $x_V$ is the restriction of the sequence $x = \{x_u \mid u \in U\}$ to the set $V \subset U$.

The inequality (4.5) is [11, (B.7)], and it suggests that the $L^1$ norm is insensitive to ‘lower order’ perturbations. We single out a more general inequality that is important to us.
4.6 Lemma. Under the hypotheses of Corollary 4.4, for $V_0 \subseteq U$, we have

\[
\left| \mathbb{E}_{x \in X_U} \prod_{V \subseteq U, |V| = |V_0|} f_V(x_V) \right| \leq \| f_{V_0} \|_{\square^V(X)}.
\]

The inequality (4.7) has a proof similar to (4.5), and we omit the proof. (Our proof of the von Neumann Lemma below could provide a proof, as we comment when we arrive there.) It has a similar interpretation to the first inequality: the $\square^V$ norm is insensitive to perturbations of the same order in distinct variables.

4.8 Corollary. For all $\epsilon > 0$ and all integers $k$, and finite sets $U$ with $|U| \geq k$ there is a $C_1 = C_1(|U|, k, \epsilon)$ for which the following holds.

Let $\{X_u\}_{u \in U}$ be a finite collection of finite non-empty sets, and $X_V = \prod_{u \in V} X_u$, for $V \subset U$. Let $\mathcal{U}_k$ be the collection of subsets of $U$ of cardinality $k$, and for each $V \in \mathcal{U}_k$ let $S_V \subset X_V$ satisfy

\[
\| S_V - \mathbb{P}(S_V) \|_{\square^V X_V} \leq \left( \frac{1}{2} \mathbb{P}(S_V) \right)^{C_1}, \quad V \in \mathcal{U}_k.
\]

Then, we have the inequality

\[
\left| \mathbb{E}_{X_U} \prod_{V \in \mathcal{U}_k} S_V - \prod_{V \in \mathcal{U}_k} \mathbb{E}_{X_V} S_V \right| \leq \epsilon \prod_{V \in \mathcal{U}_k} \mathbb{E}_{X_V} S_V.
\]

Thus, if all the sets $S_V$ are very uniform with respect to the natural Box Norms, the expectation of the products of the $S_V$ behaves as if the sets are randomly selected.

Proof. We induct on the number $w$ of elements of $V \in \mathcal{U}_k$ for which $S_V \neq X_V$. That is, we prove that for all all $\epsilon > 0$, integers $k$, and $1 \leq w \leq |\mathcal{U}_k|$ there is a $C_1(|U|, k, \epsilon, w)$ so that if for collections $S_V$, with at most $w$ choices of $V \in \mathcal{U}_k$ do we have $S_V \neq X_V$ satisfying (4.9) we have (4.10).

The case of $w = 1$ is obvious. Let us suppose that this holds for $1 \leq w < |\mathcal{U}_k|$, and prove the claim for $w + 1$. We take

\[
C_2 = C_2(|U|, k, \epsilon, w + 1) = w + 3 + \log_2 1/\epsilon + C_1(|U|, k, \epsilon/2, w).
\]

Considering the collections $S_V$ for $V \in \mathcal{U}_k$, we select $V_0$ so that $\mathbb{P}(S_{V_0})$ minimal. Thus, in particular we must have $S_{V_0} \subseteq X_{V_0}$. Write $S_{V_0} = \mathbb{P}(S_{V_0}) + f_{V_0}$. Since all the sets in $\mathcal{U}_k$ have
the same cardinality, we have the inequality
\[
\left| \mathbb{E}_{x \in X_U} f_{V_0} \prod_{V \in U - \{V_0\}} S_V \right| \leq \|f_{V_0}\|_{L^\infty(X_0)} \leq \left( \frac{1}{2} \mathbb{P}(S_{V_0}) \right)^{C_2} \leq \frac{e}{4} \prod_{V \in U_k} \mathbb{E}_{x \in X_V} S_V .
\]

The last line follows from the selection of \( V_0 \).

We can then apply the induction hypothesis to estimate
\[
\left| \mathbb{E}_{x \in X_U} \prod_{V \in U_k} S_V - \mathbb{E}_{x \in X_U} \prod_{V \in U_k} \mathbb{E}_{x \in X_V} S_V \right| \leq \frac{e}{4} \prod_{V \in U_k} \mathbb{E}_{x \in X_V} S_V 
\]
\[
+ \mathbb{P}(S_{V_0}) \left| \mathbb{E}_{x \in X_U} \prod_{V \in U_k - \{V_0\}} S_V - \mathbb{E}_{x \in X_U} \prod_{V \in U_k - \{V_0\}} \mathbb{E}_{x \in X_V} S_V \right| 
\]
\[
\leq \epsilon \prod_{V \in U_k} \mathbb{E}_{x \in X_V} S_V .
\]

So the induction is complete.

We can then conclude the Lemma by taking \( C_1(|U|, k, \epsilon) = C_2(|U|, k, \epsilon/2, |U_k|) \).

\( \square \)

We frequently use this corollary of the Gowers-Cauchy-Schwartz inequality.

4.11 Lemma. Let \( \{X_u\}_{u \in U} \) be a finite collection of finite non-empty sets. For \( V \subset U \), let \( S_V \subset X_V \).

Then, for an integer \( k \leq |U| \)

\begin{equation}
\tag{4.12}
\left| \mathbb{E}_{x \in X_U} \prod_{V \in U, |V| \leq k} S_V(x_V) - \prod_{V \in U, |V| \leq k} \mathbb{E}_{x \in X_V} S_V(x_V) \right| \leq 2^{|U|} \max_{V \in U, |V| \leq k} \|S_V - \mathbb{E}_{x \in X_V} S_V\|_{L^\infty} .
\end{equation}

Box Norms, the expectation of the products of the \( S_V \) behaves as if the sets are randomly selected. In order for this inequality to be non-trivial, we need

\[
\max_{V \in U, |V| \leq k} \|S_V - \mathbb{E}_{x \in X_V} S_V\|_{L^\infty} \leq 2^{-|U|} \prod_{V \in U, |V| \leq k} \mathbb{E}_{x \in X_V} S_V(x_V)
\]

Of course, the Lemma is trivial if \( k = 1 \), and for \( k > 1 \), this uniformity requirement is quite restrictive if the sets \( S_V \) have small probabilities. This is exactly the situation in our proof.
Proof. We view

\[ (4.13) \quad \mathbb{E}_{x \in X_U} \prod_{V \subset U \mid |V| \leq k} S_V(x_V) \]

as a multi-linear form, with the order of the multi-linearity being \( \sum_{j=1}^{k} \binom{|U|}{j} \), a term which we have crudely estimated by \( 2^{\left| U \right|} \) in (4.12). For each set \( V \subset U \), we consider the expansion of the function \( S_V \) as \( S_V = g_{V,0} + g_{V,1} \) where \( g_{V,0} = \mathbb{P}(S_V \mid X_V) \cdot X_V \), and \( g_{V,1} \) is the balanced function. We expand the term in (4.13). Let \( I \) be the collection of subsets of \( A \) of cardinality at most \( k \). We have

\[ (4.13) = \sum_{\varepsilon \in \{0,1\}^I} \mathbb{E}_{x_I \in X_I} \prod_{V \subset U \mid |V| \leq k} g_{\varepsilon(V)}(x_V). \]

The leading term arises from the choice of \( \varepsilon_0 \) which takes the value 0 for all choices of sets \( V \). For this function we have

\[ \mathbb{E}_{x_I \in X_I} \prod_{V \subset U \mid |V| \leq k} g_{\varepsilon_0(V)}(x_V) = \prod_{x_V \in X_V} S_V(x_V), \]

which is part of the expression on the left in (4.12). Let \( B_1 \subset A \) be a maximal cardinality set for which \( \varepsilon(B_1) = 1 \). Then, for any subset \( V \subset U \) with \( |B_1| < |V| \leq k \), we have \( \varepsilon(V) = 0 \), so that \( g_{\varepsilon(V)} \) is a constant function, taking a value of at most one. It follows from (4.7) that we have

\[ \left| \mathbb{E}_{x_I \in X_I} \prod_{V \subset U \mid |V| \leq k} g_{\varepsilon(V)}(x_V) \right| \leq \left| \mathbb{E}_{x_I \in X_I} \prod_{V \subset U \mid |V| \leq |B_1|} g_{\varepsilon(V)}(x_V) \right| \leq \| g_{V,1} \|_{\ell^p(X_U)}. \]

From this, (4.12) follows. \( \square \)

We note the following Corollary to the proof above, with the main distinction being that some of the functions are indicators of uniform sets as before, while others are arbitrary bounded functions. The conclusion is that the uniform sets matter little to the computation of the expectation.

4.14 Corollary. Let \( \{X_u\}_{u \in U} \) be a finite collection of finite non-empty sets and let \( k \) be a non-zero integer. Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be two collections of subsets of \( U \), with all members of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) having cardinality at most \( k \). For \( V \in \mathcal{V}_1 \), let \( S_V \subset X_V \). For \( W \in \mathcal{V}_2 \) let \( f_W : X_W \rightarrow [-1,1] \) be a
bounded function. Then,
\[
\left| \mathbb{E}_{x \in \mathcal{U}} \prod_{V \in \mathcal{V}_1} S_V(x_V) \prod_{V \in \mathcal{V}_2} f_W(x_W) - \prod_{V \in \mathcal{V}_1} \mathbb{E}_{x \in X_V} S_V(x_V) \times \prod_{V \in \mathcal{V}_2} f_W(x_W) \right|
\leq 2^{\| \mathcal{U} \|} \cdot \max_{V \in \mathcal{V}_1} \| S_V - \mathbb{E}_{x \in X_V} S_V \|_{\Box(X_V)}.
\]

We turn to a more complicated version of these Lemmas and Corollaries.

**4.15 Lemma.** Let \( \mathcal{U} \) be a finite set, and \( X_u \) for \( u \in \mathcal{U} \) another finite set. Fix \( 1 < k < |\mathcal{U}| \), and let \( \mathcal{V} \) be a collection of subsets of \( \mathcal{U} \) of cardinality at most \( k \). Let \( S_{\mathcal{U}} \subset X_{\mathcal{U}} \), and write \( \delta = \mathbb{P}(S_{\mathcal{U}}) \). Assume that
\[
\sup_{V \in \mathcal{V}_1} \mathbb{E}_{x \in X_{\mathcal{U}} \setminus V} \| f_U(x_{\mathcal{U}}) \|_{\Box(X_{\mathcal{U}})} = \tau < \delta |\mathcal{V}|^{-1}, \quad f_U := S_{\mathcal{U}} - \delta.
\]

We emphasize that, in the expansion of the Box Norm above, the Box Norm is taken over the variables associated to \( V \) and the expectation is taken over all variables in \( \mathcal{U} \). The conclusion is that we have the inequality below.

\[
(4.16) \quad \mathbb{E}_{x_{\mathcal{U}}} \left| \delta |\mathcal{V}| - \mathbb{E}_{x_{\mathcal{U}}} \prod_{V \in \mathcal{V}} S_U(x_V) \right| \leq \tau.
\]

The implied constant depends upon \( |\mathcal{V}| \). Above, by very slight abuse of notation, we mean
\[
x_{\mathcal{U}}^V = \begin{cases} x_v^1 & v \in V \\ x_v^0 & v \notin V \end{cases}
\]

This is a ‘conditional’ version of Corollary 4.14. In particular, note that in (4.16), we impose the Box Norms in the variables \( X_V \), and take the expectation over all of \( X_{\mathcal{U}} \). The conclusion is again that if the set is suitably small with respect to a family of relevant Box Norms, then a range of products of these sets behave as if the set were randomly selected.

**Proof.** Let us begin by noting that for \( V \in \mathcal{V} \), the monotonicity of the Box Norms as the variables increase imply that
\[
\mathbb{E}_{x_{\mathcal{U}}} |\delta - \mathbb{E}_{x_{\mathcal{U}}} S_V(x_V)| \leq \| S_{\mathcal{U}} - \delta \|_{\Box(X_{\mathcal{U}})} \leq \tau.
\]
It follows by the assumption on the magnitude of $\tau$ that we can estimate
\[
|\delta^{|V|} - \prod_{V \in \mathcal{V}} \mathbb{E}_{x_{ij}} S_{V}(x_{ij}^{V})| \leq (\delta + \tau)^{|V|} - \delta^{|V|} \\
\leq \delta^{|V|} \left[ (1 + \tau \delta^{-1})^{|V|} - 1 \right] \leq \tau
\]
Also note that we can estimate, using Lemma [4.11]
\[
\mathbb{E}_{x_{ij}} \left| \prod_{V \in \mathcal{V}} S_{U}(x_{ij}^{V}) - \prod_{V \in \mathcal{V}} \mathbb{E}_{x_{ij}} S_{U}(x_{ij}^{V}) \right| \leq \mathbb{E}_{x_{ij}} \sup_{V \in \mathcal{V}} \left\| S_{U}(x_{ij}) - \mathbb{E}_{x_{ij}} S_{U}(x_{ij}^{V}) \right\|_{\infty} \leq 2\tau.
\]
Putting these inequalities together proves the Lemma.

\[\square\]

5 Linear Forms for the Analysis of Box Norms

Box Norms, and counting corners in sets are examples of multi-linear forms that we will work with. Their analysis will lead to forms in as many as 24 functions, leading to the need for some general remarks on such objects. Moreover, we are analyzing these forms on objects that are far from tensor products. This is the primary focus of this section.

We will be making a wide variety of approximations to different expectations. In order to codify these approximations, let us make this definition.

5.1 Definition. Fix $0 < \nu < 3^{-28}$ be a small constant. For $A, B > 0$ we will write $A \equiv B$ if $|A - B| < \nu A$. (We stack a ‘$u$’ on the equality, as this relation will always come about from uniformity.) In those (few) instances, where it is important emphasize the role of $\nu$, we will write $A \equiv^\nu B$.

We will only use the notation for quantities between 0 and 1. Observe the following.
Let $0 < A, B, \alpha, \beta < 1$. If $A \equiv^\nu \alpha$ and $B \equiv^\nu \beta$, then we have
\[
|A - \alpha \cdot \beta| \leq |A - \alpha B| + \alpha \cdot |\beta - B| \\
\leq \nu A + \nu \alpha B \leq 3\nu A.
\]
Thus, we can write $A^u_\alpha \beta$, that is this relationship is weakly transitive. We will need to use a finite chain of inequalities of this type, with the longest chain associated with the analysis of a $2^8$-linear form in Lemma 7.25 below. By abuse of notation, we will adopt the convention $A \overset{u}{=} B$ and $B \overset{u}{=} C$ implies $A \overset{u}{=} C$. This transitivity will only be applied a finite number of times, so that taking an initial $\nu$ in Definition 5.1 will lead to a meaningful inequality at every stage of our proof.

A second situation we will have is this. Suppose that $A \overset{u}{=} A'$ and $B \overset{u}{=} B'$. Then,

$$|AA' - BB'| \leq |A - B|A' + |A' - B'B|$$

$$\leq \nu(AA' + A'B) \leq 3\nu AA'.$$

Thus, we can write $AA' \overset{u}{=} BB'$, thus this relationship is weakly multiplicatively transitive.

We will need to use a finite chain of these inequalities, mostly related to computing conditional expectations. By abuse of notation, we will adopt the convention that $A \overset{u}{=} A'$ and $B \overset{u}{=} B'$ implies $AA' \overset{u}{=} BB'$. This observation is closely linked with the fact that our definition of admissibility, Definition 3.4 includes relative measures of uniformity.

Our Lemmas and Definitions should be coordinate-free, but to ease the burden of notation, we state them distinguishing the coordinate $x_4$ for a special role. They will be applied in their more general formulations, which are left to the reader.

We are concerned with the evaluation of certain multi-linear forms, especially those associated with Box Norms. For a collection of maps $\Omega \subset \{0, \ldots, \lambda - 1\}^{[1,2,3]}$, where $\lambda \geq 2$ is an integer, let $\{f_\omega \mid \omega \in \Omega\}$ be a collection of functions. The linear forms we are interested in are

$$L(f_\omega \mid \Omega) = \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}, 0 \leq \ell \leq 2} \prod_{\omega \in \Omega} f_\omega(x_{1,2,3}).$$

This next definition is concerned with the uniform evaluation of forms of this type, where the $f_\omega$ are particularly simple.

**5.2 Definition.** Let $\lambda \geq 3$ be an integer, and $0 < \delta < 1$. A subset $U \subset T_4$ is called $(\lambda, \delta, 4)$-uniform if the following holds. Set $\Omega_{3\ldots1} = [0, \ldots, \lambda - 1]^{[1,2,3]}$. For any subset $\Omega \subset \Omega_{3\ldots1}$ we have the inequalities

$$(5.3) \quad L_{\Omega}(U \mid \Omega) \overset{u}{=} \left[\delta_4 \delta U_{[4]} \right]^{\Omega} \prod_{1 \leq j < k \leq 3} \delta_{[\ell]{4}\{\omega_{[j,k]} \mid \Omega\}}^{[\ell]}$$
Here, $\delta_{U|4} = \mathbb{P}(U | T_4)$. That is, the percentage error between the two terms is at most $\vartheta$.

It is an important point that we index this notion on the number of linearities that we permit the form to have, as we must provide an upper bound on this notion of complexity. Our primary objective is that $T$ be well-behaved with respect to the Box Norm, in particular that Lemma 5.3 holds. This will require that $T$ be $(4, \vartheta_1, 4)$-uniform, where $\vartheta_1$ is specified in that Lemma. But this will in turn require us to require $T_4$ is $(12, \vartheta_2, 4)$-uniform. It is one purpose of this section to explain this relationship. See Lemma 5.4.

While we will use these results several times, there are two points where either these results apply, but would lead to an increased order of complexity, as in the proof of (7.31), or the results of this section are not stated in enough generality, as in the proof of (8.23). A full understanding of these issues would likely be an aid to extending this argument to higher dimensions.

In this definition, examining the product of densities, we see that $\delta_{U|4} = \mathbb{P}(U | T_4)$ has the power $|\Omega|$, that is the total number of terms in the product. The power on the density $\delta_{j,k}$ is the number of distinct maps of the form $\omega$, restricted to $\{j,k\}$ in the set $\Omega$. To set out an example, a typical term to which we will apply this definition is to the set $U = T_4$, in

$$\mathbb{E} \prod_{x_1 \in S_1, \epsilon_2, \epsilon_3 \in \{0,1\}^{[2,3]} T_4(x_1, x_{2,3}^\epsilon)$$

Here, it is clear that $|\Omega| = 4$, while

$$|\omega|_{[1,2]} | \Omega| = 2, \quad |\omega|_{[1,3]} | \Omega| = 2, \quad |\omega|_{[2,3]} | \Omega| = 4.$$

The parameter $\vartheta$ appears on the right in (5.3), and represents how close, in terms of percentages, the expectation behaves with respect to its expected behavior.

A set $U$ is $(\lambda, \vartheta, 4)$-uniform if a wide set of expectations of $U$ behave as expected.’ It is hardly obvious that even the set $T_4$ satisfies this definition, but it does, and we prove in Lemma 5.4 that both $T_4$ and $T$ are uniform.

**5.4 Lemma.** We have the following two assertions. For constants $C_1 > C_0 > 0$ that depend only on $C_{\text{admiss}}$ in Definition 3.4 the following are true.

1. For $\vartheta = \delta_{T|T_4}^{C_0}$, the set $T_4$ is $(12, \vartheta, 4)$-uniform.
2. For $\delta = \delta_{2|4}^{C_1}$ the set $T$ is $(6, 8, 4)$-uniform.

In fact, $C_1, C_0$ can be taken to be a small constant multiple of $C_{\text{admiss}}$.

As the statement of the Lemma indicates, there is a link between the complexity of the linear forms we need to consider for $T$ and $T_4$.

Proof. Let us discuss $T_4$ first. Note that by (3.7) and (4.7),

$$L(T_4 | \Omega) = E_{x_{1,2,3} \in S_{1,2,3}} \prod_{0 \leq f \leq 11} T_4(x_{1,2,3}^f)$$

(5.5)

$$= E_{x_{1,2,3} \in S_{1,2,3}} \prod_{0 \leq f \leq 11} E_{\omega \in \Omega} (S_4(x_{1}^{\omega(1)} + x_2^{\omega(2)} + x_3^{\omega(3)}) \prod_{1 \leq j < k \leq 3} S_{j,k}(x_{j,k}^\omega))$$

(5.6)

$$= \delta_{T_4}^{[\Omega]} \cdot E_{x_{1,2,3} \in S_{1,2,3}} \prod_{0 \leq f \leq 11} \prod_{\omega \in \Omega} S_{j,k}(x_{j,k}^\omega) + O(\mathbb{P}(T | H \times H \times H)^{C_{\text{admiss}}-12}).$$

The power on $\mathbb{P}(T | H \times H \times H)$ accounts for the fact that implicitly the condition (3.7) is an expectation over $H$, while above we are taking integration over $S_{1,2,3}$.

We continue with the analysis of the expectation above. We can use (4.7) and (3.6) to estimate

(5.7) $E_{x_{1,2,3} \in S_{1,2,3}} \prod_{0 \leq f \leq 11} \prod_{\omega \in \Omega} S_{j,k}(x_{j,k}^\omega) = \prod_{1 \leq j < k \leq 3} \delta_{j,k}^{[\omega(\omega) | \Omega]} + O(\mathbb{P}(T | S_{1,2,3,4})^{C_{\text{admiss}}})$.

