UNRIVALED QUANTUM VACUUM IN THE PRIMORDIAL UNIVERSE

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ABSTRACT

In a primordial universe after the inflation ended, there could be phases of early universe made of cold gas baryons, radiation and early post inflationary cosmological constant. I showed that in the baryonic epoch, the quantum vacuum been unique. By using the standard quantization scheme for a massive minimally coupled scalar field in the classical conformal flat spacetime I demonstrated that the scalar modes posses vanish (or uniform) effective mass $m_{\text{eff}}^2 \approx 0$ (or $m_{\text{eff}}^2 \approx \text{constant}$). This argument is validating when the conformal time is keeping close to the inflation ending time $\eta = \eta_c$. The energy density of the baryonic matter diverges at the inflation border and vanishes at the late time future. Furthermore I argue that at very early accelerating epoch when the radiation was the dominant part in the close competition with the early time cosmological constant, fine tuned mass of the scalar field $m \propto \sqrt{\Lambda}$ also provides a unique quantum vacuum. The reason is vanishing effective mass similar to the cold baryonic epoch. A remarkable observation is that all the other possible vacuum states “squeezed” eternally.

Keywords First keyword · Second keyword · More

Introduction

Vacuum in quantum field theory in curved spacetimes doesn’t have a unique definition. As a standard method in a typical model of scalar field on curved background, once we determine the pair of creation and annihilation operators defined as $a^{-\dagger}_k$ in the momentum space $k$, the possible (and not the physical) vacuum state $|0>$ is defined as the eigenstate of all annihilation operators $a^{-}_k|0> = 0$, $\forall k$. The other alternative approaches to find the vacuum state are the minimization of instantaneous energy or minimize the average for a certain period of time or the number of particles with respect to some other vacuum states. In addition to the commutation relations between operators we also need to choose an appropriately selection of the mode functions. If we find two sets of annihilation operator, we will have two copies of the corresponding vacua. As a result, the particle interpretation is strictly depending on the choice of mode functions. This statement goes back mainly to the non existence of a plane wave as a general solution for a simple scalar field wave equation in an arbitrary curved spacetime. Only in the Minkowski , mode frequencies are time-independent and we can recognize a unique basic function (the plane waves) as quantum vacuum and only in Mikowski spacetime this mode function remains the vacuum mode at all times. Here it raising an interesting question : does exist any non flat cosmological spacetime with only one possible vacuum state?

In this letter I addressed this question by considering a very specific cosmological model in the primordial universe. This era initiated after the inflation ended. We suppose that it could be made of a cold gas of baryons. It was Zel’dovich who showed that the general equation of the state (EoS) for such Universe can be described by the stiff matter era $p = \rho$, where $p$ is the pressure and $\rho$ is the energy density [1],[2]. The cosmological epoch can be considered as the

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ultra-relativistic limit of a general class of the EoS for a fluid at \( T = 0 \) (see \cite{3,4} for a complete analysis of exact cosmological solutions in this family of fluids). The aim is to quantize a simple scalar field theory on the classical background arisen from the Zel’dovich’s matter content. The quantization scheme completed in short time interval after the inflation ended. Furthermore, I investigate a mixed phase of early universe, when the universe filled by the radiation and undergoes an acceleration expansion. To take into the account the post inflationary acceleration I consider cosmological constant term. As we knew, cosmological constant term \( \Lambda \) is given as the vacuum energy for the vacuum spacetime. A simple reason to consider \( \Lambda \) as a potential candidate for vacuum is the energy-momentum tensor of vacuum expressed in the unique Lorentz invariant form as \( T_{\mu \nu} = \Lambda g_{\mu \nu} \) here \( g_{\mu \nu} \) defines the Lorentzian metric for curved manifold of the cosmological spacetime. Such early time cosmological constant appeared in the induced gravity scenario in the form of an regulated formulae for \( \Lambda \) (see \cite{5} for modern translation of this idea). The aim is to prove that quantum vacuum has a unique definition for both baryonic and radiation-cosmological constant epochs. I use the simple quantization method for massive non minimally couple scalar field on the classical background of the conformally flat metric. The scale factor governs by the Einstein field equation and the scalar field equation of motion simple reduce to the wave equation with effective potential. In the first example I show that cold baryonic matter dominated post inflationary epoch provides a negligible effective mass at cosmological time close to the ending of inflation. Consequently the eigenmodes are reduced to Minkowskian flat modes. In the second example I investigate the effective mass for radiation-\( \Lambda \) post inflationary model. It also gives us a tiny effective mass for some values of the scalar field mass. I prove that if the mass of scalar field opted as the fine tuned \( \Lambda^{1/2} \), then the effective mass again vanishes. As a result we will end by a unique quantum vacuum.

