FISHER WAVES IN AN EPIDEMIC MODEL

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Abstract. The existence of Fisher type monotone traveling waves and the minimal wave speed are established for a reaction-diffusion system modeling man-environment-man epidemics via the method of upper and lower solutions as applied to a reduced second order ordinary differential equation with infinite time delay.

1. Introduction. The geographic spread of epidemics is an important subject in mathematical epidemiology. In 1981, Capasso and Maddalena [3] proposed and analysed a reaction diffusion system modeling the spatial spread of a class of bacterial and viral diseases. In this model, there is a positive feedback interaction between the infective human population and the infectious agent in the environment. The human population, once infected, acts as a multiplier of the infectious agent, which is then returned to the environment in fecal excretion; on the other hand, the infectious agent is transmitted to the human population via contaminated food consumptions. In the case where the function of the force of infection on the human population is strictly increasing and concave, the system admits a threshold dynamics (see, e.g., [5]): when the trivial equilibrium is linearly stable, then it is globally asymptotically stable; when the trivial equilibrium is linearly unstable, then the unique positive equilibrium is globally asymptotically stable (with respect to positive initial values). As pointed out in [20], this result is still valid if the strict sublinearity is assumed instead of the concavity. In the case where there are two nontrivial equilibria, the model may admit a saddle point behavior ([4, 5, 6]).

Traveling wave solutions have been widely studied for nonlinear reaction diffusion equations modeling a variety of physical and biological phenomena (see, e.g., [1, 12, 9, 14, 17, 11, 10] for the case without time delay, and [15, 18, 19, 16, 13] for the case with time delay). The purpose of this paper is to study the Capasso and Maddalena’s model in the monostable case (one equilibrium is unstable and the other is stable) from the viewpoint of epidemic waves (see [14]). Our main result

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shows that there is a minimal wave speed \( c^* > 0 \) such that this model system has a monotone traveling wave solution with speed \( c \) if \( c \geq c^* \), and no such solution if \( 0 < c < c^* \). More precisely, we consider the following reaction-diffusion system

\[
\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) &= d \frac{\partial^2}{\partial x^2} u_1(x, t) - a_{11} u_1(x, t) + a_{12} u_2(x, t) \\
\frac{\partial}{\partial t} u_2(x, t) &= -a_{22} u_2(x, t) + g(u_1(x, t))
\end{align*}
\]  

where \( d, a_{11}, a_{12} \) and \( a_{22} \) are positive constants, \( u_1(x, t) \) denotes the spatial density of infectious agent at a point \( x \) in the habitat at time \( t \geq 0 \), and \( u_2(x, t) \) denotes the spatial density of the infective human population at time \( t \). \( 1/a_{11} \) is the mean lifetime of the agent in the environment, \( 1/a_{22} \) is the mean infectious period of the human infectives, \( a_{12} \) is the multiplicative factor of the infectious agent due to the human population, and \( g(z) \) is the force of infection on the human population due to a concentration \( z \) of the infectious agent. Note that system (1) models random dispersal of the pollutant while ignoring the small mobility of the infective human population. Mathematically it suffices to study the following dimensionless system

\[
\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) &= d \frac{\partial^2}{\partial x^2} u_1(x, t) - u_1(x, t) + \alpha u_2(x, t) \\
\frac{\partial}{\partial t} u_2(x, t) &= -\beta u_2(x, t) + g(u_1(x, t))
\end{align*}
\]  

where

\[ \alpha = \frac{a_{12}}{a_{11}}, \quad \beta = \frac{a_{22}}{a_{11}}. \]

In order to get a monostable case, we make the following assumptions on the function \( g \):

(G1) \( g \in C^1([0, \infty), \mathbb{R}) \), \( g(0) = 0 \), \( g''(0) \) exists, and \( g'(z) > 0 \), \( \forall z \geq 0 \);  
(G2) \( \frac{g''(0)}{2} > 1 \), and there is a \( \varepsilon > 0 \) such that \( g(z) \leq \frac{z^2}{2\varepsilon} \);  
(G3) \( g(z) \) is strictly sublinear on \( \mathbb{R}^+ \) in the sense that \( g(sz) > sg(z), \forall z > 0, s \in (0, 1) \).

