THE MODELING ERROR OF WELL TREATMENT FOR UNSTEADY FLOW IN POROUS MEDIA

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Abstract. In petroleum engineering, the well is usually treated as a point or line source, since its radius is much smaller than the scale of the whole reservoir. In this paper, we consider the modeling error of this treatment for unsteady flow in porous media.

1. Introduction. In many practical applications, especially in resistivity well-logging in petroleum exploitation, a kind of boundary value problem with equivalued surface is formulated. It is a kind of nonlocal boundary value condition, which can be also used to give mathematical descriptions for other problems in physics and mechanics (cf. [11, 10, 5, 6]). From the physical perspective, the equivalued surface boundary value condition corresponds to a source.

In resistivity well-logging, we are most concerned about two quantities: the bottom-hole pressure (BHP) and the rate of production (injection). The BHP is often difficult to calculate since the well radius is much smaller than the scale of the whole reservoir. In practical calculation, the variation of solutions near the well is quite large, and in finite element procedure, it is necessary to have a refined partition of elements near the well. This causes a complexity in computation [12]. To get rid of this difficulty, the well can be approximately regarded as a point and corresponding boundary value problems with equivalued boundary surface on the well boundary can be approximately replaced by the boundary value problems with Dirac function. Our focus is to estimate the modeling error from the above approximation for unsteady flow. Of course, numerical well model for unsteady flow will be studied in the subsequent paper.

There have been some previous works on the steady flow in porous media. In [5, 6], the authors considered the limit behavior of solutions for elliptic problems with equivalued surface boundary value conditions as the radius of well tends to zero. In [13], the authors presented the modeling error analysis for steady flow.

In this paper, we consider unsteady flow in porous media with a well. Let Ω be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \Gamma \). Denote by \( B(x_0, \delta) \) the disk...
centered at $x_0$ with radius $\delta > 0$ occupied by a well and by $\Omega_\delta = \Omega \setminus (B(x_0, \delta))$. We consider the following governing equation for unsteady flow in porous media
\[
\frac{\partial u_\delta}{\partial t} - \text{div}(K(x)\nabla u_\delta) = 0 \quad \text{in} \quad \Omega_\delta \times (0, T),
\]
where $u_\delta$ is the pressure, $K$ the permeability.

On the well boundary $\gamma = \partial B(x_0, \delta)$, two quantities are of particular importance in practical applications as we have already emphasized as before: the well bore pressure $u_\delta|_\gamma$ and the well flow rate $\int_\gamma K(x) \frac{\partial u_\delta}{\partial \nu} \, ds$, where $\nu$ is the unit outer normal to $\partial \Omega_\delta$. The boundary condition to be imposed on $\gamma$ is either the following mixed boundary condition which fixes the well flow rate $q_0(t)$
\[
u_\delta = c(t)(\text{unknown}) \quad \text{on} \quad \gamma \times (0, T), \quad \int_\gamma K(x) \frac{\partial u_\delta}{\partial \nu} \, ds = q_0(t) \quad \text{a.e.} \quad t \in (0, T),
\]
or the Dirichlet boundary condition which fixes the well bore pressure $\alpha(t)$ (a function with respect only to $t$)
\[
u_\delta = \alpha(t) \quad \text{on} \quad \gamma \times (0, T).
\]

Let $Q_\delta = \Omega_\delta \times (0, T), Q = \Omega \times (0, T)$, we consider the following two equivalued surface boundary value problems:
\[
\begin{cases}
\frac{\partial u_\delta}{\partial t} - \text{div}(K(x)\nabla u_\delta) = 0 & \text{in} \quad \Omega_\delta \times (0, T), \\
u_\delta(x, 0) = 0 & \text{in} \quad \Omega_\delta, \\
u_\delta|_{\Gamma} = 0, \\
u_\delta|_\gamma = c(t)(\text{unknown}), \int_\gamma K(x) \frac{\partial u_\delta}{\partial \nu} \, ds = q_0(t) & \text{a.e.} \quad t \in (0, T),
\end{cases}
\tag{1.1}
\]
which is a fixed rate well; and
\[
\begin{cases}
\frac{\partial u_\delta}{\partial t} - \text{div}(K(x)\nabla u_\delta) = 0 & \text{in} \quad \Omega_\delta \times (0, T), \\
u_\delta(x, 0) = 0 & \text{in} \quad \Omega_\delta, \\
u_\delta|_{\Gamma} = 0, \\
u_\delta|_\gamma = \alpha(t),
\end{cases}
\tag{1.2}
\]
which is a fixed pressure well.

In these problems, there exist two separated scales $O(1)$ and $\delta$, where $O(1)$ represents the typical length of the reservoir, $\delta$ is the radius of the well, and $O(1) \gg \delta$. The radius $\delta$ is so small that the well is usually treated as an ‘point’ source. Then the approximation of problem (1.1) and (1.2) are as follows
\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div}(K(x)\nabla u) = q_0(t)\delta_{x_0} & \text{in} \quad \Omega \times (0, T), \\
u(x, 0) = 0 & \text{in} \quad \Omega, \\
u|_{\Gamma} = 0,
\end{cases}
\tag{1.3}
\]
where $q_0(t)$ is the well flow rate mentioned in (1.1); and
\[
\begin{cases}
\frac{\partial u}{\partial t} - \text{div}(K(x)\nabla u) = \tilde{q}_0(t)\delta_{x_0} & \text{in} \quad \Omega \times (0, T), \\
u(x, 0) = 0 & \text{in} \quad \Omega, \\
u|_{\Gamma} = 0, \\
(u - \tilde{\phi})(x_0, t) + \tilde{\phi}|_\gamma = \alpha(t),
\end{cases}
\tag{1.4}
\]
where the well flow rate $\tilde{q}_0(t)$ is unknown and it is implicitly determined through the condition $(u - \tilde{\phi})(x_0, t) + \tilde{\phi}|_\gamma = \alpha(t)$ which is actually a first-kind Volterra
integral equation with respect to \( \tilde{q}_0(t) \) (will be addressed later in Section 4) and is an approximation of the boundary condition \( u_\delta|_\gamma = \alpha(t) \). Here

\[
\tilde{\phi}(x,t) = \int_0^t \tilde{q}_0(s) e^{-\frac{|x-x_0|^2}{4\pi K_0(t-s)}} ds
\]

is the solution of problem

\[
\begin{cases}
\frac{\partial \tilde{\phi}}{\partial t} - \text{div}(K_0 \nabla \tilde{\phi}) = \tilde{q}_0(t) \delta_{x_0} & \text{in } \mathbb{R}^2 \times (0, T), \\
\tilde{\phi}(x, 0) = 0 & \text{in } \mathbb{R}^2,
\end{cases}
\]

with \( K_0 = K(x_0) \).

Note that \( \tilde{q}_0(t) \) depends on \( \delta \) which is determined by the condition \( (u - \tilde{\phi})(x_0, t) + \tilde{\phi}|_\gamma = \alpha(t) \). Problem (1.4) is not dependent on \( \delta \) once \( \tilde{q}_0(t) \) is solved by appropriate method.

