EXACT PATHWISE AND MEAN–SQUARE ASYMPTOTIC
BEHAVIOUR OF STOCHASTIC AFFINE VOLterra AND
FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The almost sure rate of exponential-polynomial growth or decay of affine stochastic Volterra and affine stochastic finite-delay equations is investigated. These results are achieved under suitable smallness conditions on the intensities of the deterministic and stochastic perturbations diffusion, given that the asymptotic behaviour of the underlying deterministic resolvent is determined by the zeros of its characteristic equation. The results rely heavily upon a stochastic variant of the admissibility theory for linear Volterra operators.

1. INTRODUCTION

Interest in stochastic functional differential equations, including stochastic differential equations with delay, and stochastic Volterra equations, has increased in recent years, in part because of their attraction for modelling real-world systems in which the change in the state of a system is both random and depends on the path of the process in the past. Examples include population biology (Mao [46], Mao and Rassias [47, 48]), neural networks (cf. e.g. Blythe et al. [18]), viscoelastic materials subjected to heat or mechanical stress Drozdov and Kolmanovskii [28, Caraballo et al. [20], Mizel and Trutzer [51, 52]), or financial mathematics Anh et al. [12, Appleby et al. [13, Appleby and Daniels [5], Arrojas et al. [16], Hobson and Rogers [39], and Bouchaud and Cont [19].

Naturally, in all these disciplines, there is a great interest in understanding the long-run behaviour of solutions. In disciplines such as engineering and physics it is often of great importance to know that the system is stable, in the sense that the solution of the mathematical model converges in some sense to equilibrium. Consequently, a great deal of mathematical activity has been devoted to the question of stability of point equilibria of stochastic functional differential equations and also to the rate at which solutions converge. The literature is extensive, but a flavour of the work can be found in the monographs of Mao [44, 45, Mohammed [53], and Kolmanovskii and Myshkis [31]. Results are known concerning the asymptotic behaviour of affine stochastic Volterra equations, including rates of convergence (see [13, 12]), but generally upper bounds on the solutions are found, rather than
exact rates of decay. In this paper, we investigate not only the exact rate of convergence of solutions to point equilibria, but also the exact rate of growth of solutions of affine equations, which are of interest in studying the explosive growth or collapse of asset prices in financial market models. This develops results established in [14].

To determine the precise asymptotic results we require, it proves efficient and instructive to ask first a more general question concerning the asymptotic behaviour of stochastic integrals of the form

$$(Hf)(t) := \int_0^t H(t, s)f(s) \, dB(s)$$

(1.1)

where $H$ is a deterministic Volterra kernel and $f$ is a deterministic function on $[0, \infty)$. There is a deterministic theory of admissible operators which enables one to give precise asymptotic information regarding the solutions of integral and differential equations. As part of the analysis of such theory one encounters deterministic counterparts of (1.1). It is then unsurprising to see (1.1) in the study of affine stochastic differential equations. The admissibility theory is often useful when any forcing terms are of the same or smaller order to the solution of the unperturbed equation.

This theory is examined in depth in Appleby et al. [6], the chief results are summarised in Section 2.2. It is supposed in [6] that there exists a $H_\infty: \mathbb{R} \to \mathbb{R}$ such that (1.1) converges almost surely according to

$$\lim_{t \to \infty} \int_0^t H(t, s)f(s) \, dB(s) = \int_0^\infty H_\infty(s)f(s) \, dB(s).$$

This paper largely employs the admissibility theory of [6]. However as [6] does not provide a method of constructing such a $H_\infty$, we remark when hypothesising the precise form of $H_\infty$ it often proves useful to examine $\lim_{t \to \infty} H(t, s)$. If there is any growing or indeed oscillating component in $t \mapsto H(t, s)$ one may use this to deduce the correct form of $H_\infty$.

Once we have developed some general results concerning the asymptotic behaviour of $Hf$, the majority of the paper is devoted to applying this theory to describe the fine structure of the asymptotic behaviour of affine stochastic functional differential equations of the form

$$dX(t) = L(t, X_t) \, dt + \Sigma(t) \, dB(t)$$

where $L = L(\phi)$ is a linear functional from $C([-\tau, 0])$ to $\mathbb{R}^d$, or $L(t, \phi_t)$ is a linear convolution Volterra functional from $C([0, \infty))$ to $\mathbb{R}^d$. Therefore, we are chiefly interested in the effect of time–dependent stochastic perturbations on the asymptotic behaviour of autonomous (or asymptotically autonomous) linear functional differential equations. It is assumed that the asymptotic behaviour of solutions of the underlying fundamental solution of differential resolvent can be described in terms of the solutions of the characteristic equation, and that such solutions lie in the region of existence of the transform of the resolvent.

Results of Mohammed and Scheutzow [54] show that with respect to white noise perturbations, the Liapunov spectrum of deterministic functional differential equations is preserved, to the extent that the leading positive Liapunov exponent of the deterministic equation becomes the a.s. leading Liapunov exponent of the stochastic equation. However, it is also of interest to ask whether oscillation, or multiplicity of the characteristic equations are preserved when the noise intensity is sufficiently small (or does not grow too rapidly, or decay to slowly, relative to the exponential rate of growth or decay of the resolvent). It is known from [14] in the case of a particular scalar functional differential equation with finite delay, for which the
solution of the characteristic equation with largest real part is real and simple, and for which the noise intensity is constant, that the solution of the stochastic equation inherits exactly the rate of growth of the resolvent. It is natural to ask whether a result of this kind can be generalised to deal with finite dimensional equations, of both finite delay and Volterra type, for which there may be many solutions of the characteristic equation which have the same real part, need not be simple, nor even be real solutions.

It is a longstanding theme in the asymptotic theory of differential equations, and especially of linear equations, to ask the question: how large can a forcing or perturbation term be, so that the perturbed differential system preserves the asymptotic behaviour of the underlying unperturbed equation. Investigations of this type were systematically initiated by Hartman and Wintner in the 1950’s [35, 36, 37, 38]. More recently, there have been many interesting contributions concerning the asymptotic behaviour of functional differential equations: the literature is quite large, but some important and representative papers include Cruz and Hale [27], Haddock and Sacker [34], Castillo and Pinto [21], Győri and Pituk [31], Pituk [55, 56], and Győri and Hartung [30] among many others. Already, some results for stochastic Volterra equations with state–independent perturbations suggest that results of this type may also be available in the random case Appleby [4].

It is one of the goals of this paper to demonstrate that very sharp conditions can be identified on the intensity of the perturbations under which the asymptotic behaviour of the deterministic equations is preserved. Moreover, we show that the results apply to a wide class of affine stochastic functional differential equation, and examples and underlying admissibility results show that there is the potential for our work to apply to a wider class yet.

Our results for the solution $X$ of functional differential equations have the form

$$\lim_{t \to \infty} \left\{ \frac{X(t, \omega)}{\gamma(t)} - S(t, \omega) \right\} = 0, \quad \text{a.s. and in mean square} \quad (1.2)$$

where $\gamma : (0, \infty) \to (0, \infty)$ is a deterministic real exponential polynomial, and $S$ is a random sinusoidal vector, whose “frequencies” are deterministic but whose “amplitudes” or “multipliers” are multidimensional normal random variables which are path–dependent (in the case where the zeros of the characteristic equation with largest real part are real, $S$ is a constant random vector). These “multipliers” turn out to be identifiable linear functionals of the Brownian motion, the noise intensity $\Sigma$, and of the initial function or condition, because we have an explicit formula for these multipliers in terms of the solutions of the characteristic equation with largest real part. Similar multipliers emerge in papers of Appleby, Devin and Reynolds on stochastic Volterra equations whose solutions have Gaussian limits [7, 8]. Moreover, the joint distribution of these random limits is known exactly, because the mean and covariance matrix of the Gaussian limit can be computed explicitly in terms of the components of the random vector. This has already proved of interest in [14] where the form of the multiplier can be used to describe the mechanism by which financial market bubbles can start. Our results here are also superior to those in Appleby and Daniels [5] (i.e. Chapter 5) in which a limit formula for asset returns of the form (1.2) is found for a nonautonomous stochastic functional differential equation. The method of asymptotic analysis, which applies the deterministic admissibility theory pathwise, shows that the distribution of $S$ is Gaussian, but does not enable a formula for the variance to be determined. These examples from finance demonstrate the utility of an authentically stochastic admissibility theory in finding the exact form of the limiting multiplier.
2.1. Notation and terminology. Let \(Z\) be the set of integers, \(\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}\) and \(\mathbb{R}\) the set of real numbers. We denote by \(\mathbb{R}_+\) the half-line \([0, \infty)\). The complex plane is denoted by \(\mathbb{C}\) and \(\mathbb{C}_0 := \{z \in \mathbb{C} : \Re(z) \geq 0\}\), where \(\Re(z)\) and \(\Im(z)\) denote the real and imaginary parts of any complex number \(z\). If \(d\) is a positive integer, \(\mathbb{R}^d\) is the space of \(d\)-dimensional column vectors with real components and \(\mathbb{R}^{d_1 \times d_2}\) is the space of all \(d_1 \times d_2\) real matrices. The identity matrix on \(\mathbb{R}^{d \times d}\) is denoted by \(I_d\), while \(0_{d_1, d_2}\) represents the matrix of zeros in \(\mathbb{R}^{d_1 \times d_2}\). Let \(A \in \mathbb{R}^{d \times d}\) then \(\det(A)\) denotes the determinant of the square matrix \(A\). \(A^T\) denotes the transpose of any \(A \in \mathbb{R}^{d_1 \times d_2}\). The absolute value of \(A = A_{i,j}\) in \(\mathbb{R}^{d_1 \times d_2}\) is the matrix given by \((|A|)_{i,j} = |A_{i,j}|\).

We employ the standard Landau notation: if \(f : \mathbb{C} \rightarrow \mathbb{C}\) and \(g : \mathbb{C} \rightarrow \mathbb{R}\), we write \(f = O(g)\) as \(|z| \rightarrow \infty\) if there exist \(z_0 > 0\) and \(M > 0\) such that \(|f(z)| \leq M|g(z)|\) for all \(|z| > z_0\), for a matrix valued function the Landau notation is applied element-wise. For any two functions \(U : \mathbb{R}_+ \rightarrow \mathbb{R}^{d_1 \times d_2}\) and \(V : \mathbb{R}_+ \rightarrow \mathbb{R}^{d_2 \times d_3}\), we define the convolution of \(\{(U * V)(t)\}_{t \geq 0}\) by

\[
(U * V)(t) = \int_0^t U(t-s)V(s) \, ds, \quad t \geq 0.
\]

In this paper the Laplace transform of a sequence \(U\) in \(\mathbb{R}^{d_1 \times d_2}\) is the function defined by

\[
\tilde{U}(\lambda) = \int_0^\infty e^{-\lambda s} U(s) \, ds,
\]

provided \(\lambda\) is a complex number for which the integral converges absolutely. A similar definition pertains for the Laplace transform of a measure, \[29\], Definitions 2.1, 2.2] and for functions with values in other spaces.

Let \(BC(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2})\) denote the space matrices whose elements are bounded continuous functions. The abbreviation \(a.e.\) stands for \textit{almost everywhere}. The space of continuous and continuously differentiable functions on \(\mathbb{R}_+\) with values in \(\mathbb{R}^{d_1 \times d_2}\) is denoted by \(C(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2})\) and \(C^1(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2})\) respectively. While \(C^i,0(\Delta; \mathbb{R}^{d_1 \times d_2})\) represents the space of functions which are \(i\)-times continuously differentiable in their first argument and continuous in their second argument, over some two-dimensional space \(\Delta\). For any scalar function \(\varphi\), the space of weighted \(p^{th}\) integrable functions is denoted by

\[
L^p(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2}; \varphi) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^{d_1 \times d_2} : \int_0^\infty \varphi(s)|f(s)_{i,j}|^p \, ds < +\infty, \text{ for all } i,j\},
\]

when \(\varphi = 1\), we do not include it in our notation, i.e. \(L^p(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2}; 1) = L^p(\mathbb{R}_+; \mathbb{R}^{d_1 \times d_2})\).

For any vector \(x \in \mathbb{R}^d\) the norm \(|\cdot|\) denotes the Euclidean norm, \(|x|^2 = \sum_{j=1}^d x_j^2\) and the infinity norm, \(|\cdot|_\infty\), is defined by \(|x|_\infty = \max_{i=1,...,d}(|x_1|,...,|x_d|)|\).

While for a matrix norm we use the Frobenius norm, for any \(A = (a_{i,k}) \in \mathbb{R}^{n \times d}\)

\[
\|A\|_F^2 = \sum_{i=1}^n \sum_{k=1}^d |a_{i,k}|^2.
\]

As both \(\mathbb{R}^d\) and \(\mathbb{R}^{d \times d}\) are finite dimensional Banach spaces all norms are equivalent in the sense that for any other norm, \(|\cdot|\), one can find universal constants \(d_1(n, d) \leq d_2(n, d)\) such that

\[
d_1 \|A\|_F \leq \|A\| \leq d_2 \|A\|_F.
\]

Thus there is no loss of generality in using the Euclidean and Frobenius norms, which for ease of calculation, are used throughout the proofs of this paper. Moreover
we remark that the Frobenius norm is a consistent matrix norm, i.e. for any \( A \in \mathbb{R}^{n_1 \times n_2}, B \in \mathbb{R}^{n_2 \times n_3} \)
\[ \| AB \|_F \leq \| A \|_F \| B \|_F. \]

For any matrix \( C \in \mathbb{R}^{n \times d} \) we say \( C \geq 0 \) if \((C)_{i,j} \geq 0\) for all \( i, j \). Also, we say for any matrices \( A, B \in \mathbb{R}^{n \times d} \) that \( A \leq B \) if \( B - A \geq 0 \). We will use the fact that \( \| A \| \leq \| B \| \) whenever \( 0 \leq A \leq B \).

We also will require some notation and results regarding finite measures on sub-intervals of the real line. Let \( M(J, \mathbb{R}^{d \times d'}) \) be the space of finite Borel measures on \( J \) with values in \( \mathbb{R}^{d \times d'} \), where \( J \) shall be either \( \mathbb{R}_+ \) or \([-\tau, 0]\). The total variation of a measure \( \nu \) in \( M(J, \mathbb{R}^{d \times d'}) \) on a Borel set \( B \subseteq \mathbb{R}_+ \) is defined by
\[ |\nu|(B) := \sup \sum_{i=1}^N |\nu|(E_i), \]
where \((E_i)_{i=1}^N\) is a partition of \( B \) and the supremum is taken over all partitions. The total variation defines a positive scalar measure \( |\cdot| \) in \( M(J, \mathbb{R}) \).

If one specifies temporarily the norm \(|\cdot|\) as the \( l^1 \)-norm on the space of real-valued sequences and identifies \( \mathbb{R}^{d \times d'} \) by \( \mathbb{R}^{dd'} \) one can easily establish for the measure \( \nu = (\nu_{i,j})_{i,j=1}^{d,d'} \) the inequality
\[ |\nu|(B) \leq C \sum_{i=1}^d \sum_{j=1}^{d'} |\nu_{i,j}|(B) \]
for every Borel set \( B \subseteq \mathbb{R}_+ \) (2.1) with \( C = 1 \). Then, by the equivalence of every norm on finite-dimensional spaces, the inequality (2.1) holds true for the arbitrary norms \(|\cdot|\) and some constant \( C > 0 \).