It then follows that the corresponding reaction system

\[
\begin{align*}
\dot{u}_1 &= -u_1 + \alpha u_2 \\
\dot{u}_2 &= -\beta u_2 + g(u_1)
\end{align*}
\]  

admits only two equilibria \((0, 0)\) and \((u^*_1, u^*_2)\), and \((u^*_1, u^*_2)\) is globally asymptotically stable (see [20, Proposition 4.1]). We are interested in traveling wave solutions of (2) connecting these two equilibria.

The remaining part of this paper is organized as follows. In Section 2, we first reduce the existence of traveling waves to that of solutions of a second-order ordinary differential equation with infinite time delay. Motivated by the monotone iteration and fixed point methods developed in [2, 8, 19, 13], we further reduce the problem to the construction of an ordered pair of upper and lower solutions to the resulting scalar equations with the wave speed \( c \) being a parameter. In Section 3, we prove the existence of monotone waves and the minimal wave speed by the established reduction theorem in Section 2 and the linearization method. Our construction of an ordered pair of upper and lower solutions was highly motivated by the super- and sub-solutions introduced by Diekmann [8]. In Section 4, we make some numerical simulations and discussions.
2. **Reduction to scalar equations.** The objective of this section is to reduce the existence of traveling wave solutions of system (2) to that of a pair of upper and lower solutions of a second order ordinary differential equation with infinite time delay.

Let \((u_1(x, t), u_2(x, t)) = (U_1(z), U_2(z)), z = x + ct\), be a traveling wave front solution of (2) with positive wave speed \(c\). Substituting this special solution into (2), we then obtain

\[
\begin{align*}
\begin{cases}
  cU_1' &= dU_1'' - U_1 + \alpha U_2 \\
  cU_2' &= g(U_1) - \beta U_2.
\end{cases}
\end{align*}
\]

(4)

Since we are interested in the traveling waves connecting \((0, 0)\) and \((u_1^*, u_2^*)\), we impose the following boundary condition on \((U_1, U_2)\)

\[
U_i(-\infty) := \lim_{z \to -\infty} U_i(z) = 0, \quad U_i(+\infty) := \lim_{z \to +\infty} U_i(z) = u_i^*, \quad 1 \leq i \leq 2.
\]

(5)

By the second equation of (4), we have

\[
U_2(t) = e^{-\frac{\beta}{c}(t-t_0)}U_2(t_0) + \frac{1}{c} \int_{t_0}^{t} e^{-\frac{\beta}{c}(t-s)} g(U_1(s))ds, \quad \forall t_0 \in \mathbb{R}, \ t \geq t_0.
\]

Since \(U_2(t)\) and \(g(U_1(t))\) are bounded functions on \(\mathbb{R}\), by taking \(t_0 \to -\infty\), we obtain

\[
U_2(t) = \frac{1}{c} \int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} g(U_1(s))ds, \quad \forall t \in \mathbb{R}.
\]

(6)

Substituting (6) into the first equation of (4), we get

\[
cU_1'(t) = dU_1''(t) - U_1(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} g(U_1(s))ds, \quad \forall t \in \mathbb{R}.
\]

(7)

Assume that \(U_1(t)\) is a monotone increasing solution of (7) with

\[
U_1(-\infty) = 0, \quad U_1(+\infty) = u_1^*.
\]

(8)

Let \(U_2(t)\) be defined by (6). It then easily follows that \((U_1(t), U_2(t))\) is a solution of (4), and \(U_2(t)\) is also a monotone increasing function with

\[
U_2(-\infty) = 0, \quad U_2(+\infty) = u_2^*.
\]

(9)

Consequently, it suffices to consider monotonic solutions of problem (7) subject to (8). In order to use the method of upper and lower solutions, we start with a technical result.

**Lemma 2.1.** Suppose that \(u \in C^2(\mathbb{R}, \mathbb{R})\) and \(u, u'\) and \(u''\) are bounded on \(\mathbb{R}\). If

\[
cu'(t) \geq du''(t) - u(t), \quad \forall t \in \mathbb{R},
\]

then \(u(t) \geq 0, \forall t \in \mathbb{R}\).