The leading terms of the expectations are exactly as desired. The two error terms in (5.6) and (5.7) should be as small as desired, namely that they contribute at most $\delta L(T_4 | \Omega)$. But it is straightforward to see that we can take $C_0$ of the Lemma to be $C_{\text{admiss}} - 12 - |\Omega| \geq C_{\text{admiss}} - 12 - 3^{12}$, with $3^{12}$ being the cardinality of $\Omega_{3 \rightarrow 12} = \{0, \ldots, 11\}^{1,2,3}$.

We turn to the second conclusion of the Lemma. Let $\Omega \subset \Omega_{3 \rightarrow 6}$, and consider the multi-linear expression $L(T | \Omega)$. Each occurrence of $T$ is expanded as $T = f_1 + f_0$ where $f_1 = \delta_{T | 4} T_4$. The leading term is when each $T$ is replaced by $f_1$, which leads to $\delta_{T | 4}$ times the expectation in (5.5). There are $2^{[\Omega]} - 1$ terms remaining. Each of them has an occurrence of $f_0$. All of these terms can be controlled by the assumption (3.5), and importantly, the inequality (5.20) below. (We have not yet proved (5.20), part of Lemma 5.17, but its proof is independent of this argument.) This last Lemma is applied with $\lambda = 6, V = T_4$, which as...
we have just seen in the first half of the proof, is \((12, \vartheta', 4)\)-uniform, for a very small choice of \(\vartheta'\). This gives us

\[
\left| L(T \mid \Omega) - \delta_T^{1|\Omega|} L(T_4 \mid \Omega) \right| \leq 2^{1|\Omega|+1} L(T_4 \mid \Omega) \cdot \frac{\|f_0\|_{\square_{1,2,3} S_1,2,3}}{\|T_4\|_{\square_{1,2,3} S_1,2,3}} \\
\leq 2^{1|\Omega|+1} \delta_{\text{admiss}}^{T_4} L(T_4 \mid \Omega).
\]

And this completes the proof. \qed

Here is a corollary to the previous Lemma that is certainly relevant for us.

5.8 Lemma. We have this estimate

\[
\|T_4\|_{\square_{1,2,3} H_{1,2,3}}^8 = \mathbb{E}_{x_{1,2,3} \sim \mathbb{P}_{1,2,3} \in H_{1,2,3}} \prod_{\omega \in \{0,1\}^{1,2,3}} T_4 \circ \lambda_4(x_{1,2,3}^{\omega}) \\
= \mathbb{E}_{x_{1,2,3} \sim \mathbb{P}_{1,2,3} \in H_{1,2,3}} \prod_{\omega \in \{0,1\}^{1,2,3}} S_4 \circ \lambda_4(x_{1,2,3}^{\omega}) \prod_{1 \leq j < k \leq 3} S_{jk}(x_{1,2,3}^{\omega}) \\
= \mu \prod_{j=1}^{3} \delta_j^2 \cdot \delta_4^8 \cdot \prod_{1 \leq j < k \leq 3} \delta_{jk}^2.
\]

We return to general considerations, and make a remark that we will refer to several times. Let \(V \subset T_4\) be \((\lambda, \vartheta, 4)\)-uniform. Let \(\Omega \subset \Omega_{3-\lambda-1}\), and assume that the set \(\Omega_{1-0}\) is non-trivial.

\[
\Omega_{1-0} = \{\omega \in \Omega \mid \omega(1) = 0\}, \quad \Omega_{1\neq0} = \Omega - \Omega_{1-0}.
\]

Consider the estimate below obtained by applying the Cauchy-Schwartz inequality in all variables except \(x_1^0\).

\[
(5.9) \quad \mathbb{L}(V \mid \Omega) \leq \left[ \mathbb{L}(\Omega_{1\neq0}) \cdot U_2 \right]^{1/2} \\
U_2 = \mathbb{E} \prod_{\omega \in \Omega_{1\neq0}} V(x_{1,2,3}^{\omega}) \cdot |\mathbb{E}_{x_{1}^{0} \in S_1} \prod_{\omega \in \Omega_{1-0}} \prod_{\omega \in \Omega_{1-0}} V(x_{1,2,3}^{\omega})|^2.
\]

Use (7.11) to write the last term as \(U_2 = \mathbb{L}(V \mid \Omega^1)\), where we define

\[
\overline{\omega}(j) = \begin{cases} 
\lambda & j = 1 \\
\omega(j) & j = 2, 3
\end{cases}
\]

\[
(5.10) \quad \Omega^1 = \Omega_{1\neq0} \cup \{\omega, \overline{\omega} \mid \omega \in \Omega_{1-0}\}.
\]
5.11 First Proposition on Conservation of Densities. If $V \subset T_4$ be ($\lambda, \varnothing, 4$)-uniform, $\Omega \subset \Omega_{3-\lambda-1}$, with the notation in (5.9)—(5.10) we have the equality

\[(5.12) \quad L(V \mid \Omega)^{\sqrt{\varnothing}} = L(V \mid \Omega_{1\neq 0})^{1/2} \cdot L(V \mid \Omega^1)^{1/2}.
\]

Proof. The proof is almost trivial. Each $\omega \in \Omega$ on the contributes 1 to the densities $\delta_{V|4}, \delta_4, \delta_{jk}$ for $1 \leq j < k \leq 3$. If $\omega(1) \neq 0$, it contributes to both terms on the right, so the square root makes contribution 1. If $\omega(1) = 0$, then it contributes nothing to $L(V \mid \Omega_{1\neq 0})$, but contributes 2 to the other term $L(V \mid \Omega^1)$. \[\square\]

The previous Lemma plays a decisive role in all our applications of the Cauchy-Schwartz inequality, to prove our weighed versions of these inequalities. This Conservation of Densities has an essentially equivalent formulation, also important to us, that we give here. With the notation of (5.9)—(5.10), set

\[(5.13) \quad Z[\Omega_{1\neq 0} : \Omega_{1\rightarrow 0}] = \mathbb{E}_{x_1^0 \in S_1} \prod_{\omega \in \Omega_{1\neq 0}} V(x_{1,2,3}^\omega)
\]

5.14 Lemma. Let $\lambda = 1, \ldots, 6$. Suppose that the set $V \subset T_4$ is ($\lambda, \varnothing, 4$)-uniform, where $\varnothing \leq \mathbb{P}(V \mid T_4)^{2,3}$. Then, for all choices of $\Omega \subset \Omega_{3-\lambda-1}$ as above, we have

\[(5.14) \quad \text{Var}_{x_j \in \Omega}
\big(Z[\Omega_{1\neq 0} : \Omega_{1\rightarrow 0}] \mid \prod_{\omega \in \Omega_{1\neq 0}} V(x_{1,2,3}^\omega)\big)
\]

\[(5.15) \quad \leq K \sqrt{\varnothing} \cdot \mathbb{E}
\big(Z[\Omega_{1\neq 0} : \Omega_{1\rightarrow 0}] \mid \prod_{\omega \in \Omega_{1\neq 0}} V(x_{1,2,3}^\omega)\big)^2.
\]

Here, $K$ is an absolute constant.

Of course the conditional expectation of $Z$ can be computed.

Proof. We use the standard formula for the variance of a random variable $W$ supported on a set $Y$.

\[(5.16) \quad \text{Var}(W \mid Y) = \mathbb{P}(Y)^{-1} \mathbb{E}W^2 - (\mathbb{P}(Y)^{-1} \cdot \mathbb{E})W^2
\]

The conditional variance will be small if we have

\[
\mathbb{E}
\big(Z[\Omega_{1\neq 0} : \Omega_{1\rightarrow 0}] \mid \prod_{\omega \in \Omega_{1\neq 0}} V(x_{1,2,3}^\omega)\big)^2 = \mathbb{E}
\big(Z[\Omega_{1\neq 0} : \Omega_{1\rightarrow 0}] \mid \prod_{\omega \in \Omega_{1\neq 0}} V(x_{1,2,3}^\omega)\big)^2.
\]
But this is a recasting of (5.12). Namely, using the notation of (5.12), we can write the equation above as

\[
\frac{L(V \mid \Omega^1)}{L(V \mid \Omega_{1 \rightarrow 0})} = \frac{L(V \mid \Omega^2)}{L(V \mid \Omega_{1 \rightarrow 0})^2}
\]

which is (5.12).

□

We are interested in refinements of the Gowers Box Norms, in which we estimate L in terms of a Box Norm of one of its arguments, but do so in a more efficient manner, just as in the proof of Lemma 3.13, which is presented in §7. For this Lemma, let us consider selections of \( f_\omega \) where \( f_\omega \in \{ f, V \} \), and \( f \) is a fixed function supported on \( V \) and at most one in absolute value. In application, \( f \) is a balanced function.

In this Lemma, we will single out the first and second coordinates for a distinguished role, which is done just for simplicity.

**5.17 Lemma.** Let \( \lambda = 2, \ldots, 6 \). Suppose that \( V \) is \((2\lambda, \theta, 4)\)-Uniform, where \( \theta < \mathbb{P}(V \mid T_4)^{2^{3\lambda}} \). Let \( \Omega \subset \Omega_{3 \rightarrow \lambda} \), where the value of \( \lambda \) is half of the uniformity assumption imposed on \( V \). Let \( \{ f_\omega \mid \Omega \} \) be a selection of functions which are either equal to \( V \) or a fixed function \( f \) which is supported on \( V \) and bounded by one in absolute value. (In application, \( f \) will be a balanced function.)

1. Suppose that there is an \( \omega_0 \in \Omega \) with \( f_{\omega_0} = f \), and \( \omega_0(1) \neq \omega(1) \) for all other \( \omega \in \Omega \) with \( f_\omega = f \). Then, we have the estimate

\[
(5.18) \quad |L(f_\omega \mid \Omega)| < 4 L(V \mid \Omega) \cdot \left[ O(\delta) + \frac{\mathbb{E}_{x_2,x_3 \in \mathbb{S}_{2,3}} \| f \|^2_{\mathbb{Q}_{1\mid S_1}}}{\mathbb{E}_{x_2,x_3 \in \mathbb{S}_{2,3}} \| V \|^2_{\mathbb{Q}_1 \mid S_1}} \right]^{1/2}.
\]

2. Suppose that there is an \( \omega_0 \in \Omega \) with \( f_{\omega_0} = f \), and \( (\omega_0(1), \omega_0(2)) \neq (\omega(1), \omega(2)) \) for all other \( \omega \in \Omega \) with \( f_\omega = f \). Then, we have the estimate

\[
(5.19) \quad |L(f_\omega \mid \Omega)| < 4 L(V \mid \Omega) \cdot \left[ O(\delta) + \frac{\mathbb{E}_{x_3 \in \mathbb{S}_{2,3}} \| f \|^4_{\mathbb{Q}_{1\mid S_1 \mid 2,12}}}{\mathbb{E}_{x_3 \in \mathbb{S}_{2,3}} \| V \|^4_{\mathbb{Q}_1 \mid S_1 \mid 2,12}} \right]^{1/4}.
\]

3. If there is at least one \( \omega_0 \in \Omega \) with \( f_{\omega_0} = f \), we have

\[
(5.20) \quad |L(f_\omega \mid \Omega)| < 8 L(V \mid \Omega) \cdot \left[ O(\delta) + \frac{\mathbb{E}_{x_3 \in \mathbb{S}_{2,3}} \| f \|^8_{\mathbb{Q}_{1\mid S_1 \mid 2,12}}}{\mathbb{E}_{x_3 \in \mathbb{S}_{2,3}} \| V \|^8_{\mathbb{Q}_1 \mid S_1 \mid 2,12}} \right]^{1/8}.
\]
Of course the estimate (5.20) applies in the first two cases of the Lemma. But we will be in situations, in the proof of Lemma 8.3, where we do not wish to use the estimate (5.20).

We remark that one could read the proof of Lemma 3.13 in §7 before the one below. This proof in §7 is independent of the proof below. It treats a more complicated situation, in that all the $T_j$ have to be considered, but is only discussed in a single concrete instance.

Proof. We can read off a good estimate for $L(V \mid \Omega)$ from (5.3), in all cases (1)—(3) above. For each of the three cases, we assume that the choice of $\omega_0$ specified in each of the three cases satisfies $\omega_0 \equiv 0$.

In case (1), we will apply the Cauchy-Schwartz inequality in all other variables. To set notation for this, let

$$
\Omega_{1 \rightarrow 0} = \{ \omega \in \Omega \mid \omega(1) = 0 \}, \quad \Omega_{1 \rightarrow 0} = \{ \omega \in \Omega \mid \omega(1) \neq 0 \},
$$

and let $X' = \{ x'_j \mid 1 \leq j \leq 3, 0 \leq \ell \leq \lambda - 1 \} - \{ x'_1 \}$. Then, we apply the Cauchy-Schwartz inequality to estimate

$$
(L(f_\omega \mid \Omega)) \leq \left[ L(V \mid \Omega_{1 \rightarrow 0}) \cdot W_1 \right]^{1/2} \tag{5.21}
$$

$$
W_1 = \mathbb{E}_{x'_j \in X'} \prod_{\omega' \in \Omega_{1 \rightarrow 0}} V(x'_{1,2,3}) \prod_{\omega \in \Omega_{1 \rightarrow 0}} f_\omega(x'_{1,2,3}) \tag{5.22}
$$

We continue the analysis of $W_1$. It follows from the assumption in part (1) of the Lemma, that $\omega_0 \in \Omega_1$, and $f_{\omega_0} = f$, but for all other choices of $\omega \in \Omega_{1 \rightarrow 0}$ we have $f_\omega = V$. In order to expand the square of the expectation, using (7.11), let us define a new class of maps as follows. For $\omega \in \Omega_1$, define

$$
\overline{\omega}(j) = \begin{cases} 
\omega(j) & j \neq 1 \\
\lambda & j = 1
\end{cases}
$$

$$
\Omega_{1 \rightarrow \lambda} = \{ \overline{\omega} \mid \omega \in \Omega_{1 \rightarrow 0} \}, \quad \Omega^1 = \Omega_{1 \rightarrow 0} \cup \Omega_{1 \rightarrow 0} \cup \Omega_{1 \rightarrow \lambda},
$$

$$
\Omega_{\{1\} \rightarrow \{0, \lambda - 1\}} = \{ \omega \in \Omega^1 \mid \omega(1) = 0 \}.
$$

Notice that $\Omega_{\{1\} \rightarrow \{0, \lambda - 1\}} = \{ \omega_0, \overline{\omega_0} \}$, by assumption on $\Omega$ that holds in this case.

Here and below, we are expanding the set $\Omega$. We take $f_\omega = V$ for all $\omega \notin \Omega$. 

30
We can write

\begin{equation}
W_1 = \mathbb{E}_{x_1} \mathbb{E}_{x_2} \prod_{\alpha' \in \Omega_{1} \cup \Omega_{1} \cup \Omega_{1} \cup \Omega_{1}} V(\alpha') \prod_{\alpha \in \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]}} f(\alpha)
\end{equation}

where the last term is defined in (5.13).

It follows from Lemma 5.14 that \(Z[\Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]}] \) is essentially constant on \(V(x_{1,2,3} \cup \Omega_{1})\). Namely,

\[\mathbb{E}(Z[\Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]}] \mid V(x_{1,2,3} \cup \Omega_{1})) = \frac{L(V \mid \Omega_{1})}{L(V \mid \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]})}.
\]

The implied \(\kappa\) in the \(\tilde{\mu}\) is \(\kappa = \sqrt{g}\), see Definition 5.1. Similar comment applies to other uses of the the symbol \(\tilde{\mu}\) below. And the variance of \(Z[\Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]}] \) is very small. Note that \(L(V \mid \Omega_{1}) \) can be estimated

\begin{equation}
W_1 \leq 2 L(V \mid \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]} \cup \Omega_{[1]}) \mathcal{O}(\sqrt{g}) + \frac{\mathbb{E}_{x_2, x_3} ||f||^2_{\Omega_{[1]} S_1}}{\mathbb{E}_{x_2, x_3} ||V||^2_{\Omega_{[1]} S_1}}.
\end{equation}

We combine (5.21) to conclude that

\[|L(f_0 \mid \Omega)| \leq 2 \left[ L(V \mid \Omega_{1,4}) \cdot L(V \mid \Omega_{1}) \right]^{1/2} \times \mathcal{O}(\sqrt{g}) + \frac{\mathbb{E}_{x_2, x_3} ||f||^2_{\Omega_{1} S_1}}{\mathbb{E}_{x_2, x_3} ||V||^2_{\Omega_{1} S_1}}^{1/2}.
\]

And so the proof of (5.18) will follow from the inequality

\[L(V \mid \Omega_{1,4}) \cdot L(V \mid \Omega_{1}) \leq 2 L(V \mid \Omega)^2.
\]

This is Conservation of Densities Proposition, Proposition 5.11.

We turn to the proof of the second part, namely (5.19). The initial stage of the argument follows the lines of the argument above. Namely, we use the estimate (5.21) and (5.22). The term \(W_1\) is expanded as in (5.24), with the same notation that we have in (5.23). But,
under the assumptions on \( \Omega \) that hold in this case, \( \Omega_{[1]} \rightarrow [0, \lambda - 1] \) need not consist of just two maps \( \omega \).

We apply the Cauchy-Schwartz inequality to \( W_1 \). To do this, we make these definitions, recalling that \( \Omega^1 \) is defined in (5.23).

\[
\Omega^1_{2 \rightarrow 0} = \{ \omega \in \Omega^1 \mid \omega(2) \neq 0 \}, \quad \Omega^1_{2 \rightarrow 0} = \{ \omega \in \Omega^1 \mid \omega(2) = 0 \},
\]

\[
X'' = \{ x'_1 \mid 0 \leq \ell \leq \lambda \} \cup \{ x'_2 \mid 1 \leq \ell \leq \lambda - 1 \} \cup \{ x'_3 \mid 0 \leq \ell \leq \lambda - 1 \}.
\]

Here, the point is that the only variable omitted from \( X'' \) is \( x''_2 \). Then, we can estimate

(5.26) \[
W_1 \leq \left[ L(V \mid \Omega^1_{2 \rightarrow 0}) \cdot W_2 \right]^{1/2}
\]

(5.27) \[
W_2 = \mathbb{E}_{x'' \in X''} \prod_{\omega \in \Omega^1_{2 \rightarrow 0}} V(x''_{1,2,3}) \left| \mathbb{E}_{x'' \in S_2} \prod_{\omega \in \Omega_{2 \rightarrow 0}} f_\omega(x''_{1,2,3}) \right|^2.
\]

To expand the square in the definition of \( W_2 \), we set

\[
\tilde{\omega}(j) = \begin{cases} \omega(j) & j \neq 2 \\ \lambda & j = 2 \end{cases}
\]

\[
\Omega^1_{2 \rightarrow 1} = \{ \tilde{\omega} \mid \omega \in \Omega^1_{2 \rightarrow 0} \}, \quad \Omega^2 = \Omega^1_{2 \rightarrow 0} \cup \Omega^1_{2 \rightarrow 0} \cup \Omega^1_{2 \rightarrow 1},
\]

\[
\Omega_{[1,2] \rightarrow [0, \lambda - 1]} = \{ \omega \in \Omega^2 \mid \omega(1), \omega(2) \in [0, \lambda - 1] \}.
\]

Observe that \( \Omega_{[1,2] \rightarrow [0, \lambda - 1]} = \{ \omega_0, \tilde{\omega}_0, \tilde{\omega}_0, \tilde{\omega}_0 \} \). Then, we can write

(5.28) \[
W_2 = \mathbb{E}_{x'' \in Y''} \prod_{\omega \in \Omega_{[1,2] \rightarrow [0, \lambda - 1]}} f(x''_{1,2,3}) \times Z[\Omega_{[1,2] \rightarrow [0, \lambda - 1]} : \Omega^2 - \Omega_{[1,2] \rightarrow [0, \lambda - 1]}].
\]

where \( Y'' = \{ x''_0, x''_1, x''_2, x''_3 \} \), and \( Z[\Omega_{[1,2] \rightarrow [0, \lambda - 1]} : \Omega^2 - \Omega_{[1,2] \rightarrow [0, \lambda - 1]}] \) is defined in (5.13). (We assumed that \( \omega_0 \equiv 0 \).)

Using Lemma 5.14 and the assumption of \((2\lambda, 3, 4)\)-uniformity on \( V \), we can estimate

\[
\mathbb{E}_{x'' \in Y''} \left( Z[\Omega_{[1,2] \rightarrow [0, \lambda - 1]} : \Omega^2 - \Omega_{[1,2] \rightarrow [0, \lambda - 1]}] \mid \prod_{\omega \in \Omega_{[1,2] \rightarrow [0, \lambda - 1]}} V(x''_{1,2,3}) \right)
\]

\[
\leq \frac{L(V \mid \Omega^2)}{L(V \mid \Omega_{[1,2] \rightarrow [0, \lambda - 1]})}
\]

32
and the conditional variance of $Z[\Omega_{[1,2] \rightarrow [0,1-1]} : \Omega^2 - \Omega_{[1,2] \rightarrow [0,1-1]}]$ is very small. Thus, we can estimate

$$W_2 = 2L(V | \Omega^2) \times \left[ O(\sqrt{\theta}) + \frac{\mathbb{E}_{\omega \in S_3} \|f\|^4_{\Omega_{1,2} S_{1,2}}}{\mathbb{E}_{\omega \in S_3} \|V\|^4_{\Omega_{1,2} S_{1,2}}} \right].$$

Combining (5.21), (5.22), (5.26), (5.27), and (5.29), we see that

$$|L(f_\omega \mid \Omega)| \leq 2L(V | \Omega_{1 \times 0})^{1/2} \cdot L(V | \Omega_{2 \times 0})^{1/4} \cdot L(V | \Omega^2)^{1/4} \cdot W_2^{1/2}.$$  

The last step in the proof of (5.19) is to verify that

$$L(V | \Omega_{1 \times 0})^{1/2} \cdot L(V | \Omega_{2 \times 0})^{1/4} \cdot L(V | \Omega^2)^{1/4} \leq 2 L(V | \Omega).$$

This is again the Conservation of Densities Proposition, Proposition 5.11.

