Randomness vs Non Locality and Entanglement

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Abstract

According to quantum theory, the outcomes obtained by measuring an entangled state necessarily exhibit some randomness if they violate a Bell inequality. In particular, a maximal violation of the CHSH inequality guarantees that 1.23 bits of randomness are generated by the measurements. However, by performing measurements with binary outcomes on two subsystems one could in principle generate up to two bits of randomness. We show that correlations that violate arbitrarily little the CHSH inequality or states with arbitrarily little entanglement can be used to certify that close to the maximum of two bits of randomness are produced. Our results show that non-locality, entanglement, and the amount of randomness that can be certified in a Bell-type experiment are inequivalent quantities. From a practical point of view, they imply that device-independent quantum key distribution with optimal key generation rate is possible using almost-local correlations and that device-independent randomness generation with optimal rate is possible with almost-local correlations and with almost-unentangled states.

Two of the most remarkable features of quantum theory are its intrinsic randomness and its non-local character. The conclusion that measurements on quantum systems yield random results was first reached by Born and is now one of the basic axioms of the theory. The intuition that measurements on entangled quantum systems give rise to correlations that exhibit some form of non-locality was made precise by Bell, whose work led the way to the introduction of a series of inequalities that must be satisfied by any locally causal theories, but which are violated by quantum theory \cite{Bell1964}.

These two – a priori independent – properties of quantum theory are related through a third one, the no-signalling principle. The no-signalling principle states that the outcomes of measurements on separated systems cannot be used to send instantaneous signals. Any theory that satisfies the no-signalling principle and which is non-local, is also necessarily intrinsically random \cite{Cabello2006, Peres2005, Hensen2015}. More precisely, if the measurement outcomes of a Bell-type experiment violate a Bell inequality, then these outcomes cannot be perfectly predicted within a no-signalling (hence within quantum) theory. This conclusion holds independently of any hypothesis on the type of measurements performed or on the quantum systems (it even hold in non-quantum theories provided they satisfy no-signalling). Conversely, if no Bell inequalities are violated in a Bell experiment, then the experimental results admits a purely deterministic explanation if no additional hypothesis are made on the underlying system \cite{Gisin2007}.

The quantitative aspects of this fundamental connection between non-locality and randomness have hardly been explored. Here we address this problem within the quantum formalism (i.e. we do not look at post-quantum theories) and investigate the relation between non-locality, entanglement (which is necessary to produce non-local correlations), and the amount of randomness necessarily present in a Bell experiment.
Beyond its fundamental interest, this question is also motivated by the recent development of device-independent randomness generation (DIRNG) [6, 7] and quantum key distribution (DIQKD) [3, 8, 9]. The observation that the outcomes of measurements performed on two separate quantum systems are necessarily random if they violate a Bell inequality can be exploited to certify the randomness of strings or the secrecy of shared keys generated in quantum cryptographic protocols without the necessity to model the quantum state or the measurement devices, thereby notably increasing the security of such protocols. The quantitative study of the relation between entanglement, non-locality, and randomness allows to determine the minimal resources needed for DIRNG and DIQKD.

We focus here mostly on the simplest case in which two measurements with binary outcomes can be applied to each one of two separated quantum systems. In this context, the only facet-defining Bell inequality is the CHSH inequality [10, 11], whose violation is thus a necessary condition to certify the presence of randomness. The amount of violation of the CHSH inequality can also be considered a natural measure of non-locality: it uniquely determines for instance the maximum local weight of a given set of correlations or the minimal amount of communication required to simulate them [12]. Naively, one would thus expect a direct relation between the amount of CHSH violation, and therefore of entanglement, and the randomness produced in a Bell-type experiment, i.e., the less CHSH violation or entanglement, the less randomness.

Our analysis, however, show that this intuition is not correct and that the relation between these three concepts is much subtler than expected. In our scenario, where two subsystems are measured and where each measurement results in one out of two possible outcomes, the maximal amount of “local randomness” characterizing an individual outcome is 1 bit, while the maximal amount of “global randomness” characterizing the joint pair of outcomes is 2 bits. We introduce here non-local correlations that are arbitrarily close to the local region (i.e., which violate arbitrarily little the CHSH inequality) or that arise from states with arbitrarily little entanglement, yet which necessarily imply that (arbitrarily close to) the maximal amounts of local or global randomness are generated each time the system is measured. To obtain these results we partially characterize the boundary of the set of quantum correlations by introducing a family of Bell inequalities and by determining the quantum points that maximally violate them, i.e., by computing their “Tsirelson bounds” [13].

Before presenting our main results, we introduce the notation and definitions that will be used in the remainder of the paper and establish three useful technical facts.

