Abstract

Here we study non-convex composite optimization: first, a finite-sum of smooth but non-convex functions, and second, a general function that admits a simple proximal mapping. Most research on stochastic methods for composite optimization assumes convexity or strong convexity of each function. In this paper, we extend this problem into the non-convex setting using variance reduction techniques, such as prox-SVRG and prox-SAGA. We prove that, with a constant step size, both prox-SVRG and prox-SAGA are suitable for non-convex composite optimization, and help the problem converge to a stationary point within $O(\frac{1}{\epsilon})$ iterations. That is similar to the convergence rate seen with the state-of-the-art RSAG method and faster than stochastic gradient descent. Our analysis is also extended into the min-batch setting, which linearly accelerates the convergence. To the best of our knowledge, this is the first analysis of convergence rate of variance-reduced proximal stochastic gradient for non-convex composite optimization.

1 Introduction

We study a more general non-convex problem with composite objectives:

$$\min_{x \in \mathbb{R}^d} p(x) := f(x) + h(x),$$

where $f(x)$ is the average sum of many smooth but non-convex component functions $f_i(x), \forall i \in \{1, 2, \cdots, n\}$, i.e.

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

and in our paper, $h(x)$ is relatively simple but can be non-differentiable. We define this general class of functions $p(x)$ as $\mathcal{F}_n$ with only one simple assumption.

Assumption 1 For $\forall i \in \{1, 2, \cdots, n\}$, $f_i$ is non-convex and $L$-smooth, that is, there is exist a constant $L$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^d,$$

and this is equivalent to

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|^2.$$

Problems taking the form $[1]$ are very general and popular in machine learning, such as for regularized empirical risk minimization (regularized ERM). For instance, $f_i(x)$ can be the sigmoid loss function $\frac{1}{1 + e^{-x}}$ for binary classification. The regularization term $h(x)$ can be the $l_1$-norm $\lambda_1\|x\|_1$, $l_2$ norm $\frac{\lambda_2}{2}\|x\|_2^2$ or elastic net $\lambda_1\|x\|_1 + \frac{\lambda_2}{2}\|x\|_2^2$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. More recently, deep neural networks (DNNs) have become popular in machine learning. It can also be cast into this form if
We evaluate our methods on regularized empirical risk minimization problems to demonstrate the

\[ \hat{x} = \text{prox}_{\eta h}(x - \eta t v_{t-1}) \]

where \( \eta t \) is the step size and \( \text{prox}_h(y) \) is the proximal mapping operator,

\[ \text{prox}_h(y) = \arg \min_{x \in \mathbb{R}^d} \left\{ \frac{1}{2} \|x - y\|^2 + h(x) \right\}; \]

Here, \( v_{t-1} = \nabla f(x_{t-1}) \) in the full gradient method; \( v_{t-1} = \nabla f_i(x_{t-1}) - \nabla f_i(\tilde{x}) + \nabla f(\tilde{x}) \) in proximal stochastic variance-reduced gradient (prox-SVRG) \cite{ prox-SVRG } ; and \( v_{t-1} = \nabla f_i(x_{t-1}) - \nabla f_i(z^*_{i_t}) + \frac{1}{n} \sum_{i=1}^n \nabla f_i(z^*_{i_t}) \) in proximal SAGA (prox-SAGA) \cite{ prox-SAGA }, where \( i_t \) is uniformly randomly selected from \( \{1, 2, \ldots, n\} \). The other variables are defined below.

In many practical situations, for composite objectives with a large sample size \( n \), computation costs of the proximal stochastic gradient (prox-SG) method is usually less expensive than that of the proximal full gradient method (prox-FG). This is because prox-SG only estimates the gradient of a single one rather than \( n \) component functions at each iteration. However, its convergence rate is limited by variances in the stochastic process. Therefore, variance reduction techniques like SVRG \cite{ SVRG } and SAGA \cite{ SAGA } are applied into proximal stochastic gradient methods to accelerate its convergence. These techniques have proven to be very efficient in optimizing convex or strongly convex problems, and can help proximal stochastic gradient methods attain linear convergence rates.

However, there has been only limited analysis of non-convex composite optimization problems solved by proximal stochastic gradient descent. Such analysis would be useful in many practical non-convex problems, such as, in DNNs. Therefore, it is not trivial. To the best of our knowledge, this is the first analysis of the convergence rate of variance-reduced proximal stochastic gradient methods for non-convex composite optimization problems. With the only smooth-ness assumption and using a constant step-size \( \eta \), the variance-reduced proximal stochastic methods, prox-SVRG and prox-SAGA, can produce a \( \epsilon \)-stationary point in only \( O\left( \frac{L}{\epsilon^2} \right) \) iterations. This is faster than the convergence rate of the typical SGD of \( O\left( \frac{1}{\epsilon} \right) \) \cite{ SGD }, and similar to that of the mini-batch RSAG method \cite{ RSAG, RSAG2 }. Although the RSAG method was probably the first to analyze the convergence of non-convex composite objectives to produce a rate of \( O(L_2^2/\epsilon^2 + L_p L_1/\epsilon) \) ( \( L \) is the Lipschitz constant). Their analysis was directly based on the mini-batching and accelerating strategies which, in our analysis, further accelerates our proposed method. Moreover, the complexity of both SGD and mini-batch RSAG is \( O(\frac{1}{\epsilon^2}) \). When \( \epsilon \) is relatively small, more component gradient evaluations are required than in our method which requires \( O(n/\epsilon) \).

Another contribution of this paper is that we extend our analysis to mini-batch variants. In doing so, variance-reduced proximal stochastic gradient methods might benefit from mini-batching. Specifically, the convergence rate of the mini-batch method is faster by a factor of \( b \), where \( b \) is the batch size. We evaluate our methods on regularized empirical risk minimization problems to demonstrate the effectiveness of the proposed methods.

**Preliminaries** To analyze the convergence rate in the non-convex setting, we do not use \( p(x) - p(x^*) \) or \( \|x - x^*\| \) because the global minimizer \( x^* \) is difficult to find. Similar to \cite{prox-SVRG}, a general and important quantity \( \|g(x, v, \eta)\|^2 \) is used in our convergence analysis, and \( g(x, v, \eta) \) is defined as

\[ g(x, v, \eta) = \frac{1}{\eta}(x - \tilde{x}), \]

where \( \tilde{x} = \text{prox}_{\eta h}(x - \eta v) \). This works because the proximal mapping operator approaches to a stationary point as \( g(x, v, \eta) \) vanishes \cite{prox-SVRG}. Plus, this quantity is the general case of the popular \( \|\nabla f(x)\|^2 \). We can see that, when \( b(x) = 0 \) and \( dv = \nabla f(x) \), \( g(x, v, \eta) = \nabla f(x) \). In the rest of this paper, we use \( g \) to denote \( g(x, v, \eta) \) for simplicity. Before our proceed, we must also define the \( \epsilon \)-stationary point as follows,
Definition 1. A point $x$ is said to be $\epsilon$-accurate if $\|g(x,v,\eta)\|^2 \leq \epsilon$; a stochastic iteration algorithm is said to achieve $\epsilon$-accuracy in $t$ iterations if $\mathbb{E}\|g_t\|^2 \leq \epsilon$, where the expectation is taken over all the stochasticity of the algorithm.

2 Related Works

In this paper, the composite optimization problem is solved by variance-reduced proximal stochastic gradient method. In these methods, past gradient information is used to reduce variance. If there is sufficient covariance between the current stochastic gradient descent direction and past gradient directions, the variance of stochastic gradients will be reduced by removing this covariance. This results in faster convergence. Although this kind of variance reduction techniques has been applied to both convex and non-convex problems. But their properties have not been analyzed for non-convex composite optimization.

Convex  In convex optimization, [7] proposed the stochastic average gradient (SAG) method with a biased stochastic oracle. [4] proposed a stochastic variance-reduced gradient (SVRG) method which used past gradient information from a snapshot vector $\tilde{x}$ to reduce the variance. Instead of updating all the component gradients after a whole epoch like in SVRG, a fast incremental gradient method has been proposed [3] that updates the gradient of a randomly selected component at each iteration. Although these methods used different updating strategies, they can all be formulated in a variance reduction framework using past gradient information. Thus, [8] unified SVRG and SAGA into a memorization algorithm framework and proposed an approximation algorithm to share gradient between neighbors. Other literature also examines variance reduction. For example, the stochastic dual coordinate ascent (SDCA) method [9] has shown to have variance reduction properties.

To solve composite optimization, [1] extended the SVRG algorithm and developed a proximal SVRG method. [10] proposed a proximal SDCA algorithm to handle more general problems with regularizers. SAGA represents natural stochastic method for composite objectives. Most of these methods achieved a linear convergence rate, and can be further accelerated [11]. Overall, these studies demonstrate the effectiveness of variance-reduced stochastic gradient methods compared to the sublinear convergence rate produced by traditional methods. However, none of these studies examined the non-convex case.

non-convex  Several researchers have recently extended variance reduction techniques into non-convex optimization. [12] studied how variance reduction can improve the convergence of stochastic principal component analysis and demonstrated a linear convergence rate. Variance reduction has also been applied to stochastic optimization of non-convex sparse learning [13]. These methods exhibited fast convergence by exploring the properties of the specific problems.

Both [14,15] and [16] analyzed the convergence of variance-reduced stochastic optimization of general finite-sum problems and proved that the problems can converge to a stationary point in $O(1/\epsilon)$ iterations. Our analysis is most related to [14,15], but extended into an even more general non-convex problem with composite objectives. This analysis is nontrivial because composite optimization problems are very common in fields such as regularized ERM and neural networks. [5] first presented a nonasymptotic convergence rate analysis of proximal stochastic gradients and showed that accelerated randomized min-batch gradients ensured a $O(L^2_f/\epsilon^2 + L_f L_f/\epsilon)$ convergence rate and needed $O(L^2 f L_f)$ component gradient evaluations. Our analysis gives a similar $O(1/\epsilon)$ convergence rate but is the first to analyze the convergence for the variance-reduced proximal stochastic gradient descent method.

3 non-convex Prox-SVRG

We next consider the prox-SVRG for non-convex composite objectives. As shown in Algorithm [11], we choose a multistage scheme analogous to prox-SVRG [1] for the convex setting. At the end of each epoch, we estimate the full gradient of a snapshot vector $\tilde{x}^{s+1}$ as past gradient information. To achieve this, we may need to memorize all $x^{s+1}_t, \forall 0 \leq t \leq m$ to compute the snapshot vector. During implementation, we can simplify this using a moving average strategy with respect to the
We firstly consider two simple but important lemmas: the first to upper bound the variance; and the after obtaining this bound, we can describe the expected decrease in the objective function in a single epoch such that we can ensure an expected decrease in the composite objective.

A key step in Algorithm 1 is replacing the full gradient $\nabla f(x_t)$ with $v_t = \nabla f_i(x_t) - \nabla f_i(\tilde{x}) + \nabla f(\tilde{x})$, and then updating $x_t$ using a proximal mapping operator. Here and in the rest of the paper, we omit the superscripts for simplicity. We can see that, $v_t$ is unbiased; that is, $\mathbb{E} v_t = \nabla f(x_t)$. Further, for $v_t$, the past gradient information $\nabla f_i(\tilde{x})$ is supposed to have large covariance with the current gradient direction $\nabla f_i(x_t)$. Thus removing this covariance helps to reduce the variance.

