Laplace transformation updated

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Abstract

The traditional theory of Laplace transformation (TLT) as it was put forward by Gustav Doetsch was principally intended to provide an operator calculus for ordinary derivable functions of the $t$-domain. As TLT does not account for the behavior of the inverse L-transform at $t = 0$ its validity is essentially confined to $t > 0$. However, from solutions of linear differential equations (DEs) one can discern that the behavior of functions for $t \leq 0$ actually is significant. It turns out that in TLT several fundamental features of Laplace transformation (LT) are not consistently accounted for. To get LT consistent one has to make it consistent with the theory of Fourier transformation, and this requires that the behavior of both the original function and of the pertinent inverse L-transform has to be accounted for in the entire $t$-domain, i.e., for $-\infty < t < +\infty$. When this requirement is observed there emerges a new approach to LT which is liberated from TLT’s deficiencies and reveals certain implications of LT that previously have either passed unnoticed or were not taken seriously. The new approach is described; its implications are far-reaching and heavily affect, in particular, LT’s theorems for derivation/integration and the solution of linear DEs.

1 Introduction

The mathematical concept which customarily is addressed by the term Laplace Transformation (LT) was primarily designed as a method for the solution of linear differential equations (DEs), i.e., by a kind of operator calculus. Though the theory of LT dates back to Leibniz, Euler, Laplace, Petzval, and many 20th-century authors, its presently prevalent form, for example in [7 5 14], was essentially worked out by Gustav Doetsch [2 3 4 5]. This version of the theory is in the present article addressed as the traditional theory of LT, TLT.

The theory of Laplace transformation is based on the pair of integral transformations

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st} \, dt; \quad s = \sigma + i\omega; \quad \sigma > 0;$$

(1)

$$L^{-1}\{F(s)\} = \varphi(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st} \, ds,$$

(2)
where \( f(t) \) is presumed to be a real function of the real variable \( t \). The L-transform \( F(s) \) is a complex function of the complex variable \( s = \sigma + i\omega \). In typical applications, \( t \) denotes time and \( \omega \) denotes circular frequency.

In TLT the behavior of the inverse transform is customarily characterized by

\[
\varphi(t) = \begin{cases} 
0 & \text{for } t < 0, \\
 f(t) & \text{for } t > 0;
\end{cases}
\]

(3)

at \( t = 0 \) it is left undefined. Indeed, in the realm of ordinary mathematical functions – which originally was envisaged by TLT – it is impossible to describe the inverse L-transform’s behavior at \( t = 0 \), except if \( f(0) = 0 \); cf. Sect. 2.

By contrast, the original function \( f(t) \) is – and has to be – defined at least for \( 0 \leq t \), i.e., including \( t = 0 \). If \( f(t) \) were at \( t = 0 \) insufficiently defined the integral (1), i.e., the L-transform \( L\{f(t)\} \), would not exist.

Hence, in TLT the definition interval of the original function \( f(t) \) does not match that of the pertinent inverse transform \( \varphi(t) \). In the tradition of Doetsch’s theory of LT it is widely believed that this mismatch of definition intervals is irrelevant, provided that application of Laplace transformation is confined to the interval \( t > 0 \) – which in TLT is therefore envisaged. However, this assumption is not tenable. For example, the solution \( y(t) \) of an inhomogeneous linear DE of the type \( f[y'(t), y''(t), \ldots] = x(t) \), even when envisaged only for \( t > 0 \), in general depends on the behavior of \( x(t) \) for \( -\infty < t \). Thus, when the solution \( y(t) \) is to be obtained by LT it is crucial which kind of function is by \( L\{x(t)\} \) actually represented in the interval \( -\infty < t \). This fact is in Sect. A.1 illustrated by an example.

The most obvious symptoms of TLT’s inconsistency may be listed, as follows.

a) The mismatch of definition intervals appears to disallow concatenation of L-transforms. As the inverse transform \( \varphi(t) \) is at \( t = 0 \) undefined it can, rigorously, not be allowed to be subject to another L-transformation. The fact that concatenation turns out actually to be possible does not make TLT formally consistent in this respect. In Sect. 2.4 it is explained why concatenation of L-transforms is possible.

b) The mismatch of definition intervals appears to exclude impulse functions at \( t = 0 \) – such as \( \delta(t) \) – from L-transformation. The delta-impulse \( \delta(t) \) is only at \( t = 0 \) different from null; thus, according to TLT \( \delta(t) \) can not exist in the inverse L-transform. The fact that the impulse functions actually can be retrieved from their L-transforms does not cure this formal inconsistency of TLT. In Sect. 2.2 it is explained why impulse functions are LT-consistent.

c) TLT’s derivation theorem

\[
L\{f'(t)\} = sL\{f(t)\} - f(0)
\]

(4)

is in conflict with the definition interval of TLT, i.e., \( t > 0 \). The real constant \( f(0) \) that in (4) appears in the L-domain represents a \( t \)-domain function of its own, namely, the
delta impulse $f(0)\delta(t)$ at $t = 0$. This impulse is outside TLT’s definition interval and therefore should be regarded as irrelevant [12]. Inconsequently, TLT praises the “initial value” $f(0)$ as one of its most advantageous features, as that value eventually appears in the solutions of linear DEs and is utilized to account for a system’s initial state at $t = 0$. By its inconsequent treatment of the initial value TLT implicitly admits that the behavior at $t = 0$ of functions and of their L-transforms actually is significant.

d) TLT’s derivation theorem (4) is inconsistent with TLT’s integration theorem

$$L\{f(t)\} = \frac{1}{s}L\{f'(t)\}. \quad (5)$$

The two theorems differ by the initial value $f(0)$. In TLT, this unexpected and unexplained discrepancy is customarily tolerated. In Sect. 3.4 the relationships between the theorems for derivation and integration are outlined and the origin of the conflict between Eqs. (4) and (5) is revealed.

e) TLT does not in general keep its promise to provide the solution of linear DEs, i.e., for $t > 0$. There are discrepancies involved which tend to be disguised by formal pseudo-consistency. This notion just restates what was said above about solution of linear DEs. In Sect. A.1 an example is described of this kind of failure.

f) In particular, TLT’s general solution of the linear DE suffers from the so-called initial-value conflict, In TLT it has become customary to work around this conflict by “patching” the original solution. The initial-value conflict is discussed and explained in Sect. 4.4.

These observations indicate that in TLT certain fundamental aspects of LT’s behavior are not consistently accounted for. The mismatch of definition intervals and its consequences need to be resolved rather than circumvented and/or ignored. Laplace transformation needs to become updated.

The present article offers an outline of a new, alternative approach to LT. The problem of LT-consistency of $t$-domain functions is re-inspected and resolved. From the results such obtained there emerge new methods for the solution by LT of both the linear inhomogeneous and the linear homogeneous DE.

Crucial results and observations are noted and emphasized as theorems. Several familiar theorems of LT, such as, e.g. the fundamental

**superposition theorem**

$$L\{c_1f_1(t) + c_2f_2(t) + \ldots\} = L\{c_1f_1(t)\} + L\{c_2f_2(t)\} + \ldots$$

$$= c_1L\{f_1(t)\} + c_2L\{f_2(t)\} + \ldots, \quad (6)$$

are not affected by the new insights. Another group of theorems become restated and re-justified without assuming a new form. A third group includes theorems that become more or less drastically modified as compared to their familiar form. Eventually, there is
a fourth group, i.e., of theorems which may be regarded as new – at least in so far as in
TLT they do not play a role.

Several complementary explanations and examples are exiled into an appendix. This
article is based on, and complements, earlier related work of the present author [11, 12, 13].

2 Getting Laplace transformation consistent

TLT’s heel of Achilles lies at \( t = 0 \). For LT to be consistent it is not sufficient that
\( \varphi(t) = f(t) \) for \( t > 0 \); rather, the condition

\[
\varphi(0) = f(0)
\]  

must also be fulfilled. To suggest the implications of this requirement, the behavior of
the inverse L-transform pertinent to a continual function \( f(t) \) is illustrated in Fig. 1.

From the figure it becomes apparent that for functions which at \( t = 0 \) assume a
definite unique value \( f(0) \neq 0 \) the inverse L-transform \( \varphi(t) \) includes at \( t = 0 \) an abrupt
transitional section, i.e., from 0 to \( f(0) \). As a consequence, such type of function can
not meet the criterion (7), because a section of \( \varphi(t) \) at \( t = 0 \) can not be equal to the
definite unique value \( f(0) \). This kind of non-LT-consistency applies, in particular, to the
prominent class of continual derivable functions. By contrast, if \( f(t) \) is a priori defined
as a causal function, i.e., \( f(t) = 0 \) for \( t < 0 \), the criterion may actually be met, i.e., for
certain conditions which will be discussed below.

For brevity and simplicity, the discussion of LT-consistency is in the present article
focussed on the dichotomy between causal functions and d-functions. The term d-function
denotes the class of bilateral continual functions that are in the ordinary sense derivable
as many times as required. It should be kept in mind, in particular, that any solution of
a linear homogeneous DE of finite order is a d-function.

To achieve an explicit account of LT’s behavior at \( t = 0 \), the definition interval must
obviously be expanded from \( t > 0 \) into \( t \leq 0 \). As the scope of LT’s formula for inverse
transformation [2] already encompasses the entire \( t \)-domain, it is the scope of the formula
for L-transformation [1] which has to be expanded. To accomplish this kind of expansion
it is not required to challenge the definitions of LT; the bilateral scope is already implied
in Eq. (1).
2.1 Laplace transformation by Fourier transformation

The implications of the latter notion become apparent when one exploits the intimate relationship that exists between Laplace transformation and Fourier transformation. Equation (1) is equivalent to the following “unilateral” Fourier-transformation formula for the function \( f(t) \exp(-\sigma t) \), i.e.,

\[
L\{f(t)\} = F(s) = F(\omega, \sigma) = \int_{0}^{\infty} f(t) e^{-\sigma t} e^{-i\omega t} \, dt.
\]  

(8)

However, there is not really such a thing as unilateral Fourier transformation; Fourier transformation is inherently bilateral. The actual analysis interval (the “scope”) of the transformation (8), and thus of (1), is not determined by the integral’s limits but by the reciprocal of the frequency spacing \( df = d\omega/(2\pi) \) of the corresponding Fourier-integral representation [11, 13]. The actual analysis interval of both (1) and (8) encompasses the entire \( t \)-domain \(-\infty < t < +\infty\), and the low limit \( t = 0 \) of the integral (1) corresponds to the center of the analysis interval. Though these implications of Fourier transformation are fairly elementary, apprehension of them appears to be scarce; cf. Sect. A.2.