1 Notation and definitions

**Bell experiments.** We consider measurements on two distinct systems, denoted A and B. On system A, one of two possible measurements \( u \in \{1, 2\} \) are carried out, resulting in one of two possible outcomes \( a \in \{-1, 1\} \). Similarly, measurements \( v \in \{1, 2\} \) are carried out on system B, yielding outcomes \( b \in \{-1, 1\} \). We denote \( P(ab|uv) \) the probability to obtain the pair of outcomes \( (a, b) \) when the measurement settings \( (u, v) \) are used. We focus here on quantum probabilities, i.e., we assume that the distribution \( P \) is of the form

\[
P(ab|uv) = \text{tr}[M_{a|u} \otimes M_{b|v} \rho],
\]

where \( \rho \) is a quantum state in some arbitrary Hilbert space \( H_A \otimes H_B \) and \( M_{a|u} \) and \( M_{b|v} \) are measurement operators, i.e., they are positive and sum to the identity on \( H_A \) and \( H_B \), respectively. By increasing the dimension of the Hilbert spaces \( H_A \) and \( H_B \), we can without loss of generality assume the positive operators \( M_{a|u} \) and \( M_{b|v} \) to be projections. The measurements on system A and B can thus be described by hermitian observables \( A_u = M_{0|u} - M_{1|u} \) and \( B_v = \ldots \)
We say that the state $\rho$ and the observables $M = \{A_u, B_v\}$ form a quantum realization $\{\rho, M\}$ for $P$. In term of the expectation values of the measurements $A_u$ and $B_v$, the probabilities $P(ab|uv)$ can be expressed as
\[
P(ab|uv) = \frac{1}{4} (1 + a\langle A_u \rangle + b\langle B_v \rangle + ab\langle A_u B_v \rangle).
\] (2)

A Bell inequality is a linear constraint $I = \sum_{abuv} I_{abuv} P(ab|uv) \leq I_L$ on $P$ that is satisfied by every locally causal distribution, but which can be violated by quantum distributions. The bound $I_L$ is called the local bound of the inequality. We say that $P$ is non-local if it violates a Bell inequality, i.e., $I > I_L$. In our scenario (two binary measurement per party), there is a unique (up to relabelling of the measurement outcomes and settings) facet inequality, the CHSH inequality
\[
I = \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2.
\] (3)

That is, an arbitrary Bell inequality is violated only if the CHSH inequality is also violated. Furthermore, the amount of violation $I > I_L$ of the CHSH inequality can be viewed as a proper measure of the non-locality of a given distribution $P$: for instance the optimal amount of average communication $C$ required to simulate classically a non-local distribution is directly related to the CHSH violation through $C = I/2 - 1$ [12].

**Randomness.** Within the quantum formalism, two types of randomness have to be distinguished: the genuine, intrinsic randomness of pure states and the randomness of mixed states, which merely represents a lack of knowledge about the definite state of the system. It is the first type of randomness that we want to characterize here.

We quantify the randomness of the outcome pair $(a, b)$ resulting from the measurement of the observables $A$ and $B$ on a given pure state $|\psi\rangle \in H_A \otimes H_B$ through the guessing probability
\[
G(\psi, A, B) = \max_{ab} P(ab|\psi, A, B).
\] (4)

where $P(ab|\psi, A, B) = \frac{1}{4} (1 + a\langle A \rangle_\psi + b\langle B \rangle_\psi + ab\langle AB \rangle_\psi)$ are the corresponding joint outcome probabilities. The quantity (4) corresponds to the probability to guess correctly the outcome pair $(a, b)$, since the best guess that one can make is simply to output the most probable pair. The guessing probability can be expressed in bits and is then known as the min-entropy $H_\infty(\psi, A, B) = -\log_2 G(\psi, A, B)$. If a specific pair of outcomes $(a, b)$ is certain to occur, then the guessing probability takes its maximal value 1 corresponding to 0 bits of min-entropy, while if all four possible pairs of outcomes are equally probable, it takes its minimal value 1/4 corresponding to 2 bits of min-entropy.

**Example:** Let $\psi = (|00\rangle + |11\rangle)/\sqrt{2}$, $A = \sigma_z$, $B = \sigma_x$. Then $P(ab|\psi, A, B) = 1/4$ for all $a, b = 0, 1$, hence $G(\psi, A, B) = 1/4$.

For a mixed state $\rho \in B(H_A \otimes H_B)$, we define the guessing probability associated to measurements $A$ and $B$ as
\[
G(\rho, A, B) = \max_{ab} \sum_\lambda q_\lambda G(\psi_\lambda, A, B).
\] (5)

where the maximum is taken over all pure state decompositions $\rho = \sum_\lambda q_\lambda |\psi_\lambda\rangle\langle \psi_\lambda|$. This corresponds to the maximal average guessing probability given the knowledge of which underlying state $|\psi_\lambda\rangle$ in the ensemble has been prepared. Equivalently, it corresponds to the maximal guessing probability of someone that possess a quantum system correlated with $\rho$ and who can perform measurements on his system to guess the outcomes $(a, b)$.

**Example:** Let $\rho = (|00\rangle|00\rangle + |11\rangle|11\rangle)/2$, $A = \sigma_z$, $B = \sigma_x$. Then $P(ab|\rho, A, B) = 1/4$ for all $a, b = 0, 1$ as above, but $G(\psi, A, B) = 1/2$. 

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Our aim here is to analyze the fundamental constraints on the randomness of a joint probability distribution \( P(ab|uv) \) that follow from its non-local properties alone, independently of any particular quantum realization. Given a quantum distribution \( P \), we thus define the (realization-independent) guessing probability of the outcome pair \((a, b)\) associated to the measurement choices \((u,v)\) as

\[
G(P, u, v) = \max_{\{\rho, M\} \rightarrow P} G(\rho, A_u, B_v)
\]

where the maximum is taken over all quantum realisations \(\{\rho, M\}\) compatible with \( P \), i.e., satisfying \( P(ab|uv) = \text{tr}[M_u \otimes M_v \rho] \).