Problems (1.1) and (1.2) are the parabolic boundary value problems with non-local equivalent surface. They are different from the usual parabolic equations with the typical boundary conditions. The existence and uniqueness of weak solutions to problem (1.1) can be found in [11].

For fixed rate well, the most significant work is that A.Damlamian and Ta-Tsien Li discussed the relationship of solutions between problem (1.1) and problem (1.3) in 1982. Their main result is the following theorem only for the fixed rate well, which can be found in [2, 3].

We first make the following hypothesis:

(H1) There exists a positive constant \( \lambda \) such that \( \lambda \leq K(x) \leq \lambda^{-1} \).

**Theorem 1.1** ([2, 3]). Under the hypothesis of (H1) and \( q_0 \in L^2(0, T) \), for the solution \( u_\delta \) of problem (1.1), if we set

\[
\tilde{u}_\delta = \begin{cases} 
  u_\delta, & \text{in } Q_\delta, \\
  0 \text{ or } c(t), & \text{in } Q \setminus Q_\delta,
\end{cases}
\]

then

\( \tilde{u}_\delta \to u \) strongly in \( L^2(Q) \), as \( \delta \to 0 \),

where \( u \) is the solution of problem (1.3).

To the best of our knowledge, there are few literature discussing the fixed pressure well, even if the convergence of the solutions \( u_\delta \) to \( u \) in a certain sense as \( \delta \to 0 \).

In this paper, we consider the error estimate for this modeling treatment. Before stating our main results, we need the following hypotheses

(H2) \( |q_0(t)| + |\tilde{q}_0(t)| \leq C, \forall t \in [0, T], \) with \( C > 0 \) independent of \( \delta \),

(H3) (Compatibility) \( \alpha(0) = 0, \ q_0(0) = 0. \)

The main results are as follows.

For fixed rate well and homogeneous media, we get

**Theorem 1.2.** Assume that \( K(x) \equiv K_0 > 0 \), (H2) and (H3) are satisfied. Let \( u_\delta \) and \( u \) be the solutions of (1.1) and (1.3) respectively. Then there exists a positive constant \( C \) independent of \( \delta \) such that

\[
\max_{(x,t) \in Q \times [0,T]} |u - u_\delta| \leq C \delta (1 + |\ln \delta|).
\]

For fixed pressure well and homogeneous media, we obtain
Theorem 1.3. Assume that $K(x) \equiv K_0 > 0$, $\alpha \in C^2([0, T])$ and (H3) is satisfied. Let $u_\delta$ be the solution of (1.2) and $q_\delta^0(t) = -\int_\gamma K_0 \frac{\partial u_\delta}{\partial \nu} ds$. Let $(u, \tilde{q}_0(\cdot))$ be the solution of (1.4) and $q_0^0(\cdot)$ be bounded uniformly with respect to $\delta$. Then for sufficiently small $\delta > 0$, there exists a positive constant $C$ independent of $\delta$ such that

$$\max_{(x,t) \in \Omega \times [0,T]} |u - u_\delta| \leq C\delta,$$

and

$$\max_{t \in [0,T]} |\tilde{q}_0(t) - q_0^0(t)| \leq C\delta.$$

For fixed rate well and heterogeneous media, we have

Theorem 1.4. Let the assumptions (H1), (H2) and (H3) be satisfied, and $K \in C^{0,1}(\tilde{\Omega})$. Let $u_\delta$ and $u$ be the solutions of (1.1) and (1.3), respectively. Then there exists a positive constant $C$ independent of $\delta$ such that

$$\max_{(x,t) \in \Omega \times [0,T]} |u - u_\delta| \leq C\delta^{1-\frac{\alpha}{p}}(1 + |\ln \delta|),$$

with $p > 3$ and $C = C(p)$.

For fixed pressure well and heterogeneous media, we get

Theorem 1.5. Let the assumptions (H1) and (H3) be satisfied, and $K \in C^{0,1}(\tilde{\Omega})$, $\alpha \in C^2([0, T])$. Let $u_\delta$ be the solution of (1.2) and $q_\delta^0(t) = -\int_\gamma K_0 \frac{\partial u_\delta}{\partial \nu} ds$. Let $(u, \tilde{q}_0(\cdot))$ be the solution of (1.4) and $q_0^0(\cdot)$ be bounded uniformly with respect to $\delta$. Then for sufficiently small $\delta > 0$, there exists a positive constant $C$ independent of $\delta$ such that

$$\max_{(x,t) \in \Omega \times [0,T]} |u - u_\delta| \leq C\delta^{1-\frac{1}{p}},$$

$$\max_{t \in [0,T]} |\tilde{q}_0(t) - q_0^0(t)| \leq C\delta^{1-\frac{1}{p}}$$

where $p > 3$, $C = C(p)$.

Remark 1. We guess that the constant $C = C(p)$ in Theorems 1.4 and 1.5 has the form as $C(p) = c_1 p^\beta$ for some $\beta > 0$ and $c_1$ is independent of $p$ and $\delta$. If so, we may take $p = |\ln \delta|$, then $\delta^{-\beta/p} = e^3$ for $\delta < 1$. Hence, the result in Theorem 1.4 becomes that for sufficiently small $\delta > 0$,

$$\max_{(x,t) \in \Omega \times [0,T]} |u - u_\delta| \leq C\delta \ln |\ln \delta|^{\beta+1},$$

and the results in Theorem 1.5 turn into that for sufficiently small $\delta > 0$,

$$\max_{(x,t) \in \Omega \times [0,T]} |u - u_\delta| \leq C\delta |\ln \delta|^{\beta},$$

$$\max_{t \in [0,T]} |\tilde{q}_0(t) - q_0^0(t)| \leq C\delta |\ln \delta|^\beta.$$

This paper is organized as follows. In Section 2, some preliminary works are introduced; then in Section 3–Section 6, we give the proof of Theorems 1.2–1.5 respectively.
2. **Some preliminary works.** We first introduce the following lemma which plays an important role in the proof of the main results.

**Lemma 2.1** ([13]). *Under the assumption of (H1), if \( v \) is the solution of the following problem*

\[
\begin{aligned}
\begin{cases}
- \text{div}(K(x) \nabla v) = 0 & \text{in } \Omega, \\
v|_{\Gamma} = 0, \\
v|_{\gamma} = 1,
\end{cases}
\end{aligned}
\]

*then there exists a constant \( C \) depending only on \( \lambda \) such that*

\[
2\pi \lambda (\ln \frac{D}{\delta})^{-1} \leq \int_{\gamma} K(x) \frac{\partial v}{\partial n} ds \leq C(\ln \frac{d}{\delta})^{-1},
\]

*where \( d = \frac{1}{2} \min_{x \in \Gamma} |x - x_0| \) and \( D = \max_{x \in \Gamma} |x - x_0| \).*

To prove our main results, we first derive the equation for error between problems (1.1) and (1.3).

