Optimal dividend problem for a generalized compound Poisson risk model

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May 9, 2013

Abstract

In this note we study the optimal dividend problem for a company whose surplus process, in the absence of dividend payments, evolves as a generalized compound Poisson model in which the counting process is a generalized Poisson process. This model including the classical risk model and the Pólya-Aeppli risk model as special cases. The objective is to find a dividend policy so as to maximize the expected discounted value of dividends which are paid to the shareholders until the company is ruined. We show that under some conditions the optimal dividend strategy is formed by a barrier strategy.

Key Words: Barrier strategy, Optimal dividend strategy, Generalized compound Poisson risk model, Stochastic control.

Mathematics Subject Classification (2000): 60J99 · 93E20 · 60G51
1 INTRODUCTION

The Poisson distribution and the corresponding Poisson process are the most basic and widely used stochastic model for modeling discrete data, it may provide a poor fit in the presence of over-dispersion. For example, the use of the Poisson distribution as a model describing the number of claims caused by individual policyholders (e.g. in automobile insurance) during a certain period is usually rejected, since in practice the behavior of policyholders is heterogeneous. In such a case the standard Poisson model is inappropriate. For example, in collective risk theory, it is assumed that claims occur in bulk, where the number of bulks \( M_t \) occurring in \((0,t]\) follows a Poisson process with parameter \( \lambda \).

Each bulk consists of a random number of claims so that the total number of claims is of the form \( N_t = \sum_{i=1}^{M_t} X_i \), where \( \{X_i, i \geq 1\} \) denotes the number of claims in the \( i \)-th bulk.

The aggregate claim payments made up to time \( t \), called the generalized Poisson process, is given by \( \sum_{i=1}^{N_t} Y_i \), where \( \{Y_i, i \geq 1\} \) representing the individual claim amounts. In this paper, we formulate and solve an optimal dividends problem for a generalized Poisson risk model in which the aggregate claim payments is defined by a generalized Poisson process.

2 The model and main result

Consider the compound Poisson process \( \{N_t; t \geq 0\} \) defined on the filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P}) \) and, \( N_t = \sum_{i=1}^{M_t} X_i \), where \( \{X_i, i \geq 1\} \) are discrete independent and identically distributed random variables whose probability distribution is given by \( P(X_i = k) = p_k, k = 1, 2, \ldots \), and \( \{M_t, \geq 0\} \) is a homogeneous Poisson process with intensity \( \lambda > 0 \). Moreover, it is assumed that \( \{M_t\} \) and \( \{X_i\} \) are independent. In particular, when \( P(X_i = 1) = 1 \), the process \( \{N_t\} \) reduced to the homogeneous Poisson process with intensity \( \lambda > 0 \). The probability mass function of \( N_t \) is given by

\[
P(N_t = n) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} p_k^n, \quad n = 0, 1, 2 \ldots ,
\]

where \( p_k^n \) is the \( n \)-fold convolution of \( \{p_k\} \). In a few special cases it is possible to determine the probabilities \( P(N_t = n) \)'s explicitly.
Example 2.1 Suppose that $X_1, X_2, \cdots$ are geometrically distributed with parameter $1 - \rho$, where $\rho \in (0, 1)$, i.e.
\[
P(X_i = k) = (1 - \rho)\rho^{k-1}, k = 1, 2, \cdots.
\]
Then the compound Poisson process by geometric compounding leads to the Pólya-Aeppli process $\{N_t, t \geq 0\}$ with parameters $\lambda$ and $\rho$ (cf. Minkova (2004)). That is for all $t \geq 0$,
\[
P(N_t = n) = \begin{cases} 
e^{-\lambda t}, & \text{if } n = 0, \\ ne^{-\lambda t} \sum_{i=1}^{n} \left( \frac{n-1}{i-1} \right) \left( \frac{1-\rho}{\rho} \right)^i \rho^{n-i}, & \text{if } n = 1, 2, \cdots. \end{cases} \tag{2.2}
\]
Note that the Pólya-Aeppli process is a time-homogeneous process, it is also called Poisson-geometric process in Chinese literature, for example see Mao and Liu (2005), where the ruin probability was studied for compound Poisson-geometric process. In the case of $\rho = 0$, the Pólya-Aeppli process becomes a homogeneous Poisson process.

