On the Best Approximation of Certain Classes of Periodic Functions by Trigonometric Polynomials

A. Serdyuk*, I. Ovsii

Abstract. We obtain the estimates for the best approximation in the uniform metric of the classes of $2\pi$-periodic functions whose $(\psi, \beta)$-derivatives have a given majorant $\omega$ of the modulus of continuity. It is shown that the estimates obtained here are asymptotically exact under some natural conditions on the parameters $\psi$, $\omega$ and $\beta$ defining the classes.

Key Words and Phrases: Best approximation, Modulus of continuity, Asymptotic formula, $(\psi, \beta)$-derivative, Convolution

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1. Introduction

Let $L$ be the space of $2\pi$-periodic functions summable over the period with the norm $\|f\|_1 = \int_{-\pi}^{\pi} |f(t)| \, dt$ and let $C$ be the space of $2\pi$-periodic continuous functions $f$ with the norm $\|f\|_C = \max_t |f(t)|$. Suppose $f \in L$ and

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is its Fourier series. Suppose also that $\psi(k)$ is an arbitrary numerical sequence and $\beta$ is a fixed real number ($\beta \in \mathbb{R}$). If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k \sin \left( kx + \frac{\beta \pi}{2} \right) \right)$$

is the Fourier series of a certain function $\varphi \in L$, then $\varphi$ is called (see, e.g., [10, 11]) the $(\psi, \beta)$-derivative of the function $f$ and is denoted by $f_{\beta}^{\psi}$. The set of continuous functions $f(x)$ having $(\psi, \beta)$-derivatives such that $f_{\beta}^{\psi} \in H_\omega$, where

$$H_\omega = \{ \varphi \in C : |\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|) \ \forall t', t'' \in \mathbb{R} \},$$

*Corresponding author.
and $\omega(t)$ is a fixed modulus of continuity is usually denoted by $C^\psi_\beta H_\omega$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C^\psi_\beta H_\omega$ become the well-know Weyl-Nagy classes $W^r_\beta H_\omega$ which, in turn, for $\beta = r$ coincide with the Weyl classes $W^r_\beta H_\omega$ (see, e.g., [11, Chap. 3, Sec. 4, 6]). For natural numbers $r$ and $\beta = r$ we obtain the classes of periodic functions whose $r$-th derivatives are in the class $H_\omega$.

Let $M^\prime$ be the set of all continuous functions $\psi(t)$ convex downwards for $t \geq 1$ and satisfying the condition $\lim_{t \to \infty} \psi(t) = 0$.

If $\psi \in M^\prime$, where

$$M^\prime := M^\prime(\beta) = \{ \psi : \psi \in M \text{ when } \sin \frac{\beta \pi}{2} = 0 \text{ or } \psi \in M \text{ and } \int_1^\infty \frac{\psi(t)}{t} dt < \infty \text{ when } \sin \frac{\beta \pi}{2} \neq 0 \},$$

then the classes $C^\psi_\beta H_\omega$ coincide with the classes of functions $f(x)$, which are representable by the convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x + t) \Psi_\beta(t) dt, \quad \phi \in H^0_\omega, \quad x \in \mathbb{R}, \quad (2)$$

(see, e.g., [10, p. 31]), where $H^0_\omega = \{ \phi \in H_\omega : \int_{-\pi}^{\pi} \phi(t) dt = 0 \}$, and $\Psi_\beta(t)$ is a summable function, whose Fourier series have the form $\sum_{k=1}^{\infty} \psi(k) \cos(k t \pi / 2)$.

The set $M^\prime$ is very inhomogeneous in the rate of convergence of functions $\psi(t)$ to zero as $t \to \infty$. This is why it was suggested in [10, pp. 115, 116] (see also [13, Subsec. 1.3]) to select subsets $M^\prime_0$ and $M^\prime_C$ from $M^\prime$ as follows:

$$M^\prime_0 = \{ \psi \in M : 0 < \mu(t) \leq K < \infty, \quad \forall t \geq 1 \},$$

$$M^\prime_C = \{ \psi \in M : 0 < K_1 \leq \mu(t) \leq K_2 < \infty, \quad \forall t \geq 1 \},$$

where $\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}$, $\eta(t) = \eta(\psi; t) = \psi^{-1}(\psi(t)/2)$, $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$, and $K$, $K_1$, $K_2$ are positive constants (possibly dependent on $\psi(\cdot)$). The function $\mu(\psi; t)$ is called the modulus of half-decay of the function $\psi(t)$. It is obvious that $M^\prime_C \subset M^\prime_0$.

Typical representatives of the set $M^\prime_C$ are the functions $t^{-r}$, $r > 0$, representatives of the set $M^\prime_0 \setminus M^\prime_C$ are the functions $\ln^{-\alpha}(t + 1)$, $\alpha > 0$. Let $M^\prime_0 = M^\prime \cap M^\prime_0$. Natural representatives of the set $M^\prime_0$ are the functions $\ln^{-\alpha}(t + 1)$, $\alpha > 1$. It is easy to see that if $\beta = 2l$, $l \in \mathbb{Z}$, the set $M^\prime_0$ coincide with $M^\prime_0$. Moreover, since for all $\psi \in M^\prime_C$

$$\int_1^{\infty} \frac{\psi(t)}{t} dt \leq K \psi(n), \quad n \in \mathbb{N}, \quad (3)$$

(see [11, p. 204]) then $M^\prime_C \subset M^\prime_0$. Throughout the paper we denote the positive constants that may be different in different relations by $K$, $K_i$, $i = 1, 2$. 


