Uniqueness of the critical probability for percolation in the two dimensional Sierpiński carpet lattice

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Abstract We prove that the critical probability for the Sierpiński carpet lattice in two dimensions is uniquely determined. The transition is sharp. This extends the Kumagai’s result [5] to the original Sierpiński carpet lattice.

1 Introduction

We consider bond percolation problem on the original two dimensional Sierpiński carpet lattice. Let $T$ be given by $\{0, 1, 2\}^2 \setminus \{(1, 1)\}$, and for each $(i, j) \in T$ we put

$$\psi_{i,j}(x, y) = 3^{-1}(i, j) + 3^{-1}(x_1, x_2) \quad \text{for } (x_1, x_2) \in [0, 1]^2. \quad (1)$$

The Sierpiński carpet is the closed subset $K^T$ of $[0, 1]^2$ which satisfies

$$K^T = \bigcup_{(i,j) \in T} \psi_{i,j}(K^T). \quad (2)$$

This set is a decreasing limit of

$$K^T_n = \bigcup_{(i_n,j_n) \in T} \ldots \bigcup_{(i_1,j_1) \in T} \psi_{i_n,j_n} \circ \ldots \circ \psi_{i_1,j_1} ([0, 1]^2). \quad (3)$$

The pre-Sierpiński carpet lattice $G^T$ is the subgraph of $\mathbb{Z}^2$, which is the increasing limit of

$$G^T_n = \mathbb{Z}^2 \cap 3^n K^T_n. \quad (4)$$

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To be more precise, (4) determines the vertex set of $G_T^n$, and the edge set of $G_T^n$ is given in an obvious way:

$$\mathcal{E}(G_T^n) = \left\{ \{x, y\} \subset G_T^n : |x - y| = 1 \right\},$$

where $|x - y|$ denotes the Euclidean distance of $x$ and $y$. We will abuse the notation $G_T^n$ and $G_T$ both for the graphs defined above and their vertex sets. This will not cause a problem. We define the full Sierpiński carpet lattice $S_T$, which we sometimes call the original Sierpiński carpet lattice, by

$$S_T = G_T \cup \Phi_1(G_T) \cup \Phi_2(G_T) \cup \Phi_1 \circ \Phi_2(G_T),$$

where $\Phi_j$ is the reflection with respect to the $x_j$-axis for $j = 1, 2$, respectively. Each edge $e$ of $S_T$ takes independently two states, open or closed, and the probability that $e$ is open is $p \in [0, 1]$. The distribution of states of all edges is denoted by $P_p$. This can be regarded as a restriction of Bernoulli probability measure $P_p$ on states of all edges of $\mathbb{Z}^2$ to those of edges of $S_T$.

Also let $S_T^*$ be the dual graph of $S_T$. Namely, a vertex of $S_T^*$ is the central point of a face of $S_T$, and an edge of $S_T^*$ is a pair of vertices of $S_T^*$ such that the corresponding faces of $S_T$ have a common edge of $S_T$ in their boundaries. For every edge $e$ of $S_T$ there exists uniquely an edge $e^*$ of $S_T^*$ such that $e$ and $e^*$ cross each other. As usual we say that $e^*$ is open (closed) if $e$ is open (closed).

Put

$$p_c(G) = \inf \left\{ p \in [0, 1] : P_p[\text{there exists an infinite open cluster in } G] > 0 \right\}$$

for $G = S_T, G_T$ and $S_T^*$. First, we show the following theorem by a standard percolation argument in section 2.

**Theorem 1.1**

1. If $p > p_c(S_T)$, then the infinite cluster is unique a.s.
2. If $p > p_c(G_T)$, then

$$\inf_{x, y \in S_T} \tau_p(x, y) > 0,$$

where,

$$\tau_p(x, y) := \left\{ x \text{ and } y \text{ are in the same open cluster in } S_T \right\}.$$

3. If $p < 1 - p_c(S_T^*)$, then $\tau_p(x, y)$ decays exponentially as $|x - y| \to \infty$.

**Remark 1.1** Kumagai considered percolation problem on Sierpiński carpet lattice in a general setting. Namely, let $T \subset \{0, 1, 2, \ldots, L-1\}^2$, and for $(i, j) \in T$, put

$$\psi_{i,j}(x_1, x_2) = L^{-1}(i, j) + L^{-1}(x_1, x_2) \quad (x_1, x_2) \in [0, 1]^2.$$

Then there exists an unique closed set $E^T$ such that

$$E^T = \bigcup_{(i, j) \in T} \psi_{i,j}(E^T).$$

Assume that $T$ satisfies the following conditions:
1. $E^T$ is connected,
2. If $(i, j) \in T$ then both $(j, i)$ and $(i, L - 1 - j)$ are in $T$.
3. $\{(0, j); 0 \leq j \leq L - 1\} \subset T$.

Then he proved the uniqueness of the critical probability for the general Sierpiński carpet lattice $G^T$ generated by $T$, under a condition related to the crossing probabilities, which we introduce in section 2. However, it is not clear whether our (original) Sierpiński carpet lattice $G^T$ or $S^T$ satisfies his condition and the problem remained open for $T = \{0, 1, 2\} \setminus \{(1, 1)\}.$

Theorem 1 summarizes where the essential problem lies. Also the proof of this theorem is a preparation for the argument to obtain the following final result.

**Theorem 1.2**  

1. $p_c(S^T) = p_c(G^T) = 1 - p_c(S^{T^*}).$
2. The percolation probability
   \[
   \theta(p) = P_p \left[ \text{there is an open path from the origin to infinity in } S^T \right]
   \]
   is continuous at $p_c(S^T)$.

Combining this theorem with the result in [10], we obtain that the central limit theorem for the number of open clusters in $G^T_n$ holds for every $p \in (0, 1)$.

## 2 Sponge percolation probabilities

For the proof of our theorems, the sponge percolation probabilities play important roles as usual. For integers $\ell, k \geq 1$ and $n \geq 1$, let $G_n(\ell, k)$ be the union of shifts of $\ell \times k G^T_n$'s, with $k$ rows and $\ell$ columns. The origin is located at the lower left corner of $G_n(\ell, k)$. To be more precise,

\[
G_n(\ell, k) = \bigcup_{0 \leq i \leq \ell - 1} \bigcup_{0 \leq j \leq k - 1} [G^T_n + (i \cdot 3^n, j \cdot 3^n)].
\]

These may not be subgraphs of $S^T$, but we can consider them as subgraphs of $\mathbb{Z}^2$, and therefore we can also consider $P_p$ probabilities of events on edges of these graphs. Let $A_n(\ell, k)$ denote the event that there is an open left-right crossing in $G_n(\ell, k)$, and let $B_n(\ell, k)$ denote the event that there exists an open up-down crossing in $G_n(\ell, k)$. Let $G_n^*(\ell, k)$ denote the dual graph of $G_n(\ell, k)$. By this, we mean the following graph. First we put a vertex at the center of every finite face of $G_n(\ell, k)$, and connect each pair $u^*, v^*$ of these vertices if the corresponding faces of $G_n(\ell, k)$ have an edge of $G_n(\ell, k)$ in common on their boundaries. Next, we add edges of $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ if they connect a finite face of $G_n(\ell, k)$ to the unique infinite face of $G_n(\ell, k)$. Thus, the total edges obtained above form the edge set of $G_n^*(\ell, k)$, and the set of all points incident to some
of these edges is the vertex set of $G_n^\ast(\ell, k)$. We write $A_n^\ast(\ell, k)$ for the event that there is a dual closed left-right crossing in $G_n^\ast(\ell, k)$, and $B_n^\ast(\ell, k)$ for the event that there is a dual closed up-down crossing in $G_n^\ast(\ell, k)$.

