Congruence kernels around affine curves

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Abstract

Let $S$ be a smooth affine algebraic curve, and let $\hat S$ be the Riemann surface obtained by removing a point from $S$. We provide evidence for the congruence subgroup property of mapping class groups by showing that the congruence kernel

$$\ker \left( \hat{\text{Mod}}(\hat S) \to \text{Out}(\hat\pi_1(\hat S)) \right)$$

lies in the centralizer of every braid in $\text{Mod}(\hat S)$. We also obtain a new proof of a theorem of Matsumoto which says that the congruence kernel depends only on the genus in the affine case.

1 Introduction

Let $S$ be a Riemann surface of finite type, and let $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))$ be its mapping class group. If $C$ is a finite index characteristic subgroup of $\pi = \pi_1(S)$, there is a natural map

$$\text{Mod}(S) \to \text{Out}(\pi/C),$$

and a subgroup containing the kernel of such a map is called a congruence subgroup of $\text{Mod}(S)$—the kernels themselves are principal congruence subgroups. Ivanov’s congruence subgroup problem asks if every finite index subgroup of $\text{Mod}(S)$ is congruence, and we say that $\text{Mod}(S)$ has the congruence subgroup property if they are—this is Problem 2.10 of Kirby’s List [1], see [19, 16, 17]. The mapping class group is known to possess this property when $S$ has genus no more than two. In genus zero, the theorem is due to Diaz, Donagi, and Harbater [11] (see also [24] and Section 4 here); in genus one, to Asada [4] (see also [10]); and, in genus two, to Boggi [7, 8].

If $G$ is a group, we let $\hat G$ denote its profinite completion. There is a natural map

$$\hat{\text{Mod}}(S) \to \text{Out}(\hat\pi)$$

whose kernel $K = K(S)$ is the congruence kernel. Vanishing of $K$ is equivalent to the congruence subgroup property.

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Let $\Gamma(S)$ be the congruence subgroup of $\text{Mod}(S)$ consisting of those mapping classes acting trivially on $H_1(S;\mathbb{F}_3)$. As intersections of congruence subgroups are congruence (see Lemma 1), a finite index subgroup of $\text{Mod}(S)$ contains a congruence subgroup if and only if its intersection with $\Gamma(S)$ does. So, letting $\hat{\Gamma}(S) = \Gamma(S)$, we need only check injectivity of 

$$\hat{\Gamma}(S) \to \text{Out}(\hat{\pi})$$

to establish the congruence subgroup property. Thanks to a theorem of Grossman [13], the group $\Gamma(S)$ injects into $\text{Out}(\hat{\pi})$, and we let $\tilde{\Gamma}(S)$ be the closure of its image. By the universal property of profinite completions, $\hat{\Gamma}(S)$ is the image of $\Gamma(S) \to \text{Out}(\hat{\pi})$.

Let $\mathcal{C}(S)$ be the curve complex of $S$—the simplicial flag complex whose vertices are isotopy classes of nonperipheral simple loops on $S$ joined by an edge if they may be realized as disjoint loops on $S$. In his attack on the congruence subgroup problem [6], Boggi introduced profinite versions of this complex, subsequently studied in [9, 21, 22], which we now discuss.

Let $\mathcal{A} = \mathcal{A}(S)$ be the inverse system of all finite index subgroups of $\Gamma(S)$ under inclusion and let $\mathcal{H} = \mathcal{H}(S)$ be the inverse system of all congruence subgroups of $\Gamma(S)$. The group $\Gamma(S)$ is torsion–free [31]. Not only that, but $\Gamma(S)$ is pure, meaning that its elements fix a simplex of $\mathcal{C}(S)$ if and only if they fix all of the vertices of that simplex, see Corollary 1.8 of [18]. It follows that if $a$ lies in $\mathcal{A}$, the quotient

$$\mathcal{C}^a = \mathcal{C}(S)/a$$

is naturally a finite simplicial complex. The **profinite curve complex** is the limit

$$\hat{\mathcal{C}}(S) = \lim_{\mathcal{A}} \mathcal{C}^a$$

and the **procongruence curve complex** is

$$\check{\mathcal{C}}(S) = \lim_{\mathcal{H}} \mathcal{C}^\kappa.$$  

These limits may be taken in the topological category, but the resulting spaces are quite unruly. A natural solution, that we adopt, is to consider the $\mathcal{C}^a$ simplicial finite sets and take limits in the category of simplicial profinite sets. The objects of this category are simplicial objects in the category of profinite sets, called **profinite spaces**, and the morphisms are the natural transformations between them. See Section 5 and [6, 22, 27, 26].

Passing to limits, the action of $\Gamma(S)$ on $\mathcal{C}(S)$ yields an action of $\hat{\Gamma}(S)$ on $\hat{\mathcal{C}}(S)$ and of $\tilde{\Gamma}(S)$ on $\check{\mathcal{C}}(S)$. The zero–skeleton of $\mathcal{C}(S)$ injects into the zero–skeleta of $\hat{\mathcal{C}}(S)$ and $\check{\mathcal{C}}(S)$, and we let context determine of which space a simple loop is to be considered a vertex.

If $G$ is a group acting on a set $X$, we let $G_x$ denote the stabilizer in $G$ of an element $x$ in $X$. Most often, $X$ is the curve complex or one of its profinite cousins, and $x$ is a vertex. If $X$ and $Y$ are subsets of $\tilde{\Gamma}(S)$ and $\check{\Gamma}(S)$, we let $\overline{X}$ and $\overline{Y}$ denote their closures, respectively.

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Let \( \hat{S} \) denote the surface obtained from \( S \) by removing a point, let \( \bar{S} \) denote the surface obtained from \( \hat{S} \) by removing a point, and let \( \bar{\pi} = \pi_1(\hat{S}) \).

Our main theorem is the following, which provides some evidence for the congruence subgroup problem. See Section 8.

**Theorem.** Let \( S \) be a smooth affine curve. Let \( \mathcal{P} \) be the set of nonperipheral simple loops on \( \hat{S} \) that are peripheral in \( S \). Then

\[
K(\hat{S}) \subset \bigcap_{\gamma \in \mathcal{P}} \Gamma(\bar{S})_{\gamma}.
\]

In particular, the kernel \( K(\hat{S}) \) lies in the centralizer of every braid in \( \text{Mod}(\hat{S}) \).

Our techniques provide a new proof of the following theorem of M. Matsumoto (which follows from Theorem 2.2 of [23]). See Section 9.

**Theorem (Matsumoto).** For affine curves, the congruence kernel depends only on the genus. In other words, \( K(\hat{S}) \cong K(S) \) when \( S \) is affine. In this case, there is a short exact sequence

\[
1 \to \hat{\pi} \to \hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(S) \to 1.
\]

It is a more difficult theorem of Boggi [8] and, independently, Hoshi–Mochizuki [15] that this is still the case when \( S \) is projective. Note that Matsumoto’s theorem immediately implies that the congruence subgroup property holds when the genus is zero, as a thrice–punctured sphere has trivial mapping class group. In fact, and as first shown by Asada [4], one does not even need the full strength of Matsumoto’s theorem to obtain the congruence subgroup property in genus zero, see Section 4.

