Quantum communication through Jaynes-Cummings-Hubbard arrays

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Abstract
We study the dynamics of an one dimensional array of Jaynes-Cummings-Hubbard system of arbitrary number of coupled cavities, each containing a two level atom that interacts with a field mode. In particular, we consider propagation of a single excitation quantum state for two different couplings of the photonic modes of the adjacent cavities, namely, a translation invariant closed chain of uniformly coupled cavities, and also a linear chain with nonuniform parabolically varying intercavity coupling where the interaction Hamiltonian is associated with the Jacobi matrix of the Krawtchouk polynomials. Using a description via the delocalized atomic and field modes we observe that for a large detuning of these two degrees of freedom atomic excitations propagate without populating the field modes, and \textit{vice versa}. For the near-resonance scenario between these modes the atomic excitations, say, while propagating mix with the photonic states. In the context of the parabolic coupling between photons of adjacent cavities an arbitrary element of the time-dependent correlation function between two arbitrary cavities may be expressed in closed form for dominant values of the detuning parameter, when an exact transmission of the quantum state at pre-specified times is realized.

I Introduction
Recently considerable theoretical and experimental attention has been devoted to a class of models involving coupled optical cavities where photons interact with embedded two level atoms. On the experimental side these models have been facilitated by experimental advance in photonic crystals \cite{1}, optical microcavities \cite{2}, and superconducting devices \cite{3}. These coupled cavity structures have potential applications as quantum optical simulators \cite{4,5} of condensed-matter phenomena making it feasible to study many-body effects such as phase transitions where the particles of interest are photons. The coupling between cavities also provides a setting for the system being considered as a wave guide in the context of distributed quantum information processing \cite{6}. Allowing for control and measurement at individual lattice sites proposals have been put forward for generating entangled photonic states \cite{7}, creation of cluster states \cite{8,9}, and transfer of a quantum state along an array of polaritonic qubits \cite{10}. Towards investigating the transmission of quantum states in an one-dimensional coupled array of cavities much study has been done in the context of Jaynes-Cummings-Hubbard model \cite{11}. The atom-cavity photon interaction is described by the well-understood Jaynes-Cummings model \cite{12} that relies on rotating wave approximation. Tunneling of photons between adjacent cavities is introduced via a hopping parameter. As the number of excitations in the system remain conserved it is, from the point of view of propagation of quantum states, of importance to develop a detailed understanding of the time evolution of single excitation states. Studying a system of two coupled cavities the authors of Ref. \cite{13} introduced delocalized modes to reveal that the detuning parameters between the atoms and these modes govern the dynamics of the propagation. The possibility of control of the individual coupling constants between the adjacent cavities has been investigated in \cite{14}. For the choice of a coupling that varies parabolically on the cavity sites a dispersion-free propagation of the single excitation states has been observed \cite{14}. Following the procedure of \cite{13} here we study the propagation of atomic and photonic states between two sites of an array of cavities with two different choices of coupling coefficients linking photons of adjacent cavities. Generalizing the results of \cite{13} we first consider a translation invariant closed chain of $N$ identical cavities with periodic boundary condition where the coupling coefficient between the photons of nearest-neighbor cavities is held uniform. Introducing the delocalized atomic and photonic modes we follow a
time-averaging procedure \cite{15, 13} to perturbatively evaluate the effective Hamiltonian for the three limiting cases: (i) dominant hopping parameter, (ii) large detuning between the atomic and photonic frequencies, and (iii) the case when the resonance between the atomic frequency and an eigenfrequency of a delocalized photonic mode is realized. It is interesting to note that in terms of delocalized atomic coordinates a diagonal spin-spin effective interaction that directly transfers energy without the intermediacy of a photonic process is introduced.

Allowing for the precise control of the coupling constant between the adjacent sites the authors of Ref. \cite{14} introduced the parabolic coupling scenario. In the context of one dimensional spin chains \cite{16} it has been observed \cite{17} that the parabolic coupling between nearest-neighbor spins leads to exact transmission of a single excitation quantum state. In the limit of a dominant hopping parameter between the photons of the adjacent sites perfect transfer of a quantum state at pre-specified times is also realized in the present case. In the model studied here the atom-photon coupling at individual cavities introduces a dispersive effect that prohibits perfect transfer of a quantum state for large values of the said coupling. However, in another limiting case when the detuning parameter is large compared to both the hopping parameter and the atom-cavity photon coupling constant, exact propagation of the quantum state is also obtained with an increment in the required time. For these cases the exact time-dependent correlation function of the quantum state between two arbitrary cavities is obtained in a closed form. The plan of the paper is as follows. In Sec. II we discuss the case of uniform coupling between the photons of adjacent cavities arranged periodically on a closed loop. Our discussion of the case of parabolic coupling between the photons of adjacent cavities is contained in Sec. III. Following this we conclude.

II Cyclic cavities with uniform coupling

Here we consider a cyclic chain of $N$ identical cavities obeying periodic boundary condition, and each containing a two-level atom that is coupled to a localized photonic mode modeled as a harmonic oscillator. The transition frequency $\varepsilon$ of the atoms and the frequency $\Omega$ of the oscillators are held uniform over the chain. The atom-cavity photon interaction is described by Jaynes-Cummings model with the adoption of rotating wave approximation. The adjacent cavities are interlinked via photon hopping with uniform couplings between neighboring cavities. The Hamiltonian may be expressed as a sum of the Jaynes-Cummings Hamiltonians for identical cavities and a nearest-neighbor photon hopping term:

$$H = \sum_{j=0}^{N-1} H_{j}^{JC} + H_{\text{hop}},$$

where the Jaynes-Cummings Hamiltonian for the $j$-th cavity and the hopping term between adjacent cavities, respectively, read:

$$H_{j}^{JC} = \varepsilon \sigma_{j}^{+} \sigma_{j}^{-} + \Omega a_{j}^{\dagger} a_{j} + g \left( \sigma_{j}^{+} a_{j} + \sigma_{j}^{-} a_{j}^{\dagger} \right),$$

$$H_{\text{hop}} = \kappa \sum_{j,k=0}^{N-1} a_{j}^{\dagger} C_{j,k} a_{k}, \quad C_{j,k} = \delta_{j,k+1} + \delta_{j+1,k}. \quad (2.2)$$

In the above expression the $N$-th and the 0-th degrees of freedom are identified. The standard commutation relations for the photonic and the atomic modes pertaining to individual cavities are given below:

$$[a_{j}, a_{k}^{\dagger}] = \delta_{jk}, \quad [\sigma_{j}^{+}, \sigma_{k}^{-}] = \pm 2 \delta_{jk} \sigma_{k}^{\pm}, \quad [\sigma_{j}^{+}, \sigma_{k}^{+}] = \delta_{jk} \sigma_{k}^{\pm}. \quad (2.3)$$

The operator $N$ that represents the total number of atomic and photonic excitations of the combined system commutes with the Hamiltonian (2.1):

$$N = \sum_{j=0}^{N-1} \left( \sigma_{j}^{+} \sigma_{j}^{-} + a_{j}^{\dagger} a_{j} \right), \quad [H, N] = 0. \quad (2.4)$$

In this work we focus on the time-evolution of single excitation states that may be arbitrary superpositions of one excitation states of atoms and photons. The ground states of the photonic and atomic systems may
be listed as \(|0\rangle \equiv \{|0_j\rangle\}, \{|G\rangle\} \equiv \{|g_j\rangle\}\) for \(j = 0, 1, \ldots, N - 1\). The photonic and atomic one excitation states localized in \(j\)-th cavity are created as follows:

\[
a_j^\dagger |0\rangle = |1_j\rangle, \quad a_j |1_k\rangle = \delta_{j,k} |0\rangle, \quad \sigma^+_j |G\rangle = |e_j\rangle, \quad \sigma^-_j |e_k\rangle = \delta_{j,k} |G\rangle.
\]

(2.5)

To diagonalize the photonic part of the Hamiltonian we need to introduce delocalized coordinates unitarily related to their local analogs pertaining to a cavity. For \(\omega\) being a root of unity the unitary transformation reads

\[
U_{jk} = \frac{1}{\sqrt{N}} \omega^{jk}, \quad \forall j, k = 0, 1, \ldots, N - 1, \quad \omega^N = 1 \Rightarrow \omega = \exp (2\pi i/N),
\]