Let \( \phi \) be the solution of

\[
\begin{aligned}
\begin{cases}
\frac{\partial \phi}{\partial t} - \text{div}(K_0 \nabla \phi) = q_0(t) \delta_{x_0} & \text{in } \mathbb{R}^2 \times (0, T), \\
\phi(x, 0) = 0 & \text{in } \mathbb{R}^2,
\end{cases}
\end{aligned}
\]

with \( K_0 = K(x_0) \). Then

\[
\phi(x, t) = \int_0^t \frac{q_0(s)}{4\pi K_0(t - s)} e^{-\frac{|x - x_0|^2}{4K_0(t - s)}} ds = \int_0^t \frac{q_0(t - s)}{4\pi K_0 s} e^{-\frac{|x - x_0|^2}{4K_0 s}} ds.
\]

If we denote \( z_\delta = u_\delta - \phi, z = u - \phi \), then \( v_\delta = u_\delta - z = z_\delta - z \) satisfies

\[
\begin{aligned}
\begin{cases}
\frac{\partial v_\delta}{\partial t} - \text{div}(K(x) \nabla v_\delta) = 0 & \text{in } \Omega_\delta \times (0, T), \\
v_\delta(x, 0) = 0 & \text{in } \Omega, \\
v_\delta|_{\Gamma} = 0, \\
v_\delta|_{\gamma} = (u_\delta - z)|_{\gamma} = c(t) - z|_{\gamma} - \phi|_{\gamma},
\end{cases}
\end{aligned}
\]

In order to derive the estimate for the error term \( v_\delta \), we always split it into two parts \( v_\delta = v_1 + v_2 \), where \( v_1 \) satisfies the problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial v_1}{\partial t} - \text{div}(K(x) \nabla v_1) = 0 & \text{in } \Omega_\delta \times (0, T), \\
v_1(x, 0) = 0 & \text{in } \Omega, \\
v_1|_{\Gamma} = 0, \\
v_1|_{\gamma} = z(x_0, t) - z|_{\gamma},
\end{cases}
\end{aligned}
\]

and \( v_2 \) satisfies the problem

\[
\begin{aligned}
\begin{cases}
\frac{\partial v_2}{\partial t} - \text{div}(K(x) \nabla v_2) = 0 & \text{in } \Omega_\delta \times (0, T), \\
v_2(x, 0) = 0 & \text{in } \Omega_\delta, \\
v_2|_{\Gamma} = 0, \\
v_2|_{\gamma} = c(t) - z(x_0, t) - \phi|_{\gamma} (c(t) \text{ unknown}), \\
\int_{\gamma} K(x) \frac{\partial v_2}{\partial n} ds = -\int_{\gamma} K(x) \frac{\partial v_1}{\partial n} ds + \int_{B_0} \frac{\partial v_2}{\partial n} dx,
\end{cases}
\end{aligned}
\]

where \( B_0 = B(x_0, \delta) \).
3. Proof of Theorem 1.2. We consider the case of the fixed rate well and homogeneous media, i.e. \( K = K_0 \). Our plan is as follows: To prove Theorem 1.2, we first estimate \( \|v_1\|_{L^\infty(Q_t)} \) in (2.1). By maximum principle, it is bounded by the boundary value at the well: \( \max_{(x,t) \in \gamma \times (0,T)} |z(x_0, t) - z(x, t)| \). Secondly, we prove that the flow rate \( \int_\gamma K_0 \partial v_2/\partial \nu ds \) is small if \( \|v_1\|_{L^\infty(Q_t)} \) is small. At last, we prove that \( \|v_2\|_{L^\infty(Q_t)} \) in (2.2) is small provided the flow rate \( \int_\gamma K_0 \partial v_2/\partial \nu ds \) is small.

Proof. **Step 1.** We first give the bound of \( \|v_1\|_{L^\infty(Q_t)} \) in (2.1). To this purpose, we only need to estimate \( \max_{(x,t) \in \gamma \times (0,T)} |z(x_0, t) - z(x, t)| \).

Note that \( z = u - \phi \) satisfies
\[
\begin{cases}
\partial _t \phi - \text{div}(K_0 \nabla \phi) = 0 & \text{in } \Omega \times (0, T), \\
z(x, 0) = 0 & \text{in } \Omega, \\
z|_{\Gamma} = -\phi|_{\Gamma}.
\end{cases}
\]

By the regularity theorem [1], we easily get that \( z \in C^2(\Omega \times (0, T)) \), since \( \phi(x, 0)|_{\Gamma} = 0 \). Then we get
\[
\max_{(x,t) \in \gamma \times (0,T)} |z(x_0, t) - z(x, t)| \leq C_1 \delta \| \nabla z \|_{C^0(0,0,T)},
\]
\[
\leq C_1 C_2 \delta \| \nabla z \|_{C^0(\Omega \times (0,T))}
\]
\[
\leq C \delta.
\]

By the Maximum Principle, we get that
\[
\|v_1\|_{L^\infty(Q_t)} \leq \max_{(x,t) \in \gamma \times (0,T)} |z(x_0, t) - z(x, t)| \leq C \delta.
\]

**Step 2.** Next we give the bound of the flow rate \( \int_\gamma K_0 \partial v_2/\partial \nu ds \) for \( v_2 \) in (2.2). We just aim at estimating the term \(- \int_\gamma K_0 \partial v_2/\partial \nu ds \) and \( \int_{B_0} \partial u/\partial t dx \) owing to the formula:
\[
\int_\gamma K_0 \partial v_2/\partial \nu ds = - \int_\gamma K_0 \partial v_1/\partial \nu ds + \int_{B_0} \partial u/\partial t dx. \tag{3.1}
\]

To estimate the first term at the right hand side of (3.1), we introduce the corresponding steady problem of (2.1)
\[
\begin{cases}
-\text{div}(K_0 \nabla \rho_1) = 0 & \text{in } \Omega_s, \\
\rho_1|_{\Gamma} = 0, \\
\rho_1|_{\gamma} = 1.
\end{cases} \tag{3.2}
\]

Multiplying (2.1) by the solution \( \rho_1 \) of (3.2) and integrating on \( \Omega_s \), we get
\[
\int_{\Omega_s} \partial v_1/\partial t \rho_1 dx - \int_\gamma K_0 \partial v_1/\partial \nu \rho_1 ds + \int_{\Omega_s} K_0 \nabla v_1 \cdot \nabla \rho_1 dx = 0.
\]

Multiplying (3.2) by the solution \( v_1 \) of (2.1) and integrating on \( \Omega_s \), we have
\[
- \int_\gamma K_0 \partial \rho_1/\partial \nu v_1 ds + \int_{\Omega_s} K_0 \nabla \rho_1 \cdot \nabla v_1 dx = 0.
\]

Hence,
\[
\int_{\Omega_s} \partial v_1/\partial t \rho_1 dx - \int_\gamma K_0 \partial v_1/\partial \nu \rho_1 ds + \int_\gamma K_0 \partial \rho_1/\partial \nu v_1 ds = 0.
\]