Boggi has observed that \( \lim_{\mathcal{X}} \hat{\pi}_1(\mathcal{C}^{\mathcal{X}}(S)) \) vanishes for any surface \( S \), see Theorem 11. The proof of our main theorem produces the analogous theorem for \( \mathcal{C}(\hat{S}) \) when \( S \) is affine, see Theorem 12.

**Theorem.** Let \( S \) be a smooth affine algebraic curve. Then

\[
\lim_{\mathcal{X}} \hat{\pi}_1(\mathcal{C}^\mathcal{X}(\hat{S})) = 1.
\]

**Corollary.** If \( S \) is affine, then, for any prime number \( p \),

\[
H_1(\mathcal{C}(\hat{S}); \mathbb{F}_p) := \lim_{\mathcal{X}} H_1(\mathcal{C}^\mathcal{X}(\hat{S}); \mathbb{F}_p) = 0.
\]

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2 Point pushing

We make extensive use of the Birman exact sequence [5]:

\[
1 \rightarrow \pi \xrightarrow{\iota} \Gamma(\hat{S}) \xrightarrow{F} \Gamma(S) \rightarrow 1.
\]

The map \(F\) is the natural map obtained by “forgetting the puncture,” or, in other words, by extending homeomorphisms from \(\hat{S}\) to \(S\). The image of \(\pi\) in \(\Gamma(\hat{S})\) is the subgroup \(B\) of elements represented by point–pushing homeomorphisms, defined as follows—see Section 4.2 of [12] for more detail. Let \(\{x\} = S - \hat{S}\), and pick an isomorphism \(\pi_1(S) \cong \pi_1(S,x)\). A loop \(h: [0,1] \rightarrow S\) based at \(x\) is an isotopy of the inclusion map \(\{x\} \rightarrow S\). Such an isotopy may be extended to an ambient isotopy \(H: S \times [0,1] \rightarrow S\). The homeomorphism \(H(\cdot,1)\) at time one is a homeomorphism of \(S\) fixing \(x\), which induces a homeomorphism \(H\) of \(\hat{S}\). We call \(H(\cdot,1)\) and \(H\) point–pushing homeomorphisms. The isotopy class of \(H\) depends only on the pointed homotopy class of \(h\), and so yields a well defined element of \(\text{Mod}(\hat{S})\). This mapping class acts trivially on homology, and so lies in \(\Gamma(\hat{S})\).

The Birman exact sequence fits naturally into a commutative diagram

\[
\begin{array}{cccc}
1 & \rightarrow & \pi & \xrightarrow{\iota} & \Gamma(\hat{S}) & \xrightarrow{F} & \Gamma(S) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \text{Inn}(\pi) & \xrightarrow{} & \text{Aut}(\pi) & \xrightarrow{} & \text{Out}(\pi) & \rightarrow & 1
\end{array}
\]

where the vertical maps are injections and the map \(\pi \rightarrow \text{Inn}(\pi)\) is the natural isomorphism. A consequence is that every inner automorphism of \(\pi_1(S,x)\) is realized by a point–pushing homeomorphism fixing \(x\).

3 Pulling and pushing congruence subgroups

Lemma 1. Finite intersections of congruence subgroups are congruence.

Proof. It suffices to prove that the intersection of two principal congruence subgroups is congruence.

Let \(C\) and \(D\) be finite index characteristic subgroups of \(\pi\), and let

\[
E = \bigcap_{\varphi \in \text{Aut}(\pi)} \varphi(C \cap D)
\]

be the characteristic core of \(C \cap D\). Our maps to \(\text{Out}(\pi/C)\) and \(\text{Out}(\pi/D)\) both factor through \(\text{Out}(\pi/E)\):

\[
\begin{array}{cccc}
\Gamma(S) & \xrightarrow{} & \text{Out}(\pi/E) & \xrightarrow{} & \text{Out}(\pi/C) \\
& & \xrightarrow{} & & \text{Out}(\pi/D)
\end{array}
\]
and so \( \ker \left( \Gamma(S) \to \text{Out}(\pi/E) \right) \) lies in the intersection of \( \ker \left( \Gamma(S) \to \text{Out}(\pi/C) \right) \) and \( \ker \left( \Gamma(S) \to \text{Out}(\pi/D) \right) \).

Lemma 1 shows that \( \mathcal{X} \) forms an inverse system and allows us to replace \( \text{Mod}(S) \) with \( \Gamma(S) \) when considering the congruence subgroup problem, replacing congruence subgroups of \( \text{Mod}(S) \) with their intersections with \( \Gamma(S) \), called congruence subgroups of \( \Gamma(S) \).

**Lemma 2.** If \( \kappa \) is a congruence subgroup of \( \Gamma(S) \), then \( F^{-1}(\kappa) \) is a congruence subgroup of \( \Gamma(\hat{S}) \).

A loop on a Riemann surface of finite type is peripheral if it is freely homotopic into a subsurface that is conformally equivalent to a punctured disk. An element of the fundamental group is peripheral if its representatives are. Note that we consider a null–homotopic loop to be peripheral. Letting \( \iota: \hat{S} \to S \) be the inclusion map, we have the following short exact sequence

\[
1 \longrightarrow \mathcal{N} \longrightarrow \hat{\pi} \longrightarrow \pi \longrightarrow 1
\]

(2)

where \( \mathcal{N} \) is the normal closure of a peripheral element corresponding to the distinguished puncture of \( \hat{S} \) (the puncture located at \( \{x\} = S - \hat{S} \)).

**Proof of Lemma 2.** Pick a point \( z \) on \( \hat{S} \). Identify \( \hat{\pi} \) with \( \pi_{1}(\hat{S}, \hat{z}) \) and \( \pi \) with \( \pi_{1}(S, \hat{z}) \).

Let \( D \) be a characteristic subgroup of \( \pi \) and let \( \kappa \) be the associated principal congruence subgroup.