Let us denote the best approximation of the classes $C_{\beta}^{\psi} H_\omega$ by trigonometric polynomials $t_{n-1}(\cdot)$ of order not more than $n - 1$ by $E_n(C_{\beta}^{\psi} H_\omega)$, that is

$$E_n(C_{\beta}^{\psi} H_\omega) = \sup_{f \in C_{\beta}^{\psi} H_\omega} \inf_{t_{n-1}} \| f(\cdot) - t_{n-1}(\cdot) \|_C.$$  \hspace{1cm} (4)

As is shown in [10, p. 330] if $\omega(t)$ is an arbitrary modulus of continuity and $\psi \in M_C$, $\beta \in \mathbb{R}$ or $\psi \in M_0$, $\beta = 0$, then the following estimates hold for the quantity $E_n(C_{\beta}^{\psi} H_\omega)$:

$$K_1 \psi(n) \omega(1/n) \leq E_n(C_{\beta}^{\psi} H_\omega) \leq K_2 \psi(n) \omega(1/n).$$  \hspace{1cm} (5)

When $\psi(k) = k^{-r}$, $r > 0$, $\beta \in \mathbb{R}$, the orders of decrease of quantity (4) have been known earlier [3] (see also [15, p. 508]).

It should be noted that unlike order estimates, exact values for the quantity $E_n(C_{\beta}^{\psi} H_\omega)$ have been found for $\psi(k) = k^{-r}$, $r \in \mathbb{N}$, $\beta = r$ and for the convex upwards modulus of continuity by Korneichuk [5] (see also [6, p. 319], [2, p. 344]). The similar problem on the class of real-valued functions defined on the entire real axis and having the $r$-th continuous derivatives $f^{(r)}$ such that $\omega(f^{(r)}; t) \leq \omega(t)$, $t \in [0, \infty)$, is solved in the paper of Ganzburg [4].

The aim of the present work is to study the rate of decrease of quantity (4) when $\psi \in M_0$ and $\beta \in \mathbb{R}$.

2. Main Results

The following statements are true.

**Theorem 1.** Let $\psi \in M_0$, $\beta \in \mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. Then, as $n \to \infty$,

$$E_n(C_{\beta}^{\psi} H_\omega) = \frac{\theta_\omega(n)}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1) \psi(n) \omega(1/n),$$

(6)

where $\theta_\omega(n) \in [2/3, 1]$ and $O(1)$ is a quantity uniformly bounded in $n$ and $\beta$. If $\omega(t)$ is a convex upwards modulus of continuity, then $\theta_\omega(n) = 1$.

We give an example of functions $\psi$ and $\omega$ for which (6) is an asymptotic formula.

**Example 1.** Let $\psi(t) = \ln^{-\gamma}(t + 1)$, $\gamma > 1$, $\beta \neq 2l$, $l \in \mathbb{Z}$ and

$$\omega(t) = \begin{cases} 0, & t = 0, \\ \ln^{-\alpha} \left(\frac{1}{t} + 1\right), & t > 0, \ 0 < \alpha \leq 1. \end{cases}$$

Then by virtue of (6) the following asymptotic formula holds as $n \to \infty$:

$$E_n(C_{\beta}^{\psi} H_\omega) = \ln^{-(\gamma+\alpha)}(n + 1) \left(\frac{1}{\pi(\gamma + \alpha - 1)}\right) \sin \frac{\beta \pi}{2} \ln n + O(1),$$

where $O(1)$ is a quantity uniformly bounded in $n$ and $\beta$. 

Note that if
\[
\lim_{n \to \infty} \frac{|\psi'(n)|n}{\psi(n)} = 0, \quad \psi'(n) := \psi'(n+),
\]
and
\[
\lim_{n \to \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0, \quad \omega'(1/n) := \omega'(1/n+),
\]
then equalities
\[
\lim_{n \to \infty} \psi(n)\omega(1/n) \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} \, dt = \lim_{n \to \infty} \frac{|\psi'(n)|n}{\psi(n)} + \lim_{n \to \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0,
\]
are valid.

Therefore from Theorem 1 we obtain

**Corollary 1.** Assume that \( \psi \in \mathfrak{M}_0 \), \( \beta \neq 2l, l \in \mathbb{Z} \), \( \omega(t) \) is a convex upwards modulus of continuity and conditions (7) and (8) are fulfilled. Then the following asymptotic formula holds as \( n \to \infty \):
\[
E_n(C^\psi_H \omega) = \frac{1}{\pi} \sin \frac{\beta \pi}{2} \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} \, dt + O(1)\psi(n)\omega(1/n),
\]
where \( O(1) \) is a quantity uniformly bounded in \( n \) and \( \beta \).

The functions \( \psi \) and \( \omega \) from Example 1 can serve as an example of the functions which satisfy conditions (7) and (8), respectively.

