Explicit Time-Optimal Acceleration-Constrained Speed Profiles for Planar Paths with Monotone Curvature

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Abstract
It is shown how to construct minimum-time speed profiles for planar paths with smooth strictly-monotonic signed curvature, subject to constraints on velocity, normal acceleration and tangential acceleration. The construction is explicit and exact, and global optimality is rigorously established from first principles under mild regularity conditions on the path. Free, fixed, and inequality-constrained boundary speeds are all accommodated. Numerical implementation is straightforward.

Keywords: optimal control; calculus of variations; optimisation; velocity planning; path tracking

1. Introduction
Minimum-time traversal of a predefined path is a well-studied problem. This work considers time-optimal speed profile synthesis for the path, without assuming a particular model for agent kinetics. Only kinematic constraints on velocities and accelerations are imposed: maximum values for tangential acceleration and deceleration, normal acceleration, and speed, along with boundary conditions on the initial and terminal speeds. The resulting speed profile can, for example, be provided to a velocity-tracking controller on board a mobile robot, or used to predict travel times for path-planning. The novelty of the proposed method is that it yields a solution that is explicit, exact, and globally optimal. To achieve this, the scope is restricted to planar paths that have smooth strictly-monotonic signed curvature. The results presented are a first step towards exact minimum-time velocity planning for more general paths.

A portion of the robotics literature is devoted to time-optimal velocity planning for robotic manipulators, subject to limits on the forces and torques at the manipulator joints. Examples include [1–5], which rely on models of the manipulator kinetics. Force and torque limits lead to more complex constraints on tangential acceleration than the constant bounds imposed here, but those works leave velocity and normal acceleration unconstrained. Given a path parametrised by a scalar $s$, computation of the maximum velocity curve in the $(s, \dot{s})$-plane via numerical search lies at the core of [1–4]. A more computationally efficient solution is proposed in [5]. These techniques are extended in [6–9] to accommodate velocity constraints, resulting in an optimisation problem that captures the one herein as a special case, but the algorithms proposed do not yield explicit solutions. Other approaches to solving the minimum-time robotic manipulator velocity planning problem include path discretisation [10] and dynamic programming [11, 12].

An explicit time-optimal velocity profile is presented in [13] for a point-mass vehicle subject to a net force constraint. A simple kinetic model for the normal and tangential force components is adopted, and an upper bound imposed on the magnitude of the vector sum. This leads to a mixed constraint, involving both velocity and tangential acceleration in the same equation. An optimal control problem is formulated, and Pontryagin’s Maximum Principle (PMP) applied to derive a solution. Herein, by contrast, the normal and tangential acceleration constraints are decoupled. The corresponding optimal control problem involves pure state inequality constraints, which demand more advanced techniques than in the case of mixed constraints [14, Section 4]. Depending on the path, PMP may require a large number of jump conditions to be resolved, which hinders the discovery of an explicit solution. A first principles approach is adopted here instead. The minimum-time problem considered is mathematically equivalent to that of [15], in which the path is discretised to obtain an approximate numerical solution, albeit without requiring the signed curvature to be monotone.

Paths with strictly-monotone signed curvature constitute an important class of curves. Clothoids are used for highway and railway track design [16], and in path-planning for autonomous vehicles [17–19], because they have linear signed curvature. This allows vehicles to transition smoothly between two paths. A method for generating feasible velocity profiles along clothoids is proposed in [17], but time-optimality is not claimed. Since the Cartesian parametric equations of a clothoid require the eval-

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nation of Fresnel Integrals, they are often approximated with polynomial series expansions instead [16, 18]. The strict monotonicity assumption on path curvature here admits both clothoids and their polynomial approximations. Beyond these, any path of practical use can be decomposed into a finite number of segments with either strictly-monotonic or constant signed curvature. Since velocity profiling for the latter is trivial, and velocity profiling for the former is treated here, a time-optimal velocity profile for the entire path can be constructed by choosing the right boundary conditions for each segment. This reduces an infinite-dimensional optimisation problem to a finite-dimensional one, without introducing any approximations. Extensions in this direction are the subject of ongoing work.

2. Preliminaries

2.1. Notation

The (non-strict) subset relation is denoted by $\subset$. The function $g \circ f : F \to H$ denotes the composition of $f : F \to G$ with $g : G \to H$, and $f^{-1}[Y] \subset F$ the preimage of $Y \subset G$ under $f$. The real numbers are denoted by $\mathbb{R}$, and the natural numbers by $\mathbb{N} := \{0, 1, \ldots\}$. For $i, j \in \mathbb{N}$, $[i : j] := \{k \in \mathbb{N} | i \leq k \leq j\}$. By convention, $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$. For $n \in \{1, 2, \ldots\}$, elements of $\mathbb{R}^n$ are considered column vectors. Given $a < b$, the derivative of $h : [a, b] \to \mathbb{R}^n$ at $s \in [a, b]$ is denoted $h'(s)$ wherever it exists. The right derivative of $h$ at $s$ is given by

$$h^+(s) := \lim_{\tau \downarrow 0} \frac{h(s + \tau) - h(s)}{\tau},$$

wherever the one-sided limit exists. Observe, $h'(a) = h^+(a)$ at the left endpoint.

2.2. Technical results

Subsequent analysis hinges on the following propositions about continuous functions. Proofs are provided for completeness.

**Proposition 1.** For any continuously differentiable $f : [0, 1) \to \mathbb{R}$,

$$\lim_{s \to 1} f(s) = \infty \implies \lim_{s \to 1} f'(s) = \infty.$$  

**Proof.** If $\lim_{s \to 1} f'(s) < \infty$, then $c := \sup\{f'(s) | s \in [0, 1)\} < \infty$, and

$$\forall s \in [0, 1), \quad f(s) = f(0) + \int_{0}^{s} f'(\tau) \, d\tau \leq f(0) + \int_{0}^{s} c \, d\tau = f(0) + cs,$$

by which

$$\lim_{s \to 1} f(s) \leq \lim_{s \to 1} (f(0) + cs) = f(0) + c < \infty.$$  

**Proposition 2.** Let $f : [0, 1) \to \mathbb{R}$ be continuous, and

$$a := \inf\{s \in [0, 1) | f(s) > 0\}.$$  

If $0 < a < \infty$, then $a \in [0, 1) \cap f(a) \geq 0$. Moreover, if $0 < a < \infty$, then $f(s) \leq 0$ for all $s \in [0, a)$, and in particular, $f(a) = 0$.

