IIA String Instanton Corrections
to the Four-Fermion Correlator in
the Intersection of Del Pezzo Surfaces

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Abstract

The Becker-Becker-Strominger formula, describing the string world-sheet instanton corrections to the four-fermion correlator in the Calabi-Yau compactified type-IIA superstrings, is calculated in the special case of the Calabi-Yau threefold realized in the intersection of two Del Pezzo surfaces. We also derive the selection rules in the supersymmetric GUT of the Pati-Salam type associated with our construction.

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1 Introduction

One of the central problems in modern string and field theories is a calculation of strong-coupling effects. A calculation of the instanton corrections to various physical quantities is the important part of this problem. A study of the non-perturbative corrections in string theory due to the M-Theory branes was pioneered by Becker, Becker and Strominger [1]. The simplest instantons are the so-called string world-sheet instantons whose contributions are independent upon the string coupling. The string world-sheet instantons were extensively studied in the past [2] even before ref. [1]. In the context of the IIA superstring compactification, the existence of the world-sheet string instantons can be related to the holomorphic curves in the internal Calabi-Yau (CY) space [4].

In the context of eleven-dimensional M-Theory [3], the ten-dimensional IIA superstring theory arises from the M-Theory compactification on a circle $S^1$, whereas the IIA superstrings themselves can be understood as the double compactified (in spacetime as well as in worldvolume) M2-branes [4].

The M2-branes can be wrapped about the $S^1$ and a CY (supersymmetric) 2-cycle $C_2$. They give rise to instantons in four (uncompactified) spacetime dimensions, whose effects can be computed by the standard methods of quantum field theory [3]. The low-energy effective four-dimensional field theory of the CY compactified type-IIA superstrings is given by the N=2 supergravity interacting with $h_2,1$ hypermultiplets and $h_{1,1}$ vector N=2 multiplets, where $h_{2,1}$ and $h_{1,1}$ are the Hodge numbers of CY [4]. The moduli space $\mathcal{M}$ of the compactified theory is given by a direct product of the hypermultiplet moduli space $\mathcal{M}_H$ and the N=2 vector multiplet moduli space $\mathcal{M}_V$, while the $S^1\times C_2$-wrapped M2-branes correct the geometry of $\mathcal{M}_V$ only. The BPS (or the supersymmetric map) condition on these wrapped M2-brane configurations just amounts to the holomorphy condition on the world-sheet instantons [2]. The same conclusion was rederived in ref. [1] by requiring the equivalence between a global supersymmetry transformation and a kappa-transformation of the Green-Schwarz supersring action,

$$\partial X^m = 0, \quad \text{or} \quad \bar{\partial} X^m = 0,$$

where $\partial$ is the holomorphic string world-sheet exterior derivative, and $X^m$ are the complex coordinates in CY, $m = 1, 2, 3$.

The topological equation formally describing the string world-sheet instanton corrections to the four-point fermion (gaugino) correlator $\mathcal{F}_{IJKL}$, where $I, J, K, L =$
1, 2, \ldots , h_{1,1}, was obtained by Becker, Becker and Strominger [1],

\[ \Delta_{C_2} F_{IJKL} = N e^{-\int_{C_2} J - i \int_{C_2} B} \int_{C_2} b_I \int_{C_2} b_J \int_{C_2} b_K \int_{C_2} b_L , \]  

(1.2)

where \( C_2 \) is the homology class of the instanton, \( \{ b_I \} \) is the orthonormal basis of harmonic \((1,1)\) forms in CY, \( J \) is the Kähler \((1,1)\) form of CY, \( B \) is the (closed) Neveu-Schwarz \((1,1)\) form, and \( N \) is the normalization factor independent upon \( b_I \).

Like any other \((1,1)\) form, the form \( J + iB \) can be decomposed with respect to the cohomology basis \( \{ b_I \} \),

\[ J + iB = \sum_{I=1}^{h_{1,1}} z^I b_I , \]  

(1.3)

where the complex coefficients \( \{ z^I \} \) are called CY moduli. Integrating eq. (1.3) once with respect to the modulus \( z^I \) yields the famous topological formula for the world-sheet instanton corrections to the Yukawa couplings \( F_{IJK} \) [6]. In the case of Yukawa couplings, mirror symmetry is known to confirm the topological equation on them [7]. This fact indirectly supports the more general equation (1.2) also [1].

Like the similar equation on the Yukawa couplings, the topological eq. (1.2) is merely a formal equation since one still has to specify how the integrals on the right-hand-side of the equation are to be calculated. Their calculation for a generic CY space represents the important technical problem whose solution is unknown, to the best of our knowledge. There are, nevertheless, some explicit calculations of the Yukawa couplings in the literature for the special CY spaces realized as the complete intersections in a product of the projective spaces [7].

Our main purpose in this paper is to calculate eq. (1.2) in the case of the special CY to be defined in the intersection of Del Pezzo surfaces. The Del Pezzo surface is a manifold of complex dimension 2 with a positive first Chern class [8]. We use the ‘old’ geometrical methods first developed in ref. [9] for computing the Yukawa couplings in a superstring model with three generations of quarks and leptons, see also ref. [10]. The geometrical approach is based on Poincaré duality and a knowledge of the homology group basis of CY.