Example 2.2 (Quenouille (1949)) Let $\{X_i, i \geq 1\}$ denote a sequence of independent and identically distributed random variables, each one having the logarithmic distribution (also known as the logarithmic series distribution) $\ln(\theta)$, with probability mass function
\[
P(X_i = n) = \theta^n / (-n \ln(1 - \theta)), n = 1, 2, \cdots, 0 < \theta < 1.
\]
Suppose that $M_t$ has a Poisson process with parameter $\lambda = -r \ln(1 - \theta)$. Then the random sum
\[
N_t = \sum_{i=1}^{M_t} X_i,
\]
has the negative binomial distribution $NB(rt, \theta)$:
\[
P(N_t = n) = \binom{n + rt - 1}{n} (1 - \theta)^rt \theta^n, n = 0, 1, 2, \cdots. \tag{2.3}
\]
In this way, the negative binomial distribution is seen to be a compound Poisson distribution.

Consider the risk model $\{X(t), t \geq 0\}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, P)$ and,
\[
X(t) = x + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i, \tag{2.4}
\]
where \( \{W_t; t \geq 0\} \) is a standard Brownian motion with \( W_0 = 0 \), \( \{N_t; t \geq 0\} \) is a generalized Poisson process defined as (2.1) and the claim sizes \( \{Y_i; i \geq 1\} \) are positive independent and identically distributed random variables whose probability distribution function is given by \( P(y) \). Moreover, it is assumed that \( \{W_t\}, \{N_t\} \) and \( \{Y_i\} \) are mutually independent.

Note that
\[
\psi(s) := \ln E e^{s(X(1) - x)} = cs + \frac{1}{2} \sigma^2 s^2 + \lambda \int_0^\infty (e^{-sz} - 1) dF(z), \Re(s) \geq 0,
\]
where
\[
F(z) = \sum_{k=1}^\infty p_k P^{*k}(z). \tag{2.5}
\]
Here \( P^{*k} \) is the \( k \)-fold convolution of \( P \) with itself. Obviously, \( F \) is the distribution function of random sum \( \sum_{k=1}^\nu Y_k \), where \( \nu \) is positive integer random variable distributed as \( X_1 \) and independent of \( \{Y_k, k \geq 1\} \).

We now consider the classical optimal dividend control problem. Let \( \pi \) be a dividend strategy consisting of a non-decreasing left-continuous \( \mathbb{F} \)-adapted process \( \pi = \{L_t^\pi, t \geq 0\} \) with \( L_0^\pi = 0 \), where \( L_t^\pi \) represents the cumulative dividends paid out by the company till time \( t \) under the control \( \pi \). We define the controlled risk process \( U_\pi^t = \{U_t^\pi, t \geq 0\} \) by
\[
U_t^\pi = X(t) - L_t^\pi.
\]
Let \( \tau^\pi = \inf\{t > 0 : U_t^\pi < 0\} \) be the ruin time and define the value function of a dividend strategy \( \pi \) by
\[
V_\pi(x) = E \left[ \int_0^{\tau^\pi} e^{-qt} dL_t^\pi(s) \bigg| U_0^\pi = x \right],
\]
where \( q > 0 \) is an interest force for the calculation of the present value. Let \( \Xi \) be the set of all admissible dividend strategies, that is all strategies \( \pi \) such that \( L_t^\pi - L_t^{\pi_0} \leq U_t^\pi \) for \( t < \tau^\pi \). The objective is to solve the following stochastic control problem:
\[
V(x) = \sup_{\pi \in \Xi} V_\pi(x), \tag{2.6}
\]
and to find an optimal policy \( \pi^* \in \Xi \) that satisfies \( V(x) = V_{\pi^*}(x) \) for all \( x \geq 0 \).

This optimization problem goes back to de Finetti (1957), who considered a discrete time random walk with step sizes \( \pm 1 \). For the compound Poisson model, this problem
was solved by Gerber (1969), identifying so-called band strategies as the optimal ones. For exponentially distributed claim sizes this strategy simplifies to a barrier strategy. Azcue and Nuler (2005) follows a viscosity approach to investigate optimal reinsurance and dividend strategies in the Cramér-Lundberg model. Albrecher and Thonhauser (2008) showed that the optimality of barrier strategies in the classical model with exponential claims still holds if there is a constant force of interest. Avram et al. (2007) considered the case where the risk process is given by a general spectrally negative Lévy process and gave a sufficient condition involving the generator of the Lévy process for optimality of the barrier strategy. Loeffen (2008) showed that barrier strategy is optimal among all admissible strategies for general spectrally negative Lévy risk processes with completely monotone jump density, and Kyprianou et al. (2010) relaxed this condition on the jump density to log-convex. An alternative proof is given in Yin and Wang (2009). Loeffen and Renaud (2010) pushed this result further by assuming the weaker condition that the Lévy measure has a density which is log-convex. Azcue and Muler (2010) examines the analogous questions in the compound Poisson risk model with investment.

