Structure of solutions near the initial singularity for the surface-symmetric Einstein-Vlasov system

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Abstract

Results on the behaviour in the past time direction of cosmological models with collisionless matter and a cosmological constant $\Lambda$ are presented. It is shown that under the assumption of non-positive $\Lambda$ and spherical or plane symmetry the area radius goes to zero at the initial singularity. Under a smallness assumption on the initial data, these properties hold in the case of hyperbolic symmetry and negative $\Lambda$ as well as in the positive $\Lambda$ case. Furthermore in the latter cases past global existence of spatially homogeneous solutions is proved for generic initial data. The early-time asymptotics is shown to be Kasner-like for small data.

1 Introduction

Global existence and asymptotics in the future for the surface-symmetric Einstein-Vlasov system with cosmological constant have been obtained in [11]-[12]. The present paper deals with the analysis in the past time direction.

In the contracting direction the main result in [8] was that solutions of the surface-symmetric Einstein-Vlasov system with vanishing cosmological constant exist up to $t = 0$ for small initial data, and then the nature of the initial singularity was analyzed. In the following these results are generalized to the case with positive cosmological constant or even negative cosmological constant and hyperbolic symmetry. Also in the present investigation we show that these results can be strengthened a lot for the plane or spherically symmetric case with $\Lambda \leq 0$. We prove in these cases that solutions of the Einstein-Vlasov system exist on the whole interval $(0, t_0]$ for general initial data. This is the main result of this paper. An important tool of the proof is a change of variables inspired by one done by M. Weaver in [15] where she showed existence up to $t = 0$ for a certain class of $T^2$ symmetric solutions of the Einstein-Vlasov system with vanishing cosmological constant. In her paper, the general strategy of [6], using areal coordinates directly in the contracting direction rather than conformal coordinates (as in [11]-[12]), is used to so sharpen global existence results obtained
in [1] and [3] for Einstein-Vlasov initial data on $T^3$ with $T^2$ symmetry. We use the same strategy to sharpen results previously obtained in [8] and [2].

Now let us recall the formulation of the Einstein-Vlasov system which governs the time evolution of a self-gravitating collisionless gas in the context of general relativity. All the particles are assumed to have the same rest mass, normalized to unity, and to move forward in time so that their number density $f$ is a non-negative function supported on the mass shell

$$PM := \{ g_{\alpha\beta} p^\alpha p^\beta = -1, \quad p^0 > 0 \},$$

a submanifold of the tangent bundle $TM$ of the space-time manifold $M$ with metric $g$ of signature $-++$. We use coordinates $(t, x^a)$ with zero shift and corresponding canonical momenta $p^\alpha$; Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On the mass shell $PM$ the variable $p^0$ becomes a function of the remaining variables $(t, x^a, p^b)$:

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ab} p^a p^b}.$$

The Einstein-Vlasov system now reads

$$\partial_t f + \frac{p^\alpha}{p^0} \partial_{x^a} f - \frac{1}{p^0} \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0$$

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0},$$

where $p_\alpha = g_{\alpha\beta} p^\beta$, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, $|g|$ denotes the determinant of the metric $g$, $G_{\alpha\beta}$ the Einstein tensor, $\Lambda$ the cosmological constant, and $T_{\alpha\beta}$ is the energy-momentum tensor.

Here we adopt the definition of spacetimes with surface symmetry, i.e., spherical, plane or hyperbolic symmetry given in [9]. We write the system in areal coordinates, i.e., coordinates are chosen such that $R = t$, where $R$ is the area radius function on a surface of symmetry. The circumstances under which coordinates of this type exist are discussed in [2] for the Einstein-Vlasov system with vanishing $\Lambda$, and in [12] and [13] for the case with $\Lambda$. In such coordinates the metric takes the form

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + t^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1.1)$$

where

$$\sin_k \theta := \begin{cases} 
\sin \theta & \text{if } k = 1 \\
1 & \text{if } k = 0 \\
\sinh \theta & \text{if } k = -1 
\end{cases}$$

Here $t > 0$, the functions $\lambda$ and $\mu$ are periodic in $r$ with period 1. It has been shown in [7] and [2] that due to the symmetry $f$ can be written as a function of $t, r, w := e^\lambda p^1$ and $F := t^4 (p^2)^2 + t^4 \sin^2 \theta (p^3)^2$, with $r, w \in \mathbb{R} \setminus \{0\}$, $F \in [0, +\infty[$.
i.e. \( f = f(t, r, w, F) \). In these variables we have \( p^0 = e^{-\mu} \sqrt{1 + w^2 + F/t^2} \). After calculating the Vlasov equation in these variables, the non-trivial components of the Einstein tensor, and the energy-momentum tensor and denoting by a dot or by prime the derivation of the metric components with respect to \( t \) or \( r \) respectively, the complete Einstein-Vlasov system reads as follows:

\[
\partial_t f + \frac{e^{\mu - \lambda} w}{\sqrt{1 + w^2 + F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu - \lambda} \mu' \sqrt{1 + w^2 + F/t^2}) \partial_w f = 0 \quad (1.2)
\]

\[
e^{-2\mu}(2t\dot{\lambda} + 1) + k - \Lambda t^2 = 8\pi t^2 \rho \quad (1.3)
\]