Let $E^2$ be the edge set of $\mathbb{Z}^2$. The space of total edge configurations in $E^2$ is denoted by

$$\Omega_{E^2} = \{0, 1\}^{E^2}.$$ 

Here, we can define shifts;

$$\tau_x \omega := \omega(b - x), \quad x \in \mathbb{Z}^2.$$

Although shifts are not possible to define in the edge configuration space on $S^T$, $G^T$ or $S^T^\ast$, we can define shifts of local edge events there. Namely, let $A$ be an edge event on a finite subgraph $V$ of $\mathbb{Z}^2$. If for some $x \in \mathbb{Z}^2$, $V + x$ is a subgraph of $S^T$, then we can regard $\tau_x A$ as an edge event in $S^T$. We will use this convention hereafter.

The Kumagai’s condition mentioned in the previous section is as follows.

$$\limsup_{n \to \infty} P_p[A_n(3L, 1)] < 1 \iff \limsup_{n \to \infty} P_p[A_n(3L, 2)] < 1$$

for all $p \in [0, 1]$, where in our case, $L = 3$. We are not going to check this condition directly, but investigate crossing probabilities in a more precise manner.

First, we give an RSW type result which is valid for both original and dual connection.

**Lemma 2.1** Let $0 < a, b < 1$. For every $k \geq 1$, there exist continuous increasing functions $f_k(x), g_k(x, y)$ such that

1. If $P_p[A_n^\ast(2, 2)] \geq a$, then

$$P_p\left[A_n^\ast(k, 2)\right] \geq f_k(a).$$

2. If $P_p[A_n^\ast(2, 2)] \geq a$, and $P_p[A_{n+1}(1, 1)] \geq b$, then

$$P_p\left[A_{n+1}^\ast(1, 1)\right] \geq g_k(a, b).$$

3. $f_k(x) > 0$ if $x > 0$, and $g_k(x, y) > 0$ if $x > 0, y > 0$. Further, $f_k(0) = g_k(0, 0) = g_k(x, 0) = 0$ for $(x, y) \in [0, 1]^2$, and $f(1) = g(1, 1) = 1$.

The proof of above lemma is about the same as that of the RSW theorem in $\mathbb{E}^2$ (see e.g. [8]). The functions $f_k(x)$ can be taken as

$$f_1(x) = f_2(x) = x, \quad f_3(x) = (1 - \sqrt{1 - x})^3,$$

$$f_{2k+1}(x) = x \cdot f_{k+1}(x) \cdot f_{k+2}(x),$$

$$f_{2k}(x) = x \cdot f_{k+1}(x)^2.$$
for $k \geq 2$. As for the functions $g_k$, we give brief sketch of the proof of (2). We look at $G_{n+1}^T(2, 1)$. By our assumption and the square root trick, the probability that there is an open (closed dual) left-right crossing in $G_{n+1}^T$ passing below $(2 \cdot 3^n, \frac{1}{2}3^{n+1})$ is not less than $1 - \sqrt{1-a}$. Let $c$ be the lowest of such a left-right crossing and let $c'$ be its reflection with respect to the line $\{x_1 = 3^{n+1}\}$. Then the probability that there is an open (closed dual) up-down crossing in the region above $c \cup c'$ and in $H = G_n(2, 3) + (2 \cdot 3^n, 0)$, that connects the top side of $H$ with $c$, is not less than $1 - \sqrt{1-f_3(a)}$. Further the probability that there is an open (closed dual) left-right crossing in $H$ starting above $(2 \cdot 3^n, \frac{1}{2}3^{n+1})$ is not less than $1 - \sqrt{1-a}$. Thus, by the FKG inequality and by the usual RSW argument the probability that there is an open (closed dual) left-right crossing in $G_{n+1}^T \cup H$ is not less than

$$(1 - \sqrt{1-b}) (1 - \sqrt{1-f_3(a)}) (1 - \sqrt{1-a}).$$

The same is true for the reflected region of $G_{n+1}^T \cup H$ with respect to the line $\{x_1 = 3^{n+1}\}$. The probability of the intersection of these two events and the event that there is an open (closed) up-down crossing in $H$ can be therefore not less than

$$(1 - \sqrt{1-b})^2 (1 - \sqrt{1-f_3(a)})^2 (1 - \sqrt{1-a})^2 \cdot f_3(a),$$

which we take as $g_2(a, b)$. For general $k \geq 2$, we can take

$$g_{k+1}(a, b) = b \cdot g_k(a, b) g_2(a, b).$$

On the other hand, the following lemma is specially for $S$ or $G^T$.

**Lemma 2.2**

1. If $\lim_{n \to \infty} P_p[A_n(1, 1)] = 1$, then $\lim_{n \to \infty} P_p[A_n(k, 1)] = 1$ for $k \geq 1$.

2. If $\lim_{n \to \infty} P_p[A_n^*(2, 2)] = 1$, then $\lim_{n \to \infty} P_p[A_n^*(k, 1)] = 1$ for $k \geq 1$.

**Proof.**

1. If there is an open left-right crossing in $G_n^T$, then this open crossing avoids the central hole $[3^{n-1}, 2 \cdot 3^{n-1}]$. By symmetry and the square root trick, the probability that there is an open left-right crossing in $G_n^T$ passing below the central hole is not less than $1 - \sqrt{1 - P_p[A_n(1, 1)]}$. Taking intersection of rotations of this event and by the FKG inequality, the probability that there is an open circuit in $G_n^T$ surrounding the central hole is not less than

$$\left(1 - \sqrt{1 - P_p[A_n(1, 1)]}\right)^4.$$

So, with this probability, we can find an open path in $G_{n-1}(3, 1)$ connecting $[0, 3^{n-1}] \times \{3^{n-1}\}$ with $[2 \cdot 3^{n-1}, 3^n] \times \{3^{n-1}\}$ without touching $(3^{n-1}, 2 \cdot 3^{n-1}) \times \{3^{n-1}\}$. Let $E_n$ denote this event. Then there exists an open left-right crossing in $G_{n-1}(k, 1)$ if

$$\tau_{(-3^{n-1}, 0)} E_n \cap E_n \cap \tau_{(3^{n-1}, 0)} E_n \cap \ldots \cap \tau_{((k-2)3^{n-1}, 0)} E_n$$
occurs. The probability of this event is not less than

\[ P_p[A_n(k, 1)] \geq \left(1 - \sqrt{1 - P_p[A_n(1, 1)]}\right)^{4k}, \]

which converges to 1 if \( \lim_{n \to \infty} P_p[A_n(1, 1)] = 1 \).

(2) By (1) of Lemma 2.1, we have

\[ \lim_{n \to \infty} P_p[A_n^*(k, 2)] = 1 \]

for every \( k \geq 1 \). But for any \( k \geq 1 \), we always have

\[ P_p[A_n^*(3k, 2)] \leq P_p(A_{n+1}^*(k, 1)) \]

by comparison. Thus we get (2).

The crossing probabilities are related to the critical probabilities introduced in section 1 in the following manner.