**Proof.** Let $S := \{s \in [0, 1) | f(s) > 0\} = f^{-1}([0, \infty))$. Since $S \subset [0, 1)$, $a = \inf S \geq \inf [0, 1) = 0$. If $S = \emptyset$, then $a = \infty$, by convention. Suppose that $a < \infty$, which implies $S \neq \emptyset$. Then there exists $\tau \in [0, 1)$ such that $f(\tau) > 0$. This follows that $a \leq \tau < 1$, and thus, $a \in [0, 1) = \operatorname{dom} f$.

Now, let $T := \{s \in [0, 1) | f(s) \leq 0\} = f^{-1}((\infty, 0])$. Both $S$ and $T$ are open relative to $[0, 1)$, because $f$ is continuous. Suppose $a \in T$. Then there exists a neighbourhood $U \subset [0, 1)$ of $s$ such that $U \subset T$. Since $S$ and $T$ are disjoint, $U \cap S = \emptyset$. But $a = \inf S$ is a boundary point of $S$ by definition, which implies the contradiction $U \cap S \neq \emptyset$. Thus, $a \notin T$.

Finally, to again obtain a contradiction, suppose that $f(\tau) > 0$ for some $\tau \in [0, s] \subset [0, 1)$, where $s > 0$. Then $\tau \in S$ by definition, which implies $a = \inf S \leq \tau$. Since $\tau \leq s$ by hypothesis, $a = \tau \in S$, and thus, $\inf S \leq S$. But $S$ is open and therefore contains none of its boundary points, including $\inf S$, leading to a contradiction.

**Remark 1.** The preceding results extend to any $f : [a, b) \to \mathbb{R}$, with $a < b$, because $[a, b)$ can be mapped smoothly to $[0, 1)$; e.g., $s \mapsto \frac{a + b}{b - a} s$.

**Proposition 3.** Let $a < b$, and $f, g : [a, b] \to \mathbb{R}$ be Lipschitz. Then $h : [a, b] \to \mathbb{R}, \quad h(s) := \min\{f(s), g(s)\}$ is also Lipschitz. Let $F \subset \mathbb{R}$ be such that $f'(s) \in F$ for almost every $s \in [a, b]$, and $G$ such that $g'(s) \in G$ for almost every $s \in [a, b]$. If $f$ is strictly increasing and $g$ non-increasing, then

$$h'(s) \in F \cup G \quad (1)$$

for almost every $s \in [a, b]$. The same holds if $f$ is strictly decreasing and $g$ non-increasing.

**Proof.** That $h$ is Lipschitz follows from [20, Proposition 2.3.9]. If $f$ is strictly increasing and $g$ non-increasing, then there is at most one point at which $f(s) = g(s)$. The same holds if $f$ is strictly decreasing and $g$ non-decreasing. In either case, the set $E := \{s \in [a, b] | f(s) = g(s)\}$ is a singleton.

Let $S^c$ denote $[a, b] \setminus S$ for any $S \subset [a, b]$. Define $\hat{F} := \{s \in [a, b] | f'(s) \in F\}$ and $\hat{G} := \{s \in [a, b] | g'(s) \in G\}$. Choose any $s \in \hat{F} \cap \hat{G} \cap E^c$, recalling that $f$ and $g$ are both continuous. Since $f(s) \neq g(s)$, either $f(s) < g(s)$ or $g(s) < f(s)$. If the former, then there exists a neighbourhood $U \subset [a, b]$ of $s$ such that $h(\tau) = f(\tau)$ for all $\tau \in U$, and therefore $h'(s) = f'(s) \in F$. If the latter, then $h'(s) = g'(s) \in G$ for the same reasons. This establishes (1) for all $s \in \hat{F} \cap \hat{G} \cap E^c$. If (1) does not hold, then $s \in (\hat{F} \cap \hat{G} \cap E^c)^c = \hat{F}^c \cup \hat{G}^c \cup E$, which has measure zero by hypothesis.
3. Problem formulation

Consider a thrice continuously differentiable curve \( r : [0, L] \to \mathbb{R}^2 \), parametrised by arc-length \( s \). Its signed curvature
\[
\kappa : [0, L] \to \mathbb{R}, \quad \kappa(s) = \det \begin{bmatrix} r''(s) & r''(s) \\ r''(s) & r''(s) \end{bmatrix}
\]
is then continuously differentiable, and \( ||r''(s)|| = 1 \) for all \( s \in [0, L] \). The goal of this paper is to construct a continuous speed profile \( v : [0, L] \to \mathbb{R} \) that minimises travel-time along \( r \), subject to a finite maximum tangential acceleration \( A > 0 \), maximum braking deceleration \( B > 0 \), maximum normal acceleration \( C > 0 \), and maximum speed \( V > 0 \). This is encoded in the optimisation problem below, where \( v_0, v_L \geq 0 \) are initial and terminal speeds, and \( x(s) = v(s)^2 \) is the squared-speed at distance \( s \in [0, L] \) along the path. The tangential acceleration at \( s \) is then given by \( v(s) v''(s) = \frac{1}{2} \frac{dv(s)^2}{ds} = \frac{x'(s)}{x(s)} \), and the normal acceleration by \( x(s) \kappa(s) \).

**Problem 1** (Time-optimal speed-squared profile). Given \( A, B, C, V, L > 0, \quad \kappa : [0, L] \to \mathbb{R} \) continuously differentiable, and \( v_0, v_L \geq 0 \), find an absolutely continuous \( x : [0, L] \to [0, \infty) \) that minimises
\[
J(x) = \int_0^L \frac{1}{\sqrt{x(s)}} \, ds,
\]
subject to
\[
-2B \leq x'(s) \leq 2A \quad \forall s \in [0, L], \quad \text{(3)}
\]
for almost every \( s \in [0, L] \), and
\[
x(0) \leq v_0^2
\]
\[
x(L) \leq v_L^2
\]
\[
\forall s \in [0, L], \quad x(s) \leq \min \left\{ \frac{C}{|\kappa(s)|}, V^2 \right\}.
\]

**Definition 1** (Feasibility). An absolutely continuous \( x : [0, L] \to [0, \infty) \) is **feasible** iff (3) holds for almost every \( s \in [0, L] \), and it satisfies (4)–(6).