**Geometry for cold gas baryon phase of the early universe**

To have proceed, we have to opt our geometry of the spacetime. We are writing down the set of Friedmann equations for curved, homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime characterized by the metric in conformal time \( \eta \):

\[
    ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2) \tag{1}
\]

where \( x^\mu \equiv (\eta, \vec{x}) \). Because our scalar field started after the inflation ended, we suppose that \( \eta \geq \eta_c \). We end to the pair of nonlinear second order differential equations for scale factor and continuity equation,

\[
    \left( \frac{a'}{a} \right)^2 = \frac{\kappa^2}{3} a^2 \rho(\eta) \tag{2}
\]

\[
    \rho' + \frac{6a'}{a} \rho(\eta) = 0 \tag{3}
\]

where the prime ‘\( \prime \)’ denotes derivative with respect to \( \eta \) and \( \kappa^2 = 8\pi G \) is gravitational coupling constant. To compare it with the standard FLRW equation in the cosmological time \( t \), we remember that the cosmological Hubble parameter \( H = \frac{\dot{a}}{a} \) turns to the \( H = \frac{a'}{a} = \frac{\dot{H}}{a} \) here \( H \) we mean Hubble parameter in the conformal time \( \eta \) defined as \( H = \frac{a'}{a} \). The \( \eta-\eta \) component for Ricci tensor \( R_{\eta\eta} = -3(\ln a)'' \) and the Ricci scalar \( R = -\frac{6}{a'}((\ln a)'' + (\ln a)'^2) \), consequently the \( \eta\eta \) component of the Einstein tensor \( G_{\eta\eta} = R_{\eta\eta} - \frac{R}{2} g_{\eta\eta} = 3(\ln a)^2 \), furthermore the \( \eta-\eta \) component for the energy-momentum tensor component \( T_{\eta\eta} = g_{\eta\eta} \rho(\eta) \) (see \cite{6} for a complete study of the geometrical quantities for a conformal cosmological metric). With these expressions we can find the first Friedmann equation as given in eq. (2).

We notify that the continuity equation doesn’t change by passing from cosmological to the conformal time.

If we integrate the second differential equation, we obtain

\[
    \rho(\eta) = \rho_c \left( \frac{a_c}{a} \right)^6 \tag{4}
\]

here \( \rho_c = \rho(a_c) = \rho(\eta = \eta_c) \) denotes the value of density at the inflation ending time. Note that at the early epoch this baryonic matter is the dominant part at least for short time interval i.e. when \( \eta \rightarrow \eta + \epsilon + \delta \eta \) where \( \delta \eta \ll \eta_c \). By plugging it into the eq. (2) we can integrate the latter differential equation. The general exact solution for the scale factor can be obtained

\[
    a(\eta) = \sqrt{\frac{2\sqrt{\rho_c a_c^3}}{3}} |\eta - \eta_c|^2. \tag{5}
\]

The short early universe covered by \( \eta \rightarrow \eta_c \) when \( a \rightarrow a_c \) and late future \( a \rightarrow \infty \) will be happen at time \( \eta \rightarrow \infty \). Note that the density function eq. (4) behaves as \( \rho(\eta) \propto |\eta - \eta_c|^{-3} \) diverges at early times \( \eta \in (\eta_c, \eta_c + \delta \eta) \) and has finite

\[
    a(\eta) = \sqrt{\frac{2\sqrt{\rho_c a_c^3}}{3}} |\eta - \eta_c|^2. \tag{5}
\]
value \( \rho(\eta) \propto \eta_c^{-3} \) at the present time and will vanish in future. If such phase of the early universe existed, contained the radiation era with \( p = \frac{1}{3} \rho \), the dust matter era \( p = 0 \), and the dark energy era \( p = -\rho \). A remarkable observation is that the baryonic matter density dominated over all the other contents. This type of stiff matter cosmological model is strictly connected to a scenario for early dark matter in which dark is composed of relativistic self-gravitating BECs. Note that in this cosmological model, the energy density of the stiff matter \( \rho \) can be positive or negative. The sign of energy density is depending on the type of self-interaction. Furthermore this cosmological era corresponds to a Universe filled a perfect fluid at zero temperature \( T = 0 \) (or some low regimes). Effective thermodynamics model for such Universe described by a polytropic EoS. If we determine the relation between energy density and the rest-mass, we observe that the EoS reduces to a stiff EoS \( p = \rho \) describes a cold gas of baryons in Zel’dovich model. A remarkable fact about Zel’dovich model is that the velocity of sound is equal to that of light. Consequently the model naturally avoid from any ghost or superluminal propagating mode.