**Lemma 2.3**

(1) If \( p > p_c(S^T) \), then

\[ \lim_{n \to \infty} P_p[A_{n}(2, 2)] = 1. \]

(2) If \( p > p_c(G^T) \), then

\[ \lim_{n \to \infty} P_p[A_{n}(1, 1)] = 1, \]

moreover, there exist constants \( C > 0 \) and \( 0 < \alpha < 1 \) such that

\[ P_p[A_{n}(3, 1)] \geq 1 - C\alpha^{2n} \]

for sufficiently large \( n \)'s.

(3) If \( p < 1 - p_c(S^{T^*}) \), then

\[ \lim_{n \to \infty} P_p[A_{n}^*(2, 2)] = 1. \]

Further, there exist constants \( C^* > 0 \) and \( 0 < \alpha^* < 1 \) such that

\[ P_p[A_{n}^*(3, 1)] \geq 1 - C^*(\alpha^*)^{3n}. \] (7)

**Proof.** (1) As in the proof of (2) of the previous lemma, the probability that there is a closed dual path in \( G_{n-2}(6, 2) + (-3^{n-1} - 2 \cdot 3^{n-2}) \) connecting the holes \([-2 \cdot 3^{n-1}, 2 \cdot 3^{n-2}] \times [3^{n-1}, 2 \cdot 3^{n-1}] \) and \([3^{n-1}, 2 \cdot 3^{n-1}]^2 \) is not less than

\[ \left(1 - \sqrt{1 - f_6(P_p(A_{n-2}^*(2, 2)))}\right)^2. \]

Taking intersection of rotations of this event, we see that the probability that there is a dual closed circuit in \( S^T \cap [-3^n, 3^n]^2 \) surrounding \([-3^{n-2}, 3^{n-2}]^2 \) is not less than

\[ \left(1 - \sqrt{1 - f_6(P_p(A_{n-2}^*(2, 2)))}\right)^8. \]
Therefore, if

\[
\limsup_{n \to \infty} P_p [A_n^* (2, 2)] = \limsup_{n \to \infty} (1 - P_p [A_n (2, 2)]) > 0,
\]

then by the second Borel-Cantelli’s lemma we find \( P_p \)-a.s. infinitely many disjoint closed dual circuits surrounding the origin. Thus, there is no infinite open cluster in \( S^T \), which implies that \( p \leq p_c (S^T) \).

(2) If \( p > p_c (G^T) \), \( P_p \)-a.s. for sufficiently large \( n \geq 1 \), the infinite open cluster in \( G^T \) intersects \( G_n^T \). This open cluster will go out of \( G_{n+1}^T \), avoiding the central hole \( [3^n, 2 \cdot 3^n]^2 \). Therefore it must contain an open left-right crossing of \( G_n^T + (3^n, 0) \) or an open up-down crossing of \( G_n^T + (0, 3^n) \). By symmetry and the square root trick these probabilities go to 1 as \( n \to \infty \). Thus we have

\[
\lim_{n \to \infty} P_p [A_n (1, 1)] = 1.
\]

The remaining estimate comes from a simple scaling argument introduced in [1]. By symmetry and the FKG inequality we have

\[
P_p [A_n (9, 1)] \geq P_p [A_n (3, 1)]^5,
\]

because \( A_n (9, 1) \) occurs when there is an open left-right crossing in each \( A_n (3, 1) + (2j \cdot 3^n, 0) \) for \( j = 0, 1, 2, 3 \) and also there is an open up-down crossing in \( A_n (1, 1) + (2j \cdot 3^n, 0) \) for \( j = 1, 2, 3 \). By independence, we have

\[
P_p [A_{n+1} (3, 1)] \geq \varphi \left( P_p [A_n (9, 1)] \right),
\]

where \( \varphi (x) = 1 - (1 - x)^2 \). These together imply that

\[
P_p [A_{n+1} (3, 1)] \geq \psi \left( P_p [A_n (3, 1)] \right),
\]

where \( \psi (x) = 1 - (1 - x^5)^2 \). If \( 1 > x > 1 - 5^{-2} \theta \) for some \( 0 < \theta < 1 \), then \( \psi (x) \geq 1 - 5^{-2} \theta^2 \), therefore once we have

\[
P_p [A_n (3, 1)] \geq 1 - 5^{-2} \theta,
\]

for some \( n \), we have

\[
P_p [A_{n+k} (3, 1)] \geq 1 - 5^{-2} \theta^{2k}
\]

for all \( k \geq 0 \) and the statement of (2) is true since by Lemma 2.2 (1),

\[
P_p [A_n (3, 1)] \to 1.
\]

The proof of (3) is done by usual argument (see e.g. Theorem 8.97 of [2]). Namely, if \( p < 1 - p_c (S^T^*) \), then \( P_p \)-a.s. there is an infinite closed dual cluster. Putting

\[
\theta^*(p) = P_p \left[ \text{one of } (\pm \frac{1}{2}, \pm \frac{1}{2}) \text{ is in an infinite closed dual cluster} \right],
\]
we have by symmetry and the FKG inequality
\[ P_p \left[ A_n^*(2, 2) \right] \geq \left( \frac{1}{4} \theta^*(p) \right)^2 (1 - p)^4. \]

Then by Lemma 2.1 and by the same argument in the proof of (1), we have \( P_p \)-a.s. for every finite \( \Lambda \ni 0 \), there is a closed dual circuit surrounding \( \Lambda \). Since an infinite closed dual cluster intersects large \( \Lambda \ni 0 \) with probability close to 1, by symmetry and the FKG inequality we have
\[ P_p \left[ A_n^*(2, 2) \right] \to 1. \]

As for the exponential estimate (7), note that \( A_n^*(3, 1) \) occurs if there exists a closed left-right crossing in one of \( G_n(9, 1), G_n(9, 1) + (0, 3^n) \) and \( G_n(9, 1) + (0, 2 \cdot 3^n) \). Each probability is not less than \( P_p[A_n^*(9, 1)] \), and hence we have desired estimate by the same scaling argument as in the proof of (2) of the lemma.

Proof of Theorem 1.1

(1) Let \( p > p_c(S^T) \). Then by Lemma 2.3,
\[ \lim_{n \to \infty} P_p[A_n(2, 2)] = 1. \]

If there are more than one infinite open clusters \( P_p \)-a.s., then there exists an infinite closed dual cluster separating different infinite open clusters \( P_p \)-a.s., which implies that
\[ \lim_{n \to \infty} P_p[A_n^*(2, 2)] = 1 \]
by the same argument as in the proof of Lemma 2.3 (3). This is a contradiction, since \( P_p[A_n^*(2, 2)] = 1 - P_p[A_n(2, 2)] \).