Hence, the unilaterality of the integration interval in both (8) and (1) is not equivalent to unilaterality of the analysis interval but rather indicates causality of the transformed function. Equation (8) has to be equivalent to ordinary, i.e., bilateral Fourier transformation of a causal, i.e., bilaterally defined function \( f_c(t) \exp(-\sigma t) \), such that

\[
L\{f(t)\} = F(\omega, \sigma) = \int_{-\infty}^{+\infty} f_c(t) e^{-\sigma t} e^{-i\omega t} \, dt = \int_{-\infty}^{+\infty} f_c(t) e^{-\sigma t} e^{-i\omega t} \, dt.
\]  

(9)

For LT to be consistent with Fourier transformation it is necessary that the causal function \( f_c(t) \) be defined in such a way that the two integrals in (9) are fully equivalent. At first sight this requirement is met, e.g., by the definition \( f_c(t) = 0 \) for \( t < 0 \); \( f_c(t) = f(t) \) for \( t \geq 0 \). However, this definition of the causal function is not actually sufficient, because it leaves the transition from \( f_c(-0) \) to \( f_c(+0) \) undefined. There is another condition involved: The second (bilateral) integral in (9) has to be consistent with the pertinent Fourier-integral representation, i.e., the “inverse Fourier transform”

\[
\varphi(t) = L^{-1}\{L\{f(t)\}\} = \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^{+\infty} F(\omega, \sigma) e^{i\omega t} \, d\omega \quad \text{for} \quad -\infty < t.
\]  

(10)

As is suggested in (10), this expression is equivalent to (2). The Fourier-integral representation (10) of \( \varphi(t) \) is continuously defined for \(-\infty < t < +\infty\). In particular, while \( \varphi(t) \) is causal, the transition from \( \varphi(-0) \) to \( \varphi(+0) \) is not undefined but either includes the so-called connect interval (see below, in particular Sect. 2.3), or its derivative(s), i.e., the impulse functions \( \delta^{(n)}(t) \ (n = 0, 1, \ldots) \). The Fourier integral’s evident capability to represent impulse functions proves that neither the inverse Fourier transform nor the inverse L-transform are at \( t = 0 \) undefined. As a consequence, for \( \varphi(0) = f_c(0) \) to hold \( f_c(0) \) must not be left insufficiently defined. In TLT this requirement is ignored.
Below, the consistent definition of \( f_c(t) \) and thus the consistent expansion of LT’s transformation formula (1, 8) into an equivalent bilateral form, is approached by three steps, advancing from impulse functions to the unit step function and finally to the entire class of causal functions.

### 2.2 Impulse functions

The requirements just outlined for \( f_c(t) \) are \textit{a priori} met by the delta impulse and its derivatives. Thus, Eq. (9) applies to \( f_c(t) = \delta^{(n)}(t) \) and one obtains

\[
L\{\delta^{(n)}(t)\} = s^n; \quad n = 0, 1, \ldots
\]  

As the impulse functions are only at \( t = 0 \) different from null, inverse transformation is by (10) achieved for \( \exp(st) = 1 \), i.e., by inverse Fourier transformation of the function \((i\omega)^n\). One obtains

\[
L^{-1}\{L\{\delta^{(n)}(t)\}\} = \delta^{(n)}(t); \quad n = 0, 1, \ldots
\]

It is thus established by the theory of Fourier transformation that both the L-transform of \( \delta^{(n)}(t) \) and its inverse L-transform exist, and that the latter is identical to \( \delta^{(n)}(t) \). The impulse functions are LT-consistent.

As \( \delta^{(n)}(t) \) is defined for \(-\infty < t < +\infty\), while \( \delta^{(n)}(t) = 0 \) for \( t \neq 0 \), existence of the L-transform \( L\{\delta^{(n)}(t)\} \) is consistent with the condition that for the “unilateral” transformations (1) and (8) to apply \( f(t) \) must be defined for \( t \geq 0 \). Thus, even when one sticks to the “unilateral” form of the transformation there is no conflict. It is only the inverse L-transform of impulse functions that is in conflict with TLT’s assumptions.

### 2.3 The unit step function and the connect function

Also the unit step function is causal and may be regarded as just another member of the class of impulse functions. The unit step function can be defined as the integral of the delta impulse at \( t = 0 \), i.e., by

\[
u(t) := \int_{-\infty}^{t} \delta(\tau) \, d\tau; \quad -\infty < t < +\infty.
\]  

Equation (13) may be regarded as an \textit{implicit} definition of \( u(t) \). When the unit step function is explicitly defined it is important to preserve its definition at \( t = 0 \). This can be accomplished in the form

\[
u(t) = \begin{cases} 
0 & \text{for } t < 0, \\
u_0(t) & \text{for } t = 0; \quad u_0(t) = \{0 \ldots 1\}, \\
1 & \text{for } t > 0.
\end{cases}
\]  

(14)
This definition accounts for the abrupt transition from 0 to 1 that occurs at $t = 0$, namely, by inclusion of the so-called unit connect function, $u_0(t)$. The pseudo-function $u_0(t)$ may be conceived of as an infinite set $\{0 \ldots 1\}$ of real numbers (a distribution) that exists at $t = 0$.

Inclusion of $u_0(t)$ in $u(t)$ is indispensable not only for LT-consistency of $u(t)$ but also for $u(t)$ to be consistent with the concept of the delta impulse. LT-consistency of $u(t)$ requires that the condition (7) is met, i.e., that $L^{-1}\{L\{u(t)\}\} = u(t)$ at $t = 0$. When the interval extending from $u(-0) = 0$ to $u(+0) = 1$ is left undefined, fulfilment of the condition (7) remains undecided.

The existence of the delta impulse, which is conceptualized as the first (non-ordinary) derivative of $u(t)$, crucially depends on existence of the connect function $u_0(t)$. With respect to (14) there holds

$$\delta(t) = u'_0(t) = u'(t).$$

Thus, indeed, inclusion of $u_0(t)$ in $u(t)$ is indispensable. One can not in earnest conceive the delta impulse to be the derivative of a gap.

As the unit step function is the integral of the delta impulse, it follows from the results depicted in Sect. 2.2 that the unit step function as defined by (13) and (14) is indeed LT-consistent; i.e., there holds

$$L^{-1}\{L\{u(t)\}\} = u(t) \quad \text{for} \quad -\infty < t < +\infty.$$  

Notably, the L-transform of the unit step function is equivocal. There holds

$$L\{u(t)\} = L\{1\} = 1/s.$$  

With respect to the interval $t \geq 0$ the functions $f(t) = u(t)$ and $f(t) = 1$ differ by the connect function $u_0(t)$ which is included in $f(t) = u(t)$ but not in $f(t) = 1$. The connect function does not become explicitly apparent in the L-transform. As a consequence, from the L-transform $1/s$ one can not tell whether it was obtained from $f(t) = u(t)$ or from $f(t) = 1$.

From the fact that both $u(t)$ and its derivatives (the impulse functions) are LT-consistent one can conclude that $u_0(t)$ invariably is contained in the inverse L-transform. According to (16, 17) there holds

$$L^{-1}\{L\{u(t)\}\} = L^{-1}\{L\{1\}\} = u(t),$$

and $u(t)$ contains $u_0(t)$. Thus, for a real function $f(t)$ to be LT-consistent it is indispensable that it contains the connect function. This is why $f(t) = u(t)$ is LT-consistent whereas $f(t) = 1$ is not.

The somewhat confusing behavior of $u_0(t)$ emerges from the fact that $u_0(t)$ is a null-function, i.e.,

$$\int_{-T}^{+T} u_0(t) \, dt = 0; \quad T \geq 0;$$

$$\int_{-T}^{+T} u_0(t) \, dt = 0;$$

$$T \geq 0;$$

(19)
and, therefore,

\[ L\{u_0(t)\} = 0. \tag{20} \]

Whereas \( u_0(t) \) as such does not become apparent in L-transforms, its derivatives do. The derivatives

\[ u_0^{(n)}(t) = u^{(n)}(t) = \delta^{(n-1)}(t); \quad n = 1, 2, \ldots \tag{21} \]

have the L-transforms

\[ L\{u_0^{(n)}(t)\} = L\{u^{(n)}(t)\} = L\{\delta^{(n-1)}(t)\} = s^{n-1}, \quad n = 1, 2, \ldots \tag{22} \]

and these are different from null.

The occurrence of the connect function in the inverse L-transform was implicitly pointed out, e.g., by Doetsch [2, 4, 5]. He proved that the inverse L-transform is identical to the original function except for a null-function, i.e., a function whose integral is null. Thus, Doetsch in effect anticipated the involvement of the connect function. However, the fact that the null function does not become apparent in L-transforms led him to define inverse L-transforms that differ only by a null-function to be identical. As a consequence, in TLT the connect function is being ignored. This is a serious mistake, i.e., for the following reasons.

a) Without inclusion of \( u_0(t) \) one can not obtain a solid conceptual definition of LT-consistency, as was pointed out above; cf. Fig. 1.

b) Without inclusion of \( u_0(t) \) one can not obtain a solid formal definition of LT-consistency, because the unit-step redundancy theorem \( u(t) \cdot u^{(n)}(t) = u^{(n)}(t) \) does not hold; cf. Sects. 2.4, A.3.

c) Without inclusion of \( u_0(t) \) neither the derivatives of \( u(t) \), i.e., the impulse functions \( \delta^{(n)}(t) \), nor their L-transforms are sufficiently defined.

### 2.4 Causal functions

Utilizing the consistent definition of the unit step function, finally the causal type of function \( f_c(t) \) can be defined in the familiar way, i.e.,

\[ f_c(t) = u(t)f(t). \tag{23} \]

It is inclusion of the unit connect function \( u_0(t) \) in the unit step function \( u(t) \) that makes the expression universal. Equation (23) holds irrespective of whether or not \( f(t) \) itself is causal. \( f(t) \) may be either a d-function \( f_d(t) \); a causal function of the form \( [u(t)f_d(t)] \); or an impulse function \( \delta^{(n)}(t) \). This follows from the identities outlined in Sect. A.3 which can be subsumed by the
unit-step redundancy theorem

\[ u(t) \cdot u^{(n)}(t) = u^{(n)}(t); \quad n = 0, 1, \ldots \]  

(24)

By this theorem Eq. (23) remains in effect unchanged when both sides are multiplied by \( u(t) \); in particular, there holds

\[ u(t)f_c(t) = f_c(t). \]  

(25)

Multiplication by \( u(t) \) of any kind of causal function is redundant.

On the basis of these observations the bilateral formula for L-transformation envisaged in (9) can be restated, and its equivalence to (1) can be expressed by the

bilaterality theorem

\[ L\{f(t)\} = \int_{-\infty}^{+\infty} u(t)f(t)e^{-st} \, dt. \]  

(26)

It is the LT-consistent definition (14) of \( u(t) \) which renders Eq. (26) consistent, i.e., by consistency with Fourier transformation; and Eq. (1) becomes consistent with Fourier transformation by its equivalence to (26). The mathematical implications of (1) are the same as those of (26).

While the above three-step approach to the formula (26) is helpful by its elucidating implications, it should be noticed that formally the expansion of (1) into (26) can be obtained by one single step, namely,

\[ L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st} \, dt = \lim_{T \to 0} \int_{-\infty}^{+\infty} r(t, T)f(t)e^{-st} \, dt, \]  

(27)

where \( r(t, T) \) denotes the unit ramp function of which an example is illustrated in Fig. 2.

![Fig. 2. The unit ramp function](image)

By letting \( T \to 0 \) the unit step function \( u(t) \) emerges from the unit ramp function \( r(t, T) \), and the unit connect function \( u_0(t) \) emerges from \( r_0(t) \).