\[
e^{-2\mu}(2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p \quad (1.4)
\]

\[
\mu' = -4\pi te^{\lambda+\mu} j \quad (1.5)
\]

\[
e^{-2\lambda} (\mu'' + \mu' (\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + (\dot{\lambda} - \dot{\mu})(\dot{\lambda} + \frac{1}{t})) + \Lambda = 4\pi q \quad (1.6)
\]

where

\[
\rho(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw = e^{-2\mu} T_{00}(t, r) \quad (1.7)
\]

\[
p(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw = e^{-2\lambda} T_{11}(t, r) \quad (1.8)
\]

\[
j(t, r) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_{0}^{\infty} w f(t, r, w, F) dF dw = -e^{\lambda+\mu} T_{01}(t, r) \quad (1.9)
\]

\[
q(t, r) := \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{F}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw = \frac{2}{t^2} T_{22}(t, r) \quad (1.10)
\]

We prescribe initial data at some time \( t = t_0 > 0 \),

\[
f(t_0, r, w, F) = \bar{f}(r, w, F), \quad \lambda(t_0, r) = \bar{\lambda}(r), \quad \mu(t_0, r) = \bar{\mu}(r),
\]

and want to show that the corresponding solution exists for all \( t \in [0, t_0] \). The paper is organized as follows. In section 2 we first prove that for the cases \( \Lambda \leq 0 \) and \( k \geq 0 \) the solutions obtained in Proposition 2.1 below exist on the whole interval \( [0, t_0] \). Next for \( \Lambda < 0 \) and \( k = -1 \), and in the case \( \Lambda > 0 \), those solutions exist on \( [0, t_0] \) provided the initial data are sufficiently small. Later on we investigate the spatially homogeneous case. In section 3 the asymptotic behaviour of solutions as \( t \to 0 \) is investigated for small data. The last section summarizes all the results which are known on the contracting direction, for all values of \( k \) and \( \Lambda \), including \( \Lambda = 0 \) and both homogeneous and inhomogeneous models.
2 On past global existence

2.1 The inhomogeneous case

In this section we make use of the continuation criterion in the following local existence result:

**Proposition 2.1** Let \( \overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0, \infty[) \) with \( \overset{\circ}{f}(r+1, w, F) = \overset{\circ}{f}(r, w, F) \) for \((r, w, F) \in \mathbb{R}^2 \times [0, \infty[\), \( f \geq 0 \), and

\[
\begin{aligned}
w_0 & := \sup \{|w||(r, w, F) \in \text{supp}\overset{\circ}{f}\} < \infty \\
F_0 & := \sup \{|F||(r, w, F) \in \text{supp}\overset{\circ}{f}\} < \infty 
\end{aligned}
\]

\( \overset{\circ}{\lambda} \in C^1(\mathbb{R}), \overset{\circ}{\mu} \in C^2(\mathbb{R}) \) with \( \overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1), \overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1) \) for \( r \in \mathbb{R} \), and

\[
\begin{aligned}
\overset{\circ}{\mu}'(r) & = -4\pi t_0 e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} j(r) = -\frac{4\pi^2}{t_0} e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} \int_{-\infty}^{\infty} \int_0^\infty w \overset{\circ}{f}(r, w, F) dF dw, \quad r \in \mathbb{R} 
\end{aligned}
\]

Let \( \overset{\circ}{\lambda} \in C^1(\mathbb{R}), \overset{\circ}{\mu} \in C^2(\mathbb{R}) \) with \( \overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1), \overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1) \) for \( r \in \mathbb{R} \), and

Then there exists a unique, left maximal, regular solution \((f, \lambda, \mu)\) of (1.3)-(1.7) with \((f, \lambda, \mu)(t_0) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})\) on a time interval \(]-T, t_0]\) with \(T \in [0, t_0[\). If

\[
\begin{aligned}
sup \{|w||(t, r, w, F) \in \text{supp}f\} < \infty \\
sup \{e^{\mu(t, r)}|r \in \mathbb{R}, t \in ]T, t_0]\} < \infty
\end{aligned}
\]

and then \( T = 0 \).

This is the content of theorems 3.1 and 3.2 in [11]. For a regular solution all derivatives which appear in the system exist and are continuous by definition (cf. [11]). We prove the following result:

**Theorem 2.2** Consider a solution of the Einstein-Vlasov system with \(k \geq 0\) and \(\Lambda \leq 0\) and initial data given for \(t = t_0 > 0\). Then this solution exists on the whole interval \(0, t_0)\).

**Proof** Observe that since there are two choices between two alternatives, this covers four cases in total namely

\[
(\Lambda, k) \in \{(0, 0), (0, 1)\}, \ (\Lambda < 0, k = 0) \ or \ (\Lambda < 0, k = 1). 
\]

In the case \(\Lambda = 0, k = 0\) the theorem is a special case of what was proved by M. Weaver in [15]. We then have to prove the other three cases.

The strategy of the proof is the following: suppose we have a solution on an interval \((t_1, t_0)\) with \(t_1 > 0\). We want to show that the solution can be extended
to the past. By consideration of the maximal interval of existence this will prove the assertion.

Firstly let us prove that under the hypotheses of the theorem, $\mu$ is bounded above.