(2) It is sufficient to prove that the inequality \( p > p_c(G^T) \) guarantees that
\[ \inf_{x \in G^T} P_p[x \text{ is connected to the origin by an open path in } G^T] > 0. \] (9)

Let \( p > p_c(G^T) \). Then by Lemmas 2.2 and 2.3, we have
\[ \lim_{n \to \infty} P_p[A_n(3, 1)] = 1. \]

Put
\[ \Delta_n = A_{n-1}(3, 1) \cap B_{n-1}(1, 3) \cap \tau_{(0, 2 \cdot 3^{n-1})} A_{n-1}(3, 1) \cap \tau_{(2 \cdot 3^{n-1}, 0)} B_{n-1}(1, 3). \]

Then by the FKG inequality we have
\[ P_p[\Delta_n] \geq (P_p[A_{n-1}(3, 1)])^4. \]

On \( \Delta_n \), there is a big open cluster in \( G_n^T \) such that every open crossing in \( G_{n-1}(3, 1), G_{n-1}(1, 3), G_{n-1}(3, 1) + (0, 2 \cdot 3^{n-1}) \) and \( G_{n-1}(1, 3) + (2 \cdot 3^{n-1}, 0) \) connecting shorter sides of the rectangle is a subset of this big open cluster. We
call this open cluster as the spanning cluster in $G^T_n$. Take a subgraph of $G^T_n$ such that it is a shift of $G^T_{n-1}$. Then it is written as

$$G^T_{n-1} + 3^{n-1}(i,j)$$

for some $(i,j) \in T$. Note that the spanning cluster of $G^T_n$ includes the spanning cluster of $G^T_{n-1} + 3^{n-1}(i,j)$ if the event $\tau_{(i3^{n-1},j3^{n-1})}\Delta_{n-1} \cap \Delta_n$ occurs. For $x \in G^T$, we define the level of $x$ as $\ell(x) = \inf\{n : x \in G^T_n\}$. Also we define $G^T_n(x)$, the $n$-th box of $x$, by the shift of $G^T_n$ such that it is a subgraph of $G^T$ and it contains $x$. If there are more than one such boxes, then we take the nearest one to the origin as $G^T_n(x)$. For convenience, let us write

$$G^T_n(x) = G^T_n + w_n(x).$$

Note that $w_n(0) = 0$ for every $n \geq 1$. Since $p > p_c(G^T)$, by (2) of Lemma 2.3 we have

$$P_p[A_n(3,1)] \geq 1 - C\alpha 2^n > 0 \quad n \geq m_0$$

if $m_0$ is sufficiently large. If $\ell(x) \leq m_0$, then we have

$$\tau_p(0, x) \geq P_p[\text{all edges of } G^T_{m_0} \text{ are open }] := \xi(p) > 0.$$ 

If $\ell(x) > m_0$, then consider the event

$$\bigcap_{n=m_0+1}^{\ell(x)} \Delta_n \cap \bigcap_{n=m_0+1}^{\ell(x)} \tau_{w_n(x)}\Delta_n \cap \{ \text{all edges of } G^T_{m_0} \cup G^T_{m_0}(x) \text{ are open } \}.$$ 

On this event $x$ and the origin are connected by an open path in $G^T$, and by the FKG inequality this probability is bounded by

$$\left( \xi(p) \prod_{n=m_0+1}^{\infty} (1 - C\alpha 2^n) \right)^2 > 0$$

from below.

(3) Let $p < 1 - p_c(S^T_c)$ and let $x, y \in S^T$. We define $n(x, y)$ by

$$n(x, y) = \begin{cases} \max\{n : d(G^T_n(x), G^T_n(y)) > 0\}, & \text{if } |x-y| > 6\sqrt{2}, \\ 0, & \text{if } |x-y| \leq 6\sqrt{2}. \end{cases}$$

Then we have

$$|x-y| \leq 6\sqrt{2} \cdot 3^{n(x,y)}$$

Now consider the following 4 rectangles surrounding $G^T_{n(x,y)}(x)$.

$$R^1_n(x) = [-3^n, 2\cdot 3^n] \times [3^n, 2\cdot 3^n] + w_n(x),$$
$$R^2_n(x) = [-3^n, 2\cdot 3^n] \times [-3^n, 0] + w_n(x),$$
$$R^3_n(x) = [3^n, 2\cdot 3^n] \times [-3^n, 2\cdot 3^n] + w_n(x),$$
$$R^4_n(x) = [-3^n, 0] \times [-3^n, 2\cdot 3^n] + w_n(x),$$
where \( n = n(x, y) \). Let \( \Delta^*_n(x, y) \) be the event that there exists a closed dual crossing in each of above rectangles, connecting shorter sides of each \( R_i^*(x) \), \( i = 1, 2, 3, 4 \). Then the probability of \( \Delta^*_n(x, y) \) is not less than

\[
\left( 1 - C^* \cdot (\alpha^*)^{3n} \right)^4,
\]

by Lemma 2.3. Finally, note that there is a closed dual circuit surrounding \( x \), such that \( y \) is outside of this circuit.

Now we proceed to the proof of Theorem 1.2. Let \( C_{n, bl} \) be the event that satisfies all of the following conditions:

1. In the rectangle
   \( G_n(6, 2) + (-3^{n+1}, 2 \cdot 3^n) \),
   there exists an open left-right crossing which ends at the boundary of the central hole \( [3^{n+1}, 2 \cdot 3^{n+1}] \) of \( G_{n+2}^T \),

2. in the rectangle
   \( G_n(2, 3) + (2 \cdot 3^n, 0) \),
   there exists an open up-down crossing which ends at the boundary of the central hole of \( G_{n+2}^T \), and

3. any pair of above open crossings are connected by an open path in
   \( G_{n+2}^T \cap \left\{ [G_n(6, 2) + (-3^{n+1}, 2 \cdot 3^n)] \cup [G_n(2, 3) + (2 \cdot 3^n, 0)] \right\} \).

The symbol “\( bl \) “ stands for “bottom and left”.

The key to the proof of the first equality of (1) of Theorem 1.2 is the following lemma.

**Lemma 2.4** If \( \lim_{n \to \infty} P_p[A_n(2, 2)] = 1 \), then

\[
\lim_{n \to \infty} P_p[C_{n, bl}] = 1.
\]

We will give proof of this lemma in the next section. By this lemma, we can show that

**Lemma 2.5** If \( \lim_{n \to \infty} P_p[A_n(2, 2)] = 1 \), then

\[
\lim_{n \to \infty} P_p[A_n(1, 1)] = 1,
\]

which means that \( p \geq p_c(G^T) \).
Combining this lemma with (1) of Lemma 2.3 we obtain the equality
\[ p_c(S^T) = p_c(G^T). \]

**Proof of Lemma 2.5** Let \( C_{n,br} \) be the reflected event of \( C_{n,b\ell} \) with respect to the line \( \{ x^1 = \frac{3^n \cdot 2}{2} \} \), and \( C_{n,t\ell} \) be the reflected event of \( C_{n,b\ell} \) with respect to \( \{ x^2 = \frac{3^n \cdot 2}{2} \} \). Also, we define \( C_{n,tr} \) as the reflected event of \( C_{n,b\ell} \) with respect to the point \( (\frac{3^n \cdot 2}{2}, \frac{3^n \cdot 2}{2}) \). Now, let
\[ C_{n,all} := C_{n,b\ell} \cap C_{n,br} \cap C_{n,t\ell} \cap C_{n,tr} \]
and let
\[ C_{n,all}' := C_{n,all} + (3^n \cdot 2, 0). \]

Let \( D_n \) be the event that there is an open left right crossing in \( G^T_n \). Then \( \omega \in D_{n+2} \cap C_{n,all} \) implies that there is an open left right crossing in the rectangle
\[ [-3^{n+1}, 3^{n+2} + 3^{n+1}] \times [0, 3^n]. \]

Let
\[ C_{n,bottom} = C_{n,b\ell} \cap C_{n,br} \quad C_{n,top} = C_{n,t\ell} \cap C_{n,tr}. \]

Then we have
\[ \{ C_{n,all} \cap C_{n-1,all} \cap D''_{n+1} \} \cup \{ C_{n,bottom} \cap C_{n-1,all} \cap D'_{n+1} \} \subset D_{n+2}, \] (10)
where \( C_{n-1,all}' \) and \( D_{n+1}' \) are shifts of \( C_{n-1,all} \) and \( D_{n+1} \) by \( (3^{n+1} \cdot 2, 3^{n+1}) \), and \( C_{n-1,all}' \) and \( D_{n+1}' \) are shifts of \( C_{n-1,all} \) and \( D_{n+1} \) by \( (3^{n+1}, 0) \). Since \( C_{n,all} \cap C_{n-1,all} \cap D_{n+1}' \) and \( C_{n,bottom} \cap C_{n-1,all} \cap D_{n+1}' \) are independent, (10) implies the following inequality
\[ P_p[D_{n+2}] \geq f(P_p[C_{n,bottom} \cap C_{n-1,all} \cap D_{n+1}']), \]
where \( f(t) = 2t - t^2 \), which is an increasing function in the interval \([0, 1]\).