Thanks to \( \rho_1|_{\gamma} = 1 \), we have
\[
\int_\gamma K_0 \partial v_1/\partial \nu ds = \int_{\Omega_s} \partial v_1/\partial t \rho_1 dx + \int_\gamma K_0 \partial \rho_1/\partial \nu v_1 ds. \tag{3.3}
\]
For problem (3.2), by Maximum Principle and the Hopf’s Lemma, we have
\[ 0 \leq \rho_1 \leq 1 \text{ in } \Omega_\delta, \text{ and } \frac{\partial \rho_1}{\partial \nu} > 0 \text{ on } \gamma. \]

Then we have by Lemma 2.1 that
\[
\left| \int_\gamma K_0 \frac{\partial \rho_1}{\partial \nu} v_1 ds \right| \leq \|v_1\|_{L^\infty(Q_\delta)} \left| \int_\gamma K_0 \frac{\partial \rho_1}{\partial \nu} ds \right| \leq C\delta \int_\gamma K_0 \frac{\partial \rho_1}{\partial \nu} ds \leq C \frac{\delta}{|\ln \delta|}. \quad (3.4)
\]

To bound the first term at the right hand side of (3.3), we know by the compatibility that \( \tilde{v}_1 = \frac{\partial v_1}{\partial t} \) satisfies
\[
\begin{cases}
\frac{\partial \tilde{v}_1}{\partial t} - \text{div}(K_0 \nabla \tilde{v}_1) = 0 & \text{in } \Omega_\delta \times (0, T), \\
\tilde{v}_1(x, 0) = 0 & \text{in } \Omega_\delta, \\
\tilde{v}_1|_{\Gamma} = 0, \\
\tilde{v}_1|_{\gamma} = \frac{\partial z}{\partial t}(x_0, t) - \frac{\partial z}{\partial t} \big|_{\gamma}.
\end{cases}
\]

By the regularity of \( z \in C^2(\Omega \times (0, T)) \) and Maximum Principle, we get that
\[ \|\frac{\partial v_1}{\partial t}\|_{L^\infty(Q_\delta)} \leq C\delta. \]

Combining with (3.3) and (3.4), we obtain
\[
\left| \int_\gamma K_0 \frac{\partial v_1}{\partial \nu} ds \right| \leq \left| \int_\Omega \frac{\partial v_1}{\partial t} \rho_1 dt \right| + \left| \int_\gamma K_0 \frac{\partial \rho_1}{\partial \nu} v_1 ds \right| \leq C\delta \left( 1 + \frac{1}{|\ln \delta|} \right). \quad (3.5)
\]

For the second term at the right hand side of (3.1),
\[
\int_{B_0} \frac{\partial u}{\partial t} dx = \int_{B_0} \frac{\partial (u - \phi)}{\partial t} dx + \int_{B_0} \frac{\partial \phi}{\partial t} dx = \int_{B_0} \frac{\partial z}{\partial t} dx + \int_{B_0} \frac{\partial \phi}{\partial t} dx \triangleq \text{I} + \text{II}.
\]

The regularity that \( z \in C^2(\Omega \times (0, T)) \) implies
\[ |\text{I}| = \left| \int_{B_0} \frac{\partial z}{\partial t} dx \right| \leq C|B_0| \leq C\delta^2. \quad (3.6)
\]

For the second term, we have
\[
|\text{II}| = \left| \int_{B_0} \int_0^t \frac{q'_0(t-s)}{4\pi K_0 s} e^{-\frac{x^2}{4\pi K_0 s}} ds dx \right|
\leq C \left| \int_0^t \left( 1 - e^{-\frac{s^2}{4\pi K_0 s}} \right) ds \right|
\leq C \left( \int_0^\delta (1 - e^{-\frac{s^2}{4\pi K_0 s}}) ds + \int_\delta^t (1 - e^{-\frac{s^2}{4\pi K_0 s}}) ds \right)
\leq C(\delta + \delta^2|\ln \delta|) \leq C\delta(1 + |\ln \delta|),
\]
where we have used the hypothesis (H2) and the elementary inequality \( 1 - e^{-x} \leq x \), for \( x \geq 0. \)

From (3.5), (3.6) and (3.7), we obtain the bound for the flow rate of \( v_2 \),
\[ \left| \int_\gamma K_0 \frac{\partial v_2}{\partial \nu} ds \right| \leq C\delta \left( 1 + |\ln \delta| \right). \]
Step 3. At the last step, we give the bound of \(\|v_2\|_{L^\infty(Q_\delta)}\) in (2.2).

Set \(r(t) = \int_\gamma K_0 \frac{\partial v_2}{\partial \nu} ds = r^+(t) - r^-(t)\) with \(r^\pm(t) = \max\{\pm r(t), 0\}\). It follows that \(|r^\pm(t)| \leq C\delta (1 + |\ln \delta|)\). Let \(v_2 = v_{2,1} + v_{2,2}\), where \(v_{2,1}\) satisfies
\[
\begin{cases}
\frac{\partial v_{2,1}}{\partial t} - \text{div}(K_0 \nabla v_{2,1}) = 0 & \text{in } \Omega_\delta \times (0, T), \\
v_{2,1}(x, 0) = 0 & \text{in } \Omega_\delta, \\
v_{2,1}|_{\Gamma} = 0, \\
v_{2,1}|_{\gamma} = c(t)\text{(unknown)}, \quad \int_\gamma K_0 \frac{\partial v_{2,1}}{\partial \nu} ds = +r^+(t),
\end{cases}
\]
and \(v_{2,2}\) satisfies
\[
\begin{cases}
\frac{\partial v_{2,2}}{\partial t} - \text{div}(K_0 \nabla v_{2,2}) = 0 & \text{in } \Omega_\delta \times (0, T), \\
v_{2,2}(x, 0) = 0 & \text{in } \Omega_\delta, \\
v_{2,2}|_{\Gamma} = 0, \\
v_{2,2}|_{\gamma} = c(t)\text{(unknown)}, \quad \int_\gamma K_0 \frac{\partial v_{2,2}}{\partial \nu} ds = -r^-(t).
\end{cases}
\]
Let \(\tilde{v} = v_{2,1} - v_{2}\), we get that \(\tilde{v}\) satisfies the problem
\[
\begin{cases}
\frac{\partial \tilde{v}}{\partial t} - \text{div}(K_0 \nabla \tilde{v}) = 0 & \text{in } \Omega_\delta \times (0, T), \\
\tilde{v}(x, 0) = 0 & \text{in } \Omega_\delta, \\
\tilde{v}|_{\Gamma} = 0, \\
\tilde{v}|_{\gamma} = c(t)\text{(unknown)}, \quad \int_\gamma K_0 \frac{\partial \tilde{v}}{\partial \nu} ds \geq 0,
\end{cases}
\]
then we can get that \(\tilde{v} \geq 0\) in \(\bar{\Omega}_\delta \times [0, T]\). We prove it by contraction. Suppose that \(\tilde{v}(y_0, t_0) < 0\) for some \((y_0, t_0) \in \Omega_\delta \times (0, T)\), then its minimum in \(\Omega_\delta \times (0, T)\) is attained only on \(\gamma \times (0, T)\) by Maximum Principle, say at \((x^*, t^*) \in \gamma \times (0, T)\). Again Hopf’s Lemma implies \(\frac{\partial \tilde{v}}{\partial \nu}(x^*, t^*) < 0\). Hence \(\int_\gamma K_0 \frac{\partial \tilde{v}}{\partial \nu} ds < 0\) at \(t = t^*\). This is a contradiction. Therefore we have \(v_2 \leq v_{2,1}\). Similarly, there holds \(v_{2,2} \leq v_2\). Then we obtain
\[v_{2,2} \leq v_2 \leq v_{2,1}.
\]
Take \(q = \max_{t \in [0,T]} r^+(t)\), then we have \(0 \leq q \leq C\delta (1 + |\ln \delta|)\). Next we introduce the auxiliary problem
\[
\begin{cases}
-\text{div}(K_0 \nabla \bar{v}_{2,1}) = 0 & \text{in } \Omega_\delta \\
\bar{v}_{2,1}|_{\Gamma} = 0, \\
\bar{v}_{2,1}|_{\gamma} = c(\text{unknown}), \quad \int_\gamma K_0 \frac{\partial \bar{v}_{2,1}}{\partial \nu} ds = q.
\end{cases}
\]
Similar argument implies \(v_{2,1} \leq \bar{v}_{2,1}\). And we can get by Hopf’s Lemma that \(\bar{v}_{2,1} \geq 0\) in \(\Omega_\delta\).