**Proof.** Let

\[
h(t) = cu'(t) - du''(t) + u(t), \quad t \in \mathbb{R}.
\]

Then \(h(t)\) is a nonnegative, continuous and bounded function on \(\mathbb{R}\), and \(u(t)\) satisfies the following linear equation

\[
du''(t) - cu'(t) - u(t) + h(t) = 0, \quad t \in \mathbb{R}.
\]

(10)
By the theory of second order linear ordinary differential equations, it follows that
\[ u(t) = e^{\gamma_1 t} + e^{\gamma_2 t} + \frac{1}{d(\gamma_2 - \gamma_1)} \left( \int_{-\infty}^{t} e^{\gamma_1 (t-s)} h(s) ds + \int_{t}^{\infty} e^{\gamma_2 (t-s)} h(s) ds \right) \] (11)
where
\[ \gamma_1 = \frac{c - \sqrt{c^2 + 4d}}{2d} < 0, \quad \gamma_2 = \frac{c + \sqrt{c^2 + 4d}}{2d} > 0. \]
Since both \( u(t) \) and \( h(t) \) are bounded on \( \mathbb{R} \), we have \( c_1 = c_2 = 0 \). It follows from the nonnegativity of \( h(t) \) on \( \mathbb{R} \) that \( u(t) \geq 0, \forall t \in \mathbb{R} \).

Let \( X = BUC(\mathbb{R}, \mathbb{R}) \) be the Banach space of all bounded and uniformly continuous functions from \( \mathbb{R} \) into \( \mathbb{R} \) with the usual supremum norm. Define a continuous mapping \( S : X \rightarrow X \) by
\[
S(\phi)(t) = \frac{\alpha}{c(\gamma_2 - \gamma_1)} \left[ \int_{-\infty}^{t} e^{\gamma_1 (t-s)} ds \int_{-\infty}^{s} e^{-\frac{\alpha}{c}(s-\theta)} g(\phi(\theta)) d\theta + \int_{t}^{\infty} e^{\gamma_2 (t-s)} ds \int_{-\infty}^{s} e^{-\frac{\alpha}{c}(s-\theta)} g(\phi(\theta)) d\theta \right].
\] (12)
By direct calculations we see that the first and second order derivatives of \( S(\phi)(t) \) with respect to \( t \) are bounded on \( \mathbb{R} \) and \( S(\phi)(t) \) is the unique bounded solution on \( \mathbb{R} \) to the following linear ordinary differential equation
\[ du' = cu' - u + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\alpha}{c}(t-s)} g(\phi(s)) ds = 0. \] (13)

It is clear that any fixed point of \( S \) in \( X \) is a solution of (7). By the monotonicity of \( g \) on \( \mathbb{R} \) and Lemma 2.1, as applied to \( u(t) := S(\phi)(t + s) - S(\phi)(t) \) with \( s > 0 \), it follows that \( S \) has the following properties:

(P1) \( S \) is a monotonic increasing operator on \( \mathbb{R} \) with respect to the pointwise ordering;
(P2) If \( \phi \in X \) is monotone increasing on \( \mathbb{R} \), so is \( S(\phi) \).

Motivated by the iteration method for monotone operators, we introduce the following definition.

**Definition 2.1.** A function \( \phi \in X \) is called an upper solution of (7) if \( S(\phi)(t) \leq \phi(t), \forall t \in \mathbb{R} \). A lower solution of (7) is defined by reversing the inequality.

Note that if \( \phi \in X \) is twice continuously differentiable on \( \mathbb{R} \) except finite many points \( t_i \) with \( \phi'(t_i+) \leq \phi'(t_i-), 1 \leq i \leq m \), and satisfies
\[
d\phi''(t) - c\phi'(t) - \phi(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\alpha}{c}(t-s)} g(\phi(s)) ds \leq 0, \forall t \neq t_i, 1 \leq i \leq m, \] (14)
it then easily follows that \( \phi \) is an upper solution of (7) (see, e.g., the proof of [13, Lemma 2.5]). A similar note applies to lower solutions of (7) if we reverse the afore-mentioned two inequalities.

Now we are ready to establish the existence of monotone solutions to (7)-(8).

**Theorem 2.1.** Suppose that (7) admits an upper solution \( \bar{\phi}(t) \) and a lower solution \( \underline{\phi} \) such that
Let $\rho(t)$ be monotone increasing on $\mathbb{R}$ and $\rho(-\infty) = 0, \rho(+\infty) = u^*_1$.

Then (7)-(8) has a monotone increasing solution on $\mathbb{R}$.

