Applying incomplete statistics to nonextensive systems with different $q$ indices

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Abstract

The nonextensive statistics based on the $q$-entropy $S_q = -\frac{\sum_i^n (p_i - p_i^q)}{1-q}$ has been so far applied to systems in which the $q$ value is uniformly distributed. For the systems containing different $q$’s, the applicability of the theory is still a matter of investigation. The difficulty is that the class of systems to which the theory can be applied is actually limited by the usual nonadditivity rule of entropy which is no more valid when the systems contain non uniform distribution of $q$ values. In this paper, within the framework of the so called incomplete information theory, we propose a more general nonadditivity rule of entropy prescribed by the zeroth law of thermodynamics. This new nonadditivity generalizes in a simple way the usual one and can be proved to lead uniquely to the $q$-entropy.

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1 Introduction

This work addresses the problem of the application of the nonextensive statistics based on a $q$-entropy to systems containing non uniform distribution of
values, i.e., containing subsystems which have different $q$’s. The $q$-entropy for open systems is given by

$$S_q(p_1, p_2, \ldots, p_v, q) = -\frac{\sum_{i=1}^{v} (p_i - p_i^q)}{1 - q}, \quad q \geq 0$$

(1)

where $p_i$ is the probability that the system is found at the state $i$, and $v$ is the total number of states occupied by the system (let Boltzmann constant $k$ be unity). This entropy is proposed first by Havrda, Charvat[1, 2] and Daroczy[3], who deleted the additivity requirement from the entropy axioms of Shannon[4], as a possible generalization of Shannon entropy $S_1 = -\sum_{i=1}^{v} p_i \ln p_i$. Daroczy[3] further discussed in detail the properties of this entropy, including its nonadditivity for a composite system $A + B$ containing independent subsystems $A$ and $B$ :

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B).$$

(2)

In the last decades, a nonextensive statistical mechanics (NSM) is derived by Tsallis and coworkers from maximizing $S_q$[5, 6]. A main character of NSM is the $q$-exponential probability distribution :

$$p_i = \frac{1}{Z_q} [1 - a \beta E_i]^{1/a} \quad [\cdot] > 0.$$ 

(3)

where $a$ equals $1 - q$ or $q - 1$ according to the maximization process of $S_q$[7] and $E_i$ can be the energy (or any other thermodynamic quantity introduced into the process of maximum entropy) of the system at the state $i$. The validity range of NSM is still under investigation. Actually, a point of view is widely accepted that NSM can be used to describe nonequilibrium systems at stationary state whose probability distribution is time independent. This position is supported by the fact that all the actual applications of NSM and its $q$-exponential distribution in Eq.(3) are related to critical, nonequilibrium and chaotic systems.

In this paper, we will focalize our attention on an important question about the applicability of NSM to nonextensive systems containing subsystems with different $q$’s. For this purpose, we take it for granted that : 1) the $q$-entropy, as an information or uncertainty measure with respect to the probability $p_i$, is valid for some nonequilibrium nonextensive systems in steady...
or stationary states with time independent probability distribution $p_i$; 2) the Janyes’ inference method of maximum entropy[8, 9] applies for these states, i.e., the time independent probability distribution on these states can be deduced from maximizing the $q$-entropy.

The above assumptions are necessary for NSM to be applied to nonequilibrium systems with the distribution given by Eq.(3). Due to the maximum $q$-entropy, the framework of the conventional equilibrium thermodynamics (CET) can be copied for stationary nonequilibrium systems. Essential for this work is the establishment of the zeroth law or the equality between the intensive variable $\beta$ at state maximizing entropy $S_q$. It is worth noticing that, due to the different formalisms of NSM proposed in the past 15 years, the intensive parameter $\beta$ has several definitions and interpretations which sometimes cause confusion. A comment on this subject was given in ref.[10]. In this work, we choose a formalism allowing $\beta = \frac{\partial S_q}{\partial U_q}$ (we will call it inverse temperature from now on), here $U_q$ is the expectation of $E_i$ discussed below.

The application of NSM to systems without uniform $q$ value distribution is fundamental because a large number of systems in Nature, especially the systems far from equilibrium, the chaotic and fractal systems, are inhomogeneous and may be divided into subsystems with different nonextensive property and $q$ values. If NSM does not allow treatments of such systems, there would be an handicap in its theoretical formulation. In other words, if NSM can separately treat $A$ and $B$ each having its own $q$, it is also expected to be valid for the total system $A + B$ and to provide method to derive $q_{A+B}$ from $q_A$ and $q_B$. This problem was recently discussed by many scientists[11, 12, 13] on the basis of Eq.(2). The attempts were interesting but a rigorous mathematical formulation of zeroth law relating $\beta(A)$ and $\beta(B)$ is still missing. This result is not surprising because Eq.(2) is only the nonadditivity rule for the systems containing subsystems with same $q$. Its application to different $q$-systems is in fact forbidden.