For
\[
\frac{d}{dt} (te^{-2\mu}) = -k + \Lambda t^2 - 8\pi t^2 p \leq 0.
\] (2.1)

So $te^{-2\mu}$ cannot increase towards the future, i.e. it cannot decrease towards the past. Thus on $(t_1, t_0]$, $e^{-2\mu}$ must remain bounded away from zero and hence $\mu$ is bounded above.

Recalling that the analogue of Proposition 2.1 for $\Lambda = 0$ was proved by G. Rein in [8], we can deduce that for all three cases being considered it is enough to bound $w$ to get existence up to $t = 0$, using Proposition 2.1.

So let us prove that $w$ is bounded.

Consider the following rescaled version of $w$, called $u_1$, which has been inspired by the work of M. Weaver [15, p. 1090]:

\[
u_1 = \frac{e^\mu}{2t} w.
\]

If we prove that $\mu$ is bounded below then the boundedness of $u_1$ will imply the boundedness of $w$. So let us show that $\mu$ is bounded below under the assumption that $u_1$ is bounded.

Using the first equality in (2.1) and transforming the integral defining $p$ to $u_1$ as an integration variable instead of $w$ yields

\[
p = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{8\pi t e^{-3\mu} u_1^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^4 + F/t^2}} f dF du_1,
\]

the integrand can then be estimated by $4\pi e^{-2\mu} |u_1|$. Thus, using the bound for $u_1$, $p$ can be estimated by $Ce^{-2\mu}$ and so (2.1) implies that

\[|\frac{d}{dt} (te^{-2\mu})| \leq C(1 + te^{-2\mu}),\]

integrating this with respect to $t$ over $[t, t_0]$ yields

\[te^{-2\mu}(t, r) \leq t_0 e^{-2\mu}(t_0, r) + \int_{t}^{t_0} C \left(1 + se^{-2\mu}(s, r)\right) ds,
\]

which implies by the Gronwall inequality that $te^{-2\mu}$ is bounded on $(t_1, t_0]$ that is $\mu$ is bounded below on the given time interval.

The next step is to prove that $u_1$ is bounded. To this end, it suffices to get a suitable integral inequality for $\bar{u}_1$, where $\bar{u}_1$ is the maximum modulus of $u_1$ on support of $f$ at a given time. In the vacuum case there is nothing to be proved and therefore we can assume without loss of generality that $\bar{u}_1 > 0$.

We can compute $\dot{u}_1$:

\[
\dot{u}_1 = -\frac{e^\mu}{2t^2} w + \frac{e^\mu}{2t} w (\dot{\mu} + \dot{\mu}) + \frac{e^\mu}{2t} \ddot{w}
\]
i.e.
\[
\dot{u}_1 = \left( \dot{\mu} + \dot{r} \mu' - \frac{1}{t} \right) u_1 + \frac{\epsilon}{2t} \dot{w}
\]  
(2.2)

but we have
\[
\mu' = -4\pi t e^{\mu + \lambda} j,
\]
\[
\dot{r} = \frac{e^{\mu - \lambda w}}{\sqrt{1 + w^2 + F/t^2}}
\]

and
\[
\dot{w} = 4\pi t e^{2\mu} (j \sqrt{1 + w^2 + F/t^2} - \rho w) + \frac{1 + ke^{2\mu}}{2t} w - \frac{\Lambda}{2} e^{2\mu} w
\]

so that multiplying equation (2.2) by \(2u_1\) yields the following :
\[
\frac{d}{dt} (u_2^2) = 2e^{2\mu} \left[ -4\pi t (\rho - p) + \frac{k}{t} - 4\pi e^{3\mu} j \right] u_1 + 4\pi e^{3\mu} j \frac{u_1 (1 + F/t^2)}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}}
\]

(2.3)

Now the modulus of the first term on the right hand side of equation (2.3) will be estimated. What we need to estimate is \(e^{2\mu} (\rho - p) \bar{u}_1^2\). For convenience let \(\log_+ \) be defined by \(\log_+ (x) = \log x\) when \(\log x\) is positive and \(\log_+ (x) = 0\), otherwise. Then estimating the integral defining \(\rho - p\) shows that
\[
\rho - p \leq C (1 + \log_+ (\bar{u}_1) - \mu).
\]

The expression \(-\mu\) is not under control ; however the expression we wish to estimate contains a factor \(e^{2\mu}\). The function \(\mu \mapsto -\mu e^{2\mu}\) has an absolute maximum at \(-1/2\) where it has the value \((1/2)e^{-1}\). Thus the first term on the right hand side of equation (2.3) can be estimated in modulus by \(C \bar{u}_1^2 (1 + \log_+ (\bar{u}_1))\).