Proof. Let $\phi_m = S^m(\rho), \forall m \geq 0$. By Definition 2.1 and (P1), it follows that

$$\rho(t) \leq \phi_m(t) \leq \phi_{m+1}(t) \leq \rho(t), \quad \forall t \in \mathbb{R}, \ m \geq 1.$$  \hspace{1cm} (15)

In particular, for each $t \in \mathbb{R}$, the sequence $\{\phi_m(t)\}$ is decreasing. Thus $\phi(t) = \lim_{m \to \infty} \phi_m(t)$ exists and

$$\rho(t) \leq \phi(t) \leq \rho(t), \quad \forall t \in \mathbb{R}.$$  \hspace{1cm} (16)

By property (P2), we see that the sequence $\{\phi_m(t)\}$ is increasing in $t$ for each $m$. It follows that $\phi(t)$ is increasing in $t$ and

$$0 = \phi(-\infty) \leq \phi(\infty) \leq u^*_1.$$  \hspace{1cm} (17)

For each fixed $t$, since $\phi_m(t) = S(\phi_{m-1})(t)$, by Lebesque’s convergence theorem, we obtain $\phi(t) = S(\phi)(t)$. Hence, $\phi$ is a fixed point of $S$. This means that $\phi(t)$ is a monotone solution of (7). We are now left to show $\phi(\infty) = u^*_1$. Note that

$$\rho(t) \leq \phi(t) \leq \rho(t), \quad \forall t \in \mathbb{R},$$  \hspace{1cm} (18)

and $\rho(t) \neq 0$. We then have $0 < \phi(\infty) \leq u^*_1$. By (7), it follows that

$$\phi(\infty) = \frac{\alpha}{\beta}g(\phi(\infty)).$$

Thus the uniqueness of positive equilibrium of (3) implies that $\phi(\infty) = u^*_1$. \hfill $\square$

3. Monotone waves and minimal wave speed. In this section, we discuss the existence of monotone waves and minimal wave speed for the model system (2).

In order to construct appropriate upper and lower solutions to (7), we linearize (7) at $U_1 = 0$ to obtain

$$cU_1'(t) = dU_1''(t) - U_1(t) + \frac{\alpha g'(0)}{c} \int_{-\infty}^{t} e^{-\frac{s}{c}(t-s)}U_1(s)ds.$$  \hspace{1cm} (19)

By substituting $U_1(t) = e^{\lambda t}$ into (19), we get a characteristic equation

$$P(\lambda) := \lambda^3 + \left( \frac{\beta}{c} - \frac{d}{a} \right) \lambda^2 - \frac{1 + \beta}{d} \lambda + \frac{\alpha g'(0) - \beta}{d} = 0.$$  \hspace{1cm} (20)

By (G2), it is easy to see that (17) has a negative root. By Routh-Hurwitz method, we see that (17) has two roots with positive real parts. In order to identify the conditions under which these two roots are positive real numbers, let us consider

$$P_1(\lambda) := P'(\lambda)/\beta = \lambda^2 - \frac{2}{3} \frac{(c^2 - d \beta)}{d} \lambda - \frac{1}{3} \frac{c \beta + c}{d}.$$  \hspace{1cm} (21)

It is easy to see that $P'(\lambda) = 0$ has a unique positive root

$$\lambda^* = \frac{1}{3} \frac{c^2 - d \beta + \sqrt{d \beta c^2 + 3dc^2 + c^4 + d^2 \beta^2}}{dc}.$$  \hspace{1cm} (22)

Because $P(0) > 0$, (17) has two positive roots if and only if

$$P(\lambda^*) < 0,$$  \hspace{1cm} (23)

and has two complex roots with positive real parts if $P(\lambda^*) > 0$. 


We now transform (19) so that it is expressed in terms of the parameters. First, we find conditions under which $P(\lambda^*) = 0$ and $P'(\lambda^*) = 0$. Set

$$
P(\lambda) = P_1(\lambda)Q_1(\lambda) + R_1(\lambda)$$

$$
P_1(\lambda) = R_1(\lambda)Q_2(\lambda) + R_2(c, g'(0))$$

where $Q_1(\lambda)$ and $R_1(\lambda)$ are the quotient and remainder of $P(\lambda)$ divided by $P_1(\lambda)$, and $Q_2(\lambda)$ and $R_2$ are the quotient and remainder of $P_1(\lambda)$ divided by $R_1(\lambda)$, respectively. By direct calculations, we see that the sign of $-R_2(c, g'(0))$ is determined by

$$
P_2(c, g'(0)) := b_0c^6 + b_1c^4 + b_2c^2 + b_3$$

where

$$
b_0 = (\beta - 1)^2 + 4\alpha g'(0),$$

$$
b_1 = 2d(\beta^3 + 2 + 9\alpha g'(0) + 3\alpha g'(0)\beta - 4\beta + \beta^2),$$

