Method of integral transforms for few–body reactions

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Abstract

The method of integral transforms is reviewed. In the framework of this method reaction observables are obtained with the bound–state calculation techniques. New developments are reported.
I. INTRODUCTION

The approach discussed in the paper simplifies solving reaction problems. In particular, substantial simplifications occur at intermediate energy when many reaction channels are effective. No specification of reaction channels at solving the dynamic equations is required, and to obtain reaction observables only the bound–state type calculations are to be performed. One avoids the consideration of the complicated configuration–space asymptotic behavior of continuum wave functions or the complicated momentum–space singularities. In case of perturbation–induced inclusive reactions the cross section is obtained in a closed form, while the summation of contributions from various final states with the same energy is avoided. The long–range Coulomb final state interaction is automatically taken into account in this case.

In [1] it was suggested to compare with experiment the Stiltjes transform of response functions for perturbation–induced reactions, such as the \((e,e')\) reaction. The transform is directly obtained with the bound state few–body methods. A qualitative study of the Fourier transform of \((e,e')\) response functions was performed at the same time [2,3] but with no intention for a complete quantitative evaluation. In [4] the general procedure presented below was formulated for perturbation–induced inclusive and exclusive reactions, and for nonperturbative processes. Despite the successful solution of simple models, use of real values of the transform variable in that formulation imposed too high requirements to precision of the transform [5]. In [6] the real transform variable was changed to the complex one which substantially improved the accuracy of the inversion. The application of the method separately to various multipoles also increased the accuracy [6]. Successful applications to three– and four–nucleon \((e,e')\) and total photodisintegration cross sections were given in [7–12], and the four–nucleon spectral function was obtained along these lines [13]. In the response function case, a similar procedure accompanied with the statistical averaging is attempted in condensed–matter physics, see e.g. [14]. Different from the transforms considered below it is based on the Laplace transform calculable with the Monte–Carlo path integration which reliable inversion is a very difficult task.

II. OUTLINE OF THE METHOD

Let us consider "\(E\)–overlaps" defined as

\[
 r(E) = \int d\gamma \langle \chi_2 | \Psi_\gamma \rangle \langle \Psi_\gamma | \chi_1 \rangle \delta(E_\gamma - E) \tag{1}
\]

where the state \(\chi_1\) (but not necessarily \(\chi_2\)) is localized, i.e. its conventional norm is finite. The functions \(\Psi_\gamma\) are solutions to the Schrödinger equation: \((\hat{H} - E_\gamma)\Psi_\gamma = 0\), where \(\hat{H}\) is the Hamiltonian of a system. The set \(\Psi_\gamma\) is supposed to be complete and orthogonal. The notation \(\int d\gamma\) implies an integration over the continuum states plus summation over the bound states in the set. We thus have \(\int d\gamma |\Psi_\gamma\rangle \langle \Psi_\gamma | = 1\). The \(E\)–overlaps [4] are the basic quantities of the approach, and in this sense they replace continuum wave functions the conventional approach deals with.

The main points are the following. 1. Reaction observables are expressed in terms of the \(E\)–overlaps \(r(E)\). 2. The latter are computed indirectly\(^1\) with the bound state calculation techniques. In the next section we explain the first of these points, while in Sec. 4 and 5 the

\(^1\)The functions \(\Psi_\gamma\) are not used for obtaining \(r(E)\), and continuum wave functions thus do not enter the calculation at all.
second one is explained. Few–body calculations done so far with the method are outlined in Sec. 7. In Sec. 6 other non–conventional approaches having some common features with the present one are discussed.

Eq. (1) includes the case of response functions for perturbation–induced reactions. They arise if one sets in (1) $\chi_1 = \hat{O}_1 \Psi_0$, $\chi_2 = \hat{O}_2 \Psi_0$, and $E = E_0 + \epsilon$, where $\hat{O}_{1,2}$ are the transition operators, and $\Psi_0$ and $E_0$ are the initial–state wave function and energy:

$$R(\epsilon) = \langle 1/2 \rangle (x + x^*) ,$$

$$x = \int d\gamma \langle \hat{O}_2 \Psi_0 | \Psi_\gamma \rangle \langle \Psi_\gamma | \hat{O}_1 \Psi_0 \rangle \delta (E_\gamma - E_0 - \epsilon) .$$

In Eq. (2) $\Psi_\gamma$ and $E_\gamma$ are the final state wave functions and energies, respectively. The $\hat{O}_1 = \hat{O}_2$ case is typical.

The method proceeds in the following steps [4]. First some integral transform with a smooth kernel $K$ of the $E$–overlap $r(E)$, (or of the response $R(\epsilon)$),

$$\int K(\sigma, E) r(E) dE = \Phi(\sigma) ,$$

is calculated instead of $r(E)$ itself. To this aim, let us multiply Eq. (1) by $K(\sigma, E)$ and integrate over $E$. One obtains

$$\Phi(\sigma) = \int d\gamma \langle \chi_2 | \Psi_\gamma \rangle K(\sigma, E_\gamma) \langle \Psi_\gamma | \chi_1 \rangle$$

$$= \int d\gamma \langle \chi_2 | \hat{\hat{H}} \Psi_\gamma \rangle \langle \Psi_\gamma | \chi_1 \rangle .$$

Using the closure property one gets

$$\Phi(\sigma) = \langle \chi_2 | K(\sigma, \hat{\hat{H}}) \chi_1 \rangle .$$

Eq. (4) may be viewed as a generalized sum rule depending on a continuous parameter. The expression (4) is evaluated with the bound–state calculation techniques as described in Sec. 4. At the second step, Eq. (3) is considered as an integral equation to invert and thus one gets $r(E)$ from $\Phi(\sigma)$. In case (2) this provides a final result since the response function is an observable quantity. In the general case reaction observables are expressed as some quadratures in terms of $E$–overlaps (1) thus obtained, see the next section. (The reader interested only in the calculation of response functions may skip the next section except, perhaps, Eqs. (11)–(16). Note also that in the response function case the transform variable is changed to $\epsilon = E - E_0$.)

In the non–relativistic case, all the considered states, or operators may be viewed as ”intrinsic” ones.

III. REACTION OBSERVABLES IN TERMS OF $E$–OVERLAPS

First let us consider the case of transitions to continuum induced by a perturbation $\hat{O}$. The matrix element

$$M_{fi} = \langle \Psi_f(E) | \hat{\hat{O}} | \Psi_0 \rangle ,$$

is to be calculated where $\Psi_0$ is a localized initial state wave function, while $\Psi_f$ belongs to continuum. We set

$$\Psi_f(E) = \phi_f(E) + (E - \hat{\hat{H}} - i\eta)^{-1} \tilde{\phi}_f(E) .$$
Using Eq. (6), we rewrite (5) as

\[ \hat{\phi}_f = (\hat{H} - E)\phi_f. \]  

Here \( \phi_f \) is either the "channel" function that consists of a product of wave functions of fragments pertaining to a given reaction channel\(^2\) or the sum over such channel functions corresponding to (anti)symmetrization \([15]\). In the normal non-relativistic case one may use intrinsic channel functions that are angular momentum coupled products of intrinsic wave functions of fragments and of the relative motion function with a given orbital momentum. The functions \( \tilde{\phi}_f \) are

\[ \tilde{\phi}_f = (\hat{H} - E)\phi_f. \]  

Using Eq. (6), we rewrite (5) as

\[ M_{fi} = \langle \phi_f(E)|\hat{\Psi}_i \rangle + \langle \tilde{\phi}_f(E)|(E - \hat{H} + i\eta)^{-1}\hat{\Psi}_i \rangle. \]  

The first, Born, term in (8) is computed directly. The second term takes into account the final state interaction. We present this term in the form

\[ \sum_n (E - E_n)^{-1}F_{fi}(E_n) + \int_{E_{th}}^\infty (E - E' + i\eta)^{-1}F_{fi}(E')dE' \]

\[ = \sum_n (E - E_n)^{-1}F_{fi}(E_n) - i\pi F_{fi}(E) + P \int_{E_{th}}^\infty (E - E' - E_n)^{-1}F_{fi}(E')dE', \]  

where the form factor \( F_{fi} \) is defined as

\[ F_{fi}(E') = \int d\gamma\langle \tilde{\phi}_f(E)|\Psi_\gamma \rangle \langle \Psi_\gamma|\hat{\Psi}_i \rangle\delta(E_\gamma - E'). \]  

Eq. (10) is a special case of Eq. (1), and the reaction amplitude (5) is thus expressed in terms of a simple quadrature (8) over a \( E \)-overlap (4).