We turn to the third point of the Lemma, namely the inequality (5.20) is true. We can use earlier parts of the argument. Let us combine (5.21), (5.24), (5.26), and (5.27). We have

$$|L(f_\omega \mid \Omega)| \leq 2L(V | \Omega_{1 \times 0})^{1/2} \cdot L(V | \Omega_{2 \times 0})^{1/4} \cdot W_2^{1/4},$$

where $W_2$ is defined in (5.28).

The strategy is to repeat an application of the Cauchy-Schwarz inequality in all variables except $x_3^{0}$. To do this, we define

$$\Omega_{3 \times 0}^2 = \{\omega \in \Omega^2 \mid \omega(3) \neq 0\}, \quad \Omega_{3 \rightarrow 0}^2 = \{\omega \in \Omega^2 \mid \omega(3) = 0\},
X''' = \{x_j^\ell \mid j = 1, 2, 0 \leq \ell \leq \lambda \} \cup \{x_3^\ell \mid 1 \leq \ell \leq \lambda - 1\}.$$  

Here, the point is that the only variable omitted from $X'''$ is $x_3^{0}$. Then, we can estimate

$$W_2 \leq \left[ L(V | \Omega_{3 \times 0}^2) \cdot W_3 \right]^{1/2}$$

and

$$W_3 = \mathbb{E}_{x_j^\ell \in X'''_{\omega \in \Omega_{3 \times 0}^2}} \mathbb{E}_{x_3^\ell \in S_3} \prod_{\omega \in \Omega_{3 \rightarrow 0}^2} V(x_{1,2,3}^\omega) \left| \mathbb{E}_{x_3^\ell \in S_3} \prod_{\omega \in \Omega_{3 \rightarrow 0}^2} f_\omega(x_{1,2,3}^\omega) \right|^2.$$
In the product over $\Omega^2_{3\to\lambda}$ it is important to observe that if $f_\omega = f$, it must follow that $(\omega(1), \omega(2)) \in [0, \lambda]^{1,2}$. For if this is not the case, an earlier step would have switched $f_\omega$ to $V$.

To expand the square, we define

$$\omega(j) = \begin{cases} 
\omega(j) & j \neq 3 \\
\lambda & j = 3
\end{cases}$$

$$\Omega_{3\to\lambda} = \{ \omega \mid \omega \in \Omega^2, \omega(3) = 0 \}, \quad \Omega^3 = \Omega^2 \cup \Omega_{3\to\lambda},$$

$$\Omega_{[1,2,3] \to [0,\lambda]} = [0, \lambda]^{1,2,3}.$$ 

Then, we can write

$$W_3 = \mathbb{E}_{x_{1,2,3} \in \Omega_{1,2,3}} \prod_{\omega \in \Omega_{[1,2,3] \to [0,\lambda]}} f(x_\omega^0) \times Z[\Omega_{[1,2,3] \to [0,\lambda]} : \Omega^3 - \Omega_{[1,2,3] \to [0,\lambda]}].$$

Now, the term $Z$ is nearly constant, by Lemma 5.14, and we have

$$\mathbb{E}\left(Z[\Omega_{[1,2,3] \to [0,\lambda]} : \Omega^3 - \Omega_{[1,2,3] \to [0,\lambda]}] \prod_{\omega \in \Omega_{[1,2,3] \to [0,\lambda]}} V\right) = \frac{L(V \mid \Omega^3)}{L(V \mid \Omega_{[1,2,3] \to [0,\lambda]})}$$

Therefore, we can estimate

$$W_3 = \left[O(\sqrt{\delta}) + \frac{\|f\|_{\Omega^1_{[1,2,3]S_{1,2,3}}}^8}{\|V\|_{\Omega^1_{[1,2,3]S_{1,2,3}}}^8}\right] \times L(V \mid \Omega^3).$$

Combine (5.30), (5.31), (5.32), and (5.33) to conclude that

$$|L(f_\omega \mid \omega \in \Omega)| \leq 2 L(V \mid \Omega_{1\neq0})^{1/2} \cdot L(V \mid \Omega_{2\neq0})^{1/4} \cdot L(V \mid \Omega_{3\neq0})^{1/8} \times L(V \mid \Omega^3)^{1/8} \cdot \left[O(\sqrt{\delta}) + \frac{\|f\|_{\Omega^1_{[1,2,3]S_{1,2,3}}}^8}{\|V\|_{\Omega^1_{[1,2,3]S_{1,2,3}}}^8}\right]^{1/8}.$$ 

Therefore, it remains for us to check that

$$L(V \mid \Omega_{1\neq0})^{1/2} \cdot L(V \mid \Omega_{2\neq0})^{1/4} \cdot L(V \mid \Omega_{3\neq0})^{1/8} \cdot L(V \mid \Omega^3)^{1/8} \leq 2 L(V \mid \Omega).$$

This again follows from Proposition 5.11.

\[\square\]
6 Linear Forms for the Analysis of Corners

In this section, we reprise the initial portion of the previous section, though our needs are not quite as significant. For the uses of this discussion, let us make the definition

\[ \widetilde{T}_\ell = \prod_{1 \leq j < k \leq 4 \atop j, k \neq \ell} R_{jk}. \]

This is the same definition as for \( T_\ell \), but the set \( S_\ell \) is missing.

For \( \Omega \subset \Omega_{4 \to 3} \), where \( \lambda \leq 3 \), and choices of functions \( F_\omega \in \{ T_\ell, \widetilde{T}_\ell \mid 1 \leq \ell \leq 4 \} \), we have the linear form

\[ \Lambda(F_\omega \mid \Omega) = \mathbb{E}_{x_{1,2,3,4} \in S_{1,2,3,4}} \prod_{\omega \in \Omega} F_\omega(x_{1,2,3,4}). \]

Here, any \( S_j \) that occurs in this expectation is composed with \( \lambda_j \). Our first Lemma states that we can easily estimate the values of these forms.

**6.1 Lemma.** For \( \Omega \) and choices of \( F_\omega \) as above we have

\[ \Lambda(F_\omega \mid \Omega) = \prod_{\ell=1}^{4} \prod_{\omega \in \Phi(\ell)} \delta_{\ell} \cdot \prod_{1 \leq j < k \leq 4} \prod_{\omega \in \Psi(j,k)} \delta_{jk}. \]

\[ \Phi(\ell) = |\{ \omega \mid F_\omega = T_\ell \}|, \quad \Psi(j,k) = |\{ \omega \mid j,k \mid \omega \in \Omega \}|. \]

In the last display we are counting the number of distinct maps there are when \( \omega \) is restricted to the sets \( \{ j, k \} \).

**Proof.** We have

\[ \prod_{\omega \in \Omega} F_\omega(x_{1,2,3,4}) = \prod_{\ell=1}^{4} \prod_{\omega \in \Phi(\ell)} S_\ell \circ \lambda(x_{1,2,3,4}) \times \prod_{1 \leq j < k \leq 4} \prod_{\omega \in \Psi(j,k)} S_{jk} \circ \lambda(x_{jk}). \]

where \( \psi(\ell) = \{ \omega \mid F_\omega = T_\ell \} \), and \( \psi(j,k) = \{ \omega \mid j,k \mid \omega \in \Omega \} \). The Lemma then follows from the assumptions of admissibility, namely (3.7) and (3.6), with application of (4.5). \( \square \)

We need an analog of the Conservation of Densities Lemma, Proposition 5.11. Let \( \Omega \subset \Omega_{4 \to 3} \), and assume that for the set \( \Omega_{1 \to 0} \) below is not empty.

\[ \Omega_{1 \to 0} = \{ \omega \in \Omega \mid \omega(1) = 0, F_\omega \neq \widetilde{T}_1 \}, \quad \Omega_{1 \to 0} = \Omega - \Omega_{1 \to 0}. \]
Here, we exclude $\tilde{T}_1$, as its expectation does not include any $\delta_1$.

Consider the estimate below obtained by applying the Cauchy-Schwartz inequality in all variables except $x_1^0$.

\begin{equation}
\Lambda(F_\omega \mid \Omega) \leq \left[ \Lambda(\Omega_{1,\not\rightarrow 0}) \cdot U_2 \right]^{1/2},
\end{equation}

\begin{equation}
U_2 = \mathbb{E} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega) \cdot \left| \mathbb{E}_{\omega_1 \in S_1} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega) \right|^2.
\end{equation}

Use (7.11) to write the last term as $U_2 = \Lambda(F_\omega \mid \Omega^1)$, where we define

\begin{equation}
\mathbb{E}_{x_0 \in S_1} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega) = \Lambda(F_\omega \mid \Omega^1).
\end{equation}

And we define $F_{\sigma} = F_\omega$.

6.4 Second Proposition on Conservation of Densities. If if $\Omega \subset \Omega_{3,\rightarrow 1}$, with the notation in (6.2)—(6.3) we have the equality

\begin{equation}
\Lambda(F_\omega \mid \Omega) \overset{U}{=} \Lambda(F_\omega \mid \Omega_{1,\not\rightarrow 0})^{1/2} \cdot \Lambda(F_\omega \mid \Omega^1)^{1/2}.
\end{equation}

Proof. Each $\omega \in \Omega$ be such that it contributes $1$ to the density $\delta_\ell$, for $2 \leq \ell \leq 4$ on the left-hand-side of (6.5). Thus, $\omega \in \Omega_{1,\not\rightarrow 0}$, and it contributes a $1/2$ to this same density in each of the two terms on the right-hand side. Let $\omega \in \Omega_{1,\not\rightarrow 0}$. Then, it contributes a $1$ to the density of $\delta_1$ on the left-hand side, while on the right hand-side, there is no contribution from the first term, while the second term contributes a $2 \cdot 1/2 = 1$, since the there is a new variable $x_4^1$.

If one considers a density $\delta_{j,k}$ where $2 \leq j < k \leq 4$, it is accounted for much as the case of $\delta_2$ above. And a density $\delta_{1,j}$, with $j = 2, 3, 4$, is accounted for as is $\delta_1$ above. \hfill \Box

This Conservation of Densities has an essentially equivalent formulation, also important to us, that we give here. With the notation of (6.2)—(6.3), set

\begin{equation}
Z[\Omega_{1,\not\rightarrow 0} : \Omega_{1,\not\rightarrow 0}] = \mathbb{E}_{x_1^0 \in S_1} \prod_{\omega \in \Omega_{1,\not\rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega)
\end{equation}

36
6.6 Lemma. For all choices of $\Omega \subset \Omega_{4 \rightarrow 3}$ as above, we have
\[
\text{Var}_{\chi' \in \Omega}(Z[\Omega_{1 \not
rightarrow 0} : \Omega_{1 \rightarrow 0}] | \prod_{\omega \in \Omega_{1 \not
rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega))
\leq K \sqrt{8} \cdot [\mathbb{E}(Z[\Omega_{1 \not
rightarrow 0} : \Omega_{1 \rightarrow 0}] | \prod_{\omega \in \Omega_{1 \not
rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega))]^2.
\]
Here, $K$ is an absolute constant.

Of course the conditional expectation of $Z$ can be computed.

Proof. We use the standard formula for the variance of a random variable $W$ supported on a set $Y$ given in (5.16). The conditional variance will be small if we have
\[
\mathbb{E}(Z[\Omega_{1 \not
rightarrow 0} : \Omega_{1 \rightarrow 0}] | \prod_{\omega \in \Omega_{1 \not
rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega)) \leq \mathbb{E}(Z[\Omega_{1 \not
rightarrow 0} : \Omega_{1 \rightarrow 0}] | \prod_{\omega \in \Omega_{1 \not
rightarrow 0}} F_\omega(x_{1,2,3,4}^\omega))^2.
\]
But this is a recasting of (6.5). □

There is a variant of the inequality (5.20) which holds. Let us formulate it.

6.7 Lemma. Let $\Omega \subset \Omega_{4 \rightarrow 3}$, and let $F_\omega \in \{T_1, T_2, T_3, T_4\}$. Let $f_\omega$ be a choice of function satisfying $|f_\omega| \leq F_\omega$. Then, we have the following inequality. Suppose, for the sake of simplicity that for $\omega_0 \in \Omega$ we have $F_{\omega_0} = T_1$

\[
|\Lambda(f_\omega | \Omega)| \leq 2|\Lambda(F_\omega | \Omega)| \times \left\{ u + \frac{\|f_{\omega_0}\|_{L_{2,3,4}}^{8} H_{2,3,4}}{\|T_1\|_{L_{2,3,4}}^{8} H_{2,3,4}} \right\}^{1/8}
\]

In view of the fact that we have the Second Conservation of Densities Proposition, Proposition 6.4 and the variance principle Lemma 6.6, the proof of this inequality is just an iteration of the proof of (5.20) above, as well as the proof of Lemma Z.1 below. Accordingly we omit it.
7 Proof of the von Neumann Lemma

This is a careful application of weighted Gowers-Cauchy-Schwartz inequality, which does not seem to follow from any standard inequality in the literature. The primary difference with the weighted inequalities of the work of Green and Tao, [8, 11] is the absence of the von Mangoldt function with its uniformity properties, a difference overcome by the enforced uniformity, an argument invented by Shkredov [18].

In our setting, the sets $X_a$ will most frequently be $H$, the copy of the finite field. The set $U$ will for the most part be $\{1, 2, 3, 4\}$, though there are larger sets $U$, as large as 24 elements, that occurs in the analysis of different terms below.

We introduce the following 4-linear form. For four functions $f_j : H \times H \times H \to \mathbb{C}$, for $1 \leq j \leq 4$, define

$$Q(f_1, f_2, f_3, f_4) \overset{\text{def}}{=} \mathbb{E}_{y, x_j \in H} f_4(x_1, x_2, x_3) f_3(x_1, x_2, x_3 + y) \times f_2(x_1, x_2 + y, x_3) f_1(x_1 + y, x_2, x_3)$$

If $A \subset H \times H \times H$, it follows that $Q(A, A, A, A)$ is the expected number of corners in $A$. It is an important remark that this is defined as an average over copies of $H$, whereas earlier sections have been defined over e.g., $S_{1,2,3,4}$. This fact introduces extra factors of $\delta_\ell$ below.

We are deliberately choosing a definition that is slightly asymmetric with respect to the subscripts on the $f_j$ on the right above, to make the next display more symmetric. Using the change of variables $y = x_4 - (x_1 + x_2 + x_3)$, this is

$$Q(f_1, f_2, f_3, f_4) = \mathbb{E}_{x_j \in H} \prod_{1 \leq j \leq 4} f_j \circ \lambda_j,$$

$$\lambda_j(x_1, x_2, x_3, x_4) = \sum_{k : k \neq j} x_k e_k, \quad 1 \leq j \leq 4.$$

The point which dominates the analysis below is that the functions $f_j \circ \lambda_j$ is a function of $\{x_\ell \mid 1 \leq \ell \neq j \leq 4\}$, i.e., is not a function of $x_j$.

We will write, by small abuse of notation, $\lambda_1(x_{1,2,3,4}^\ell) = x_{2,3,4}^\ell$. This is allowed, as $\lambda_1(x_{1,2,3,4}^\ell)$ is not a function of $x_1^\ell(1)$. This will allow us reduce the complexity of some formulas below.
We codify the result of the application of the proof of the Gowers-Cauchy-Schwartz Inequality for the operator $Q$ into the results of the following Lemma. This technical result codifies the results that we need to understand about the set $T$, and $A$ to conclude Lemma 3.13.

In this Lemma, we single out for a distinguished role the function that falls in the last place of $Q$, but there is a corresponding estimate for all the other three functions.

**7.1 Lemma.** Let $T_j$ either be identically $T$, or $T_j = T_j$ for all $1 \leq j \leq 4$. Let $f_j : T_j \rightarrow [-1, 1]$ be functions. We have the following estimate.

\[
(7.2) \quad |Q(f_1, f_2, f_3, f_4)| \leq U_1^{1/2} \cdot U_2^{1/4} \cdot U_3^{1/8} \cdot U_4^{1/8},
\]

\[
(7.3) \quad U_1 = U_1(T_1) = \mathbb{E}_{x_2, x_3, x_4 \in H} \sum_{x_1 \in H} T_1(x_2, x_3, x_4),
\]

\[
(7.4) \quad U_2 = U_2(T_2) = \mathbb{E}_{x_3, x_4 \in H} \prod_{x_1 \in H} T_2(x_1^{(1, 3, 4)}),
\]

\[
(7.5) \quad U_3 = U_3(T_3) = \mathbb{E}_{x_4 \in H} \prod_{x_1, x_2 \in H} T_3(x_1^{(1, 2, 4)}),
\]

\[
(7.6) \quad U_4 = U_4(f_4, T_1, T_2, T_3) = \mathbb{E}_{x_1 \in H} \prod_{x_1, x_2, x_3 \in H} T_4(x_1^{(1, 2, 3)}),
\]

\[
(7.7) \quad Z = Z(T_1, T_2, T_3) = \mathbb{E}_{x_1 \in H} \prod_{x_1, x_2, x_3 \in H} T_j \circ A_j(x_1^{(1, 2, 3, 4)}).
\]

This Lemma makes it clear that we need to understand the linear forms $U_1, U_2, U_3$, and $Z$ for both the $T_j$ and for $T$.

**7.8 Remark.** The presence of the term $Z$ in (7.14) can be seen in the argument of [15], but it is not needed in Shkredov’s approach [18]. However, this term is much more subtle in the three dimensional case. Similar terms will arise in §8 are dealt with systematically in Lemma 5.14.

**Proof.** The method of proof is to follow the proof of the Gowers-Cauchy-Schwartz inequality, especially in the case of (4.7), but keeping track of the additional information that follows from terms that are neglected in the usual proofs of this inequality. All earlier applications of the Gowers-Cauchy-Schwartz inequality has in some sense ‘lost units of density.’ In the present argument, we recover these lost units by the mechanism of the various functions of $T$ that appear in the definitions of $U_1, U_2$ and $U_3$ above.
Estimate the left-hand side of (7.2) by

\[ |Q(f_1, f_2, f_3, f_4)| \leq \left| U_1 \cdot U_1 \right|^{1/2}, \]

\[ U_1 = \mathbb{E}_{x_2, x_3, x_4 \in H} |f_1 \circ \lambda_1|^2 \leq \mathbb{E}_{x_2, x_3, x_4 \in H} T_1(x_2, x_3, x_4), \]

\[ U_{1,2} = \mathbb{E}_{x_2, x_3, x_4 \in H} T_1(x_{[2,3,4]}) \left| \mathbb{E}_{x_3} \prod_{j=1}^{3} f_{\lambda(j)} \circ \lambda_j(x_{[1,2,3,4]}) \right|^2. \]

We use the Cauchy-Schwartz inequality in the variables \( x_2, x_3, x_4 \). The term in (7.9) proves (7.3). In the last line, we are using the notation of the general Gowers-Cauchy-Schwartz Inequalities, so that \( x_{[1,2,3,4]} = (x_1, x_2, x_3, x_4) \). This will be helpful in the steps below.

For \( U_{1,2} \), we use the elementary fact that

\[ \mathbb{E}_{x \in X} g(x) \left| \mathbb{E}_{y \in Y} f(x, y) \right|^2 = \mathbb{E}_{x \in X, y \in Y} g(x) \prod_{e=0}^{2} f(x, y^e). \]

This is in fact crucial to the proof of the Gowers-Cauchy-Schwartz inequality. In particular, it is essential that we insert the \( T_1(x_{[2,3,4]}) \) on the right in (7.10). Thus,

\[ U_{1,2} = \mathbb{E}_{x_2, x_3, x_4 \in H} T_1(x_{[2,3,4]}) \prod_{\omega \in \{0,1\}^{[1]} \times \{0\}^{[2,3,4]}} \prod_{j=2}^{4} f_{\lambda(j)} \circ \lambda_j(x_{[1,2,3,4]}). \]

We refer to this identity as ‘passing \( x_1 \) through the square.’ With this notation, it is clear that the variables \( x_2, x_3, x_4 \) will also need to ‘pass through the square’.

Thus, we write as below, using the Cauchy-Schwartz inequality in the variables \( x_1^0, x_1^1, x_3^0, \) and \( x_4^0 \).

\[ U_{1,2} \leq \left[ U_2 \cdot U_{2,2} \right]^{1/2}, \]

\[ U_2 = \mathbb{E}_{x_3, x_4 \in H} T_2 \circ \lambda_2(x_{[1,2,3,4]}), \]

\[ U_{2,2} = \mathbb{E}_{x_3, x_4 \in H} \prod_{\omega \in \{0,1\}^{[1]} \times \{0\}^{[3,4]}} T_2(x_{[1,3,4]}), \]

\[ \times \left| \mathbb{E}_{x_2 \in H} T_1(x_{[2,3,4]}) \prod_{\omega \in \{0,1\}^{[1]} \times \{0\}^{[2,3,4]}} \prod_{j=3}^{4} f_{\lambda(j)} \circ \lambda_j(x_{[1,2,3,4]}) \right|^2. \]
The term in (7.12) is (7.4).