It is illustrative to show that metric with the scale factor eq. (5) provides a positive variable (differ from the de Sitter scenario) on the flat space. Note that although our background metric is non flat but the reduced action still is defined on Minkowski spacetime and with vanishing effective mass gives massless scaleron reduces to the massless scalar field theory on a flat space. Note that our background metric with scale factor \( \eta \) is non flat and distinct from the de Sitter exact cosmological model. This metric provides our classical background where the quantum scalar field propagates. To investigate dynamics of the scalar field on this classical background we follow the method proposed in [7].

**Quantum Scalar field on classical homogeneous and isotropic backgrounds**

Let us start considering a minimally coupled scalar field \( \phi(x^\mu) \) in a curved cosmological background presented in eq. (1) with scale factor eq. (5), the action is

\[
S = \int \sqrt{-g} d^4x \left( \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - \frac{1}{2} m^2 \phi^2 \right). \tag{6}
\]

here \( \phi_{,\mu} = \partial_{\mu} \phi = \nabla_{\mu} \phi \) and \( m \) is mass of the scalar field (or scaleron). The field equation using an auxiliary field \( \chi = a(\eta) \phi \) reduces to the equation of motion for auxiliary field \( \chi \) in the Minkowski spacetime for a time dependent effective mass

\[
m_{eff}^2 = m^2 \frac{a''}{a}. \tag{7}
\]

By plugging the scale factor given in eq. (5) in the above equation we can show that for \( m = 0 \) we can vanish the effective mass at \( \eta = \eta_c \). Since the baryons produced after the inflation ended, the massless scalar field in our scenario lives on the border when time scale of inflation is identified as \( \eta_c \). As a result, scalar field theory with a massless scaleron reduces to the massless scalar field theory on a flat space. Note that although our background metric is non flat but the reduced action still is depending on Minkowski spacetime and with vanishing effective mass gives modes with uniform frequency and independent from the time \( \eta \). There is another possibility to vanish the effective mass for all times close to \( \eta - \eta_c \propto m^{-3/4} \) but we are interesting to have a scale invariant scenario with no specified time scale, we prefer a massive scalar field instead of finite flatness condition where the effective mass vanish at conformal time scales.

One can set \( m_{eff} \approx 0 \) and then by expanding the auxiliary field \( \chi \) in Fourier modes

\[
\chi(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \chi_k(\eta) e^{i\vec{k} \cdot \vec{x}}. \tag{8}
\]

the decoupled equations of motion for the mode \( \chi_k(\eta) \),

\[
\chi_k'' + k^2 \chi_k = 0. \tag{9}
\]

here \( k = |\vec{k}| \) (the Euclidean norm not the curved one!). This is a harmonic oscillator equation. All modes \( \chi_k(\eta) \) with \( k = |\vec{k}| \) are complex exponential solutions. If we choose a normalized mode function \( v_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2}} \) subjected to the normalization condition \( \sqrt{3}(v_k v_k^\dagger) = 1 \) (by * we mean the complex conjugate), we can show that the exact solution for \( \chi_k(\eta) \) can be expressed as

\[
\chi_k(\eta) = \frac{1}{\sqrt{2}} \left( a_k v_k^\dagger(\eta) + a_k^\dagger v_k(\eta) \right). \tag{10}
\]

Note that since \( \chi_k \) is a real function, \( \chi_k' = -\chi_k \) and \( a_k^\dagger = (a_k^-)^* \). A complete form of the solution for \( \chi(\vec{x}, \eta) \) can be represented as follow
\[ \chi_k(\vec{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{|k|} \left( a^+_k e^{i(k\eta - k \cdot \vec{x})} + a^-_k e^{-i(k\eta - k \cdot \vec{x})} \right) \] (11)