3.1 Convergence Analysis

As shown in Algorithm 1, our analysis is a multistage scheme. At the first stage, the objective will exhibit an expected descent after each epoch and the quantity of the descent is about $O(\gamma \mathbb{E} \|g_t\|^2)$, where $\gamma$ is a constant depending on the step size $\eta$. At the second stage, the descent at each epoch will force the point sequences $\{x_t\}_{t=0}^S$ to approach the stationary point at a rate of $O(1/T)$, where $T$ is the number of total iterations.

In prox-SVRG, $v_t$ is used to reduce the variance. Crucially, its variance must first be upper bounded. To prove our result, we upper bound the variance with $O(\|x_t - \tilde{x}\|^2)$ rather than the popular $O(p(x_t) - p(x^*))$ or $O(\|x_t - x^*\|^2)$ used in convex optimization because the gradient descent method for non-convex optimization is difficult to converge to the global minimum. Hopefully, as number of iterations increases, both $x_t$ and $\tilde{x}$ approach the stationary point, and the stochastic gradient variance vanishes. In our analysis, we eliminate the effect of variance at the end of each epoch such that we can ensure an expected decrease in the composite objective.

We firstly consider two simple but important lemmas: the first to upper bound the variance; and the second is the expected decrease of the objective at each iteration. These two lemmas are commonly used for convergence analysis in both convex and non-convex optimization but with slightly different forms.

**Lemma 1** (upper bounds for variance). Suppose $p(x) \in \mathcal{F}_n$. For Algorithm 1 we set $\Delta_t = v_t - \nabla f(x_t)$, where $v_t = \nabla f_i(x_t) - \nabla f_i(\tilde{x}) + \nabla f(\tilde{x})$ is the stochastic gradient estimator. Then we have

$$\mathbb{E} \|\Delta_t\|^2 \leq L^2 \|x_t - \tilde{x}\|^2,$$

where $\mathbb{E} \|\Delta_t\|^2$ denotes the variance of stochastic gradients.

After obtaining this bound, we can describe the expected decrease in the objective function in a single iteration by the following lemma:

**Lemma 2** (expected decrease in the objective). For objective function $p(x)$ in problem 1 and $p(x) \in \mathcal{F}_n$, after each iteration, Algorithm 1 will ensure
Algorithm 1 ensures that

$$E_p(x_t) \leq p(x_{t-1}) + \frac{\eta_t^2 L^2 \lambda_t}{2} \|x_{t-1} - \tilde{x}\|^2 - \left(\frac{\eta_t}{2} - \frac{\eta_t^2 - \alpha_t}{2\lambda_t} - \frac{Lp_t^2}{2}\right)E\|g_t\|^2,$$

where $\lambda_t > 0$, $\alpha_t \in \mathbb{R}$ and $\forall t \in \{1, 2, \cdots, m\}$. Ideally, if the variance is zero, $\|x_{t-1} - \tilde{x}\|^2$ can be ignored. Then, a constant step size $\eta$ exists such that the decrease in the objective will be $O(\gamma E\|g_t\|^2)$ where $\gamma$ is dependent on $\eta$. In this way, $E\|g_t\|^2 \leq \frac{p(x_{t-1}) - E_p(x_t)}{\gamma}$ and $\frac{1}{T}\sum_{t=0}^{T-1}E\|g_t\|^2 \leq \frac{p(x_0) - E_p(x)}{\gamma}$. This ensures the final convergence of this problem. However, in this lemma, we observe that the expected decrease in the objective is not explicit due to the term $\|x_{t-1} - \tilde{x}\|^2$ resulted from the variance. A more complex technique is therefore needed to eliminate the variance. Then, using similar idea above, we can complete our proof. Here, we introduce the Lyapunov function method,

$$R_t := E[p(x_t) + c_t \|x_t - \tilde{x}\|^2].$$

Using the Lyapunov function, by choosing a proper $\eta$, we can design a decreasing sequence $\{c_t\}_{t=0}^m$ and $c_m = 0$. That is, at the end of each epoch, we can eliminate the effect of variance and ensure an expected descent of the composite objective. Before presenting this result, we define

$$\Gamma_t = \left(\frac{\eta_t}{2} - \frac{Lp_t^2}{2} - \frac{c_t \eta_t^2 - \rho_t}{\beta_t} - \frac{\eta_t^2 - \alpha_t}{2\lambda_t}\right),$$

where $\lambda_t, \beta_t > 0$ and $\rho_t, \alpha_t \in \mathbb{R}$ are defined as in our proof provided in the Supplementary Materials. The following lemma shows the descent of the Lyapunov function at each iteration:

**Lemma 3.** For $c_{t-1}, c_t > 0$, $\forall t \in \{1, 2, \cdots, m\}$, suppose we have

$$c_{t-1} = c_t + c_t \eta_t^\rho t \beta_t + \frac{L^2 \lambda_t \eta_t^{\lambda t}}{2}.$$

Let $c_t, \beta_t, \rho_t, \alpha_t, \lambda_t$ be chosen such that $\Gamma_t > 0$. Then the iterations of Algorithm 1 satisfy the bound

$$E\|g_t\|^2 \leq \frac{R_{t-1} - R_t}{\Gamma_{t-1}}.$$

Now, we assume that $\Gamma_t, \forall t \in \{0, \cdots, m - 1\}$ is lower bounded by a universal constant $\gamma$. We observe that $\{c_t\}_{t=0}^m$ is indeed a decreasing sequence for each epoch. As noted above, we let $c_m = 0$ for each epoch. Then, for epoch $s$, the $R_m$ will be equal to $p(x_m^s)$ and $\tilde{x}^s+1 = x_m^s$ as defined in Algorithm 1. $R_0 = p(x_0^s)$ because $x_0^s = \tilde{x}^s$. Then, the expected decrease in the composite objective will remove the effects of variance and be bounded by $O(\gamma E\|g_t\|^2)$ or, equivalently, the quantity $E\|g_t\|^2$ is bounded by $O(\frac{p(x^*) - p(\tilde{x}^s+1)}{\gamma})$ in each epoch.

We can now present our main result.

**Theorem 1.** Suppose $p(x) \in \mathcal{F}_n$. Let $c_m = 0$, there exist $\eta_t = \eta > 0$, $\beta_t = \beta > 0$, $\lambda_t = \lambda > 0$, $\alpha_t = \alpha$, $\rho_t = \rho$, and $c_{t-1} = c_t + c_t \eta_t^\rho \beta + \frac{L^2 \lambda_t \eta_t^{\lambda t}}{2}$ such that $\Gamma_t > 0$ for $0 \leq t \leq m - 1$. Define $\gamma_n := \min_t \Gamma_t$. Further, let $p_t = 0$ for all $0 \leq i < m$ and $p_m = 1$, and let $T$ be a multiple of $m$. Algorithm 1 ensures that

$$E\|g_a\|^2 \leq \frac{p(x_0^s) - p(x^*)}{T \gamma_n},$$

where $x^*$ is an optimal solution to Eq (1) and $g_a$ is defined in Algorithm 1.

**Proof.** According to Lemma 3 we have

$$E\|g_t\|^2 \leq \frac{R_{t-1} - R_t}{\Gamma_{t-1}} \leq \frac{R_{t-1} - R_t}{\gamma_n}.$$

Then the inequality implies that

$$\sum_{t=0}^{m-1}E\|g_t\|^2 \leq \frac{p(x^*) - p(\tilde{x}^s+1)}{\gamma_n},$$

where $\tilde{x}^s$.
where \( x_0 = \hat{x}^s \) and \( x_m = \hat{x}^{s+1} \). Now summing over all epochs combined with \( p(x^S_m) \geq p(x^*) \), we obtain,
\[
\frac{1}{T} \sum_{s=0}^{S} \sum_{t=0}^{m-1} \mathbb{E}[\|g_{t+1}\|^2] \leq \frac{p(x_0) - p(x^*)}{T \gamma_n}.
\]
With the definition of \( g_a \) in Algorithm 1 we can complete the proof.

Finally, we need to analyze the lower bound \( \gamma_n \) with a universal constant independent of \( n \). We can easily have

**Theorem 2.** Suppose \( p \in F_n \). Let \( \eta = \frac{u_0}{L m} \) \((0 < u_0 < 1)\), \( \beta = \frac{L}{u_0} \), \( \lambda = \frac{u^n - \alpha}{u_0} \) \( \forall \alpha \in \mathbb{R} \), \( \rho = 1 \), \( m = n^c \), and \( T \) is some multiple of \( m \). Then there exists universal constants \( u_0, v > 0 \) such that we have following: \( \gamma_n \geq \frac{1}{L n^c} \) in Theorem 1 and
\[
\mathbb{E}[\|g_a\|^2] \leq \frac{L n^c [p(x^0) - p(x^*)]}{T v}.
\]

Then if we want to obtain an \( \epsilon \)-accurate solution, \( T \) is bounded by \( \frac{L n^c [p(x^0) - p(x^*)]}{\epsilon v} \). This result is faster than traditional SGD when \( \epsilon \) is relatively small. It is also comparable to the results produced SVRG and SAGA for non-convex problems, and the result of min-batch RSAG method. Here, we utilize the concept of IFO complexity in \([14]\) to describe the total number of component gradient evaluations. For each epoch, the number of component gradient evaluations is \( O(n + m) \), where \( m = n^c \). We know that the number of epochs \( S \) is \( O(n^c/(em)) \), so the total IFO complexity is \( O((n + n^c)/\epsilon) \). When \( \kappa \leq 1 \), the total complexity is \( O(n/\epsilon) \). Therefore, we have:

**Corollary 1.** Suppose \( p(x) \in F_n \), \( \eta = \frac{u_0}{L m} \), \( \beta = \frac{L}{u_0} \), \( m = n \), and \( T \) is some multiple of \( m \). Then there exists universal constants \( u_0, v > 0 \) such that the IFO complexity of Algorithm 1 to obtain a \( \epsilon \)-accuracy solution is \( O(n/\epsilon) \).

This result could be lower than \( O(\frac{1}{\epsilon^2}) \) of SGD and the min-batch RSAG method when we demand a higher accuracy.

### 3.2 Min-batch Variant

In this section, we study the min-batch version of prox-SVRG. The key difference in the min-batch variant is that \( v_t = \frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_i) - \nabla f_i(\hat{x})) + \nabla f(\hat{x}) \), where \( I_t \) is a randomly selected min-batch. In stochastic gradient descent, min-batching helps to exploit parallelism and can reduce the communication costs. Moreover, min-batching can help to further reduce variance at each iteration and therefore, produce linear acceleration of the convergence rate without increasing the IFO complexity. For brevity, we only demonstrate the main result,

**Theorem 3.** Suppose \( p \in F_n \). With min-batch size \( b < n^c \), let \( \eta = \frac{u_0}{L n^c} \) \((0 < u_0 < 1)\), \( \beta = \frac{L}{u_0} \), \( \lambda = \frac{u^n - \alpha}{u_0} \) \( \forall \alpha \in \mathbb{R} \), \( \rho = 1 \), \( m = n^c \), and \( T \) is some multiple of \( m \). Then there exists universal constants \( u_0, v > 0 \) such that we have following: \( \gamma_n \geq \frac{b u}{L n^c} \) and
\[
\mathbb{E}[\|g_a\|^2] \leq \frac{L n^c [p(x^0) - p(x^*)]}{b Tv}.
\]

Therefore, the min-batch method needs only \( T \geq \frac{L n^c [p(x^0) - p(x^*)]}{\epsilon b v} \) iterations to obtain a \( \epsilon \)-accurate solution. It is a linear acceleration with a factor of \( b \) in the parallel setting. Fortunately, min-batching does not increase the IFO complexity:

**Corollary 2.** Suppose \( p(x) \in F_n \), \( \eta = \frac{u_0}{L n^c} \), \( \beta = \frac{L}{u_0} \), \( m = n \), and \( T \) is some multiple of \( m \). Then there exists universal constants \( u_0, v > 0 \), such that the IFO complexity of min-batch prox-SVRG to obtain a \( \epsilon \)-accurate solution is \( O(n/\epsilon) \).

