Abstract

Let $M$ be a matroid on $E$, representable over a field of characteristic zero. We show that $h$-vectors of the following simplicial complexes are log-concave:

1. The matroid complex of independent subsets of $E$.
2. The broken circuit complex of $M$ relative to an ordering of $E$.

The first implies a conjecture of Colbourn on the reliability polynomial of a graph, and the second implies a conjecture of Hoggar on the chromatic polynomial of a graph. The proof is based on the geometric formula for the characteristic polynomial of Denham, Garrousian, and Schulze.

1. Introduction and results

A sequence $e_0, e_1, \ldots, e_n$ of integers is said to be log-concave if for all $0 < i < n$,

$$e_{i-1}e_{i+1} \leq e_i^2,$$

and is said to have no internal zeros if there do not exist $i < j < k$ satisfying

$$e_i \neq 0, \quad e_j = 0, \quad e_k \neq 0.$$

Empirical evidence has suggested that many important enumerative sequences are log-concave, but proving the log-concavity can sometimes be a non-trivial task. See [Bre94, Sta89, Sta00] for a wealth of examples arising from algebra, geometry, and combinatorics. The purpose of this paper is to demonstrate the use of an algebro-geometric tool to the log-concavity problems.

Let $X$ be a complex algebraic variety. A subvariety of $X$ is an irreducible closed algebraic subset of $X$. If $V$ is a subvariety of $X$, then the top dimensional homology group $H_{2\dim(V)}(V; \mathbb{Z}) \simeq \mathbb{Z}$ has a canonical generator, and the closed embedding of $V$ in $X$ determines a homomorphism

$$H_{2\dim(V)}(V; \mathbb{Z}) \rightarrow H_{2\dim(V)}(X; \mathbb{Z}).$$

The image of the generator is called the fundamental class of $V$ in $X$, denoted $[V]$. A homology class in $H_*(X; \mathbb{Z})$ is said to be representable if it is the fundamental class of a subvariety.

Hartshorne asks in [Har74, Question 1.3] which even dimensional homology classes of $X$ are representable by a smooth subvariety. Although the question is exceedingly difficult in general, it has a simple partial answer when $X$ is the product of complex projective spaces $\mathbb{P}^m \times \mathbb{P}^n$. Note in this case that the $2k$-dimensional homology group of $X$ is freely generated by the classes of subvarieties of the form $\mathbb{P}^{k-1} \times \mathbb{P}^1$.

Representable homology classes of $\mathbb{P}^m \times \mathbb{P}^n$ can be characterized numerically as follows [Huh12a, Theorem 20].

2010 Mathematics Subject Classification 05B35, 52C35

Keywords: matroid, hyperplane arrangement, $f$-vector, $h$-vector, log-concavity, characteristic polynomial.
Theorem 1. Write $\xi \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n; \mathbb{Z})$ as the integral linear combination

$$
\xi = \sum_i e_i [\mathbb{P}^{k-i} \times \mathbb{P}^i].
$$

(i) If $\xi$ is an integer multiple of either

$$
[\mathbb{P}^m \times \mathbb{P}^n], [\mathbb{P}^m \times \mathbb{P}^0], [\mathbb{P}^0 \times \mathbb{P}^n], [\mathbb{P}^0 \times \mathbb{P}^0],
$$

then $\xi$ is representable if and only if the integer is 1.

(ii) If otherwise, some positive integer multiple of $\xi$ is representable if and only if the $e_i$ form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

In short, subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ correspond to log-concave sequences of nonnegative integers with no internal zeros. Therefore, when trying to prove the log-concavity of a sequence, it is reasonable to look for a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ which witnesses this property. We demonstrate this method by proving the log-concavity of $h$-vectors of two simplicial complexes associated to a matroid, when the matroid is representable over a field of characteristic zero. Other illustrations can be found in [Huh12a, HK12, Len12].

In order to fix notations, we recall from [Bjo92] some basic definitions on simplicial complexes associated to a matroid. We use Oxley's book as our basic reference on matroid theory [Oxl11].