(2.6)

and the unitarity constraint readily follows:

\[
\frac{1}{N} \sum_{j=0}^{N-1} \omega^{j(k-l)} = \delta_{kl} \Rightarrow UU^\dagger = U^\dagger U = I.
\]

(2.7)

We note that here and elsewhere in this section we follow the mod \(N\) arithmetic. Now the delocalized photonic coordinates obeying canonical commutation relations are given by

\[
A_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} a_k, \quad A_j^\dagger = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} a_k^\dagger \Rightarrow [A_j, A_k^\dagger] = \delta_{j,k}.
\]

(2.8)

It turns out that the atomic coordinates that couple with the delocalized photonic coordinates in the reconstructed interaction Hamiltonian are also delocalized in nature. These collective spin variables are introduced as discrete Fourier transforms

\[
S_j^\pm = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} \sigma_k^\pm, \quad S_j^z = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} \sigma_k^z, \quad (S_j^\pm)^\dagger = S_j^\mp, \quad (S_j^z)^\dagger = S_j^z.
\]

(2.9)

obeying a closed algebra:

\[
[S_j^z, S_k^\pm] = \pm \frac{2}{N} S_k^\pm, \quad [S_j^+, S_k^-] = S_j^z.
\]

(2.10)

The unitary operator \((2.6)\) diagonalizes the coupling matrix between adjacent cavities given in \((2.2)\):

\[
(U C U^\dagger)_{jk} = (\omega^j + \omega^{-j}) \delta_{j,k} = 2 \cos \left(\frac{2\pi}{N}\right) \delta_{j,k}.
\]

(2.11)

Recasting the Hamiltonian \((2.1)\) via the delocalized coordinates we obtain

\[
H = H_0 + H^{\text{int}}, \quad H_0 = H_0^{\text{cavity}} + H_0^{\text{photon}},
\]

(2.12)

where

\[
H_0^{\text{cavity}} = \varepsilon \sum_{j=0}^{N-1} S_j^+ S_j^- - \sum_{j=0}^{N-1} \Omega_j A_j^\dagger A_j, \quad \Omega_j = \Omega + 2\kappa \cos \left(\frac{2\pi}{N}\right).
\]

(2.13)

For \(N > 2\) the eigenfrequencies \(\Omega_j\) of the delocalized modes \(A_j\) are degenerate: \(\Omega_j = \Omega_{N-j}\) for \(j \neq (N-j) \mod N\). The non-degenerate eigenfrequencies correspond to the ‘center-of-mass’ mode \(A_0\) and ‘alternating’ mode \(A_{N/2}\) for even \(N\). The delocalized coordinates introduced above maintains the structure that reflects the rotating wave approximation for the atom-cavity photon interaction term \(H^{\text{int}}\) in \((2.12)\):

\[
H^{\text{int}} = g \sum_{j=0}^{N-1} \left(S_j^+ A_j + S_j^- A_j^\dagger\right).
\]

(2.14)

The total number of excitation operator \((2.4)\) now assumes the form

\[
N = \sum_{j=0}^{N-1} \left(S_j^+ S_j^- + A_j^\dagger A_j\right).
\]

(2.15)
The delocalized photonic and atomic single-excitation states may be introduced via invertible unitary transformation as follows:

\[
|\tilde{1}_j\rangle = A_j^\dagger |0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} |1_k\rangle, \quad A_j |\tilde{1}_k\rangle = \delta_{jk} |0\rangle, \quad \langle \tilde{1}_j |\tilde{1}_k\rangle = \delta_{jk},
\]

\[
|\vartheta_j\rangle = S_j^+ |\Gamma\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} |\vartheta_k\rangle, \quad S_j^- |\vartheta_k\rangle = \delta_{jk} |\Gamma\rangle, \quad \langle \vartheta_j |\vartheta_k\rangle = \delta_{jk}.
\] (2.16)

The localized and the delocalized one excitation states introduced in (2.5) and (2.16) are two sets of mutually unbiased bases:

\[
\langle 1_j |\tilde{1}_k\rangle = \frac{1}{\sqrt{N}} \omega^{-jk}, \quad \langle e_j |\vartheta_k\rangle = \frac{1}{\sqrt{N}} \omega^{-jk} \Rightarrow |\langle 1_j |\tilde{1}_k\rangle| = |\langle e_j |\vartheta_k\rangle| = \frac{1}{\sqrt{N}}
\] (2.17)

The distinction between the localized and delocalized states may be summarized by introducing a unitary shift operator that translates localized photonic and atomic states through a single cavity. Specifically, it acts on the single excitation states as follows:

\[
\tau |1_j\rangle = |1_{j+1}\rangle, \quad \tau^{-1} |1_j\rangle = |1_{j-1}\rangle, \quad \tau |e_j\rangle = |e_{j+1}\rangle, \quad \tau^{-1} |e_j\rangle = |e_{j-1}\rangle, \quad \tau^\dagger = \tau^{-1}.
\] (2.18)

The construction (2.16) makes it apparent that the delocalized states are invariant under the action of the shift operator \(\tau\) where the eigenvalues correspond to the root of unity phase angles:

\[
\tau |\tilde{1}_j\rangle = \omega^j |\tilde{1}_j\rangle, \quad \tau |\vartheta_j\rangle = \omega^j |\vartheta_j\rangle.
\] (2.19)

The delocalized photonic and atomic operators introduced in (2.8) and (2.9) transform as follows:

\[
\tau A_j \tau^{-1} = \omega^{-j} A_j, \quad \tau A_j^\dagger \tau^{-1} = \omega^j A_j^\dagger, \quad \tau S_j^\pm \tau^{-1} = \omega^{\pm j} S_j^\pm.
\] (2.20)

It is observed that the interaction Hamiltonian (2.11) is invariant under the shift transformation:

\[
\tau H^\text{int} \tau^{-1} = H^\text{int}.
\] (2.21)

The most general one excitation state may be equivalently expressed as linear compositions of either the localized or the delocalized basis states. These alternate expansions are listed, respectively, below:

\[
|\Psi(t)\rangle = \sum_{j=0}^{N-1} (a_j(t) |G\rangle \otimes |1_j\rangle + b_j(t) |e_j\rangle \otimes |0\rangle),
\] (2.22)

\[
|\Psi(t)\rangle = \sum_{j=0}^{N-1} (\alpha_j(t) |G\rangle \otimes |\tilde{1}_j\rangle + \beta_j(t) |\vartheta_j\rangle \otimes |0\rangle).
\] (2.23)

Invertible unitary transformations interrelate the coefficients of the above expansions (2.22) and (2.23):

\[
\alpha_j(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} a_k(t), \quad \beta_j(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{jk} b_k(t),
\] (2.24)

\[
a_j(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} \alpha_k(t), \quad b_j(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-jk} \beta_k(t).
\] (2.25)

The action of the Hamiltonian (2.12) on the two dimensional subspace \(|G\rangle \otimes |\tilde{1}_j\rangle, |\vartheta_j\rangle \otimes |0\rangle\) reads

\[
H |G\rangle \otimes |\tilde{1}_j\rangle = \Omega_j |G\rangle \otimes |\tilde{1}_j\rangle + g |\vartheta_j\rangle \otimes |0\rangle,
\]

\[
H |\vartheta_j\rangle \otimes |0\rangle = \varepsilon |\vartheta_j\rangle \otimes |0\rangle + g |G\rangle \otimes |\tilde{1}_j\rangle.
\] (2.26)
The Schrödinger equation for the state $|\Psi(t)\rangle$ now block diagonalizes the one excitation subspace of the Hamiltonian (2.12) in $2 \times 2$ blocks with the $j$-th block given by

$$H_j = \begin{pmatrix} \Omega_j & g \\ g & \varepsilon \end{pmatrix}. \tag{2.27}$$

The eigenvalues of the matrix $H_j$ and the corresponding eigenvectors in the two dimensional subspace $\{|G\rangle \otimes |\tilde{l}_j\rangle, |\vartheta_j\rangle \otimes |0\rangle\}$ read, respectively, as

$$\Omega_{j, \pm} = \frac{\Omega_j + \varepsilon}{2} \pm \chi_j, \quad \chi_j = \left( \left( \frac{\Delta_j}{2} \right)^2 + g \right)^{1/2}, \quad \Delta_j = \varepsilon - \Omega_j = \delta - 2\kappa \cos \left( j \frac{2\pi}{N} \right), \quad \delta = \varepsilon - \Omega, \quad \tag{2.28}$$

$$|\Omega_{j, \pm}\rangle = \left( (\Omega_{j, \pm} - \varepsilon)^2 + g^2 \right)^{-\frac{1}{2}} \left( (\Omega_{j, \pm} - \varepsilon) \ |G\rangle \otimes |\tilde{l}_j\rangle + g |\vartheta_j\rangle \otimes |0\rangle \right). \tag{2.29}$$