$$
b_2 = -d^2(8\beta^2 - \beta^4 - 36\alpha g'(0)\beta + 27\alpha^2(g'(0))^2 + 6\beta^2\alpha g'(0) - 8\beta^3),$$

$$
b_3 = -4d^3\beta^3(-\beta + \alpha g'(0)).$$

Clearly, we must have

$$
P_2(c, g'(0)) = 0$$

so that $P(\lambda^*) = 0$ and $P'(\lambda^*) = 0$. Note that $b_0 > 0, b_1 > 0$ and $b_3 < 0$. By Descarte’s rule of signs ([14, Appendix A.2.2]), it follows that there is a unique $c^* > 0$ such that $P_2(c^*, g'(0)) = 0$, which implies that

$$
P_2(c, g'(0)) \begin{cases} < 0 & \text{if } 0 < c < c^* \\ = 0 & \text{if } c = c^* \\ > 0 & \text{if } c > c^* \end{cases}$$

By direct calculations, we can verify that $P(\lambda^*) = 0$ and $P'(\lambda^*) = 0$ are really valid when $c = c^*$. Note that $P(\lambda)$ is a decreasing function of $c$. We conclude that $P(\lambda)$ has two positive roots if $c > c^*$, two complex roots with positive real parts if $0 < c < c^*$, and only one positive root if $c = c^*$.

Now we are in a position to prove our main result.

**Theorem 3.1.** Assume (G1)-(G3) hold, and let $c^*$ be defined as in (21). Then (2) has a monotone traveling wave connecting $(0, 0)$ and $(u_1^*, u_2^*)$ with speed $c$ if $c \geq c^*$, and no such wave if $0 < c < c^*$.

**Proof.** In the case where $c > c^*$, (17) has two positive roots $\lambda_1 < \lambda_2$. Based on these two roots, we define

$$
\bar{\rho} = \min\{u_1^* e^{\lambda_1 t}, u_1^*\}.\tag{22}
$$
We now verify that \( \bar{\rho} \) is an upper solution of (7). If \( t < 0 \), then \( \bar{\rho} = u_1^* e^{\lambda_1 t} \). It is easy to obtain
\[
d\bar{\rho}'(t) - c\bar{\rho}'(t) - \bar{\rho}(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(\bar{\rho}(s)) ds
\]
\[
= u_1^* e^{\lambda_1 t} [d\bar{\lambda}_1^2 - c\lambda_1 - 1] + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(u_1^*) ds
\]
\[
\leq u_1^* e^{\lambda_1 t} [d\bar{\lambda}_1^2 - c\lambda_1 - 1] + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} u_1^* e^{\lambda_1 s} ds
\]  
(23)
\[
= u_1^* e^{\lambda_1 t} [d\bar{\lambda}_1^2 - c\lambda_1 - 1 + \frac{\alpha g'(0)}{c(\lambda_1 + \beta/\gamma)}]
\]
\[
= u_1^* e^{\lambda_1 t} \frac{1}{\alpha(c\lambda_1 + \beta)} P(\lambda_1) = 0,
\]
where the inequality is due to the fact that \( g(u) \leq g'(0)u, \forall u \geq 0 \), which is implied by assumption (G3).

If \( t > 0 \), then \( \bar{\rho} = u_1^* \). It is easy to obtain
\[
d\bar{\rho}'(t) - c\bar{\rho}'(t) - \bar{\rho}(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(\bar{\rho}(s)) ds
\]
\[
\leq -u_1^* + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(u_1^*) ds
\]  
(24)
\[
= -u_1^* + \frac{\alpha g(u_1^*)}{c} = 0.
\]

Hence, as noted in previous section, \( \bar{\rho} \) is an upper solution of (7).