### 4 non-convex Prox-SAGA

Prox-SAGA is also suitable for solving composite optimization problems. Unlike prox-SVRG which uses a multistage scheme and updates the full gradient of a snapshot vector \( \hat{x} \) as past information in
We next present the main result for brevity:

\[ \text{Algorithm 2 Prox-SAGA} \quad \begin{array}{l}
\text{Input:} \quad x_0 \in \mathbb{R}^d, z_i^0 = x_i \text{ for } i \in \{1, 2, \cdots, n\}, \text{ number of iterations } T, \text{ step size } \{\eta_t > 0\}_{t=0} \quad \\
\text{Output:} \quad \text{Iterate } x_a \text{ chosen uniformly random from } \{x_i\}_{i=0}^{T-1}, \text{ and } g_a \text{ chosen uniformly random from } \\
\{\frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i+1})\}_{i=0}^{T-1} \\
1: \phi_0 = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_i^0) \\
2: \text{for } t = 0 \text{ to } T - 1 \text{ do} \\
3: \quad \text{Uniformly randomly pick } i, j \text{ from } \{1, 2, \cdots, n\} \\
4: \quad v_t = \nabla f_i(x_t) - \nabla f_i(z_i^t) + \phi_t \\
5: \quad x_{t+1} = \text{prox}_{\eta_t h}(x_t - \eta_t v_t) \\
6: \quad z_{jt}^{t+1} = x_t \text{ and } z_{jt}^{t+1} = z_{jt}^t \text{ for } j \neq j_t \\
7: \quad \phi_{t+1} = \phi_t - \frac{1}{n}(\nabla f_j(z_j^t) - \nabla f_j(z_j^{t+1})) \\
8: \text{end for} \\
\end{array} \]

In this section, we empirically test the proposed method. Our experiments study the problem of regularized ERM focusing on binary classification. Given a set of training samples
\{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}, \text{ where } a_i \in \mathbb{R}^d, b_i \in \{-1, +1\}, \forall i \in \{1, 2, \cdots, n\}, \text{ we try to optimize the following objective and find an optimal predictor } x \in \mathbb{R}^d,
\[
\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \lambda_1 \|x\|_1 + \frac{\lambda_2}{2} \|x\|_2^2,
\]
where \(\lambda_1, \lambda_2 > 0;\) and \(f_i(x)\) can be \(\frac{1}{1+\exp(b_i a_i^T x)}\) for sigmoid loss, \((b_i - a_i^T x)^2\) for square loss, \(\log(1 + \exp(-b_i a_i^T x))\) for logistic loss, and \(\max(0, 1 - b_i a_i^T x)\) for hinge loss.

**Experimental setup** We use three publicly available datasets [17] to evaluate our approach: adult (a9a), web (w8a), and mnist (class 1). In our experiments, we set \(\lambda_1 = 10^{-5}\) and \(\lambda_2 = 10^{-4}\). As suggested in Theorem 2 for prox-SVRG, we set \(m = n\) for non-convex loss and \(m = 2n\) for convex loss [4]. Our analysis and comparison are all based on the number of effective passes, where each effective pass includes \(n\) component gradients. For convex prox-SVRG, we set \(\eta = 0.1/L\) [1]. For non-convex prox-SVRG, as suggested in our theoretical analysis, we set \(\eta = \frac{10}{u_0}\) and carefully choose \(u_0\) from all powers of 10. The best result is chosen. For SGD, we choose learning rate \(\alpha(1 + T/n)^{\beta}\) among sets of \(\alpha\) and \(\beta\). For prox-SG, we use a constant step size that delivers the best performance of all powers of 10 [1].

**Accuracy Experiments** In the first experiment, we apply prox-SVRG to this regularized ERM with four loss functions: logistic loss, square loss, hinge loss and sigmoid loss for the non-convex case. Our goal is to show how the testing accuracy of non-convex loss compares to that of convex ones. By applying prox-SVRG to a general non-convex problem with well-chosen regularizations, the proposed method achieves higher accuracy in many cases.

Figure 1: Testing accuracy comparison between prox-SVRG with sigmoid loss, square loss, logistic loss and hinge loss. In (b) and (c), we randomly flip the labels in the training set to add some noises [16].

**Run time Experiments** In the second experiment, we only consider sigmoid loss. We compare the run time of the proposed method with SGD and prox-SG. We can see that prox-SVRG delivers superior performance. The quantity \(\mathbb{E}\|g_n\|^2\) converges to zero much faster than that of SGD and prox-SG.

Figure 2: Run time comparison between prox-SVRG, prox-SG and SGD. The y axis presents \(\log \|g\|^2\) and the x axis represents the number of effective passes.
6 Conclusion

In this paper, we extend the variance reduction algorithms, SVRG and SAGA, into the proximal stochastic gradient method for a more general problem: composite optimization. Our analysis shows that prox-SVRG and prox-SAGA are suitable for optimizing the general composite objectives, and can ensure a similar $O(\frac{1}{\epsilon})$ convergence rate to the state-of-the-art min-batch RSAG method. It is also faster than traditional SGD method. We also develop min-batch variants for these variance reduction techniques and accelerate the convergence rate by a factor of $b$ in parallel setting.

References

[1] Lin Xiao and Tong Zhang. A proximal stochastic gradient method with progressive variance reduction. *SIAM Journal on Optimization*, 24(4):2057–2075, 2014.

[2] Neal Parikh and Stephen P Boyd. Proximal algorithms. *Foundations and Trends in optimization*, 1(3):127–239, 2014.

[3] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in Neural Information Processing Systems*, pages 1646–1654, 2014.

[4] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in Neural Information Processing Systems*, pages 315–323, 2013.

[5] Saeed Ghadimi and Guanghui Lan. Accelerated gradient methods for nonconvex nonlinear and stochastic programming. *Mathematical Programming*, pages 1–41, 2015.

[6] Saeed Ghadimi, Guanghui Lan, and Hongchao Zhang. Mini-batch stochastic approximation methods for nonconvex stochastic composite optimization. *Mathematical Programming*, 155(1-2):267–305, 2016.

[7] Mark Schmidt, Nicolas Le Roux, and Francis Bach. Minimizing finite sums with the stochastic average gradient. *arXiv preprint arXiv:1309.2388*, 2013.

[8] Thomas Hofmann, Aurelien Lucchi, Simon Lacoste-Julien, and Brian McWilliams. Variance reduced stochastic gradient descent with neighbors. In *Advances in Neural Information Processing Systems*, pages 2296–2304, 2015.

[9] Shai Shalev-Shwartz and Tong Zhang. Stochastic dual coordinate ascent methods for regularized loss. *The Journal of Machine Learning Research*, 14(1):567–599, 2013.

[10] Shai Shalev-Shwartz and Tong Zhang. Accelerated proximal stochastic dual coordinate ascent for regularized loss minimization. *Mathematical Programming*, 155(1-2):105–145, 2016.

[11] Atsushi Nitanda. Stochastic proximal gradient descent with acceleration techniques. In *Advances in Neural Information Processing Systems*, pages 1574–1582, 2014.

[12] Ohad Shamir. A stochastic pca and svd algorithm with an exponential convergence rate. *arXiv preprint arXiv:1409.2848*, 2014.

[13] Xingguo Li, Tuo Zhao, Raman Arora, Han Liu, and Jarvis Haupt. Stochastic variance reduced optimization for nonconvex sparse learning. *arXiv preprint arXiv:1605.02711*, 2016.

[14] Sashank J Reddi, Ahmed Hefny, Suvrit Sra, Barnabas Poczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. *arXiv preprint arXiv:1603.06160*, 2016.

[15] Sashank J Reddi, Suvrit Sra, Barnabas Poczos, and Alex Smola. Fast incremental method for nonconvex optimization. *arXiv preprint arXiv:1603.06159*, 2016.

[16] Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster nonconvex optimization. *arXiv preprint arXiv:1603.05643*, 2016.

[17] Rong-En Fan and Chih-Jen Lin. Libsvm data: Classification, regression, and multi-label.
7 Appendix

7.1 Analysis of Nonconvex Prox-SVRG

In this paper, we set $g_t^{i+1} = \frac{1}{n_t}(x_t^{i+1} - x_t^{i+1})$, $v_t^{i+1} = \nabla f_{i_t}(x_t^{i+1}) - \nabla f_{i_t}(\hat{x}) + \nabla f(\tilde{x})$, and $\Delta_t^{i+1} = v_t^{i+1} - \nabla f(x_t^{i+1})$. And we denote them as $g_t, v_t, \Delta_t,$ and denote $\hat{x}$ rather than $\tilde{x}$ for simplicity.

**Lemma 4** (upper bounds for variance). Suppose $p(x) \in F_n$. For Algorithm 4 we set $\Delta_t = v_t - \nabla f(x_t)$, where $v_t = \nabla f_{i_t}(x_t) - \nabla f_{i_t}(\hat{x}) + \nabla f(\tilde{x})$ is the stochastic gradient estimator. Then we have

$$E\|\Delta_t\|^2 \leq L^2 \|x_t - \hat{x}\|^2,$$

where $E\|\Delta_t\|^2$ denotes the variance of stochastic gradients.

**Proof.**

$$E\|\Delta_t\|^2 = E\|\nabla f_{i_t}(x_t) - \nabla f_{i_t}(\hat{x}) + \nabla f(\tilde{x}) - \nabla f(x_t)\|^2$$

$$= E\|\nabla f_{i_t}(x_t) - \nabla f_{i_t}(\hat{x})\|^2 - \|\nabla f(x_t) - \nabla f(\tilde{x})\|^2$$

$$\leq E\|\nabla f_{i_t}(x_t) - \nabla f_{i_t}(\hat{x})\|^2$$

$$\leq L^2 \|x_t - \hat{x}\|^2$$

where the first equality is due to $E(\zeta - E\zeta)^2 = E\zeta^2 - (E\zeta)^2$ for any random variable. \qed

**Lemma 5** (expected decrease of objective). For objective function $p(x)$ in problem 4 and $p(x) \in F_n$, after one iteration, Algorithm 4 will ensure

$$E[p(x_t)] \leq p(x_{t-1}) + \eta_t^2 \frac{L^2 \lambda_t}{2} \|x_{t-1} - \hat{x}\|^2 + \eta_t \|x_t - x_{t-1}\|^2$$

where $\lambda_t > 0, \alpha_t \in \mathbb{R}$ and $\forall t \in \{1, 2, \ldots, m\}$.