From this approach it becomes apparent that the transition from \( u(-0)f(t) = 0 \) to \( u(+0)f(t) = f(+0) \) is not empty but is a vertically ascending continuous function, i.e., \( u_0(t) \). From this notion there emerges another definition of \( u_0(t) \), namely,

\[ u_0(t) = \lim_{T \to 0} r_0(t, T). \]  

(28)
The function $r_0(t)$ does not necessarily have to be linear – such as in Fig. 2 – but may be represented by any kind of real continuous function that in the interval $0 \leq T$ rises monotonically from 0 to 1.

Comparison of (1) to (26) reveals a crucial feature of LT which in TLT is ignored, namely, LT’s inherent ambivalence. There holds the

**ambivalence theorem**

$$L\{f(t)\} = L\{u(t)f(t)\} = F(s). \quad (29)$$

From an L-transform $F(s)$ one can not tell whether it was obtained from $f(t)$ or from $u(t)f(t)$. This kind of ambivalence corresponds to the ambivalence of $L\{u(t)\}$ that was noted by (17). In both cases the ambivalence is significant by the presence of $u_0(t)$ in $u(t)$ and by the fact that $L\{u_0^{(n)}(t)\} = s^{n-1} \neq 0 \, (n = 1, 2, \ldots)$.

As a consequence of the equivalence of (26) to the formula for Fourier-transformation, the inverse L-transform obtained by (2) is invariably and unequivocally identical to $u(t)f(t)$. There holds the

**causality theorem**

$$\varphi(t) = L^{-1}\{L\{f(t)\}\} = L^{-1}\{L\{u(t)f(t)\}\} = u(t)f(t) \quad \text{for } -\infty < t < +\infty. \quad (30)$$

The inverse L-transform is the *causal companion* of the original function $f(t)$.

The latter two theorems warrant consistency of *concatenation* of L-transformations. From (29) and (30) there follows the

**concatenation theorem**

$$L\{\varphi(t)\} = L\{f(t)\}. \quad (31)$$

In TLT this theorem can not exist because $\varphi(0)$ is undefined such that, rigorously, $L\{\varphi(t)\}$ can not be supposed to exist.

As the behavior of the L-transform $L\{f(t)\}$ is in every respect characterized by the causal inverse L-transform $\varphi(t)$, in a sense the L-transform of $f(t)$ *actually* constitutes the L-transform of the causal function $u(t)f(t)$. In many contexts – e.g., that of the derivation/integration theorem (cf. Sect. 3) – it is indeed quite helpful to observe the

**alias theorem:**

The notation $L\{f(t)\}$ of an L-transform ultimately is an alias for $L\{u(t)f(t)\}$.

From the above insights there emerges a concise proof of the

**convolution theorem**

$$L\{f_1(t)\} \cdot L\{f_2(t)\} = L\{\int_{0}^{t} f_1(t - \tau)f_2(\tau) \, d\tau\}. \quad (32)$$
The proof of (32) can be based on the identity
\[ \int_{-\infty}^{T} g_1(t) \, dt \cdot \int_{-\infty}^{T} g_2(t) \, dt = u(t) \int_{0}^{T} \int_{0}^{t} g_1(t-\tau) g_2(\tau) \, d\tau \, dt; \quad T > 0; \quad t \leq T, \tag{33} \]
which holds for any pair of causal functions \( g_{1,2}(t) \). Letting \( g_{1,2}(t) = u(t) f_{1,2}(t) \exp(-st) \) and \( T \to \infty \), one obtains from (33)
\[ \int_{0}^{\infty} f_1(t) e^{-st} \, dt \cdot \int_{0}^{\infty} f_2(t) e^{-st} \, dt = \int_{0}^{\infty} e^{-st} \int_{0}^{t} f_1(t-\tau) f_2(\tau) \, d\tau \, dt, \tag{34} \]
where the “factor” \( u(t) \) is appropriately accounted for. By (11) Eq. (34), indeed, is equivalent to (32).

2.5 Testing LT-consistency

For a function \( f(t) \) to be LT-consistent it is with regard to (30) crucial that
\[ u(0) f(0) = f(0). \tag{35} \]
In this equation neither \( u(0) \) nor \( f(0) \) necessarily denote unique definite values. The symbol \( u(0) \) is just a synonym for the pseudo-function \( u_0(t) \). As \( f(t) \) may be any linear combination of d-functions, causal functions, and derivatives, \( f(0) \) may actually denote any linear combination of definite unique values, connect functions, and derivatives of the latter, i.e., impulse functions. Keeping this in mind, and observing the unit-step redundancy theorem (24), the criterion for LT-consistency of any type of real function \( f(t) \) can be universally expressed by the

**LT-consistency theorem:** The function \( f(t) \) is LT-consistent if, and only if,
\[ u(t) f(t) = f(t) \quad \text{for} \quad -\infty < t < +\infty. \tag{36} \]

The criterion (36) provides for the formal proof of the conclusion that was already drawn from Fig. 1, namely, that d-functions \( f_d(t) \) are non-LT-consistent: \( u(t) f_d(t) \neq f_d(t) \). By contrast, causal functions \( f_c(t) \) are LT-consistent, because by the unit-step redundancy theorem there holds \( u(t) f_c(t) = f_c(t) \).

A useful application of the LT-consistency theorem is verification of LT-consistency of shifted functions. From the formula for inverse L-transformation (2) one obtains in the familiar way the relationship
\[ \varphi(t-\tau) = L^{-1}\{L\{f(t)\} \cdot e^{-st}\}, \tag{37} \]
which indicates that multiplication of \( L\{f(t)\} \) by \( \exp(-s\tau) \) shifts the causal inverse transform by any positive or negative amount \( \tau \). Utilizing (30) one obtains from (37) by L-transformation the
This expression holds for any type of real function \( f(t) \) and for any \( \tau \). However, it is only for \( \tau \geq 0 \) that (38) is consistent with the alias theorem, indicating that (38) is only for \( \tau \geq 0 \) LT-consistent. Indeed, for a shifted causal function \( f_c(t - \tau) \) the criterion (36) reads

\[
u(t) \cdot f_c(t - \tau) = f_c(t - \tau),\tag{39}\]

and this condition is met only for \( \tau \geq 0 \).

The shifting theorem makes particularly apparent that Laplace transformation virtually is confined to causal functions. TLT’s attempt to express the shifting theorem in terms of ordinary derivable functions is awkward and incoherent [4, 5].

Yet, Laplace transformation of non-causal functions, in particular, δ-functions, does not entirely have to be ruled out. L-transformation of δ-functions can be sensible and useful; cf. Sects. 3.2, 3.3, 4.2. One just has to keep in mind that the L-transform of a δ-function, \( L\{f_d(t)\} \), actually is the L-transform of \( u(t)f_d(t) \), i.e., \( L\{u(t)f_d(t)\} \).

### 2.6 Summary

Laplace transformation of a real function \( f(t) \) is said to be consistent if the pertinent inverse L-transform is identical to \( f(t) \). For \( t > 0 \) LT is in this sense consistent for any type of real function that can be L-transformed at all. However, such confined consistency is not sufficient for LT to provide a coherent system of operator calculus. Even in the realm of TLT the behavior of functions at \( t = 0 \) turns out in effect to be involved. Explicit inclusion of the point \( t = 0 \) into LT’s definition interval requires

a) appreciation of the implicit bilaterality of the L-transformation formula (1); and

b) mathematical description of the behavior at \( t = 0 \) of both \( f(t) \) and \( \varphi(t) \), i.e., by the connect function \( u_0(t) \) and/or its derivatives.

From a) there emerges a bilateral equivalent of the L-transformation formula (1) that makes LT compatible with the theory of Fourier-transformation. This formula in turn is dependent on consistent definition of the unit step function, i.e., according to b). L-transformation turns out to be ambivalent, i.e., there holds \( L\{f(t)\} = L\{u(t)f(t)\} \). Inverse L-transformation is unequivocal; the inverse transform has the form \( u(t)f(t) \). Therefore, only causal functions are LT-consistent.
The derivation/integration theorem

There are two fundamentally different approaches to obtaining LT-theorems such as those for derivation/integration:

a) Determination by (1) of the L-domain operation that corresponds to derivation/integration of the original function \( f(t) \).

b) Determination by (2) of the L-domain operation that corresponds to derivation/integration of the inverse L-transform \( \varphi(t) \).

L-transformation by (1) or (26) is ambivalent, whereas inverse L-transformation by (2) is unequivocal. Moreover, any kind of application of LT, in particular, to the solution of linear DEs, is in the first place dependent on the inverse L-transform \( \varphi(t) \) – as opposed to the original function \( f(t) \). Therefore, only the approach b) is adequate.

TLT’s derivation theorem (4) is based on the inadequate approach a). In TLT, the derivative \( f'(t) \) of \( f(t) \) is presupposed to exist in the ordinary mathematical sense and the L-transform of \( f'(t) \) is expressed by (1). The theorem for the first derivative then emerges from integration by parts:

\[
L\{f'(t)\} = \int_0^\infty f'(t)e^{-st} \, dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st} \, dt = sL\{f(t)\} - f(0). \tag{40}
\]

Although this kind of mathematical reasoning is formally correct, the result is not LT-consistent. The theorem holds only for d-functions \( f(t) = f_d(t) \), i.e., functions that at \( t = 0 \) are derivable in the ordinary sense; d-functions are non-LT-consistent because \( u(t)f_d(t) \neq f_d(t) \); cf. (36).

Below, the LT-consistent theorems for derivation and integration are obtained by the approach b). These theorems are termed the primary derivation and integration theorem, respectively. They hold for derivatives/integrals of the form \( [u(t)f(t)]^{(\pm n)} \). There also exist secondary theorems; these hold for \( f^{(\pm n)}(t) \). TLT’s derivation theorem turns out to be of the secondary type. (In TLT the distinction between \( f^{(\pm n)}(t) \) and \( [u(t)f(t)]^{(\pm n)} \) is ignored.) The primary theorems for derivation/integration have the same form such that they can be unified. The unified theorem can be generalized for non-integer order of derivation/integration. The secondary theorems turn out to be virtually irrelevant.

3.1 The primary derivation theorem

As the inverse transform \( \varphi(t) \) is a causal and non-ordinary function, its derivatives are non-ordinary and causal, as well. By contrast, the formula (2) can within the integral be
derived for $t$ in the ordinary sense, and the order of derivation is unlimited, as only the function $\exp(st)$ needs to be derived. For the first derivative of $\varphi(t)$ one obtains

$$\varphi'(t) = L^{-1}\{sF(s)\} = L^{-1}\{sL\{f(t)\}\} = L^{-1}\{sL\{u(t)f(t)\}\}. \quad (41)$$

Observing the concatenation theorem (31), one obtains for the second derivative

$$\varphi''(t) = L^{-1}\{sL\{\varphi'(t)\}\} = L^{-1}\{s^2L\{f(t)\}\}, \quad (42)$$

Continuing with this kind of reasoning to obtain higher-order derivatives, and observing (30), one obtains the

**primary derivation theorem**

$$L\{[u(t)f(t)]^{(n)}\} = s^nL\{f(t)\}; \quad n = 0, 1, \ldots \quad (43)$$

Equation (43) holds for any type of real function $f(t)$ and for any $n$ for which the inverse transform of $s^nL\{f(t)\}$ exists.