**Remark 2** (Boundary conditions). Inequality constraints are imposed on the boundary values in Problem 1, because unconstrained and equality-constrained boundary values can also be addressed within this framework. To make the initial speed free, set \( v_0 \geq \min \{ \sqrt{C|\kappa(0)|^{-1}}, V \} \), and setting \( v_L \geq \min \{ \sqrt{C|\kappa(L)|^{-1}}, V \} \) makes the final speed free. It is ultimately established in Corollary 10 that the solution proposed in Theorem 1 satisfies (4) and (5) with equality, if such a feasible profile exists.

Note that absolutely continuous functions are differentiable almost everywhere, and satisfy the Fundamental Theorem of Calculus [21, Chapter 6.5]. An explicit solution to Problem 1 is constructed next, under the following assumption.

**Assumption 1.** The signed curvature \( \kappa : [0, L] \to \mathbb{R} \) is **strictly-monotonic**.

4. Construction of an optimal speed-squared profile

Algorithm 1 is used to explicitly construct a solution to Problem 1. It is invoked twice to process curvature information in two ‘sweeps’ along the path. The forward sweep progresses in the direction of increasing \( s \) to yield a speed-squared profile \( x_F : [0, L] \to [0, \infty] \) that switches between the curvature constraint boundary associated with (6), and the maximum tangential acceleration boundary in (3). Line 11 defines the points \( q_\ell \) at which it enters the latter, and Line 14 the points \( c_k \) at which it switches back to the former. Pathological paths requiring an infinite number of switches over a finite distance are excluded by the following assumption. (Cf. [2, Hypothesis 2], the final paragraph of [6, Section 2], and the discussion immediately preceding [3, Theorem 1].)

**Assumption 2** (Forward sweep termination). The given constants \( A, C > 0 \) and curvature \( \kappa \) in Problem 1 are such that \( \text{Sweep}(\kappa, A, C, v_0) \) returns a result; i.e., the **while-loop** (Lines 10–19) in Algorithm 1 terminates.

The reverse sweep progresses in the opposite direction to yield a profile \( x_R : [0, L] \to [0, \infty] \) that switches between the curvature constraint boundary and maximum tangential deceleration boundary. A similar path curvature regularity assumption is needed to guarantee termination of the algorithm.

**Assumption 3** (Reverse sweep termination). **Define the transformation** \( \gamma : [0, L] \to [0, L], \quad \gamma(s) := L - s \). The given constants \( B, C > 0 \) and curvature \( \kappa \) in Problem 1 are such that \( \text{Sweep}(\kappa \circ \gamma, B, C, v_L) \) returns a result; i.e., the while-loop (Lines 10–19) in Algorithm 1 terminates.

The pointwise minimum of \( x_F, x_R, \) and \( V^2 \), is an optimal profile. This result is the main contribution of the paper.

**Theorem 1.** Under Assumptions 1–3, define
\[
x_F := \text{Sweep}(\kappa, A, C, v_0),
\]
\[
x_R := \text{Sweep}(\kappa \circ \gamma, B, C, v_L) \circ \gamma.
\]
The speed-squared profile
\[
x^* : [0, L] \to [0, \infty), \quad x^*(s) := \min\{x_F(s), x_R(s), V^2\}. \]
solves Problem 1.

**Remark 3.** Lemma 10 establishes that the proposed solution \( x^* \) is Lipschitz, which implies (but is not necessary for) the absolute continuity required in Definition 1. This stronger property is desirable, because it ensures the tangential acceleration is bounded wherever it exists. It comes without sacrificing optimality.

The remainder of this section is devoted to proving Theorem 1. This involves verifying that the profiles \( x_F \) and \( x_R \) generated by Algorithm 1 are well defined, before establishing the feasibility and global optimality of \( x^* \).
1 Function Sweep($\kappa, A, C, v_0$):  
   Input: $\kappa: [0, L] \to \mathbb{R}, A > 0, C > 0, v_0 \geq 0$  
   Output: $x_F: [0, L] \to [0, \infty]$  
   $s_0 := \arg \min_{s \in [0, L]} |\kappa(s)|$ // Minimiser is unique under Assumption 1  
   $c_0 := 0, k := 0$  
   if $v_0^2 < C|\kappa(0)|^{-1}$ then // If initial speed below curvature boundary  
      $a_0 := 0$ // Start at max acceleration  
      set $x_0: [0, L] \to \mathbb{R}, x_0(s) := v_0^2 + 2As$ // Initial max acceleration profile  
      $c_1 := \inf \{ s \in [0, s_0) \mid C|\kappa(s)|^{-1} < x_0(s) \}$ // First switch to curvature constraint  
      $k := 1$  
   end  
   while $c_k < \infty$ do  
      $a_k := \inf \{ s \in [c_k, s_0) \mid \frac{C|\kappa(s)|}{|\kappa(s)|^2} > 2A \}$ // Switch to max acceleration  
      if $a_k < \infty$ then  
         set $x_k: [0, L] \to \mathbb{R}, x_k(s) := C|\kappa(a_k)|^{-1} + 2A \cdot (s - a_k)$ // Max acceleration profile  
         $c_{k+1} := \inf \{ s \in [a_k, s_0) \mid C|\kappa(s)|^{-1} < x_k(s) \}$ // Switch to curvature constraint  
      else  
         $c_{k+1} := a_k$  
      end  
      $k \leftarrow k + 1$  
   end  
   set $x_F: [0, L] \to [0, \infty], x_F(s) := \begin{cases} 
   C|\kappa(s)|^{-1}, & \text{if } s \in \bigcup_{n=0}^{k-1} [c_n, a_n) \\
   x_{\max\{n|c_n \leq s\}}(s), & \text{otherwise} 
\end{cases}$  
   return $x_F$  

Algorithm 1: Constructs a speed-squared profile $x_F$ given an initial speed $v_0$, maximum tangential and normal acceleration limits $A$ and $C$, respectively, and path curvature $\kappa$ parametrised by arc-length.

4.1. The point of minimum unsigned curvature

Assumption 1 requires the signed curvature $\kappa$ to be either strictly increasing or strictly decreasing. The unsigned curvature $|\kappa(s)|$ may, however, have a turning point. Line 2 of Algorithm 1 defines $s_0 \in [0, L]$ as the minimiser of unsigned curvature along the path:

$$s_0 := \arg \min_{s \in [0, L]} |\kappa(s)|. \quad (7)$$

The forward sweep deals with the decreasing portion of $|\kappa(s)|$ over $s \in [0, s_0]$, and the reverse sweep with the increasing portion over $s \in [s_0, L]$.

