We prove the irreducibility of the universal non-degenerate Whittaker modules for the affine Lie algebra \( \hat{\mathfrak{sl}}_2 \) of type \( A^{(1)}_1 \) with noncritical level which are also irreducible Whittaker modules over \( \tilde{\mathfrak{sl}}_2 = \hat{\mathfrak{sl}}_2 + \mathbb{C}d \) with the same Whittaker function and central charge. We have to modulo a central character for \( \mathfrak{sl}_2 \) to obtain irreducible degenerate Whittaker \( \hat{\mathfrak{sl}}_2 \)-modules with noncritical level. In the case of critical level the universal Whittaker module is reducible. We prove that the quotient of universal Whittaker \( \hat{\mathfrak{sl}}_2 \)-module by a submodule generated by a scalar action of central elements of the vertex algebra \( V_{-2}(\mathfrak{sl}_2) \) is irreducible as \( \hat{\mathfrak{sl}}_2 \)-module. We also explicitly describe the simple quotients of universal Whittaker modules at the critical level for \( \tilde{\mathfrak{sl}}_2 \). Quite surprisingly, with the same Whittaker function and the same central character of \( V_{-2}(\mathfrak{sl}_2) \), some irreducible \( \tilde{\mathfrak{sl}}_2 \) Whittaker modules can have semisimple or free action of \( d \). At last, by using vertex algebraic techniques we present a Wakimoto type construction of a family of generalized Whittaker irreducible modules for \( \hat{\mathfrak{sl}}_2 \) at the critical level. This family includes all classical Whittaker modules at critical level. We also have Wakimoto type realization for irreducible degenerate Whittaker modules for \( \hat{\mathfrak{sl}}_2 \) at noncritical level.

1. Introduction

Recently, Whittaker modules over various Lie algebras are attracting a lot of attentions from mathematicians. See [MZ2] and the references there. In Block’s classification of all irreducible modules for the three-dimensional simple Lie algebra \( \mathfrak{sl}_2 \), they fall into two families: highest (lowest) weight modules, and a family which are irreducible modules over a Borel subalgebra of \( \mathfrak{sl}_2 \) including Whittaker modules (see [B]). Whittaker modules are an important class of irreducible modules. Kostant defined and systematically studied in [Ko] Whittaker modules for an arbitrary finite dimensional complex semisimple Lie algebra \( \mathfrak{g} \). He showed that these modules with a fixed regular Whittaker function (Lie homomorphism) on a nilpotent radical are (up to isomorphism) in bijective correspondence with central characters of \( U(\mathfrak{g}) \).

McDowell studied in [M4] a category of modules for an arbitrary finite-dimensional complex semisimple Lie algebra \( \mathfrak{g} \) which includes the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) as well as those Whittaker modules where the Whittaker function on a nilpotent radical may be irregular (degenerate). The irreducible objects in this category are constructed by inducing over a parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) from an irreducible Whittaker module (in Kostant’s sense) or from a highest weight module for the reductive Levi factor of \( \mathfrak{p} \).

2000 Mathematics Subject Classification. Primary 17B69, Secondary 17B67, 17B68, 81R10.

Key words and phrases. affine Lie algebras, vertex algebras, Wakimoto modules, Whittaker modules, critical level, Virasoro algebra.
Affine Lie algebras are the most extensively studied and most useful ones among the infinite-dimensional Kac-Moody algebras. The integrable highest weight modules were the first class of representations over affine Kac-Moody algebras being extensively studied, see [K] for detailed discussion of results and further bibliography. In [Ch] V. Chari classified all irreducible integrable weight modules with finite-dimensional weight spaces over the untwisted affine Lie algebras. Chari and Pressley, [CP2], then extended this classification to all affine Lie algebras. Verma-type modules were first studied by Jakobsen and Kac [JK], and then by Futorny [Fu1, Fu2].

Very recently, a complete classification for all irreducible weight modules with finite-dimensional weight spaces over affine Lie algebras were obtained in [DG]. Naturally, the next important task is to study irreducible weight modules with infinite-dimensional weight spaces and irreducible non-weight modules. Besides the irreducible modules constructed in [CP1], a class of irreducible weight modules over affine Lie algebras with infinite-dimensional weight spaces were constructed in [BBFK]. A complete classification for all irreducible (weight and non-weight) modules over affine Lie algebras with locally nilpotent action of the nilpotent radical were obtained in [MZ]. All irreducible (weight and non-weight) modules over untwisted affine Lie algebras with locally finite action of the nilpotent radical were classified in [GZ], where the structure of simple Whittaker modules are unclear.

A class of irreducible modules for non-twisted affine Lie algebras from irreducible Whittaker modules over the subalgebra generated by imaginary root spaces (isomorphic to an infinite dimensional Heisenberg algebra) were constructed in [Ch1]. These modules are called imaginary Whittaker modules since they are different from the above Whittaker modules in nature.

It is natural to investigate usual Whittaker modules over affine Kac-Moody algebras in the more general setting as in [Mc], i.e., the Whittaker function on the nilpotent radical may be irregular. The objective of the present paper is to completely determine the structure of Whittaker modules over the affine Kac-Moody algebra $A^{(1)}_1$ (including $\tilde{sl}_2$ and $\hat{sl}_2$). Let us first recall the related affine Lie algebra theory and vertex algebra theory from [FB, K, LL, DL].

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot, \cdot)$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ be a triangular decomposition for $\mathfrak{g}$.

The affine Lie algebra $\tilde{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is the canonical central element [K] and the Lie algebra structure is given by

$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m, 0}c,$

$[d, x \otimes t^n] = nx \otimes t^n$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. We will write $x(n)$ for $x \otimes t^n$.

The Cartan subalgebra $\tilde{\mathfrak{g}}_0$ of $\tilde{\mathfrak{g}}$ are defined by

$\tilde{\mathfrak{g}}_0 = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,$

$\tilde{\mathfrak{g}}_\pm = \mathfrak{g} \otimes t^\pm \mathbb{C}[t^\pm] + n_\pm \otimes \mathbb{C}.$

Denote $\hat{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_-$, where $\tilde{\mathfrak{g}}_0 = \mathfrak{h} \oplus \mathbb{C}c$, $\tilde{\mathfrak{g}}_\pm = \tilde{\mathfrak{g}}_\pm$. Let $\hat{\mathfrak{b}}$ be the subalgebra generated by $e(n), h(n)$ for $n \in \mathbb{Z}$.

Let $P = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$. For every $\kappa \in \mathbb{C}$, let $P_{\kappa}$ be 1-dimensional $P$–module such that the subalgebra $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially, and the central element $c$ acts as
Let Theorem 1.1. Assume that \( \hat{\text{irr}} \) irreducible Whittaker modules for \( \mathfrak{sl}_2 \) \( V \) satisfies the commutation relation for the Virasoro algebra of central charge \( \mathfrak{v} \) such that \( e \mathfrak{v} \) \( \lambda \in \mathbb{C} \). Define the generalized Verma module \( V_\kappa(\mathfrak{g}) \) as
\[
V_\kappa(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C} V_\kappa.
\]
Then \( V_\kappa(\mathfrak{g}) \) has a natural structure of a vertex algebra generated by fields
\[
x(z) = Y(x(-1)1, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}, \quad (x \in \mathfrak{g}),
\]
where \( 1 = 1 \otimes v_\kappa \) is the vacuum vector.
A \( \mathfrak{g} \)-module \( M \) is called restricted if for every \( w \in M \) and \( x \in \mathfrak{g} \) we have
\[
x(z)w \in \mathbb{C}((z)).
\]
A restricted \( \mathfrak{g} \)-module of level \( \kappa \) has the structure of a module over vertex algebra \( V_\kappa(\mathfrak{g}) \).
Let now \( \mathfrak{g} = \mathfrak{sl}_2 \) with standard generators \( e, f, h \); and \( \mathfrak{b} = \mathbb{C} h + \mathbb{C} e \).
Assume first that \( \kappa \neq -2 \). Let
\[
\omega = \frac{1}{2(\kappa + 2)} (e(-1)f(-1) + f(-1)e(-1) + 1/2h(-1)^2) 1
\]
be the canonical Sugawara Virasoro vector in the vertex algebra \( V_\kappa(\mathfrak{sl}_2) \). Then the components of the field
\[
Y(\omega, z) = L(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}
\]
satisfies the commutation relation for the Virasoro algebra of central charge \( c_\kappa = \frac{3\kappa}{\kappa + 2} \). Then every module over \( V_\kappa(\mathfrak{sl}_2) \) becomes a module for the Virasoro algebra. Recall also that
\[
[L(n), x(m)] = -mx(n + m) \quad \text{for} \quad x \in \{e, f, h\}.
\]
In particular,
\[
[L(n), x(0)] = 0 \quad \text{for} \quad x \in \{e, f, h\}.
\]
Let \( \kappa = -2 \). Let \( t = \frac{1}{2}(e(-1)f(-1) + f(-1)e(-1) + \frac{1}{2}h(-1)^2) 1 \in V_{-2}(\mathfrak{sl}_2) \) and
\[
T(z) = Y(t, z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-2}.
\]
Then
\[
[T(n), x(m)] = 0 \quad \forall m, n \in \mathbb{Z},
\]
i.e., \( T(n) \) are central elements. In particular, \( t \) generates the center of the vertex algebra \( V_{-2}(\mathfrak{sl}_2) \) (cf. [F], [FB]). The center is a commutative vertex algebra \( M_T(0) \) which is as vector spaces isomorphic to the polynomial algebra \( \mathbb{C}[T(-n) \mid n \geq 2] \).

Let us here describe the main results of the paper. Let \( (\lambda, \mu) \in \mathbb{C}^2 \), and \( V_\kappa(\mathfrak{sl}_2, \lambda, \mu, \kappa) \) be the universal Whittaker module of level \( \kappa \) generated by vector \( w_{\lambda, \mu, \kappa} \) such that \( e(0)w_{\lambda, \mu, \kappa} = \lambda w_{\lambda, \mu, \kappa} \), \( f(1)w_{\lambda, \mu, \kappa} = \mu w_{\lambda, \mu, \kappa} \) (for details see Section 2). If \( \lambda \cdot \mu \neq 0 \) we call such Whittaker module non-degenerate. We describe all the irreducible Whittaker modules for \( \mathfrak{sl}_2 \) and \( \mathfrak{sl}_2 \).

**Theorem 1.1.** Assume that \( \lambda \in \mathbb{C}^* \), \( \mu \in \mathbb{C} \).
(i) Let \( \mu \neq 0 \). Then \( V_\kappa(\mathfrak{sl}_2, \lambda, \mu, \kappa) \) is an irreducible \( \mathfrak{sl}_2 \)-module at level \( \kappa \neq -2 \).
(i') Let $\mu \neq 0$ and $d = -L(0) + a$, $a \in \mathbb{C}$. Then $V_{\widehat{sl}_2}(\lambda, \mu, \kappa)$ is an irreducible $\widehat{sl}_2$-module at level $\kappa \neq -2$.

(ii) For every $c(z) = \sum_{n \leq 0} c_n z^{-n-2} \in \mathbb{C}(z)$,

$$V_{\widehat{sl}_2}(\lambda, \mu, -2, c(z)) = V_{\widehat{sl}_2}(\lambda, \mu, -2)/<(T(n) - c_n)w_{\lambda, \mu, -2}, n \leq 0>$$

is an irreducible $\widehat{sl}_2$-module at the critical level.

(iii) Let $\mu \neq 0$. The induced module

$$\text{Ind}^{\widehat{sl}_2}_{\widehat{sl}_2} V_{\widehat{sl}_2}(\lambda, \mu, -2, c(z))$$

is irreducible Whittaker module at the critical level.

Modules described above provide a complete list of non-degenerate Whittaker modules for $\widehat{sl}_2$ and $sl_2$.

In the proof of these results we show that irreducible Whittaker modules for $\widehat{sl}_2$ are irreducible modules for the Lie algebra $\hat{b} \times \text{Vir}$ in the non-critical case and for the Lie algebra $\hat{b} \times \mathcal{F}$ in the case of critical level.

The only case which is not completely described in the above theorem is the case of degenerate Whittaker modules outside the critical level. We prove in Theorem 3.2 that simple degenerate Whittaker modules of type $(\lambda, 0)$ without critical level ($\kappa \neq -2$) are simple quotients of the $\widehat{sl}_2$-module

$$M_{\widehat{sl}_2}(\lambda, 0, \kappa, a) := V_{\widehat{sl}_2}(\lambda, 0, \kappa)/<(L(0) - a)), (a \in \mathbb{C})$$

Irreducible degenerate Whittaker $\widehat{sl}_2$-modules with critical level are quite complicated which are completely determined in Theorem 4.7. Surprisingly, with the same Whittaker function and the same central character of $V_{\widehat{sl}_2}(\lambda, 0)$, irreducible $\widehat{sl}_2$ Whittaker modules can have semisimple or free action of $d$.

The second part of our paper is devoted to the bosonic realizations of Whittaker modules. We use the concept of Wakimoto modules for $\widehat{sl}_2$ and the theory of vertex algebras. At the critical level we present explicit realization of a family of irreducible Whittaker modules which include all classical (degenerate and non-degenerate) irreducible Whittaker modules. Let us present this result in more details.

Let $M$ be a Weyl vertex algebra generated by the fields $a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$, $a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n)z^{-n-1}$ such that the components of these fields satisfies the commutation relations [5] for the infinite dimensional Weyl algebra. Let $\pi^{2(\kappa+2)}$ be Heisenberg vertex algebra associated to the representations of the Heisenberg Lie algebra of level $2(\kappa + 2)$. Construction of Wakimoto modules is based on the embedding of the affine vertex algebra $V(\widehat{sl}_2)$ into $M \otimes \pi^{2(\kappa+2)}$ (cf. [F], [W]). This implies that for any $M$-module $M_1$ and any $\pi^{2(\kappa+2)}$-module $N_1$ the tensor product $M_1 \otimes N_1$ is a module for $V(\widehat{sl}_2)$. This construction was usually applied on highest weight modules for $M$ and $\pi^{2(\kappa+2)}$. In the present paper we shall apply this construction of Whittaker modules for $M$ and $\pi^{2(\kappa+2)}$.