By the FKG inequality and by the equality \( P_p[D_{n+1}'] = P_p[D_{n+1}] \), we have
\[ P_p[C_{n,bottom} \cap C_{n-1,all} \cap D_{n+1}'] \geq P_p[C_{n,bottom}] P_p[C_{n-1,all}] P_p[D_{n+1}]. \]

By Lemma 2.4 if \( \lim_{n \to \infty} P_p[A_n(2, 2)] = 1 \), then we have
\[ \lim_{n \to \infty} P_p[C_{n,bottom}] = \lim_{n \to \infty} P_p[C_{n,all}] = 1. \]

Therefore for every \( \varepsilon > 0 \), there exists some \( N > 0 \) such that
\[ P_p[C_{n,bottom}] \geq P_p[C_{n,all}] > 1 - \varepsilon \]
for every \( n > N \). Combining (12), (13), we have
\[ P_p[D_{n+2}] \geq f((1 - \varepsilon)^2 P_p[D_{n+1}]) \]
(14)
for every $n > N + 1$. This implies that
\[
\liminf_{n \to \infty} P_p[D_n] \geq x_\varepsilon,
\]
where $x_\varepsilon$ is the unique solution to
\[
t = f((1 - \varepsilon)^2 t),
\]
which converges to 1 as $\varepsilon \to 0$. Since $D_n = A_n(1, 1)$, we have
\[
\lim_{n \to \infty} P_p[A_n(1, 1)] = \lim_{n \to \infty} P_p(D_n) = 1.
\]
By Lemma 2.2 and by the proof of Theorem 1.1 (2), this implies the inequality (10). But then we have
\[
P_p[\text{the open cluster of the origin is an infinite cluster}]
\geq \liminf_{n \to \infty} P_p[\text{the origin is connected to }(3^n, 0) \text{ by an open path in } G_n^T].
\]
The right hand side of the above inequality is positive by (10).

The proof of the second equality in (1) of Theorem 1.2 and the proof of (2) of Theorem 1.2 are postponed to the last section.

3 Branching argument

In this section, we prove Lemma 2.4. Before going into the detail, we give rough idea of the proof. By the condition that
\[
\lim_{n \to \infty} P_p[A_n(2, 2)] = 1
\]
and closer. Therefore with high probability, we can find many pairs of open branches of $\gamma$ and $\delta$ which are very close. Finally, connecting one of such pair of branches costs loss of only small probability.
In the actual procedure, we have to choose $\gamma$ and $\delta$ so that they are also close to each other. For this, we will use site percolation on a rooted binary tree. Now, let us begin with some notations. By $\mathbb{T}_2$, we mean a rooted binary tree. The origin of $\mathbb{T}_2$ is denoted by $0$. A point of $\mathbb{T}_2 \setminus \{0\}$ is denoted by $j = (j_1, \ldots, j_n)$ with $j_1, \ldots, j_n \in \{1, 2\}$. The point $0$ is the first child of $0$, and $(1, 2)$ is the second child of $(1)$, and so on.

Let $N$ be sufficiently large and fixed. We will specify later how large $N$ should be. Let

$$V_0^{(N)} = [-3^N, 3^N] \times [-3^N, 2 \cdot 3^{N+1} + 3^{N-1}]$$

and

$$J_0^{(N)} = [-3^N - 3^{N-1}, 3^N + 3^{N-1}] \times [2 \cdot 3^{N+1} - 3^{N-1}, 2 \cdot 3^{N+1} + 3^{N-1}]$$

and set

$$\tilde{V}_0^{(N)} = V_0^{(N)} \cup J_0^{(N)}.$$  

This is the mother shape for the straight connection. Further, we introduce

$$J_{0,t}^{(N)} = [-3^N - 3^{N-1}, -3^N + 3^{N-1}] \times [2 \cdot 3^{N+1} - 3^{N-1}, 2 \cdot 3^{N+1} + 3^{N-1}],$$

$$J_{0,b}^{(N)} = [3^N - 3^{N-1}, 3^N + 3^{N-1}] \times [2 \cdot 3^{N+1} - 3^{N-1}, 2 \cdot 3^{N+1} + 3^{N-1}],$$

(17)

For the branching connection, the mother shape is different. Let

$$\Lambda_0^{(N)} = [-3^{N+1} - 3^{N-1}, 3^N] \times [-3^N, 3^N],$$

(18)

$$I_0^{(N)} = [-3^{N+1} - 3^{N-1}, -3^{N+1} + 3^{N-1}] \times [-3^N - 3^{N-1}, 3^N + 3^{N-1}]$$

and

$$\tilde{\Lambda}_0^{(N)} = \Lambda_0^{(N)} \cup I_0^{(N)},$$

(19)

which is the mother shape for the branching connection. Further, we introduce

$$J_{0,t}^{(N)} = [-3^N + 3^{N-1}, 3^{N-1}] \times [3^N - 3^{N-1}, 3^N + 3^{N-1}]$$

$$J_{0,b}^{(N)} = [-3^N - 3^{N-1}, -3^{N+1} + 3^{N-1}] \times [-3^N - 3^{N-1}, -3^N + 3^{N-1}].$$

(20)

The scaled shapes $\tilde{V}_k^{(N)}$ and $\tilde{\Lambda}_k^{(N)}$ are defined by

$$\tilde{V}_k^{(N)} = 3^{-k} \tilde{V}_0^{(N)}, \quad \tilde{\Lambda}_k^{(N)} = 3^{-k} \tilde{\Lambda}_0^{(N)}.$$  

(21)

In the same way, we define

$$V_k^{(N)} = 3^{-k} V_0^{(N)}, \quad J_k^{(N)} = 3^{-k} J_0^{(N)}, \quad J_{k,*}^{(N)} = 3^{-k} J_{0,*}^{(N)} \quad * = t, r.$$  

(22)
and 

\[ \Lambda_k^{(N)} = 3^{-k} \Lambda_0^{(N)}, \quad I_k^{(N)} = 3^{-k} I_0^{(N)}, \quad I_k^{*} = 3^{-k} I_0^{*}, \quad ** = t, b. \]

Let \( \theta \) denote the rotation of 90 degrees with respect to the origin. We put

\[
B_0^{(N)} = \tilde{V}_0^{(N)} + (3^N + 2, 0) \\
B_0^{(N)\dagger} = \theta^{-1} V_0^{(N)} + (0, 3^N + 2)
\]

and \( B_0^{(N)} = B_0^{(N)} \cup B_0^{(N)\dagger} \). Note that \( B_0^{(N)\dagger} \) is the reflection of \( B_0^{(N)} \) with respect to the line \( \{ x_1 = x_2 \} \), and that \( B_0^{(N)} \) is a subgraph of \( S^T \). Let \( i = (i_1, \ldots, i_n) \in T_2 \setminus \{ 0 \} \). We introduce the following notations for \( i \).

