Full description of the eigenvalue set of the $(p,q)$-Laplacian with a Steklov-like boundary condition

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Abstract In this paper we consider in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary an eigenvalue problem for the negative $(p,q)$-Laplacian with a Steklov-like boundary condition, where $p,q \in (1,\infty)$, $p \neq q$, including the open case $p \in (1,\infty)$, $q \in (1,2)$, $p \neq q$. A full description of the set of eigenvalues of this problem is provided. Our results complement those previously obtained by Abreu and Madeira [1], Barbu and Moroşanu [4], FărăŞeanu, Mihăilescu and Stancu-Dumitru [8], Mihăilescu [13], Mihăilescu and Moroşanu [14].

Keywords Eigenvalues · $(p,q)$-Laplacian · Sobolev space · Nehari manifold · variational methods.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$. Consider in $\Omega$ the eigenvalue problem

$$\begin{aligned}
Au := -\Delta_p u - \Delta_q u &= \lambda a(x) |u|^{q-2} u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \omega} &= \lambda b(x) |u|^{q-2} u \quad \text{on } \partial \Omega,
\end{aligned}$$

(1)

under the following hypotheses

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\[(h_{pq})\quad p, q \in (1, \infty), p \neq q;\]

\[(h_{ab})\quad a \in L^\infty(\Omega) \text{ and } b \in L^\infty(\partial \Omega) \text{ are given nonnegative functions satisfying}\]
\[
\int_\Omega a(x) \, dx + \int_{\partial \Omega} b(\sigma) \, d\sigma > 0.\tag{2}
\]

We have used above the notation
\[
\frac{\partial u}{\partial \nu_A} := (| \nabla u |^{p-2} + | \nabla u |^{q-2}) \frac{\partial u}{\partial \nu},
\]
where \(\nu\) is the unit outward normal to \(\partial \Omega\). As usual, \(\Delta_p\) denotes the \(p\)-Laplacian, i.e.,
\[
\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u).
\]

The operator \((\Delta_p + \Delta_q)\), called \((p,q)\)-Laplacian, occurs in many applications in physics and related sciences such as biophysics (see \cite{9}, \cite{15}), quantum and plasma physics (see \cite{2}, \cite{19}), solid state physics (\cite{16}), chemical reaction design (see \cite{3}), etc.

The solution \(u\) of \((1)\) is understood in a weak sense, as an element of the Sobolev space \(W := W^{1, \max\{p,q\}}(\Omega)\) satisfying equation \((1)_1\) in the sense of distributions and \((1)_2\) in the sense of traces.

\[\text{Definition 1}\quad \lambda \in \mathbb{R} \text{ is an eigenvalue of problem } (1) \text{ if there exists } u_\lambda \in W \setminus \{0\} \text{ such that}
\]
\[
\int_{\Omega} \left( | \nabla u_\lambda |^{p-2} + | \nabla u_\lambda |^{q-2} \right) \nabla u_\lambda \cdot \nabla w \, dx
\]
\[
= \lambda \left( \int_{\Omega} a | u_\lambda |^{q-2} u_\lambda w \, dx + \int_{\partial \Omega} b | u_\lambda |^{q-2} u_\lambda w \, d\sigma \right) \forall w \in W.\tag{3}
\]

According to a Green type formula (see \cite{7}, p. 71), \(u \in W\) is a solution of \((1)\) if and only if it satisfies \((3)\).

Choosing \(w = u_\lambda\) in \((3)\) shows that the eigenvalues of problem \((1)\) cannot be negative. It is also obvious that \(\lambda_0 = 0\) is an eigenvalue of this problem and the corresponding eigenfunctions are the nonzero constant functions. So any other eigenvalue belongs to \((0, \infty)\).

If we assume that \(\lambda > 0\) is an eigenvalue of problem \((1)\) and choose \(w \equiv 1\) in \((3)\) we deduce that every eigenfunction \(u_\lambda\) corresponding to \(\lambda\) satisfies the equation
\[
\int_{\Omega} a | u_\lambda |^{q-2} u_\lambda \, dx + \int_{\partial \Omega} b | u_\lambda |^{q-2} u_\lambda \, d\sigma = 0.\tag{4}
\]

So all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set
\[
\mathcal{C} := \left\{ u \in W; \int_{\Omega} a | u |^{q-2} u \, dx + \int_{\partial \Omega} b | u |^{q-2} u \, d\sigma = 0 \right\}.\tag{5}
\]

This set is a symmetric cone and for \(q = 2\) it is a linear subspace of \(W\).

In the particular case \(q = 2, a \equiv 1, b \equiv 0\), the set of eigenvalues for problem \((1)\) was completely described by M. Mihăilescu \cite{13} (for \(p > 2\)) and M. Fărăcaseanu, M. Mihăilescu and D. Stancu-Dumitru \cite{8} (for \(p \in (1, 2)\)). Problem \((1)\) with \(q = 2, p \in (1, \infty) \setminus \{2\}\) has been studied by J. Abreu and G. Madeira \cite{1}. We point
out that in the case \( p = 2 \) the techniques employed in the papers just mentioned are not applicable to the case of the \((p, q)\)-Laplacian with \( q \neq 2 \) since in this situation \( C \) is no longer a linear subspace of \( W \). Note that problem \( \text{I} \) with \( p \in (1, \infty) \), \( q \in (2, \infty) \), \( p \neq q \), \( a \equiv 1 \), \( b \equiv 0 \) has been investigated by M. Mihăilescu and G. Moroşanu in [14]; also, problem \( \text{I} \) with \( p \in (1, \infty) \), \( q \in (2, \infty) \), \( p \neq q \) has been solved by L. Barbu and G. Moroşanu [4]. The strategy employed in these two papers, based on the Lagrange Multipliers Rule, cannot be applied to the case of the \((p, q)\)-Laplacian with \( p \neq q \), since the constraint set \( C \) defined in (5) is no longer a \( C^1 \) manifold. This case requires separate analysis and some difficulties that occur within the new framework have to be overcome. We shall make use of the so-called direct methods in the Calculus of Variations. In fact, the arguments we shall use work for all \( q \in (1, \infty) \), not just for \( q \in (1, 2) \).