By assumption (G1), it follows that there exist \( \kappa > 0 \) and \( \delta \in (0, u_1^*) \) such that
\[
g(s) \geq g'(0)z - k\delta^2, \quad \forall z \in [0, \delta].
\]  
(25)

In view of \( 0 < \lambda_1 < \lambda_2 \), we first fix an \( \epsilon \in (0, \lambda_1) \) such that \( \lambda_1 + \epsilon < \lambda_2 \), and then define
\[
\bar{\rho}(t) = \max\{0, \delta(1 - M e^{\epsilon t}) e^{\lambda_1 t}\},
\]  
(26)
where the constant \( M \geq 1 \) is to be determined. Since \( \delta < u_1^* \), \( t_0 := -\frac{\ln M}{\epsilon} \leq 0 \), and \( 0 < \epsilon \leq \lambda_1 \), it is easy to see that
\[
0 \leq \bar{\rho}(t) \leq \bar{\rho}(t), \quad \bar{\rho}^2(t) \leq (u_1^*)^2 e^{(\lambda_1 + \epsilon)t}, \quad \forall t \in \mathbb{R}.
\]  
(27)

If \( t > t_0 \), then \( \bar{\rho}(t) = 0 \). It follows that
\[
d\bar{\rho}'(t) - c\bar{\rho}'(t) - \bar{\rho}(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(\bar{\rho}(s)) ds
\]
\[
= \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\bar{\rho}}{2}(t-s)} g(\bar{\rho}(s)) ds \geq 0.
\]  
(28)

Let \( L \) and \( H \) be two linear operators defined by
\[
L(\phi)(t) := d\phi''(t) - c\phi'(t) - \phi(t), \text{ and } H(\phi)(t) := \frac{\alpha g'(0)}{c} \int_{-\infty}^{t} e^{-\frac{\phi}{2}(t-s)} g(s) ds.
\]

It then easily follows that
\[
L(e^{\lambda t})(t) = (d\lambda^2 - c\lambda - 1)e^{\lambda t}, \quad \forall t \in \mathbb{R}, \ \lambda \in \mathbb{R};
\]
\[
L(e^{\lambda t})(t) + H(e^{\lambda t})(t) = \frac{P(\lambda)}{dc(c\lambda + \beta)} e^{\lambda t}, \quad \forall t \in \mathbb{R}, \ \lambda \geq 0.
\]  
(29)

If \( t < t_0 \), we then have
\[
\bar{\rho}(t) = \delta e^{\lambda_1 t} - \delta Me^{(\lambda_1 + \epsilon)t}.
\]
This, together with (25), (27) and (29), implies that
\[
\frac{d\rho''(t)}{dt} - \frac{c\rho'(t)}{c} - \rho(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} g(\rho(s)) \, ds
\]
\[
\geq \frac{d\rho''(t)}{dt} - \frac{c\rho'(t)}{c} - \rho(t) + \frac{\alpha}{c} \int_{-\infty}^{t} e^{-\frac{\beta}{c}(t-s)} [g'(0)\rho(s) - kg^2(s)] \, ds
\]
\[
\geq \frac{d\rho''(t)}{dt} - \frac{c\rho'(t)}{c} - \rho(t) + H(\rho(t) - \frac{k(u_1^*)^2}{g'(0)}(e^{(\lambda_1+\epsilon)})t)
\]
\[
= \delta \left[ L(e^{(\lambda_1)}) + H(e^{(\lambda_1)})t \right] - \delta M \left[ L(e^{(\lambda_1+\epsilon)})t + H(e^{(\lambda_1+\epsilon)})t \right]
\]
\[
- \frac{k(u_1^*)^2}{g'(0)}(e^{(\lambda_1+\epsilon)})t
\]
\[
= \frac{\delta P(\lambda_1)}{dc(\lambda_1 + \beta)} e^{\lambda_1 t} - \frac{\delta M P(\lambda_1 + \epsilon)}{dc(\lambda_1 + \epsilon + \beta)} e^{(\lambda_1 + \epsilon)t}
\]
\[
k(u_1^*)^2 \frac{P(\lambda_1 + \epsilon)}{g'(0)} \left[ \frac{d(\lambda_1 + \epsilon) - c(\lambda_1 + \epsilon)}{\lambda_1 + \epsilon - 1} \right] e^{(\lambda_1 + \epsilon)t}
\]
\[
> 0,
\]
provided \( M \) is sufficiently large. Note that in (30) we have used the fact that \( P(\lambda_1) = 0 \) and \( P(\lambda_1 + \epsilon) < 0 \). By (28) and (30), it then follows that there exist positive numbers \( \delta, \epsilon \) and \( M = M(\delta, \epsilon) \) such that \( \rho(t) \) is a lower solution of (7). Thus the existence of monotone traveling wave of (2) with speed \( c \) follows from Theorem 2.1.