For $(\lambda, \mu) \in \mathbb{C}^2$ let $M_1(\lambda, \mu)$ be the module for the Weyl algebra generated by the Whittaker vector $v_1$ such that

$a(0)v_1 = \lambda v_1$, $a^*(1)v_1 = \mu v_1$, $a(n+1)v_1 = a^*(n+2)v_1 = 0$ (n ≥ 0).

By using Whittaker module $M_1(\lambda, \mu)$ and certain Whittaker modules for the Heisenberg vertex algebra we construct a family of modules on arbitrary level (see
Section [9]. In the case of critical level, $\pi^0$ is a commutative vertex algebra and their irreducible (Whittaker) representations are 1–dimensional, so our Wakimoto modules will be actually realized on $M_1(\lambda, \mu)$. We prove:

**Theorem 1.2.** For every $\chi(z) \in \mathbb{C}((z))$, $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda \neq 0$ there exists irreducible $\mathfrak{sl}_2$–module $M_{\text{Wak}}(\lambda, \mu, -2, \chi(z))$ realized on the $M$–module $M_1(\lambda, \mu)$ such that

$$
eq a(z);$$

$$h(z) = -2 : a^*(z)a(z) : + \chi(z);$$

$$f(z) = - : a^*(z)^2a(z) : - 2\partial_z a^*(z) + a^*(z)\chi(z)$$

In the case when $\mu = 0$ and suitable $\chi(z)$ the Wakimoto modules above provide a complete list of irreducible Whittaker modules of type $(\lambda, \mu)$. But it is interesting that Wakimoto modules don’t present a construction of classical non-degenerate Whittaker modules of type $(\lambda, \mu)$.

In order to present a realization of non-degenerate Whittaker modules at the critical level we slightly generalize the concept of Wakimoto modules. Then for every $\lambda \in \mathbb{C}$ and $\chi(z) \in \mathbb{C}((z))$ we consider Whittaker modules $\Pi_\lambda$ for $\Pi(0)$ and 1–dimensional representation $M_T(\chi(z))$ of $M_T(0)$ such that $T(n)$ acts as $\chi_n \in \mathbb{C}$.

We prove:

**Theorem 1.3.** Let $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C}$ and $c(z) = \sum_{n \leq 0} c(n)z^{-n-2} \in \mathbb{C}((z))$. Set $\chi(z) = \frac{\mu}{\eta} + c(z)$. Then we have:

$$V^\vee_{\mathfrak{sl}_2}(\lambda, \mu, -2, \chi(z)) \cong M_T(\chi(z)) \otimes \Pi_\lambda.$$  

In our forthcoming papers we plan to generalize our results on higher level affine Lie algebras.

## 2. Whittaker modules

In this section we will recall Whittaker modules over a Lie algebra with a triangular decomposition and some related results.

Let $\mathcal{G}$ be a complex Lie algebra with triangular decomposition $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_0 \oplus \mathcal{G}_-$ in the sense of [MP]. Let $\eta : \mathcal{G}_+ \to \mathbb{C}$ be a Lie algebra homomorphism which will be called a Whittaker function, and $V$ be a $\mathcal{G}$–module. Note that $\eta([\mathcal{G}_+, \mathcal{G}_+]) = 0$. A nonzero vector $v \in V$ is called a Whittaker vector of type $\eta$ if $xv = \eta(x)v$ for all $x \in \mathcal{G}_+$. The module $V$ is said to be a type $\eta$ Whittaker module for $\mathcal{G}$ if it is generated by a type $\eta$ Whittaker vector. We say that $\mathcal{G}_+$ acts on $V$ locally nilpotently if for any $v \in V$ there is $s \in \mathbb{N}$ depending on $v$ such that $x_1x_2...x_sv = 0$ for any $x_1, x_2, ..., x_s \in \mathcal{G}_+$. Let $\mathcal{G}_+^{(n)} = \{x - \eta(x) | x \in \mathcal{G}_+\}$ which is a Lie subalgebra of $U(\mathcal{G}_+)$. The following result is somewhat known in some cases.

**Lemma 2.1.** Let $V$ be a type $\eta$ Whittaker module for $\mathcal{G}$. Suppose that $\mathcal{G}_+$ acts locally nilpotently on $\mathcal{G}/\mathcal{G}_+$. Then
(i). \( \mathcal{G}_+^{(n)} \) acts locally nilpotently on \( V \). In particular, \( x - \eta(x) \) acts locally nilpotently on \( V \) for any \( x \in \mathcal{G}_+ \);

(ii). any nonzero submodule of \( V \) contains a Whittaker vector of type \( \eta \);

(iii). if the Whittaker vectors of \( V \) is 1-dimensional, then \( V \) is irreducible.

Proof. Let \( V = U(\mathcal{G})v \) where \( v \) is a Whittaker vector.

(i). Let \( w = y_1y_2...y_rv \) be a nonzero vector in \( V \) where \( y_i \in \mathcal{G} \). It is enough to show that \( \mathcal{G}_+^{(n)} \) is locally nilpotent on \( w \). We shall do this by induction on \( r \). This is trivial for \( r = 0 \). Now suppose \( \mathcal{G}_+^{(n)} \) is locally nilpotent on \( w_1 = y_2...y_rv \). There exists \( s \in \mathbb{N} \) such that \( \prod_{i=1}^s (\text{ad}(x_i))y_1 \in [\mathcal{G}_+, \mathcal{G}_+] \) and \( \prod_{i=1}^s (x_i - \eta(x_i))w_1 = 0 \) for any \( x_1, x_2, ..., x_s \in \mathcal{G}_+ \). for any \( x_1, x_2, ..., x_3s \in \mathcal{G}_+ \) we have

\[
\prod_{i=1}^{3s} (x_i - \eta(x_i))w = \prod_{i=1}^{3s} (x_i - \eta(x_i)) \prod_{i=1}^{2s} (x_i - \eta(x_i))y_1w_1
\]

\[
= \prod_{i=1}^{3s} (x_i - \eta(x_i)) \sum_{p=0}^{2s} \left( \prod_{j=p+1}^{2s} (\text{ad}(x_{i_j}))y_1 \right) \prod_{j=1}^{p} (x_{i_j} - \eta(x_{i_j}))w_1
\]

\[
= \prod_{i=1}^{3s} (x_i - \eta(x_i)) \sum_{p=0}^{s-1} \left( \prod_{j=p+1}^{2s} (\text{ad}(x_{i_j}))y_1 \right) \prod_{j=1}^{p} (x_{i_j} - \eta(x_{i_j}))w_1 = 0.
\]

We have used the fact that \( \eta(\sum_{p=0}^{s-1}(\prod_{j=p+1}^{2s}(\text{ad}(x_{i_j})))y_1) = 0 \).

Thus, \( \mathcal{G}_+^{(n)} \) is locally nilpotent on \( w \).

(ii). Let \( W \) be a nonzero \( \mathcal{G} \)-submodule of \( V \) and \( w = yv \) is a nonzero vector in \( W \) where \( y \in U(\mathcal{G}) \). From (i) we know that there is \( n \in \mathbb{Z}_{>0} \) such that

\[
\prod_{i=1}^{n} (x_i - \eta(x_i))yv = 0, \text{ for any } x_1, ..., x_n \in \mathcal{G}_+.
\]

Take \( n \) minimal. There exist \( x_2, x_3, ..., x_n \in \mathcal{G}_+ \) such that \( w' = \prod_{i=2}^{n} (x_i - \eta(x_i))yv \neq 0 \) but \( (x - \eta(x))w' = 0 \) for all \( x \in \mathcal{G}_+ \). That is \( w' \) is a Whittaker vector in \( W \).

Part (iii) follows from (ii). \( \square \)

We remark that the convers of (iii) is in general not true, see Theorem 4.7.

Note that for \( \widetilde{\mathfrak{sl}_2} \) and \( \mathfrak{sl}_2 \), \( (\mathfrak{sl}_2)_+ = (\mathfrak{sl}_2)_+ \) is generated by \( e(0) \) and \( f(1) \). Thus the Whittaker function \( \eta \) is uniquely determined by \( (\lambda, \mu) = (\eta(e(0)), \eta(f(1))) \). A type \( \eta \) Whittaker module for \( \mathfrak{sl}_2 \) or \( \mathfrak{sl}_2 \) is also called a Whittaker module of type \( (\lambda, \mu) = (\eta(e(0)), \eta(f(1))) \).

For any \( \lambda, \mu, \kappa \in \mathbb{C} \), let \( J(\lambda, \mu, \kappa) \) be the left ideal of \( U(\mathfrak{sl}_2) \) generated by

\[
\{ f(i+1), e(i), f(1) - \mu, e(0) - \lambda, h(i), c - \kappa | i \in \mathbb{Z}_{>0} \}.
\]

So we have the universal level \( \kappa \) Whittaker module \( V_{\mathfrak{sl}_2}(\lambda, \mu, \kappa) := U(\mathfrak{sl}_2)/J(\lambda, \mu, \kappa) \) and denote the image of \( 1 \) by \( w_{\lambda, \mu, \kappa} \).

Similarly one may define \( V_{\mathfrak{sl}_2}(\lambda, \mu, \kappa) := U(\mathfrak{sl}_2)/\widetilde{J}(\lambda, \mu, \kappa) \), where \( \widetilde{J}(\lambda, \mu, \kappa) \) is the left ideal in \( U(\mathfrak{sl}_2) \) generated by the set \( \{ \} \).

It is important to notice that the left ideal \( \widetilde{J}(\lambda, \mu, \kappa) \) is \( d \)-invariant if and only if \( \mu = 0 \).
In this paper, we will determine all simple Whittaker modules for \( \tilde{\mathfrak{sl}}_2 \) and \( \tilde{\mathfrak{sl}}_2 \), i.e., simple quotients of \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \) and \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \).

3. Classical Whittaker modules for \( \tilde{\mathfrak{sl}}_2 \) with noncritical level. In this subsection we assume that \( \kappa \neq -2 \). Note that \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \) is a restricted module. Therefore it is a module over the universal affine vertex algebra \( V(\tilde{\mathfrak{sl}}_2) \). Recall that every module over \( V(\tilde{\mathfrak{sl}}_2) \) for \( \kappa \neq -2 \) is a module for the Virasoro algebra generated by the components of the field \( L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \) and that

\[
[L(n), x(m)] = -mx(n + m) \quad \text{for } x \in \{e, f, h\}.
\]

Denote by \( \mathbb{M} \) the set of all (infinite) vectors of the form \( i := (\ldots, i_2, i_1) \) with entries in \( \mathbb{Z}_{\geq 0} \), such that only finitely many entries nonzero. Let \( 0 = (\ldots, 0, 0), \varepsilon_n = (\ldots, \delta_{1,n}, \ldots, \delta_{1,n}) \), and \( |i| = \sum_{l=1}^{+\infty} i_l \) for all \( i \in \mathbb{M} \). For any \( i, j, k \in \mathbb{M} \), denote \( u_{i,j,k} = (e(-n)^{i_1} \ldots e(-1)^i)(-(-n)^{j_1} \ldots h(0)^j)(\ldots f(-n)^{k_1} \ldots f(0)^k) \) in \( U(\tilde{\mathfrak{sl}}_2) \). It is clear that

\[
B = \{ u_{i,j,k} w_{\lambda,\mu,\kappa} | i, j, k \in \mathbb{M} \}
\]

is a basis of \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \). For any \( i, j, k \in \mathbb{M} \), denote

\[
U_{i,j,k} = (e(-n)^{i_1} \ldots e(-1)^i)(-(-n)^{j_1} \ldots h(0)^j)(\ldots L(-n)^{k_1} \ldots L(0)^k),
\]

\[
B = \{ U_{i,j,k} w_{\lambda,\mu,\kappa} | i, j, k \in \mathbb{M} \}.
\]

Then we have the following

Lemma 3.1. Let \( \lambda \in \mathbb{C}^*, \mu, \kappa \in \mathbb{C} \) with \( \kappa \neq -2 \). Then \( B \) is a basis of the Whittaker module \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \).

Proof. We define the total order \( " \succ " \) on \( \mathbb{M} \) by \( i > j \) if and only if one of the following conditions is satisfied:

- \( |i| > |j| \);
- \( |i| = |j| \) and there exists some \( k_0 \in \mathbb{Z}_{>0} \) such that \( i_{k_0} > j_{k_0} \) and \( i_k = j_k \) for all \( k > k_0 \).

And the total order \( " \succ \) on \( \mathbb{M} \) is defined by \( (i, j, k) > (i', j', k') \) if and only if one of the following conditions is satisfied:

- \( k > k' \);
- \( k = k' \) and \( j > j' \);
- \( k = k' \) and \( j = j' \) and \( i > i' \).

Now every element \( v \) of \( V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \) can be uniquely written in the form

\[
v = \sum_{i,j,k \in \mathbb{M}} u_{i,j,k} w_{i,j,k} w_{\lambda,\mu,\kappa},
\]

where only finite many \( u_{i,j,k} \in \mathbb{C} \) are nonzero. We denote by \( \text{supp}(v) \) the set of all \( (i, j, k) \) with \( u_{i,j,k} \neq 0 \). For any \( 0 \neq v \in V_{\tilde{\mathfrak{sl}}_2}(\lambda, \mu, \kappa) \), let \( \text{deg}(v) \) denote the maximal (with respect to \( > \) ) element of \( \text{supp}(v) \), called the degree of \( v \). For convenience we define \( \text{deg}(0) = -\infty \). It is clear that

\[
\text{deg}(u_{i,j,k}v) = (i, j, k) + \text{deg}(v).
\]

Let us first prove by induction on \( |k| \) that
Claim. If \((i', j', k') \in \text{supp}(U_{i,j,k}w_{\lambda,\mu,\kappa} - (\frac{\lambda}{\kappa+2})|k|u_{i,j,k}w_{\lambda,\mu,\kappa})\), then \(k > k'\).

It is trivial for \(|k| = 0\). Now suppose that \(|k| > 0\). Let \(j\) be the minimal non-negative integer such that \(k_{j+1} \neq 0\).