**Example:** Let \( P \) be the distribution that arises from the measurements \( A_1, B_1 = \sigma_z \), \( A_2 = B_2 = \sigma_x \) on the state \( \psi = (|00\rangle + |11\rangle)/\sqrt{2} \). Then we have \( P(ab|1, 2) = 1/4 \) for all \( a, b = 0, 1 \), as in the two examples above. However \( G(P, 1, 2) = 0 \) since we cannot reproduce the entire distribution \( P \) by measuring the \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) state \( \rho_{AB} = \frac{1}{4} \sum_{z_0, z_1 = 0}^1 (|z_0 z_1\rangle \langle z_0 z_1|)_A \otimes (|z_0 z_1\rangle \langle z_0 z_1|)_B \) with the observables \( A_1 = B_1 = \sigma_z \otimes I, A_2 = B_2 = I \otimes \sigma_z \).

In the same way as above, we can also define the (realization-independent) guessing probability \( G(P, u) \) of the single outcome \( a \) associated to the measurement choice \( u \), which has corresponding min-entropy comprised between 0 and 1 bits. In the following, we will often write \( G_{uw} \) and \( G_u \) for \( G(P, u, v) \) and \( G(P, u) \) to shorten the notation.

**Randomness and non-locality.** A distribution \( P \) is said to be local deterministic if a measurement of \( u \) always return an outcome \( a = \alpha_u \) and a measurement of \( v \) always return an outcome \( b = \beta_v \), i.e., if \( P(ab|uv) = \delta(\alpha_u, \alpha_v)\delta(b_v, \beta_v) \). Clearly a local deterministic distribution admits a pure-state quantum realization with guessing probability 1 (take for instance \( |\psi\rangle = |\alpha_1, \alpha_2\rangle \otimes |\beta_1, \beta_2\rangle, A_u = \sum_{\alpha_u} \alpha_u |\alpha_u\rangle \langle \alpha_u| \) and similarly for \( B_v \). Since a distribution is local if and only if it can be written as a convex sum of local deterministic distributions [5], the violation of a Bell inequality is a necessary condition for the guessing probabilities \( G_{uw} \) and \( G_u \) to be different from 1. Furthermore, it is also a sufficient condition, since non-local correlations cannot be reproduced deterministically in quantum theory [2, 3, 4]. The guessing probabilities \( G_{uw} \) and \( G_u \) are thus different from 1 if and only if \( P \) violates a Bell inequality, that is in our scenario, if and only if it violates the CHSH inequality.

In general, for any given Bell inequality, one can derive bounds \( G_{uw} \leq f_{uw}(I) \) and \( G_u \leq g_u(I) \) on the guessing probabilities as a function of the amount of Bell violation \( I \) [7]. Here, we will characterize the amount of randomness associated to the Bell expressions

\[
I_{\alpha}^\beta = \beta \langle A_1 \rangle + \alpha \langle A_1 B_1 \rangle + \alpha \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle,
\]

which depend on two parameters \( \alpha \) and \( \beta \). Without loss of generality, we assume that \( \alpha \geq 1 \) and \( \beta \geq 0 \) (the expressions where either \( \alpha < 1 \) or \( \beta < 0 \) can be shown to be equivalent to the expressions with \( \alpha \geq 1 \) and \( \beta \geq 0 \) by relabelling the measurement settings and outcomes). To simplify the notation we denote by \( I_{\alpha} \) the Bell expression \( I_{\alpha} = I_{\alpha}^0 = \alpha \langle A_1 B_1 \rangle + \alpha \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \). When \( \alpha = 1 \), \( I_{\alpha} \) coincides with the CHSH expression. The local bound of \( I_{\alpha}^\beta \) is easily found to be \( \beta + 2\alpha \).

In the following, we will be interested in the maximal amount of randomness that can in principle be guaranteed by the Bell expressions \( I_{\alpha}^\beta \), that is, we will be interested in the guessing probabilities \( G_{uw} \) and \( G_u \) under the constraint that \( I_{\alpha}^\beta \) is maximally violated.

2 Technical preliminaries

We start by presenting three useful technical results.
Reduction to two dimensions. First, note that in our scenario (two observables with binary outcomes per system), it is sufficient to restrict the analysis to pure two-qubit states. More precisely, let $G(\Psi, A_u, B_v) \leq f_{uv}(I)$ and $G(\Psi, A_u) \leq g_u(I)$ be bounds on the guessing probabilities that are satisfied by any pure two-qubit state

$$|\Psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle$$

and non-degenerate Pauli observables

$$A_u = \vec{a}_u \cdot \vec{\sigma}, \quad B_v = \vec{b}_v \cdot \vec{\sigma}$$

yielding a Bell violation $I$. In the above expressions, $\theta$ is an angle satisfying $0 \leq \theta \leq \pi/4$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices, and $\vec{a}_u = (a^1_u, a^2_u, a^3_u)$ and $\vec{b}_v = (b^1_v, b^2_v, b^3_v)$ are unit vectors. Without loss of generality suppose that the functions $f_{uv}(I)$ and $g_u(I)$ are concave (if not take their concave hull). Then for any quantum distribution $P$ with Bell violation $I$, it holds that $G_{uv} \leq f_{uv}(I)$ and $G_u \leq g_u(I)$.