Next the modulus of the second term on the right hand side of equation (2.3) will be estimated. Using equation (1.9) defining \(j\) it can be estimated by \(C \bar{w}^2\), i.e.
\[
j \leq C \bar{u}_1^2 e^{-2\mu},
\]

so that it suffices to estimate the quantity
\[
\bar{u}_1^2 (1 + F/t^2) |u_1| \frac{1}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}}
\]

(2.4)
in order to estimate the modulus of the second term on the right hand side of equation (2.3). But since \(\mu\) and \(t^{-1}\) are bounded on the interval being considered, the quantity (2.4) can be estimated by \(C \bar{u}_1^2\). Thus adding the estimates for the modulus of the first and second terms on the right hand side of (2.3) allows us to deduce from (2.3), since \(\log_+ x = (1/2) \log_+ (x^2)\), that
\[
\left| \frac{d}{dt} (u_2^2) \right| \leq C \bar{u}_1^2 (1 + \log_+ (\bar{u}_1^2)).
\]

(2.5)

Integrating this in \(t\) and using the estimates gives the following integral inequality for \(\bar{u}_1^2\)
\[
\bar{u}_1^2 (t) \leq \bar{u}_1^2 (t_0) + C \int_{t_0}^t \bar{u}_1^2 (s) (1 + \log_+ (\bar{u}_1^2) (s)) \, ds.
\]

(2.6)
Now let us prove that this integral inequality allows $\bar{u}_1^2$ and hence $\bar{u}_1$ to be bounded. The integral inequality \(2.6\) is of the form

$$v(t) \leq v(t_0) + C \int_t^{t_0} v(s) \left(1 + \log_+(v(s))\right) ds$$

where we have written $v = \bar{u}_1^2$. By the comparison principle for solutions of integral equations it is enough to show that the solution of the integral equation

$$v_1(t) = v_1(t_0) + C \int_t^{t_0} v_1(s) \left(1 + \log_+(v_1(s))\right) ds$$

is bounded. This solution $v_1(t)$ is a non-increasing function. Thus either $v_1(t) \leq e$ everywhere, in which case the desired conclusion is immediate, or there is some $t_2$ in $(t_1, t_0)$ such that $v_1(t) \geq e$ on $(t_1, t_2]$, we take $t_2$ maximal with that property. In this second case it follows that on the interval $(t_1, t_2]$ the inequality

$$v_1(t) \leq C \left(1 + \int_t^{t_2} v_1(s) \log v_1(s) ds\right)$$

holds for a constant $C$. The boundedness of $v_1(t)$ follows from that of the solutions of the differential equation $\dot{v}_2(t) = Cv_2(t) \log v_2(t)$. In fact we get a bound like $\exp(\exp t)$ for $v_1(t)$. Either case $v_1(t)$ is bounded. Thus we conclude that $\bar{u}_1^2$ and hence $u_1$ is bounded i.e. $w$ is bounded and the proof of the theorem is complete.\]

It is important to note that in the case $(\Lambda = 0, k = 1)$ the result proved in Theorem 2.2 is new and so strengthens the existence up to $t = 0$ for small data obtained in [8].

Next we have the following result which generalizes Theorem 4.1 in [8] to the case with non-zero cosmological constant $\Lambda$. Since there are only minor changes in the proof, we omit it here.

**Theorem 2.3** Let $(\tilde{f}, \tilde{\lambda}, \tilde{\mu})$ be initial data as in Proposition 2.1 and assume that $e^{-2\tilde{\mu}(r)} - \frac{\Lambda}{4} \tilde{M}_0^2 - 2 > 0$ for $r \in \mathbb{R}$ and $c > 0$ with

$$c := \frac{1}{2} \left(1 - \frac{\Lambda \tilde{\mu}}{1 - \frac{\Lambda \tilde{\mu}}{e^{2\tilde{\mu}}}}\right) - 10\pi^2 w_0 F_0 \sqrt{1 + \frac{\tilde{w}_0^2 + F_0 / \tilde{t}_0^2}{1 - \frac{\tilde{w}_0^2}{e^{2\tilde{\mu}}}}}$$

and for $\Lambda > 0$

$$c := \begin{cases} \frac{1}{2} \left(1 - \frac{\Lambda \tilde{\mu}}{1 - \frac{\Lambda \tilde{\mu}}{e^{2\tilde{\mu}}}}\right) - 10\pi^2 w_0 F_0 \sqrt{1 + \frac{\tilde{w}_0^2 + F_0 / \tilde{t}_0^2}{1 - \frac{\tilde{w}_0^2}{e^{2\tilde{\mu}}}}} & \text{if } k = 0 \text{ or } k = 1, \\ \frac{1}{2} \left(1 - \frac{(\Lambda \tilde{\mu} + 1) \tilde{\mu}}{1 - (\frac{\Lambda \tilde{\mu} + 1}{e^{2\tilde{\mu}}})}\right) - 10\pi^2 w_0 F_0 \sqrt{1 + \frac{\tilde{w}_0^2 + F_0 / \tilde{t}_0^2}{1 - (\frac{\tilde{w}_0^2}{e^{2\tilde{\mu}}})}} & \text{if } k = -1. \end{cases}$$

Then the corresponding solution exists on the interval $[0, t_0]$, and

$$|w| \leq w_0 t^c, \ (r, w, F) \in \text{supp} f(t), \ t \in [0, t_0].$$

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Remark 2.4  Note that in the case $\Lambda < 0$ and $k \geq 0$ it suffices to let 
\[
c := \frac{1}{2} - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0/t_0^2} \parallel \int \parallel e^{2\mu} \parallel
\]
in the hypotheses of Theorem 2.3 in order to obtain the same conclusion in the latter theorem.

