ON AN INEQUALITY RELATED TO THE RADIAL GROWTH OF QUASINEARLY SUBHARMONIC FUNCTIONS IN LOCALLY UNIFORMLY HOMOGENEOUS SPACES

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Abstract

We begin by recalling the definition of nonnegative quasinearly subharmonic functions on locally uniformly homogeneous spaces. Recall that these spaces and this function class are rather general: Among others subharmonic, quasisubharmonic, and nearly subharmonic functions on domains of Euclidean spaces $\mathbb{R}^n$, $n \geq 2$, are included. The following result of Gehring and Hallenbeck is classical: Every subharmonic function, defined and $L^p$-integrable for some...
$p, 0 < p < +\infty$, on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ is for almost all $\theta$ of the form $o((1 - |\theta|^{-1/p})$, uniformly as $z \to e^{i\theta}$ in any Stolz domain. Recently, both Pavlović and Riihentaus have given related and partly more general results on domains of $\mathbb{R}^n, n \geq 2$. Now, we extend one of these results to quasinearly subharmonic functions on locally uniformly homogeneous spaces.

1. Introduction

1.1. Locally uniformly homogeneous spaces. The definition of locally uniformly homogeneous spaces was given in [22]. However, for the convenience of the reader, we recall it here, too. A set $X$ is a locally uniformly homogeneous space, if the following conditions are satisfied:

(i) $X$ is a topological space.

(ii) There is a Borel measure $\mu$ defined on $X$.

(iii) There is a quasimetric (quasidistance) on $X$, that is, there is a constant $K \geq 1$ and a mapping $d_K : X \times X \to [0, +\infty)$ such that

1° $d_K(x, y) = d_K(y, x)$ for all $x, y \in X$,

2° $d_K(x, y) = 0$, if and only if $x = y$,

3° $d_K(x, y) \leq K[d_K(x, z) + d_K(z, y)]$ for all $x, y, z \in X$,

4° the ((K-)quasi) balls $B_K(x, r)$,

$$B_K(x, r) := \{y \in X : d_K(x, y) < r\},$$

centered at $x$ and of radii $r > 0$, form a basis of open neighborhoods at the point $x \in X$,

5° $0 < \mu(B_K(x, r)) < +\infty$, for all $x \in X$ and $r > 0$,

6° there exist absolute constants $A = A(K) \geq 1$ and $\rho_0 = \rho_0(K) > 0$ such that

$$\mu(B_K(x, r)) \leq A\mu(B_K(x, \frac{r}{2})), $$

for all $x \in X$ and all $r, 0 < r \leq \rho_0$. 
Remark 1.1.1. Locally uniformly homogeneous spaces are slightly more general than spaces of homogeneous type, defined and considered by Coifman and Weiss [1], pp. 66-68, and [2], pp. 587-590. As a matter of fact, the only difference with their definition is that, instead of the above condition $6^o$, Coifman and Weiss use the stronger condition:

$6^o$ There exists an absolute constant $A = A(K) \geq 1$ such that

$$\mu(B_K(x, r)) \leq A\mu(B_K(x, \frac{r}{2})),$$

for all $x \in X$ and all $r > 0$.

For a list of examples of spaces of homogeneous type, see [2], pp. 588-590.

Remark 1.1.2. In order to be able to consider Hausdorff measures on locally uniformly homogeneous spaces, we make the following additional assumption (cf. [15], p. 54): Let $X$ be a locally uniformly homogeneous space. Suppose that $X$ satisfies the following additional condition:

For every $\delta > 0$, there are $E_j \in \mathcal{F}$, $j = 1, 2, \ldots$, such that

$$X = \bigcup_{j=1}^{+\infty} E_j \quad \text{and} \quad d_K(E_j) \leq \delta,$$

(1)

where $\mathcal{F} = \{B_K(x, r) : x \in X, r > 0\}$. Then for each $d > 0$, one can define in $X$ a $d$-dimensional Hausdorff (outer) measure $\mathcal{H}_K^d$, which is a $(K$-quasi)$\text{metric(outer) measure in the following sense: If } A, B \subset X \text{ such that } d_K(A, B) > 0, \text{ then } \mathcal{H}_K^d(A \cup B) = \mathcal{H}_K^d(A) + \mathcal{H}_K^d(B). \text{ As a matter of fact, in the standard definition (see, e.g., [14], pp. 125-126), just work with the quasimetric } d_K \text{, instead of the metric } d \text{ (or } \rho \text{). One sees also, that all Borel sets of } X \text{ are } \mathcal{H}_K^d \text{-measurable. Above, we have used the following notation: If } A, B \subset X, \text{ then}$
\[ d_K(A) := \sup \{ d_K(x, y) : x, y \in A \}, \]

and

\[ d_K(A, B) := \inf \{ d_K(x, y) : x \in A, y \in B \}. \]

1.2. **Quasinearly subharmonic functions.** Though the definition of quasinearly subharmonic functions in locally uniformly homogeneous spaces was given in [22], we recall it also here for the convenience of the reader. Let \( X \) be a locally uniformly homogeneous space. Let \( u : X \to [0, +\infty) \) be Borel measurable. Let \( C \geq 1 \). Then \( u \) is \( C \)-**quasinearly subharmonic in** \( X \), if there is a constant \( \varepsilon_0 = \varepsilon_0(u) \) (depending on the considered function \( u \)), \( 0 < \varepsilon_0 < 1 \), such that for each open set \( \Omega \subset X, \Omega \neq X \), for each \( x \in \Omega \) and each \( r, 0 < r \leq \min \{ \rho_0, \varepsilon_0 \delta_K^\Omega(x) \} \), one has \( u \in L^1(B_K(x, r)) \) and

\[
    u(x) \leq \frac{C}{\mu(B_K(x, r))} \int_{B_K(x, r)} u(y) d\mu(y).
\]

The function \( u \) is **quasinearly subharmonic in** \( X \), if \( u \) is \( C \)-quasinearly subharmonic for some \( C \geq 1 \).