Relation (6) implies that if \( \psi \in \mathfrak{M}_0 \) and
\[
|\sin \frac{\beta \pi}{2} \int_0^{1/n} \omega(t) dt = O(1)\omega(1/n), \quad \beta \in \mathbb{R},
\]
or \( \psi \in \mathfrak{M}_C \) (see (3)), then
\[
E_n(C^\psi_H \omega) = O(1)\psi(n)\omega(1/n).
\]
Taking into account that function \( \psi(t) \) is monotonically decreasing for \( t \geq 1 \) and using the estimate
\[
E_n(C^\psi_H \omega) \geq K\psi(n)\omega(1/n), \quad \forall \psi \in \mathfrak{M}',
\]
(see [10, pp. 329, 330]), by virtue of relation (6) we arrive at the following statement:

**Corollary 2.** Let \( \beta \in \mathbb{R} \) and let \( \omega(t) \) be an arbitrary modulus of continuity. If \( \psi \in \mathfrak{M}_C \) or \( \psi \in \mathfrak{M}_0 \) and \( \omega(t) \) satisfies condition (9), then
\[
K_1\psi(n)\omega(1/n) \leq E_n(C^\psi_H \omega) \leq K_2\psi(n)\omega(1/n),
\]
where \( K_1 \) and \( K_2 \) are positive constants.

Thus, estimates (5) obtained by Stepanets [10, p. 330] (see also [11, Chap. 5, Sec. 22; Chap. 7, Sec. 4]) for the arbitrary modulus of continuity \( \omega(t) \) and for \( \psi \in \mathfrak{M}_C, \beta \in \mathbb{R} \) or for \( \psi \in \mathfrak{M}_0, \beta = 0 \), hold also in the case when \( \psi \in \mathfrak{M}_0, \beta \neq 0 \) and \( \omega(t) \) satisfies condition (9). For example, the function \( \omega(t) = t^\alpha, \ 0 < \alpha < 1, \) satisfies (9).
3. Proof of Theorem 1

Suppose that all conditions of the theorem are satisfied. Let us carry out the proof in two stages.

1. We shall find an upper estimate for $E_n(C^\psi_\beta H_\omega)$.

We set

$$U_{n-1}(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{\psi(n) k^2}{\psi(k) n^2}\right) (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N},$$

where $a_k$ and $b_k$ are the Fourier coefficients of a function $f \in C^\psi_\beta H_\omega$. Show that for the quantity

$$E_n(C^\psi_\beta H_\omega) = \sup_{f \in C^\psi_\beta H_\omega} \|f(\cdot) - U_{n-1}(f; \cdot)\|_C,$$

the inequality

$$E_n(C^\psi_\beta H_\omega) \leq \frac{1}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \omega\left(\frac{1}{t}\right) dt + O(1)\psi(n)\omega(1/n),$$

is true. Since

$$E_n(C^\psi_\beta H_\omega) \leq E_n(C^\psi_\beta H_\omega),$$

then the required upper estimate for $E_n(C^\psi_\beta H_\omega)$ follows from (13).

For further reasoning, we need the one statement, which follows from the results of book [10, p. 65]. We will give a few notations before formulating it. Let $f$ be a summable function, whose Fourier series have the form (1). Further, let $\lambda_n = \{\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)\}$ be a collection of continuous functions on $[0, 1]$ such that $\lambda(k/n) = \lambda_k^{(n)}$, $k = 0, n$, $n \in \mathbb{N}$, where $\lambda_k^{(n)}$ are elements of the triangular matrix $\Lambda = \|\lambda_k^{(n)}\|$, $k = 1, n$, $\lambda_0^{(n)} = 1$, that determine a polynomial of the form

$$U_n(f; x; \Lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}.$$

The following statement is true:

**Lemma A** [10, p. 65]. Suppose that $f \in C^\psi_\beta H_\omega$ and $\tau_n(u)$ is the continuous function defined by relation

$$\tau_n(u) = \tau_n(u; \lambda; \psi) = \begin{cases} (1 - \lambda_n(u))\psi(1)u, & 0 \leq u \leq \frac{1}{n}, \\ (1 - \lambda_n(u))\psi(1/n), & \frac{1}{n} \leq u \leq 1, \\ \psi(1/u), & u \geq 1, \end{cases}$$

and such that its Fourier transform

$$\hat{\tau}_n(t) := \hat{\tau}_n(t; \beta) = \frac{1}{\pi} \int_0^\infty \tau_n(u) \cos \left(ut + \frac{\beta \pi}{2}\right) du, \quad \beta \in \mathbb{R},$$
is summable on the whole real line, i.e. \( \int_{-\infty}^{\infty} |\hat{\tau}_n(t)| \, dt < \infty \). Then at any point \( x \) the following equality holds:

\[
f(x) - U_n(f; x; \Lambda) = \int_{-\infty}^{\infty} f_\beta(x + \frac{t}{n}) \tilde{\tau}_n(t) \, dt, \quad n \in \mathbb{N}.
\]

(17)

Using Lemma A, let us show that

\[
f(x) - U^{\psi}_{n-1}(f; x) = \int_{-\infty}^{\infty} f_\beta(x + \frac{t}{n}) \tilde{\tau}_n(t) \, dt, \quad \forall f \in C^\varphi_{\beta}H_\omega, \quad n \in \mathbb{N},
\]