For causal functions $f_c(t)$ the primary derivation theorem assumes the form

$$L\{f_c^{(n)}(t)\} = s^nL\{f_c(t)\}; \quad n = 0, 1, \ldots \quad (44)$$

The form (44) applies, in particular, to the impulse functions $f_c(t) = \delta^{(n)}(t)$, and therefore Eqs. (11, 22) are consistent with (44). In contrast to a widespread misconception, the reason why the form (44) holds for impulse functions is not because these are distributions but because they are causal.

The $t$-domain implications of the operation $s^nL\{f(t)\}$ are quite different for causal functions *versus* d-functions. As for causal functions $f_c(t)$ there holds $\varphi(t) = f_c(t)$, there follows from the above deduction of (43)

$$L^{-1}\{s^nL\{f_c(t)\}\} = f_c^{(n)}(t). \quad (45)$$

For d-functions the corresponding relationship is considerably more complicated. To demonstrate this, functions of the type

$$f_{ud}(t) = u(t)f_d(t) \quad (46)$$

are taken into consideration, where $f_d(t)$ denotes a d-function. The $n$-th derivative of the so-called ud-function $f_{ud}(t)$ is depicted by the

**dud-theorem**

$$f_{ud}^{(n)}(t) = [u(t)f_d(t)]^{(n)} = u(t)f_d^{(n)}(t) + \sum_{\nu=0}^{n-1} f_d^{(n-1-\nu)}(0) \cdot \delta^{(\nu)}(t); \quad n = 1, 2, \ldots \quad (47)$$
The dud-theorem (derivation of $u_d$-function) expresses the $n$-th non-ordinary derivative of $u(t)f_d(t)$ by the ordinary derivatives of $f_d(t)$, i.e., at the expense of getting impulse functions involved as depicted by (47). The dud-theorem is explained in Sect. A.4.

By (47), utilizing (43), the effect of the operation $s^n L\{f_d(t)\}$ gets depicted by

$$L^{-1}\{s^n L\{f_d(t)\}\} = u(t)f_d^{(n)}(t) + \sum_{\nu=0}^{n-1} f_d^{(n-1-\nu)}(0) \cdot \delta^{(\nu)}(t); \quad n = 1, 2, \ldots$$

(48)

As an example, consider the operation $sL\{\cos \omega t\}$. From (48) one obtains

$$L^{-1}\{sL\{\cos \omega t\}\} = -u(t) \cdot \omega \sin \omega t + \delta(t).$$

(49)

The same result is obtained from the LT-correspondences (133, 134), i.e.,

$$L^{-1}\{sL\{\cos \omega t\}\} = L^{-1}\{s \cdot \frac{s}{s^2 + \omega^2}\} = L^{-1}\{1 - \frac{\omega^2}{s^2 + \omega^2}\} = \delta(t) - u(t) \cdot \omega \sin \omega t. \quad (50)$$

The most important domain of application of the primary derivation theorem is determination of the evoked solution of the inhomogeneous linear DE; cf. Sect. 4.1.

### 3.2 The secondary derivation theorem

Although d-functions and their derivatives are non-LT-consistent, the question is legitimate how the L-transform of $f_d^{(n)}(t)$ can be expressed by the L-transform of $f_d(t)$. The answer is already implied in (48). From that expression one obtains by L-transformation the secondary derivation theorem

$$L\{f_d^{(n)}(t)\} = s^n L\{f_d(t)\} - \sum_{\nu=0}^{n-1} f_d^{(n-1-\nu)}(0) \cdot s^\nu; \quad n = 1, 2, \ldots$$

(51)

The secondary derivation theorem (51) turns out to be identical to TLT’s derivation theorem. This is a consequence of TLT’s original endeavour to provide an operator calculus essentially for d-functions; cf. Eq. (40).

The secondary derivation theorem is not a self-contained derivation theorem as it just emerges from application of the primary derivation theorem to d-functions.

### 3.3 The primary integration theorem

According to (2) the inverse transform’s integral function can be depicted by

$$\varphi^{(-1)}(t) = L^{-1}\{s^{-1}F(s)\} = L^{-1}\{s^{-1}L\{f(t)\}\} = L^{-1}\{s^{-1}L\{u(t)f(t)\}\}. \quad (52)$$
As \( \varphi(t) \) is causal and LT-consistent, the integral function \( \varphi^{(-1)}(t) \) is causal and LT-consistent, as well. Thus for the second-order integral there holds

\[
\varphi^{(-2)}(t) = L^{-1}\{s^{-1}L\{\varphi^{(-1)}(t)\}\} = L^{-1}\{s^{-2}\{L\{\varphi(t)\}\}\} = L^{-1}\{s^{-2}L\{f(t)\}\};
\]

and so on for higher-order integrals. Utilizing (30) one obtains the primary integration theorem

\[
L\{[u(t)f(t)]^{(-n)}\} = s^{-n}L\{f(t)\}; \quad n = 0, 1, \ldots
\]

which for causal functions \( f(t) = f_c(t) \) reads

\[
L\{f_c^{(-n)}(t)\} = s^{-n}L\{f_c(t)\}.
\]

The operation \( s^{-n}L\{f(t)\} \) is equivalent to \( n \)-fold integration of the causal function \([u(t)f(t)]\), which implies \((n-1)\)-fold iteration of the \( t \)-domain operation

\[
[u(t)f(t)]^{(-1)} = \int_{-\infty}^t u(\tau)f(\tau) \, d\tau = u(t)\int_0^t f(\tau) \, d\tau.
\]

As \( u(t)f(t) \) and the integral functions are causal, the \( n \)-fold integral can be expressed by the formula

\[
[u(t)f(t)]^{(-n)} = \frac{u(t)}{(n-1)!}\int_0^t (t-\tau)^{n-1}f(\tau) \, d\tau; \quad n = 1, 2, \ldots
\]

The consistency of this formula with the primary integration theorem (54) can be verified by L-transformation and application of the convolution theorem (32), utilizing the LT-correspondence (130).

The \( t \)-domain implications of the operation \( s^{-n}L\{f(t)\} \) are just as different for causal functions versus \( d \)-functions as was found for derivation. For causal functions \( f_c(t) \) one obtains

\[
L^{-1}\{s^{-n}L\{f_c(t)\}\} = f_c^{(-n)}(t).
\]

When \( f(t) = f_d(t) \) is a \( d \)-function such that \( f_d(t) \) is the \( n \)-th ordinary derivative of \( f_d^{(-n)}(t) \), there holds

\[
[u(t)f_d(t)]^{(-1)} = \int_{-\infty}^t u(t)f_d(\tau) \, d\tau = u(t)\int_0^t f_d(\tau) \, d\tau = u(t)[f_d^{(-1)}(t) - f_d^{(-1)}(0)].
\]

By iteration of (59) one obtains the so-called iud-theorem (integration of ud-function)

\[
f_{ud}^{(-n)}(t) = [u(t)f_d(t)]^{(-n)} = u(t)f_d^{(-n)}(t) - u(t)\sum_{\nu=0}^{n-1} f_d^{(-n+\nu)}(0) \cdot \frac{\nu^\nu}{\nu!}; \quad n = 1, 2, \ldots
\]
For the effect of the operation $s^{-n}L\{f_d(t)\}$ one obtains from (54) and (60)

$$L^{-1}\{s^{-n}L\{f_d(t)\}\} = u(t)f_d^{(-n)}(t) - u(t)\sum_{\nu=0}^{n-1} f_d^{(-n+\nu)}(0) \cdot \frac{t^\nu}{\nu!}; \quad n = 1, 2, \ldots \quad (61)$$

As an example, consider the operation $s^{-1}L\{\exp(-at)\}$. From (61) one obtains

$$L^{-1}\{s^{-1}L\{\exp(-at)\}\} = \frac{u(t)}{a} \cdot e^{-at} + \frac{u(t)}{a}. \quad (62)$$

Using the LT-correspondences (131, 132) one obtains the same result, i.e.,

$$L^{-1}\{s^{-1}L\{\exp(-at)\}\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s + a}\right\} = \frac{u(t)}{a}(1 - e^{-at}). \quad (63)$$

### 3.4 The secondary integration theorem

By analogy to the secondary derivation theorem, Eq. (61) enables for d-functions expression of the L-transform of the $n$-th order integral in terms of the L-transform of the d-function itself. By L-transformation of (61), utilizing (130), there emerges the secondary integration theorem

$$L\{f_d^{(-n)}(t)\} = s^{-n}L\{f_d(t)\} + \sum_{\nu=0}^{n-1} f_d^{(-n+\nu)}(0) \cdot s^{-\nu-1}; \quad n = 1, 2, \ldots \quad (64)$$

The secondary integration theorem is complementary to the secondary derivation theorem. For $n = 1$ the two theorems are equivalent; indeed, from (64) one obtains

$$L\{f_d^{(-1)}(t)\} = s^{-1}L\{f_d(t)\} + f_d^{(-1)}(0) \cdot s^{-1}. \quad (65)$$

Taking into account that by definition $f_d(t)$ is the first ordinary derivative of $f_d^{(-1)}(t)$, Eq. (65) is identical to the secondary derivation theorem and thus to TLT’s derivation theorem.

As TLT’s derivation theorem (4, 40) is identical to the secondary derivation theorem (51), one would expect TLT’s integration theorem to comply with the secondary integration theorem (64). However, TLT’s integration theorem (5) actually is identical to the primary integration theorem (54). The latter theorem holds only for causal functions; cf. Eq. (55); its application to d-functions – such as is customary in TLT – will in general yield erroneous results.

Thus, it turns out that the alleged mathematical consistency of TLT’s theorems for derivation/integration is delusive. TLT’s derivation theorem holds only for d-functions whereas TLT’s integration theorem holds only for causal functions. In TLT both theorems
are ordinarily used in an untenable way: TLT’s derivation theorem is regarded as the derivation theorem of LT although it holds only for the non-LT-consistent d-functions. TLT’s integration theorem is ordinarily utilized for d-functions although it does not apply to this type of function. Eventually, these observations explain why there is a formal conflict between TLT’s theorems for derivation and integration.

The secondary integration theorem is not a self-contained integration theorem as it just emerges from application of the primary integration theorem to d-functions.

3.5 The generalized derivation/integration theorem

The LT-consistent theorems for derivation and integration, i.e., (43) and (54), can obviously be unified into one formula, i.e.,

\[ L\{[u(t)f(t)]^{(n)}\} = s^n L\{f(t)\} \text{ } n = 0, \pm 1, \pm 2, \ldots \]  

Thus, ultimately \( t \)-domain integration is in the L-domain merely “reciprocal” to \( t \)-domain derivation. The \( t \)-domain implications of the operation \( s^n L\{f(t)\} \) that are outlined in Sects. 3.1, 3.3 must be observed.