Specifically, our goal here is to determine the set of all eigenvalues of problem \( \text{I} \) under \((h_pq) \) and \( (h_{ab}) \). As we have already mentioned, in \[ \text{I} \] Theorem 1.1] and \[ \text{I} \] Theorem 3.1] it was proved that in the cases \( p \in (1, \infty) \), \( q \in (1, 2) \), \( p \neq q \), respectively, the set of eigenvalues of problem \( \text{I} \) is given by \( 0 \cup (\lambda_1, \infty) \), where \( \lambda_1 \) is given by

\[
\lambda_1 := \inf_{w \in C \\setminus \{0\}} \frac{\int_{\Omega} a \left| \nabla w \right|^q dx}{\int_{\Omega} w \left| \nabla w \right|^q dx + \int_{\partial\Omega} b \left| w \right|^q d\sigma} \tag{6}
\]

Note that the denominators of the above fractions may equal zero for some \( w \)'s in \( C \setminus \{0\} \) and in such cases the corresponding numerators are obviously \( > 0 \), thus the values of those fractions are considered \( \infty \) so they do not contribute to \( \lambda_1 \).

Let us now state the main result of this paper (which covers the open case \( p \in (1, \infty) \), \( q \in (1, 2) \), \( p \neq q \)).

**Theorem 1** Assume that \((h_pq)\) and \((h_{ab})\) above are fulfilled. Then the set of eigenvalues of problem \( \text{I} \) is precisely \( \{0\} \cup (\lambda_1, \infty) \), where \( \lambda_1 \) is the positive constant defined by (6).

The conclusion that the eigenvalue set contains an interval is due to the fact that the operator \( A \) is nonhomogeneous \( (p \neq q) \). Note also that Theorem \( \text{I} \) provides a full description of the eigenvalue set of \( A \).

On the other hand, a complete description of the eigenvalue set in the homogeneous case \( p = q \) is not known even in particular cases. For example, if \( p = q > 1 \), \( a \equiv 1 \), \( b \equiv 0 \), then the eigenvalue set of the corresponding problem is fully known only if \( p = q = 2 \) (i.e., \( A = -2\Delta \)); otherwise, i.e. if \( p = q \in (1, \infty) \), \( p \neq 2 \), then it is only known, as a consequence of the Ljusternik-Schnirelman theory, that there exists a sequence of positive eigenvalues of problem \( \text{I} \) with \( A = -2\Delta \) (see, e.g., [11] Chap. 6)], but this sequence may not constitute the whole eigenvalue set.

### 2 Preliminary results

Let \( q \in (1, \infty) \) be arbitrary but fixed. As we have pointed out in Introduction, all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

\[
\mathcal{C} := \left\{ u \in W; \int_{\Omega} a \left| u \right|^{q-2} u \, dx + \int_{\partial\Omega} b \left| u \right|^{q-2} u \, d\sigma = 0 \right\}.
\]
This is a symmetric cone. Moreover, \( C \) is a weakly closed subset of \( W \). Indeed, let \( (u_n)_n \subset C \) such that \( u_n \rightharpoonup u_0 \) in \( W \). Since \( W \hookrightarrow L^q(\Omega) \) and \( W \hookrightarrow L^q(\partial \Omega) \) compactly, there exists a subsequence of \( (u_n)_n \), also denoted \( (u_n)_n \), such that
\[
 u_n \rightharpoonup u_0 \text{ in } L^q(\Omega), \quad u_n \rightharpoonup u_0 \text{ in } L^q(\partial \Omega).
\]

By Lebesgue’s Dominated Convergence Theorem (see also [3, Theorem 4.9]) we obtain \( u_0 \in C \). In addition, \( C \) has nonzero elements (see [4, Section 2]).

Now, for \( r > 1 \) define the set
\[
 C_r := \left\{ u \in W^{1,r}(\Omega); \int_{\Omega} a |u|^{r-2} u \, dx + \int_{\partial \Omega} b |u|^{r-2} u \, d\sigma = 0 \right\}.
\]

Arguing as before, we infer that for all \( r > 1 \), \( C_r \) is a symmetric, weakly closed (in \( W^{1,r}(\Omega) \)) cone, containing infinitely many nonzero elements. Note also that \( C = C_q \) if \( q > p \), otherwise (i.e., if \( q < p \)), then \( C \) is a proper subset of \( C_q \).

Next, for \( q > 1 \), we consider the eigenvalue problem
\[
 \begin{align*}
 -\Delta q u &= \lambda a(x) |u|^{q-2} u \text{ in } \Omega, \\
 |\nabla u|^{q-2} \nabla u &\cdot \nabla w = \lambda b(x) |u|^{q-2} u \text{ on } \partial \Omega.
\end{align*}
\]

As usual, the number \( \lambda \in \mathbb{R} \) is said to be an eigenvalue of problem (7) if there exists a function \( u_{\lambda} \in W^{1,q}(\Omega \setminus \{0\}) \) such that
\[
 \int_{\Omega} |\nabla u_{\lambda}|^{q-2} \nabla u_{\lambda} \cdot \nabla w \, dx \\
 = \lambda \left( \int_{\Omega} a |u_{\lambda}|^{q-2} u_{\lambda} w \, dx + \int_{\partial \Omega} b |u_{\lambda}|^{q-2} u_{\lambda} w \, d\sigma \right) \quad \forall w \in W^{1,q}(\Omega).
\]

Obviously, \( \lambda_0 = 0 \) is an eigenvalue of problem (7) and any other eigenvalue belongs to \( (0, \infty) \). Moreover, if we consider an eigenvalue \( \lambda > 0 \) of (7) and choose \( w \equiv 1 \) in (7), we deduce that every eigenfunction \( u_{\lambda} \) corresponding to \( \lambda \) belongs to \( C_q \setminus \{0\} \). We also define
\[
 \lambda_{1q} := \inf_{u \in C_q \setminus \{0\}} \frac{\int_{\Omega} |\nabla w|^{q} \, dx}{\int_{\Omega} a |w|^{q} \, dx + \int_{\partial \Omega} b |\nabla w|^{q} \, d\sigma}.
\]

Now, let us consider the functional
\[
 J : W^{1,q}(\Omega) \to \mathbb{R}, \quad J(w) := \int_{\Omega} |\nabla w|^{q} \, dx,
\]
which is positively homogeneous of order \( q \). By standard arguments we can infer that functional \( J \) is convex and weakly lower semicontinuous for all \( q > 1 \).