Let $\Delta$ be an abstract simplicial complex of dimension $r$. The $f$-vector of $\Delta$ is a sequence of integers $f_0, f_1, \ldots, f_{r+1}$, where

$$
f_i = \text{the number of } (i-1)\text{-dimensional faces of } \Delta.
$$

For example, $f_0$ is one, $f_1$ is the number of vertices of $\Delta$, and $f_{r+1}$ is the number of facets of $\Delta$. The $h$-vector of $\Delta$ is defined from the $f$-vector by the polynomial identity

$$
\sum_{i=0}^{r+1} f_i (q-1)^{r+1-i} = \sum_{i=0}^{r+1} h_i q^{r+1-i}.
$$

When there is a need for clarification, we write the coefficients by $f_i(\Delta)$ and $h_i(\Delta)$ respectively.

Let $M$ be a matroid of rank $r+1$ on an ordered set $E$ of cardinality $n+1$. We are interested in the $h$-vectors of the following simplicial complexes associated to $M$:

1. The matroid complex $IN(M)$, the collection of subsets of $E$ which are independent in $M$.
2. The broken circuit complex $BC(M)$, the collection of subsets of $E$ which do not contain any broken circuit of $M$.

Recall that a broken circuit is a subset of $E$ obtained from a circuit of $M$ by deleting the least element relative to the ordering of $E$. We note that the isomorphism type of the broken circuit complex does depend on the ordering of $E$. However, the results of this paper will be independent of the ordering of $E$.

Remark 2. A pure $r$-dimensional simplicial complex is said to be shellable if there is an ordering of its facets such that each facet intersects the complex generated by its predecessors in a pure $(r-1)$-dimensional complex. $IN(M)$ and $BC(M)$ are pure of dimension $r$, and are shellable. As a consequence, the $h$-vectors of both complexes consist of nonnegative integers [Bjo92]. This nonnegativity is recovered in Theorem 3 below.

Dawson conjectured that the $h$-vector of a matroid complex is a log-concave sequence [Daw84, Conjecture 2.5]. Our main result verifies this conjecture for matroids representable over a field of characteristic zero.
Theorem 3. Let $M$ be a matroid representable over a field of characteristic zero.

(i) The $h$-vector of the matroid complex of $M$ is a log-concave sequence of nonnegative integers with no internal zeros.

(ii) The $h$-vector of the broken circuit complex of $M$ is a log-concave sequence of nonnegative integers with no internal zeros.

Indeed, as we explain in the following section, there is a subvariety of a product of projective spaces which witnesses the validity of Theorem 3.

It can be shown that the log-concavity of the $h$-vector implies the strict log-concavity of the $f$-vector:

$$f_i - f_{i+1} f_i < f_i^2, \quad i = 1, 2, \ldots, r.$$ 

See [Len12, Lemma 5.1]. Therefore Theorem 3 implies that the two $f$-vectors associated to $M$ are strictly log-concave. The first statement of the following corollary recovers [Len12, Theorem 1.1].

Corollary 4. Let $M$ be a matroid representable over a field of characteristic zero.

(i) The $f$-vector of the matroid complex of $M$ is a strictly log-concave sequence of nonnegative integers with no internal zeros.

(ii) The $f$-vector of the broken circuit complex of $M$ is a strictly log-concave sequence of nonnegative integers with no internal zeros.

The main special cases of Theorem 3 and Corollary 4 are treated in the following subsections.

Remark 5. A pure simplicial complex is a matroid complex if and only if every ordering of the vertices induces a shelling [Bjo92, Theorem 7.3.4]. In view of this characterization of matroids, one should contrast Theorem 3 with examples of other ‘nice’ shellable simplicial complexes whose $f$-vector and $h$-vector fail to be log-concave. In fact, the unimodality of the $f$-vector already fails for simplicial polytopes in dimension $\geq 20$ [BL81, Bjo81].

These shellable simplicial complexes led to suspect that various log-concavity conjectures on matroids might not be true in general [Sta00, Wag08]. Theorem 3 shows that there is a qualitative difference between the $h$-vectors of

1. matroid complexes and other shellable simplicial complexes, and/or
2. matroids representable over a field and matroids in general.

See [Sta80] and [Sta77] for characterizations of $h$-vectors of simplicial polytopes and, respectively, shellable simplicial complexes in general. We note that the method of the present paper to prove the log-concavity crucially depends on the assumption that the matroid is representable over a field.