To show this, we recall the well-known fact that in our scenario any distribution $P$ arising by measuring a state $\rho \in B(H_A \otimes H_B)$, where the dimensions $\dim(H_A)$ and $\dim(H_B)$ are in principle arbitrary, can always be expressed as a convex combination $P = \sum_p p_c P_c$ of distributions $P_c$ arising from measurements on systems with $\dim(H_A) \leq 2$ and $\dim(H_B) \leq 2$ [9, 14]. Further, by convexity, it is sufficient to consider the case where each $P_c$ arise from measuring a pure state. Note that if either $\dim(H_A) = 1$, or $\dim(H_B) = 1$, or one of the operators $A_u$ or $B_v$ is degenerate (e.g. $A_u = \pm I$), then the corresponding point $P_c$ is necessary local; but any local distribution can be expressed as a convex combination of points obtained by measuring the state $|00\rangle$ with $\pm \sigma_z$ observables. It is therefore completely general to assume that each $P_c$ admits a realization in term of a pure two-qubit state, which can always be written as [8] in the Schmidt basis, and measurements corresponding to non-degenerate Pauli observables of the form [3].

Now let $\psi$ be an arbitrary (not necessarily two-qubit) pure state and $A_u, B_v$ observables yielding a violation $I$. We then have using the above observation that $G(\psi, A_u, B_v) = \max_{ab} P(ab|\psi, A_u, B_v) = \max_{ab} \sum_c p_c P_c(ab|uv) \leq \sum_c p_c \max_{ab} P_c(ab|uv) = \sum_c p_c G(\psi, A_{u,c}, B_{v,c}) \leq \sum_c p_c f(I_c) \leq f_w(\sum_c p_c I_c) \leq f_{uv}(I)$, where we have expressed the probabilities $P(ab|\psi, A_u, B_v)$ as a convex sum of probabilities $P_c(ab|uv)$ arising from pure two-qubit states $\psi_c$ and non-degenerate Pauli observables $A_{u,c}, B_{v,c}$ in the second equality, have used the bound on the guessing probability valid for pure two-qubit states in the second inequality, and the concavity of the function $f_{uv}$ in the third inequality. Since the bounds $G(\psi, A_u, B_v) \leq f_{uv}(I)$ hold for any pure state, it follows from the definitions [5] and [6] and again the concavity of $f_{uv}$ that $G_{uv} \leq f(I)$ hold for any distribution $P$. The same reasoning applies to $G_u$.

Bound on predictability. Second, note that by measuring the state $|\Psi\rangle$ with the observables [9], one necessarily has

$$-\cos 2\theta \leq \langle A_u \rangle \leq \cos 2\theta,$$

the extremal values being obtained when $A_u = \pm \sigma_z$. One finds similarly $-\cos 2\theta \leq \langle B_v \rangle \leq \cos 2\theta$.

Optimal violation of $I_\alpha$ for $2 \times 2$ systems. Finally, for any set of measurements [9] performed on the state $|\Psi\rangle$, the following inequality necessarily holds

$$I_\alpha \leq 2\sqrt{\alpha^2 + \sin^2 2\theta}.$$
Furthermore, if $\theta > 0$ there are only two probability distributions $P$ saturating this inequality defined by the expectation values

\begin{align}
\langle A_1 \rangle &= \pm \cos 2\theta, \\
\langle A_2 \rangle &= 0, \\
\langle B_1 \rangle &= \langle B_2 \rangle = \pm \cos \mu \cos 2\theta, \\
\langle A_1 B_1 \rangle &= \langle A_1 B_2 \rangle = \cos \mu, \\
\langle A_2 B_1 \rangle &= -\langle A_2 B_2 \rangle = \sin 2\theta \sin \mu,
\end{align}

where $\tan \mu = \sin 2\theta/\alpha$. These two points are obtained using the Pauli observables

\begin{align}
A_1 &= \pm \sigma_z, \\
A_2 &= \cos \varphi \sigma_x + \sin \varphi \sigma_y, \\
B_1 &= \pm \cos \mu \sigma_z + \sin \mu (\cos \varphi \sigma_x - \sin \varphi \sigma_y), \\
B_2 &= \pm \cos \mu \sigma_z - \sin \mu (\cos \varphi \sigma_x - \sin \varphi \sigma_y),
\end{align}

where $\varphi \in [0, 2\pi]$ is a free parameter.

To show this, it is convenient to rewrite the state (8) as $\rho = |\Psi\rangle \langle \Psi|$ with

\[ \rho = \frac{I}{4} + \cos 2\theta \frac{\sigma_z \otimes I}{4} + \cos 2\theta \frac{I \otimes \sigma_z}{4} + \sum_{ij} T_{ij} \sigma_i \otimes \sigma_j, \]

where the $3 \times 3$ real matrix $T$ has components

\[ T_{xx} = \sin 2\theta, \quad T_{yy} = -\sin 2\theta, \quad T_{zz} = 1, \quad T_{ij} = 0 \text{ for } i \neq j. \]

Following the method exposed in [15], we introduce two normalised mutually orthogonal vectors $\vec{c}_1$ and $\vec{c}_2$ by

\[ \vec{b}_1 + \vec{b}_2 = 2 \cos \mu \vec{c}_1; \quad \vec{b}_1 - \vec{b}_2 = 2 \sin \mu \vec{c}_2, \]

where $\mu \in [0, \frac{\pi}{2}]$. We can then write

\[ I_\alpha = 2\alpha \cos \mu \left( \vec{a}_1 \cdot T \vec{c}_1 \right) + 2 \sin \mu \left( \vec{a}_2 \cdot T \vec{c}_2 \right). \]