Above (and below), we have used the following notation: \( \delta_K^\Omega(x) \), or shortly \( \delta_K(x) \), is the \( (K) \)-quasidistance from \( x \in \Omega \) to \( \partial \Omega \), and thus defined by

\[
    \delta_K(x) := \delta_K^\Omega(x) := \inf \{ d_K(x, y) : y \in \Omega^c \},
\]

where \( \Omega^c \) is the complement of \( \Omega \), taken in \( X \).

**Examples 1.2.1.** Quasinearly subharmonic functions, especially nearly subharmonic, quasisubharmonic, and subharmonic functions in an open subset \( D \) of an Euclidean space \( \mathbb{R}^n, n \geq 2 \), give examples of quasinearly subharmonic functions in a locally uniformly homogeneous space. As an additional example, we recall that \( B^{2n} \), the unit ball of
\( C^n, n \geq 1 \), is locally uniformly homogeneous, and nonnegative \( C \)-subharmonic functions on \( B^{2n} \) (see, e.g., [36], p. 31, and [37], p. 3774) are \( l \)-quasinearly subharmonic. For further examples, see [22].

For the definition, examples and properties of quasinearly subharmonic functions (sometimes, however, perhaps with a different terminology) in domains of an Euclidean space \( \mathbb{R}^n, n \geq 2 \), see, e.g., [4], pp. 2-6, [5], [12], pp. 243-244, [18], pp. 18-19, [19], pp. 15-16, [20], pp. 90-91, [24], p. 233, [26], p. 171, [27], pp. 196-197, [28], p. 28, [29], p. 158, [30], p. 52, [31], pp. 2-3, [32], pp. 129-130, [33], p. 2614, and the references therein. In this connection, see also [40], pp. 259, 263.

1.3. Weighted boundary behavior. The following theorem is a special case of the original result of Gehring [7], Theorem 1, p. 77, and of Hallenbeck [9], Theorems 1 and 2, pp. 117-118, and of the later and more general results of Stoll [38], Theorems 1 and 2, pp. 301-302, 307:

**Theorem 1.4.** If \( u \) is a function harmonic in the unit disk \( \mathbb{D} \) of the complex plane \( \mathbb{C} \) such that

\[
I(u) := \int_{\mathbb{D}} |u(z)|^p (1 - |z|)^{\beta} dm_2(z) < +\infty,
\]

where \( p > 0, \beta > -1 \), then

\[
\lim_{r \to 1^-} |u(re^{i\theta})|^p (1 - r)^{\beta+1} = 0,
\]

for almost all \( \theta \in [0, 2\pi) \). Above \( m_2 \) is the Lebesgue measure in \( \mathbb{R}^2 \).

Observe that Gehring, Hallenbeck, and Stoll considered in fact subharmonic functions and that the limit in (3) was uniform in Stolz approach regions, in Stoll’s result in even more general regions. For more general results, see [24], Theorem, p. 233, [16], Theorem 2, p. 73, [26], Theorem 2, pp. 175-176, [27], Theorem 3.4.1, pp. 198-199, [28], Theorem, p. 31, and [20], Theorem 4, p. 102.
Gehring’s proof was based on Hardy-Littlewood inequality, whereas the other authors based their proofs, more or less, on certain generalized mean value inequalities for subharmonic functions. For such inequalities and related properties, see [6], Lemma 2, p. 172, [13], Theorem 1, p. 529, [39], pp. 188-190, [17], pp. 53, 64-65, [23], Lemma, p. 69, [9], Lemma 1, p. 113, [16], p. 68, [25], Theorem, p. 188, and the references therein.

With the aid of the following Theorem 1.5, see [21], Theorem 1, pp. 433-434, Pavlović showed that the convergence in (3) is dominated. At the same time, he pointed out that whole Theorem 1.4 follows from his result:

**Theorem 1.5.** If $u$ is a function harmonic in $D$ satisfying (2), where $p > 0$, $\beta > -1$, then

$$J(u) := \sup_{0<r<1} \int_{0}^{2\pi} |u(re^{i\theta})|^p (1-r)^{\beta+1} d\theta < +\infty.$$  

Moreover, there is a constant $C = C_{p,\beta}$ such that $J(u) \leq CI(u)$.

In [32], Theorems 1 and 2, pp. 131-132, we extended Theorem 1.5 to the case, where, instead of absolute values of harmonic functions in the unit disk $D$ of the complex plane $C$, one considers more generally nonnegative quasinearly subharmonic functions in rather general domains of $\mathbb{R}^n$, $n \geq 2$. Now, our aim is to extend this cited Theorem 1 even further: We will give a related result for quasinearly subharmonic functions in locally uniformly homogeneous spaces, satisfying the above additional assumption (1), see Theorem 2.5 below. As an application, we get in Corollary 2.6 below a weighted boundary behavior result in our rather general setup of locally uniformly homogeneous spaces.

**1.6. Notation.** Our notation is rather standard, see, e.g., [10], [20], and [22]. However, for the convenience of the reader, we recall the following. The common convention $0 \cdot \infty = 0$ is used. Below, $X$ is always a locally uniformly homogeneous space, and $\Omega$ always a domain in $X$, $\Omega \neq X$, whose boundary $\partial \Omega$ is Ahlfors-regular from above, with dimension $d > 0$
and with constant \( C_4 > 0 \) (for the definition of this, see 1.9 below). For \( \rho > 0 \), write \( \Omega_\rho = \{ x \in \Omega : \delta_K(x) < \rho \} \). \( B_K(x, r) \) is the \((K\text{-}quasi)\) ball in \( X \), with center \( x \) and radius \( r \), and \( B_K(x) = B_K(x, \frac{1}{3K}\delta_K(x)) \). The \( d \)-dimensional Hausdorff (outer) measure in \( X, d > 0 \), constructed with the aid of the \( K\text{-}quasimetric} \( d_K \), is denoted by \( \mathcal{H}_d^{K} \), see Remark 1.1.2 above. \( C_0 \) and \( r_0 \) are fixed constants, which are involved with the used (and thus fixed) admissible function \( \phi \) (see (4) below in 1.7). Similarly, if \( \alpha > 0 \) is given, \( C'_1 = C'_1(\alpha, C_0, K), C'_2 = C'_2(\alpha, C_0, K), \) and \( C'_3 = C'_3(\alpha, C_0) \) are fixed constants, coming directly from Lemmas 2.1, 2.3, and 2.4 below. (Compare these with the related constants \( C_1, C_2, \) and \( C_3 \) in [24], p. 234, [28], pp. 32-33, and [32], p. 129.)