Our paper is organized as follows: in sect. 2 we introduce into the special CY spaces realized as the complete intersections of five quadrics in a product of two projective spaces \( \mathbb{P}^4 \times \mathbb{P}^4 \). The main body of our paper is given by sect. 3 where we formulate the mathematical instruments allowing us to calculate the integrals in eq. (1.2). Some explicit examples are given in sect. 4. We conclude with sect. 5 where some connections between our work and the recent literature are outlined.
2 Quadrics, Del Pezzo and CY

Let’s consider the compact CY spaces realized as the complete intersections in a product of two projective spaces, $\mathcal{P}^4 \times \mathcal{P}^4$, with the configuration matrix:

$$
\begin{pmatrix}
4 & 2 & 2 & 0 & 0 & 1 \\
4 & 0 & 0 & 2 & 2 & 1
\end{pmatrix}
$$

To decipher this matrix, we introduce two sets of homogeneous coordinates, $x$ and $y$, in each $\mathcal{P}^4$, define two Del Pezzo surfaces $K_x$ and $K_y$, and a hypersurface $S$ in $\mathcal{P}^4 \times \mathcal{P}^4$ by the following constraints:

$$
K_x = \left\{ x \in \mathcal{P}^4 : P_1(x) = \sum_{i=0}^{4} x_i^2 = 0 , \quad P_2(x) = \sum_{i=0}^{4} a_i x_i^2 = 0 \right\},
$$

$$
K_y = \left\{ y \in \mathcal{P}^4 : P_3(y) = \sum_{i=0}^{4} y_i^2 = 0 , \quad P_4(x) = \sum_{i=0}^{4} b_i y_i^2 = 0 \right\},
$$

$$
S = \left\{ (x, y) \in \mathcal{P}^4 \times \mathcal{P}^4 : P_5(x, y) = \sum_{i,j=0}^{4} c_{ij} x_i y_j = 0 \right\}.
$$

The sum of entries in each line of the matrix (2.1) to the right of $\|$ exceeds exactly by one the dimension of the embedding space $\mathcal{P}^4$ to the left of $\|$, so that

$$
K_0 = (K_x \times K_y) \cap S
$$

appears to be Kähler and of the vanishing first Chern class, i.e. $K_0$ is a CY space, in agreement with the theorem of Greene, Vafa and Warner. We assume that the real coefficients of the quadrics $P_2$, $P_4$ and $P_5$ in eq. (2.2) are chosen to obey the transversality condition for all hypersurfaces in the definition (2.3) of $K_0$, i.e.

$$
dP_1 \wedge dP_2 \wedge dP_3 \wedge dP_4 \wedge dP_5 \neq 0.
$$

This equation guarantees the smoothness of the simply connected manifold $K_0$.

The first column in eq. (2.1) thus indicates that we consider a CY in the product $\mathcal{P}^4 \times \mathcal{P}^4$, whereas the other columns denote bi-powers of the polynomials of $x$ and $y$ in eq. (2.2).

The non-trivial Hodge numbers of $K_0$ are given by

$$
h_{2,1}(K_0) = 28 \quad \text{and} \quad h_{1,1}(K_0) = 12,
$$

\[3\] We use the standard notation [12].
so that its Euler characteristic \((K_0)\) is

\[
\frac{1}{2}(K_0) = h_{1,1} - h_{2,1} = -16 .
\] (2.6)

It is not easy to construct the complete intersection CY spaces with the physically interesting values of the Euler characteristic, \(\mp 6, \pm 8\). However, it is easily becomes possible through the so-called ‘orbifoldization’ process [11]. In our case, we can introduce the quotient \(K\) of the manifold \(K_0\) with respect to a discrete symmetry subgroup \(G_F = \mathbb{Z}_2^2\) of \(G\), which acts freely in \(K_0\). This yields the CY space \(K\) of the Euler characteristic \((K) = -8\). The \(G\)-group elements generating \(G_F\) can be chosen as follows:

\[
g_1 = \text{diag}(1, 1, 1, -1, -1) , \quad g_2 = \text{diag}(1, 1, -1, 1, -1) ,
\] (2.7)

so that the action of \(g_1\) reads

\[
g_1 : (x_0, x_1, x_2, x_3, x_4; y_0, y_1, y_2, y_3, y_4) \rightarrow (x_0, x_1, x_2, -x_3, -x_4; y_0, y_1, y_2, -y_3, -y_4) ,
\] (2.8)

and similarly for \(g_2\). The manifold \(K_0\) has the hidden discrete symmetry group \(G\) isomorphic to \(\mathbb{Z}_2^5\), whose action is given by

\[
\begin{align*}
\mathbb{Z}_2(A) : & \quad A = \text{diag}(-1, 1, 1, 1, 1) , \\
\mathbb{Z}_2(B) : & \quad B = \text{diag}(1, -1, 1, 1, 1) , \\
\mathbb{Z}_2(C) : & \quad C = \text{diag}(1, 1, -1, 1, 1) , \\
\mathbb{Z}_2(D) : & \quad D = \text{diag}(1, 1, 1, -1, 1) , \\
\mathbb{Z}_2(S) : & \quad S(x_i) = y_i , \quad S(y_j) = x_j .
\end{align*}
\] (2.9)

