Lectures on Reduce and Maple at UAM-I, Mexico

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In order to get an idea what facilities CA provides it might be sensible to ask what we expect from CA. I suggest that

CA is supposed to **solve or simplify any algebraic problem** that we can state explicitly.

So, CA has to provide methods to simplify algebraic expression and to solve equation systems. This, of course, is not enough because it also has enable us to input our problem and has to generate some output of the result (also graphical, plots, LaTeX, etc.). I want to differentiate the needs for the input a little bit more: On the one hand, CA has to offer some language or formalism that includes many operators to form expressions (+, *, sin, ∫, ∑, etc.) and on the other it should provide methods to make definitions, declarations, substitutions, and assumptions. Thus, we split the facilities we expect from CA into four groups:

- **A** operators to form expressions
- **B** methods to make definitions, declarations, substitution, and assumptions
- **C** methods to simplify and solve
- **D** methods to generate output

The reason why I suggest this splitting of the commands is the following: It seems that the discussion of group A dominates most handbooks on CA only because it is by far the largest group of commands and operators. I find though, that the commands in group B and C are more elementary and important to understand whereas for the commands in group A one can well refer to the online user’s manual. The commands in group D are also important for efficient work with CA. This is why these lectures try to emphasize the commands in groups B, C, and D a little bit more than usual.
Chapter 1

Reduce I

1.1 The simplification principle

As stated, I think the simplification and solving methods of a CA system to be most elementary and important. Hence we start to discuss these in the case of Reduce.

Reduce is an input-output machine. One could say that all that Reduce does is to reformulate your input expression obeying certain rules! One of these rules is, e.g., to execute all operators and commands in the expression. For example, Reduce reformulates $1 + 1$ by executing the + operator and answering 2. Fortunately, the rules for reformulation are such that - usually - this reformulation means a simplification. These rules can be influenced by the user: Either by switching on and off the rule switches or by introducing new rules. Most important, Reduce offers the rule switches collected in table 1.1.

As one can see, by switching on the exp-rule, Reduce reformulates and thus simplifies the expression $(x + 1)(x - 1)$ as $x^2 - 1$. This principle is quite different to other CA systems. The combination of such rules can be very powerful. The rule switches turned on at default are: allfac, exp, mcd, lcm, ratpri, pri, nat. This means that, at default, for any expression (subexpression, component, etc.), Reduce expands the multiplication of two larger terms, interprets all divisions as rationals, tries to divide common factors in numerators and denominators, and finally tries to factorize simple factors. This is already quite a powerful simplification scheme.
### CHAPTER 1. REDUCE I

| switch | description                                           | e.g.                  |
|--------|-------------------------------------------------------|-----------------------|
| * allfac | factorize simple factors                              | $2x + 2 \rightarrow 2(x + 1)$ |
| div    | divide by the denominator                             | $(x^2 + 2)/x \rightarrow x + 2/x$ |
| * exp  | expand all expressions                                | $(x + 1)(x - 1) \rightarrow x^2 - 1$ |
| * mcd  | make (common) denominators                            | $x^{-1} \rightarrow 1/x$ |
| * lcm  | cancel least common multiples                          |                        |
| gcd    | cancel greatest common divisor                         |                        |
| rat    | display as polynomial in factor                       |                        |
| * ratpri| display rationals as fraction                         |                        |
| * pri  | dominates allfac, div, rat, revpri                    |                        |
| revpri | display polynomials in opposite order                 |                        |
| rounded| calculate with floats                                  |                        |
| complex| simplify complex expressions                          |                        |
| nero   | don’t display zero results                            |                        |
| * nat  | display in Reduce input format                        |                        |
| msg    | suppress messages                                     |                        |
| fort   | display in Fortran format                             |                        |
| tex    | display in TeX format                                 |                        |

Table 1.1: Switches for Reduce’s reformulation rules. Those marked with * are turned on at default.

### 1.2 The interface

Starting Reduce on Unix it displays about

```
Loading image file:
/vol/sc/lib/reduce3.6_patch980830/reduce.img REDUCE 3.6, 15-Jul-95, patched to 30 Aug 98 ...

1:
```

The first three lines are an artifact of Reduce being implemented in LISP: The program code (image) of Reduce is passed to the LISP interpreter. The prompt `1:` expects a command as input. **Reduce ignores all upper case letters!** A command consists out of a statement and a terminator. The terminator decides whether the Reduce’s response is displayed (; terminator) or not ($ terminator). The following lines are command lines only, the response of Reduce is (sometimes) indicated after the comments sign %. All examples in this chapter are collected in the file `red1` at [5].

```
% file "red1"

off factor, exp, mcd, ratpri$
(x+1)**3/3 - x; %-> 1/3*(x + 1)**3 - x
on mcd$ ws; %-> ((x + 1)**3 - 3*x)/3
on ratpri$ ws; %->
```
1.3 User definitions

1.3.1 Names and assignments

A string can be used as a name (or identifier) for variables, procedures, etc. The first character of a name has to be alphabetic, whereas others can be numbers or ' ', e.g. 'a11' and 'ric_scalar0' are proper names. A name should not coincide with a reserved variable name (e i infinity nil pi t) or the command names in table 1.2.

As long as a name is not assigned to some (symbolic) value it is considered to be a clear name. The assignment operator

\[ name := expr; \]

assigns the (reformulated) expression on the rhs to the name on the lhs. To withdraw an assignment one uses the command

\[ clear name; \]

This command also deallocates the memory resource that is associated with the name.

For understanding Reduce, it is important to distinguish between the expression assigned to a name and the value that Reduce displays when evaluating the name. The last of which was reformulated by Reduce with current rules and definitions:

\[ a := b $ a; \quad \%-> b \]
\[ b := 1$ a; \quad \%-> 1 \]
\[ a := a$ b := 2$ a; \quad \%-> 1 \]
\[ clear a, b; \]

In the first line, \( a \) is assigned to \( b \) and is, of course, also evaluated to be \( b \). In the second line, \( a \) is still assigned to \( b \) but is evaluated to be \( 1 \) (because \( b \) is assigned to \( 1 \)). In the third line, \( a \) is assigned to \( 1 \) (because the rhs \( a \) is evaluated to be \( 1 \)) and an thus it is also evaluated to be \( 1 \) even after \( b \) is assigned to \( 2 \). We see that the reassignment \( a := a$ \) is not trivial at all! Analogously, the following example shows the importance of the reassignment if rules have changed:
CHAPTER 1. REDUCE I

off mcd$ a:=(x**2-1) / (x-1); %-> a := (x**2 - 1)*(x - 1)**(-1)
on mcd$  a := a$ a;   %-> x + 1
off mcd$ a;            %-> x + 1

Here, in the end, a is assigned to x+1. Without the reassignment a:=a$, a would still be assigned to (x**2 - 1)*(x - 1)**(-1) and also evaluated to be such in the last line.

The reassignment is one of the most important tools to apply new definitions and to use the simplification methods of Reduce.

1.3.2 Aliases

To define aliases for any expression, one can use the define command. The syntax is

\[
\text{define alias}=\text{expression};
\]

Note that the rhs expression is evaluated before it is assigned to the alias. However: An alias can never ever be changed or unassigned during a Reduce session! If an alias appears in an expression it is replaced by its associated value before Reduce applies any other rules. Example:

\[
\begin{align*}
\text{define isprime} &= \text{primep}; \\
\text{for } x:=0:20 \text{ do if isprime(x) then write } x, \text{" is a prime!"}; & \% \text{ throws 2 3 5 7 11 13 17 19} \\
\text{define is_assigned_to} &= :=; \\
x \text{ is_assigned_to 5}; & \%-> x := 5
\end{align*}
\]

1.3.3 Substitutions

Actually, we already noticed how to substitute identities into expressions: Define the identity with an assignment statement or introduce a rule for this identity. Then reevaluate the expression. Say, e.g., we want to substitute \( x = 0 \) into \( f(x) = \cos(x) \), then we could write one of the following three lines:

\[
\begin{align*}
f := \cos(x) & \quad x:=0$ \quad f; \text{ clear x}$ \quad %-> 1 \\
f := \cos(x) & \quad \text{let } \{ x=>0 \}$ \quad f; \text{ clear x}$ \quad %-> 1 \\
f := \cos(x) & \quad \text{where } \{ x=>0 \}; \quad %-> 1 \\
f := \cos(x) & \quad \text{sub}(x=0,f); \quad %-> 1
\end{align*}
\]

The last possibility is the most beautiful, because one does not have to clear the assignment \( x:=0 \) or the rule \( x=>0 \) hereafter. The where command only introduces a local rule. After these substitutions, \( f \) is still assigned to \( \cos(x) \) but was only locally evaluated to be 1. Reduce also offers the match command to substitute expressions for polynomials instead of names.
1.3. Rules

In the upper example we already introduced the \texttt{let} command that introduces new rules. In general, a rule has syntax

$$\texttt{expr} \Rightarrow \texttt{expr} \ [\texttt{when boolean}]$$

and the \texttt{let} command accepts a list of rules as parameter. Rules can be of explicit nature where any literal occurrence of the lhs expression is replaced by the rhs expression. Or they can be of parametric nature where the lhs expression includes formal names (like parameters) that represent any expressions. These formal names are marked with a twiddle such as $\sim x$:

\begin{verbatim}
\texttt{sin(x)**2 + cos(x)**2; \%-> \cos(x)**2 + \sin(x)**2}
\texttt{trig\_rules := {cos(\sim x)*\cos(\sim y) \Rightarrow (1/2)*(\cos(x+y)+\cos(x-y)),}
\texttt{sin(\sim x)**2 + cos(\sim x)**2 \Rightarrow 1 }}$
\texttt{let trig\_rules$
\texttt{sin(x+1)**2 + cos(x+1)**2; \%-> 1}
\texttt{cos(pi/3-1) * cos(1); \%-> (2*cos((pi - 6)/3) + 1)/4}
\texttt{showrules cos; \% shows ALL rules associated with cos!}
\texttt{clearrules trig\_rules;
\end{verbatim}

As already mentioned, the \texttt{where} command applies rules only one the preceding expression. The command

\texttt{showrules name;}

displays all rules associated with the name. This gives an insight in how Reduce defines functionals:

\begin{verbatim}
1: showrules log;
{log(1) => 0, log(e) => 1, $\sim x$
\texttt{log(e ) => x,}
\texttt{df(log($\sim x$),$\sim x$) => $\frac{1}{x}$
\end{verbatim}