In the case where \( c = c^* \), we choose a sequence \( \{c_n\} \) such that \( c_n \in (c^*, c^* + 1] \) and \( \lim_{n \to \infty} c_n = c^* \). Let \( U_{1n}(t) \) be the monotone solution of (7) with \( c = c_n \) subject to (8). Since each \( U_{1n}(t+h), h \in \mathbb{R} \), is also such a solution, we can assume that \( U_{1n}(0) = \frac{1}{2}u_1^*, \forall n \geq 1 \). Clearly, \( |U_{1n}(t)| \leq u_1^*, \forall t \in \mathbb{R}, n \geq 1 \), and \( U_{1n} \) satisfies
\[
c_n U''_{1n}(t) = dU''_{1n}(t) - U_{1n}(t) + \frac{\alpha}{c_n} \int_{-\infty}^{t} e^{-\frac{\beta}{c_n}(t-s)} g(U_{1n}(s)) \, ds,
\forall t \in \mathbb{R}.
\]
Then \( U_{1n}(\cdot) \) is a fixed point of the mapping \( S_n : X \to X \) defined by (12) with \( c = c_n \), which implies that there exists \( b_1 = b_1(c^*) > 0 \) such that \( |U'_{1n}(t)| \leq b_1, \forall t \in \mathbb{R}, n \geq 1 \). By equation (31), it follows that there exists \( b_2 = b_2(c^*) > 0 \) such that \( |U''_{1n}(t)| \leq b_2, \forall t \in \mathbb{R}, n \geq 1 \). Differentiating both sides of (31) with respect to \( t \), we then get
\[
c_n U'''_{1n}(t) = dU'''_{1n}(t) - U'_{1n}(t) + \frac{\alpha}{c_n} \int_{-\infty}^{t} e^{-\frac{\beta}{c_n}(t-s)} g(U_{1n}(s)) \, ds,
\]
which implies that there exists \( b_3 = b_3(c^*) > 0 \) such that \( |U'''_{1n}(t)| \leq b_3, \forall t \in \mathbb{R}, n \geq 1 \). Consequently, \( U_{1n}(t), U'_{1n}(t) \) and \( U''_{1n}(t) \) are equi-continuous and uniformly bounded sequences of functions on \( \mathbb{R} \). By Ascoli’s theorem and a nested subsequence argument, it follows that there exists a subsequence of \( \{c_n\} \), still denoted by \( \{c_n\} \), such that \( \lim_{n \to \infty} c_n = c^* \), and \( U_{1n}(t), U'_{1n}(t) \) and \( U''_{1n}(t) \) converge
uniformly on every bounded interval, and hence pointwise on \( \mathbb{R} \) to \( W(t), W_1(t) \) and \( W_2(t) \), respectively. Then \( W(t) \) and \( W_1(t) \) are differentiable, and \( W''(t) = W_1(t), W''(t) = W_1'(t) = W_2(t), \forall t \in \mathbb{R} \). Letting \( n \to \infty \) in (31) and using the dominated convergence theorem, we get

\[
c^* W_1(t) = dW_2(t) - W(t) + \frac{\alpha}{c^*} \int_{-\infty}^{t} e^{-\frac{c^*}{\alpha}(t-s)} g(W(s)) ds, \forall t \in \mathbb{R},
\]

that is,

\[
c^* W''(t) = dW''(t) - W(t) + \frac{\alpha}{c^*} \int_{-\infty}^{t} e^{-\frac{c^*}{\alpha}(t-s)} g(W(s)) ds, \forall t \in \mathbb{R}. \tag{32}
\]

Then \( W(t) \) is a solution of (7) with \( c = c^* \). Clearly, \( W(\cdot) \) is monotone increasing on \( \mathbb{R} \) and \( W(0) = \frac{1}{2} u_1^* \). Since both \( W(-\infty) \) and \( W(+\infty) \) exist, there holds \( W''(\pm \infty) = 0 \) and \( W''(\pm \infty) = 0 \). Letting \( t \to -\infty \) and \( t \to \infty \) in (32), respectively, we then get \( W(-\infty) = 0 \) and \( W(\infty) = \frac{1}{2} g(W(\infty)) \). Since \( \frac{1}{2} u_1^* \leq W(\infty) \leq u_1^* \), the uniqueness of positive equilibrium of (3) in the order interval \([0, u^*]\) implies that \( W(\infty) = u_1^* \).

Consequently, (2) has a monotone traveling wave connecting \((0, 0)\) and \((u_1^*, u_2^*)\) with speed \( c^* \).