1.7. Admissible functions. A function \( \varphi : [0, +\infty) \rightarrow [0, +\infty) \) is admissible, if it is strictly increasing, surjective, and there are constants \( C_0 = C_0(\varphi) \geq 1 \) and \( r_0 = r_0(\varphi) > 0 \) such that

\[
\varphi(2t) \leq C_0 \varphi(t) \text{ and } \varphi^{-1}(2s) \leq C_0^{-1}(s),
\]

for all \( s, t, 0 \leq s, t \leq r_0 \).

Examples of admissible functions are: Functions \( \varphi_1(t) = t^p, p > 0 \), nonnegative, increasing surjective functions \( \varphi_2(t) \) satisfying the \( \Delta_2 \)-condition and for which the functions \( t \mapsto \frac{\varphi_2(t)}{t} \) are increasing, and functions \( \varphi_3(t) = c t^\alpha \left[ \log(\delta + t^\gamma) \right]^\beta \), where \( c > 0, \alpha > 0, \delta \geq 1, \) and \( \beta, \gamma \in \mathbb{R} \) are such that \( \alpha + \beta \gamma > 0 \).

1.8. Approach sets. Let \( \varphi : [0, +\infty) \rightarrow [0, +\infty) \) be an admissible function and let \( \alpha > 0 \). Let \( X \) be a locally uniformly homogeneous space. Let \( \Omega \) be a domain in a component \( X_1 \) of \( X, \Omega \neq X_1 \). For \( \zeta \in \partial \Omega \), write

\[
\Gamma_\varphi(\zeta, \alpha) := \{ x \in \Omega : \varphi(d_K(x, \zeta)) < \alpha \delta_K(x) \},
\]
and call it a \((\varphi, \alpha)\)-\textit{approach set (region)}, shortly an \textit{approach set (region)}, in \(\Omega\) at \(\zeta\). Observe that though \(\partial\Omega\) is surely nonempty, the approach set \(\Gamma_\varphi(\zeta, \alpha)\) may, in certain cases, be empty. Anyway in the case of the unit disk \(\mathbb{D}\) of the complex plane \(\mathbb{C}\), the choice \(\varphi(t) = t\) gives the familiar Stolz approach regions. Choosing \(\varphi(t) = t^\tau, \tau \geq 1\), say, one gets more general approach regions, see [38], p. 301.

For \(x \in \Omega\) and \(\alpha > 0\), we also write
\[
\tilde{\Gamma}_\varphi(x, \alpha) := \{\xi \in \partial\Omega : x \in \Gamma_\varphi(\xi, \alpha)\}.
\]
Moreover, for \(\rho > 0\), we write
\[
\Gamma_{\varphi, \rho}(\zeta, \alpha) := \{x \in \Gamma_\varphi(\zeta, \alpha) : \delta_K(x) < \rho\}.
\]
One says that \(\zeta \in \partial\Omega\) is \((\varphi, \alpha)\)-\textit{accessible}, shortly \textit{accessible}, if \(\Gamma_\varphi(\zeta, \alpha) \cap B_K(\zeta, \rho) \neq \emptyset\) for all \(\rho > 0\).

1.9. \textbf{Ahlfors-regular sets.} Let \(d > 0\). A set \(E \subset X\) is \textit{Ahlfors-regular from above, with dimension} \(d\) \textit{and with constant} \(C_4 > 0\), shortly \textit{Ahlfors-regular from above}, if it is closed and
\[
\mathcal{H}^d_K(E \cap B_K(x, r)) \leq C_4 r^d,
\]
for all \(x \in E\) and \(r > 0\). The smallest constant \(C_4\) is called the \textit{regularity constant} for \(E\). A set \(E \subset X\) is \textit{Ahlfors-regular, with dimension} \(d\) \textit{and with constant} \(C_4 > 0\), shortly \textit{Ahlfors-regular}, if it is closed and
\[
C_4^{-1} r^d \leq \mathcal{H}^d_K(E \cap B_K(x, r)) \leq C_4 r^d,
\]
for all \(x \in E\) and \(r > 0\).

Simple examples of Ahlfors-regular sets in \(\mathbb{R}^n\), \(n \geq 2\), are \(d\)-planes and \(d\)-dimensional Lipschitz graphs. Also certain Cantor sets and self-similar sets are Ahlfors-regular. For more details, see [3], pp. 9-10.
2. Boundary Integral Inequalities

We begin with four lemmas. Recall that \( X \) is always a locally uniformly homogeneous space. \( X_1 \) will be an arbitrary component of \( X \), and \( \Omega \) a domain in \( X_1 \), \( \Omega \neq X_1 \). Moreover, \( \varphi : [0, + \infty) \to [0, + \infty) \) is an admissible function, with constants \( r_0 \) and \( C_0 \).

Let \( x \in \Omega \). It is easy to see that

\[
\frac{2}{3K} \delta_K(x) \leq \delta_K(y) \leq (K + \frac{1}{3})\delta_K(x),
\]

for all \( y \in B_K(x) \). Let \( \alpha > 0 \), write

\[
\hat{\rho}_0 := \min \{ \frac{r_0}{2K^{\frac{1}{3}}}, \frac{r_0}{2^{\alpha+1}}, \frac{r_0}{2^{3\alpha}C_0^{K+\frac{1}{3}}}, \frac{1}{2^{\alpha+1}} \varphi \left( \frac{r_0}{2(K + \frac{1}{3})} \right), \rho_0 \}.
\]

**Lemma 2.1.** Let \( \zeta \in \partial \Omega \) and \( x_0 \in \Gamma_{\varphi, \rho}(\zeta, \alpha) \). Let \( C_1 \geq 1 \) be an arbitrary. Then for \( C_2 = \frac{C_0^\alpha}{3} + KC_0^{\frac{1}{3}} \) and for all \( x \in B_K(x_0) \),

\[
B_K(x, C_1 \varphi^{-1}(\delta_K(x))) \subset B_K(x_0, C_1C_2 \varphi^{-1}(\delta_K(x_0))),
\]

provided \( 0 < \rho \leq \hat{\rho}_0 \).