As a consequence of the formal equivalence of derivation and integration in the L-domain description, the theorem can be generalized for non-integer order of derivation/integration. For any real \( r \geq 0 \) there holds (Riemann, Liouville, Cauchy)

\[ D^{-r}\{u(t)f(t)\} = \frac{u(t)}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} f(\tau) \, d\tau; \quad r \geq 0. \]  

The operator \( D \) denotes generalized derivation/integration, while the negative exponent \(-r\) indicates \( r \)-th-order integration. As the integral in (67) is equivalent to the convolution \( t^{r-1} * f(t) \), Eq. (67) can by the convolution theorem (32) be L-transformed, and when the LT-correspondence (141) is utilized one obtains

\[ L\{D^{-r}\{u(t)f(t)\}\} = s^{-r}L\{f(t)\}; \quad r \geq 0. \]  

Equation (69) depicts the generalized integration theorem. This theorem suffices to obtain the same kind of generalization for derivation, namely, by concatenation of (69) and (43). The sequence of \( r \)-th order integration and \( n \)-th order derivation yields \((n-r)\)th order derivation or integration, depending on whether \( n > r \) or \( n < r \). Denoting \( n-r = \alpha \) one obtains the

\[ gdi-theorem \text{ (generalized derivation/integration theorem)} \]

\[ L\{D^\alpha\{u(t)f(t)\}\} = s^\alpha L\{f(t)\}; \quad \alpha \in \mathbb{R}. \]  

(70)
In the form (70) the theorem holds for any type of real function \( f(t) \). For integer values of \( \alpha \) Eq. (70) is equivalent to (66).

The concept of fractional calculus, i.e., utilization of derivatives of non-integer order, is essentially based on the formula (67), and this formula holds only for causal functions. In the realm of TLT derivatives of non-integer order can not be consistently expressed at all, as TLT’s derivation theorem neither holds for causal functions nor for non-integer order of derivation. When TLT’s derivation theorem is used anyway, initial values become involved which merely are a nuisance. It is by the present approach – which provides for the generalized theorem (66) – that Laplace transformation becomes a consistent and efficient tool for doing fractional calculus.

Examples for the application of (70), i.e., for \( \alpha = 1/2 \), are listed in Sect. A.5.

3.6 Summary

LT-consistent theorems for t-domain derivation and integration are obtained by deduction from the behavior of the inverse L-transform. These theorems – the primary theorems – are formally congruent and thus can be unified. The unified primary theorem is generalized for non-integer order of derivation/integration. The generalized theorem (the gdi-theorem) accounts in the L-domain for both integer-order and non-integer-order of \( t \)-domain derivation/integration. As the inverse transform is causal the primary theorems and the gdi-theorem hold for causal functions.

The respective secondary theorems emerge from application of the primary ones to the L-transforms of ordinary derivable functions (d-functions). TLT’s derivation theorem is identical to the secondary derivation theorem. This theorem can not be regarded as the derivation theorem of Laplace transformation because it applies only to the class of d-functions, which is non-LT-consistent. TLT’s integration theorem is identical to the primary integration theorem and therefore holds only for causal functions; its utilization for d-functions – such as in TLT – is untenable.

4 Solution of linear differential equations

Below, the new approach to the solution of linear DEs by LT is demonstrated for a fairly general form of the ordinary linear DE, namely,

\[
\sum_{n=0}^{N} a_n y^{(n)}(t) = \sum_{m=0}^{M} b_m x^{(m)}(t); \quad N = 1, 2 \ldots; \quad M = 0, 1 \ldots \tag{71}
\]

The coefficients \( a_n, b_m \) are presupposed to be real constants. Two functions are involved, namely, the excitation function \( x(t) \) and the response function \( y(t) \). The form (71) is
more general than that accounted for by the TLT method, as in (71) derivatives of the excitation function are allowed to be included, i.e., corresponding to \( M > 0 \).

A clear distinction is made between the evoked solution and the spontaneous solution of the inhomogeneous DE [13]. The evoked solution (the pertinent system’s evoked response to \( x(t) \)) is that part of the total response that is elicited by the excitation function \( x(t) \) alone. The spontaneous solution (the system’s “spontaneous” response), if it exists, is ascribed to a pre-excited initial state of the system. Mathematically, the evoked solution is the DE’s total (general) solution minus the general solution of the pertinent homogeneous DE. The spontaneous solution is identical and synonymous to the general solution of the homogeneous DE.

4.1 The evoked response: 
Particular solution of the inhomogeneous DE

For the excitation function \( x(t) \) to be LT-consistent it must be defined as a causal function. As a consequence, its derivatives have the form \( u(t)x(t)^{(m)} \). Another consequence is that the evoked response \( y_e(t) \) is causal as well; thus, its derivatives implicitly assume the form \( u(t)y_e(t)^{(n)} \). Therefore, conversion of (71) into the L-domain is governed by the primary derivation theorem (43). The L-domain representation of (71) reads

\[
\sum_{n=0}^{N} a_n s^n L\{y_e(t)\} = \sum_{m=0}^{M} b_m s^m L\{x(t)\}; \quad N = 1, 2 \ldots; \quad M = 0, 1 \ldots, \tag{72}
\]

and one eventually obtains the evoked solution

\[
y_e(t) = L^{-1}\left\{ \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} \cdot L\{x(t)\} \right\}. \tag{73}
\]

In contrast to the TLT method, the evoked response is by (73) obtained without involvement of initial values, i.e., additional constants. To obtain the evoked response one does not have to pretend that the pertinent system is in a non-preexcited initial state. With respect to the evoked response the system’s initial state is irrelevant.

For the operation \( L^{-1} \), i.e., inverse L-transformation, there exist two alternatives. The first of them is taking (73) at face value, which implies that the L-transform of \( x(t) \) has to be included. The second alternative exploits the convolution theorem (32), whereby inverse L-transformation can be confined to the first factor in (73):

\[
h(t) = L^{-1}\left\{ \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} \right\}. \tag{74}
\]

The function \( h(t) \) depicts the evoked response to the excitation function \( x(t) = \delta(t) \), i.e., the impulse response. As (73) has the form

\[
y_e(t) = L^{-1}\{L\{h(t)\} \cdot L\{x(t)\}\}, \tag{75}
\]
by (32) there emerges the familiar convolution formula

\[ y_e(t) = u(t) \int_0^t h(t - \tau)x(\tau) \, d\tau. \]  

For example, for \( N = 1, M = 0 \) one obtains the impulse response

\[ h(t) = L^{-1}\left\{ \frac{b_0}{a_0 + a_1 s} \right\} = u(t)\beta e^{-\alpha t}; \quad \alpha = a_0/a_1; \quad \beta = b_0/a_1. \]  

Utilizing (77) and (76), one obtains the 

**evoked solution of the first-order linear DE**

\[ y_e(t) = u(t)\beta \int_0^t x(\tau)e^{-\alpha(t-\tau)} \, d\tau = u(t)\beta e^{-\alpha t} \int_0^t x(\tau)e^{\alpha \tau} \, d\tau. \]  

Though this solution is entirely expressed in the \( t \)-domain, it nevertheless is based on LT, which implies that (78) is correct only for causal excitation functions \( x(t) \).

It may be noted that it is the independent LT-based expression of the evoked solution (73) that provides a solid basis to *operator calculus* as it is customarily employed in the theories of linear control systems and electrical circuits.

### 4.2 The spontaneous response: General solution of the homogeneous DE

When the evoked response is by LT determined as just described, for the solution of the homogeneous DE still any method available can be chosen. From the present approach there emerges a new LT-based method which exploits the facts that

a) the spontaneous response \( y_s(t) \) and its derivatives are d-functions; their L-transforms have the form \( L\{y^{(n)}_s(t)\} = L\{u(t)y^{(n)}_s(t)\} \);

b) the homogeneous DE can by the dud-theorem be converted into an equivalent inhomogeneous DE.

To get the homogeneous DE made up for conversion into an equivalent inhomogeneous DE, it is multiplied by \( u(t) \); this yields

\[ a_0u(t)y_s(t) + a_1u(t)y'_s(t) + \ldots + a_Nu(t)y^{(N)}_s(t) = 0. \]  

Because of LT’s ambivalence, the L-transform of (79) is identical to the L-transform of the original homogeneous DE. Application of the dud-theorem (47) converts (79) into the form

\[
\begin{align*}
& a_0u(t)y_s(t) + a_1\left\langle [u(t)y_s(t)]' - y_s(0) \cdot \delta(t) \right\rangle + a_2\left\langle [u(t)y_s(t)]'' - y_s'(0) \cdot \delta(t) - y_s(0) \cdot \delta'(t) \right\rangle \\
& \quad + \ldots + a_N\left\langle [u(t)y_s(t)]^{(N)} - \sum_{\nu=0}^{N-1} y^{(N-1-\nu)}_s(0) \cdot \delta^{(\nu)}(t) \right\rangle = 0.
\end{align*}
\]  

(80)
Equation (80) is equivalent to (79). However, (80) actually is an inhomogeneous DE. The impulse functions included in (80) play the role of virtual excitation functions [13]. This becomes particularly apparent when (80) is expressed in the form

$$
\sum_{n=0}^{N} a_n [u(t)y_s(t)]^{(n)} = \sum_{\mu=0}^{N-1} c_\mu \delta^{(\mu)}(t),
$$

(81)

where the coefficients $c_\mu$ are depicted by

$$
c_\mu = \sum_{\nu=0}^{N-1-\mu} a_{\mu+\nu+1} \cdot y_s^{(\nu)}(0).
$$

(82)

From (81) the solution of the homogeneous DE can be obtained as an evoked solution, i.e., as described in Sect. 4.1. One eventually obtains

$$
u(t)y_s(t) = L^{-1} \left\{ \frac{\sum_{\mu=0}^{N-1} c_\mu s^\mu}{\sum_{n=0}^{N} a_n s^n} \right\}.
$$

(83)

The notation $u(t)y_s(t)$ is not redundant, because $y_s(t)$ is a d-function. Equation (83) in fact depicts the evoked response of a virtual system, i.e., to the excitation function $\delta(t)$. The transmission function of that virtual system is defined by the quotient in (83), whose numerator essentially originates from the impulse functions contained in (81).

Finally, the spontaneous solution $y_s(t)$ itself can be extrapolated into $t \leq 0$ be obtained from (83). As according to the causality theorem (30) the operation $L^{-1}$ on the right side of (83) yields either explicitly or implicitly a function of the form $u(t)y_s(t)$, that kind of extrapolation is equivalent to cancellation of $u(t)$ on both sides of (83), i.e., after inverse L-transformation.

For example, for $N = 1$ one obtains from (83) and (82)

$$
u(t)y_s(t) = L^{-1} \left\{ \frac{a_1 y_s(0)}{a_0 + a_1 s} \right\} = u(t)y_s(0)e^{-\alpha t}; \quad \alpha = a_0/a_1.
$$

(84)

By cancellation of $u(t)$ there emerges from (83) the 

**spontaneous solution of the first-order linear DE**

$$y_s(t) = y_s(0)e^{-\alpha t} \text{ for } -\infty < t.
$$

(85)

The described deduction of (83) ultimately constitutes a new method for obtaining the general solution of the linear homogeneous DE of finite order $N$, i.e., by Laplace transformation. The method involves a) determination by (82) of the coefficients $c_\mu$, i.e., from the DE’s coefficients $a_n$; b) insertion of the coefficients $c_\mu$ into (83); c) inverse L-transformation; and d) cancellation of the factor $u(t)$. The $N$ arbitrary constants which
invariably get involved are provided by the initial values at \( t = 0 \) of the solution \( y_h(t) \) itself and of the latter’s \( N - 1 \) derivatives. The method is straightforward; yet, mathematical intricacies may occur in step c), i.e., inverse L-transformation.