Lemma 1. Under Assumption 1, there exists a unique minimiser of $|\kappa(s)|$ over $s \in [0, L]$.

Proof. The continuity of $\kappa$ and the compactness of $[0, L]$ guarantee the existence of a minimiser, by the extreme value theorem of Bolzano and Weierstrass. Suppose $\kappa$ changes sign. Then there exists $s_0 \in (0, L)$ such that $\kappa(s_0) = 0$ by the intermediate value theorem. It is unique by Assumption 1. Thus, $s_0$ is the unique minimiser of $|\kappa(s)|$. Suppose now that $\kappa$ does not change sign, and let $s_0, s_1 \in [0, L]$ be minimisers of $|\kappa(s)|$. Then $|\kappa(s_0)| = |\kappa(s_1)|$, and since $\kappa(s_0)$ and $\kappa(s_1)$ have the same sign, it follows that $\kappa(s_0) = \kappa(s_1)$. If $s_0 \neq s_1$, this contradicts the strict monotonicity of $\kappa$.

The value $\kappa(s_0)$ can be non-zero only if $s_0$ is at the start or end of the path, in which case the unsigned curvature $|\kappa(s)|$ is also strictly-monotonic, as demonstrated below.

Lemma 2. Under Assumption 1, if $s_0 \in (0, L)$, then $\kappa(s_0) = 0$.

Proof. Suppose $s_0 \in (0, L)$ and $|\kappa(s_0)| > 0$. Since $\kappa$ is continuous, there exists a neighbourhood $U \subset (0, L)$ of $s_0$ such that $|\kappa(s)| > 0$ for all $s \in U$. The strict monotonicity of $\kappa$ then implies the existence of some $s_1 \in U$ such that $|\kappa(s_1)| < |\kappa(s_0)|$, which contradicts the definition of $s_0$.

Corollary 1. Under Assumption 1, if $\kappa(s_0) \neq 0$, then $s_0 \in \{0, L\}$.

Remark 4. The map $s \mapsto |\kappa(s)|$ is strictly decreasing iff $s_0 = L$, and strictly increasing iff $s_0 = 0$.

The maximum speed-squared permitted by the normal acceleration constraint at $s \in [0, L]$ is given by $\xi(s) := C|\kappa(s)|^{-1}$. Below is an expression for its derivative over the initial portion of the path $[0, s_0)$, which has decreasing unsigned curvature.
Lemma 3. Suppose $s_0 > 0$ and let $\xi : [0, s_0) \to \mathbb{R}$, $\xi(s) := \frac{C|\kappa(s)|}{|\kappa(s)|}$. Under Assumption 1,

$$\forall s \in [0, s_0), \xi'(s) = \frac{C|\kappa'(s)|}{|\kappa(s)|^2}.$$  

Proof. Applying the chain rule, $\xi'(s) = -\text{sgn}(\kappa(s)) \frac{C\kappa'(s)}{|\kappa(s)|^2}$ for all $s < s_0$. By Lemma 1,

$$\forall s < s_0, |\kappa(s)| > |\kappa(s_0)| \geq 0. \quad (8)$$

Suppose $\kappa$ is strictly increasing. Then $\kappa(s) < 0$ for all $s < s_0$, by which $-\text{sgn}(\kappa(s)) = 1$. Moreover, $\kappa'(s) \geq 0$, which implies $\kappa'(s) = |\kappa'(s)|$. If, instead, $\kappa$ is strictly decreasing, then $\kappa'(s) = -|\kappa'(s)| \leq 0$, and $\kappa(s) > 0$ for all $s < s_0$, by which $-\text{sgn}(\kappa(s)) = -1$. In both cases, $-\text{sgn}(\kappa(s))\kappa'(s) = |\kappa'(s)|$ for all $s < s_0$, as claimed.

4.2. Forward sweep

In this subsection, the objects generated by Algorithm 1 during the forward sweep

$$x_F := \text{Sweep}(\kappa, A, C, v_0),$$

are examined under Assumptions 1 and 2. Lines 3 – 19 of Algorithm 1 yield the chain of inequalities

$$\forall k \in [0 : N], \quad 0 \leq c_k \leq a_k \leq c_{k+1}, \quad (9)$$

with Assumption 1 guaranteeing the existence of

$$N := \max\{k \mid c_k < \infty\}, \quad (10)$$

the index of the largest finite $c_k$. By this definition, $c_k, a_k \in [0, s_0)$ for each $k \leq N$, $C_{N+1} = \infty$, and $a_k \in [0, s_0) \cup \{\infty\}$. To facilitate the subsequent analysis, define

$$\eta : [0, L] \to [0 : N], \quad \eta(s) := \max\{k \in [0 : N] \mid c_k \leq s\}, \quad (11)$$

which returns the unique index $\eta(s) = k$ such that $s \in [c_k, c_{k+1})$. Line 20 can then be written as

$$x_F(s) = \begin{cases} 
C|\kappa(s)|^{-1}, & \text{if } s \in \bigcup_{k=0}^{N} [c_k, a_k) \\
x_{\eta(s)}(s), & \text{if } s \in \bigcup_{k=0}^{N} (c_k, c_{k+1})
\end{cases} \quad (12)$$

$$= \begin{cases} 
C|\kappa(s)|^{-1}, & \text{if } s \in \bigcup_{k=0}^{N} [c_k, a_k) \\
x_0(s), & \text{if } s \in (a_0, c_1) \\
x_1(s), & \text{if } s \in (a_1, c_2) \\
\vdots & \vdots \\
x_N(s), & \text{if } s \in (a_N, c_{N+1})
\end{cases} \quad (13)$$

Remark 5 (Non-negativity). For all $s$, $C|\kappa(s)|^{-1} > 0$. Moreover, since $A > 0$, every $x_k(s) \geq 0$ for all $s \geq a_k$, by Lines 6 and 13. This makes it clear from (13) that $x_F$ is non-negative, taking a possibly infinite value at $s_0$ when $\kappa(s_0) = 0$.

First consider the output of Algorithm 1 when $s_0 = 0$, which is the simplest case.