**Proof.** Take \( z \in B_K(x, C_1 \varphi^{-1}(\delta_K(x))) \) arbitrarily. Then

\[
d_K(x_0, z) \leq K [d_K(x_0, x) + d_K(x, z)]
\]

\[
< K \left[ \frac{\delta_K(x_0)}{3K} + C_1 \varphi^{-1}(\delta_K(x)) \right]
\]

\[
< K \left[ \frac{\delta_K(x_0)}{3K} + C_1 \varphi^{-1}((K + \frac{1}{3})\delta_K(x_0)) \right]
\]

\[
< K \left[ \frac{\delta_K(x_0)}{3K} + C_1 \varphi^{-1}(2^{K+\frac{1}{3}}\delta_K(x_0)) \right]
\]

\[
< K \left[ \frac{\delta_K(x_0)}{3K} + C_1C_0^{K+\frac{1}{3}} \varphi^{-1}(\delta_K(x_0)) \right]
\]
where \( C_2 = \frac{C_0^a}{3} \). Hence, \( x \in B_K(x_0, C_1 C_2^{-1}(\delta_K(x_0))) \). Above, we have used the facts that \( 2^{K+\frac{1}{3}} \rho_0 \leq r_0 \) and \( 2^{a+1} \rho_0 \leq r_0 \), which follow from the definition of \( \hat{\rho}_0 \).

\[ \text{Lemma 2.2.} \ Let \ \zeta \in \partial \Omega \ and \ x_0 \in \Gamma_{\phi, \rho}(\zeta, \alpha). \ Then \ B_K(x_0) \subset \Gamma_{\phi, \rho'}(\zeta, \alpha'), \ where \ \rho' = (K + \frac{1}{3}) \rho \ and \ \alpha' = \frac{3aK + 1}{2} C_0, \ provided \ 0 < \rho \leq \hat{\rho}_0. \]

\[ \text{Proof.} \ \text{Take} \ x \in B_K(x_0) \ \text{arbitrarily. Then} \ d_K(x_0, x) < \frac{\delta_K(x_0)}{3K} \]

Since \( \phi(d_K(x_0, \zeta)) < a\delta_K(x_0) \), we have

\[ \phi(d_K(x, \zeta)) < \phi(\phi(d_K(x_0, \zeta) + d_K(x_0, \zeta))) \leq \phi(\phi(\phi(\phi(\phi^{-1}(d_K(x_0, \zeta))) + d_K(x_0, \zeta))) + d_K(x_0, \zeta))) < \phi(\phi(\phi(\phi(\phi^{-1}(d_K(x_0, \zeta))) + d_K(x_0, \zeta))) + d_K(x_0, \zeta))) \]

\[ \leq \phi(2^{K+\frac{1}{3}} d_K(x_0, \zeta)) < C_0^{K+\frac{1}{3}} \phi(d_K(x_0, \zeta)) \]

\[ < C_0^{K+\frac{1}{3}} a\delta_K(x_0) \leq \frac{C_0^{K+\frac{1}{3}} 3aK}{2} \delta_K(x), \]

provided that \( 2^{K+\frac{1}{3}} d_K(x_0, \zeta) \leq r_0 \). But, this surely holds, since \( x_0 \in \Gamma_{\phi, \rho}(\zeta, \alpha) \) and \( \rho \leq \hat{\rho}_0 \leq \frac{1}{2^{a+1}} \phi\left(\frac{r_0}{2^{K+\frac{1}{3}}}\right) \). Hence, \( x \in \Gamma_{\phi, \rho'}(\zeta, \alpha'). \)

\[ \square \]
Lemma 2.3. Let $C'_1 = \frac{3aK}{2}C_0^{K + \frac{1}{3} + 1}$ and $a' = \frac{3aK}{2}C_0^{K + \frac{1}{3}}$. Then for all $x \in \Omega_{\rho'}$, where $\rho' = (K + \frac{1}{3})\rho$, one has $\Gamma_\varphi(x, \alpha') \subset B_K(x, C'_1 \varphi^{-1}(\delta_K(x)))$, provided $0 < \rho \leq \hat{\rho}_0$.

Proof. Suppose that $\Gamma_\varphi(x, \alpha') \not= \emptyset$ and take $\xi \in \Gamma_\varphi(x, \alpha')$ arbitrarily. But, then $x \in \Gamma_\varphi(\xi, \alpha')$, that is, $\varphi(d_K(x, \xi)) < \alpha'\delta_K(x)$. Hence,

$$d_K(x, \xi) \varphi^{-1}(\alpha'\delta_K(x)) < \varphi^{-1}(2^{\alpha'+1}\delta_K(x)) \leq C_0^{a'+1}\varphi^{-1}(\delta_K(x)),$$

provided that $2^{\alpha'+1}(K + \frac{1}{3})\hat{\rho}_0 \leq \eta_0$. But this holds, since, by assumption,

$$\hat{\rho}_0 \leq \frac{\eta_0}{2^{\alpha'+1}(K + \frac{1}{3})}.$$ 

Thus $\xi \in B_K(x, C'_1 \varphi^{-1}(\delta_K(x)))$. $\square$

Lemma 2.4. Let $\zeta \in \partial\Omega$ and $x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)$. Let $C'_1$ and $C'_2$ be as above. Then for $C'_3 = K\left(1 + \frac{C_0^{a'+1}}{C'_1C'_2}\right)$, one has

$$B_K(x, C'_1C'_2\varphi^{-1}(\delta_K(x))) \subset B_K(\zeta, C'_1C'_2C'_3\varphi^{-1}(\delta_K(x))),$$

provided $0 < \rho \leq \hat{\rho}_0$.