For the term (7.13), we write

\[
U_{2,2} = \mathbb{E} \left[ \prod_{x^0, x^1, x^2 \in H} \prod_{\omega \in [0,\omega^{[2]} \times [0,\omega^{[3]}]} \left[ T_2(x_{[1,3],4}^\omega) T_1(x_{[2,3],4}^\omega) \right] \right]
\]

We estimate using the Cauchy-Schwartz inequality in the variables \(x_{1,2}^0, x_{1,2}^1\) and \(x_{4}^0\).

\[
U_{2,2} \leq \left[ U_3 \cdot U_{3,2} \right]^{1/2},
\]

\[
U_3 = \mathbb{E} \prod_{x^0, x^1, x^2 \in H} \prod_{\omega \in [0,\omega^{[2]} \times [0,\omega^{[3]}]} \left[ T_3(x_{[1,2],4}^\omega) \right] \]

\[
U_{3,2} = \mathbb{E} \prod_{x^0, x^1, x^2 \in H} \prod_{\omega \in [0,\omega^{[2]} \times [0,\omega^{[3]}]} \left[ T_2(x_{[1,3],4}^\omega) T_1(x_{[2,3],4}^\omega) \right] \]

\[
\times \prod_{j=2}^4 f_{\epsilon(j)} \circ \lambda_j(x_{[1,2,3],4}^\omega) \right]^{1/2}
\]

The term \(U_3\) is (7.5).

We write \(U_{3,2}\) as follows, after application of (7.11), and recalling the definition of \(Z\) in (7.7).

\[
(7.14) \quad U_{3,2} = \mathbb{E} \prod_{x^0, x^1, x^2 \in H} \prod_{\omega \in [0,\omega^{[2]} \times [0,\omega^{[3]}]} Z \cdot f_4 \circ \lambda_4(x_{[1,2,3],4}^\omega) \]

This completes the proof.

\[
\square
\]

We now provide the estimates that the previous Lemma calls for, in the case of the sets \(T_j\).

7.15 Lemma. For the terms \(U_1, U_2, U_3\) and \(Z\) as defined in (7.3)—(7.5) and (7.7), and \(T_j = T_j\) we have these estimates.

\[
(7.16) \quad Q(T_1, T_2, T_3, T_4) \leq U_1(T_1)^{1/2} \cdot U_2(T_2)^{1/4} U_3(T_3)^{1/8} \cdot U_4(T_4, T_3, T_2, T_1)^{1/8}.
\]
The constant $\vartheta$ in the definition of $\bar{u}$, see Definition 5.2, can be taken to be $\vartheta = P(T \mid H \times H \times H)^C$, where $C$ is a large constant, depending only on $C_{\text{admiss}}$ in Definition 3.4. And for $Z(T_1, T_2, T_3)$, we have this inequalities on conditional variance.

\begin{equation}
\text{Var}(Z(T_1, T_2, T_3) \mid \prod_{\omega \in \{0\}^{[1,2,3]} \times \{0\}^{[4]}} T_4(x^\omega_{[1,2,3]}) \leq \vartheta P(A \mid H \times H \times H)^C.
\end{equation}

\textbf{Proof.} The first claim (7.16) follows from (an iteration of) the Second Proposition on Conservation of Densities, Proposition 6.4. The second from Lemma 6.6. \qed

The content of the next Lemma is that in the case where $A \subset T$ has full probability, that $A$ has the expected number of corners.

\textbf{7.18 Lemma.} Let $A$ be an admissible corner system. Then, we have

\begin{equation}
Q(T, T, T, T) = \prod_{\ell=1}^4 \delta_{T \mid \ell} \times Q(T_1, T_2, T_3, T_4).
\end{equation}

Here, the constant $\vartheta$ implicit in the $\bar{u}$ can be taken to be $\vartheta = \kappa \epsilon$, where these two constants are determined by $\kappa_{\text{admiss}}$ and $\epsilon_{\text{admiss}}$ in Definition 3.4 and can be made arbitrarily small.

\textbf{Proof.} One considers the expression in (7.19) is a 4-linear form, and expand $T$ as $T = f_{j,1} + f_{j,0}$, where $f_{j,1} = \delta_{T \mid j} T_j$. This leads to an expansion of $Q(T, T, T, T)$ into $2^4$ terms, of which the leading term is

\begin{equation}
Q(f_{1,1}, f_{2,1}, f_{3,1}, f_{4,1}) = \prod_{j=1}^4 \delta_{T \mid j} \cdot Q(T_1, T_2, T_3, T_4).
\end{equation}

The remaining $2^4 - 1$ terms all have at least one $f_{j,0}$. We can show that all of these terms is at most a small constant times the expression above by appealing to (3.5) and (4.7). In particular, we show that we can estimate

\begin{equation}
\left|Q(f_{1,(1)}, f_{2,(2)}, f_{3,(3)}, f_{4,0})\right| \leq 2 Q(T_1, T_2, T_3, T_4) \cdot \left[\nu + \frac{||f_{4,0}||_{\Delta_{[1,2,3]S_{1,2,3}}}^8}{||T_4||_{\Delta_{[1,2,3]S_{1,2,3}}}^8} \right]^{1/8}.
\end{equation}
By (3.5), this proves that this term is very small. This inequality singles out the fourth coordinate for a special role, but the proof, presented in full in this case, holds in full generality, so completes this case.

Apply Lemma 7.1 with \(T_j = T_j\) and \(f_j = f_j(e(j))\) as above. The estimate we get from this Lemma is (7.2), with the terms in (7.3)—(7.7) estimated in Lemma 7.15. The particular point to observe is that the function \(Z\) has a small conditional variance (7.17). These conditional estimates hold on the support of the product that occurs in (7.6). Hence, we can estimate

\[
\left| Q(f_1,e(1), f_2,e(2), f_3,e(3), f_4,0) \right| \leq U_1(T_1)^{1/2} \cdot U_2(T_2)^{1/4} \cdot U_3(T_3)^{1/8} \cdot U_4(T_1, T_2, T_3, f_4,0)^{1/8}
\]

\[
= U_1(T_1)^{1/2} \cdot U_2(T_2)^{1/4} \cdot U_3(T_3)^{1/8} \cdot \mathbb{E} \left( Z(T_1, T_2, T_3) \mid T_4(x_{[1,2,3]}^\omega)^{1/8} \right)
\]

\[
\times \| T_4 \|_{\mathbb{D}^{1,2,3}_{H_{1,2,3}}} \cdot \left\{ v + \frac{\| f_4,0 \|_{\mathbb{D}^{1,2,3}_{H_{1,2,3}}}}{\| T_4 \|_{\mathbb{D}^{1,2,3}_{H_{1,2,3}}}} \right\}
\]

In the last line, \(v\) is a small quantity arising from the conditional variance estimate (5.15).

The key identity is (7.16). In it, observe that

\[
U_4(T_4, T_3, T_2, T_1) \overset{\mu}{=} \| T_4 \|_{\mathbb{D}^{1,2,3}_{H_{1,2,3}}} \cdot \mathbb{E} \left( Z(T_1, T_2, T_3) \mid T_4(x_{[1,2,3]}^\omega) \right).
\]

Therefore, we have

\[
Q(T_1, T_2, T_3, T_4) \overset{\mu}{=} U_1(T_1)^{1/2} \cdot U_2(T_2)^{1/4} \cdot U_3(T_3)^{1/8}
\]

\[
\times \mathbb{E} \left( Z(T_1, T_2, T_3) \mid T_4(x_{[1,2,3]}^\omega) \right)^{1/8}
\]

\[
\times \left\{ v + \| T_4 \|_{\mathbb{D}^{1,2,3}_{H_{1,2,3}}} \right\}
\]

And this completes the proof of (7.20) and hence the Lemma. \(\square\)

To apply Lemma 7.1 to prove Lemma 3.13, we will need estimates for the terms in (7.3)—(7.5). We turn to this next, discussing the estimates for the terms \(U_j\). The estimates for \(Z(T, T, T, T)\) as defined in (7.7) we discuss in the next Lemma.
7.21 Lemma. We have the estimates below for the forms $U_j$ defined in (7.3)—(7.6).

\[
\begin{align*}
U_1(T) & = \delta_{T | 1} U_1(T), \\
U_2(T) & = \delta_{T | 2} U_2(T), \\
U_3(T) & = \delta_{T | 3} U_3(T), \\
\|T\|_{\Omega_{1,2,3}}^8 & = \delta_{T | 4}^8 \cdot \|T_4\|_{\Omega_{1,2,3}}^8
\end{align*}
\]

The implied constant $\delta$ in the definition of $\|\cdot\|$ can be taken to be $\mathbb{P}(T | H \times H \times H)$ to some large power.

Proof. The equality (7.24) is a corollary to part 2 of Lemma 5.4 and Definition 5.2. The other parts of the Lemma are also corollaries to the same fact, but not as stated, but with the role of $T_4$ in Definition 5.2 replaced by that of $T_2$ for (7.22), and $T_3$ for (7.23). \(\Box\)

We turn to the analysis of the term $Z(T, T, T)$ as defined in (7.7).

7.25 Lemma. We have the estimates below where $Z = Z(T, T, T)$.

\[
(7.26) \quad \mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} (Z \mid U) \leq \prod_{j=1}^{3} \delta_{T | j} \times \mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} (Z(T_1, T_2, T_3) \mid U),
\]

\[
\text{Var}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} (Z \mid U) \leq \delta_{A \mid T}^{12},
\]

where $U = \prod_{\omega \in \{0,1\}^{1,2,3}} \prod_{1 \leq j < k \leq 3} R_{j,k}(x_{j,k}^\omega)$.

The implied constant in $\|\cdot\|$ can be taken as in Lemma 7.18.

Here, note that we are using the conditional expectation notation. As the random variable $Z$ is supported on the event $U \subset H_{1,2,3}^0 \times H_{1,2,3}^1$, we have

\[
(7.27) \quad \mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} (Z \mid U) = \frac{\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} Z}{\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} U}
\]

\[
(7.28) \quad \text{Var}(Z \mid U) = \frac{\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} Z^2 - (\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} Z)^2}{\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} U} \left(\mathbb{E}_{\chi_{1,2,3}^0}^{\chi_{1,2,3}^1} e_{H_{1,2,3}} U\right)^{-1}
\]

44
And the point of the Lemma is that the random variable $Z$ is nearly constant on the set $U$, and we can compute that constant.

**Proof.** We first calculate the denominator in (7.27) and (7.28). This is relatively simple as the sets $R_{jk}$ are uniform in $S_j \times S_k$, so that we can estimate

$$E_{x_{(1,2,3)}^j \in H_{(1,2,3)}^j \times \Omega_{(1,2,3)}} U = \prod_{j=1}^3 \delta_j^2 \prod_{1 \leq j < k \leq 3} \delta_{jk}^4. \tag{7.29}$$

We now turn to the numerator in (7.27). The expectation of $Z$ in (7.27) is thought of as a 12-linear form. Set

$$\Omega_{\neq j} = \{0,1\}_{1 \leq k \neq j \leq 3} \times \{0\}^4, \quad 1 \leq j \leq 3.$$ Set $\Omega_{\neq} = \bigcup_{j=1}^3 \Omega_{\neq j}$. For functions $\{f_\omega \mid \omega \in \Omega_{\neq}\}$ define

$$L(f_\omega \mid \Omega_{\neq}) = E_{x_{(1,2,3)}^j \in H_{(1,2,3)}^j \times \omega \in \Omega_1} \prod_{\omega \in \Omega_1} f_\omega.$$ We are to prove the estimate

$$L(T \mid \Omega_{\neq}) \equiv \prod_{j=1}^3 \delta_1^{4j} \cdot L(T_j \mid \Omega_{\neq j}, 1 \leq j \leq 3). \tag{7.30}$$

Expand $T \circ \lambda_j = f_{j,1} - f_{j,0}$, where $f_{j,1} = \delta T \mid T_j$. The leading term is then when $f_{j,1}$ occurs in all twelve positions. But, then we have the Second Conservation of Densities Proposition at our disposal, so that (7.30) follows from Proposition 6.4.

The ratio of (7.30) and (7.29) proves (7.26), provided the other terms arising from the expansion of the 12-linear form are all sufficiently small. That is, we should see that for all $2^{12} - 1$ selections of $f_{j,\epsilon(\omega)} \in \{f_{j,0}, f_{j,1}\}$ for $\omega \in \Omega_{\neq j}$, $1 \leq j \leq 3$, with at least one $f_{j,\epsilon(\omega)} = f_{j,0}$ we have

$$\|L(f_{j,\epsilon(\omega)} \mid \Omega_{\neq})\| \leq \kappa L(T \mid \Omega_{\neq}), \tag{7.31}$$

for a suitably small constant $\kappa$.

If we use the same line of reasoning that we have before, this would lead to a (yet) longer multi-linear form. We therefore present the following variant of the argument

45
used thus far. We prove (7.31) under the following assumptions. For some \( \omega \in \Omega_{\neq 1} \), we have \( f_{1,\epsilon(\omega)} = f_{1,0} = T - \delta_{T|1} T_1 \). Moreover, this happens for \( \omega \equiv 0 \), which we can assume after a change of variables. Finally, let \( J_{\text{small}} = \{ j = 2, 3 \mid \delta_{T|j} < \delta_{T|1} \} \). We assume that \( f_{j,\epsilon(\omega)} = \delta_{T|j} T_j \) for all \( j \in J_{\text{small}} \). This can also be assumed, after a permutation of the coordinates. We now prove the inequality

\[
(7.32) \quad \left| L(f_{j,\epsilon(\omega)} | \Omega_{\neq 1}) \right| \leq \prod_{j \in J_{\text{small}}} \delta_{T|j}^4 \cdot L(T_j | \Omega_{\neq j}, 1 \leq j \leq 3) \cdot \left[ v + \frac{\|f_{1,0}\|^8_{\square[2,3,4]}}{\|T_1\|^8_{\square[2,3,4]}} \right]^{1/8}.
\]

Here, \( v \) will be a very small positive constant. Our assumption (3.5), together with the assumption about \( J_{\text{small}} \) permits us to conclude (7.31) from this inequality. In particular, we can accumulate a large number of powers of \( \delta_{T|1} \) from (3.5). The essential point, is that we accumulate the correct power on the densities \( \delta_{T|j} \) for \( j \in J_{\text{small}} \), as there is no a priori reason that the different densities \( \delta_{T|j} \) need be comparable.

But, (7.32) follows from application of the inequality (6.8), and so our proof of the Lemma is complete.

\[\Box\]

Proof of Lemma 3.13. Write \( A = f_0 + f_1 \) where \( f_1 = \delta_{A|T} T \). We expand

\[
Q(A, A, A, A) = \sum_{\epsilon \in M_4} Q(f_{\epsilon(1)}, f_{\epsilon(2)}, f_{\epsilon(3)}, f_{\epsilon(4)}).
\]

The leading term is for the function \( \epsilon \equiv 1 \). It is \( \delta_{A|T}^4 Q(T, T, T, T) \), with the latter expression estimated in (7.19).

All other choices of \( \epsilon \) have at least one choice choice of \( 1 \leq j \leq 4 \) for which we have \( \epsilon(j) = 0 \). We claim that for all of these we have the estimate

\[
(7.33) \quad |Q(f_{\epsilon(1)}, f_{\epsilon(2)}, f_{\epsilon(3)}, f_{\epsilon(4)})| \leq \kappa \delta_{A|T}^4 Q(T, T, T, T).
\]

This depends upon the assumption (3.15). For \( \kappa < 2^{-32} \), this will show that \( Q(A, A, A, A) \geq \frac{1}{4} \delta_{A|T}^4 Q(T, T, T, T) \). From this, we conclude that the number of corners in \( A \) is at least

\[
Q(A, A, A, A)|H|^4 - |A| \geq \frac{1}{4} \delta_{A|T}^4 Q(T, T, T, T)|H|^4 - |A| > 0.
\]

Here, we subtract off \( |A| \), as the average \( Q(A, A, A, A) \) includes the ‘trivial corners’ where all four points in the corner are the same. The inequality holds by (3.14), and this completes the proof.
We prove (7.33) for \( \epsilon(4) = 0 \), with the other cases following by symmetry. Apply Lemma [7.1] with \( T_j = T \), and \( f_4 = f_0 \). This gives us the inequality

\[
|Q(f_{\epsilon(1)}, f_{\epsilon(2)}, f_{\epsilon(3)}, f_0)| \leq U_1(T)^{1/2} \cdot U_2(T)^{1/4} \cdot U_3(T)^{1/8} \cdot U_4(f_0, T, T, T)^{1/8}.
\]

The terms \( U_j(T) \) for \( j = 1, 2, 3 \) are estimated in Lemma [7.21]. The definition of \( U_4(f_0, T, T, T) \) in (7.6) depends upon \( Z \), which has its properties listed in Lemma [7.25]. This leads us to the estimate

\[
Q(f_{\epsilon(1)}, f_{\epsilon(2)}, f_{\epsilon(3)}, f_0) \leq U_1(T)^{1/2} \cdot U_2(T)^{1/4} \cdot U_3(T)^{1/8} \cdot \mathbb{E}(Z \mid U)^{1/8} \\
\times \frac{\|f_0\|_{C[1,2,3]}}{\|T\|_{C[1,2,3]}} \cdot \left[ v + \frac{\|f_0\|_{C[1,2,3]}}{\|T\|_{C[1,2,3]}} \right] \\
\leq \prod_{\ell=1}^{4} \delta_{T \mid \ell} \times U_1(T_1)^{1/2} \cdot U_2(T_2)^{1/4} \cdot U_3(T_3)^{1/8} \\
\times \mathbb{E}(Z(T_1, T_2, T_3) \mid U)^{1/8} \\
\times \frac{\|T_4\|_{C[1,2,3]}}{\|T\|_{C[1,2,3]}} \cdot \left[ v + \frac{\|f_0\|_{C[1,2,3]}}{\|T\|_{C[1,2,3]}} \right] \\
\leq Q(T, T, T) \left[ v + \frac{\|f_0\|_{C[1,2,3]}}{\|T\|_{C[1,2,3]}} \right].
\]

Our proof is complete. \( \square \)

8 The Paley-Zygmund Inequality for the Box Norm and the set \( T \)

Let us recall the following classical result.

**8.1 The Paley-Zygmund Inequality.** There is a \( 0 < c < 1 \) so that for all random variables \(-1 < Z < 1 \) with \( \mathbb{E}Z = 0 \) we have \( \mathbb{P}(Z > c\mathbb{E}Z^2) \geq c\mathbb{E}Z^2 \).

Our central purpose in this section is to provide extensions of this result to the case where the assumption on the standard deviation of the random variable is replaced by an assumption on the Box Norm. Extensions are provided into two different settings, an
'unweighted' and a 'weighted' one. Indeed, in the unweighted case, we will only require the two dimensional version of this inequality.

8.2 The Paley-Zygmund Inequality for the Box Norm. There is a constant $c(2)$, and $t(2) > 1$ so that the following holds. For all finite sets $X_t$, $1 \leq t \leq 2$, and subsets $A \subset X_{[1,2]}$, set $\delta = |A - \mathbb{P}(A)|_{[2]}$ and $\sigma = |A - \mathbb{P}(A)||_{[1.2]}$. There are subsets $X_t^i \subset X_i$, $i = 1, 2$, $\mathbb{P}(X_t^i) \geq c(2)(\sigma \delta)^{t(2)}$, $\mathbb{P}(A \mid X_{t,2}) \geq \delta + c(2)(\delta \sigma)^{t(2)}$.

We refer the reader to [9, Proposition 5.7] or [15, Lemma 3.4] for a proof of this Lemma.

We need a more general version of the Paley-Zygmund Inequality for the Box Norm, based upon the properties of the sets $A \subset T \subset T_i$. We need two Lemmas, with very similar proofs, accordingly we state one Lemma. Our Lemmas should be coordinate-free, but to ease the burden of notation, we state them distinguishing the coordinate $x_4$ for a special role.

8.3 Lemma. There are constants $c > 0$ and $C, p > 1$ so that the following holds. Suppose that $T$ is a $T$-system as in (3.3), which satisfies (3.7) and (3.6). Let $U \subset V \subset T_4$. Assume that $V \in \{T_4, T_1\}$.

$$\frac{\|U - \mathbb{P}(U \mid V) V\|_{[1.2]} \mid_{[1.2]}}{\|V\|_{[1.2]} \mid_{[1.2]}} \geq \tau$$

and that $V$ is $(4, \delta, 4)$-uniform, (Recall Definition 5.2) where

$$\delta = (\tau |\mathbb{P}(U \mid V)|)^C.$$

Then, there is a $T$-system

$$T' = \{H', S'_k, R'_{k,\ell}, T' \mid 1 \leq k, \ell \leq 4, k < \ell\}$$

and a set $V' \subset T_4'$ which satisfy

$$\begin{cases} V' = T'_4 & V = T_4 \\ V' \subset V & V = T \end{cases}$$

$$\begin{cases} |\mathbb{P}(T'_4 \mid T_4) \geq (\tau |\mathbb{P}(U \mid T_4)|^p) V = T_4 \\ |\mathbb{P}(T' \mid T) \geq (\tau |\mathbb{P}(U \mid T)|^p) & V = T \end{cases}$$

$$\mathbb{P}(U \mid T' \cap V) \geq \mathbb{P}(U \mid V) + c(\tau \cdot |\mathbb{P}(U \mid V)|^p).$$
The point of these estimates is that we have a little information about the new data, in \((8.7)\). There are some lower bounds on the probabilities of the elements of the new \(T\)-system given by the estimate \((8.8)\). And in \((8.9)\), we have that \(U\) has a slightly larger probability in \(T' \cap V\). Note that we certainly do not assume that the new \(T\)-system \(T'\) satisfies the uniformity assumptions in the definition of admissibility, Definition 3.4.