1.3.5 Operators

An operator is any name that accepts parameters. In Reduce, one can declare a name to be an operator without at all specifying how many parameters the operator expects and what the functionality of the operator is! This leaves the user large freedom to handle and use operators. E.g., one can introduce indexed coordinates $x^i$ by declaring $x$ to be an
operator with the index as only parameter and without any further properties. In fact, most other, more specialized declarations (like matrix, procedure, etc.) can be thought of special kinds of operator declarations. And also functions (such as sin, log, etc.) can be introduced as operators with certain properties. The properties of an operator are declared by specifying the rules how Reduce has to handle them. Example:

```reduce
operator eps$
antisymmetric eps$
let { eps(0,1,2,3) => 1 }$
eps(3,1,2,0); eps(2,0,1,1); %-> -1 and 0
showrules eps; %-> {eps(3,2,1,0) => 1}$

clear my_sin$ operator my_sin$
my_sin_rules:=
   my_sin(~x) => - my_sin(-x) when numberp(x/pi) and x/pi<0,
   my_sin(~x) => my_sin(x - 2*pi) when numberp(x/pi) and x/pi>2,
   my_sin(~x) => - my_sin(x - pi) when numberp(x/pi) and x/pi>1,
   my_sin(~x) => my_sin(pi - x) when numberp(x/pi) and x/pi>1/2,
   my_sin(0) => 0,
   my_sin(pi/6) => 1/2,
   my_sin(pi/4) => 1/2*sqrt(2),
   my_sin(pi/3) => 1/2*sqrt(3),
   my_sin(pi/2) => 1
}$
let my_sin_rules$
my_sin(-124*pi/6); %-> - sqrt(3)/2
clearrules my_sin_rules;

1.4 Solving

solve(x**3-3*x**2-61*x+63,x); %-> {x=9,x=1,x=-7} which are the nulls of this polynomial
solve({x+y-9,-2*y**2+3*x},{x,y}); %-> { {y= - 9/2,x=27/2 } , {y=3,x=6} }

Reduce tries to solve algebraic equation systems with

```
solve(expr_list, name_list);
```

Without the switch multiplicities Reduce display multiple solutions only once.

Reduce has no build-in routines to solve differential equations. However, there exist packages for this problem. One should especially mention the CATHODE 2 project (Computer Algebra Tools for Handling Ordinary Differential Equations) which develops routines to handle differential equation systems also for Reduce. The members of this group are M. MacCallum (London), V. Fairen (Madrid), E. Tourner (Grenoble), Th. Mulders (Zurich), M. van der Puf (Groningen), F. Schwarz (Bonn), L. Brenig (Brussel). You find their packages at [10], especially Crack and ODEsolve.
1.5 Commands and references

Most of the commands Reduce offers are to formulate expressions. I tried to collect most in table 1.2. The point of this table is to give you an idea what commands there exist and to briefly describe the syntax. One should read through this table once. For detailed explanations we refer to the user's manual which can be accessed in the internet at [6]. I prepared more online references at [5].

for l:=1:50 sum l; %-> 1275
for l:=1:100 product l; % gives the huge number 100!
int(sin(x)**2,x); %-> x/2 + (-cos(x)*sin(x))/2
depend (f,x)$ df(f,x); int(f,x); %-> df(f,x) int(f,x)
nodepend (f,x)$df(f,x);int(f,x); %-> 0 x*f

matrix m(3,2)$ % declare matrix
m:=mat((1,2,3),(2,1,2),(3,2,1))$ % construct matrix
det m; %->8 determinante
1/m; % gives the inverse
matrix v(3,1)$ % declare a vector
v:=mat((1),(2),(3))$
1/m * v; %->(1,0,0) matrix multiplication

procedure fac1(k)$
  for l:=1:k product l$ % defines a new factorial procedure:
fac1(16); %-> 20922789888000 which is 16!

operator fac2$
let { fac2(1) => 1 ,
    fac2(~n) => n*fac2(n-1) when numberp n and n > 1}$
fac2(16); %-> 20922789888000

write "next ""prime > 500"" is ",nextprime(500)$ %-> next "prime > 500" is 503

showtime$ % milli seconds since last showtime
in "my_file"$ % executes all lines in "my_file" as input
quit$ % exits Reduce

1.6 Exercises

1. Get used to Reduce’s interactive interface. Try control commands like showtime write quit input ws clear %.

2. Turn all switches (for the reformulation rules) off:
   off allfac,div,exp,mcd,lcd,ratpri,nat$
Now type in term \((x**2 - 1)/((x+1)*(x-1))\); Which switches do you have to turn on additionally to get the following reformulations by Reduce:
control | quit showtime pause cont; ws input prompt_number;
write exprs; in out shut load_package "filename";
rederr "message"; \% comment

assignment | name := expr; set(name,expr); define name:=expr;

unassignment | clear names;

substitutions | let rule_list; sub match (eqn_list,expr); expr where rule_list;

rules | expr => expr; let clearrules rule_list; showrules name;

elementary | expr + - * / ** expr;

logical | expr = neq > >= <= < expr; ordp (names);
 boolean and or boolean; not boolean;
 numberp fixp evenp primep (expr); freeof (expr,name);

selection | rhs lhs eqn; num den expr; coeff (expr,name);
 coeffn (expr,name,degree);

functional | cos sin tan csc sec cot a^~h atan2 abs sqrt exp ln log
 logb log10 hypot factorial fix ceiling floor nextprime conj
 impart repart random round sign (expr); max min (expr_list);

calculus | int df (expr,names); depend nodepend (name,names);

matrix | mat (components); tp trace det rank mateigen (matrix);

lists | {elements}; first second third rest reverse length (list);
 append (list,list); expr . cons list;

loops | for (see section 2.3.2); while boolean do statement;
 repeat statement until boolean;

conditions | if boolean then statement [else statement];

groups | begin scalar names$ statements return expr$ end;
<< statements >>;

ordering | order korder factor names;

solving | solve (expr_list,name_list);

declarations | operator matrix names; array name(dimensions);
 procedure name(parameter_names)$ statement;

properties | noncom symmetric antisymmetric even odd linear operators;

Table 1.2: Standard commands in Reduce. **Explanation:** Each group of commands which are only separated by blanks have the same syntax. These groups are separated by ';' and the syntax in specified only once for all commands in one group (think of the blanks as 'or'). The plural of italic words means that there can be many of them separated by commas. Terms in brackets '[ ]' are optional. The twiddle ~ means one of sin, cos, etc. expr=expression, eqn=equation (which has syntax expr=expr)
1.6. EXERCISES

a) \((x^2 - 1)*(x + 1)^{-1}(x - 1)^{-1}\)$

b) \((x^2 - 1)^{-1}*x^2 - (x^2 - 1)^{-1}\)$

c) \((x^2 - 1)^{-1}*(x^2 - 1)\)$

d) \(1\)$

We see that \texttt{mcd} is absolutely necessary to calculate with rationals. The \texttt{exp} switch is important when it is not obvious that terms cancel in large expressions. Similarly, the \texttt{gcd} switch improves the performance in canceling a common divisor of numerator and denominator (which \texttt{mcd} does also in easier cases).

3. What is the value of

\[ a(a + 2) + c(c - 2) - 2ac \]  \hspace{1cm} (1.1)

for \(a - c = 7\)?

4. Get used to the user’s manual at [http://www.uni-koeln.de/REDUCE/3.6/doc/reduce/reduce.html](http://www.uni-koeln.de/REDUCE/3.6/doc/reduce/reduce.html) and to the file execution with the \texttt{in} command (See also section 2.1). Find out the syntax of the commands \texttt{limit factorial int df} for. Thereby get answers for

\[
\lim_{x \to 0} \frac{\sin(x)}{x}, \quad 15!, \quad \int f'(x) \, dx, \quad \int \exp(f(x)) \, f'(x) \, dx, \quad 1 - \sum_{i=1}^{10} \frac{1}{2^i}.
\]

Generate the list \(\{2^0, 2^1, \ldots, 2^{10}\}\) (use \texttt{for} and \texttt{collect}).

5. Does Reduce confirm

\[
\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = ae^{-z} \, ?
\]

Teach Reduce the appropriate rule for such problems!

6. Generate a list of all prime numbers smaller than 1000 and being a divisor of 606353.

7. Write a procedure that counts the zeros at the end of the number \(k!\) for any positive integer \(k\).

8. Develop a procedure to determine the \(n\)th order Taylor expansion of some arbitrary function \(f(x)\) at one arbitrary point \(x = x_0\). Test the procedure by considering \(e^x\) and \(\sin x\). Confirm \(e^{iz} = \cos z + i \sin z\) up to 5th order.

9. Prove by complete induction that

\[
\sum_{k=2}^{n} \frac{1}{(k-1)k} = \frac{n-1}{n}.
\]
10. Write a procedure that return the characteristic polynomial and the eigen values of a square matrix. (Check with but don’t use mateigen.) Implement operators that return the scalar and vector product, \( a \cdot b \) and \( a \times b \), for two vectors \( a, b \). Use arrays for these problems if you want indices to run from 0.
Chapter 2

Reduce II

2.1 Running files with Reduce and packages

For most purposes it is convenient to write all commands to be executed by Reduce into a separate file edited with your favorite editor. The in command causes Reduces to run this file as if all the lines were typed in during an interactive session. Use in "filename"; if you want Reduce to display each line of the file and in "filename"$ if not. The last line of such a file should be end$. Other useful command in this context are pause, cont, demo.

On Unix systems one can also use the following command line to pipe your Reduce file sample.rei into reduce and collect all output in the file sample.reo

reduce < sample.rei > sample.reo

Just like running own file you can read packages that implement new commands or whole calculi with the command load_package. Table 2.1 briefly displays the packages available for Reduce. In principle, one can think of packages as ordinary Reduce code which is precompiled into a fast loading file. (Such files can be generated with the commands faslout "filename"; faslend;). Later, we will describe the Excalc package implementing the exterior calculus. You find a list of all packages and links to online references in the table of contents of the Reduce user’s manual at [6]

Also, the ZIB (Berlin, Germany) offers a set of references (in PDF format) for all packages at [9].
| Package   | Description                                                                 |
|-----------|-----------------------------------------------------------------------------|
| ALGINT    | integration for functions involving roots (James H. Davenport)             |
| ARNUM     | algebraic numbers (Eberhard Schrfer)                                       |
| ASSIST    | useful utilities for various applications (Hubert Caprasse)                 |
| AVECTOR   | vector algebra (David Harper)                                              |
| CALI      | package for computational commutative algebra (Hans-Gert Graebe)           |
| CAMAL     | calculations in celestial mechanics (John Fitch)                           |
| CHANGEVAR | transformation of variables in differential equations (G. oluk)            |
| COMPACT   | condensing of expressions with polynomial side relations (Anthony C. Hearn) |
| CRACK     | package for solving overdetermined systems of PDEs or ODEs (Andreas Brand, Thomas Wolf) |
| CVIT      | Dirac gamma matrices (V.Ilyin, A.Kryukov, A.Rodionov, A.Taranov)           |
| DESIR     | differential equations and singularities (C. Dicrescenzo, F. Richard-Jung, E. Tournier) |
| EXCALC    | calculus for differential geometry (Eberhard Schrfer)                      |
| FIDE      | code generation for finite difference schemes (Richard Liska)              |
| GENTRAN   | code generation in FORTRAN, RATFOR, C (Barbara Gates)                     |
| GNUPLOT   | display of functions and surfaces (Herbert Melenk)                         |
| GROEBNER  | computation in multivariate polynomial ideals (Herbert Melenk, H.Michael Miller, Winfried Neun) |
| HEPHYS    | high energy physics (Anthony C. Hearn)                                     |
| IDEALS    | Arithmetic for polynomial ideals (Herbert Melenk)                          |
| LAPLACE   | Laplace and inverse Laplace transform (C. Kazasov et al.)                  |
| LIE       | functions for the classification of real n-dimensional Lie algebras (Carsten, Franziska Schbel) |
| LIMITS    | a package for finding limits (Stanley L. Kameny)                           |
| LININEQ   | linear inequalities and linear programming (Herbert Melenk)                 |
| NUMERIC   | solving numerical problems using rounded mode (Herbert Melenk)             |
| ODESOLVE  | ordinary differential equations (Malcolm MacCallum et al.)                 |
| ORTHOVEC  | calculus for scalar and vector quantities (J.W. Eastwood)                  |
| PHYSOP    | additional support for non-commuting quantities (Mathias Warnings)         |
| PM        | general algebraic pattern matcher (Kevin McIsaac)                          |
| REACTEQN  | manipulation of chemical reaction systems (Herbert Melenk)                 |
| RLFI, TRI | TeX and LaTeX output (Richard Liska, Ladislav Drska, Werner Antweiler)     |
| ROOTS     | roots of polynomials (Stanley L. Kameny)                                  |
| SCOPE     | optimization of numerical programs (J. A. van Hulzen)                     |
| SPDE      | symmetry analysis for partial differential equations (Fritz Schwarz)       |
| SPECFN    | package for special functions (Chris Cannon et al.)                        |
| SPECFN2   | package for special special functions (Victor Adamchik, Winfried Neun)    |
| SYMMETRY  | symmetry-adapted bases and block diagonal forms of symmetric matrices (Karin Gatermann) |
| SUM       | sum and product of series (Fuji Kako)                                      |
| TAYLOR    | multivariate Taylor series (Rainer Schpf)                                  |
| TPS       | univariate Taylor series with indefinite order (Alan Barnes, Julian Padget) |
| WU        | Wu Algorithm for polynomial systems (Russell Bradford)                     |

Table 2.1: Reduce packages
2.2 Examples

2.2.1 Generating a Julia set

% file "red3"

on comp$

procedure julia(c,s,file)$ begin
  on complex,rounded$
  precision(6)$
  l:={}$
  for x:=-3/2*s:3/2*s do << write x$ for y:=-3/2*s:3/2*s do <<
    z:=x/s+i*y/s$
    j:=0$ repeat z:=z**2+c until
      abs(repart(z))>2 or abs(impart(z))>2 or (j:=j+1)=50$
    if j=50 then l:={x,y}.l$
    >> >>$
  off complex,rounded$

  out file$
  for each point in l do
    write first(point)," ",second(point)$
  shut file$
end$

showtime$
julia(-0.11+0.67*i,100,"l2.red")$
showtime;

% time: 780730 ms

% in gnuplot:
% set data style dots
% set nokey; set noxtics; set noytics; set size square; set noborder
% plot "l2.red"
% plot "julia1.ps"
% replot

2.2.2 Proving the Casimirs of the Poincaré group

% file "red2"

% defining the epsilon and the Minkowski metric:
operator eps$ antisymmetric eps$
let eps(0,1,2,3)=>1$
array mink(3,3)$
mink(0,0)::=1$ mink(1,1)::=mink(2,2)::=mink(3,3)::=-1$
Figure 2.1: The Julia set for $z \to z^2 + (-0.11 + 0.67i)$. (File julia1.ps).

% declaring the generators of the Poincare group:
operator j,p$ noncom j,p$ antisymmetric j$

% the algebra of the Poincare group and general rules for the Lie bracket:
% note that the order of these rules is very important!
infix lie$
let {
  ( p(\~a) \ lie \ p(\~b) ) \ => \ 0,
  ( p(\~a) \ lie \ j(\~b,\~c) ) \ => \ \text{mink}(a,b)*p(c) - \text{mink}(a,c)*p(b),
  (j(\~a,\~b) \ lie \ p(\~c) ) \ => \ - (p(c) \ lie \ j(a,b)),
  (j(\~a,\~b) \ lie \ j(\~c,\~d) ) \ => \ - \text{mink}(a,c)*j(b,d) - \text{mink}(b,d)*j(a,c)
       + \text{mink}(a,d)*j(b,c) + \text{mink}(b,c)*j(a,d),
  (\~x \ lie \ \~y) \ => \ - (x \ lie \ y),
  (\~x \ lie \ -\~y) \ => \ - (x \ lie \ y),
  (\~x \ lie \ \~y) \ => \ x * y - y * x
}$

% rules for applying commutator to order products of operators
l:=oplist:={p(0),p(1),p(2),p(3),j(0,1),j(0,2),j(0,3),j(2,3),j(3,1),j(1,2)}$
for each x in 1 do
  for each y in (l:=rest l) do let y * x => x * y - (x lie y)$

% the momentum square (p2), the Pauli-Lubanski (pl), and its square (pl2):
operator p2,pl,pl2$
p2:=for a:=0:3 \ sum for b:=0:3 \ sum \text{mink}(a,b)*p(a)*p(b)$$
for a:=0:3 do pl(a) := (-1/2)*
  for b:=0:3 \ sum for c:=0:3 \ sum for d:=0:3 \ sum \text{eps}(a,b,c,d)*j(b,c)*p(d)$$
pl2:=for a:=0:3 \ sum for b:=0:3 \ sum \text{mink}(a,b)*pl(a)*pl(b)$$

% the commutators of p2 and pl2 with all generators:
for each x in oplist do write "[ p2 , ",",x," ] = ",p2 lie x$
for each x in oplist do write "[ pl2 , ",",x," ] = ",pl2 lie x;

2.3 Advanced structures

2.3.1 Lists

Lists are ordered sets. They are constructed via

{ elements };

The meaning of the commands to manipulate list should become clear from the following example:

li:={a,b,c,d};
first li; %-> a
second li; %-> b
third li; %-> c
part (li,4); %-> d
rest li; %-> {b,c,d}
reverse li; %-> {d,c,b,a}
length li; %-> 4
append (li,{e,f,g}); %-> {a,b,c,d,e,f,g}
0 . li; %-> {0,a,b,c,d}
0 . 1 . 2 . 3 . li; %-> {0,1,2,3,a,b,c,d}

2.3.2 Loops and conditions

The syntax for the loop and conditional commands are

for name := start:stop [do|sum|product|collect|join] statement;
for name := start step stepsize until stop [do|sum|product|collect|join] statement;
for each name in list [do|sum|product|collect|join] statement;
while boolean do statement;
repeat statement until boolean;
if boolean then statement;
if boolean then statement else statement;

The do action simple executes the statement in each iteration. With the sum, product actions, the hole for statement returns the sum or product of all statements in the iteration. Analogously, the collect action returns a list of all statements and the join action returns the union of all statements (these have to be lists in this case!).

Examples:
for x:=1:5 do write x; % writes the numbers from 1 to 5
for each x in {2,4,8} sum x/2; %-> 7
for each x in {a,b,c} join {x,2*x}; %-> {a,2*a,b,2*b,c,2*c}
if x neq 0 and not x>0 then write "impossible!"; % returns nothing

As we can see, a boolean is an expression formed by the logical operators = neq > >= <= < and or not.

2.3.3 Groups

Instead of executing only one statement in each loop or in some conditional case, we can execute an arbitrary set of statements by embracing them to a group. A simple way to form a group is

$$\langle \text{1st statement; 2nd statement; ... ; last statement} \rangle$$

This group is again a statement with the value of the last statement (if one did not put a terminator after the last statement). A more robust way to form groups is

begin [scalar local names;] statements [return value;] end;

Here, local names are such that have no effect outside of this group. They are initialized with 0. This group returns the specified value. To produce an output within a group one has to use the write command.

2.3.4 Procedures

Procedures are operators with an explicitly defined set of parameters and an explicitly defined functionality. The syntax for defining procedures is:

procedure name (parameter_names); statement;

Here, statement defines the (return) value of the procedure and is usually a group statement.

2.3.5 Operators

Operators can be declared prefix or infix by:

operator names;
infix names;

The declaration

precedence name, next_lower_precedence_operator;
specifies the precedence of this operator: The declared operator has just higher precedence than the operator specified.

Operators can have one of the following properties:
- totally symmetric or antisymmetric,
- even or odd with respect to parity of the parameter,
- linear, or
- noncom-muting under the multiplication *

All of these properties can be declared by