The above formulation is applicable when the long-range Coulomb interaction of fragments in a final state is neglected. In order to take into account this interaction one can proceed as follows. To simplify notation we omit (anti)symmetrization of the channel wave functions. (Changes to include (anti)symmetrization are obvious.) Let us write \( \hat{H} = T_f + V_f \) where \( V_f \) is the interaction of fragments in the final state. Let us subtract from \( V_f \) a part \( U_f \) that includes the long-range Coulomb interaction: \( V_f = \tilde{V}_f + U_f \). Let us denote \( \varphi_f \) the "plain wave plus going wave" continuum wave functions of the Coulomb-like Hamiltonian \( \hat{H}_f = T_f + U_f \). Instead of Eq. (9) let us use (cf. [15], Chap. 5, Eq. (108))

\[ \Psi_f = \varphi_f - (E - H + i\eta)^{-1}\tilde{V}_f\varphi_f. \]  

One can proceed as above with the replacement of the channel function \( \phi_f \) by the "Coulomb" channel function \( \varphi_f \), and \( \tilde{\phi}_f \) by \( \tilde{V}_f\varphi_f \), or, more generally, by \( (H - E)\varphi_f \) if \( \varphi_f \) is (anti)symmetrized. This procedure requires constructing \( \varphi_f \) which can be done exactly when the Coulomb interaction in only one pair of fragments is present.

The arguments similar to those presented above are valid if one replaces \( \hat{\Psi}_0 \) with any other localized state. Consider, for example, a breakup reaction amplitude in the framework of the Glauber approximation. It is given by the matrix element of the (approximate) fixed-center scattering amplitude between the initial and final state of the target. We thus encounter again a matrix element of the form \( \langle \Psi_f|\chi \rangle \), \( \Psi_f \) and \( \chi \) being the final state wave function and a known localized function, respectively. Proceeding as above in connection with Eq. (5), and

\(^2\)The quantity (8) equals \( V_f\phi_f \) in this case.
putting $\chi$ instead of $\hat{O}\Psi_0$ one replaces the calculation of the continuum wave function $\Psi_f$ by that of a $E$–overlap.

In Ref. [13] the spectral function of an $A$–body system is expressed in terms of an $(A-1)$–body $E$–overlap. Below the formulas are presented in more detail. The spectral function $P(k, E)$ represents the joint probability of finding a nucleon with momentum $k$ and a residual subsystem of $A - 1$ nucleons with energy $E$ in the ground state $\Psi_0^A$ at rest. The energy $E$ is calculated with respect to the ground state energy:

$$P(k, E) = (2J_0 + 1)^{-1} \sum_{M_0 = -J_0}^{J_0} \sum_{s_z = \pm 1/2} \int df dK_0 \left| \langle \psi_{fK_0}^{A-1}; k, s_z t_z | \Psi_0^A (J_0M_0) \rangle \right|^2 \delta(E - (E_f^{A-1} - E_0^A)).$$

(11)

Here $s_z$ and $t_z$ are the third components of the nucleon spin and isospin, $E_f^{A-1} = M_f^{A-1} - (A - 1)M$, $E_0^A = M_0^A - AM$, where $M_f^{A-1}$ and $M_0^A$ are the masses of the final residual state and the initial state, and $M$ is the nucleon mass. The functions $\psi_{fK_0}^{A-1}$ form the complete set of eigenstates of the $A - 1$ subsystem with the total momentum $K_0$:

$$\psi_{fK_0}^{A-1} = \int dk_1 \ldots dk_{A-1} \varphi_{f}^{A-1}(k_1, \ldots, k_{A-1}) \delta(\sum_{i=1}^{A-1} k_i - K_0) |k_1 \ldots k_{A-1}).$$

(12)

The ground state can be written as

$$\Psi_0^A = \int f(P) dP \int dk_1 \ldots dk_A \varphi_0^A(k_1, \ldots, k_A) \delta(\sum_{i=1}^{A} k_i - P) |k_1 \ldots k_A),$$

(13)

where $f(P)$ represents a narrow peak concentrated in the vicinity of $P = 0$, and $\int |f(P)|^2 dP = 1$. In Eqs. (12), (13) and below we omit spin and isospin variables.

In terms of the wave functions entering (12), (13) the spectral function can be written as

$$P(k, E) = (2J_0 + 1)^{-1} \sum_{M_0 = -J_0}^{J_0} \int df \delta(E - (E_f^{A-1} - E_0^A))W_f(k),$$

(14)

$$W_f(k) = \left| \int dk_1 \ldots dk_{A-1} \varphi_{f}^{A-1*}(k_1, \ldots, k_{A-1}) \varphi_0^A(k_1, \ldots, k_{A-1}, k_A) \delta(\sum_{i=1}^{A-1} k_i + k) \right|^2$$

(15)

with $k_A = k$.

For the $A - 1$ subsystem and for the $A$ system in Eq. (15) let us perform orthogonal transformations to Jacobi momenta

$$\pi_1, \ldots, \pi_{A-2}, \Pi_{A-1} = (A - 1)^{-1/2} \sum_{i=1}^{A-1} k_i; \quad \text{and} \quad \pi_1, \ldots, \pi_{A-2}, \pi_{A-1}, \Pi_A = A^{-1/2} \sum_{i=1}^{A} k_i,$$

respectively. Here $\pi_{A-1} = [(A - 1)/A]^{1/2}[k_A - (A - 1)^{-1} \sum_{i=1}^{A-1} k_i]$. Eq. (13) takes the form

$$W_f(k) = [A/(A - 1)]^{3/2} \left| \int d\pi_1 \ldots d\pi_{A-2} \tilde{\varphi}_{f}^{A-1*}(\pi_1, \ldots, \pi_{A-2}, -k) \tilde{\varphi}_{0}^{A}(\pi_1, \ldots, \pi_{A-2}, \pi_{A-1}; 0) \right|^2$$

(16)

with $\pi_{A-1} = [A/(A - 1)]^{1/2} k$. Here $\tilde{\varphi}_{f}^{A-1} = (A - 1)^{-3/4} \varphi_{f}^{A-1}, \tilde{\varphi}_{0}^{A} = A^{-3/4} \varphi_{0}^{A}$, and

$$\int d\pi_1 \ldots d\pi_{A-2} \tilde{\varphi}_{f}^{A-1*} \varphi_{f}^{A-1} = \delta(f - f'), \quad \int d\pi_1 \ldots d\pi_{A-1} |\varphi_{0}^{A}|^2 = 1.$$
The last arguments of the functions $\tilde{\phi}_A^{A-1}$ and $\tilde{\phi}_0^A$ are total momenta of the $A - 1$ subsystem and $A$ system, respectively. In the non–relativistic limit $\tilde{\phi}_A^{A-1}$ and $\tilde{\phi}_0^A$ cease to depend on these arguments. From Eqs. (14), (16) one sees that the spectral function is a parameter $k$ depending $E$–overlap in the subspace of $A - 1$ particle.