From the definition of \(L(n)\) we have

\[
L(-j)w_{\lambda,\mu,\kappa} = \frac{1}{\kappa+2}(\lambda f(-j) + \sum_{i=0}^{j-1} e(-j+i)f(-i) + \mu e(-j-1) + u')w_{\lambda,\mu,\kappa},
\]

for some \(u' \in U(\hat{h}^{\leq 0})\), where \(\hat{h}^{\leq 0} = \text{span}_C\{h(-n)|n \geq 0\}\). In particular, the Claim also holds for \(|k| = 1\). So we have only need to consider \(|k| > 1\). Now we may write

\[
U_{0,0,k}w_{\lambda,\mu,\kappa} = \frac{1}{\kappa+2}\mathcal{U}_{0,0,k-\varepsilon_{j+1}}(\lambda f(-j) + \sum_{i=0}^{j-1} e(-j+i)f(-i) + \mu e(-j-1) + u')\mathcal{U}_{0,0,k-\varepsilon_{j+1}}w_{\lambda,\mu,\kappa},
\]

\[
v_1 = \frac{1}{\kappa+2}(\lambda f(-j) + \sum_{i=0}^{j-1} e(-j+i)f(-i) + \mu e(-j-1) + u')\mathcal{U}_{0,0,k-\varepsilon_{j+1}}w_{\lambda,\mu,\kappa},
\]

\[
v_2 = \frac{1}{\kappa+2}\mathcal{U}_{0,0,k-\varepsilon_{j+1},\lambda f(-j) + \sum_{i=0}^{j-1} e(-j+i)f(-i) + \mu e(-j-1) + u'}w_{\lambda,\mu,\kappa}.
\]

Then \(\mathcal{U}_{0,0,k}w_{\lambda,\mu,\kappa} = v_1 + v_2\). Note that \(v_1, v_2\) depend on \(k\). From induction hypothesis, we have \(\mathcal{U}_{0,0,k-\varepsilon_{j+1}}w_{\lambda,\mu,\kappa} \in \sum C_{\varepsilon_{j+1},k'}w_{\lambda,\mu,\kappa}\)

to give

\[
v_1 \in \left(\frac{\lambda}{\kappa+2}\right)^{|k|} U_{0,0,k}w_{\lambda,\mu,\kappa} + \sum C_{\varepsilon_{j+1},k'}w_{\lambda,\mu,\kappa}.
\]

By using the fact \([L(n), x(m)] = -mx(n + m)\) for all \(x \in \mathfrak{sl}_2\), we have

\[
v_2 \in \sum \sum C_{\varepsilon_j,k', s \in \mathbb{Z}} f(s) + C e(-s+1)f(-t) + U(\hat{h}^{\leq 0})U_{0,0,k}w_{\lambda,\mu,\kappa}.
\]

Again from induction hypothesis, we have \(\text{deg}(v_2) < (0, 0, k)\).

So we have proved the claim for \((i, j, k) = (0, 0, k)\). Using the fact that

\[
\text{deg}(u_{i,j,0}u_{i',j',k}) = \text{deg} u_{i+i',j+j',k},
\]

we may easily see that the claim holds for all \((i, j, k)\).

From the Claim we see that \(\text{deg}(U_{i,j,k}w_{\lambda,\mu,\kappa}) = (i, j, k)\). Then the linear independence of \(\mathcal{B}\) follows. And from the Claim and by induction on \((k, >)\), we may deduce that \(u_{i,j,k}w_{\lambda,\mu,\kappa}\) is a linear combination of \(\mathcal{B}\). This completes the proof. \(\square\)

Lemma 3.1 shows that \(V_{\mathfrak{sl}_2}(\lambda, \mu, \kappa)\) is a cyclic module for the Lie algebra \(\widehat{\mathfrak{b}} \rtimes \text{Vir}\) generated by

\[
L(n), e(n), h(n), \quad n \in \mathbb{Z}
\]

satisfying commutation relations

\[
[L(n), L(m)] = (m-n)L(n+m) + \frac{m^3-m}{12}\delta_{n+m,0}\frac{3\kappa}{\kappa+2}
\]

\[
[h(n), h(m)] = 2n\delta_{n+m,0}\kappa,
\]

\[
[h(n), e(m)] = 2e(n+m), \quad [e(n), e(m)] = 0.
\]
The structure of the module $V_{\mathfrak{sl}_2}(\lambda, \mu, \kappa)$ is completely determined in the following theorem which will be proved in rest of this subsection.

**Theorem 3.2.** Assume that $\lambda, \mu, \kappa \in \mathbb{C}$ with $\kappa \neq -2$.

1. If $\lambda \cdot \mu \neq 0$, then $V_{\mathfrak{sl}_2}(\lambda, \mu, \kappa)$ is an irreducible $\mathfrak{sl}_2$-module.
2. If $\lambda \neq 0$, then for any $a \in \mathbb{C}$,
   
   \[ M_{\mathfrak{sl}_2}(\lambda, 0, \kappa, a) := V_{\mathfrak{sl}_2}(\lambda, 0, \kappa)/U(\mathfrak{sl}_2)(L(0) - a)w_{\lambda, 0, \kappa} \]
   
   has a unique simple quotient, which we denote by $\hat{\mathfrak{sl}}_{\mathfrak{sl}_2}(\lambda, 0, \kappa, a)$. Moreover, any simple quotient of $V_{\mathfrak{sl}_2}(\lambda, 0, \kappa)$ is isomorphic to a $\hat{\mathfrak{sl}}_{\mathfrak{sl}_2}(\lambda, 0, \kappa, a)$ for some $a \in \mathbb{C}$.
3. If $\mu 
eq 0$, then for any $a \in \mathbb{C}$,
   
   \[ M_{\mathfrak{sl}_2}(0, \mu, \kappa, a) := V_{\mathfrak{sl}_2}(0, \mu, \kappa)/U(\mathfrak{sl}_2)(h(0)/2 - L(0) - a)w_{0, \mu, \kappa} \]
   
   has a unique simple quotient, which we denote by $\hat{\mathfrak{sl}}_{\mathfrak{sl}_2}(0, \mu, \kappa, a)$. Moreover, any simple quotient of $V_{\mathfrak{sl}_2}(0, \mu, \kappa)$ is isomorphic to a $\hat{\mathfrak{sl}}_{\mathfrak{sl}_2}(0, \mu, \kappa, a)$ for some $a \in \mathbb{C}$.
4. The simple quotient of $V_{\mathfrak{sl}_2}(0, 0, \kappa)$ is obtained by restricting from the simple highest weight $\hat{\mathfrak{sl}}_{\mathfrak{sl}_2}$-module of level $\kappa$.

3.1.1. Whittaker $\hat{\mathfrak{b}} \rtimes \text{Vir}$–modules. In order to prove Theorem 3.2 (1) we need to develop the theory of Whittaker modules for the Lie algebra $\hat{\mathfrak{b}} \rtimes \text{Vir}$. So let

\[ \mathcal{L} = \hat{\mathfrak{b}} \rtimes \text{Vir} = \text{span}\{L(i), h(i), e(i), c_1, c | i \in \mathbb{Z}\} \]

be the Lie algebra with the Lie brackets

\[
\begin{align*}
[L(i), L(j)] &= (i - j)L(i + j) + \frac{j^3 - j}{12}c_1, \\
[L(i), h(j)] &= -jh(i + j), \quad [L(i), e(j)] = -je(i + j) \\
h(i), h(j) &= \delta_{i,j}02ic, \quad [h(i), e(j)] = 2e(i + j), \\
[e(i), e(j)] &= 0, \\
[c_1, \mathcal{L}] &= [c, \mathcal{L}] = 0.
\end{align*}
\]

Now we consider a $\mathbb{Z}$-gradation on $\mathcal{L}$ different from that by $L_0$. Let $\mathcal{L}_0 = \text{span}\{e(-1), h(0), L(0), c_1, c\}$ and $\mathcal{L}_i = \text{span}\{e(-i - 1), h(-i), L(-i)\}$ for all $i \neq 0$. Then $[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$, and $\mathcal{L}$ and $U(\mathcal{L})$ are $\mathbb{Z}$-graded. Denote $D(x) = i$ if $0 \neq x \in U(\mathcal{L})_i$.

For any $i \in \mathbb{M}$, denote

\[ u_i = \cdots (h(-n)^{3n+3}L(-n)^{3n+2}e(-n-1)^{3n+1}) \cdots (h(0)^{3i}L(0)^{3i}e(-1)^{3i}) \in U(\mathcal{L})_{\geq 0}. \]

Let $D(i) = D(u_i) = \sum_{j=1}^{3i} \sum_{k=0}^{\infty} k^j \varepsilon_{3k + j}$.

We are going to define another total order on the set $\mathbb{M}$. This total order will be used in the rest of this section, and is different from that defined in the proof of Lemma 3.1.

Denote by $<$ the reverse lexicographic total order on $\mathbb{M}$, defined recursively (with respect to the degree) as follows: 0 is the minimum element; and for different nonzero $i, j \in \mathbb{M}$ we have $i < j$ if and only if one of the following conditions is satisfied:

- $\min\{s : i_s \neq 0\} > \min\{s : j_s \neq 0\}$;
- $\min\{s : i_s \neq 0\} = \min\{s : j_s \neq 0\} = k$ and $i - \varepsilon_k < j - \varepsilon_k$. 

Define the principal total order $\prec$ on $\mathbb{M}$ as follows: for different $i, j \in \mathbb{M}$ set $i \prec j$ if and only if one of the following conditions is satisfied:

- $D(i) < D(j)$;
- $D(i) = D(j)$ and $|i| < |j|$;
- $D(i) = D(j)$ and $|i| = |j|$, but $i < j$.

For any $(\lambda, \mu, \kappa_1, \kappa) \in \mathbb{C}^4$, let

$$V(\lambda, \mu, \kappa_1, \kappa) = U(\mathcal{L})/\langle e(0) - \lambda, L(1) - \mu, L(i + 1), h(i), e(i), c_1 - \kappa_1, c - \kappa | i > 0 \rangle,$$

where $w_{\lambda, \mu, \kappa_1, \kappa}$ is the image of 1. It is clear that

$$B = \{u_1 w_{\lambda, \mu, \kappa_1, \kappa} | i \in \mathbb{M}\}$$

is a basis of $V(\lambda, \mu, \kappa_1, \kappa)$. Now every element $v$ of $V(\lambda, \mu, \kappa_1, \kappa)$ can be uniquely written in the form

$$v = \sum_{i \in \mathbb{M}} u_1 u_i w_{\lambda, \mu, \kappa_1, \kappa},$$

where only finitely many $v_i \in \mathbb{C}$ are nonzero. We denote by $\text{supp}(v)$ the set of all $i$ with $v_i \neq 0$. For a nonzero $v \in V(\lambda, \mu, \kappa_1, \kappa)$ let $l(v)$ denote the maximal (with respect to $\prec$) element of $\text{supp}(v)$, called the leading term of $v$. Let $D(v) = D(l(v))$, and $D(0) = -\infty$. For any $k \in \mathbb{Z}_{\geq 0}$, set $\text{supp}_k(v) = \{i \in \text{supp}(v) | D(i) = k\}$.

**Lemma 3.3.** For any $n \in \mathbb{Z}_{\geq 0}$, let $x = h(n)$, or $L(n) - \delta_{n,1}\mu$, or $e(n - 1) - \delta_{n,1}\lambda$. Then, for any $v \in V(\lambda, \mu, \kappa_1, \kappa)$ with $k = D(v)$ we have

(i) $D(xv) \leq k - n + 1$;

(ii) $\text{supp}_{k-n+1}(xv) \subset \{i - j | i \in \text{supp}(v), D(j) = n - 1\}$.

**Proof.** We may assume that $v = u_1 w_{\lambda, \mu, \kappa_1, \kappa}$ with $k = D(i)$. For any fixed $x = h(n), L(n) - \delta_{n,1}\mu, e(n - 1) - \delta_{n,1}\lambda$, by commuting $x$ with some terms of $u_1$ in all possible ways, we may transfer the only negative degree term in $[x, u_1]$ to the right side, i.e., $[x, u_1] \in \sum_{i \in \{k-n, \ldots, k\}} U(\mathcal{L}_{\geq 0}) \mathcal{L}_{k-n-i}$. Hence

$$xv = [x, u_1] w_{\lambda, \mu, \kappa_1, \kappa} = (u_{k-n} + \sum_{j \in \{k | i-k \in \mathbb{M}, D(k-n-1)\}} u_{i-j} c_j) w_{\lambda, \mu, \kappa_1, \kappa},$$

for some $c_j \in \mathbb{C}$ and $u_{k-n} \in U(\mathcal{L}_{\geq 0})_{k-n}$. Hence $D(xu_1 w_{\lambda, \mu, \kappa_1, \kappa}) \leq D(u_1 w_{\lambda, \mu, \kappa_1, \kappa}) - n + 1$. The lemma follows.

**Lemma 3.4.** Let $i \in \mathbb{M}$ with $n = \min\{k | i_k \neq 0\} > 0, \lambda \mu \neq 0$.

(a) If $n = 3k + 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then

(i) $\langle h(k + 1)u_1 w_{\lambda, \mu, \kappa_1, \kappa} \rangle = \hat{i} - \varepsilon_n$,

(ii) $i - \varepsilon_n \notin \text{supp}(h(k + 1)u_1 w_{\lambda, \mu, \kappa_1, \kappa})$ for all $i' < i$.

(b) If $n = 3k + 2$ for some $k \in \mathbb{Z}_{\geq 0}$, then

(i) $\langle (L(k + 1) - \delta_{k,0}\mu)u_1 w_{\lambda, \mu, \kappa_1, \kappa} \rangle = \hat{i} - \varepsilon_n$,

(ii) $i - \varepsilon_n \notin \text{supp}((L(k + 1) - \delta_{k,0}\mu)u_1 w_{\lambda, \mu, \kappa_1, \kappa})$ for all $i' < i$.

(c) If $n = 3k + 3$ for some $k \in \mathbb{Z}_{\geq 0}$, then

(i) $\langle (e(k) - \delta_{k,0}\lambda)u_1 w_{\lambda, \mu, \kappa_1, \kappa} \rangle = \hat{i} - \varepsilon_n$,

(ii) $i - \varepsilon_n \notin \text{supp}((e(k) - \delta_{k,0}\lambda)u_1 w_{\lambda, \mu, \kappa_1, \kappa})$ for all $i' < i$.

**Proof.** (a) (i). Write $h(k + 1)u_1 w_{\lambda, \mu, \kappa_1, \kappa}$ as in (4). It is clear that the only way to obtain the term $u_1 - \varepsilon_n w_{\lambda, \mu, \kappa_1, \kappa}$ is to commute $h(k + 1)$ with an $e(-k - 1)$, which implies $i - \varepsilon_n \in \text{supp}(h(k + 1)u_1 w_{\lambda, \mu, \kappa_1, \kappa})$. Note that

$$[h(k + 1), L(-k)] w_{\lambda, \mu, \kappa_1, \kappa} = [h(k + 1), h(-k)] w_{\lambda, \mu, \kappa_1, \kappa} = 0.$$
Combining this with Lemma 3.3 it is easy to see that to \( \{(h(k + 1)u_t w_{\lambda, \mu, \kappa, \kappa}) = i - \varepsilon_n \). 

(ii). Assume that \( D(i') < D(i) \), then from lemma 3.3 we have 
\[
D(h(k + 1)u_t w_{\lambda, \mu, \kappa, \kappa}) \leq D(i') - k < D(i - \varepsilon_n) = D(i) - k.
\]
So (b) holds in this case. Assume that \( D(i') = D(i) = s \), and \(|i'| < |j| \). From Lemma 3.3(2), we have for any \( j \in \text{supp}_{s-k}(h(k + 1)u_t w_{\lambda, \mu, \kappa, \kappa}) \) we have \(|j| \leq |i'| - 1 < |i| \). So (b) also holds in this case.

Assume that \( n' = \min(k_i^i \neq 0) \). If \( n' = n \), then from (a), \( t(h(k + 1)u_t w_{\lambda, \mu, \kappa, \kappa}) = i' - \varepsilon_n < i - \varepsilon_n \). we also have (b) in this case.

Now we only need to consider the case \( D(i') = D(i) = s \), and \(|i'| = |i| \), and \( n' > n \). Then from lemma 3.3 it is easy to see that \( D(h(k + 1)u_t w_{\lambda, \mu, \kappa, \kappa}) < s - k = D(i - \varepsilon_n) \), which completes the proof of (ii).

(b) (i). Write \( (L(k + 1) - \delta_{k,0} \mu)u_t w_{\lambda, \mu, \kappa, \kappa} \) as in (1). Similarly, the only way to obtained the term \( u_{i - \varepsilon_n} w_{\lambda, \mu, \kappa, \kappa} \) is to commute \( (L(k + 1) - \delta_{k,0} \mu) \) with an \( L(-k) \), which implies \( i - \varepsilon_n \in \text{supp}((L(k + 1) - \delta_{k,0} \mu)u_t w_{\lambda, \mu, \kappa, \kappa}) \). Note that \( [L(k + 1) - \delta_{k,0} \mu, h(-k)]w_{\lambda, \mu, \kappa, \kappa} = 0 \) and \( e(-k - 1) \) does not occur in \( u_t \). And combining with Lemma 3.3 it is easy to see that to \( t((L(k + 1) - \delta_{k,0} \mu)u_t w_{\lambda, \mu, \kappa, \kappa}) = i - \varepsilon_n \).

(ii). The proof is similar with (1)(b). We omit the details.

Similarly as (a) and (b), we have (c).

From lemma 3.3 we have

**Corollary 3.5.** Suppose that \( \lambda \mu \neq 0 \). Let \( 0 \neq v \in V(\lambda, \mu, \kappa, \kappa) \) with \( n = \min(k|t(v)_k \neq 0| > 0 \).

(i). If \( n = 3k + 1 \) for some \( k \in \mathbb{Z}_{\geq 0} \), then \( h(k + 1)v \neq 0 \).

(ii). If \( n = 3k + 2 \) for some \( k \in \mathbb{Z}_{\geq 0} \), then \( (L(k + 1) - \delta_{k,0} \mu)v \neq 0 \).

(iii). If \( n = 3k + 3 \) for some \( k \in \mathbb{Z}_{\geq 0} \), then \( e(k - \delta_{k,0} \lambda)v \neq 0 \).

**Proposition 3.6.** If \( \lambda \mu \neq 0 \), the \( \mathcal{L} \)-module \( V(\lambda, \mu, \kappa, \kappa) \) is simple.

**Proof.** From corollary 3.5 any Whittaker vector has to be contained in \( \mathbb{C} w_{\lambda, \mu, \kappa, \kappa} \). From Lemm 2.1 (iii) we see that \( V(\lambda, \mu, \kappa, \kappa) \) is a simple \( \mathcal{L} \)-module. □

### 3.1.2. Some results from [Ch2].

We first recall some notations and results from Chapter 4 in [Ch2]. Let 
\[
\Omega = 2(c+2)d + \frac{1}{2} h(0)^2 + h(0) + 2f(0)e(0) + 2 \sum_{n=1}^{+\infty} (e(-n)f(n) + f(-n)e(n) + \frac{1}{2} h(-n)h(n))
\]
be the Casimir operator for \( \mathfrak{sl}_2 \).

Note that \( \Omega = 2(\kappa + 2)(d + L(0)) \) as operators on any restricted \( \mathfrak{sl}_2 \) module of level \( \kappa \neq -2 \).

Let 
\[
z = \frac{1}{2} h(0)^2 + h(0) + 2f(0)e(0)
\]
be the Casimir element for \( \mathfrak{g} = \mathbb{C} e(0) + \mathbb{C} h(0) + \mathbb{C} f(0) \). For any \( (\bar{d}, \bar{z}) \in \mathbb{C}^2 \), Denote by 
\[
V_{\mathfrak{sl}_2}(\lambda, 0, \kappa, \bar{d}, \bar{z}) = V_{\mathfrak{sl}_2}(\lambda, 0, \kappa)/(U(\mathfrak{sl}_2)(d - \bar{d}) + U(\mathfrak{sl}_2)(z - \bar{z}))w_{\lambda,0,0}.
\]

**Lemma 3.7.** Assume that \( \lambda \neq 0 \) and \( \kappa \neq -2 \). Let \( V = V_{\mathfrak{sl}_2}(\lambda, 0, \kappa, \bar{d}, \bar{z}) \). Then
V have a unique simple quotient;
(ii). \( \Omega(V) = (2(k + 2)\hat{d} + \hat{z})\hat{d}V \);
(iii). If \( \hat{z} \neq \frac{(\hat{d}(k+2)-m)^2-1}{2} \) for any \( i, m \in \mathbb{Z}_{>0} \), then \( V_{\hat{sl}_2}(\lambda, 0, \kappa, \hat{d}, \hat{z}) \) is simple;
(iv). Let \( M \) be any simple Whittaker module of level \( \kappa \) for \( \hat{sl}_2 \) with a Whittaker vector \( w \) of type \( (\lambda, 0) \). If there exists some nonzero polynomial \( p(x) \) such that \( p(d)w = 0 \), then \( M \) is isomorphic to the simple quotient of \( V_{\hat{sl}_2}(\lambda, 0, \kappa, \hat{d}, \hat{z}) \) for some \( \hat{d}, \hat{z} \in \mathbb{C} \).

Proof. Statements (i)-(iv) are the immediately consequences of Proposition 3.15, Corollary 4.4, Corollary 4.16 and Corollary 3.29 in [Ch2], respectively. We remark that the typo, from the proof of Corollary 4.16 in [Ch2], respectively. We remark that the typo, from the proof of Corollary 4.16 in [Ch2], respectively. We remark that the typo, from the proof of Corollary 4.16 in [Ch2], respectively. We remark that the typo, from the proof of Corollary 4.16 in [Ch2], respectively.

Lemma 3.8. Assume that \( \kappa \neq -2 \). For any \( a \in \mathbb{C} \) and any simple Whittaker \( \hat{sl}_2 \)-module \( V \), we can make \( V \) into an \( \hat{sl}_2 \)-module by defining \( d = -L(0) + a \) on \( V \). Moreover, any simple Whittaker \( \hat{sl}_2 \)-module with central charge \( \kappa \neq -2 \) is of this form.

Proof. The first part of the claim is obvious. The second part follows from the fact that the generalized Casimir element \( \Omega \) commutes with \( \hat{sl}_2 \) and acts as a scalar on each simple Whittaker \( \hat{sl}_2 \) module.

3.1.3. Proof of Theorem 3.2. From Lemma 3.1, it is easy to see that \( V_{\hat{sl}_2}(\lambda, \mu, \kappa) \cong V(\lambda, \lambda\mu, \frac{\lambda}{\kappa}, \kappa) \) as \( \hat{b} \times \hat{Vir} \)-module.

(1) From Proposition 3.6, \( V_{\hat{sl}_2}(\lambda, \mu, \kappa) \) is an irreducible \( \hat{sl}_2 \)-module.

(2) Note that \( \hat{\mathfrak{g}} \) hence \( U(\hat{\mathfrak{g}}) \) is \( \mathbb{Z} \)-graded with respect to \( d \). Denote \( U(\hat{\mathfrak{g}})|_{d, u} = \{ u \in U(\hat{\mathfrak{g}})[d, u] = iu \} \). Then \( V_{\hat{sl}_2}(\lambda, 0, \kappa) = \oplus_{i \in \mathbb{Z}} U(\hat{\mathfrak{g}})^{m, \lambda, 0, \kappa} \). Since \( \mu = 0 \), from Lemma 3.1 we see that \( U(\hat{\mathfrak{g}})L(0) - a \) \( w_{\lambda, 0, \kappa} \) is a nonzero submodule of \( V_{\hat{sl}_2}(\lambda, 0, \kappa) \) for any \( a \in \mathbb{C} \). Suppose that \( V \) is a proper maximal submodule of \( V_{\hat{sl}_2}(\lambda, 0, \kappa) \). Assume that \( V \neq 0 \). Then \( V \) must contain a Whittaker vector \( v = \sum_{i=0}^r a_i u_{-r} w_{\lambda, 0, \kappa} \) where \( u_{-r} \in U_{-r} \), and \( a_r u_{-r} w_{\lambda, 0, \kappa} \neq 0 \). It is clear that \( u_{-r} w_{\lambda, 0, \kappa} \) is also a Whittaker vector (which may be not in \( V \)).

If \( r > 0 \), from Lemma 3.1, we know that the image of \( u_{-r} w_{\lambda, 0, \kappa} \) is a nonzero Whittaker vector in \( V_{\hat{sl}_2}(\lambda, 0, \kappa)/U(\hat{\mathfrak{g}})(L(0) - a)w_{\lambda, 0, \kappa} \) for all but finitely many \( a \in \mathbb{C} \) since \( u_{-r} \) has finitely many factors. Therefore the Whittaker module \( V_{\hat{sl}_2}(\lambda, 0, \kappa)/U(\hat{\mathfrak{g}})(L(0) - a)w_{\lambda, 0, \kappa} \) (which is graded with respect to the action of \( L(0) \)) is not simple for all but finitely many \( a \in \mathbb{C} \). Now we may regarded \( V_{\hat{sl}_2}(\lambda, 0, \kappa)/U(\hat{\mathfrak{g}})(L(0) - a)w_{\lambda, 0, \kappa} \) as a \( \hat{sl}_2 \)-module by defining \( d = -L(0) \), which is of course not simple too for all but finite many \( a \in \mathbb{C} \). However, from Lemma 3.7 (3), For any given \( \kappa \neq -2 \), we have

\[
V_{\hat{sl}_2}(\lambda, 0, \kappa)/U(\hat{\mathfrak{g}})(L(0) - a)w_{\lambda, 0, \kappa} \cong V_{\hat{sl}_2}(\lambda, 0, \kappa, -a, 2(\kappa + 2)a)
\]

is simple as \( \hat{sl}_2 \) module for all but at most countably many \( a \), a contradiction. So we have \( r = 0 \).

Now \( v \in \mathbb{C}[h(0), L(0)]w_{\lambda, 0, \kappa} \). From \( (e(0) - \lambda)v = 0 \), we have \( v \in \mathbb{C}[L(0)]w_{\lambda, 0, \kappa} \). Now \( V_{\hat{sl}_2}(\lambda, 0, \kappa)/V \) satisfies the condition of Lemma 3.7 (4), which completes the proof.
(3) Let \( \sigma \) be the automorphism of \( \widehat{\mathfrak{sl}}_2 \) defined by
\[
\sigma(e(i)) = f(i + 1), \sigma(f(i)) = e(i - 1), \sigma(h(i)) = -h(i) + \delta_{i,0}e,
\]
\[
\sigma(c) = c, \sigma(d) = d + \frac{h(0)}{2}.
\]
Then \( V_{\widehat{\mathfrak{sl}}_2}(0, \mu, \kappa) \) are equivalent to \( V_{\widehat{\mathfrak{sl}}_2}(\mu, 0, \kappa) \) via \( \sigma \). Now (3) follows from (2).

(4) Again any simple \( \widehat{\mathfrak{sl}}_2 \) quotient of \( V_{\widehat{\mathfrak{sl}}_2}(0, 0, \kappa) \) is also a simple \( \widehat{\mathfrak{sl}}_2 \) module by taking \( d = -L(0) \). And (4) follows from Theorem 1.1 in [MZ]. \( \square \)

3.2. Classical Whittaker modules for \( \widehat{\mathfrak{sl}}_2 \) at the critical level. We will determine all simple quotients of \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2) \) in this subsection.

Since \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2) \) is a restricted module for the affine Lie algebra \( \widehat{\mathfrak{sl}}_2 \), it is a module for the universal affine vertex algebra \( V_{-2}(\mathfrak{sl}_2) \). Let
\[
t = \frac{1}{2}(e(-1)f(-1) + f(-1)e(-1) + \frac{1}{2}h(-1)^2)1 \in V_{-2}(\mathfrak{sl}_2),
\]
\[
T(z) = Y(t,z) = \sum_{n \in \mathbb{Z}} T(n)z^{-n-1}.
\]
Then
\[
[T(n), x(m)] = 0, \text{ for all } m, n \in \mathbb{Z}, x \in \widehat{\mathfrak{sl}}_2,
\]
i.e., \( T(n) \) are central elements.

Denote \( w = w_{\lambda, \mu, -2} \) in \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2) \).

**Lemma 3.9.** Let \( \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}, c(z) = \sum_{n \leq 0} c_n z^{-n-2} \in \mathbb{C}[[z]] \).

(a) The Whittaker module \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2) \) has a basis \( \{V_{i,j,k} | i, j, k \in \mathbb{M} \} \) where
\[
V_{i,j,k} = \langle \cdots e(-n)^{i_n} \cdots e(-1)^{i_1} \cdots h(-n)^{j_{n+1}} \cdots h(0)^{j_1} \cdots T(-n)^{k_{n+1}} \cdots T(0)^{k_1} \rangle w.
\]

(b) The Whittaker module \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \) is \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu)/\langle T_n - c_n | n \leq 0 \rangle \) has a basis
\[
\{ \langle \cdots e(-n)^{i_n} \cdots e(-1)^{i_1} \cdots h(-n)^{j_{n+1}} \cdots h(0)^{j_1} \rangle \bar{w} | i, j \in \mathbb{M} \}
\]
where \( \bar{w} \) is the image of \( w \).

(c) \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \) is simple as \( U(\mathfrak{h})\)-module.

Proof. (a) The proof is similar as that of Lemma 3.1 We omit the details.

Part (b) follows from (a).

(c) By using completely analogous proof, we may deduce a similar result of Corollary 3.5(i) and (iii) for \( \mathfrak{h} \)-module \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \). Then similarly as Proposition 3.6, we have \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \) is irreducible as \( \mathfrak{h} \)-module. \( \square \)

**Theorem 3.10.** Let \( c(z) = \sum_{n \leq 0} c_n z^{-n-2}, c'(z) = \sum_{n \leq 0} c'_n z^{-n-2} \in \mathbb{C}[[z]] \), and \( \lambda, \lambda' \in \mathbb{C}^*, \mu, \mu' \in \mathbb{C} \).

(a) The Whittaker module \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \) is a simple \( \widehat{\mathfrak{sl}}_2 \) module.

(b) Any simple quotient of \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2) \) is of the form \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \) for some \( c(z) \).

(c) \( V_{\widehat{\mathfrak{sl}}_2}(\lambda, \mu, -2, c(z)) \cong V(\lambda', \mu', c'(z)) \) if and only if \( \lambda = \lambda', \mu = \mu', c(z) = c'(z) \).
Proof. (a) follows from Lemma 3.9 (c)
(b) Let \( \hat{V} \) be any simple quotient of \( V_{\hat{s}l_2}(\lambda, \mu, -2) \). From \( [T(n), \hat{s}l_2] = 0 \) on \( V_{\hat{s}l_2}(\lambda, \mu, -2) \), we have \( T(n) \) act as scalar on \( \hat{V} \) for any \( n \in \mathbb{Z} \). Say \( \sum_{n \geq 0} T(n)z^{-n-2} = c(z) \) on \( \hat{V} \). Then \( \hat{V} \) has to isomorphic to \( V_{\hat{s}l_2}(\lambda, \mu, -2, c(z)) \).
(c) It follows from the fact that \( T(z) = \lambda \mu z^{-3} + c(z) \) on \( V_{\hat{s}l_2}(\lambda, \mu, -2, c(z)) \). \( \square \)

Remark 1. A different proof of irreducibility of \( V_{\hat{s}l_2}(\lambda, \mu, -2, c(z)) \) will be presented in Section 8.4 where we shall realize this module using an explicit bosonic realization.

4. WHITTAKER MODULES FOR THE AFFINE LIE ALGEBRA \( \hat{sl}_2 \)

From Lemma 3.9 and Theorem 3.2 we know that Whittaker modules for the affine Lie algebra \( \hat{sl}_2 \) with noncritical level are completely determined. So we only need to consider the critical case, i.e., \( \kappa = -2 \).

We shall start with one general method for constructing irreducible \( \hat{sl}_2 \)-modules from irreducible \( \hat{sl}_2 \)-modules:

Theorem 4.1. Assume that \( M \) is any irreducible \( V_{-\hat{sl}_2}(\lambda, \mu, -2) \)-module (i.e., irreducible restricted \( \hat{sl}_2 \) module at the critical level) such that \( T(k_0) \neq 0 \) on \( M \) for certain \( k_0 \neq 0 \). Then
\[
\widetilde{M} = \text{Ind}_{\hat{sl}_2}^{\hat{sl}_2} M
\]
is an irreducible \( \hat{sl}_2 \)-module.

Proof. Since \( M \) is irreducible, \( T(k_0) \) acts on \( M \) as a non-zero scalar \( c_{k_0} \). For any \( 0 \neq v \in \text{Ind}_{\hat{sl}_2}^{\hat{sl}_2} M \), we have
\[
0 \neq (T(k_0) - c_{k_0})^i v \in 1 \otimes M
\]
for some \( i \in \mathbb{Z}_{\geq 0} \). Combining with the simplicity of \( M \), we have \( U(\hat{sl}_2)v = \text{Ind}_{\hat{sl}_2}^{\hat{sl}_2} M. \) So
\[
\text{Ind}_{\hat{sl}_2}^{\hat{sl}_2} M
\]
is simple as \( \hat{sl}_2 \) module. \( \square \)

Remark 2. Previous theorem can be applied on irreducible modules constructed in \( \text{[A1]} \) and \( \text{[A2]} \).

We shall first classify all irreducible quotients of the universal Whittaker modules at the critical level in non-degenerate case.

Theorem 4.2. Let \( \lambda, \mu \in \mathbb{C}^* \). Then any simple quotient of \( V_{\hat{s}l_2}(\lambda, \mu, -2) \) is isomorphic to
\[
\text{Ind}_{\hat{s}l_2}^{\hat{s}l_2} \left( V_{\hat{s}l_2}(\lambda, \mu, -2, c(z)) \right)
\]
for some \( c(z) = \sum_{n \leq 0} c_n z^{-n-2} \).

Proof. Note that \( T(1) \) acts on \( V_{\hat{s}l_2}(\lambda, \mu, -2) \) as scalar \( \lambda \mu \neq 0 \). Irreducibility of the induced module \( \text{Ind}_{\hat{s}l_2}^{\hat{s}l_2} \left( V_{\hat{s}l_2}(\lambda, \mu, -2, c(z)) \right) \) follows from Theorem 4.1

Now let \( W \) be any maximal submodule of \( V_{\hat{s}l_2}(\lambda, \mu, -2) \). Note that \( T(1)T(-i) \), \( i \in \mathbb{Z}_{\geq 0} \) commute with \( \hat{sl}_2 \) as operators on \( V_{\hat{s}l_2}(\lambda, \mu, -2)/W \). So for any \( i \in \mathbb{Z}_{\geq 0} \),
Lemma 4.3. Let \( \lambda \in \mathbb{C}^* \). Then

1. \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \cong V_{\mathfrak{sl}_2}(\lambda, 0, -2, 0) \otimes (U(\mathfrak{T})w_{\lambda, 0, -2})^{\mathfrak{b} \ltimes \mathfrak{T}} \) as \( \mathfrak{b} \ltimes \mathfrak{T} \)-modules.

2. Any \( \mathfrak{b} \ltimes \mathfrak{T} \) submodule of \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) is of the form \( U(\mathfrak{b} \ltimes \mathfrak{T})X = U(\mathfrak{b})X \), where \( X \) is a \( \mathfrak{T} \)-submodule of \( U(\mathfrak{T})w_{\lambda, 0, -2} \).

3. Any \( \mathfrak{b} \ltimes \mathfrak{T} \) simple quotient of \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) is isomorphic to \( V_{\mathfrak{sl}_2}(\lambda, 0, -2, 0) \otimes X^{\mathfrak{b} \ltimes \mathfrak{T}} \) for some \( \mathfrak{T} \)-module \( X \) of \( U(\mathfrak{T})w_{\lambda, 0, -2} \).

4. Any \( \mathfrak{b} \ltimes \mathfrak{T} \) simple quotient of \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) is a \( \mathfrak{sl}_2 \) simple quotient.

Proof. (1) From Lemma 4.2, \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) has a basis:

\[
(\ldots e(-n)^i \cdots e(-1)^i)(\ldots h(-n)^{j_{n+1}} \cdots h(0)^{j_1})d^i(\ldots T(-n)^{k_{n+1}} \cdots T(0)^{k_1})w
\]

for all \( i \in \mathbb{Z}_{\geq 0}, i, j, k \in \mathbb{N}, \) where \( w = w_{\lambda, 0, -2} \).

Using this basis it is easy to check that

\[
\psi((\ldots e(-n)^i \cdots e(-1)^i)(\ldots h(-n)^{j_{n+1}} \cdots h(0)^{j_1})d^i(\ldots T(-n)^{k_{n+1}} \cdots T(0)^{k_1})w)
= (\ldots e(-n)^i \cdots e(-1)^i)(\ldots h(-n)^{j_{n+1}} \cdots h(0)^{j_1})\bar{w} \otimes \]

\[
(d^i(\ldots T(-n)^{k_{n+1}} \cdots T(0)^{k_1})w)
\]

defines an isomorphism of \( \mathfrak{b} \ltimes \mathfrak{T} \) modules from \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) to \( V_{\mathfrak{sl}_2}(\lambda, 0, -2, 0) \otimes (U(\mathfrak{T})w)^{\mathfrak{b} \ltimes \mathfrak{T}} \).
(2) and (3). Note that from Lemma 3.9 \( V_{\widetilde{s}_2}^{\sim}(\lambda, 0, -2, 0) \) is simple as \( \widetilde{b} \) module and \( \widetilde{b}(U(\widehat{T}))w^{b}_{\sim} = 0 \). Now claim (2) and (3) follow from (1) and Theorem 7 in [\( I\bar{Z} \)].

(4) From (2), we only need to verify that \( U(\hat{b})X = U(\tilde{s}_2)X \) for any \( \widetilde{T} \) submodule \( X \) of \( U(\tilde{T})w_{\lambda, 0, -2} \). Note that for any \( v \neq x \in X \), it is clear that \( v \) is also a Whittaker vector in \( V_{\widetilde{s}_2}^{\sim}(\lambda, 0, -2) \). Thus from Lemma 3.9 (a), \( f(i)v \in U(\hat{b} \times \widetilde{T})v \subset U(\hat{b}T) \).

This completes the proof. \( \Box \)

From the Lemma 4.3 and Lemma 4.4 (a), for any \( \lambda \neq 0 \) and any simple \( \widetilde{T} \) quotient \( X \) of \( U(\tilde{T})w_{\lambda, 0, -2} \), we have \( V_{\widetilde{s}_2}^{\sim}(\lambda, 0, -2, 0) \otimes X^{b_{\sim}T} \) is also a simple \( \widetilde{s}_2 \)-module with the actions of \( \{ f(i), i \in \mathbb{Z} \} \) determined by the action of \( b_{\sim} \).

Since \( U(\tilde{T})w_{\lambda, 0, -2} \cong U(\tilde{T}_{-}) \) one sees that there is one to one correspondence between simple \( \tilde{T} \) quotients of \( U(\tilde{T})w_{\lambda, 0, -2} \) and simple \( \tilde{T}_{-} \)-modules. We get

**Corollary 4.4.** There is one to one correspondence between simple \( \tilde{T}_{-} \)-modules \( X \) and simple \( \tilde{s}_2 \)-quotients of \( V_{\widetilde{s}_2}^{\sim}(\lambda, 0, -2) \).

Now we shall describe all simple \( \tilde{T}_{-} \)-modules.

Let \( s \) be any nonempty subset of \( \mathbb{Z}_{>0} \) and \( r_\mathbb{Z} (r_\mathbb{Z} > 0) \) be the additive subgroup of \( \mathbb{Z} \) generated by \( s \). For any map \( \chi_s : s \cup \{ 0 \} \to \mathbb{C}^* \), we define an associative algebra homomorphism \( \phi_{\chi_s} : U(\tilde{T}) \to U(b) \) by

\[
\phi_{\chi_s}(d) = -\frac{r_s}{2} h, \phi_{\chi_s}(T(i)) = 0, \forall i \notin s \cup \{ 0 \},
\]

\[
\phi_{\chi_s}(T(i)) = \chi_s(i) e^{-i/r_s}, \forall i \in s \cup \{ 0 \}.
\]

Then for any \( \phi_{\chi_s} \) and \( b \) module \( M \), we have a \( \tilde{T}_{-} \) module \( M^{\phi_{\chi_s}} = M \) with the action \( xv = \phi_{\chi_s}(x)v, \forall x \in \tilde{T}_{-}, v \in M \). Recall that all simple modules over \( b \) are classified in [13].

**Lemma 4.5.** \( M^{\phi_{\chi_s}} \) is a simple \( \tilde{T}_{-} \) module if and only if \( M \) is a simple \( b \) module.

**Proof.** The necessity is obvious. Now suppose that \( M \) is a simple \( b \) module. If \( M \) is finite dimension, then \( M \) is 1-dimensional and the claim follows. Now we assume that \( M \) is infinite dimensional simple \( b \) module. Since \( eM \) and \( \{ v \in M | ev = 0 \} \) are submodules of \( M \), we have \( e \) acts bijectively on \( M \). From the definition it is easy to see that there exists a \( m \in \mathbb{Z}_{>0} \) such that \( \mathbb{C}[e^{m+i}, h]_{i \geq 0} \subset \phi_{\chi_s}(\tilde{T}) \). So we only need to prove that \( M \) is also simple as \( \mathbb{C}[e^{m+i}, h]_{i \geq 0} \) module. For any \( 0 \neq v_1, v_2 \in M \), we have \( e^{m_1}v_1 \neq 0 \) and \( v_2 = u e^{m_1}v_1 \in \mathbb{C}[e^{m+1}, h]_{i \geq 0}v_1 \) for some \( u \in \mathbb{C}[e, h] \), which completes the proof. \( \Box \)

**Lemma 4.6.** Let \( \lambda \neq 0 \), and \( X \) be any simple \( \tilde{T} \) quotient of \( U(\tilde{T})w_{\lambda, 0, -2} \). Then one of the following holds:

1. \( X \) is 1-dimensional.
2. \( X \cong M^{\phi_{\chi_s}} \) for some \( s \subseteq \mathbb{Z}_{>0} \) and infinite dimensional simple \( b \)-module \( M \).

**Proof.** For any \( i \in \mathbb{Z} \), since both \( T(i)X \) and \( \{ v \in X | T(i)v = 0 \} \) are submodules of the simple module \( X \), we have \( T(i) \) acts on \( X \) either bijectively or as zero. Note that \( T(0) \) is the central element and \( T(i)X = 0 \) for all \( i > 0 \). Let \( s = \{ i \in \mathbb{Z}_{>0} | T(i)X \neq 0 \} \). If \( s = \emptyset \), then \( \dim X = 1 \). So we assume that \( s \neq \emptyset \). Then \( X \) is
also a simple \( A = \mathbb{C}[T(i), T(i)^{-1}, d|i \in \mathfrak{s}] \) module. Fix an \( x = \Pi_{i,j \in \mathfrak{s}, \gamma \in \mathfrak{s}} T(ik)^{jk} \in A \) with \( \sum \gamma_{i,j} = -r_a \). Then for any \( i \in \mathfrak{s} \), we have \( T(i)x^{1/r_s} \in Z(A) \), which acts on \( X \) as a nonzero scalar \( a_i \). Now \( X \) is a simple module over \( A/(T(i)x^{1/r_s} - a_i|i \in \mathfrak{s}) \). It is clear that we have a homomorphism

\[
\phi_{x*} : A \rightarrow \mathbb{C}[h, e, e^{-1}]
\]

\[
\phi_{x*}(T(i)) = a_i e^{-i/r_s}, \quad \phi_{x*}(d) = -\frac{r_a}{2} h.
\]

Now it is easy to check that \( \ker \phi_{x*} = (T(i)x^{1/r_s} - a_i|i \in \mathfrak{s}) \) and we have the induced isomorphism \( \bar{\phi}_{x*} : A/(T(i)x^{1/r_s} - a_i|i \in \mathfrak{s}) \rightarrow \mathbb{C}[e, e^{-1}, h] \). Now \( X \) can be regarded as \( \mathbb{C}[e, e^{-1}, h] \) module via the isomorphism. Then from definition of \( \mathfrak{sl}_2 \) structure on \( X \), \( X \) is simple as \( \phi_{x*}((\mathfrak{T})) = \mathbb{C}[e^2, h] - iv_s \in \mathfrak{s} \) module. So \( X \) is a simple \( \mathbb{C}[e, h] \) module, which gives (2).

Now we may summarize the main results in this section

**Theorem 4.7.** Let \( \lambda \in \mathbb{C}^* \).

1. Any simple quotient \( W \) of \( V_{\mathfrak{sl}_2}(\lambda, 0, -2) \) is isomorphic to

\[
V_{\mathfrak{sl}_2}(\lambda, 0, -2, 0) \otimes X^{\mathfrak{b} \times \bar{\mathfrak{T}}},
\]

where \( X \) is a simple \( \bar{\mathfrak{T}} \)-module determined in Lemma 4.6.

2. If \( \mu \in \mathbb{C}^* \), then any simple quotient of \( V_{\mathfrak{sl}_2}(\lambda, \mu, -2) \) is isomorphic to

\[
\text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_2}(V_{\mathfrak{sl}_2}(\lambda, \mu, -2, c(z)))
\]

for some \( c(z) = \sum_{n < 0} c_n z^{-n-2} \).

**Remark 3.** An interesting observation is that the action of \( d \) can be semisimple or not semisimple on different degenerate simple \( \mathfrak{sl}_2 \) Whittaker modules of critical level with the same Whittaker function, which makes a striking difference from the case of noncritical central charge.

**Remark 4.** The result for \( \lambda = 0 \) and \( \kappa = -2 \) may be obtained similarly as Theorem 4.2 (3) and (4). We omit the details.

5. **Wakimoto modules for \( \mathfrak{sl}_2 \)**

In this section we shall review the construction of Wakimoto modules for \( \mathfrak{sl}_2 \) (cf. [W]). Details on the construction of Wakimoto modules using concepts of vertex algebras can be found in [F].

5.1. **Weyl vertex algebra.** Recall that the Weyl algebra is an associative algebra with generators

\[
a(n), a^*(n) \quad (n \in \mathbb{Z})
\]

and relations

\[
[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0 \quad (n, m \in \mathbb{Z}).
\]

Let \( M \) denotes the irreducible Weyl module generated by the cyclic vector \( 1 \) such that

\[
a(n)1 = a^*(n+1)1 = 0 \quad (n \geq 0).
\]

As a vector space

\[
M \cong \mathbb{C}[a(-n), a^*(-m) \mid n < 0, \ m \leq 0].
\]
Let us describe the basis of $M$. Recall that a partition is a sequence of non-negative integers

$$(\lambda_1, \lambda_2, \cdots)$$

such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \quad \text{and} \quad \lambda_n = 0 \quad \text{for n sufficiently large}.$$ 

Let $\mathcal{P}$ be the set of all partitions. Define the length $\ell(\lambda)$ and size $|\lambda|$ of partitions by

$$\ell(\lambda) = \max\{ n \mid \lambda_n \neq 0 \} \quad |\lambda| = \lambda_1 + \lambda_2 + \cdots.$$ 

If $\ell(\lambda) = \ell$ we write $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Let $\mathcal{P}_\ell$ be the set of all partitions of length $\ell$. Let $\phi$ be the partition with all the entries being zero. Then we write $\ell(\phi) = 0$.

We shall also need a total order on the set $\mathcal{P}_\ell$ such that

$$\lambda > \mu \quad \text{if} \quad \lambda_1 = \mu_1, \ldots, \lambda_{i-1} = \mu_{i-1} \quad \text{and} \quad \lambda_i > \mu_i \quad \text{for some} \quad i.$$ 

For partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ we set

$$u_{\lambda,\mu} := a(-\lambda_1) \cdots a(-\lambda_r) a^*(-\mu_1 + 1) \cdots a^*(-\mu_s + 1),$$ 

$$u_{\lambda,\phi} := a(-\lambda_1) \cdots a(-\lambda_r), \quad u_{\phi,\mu} := a^*(-\mu_1 + 1) \cdots a^*(-\mu_s + 1),$$ 

$$u_{\phi,\phi} = 1.$$ 

Then the set

$$\{u_{\lambda,\mu}1 \mid (\lambda, \mu) \in \mathcal{P} \times \mathcal{P}\}$$

is a basis for $M$.

There is a unique vertex algebra $(M, Y, 1)$ where the vertex operator map is

$$Y : M \to \text{End}[[z, z^{-1}]]$$

such that

$$Y(a(-1)1, z) = a(z), \quad Y(a^*(0)1, z) = a^*(z)$$

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n}.$$ 

In particular we have:

$$Y(a(-1)a^*(0)1, z) = a(z)^+ a^*(z) + a^*(z) a(z)^-;$$

$$Y(a(-1)a^*(0)^21, z) = a(z)^+ (a^*(z))^2 + (a^*(z))^2 a(z)^-$$

where

$$a(z)^+ = \sum_{n \leq -1} a(n) z^{-n-1}, \quad a(z)^- = \sum_{n \geq 0} a(n) z^{-n-1}.$$
5.2. Wakimoto modules. Let $\mathfrak{h} = \mathbb{C}b$ be 1–dimensional commutative Lie algebra with a symmetric bilinear form fixed that $(b, b) = 2$, and $\hat{\mathfrak{h}} = h \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ be its affinization. Set $b(n) = b \otimes t^n$. Let $\pi^{\kappa+2}$ denotes the irreducible $\hat{\mathfrak{h}}$–module of level $\kappa + 2$ generated by the vector $1$ such that

$$b(n)1 = 0 \quad \forall n \geq 0.$$ 

As a vector space

$$\pi^{\kappa+2} = \mathbb{C}[b(n) | n \leq -1],$$

Then $\pi^{\kappa+2}$ has the unique structure of a vertex algebra generated by the field $b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n-1}$ such that

$$[b(n), b(m)] = 2(\kappa + 2) n \delta_{n+m,0}.$$ 

Let $V_{\kappa}(\mathfrak{sl}_2)$ be the universal vertex algebra of level $\kappa$ associated to the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. There is a injective homomorphism of vertex algebras $\Phi : V_{\kappa}(\mathfrak{sl}_2) \to M \otimes \pi^{\kappa+2}$ generated by

$$e = a(-1)1, \quad h = -2a^*(0)a(-1)1 + b(-1), \quad f = -a^*(0)^2 a(-1)1 + ka^*(-1)1 + a^*(0)b(-1)1.$$ 

For $x \in \{e, f, z\}$ we set $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$. We have

$$e(z) = Y(e, z) = a(z); \quad h(z) = Y(h, z) = -2 : a^*(z)a(z) : + b(z);$$

$$f(z) = Y(f, z) = - : a^*(z)^2 a(z) : + k \partial_a a^*(z) + a^*(z)b(z) = - (a(z)^+ a^*(z)^- + (a^*(z))^2) + k \partial_a a^*(z) + a^*(z)b(z).$$

The following proposition is a standard result in the theory of vertex algebras (cf. [K], [LL], [FB]) applied on the vertex algebra $M \otimes \pi^{\kappa+2}$.

**Proposition 5.1.** Assume that $M_1$ is restricted module for the Weyl algebra and $N_1$ is restricted module of level $\kappa + 2$ for the Heisenberg algebra $\mathfrak{h}$, i.e., for every $u \in M_1$ and $v \in N_1$ there is $N \in \mathbb{Z}_{\geq 0}$ such that

$$a(n)u = 0, \quad b(n)v = 0 \quad \text{for } n \geq N.$$ 

Then $M_1 \otimes N_1$ is a $M \otimes \pi^{\kappa+2}$–module, and therefore $V_{\kappa}(\mathfrak{sl}_2)$–module.

Assume first that $\kappa \neq -2$. We have the natural action of the Virasoro algebra generated by the Sugawara Virasoro vector

$$\omega = \frac{1}{2(k + 2)} (e(-1)f(-1) + f(-1)e(-1) + \frac{1}{2} h(-1)^2)1 = a(-1)a^*(-1) + \frac{1}{4(k + 2)} (b(-1)^2 - 2b(-2)),$$

$$L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$
Assume next that \( \kappa = -2 \) (critical level). The center of \( V_{-2}(\mathfrak{sl}_2) \) is generated by the field
\[
T(z) = Y\left(\frac{1}{2} (e(-1)f(-1) + f(-1)e(-1) + 1/2h(-1)^2)1, z\right) = \sum_{n \in \mathbb{Z}} T(n)z^{-n-2}.
\]
In our case we have that
\[
T(z) = Y\left(\frac{1}{2} (b(-1)^2 + 2b(-2))1, z\right) = \frac{1}{2} (b(z)^2 - 2\partial_z b(z)).
\]
For details see [FB].

6. Whittaker modules from Wakimoto modules

Proposition 5.1 gives a very useful method for a construction of representations of affine Lie algebra of certain level which uses Wakimoto modules. So we just need to construct modules for the vertex algebra \( M \otimes \pi^{\kappa+2} \). Since \( V_{\kappa}(\mathfrak{sl}_2) \) is a subalgebra of \( M \otimes \pi^{\kappa+2} \), every \( M \otimes \pi^{\kappa+2} \)-module becomes a module for the vertex algebra \( V_{\kappa}(\mathfrak{sl}_2) \) and therefore for the affine Lie algebra \( \mathfrak{sl}_2 \) of level \( \kappa \). In this section we shall use this fact and construct certain Whittaker modules for \( \mathfrak{sl}_2 \). The main new idea in our approach will be in considering certain Whittaker modules for the Weyl algebra as modules for the vertex algebra \( M \).

We shall first study a simple case of Whittaker modules for Weyl and Heisenberg algebras. Let \( \lambda, \mu \in \mathbb{C} \). Then there is a unique irreducible module \( M_1(\lambda, \mu) \) for the Weyl algebra \( \text{Weyl} \) generated by the vector \( v_1 \) such that
\[
\begin{align*}
  a(0)v &= \lambda v_1, & a(n)v &= 0 & \forall n \geq 1 \\
  a^*(1)v &= \mu v_1, & a^*(m)v &= 0 & \forall m \geq 2
\end{align*}
\]
Note that as a vector space \( M_1(\lambda, \mu) \cong M \).

Since \( M_1(\lambda, \mu) \) is a restricted module for the Weyl algebra we have that \( M_1(\lambda, \mu) \) is an irreducible module over vertex algebra \( M \).

Similarly for \( \chi_0, \chi_1 \in \mathbb{C} \) let \( N_{1}(\chi_0, \chi_1) \) be a module over Heisenberg algebra \( \hat{\mathfrak{h}} \) generated by the vector \( v_2 \) such that
\[
\begin{align*}
  c v &= (\kappa + 2)v_2, & b(0)v &= \chi_0 v_2, & b(1)v &= \chi_1 v_2, & b(m)v &= 0 & \forall m \geq 2
\end{align*}
\]
This module is also restricted, and therefore it is a module over Heisenberg vertex algebra \( \pi^{\kappa+2} \).

So we have \( M \otimes \pi^{\kappa+2} \)-module \( M_{Wak}(\lambda, \mu, \kappa, \chi_0, \chi_1) := M_1(\lambda, \mu) \otimes N_{1}(\chi_0, \chi_1) \). Then \( M_{Wak}(\lambda, \mu, \kappa, \chi_0, \chi_1) \) is an irreducible \( M \otimes \pi^{\kappa+2} \)-module iff \( \kappa \neq -2 \).

If \( \kappa = -2 \) we set
\[
\overline{M_{Wak}(\lambda, \mu, \kappa, \chi(z))} = M_{Wak}(\lambda, \mu, \kappa, \chi_0, \chi_1)/\langle b(-n) - \chi, \neg n, n \geq 1 \rangle
\]
where
\[
\chi(z) = \frac{\chi_1}{z^2} + \frac{\chi_0}{z} + \sum_{n=1}^{\infty} \chi \neg n z^n - 1 \in \mathbb{C}((z)).
\]
Then \( \overline{M_{Wak}(\lambda, \mu, -2, \chi(z))} \) is an irreducible \( M \otimes \pi^0 \)-module. Note that as a vector space \( \overline{M_{Wak}(\lambda, \mu, -2, \chi(z))} \) is naturally isomorphic to \( M \).

Let \( v = v_1 \otimes v_2 \).
Lemma 6.1. We have:
\[
\begin{align*}
    f(1)v &= (\chi_1 - 2\mu\lambda)a^*(0)v + \mu(\chi_0 - \kappa)v + \mu^2a(-1)v \\
    f(2)v &= \mu(\chi_1 - \lambda\mu)v \\
    e(0)v &= \lambda v \\
    h(1)v &= (\chi_1 - 2\mu\lambda)v \\
    e(n)v &= h(1 + n)v = f(2 + n)v = 0 \quad \forall n \geq 1.
\end{align*}
\]

Proof. By direct calculation. We have
\[
\begin{align*}
    f(1)v &= -2a^*(1)a^*(0)a(0)v + a(-1)a^*(1)^2v - \kappa a^*(1)v + b(0)a^*(1)v + a^*(0)b(1)v \\
    &= (-2\mu\lambda + \chi_1)a^*(0)v + \mu(\chi_0 - k)v + \mu^2a(-1)v \\
    f(2)v &= -a^*(1)^2a(0)v + a^*(1)b(1)v = (-\lambda\mu^2 + \mu\chi_1)v = \mu(\chi_1 - \lambda\mu)v \\
    h(1)v &= -2a^*(1)a(0)v + b(1)v \\
    &= (-2\mu\lambda + \chi_1)v.
\end{align*}
\]

Proof follows. \(\Box\)

Remark 5. Unfortunately for \(\lambda \cdot \mu \neq 0\) vector \(v\) is not Whittaker vector and therefore \(U(\hat{sl}_2)\) is not a classical Whittaker module. There are some hope that we can construct Whittaker vectors in Wakimoto modules using methods developed in the case of Virasoro algebra in [Y]. We will see below that our Wakimoto modules only gives a realization of degenerate classical Whittaker modules of type \((\lambda, 0)\) and \((0, \mu)\). Let us note here that in the tensor product modules we can construct classical non-degenerate Whittaker vectors. Let \(L_{\hat{sl}_2}(\lambda, 0, \kappa_1, a)\) and \(L_{\hat{sl}_2}(0, \mu, \kappa_2, b)\) be simple Whittaker modules generated by Whittaker vectors \(v_1\) and \(v_2\). Then \(v_1 \otimes v_2\) is a Whittaker vector of type \((\lambda, \mu)\) and we have
\[
V_{\hat{sl}_2}(\lambda, \mu, \kappa_1 + \kappa_2) \cong U(\hat{sl}_2)(v_1 \otimes v_2) \subset L_{\hat{sl}_2}(\lambda, 0, \kappa_1, a) \otimes L_{\hat{sl}_2}(0, \mu, \kappa_2, b).
\]

Our previous lemma is useful for a realization of degenerate Whittaker module. First we shall consider the case when \(\kappa \neq -2\).

Proposition 6.2. Assume that \(\mu = 0\) and \(\chi_1 = 0\). Then \(v\) is a Whittaker vector in \(M(\lambda, 0, \chi_0, 0)\); i.e.,
\[
e(0)v = \lambda v, \quad f(1)v = 0.
\]

If \(\kappa \neq -2\) then
\[
L(0)v = \frac{\chi_0(\chi_0 + 2)}{4(\kappa + 2)}v,
\]
and there is a non-trivial \(\hat{sl}_2\)-homomorphism
\[
M_{\hat{sl}_2}(\lambda, 0, \kappa, \frac{\chi_0(\chi_0 + 2)}{4(\kappa + 2)}) \rightarrow M_{Wak}(\lambda, 0, \kappa, \chi_0, 0).
\]

Now we shall consider the case of critical level and \(\hat{sl}_2\)-module \(\overline{M_{Wak}}(\lambda, \mu, -2, \chi(z))\) where
\[
\chi(z) = \frac{\chi_1}{z^2} + \frac{\chi_0}{z} + \sum_{n=1}^{\infty} \chi_n z^{n-1}.
\]

Theorem 6.3. Let \(\lambda, \mu \in \mathbb{C}\) with \(\lambda \neq 0\).

1. The \(U(\hat{sl}_2)\)-module \(\overline{M_{Wak}}(\lambda, \mu, -2, \chi(z))\) is irreducible.
The $U(\mathfrak{sl}_2)$–module $\overline{W}_{\text{Wak}}(\lambda, \mu, -2, \chi(z))$ has the following basis:

$$e(-n_1 - 1) \cdots e(-n_r - 1)h(-m_1) \cdots h(-m_s)v$$

where $n_1 \geq \cdots \geq n_r \geq 0$, $m_1 \geq \cdots \geq m_s \geq 0$, $r, s \in \mathbb{N}$.

Proof will be presented in a more general setting in Section 7.

In particular, we have obtained explicit realization of irreducible degenerate Whittaker modules:

**Corollary 6.4.** Assume that $\lambda \neq 0$, $\mu = 0$ and $\chi_1 = 0$. Then the $\mathfrak{sl}_2$–module $\overline{M}_{\text{Wak}}(\lambda, 0, -2, \chi(z))$ is an irreducible Whittaker module; i.e.,

$$V_{\mathfrak{sl}_2}(\lambda, 0, -2, c(z)) \cong \overline{M}_{\text{Wak}}(\lambda, \mu, -2, \chi(z)),$$

where

$$c(z) = \frac{1}{2}(\chi(z)^2 - 2\partial_z \chi(z)).$$

**Remark 6.** From our proof of irreducibility of $\overline{M}_{\text{Wak}}(\lambda, \mu, -2, \chi(z))$ we will see that $\overline{M}_{\text{Wak}}(\lambda, \mu, -2, \chi(z))$ is irreducible as a module for the parabolic subalgebra $\widehat{b} = b \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ of $\mathfrak{sl}_2$ where $b = \mathbb{C}e + \mathbb{C}h$. In fact as a $\widehat{b}$–module $\overline{M}_{\text{Wak}}(\lambda, \mu, -2, \chi(z))$ is isomorphic to the Whittaker module generated by $v$ such that

$$e(0)v = \lambda v, \ h(n + 1)v = e(n)v = 0 \quad \forall n \geq 1.$$

So our results are analogous to that of W.R. Wallach [Wa] and of D. Miličić and W. Sorgel [MS].

We have constructed a family of extensions of this Whittaker modules to $\mathfrak{sl}_2$.

**7. Proof of irreducibility and more general examples**

In the vertex algebra $(M, Y, \mathbf{1})$, let $e = a(-1)\mathbf{1}$, $\varphi = -2a(-1)\ast(0)\mathbf{1} \in M$, $b_1 := \mathbb{C}e + \mathbb{C} \varphi \subset M$. Then $b_1$ has the structure of 2–dimensional complex Lie algebra with bracket

$$[\varphi, e] = \varphi_0 e = 2e.$$

Let $\widehat{b}_1 = b_1 \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ be its affinization.

Let $\varphi(z) = Y(\varphi, z) = \sum_{n \in \mathbb{Z}} \varphi(n) z^{-n-1}$. Then

$$\varphi(n) = -2 \sum_{k \in \mathbb{Z}} a^\ast(k)a(n - k) \quad \text{for} \quad n \neq 0$$

$$\varphi(0) = -2 \left( \sum_{k \leq -1} a(k)a^\ast(-k) + \sum_{k \geq 0} a^\ast(-k)a(k) \right).$$

By using commutator formula in the vertex algebra $M$ we get the following relations:

$$[\varphi(n), \varphi(m)] = -4n\delta_{n+m,0}, \quad [\varphi(n), e(m)] = 2e(n + m).$$

We shall now prove that $M$–module $M_1(\lambda, \mu)$ is irreducible as $\widehat{b}_1$–module of level $\kappa = -2$.

**Lemma 7.1.** We have:

$$M_1(\lambda, \mu) = U(\widehat{b}_1)v,$$

i.e., $v_1$ is a cyclic vector.
Proof. In order to prove that $v_1$ is a cyclic vector it is enough to verify that
\[ C[a^*(n) \mid n \leq 0]v \subset U(\mathfrak{b}_1)v_1. \]

Other basis vector can be constructed by using action of $a(n) = e(n)$, $n \leq -1$.

By using the definition of the action of operator $\varphi(0)$ one can easily see that
\[ \text{span}_\mathbb{C}\{\varphi(0)^n v_1 \mid n \in \mathbb{N}\} = \text{span}_\mathbb{C}\{a^*(0)^n v_1 \mid n \in \mathbb{N}\}. \]
(This is also known from the theory of Whittaker modules for $\mathfrak{sl}_2$).

So it remains to prove that
\[ a^*(-n_1) \cdots a^*(-n_r) v \in U(\mathfrak{b}_1)v_1 \]
for all $r \in \mathbb{Z}_{\geq 0}$, and $n_1 \geq n_2 \cdots \geq n_r \geq 1$.

By the definition of action of $\varphi(-n)$, $n > 0$, we get:
\[ \varphi(-n_1) \cdots \varphi(-n_r) v_1 = Da^*(-n_1) \cdots a^*(-n_r) v_1 + w \quad (D \neq 0) \]
where
\[ w = \sum_{(\lambda, \mu) \in P \times P} C_{\lambda, \mu} u_{\lambda, \mu} v_1 \quad (C_{\lambda, \mu} \in \mathbb{C}) \]
and $\mu_0 = (n_1 + 1, n_2 + 1, \ldots, n_r + 1) > \mu$ if $C_{\lambda, \mu} \neq 0$. Proof now follows by induction. \qed

**Proposition 7.2.** For every $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, $M_1(\lambda, \mu)$ is an irreducible $\mathfrak{b}_1$–module.

*Proof.* It is enough to prove that every vector $w$ in $M_1(\lambda, \mu)$ is cyclic. By applying the action of $a(n)$ we can eliminate basis elements which contain products of $a^*(-n)$. Then we can assume that $w$ belongs to the space:
\[ C[a(-n) \mid n \geq 1]v_1. \]

So $w$ can be written in the form
\[ w = \sum_{\nu \in P} C_{\nu} u_{\nu, \phi} v_1 \quad (C_{\nu} \in \mathbb{C}) \]
(7)

Let
\[ N = \max\{|\nu| \mid C_{\nu} \neq 0\} \quad r = \min\{\ell(\nu) \mid C_{\nu} \neq 0, |\nu| = N\}. \]

Take any $\nu_0 = (i_1, \ldots, i_r) \in P$ such that $C_{\nu_0} \neq 0$, $\ell(\nu_0) = r$, $|\nu| = N$. (So we choose the shortest possible basis element which appear in (7) of maximal size).

We have
\[ \varphi(i_1) \cdots \varphi(i_r) a(-i_1) \cdots a(-i_r) v_1 = Da(0)^r v_1 = D\lambda^r v_1 \quad (D \neq 0). \]
Moreover if $C_{\nu} \neq 0$ and $\nu \neq \nu_0$ one easily sees that
\[ \varphi(i_1) \cdots \varphi(i_r) u_{\nu, \phi} v_1 = 0. \]
Therefore
\[ \varphi(i_1) \cdots \varphi(i_r) w = D\lambda^r v_1 \quad (D \neq 0). \]
The proof follows. \qed

Let now $\chi(z) = \sum_{n \in \mathbb{Z}} \chi_n z^{-n-1} \in \mathbb{C}(z)$ is arbitrary. Let $\mathbb{C}1_\chi$ be 1–dimensional $\pi^0$–module with the property that $b(n)$ acts on $\mathbb{C}1_\chi$ by multiplication with $\chi_n$.

Let $h = \phi + b(-1)1$, $b = \mathbb{C}h + C$ be as above. Let $M_{\text{Wh}}(\lambda, \mu, -2, \chi(z)) := M_1(\lambda, \mu) \otimes \mathbb{C}1_\chi$. (When $\chi(z)$ is as in (3) this module was already defined). As before we set $v = v_1 \otimes 1_\chi$. 

Lemma 7.3. Let $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, $\chi(z) \in \mathbb{C}(z)$. Then $M_1(\lambda, \mu) \otimes C_1\chi$ is an irreducible $\hat{\mathfrak{b}}$-module.

Proof. In this case, $h(z) = \varphi(z) + \chi(z)$. One can easily see that irreducibility from the Proposition 7.2 is not affected if we twist the action of $\varphi(z)$ by $\chi(z)$. The proof follows. $\square$

By using the Wakimoto realization we can extend the action of $\hat{\mathfrak{b}}$ to the action of affine Lie algebra $\hat{\mathfrak{sl}_2}$. Since $M_1(\lambda, \mu) \otimes C_1\chi$ is irreducible $\hat{\mathfrak{b}}$-module it is also irreducible as a module for the affine Lie algebra. We have:

Theorem 7.4. For every $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, $\chi(z) \in \mathbb{C}(z)$,

$$M_{Wak}(\lambda, \mu, -2, \chi(z)) = M_1(\lambda, \mu) \otimes C_1\chi$$

is an irreducible $\hat{\mathfrak{sl}_2}$-module at the critical level.

Assume that $\chi(z) = \sum_{k=-p}^{\infty} \chi_k z^{k-1}$, $\chi_p \neq 0$. The case $p = 1$ was already discussed above. So let us consider the case $p \geq 2$. We get condition of (generalized) Whittaker type

\[
eq (8) \quad e(0)v = \lambda v, \\
\neq (9) \quad h(1)v = (-2\lambda \mu + \chi_1)v, \quad h(k)v = \chi_k v \text{ (} k = 2, \ldots, p \text{)} \\
\neq (10) \quad f(p + 1)v = \mu \chi_p v \\
\neq (11) \quad e(n)v = h(n + p)v = f(n + p + 1)v = 0 \quad \forall n \in \mathbb{Z}_{\geq 0}.
\]

Modules constructed above are non-isomorphic. This can be proved by using the action of the center of $V_{-2}(\mathfrak{sl}_2)$ which is generated by components of the field $T(z)$. In our case

$$T(z) = \frac{1}{2}(\chi(z)^2 - 2\partial_z \chi(z)) \text{ on } M_{Wak}(\lambda, \mu, -2, \chi(z)).$$

Theorem 7.5. Assume that $\lambda \neq 0$ and $\chi(z) = \sum_{k=-p}^{\infty} \chi_k z^{k-1}$, $\chi_p \neq 0$, $p \geq 2$. Then

$$M_{Wak}(\lambda, \mu, -2, \chi(z)) \cong M_{Wak}(\lambda', \mu', -2, \chi'(z))$$

if and only if

$$\lambda = \lambda', \mu = \mu', \chi(z) = \chi'(z).$$

Proof. Assume that

$$M_{Wak}(\lambda, \mu, -2, \chi(z)) \cong M_{Wak}(\lambda', \mu', -2, \chi'(z)).$$

Then conditions (8)-(11) easily imply that

\[
eq (12) \quad \lambda = \lambda', \mu = \mu', \chi_k = \chi'_k \forall k \geq 1.
\]

On the other hand the action of the central elements should be the same on both modules. Therefore

\[
eq (13) \quad \chi(z)^2 - 2\partial_z \chi(z) = \chi'(z)^2 - 2\partial_z \chi'(z).
\]

By straightforward calculation, using (12) and identifying coefficients in the Laurent expansions of (13) we get that $\chi(z) = \chi'(z)$. The proof follows. $\square$
Let $\pi_s$ be the automorphism of $U(\hat{\mathfrak{sl}}_2)$ such that

$$
\pi_s(e(n)) = e(n-s), \quad \pi_s(f(n)) = f(n+s), \quad \pi_s(h(n)) = h(n) - s\delta_{n,0}c
$$

(cf. [A1]). Then for every $s \in \mathbb{Z}$ $\pi_s(M_{Wh}^{ak}(\lambda, \mu, -2, \chi(z)))$ is also irreducible $\hat{\mathfrak{sl}}_2$–module at the critical level generated by vector $v$ such that

$$
e(s)v = \lambda v, \quad h(1)v = (-2\lambda \mu + \chi_1)v, \quad h(k)v = \chi_k v \quad (k = 2, \ldots, p),
$$

$$
f(p + 1 - s)v = \mu \chi_p v, \quad e(n+s)v = h(n+p)v = f(n+p + 1 - s)v = 0 \quad \forall n \in \mathbb{Z}_{\geq 0}.
$$

8. **Bosonic Realization of non-degenerate classical Whittaker modules at the critical level**

In this section we shall present a bosonic realization of the classical Whittaker modules in the non-degenerate case. It turns out that we need to extend Wakimoto modules to a larger space. The main idea is to replace Weyl vertex algebra $M$ with a larger vertex algebra $\Pi(0)$ obtained using localization of $M$ with respect to $a(\alpha^{-1})$.

Let $L$ be the lattice $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$, $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1$, $\langle \alpha, \beta \rangle = 0$, and $V_L = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$ the associated lattice vertex superalgebra, where $M_{\alpha,\beta}(1)$ is the Heisenberg vertex algebra generated by fields $\alpha(z)$ and $\beta(z)$ and $\mathbb{C}[L]$ is the group algebra of $L$. We have its subalgebra $\Pi(0) = M_{\alpha,\beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \subset V_L$.

The Weyl vertex algebra $M$ can be realized as a subalgebra of $\Pi(0)$ generated by $a = e^{\alpha+b}, \quad a^* = -\alpha(-1)e^{-\alpha-b}$.

Recall that (cf. [A1], [F]):

$$M = \text{Ker}_{\Pi(0) \oplus}^\alpha.
$$

This vertex algebra can be described as a localization of Weyl vertex algebra with respect to $a(-1)$, $\Pi(0) = M[(a(-1)^{-1})]$. Let us write $a^{-1} := e^{-\alpha-b}$ and $a^{-1}(n) := e^{-\alpha-b}_n$. We have the expansion

$$
Y(a^{-1}, z) = \sum_{n \in \mathbb{Z}} a^{-1}(n)z^{-n+1}.
$$

Choose the Virasoro vector

$$\omega = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \beta(-2)).
$$

Then $\omega$ defines a $\mathbb{Z}$–graduation on $\Pi(0)$ so that deg $a = 1$ and deg $a^{-1} = -1$. In particular on any $\Pi(0)$–modules we have:

(14) $[L(0), a(n)] = -n a(n), \quad [L(0), a^{-1}(n)] = -n a^{-1}(n)$.

The following theorem shows the existence of a Whittaker module for the vertex algebra $\Pi(0)$. 

Theorem 8.1. Assume that \( \lambda \neq 0 \). There is a \( \Pi(0) \)-module \( \Pi_\lambda \) generated by the cyclic vector \( w_\lambda \) such that
\[
a(0)w_\lambda = \lambda w_\lambda, \quad a^{-1}(0)w_\lambda = \frac{1}{\lambda} w_\lambda, \quad a(n)w_\lambda = a^{-1}(n)w_\lambda = 0 \quad \text{for } n \geq 1.\]
As a vector space
\[
\Pi_\lambda \cong \mathbb{C}[d(-n), c(-n-1) \mid n \geq 0] = \mathbb{C}[d(0)] \otimes M_{\alpha, \beta}(1),
\]
where \( c = \alpha + \beta, \quad d = \alpha - \beta. \)
The module \( \Pi_\lambda \) is \( \mathbb{Z}_{\geq 0} \)-graded
\[
\Pi_\lambda = \bigoplus_{n \in \mathbb{N}} \Pi_\lambda(n)
\]
and its lowest component is isomorphic to \( \mathbb{C}[d(0)]. \)
Proof. The proof is based on the construction presented in [BDT] (see also [LW]). We shall omit details. Let \( \mathcal{A} \) be associative unital algebra generated by generators
\[
d, \quad e^{nc}, \quad n \in \mathbb{Z}
\]
and relations
\[
[d, e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c} \quad (n, m \in \mathbb{Z}).
\]
The results from Section 4 of [BDT] implies that for any \( \mathcal{A} \)-module \( U \) and any \( \gamma \in \frac{1}{2} \mathbb{Z}d \) on the vector space
\[
L_\gamma(U) = U \otimes M_{\alpha, \beta}(1)
\]
there exists the unique \( \Pi(0) \)-module structure. Moreover, \( U \otimes M_{\alpha, \beta}(1) \) is irreducible \( \Pi(0) \)-module if and only if \( U \) is irreducible \( \mathcal{A} \)-module. On \( L_\gamma(U) \) we have
\[
d(0) = d \otimes \text{Id}, \quad c(0) = (c, \gamma) \text{Id}.
\]
Let \( U_\lambda \) be \( \mathcal{A} \)-module generated by vector \( v_1 \) such that
\[
e^{nc}v_1 = \lambda^n v_1 \quad (n \in \mathbb{Z})
\]
and that \( d \) acts freely. Then \( U_\lambda \) is an irreducible \( \mathcal{A} \)-module and \( U_\lambda = \mathcal{A}.v_1 \cong \mathbb{C}[d] \) as a vector space.
Let \( \gamma = -\frac{1}{2}d. \) Then
\[
\Pi_\lambda := L_\gamma(U_\lambda)
\]
is an irreducible \( \Pi(0) \)-module. Let \( v = v_1 \otimes 1. \) By construction we have:
\[
a(0)v = \lambda v, \quad a^{-1}(0)v = \frac{1}{\lambda} v, \quad a(n)v = a^{-1}(n)v = 0 \quad (n \geq 1).\]
Since
\[
L(0)v = \frac{1}{2}(c(0)d(0) + d(0))v = 0
\]
relations (14) and (15) imply that \( L(0) \) acts semisimply on \( \Pi_\lambda \) and it defines a required \( \mathbb{Z}_{\geq 0} \)-gradation. The proof follows. \( \square \)
Remark 7. Note that \( \Pi(0) \) is \( \mathbb{Z} \)-graded, but it is not \( \mathbb{Z}_{\geq 0} \)-graded. But one can also apply Zhu’s algebra theory to construct and classify its \( \mathbb{Z}_{\geq 0} \)-graded modules. One can show that in our case Zhu’s algebra \( A(\Pi(0)) \) is isomorphic to the associative algebra \( \mathcal{A} \) and therefore \( U \) has the structure of an irreducible module for \( A(\Pi(0)). \)
So the existence of the module \( \Pi_\lambda \) also follows from Zhu’s algebra theory.
Since \( M \subset \Pi(0) \) we can consider \( \Pi_\lambda \) as a \( \hat{\mathfrak{b}}_1 \)-module.
Lemma 8.2. Let \( \lambda \neq 0 \). Then we have:

(i) \( \Pi_\lambda \cong M_1(\lambda,0) \) as \( M \)-modules

(ii) \( \Pi_\lambda \) is irreducible \( \mathfrak{b}_1 \)-module.

Proof. We shall first consider \( \Pi_\lambda \) as a module over Weyl vertex algebra \( M \). We directly see that

\[
a(0)w_\lambda = \lambda w_\lambda, \quad a(n+1)w_\lambda = a^*(n+1)w_\lambda = 0 \quad (n \geq 0).
\]

Therefore the cyclic \( M \)-submodule \( M.w_\lambda \) of \( \Pi_\lambda \) is isomorphic to \( M_1(\lambda,0) \).

On the other we have relation

\[
a(z)a^{-1}(z) = Y(a(-1)a^{-1}1, z) = Y(1, z) = Id
\]

which easily implies that

\[
\mathbb{C}[a^{-1}(-n) \mid n \geq 0]w_\lambda \subset \mathbb{C}[a(-n) \mid n \geq 0]w_\lambda \subset M.w_\lambda.
\]

This proves that \( M.w_\lambda = \Pi_\lambda \), and assertion (i) holds. Assertion (ii) follows from (i) and Proposition 7.2.

Let \( M_T(0) \) be the commutative vertex subalgebra of \( V_{-2}(sl_2) \) generated by \( T(z) = \sum_{n \in \mathbb{Z}} T(n)z^{-n-2} \).

By using standard calculations in vertex algebras we get:

**Proposition 8.3.** There is an embedding of vertex algebras

\[
V_{-2}(sl_2) \rightarrow M_T(0) \otimes \Pi(0)
\]

such that

\[
e = a, \quad h = -2\beta(-1) = -2a^*(0)a(-1)1, \quad f = [T(-2) - (\alpha(-1)^2 - \alpha(-2))a^{-1}]
\]

\[
= -a^*(0)^2a(-1)1 - 2a^*(-1)1 + T(-2)a^{-1}.
\]

For any \( \chi(z) = \sum_{n \in \mathbb{Z}} \chi(n)z^{-n-2} \in \mathbb{C}(z) \) let \( M_T(\chi(z)) \) be 1-dimensional \( M_T(0) \)-module such that \( T(n) \) acts as multiplication with \( \chi(n) \in \mathbb{C} \).

We have:

**Theorem 8.4.** Let \( \lambda \neq 0 \). Let

\[
\chi(z) = \frac{\lambda \mu}{z^3} + c(z), \quad c(z) = \sum_{n \leq 0} \chi(n)z^{-n-2} \in \mathbb{C}(z).
\]

Then we have:

\[
V_{sl_2}(\lambda, \mu, -2, c(z)) \cong M_T(\chi(z)) \otimes \Pi_\lambda.
\]

Proof. Irreducibility of \( M_T(\chi(z)) \otimes \Pi_\lambda \) as a \( \mathfrak{b}_1 \)-module follows from Lemma 8.2. Therefore \( M_T(\chi(z)) \otimes \Pi_\lambda \) is irreducible \( \mathfrak{sl}_2 \)-modules at the critical level on which \( T(z) \) acts as \( \chi(z) \). It remains to prove that \( M_T(\chi(z)) \otimes \Pi_\lambda \) is generated by Whittaker vector of type \( \lambda \).

By construction we have

\[
e(0)(1 \otimes w_\lambda) = 1 \otimes a(0)w_\lambda = \lambda(1 \otimes w_\lambda)
\]

\[
f(1)(1 \otimes w_\lambda) = T(1).1 \otimes a^{-1}(0)w_\lambda = \mu(1 \otimes w_\lambda)
\]

\[
e(n+1)(1 \otimes w_\lambda) = h(n+1)(1 \otimes w_\lambda) = f(n+2)(1 \otimes w_\lambda) = 0 \quad (n \geq 0).
\]
Therefore $1 \otimes w_\lambda$ is a Whittaker vector of type $(\lambda, \mu)$. The proof follows. \hfill \Box

If we put $d = -Id \otimes L(0)$ we will get that $[d, x(n)] = nx(n)$ for $x \in \mathfrak{sl}_2$ iff $\chi(z) = a/z^2$. So we have:

**Corollary 8.5.** Assume that $\lambda \neq 0$ and $\chi(z) = a/z^2$, $a \in \mathbb{C}$. Then $M_T(\frac{a}{z^2}) \otimes \Pi_\lambda$ is the irreducible $\tilde{\mathfrak{sl}}_2$ Whittaker module $V_{\tilde{\mathfrak{sl}}_2}(\lambda, 0, -2, 0) \otimes X^{\mathbf{b} \times \tilde{T}}$ for the $1$-dimensional irreducible $\tilde{T}$-module $X$ (see Theorem 4.7).

**Acknowledgments.** Part of the research presented in this paper was carried out during the visit of the first two authors to Wilfrid Laurier University in April of 2014 and of the second author to University of Waterloo in 2014. The second author thanks professors Wentang Kuo and Kaiming Zhao for sponsoring his visit, and University of Waterloo for providing excellent working conditions. The first author thanks Kaiming Zhao and Wilfrid Laurier University for hospitality during his visit.

D.A. is partially supported by the Croatian Science Foundation under the project 2634.

R.L. is partially supported by NSF of China (Grant 11471233, 11371134) and Jiangsu Government Scholarship for Overseas Studies (JS-2013-313).

K.Z. is partially supported by NSF of China (Grant 11271109, 11471233) and NSERC.

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