**Proof.**

$$p(x_t) = f(x_t) + h(x_t)$$

$$\leq f(x_{t-1}) + \nabla f(x_{t-1})^T(x_t - x_{t-1}) + \frac{L}{2} \|x_t - x_{t-1}\|^2 + h(x_t)$$

$$= f(x_{t-1}) + v_{t-1}^T(x_t - x_{t-1}) + \frac{1}{2\eta_t} \|x_t - x_{t-1}\|^2 + h(x_t) + \nabla f(x_{t-1}) - v_{t-1})^T(x_t - x_{t-1})$$

$$+ \left(\frac{L}{2} - \frac{1}{2\eta_t}\right) \|x_t - x_{t-1}\|^2$$

$$\leq f(x_{t-1}) + h(x_{t-1}) + (\nabla f(x_{t-1}) - v_{t-1})^T(x_t - x_{t-1}) + \left(\frac{L}{2} - \frac{1}{2\eta_t}\right) \|x_t - x_{t-1}\|^2$$

$$= p(x_{t-1}) + \eta_t \lambda_T \|x_t - x_{t-1}\|^2 + \eta_t \|x_t - x_{t-1}\|^2$$

$$\leq p(x_{t-1}) + \eta_t^2 \frac{\lambda_t}{2} \|\Delta_t\|^2$$

Taking expectation conditioned on information $i_t$, we have

$$E[p(x_t)] \leq p(x_{t-1}) + \lambda_t \eta_t^2 \frac{L^2}{2} \|x_t - x_{t-1}\|^2$$

where $\lambda_t > 0$ and $\alpha_t \in \mathbb{R}$; and the first inequality is due to the $L$-smoothness of $f$; the second inequality is due to $x_t$ is the minimizer of Eq (3); the third inequality is due to Cauchy–Schwarz and Young’s inequality; the fourth inequality is due to Lemma 4. \qed
Before our proof, we define

$$\Gamma_{t-1} = \left( \eta_t - \frac{L \eta_t^2}{2} - c_t \eta_t^2 - \frac{c_t \eta_t^{2-\rho_t}}{\beta_t} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} \right),$$

for some parameter $c_t, \beta_t, \rho_t, \lambda_t$ which will be defined shortly.

**Lemma 6.** For $c_{t-1}, c_t > 0$, $\forall t \{1, 2, \cdots, m\}$, suppose we have

$$c_{t-1} = c_t + c_t \eta_t^{\rho_t} \beta_t + \frac{L^2 \lambda_t \eta_t^{\rho_t}}{2}.$$

Let $c_t, \beta_t, \rho_t, \alpha_t, \lambda_t$ be chosen such that $\Gamma_{t-1} > 0$. Then the iterations of Algorithm 1 satisfy the bound

$$E\|g_t\|^2 \leq \frac{R_{t-1} - R_t}{\Gamma_{t-1}}.$$

**Proof.** We firstly define following Lyapunov function

$$R_t := E[p(x_t) + c_t \|x_t - \bar{x}\|].$$

Consider $E\|x_t - \bar{x}\|^2$, we have

$$E\|x_t - \bar{x}\|^2 = E\|x_t - x_{t-1} + x_{t-1} - \bar{x}\|^2$$

$$= E\|x_t - x_{t-1}\|^2 + 2E(x_t - x_{t-1})^T (x_{t-1} - \bar{x}) + \|x_{t-1} - \bar{x}\|^2$$

$$= E[\eta_t^2 \|g_t\|^2 + 2\eta_t^{\rho_t} - \eta_t^{1-\rho_t} \|g_t\|^2]$$

$$\leq E[\eta_t^2 \|g_t\|^2 + 2\eta_t^{\rho_t} - \eta_t^{1-\rho_t} \|g_t\|^2]$$

$$= (\eta_t^2 + \frac{\eta_t^{2-\rho_t}}{\beta_t})E[\|g_t\|^2] + (1 + \beta_t \eta_t^{\rho_t}) \|x_{t-1} - \bar{x}\|^2.$$

where $\beta_t > 0$ and the first inequality is due to Cauchy–Schwarz and Young’s inequality. Because we have,

$$E[p(x_t)] \leq E[p(x_{t-1}) + \frac{\eta_t^{\alpha_t} L^2 \lambda_t}{2} \|x_{t-1} - \bar{x}\|^2 - \left( \eta_t \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L \eta_t^2}{2} \right)E\|g_t\|^2;$$

then combining these two inequalities, we can have

$$R_t = E[p(x_t) + c_t \|x_t - \bar{x}\|]$$

$$\leq p(x_{t-1}) + (c_t + c_t \beta_t \eta_t^{\rho_t} + \frac{\eta_t^{\rho_t} L^2 \lambda_t}{2}) \|x_{t-1} - \bar{x}\|^2 - \left( \eta_t \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L \eta_t^2}{2} - c_t \eta_t^2 + \frac{c_t \eta_t^{2-\rho_t}}{\beta_t} \right)E\|g_t\|^2$$

$$= p(x_{t-1}) + c_t \|x_{t-1} - \bar{x}\|^2 - \Gamma_{t-1} E\|g_t\|^2.$$

that is,

$$E\|g_t\|^2 \leq \frac{R_{t-1} - R_t}{\Gamma_{t-1}}.$$

**Theorem 6.** Suppose $p(x) \in \mathcal{F}_\alpha$. Let $c_m = 0$, there exist $\eta_t = \eta > 0, \beta_t = \beta > 0, \lambda_t = \lambda > 0, \alpha_t = \alpha, \rho_t = \rho, \text{ and } c_{t-1} = c_t + c_t \eta_t^{\rho_t} \beta_t + \frac{L^2 \lambda_t \eta_t^{\rho_t}}{2}$ such that $\Gamma_t > 0$ for $0 \leq t \leq m - 1$. Define $\gamma_n := \min_t \Gamma_t$. Further, let $p_t = 0$ for all $0 \leq t < m$ and $p_m = 1$, and let $T$ be a multiple of $m$. Then Algorithm 1 will ensure

$$E[\|g_a\|^2] \leq \frac{p(x_0) - p(x^*)}{T \gamma_n},$$

where $x^*$ is an optimal solution to Eq (1) and $g_a$ is defined in Algorithm 1.

**Proof.** Since $\eta_t = \eta$ for all $t \in \{0, 1, \cdots, m - 1\}$, and $\gamma_n = \min_t \Gamma_t$, we have

$$\sum_{t=0}^m E\|g_t\|^2 \leq \frac{R_0 - R_m}{\gamma_n}.$$
where \( R_0 = \mathbb{E}[p(\hat{x})] \) because \( x_0 = \hat{x}^* \); and \( R_m = \mathbb{E}[p(\hat{x}^{s+1})] \) because, for each epoch, \( c_m = 0, p_m = 1 \) and \( p_i = 0, \forall i < m \), then we obtain,

\[
\sum_{t=0}^{m} \mathbb{E}[\|g_t\|^2] \leq \frac{p(\hat{x}^s) - p(\hat{x}^{s+1})}{\gamma_n},
\]

that is,

\[
\frac{1}{T} \sum_{s=0}^{T-1} \sum_{t=0}^{m-1} \mathbb{E}[\|g_t^s\|^2] \leq \frac{p(x_0) - p(x^s)}{T\gamma_n}.
\]

Using this inequality and the definition of \( g_a \), we can complete the proof. \( \square \)

**Theorem 7.** Suppose \( p \in \mathcal{F}_u \). Let \( \eta = \frac{u}{Ln} (0 < u_0 < 1), \beta = \frac{L}{w}, \lambda = \frac{\eta^{1-\alpha}}{u_0} \forall \alpha \in \mathbb{R}, \rho = 1, m \geq n^\kappa, \) and \( T \) is some multiple of \( m \). Then there exists universal constants \( u_0, v > 0 \) such that we have following: \( \gamma_n \geq \frac{v}{Ln} \) in Theorem 6 and

\[
\mathbb{E}[\|g_a\|^2] \leq \frac{Ln^\kappa [p(x^0) - p(x^s)]]}{Tv}.
\]

**Proof.** Firstly, we give a upper bound for \( c_0 \). Because \( \eta_t = \eta, \rho_t = \rho, \alpha_t = \alpha, \beta_t = \beta > 0 \) and \( \lambda_t = \lambda > 0 \),

\[
c_t = c_t - c_t - 1 + \beta \gamma^\alpha + \frac{\eta^\alpha L^2 \lambda}{2}.
\]

that is, if we set \( \theta = \beta \gamma^\alpha \) and \( b = \frac{\eta^\alpha L^2 \lambda}{2} \), then we have,

\[
c_t = \frac{1}{1 + \theta} c_{t-1} - \frac{1}{1 + \theta} b
\]

\[
= \left( \frac{1}{1 + \theta} \right)^2 c_{t-2} - \left( \frac{1}{1 + \theta} \right)^2 b
\]

\[
= \left( \frac{1}{1 + \theta} \right)^m c_0 - \sum_{i=1}^{m} \left( \frac{1}{1 + \theta} \right)^i b
\]

\[
= \frac{1}{1 + \theta} c_0 - \frac{1 - \left( \frac{1}{1 + \theta} \right)^m}{\theta} b.
\]

Because \( c_m = 0 \), then

\[
0 = c_m = \left( \frac{1}{1 + \theta} \right)^m c_0 - \frac{1 - \left( \frac{1}{1 + \theta} \right)^m}{\theta} b,
\]

we have

\[
c_0 = \frac{(1 + \theta)^m - 1}{\theta} b.
\]

When we set \( m = \frac{1}{\theta} \), then \( c_0 \leq \frac{e-1}{\theta} b \), that is,

\[
c_0 \leq \frac{(e-1)\eta^\alpha L^2 \lambda}{2\theta} = \frac{(e-1)\eta^{(\alpha-\rho)} L^2 \lambda}{2\beta},
\]

where the inequality is due to \((1 + a)^\frac{1}{a} \to e \), when \( a \to 0, a > 0 \). Then we can lower bound \( \gamma_n \),

\[
2n = \min_t \left( \eta - \frac{\eta^{2-\alpha}}{\lambda} - L\eta^2 - 2c_t \eta^2 - \frac{2c_t \eta^{2-\rho}}{\beta} \right)
\]

\[
\geq \eta - \frac{\eta^{2-\alpha}}{\lambda} - L\eta^2 - 2c_t \eta^2 - \frac{2c_t \eta^{2-\rho}}{\beta}
\]

\[
\geq \eta - \frac{\eta^{2-\alpha}}{\lambda} - L\eta^2 - \frac{(e-1)\eta^{(2+\alpha-\rho)} L^2 \lambda}{\beta} - \frac{(e-1)\eta^{(2+\alpha-2\rho)} L^2 \lambda}{\beta^2}
\]

\[
= \eta(1 - \frac{\eta^{1-\alpha}}{\lambda} - L\eta - \frac{(e-1)\eta^{(1+\alpha-\rho)} L^2 \lambda}{\beta} - \frac{(e-1)\eta^{(1+\alpha-2\rho)} L^2 \lambda}{\beta^2}).
\]
where $\lambda > 0$ and $\beta > 0$; the first inequality is due to that $c_t$ is decreasing with $t$. Now, we set $\rho = 1$. Because $\eta = \frac{u_0}{Ln^0}$, $\lambda = \frac{2^{1/\alpha}}{u_0}$, $\beta = \frac{L}{u_0}$, where $0 < \kappa \leq 1$, then we have,

$$2\gamma_n \geq \eta(1 - \frac{u_0}{n^\kappa} - \frac{u_0(e - 1)}{n^\kappa} - u_0^2(e - 1)).$$

Because $n \geq 1$, therefore, we can conclude that there exist a universal constant $\nu$ such that

$$\gamma_n \geq \frac{\nu}{Ln^\kappa}.$$ 

Then we can get the final result. \(\Box\)

7.2 Analysis of Min-batch Prox-SVRG

| Algorithm 3 Min-batch Prox-SVRG ($x^0$, $T$, $m$, $b$, $\{p_i\}_{i=0}^m$, $\{\eta_t\}_{t=0}^m$) |
|---------------------------------------------------------------|
| **Input:** $x^0 = x_0^m = x_0 \in \mathbb{R}^d$, epoch length $m$, step size $\{\eta_t > 0\}_{t=0}^m$, $S = \lceil T/m \rceil$, discrete probability distribution $\{p_i\}_{i=0}^m$. |
| **Output:** Iterate $x_n$ chosen uniformly random from $\{\{x^{t+1}_{t=0} \}_{s=0}^{m-1}\}_{t=0}^{S-1}$, and $g_0$ chosen uniformly random from $\{\{\frac{1}{n} \Sigma_{i=1}^n \nabla f_t(x^s)\}_{s=1}^{m-1}\}_{t=0}^{S-1}$. |
| 1: for $s = 0$ to $S - 1$ do |
| 2: $x_0^{s+1} = x^m$ |
| 3: $\nabla f(\hat{x}^s) = \frac{1}{n} \Sigma_{i=1}^n \nabla f_i(\hat{x}^s)$ |
| 4: for $t = 0$ to $m - 1$ do |
| 5: Uniformly randomly choose min-batch $I_t$ from $\{1, 2, \cdots, n\}$ |
| 6: $v_t^{s+1} = \frac{1}{b} \Sigma_{i \in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(\hat{x}^s)) + \nabla f(\hat{x}^s)$ |
| 7: $x_t^{s+1} = \text{prox}_{\eta_t h}(x_t^{s+1} - \eta_t v_t^{s+1})$ |
| 8: end for |
| 9: $\tilde{x}^{s+1} = \Sigma_{t=0}^m p_i x_t^{s+1}$ |
| 10: end for |

Now we extend our analysis to min-batch setting. As we know, min-batching is widely applied strategies in large-scale optimisation problems because it could be easily parallelled and distributed. Our algorithm is shown in Algorithm 3. We can see when the batch size $b = 1$, it’s the case of original prox-SVRG. Our analysis shows that min-batching could help us reduce more variances and accelerate our algorithm by a factor of $b$.