Since \(\bar{v}_{2,1}|_{\gamma} = \text{const}\), we may apply Lemma 2.1 to \(\bar{v}_{2,1}/(\bar{v}_{2,1}|_{\gamma})\) and obtain
\[
\frac{1}{\bar{v}_{2,1}|_{\gamma}} \int_\gamma K_0 \frac{\partial \bar{v}_{2,1}}{\partial \nu} ds \geq C \left(\ln \frac{D}{\delta}\right)^{-1}
\]
which yields
\[
\bar{v}_{2,1}|_{\gamma} \leq C\delta |\ln \delta| (1 + |\ln \delta|).
\]
To bound \(v_{2,2}\), we introduce the auxiliary problem
\[
\begin{cases}
-\text{div}(K_0 \nabla \bar{v}_{2,2}) = 0 & \text{in } \Omega_\delta \\
\bar{v}_{2,2}|_{\Gamma} = 0, \\
\bar{v}_{2,2}|_{\gamma} = c(\text{unknown}), \quad \int_\gamma K_0 \frac{\partial \bar{v}_{2,2}}{\partial \nu} ds = -q.
\end{cases}
\]
Proceed as before and obtain that $\bar{v}_{2,2} \leq v_{2,2}$. Moreover, Hopf’s Lemma implies that $\bar{v}_{2,2} \leq 0$ in $\Omega_\delta$. Similar to argument in (3.8), we can show that

$$\bar{v}_{2,2}|_\gamma \geq -C\delta |\ln \delta| (1 + |\ln \delta|).$$  \hspace{1cm} (3.10)

Combining (3.9), (3.10) into the inequality $\bar{v}_{2,2} \leq v_{2,2} \leq v_2 \leq \bar{v}_{2,1} \leq \bar{v}_{2,1}$, we get

$$\|v_2\|_{L_\infty(Q_\delta)} \leq C\delta(1 + |\ln \delta|).$$

This completes the proof of Theorem 1.2, since $u_\delta - u = v_3 = v_1 + v_2$. \hfill $\Box$

4. Proof of Theorem 1.3. In this section, we deal with the case of the fixed pressure well and homogeneous media i.e. $K(x) \equiv K_0$. To prove Theorem 1.3, we first give some comments on the additional condition at the well boundary in problem (1.4)

$$(u - \hat{\phi})(x_0, t) + \hat{\phi}|_\gamma = \alpha(t)$$  \hspace{1cm} (4.1)

which is an approximation of the fixed well bore pressure $u_\delta|_\gamma = \alpha(t)$ in problem (1.2).

Actually, (4.1) can be rewritten into a first-kind Volterra integral equation with respect to $\hat{q}_0(\cdot)$ as follows

$$\int_0^t \hat{q}_0(y)f(t - y)dy = \alpha(t), \quad t \in [0, T],$$  \hspace{1cm} (4.2)

where the explicit form of the kernel $f(\cdot)$ will be derived in the following. Set $z = u - \hat{\phi}$, we know that $z$ satisfies the problem

$$\begin{cases}
\frac{\partial z}{\partial t} - \text{div}(K_0 \nabla z) = 0 & \text{in } \Omega \times (0, T),
z(x_0, 0) = 0 & \text{in } \Omega,
z|_\Gamma = -\hat{\phi}|_\Gamma.
\end{cases}$$  \hspace{1cm} (4.3)

By Green formula, the solution of problem (4.3) can be expressed as

$$z(x, t) = \int_0^t \int_\Gamma \hat{\phi}_K \frac{\partial G}{\partial \nu}(x - \xi, t - \tau)dsd\tau$$

$$= \int_0^t \int_\Gamma \left( \int_0^t e^{-\frac{|x - y|^2}{4\pi K_0 (t - \tau)}} \frac{\partial \hat{q}_0(y)}{\partial \nu} K_0 \frac{\partial G}{\partial \nu}(x - \xi, t - \tau) dy \right) dsd\tau,$n

$$= \int_0^t \hat{q}_0(y) \int_y^t \frac{1}{4\pi K_0(t - \tau)} \int_\Gamma e^{-\frac{|x - y|^2}{4\pi K_0(t - \tau)}} K_0 \frac{\partial G}{\partial \nu}(x - \xi, t - \tau) dsd\tau dy,$n

where Green’s function $G(x - \xi, t - \tau)$ is defined as

$$\begin{cases}
\frac{\partial G}{\partial \nu} - \text{div}(K_0 \nabla G) = \delta(x - \xi, t - \tau) & \text{in } \Omega \times (0, T),
G(x, 0) = 0 & \text{in } \Omega,
G|_\Gamma = 0.
\end{cases}$$

Thus the kernel $f(\cdot)$ of (4.2) has the form as

$$f(t) = \int_0^t \frac{1}{4\pi K_0 t} \int_\Gamma e^{-\frac{|x - y|^2}{4\pi K_0 t}} K_0 \frac{\partial G}{\partial \nu}(x_0 - \xi, t - \tau) dsd\tau + \frac{1}{4\pi K_0} e^{-\frac{\tau^2}{4\pi K_0}}.$$

The well-posedness of a first-kind Volterra integral equation is a tough topic. Due to $f(0) = 0$ and $\frac{\partial^nf}{\partial t^n}(0) = 0$, $n = 1, 2, \cdots$, the equation (4.2) falls into a class
of ill-posed problem in general (see [8, 7, 9]). Here we do not present the well-posedness of the equation (4.2) in the classical sense. However, equation (4.2) has a special feature as
\[ f(\delta) \sim O(1/\delta) \text{ and } f(\delta^2) \sim O(1/\delta^2), \tag{4.4} \]
which makes it quite different from general ill-posed problems. Then we will prove the solvability of the linear system which is generated by the discretization of equation (4.2).