Let us now maximise $I_\alpha$ over all measurements $A_1, A_2, B_1, B_2$, while keeping the state (i.e. $T$) fixed. We find

\[ \max_{\vec{a}_1, \vec{a}_2, \vec{c}_1, \vec{c}_2, \mu} I_\alpha = \max_{\vec{c}_1, \vec{c}_2} 2\alpha \cos \mu |T \vec{c}_1| + 2 \sin \mu |T \vec{c}_2| \]

\[ = \max_{\vec{c}_1, \vec{c}_2} 2\sqrt{\alpha^2 |T \vec{c}_1|^2 + |T \vec{c}_2|^2}, \]

where the first equality obtains when $\vec{a}_u = T \vec{c}_u/|T \vec{c}_u|$ and the second equality when $\tan \mu = \sin 2\theta/\alpha$. Since $\vec{c}_1$ and $\vec{c}_2$ are orthogonal, and since $\alpha > 1$, the maximum of (16) is obtained when $\vec{c}_1 = \pm \vec{1}_z$ lies along the direction of the largest eigenvalue of $T$, and $\vec{c}_2$ lies in the $x, y$ plane along some arbitrary direction $\varphi$. We thus find that for the state (8), $I_\alpha \leq 2\sqrt{\alpha^2 + \sin^2 2\theta}$, where this inequality is saturated using measurements given in eq. [13]. Such measurements yield the expectation values (12) for any value of the free parameter $\varphi$. 

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3 Results

Arbitrarily high randomness from arbitrarily low non-locality. From \( \text{(11)} \), we deduce that the maximal quantum violation of the \( I_\alpha \) inequality is \( 2\sqrt{\alpha^2 + 1} \) and that it can be obtained by measuring a maximally entangled state, i.e. \( \theta = \pi/4 \) in \( \text{(8)} \). Further note from \( \text{(12)} \) that there exists a unique pure two-qubit quantum probability distribution achieving this maximum defined by

\[
\langle A_u \rangle = \langle B_v \rangle = 0 \\
\langle A_1 B_v \rangle = \frac{\alpha}{\sqrt{1 + \alpha^2}}, \quad \langle A_2 B_v \rangle = \frac{(-1)^v}{\sqrt{1 + \alpha^2}}
\]

(17)

By the convex reduction of general quantum distribution to two-qubit distributions, this probability distribution is actually the unique quantum distribution reaching the maximal quantum value \( 2\sqrt{\alpha^2 + 1} \). The guessing probabilities \( G_{uv} \) and \( G_u \) at the point of maximal violation thus simply correspond to the guessing probabilities of the distribution \( \text{(17)} \).

In the case \( \alpha = 1 \), we recover the well known properties of the CHSH expression. It is bounded by \( I_1 \leq 2\sqrt{2} \) (the Tsirelson bound) and at the maximum the measurements \( A_u \) and \( B_v \) are locally completely uncertain, i.e. \( G_u = G_v = 1/2 \). While Alice’s and Bob’s outcomes are locally completely random, they are not completely uncorrelated and one finds \( G_{uw} = 1/4 + \sqrt{2}/8 \approx 0.177 \), corresponding to \( -\log_2 G_{uw} \approx 2 \) bits of global randomness in the pair \((a,b)\).

Let us now consider the \( I_\alpha \) inequality with \( \alpha > 1 \). As in the CHSH case, we see from \( \text{(17)} \) that at the point of maximal violation, the outcomes of \( A_u \) and \( B_v \) are locally completely uncertain, i.e. \( G_u = G_v = 1/2 \). Note, however, that when the \( I_\alpha \) inequality is maximally violated, the CHSH inequality has the value \( I_1 = 2(\alpha + 1)/\sqrt{\alpha^2 + 1} \approx 2 + 2/\alpha \), i.e., for large \( \alpha \), the CHSH violation is arbitrarily small. Thus we see that perfect local randomness can be obtained with points that are arbitrarily close to the local region.

A stronger results holds if we consider the global randomness. From \( \text{(17)} \), we see that at the point of maximal violation of the \( I_\alpha \) inequality, the guessing probability \( G(P,2,v) = 1/4 \times (1 + 1/\sqrt{\alpha^2 + 1}) \) which is smaller (i.e. it corresponds to more randomness) than the CHSH guessing probability for any \( \alpha > 1 \). Furthermore, for large \( \alpha \), we have \( -\log_2 G(P,2,v) \approx 2 - \ln(2)/\alpha \), which is arbitrarily close to the optimal value of 2 bits of global randomness even though the quantum point becomes arbitrarily close to the local region.

Note that in the case of the local randomness, we can characterize the guessing probability \( G_u \) not only for the point of maximal violation but for any degree of violation. That is we obtain the complete curve \( G_u \leq g(I_\alpha) \), which takes the form

\[
G_u \leq \frac{1}{2} + \frac{1}{2} \sqrt{1 + \alpha^2 - \frac{I_\alpha^2}{4}}.
\]

(18)

This bound is tight in the case \( u = 1 \). It was derived previously in the CHSH case \( (\alpha = 1) \) for qubits (i.e. not in a device-independent way) in \([16]\), and in the device-independent scenario in \([14,17]\).