1.1 The reliability polynomial of a graph

The reliability of a connected graph $G$ is the probability that the graph remains connected when each edge is independently removed with the same probability $1 - p$. If the graph has $e$ edges and $v$ vertices, then the reliability of $G$ is the polynomial

$$\text{Rel}_G(p) = \sum_{i=0}^{e-v+1} f_i p^e - (1-p)^i.$$
where \( f_i \) is the number of cardinality \( i \) sets of edges whose removal does not disconnect \( G \). For example, \( f_0 \) is one, \( f_1 \) is the number of edges of \( G \) that are not isthmuses, and \( f_{e-v+1} \) is the number of spanning trees of \( G \). The \( h \)-sequence of the reliability polynomial is the sequence \( h_i \) defined by the expression

\[
\text{Rel}_G(p) = p^{v-1} \left( \sum_{i=0}^{e-v+1} h_i (1 - p)^i \right).
\]

In other words, the \( h \)-sequence is the \( h \)-vector of the matroid complex of the cocycle matroid of \( G \). Since the cocycle matroid of a graph is representable over every field, Theorem 3 confirms a conjecture of Colbourn that the \( h \)-sequence of the reliability polynomial of a graph is log-concave \[CoI87\].

**Corollary 6.** The \( h \)-sequence of the reliability polynomial of a connected graph is a log-concave sequence of nonnegative integers with no internal zeros.

It has been suggested that Corollary 6 has practical applications in combinatorial reliability theory \[BC94\].

### 1.2 The chromatic polynomial of a graph

The **chromatic polynomial** of a graph \( G \) is the polynomial defined by

\[
\chi_G(q) = (\text{the number of proper colorings of } G \text{ using } q \text{ colors}).
\]

The chromatic polynomial depends only on the cycle matroid of the graph, up to a factor of the form \( q^c \). More precisely, the absolute value of the \( i \)-th coefficient of the chromatic polynomial is the number of cardinality \( i \) sets of edges which contain no broken circuit \[Whi32\]. Since the cycle matroid of a graph is representable over every field, Corollary 4 confirms a conjecture of Hoggar that the coefficients of the chromatic polynomial of a graph form a strictly log-concave sequence \[Hog74\].

**Corollary 7.** The coefficients of the chromatic polynomial of a graph form a sign-alternating strictly log-concave sequence of integers with no internal zeros.

Corollary 7 has been previously verified for all graphs with \( \leqslant 11 \) vertices \[LM06\].

### 2. Proof of Theorem 3

We shall assume familiarity with the Möbius function \( \mu(x, y) \) of the lattice of flats \( \mathcal{L}_M \). For this and more, we refer to \[Aig87\] \[Zas87\]. An important role will be played by the **characteristic polynomial** \( \chi_M(q) \). For a loopless matroid \( M \), the characteristic polynomial is defined from \( \mathcal{L}_M \) by the formula

\[
\chi_M(q) = \sum_{x \in \mathcal{L}_M} \mu(\emptyset, x) q^{r+1-\text{rank}(x)} = \sum_{i=0}^{r+1} (-1)^i w_i q^{r+1-i}.
\]

If \( M \) has a loop, then \( \chi_M(q) \) is defined to be the zero polynomial. The nonnegative integers \( w_i \) are called the **Whitney numbers of the first kind**. The characteristic polynomial is always divisible by \( q-1 \), defining the **reduced characteristic polynomial**

\[
\bar{\chi}_M(q) = \chi_M(q)/(q-1).
\]
2.1 Brylawski’s theorem I
We need to quote a few results from Brylawski’s analysis on the broken circuit complex \[\text{Bry77}\]. The first of these says that the Whitney number \(w_i\) is the number of cardinality \(i\) subsets of \(E\) which contain no broken circuit relative to any fixed ordering of \(E\) \[\text{Bry77, Theorem 3.3}\]. This observation goes back to Hassler Whitney, who stated it for graphs \[\text{Whi32}\].

Fix an ordering of \(E\), and let 0 be the smallest element of \(E\). We write \(\text{BC}(M)\) for the reduced broken circuit complex of \(M\), the family of all subsets of \(E \setminus \{0\}\) that do not contain any broken circuit of \(M\). Since the broken circuit complex is the cone over \(\text{BC}(M)\) with apex 0, the above quoted fact says that
\[
\chi_M(q) = \sum_{i=0}^{r} (-1)^i f_i(\text{BC}(M)) q^{r-i}.
\]
In terms of the \(h\)-vector, we have
\[
\chi_M(q+1) = \sum_{i=0}^{r} (-1)^i h_i(\text{BC}(M)) q^{r-i} = \sum_{i=0}^{r} (-1)^i h_i(\text{BC}(M)) q^{r-i}, \quad h_{r+1}(\text{BC}(M)) = 0.
\]
Therefore the second assertion of Theorem 3 is equivalent to the statement that the coefficients of \(\chi_M(q+1)\) form a sign-alternating log-concave sequence with no internal zeros.