Consider the minimization problem
\[
 \inf_{w \in C_1} J(w),
\]
where
\[
 C_1 := C_q \cap \left\{ u \in W^{1,q}(\Omega); \int_{\Omega} a |u|^{q} \, dx + \int_{\partial \Omega} b |u|^{q} \, d\sigma = 1 \right\}.
\]

The next result states that \( J \) attains its minimal value over the set \( C_{1q} \), this value is positive and is equal to \( \lambda_{1q} \).
Lemma 1 If $q \in (1, \infty)$, then there exists $u_* \in \mathcal{C}_q$ such that

$$
\mu := J(u^*) = \inf_{w \in \mathcal{C}_q} J(w) > 0.
$$

Moreover, $\mu = \lambda_{1q}$ and it is the lowest positive eigenvalue of problem (7) with eigenfunction $u_*$. 

Proof It is well-known that functional $J$ is of class $C^1$ on $W^{1,q}(\Omega)$ and obviously $J$ is bounded below. Let $(u_n)_n \subset \mathcal{C}_q$ be a minimizing sequence for $J$, i.e.,

$$
J(u_n) \to \inf_{w \in \mathcal{C}_q} J(w) = \mu.
$$

Let us prove that $(u_n)_n$ is bounded in $W^{1,q}(\Omega)$. Assume the contrary, that there exists a subsequence of $(u_n)_n$, again denoted $(u_n)_n$, such that $\|u_n\|_{W^{1,q}(\Omega)} \to \infty$ as $n \to \infty$. Define

$$
v_n := \frac{u_n}{\|u_n\|_{W^{1,q}(\Omega)}} \quad \forall n \in \mathbb{N}.
$$

Clearly, the sequence $(v_n)_n$ is bounded in $W^{1,q}(\Omega)$ so there exist a $v \in W^{1,q}(\Omega)$ and a subsequence of $(v_n)_n$, again denoted $(v_n)_n$, such that

$$
v_n \to v \text{ in } W^{1,q}(\Omega).
$$

Since $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ and $W^{1,q}(\Omega) \hookrightarrow L^q(\partial \Omega)$ compactly, we have up to a subsequence

$$
v_n \to v \text{ in } L^q(\Omega), \ v_n \to v \text{ in } L^q(\partial \Omega).
$$

As $\|v_n\|_{W^{1,q}(\Omega)} = 1 \ \forall n \in \mathbb{N}$, we have $\|v\|_{W^{1,q}(\Omega)} = 1$, and

$$
\int_{\Omega} |\nabla v|^q \, dx \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla v_n|^q \, dx = \liminf_{n \to \infty} \frac{1}{\|u_n\|_{L^q(\Omega)}} J(u_n) = 0,
$$

which shows that $v$ is a constant function. On the other hand, since $(v_n)_n \subset \mathcal{C}_q$ and $\mathcal{C}_q$ is weakly closed in $W^{1,q}(\Omega)$, we infer that $v \in \mathcal{C}_q$, hence $v \equiv 0$. But this contradicts the fact that $\|v\|_{W^{1,q}(\Omega)} = 1$. Therefore, $(u_n)_n$ is indeed bounded in $W^{1,q}(\Omega)$, hence there exist $u_* \in W^{1,q}(\Omega)$ and a subsequence of $(u_n)_n$, also denoted $(u_n)_n$, such that

$$
u_n \to u_* \text{ in } W^{1,q}(\Omega),
$$

$$
u_n \to u_* \text{ in } L^q(\Omega), \ u_n \to u_* \text{ in } L^q(\partial \Omega).
$$

By Lebesgue’s Dominated Convergence Theorem we obtain $u_* \in \mathcal{C}_q$, so the weak lower semicontinuity of $J$ leads to $\mu = J(u_*)$. In addition, $J(u_*) > 0$. Indeed, assuming by contradiction that $J(u_*) = 0$ would imply that $u_* \equiv \text{Const.}$, which is impossible because $u_* \in \mathcal{C}_q$ (see also assumption $(h)_{ab}$).

Since the functional $J$ is positively homogeneous of order $q$, we have

$$
\mu = \inf_{w \in \mathcal{C}_q \setminus \{0\}} \frac{\int_{\Omega} |\nabla w|^q \, dx}{\int_{\Omega} a \, |w|^q \, dx + \int_{\partial \Omega} b \, |w|^q \, d\sigma}.
$$

thus, we derive from (8) $\mu = \lambda_{1q}$. 

We are now going to prove that $\mu = \lambda_{1q}$ is the lowest positive eigenvalue of problem (7) with corresponding eigenfunction $u_*$. For $q \in [2, \infty)$ the result has been proved in [3] Remark 3.2. If $q \in (1, 2)$, since the constraint set $C_q$ is no longer a $C^1$ manifold, we cannot use the Lagrange Multipliers Theorem as in [3]. In order to overcome this inconvenience, let us define $J_\mu : W^{1,q}(\Omega) \to \mathbb{R}$,

$$J_\mu(u) = \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{\mu}{q} \left( \int_{\Omega} a \, |u|^q \, dx + \int_{\partial \Omega} b \, |u|^q \, d\sigma \right) \forall u \in W^{1,q}(\Omega),$$

(12)

which is a $C^1$ functional whose derivative is given by

$$\langle J'_\mu(u), w \rangle = \int_{\Omega} |\nabla u|^{q-2} \nabla u \cdot \nabla w \, dx - \mu \left( \int_{\Omega} a \, |u|^{q-2} \, uw \, dx + \int_{\partial \Omega} b \, |u|^{q-2} \, uw \, d\sigma \right)$$

(13)

for all $u, w \in W^{1,q}(\Omega)$. In order to prove that $\mu = \lambda_{1q}$ is an eigenvalue of problem (7) with eigenfunction $u_* \neq 0$, it is sufficient to show that $J'_\mu(u_*) = 0$. In this case we make use of an argument in [5, Lemma 5.8].