The detuning parameter $\delta$ measures the difference of the atomic frequency and that of the cavity photon, whereas the corresponding differences for the delocalized modes are given by the parameters $\{\Delta_j | j = 0, 1, \ldots, N - 1\}$. The time evolution of the coefficients in the expansion (2.24) of the state $|\Psi(t)\rangle$ in the delocalized basis is given by

$$\begin{pmatrix} \alpha_j(t) \\ \beta_j(t) \end{pmatrix} = \exp(-iH_jt) \begin{pmatrix} \alpha_j(0) \\ \beta_j(0) \end{pmatrix}, \tag{2.30}$$

where the explicit construction reads

$$\alpha_j(t) = \exp \left( -i \frac{\Omega_j + \varepsilon}{2} t \right) \begin{pmatrix} \cos(\chi_j t) + i \left( \frac{\Delta_j}{2} \right) \sin(\chi_j t) \chi_j & a_j(0) - i g \frac{\sin(\chi_j t)}{\chi_j} \beta_j(0) \end{pmatrix},$$

$$\beta_j(t) = \exp \left( -i \frac{\Omega_j + \varepsilon}{2} t \right) \begin{pmatrix} \cos(\chi_j t) - i \left( \frac{\Delta_j}{2} \right) \sin(\chi_j t) \chi_j & \beta_j(0) - i g \frac{\sin(\chi_j t)}{\chi_j} a_j(0) \end{pmatrix}. \tag{2.31}$$

To derive the effective Hamiltonian that discards the rapidly oscillating high-frequency components on account of a time-averaging process, we use the interaction-picture scenario. The interaction Hamiltonian given in (2.12) now assumes the form:

$$\mathcal{H}_{\text{int}}(t) = \exp(iH_0t) \mathcal{H} \exp(-iH_0t). \tag{2.32}$$

Its explicit evaluation following from (2.12,2.14) reads

$$\mathcal{H}_{\text{int}}(t) = g \sum_{j=0}^{N-1} \left( \exp(i\Delta_j t) S_j^+ A_j + \exp(-i\Delta_j t) S_j^- A_j^\dagger \right). \tag{2.33}$$

As we are interested in dynamical processes occurring at low frequencies, all high frequency components of the Hamiltonian are assumed to average out to zero resulting in a ‘coarse-grained’ effective Hamiltonian. Following the recipe given in [15,13] the time-averaged effective Hamiltonian up to the order $O(g^2)$ reads:

$$\mathcal{H}_{\text{eff}} = H_0 + \left\langle \mathcal{H}_{\text{int}}(t) \right\rangle + \frac{1}{2} \left\langle \left[ \mathcal{H}_{\text{int}}(t), \mathcal{V}(t) \right] \right\rangle - \frac{1}{2} \left\langle \left[ \mathcal{H}_{\text{int}}(t), [\mathcal{V}(t), \mathcal{V}(t)] \right] \right\rangle, \tag{2.34}$$

where the averaged value of a dynamical quantity $\mathcal{O}(t)$ with a suitable probability density function $f(t)$ is defined [15,13] as $\langle \mathcal{O}(t) \rangle = \int_{-\infty}^{\infty} f(t - t') \mathcal{O}(t') dt'$. The generating element $\mathcal{V}(t)$ of the order $O(g^2)$ contributions in the rhs of (2.34) may be obtained à la [15,13]:

$$\mathcal{V}(t) = g \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( \exp(-i\Delta_j t) S_j^+ A_j - \exp(i\Delta_j t) S_j^- A_j^\dagger \right). \tag{2.35}$$

We now study various limiting cases of the effective Hamiltonian (2.34).

The large hopping limit is dominated by the coupling between the photons of adjacent cavities: $\kappa \gg \delta, g$. We also assume $\Delta_j \gg g \ \forall j \in \{0, 1, \ldots, N - 1\}$. These limits ensure vanishing of the following time-averaged
averaged spin coordinate of the field modes. The expression (2.38) of the Hamiltonian via the de-localized modes makes it evident that the spin-spin interaction term in the effective Hamiltonian. The second sum is present due to the degeneracy of the photonic modes. It is interesting to note that the delocalized modes maintain the diagonal structure of allows us to obtain the effective Hamiltonian up to the order \( N > 2 \), and has been referred to in the context of (2.13). Explicit evaluation of the commutators following from (2.8) and (2.10)

\[
\begin{aligned}
[S_j^+A_j, S_j^-A_j^+] &= \frac{1}{2} \left( S_0^z \left( A_j^+ A_j + A_j A_j^+ \right) + (S_j^+ S_j^- + S_j^- S_j^+) \mathbb{1} \right), \\
[S_j^+A_j, S_{N-j}^+A_{N-j}^+] &= S_{2j-N}^z A_j A_{N-j}^+ \quad j \neq N-j \mod N 
\end{aligned}
\]  

(2.37)

allows us to obtain the effective Hamiltonian up to the order \( O(g^2) \): 

\[
\mathcal{H}_{\text{eff}} = H_0 + \frac{g^2}{2} \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_0^z \left( A_j^+ A_j + A_j A_j^+ \right) + (S_j^+ S_j^- + S_j^- S_j^+) \mathbb{1} \right) \\
+ g^2 \sum_{\Delta_j > j > 0} \frac{1}{\Delta_j} \left( S_{2j-N}^z A_j A_{N-j}^+ + S_{N-2j}^z A_j A_{N-j} \right). 
\]  

(2.38)

The second term in the rhs of (2.36) owes its origin to the degeneracy of the eigenfrequencies that is present for \( N > 2 \), and has been referred to in the context of (2.13). Explicit evaluation of the commutators following from (2.8) and (2.10)

\[
\begin{aligned}
[S_j^+A_j, S_j^-A_j^+] &= \frac{1}{2} \left( S_0^z \left( A_j^+ A_j + A_j A_j^+ \right) + (S_j^+ S_j^- + S_j^- S_j^+) \mathbb{1} \right), \\
[S_j^+A_j, S_{N-j}^+A_{N-j}^+] &= S_{2j-N}^z A_j A_{N-j}^+ \quad j \neq N-j \mod N 
\end{aligned}
\]  

(2.37)

allows us to obtain the effective Hamiltonian up to the order \( O(g^2) \): 

\[
\mathcal{H}_{\text{eff}} = H_0 + \frac{g^2}{2} \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_0^z \left( A_j^+ A_j + A_j A_j^+ \right) + (S_j^+ S_j^- + S_j^- S_j^+) \mathbb{1} \right) \\
+ g^2 \sum_{\Delta_j > j > 0} \frac{1}{\Delta_j} \left( S_{2j-N}^z A_j A_{N-j}^+ + S_{N-2j}^z A_j A_{N-j} \right). 
\]  

(2.38)

The first term in the rhs of (2.35) that may be read from (2.12) describes the energy of the bare delocalized atomic and field modes. The first sum represents the contribution due the Stark shift, and an induced interaction of order \( O(g^2) \) operating directly between the delocalized spin modes without the intermediacy of the photonic modes. It is interesting to note that the delocalized modes maintain the diagonal structure of the spin-spin interaction term in the effective Hamiltonian. The second sum is present due to the degeneracy of the field modes. The expression (2.38) of the Hamiltonian via the delocalized modes makes it evident that there is no transfer of energy between the atomic and the photonic modes. Employing the reduction of the ‘averaged’ spin coordinate \( S_0^z \) on the projectors \( S_0^z = \frac{1}{N} (P_e - P_g) \), where \( P_e = \sum_{j=0}^{N} |x_j \rangle \langle x_j| \forall x \in \{g,e\} \), and with a suitable rearrangement of the terms in the first sum in (2.38) we recast \( \mathcal{H}_{\text{eff}} \) as follows:

\[
\mathcal{H}_{\text{eff}} = H_0 + \frac{g^2}{N} \left( P_e \sum_{j=0}^{N-1} \frac{1}{\Delta_j} A_j A_j^+ - P_g \sum_{j=0}^{N-1} \frac{1}{\Delta_j} A_j A_j^+ + \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \sum_{k,l=0}^{N-1} \sum_{k \neq l} \omega^{-j(k-l)} \sigma_k^+ \sigma_l^- \mathbb{1} \\
+ \sum_{\Delta_j > j > 0} \frac{1}{\Delta_j} \sum_{k=0}^{N-1} \sigma_k^z \left( \omega^{-2jk} A_j A_{N-j}^+ + \omega^{2jk} A_j^+ A_{N-j} \right) \right). 
\]  