2.2 Brylawski’s theorem II
We show that the first assertion of Theorem 3 is implied by the second. This follows from the fact that the matroid complex of \(M\) is the reduced broken circuit complex of the free dual extension of \(M\) \[\text{Bry77, Theorem 4.2}\]. We note that not every reduced broken circuit complex can be realized as a matroid complex \[\text{Bry77, Remark 4.3}\]. The second assertion of Theorem 3 is strictly stronger than the first in this sense.

Recall that the free dual extension of \(M\) is defined by taking the dual of \(M\), placing a new element \(p\) in general position (taking the free extension), and again taking the dual. In symbols,
\[
M \times p := (M^* + p)^*.
\]
If \(M\) is representable over a field, then \(M \times p\) is representable over some finite extension of the same field. Choose an ordering of \(E \cup \{p\}\) such that \(p\) is smaller than any other element. Then, with respect to the chosen ordering,
\[
\text{IN}(M) = \text{BC}(M \times p).
\]
For more details on the free dual extension, see \[\text{Bry77, Bry86, Len12}\].

2.3 Reduction to simple matroids
A standard argument shows that it is enough to prove the assertion on \(\chi_M(q+1)\) when \(M\) is simple:

1. If \(M\) has a loop, then the reduced characteristic polynomial of \(M\) is zero, so there is nothing to show in this case.
2. If \(M\) is loopless but has parallel elements, replace \(M\) by its simplification \(\overline{M}\) as defined in \[\text{Oxl11, Section 1.7}\]. Then the reduced characteristic polynomials of \(M\) and \(\overline{M}\) coincide because \(\mathcal{L}_M \simeq \mathcal{L}_{\overline{M}}\).

Hereafter \(M\) is assumed to be simple of rank \(r+1\) with \(n+1\) elements, representable over a field of characteristic zero.
2.4 Reduction to complex hyperplane arrangements

We reduce the main assertion to the case of essential arrangements of affine hyperplanes. We use the book of Orlik and Terao as our basic reference in hyperplane arrangements [OT92].

Note that the condition of representability for matroids of given rank and given number of elements can be expressed in a first-order sentence in the language of fields. Since the theory of algebraically closed fields of characteristic zero is complete [Mar02, Corollary 3.2.3], a matroid representable over a field of characteristic zero is in fact representable over $\mathbb{C}$.

Let $\tilde{A}$ be a central arrangement of $n+1$ distinct hyperplanes in $\mathbb{C}^{r+1}$ representing $M$. This means that there is a bijective correspondence between $E$ and the set of hyperplanes of $\tilde{A}$ which identifies the geometric lattice $\mathcal{L}_M$ with the lattice of flats of $\tilde{A}$. Choose any one hyperplane from the projectivization of $\tilde{A}$ in $\mathbb{P}^r$. The decone of the central arrangement, denoted $A$, is the essential arrangement of $n$ hyperplanes in $\mathbb{C}^r$ obtained by declaring the chosen hyperplane to be the hyperplane at infinity. If $\chi_{\tilde{A}}(q)$ is the characteristic polynomial of the decone, then

$$\chi_{\tilde{A}}(q) = \chi_M(q).$$

Therefore it suffices to prove that the coefficients of $\chi_{\tilde{A}}(q+1)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

2.5 The variety of critical points

Finally, the geometry comes into the scene. We are given an essential arrangement $A$ of $n$ affine hyperplanes in $\mathbb{C}^r$. Our goal is find a subvariety of a product of projective spaces, whose fundamental class encodes the coefficients of the translated characteristic polynomial $\chi_{\tilde{A}}(q+1)$.

The choice of the subvariety is suggested by an observation of Varchenko on the critical points of the master function of an affine hyperplane arrangement [Var95]. Let $L_1, \ldots, L_n$ be the linear functions defining the hyperplanes of $\mathbb{A}$. A master function of $\mathbb{A}$ is a nonvanishing holomorphic function defined on the complement $\mathbb{C}^r \setminus \mathbb{A}$ as the product of powers

$$\psi_\mathbf{u} := \prod_{i=1}^n L_i^{u_i}, \quad \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n.$$

VARCHENKO’S CONJECTURE. If the exponents $u_i$ are sufficiently general, then all critical points of $\psi_\mathbf{u}$ are nondegenerate, and the number of critical points is equal to $(-1)^r \chi_\mathbb{A}(1)$.