Let \( \varphi \) lie in \( \Gamma(\hat{S}) \) and let \( H: S \to S \) be a homeomorphism fixing \( x \) and \( \bar{z} \) such that \( h = H|_{\bar{z}} \) represents \( \varphi \). Let \( c \) lie in \( t_{\bar{z}}^{-1}(D) \). We have \( t_{\bar{z}}h_{c}(c) = H_{c}t_{\bar{z}}(c) \). Since \( D \) is characteristic, \( H_{c}t_{\bar{z}}(c) \) lies in \( D \), and so \( h_{c}(c) \) lies in \( t_{\bar{z}}^{-1}(D) \). Therefore \( t_{\bar{z}}^{-1}(D) \) is \( h_{c} \)-invariant, and so the conjugacy class of \( t_{\bar{z}}^{-1}(D) \) is \( \Gamma(\hat{S}) \)-invariant. We thus have a representation

\[
\rho: \Gamma(\hat{S}) \to \text{Out}(\hat{\pi}/t_{\bar{z}}^{-1}(D))
\]

that fits into a commuting diagram:

\[
\begin{array}{ccc}
F^{-1}(\kappa) & \longrightarrow & \Gamma(\hat{S}) \\
\downarrow & & \downarrow \rho \\
1 & \longrightarrow & \kappa \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{Out}(\hat{\pi}/t_{\bar{z}}^{-1}(D)) \\
\downarrow F & & \downarrow \end{array}
\]

\[
\begin{array}{ccc}
\Gamma(\hat{S}) & \longrightarrow & \text{Out}(\pi/D) \\
\end{array}
\]

Exactness of the bottom row yields \( F^{-1}(\kappa) \supset \ker \rho \). So \( F^{-1}(\kappa) \) is congruence. \[\square\]

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1While the conjugacy class of \( t_{\bar{z}}^{-1}(D) \) is \( \Gamma(\hat{S}) \)-invariant, it is not necessarily characteristic. To see that this is not an issue, consider the characteristic core \( \mathcal{D} \) of \( t_{\bar{z}}^{-1}(D) \). Our kernel \( \ker \rho \subset F^{-1}(\kappa) \) contains the kernel of the representation \( \Gamma(\hat{S}) \to \text{Out}(\hat{\pi}/\mathcal{D}) \), which is a principal congruence subgroup in the usual sense.
If $M$ and $N$ are subgroups of a group $G$, we let

$$MN = M \cdot N = \{mn | m \in M \text{ and } n \in N\}.$$ 

If $G$ acts on a set $X$, our notation for stabilizers becomes ambiguous when applied to products $MN \subseteq G$. To remedy this, we adopt the convention that, whenever we write $MN$, we always mean $MN = M \cdot (N \gamma)$, and not $(MN) \gamma$.

**Theorem 3.** Let $S$ be a smooth affine curve, and let $\gamma$ be a nonperipheral simple loop in $\hat{S}$ that is peripheral in $S$. Let $\mathcal{B} = \kappa \cap \mathcal{B}$ and $\kappa \gamma = \kappa \cap \Gamma(\hat{S})$. Then $\mathcal{B} \kappa = \mathcal{B} \cdot (\kappa \gamma)$ is a congruence subgroup.

As $\mathcal{B} \kappa \kappa \gamma \subseteq \kappa$, the following corollary is immediate.

**Corollary 4.** For $S$ a smooth affine curve, the inverse system of subgroups

$$\{\mathcal{B} \kappa \kappa \gamma | \kappa \text{ in } \mathcal{H}(\hat{S})\}$$

is cofinal in $\mathcal{H}(\hat{S})$.

**Lemma 5.** Let $S$ be a smooth affine curve, and let $\gamma$ be a nonperipheral simple loop in $\hat{S}$ that is peripheral in $S$. Let $\kappa$ be a congruence subgroup of $\Gamma(\hat{S})$. Then $\mathcal{B} \kappa \gamma$ is a congruence subgroup of $\Gamma(\hat{S})$.

**Proof.** As $\mathcal{B}$ is normal in $\Gamma(\hat{S})$, the set $\mathcal{B} \kappa \gamma$ is a subgroup for any $\kappa$ in $\mathcal{H}$. Let $\kappa$ be a congruence subgroup containing a principal congruence subgroup $p = \ker (\Gamma(\hat{S}) \to \Out(\hat{\pi}/C))$ for some characteristic subgroup $C$ of $\hat{\pi}$.

Let $\iota : \hat{S} \to S$ be the inclusion map. By our choice of $\gamma$, there is a $\pi_1$–injective subsurface $\Sigma \subseteq \hat{S}$ with $\partial \Sigma = \gamma$ such that

$$j = t_\kappa|_{\pi_1(\Sigma)} : \pi_1(\Sigma) \to \pi$$

is an isomorphism. Here we choose a basepoint $z$ in $\gamma$ and identify $\hat{\pi}$ and $\pi$ with $\pi_1(\hat{S}, z)$ and $\pi_1(S, z)$ respectively.

We lift $\Gamma(\hat{S})\gamma$ to $\Aut(\hat{\pi})$ by identifying the former with the group of homeomorphisms of $\hat{S}$ that are the identity on $\hat{S} - \Sigma$ up to isotopy relative to $\hat{S} - \Sigma$. Since $C$ is characteristic and $\pi_1(\Sigma, z)$ is $\Gamma(\hat{S})\gamma$–invariant, the subgroup $D = C \cap \pi_1(\Sigma, z)$ is $\Gamma(\hat{S})\gamma$–invariant. It follows that the conjugacy class of $t_\kappa(D)$ is $\Gamma(\hat{S})$–invariant, and we have an exact sequence

$$1 \to \mathcal{B} \to \Gamma(\hat{S}) \to \Out(\pi/t_\kappa(D)).$$

The subgroup $\mathcal{B}$ is a $\Gamma(\hat{S})$–congruence subgroup, and we have the following claim.

**Claim.** $\mathcal{B} \subseteq F(p\gamma)$. 

**Proof of claim.** Let \( \psi \) be an element of \( \mathcal{D} \). Choose a homeomorphism \( h : S \to S \) that fixes \( z \) and represents \( \psi \). As mentioned above, the conjugacy class of \( t_s(D) \) is \( \Gamma(S) \)–invariant. By postcomposing with a point–pushing homeomorphism fixing \( z \), we may assume that \( h \) represents an automorphism \( h_s \) of \( \pi_1(S, z) \) that preserves \( t_s(D) \). Since \( \psi \) is an element of \( \mathcal{D} \), this automorphism \( h_s \) descends to an inner automorphism of \( \pi_1(S, z)/t_s(D) \). By further postcomposing with a point–pushing homeomorphism fixing \( z \), we may, and do, assume that \( h_s \) in fact represents the trivial automorphism of \( \pi_1(S, z)/t_s(D) \). We now isotope \( h \) relative to \( z \) so that \( h \) is the identity on \( S - t(\Sigma) \).

Restricting \( h \) to \( t(\hat{S}) \) yields a homeomorphism \( H : \hat{S} \to \hat{S} \) that fixes \( \gamma \) pointwise, and hence represents an element \( \varphi \) of \( \Gamma(\hat{S})_{\gamma} \) that maps to \( \psi \) under the forgetful map \( F : \Gamma(\hat{S}) \to \Gamma(S) \).

