Integrability of three-particle evolution equations in QCD

V.M. Braun
NORDITA, Blegdamsvej 17, DK–2100 Copenhagen, Denmark

S.É. Derkachov
Department of Mathematics, St.-Petersburg Technology Institute, St.-Petersburg, Russia

A.N. Manashov
Department of Theoretical Physics, Sankt-Petersburg State University, St.-Petersburg, Russia

Abstract:
We show that the Brodsky-Lepage evolution equation for the spin 3/2 baryon distribution amplitude is completely integrable and reduces to the three-particle $XXX_{s=-1}$ Heisenberg spin chain. Trajectories of the anomalous dimensions are identified and calculated using the $1/N$ expansion. Extending this result, we prove integrability of the evolution equations for twist 3 quark-gluon operators in the large $N_c$ limit.

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A QCD description of hard exclusive processes invokes the concept of hadron distribution amplitudes (DAs) given by matrix elements of nonlocal light-cone operators between the vacuum and the hadron state. It was realized long ago that one-loop renormalization group (RG) equations for the leading twist meson distributions are diagonalized in the conformal basis: The mixing matrix for the corresponding local operators becomes diagonal and the anomalous dimensions coincide with those familiar from studies of the deep-inelastic scattering. The situation with baryon DAs is much more complicated. In this case one has to consider renormalization of three-quark operators of the type:

\[ B(a, b, c) = \epsilon_{ijk} q^i (au) q^j (bu) q^k (cu). \]

(1)

Here \( q \) is a quark field and \( u \) is an auxiliary light-like vector \( u^2 = 0 \); taking the leading-twist part and insertion of gauge factors is implied. Conformal symmetry allows to eliminate all mixing with operators including total derivatives but is not sufficient to diagonalize the mixing matrix. In the usual expansion of baryon DAs in Appell polynomials only the lowest-order term (asymptotic DA) is renormalized multiplicatively while in general one is left with a nontrivial \( (N+1) \times (N+1) \) mixing matrix, \( N \) being the number of derivatives, and has to diagonalize it explicitly order by order, see [2–4]. The analytic structure of the spectrum was not known and further analytic results were not expected.

Main result of this letter is that the 3-particle Schrödinger equation describing the renormalization of spin 3/2 baryon operators is completely integrable, i.e. has an additional integral of motion. Our result is similar in spirit to the recent discovery [5,6] of integrability of the system of interacting reggeized gluons in QCD, but is obtained in a different physical context. In particular, the ERBL evolution equation for spin 3/2 baryon operators appears to be mathematically equivalent to the equation for the odderon trajectory, and the results obtained in the latter context [7,8] can be adapted to unravel the spectrum of baryon operators.

Extending this result, we prove integrability of the RG equations for twist 3 quark-gluon operators

\[
S^\pm_\mu(a, b, c) = \bar{q}(au)[iG_{\mu\nu}(bu) \pm \bar{G}_{\mu\nu}(bu) \gamma_5]u^\nu q^\mu q(cu),
\]

(2)

\[
T(a, b, c) = \bar{q}(au)u^\mu u^\nu \sigma_\mu^\nu G_{\nu\rho}(bu) \Gamma q(cu),
\]

(3)

where \( \Gamma = \{ I, i\gamma_5 \} \), in the limit of large number of colors \( N_c \). Such operators give rise to twist 3 nucleon parton distributions and have attracted considerable interest recently, see e.g. [9,10]. Further results will be presented elsewhere [11].

To one-loop accuracy, the divergent part of the nonlocal operator \( \Phi(a_1, a_2, a_3) \) where \( \Phi = B, S^\pm, T, \) has the form \((1/\epsilon)H \Phi(a_i)\), and the explicit expression for the integral operator \( H \) is known for all cases under consideration [3,4,10]. An arbitrary local operator \( O \) with \( N \) covariant derivatives can be represented by the associated polynomial in three variables \( \psi(a_1, a_2, a_3) \) of degree \( N \) such that \( O_{\psi} = \psi(\partial_a, \partial_b, \partial_c) \Phi(a_1, a_2, a_3) \) where \( \partial_a = \partial/\partial a \) etc. In order to find multiplicatively renormalizable local operators one has to solve the Schrödinger equation for the \( \psi \)-functions, \( H \psi = \mathcal{E} \psi \), where \( H \) is easy to find if \( H \) is given.