(2.39)

In the above expression the Stark shift depends upon the population of the delocalized photonic modes. The coupling of spins between any two arbitrary distinct site exist at the order \( O(g^2) \). For \( N = 2 \) the effective Hamiltonian (2.39) agrees with the result obtained in [13] at the corresponding limit. To make the process transparent we introduce a discrete Fourier transform on the reciprocal of the detuning parameters of the delocalized photonic modes:

\[
G(j) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\Delta_k} \omega^{-jk}, 
\]  

(2.40)

and thereby reexpress (2.39) in the following form:

\[
\mathcal{H}_{\text{eff}} = H_0 + \frac{g^2}{N} \left( P_e \sum_{j,k=0}^{N-1} a_j G(j-k) a_k^+ - P_g \sum_{j,k=0}^{N-1} a_j^+ G(j-k) a_k + N \sum_{j,k=0}^{N-1} \sigma_k^+ G(j-k) \sigma_j^- \mathbb{1} \\
+ \sum_{\Delta_j > j > 0} \frac{1}{\Delta_j} \sum_{k=0}^{N-1} \sigma_k^z \left( \omega^{-2jk} A_j A_{N-j}^+ + \omega^{2jk} A_j^+ A_{N-j} \right) \right). 
\]  

(2.41)
It is evident from the structure of (2.41) that the Fourier transform \( G(j) \) acts as the propagator of the atomic and the photonic excitations. The transition amplitudes of these excitations proceeding on the loop are proportional to the magnitude of the propagator.

To further examine the issue of transfer of the excitation modes we turn to the solutions (2.31). The following limits valid in the present approximation

\[
\frac{\Delta j}{2\chi_j \, g \ll |\Delta j|} \rightarrow \frac{\Delta j}{|\Delta j|}, \quad \frac{g}{\chi_j \, g \ll |\Delta j|} \rightarrow 0,
\]  

(2.42)
yields the structure

\[
\alpha_j(t) = \exp \left( -i \left( \Omega_j - \frac{g^2}{\Delta j} \right) t \right) \alpha_j(0), \quad \beta_j(t) = \exp \left( -i \left( \varepsilon + \frac{g^2}{\Delta j} \right) t \right) \beta_j(0).
\]  

(2.43)

It follows from the above equations for the delocalized modes that the atomic and photonic excitations remain decoupled from each other. Employing these solutions with the Fourier expansions (2.24, 2.25) we, in this limit, obtain the time-evolution equation for the coefficients of the localized single excitation states:

\[
a_j(t) = \frac{1}{N} \sum_{k, \ell = 0}^{N-1} \exp \left\{ -i \left( \frac{2\pi}{N} (j-k) \ell + \left( \Omega_\ell - \frac{g^2}{\Delta_\ell} \right) t \right\} a_k(0),
\]

\[
b_j(t) = \frac{1}{N} \sum_{k, \ell = 0}^{N-1} \exp \left\{ -i \left( \frac{2\pi}{N} (j-k) \ell + \left( \varepsilon + \frac{g^2}{\Delta_\ell} \right) t \right\} b_k(0).
\]  

(2.44)

Equation (2.44) indicates that subject to the approximation (2.42) transfer of excitations from the atomic to the photonic modes or vice versa does not take place, and, consequently, the probability density in each kind of degree of freedom is conserved:

\[
\sum_{j=0}^{N-1} |a_j(t)|^2 = \sum_{j=0}^{N-1} |a_j(0)|^2, \quad \sum_{j=0}^{N-1} |b_j(t)|^2 = \sum_{j=0}^{N-1} |b_j(0)|^2.
\]  

(2.45)

For definiteness, we assume that at \( t = 0 \) only the 0-th atom is in the excited state and all other degrees of freedom are in the ground state: \( a_j(0) = 0, \quad b_j(0) = \delta_{j0} \). Evolution equations in (2.44) now reduce to

\[
a_j(t) = 0, \quad b_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} \exp \left\{ -i \left( j k \frac{2\pi}{N} + \left( \varepsilon + \frac{g^2}{\Delta_k} \right) t \right\}.
\]  

(2.46)

For \( N \neq 2 \) exact transmission of the excitation does not take place.

We next study the large detuning limit: \( \delta \gg \kappa, g \) when the atoms are highly detuned from the photonic modes. In this limit also the following time-averaged properties hold: \( \langle \mathcal{H}^{\text{int}}(t) \rangle = 0, \quad \langle V(t) \rangle = 0 \). Towards obtaining the effective Hamiltonian we first evaluate the time-averaged commutator in this limit:

\[
\left\langle \left[ \mathcal{H}^{\text{int}}(t), V(t) \right] \right\rangle = g^2 \left\{ 2 \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left[ S_j^+ A_j, S_j^- A_j \right] + \sum_{j, k=0 \atop j \neq k}^{N-1} \left( \frac{1}{\Delta_j} + \frac{1}{\Delta_k} \right) \left[ S_j^+ A_j, S_k^- A_k \right] \right\}.
\]  

(2.47)

Employing the first equation in (2.37) and rearranging terms on the rhs of (2.47) the said commutator is reexpressed as

\[
\left\langle \left[ \mathcal{H}^{\text{int}}(t), V(t) \right] \right\rangle = g^2 \left\{ \sum_{j, k=0}^{N-1} \frac{1}{\Delta_j} \left( S_{j-k}^z A_j A_k^\dagger + S_{j-k}^\dagger A_j^\dagger A_k \right) + \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_j^+ S_j^- + S_j^z S_j^+ \right) \right\}.
\]  

(2.48)

The effective Hamiltonian up to the order \( O(g^2) \) may now be obtained via (2.34) and (2.48):

\[
\mathcal{H}^{\text{eff}} = H_0 + \frac{g^2}{2} \left\{ \sum_{j, k=0}^{N-1} \frac{1}{\Delta_j} \left( S_{j-k}^z A_j A_k^\dagger + S_{j-k}^\dagger A_j^\dagger A_k \right) + \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_j^+ S_j^- + S_j^z S_j^+ \right) \right\}.
\]  

(2.49)
In the $N = 2$ case the above expression reduces to the results obtained in \[3\] in the large detuning limit. The effective Hamiltonian \[49\] may be recast using the propagator in the Fourier space \[40\], and the transforms \[5, 8, 9\]:

$$
\mathcal{H}^{\text{eff}} = H_0 + \frac{g^2}{2} \sum_{j,k=0}^{N-1} \left( \sigma_k^+ \left( a_j^\dagger G(j-k) a_k + a_j G(j-k) a_k^\dagger \right) \right.
$$

$$
+ \left. \left( \sigma_j^+ G(j-k) \sigma_k^- + \sigma_j^- G(j-k) \sigma_k^+ \right) \right) \mathbf{1}.
$$

In this approximation also the atomic and the photonic modes remain decoupled from each other forbidding any transfer of energy up to the order $O(g^2)$. The time-evolution of the coefficients $a_j(t), b_j(t)$ of the localized single-excitation states reduces to previously described form \[44\]. Assuming dominant value of the detuning parameter $\delta \gg \kappa, g$ we now study the asymptotic $N \to \infty$ limit of \[44\] for the boundary condition $a_j(0) = \delta_{j0}, b_j(0) = 0$. Using the Fourier transform

$$
\exp(\pm ix \cos \theta) = \sum_{n=-\infty}^{\infty} (\pm i)^n J_n(x) \exp(\pm in\theta), \quad J_n(x) = (-1)^n J_n(x)
$$

via the Bessel functions $J_n(x)$, and employing the said boundary condition the time evolution of the coefficients of the localized states may be expressed as

$$
a_j(t) = \exp \left( -i \left( \Omega - \frac{g^2}{\delta} \right) t \right) \sum_{\nu=-\infty}^{\infty} (-i)^{\nu N} J_{\nu N} \left( 2\kappa \left( 1 - \frac{g^2}{\delta^2} \right) t \right), \quad b_j(t) = 0.
$$

In deriving \[52\] we have used the summation \[7\]. The asymptotic expansion of the Bessel function of large order \[13\] $J_N(N)|_{N \to \infty} \sim N^{-1/3}$ allows us to obtain an asymptotic scaling limit of the magnitude of the transition coefficient $|a_{N/2}(t \sim N)|_{N \to \infty} \sim N^{-1/3}$.