\[
| i | = n = \text{ the generation that } i \text{ belongs}, \quad | 0 | = 0, \\
N_2(i) = \# \{ \alpha \in \{ 1, \ldots, n \} : i_\alpha = 2 \}, \quad N_2(0) = 0 \\
\epsilon(i) = N_2(i) \pmod{2}
\]

\[
\begin{cases}
\tau_0 = 0, \\
\tau_\mu = \begin{cases}
\min \{ \tau_{\mu-1} < \alpha \leq n : i_\alpha = 2 \}, \\
\infty & \text{if the above set is empty}
\end{cases}, \\
\ell_v(0) = 2 \cdot 3^{N+1} \\
\ell_h(0) = 0 \\
\ell_v(i) = \frac{\lfloor N / 2 \rfloor}{2} \sum_{\alpha = \tau_\mu}^{(\tau_\mu + 1) \wedge n} 2 \cdot 3^{N-\alpha + 1} \\
\ell_h(i) = \frac{\lfloor N / 2 \rfloor}{2} \sum_{\alpha = \tau_\mu - 1}^{(\tau_\mu - 1) \wedge n} 2 \cdot 3^{N-\alpha + 1}
\end{cases}
\tag{23}
\tag{24}
\]

where \(| x | \) denotes the largest integer not larger than \( x \), and \( a \wedge b = \min\{a, b\} \).

For a point \( j = (j_1, \ldots, j_n) \in T_2 \setminus \{ 0 \} \), we define vectors \( x(j) \), \( x^\dagger(j) \) and the sets \( B_j^{(N)} = B_j^{(N)} \cup B_j^{(N)\dagger} \) in the following way.

\[ x(j) = \begin{cases}
(3^{N+2} - \ell_h(j^*), \ell_v(j^*)) & \text{if } j_n = 1, \\
(3^{N+2} - \ell_h(j^*) - \delta_v(j^*), \ell_v(j^*) + \delta_v(j^*)) & \text{if } j_n = 2,
\end{cases} \]

where \( j^* \) is the parent of \( j \), i.e. \( j^* = (j_1, \ldots, j_n-1) \), and

\[
\delta_v(0) = 0, \quad \delta_v(j) = 3^{N-|j|} \epsilon(j), \\
\delta_h(0) = 3^N, \quad \delta_h(j) = 3^{N-|j|} (1 - \epsilon(j)),
\]

and we write \( x^\dagger(j) \) for the symmetric point of \( x(j) \) with respect to the line \( \{ x_1 = x_2 \} \).

1. If \( j_n = 1 \), put

\[ B_j^{(N)} = \theta^{(j^*)} \tilde{V}_0^{(N)} + x(j). \]
2. If \( j_n = 2 \), put
\[
B_j^{(N)} = \theta^{-\epsilon(j)} \hat{\Lambda}_n^{(N)} + x(j).
\]
We define \( B_j^{(N)} \) as the reflection of \( B_j^{(N)} \) with respect to the line \( \{ x_1 = x_2 \} \).

Note that \( B_j^{(N)} \) is a subgraph of \( S^T \). For a rectangle \( R \subset S^T \), we say that there exists an open traversing in \( R \) if there exists an open path in \( R \) which connects shorter sides of \( R \). Let
\[
T(R) = \{ \text{there exists an open traversing in } R \}.
\]
Further, we define
\[
S(J_n^{(N)}) = T(J_n^{(N)}) \cap T(I_{n,b}^{(N)}) \cap T(I_{n,t}^{(N)}),
\]
where the traversing direction of \( T(J_n^{(N)}) \) is chosen to be the same direction as the traversing of \( V_n^{(N)} \). Similarly, let
\[
S(I_n^{(N)}) = T(I_n^{(N)}) \cap T(I_{n,t}^{(N)}) \cap T(I_{n,b}^{(N)}),
\]
where the traversing direction of \( T(I_n^{(N)}) \) is chosen to be the same direction as the traversing of \( \tilde{\Lambda}_n^{(N)} \). Note that these are all edge events of \( S^T \). Let \( \theta \) denote the induced transformation on \( \mathbb{E}^2 \) by the rotation \( \theta \), i.e.,
\[
\theta \omega(b) = \omega(\theta^{-1}b) \quad b \in \mathbb{E}^2.
\]
Let
\[
S(B_0^{(N)}) = \tau_{(3N+2,0)} S(V_0^{(N)}), \quad S(B_0^{(N)}) = \tau_{(0,3N+2)} \theta^{-1} S(V_0^{(N)}),
\]
and for \( j \in \mathbb{T}_2 \setminus \{ 0 \} \), we define an edge event \( S(B_j^{(N)}) \) by
\[
S(B_j^{(N)}) = \begin{cases} \tau_x(j) \theta^{-\epsilon(j)} S(V_j^{(N)}), & \text{if } j = (j^*, 1), \\ \tau_x(j) \theta^{-\epsilon(j)} S(\tilde{\Lambda}_j^{(N)}), & \text{if } j = (j^*, 2), \end{cases}
\]
where \( j^* \) is the parent of \( j \). This is actually an edge event on \( B_j^{(N)} \). In the same way, we define an edge event \( S(B_j^{(N)}) \) by the reflected event of \( S(B_j^{(N)}) \) with respect to the line \( \{ x_1 = x_2 \} \), i.e.,
\[
S(B_j^{(N)}) = \begin{cases} \tau_x(j) \theta^{-1-\epsilon(j)} S(V_j^{(N)}), & \text{if } j = (j^*, 1), \\ \tau_x(j) \theta^{1+\epsilon(j)} S(\tilde{\Lambda}_j^{(N)}), & \text{if } j = (j^*, 2). \end{cases}
\]
For convenience, we introduce site variables \( X(t), X^1(t) \) and \( Z(t) \) for \( t \in \mathbb{T}_2 \) by
\[
X(t) = \begin{cases} 1, & \text{if } S(B_t^{(N)}) \text{ occurs}, \\ 0, & \text{otherwise}, \end{cases}
\]

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\[
X^\dagger(t) = \begin{cases} 
1, & \text{if } S(B^\dagger_t(N)) \text{ occurs,} \\
0, & \text{otherwise,}
\end{cases}
\]

and

\[Z(t) = X(t)X^\dagger(t).\]

Let us start with simple facts that can be derived from the assumption of Lemma 2.4. By Lemma 2.1 and the assumption of Lemma 2.4, we know that for every \(\varepsilon > 0\), we can find \(m_0 \geq 1\) such that

\[P_p \left[ A_n(8, 2) \right] \geq 1 - \varepsilon \quad (25)\]

for every \(n \geq m_0\). Therefore if \(N \geq m \geq m_0 + 1\), then by the FKG inequality we have

\[P_p \left[ S(\tilde{\Lambda}_N(N)) \right] \geq (1 - \varepsilon)^4, \quad P_p \left[ S(\tilde{\Lambda}_N(N)) \right] \geq (1 - \varepsilon)^4.\]

This means that for \(N \geq m \geq m_0 + 1\),

\[P_p[Z(t) = 1] \geq (1 - \varepsilon)^8\]

for every \(t \in T_2\) with \(|t| \leq N - m\). Note that \(\{X_j, |j| \leq N - m\}\) and \(\{X^\dagger_j, |j| \leq N - m\}\) are independent. Further it is easy to see that

\[P_p[Z_j = 1 | Z_t = \varepsilon_\ell, |t - j| > 1] = P_p[Z_j = 1] \geq (1 - \varepsilon)^8,\]

where \(|t - j|\) is the graph distance of \(t\) and \(j\) in \(T_2\). In this sense \(Z_j\)'s are 1-dependent. Then by [6], p.14, Theorem B26, the distribution of \(\{Z(t), |t| \leq N - m\}\) dominates that of Bernoulli random variables \(\{W(t), |t| \leq N - m\}\) with

\[P[W(t) = 1] = p_\varepsilon,\]

where \(p_\varepsilon\) is given by the unique positive solution to

\[1 - (1 - \sqrt{p})^4 = (1 - \varepsilon)^8.\]

Note that \(p_\varepsilon \leq (1 - \varepsilon)^8\) and that \(p_\varepsilon \to 1\) as \(\varepsilon \to 0\).