```
property operator_names;
```

where `property` is one of `symmetric`, `antisymmetric`, `even`, `odd`, `linear`, `noncom`.

### 2.3.6 Arrays, matrices

For handling multicomponent objects Reduce offers the following declarations:

```
array(dimensions-1);
matrix(co_dimension, contra_dimension);
```

Both declarations initialize all components to be zero. The array can have arbitrary many indices (slots) and they are counted starting at 0. There are no special operators defined for handling arrays – all operations have to be done by hand, mostly with `for` loops running over the indices.

The matrix has exactly two indices which start counting at 1! A matrix can also be constructed by the `mat` command where the components have to be structured in tuples: e.g. `mat ((1,2,3),(4,5,6));` constructs a 2×3-matrix that Reduce displays as

```
[1 2 3]
[   ]
[4 5 6]
```

Reduce offers the following routines to handle matrices:

```
tp trace det rank mateigen (matrix);
```

They calculate the transpose, trace, determinate, rank, and the eigenvalue equation and eigenvectors of the given `matrix`, respectively. Also, matrices can be added, multiplied, and divided. (A division means multiplication with the inverse.) A matrix can be constructed via the
2.4 Export / import

2.4.1 Storing results

Storing results is very important for large problems. Reduce offers no extra command for storing expressions but it is quite easy to write a little procedure that stores e.g. an arbitrary list of names together with their values. For this we write appropriate assignments into an extra file with the nat switch turned off. Such a procedure could read

```
procedure store(filename,namelist,valuelist)$
begin
  off nat$
  out filename$
  for each name in namelist do <<
    write name,"=":first valuelist$
    valuelist:=rest valuelist$
  >>$
  write "end$
  shut filename$
  on nat$
end$

With this procedure defined we can execute the lines

\[
f:=\sin(x)\quad g:=\cos(x)\]

store("FandG","f","g",f,g)$
clear f,g$
in "FandG"$
f;g;
```

where, the filename "FandG", the name list \{"f","g"\}, and their values \{f,g\} is passed to the procedure store. This produces the file "FandG" in the current directory reading

\[
f:=\sin(x)\quad g:=\cos(x)
end$
```

such that the in command reassings their values to f and g.

2.4.2 TeX

To export algebraic expression to TeX you need to load the package tri. This packages provides two switches tex and texbreak which causes Reduce to format any output in TeX syntax (texbreak also breaks the line in long equations). Example:

```
1: load_package tri;
*** global ‘metricu!*’ cannot become fluid
*** global ‘indxl!*’ cannot become fluid
% TeX-REDUCE-Interface 0.50
```
For some TeX-symbols Reduce uses own macros which can be included in TeX with an \input{tridefs} command. (You can find the tridefs.tex file at [5].) To export all the output of a Reduce file in one file in TeX format one could add the lines

\begin{verbatim}
load_package tri$
off msg$
on texbreak$
out "sample.out.tex$
\end{verbatim}

at the beginning of the Reduce file and the following lines at the end

\begin{verbatim}
shut "sample.out.tex$
quit$
\end{verbatim}

Turning off the msg switch prevents errors. For further information on the tri package see http://www.uni-koeln.de/REDUCE/3.6/doc/tri.ps

2.4.3 Fortran

Exporting to Fortran is very similar to exporting to TeX. Reduce offers the fort switch to display all output in Fortran syntax. Unfortunately, Reduce can not generate whole Fortran procedures - as Maple does, e.g..