Proof of Lemma 3.16. To prove Lemma 3.16, apply Lemma 8.3 with \(V = T, U = A,\) and \(\tau = \kappa \delta^4\) where \(\kappa\) is as in \((3.15)\). The conclusions of Lemma 8.3 then imply those of Lemma 3.16. \(\square\)

8.1 One-Dimensional Obstructions

We carry out the proof of Lemma 8.3. Throughout, we use the expansion \(U = f_1 + f_0\) where \(f_1 = \delta_{U \mid V} V\) where \(\delta_{U \mid V} = \mathbb{P}(U \mid V)\). We will also use the notation \(\delta_{V \mid 4} = \mathbb{P}(V \mid T_4)\). The key assumption \((8.4)\), which could hold due to lower-dimensional obstructions, and so there are two initial stages in which we address these obstructions.

We begin by considering the possibility that \((8.4)\) holds for some one-dimensional reason. Namely, let us assume that, for instance, we have

\[
(8.10) \quad \mathbb{E}_{x_2, x_3 \in S_2, 3} \left| \mathbb{E}_{x_1 \in S_1} f_0(x_1, x_2, x_3) \right|^2 \geq \frac{1}{2} c_1 (\delta_{U \mid V} \tau)^{t_1} \cdot \delta^2 \cdot \delta_{V \mid 4}^2 \cdot \delta_{1,2}^2 \cdot \delta_{1,3}^2.
\]

Note that the last expectation is estimated by virtue of our assumption on \((4, \theta, 4)\)-uniformity, recall \((5.3)\). Here, \(c_1 > 0\) and \(t_1 > 1\) are constants that we will specify below, based upon considerations in the next two stages of our argument.

Let us rephrase \((8.10)\) as

\[
(8.11) \quad \mathbb{E}_{x_2, x_3 \in R_{2,3}} \left| \mathbb{E}_{x_1 \in S_1} f_0(x_1, x_2, x_3) \right|^2 \geq \frac{1}{2} c_1 (\delta_{U \mid V} \tau)^{t_1} \cdot \delta^2 \cdot \delta_{V \mid 4}^2 \cdot \delta_{1,2}^2 \cdot \delta_{1,3}^2
\]

where we have replaced the expectation over \(S_{2,3} = S_2 \times S_3\) by expectation over the smaller set \(R_{2,3}\). Of course, we have \(|\mathbb{E}_{x_1 \in S_1} f_0(x_1, x_2, x_3)| \leq \mathbb{E}_{x_1 \in S_1} V(x_1, x_2, x_3)\). But, the variance of this last random variable over \(R_{2,3}\) is nearly constant. Namely,

\[
(8.12) \quad \text{Var}_{x_2, x_3 \in R_{2,3}} \left( \mathbb{E}_{x_1 \in S_1} V(x_1, x_2, x_3) \right) \leq K \tau \mathbb{E} \left[ \left( \mathbb{E}_{x_1 \in S_1} V(x_1, x_2, x_3) \right)^2 \right].
\]
This follows from assumption and (5.3). Moreover, by (8.12), the random variable (8.11) and (8.13): the Paley-Zygmund inequality, we can use the normalized variance given by the ratio

\[ E_{x_1 \in S_1} V(x_1, x_2, x_3) = \delta_{V|4} \cdot \delta_{1,2} \cdot \delta_{1,3} \cdot \delta_4. \]

This follows from assumption and (5.3). Moreover, by (8.12), the random variable \( E_{x_1 \in S_1} V(x_1, x_2, x_3) \) has very small variance on \( R_{2,3} \), so that except for a negligible probability, it is dominated by, say, twice its expectation. The key point here, is that in applying the Paley-Zygmund inequality, we can use the normalized variance given by the ratio (8.11) and (8.13):

\[
\frac{E_{x_2,3 \in S_2,3} E_{x_1 \in S_1} f_0(x_1, x_2, x_3)}{[E_{x_1 \in S_1} V(x_1, x_2, x_3)]^2} \geq \frac{\frac{1}{2} c_1 (\delta_{U|V^T})^l \delta_{V|4}^2 \delta_{1,2}^2 \delta_{1,3}^2}{\delta_{V|4} \cdot \delta_{1,2} \cdot \delta_{1,3}^2}
\]

Thus, we can estimate

\[
R_{2,3}' = \{ x_{2,3} \in R_{2,3} | E_{x_1 \in S_1} f_0(x_1, x_2, x_3) \geq \frac{c_1}{40} (\delta_{U|V^T})^l E_{x_1 \in S_1} V(x_1, x_2, x_3) \},
\]

(8.14)

\[ \mathbb{P}(R_{2,3}' | R_{2,3}) \geq \frac{1}{10} c_1 (\delta_{U|V^T})^l. \]

We conclude the Lemma by taking the set \( R_{2,3}' \) in (8.6) as above, \( T' = T \cap R_{2,3}' \), and the other data is unchanged. If \( V = T_4 \), the new set \( V' = V \cdot R_{2,3}' \), so that (8.7) holds. That (8.8) holds follows from (8.14), and several applications of (4.7). And that (8.9) holds follows from construction of \( R_{2,3}' \).

### 8.2 Two-Dimensional Obstructions

We continue the proof assuming that (8.10) fails as written, and also fails under any permutation of the variables \( x_1, x_2, \) and \( x_3 \). The potential lower dimensional obstruction are now two-dimensional in nature. We could have for instance

\[ E_{x_1 \in S_1} \| f_0 \|_{L^2_{S_2}}^4 \geq c_2 (\delta_{U|V^T})^l E_{x_1 \in S_1} \| V \|_{L^2_{S_2}}^4. \]

50
Here, \( t_2, c_2 > 0 \) are constants that are to be specified, based upon considerations in the next stage of the argument. The last expectation can be computed exactly, and is

\[
\mathbb{E}_{x_1 \in S_1} \|V\|^4_{\ell^2 S_2 S_3} = \mathbb{E}_{x_1 \in S_1} \prod_{x_{2,3}^0, x_{2,3}^1, x_{2,3}^3 \in S_{2,3}} V(x_1, x_{2,3}^\ell)
\]

(8.16)

\[
\equiv [\delta_4 \delta_{\ell/4}]^4 \prod_{1 \leq j < k \leq 3} \delta_{jk}^2.
\]

Of course we have \(\|f_0\|_{\ell^2 S_2 S_3} \leq \|V\|_{\ell^2 S_2 S_3}^4 \). Still, the deduction of the Lemma in this case doesn’t follow from a a straight forward application of Lemma 8.2 in two dimensions, as we are in the weighted case. This argument is the one that relates the constants \( c_1, t_1 \) and constants \( c_2, t_2 \).

Following notation used in the proof of Lemma 8.2, we define a four linear term which arises from (8.15).

\[
B_4(f_0, f_0, f_1, f_1) = \mathbb{E}_{x_1 \in S_1} \prod_{x_{2,3}^0, x_{2,3}^1, x_{2,3}^3 \in S_{2,3}} f_\ell(x_1, x_{2,3}^\ell).
\]

(8.17)

Note that the left-hand-side of (8.15) is \( B_4(f_0, f_0, f_0, f_0) \), and that \( \mathbb{E}_{x_1 \in S_1} \|V\|^4_{\ell^2 S_2 S_3} = B_4(V, V, V, V) \), which is given in (8.16).

Our central claims are these inequalities, which hold for \( c_1, t_1 \) sufficiently large, in terms of \( c_2, t_2 \).

\[
\frac{B_4(U, U, U, U)}{B_4(V, V, V, V)} \geq \delta_{\ell/4}^4 + \frac{1}{2} c_2 (\delta_{\ell/4} t_2)^2,
\]

(8.18)

\[
\left| \delta_{\ell/4} - \frac{B_4(U, U, U, V)}{B_4(V, V, V, V)} \right| \leq 8c_1 (\delta_{\ell/4} t_1)^t,
\]

(8.19)

\[
Z_V := \mathbb{E}_{x_1 \in S_1} V(x_1, x_{2,3}^0) V(x_1, x_{2,3}^1) V(x_1, x_{2,3}^2) V(x_1, x_{2,3}^3),
\]

(8.20)

\[
\mathbb{E}_{x_{2,3} \in S_{2,3}} (Z_V) = B_4(V, V, V, V),
\]

(8.21)

\[
\text{Var}_{x_{2,3} \in S_{2,3}} (Z_V) \leq \sqrt{\delta} \cdot B_4(V, V, V)^2.
\]

(8.22)
\[
\text{Var}_{\mathbb{Z}^U} (Z_U) \leq 32c_1 (\delta_{U|V} \tau)^{t_1} B_4 (V, V, V)^2.
\]

Notice that the constant \(t_1\) of (8.10) appears in the estimates (8.19) and (8.23). We take \(t_1 > 2t_2 + 3\). In (8.23), note that we have three occurrences of \(U\) and one of \(V\). The expectation of \(Z\) is the term in (8.19).

**Proof of (8.18).** The denominator on the left-hand-side is estimated in (8.16). So we estimate the numerator. We use the expansion \(U = f_1 + f_0\) four times to write \(B_4(U, U, U, U)\) as a sum of sixteen terms.

\[
B_4(U, U, U, U) = \sum_{\epsilon \in M_4} B_4(f_{\epsilon(0,0)}, f_{\epsilon(0,1)}, f_{\epsilon(1,0)}, f_{\epsilon(1,1)})
\]

where \(M_4\) denotes the collection of sixteen maps from \(\{0, 1\}^2\) into \(\{0, 1\}\). The two significant terms are associated to the maps \(\epsilon \equiv 0\) and \(\epsilon \equiv 1\).

\[
\begin{align*}
B_4(f_1, f_1, f_1, f_1) &= \delta_{U|V}^4 B_4(V, V, V) \\
B_4(f_0, f_0, f_0, f_0) &\geq c_2 (\delta_{U|V} \tau)^{t_2} B_4(V, V, V)
\end{align*}
\]

The first is by definition of \(f_1 = \delta_{U|V} V\), while the second is by assumption (8.15). We should argue that the sum of the remaining fourteen choices of \(\epsilon\) are small. But this follows from the fact that (8.11) fails, and the inequality (5.18). For any choice of \(\epsilon \not\equiv 0, 1\), the central hypothesis leading to that inequality holds. Of course, it is important to use the fact that the one-dimensional obstructions are not in place at this point.

\(\square\)

**Proof of (8.19).** In \(B_4(U, U, U, V)\), expand each \(U\) as \(f_1 + f_0\). The leading term is when each \(U\) is replaced by \(f_0\), giving us

\[
B_4(f_1, f_1, f_1, V) = \delta_{U|V}^3 B_4(V, V, V).
\]

The remaining seven terms are of the form \(B_4(f_{\epsilon(0,0)}, f_{\epsilon(0,1)}, f_{\epsilon(1,0)}, V)\), where \(\epsilon \not\equiv 1\). But then, the estimate (5.18) applies, so this proof is finished.

\(\square\)

**Proof of (8.20) and (8.21).** The equation (8.20) is by definition, and (8.21) is a consequence of assumption on \(V\) and Lemma 5.14.

\(\square\)
Proof of (8.22) and (8.23). The equation (8.22) is by definition of \( Z_U \). The inequality (8.23) is very similar in spirit to Lemma 5.14, but does not explicitly follow from that Lemma.

To compute the variance of \( Z_U \), we need the following 8-linear form.

\[
L_8(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) = \mathbb{E}_{x_1, x_2, x_3} g_1(x_1) g_2(x_2) g_3(x_3) g_4(x_1, x_2, x_3) g_5(x_1, x_2, x_3) g_6(x_1, x_2, x_3) g_7(x_1, x_2, x_3) g_8(x_1, x_2, x_3)
\]

The point of this definition is that \( \mathbb{E}_{x_1, x_2, x_3} Z_U^2 = L_8(U, U, U, V, U, U, U, V) \), and we want to establish the estimate

\[
\mathbb{E}_{x_1, x_2, x_3} Z_U^2 - (\mathbb{E}_{x_1, x_2, x_3} Z_U)^2 \leq 20c_1(\delta_{U | V})^t \mathbb{E}_{x_1, x_2, x_3} Z_U^2.
\]

We already have (8.19), which gives us an estimate of \( \mathbb{E}_{x_1, x_2, x_3} Z_U \). It follows from \( V \) being \((4, 8, 4)\)-uniform that we have

\[
\delta_{U \mid V}^6 \mathbb{L}_8(V, V, V, V, V, V, V, V) \leq \delta_{U \mid V}^3 \cdot B_4(V, V, V, V)^2
\]

And so, we should verify that

\[
(8.24) \quad \left| L_8(U, U, U, V, U, U, U, V) - \delta_{U \mid V}^6 \mathbb{L}_8(V, V, V, V, V, V, V, V) \right| \leq 20c_1(\delta_{U \mid V})^t \mathbb{L}_8(V, V, V, V, V, V, V, V).
\]

The key assumption is that (8.10) fails, which in turn suggests that we appeal to the inequality (5.18). But, in the definition of \( L_8 \), no single variable occurs in just one function, the key hypothesis needed to apply (5.18). This fact brings us to the observation that, for instance, in the definition of \( L_8 \), only \( g_7 \) and \( g_8 \) are functions of \( x_1^2 \). Moreover, we are interested in the case where \( g_8 = V \), a ‘highly uniform’ function, and \( g_7 = U = f_1 + f_0 \). Thus, our strategy is to selectively replace occurrences of \( U \) in \( L_8(U, U, U, V, U, U, U, V) \) in such a way that at each stage, there is single occurrence of \( f_0 \), and that there is a variable in \( f_0 \) which is only occurs in instances of \( V \).

Specifically, we write

\[
L_8(U, U, U, V, U, U, U, V) - \delta_{U \mid V}^6 \mathbb{L}_8(V, V, V, V, V, V, V, V) = \sum_{m=1}^6 D_m,
\]

53
\[ D_1 = \mathbb{L}_8(U, U, U, V, U, U, f_0, V), \quad D_2 = \mathbb{L}_8(U, U, U, V, U, f_0, V, V), \]
\[ D_3 = \mathbb{L}_8(U, U, U, V, f_0, V, V), \quad D_4 = \mathbb{L}_8(U, U, f_0, V, V, V, V), \]
\[ D_5 = \mathbb{L}_8(U, f_0, V, V, V, V, V), \quad D_6 = \mathbb{L}_8(f_0, V, V, V, V, V). \]

Then, (8.24) will follow from the estimate
\[(8.25) \quad |D_m| \leq 3c_1(\delta_{U|V})^1 \mathbb{L}_8(V, V, V, V, V, V), \quad 1 \leq m \leq 6.
\]

Each of the six inequalities in (8.25) follow from the same principle, and so we will only explicitly discuss the estimate for \(D_1\). Write
\[ D_1 = \mathbb{E}_{x_{1,2,3}} \mathbb{E}_{x_{1,2,3}} \mathbb{E}_{x_{1,2,3}} \mathbb{E}_{x_{1,2,3}} U(x_1^0, x_2^0, x_3^0)U(x_1^0, x_2^0, x_3^1)U(x_1^0, x_1^2, x_3^0)V(x_1^0, x_2^1, x_3^1)
\times U(x_1^0, x_2^1, x_3^3)U(x_1^0, x_2^2, x_3^3) \cdot \mathbb{E}_{x_1^0, x_2^0, x_3^0} f_0(x_1^0, x_2^1, x_3^3) V(x_1^0, x_2^1, x_3^3).
\]

Apply the Cauchy-Schwartz inequality in all variables except \(x_2^2 \in S_2\). In so doing, apply the First Proposition on Conservation of Densities, Proposition 5.11, and the assumption of \(V\) being \((4, 8, 4)\)-uniform to conclude that
\[(8.26) \quad |D_1| \leq \mathbb{L}_8(V, V, V, V, V, V) \left( \sqrt{8} + \frac{\mathbb{L}_4(f_0, f_0, V)}{\mathbb{L}_4(V, V, V, V)} \right)^{1/2}.
\]
\[ \mathbb{L}_4(g_1, g_2, g_3, g_4) = \mathbb{E}_{x_1^0, x_2^0, x_3^0, x_4^0} g_1(x_1^0, x_2^0, x_3^0) g_2(x_1^0, x_2^0, x_3^0) g_3(x_1^0, x_2^0, x_3^0) g_4(x_1^0, x_2^0, x_3^0).
\]

In the right-hand-side of (8.26), observe that we can write
\[ \mathbb{L}_4(f_0, f_0, V) = \mathbb{E}_{x_1^0, x_2^0, x_3^0} f_0(x_1^0, x_2^0, x_3^0) f_0(x_1^0, x_2^0, x_3^0) \cdot Y
\]
\[ Y = Y(x_1^0, x_2^0, x_3^0) \mathbb{E}_{x_1^0, x_2^0, x_3^0} V(x_1^0, x_2^0, x_3^0) V(x_1^0, x_2^0, x_3^0).
\]

It follows from Lemma 5.14 and assumption on \(V\), that \(Y\) is a random variable with non-zero mean and very small variance on the event \(V(x_1^0, x_2^0, x_3^0) V(x_1^0, x_2^0, x_3^0)\). Hence,
\[ \frac{\mathbb{L}_4(f_0, f_0, V)}{\mathbb{L}_4(V, V, V, V)} \leq \sqrt{3} + \frac{\mathbb{L}_4(f_0, f_0, 1, 1)}{\mathbb{L}_4(V, V, V, V)}.
\]

But the last ratio is controlled by the failure of (8.10), so our proof of (8.25), and hence (8.23) is complete. \(\square\)
We need to conclude the proof of the Lemma, assuming the inequalities (8.18)–(8.23). Select a point $x^0_{2,3} \in S_{2,3}$ at random, and define the data in (8.6) as follows.

\[
\begin{align*}
S'_1(x^0_{2,3}) &= \{x_1 \mid (x_1, x^0_2, x^0_3) \in U\}, \\
S'_{1,2}(x^0_{2,3}) &= \{(x_1, x^1_2) \mid (x_1, x^0_2, x^0_3), (x_1, x^1_2, x^0_3) \in U\}, \\
S'_{1,3}(x^0_{2,3}) &= \{(x_1, x^1_3) \mid (x_1, x^0_2, x^0_3), (x_1, x^1_3, x^0_3) \in U\}, \\
T'(x^0_{2,3}) &= \{(x_1, x^1_2, x^1_3) \mid (x_1, x^0_2, x^0_3), (x_1, x^1_2, x^1_3) \in U, (x_1, x^1_2, x^1_3) \in V\}.
\end{align*}
\]

With this definition, it is clear that (8.7) holds, namely if $V = T_4$, we have $V' = T'_4 = T'(x^0_{2,3})$. No change is made to the data not listed here, namely $S_2, S_3$ and $S_{2,3}$. The point of these definitions is that we have

\[
\mathbb{E}_{x_1 \in S_1} T'(x^0_{2,3}) = B_4(U, U, U, V),
\]

and $\mathbb{P}_{x_1 \in S_1} (T'(x^0_{2,3})) = Z_U(x^0_{2,3}) = Z_U$, in the notation of (8.22) and (8.23).

Define the event

\[
\tilde{S}_{2,3} = \{x^0_{2,3} \in S_{2,3} \mid |Z_U - B_4(U, U, U, V)| < [c_2(\delta_U \mid V)]^{t_{1/2}} B_4(V, V, V) \\
\quad \quad |Z_V - B_4(V, V, V)| < [c_2(\delta_U \mid V)]^{t_{1/2}} B_4(V, V, V)\}.
\]

It follows from (8.20)–(8.23) that we have

\[
\mathbb{P}(S_{2,3} - \tilde{S}_{2,3}) < 32[c_2(\delta_U \mid V)]^{t_{1/2}}.
\]

Moreover, for $t_1 > 4t_2$, notice that we would have inequalities that look quite similar to (8.18) and (8.19). In particular, we will have

\[
|\mathbb{E}_{x^0_{2,3} \in S_{2,3}} Z_U - B_4(U, U, U, V)| \leq [c_2(\delta_U \mid V)]^{t_{1/2}} B_4(V, V, V),
\]

with a similar inequality for $Z_V$. Hence, we can conclude the proof of the Lemma, by noting that

\[
\sup_{x^0_{2,3} \in S_{2,3}} \frac{Z_U}{Z_V} \geq \frac{\mathbb{E}_{x^0_{2,3} \in S_{2,3}} Z_U}{\mathbb{E}_{x^0_{2,3} \in S_{2,3}} Z_V} \geq \delta_U \mid V + \frac{1}{4}(\delta_U \mid V)^{t_2}.
\]
8.3 Three-Dimensional Obstructions

We proceed under the assumption that both (8.10) and (8.15) fail, as written and under all permutations of coordinates. We have specified \( c_1, t_1 \) as functions of \( c_2, t_2 \), and this argument will specify these last two constants.