The case of comparable hopping and detuning is of particular interest. Turning towards this near-coherence scenario where the atomic frequency approximately equals to the eigenfrequency of, say the $\ell$-th delocalized photonic mode: $\varepsilon \approx \Omega_{\ell} \Rightarrow \Delta_{\ell} \approx 0$, we observe that the degeneracy \[13\] of the photonic eigenmodes present when $\ell \neq N - \ell \mod N$ needs to be taken into account. We also assume that the detuning parameters for all the nonresonant modes are large: $\Delta_j \gg 0 \forall j \notin \{ \ell, N - \ell (\neq \ell \mod N) \}$. Unlike the previous cases considered the present near-resonance limit allows a direct transfer of the excitation between the atomic and the photonic modes. The time-averaged contribution of the near-resonance mode to the $\mathcal{H}^{\text{int}}$ at the order $O(g)$ may be read from \[53\]:

$$
\left\langle \mathcal{H}^{\text{int}}(t) \right\rangle = g \left( \varepsilon A_\ell + \varepsilon \mathbf{1} \right) \left( S_{N-\ell}^+ A_{N-\ell} + S_{N-\ell}^+ A_{N-\ell}^\dagger \right),
$$

whereas its contribution to $\left\langle \mathcal{V}(t) \right\rangle$ follows from \[55\]:

$$
\left\langle \mathcal{V}(t) \right\rangle = \frac{g}{2\varepsilon} \left( \varepsilon A_\ell - \varepsilon \mathbf{1} \right) \left( S_{N-\ell}^+ A_{N-\ell} - S_{N-\ell}^+ A_{N-\ell}^\dagger \right).
$$

When a near-resonance scenario involving these degenerate eigenmodes are realized the said modes give rise to independent contributions associated with them, respectively. The commutator of the above time-averaged quantities reads

$$
\left[ \left\langle \mathcal{H}^{\text{int}}(t) \right\rangle, \left\langle \mathcal{V}(t) \right\rangle \right] = \frac{2g^2}{\Delta_{\ell}} \left( \varepsilon A_\ell, \varepsilon \mathbf{1} \right) \left( 1 - \delta_{\ell,N-\ell \mod N} \right) \Lambda_{\ell},
$$

$$
\Lambda_{\ell} = \left[ S_{N-\ell}^+ A_{N-\ell}, S_{N-\ell}^+ A_{N-\ell}^\dagger \right] + S_{2\ell-N}^+ A_{\ell} A_{N-\ell}^\dagger + S_{N-2\ell}^+ A_{\ell} A_{N-\ell}.
$$

Combining the previous results the effective Hamiltonian \[34\] in the near-resonance case up to the order
\( O(g^2) \) assumes the form

\[
\mathcal{H}_{\text{eff}} = H_0 + g \left( (S^+_\ell A_\ell + S^-_\ell A^\dagger_\ell) + (1 - \delta_{\ell, N-\ell} \mod N) \left( S^+_{N-\ell} A_{N-\ell} + S^-_{N-\ell} A^\dagger_{N-\ell} \right) \right) + g^2 \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S^+_j A_j A^\dagger_j + S^-_j S^-_j + S^+_j S^+_j \right) + g^2 \sum_{j' > j > 0} \frac{1}{\Delta_j} \left( S^+_{N-j} A_j A^\dagger_{N-j} + S^-_{N-2j} A^\dagger_j A_{N-j} \right),
\]

where the first sum on the rhs excludes the resonance modes \( \{ \ell, (N-\ell) \text{ when } \ell \not\equiv (N-\ell) \mod N \} \), and the resonance mode (\( \ell \)) lying in the domain of the index of the second sum is eliminated. The term \( O(g) \) in this Hamiltonian describes a transfer of energy between the near-resonant delocalized photon modes and their spin excitation partners. The non-resonant modes contributing in the first sum at order \( O(g^2) \) represent the Stark shift of the atoms and a direct transfer of excitations between the atomic modes without the intermediacy of the field modes. In the near-resonance condition we assume the detuning parameter of the resonant modes are negligible compared to the atom-cavity photon coupling: \(|\frac{\Delta}{\Delta_j}| \ll 1 \forall j \in \{ \ell, (N-\ell) \text{ when } \ell \not\equiv (N-\ell) \mod N \} \). Excitations of the near-resonant field modes and that of their coupled spin partners may be read from (2.31):

\[
\alpha_j(t) = \exp \left( -i \left( \varepsilon - \frac{\Delta_j}{2} \right) t \right) \left( \cos \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \alpha_j(0) - i \sin \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \beta_j(0) \right),
\]

\[
\beta_j(t) = \exp \left( -i \left( \varepsilon - \frac{\Delta_j}{2} \right) t \right) \left( \cos \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \beta_j(0) - i \sin \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \alpha_j(0) \right). \tag{2.57}
\]

Assuming that the detuning parameters for the non-resonant modes satisfy \(|\frac{\Delta}{\Delta_j}| \ll 1 \), where \( j \not\equiv \ell, (N-\ell) \) when \( \ell \not\equiv (N-\ell) \mod N \), the corresponding excitations assume the form (2.33).

The above evolution equations for the delocalized modes in conjunction with the Fourier transforms (2.24) now yield the single-excitation states of the atoms and the cavity photons. We first assume that the near resonant \( \ell \)-th mode is nondegenerate: \( \ell = (N-\ell) \mod N \). The \( \ell = 0, N/2 \) (for even \( N \)) modes satisfy this property. For the choice of the initial excitations being atomic in nature: \( a_j(0) = 0 \), the time evolution of the coefficients are given by

\[
a_j(t) = -\frac{i}{N} \exp \left( -i \left( \varepsilon - \frac{\Delta_j}{2} \right) t \right) \sin \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \sum_{k=0}^{N-1} \omega^{-\ell(j-k)} b_k(0),
\]

\[
b_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{n=0}^{N-1} \omega^{-n(j-k)} \exp \left( -i \left( \varepsilon + \frac{g}{\Delta_n} \right) t \right) \right.
\]

\[
+ \omega^{-\ell(j-k)} \exp \left( -i \left( \varepsilon - \frac{\Delta_j}{2} \right) t \right) \cos \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \left. \right) b_k(0). \tag{2.58}
\]

In the exact resonance condition \( \Delta_j = 0 \), and in a scenario where the dispersive effects are negligible \(|\frac{\Delta}{\Delta_j}| \approx 0 \forall j \not\equiv \ell \) the above solutions assume the form

\[
a_j(t) = -\frac{i}{N} \exp(-i\varepsilon t) \sin(gt) \sum_{k=0}^{N-1} \omega^{-\ell(j-k)} b_k(0),
\]

\[
b_j(t) = \exp(-i\varepsilon t) \left( b_j(0) - \frac{2}{N} \sin^2 \left( \frac{gt}{2} \right) \sum_{k=0}^{N-1} \omega^{-\ell(j-k)} b_k(0) \right). \tag{2.59}
\]

With the further assumption that only the 0-th atom is initially excited whereas all other atoms remain in the ground state \( b_j(0) = \delta_{j0} \forall j \in \{0, 1, \ldots, N-1\} \), the coefficients of the localized single-excitation states
read
\[ a_j(t) = -\frac{i}{N} \sin(gt) \exp \left( -i \left( j\ell \frac{2\pi}{N} + \varepsilon t \right) \right), \]
\[ b_j(t) = \exp(-i\varepsilon t) \delta_{j0} - \frac{2}{N} \sin^2 \left( \frac{gt}{2} \right) \exp \left( -i \left( j\ell \frac{2\pi}{N} + \varepsilon t \right) \right). \] (2.60)

For \( N = 2 \) the magnitude of the localized atomic excitation \( b_j(t) \) assumes unit value for both choices of the resonance modes \( \ell = 0, 1 \) at the specified time \( t = (2n + 1)\pi/\varepsilon \) \( \forall n \in \mathbb{Z}_+ \), making the transmission of the state exact.