Proof. Take $z \in B_K(x, C'_1C'_2\varphi^{-1}(\delta_K(x)))$ arbitrarily. Then clearly,

$$d_K(z, \zeta) \leq K [d_K(x, z) + d_K(x, \zeta)] < K [C'_1C'_2\varphi^{-1}(\delta_K(x)) + d_K(x, \zeta)].$$

Now, $x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)$, thus $\varphi(d_K(x, \zeta)) < a\delta_K(x)$, and also

$$d_K(x, \zeta) \varphi^{-1}(a\delta_K(x)) \leq \varphi^{-1}(2^{a'+1}\delta_K(x)) \leq C_0^{a'+1}\varphi^{-1}(\delta_K(x)),$$
provided that \( 2^{a+1} \hat{\rho}_0 \leq r_0 \), which again holds, since, by assumption,
\[ \hat{\rho}_0 \leq \frac{r_0}{2^{a+1}}. \]
Hence,
\[
d_K(z, \zeta) < K[C_1 C_2 \varphi^{-1}(\delta_K(x)) + C_0^{a+1} \varphi^{-1}(\delta_K(x))]
\]
\[
< K(C_1 C_2 + C_0^{a+1}) \varphi^{-1}(\delta_K(x))
\]
\[
= C_1 C_2 K \left( 1 + \frac{C_0^{a+1}}{C_1 C_2} \right) \varphi^{-1}(\delta_K(x)),
\]
and so \( z \in B_K(\zeta, C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))) \). \( \square \)

Then our result, an extension to Theorem 1, pp. 131-132, of [32]:

**Theorem 2.5.** Let \( X \) be a locally uniformly homogeneous space satisfying the condition (1). Suppose that \( d_K : X \times X \to [0, +\infty) \) is separately continuous and a Borel function. Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be an admissible function, with constants \( r_0 \) and \( C_0 \). Let \( a > 0, \gamma \in \mathbb{R}, d > 0, \) and \( C_4 > 0 \) be arbitrary. Let \( u : X \to [0, +\infty) \) be a \( K_1 \)-quasinearly subharmonic function. Then, there is a constant \( C = C(a, \gamma, \varepsilon_0, d, A, C_0, C_4, K, K_1) \) such that for each component \( X_1 \) of \( X \) and for each domain \( \Omega \subset X_1, \Omega \neq X_1 \), whose boundary \( \partial \Omega \) is Ahlfors-regular from above, with dimension \( d \) and with constant \( C_4 \),

\[
\int_{\partial \Omega} \sup_{x \in \Gamma_{\varepsilon_0}(\zeta, \alpha)} \{ \delta_K(x)^\gamma \mu(B_K(x))[\varphi^{-1}(\delta_K(x))]^{-d} u(x) \} dH^d_K(\zeta)
\]
\[
\leq C \int_{\Omega_{\rho'}} \delta_K(x)^\gamma u(x) d\mu(x), \tag{5}
\]

for all \( \rho, 0 < \rho \leq \hat{\rho}_0 \). Here \( \rho' = (K + \frac{1}{3})\rho \) and \( \hat{\rho}_0 \) is as above.
Remark 2.5.1. Above and below, we use the following (maybe unstandard, but in the considered situation of nonnegative functions nevertheless natural) convention: Let \( A \subset X, B \subset A, \) and \( g : A \to [0, +\infty]. \) If \( B = \emptyset, \) then we define \( \sup_{x \in B} \{ g(x) \} = 0. \)

Remark 2.5.2. Though the constant \( C \) above in (5) does depend on \( K_1 \) and on \( \varepsilon_0, \) it is, nevertheless, otherwise independent of the \( K_1 \)-quasinearly subharmonic function \( u. \)

Proof. Suppose \( 0 < \rho \leq \hat{\rho}_0. \) Write
\[
E := \{ \zeta \in \partial \Omega : \Gamma_{\phi, \rho}(\zeta, \alpha) \neq \emptyset \}.
\]
Using the fact that \( d_K(\cdot, \cdot) \) is separately upper semicontinuous, one sees easily that \( E \) is open in \( \partial \Omega. \)

Take \( \zeta \in E \) and \( x_0 \in \Gamma_{\phi, \rho}(\zeta, \alpha) \) arbitrarily. Since \( u \) is quasinearly subharmonic and
\[
\frac{\varepsilon_0 \delta_K(x_0)}{3K} < \varepsilon_0 \delta_K(x_0) < \delta_K(x_0) \leq \rho \leq \hat{\rho}_0 \leq \rho_0,
\]
one obtains
\[
u(x_0) \leq \frac{K_1}{\mu(B_K(x_0, \varepsilon_0 \delta_K(x_0) / 3K))} \int_{B_K(x_0, \varepsilon_0 \delta_K(x_0) / 3K)} u(x) d\mu(x)
\]
\[
\leq \frac{K_1}{\mu(B_K(x_0, \varepsilon_0 \delta_K(x_0) / 3K))} \int_{B_K(x_0, \delta_K(x_0) / 3K)} u(x) d\mu(x).
\]
Choose \( n_0 \in \mathbb{N} \) such that
\[
2^{n_0 - 1} < \frac{1}{\varepsilon_0} \leq 2^{n_0}.
\]
Then
\[
\mu(B_K(x_0, \frac{\delta_K(x_0)}{3K})) = \mu(B_K(x_0, \frac{1}{\varepsilon_0} \cdot \varepsilon_0 \delta_K(x_0) \frac{3K}{3K}))) \\
\leq \mu(B_K(x_0, 2^n_0 \cdot \varepsilon_0 \delta_K(x_0))) \\
\leq A^{n_0} \mu(B_K(x_0, \frac{\varepsilon_0 \delta_K(x_0)}{3K})) \\
\leq A^{1-\log_2 \varepsilon_0} \mu(B_K(x_0, \frac{\varepsilon_0 \delta_K(x_0)}{3K})).
\]

Hence,
\[
u(x_0) \leq \frac{K_1 A^{1-\log_2 \varepsilon_0}}{\mu(B_K(x_0, \frac{\delta_K(x_0)}{3K}))} \int_{B_K(x_0, \frac{\delta_K(x_0)}{3K})} u(x) d\mu(x)
\]
\[
= \frac{K_1 A^{1-\log_2 \varepsilon_0}}{\mu(B_K(x_0))} \int_{B_K(x_0)} u(x) d\mu(x).
\]