We need the 8-linear form, the analog of (8.17) given by

\[
B_8(f_\epsilon | \epsilon \in \{0, 1\}^{1,2,3}) = \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} \prod_{\epsilon \in \{0, 1\}^{1,2,3}} f_\epsilon(x_{1,2,3}^\epsilon).
\]

The relevant facts we need about this form concern these values. Set

\[
B_8[W] = B_8(W | \epsilon \in \{0, 1\}^{1,2,3}), \quad W = U, V
\]

\[
B_8[U, V] = B_8(U, \ldots, U, V | \epsilon \in \{0, 1\}^{1,2,3}),
\]

where the lone \( V \) occurs in the \( \{1\}^{1,2,3} \) position. Indeed, note that \( B_8[U] = \|U\|_{\square,1,2,3,S_{1,2,3}} \).

The facts we need are these.

\[
\begin{align*}
(8.27) & \quad \frac{B_8[U]}{B_8[V]} \geq \delta_{U \mid V}^8 + \frac{1}{2} \tau^8, \\
(8.28) & \quad \left| \delta_{U \mid V}^7 - \frac{B_8[U, V]}{B_8[V]} \right| \leq \frac{1}{20} (\delta_{U \mid V} \tau)^{30}, \\
& \quad Z = \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} V(x_{1,2,3}^1) \prod_{\epsilon \in \{0, 1\}^{1,2,3}} U(x_{1,2,3}^\epsilon),
\end{align*}
\]

\[
(8.29) \quad \mathbb{E}(Z | U) = \frac{B_8[U, V]}{\mathbb{P}(U)},
\]

\[
(8.30) \quad \text{Var}_{x_{1,2,3} \in S_{1,2,3}} (Z | U) \leq \frac{1}{20} (\delta_{U \mid V} \tau)^{30} B_8[V]^2.
\]

Proof of (8.27). Consider \( B_8[U] \). Expand each occurrence of \( U \) as \( f_1 + f_0 \), where \( f_1 = \delta_{U \mid V} V \). This leads to

\[
(8.31) \quad B_8[U] = \sum_{\rho \in M_8} B_8(f_\rho(\epsilon) | \epsilon \in \{0, 1\}^{1,2,3})
\]

where \( M_8 \) is the class of maps from \( \{0, 1\}^{1,2,3} \) into \( \{0, 1\} \). The leading term is \( \rho \equiv 1 \), which is

\[
(8.32) \quad \delta_{U \mid V}^8 B_8[V] = \delta_{U \mid V}^8 \|V\|_{\square,1,2,3,S_{1,2,3}}^8.
\]
The other significant term is \( \rho \equiv 0 \), which is

\[ B_8(f_0 \mid \epsilon \in \{0, 1\}^{1,2,3}) = \|f_0\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8 \geq \tau^8 \|V\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8. \]

The last inequality follows from (8.4).

That leaves \( 2^8 - 2 \) additional terms in \( M_8 \) to consider. For each \( \rho \in M_8 \) which is not equivalent to 0 or 1, the assumption for the inequality (5.19) holds. Namely, there is a choice of \( \epsilon \in \{0, 1\}^{1,2,3} \), and choice of distinct \( j, k \in \{1, 2, 3\} \) so that \( \rho(\epsilon) = 0 \), and for every other \( \epsilon' \), we have either \( \epsilon(j) \neq \epsilon'(j) \) or \( \epsilon(k) \neq \epsilon'(k) \). Therefore, the inequality (5.19) holds. Combining this inequality with our assumption that (8.15) fails, we see that this holds.

(8.33) \[ |B_8(f_{\rho(\epsilon)} \mid \epsilon \in \{0, 1\}^{1,2,3})| \leq c_2(\delta_{U \mid V} \tau t)^{t_2} \times \|V\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8. \]

For \( c_2 \) sufficiently small, and \( t_2 \geq 8 \), this completes the proof of (8.27).

\[ \square \]

**Proof of (8.28).** Keeping the notation of (8.31), we have

\[ B_8[U, V] = \delta_{U \mid V}^{-1} \sum_{\rho \in M'_8} B_8(f_{\rho(\epsilon)} \mid \epsilon \in \{0, 1\}^{1,2,3}) \]

where \( M'_8 \) is the class of maps \( \rho \in M_8 \) such that \( \rho(1^{1,2,3}) = 1 \). The leading term is again \( \rho \equiv 1 \), which is (8.32) above. The remaining \( 2^8 - 1 \) terms all admit the bound (8.33). Therefore,

\[ |B_8[U, V] - \delta_{U \mid V} - \delta_7_{U \mid V} \|V\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8| \leq 2^8(\delta_{U \mid V} \tau)^{t_2 - 1} \times \|V\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8. \]

This proves (8.28) for \( c_2 \) sufficiently small, and \( t_2 \geq 31 \).

\[ \square \]

**Proof of (8.29) and (8.30).** The equation (8.29) is just the definition of conditional expectation. Note that as \( V \) is \((4, 3, 4)-\)uniform, we have

\[ \mathbb{E}_{x_{1,2,3}^j, x_{1,2,3}^k \in S_{1,2,3}} Z \cdot U = B_8[U, V] \]

\[ = \delta_{U \mid V}^{7} \|V\|_{\Omega^{1,2,3}_{S_{1,2,3}}}^8 + \epsilon, \]

(8.34) \[ = \delta_{U \mid V}^{7} \delta_{V \mid 4} \prod_{1 \leq j < k \leq 3} \delta_{j,k}^{4} + \epsilon, \]

57
\[ |e| \leq \frac{1}{20}(\delta_{U|V}\tau)^{30} B_8[V], \]

by (8.28), and (5.3).

The inequality (8.30) is clearly a relative of Lemma 5.14, but does not follow from any principal like that which we have stated. Indeed, we will see that (8.15) is instrumental to this inequality, as it has been to the prior inequalities. Recalling (5.16), we see that we need to estimate \( \mathbb{E}Z^2 \cdot U \). This is a linear form on \( U \) and \( V \), which we now specify. Take \( \Omega \subset \{0, 1, 2\}^{1, 2, 3} \) be set of maps \( \epsilon : \{1, 2, 3\} \to \{0, 1, 2\} \) such that the range of \( \epsilon \) does not include both 1 and 2. Then,

\[
\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} Z^2 \cdot U = \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} V(x_{1,2,3}^1) V(x_{1,2,3}^2) \prod_{\epsilon \in \Omega, \epsilon \neq 1, 2} U(x_{1,2,3}^\epsilon). 
\]

(8.36)

There are 13 occurrences of \( U \) in this expression. (Of the 7 occurrences of \( U \) in \( B_8[U, V] \), all but one get ‘doubled’ in the expression above.) Each occurrence is expanded as as \( f_1 + f_0 \), where \( f_1 = \delta_{U|V} \). The leading term is when each occurrence of \( U \) is replaced by \( f_1 \). This leads to

\[
\delta^{13}_{U|V} \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} V(x_{1,2,3}^\epsilon) \prod_{\epsilon \in \Omega} \delta_{\epsilon | j, k} = \delta^{13}_{U|V} \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} V(x_{1,2,3}^\epsilon) \prod_{\epsilon \in \Omega} \delta_{\epsilon | j, k}.
\]

(8.37)

Recall that this last expectation can be estimated by assumption that \( V \) is \((4, \vartheta, 4)\)-uniform, see (5.3).

In each of the \( 2^{13} - 1 \) remaining terms, there is at least one occurrence of \( U \) which is replaced by \( f_0 \). As in the previous two proofs, we are again in a situation in which (5.19) applies. Therefore, as (8.15) fails, each of these terms is at most

\[
2L_V \left\{ \delta' + c_2(\delta_{U|V}\tau)^{t_2} \right\}.
\]

(8.38)

Therefore, for \( c_2 \) sufficiently small, and \( t_2 \) sufficiently large, we can combine (8.38), (8.37) and (8.36) to conclude that

\[
\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} Z^2 \cdot U = \delta^{13}_{U|V} L_V + \epsilon'
\]

(8.39)

\[ |\epsilon'| \leq c_2 L_V (\delta_{U|V}\tau)^{t_2} . \]
Here, the implied constant in ‘$\mu$’ depends upon the failure of the inequality (8.15), and $L_V$ is defined in (8.37).

Now observe that combining (8.34) and (8.36) and (8.37), we have

$$
\mathbb{P}(U \mid T_4) \cdot \mathbb{E}Z^2 \cdot U = \delta_{U \mid V}^{14} \delta_{V}^{16} \prod_{1 \leq j < k \leq 3} \delta_{j,k}^8 + \epsilon' \cdot \mathbb{P}(U \mid T_4)
$$

(8.40)

$$
|\epsilon'| \leq \epsilon' \cdot B_8[V]^2 \cdot (\delta_{U \mid V}^1)^2 + \frac{1}{2} (\delta_{U \mid V}^3)^2.
$$

(8.41)

In the last line, we have used (8.35) and (8.39). Dividing (8.40) by $\mathbb{P}(U \mid T_4)^2$, and using the estimate in (8.41) completes the proof of (8.30).

\[\Box\]

We can complete the proof of Lemma 8.3 assuming the inequalities (8.27)—(8.30). For a suitably generic point $x_{1,2,3}^0 \in U$, we define the new data in (8.6) to be

$$
S'_1(x_{1,2,3}^0) = \{x_{1,2,3}^1 : x_{1,2,3}^1 \in U \},
$$

with a corresponding definition for $S'_2(x_{1,2,3}^0)$ and $S'_3(x_{1,2,3}^0)$. The set $S'_{1,2}(x_{1,2,3}^0)$ is defined as

$$
S'_{1,2}(x_{1,2,3}^0) = \{x_{1,2}^1 : x_{1,2}^1 \in S'_1(x_{1,2,3}^0) \times S'_2(x_{1,2,3}^0) \mid x_{1,2,3}^1 \in U \},
$$

with a corresponding definition for $S'_{1,3}(x_{1,2,3}^0)$ and $S'_{2,3}(x_{1,2,3}^0)$. Last of all, the set $T'(x_{1,2,3}^0)$ is taken to be

$$
T'(x_{1,2,3}^0) = \{x_{1,2,3}^0 \in V \mid x_{1,1,0}^1 \in S'_{1,2}(x_{1,2,3}^0), x_{1,0,1}^1 \in S'_{1,3}(x_{1,2,3}^0), x_{0,1,1}^1 \in S'_{2,3}(x_{1,2,3}^0) \}.
$$

With these definitions, note that (8.7) holds, that is if $V = T_4$, then $V' = T'(x_{1,2,3}^0) = T_4$ in the new $T$-system. The point of this definition is that

$$
\mathbb{E}_{x_{1,2,3}^0 \sim S_{1,2,3}} U(x_{1,2,3}^0) T'(x_{1,2,3}^0) = B_8[U, V],
$$

with the last expression found in (8.28).

Now, set

$$
U' = \{x_{1,2,3}^0 \in U \mid \mathbb{P}_{x_{1,2,3}^0 \sim S_{1,2,3}} (T'(x_{1,2,3}^0)) \geq \frac{1}{4} \delta_{U \mid V} B_8[V] \}.
$$
It follows from (8.29) and (8.30) that we have
\[ \mathbb{P}_{x_{1,2,3} \in S_{1,2,3}}(U - U') \leq \mathbb{P}(U) \cdot \left( (\tau \delta_{U | V})^7 B_8[V] \right)^2 \text{Var}(Z | U) \leq \mathbb{P}(U)(\tau \delta_{U | V})^{14}. \]

Now, it will follow from the (4, 8, 3)-uniformity of \( V \), and Lemma 5.14 that we have
\[ \text{Var}_{x_{1,2,3} \in S_{1,2,3}} \left( \mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} \prod_{e \in [0,1]^{1,2,3}} V(x_{e,1,2,3}) | V(x_{0,1,2,3}) \right) \leq \delta B_8[V]^2. \]

Here, \( \delta \) is as in (8.5). Therefore, it will follow that in the formula (8.27), we can change the leading \( U(x_{1,2,3}^0) \) by \( U'(x_{1,2,3}^0) \). Namely, we have
\[ B_8[U - U', U, \ldots, U] \leq B_8[U - U', V, \ldots, V] \leq 2(\tau \delta_{U | V})^{14} B_8[V]. \quad (8.42) \]

We can conclude this proof by estimating as follows: For element \( x_{1,2,3}^0 \in U' \), we have
\[ \sup_{x_{1,2,3} \in U'} \mathbb{P}(U | T) = \frac{\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} \prod_{e \in [0,1]^{1,2,3}} U(x_{e,1,2,3})}{\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} V(x_{1,2,3}) \prod_{e \in [0,1]^{1,2,3}} U(x_{e,1,2,3})} \geq \frac{\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} U'(x_{1,2,3}) \prod_{e \in [0,1]^{1,2,3}} U(x_{e,1,2,3})}{\mathbb{E}_{x_{1,2,3} \in S_{1,2,3}} U'(x_{1,2,3}) V(x_{1,2,3}) \prod_{e \in [0,1]^{1,2,3}} U(x_{e,1,2,3})} \geq \delta_{U | V} + \frac{1}{4} \tau^8. \]

The last line follows by combining (8.27), (8.28), and (8.42), with this last inequality showing that modifications of (8.27) and (8.28) hold, with the leading \( U(x_{1,2,3}^0) \) replaced by \( U'(x_{1,2,3}^0) \).

### 9 Proof of Uniformizing Lemma

We marshal several facts, and set some notations, before beginning the main lines of the proof of the Information Lemma 3.17.
9.1 Martingales

We will use basic facts about martingales. Let $Z$ be a real-valued random variable on a probability space $\Omega$, bounded by one. And let $\mathcal{P}$ be a finite partition of $\Omega$. Elements of the partition we refer to as atoms. The conditional expectation of $Z$ relative to $\mathcal{P}$ is

$$\mathbb{E}(Z \mid \mathcal{P}) := \sum_{A \in \mathcal{P}} A \cdot \mathbb{P}(A)^{-1} \mathbb{E}(Z \cdot A).$$

Partition $\mathcal{P}$ refines $\mathcal{Q}$ iff each element of $\mathcal{Q}$ is a finite union of elements of $\mathcal{P}$. In our application, all partitions will be a finite collection of sets. Let $\mathcal{P}_n$ be a sequence of refining partitions of $\Omega$, that is, $\mathcal{P}_n$ is a refining sequence of partitions means that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for all integers $n$. We will take $\mathcal{P}_0$ to be the trivial partition, namely $\mathcal{P}_0 = \{\Omega\}$.

The sequence of random variables $\mathbb{E}(Z \mid \mathcal{P}_n)$ is an example of a martingale. The sequence of random variables $\Delta Z_n = \mathbb{E}(Z \mid \mathcal{P}_n) - \mathbb{E}(Z \mid \mathcal{P}_{n-1})$ for $n \geq 1$ is a martingale difference sequence. Then, the sum below is telescoping

$$\mathbb{E}(Z \mid \mathcal{P}_n) = \mathbb{E}(Z \mid \mathcal{P}_0) + \sum_{m=1}^{n} \Delta Z_m.$$

Observe that the martingale difference sequence is a sequence of pairwise orthogonal random variables. That is, for $m < n$,

$$\mathbb{E}\Delta Z_m \cdot \Delta Z_n = 0.$$  

Indeed, as the partitions $\mathcal{P}_n$ are refining, and $m < n$, for each element $E \in \mathcal{P}_m$, the random variable $\Delta Z_m$ is constant on $E$, while $\mathbb{E}\Delta Z_n \cdot E = 0$. This leads us to:

**9.2 Proposition.** Let $0 < u < 1$. Suppose that $Z$ is a random variable bounded by 1, and that $\mathcal{P}_n$ is the sequence of refining partitions such that for an increasing sequence of integers $t_m$ we have

$$\mathbb{E}[\mathbb{E}(Z \mid \mathcal{P}_{t_{m-1}})]^2 + u \leq \mathbb{E}[\mathbb{E}(Z \mid \mathcal{P}_{t_m})]^2, \quad 1 \leq m < M.$$

Then, $M \leq u$.

**9.3 Remark.** Below, we will refer to an increasing sequence of integers as ‘stopping times.’ An extension of this definition, to make the stopping times certain sequences of measurable functions, is an essential tool in martingale theory.
Proof. Notice that the assumption tells us that $\mathbb{E}(\Delta Z_{tm})^2 \geq u$. Indeed, since $\mathbb{E}(Z | P_{tm}) = \mathbb{E}(Z | P_{tm-1}) + \Delta Z_{tm}$, and orthogonality of martingale difference sequences,

$$
\mathbb{E}(\Delta Z_{tm})^2 = \mathbb{E}[\mathbb{E}(Z | P_{tm})^2] - 2\mathbb{E}[\mathbb{E}(Z | P_{tm}) \cdot \mathbb{E}(Z | P_{tm-1})] + \mathbb{E}[\mathbb{E}(Z | P_{tm-1})^2]
$$

$$
= \mathbb{E}[\mathbb{E}(Z | P_{tm})^2] - \mathbb{E}[\mathbb{E}(Z | P_{tm-1})^2]
$$

$$
\geq u.
$$

We then have

$$
1 \geq EZ^2 \geq \sum_{m=1}^{M} \mathbb{E}[\mathbb{E}(\Delta Z_{tm})^2] \geq Nu.
$$

We will use the extension of the previous proposition.

9.4 Corollary. Suppose that $\Omega' \subset \Omega$, where $(\Omega, \mathbb{P})$ is a probability space. Let $P$ be a partition of $\Omega'$ into a finite number of sets. Let $P_{tm}$ be a sequence of refining partitions of $p$, and $t_m(p)$, for $p \in P$, be a set of stopping times so that for all $1 \leq m \leq M(p)$ we have

$$
\mathbb{E}[\mathbb{E}(p | P_{tm(p-1)})^2] + u \leq \mathbb{E}[\mathbb{E}(Z | P_{tm(p)})^2], \quad p \in P, \ 1 \leq m < M(p).
$$

Then,

$$
\sum_{p \in P} M(p) \leq u^{-1}.
$$

Proof. We have

$$
1 \geq \sum_{p \in P} \mathbb{P}(p) \geq \sum_{p \in P} \sum_{m=1}^{M(p)} \mathbb{E}[\mathbb{E}(\Delta p_{tm})^2] \geq \sum_{p \in P} \sum_{m=1}^{M(p)} u.
$$

And this proves our Corollary. □

Here is an extension of the previous propositions, where the conditional variance increment is permitted to be much smaller.
9.5 Proposition. Let $0 < u, \tau < 1$, and $C \geq 1$. Suppose that $0 \leq Z \leq 1$ is a random variable, and that $P_m$ is the sequence of refining partitions, and that $t_m$ is a sequence of stopping times such that for all $1 \leq m \leq M$,

$$
E[Z \cdot E_m] \geq \tau
$$

$$
E_m := \left\{ p \in P_{t_m-1} \mid E[E(Z \cdot p \mid P_{t_m})]^2 \geq E(Z \mid p)^2 + uE(Z \mid p)^C \right\}
$$

Then, $M \leq u^{-2}\tau^{-C}$.

Proof. Observe that for $\Delta_m := E(Z \mid P_{t_m}) - E(Z \mid P_{t_m-1})$ we have the estimate

$$
E[\Delta_m^2 \cdot E_m] \geq u^2E[E(Z \mid P_{t_m-1})^CE_m].
$$

Therefore, using Jensen’s inequality, available to us as $C \geq 1$,

$$
1 \geq \sum_{m=1}^{M} E[\Delta_m^2] \geq \sum_{m=1}^{M} E[\Delta_m^2 E_m] \geq \sum_{m=1}^{M} u^2E[E(Z \mid P_{t_m-1})^CE_m]
$$

$$
\geq \sum_{m=1}^{M} u^2E[E(Z \mid P_{t_m-1})E_m]^C \geq Mu^2\tau^C.
$$

This proves the Proposition. \qed

9.2 Partitions

We need several partitions, which ‘fit together’ in an appropriate way.

Let $\Omega$ be a set with partition $P$. Let $\Omega' \subset \Omega$ have partition $P'$. Say that $P'$ is subordinate to $P$ iff each atom $p' \in P'$ is contained in some atom $p \in P$. We do not insist that every atom of $P$ be a union of atoms from $P'$, that is, we do not require that $P'$ refine $P$.

The minimum of two partitions $P$ and $P'$ of the same set $\Omega$ is

$$
P \wedge P' = \{ A \cap B \mid A \in P, B \in P' \}.
$$

If $P'$ is a partition of a subset $\Omega' \subset \Omega$, we use the same notation $P \wedge P'$ for a (maximal) partition of $\Omega'$ subordinate to both $P$ and $P'$. 

63
Suppose that \( P \) is a partition in \( \Omega \), and that \( P' \) is a partition of \( \Omega' \subset \Omega \), that is subordinate to \( P \). We define

\[
\text{multi}(P' \mid P) = \sup_{p \in P} \#\{p' \in P' \mid p' \subset p\}.
\]

### 9.3 Useful Propositions

This general proposition provides the motivation for the overall approach we take.

**9.7 Proposition.** Let \( 0 < \nu < \delta < 1 \). Let \( A \subset T \subset X \) be finite sets with \( \mathbb{P}(A \mid T) \geq \delta + \nu \). Let \( P \) be a partition of \( X \), and let \( P' \subset P \) be any subset of \( P \) for which

\[
\mathbb{P}\left(\bigcup_{p \in P'} p\right) \leq \nu/4.
\]

Then, there is some element \( p \in P - P' \) with

\[
\mathbb{P}(T \mid p) \geq \frac{\nu}{2} \mathbb{P}(T \mid X), \quad \mathbb{P}(A \mid T \cap p) \geq \delta + \frac{\nu}{2}.
\]

**Proof.** Take \( P'' \) to be all those elements \( p \in P \) which are in \( P' \) or \( \mathbb{P}(T \mid P) \leq \frac{\nu}{4} \mathbb{P}(T \mid X) \). It is clear that we have

\[
\mathbb{P}\left(A \cap \bigcup_{p \in P''} p \mid T\right) \leq \frac{\nu}{2}.
\]

Applying the pigeonhole principle to those elements of \( P - P'' \) proves the Proposition. \( \Box \)

The ‘energy increment’ steps we take are governed by these two general propositions.