In this respect, we fix $v \in \text{Lip}(\Omega)$ arbitrarily and try to construct a sequence $(u_n)_n \subset C_q$ such that $u_n \to u_*$ in $W^{1,q}(\Omega)$ as $n \to \infty$. To this aim, let us define $I : W^{1,q}(\Omega) \to \mathbb{R}$,

$$I(w) := \int_{\Omega} a \, |w|^{q-2} \, w \, dx + \int_{\partial \Omega} b \, |w|^{q-2} \, w \, d\sigma \forall w \in W^{1,q}(\Omega),$$

and for each $n \in \mathbb{N}^*$,

$$g_n : \mathbb{R} \to \mathbb{R}, \quad g_n(s) := I\left(u_* + \frac{1}{n} v + s\right) \forall s \in \mathbb{R}. \quad (14)$$

Since the function $s \mapsto |w + s|^{q-2} (w + s)$ is strictly increasing on $\mathbb{R}$, $g_n$ is increasing on $\mathbb{R}$. In fact, $g_n$ is strictly increasing on $\mathbb{R}$ since, by virtue of $(h_{ab})$, we see that [2] implies that either $\{|x \in \Omega; a(x) > 0\}|_N > 0$ or $a = 0$ a.e. in $\Omega$ and $|\{x \in \partial \Omega; b(x) > 0\}|_{N-1} > 0$. Here $|\cdot|_N, |\cdot|_{N-1}$ denote the Lebesgue measures of the corresponding sets.

In order to show that for all $n \in \mathbb{N}^*$ there exists $s_n \in \mathbb{R}$ such that $g_n(s_n) = 0$, i.e. $u_* + \frac{1}{n} v + s_n \in C_q$, we also define $h_n : \mathbb{R} \to \mathbb{R}$,

$$h_n(s) = \int_{\Omega} a \, |u_* + \frac{1}{n} v + s|^q \, dx + \int_{\partial \Omega} b \, |u_* + \frac{1}{n} v + s|^q \, d\sigma \forall n \in \mathbb{N}^* \forall s \in \mathbb{R}. \quad (15)$$

It is easily seen that $h_n$ is coercive, because

$$h_n(s) \geq 2^{-q} \, |s|^q \left( \|a\|_{L^\infty(\Omega)} |\Omega|_N + \|b\|_{L^\infty(\partial \Omega)} |\partial \Omega|_{N-1} \right) - \int_{\Omega} a \, |u_* + \frac{1}{n} v|^q \, dx - \int_{\partial \Omega} b \, |u_* + \frac{1}{n} v|^q \, d\sigma.$$

Here, we have also used the inequality

$$|x|^q \leq (|x + y| + |y|)^q \leq 2^q (|x + y|^q + |y|^q) \forall x, y \in \mathbb{R}, \quad q > 1.$$
Moreover, $h_n$ is continuously differentiable, $h'_n = g_n$ (see [10] Theorem 2.27) and convex (its derivative $g_n$ is an increasing function). Therefore, for all $n \in \mathbb{N}^*$, $h_n$ has a minimizer $s_n$, such that $h'_n(s_n) = g_n(s_n) = 0$.

Next, we want to show that the sequence $(ns_n)_n$ is bounded. Arguing by contradiction, let us assume that, after passing to a subsequence if necessary, $ns_n \to \infty$ or $ns_n \to -\infty$ as $n \to \infty$. Since $v \in \text{Lip}(\Omega)$, there exists $N_1$ large enough such that, we have either $v(\cdot) + ns_n > 0$ in $\Omega$, or $v(\cdot) + ns_n < 0$ in $\Omega \forall n \geq N_1$.

Set

$$u_n := u_* + \frac{1}{n} v + s_n \forall n \in \mathbb{N}^*. \quad (16)$$

Obviously, $(u_n)_n \subset C_q$.

Since the functions $g_n$, $n \geq N_1$, are strictly increasing on $\mathbb{R}$, we have

$$0 = g_n(u_n) > g(u_*) = 0 \quad \forall n \geq N_1, \quad (17)$$

if $v(\cdot) + ns_n > 0$ in $\Omega$, or the reverse inequality in the latter case, when $v(\cdot) + ns_n < 0$ in $\Omega$. So, in both cases we get a contradiction.

Consequently, the sequence $(ns_n)_n$ is indeed bounded. This implies that there exists $S \in \mathbb{R}$ such that, on a subsequence, $ns_n \to S$ as $n \to \infty$. Therefore, on a subsequence, we have

$$n( u_n - u_*) \to v + S \text{ and } u_n \to u_* \text{ in } W^{1,q}(\Omega) \text{ as } n \to \infty. \quad (18)$$

In addition, there exists $N_2 \in \mathbb{N}^*$ such that $u_n \not\equiv 0 \forall n \geq N_2$. Now, making use of [11] and [12] it is easy to observe that $u_*$ minimizes functional $J_\mu$ over $C_q \setminus \{0\}$. By using the minimality of $u_*$ and the fact that $u_n \in C_q \setminus \{0\}$, we obtain that

$$0 \leq \lim_{n \to \infty} \frac{J_\mu(u_n) - J_\mu(u_*)}{(1/n)}. \quad (19)$$

On the other hand,

$$n(J_\mu(u_n) - J_\mu(u_*)) = \langle J_\mu'(u_*), n(u_n - u_*) \rangle + o(n; u_*, v), \quad (20)$$

where $o(n; u_*, v)$ is a notation for the term which tends to zero in the definition of the Fréchet differential of $J_\mu$ at $u_*$, that is $o(n; u_*, v) \to 0$ as $n \to \infty$. It follows from [19] - [20] in combination with $u_* \in C_q$ that

$$0 \leq \lim_{n \to \infty} n(J_\mu(u_n) - J_\mu(u_*)) = \lim_{n \to \infty} \langle J_\mu'(u_*), n(u_n - u_*) \rangle + o(n; u_*, v)$$

$$= \langle J_\mu'(u_*), v + S \rangle = \langle J_\mu'(u_*), v \rangle. \quad (21)$$

A similar reasoning with $-v$ instead of $v$ shows that $\langle J_\mu(u_*), v \rangle = 0$ for every Lipschitz test function $v$. Taking into account the density of Lipschitz functions in $W^{1,q}(\Omega)$, which is true since $\partial \Omega$ is smooth (hence Lipschitz, see [12] Theorem 3.6), we obtain that $u_*$ is an eigenfunction of problem (7) corresponding to eigenvalue $\mu = \lambda_q > 0$.