It proves convenient to define the integral transformation [13] \( \psi(a_i) \to \hat{\psi}(z_i) \) by

\[
\hat{\psi}(z_i) \equiv \prod_i \int_0^\infty dt_i e^{-t_i z_i^{l+i+s}-1} \psi(z_1 t_1, ..., z_3 t_3)
\]

(4)
where \( l_i \) and \( s_i \) are the canonical dimension and spin projection of the \( i \)-th field, respectively: \( l = 3/2 \), \( s = 1/2 \) for quarks (antiquarks) and \( l = 2 \), \( s = 1 \) for gluons. We can reformulate the above eigenvalue problem in terms of \( \hat{\psi} \) functions and it is easy to check that the corresponding Hamiltonian \( \hat{H} \) coincides with the initial Hamiltonian for the nonlocal operator, \( \hat{H} \equiv H \). As a trivial consequence of topology of one-loop Feynman diagrams \( H \) has a two-particle structure, \( H = \sum l_i,k H_{i,k} \). Conformal invariance implies that the two-particle Hamiltonians \( H_{i,k} \) commute with \( SL(2) \) generators \( J_{i,k} = \sum_{i=1}^{3} J_{i,k}^{\pm,3} \), where

\[
J_i^+ = z_i^2 \partial_i + (l_i + s_i)z_i, \quad J_i^- = -\partial_i, \\
J_i^3 = z_i \partial_i + (l_i + s_i)/2,
\]

and are hermitean with respect to the scalar product

\[
\langle \hat{\psi}_1 | \hat{\psi}_2 \rangle = \psi_1(\partial_1, \ldots, \partial_3) \hat{\psi}_2(z_1, \ldots, z_3) \bigg|_{z_i=0}.
\]

Thus, the equation \( H \hat{\psi} = E \hat{\psi} \) decays into the set of eigenvalue problems on the subspaces of functions with fixed value of \( J_3, J_3 \hat{\psi} = j_3 \hat{\psi} \) and annihilated by \( J_-, J_- \hat{\psi} = 0 \). The advantage of the \( \hat{\psi} \)-representation' is that in this basis the latter condition is simply shift-invariance \( \hat{\psi}_i \). Therefore, eigenfunctions of two-particle Hamiltonians are given by simple powers \( \psi_i = (z_i - z_k)^l \) instead of Jacobi polynomials in standard variables \( \hat{\psi}_i \).

The \( SL(2) \) invariance imposes stringent restrictions on the form of two-particle operators, so that only a few structures are allowed. One such structure corresponds to the 'vertex correction' involving the gluon field from (one of) the covariant derivatives (in Feynman gauge):

\[
H^v_{12}(\z) = -\int_0^1 \frac{d\alpha}{\alpha} \left\{ \frac{1}{\alpha_{1}^{l_1 + s_1 - 1}} \left[ \hat{\psi}(\z_{12}^{\alpha}, z_2, z_3) - \hat{\psi}(\z) \right] \\
+ \frac{1}{\alpha_{2}^{l_2 + s_2 - 1}} \left[ \hat{\psi}(\z_2, z_{21}^{\alpha}, z_3) - \hat{\psi}(\z) \right] \right\},
\]

where \( \z \equiv \{z_1, z_2, z_3\}, \ z_{ik}^{\alpha} = z_i \alpha + z_k \alpha \) and \( \alpha = 1 - \alpha \). Another structure originates from gluon exchange between quarks (or between a quark and a gluon):

\[
H^c_{12}(\z) = 2 \int_0^1 \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{1}{\alpha_1 \alpha_2} \hat{\psi}(z_{12}^{\alpha_1}, z_{21}^{\alpha_2}, z_3),
\]

where \( D \alpha = \prod_{i=1}^{3} d\alpha_i \delta(1 - \sum \alpha_i) \).

Due to \( SL(2) \) invariance the two-particle Hamiltonians must depend on the corresponding Casimir operators \( L_{i,k} \equiv (\bar{J}_i + \bar{J}_k)^2 = J_{i,k}(J_{i,k} + 1) \) only. This dependence can be easily reconstructed from the spectrum of \( H_{i,k} \). Since the form of the eigenfunctions is known \( \psi_i = (z_i - z_k)^l \), it is straightforward to derive

\[
H_{i,k}^{(qq)} = 2 [\psi(J_{i,k} + 1) - \psi(2)], \\
H_{i,k}^{(qq)} = \psi(J_{i,k} + 3/2) + \psi(J_{i,k} + 1/2) - \psi(3) - \psi(2),
\]

where \( \psi(x) \equiv \Gamma'(x)/\Gamma(x) \). The superscripts \( (qq) \) and \( (qq) \) indicate quark-quark and quark-gluon operators, respectively. Similarly, we obtain