Lemma 4. Suppose Assumption 1 holds and $s_0 = 0$. If $v_0^2 < C|\kappa(0)|^{-1}$, then

$$\forall s \in [0, L], \quad x_F(s) = v_0^2 + 2As.$$  

Otherwise,

$$\forall s \in [0, L], \quad x_F(s) = C|\kappa(s)|^{-1}. \quad (14)$$

Proof. If $v_0^2 < C|\kappa(0)|^{-1}$, then $a_0 = 0$ by Line 5, and $c_1 = \infty$ by Line 7, because $[0, s_0) = \emptyset$. Otherwise, $c_0 = 0$, and $a_0 = \infty$ by Line 11. In both cases, Assumption 2 is satisfied automatically, and the result follows from (12).

Now consider the case $s_0 > 0$. First, a technical result.

Lemma 5. Suppose Assumption 1 holds and $s_0 > 0$. If $c_k < \infty$, then $a_k < s_0$ or $\kappa(s_0) = 0$.

Proof. Suppose that $c_k < \infty$ and $\kappa(s_0) = 0$. If $k = 0$, then $0 = c_0 < s_0$ by hypothesis; otherwise $k > 0$, and $c_k < s_0$ by Lines 7 and 14. Either way, $c_k < s_0$. Recalling $\xi : [0, s_0) \to \mathbb{R}$ defined in Lemma 3, $\limsup_{s \to s_0}\xi(s) = \limsup_{s \to s_0}C|\kappa(s)|^{-1} = \infty$ by Proposition 1, and thus, $a_k < s_0$ by Lines 5 and 11.

Recall that $x_F$ enters the curvature constraint boundary at each $c_k$ and leaves it at $a_k$. In view of Lemma 3, the tangential acceleration does not exceed $A$ while travelling across this boundary.

Lemma 6. Suppose Assumption 1 holds and $s_0 > 0$. Then $C|\kappa(s)|^{-1} \leq 2A$ for all $s \in [c_k, a_k) \cap [0, L]$. Moreover, if $c_k < a_k < s_0$, then $C|\kappa'(a_k)|^{-1} = 2A$.

Proof. Consider the following cases.

- If $c_k = a_k$, then $[c_k, a_k) = \emptyset$, and the result holds vacuously.
- If $c_k < a_k < s_0$, then the result follows from Proposition 2 with $f(s) := C|\kappa'(s)|^{-1} - 2A$, which derives its continuity from that of $\kappa$.

- Suppose $c_k < a_k = \infty$. Then Line 11 gives

$$\forall s \in [c_k, s_0), \quad \frac{C|\kappa'(s)|}{|\kappa(s)|^2} \leq 2A. \quad (14)$$

Moreover, since $s_0 > 0$, it follows from Lines 3 and 14 that $c_k < s_0 \in [0, L]$. Further, since $a_k = \infty$ here, $\kappa(s_0) \neq 0$ by Lemma 5, and thus $s_0 = L$ by Corollary 1, which establishes the result.

This exhausts all possibilities for the values of $c_k$ and $a_k$.

The normal acceleration limit $C$ is also not exceeded while travelling at maximum tangential acceleration between $a_k$ and $c_{k+1}$.
Lemma 7. Suppose Assumption 1 holds and \( s_0 > 0 \). Then \( x_k(s) \leq C|\kappa(s)|^{-1} \) for all \( s \in [a_k, c_{k+1}] \cap [0, s_0] \). Moreover, if \( a_k < c_{k+1} < s_0 \), then \( x_k(c_{k+1}) = C|\kappa(c_{k+1})|^{-1} \).

Proof. Consider the following cases.

- If \( a_k = c_{k+1} \), then \([a_k, c_{k+1}] = \emptyset\), and the result holds vacuously.
- Suppose \( a_k < c_{k+1} < s_0 \). Lines 7 and 14 can both be written as
  \[ c_{k+1} = \inf \{ s \in [a_k, s_0] : |x_k(s) - C|\kappa(s)|^{-1}| > 0 \} \]
  Proposition 2 then implies that \( x_k(s) - C|\kappa(s)|^{-1} \leq 0 \) for all \( s \in [a_k, c_{k+1}] \), and that \( x_k(c_{k+1}) = C|\kappa(c_{k+1})|^{-1} = 0 \), as claimed.
- If \( a_k < c_{k+1} = \infty \), then \( a_k = s_0 \) by Lines 5 and 11, whereby Lines 7 and 14 give
  \[ \forall s \in [a_k, s_0], \ x_k(s) \leq C|\kappa(s)|^{-1}. \] (15)
  Since \([a_k, s_0] = [a_k, c_{k+1}] \cap [0, s_0] \), it only remains to check the inequality at \( s_0 \). If \( \kappa(s_0) = 0 \), then clearly \( x_F(s_0) = C|\kappa(s_0)|^{-1} = \infty \). Otherwise, (15) implies
  \[ x_k(s_0) = \lim_{s \to s_0} x_k(s) \leq \lim_{s \to s_0} C|\kappa(s)|^{-1} = C|\kappa(s_0)|^{-1}, \]
  because \( \kappa \) and \( x_k \) are continuous.

Lines 5, 7, 11 and 14 permit no other possibilities for the values of \( a_k \) and \( c_{k+1} \).

It follows that \( x_F \) satisfies the curvature constraint throughout the portion of the path with strictly decreasing unsigned curvature.

Corollary 2. Under Assumptions 1–2, \( x_F : [0, L] \to [0, \infty] \) satisfies

\[ \forall s \in [0, s_0], \ x_F(s) \leq C|\kappa(s)|^{-1}. \] (16)

Proof. If \( s_0 = 0 \), then \([0, s_0] = \{0\}\), and the result follows from Lemma 4. Suppose \( s_0 > 0 \), and choose any \( s \in [0, s_0] \). Let \( k := \eta(s) \), where \( \eta \) is defined in (11). Then \( s \in [c_k, c_{k+1}] = [c_k, a_k] \cup [a_k, c_{k+1}] \). If \( s \in [c_k, a_k] \), then \( x_F(s) = C|\kappa(s)|^{-1} \) by Line 20. Otherwise, \( s \in [a_k, c_{k+1}] \), and \( x_F(s) \leq C|\kappa(s)|^{-1} \) by Lemma 7.

The boundary condition (4) is also satisfied by \( x_F \).

Lemma 8. Under Assumptions 1–2, \( x_F(0) \leq v_0^2 \).

Proof. If \( C|\kappa(0)|^{-1} \leq v_0^2 \), then the result follows from Corollary 2. Suppose, instead, that \( v_0^2 < C|\kappa(0)|^{-1} \). Then \( a_0 = 0 \) by Line 5. If \( c_1 = 0 \), then Proposition 2 implies \( C|\kappa(0)|^{-1} = C|\kappa(c_1)|^{-1} \leq x_0(c_1) = v_0^2 \), which is a contradiction. Thus, \( c_1 > 0 \), \( 0 \in [a_0, c_1] \), and in view of (13), \( x_F(0) = x_0(0) = v_0^2 \) by Line 6.