It may be asked what happens for general data. In order to answer this question we begin here by examining the spatially homogeneous case. Note that for $\Lambda = 0$ more information is available in the homogeneous case in [10].

2.2 The spatially homogeneous case

In this subsection we want to prove that spatially homogeneous solutions (i.e. solutions which are independent of $r$) exist on the whole interval $(0, t_0]$ for general initial data in the case of hyperbolic symmetry and negative $\Lambda$ as well as in the positive $\Lambda$ case.

To this end we use the continuation criterion stated in Proposition 2.1. Since the proof for the boundedness of $w$ in Theorem 2.2 depends neither on the sign of $k$ nor that of $\Lambda$, the only thing to do here is to bound $\mu$ in order to get existence up to $t = 0$. We prove in three steps that $\mu$ is bounded above.

Step 1 As a first step we use the notations and the method of the proof for Lemma 3 in [15] to obtain a 'lower bound' of the quantity
\[
\int_{\mathbb{R}^3} f \frac{v_2^2}{|v_0|} dv_1 dv_2 dv_3.
\]
By equation (4) in [15],
\[
v_0 = -\alpha e^{2\nu-2U} + \alpha v_1^2 + \alpha e^{2\nu-4U} v_2^2 + \alpha \tilde{t}^{-2} e^{2\nu} (v_3 - Av_2)^2,
\]
in order to save some notation we have denoted by $\tilde{t}$ the area of the symmetry orbits in [15]. Taking $A = 0$ we obtain the following inequality, since $|v_2| \leq \bar{v}_2$ and $|v_3| \leq \bar{v}_3$ :
\[
|v_0| \leq \sqrt{K + \alpha v_1^2},
\]
with
\[
K = \alpha e^{2\nu-2U} + \alpha e^{2\nu-4U} \bar{v}_2^2 + \alpha \tilde{t}^{-2} e^{2\nu} \bar{v}_3^2,
\]
so that
\[
\frac{v_1^2}{|v_0|} \geq \frac{v_1^2}{\sqrt{K + \alpha v_1^2}}.
\]
If $|v_1| > \delta$ then it follows that
\[
\frac{v_1^2}{\sqrt{K + \alpha v_1^2}} \geq \frac{\delta^2}{\sqrt{K + \alpha \delta^2}}.
\]
Now the argument of the proof for Lemma 3 in [15] applies here and the only thing we need to change there is that we replace the displayed equation leading to equation (10) by what follows:

\[
\int_{\mathbb{R}^3} f \frac{v_1^2}{|v_0|} dv_1 dv_2 dv_3 = \int_{\mathbb{R}^2} \int_{|v_1|<\delta} f \frac{v_1^2}{|v_0|} dv_1 dv_2 dv_3 + \int_{\mathbb{R}^2} \int_{|v_1|>\delta} f \frac{v_1^2}{|v_0|} dv_1 dv_2 dv_3 \\
\geq \frac{\delta^2}{\sqrt{K + \alpha \delta^2}} \int_{\mathbb{R}^2} \left( \int_{[v_1]>\delta} f dv_1 \right) dv_2 dv_3 \\
\geq \frac{\delta^2 b}{\sqrt{K + \alpha \delta^2}} \\
\geq \frac{C}{\sqrt{K + \alpha}} \tag{2.7}
\]

where \( C \) is a positive constant.

**Step 2** Now we translate the latter inequality into our familiar notation of plane symmetry. We have

\[
v_0 = -e^{2\mu} p_0 / 2t, \quad v_1 = e^{2\lambda} p_1, \quad v_2 = t^2 p^2, \quad v_3 = t^2 p^3, \quad \tilde{t} = t^2, \quad U = \log t, \quad \nu = \lambda + \log t \quad \text{and} \quad \alpha = (1/4)t^{-2} e^{2(\mu - \lambda)}.
\]

Then

\[
K + \alpha = (1/4)t^{-2} e^{2(\mu - \lambda)} - t^2 + \tilde{v}_2 + \tilde{v}_3, \tag{2.8}
\]

so that

\[
\frac{1}{\sqrt{K + \alpha}} \geq \frac{Ct^2 e^{2\mu}}{e^{2\lambda} + 1}.
\]

On the other hand the jacobian determinant of the transformation \((p_1, p_2, p_3) \mapsto (v_1, v_2, v_3)\) gives

\[
\det \left( \frac{\partial (v_1, v_2, v_3)}{\partial (p_1, p_2, p_3)} \right) = t^4 e^{2\lambda},
\]

it follows that

\[
\int_{\mathbb{R}^3} f \frac{v_1^2}{|v_0|} dv_1 dv_2 dv_3 = 2 \int_{\mathbb{R}^3} t^5 e^{6\lambda} - 2\mu \frac{(p_1)^2}{|p_0|} dp^1 dp^2 dp^3;
\]

but we have \( w := e^{\lambda} p_1, \quad F := t^4 [(p_2)^2 + (p_3)^2] \) so that

\[
|\det \left( \frac{\partial (w, F, p_3)}{\partial (p_1, p_2, p_3)} \right) | = 2t^4 e^{\lambda} \sqrt{F^2 - (p_3)^2},
\]

therefore calculating the latter integral implies that

\[
\int_{\mathbb{R}^3} f \frac{v_1^2}{|v_0|} dv_1 dv_2 dv_3 = (1/2)t^3 e^{3\lambda - \mu} p. \tag{2.9}
\]

Thus using equations (2.7), (2.8) and (2.9) yields the following estimate:

\[
p \geq \frac{Ct^{-1} e^{-2\lambda}}{e^{2\lambda} + 1}. \tag{2.10}
\]
Step 3 Boundedness for $\mu$.