To show \( \text{(18)} \), remember that it is sufficient to establish this relation for the case of pure two-qubit states \( \text{(8)} \), as discussed previously. From Eq. \( \text{(11)} \), it follows that the only states compatible with a given value of \( I_\alpha \), are those satisfying \( \cos 2\theta \leq \sqrt{1 + \alpha^2 - I_\alpha^2/4} \). Using this inequality in \( \text{(10)} \), we obtain \( \text{(18)} \). Note that this bound is tight in the case \( u = 1 \), since the correlations \( \text{(12)} \) saturating \( \text{(11)} \) for fixed \( \theta \), also saturate \( \text{(10)} \).

Perfect local randomness from any partially entangled state. Let us now characterize the point of maximal violation of the inequality \( I_\beta \) with \( \beta > 0 \). As usual, it is
sufficient to consider pure-state two-qubit correlations. Combining Eq. \([10]\) and \([11]\), we find the inequality \(\langle A_1 \rangle \leq \sqrt{1 + \alpha^2 - I^2_\alpha}/4\). Inserting this bound for \(\langle A_1 \rangle\) in \([7]\), we obtain \(I^\beta_\alpha \leq I_\alpha + \beta \sqrt{1 + \alpha^2 - I^2_\alpha}/4\). This expression is easily seen to be maximized when \(I_\alpha = 2\sqrt{1 + \alpha^2}/\sqrt{1 + \beta^2}/4\), implying that the maximal quantum violation of the inequality \(I^\beta_\alpha \leq 2\sqrt{(1 + \alpha^2)/(1 + \beta^2)/4}\). Furthermore, provided that \(\beta \neq 2/\alpha\), the inequality \(\langle A_1 \rangle \leq \sqrt{1 + \alpha^2 - I^2_\alpha}/4\) that we used in the derivation of this bound is uniquely saturated by the quantum point \([12]\) (with the + sign) and thus we necessarily have at the point of maximal violation that \(\langle A_2 \rangle = 0\), i.e., \(G_2 = 1/2\) for any \(\beta \neq 2/\alpha\). Note further that the maximal violation is obtained by measuring the state \([8]\) with \(\theta\) such that \(I_\alpha = 2\sqrt{1 + \alpha^2}/\sqrt{1 + \beta^2}/4\) becomes uncorrelated random bits in the limit. The main difficulty is that since this Bell experiment involves four measurements per party, we cannot directly reduce the analysis to two-qubit states.

**Arbitrarily high global randomness from almost unentangled states.** We showed above that using the \(I_\alpha\) inequality in the limit \(\alpha \to 0\) one can certify that arbitrarily close to 2 bits of randomness are produced from a \(2 \times 2\) maximally entangled quantum system. We now show that one can also certify that arbitrarily close to 2 bits of randomness are produced in the limit where the \(2 \times 2\) system tends towards an unentangled state.

To this end, we consider a slightly more complex situation than the one analyzed so far, in which Alice and Bob each have four two-outcome measurements \(A_1, A_2, A_1', A_2'\) and \(B_1, B_2, B_1', B_2'\) (since the measurements have binary outcomes, the maximal randomness associated to a pair of joint measurements is still 2 bits). Let \(I^\beta_\alpha\) and \(I'^\beta_\alpha\) denote the Bell expression \([7]\) obtained using the unprimed measurements \(A_u, B_u\), or the primed measurements \(A_u', B_u'\) respectively, where for \(I'^\beta_\alpha\) the roles of Alice and Bob are reversed. Suppose that both \(I^\beta_\alpha\) and \(I'^\beta_\alpha\) are maximally violated, i.e., \(I^\beta_\alpha = I'^\beta_\alpha = 2\sqrt{(1 + \alpha^2)/(1 + \beta^2)/4}\). To determine the corresponding guessing probability \(G_{uw}\), it is clearly sufficient to characterize the maximal guessing probability \(G(\Psi, A_u, B_u)\) for all pure states \(\Psi\) and all observables for which both \(I^\beta_\alpha\) and \(I'^\beta_\alpha\) are maximally violated. From the previous results, we know (provided that \(\beta \neq 2/\alpha\)) that the outcomes of \(A_2\) and \(B_2\) are locally completely random, i.e., \(\langle A_2 \rangle = \langle B_2 \rangle = 0\). We now show that the results of \(A_2\) and \(B_2\) are almost not correlated for \(\beta \to 2/\alpha\), more precisely we show that \(\langle A_2 B_2 \rangle = \sqrt{(1 - \alpha^2)(1 + \beta^2)/4}\) for all \(\Phi\). If we take, e.g., \(\alpha = 1, \beta = 2 - \epsilon\), this implies that \(G(\Psi, A_2, B_2') = (1 + \langle A_2 \rangle \) + \langle B_2' \rangle \) + \langle A_2 B_2' \rangle \) \(\langle \Phi \rangle \approx 1/4 + 1/4 + \epsilon/2\) and thus \(G_{22} \approx 1/4 + 1/4 + \epsilon/2\). Moreover, a maximal violation of both \(I^\beta_\alpha\) and \(I'^\beta_\alpha\) can be obtained by measuring a state \([8]\) with \(\sin 2\theta \approx \sqrt{\epsilon/2}\). We thus see that arbitrarily close to 2 bits of global randomness can be certified using states that are almost unentangled.

The intuition behind this result is that in the limit \(\theta \to 0\) the state tends to \(\ket{\Psi} \to \ket{00}\) while \(A_2\) and \(B_2'\) are both measurements in the \(x, y\) plane (see eq. \([13]\)). Hence the the corresponding measurement outcomes become uncorrelated random bits in the limit. The main difficulty is that since this Bell experiment involves four measurements per party, we cannot directly reduce the analysis to two-qubit states.