The next result establishes, among other things, that under conditions on \( s_0 \) and \( v_0 \), the forward speed-squared profile \( x_F \) satisfies the tangential acceleration constraint wherever its derivative exists.

Lemma 9. Suppose Assumptions 1 and 2 hold. If \( s_0 > 0 \) and \( v_0^2 < C|\kappa(0)|^{-1} \), then \( x_F \) is real-valued, Lipschitz, strictly increasing, and

\[ \forall s \in (0, L) \setminus \bigcup_{k=0}^N \{c_k, a_k\}, \ 0 \leq x_F'(s) \leq 2A. \] (17)

Proof. If \( v_0^2 < C|\kappa(0)|^{-1} \) and \( s_0 = 0 \), then the result follows immediately from the first part of Lemma 4, whereby \( x_F'(s) = 2A > 0 \) for all \( s \in [0, L] \).

Suppose, now, that \( s_0 > 0 \). From Lines 6 and 13 of Algorithm 1, \( x_k \) is only defined if \( a_k < \infty \), by which \( a_k < s_0 \). Since \( \kappa(s) \neq 0 \) for any \( s \neq s_0 \), every \( x_k \) is real-valued. If \( \kappa(s_0) \neq 0 \), then \( C|\kappa(s)|^{-1} \leq \infty \) for all \( s \in [0, L] \), and so it follows from Line 20 that \( x_F(s) < \infty \) for all \( s \in [0, L] \). Suppose, instead, that \( \kappa(s_0) = 0 \), and choose any \( s \in [0, L] \). Then \( s \in [c_k, c_{k+1}] \), where \( \kappa := \eta(s) \). If \( s \in [c_k, a_k] \), then \( c_k < \kappa < a_k < s_0 \) by Lemma 5, and \( x_F(s) = C|\kappa(s)|^{-1} \) by Line 20. Otherwise, \( s \in ([a_k, c_{k+1}], \) in which case \( x_F(s) = x_k(s) \in \mathbb{R} \). This establishes that \( x_F \) is real valued. The rest of the proof establishes the other properties claimed.

If \( s \in (0, L) \setminus \bigcup_{k=0}^N \{c_k, a_k\} \), then either \( s \in (c_k, a_k) \) or \( s \in (a_k, c_{k+1}) \), where \( k = \eta(s) \). The latter implies that \( x_F'(s) = x_k'(s) = 2A > 0 \) by (13). In the case of the former, \( s < a_k < s_0 \) by Lemma 5, and therefore \( x_F'(s) = \frac{C|\kappa(s)|}{\kappa(s)} \in [0, 2A] \) by Lemmas 3 and 6. This establishes (17).

It is clear \( x_F \) is piecewise continuous, because \( \kappa \) is continuous and so is every \( x_k \). All that remains is to test \( x_F \) for continuity at \( c_1, \ldots, c_N \) and \( a_0, \ldots, a_{N-1} \), and finally at \( a_N \) if finite. From (12), \( x_F \) is right-continuous at every finite \( c_k \) and \( a_k \). If \( a_k < c_{k+1} < \infty \), then by Lemma 7,

\[ \lim_{s \to c_{k+1}} x_F(s) = x_k(c_{k+1}) = C|\kappa(c_{k+1})|^{-1} = x_F(c_{k+1}), \]

which establishes left-continuity at \( c_1, c_2, \ldots, c_N \). If \( c_k < a_k \), then

\[ \lim_{s \to a_k} x_F(s) = C|\kappa(a_k)|^{-1} = x_k(a_k) = x_F(a_k), \]

which establishes left-continuity at every finite \( a_k \). Thus \( x_F \) is continuous.

To demonstrate that \( x_F \) is Lipschitz, consider the right-derivative \( x_F^+(s) \), which is equal to \( x_k'(s) \) wherever the latter exists. By (17), \( x_F^+(s) \in [0, 2A] \) for all \( s \in (0, L) \setminus \bigcup_{k=0}^N \{c_k, a_k\} \). The existence of \( x_F^+(s) \) at \( s \in \bigcup_{k=0}^N \{c_k, a_k\} \cap [0, L] \) is now verified. If \( a_k < c_{k+1} \), (13) implies \( x_F^+(a_k) = x_k'(a_k) = 2A \). Similarly, if \( c_k < a_k \), then \( c_k < s_0 \), and with \( \xi : [0, s_0) \to \mathbb{R} \) defined as in Lemma 3,

\[ x_F^+(c_k) = \xi(c_k) = \frac{C|\kappa(c_k)|}{\kappa(c_k)^2} \in [0, 2A], \]
by Lemma 6. This establishes that \(x_F^+(s)\) exists and is bounded for all \(s \in [0, L)\), which implies \(x_F\) is Lipschitz by [20, Proposition 2.2.3].

Finally, strict monotonicity is confirmed. Every \(x_k\) is strictly increasing because \(A > 0\), and \(|\kappa(s)|\) is strictly decreasing over \([0, s_0)\). If \(s_0 = L\), then clearly

\[
[0, L] \cap \bigcup_{k=0}^{N} [c_k, a_k] \subset [0, s_0].
\]  

(18)

Otherwise \(s_0 \in (0, L)\), and \(\kappa(s_0) = 0\) by Lemma 2. Since \(c_N < \infty\) by definition in (10), Lemma 5 then implies \(a_N < s_0 < L\), and therefore (18) still holds. Thus, \(C|\kappa(s)|^{-1}\) is strictly decreasing over \([0, L] \cap \bigcup_{k=0}^{N} [c_k, a_k]\). Since the continuity of \(x_F\) has been established, and its piecewise definition consists of only strictly decreasing functions, it follows that \(x_F\) is strictly increasing.