(18)

where \( \tilde{\tau}_n(t) \) is the Fourier transform of the function

\[
\tau_n(u) = \tau_n(u; \psi) = \begin{cases} 
\psi(n)u^2, & 0 \leq u \leq 1, \\
\psi(nu), & u \geq 1.
\end{cases}
\]

(19)

Since polynomial (12) can be represented in the form

\[
U^{\psi}_{n-1}(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda^{\psi}(k/n)(a_k \cos kx + b_k \sin kx),
\]

where \( \lambda^{\psi}(k/n) \) are the values of continuous function

\[
\lambda^{\psi}(u) = \lambda^{\psi}_n(u) = \begin{cases} 
1 - \frac{\psi(n)}{\psi(1)} \frac{u}{n}, & 0 \leq u \leq \frac{1}{n}, \\
1 - \frac{\psi(n)}{\psi(n)} u^2, & \frac{1}{n} \leq u \leq 1,
\end{cases}
\]

(20)

at the points \( u = k/n \) and

\[
\tau_n(u) = \tau_n(u; \psi) = \begin{cases} 
(1 - \lambda^{\psi}(u))\psi(1)nu, & 0 \leq u \leq \frac{1}{n}, \\
(1 - \lambda^{\psi}(u))\psi(nu), & \frac{1}{n} \leq u \leq 1, \\
\psi(nu), & u \geq 1,
\end{cases}
\]

then it follows from Lemma A that for proving (18) it is sufficient to establish the inequality

\[
\int_{-\infty}^{\infty} |\tilde{\tau}_n(t)| \, dt < \infty.
\]

(21)

With this aim we put

\[
\mu_n(u) = \begin{cases} 
\psi(n)(u^2 - u), & 0 \leq u \leq 1, \\
0, & u \geq 1,
\end{cases} \quad \nu_n(u) = \tau_n(u) - \mu_n(u).
\]

Integrating twice by parts, we get

\[
\hat{\mu}_n(t) := \hat{\mu}_n(t; \beta) = \frac{1}{\pi} \int_{0}^{\infty} \mu_n(u) \cos \left( ut + \frac{\beta\pi}{2} \right) \, du = \frac{O(1)}{t^2}, \quad t > 0,
\]
which yields
\[ \int_{|t| \geq 1} |\hat{\mu}_n(t)| \, dt < \infty. \] (22)

It is obvious that
\[ \int_{|t| \leq 1} |\hat{\mu}_n(t)| \, dt < \infty. \] (23)

Taking (22), (23) together and using the estimates
\[ \int_{-\infty}^{\infty} |\hat{\nu}_n(t)| \, dt < \infty \quad \forall \psi \in \mathcal{M}_0', \]
(see, e.g., [11, p. 174]) and
\[ |\hat{\tau}_n(t)| \leq |\hat{\mu}_n(t)| + |\hat{\nu}_n(t)|, \]
we obtain (21).

Furthermore, since the function \( \tau_n(u) \) satisfies all conditions of Lemma 3 from [14] according to which
\[ \tau_n(u) = \int_{-\infty}^{\infty} \cos \left( ut + \frac{\beta \pi}{2} \right) \hat{\tau}_n(t) \, dt, \quad u \geq 0, \]
we have
\[ \int_{-\infty}^{\infty} \hat{\tau}_n(t) \, dt = \frac{\tau_n(0)}{\cos \frac{\beta \pi}{2}} = 0, \quad \beta \neq 2l - 1, \quad l \in \mathbb{Z}. \]
If \( \beta = 2l - 1, \quad l \in \mathbb{Z} \), the equality \( \int_{-\infty}^{\infty} \hat{\tau}_n(t) \, dt = 0 \) is obvious, because \( \hat{\tau}_n(t) \) is odd. Hence, starting from (18) we can write
\[ f(x) - U_{n-1}^\psi(f; x) = \int_{-\infty}^{\infty} \left( f^\psi_{\beta}(x + \frac{t}{n}) - f^\psi_{\beta}(x) \right) \hat{\tau}_n(t) \, dt \quad \forall f \in C^\psi_{\beta} H_\omega, \quad n \in \mathbb{N}. \] (24)

Since \( f^\psi_{\beta} \in H_\omega^0 \) and, as it is not hard to see, for every \( \varphi \in H_\omega^0 \) function \( \varphi_1(u) = \varphi(u + h), \quad h \in \mathbb{R}, \) also belongs to \( H_\omega^0 \), then using the notation
\[ \delta(t, \varphi) = \varphi(t) - \varphi(0), \]
it follows from (24) that
\[ \mathcal{E}_n(C^\psi_{\beta} H_\omega) \leq \sup_{\varphi \in H_\omega^0} \left| \int_{-\infty}^{\infty} \left( \varphi \left( \frac{t}{n} \right) - \varphi(0) \right) \hat{\tau}_n(t) \, dt \right| = \sup_{\varphi \in H_\omega^0} \left| \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \hat{\tau}_n(t) \, dt \right|. \] (25)