Now we have

$$2te^{-2\mu}(\dot{\lambda} + \dot{\mu}) = 8\pi t^2(\rho + p) \geq 0$$

which implies that $\lambda + \mu$ is increasing and so in the past time direction the estimate $e^{\lambda + \mu} \leq C$ i.e. $e^{-2\lambda} \geq Ce^{2\mu}$ holds. Equation (2.10) then implies

$$p \geq C \frac{e^{2\mu}}{e^{-2\mu} + 1},$$

(2.11)

The estimate (2.11) shows that if $\mu$ is arbitrarily large, $p$ will become also large and will dominate the other terms of the right hand side in the equality

$$\frac{d}{dt}(te^{-2\mu}) = -k + \Lambda t^2 - 8\pi t^2 p$$

and then $\frac{d}{dt}(te^{-2\mu})$ will become negative, i.e. there is a constant $L$ such that the following implication holds

$$\mu \geq L \implies \frac{d}{dt}(te^{-2\mu}) < 0.$$ 

Now fix $t_1$ in any interval $(t_*, t_0]$. Then either $\mu(t_1) \leq L$ or $\mu(t_1) > L$. In this second case define

$$t_2 := \sup \{ t \in (t_*, t_0] : \mu \geq L \text{ on } [t_1, t] \},$$

we thus have either $t_2 < t_0$ or $t_2 = t_0$. Since $\mu(t) \geq L$ for all $t \in [t_1, t_2]$, it follows that

$$t_1 e^{-2\mu(t_1)} \geq t_2 e^{-2\mu(t_2)}$$

that is

$$e^{-2\mu(t_1)} \leq e^{-2\mu(t_2)}(t_1/t_2)$$

$$\leq \begin{cases} e^{2L}(t_1/t_2) & \text{if } t_2 < t_1 \\ e^{2\mu(t_0)}(t_1/t_0) & \text{if } t_2 = t_0 \end{cases}$$

either case, $e^{2\mu}$ and so $\mu$ is uniformly bounded in $t$ in any interval $(t_*, t_0]$.

We have proven the following

**Theorem 2.5** Consider a spatially homogeneous solution of the Einstein-Vlasov system with $\Lambda > 0$ or $(\Lambda < 0, k = -1)$, and initial data given for $t = t_0 > 0$. Then this solution exists on the whole interval $(0, t_0]$.

3 On past asymptotic behaviour

In this section we examine the behaviour of solutions as $t \rightarrow 0$. 

10
3.1 The initial singularity

First we analyze the curvature invariant \( R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \) called the Kretschmann scalar in order to prove that there is a spacetime singularity.

**Theorem 3.1** Let \((f, \lambda, \mu)\) be a regular solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval \([0, t_0]\) with small initial data as described in Theorem 2.3 and in Remark 2.4. Then

\[
\lim_{t \to 0} (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) = \infty,
\]

uniformly in \( r \in \mathbb{R} \).

**Proof** We can compute the Kretschmann scalar as in [8] and obtain

\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 4[e^{-2\lambda}(\mu'' + \mu'(\lambda' - \lambda')) - e^{-2\mu}(\dot{\lambda} + \dot{\lambda}(\lambda - \mu))]^2
+ \frac{8}{t^2}[e^{-4\mu}\dot{\lambda}^2 + e^{-4\mu}\ddot{\mu}^2 - 2e^{-2(\lambda+\mu)}(\mu')^2]
+ \frac{4}{t^2}(e^{-2\mu} + k)^2
=: K_1 + K_2 + K_3
\tag{3.1}
\]

Since \( K_1 \) is nonnegative it can be dropped.

Now let us distinguish the cases \( \Lambda > 0 \) and \( \Lambda < 0 \).

Case \( \Lambda > 0 \) :

In this case we use the same argument as in [8] to estimate \( K_2 \). We then obtain

\[
K_2 \geq \frac{8}{t^2} \left( \left( 4\pi t (\rho - p) \frac{k + e^{-2\mu}}{2t} \right)^2 + \left( \frac{k + e^{-2\mu}}{2t} - \Lambda t \right)^2 - \frac{\Lambda^2}{2} t^2 + 4\pi t^2 \Lambda (\rho - p) \right)
\geq -4\Lambda^2
\tag{3.2}
\]

since \( 4\pi t^2 \Lambda (\rho - p) \geq 0 \).