In fact, denote by
\[ I_h = \{t_i|0 = t_0 < t_1 < \cdots < t_n = T\} \]
a partition of \([0, T]\) with \(h = T/n\), and \(t_i = ih, \forall 0 \leq i \leq n\). Taking \(t_i = ih (i = 1, 2, \cdots, n)\) and discretizing the left hand side of the integral equation (4.2) by trapezoidal rule, we obtain a linear system with a lower triangular matrix
\[ Fq = \alpha, \tag{4.5} \]
where
\[ F = \begin{pmatrix} f_{11} & 0 & \cdots & 0 \\ f_{21} & f_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} \tag{4.6} \]
with
\[ f_{ij} = \begin{cases} \frac{h}{2} f((i - j - 1)h), & \text{for } j = 1, i = 1, 2, \cdots, n; \\ h f((i - j)h), & \text{for } 2 \leq j \leq n - 1, \ 2 \leq i \leq n; \\ \frac{h}{2} f((i - j - 1)h), & \text{for } i, j = n, \end{cases} \]
and
\[ q = (q_0, q_1, \cdots, q_{n-1})^T, \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})^T, \]
where \(q_i = q(t_i), \alpha_i = \alpha(t_i) (i = 0, 1, \cdots, n-1)\).

Take \(h = \delta^2\), then for sufficiently small \(\delta\), the associated matrix \(F\) for the linear system of equations (4.5) is diagonally concentrated
\[ f_{ii} = \frac{h}{2} f(h) = O(1), \ \forall \ 1 \leq i \leq n. \]
Therefore, the linear system (4.5) is stable and easy to be solved.

**Remark 2.** For general smooth kernel with \(f(0) = 0\), we usually have \(f(\delta^2) \sim O(\delta^2)\), then \(f_{ii} = \frac{h}{2} f(h) = O(\delta^4)\). So the determinant of \(F\) is
\[ \det(F) = O(\delta^{4n}), \]
which means that \(F\) is almost singular and the linear system (4.5) can hardly be solved.

In the following, we give a priori error estimate for the flow rate \(\tilde{q}_0(\cdot)\) under the condition that \(\tilde{q}_0(\cdot)\) is bounded uniformly with respect to \(\delta\).

**Proof.** Let \(z_\delta = u_\delta - \tilde{\phi}\), then \(\tilde{v}_\delta = u_\delta - u = z_\delta - z\) satisfies
\[ \begin{cases} \frac{\partial \tilde{v}_\delta}{\partial t} - \text{div}(K_0 \nabla \tilde{v}_\delta) = 0 & \text{in } \Omega_\delta \times (0, T), \\ \tilde{v}_\delta(x, 0) = 0 & \text{in } \Omega_\delta, \\ \tilde{v}_\delta|_{\Gamma} = 0, \\ \tilde{v}_\delta|_{\gamma} = (u_\delta - u)|_{\gamma} = \alpha(t) - u|_{\gamma}. \end{cases} \]
By (4.1), we have
\[ \alpha(t) - u|_{\gamma} = (u - \tilde{\phi})(x_0, t) + \tilde{\phi}|_{\gamma} - u|_{\gamma} = z(x_0, t) - z|_{\gamma}. \]

In (4.3), we have \( z \in C^2(\Omega \times (0, T)) \), since we have \( \tilde{\phi}(x, 0)|_{\Gamma} = 0 \). The estimate for \( \| \tilde{v}_s \|_{L^\infty(Q_{\delta})} \) can be obtained by Maximum Principle,
\[
\max_{(x, t) \in \Omega \times (0, T)} |u - u_s| = \| \tilde{v}_s \|_{L^\infty(Q_{\delta})} \leq \max_{(x, t) \in \gamma \times (0, T)} |z(x_0, t) - z(x, t)| \leq C\delta,
\]
and similar to the argument for (3.5), we get
\[
\left| \int_{\gamma} K_0 \frac{\partial \tilde{v}_s}{\partial \nu} ds \right| \leq C\delta \left( 1 + \frac{1}{\ln \delta} \right). \quad (4.7)
\]

Now,
\[
\tilde{q}_0(t) - \tilde{q}_0^0(t) = \int_{B_0} \frac{\partial u}{\partial t} dx + \int_{\gamma} K_0 \frac{\partial u}{\partial \nu} ds - \int_{\gamma} K_0 \frac{\partial u_s}{\partial \nu} ds = \int_{B_0} \frac{\partial u}{\partial t} dx - \int_{\gamma} K_0 \frac{\partial \tilde{v}_s}{\partial \nu} ds. \quad (4.8)
\]

For the first term at the right hand side of (4.8),
\[
\int_{B_0} \frac{\partial u}{\partial t} dx = \int_{B_0} \frac{\partial (u - \tilde{\phi})}{\partial t} dx + \int_{B_0} \frac{\partial \tilde{\phi}}{\partial t} dx = \int_{B_0} \frac{\partial z}{\partial t} dx + \int_{B_0} \frac{\partial \tilde{\phi}}{\partial t} dx \triangleq I + II.
\]

The regularity of \( z \in C^2(\Omega \times (0, T)) \) implies that
\[
|I| = \left| \int_{B_0} \frac{\partial z}{\partial t} dx \right| \leq C\delta^2. \quad (4.9)
\]

To bound II, we firstly show a fact that \( \tilde{q}_0(0) = O(\delta^2) \). Actually, differentiating with respect to \( t \) on both sides of (4.2), we have
\[
\alpha'(t) = \tilde{q}_0(0)f(t) + \int_{0}^{t} \tilde{q}_0'(t - s)f(s)ds.
\]
Taking \( t = \delta^2 \), we get from \( f(\delta^2) = O(1/\delta^2) \) and the regularity of \( \tilde{q}_0 \) that
\[
\tilde{q}_0(0) = (\alpha'(\delta^2) + C)/f(\delta^2) = O(\delta^2).
\]