It remains to show that there is no eigenvalue of problem (7) in the open interval $(0, \lambda_q)$. 

Remark 2

Assume by way of contradiction that there exists \( \lambda \in (0, \lambda_1) \) for which (7) possesses a solution \( u_\lambda \in C \setminus \{0\} \). It follows from (3) with \( w = u_\lambda \) and (9) that

\[
0 < (\lambda_1 - \lambda) \left( \int_{\Omega} a \ | u_\lambda |^q \ dx + \int_{\partial \Omega} b \ | u_\lambda |^q \ d\sigma \right) \leq \int_{\Omega} | \nabla u_\lambda |^q \ dx
\]

\[
- \lambda \left( \int_{\Omega} a \ | u_\lambda |^q \ dx + \int_{\partial \Omega} b \ | u_\lambda |^q \ d\sigma \right) = 0,
\]

which is a contradiction. This concludes the proof.

Remark 1

If \( u_\lambda \) is an eigenfunction corresponding to an eigenvalue \( \lambda > 0 \), then we have from (3)

\[
\int_{\Omega} \left( | \nabla u_\lambda |^p + | \nabla u_\lambda |^p \right) \ dx = \lambda \left( \int_{\Omega} a \ | u_\lambda |^q \ dx + \int_{\partial \Omega} b \ | u_\lambda |^q \ d\sigma \right),
\]

thus \( u \) cannot be a constant function (see (2) and so

\[
\int_{\Omega} a \ | u_\lambda |^q \ dx + \int_{\partial \Omega} b \ | u_\lambda |^q \ d\sigma > 0.
\]

Therefore, denoting

\[
\Gamma_1(u_\lambda) := \{ x \in \Omega; \ a(x)u_\lambda(x) \neq 0 \}, \quad \Gamma_2(u_\lambda) := \{ x \in \partial \Omega; \ b(x)u_\lambda(x) \neq 0 \},
\]

we see that either \( |\Gamma_1(u_\lambda)| > 0 \) or \( |\Gamma_2(u_\lambda)| > 0 \). Obviously, \( u_\lambda \) corresponding to any eigenvalue \( \lambda > 0 \) cannot be a constant function (see (3) with \( v = u_\lambda \) and (2)).

Remark 2

Note that the infimum on \( C \setminus \{0\} \) of the Rayleigh-type quotient associated to the eigenvalue problem (1) is given by

\[
\bar{\lambda}_1 := \inf_{w \in C \setminus \{0\}} \frac{\frac{1}{p} \int_{\Omega} | \nabla w |^p \ dx + \frac{1}{q} \int_{\partial \Omega} | \nabla w |^q \ dx}{\int_{\Omega} a \ | w |^q \ dx + \int_{\partial \Omega} b \ | w |^q \ d\sigma} \quad (22)
\]

In fact, \( \bar{\lambda}_1 = \lambda_1 \). Indeed, it is obvious that \( \lambda_1 \leq \bar{\lambda}_1 \) and for the converse inequality we note that, \( \forall v \in C \setminus \{0\} \), \( t > 0 \), we have \( tv \in C \setminus \{0\} \) and

\[
\bar{\lambda}_1 \leq \inf_{w \in C \setminus \{0\}} \frac{\frac{1}{p} \int_{\Omega} | \nabla v |^p \ dx + \frac{1}{q} \int_{\partial \Omega} | \nabla v |^q \ dx}{\int_{\Omega} a \ | v |^q \ dx + \int_{\partial \Omega} b \ | v |^q \ d\sigma} \leq \frac{\int_{\Omega} | \nabla v |^p \ dx}{\int_{\Omega} a \ | v |^q \ dx + \int_{\partial \Omega} b \ | v |^q \ d\sigma} + tp^{-q} \frac{q \int_{\Omega} | \nabla v |^p \ dx}{p(\int_{\Omega} a \ | v |^q \ dx + \int_{\partial \Omega} b \ | v |^q \ d\sigma)}.
\]

Now letting \( t \to \infty \) if \( q > p \), and \( t \to 0 \) if \( q < p \), then passing to infimum for \( v \in C \setminus \{0\} \) we get the desired inequality. Hence \( \lambda_1 \) can be expressed in two different ways (see (22) and (22)).

Remark 3

As a consequence of Lemma 1 we have \( \lambda_1 > 0 \). Indeed, from (6) we have

\[
\lambda_1 := \inf_{w \in C_1} \int_{\Omega} | \nabla w |^q \ dx,
\]

where \( C_1 = \{ v \in C; \int_{\Omega} a \ | v |^q \ dx + \int_{\partial \Omega} b \ | v |^q \ d\sigma = 1 \} \). So \( \lambda_1 = J(u^*) \) for \( p \leq q \) and \( \lambda_1 \geq J(u^*) \) if \( p > q \). Thus in both cases \( \lambda_1 > 0 \).
3 Proof of the main result

We have already stated that \( \lambda_0 = 0 \) is an eigenvalue of problem (1) and any other eigenvalue of this problem belongs to \((0, \infty)\). We verify next that no eigenvalue belongs to \((0, \lambda_1)\). To argue by contradiction, assume that problem (1) possesses an eigenvalue \( \lambda \in (0, \lambda_1) \) with a corresponding eigenfunction \( u_\lambda \). Then, from (3)

\[
\int_\Omega \left( |\nabla u_\lambda|^p + |\nabla u_\lambda|^q \right) \, dx = \lambda \left( \int_\Omega a |u_\lambda|^q \, dx + \int_{\partial\Omega} b |u_\lambda|^q \, d\sigma \right).
\]

Note that \( \int_\Omega a |u_\lambda|^q \, dx + \int_{\partial\Omega} b |u_\lambda|^q \, d\sigma \neq 0 \), otherwise \( u_\lambda \equiv \text{Const.} \), which is impossible (see Remark 1). On the other hand, as \( u_\lambda \in C \setminus \{0\} \), we derive from (1) and (23)

\[
\lambda \leq \lambda_1 \leq \frac{\int_\Omega |\nabla u_\lambda|^q \, dx}{\int_\Omega a |u_\lambda|^q \, dx + \int_{\partial\Omega} b |u_\lambda|^q \, d\sigma} \leq \lambda \cdot \frac{\lambda \int_\Omega a |u_\lambda|^q \, dx + \int_{\partial\Omega} b |u_\lambda|^q \, d\sigma}{\int_\Omega a |u_\lambda|^q \, dx + \int_{\partial\Omega} b |u_\lambda|^q \, d\sigma} < \lambda,
\]

which is a contradiction.