Recalling the expression for \( e^{-2\mu} \) we get

\[
e^{-2\mu} + k = \frac{t_0(e^{-2\mu}(r) + k)}{t} + \frac{8\pi}{t} \int_{t_0}^{t_0} s^2 p(s, r)ds + \frac{\Lambda}{3t} (t^3 - t_{0}^3)
\geq t_0 (\inf e^{-2\mu} + k - \frac{\Lambda}{3t_0} t_0^2)
\tag{3.3}
\]

thus

\[
K_3 = \frac{4}{t^4}(e^{-2\mu} + k)^2 \geq \frac{4t_0^2}{t_0^6} \left( \inf e^{-2\mu} + k - \frac{\Lambda}{3t_0} \right)^2
\]

and so

\[
(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) \geq \frac{4t_0^2}{t_0^6} \left( \inf e^{-2\mu} + k - \frac{\Lambda}{3t_0} \right)^2 - 4\Lambda^2, \quad t \in [0, t_0], \quad r \in \mathbb{R}, \quad \Lambda > 0
\]
and the assertion is proved for $\Lambda > 0$.

Case $\Lambda < 0$

We have by (3.2):

$$K_2 \geq -4\Lambda^2 + 32\pi\Lambda(\rho - p).$$

Now we use the estimate for $w$ in Theorem 2.3 so that

$$(\rho - p)(t, r) \leq \rho(t, r) = \frac{\pi}{t^2} \int_{-p(t)}^{P(t)} \int_{0}^{F_0} \sqrt{1 + w^2 + F/t^2 f(t, r, w, F)dFdw} \leq Ct^{-3+c},$$

where $c$ is defined in Theorem 2.3 and in Remark 2.4.

Thus

$$K_2 \geq -4\Lambda^2 + C\Lambda t^{-3+c}.$$ (3.3)

becomes in this case ($\Lambda < 0$)

$$e^{-2\mu} + k \geq \frac{t_0(\inf e^{-2\mu} + k)}{t},$$

therefore

$$K_3 \geq \frac{4t_0^2}{t^6} \left(\inf e^{-2\mu} + k\right)^2$$

so that

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) \geq \frac{4t_0^2}{t^6} \left(\inf e^{-2\mu} + k\right)^2 + C\Lambda t^{-3+c} - 4\Lambda^2, \; t \in [0, t_0], \; r \in \mathbb{R},$$

that is the assertion in the theorem holds for $\Lambda < 0$ as well, and the proof is complete. □

Next we prove that the singularity at $t = 0$ is a crushing singularity i.e. the mean curvature of the surfaces of constant $t$ (cf. 4 (1.0.2) for a definition) blows up, also it is a velocity dominated singularity i.e the generalized Kasner exponents have limits as $t \to 0$. We have the same results as in the vanishing cosmological constant case 8, the argument of the proof in that case also applies here.

**Theorem 3.2** Let $(f, \lambda, \mu)$ be a solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval $]0, t_0]$, assume that the initial data satisfy $e^{-2\mu(r)} - \frac{4t_0^2}{3\Lambda} - 1 > 0$ for $r \in \mathbb{R}$. Let

$$K(t, r) := -e^{-\mu} \left(\dot{\lambda}(t, r) + \frac{2}{t}\right)$$

which is the mean curvature of the surfaces of constant $t$. Then

$$\lim_{t \to 0} K(t, r) = -\infty,$$

uniformly in $r \in \mathbb{R}$.
Proof In fact we use the same argument as in [8] and obtain the following inequalities:
if \( \Lambda > 0 \) then
\[
K(t, r) \leq -\frac{t_0^{1/2} \left( \inf e^{-2\dot{\mu}} + k - \frac{\Lambda t_0^2}{3} \right)}{t^{3/2}} \quad \text{for } k = 0 \text{ or } k = -1,
\]
and
\[
K(t, r) \leq -\frac{t_0^{1/2} \left( \inf e^{-2\dot{\mu}} - \frac{\Lambda t_0^2}{3} \right)}{t^{3/2}} \quad \text{for } k = 1;
\]
whereas if \( \Lambda < 0 \),
\[
K(t, r) \leq -\frac{3t_0^{1/2} \left( \inf e^{-2\dot{\mu}} + k \right)}{2} - \frac{\Lambda t_0^{-1/2}(\inf e^{-2\dot{\mu}} + k)^{-1/2}}{t^{-3/2}} \quad \text{for } k = 0 \text{ or } k = -1,
\]
and
\[
K(t, r) \leq -\frac{e^{-\mu}}{t} - \frac{\Lambda}{2} te^{\mu} \leq -\frac{t_0^{1/2} \left( \inf e^{-2\dot{\mu}} \right)}{t^{3/2}} - \frac{\Lambda t_0^{-1/2}(\inf e^{-2\dot{\mu}})^{-1}}{t^{-3/2}} \quad \text{for } k = 1. \Box
\]

Theorem 3.3 Let \((f, \lambda, \mu)\) be a regular solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval \([0, t_0] \) with small initial data as described in Theorem 2.3 and in Remark 2.4. Then
\[
\lim_{t \to 0} \frac{K_1(t, r)}{K(t, r)} = -\frac{1}{3}; \lim_{t \to 0} \frac{K_2(t, r)}{K(t, r)} = \lim_{t \to 0} \frac{K_3(t, r)}{K(t, r)} = \frac{2}{3},
\]
uniformly in \( r \in \mathbb{R} \),
where
\[
\frac{K_1(t, r)}{K(t, r)} + \frac{K_2(t, r)}{K(t, r)} + \frac{K_3(t, r)}{K(t, r)} = 2.
\]

are the generalized Kasner exponents.