Direct calculation yields
\[
|II| = \left| \int_{B_0} \left( \int_{0}^{t} \frac{\tilde{q}_0'(t - s)}{4\pi K_0 s} e^{-\frac{(x-x_0)^2}{4\pi K_0 s}} ds + \frac{\tilde{q}_0(0)}{4\pi K_0 t} e^{-\frac{|x-x_0|^2}{4\pi K_0 t}} \right) dx \right|
\leq \left| \int_{0}^{t} \int_{0}^{\infty} \int_{\delta^2}^{\infty} \frac{\tilde{q}_0'(t - s)}{4\pi K_0 s} e^{-\frac{r^2}{2\pi \sigma^2}} rdrd\theta ds \right| + \left| \int_{0}^{t} \int_{\delta^2}^{\infty} \frac{\tilde{q}_0(0)}{4\pi K_0 t} e^{-\frac{r^2}{2\pi \sigma^2}} rdrd\theta \right|
\leq C_1 \left( \int_{0}^{t} (e^{-\frac{r^2}{2\pi \sigma^2}} - 1) ds + C_2|\tilde{q}_0(0)| e^{-\frac{r^2}{2\pi \sigma^2}} - 1 \right)
\leq C_1 \left( \int_{0}^{t} (1 - e^{-\frac{r^2}{2\pi \sigma^2}}) ds + \int_{\delta}^{t} (1 - e^{-\frac{r^2}{2\pi \sigma^2}}) ds \right) + C_2|\tilde{q}_0(0)|(1 - e^{-\frac{r^2}{2\pi \sigma^2}})
\leq C_1(\delta + \delta^2|\ln \delta|) + C_2|\tilde{q}_0(0)|
\leq C\delta(1 + \delta|\ln \delta|).
\quad (4.10)
\]
Combining (4.7), (4.9) and (4.10) into (4.8), we can show that
\[
\max_{t \in [0,T]} |\tilde{q}_0(t) - \tilde{q}_0^\delta(t)| \leq \left| \int_{B_0} \frac{\partial u}{\partial t} \, dx \right| + \left| \int_{\gamma} K_0 \frac{\partial \tilde{q}_0^\delta}{\partial \nu} \, ds \right| \leq C \delta.
\]
This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.4. In this section, we consider the case of fixed rate well and heterogeneous media. The proof is almost the same as that of Theorem 1.2, except for that we do not have the regularity \( z \in C^2(\Omega \times [0,T]) \) in (2.1) as for the homogenous media case in Theorem 1.2. Due to the loss of regularity, we can prove a relatively weak results.

Proof. Step 1. We first give the bound of \( \|v_1\|_{L^\infty(Q_\delta)} \) in (2.1). To this purpose, we only need to estimate \( \max_{(x,t) \in \gamma \times [0,T]} |z(x_0,t) - z(x,t)| \).

Observe that \( z = u - \phi \) satisfies
\[
\begin{cases}
\frac{\partial z}{\partial t} - \text{div}(K(x) \nabla z) = \text{div}((K(x) - K_0) \nabla \phi) \equiv \text{div} h \quad \text{in} \quad \Omega \times (0,T), \\
z(x,0) = 0 \quad \text{in} \quad \Omega, \\
z|_\Gamma = -\phi|_\Gamma.
\end{cases}
\]
Since \( K \in C^{0,1}(\bar{\Omega}) \), there exists a constant \( \Lambda > 0 \) such that
\[
|K(x) - K(y)| \leq \Lambda |x - y|, \quad \forall \ x, y \in \bar{\Omega},
\]
and by (H2), we know that
\[
\|h\|_{0,p,\Omega}^p \leq \int_\Omega |(K - K_0) \nabla \phi|^p \, dx
\]
\[
\leq \int_\Omega \left( \Lambda |x - x_0| |\nabla_x^2 \phi + \nabla_y^2 \phi| \right)^p \, dx
\]
\[
\leq C_1 \int_\Omega \left| \int_0^t \frac{q_0(t-s)}{4 \pi K_0} e^{-\frac{|x-x_0|^2}{4 K_0 s}} \left( \frac{|x - x_0|^2}{4 K_0 s^2} \right) \, ds \right|^p \, dx
\]
\[
\leq C_2 \int_\Omega \left| \int_0^t e^{-\frac{|x-x_0|^2}{4 K_0 s}} \left( -\frac{|x - x_0|^2}{4 K_0 s} \right)^p \, ds \right| \, dx
\]
\[
\leq C_2 \int_\Omega e^{-\frac{p|x-x_0|^2}{4 K_0}} \, dx \leq C_2 |\Omega|
\]
which implies \( h(x,t) \in L^\infty((0,T), L^p(\Omega)) \) for all \( p > 1 \). Then by [4] (Theorem 2.1, Page 925), we have that \( z \in C^{0,\alpha}(\Omega \times (0,T)) \) with \( \alpha = 1 - \frac{3}{p} \), \( p > 3 \) and
\[
\max_{(x,t) \in \gamma \times [0,T]} |z(x_0,t) - z(x,t)| \leq C_3 \delta^{\alpha} \|z\|_{C^{0,\alpha}(B_0 \times (0,T))}
\]
\[
\leq C_3 C_4 \delta^{\alpha} \|(K(x) - K_0) \nabla \phi\|_{L^p((0,T) \times B_0)}
\]
\[
\leq C_3 C_4 \delta^{\alpha} \|(K(x) - K_0) \nabla \phi\|_{L^\infty((0,T), L^p(B_0))}
\]
\[
\leq C \delta^{\alpha},
\]
where \( C = C(p) \).

By the Maximum Principle, we get
\[
\|v_1\|_{L^\infty(Q_\delta)} \leq \max_{(x,t) \in \gamma \times [0,T]} |z(x_0,t) - z(x,t)| \leq C \delta^{\alpha},
\]
where $C = C(p)$ is independent of $\delta$ and $\alpha = 1 - \frac{3}{p}$, $p > 3$.

**Step 2.** Next we give the bound of the flow rate $\int_\gamma K \frac{\partial v_2}{\partial \nu} \, ds$ for $v_2$ in (2.2).

By the similar argument to **Step 2** in Section 3, we get that

$$\left| \int_\gamma K(x) \frac{\partial v_2}{\partial \nu} \, ds \right| \leq C \delta^{1 - \frac{2}{\pi}} \left( 1 + \delta |\ln \delta| + \frac{1}{|\ln \delta|} \right).$$

**Step 3.** Finally, we give the bound of $\|v_2\|_{L^\infty(Q_\delta)}$ in (2.2).

The argument is the same as **Step 3** in Section 3, since only Maximum Principle and Hopf’s Lemma are multiple used in Section 3 and the assumption for the regularity of $K$ does not affect their application to a certain extent. Hence,

$$\|v_2\|_{L^\infty(Q_\delta)} \leq C \delta^{1 - \frac{2}{\pi}} (1 + |\ln \delta|).$$

\[ \square \]

6. **Proof of Theorem 1.5.** In this section, we deal with the fixed pressure well and heterogeneous media. Similar to the proof of theorem 1.3, the additional condition

$$\langle u - \phi \rangle(x_0, t) + \tilde{\phi}|_{\gamma} = \alpha(t) \tag{6.1}$$

in problem (1.4) can also be written into a first-kind Volterra integral equation with respect to $\tilde{\phi}_0(\cdot)$ that

$$\int_0^t \tilde{\phi}_0(y) g(t, y) dy = \alpha(t), \quad t \in [0, T], \tag{6.2}$$

where the explicit form of the kernel $g(\cdot)$ will be derived in the following. Set $z = u - \phi$, we know that $z$ satisfies the problem

$$\left\{ \begin{array}{l}
\frac{\partial z}{\partial 	au} - \text{div}(K(x) \nabla z) = \text{div}((K(x) - K_0) \nabla \tilde{\phi}) = \text{div} h \quad \text{in} \quad \Omega \times (0, T), \\
z(x, 0) = 0 \quad \text{in} \quad \Omega, \\
z|_{\Gamma} = -\tilde{\phi}|_{\Gamma}.
\end{array} \right. \tag{6.3}$$