With the aid of the fact that \(\delta_K(x_0)^\gamma \leq (\frac{3K}{2})^d \delta_K(x)^\gamma\) for all \(x \in B_K(x_0)\), and that \([\varphi^{-1}(\delta_K(x_0))]^d \geq \frac{1}{C_0} \int_{B_K(x_0, C_1C_2^{-1}(\delta_K(x_0)))} d\mu(x)\) for all \(x \in B_K(x_0)\), we get, with the aid of Lemma 2.1 above, from (6) above, for all \(C_1 \geq 1\),
\[
\frac{\delta_K(x_0)^\gamma \mu(B_K(x_0))u(x_0)}{[\varphi^{-1}(\delta_K(x_0))]^d + \mathcal{H}_K^d(B_K(x_0, C_1C_2^{-1}(\delta_K(x_0))))} \leq K_1A^{1-\log_2 \varepsilon_0}
\]
\[
\times \int_{B_K(x_0)} \frac{\delta_K(x_0)^\gamma u(x)}{[\varphi^{-1}(\delta_K(x_0))]^d + \mathcal{H}_K^d(B_K(x_0, C_1C_2^{-1}(\delta_K(x_0))))} d\mu(x)
\]
\[
\leq K_1A^{1-\log_2 \varepsilon_0}
\]
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\[ \times \int_{B_K(x_0)} \frac{(3K/2)b|\delta_K(x)\cdot u(x)}{d(\frac{K+1}{3})} d\mu(x) \]

\[ \leq \left( \frac{3K}{2} \right)^{|b|} K_1 A^{1-\log_2 \epsilon_0} C_0^d(K+\frac{1}{3}) \]

By Lemma 2.2, \( B_K(x_0) \subset \Gamma_{\varphi, \rho}(\zeta, \alpha) \), where \( \rho' = (K + \frac{1}{3}) \rho \) and \( \alpha' = \frac{3\alpha K}{2} C_0^{-\frac{1}{3}} \). Thus

\[ \frac{\delta_K(x_0) \cdot u(B_K(x_0)) u(x_0)}{d(\frac{K+1}{3})} d\mu(x) \]

\[ \leq \left( \frac{3K}{2} \right)^{|b|} K_1 A^{1-\log_2 \epsilon_0} C_0^d(K+\frac{1}{3}) \]

\[ \times \int_{\Gamma_{\varphi, \rho}(\zeta, \alpha')} \frac{\delta_K(x) \cdot u(x)}{d(\frac{K+1}{3})} d\mu(x). \]

Taking then the supremum on the left hand side over \( x_0 \in \Gamma_{\varphi, \rho}(\zeta, \alpha) \), we get

\[ \sup_{x_0 \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} \frac{\delta_K(x_0) \cdot u(B_K(x_0)) u(x_0)}{d(\frac{K+1}{3})} d\mu(x) \]

\[ \leq \left( \frac{3K}{2} \right)^{|b|} K_1 A^{1-\log_2 \epsilon_0} C_0^d(K+\frac{1}{3}) \]

\[ \times \int_{\Gamma_{\varphi, \rho}(\zeta, \alpha')} \frac{\delta_K(x) \cdot u(x)}{d(\frac{K+1}{3})} d\mu(x). \]
Next, integrate on both sides with respect to $\zeta$ over $E$ and use Fubini’s theorem:

$$\int \sup_{x_0 \in \Gamma_{\phi', \rho}(\zeta, a')} \delta_{K}(x_0)^{\frac{\nu}{2}} \mu(B_{K}(x_0))u(x_0) \frac{d\mathcal{H}^d_{E}(\zeta)}{[\phi^{-1}(\delta_{K}(x_0))]^d + \mathcal{H}^d_{K}(B_{K}(x_0, C_1 C_2 \phi^{-1}(\delta_{K}(x_0))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$

$$\leq \left( \frac{3K}{2} \right)^{\frac{3}{2}} K_1 A^{1 - \log_2 \varepsilon_0} C_0^{d(K + \frac{1}{3})} \int \int \left\{ \int \int_{\Omega_{\phi'}} \chi_{\Gamma_{\phi}(\zeta, \alpha')(x)}(x) \right\} d\mathcal{H}^d_{K}(\zeta) \, d\mu(x)$$

$$\times \frac{\delta_{K}(x)^{\gamma} u(x)}{[\phi^{-1}(\delta_{K}(x))]^d + \mathcal{H}^d_{K}(B_{K}(x, C_1 \phi^{-1}(\delta_{K}(x))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$

$$\leq \left( \frac{3K}{2} \right)^{\frac{3}{2}} K_1 A^{1 - \log_2 \varepsilon_0} C_0^{d(K + \frac{1}{3})} \int \int \left\{ \int \int_{\Gamma_{\phi}(\zeta, \alpha')(\zeta)} \right\} d\mathcal{H}^d_{K}(\zeta) \, d\mu(x)$$

$$\times \frac{\delta_{K}(x)^{\gamma} u(x)}{[\phi^{-1}(\delta_{K}(x))]^d + \mathcal{H}^d_{K}(B_{K}(x, C_1 \phi^{-1}(\delta_{K}(x))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$

$$\leq \left( \frac{3K}{2} \right)^{\frac{3}{2}} K_1 A^{1 - \log_2 \varepsilon_0} C_0^{d(K + \frac{1}{3})} \int \int \left\{ \int \int_{\Omega_{\phi'}} \chi_{\Gamma_{\phi}(\zeta, \alpha')(\zeta)}(\zeta) \right\} d\mathcal{H}^d_{K}(\zeta) \, d\mu(x)$$

$$\times \frac{\delta_{K}(x)^{\gamma} u(x)}{[\phi^{-1}(\delta_{K}(x))]^d + \mathcal{H}^d_{K}(B_{K}(x, C_1 \phi^{-1}(\delta_{K}(x))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$