We also define the limits
\[ \alpha_{\varphi} := - \lim_{t \to -\infty} \frac{\ln(\varphi(t))}{t}, \quad \omega_{\varphi} := - \lim_{t \to \infty} \frac{\ln(\varphi(t))}{t}. \]
Which always exist when \( \varphi \) is a submultiplicative function, c.f. [29, Lemma 4.1].

We define the following modes of convergence:

**Definition 2.** The \( \mathbb{R}^n \)-valued stochastic process \( \{X(t)\}_{t \geq 0} \) converges in mean-square to \( X_\infty \) if
\[ \lim_{t \to \infty} \mathbb{E}||X(t) - X_\infty||^2 = 0. \]

**Definition 3.** If there exists a \( \mathbb{P} \)-null set \( \Omega_0 \) such that for every \( \omega \notin \Omega_0 \) the following holds
\[ \lim_{t \to \infty} X(t, \omega) = X_\infty(\omega), \]
then we say \( X \) converges almost surely (a.s.) to \( X_\infty \).
2.2. Admissibility theory for linear stochastic Volterra operators. The main results in this paper are established using convergence results proven in [6] for linear stochastic Volterra operators. Since these results are used extensively throughout, they are stated here for the convenience of the reader. A important corollary of these results, which is of especial use in our asymptotic analysis of affine stochastic equations, is given in the next section.

We consider the following hypotheses: let $\Delta \subset \mathbb{R}^2$ be defined by

$$\Delta = \{(t, s) : 0 \leq s \leq t < +\infty\}$$

and

$$H : \Delta \rightarrow \mathbb{R}^{n \times n} \text{ is continuous.} \quad (2.2)$$

We first characterise, for $f \in C([0, \infty); \mathbb{R}^{n \times d})$ with bounded norm, the convergence of the stochastic process $X_f = \{X_f(t) : t \geq 0\}$ defined by

$$X_f(t) = \int_0^t H(t, s)f(s)dB(s), \quad t \geq 0$$

to a limit as $t \rightarrow \infty$ in mean-square, where $B(t) = \{B_1(t), B_2(t), ..., B_d(t)\}$ is a vector of mutually independent standard Brownian motions. For the definition of a stochastic integral in higher dimensions and the result corresponding to Itô’s isometry we refer the reader to [45, Definition 1.5.20 and Theorem 1.5.21].

Before discussing the convergence in mean square, we note that (2.2) is sufficient to guarantee that $X_f(t)$ is a well-defined random variable for each fixed $t$. Therefore the family of random variables $\{X_f(t) : t \geq 0\}$ is well-defined, and $X_f$ is indeed a process, and for each fixed $t$ the random variable $X_f(t)$ is $\mathcal{F}^B(t)$-adapted. Condition (2.2) also guarantees that $\mathbb{E}[X_f(t)^2] < +\infty$ for each $t \geq 0$. Since $f \rightarrow X_f$ is linear, and the family $(X_f(t))_{t \geq 0}$ is Gaussian for each fixed $f$, the limit should also be Gaussian and linear in $f$, as well as being an $\mathcal{F}^B(\infty)$-measurable random variable. Therefore, a reasonably general form of the limit should be

$$X_f^* := \int_0^\infty H_\infty(s)f(s)dB(s),$$

where we would expect $H_\infty$ to be a function independent of $f$. Our main result, which is proven in [6], characterises the conditions under which $X_f(t) \rightarrow X_f^*$ in mean square as $t \rightarrow \infty$ for each $f$.

**Theorem 4.** Suppose that $H$ obeys (2.2). Then the statements

(A) There exists $H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})$ such that $\int_0^\infty \|H_\infty(s)\|^2 ds < +\infty$ and

$$\lim_{t \rightarrow \infty} t \int_0^t \|H(t, s) - H_\infty(s)\|^2 ds = 0. \quad (2.3)$$

(B) There exists $H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})$ such that for each $f \in BC(\mathbb{R}_+; \mathbb{R}^{n \times d})$,

$$\lim_{t \rightarrow \infty} \mathbb{E}\left[\left\|\int_0^t H(t, s)f(s)dB(s) - \int_0^\infty H_\infty(s)f(s)dB(s)\right\|^2\right] = 0 \quad (2.4)$$

are equivalent.

We now consider the almost sure convergence of $X_f(t)$ as $t \rightarrow \infty$ to a limit. Our next main result states that if we have convergence in an a.s. sense, we must also have convergence in a mean square sense.

**Theorem 5.** Suppose that $H$ obeys (2.2) and there exists $H_\infty \in C([0, \infty); \mathbb{R}^{n \times n})$ such that for each $f \in BC([0, \infty); \mathbb{R}^{n \times d})$,

$$\lim_{t \rightarrow \infty} \int_0^t H(t, s)f(s)dB(s) = \int_0^\infty H_\infty(s)f(s)dB(s), \quad a.s. \quad (2.5)$$
Then (2.3) and (2.4) hold.

(2.3) is a necessary condition for a.s. convergence. It is of course natural to then ask whether (2.3) is sufficient. By means of examples, it is shown in [6] that in general additional conditions are needed in order for (2.3) to hold. We now state our main result which guarantees a.s. convergence of the stochastic integral.

**Theorem 6.** Suppose that $H$ obeys (2.2) and also that $H \in C^{1,0}(\Delta; \mathbb{R}^{n \times n})$. Suppose also that there exists $H_\infty \in C([0,\infty); \mathbb{R}^{n \times n})$ such that \( \int_0^\infty \|H_\infty(s)\|^2 \, ds < +\infty \) and

\[
\lim_{t \to \infty} \int_0^t \|H(t, s) - H_\infty(s)\|^2 \, ds \cdot \log t = 0, \quad (2.6)
\]

and

There exists $q \geq 0$ and $c_q > 0$ such that

\[
\int_0^t \|H_1(t, s)\|^2 \, ds \leq c_q (1 + t)^{2q}, \quad \|H(t, t)\|^2 \leq c_q (1 + t)^{2q}. \quad (2.7)
\]

Then $H$ obeys (2.5).

**Remark 1.** We notice that (2.6) implies a given rate of decay to zero of \( \int_0^t (H(t, s) - H_\infty(s))^2 \, ds \) as $t \to \infty$. This strengthens the hypothesis (2.3) which is known, by Theorem 5, to be necessary.

**Remark 2.** While the pointwise bound on $H(t, t)$ given by \( \|H(t, t)\|^2 \leq c_q (1 + t)^{2q} \) in Theorem 6 may appear quite mild, one may prefer an integral condition to this pointwise bound as this would allow for $H(t, t)$ to potentially have “thin spikes” of larger than polynomial order. In [6] it is pointed out that this pointwise condition can be replaced by

\[
\lim_{k \to \infty} \int_{(k+1)^\theta}^{(k+1)\theta} \|H(s, s)\|^2 \, ds \cdot \log k = 0, \quad \text{for } 0 < \theta < 1/(1+2q), \quad (2.8)
\]

where the limit is taken through the integers. Nevertheless for simplicity we retain the condition on $H(t, t)$ in the statement of Theorem 5.

### 2.3. Asymptotic behaviour of a stochastic convolution integral

In this section we state a key theorem which will be used to determine the asymptotic behaviour of solutions of Volterra linear SFDEs and linear SFDEs with finite delay with state–independent noise intensity. This theorem is a consequence of the stochastic admissibility results stated in Section 2.2.

To see the connection between these admissibility results and the asymptotic behaviour of such affine equations, we note that both classes of equations can be written in the form

\[
dX(t) = \left( f(t) + L(X_1) \right) dt + \Sigma(t) \, dB(t), \quad t \geq 0,
\]

where $L$ is a linear functional, $\Sigma \in C([0,\infty); \mathbb{R}^{d \times d'})$, $f \in C([0,\infty); \mathbb{R}^d)$, $B$ is a standard $d'$-dimensional Brownian vector and the solution $X$ lies in $\mathbb{R}^d$. For any $y : \mathbb{R} \to \mathbb{R}^{d \times n}$ we define the segment $y_t : \mathbb{R} \to \mathbb{R}^{d \times n} : s \mapsto y(t + s)$ for any $n, d \in \mathbb{Z}^+$. An appropriate initial condition is also imposed. The associated deterministic equation is

\[
x'(t) = L(x_t), \quad t \geq 0,
\]

with the same initial value as the stochastic equation. Also defining the differential resolvent, $r$,

\[
r'(t) = L(r_t), \quad t \geq 0, \quad r(0) = I_d, \quad (2.9)
\]
allows one to write the variation of parameters formula, for \( t \geq 0 \),

\[
X(t) = x(t) + \int_0^t r(t - s) f(s) \, ds + \int_0^t r(t - s) \Sigma(s) \, dB(s).
\]

The asymptotic behaviour of \( x \) and \( r \) is primarily known from the theory of deterministic linear differential equations and so one may now apply the admissibility theory of Section 2.2 to determine the asymptotic behaviour of the stochastic convolution integral, \( \int_0^t r(t - s) \Sigma(s) \, dB(s) \), and hence of \( X \), providing that the diffusion, \( \Sigma \), does not grow too rapidly.

**Proposition 1.** Let \( \alpha \in \mathbb{R}, N \) be some finite positive integer, \( \{\beta_j\}_{j=1}^N \) be a sequence of some real constants and \( (P_j)_{j=1}^N \) and \( (Q_j)_{j=1}^N \) be sequences of \( d \times d \) matrix polynomials of degree \( n \), for some positive integer \( n \), and in particular

\[
P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}).
\]

where at least one of \( P_j^*, Q_j^* \) \( \neq 0 \) for all \( j \in \{1, \ldots, N\} \). Suppose \( R \) is a.e. absolutely continuous and is defined such that it obeys, for some \( \epsilon > 0 \), the asymptotic estimates

\[
R(t) = \begin{cases} 
O(e^{\alpha t}), & \text{if } n = 0 \\
O(e^{\alpha t} t^{n-1}), & \text{if } n \geq 1
\end{cases}, \quad \text{as } t \to \infty,
\]

and suppose that \( r \) is given by

\[
r(t) = \sum_{j=1}^N e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \} + R(t), \quad t \geq 0.
\]

Let \( \Sigma \in C([0, \infty); \mathbb{R}^{d \times d}) \) be continuous with

\[
\int_0^\infty e^{-2\alpha t} \|\Sigma(t)\|^2 \, dt < +\infty.
\]

Let \( Y \) be the process defined by

\[
Y(t) = \int_0^t r(t - s) \Sigma(s) \, dB(s), \quad t \geq 0, \quad Y(0) = 0.
\]

Then

\[
\lim_{t \to \infty} \left( \frac{Y(t)}{t^n e^{\alpha t}} - \sum_{j=1}^N \{ L_{1,j} \sin(\beta_j t) + L_{2,j} \cos(\beta_j t) \} \right) = 0, \quad \text{a.s.}
\]

where

\[
L_{1,j} := \int_0^\infty e^{-\alpha s} \{ P_j^* \sin(\beta_j s) + Q_j^* \cos(\beta_j s) \} \Sigma(s) \, dB(s),
\]

\[
L_{2,j} := \int_0^\infty e^{-\alpha s} \{ P_j^* \cos(\beta_j s) - Q_j^* \sin(\beta_j s) \} \Sigma(s) \, dB(s).
\]

The square integrability, \( L^2(0, \infty) \), of the noise term, i.e. (2.13), is a usual condition to have when dealing with stochastic terms. When ascertaining asymptotic behaviour of deterministic forcing functions it is more typical to require an absolute integrability condition, \( L^1(0, \infty) \). This is indeed what is required in Corollary 1, i.e. (2.17). Proposition 1 is shown to be robust with respect to deterministic perturbations.
Corollary 1. Let $\alpha \in \mathbb{R}$, $N \in \mathbb{Z}^+$, and let $f \in C([0,\infty),\mathbb{R}^d)$ with (2.13) holding. Let $f \in C([0,\infty),\mathbb{R}^d)$ with $\sqrt{\int_0^\infty \text{e}^{-\alpha t} |f(t)| dt} < +\infty$. (2.17)

Let $V$ be the process defined by

$$V(t) = \int_0^t r(t-s) f(s) ds + Y(t), \quad t \geq 0, \quad V(0) = 0. \quad (2.18)$$

Then

$$\lim_{t \to +\infty} \left( \frac{V(t)}{t \text{e}^{\alpha t}} - \sum_{j=1}^N \{M_{1,j} \sin(\beta_j t) + M_{2,j} \cos(\beta_j t)\} \right) = 0, \quad \text{a.s.}$$

where

$$M_{1,j} = L_{1,j} + \int_0^\infty e^{-\alpha s} \{P_j^* \sin(\beta_j s) + Q_j^* \cos(\beta_j s)\} f(s) ds,$$

$$M_{2,j} = L_{2,j} + \int_0^\infty e^{-\alpha s} \{P_j^* \cos(\beta_j s) - Q_j^* \sin(\beta_j s)\} f(s) ds$$

and where $L_{1,j}$ and $L_{2,j}$ are given by Proposition 1.

3. Affine Stochastic Functional Differential Equations

The organisation of this section is as follows: in the first part of this section, we discuss the structure of solutions of affine stochastic Volterra functional, and show that the resolvent of the underlying deterministic equation can play the role of the function $r$ introduced in the statement of Proposition 1 modulo some deterministic asymptotic estimates. The second part of the section contains a parallel discussion for the solution of the affine stochastic functional differential equation with finite delay. These preliminary discussions pave the way for main asymptotic results for both stochastic Volterra and finite delay equations which are stated in Section 4.

3.1. Volterra linear functional equations. A Shea-Wainger theorem is developed in [13], which relates the location of the roots of a characteristic equation to the solution of a Volterra linear SFDE lying in a weighted $L^p$-space. We reproduce the set-up of those equations here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$, and let $(B(t))_{t \geq 0}$ be a standard $d'$-dimensional Brownian motion on this probability space. Consider the stochastic integro-differential equation with stochastic perturbations of the form

$$dX(t) = \left( f(t) + \int_{[0,t]} \mu(ds) X(t-s) \right) dt + \Sigma(t) dB(t) \quad \text{for } t \geq 0, \quad (3.1)$$

where $\mu$ is a measure in $M(\mathbb{R}_+, \mathbb{R}^{d \times d})$, $\Sigma \in C(\mathbb{R}_+; \mathbb{R}^{d \times d'})$, $f \in C(\mathbb{R}_+; \mathbb{R}^d)$. The initial condition $X_0$ is an $\mathbb{R}^d$-valued, $\mathcal{F}_0$-measurable random variable with $\mathbb{E} |X_0|^2 < \infty$. The existence and uniqueness of a continuous solution $X$ of (3.1) with $X(0) = X_0$ $\mathbb{P}$-a.s. is covered in Berger and Mizel [17], for instance. Independently, the existence and uniqueness of solutions of stochastic functional equations was established in Itô and Nisio [40] and Mohammed [53].
The so-called fundamental solution or resolvent of (3.1) is the matrix-valued function \( r : \mathbb{R}^+ \to \mathbb{R}^{d \times d} \), which is the unique solution of
\[
r'(t) = \int_{[0,t]} \mu(ds) r(t-s) \quad \text{for } t \geq 0, \quad r(0) = I_d. \tag{3.2}
\]
In the following Proposition, we give a variation of constants formula for the solution of (3.1) in terms of the solution \( r \) of (3.2). The proof is a simple adaptation of a result of Reiß, Riedle and van Gaans [28, Lemma 6.1].