Now,

\[
\pi_1(\hat{S}, z) \cong A \ast \mathbb{Z} B
\]

where \( A \) is the free group \( \pi_1(S - \Sigma, z) \), \( B = \pi_1(\Sigma, z) \), and the amalgamating \( \mathbb{Z} \) is \( \pi_1(\gamma, z) \).

Each element of \( \pi_1(\hat{S}, z) \) may then be written in a normal form

\[
a_1 b_1 \cdots a_n b_n
\]

where each \( a_i \) lies in \( A \) and each \( b_i \) lies in \( B \). The homeomorphism \( H \) induces an automorphism \( H_s \) of \( \pi_1(\hat{S}, z) \), and since \( H \) is the identity on \( S - \Sigma \), we have

\[
H_s(a_1 b_1 \cdots a_n b_n) = a_1 H_s(b_1) \cdots a_n H_s(b_n).
\]

Now, by construction, \( j H_s \big|_B = h_s \), and so \( H_s \big|_B \) is trivial in \( \text{Aut}(B/D) \). So, for each \( b \) in \( B \), there is a \( d \) in \( D \) such that \( H_s(b) = bd \). So, for each \( i \) there is a \( d_i \) in \( D \) such that

\[
H_s(a_1 b_1 \cdots a_n b_n) = a_1 H_s(b_1) \cdots a_n H_s(b_n)
= a_1 (b_1 d_1) \cdots a_n (b_n d_n).
\]

Since \( D \subset C \), we conclude that the automorphism \( H_s \) induces the trivial automorphism of \( \pi_1(\hat{S}, z)/C \). But this means that \( \varphi \) lies in \( p \), and as it also lies in \( \Gamma(\hat{S})_{\gamma} \), it lies in \( p_{\gamma} \).

Since \( F(\varphi) = \psi \), we conclude that \( \psi \) is in \( F(p_{\gamma}) \). Since \( \psi \) was arbitrary, we conclude that \( \mathcal{D} \subset F(p_{\gamma}) \).

Since \( \mathcal{D} \) is congruence, and

\[
\mathcal{R}_p \kappa_{\gamma} \supset \mathcal{R}_p p_{\gamma} = F^{-1}(F(p_{\gamma})) \supset F^{-1}(\mathcal{D}),
\]

the subgroup \( \mathcal{R}_p \kappa_{\gamma} \) is congruence by Lemma 2.

The claim in the proof of Lemma 5 gives us the following.

**Scholium 6.** If \( \kappa \) is a congruence subgroup of \( \Gamma(\hat{S}) \), then \( F(\kappa_{\gamma}) \) is a congruence subgroup of \( \Gamma(S) \).
Proof of Theorem 3. The set $B^\kappa \kappa_\gamma$ lies in $\kappa$, and hence in $\kappa \cap B \kappa_\gamma$. Note that, as $B^\kappa$ is not necessarily normal in $\Gamma(\hat{S})$, it is not clear that $B^\kappa \kappa_\gamma$ is a subgroup of $\Gamma(\hat{S})$—we are not requiring $\kappa$ to be normal. We claim that $B^\kappa \kappa_\gamma = \kappa \cap B \kappa_\gamma$, which will establish that $B^\kappa \kappa_\gamma$ is in fact a subgroup, and, by Lemma 5, a congruence one. To see this, let $b$ in $B$ and $k$ in $\kappa_\gamma$ be such that $bk$ lies in $\kappa$. Since $\kappa_\gamma$ lies in $\kappa$, the element $b$ must lie in $\kappa$. So $b$ lies in $B^\kappa = \kappa \cap B$.

4 The congruence topology on the Birman kernel

The group $\Gamma(\hat{S})$ not only embeds in $\text{Out}(\hat{\pi})$, but also as a subgroup of $\text{Aut}(\hat{\pi})$. The first embedding gives rise to the geometric completion $\hat{\Gamma}(\hat{S})$, by taking the closure of $\Gamma(\hat{S})$ in $\text{Out}(\hat{\pi})$. The second embedding gives us a completion $\hat{\tilde{\Gamma}}(\hat{S})$, by taking the closure in $\text{Aut}(\hat{\pi})$.

By Proposition 3 of [2], if $G$ is any finitely generated group, the kernel of $\hat{G} \to \text{Aut}(\hat{G})$ lies in the center of $\hat{G}$. Now, the center of $\hat{\pi}$ is trivial—thanks to a theorem of Anderson, Proposition 18 of [2]; and, independently, Nakamura, Corollary 1.3.4 of [25]. So there is a natural short exact sequence

\[ 1 \to \hat{\pi} \to \hat{\pi} \to \hat{\pi} \to 1. \]

This gives us a short exact sequence

\[ 1 \to \hat{\pi} \to \hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(\hat{S}) \to 1. \] (3)

When $\hat{S}$ is affine, it is a theorem of Matsumoto that $\hat{\Gamma}(\hat{S}) \cong \hat{\Gamma}(\hat{S})$, and hence that there is a short exact sequence

\[ 1 \to \hat{\pi} \to \hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(\hat{S}) \to 1. \]

This follows from Theorem 2.2 of [23], and we will provide a new proof of this in Section 9. We need the fact that $\hat{\pi}$ injects into $\hat{\Gamma}(\hat{S})$—in other words, that the closure $\hat{\pi}^*$ of $\hat{\pi}$ in $\hat{\Gamma}(\hat{S})$ is isomorphic to $\hat{\pi}$. For this, one constructs a natural epimorphism $\hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(\hat{S})$.

Theorem 7 (Matsumoto). The closure $\hat{\pi}^*$ of $\hat{\pi}$ in $\hat{\Gamma}(\hat{S})$ is isomorphic to $\hat{\pi}$.

Proof (Asada). Let $\tau: \pi \to \Gamma(\hat{S})$ be as in the Birman exact sequence (1), and let $\hat{\tau}: \hat{\pi} \to \hat{\Gamma}(\hat{S})$ be its natural extension. The image $\hat{\pi}$ is $\hat{\pi}^*$. We claim that $\hat{\tau}: \hat{\pi} \to \hat{\Gamma}(\hat{S})$ factors through $\hat{\tau}: \hat{\pi} \to \hat{\Gamma}(\hat{S})$. This will imply that $\hat{\tau}$ is injective, and hence that $\hat{\pi}^* \cong \hat{\pi}$. So we find a commutative triangle

\[ \begin{array}{ccc}
\hat{\pi} & \xrightarrow{\hat{\tau}} & \hat{\Gamma}(\hat{S}) \\
\downarrow{\xi} & & \downarrow{\xi} \\
\hat{\Gamma}(\hat{S}) & \to & \hat{\Gamma}(\hat{S})
\end{array} \]
The argument here is borrowed from the proof of Theorem 1 of [4].