Now we shall simplify the integral in the right-hand side of (25) without loss of its principal value. The following relations are true:
\[ \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \hat{\tau}_n(t) \, dt = \]
\[= \cos \frac{\beta \pi}{2} \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \int_{0}^{\infty} \tau_{n}(u) \cos ut du dt - \sin \frac{\beta \pi}{2} \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \int_{0}^{\infty} \tau_{n}(u) \sin ut du dt = \]

\[= \cos \frac{\beta \pi}{2} \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \int_{0}^{\infty} \tau_{n}(u) \cos ut du dt - \sin \frac{\beta \pi}{2} \left( \int_{|t| \geq 1} \delta \left( \frac{t}{n}, \varphi \right) \int_{0}^{\infty} \tau_{n}(u) \sin ut du dt \right) \]

Integrating by parts, taking into account the equality \( \tau_{n}(0) = \tau_{n}(\infty) = 0 \) and assuming that \( \psi'(u) := \psi'(u+) \), we have

\[\int_{0}^{\infty} \tau_{n}(u) \cos ut du = -\frac{1}{t} \int_{0}^{\infty} \tau_{n}'(u) \sin ut du = \]

\[= -\frac{2\psi(n)}{t} \int_{0}^{1} u \sin ut du - \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \sin ut du, \]

and similarly

\[\int_{0}^{\infty} \tau_{n}(u) \sin ut du = \frac{2\psi(n)}{t} \int_{0}^{1} u \cos ut du + \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \cos ut du. \]

Combining (26)–(28), we obtain

\[\int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \tilde{\tau}_{n}(t) dt = \]

\[= -\sin \frac{\beta \pi}{2} \int_{|t| \leq 1} \delta \left( \frac{t}{n}, \varphi \right) \int_{1}^{\infty} \psi(nu) \sin ut du dt + r_{n}(\psi, \varphi, \beta), \quad \varphi \in H_{0}^{0}, \quad n \in \mathbb{N}, \]

where

\[r_{n}(\psi, \varphi, \beta) = \cos \frac{\beta \pi}{2} \left( -2\psi(n) \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \frac{1}{t} \int_{0}^{1} u \sin ut du dt - \right. \]

\[-n \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \frac{1}{t} \int_{1}^{\infty} \psi'(nu) \sin ut du dt \] -

\[-\sin \frac{\beta \pi}{2} \left( 2\psi(n) \int_{|t| \geq 1} \delta \left( \frac{t}{n}, \varphi \right) \frac{1}{t} \int_{0}^{1} u \cos ut du dt + \right. \]

\[+n \int_{|t| \geq 1} \delta \left( \frac{t}{n}, \varphi \right) \frac{1}{t} \int_{1}^{\infty} \psi'(nu) \cos ut du dt + \]

\[+ \int_{|t| \leq 1} \delta \left( \frac{t}{n}, \varphi \right) \int_{0}^{1} \tau_{n}(u) \sin ut du dt \] = \frac{\cos \frac{\beta \pi}{2}}{\pi} \sum_{i=1}^{2} J_{i,n} - \frac{\sin \frac{\beta \pi}{2}}{\pi} \sum_{i=3}^{5} J_{i,n}. \]
Let us show that
\[ r_n(\psi, \varphi, \beta) = O(1)\psi(n)\omega(1/n). \] (31)

Since for \( t \in [-1, 1] \) the quantity
\[ \frac{1}{t} \int_0^1 u \sin ut \, du, \]
is bounded by a constant, then using the inequality \(|\delta(t, \varphi)| \leq \omega(|t|)\), we get
\[ J_{1,n} = -2\psi(n) \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_0^1 u \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n). \] (32)

To estimate the integral in (32) we establish the following auxiliary statements.

**Lemma 1.** On every interval \((x_k^{(i)}, x_{k+1}^{(i)})\), \(x_k^{(i)} = (2k - 1 + i)\pi/2a\), \(i = 0, 1\), \(k \in \mathbb{N}\), \(a > 0\), the function
\[ \int_x^\infty \frac{1}{t} \int_0^a u^s \sin \left(ut + \frac{i\pi}{2}\right) du \, dt, \quad x > 0, \quad s \geq 1, \]
has at least one zero.

**Proof.** We will give a proof of the lemma only for the case \(i = 0\), because the proof in case \(i = 1\) is similar. On the basis of the estimate \(|\int_x^\infty \sin t \, dt| \leq \frac{2}{x}, \quad x > 0\) (see, e.g., [1, p. 5], [9, p. 343]) it is simple to see that the integral
\[ \int_x^\infty \frac{u^s \sin ut}{t} \, dt = u^s \int_0^\infty \frac{\sin t}{t} \, dt, \]
converges uniformly with respect to \(u \in [0, a), \ \ a > 0\). Therefore, changing the order of integration, we obtain
\[ S(x) := \int_x^\infty \frac{1}{t} \int_0^a u^s \sin ut \, du \, dt = \int_0^a u^s \int_x^\infty \frac{\sin ut}{t} \, dt \, du. \]

Making the change of variables and integrating by parts, we have
\[ S(x) = \int_0^a u^s \int_x^{1/u} \frac{\sin t}{t} \, dt \, du = \frac{1}{s+1} \left( \int_x^\infty \frac{\sin t}{t} \, dt + \int_0^a u^s \sin ux \, du \right) = \frac{1}{s+1} \left( a^{s+1} \int_x^\infty \frac{\sin t}{t} \, dt + \int_0^a u^s \sin ux \, du \right) = a^{s+1} \int_x^\infty \frac{\sin t}{t} \, dt - a^s \cos ax + \frac{s}{x} \int_0^a u^{s-1} \cos ux \, du. \]