The amplitudes of nonperturbative reactions have been expressed in terms of $E$–overlaps in Ref. [4]. Usual processes with two fragments in an initial state and any number of fragments in final states were considered. Similar to Eq. (7) we shall use the functions
\[ \tilde{\phi}_i = (\hat{H} - E)\phi_i \quad \tilde{\phi}_f = (\hat{H} - E)\phi_f, \]

where $\phi_i$ and $\phi_f$ are defined in the same way as $\phi_f$ in (7). We rewrite the expression [15] for the $i \rightarrow f$ transition $T$ matrix in terms of $\tilde{\phi}_{i,f}$:
\[ T_{fi}(E) = T_{fi}^{\text{Born}} + \langle \tilde{\phi}_f(E) | (E - \hat{H} + i\eta)^{-1} \tilde{\phi}_i(E) \rangle, \]

where
\[ T_{fi}^{\text{Born}} = \langle \tilde{\phi}_f(E) | \phi_i(E) \rangle = \langle \phi_f(E) | \tilde{\phi}_i(E) \rangle \]
is the Born term.

We need to calculate the second term in Eq. (18) representing the final–state interaction correction. Similar to Eq. (14) we define a form factor
\[ F_{fi}(E, E') = \int d\gamma \langle \tilde{\phi}_f(E) | \Psi_\gamma \rangle \langle \Psi_\gamma | \tilde{\phi}_i(E) \rangle \delta(E_\gamma - E'), \]

and, finally, as in Eq. (8), we express the contribution in Eq. (18) we consider in terms of the quantity (13). Again, Eq. (19) is a special case of Eq. (1). Because there are two colliding fragments in the entrance channel the function $\tilde{\phi}_i$ is localized. The method is directly applicable when the long–range Coulomb interaction is neglected.

**IV. EVALUATION OF THE TRANSFORMS AND THE CHOICE OF THE TRANSFORM KERNEL**

Having expressed reaction observables in terms of $E$–overlaps, now we consider the calculation of the latter quantities using Eqs. (3), (4) of Sec. 2. One possibility is the following. Consider some complete set of localized states and denote $P_N$ the projector onto the subspace of the first $N$ states in the set. As $N$ tends to infinity at some conditions one has
\[ \langle \chi_2 | P_N K(\sigma, \hat{H}) P_N | \chi_1 \rangle \rightarrow \langle \chi_2 | K(\sigma, \hat{H}) | \chi_1 \rangle \]
uniformly with respect to $\sigma$. Therefore Eq. (3) can be approximated with the left–hand side of Eq. (20). Correspondingly, if one denotes by $\psi_n(N)$ the states that diagonalize the Hamiltonian in the subspace determined by the projector $P_N$ and by $E_n(N)$ the corresponding eigenenergies, then one has
\[ \Phi_N(\sigma) = \sum_{n=1}^N \langle \chi_2 | \psi_n(N) \rangle \langle \psi_n(N) | \chi_1 \rangle K(\sigma, E_n(N)) . \]

This expression can be used as an approximate right–hand side in the integral equation (3). Thus, the input to the integral equation is obtained with the bound–state calculation technique.
Eq. (21) is valid when both $\chi_1$ and $\chi_2$ are localized, and the kernel $K$ is bounded uniformly with respect to $\sigma$: $|K(\sigma, E)| < C$. If the latter is valid then the state $K(\sigma, \hat{H})\chi_1$ is localized along with $\chi_1$. Indeed, the norms of these two states are $\int d\gamma |K(\sigma, E, \gamma)|^2|\Psi_\gamma(\phi_1)|^2$ and $\int d\gamma |\Psi_\gamma(\phi_1)|^2$, respectively. Since the latter is finite, the former is finite as well. Taking this into account, one sees that the quantities $\langle \phi_2|Q_N K\phi_1\rangle$, $\langle \phi_2|KQ_N\phi_1\rangle$, and $\langle \phi_2|Q_N KQ_N\phi_1\rangle$ tend to zero at $N \to \infty$, $Q_N$ being $1 - P_N$. This gives Eq. (21).

The exact solution with the right-hand side (24) is

$$r_N(E) = \sum_{n=1}^N \langle \chi_2|\psi^{(N)}_n\rangle \langle \psi^{(N)}_n|\chi_1\rangle \delta(E^{(N)}_n - E).$$

While $\Phi_N(\sigma) \to \Phi(\sigma)$ as $N$ tends to infinity, the amplitudes entering Eq. (22) are rather chaotic in the continuous spectrum region $E > E_{th}$, and $r_N(E)$ thus does not tend to the true $r(E)$ in the normal sense. However, if approximate solutions to the integral equation are sought for in the class of smooth functions then the sequence of the approximate solutions $r_N^{smooth}(E)$ will tend to the true $r(E)$ as $N$ goes to infinity. (See also the next section.)

In other words, if one averages both $r(E)$ and $r_N(E)$ over $E$ using a smooth kernel $K(\sigma, E)$ then the corresponding ”averaged $E$–overlaps” $\Phi$ and $\Phi_N$ should be close to each other provided that $N$ is high enough. If $K$ effectively vanishes for $E$ values beyond some range $\Delta E$ (depending on $\sigma$), the case most interesting for us, one can expect that the two averaged $E$–overlaps will be close to each other when a sufficient number of levels $E^{(N)}_n$ fall into $\Delta E$. Then one can use the averaged discretized $E$–overlap $\Phi_N(\sigma)$ to reconstruct the true $E$–overlap $r(E)$ via solving Eq. (3) in an appropriate way.

The procedure is applicable with any smooth $K(\sigma, E)$, say $K = \exp\left[-(E - \sigma)^2/\sigma_0^2\right]$. On the other hand, it is restricted to the case when both $\chi_1$ and $\chi_2$ are localized. Besides, the complete diagonalization of a Hamiltonian is not manageable for larger systems. Let us consider the evaluation of the transform with the help of Eq. (4) without the above restriction and without having recourse to the diagonalization of a Hamiltonian. Unlike the considerations above, this evaluation will be done for the kernels of a special form. We set first

$$K(\sigma, E) = (E + \sigma)^{-1}.$$ (23)

According to Eq. (4), this leads to the transform

$$\Phi(\sigma) = \langle \chi_2|\hat{H} + \sigma\rangle^{-1}\chi_1\rangle.$$ (24)

Eq. (24) can be written as

$$\hat{\Phi}(\sigma) = \langle \chi_2|\hat{\Psi}(\sigma)\rangle,$$ (25)

where $\hat{\Psi}$ is the solution to the Schrödinger–like equation with a source,

$$\hat{\hat{H}} + \sigma \hat{\Psi} = \chi_1.$$ (26)

The $\sigma$ values we use are such that $-\sigma$ lie apart from the spectrum of the Hamiltonian so that $K$ is bounded. Due to this reason $\hat{\Psi}$ is localized along with $\chi_1$ as it is explained above in connection with Eq. (20). Due to the same reason there exists only one localized solution to Eq. (24): the difference of any two localized solutions would satisfy the homogeneous equation that has no solutions yet.

One can also use

$$K(\sigma, E) = [(E + \sigma^*) (E + \sigma)]^{-1}.$$ (27)
We consider here the case of the response function (2) and replace \( E \) with \( \epsilon = E - E_0 \) in Eq. (27). According to Eq. (4) the transform is

\[
\Phi(\sigma) = (1/2) \left[ \langle \hat{O}_2 \Psi_0 | (\hat{H} - E_0 + \sigma^*)^{-1} (\hat{H} - E_0 + \sigma)^{-1} \hat{O}_1 | \Psi_0 \rangle + \text{c.c.} \right].
\]  

(28)

The quantity (28) can be calculated as

\[
\Phi(\sigma) = (1/2) \left[ \langle \tilde{\Psi}_2(\sigma) | \tilde{\Psi}_1(\sigma) \rangle + \langle \tilde{\Psi}_1(\sigma) | \tilde{\Psi}_2(\sigma) \rangle \right],
\]  

(29)

where \( \tilde{\Psi}_i \) are solutions to

\[
(\hat{H} - E_0 + \sigma) \tilde{\Psi}_1 = \hat{O}_1 \Psi_0 \quad (\hat{H} - E_0 + \sigma) \tilde{\Psi}_2 = \hat{O}_2 \Psi_0.
\]  

(30)

The first of Eqs. (30) is, of course, Eq. (26) in the different notation. From Eqs. (29) at \( \hat{O}_1 = \hat{O}_2 \) it could be inferred once more that \( \tilde{\Psi}_i \) are localized. The general method we discuss has been given in Ref. [4] with use of the kernels (23) and (27) for real \( \sigma \) values. The corresponding integral transforms are called the Stieltjes and generalized Stieltjes transform [10]. We also note that if Eqs. (31) are solved with the help of an expansion of \( \tilde{\Psi} \) over a set of known functions then e.g. the quantity (28) thus obtained coincides with that in Eq. (21) with the kernel (27) provided that the diagonalization of a Hamiltonian is done in the subspace of retained basis functions.

So far we chose the kernel in a way that the evaluation of the transform, Eq. (4), is possible. Now let us discuss the choice of the kernel from the point of view of the stable reconstructing the response from the integral equation (4). At the same accuracy in \( \Phi \), the inversion of the transform (4) will be the most stable if the kernel is chosen as ”narrow” as possible. Indeed, at integrating in Eq. (4), variations of \( r \) occurring at intervals \( \Delta E \) that are smaller than the range of the kernel may smear out and even tend to cancel at some \( \sigma \). It then may be hard to reconstruct such changes in presence of numerical inaccuracies in \( \Phi \). The kernels (23) and (27) with real \( \sigma \) do not possess a ”finite” range and thus are unfavorable from this point of view. Use of complex \( \sigma = -\sigma_R + i\sigma_I \) in Eq. (27) has been suggested in the response function case [3] to increase the stability. The kernel then takes the ”Lorentz” form, and

\[
\sum_n \frac{R_n}{(\sigma_R - \epsilon_n)^2 + \sigma_I^2} + \int_{\epsilon_{th}}^{\infty} \frac{R(\epsilon)}{(\sigma_R - \epsilon)^2 + \sigma_I^2} d\epsilon = \Phi(\sigma).
\]  

(31)

Here the contribution from discrete levels is written down explicitly. For a stable reconstruction of the continuous part of the response it is advantageous to use mostly \( \sigma_R \geq \epsilon_{th} \), and \( \sigma_I \) should be chosen ”sufficiently small”. In some sense \( \sigma_I \) plays the role of the energy resolution: any two responses that noticeably differ from each other at \( \Delta \epsilon \) intervals not much smaller than \( \sigma_I \) lead to noticeably different transforms \( \Phi \) and thus can be discriminated even in presence of numerical uncertainties in \( \Phi \). But in order to distinguish between responses differing from each other in regions small compared to \( \sigma_I \) a higher numerical accuracy in \( \Phi \) than that required in \( R \) might be necessary.

The procedure similar to that used above to calculate the Lorentz transform in the response function case can also be used for other \( E \)-overlaps provided that in Eq. (4) both \( \chi_1 \) and \( \chi_2 \) are localized. The following procedure can be suggested in the general case. It is applicable for breakup reactions with three or more fragments in the final state when only \( \chi_1 \) is localized. Ons writes down the kernel (27) as

\[
K(\sigma, E) = (2i\sigma_I)^{-1} [(E + \sigma^*)^{-1} - (E + \sigma)^{-1}],
\]  

(32)

and, similar to Eq. (25), one has
\[ \Phi(\sigma) = (2i\sigma_I)^{-1}[(\langle \chi_2|\tilde{\Psi}'\rangle - \langle \chi_2|\tilde{\Psi}\rangle), \]

where \( \tilde{\Psi}' \) and \( \tilde{\Psi} \) are, respectively, the localized solutions to the equations

\[ (\hat{H} + \sigma^*)\tilde{\Psi}' = \chi_1 \quad (\hat{H} + \sigma)\tilde{\Psi} = \chi_1. \quad (33) \]

In the response function case one may replace \( E \) by \( \epsilon = E - E_0 \), and \( \hat{H} \) by \( \hat{H} - E_0 \) in Eqs. (32) and (33), respectively. The kernel (23) with complex \( \sigma = -\sigma_R + i\sigma_I \):

\[ K(\sigma, E) = \frac{[(E - \sigma_R) - i\sigma_I]/[(E - \sigma_R)^2 + \sigma_I^2]}{(34)} \]

is also presumably good from the point of view of the inversion stability.

We note that no information on exit channels of a reaction is required in order to solve the above listed dynamic equations. In the response function case, the long-range Coulomb interaction manifests itself only in Eqs. (30) and does not cause any problems.

To calculate the response function, Eq. (2), it is convenient to use the multipole expansion of the transform. This allows taking into account contributions from various projections of angular momenta in a general form. We perform the calculation in the reference system where the initial state \( \Psi_0 \) is at rest and carries a given total spin \( J_0 \) and its projection \( M_0 \). It will be denoted \( \Psi_{J_0M_0} \) here. We set \( \hat{O}_1 = \hat{O}_2 = \hat{O} \) in Eq. (2) and consider the response \( \bar{R} \) averaged over \( M_0 \) along with the corresponding transform \( \bar{\Phi} \):

\[ \bar{R} = (2J_0 + 1)^{-1} \sum_{M_0=-J_0}^{J_0} R(M_0) \quad \bar{\Phi} = (2J_0 + 1)^{-1} \sum_{M_0=-J_0}^{J_0} \Phi(M_0). \quad (35) \]

We expand the operator \( \hat{O} \) in a sum of irreducible tensor operators \( \hat{O}_{jm} \):

\[ \hat{O} = \sum_{jm} a_{jm} \hat{O}_{jm}, \quad (36) \]

where \( a_{jm} \) are the expansion coefficients. We define the partial right-hand sides

\[ q_{jM}^j = \sum_{m+M_0=M} C_{jmJ_0M_0}^J \hat{O}_{jm} \Psi_{J_0M_0}, \]

the partial solutions \( \tilde{\Psi}_{jM}^j \):

\[ (\hat{H} - E_0 + \sigma)\tilde{\Psi}_{jM}^j = q_{jM}^j, \quad (37) \]

and the partial transforms: \( \Phi_{jJ} = \langle q_{jM}^j|\Psi_{jM}^j\rangle \), or \( \Phi_{jJ} = \langle \tilde{\Psi}_{jM}^j|\tilde{\Psi}_{jM}^j\rangle \) for the case of Eq. (23) and Eq. (27), respectively. The latter quantities do not depend on \( M \). (These quantities are the corresponding transforms of the components \( R_{jJ} \) of response functions which correspond to transitions to the continuum states with given \( J M \) induced by the \( \hat{O}_{jm} \) multipole operators.)

One has

\[ \hat{O}\Psi_{J_0M_0} = \sum_{jmJ_0M_0} a_{jm} C_{jmJ_0M_0}^J q_{jM}^j \quad \Psi = \sum_{jmJ_0M_0} a_{jm} C_{jmJ_0M_0}^J \tilde{\Psi}_{jM}^j. \]

\[ ^3 \text{It is easy to conclude from Eqs. (32), (33) that in the response function case the ”squared Lorentz” transform } K^2(\sigma, E) \equiv [(E - E_0 + \sigma^*)(E - E_0 + \sigma)]^{-2} \text{ is also calculable in a simple form. The kernel } K^2 \text{ is more sharp peaked than } K \text{ which facilitates the inversion.} \]
Putting these expressions into Eq. (25) or (29), substituting this into the second of Eqs. (35), and then performing the summation of the products of Clebsh–Gordan coefficients one finally gets

$$\bar{\Phi} = (2J_0 + 1)^{-1} \sum_j (2J + 1) \Phi_j \sum_m (2j + 1)^{-1} |a_{jm}|^2.$$  

(38)

We also note that in Eq. (37)

$$\langle \varphi_{JM} | q^J_{JM} \rangle = (2J + 1)^{-1/2} (\varphi_{J} || \hat{O}_J || \Psi_{J0})$$

where \( \varphi_{JM} \) is a (basis) state with given \( J, M \) values, and the right–hand side includes a reduced matrix element (defined as in [17]).

If \( \hat{O}(q) \) is the charge operator then one has in Eq. (36) \( a_{jm} = Y_{jm}^* (\hat{q}) \), and the sum over \( m \) in Eq. (38) equals \((4\pi)^{-1}\). Let us consider the transversal current operator \( J_t(q) \). Eq. (36) should be modified to

$$J_t(q) = \sum_{jm, \lambda} Y_{jm}^{\lambda*} (\hat{q}) \hat{O}_{jm}^{\lambda*}$$

where \( Y_{jm}^{\lambda} \) are vector spherical harmonics [17], and the \( \lambda \) superscript signifies "electric" or "magnetic". The quantities defined above acquire the additional superscript \( \lambda \), and Eq. (38) takes the form

$$\bar{\Phi} = [4\pi(2J_0 + 1)]^{-1} \sum_{J, j, \lambda} (2J + 1) \Phi_{jJ}^{\lambda*}.$$  

Here the orthogonality property of vector spherical harmonics was used.

It was implied in the formulas of this section that the transition operators \( \hat{O}_{1,2} \) in Eq. (4) do not depend on the energy transfer \( \epsilon \). If a dependence on \( \epsilon \) occurs it leads to a dependence of \( \hat{O}_{1,2} \) on \( \hat{H} \) after performing the closure. Then the above methods for calculating the transform should be modified. In most of the cases the \( \epsilon \) dependence emerges only as a dependence of nucleon form factors on \( \epsilon \). The nucleon form factors can be factored out from a calculation, and the formulas of the above type then become applicable. Often it is sufficient to divide out only one general factor. For single–particle nuclear currents this takes place if one neglects the \( \epsilon \) dependence of the form factor ratio \( G_E^n/G_E^p \) or uses the similarity of the form factors \( G_M^n \) and \( G_M^p \). In case of zero isospin nuclei there is no interference terms between isovector and isoscalar components of the transition operator, and the same can be achieved without approximations by dividing out the isovector and isoscalar nucleon form factors.

**V. INVERSION OF THE TRANSFORM**

One could compare the calculated transform of a response function with the transform of experimental data on the response function [1,18,19], and this does not require inversion of the transform. However at such a comparison physics might be obscured by the fact that parts of different nature of a response are mixed up in the transform. In addition, with a given approximate Hamiltonian, often the comparison of a low–energy part of a response with experiment is only sensible. At the transform level, the comparison is then obscured by high–energy contributions. (These difficulties are less pronounced in case of sharply peaked kernels. But for such kernels inversion is also an easier task.) Therefore performing inversion is preferable, and below we consider procedures for solving Eq. (3).
First we disregard discrete contributions to the solution and thus consider the integral equation
\[ \int_{E_{th}}^{\infty} K(\sigma, E) r(E) dE = \Phi(\sigma). \] (39)

Afterwards we shall remove this restriction. In the following we assume that Eq. (39) has only one solution. This is obviously true for the kernel (23). Let us demonstrate that this is also the case for the Lorentz kernel. The difference \( \Delta r(E) \) of two possible solutions would satisfy the homogenous equation of the form
\[ \int_{a}^{\infty} dx \left[ (x - y)^2 + \beta^2 \right]^{-1} \Delta r(x) = 0. \]
This leads to
\[ \int_{a}^{\infty} dx \int_{a}^{\infty} dy \left[ (x - y)^2 + \beta^2 \right]^{-1} \Delta r(x) \Delta r(y) = 0. \] (40)

Using the representation
\[ (x^2 + \beta^2)^{-1} = (2\beta)^{-1} \int_{-\infty}^{\infty} dk \exp(-\beta|k| + ikx), \]
\( \beta > 0 \), one can rewrite the left-hand side of Eq. (40) as
\[ (2\beta)^{-1} \int_{-\infty}^{\infty} dk \exp(-\beta|k|)|\varphi(k)|^2 \] (41)
with
\[ \varphi(k) = \int_{-\infty}^{\infty} dx \exp(ikx) \theta(x - a) \Delta r(x). \]

The quantity (41) can equal to zero only if \( \varphi(k) = 0 \) that leads to \( \Delta r = 0 \).

In the calculations performed so far [4–13] the inversion procedure was the following. The solution was sought for in the form
\[ r(E) = \sum_{n=1}^{N} c_n \varphi_n(E, \alpha) \] (42)
where \( \chi_n \) are known functions forming a complete set. They include non-linear parameters \( \alpha \). If one substitutes the expansion (42) into the left-hand side of Eq. (3) one obtains \( \sum_{n=1}^{N} c_n \bar{\varphi}_n(\sigma, \alpha) \) where \( \bar{\varphi}_n \) are transforms of the basis functions. The expansion coefficients \( c_n \) and the parameters \( \alpha \) are obtained from the best fit requirement
\[ \sum_{k=1}^{K} \left| \Phi(\sigma_k) - \sum_{n=1}^{N} c_n \bar{\varphi}_n(\sigma_k, \alpha) \right|^2 = \text{min}. \] (43)

At fixed \( \alpha \) this leads to a system of linear equations for the expansion coefficients. (At large \( N \) this system may become ill-defined. Instead of solving it, the SVD algorithm for a direct minimization in (43), see e.g. [20], may be applied.)

In the Lorentz case, see [6–13] the values \( \sigma_k = -\sigma^k_R + i\sigma^k_I \) in (43) were chosen such that the \( \sigma^k_R \) points covered the interval \( \epsilon_{th} \leq \epsilon \leq \epsilon_{max} \) where \( \epsilon_{max} \) is such that \( R(\epsilon > \epsilon_{max}) \) is already very small. The \( \sigma^k_I \) points covered also some interval with \( \epsilon_{min} \leq \epsilon \leq \epsilon_{th} \) aiming a better description of the low-energy part of the response. The \( \sigma_I \) values were chosen taking
into account the following. When $\sigma_I$ is tending to zero $K(\sigma, \epsilon) \to (\pi/|\sigma_I|)\delta(\sigma_R - \epsilon)$, and $\Phi(\sigma) \to (\pi/|\sigma_I|)R(\sigma_R)$. Hence for sufficiently small $\sigma_I$ the required relative accuracy in $R$ is the same as in $\Phi$, and no additional uncertainty in $R$ arises due to the inversion procedure. On the other hand, the smaller $\sigma_I$ is the harder is to achieve this accuracy at solving the dynamic equation (30) with the bound–state type methods. Indeed, at $\sigma_I \to 0$ the scattering regime is recovered. For few–nucleon systems use of $\sigma_I$ values which are comparable to the widths of response functions and fall into the range between several MeV and 20 MeV was found to be the most convenient. (Of course, the final results should not depend on $\sigma_I$ that can serve as a possible check.) Values of $\sigma$ such that $\sigma = -\sigma_R + i\sigma_I(\sigma_R)$ with $\sigma_I$ increasing as $\sigma_R$ increases were found to yield good results, in particular.

Solutions to general equations of the form (30) are known to be unstable with respect to high frequency oscillations. Indeed, if $r(E)$ is the solution with an exact $\Phi$ then, for example, $r_a(E) = r(E) + \lambda \sin Et$ is the solution with the right–hand side

$$\Phi_a(\sigma) = \Phi(\sigma) + \lambda \int_{E_{th}}^{\infty} dE K(\sigma, E) \sin Et.$$  

At any $\lambda$ the variation $\Phi_a - \Phi$ of the right–hand side becomes indefinitely small at sufficiently large $t$. However, the corresponding variation $r_a - r = \lambda \sin Et$ of the solution may take values as large as $\lambda$.

Because of the above instability one should not seek for the exact solution of the equation with a given approximate right–hand side. Instead, as it is well–known, a regularization procedure should be applied at finding the solution. The regularization suppresses a quickly oscillating component in the solution with an approximate right–hand side but do not influence much a slowly varying component in the solution. If $r_a$ and $r$ are solutions with right–hand sides $\Phi_a$ and $\Phi$, respectively, and the difference $\Phi_a - \Phi$ is small then only a quickly oscillating component in the difference $r_a - r$ may have large amplitudes. Due to this the regularization guarantees the closeness of a regularized ”solution” to the exact one.

When one looks for a solution in the form (42) the number $N$ of retained functions plays the role of a regularization parameter. If a right–hand side $\Phi_a$ is accurate enough then, normally, the accuracy in the solution $r(E)$ first increases when $N$ increases, and the results look as being ”convergent” with respect to $N$. But at a further increase in $N$ the ”solution” obtained inevitably acquires unphysical oscillations. The higher accuracy in $\Phi$ is, the higher $N$ values are at which the oscillating regime starts, and the higher accuracy in $r$ achieved at lower $N$ values is.

Stability of inversion results with respect to $N$ served both for choosing $N$ values and as the main quality criterion of an approximate solution. To check the results, responses obtained with different sets of basis functions were compared to each other as well. (Such a comparison implies that the stability ranges $\Delta N$ exist for each of the sets). Stability with respect to an accuracy in the input $\Phi$, was, of course, also checked. The quality criterion related to sum rules was applied in addition. The following sum rules should approximately be satisfied by the solution $r(E)$:

$$\int_{E_{th}}^{\infty} r(E)dE = \langle \phi_2 | \phi_1 \rangle, \quad \int_{E_{th}}^{\infty} E r(E)dE = \langle \phi_2 | \hat{H} | \phi_1 \rangle. \quad (44)$$

Incorporation of the known threshold behavior of $r(E)$ into basis functions entering Eq. (42) increased the accuracy of inversion. Other features of the solution such as positions, or

\[^4\text{For e.g. three–cluster }^{11}\text{Li response convenient }\sigma_I\text{ values proved also be comparable to the width of the peak.}\]
widths, of narrow resonances (if known) should similarly be taken into account for the same purpose.

In some cases the accuracy can also be enhanced in the following way [6]. Let us consider, for example, the longitudinal response function for the \((e,e')\) reaction. It may behave in a rather peculiar way at small \(\epsilon\). The origin is that the lowest \(l\) multipole contributions \(R_l\) to \(R = \sum_l R_l\) have maxima in the threshold region while other multipoles exhibit maxima in the quasielastic peak region. It is helpful to apply the whole procedure separately to those lower \(R_l\) and to the sum of all other \(R_l\). The reason for an increase in accuracy is that with use of simple functions \(\phi_n\) in Eq. (42) one can better describe the behavior of those pieces of the response function at sufficiently low \(N\) values than that of the total response function.

The accuracy of the inversion results has been assessed in Ref. [6] using the two–nucleon \(d(e,e')\) longitudinal response as an example. An approximate Lorentz transform with a several per cent oscillating error was taken as an input. In Fig. 1 the response obtained via inversion of the transform at \(\sigma_I = 5\) MeV is shown (dashed curve) along with the exact response (solid curve) calculated in the conventional way. The quality of the inversion is excellent.

Regularization procedures different from that discussed above to solve the general integral equation of the first kind were applied in the literature, see e.g. [20–22]. In our case the following one, see [21,22], seems to be promising. Let us use the quantity

\[
\rho(f_1, f_2) = \left\{ \int_{\sigma_{\text{min}}}^{\sigma_{\text{max}}} |f_1(\sigma) - f_2(\sigma)|^2 d\sigma \right\}^{1/2},
\]

as a measure of proximity between \(f_1\) and \(f_2\). Let \(\Phi_a(\sigma)\) be an approximate right–hand side used as the inversion input. We suppose that

\[
\rho(\Phi_a, \Phi) \simeq \delta. \tag{45}
\]

The procedure requires estimating \(\delta\). For this purpose one can proceed as follows in our case. Let \(\nu\) be some parameter (e.g. the number of basis functions at solving Eqs. (26), (30)) that determines an accuracy in \(\Phi_a\) at a given \(\sigma\) value:

\[
\Phi_a(\sigma) = f(\nu), \quad \Phi(\sigma) = f(\infty).
\]

One can estimate the difference \(f(\nu) - f(\infty)\) and hence the quantity (45) performing the calculation at various \(\nu\) values, then approximating the results with some analytic formula, such as e.g. \(f(\nu) = f(\infty) + C\nu^{-\gamma}\), and fitting the parameters of the formula that include \(f(\infty)\). It is supposed that inversion results are not sensitive to a precision in \(\delta\) which normally is the case.

There is no sense to solve Eq. (39) with an accuracy higher than \(\delta\). Therefore one seeks for the solution in the class of functions \(r(E)\) satisfying the condition

\[
\rho(Kr, \Phi_a) \leq \delta \tag{46}
\]

where \(Kr\) is the short notation for the left–hand side of Eq. (39). We consider \(\Phi_a\) such that \(\int_{\sigma_{\text{min}}}^{\sigma_{\text{max}}} |\Phi_a(\sigma)|^2 d\sigma > \delta^2\), so that \(r = 0\) does not belong to the class (46) of functions. One wants to find an approximation to the true \(r(E)\) that is stable with respect to small variations in \(\Phi\). To this aim, the functional

\[
\Omega[r] = \int_{E_{\text{th}}}^\infty \left[ q(E) |r(E)|^2 + p(E) |dr/dE|^2 \right] dE \tag{47}
\]

is used, where \(q\) and \(p\) are non–negative functions. The function \(r(E)\) from the class (46) at which the functional \(\Omega[r]\) takes its minimal value is chosen as the approximate solution.
(The minimal value does exist.) If one adds a quickly oscillating component to the true \( r(E) \) then the contributions to \( \Omega[r] \) from both terms in the right-hand side increase. Therefore the function \( r(E) \) giving a minimum to \( \Omega[r] \) belongs to the “smoothest ones” among those satisfying Eq. (16). (In Eq. (17) the simplest choices \( q = 1, p = 0 \), or \( q = 0, p = 1 \) are, in general, reasonable.) It can be shown [22] that if one takes a sequence of \( \delta \) values tending to zero then the corresponding sequence of approximate solutions found in the above way would tend to the exact \( r(E) \). Thus at sufficiently small \( \delta \) closeness between the approximate \( r(E) \) constructed as described here and the exact \( r(E) \) is guaranteed.

It is easy to show [22] that \( r(E) \) giving a minimum to \( \Omega[r] \) satisfies the condition (46) with the equality sign:

\[
\rho(Kr, \Phi_a) = \delta. \tag{48}
\]

Therefore one can seek for the minimum of \( \Omega[r] \) at the additional condition (18). This is a classical variational problem which can be solved using the Lagrange multiplier method. One forms the functional

\[
\rho^2(Kr, \Phi_a) + \mu \Omega[r], \tag{49}
\]

finds its minima \( r_\mu \) at given \( \mu \) values and, finally, fixes \( \mu \) from the equation (18) with \( r = r_\mu \). One can show [22] that the corresponding solution \( \mu = \mu(\delta) \) to this equation necessarily exists.

In our case it is convenient to seek for minima of the functional (49) using the expansion of Eq. (12) type. At given \( \mu \) this leads to a system of linear equations for the expansion coefficients. If one puts \( \mu \) equal to zero this system of equations turns to that following from Eq. (13). In contrast to the case of Eq. (13), in the present case one can retain as many basis functions in the expansion as desired. As above to increase the accuracy of the inversion one should incorporate known features of a solution into basis functions.

Since the \( E \)-overlap (1) may include discrete contributions \( \sim \delta(E - E_n) \), the sum of the form \( \sum_n K(\sigma, E_n) r(E_n) \) is present in general in the left-hand side of Eq. (31). A good strategy is to calculate this sum beforehand which leads to Eq. (34) with a modified right-hand side.

It is convenient to calculate the required \( r(E_n) \) from the equation of (30) type

\[
(\hat{H} - E)\tilde{\psi} = \chi_1
\]

with real \( E \) ranging in the vicinity of the level \( E_n \). Then at \( E \to E_n \)

\[
\langle \chi_2 | \tilde{\psi}(E) \rangle \simeq (E_n - E)^{-1} r(E_n).
\]

Another possibility applicable in the framework of the method (13) is to solve the corresponding equation of Eq. (31) type directly with inclusion of the basis functions \( \delta(E - E_n) \) into the expansion (12).

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5This can be seen as follows. Suppose that \( r = r_0 \) gives a minimum to \( \Omega[r] \), and \( \rho(Kr_0, \Phi_a) < \delta \). Then there exist surroundings \( (\rho_E(r, r_0) < \epsilon) \) of \( r_0 \) such that \( \rho(Kr, \Phi_a) < \delta \) for any \( r \) belonging to the surroundings. For the function \( r = \gamma r_0, |\gamma| < 1 \), belonging to these surroundings one has \( \Omega[r] = \gamma^2 \Omega[r_0] < \Omega[r_0] \). Therefore the conditions \( \Omega[r_0] = \min \) and \( \rho(Kr_0, \Phi_a) < \delta \) are incompatible.
VI. SOME OTHER NON–CONVENTIONAL APPROACHES

In Refs. [23,24] the following method for calculating elastic scattering and ionization processes in atomic physics was suggested and used. One calculates the $T$–matrix (18) for a set of complex energies $E$ and then performs an extrapolation to real $E$ values. This leads to the dynamic equation of Eq. (26) form at a sequence of $\sigma_I$ values tending to zero. Such a method probably requires more labour to attain accurate results then the method of integral transforms. In contrast to what was said above concerning Eq. (42), within such a framework one cannot use the known information about a solution in order to improve accuracy. Probably this is the reason for unsatisfactory results obtained in [24] in the threshold region. Besides, while one needs the extrapolation $\sigma_I \to 0$, difficulties in obtaining accurate solutions to Eq. (26) with bound–state methods increase as $\sigma_I$ decreases.

In Ref. [25] the following method for calculating response functions is suggested. One can represent the $\delta$ function in Eq. (2) as $-(\pi)^{-1}\text{Im}(E - E_\gamma + i\eta)^{-1}$, and, correspondingly, the quantity (2) as (cf. (24), (25))

$$- (\pi)^{-1} \text{Im} \langle \hat{O}_2 \psi_0 | \bar{\psi}_1 \rangle,$$

$$\bar{\psi}_1 = (E - \hat{H} + i\eta)^{-1} \hat{O}_1 \psi_0.$$  

(50)

Here $E = E_0 + \epsilon$. Like the method of integral transforms, this procedure allows avoiding summation over final states at the calculation of a response function. However, in contrast to the state (26), or (30), the state (50) is not localized, and at large distances it includes outgoing waves in all open channels. The continuum-state calculation techniques are necessary to find this state which requires more effort then obtaining the localized state (26). The method was successfully applied for calculating three–nucleon responses [25,26].

VII. FEW–NUCLEON RESPONSES FROM THE LORENTZ TRANSFORM

These responses are smooth functions with single peaks and without pronounced additional structures which facilitates calculations. In Ref. [1] the longitudinal three–nucleon $(e,e')$ response functions were studied. Eq. (31) (for $\hat{O}_1 = \hat{O}_2$) was cast into the form of inhomogenous Faddeev–like equations, and the source term was split into the three parts in a way preserving the permutational symmetries of Faddeev components. Similar to Ref. [6], inversion was performed separately for the monopole contribution and for the sum of all other contributions to the response functions. The responses thus obtained compared well with those obtained from a conventional lengthy calculation, however the second above mentioned piece of the response function exhibited a small unphysical wiggle at low energy. When the wiggle was removed ”by hand” the comparison with the conventional calculation became perfect. Probably one could have come to the same results if the proper threshold behavior of that piece of the response ($((\epsilon - \epsilon_{th})^3/2$ in the $^3$H case) had been imposed. It was demonstrated that the final results are nearly independent of $\sigma_I$, as they should.

In Refs. [8,13] the three– and four–nucleon responses were obtained using an expansion technique in the dynamic equation (30) ($\hat{O}_1 = \hat{O}_2$). The $^4$He$(e,e')$ longitudinal response function [8], the total $^4$He [8], and tri–nucleon [10,12], photodisintegration cross sections, and the $^4$He spectral function along with the corresponding approximate $^4$He$(e,e')$ response [13] were studied. Both central semi–realistic [8,11,13] and realistic [11,12] NN interactions supplemented with a 3N force [12] were used. In [8,11,13] the correlated hyperspherical expansion [27] was applied to solve the problem. The basis functions were taken as products of the Jastrow factor, properly symmetrized hyperspherical harmonics coupled with the corresponding spin–isospin functions, and hyperradial functions. In Ref. [12] the basis function are constructed as
above mentioned products of spin–isospin–hyperspherical and hypperradial functions to which
a Jastrow type spin–isospin dependent operator is applied. The prescription for selection of
basis hyperspherical harmonics \[28\], used in older work with no Jastrow correlations, is applied
in conjunction with these correlations and is found to be very effective both for bound states
and at solving Eq. (30) [12].

It was found that the calculation of the transforms with an accuracy at a per cent level
provides stable response functions. Achievement of such an accuracy in the transforms proved
to be not requiring more effort than the calculation of binding energies with a comparable
accuracy. As an example, in Fig.2 the dependence of the Lorentz transform on the maximal
value of the hyperspherical number \( K \) retained in the calculation is shown in the case of the
four–nucleon photodisintegration response function. The calculation was done with the semi–
realistic Malfliet–Tjon potential. The transforms at \( K_{max} = 7 \) and at \( K_{max} = 5 \) are already
very close to each other. The sum rule checks (44) were satisfied by the responses obtained
with a per cent accuracy. At the same time, let us mention that all possible checks of inversion
results should be done, while the criterion of intermediate stability of the results with respect
to the number \( N \) of basis functions in the expansion (42) may prove to be insufficient by
itself [11]. Convergent 3–nucleon photodisintegration results with a realistic NN interaction
supplemented with a 3N force are presented in [12].

Finding practical ways to perform the calculations for larger systems, applying the approach
to exclusive processes and to responses with more complicated structures, taking into account
the long–range Coulomb interaction for exclusive reactions remains for future.

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REFERENCES

[1] Éfros, V.D., *Ukr. Fiz. Zh.*, 1980, vol. 25, p. 907 [Ukr. Phys. J.].
[2] Rosenfelder, R., *Phys. Lett. B*, 1978, vol. 79, p. 15; *Ann. Phys. (N.Y.)*, 1980, vol. 128, p. 188.
[3] Korchin, A.Yu. and Shebeko, A.V., *Z. Phys. A*, 1981, vol. 299, p. 131.
[4] Éfros, V.D., *Yad. Fiz.*, 1985, vol. 41, p. 1498 [Sov. J. Nucl. Phys. 41, 949 (1985)], for a shorter discussion see Efros, V.D., *Yad. Fiz.*, 1993, vol. 56, p. 22 [Phys. At. Nucl. 56, 869 (1993)].
[5] Efros, V.D., Leidemann, W., and Orlandini, G., *Few–Body Sys.*, 1993, vol. 14, p. 151.
[6] Efros, V.D., Leidemann, W., and Orlandini, G., *Phys. Lett. B*, 1994, vol. 338, p. 130.
[7] Martinelli, S., Kamada, H., Orlandini, G., and Glöckle, W., *Phys. Rev. C*, 1995, vol. 52, p. 1778.
[8] Efros, V.D., Leidemann, W., and Orlandini, G., *Phys. Rev. Lett.*, 1997, vol. 78, p. 432.
[9] Efros, V.D., Leidemann, W., and Orlandini, G., *Phys. Rev. Lett.*, 1997, vol. 78, p. 4015.
[10] Efros, V.D., Leidemann, W., and Orlandini, G., *Phys. Lett. B*, 1997, vol. 408, p. 1.
[11] Efros, V.D., Leidemann, W., and Orlandini, G., *Few–Body Sys.*, (in press).
[12] Leidemann W. et al. Proc. of the Conf. on Nuclear and Particle Physics with CEBAF at Jefferson Lab., Nov. 1998, Fizika (in press); Orlandini G. et al. Proc. of the 10th Amsterdam Mini–Conference, Jan. 1999 (in press), Efros V.D. et al. Proc. of the 2nd Int. Conf. on Perspectives in Hadronic Physics, May 1999 (in press).
[13] Éfros, V.D., Leidemann, W., and Orlandini, G., *Phys. Rev. C*, 1998, vol. 58, p. 582.
[14] Gubernatis, J.E., Jarrel, M., Silver, R.N., and Sivia, D.S., *Phys. Rev. B*, 1991, vol. 44, p. 6011; Boninsegni, M. and Ceperley, D.M., *J. Low. Temp. Phys.*, 1996, v. 104, p. 339.
[15] Goldberger, M.L. and Watson, K.W., *Collision Theory*, Wiley, New York, 1964.
[16] Widder, D.V., *The Laplace Transform*, Princeton University Press, Princeton, 1946 (Chap. 8).
[17] Varshalovich, D.A., Moskalev, A.N., and Khersonskii, V.K., *Quantum theory of angular momentum*, Nauka, Leningrad, 1975.
[18] Carlson, J. and Schiavilla, R., *Phys. Rev. C*, 1994, vol. 49, p. R2880.
[19] Dobretsov, V.Yu., Éfros, V.D., and Shao, Bin, *Yad. Fiz.*, 1995, vol. 58, p. 1601 [Phys. At. Nucl. 58, 1509 (1995)].
[20] Press, W.H., Teukolsky, S.A., Vettering, W.T., and Flannery, B.P., *Numerical recipies*, Cambridge University Press, Cambridge, 1992.
[21] Phillips, D.L., *J. Assoc. Comput. Mach.*, 1962, vol. 9, p. 84.
[22] Tikhonov, A.N. and Arsenin, V.Ya., *Solutions of ill–posed problems*, Halsted, New York, 1977 [Russ. original, later edition, Nauka, Moscow, 1986].
[23] Schlessinger, L. and Schwartz, C., *Phys. Rev. Lett.*, 1966, vol. 16, p. 1173; Schlessinger, L., *Phys. Rev.*, 1968, vol. 171, p. 1523; McDonald, F.A. and Nuttal, J., *Phys. Rev. Lett.*, 1969, vol. 23, p. 361.
[24] Doolen, G., McCarter, G., McDonald, F.A., and Nuttal, J., *Phys. Rev. A*, 1971, vol. 4, p. 108.
[25] Ishikawa, S., Kamada, H., Glöckle, W., Golak, J., and Witala, H., *Phys. Lett. B*, 1994, vol. 339, p. 293.
[26] Golak, J., Witala, H., Kamada, H., Hüber, D., Ishikawa, S., and Glöckle, W., *Phys. Rev. C*, 1995, vol. 52, p. 1216; Ishikawa, S., Golak, J., Witala, H., Kamada, H., Glöckle, W., and Hüber, D., *Phys. Rev. C*, 1998, vol. 57, p. 39.
[27] Fenin, Yu.I. and Éfros, V.D., *Yad. Fiz.*, 1972, vol. 15, p. 887. [Sov. J. Nucl. Phys. 15 497 (1972)].
[28] Éfros, V.D., *Yad. Fiz.*, 1972, vol. 15, p. 226 [Sov. J. Nucl. Phys. 15 128 (1972)].
FIGURES

The $d(e,e')$ longitudinal response function. Solid curve: the conventional two-nucleon calculation. Dashed curve: calculation from the inversion of the Lorentz integral transform ($\sigma_I = 5$ MeV).

The Lorentz transform $L(\sigma = -\sigma_R + i\sigma_I(\sigma_R))$ of the $^4$He photodisintegration response function at various $K_{max}$ values. The calculations were done for the Malfliet–Tjon NN potential. The $\sigma_I$ value ranged from 5 MeV to 20 MeV as $\sigma_R$ increased.
$R(E_{np}) \left[ 10^{-3} \text{ MeV}^{-1} \right]$