Here, we also analyse the convergence rate of $g_t^{s+1} = \frac{1}{m} (x_t^{s+1} - x_{t+1}^{s+1})$, and the difference is that $v_t^{s+1} = \frac{1}{b} \Sigma_{i \in I_t} (\nabla f_i(x_t^{s+1}) - \nabla f_i(\hat{x}^s)) + \nabla f(\hat{x}^s)$, but it’s still the unbiased estimation of full gradient $\nabla f(\hat{x}_t)$. And denote $\Delta_t^{s+1} = v_t^{s+1} - \nabla f(x_t^{s+1})$. And we drop some superscripts of these variables and represent them using $g_t, v_t, \Delta_t$, and denote $\hat{x}$ for simplicity.

**Lemma 7** (upper bounds for variance). For problem [1] and $p(x) \in \mathcal{F}_n$, we set $\Delta_t = v_t - \nabla f(\hat{x}_t)$, then we have

$$\mathbb{E}\|\Delta_t\|^2 \leq \frac{L^2}{b} \|x_t - \hat{x}\|^2,$$

where we denote $\mathbb{E}\|\Delta_t\|^2$ as variance of stochastic gradients.
Proof.

\[ \mathbb{E}\|\Delta_t\|^2 = \mathbb{E}\left\| \frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_t) - \nabla f_i(\bar{x})) + \nabla f(\bar{x}) - \nabla f(x_t) \right\|^2 \\
= \mathbb{E}\left\| \frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x_t) - \nabla f_i(\bar{x}) + \nabla f(\bar{x}) - \nabla f(x_t)) \right\|^2 \\
= \frac{1}{b^2} \mathbb{E} \left\| \sum_{i \in I_t} (\nabla f_i(x_t) - \nabla f_i(\bar{x}) + \nabla f(\bar{x}) - \nabla f(x_t)) \right\|^2 \\
\leq \frac{1}{b} \mathbb{E} \| \nabla f_i(x_t) - \nabla f_i(\bar{x}) + \nabla f(\bar{x}) - \nabla f(x_t) \|^2 \\
\leq \frac{1}{b} \mathbb{E} \| \nabla f_i(x_t) - \nabla f_i(\bar{x}) \|^2 \\
\leq \frac{L^2}{b} \| x_t - \bar{x} \| \\
\]

where the second equality is due to \( \mathbb{E}(\zeta - \mathbb{E}\zeta)^2 = \mathbb{E}\zeta^2 - (\mathbb{E}\zeta)^2 \) for any random variable. \( \square \)

Lemma 8. For problem \( \mathcal{P} \) and \( p(x) \in \mathcal{F}_n \), after one step, Algorithm 3 will ensure

\[ \mathbb{E} p(x_t) \leq p(x_{t-1}) + \frac{\eta_t^2 \alpha t L^2}{2b} \| x_{t-1} - \bar{x} \|^2 - \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2} \mathbb{E} \| g_t \|^2, \]

where \( \lambda_t > 0 \), \( \alpha_t \in \mathbb{R} \) and \( \forall \{1, 2, \ldots, m\} \).

Proof.

\[ p(x_t) = f(x_t) + h(x_t) \]
\[ \leq f(x_{t-1}) + \nabla f(x_{t-1})^T (x_t - x_{t-1}) + \frac{L}{2} \| x_t - x_{t-1} \|^2 + h(x_t) \]
\[ = f(x_{t-1}) + v_{t-1}^T (x_t - x_{t-1}) + \frac{1}{2\eta_t} \| x_t - x_{t-1} \|^2 + h(x_t) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1}) \\
+ (\frac{L}{2} - \frac{1}{2\eta_t}) \| x_t - x_{t-1} \|^2 \]
\[ \leq f(x_{t-1}) + h(x_{t-1}) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1}) + (\frac{L}{2} - \frac{1}{2\eta_t}) \| x_t - x_{t-1} \|^2 \\
= p(x_{t-1}) + \eta_t^{\alpha_t} (\nabla f(x_{t-1}) - v_{t-1})^T (\eta_t^{1-\alpha_t} (-g_t)) + (\frac{L\eta_t^2}{2} - \frac{\eta_t^4}{2}) \| g_t \|^2 \\
\leq p(x_{t-1}) + \eta_t^{\alpha_t} (\frac{\lambda_t}{2} \| \Delta_t \|^2 + \frac{\eta_t^{2-2\alpha_t}}{2\lambda_t} \| g_t \|^2) + (\frac{L\eta_t^2}{2} - \frac{\eta_t^4}{2}) \| g_t \|^2 \]

Taking expectation conditioned on information \( I_t \), we have

\[ \mathbb{E}[p(x_t)] \leq p(x_{t-1}) + \frac{\lambda_t \eta_t^{\alpha_t} L^2}{2b} \| x_{t-1} - \bar{x} \|^2 - \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2} \mathbb{E} \| g_t \|^2. \]

where \( \lambda_t > 0 \); and the first inequality is due to the \( L \)-smoothness of \( f \); the second inequality is due to \( x_t \) is the minimizer of Eq (3); the third inequality is due to Cauchy–Schwarz and Young’s inequality; the fourth inequality is due to Lemma 7. \( \square \)

Before our proof, we define

\[ \Gamma_{t-1} = (\frac{\eta_t}{2} - \frac{L\eta_t^2}{2} - c_t \eta_t^2 - \frac{c_t \eta_t^{2-\rho_t}}{\beta_t} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t}) , \]

for some parameter \( c_t, \beta_t, \rho_t, \lambda_t \) which will be defined shortly.

**Theorem 8.** Suppose \( p \in \mathcal{F}_n \). Let \( c_m = 0 \), there exist \( \eta_t = \eta > 0 \), \( \beta_t = \beta > 0 \), \( \lambda_t = \lambda > 0 \), \( \alpha_t = \alpha \), \( \rho_t = \rho \), and \( c_{t-1} = c_t + c_t \eta_t^\beta + \frac{L^2 \lambda_t \eta_t^\alpha}{2b} \) such that \( \Gamma_t > 0 \) for \( 0 \leq t \leq m - 1 \). Define
We firstly define following Lyapunov function

Using this inequality and the definition of $\gamma$, we have

where $\beta_i > 0$ and the first inequality is due to Cauchy–Schwarz and Young’s inequality. Because we have

then combining these two inequalities, we can have

that is,

Since $\eta_t = \eta$ for all $t \in \{0, 1, \cdots, m - 1\}$, and $\gamma_n = \min_t \Gamma_t$, we have

where $R_0 = \mathbb{E}[p(\hat{x}_1)]$ and $R_m = \mathbb{E}[p(\hat{x}_2)]$ because, for each epoch, $c_0 = 0, p_m = 1$ and $p_i = 0, \forall i < m$, then we obtain,

that is,

Using this inequality and the definition of $g_a$, we can complete the proof. \qed

**Theorem 9.** Suppose $p \in \mathcal{F}_n$. With min-batch size $b < n^\epsilon$, let $\eta = \frac{bu_0}{\ln^\alpha}$ ($0 < u_0 < 1$), $\beta = \frac{L}{u_0}$, $\lambda = \frac{a^{1-n}}{m}$ $\forall a \in \mathbb{R}$, $\rho = 1$, $m \geq n^\epsilon$, and $T$ is some multiple of $m$. Then there exists universal constants $u_0, \nu > 0$ such that we have following: $\gamma_n \geq \frac{u_0}{\ln^\alpha}$ and

$$\mathbb{E}[\|g_a\|^2] \leq \frac{bT\nu}{L\ln^\alpha}.\]
Proof. Firstly, we give an upper bound for $c_0$. Because $\eta_t = \eta, \rho_t = \rho, \alpha_t = \alpha, \beta_t = \beta > 0$ and $\lambda_t = \lambda > 0$,

$$c_{t-1} = c_t(1 + \beta \eta^a) + \frac{\eta^a L^2 \lambda}{2b}.$$ 

that is, if we set $\theta = \beta \eta^a$ and $a = \frac{\eta^a L^2 \lambda}{2b}$, then we have,

$$c_t = \frac{1}{1 + \theta} c_{t-1} - \frac{1}{1 + \theta} a$$

$$= \left(\frac{1}{1 + \theta}\right)^2 c_{t-2} - \left(\frac{1}{1 + \theta}\right)^2 a$$

$$= \left(\frac{1}{1 + \theta}\right)^t c_0 - \sum_{i=1}^t \left(\frac{1}{1 + \theta}\right)^i a$$

$$= \left(\frac{1}{1 + \theta}\right)^t c_0 - \frac{1 - \left(\frac{1}{1 + \theta}\right)^t}{\theta} a.$$ 

Because $c_m = 0$, then

$$0 = c_m = \left(\frac{1}{1 + \theta}\right)^m c_0 - \frac{1 - \left(\frac{1}{1 + \theta}\right)^m}{\theta} a,$$

we have

$$c_0 = \frac{(1 + \theta)^m - 1}{\theta} a.$$ 

Because we set $m = \frac{1}{\theta}$, then $c_0 \leq \frac{e - 1}{\theta} a$, that is,

$$c_0 \leq \frac{(e - 1) \eta^a L^2 \lambda}{2\theta} = \frac{(e - 1) \eta^{(\alpha - \rho)} L^2 \lambda}{2b \beta},$$

where the inequality is due to $(1 + y)^\frac{1}{\theta} < e$, when $y \to 0, y > 0$. Then we can lower bound $\gamma_n$,

$$2\gamma_n = \min_t (\eta - \frac{\eta^{2 - \alpha}}{\lambda} - L\eta^2 - 2c_0 \eta^2 - \frac{2c_0 \eta^{2 - \rho}}{\beta})$$

$$\geq \eta - \frac{\eta^{2 - \alpha}}{\lambda} - L\eta^2 - 2c_0 \eta^2 - \frac{2c_0 \eta^{2 - \rho}}{\beta}$$

$$\geq \eta - \frac{\eta^{2 - \alpha}}{\lambda} - L\eta^2 - \frac{(e - 1)\eta^{(2 + \alpha - \rho)} L^2 \lambda}{b \beta} - \frac{(e - 1)\eta^{(2 + \alpha - 2\rho)} L^2 \lambda}{b \beta^2}$$

$$= \eta(1 - \frac{\eta^{1 - \alpha}}{\lambda} - L\eta - \frac{(e - 1)\eta(1 + \alpha - \rho) L^2 \lambda}{b \beta} - \frac{(e - 1)\eta(1 + \alpha - 2\rho) L^2 \lambda}{b \beta^2},$$

where $\lambda > 0$ and $\beta > 0$; the first inequality is due to that $c_t$ is decreasing with $t$. Now, we set $\rho = 1$.