In what follows we shall prove that every \( \lambda > \lambda_1 \) is an eigenvalue of problem (1). To this purpose we fix such a \( \lambda \) and define \( J_\lambda : W \to \mathbb{R} \),

\[
J_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx - \lambda \left( \int_\Omega a |u|^q \, dx + \int_{\partial\Omega} b |u|^q \, d\sigma \right),
\]

which is a \( C^1 \) functional whose derivative is given by

\[
(J'_\lambda(u), w) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx + \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla w \, dx - \lambda \left( \int_\Omega a |u|^{q-2} \, dx + \int_{\partial\Omega} b |u|^{q-2} \, d\sigma \right) \forall u, w \in W.
\]

So, according to Definition 1, \( \lambda > \lambda_1 \) is an eigenvalue of problem (1) if and only if there exists a critical point \( u_\lambda \in W \setminus \{0\} \) of \( J_\lambda \), i.e. \( J'_\lambda(u_\lambda) = 0 \).

The proof of Theorem 1 will follow as a consequence of several intermediate results. We shall discuss two cases which are complementary to each other.

**Case 1:** \( q \in (1, \infty) \), \( p > q \).

In this case we have \( W = W^{1,p}(\Omega) \). The following lemma shows, essentially, that the functional defined in (25) is coercive for every \( \lambda > \lambda_1 \) restricted to the subset \( C \subset W = W^{1,p}(\Omega) \).

**Lemma 2** Let \( q \in (1, \infty) \), \( p > q \). For every \( \lambda > \lambda_1 \), we have

\[
\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty, u \in C} J_\lambda(u) = \infty.
\]

For the proof of this lemma we refer the reader to L. Barbu and G. Moroșanu [4, Case 1].
Lemma 3 Let \( q \in (1, \infty), \ p > q. \) Every number \( \lambda \in (\lambda_1, \infty) \) is an eigenvalue of problem (1).

Proof Note that \( \mathcal{C} \) is a weakly closed subset of the reflexive Banach space \( W = W^{1,p}(\Omega) \), and functional \( J_\lambda \) is coercive (see Lemma 2) and weakly lower semicontinuous on \( \mathcal{C} \) with respect to the norm of \( W^{1,p}(\Omega) \). Standard results in the calculus of variations (see, e.g., [17, Theorem 1.2]) ensures the existence of a global minimizer \( z_* \in \mathcal{C} \) for \( J_\lambda \), i.e., \( J_\lambda(z_*) = \min_{\mathcal{C}} J_\lambda \).

From Remark 2 we know that \( \lambda_1 = \lambda_1 \), hence \( \lambda > \lambda_1 = \bar{\lambda}_1 \). Then (by (29)) there exists \( u_{0\lambda} \in \mathcal{C} \setminus \{0\} \) such that \( J_\lambda(u_{0\lambda}) < 0 \). It follows that

\[ J_\lambda(z_*) \leq J_\lambda(u_{0\lambda}) < 0, \]

which shows that \( z_* \neq 0 \).

Next, we are going to show that the global minimizer \( z_* \) for \( J_\lambda \) restricted to \( \mathcal{C} \) is a critical point of \( J_\lambda \) considered on the whole space \( W^{1,p}(\Omega) \), i.e., \( J_\lambda(z_*) = 0 \), in other words, \( z_* \) is an eigenfunction of problem (1) corresponding to \( \lambda \).

In fact, \( z_* \) is a solution of the minimization problem

\[ \min_{w \in W} J_\lambda(w), \]

under the restriction

\[ g(w) := \int_{\Omega} a \ | w |^{q-2} w \ dx + \int_{\partial \Omega} b \ | w |^{q-2} w \ d\sigma = 0. \]

If \( q \in [2, \infty), \ p > q \), we have proved in [4, Case 1], by using the Lagrange Multipliers Rule, that \( J'_\lambda(z_*) = 0 \). For \( q \in (1, 2) \), \( g \) is no longer a \( C^1 \) function on \( W \), so we cannot use the same reasoning to prove our assertion. Fortunately, we can use a technique similar to that used in the proof of Lemma [1] It is worth mentioning that this technique works for the case \( q \in [2, \infty), \) too.

Since \( p > q \), the inclusions \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) and \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \) are compact. As in the proof of Lemma [4] let us fix an arbitrary \( v \in \text{Lip}(\Omega) \) and construct the sequence

\[ u_n := z_* + \frac{1}{n} v + s_n \ \forall n \in \mathbb{N}^*, \tag{26} \]

such that \( \{u_n\}_n \subset \mathcal{C} \).

Similar arguments as in the proof of Lemma [4] can be used in order to prove that the sequence \( (n s_n)_n \) is also bounded, hence it converges on a subsequence to some \( S \in \mathbb{R} \) and so, on a subsequence,

\[ n(u_n - z_*) \to v + S \text{ and } u_n \to z_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{27} \]

Since \( z_* \) minimizes functional \( J_\lambda \) over \( \mathcal{C} \) and \( (u_{n\lambda})_n \subset \mathcal{C} \), we have

\[ 0 \leq \lim_{n \to \infty} \frac{J_\lambda(u_n) - J_\lambda(z_*)}{n}. \tag{28} \]

We also have

\[ n(J_\lambda(u_n) - J_\lambda(z_*)) = \langle J'_\lambda(z_*), n(u_n - z_*) \rangle + o(n; z_{\lambda_*}, v), \tag{29} \]
with \( o(n; z_*,v) \to 0 \) as \( n \to \infty \). From \([27]-[29]\), combined with \( z_* \in \mathcal{C} \), we get
\[
0 \leq \lim_{n \to \infty} n (J_\lambda(u_n) - J_\lambda(z_*)) = \lim_{n \to \infty} \langle J'_\lambda(z_*), n(u_n - z_*) \rangle + o(n; z_*, v) = \langle J'_\lambda(z_*), v + S \rangle = \langle J'_\lambda(z_*), v \rangle.
\]

A similar reasoning with \(-v\) instead of \(v\) and the density of Lipschitz functions in \(W^{1,p}(\Omega)\) yield \( J'_\lambda(z_*) = 0 \), which concludes the proof.