Turning towards the resonance of the atomic mode with a conjugate pair of degenerate delocalized photonic modes \( \{ \ell, (N - \ell) \} \), where \( \ell \neq (N - \ell) \mod N \), we, following the preceding recipe, construct the time-evolutions of the excitations of the atoms and the cavity-photons:

\[ a_j(t) = -\frac{2i}{N} \exp \left( -i \left( \varepsilon - \frac{\Delta \ell}{2} \right) t \right) \sin \left( \left( g + \frac{\Delta^2}{8g} \right) t \right) \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} (j - k) \right) b_k(0), \]
\[ b_j(t) = \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{n \neq 0, \ell} \omega^{-n(j-k)} \exp \left( -i \left( \varepsilon + \frac{\Delta^2}{\Delta} \right) t \right) \right. \]
\[ + 2 \exp \left( -i \left( \varepsilon - \frac{\Delta \ell}{2} \right) t \right) \cos \left( \left( g + \frac{\Delta^2}{8g} \right) t \right) \cos \left( \frac{2\pi}{N} (j - k) \right) \left. \right) b_k(0). \] (2.61)

In the above equation we have assumed that the initial excitations are only atomic in nature. For the exact resonance condition \( \Delta \ell = 0 \) and in the dispersion-free limit \( \frac{\varepsilon}{\Delta} \approx 0 \) \( \forall j \notin \{ \ell, (N - \ell) \} \) the excitations reduce to the form

\[ a_j(t) = -\frac{2i}{N} \exp(-i\varepsilon t) \sin(gt) \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} (j - k) \right) b_k(0), \]
\[ b_j(t) = \exp(-i\varepsilon t) \left( b_j(0) - \frac{4}{N} \sin^2 \left( \frac{gt}{2} \right) \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} (j - k) \right) b_k(0) \right). \] (2.62)

Restraining to the case where only the 0-th atom is initially excited \( b_j(0) = \delta_{j0} \) \( \forall j \in \{0, 1, \ldots, N - 1\} \) the time evolution of the coefficients read

\[ a_j(t) = -\frac{i}{N} \sin(gt) \left( \exp \left( i \left( j\ell \frac{2\pi}{N} - \varepsilon t \right) \right) + \exp \left( -i \left( j\ell \frac{2\pi}{N} + \varepsilon t \right) \right) \right), \]
\[ b_j(t) = \exp(-i\varepsilon t) \delta_{j0} - \frac{2}{N} \sin^2 \left( \frac{gt}{2} \right) \left( \exp \left( i \left( j\ell \frac{2\pi}{N} - \varepsilon t \right) \right) + \exp \left( -i \left( j\ell \frac{2\pi}{N} + \varepsilon t \right) \right) \right). \] (2.63)

where a superposition of the clockwise and the anticlockwise modes takes place. For \( N = 4 \) and with the choice of the resonance mode \( \ell = 1 \) the magnitude of the excitation \( b_2(t) \) assumes unit value at \( t = (2n + 1)\pi/\varepsilon \) \( \forall n \in \mathbb{Z}_+ \).

### III Linear chain with parabolic coupling

Interesting situations arise if one considers nearest-neighbor couplings between the cavities to be nonuniform in nature. In particular it has been observed [14] that a parabolic coupling between the photons of adjacent cavities gives rise to dispersion free transmission of the excitations along one dimensional chain of cavities. The inter-cavity photonic hopping term of the Hamiltonian now reads

\[ H_{\text{hop}} = \kappa \sum_{j,k=0}^{N-1} a_j^\dagger C_{j,k} a_k, \quad C_{j,k} = \sqrt{j(N-j)} \delta_{j,k+1} + \sqrt{k(N-k)} \delta_{j+1,k}. \] (3.1)

The photonic part of the interaction Hamiltonian relates to the tridiagonal Jacobi matrix of the Krawtchouk polynomials [19]. Therefore the photonic degrees of freedom may be diagonalized using delocalized wave
functions expressed via these discrete orthogonal polynomials. Very briefly we now introduce the standard notations on the Krawtchouk polynomials. Detailed discussions may be obtained from Ref. [19].

The Krawtchouk polynomial of degree \( n \) \((n = 0, 1, \ldots, N)\) in the variable \( x \), with parameter \( 0 < p < 1 \) is given by

\[
K_n(x) \equiv K_n(x; p, N) = \frac{\sqrt{w(x)K_n(x)}}{\sqrt{d_n}}.
\]

The function \( 2F_1 \) is the classical hypergeometric series [20,21], and in this case it is a terminating series because of the appearance of the negative integer \((-n)\) as a numerator parameter. It is convenient to introduce orthonormal Krawtchouk polynomials by

\[
\tilde{K}_n(x) \equiv \frac{\sqrt{w(x)K_n(x)}}{\sqrt{d_n}},
\]

where \( w(x) \) is the weight function in \( x \), and \( d_n \) is a function depending on \( n \):

\[
w(x) = \left( \frac{N}{x} \right)^p (1 - p)^{N-x} \quad \forall x = 0, 1, \ldots, N; \quad d_n = \frac{1}{(\frac{N}{n})} \left( \frac{1 - p}{p} \right)^n.
\]

The scaled polynomials \( \tilde{K}_n(x) \) satisfy a discrete orthogonality relation [19]:

\[
\sum_{x=0}^{N} \tilde{K}_n(x)\tilde{K}_m(x) = \delta_{nm}.
\]

In the present scenario of parabolic coupling of the photons of adjacent cavities, the Krawtchouk polynomials of parametric value \( p = 1/2 \) play an essential role in diagonalizing the photonic part of the Hamiltonian. Following [22] we introduce a symmetric orthogonal \( N \times N \) matrix that is comprised of Krawtchouk polynomials as

\[
U_{jk} = \tilde{K}_k \left( j; \frac{1}{2}; N - 1 \right) \quad \forall \ j, k = 0, 1, \ldots, N - 1, \quad U = U^T, \quad UU^T = U^T U = I.
\]

The adjacency matrix \( C \) given in (3.1) may now be diagonalized via the orthogonal matrix \( U \) as follows:

\[
U C U^T = D, \quad D = \text{diag}(N - 1, N - 3, \ldots, -(N - 1)).
\]

We note that the parity symmetry of the scaled Krawtchouk polynomials described below

\[
\tilde{K}_j \left( j; \frac{1}{2}; N \right) = (-1)^j \tilde{K}_{N-j} \left( j; \frac{1}{2}; N \right), \quad \tilde{K}_\ell \left( j; \frac{1}{2}; N \right) = (-1)^\ell \tilde{K}_{\ell-j} \left( N-j; \frac{1}{2}; N \right) \quad \forall j \neq N-j
\]

interrelates the components of the orthogonal matrix \( U \):

\[
U_{N-j} = (-1)^j U_{0-j}.
\]

The delocalized collective modes may now be introduced as transform of the corresponding localized degrees of freedom via the orthogonal symmetric matrix \( U \) as given below:

\[
A_j = \sum_{k=0}^{N-1} U_{jk} a_k, \quad A_j^\dagger = \sum_{k=0}^{N-1} U_{jk} a_k^\dagger \quad \Rightarrow \quad [A_j, A_k^\dagger] = \delta_{jk},
\]

\[
S_j^+ = \sum_{k=0}^{N-1} U_{jk} \sigma_k^+, \quad S_j^- = \sum_{k=0}^{N-1} U_{jk} \sigma_k^-, \quad S_j^z = \sum_{k=0}^{N-1} U_{jk} U_{k\ell} \sigma_\ell \quad \Rightarrow \quad [S_j^+, S_k^-] = \delta_{jk}.
\]

The photonic degrees of freedom \( \{ A_j | j \in 0, 1, \ldots, N - 1 \} \) satisfy the Heisenberg algebra. We now recast the Hamiltonian employing these delocalized variables. The rotating wave structure of its atom-photon interaction term is maintained in terms of the delocalized modes:

\[
H_0 = \varepsilon \sum_{j=0}^{N-1} S_j^+ S_j^- + \sum_{j=0}^{N-1} \Omega_j A_j^\dagger A_j, \quad H^\text{int} = g \sum_{j=0}^{N-1} \left( S_j^+ A_j + S_j^- A_j^\dagger \right),
\]