**Proposition 2.** Let \( \mu \in M(\mathbb{R}^+; \mathbb{R}^{d \times d}) \), \( \Sigma \in C(\mathbb{R}^+; \mathbb{R}^{d \times d}) \), \( f \in C(\mathbb{R}^+; \mathbb{R}^d) \), and suppose that \( r \) is the unique continuous solution of (3.2). Then the unique continuous adapted process \( X \) which obeys (3.1) is given by
\[
X(t) = r(t)X_0 + \int_0^t r(t-s)f(s)ds + \int_0^t r(t-s)\Sigma(s)dB(s) \quad \text{P-a.s.} \tag{3.3}
\]

The proof is given in Section 1.

The chief difficulty in estimating the asymptotic behaviour of \( X \) therefore lies in determining the asymptotic behaviour of the stochastic convolution integral on the right–hand side of (3.3). However, we argue below that the solution \( r \) of (3.2) can be decomposed as in (2.12), with leading order exponential polynomial behaviour and the remainder terms obeying the growth estimates of the form (2.10) and (2.11). Then, the last term on the righthand side of (3.3) is of the form of the process \( Y \) defined in (2.13), and therefore, under appropriate growth conditions on \( \Sigma \), Proposition 1 can be applied to this term.

In order to do this, we start by defining the real number \( \alpha^* \) by
\[
\alpha^* = \inf\{a \in \mathbb{R} : \int_{(0,\infty)} e^{-as} |\mu|(ds) \text{ is well–defined and finite}\}. \tag{3.4}
\]
Then the function \( h_\mu : \mathbb{C} \to \mathbb{C} \) defined by
\[
h_\mu(\lambda) = \det \left( \lambda I_d - \int_{[0,\infty)} e^{-\lambda s} \mu(ds) \right).
\]
is well–defined for \( \Re(\lambda) > \alpha^* \).

Define also the set
\[
\Lambda = \{ \lambda \in \mathbb{C} : h_\mu(\lambda) = 0 \}.
\]
The function \( h_\mu \) is analytic, and so the elements of \( \Lambda \) are isolated. Define
\[
\alpha := \sup\{ \Re(\lambda) : h_\mu(\lambda) = 0 \} \tag{3.5}
\]
It is always the case that such an \( \alpha \) is finite, we assume however that \( \alpha^* < \alpha \).

Because the solution \( r \) obeys an exponentially growing or decaying upper bound, this is equivalent to assuming that there exists \( \lambda \in \mathbb{C} \) with \( \Re(\lambda) > \alpha^* \) for which
\[
h_\mu(\lambda) = 0.
\]
With the assumption \( \alpha^* < \alpha \), there exists \( \delta \in (0, \alpha - \alpha^*) \). By the Riemann–Lebesgue lemma, cf. e.g. [29, Thm. 2.2.7 (i)], for such a \( \delta > 0 \) there exists \( M = M(\delta) > 0 \) such that \( h_\mu(\lambda) \neq 0 \) for all \( \lambda \in \mathbb{C} \) such that \( \alpha^* + \delta \leq \Re(\lambda) \leq \alpha + \delta \) and \( |\Im(\lambda)| \geq M(\delta) \). If \( K = \{ \lambda \in \mathbb{C} : 0 < |\Re(\lambda) - \alpha| < \delta, |\Im(\lambda)| \leq M(\delta) \} \), the fact that \( h_\mu \) is analytic ensures that there are at most finitely many zeros of \( h_\mu \) in \( K \). Therefore, there exists a minimal \( \varepsilon \in (0, \delta] \) such that \( h_\mu(\lambda) \neq 0 \) for all \( \alpha - \varepsilon \leq \Re(\lambda) < \alpha \), and therefore there exists \( \delta' = \alpha - \varepsilon \) such that \( h_\mu(z) \neq 0 \) for all \( \Re(z) = \delta' \). Define \( \varphi(t) = e^{-\delta't} \) for \( t \in \mathbb{R} \). Then \( \varphi \) is a submultiplicative weight function on \( \mathbb{R} \) for which \( \omega_\varphi = \alpha_\varphi = \delta' = \alpha - \varepsilon \). Define \( \Lambda_\varepsilon = \{ \lambda \in \Lambda : \Re(\lambda) > \alpha - \varepsilon \} \). Clearly \( \Lambda_\varepsilon \) is a set with only finitely many elements, as is \( \Lambda' = \{ \lambda \in \Lambda : \Re(\lambda) = \alpha \} \).
Then by Theorem 7.2.1 in [29], there exists an a.e. absolutely continuous function $q$ such that $q, q' \in L^1(\mathbb{R}^+; \varphi; \mathbb{R}^{d \times d})$ and

$$r(t) = \sum_{\lambda_j \in \Lambda^e, \Im(\lambda_j) \geq 0} e^{\alpha_j t} \{P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t)\} + q(t), \quad t \geq 0. \quad (3.6)$$

where $\Re(\lambda_j) = \alpha_j$ and $\Im(\lambda_j) = \beta_j$, and where $P_j$ and $Q_j$ are matrix–valued polynomials of degree $n_j$, with $n_j + 1$ being the order of the pole $\lambda_j = \alpha_j + i \beta_j$ of $[h_{\mu}]^{-1}$. We remark that $n_j$ (the ascent of $\lambda_j$) is less than or equal to the multiplicity of the zero $\lambda_j = \alpha_j + i \beta_j$ of $h_{\mu}$.

Let $n$ denote the highest degree of all polynomials associated with roots in $\Lambda^e$ and let $\lambda_1, ..., \lambda_N$ be the finitely many roots in $\Lambda^e$ which have associated polynomials of this degree and have $\Im(\lambda_j) = \beta_j \geq 0$. We associate with each such $\lambda_j = \alpha_j + i \beta_j$ the matrix polynomials $P_j$ and $Q_j$ of degree $n$ in (3.6). Therefore we may write

$$P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}). \quad (3.7)$$

where at least one of $P_j^*$ and $Q_j^*$ are not equal to the zero matrix, for each $j \in \{1, ..., N\}$. The precise values of $P_j^*$ and $Q_j^*$ can be determined from the Laurent series of the inverse of the characteristic function, $h_{\mu}$, expanded about $\lambda_j$, i.e.

$$\left[\lambda I_d - \int_{(0, \infty)} e^{-\lambda s} \mu(ds)\right]^{-1} = \sum_{m=0}^n \frac{m! K_{j,m}}{(\lambda - \lambda_j)^m} + \hat{q}_j(\lambda), \quad (3.8)$$

where the remainder term $\hat{q}_j(\lambda)$ is analytic at $\lambda_j$. If $\lambda_j$ is real then $P_j^* = K_{j,n}$, otherwise $P_j^* := 2 \Re(K_{j,n})$ and $Q_j^* := -2 \Im(K_{j,n})$. We note that (3.8) defines the value of $n$.

Now define

$$R(t) = r(t) - \sum_{j=1}^N e^{\alpha_j t} \{P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t)\}, \quad t \geq 0. \quad (3.9)$$

It is clear that $R$ is a.e. absolutely continuous. Therefore, by virtue of the decomposition in (2.12) in the statement of Proposition 1, if the growth estimates (2.10) and (2.11) can be established for $R$ defined by (3.9), we will be in a excellent position to apply Proposition 11 to the stochastic convolution term on the right–hand side of (3.3). The relevant estimates will be provided in Lemma 11 which is stated in Section 3.

### 3.2. Finite delay linear functional equations.

The exact rate of growth of the running maxima of solutions of affine SFDEs with finite memory is discussed in [11]. We reproduce the set-up of those equations here.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$, and let $(B(t))_{t \geq 0}$ be a standard $d'$-dimensional Brownian motion on this probability space. Consider the stochastic integro-differential equation of the form

$$dX(t) = \left( f(t) + \int_{[-\tau,0]} \nu(ds) X(t + s) \right) dt + \Sigma(t) dB(t) \quad \text{for } t \geq 0, \quad (3.10)$$

$$X(t) = \phi(t), \quad t \in [-\tau,0],$$

where $\nu$ is a measure in $M([-\tau,0], \mathbb{R}^{d \times d})$, $\Sigma \in C(\mathbb{R}^+; \mathbb{R}^{d \times d})$, $f \in C(\mathbb{R}^+; \mathbb{R}^d)$. For every $\phi \in C([-\tau,0], \mathbb{R}^d)$ there exists a unique, adapted strong solution $(X(t, \phi) : t \geq -\tau)$ with finite second moments of (3.10) (cf., e.g., Mao [15]). The dependence of the solution on the initial condition $\phi$ is neglected in our notation in what follows; that is, we will write $X(t) = X(t, \phi)$ for the solution of (3.10).
Turning our attention to the deterministic equation in $\mathbb{R}^d$ underlying (3.11). For fixed constant $\tau \geq 0$:

$$x'(t) = \int_{[-\tau, 0]} \nu(ds) x(t + s) \quad \text{for } t \geq 0, \quad x(t) = \phi(t) \quad t \in [-\tau, 0]. \tag{3.11}$$

For every $\phi \in C([-\tau, 0], \mathbb{R}^d)$ there is a unique $\mathbb{R}^d$-valued function $x = x(\cdot, \phi)$ which satisfies (3.11).

The so-called fundamental solution or resolvent of (3.10) is the matrix-valued function $r : \mathbb{R}_+ \to \mathbb{R}^{d \times d}$, which is the unique solution of

$$r'(t) = \int_{[\tau]} \nu(ds) r(t + s) \quad \text{for } t \geq 0, \quad r(0) = I_d. \tag{3.12}$$

For convenience one could set $r(t) = 0_{d,d}$ for $t \in [-\tau, 0)$.

The solution $x(\cdot, \phi)$ of (3.11) for an arbitrary initial segment $\phi$ exists, is unique, and can be represented as

$$x(t, \phi) = r(t)\phi(0) + \int_0^t \int_{[-\tau, u]} \nu(ds)r(t + s - u)\phi(u)du, \quad \text{for } t \geq 0;$$

cf. Diekmann et al. [26, Chapter I].

By Reiß, Riedle and van Gaans [58, Lemma 6.1] the solution $(X(t) : t \geq -\tau)$ obeys a variation of constants formula:

$$X(t) = \begin{cases} x(t) + \int_0^t r(t-s)f(s)ds + \int_0^t r(t-s)\Sigma s dB(s), & t \geq 0, \\ \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3.13}$$

The process $X$ defined by (3.13) obeys (3.10) pathwise on an almost sure event.

In order to determine the asymptotic behaviour of the solution $X$ of (3.10), we argue below, in a very similar manner to that given in Section 3.1, that the asymptotic behaviour of the stochastic convolution term on the right-hand side of (3.13) can be tackled by identifying the resolvent $r$ in (3.12) with the function in (2.12) and the convolution term with the integral $Y$ defined by (2.14).

Towards this end, we start by defining the function $g_\nu : \mathbb{C} \rightarrow \mathbb{C}$ by

$$g_\nu(\lambda) = \det \left( \lambda I_d - \int_{[-\tau, 0]} e^{\lambda s} \nu(ds) \right);$$

and also the set of its zeros

$$\Lambda = \{ \lambda \in \mathbb{C} : g_\nu(\lambda) = 0 \}.$$

The function $g_\nu$ is analytic, and so the elements of $\Lambda$ are isolated. Define

$$\alpha := \sup \{ \Re(\lambda) : g_\nu(\lambda) = 0 \}. \tag{3.14}$$

Once again $\alpha$ is finite. Furthermore the cardinality of $\Lambda' = \{ \Re(\lambda) = \alpha : \lambda \in \Lambda \}$ is finite. Then, following a similar argument as in Subsection 3.1 there exists $\varepsilon_0 > 0$ such that $g_\nu(\lambda) \neq 0$ for $\alpha - \varepsilon_0 < \Re(\lambda) < \alpha$ and hence $g_\nu(\lambda) \neq 0$ on the line $\Re(\lambda) = \varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$. Thus we have

$$r(t)e^{-\alpha t} = \sum_{\lambda_j \in \Lambda', \Im(\lambda_j) \geq 0} \left( \tilde{P}_j(t) \cos(\beta_j)t + \tilde{Q}_j(t) \sin(\beta_j)t \right) + o(e^{-\alpha t}), \quad t \to \infty, \tag{3.15}$$

where $\Re(\lambda_j) = \alpha$ and $\Im(\lambda_j) = \beta_j$, and where $\tilde{P}_j$ and $\tilde{Q}_j$ are matrix-valued polynomials of degree $n_j$, with $n_j + 1$ being the order of the pole $\lambda_j = \alpha + i\beta_j$ of $[g_\nu]^{-1}$. This is a restatement of Diekmann et al [26, Theorem 5.4].

Let $\alpha$ denote the highest degree of all polynomials associated with roots in $\Lambda'$ and let $\lambda_1, \ldots, \lambda_N$ be the finitely many roots in $\Lambda'$ which have associated polynomials of
this degree and have \( \Im(\lambda_j) = \beta_j \geq 0 \). We associate with each characteristic root \( \lambda_j = \alpha + i\beta_j \) the matrix polynomials \( P_j \) and \( Q_j \) in (3.15) above, each of which has degree \( n \). Therefore we may write

\[
P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}).
\]

where at least one of \( P_j^* \) and \( Q_j^* \) are not equal to the zero matrix, for each \( j \in \{1, \ldots, N\} \). The precise values of \( P_j^* \) and \( Q_j^* \) can be determined from the Laurent series of the inverse of the characteristic function, \( g_\nu \), expanded about \( \lambda_j \), c.f. [26, pp.31] i.e.

\[
\left[ \lambda I - \int_{[-\tau,0]} e^{\lambda s} \nu(ds) \right]^{-1} = \sum_{m=0}^{n} \frac{m! K_{j,\alpha}}{(\lambda - \lambda_j)^{m+1}} + \hat{q}_j(\lambda),
\]

where the remainder term \( \hat{q}_j(\lambda) \) is analytic at \( \lambda_j \). If \( \lambda_j \) is real then \( P_j^* = K_{j,n} \), otherwise \( P_j^* := 2 \Re(K_{j,n}) \) and \( Q_j^* := -2 \Im(K_{j,n}) \). We note that (3.17) defines the value of \( n \).

Finally, we define

\[
R(t) = r(t) - \sum_{j=1}^{N} e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \}, \quad t \geq 0.
\]

It is clear that \( R \) is a.e. absolutely continuous. Therefore, by virtue of the decomposition in (2.12) in the statement of Proposition 1 if the growth estimates (2.11) and (2.11) can be established for \( R \) defined by (3.18), we will be in a excellent position to apply Proposition 1 to the stochastic convolution term on the right-hand side of (3.4). The relevant estimates will be provided in Lemma 2 which is stated in Section 3.

4. Main Results

In this section, we state the main results of the paper, which concern the pathwise and mean–square asymptotic behaviour of the stochastic Volterra and finite delay equations introduced in the previous section. In order to do so, the decomposition of the resolvents and variation of constants formulae established in the previous section must be aligned with the hypothesis of Proposition 1. The missing ingredient in each of the proofs is an asymptotic estimate on the remainder terms defined in equations (3.19) and (3.18), and once these are supplied, the main results follow directly. In addition, this section contains a number of remarks on the scope and ramifications of these main asymptotic results.

We now state the main results for the Volterra equation and affine SFDE with finite memory.

**Theorem 7.** Let \( \alpha^* \) and \( \alpha \), as defined by (3.4) and (3.5) respectively, obey \( \alpha^* < \alpha \). Let \( n \) be given by (3.8) (i.e. \( n + 1 \) denotes the highest order of all roots in \( \Lambda'' = \Lambda \cap \{ \Re(\lambda) = \alpha, \Im(\lambda) \geq 0 \} \) and let \( (\lambda_j)_{j=1}^{N} \) be the finitely many roots in \( \Lambda'' \) with this order. Define \( \beta_j = \Im(\lambda_j) \), \( j = 1, \ldots, N \). Suppose that \( P_j^*, Q_j^* \) for \( j = 1, \ldots, N \) are given by (3.7). Let \( f \in C([0, \infty); \mathbb{R}^d) \) be such that

\[
\int_{0}^{\infty} e^{-\alpha t} |f(t)| \, dt < +\infty
\]

and let \( \Sigma \in C([0, \infty); \mathbb{R}^{d \times d}) \) be such that

\[
\int_{0}^{\infty} e^{-2\alpha t} \|\Sigma(t)\|^2 \, dt < +\infty.
\]
Let $X$ be the unique solution of (3.1). Then

$$
\lim_{t \to \infty} \left( \frac{X(t)}{t^\alpha e^{\alpha t}} - \sum_{j=1}^{N} \{(Q_j^*X_0 + M_{1,j})\sin(\beta_j t) + (P_j^*X_0 + M_{2,j})\cos(\beta_j t)\} \right) = 0, \quad \text{a.s. (4.3)}
$$

where $M_{1,j}$ and $M_{2,j}$ are given by Corollary 1.

We are now in a position to prove this Theorem. As already pointed out, in order to do this, estimates are needed on the asymptotic behaviour of (3.5) and (3.6). This condition is also required in determining the deterministic resolvent, (3.6). This condition is also required in determining the asymptotic behaviour in this case is examined in great depth in Jordan et al. [33], Kriszten and Terjéki [42] and Miller [50]. In particular, in order to apply successfully our stochastic admissibility results, we need good asymptotic information about both the resolvent and its derivative. For the cases covered here, existing deterministic results for the resolvent suffice, but new work has been required, and is supplied, for the derivative. Thus, in this case the stochastic theory as described by Theorem 7 would not necessarily hold.

Some articles which examine the case when the line containing the leading characteristic exponents of the characteristic equation co-incides with the boundary of the domain of the transform of the measure are e.g. [29] Chapter 7.3, [42] for deterministic theory and [7], [8], for stochastic theory.

The corresponding result for the affine SFDE is as follows.

**Theorem 8.** Let $\alpha$ be as defined by (3.14). Let $n$ be given by (3.8) (i.e. $n + 1$ denotes the highest order of all roots in $\Lambda'' = \Lambda \cap \{\Re(\lambda) = \alpha, \Im(\lambda) \geq 0\}$) and let $(\lambda_j)_{j=1}^{N}$ be the finitely many roots in $\Lambda''$ with this order. Define $\beta_j = \Im(\lambda_j)$, $j = 1, \ldots, N$. Suppose that $P_j^*, Q_j^*$ for $j = 1, \ldots, N$ are given by (3.14). Let $f \in C([0, \infty); \mathbb{R}^d)$ be such that

$$
\int_{0}^{\infty} e^{-\alpha t} |f(t)| dt < +\infty
$$
Let \( X \) be the unique solution of (3.10). Then
\[
\lim_{t \to \infty} \left( \frac{X(t)}{t^{\nu}e^{\alpha t}} - \frac{1}{N} \sum_{j=1}^{N} \{ J_{1,j} \sin(\beta_j t) + J_{2,j} \cos(\beta_j t) \} \right) = 0, \quad a.s. \tag{4.5}
\]
where
\[
J_{1,j} = Q^*_j \phi(0) + G_{1,j} + M_{1,j}, \quad J_{2,j} = P^*_j \phi(0) + G_{2,j} + M_{2,j},
\]
\[
G_{1,j} = \int_{-\tau}^{0} \int_{[-\tau,u]} e^{\alpha u} \nu(ds) \{ Q^*_j \cos(\beta_j u) - P^*_j \sin(\beta_j u) \} \phi(s-u) du,
\]
\[
G_{2,j} = \int_{-\tau}^{0} \int_{[-\tau,u]} e^{\alpha u} \nu(ds) \{ P^*_j \cos(\beta_j u) + Q^*_j \sin(\beta_j u) \} \phi(s-u) du,
\]
and where \( M_{1,j} \) and \( M_{2,j} \) are given by Corollary 7.

As in the case of Theorem 4, we are now ready to prove this Theorem. As indicated earlier, we can do this once appropriate estimates are available for the asymptotic behaviour of \( R \) defined by (3.18). These estimates are supplied in the following result, whose proof is deferred to Section 8.

**Lemma 2.** Let \( R \) be defined by (3.18). Suppose that \( \alpha \) is as defined by (3.14). Then there exists \( \varepsilon > 0 \) such that

(i) If \( n = 0 \), then \( R(t) = O(e^{(\alpha - \varepsilon)t}) \) as \( t \to \infty \).

(ii) If \( n = 0 \), then \( R'(t) = O(e^{(\alpha - \varepsilon)t}) \) as \( t \to \infty \).

(iii) If \( n \geq 1 \), then \( R(t) = O(t^{n-1}e^{\alpha t}) \) as \( t \to \infty \).

(iv) If \( n \geq 1 \), then \( R'(t) = O(t^{n}e^{\alpha t}) \), as \( t \to \infty \).

Observe that Lemma 2 states that \( R \) defined by (3.18) obeys equations (2.10) and (2.11). Also a rearrangement of \( r \) given by (3.18) yields the form of \( r \) in (2.12). Thus, the proof of Theorem 8 is an immediate consequence of Lemma 2, Corollary 7, and Remark 6.

**Remark 4.** Theorem 8 differs from Theorem 7 with respect to the region of existence of the characteristic equation \( \gamma_{\nu} \), i.e. \( \int_{[-\tau,0]} e^{\alpha u} \mu(ds) \) exists for all \( \alpha \in (-\infty, \infty) \) and thus the condition \( \alpha^* < \alpha \), present in Theorem 7, has no analogue in Theorem 8.

**Remark 5.** While Theorems 7 and 8 give a rate of growth or decay in an almost sure sense, it is observed, via Theorem 8, that this convergence also holds in mean square. That is, for the solution of the Volterra equation (3.1), with the assumptions of Theorem 7
\[
\lim_{t \to \infty} \mathbb{E} \left[ \left( \frac{X(t)}{t^{\nu}e^{\alpha t}} - \frac{1}{N} \sum_{j=1}^{N} \{ (Q^*_j X_0 + M_{1,j}) \sin(\beta_j t) + (P^*_j X_0 + M_{2,j}) \cos(\beta_j t) \} \right)^2 \right] = 0.
\]

Also, for the solution of the finite delay equation (3.10), with the assumptions of Theorem 8
\[
\lim_{t \to \infty} \mathbb{E} \left[ \left( \frac{X(t)}{t^{\nu}e^{\alpha t}} - \frac{1}{N} \sum_{j=1}^{N} \{ J_{1,j} \sin(\beta_j t) + J_{2,j} \cos(\beta_j t) \} \right)^2 \right] = 0.
\]
Remark 6. The asymptotic behaviour of the deterministic functional differential equations (3.2) or (3.12), each of which obey

$$\lim_{t \to \infty} \left( \frac{r(t)}{t^{\alpha t}} - \sum_{j=1}^{N} \{Q_j^* \sin(\beta_j t) + P_j^* \cos(\beta_j t)\} \right) = 0,$$

where $P_j^*$ and $Q_j^*$ are determined by (3.7) (in the case of the Volterra equation) and (3.16) (for the equation with finite delay) is analogous to the asymptotic behaviour of $X$ as given by (4.3) and (4.5) respectively.

It can therefore be seen, despite the presence of the stochastic integral, that $X$ inherits the asymptotic behaviour of $r$, provided that the intensity of the noise perturbation does not grow too rapidly.

Regarding the multipliers of the trigonometric terms we remark that $M_{1,j}$ and $M_{2,j}$ are Gaussian distributed random variables and hence their values and, in particular, sign will depend upon the sample path. Moreover these random variables depend on the coefficients of the trigonometric terms in (4.6) i.e. $P_j^*$ and $Q_j^*$.

Remark 7. The conditions (2.13) and (2.17) on the growth of $\Sigma$ and $f$ are, in some sense, unimprovable if the asymptotic behaviour of $X$ is to be recovered.

Consider, for example, the scalar ordinary affine stochastic equation

$$dX(t) = \left( \alpha X(t) + f(t) \right) dt + \Sigma(t) dB(t), \quad t \geq 0, \quad X(0) = X_0 \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$, $\Sigma \in C([0, \infty); \mathbb{R})$ and $f$ is a non-negative function, i.e. $f \in C([0, \infty); [0, \infty))$. Then we have the following equivalent conditions:

(i) (2.13) and (2.17) hold.

(ii) There exists an a.s. finite random variable $L$ such that

$$\mathbb{P} \left[ \lim_{t \to \infty} e^{-\alpha t} X(t) = L \in (-\infty, \infty) \right] > 0.$$  \hspace{1cm} (4.7)

(iii) There exists an a.s. finite random variable $L$ such that

$$\lim_{t \to \infty} e^{-\alpha t} X(t) = L, \quad \text{a.s.}$$  \hspace{1cm} (4.8)

The proof of Remark 7 is deferred to Section 9.

Remark 8. The asymptotic behaviour of the solution of (3.10) in the case when $\alpha < 0$ and the diffusion coefficient is time independent, i.e. $\Sigma(t) = \Sigma \in \mathbb{R}^{d \times d}$ for all $t \geq 0$, is considered in [11]. It is argued that asset prices in financial markets fluctuate and therefore it is of interest to describe the order of the oscillations about the mean in particular the rate of growth of the running maximum of this asset price. In this case the resolvent function decays exponentially to zero resulting in the process $X$ behaving asymptotically like a Gaussian process. Specifically, it is shown that

$$\lim_{t \to \infty} \frac{|X(t)|}{\sqrt{2 \log t}} = \max_{i=1, \ldots, d} \sqrt{\sum_{k=1}^{m} \frac{(r(s) \Sigma)_{i,k}^2}{s}} ds, \quad \text{a.s.}$$

However for constant coefficient of diffusion, condition (4.4) is violated and hence Theorem 8 does not apply.

Remark 9. The asymptotic behaviour of the solution of the scalar equation (3.10), with $d = 1$, is considered in [13] with $\alpha \geq 0$, the zero of $\Sigma$ which has this real part is a simple real zero and all other zeros of $\Sigma$ have real parts less than $\alpha$. Thus [14, Theorem 3.1 (b)], which considers the case of $\alpha > 0$, is a special case of Theorem 8. Moreover, as in practice it is quite difficult to determine the zeroes of $\Sigma$ a subclass of measures is looked at which give the desired properties on the zeroes of $\Sigma$. Also, the economic interpretations of these impositions are discussed. To summarise the
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results: it is shown that if $\alpha = 0$ then the market behaves similar to a Black-Scholes model, in particular $X$ undergoes fluctuations according to the law of the iterated logarithm.

$$\limsup_{t \to \infty} \frac{X(t)}{\sqrt{2t \log \log t}} = -\liminf_{t \to \infty} \frac{X(t)}{\sqrt{2t \log \log t}} = C_1,$$

where $C_1$ is a positive constant. On the other hand, the case $\alpha > 0$ gives

$$\lim_{t \to \infty} e^{-\alpha t} X(t) = C_2,$$

where $C_2$ is a random variable. This regime is interpreted as the market undergoing a bubble or crash, depending upon the sign of $C_2$, with both events being possible.

However the case $\alpha = 0$ studied in [14] also has a constant diffusion coefficient, thus (4.4) is not satisfied and so Theorem 8 does not apply.

5. Examples

We give some illustrative examples of Theorems 7 and 8 and Proposition 1. The first three examples consider the situation where the resolvent is of the especially simple form

$$\mu(ds) = A\delta_0(ds),$$

where $A$ is a $d \times d$ matrix with real entries. In this case, the resolvent is nothing other than the principal matrix solution

$$r'(t) = Ar(t), \quad r(0) = I_d$$

and the stochastic equation is just the affine stochastic differential equation

$$dX(t) = AX(t) dt + \Sigma(t) dB(t), \quad t \geq 0; \quad X(0) = \xi.$$  

Since there are no more than $d$ eigenvalues, the resolvent $r$ and its derivative can be expressed as finite sums, and so there is no need for a detailed analysis of remainder terms.

Our first example looks at the case when the leading eigenvalue (or zero of the characteristic equation) has algebraic multiplicity equal to the geometric multiplicity.

Example 9. Suppose that $A = \gamma I$ where $I$ is the $2 \times 2$ identity matrix. Then $Y(t) = e^{-\gamma t}X(t)$ obeys $dY(t) = e^{-\gamma t}\Sigma(t) dB(t)$, so

$$Y(t) = \xi + \int_0^t e^{-\gamma s}\Sigma(s) dB(s), \quad t \geq 0.$$  

In this case, applying our results to $Y$, we have $\alpha = 0$. If $s \mapsto e^{-\gamma s}\Sigma(s) \in L^2(0, \infty)$, by the martingale convergence theorem we have

$$\lim_{t \to \infty} \frac{X(t)}{e^{\gamma t}} = \lim_{t \to \infty} Y(t) = \xi + \int_0^\infty e^{-\gamma s}\Sigma(s) dB(s), \quad a.s.$$  

Let $\lambda_j = 0$. Since $A - \gamma I = 0$, we see that, with $n = 0$, $K_{j,0} = I$ and $\hat{q}_j(\lambda) = 0$, we have

$$(\lambda I - (A - \gamma I))^{-1} = \lambda^{-1} I = \sum_{m=0}^\infty \frac{m! K_{j,m}}{\lambda^{m+1}} + \hat{q}_j(\lambda).$$

Thus, we may set $P_j = I$, and therefore the limit for $Y$ has the form predicted by Theorem 8 with $\alpha = 0$.

We now demonstrate the resulting asymptotic behaviour of the solution of the stochastic equation when the leading eigenvalue has geometric multiplicity less than the algebraic multiplicity.
Example 10. Suppose that
\[ A = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix} . \]
Consider \( Y(t) = e^{-\gamma t}X(t) \). Then
\[ dY(t) = (A - \gamma I)Y(t) \, dt + e^{-\gamma t} \Sigma(s) \, dB(t) . \]
Then, applying our theory to \( Y \), we find that \( \alpha = 0 \), because \( \lambda = 0 \) is an eigenvalue of multiplicity 2. In this case \( r \) is given by
\[ r(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \]
Since \( \text{det}(r(t)) = 1 \) for all \( t \geq 0 \), \( r(t) \) is invertible, and we may write \( r(t-s) = r(t)r^{-1}(s) = r(t)r(-s) \) for all \( 0 \leq s \leq t \). Therefore
\[ Y(t) = r(t)\xi + \int_0^t r(t-s)e^{-\gamma s}\Sigma(s) \, dB(s) = r(t)\xi + r(t)\int_0^t r(-s)e^{-\gamma s}\Sigma(s) \, dB(s) . \]
Notice that \( Y(t) = I_d + t(A - \gamma I) \) and \( (A - \gamma I)r(-s) = A - \gamma I \). Then
\[ \frac{r(t)}{t} \int_0^t r(-s)e^{-\gamma s}\Sigma(s) \, dB(s) \]
\[ = \frac{1}{t} \int_0^t r(-s)\Sigma(s) \, dB(s) + \int_0^t (A - \gamma I)r(-s)e^{-\gamma s}\Sigma(s) \, dB(s) \]
\[ = \frac{1}{t} \int_0^t r(-s)e^{-\gamma s}\Sigma(s) \, dB(s) + (A - \gamma I) \int_0^t e^{-\gamma s}\Sigma(s) \, dB(s) . \]
Using Lemma 3, the first term has zero limit as \( s \to e^{-\gamma s}\Sigma(s) \) is in \( L^2(0, \infty) \), and \( r(-s)/s \to -(A - \gamma I) \) as \( s \to \infty \). The second term converges by the martingale convergence theorem. Thus
\[ \lim_{t \to \infty} \frac{X(t)}{t e^{\gamma t}} = (A - \gamma I)\xi + (A - \gamma I) \int_0^\infty e^{-\gamma s}\Sigma(s) \, dB(s) . \]
This is exactly the form of the limit predicted in Theorem 8 because for \( \lambda_j = 0 \) with \( n = 1 \), we have
\[ P^*_j = K_{j,1} = \lim_{\lambda \to 0} \lambda^2 (\lambda I_d - (A - \gamma I))^{-1} = A - \gamma I . \]
This next example demonstrates the case when the leading eigenvalues are complex solutions of the characteristic equation.

Example 11. Suppose that
\[ A = \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix} . \]
Suppose that \( Y(t) = e^{-\gamma t}X(t) \). If \( J = A - \gamma I \), then
\[ dY(t) = JY(t) \, dt + e^{-\gamma t} \Sigma(t) \, dB(t) . \]
For the equation solved by \( Y \), we have \( \alpha = 0 \), because \( \lambda = \pm i \) are eigenvalues of multiplicity 1. In this case \( r \) is given by
\[ r(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} . \]
Since \( \text{det}(r(t)) = 1 \) for all \( t \geq 0 \), \( r(t) \) is invertible, and we may write \( r(t-s) = r(t)r^{-1}(s) = r(t)r(-s) \) for all \( 0 \leq s \leq t \). Therefore
\[ X(t) = r(t)\xi + \int_0^t r(t-s)e^{-\gamma s}\Sigma(s) \, dB(s) = r(t)\xi + r(t)\int_0^t r(-s)e^{-\gamma s}\Sigma(s) \, dB(s) . \]
Since \( r(-s) \) is bounded, and \( s \mapsto e^{-\gamma^s} \Sigma(s) \in L^2(0, \infty) \), it follows that
\[
\lim_{t \to \infty} \int_0^t r(-s)e^{-\gamma^s} \Sigma(s) \, dB(s) = \int_0^\infty r(-s)e^{-\gamma^s} \Sigma(s) \, dB(s), \quad \text{a.s.}
\]
Therefore
\[
\lim_{t \to \infty} \left\{ Y(t) - r(t) \left( \xi + \int_0^\infty r(-s) \Sigma(s) \, dB(s) \right) \right\} = 0, \quad \text{a.s.}
\]
We now see that \( r(t) = \cos(t)I + \sin(t)J \), and so the following limit holds almost surely:
\[
\lim_{t \to \infty} \left\{ \frac{X(t)}{e^{\gamma t}} - (\cos(t)I + \sin(t)J) \left( \xi + \int_0^\infty (\cos(s)I - \sin(s)J)e^{-\gamma^s} \Sigma(s) \, dB(s) \right) \right\} = 0.
\]
Setting
\[
G_c = \int_0^\infty \cos(s)e^{-\gamma^s} \Sigma(s) \, dB(s), \quad G_s = \int_0^\infty \sin(s)e^{-\gamma^s} \Sigma(s) \, dB(s)
\]
and noting that \( J^2 = -I \), we get
\[
\lim_{t \to \infty} \left\{ \frac{X(t)}{e^{\gamma t}} - \cos(t)(\xi + G_c - JG_s) - \sin(t)(J\xi + JG_c + G_s) \right\} = 0, \quad \text{a.s.}
\]
To show that this asymptotic expansion agrees exactly with formula (15) derived in Theorem 8, we notice for \( \lambda_j = (-1)^{j-1}i \) for \( j = 1, 2 \) where each of which has multiplicity \( n + 1 = 1 \), that
\[
K_{j,0} = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j) (\lambda I - J)^{-1} = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j) \frac{1}{1 + \lambda^2} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}.
\]
Since \( (\lambda - \lambda_j)(\lambda - \overline{\lambda_j}) = 1 + \lambda^2 \), we have
\[
K_{j,0} = \frac{1}{\lambda_j - \overline{\lambda_j}} \begin{pmatrix} \lambda_j & -1 \\ 1 & \lambda_j \end{pmatrix} = \frac{1}{2\lambda_j} \begin{pmatrix} \lambda_j & -1 \\ 1 & \lambda_j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \lambda_j \\ -\lambda_j & 1 \end{pmatrix}.
\]
Hence \( 2K_{1,0} = I - iJ \) and \( 2K_{2,0} = I + iJ \). Therefore \( P_1^* = I \) and \( Q_1^* = J \).

We provide an example of a convolution Volterra integro–differential equation where the zeros of the characteristic equation do not lie in the domain of the transform of the measure, i.e. \( \alpha^* > \alpha \). Nevertheless an explicit formula for the resolvent may obtained and hence one may deduce the asymptotic behaviour of the solution of the stochastic equation.

**Example 12.** Let \( X \) be the unique solution of
\[
\frac{dX(t)}{dt} = \int_{[0,t]} \mu(ds)X(t - s) \, dt + \Sigma(t) \, dB(t), \quad t \geq 0
\]
where \( X(0) = X_0 \in \mathbb{R}^d \) and \( \mu(ds) = -6 \delta_0(ds)I_d - 4e^{-s} \, dsI_d \). Hence \( \alpha^* = -1 \) and \( h \) is given by
\[
h(\lambda) = \det \left( \lambda I_d - \int_{[0,\infty)} \mu(ds)e^{-\lambda s} I_d \right) = \frac{(\lambda + 2)^4(\lambda + 5)^d}{(\lambda + 1)^d}.
\]
Thus \( \alpha^* = -1 > -2 = \alpha \) and so we cannot apply Theorem 7 to this problem.

Nevertheless, the differential resolvent, (6.2), may rewritten as the solution of a second order equation and solved to give
\[
r(t) = -\frac{1}{3}3^{-2t}I_d + \frac{4}{3}e^{-5t}I_d.
\]
Therefore $n = 0$ and $P_1^* = -1/3$ and one can now apply Proposition 4 to determine the asymptotic behaviour of $X$, i.e.
\[
\lim_{t \to \infty} \frac{X(t)}{e^{-at}} = -\frac{1}{3}X_0 - \frac{1}{3} \int_0^\infty e^{2s}\Sigma(s)dB(s).
\]

Thus, in instances where Theorem 7 does not apply, providing that the asymptotic behaviour of $r$ may be estimated to agree with (2.12), then via Proposition 4 the asymptotic behaviour of the solution of the stochastic equation can still be recovered.

We finish with an example where the underlying deterministic functional differential equation is not equivalent to a linear ordinary differential equation, but for which it is possible, owing to the special structure of the equation, to determine exactly the leading order asymptotic behaviour.

**Example 13.** Suppose that $X$ obeys
\[
dX(t) = a(X(t) - X(t - 1/3))\,dt + \Sigma(s)\,dB(t), \quad t \geq 0,
\]
where $\Sigma \in C([\mathbb{R}_+; \mathbb{R}^{1 \times d}])$, $X(t) = \phi(t)$ for $t \in [-1/3, 0]$, where $\phi \in C([-1/3, 0], \mathbb{R})$. Let $a = 3/(1 - 1/e) > 0$. This is equivalent to choosing $\tau = 1/3$ and the finite measure $\nu(ds) = a\delta_0(ds) - a\delta_{-1/3}(ds)$. Then it can be shown that $\nu([-t, 0]) \geq 0$ for all $t \in [0, 1/3]$ with $\nu([-1/3, 0]) = 0$. Also
\[
\int_{[-1/3, 0]} s\,\nu(ds) = \frac{1}{1 - e^{-1}} > 1.
\]
Consequently, all the conditions of part (i), Theorem 3.3 in [14] hold, and therefore there is a unique positive real solution $\lambda_1 > 0$ of $g_\nu(\lambda_1) = 0$ where $g_\nu(\lambda) = \lambda - a + ae^{-\lambda/3}$, and moreover $a = \lambda_1$. Since $a = 3/(1 - 1/e)$, it is easily verified that $\lambda = \lambda_1 = 3$. Furthermore, as $g_\nu'(\lambda_1) = 1 - ae^{-\lambda_1/3} \neq 0$, it can be shown that $n = 0$ in Theorem 8 and moreover by l'Hôpital’s rule that
\[
P_1^* = \lim_{\lambda \to \lambda_1} \frac{\lambda - \lambda_1}{g_\nu'(\lambda)} = \frac{1 - e^{-1}}{1 - 2e^{-1}}.
\]
Therefore, assuming (4.3) holds, then all the conditions of Theorem 8 apply, we have that
\[
\lim_{t \to \infty} \frac{X(t)}{e^{at}} = P_1^*\phi(0) + P_1^* \int_0^\infty \int_{[-\tau, u]} e^{\mu(s)}\nu(ds)\phi(s - u)du + P_1^* \int_0^\infty e^{2s}\Sigma(s)dB(s).
\]

6. **Proofs of Supporting Results**

This section contains the proofs of some supporting results: the first part of this section concerns the variation of constants formula (3.3) in Proposition 2, the rest of the section is devoted to the a.s. convergence to zero of stochastic integrals whose integrands involve $t$–dependence, but have special features.

6.1. **Proof of Proposition 2** Define $w$ to be the unique continuous solution of
\[
w''(t) = f(t) + \int_{[0,t]} \mu(ds)w(t - s), \quad t \geq 0, \quad w(0) = 0.
\]
Then $w(t) = \int_0^t r(t - s)f(s)\,ds$ for $t \geq 0$. Noting that $X$ is the unique continuous adapted process which obeys (3.1), we may defined the continuous adapted process $Z = \{Z(t) : t \geq 0\}$ by $Z(t) := X(t) - w(t)$ for $t \geq 0$. Then $Z$ is a semimartingale, and is represented by
\[
dZ(t) = \int_{[0,t]} \mu(ds)Z(t - s) + \Sigma(t)dB(t), \quad t \geq 0, \quad Z(0) = X_0.
\]
Lemma 3. \[ Z(t) = r(t)X_0 + \int_0^t r(t - s)\Sigma(s)dB(s), \quad P\text{-a.s.} \quad t \geq 0, \]

which rearranges to yield \[ (3.3). \]

6.2. Stochastic limit results. Parts of the proofs of our main results involve the proof of some subsidiary results which may themselves be of independent interest. They are stated and proven here. We start with the proof of a preliminary lemma, which will be used in the proof of Proposition 4.1.

Lemma 3. Suppose \( f \in L^2([0, \infty), \mathbb{R}^{d \times r}) \). If \( k > 0 \), then

\[
\lim_{t \to \infty} \frac{1}{(1 + t)^k} \int_0^t s^k f(s) dB(s) = 0, \quad a.s.
\]

Proof. Define

\[
K(t) = \frac{1}{(1 + t)^k} \int_0^t s^k f(s) dB(s), \quad t \geq 0.
\]

Then \( dK(t) = -k(1 + t)^{-1}K(t) dt + (1 + t)^{-k} f(t) dB(t) \). Hence for \( i = 1, \ldots, d \) with \( K_i(t) := \langle K(t), e_i \rangle \), we have

\[
dK_i(t) = -k(1 + t)^{-1}K_i(t) dt + \sum_{j=1}^r \frac{t^k}{(1 + t)^k} f_{ij}(t) dB_j(t).
\]

Therefore

\[
d\|K(t)\|^2 = \left(-2k(1 + t)^{-1}\|K(t)\|^2 + \frac{t^{2k}}{(1 + t)^{2k}}\|f(t)\|^2_{\mathbb{F}}\right) dt + \sum_{i=1}^d 2K_i(t) \sum_{j=1}^r \frac{t^k}{(1 + t)^k} f_{ij}(t) dB_j(t).
\]

Now define the non-decreasing processes \( A_1 \) and \( A_2 \) by

\[
A_1(t) = \int_0^t \frac{s^{2k}}{(1 + s)^{2k}} \|f(s)\|^2_{\mathbb{F}} ds, \quad A_2(t) = \int_0^t 2k(1 + s)^{-1}\|K(s)\|^2 ds.
\]

and the martingale \( M \) by

\[
M(t) = \sum_{j=1}^r \int_0^t \sum_{i=1}^d 2K_i(s) \frac{s^k}{(1 + s)^k} f_{ij}(s) dB_j(s).
\]

Then we have

\[
\|K(t)\|^2 = A_1(t) - A_2(t) + M(t), \quad t \geq 0.
\]

Since \( f \) is in \( L^2(0, \infty) \), we notice that \( A_1(t) \) tends to a finite limit as \( t \to \infty \). Therefore, we have that \( \|K(t)\|^2 \to \kappa \) as \( t \to \infty \) a.s where \( \kappa \in [0, \infty) \) a.s. (It is known that \( \lim_{t \to \infty} \|K(t)\|^2 \) exists and is finite due to \[ 43 \] Theorem 7, pp.139). Then by l’Hôpital’s rule we have

\[
\lim_{t \to \infty} \frac{A_2(t)}{\log t} = 2k\kappa.
\]

Notice now that \( M \) has quadratic variation

\[
\langle M \rangle(t) = \int_0^t \sum_{j=1}^r \left( \sum_{i=1}^d 2K_i(s) \frac{s^k}{(1 + s)^k} f_{ij}(s) \right)^2 ds.
\]
Therefore by the Cauchy–Schwartz inequality
\[
(M(t)) \leq \int_0^t \sum_{j=1}^r \sum_{l=1}^d K_j^2(s) \sum_{i=1}^d \frac{s^{2k}}{(1+s)^2k^2} f_j^2(s) \, ds \leq 4 \int_0^t \|K(s)\|^2 \|f(s)\|^2 \, ds.
\]
Since \( f \) is in \( L^2(0,\infty) \), we see that \( \lim_{t \to \infty} (M(t)) \) is finite and hence that \( M \) tends to a finite limit a.s. Let \( A = \{ \omega : \kappa(\omega) > 0 \} \) and suppose that \( \mathbb{P}[A] > 0 \). Then on \( A \) we have \( \lim_{t \to \infty} \|K(t,\omega)\|^2 = -\infty \), which is a contradiction. Hence \( \mathbb{P}[A] = 0 \), or \( \kappa = 0 \) a.s. Therefore \( K(t) \to 0 \) as \( t \to \infty \), a.s., as required.

The proof of Proposition\( \text{[1]} \) in the case \( n = 0 \), uses Lemma 3 from Appleby \[3\]; this lemma is used in the proof of the next supporting convergence result, so is stated for completeness.

**Lemma 4.** Suppose \( x : \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous, integrable function, and \( \eta > 0 \) is any fixed constant. Then, the sequence \( \{a_n\}_{n=0}^\infty \) given by \( a_0 = 0 \) and
\[
a_{n+1} = \inf \left\{ t \in [a_n + \eta/2, a_n + 3\eta/4] : x(t) = \min_{a_n + \eta/2 \leq s \leq a_n + 3\eta/4} x(s) \right\}, \quad n \in \mathbb{Z}^+,\]

satisfies
\[
\frac{\eta}{4} < a_{n+1} - a_n < \eta \quad \text{for all } n \in \mathbb{Z}^+ , \quad \lim_{n \to \infty} a_n = \infty ,
\]

together with
\[
\sum_{n=0}^\infty x(a_n) < \infty.
\]

The following lemma which will be used in the proof of Proposition\( \text{[1]} \)(\( n = 0 \)), is a mild adaptation of Lemma 5.2 from \[14\].

**Lemma 5.** Let \( k : \mathbb{R}_+ \to \mathbb{R} \) be such that \( k, k' \in L^2([0,\infty);\mathbb{R}) \). Define for \( f \in L^2([0,\infty);\mathbb{R}) \) the Gaussian process \( \{K(t) : t \geq 0\} \) by
\[
K(t) = \int_0^t k(t-s) f(s) \, dB(s).
\]

Then \( \lim_{t \to \infty} K(t) = 0 \), a.s.

**Proof.** We re-express \( K \), using the stochastic Fubini Theorem, e.g. [57], Theorem 4.6.64, pp.210–211, which leads to
\[
K(t) = \int_0^t \left( k(0) + \int_0^{t-s} k'(u) \, du \right) f(s) \, dB(s)
\]
\[
= \int_0^t k(0) f(s) \, dB(s) + \int_0^t \int_0^s k'(v-s) \, dv \, f(s) \, dB(s)
\]
\[
= k(0) \int_0^t f(s) \, dB(s) + \int_0^t \int_0^s k'(v-s) \, f(s) \, dB(s) \, dv.
\]

Then for any increasing sequence \( \{a_n\}_{n=0}^\infty \) we have, for \( t \in [a_n, a_{n+1}] \),
\[
K(t) = K(a_n) + k(0) \int_{a_n}^t f(s) \, dB(s) + \int_{a_n}^t \int_0^s k'(v-s) \, f(s) \, dB(s) \, dv.
\]
Squaring, taking suprema and finally an expectation across this inequality gives
\[
E \left[ \sup_{a_n \leq t \leq a_{n+1}} |K(t)|^2 \right] \leq 3 \mathbb{E} \left[ K(a_n)^2 \right] + 3 k(0)^2 \mathbb{E} \left[ \sup_{a_n \leq t \leq a_{n+1}} \left| \int_{a_n}^t f(s) \, dB(s) \right|^2 \right]
\]
\[
+ 3 \mathbb{E} \left[ \sup_{a_n \leq t \leq a_{n+1}} \left( \int_{a_n}^t \int_0^s k'(v-s) \, f(s) \, dB(s) \, dv \right)^2 \right].
\]

(6.1)
We consider each term on the right–hand side separately. Now for the second term, applying Doob’s inequality, c.f. e.g. [45, Theorem 1.38] yields
\[ E \left[ \sup_{a_n \leq t \leq a_{n+1}} \left| \int_{a_n}^{t} f(s) dB(s) \right|^2 \right] \leq 4 \int_{a_n}^{a_{n+1}} f(s)^2 \, ds \]
and thus
\[ \sum_{n=0}^{\infty} E \left[ \sup_{a_n \leq t \leq a_{n+1}} \left| \int_{a_n}^{t} f(s) dB(s) \right|^2 \right] < +\infty. \] (6.2)
For the third term, applying the Cauchy–Schwarz inequality gives
\[ E \left[ \sup_{a_n \leq t \leq a_{n+1}} \left| \int_{a_n}^{t} k'(v-s) f(s) dB(s) \, dv \right|^2 \right] \]
\[ \leq E \left[ \sup_{a_n \leq t \leq a_{n+1}} (t-a_n) \int_{a_n}^{t} \left| \int_{0}^{v} k'(v-s) f(s) dB(s) \right|^2 \, dv \right] \]
\[ = (a_{n+1} - a_n) \int_{a_n}^{a_{n+1}} E \left[ \left| \int_{0}^{v} k'(v-s) f(s) dB(s) \right|^2 \right] dv \]
\[ = (a_{n+1} - a_n) \int_{a_n}^{a_{n+1}} k'(v-s)^2 f(s)^2 \, ds \, dv. \]

Now suppose that \( 0 < a_{n+1} - a_n < \eta \) for some \( \eta > 0 \), then
\[ \sum_{n=1}^{\infty} E \left[ \sup_{a_n \leq t \leq a_{n+1}} \left| \int_{a_n}^{t} k'(v-s) f(s) dB(s) \, dv \right|^2 \right] \]
\[ \leq \eta \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} k'(v-s)^2 f(s)^2 \, ds \, dv < +\infty. \] (6.3)

Now the first term, \( t \rightarrow x(t) = E[K(t)^2] \), is continuous and non–negative, also
\[ \int_{0}^{\infty} x(t) \, dt = \int_{0}^{\infty} k(t)^2 \, dt \int_{0}^{\infty} f(s)^2 \, ds < +\infty. \]
Therefore by Lemma [4] for all \( \eta > 0 \) there exists a sequence \( \{a_n\}_{n=0}^{\infty} \) such that
\[ \sum_{n=0}^{\infty} x(a_n) = \sum_{n=0}^{\infty} E[K(a_n)^2] < +\infty. \] (6.4)

So, using (6.2), (6.3) and (6.4) in (6.1) yields
\[ \sum_{n=0}^{\infty} E \left[ \sup_{a_n \leq t \leq a_{n+1}} |K(t)|^2 \right] < +\infty. \]

By the Monotone Convergence Theorem, c.f. e.g. [63, Theorem 5.3],
\[ E \sum_{n=0}^{\infty} \sup_{a_n \leq t \leq a_{n+1}} |K(t)|^2 < +\infty. \]
and hence
\[ \sum_{n=0}^{\infty} \sup_{a_n \leq t \leq a_{n+1}} |K(t)|^2 < +\infty, \quad \text{a.s.} \]

Thus,
\[ \lim_{n \rightarrow \infty} \sup_{a_n \leq t \leq a_{n+1}} |K(t)|^2 = 0, \quad \text{a.s.} \]
and therefore \( \lim_{t \rightarrow \infty} K(t) = 0, \quad \text{a.s.} \) \( \Box \)
7. Proof of Proposition \[ \text{1} \] and Corollary \[ \text{1} \]

In this section, we give the proofs of the limiting behaviour of the stochastic and deterministic convolutions which were stated in Section \[ \text{2.3} \].

7.1. Proof of Proposition \[ \text{1} \] for \( n \geq 1 \). In the following \( M \) denotes a positive constant whose value may change from line to line. Using (2.12) and (2.14) we may write

\[
Y(t) = \int_0^t S(t-s)\Sigma(s)\,dB(s) + \int_0^t R(t-s)\Sigma(s)\,dB(s), \quad t \geq 0,
\]

where

\[
S(t) = \sum_{j=1}^N e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \}.
\]

Thus,

\[
Y(t) = \int_0^t \frac{S(t-s)}{t^{2n} e^{2\alpha t}} \Sigma(s)\,dB(s) + \int_0^t \frac{R(t-s)}{t^{2n} e^{2\alpha t}} \Sigma(s)\,dB(s).
\]

(7.1)

We show using Theorem \[ \text{6} \] that the second stochastic integral term on the right–hand side above converges to zero almost surely. So in the notation of Section \[ \text{2.2} \] we define

\[
H(t,s) := \frac{R(t-s)}{t^{2n} e^{2\alpha t}} \Sigma(s).
\]

Now as \( R(t) = O(t^{n-1} e^{\alpha t}) \) as \( t \to \infty \) from (2.10) it is natural to choose \( H_\infty(s) = 0_{n,d} \). Thus we need only verify conditions (2.6) and (2.7). Now, from (2.10) we have

\[
\int_0^t \|H(t,s)\|_F^2 \,ds \leq \left( \frac{1 + t}{t} \right)^{2n} \frac{1}{(1 + t)^{2n} e^{2\alpha t}} \int_0^t M(1 + t-s)^{2n-2} e^{2\alpha (t-s)} \|\Sigma(s)\|_F^2 \,ds,
\]

for some \( M > 0 \). Hence for \( t \geq 1 \) we have

\[
\int_0^t \|H(t,s)\|_F^2 \,ds \leq 2^{2n} M \frac{1}{(1 + t)^{2n}} \int_0^t (1 + t-s)^{2n-2} e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds
\]

\[
\leq 2^{2n} M \frac{1}{(1 + t)^2} \int_0^t e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds
\]

\[
\leq 2^{2n} M \frac{1}{(1 + t)^2} \int_0^\infty e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds,
\]

where we use the fact that \( \int_0^\infty e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds \) is finite. Therefore

\[
\lim_{t \to \infty} \int_0^t \|H(t,s)\|_F^2 \,ds \cdot \log t = 0.
\]

Next, we consider

\[
\int_k^{(1+k)^\theta} \|H(s,s)\|_F^2 \,ds \leq \int_k^{(1+k)^\theta} Ks^{-2n} e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds
\]

\[
\leq Kk^{-2n\theta} \int_k^{(1+k)^\theta} e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds,
\]

for some \( K > 0 \). Since \( n \geq 1, \theta > 0 \) and \( \int_0^\infty e^{-2\alpha s} \|\Sigma(s)\|_F^2 \,ds \) is finite, we have that

\[
\lim_{k \to \infty} \int_k^{(1+k)^\theta} \|H(s,s)\|_F^2 \,ds \cdot \log k = 0.
\]
Turning then to the derivative condition of (2.7) we see

\[ H_1(t, s) = t^{-n}e^{-\alpha t}R'(t - s)\Sigma(s) - \alpha t^{-n}e^{-\alpha t}R(t - s)\Sigma(s) - n t^{-n-1}e^{-\alpha t}R(t - s)\Sigma(s). \]  

(7.2)

Therefore we have

\[
\|H_1(t, s)\|_F \leq t^{-n}e^{-\alpha t} \left( \|R'(t - s)\|_F + |\alpha|\|R(t - s)\|_F \right) + nt^{-1}\|R(t - s)\|_F \|\Sigma(s)\|_F,
\]

and so as \(\|R(t)\|_F \leq M(1 + t)^{n-1}e^{\alpha t}\), \(\|R'(t)\|_F \leq M(1 + t)^n e^{\alpha t}\) we have for \(t \geq 1\)

\[
\|H_1(t, s)\|_F \leq Mt^{-n}e^{-\alpha s} \left( (1 + t - s)^n + |\alpha|(1 + t - s)^{n-1} \right) + nt^{-1} \left( (1 + t - s)^{n-1} \right) \|\Sigma(s)\|_F
\leq Mt^{-n}(1 + t - s)^n (1 + (|\alpha| + n)(1 + t - s)^{-1}) e^{-\alpha s}\|\Sigma(s)\|_F 
\leq M (1 + |\alpha| + n) \cdot t^{-n}(1 + t - s)^n e^{-\alpha s}\|\Sigma(s)\|_F.
\]

Thus for \(t \geq 1\) we have

\[
\int_0^t \left( \|H_1(t, s)\|_F^2 \right) ds = Mt^{-2n} \int_0^t (1 + t - s)^{2n} e^{-2\alpha s}\|\Sigma(s)\|_F^2 ds 
\leq M^2 \left( \frac{1 + t}{t} \right)^{2n} \int_0^t e^{-2\alpha s}\|\Sigma(s)\|_F^2 ds 
\leq M^2 2^{2n} \int_0^\infty e^{-2\alpha s}\|\Sigma(s)\|_F^2 ds.
\]

Hence \(\int_0^t \|H_1(t, s)\|_F^2 ds\) may easily be bounded above by a polynomially growing function. So we have shown that

\[
\lim_{t \to \infty} \int_0^t \frac{R(t - s)}{t^n e^{\alpha t}} \Sigma(s) dB(s) = 0, \quad a.s.
\]

(7.3)

Next write

\[ P_j(t) = t^n P_j^* + P_{j,n-1}(t) \quad \text{and} \quad Q_j(t) = t^n Q_j^* + Q_{j,n-1}(t), \]

where \(P_{j,n-1}\) and \(Q_{j,n-1}\) are matrix polynomials of order \(n - 1\). Then \(S\) can be expressed according to

\[
S(t) = \sum_{j=1}^N e^{\alpha t} t^n \{ P_j^* \cos(\beta_j t) + Q_j^* \sin(\beta_j t) \}
\]

\[
+ \sum_{j=1}^N e^{\alpha t} \{ P_{j,n-1}(t) \cos(\beta_j t) + Q_{j,n-1}(t) \sin(\beta_j t) \}.
\]
Thus,
\[
\int_0^t \frac{S(t-s)}{t^n e^{at}} \Sigma(s) dB(s)
\] (7.4)
\[
= \int_0^t \sum_{j=1}^N \frac{e^{-as} (t-s)^n \{ P_j^n \cos(\beta_j(t-s)) + Q_j^n \sin(\beta_j(t-s))\} \Sigma(s)}{t^n} dB(s)
\]
\[
+ \int_0^t \sum_{j=1}^N \frac{e^{-as} P_{j,n-1}(t-s) \cos(\beta_j(t-s)) \Sigma(s)}{t^n} dB(s)
\]
\[
+ \int_0^t \sum_{j=1}^N \frac{e^{-as} Q_{j,n-1}(t-s) \sin(\beta_j(t-s)) \Sigma(s)}{t^n} dB(s).
\]

We now argue that the second and third stochastic integrals on the right–hand side in (7.4) tend to zero as \( t \to \infty \). We focus on the second integral. Note that it suffices to show for any degree \( n-1 \) polynomial \( P \) that
\[
\int_0^t \frac{P(t-s)}{(1+t)^n} \cos(\beta(t-s)) e^{-as} \Sigma dB(s) \to 0, \quad \text{as} \quad t \to \infty, \quad \text{a.s.}
\]
By recalling the trigonometric identity, for any \( a_1, a_2 \in \mathbb{R} \),
\[
\cos(a_1 - a_2) = \cos(a_1) \cos(a_2) + \sin(a_1) \sin(a_2), \quad (7.5)
\]
\[
\sin(a_1 - a_2) = \sin(a_1) \cos(a_2) - \cos(a_1) \sin(a_2),
\]
we see that it suffices to show that the process
\[
a(t) = \int_0^t \frac{P(t-s)}{(1+t)^n} f(s) dB(s),
\]
obey \( a(t) \to 0 \) as \( t \to \infty \) where \( f \) is in \( L^2(\mathbb{R}_+; \mathbb{R}^{d \times d} ) \) and \( P \) is a matrix–valued polynomial of degree \( n-1 \). Define \( H(t,s) = P(t-s)(1+t)^{-n} f(s) \). Define \( H_\infty(s) = 0 \). Since \( P \) is a polynomial, there exists \( M \) such that \( |P(t)| \leq M(1+t)^{n-1} \) and \( |P'(t)| \leq M(1+t)^{n-1} \) for all \( t \geq 0 \).

Using Theorem 6 and the same procedure as used to establish (7.3), we get
\[
\lim_{t \to \infty} \int_0^t \sum_{j=1}^N \frac{e^{-as} P_{j,n-1}(t-s) \cos(\beta_j(t-s)) \Sigma(s)}{t^n} dB(s) = 0, \quad \text{a.s.}
\]
One can argue similarly that
\[
\lim_{t \to \infty} \int_0^t \sum_{j=1}^N \frac{e^{-as} Q_{j,n-1}(t-s) \cos(\beta_j(t-s)) \Sigma(s)}{t^n} dB(s) = 0, \quad \text{a.s.}
\]

We now turn our attention to the first integral term on the right–hand side of (7.4). Consider the integral
\[
A_j(t) = \int_0^t e^{-as} (t-s)^n \{ P_j^n \cos(\beta_j(t-s)) \Sigma(s) \} dB(s), \quad (7.6)
\]
and define
\[
A_{j,0}(t) = P_j^n \cos(\beta_j t) \int_0^t \cos(\beta_j s) e^{-as} \Sigma(s) dB(s)
\]
\[
+ P_j^n \sin(\beta_j t) \int_0^t \sin(\beta_j s) e^{-as} \Sigma(s) dB(s).
\]
Since $s \mapsto e^{-\alpha s} \Sigma(s)$ is in $L^2(\mathbb{R}_+; \mathbb{R}^{d \times d'})$, if we define

$$A_{j,0}^*(t) = P_j^* \cos(\beta_j t) \int_0^\infty \cos(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s)$$

$$+ P_j^* \sin(\beta_j t) \int_0^\infty \sin(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s). \quad (7.7)$$

we have that $A_{j,0}(t) - A_{j,0}^*(t) \to 0$ as $t \to \infty$ a.s. By Newton’s binomial expansion theorem $(t-s)^n = \sum_{m=0}^n \binom{n}{m} t^m (-s)^{n-m}$ and using (7.5), we get

$$A_j(t) = \sum_{m=0}^n P_j^* (-1)^{n-m} \binom{n}{m} \frac{1}{t^{n-m}} \int_0^t s^{n-m} \cos(\beta_j (t-s)) e^{-\alpha s} \Sigma(s) \, dB(s)$$

where we have defined for $k = 1, \ldots, n$

$$A_{j,k}(t) = \frac{1}{tk} \int_0^t s^k \cos(\beta_j t) \cos(\beta_j s) + \sin(\beta_j t) \sin(\beta_j s)) e^{-\alpha s} \Sigma(s) \, dB(s).$$

This can be expressed as

$$A_{j,k}(t) = \cos(\beta_j t) \frac{1}{tk} \int_0^t s^k \cos(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s)$$

$$+ \sin(\beta_j t) \frac{1}{tk} \int_0^t s^k \sin(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s).$$

Now by applying Lemma 3 to each of the terms on the righthand side, we get

$$\lim_{t \to \infty} A_{j,k}(t) = 0, \quad \text{a.s.}$$

Therefore we see that

$$A_j(t) - A_{j,0}^*(t) \to 0, \quad \text{as } t \to \infty \text{ a.s.} \quad (7.8)$$

Define

$$C_j(t) = \int_0^t e^{-\alpha s} \frac{(t-s)^n}{t^n} Q_j^* \sin(\beta_j (t-s)) \Sigma(s) \, dB(s) \quad (7.9)$$

and

$$C_{j,0}(t) = Q_j^* \int_0^t \sin(\beta_j (t-s)) e^{-\alpha s} \Sigma(s) \, dB(s).$$

Then

$$C_{j,0}(t) = Q_j^* \sin(\beta_j t) \int_0^t \cos(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s)$$

$$- Q_j^* \cos(\beta_j t) \int_0^t \sin(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s),$$

and define

$$C_{j,0}^*(t) = Q_j^* \sin(\beta_j t) \int_0^{\infty} \cos(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s)$$

$$- Q_j^* \cos(\beta_j t) \int_0^{\infty} \sin(\beta_j s) e^{-\alpha s} \Sigma(s) \, dB(s). \quad (7.10)$$

Then $C_{j,0}(t) - C_{j,0}^*(t) \to 0$ as $t \to \infty$ a.s., and by proceeding as before we obtain

$$C_j(t) - C_{j,0}^*(t) \to 0, \quad \text{as } t \to \infty \text{ a.s.} \quad (7.11)$$
Therefore, returning to (7.4) and using (7.6), (7.9) we have
\[\int_0^t S(t-s) \sum_{n=1}^N \{A^*_j(t) + C^*_j(t)\} dB(s) = \sum_{j=1}^N \{A_j(t) - A^*_j(t)\} + \sum_{j=1}^N \{C_j(t) - C^*_j(t)\}\]
\[+ \int_0^t \sum_{j=1}^N e^{-\alpha s} \frac{P_{j,n-1}(t-s)}{n} \cos(\beta_j(t-s))\Sigma(s) dB(s)\]
\[+ \int_0^t \sum_{j=1}^N e^{-\alpha s} \frac{Q_{j,n-1}(t-s)}{n} \sin(\beta_j(t-s))\Sigma(s) dB(s),\]
so by (7.8) and (7.11) we have
\[\lim_{t \to \infty} \left( \int_0^t S(t-s) \sum_{n=1}^N \{A^*_j(t) + C^*_j(t)\} dB(s) = 0, \right. \text{ a.s. (7.13)}\]

Using (7.1), (7.3), (7.13) together with the definitions (7.7) and (7.10), we have
\[\lim_{t \to \infty} \left( Y(t) = \int_0^t S(t-s)\Sigma(s) dB(s) + \int_0^t R(t-s)\Sigma(s) dB(s), \right. \text{ a.s. (7.14)}\]

where \(L_{1,j}\) and \(L_{2,j}\) are given by (2.16a) and (2.16b), which is (2.15).

7.2. Proof of Proposition 1 for \(n = 0\). Using (2.12) and (2.14) we may write
\[Y(t) = \int_0^t S(t-s)\Sigma(s) dB(s) + \int_0^t R(t-s)\Sigma(s) dB(s), \quad t \geq 0,\]
where
\[S(t) = \sum_{j=1}^N e^{\alpha t} \{P^*_j \cos(\beta_j t) + Q^*_j \sin(\beta_j t)\}.\]

Thus,
\[e^{-\alpha t} Y(t) = \int_0^t e^{-\alpha t} S(t-s)\Sigma(s) dB(s) + \int_0^t e^{-\alpha t} R(t-s)\Sigma(s) dB(s). \quad (7.15)\]

Defining \(k(t) = e^{-\alpha t} R(t)\), then from (2.10) and (2.11), \(k(t) = O(e^{-\epsilon t})\) and
\[|k'(t)| \leq |\alpha| |k(t)| + e^{-\alpha t} |R'(t)| = O(e^{-\epsilon t})\]

Thus
\[\int_0^t e^{-\alpha t} R(t-s)\Sigma(s) dB(s) = \int_0^t k(t-s)e^{-\alpha s}\Sigma(s) dB(s)\]
and so Lemma 5 applied element-wise gives
\[\lim_{t \to \infty} \int_0^t e^{-\alpha t} R(t-s)\Sigma(s) dB(s) = 0 \quad \text{a.s. (7.16)}\]
Moreover,
\[
\lim_{t \to \infty} \left( \int_0^t e^{-\alpha s} S(t-s) \Sigma(s) dB(s) \right) = 0 \tag{7.17}
\]
\[
- \cos(\beta_j t) \int_0^\infty e^{-\alpha s} \{ P_j^* \cos(\beta_j s) - Q_j^* \sin(\beta_j s) \} \Sigma(s) dB(s) \\
- \sin(\beta_j t) \int_0^\infty e^{-\alpha s} \{ P_j^* \sin(\beta_j s) + Q_j^* \cos(\beta_j s) \} \Sigma(s) dB(s) \right) = 0.
\]
Using (7.16) and (7.17) in (7.15), gives the required result.

7.3. Proof of Corollary \[1\] In order to prove Corollary \[1\] the following simple asymptotic estimate is needed. It may be considered as a deterministic analogue of Lemma \[3\]

Lemma 6. For any \( \phi \in L^1([0, \infty); \mathbb{R}^d) \),
\[
\lim_{t \to \infty} \frac{1}{t^j} \int_0^t s^j \phi(s) ds = 0, \quad j = 1, ..., n.
\]

Proof. For any \( \theta \in (0, 1) \),
\[
\left| \frac{1}{t^j} \int_0^t s^j \phi(s) ds \right| \leq \frac{1}{t^j} \int_0^\theta t s^j |\phi(s)| ds + \frac{1}{t^j} \int_\theta^t s^j |\phi(s)| ds \\
\leq \theta^j \int_0^\infty |\phi(s)| ds + \int_\theta^\infty |\phi(s)| ds
\]
Thus,
\[
\limsup_{t \to \infty} \left| \frac{1}{t^j} \int_0^t s^j \phi(s) ds \right| \leq \theta^j \int_0^\infty |\phi(s)| ds.
\]
Letting \( \theta \to 0 \) gives the result. \( \square \)

We are now in a position to proceed with the proof of Corollary \[1\] Firstly consider the case \( n \geq 1 \). The asymptotic behaviour of \( Y \) is known from Proposition \[1\] Thus we concentrate solely upon the term \( \int_0^t r(t-s) f(s) ds \) in (2.18) in determining the asymptotic behaviour of \( V \). Defining
\[
S(t) = \sum_{j=1}^N e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \}, \quad t \geq 0.
\]
Then we have
\[
\int_0^t r(t-s) \frac{f(s)}{t^n e^{\alpha t}} ds = \int_0^t \frac{S(t-s)}{t^n e^{\alpha t}} f(s) ds + \int_0^t \frac{R(t-s)}{t^n e^{\alpha t}} f(s) ds.
\]
Then,
\[
\left| \int_0^t \frac{R(t-s)}{t^n e^{\alpha t}} f(s) ds \right| \leq \frac{1}{(1+t)^n} M \int_0^t (1+t-s)^n-1 e^{-\alpha s} |f(s)| ds \\
\leq \frac{1}{1+t} M \int_0^t e^{-\alpha s} |f(s)| ds.
\]
Taking the limit superior, as \( t \to \infty \), over this inequality yields,
\[
\lim_{t \to \infty} \frac{1}{t^n e^{\alpha t}} \int_0^t R(t-s) f(s) ds = 0.
\]
In analysing the term $S(t - s)$ one may decompose the trigonometric terms via (7.5), whilst the polynomial terms, $P_j$ and $Q_j$ may be dealt with using Newton’s binomial expansion, i.e.

$$(t - s)^n = \sum_{m=0}^{n} \binom{n}{m} t^{n-m} (-s)^m.$$ 

This, together with Lemma 6, yields

$$\lim_{t \to \infty} \left( \int_0^t \frac{r(t - s)}{t^n e^{\alpha t}} f(s) \, ds - \sum_{j=1}^{\infty} \{\sin(\beta_j t) D_{1,j} + \cos(\beta_j t) D_{2,j}\} \right) = 0.$$ 

Combining this with Proposition 1 yields the result for $V$.

For the case $n = 0$, the proof follows as for the case $n \geq 1$. However in the analysis of the remainder term, $R$, it is required to understand the asymptotic behaviour of the integral

$$\int_0^t e^{-\varepsilon (t-s)} e^{-\alpha s} f(s) \, ds.$$ 

This integral is the convolution of a term in $L^1(0, \infty)$ with a term which tends to zero. Hence this integral itself tends to zero. [29 Theorem 2.2.2 (i)].

8. Proof of Lemmas 1 and 2

This section contains the asymptotic estimates needed for the remainder terms $R$ defined in (3.9) and (3.18).

8.1. Proof of Lemma 1. We start with the proof of a preliminary lemma.

Lemma 7. Let $K_{j,0}$ be defined by (3.8) with $n = 0$. Then

$$\left( \lambda_j I_d - \int_{[0,\infty)} e^{-\lambda_j s} \mu(ds) \right) K_{j,0} = 0_{d,d},$$

where $\lambda_j \in \Lambda'$ are zeroes of $h_\mu(\lambda)$.

A corresponding result can be shown for the zeroes of the characteristic equation, $g_\nu$, of the finite delay equation using (3.17) and is omitted.

Proof of Lemma 7. Multiply (3.8) on the left by $(\lambda - \lambda_j) \left( \lambda I_d - \int_{[0,\infty)} e^{-\lambda s} \mu(ds) \right)$ to get

$$(\lambda - \lambda_j) I_d = \left( \lambda I_d - \int_{[0,\infty)} e^{-\lambda s} \mu(ds) \right) K_{j,0}$$

$$+ (\lambda - \lambda_j) \left( \lambda I_d - \int_{[0,\infty)} e^{-\lambda s} \mu(ds) \right) \hat{q}_j(\lambda).$$

Now let $\lambda \to \lambda_j$, recalling that $\hat{q}_j(\lambda)$ is analytic at $\lambda_j$, to get the result. \qed
We are now in a position to prove Lemma 1. To start, we define \( \tilde{q}(t) = e^{-\alpha t}q(t) \) for \( t \geq 0 \). Then \( \tilde{q} \) is differentiable a.e. and
\[
\int_0^\infty e^{\varepsilon t}|\tilde{q}(t)|\, dt < +\infty,
\]
where \( \varepsilon \) is defined as in Subsection 3.1. Also \( |\tilde{q}'(t)| \leq e^{-\alpha t}|q'(t)| + |\alpha|e^{-\alpha t}|q(t)| \) for \( t \geq 0 \). Since \( q, q' \in L^1(\mathbb{R}^+; \varphi; \mathbb{R}^{d\times d}) \), we have
\[
\int_0^\infty e^{\varepsilon t}|\tilde{q}'(t)|\, dt \leq \int_0^\infty e^{\varepsilon t}e^{-\alpha t}|q'(t)|\, dt + \int_0^\infty |\alpha|e^{\varepsilon t}e^{-\alpha t}|q(t)|\, dt < +\infty.
\]
Finally, we have that
\[
\tilde{q}(t)e^{\varepsilon t} = \tilde{q}(0) + \int_0^t \tilde{q}'(s)e^{\varepsilon s}\, ds + \varepsilon \int_0^t \tilde{q}(s)e^{\varepsilon s}\, ds,
\]
so \( |\tilde{q}(t)| \leq C e^{-\alpha t} \) for all \( t \geq 0 \).

Let \( \Lambda_n' = \{\lambda_1, \ldots, \lambda_N\} \). Then from (3.6) and (3.9), we get
\[
e^{-\alpha t}R(t) = \sum_{\lambda_j \in \Lambda_n \setminus \Lambda_n' \setminus \mathbb{N}(\lambda)} e^{-(\alpha - R(\lambda))t} \{P_j(t) \cos(\Im(\lambda_j)t) + Q_j(t) \sin(\Im(\lambda_j)t)\} + \tilde{q}(t)
\]
\[
= \sum_{\lambda_j \in \Lambda_n \setminus \Lambda_n' \setminus \mathbb{N}(\lambda)} e^{-(\alpha - R(\lambda))t} \{P_j(t) \cos(\Im(\lambda_j)t) + Q_j(t) \sin(\Im(\lambda_j)t)\} + \sum_{\lambda_j \in \Lambda_n' \setminus \mathbb{N}(\lambda)} e^{-(\alpha - R(\lambda))t} \{P_j(t) \cos(\Im(\lambda_j)t) + Q_j(t) \sin(\Im(\lambda_j)t)\} + \tilde{q}(t).
\]
If \( n = 0 \), then \( R(t) = O(e^{(\alpha - \varepsilon) t}) \) as \( t \to \infty \). If \( n \geq 1 \), and \( \Lambda_n' = \Lambda' \cap \{\mathbb{N}(\lambda) \geq 0\} \), then \( R(t) = O(t^{-1}e^{\alpha t}) \). If \( n \geq 1 \), and \( \Lambda_n' \subset \Lambda' \cap \{\mathbb{N}(\lambda) \geq 0\} \), then \( R(t) = O(t^{-1}e^{\alpha t}) \) as \( t \to \infty \). Therefore if \( n \geq 1 \), we always have \( R(t) = O(t^{n-1}e^{\alpha t}) \) as \( t \to \infty \).

We now prove the estimate on the derivative. We deal here with the case \( n \geq 1 \).

From (3.2) we know that \( r \) is differentiable and hence from (3.9) so too is \( R \).

Defining
\[
S(t) := \sum_{j=1}^{N} e^{\alpha t}\{P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t)\}
\]
and using (3.2) and (3.9) we have
\[
R'(t) = r'(t) - S'(t) = \int_{[0,t]} \mu(ds) r(t-s) - S'(t).
\]
It is clear from (3.6) that \( r(t) = O(t^{n}e^{\alpha t}) \) and from the definition of \( S(t) = O(t^{n}e^{\alpha t}) \). Therefore, it follows that \( ||r(t)\| \leq M(1+t)^{n}e^{\alpha t} \) and \( ||S'(t)|| \leq M(1+t)^{n}e^{\alpha t} \) for \( t \geq 0 \) and some \( M > 0 \). Hence as \( |\mu| \in M(\mathbb{R}^+; \mathbb{R}) \) and \( \int_{[0,\infty)} e^{-\alpha t}|\mu|(ds) < +\infty \), we have
\[
||R'(t)|| \leq \int_{[0,t]} |\mu|(ds) \|r(t-s)\| + ||S'(t)||
\]
\[
\leq \int_{[0,t]} |\mu|(ds) M(1+t-s)^{n}e^{\alpha(t-s)} + M(1+t)^{n}e^{\alpha t}
\]
\[
\leq \int_{[0,t]} |\mu|(ds) M(1+t)^{n}e^{\alpha(t-s)} + M(1+t)^{n}e^{\alpha t}
\]
\[
\leq M(1+t)^{n}e^{\alpha t} \int_{[0,\infty)} e^{-\alpha s}|\mu|(ds) + M(1+t)^{n}e^{\alpha t},
\]
and therefore \( R'(t) = O(t^{n}e^{\alpha t}) \) for \( n \geq 1 \).
For the case \( n = 0 \), we define
\[
S(t) := \sum_{j=1}^{N} e^{\alpha t} \{ P_j^* \cos(\beta_j t) + Q_j^* \sin(\beta_j t) \},
\]
then the real function \( S \) can be rewritten concisely using complex constants as
\[
S(t) = \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} K_{j,0}.
\]
As \( R(t) = r(t) - S(t) \) we have
\[
R'(t) = r'(t) - S'(t) = \int_{[0,t]} \mu(ds)\{r(t) - s\} - \lambda_j \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} K_{j,0}
\]
\[
= \int_{[0,t]} \mu(ds)R(t - s) + \int_{[0,t]} \mu(ds) \sum_{\lambda_j \in \Lambda'} e^{\lambda_j(t-s)} K_{j,0} - \sum_{\lambda_j \in \Lambda'} \lambda_j e^{\lambda_j t} K_{j,0}
\]
\[
= \int_{[0,t]} \mu(ds)R(t - s) - \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} \left( \lambda_j I_d - \int_{[0,t]} e^{-\lambda_j s} \mu(ds) \right) K_{j,0}
\]
\[
= \int_{[0,t]} \mu(ds)R(t - s) - \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} \left( \lambda_j I_d - \int_{[0,\infty)} e^{-\lambda_j s} \mu(ds) \right) K_{j,0}
\]
\[
- \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} \int_{(t,\infty)} e^{-\lambda_j s} \mu(ds) K_{j,0}.
\]
By Lemma \[\square\] the second term on the right–hand side is equal to zero, and so
\[
|R'(t)| \leq \left| \int_{[0,t]} \mu(ds)R(t - s) \right| + \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} \int_{(t,\infty)} e^{-\lambda_j s} \mu(ds) K_{j,0} \right|. \quad (8.1)
\]
Now,
\[
\left| \int_{[0,t]} \mu(ds)R(t - s) \right| \leq \int_{[0,t]} |\mu|(ds) Me^{(\alpha - \epsilon)(t-s)}
\]
\[
= e^{(\alpha - \epsilon)t} \int_{[0,t]} e^{-(\alpha - \epsilon)s} |\mu|(ds) M
\]
\[
\leq e^{(\alpha - \epsilon)t} \int_{[0,\infty)} e^{-(\alpha - \epsilon)s} |\mu|(ds) M.
\]
Thus, \( \int_{[0,t]} \mu(ds)R(t - s) = O(e^{(\alpha - \epsilon)t}) \). Recalling that \( \lambda_j = \alpha + i\beta_j \) and so \( |e^{\lambda_j t}| = e^{\alpha t} \). Thus,
\[
\left| \sum_{\lambda_j \in \Lambda'} e^{\lambda_j t} \int_{(t,\infty)} e^{-\lambda_j s} \mu(ds) K_{j,0} \right| \leq e^{\alpha t} \sum_{\lambda_j \in \Lambda'} \int_{(t,\infty)} e^{-\alpha s} |\mu|(ds) M
\]
\[
= e^{\alpha t} \sum_{\lambda_j \in \Lambda'} \int_{(t,\infty)} e^{-\epsilon s} e^{-(\alpha - \epsilon)s} |\mu|(ds) M
\]
\[
\leq e^{(\alpha - \epsilon)t} \sum_{\lambda_j \in \Lambda'} \int_{(t,\infty)} e^{-(\alpha - \epsilon)s} |\mu|(ds) M
\]
\[
\leq e^{(\alpha - \epsilon)t} M_1,
\]
where it is noted that \( \Lambda' \) contains finitely many elements. Therefore, \( \square \) gives
\[
R'(t) = O(e^{(\alpha - \epsilon)t}), \quad t \to \infty,
\]
and this completes the proof.

8.2. **Proof of Lemma 2.** We now use (3.15) to determine properties of $R$ of (3.18). From (3.18)

$$
    r(t) = \sum_{j=1}^{N} e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \} + R(t), \quad t \geq 0.
$$

In the case when $\{\lambda_1, ..., \lambda_N\} = \Lambda' \cap \{\Im(\lambda) \geq 0\}$, we have that $R(t) = e^{(\alpha - \varepsilon) t}$ for all $\varepsilon \in (0, \varepsilon_0)$. If $n \geq 1$, and $\{\lambda_1, ..., \lambda_N\} \subset \Lambda' \cap \{\Im(\lambda) \geq 0\}$, then $R(t) = O(t^{n-1} e^{\alpha t})$ as $t \to \infty$. Therefore if $n \geq 1$, we always have $R(t) = O(t^{n-1} e^{\alpha t})$ as $t \to \infty$. If $n = 0$, then $R(t) = O(e^{(\alpha - \varepsilon) t})$ as $t \to \infty$.

We deal here with the case $n \geq 1$. From (3.12) we know that $r$ is differentiable and hence from (3.18) so too is $R$. Defining

$$
    S(t) := \sum_{j=1}^{N} e^{\alpha t} \{ P_j(t) \cos(\beta_j t) + Q_j(t) \sin(\beta_j t) \}
$$

and using (3.12) and (3.18) we have

$$
    R'(t) = r'(t) - S'(t) = \int_{[-\tau,0]} \nu(ds) r(t + s) - S'(t), \quad \text{for all } t \geq \tau.
$$

It is clear from (3.15) that $r(t) = O(t^n e^{\alpha t})$ and from the definition of $S$ that $S'(t) = O(t^n e^{\alpha t})$. Thus, there exists $t_0 \geq 0$ and positive constant matrices $M_1, M_2$ such that for $t \geq t_0 + \tau$,

$$
    |R'(t)| \leq \int_{[-\tau,0]} |\nu|(ds) |r(t + s)| + t^n e^{\alpha t} M_2
    \leq \int_{[-\tau,0]} |\nu|(ds) (s + t)^n e^{\alpha(t+s)} M_1 + t^n e^{\alpha t} M_2
    \leq t^n e^{\alpha t} \int_{[-\tau,0]} e^{\alpha s} |\nu|(ds) M_1 + t^n e^{\alpha t} M_2.
$$

Thus, $R'(t) = O(t^n e^{\alpha t})$.

What remains to be covered is the case when $n = 0$. To do this, we start by making the observation that the differential resolvent of (3.12) may be regarded as the solution of a Volterra equation. To see this, define $\nu_+(E) = \nu(-E)$ where $-E = \{x : -x \in E\}$ for all sets $E$ which are subsets of the Borel sets formed from the interval $[0, \tau]$ and $\nu_+(E) = 0$ for all sets $E$ which are subsets of the Borel sets formed from the interval $(\tau, \infty)$. Then

$$
    r'(t) = \int_{[0,\tau]} \nu_+(ds) r(t - s) \text{ for } t \geq 0, \quad r(0) = I_d.
$$

For $t > \tau$,

$$
    r'(t) = \int_{[0,\tau]} \nu_+(ds) r(t - s) - \int_{(\tau, t]} \nu_+(ds) r(t - s)
    = \int_{[0,\tau]} \nu_+(ds) r(t - s)
$$

as $\nu_+ = 0$ in the second term on the right-hand side. On the other hand, for $0 \leq t \leq \tau$, it is true that max$\{-\tau, -t\} = -t$ and hence

$$
    r'(t) = \int_{[0,\tau]} \nu_+(ds) r(t - s) \text{ for } t \geq 0, \quad r(0) = I_d.
$$
The case \( n = 0 \) follows from this observation, using a similar proof to that which established Lemma 11

9. PROOF OF REMARK 4

In this case \( r(t) = e^{at} \) and \( X \) obeys, for \( t \geq 0, \)

\[
e^{-at}X(t) = X_0 + \int_0^t e^{-as} f(s) \, ds + \int_0^t e^{-as} \Sigma(s) \, dB(s). \tag{9.1}
\]

Define the Gaussian martingale \( M(t) = \int_0^t e^{-as} \Sigma(s) \, dB(s) \) and the deterministic function \( d(t) = X_0 + \int_0^t e^{-as} f(s) \, ds. \) Then from (4.7), we have on this event of positive probability that

\[
\lim_{t \to \infty} \{ M(t) + d(t) \} = L \in (-\infty, \infty).
\]

Suppose that \( \lim_{t \to \infty} \langle M \rangle(t) = +\infty. \) Consequently \( \limsup_{t \to \infty} M(t) = +\infty \) and \( \liminf_{t \to \infty} M(t) = -\infty. \) Also, \( \limsup_{t \to \infty} \{ d(t) \} = +\infty, \) otherwise, if \( d(t) \leq D \) for all \( t \geq 0, \) we have

\[
L = \liminf_{t \to \infty} \{ d(t) + M(t) \} \leq D + \liminf_{t \to \infty} M(t) = -\infty,
\]

which is a contradiction. (Similarly one can show that \( \liminf_{t \to \infty} d(t) = -\infty. \)

Then there exists a deterministic sequence \( \{ t_n \}_{n \in \mathbb{Z}^+}, \) with \( t_0 = 0 \) and \( t_n \to \infty \) as \( n \to \infty, \) such that \( d(t_{n+1}) > d(t_n) \) and \( d(t_n) \to \infty \) as \( n \to \infty. \) Then \( M(t_n) \to -\infty \) as \( n \to \infty. \)

Now,

\[
\tilde{M}(n) := M(t_n) = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} e^{-as} \Sigma(s) \, dB(s) = \sum_{j=1}^n G_j,
\]

where each \( G_j = \int_{t_{j-1}}^{t_j} e^{-as} \Sigma(s) \, dB(s) \) is a Gaussian distributed random variable with mean zero and variance \( \int_{t_{j-1}}^{t_j} e^{-2as} \Sigma(t)^2 \, dt, \) each \( G_j \) is measurable with respect to the filtration \( \mathcal{G}_n = \mathcal{F}^B(t_n), \) \( n \geq 1, \) and \( \{ G_j \}_{j \in \mathbb{Z}^+} \) are independent and \( \langle M \rangle(n) = \langle M \rangle(t_n) = \int_0^{t_n} e^{-2as} \Sigma(t)^2 \, dt \to \infty \) as \( n \to \infty. \)

Therefore by arguments akin to that used in Shiryaev 61 Section 4.1]

\[
P \left[ \limsup_{n \to \infty} \frac{\tilde{M}(n)}{\sqrt{\langle M \rangle(n)}} = +\infty \right] = 1, \quad P \left[ \liminf_{n \to \infty} \frac{\tilde{M}(n)}{\sqrt{\langle M \rangle(n)}} = -\infty \right] = 1, \quad (9.2)
\]

which implies that \( P \left[ \limsup_{n \to \infty} \tilde{M}(n) = +\infty \right] = 1 \) and so that

\[
P \left[ \limsup_{n \to \infty} M(t_n) = +\infty \right] = 1.
\]

But our assumption gave that \( \lim_{n \to \infty} M(t_n) = -\infty, \) with positive probability. Thus a contradiction. Hence \( \langle M \rangle(t) \to L^1 \in (-\infty, \infty) \) as \( t \to \infty, \) i.e.

\[
\int_0^\infty e^{-2at} \Sigma(t)^2 \, dt < +\infty.
\]

Therefore \( M(t) \to M(\infty) \in (-\infty, \infty) \) as \( t \to \infty \) a.s. and so \( \lim_{t \to \infty} d(t) = \lim_{t \to \infty} \{ d(t) + M(t) - M(t) \} = L - M(\infty) \in (-\infty, \infty). \) Hence

\[
\lim_{t \to \infty} \int_0^t e^{-as} f(s) \, ds = \lim_{t \to \infty} \{ d(t) - X_0 \} \in (-\infty, \infty).
\]
All that remains to be shown is the validity of (9.2), i.e. we need to show that
\[ A' = \left\{ \limsup_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} = +\infty \right\}, \quad A'' = \left\{ \liminf_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} = -\infty \right\} \]
are almost sure events. Let
\[ A'_c = \left\{ \limsup_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} > c \right\}, \quad A''_c = \left\{ \liminf_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} < -c \right\}. \]
Then \( A'_c \to A' \) and \( A''_c \to A'' \) as \( c \to \infty \) and \( A', A'', A'_c, A''_c \) are tail events. We show that \( \mathbb{P}[A'_c] = \mathbb{P}[A''_c] = 1 \) for all \( c > 0 \).

Using Section 4.1.5 Problem 5, pp.383 of [61] gives
\[ \mathbb{P}[A'_c] = \mathbb{P} \left[ \limsup_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} > c \right] \geq \limsup_{n \to \infty} \mathbb{P} \left[ \frac{M(n)}{\sqrt{\langle M \rangle(n)}} > c \right] = 1 - \Phi(c) > 0 \]
and
\[ \mathbb{P}[A''_c] = \mathbb{P} \left[ \liminf_{n \to \infty} \frac{M(n)}{\sqrt{\langle M \rangle(n)}} < -c \right] = \mathbb{P} \left[ \limsup_{n \to \infty} \frac{-M(n)}{\sqrt{\langle M \rangle(n)}} > c \right] \geq 1 - \Phi(c) > 0. \]
So, \( \mathbb{P}[A'_c] > 0 \) and \( \mathbb{P}[A''_c] > 0 \), then since the \( G_j \)'s are independent an application of Kolmogrov’s Zero-One Law, c.f. e.g. [61], Theorem 4.1.1, implies \( \mathbb{P}[A'_c] = \mathbb{P}[A''_c] = 1 \). Therefore \( \mathbb{P}[A'] = \lim_{c \to \infty} \mathbb{P}[A'_c] = 1 \) and \( \mathbb{P}[A''] = \lim_{c \to \infty} \mathbb{P}[A''_c] = 1 \).

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