It remains to show that (2) admits no monotone traveling wave solution in the case where \( 0 < c < c^* \). Set \( V_1 = U_1, V_2 = U_1', V_3 = U_2 \). Then (4) becomes

\[
\begin{align*}
V_1' &= V_2 \\
V_2' &= \frac{3}{2} [V_1 + c V_2 - \alpha V_3] \\
V_3' &= \frac{2}{\alpha}[g(V_1) - \beta V_3].
\end{align*}
\]

Linearizing it at \((0, 0, 0)\), we obtain

\[
\begin{align*}
V_1' &= V_2 \\
V_2' &= \frac{3}{2} [V_1 + c V_2 - \alpha V_3] \\
V_3' &= \frac{2}{\alpha}[g(0)V_1 - \beta V_3].
\end{align*}
\]

It is easy to verify that the characteristic equation associated with (34) is the same as (17). Let \( 0 < c < c^* \) be fixed. Then one root of (17) is negative and a pair of conjugate roots of (17) has positive real parts, and hence the unstable manifold of (33) at \((0, 0, 0)\) is two dimensional. Let \((U_1(t), U_2(t))\) be a solution to (4)-(5). Then \( U(t) = (U_1(t), U_1'(t), U_2(t)) \) is a solution of (33) with \( U(-\infty) = (0, 0, 0) \) and \( U(+\infty) = (u_1^*, 0, u_2^*) \). Clearly, (33) can be rewritten as

\[
\begin{align*}
V_1' &= V_2 \\
V_2' &= \frac{3}{2} [V_1 + c V_2 - \alpha V_3] \\
V_3' &= \frac{2}{\alpha}[g'(0)V_1 - \beta V_3 + G(V_1)]
\end{align*}
\]

where \( G(V_1) = \frac{1}{c}[g(V_1) - g'(0)V_1] = o(|V_1|) \). By [7, Theorems 13.4.3 and 13.4.5], there is a solution \( V(t) \) of (34) such that

\[
U(t) = V(t)(1 + o(1)) \quad \text{as} \quad t \to -\infty.
\]

Since \( V(t) \) is a spiral on the unstable manifold of (34) at \((0, 0, 0)\) as \( t \to -\infty \), \( U(t) \) is an oscillating solution of (33) as \( t \to -\infty \). Consequently, (4)-(5) admits no monotone solution.

\[ \square \]

**Remark 1.** In the case where \( c > c^* \), by inequality (25) and a direct calculation, we can prove that there exist two sufficiently small positive numbers \( \epsilon_0 \) and \( \eta_0 \) such that for each \( \eta \in (0, \eta_0) \), the function \( \rho(t) := \max(0, \eta(1 - e^{\epsilon_0 t})e^{\lambda t}) \) is a lower solution of (7).
4. **Discussions.** By the method of upper and lower solutions, we have proved the existence of Fisher type wave fronts for a reaction diffusion system modeling man-environment-man epidemics. As shown in [8, 19, 13], an upper solution gives rise to a traveling wave as the limit of monotone iterations and a lower solution enables us to prove that it does connect the trivial equilibrium to the positive one. We have also proved that there is a minimal wave speed by using linearization method. To simulate our main result, let us consider an example. Suppose $\alpha = \beta = d = 1$ and $g(z) = \frac{2z}{1+z^2}$. Then $u_{1c}^* = 1$ and (20) becomes

$$-8c^6 - 48c^4 + 47c^2 + 4 = 0.$$  

It follows that the minimal wave speed is 0.96238. We choose $c = 1.5$. Then (17) becomes

$$\lambda^3 - 0.8333333333\lambda^2 - 2\lambda + 0.6666666667 = 0.$$
It has two positive roots $\lambda_1 = 0.3083730617$ and $\lambda_2 = 1.756059537$. Based upon these parameters, we can obtain an approximate wave front by monotone iterations. For illustration purpose, we include the figures of $\bar{\rho}$ and $S(\bar{\rho})$ here.

In view of the well known properties of traveling waves in scalar Fisher reaction-diffusion equation (see, e.g., [1, 9, 14]), we may expect naturally that the minimal wave speed $c^*$ is the asymptotic speed of propagation for solutions of (2) satisfying initial values with compact support in the sense of Aronson-Weinberger [1] (see also [15, Theorem 2.12]). Moreover, the stability of these monotone traveling wave solutions is also important. We leave these open problems for further investigations.

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