To make our scenario fully quantized we apply the real-time Heisenberg commutation relations between the field \( \hat{\chi} \) and its conjugate momentum \( \pi \)
\[ [\hat{\chi}(\vec{x}, \eta), \hat{\pi}(\vec{x}, \eta)] = i\delta(\vec{x} - \vec{y}) \] (12)
here \( \pi = \frac{\partial \chi}{\partial \eta} \). An alternative method is to quantize the field \( \chi \) via mode expansion:
\[ \begin{align*}
[a^-_k, a^+_k] &= \delta(k - k') \\
[a^-_k, a^-_k] &= 0.
\end{align*} \] (13)

Because the complex field \( \chi \) is a set of two real fields, we can also represent the field \( \chi \) in the form
\[ \chi_k(\vec{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{|k|} \left( a^+_k e^{-i(k\eta - k \cdot \vec{x})} + a^+_k e^{i(k\eta - k \cdot \vec{x})} \right) \] (15)

Now I will show that there is no other vacuum state for our scalar theory in this special spacetime. If we can find two sets of isotropic mode functions \( u_k(\eta), v_k(\eta) \), because \( u_k(\eta), u'_k(\eta) \) are a basis for our scalar theory (all the Bogolyubov coefficients vanish). We mention here that there will be more than one vacuum state if we let time coordinate \( \eta \rightarrow \infty \).

In the first time interval we have a unique vacuum because the mode functions are only a set. In the remaining partition of the interval it is possible to find more than one set of basis functions in any sub interval given by \( I_i = (\eta_i + \delta \eta, \eta_i, i\eta_i) \) subjected to the normalization condition. But still one can show that the total number of the particles will diverge at limit \( \lim_{\eta \rightarrow \infty} \). Now the question is why the other vacuum states are absent?. One can answer it by mentioning that in the Zeldovich’s epoch with scale factor given in eq. (5) the normalized mode functions \( \chi_k(\eta) \) admits solutions under the form of Hankel’s functions:
\[ \chi_k(\eta) = \frac{\sqrt{\eta \eta}}{2} (\epsilon_1 H_0^{(1)}(k|\eta - \eta_c|) + \epsilon_2 H_0^{(2)}(k|\eta - \eta_c|)) \quad \epsilon_{1,2} = 0, 1. \] (18)

The set of isotropic mode functions for \( \epsilon_2 = 0 \) is
\[ v'_k(\eta) = \frac{\sqrt{\eta \eta}}{4} e^{ik\eta} \left( H_0^{(1)}(k|\eta - \eta_c|) - iH_1^{(1)}(k|\eta - \eta_c|) \right). \] (19)
Post inflationary non baryonic unique vacuum

In previous section we showed that the cold early baryonic phase leads to a unique Minkowskian quantum vacuum. Another remarkable example of early unique vacuum state is a post inflationary epoch when the universe filled with the early radiation fields with energy density $\rho \propto a^{-4}$ along an early post inflationary accelerating expansion. A possible total energy density for this mixed early phase is given by

$$\rho = \frac{3\mu^2}{2\kappa^2} + \frac{3C}{\kappa^2 a^4}. \quad (20)$$

It is easy to show that the above energy density corresponds to the FLRW model with radiation field and an effective early cosmological constant $\Lambda \propto \mu^2$. Furthermore the effective mass given in eq. (19) could be vanish if the free parameter $\mu = m \propto \sqrt{\Lambda}$. A rough estimation for $\mu$ is given as $\mu \approx M_{\text{pl}}$ in natural units. Such scalar field can generate a post inflationary accelerating epoch with an effective cosmological constant $\Lambda$. Since $\Lambda$ is considered as a fine tuned constant, the mass for the scalar field also is fine tuned. A remarkable result is that post inflationary accelerated epoch could be described by a toy scalar field with a unique quantum vacuum. In addition to the former post inflationary cold baryonic phase we have also an additional radiation-accelerating phase with unique quantum vacuum. The question about which vacuum state survives from reheating and other phases after ending inflation can not be answered easily without knowing more about the quantum vacuum and calculating cosmological parameters. To have $m_{\text{eff}} = \text{constant}$ instead of making it vanish, we demand the scale factor be adjusted along the scalar mass $m$ and effective $m_{\text{eff}}$. For this purpose we consider the case where we are looking for such a cosmological background which vanishes our effective mass $m_{\text{eff}}$. We rewrite the constant mass expression as follows,

