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On Stably Free Ideal Domains

Henri Bourlès∗

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Abstract

We define a stably free ideal domain to be a Noetherian domain whose left and right ideals are all stably free. Every stably free ideal domain is a (possibly noncommutative) Dedekind domain, but the converse does not hold. The first Weyl algebra over a field of characteristic 0 is a typical example of stably free ideal domain. Some properties of these rings are studied. A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.

1 Introduction

In a principal ideal domain (resp. a Dedekind domain), every left or right ideal is free (resp. projective). An intermediate situation is the one where every left or right ideal is stably free. A Noetherian domain with this property is called a stably free ideal domain in what follows. In a Bézout domain, every finitely generated (f.g.) left or right ideal is free. An Ore domain in which every f.g. left or right ideal is stably free is called a semistably free ideal domain in what follows. Stably free ideal domains and semistably free ideal domains are briefly studied in this paper.

2 Free ideal domains and semistably free ideal domains

Theorem and Definition 1 Let A be a ring and consider the following conditions.

(i) Every left or right ideal in A is stably-free.
(ii) Every f.g. torsion-free A-module is stably-free.
(iii) Every f.g. left or right ideal in A is stably-free.

(1) If A is a Noetherian domain, then (i)⇒(ii)⇔(iii). If these equivalent conditions hold, A is called a stably-free ideal domain.

∗Satie, ENS de Cachan/CNAM, 61 Avenue Président Wilson, F-94230 Cachan, France (henri.bourles@satie.ens-cachan.fr)
(2) If $A$ is an Ore domain, then (ii)$\Leftrightarrow$(iii). If these equivalent conditions hold, $A$ is called a semistably-free ideal domain.

**Proof.** (1) (ii)$\Rightarrow$(i): Assume that (ii) holds and let $I$ be a left ideal in $A$. Then $I$ is a f.g. torsion-free module, therefore it is stably-free.

(i)$\Rightarrow$(ii): Assume that (i) holds and let $P$ be a f.g. torsion-free $A$-module. Since every left or right ideal is projective, $A$ is a Dedekind domain. Therefore, $P \cong A^n \oplus I$ where $I$ is a left ideal and $n$ is an integer ([5], 5.7.8). Since $I$ is stably-free, say of rank $r \geq 0$, there exists an integer $q \geq 0$ such that $I \oplus A^q \cong A^{q+r}$. Therefore, $P \oplus A^q \cong A^{q+r}$ and $P$ is stably-free of rank $n+r$. (i)$\Leftrightarrow$(iii) is clear.

(2) (ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(ii): If (iii) holds, $A$ is semihereditary. Let $P$ be a torsion-free left $A$-module. Since $A$ is an Ore domain, there exists an integer $n > 0$ and an embedding $P \hookrightarrow A^{[5]}$. Therefore, there exists a finite sequence of f.g. left ideals $(I_i)_{1 \leq i \leq k}$ such that $P \cong \bigoplus_{i=1}^{k} I_i$ ([3], Thm. (2.29)). For every index $i \in \{1, ..., k\}$, $I_i$ is stably-free, therefore there exist non-negative integers $q_i$ and $r_i$ such that $I_i \oplus A^{q_i} \cong A^{q_i+r_i}$. As a consequence,$$P \oplus A^q \cong A^q$$where $q = \sum_{1 \leq i \leq k} q_i$ and $r = \sum_{1 \leq i \leq k} r_i$, and $P$ is stably-free. ■

3 Examples of stably free ideal domains

The examples below involve skew polynomials.

**Proposition 2** Let $R$ be a commutative stably free ideal domain.

(1) Assume that $R$ is a $\mathbb{Q}$-algebra and let $A = R[X; \delta]$ where $\delta$ is an outer derivation of $R$ and $R$ has no proper nonzero $\delta$-stable (left or right) ideals. Then $A$ is a stably free ideal domain.

(2) Let $A = R[X, X^{-1}; \sigma]$ where $\sigma$ is an automorphism of $R$ such that $R$ has no proper nonzero $\sigma$-stable (left or right) ideals and no power of $\sigma$ is an inner automorphism of $R$. Then $A$ is a stably free ideal domain.

**Proof.** The ring $A$ is simple ([5], 1.8.4/5), therefore it is a noncommutative Dedekind domain ([5], 7.11.2), thus every left or right ideal of $A$ is projective, and, moreover, stably free ([5], 12.3.3). ■

Thus we have the following examples:

1. Let $k$ be a field of characteristic 0. The first Weyl algebra $A_1(k)$ and the ring $A'_1(k) = k[x, x^{-1}][X; \frac{d}{dX}] \cong k[X][x, x^{-1}; \sigma]$ with $\sigma(X) = X + 1$ ([5], 1.8.7) are both stably free ideal domains.

2. Likewise, let $k = \mathbb{R}$ or $\mathbb{C}$, let $k\{x\}$ be the ring of convergent power series with coefficients in $k$, and let $A_{1c}(k) = k\{x\}[X; \frac{d}{dX}]$. This ring is a stably free ideal domain.
3. Let $\Omega$ be a nonempty open interval of the real line and let $R(\Omega)$ be the largest ring of rational functions analytic in $\Omega$, i.e. $R(\Omega) = \mathbb{C}(x) \cap \mathcal{O}(\Omega)$ where $\mathcal{O}(\Omega)$ is the ring of all $\mathbb{C}$-valued analytic functions in $\Omega$. The ring $A(\Omega) = R(\Omega)[X; \frac{d}{dx}]$ is a simple Dedekind domain [2] and, since $R(\Omega)$ is a principal ideal domain, $A(\Omega)$ is a stably free ideal domain.

Note that a commutative Dedekind domain which is not a principal ideal domain is not a stably free ideal domain ([5], 11.1.5).

4 Connection with principal ideal domains, Bézout domains, and Hermite rings

Proposition 3 (i) A ring is a principal ideal domain if, and only if it is both a stably free ideal domain and an Hermite ring.

(ii) A ring is a Bézout domain if, and only if it is both a semistably free ideal domain and an Hermite ring.

Proof. (i): The necessary condition is clear. Let us prove the sufficient condition. Let $A$ be a stably free ideal domain and let $a$ be a left ideal of $A$. This ideal is stably free. If $A$ is Hermite, $a$ is free, and since $A$ is left Noetherian, it is a principal left ideal domain ([1], Chap. 1, Prop. 2.2).

The proof of (ii) is similar, using ([1], Chap. 1, Prop. 1.7).

5 Localization

Proposition 4 Let $A$ be a stably free ideal domain (resp. a semistably free ideal domain) and let $S$ be a two-sided denominator set ([5], §2.1). Then $S^{-1}A$ is a stably free ideal domain (resp. a semistably free ideal domain).

Proof. (1) Let us consider the case of stably free ideal domains. Let $A$ be a stably free ideal domain. For any left ideal $a$ of $S^{-1}A$ there exists a left ideal $J$ of $A$ such that $a = S^{-1}J$. Since $J$ is stably free, there exist integers $q$ and $r$ such that $J \oplus A^q = A^r$, therefore $S^{-1}J \oplus S^{-1}A^q = S^{-1}A^r$, and $a$ is stably free. The same rationale holds for right ideals, and this proves that $S^{-1}A$ is a stably free ideal domain.

(2) The case of semistably free ideal domains is similar, considering f.g. ideals.

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