The aim of this paper is to show that, by using a more general nonextensive rule of entropy which generalizes Eq.(2), the establishment of zeroth law relating $\beta(A)$ and $\beta(B)$ is possible for the systems having different $q$’s. This approach was recently described within the formalism of NSM based on the usual normalized probability[14]. In this paper, we are interested in extending the approach to the formalism of incomplete statistics proposed several years ago by one of us[7, 15, 16], and providing in addition a detailed description of the calculus allowing one to determine the $q$ of composite systems
from the $q$'s of the subsystems.

Incomplete statistics is a version of NSM based on the notion of *incomplete information* associated with the normalization

$$\sum_{i=1}^{w} p_i^q = 1 \quad (q \geq 0), \quad (4)$$

here $w$ is only the number of states which are countable or accessible to our treatments and may be larger or smaller than the number $v$ of the physical states occupied by the systems\cite{16, 17}. It is proved\cite{16, 17} that this incomplete normalization may arise for the systems whose phase space is occupied in heterogeneous (hierarchical or fractal) way and coarse grained. An example of such systems is the polymers described in coarse graining way by using contact matrix which makes it impossible to completely calculate the information contained in the sequence of monomers units\cite{18}. It has been shown\cite{16, 17} that $q = d_f/d$ for the systems whose phase space volume is simple fractal of dimension $d_f$, where $d \geq 1$ is the dimension of the phase space when it is smoothly occupied. In this case, $S_q$ can be recast into $S_q = \frac{1}{1-q} \sum_{i=1}^{w} p_i [15]$. For a composite system having joint probability given by the product of the probabilities of its subsystems, i.e., $p_{ij}(A + B) = p_i(A)p_j(B)$, we still have Eq.(2) with only the change $(1 - q)$ to $(q - 1)$\cite{7}.

## 2 A nonadditivity of entropy prescribed by zeroth law

Eq.(2) has been shown to be a special case of the composition rule\cite{19}

$$H[Q(A + B)] = H[Q(A)] + H[Q(B)] + \lambda_Q H[Q(A)] H[Q(B)], \quad (5)$$

where $H[Q]$ is certain differentiable function satisfying $H[0] = 0$, $\lambda_Q$ is a constant, and $Q$ is either $S_q$ or $U_q[20]$, which allows the establishment of zeroth law of thermodynamics in nonextensive systems. As a matter of fact, Eq.(5) has been established\cite{19, 20} for the class of systems containing only subsystems having the same $q$. It must be generalized for the systems whose subsystems have different $q$'s. This generalization is straightforward if we replace the Eq.(1) of reference \cite{19}, i.e., $S(A + B) = f\{S(A), S(B)\}$
for uniform $q$, by $H_q[S_q(A + B)] = f\{H_{q_A}[S_{q_A}(A)], H_{q_B}[S_{q_B}(B)]\}$ (or by $H_q[S_q(A + B)] = H_{q_A}[S_{q_A}(A)] + H_{q_B}[S_{q_B}(B)] + g\{H_{q_A}[S_{q_A}(A)], H_{q_B}[S_{q_B}(B)]\}$), where $H_q(S_q)$ is a functional depending on $q$'s with the same form for the composite system as for the subsystems, where $q$, $q_A$ and $q_B$ are the parameters of the composite system $A + B$, the subsystems $A$ and $B$, respectively. This functional $H_q(S_q)$ is necessary for the equality to hold in view of the different $q$'s in the entropies of different subsystems. The function $f$ (or $g$) is to be determined by the zeroth law. Now repeating the mathematical treatments described in the references [19, 20], we find

$$H_q[Q(A + B)] = H_{q_A}[Q(A)] + H_{q_B}[Q(B)] + \lambda_Q H_{q_A}[Q(A)] H_{q_B}[Q(B)].$$

(6)

Eq.(5) turns out to be a special case of Eq.(6), and Eq.(2) corresponds to a $H_q[S_q]$ which is identity function. Now for $A$ and $B$ each having its own $q$, in view of the fact that the $q$-entropy of Eq.(1) must have the same form for any system, a simple choice is $H_q[S_q] = (q - 1)S_q$ for the version of $S_q$ within incomplete statistics, which means

$$(q - 1)S_q(A + B) = (q_A - 1)S_{q_A}(A) + (q_B - 1)S_{q_B}(B) + \lambda_S(q_A - 1)(q_B - 1)S_{q_A}(A)S_{q_B}(B).$$

(7)

This nonadditivity generalizes Eq.(2) which is recovered when $q = q_A = q_B$.