As a pragmatic alternative to the described deduction, the formula (83) may be obtained in a more immediate way, namely, by using for LT-conversion of the homogeneous DE the secondary derivation theorem (51). This option is enabled by the fact that the secondary derivation theorem accounts for the combination of the dud-theorem with the primary derivation theorem; cf. Sect. 3.2. As the secondary derivation theorem is identical to TLT’s derivation theorem, this option explains why TLT’s total solution of the inhomogeneous DE is quite similar to the new total solution; see the following two sections.

### 4.3 The total solution

Assuming that for the solution of the homogeneous DE the method just described is chosen, the total solution of (71) can be compactly depicted by a formula, i.e.,

\[
y(t) = L^{-1}\left\{ \sum_{m=0}^{M} b_m s^m \cdot L\{x(t)\} \right\} + L^{-1}_d \left\{ \frac{\sum_{\mu=1}^{N} a_{\mu} \sum_{\nu=0}^{\mu-1} y_h^{(\mu-1-\nu)}(0)s^{\nu}}{\sum_{n=0}^{N} a_n s^n} \right\}. \tag{86}\]

The first term on the right side of (86) depicts the evoked response and is identical to (73). The second term depicts the spontaneous response by one single expression; this term is equivalent to the combination of Eqs. (83) and (82). The operator \( L^{-1} \) indicates inverse transformation by (2). The operator \( L^{-1}_d \) indicates inverse transformation by (2) followed by extrapolation into \( t \leq 0 \), i.e., cancellation of the factor \( u(t) \).

When the inverse L-transform is looked up from a TLT-based table of LT-correspondences it must be observed that in those tables the \( t \)-domain functions are depicted in the non-causal form. In Sect. A.7 examples are listed of the LT-consistent notation of \( t \)-domain functions.

It must be kept in mind that LT-based solutions of linear DEs hold only for causal excitation functions \( x(t) = u(t)x(t) \). The solution for the “steady-state” may be obtained by asymptotical approach, i.e., for \( t \to \infty \).

An example for application of (86) is depicted in Sect. A.6.

Finally, it should be noticed that for \( N = 1, M = 0 \) the LT-based solution of (71) can be entirely expressed in the \( t \)-domain. By superposition of Eqs. (78) and (85) one obtains the

**total solution of the first-order linear DE**

\[
y(t) = y_h(t) + y_e(t) = u(t)\beta e^{-\alpha t} \int_{0}^{t} x(\tau)e^{\alpha \tau} \, d\tau + y_e(0)e^{-\alpha t}; \quad \alpha = a_0/a_1; \quad \beta = b_0/a_1. \tag{87}\]
Thus, for the special case that in (71) there is \( N = 1, M = 0 \) one does not actually have to do L-transformation at all. The solution is reduced to evaluation of the integral contained in (87) and it holds for causal excitation functions \( x(t) = u(t)x(t) \).

### 4.4 TLT’s initial-value conflict

One of the most annoying deficiencies of TLT is potential interference of the DE’s evoked solution with the spontaneous solution. The danger of this kind of interference to occur emerges from the fact that the DE’s solution obtained by TLT includes an inappropriate type of initial values. This is why the phenomenon is termed the initial-value conflict.

When the general solution of a linear DE of the form (71) is worked out by means of TLT – which is possible for \( M = 0 \) – one eventually obtains the formula

\[
y(t) = L^{-1}\left\{ \frac{b_0}{\sum_{n=0}^{N} a_n s^n} \cdot L\{x(t)\} + \sum_{\mu=1}^{N} a_{\mu} \sum_{\nu=0}^{\mu-1} y^{(\mu-\nu)}(0) s^\nu \sum_{n=0}^{N} a_n s^n \right\}.
\]

This formula is quite similar to (86), i.e., for \( M = 0 \). (The constant \( b_0 \) is in (88) included just for formal compatibility with (86); one may in both equations assume \( b_0 = 1 \).) Letting aside the difference in inverse L-transformation that is apparent by comparison of (88) to (86), there remains the crucial difference that (86) correctly contains the initial values of the spontaneous solution, \( y_s^{(\nu)}(0) \), whereas (88) contains the initial values of the total solution, \( y^{(\nu)}(0) \). The latter, inadequate kind of initial values inevitably emerge from LT-conversion of the DE (71) by TLT’s derivation theorem.

The potential initial-value conflict arises from the fact that the evoked and the spontaneous solutions are superimposed in \( y(t) \), such that

\[
y^{(\nu)}(0) = y_e^{(\nu)}(0) + y_s^{(\nu)}(0); \quad \nu = 0, 1, \ldots \quad (89)
\]

The initial values of the evoked solution, \( y_e^{(\nu)}(0) \), depend on \( x(t) \) and on the DE’s coefficients \( a_n \), as is depicted by (71). If there happens to be \( y_e^{(\nu)}(0) \equiv 0 \), i.e., for all \( \nu = 0, 1, \ldots, N - 1 \), then there holds \( y^{(\nu)}(0) \equiv y_s^{(\nu)}(0) \), and (88) will yield the same result as (86), i.e., except for the difference in inverse L-transformation, and for \( M = 0 \). However, for many types of system and of excitation function \( y_e^{(\nu)}(0) \) may turn out to be different from null, i.e., for at least one value of \( \nu \). If this occurs then the result obtained by TLT is incorrect; the initial values \( y^{(\nu)}(0) \) can no longer be freely chosen, i.e., independent of the excitation function; cf. Sect. A.1.

Obviously, (88) can easily be “patched”, i.e., by arbitrarily substituting the values \( y_s^{(\nu)}(0) \) for \( y^{(\nu)}(0) \). Most remarkably, there exists another, less trivial kind of patch, namely, arbitrary substitution in (88) of the values \( y^{(\nu)}(-0) \) for \( y^{(\nu)}(0) \). The reason why the latter kind of patch works around the initial-value conflict originates from the fact that the evoked response and its derivatives are causal (cf. Sect. 4.1), such that there holds
$y_e^{(\nu)}(-0) \equiv 0$, i.e., even when $y_e^{(\nu)}(0) \neq 0$. As the spontaneous response and its derivatives are d-functions, there holds $y_s^{(\nu)}(-0) \equiv y_s^{(\nu)}(0)$. Thus, one obtains from (89)

$$y^{(\nu)}(-0) = y_s^{(\nu)}(0) \quad \text{for } \nu = 0, 1, \ldots, N - 1. \quad (90)$$

The identity (90) holds for any type of linear DE and of excitation function. From this identity there follows that substitution in (88) of the initial point $t = -0$ for $t = 0$ renders (88) compatible with (86), such that TLT’s solution becomes essentially correct, i.e., except for the difference in inverse L-transformation. The “empirical” finding that by this kind of patch TLT’s initial-value conflict actually is worked around, becomes in this way explained.

It should be noticed that the latter kind of patch is just as arbitrary as the former. Even if one had no doubt at all about TLT’s consistency, the mere fact that TLT’s solution in general needs to be patched should have elicited that kind of doubt. One can not in earnest be satisfied with either patch’s working around TLT’s initial-value conflict.

4.5 Summary

The total solution of the inhomogeneous linear DE is obtained by two independent algorithms, i.e., one for the evoked response, another for the spontaneous response. As both the excitation function and the evoked response function are causal, the derivatives of these functions assume the form $[u(t)x(t)]^{(m)}$ and $[u(t)y_e(t)]^{(n)}$, respectively. When the inhomogeneous DE is by the primary derivation theorem converted into the L-domain one obtains a linear algebraic equation that does not contain any arbitrary constants. This equation is solved in the familiar way to obtain the evoked solution $y_e(t)$ which is defined for $-\infty < t$.

For the determination of the spontaneous response $y_s(t)$, i.e., solution of the pertinent homogeneous DE, the fact that the response function and its derivatives are d-functions is a priori taken into account. The L-transforms of these functions assume the form $L\{y_s^{(n)}(t)\} = L\{u(t)y_e^{(n)}(t)\}$. As a consequence, the homogeneous DE can be multiplied by $u(t)$ without affecting its L-transform. The homogeneous DE such modified can by the dud-theorem be converted into an equivalent inhomogeneous DE. The L-transform of the latter DE depicts a virtual linear system whose impulse response equals for $t > 0$ the spontaneous solution of the original homogeneous DE. By extrapolation into $t \leq 0$ one obtains from that impulse response the spontaneous solution $y_s(t)$ for $-\infty < t$.

For $M = 0$, i.e., if the excitation function does not include any derivative, the total solution obtained by TLT is formally similar to the correct solution depicted by the new method. However, the TLT-based solution suffers from an initial-value conflict. This conflict can be worked around by manipulation of the initial values. The necessity of patching TLT’s solution is another symptom of TLT’s deficiency.
5 Concluding remarks

From the theorems pointed out in Sect. 2 it should be apparent that the key to the consistent LT-theory lies in the hidden bilaterality of the transformation integral (1). By strictly taking that bilaterality into account there emerges a structure of LT-theory that is fundamentally different from that of TLT. Yet, many of the new theory’s theorems and expressions are familiar from TLT. Pronounced formal differences from TLT’s expressions occur, e.g., in the context of the theorems for derivation and integration. The difference between the new formula (86) for the general solution of the linear inhomogeneous DE from the corresponding formula (88) obtained by TLT, though at first sight marginal, actually is crucial, namely, for resolution of TLT’s inherent initial-value conflict.

Mathematical consistency of the new approach to LT is achieved at the expense of non-ordinary $t$-domain functions being crucially involved – primarily, the inverse L-transform $\varphi(t)$, which includes the connect function $u_0(t)$ and, possibly, its (non-ordinary) derivatives. This notion brings to mind the aspect of mathematical rigour. If one maintains that there is no rigorous mathematical account for non-ordinary functions and non-ordinary derivatives then the inverse L-transform can not in general be rigorously accounted for and a rigorous theory of Laplace transformation can not exist at all.

Rigorous treatment of non-ordinary functions may be achieved by invoking the theory of distributions. However, what makes LT-theory consistent in the first place is adequate incorporation of non-ordinary functions and non-ordinary derivatives. Mere attachment of distribution theory to TLT can not cure TLT’s inconsistency. Mathematically rigorous treatment of non-ordinary functions and derivatives may be achieved separately and subsequently – which, historically, actually has happened to the delta-impulse.

Making in this sense a distinction between mathematical consistency and rigour, one can say that the new approach to Laplace transformation outlined in the present article is consistent, even though it may not fully qualify as mathematically rigorous. This achievement is distinctly preferable to inconsistency disguised by fake mathematical rigour – which virtually is what is offered by Doetsch [4, 5] and many others.