**Case 2:** \( q \in (1, \infty) \), \( p < q \).

In this case \( W = W^{1,q}(\Omega) \) and \( \mathcal{C} = C_q \). Let \( \lambda > \lambda_1 \) be a fixed number. Under the assumption \( p < q \) we cannot expect coercivity on \( W^{1,q}(\Omega) \) of the functional \( J_\lambda \). From now on we analyse the action of \( J_\lambda \) on the Nehari type manifold (see \([18]\)) defined by
\[
\mathcal{N}_\lambda = \{ v \in \mathcal{C} \setminus \{0\} ; \langle J'_\lambda(v), v \rangle = 0 \} = \left\{ v \in \mathcal{C} \setminus \{0\} ; \int_\Omega (|\nabla v|^p + |\nabla v|^q) \, dx = \lambda \left( \int_\Omega a \, |v|^q \, dx + \int_{\partial \Omega} b \, |v|^q \, d\sigma \right) \right\}.
\]

It is natural to consider the restriction of \( J_\lambda \) to \( \mathcal{N}_\lambda \) since any possible eigenfunction corresponding to \( \lambda \) belongs to \( \mathcal{N}_\lambda \). Note that on \( \mathcal{N}_\lambda \), functional \( J_\lambda \) has the form
\[
J_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx - \frac{\lambda}{q} \left( \int_\Omega a \, |u|^q \, dx + \int_{\partial \Omega} b \, |u|^q \, d\sigma \right).
\]

Now, let us recall the following result from L. Barbu and G. Morosanu \([4]\) Case 2, Steps 1-4.1.

**Lemma 4** Let \( q \in (1, \infty) \), \( p < q \). Then there exists a point \( u_* \in \mathcal{N}_\lambda \) where \( J_\lambda \) attains its minimal value, \( m_\lambda := \inf_{w \in \mathcal{N}_\lambda} J_\lambda(w) > 0 \).

In the sequel we show that the minimizer \( u_* \), given by Lemma 4 is a critical point of \( J_\lambda \) considered on the whole space \( W^{1,q}(\Omega) \).

**Lemma 5** Let \( q \in (1, \infty) \), \( p < q \). The minimizer \( u_* \in \mathcal{N}_\lambda \) from Lemma 4 is an eigenfunction of problem \((1)\) with corresponding eigenvalue \( \lambda \).

**Proof** It suffices to prove that \( J'_\lambda(u_*) = 0 \).

In fact \( u_* \) is a minimizer of \( J_\lambda \) for \( w \in W \) subject to the restrictions
\[
g_1(w) := \int_\Omega (|\nabla w|^p + |\nabla w|^q) \, dx - \lambda \left( \int_\Omega a \, |w|^q \, dx + \int_{\partial \Omega} b \, |w|^q \, d\sigma \right) = 0, \quad (31)
\]
\[
g_2(w) := \int_\Omega a \, |w|^{q-2} w \, dx + \int_{\partial \Omega} b \, |w|^{q-2} w \, d\sigma = 0. \quad (32)
\]

In the case \( q \in [2, \infty) \), \( p < q \), the conclusion was proved in L. Barbu and G. Morosanu \([4]\) Step 5, by using the Lagrange Multipliers Rule. If \( q \in (1, 2) \), the function \( g_2 \) is not in \( C^1(W; \mathbb{R}) \), so the Lagrange Multipliers Rule is no longer applicable to this case. What we can do is to apply a reasoning similar to that used in the proofs of Lemmas 1 and 2 to show that \( J'_\lambda(u_*) = 0 \).
So, let $v \in \text{Lip}(\Omega)$ be an arbitrary but fixed function. Let $u_\ast \in \mathcal{N}_\lambda$ be the minimizer of $\mathcal{J}_\lambda$ over $\mathcal{N}_\lambda$, and consider the sequence $(u_n)_n \subset W^{1,q}(\Omega)$,

$$u_n := u_\ast + \frac{1}{n}v + s_n \ \forall \ n \in \mathbb{N}^*,$$  \hspace{1cm} (33)

with $(u_n)_n \subset \mathcal{C}_q$. Again, the sequence $(ns_n)_n$ is bounded, so it converges on a subsequence to some $S \in \mathbb{R}$. Therefore, on a subsequence, we have

$$n(u_n - u_\ast) \to v + S, \ u_n \to u_\ast \text{ in } W^{1,q}(\Omega) \text{ as } n \to \infty.$$  \hspace{1cm} (34)

Since $u_\ast \neq 0$, one can assume that $(u_n)_n \subset \mathcal{C}_q \setminus \{0\}$. Using this last subsequence of $(u_n)_n$, we shall construct a sequence $(t_n)_n \subset \mathbb{R}$ such that $(t_nu_n)_n \subset \mathcal{N}_\lambda$, for every $n$ sufficiently large, i.e.,

$$t_n^p \int_{\Omega} |\nabla u_n|^p \ dx + t_n^q \int_{\Omega} |\nabla u_n|^q \ dx = \lambda t_n^q \left( \int_{\Omega} a |u_n|^q \ dx + \int_{\partial \Omega} b |u_n|^q \ d\sigma \right),$$  \hspace{1cm} (35)

or, equivalently,

$$t_n = \left( \frac{\int_{\Omega} |\nabla u_n|^p \ dx}{\lambda \left( \int_{\Omega} a |u_n|^q \ dx + \int_{\partial \Omega} b |u_n|^q \ d\sigma \right)} \right)^{\frac{1}{q-p}}.$$  \hspace{1cm} (36)

Note that for sufficiently large $n$, both the numerator and the denominator are positive numbers. Indeed, since $u_\ast \in \mathcal{N}_\lambda$, we have

$$\int_{\Omega} |\nabla u_\ast|^p \ dx > 0 \text{ and } \int_{\Omega} |\nabla u_\ast|^q \ dx < \lambda \left( \int_{\Omega} a |u_\ast|^q \ dx + \int_{\partial \Omega} b |u_\ast|^q \ d\sigma \right).$$  \hspace{1cm} (37)