However we can simultaneously block-diagonalise in blocks of size 2 the pairs of observables \(A_1, A_2\) and \(B_1, B_2\), as well as the pairs of observables \(A_1', A_2'\) and \(B_1', B_2'\) \([14, 9]\). Consider first the block-diagonalisation of the pairs of observables \(A_1, A_2\) and \(B_1, B_2\). Let \(\{0_1, 1_1\}\) \((i = 1, \ldots, N_A)\) denote a basis for the \(N_A\) blocks on Alice’s side and similarly let \(\{0_j, 1_j\}\) \((j = 1, \ldots, N_B)\) denote a basis for the \(N_B\) blocks on Bob’s side. In each \(2 \times 2\) block, the unique state maximally violating \(I^\beta_\alpha\) is (up to local unitaries) the state \(\ket{\psi_{ij}} = \cos \theta \ket{0_1, 0_j} + \sin \theta \ket{1_1, 1_j}\) where \(\sin 2\theta = \sqrt{(1 - \alpha^2)(1 + \beta^2)/4}\). Since we suppose that the global state \(\ket{\Psi}\) violates maximally this inequality, it necessarily has the form \(\ket{\Psi} = \sum_{ij} c_{ij} \ket{\psi_{ij}}\) where \(\sum_{ij} |c_{ij}|^2 = 1\) and the observables have the form \(A_1 = \sum_i \sigma_i^x\) and \(A_2 = \sum_i (\cos \varphi (\sigma_i^x + \sin \varphi \sigma_i^y) + B_{1,2} = \sum_i (\sin \varphi (\sigma_i^x - \cos \varphi \sigma_i^y) + \rho_{ij}\ket{\psi_{ij}}\).
\[ \sum_j \cos \mu \sigma^J_j \pm \sin \mu (\cos \varphi_j \sigma^I_j - \sin \varphi_j \sigma^I_y), \] i.e., they are sums of Pauli operators acting on each block. The angles \( \varphi_j \) and \( \varphi_j \) are not independent, since if \( c_{ij} \neq 0 \) is non zero they must be equal. Therefore, for all \( i, j \) for which \( c_{ij} \neq 0 \), one can bring \( \varphi_i = \varphi_j = 0 \) by simultaneous rotation of Alice’s basis and Bob’s basis around the \( z \) axis by opposite angles. From now on we assume this is the case.

The matrix \( c_{ij} \) has a singular value decomposition \( c_{ij} = \sum_k U_{ik} c_k V_{kj} \) where \( U_{ik} \) and \( V_{kj} \) are unitary matrices and the \( c_k \geq 0 \) are non-negative real numbers. We can therefore rewrite the state as

\[ |\Psi\rangle = \sum_k c_k (\cos \theta |0_k\rangle \langle 0_k| + \sin \theta |\bar{1}_k\rangle \langle \bar{1}_k|), \] (19)

where \( |0_k\rangle_A = \sum_i U_{ik} |0_i\rangle_A, |\bar{1}_k\rangle = \sum_i U_{ik} |1_i\rangle_A, \) and \( |\bar{0}_k\rangle_B = \sum_j V_{jk} |0_j\rangle_B, |\bar{1}_k\rangle_B = \sum_j V_{jk} |1_j\rangle_B \) (here and below we add the index \( A \) or \( B \) whenever distinguishing between Alice and Bob’s states is not implicit from the notation). Importantly, it is easily checked that the operators \( A_1, A_2, B_1, B_2 \) have the same form expressed in the new basis as in the old basis. In particular \( A_1 = \sum_k \bar{\sigma}^I_k \) and \( A_2 = \sum_k \bar{\sigma}^I_{\bar{k}} \).

Let us now apply the same operations to the block diagonalisation of the pairs of observables \( A'_1, A'_2, B'_1, B'_2 \). We can bring the state to the form

\[ |\Psi\rangle = \sum_k c_k (\cos \theta |\bar{0}'_k\rangle \langle \bar{0}'_k| + \sin \theta |\bar{1}'_k\rangle \langle \bar{1}'_k|) \] (20)

with the operators \( B'_1 = \sum_k \bar{\sigma}^I_k \) and \( B'_2 = \sum_k \bar{\sigma}^I_{\bar{k}} \). Note that the coefficients \( c_k \) are the same and have the same degeneracy in eqs. (19) and (20) since the state is written in the Schmidt basis. However the basis states \( |0_k\rangle_{A,B}, |\bar{1}_k\rangle_{A,B} \) and \( |\bar{0}'_k\rangle_{A,B}, |\bar{1}'_k\rangle_{A,B} \) may differ. If the singular value \( c_k \) is non degenerate they may differ by a phase, whereas if the singular value \( c_k \) is degenerate they may differ by unitary transformations.