Before starting our main results, we introduce the definitions of log-convexity and complete monotonicity.

**Definition 2.1.** (Willmot and Lin (2001)). (1) A distribution \( \{P_n\} \) on the non-negative integers is said to be log-convex if \( P_n^2 \leq P_{n+1}P_{n-1}, n = 1, 2, \cdots \), and \( \{P_n\} \) is said to be strictly log-convex if \( P_n^2 < P_{n+1}P_{n-1}, n = 1, 2, \cdots \). A counting distribution \( \{r_n, n \geq 0\} \) is discrete completely monotone iff it is a mixture of geometric distributions, i.e.

\[
r_n = \int_0^1 (1 - \theta)\theta^ndU(\theta),
\]

where \( U \) is a probability distribution on \((0, 1)\).

(2) A function \( f : \mathbb{R} \to \mathbb{R}^+ \) is log-convex if \( \log f(x) \) is a convex function. Let \( f \in C^\infty(0, \infty) \) with \( f \geq 0 \). We say \( f \) is completely monotone if \( (-1)^nf^{(n)} \geq 0 \) for all \( n \in \mathbb{N} \).

(3) The distribution function \( G(x) \) is said to be decreasing (increasing) failure rate or DFR (IFR) if \( \frac{G(x+y)}{G(y)} \) is nondecreasing (nonincreasing) in \( y \) for fixed \( x \geq 0 \), i.e.
G(y) is log-convex (log-concave).

Note that the completely monotone class is a subclass of the log-convex. For examples of continuous log-convex or completely monotone functions can be found in Yin and Wang (2009). Now, we give a discrete example.

**Example 2.1** Let \( N \) be a logarithmic random variable with
\[
p_n = P(N = n) = \frac{\theta^{n+1}}{-(n+1)\log(1-\theta)}, \quad n = 0, 1, 2, \ldots, 0 < \theta < 1.
\]
Then \( \{p_n\} \) is completely monotone (see van Harn (1978, P. 58)). The generalized logarithmic series distribution is defined by
\[
r_n = \frac{\Gamma(\beta n+1)}{\beta n \Gamma(\beta n-n+1) \Gamma(n+1)} \theta^{n+1} / (-\log(1-\theta)), \quad n = 1, 2, \ldots,
\]
with \( \beta \geq 1 \) and \( 0 < \theta < \beta^{-1} \). Then \( \{r_n, n \geq 1\} \) is strictly log-convex (see Hansen and Willekens (1990)).

Denote by \( \pi_b = \{L_t^b, t \geq 0\} \) the constant barrier strategy at level \( b \) which is defined by
\[
L_0^b = 0 \quad \text{and} \quad L_t^b = \left( \sup_{0 \leq s < t} X(s) - b \right) \vee 0
\]
for all \( t > 0 \). We will now present the main results of this note which give sufficient conditions for optimality of the barrier strategy \( \pi_b^* \). It is important to note that various dividend strategies can be employed by an insurance company. However, we will only focus on the conditions for the optimality of a dividend barrier strategy. Define
\[
b^* = \{b \geq 0 : W^{(q)}(b) \leq W^{(q)}(x), \ x \geq 0\},
\]
where \( W^{(q)}(x) \) is the called the \( q \)-scale function defined in such a way that \( W^{(q)}(x) = 0 \) for all \( x < 0 \) and on \([0, \infty)\) its Laplace transform is given by
\[
\int_0^\infty e^{-sx} W^{(q)}(x) dx = \frac{1}{\psi(s) - q}, \quad s > \rho(q).
\]
Here, \( \rho(q) \) is the unique root of equation \( \psi(s) - q = 0 \) in the half-plane \( \Re(s) \geq 0 \).

**Theorem 2.1.** For model (2.4), if \( P \) has a completely monotone probability density function on \((0, \infty)\) and \( \{p_n, n \geq 0\} \) is discrete completely monotone, then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).
Theorem 2.2. For model (2.4), if \( \{p_n, n \geq 1\} \) is discrete completely monotone and \( P \) is DFR, then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).

Corollary 2.1. For model (2.4) with \( N_t \) given by (2.2) or (2.3), if \( P \) is DFR, then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).

Theorem 2.3. For model (2.4), if \( \{p_n, n \geq 1\} \) is a log-convex probability mass function and \( P \) is the exponential distribution function with mean \( 1/\beta \), then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).

3 Proof of main results

Before proving the main results, we give several lemmas.

Lemma 3.1. (Loeffen (2008) Suppose that the Lévy measure of a spectrally negative Lévy process \( X \) has a completely monotone density on \( (0, \infty) \), then the barrier strategy at \( b^* \) is an optimal strategy.