**Proof of Lemma 2.4**

Let us fix an integer \(m\) with \(m \geq m_0 + 1\) and take \(N \geq m\). By the above observation, we can construct 0-1 valued random variables \(\{\tilde{Z}(t), W(t); t \in T_2, |t| \leq N - m\}\) on a probability space \((\tilde{\Omega}, \tilde{F}, \tilde{P})\) such that

1. \(\{W(t); t \in T_2, |t| \leq N - m\}\) is i.i.d. with
   \[\tilde{P}(W(t) = 1) = p_\varepsilon,\]
2. the distribution of \( \{ Z(t): t \in \mathbb{T}_2, |t| \leq N - m \} \) is the same as that of \( \{ \hat{Z}(t): t \in \mathbb{T}_2, |t| \leq N - m \} \),

3. \( \hat{P}(C_0(\hat{Z}) \ni C_0(W)) = 1 \), where \( C_0(\hat{Z}) \) and \( C_0(W) \) are open clusters of \( \mathbf{0} \) in the configurations \( \{ \hat{Z}(t): t \in \mathbb{T}_2, |t| \leq N - m \} \) and \( \{ W(t): t \in \mathbb{T}_2, |t| \leq N - m \} \), respectively.

Let 
\[
Z_n(W) = \# \{ j \in \mathbb{T}_2 \setminus \{ \mathbf{0} \}: |j| = n, j \in C_0(W) \}.
\]

Then conditioned that \( W(\mathbf{0}) = 1 \), \( Z_n(W) \) is a Golton-Watoson branching process with offspring distribution:

\[
p_0 = (1 - p_\varepsilon)^2, \quad p_1 = 2p_\varepsilon(1 - p_\varepsilon), \quad p_2 = p_\varepsilon^2, \quad p_k = 0 \text{ for } k \geq 3.
\]

If \( 0 < \varepsilon \) is sufficiently small, then this branching process is supercritical and for any integer \( k \geq 1 \),

\[
\hat{P}(Z_n(W) \geq k) \to 1 - q_\varepsilon
\]
as \( n \) goes to infinity, where \( q_\varepsilon \) is the extinction probability of \( Z_n(W) \), which goes to 0 as \( \varepsilon \to 0 \) (cf. [23], p.8, Theorem I.6.1). Let \( N_k \geq m \) be so large that

\[
\hat{P}(Z_{N-m}(W) \geq k) \geq 1 - 2q_\varepsilon
\]
for every \( N \geq N_k \). Let

\[
\mathcal{B} = \bigcup \{ B_t^{(N)}: t \in \mathbb{T}_2, |t| \leq N - m \},
\]
and fix an integer \( N \geq N_k \) and a configuration on \( \mathcal{B} \) such that

\[
C_0(Z) \cap \{ t \in \mathbb{T}_2: |t| = N - m \} \neq \emptyset.
\]

This event occurs in \( \mathcal{B} \). We take a point \( j \in C_0(Z) \) with \( |j| = N - m \). Then the unique path \( \gamma \) in \( \mathbb{T}_2 \) which connects \( j \) with \( \mathbf{0} \) is included in \( C_0(Z) \), therefore there exists an open path \( \gamma_1 \) in \( \cup_{t \in \xi} B_t^{(N)} \) that connects the \( x_1 \)-axis with an open traversing in \( V_j^{(N)} \), and an open path \( \gamma_2 \) in \( \cup_{t \in \xi} B_t^{(N)} \) that connects the \( x_2 \)-axis with an open traversing in \( V_j^{(N)} \). Depending on whether \( \epsilon(j) = 0 \) or 1, we put

\[
y(j) = \begin{cases} 
(3^{N+2} - \ell_h(j^*), 3^{N+2} - \ell_h(j^*)), & \text{if } \epsilon(j) = 0, \\
(\ell_v(j^*), \ell_v(j^*)), & \text{if } \epsilon(j) = 1,
\end{cases}
\]
and \( Q(j) = (3^{m+1}, 3^{m+1})^2 + y(j) \). Then by independence, the probability that the open cluster in \( \mathcal{B} \) which contains \( \gamma_1 \) and the open cluster in \( \mathcal{B} \) which contains \( \gamma_2 \) are connected by an open path in \( Q(j) \setminus \mathcal{B} \) is not less than \( p^{c(m)} \), where \( c(m) \) is a constant depending only on \( m \). To be more precise we can take \( c(m) = 8 \cdot 3^{m+1} \). Since \( \{ Q(j): |j| = N - m \} \) are disjoint, the probability that such an open connection exists for some \( j \in C_0(Z) \) such that \( |j| = N - m \), is not less than

\[
1 - (1 - p^{c(m)})^k.
\]
We take $k$ so large that $(1 - p^{c(m)})^k < \varepsilon$. Then, we have
\[
P_p \left[ x_1 \text{-axis and } x_2 \text{-axis are connected by an open path in } (G_N(3, 2) + (0, 2 \cdot 3^N)) \cup (G_N(2, 3) + (2 \cdot 3^N, 0)) \right] 
\geq (1 - 2q_c)(1 - \varepsilon).
\]
By the FKG inequality we have finally
\[
P_p \left[ C_{N, m} \right] \geq P_p \left[ A_N(2, 2) \right]^2 \left[ 1 - \sqrt{1 - P_p[A_N(3, 2)]} \right] 
\times \left[ 1 - \sqrt{1 - P_p[A_N(6, 2)]} \right] (1 - 2q_c)(1 - \varepsilon)
\]
for sufficiently large $N$. Since $\varepsilon$ is arbitrarily small, this completes the proof.

4 Uniqueness of the critical probability

In this section we prove the second equality in the statement (1) and the statement (2) of Theorem 1.2. Since we have proven the equality $p_c(G^T) = p_c(S^T)$, we write simply $p_c$ for $p_c(G^T) = p_c(S^T)$. We first claim that
\[
\theta(p_c) = 0. \tag{26}
\]
Assume that $\theta(p_c) > 0$. Then by Lemma 2.3 (2) and Lemma 2.2 (1), we have
\[
\lim_{n \to \infty} P_{p_c}[A_n(3, 1)] = 1.
\]
Thus, for a $0 < \theta < 1$ and large $n \geq 1$, we have
\[
P_{p_c}[A_n(3, 1)] \geq 1 - 5^{-2}\theta.
\]
Then taking $\theta' \in (\theta, 1)$ and sufficiently small $\varepsilon > 0$, we have
\[
P_{p_c-\varepsilon}[A_n(3, 1)] \geq 1 - 5^{-2}\theta',
\]
which, by the scaling argument, implies that
\[
\lim_{n \to \infty} P_{p_c-\varepsilon}[A_n(3, 1)] = 1.
\]
By Lemma 2.4, this means that $p_c - \varepsilon \geq p_c$, a contradiction. Combining with Lemma 2.1 we have
\[
\liminf_{n \to \infty} P_{p_c}[A_n(2, 2)] < 1. \tag{27}
\]
This, together with Lemma 2.4 and the argument in the proof of (2) of Lemma 2.2 implies that
\[
\limsup_{n \to \infty} P_{p_c}[A^*_n(18, 1)] \geq \limsup_{n \to \infty} P_{p_c}[A^*_n(6, 2)] > 0. \tag{28}
\]
Lemma 4.1 If \( p < p_c \), then we have

\[
\liminf_{n \to \infty} P_p [A_n(2, 2)] = 0, \tag{29}
\]

\[
\liminf_{n \to \infty} P_p [A_n(1, 1)] = 0, \tag{30}
\]