2.4.4 Maple

Reduce offers, of course, no commands to export expressions and equations to Maple. However, in order to profit form the merits of both systems it is advantageous to know how to transport expression between them. Here we discuss how to export from Reduce to Maple.

The strategy I use is to collect all expression in one list. As first and last entry I insert "[null" and "null]", which are the braces to form ordered lists in Maple. Then I introduce rules to replace expressions which Maple does not understand, e.g. let \{ f=>f(x),
\[ \text{df}(f,x) \rightarrow \text{diff}(f(x),x) \]. After reformulating the list one switches \texttt{nat} off and writes the list into a file. See the following example to pass a list of two expressions to Maple

\begin{verbatim}
depend(f,x)$
e1:=int(f,x)+f+x**2$
e2:=df(f,x)+sin(x*f/3)$
maplist:={"[null",e1,e2,"null"]}$
operator f,diff$
let f=f(x),df(f,x)=diff(f(x),x)$
maplist:=maplist$
off nat$
out "sample.map"$
write "maplist:=op(maplist,);"$
shut "sample.map"$
on nat;
\end{verbatim}

which results in the file \texttt{sample.map}:

\begin{verbatim}
maplist:=op({null,
f(x) + int(f(x),x) + x**2,
diff(f(x),x) + sin((f(x)*x)/3),
null});$
\end{verbatim}

This file can be read by Maple with the command \texttt{read("sample.map")}. Although the $ symbol at the end of the file will cause an error message, the \texttt{maplist} is read correctly (the \LaTeX\ code for the following line was produced by Maple with the \texttt{latex} command):

\begin{verbatim}
maplist := [null, f(x) + \int f(x)dx + x^2, \frac{d}{dx}f(x) + \sin(1/3 f(x)x), null]
\end{verbatim}

\subsection*{2.4.5 Gnuplot}

To display functions with gnuplot and to save them as postscripts you need to load the package \texttt{gnuplot}. This package provides the commands \texttt{plot}, \texttt{gnuplot}, \texttt{plotreset}, and \texttt{plotshow}. With \texttt{plot} you will automatically open a gnuplot window that displays your graph. Example:

\begin{verbatim}
load_package gnuplot$
plot(cos(x)*cos(y),x=(-pi..pi),y=(-pi..pi),contour);
\end{verbatim}

will open a window that looks like the one in figure 2.2

The syntax of this \texttt{plot} command is similar to that in gnuplot. The command

\begin{verbatim}
gnuplot(cmd,param1,param2,...);
\end{verbatim}
2.4. EXPORT / IMPORT

is supposed to execute all other gnuplot commands (which it does not always!) The parameters of the gnuplot command `com` have to be separated by commas (not by blanks as in gnuplot). Example:

```plaintext
gnuplot(set,logscale,y)$
plot(x**2);
```

will produce a logarithmic plot of $x^2$. Finally, plotting into a file in postscript format is achieved by adding the options `terminal=postscript` and `output=filename` to the plot command. Example:

```plaintext
plot(cos(x)*cos(y),x=(-pi..pi),y=(-pi..pi),contour,terminal=postscript,output="gnutest.eps");
```

produces the postscript displayed in figure 2.2. For more information on the gnuplot package see http://www.zib.de/Symbolik/reduce/moredocs/gnuplot.pdf
Chapter 3

Maple I

3.1 The simplification principle

Just as Reduce, Maple is an input-output machine. But, as main difference to Reduce, Maple will not automatically reformulate any input obeying some rules. Instead, Maple only executes explicit commands in the input but cites expressions without commands verbally. Hence, the simplification performance of Maple is controlled with explicit commands only - not with automatic reformulation rules. This nature of Maple might stem from its origin: Maple is programmed in C and the command syntax is strongly influenced by this language. There exist no rules or switches, Maple becomes active only when a command or an operator is called.

Maple offers a variety of commands to reformulate expressions. Most important though is one command, \texttt{simplify}, which tries to apply many different simplification methods to the expression and can be controlled with some options. For example, Maple will simply cite an input \((x^2-1)/(x+1)\); as

\[
\frac{x^2 - 1}{x + 1}
\]

but the answer on \texttt{simplify(\((x^2-1)/(x+1)\));} is

\[
x - 1
\]

The general syntax of \texttt{simplify} is

\[
\texttt{simplify (expr [,simplification\_methods] [,assume=assumptions])};
\]

Without specifying the simplification methods, \texttt{simplify} applies all methods. The parameter \texttt{simplification\_methods} can be the \textit{name} of a user-defined simplification procedure
‘simplify/name’ (expr) or any of the build-in methods

BesselI, BesselJ, BesselK, BesselY, D, Ei, GAMMA, RootOf, LambertW, dilog, exp, ln, sqrt, polylog, pochhammer, trig (for trig functions), hypergeom (for hypergeometrics), radical (occurrence of fractional powers), power (occurrence of powers), exp, ln, and atsign (“@” - for operators)

One main problem with Maple’s simplification principle is, that simplify can not handle unknown (e.g. package-defined or user-defined) objects as expressions. For a tensor, e.g., the simplification command has to be applied on each component separately. Here, Reduce is clearly more elegant.

3.2 The interface

When you start a graphical version of Maple (‘xmaple’) you are offered a worksheet displaying the first prompt > . Behind the prompt, Maple expects a command consisting out of an statement and a terminator (; or :). Again, the terminator decides whether Maple’s response is displayed (;) or not (:). Note that Maple is case sensitive! All examples displayed in this chapter are collected in the file examples.mws at [5]. The LaTeX code to display the examples here was produced by Maple (see Import / export).

Simplifying

> (x**2-1)/(x+1);
\[
\frac{x^2 - 1}{x + 1}
\]

> simplify((x**2-1)/(x+1));
\[
x - 1
\]

> expand((x+y)**3);
\[
x^3 + 3x^2y + 3xy^2 + y^3
\]

> A:=[(x^2-1)/(x+1),0];simplify(A);
\[
A := \begin{bmatrix}
x^2 - 1 \\
x + 1
\end{bmatrix}, 0
\]

> A:=[0,0];simplify(A);
\[
A := [0, 0]
\]

> A:=array([[x**2-1]/(x+1),0],[0,0])); simplify(A);
\[
A := \begin{bmatrix}
x^2 - 1 \\
x + 1
\end{bmatrix}, 0
\]

> A:=array([[0,0],[0,0]]); simplify(A);
3.3. USER DEFINITIONS

\[
A := \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\frac{x-1}{x+1} & 0 \\
0 & 0
\end{bmatrix}
\]

Note that Maple makes an error in the last line because it does not handle arrays correctly! The operator \( \% \) has the value of the last response of Maple (cf. \( \texttt{ws} \) in Reduce), i.e. in our case \( \frac{x^2-1}{x+1} \). Similar, \( \%\% \) and \( \%\%\% \) represent the second and third last responses of Maple.

Usually, after typing the terminator one presses ENTER to execute the command. However, Maple offers to group commands in one so-called execution groups. To insert a line feed in this group one presses SHIFT+ENTER. Pressing ENTER somewhere in an execution group then executes all commands in this group.

3.3 User definitions

3.3.1 Names and assignments

Assignments

\[
\text{my\_name123} := \int f(x) \, dx
\]

\[
\text{my\_name123} := \int g(x) \, dx
\]

\[
\text{my\_name123} := \text{my\_name123}
\]

\[
\text{my\_name123} := x
\]

\[
A_{ijk} := 0
\]

As in Reduce any string (which is not reserved or a keyword, see table 3.1) may be used as a name. If you want some unusual string to be a name, the string has to be embraced by back(!)-quotes, e.g. \( \text{123 string} \) can be used as a name. The . sign allows to append numbers to a name.

The assignment operator is \( \text{name} := \text{expr} \). As in Reduce, the lhs name is identified with the rhs expression. There are three methods to unassign a name \( x \):

First with \( x := \text{`x'} \). The embracing by single quotes(!) means an unevaluation of the enclosed string. Hence, \( x := \text{`x'} \) means an assignment of the unevaluated string \( x \) back to
constants | Pi I infinity gamma Catalan true false
---|---
internal variables | Digits Order constants libname printlevel lasterror status
keywords | and by do done elif else end fi for from if in intersect local minus mod not od option options or proc quit read save stop then to union while

Table 3.1: Reserved names in Maple.

- Second with \texttt{unassign('x');}. The \texttt{unassign} command also needs an unevaluated string as parameter.
- Third with \texttt{restart}; which, however, unassigns all names. This methods is most secure to free the memory resource maple allocated.

As in Reduce it is important to distinguish between the expression assigned to a name and the value that Maple displays when evaluating the name. The following example shows the importance of the reassignment \texttt{a:=a;}. See also section 1.3.1.

\begin{verbatim}
> a:=b: a;
b
> b:=1: a;
1
> a:=a:b:=2: a;
1
> a:='a';
> b:='b';

a := a
b := b
\end{verbatim}

### 3.3.2 Aliases

Aliases

\begin{verbatim}
> alias(I='I'):macro(I='I'): 2*I; sqrt(-4);
2\textit{I}
2\sqrt{-1}
> macro(I=\texttt{sqrt(-1))}: 2*I; sqrt(-4);
2\sqrt{-1}
2\sqrt{-1}
> macro(I='I'): 2*I; sqrt(-4);
2\textit{I}
2\sqrt{-1}
\end{verbatim}
3.3. USER DEFINITIONS

> alias(I=sqrt(-1)): 2*I; sqrt(-4);
\[ 2I \]

> alias(I='I'): 2*I; sqrt(-4);
\[ 2\sqrt{-1} \]

Maple has two possibilities to define aliases which are quite useful (in contrast to the define command in Reduce). The first is macro and the second is alias. They have the same syntax but they have different meaning:

The macro(name=expr); command does not evaluate the expression expr before assigning it to the macro’s name. Whenever this name will appear in a future expression the first thing Maple does is to replace it by the associated value expr. However, Maple will not replace expr by the macro name in outputs. The point of a macro is to introduce shorthand notations for the input.

The alias(name=expr); command evaluates the expression expr before assigning it to the name. The value of expr cannot be a number. The effect of an alias is first, that Maple replaces all aliases by its value before evaluating expressions (as it does with macros), but second, that Maple also replaces expressions in the output by its alias if these exists one. So, the main point of an alias is introduce abbreviations in the input and output. The best example: The complex unit I is defined in Maple via: alias(I=sqrt(-1));

Macros and aliases can be changed and can be unassigned by assigning them back to their names: macro(name=name); and alias(name=name);

3.3.3 Substitutions and assumptions

Substitutions

> f:=cos(x): x:=0: f; x:=’x’:
1

> f:=cos(x): simplify(subs(x=0,f));
1

> f:='f':

As in Reduce, one way to substitute identities in expressions is to make a local definition that represents the identity and let Maple reevaluate the expression. After this, f is still assigned to cos(x) but was evaluated to be 1. However, one should clear the assignment for x again with x:=’x’:. A more elegant way to substitute is the subs command the syntax of which is:

subs(eqn_list, expr);
Eventually, one has to simplify again to take the substitution into account.