Kyprianou, Rivero and Song (2010) providing weaker conditions on the Lévy measure for the optimality of a barrier strategy. An alternative approach can be found in Yin and Wang (2009).

Lemma 3.2. Suppose that a spectrally negative Lévy process \( X \) has a Lévy density \( \pi \) on \( (0, \infty) \) that is log-convex, then the barrier strategy at \( b^* \) is an optimal strategy.

Note that for the Cramér-Lundberg model with or without a Brownian component, the requirement of log-convexity of the Lévy density \( \pi \) on \( (0, \infty) \) is equivalent to the log-convexity of the probability density function of the individual claim amount on \( (0, \infty) \). Since the Lévy measure having a log-convex (or completely monotone) density implies that tail of the Lévy measure is log-convex and the converse is not true (cf. Loeffen and Renaud (2010)), the following result improves the results in Lemmas 3.1 and 3.2.
Lemma 3.3. (Loeffen and Renaud (2010)) Suppose that the tail of the Lévy measure of a spectrally negative Lévy process $X$ is log-convex, then the barrier strategy at $b^*$ is an optimal strategy.

The following two results are of independent interest, which can be seen as a partially answer to a conjecture of Cai and Kalashnikov (2000): If $\nu$ is a discrete DFR random variable, and the distribution of $X_i$ is DFR, then the random sum $\sum_{i=1}^{\nu} X_i$ is also DFR.

Lemma 3.4. If \{\(p_n, n \geq 1\}\} is discrete completely monotone and $P$ is DFR, then $F$ is also DFR.

**Proof** It is well known that the property of DFR is preserved under the geometric sum (see Shanthikumar (1988, Corollary 3.6)), and since the sum of two log-convex functions is log-convex and the limit of a pointwise convergent sequence of log-convex functions is log-convex, it follows that $F$ is also DFR.

Lemma 3.5. If \{\(p_n, n \geq 1\}\} is a log-convex probability mass function and $P$ is the exponential distribution function with mean $1/\beta$, then $F$ has a density which is logarithmically convex on $(0, \infty)$.

**Proof** If $P$ is the exponential distribution function with mean $1/\beta$, then

$$F(z) = \sum_{k=1}^{\infty} p_k \left( 1 - e^{-\beta z} \sum_{j=0}^{k-1} \frac{(\beta z)^j}{j!} \right).$$

Therefore,

$$\overline{F}(z) = \sum_{k=1}^{\infty} p_k \left( e^{-\beta z} \sum_{j=0}^{k-1} \frac{(\beta z)^j}{j!} \right).$$

Interchanging the order of summation yields

$$\overline{F}(z) = e^{-\beta z} \sum_{j=0}^{\infty} \overline{P}_j \frac{(\beta z)^j}{j!},$$

where

$$\overline{P}_j = \sum_{i=j+1}^{\infty} p_i.$$
Note that \( 1 = \overline{P}_0 \geq \overline{P}_1 \geq \overline{P}_2 \geq \cdots \) and \( p_{k+1}/p_k \) is increasing in \( k \), it follows from Theorem 3.2 in Esary and Marshall (1973) that \( F \) has a density which is logarithmically convex on \((0, \infty)\).

**Proof of Theorem 2.1.** If \( \{p_n, n \geq 1\} \) is discrete completely monotone and \textit{P} has a completely monotone density on \((0, \infty)\), then \textit{F} has a completely monotone density on \((0, \infty)\) (cf. Chiu and Yin (2013)). The result follows from Lemma 3.1.

**Proof of Theorems 2.2 and 2.3.** The result of Theorem 2.2 follows from Lemmas 3.3 and 3.4. The result of Theorem 2.3 follows from Lemmas 3.2 and 3.5.

**Remark 3.1.** At the end of this paper, we give two conjectures. The first conjecture can be viewed as an extension of Theorem 2.3; The second conjecture can be viewed as an extension of Conjecture 1 and Theorem 2.2, which is consistent with the conjecture that have put forward by Cai and Kalashnikov (2000).

**Conjecture 1** For model (2.4), if \( \{p_n, n \geq 1\} \) is a log-convex and \textit{P} has a density \( \pi \) on \((0, \infty)\) that is log-convex, then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).

**Conjecture 2** For model (2.4), if \( \{p_n, n \geq 1\} \) is DFR and \textit{P} is DFR, then the barrier strategy at \( b^* \) is an optimal strategy for stochastic control problem (2.6).

**Acknowledgements** The research was supported by the National Natural Science Foundation of China (No.11171179) and the Research Fund for the Doctoral Program of Higher Education of China (No. 20093705110002).

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