Note that $(-1)^r \chi_\mathbb{A}(1)$ is equal to the number of bounded regions in the complement $\mathbb{R}^r \setminus \mathbb{A}$ when $\mathbb{A}$ is defined over the real numbers, and to the signed topological Euler characteristic of the complement $\mathbb{C}^r \setminus \mathbb{A}$. The conjecture is proved by Varchenko in the real case [Var95], and by Orlik and Terao in general [OT92].

In order to encode all the coefficients of $\chi_{\tilde{A}}(q+1)$ in an algebraic variety, we consider the totality of critical points of all possible (multivalued) master functions of $\mathbb{A}$. More precisely, we define the \textit{variety of critical points} $\mathfrak{X}(\mathbb{A})$ as the closure

$$\mathfrak{X}(\mathbb{A}) = \overline{\mathfrak{X}^0(\mathbb{A})} \subseteq \mathbb{P}^r \times \mathbb{P}^{n-1}, \quad \mathfrak{X}^0(\mathbb{A}) = \left\{ \sum_{i=1}^n u_i \cdot \text{dlog}(L_i)(x) = 0 \right\} \subseteq (\mathbb{C}^r \setminus \mathbb{A}) \times \mathbb{P}^{n-1},$$

where $\mathbb{P}^{n-1}$ is the projective space with the homogeneous coordinates $u_1, \ldots, u_n$. The variety of critical points first appeared implicitly in [OT95], and further studied in [CDFV12, DGST12]. See also [Huh12b, Section 2].
The variety of critical points is irreducible because $X^c(A)$ is a projective space bundle over the complement $C^r \setminus A$. The cardinality of a general fiber of the second projection

$$\text{pr}_2 : X(A) \rightarrow \mathbb{P}^{n-1}$$

is equal to $(-1)^r \chi_A(1)$, as stated in Varchenko’s conjecture. More generally, we have

$$[X(A)] = \sum_{i=0}^{r} v_i [\mathbb{P}^{r-i} \times \mathbb{P}^{n-1-r+i}] \in H_{2n-2}(\mathbb{P}^r \times \mathbb{P}^{n-1}; \mathbb{Z}),$$

where $v_i$ are the coefficients of the characteristic polynomial

$$\chi_A(q + 1) = \sum_{i=0}^{r} (-1)^i v_i q^{r-i}.$$

The previous statement is [Huh12b, Corollary 3.11], which is essentially the geometric formula for the characteristic polynomial of Denham, Garrousian, and Schulze [DGS12, Theorem 1.1], modulo a minor technical difference pointed out in [Huh12b, Remark 2.2]. A conceptual proof of the geometric formula can be summarized as follows [Huh12b, Section 3]:

1. Applying a logarithmic version of the Poincaré-Hopf theorem to a compactification of the complement $C^r \setminus A$, one shows that the fundamental class of the variety of critical points captures the characteristic class of $C^r \setminus A$.

2. The characteristic class of $C^r \setminus A$ agrees with the characteristic polynomial $\chi_A(q + 1)$, because the two are equal at $q = 0$ and satisfy the same inclusion-exclusion formula.

See [DGS12, Section 3] for a more geometric approach.

The proof of Theorem 3 is completed by applying Theorem 1 to the fundamental class of the variety of critical points of $A$.

Simple examples show that equalities may hold throughout in the inequalities of Theorem 3. For example, if $M$ is the uniform matroid of rank $r + 1$ with $r + 2$ elements, then

$$h_i(\text{IN}(M)) = h_i(\text{BC}(M)) = 1, \quad i = 1, \ldots, r.$$ 

However, a glance at the list of $h$-vectors of small matroid complexes generated in [DKK11] suggests that there are stronger conditions on the $h$-vectors than those that are known or conjectured. The answer to the interrogative title of [Wil76] seems to be out of reach at the moment.

References

Aig87 Martin Aigner, Whitney numbers, Combinatorial geometries, 139–160, Encyclopedia of Mathematics and its Applications 29, Cambridge University Press, Cambridge, 1987.

BL81 Louis Billera and Carl Lee, A proof of the sufficiency of McMullen’s conditions for $f$-vectors of simplicial convex polytopes, Journal of Combinatorial Theory Series A 31 (1981), 237–255.