Note that in this theorem some smallness assumption on the initial data is necessary (cf. [7, p.148]).

3.2 Determination of the leading asymptotic behaviour

In this subsection we determine the explicit leading behaviour of \( \lambda, \mu, \dot{\lambda}, \dot{\mu}, \mu' \) in the case of small data.

We have
\[
\frac{d}{dt} (te^{-2\mu}) = \Lambda t^2 - k - 8\pi^2 p.
\]
By Theorem 2.3 and Remark 2.4
\[
|w| \leq Ct^\epsilon. \quad (3.5)
\]
Using the expression for $p$, we have

$$p \leq Ct^{-2+3c}$$

so that

$$8\pi t^2p \leq C,$$

thus

$$\left| \frac{d}{dt}(te^{-2\mu}) \right| \leq C,$$

integrating this over $[t_1, t_2]$ yields

$$| t_2e^{-2\mu(t_2)} - t_1e^{-2\mu(t_1)} | \leq C(t_2 - t_1)$$

(3.6)

so $s \mapsto se^{-2\mu(s)}$ verifies the Lipschitz condition with Lipschitz constant $C$. Thus $te^{-2\mu(t)} \to L$ as $t \to 0$;

note that $L > 0$, using the lower bound on $e^{-2\mu(t)}$. (3.6) then implies

$$\left| te^{-2\mu(t)} - L \right| \leq Ct$$

and so

$$e^{-2\mu(t)} = \frac{L}{t} + O(1) = \frac{L}{t}(1 + O(t))$$

thus

$$e^{2\mu(t)} = L^{-1}t(1 + O(t))$$

(3.7)

that is

$$\mu = \frac{1}{2}\ln t + O(1).$$

(3.8)

Now we have

$$\dot{\lambda} = \frac{1}{2}(\Lambda t + 8\pi t \rho)e^{2\mu} - \frac{1 + ke^{2\mu}}{2t}.$$  

(3.9)

Using (3.8) and the expression for $\rho$ we can see that

$$8\pi t \rho \leq Ct^{-2+c}$$

thus (3.9) implies that

$$\dot{\lambda} = \frac{1}{2}[L^{-1}\Lambda t^2 + O(t^3) + O(t^{-1+c}) + O(t^c)] - \frac{1}{2t} - \frac{kL^{-1}}{2} + O(t)$$

that is, using the fact that $-1+c < 0$,

$$\dot{\lambda} = -\frac{1}{2t} + O(t^{-1+c})$$

(3.10)

and so

$$\lambda = -\frac{1}{2}\ln t + O(t^c).$$

(3.11)
Now using (3.5) and the expression for \( j \) we can see that
\[
j \leq Ct^{-2+c}
\]
and thus using equation \( \mu' = -4\pi t e^{\lambda+\mu} j \) we obtain
\[
\mu' = -4\pi t L^{-1/2} t^{1/2} [1 + O(t)] \left[ t^{-1/2} (1 + O(t^c)) \right] O(t^{-2+c})
\]
i.e.
\[
\mu' = O(t^{-1+c}); \quad (3.12)
\]
we have used equations (3.7) and (3.11).

Recalling that
\[
\dot{\mu} = \frac{1}{2} (-\Lambda t + 8\pi tp) e^{2\mu} + \frac{1 + ke^{2\mu}}{2t},
\]
we use (3.7) and the fact that \( 8\pi tp \leq C \) to obtain
\[
\dot{\mu} = \frac{1}{2t} + O(1). \quad (3.13)
\]

Thus we have proven the following

**Theorem 3.4** Let \((f, \lambda, \mu)\) be a solution of the surface-symmetric Einstein-Vlasov system with cosmological constant on the interval \([0, t_0]\) with small initial data as described in Theorem 2.3 and in Remark 2.4 for \( \Lambda \neq 0 \), and in [8, Theorem 4.1] for \( \Lambda = 0 \). Then the following properties hold at early times:

(3.8), (3.10), (3.11), (3.12), (3.13).

This theorem shows that the model for the dynamics of the class of solutions considered here is the Kasner solution (see [14]) with Kasner exponents \((2/3, 2/3, -1/3)\) for which \( \lambda = -\frac{1}{2} \ln t \) and \( \mu = \frac{1}{2} \ln t \).

### 4 Concluding remarks

In the contracting direction we have shown the global existence in the case \((\Lambda \leq 0, k \geq 0)\) for generic initial data. Similar results have been obtained for small data in the cases \((\Lambda < 0, k = -1)\) and \( \Lambda > 0 \). Detailed early-time asymptotics have been obtained for small data as well. It would be interesting to examine what happens if the smallness assumption on data is dropped. In this case the asymptotics need not be Kasner-like (cf. [7, p.146]).

In the homogeneous case we have proven existence up to \( t = 0 \) for generic data. Together with the results obtained in [12] this would prove strong cosmic censorship (cf. [5]) in the class of Bianchi I and III solutions of the Einstein-Vlasov system with positive cosmological constant, if we obtained curvature blow-up for generic data. In fact proving cosmic censorship requires proving inextendibility of the maximal Cauchy development in the future and in the past. Still in the homogeneous case more information about asymptotics for \( \Lambda = 0 \) is available in [10]. In the case with \( \Lambda \), such results are not available.
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