Let us assume that the singular value \( c_k \) has degeneracy \( d \). From now on we work within this \( 2d \times 2d \) block, and drop the index \( k \) (indeed all the operators commute with the projections onto these \( 2d \times 2d \) blocks). We thus consider the normalised state

\[ |\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{l=1}^d (\cos \theta |0_l\rangle \langle 0_l| + \sin \theta |1_l\rangle \langle 1_l|) \] (21)

\[ = \frac{1}{\sqrt{d}} \sum_{l=1}^d (\cos \theta |0'_l\rangle \langle 0'_l| + \sin \theta |1'_l\rangle \langle 1'_l|) \] (22)

and measurements \( A_2 = \sum_l \sigma^I_l \) and \( B'_2 = \sum_l \sigma^I_{\bar{l}} \) (where for simplicity we have omitted the bar “ - ” over the states and operators). We can rewrite \( |0'_l\rangle_B = \sum_m W_{lm} |0_m\rangle_B \) with \( W \) the unitary matrix that transform from the \( |0_l\rangle \) to the \( |0'_l\rangle \) basis. We also write \( |1'_l\rangle_B = \sum_m W_{lm} |1_m\rangle_B \), where the states \( |1'_m\rangle \) must be orthogonal to the \( |0_l\rangle \) states, hence can be written as \( |1'_m\rangle_B = \sum R_{lm} |1_m\rangle_B \) for some unitary matrix \( R \). It is easily verified using these relations that the operator \( B'_2 = \sum_l |0'_l\rangle \langle 1'_l| + |1'_l\rangle \langle 0'_l| = \sum_l |0_l\rangle \langle 1'_l| + |1_l\rangle \langle 0'_l| \). By computing explicitly the expectations value \( \langle A_2 B'_2 \rangle_{\Psi} \) using this last expression for \( B'_2 \) on the state given by Eq. (21), one finds \( \langle A_2 B'_2 \rangle_{\Psi} = \frac{1}{d} \cos \theta \sin \theta \sum (R_{ll} + R_{\bar{l}\bar{l}}) = \sin(2\theta) \), which is the desired result.

4 Discussion

In this work we have considered the relation between entanglement, non locality, and the amount of randomness that can be certified in a Bell-type experiment. These quantities are closely
A Bell expression defines a hyperplane in the space of correlations. The CHSH hyperplane \( CHSH = 2 \) separates the local region from the non-local quantum region. When the Bell inequalities \( I_\beta ^\alpha \) are maximally violated, the corresponding hyperplanes become tangent to the quantum boundary and identify one of the extremal points of \( Q \). The distribution \( P \) associated to such extremal points may be close to uniformly random for certain values of \( u, v \), but need not be highly non-local as measured by the CHSH violation (as represented on the figure) or need not originate from maximally entangled states.

related: entanglement is necessary for non locality, and non locality is necessary for certifying randomness. The quantitative relations between these concepts, however, are more subtle than expected. It was already known that entanglement and non locality are inequivalent resources \([18, 19, 20]\). Here we have shown that the amount of randomness that can be certified by a Bell-type experiment is inequivalent to either of these two resources.

Some understanding of why non-locality is inequivalent to certified randomness can be obtained by going back to the geometric picture. In our work, we have characterized part of the boundary of quantum correlations and have shown that there exists extremal quantum distributions \( P(ab|uv) \) which are arbitrarily close to the set of local correlations or which arise from partially entangled states, yet which are close to uniformly random for specific choices of \( u \) and \( v \), see Figure 1. This suggests that while non-locality is necessary to certify the presence of randomness, its quantitative aspects are related to the extremality of non-local correlations. In this sense, our work goes in the same direction as \([21]\), where extremality was identified as a key property for the security of DIQKD.

From a practical point of view our results have direct applications for DIRNG and DIQKD. The guessing probabilities \( G_{uv} \) and \( G_u \) play a central role in the recent security proofs for, respectively, DIRNG \([7]\) and DIQKD \([17, 22]\). Upper-bounds on these quantities, as a function of the amount of violation of a Bell inequality, directly translate into bounds on the amount of randomness generated in DIRNG protocols and on the key rate of DIQKD protocols. It follows in particular from our results that the CHSH inequality is not optimal for DIRNG but that higher generation rates, up to the optimal value of 2 bits per use of the system, can be obtained using other inequalities, and that randomness generation rates superior to 1 bit per use of the system are possible from any partially entangled states. In the context of DIQKD, the fact that 1 bit of local randomness can be extracted from maximally entangled states irrespectively of the amount of violation of the CHSH inequality implies that DIQKD with an optimal asymptotic rate of 1 bit of secret key per use of the system is possible using correlations that are almost local.

From a fundamental point of view, it is interesting to compare our results to those that can be established for post-quantum theories limited only by the no signalling principle. In this case,
the geometry of the space of non-local correlations corresponding to experiments involving two possible binary measurements on each subsystem is very simple since the unique extremal points are the local deterministic correlations and the Popescu-Rohlich boxes [11]. This implies that the amount of certifiable randomness is proportional to the violation of the CHSH inequality, and reaches at most 1 bit when the CHSH violation is equal to 4. On the other hand, in the quantum case the amount of certifiable randomness can be arbitrarily close to the maximal possible value of 2 bits, i.e., more randomness can be extracted from the non-local correlations of quantum theory than it would be possible in the most non-local theory compatible with no-signalling. It would be interesting to investigate if there are other no-signalling theories allowing for maximal certifiable randomness.

Finally, we have shown that arbitrarily close to 2 random bits can be certified by maximally entangled states, as well as by states with arbitrarily little entanglement. We conjecture that this value can be reached for any value of the entanglement (the parameter $\theta$ in eq. (8)). It would also be interesting to understand whether measurements beyond the projective case provide any advantage or whether two bits is the maximum amount of randomness that can be certified by $2 \times 2$ states.

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