Hence, taking into account the equation
\[ \int_x^\infty \frac{\sin t}{t} \, dt = \frac{\cos ax}{ax} - \int_x^\infty \frac{\cos t}{t^2} \, dt, \]
we get
\[ S(x) = \frac{1}{s+1} \left( -x^{s+1} \int_x^\infty \frac{\cos t}{t^2} dt + \frac{s}{x^{s+1}} \int_0^{ax} u^{s-1} \cos u \, du \right). \]  
(33)

On every interval \((t_j, t_{j+1}),\) \(t_j = (2j+1)\pi/2, j = 0, 1, \ldots,\) the function \(\int_x^\infty \frac{\cos t}{t^2} dt\) vanishes with a change of sign at some point \(\tilde{x}_j.\) Since
\[ \int_{\pi/2}^\infty \frac{\cos t}{t^2} dt = -\int_{\pi/2}^\infty \frac{\sin t}{t} dt < 0, \]
then for any \(k \in \mathbb{N}\)
\[ \text{sign} \int_{(2k-1)\pi/2}^\infty \frac{\cos t}{t^2} dt = (-1)^k. \]  
(34)

Further, we have
\[ \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \alpha_0 + \sum_{j=1}^{k-1} \alpha_j, \]
where
\[ \alpha_0 = \int_0^{\pi/2} u^{s-1} \cos u \, du, \quad \alpha_j = \int_{(2j-1)\pi/2}^{(2j+1)\pi/2} u^{s-1} \cos u \, du. \]

If \(k = 1,\) then
\[ \text{sign} \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \text{sign} \alpha_0 = 1. \]  
(35)

Let \(k = 2, 3, \ldots\) Since the function \(u^{s-1}\) does not decrease \((s \geq 1)\) for \(u \geq 0,\) we can write
\[ |\alpha_0| < |\alpha_j| \leq |\alpha_{j+1}|, \quad j \geq 1, \]
and respectively
\[ \text{sign} \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \text{sign} \int_0^{(2k-1)\pi/2} u^{s-1} \cos u \, du = (-1)^{k+1}, \quad k = 2, 3, \ldots. \]  
(36)

Taking account of (33)–(36), we have
\[ \text{sign} S\left(\frac{2k-1}{2a} \pi\right) = (-1)^{k+1}, \quad k \in \mathbb{N}, \quad a > 0. \]  
(37)

The function \(S(x)\) is continuous for any \(x > 0.\) Therefore, it follows from (37) that on every interval \((x_k, x_{k+1}),\) where \(x_k = (2k-1)\pi/2a, \quad k \in \mathbb{N}, \quad a > 0,\) the function \(S(x)\) has the required zero. Lemma 1 is proved.

**Lemma 2.** Let \(\varphi \in H_\omega,\) \(1 \leq a \leq n, \quad n \in \mathbb{N}\) and \(s \geq 1.\) Then for \(i = 0, 1,\) the following estimate holds:
\[ \int_{|t| \geq 1} \left( \varphi\left(\frac{t}{n}\right) - \varphi(0) \right) \frac{1}{t} \int_0^{ap/n} u^s \sin\left(ut + \frac{i\pi}{2}\right) \, du \, dt = O(1)\omega(1/n), \]
where \(O(1)\) is a quantity uniformly bounded in \(n, \varphi, a\) and \(s.\)
Proof. Making the change of variables, we get

\[ \int_{|t|\geq 1} \left( \varphi\left( \frac{t}{n} \right) - \varphi(0) \right) \frac{1}{t} \int_{0}^{a/n} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt = \]

\[ = \frac{1}{n^{s+1}} \int_{|t|\geq 1/n} (\varphi(t) - \varphi(0)) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt, \quad i = 0, 1. \quad (39) \]

Let us denote by \( t^{(i)}_k \) the zero of function

\[ \int_{x}^{\infty} \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt, \quad i = 0, 1, \]

on interval \((x_k^{(i)}, x_{k+1}^{(i)})\), \( x_k^{(i)} = \frac{2k-1+i}{2a} \pi \), which exists according to Lemma 1. Using the notation \( \delta(t) = \varphi(t) - \varphi(0) \), we have

\[ \left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt \right| = \left| \int_{1/n}^{t^{(i)}_k} \delta(t) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt + \right. \]

\[ + \sum_{k=1}^{\infty} \int_{t^{(i)}_k}^{t^{(i)}_{k+1}} \left( \delta(t) - \delta(t^{(i)}_k) \right) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt \right| \leq \]

\[ \leq \omega(t^{(i)}_1) \int_{1/n}^{t^{(i)}_1} \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt + \omega(\Delta_i) \int_{t^{(i)}_1}^{\infty} \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt, \quad (40) \]

where \( \Delta_i = \sup_k (t^{(i)}_{k+1} - t^{(i)}_k) \). Since \( t^{(i)}_1 < \frac{2\pi}{a} \) and \( \Delta_i < \frac{2\pi}{a} \), it follows from (40) that

\[ \left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt \right| < \omega\left( \frac{2\pi}{a} \right) \int_{1/n}^{\infty} \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt. \quad (41) \]