Maple has one quite surprising and useful tool that Reduce lacks. **Maple allows to make assumptions on unassigned names!** With this, e.g., Maple simplifies $\sqrt{x^2}$ to $x$ if one assumed $x > 0$ before:

Assumptions

```
> sqrt(x^2);
\sqrt{x^2}
> assume(x>0): about(x);
```

Originally x, renamed x^-:

```
is assumed to be: RealRange(Open(0),infinity)
```

```
> sqrt(x^-2);
x^-2
> x := 'x';
x := x
> sqrt(x^-2);
\sqrt{x^2}
```

The syntax is `assume(properties);` The twiddle `~` behind the name x indicates that there are properties assumed for x. You can ask for properties of names with `about` and `is` and add more properties with `additionally`. Properties are deleted with the reassignment `x:='x';`

### 3.3.4 Operators

Operators

```
> define(log2,log2(2)=1):
> log2(2)*log2(3);
\log2(3)
> define(fac,fac(0)=1,fac(n::posint)=n*fac(n-1)):
> fac(5);
120
> unassign(log2,fac):
```

Maple offers the `define` commands to declare new operators. The general syntax is

```
define (name,properties);
definemore (name,properties);
```
Here, *properties* are equations (rules) or any of the following key properties: *linear*, *multilinear*, *orderless*, *flat* (*means associative*). Note that it is possible to specify the type (see sections 4.2.2 and 4.2.3) of a parameter (e.g. `n::posint` to allow only positive integers for the parameter `n`). The command `definemore` adds more properties to an already defined operator.

### 3.3.5 Functionals

> f(x):=cos(x): f(x); f(0); f(y);

\[
\begin{align*}
cos(x) \\
f(0) \\
f(y)
\end{align*}
\]

> f(x):='f(x)':

Functionals

> g := x -> cos(x); g(x); g(0); g(y); D(g);

\[
\begin{align*}
g &:= \cos \\
\cos(x) \\
1 \\
\cos(y) \\
-\sin
\end{align*}
\]

> h := (x,y) -> sin(x)*sin(y); D[1](h); D[2](h);

\[
\begin{align*}
h &:= (x, y) \to \sin(x)\sin(y) \\
(x, y) &\to \cos(x)\sin(y) \\
(x, y) &\to \sin(x)\cos(y)
\end{align*}
\]

> k := h @ (x -> (x,exp(x))); k(x);

\[
\begin{align*}
k &:= h \circ (x \to (x, e^x)) \\
\sin(x)\sin(e^x)
\end{align*}
\]

> sin@exp; (sin@exp)(x); D(sin@exp); (D(sin@exp))(x);

\[
\begin{align*}
sin@exp \\
\sin(e^x) \\
\cos@exp\exp \\
\cos(e^x) e^x
\end{align*}
\]

> sin@@3; (sin@@3)(x);

\[
\begin{align*}
sin^{(3)} \\
(sin^{(3)})(x)
\end{align*}
\]

> g:=’g’: f:=’f’: h:=’h’: k:=’k’:
3.4 Solving

Maple offers the `solve` command for solving equations. The syntax is

```
solve (eqn_list, name_list);
```

For solving differential equations, use `dsolve`. For purely floating-point solutions use `fsolve`. Use `isolve` to solve for integer solutions, `msolve` to solve modulo a prime; `rsolve` for recurrences, and `linalg[linlsolve]` to solve matrix equations.

Solving

```
solve(x^4-5*x^2+6*x=2,x);
```

```
-1 + \sqrt{3}, -1 - \sqrt{3}, 1, 1
```

```
diff(y(x),x)-y(x)^2+y(x)*sin(x)-cos(x); dsolve(%);
```

```
\frac{d}{dx} y(x) - y(x)^2 + y(x) \sin(x) - \cos(x)
```

```
y(x) = - \frac{e(-\cos(x))}{C1 + \int e(-\cos(x)) \, dx} + \sin(x)
```

```
diff(x(t),t)=y(t); diff(y(t),t)=x(t)+y(t); x(0)=2,y(0)=1;
```

```
\frac{d}{dt} x(t) = y(t)
```

```
\frac{d}{dt} y(t) = x(t) + y(t)
```

```
x(0) = 2, y(0) = 1
```

```
res := dsolve({%%,%%,%},\{x(t),y(t)\}, type=numeric, output=procedurelist);
res := proc(rkf45 x) ... end
```

```
res(1);
```

```
[t = 1, x(t) = 5.582168689244844, y(t) = 7.826891137110794]
```

We see how to plot solutions (esp. of `dsolve`) in chapter 4.

3.5 Commands and references

Maple has many, many command - a lot more than Reduce. Here, I only tried to collect the most basic ones in table 3.2. Other than Reduce, Maple offers complete commands for almost any standard problem instead of providing only elementary commands that suffice to write procedures. Many of them might provide quick help. Unfortunately, this means a vast number of commands which can hardly be overviewed. Most of these one will never use anyway because they are too specialized.

Because of these innumerable commands Maple offers the most efficient way to work with Maple is by using the build-in help browser. This browser is opened via the programs menu or, more efficiently, by positioning the cursor on a word and pressing 'Crtl+F1'. To
begin with, it is useful to have a look on the 'Index...' topic (you find in the upper left window). There you find lists of expressions, functions, packages, etc. Alternatively, the 'Topic search...' in the 'Help' menu it useful to search for commands etc. alphabetically.

There is also online help in the internet available at [8]. This page is the producers address. You can find more addresses at [5].

There is also help available with in the non-graphical interface. The command \texttt{?subject} will display help and examples for the specified subject. Also the commands \texttt{index info usage example related (subject)}; provide help on the subject.

Integration / differentiation /dependencies:
\[
\begin{align*}
\text{diff(cos(5*x),x)}; & \quad -5 \sin(5 x) \\
\text{int(sin(x)**2,x)}; & \quad -\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x \\
\text{diff(f(x),x); diff(f,x); int(f(x),x); int(f,x);} & \quad \frac{\partial}{\partial x} \cos(x)(x) \\
& \quad -\sin(x) \\
& \quad \int \cos(x)(x) \, dx \\
& \quad \sin(x) \\
\text{f(x):=cos(x): f(x); f(0); f(y);} & \quad \cos(x) \\
& \quad f(0) \\
& \quad f(y) \\
\text{f(x):=’f(x)’:} & \\
\text{my_factorial:=proc(k)} & \\
\text{local x, l: x:=1:} & \\
\text{for l from 1 to k do x:=x*l: od:} & \\
\text{end; my_factorial := proc(k) local x, l; x := 1; for l to k do x := x \times l od end} & \\
\text{my_factorial(16);} & \quad 2092278988800 \\
\text{unassign(my_factorial):} & \\
\text{op( int(f(x),x) ); op( [1,2,3] ); f(x), x} & \\
\end{align*}
\]
\[ \text{CHAPTER 3. MAPLE I} \]

1, 2, 3

\[ \text{lhs}(x=1); \ \text{rhs}(f(x)=\sin(x)); \]
\[ x \]
\[ \sin(x) \]

\[ \{2,1\}; [2,1]; \]
\[ \{1, 2\} \]
\[ [2, 1] \]

\[ \text{seq}(x[i], i=1..5); x[i] \]
\[ x_1, x_2, x_3, x_4, x_5 \]
\[ x_1, x_2, x_3, x_4, x_5 \]

\[ \{1,2\} \text{ union } \{3,4\}; \{1,2\} \text{ intersect } \{1,3\}; \]
\[ \{1, 2, 3, 4\} \]
\[ \{1\} \]

\[ f := x \rightarrow x^2; \]

\[ \text{map}(f,[x,y]); \text{map}(f,x+(y*z)); \text{map}(f,x*(y+z)); \]
\[ \begin{bmatrix} x^2 \, y^2 \end{bmatrix} \]
\[ x^2 + y^2 + z^2 \]
\[ x^2 (y + z)^2 \]

Logical operators

\[ (x>3) \text{ and } (y\leq1); \]
\[ 3 - x < 0 \text{ and } y - 1 \leq 0 \]

\[ \text{subs}\{x=5,y=0\},\%; \]
\[ -2 < 0 \text{ and } -1 \leq 0 \]

\[ \text{eval}(\%); \]
\[ true \]

Inert operators

\[ \text{Sum}(1/2^i, i=1..k) = \text{sum}(1/2^i, i=1..k); \]
\[ \sum_{i=1}^{k} \frac{1}{2^i} = -2 (\frac{1}{2})^{(k+1)} + 1 \]

\[ \text{Limit}(\text{Sum}(1/2^i, i=1..k), k=\text{infinity}) = \]
\[ \text{limit}(\text{sum}(1/2^i, i=1..k), k=\text{infinity}); \]
\[ \lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{2^i} = 1 \]

\[ \text{value}(\text{Limit}(\text{Sum}(1/2^i, i=1..k), k=\text{infinity})); \]
\[ 1 \]

Float evaluation

\[ \text{evalf}(\Pi, 70); \]
\[ 3.141592653589793238462643383279502884197169399375105820974944592307816 \]
Matrices

> with(linalg): A:=matrix([[1,2,4],[2,1,2],[4,2,1]]);

Warning, new definition for norm

Warning, new definition for trace

\[
A := \begin{bmatrix}
1 & 2 & 4 \\
2 & 1 & 2 \\
4 & 2 & 1 \\
\end{bmatrix}
\]

> det(A);

9

> inverse(A);

\[
\begin{bmatrix}
-1 & 2 & 0 \\
2 & -5 & 2 \\
0 & 2 & -1 \\
\end{bmatrix}
\]

$
### Table 3.2: Elementary commands in Maple

We do not display the syntax of all commands and the innumerous functional operators \( \sin \) etc. For explanations see table 1.2. The slash in \( E/eval \) means that there exist both commands \( \text{Eval} \) and \( \text{eval} \), the first of which is the inert version (non-executing) whereas the second tries to execute the operation. The construction \( \text{eval}[a|gf|m|..] \) means that there exists \( \text{evala} \ \text{evalgf} \ \text{evalm} \) etc.
Chapter 4

Maple II

4.1 Examples

4.1.1 Generating a Julia set

# file "julia.map"

restart:

julia:=proc(c,s,file)
    local x,y,z,i,l:
    l:=[]:z:=0:
    for x from -3/2*s to 3/2*s do print(x): for y from -3/2*s to 3/2*s do
        z:=x/s+I*y/s:
        for i from 0 while i<100 and abs(Re(z))<2 and abs(Im(z))<2 do
            z:=z**2+c: od:
        if i=100 then l:=[op(l),[x,y]]: fi:
    od:od:

    save l,file:
end:

time(julia(0.3-0.5*I,200,"l6.map"));

with(plots):
read "l6.map";
plotsetup(ps,plotoutput="julia3.ps",plotoptions="noborder"): listplot(l,style=POINT,axes=NONE,symbol=POINT,scaling=CONSTRAINED);
plotsetup(default):
CHAPTER 4. MAPLE II

Figure 4.1: The Julia set for $z \rightarrow z^2 + (0.3-0.5i)$. (File julia3.ps).