(3.11)
where the spectrum of the delocalized photonic eigenmodes is given by $\tilde{\Omega}_j = \Omega + \kappa (N - 1 - 2j)$ for $j = 0, 1, \ldots, N - 1$. The delocalized single excitation atomic and photonic states are constructed by the action of the collective operators introduced in (3.10):

$$|\hat{1}_j\rangle = A_j^\dagger |0\rangle = \sum_{k=0}^{N-1} U_{jk} |1_k\rangle, \quad A_j|\hat{1}_k\rangle = \delta_{jk} |0\rangle, \quad \langle \hat{1}_j|\hat{1}_k\rangle = \delta_{jk}, \quad \langle 1_j|\hat{1}_k\rangle = U_{jk},$$

$$|\hat{\vartheta}_j\rangle = S_j^+ |G\rangle = \sum_{k=0}^{N-1} U_{jk} |e_k\rangle, \quad S_j^- |\hat{\vartheta}_k\rangle = \delta_{jk} |G\rangle, \quad \langle \hat{\vartheta}_j|\hat{\vartheta}_k\rangle = \delta_{jk}, \quad \langle e_j|\hat{\vartheta}_k\rangle = U_{jk}. \quad (3.12)$$

In the above delocalized basis an arbitrary single excitation state may be expanded as follows:

$$|\Psi(t)\rangle = \sum_{j=0}^{N-1} \left( \hat{\alpha}_j(t) |G\rangle \otimes |\hat{1}_j\rangle + \hat{\beta}_j(t) |\hat{\vartheta}_j\rangle \otimes |0\rangle \right). \quad (3.13)$$

Invertible transformation via the orthogonal symmetric matrix $U$ interrelate the coefficients of the above expansions with the the corresponding coefficients of the expansion $|2, 22}$ in the basis of the localized single excitation states:

$$\hat{\alpha}_j(t) = \sum_{k=0}^{N-1} U_{jk} a_k(t), \quad \hat{\beta}_j(t) = \sum_{k=0}^{N-1} U_{jk} b_k(t), \quad a_j(t) = \sum_{k=0}^{N-1} U_{jk} \hat{\alpha}_k(t), \quad b_j(t) = \sum_{k=0}^{N-1} U_{jk} \hat{\beta}_k(t). \quad (3.14)$$

We first consider the regime of dominant hopping parameter: $\kappa \gg \delta$, whereas the atom-photon coupling is assumed to be small: $|\frac{\Delta_j}{\kappa}| \ll 1 \ \forall j \in (0, 1, \ldots, N - 1)$. Unlike the case considered in Sec. II the parabolic coupling between the photons of adjacent cavities does not produce degenerate eigenvalues of the diagonalized photonic modes. The effective Hamiltonian that eliminates the rapidly varying frequencies may be obtained following the recipe described in Sec. II. We quote the result below:

$$\mathcal{H}_{\text{eff}} = H_0 + \frac{g^2}{2} \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_{j+} S_{j-} \left( A_j^\dagger A_j + A_j A_j^\dagger \right) + (S_j^+ S_j^- + S_j^- S_j^+) \right), \quad (3.15)$$

where the detuning parameters for the photon eigenmodes read $\Delta_j = \varepsilon - \tilde{\Omega}_j = \delta - \kappa (N - 1 - 2j)$. Towards obtaining the transmission of single excitation quantum states we study the evolution of the coefficients $a_j(t), b_j(t)$ of the localized excitations. In the present limit the equations (3.14) in conjunction with (2.31) produce the time-dependent coefficients as

$$a_j(t) = \sum_{k=0}^{N-1} K_a(j, k; t) a_k(0), \quad b_j(t) = \sum_{k=0}^{N-1} K_b(j, k; t) b_k(0), \quad (3.16)$$

where the kernels encoding the correlation functions read

$$K_a(j, k; t) = \sum_{\ell=0}^{N-1} U_{j\ell} U_{k\ell} \exp \left( -i \left( \tilde{\Omega}_\ell - \frac{g^2}{\Delta_\ell} \right) t \right),$$

$$K_b(j, k; t) = \sum_{\ell=0}^{N-1} U_{j\ell} U_{k\ell} \exp \left( -i \left( \varepsilon + \frac{g^2}{\Delta_\ell} \right) t \right). \quad (3.17)$$

Of particular significance is the case where the dispersive coupling between the atoms and the corresponding localized cavity excitations is neglected: $g = 0$. The evolution kernel of the localized photonic state reads as

$$K_a(j, k; t) = \exp \left( -i (\Omega + \kappa (N - 1)) t \right) f_{j,k}(t), \quad f_{j,k}(t) = \sum_{\ell=0}^{N-1} U_{j\ell} U_{k\ell} z^\ell \quad z = \exp(2i\kappa t). \quad (3.18)$$

By employing a classical summation formula of the hypergeometric series the time-dependent correlation between the $j$-th and $k$-th sites may be expressed in a closed form:

$$f_{j,k}(t) = \frac{1}{2^{N-1}} \sqrt{\left( \frac{N-1}{j} \right) \left( \frac{N-1}{k} \right)} (1 + z)^{j+k} (1 + z)^{N-1-j-k} \binom{-j-k}{-(N-1)}^{2F_1} \left( -\frac{4z}{(1-z)^2} \right). \quad (3.19)$$
In the context of a linear spin chain with the interaction determined by a Jacobi matrix such propagator of a single excitation state was earlier obtained in \cite{24}. Assuming that the initial state in (3.19) is given by a photon in 0-th cavity: \(a_j(0) = \delta_{j0}\), the transmission of the photonic excitation to an arbitrary cavity is described by the evolution of the said coefficient as follows:

\[
a_j(t) = (-i)^j \sqrt{\begin{pmatrix} N-1 \\ j \end{pmatrix}} \exp(-i\Omega t) \left( \sin(\Omega t) \right)^j \left( \cos(\Omega t) \right)^{N-1-j}.
\]

(3.20)

It is evident from the above structure that in the dispersion-free \(g = 0\) case the amplitude of the photonic excitation at the \((N-1)\)-th cavity is of unit magnitude \(|a_j(t)| = \delta_j \cdot N-1\) at the following predetermined times: \(t = t_n \equiv (2n+1)\pi / 2\delta \mid n = 0, 1, \ldots\).

In the presence of the coupling between the atom and the cavity photon \(g \neq 0\) it is not possible in general to obtain a closed form expression for the time-evolution of the coefficient of the single photonic excitation state. To demonstrate that an exact transmission for this case is not possible in the present limit \(\kappa \gg \delta, g\) we proceed as follows. For the choice of initial state \(a_j(0) = \delta_{j0}\) the evolution kernels given in (3.16, 3.17) may be employed to express the magnitude of the excitation in the \((N-1)\)-th site as given below:

\[
|a_{N-1}(t)| = \left| \sum_{\ell=0}^{N-1} U_{N-1 \ell} U_{0 \ell} \exp \left( i \left( 2\kappa \ell + \frac{g^2}{\Delta t} \right) t \right) \right|.
\]

(3.21)

The parity relation (3.9) allows us to recast (3.21) as

\[
|a_{N-1}(t)| = \left| \sum_{\ell=0}^{N-1} \left( U_{0 \ell} \right)^2 \exp \left( i \left( (2n+1)\pi \ell + \frac{g^2}{\Delta t} \right) t \right) \right|, \quad n = 0, 1, \ldots
\]

(3.22)

The relative phase factors of the terms in the above summand do not, in general, identically reduce to zero for any choice of time. Consequently, the magnitude of the excitation \(a_{N-1}(t)\) obeys the inequality

\[
|a_{N-1}(t)| \leq \left| \sum_{\ell=0}^{N-1} \left( U_{0 \ell} \right)^2 \right| = 1.
\]

(3.23)

Only in the vanishing limit of the coupling \(g \to 0\) between the atom and the cavity photon, and also for large detuning limit \(\delta \gg \kappa, g\) to be discussed next the excitation \(|a_{N-1}(t)|\) equals its unit limiting value for specific predetermined values of time.