**9.8 Proposition.** Let \( A \) be a subset of a probability space \((\Omega, \mathbb{P})\). Suppose that the there is a subset \( B \subset \Omega \) for which we have

\[
\mathbb{P}(A \mid B) = \mathbb{P}(A) + \nu > \mathbb{P}(A).
\]

Then, for the partition \( P_B \) of \( \Omega \) generated by \( B \), we have

\[
\mathbb{E}[\mathbb{E}(A \mid P_B)]^2 \geq \mathbb{P}(A)^2 + \mathbb{P}(B) \cdot \nu^2.
\]
In application, we will have $\nu, P(B) \geq P(A)^C$, for an absolute constant $C$. Thus, we have

$$E[E(A | P_B)]^2 \geq P(A)^2 + P(A)^3C.$$ 

**Proof.** Let us set $\alpha = P(A)$, $P(B) = \beta$ so that

$$P(A \cap B) = (\alpha + \nu)\beta, \quad P(A \cap B^c) = (1 - \beta)\alpha - \nu\beta.$$ 

We can calculate the left-hand side of (9.9) directly.

$$E[E(A | P_B)]^2 = P(B)[P(A | B)]^2 + (1 - P(B))[P(A | B^c)]^2$$
$$= P(A \cap B) \cdot P(A | B) + P(A \cap B^c)P(A | B^c)$$
$$= (\alpha + \nu)\beta + (1 - \beta)^{-1}[(1 - \beta)\alpha + \nu\beta]^2$$
$$= \alpha^2 + (1 - \beta)^{-1}\nu^2\beta$$
$$\geq \alpha^2 + \nu^2\beta.$$ 

And this proves the proposition. $\square$

This trivial extension of the previous proposition is the one that we use.

**9.10 Proposition.** Let $A$ be a subset of a probability space $(\Omega, P)$, and let $P$ be a finite partition of $\Omega$ so that this condition holds. For a subset $Q \subset P$, suppose the following holds. For each element $p \in P$, there is a further subset $p'$ so that

$$P(A | p') \geq P(A | p) + \nu, \quad p \in Q.$$ 

$$P\left(\bigcup_{p \in P} p'\right) \geq \tau.$$ 

Then, for the partition $P'$ which refines both $P$ and $\{p' | p \in Q\}$, we have the estimate

$$E[E(A | P')]^2 \geq E[E(A | P)]^2 + \tau\nu^2.$$ 

We will appeal to a simple bound for the tower notation given by

(9.11) $2 \uparrow n := 2^n$, \quad $2 \uparrow\uparrow n := 2 \uparrow (2 \uparrow\uparrow n - 1).$

In the function $2 \uparrow\uparrow n$ is called the Ackerman function, and its inverse is

(9.12) $\log^* N = \min\{n | N \leq 2 \uparrow\uparrow n\}.$
9.13 Proposition. For integers \( \ell, u, v \geq 2 \) define

\[
\psi(0, u, v) = u \cdot v, \quad \psi(\ell + 1, u, v) = 2 \uparrow (u \cdot \psi(\ell, u, v))
\]

We have the estimate

\[
\psi(\ell, u, v) \leq 2 \uparrow\uparrow [\ell + \log_2 2uv].
\]

Proof. Define

\[
\epsilon_\ell = \frac{\log_2 u}{u \psi(\ell - 1)}, \quad \epsilon_{k-1} = \frac{\log_2 u(1 + \epsilon_k)}{u \psi(k - 1)}.
\]

It is elementary to see that \( \epsilon_1 \leq 1 \).

The point of these definitions is that we have

\[
\psi(\ell, u, v) = 2 \uparrow [(1 + \epsilon_\ell)u \psi(\ell - 1)] \\
= 2 \uparrow [2 \uparrow [(1 + \epsilon_{\ell-1})\psi(\ell - 2)] \\
\vdots \\
\underbrace{2 \uparrow [\cdots 2 \uparrow [(1 + \epsilon_1)uv] \cdots]}_{\ell \text{ times}} \\
\leq 2 \uparrow\uparrow [\ell + \log_2 2uv].
\]

\[\qed\]

The following definition is used to make a quicker appeal to Lemma 8.2 and its relative Lemma 8.3.

9.14 Definition. Consider a subset \( S \) of a set \( X \), a partition \( P \), and a positive parameter \( \Delta \).

Say that \( P' \) is \((S, \Delta, P)\)-good iff \( P' \) refines \( P \) and

\[
\mathbb{E}(\mathbb{E}(S | P')^2) \geq \mathbb{E}(\mathbb{E}(S | P)^2) + \Delta.
\]

9.4 The \( U(3) \) Norm

In this section we discuss the Lemmas needed to obtain sets that are uniform with respect to the Gowers \( U(3) \) norm.
9.16 Definition. We call a partition of $H \times H \times H$ affine iff all atoms of the partition are of the form $V_1 \times V_2 \times V_3$, where $V_i$ are all translates of the same subspace $V \leq H$. This is an essential definition for us, as an affine partition, in say the basis $(e_1, e_2, e_3)$ is also affine in any choice of basis formed from these three vectors. Each atom of an affine partition is, after translation, a copy of $H \times H \times H$ with a lower dimension.

In particular, given $S_j$, $1 \leq j \leq 4$, and an affine partition $P$, for each atom $\alpha \in P$, it makes sense to compute the Gowers uniformity norm of $S_j$ relative to the atom $\alpha$. That is, the atom $\alpha$ determines an affine subspace $V_j$ in the coordinate $e_j$. After translation, we could assume that $V_j$ is actually a subspace, in which we can unambiguously compute the Gowers $U(3)$ norm. This is what we mean by

$$\|S_j - P(S_j | \alpha)\|_{U(3),\alpha}$$

The codimension of an affine partition, written as $\text{codim}(P)$ is the maximum codimension of $V_1$ in $H$, for all $V_1 \times V_2 \times V_3 \in P$. Clearly, we have

$$|P| \leq 5^{\text{codim}(P)}.$$

We need the following version of the Inverse Theorem for the $U(3)$ Norm, in a

9.17 Inverse Theorem for the Gowers $U(3)$ Norm. There are constant $0 < c < C < \infty$ so that the following holds. Let $S \subset H$ and assume that $\dim(H) > 10Cu^{-C}$ and

$$\|S - P(S | H)\|_{U(3)} > u$$

Then, there is an affine subspace $H'$ of $S$ so that $\dim(H') \geq \dim(H) - Cu^{-C}$ and

$$P(S | H') \geq P(S | H) + cu^C.$$

We emphasize that the exact value of the estimates on the co-dimensions above are important in the study of four-term progressions, but the exact form of these estimates are not important to the proof of our Main Theorem, Theorem 1.2. For this result, see [10, p. 27—28].

We will use this elementary observation: If $P, P'$ are affine partitions, then

$$\text{codim}(P \land P') \leq \text{codim}(P) + \text{codim}(P').$$

67
9.18 Proposition. There is a constant $C$ so that the following holds for all $0 < u, \tau < 1$ the following holds. Let $S_j, 1 \leq j \leq 4$ be sets in the $j$th coordinate. Then there is an affine partition $P$ of $H \times H \times H$, satisfying $\text{codim}(P) \leq (C/\tau)^C$, so that

$$
P(A \in P \mid \sup_j \|S_j\|_{l(3),A} > u) < \tau.
$$

Proof. Here is an important point in the proof. For an affine partition $P$, suppose there is an atom $A \in P$ such that

$$
\|S_j - \mathbb{P}(S_j \mid A)\|_{l(3),A} > u
$$

Let $A_j$ denote the affine subspace for coordinate $e_j$. Then, there is a partition $P_A$ of $A_j$ into affine subspaces of codimension $\leq Cu^{-C}$, for which we have

$$
\mathbb{E}_{A_j}(\mathbb{E}(S_j \cap A_j \mid P_A)^2) \geq \mathbb{E}_{A_j}(S_j \cap A_j)^2 + cu^C.
$$

A moments thought shows that there is then an affine refinement $P'$ of $P$, in which only the atom $A$ is further refined, for which we have

$$
\mathbb{E}(\mathbb{E}(S_j \mid P')^2) \geq \mathbb{E}(\mathbb{E}(S_j \mid P)^2) + cu^C \mathbb{P}(A).
$$

Indeed, since the atom $A$ is the product of translates of the same subspace $A_j$, we impose an appropriate translate of the partition $P_A$ on the two choices of the remaining coordinates. The codimension of the refining partition has increased by only $Cu^{-C}$.

Here is the principal line of the argument. We construct a sequence of refining affine partitions $P_n$, and a sequence of stopping times $\tau_{j,k}$, for $1 \leq j \leq 4$ and $k \geq 1$, which are used to running time of the recursive procedure below.

Let $P$ be an affine partition. Notice that there is some $C > 0$ so that the following is sufficient condition for the existence of a $(S_j, u^C \tau, P)$-good partition $P'$:

$$
P(A \in P \mid \|S_j\|_{l(3),A} > u) \geq \tau/4
$$

In addition, $P'$ can be taken to be affine and $\text{codim}(P') \leq \text{codim}(P) + Cu^{-C}$. This is a consequence of the discussion at the beginning of the proof. The notion of a good partition is defined in Definition 9.14.

Initialize variables

$$
P_0 \leftarrow \{H \times H \times H\}, \quad n \leftarrow 0, \quad \tau_{j,0} = 0, \quad k_j \leftarrow 0
$$

68
Likewise set $\tau_{j,0} = 0$ WHILE for some $1 \leq j \leq 4$, there is an affine $(S_j, u^C \tau / 4, P_n)$-good partition $P'$, with \( \text{codim}(P_{n+1}) \leq \text{codim}(P_n) + Cu^{-C} \), increment

\[ n \leftarrow n + 1, \quad k_j \leftarrow k_j + 1. \]

Define $\tau_{j,k_j} = n$, and $P_{n+1} = P'$.

As the underlying space is finite dimensional, this WHILE loop must stop. The sequence of stopping times $\tau_{j,1}, \ldots, \tau_{j,K}$ cannot exceed $(\tau u)^{1-C}$. Indeed, the hypotheses of Proposition 9.2 hold, proving this claim immediately. The conclusions of the Lemma are then immediate from the recursion, and the observation (9.4).

\[ \square \]

In fact, we will rely upon the following variant of the previous result.

**9.19 Lemma.** There is a constant $C$ so that the following holds for all $0 < u, \tau < 1$ the following holds. Let $S_j, 1 \leq j \leq 4$ be a collection of sets in the $j$th coordinate. Then there is an affine partition $P$ of $H \times H \times H$ of

\[ \text{codim}(P) \leq (u\tau)^{-1} \prod_{j=1}^{4} |S_j|^C \quad \text{and} \quad \mathbb{P}(A \in P \mid \sup_j \|S_j\|_{U(3),A} > u) < \tau. \]

This proof is a simple variant of the previous proof. Note that the codimension of the the partition admits a substantially worse bound. This is because we have to keep track of a running time for each possible set $S \in \bigcup_j S_j$.

**9.5 The Box Norm in Two Variables**

The goal of this subsection is Lemma 9.32, which combines the fact about the $U(3)$ norm in Lemma 9.19 with some facts about the Box Norm. We begin with some generalities on the Box Norm in two variables. Recall the definition of $P'$ being $(S, \delta, P)$-good given in (9.15) above.

**9.20 Proposition.** There is a $C_2$ so that for all $0 < u, \tau < 1$ the following holds. Let $Z \subset X \times Y$, and let $P_X, P_Y$ be partitions of $X$ and $Y$. Suppose that the following condition holds.

\[ \mathbb{P}(E \mid X \times Y) \geq \tau, \quad \text{where} \]

\[ E = \{ (p_x, p_y) \in P_X \times P_Y \mid \|Z - \mathbb{P}(Z \mid p_x \times p_y)\|_{\infty} \geq u \} . \]

Then, there are partitions \( P'_X \) and \( P'_Y \) so that
\[ P'_X \times P'_Y \text{ is } (Z, \tau u C_2, P_X \times P_Y)\text{-good.} \tag{9.21} \]
\[ \text{multi}(P'_X \mid P_X) \leq 2 \uparrow \#P_Y, \text{ and likewise for } P'_Y. \tag{9.22} \]

Here, \( C_2 \) could be taken to be 4.

Note that the estimate (9.22), recursively applied, leads to tower power style bounds.

**Proof.** For each \( (p_x, p_y) \in E \), Lemma 8.2 assures us the existence of a partition \( P_x(y) \) of \( p_x \) into two elements, and a partition \( P_y(x) \) of \( p_y \) into two elements so that \( P_x(y) \times P_y(x) \) is \((Z \cap p_x \times p_y, u C_2, p_x \times p_y)\)-good. (There is no \( \tau \) in this last assertion.)

We take
\[ P'_X = P_X \land \bigwedge_{y \in P_Y} P_y(y), \]
and likewise for \( P'_Y \). It is clear that (9.22) holds. By the assumption that \( \mathbb{P}(E) > \tau \), and the martingale property (9.1), it follows that (9.21) holds. \( \square \)

**9.23 Proposition.** There is a \( C_2 > 0 \) so that for all \( 0 < u, \tau < 1 \) the following holds. Let \( Z \subset X \times Y \), and let \( P_X, P_Y \) be partitions of \( X \) and \( Y \). Let \( P_Z \) be a partition of \( Z \) that is subordinate to \( P_X \times P_Y \). Suppose that the following condition holds.
\[ \mathbb{P}(E \mid Z) \geq \tau, \]
\[ E = \{ z \in P_Z \mid \|z - \mathbb{P}(z \mid X \times Y)\|_{\infty} \times X \times Y \geq u \}. \]

Here, \( z \subset X \times Y \), and \( X_z, Y_z \subseteq P_X \) and \( Y_z \subseteq P_Y \). \( X_z, Y_z \) must exist as \( P_Z \) is subordinate to \( P_X \times P_Y \).

Then, there is a partition \( P'_X \) and \( P'_Y \) so that
\[ P'_X \times P'_Y \text{ is } (P_Z, \tau u C_2, P_X \times P_Y)\text{-good.} \tag{9.24} \]
\[ \text{multi}(P'_X \mid P_X) \leq 2 \uparrow [\#P_Y \cdot \text{multi}(P_Z \mid P_X \times P_Y)], \text{ and likewise for } P'_Y. \tag{9.25} \]

Here, \( C_2 \) could be taken to be 4.

Note in particular the form of the tower in (9.25), with the notation as in (9.11).
Proof. For each $z \in E$, there is a partition $P'_{X_z}$ into two elements, and likewise for $P'_{Y_z}$ so that $P'_{X_z} \times P'_{Y_z}$ is $(z, u^{C_2}, \{X_z\} \times \{Y_z\})$-good. This follows from (9.21) and (9.22).

Define the partition $P'_X$ to be

$$P'_X = P_X \land \bigwedge_{z \in E} P'_{X_z}.$$ 

Observe that (9.25) follows. Indeed, for each $x \in P_X$, we could have up to $(\#P_Y) \cdot \text{multi}(P_Z \mid P_X \times P_Y)$ many sets to form the minimum partition over, leading to (9.25).

Use the basic fact about martingales, (9.1), and the assumption that $P(E) \geq \tau$ to conclude that (9.24) holds. □

We make a definition that we use in this section, and the next.

9.26 Definition. We say that the data

$$S = \{H \times H \times H, P_H, S, P_i, R_{jk}, P_{jk}, T, P_T \mid 1 \leq i \leq 4, 1 \leq j < k \leq 4\}$$

is a partition-system iff

- $P_H$ is an affine partition of $H \times H \times H$.
- $S_i \subset H$, and $P_i$ is a partition of $\overline{S}_i$ that is subordinate to $P_H$, $1 \leq i \leq 4$.
- $R_{jk} \subset S_i \times S_k$, and $P_{jk}$ is a partition of $\overline{R}_{jk}$ that is subordinate to $P_j \land \overline{S}_k$ and $\overline{S}_j \times P_k$, $1 \leq j < k \leq 4$.
- $T \subset H \times H \times H$ is such that $T \subset \overline{R}_{jk}$, $1 \leq j < k \leq 4$.
- $P_T = \bigwedge_{1 \leq j < k \leq 4} P_{jk}$.

We stress that all partitions are collections of subsets of $H \times H \times H$. Set

$$P_{T,\ell} := P_\ell \land \bigwedge_{\substack{1 \leq j < k \leq 4 \\ell \neq jk}} P_{jk}, \quad 1 \leq \ell \leq 4,$$

(9.28) $$P_1(S) = \sum_{i=1}^{4} \text{multi}(P_i \mid P_H),$$
\[ P_2(S) = \sum_{1 \leq j < k \leq 4} \text{multi}(P_i | P_{jk}), \]
\[ P_T(S) = \text{multi}(P_T | P_H), \]

These last quantities are some counting functions that we will need to keep track of.

A trivial partition-system is a partition-system in which each of the partitions are trivial. For each \( t \in P_T \), we take
\[ S_3(t) = \{ H_{i,1} \times H_{i,2} \times H_{i,3}, s_{t,i}, r_{t,jk}, t | 1 \leq i \leq 4, 1 \leq j < k \leq 4 \} \]
to be the trivial partition-system associated to \( t \). Namely, we have

- \( t \subset H_{i,1} \times H_{i,2} \times H_{i,3} \). Here, \( H_{i,1} \times H_{i,2} \times H_{i,3} \) may be the product of affine subspaces in \( H \times H \times H \), but all relevant notions extend to this setting.
- \( s_{t,jk} \in P_{jk} \), with \( s_{t,jk} \subset H_{i,1} \times H_{i,2} \times H_{i,3} \), and \( t = \bigwedge_{1 \leq j < k \leq 4} s_{t,jk} \).

This is the Lemma that will be applied in the next section.

**9.32 Lemma.** Let \( C_1 \geq 1 \) be given. There are finite functions \( \Psi_{2-\square} : [0, 1]^2 \times \mathbb{N} \to \mathbb{N} \) and \( \Psi_{\text{codim}} \ : [0, 1]^2 \times \mathbb{N}^2 \to \mathbb{N} \) so that the following holds for all \( 0 < u_2, u_3 \tau < 1 \).

For all partition-systems \( S \), as in (9.27), there is a partition-system
\[ S' = \{ H \times H \times H, P'_{iH}, S_i, P'_i, R_{jk}, P'_{jk}, T, P'_T | 1 \leq i \leq 4, 1 \leq j < k \leq 4 \} \]
which refines \( S \), so that these conditions are met. For \( 1 \leq i \leq 4 \) and \( 1 \leq j, k \leq 4 \),

\[ \text{codim}(P'_{iH}) \leq \Psi_{\text{codim}}(u_3, \tau, P_i(S), P_2(S)), \]
\[ \text{multi}(P'_i | P_i) \leq \Psi_{2-\square}(u_2, \tau, P_1(S), P_2(S)), \]
\[ \text{multi}(P'_{jk} | P_j \wedge P_k) \leq \text{multi}(P_{jk} | P_j \times P_k), \]
\[ \mathbb{P}(E_{2,jk} | S_j \times S_k) \leq \tau, \]

\[ E_{2,jk} = \left\{ r_{jk} \in P'_{jk} \mid r_{jk} \subset s_j \cap s_k, \ s_v \in P'_{v}, \ v = j, k, \right\} \]

\[ \| r_{jk} - \mathbb{P}(r_{jk} | s_j \times s_k) \|_{L^2(s_j \times s_k)} \geq u_2 [P_T(S')]^{-C_1} \]
\[ P(E_{3,j} \mid S_j) \leq \tau, \]
\[ E_{3,j} = \left\{ s_j \in P_j' \mid \| s_j - P(s_j \mid A_j)A_j \|_{\text{lip}A_j} \geq u_3[P_T(S')]^{-C_1} \right\}. \]

Finally, \( P_T(S') = P_T(S) \). We are using the notation (9.28)—(9.30).

The conclusion is that virtually all of the elements of the partitions \( P_j' \) and \( P_{j,k}' \) are uniform with respect to Gowers Norm, and the Box Norm.

We emphasize that this Lemma provides us with a tower power bound. In (9.35), we have the estimates below, where note that we have a \( \log_* \), as in (9.12), on the left.

\[ \log_*(\#P_j) \leq 2u_2^{-C_2\tau^{-1}}P_2(S)^{C_1 C_2} + \log_1 \ P_1(S). \]

Note that by (9.36), the multiplicity of the partitions \( P_{j,k}' \), defined in (9.6), are not increased in this procedure, though we get a very substantial increase in the multiplicity of the \( P_i' \), from the bound (9.35), forming the principal loss in the application of this Lemma. The sets \( s_i \notin E_{1,i} \) are ‘very uniform,’ even with respect to their probabilities in the respective cell of \( P' \). The ‘tower’ notation in (9.35) is defined in (9.11).