\(\square\)

4.3. Reverse sweep

Recall \(\gamma : [0, L] \to [0, L]\), \(\gamma(s) = L - s\). If \(\kappa\) satisfies Assumption 1, then so does \(\kappa \circ \gamma\), where \(\gamma\) reverses direction along the path. The profile

\[
x_R := \text{Sweep}(\kappa \circ \gamma, B, C, v_L) \circ \gamma
\]  

(19)

is obtained by applying the smooth co-ordinate transformation \((s \mapsto L - s)\) to both input and output of Algorithm 1. The relevant properties of \(x_R\) follow directly from those of \(x_F\) established previously. To avoid confusion, internal variables of Algorithm 1 generated by the reverse sweep (19) are adorned with a tilde. For example,

\[
\tilde{s}_0 := \arg \min_{s \in [0, L]} |\kappa \circ \gamma(s)| = \left(\gamma(s_0) = L - s_0\right).
\]

where \(s_0 := \arg \min_{s \in [0, L]} |\kappa(s)|\), as per (7). The next four results are corollaries of Lemma 4, Corollary 2, Lemma 8, and Lemma 9, respectively.

**Corollary 3.** Suppose Assumption 1 holds, and \(s_0 = L\). If \(v_L^2 < C|\kappa(L)|^{-1}\), then

\[
\forall s \in [0, L], \ x_R(s) = v_L^2 + 2B(L - s).
\]

Otherwise,

\[
\forall s \in [0, L], \ x_R(s) \in C|\kappa(s)|^{-1}.
\]

**Corollary 4.** Under Assumptions 1 and 3, \(x_R : [0, L] \to [0, \infty]\) satisfies

\[
\forall s \in [s_0, L], \ x_R(s) \leq C|\kappa(s)|^{-1}.
\]

**Corollary 5.** Under Assumptions 1 and 3, \(x_R(L) \leq v_L^2\).

**Corollary 6.** Suppose Assumptions 1 and 3 hold. If \(s_0 < L\) or \(v_L^2 < C|\kappa(L)|^{-1}\), then \(x_R\) is real-valued, Lipschitz, strictly decreasing, and

\[
\forall s \in (0, L) \backslash \bigcup_{k=0}^{N} \{L - a_k, L - c_k\}, \ -2B \leq x_R'(s) \leq 0.
\]

**Proof.** The map \(\gamma\) is Lipschitz, and the composition of Lipschitz functions remains Lipschitz [20, Proposition 2.3.1].

\(\square\)

4.4. Feasibility and global optimality

Recall the definition of \(x^*\) in Theorem 1:

\[
x^* : [0, L] \to [0, \infty), \ x^*(s) := \min\{x_F(s), x_R(s), V^2\}.
\]  

(20)

The feasibility of \(x^*\) is established first, followed by global optimality.

**Lemma 10** (Feasibility). Under Assumptions 1 - 3, \(x^*\) in Theorem 1 is Lipschitz and feasible.

**Proof.** By definition, \(x^*(s) \leq V^2\) for all \(s \in [0, L]\). Moreover, \(x^* : [0, L] \to [0, \infty]\), because \(x_F, x_R : [0, L] \to [0, \infty]\) and \(0 < V < \infty\).

To verify (4) - (9), \(x^*(0) \leq x_F(0) \leq v_0^2\) by Lemma 8, and \(x^*(L) \leq x_R(L) \leq v_L^2\) by Corollary 5. Now choose any \(s \in [0, L]\). If \(s \leq s_0\), then \(x^*(s) \leq x_F(s) \leq C|\kappa(s)|^{-1}\) by Corollary 2. Otherwise, \(s > s_0\), and \(x^*(s) \leq x_R(s) \leq C|\kappa(s)|^{-1}\) by Corollary 4. Thus, (6) is verified.

To verify (3) and absolute continuity of \(x^*\), consider the following cases.

• Suppose \(s_0 = 0\) and \(v_0 \geq C|\kappa(0)|^{-1}\). Then \(x_F(s) = C|\kappa(s)|^{-1}\) for all \(s \in [0, L]\), by Lemma 4. Since \(x_F(s) \leq C|\kappa(s)|^{-1} = x_F(s)\) for all \(s \in [0, L]\) by Corollary 4,

\[
\forall s \in [0, L], \ x^*(s) = \min\{x_R(s), V^2\},
\]  

(21)

where \(x_R\) is Lipschitz and strictly decreasing by Corollary 6, with \(x_R(s) \in [-2B, 0]\) almost everywhere.

• Suppose \(s_0 = L\) and \(v_L \geq C|\kappa(L)|^{-1}\). Then \(x_R(s) = C|\kappa(s)|^{-1}\) for all \(s \in [0, L]\) by Corollary 3. Since \(x_F(s) \leq C|\kappa(s)|^{-1} = x_R(s)\) for all \(s \in [0, L]\) by Corollary 2,

\[
\forall s \in [0, L], \ x^*(s) = \min\{x_F(s), V^2\},
\]  

(22)

where \(x_F\) is Lipschitz and strictly increasing by Lemma 9, with \(x_F^+(s) \in [0, 2A]\) almost everywhere.

• Otherwise, \(s_0 \in (0, L)\) or \(v_0 < C|\kappa(0)|^{-1}\) or \(v_L < C|\kappa(L)|^{-1}\). Then \(x_F\) is strictly increasing with \(x_F^+(s) \in [0, 2A]\) almost everywhere by Lemma 9, and \(x_R\) strictly decreasing with \(x_R'(s) \in [-2B, 0]\) almost everywhere by Corollary 6. Both \(x_F\) and \(x_R\) are Lipschitz. Equation (20) can be written as \(x^*(s) = \min\{x_R(s), g(s)\}\), where \(g(s) := \min\{x_F(s), V^2\}\) is non-decreasing.

Applying Proposition 3 to each case, \(x^*\) is Lipschitz and satisfies (3) almost everywhere. All real Lipschitz functions over a compact domain are absolutely continuous [20, Proposition 3.3.3].