After integrating by parts it is easy to see, that

\[ \left| \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \right| \leq \frac{2a^s}{t}, \quad t > 0, \quad i = 0, 1. \quad (42) \]

From (41) and (42) follows the inequality

\[ \left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt \right| < 2a^s \omega\left( \frac{2\pi}{a} \right) \int_{1/n}^{\infty} \frac{dt}{t^2} = 2a^s \omega\left( \frac{2\pi}{a} \right) n \leq \]

\[ \leq 2a^s \left( \frac{2\pi n}{a} + 1 \right) \omega\left( \frac{1}{n} \right) n < 8a^s - 1\pi n^2 \omega\left( \frac{1}{n} \right) \leq 8\pi n^{s+1} \omega\left( \frac{1}{n} \right), \quad i = 0, 1. \quad (43) \]

The estimate

\[ \int_{-\infty}^{-1/n} \delta(t) \frac{1}{t} \int_{0}^{a} u^s \sin\left( ut + \frac{i\pi}{2} \right) du \, dt = O(1)n^{s+1} \omega(1/n), \quad i = 0, 1, \quad (44) \]
is similarly proved. Comparing relations (43), (44) and (39), we obtain (38). Lemma 2 is proved.

Applying Lemma 2 to the integral in (32) and, at the same time, to $J_{3,n}$, we have

$$J_{1,n} = O(1)\psi(n)\omega(1/n),$$

$$J_{3,n} = O(1)\psi(n)\omega(1/n).$$

In the monograph [11, pp. 212, 216, see relations (4.26') and (4.42), (4.45), (4.46)] it is shown, that

$$J_{2,n} = O(1)\psi(n)\omega(1/n), \quad \forall \psi \in M_0,$$

and

$$J_{4,n} = O(1)\psi(n)\omega(1/n) \quad \forall \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z}.$$

Since $|\tau_n(u)| \leq \psi(n), \quad u \in [0,1]$, it is clear that

$$J_{5,n} = O(1)\psi(n)\omega(1/n).$$

Comparing (30), (45)–(49), we arrive at (31). Then from (29) for any function $\varphi \in H^0$ and $n \in \mathbb{N}$, we obtain

$$\int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \tau_n(t) dt = -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_{-\infty}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n) =$$

$$= -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_{0}^{1} \left( \delta\left(\frac{t}{n}, \varphi\right) - \delta\left(-\frac{t}{n}, \varphi\right) \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n), \quad \psi \in M_0, \quad \beta \in \mathbb{R}.$$

Since

$$\int_{1}^{\infty} \psi(nu) \sin ut \, du > 0, \quad t \in (0,1], \quad \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z},$$

(see, e.g., [12, p. 143]) and

$$\int_{0}^{1} \omega\left(\frac{2t}{n}\right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt =$$

$$= \int_{0}^{1/n} \psi\left(\frac{1}{t}\right) \int_{t}^{\infty} \omega\left(\frac{2t}{n}\right) dt + O(1)\psi(n)\omega(1/n), \quad \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z},$$

(see [8, p. 528]), from (25) and (50) we obtain (13). Putting together inequalities (13) and (14) we find a required estimate for quantity (4):

$$E_n(C^\psi_\beta H_\omega) \leq \frac{1}{\pi} \left|\sin \frac{\beta \pi}{2}\right| \int_{0}^{1/n} \psi\left(\frac{1}{t}\right) \int_{t}^{\infty} \omega\left(\frac{2t}{n}\right) dt + O(1)\psi(n)\omega(1/n), \quad \psi \in M'_0, \quad \beta \in \mathbb{R}. \quad (53)$$

2. Let us find a lower bound for $E_n(C^\psi_\beta H_\omega)$. 

Let \( \varphi(t) \) be an odd \( 2\pi/n \)-periodic function defined on \([0, \pi/n]\) by the equalities

\[
\varphi(t) = \begin{cases} 
\frac{c_\omega}{2} \omega(2t), & t \in [0, \pi/2n], \\
\frac{c_\omega}{2} \omega(\frac{2\pi}{n} - 2t), & t \in [\pi/2n, \pi/n], 
\end{cases}
\]

where \( c_\omega = 1 \) if \( \omega(t) \) is a convex upwards modulus of continuity and \( c_\omega = 2/3 \) otherwise.

As shown in [10, pp. 83–85] if \( \omega(t) \) is an arbitrary modulus of continuity, then

\[
|\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in [-\pi/2n, \pi/2n].
\]

This implies that

\[
|\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in \mathbb{R},
\]

and, hence, \( \varphi \in H_\omega \). We denote by \( f^*(\cdot) \) the function from the set \( C_\beta^\psi H_\omega, \psi \in \Psi' \), whose \((\psi, \beta)\)-derivative \( f^*_\beta(t) \) coincides with the function \( \varphi_n(t) \) on a period. By relations (2), such a function \( f^*(\cdot) \) exists.