Since the functionals

$$\mathcal{I}_1, \mathcal{I}_2 : W \to \mathbb{R}, \ \mathcal{I}_1(w) := \int_{\Omega} |\nabla w|^p \ dx,$$
$$\mathcal{I}_2(w) := -\int_{\Omega} |\nabla w|^q \ dx + \lambda \left( \int_{\Omega} a |w|^q \ dx + \int_{\partial \Omega} b |w|^q \ d\sigma \right) \ \forall \ w \in W$$  \hspace{1cm} (38)

are continuous on $W$ and $\mathcal{I}_1(u_\ast) > 0$, $\mathcal{I}_2(u_\ast) > 0$, (see (37)), there exists $\delta_0 > 0$ such that

$$w \in W, \ ||w - u_\ast||_W < \delta_0 \implies \mathcal{I}_1(w) > 0, \ \mathcal{I}_2(w) > 0.$$  \hspace{1cm} (39)

Since $u_n \to u_\ast$ in $W$, it follows that for $N_0$ large enough, $\mathcal{I}_1(u_n) > 0, \mathcal{I}_2(u_n) > 0 \ \forall \ n \geq N_0$, hence $t_n$ given by (36) is well defined for $n \geq N_0$. So we can define

$$z_n := t_n \left( u_\ast + \frac{1}{n}v + s_n \right) = t_n u_n \ \forall \ n \geq N_0,$$  \hspace{1cm} (39)

with $(z_n)_n \subset \mathcal{N}_\lambda$. In addition, using (36) and (39), we can see that

$$t_n \to 1 \text{ in } \mathbb{R}, \ z_n \to u_\ast \text{ in } W^{1,q}(\Omega) \text{ as } n \to \infty.$$  \hspace{1cm} (40)
In what follows we shall prove that the sequence \((n(t_n - 1))_n\) is bounded. To this purpose, let us first show that the sequence \((n(t_n^p - q - 1))_n\) is bounded. Define the functional \(L_\lambda : W \to \mathbb{R}\),

\[
L_\lambda(u) = -\int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^q \, dx + \lambda \int_\Omega a |u|^q \, dx + \int_{\partial \Omega} b |u|^q \, d\sigma \quad \forall u \in W,
\]

which belongs to \(C^1(W; \mathbb{R})\), and for \(u, w \in W\)

\[
\langle L'_\lambda(u), w \rangle = -p \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx - q \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla w \, dx + \lambda \int_\Omega a |u|^{q-2} u w \, dx + \int_{\partial \Omega} b |u|^{q-2} u w \, d\sigma.
\]

From (41) and (45) we deduce that the sequence \((u_n) \to u\) as \(n \to \infty\), since \(p < q\).

Hence, there exists \(N \in \mathbb{N}\) such that for all \(n \geq N_0\), we infer from (46) that the sequence \(n(t_n^p - q - 1)\) has a finite limit. Hence, there is \(K > 0\) such that for all \(n \geq N_0\), \(n(t_n^p - q - 1) \leq K\), which implies

\[
1 - \frac{K}{n} \leq t_n^p - q - 1 \leq 1 + \frac{K}{n} \quad \forall n \geq N_0.
\]

Since, there exists \(N_1 \in \mathbb{N}\) such that \(1 - K/n > 0 \quad \forall n \geq N_1\), we have

\[
n \left(1 + \frac{K}{n} \right)^{1/(p-q)} - 1 \leq n(t_n - 1) \leq n \left(1 - \frac{K}{n} \right)^{1/(p-q)} - 1 \quad \forall n \geq \max\{N_0, N_1\}.
\]

Taking into account the relations

\[
\lim_{x \to 0} \frac{(1 + Kx)^{1/(p-q)} - 1}{x} = K/(p - q), \quad \lim_{x \to 0} \frac{(1 - Kx)^{1/(p-q)} - 1}{x} = -K/(p - q),
\]

we infer from (46) that the sequence \((n(t_n - 1))_n\) is bounded, thus, by possibly passing to a subsequence, there exists \(T \in \mathbb{R}\), such that \(n(t_n - 1) \to T\) as \(n \to \infty\).

By using the minimality of \(u_*\) and the fact that \((z_n)_n \subset N_\lambda\) we obtain that

\[
0 \leq \lim_{n \to \infty} J_\lambda(z_n) - J_\lambda(u_*)
\]

Since functional \(J_\lambda \in C^1(W; \mathbb{R})\), we can write

\[
n(J_\lambda(z_n) - J_\lambda(u_*)) = \langle J'_\lambda(u_*), n(z_n - u_*) \rangle + o(n; u_*, v),
\]

where \(o(n; u_*, v)\) denotes a term which is of order \(o(n)\) as \(n \to \infty\).
with $o(n; u_*, v) \to 0$ as $n \to \infty$. Taking into account (39) and (40), we can see that, on a subsequence,

$$n(z_n - u_*) = n(t_n - 1)u_* + v + ns_n \to Tu_* + v + S \text{ as } n \to \infty \text{ in } W.$$  \hspace{1cm} (49)

It follows from (47) and (49) that

$$0 \leq \langle J'(\lambda)(u_*), v + S + Tu_* \rangle.$$  \hspace{1cm} (50)

Since $u_* \in \mathcal{N}_\lambda$, we obtain that

$$\langle J'(\lambda)(u_*), u_* \rangle = 0, \quad \langle J'(\lambda)(u_*), S \rangle = 0,$$

hence (50) implies

$$0 \leq \langle J'(\mu)(u_*), v \rangle.$$  \hspace{1cm} (51)

A similar reasoning with $-v$ instead of $v$ shows that the converse inequality holds, hence $0 = \langle J'(\lambda)(u_*), v \rangle$. Finally, using the density of Lipschitz functions in $W$ we obtain that $J'(\lambda)(u_*) = 0$, which concludes the proof.

Therefore, as it has already been pointed out, $\lambda = 0$ is an eigenvalue, so the conclusion of Theorem 1 follows from Lemma 3 and Lemma 5.

**Remark 4** Thus, if $q > 1$ and $1 < p < q$ then $\lambda_1 = \lambda_{1q}$, so the eigenvalue set of problem (1) is $\{0\} \cup (\lambda_{1q}, \infty)$, which is independent of $p$. If $1 < q < p$ then $\lambda_1 \geq \lambda_{1q}$.

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