Because $\eta = \frac{b u_0}{L n \kappa}$, $\lambda = \frac{n^{1 - \alpha}}{u_0}$, $\beta = \frac{L}{u_0}$, then we have,

$$2\gamma_n \geq \eta(1 - u_0 - \frac{b u_0}{n \kappa} - \frac{u_0 (e - 1)}{n \kappa} - \frac{u_0 (e - 1)}{b}).$$

Because $n \geq 1$, therefore, we can conclude that there exist a universal constant $v$ such that

$$\gamma_n \geq \frac{bv}{L n \kappa}.$$ 

Then we can get the final result. \qed

7.3 Analysis of Nonconvex Prox-SAGA

Set $v_t = \nabla f_i (x^t) - \nabla f_i (z_{i_t}^t) + \phi_t$ for a uniformly randomly selected $i_t$, and $\phi_t = \frac{1}{n} \sum_{i=1}^n \nabla f_i (z_{i_t}^t)$. Then $v_t$ is unbiased estimation of $\nabla f(x_t)$. We have following lemma,

Lemma 9 (upper bounds for variance). For problem (4), and suppose $p(x) \in \mathcal{F}_n$, we set $\Delta_t = v_t - \nabla f(x_t)$, then we have

$$\mathbb{E}\|\Delta_t\|^2 \leq \frac{L^2}{n} \sum_{i=1}^n \mathbb{E}\|x_t - z_{i_t}^t\|^2,$$

where we denote $\mathbb{E}\|\Delta_t\|^2$ as the variance of stochastic gradients.
Proof.

\[
\mathbb{E}\|\Delta_t\|^2 = \mathbb{E}\|\nabla f_t(x_t^t) - \nabla f_t(z^t_i)\|_2^2 \\
= \mathbb{E}\|\nabla f_t(x_t) - \nabla f_t(z^t_i)\|_2^2 - \|\nabla f_t(x_t) - \phi_t\|_2^2 \\
\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\|\nabla f_t(x_t) - \nabla f_t(z^t_i)\|^2 \\
\leq \frac{L^2}{n} \sum_{i=1}^{n} \mathbb{E}\|x_t - z^t_i\|_2^2
\]

where the first equality is due to \(\mathbb{E}(\zeta - \zeta) = \mathbb{E}^2 - (\mathbb{E} \zeta)^2\) for any random variable. \(\square\)

Lemma 10. For problem \(\mathcal{F}\), and suppose \(p(x) \in \mathcal{F}_n\), we have, after one step, Algorithm 2 will ensure

\[
\mathbb{E}p(x_t) \leq p(x_{t-1}) + \frac{\eta_t^2}{2} \sum_{i=1}^{n} \mathbb{E}\|x_t - z_i^{t-1}\|_2^2 - \left(\frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2}\right)\mathbb{E}\|g_t\|^2,
\]

where \(\lambda_t > 0\) and \(\alpha_t \in \mathbb{R}\).

Proof.

\[
p(x_t) = f(x_t) + h(x_t)
\leq f(x_t) + \nabla f(x_{t-1})^T (x_t - x_{t-1}) + \frac{L}{2} \|x_t - x_{t-1}\|^2 + h(x_t)
= f(x_t) + v_{t-1}^T (x_t - x_{t-1}) + \frac{1}{2\eta_t} \|x_t - x_{t-1}\|^2 + h(x_t) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1})
+ \left(\frac{L}{2} - \frac{1}{2\eta_t}\right) \|x_t - x_{t-1}\|^2
\leq f(x_t) + h(x_t) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1}) + \left(\frac{L}{2} - \frac{1}{2\eta_t}\right) \|x_t - x_{t-1}\|^2
= p(x_{t-1}) + \eta_t^2 (\nabla f(x_{t-1}) - v_{t-1})^T (\eta_t^{1-\alpha_t} (-g_t)) + \left(\frac{L\eta_t^2}{2} - \frac{\eta_t^4}{2}\right) \|g_t\|^2
\leq p(x_{t-1}) + \eta_t^2 (\lambda_t \|\Delta_t\|^2 + \frac{\eta_t^{2-2\alpha_t}}{2\lambda_t} \|g_t\|^2) + \left(\frac{L\eta_t^2}{2} - \frac{\eta_t^4}{2}\right) \|g_t\|^2
\]

Taking expectation conditioned on information at \(t - 1\), we have

\[
\mathbb{E}[p(x_t)] \leq p(x_{t-1}) + \lambda_t \eta_t^2 \sum_{i=1}^{n} \mathbb{E}\|x_t - z_i^{t-1}\|_2^2 - \left(\frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2}\right)\mathbb{E}\|g_t\|^2.
\]

where \(\lambda_t > 0\); and the first inequality is due to the \(L\)-smoothness of \(f\); the second inequality is due to \(x_t\) is the minimizer of Eq (1); the third inequality is due to Cauchy–Schwarz and Young's inequality; the fourth inequality is due to Lemma 9. \(\square\)

Before our theory, we define,

\[
\Gamma_{t-1} = \left(\frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2} - \frac{\eta_t^2}{2}\right),
\]

with some parameters \(\eta_t, \lambda_t, \beta_t, \rho_t\) which will be defined later.

Theorem 10. Suppose \(p(x) \in \mathcal{F}_n\). Let \(\alpha_t = 0, \beta_t = \beta > 0, \lambda_t = \lambda > 0, \alpha_t = \alpha, \rho_t = \rho, \) and \(c_t = c[1 - \left(\frac{1}{n} + \beta_t\eta_t^{\alpha_t}\right)] + \frac{\eta_t^2}{2} L^2 \lambda_t\) such that \(\Gamma_t > 0\) for \(0 \leq t \leq T - 1\). Define the quantity \(\gamma_n := \min_{\Gamma_t} \Gamma_t\). Then the Algorithm 2 will ensure

\[
\mathbb{E}\|g_n\|^2 \leq \left\lfloor \frac{|p(x_0) - p(x^*)|}{\Gamma_n}\right\rfloor,
\]

where \(x^*\) is the optimal of \(p(x)\).
Proof. We consider following Lyapunov function

$$R_t := \mathbb{E}[p(x_t) + \frac{c_t}{n} \Sigma_{i=1}^n \|x_t - z_t^i\|^2],$$

then we have,

$$\frac{1}{n} \Sigma_{i=1}^n \mathbb{E}[[\|x_t - z_t^i\|^2] = \frac{1}{n} \Sigma_{i=1}^n \left( \frac{1}{n} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \frac{n-1}{n} \mathbb{E}[\|x_t - z_t^{t-1}\|^2] \right)$$

$$= \frac{\eta_t^2}{n} \mathbb{E} \|g_t\|^2 + \frac{n-1}{n} \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^{t-1}\|^2]$$

And

$$\mathbb{E}[\|x_t - z_t^{t-1}\|^2] = \mathbb{E}[\|x_t - x_{t-1}\|^2 + 2(x_t - x_{t-1})^T (x_{t-1} - z_t^{t-1}) + \|x_{t-1} - z_t^{t-1}\|^2]$$

$$= \mathbb{E}[\|x_t - x_{t-1}\|^2 + 2\eta_t^2 (-\eta_t^{1})^T (x_{t-1} - z_t^{t-1}) + \|x_{t-1} - z_t^{t-1}\|^2]$$

$$\leq \mathbb{E}[\|x_t - x_{t-1}\|^2 + 2\eta_t^2 (\frac{\eta_t^2 - \eta_t^{1}}{2\beta_t}) \mathbb{E} \|g_t\|^2 + \beta_t (\|x_t - z_t^{t-1}\|^2) + \|x_{t-1} - z_t^{t-1}\|^2]$$

$$= (\eta_t^2 + \frac{\eta_t^2 - \eta_t^{1}}{\beta_t}) \mathbb{E} \|g_t\|^2 + (1 + \beta_t \eta_t^{1}) \mathbb{E}[\|x_t - z_t^{t-1}\|^2].$$

Therefore

$$\frac{1}{n} \Sigma_{i=1}^n \mathbb{E}[[\|x_t - z_t^i\|^2] \leq (\eta_t^2 + (1 - \frac{\eta_t^2 - \eta_t^{1}}{\beta_t}) \mathbb{E}[\|g_t\|^2 + \frac{1}{n} \Sigma_{i=1}^n (1 - \frac{1}{n}) (1 + \beta_t \eta_t^{1}) \mathbb{E}[\|x_t - z_t^{t-1}\|^2].$$

Because

$$\mathbb{E}[p(x_t)] \leq p(x_{t-1}) + \frac{\eta_t^2 L^2 \lambda_t}{2n} \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2] - \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right) \mathbb{E}[\|g_t\|^2],$$

then we have

$$R_t = \mathbb{E}[p(x_t) + \frac{c_t}{n} \Sigma_{i=1}^n \|x_t - z_t^i\|^2]$$

$$\leq p(x_{t-1}) + \frac{\eta_t^2 L^2 \lambda_t}{2n} \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2] - \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right) \mathbb{E}[\|g_t\|^2]$$

$$+ \frac{c_t}{n} \mathbb{E}[\|g_t\|^2] + \frac{c_t}{n} \eta_t^2 \left( \frac{\eta_t^2}{\beta_t} \right) \mathbb{E}[\|g_t\|^2] + \left( 1 + \beta_t \eta_t^{1} \right) \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2]$$

$$= p(x_{t-1}) + \left( \frac{\eta_t^2 L^2 \lambda_t}{2} + c_t (n-1)(1 + \beta_t \eta_t^{1}) \right) \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2]$$

$$- \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right) \mathbb{E}[\|g_t\|^2]$$

$$+ \frac{c_t}{n} \eta_t^2 \left( \frac{\eta_t^2}{\beta_t} \right) \mathbb{E}[\|g_t\|^2] + \left( 1 + \beta_t \eta_t^{1} \right) \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2]$$

$$\leq p(x_{t-1}) + \left[ c_t (1 + \beta_t \eta_t^{1}) - \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right) \right] \Sigma_{i=1}^n \mathbb{E}[\|x_t - z_t^i\|^2]$$

$$- \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right) \mathbb{E}[\|g_t\|^2]$$

Because $c_{t-1} = c_t (1 + \beta_t \eta_t^{1}) - \left( \frac{\eta_t^2 - \eta_t^{1}}{2\lambda_t} - \frac{L\eta_t^2}{2} \right)$, and $\gamma_n = \min_t \Gamma_t$, then we have,

$$R_t \leq R_{t-1} - \Gamma_{t-1} \mathbb{E}[\|g_t\|^2],$$

that is, $\mathbb{E}[\|g_t\|^2] \leq \frac{R_{t-1} - R_t}{\gamma_n}$.