In virtue of formula (3.4) from the book [10, Chap. 2, Subsec. 3.1] the following equality holds for any \( f \in C_\beta^\psi H_\omega, \psi \in \Psi' \):

\[
f(x) - U_{n-1}(f; x; \Lambda) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \left( \sum_{k=1}^{\infty} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) \right) dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N},
\]

(54)

where \( U_{n-1}(f; x; \Lambda) \) is a trigonometric polynomial of the form (15), such that \( \lambda_n^{(n)} = 0 \). Since function \( \varphi_n(t) \) is odd \( 2\pi/n \)-periodic, the equalities

\[
\int_{-\pi}^{\pi} \varphi_n(t) \sin kt dt = 0, \quad k = 1, 2, \ldots, n - 1, \quad n \geq 2,
\]

(55)

(see, e.g., [6, p. 159]) and

\[
\varphi_n \left( \frac{i\pi}{n} + t \right) = (-1)^i \varphi_n(t), \quad i \in \mathbb{Z},
\]

hold. Then, using relation (54) for \( f^*(\cdot) \), we obtain

\[
f^* \left( \frac{i\pi}{n} \right) - U_{n-1}(f^*; \frac{i\pi}{n}; \Lambda) = \frac{(-1)^i}{\pi} \int_{-\pi}^{\pi} \varphi_n(t) \left( \sum_{k=1}^{\infty} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) \right) dt = \frac{(-1)^i}{\pi} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=1}^{\infty} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) dt = \]

\[
= \frac{(-1)^i}{\pi} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=1}^{\infty} \psi(k) \cos \left( kt + \frac{\beta\pi}{2} \right) dt = \]
\[
(58) = \frac{(-1)^i}{\pi} \sin \frac{\beta \pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt \, dt, \quad i \in \mathbb{Z}, \quad n = 2, 3, \ldots \tag{56}
\]

It is obvious from this that there exist 2n points \( t_i = \frac{\pi i}{n}, \quad i = 0, 1, \ldots, 2n - 1 \), on the period \([0, 2\pi]\), we find
\[
f^*(x) - U_{n-1}(f^*; \Lambda),
\]
takes values with alternating signs. Then by the de la Vallée Poussin theorem [7] (see also [10, p. 312], [11, p. 491]), we find
\[
E_n(f^*) \geq \frac{1}{\pi} \sin \frac{\beta \pi}{2} \int_{-\pi}^{\pi} \varphi_n(t) \sum_{k=n}^{\infty} \psi(k) \sin kt \, dt, \quad \psi \in \mathcal{M}', \tag{57}
\]
where
\[
E_n(f^*) = \inf_{t_{n-1}} \| f^*(\cdot) - t_{n-1}(\cdot) \|_C, \quad n \in \mathbb{N}.
\]

From (56) and (57) it follows, in particular, that
\[
E_n(f^*) \geq | f^*(0) - U_{n-1}(f^*; 0; \Lambda) |, \quad n = 2, 3, \ldots \tag{58}
\]

Inequality (58) is satisfied for triangular matrix \( \Lambda = \| \lambda_k^{(n)} \|, \quad k = \overline{1, n} \), such that \( \lambda_k^{(n)} = 0 \).

Let’s define its remaining elements in the following way:
\[
\lambda_k^{(n)} = \lambda^\psi(k/n), \quad k = \overline{1, n-1}, \quad n \in \mathbb{N},
\]
where \( \lambda^\psi(\cdot) \) is defined by (20). Since in this case
\[
U_{n-1}(f^*; 0; \Lambda) = U^\psi_{n-1}(f^*; 0),
\]
then from (58) we obtain, taking the inequality \( E_n(C^\psi_{\beta} H_\omega) \geq E_n(f^*) \) into account,
\[
E_n(C^\psi_{\beta} H_\omega) \geq | f^*(0) - U^\psi_{n-1}(f^*; 0) |, \quad n = 2, 3, \ldots, \psi \in \mathcal{M}'. \tag{59}
\]

By virtue of (24) and (50)
\[
f^*(0) - U^\psi_{n-1}(f^*; 0) = \int_{-\infty}^{\infty} \left( f^\psi_{\beta} \left( \frac{t}{n} \right) - f^\psi_{\beta} (0) \right) \hat{\tau}_n(t) \, dt =
\]
\[
= -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_0^1 \varphi_n \left( \frac{t}{n} \right) - \varphi_n \left( -\frac{t}{n} \right) \psi(nu) \sin ut \, du \, dt + O(1) \psi(n)\omega(1/n) =
\]
\[
= -c_o \frac{\sin \beta \pi}{2} \int_0^1 \omega \left( \frac{2t}{n} \right) \int_1^{\infty} \psi(nu) \sin ut \, du \, dt + O(1) \psi(n)\omega(1/n), \quad \psi \in \mathcal{M}_0'. \tag{60}
\]

Combining (51), (52), (59) and (60), we arrive at the desired estimate
\[
E_n(C^\psi_{\beta} H_\omega) \geq \frac{c_o}{\pi} \sin \frac{\beta \pi}{2} \int_0^{1/n} \psi \left( \frac{1}{t} \right) \omega \left( \frac{t}{n} \right) \, dt + O(1) \psi(n)\omega(1/n), \psi \in \mathcal{M}_0', \quad \beta \neq 2l, \quad l \in \mathbb{Z}. \tag{61}
\]

From (53) and (61) we obtain formula (6). Theorem 1 is proved.
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Anatolii Serdyuk
Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine
E-mail: serdyuk@imath.kiev.ua

Ievgen Ovsii
Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine
E-mail: ievgen.ovsii@gmail.com

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