2
\[ H_{ik}^{c,(qq)} = 2J_{ik}^{-1}(J_{ik} + 1)^{-1}, \]
\[ H_{ik}^{v,(qq)} = 2(J_{ik} + 3/2)^{-1}(J_{ik} - 1/2)^{-1}. \]

We are now in a position to specify RG equations for the operators in (11) explicitly. One has to distinguish 3-quark operators belonging to (3/2,0) and (1,1/2) representations, which correspond to DAs for spin 3/2 and spin 1/2 baryons, respectively. We get (15):

\[ H_{3/2} = H_{12}^{v,(qq)} + H_{13}^{v,(qq)} + H_{23}^{v,(qq)}, \]
\[ H_{1/2} = H_{3/2} - (1/2)H_{12}^{c,(qq)} - (1/2)H_{23}^{c,(qq)}. \]

Omitting subleading in \( N \) terms, the quark-antiquark-gluon Hamiltonians are (12):

\[ H_{S^+} = H_{12}^{v,(qq)} + H_{23}^{v,(qq)} - H_{12}^{c,(qq)}, \]
\[ H_{S^-} = H_{12}^{v,(qq)} + H_{23}^{v,(qq)} - H_{23}^{c,(qq)}, \]
\[ H_{T} = H_{12}^{v,(qq)} + H_{23}^{v,(qq)} - H_{12}^{c,(qq)} - H_{23}^{c,(qq)}. \]

The properly defined anomalous dimensions are given in terms of eigenvalues of the above operators including color factors and trivial contributions of self-energy insertions:

\[ \gamma_{3/2,1/2}(N) = (1 + 1/N_c) \mathcal{E}_{3/2,1/2}(N) + (3/2) C_F, \]
\[ \gamma_{S,T}(N) = N_c \mathcal{E}_{S,T}(N) + (7/2) N_c \]

where \( C_F = (N_c^2 - 1)/(2N_c) \).

The operators \( H_{S^\pm} \) are equivalent; hereafter we consider \( H_{S^+} \). The \( 1/N_c^2 \) corrections to (12) and RG equations for 3-gluon operators involve additional structures (11) and will not be discussed here.

We have been able to find integrals of motion \( Q_i \), \( [H_i, Q_i] = 0 \), for all Hamiltonians in question except for \( H_{1/2} \). Explicit expressions for the conserved charges \( Q_i \) present the main result of this letter:

\[ Q_{3/2} = \hat{\imath} [L_{12}, L_{13}] = \hat{\imath} \partial_1 \partial_2 \partial_3 z_{12} z_{23} z_{31}, \]
\[ Q_{S^+} = \{ L_{12}, L_{23} \} - 9/2 L_{23} - 1/2 L_{12}, \]
\[ Q_T = \{ L_{12}, L_{23} \} - 9/2 L_{12} - 9/2 L_{23}, \]

where \( \{ \, , \} \) stands for an anticommutator. Remarkably, \( H_{3/2} \) coincides in our representation with the Hamiltonian of the XXX \( s=-1 \) 3-particle Heisenberg spin chain. Hence the expression in (17) for the conserved charge \( Q_{3/2} \) follows directly from the corresponding classical result, see also [6].

To check that \( [H_T, Q_T] = 0, [H_S, Q_S] = 0 \) we introduce a complete set of functions (13)

\[ \tilde{\psi}_{ik}^n(z_i, z_k; z_l), n = 0, \ldots, N \] where \( i, k \) is a fixed pair of indices (say, (1,2)) and \( z_i \) is the third variable, different from \( z_i, z_k \). The functions \( \tilde{\psi}_{ik}^n \) are obtained by the ‘hat’-transformation (4) of the set of polynomials of degree \( N \):
\[
\psi_{ik}(z_i, z_k; z_l) = Z_n \sum_{m=0}^{N-n} K_m^n(z_i + z_k)^{N-n-m} \psi_n(z_i, z_k),
\]
\[
K_m^n = \frac{(-1)^{N-n-m} C_m^n}{(l_l + s_l)(2n + l_i + l_k + s_i + s_k)(N-n-m)}. \tag{17}
\]

where \(C_m^n\) is the binomial coefficient and \((a)_n \equiv \Gamma(a + n)/\Gamma(a)\). \(\psi_n(z_i, z_k)\) is defined such that \(\psi_n(z_i, z_k) = (z_i - z_k)^n\) is an eigenfunction of \(H_{ik}\) and the factor \(Z_n\) is chosen by requiring that \(\psi_{ik}^n\) has unit norm.

The functions \(\psi_{ik}^n\) are shift invariant and mutually orthogonal with respect to the scalar product \((\cdot, \cdot)\). It is easy to check that \(J_3 \psi_{ik}^n = (N + 7/2) \psi_{ik}^n\). The operator \(L_{ik}\) is diagonal in the basis \(\psi_{ik}^n\) while the other two Casimir operators are three-diagonal (\(|n|L|n'| \neq 0\) for \(|n - n'| \leq 1\) only).

Consider matrix elements of the commutator
\[
[A = [Q_T, H_T^{(12)}] = \{L_{12} - 9/4, [H_T^{(12)}, L_{23}]\}
\]

where \(H_T^{(12)} = H_T^{(qg)} - H_T^{(gq)}\), sandwiched between the \(\psi_{12}^n\) 'states'. Since \(L_{23}\) is three-diagonal in this basis, the only nonzero elements are \(A_{n,n+1}\) and \(A_{n+1,n}\). Due to antihermiticity of \(A\) it is sufficient to consider \(A_{n,n+1}\):
\[
A_{n,n+1} = (L_{23})_{n,n+1}(E_n - E_{n+1})(L_n + L_{n+1} - 9/2),
\]
where \(E_n\) and \(L_n\) are the eigenvalues of the operators \(H_T^{(12)}\) and \(L_{12}\), respectively. Using explicit expressions for \(H_T^{(12)}, L_{12}\) it is easy to derive that \(A_{n,n+1} = 2(L_{23})_{n,n+1}(E_n - L_{n+1})\), that is in operator form
\[
[Q_T, H_T^{(12)}] = 2[L_{12}, L_{23}]. \tag{18}
\]

Similarly, we obtain that \([Q_T, H_T^{(23)}] = 2[L_{23}, L_{12}]\) and, consequently, \([Q_T, H_T] = 0\). The proof for \(S^+\) operator is analogous.

Once conserved charges are known, one can consider the eigenvalue problem for these charges instead of the Hamiltonians, which is simpler. For the Heisenberg spin chain, a detailed study exists due to Korchemsky \([7, 8]\). The spectrum of \(Q_{3/2}\) is shown in Fig. 1a. For generic integer \(N\) there exist \(N + 1\) eigenvalues which come in pairs \(\pm q\). Note that for even \(N\) \(Q_{3/2}\) has zero eigenvalue \(q = 0\). The corresponding value of energy can be calculated exactly:
\[
E_{3/2}(N, q = 0) = 4\psi(N + 3) + 4\gamma_E - 6, \tag{19}
\]
see the dotted curve in Fig. 1b. Eigenvalues of \(Q_{3/2}\) lie on trajectories which were found in \([7]\) using a 'semiclassical' expansion in the parameter \(h = N + 3\):
\[
q(N, k)/h^3 = \sum_m q^{(m)}(k)/k^m,
\]
\[
q^{(0)} = 1/\sqrt{2\pi}, \quad q^{(1)}(k) = -(k + 1)/\sqrt{3}, \ldots \tag{20}
\]
The $q^{(m)}(k)$ are polynomials of degree $m$; first eight of them are given in Eq. (5.14) in Ref. [7]. $k$ is a nonnegative integer which numerates the trajectory. The asymptotic expansion in (20) is valid for $q > 0$ only and the analytic continuation of the trajectory to $q < 0$ can be obtained by using symmetry properties of the solutions [7]:

$$q(N, k) \xrightarrow{q<0} -q(N, N-k)$$

Two examples of the trajectories with $k = 2$ and $k = 7$ are shown in Fig. 1a together with exact eigenvalues (crosses) calculated numerically. Note that the two asymptotic expansions — for positive and negative $q$ — match reasonably well. Explicit analytic formulas for the trajectories in the $q \to 0$ region are available from [8].

The low-lying eigenvalues of $q(N,k)$ can be calculated to $O(1/N)$ accuracy from the equation [11]

$$(q/N^2) \ln N - arg[\Gamma(1+i q/N^2)] = \frac{\pi}{6} (N - 2k)$$

which is valid for $k - N/2 \ll \ln N$. The lowest value of $|q|$ for odd $N$ is thus of order

$$q/N^2 = \pm \frac{\pi}{6} (\ln N + \gamma_E)^{-1} + O(1/(\ln N + \gamma_E)^4).$$

(22)

The spectrum of $H_{3/2}$ is shown in Fig. 1b. Exact eigenvalues obtained by explicit diagonalization of the mixing matrix are shown by crosses. Since $\mathcal{E}_{3/2}(q) = \mathcal{E}_{3/2}(-q)$ all eigenvalues except for the ones for $q = 0$ are double-degenerate. The energy eigenvalues lie on trajectories corresponding to the trajectories for $q$ in Fig. 1a, and, similar to the latter, can be calculated using a ‘semiclassical’ expansion [7]

$$\mathcal{E}_{3/2}(N,k) = \varepsilon^{(0)} - \sum_{m=1}^{\infty} \varepsilon^{(m)}(k)/\hbar^m,$$

$$\varepsilon^{(0)} = 6 \ln(N + 3) + 6 \gamma_E - 6 - 3 \ln 3, \ldots$$

(23)

The polynomials $\varepsilon^{(m)}(k)$ are given in Eq. (6.5) of Ref. [7] up to $m = 7$. The trajectories corresponding to $k = 2$ and $k = 7$ are shown in Fig. 1b by broken lines, whereas the solid curves correspond to the asymptotic expansion in (23) [16]. Note that the expansion diverges close to the 'deflection points' which occur at even integer $N$ and with the energy given by Eq. (19). Convergence of the $1/\hbar$ expansion is somewhat worse for the energy compared to the conserved charge $q$, but it can be improved systematically. Alternatively, one can derive analytic approximations for the conserved charge $q(N,k)$ and $\mathcal{E}_{3/2}(q)$ applicable in the $q \to 0$ region, see [8,11].

Using (22) one can derive an estimate for the lowest energy eigenvalue for odd $N$:

$$\mathcal{E}_{3/2}(N) = 4 \ln N - 6 + 4 \gamma_E + \frac{\zeta(3)}{18 \ln^2 N}.$$  

(24)

Numerically the difference to Eq.(19) is very small, compare exact eigenvalues with the dotted curve in Fig. 1b, and is probably irrelevant for phenomenological applications. One has to bear in mind, however, that an approximation of taking into account operators with
the lowest anomalous dimension only for each $N$ is theoretically inconsistent since they belong to different trajectories.

The anomalous dimensions of quark-gluon operators (15), (16) can be studied along similar lines [11]. To the $O(1/N)$ accuracy the spectrum of low-lying energy eigenvalues is given in terms of eigenvalues of the corresponding integrals of motion as

$$\mathcal{E}(\nu) = 2\ln N + 4\gamma_E - 5 + 2\Re e [\psi(3/2 + i\nu)]$$

where $2\nu_{S,T}^2 = q_{S,T} - 3/2$, and quantization conditions for the effective charges read, to the same accuracy

$$\nu_T \ln N + \Phi_1(\nu) - \Phi_3(\nu) = \frac{\pi n}{2},$$

$$\nu_S \ln N + \frac{1}{2} (\Phi_1(\nu) + \Phi_2(\nu)) - \Phi_3(\nu) = \frac{\pi n}{2}$$

where $n = 1, 2, \ldots$ and

$$\Phi_1(\nu) = (1/2) \arg [\text{hypergeom}(3/2 + i\nu, -3/2 + i\nu, 1 + 2i\nu, 1)],$$

$$\Phi_2(\nu) = \arg [\text{hypergeom}(1/2 + i\nu, -1/2 + i\nu, 1 + 2i\nu, 1)],$$

$$\Phi_3(\nu) = \arg [\Gamma(3/2 + i\nu)].$$

These formulas are not applicable to the exact solutions with minimum anomalous dimensions, found in Refs. [9,10], which correspond to imaginary $\nu$ and have to be treated separately. We can show that these special solutions are separated from the rest of the spectrum by a finite gap. A detailed study is in progress.

To summarize, we have shown that a few important three-particle evolution equations in QCD are exactly integrable, that is they possess nontrivial integrals of motion. This allows for a fairly complete description of the spectrum of anomalous dimension of baryon operators with spin 3/2, and similar techniques can be developed for other cases as well. The eigenfunctions can also be studied [8,11]. We believe that the approach based on integrability may find many phenomenological applications to studies of higher-twist parton distributions in QCD.

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[15] We assume here that all three quarks are different; otherwise the terms $H_{ik}^{e(qq)}$ in (113) have to be multiplied by the appropriate symmetrization factors, cf. [3].

[16] The intersecting trajectories pick up different eigenstates from the degenerate pairs.
FIG. 1. The spectrum of eigenvalues for the conserved charge $Q_{3/2}$ (a) and for the spin 3/2 Hamiltonian $H_{3/2}$ (b), see text.