Bjo81 Anders Björner, The unimodality conjecture for convex polytopes, Bulletin of the American Mathematical Society 4 (1981), 187–188.

Bjo92 Anders Björner, The homology and shellability of matroids and geometric lattices, Matroid Applications, 226-283, Encyclopedia Mathematics and its Applications 40, Cambridge University Press, Cambridge, 1992.

Bre94 Francesco Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Jerusalem Combinatorics '93, 71–89, Contemporary Mathematics 178, American Mathematical Society, Providence, 1994.
June Huh

BC94  Jason Brown and Charles Colbourn, On the log concavity of reliability and matroidal sequences, Advances in Applied Mathematics 15 (1994), 114–127.

Bry77  Thomas Brylawski, The broken-circuit complex, Transactions of the American Mathematical Society 234 (1977), 417–433.

Bry86  Thomas Brylawski, Constructions, Theory of Matroids, 127–223, Encyclopedia Mathematics and its Applications 26, Cambridge University Press, Cambridge, 1986.

CDFV12 Daniel Cohen, Graham Denham, Michael Falk, and Alexander Varchenko, Critical points and resonance of hyperplane arrangements, Canadian Journal of Mathematics 63 (2011), 1038–1057.

Col87  Charles Colbourn, The Combinatorics of Network Reliability, International Series of Monographs on Computer Science, The Clarendon Press, Oxford University Press, New York, 1987.

Daw84  Jeremy Dawson, A collection of sets related to the Tutte polynomial of a matroid, Graph theory, Singapore 1983, 193–204, Lecture Notes in Mathematics 1073, Springer, Berlin, 1984.

DKK11  Jesus De Loera, Yvonne Kemper, and Steven Klee, h-vectors of small matroid complexes, 2011, arXiv:1106.2576.

DGS12  Graham Denham, Mehdi Garrousian, and Mathias Schulze, A geometric deletion-restriction formula, Advances in Mathematics 230 (2012), 1979–1994.

Har74  Robin Hartshorne, Varieties of small codimension in projective space, Bulletin of the American Mathematical Society 80 (1974), 1017–1032.

Hog74  Stuart Hoggar, Chromatic polynomials and logarithmic concavity, Journal of Combinatorial Theory Series B 16 (1974), 248–254.

Huh12a  June Huh, Minors numbers of projective hypersurfaces and the chromatic polynomial of graphs, Journal of the American Mathematical Society 25 (2012), 907–927.

Huh12b  June Huh, The maximum likelihood degree of a very affine variety, 2012, arXiv:1207.0553.

HK12  June Huh and Eric Katz, Log-concavity of characteristic polynomials and the Bergman fan of matroids, Mathematische Annalen, to appear.

Len12  Matthias Lenz, The f-vector of a realizable matroid complex is strictly log-concave, Combinatorics, Probability, and Computing, to appear.

LM06  Per Håkan Lundow and Klas Markström, Broken-cycle-free subgraphs and the log-concavity conjecture for chromatic polynomials, Experimental Mathematics 15 (2006), 343–353.

Mar02  David Marker, Model Theory. An Introduction, Graduate Texts in Mathematics 217, Springer-Verlag, New York, 2002.

OT92  Peter Orlik and Hiroaki Terao, Arrangements of Hyperplanes, Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1992.

OT95  Peter Orlik and Hiroaki Terao, The number of critical points of a product of powers of linear functions, Inventiones Mathematicae 120 (1995), 1–14.

Oxl11  James Oxley, Matroid theory, Second edition, Oxford Graduate Texts in Mathematics 21, Oxford University Press, Oxford, 2011.

Sta77  Richard Stanley, Cohen-Macaulay complexes, Higher combinatorics, 51–62, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences 31, Reidel, Dordrecht, 1977.

Sta80  Richard Stanley, The number of faces of a simplicial convex polytope, Advances in Mathematics 35 (1980), 236–238.

Sta89  Richard Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Graph Theory and Its Applications: East and West (Jinan 1986), 500–535, Annals of New York Academy of Sciences 576, 1989.

Sta00  Richard Stanley, Positivity problems and conjectures in algebraic combinatorics, Mathematics: Frontiers and Perspectives, 295-319, American Mathematical Society, Providence, 2000.

Var95  Alexander Varchenko, Critical points of the product of powers of linear functions and families of bases of singular vectors, Compositio Mathematica 97 (1995), 385–401.
June Huh  junehuh@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA