Effective mass of the composite fermions and energy gaps of quantum Hall states

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Abstract

The effective mass of the quasiparticles in the fermion-Chern-Simons description of the quantum Hall state at half-filling is computed for electron-electron interactions \( V(r) \sim r^{x-2} \), for \( 0 < x < 3/2 \), following the previous work of Stern and Halperin [Phys. Rev. B 52, 5890 (1995)]. The energy gap of quantum Hall states with filling factors \( \nu = \frac{p}{2p+1} \) for \( p \gg 1 \) can then be obtained either from the effective mass at half-filling, as proposed by Halperin, et al. [Phys. Rev. B 47, 7312 (1993)], or evaluated directly from the self-energy of the system in presence of the residual magnetic field; both results are shown to agree as \( p \to \infty \). The energy gap is then given by a self-consistent equation, which asymptotic solution for \( p \gg 1 \) and short-range interactions is \( E_g(p) \sim (2p + 1)^{-\frac{3-x}{2}} \), in agreement with previous results by Kim et al. [Phys. Rev. B 52, 17275 (1995)]. The power law for the energy gap seems to be exact to all orders in the perturbation expansion. Moreover, the energy gap for systems with Coulomb interaction is recovered in the limit \( x \to 1 \).
I. INTRODUCTION

In a two-dimensional electron gas in the partially filled lowest Landau level, a predominant role is played by the interaction between the electrons, which sets the energy scale of the problem. While the qualitative properties of the quantum Hall states are mostly unaffected by the precise form of the interaction, the value of some quantities, as the energy gap in the incompressible Hall states, depends on the type of interaction. The physical relevant interaction between the electrons is the electrostatic Coulomb interaction. However, the study of systems having more general interactions of long or short ranges between the fermions is of great interest. In this work, we will consider the quantum Hall system at or near half-filling with different types of electron-electron interactions, using the fermion-Chern-Simons (FCS) approach. [1]

The Chern-Simons composite fermions are constructed attaching an even numbers $\phi$ of flux quanta to the electrons; [2] this transformation can be realized introducing an appropriate Chern-Simons gauge field. [3] As a consequence, the external magnetic field is partially screened, but a long ranged, gauge-mediated interaction between the new fermions has been introduced.

At the mean-field level, the system can be described in terms of free composite fermions in a constant effective magnetic field $\Delta B = B(1 - \phi \nu)$, where $\nu = 2\pi n \hbar c/(eB)$ is the filling factor and $n$ the electron density. Choosing $\phi = 2$ at half-filling ($\nu = 1/2$) gives $\Delta B = 0$; the ground state of the system is then the Fermi sea of composite fermions, with Fermi momentum $k_F = \sqrt{4\pi n}$. Along the main Jain series, which is determined by the filling factors $\nu = \frac{p}{2p+1}$ for integer $p$, the fractional quantum Hall states correspond to the integer quantum Hall states for composite fermions in the residual magnetic field $\Delta B = B/(2p+1)$, with $p$ Landau levels filled.

More generally, the mean-field picture predicts the occurrence of fractional quantum Hall states at filling factors that corresponds to integer filling factors for the composite Fermions. These predictions are believed to be correct, providing that the interaction belongs to a well-defined class of “hard-core” repulsive interactions. [4] In this case, the qualitative properties of the quantum Hall states do not depend on the precise form of the interactions.

We are interested in the energy gap of the incompressible quantum Hall state at $\nu = \frac{p}{2p+1}$, which corresponds to the energy required to create a quasiparticle and a quasihole infinitely...
far apart. In the mean-field approximation, the energy gap is given by the separation of the Landau energy levels in the residual magnetic field $\Delta B = \frac{B}{2p+1}$. In particular, the mean-field energy gap at $\nu = \frac{p}{2p+1}$ falls off as $E_{mf}(p) \sim 1/(2p+1)$ in the limit of large $p$; however, the energy scale of the mean-field gap is set by the cyclotron energy $\hbar eB/(cm)$, where $m$ is the band mass of the electrons, and not by the interaction energy as expected.

In order to include the effects of gauge fluctuations, Halperin, Lee and Read (HLR) \cite{5} conjectured that the energy gap of the quantum Hall states belonging to the main Jain series is given by the cyclotron energy of the composite fermions in the residual magnetic field $\Delta B$, with the bare mass of the quasiparticles replaced by their effective mass $m^*(\omega)$ at the gap energy $E_g(\nu)$,

$$E_g(\nu) = \frac{\hbar e|\Delta B|}{cm^*(E_g(\nu))},$$

(1.1)

This equation for the energy gap will be referred to as the HLR-gap equation.

Halperin \textit{et al.} have computed in their seminal paper \cite{5} the effective mass of the composite fermions for various interactions, including the gauge fluctuations up to the one-loop level. They concluded that the effective mass of the composite fermions diverges at the Fermi level for short range and Coulomb interactions, while remaining finite for long range interactions. In the case of the Coulomb interaction, the effective mass of the composite fermions is given by \cite{5, 6}

$$m^*_{Cb} = \lim_{\omega \to 0} \frac{2\varepsilon k_F}{\pi e^2} |\ln \omega| + \cdots,$$

(1.2)

where $\varepsilon$ is the dielectric constant, and the dots represent subleading terms. Stern and Halperin \cite{6} argued that this result is exact to all orders in perturbation theory, up to finite contributions from short wavelengths. For short range interactions of the form $V(r) \sim 1/r^{2-x}$ with $0 < x < 1$, the divergence of the effective mass obeys the power law \cite{5, 7}

$$m^*_{sr} \sim \lim_{\omega \to 0} \omega^{\frac{x}{x-1}}.$$  

(1.3)

Moreover, in the one-loop approximation, the imaginary part of the effective mass $m^*_{sr}$ diverges with the same rate as its real part does. These results suggest that, at half-filling, the Fermi velocity $v_F = k_F/m^*$ goes to zero, and that the lifetime of the quasiparticles is vanishingly short. \cite{7} In particular, the Fermi liquid description of the compressible Hall states at half-filling with short range interaction is expected to break down. This picture is consistent with the renormalization group analysis proposed by Nayak and Wilczek. \cite{8}
Indeed, using an approach inspired by the $\epsilon$ expansion, Nayak and Wilczek showed that the Fermi liquid description of the fermion-Chern-Simons theory is a fixed point of the renormalization group flow as long as the interaction is long ranged. For short range interactions, the theory flows away from the Fermi liquid fixed point toward strong coupling, eventually leading to a breakdown of the Fermi liquid description. The theory with Coulomb interaction is marginal; it flows toward the weak-interacting fixed point, but the two-point correlation function for the composite fermions exhibits at low energy a non-Fermi liquid behavior $G(\omega) \sim 1/(\omega \ln \omega)$.

The question of the physical meaning of the divergent effective mass has to be addressed in terms of gauge-invariant quantities. For instance, the singular self-energy leading to a divergence of the effective mass is a gauge-dependent quantity. In the gauge-invariant density-density and current-current correlation functions, important cancellations between the self-energy corrections and the vertex corrections due to Ward identities lead to a Fermi liquid behavior for all ratios of $\omega$ and $k_F|k|/m$. However, other gauge-invariant quantities such as the specific heat are sensitive to the singularity in the effective mass. In this context, Kim and Lee showed that for the Coulomb interaction, the singular part of the specific heat of the composite fermion system predicted by the quasiparticle approximation coincides with the singular part of the specific heat obtained from the free energy of the gauge field. For short range interactions, both results for the specific heat disagree, and they concluded that the Landau quasiparticle approximation is valid only for the Coulomb interaction.

Aware of the limitations of the Fermi liquid approach for short range interactions, we might still try to use it to compute some properties of the incompressible Hall states near half-filling. In particular, we may hope that the quasiparticle approximation will make valid qualitative predictions for the behavior of the energy gap along the main Jain series as function of $p$.

For the Coulomb interaction, the HLR-gap equation has been confirmed by one-loop calculations in the vicinity of half-filling by Stern and Halperin. According to formula (1.1), the energy gap of the quantum Hall states at the filling fraction $\nu = \frac{p}{2p+1}$ in the limit of large $p$ behaves as

$$E_g(\nu) \sim \frac{\pi e^2 k_F}{2 \varepsilon} \frac{1}{(2p+1)[\ln(2p+1) + C']}. \quad (1.4)$$
where the constant $C'$ depends on short wavelength contributions, and cannot be evaluated within the fermion-Chern-Simons theory. However, the coefficient of the logarithmic term is believed to be exact.

Numerical studies of the energy gap in the incompressible quantum Hall states with $\nu = 1/3, 2/5, 3/7$ and $4/9$ (corresponding to $p = 1, 2, 3$, and $4$) have been performed by Morf et al. \cite{11} for the Coulomb interaction. They have computed the energy gap of systems of up to 18 electrons on a sphere, and extrapolated the result to the limit of infinite systems. Using $C'$ as a fitting parameter, they found a good agreement between formula (1.4) and the numerical results. Moreover, the fit with Eq. (1.4) is better than it would be without the logarithmic correction.

These results were a motivation to produce a counterpart to Eq. (1.4) for long and short range interactions, in view of comparison with numerical studies. For this purpose, we have reproduced the study of Stern and Halperin \cite{6} for general interactions, in spite of the strong evidence for the breakdown of the Fermi liquid approach when the interaction is short ranged. Following Ref. \cite{6}, we first compute the effective mass at half-filling in the presence of an electron-electron interaction determined by the potential

$$V(r) \sim \frac{1}{r^{2-x}}, \quad (1.5)$$

where $x < 1$ corresponds to short range interaction, and $x = 1$ is the Coulomb interaction, while for $x > 1$, the interaction is long ranged. \cite{12} For $0 < x < 1$, we find the effective mass

$$m^* \simeq \frac{\phi^2 F(x)}{V(k_F)} \lim_{\omega \to 0} \left( \frac{4\pi\phi\omega}{k_F^2 V(k_F)} \right)^{(x-1)/(3-x)}, \quad (1.6)$$

in agreement with the effective mass previously obtained in the literature. \cite{5, 7} Here, $\hat{V}(k) \sim k^{-x}$ is the Fourier transform of the potential (1.5) and $k_F = \sqrt{4\pi n}$ the Fermi momentum. $F(x)$ is a complex function, with a real part that diverges at $x = 1$. The effective mass $m^*$ diverges following a power law for $x < 1$; in the limit $x \to 1$, the effective mass (1.2) obtained for the Coulomb interaction is recovered. Using an argument based on Ward identities to account for renormalization of the vertex function, we argue that Eq. (1.6) is exact; however, unlike in the case of Coulomb interaction, the function $F(x)$ may get contributions from all orders in the loop expansion. In other words, the contributions from higher orders do not modify the strength of the divergence.
In a second step, we have computed the energy gap at the incompressible quantum Hall states at \( \nu = \frac{p}{2p+1} \) and in the limit of large \( p \), still following Ref. 6. The self-consistent equation for the energy gap obtained in presence of the effective magnetic field \( \Delta B \) corresponds to the HLR-formula (1.1), up to subleading corrections. The asymptotic behavior of the energy gap for large \( p \) is then

\[
E_g(p) \simeq Y(x) \frac{k_F^2 \hat{V}(k_F)}{4\pi\phi} \left( \frac{2\pi}{2p + 1} \right)^{(3-x)/2}.
\]  

(1.7)

In the one-loop approximation, \( Y(x) \) is a complex function of \( x \), and the solution of the gap equation (1.1) has a nonvanishing, unphysical imaginary part, which comes probably from the fact that higher order terms in the loop expansion have been neglected.

In the case of Coulomb interaction, the imaginary part obtained solving the one-loop gap equation (1.1) is subleading, and vanishes faster than the real part as \( p \) tends to infinity; hence, for the Coulomb interaction, the energy gap obtained at the one-loop order is asymptotically exact in the limit \( p \to \infty \). For short range interactions, both the real and imaginary parts of the solution of Eq. (1.1) behave the same way in the limit \( p \to \infty \). The coefficient \( Y(x) \) in Eq. (1.7) remains, therefore, undetermined at the one-loop order for any \( p \), and only the overall power-law behavior of the energy gap as function of \( p \) is captured by the one-loop calculations.

We interpret the appearance of imaginary parts both in the effective mass and in the energy gap as a signature of the breakdown of the composite Fermion approach. The loop expansion relies on the smallness of the gauge fluctuations around its mean-field value. For long range interactions, long wavelength gauge fluctuations are suppressed, and the FCS approach is justified; the effective mass and energy gap are finite and real. However, since both long and short wavelength contributions are equally important, the FCS approach is not appropriated to give an approximation for the effective mass or the energy gap in this case.

For short range interactions, the gauge field may fluctuate strongly, eventually leading to the breakdown of the FCS picture; any prediction has, therefore, to include all orders in the loop expansion. In doing so, we could compute the power-law behavior of the most divergent part of the effective mass, and the \( p \) dependency of the energy gap; the contributions coming from short wavelengths are subleading, and may be neglected in this context. The Coulomb interaction is marginal: the long wavelength gauge fluctuations are sufficiently suppressed by
the long range interaction to allow loop expansion, but remain strong enough to produce a divergence in the effective mass, which can then be obtained exactly by a one-loop calculation including only long wavelength fluctuations.

This paper is constructed as follows. A brief description of the fermion Chern-Simons theory is given in Sec. II. The effective mass is then computed at half-filling from the one-loop approximation of the self-energy, whereby the role of higher-order contributions in the loop expansion is also discussed. In Sec. IV the gap equation and its solution are discussed.

II. FERMION-CHERN-SIMONS THEORY

In this section, we briefly review the construction of the fermion Chern-Simons theory. We start from the zero-temperature partition function for a two-dimensional electron gas in a strong magnetic field $B$ perpendicular to the plane, which is (in units where $\hbar = c = 1$)

$$Z = \int \mathcal{D}[\psi_e^+, \psi_e] e^{iS},$$

(2.1)

where $\psi_e^+$ and $\psi_e$ are the anticommuting electron fields. The action in the real-time formalism is given by

$$S = \int dt \int d^2r \left( \mathcal{L}_0 + V \right),$$

(2.2a)

with the Lagrangian density

$$\mathcal{L}_0 = \psi_e^+ \left[ i \partial_t - \frac{1}{2m} \left( -i \nabla + eA \right)^2 + \epsilon_F \right] \psi_e,$$

(2.2b)

$$V = \frac{1}{2} \int d^2r' \rho_e(t, r)V(r - r')\rho_e(t, r'),$$

(2.2c)

where $\epsilon_F$ is the chemical potential, $m$ is the band mass of the electrons, $A$ is the vector potential corresponding to the magnetic field $B = \nabla \times A$ and $\rho_e(t, r) = \psi_e^+(t, r)\psi_e(t, r)$ is the electron density. The precise form of the static electron-electron interaction potential $V$ will be discussed later.

We introduce the composite fermions $\psi$, which are related to the electron fields by a singular gauge transformation,

$$\psi^+ = \psi_e^+ \exp \left[ -i \phi \int d^2r' \arg(r - r')\rho_e(t, r') \right],$$

(2.3)

where $\arg(r)$ is the angle between the vector $r$ and the $x$ axis. The fields $\psi$ and $\psi^+$ satisfy the usual anticommutation relations, providing $\phi$ is an even integer; we will specialize to the
case \( \phi = 2 \). The composite fermion density remains the same as the electron density,
\[
\rho(t, x) = \psi^+(t, x)\psi(t, x) = \rho_e(t, x).
\] (2.4)

This transformation is associated with the gauge potential,
\[
a = \phi \int d^2 r' g(r - r')\rho(t, r'), \quad (2.5a)
g(r) = (\hat{z} \times r)/|r|^2. \quad (2.5b)
\]

Observe that the vector potential \( a \) satisfies to the constraint
\[
\nabla \times a = 2\phi\pi \rho(t, r), \quad (2.6)
\]
where \( \nabla \times a = e^{ij}\partial_i a_j \). We choose to work in the Coulomb gauge, with \( \nabla \cdot a = 0 \).

The longitudinal part of the gauge vector vanishes, and we denote \( a_\perp \) its transversal part, satisfying \( a_i = e^{ij}k_ja_\perp/|k| \) in momentum space. The constraint (2.6) can be ensured by the Lagrange multiplier field \( a_0 \), adding the term
\[
\int dt \int d^2 r a_0 \left( \frac{1}{2\phi\pi} \nabla \times a - \rho(t, r) \right) \quad (2.7)
\]
to the action. The partition function can be rewritten in terms of the composite fermion and gauge fields as
\[
Z = \int \mathcal{D}[a_0, a_\perp] \int \mathcal{D}[\psi^+, \psi] e^{iS_{\text{FCS}}}
\] (2.8)

where the fermion-Chern-Simons action is obtained from the Lagrangian density,
\[
L_{\text{FCS}} = L_{\text{CF}} + L_{\text{CS}} + \mathcal{V}. \quad (2.9a)
\]

The Lagrangian for the composite fermions is
\[
L_{\text{CF}} = \psi^+ \left[ i\partial_t - a_0 - \frac{1}{2m} \left( -i \nabla - a + eA \right)^2 + \epsilon_F \right] \psi \quad (2.9b)
\]
and the Chern-Simons Lagrangian for the gauge field in the Coulomb gauge
\[
L_{\text{CS}} = \frac{1}{2\phi\pi} \epsilon^{ij}a_0 \partial_i a_j. \quad (2.9c)
\]

In the mean-field approximation, the density of electrons is assumed to be constant, such that, in virtue of Eq. (2.6), the composite fermions are subject to the reduced magnetic field
\[
\Delta B = B - 2\pi \phi n/e = B(1 - \phi\nu), \quad (2.10)
\]
where \( n = \langle \rho \rangle \) is the mean density of particles and \( \nu = 2\pi n/(eB) \) the filling factor. In particular, at half-filling \( (\nu = 1/2) \) the residual magnetic field vanishes providing \( \phi = 2 \). Along the main Jain series determined by \( \nu = p/(2p + 1) \), the effective magnetic field becomes

\[
\Delta B^{(p)} = \frac{B}{2p + 1}.
\]

(2.11)

According to the mean-field predictions, the quantum Hall system at half-filling is in a compressible state, which may be described in terms of a Fermi liquid of composite fermions; furthermore, the incompressible quantum Hall states of the main Jain series are described in terms of the integer quantum Hall effect for composite fermions.

In order to go beyond the mean-field approximation, one has to take into account fluctuations of the gauge field. For this purpose, we expand the Lagrangian (2.9a) up to the second order in fluctuations of the Chern-Simons gauge field \( (a_0, a_\perp) \) around its mean-field value. Doing so, and using the constraint (2.6) in order to express the interaction term \( V \) of Eq. (2.9a) solely in terms of the gauge field, we obtain the effective action

\[
S_{\text{eff}} = \int dt \int d^2r \mathcal{L}_{\text{CF}} + S_{\text{gauge}},
\]

(2.12)

where in momentum space,

\[
S_{\text{gauge}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} a_\lambda(-k, -\omega)(D_0^{-1})_{\lambda\mu} a_\nu(k, \omega).
\]

(2.13)

The Greek indices run over the longitudinal and perpendicular directions, and the inverse bare gauge propagator is

\[
D_0^{-1} = \begin{pmatrix} 0 & \frac{ik}{2\phi} \\
-\frac{ik}{2\phi} & -\frac{k^2V(k)}{2\pi}\phi^2 \end{pmatrix}.
\]

(2.14)

We will consider electron-electron interactions of the form

\[
\hat{V}(k) = \frac{\hat{V}(k_F)}{(k/k_F)^x}
\]

(2.15)

in momentum space, with \( 3/2 > x > 0 \). For the Coulomb interaction, \( x = 1 \) and \( k_F \hat{V}(k_F) = 2\pi e^2/\varepsilon \) with the dielectric constant \( \varepsilon \); \( x < 1 \) corresponds to short range interactions. For \( x > 1 \), the interaction is long range, i.e., falls off slower than \( 1/r \) in position space. We assume the occurrence of the quantum Hall states for all filling factors in the Jain series; this is reasonable since the corresponding Haldane pseudopotentials exhibit hard-core-like properties, as shown in Appendix C.
We are interested in the effective mass for the composite fermions, which is given in terms of the self-energy by

\[ m^* = m \left. \frac{1 - \partial_\omega \Sigma}{1 + \partial_\epsilon k \Sigma} \right|_{\omega=0, k=k_F} \]  

(2.16)

where \( \epsilon_k = \frac{k^2}{2m} - \epsilon_F \) is the bare dispersion relation.

III. EFFECTIVE MASS OF THE COMPOSITE FERMIONS

We turn to the computation of the self-energy to the one-loop order (first order in the gauge propagator \( D \)). In the following calculations, the gauge propagator \( D \) in the random phase approximation \[5, 6\] will be used. In the Coulomb gauge, using the (0, \( \perp \))-notation introduced above,

\[
D^{-1}(\omega, k) = \left( \frac{m}{2\pi^2} \right)^2 \int d^2 k' \frac{i |k|}{k_F} \left( i \frac{\omega}{\omega_{\text{int}}} - \frac{|k|}{k_F} \right)^3 \left( \frac{\omega_{\text{int}}}{k_F^2} \right) \right)^{-1},
\]

(3.1)

The one-loop computations of the self-energy involve only the diagonal part of the gauge propagator. In the limit of long wavelengths \( k \to 0 \) and small frequencies \( \omega \to 0 \),

\[
D_{\perp\perp}(\omega, k) \simeq \frac{\phi \pi}{\omega_{\text{int}} k_F} \left[ i \frac{\omega}{\omega_{\text{int}}} - \left( \frac{|k|}{k_F} \right)^3 \right]^{-1}
\]

(3.2)

where for \( x > 0, [14] \)

\[
\omega_{\text{int}} = k_F^2 \tilde{V}(k_F)/(4\pi \phi)
\]

(3.3)

is the energy scale defined by the electron-electron interaction; for \( x = 1, \omega_{\text{int}} = e^2 k_F/(2\phi \varepsilon) \).

Further, the longitudinal gauge propagator is \( D_{00} \simeq 2\pi/m \).

A. Self-energy

To the first order in \( D \), the most important contribution to the effective mass comes from the transversal part of the self-energy

\[
\Sigma_{\perp}(\omega, k) = \frac{i}{2m} \int \frac{d\Omega}{2\pi} \int \frac{d^2 k'}{(2\pi)^2} \frac{(k \times k')^2}{(k - k')^2} D_{\perp\perp}(\Omega, k - k') G_0(\omega - \Omega, k'),
\]

(3.4)

where

\[
G_0(\omega, k) = \frac{1}{\omega - \epsilon_k + i0^+ \text{sgn}(\omega)}
\]

(3.5)
is the free propagator for the composite fermions at half-filling, with $\epsilon_k = k^2/(2m) - \epsilon_F$. The self-energy (3.4) is computed in the same way as in the case of electron-phonon coupling. [15] We compute $\Sigma_\perp(\omega, k) - \Sigma_\perp(0, k)$ and introduce the variable $q = |k - k'|$, changing the coordinates according to

$$
\int d^2k' f(k') = \int_0^\infty dk' \int_{|k-k'|}^{k'+k} \frac{dq}{|k \times k'|} f(k', q).
$$

(3.6)

The dominant contribution to the transversal self-energy comes from the region of the momentum space where $k' \simeq k_F$, due to the pole in the fermion propagator (3.5); we thus neglect the $k'$-dependency of the integrand, except in $G_0(\Omega - \omega, k')$. Furthermore, the $k$ dependency of the self-energy leads to subleading corrections, and we may assume $k \simeq k_F$. The integration over $k'$ is, finally, performed closing the contour in the upper complex half-plane. As a consequence, the integration over the frequency $\Omega$ is restricted to the range $[0, \omega]$. Integrating $D_{\perp\perp}(\Omega, q)$ over $\Omega$ leads to

$$
\Sigma_\perp(\omega, k_F) - \Sigma_\perp(0, k_F) \simeq -\frac{\phi k_F^2}{2\pi m} \frac{I(2u_{\text{max}})}{u_{\text{max}}^2},
$$

(3.7)

where we have define the integral

$$
I(v) = \int_0^v du \left\{ i \text{sgn}(\omega) \ln \sqrt{1 + u^{2x-6}} + \arctan u^{x-3} \right\}.
$$

(3.8)

The dimensionless variable $u$ is given by

$$
u = \frac{q}{n} \left( \frac{\omega_{\text{int}}}{|\omega|} \right)^{1/(3-x)},
$$

(3.9)

with $u_{\text{max}} = (\omega_{\text{int}}/|\omega|)^{1/(3-x)}$. For $\omega \to 0$, the integral (3.8) has to be evaluated for $v \gg 1$, as done in Appendix A. Inserting the approximation (A5) for $I(v)$ in the expression (3.7) for the self-energy leads to

$$
\Sigma_\perp(\omega, k_F) - \Sigma_\perp(0, k_F) \simeq |\omega| \frac{m_{\text{int}}}{m} \left\{ F_0(x)(|\omega|/\omega_{\text{int}})^{(x-1)/(3-x)} + \frac{2^x}{x-1} + \ldots \right\}
$$

(3.10)

where we have defined the interacting mass scale

$$
m_{\text{int}} = \frac{\phi k_F^2}{4\pi \omega_{\text{int}}}.
$$

(3.11)
For the Coulomb interaction, \( m_{\text{int}} = \frac{\phi^2 k_F}{2\pi e^2} \). The function \( F_0(x) \) is obtained from the integral (3.8) and is given by

\[
\text{Re } F_0(x) = \frac{\pi (3 - x)}{4} - \sum_{n \geq 0} \frac{4(3 - x)(-1)^n}{(2n + 1)[4 - (2n + 1)^2(3 - x)^2]}
\]

\[
\text{Im } F_0(x) = \text{sgn}(\omega)(3 - x) \int_0^\infty \frac{u du}{1 + u^6 - 2x}.
\]

(3.12)

Notice that the real part of \( F_0(x) \) diverges as \( x \to 1 \), while its imaginary part remains finite. For short range interactions \( (x < 1) \), both the real and imaginary parts of the self-energy are proportional to \( |\omega|^{2/(3-x)} \). In particular, the Landau criterion for the existence of the quasiparticle is not fulfilled, since \( |\text{Im } \Sigma(\omega,k)| > |\omega| \) for \( \omega \to 0 \). The prediction of the Fermi liquid theory should, therefore, be considered with care if the interaction between the electrons is short ranged, as already discussed in Ref. 10. For \( x < 1 \), the leading term of the self-energy (3.10) comes from the integration region where \( u \lesssim 1 \) in Eq. (3.8); this corresponds to momenta \( q \) smaller than \( q_0 \sim |\omega|\frac{1}{|x-1|} \). In other words, for systems with short range interactions, the divergence of the effective mass comes from small frequencies and small momenta.

In the limit \( x \to 1 \), the function \( F_0(x) \) can be replaced by its limiting value

\[
F_0(x \to 1) = -\frac{2}{x - 1} + \frac{i\pi}{2} + 2,
\]

(3.13)

and Eq. (3.10) becomes

\[
\Sigma_\perp(\omega, k_F) - \Sigma_\perp(0, k_F) \simeq
-|\omega| \frac{m_{\text{int}}}{m} \left\{ -\frac{2}{x - 1} \left( \left| \frac{\omega}{\omega_{\text{int}}} \right|^{(x - 1)/2} - 1 \right) + i \text{sgn}(\omega) \frac{\pi}{2} \right\}.
\]

(3.14)

Expanding in powers of \( x - 1 \) leads to the logarithmic behavior characteristic for the Coulomb interaction, [5, 6]

\[
\Sigma_\perp(\omega, k_F) - \Sigma_\perp(0, k_F) \simeq
-\frac{\phi^2 k_F}{2\pi e^2 m} |\omega| \left( \ln \left| \frac{k_F e^2}{2\phi e |\omega|} \right| + i \text{sgn}(\omega) \frac{\pi}{2} \right).
\]

(3.15)

The logarithm is obtained combining the leading term of Eq. (3.14), which comes from integration over small momenta \( q \), with the subleading term \( \sim 2\omega/(x - 1) \), coming from
integration over large $q$ in Eq. (3.8). Hence, as pointed out in Ref. 6, the logarithmic part of the self-energy comes from small frequencies, but from a large range of momenta. The imaginary part of the self-energy is much smaller than its real part in the limit $\omega \to 0$, justifying the quasiparticle approach for systems with Coulomb interaction.

Finally, for long range interactions ($x > 1$), the self-energy is proportional to $\omega$, and its imaginary part is of higher-order in $\omega$. The Fermi liquid picture is thus well justified in this case, as pointed out by Nayak and Wilczek. [8]

**B. Effective mass**

We may now compute the effective mass in the one-loop approximation from the self-energy obtained in Eq. (3.10). We will consider only the most divergent part of the effective mass, neglecting finite contributions from the momentum dependency of the self-energy or from the longitudinal part of the self-energy. Using formula (2.16), we get for short range interactions in the one-loop approximation

$$\frac{m^*}{m_{\text{int}}} \simeq F_0(x) \lim_{\omega \to 0} \frac{\omega}{\omega_{\text{int}}}^{(x-1)/(3-x)} + \frac{2x}{x-1} + \cdots ,$$

(3.16)

where $m_{\text{int}}$ was defined in Eq. (3.11). The effective mass thus diverges following a power-law on the Fermi surface, according to the prediction of Ref. 5. Moreover, the imaginary part of the effective mass diverges, leading to instabilities in the Fermi liquid at half-filling with short range interactions. Again, we recover the previous results by Halperin and Stern [6] in the limit $x \to 1$,

$$m^* = \frac{\phi^2}{2\pi} \frac{\varepsilon k_F}{e^2} \left( \ln \frac{k_F e^2}{2\phi \varepsilon \omega} + i \frac{\pi}{2} \right) + \text{const.}$$

(3.17)

For long range interactions ($x > 1$), we expect the effective mass to be real and finite; in this case, higher-order terms and short wavelengths contributions have to be taken into account in order to determine the value of the effective mass.

**C. Beyond the one-loop approximation**

Since there is no small expansion parameter, the result of the field theory can be trusted only if all orders in the perturbation theory can be controlled. In the case of Coulomb interaction, it is believed that the divergent part of the effective mass (3.17) is exact to
all orders in the perturbation theory. We will discuss here the effect of higher-order contributions on the divergence of the effective mass for general interactions, and eventually argue that the power-law divergence in Eq. (3.16) is the same to all orders for \( x < 1 \), whereby the one-loop function \( F_0(x) \) defined in Eq. (3.12) has to be replaced by its renormalized counterpart \( F(x) \). We use an argument relying on Ward identities similar to the argument used by Halperin and Stern for systems with a Coulomb interaction.

1. Self-consistent equation for the self-energy

A first improvement to the one-loop calculations can be obtained considering a self-consistent equation for the self-energy, replacing in Eq. (3.4) the bare fermion propagator (3.5) by

\[
G(\omega, k) = \frac{1}{G_0^{-1}(\omega, k) - \Sigma(\omega, k)}.
\]

The self-energy obtained is then the sum over rainbow diagrams. Repeating the calculations that led to the one-loop result (3.10) shows that the leading term of the self-energy, and, consequently, the divergent part of the effective mass (3.16), is not affected by this procedure; the function \( F_0(x) \) defined in Eq. (3.12) is not modified either.

2. Short wavelength contributions

Following Ref. 6, we now show that the short wavelength contributions to the self-energy do not modify the power-law behavior of the most divergent part of the effective mass. For this purpose, we decompose the momentum space into regions of short and long wavelengths, delimited by an intermediate cutoff \((\omega_0, q_0)\) with \( \omega_0 \gtrsim \omega \) and \(|q_0| \sim \omega_0^{\frac{1}{2-x}}\). The contributions to the self-energy or to the vertex functions can then be decomposed in short and long wavelength parts, following, for example, the procedure of the constructive renormalization. For instance,

\[
\Sigma(\omega, k) = \Sigma^<(\omega, k) + \Sigma^>(\omega, k),
\]

where all the internal gauge propagators in the Feynman diagrams contributing to the short wavelength part \( \Sigma^>(\omega, k) \) of the self-energy, carry a momentum larger than \((\omega_0, q_0)\). By the Ward identities, the self-energy \( \Sigma^>(\omega, k) \), which is a regular function of \( \omega \) and \( k \), determines
the renormalization by the short wavelength contributions of both the Green function and
the vertex function.

The Feynman diagrams that contribute to the remaining part $\Sigma^<(\omega, k)$ of the self-energy, referred to as the long wavelength part of the self-energy, contain at least one gauge line carrying a small momenta. If one of the gauge line carrying a small momenta is singled out, the long wavelength contribution $\Sigma^<(\omega, k)$ can be expressed in the form

$$\Sigma^<(\omega, k) = \frac{i}{m^2} \int_\Omega d\Omega \int_\mathbb{R}^2 d^2k' rac{D_{\mu\nu}(\Omega, k - k')}{(2\pi)^2} \times \Gamma_{\mu}(\omega, k; \Omega, k') \Gamma_{\mu}(\omega - \Omega, k'; \omega, k) G(\omega - \Omega, k'),$$

where $\Gamma_{\mu}$ is the dressed vertex function. In Eq. (3.20), the integration range is restricted such that the frequency $\Omega$ and the transfer momenta $q = |k - k'|$ are small.

Suppose that the leading contribution to the long wavelengths self-energy comes from its transversal part $\Sigma_{\perp}^<(\omega, k)$, involving only the transversal vertex function $\Gamma_{\perp}$. (This assumption will be justified below.) The effect of the short wavelength gauge fluctuations on the vertices $\Gamma_{\perp}$ is included in a multiplicative renormalization constant $Z_{\perp}$. Moreover, by Ward’s identities, $Z_{\perp} = (1 + \frac{\partial \Sigma^<}{\partial \epsilon_k})$. Furthermore, the integration over the internal momentum $k'$ in Eq. (3.20) yields a factor of $(1 + \frac{\partial \Sigma^<}{\partial \epsilon_k})^{-1}$. The gauge propagator, which is determined by the electron’s charge and electron density does not get renormalized by short wavelengths fluctuation. \[6\] The leading term for effective mass then becomes

$$\frac{m^*}{m} \simeq \frac{1 - Z_{\perp} \partial_\omega \tilde{\Sigma}_{\perp}^< + Z_{\perp} \partial_\epsilon \tilde{\Sigma}_{\perp}^<}{1 + \partial_\omega \Sigma^< + Z_{\perp} \partial_\epsilon \Sigma^<} \simeq - \frac{\partial_\omega \tilde{\Sigma}_{\perp}^<(0, k_F)}{1 + \partial_\epsilon \Sigma^<(0, k_F)} + \cdots,$$

where $\tilde{\Sigma}_{\perp}^<(\omega, k)$ is the self-energy obtained from Eq. (3.20) replacing the vertex functions $\Gamma_{\perp}$ by their long wavelength counterpart $\Gamma_{\perp}^\perp$. Observe that only the long wavelength contributions to the self-energy participate in the renormalization of the most divergent part of the effective mass.

3. Long wavelength contributions

We now argue that including the long wavelength vertex corrections does not modify the nature of the power-law (respectively the logarithm for $x = 1$) divergence of the effective mass. For this purpose, we show that the self-energy $\Sigma^<(\omega, k)$ behaves in the same way as the one-loop self-energy \[3.10\] does, using an induction over the number of loops. Suppose
that the leading term of the long wavelength self-energy has been computed to the \( n \)-loop order and is, up to subleading corrections,

\[
\Sigma_n^< (\omega, k) - \Sigma_n^< (0, k_F) \simeq F_n(x) \frac{m_{\text{int}}}{m} \left( \frac{|\omega|}{\omega_{\text{int}}} \right)^{2/(3-x)} \tag{3.22}
\]

with a complex function \( F_n(x) \). The self-energy \( \Sigma_n^< (\omega, k) \) can be put into the form of Eq. (3.20), with vertex functions \( \Gamma^m_\mu \) to a lower order \( m < n \) in the loop expansion.

As described in the beginning of this section, the substitution of the bare propagator \( G_0(\omega, k) \) by the random-phase-approximation (RPA) propagator (3.18) in the sunset diagram of Eq. (3.20) can be achieved using a self-consistent equation for the self-energy, without affecting the leading term of the self-energy. We should, therefore, replace \( G(\omega, k) \) in Eq. (3.20) by the bare propagator for this discussion, considering only the skeleton diagrams for the self-energy.

The leading contribution to the self-energy \( \Sigma_{n+1}^< (\omega, k) \) to the next order is then obtained replacing one of the vertex function \( \Gamma^m_\mu \) by its expression to the next order in the loop expansion. We then repeat the calculation that led to the one-loop approximation for the self-energy. We compute the self-energy at \( |k| = k_F \), neglecting the \( k' \) dependency of the vertex functions that have no pole for \( k' \to k_F \). Again, integrating over the fermion propagator restricts the frequency integration to the range \([0, |\omega|]\). We need an approximation for the vertex functions in the limit of small frequency and transfer momenta; this can be achieved using the Ward identities

\[
\lim_{\Omega \to 0} \Gamma^0_n (\omega, k; \omega + \Omega, k + q) \simeq 1 - \frac{\partial \Sigma^<_n (\omega, k)}{\partial \omega} \tag{3.23a}
\]

\[
\lim_{\Omega \to 0} \Gamma^\perp_n (\omega, k; \omega + \Omega, k + q) \simeq 1 + \frac{\partial \Sigma^<_n (\omega, k)}{\epsilon_k} \tag{3.23b}
\]

where \( \Gamma^0_n \) and \( \Gamma^\perp_n \) are the vertex functions to the \( n \)-loop order, up to subleading contributions coming from the short wavelength part of the self-energy. Inserting the ansatz (3.22) for the self-energy leads to the limiting behavior \( \Gamma^0_n \sim |\omega|^{3-x} \) and \( \Gamma^\perp_n \sim \text{const} \).

Armed with these observations, we derive an approximation for the self-energy \( \Sigma_{n+1}^< (\omega, k) \) to the next order in the loop expansion. We first consider the longitudinal part of the self-energy. The longitudinal gauge propagator is \( D_{00} \simeq 2\pi/m \) in the limit’s \( \omega \to 0 \) and \( q \to 0 \). The dominant contribution comes from the region of the momentum space where the vertex
function diverges, i.e., when $v_F q \lesssim |\omega|$. Collecting the negative powers of $\omega$ coming from the two vertex functions (3.23a), and the factor $\sim \omega^2$ coming from the integrals over $\Omega \in [0, \omega]$ and $q \lesssim |\omega|/v_F$ leads to

$$\Sigma_{0,n+1}^<(\omega, k_F) - \Sigma_{0,n+1}^<(0, k_F) \sim |\omega|^{4/(3-x)} + \cdots,$$

which is subleading as compared to the ansatz (3.22) for the $n$-loop self-energy.

We turn to the transversal part of the self-energy. According to the Ward identities (3.23b), the vertex function can be approximated by a constant in the small frequencies and long wavelength limit. The integration then works in the same way as for the one-loop calculation, and we recover the ansatz of Eq. (3.23b), with a modified function $F_{n+1}(x)$. We, thus, believe that the effective mass of the composite fermions is given by

$$\frac{m^*}{m_{\text{int}}} \approx F(x) \lim_{\omega \to 0} \left( \frac{\omega}{\omega_{\text{int}}} \right)^{(x-1)/(3-x)} + \cdots,$$

where $m_{\text{int}}$ was defined in Eq. (3.11).

We briefly discuss the case of Coulomb interaction and show that the effective mass (3.17) obtained at one loop is exact. The original argument of Stern and Halperin [6] is based on formula (3.21), and the fact that for Coulomb interaction, the leading contribution to the effective mass comes from a small range of frequencies, but from a large range of momenta $q \in [q_0, 2k_F]$, with $q_0 \sim \sqrt{|\omega|}$. It follows that the leading contribution to $\tilde{\Sigma}_{\perp}^<(\omega, k_F)$ comes solely from the one-loop diagram: According to Ref. [6], the coefficient of the divergent term in $\partial_\omega \tilde{\Sigma}_{\perp}^<(0, k_F)$ is not affected if the diagrams having two or more lines carrying long wavelength momenta $q$ are omitted. Thus, up to subleading contributions

$$\partial_\omega \tilde{\Sigma}_{\perp}^<(0, k_F) = \partial_\omega \Sigma(0, k_F)|_{\text{oneloop}}.$$

Since $\partial_{qk} \tilde{\Sigma}_{\perp}^<(0, k_F)$ is subleading, the effective mass (3.17) is exact.

**IV. EVALUATION OF THE ENERGY GAP**

In the mean-field picture described in Sec. [11] the composite fermions experience a residual magnetic field $\Delta B = B/(2p+1)$ at the filling factor $\nu = \frac{p}{2p+1}$. The fractional quantum Hall effect can then be viewed as an integer quantum Hall effect for the composite fermions filling exactly $p$ Landau levels. The mean-field energy gap is then given by the separation between
the Landau levels in the magnetic field $\Delta B$. In this approximation, the energy scale is set by the cyclotron energy of the electrons in the magnetic field $B$. However, the energy gap is expected to be determined by interaction effects; in particular, it should remain finite after projection into the lowest Landau level, i.e., in the limit $m \to 0$. In order to achieve this requirement, one can replace by hand the band mass of the electrons by the “interaction” mass $m_{\text{int}}$ defined in Eq. (3.11) and obtained equating the composite fermions kinetic energy $k_F^2/m_{\text{int}}$ with the interaction energy $\omega_{\text{int}}$ defined in Eq. (3.3). In this modified mean-field approximation, the energy gap is given by

$$E_{\text{mf}}(p) = \frac{\Delta B}{e m_{\text{int}}} = \frac{2\pi \omega_{\text{int}}}{(2p + 1)}. \quad (4.1)$$

In particular, the modified mean-field theory predicts that the energy gap decays as $1/(2p+1)$ for $p \to \infty$, for all types of interactions.

We have seen in the last section that the effect of the gauge fluctuations cannot be neglected. Halperin et al. proposed to account for these fluctuations, replacing the interacting mass $m_{\text{int}}$ by the effective mass at half-filling $m^*(\omega)$ evaluated at the energy gap $\omega = E_g(p)$. This leads to the following equation for the energy gap:

$$E_g(p) = \frac{\Delta B}{e m^*(E_g(p))}. \quad (4.2)$$

In the next section, we will discuss the solution of this equation in the one-loop approximation for different interactions.

A. Energy gap from the HLR-conjecture

The results of the Sec. III B combined with the HLR-conjecture (4.2) allow to evaluate the energy gap of the quantum Hall states of the main Jain series with filling factors $\nu = \frac{p^2}{2p+1}$.

We first consider the case of Coulomb interaction. Inserting the one-loop result (3.17) for the effective mass in the gap equation (1.1), one obtains for $\phi = 2$,

$$\Delta \omega^*_c = \frac{2\pi}{2p + 1 - \ln \left( \Delta \omega^*_c \right) + i \frac{\pi}{2} + C_1}, \quad (4.3)$$

where we have introduce the energy gap in units of the interaction energy,

$$\Delta \omega^*_c = \frac{E_g(p)}{\omega_{\text{int}}}. \quad (4.4)$$
The interaction energy $\omega_{\text{int}}$ is defined in Eq. (3.3); for the Coulomb interaction and $\phi = 2$, $\Delta \omega_c^* = 4\varepsilon E_g(p)/(k_F e^2)$. The constant $C_1$ cannot be estimated within the composite fermion approach and included short wavelength contributions as well.

Observe that the solution of Eq. (4.3) is complex. The unphysical imaginary part of the energy gap is a consequence of the approximations used to derive the gap equation. The real part of the solution of the gap equation (4.3) can be accurately approximated by

$$E_g(p) \simeq \frac{k_F e^2}{\varepsilon} \frac{\pi/2}{(2p + 1)\left[\ln(2p + 1) + C'\right]},$$

with a constant $C'$ that depends on $C_1$. The imaginary part of the solution is subleading,

$$\text{Im} \Delta \omega_c^* \sim \frac{1}{(2p + 1)\ln^2(2p + 1)}.$$

Moreover, the leading contribution to the effective mass obtained in the one-loop approximation is exact. We may, thus, conclude that the one-loop approximation (4.5) for the energy gap is asymptotically exact for $p \to \infty$.

The energy gaps of quantum Hall states at filling factors $\nu = \frac{p^2}{2p + 1}$ with $p = 1, 2, 3$ and 4 have been estimated from finite-size calculations by Morf et al. [11]. They then used formula (4.5) in order to fit their numerical results, using $C'$ as a fitting parameter. They found that the $p$ dependency of the energy gap is better explained by Eq. (4.5) than it would be by the mean-field results of Eq. (4.1). This is surprising, since the theoretical results are expected to be valid for $p \gg 1$ only. The estimation for the energy gap from the exact diagonalization of Ref. [11] is plotted together with a numerical solution of Eq. (4.3) in Fig. 1. The constant $C_1$ is chosen such that the solution of the gap equation matches the numerical result of Ref. [11] for $p = 1$.

We turn to systems with short range interactions ($x < 1$). Using the one-loop result (3.16) for the effective mass, we obtain from the HLR conjecture (4.2) the self-consistent gap equation (for $\phi = 2$)

$$\Delta \omega_c^* = \frac{2\pi}{2p + 1} \left( F(x)\left(\Delta \omega_c^*\right)^{(x-1)/(3-x)} + \frac{2x}{x - 1} + C \right)^{-1},$$

with an undetermined constant $C$ that depends on $x$. (See Fig. 2) For $p \gg 1$, the gap equation (4.7) has the asymptotic solution

$$\Delta \omega_c^* \simeq Y(x) \left( \frac{2\pi}{2p + 1} \right)^{(3-x)/2},$$
FIG. 1: (Color online) The energy gap of quantum Hall states at filling factors \( \nu = p/(2p+1) \) as function of \( p \), for systems with Coulomb interaction. The energy gap is obtained solving numerically the gap equation (4.3), with \( C_1 = 3.8 \) (plain line); the opposite of the (unphysical) imaginary part of the gap, giving an estimation of the quality of the one-loop approximation, is also plotted (dashed line). The dots represent the energy gaps for \( p = 1, 2, 3 \) and 4 calculated numerically by Morf et al. [11] from the energy gaps obtained by exact diagonalization in finite-size systems. The energy gap is given in units of \( \omega_{\text{int}} = k_F e^2/(4\varepsilon) \).

where the coefficient \( Y(x) \) cannot be determined by one-loop calculations. It is argued in Sec. III C that the one-loop approximation gives the right exponent for the effective mass as function of \( \omega \). We, thus, believe that Eq. (4.8) gives the right \( p \) dependency of the energy gap. In particular, for \( x < 1 \), the energy gap vanishes faster than the mean-field result as \( p \to \infty \).

Using the HLR conjecture (4.2) and the one-loop approximation for the effective mass, we obtain \( Y_0(x) = F_0 \frac{x^3}{(x-1)^2} \) (i.e., the solution (4.8) is complex. Moreover, the ratio between the imaginary part and the real part of the solution (4.8) is independent of \( p \), for \( p \gg 1 \). This indicates that the one-loop approximation only gives a qualitative estimation of the energy gap as function of \( p \).

The asymptotic solution (4.8) of the gap equation is only valid as long as

\[
(x-1) \ln |\Delta \omega_c^*| \gtrsim 2. \tag{4.9}
\]

For \( x \) close to 1, and \( p < \infty \), this inequality might be violated and the constant term \( 2/(x-1) \)
FIG. 2: (Color online) The energy gap of quantum Hall states at filling factors $\nu = p/(2p + 1)$ as function of $p$ for short range interaction $V(r) \sim 1/r^{3/2}$ (dotted line), Coulomb interaction $V(r) \sim 1/r$ (plain line), and long range interaction $V(r) \sim 1/\sqrt{r}$ (dashed line). The gaps are obtained solving the HLR-gap equation (4.2); only the real part of the solution is plotted. The energy gap is given in units of $\omega_{\text{int}} = k_F^2 \hat{V}(k_F)/(4\pi\phi)$.

on the right-hand side of Eq. 4.7 becomes important. In this case, the gap equation reads

$$\Delta \omega_c^* \simeq \frac{2\pi}{2p + 1} \left[ \frac{2}{x - 1} \left( 1 - (\Delta \omega_c^*)^{(x-1)/2} \right) + \frac{i\pi}{2} + C_1 \right]^{-1},$$

(4.10)

up to terms of order $(x - 1)$. Observe that the leading term [proportional to $1/(x - 1)$] in this equation is exact, i.e., is not modified by contributions from higher orders in the loop expansion, as in the case of Coulomb interaction. In particular, a solution of Eq. (4.10) should be used instead of the asymptotic solution (4.8) for comparison with exact diagonalization.

For short range interactions ($x > 1$), the effective mass $m^*$ is finite and real. The HLR-conjecture (4.2), thus, predicts for $p \gg 1$ the energy gap

$$\Delta \omega_c^* \simeq \frac{2\pi C'}{2p + 1},$$

(4.11)

with an undetermined constant $C'$ (See Fig. 2). In particular, the energy gap falls off as function of $p$ according to the mean-field prediction.

**B. Direct evaluation of the energy gap**

In this section we evaluate the energy gap of the quantum Hall states at filling factors $\nu = \frac{p}{2p + 1}$ directly, using the Chern-Simons composite fermion approach. [6] Gauge fluctuations
will be included to the one-loop order. We will, in particular, verify the conjecture (1.1) of Halperin et al. [5] for short and long range interactions.

The starting point is the action defined in Eq. (2.8) for the composite fermions in a weak magnetic field, in which case the propagator for the composite fermions is not diagonal in momentum space. However, following Stern and Halperin, [6] the propagator can be diagonalized in the Landau levels’ representation: As proposed by Haldane, [18] the problem of a two-dimensional electrons gas in a magnetic field perpendicular to the plane can be studied, considering a system on a sphere with a magnetic field perpendicular to the surface. In the spherical geometry, the states in the $n$th Landau level are eigenstates of the angular momentum operator $L^2$. States within a Landau levels can then be distinguished as being different eigenstates of the operator $L_z$. By conservation of the angular momentum, the propagator for the composite fermions on a sphere of infinite radius is diagonal, and has the general form

$$G(\omega, l) = \frac{1}{\omega - \epsilon(l) - \Sigma(\omega, l) + i\delta_+ \text{sgn}(\omega)},$$

where $l$ labels the Landau levels in the residual magnetic field $\Delta B$. The dispersion relation is defined by

$$\epsilon(l) = E_0 \cdot (l + 1/2) - \epsilon_F(p), \quad E_0 = \frac{\Delta B}{em},$$

and $\epsilon_F(p)$ is the chemical potential for which exactly $p$ Landau levels are filled, and such that adding a fermion to the system would start filling the $p + 1$th Landau level. Observe that we work at fixed $\Delta B$, such that the cyclotron frequency $E_0 = \omega_c/(2p+1)$ is independent of $p$, as discussed in Ref. 6. By definition, the propagator (4.12) has a pole at $\omega = 0$ for $l = p$ corresponding to the Fermi level. Hence,

$$\epsilon_F(p) = E_0 \cdot (p + 1/2) + \Sigma(0, p).$$

The next Landau level corresponds to a pole in Eq. (4.12) located at $l = p + 1$ and $\omega = E_g(p)$, where

$$E_g(p) = E_0 \cdot (p + 3/2) - \epsilon_F(p) + \Sigma(E_g(p), p + 1) \quad (4.15)$$

$$= E_0 + \Sigma(E_g(p), p + 1) - \Sigma(0, p).$$

As discussed in Ref. 6, the quantity $E_g(p)$ is the energy gap of the fractional quantum Hall state at $\nu = \frac{2p + 1}{2p+1}$. In the limit $\omega_{int}/\omega_c \rightarrow 0$, which corresponds to the projection of the
original electrons gas into the lowest Landau level, and for \( p \gg 1 \), the gap equation can be approximated by

\[
\Sigma(E_g(p), p) - \Sigma(0, p) \simeq -\frac{\omega_c}{2p + 1},
\]  

(4.16)

where we have assumed \( \partial_{\epsilon(p)} \Sigma(\omega, p) \simeq 0 \). We now compute the self-energy \( \Sigma(\omega, l) \) for \( l \simeq p \) and \( \omega \) small, and will show that the gap equation (4.16) leads to the same equation as derived from the HLR-conjecture (4.2), up to subleading terms.

C. Evaluation of the self-energy

We now evaluate the self-energy in the residual magnetic field \( \Delta B \) at the one-loop order. As discussed in Ref. [6], the RPA-gauge propagator \( D_p(\Omega, q) \) in the presence of the residual magnetic field \( \Delta B \) can be approximated by its limiting form (3.1) at half-filling, i.e., for \( p \to \infty \). This approximation is justified since the main contribution to the one-loop self-energy comes from transfer momenta satisfying \( q > k_F/(2p + 1) \). [See Eq. (4.21).] In this formalism, the transversal self-energy is

\[
\Sigma_\perp(\omega, l) = i \sum_{j \geq 1} \int \frac{d\Omega}{2\pi} \int \frac{d^2 q}{(2\pi)^2} M_{lj}(|q|) \times G_0(\omega - \Omega, j) D(\Omega, |q|).
\]  

(4.17)

The matrix elements for the transversal current operator are

\[
M_{lj}(|q|) = \frac{1}{N_\phi} \sum_{g, g'} |\langle lg | J_\perp(q) | lg' \rangle|^2,
\]  

(4.18)

where \( l \) and \( j \) label Landau levels and \( N_\phi \) is the degeneracy of the Landau levels labeled by \( g \) and \( g' \). The matrix elements \( M_{pp}(|q|) \) are evaluated in the Appendix [B].

As at half-filling, we compute the difference \( \Sigma(\omega, l) - \Sigma(0, l) \) rather than the self-energy directly. In the limit of large \( p \), most contributions to the sum over the Landau levels come from \( j \simeq p \). Since the matrix elements depend smoothly on \( j \), we may neglect their \( j \) dependency and set \( M_{lj}(q) \simeq M_{pp}(q) \) for \( l \simeq p \) and \( p \gg 1 \). Further, we approximate the sum over \( j \) by an integral that is performed closing the contour in the upper complex half-plane; the result of the integration is to restrict the frequency integration to \( \Omega \in [0, \omega] \). Performing the frequency integration, we obtain for the transversal part of the self-energy,

\[
\Sigma_\perp(\omega, l) - \Sigma_\perp(0, l) \simeq -\frac{k_F^2}{E_0} \int_0^\infty du u^2 M_{pp}(k_F u) H(u),
\]  

(4.19)
where we have defined
\[ H(u) = \frac{i}{2} \ln \left[ 1 + \left( u^{x-3} \frac{\omega}{\omega_{\text{int}}} \right)^2 \right] + \arctan \left[ u^{x-3} \frac{\omega}{\omega_{\text{int}}} \right], \tag{4.20} \]
with the dimensionless momentum \( u = q/k_F \). As explained in Appendix B, the matrix elements \( M_{pp}(q) \) are non-negligible only for momenta satisfying
\[ q/k_F \in \left[ (2p + 1)^{-1}, 1 \right]. \tag{4.21} \]
In this range, we have from Eq. (B10)
\[ M_{pp}(q) \simeq \frac{k_F^2}{\pi m^2 (2p + 1)q}. \tag{4.22} \]
Inserting this approximation in Eq. (4.19) gives
\[ \Sigma \perp (\omega, l) - \Sigma \perp (0, l) \simeq -\frac{k_F^2}{\pi m} \frac{I(u_{\text{max}}) - I(u_{\text{min}})}{u_{\text{max}}^2}, \tag{4.23} \]
where \( I(v) \) is defined in Eq. (3.8) and computed in Appendix A. We have introduced
\[ u_{\text{max}} = \left( \frac{\omega_{\text{int}}}{\omega} \right)^{1/(3-x)} \quad \text{and} \quad u_{\text{min}} = \frac{u_{\text{max}}}{2p + 1}. \tag{4.24} \]
We are interested in the value of the self-energy at the energy gap, \( \omega = E_g(p) \). Anticipating the results from the HLR conjecture, we expect \( u_{\text{max}} \sim \sqrt{2p + 1} \) to be large for \( p \gg 1 \), while \( u_{\text{min}} \sim 1/\sqrt{2p + 1} \) is expected to be small. Inserting the approximations (A3) and (A5) of \( I(v) \) for small and large arguments in Eq. (4.23) gives at the gap energy \( \omega = E_g(p) \) with \( \Delta \omega_c^* = E_g(p)/\omega_{\text{int}} \)
\[ \Sigma \perp (E_g(p), p) - \Sigma \perp (0, p) \simeq -\frac{k_F^2}{2\pi m} \left\{ F_0(x) \left( \Delta \omega_c^* \right)^{2/(3-x)} + \frac{2\Delta \omega_c^*}{x - 1} + \cdots \right\}. \tag{4.25} \]
Inserting the result \( \Delta \omega_c^* \sim (2p + 1)^{-(3-x)/2} \) expected from the HLR-conjecture indicates that the first term in curly brackets on the right-hand side of Eq. (4.25) is of order \( 1/(2p + 1) \). The second term is proportional to \( (2p + 1)^{(x-3)/2} \) and leads to the logarithmic behavior for \( x \to 1 \).

Inserting the obtained expression for the self-energy in the gap equation (4.16) gives after taking the limit of weak interaction \( \omega_{\text{int}}/\omega_c \to 0 \),
\[ \Delta \omega_c^* \simeq \frac{2\pi}{2p + 1} \left[ F_0(x) \left( \Delta \omega_c^* \right)^{(x-1)/(3-x)} + \frac{2}{x - 1} + \cdots \right]^{-1}, \tag{4.26} \]
where the dots stand for subleading terms in $p$, which corresponds to the gap equation (4.8).

For systems with Coulomb interaction, we obtain from Eq. (4.23)

$$
\Sigma_\perp (E_g(p), l) - \Sigma_\perp (0, l) \simeq \frac{k_F^2}{2\pi m} \left\{ \left( -\ln \left( \Delta \omega^*_c \right) + \frac{i\pi}{2} + C_1 \right) \Delta \omega^*_c + \cdots \right\},
$$

(4.27)

where we have neglected terms of order $\ln \Delta \omega^*_c/(2p + 1)^2$ and $\ln(2p + 1)/(2p + 1)^2$. The gap equation is then

$$
\Delta \omega^*_c \simeq \frac{2\pi}{2p + 1} \left[ -\ln \left( \Delta \omega^*_c \right) + \frac{i\pi}{2} + C_1 + \cdots \right]^{-1},
$$

(4.28)

where the dots represent subleading terms in $p$. Again, the gap equation (4.3) is recovered for large $p$.

V. CONCLUSION

In this work, we have computed the long wavelength, low energy contribution to the effective mass of the composite fermions at half-filling for different electron-electron interactions. Our results, obtained in the fermion-Chern-Simons approach, are consistent with the effective mass obtained previously in the literature. [5, 7, 8]

For short range interactions, we find an effective mass that diverges following a power-law. The diverging imaginary part of the effective mass at the one-loop level indicates that the Fermi liquid picture breakdown [7, 8] for short range interactions. Nevertheless, we argued that the power-law behavior found in the one-loop approximation is exact, i.e., is not modified by higher-order terms. However, unlike in the case of Coulomb interaction, higher orders contribute to the most divergent part of the effective mass. The logarithmic divergence of the effective mass obtained for the Coulomb interaction [5, 6] is recovered as the interaction tends to the Coulomb interaction. For long range interactions, the effective mass is finite, and presumably real if higher-order contributions are properly included.

In the second part of this work, we have considered the energy gap, which can be either computed from the effective mass of the composite fermions using the HLR conjecture [5] or directly within the fermion-Chern-Simons picture in a reduced magnetic field $\Delta B$. [6] Both approaches lead to the same self-consistent equation for the energy gap. For short range interaction, the energy gap tends to zero faster than the mean-field prediction when the filling factor approaches $1/2$. Moreover, as for the effective mass, the power-law behavior
predicted at the one-loop level turns out to be exact to all orders in perturbation theory, but the prefactor of the leading term of the energy gap gets contributions from all orders in perturbation theory.

At the one-loop level, the computed value for the energy gap has a large imaginary part for short range interactions. This imaginary part is unphysical since the quantum Hall states are gaped, i.e., cannot decay. Thus, including properly all terms in the perturbation expansion should lead to the exact energy gap, which behavior is described by the power-law computed at the one-loop level.

As already pointed out in the introduction, the leading contributions to the self-energy arise from strong, long wavelength gauge fluctuations that are not suppressed at large distances if the interaction is short ranged. The system is then described by a gas of strongly coupled composite fermions; in particular, as predicted by renormalization theory, the perturbative Fermi liquid description of the system is not valid. In this context, the question whether the HLR conjecture under the form of Eq. (1.1) is valid remains open. For long range interaction \((x > 1)\), the HLR conjecture is probably exact, since the Fermi liquid behavior at half-filling is well established by renormalization analysis; for Coulomb interaction, the (exact) one-loop results of Stern and Halperin confirm this conjecture. The present work established that for short range interactions, the HLR conjecture gives the right power-law behavior for the energy gap. However, the HLR conjecture in the form of Eq. (1.1) cannot be exact if the effective mass of the composite fermions at half-filling is not real. In this case, one may have to modify the HLR conjecture, replacing, for instance, the effective mass by its real part in Eq. (1.1). Alternatively, an imaginary cutoff in the effective mass might be used, the phase of the cutoff being fixed by the requirement that the solution of the gap equation has to be real.

As established in previous works, the agreement between the energy gap computed in the fermion-Chern-Simons approach with Coulomb interaction, and the same quantity obtained by exact diagonalization for the filling factors \(\nu = 1/3, 2/5, 3/7 \text{ and } 4/9\) is surprisingly good. The same numerical computations have been repeated for the same filling factors, but for different interactions of short and long ranges by Morf. The results for the energy gap are not consistent with our predictions. In particular, the energy gap as function of the filling factor decays slower for the short range interaction, than it does for the Coulomb interactions, and slower for Coulomb interaction, than for the long range interaction, as
half-filling is approached from below.  

The origin of the discrepancy between the numerical results of Morf and our theoretical calculations can have different explanations. First, recall that our theoretical calculations are valid in the long wavelength limit, and for filling factors asymptotically approaching half-filling ($p \to \infty$). In exact diagonalization studies, however, small systems with filling factor up to $\nu = 4/9$, i.e., $p \leq 4$, are considered. One cannot exclude that, for small $p$'s, the self-energy is dominated by the short wavelength rather than by the long wavelength contributions.

Further, numerical studies of small samples may probe mainly the contributions coming from the short range part of the interaction potential, which is not accessible within our long wavelength approach, leading to the observed discrepancies. In order to circumvent this problem, one might perform exact diagonalization computations for different interaction potentials, all having the same short range part (up to a distance of order of a few magnetic lengths), but with different (power-law) behaviors for the long range part. If numerical studies are oversensitive to the short range part of the interaction, this procedure would introduce a systematic error, independent of the interaction type.

More general questions concerning the quantum Hall states in the vicinity of half-filling remain open. For instance, the sum of the jumps in the chemical potential occurring at each filling factor $\nu = p/(2p + 1)$ diverges. This suggests that the Jain series could be interrupted for some value of $p$; in particular, for short range interactions, phase separation might occur. In real sample, phase separation might be triggered even in the case of long range or Coulomb interactions by density variations induced by disorder. However, the FCS approach is probably not appropriated to answer such questions, which could be studied by Monte-Carlo simulations.