$$\leq \left( \frac{3K}{2} \right)^{\frac{3}{2}} K_1 A^{1 - \log_2 \varepsilon_0} C_0^{d(K + \frac{1}{3})} \int \int \left\{ \int \int_{\Omega_{\phi'}} \chi_{\Gamma_{\phi}(\zeta, \alpha')(\zeta)}(\zeta) \right\} d\mathcal{H}^d_{K}(\zeta) \, d\mu(x)$$

$$\times \frac{\delta_{K}(x)^{\gamma} u(x)}{[\phi^{-1}(\delta_{K}(x))]^d + \mathcal{H}^d_{K}(B_{K}(x, C_1 \phi^{-1}(\delta_{K}(x))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$

$$\leq \left( \frac{3K}{2} \right)^{\frac{3}{2}} K_1 A^{1 - \log_2 \varepsilon_0} C_0^{d(K + \frac{1}{3})} \int \int \left\{ \int \int_{\Omega_{\phi'}} \chi_{\Gamma_{\phi}(\zeta, \alpha')(\zeta)}(\zeta) \right\} d\mathcal{H}^d_{K}(\zeta) \, d\mu(x)$$

$$\times \frac{\delta_{K}(x)^{\gamma} u(x)}{[\phi^{-1}(\delta_{K}(x))]^d + \mathcal{H}^d_{K}(B_{K}(x, C_1 \phi^{-1}(\delta_{K}(x))) \cap \partial \Omega)} \, d\mu(x) \, d\mathcal{H}^d_{K}(\zeta)$$
Choosing $C_1 = C_1'$ and using then Lemma 2.3, we get

$$\int \sup_{x_0 \in \Gamma_{p, \rho}(\zeta, \alpha)} \frac{\delta_K(x_0) \mu(B_K(x_0)) u(x_0)}{[\varphi^{-1}(\delta_K(x_0))]^d + \mathcal{H}_K^d(B_K(x_0, C_1 C_2 \varphi^{-1}(\delta_K(x_0))) \cap \partial \Omega)} \, d\mathcal{H}_K^d(\zeta)$$

\[ \leq \left( \frac{3K}{2} \right)^{\|h\|} K_1 A \{ 1 - \log_2 \varepsilon_0 \} C_0^{d(K + \frac{1}{3})} \]

\[ \times \int_{\Omega_p} \frac{\mathcal{H}_K^d(B_K(x, C_1 C_2 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega) \delta_K(x)^\gamma u(x)}{\mathcal{H}_K^d(B_K(x, C_1 C_2 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega)} \, d\mu(x) \]

\[ \leq \left( \frac{3K}{2} \right)^{\|h\|} K_1 A \{ 1 - \log_2 \varepsilon_0 \} C_0^{d(K + \frac{1}{3})} \int_{\Omega_p} \delta_K(x)^\gamma u(x) \, d\mu(x). \]

On the other hand, by Lemma 2.4, we get

$$\sup_{x \in \Gamma_{p, \rho}(\zeta, \alpha)} \frac{\delta_K(x)^\gamma \mu(B_K(x)) u(x)}{[\varphi^{-1}(\delta_K(x))]^d + \mathcal{H}_K^d(B_K(x, C_1 C_2 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega)}$$

\[ \geq \sup_{x \in \Gamma_{p, \rho}(\zeta, \alpha)} \frac{\delta_K(x)^\gamma \mu(B_K(x)) u(x)}{[\varphi^{-1}(\delta_K(x))]^d + \mathcal{H}_K^d(B_K(x, C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega)} \cdot \]

Since $\partial \Omega$ is Ahlfors-regular from above, one has

$$\mathcal{H}_K^d(B_K(\zeta, C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega) \leq C_4 [C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))]^d < +\infty.$$ 

Therefore,

$$\sup_{x \in \Gamma_{p, \rho}(\zeta, \alpha)} \frac{\delta_K(x)^\gamma \mu(B_K(x)) u(x)}{[\varphi^{-1}(\delta_K(x))]^d + \mathcal{H}_K^d(B_K(x, C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega)}$$

\[ \geq \sup_{x \in \Gamma_{p, \rho}(\zeta, \alpha)} \frac{\delta_K(x)^\gamma \mu(B_K(x)) u(x)}{[\varphi^{-1}(\delta_K(x))]^d + C_4 [C_1 C_2 C_3 \varphi^{-1}(\delta_K(x))]^d} \]

\[ = \frac{1}{1 + C_4 (C_1 C_2 C_3)^d} \sup_{x \in \Gamma_{p, \rho}(\zeta, \alpha)} \{ \delta_K(x)^\gamma \mu(B_K(x))}[\varphi^{-1}(\delta_K(x))]^{-d} u(x) \}.$
Thus, we have:

\[
\int \sup_{x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} |\delta_K(x)^\gamma \mu(B_K(x))| [\varphi^{-1}(\delta_K(x))]^{-d} u(x) d\mathcal{H}^d_K(\zeta) \leq C \int_{\Omega_{\nu'}} \delta_K(x)^\gamma u(x) d\mu(x),
\]

where

\[
C = \left( \frac{3K}{2} \right) K_1 A^{1-\log_2 \varepsilon_0} C_0 d(K + \frac{1}{3}) [1 + C_4 (C_1 C_2 C_3)^d],
\]

and

\[
C_1' = C_0^{-\frac{2}{3}} C_0^{K + \frac{1}{3} + 1}, \quad C_2' = C_0^{-\frac{2}{3}} + K C_0^{K + \frac{1}{3}}, \quad C_3' = K \left( 1 + \frac{C_0^{\alpha+1}}{C_1 C_2} \right).
\]

To conclude the proof, observe the following. First, since \( \Gamma_{\varphi, \rho}(\zeta, \alpha) = \emptyset \) for all \( \zeta \in \tilde{\Omega} \setminus E \), we can, just using our convention in Remark 2.5.1, replace (7) by the desired inequality:

\[
\int \sup_{x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} |\delta_K(x)^\gamma \mu(B_K(x))| [\varphi^{-1}(\delta_K(x))]^{-d} u(x) d\mathcal{H}^d_K(\zeta) \leq C \int_{\Omega_{\nu'}} \delta_K(x)^\gamma u(x) d\mu(x).
\]

Second, the functions

\[
\tilde{\partial} \Omega \ni \zeta \mapsto \sup_{x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} \frac{\delta_K(x)^\gamma \mu(B_K(x)) u(x)}{[\varphi^{-1}(\delta_K(x))]^d + \mathcal{H}^d_K(B_K(x, C_1 C_2 \varphi^{-1}(\delta_K(x))) \cap \tilde{\partial} \Omega) \in [0, + \infty],
\]

and
\[ \partial \Omega \ni \zeta \mapsto \sup_{x \in \Gamma_{\varphi, \rho}(\zeta, \alpha)} \{ \delta_K(x)^{\gamma} \mu(B_K(x)) [\varphi^{-1}(\delta_K(x))]^{-d} u(x) \} \in [0, + \infty], \]

are lower semicontinuous. Thus, the above integrations on “the left hand sides” are justified.