4.2 Advanced structures

4.2.1 Loops and conditions

The syntax for loop and conditional commands are

- if boolean then statements [elif boolean then statements] [else statements] fi;
- for name from start [by step] to stop do statements od;
- for name from start [by step] while boolean do statements od;
- for name in list do statements od;
- while boolean do statements od;

4.2.2 Procedures

The syntax for defining procedures in Maple is

- name := proc (parameters::types) local names; global names; statements end;
The last of the statements needs no terminator and its value is automatically the return value of the procedure. One can specify the type of each parameter by adding ::type to the parameters name. Example:

\[
\begin{align*}
f := \text{proc}(n::\text{integer}) \text{factorial}(n) \text{ end;}
\end{align*}
\]

only excepts integers as parameters. Error messages or warnings can be produced with \texttt{ERROR} and \texttt{WARNING}. To use other procedures within a procedure it is sometimes useful to \texttt{catch} their error messages to prevent the procedure to be aborted. This is done be using the \texttt{trapererror} command.

### 4.2.3 Types

There are innumerable different types defined in Maple. The main point of these is that procedures and operators (also user defined ones) can check if the given parameters have the correct type and throw error messages if not. E.g., if you pass an integer to the display command \texttt{display(5);} it says:

\[
\begin{align*}
\text{Error, (in display) invalid argument, 5, must be a plot structure, or a list/set/array thereof}
\end{align*}
\]

This error message if quite useful. Also, if you want to tinker plot structures yourself and pass them to display, it is of course useful to find the exact description/syntax of the \texttt{plot} type in Maple’s help page \texttt{plot, structure}. With this knowledge you could also manipulate plots.

The commands

\[
\begin{align*}
\text{\texttt{whattype}}(\text{expr});
\text{\texttt{type}}(\text{expr,}\text{type});
\text{\texttt{convert}}(\text{expr,}\text{type});
\end{align*}
\]

query the type of \texttt{expr}, check if \texttt{expr} has the type \texttt{type}, and convert \texttt{expr} such as to have the type \texttt{type}, respectively. See the help for the \texttt{type} for a list of all types Maple provides.

### 4.3 Export / import

For export and import there are four groups of commands. The first is very useful to store expressions. They are:

\[
\begin{align*}
\text{\texttt{save}} \text{ names,\texttt{filename}}; \\
\text{\texttt{read}} \text{ \texttt{filename}};
\end{align*}
\]

Example:

\[
\begin{align*}
f:=\text{diff(sin(x),x)};
\text{save f,\texttt{sample.mai}};
\text{restart};
\text{read \texttt{sample.mai}};
\end{align*}
\]
Here, the \texttt{save} command produces the file \texttt{sample.mai} which simply reads:

\begin{verbatim}
f := \cos(x);
\end{verbatim}

The following \texttt{restart} clears all assignments and cleans the memory buffer. The \texttt{read} command read the file as if it were typed in.

The second group of commands is similar to the \texttt{out} command in Reduce and only write into files. The commands are

\begin{verbatim}
\texttt{writeto("filename");}
\texttt{appendto("filename");}
\texttt{writeto(terminal);}
\end{verbatim}

They send all Maple’s output to the specified file. \texttt{writeto(terminal)}; closes the file and redirects the output to the worksheet.

The third group of commands allows to read and write ordinary ascii data sheets as. Lists are used as buffer of such files:

\begin{verbatim}
\texttt{readdata("filename" [, format_list] [, size_of_one_data_tuple]);}
\texttt{writedata("filename", list);}
\end{verbatim}

\texttt{readdata} returns the data as list. The \texttt{format_list} can, e.g., be \texttt{[integer,float,integer]} for producing 3-tuples of data.

The fourth group of commands implements the syntax of the programming language C. They are

\begin{verbatim}
f := fopen("filename",READ|WRITE|APPEND,TEXT|BINARY);
\texttt{fprintf(f,structure_string,exprs);
\texttt{fscanf(f,structure_string,variables);
\texttt{fclose(f);
\end{verbatim}

The meaning should be clear if one knows C. The commands \texttt{feof}, \texttt{fflush}, \texttt{fremove}, \texttt{filepos} also belong to this group. With these commands you can produce files with arbitrary content.

### 4.3.1 \LaTeX, Fortran, and C

The commands \texttt{latex(expr)}, \texttt{fortran(expr)}, and \texttt{C(expr)} are nice facilities to export to \LaTeX, Fortran of the programming language C. See these examples:

\begin{verbatim}
> restart: readlib(C):

> latex(Limit(Sum(1/2^i,i=1..k),k=infinity));
\texttt{lim _{k\rightarrow \infty} \sum _{i=1}^{k}\left (\frac{1}{2^i}\right )^{-1}}

> f:= x \rightarrow \Pi*ln(x^2)-\sqrt{2}*ln(x^2)^2;
\end{verbatim}
4.3. EXPORT / IMPORT

> C(f, optimized);
/* The options were : operatorarrow */
#include <math.h>
double f(x)
double x;
{

double t2;
double t1;
double t4;
double t5;
{

t1 = x*x;
t2 = log(t1);
t4 = sqrt(2.0);
t5 = t2*t2;
return(0.3141592653589793E1*t2-t4*t5);
}
}

> fastgrow:=proc(k)
> local x,i;
> x:=1;
> for i from 1 to k do x:=x+x**x; od;
> x;
> end;

> fortran(fastgrow);

real function fastgrow(k)
real k

integer i
real x

x = 1
do 1000 i = 1,k,1
x = x+x**x
1000 continue
fastgrow = x
return
end

A possibility to export whole Maple worksheet to Latex is with the menu File – Export As – LaTeX. I used this method in this paper. This will produce a LaTeX document reproducing the appearance of the worksheet. The Maple style files have to be installed for this. One can get them at http://www.maplesoft.com/latex_patch.html and provisionally copy them in the same directory as the LaTeX file.
4.3.2 Plotting

Again, Maple offers innumerable commands to generate plots. Here, I present only those that I find most elementary:

\[
\text{plot}([\text{function list} [,\text{horizontal range}] [,\text{vertical range}] [,\text{plot options}]]);
\text{textplot}([[\text{x coord}, \text{y coord}, "\text{text}"],..] [,\text{plot options}]);
\text{display}([\text{plot structure list} [,\text{plot options}]]);
\text{with}();
\text{odeplot}(\text{numeric dsolve output}, \text{function list}, \text{x range} [,\text{plot options}]);
\text{plotsetup}(\text{device type} [,\text{options}]);
\]

For handling plots it is important to realize that all plot commands return a plot structure of type \text{PLOT}. See this example:

\[
\text{plt}:=\text{plot}(\sin(x),x=-\pi..\pi):
\text{save plt, "sample.plt":}
\text{restart:}
\text{read "sample.plt":}
\text{display(plt);}
\]

Here, we assign the plot of \(\sin(x)\) to the variable \text{plt} and store this variable in the \text{sample.plt} file. This is very useful when it takes long to produce plots (e.g. with \text{odeplot}!). The following commands produce the postscript in figure 4.2 and write it in the file \text{mapplot.ps}.

\[
\text{with(plots):}
\text{solution} := \text{dsolve}([[\text{diff}(y(x),x) = \sin(x*y(x)), y(0)=2], y(x), \text{type}=\text{numeric}]):
\text{plt} := \text{odeplot}(\text{solution}, [x,y(x)], 0..6, \text{labels}=[x,y], \text{labelfont}=[\text{TIMES,ITALIC,22}]);
\text{txt} := \text{textplot}([[2.5,2.5, "\text{Solution of y'=sin(x*y), y(0)=2}"], \text{align=RIGHT, font=[TIMES,ROMAN,22]}]);
\text{plotsetup(ps, plotoutput= "mapplot.ps", plotoptions= "noborder");}
\text{display}([\text{plt, txt}], \text{axesfont}=[\text{TIMES,ROMAN,18}]);
\text{plotsetup(default):}
\]

![Figure 4.2: Plot produced by Maple](image)
Chapter 5

Tensor type calculi on Reduce and Maple

5.1 Maple

5.1.1 The tensor package

Maple provides a the tensor package to implement the tensor calculus. This package is strongly oriented to perform the standard calculations in Einsteinian gravity, i.e. to calculate the Christoffel symbol, the Riemannian curvature tensor, the Ricci tensor, etc. from a given symmetric metric. For this special purpose the package is quite useful. Of course, the package provides also elementary operations to work with tensors: adding tensors, the tensor product, raising, lowering, contracting, and (anti-) symmetrizing indices raise, lower, contract, antisymmetrize, symmetrize. However, as one can see from table 5.1, the syntax of these commands is not at all in analogy to the usual notation of tensor calculus: To setup a general 3rd rank tensor \( A \) in 2D space with two covariant and one contravariant indices one has to declare

```maple
restart:with(tensor):
A:=create([-1,-1,1],array(          
  [[[a.1.1.1(x.1,x.2),a.1.1.2(x.1,x.2)],[a.1.2.1(x.1,x.2),a.1.2.2(x.1,x.2)]],
   [[a.2.1.1(x.1,x.2),a.2.1.2(x.1,x.2)],[a.2.2.1(x.1,x.2),a.2.2.2(x.1,x.2)]]]
));
```

To specify the tensor type, '-1' means a covariant index whereas '1' means a contravariant (in contrast to what is said in the help page of create). Maple starts counting indices from 1. As you see, one has to introduce general names for all components oneself (that include the dependency on coordinates). The following examples show other tensor manipulations:
CHAPTER 5. TENSOR TYPE CALCULI ON REDUCE AND MAPLE

declarations
create (tensor_type, components_array); entermetric

elementary ops
lin_com (coefficients_and_tensors);
prod (1st_tensor, 2nd_tensor, index_pairs_to_contract);
raise lower (metric, tensor, indices);
contract (tensor, index_pairs);
symmetrize antisymmetrize (tensor, index_array);
get_compts get_char get_rank (tensor);

differentiation
partial_diff exterior_diff (tensor, coord_array);
cov_diff (tensor, coord_array, Christoffel_symbol);
Lie_diff directional_diff (tensor, vector, coord_array);

more ops
compare invert dual permute_indices exterior_prod act
conj frame commutator

certain tensors
Levi_Civita dimetric d2metric

coord trans
Jacobian change_basis transform

GR tensors
tensorsGR display_allGR displayGR Christoffel1
Christoffel2 Einstein Ricci Ricciscalar Riemann Weyl
connexF RiemannF invars petrov

Newmann-P.
convertNP npcurve npspin

extract eqns
Killing_eqns geodesic_eqns

Table 5.1: The commands of Maple’s tensor package. The syntax is only specified for elementary commands.

| usual notation | tensor package syntax |
|----------------|-----------------------|
| \( 5A_{ij}^k + B_{ij}^k + 2C_{ij}^k \) | lin_com(5,A,B,2,c); |
| \( A_{ijk}B_{lm}^i \) | prod(A,B,[1,3]); |
| \( A_{ij}^i \) | contract(A,[1,3]); |
| \( \partial_i A_{jk} \) | partial_diff(A,[x.1,x.2,x.3,x.4]); |

In contrast, we will see the Excalc allows tensor manipulations that are a direct mirror of the usual notation. The entermetric command allows to enter a metric interactively. Personally, I find this unacceptable for a CA system.