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APPENDIX A: THE INTEGRAL $I(v)$

We shortly evaluate the integral

$$I(v) = \int_0^v du \left\{ \frac{i}{2} \ln \left[ 1 + u^{2x-6} \right] + \arctan u^{x-3} \right\}$$

(A1)

defined in Eq. (3.8). In particular, for $x = 1$,

$$I(v) = \frac{i}{2} \arctan v^2 + \frac{i}{4} v^2 \ln \left[ 1 + v^{-4} \right]$$

$$+ \frac{v^2}{2} \arctan v^{-2} + \frac{1}{4} \ln \left( 1 + v^4 \right).$$

(A2)

For small arguments $v \ll 1$, the integral $I(v)$ is small since

$$I(v) \simeq -\frac{3 - x}{2} v^2 \left( \ln v - \frac{1}{2} + i \frac{\pi}{2(3 - x)} \right)$$

$$+ \frac{3 - x}{5 - x} v^{3-x} + \mathcal{O}(v^{5-x}, iv^{8-2x}).$$

(A3)

In the limit $x \to 1$, we obtain, for $v \ll 1$,

$$I(v) \xrightarrow{x=1} -iv^2 \left( \ln v - \frac{1}{2} + i \frac{\pi}{4} + i \frac{1}{2} \right) + \mathcal{O}(v^4).$$

(A4)

We now turn to large arguments, $v \gg 1$. In order to expand the integrand of (A1) in powers of $u^{x-3}$, we have to split the integration over $u$ in two regions delimited by an arbitrary constant $V \geq 1$. For the imaginary part of $I(v)$, we may choose $V \to \infty$, while for the real part of $I(v)$, we set $V = 1$. The first terms in the expansion in powers of $1/v$ are

$$I(v) \simeq \frac{F_0(x)}{2} + \frac{v^{x-1}}{x-1} + iv^{2x-4} + \frac{v^{2x-4}}{4x-8} + \mathcal{O}(v^{3x-7}, iv^{4x-10})$$

(A5)

where

$$\text{Re } F_0(x) = \frac{\pi(3-x)}{4}$$

$$- \sum_{n \geq 0} \frac{4(3-x)(-1)^n}{(2n+1)(4-(2n+1)^2)(3-x)^2}$$

$$\text{Im } F_0(x) = (3-x) \int_0^\infty \frac{u \, du}{1+u^{6-2x}}.$$

(A6)

In the limit $x \to 1$, this function tends to

$$F_0(x \to 1) \simeq -\frac{2}{x-1} + i \frac{\pi}{2}$$

(A7)

Hence, dropping all terms of order $(x-1)$ in Eq. (A5) leads to

$$I(v) \xrightarrow{x=1} \frac{v^{x-1}}{x-1} + i \frac{\pi}{4} - i \frac{1}{4v^2} + \mathcal{O}(v^{-4}, iv^{-6})$$

(A8)
The matrix elements
\[ M_{ll'}(q) = \frac{1}{N_\phi} \sum_{g,g'} |\langle l| J_\perp(q)|l'\rangle|^2 \] (B1)
can be evaluated choosing the \( x \) axis in the \( q \) direction, such that the transversal current operator becomes
\[ J_\perp(q) = v_y e^{i|q|x}, \] (B2)
where \( x \) and \( v_y \) are the position and velocity operators. In terms of raising and lowering operators, they have the representation
\[ x = \frac{1}{\sqrt{2mE_0}} \left( a^+ + a + b^+ + b \right) \]
\[ v_y = \sqrt{\frac{E_0}{2m}} (a^+ + a) \] (B3)
where \( a^+ \) and \( a \) are the inter-Landau level creation and annihilation operators, while \( b^+ \) and \( b \) are the intralevel creation and annihilation operators. Defining \( x_a = (a^+ + a) / \sqrt{2mE_0} \), we may decompose \( J_\perp(q) \) in the product of the operators \( v_y e^{i|q|x_a} \) and \( e^{i|q|\bar{x}_b} \), where the inter- and intralevel operators are well separated. After a simple calculation, we find
\[ M_{ll'}(|q|) = |\langle l| v_y e^{i|q|x_a} |l'\rangle|^2 \]
\[ = E_0^2 \left| \frac{\partial}{\partial |q|} \langle l| e^{i|q|x_a} |l'\rangle \right|^2. \] (B4)

We need to evaluate the matrix element \( M_{pp}(q) \) for large \( p \). Using the a semiclassical approximation for the harmonic oscillator,
\[ \langle p| e^{i|q|x_a} |p\rangle \simeq \frac{E_0}{2\pi} \int_0^{2\pi} dt e^{iqR_c \sin(E_0t)} = J_0(R_c q), \] (B5)
where \( J_0(x) \) is the zeroth Bessel’s function, and
\[ R_c = \sqrt{\frac{2p + 1}{mE_0}} = (2p + 1)/k_F \] (B6)
is the cyclotron radius in the \( p \)th Landau level. The semiclassical approximation is valid only for
\[ q \lesssim \sqrt{mE_0(2p + 1)} = k_F. \] (B7)
For bigger $q$, the matrix element $M_{pp}(q)$ falls off exponentially. In the range of validity of the semiclassical approximation, Eq. (B4) becomes

$$M_{pp}(q) \simeq E_0^2 R_c^2 J_1^2(R_cq).$$

(B8)

Using the asymptotic forms of the Bessel’s function for small and large arguments,

$$J_1(x) \simeq \begin{cases} 
  x/2, & x \ll 1 \\
  \sqrt{\frac{2}{\pi x}} \cos(x - 3\pi/4), & x \gg 1,
\end{cases}$$

(B9)

finally, gives (using $E_0 R_c = k_F/m$ and replacing the fast oscillating cosine by its average value)

$$M_{pp}(q) \simeq \begin{cases} 
  \frac{k_F}{4 \pi m} (2p+1)^2(q/k_F)^2, & q/k_F \ll 1/(2p+1) \\
  \frac{k_F}{\pi m^2} \frac{k_F}{(2p+1)q}, & 1/(2p+1) \ll q/k_F \lesssim 1 \\
  0, & q \geq k_F.
\end{cases}$$

(B10)

Observe that this approximation corrects Eq. (38) of Ref. 6; however, the final result for the energy gap found by Stern and Halperin was not affected.

**APPENDIX C: HALDANE PSEUDOPOTENTIAL**

The Haldane pseudopotentials [18] are defined as the interaction energy of electron pair with given relative angular momentum $m$.

The two-body states in the lowest Landau level are described by the wave functions, [22]

$$\psi_m(z, \bar{z}) = \frac{z^m e^{-z\bar{z}/8\ell^2}}{2^{m+1}(m+1)! \sqrt{\pi m!}},$$

(C1)

where $z = x - iy$ are the relative coordinates of the two electrons, and $m$ is their relative angular momentum. The Haldane pseudopotentials are the expectation value of the potential energy in a state of fixed angular momentum:

$$V_m^{(x)} = \int d^2r \, |\psi_m(r)|^2 V_x(r)$$

$$= \frac{V_x(2\ell)}{m!} \Gamma(m + \frac{x}{2}),$$

(C2)

where $\ell$ is the magnetic length. For Coulomb repulsion between the electrons, the Haldane pseudopotentials are

$$V_m^{Cb} = \frac{e^2}{\varepsilon 2\ell} \sqrt{\pi} \frac{(2m-1)!!}{2^m m!}.$$
It can be easily verified that the interactions $V_x(r)$ for $0 < x < 3/2$ satisfy to the inequalities

\begin{align}
V_m > V_{m+2} \\
V_{m-2} - V_m > V_m - V_{m+2}.
\end{align}

Moreover, the highest slope $V_1^{(x)} - V_3^{(x)}$ is significantly bigger than the second highest slope $V_3^{(x)} - V_5^{(x)}$. These "hard-core" properties \cite{Wojys} seems to be necessary to the formation of quantum Hall states. For the interactions considered, i.e., for $0 < x < 3/2$, the occurrences of quantum Hall states at the filling factors $\nu = 1/3, 2/5, 3/7, \text{ and } 4/9$ have been tested numerically. \cite{Jain1989PhysRevLett63199}

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  \item [13] The Fourier transform of the interaction is $\hat{V}(k) = \alpha(x)\hat{V}(k_F)k_F^x/|k|^x$, with $\alpha(x) = \int_0^{\infty} du u^{x-1} J_0(u)$, where $J_0$ is Bessel’s function and $u = r|k|$. For $0 < x < 3/2$, the integral is well-defined. For $x = 3/2$, the integral $\alpha(x)$ diverges at large distance, while for $x = 0$, the integral diverges for $u \to 0$. We will not consider these cases in the present work.
\end{itemize}
The short range interactions considered in this work are rather unusual, being of the form $V(r) \sim 1/r^\eta$, where $1 < \eta < 2$. Interactions falling off as $1/r^2$ or faster, the short wavelength part of the gauge propagator has to be taken into account. The energy scale is then fixed by

$$\omega_{int} = \frac{k_F^2 V(k_F)}{4\pi\phi} + \frac{k_F^2}{2m}(\phi/6 - 1/\phi).$$

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For the Coulomb interaction, one has to separate between small and large frequencies, but consider the whole range of momenta. This is due to the fact that the divergent contribution to the effective mass comes from small frequencies, but from a wide range of momenta.

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