3 Uniqueness of $S_q$

Eq.(2) has been shown to lead uniquely to the $q$-entropy of Eq.(1)[21, 22] when the product joint probability holds. This result is not complete because it was obtained only for the systems of uniform $q$ value distribution. The reasoning can also be straightforwardly generalized to inhomogeneous systems containing different $q$'s. In order to do this, the axiom $[\Pi]^*$ of the reference [22], i.e., Eq.(2), should be replaced by Eq.(7), with other axioms unchanged. In this way, the Eq.(16) of [22], i.e., $L_q(r^m) = \frac{1}{(1-q)^m} \{1 + (1-q)L_q(r)^m - 1\}$ should be replaced by $[1 + (q - 1)L_q(r^m)] = \prod_{l=1}^{m}[1 + (q_l - 1)L_q(r)]$ derived from Eq.(7), where $L_q$ is the functional of the $q$-entropy, $q$ and $q_l$ are respectively the parameter of the composite system and of the subsystems labelled by $l$ ($l = 1, 2, ... m$), and $r$ the number of states in each subsystem. This relationship can be written as $\ln[1 + (q - 1)L_q(r^m)] = \sum_{l=1}^{m} \ln[1 + (q_l - 1)L_q(r)]$. 

5
The same mathematics as in [21, 22] leads to $S_q(1/r, q_l) = L_q(r) = \frac{q^{q-1}-1}{q-1}$ and $S_q(p_1, p_2, \ldots p_v, q_l) = \frac{\sum_{i=1}^v p_i}{q-1}$.

This uniqueness of $S_q$ implies that Eq.(7) is intrinsically a possible composition rules of the $q$-entropy which becomes in this way invariant with respect to the inhomogeneity of $q$ values in the systems of interest. The $q$ value in the entropy of the composite system can be determined from the $q$’s of the subsystems, which will be described below.

4 A zeroth law

Now from Eq.(7), according to our starting assumption that the stationary state of the composite system $A+B$ maximizes its $q$-entropy, i.e., $dS(A+B) = 0$, we get:

$$\frac{(q_A-1)dS(A)}{1+(q_A-1)S(A)} + \frac{(q_B-1)dS(B)}{1+(q_B-1)S(B)} = 0 \quad (8)$$

or

$$\frac{(q_A-1)dS(A)}{\sum_i p_i(A) \cdot \frac{dS(A)}{p_i(A)}} + \frac{(q_B-1)dS(B)}{\sum_i p_i(B) \cdot \frac{dS(B)}{p_i(B)}} = 0.$$

As to the nonadditivity of the quantity $U_q$, it is in fact uniquely determined by the product joint probability $p_{ij}(A + B) = p_i(A)p_j(B)$ which implies that, if we want to split the thermodynamics of the composite system into those of the subsystems, the expectation should be defined by $U_q = \sum_i p_i E_i$ or $U_q' = \sum_i p_i E_i/\sum_i p_i$ (it is evident that the normalized expectation $U_q = \sum_i p_i E_i$ is no more factorizable). It is easy to show that, using $U_q$ and Eq.(4) as constraints, the maximization of $S_q$ gives:

$$p_i = \frac{1}{Z_q} [1 - (q - 1) \beta E_i]^{1/q - 1} \quad [1] > 0 \quad (9)$$

where the partition function is given by $Z_q = \{\sum_i [1 - (q - 1) \beta E_i]^{q/q - 1}\}^{1/q}$. With some mathematics, we find

$$\sum_i p_i = Z_q^{q-1} + (q - 1) \beta U_q. \quad (10)$$

Then combining the product probability and Eq.(10), we get

$$Z_q^{1-q}(A + B) + (q - 1) \beta (A + B) U_q(A + B) \quad (11)$$

$$= [Z_q^{1-qA}(A) + (q_A - 1) \beta(A) U_q(A)] [Z_q^{1-qB}(B) + (q_B - 1) \beta(B) U_q(B)].$$
The energy conservation law of the total system \( dU_q(A + B) = 0 \) leads to

\[
\frac{(q_A - 1)\beta(A)dU_{q_A}(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)\beta(B)dU_{q_B}(B)}{\sum_i p_i(B)} = 0
\]  

(12)

which suggests the following nonadditivity :

\[
\frac{(q_A - 1)dU_{q_A}(A)}{\sum_i p_i(A)} + \frac{(q_B - 1)dU_{q_B}(B)}{\sum_i p_i(B)} = 0.
\]

(13)

This relationship should be considered as a generalization of the additivity rule \( dU(A) + dU(B) = 0 \) of CET. From Eq.(8) to Eq.(13), we see the necessity to choose the unnormalized expectation in order to split the thermodynamics of the composite system into those of the subsystems via the energy nonadditivity given by Eq.(13). By splitting thermodynamics, we means that the thermodynamics of the subsystems can be formulated exactly in the same way and with the same mathematical definition of all the thermodynamic variables and functions as for the total system. Without this splitting, the establishment of zeroth law would be impossible.