From this lemma it is easy to obtain the final equality \( p_c = 1 - p_c(S^{T*}) \). For this, it is sufficient to see the inequality \( p_c \leq 1 - p_c(S^{T*}) \), since by Theorem 1.1, we have \( 1 - p_c(S^{T*}) \leq p_c \). But if \( p < p_c \), by (29) and by the scaling argument we have \( P_p(A_{n_0}^*(3, 1)) \) converges to 1 exponentially fast, which implies that \( 1 - p \geq p_c(S^{T*}) \).

The proof of Lemma 4.1 is essentially the same as Kesten’s original argument. Here, we sketch the proof of (29). Let \( \delta > 0 \) be a positive number such that

\[
\delta < \limsup_{n \to \infty} P_{p_c} [A_{n_k}^*(18, 1)]. \tag{31}
\]

Then we can find a subsequence \( \{n_k\} \) such that

\[
P_{p_c} [A_{n_k}^*(18, 1)] > \delta. \tag{31}
\]

By Russo’s formula we have

\[
\frac{d}{dp} P_p [A_n(2, 2)] = E_p [N_{A_n(2, 2)}] \tag{32}
\]

where \( N_{A_n(2, 2)} \) denotes the number of pivotal edges for \( A_n(2, 2) \). Let us recall that an edge \( e \) is pivotal for an event \( A \) in a configuration \( \omega \) if and only if either of the followings holds;

1. \( \omega \in A \) and \( \omega^e \notin A \), or
2. \( \omega \notin A \) and \( \omega^e \in A \),

where

\[
\omega^e(f) = \begin{cases} 
\omega(f) & \text{if } f \neq e, \\
1 - \omega(e) & \text{if } f = e.
\end{cases}
\]

By (32), we have

\[
\frac{d}{dp} P_p [A_n(2, 2)] \geq E_p [N_{A_n(2, 2)}|A_n(2, 2)] P_p [A_n(2, 2)].
\]

Integrating this from \( p \) to \( p_c \), we obtain

\[
P_{p_c} [A_n(2, 2)] \geq P_p [A_n(2, 2)] \exp \left\{ \int_p^{p_c} E_q [N_{A_n(2, 2)}|A_n(2, 2)] dq \right\}. \tag{33}
\]

This is valid for all \( n \geq 1 \). We will show that for every \( q \in (p, p_c) \),

\[
E_q [N_{A_{n_k+2}(2, 2)}|A_{n_k+2}(2, 2)] \geq k\delta^5,
\]

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where \( \{ n_k \} \) satisfies the inequality (31). Clearly this together with (33) proves (29). We divide \( A_{n_k+2} \) into the sets that specify the lowest open left-right crossing \( r \) of \( G_{n_k+2}(2, 2) \). For a path \( r \) in \( G_{n_k+2}(2, 2) \) connecting left side of \( G_{n_k+2}(2, 2) \) with its right side, let

\[
E(r) = \{ \text{r is the lowest open left-right crossing in } G_{n_k+2}(2, 2) \},
\]

and

\[
E^*(r) = \left\{ \begin{array}{ll}
\text{there exists a closed dual path in } G_n^*(1, 18) & \\
\text{connecting the top side of } G_{n_k+2}^*(2, 2) \text{ with a dual edge which crosses } r
\end{array} \right. \]

Then, by (31) for \( 0 < q < p_c \) we have

\[
P_q \left[ E^*(r) \mid E(r) \right] > \delta,
\]

by the FKG inequality since \( E^*(r) \) is a decreasing event. For \( \omega \in E^*(r) \), let \( \psi \) denote the left-most closed dual path connecting the top side of \( G_n^*(1, 18) \) with a dual edge crossing \( r \). Further, let \( e_\psi \) denote the edge in \( r \) whose dual edge \( e^*_\psi \) is connected to \( \psi \). Apparently \( e_\psi \) is then a pivotal edge for \( A_{n_k+2}(2, 2) \). Let \( r_+ \) denote the part of \( r \) to the right of \( e_\psi \), i.e. \( r_+ \) connects \( e_\psi \) with the right side of \( G_{n_k+2}(2, 2) \).

For \( j = 1, \ldots, k-1 \), let \( G_{n_j}(e_\psi) \) denote the subgraph of \( G_{n_k+2}(2, 2) \) such that it is a shift of \( G_{n_j+2}(1, 1) \) and it contains \( e_\psi \). We write

\[
G_{n_j}(e_\psi) = G_{n_j+2} + x_j(e_\psi)
\]

so that \( x_j(e_\psi) \) is the lower left corner point of \( G_{n_j}(e_\psi) \). Consider an annulus

\[
H_{n_j} = [-3^{n_j+1}, 3^{n_j+2} + 3^{n_j+1}]^2 \setminus (-2 \cdot 3^{n_j}, 3^{n_j+2} + 2 \cdot 3^{n_j})^2,
\]

and let

\[
H_{n_j}(e_\psi) = H_{n_j} + x_j(e_\psi).
\]

Note that \( \{ H_{n_j}(e_\psi) \}_{1 \leq j \leq k-1} \) are disjoint. Let

\[
F(r, \psi) = \left\{ \begin{array}{ll}
\psi \text{ is the left-most closed dual path connecting the top side} & \\
\text{of } G_n^*(1, 18) \text{ with a dual edge crossing } r
\end{array} \right. \]

and for given \( r \) and \( \psi \), let

\[
C_j(r, \psi) = \left\{ \begin{array}{ll}
\text{there is a dual closed path in } H_{n_j}(e_\psi), \text{ located entirely} & \\
\text{above } r \text{ and to the right of } \psi, \text{ connecting } \psi \text{ with a} & \\
\text{dual edge which crosses } r
\end{array} \right. \]

Then for \( 0 < q < p_c \), by the FKG inequality we have

\[
P_q \left[ C_j(r, \psi) \mid F(r, \psi) \right] > \delta^4.
\]
Note that for $\omega \in F(r, \psi) \cap C_j(r, \psi)$, there exists a pivotal edge for $A_{n_k+2}(2, 2)$ in $H_{n_j}(e_\psi)$. Therefore we have

$$E_q \left[ N_{A_{n_k+2}(2,2)} \big| F(r, \psi) \right] \geq 1 + \sum_{j=1}^{k-1} E_q \left[ 1_{C_j(r,\psi)} \big| F(r, \psi) \right] \geq 1 + (k-1)\delta^4,$$

since $e_\psi$ is pivotal on $F(r, \psi)$. Therefore, we have

$$E_q(N_{A_{n_k+2}(2,2)} \mid r \text{ is the lowest left-right open crossing }) \geq k\delta^5.$$

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