Let \( \hat{F} : \hat{\Gamma}(\hat{S}) \to \Gamma(\hat{S}) \) be the natural projection. Given an element \( \varphi \) of \( \hat{\Gamma}(\hat{S}) \subset \text{Out}(\hat{\pi}) \), we may lift it to an element \( \tilde{\varphi} \) of \( \tilde{\Gamma}(\hat{S}) \subset \text{Aut}(\hat{\pi}) \) that fixes the element \( c \) corresponding to the new puncture of \( \hat{S} \), and then project to an element \( \hat{F}(\tilde{\varphi}) \) of \( \text{Aut}(\hat{\pi}) \).

Any two lifts \( \tilde{\varphi} \) and \( \tilde{\varphi}' \) of \( \varphi \) differ by an inner automorphism centralizing \( c \). This centralizer is topologically generated by \( c \), see Lemma 2.1.2 of [25], and so we have a well-defined map \( \xi : \hat{\Gamma}(\hat{S}) \to \Gamma(\hat{S}) \) given by \( \xi(\varphi) = \hat{F}(\tilde{\varphi}) \).

The map \( \xi \) is clearly a homomorphism. To see that it is continuous, take a sequence \( \varphi_n \) in \( \hat{\Gamma}(\hat{S}) \) converging to some \( \varphi \). Pick lifts \( \tilde{\varphi}_n \) of the \( \varphi_n \) to \( \text{Aut}(\hat{\pi}) \) fixing \( c \). After passing to a subsequence, the \( \tilde{\varphi}_n \) converge to some \( \tilde{\varphi}_\infty \). Since the projection \( \hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(\hat{S}) \) is continuous, the element \( \varphi_\infty \) is a lift of \( \varphi \). By continuity of multiplication, this \( \varphi_\infty \) fixes \( c \), and since \( \hat{F} \) is continuous, we conclude that \( \xi \) is continuous.

Now, every inner automorphism of \( \pi \) is realized by a point–pushing homeomorphism of \( \hat{S} \) which is the identity in a neighborhood of the basepoint. Deleting the fixed points of such homeomorphisms and passing to isotopy classes produces the subgroup \( \tau(\pi) = \mathcal{B} \subset \hat{\Gamma}(\hat{S}) \), and so

\[
\xi \circ \tau|_\pi = \tau|_\pi.
\]

By the universal property of profinite completions, we have \( \xi \circ \tau = \tau \).

Since \( \xi \) is continuous, the closure \( \mathcal{B}^* \) of \( \mathcal{B} = \tau(\pi) \) in \( \hat{\Gamma}(\hat{S}) \) surjects the closed subgroup \( \overline{\tau(\pi)} \) of \( \text{Aut}(\hat{\pi}) \), as \( \tau(\pi) \) is dense in the latter. As \( \mathcal{B}^* = \overline{\tau(\pi)} \) and \( \tau \) is injective, we have \( \mathcal{B}^* \cong \hat{\pi} \).

\( \square \)

See [24] for a different proof.

The continuous epimorphism \( \hat{\Gamma}(\hat{S}) \to \hat{\Gamma}(\hat{S}) \) and the short exact sequences

\[
1 \longrightarrow \hat{\pi} \longrightarrow \hat{\Gamma}(\hat{S}) \longrightarrow \hat{\Gamma}(\hat{S}) \longrightarrow 1
\]

and (3) are all that is needed to establish the congruence subgroup property for mapping class groups of punctured spheres. See the proof of Theorem 1 in [4], Section 4 of [6], or Section 6 of [22].

5 Profinite spaces

We quickly review some of the notions from [27]. See also [6, 22, 26].

A **profinite set** is an inverse limit (in the topological category) of discrete finite sets. Profinite sets form a category \( \mathcal{E} \) with continuous maps as morphisms. A **profinite space** is a simplicial object in this category—a contravariant functor from the simplex category \( \Delta \) to \( \mathcal{E} \). Profinite spaces form a category \( \mathcal{F} \) whose morphisms are the natural transformations. We let \( \mathcal{S} \) denote the category of simplicial sets. There is a forgetful functor \( | \cdot | : \mathcal{F} \to \mathcal{S} \) that sends a profinite space to its underlying simplicial set.

A profinite space \( \mathcal{X} \) may be considered a sequence \( \mathcal{X}_n \) of profinite sets \( \{ \mathcal{X}_n \}_{n=0}^{\infty} \), called the **skeleta**, together with all compositions of face \( d_i : \mathcal{X}_n \to \mathcal{X}_{n-1} \) and degeneracy maps \( s_j : \mathcal{X}_n \to \mathcal{X}_{n+1} \). A group \( G \) acts on \( \mathcal{X} \) if it acts on the \( \mathcal{X}_n \) equivariantly.
respecting face and degeneracy maps. If $G$ is a topological group, we say that a $G$–action is continuous if $G$ acts continuously on the $X$. Since singletons are closed in profinite sets, stabilizers $G_x$ are closed for continuous actions.

The action of $\Gamma(S)$ on the simplicial set $\mathcal{C}(S)$ extends naturally to continuous actions of $\hat{\Gamma}(S)$ on $\hat{\mathcal{C}}(S)$ and $\check{\mathcal{C}}(S)$. The forgetful functor $| \cdot | : \check{\mathcal{C}} \to \check{\mathcal{F}}$ called profinite completion. We warn the reader that the profinite completion $\mathcal{C}(S)$ is not the same as Boggi’s profinite curve complex $\mathcal{C}(S)$.

In [27], the homotopy theory of profinite spaces is developed by giving $\hat{\mathcal{F}}$ the structure of a model category, which allows us to discuss the homotopy type of the profinite spaces $\mathcal{C}(S)$ and $\check{\mathcal{C}}(S)$. Quick defines profinite homotopy groups $\Pi_n(\cdot)$ on the category $\check{\mathcal{F}}$, and we say that a profinite space $\mathcal{F}$ is simply–connected if $\Pi_0(\mathcal{F})$ and $\Pi_1(\mathcal{F})$ vanish.  

### 6 Stabilizers

Any vertex $\sigma$ of $\mathcal{C}(S)$ may be viewed as a vertex of $\hat{\mathcal{C}}(S)$ or of $\check{\mathcal{C}}(S)$, and we let context determine which is meant.

There is ambiguity in the notation $\hat{\Gamma}(\hat{S})_{\gamma}$, as it could denote the stabilizer of $\gamma$ in the group $\hat{\Gamma}(\hat{S})$, or the profinite completion of $\Gamma(S)$ for arbitrary $\sigma$. For arbitrary $\sigma$, it remains unknown if these two groups coincide (but this would follow from the congruence subgroup property). In the cases we consider here, there is no ambiguity, and we record the following lemma to alleviate any uneasiness. See Proposition 6.5 of [6] for a more general statement.