$$\Sigma_{t=0}^{T-1} \mathbb{E}[\|g_t\|^2] \leq \frac{R_0 - R_T}{\gamma_n},$$
where we use \( c_T = 0 \), then \( R_T = \mathbb{E}p(x_T) \); and \( R_0 = p(x_0) \), we can obtain
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|g_t\|^2 \leq \frac{p(x_0) - \mathbb{E}p(x_T)}{T\gamma_n} \leq \frac{p(x_0) - p(x^*)}{T\gamma_n}.
\]

\[\text{Theorem 11.} \supseteq \text{Suppose } p(x) \in \mathcal{F}^n. \text{Let } \eta = \frac{u_0}{L\eta} (0 < u_0 < 1), \beta \eta^\rho = \frac{1}{2n}, \lambda = \frac{\eta^{1-\alpha}}{u_0}, \forall \alpha \in \mathbb{R}, \text{and } T \text{ the total number of iterations. Then there exists universal constants } u_0, v > 0 \text{ such that we have following: } \gamma_n \geq \frac{v}{L \eta} \text{ and}
\]
\[
\mathbb{E} \|g_a\|^2 \leq \frac{L n [p(x_0) - p(x^*)]}{T v}.
\]

\[\text{Proof.} \text{ Firstly, we give a upper bound for } c_0. \text{ Because } \eta_t = \eta, \rho_t = \rho, \alpha_t = \alpha, \beta_t = \beta > 0 \text{ and } \lambda_t = \lambda > 0,
\]
\[
c_t-1 = c_t(1 + \beta \eta^\rho - \frac{1}{n}) + \eta^\alpha L^2 \lambda \frac{2}{2n}.
\]

that is, if we set \( \theta = -\beta \eta^\rho + \frac{1}{n} > 0 \) and \( b = \eta^\alpha L^2 \lambda \frac{2}{2n} \), then we have,
\[
c_t-1 = c_t(1 - \theta) + b.
\]

Because \( c_T = 0 \), then we have
\[
c_t = \frac{1 - (1 - \theta)^{T-t}}{\theta} b.
\]

That is,
\[
c_t = \frac{1 - (1 - \theta)^{T-t}}{\theta} b \leq \frac{b}{\theta}.
\]

Then we can lower bound \( \gamma_n \). Because we have
\[
\Gamma_{t-1} = \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha}}{2 \lambda_t} - \frac{L \eta_t^2}{2} - c_t \eta_t^2 = \frac{c_t \eta_t^{2-\alpha}}{\beta_t}
\]
\[
= \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha}}{2 \lambda} - \frac{L \eta_t^2}{2} - c_t \eta_t^2 = \frac{c_t \eta_t^{2-\rho}}{\beta}
\]
\[
\]
Then
\[
2 \gamma_n = \min_{t} \eta_t - \frac{\eta_t^{2-\alpha}}{\lambda} - L \eta_t^2 - 2 c_t \eta_t^2 - \frac{2 c_t \eta_t^{2-\rho}}{\beta}
\]
\[
\geq \eta_t - \frac{\eta_t^{2-\alpha}}{\lambda} - L \eta_t^2 - \frac{2 b \eta_t^2}{\theta} - \frac{2 b \eta_t^{2-\rho}}{\theta \beta}
\]
\[
= \eta_t - \frac{\eta_t^{2-\alpha}}{\lambda} - L \eta_t^2 - \frac{L^2 \lambda \eta_t^{1+\alpha}}{\theta} - \frac{L^2 \lambda \eta_t^{1+\alpha-\rho}}{\theta \beta}
\]
\[
= \eta_t(1 - \frac{\eta_t^{1-\alpha}}{\lambda} - L \eta_t - \frac{L^2 \lambda \eta_t^{1+\alpha}}{\theta} - \frac{L^2 \lambda \eta_t^{1+\alpha-\rho}}{\theta \beta})
\]

Set \( \beta \eta^\rho = \frac{1}{2n} \), then, \( \theta = \frac{1}{2n} \), \( \eta = \frac{u_0}{L \eta} \), and \( \lambda = \frac{\eta^{1-\alpha}}{u_0} \), then we have,
\[
\frac{\eta^{1-\alpha}}{\lambda} = u_0;
\]
\[
L \eta \leq u_0;
\]
\[
\frac{L^2 \lambda \eta^{1+\alpha}}{\theta} = \frac{2 u_0}{n};
\]
\[
\frac{L^2 \lambda \eta^{1+\alpha-\rho}}{\theta \beta} = 4 u_0.
\]
Thus we can conclude that we can find a universal constant \( v \) such that
\[
\gamma_n \geq \frac{v}{L n}.
\]
Then we complete our proof.

7.4 Analysis of Min-batch Nonconvex SAGA

**Algorithm 4** Min-batch Prox-SAGA \((x_0, T, \{\eta_t\}_{t=0}^{T-1})\)

**Input:** \( x_0 \in \mathbb{R}^d, z_i^0 = x_0 \) for \( i \in \{1, 2, \cdots, n\} \), number of iterations \( T \), step size \( \{\eta_t > 0\}_{t=0}^{T-1} \)

**Output:** Iterate \( x_a \) chosen uniformly random from \( \{x_i\}_{i=0}^{T-1} \), and \( g_a \) chosen uniformly random from \( \{\frac{1}{n} \sum_{t=1}^{n} (x_t - x_{t+1})\}_{t=0}^{T-1} \)

1: \( \phi_0 = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_i^0) \)
2: for \( t = 0 \) to \( T - 1 \) do
3: Uniformly randomly pick a min-batch \( I_t, J_t \) from \( \{1, 2, \cdots, n\} \)
4: \( v_t = \frac{1}{n} \sum_{i \in I_t} (\nabla f_i(x^t) - \nabla f_i(z_i^t)) + \phi_t \)
5: \( x_{t+1} = \text{prox}_{\eta_t} (x_t - \eta_t v_t) \)
6: \( z_{t+1} = x_t \) for \( j_t \in J_t \) and \( z_{t+1} = z_i^t \) for \( j_t \neq J_t \)
7: \( \phi_{t+1} = \phi_t - \frac{1}{n} (\nabla f_{j_t}(z_i^t) - \nabla f_{j_t}(z_{t+1}^t)) \)
8: end for

Set \( v_t = \frac{1}{n} \sum_{i \in I_t} (\nabla f_i(x^t) - \nabla f_i(z_i^t)) + \phi_t \) for a uniformly randomly selected min-batch \( I_t \), and \( \phi_t = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(z_i^t) \). Then \( v_t \) is unbiased estimation of \( \nabla f(x_t) \). We have following lemma.

**Lemma 11** (upper bounds for variance). For problem \((7)\), and suppose \( p(x) \in \mathcal{F}_n \), we set \( \Delta_t = v_t - \nabla f(x_t) \), then we have
\[
E\|\Delta_t\|^2 \leq \frac{L^2}{bn} \sum_{t=1}^{n} E\|x_t - z_t^t\|^2,
\]
where we denote \( E\|\Delta_t\|^2 \) as the variance of stochastic gradients.

**Proof.**
\[
E\|\Delta_t\|^2 = E\left\| \frac{1}{b} \sum_{i \in I_t} (\nabla f_i(x^t) - \nabla f_i(z_i^t)) + \phi_t - \nabla f(x_t) \right\|^2
\]
\[
= \frac{1}{b^2} E\|\sum_{i \in I_t} \nabla f_i(x_t) - \nabla f_i(z_i^t)\|^2 - \|\nabla f(x_t) - \phi_t\|^2
\]
\[
\leq \frac{1}{b^2} E\|\sum_{i \in I_t} \nabla f_i(x_t) - \nabla f_i(z_i^t)\|^2
\]
\[
\leq \frac{1}{b^2 n} \sum_{i=1}^{n} E\|\nabla f_i(x_t) - \nabla f_i(z_i^t)\|^2
\]
\[
\leq \frac{L^2}{bn} \sum_{t=1}^{n} E\|x_t - z_t^t\|
\]
where the first equality is due to \( E(\zeta - E\zeta)^2 = E\zeta^2 - (E\zeta)^2 \) for any random variable.

**Lemma 12.** For problem \((7)\), and suppose \( p(x) \in \mathcal{F}_n \), we have, after one step, **Algorithm 4** will ensure
\[
E p(x_t) \leq E p(x_{t+1}) + \frac{\eta_t^2}{2} \frac{L^2}{2 n} E\|x_{t+1} - z_t^t\|^2 - \frac{\eta_t^2}{2} \frac{L^2}{2 n} E\|g_t\|^2,
\]
where \( \lambda_t > 0 \) and \( \alpha_t \in \mathbb{R} \).
Proof.

\[ p(x_t) = f(x_t) + h(x_t) \]
\[ \leq f(x_{t-1}) + \nabla f(x_{t-1})^T (x_t - x_{t-1}) + \frac{L}{2} ||x_t - x_{t-1}||^2 + h(x_t) \]
\[ = f(x_{t-1}) + v_{t-1}^T (x_t - x_{t-1}) + \frac{1}{2\eta_t} ||x_t - x_{t-1}||^2 + h(x_t) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1}) \]
\[ + \left( \frac{L}{2} - \frac{1}{2\eta_t} \right) ||x_t - x_{t-1}||^2 \]
\[ \leq f(x_{t-1}) + h(x_{t-1}) + (\nabla f(x_{t-1}) - v_{t-1})^T (x_t - x_{t-1}) + \left( \frac{L}{2} - \frac{1}{2\eta_t} \right) ||x_t - x_{t-1}||^2 \]
\[ = p(x_{t-1}) + \eta_t^\alpha_t (\nabla f(x_{t-1}) - v_{t-1})^T (\eta_t^{1-\alpha_t} (-g_t)) + \left( \frac{L\eta_t^2}{2} - \frac{\eta_t^2}{2} \right) ||g_t||^2 \]
\[ \leq p(x_{t-1}) + \eta_t^\alpha_t \left( \frac{\lambda_t}{2} ||\Delta_t||^2 + \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} ||g_t||^2 \right) + \left( \frac{L\eta_t^2}{2} - \frac{\eta_t^2}{2} \right) ||g_t||^2 \]

Taking expectation conditioned on information at \( t - 1 \), we have
\[ \mathbb{E}[p(x_t)] \leq p(x_{t-1}) + \frac{\lambda_t \eta_t^\alpha_t L^2}{2\beta_t} \sum_{i=1}^n \mathbb{E} ||x_{t-1} - z_i^{t-1}||^2 - \frac{\eta_t^2}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2} \mathbb{E} ||g_t||^2, \]

where \( \lambda_t > 0 \); and the first inequality is due to the \( L \)-smoothness of \( f \); the second inequality is due to \( x_t \) is the minimizer of Eq (1); the third inequality is due to Cauchy–Schwarz and Young’s inequality; the fourth inequality is due to Lemma [1]. \( \square \)

Before our theory, we define,
\[ \Gamma_{t-1} = \left( \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha_t}}{2\lambda_t} - \frac{L\eta_t^2}{2} - c_t \frac{\eta_t^{2-\rho_t}}{\beta_t} \right), \]

with some parameters \( \eta_t, \lambda_t, \beta_t, \rho_t \) which will be defined later.

**Theorem 12.** Suppose \( p(x) \in \mathcal{F}_n \). Let \( c_T = 0 \), \( \beta_t = \beta > 0 \), \( \alpha_t = \lambda_t = \rho_t = \alpha_t = \lambda_t = \rho_t = 0 \), \( c_{t-1} = c_t [1 - (1/n + \beta_t \eta_t^\alpha_t - \beta_t \eta_t^{1-\alpha_t})] + \frac{\eta_t L^2}{n} \) such that \( \Gamma_t > 0 \) for \( 0 \leq t \leq T - 1 \). Define the quantity \( \gamma_n := \min_1 \Gamma_t \). Then the Algorithm [2] will ensure
\[ \mathbb{E} ||g_n||^2 \leq \frac{[p(x_0) - p(x^*)]}{T \gamma_n}, \]

where \( x^* \) is the optimal of \( p(x) \).