Proceeding parallel to the derivation in Sec. II the effective Hamiltonian in the limit of large detuning parameter \(\delta \gg \kappa, g\) may be obtained. Up to the order \(O(g^2)\) the effective Hamiltonian reads

\[
\mathcal{H}^{\text{eff}} = H_0 + \frac{g^2}{2} \left( \sum_{j,k=0}^{N-1} \frac{1}{\Delta_j} \left( S_j^+ A_k^+ + A_k^+ A_j \right) \right) + \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left( S_j^+ S_j^+ + S_j^+ S_j^+ \right) \mathbb{1}
\]

(3.24)

The propagation of single excitation states follows the description given in (3.16, 3.17). To evaluate the propagator we expand the exponent in (3.17) retaining terms up to the order \(O(\frac{g}{\delta})\), and neglecting higher order terms in the said coefficient. The kernel \(K_{a}(j,k;t)\) given in (3.17) may now be evaluated exactly as before reproducing the result (3.13, 3.14) with a redefinition of the constants:

\[
\Omega \rightarrow \Omega' = \Omega \left(1 - \frac{g^2}{\delta} \right), \quad \kappa \rightarrow \kappa' = \kappa \left(1 - \left( \frac{g}{\delta} \right)^2 \right).
\]

(3.25)

We reproduce the kernel \(K_{a}(j,k;t)\) as follows:

\[
K_{a}(j,k;t) = \exp \left( -i(\Omega' + \kappa'(N - 1))t \right) \sum_{\ell=0}^{N-1} U_{j\ell} U_{k\ell} \left( z' \right)^\ell, \quad z' = \exp(i2\kappa't).
\]

(3.26)

Therefore, as in the dispersion free limit discussed in (3.20), the exact transmission of a single photon state from the 0-th to \((N - 1)\)-th cavity also takes place when in the exponent of the relative phases in
the correlation function (3.17) we retain only the linear terms in the parameter $\frac{2}{g}$. The time interval of exact propagation of the state now assumes the value $t_n = \frac{(2n+1)\pi}{2g} \quad \forall n = 0, 1, \ldots$. In the context of the approximation used here the effect of a nonvanishing value of the coupling constant $g \neq 0$ is to increase the time of transmission of the single excitation state $t_n$. Consequently, the summation no longer remains of hypergeometric type, and therefore a general closed form expression is not obtained by the present technique.

Lastly we study the resonance limit $\varepsilon = \tilde{\Omega}_{\ell} \Rightarrow \tilde{\Delta}_{\ell} \approx 0$ for the $\ell$-th delocalized photon eigenstate in the presence of the parabolic coupling between the photons of adjacent cavities. We also simultaneously assume that the following hierarchy of values holds: $\frac{\delta}{\tilde{\Delta}_{\ell}} \ll 1$, $\frac{\Delta_{n}}{\tilde{\Delta}_{\ell}} \ll 1 \forall j \neq \ell$. In contrast to our description of the resonance scenario given in Sec. III the delocalized photon eigenstates here are not degenerate. In the resonance limit the effective Hamiltonian has contributions of order $O(\tilde{g})$ that couples atomic excitations with the photonic excitations:

$$\hat{H}_{\text{eff}} = H_0 + g \left( S_{\ell}^+ A_{\ell} + S_{\ell}^- A_{\ell}^\dagger \right) + \frac{g^2}{2} \sum_{j=0 \atop j \neq \ell}^{N-1} \frac{1}{\Delta_j} \left( S_{jj}^+ (A_j A_j^\dagger + A_j^\dagger A_j) + (S_{jj}^+ S_{jj}^- + S_{jj}^- S_{jj}^\dagger) \right). \quad (3.27)$$

Assuming that the initial excitation is only atomic in nature: $a_j(0) = 0$ $\forall j \in \{0, 1, \ldots, N-1\}$ the time-evolution of the coefficients may be obtained via (3.14, 2.31) as given below

$$a_j(t) = -i \exp \left( -i \left( \varepsilon - \frac{\Delta_j}{2} \right) t \right) \sin \left( \left( g + \frac{\Delta_j^2}{8g} \right) t \right) \sum_{k=0}^{N-1} U_{\ell k} U_{k \ell} b_k(0),$$

$$b_j(t) = \sum_{k=0}^{N-1} \left( \sum_{n=0 \atop n \neq \ell}^{N-1} U_{jn} U_{kn} \exp \left( -i \left( \varepsilon + \frac{g^2}{\Delta_n} \right) t \right) + U_{\ell j} U_{k \ell} \exp \left( -i \left( \varepsilon - \frac{\Delta_{\ell}}{2} \right) t \right) \cos \left( \left( g + \frac{\Delta_{\ell}^2}{8g} \right) t \right) \right) b_k(0). \quad (3.28)$$

For the exact resonance case $\Delta_{\ell} = 0$, and in the absence of dispersive effects $\frac{\delta}{\Delta_j} \sim 0$ $\forall j \neq \ell$ the above evolution equations with the initial condition $b_j(0) = \delta_{j,0}$ read

$$a_j(t) = -i U_{0 \ell} U_{j \ell} \exp(-i\varepsilon t) \sin(g t), \quad b_j(t) = \exp(-i\varepsilon t) \left( \delta_{j,0} - 2 U_{0 \ell} U_{j \ell} \sin^2 \left( \frac{gt}{2} \right) \right). \quad (3.29)$$

In particular the excitation of the $(N-1)$-th atom reads

$$b_{N-1}(t) = -2 \exp(-i\varepsilon t) U_{0 \ell} U_{N-1 \ell} \sin^2 \left( \frac{gt}{2} \right) + 2 (-1)^{\ell+1} \exp(-i\varepsilon t) (U_{0 \ell})^2 \sin^2 \left( \frac{gt}{2} \right), \quad (3.30)$$

where in the second equality we have used the parity relation (3.19). As it may be observed from the construction (3.6, 3.3) of the orthogonal matrix $U$ that in the exact resonance regime for the parabolic coupling a perfect transmission of the quantum state is realized for the length of the array $N = 3$, and for the resonance mode $\ell = 1$ at the time $\frac{2n+1}{g} \forall n \in \{0, 1, \ldots\}$.

**IV Conclusion**

Here we have considered one dimensional array of optical cavities with a nearest-neighbor hopping interaction of the cavity photons described by Jaynes-Cummings-Hubbard model. In particular we have studied the time evolution of single excitation states in such chains of coupled cavities as models for quantum communication. Two different choices of coupling coefficients linking photons of adjacent cavities have been investigated. Employing delocalized collective photonic and atomic modes we first study a translation invariant closed
chain of an arbitrary number of $N$ identical cavities with uniform coupling constant. Following this we have considered a linear chain of cavities where the non-uniform parabolic hopping term of the Hamiltonian is related to the tridiagonal Jacobi matrix associated with the Krawtchouk polynomials. For both of these cases we obtain the effective Hamiltonian in various simplifying limits. For dominant value of photonic hopping parameter, as well as for large detuning parameter between the atomic and the photonic frequencies the excitations are transferred between, say, the atoms without populating the field modes. Expressed via the delocalized atomic modes the spin-spin interaction term in the effective Hamiltonian is diagonalized. The atoms experience a Stark shift dependent upon the population of the field modes. In the case of resonance between the atoms and a delocalized photonic mode propagation of excitation between the atoms requires intermediate excitation of a photonic mode. In the case of non-uniform parabolic coupling between the photons of adjacent cavities the transmission of single-excitation states is exact for the limiting value $g \to 0$ of the atom-cavity photon coupling constant. However in the large detuning limit where we retain only the linear terms in the parameter $\delta$ the propagation of the single excitation states remains exact with an increase in the time of transmission. This is observed in our evaluation of the time-dependent correlation function of the one-excitation states.

The analysis developed here may have applications in certain contexts. The idea of having optical lattice systems with pre-engineered coupling constants between individual lattice sites may give rise to interesting physical situations. In particular the resonance situation where mixing of atomic and field modes is realized merits attention. In the present model it may be possible to generate multipartite entangled states of polaritonic qubits and study the time variation in the entanglement in the presence of dissipative atom-cavity photon coupling. Moreover, in the context of one dimensional spin chains it has recently been observed [25] that coupling between lattice sites determined by the Jacobi matrices of $q$-deformed Krawtchouk polynomials ensure perfect transmission of quantum states. It should be useful to study the problem in the context of Jaynes-Cummings-Hubbard lattices where coupled atomic and photonic modes are considered. The delocalized collective modes discussed here may allow easy extraction of the effective Hamiltonian at the desired perturbative level. Another possible application of the present method may lie in the quantum phase transitions in coupled array of atom-cavity photon systems [26]. Specific pre-engineered coupling between the cavity sites may give rise to desirable properties for a quantum transition between the Mott insulator and the superfluid states.

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