**Proposition 8.** Let $\gamma$ be a nonperipheral simple loop in $\hat{S}$ that is peripheral in $S$. Then

$$\left( \hat{\Gamma}(\hat{S}) \right)_{\gamma} \cong \overline{\Gamma(\hat{S})}_{\gamma} \cong \overline{\Gamma(S)}_{\gamma},$$

and there is a short exact sequence

$$1 \rightarrow \hat{Z} \rightarrow \hat{\Gamma}(\hat{S})_{\gamma} \rightarrow \hat{\Gamma}(S) \rightarrow 1.$$  

**Proof.** We first establish the isomorphism $\overline{\Gamma(\hat{S})}_{\gamma} \cong \overline{\Gamma(S)}_{\gamma}$. Consider the short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma(\hat{S})_{\gamma} \xrightarrow{i} \Gamma(S) \rightarrow 1 \quad (4)$$

where $\Phi$ is the restriction of the forgetful map $F$ to the stabilizer $\Gamma(\hat{S})_{\gamma}$ and the subgroup $t(\mathbb{Z})$ is generated by a Dehn twist in $\gamma$, see . Taking profinite completions is right exact

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2Quick [27] uses the notation $\hat{\cdot}$ for this functor, which conflicts with our use of that notation.

3Quick uses the notation $\pi_n(\cdot)$ for his homotopy groups, but, to avoid confusion, we do not.
Theorem 9 (Harer [14]). Then $\mathcal{C}(S)$ is homotopy equivalent to a wedge of spheres of dimension $h$. In particular, if $\dim \mathcal{C}(S) = 3g - 3 + n \geq 2$, then $\mathcal{C}(S)$ is simply-connected.
Theorem 10 (M. A. Armstrong [3]). Let $\mathcal{X}$ be a simply connected simplicial complex. Let $G$ be a group of simplicial homeomorphisms of $\mathcal{X}$ and let $G_*$ be the normal subgroup of $G$ generated by elements with nonempty fixed–point set. Then $\pi_1(\mathcal{X}/G) \cong G/G_*$. 

This theorem allows us to view $\pi_1(\mathcal{C}^a)$ as a quotient of $\mathcal{C}$ by the subgroup generated by its reducible elements.

8 Cornering the congruence kernel

The following theorem was first observed by Boggi.

Theorem 11 (Boggi).

$$\lim_{\mathcal{X}} \pi_1(\mathcal{C}^a(S)) = 1.$$ 

Proof. By Theorem 10, we have a surjection $\hat{\pi}_1(\mathcal{C}^a) \twoheadrightarrow \hat{\pi}_1(\mathcal{C}^a(S))$. Since inverse limit functors are exact on profinite groups (see Proposition 2.2.4 of [28]), we have

$$\lim_{\mathcal{X}} \hat{\pi}_1(\mathcal{C}^a) \twoheadrightarrow \lim_{\mathcal{X}} \pi_1(\mathcal{C}^a(S)).$$

But $\lim_{\mathcal{X}} \hat{\pi}_1 = 1$. \qed

To our knowledge, it remains unknown if $\mathcal{C}(S)$ is simply–connected.

Given $a$ in $\mathcal{X}$, we let $a_*$ denote the subgroup of $a$ generated by reducible elements.

The rest of this section is devoted to the proof of the following theorem.

Theorem 12. Let $S$ be a smooth affine algebraic curve. Let $\mathcal{P}$ be the set of nonperipheral simple loops on $\hat{\mathcal{S}}$ that are peripheral in $S$. Then

$$K(\hat{\mathcal{S}}) \subset \bigcap_{\gamma \in \mathcal{P}} \hat{\Gamma}(\hat{\mathcal{S}})_{\gamma}$$

and $\lim_{\mathcal{X}} \hat{\pi}_1(\mathcal{C}^\kappa(\hat{\mathcal{S}})) = 1$.

Unfortunately, we are unable to show that $\mathcal{C}(S)$ is simply–connected.

Lemma 13.

$$\lim_{\mathcal{X}} \hat{\kappa} = K.$$ 

Proof. The closure $\overline{\kappa}$ of $\kappa$ in $\hat{\Gamma}(\hat{\mathcal{S}})$ is isomorphic to $\hat{\kappa}$, and so $\lim_{\mathcal{X}} \hat{\kappa} = \lim_{\mathcal{X}} \overline{\kappa}$. This limit is the intersection of all of the finite index subgroups of $\hat{\Gamma}(\hat{\mathcal{S}})$ obtained by pulling back finite index subgroups of $\overline{\Gamma}(\hat{\mathcal{S}})$, which is precisely the congruence kernel $K$. \qed

Lemma 14. For any $\eta$ in $\mathcal{X}$, we have $\lim_{\kappa \in \mathcal{X}} \overline{\mathcal{B}^\kappa \eta} = \overline{\eta}$.
Proof. By Theorem 7,
\[ \lim_{\kappa \in \mathcal{K}} \overline{B^\kappa h}\gamma = \lim_{a \in A} \overline{B^a h}\gamma, \]
and clearly
\[ \lim_{a \in A} \overline{B^a h}\gamma = \bigcap_{a \in A} \overline{B^a h}\gamma \supset h\gamma. \]

On the other hand, if \( H \) is a finite index subgroup of \( \hat{\Gamma}(\hat{S}) \) that contains \( h\gamma \), then there is a finite index subgroup of \( H \) of the form \( \overline{B^a h}\gamma \), obtained by letting \( a = H \cap \Gamma(\hat{S}) \). But since \( h\gamma \) is closed, it is the intersection of all of the finite index subgroups containing it, and we conclude that
\[ h\gamma = \bigcap_{a \in A} \overline{B^a h}\gamma. \]

Proof of Theorem 12. Consider the short exact sequence
\[ 1 \to \kappa_s \to \kappa \to \pi_1(C^\kappa) \to 1 \]
given by Theorem 10. Passing to profinite completions is right exact (see Proposition 3.2.5 of [28]), and so we have an exact sequence
\[ \hat{\kappa}_s \to \hat{\kappa} \to \hat{\pi}_1(C^\kappa) \to 1. \]

Inverse limits are exact on profinite groups (Proposition 2.2.4 of [28]), and so, by Lemma 13, we have an exact sequence
\[ \lim_{\mathcal{K}} \hat{\kappa}_s \to \hat{\kappa} \to \hat{\pi}_1(C^\kappa) \to 1. \]