In the embedding space \(\mathcal{P}^4 \times \mathcal{P}^4\) we have

\[
g_1 = ABC \quad \text{and} \quad g_2 = ABD .
\] (2.10)

The CY manifold \(K_0\) is the simply connected covering space of the CY space \(K\). The latter still possesses some hidden symmetries that survive after its factorization by \(G_F\). These discrete symmetries are

\[
G_H = \frac{\mathbb{Z}_2(A) \times \mathbb{Z}_2(B) \times \mathbb{Z}_2(C) \times \mathbb{Z}_2(D) \times \mathbb{Z}_2(S)}{\mathbb{Z}_2(g_1) \times \mathbb{Z}_2(g_2)} = \mathbb{Z}_2(A) \times \mathbb{Z}_2(B) \times \mathbb{Z}_2(S) .
\] (2.11)

In the context of the IIA superstring compactification, the CY space \(K\) gives rise to the four-dimensional unified model with four generations of quarks and leptons, and an \(E_6\) gauge group. Further breaking of \(E_6\) by the standard mechanism of the vacuum Wilson loops yields the Pati-Salam-like unified model with a gauge group \(SU(4)_c \times SU(2)_L \times SU(2)_R \times U(1)\). The Yukawa couplings in this four-generation superstring model were calculated in ref. [12].
3 Instantons in Del Pezzo

Equation (1.1) implies that the CY-compactified IIA superstring world-sheet instantons are described by the isolated holomorphic curves in CY. A single instanton corresponds to a curve of genus zero. In the case of $K_0$, there are 256 holomorphic or $CP(1)$ curves. A derivation of this number was given, for example, in ref. [15] where it appeared as the leading term in the series expansion of the fundamental period as a solution to the Picard-Fuchs equation for the given CY. A geometrical derivation of the same result is given below in this section. However, first we need more information about the geometrical structure of the space $K_0$ defined by eq. (2.3) and the topology of the Del Pezzo surfaces $K_x$ and $K_y$.

As is well known in algebraic geometry [16], a smooth intersection of two quadrics in $\mathbb{P}^4$ is biholomorphic equivalent to the projective plane with five different blown-up points. Since the Hodge number $h_{1,1}$ of $\mathbb{P}^2$ is equal to one, after blowing up at five points $h_{1,1}$ equals to $1 + 5 = 6$, while the other Hodge numbers remain unchanged. Next, the Del Pezzo surface $K_x$ possesses exactly 16 complex lines $\{C_x\}$ that can be described by the relations

$$a_{42}a_{32}a_{10}x_2 - \varepsilon_1 a_{40}a_{30}a_{42}x_0 - i\varepsilon_2 a_{41}a_{31}a_{20}x_1 = 0,$$

$$a_{43}a_{32}a_{10}x_3 - \varepsilon_3 a_{40}a_{30}a_{20}x_0 - \varepsilon_4 a_{41}a_{30}a_{21}x_1 = 0,$$

$$a_{43}a_{42}a_{10}x_4 - \varepsilon_5 a_{41}a_{30}a_{20}x_0 - i\varepsilon_6 a_{40}a_{21}a_{31}x_1 = 0,$$  \(3.1\)

where $a_{kl} = \sqrt{a_k - a_l}$, $0 \leq l < k \leq 4$, and $\varepsilon_j = \pm 1$, $j = 1, 2, \ldots, 6$. The sign coefficients $\varepsilon_j$ and our notation for the complex lines on the Del Pezzo surfaces are collected in Table I.

The homology class of the Kähler form on Del Pezzo $K_x$ can be represented by the intersection of the hyperplane $S$ with $K_x$,

$$H = \{x_0 = 0\} \subset K_x.$$  \(3.2\)

Under the symmetry group $G$ the 16 lines on the Del Pezzo surface $K_x$ are naturally decomposed into three classes: (i) the five lines $E_i$, $i = 1, 2, 3, 4, 5$, that (pairwise) do not intersect with each other and thus represent five linearly independent homology classes of $H_2(K_x, \mathbb{R})$; together with the hyperflat section $H$ (dual to a Kähler form of the Del Pezzo surface $K_x$) they form a basis in $H_2(K_x, \mathbb{R})$, (ii) ten lines $F_{ij}$ that have intersections only with $E_i$ and $E_j$, and (iii) one line $G$ intersecting with all $E_i$ (see Table I too). The 256 holomorphic curves are then decomposed with respect
to the $G_F = \mathbb{Z}_2 \times \mathbb{Z}_2$ discrete symmetry group of order 4 into four classes that are cyclically symmetric with respect to their interchanging.

Table 1. Complex lines in the Del Pezzo surface $K_x (K_y)$ and the sign factors $\varepsilon_j$.

| line | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | $\varepsilon_5$ | $\varepsilon_6$ | line | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | $\varepsilon_5$ | $\varepsilon_6$ |
|------|----------------|----------------|----------------|----------------|----------------|----------------|------|----------------|----------------|----------------|----------------|----------------|----------------|
| $E_1$ | +   | +   | −   | +   | +   | +   | $E_14$ | −   | −   | −   | +   | +   | +   |
| $E_2$ | −   | +   | +   | +   | +   | −   | $E_15$ | +   | −   | −   | −   | +   | +   |
| $E_3$ | +   | +   | +   | −   | −   | −   | $E_{23}$ | −   | +   | −   | −   | +   | −   |
| $E_4$ | −   | −   | −   | +   | −   | −   | $E_{34}$ | −   | +   | −   | −   | −   | −   |
| $E_5$ | +   | −   | −   | −   | −   | +   | $E_{25}$ | −   | +   | −   | −   | +   | −   |
| $G$   | +   | +   | +   | +   | −   | −   | $E_{25}$ | −   | +   | −   | −   | −   | −   |
| $F_{12}$ | −   | +   | +   | +   | +   | +   | $F_{35}$ | +   | −   | +   | −   | +   | −   |
| $F_{13}$ | +   | +   | +   | +   | +   | +   | $F_{45}$ | −   | −   | −   | −   | −   | −   |

Accordingly, we get the following matrix of the intersection indices:

$$ (E_i, E_j) = -\delta_{ij} , \quad (E_i, F_{jk}) = \delta_{ij} + \delta_{ik} , \quad (E_i, G) = 1 , \quad (E_i, H) = 1 , \quad (H, H) = 4 . $$ (3.3)

Having obtained the holomorphic curves and the homology basis explicitly, it is not difficult to determine the action of the discrete symmetry group on the latter. The generating elements $(g_1, g_2, A, B)$ of the group $G$ act as follows:

$$ g_1(E_1, E_2, E_3, E_4, E_5, H) = (E_3, F_{45}, E_1, F_{25}, F_{24}, H) , $$
$$ g_2(E_1, E_2, E_3, E_4, E_5, H) = (E_4, F_{35}, F_{25}, E_1, F_{23}, H) , $$
$$ A(E_1, E_2, E_3, E_4, E_5, H) = (F_{12}, G, F_{23}, F_{24}, F_{25}, H) , $$
$$ B(E_1, E_2, E_3, E_4, E_5, H) = (F_{15}, F_{25}, F_{35}, F_{45}, G, H) . $$ (3.4)

For example, to get $g_1(E_2) = F_{45}$, we choose for definiteness $a_0 = 0, a_1 = 1, a_3 = 3$ and $a_4 = 4$ in eq. (3.1) where the coefficients $\varepsilon_j$ are given by Table 1. We find

$$ g_1(E_2) = g_1 \left\{ x_2 + \sqrt{6} x_0 - i \sqrt{6} = x_3 - i 4 x_0 - 3 x_1 = x_4 - 3 x_0 + 2 x_1 = 0 \right\} $$
$$ = \left\{ x_2 + \sqrt{6} x_0 - i \sqrt{6} x_1 = x_3 + i 4 x_0 + 3 x_1 = x_4 + 3 x_0 - i 2 x_1 = 0 \right\} = F_{45} . $$ (3.5)

The curve $H$ is invariant under all these symmetries, whereas each line $E_i$ goes into one of the 16 lines lying in the intersection of quadrics. The symmetry transformations act independently on each factor $\mathcal{P}^4$ in a product $\mathcal{P}^4 \times \mathcal{P}^4$, so that it is enough to
consider only one projective space $\mathcal{P}^4$. The action of the $S$ symmetry of $G$ just replaces each $(1, 1)$ form on Del Pezzo $K_x$ by the corresponding $(1, 1)$ form on $K_y$. We find the following decompositions:

$$F_{ij} = \frac{1}{3} \left( \sum_{i=1}^{5} E_i + H \right) - E_i - E_j , \quad (3.6)$$

and

$$G = \frac{1}{3} \left( 2H - \sum_{i=1}^{5} E_i \right) . \quad (3.7)$$

For example, to prove eq. (3.6), we begin with a decomposition

$$F_{ij} = \sum_{i=1}^{5} c_i E_i + c_6 H \quad (3.8)$$

whose coefficients $(c_i, c_6)$ are to be determined. Let’s now consider the intersections of $F_{ij}$ with $H$, $E_i$, $E_j$ and $M = \sum_{i=1}^{5} E_i$ by using the index intersection matrix (3.3). We find

$$(F_{ij}, H) = c_1 + c_2 + c_3 + c_4 + c_5 + 4c_6 = 1 ,$$

$$(F_{ij}, E_i) = -c_i + c_6 = 1 ,$$

$$(F_{ij}, E_j) = -c_j + c_6 = 1 ,$$

$$(F_{ij}, \sum_{i=1}^{5} E_i) = -(c_1 + c_2 + c_3 + c_4 + c_5) + 5c_6 = 2 . \quad (3.9)$$

Hence, the coefficients in eq. (3.8) are given by

$$c_6 = 1/3 \quad \text{and} \quad c + i = c_j = -2/3 . \quad (3.10)$$

Equation (3.7) is obtained similarly.

In the Grand Unification Theories of the Pati-Salam type, based on the gauge group $E_6$ that is supposed to be broken by Wilson lines as

$$E_6 \rightarrow SU(4)_{c} \times SU(2)_{L} \times SU(2)_{R} \times U(1) , \quad (3.11)$$

the representation $27$ of $E_6$ is decomposed as follows:

$$27 = [(q, l) = (4c, 2L, 1R)] + [g^c, l^c = (4c, 1L, 2R)]$$

$$+ [H = (1c, 2L, 2R)] + [g, g^c = (6c, 1L, 1R)] + [n = (1c, 1L, 1R)] , \quad (3.12)$$

where $(q_{R,L}, l_{R,L})$ stand for the quark-lepton families, $H$ are the new leptons, $g$ are the new quarks and $n$ is the singlet.
As was demonstrated in ref. [14], the particle spectrum corresponding to the \((2, 1)\) forms in \(K\) is given by

\[
h_{2,1} : \quad 10(n, g, g^c) + 6(f, H),
\]

(3.13)

where \(f\) stands for \((q, l, q^c, l^c)\). The anti-particles corresponding to the \((1, 1)\) forms in \(K\) are given by

\[
h_{1,1} : \quad 6(\bar{n}, \bar{g}, \bar{g}^c) + 2(\bar{f}, H).
\]

(3.14)

The transformation properties of the fields, in accordance with the decomposition (3.12), are collected in Table 2.

Equations (3.4) also allow us to identify the special combinations of the basic \((1, 1)\) homology elements that are invariant under the CY symmetry group \(G_H\) of eq. (2.11). They are

\[
F_{x,y}^i = H_{x,y} + 6E_{x,y} - 2 \sum_{i=1}^{5} E_{x,y}^i,
\]

\[
H^\pm = H^x \pm H^y, \quad F_i^\pm = F_i^x \pm F_i^y,
\]

(3.15)

where \(i = 1, 2, 3, 4, 5\).

In the context of the CY superstring compactification, the invariant elements of the \((1, 1)\) cohomology basis correspond to the physical matter fields transforming in \(27\) of \(E_6\). A direct calculation yields the following set of twelve invariant combinations in the given homology basis of \(H_2(K^x, \mathbf{R})\) dual to the cohomology group \(H^{1,1}\):

\[
(\bar{n}, \bar{g}, \bar{g}^c)_1 = H^x + H^y,
\]

\[
(\bar{n}, \bar{g}, \bar{g}^c)_2 = F_2^+, \quad (\bar{n}, \bar{g}, \bar{g}^c)_3 = F_5^+,
\]

\[
(\bar{n}, \bar{g}, \bar{g}^c)_4 = H^x - H^y \equiv H^-,
\]

\[
(\bar{n}, \bar{g}, \bar{g}^c)_5 = F_2^-, \quad (\bar{n}, \bar{g}, \bar{g}^c)_6 = F_5^-,
\]

(3.16)

\[
(\bar{g}, \bar{l})_1 = F_3^+ , \quad (\bar{g}, \bar{l})_2 = F_3^- ,
\]

\[
(\bar{g}^c, \bar{l}^c)_1 = F_4^+ , \quad (\bar{g}^c, \bar{l}^c)_2 = F_4^- ,
\]

\[
\bar{H}_1 = F_1^+ , \quad \bar{H}_2 = F_1^-,
\]

where the quark-lepton families \((q, l)\) and extra leptons \((H)\) are merely considered here as the formal notation. There are no two different combinations of the cycles that would have the same transformation properties under the discrete symmetries.

9
We verified this statement by a straightforward calculation (see Table 2). This means that our identification of cycles is unique.

Table 2. The transformation properties of the $(1,1)$ forms in $\overline{E}_6$ of $E_6$ under the discrete symmetries.

| fields       | $g_1$ | $g_2$ | $A$ | $B$ | $S$ |
|--------------|-------|-------|-----|-----|-----|
| $(\bar{n}, \bar{g}, \bar{g}^c)_1$ | 1     | 1     | 1   | 1   | 1   |
| $(\bar{n}, \bar{g}, \bar{g}^c)_2$ | 1     | 1     | 1   | -1  | 1   |
| $(\bar{n}, \bar{g}, \bar{g}^c)_3$ | 1     | 1     | -1  | 1   | 1   |
| $(\bar{n}, \bar{g}, \bar{g}^c)_4$ | 1     | 1     | -1  | 1   | -1  |
| $(\bar{n}, \bar{g}, \bar{g})_5$   | 1     | 1     | 1   | -1  | -1  |
| $(\bar{n}, \bar{g}, \bar{g}^c)_6$ | 1     | 1     | -1  | 1   | -1  |
| $(\bar{g}, \bar{l})_1$            | -1    | 1     | -1  | -1  | 1   |
| $(\bar{g}, \bar{l})_2$            | -1    | 1     | -1  | -1  | -1  |
| $(\bar{g}, \bar{l}, \bar{g}^c)_1$ | 1     | -1    | -1  | -1  | 1   |
| $(\bar{g}, \bar{l})_2$            | 1     | -1    | -1  | -1  | -1  |
| $\bar{H}_1$                       | -1    | -1    | -1  | -1  | 1   |
| $\bar{H}_2$                       | -1    | -1    | -1  | -1  | -1  |

The instantons in the Del Pezzo intersection have the form $C_x \times C_y$ that yields $16 \times 16 = 256$, as it should. The intersection of these 256 surfaces with the hyperplane $S$ in eq. (2.3) yields 128 complex curves of genus zero on one of the Del Pezzo surfaces $\times$ point on the other Del Pezzo surface. Accordingly, there are two ways of choosing on which Del Pezzo surface we take the line to lie on, while there are four ways of choosing a point on the other Del Pezzo surface. This yields in total $2 \times 4 \times 16 = 128$ different instantons of the type $line \times point$, and, in addition, 128 different instantons of the type $line \times line$. Unlike ref. [1], where a similar problem was solved in the case of the cubic Del Pezzo intersection in $\mathcal{P}^3 \times \mathcal{P}^3$, we have a more degenerate (and more symmetric) situation.

Each holomorphic curve corresponding to an instanton is thus given by an intersection of $C_x \times C_y$ with the hyperplane $S$ in accordance with eq. (2.3), where $C_x$ are 16 lines in the Del Pezzo surface $K_x$ and similarly for $K_y$,

$$\mathcal{L} = (C_x \times C_y) \cap S.$$  \hspace{1cm} (3.17)

There are four classes amongst the 256 instantons that are cyclically connected in our case. To calculate eq. (1.2) we have to choose a representative $\mathcal{L}$ from each class. The
holomorphic curve $L$ is the image of the string world-sheet in the CY space under the instanton map.

Now, on the one hand side, the integral of any closed form $\varnothing$ of the maximal degree over $K_0 = (K_x \times K_y) \cap S$ can be represented by the value of the cohomology class of $\varnothing$ on the cycle $K_0$. On the other hand side, the cycle intersection in the homologies is dual to the exterior multiplication in the cohomologies. Hence, we have

$$\int_{K_0} \varnothing = (w \wedge H)[K_x \times K_y],$$

where we have introduced the class of cohomologies $w$ of $\varnothing$, and the image $H$ of the cohomology class of the Kähler form in $\mathcal{P}^{24}$ dual to the hyperplane $S$. The hyperplane $S$ represents the hypersurface $S$ after Segre embedding [14] restricted on $K_x \times K_y$. The Segre embedding in this case means the embedding of a product $\mathcal{P}^4 \times \mathcal{P}^4$ into the projective space $\mathcal{P}^{24}$ by the coordinate identification $w_{ij} = x_iy_j$ where $w_{ij}$ are the homogeneous coordinates of $\mathcal{P}^{24}$.

It is worth mentioning that only one component $H^x(H^y)$ remains in what follows from $H = H^x + H^y$. Hence, our problem reduces to a calculation of the intersection indices from the homology group $H_2(K_x, \mathbb{Z})$ only. In general, the Poincaré duality establishes the isomorphism between the closed (DeRham) homologies and compact-dual cohomology classes, as well as the isomorphism between compact (DeRham) homologies and cohomologies [15]. In the compact CY case we consider, there is no difference between the closed and compact classes.

Taken together, this allows us to replace the integral of the $(1, 1)$ form $b_I$ along the curve $L$ in eq. (1.2) by the intersection index of this curve $L$ with the cycle $F_I$ that is Poincaré dual to $b_I$,

$$\int_{L} b_I = (F_I \cdot L).$$

(3.19)

As a result, the instanton correction to the four-fermion correlator in eq. (1.2) is proportional to a product

$$\int_{L} b_I \int_{L} b_J \int_{L} b_K \int_{L} b_L = (F_I \cdot L)(F_J \cdot L)(F_K \cdot L)(F_L \cdot L),$$

(3.20)

where the brackets with dots stand for the intersection indices of the corresponding cycles, which are to be determined from the matrix (3.3) of our basic curves ($E^x_i, H^x, E^y_i, H^y$) — see the next sect. 4 for some explicit examples.

We label our 256 curves by the corresponding lines in one of the following form: line $\times$ line, point $\times$ point and line $\times$ line in the cover $K_0$. The validity of the
cycle intersection matrix for both Del Pezzo surfaces $K_x$ and $K_y$ is justified by the fundamental commutative diagram:

$$
\begin{array}{ccc}
K_x & \xrightarrow{p_x} & K_x \times K_y \\
\downarrow{p_y} & & \downarrow{p_y} \\
K_y & \xleftarrow{p_y} & K_x \\
\end{array}
$$

In a more explicit notation we just get

$$
\int_{\mathcal{L}} b^x_I \equiv \int_{\mathcal{L}} p_x^* b^x_I = \int_{\mathcal{L}} (p_x \circ i)^* b^x_I = \int_{(p_x \circ i)_\ast \mathcal{L}} b^x_I x \equiv \int_{\mathcal{L}_x} b^x_I = (F_x, \mathcal{L}_x) ,
$$

where the projection $p_x$ lifts the form $b_I$ on $K_x$ to $K_x \times K_y$ with its simultaneous restriction on the curve $\mathcal{L}_x$. The notation $(i)$ in eq. (3.21) stands for the embedding of $\mathcal{L}$ into $K_x \times K_y$.

4 Examples

Equations (3.19) and (3.20) reduce a calculation of eq. (1.2) to a calculation of the homology intersection indices. In turn, this can be easily done when instantons are labelled by lines in the intersection of quadrics. As was demonstrated in sect. 3, there are the 128 holomorphic curves of the type $\text{point} \times \text{line}$ and $\text{line} \times \text{point}$, and the 128 curves of the type $\text{line} \times \text{line}$. The intersection of these curves with our homology basis is given by eq. (3.3), in agreement with eq. (3.18). The 16 lines on Del Pezzo are divided into three classes: one $G$, five $E_i^x$ that intersect with $G$, and ten $F_{ij}^x$ (see Table 1). Hence, each type of lines receives further classification according to its class. It is worth mentioning that the instanton contributions of the type $\text{point} \times \text{line}$ and $\text{line} \times \text{point}$ do not always coincide.

As our first (simple) example, let’s consider the correlator given by

$$
\int H^+ \int H^+ \int H^+ \int H^+ = (H^+ \cdot \mathcal{L})^4
$$

over $\mathcal{L} = G^x \times G^y$. Let’s recall (sect. 3) that $H^+ = H^x + H^y$, while

$$
\int_{G^x \times \{\text{point}\}} H^+ = \int_{\{\text{point}\} \times G^y} H^+ = 1
$$

and

$$
\int_{G^x \times \{\text{point}\}} H^- = 1 , \quad \int_{\{\text{point}\} \times G^y} H^- = -1 , \quad \int_{G^x \times G^y} H^- = 0 .
$$
Therefore, we obtain
\[
(H^+ \cdot L)^4 = [(H^x + H^y) \cdot (G^x + G^y)]^4 = (H^x \cdot G^x + H^y \cdot G^y)^4 = (1 + 1)^4 = 2^4. \tag{4.4}
\]

Similarly, we find
\[
\int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^- = (H^- \cdot [G^x \times G^y])^4 = [(H^x - H^y) \cdot (G^x \times G^y)]^4 = (H^x \cdot G^x - H^y \cdot G^y)^4 = (1 - 1)^4 = 0. \tag{4.5}
\]

In fact, any correlator containing the factor \( \int H^- \) vanishes. Thus we see that some fermionic correlators are equal to zero despite of the fact that the discrete symmetries allow non-vanishing values for them.

As another (less trivial) example, let's consider the factor
\[
\int_{F_{ij} \times \{\text{point}\}} H^+ = \left( [H^x + H^y] \cdot \frac{1}{3} \left( \sum E_i + H - E_i - E_j \right) \right) = \frac{5}{3} + \frac{4}{3} - 2 = 1. \tag{4.6}
\]

Similarly, we find
\[
\int_{\{\text{point}\} \times F_{ij}^y} H^+ = \int_{F_{ij}^x \times \{\text{point}\}} H^+ = 1, \quad \int_{F_{ij}^x \times F_{ij}^y} H^+ = 2, \tag{4.7}
\]

and
\[
\int_{F_{ij}^x \times \{\text{point}\}} H^- = \int_{\{\text{point}\} \times F_{ij}^y} H^- = -1, \quad \int_{F_{ij}^x \times F_{ij}^y} H^- = 0. \tag{4.8}
\]

A more complicated example is given by
\[
\int_{G^x \times \{\text{point}\}} F_{5}^+ \equiv (F_{5}^+, G^x \times \{\text{point}\}) = ((F_{5}^x + F_{5}^y), (G^x \times \{\text{point}\}))
= (F_{5}^x \cdot G^x) = (H^x + 6E^x_5 - 2 \sum_i E^x_i) \cdot G^x
= (H^x + 6E^x_5 - 2 \sum_i E^x_i) \cdot \frac{1}{3} (2H^x - \sum_i E^x_i)
= \frac{2}{3} H^x \cdot H^x + 4H^x \cdot E^x_5 - \frac{4}{3} \sum_i E^x_i \cdot H^x
- \frac{1}{3} H^x \cdot \sum_i E^x_i - 2E^x_5 \cdot \sum_i E^x_i + \frac{2}{3} \sum_i E^x_i \cdot \sum_j E^x_j
= \frac{8}{3} + 4 - \frac{20}{3} - \frac{5}{3} + 2 - \frac{10}{3} = -3. \tag{4.9}
\]

Similarly, we find
\[
\int_{\{\text{point}\} \times G^y} F_{5}^+ = -3, \tag{4.10}
\]
and, hence,
\[
\left( \int_{G^x \times G^y} F_5^+ \right)^4 = (-3 - 3)^4 = 6^4 .
\]  \hfill (4.11)

We find, in addition,
\[
\int_{G^x \times \{\text{point}\}} F_i^- = \int_{G^x \times \{\text{point}\}} F_i^+ = -3 ,
\]  \hfill (4.12)

and
\[
\int_{\{\text{point}\} \times G^y} F_i^- = -\int_{\{\text{point}\} \times G^y} F_i^+ = +3 .
\]  \hfill (4.13)

The similar contributions are given by
\[
\int_{G^x \times G^y} F_i^- = \int_{F_{ij}^x \times F_{ij}^y} F_i^- = 0 ,
\]  \hfill (4.14)

\[
\int_{\{\text{point}\} \times E_{ij}^y} F_i^- = -\int_{E_{ij}^x \times \{\text{point}\}} F_i^- = -\int_{E_{ij}^x \times \{\text{point}\}} F_i^- = -7 + 6\delta_{ij} ,
\]  \hfill (4.15)

and
\[
\int_{F_{ij}^x \times E_{ij}^y} F_i^+ = 14 - 12\delta_{ij} .
\]  \hfill (4.16)

The rest of the integrals is given by
\[
\int_{E_{ij}^x \times E_{ij}^y} F_i^+ = \int_{G^x \times F_{ij}^y} F_i^+ = 4 - 6\delta_{ij} ,
\]
\[
\int_{\{\text{point}\} \times E_{ij}^y} F_i^- = -\int_{\{\text{point}\} \times E_{ij}^y} F_i^- = -\int_{E_{ij}^x \times \{\text{point}\}} F_i^- = -\int_{E_{ij}^x \times \{\text{point}\}} F_i^- = 3 ,
\]
\[
\int_{E_{ij}^x \times E_{ij}^y} F_i^+ = 9 , \quad \int_{E_{ij}^x \times E_{ij}^y} F_i^- = 0 , \quad \int_{E_{ij}^x \times E_{ij}^y} F_i^- = -6 + 6\delta_{ij} ,
\]
\[
\int_{E_{ij}^x \times \{\text{point}\}} F_i^+ = -\int_{\{\text{point}\} \times E_{ij}^y} F_i^- = 3 - 6\delta_{ij} , \quad \int_{E_{ij}^x \times E_{ij}^y} F_i^- = 6 - 6\delta_{ij} ,
\]
\[
\int_{E_{ij}^x \times \{\text{point}\}} H^+ = \int_{\{\text{point}\} \times E_{ij}^y} H^+ = \int_{E_{ij}^x \times \{\text{point}\}} H^- = -\int_{\{\text{point}\} \times E_{ij}^y} H^- = 1 ,
\]
\[
\int_{E_{ij}^x \times E_{ij}^y} H^+ = \int_{E_{ij}^x \times G^y} H^+ = \int_{G^x \times E_{ij}^y} H^+ = 2 ,
\]
\[
\int_{G^x \times F_{ij}^y} H^+ = \int_{F_{ij}^x \times E_{ij}^y} H^+ = 2 ,
\]
\[
\int_{G^x \times F_{ij}^y} H^- = \int_{E_{ij}^x \times F_{ij}^y} H^- = \int_{F_{ij}^x \times E_{ij}^y} H^- = 0 .
\]  \hfill (4.17)

The fermionic correlators in eq. (1.2) are given by various products of four factors calculated above.
5 Conclusion

It is surprising, from the mathematical viewpoint, that the fermionic correlators (1.2) are entirely determined by topology so that they can be explicitly calculated. Their physical significance is yet to be understood. At the very least, however, all the fermionic correlators (1.2) vanish in the (classical) tree approximation by index theorems [3], so that the instanton corrections obtained are actually the leading contributions to these correlators. This leads to the highly non-trivial selection rules for the physical processes described by the fermionic correlators in the CY compactified type-II strings. For example, some correlators exactly vanish (sect. 4) even though the discrete symmetries of CY allow non-vanishing values for them.

Though our geometrical approach is similar to the one used earlier for other string models in refs. [2, 4], there are also some conceptual differences. We find it simpler to consider instantons in the simply connected covering space (CY) manifold $K_0$ of $K$, instead of the CY space $K$. Each instanton in $K$ has four representatives in $K_0$, which are all equivalent as regards the $G$-invariant real quantities.

We merely discussed the one-instanton corrections to the four-fermion correlators. One may expect the existence of the multi-instanton corrections from the maps of higher degree (more than one). Unfortunately, the status of multi-instantons in the context of type-IIA superstrings is not quite clear [1] (see, however, ref. [18]).

One might also think that the intersection of Del Pezzo surfaces is the very special case of the type-IIA string/M-Theory compactification. In fact, as was recently noticed in ref. [19], there is a non-trivial duality between toroidal compactifications of M-Theory and del Pezzo surfaces. According to ref. [19], a group of the classical symmetries of Del Pezzo (i.e. the global diffeomorphisms preserving the canonical class of Del Pezzo) corresponds to the U-duality symmetries of the toroidally compactified M-Theory. The M-Theory (BPS) branes are mapped under this ‘mysterious duality’ to rational curves on Del Pezzo, so that the electric-magnetic duality of M-Theory receives a nice geometrical description in terms of the Del Pezzo surfaces [19]. In particular, the bound states of the (1/2)-BPS branes in M-Theory can be related to the intersections of spheres in Del Pezzo [19]. Further developments of this new duality require counting intersections of the holomorphic curves of higher genus in CY [20].

In the context of Horava-Witten theory [21], similar calculations of instanton corrections are needed when one considers a torus-fibered CY threefold $Z$ over the Del Pezzo base, with the non-trivial first homotopy group $\pi_1(Z) = \mathbb{Z}_2$. When a gauge vacuum on the hidden brane is trivial, the threefold $Z$ admits three families of
the semi-stable holomorphic vector bundles associated with an N=1 supersymmetric
gauge theory having three (chiral) quark-lepton families and the GUT group SU(5) in
the observable brane [20]. Both five-branes in this Horawa-Witten type construction
are wrapped about holomorphic curves in Z whose homology classes are exactly
calculable.

Our investigation is also relevant for studying the supersymmetric Pati-Salam-
type models from intersecting D-branes (see, e.g., ref. [22]), and the non-perturbative
flipped SU(5) vacua in heterotic M-Theory [23, 24].

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