**Proof.** We define a sequence of partition-systems. They are

\[ S(m) = \{ H \times H \times H, P_H(m), S_i, P_i(m), R_{j,k}, P_{j,k}(m), T, P_T(m) \} \]

where \( S(0) \) is the partition-system given to us by assumption. These partition-systems are refining, in the sense that the corresponding sequences of partitions are refining.

In this process, the only incremental change to the partitions \( P_T(m) \) that are made are to make them subordinate to the other partitions. Thus, quantities that appear in (9.37) and (9.38) are constant. Namely, \( Q = P_T(S(m)) \) is independent of \( m \).

We also define a sequence of stopping times \( \sigma(j,k;m) \), and \( m(j,k) \) for \( 1 \leq j < k \leq 4 \), and \( m \geq 0 \). Initialize these stopping times as follows, where \( 1 \leq j < k \leq 4 \).

\[ m \leftarrow 0, \quad \sigma(j,k;0) \leftarrow 0, \quad m(j,k) \leftarrow 0. \]
We choose $C_2$ as in Proposition 9.23. The main recursion is this: Set

\[(9.41) \quad \Delta = u_2^{C_2} \tau = u_2^{C_2} \tau Q^{-C_1} C_2 \]

\[\text{WHILE there are } 1 \leq j < k \leq 4 \text{ so that there is are two partitions } P'_j \text{ and } P'_k \text{ which satisfy (9.24) and (9.25)} \]

above for the quantity $\Delta$. Namely,

1. $P'_j \land P'_k$ is $(P_{j,k}(m), \Delta, P_j(m) \land P_k(m))$-good.

2. The multiplicity of $P'_j$ satisfies

\[(9.42) \quad \text{mult}(P'_j \mid P_j(m)) \leq 2 \uparrow \left[ \text{mult}(P_k(m) \mid P_H(m)) \cdot \text{mult}(P_{j,k}(m) \mid P_j(m) \times P_k(m)) \right] \]

\[\leq 2 \uparrow \left[ \text{mult}(P_k(m) \mid P_H(m)) \cdot \text{mult}(P_{j,k}(0) \mid P_j(0) \times P_k(0)) \right], \]

and likewise for $P'_k$.

We take these steps. Update

1. (Keep track of stopping times.)

   $m \leftarrow m + 1, \quad m(j,k) \leftarrow m(j,k) + 1, \quad \sigma(j,k; m(j,k)) \leftarrow m$.

2. (Select affine partition.) To each element of the affine partition $P_H(m)$, apply Lemma 9.19 to $P'_j \mid P'_j$ for $1 \leq j \leq 4$, with the parameter $\tau$ that is given to us, and the value of $u$ in Lemma 9.19 equal to $u = u_3 Q^{-C_1}$. Set the partition that Lemma 9.19 supplies to us to be $P_H(m + 1)$. Observe that

\[(9.43) \quad \text{codim}(P_H(m + 1)) \leq \text{codim}(P_H(m)) + \left[ (u_3 \tau)^{-1} Q \right] D \]

This follows from Lemma 9.19 and (9.22), for appropriate choice of constant $D$. Note that the term $\text{mult}(P'_j \mid P_H(m))$ is bounded in (9.42).

3. (Updating the remaining partitions.) Set $P_j(m + 1)$ to be the maximal partition which refines $P'_j$ and is subordinate to $P_H(m + 1)$. Set $P_{j,k}(m + 1)$ to be the maximal partition which refines $P_{j,k}(m)$, and is subordinate to both $P_j(m + 1)$ and $P_k(m + 1)$. The last partition $P_T(m + 1)$ is then defined.
At the conclusion of the WHILE loop, return this data: For $1 \leq j < k \leq 4$,

- $m$, the integers $m(j,k)$.
- The sequence of stopping times $\sigma(j,k;\lambda)$, for $0 \leq \lambda \leq m(j,k)$.

It remains to argue that the partitions returned satisfy the conclusions of the Lemma. We must have (9.37), else by the definition of $\Delta$ in (9.41) and Proposition 9.23, the routine would not have stopped. The conclusion (9.36) follows from the construction. The conclusion (9.38) follows from the manner in which we apply Lemma 9.19 in in particular the point (2) above. The remaining conclusions (9.34) and (9.35) require us to know how many recursions were performed. We turn to this next.

We claim that

$$m \leq \Delta^{-1} = u^{-C_2} \tau^{-1} Q^{C_1 C_2}.$$

But this follows from Corollary 9.4 applied to the construction, the sets in $P_{jk}$, and the stopping times $\sigma([j,k], r_{jk}, \lambda)$.

Therefore, we have, by induction, and (9.42), we have

$$\text{multi}(P'_{i} \mid P') = \text{multi}(P_{i}(m) \mid P(m))$$

$$\leq 2 \uparrow [P_2 \cdot \text{multi}(P_{i}(m-1) \mid P(m-1))]$$

$$\leq 2 \uparrow [P_2 \cdot 2 \uparrow [P_2 \cdot 2 \uparrow P_2 \cdot P_1 \cdot \cdots]] = \psi(m, P_1, P_2),$$

Here, the notation is from (9.28), (9.29), and Proposition 9.13, which provides crude bound given in (9.39). This proves (9.35). The final conclusion (9.34) follows from this last bound and (9.43).

\(\square\)

9.6 The Box Norm in Three Variables

The goal of this section is to add the considerations about the Box Norm in three variables into our Lemmas, to build up an analog of Lemma 9.32 which also stipulates facts about the partition $P_T$, which as of yet we have not made any statements about.
9.44 Lemma. There are finite functions $\Psi_{\text{codim}}, \Psi_T : [0, 1]^2 \times \mathbb{N}^2 \to \mathbb{N}$ so that the following holds for all $0 < u_T, \tau_T < 1$.

For all trivial partition-systems $S$ there is a partition-system $S'$ as in (9.33), such that
\begin{align}
\text{(9.45)} & \quad \text{codim}(S') \leq \Psi_{\text{codim}}(u_T, \tau_T, \mathbb{P}(T | H \times H \times H)), \\
\text{(9.46)} & \quad \mathbb{P}_T(S') \leq \Psi_T(u_T, \tau_T, \mathbb{P}(T | H \times H \times H)), \\
\text{(9.47)} & \quad \mathbb{P}(E | H \times H \times H) \leq \tau_T,
\end{align}
where $E := \{ t \in \mathbb{P}_T^r | S_3(t) \text{ is not } u_T\text{-admissible} \}.$

Here, $S_3(t)$ is the trivial partition system associated with $t$, as defined in (9.31).

In (9.47), admissibility is as in Definition 3.4. This proof will generate a second tower power in our estimate for the codimension in (9.46), but we don’t detail this particular fact.

Proof. For this proof, we define a sequence of partition-systems $S(m)$ as in (9.40). These partition-systems are refining in the sense that the corresponding sequences of partitions are refining. We take $S(0)$ to be the trivial partition-system given by the hypothesis of the Lemma.

We also define a sequence of stopping times $\sigma(\ell, p_\ell)$ for $1 \leq \ell \leq 4$, with counters $p_\ell \geq 0$. Initialize these variables $\sigma(\ell, 0) \leftarrow 0$ and $p_\ell \leftarrow 0$, where $1 \leq \ell \leq 4$.

Here is the recursive algorithm. If $m$ is even, apply of Lemma 9.32 to $S(m)$, with the values $\kappa(\frac{1}{8} u_T \tau_T)^C$ and $\frac{1}{100} \tau_T$ specified at the beginning of Lemma 9.44 the Lemma we are proving. The value of $C_1$ in Lemma 9.32 is the value of $C + 1$, where the constants $\kappa$ and $C$ are as in the definition of admissible, Definition 3.4.

We then update $m \leftarrow m + 1$, and take the updated data $S(m)$ to be the partition-system from Lemma 9.32. Observe that from (9.35) we have the estimates:
\begin{align}
\text{(9.48)} & \quad \text{multi}(P_i(m) | P_i(m - 1)) \leq \Psi_{2-C}(u_T, \frac{1}{2} \tau_T, P_1(m - 1), P_2(m - 1)).
\end{align}

If $m$ is odd, by the previous step, the conclusions of Lemma 9.32 are in force. The observation to make is that we have this condition. For the event $B$ defined below, we have $\mathbb{P}(B) \leq \frac{1}{8} \tau_T$.

\begin{align}
B = \{ t \in \mathbb{P}_T(m) | S_3(t) \text{ satisfies (3.6) and } (3.7) \text{ in the definition of } u_T\text{-admissible}. \}
\end{align}

76
Recall that $\mathcal{S}_3(t)$ is given in (9.31). That is, with very high probability, if the trivial partition-system $\mathcal{S}_3(t)$ fails $u_T$-admissibility, it must be the condition (3.5) that fails.

Let us see that this observation is true. The conditions (9.37) and (9.38) applied to $S(m)$ hold. Thus, except on a set of probability at most $\frac{1}{10} \tau_T$, we have, using the notation of (9.31),

$$\|r_{t;jk} - \mathbb{P}(r_{t;jk} \mid s_{t;j} \times s_{t;k})\|_{\infty} \leq \kappa \left( \frac{1}{8} \tau_T u_T \right)^C [\mathbb{P}_T(S(m))]^{-C/2},$$

$$\|s_{t;j} - \mathbb{P}(s_{t;j} \mid H_{t;j})\|_{H(3)} \leq \kappa \left( \frac{1}{8} \tau_T u_T \right)^C [\mathbb{P}_T(S(m))]^{-C/2}.$$

Therefore, if the trivial partition-system $\mathcal{S}_3(t)$ fails either (3.6) or (3.7) in the definition of $u_T$-admissibility, it must follow that $t$ has very small probability in its affine cell. Namely, we must have

$$\mathbb{P}(t \mid H_{t:1} \times H_{t:2} \times H_{t:3}) \leq \frac{1}{8} \mathbb{P}_T(S(m))^{-1} \tau_T.$$

But certainly, by the definition of $\mathbb{P}_T(S(m)$ in (9.30), we have

$$\sum_{t : t \text{ satisfies } (9.50)} \mathbb{P}(t \mid H \times H \times H) \leq \frac{1}{8} \tau_T.$$

This means that $\mathbb{P}(B) \leq \frac{1}{8} \tau_T$ for $B$ as in (9.49).

If there is an $1 \leq \ell \leq 4$ for which we have

$$\mathbb{P}(F_\ell \mid H \times H \times H) \geq \frac{1}{8} \tau_T,$$

$$F_\ell : = \{ t \in \mathbb{P}_T(m) - B \mid \mathcal{S}_3(t) \text{ does not satisfy (3.5) for this value of } \ell \}.$$

For such a choice of $\ell$, update $p_\ell \leftarrow p_\ell + 1$, and set $\sigma(\ell, p_\ell) \leftarrow m$. For each $t \in F_\ell$, we can apply Lemma 8.3. Write

$$t_\ell = s_{t_\ell} \prod_{1 \leq j < k \leq 4 \atop j \neq \ell} r_{t;jk}.$$

Apply Lemma 8.3 with $V = t_\ell$, $U = t$, and $\tau = \kappa u_T^C$. Since $t \notin B$, it follows that $V = t_\ell$ satisfies the hypothesis of that Lemma, namely that $V = t_\ell$ is $(4, \varnothing, \ell)$-uniform, with $\varnothing$ as in (8.5).
Then, from the conclusion of Lemma 8.3, we read this. There are partitions $P(s_{t;j}, t_\ell)$,  $1 \leq j \leq 4$, of $s_{t;j}$ into two sets, and partitions $P(r_{t;j,k}, t_\ell)$,  $1 \leq j < k \leq 4$,  $j, k \neq \ell$ of $r_{t;j,k}$ into two sets, so that the there is an atom $V'$ in the partition $P(s_{t;j}, t_\ell) \land \bigwedge_{1 \leq j < k \leq 4 \atop j,k \neq \ell} P(r_{t;j,k}, t_\ell)$ which has a higher correlation with $t_\ell$. Namely,

$$P(V' \mid t) \geq c \left[ \kappa u_C^i P(t \mid t_\ell) \right]^p,$$

$$P(t \mid V') \geq P(t \mid t_\ell) + c \left[ \kappa u_C^i P(t \mid t_\ell) \right]^p.$$

Let

$$P(t_\ell) = \bigwedge_{1 \leq j < k \leq 4 \atop j,k \neq \ell} P(r_{t;j,k}, t_\ell).$$

It follows that we have

$$(9.51) \quad \mathbb{E}[\mathbb{E}(T \cap t_\ell \mid P(t_\ell))]^2 \geq P(T \mid t_\ell)^2 + u_C^i P(T \mid t_\ell)^2.$$

We update

$$P_i(m + 1) \leftarrow P_i(m), \quad i \neq \ell,$$

$$P(R_{j,k}, m) \land \bigwedge_{t_\ell \in F_\ell} P(R_{j,k}, t_\ell), \quad 1 \leq j < k \leq 4, \quad j, k \neq \ell.$$

It is this last two steps that create a second tower. Observe that we have, using the notation of (9.28) and (9.29),

$$(9.52) \quad P_u(S(m)) \leq P_u(S(m - 1)) 2 \uparrow [2P_2(S(m - 1))^6] \quad u = 1, 2.$$

It follows from (9.51) that we have

$$(9.53) \quad \mathbb{E} \left[ \mathbb{E}(T \mid P_{T_\ell}(m)) \right]^2 \geq \mathbb{E} \left[ \mathbb{E}(T \mid P_{T_\ell}(m - 1)) \right]^2 + \tau_T u_C^i P(T \mid T_\ell)^2.$$

78
The recursion then loops.

Once the recursion has stopped, it follows from the construction, in particular (9.53), and Proposition 9.5 that we must have

\begin{equation}
\sum_{\ell=1}^{4} p_{\ell} \leq \tau_{T}^{-2} u_{T}^{-2c}.
\end{equation}

The sum \(2 \sum_{\ell=1}^{4} p_{\ell}\) bounds the running time.

At the end of the recursion, the conclusion (9.47) holds. The other conclusions are appropriate upper bounds on the multiplicities in terms of some (very quickly growing) function of \(u_{T}, \tau_{T}\), and the multiplicities of the given partitions. These estimates follow from (9.48), and (9.52).

To supply some details, let us set

\begin{align*}
\Gamma(1) &:= \Psi_{3}(u_{T}, \frac{1}{2} \tau_{T}, P_1(S), P_2S) \times [2 \uparrow [2P_{2}^3]], \\
\Gamma(p + 1) &:= \Psi_{3}(u_{T}, \frac{1}{2} \tau_{T}, \Gamma(p), \Gamma(p)) \times [2 \uparrow [2\Gamma(p)^{p}]].
\end{align*}

From (9.28), (9.29), (9.48), (9.52), and (9.54), we have

\[ \text{mult}(P_i(m) \mid P(m)) \leq \Gamma(m) \leq \Gamma(8 \tau_{T}^{-2} u_{T}^{-2c}), \quad i = 1, 2. \]

Since \(\Psi_{3}\) is itself a power-tower, defined in terms of the \(2 \uparrow \uparrow J\) function, we thus, have a second power-tower from this estimate. Since the partition \(P_{T}\) is generated from the prior partitions, this last estimate proves (9.46). The estimate (9.45) follows from similar considerations, and the estimate (9.34).

\section{9.7 Proof of Lemma 3.17}

Recall that \(A \subset T\), by assumption, and that \(P(A \mid T) \geq \delta + \nu\). Apply Lemma 9.44 to the corner system \(A\) as in (3.2). This Lemma also takes the parameters

\[ u_{T} = \delta, \quad \tau_{T} = cv^{C_T} \mathbb{P}(T \mid H \times H \times H). \]

Here the constant \(C_T\) is the constant that appears Lemma 8.3, see (8.9). Let \(S'\) be the partition-system given to us by this Lemma, satisfying (9.46) and (9.47).
Also consider the set
\[ E' := \{ t \in P_T' \mid \mathbb{P}(t \mid H_{t,1} \times H_{t,2} \times H_{t,3}) \leq v[P_T(S')]^{-1}\mathbb{P}(T \mid H \times H \times H) \} \]

Here, we are using the notation of (9.31) and (9.30). Then, it is clear that \( \mathbb{P}\left( \bigcup \{ t \mid t \in E' \} \right) \leq \tau_T \). Hence, by the pigeonhole principle (See Proposition 9.7.) we can select \( t \in P_T' \) so that \( t \notin E' \), and the \( T \)-system \( S_3(t) \) is \( \delta \)-admissible, which is (3.20) and \( \mathbb{P}(A \mid T) \geq \delta + \nu/4 \) which is (3.19). The estimate (3.18) follows from the estimate (9.45).

### 10 The Algorithm to Conclude the Main Theorem

This is a well-known argument. To prove our main Theorem, we should show that for any \( 0 < \delta < 1 \) there is an \( n(\delta) \) so that if \( \dim(H) \geq n(\delta) \), and \( A \subset H \times H \times H \) with \( \mathbb{P}(A \mid H \times H \times H) \geq \delta \), then \( A \) contains a corner.

We recursively construct a sequence of corner-systems
\[ \mathcal{A}(m) = \{ H, S_i(m), R_{i,j}(m), T(m), A(m) \mid 1 \leq i, j \leq 4 \} \]

\( \mathcal{A}(0) \) is the ‘trivial’ corner-system
\[ R_i(0) = H, \quad S_{i,j}(0) = H \times H, \quad T = H \times H \times H, \quad A(0) = A. \]

Moreover, at each stage, \( A(m) \subset A \), so that a corner in \( A(m) \) is a corner in \( A \).

The point is that the recursion, when it stops, provides us with a corner-system \( \mathcal{A}(m_0) \) so that (1) \( \mathbb{P}(A(m_0) \mid T(m_0)) \geq \delta \), (2) \( \mathcal{A}(m_0) \) is \( \mathbb{P}(A(m_0) \mid T(m_0))-\text{admissible} \), (3) \( \mathcal{A}(m_0) \) satisfies (3.15),

\[
\begin{align*}
\dim(H(m_0)) & \geq \dim(H) - \Phi_{\text{dim}}(\delta), \\
\mathbb{P}(T(m_0) \mid H(m_0) \times H(m_0) \times H(m_0)) & \geq \Phi_{A,T}(\delta).
\end{align*}
\]

Here, \( \Phi_{\text{dim}} \) is a map from \([0,]\) to \( \mathbb{N} \), and \( \Psi_{A,T}(\delta) \) is a finite function from \([0,1]\) to itself. Then, it follows that Lemma 3.13 implies \( A(m_0) \) has a corner provided (3.14) holds, that is
\[ |H(m_0)|^4 \geq 100\Psi_{A,T}(\delta)^3. \]
By (10.1), this will clearly hold provided $\dim(H) > n(\delta)$, for a computable function $n(\delta)$. Thus, our Main Theorem is proved.

The recursion is this: Given the corner-system $\mathcal{A}(m)$, it will be $\mathbb{P}(A(m) \mid T(m))$-admissible. If it does not satisfy (3.13), then we apply Lemma 3.16 to conclude the existence of an corner-system

$$\mathcal{A}'(m) = \{H'(m), S'_i(m), R'_{ij}(m), T'(m), A'(m) \mid 1 \leq i, j \leq 4\}$$

satisfying these conditions: $A'(m) \subset A(m)$,

$$\mathbb{P}(T'(m) \mid T(m)) \geq \kappa[\mathbb{P}(A(m) \mid T(m))]^{1/\kappa},$$

$$\mathbb{P}(A'(m) \mid T'(m)) \geq \mathbb{P}(A(m) \mid T(m)) + \kappa[\mathbb{P}(A(m) \mid T(m))]^{1/\kappa}.$$ 

These are the conclusions of Lemma 3.16.

The corner-system $\mathcal{A}'(m)$ need not be $\mathbb{P}(A'(m) \mid T'(m))$-admissible, therefore, we apply Lemma 3.17, with

$$\delta = \mathbb{P}(A(m) \mid T(m)), \quad \nu = \kappa[\mathbb{P}(A(m) \mid T(m))]^{1/\kappa}.$$

The conclusion of this Lemma gives us a new corner-system $\mathcal{A}(m + 1)$, which satisfies

$$\mathbb{P}(A(m + 1) \mid T(m + 1)) \geq \mathbb{P}(A(m) \mid T(m)) + \kappa[\mathbb{P}(A(m) \mid T(m))]^{1/\kappa}$$

$$\geq \delta + \kappa \delta^{1/\kappa}$$

$$\mathbb{P}(T(m + 1) \mid H(m + 1) \times H(m + 1) \times H(m + 1)))$$

$$\geq \widetilde{\Psi}_T(\mathbb{P}(A(m) \mid T(m)), \mathbb{P}(T(m) \mid H(m) \times H(m) \times H(m))),$$

$$\codim(H(m + 1)) \leq \Psi_{\text{codim}}(\mathbb{P}(A(m) \mid T(m)), \mathbb{P}(T(m) \mid H(m) \times H(m) \times H(m))).$$

The functions $\Psi_{\text{codim}}$ and $\widetilde{\Psi}_T$ are derived from those in (3.18) and (3.21) by a change of variables.

Note that (10.3) implies that the recursion can continue for at most $m_0 \leq 4(\kappa \delta^{1/\kappa})^{-1}$ times before it must stop, as the density of $A(m)$ in $T(m)$ can never be more than 1. Note that initially, we have $T(0) = H(0) \times H(0) \times H(0)$, therefore the iteration of the estimate (10.4) can be phrased completely in terms of a fixed function of $\delta = \mathbb{P}(A(0))$, therefore the estimate (10.2) holds. A similar argument applies to prove the estimate (10.1), completing the proof of our Main Theorem.
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