We let \( \mathcal{R}_s \) and \( \mathcal{R}_\gamma \subset \mathcal{R}_s \) be the images in \( \mathcal{K} \) of \( \lim_{\mathcal{K}} \hat{\kappa}_s \) and \( \lim_{\mathcal{K}} \hat{\kappa}_\gamma \), respectively. By Corollary 4, the system of subgroups
\[ \{ B^\kappa \mathcal{K}_\gamma \mid \kappa \in \mathcal{K} \text{ and } \mathcal{B}^\kappa = \kappa \cap \mathcal{B} \} \]
is cofinal in \( \mathcal{K} \). Now, \( \mathcal{B}^\kappa \cong \mathcal{K} \), and making use of the forgetful map \( \Gamma(\hat{S}) \to \Gamma(S) \), it is easily seen that \( \mathcal{K}_\gamma \cong \mathcal{K}_\gamma \). We also have \( \mathcal{B}^\kappa \mathcal{K}_\gamma \cong \mathcal{B}^\kappa \mathcal{K}_\gamma \subset \hat{\Gamma} \), and since \( \mathcal{B}^\kappa \mathcal{K}_\gamma \) is dense in \( \mathcal{B}^\kappa \mathcal{K}_\gamma \), the two are equal, as both are closed. The congruence kernel \( \mathcal{K} \) is then
\[ \mathcal{K} = \lim_{\kappa \in \mathcal{K}} \mathcal{B}^\kappa \mathcal{K}_\gamma = \lim_{\kappa \in \mathcal{K}} \mathcal{B}^\kappa \mathcal{K}_\gamma. \]

Now, for a fixed \( h \) in \( \mathcal{K} \), the subset \( \{ h \cap \kappa \mid \kappa \in \mathcal{K} \} \) is cofinal in \( \mathcal{K} \), by Lemma 1. It follows that, for any such \( h \),
\[ \lim_{\kappa \in \mathcal{K}} \mathcal{B}^\kappa \mathcal{K}_\gamma \subset \lim_{\kappa \in \mathcal{K}} \mathcal{B}^\kappa \mathcal{K}_\gamma \]
(6)
and we have
\[ K = \lim_{\kappa \in \mathcal{K}} \lim_{b \in \mathcal{X}} B^b_\kappa \]
\[ \subset \lim_{b \in \mathcal{X}} \lim_{\kappa \in \mathcal{K}} B^b_\kappa \]  
\[ \subset \lim_{b \in \mathcal{X}} \hat{b}_\kappa \]  
\[ \subset \mathcal{M}_\gamma \]
\[ \subset \mathcal{M}_s \]

Together with (5), we have the short exact sequence
\[ K \rightarrow K \rightarrow \lim_{\mathcal{X}} \hat{\pi}_1(\mathcal{C}_\gamma) \rightarrow 1. \]

**Corollary 15.** If \( S \) is affine, then, for any prime number \( p \),
\[ H_1(\hat{\mathcal{C}}(\hat{S}); \mathbb{F}_p) := \lim_{\mathcal{X}} H_1(\mathcal{C}_\gamma(\hat{S}); \mathbb{F}_p) = 0. \]

**Proof.** The surjections \( \hat{\pi}_1(\mathcal{C}_\gamma(\hat{S})) \rightarrow H_1(\mathcal{C}_\gamma(\hat{S}); \mathbb{F}_p) \), exactness of inverse limits on profinite groups, and Theorem 12 prove the corollary.

Let \( S \) be an affine curve and let \( \Sigma \) be its projective completion. The complement \( \Sigma - S \) is a finite set of \( n \) points, for some \( n \). There is a map \( \Gamma(S) \rightarrow \Gamma(\Sigma) \) obtained by extending homeomorphisms from \( S \) to \( \Sigma \). The kernel \( b(S) \) of this map is the **braid group of \( \Sigma \) on \( n \) strands**. The elements of \( b(S) \) are called **braids**.

If \( \gamma \) is a nonperipheral simple closed loop in \( \hat{S} \) that is peripheral in \( S \), then a Dehn twist \( T_\gamma \) in \( \gamma \) lies in \( b(\hat{S}) \). In fact, the Dehn twists in such curves generate \( b(\hat{S}) \). Since \( \hat{\Gamma}(\hat{S})_\gamma \) lies in the centralizer of \( T_\gamma \), we have the following corollary of Theorem 12.

**Corollary 16.** The congruence kernel \( K(\hat{S}) \) lies in the centralizer of \( b(\hat{S}) \).

### 9 Kernels depend only on the genus of affine curves

If \( S \) is affine, it follows from Matsumoto’s exact sequence
\[ 1 \rightarrow \hat{\pi} \rightarrow \hat{\Gamma}(\hat{S}) \rightarrow \hat{\Gamma}(S) \rightarrow 1 \]

that \( K(\hat{S}) \) and \( K(S) \) are isomorphic, and so the congruence subgroup problem depends only on the genus in the affine case. It is a deeper theorem of Boggi [8] and, independently, Hoshi and Mochizuki [15], that the congruence kernel is independent of genus in general.

We provide a new proof of this fact in the affine case. We are grateful to Marco Boggi for suggesting that our techniques should accomplish this.
Theorem 17 (Matsumoto). If $S$ is affine, then $K(\hat{S}) \cong K(S)$ and there is a short exact sequence

$$1 \longrightarrow \hat{\pi} \longrightarrow \hat{\Gamma}(\hat{S}) \longrightarrow \hat{\Gamma}(S) \longrightarrow 1.$$ 

Proof of Theorem 17. By Theorem 12, we have $K(\hat{S}) \subset \hat{\Gamma}(\hat{S})$, where $\gamma$ bounds a pair of pants in $\hat{S}$.

By Proposition 8, we have the short exact sequence

$$1 \longrightarrow \hat{Z} \longrightarrow \hat{\Gamma}(\hat{S}) \longrightarrow \hat{\Gamma}(S) \longrightarrow 1 \quad (7)$$

where $\hat{\Phi}$ is the restriction of the forgetful map $\hat{F}$ to $\hat{\Gamma}(\hat{S})$. Exactness on the left follows from Theorem 7 and Scott’s theorem that $\pi$ is subgroup separable (see the proof of Proposition 8). In fact, these two theorems imply that the $\hat{Z}$ in this exact sequence injects into $\hat{\Gamma}(\hat{S})$. In particular, $K(\hat{S})$ intersects this $\hat{Z}$ trivially. So $K(\hat{S})$ injects into $\hat{\Gamma}(S)$.

Given a congruence subgroup $\eta \subset \Gamma(S)$, the subgroup $\hat{F}^{-1}(\eta)$ is a congruence subgroup of $\Gamma(\hat{S})$, by Lemma 2. So,

$$K(\hat{S}) = \bigcap_{x(\hat{S})} x \cap \bigcap_{x(S)} \hat{F}^{-1}(x) = \hat{F}^{-1} \bigcap_{x(S)} x = \hat{F}^{-1}(K(S))$$

and so $\hat{F}(K(\hat{S})) \subset K(S)$.