By Green formula, the solution of problem (6.3) can be expressed as

$$z(x, t) = -\int_0^t \int_\Omega (K(\xi) - K_0) \nabla \tilde{\phi} \nabla G(x - \xi, t - \tau) d\xi d\tau$$

$$+ \int_0^t \int_\Gamma (K(\xi) \frac{\partial G}{\partial \nu}(x - \xi, t - \tau) \tilde{\phi} d\nu d\tau$$

$$= -\int_0^t \int_\Omega (K(\xi) - K_0) \int_0^\tau \frac{\tilde{\phi}_0(y)}{4\pi K_0(\tau - y)} \nabla e^{-\frac{|\xi - x_0|^2}{4\pi K_0(\tau - y)}}$$

$$\cdot \nabla G(x - \xi, t - \tau) dy d\xi d\tau$$

$$+ \int_0^t \int_\Gamma \frac{\tilde{\phi}_0(y)}{4\pi K_0(\tau - y)} e^{-\frac{|\xi - x_0|^2}{4\pi K_0(\tau - y)}} dy K(\xi) \frac{\partial G}{\partial \nu}(x - \xi, t - \tau) d\nu d\tau$$

$$= -\int_0^t \tilde{\phi}_0(y) \int_\Omega (K(\xi) - K_0) \int_y^t \frac{1}{4\pi K_0(\tau - y)} \nabla e^{-\frac{|\xi - x_0|^2}{4\pi K_0(\tau - y)}}$$

$$\cdot \nabla G(x - \xi, t - \tau) d\tau d\xi dy$$

$$+ \int_0^t \tilde{\phi}_0(y) \int_y^t \frac{1}{4\pi K_0(\tau - y)} \int_\Gamma e^{-\frac{|\xi - x_0|^2}{4\pi K_0(\tau - y)}} K(\xi) \frac{\partial G}{\partial \nu}(x - \xi, t - \tau) d\nu d\tau dy.$$. 
here Green’s function $G(x - \xi, t - \tau)$ is defined as

$$\begin{cases}
\frac{\partial G}{\partial x} - \text{div}(K(x)\nabla G) = \delta(x - \xi, t - \tau) & \text{in } \Omega \times (0, T), \\
G(x, 0) = 0 & \text{in } \Omega, \\
G|_{\Gamma} = 0. 
\end{cases}$$

The kernel $g(\cdot)$ of (6.2) is

$$g(t) = -\int_{\Omega} (K(\xi) - K_0) \int_0^t \frac{1}{4\pi K_0 \tau} \nabla e^{-\frac{|\xi - \tau\alpha|^2}{4\kappa^2 \tau}} \cdot \nabla G(x_0 - \xi, t - \tau) d\tau d\xi$$

$$+ \int_0^t \frac{1}{4\pi K_0 \tau} \int_{\Gamma} e^{-\frac{|\xi - \tau\alpha|^2}{4\kappa^2 \tau}} K(\xi) \frac{\partial G}{\partial \nu}(x_0 - \xi, t - \tau) ds d\tau$$

$$+ \frac{1}{4\pi K_0} e^{-\frac{x^2}{4\kappa^2}}.$$

Due to $g(0) = 0$ and $\frac{\partial g}{\partial \nu}(0) = 0$, $n = 1, 2, \cdots$, the equation (6.2) is also ill-posed in general as (4.2). However, $g(\cdot)$ has the similar features as (4.4) to the kernel $f(\cdot)$ in (4.2), which makes the integral equation (6.2) easy to be solved numerically as (4.5)–(4.6). And all the properties of $\tilde{q}_0(\cdot)$ in Section 4 still hold in this section.

In the following, we give a priori estimate for the flow rate $\tilde{q}_0(\cdot)$.

Proof. Let $z_\delta = u_\delta - \tilde{\phi}$, then $\tilde{v}_\delta = u_\delta - u = z_\delta - z$ satisfies

$$\begin{cases}
\frac{\partial \tilde{v}_\delta}{\partial t} - \text{div}(K(x)\nabla \tilde{v}_\delta) = 0 & \text{in } \Omega \times (0, T), \\
\tilde{v}_\delta(x, 0) = 0 & \text{in } \Omega, \\
\tilde{v}_\delta|_{\Gamma} = 0, \\
\tilde{v}_\delta|_{\gamma} = (u_\delta - u)|_{\gamma} = \alpha(t) - u|_{\gamma}.
\end{cases}$$

From (6.1) we have

$$\alpha(t) - u|_{\gamma} = (u - \tilde{\phi})(x_0, t) + \tilde{\phi}|_{\gamma} - u|_{\gamma} = z(x_0, t) - z|_{\gamma}.$$

In (6.3), direct calculation yields $\tilde{\phi}(x, 0)|_{\Gamma} = 0$ and $\tilde{h} \in L^\infty((0, T), L^p(\Omega))$ for $p > 3$, then we have $z \in C^{0, 1/\hat{p}}(\Omega \times (0, T))$ [4]. And by Maximum Principle, we have

$$\max_{(x, t) \in \Omega \times [0, T]} |u - u_\delta| = ||\tilde{v}_\delta||_{L^\infty(\Omega)} \leq \max_{(x, t) \in \Omega \times [0, T]} \max_{|\gamma| = 1, T} |z(x_0, t) - z(x, t)| \leq C \delta^{1/\hat{p}},$$

where $C = C(p)$, and $p > 3$.

Now,

$$\tilde{q}_0(t) - \tilde{q}_0^\delta(t) = \int_{B_0} \frac{\partial u}{\partial t} dx + \int_{\gamma} K(x) \frac{\partial u}{\partial \nu} ds - \int_{\gamma} K(x) \frac{\partial u_\delta}{\partial \nu} ds$$

$$= \int_{B_0} \frac{\partial u}{\partial t} dx - \int_{\gamma} K(x) \frac{\partial \tilde{v}_\delta}{\partial \nu} ds.$$

By the same argument as Section 4, we have that

$$\max_{t \in [0, T]} |\tilde{q}_0(t) - \tilde{q}_0^\delta(t)| \leq \left| \int_{B_0} \frac{\partial u}{\partial t} dx \right| + \left| \int_{\gamma} K(x) \frac{\partial \tilde{v}_\delta}{\partial \nu} ds \right| \leq C \delta^{1/\hat{p}}.$$

\(\square\)
Conclusion. This paper focuses on the modeling error of well treatment for unsteady flow in porous media. The well is usually treated as a ‘point’ source since the radius $\delta$ is much smaller than the scale of reservoir. For fixed flow rate well, we derive the modeling error for the well-bore pressure; while for fixed pressure well, we give a priori estimate for the flow rate of the well.

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