Finally, we show how simple it is to calculate the standard GR tensors with the tensor package for a given metric. In the example we derive the Schwarzschild metric as solution of the Einstein equation for a general static, spherically symmetric ansatz.

> restart:with(tensor):
5.1. MAPLE

> gdd:=create([-1,-1],array(symmetric,[
> [f(r),0,0,0],
> [0,-g(r)/f(r),0,0],
> [0,0,-r^2,0],
> [0,0,-r^2*sin(theta)^2]]);
> ]));

\[
gdd := \text{table}[\text{compts} = \begin{bmatrix}
f(r) & 0 & 0 & 0 \\
0 & -\frac{g(r)}{f(r)} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2\sin(\theta)^2
\end{bmatrix}, \text{index_char} = [-1,-1]
\]

> tensorsGR([t,r,theta,phi],gdd,
> guu,gdet,chris1,chris2,riem,ric,ric_scalar,einstein,weyl):
> eqs:=get_compts(ric);

\[
eqs := \begin{bmatrix}
\frac{1}{4} f(r) (-2 r g(r) \left(\frac{\partial}{\partial r} f(r)\right) + r \left(\frac{\partial}{\partial r} g(r)\right) \left(\frac{\partial}{\partial r} f(r)\right) - 4 g(r) \left(\frac{\partial}{\partial r} f(r)\right)) & 0, 0, 0 \\
0, -\frac{1}{4} -2 r g(r) \left(\frac{\partial}{\partial r} f(r)\right) + r \left(\frac{\partial}{\partial r} g(r)\right) \left(\frac{\partial}{\partial r} f(r)\right) + 4 \left(\frac{\partial}{\partial r} g(r)\right) f(r) - 4 g(r) \left(\frac{\partial}{\partial r} f(r)\right)) & 0, 0 \\
0, 0, \frac{1}{2} \frac{r f(r) g(r)}{g(r)^2} & 0 \\
0, 0, 0, \frac{1}{2} \frac{\sin(\theta)^2 \left(2 \left(\frac{\partial}{\partial r} f(r)\right) r g(r) - r \left(\frac{\partial}{\partial r} g(r)\right) f(r) - 2 g(r)^2 + 2 f(r) g(r)\right)}{g(r)^2}
\end{bmatrix}
\]

> dsolve({eqs[1,1],eqs[2,2]},\{f(r),g(r)\});

\{f(r) = C2 + \frac{C3}{r}\}, \{g(r) = C1\}

5.1.2 The GRTensor package
> restart:
> read "/home/mt/usr/maple/lib5/grii.m":
> grtensor():
> makeg(ss):

Makeg 2.0: GRTensor metric/basis entry utility

To quit makeg, type 'exit' at any prompt.\large

Do you wish to enter a 1) metric \([g(dn,dn)]\),
2) line element \([ds]\),
3) non-holonomic basis \([e(1)\ldots e(n)]\), or
4) NP tetrad \([l,n,m,mbar]\)?

> 2:

Enter coordinates as a LIST (e.g. \([r,\theta,\phi,t]\)):
> [t,r,theta,phi]:

Enter the line element using \(d[\text{coord}]\) to indicate differentials.
(for example, \(r^2*(d[\theta]^2 + \sin(\theta)^2*d[\phi]^2)\))
[Type 'exit' to quit makeg]
\[
ds^2 = 
\]
> f(r)*d[t]^2 - \frac{g(r)}{f(r)}*d[r]^2 - r^2*(d[\theta]^2 + \sin(\theta)^2)*
> d[phi]^2:

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. \{ z, psi \} ).

Complex quantities [default=\{\}]:
>\{\}:

The values you have entered are :
Coordinates = \([t, r, \theta, \phi]\]

\[
\begin{bmatrix}
    f(r) & 0 & 0 & 0 \\
    0 & -\frac{g(r)}{f(r)} & 0 & 0 \\
    0 & 0 & -r^2 & 0 \\
    0 & 0 & 0 & -r^2\sin(\theta)^2
\end{bmatrix}
\]
5.1. MAPLE

You may choose to
0) Use the metric WITHOUT saving it,
1) Save the metric as it is,
2) Correct an element of the metric,
3) Re-enter the metric,
4) Add/change constraint equations,
5) Add a text description, or
6) Abandon this metric and return to Maple.

> 0:

Calculated ds for ss (.020000 sec.)

Default spacetime = ss
For the ss spacetime:
Coordinates
x(up)
\[ x^a = [t, r, \theta, \phi] \]
Line element
\[ ds^2 = f(r) \, dt^2 - \frac{g(r) \, dr^2}{f(r)} - r^2 \, d\theta^2 - r^2 \sin(\theta)^2 \, d\phi^2 \]

makeg() completed.

> grcalc( R(dn,dn) ):

Calculated detg for ss (.010000 sec.)
Calculated g(up,up) for ss (.020000 sec.)
Calculated g(dn,dn,pdn) for ss (.020000 sec.)
Calculated Chr(dn,dn,dn) for ss (.020000 sec.)
Calculated Chr(dn,dn,up) for ss (.040000 sec.)
Calculated R(dn,dn) for ss (.040000 sec.)

CPU Time = .150

> grdisplay( R(dn,dn) ):

For the ss spacetime:
Covariant Ricci
\[ R_{a \, b} = \begin{bmatrix}
-\frac{1}{4} f(r) \left( \left( \frac{\partial}{\partial r} f(r) \right) (\frac{\partial}{\partial r} g(r)) r - 2 \frac{\partial^2}{\partial r^2} f(r) g(r) r - 4 g(r) \left( \frac{\partial}{\partial r} f(r) \right) \right), & 0, & 0, & 0 \\
0, & 0, & 1 & \frac{1}{2} -2 \frac{\partial^2}{\partial r^2} f(r) r g(r) + f(r) \left( \frac{\partial}{\partial r} f(r) \right) + 2 g(r)^2 - 2 f(r) g(r), & 0 \\
0, & 0, & 1 & \frac{1}{2} \sin(\theta)^2 \left( -2 \frac{\partial}{\partial r} f(r) r g(r) + f(r) \left( \frac{\partial}{\partial r} g(r) \right) + 2 g(r)^2 - 2 f(r) g(r) \right), & 0 \\
\end{bmatrix} \]
who doubts about the beauty of the exterior calculus?! Reduce offers a package Excalc that provides very handy methods to handle exterior forms as well as tensors. Fortunately, this package implements elementary declarations and operators that let all freedom to the user to define quantities - it does not offer complete GR routines as the tensor of GRTensor packages in Maple that impose their notations and conventions. Excalc really generally implements the tensor and exterior calculus. Tensors can also be handled by Reduce without the Excalc package – you can find some explanation for this in the appendix. However, since with Excalc it is a lot easier and more beautiful, I advice to use the Excalc package to manipulate tensors.

The Excalc package is loaded with load_package excalc$. It provides the commands and operators summarized in table 5.2. The table should explain most commands. There is online help available at [7]. We now give more detailed explanations on the basic commands.

The most important declaration is pform which allows to introduce exterior forms with additional tensor indices. E.g., with the following we introduce a 2rd rank tensor \( t(i,j) \), a 4th rank tensor \( r(i,j,k,l) \), and a 2-form \( p \) with one index \( (p^i \in \Lambda^2) \):

\[
pform t(i,j)=0, r(i,j,k,l)=0, p(i)=2$
\]
### Table 5.2: The commands of Excalc.

A form is in general an exterior form declared with `pform` which can have tensor indices also. For the meaning of `index_pairs`, `coframe_def`, `metric_def`, and `signature` see the text.

| Command                      | Example                                                                 |
|-------------------------------|-------------------------------------------------------------------------|
| Declare forms/tensors        | `pform name[(tensor_indices)]=degree;`                                   |
| Declare vectors              | `tvector names;`                                                         |
| Declare dependence           | `fdomain name=name(dependency);`                                         |
| Tensor symmetries            | `index_symmetries tensor: [symmetric [in index_pairs]]`                   |
| Get form degree              | `exdegree(form);`                                                        |
| Exterior derivative          | `d(form);`                                                               |
| Partial differentiation      | `@ (form, coord);`                                                       |
| Inner product                | `vector _.form;`                                                         |
| Hodge dual                   | `# (form);`                                                              |
| Lie derivative               | `vector \_ form;`                                                        |
| Variational derivative       | `vardf (form, non_tensor_form);`                                         |
| Declare coframe              | `coframe coframe_def [with metric metric_def | with signature signature];`       |
| Declare frame                | `frame name;`                                                            |
| Display coframe              | `displayframe;`                                                          |
| Indices                      | `indexrange numbers_or_names;`                                           |
| Set space dimension          | `spacedim number;`                                                       |
| Tangent vector               | `@ (coord);`                                                             |
| Epsilon tensor               | `eps (indices);`                                                         |
| Christoffel symbol           | `riemannconx name;`                                                      |
| Others                       | `noether keep nosum renosum noxpnd forder remforder xpnd`                |

After this declaration, these tensors/forms can be referred to by `t(1st_index_name, 2nd_index_name)`, etc., i.e., we can use arbitrary names for the indices. Specifying the index range lets Excalc run the index names over this range in all expression.

Putting a minus sign in front of the index name lowers the index, without a minus sign the index is a contravariant index. **Excalc knows the summation convention** summing over identical lower and upper tensor indices. E.g., the following examples calculate the trace $t_{i}^{i}$ and perform the assignment $t_{ij} := r_{kij}^{k}$:

```
1: load_package excalc$
   *** ^ redefined
2: pform t(i,j)=0,r(i,j,k,l)=0,p(i)=2$
3: t(-i,i);
```
54  CHAPTER 5. TENSOR TYPE CALCULI ON REDUCE AND MAPLE

\[ t(i,j) := r(-k,i,j,k); \]

\[ i j i j k \]
\[ t := r \]
\[ k \]

5: indexrange 0,1,y,z$

6: t(-i,i);

\[ 0 1 y z \]
\[ t + t + t + t \]
\[ 0 1 y z \]

7: t(0,0):=a$ t(0,0);

a

9: t (-0,0);

0

If we have defined a metric Excalc automatically calculates the (contra-)covariant components from assignments to (co-)contravariant components. For a diagonal metric, the response to input 9 would then be \( g_{00} a \). Specifying the metric structure is a little bit involved. Roughly, the syntax to introduce a frame and metric is

\[ \text{coframe coframe_def [with metric metric_def | with signature signature];} \]

The following examples should make clear what the parameters mean:

\% file "excl"
load_package excalc$

\% symbolic 3D coframe with Euclidean metric:
coframe o(0),o(1),o(2)$

\% polar coframe with Euclidean metric:
coframe o(r)=d r,o(p)=r * d phi$

\% Schwarzschild metric in anholonomic/diagonalized form:
coframe o(0)=f * d t,
o(1)=1/f * d r,
o(2)=r * d theta,
o(3)=r*sin(theta) * d phi
with signature (+1,-1,-1,-1)$