Third, the functions
\[ \Omega_{\rho'} \ni x \mapsto \frac{\delta_K(x)^{\gamma} u(x)}{[\varphi^{-1}(\delta_K(x))]^{d} + \mathcal{H}_K^d(B_K(x, C_1 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega) \in [0, + \infty), \]

and
\[ \Omega_{\rho'} \times \partial \Omega \ni (x, \zeta) \mapsto \chi_{\Gamma_{\varphi}(\zeta, \alpha')}(x) \]

\[ \times \frac{\delta_K(x)^{\gamma} u(x)}{[\varphi^{-1}(\delta_K(x))]^{d} + \mathcal{H}_K^d(B_K(x, C_1 \varphi^{-1}(\delta_K(x))) \cap \partial \Omega) \in [0, + \infty), \]

are Borel measurable. Hence, the integrations and the use of Fubini’s theorem on “the right hand sides” are justified, too. Observe that here, we use our additional assumption that the \( K \)-quasimetric \( d_K \) is Borel measurable.

\[ \square \]

**Remark 2.5.3.** At present, we do not know whether our assumption that the \( K \)-quasimetric \( d_K : X \times X \to [0, + \infty) \) is separately continuous and Borel measurable, is really necessary or not. Observe anyway that a quasimetric \( d_K \) is separately upper semicontinuous, see (iii) 4° above. But a separately upper semicontinuous function need not, however, be measurable, see [35] and, e.g., [8], Example 1, p. 11. On the other hand, if \( K = 1 \), that is, if \( X \) is a metric space, and the function \( d_1 \) is separately continuous, then \( d_1 \) is Borel measurable by a result of Kuratowski, see [34], p. 742, and the references therein, say. Observe also that, if the considered locally uniformly homogeneous space \( X \) is moreover locally compact, then by a result of Johnson, see [11], Theorem 2.2, p. 422, and again [34], p. 742, \( d_K \) is indeed Borel measurable.
Corollary 2.6. Let \( X \) be a locally uniformly homogeneous space satisfying the condition (1). Suppose that \( d_K : X \times X \to [0, +\infty) \) is separately continuous and a Borel function. Let \( \varphi : [0, +\infty) \to [0, +\infty) \) be an admissible function, with constants \( r_0 \) and \( C_0 \). Let \( \alpha > 0, \gamma \in \mathbb{R}, d > 0, \) and \( C_4 > 0 \) be arbitrary. Let \( u : X \to [0, +\infty) \) be a \( K_1 \)-quasinearly subharmonic function. Suppose that \( X_1 \) is an arbitrary component of \( X \) and that \( \Omega \subset X_1, \Omega \neq X_1, \) is a domain, whose boundary \( \partial \Omega \) is Ahlfors-regular from above, with dimension \( d \) and with constant \( C_4 \), and that
\[
\int_{\Omega} \delta_K(x)^\gamma u(x) \, d\mu(x) < +\infty. \tag{8}
\]

Then for \( \mathcal{H}^d_K \)-almost every \((\varphi, \alpha)\)-accessible point \( \zeta \in \partial \Omega \),
\[
\lim_{\delta \to 0} \left( \sup_{x \in \Gamma_{\varphi, \alpha}(\zeta)} \{ \delta_K(x)^\gamma \mu(B_K(x))[\varphi^{-1}(\delta_K(x))]^{-d} u(x) \} \right) = 0.
\]

Remark 2.6.1. If instead of a locally uniformly homogeneous space \( X \), one works in an Euclidean space \( \mathbb{R}^n, n \geq 2, \) then slightly better results hold: Namely, one can omit the assumed Ahlfors-regularity condition of \( \partial \Omega \), see the already cited results [24], Theorem, p. 233, [26], Theorem 2, pp. 175-176, [27], Theorem 3.4.1, pp. 198-199, [28], Theorem, p. 31, and [20], Theorem 4, p. 102. Observe, however, that the possibility for this omission in the Euclidean setup is based essentially on a well-known density estimate result for Hausdorff measures (which in turn is based, among others, on Vitali’s covering theorem and thus is of “very Euclidean space-type”), see, e.g., [15], Theorem 6.2, p. 89.

Proof. By Theorem 2.5,
\[
\int_{\partial \Omega} \sup_{x \in \Gamma_{\varphi, \alpha}(\zeta)} \{ \delta_K(x)^\gamma \mu(B_K(x))[\varphi^{-1}(\delta_K(x))]^{-d} u(x) \} d\mathcal{H}^d_K(\zeta)
\]
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\[ \leq C \int_{\Omega_0'} \delta_K(x)^\ell u(x) d\mu(x), \]

where \( C = C(\alpha, \gamma, \varepsilon_0, d, A, C_0, C_4, K, K_1) \). Using then just Fatou's lemma and (8), one sees that

\[
\int_{\partial \Omega} \liminf_{\rho \to 0} \left( \sup_{x \in \Gamma_{\rho, \rho/2}(\zeta_0, a)} \{ \delta_K(x)^\ell \mu(B_K(x)) \left[ \varphi^{-1}(\delta_K(x)) \right]^{-d} u(x) \} \right) d\mathcal{H}^d_K(\zeta) \\
\leq C \liminf_{\rho \to 0} \int_{\Omega_0'} \delta_K(x)^\ell u(x) d\mu(x) = 0.
\]

Thus, the claim follows. \( \square \)

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