Once one has developed some scepticism about TLT one may be prepared to realizing that Doetsch’s books are written in a somewhat defensive and dogmatic style, and that Doetsch devoted considerable portions of text to reasoning by plausibility and to explaining away apparent discrepancies. From these observations it may be concluded that Doetsch virtually was aware of TLT’s inherent problems. Indeed, in the preface to [3] he suggested the need for a fundamental redesign of TLT. However, until the end of his life (1977) he stuck to the original layout of TLT, i.e., with some distribution theory as a complement. In the two volumes of his Handbook [2, 3], Doetsch assembled a wealth of mathematics on LT – which, however, may distract from TLT’s fundamental flaws.

To obtain a more thorough understanding of TLT’s unfortunate history it is probably helpful to take notice of Doetsch’s biography [6, 9].
A APPENDIX

A.1 Failure of TLT: An example DE

TLT promises to provide a method for obtaining the general solution of the linear inhomogeneous DE, i.e., for any kind of excitation function \( x(t) \). Below, this promise is challenged by an example, i.e., for the first-order DE

\[ ay(t) + y'(t) = \hat{x} \sin \omega t, \quad (91) \]

where \( \hat{x} \sin \omega t \) denotes the excitation function and \( y(t) \) denotes the response function.

From the theory of linear DE’s one obtains the general solution

\[ y(t) = \frac{\hat{x}}{a^2 + \omega^2}(a \sin \omega t - \omega \cos \omega t) + y_s(0)e^{-at}, \quad (92) \]

where \( y_s(0) \) is an arbitrary real constant which specifies any particular solution of the pertinent homogeneous DE.

By the TLT method one obtains the solution

\[
y(t) = L^{-1}\left\{ \frac{\hat{x}\omega}{s^2 + \omega^2} \cdot \frac{1}{s + a} \right\} + L^{-1}\left\{ \frac{y(0)}{s + a} \right\} \\
= \frac{\hat{x}}{a^2 + \omega^2}(\omega e^{-at} + a \sin \omega t - \omega \cos \omega t) + y(0)e^{-at}; \quad \text{for } t > 0. \quad (93)
\]

The result (93) differs from the correct solution (92) in two significant respects:

a) As compared to the correct evoked response

\[ y_e(t) = \frac{\hat{x}}{a^2 + \omega^2}(a \sin \omega t - \omega \cos \omega t) \quad (94) \]

Eq. (93) includes an extra decaying exponential.

b) As compared to the correct spontaneous response

\[ y_s(t) = y_s(0)e^{-at} \quad (95) \]

Eq. (93) includes the initial value \( y(0) \) of the total response \( y(t) = y_e(t) + y_s(t) \) instead of that of the spontaneous response alone, i.e., \( y_s(0) \).

The solutions (92) and (93) both are consistent with the DE (91). Yet only (92) represents the DE’s general solution. This becomes apparent when one attempts formal reconciliation of (93) with (92). Reconciliation can only be obtained by suitable choice of
a particular “initial state”. For instance, for \( y_s(0) = 0 \) Eqs. (93, 92) can only be reconciled by setting

\[ y(0) = \frac{-\hat{x}\omega}{a^2 + \omega^2}. \quad (96) \]

Hence, \( y(0) \) is not really a free parameter of the spontaneous response. While the consistency of (93) with (91) suggests that the TLT-based solution is correct, comparison with (92) reveals that (93) actually is only a particular solution of (91) and that \( y(0) \) is not a freely assignable constant.

In point of fact, the correct solution of (91) can by L-transformation not be obtained at all, because the L-transform of \( \hat{x}\sin\omega t \) is just an alias for \( L\{u(t) \cdot \hat{x}\sin\omega t\} \); cf. Sect. 2.4.

The evoked part of the TLT-based solution (93) inevitably depicts for \( t > 0 \) the response to \( u(t) \cdot \hat{x}\sin\omega t \) instead of to \( \hat{x}\sin\omega t \). The above observations reveal that LT actually is sensitive to the difference between \( x(t) \) and \( u(t) \cdot x(t) \), i.e., if there is \( x(t) \neq u(t) \cdot x(t) \).

From a more general point of view, it should be appreciated that any “causal” physical system which is governed by a linear DE of finite order has a “memory” which becomes manifest in the length of the system’s impulse response. This notion suffices to explain that the evoked response \( y_e(t) \) to an excitation function \( x(t) \) in general depends on the behavior of \( x(t) \) for \( -\infty < t < \infty \). When the response is determined by LT, this fact is automatically accounted for, namely, by the implicit bilaterality of (1); cf. Sects. 2.1, 2.4, A.2.

The bilaterality of \( \varphi(t) \) is indispensable for the evoked response to be correct, i.e., whether or not \( \varphi(t) \) is causal. L-transformation by (1) just provides for \( \varphi(t) \) being always causal. As a consequence, the evoked response complies with \( x(t) \) only if \( x(t) \) was a priori defined to be causal.

### A.2 Analysis interval versus integration interval

In Sect. 2.1 it is pointed out that the t-domain interval which actually is encompassed both by Fourier- and Laplace-transformation invariably is infinite, i.e., \( -\infty < t < +\infty \). This interval is termed the respective transformation’s analysis interval; it has to be distinguished from the integration interval, i.e., the interval of \( t \) which is determined by the transformation-integral’s limits, e.g. in Eqs. (1) and (8). The relationship between these two kinds of interval can be made apparent by regressing to discrete Fourier- and Laplace-transformation. The discrete transformations can be deduced from the Fourier-series representation of a real function \( f(t) \), i.e.,

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t; \quad n \text{ integer; } \quad (97)
\]

where

\[
a_0 = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) \, dt; \quad (98)
\]
\[ a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \cos \omega_n t \, dt; \quad (99) \]
\[ b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \sin \omega_n t \, dt; \quad (100) \]
\[ \omega_n = \frac{2\pi n}{T}. \quad (101) \]

The function \( f(t) \) such represented is periodic with the period length \( T = 2\pi/\omega_1 \).

This set of formulas can be regarded and employed as a complementary pair of transformations \[11]. By Eqs. (98-100) a particular section of \( f(t) \) becomes transformed into an infinite set of real coefficients \( \{a_n, b_n\} \) each of which pertains to a discrete frequency \( \omega_n \). The section of \( f(t) \) that extends from \(-T/2\) to \(+T/2\) plays the role of an analysis interval, \( T \). The function \( f(t) \) needs to be defined for \(-T/2 \leq t \leq +T/2\). Inverse transformation is achieved by (97). In general, the inverse transform matches the original function \( f(t) \) only for \(-T/2 < t < +T/2\). By the notation chosen in Eqs. (98-100) the center of the analysis interval becomes implicitly denoted \( t = 0 \).

To elucidate the implications of this approach it is helpful to express the above formulas in complex notation. Using Euler’s formula, one obtains from Eqs. (99, 100)
\[ a_n \pm ib_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(t)e^{\pm i\omega_n t} \, dt; \quad n = 1, 2, \ldots \quad (102) \]

By definition of the discrete Fourier transform
\[ F(\omega_n, T) = \frac{T}{2}(a_n - ib_n) \quad (103) \]

one obtains from (102)
\[ F(\omega_n, T) = \int_{-T/2}^{+T/2} f(t)e^{-i\omega_n t} \, dt. \quad (104) \]

Equation (97) can be expressed in the form
\[ f(t) = a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n)e^{i\omega_n t} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n)e^{-i\omega_n t}. \quad (105) \]

Observing that by (98, 104) there holds \( a_0 = F(0, T)/T \), one obtains from (105) and (103) for \( n = \ldots, -1, 0, +1, \ldots \) the formula for inverse transformation
\[ \varphi(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} F(\omega_n, T) \cdot e^{i\omega_n t} = f(t) \quad \text{for} \quad -T/2 < t < +T/2. \quad (106) \]

The transformation (104) is governed by, and confined to, the analysis interval \( T \). This interval can either be regarded as preset or as being determined by the choice of \( \omega_1 \) and thus of the spacing of the discrete frequencies \( \omega_n \), cf. (101). As the center of the analysis
interval is for convenience denoted \( t = 0 \) there is only one way in which the limits of the integral \((104)\) may become different from \( \pm T/2 \), namely, by \( f(t) \) being null within a sub-interval of \( T \) that borders \( t = -T/2 \) and/or \( t = +T/2 \). Vice versa, when one or both of the integral’s limits are different from \( \pm T/2 \) this invariably indicates that the function \( f(t) \) contains a null interval. In particular, when \( f(t) \) is replaced with the causal function \( f_c(t) = u(t)f(t) \) the discrete Fourier transform gets expressed by

\[
F(\omega_n, T) = \int_{-T/2}^{T/2} u(t)f(t)e^{-i\omega nt} dt = \int_0^{T/2} f(t)e^{-i\omega nt} dt = \int_0^{T/2} f_c(t)e^{-i\omega nt} dt. \quad (107)
\]

The difference from \(-T/2\) of the integral’s low limit neither affects the analysis interval \( T \) nor does it make the transformation unilateral; it just indicates that \( f(t) \) is causal.

The discrete variant of Laplace transformation emerges from \((104)\) by including in the integral the factor \( \exp(-\sigma t) \) and by setting the integral’s low limit to \( t = 0 \). This yields

\[
L_T \{ f(t) \} = \int_0^{T/2} f(t)e^{-st} dt, \quad (108)
\]

where \( L_T \) denotes discrete L-transformation, and

\[
s_n = \sigma + i\omega_n. \quad (109)
\]

Regarding the relationship between analysis interval and integration interval, the same applies as was just pointed out above: The integral’s \((108)\) low limit \( t = 0 \) indicates that the function actually transformed is causal while the low limit marks the center of the analysis interval.

The inverse discrete L-transform \( \varphi(t) \) is determined by the expression

\[
\varphi(t) = L_T^{-1} \{ L_T \{ f(t) \} \} = \frac{1}{T} \sum_{n=-\infty}^{+\infty} L_T \{ f(t) \} \cdot e^{s_nt}; \quad -T/2 < t < +T/2. \quad (110)
\]

When \((108)\) is employed for transformation of a non-causal function \( f(t) \), the inverse transform \( \varphi(t) \) still is defined for \(-T/2 < t < +T/2\); however, validity of \( \varphi(t) = f(t) \) is confined to the interval \( 0 < t < T/2 \). In this sense, the transformation is for non-causal functions inconsistent. For causal functions \( f(t) = f_c(t) \) the transformation is consistent, as there holds \( \varphi(t) = f_c(t) \) for \(-T/2 < t < +T/2\).

Laplace transformation as defined by \((1, 2)\) emerges from \((108)\) \((110)\) by letting \( T \to \infty \), implying that the spacing of analysis frequencies becomes infinitesimally small such that the sum \((110)\) becomes an integral, i.e., \((2)\). The transformation integral’s \((1)\) low limit, i.e., \( t = 0 \), marks the center of the infinite analysis interval.

### A.3 Redundancy of multiplication by \( u(t) \)

When the unit step function is defined without inclusion of the connect function \( u_0(t) \), i.e., \( u(t) = 0 \) for \( t < 0 \); \( u(t) = 1 \) for \( t \geq 0 \), there evidently holds \( u^n(t) = u(t) \) \( (n = 1, 2, \ldots) \).
Hence, multiplication of $u(t)$ by itself is redundant. As $u(t)$ actually includes $u_0(t)$ it must be verified that this kind of redundancy also holds when $u(t)$ is defined according to (14).