**Lemma 11.** Suppose Assumptions 1 and 2 hold, and let \(x : [0, L] \to \mathbb{R}\) be absolutely continuous. If

\[
x(0) \leq v_0^2,
\]  

(23)

\[
\forall s \in [0, L], \ x(s) \leq C|\kappa(s)|^{-1},
\]  

(24)

and \(x'(s) \leq 2A\) for almost every \(s \in [0, L]\), then \(x(s) \leq x_F(s)\) for all \(s \in [0, L]\).
Proof. Choose any \( s \in [0, L] \). Then \( s \in [a_k, c_k+1) \), where \( k := \eta(s) \). If \( s \in [a_k, c_k) \), then (24) and (12) imply \( x(s) \leq C|\kappa(s)|^{-1} = x_F(s) \). Suppose, instead, that \( s \in [a_k, c_k+1) \). If \( k = 0 \) and \( v_0^2 < C|\kappa(0)|^{-1} \), then \( a_k = 0 \) by Line 5, and (23) implies
\[
x(a_0) = x(0) \leq v_0^2 = x_0(0),
\]
by Line 6. Otherwise, (24) implies
\[
x(a_k) \leq C|\kappa(a_k)|^{-1} = x_k(a_k),
\]
by Line 13. Either way, \( x(a_k) \leq x_k(a_k) \). Since \( x \) is absolutely continuous with \( x'(s) \leq 2A \) almost everywhere,
\[
x(s) = x(a_k) + \int_{a_k}^{s} x'(\tau) \, d\tau \\
\leq x_k(a_k) + 2 \int_{a_k}^{s} \, A \, d\tau \\
= x_k(a_k) + 2A(s - a_k) \\
= x_k(s) = x_F(s),
\]
by (13), recalling that \( s \in [a_k, c_k+1) \).

\textbf{Corollary 7.} Suppose Assumptions 1 and 3 hold, and let \( x : [0, L] \to \mathbb{R} \) be absolutely continuous. If
\[
x(L) \leq v_L^2, \\
\forall s \in [0, L], x(s) \leq C|\kappa(s)|^{-1},
\]
and \( x'(s) \geq -2B \) for almost every \( s \in [0, L] \), then \( x(s) \leq x_R(s) \) for all \( s \in [0, L] \).

\textbf{Proof.} Recall \( \gamma(s) := L - s \). If \( x \) satisfies the hypotheses, then \( x \circ \gamma \) satisfies \( (x \circ \gamma)'(s) \leq 2B \) almost everywhere, \( x \circ \gamma(0) \leq v_L^2 \), and (24). Thus \( x \circ \gamma(s) \leq x_F(s) \) for all \( s \) by Lemma 11, where \( x_F(s) := \text{Sweep}(\kappa \circ \gamma, B, C, v_L) \). Since \( x_R = x_F \circ \gamma \), the result follows.

\textbf{Corollary 8. Under Assumptions 1–3, for any feasible \( x \), \( x(s) \leq x^*(s) \) for all \( s \in [0, L] \).}

\textbf{Proof.} This follows directly from Lemma 11, Corollary 7, Definition 1 and (20).

\textbf{Corollary 9 (Global optimality). Under Assumptions 1–3, \( J(x) \geq J(x^*) \) for any feasible \( x \).}

\textbf{Proof.} If \( x : [0, L] \to \mathbb{R} \) is feasible, Corollary 8 implies \( x(s) \leq x^*(s) \) for all \( s \), and the result follows because the running cost \( \frac{1}{\sqrt{x'(s)}} \) in (2) is a strictly decreasing function of \( x(s) \).

Theorem 1 has been proved. Now consider tightening boundary conditions (4), (5), or both, to equality in Problem 1. If a solution exists, then such is \( x^* \).

\textbf{Corollary 10 (Equality-constrained boundary values). Under the hypotheses of Theorem 1:}

\begin{enumerate}
  \item[i)] If there exists a feasible speed-squared profile \( x \) with \( x(0) = v_0^2 \), then \( x^*(0) = v_0^2 \).
  \item[ii)] If there exists a feasible speed-squared profile \( x \) with \( x(L) = v_L^2 \), then \( x^*(L) = v_L^2 \).
\end{enumerate}

\textbf{Proof.} By Corollary 8, \( x(0) \leq x^*(0) \leq v_0^2 \) and \( x(L) \leq x^*(L) \leq v_L^2 \). \qed

5. Numerical example

Figure 1 depicts the cubic path
\[
r : [-1, 1] \to \mathbb{R}^2, \quad r(\tau) = \left[ \frac{\beta \tau}{\tau^3} \right],
\]
with \( \beta := 3\sqrt{3} \) chosen to ensure strictly increasing signed curvature over its domain. It is an example of a low-order clothoid approximation. Although \( r \) is not parametrised by arc-length, the re-parametrisation
\[
\left( \tau \mapsto s = \int_{-1}^{\tau} \|r'(y)\| \, dy \right)
\]
can be performed numerically, before invoking Theorem 1 to obtain optimal speed and tangential acceleration profiles. The profiles are plotted in Figure 2, under the stated speed and acceleration limits, and for a variety of boundary conditions. As is characteristic of time-optimal control, at every point along the path, some constraint is active. The simplest profile, obtained by setting both boundary speeds to zero, is plotted in Figure 2a. Only a single switch is required, from maximum tangential acceleration to maximum deceleration. In Figure 2b, the initial speed is increased, causing the speed profile to make contact with the normal acceleration and maximum speed boundaries. The profile in Figure 2c is obtained by leaving both boundary speeds free (see Remark 2). The profile with the greatest number of switches is plotted in Figure 2d. It begins at maximum tangential acceleration and finishes at maximum deceleration, to achieve the boundary speeds imposed.

6. Conclusion

Problem 1 encodes the minimum-time speed planning task subject to limits on tangential acceleration and deceleration, normal acceleration, and speed. Boundary value constraints can be removed by setting sufficiently high boundary speeds. Theorem 1 presents an exact solution for paths with smooth strictly-monotonic signed curvature, under the mild regularity conditions in Assumptions 2 and 3. Similar regularity conditions appear in \cite{2, 3, 6}. The solution is constructed explicitly by invoking Algorithm 1 in two sweeps. The forward sweep progresses in the direction of increasing path parameter, to yield a speed-squared profile that switches between the maximum tangential acceleration and normal acceleration boundaries. The reverse
Figure 1: Cubic path with strictly increasing signed curvature.

Figure 2: Time-optimal speed and tangential acceleration profiles (for $A = 1.5 \text{ m/s}^2$, $B = 2 \text{ m/s}^2$, $C = 1 \text{ m/s}^2$, $V = 5 \text{ m/s}$).
sweep proceeds in the opposite direction to give a speed-squared profile that switches between the maximum deceleration and normal acceleration boundaries. The point-wise minimum of the two profiles and the speed-squared limit is globally time-optimal. In particular, any feasible speed-squared profile is uniformly less than or equal to this optimiser, thereby taking at least as much time to traverse. All paths with strictly-monotonic unsigned curvature have strictly-monotonic signed curvature, and can therefore be treated as special cases. Future work will address the decomposition of general paths into segments of monotone curvature, and the choice of boundary conditions to stitch the segment profiles together without losing exactness or optimality.

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