Proof. We consider following Lyapunov function
\[ R_t := \mathbb{E}[p(x_t) + \frac{c_t}{n} \sum_{i=1}^n ||x_t - z_i^t||^2], \]
then we have,
\[ \frac{1}{n} \sum_{i=1}^n \mathbb{E} [||x_t - z_i^t||^2] = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{n} \mathbb{E} [||x_t - x_{t-1}||^2] + \frac{n-1}{n} \mathbb{E} [||x_t - z_i^{t-1}||^2] \right) \]
\[ = \frac{\eta_t^2}{n} \mathbb{E} ||g_t||^2 + \frac{n-1}{n} \sum_{i=1}^n \mathbb{E} ||x_t - z_i^{t-1}||^2 \]

And
\[ \mathbb{E} ||x_t - z_i^{t-1}||^2 = \mathbb{E} ||x_t - x_{t-1} + x_{t-1} - z_i^{t-1}||^2 \]
\[ = \mathbb{E} [||x_t - x_{t-1}||^2 + 2(x_t - x_{t-1})^T (x_{t-1} - z_i^{t-1}) + ||x_{t-1} - z_i^{t-1}||^2] \]
\[ = \mathbb{E} [||x_t - x_{t-1}||^2 + 2\eta_t^\alpha_t (-\eta_t^{1-\alpha_t} g_t)^T (x_{t-1} - z_i^{t-1}) + ||x_{t-1} - z_i^{t-1}||^2] \]
\[ \leq \mathbb{E} [||x_t - x_{t-1}||^2 + 2\eta_t^\alpha_t \left( \frac{\eta_t^{2-\alpha_t}}{2\beta_t} ||g_t||^2 + \frac{\beta_t}{2} ||x_{t-1} - z_i^{t-1}||^2 \right) + ||x_{t-1} - z_i^{t-1}||^2] \]
\[ = (\eta_t^2 + \frac{\eta_t^{2-\rho_t}}{\beta_t}) \mathbb{E} ||g_t||^2 + (1 + \beta_t \eta_t^{1-\alpha_t}) \mathbb{E} ||x_{t-1} - z_i^{t-1}||^2. \]
Therefore
\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|x_t - z_i\|^2] \leq \left( \eta_t^2 + \left(1 - \frac{1}{n} \right) \frac{\eta_t^{2 - \rho_t}}{\beta_t} \right) \mathbb{E}[\|g_t\|^2] + \frac{1}{n} \sum_{i=1}^{n} \left(1 - \frac{1}{n} \right) \frac{\eta_t^{2 - \alpha_t} + \frac{L_t^2}{\lambda_t}}{2} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2]. \]

Because
\[ \mathbb{E}[p(x_t)] \leq p(x_{t-1}) + \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2bn} \sum_{i=1}^{n} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2] - \frac{\eta_t}{2} \frac{\eta_t^{2 - \alpha_t}}{2\lambda_t} \mathbb{E}[\|g_t\|^2], \]
then we have
\[
R_t = \mathbb{E}[p(x_t) + \frac{c_t}{n} \sum_{i=1}^{n} \|x_t - z_i\|^2] \
\leq p(x_{t-1}) + \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2bn} \sum_{i=1}^{n} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2] - \frac{\eta_t}{2} \frac{\eta_t^{2 - \alpha_t}}{2\lambda_t} \mathbb{E}[\|g_t\|^2] \
+ \frac{c_t}{n} \mathbb{E}[\|g_t\|^2] + \frac{c_t (n-1)}{n} \left[ (\eta_t^2 + \frac{\eta_t^{2 - \rho_t}}{\beta_t}) \mathbb{E}[\|g_t\|^2] + (1 + \beta_t \eta_t^{\rho_t}) \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2] \right] \
= p(x_{t-1}) + \left( \eta_t^{2 - \alpha_t} L_t^2 \lambda_t \right) \frac{2b}{n} + \frac{c_t (n-1) (1 + \beta_t \eta_t^{\rho_t})}{n} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2] \
- \frac{\eta_t}{2} \frac{\eta_t^{2 - \alpha_t}}{2\lambda_t} - \frac{L_t^2}{2} - \frac{c_t \eta_t^2}{n} - \frac{c_t (n-1) (\eta_t^2 + \frac{\eta_t^{2 - \rho_t}}{\beta_t})}{n} \mathbb{E}[\|g_t\|^2] \
\leq p(x_{t-1}) + \left( \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2b} + \frac{c_t (1 + \beta_t \eta_t^{\rho_t})}{n} - \frac{\beta_t \eta_t^{\rho_t}}{n} \right) + \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2b} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|x_{t-1} - z_i^{t-1}\|^2] \
- \frac{\eta_t}{2} \frac{\eta_t^{2 - \alpha_t}}{2\lambda_t} - \frac{L_t^2}{2} - \frac{c_t \eta_t^2}{n} - \frac{c_t \eta_t^{2 - \rho_t}}{\beta_t} \mathbb{E}[\|g_t\|^2].
\]

Because \( c_{t-1} = c_t (1 + \beta_t \eta_t^{\rho_t} - \frac{1}{n} - \beta_t \eta_t^{\rho_t}) + \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2b}, \) and \( \gamma_n = \min_i \Gamma_i, \) then we have,
\[
R_t \leq R_{t-1} - \Gamma_{t-1} \mathbb{E}[\|g_t\|^2],
\]
that is, \( \mathbb{E}[\|g_t\|^2] \leq \frac{R_{t-1} - R_T}{\gamma_n}. \) Then
\[
\sum_{i=1}^{T-1} \mathbb{E}[\|g_t\|^2] \leq \frac{R_0 - R_T}{\gamma_n},
\]
where we use \( c_T = 0, \) then \( R_T = \mathbb{E}[p(x_T)]; \) and \( R_0 = p(x_0), \) we can obtain
\[
\frac{1}{T} \sum_{i=1}^{T-1} \mathbb{E}[\|g_t\|^2] \leq \frac{p(x_0) - \mathbb{E}[p(x_T)]}{T \gamma_n} \leq \frac{p(x_0) - p(x^*)}{T \gamma_n}. \]

\[ \square \]

**Theorem 13.** Suppose \( p(x) \in F_n. \) Let \( \eta = \sqrt{\frac{\log(\lambda \alpha)}{\alpha \log(1/\alpha) \lambda}}, \) \( \beta \eta^\rho = \frac{1}{\alpha}, \) \( \lambda = \frac{n^{\frac{1}{2}}}{\alpha}, \) \( \gamma > 0, \) and \( T \) the total number of iterations. Then there exists universal constants \( u_0, v > 0 \) such that we have following: \( \gamma_n \geq \sqrt{\frac{v}{T \alpha n}} \) and
\[
\mathbb{E}[\|g_0\|^2] \leq \frac{L n [p(x_0) - p(x^*)]}{\sqrt{bTv}}.
\]

**Proof.** Firstly, we give a upper bound for \( c_0. \) Because \( \eta_t = \eta, \rho_t = \rho, \alpha_t = \alpha, \beta_t = \beta > 0 \) and \( \lambda_t = \lambda > 0, \)
\[
c_{t-1} = c_t (1 + \beta_t \eta^\rho - \frac{1}{n} - \beta_t \eta^\rho) + \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2b}.
\]
that is, if we set \( \theta = \frac{1}{n} + \frac{\beta_t \eta^\rho}{n} - \beta_t \eta^\rho > 0 \) (we can get \( \theta = \frac{1}{n} \) if we set the values as those in Theory 12 and \( a = \frac{\eta_t^{2 - \alpha_t} L_t^2 \lambda_t}{2b}, \) then we have,
\[
c_{t-1} = c_t (1 - \theta) + a.
\]
Because \( c_T = 0 \), then we have
\[
\begin{align*}
c_T &= 0; \\
c_{T-1} &= (1 - \theta)c + a = a; \\
c_{T-2} &= (1 - \theta)a + a; \\
&\quad \vdots \\
c_t &= 1 - (1 - \theta)^{T-t} a.
\end{align*}
\]
That is,
\[
\begin{align*}
c_t &= 1 - (1 - \theta)^{T-t} a \leq \frac{a}{\theta}.
\end{align*}
\]
Then we can lower bound \( \gamma_n \). Because we have
\[
\Gamma_{t-1} = \frac{\eta_t}{2} - \frac{\eta_t^{2-\alpha}}{2\lambda_t} - \frac{L\eta_t^2}{2} - c_t\eta_t^2 - \frac{c_t\eta_t^{2-\rho}}{\beta_t}
\]
Then
\[
2\gamma_n = \min_t \eta - \frac{\eta_t^{2-\alpha}}{\lambda} - L\eta_t^2 - 2c_t\eta_t^2 - \frac{c_t\eta_t^{2-\rho}}{\beta}
\]
\[
\geq \eta - \frac{\eta_t^{2-\alpha}}{\lambda} - L\eta_t^2 - 2\alpha\eta_t^2 - \frac{2\alpha\eta_t^{2-\rho}}{\theta\beta}
\]
\[
= \eta - \frac{\eta_t^{2-\alpha}}{\lambda} - L\eta_t^2 - \frac{L^2\lambda_t^{1+\alpha}}{b\theta} - \frac{L^2\lambda_t^{1+\alpha-\rho}}{\theta b\beta}
\]
\[
= \eta(1 - \frac{\eta_t^{2-\alpha}}{\lambda} - L\eta_t^2 - \frac{L^2\lambda_t^{1+\alpha}}{b\theta} - \frac{L^2\lambda_t^{1+\alpha-\rho}}{\theta b\beta})
\]
Set \( \rho = 1, \eta = \frac{u_0}{L\eta}, \beta\eta^\rho = \frac{1}{2n}, \theta \geq \frac{1}{2n}, b \leq n \) then,
\[
\eta_t^{1-\alpha} = u_0; \\
L\eta = \frac{u_0}{n}; \\
\frac{L^2\lambda_t^{1+\alpha}}{\theta} = \frac{2u_0}{n}; \\
\frac{L^2\lambda_t^{1+\alpha-\rho}}{\theta b\beta} = 4u_0.
\]
Thus we can conclude that we can find a universal constant \( v \) such that
\[
\gamma_n \geq \frac{\sqrt{b}\eta}{L\eta}.
\]
Then we complete our proof.

7.5 Experiments

**Running-Time Experiments** Here we show how fast the objective value decreases for each method. We choose the parameters as stated in the paper. We can see, the prox-SVRG could enjoy less variance and help the problem converge faster to the stationary point.
Figure 3: Objective value decrease comparison between prox-SVRG, prox-SG and SGD. The $y$ axis presents objective values and the $x$ axis represents the number of effective passes.

**Accuracy Experiments** Here we show the accuracy of all three datasets on each method. For mnist and web datasets, we set $\lambda_1 = 10^{-6}$ and $\lambda_2 = 10^{-6}$, and choose the step size as stated in the paper. We will see higher accuracy for this datasets. Nonconvex prox-SVRG could also be quickly stacked into a stationary point which is not good enough and thus, for some datasets, it may not achieve higher accuracy.

Figure 4: Testing accuracy comparison between prox-SVRG with sigmoid loss, square loss, logistic loss and hingeloss.

Figure 5: Testing accuracy comparison between prox-SVRG with sigmoid loss, square loss, logistic loss and hingeloss.
Figure 6: Testing accuracy comparison between prox-SVRG with sigmoid loss, square loss, logistic loss and hinge loss.