For $u^n(t) = u(t)$ to hold it is necessary and sufficient that $u^n_0(t) = u_0(t)$. As was outlined in Sect. 2.3 the unit connect function $u_0(t)$ is sufficiently characterized by saying that it is an infinite set of real numbers $\{0 \ldots 1\}$ that exists at $t = 0$ and fills the gap which otherwise exists between $u(-0) = 0$ and $u(+0) = 1$. From this definition it follows that any integer power of $u_0(t)$ is characterized by precisely the same criterion: $u^n_0(t)$ for any $n = 2, 3, \ldots$ is also an infinite set of real numbers $\{0 \ldots 1\}$. Thus, any power of $u_0(t)$ can be renamed $u_0(t)$. Therefore, there actually holds the identity

$$u^n(t) = u(t); \quad n = 1, 2, \ldots \tag{111}$$

Another important question is whether multiplication by $u(t)$ of a derivative of $u(t)$, i.e., of an impulse function, is also redundant such that there holds

$$u(t)u^{(n)}(t) = u(t)\delta^{(n-1)}(t) = u^{(n)}(t) = \delta^{(n-1)}(t); \quad n = 1, 2, \ldots \tag{112}$$

The identity (112) turns out to be an inevitable consequence of the fact that the impulse functions are LT-consistent (Sect. 2.2, Eq. (12)), in combination with the causality theorem (30). When in (30) one lets $f(t) = \delta^{(n)}(t)$ one obtains from (12) and (30)

$$L^{-1}\{L\{\delta^{(n)}(t)\}\} = \delta^{(n)}(t) = u(t)\delta^{(n)}(t). \tag{113}$$

Thus, the identity (112) actually is implied in LT’s fundamental definitions and features. The identity can be made plausible by characterizing $\delta^{(n)}(t)$ as a set of pseudo-functions the type of which depends on $n$ and which only at $t = 0$ are different from 0. Then it may be concluded that multiplication of $\delta^{(n)}(t)$ by $u_0(t)$ does not alter the type of function denoted $\delta^{(n)}(t)$ such that, indeed, $u(t)\delta^{(n)}(t)$ can be renamed $\delta^{(n)}(t)$. If more mathematical rigour is desired, the theory of distributions may be invoked.

In Sect. 2.4 the identities (111) and (112) are subsumed by the unit-step redundancy theorem (24).

### A.4 The dud-theorem

The dud-theorem depicts the $n$-th derivative of the ud-function $f_{ud}(t)$ such as defined by (46), as follows.

With regard to the definition (14) of the unit step function the first derivative can be expressed by

$$f'_{ud}(t) = [u(t)f_d(t)]' = 0 \quad \text{for } t < 0, \quad \text{for } t = 0, \quad \text{for } t > 0. \tag{114}$$
To obtain the derivatives of higher order, it is helpful that (114) can be converted into a more convenient form, as follows. Utilizing the definition (14) of \( u(t) \), one can replace the third line on the right side of (114) with the expression

\[
u(t)f_d'(t) - f_d'(0)u_0(t) \quad \text{for } -\infty < t.
\]

This converts (114) into a superposition of causal functions which are known to be LT-consistent. As these terms are LT-consistent they can be replaced with the pertinent inverse L-transforms without affecting the validity of \( f_{ud}'(t) \). This has the effect that the term \( f_d'(0)u_0(t) \) gets eliminated from (115). As a result, \( f_{ud}'(t) \) can be expressed in the form

\[
f_{ud}'(t) = u(t)f_d'(t) + f_d(0)u_0'(t) \quad \text{for } -\infty < t.
\]

By comparison of (116) to (46) the scheme becomes apparent according to which the second derivative emerges from the first, the third from the second, and so on. For instance, when (116) is derived to obtain the second derivative of \( f_{ud}(t) \), the term \( u(t)f_d'(t) \) gets converted into \( u(t)f_d''(t) + f_d'(0)u_0'(t) \); and the second term \( f_d(0)u_0'(t) \) gets converted into \( f_d(0)u_0''(t) \). By this scheme one eventually obtains for the \( n \)-th derivative the expression

\[
f_{ud}^{(n)}(t) = u(t)f_d^{(n)}(t) + f_d^{(n-1)}(0)u_0'(t) + f_d^{(n-2)}(0)u_0''(t) + \ldots + f_d(0)u_0^{(n)}(t); \quad n = 1, 2, \ldots
\]

Equation (117) is by (47) expressed as the dud-theorem.

### A.5 Derivatives of order 1/2

From the derivatives of non-integer order of causal functions those of order 1/2 are of particular interest \[10\]. Below, a number of derivatives are listed which were determined by the gdi-theorem \[70\], using the table of LT-correspondences Sect. A.7.

\[
D^{1/2}\{u(t)\} = L^{-1}\{\sqrt{s} \cdot \frac{1}{s}\} = u(t) \cdot \frac{1}{\sqrt{\pi t}}
\]

\[
D^{1/2}\{\frac{u(t)}{\sqrt{t}}\} = L^{-1}\{\sqrt{s} \cdot \sqrt{\pi/s}\} = \delta(t) \cdot \sqrt{\pi}
\]

\[
D^{1/2}\{u(t) \cdot \sqrt{t}\} = L^{-1}\{\sqrt{s} \cdot \sqrt{\pi/2s}\} = u(t) \cdot \frac{\sqrt{\pi}}{2}
\]

\[
D^{1/2}\{u(t) \cdot e^t\} = L^{-1}\{\sqrt{s} \cdot \frac{1}{s-1}\} = u(t) \cdot [1/\sqrt{\pi t} + e^t \cdot \text{erf}(\sqrt{t})]
\]

\[
D^{1/2}\{u(t) \cdot \ln t\} = L^{-1}\{\sqrt{s} \cdot \frac{-\ln s - C_E}{s}\} = u(t) \cdot \frac{\ln(4t)}{\sqrt{\pi t}}
\]

\[
D^{1/2}\{\frac{u(t)}{\sqrt{t}} \cdot e^{-a^2/(4t)}\} = L^{-1}\{\sqrt{\pi} \cdot e^{-a\sqrt{s}}\} = u(t) \cdot \frac{a}{2} \cdot t^{-3/2} \cdot e^{-a^2/(4t)}
\]
\[
D^{1/2}\{u(t) \cdot J_0(2\sqrt{at})\} = L^{-1}\left\{\frac{1}{\sqrt{s}} \cdot e^{-a/s}\right\} = u(t) \cdot \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}
\]  

(124)

A.6 Linear DE that includes a derivative of the excitation function

When the linear first-order DE (91) is modified into the form

\[ay(t) + y'(t) = x'(t),\]  

(125)

the response \(y(t)\) denotes the electrical current through the capacitor of the electrical circuit shown in Fig. 3, while \(x(t)\) denotes the current exerted on the circuit by the source Q. The constant \(a\) equals \(a = 1/(RC)\).

![Fig. 3. Electrical circuit which is accounted for by Eq. (125). \(x(t)\): electrical current from source Q; \(\eta(t)\): voltage at condenser C; \(y(t)\): current through C](image3)

The DE (125) is of the type which by Doetsch was banished from his theory because it includes a derivative of the excitation function. In terms of (71) this DE corresponds to \(N = 1, M = 1\).

The new method, i.e., the formula (86), provides for an algorithmic, straightforward solution. Assuming, as an example, \(x(t) = \hat{x}u(t)\), and taking into account the parameters \(N = M = 1; a_0 = a; a_1 = 1; b_0 = 0; b_1 = 1\), one obtains

\[
y(t) = L^{-1}\left\{\frac{\hat{x}}{s + a}\right\} + L^{-1}\left\{\frac{y_s(0)}{s + a}\right\} = [\hat{x}u(t) + y_s(0)] \cdot e^{-at} \text{ for } -\infty < t.
\]

(126)

Any particular initial state can be freely accounted for by setting \(y_s(0)\) accordingly. The two components of the total solution are illustrated in Fig. 4.

![Fig. 4. The two components of the solution of (125). Left: evoked response for \(x(t) = \hat{x}u(t)\). Right: spontaneous response. This solution is depicted by Eq. (126)](image4)
A.7 LT-correspondences

Below, LT-correspondences are listed that are used in the present article. The $t$-domain functions are identical to those included in customary tables of TLT, except for the factor $u(t)$, which is required to make the correspondences valid in both directions. Notice that the first correspondence (127) holds only in one direction.

\begin{align*}
1 & \Rightarrow 1/s \\
u(t) & \Leftrightarrow 1/s \\
\delta^{(n)}(t) = u^{(n+1)}(t) = u^{(0+1)}(t) & \Leftrightarrow s^n \quad (n = 0, 1, \ldots) \\
u(t) \cdot t^n & \Leftrightarrow \frac{n!}{s^{n+1}} \quad (n = 0, 1, \ldots) \\
u(t) \cdot e^{-at} & \Leftrightarrow \frac{1}{s + a} \\
u(t) \left(1 - e^{-at}\right) & \Leftrightarrow \frac{1}{s(s + a)} \\
u(t) \cdot \sin \omega t & \Leftrightarrow \frac{\omega}{s^2 + \omega^2} \\
u(t) \cdot \cos \omega t & \Leftrightarrow \frac{s}{s^2 + \omega^2} \\
u(t) \left(\omega e^{-at} + a \sin \omega t - \omega \cos \omega t\right) & \Leftrightarrow \frac{\omega}{(s + a)(s^2 + \omega^2)} \\
u(t) \cdot \sqrt{t} & \Leftrightarrow \sqrt{\frac{\pi}{s}} \\
u(t) \cdot \sqrt{t} & \Leftrightarrow \frac{\sqrt{\pi/s}}{2s} \\
u(t) \cdot e^t \cdot \text{erf}(\sqrt{t}) & \Leftrightarrow \frac{1}{(s - 1)\sqrt{s}} \\
u(t) \cdot \ln t & \Leftrightarrow -\frac{\ln s - C_E}{s} \quad (C_E = 0.577215 \ldots) \\
u(t) \cdot \frac{\ln t}{\sqrt{t}} & \Leftrightarrow -\sqrt{\frac{\pi}{s}} \cdot (\ln 4s + C_E) \\
u(t) \cdot t^r & \Leftrightarrow \frac{\Gamma(r + 1)}{s^{r+1}} \quad (r \in \mathbb{R}; \ r > -1) \\
u(t) \cdot \frac{e^{-a^2/(4t)}}{\sqrt{\pi t}} & \Leftrightarrow \frac{1}{\sqrt{s}} \cdot e^{-a\sqrt{s}} \\
u(t) \cdot a \cdot t^{-3/2} \cdot e^{-a^2/(4t)} & \Leftrightarrow 2\sqrt{\pi} \cdot e^{-a\sqrt{\pi}}
\end{align*}
\[
\frac{u(t)}{\sqrt{\pi t}} \cdot \cos 2\sqrt{at} \leftrightarrow \frac{1}{\sqrt{s}} \cdot e^{-a/s}
\] (144)

\[
u(t) \cdot J_0(2\sqrt{at}) \leftrightarrow \frac{1}{s} \cdot e^{-a/s}
\] (145)

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