\% symbolic 4D coframe with arbitrary metric:
5.2. EXCALC

operator gll,o$ % introduce general notations
for i:=0:3 do for j:=i+1:3 do gll(i,j):=gll(j,i)$ % symmetrize metric
let { metric_def => for i:=0:3 sum for j:=0:3 sum gll(i,j)*o(i)*o(j) };
coframe o(0),o(1),o(2),o(3) with metric g = metric_def$

% holonomic 4D coframe with arbitrary metric and coordinates x(a):
operator gll,o$
for i:=0:3 do for j:=i+1:3 do gll(i,j):=gll(j,i)$
let { metric_def => for i:=0:3 sum for j:=0:3 sum gll(i,j)*o(i)*o(j) };
coframe o(0)=d x0,o(1)=d x1,o(2)=d x2,o(3)=d x3 with metric g = metric_def$
pform x(i)=0$ % introduce coordinates as indexed 0-form defined as:
x(-0):=x0$ x(-1):=x1$ x(-2):=x2$ x(-3):=x3$
frame e$ % generate a frame e(-a) dual to o(a)
e(-a) _| o(-b); %-> gll(a,b)

The first example introduces a coframe index by indices running over 0, 1, 2 in the Euclidean (because no metric is specified) 3D space. The second example introduces alphabetic(!) indices running over r, p and defines the anhonomic coframe with respect to holonomic coordinate derivatives (dr,dφ). Thus, also r and phi are introduces as coordinates (e.g. enabling to write @ r for the radial vector). The 3rd example introduces the Schwarzschild geometry with diagonal metric and anholonomic coframe.

The 4th example introduces a general coframe with arbitrary metric. The metric (low-low) components have been predefined as symmetric operator. Since the coframe command does not accept for constructions, the variable metric_def was used as alias for the metric expression. It is better to use let rules to predefine expressions that will be used in the coframe definition. The 5th example introduces a holonomic coframe with arbitrary metric, coordinate basis x(i), and holonomic frame @ x(a) which coincides with e(-a).

The command

frame e$

introduces the name e for the respective frame (defined via the metric by $e_i$|$o^j$ = $\delta^j_i$ or $e_i$|$o_j$ = $g_{ij}$). Thus, the last example of the coframe definitions will yield

e(-a) _| o(-b) % throws gll(a,b)

Specifying the symmetries of tensors (or tensor indices on forms) saves Reduce from calculating redundant components and thus saves time. Excalc provides the index_symmetries command for this. Note that if no metric is specified before the index_symmetries command assumes an Euclidean metric! The syntax given in the table is

index_symmetries tensor: [symmetric [in index_pairs]] [antisymmetric [in index_pairs]];

Here, index_pairs can be {i,j} etc. but also pairs of pairs of indices, i.e. {{i,j},{k,l}} which means that the pair {i,j} (anti-)commutes with the pair {k,l}. The following example will clarify this:
INDEX SYMMETRIES $t(i,j)$: symmetric

INDEX SYMMETRIES $r(k,l,m,n)$: symmetric in $\{k,l\}, \{m,n\}$ antisymmetric in $\{k,l\}, \{m,n\}$

These commands mean $t^i_j = t^j_i$ and $r^{klmn} = r^{lkmn} = r^{klnm} = -r^{mnkl}$.

The rest of the commands work rather canonical and can be understood by looking at some examples: The first example implements the Schwarzschild metric in the tensor calculus:

```
% file "exc2"
% The Schwarzschild metric in tensor formalism

load_package excalc$

% introducing the metric (the symbolic coframe is of no importance):
depend(f,r)$ depend(g,r)$
coframe o(0), o(1), o(2), o(3) with metric
g = f**2*o(0)**2 - g**2/f**2*o(1)**2 - r**2*o(2)**2 - r**2*sin(theta)**2*o(3)**2$

% introducing coordinates with running indices:
pform x(i)=0$
x(-0):=t$
x(-1):=r$
x(-2):=theta$
x(-3):=phi$

% calculating the Christoffel symbol:
pform chris(i,j,k)=0$
index_symmetries chris(i,j,k): symmetric in $\{i,j\}$
chris(-i,-j,-k) :=
(1/2)* (@(g(-j,-k),x(-i)) + @(g(-k,-i),x(-j)) - @(g(-i,-j),x(-k)))$

% the Riemannian curvature:
pform riem(i,j,k,l)=0$
index_symmetries riem(i,j,k,l): antisymmetric in $\{i,j\}, \{k,l\}$ symmetric in $\{i,j\}, \{k,l\}$
riem(-i,-j,-k,l) :=
@ (chris(-j,-k,l),x(-i)) + chris(-i,-m,l)*chris(-j,-k,m)
- @ (chris(-i,-k,l),x(-j)) - chris(-j,-m,l)*chris(-i,-k,m)$

% the Ricci tensor:
pform ric(i,j)=0$
index_symmetries ric(i,j): symmetric
ric(-i,-j) := riem(-m,-i,-j,m)$

% the Ricci scalar:
pform ricscalar=0$
ricscal := ric(-i,i);

let {g=>1,f=>sqrt(1-mm/r)}$
ric(-i,-j);

end$
```

The next example proves the Bianchi identity in a purely symbolic manner:

```
% file "exc3"
```
% Proving the Bianchi identity in the exterior calculus
%
load_package excalc$

indexrange 0,1,2,3$

% general introduction of a connection:
pform gamma(a,b)=1$

% definition of the curvature:
pform curv(a,b)=2$
curv(-a,b):=d gamma(-a,b) - gamma(-a,c) ^ gamma(-c,b)$$

% Bianchi identity:
d curv(-a,b) - gamma(-a,c) ^ curv(-c,b) + gamma(-c,b) ^ curv(-a,c);
% -> yields zero!
end$

The next example implements the Taub-NUT solution with electric charge in the exterior calculus (in the Kaluza-Klein formalism):

% nut.exi, Marc Toussaint, Cologne
% 99-Oct-19
%
% The Taub-NUT solution with electric charge

load_package excalc$
off gcd,exp$

% COFRAME %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear t,r,theta,phi,o,e,f,rho,dtau,dsigma,m,n,q$
pform f=0,rho=0$
fdomain f=f(r),rho=rho(r)$
let { dtau => d t - 2*n*cos(theta)*d phi,
   dsigma => (r**2+n**2)*d phi }$
dim:=5$
coframe o(0) = f/rho* dtau,
   o(1) = rho/f* d r,
   o(2) = rho * d theta,
   o(3) = 1/rho*sin(theta)* d sigma,
   o(5) = d w + r/rho**2* q*dtau
with signature (1,-1,-1,-1,-1)$
frame e$

% ANSATZ %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
f := sqrt(r**2-2*r*m-n**2+q**2/4)$
\[ \rho := \sqrt{r^2 + n^2} \]

\% FIELD EQUATION

\% Christoffel symbol:
\[
\text{pform} \ chris1(a, b) = 1$
\]
\[
\text{Riemannconx} \ chris1$
\]
\[
chris1(a, b) := chris1(b, a)$
\]

\% Riemannian curvature and Einstein 3-form:
\[
\text{pform} \ rie2(a, b) = 2, \text{einstein3}(a) = \text{dim} - 1$
\]
\[
rie2(-a, b) := d \ chris1(-a, b) + chris1(-c, b)^{chris1(-a, c)}$
\]
\[
einstein3(a) := 1/2*#(o(a)^o(b)^o(c))^{rie2(-b, -c)}$
\]

\% Result:
\[
einstein3(a); \quad \rightarrow \text{zero (except the (5,5)-component)}$
\]

5.3 The tensor calculus in Reduce

The tensor calculus is realized in Reduce by manipulating arrays with elementary commands. Usually, the first step is to introduce countable (i.e. indexed) coordinates as an operator (giving the coordinate for each index). Next one can introduce a metric and calculate the inverse to be able to raise and lower indices. Then tensors are introduced as arrays and all operations are performed by explicitly running over the indices (with the for command). Whether the array contains covariant or contravariant components is specified by adding the characters 'l' for lower and 'u' for upper indices to the name. The following example will follow this scheme to calculate the curvature of a general static, symmetric geometry in polar coordinates and thereby derive the Schwarzschild metric.

\% indexed polar coordinates:
\[
\text{operator} \ x$
\]
\[
x(0) := t, x(1) := r, x(2) := \theta, x(3) := \phi$
\]

\% the metric (gll) with two lower and its inverse (guu) with two upper indices:
\[
\text{array} \ gll(3, 3), guu(3, 3)$
\]
\[
\text{static, spherically symmetric ansatz, signature: } + - - - :$
\]
\[
depend(f, r)$
\]
\[
depend(g, r)$
\]
\[
gll(0, 0) := f^2, \ gll(1, 1) := -g^2/f^2, \ gll(2, 2) := -r^2, \ gll(3, 3) := -r^2*(\sin(\theta))^2$
\]

\% calculating the inverse by using Reduce’s matrix calculus:
\[
\text{matrix} \ m(4, 4)$
\]
\[
\text{for} \ i := 0:3 \ \text{do for} \ j := i:3 \ \text{do} \ m(j+1, i+1) := m(i+1, j+1) := gll(i, j)$
\]
\[
m := 1/m$
\]
\%

% calculates the inverse

% results

\[
einstein3(a); \quad \rightarrow \text{zero (except the (5,5)-component)}$
\]
for i:=0:3 do for j:=i:3 do guu(j,i):=guu(i,j):=m(i+1,j+1)$
clear m$

% calculating the Christoffel symbol of type low-low-low and low-low-up:
array chrislll(3,3,3),chrisllu(3,3,3)$
for i:=0:3 do for j:=i:3 do <<
  for k:=0:3 do chrislll(j,i,k) := chrislll(i,j,k) :=
    (1/2) * (df(gll(j,k),x(i)) + df(gll(k,i),x(j)) - df(gll(i,j),x(k)))$
  for k:=0:3 do chrisllu(j,i,k) := chrisllu(i,j,k) :=
    for m:=0:3 sum guu(k,m) * chrislll(i,j,m)$
>>$

% calculating the Riemannian curvature (neglecting symmetries \rightarrow inefficient!):
array riemlllu(3,3,3,3)$
for i:=0:3 do for j:=0:3 do for k:=0:3 do for l:=0:3 do
  riemlllu(i,j,k,l):= df(chrisllu(j,k,l),x(i)) - df(chrisllu(i,k,l),x(j))
  + for m:=0:3 sum ( chrisllu(i,m,l)*chrisllu(j,k,m)
    - chrisllu(j,m,l)*chrisllu(i,k,m) )$

% the Ricci tensor:
array ricll(3,3)$
for i:=0:3 do for j:=0:3 do
  ricll(i,j) := for k:=0:3 sum riemlllu(k,i,j,k)$

% the Ricci scalar:
ricscalar := for k:=0:3 sum for l:=0:3 sum guu(l,k)*ricll(k,l)$

{ricll(0,0),ricll(1,1),ricll(2,2),ricll(3,3)} where {g=>1,f=>sqrt(1-mm/r)};
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