$$a'' = -\frac{d}{da} V_{\text{eff}}(a), \quad (21)$$

$$V_{\text{eff}}(a) \equiv \frac{1}{2} \left( m_{\text{eff}}^2 a^2 - \frac{m^2}{2} a^4 \right) \quad (22)$$

note that in the above effective classical representation, the potential has three equilibrium points $a_{\pm,0} = \pm |\frac{m_{\text{eff}}}{m}|$, $0$, the roots $a^k$ correspond to unstable vacuum states and only the origin gives stability with quantum energy given by $E_0 \approx \frac{m_{\text{eff}}}{\sqrt{2}}$. We conclude from the above figure that the inclusion of the scaler mass $m^2$ term reduces the initial energy density $\rho_0$ in eq. (2).

As a result one can develop a perturbative scale factor $a(\eta)$ up to the first order,

$$a(\eta) \approx A \cos(m_{\text{eff}} \eta + \phi_0) + B \frac{m}{m_{\text{eff}}} \cos(\sqrt{2} m_{\text{eff}} \eta + \phi_1) \quad (23)$$

here $\phi_{0,1}$ are initial phases, $B = O(A)$ is a constant and for sake of the perturbative regime we assume that $|\frac{m_{\text{eff}}}{m}| \gg 1$, it implies that the scalar field is super light and loss the mass during epoch. This scenario is the opposite side of the previous where we adopt a non flat but still vacuum classical background with a super massive scaleron. This cosmological scenario with bouncing scale factor given in eq. (23) reviewed in [10]. A remarkable observation is that our proposed scale factor leads to a non vanishing Ricci scalar $R \propto a^{-2}(m_{\text{eff}}^2 - m^2 a^2)$. At the vicinity of unstable vacuum state we obtain $R \approx 0$ and spacetime still remains close to flat and to the stable vacuum epoch, when $a \rightarrow a_c \rightarrow 0$, the space time suffers from a initial singularity $R \rightarrow \infty$ as we expected from the big bang scenario as well an infinite amount for the initial energy density $\rho$, in eq. (2) as it was expected from the primordial model with Zeldovich’s EoS presented in this letter. The quantum vacuum fluctuations of the single scalar degree of freedom around the stable critical point $a_c = 0$ are compatible with inflationary scenario as a non singular bounce model [11].

Discussion

In summary I investigated the problem of uniqueness the vacuum state in curved cosmological background. There is ambiguity to define vacuum state in curved spacetime. It refers to the usual definitions of the vacuum and of “particles with momentum $k$” in the Minkowski spacetime. In flat manifolds the mode decomposition of the scalar field equation of the motion is based on the plane waves solutions of the ordinary quantum mechanics. According to the Heisenberg’s uncertainty principle, the wave function for a free particle with momentum $p$ is a wave packet with standard variance $\Delta p$. The spread of the wave packet should be sufficiently small $\Delta p \ll p$. This condition is necessary to define an observable momentum of the particle. According to the uncertainty principle, the spatial size $l$ of the wave packet satisfies $l \Delta p \sim 1$, as a result we have $l \gg p^{-1}$. When we try to quantize a scalar field on a curved background (here a cosmological metric), there is a risk to have a significant variation of the geometry (for example the scalar Ricci)
across a region of size $l$. Even if the chance be very low still we can conclude that the plane waves are becoming a poor approximation as solutions for the scalar field on such curved backgrounds. Furthermore, we understand that the particles with momentum $p$ cannot be defined similar to the Minkowskian case. The notion of a free particle with momentum $p$ is failed. The only possible meaningful way to keep the concept of the free particle is to preserve the metric of curved spacetime very close to the Minkowski at the scales $p^{-1}$. In this letter we demonstrated that there is a curved background which is different from the Minkowski spacetime and still it has plane wave as a vacuum state. The only constraint is that the mass of the scalar field should be matched to the free parameter (probably some type of the charge or mass) of the cosmological background. Because the geometry corresponds to a very early cold era of the Universe, our result suggested that the other vacuum states squeezed smoothly during this cosmological evolutionary epoch. We conclude that although the “vacuum” and “particles” are approximate concepts and for a generic curve background are ambiguous, in the primordial epoch of the universe, all observers are agree on one type of the vacuum.

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