On the other hand, by Scholium 6, for every $x$ in $x(\hat{S})$, the subgroup $\hat{F}(x)$ is a congruence subgroup of $\Gamma(S)$. So

$$K(S) = \bigcap_{x(S)} x \supset \bigcap_{x(\hat{S})} \hat{F}(x)$$

which yields

$$\Phi^{-1}(K(S)) \subset \hat{F}^{-1} \bigcap_{x(\hat{S})} \hat{F}(x)$$

$$= \bigcap_{x(\hat{S})} \hat{F}^{-1}(x) = \bigcap_{x(S)} \hat{Z}.$$
Now, let $x$ be an element of $\bigcap_{\mathcal{X}(\hat{S})} \hat{\mathbb{Z}}_{\mathcal{X}}$. So for each $\kappa$ in $\mathcal{X}(\hat{S})$, there are $z_\kappa$ in $\hat{\mathbb{Z}}$ and $x_\kappa$ in $\mathbb{K}_\kappa$ such that $x = z_\kappa x_\kappa$. By Lemma 1, we may enumerate the elements of $\mathcal{X}(\hat{S})$ and take intersections to obtain a nested sequence $\{\kappa^n\}$ in $\mathcal{X}(\hat{S})$ such that

$$\bigcap_{n=1}^{\infty} \mathbb{K}_{\kappa^n} = \bigcap_{\mathcal{X}(\hat{S})} \mathbb{K}_{\mathcal{X}(\hat{S})} = \mathbb{K}(\hat{S}).$$

Let $z_n = z_{\kappa_n}$ and $x_n = x_{\kappa_n}$. After passing to a subsequence, the $z_n$ converge to some $z$ in $\mathbb{Z}$, since $\hat{\mathbb{Z}}$ is compact. Since the $\mathbb{K}_{\kappa^n}$ are compact and nested, we may pass to a further subsequence so that the $x_n$ converge to some $x_{\infty}$ in $\mathbb{K}(\hat{S})$. We conclude that $x = x_{\infty}$, which transparently lies in $\hat{\mathbb{Z}} \mathbb{K}(\hat{S})$. So

$$\bigcap_{\mathcal{X}(\hat{S})} \hat{\mathbb{Z}}_{\mathcal{X}} = \hat{\mathbb{Z}} \cdot \bigcap_{\mathcal{X}(\hat{S})} \mathbb{K}_{\mathcal{X}(\hat{S})} = \hat{\mathbb{Z}} \mathbb{K}(\hat{S}).$$

All together, we have

$$\Phi^{-1}(\mathbb{K}(\mathcal{S})) \subset \bigcap_{\mathcal{X}(\hat{S})} \hat{\mathbb{Z}}_{\mathcal{X}} = \hat{\mathbb{Z}} \cdot \bigcap_{\mathcal{X}(\hat{S})} \mathbb{K}_{\mathcal{X}(\hat{S})} = \hat{\mathbb{Z}} \mathbb{K}(\hat{S})$$

and so

$$\mathbb{K}(\mathcal{S}) \subset \hat{F}(\hat{\mathbb{Z}} \mathbb{K}(\hat{S})) = \hat{F}(\mathbb{K}(\hat{S})).$$

We conclude that $\hat{F}|_{\mathbb{K}(\mathcal{S})} : \mathbb{K}(\mathcal{S}) \to \mathbb{K}(\mathcal{S})$ is an isomorphism.

To see that we recover the exact sequence, consider the diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \hat{\mathbb{R}} & \longrightarrow & \Gamma(\hat{S}) & \longrightarrow & \Gamma(\hat{S}) & \longrightarrow & 1 \\
& & \downarrow{\hat{F}} & & \downarrow{F} & & \downarrow{F} & & \\
1 & \longrightarrow & \hat{\pi} & \longrightarrow & \Gamma(S) & \longrightarrow & \Gamma(S) & \longrightarrow & 1 \\
& & \downarrow{\xi} & & & & \downarrow{q} & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
$$

where the diagonal map $\xi$ is the map from the proof of Theorem 7 and $\hat{F} = q \circ \xi$. Other than the right–most square, the diagram commutes by definition. To see that this square commutes, note that

$$\hat{F} \circ \rho|_{\Gamma(S)} = F = \rho \circ \hat{F}|_{\Gamma(\hat{S})},$$

and that, by the universal property of the profinite completion, there is a unique continuous homomorphism to $\Gamma(\hat{S})$ extending $\hat{F}$, which must be $\hat{F} \circ \rho|_{\Gamma(S)} = \rho \circ \hat{F}|_{\Gamma(\hat{S})}$.

Suppose that $\varphi$ is an element of $\ker(\hat{F})$ in $\mathcal{B}^*$. Pick an element $\varphi'$ in $\Gamma(\hat{S})$ in the preimage of $\varphi$. Since the diagram commutes, we know that $\hat{F}(\varphi')$ lies in $\mathbb{K}(\mathcal{S})$. Since

$$\hat{F}|_{\mathbb{K}(\mathcal{S})} : \mathbb{K}(\mathcal{S}) \to \mathbb{K}(\mathcal{S})$$

is an isomorphism, we may choose an element $\psi$ of $\mathcal{B} = \ker(\hat{F})$ such that $\psi \varphi'$ lies in $\mathbb{K}(\mathcal{S})$. But this means that we may multiply $\varphi$ by an element of $\mathcal{B}^*$ to obtain the trivial element, contradicting our choice of $\varphi$. We conclude that $\ker(\hat{F}) = \mathcal{B}^* \cong \hat{\pi}$.
10 Thoughts on the fundamental group at each level

The simplicial complex $\mathcal{C}(S)$ comes equipped with the weak topology. There is another natural topology; the topology induced by declaring each simplex to be a regular Euclidean simplex with edges of length one, and taking the induced path metric.

Let $S$ be such that $\dim \mathcal{C}(S) \geq 2$.

Let $a$ be in $\mathcal{A}$ and $\sigma$ a nonperipheral simple loop in $S$. The group $\Gamma(S)\sigma$ has finite index in $\Gamma(S)$, which acts cocompactly on the star of $\sigma$, and so the simplicial complex $\mathcal{C}^a\sigma$ is locally finite. In particular, the complex $\mathcal{C}^a\sigma$ is compact if and only if it has finite diameter in the metric induced by that of $\mathcal{C}(S)$. By Armstrong’s theorem, $\pi_1(\mathcal{C}^a\sigma)$ is finite if and only if $\mathcal{C}^a\sigma$ is compact.

There is a Serre fibration $\mathcal{C}(\tilde{S}) \to \mathcal{C}(S)$ whenever $S$ is projective [20]. When $S$ is affine, there is no map from $\mathcal{C}(\tilde{S})$ to $\mathcal{C}(S)$, but there is a 1–dense subcomplex $\mathcal{D}(\tilde{S})$ for which there is a map

$$p: \mathcal{D}(\tilde{S}) \to \mathcal{C}(S),$$

see [20]. Each fiber of this map is 1–dense in $\mathcal{C}(\tilde{S})$, and so $\pi_1(\mathcal{C}^a\sigma)$ will be finite if and only if a fiber $\mathcal{T}$ of $p$ projects to a set of finite diameter in $\mathcal{C}^a\sigma$.

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