Now comparing Eq.(8) and Eq.(13), we obtain

\[
\beta(A) = \beta(B)
\]

(14)

with \( \beta(A) = \frac{\partial S_A}{\partial U_{q_A}(A)} \) and \( \beta(B) = \frac{\partial S_B}{\partial U_{q_B}(B)} \) which can be straightforwardly derived from Eq.(1) and Eq.(10). So this zeroth law is independent of whether or not the subsystems have the same \( q \).

## 5 How to determine the composite \( q \)

If \( q_A, q_B, p_i(A) \) and \( p_j(B) \) of the subsystems are well known, the parameter \( q \) of the composite system is uniquely determined by the product joint probability. Using the incomplete normalization of the joint probability, we obtain :

\[
\sum_{i=1}^{w_A} p_i^q(A) \sum_{i=1}^{w_B} p_i^q(B) = 1
\]

(15)

or

\[
\sum_{i=1}^{w_A} (p_i^{qA})^{q/q_A}(A) \sum_{i=1}^{w_B} (p_i^{qB})^{q/q_B}(B) = 1
\]

(16)
which means $q_A < q < q_B$ if $q_A < q_B$ and $q = q_A = q_B$ if $q_A = q_B$. In what follows, the above result will be generalized to more complicated composite system containing $N > 2$ subsystems each having its own $q$. Suppose an integer ensemble $\aleph_m = \mathbb{N} \cap [1, m] = \{1, 2, \ldots, m\}$. The subsystems are characterized by $[p_{i_k}, q_k, n_k]$ where $k \in \aleph_N$ and $i_k \in \aleph_{n_k}$. The joint probability is given by

$$p_{i_1, i_2 \ldots i_N} = \prod_{k=1}^{N} p_{i_k}$$

(17)

where

$$\sum_{i_k=1}^{n_k} p_{i_k}^{q_k} = 1 \quad \forall k \in \aleph_N.$$  

(18)

Suppose that there is a $q$ such that the joint probability is normalized as follows

$$\sum_{i_1, i_2 \ldots i_N = 1}^{n_1 n_2 \ldots n_N} p_{i_1, i_2 \ldots i_N}^q = 1,$$

(19)

then $q$ is determined by

$$\prod_{k=1}^{N} \sum_{i_k=1}^{n_k} p_{i_k}^q = 1.$$  

(20)

To see how to estimate $q$, let us first suppose that each subsystem has equiprobable states. One gets, for the $k^{th}$ subsystem, $p_k = (1/n_k)^{1/q_k}$. From Eqs.(17) to (20), we obtain

$$(\prod_{k=1}^{N} n_k)^{1/q} = \prod_{k=1}^{N} (n_k^{1/q_k}).$$

(21)

Property 1 : $q \in [\min q_k, \max q_k] \quad \forall k \in \aleph_N$.

Proof : Eq.(21) can be recast into $(1/q) \sum_k \ln n_k = \sum_k (1/q_k) \ln n_k$ or $(1/q) = [\sum_k (1/q_k) \ln n_k]/[\sum_k \ln n_k]$. So $1/q$ can be seen as a barycenter of the terms $1/q_k \geq 0$ with the coefficients $\ln n_k \geq 0$. Hence we have $q \in [\min q_k, \max q_k]$. The property 1 is proved.

We see that, if all the subsystems have the same number of states, we get

$$\left(1/q\right) = \frac{1}{N} \sum_{k=1}^{N} \left(1/q_k\right).$$

(22)
Property 2: The property 1 can be generalized to subsystems having non-equiprobable states.

Proof: Suppose $q > \max_k q_k \forall k \in \mathbb{N}_N$, we have $\sum_{i_k} p_{ik}^q < \sum_{i_k} p_{ik}^{q_k} = 1$ and, as a consequence, $\prod_k \sum_{i_k} p_{ik}^q = \sum_{i_1,i_2,...,i_N} p_{i_1,i_2,...,i_N}^q < 1$ which is at variance with the normalization of the joint probability. So $q \leq \max_k q_k \forall k \in \mathbb{N}_N$. In the same way, it can be shown that $q \geq \min_k q_k$. Property 2 is proved.

6 Conclusion

A general entropy composition rule prescribed by the zeroth law of thermodynamics is proposed for the systems containing subsystems with different $q$’s. The usual entropy composition rule of NSM can be recovered when $q$ is the same for all subsystems. The zeroth law, or the equality between the intensive parameters $\beta$ of each subsystems, is proved to be independent of the $q$ values of subsystems. It is shown that $q$-entropy $S_q$ is unique in the context of this general nonadditivity. A calculus is provided for deriving the $q$ value characterizing the composite system from the $q$ values and the probability distributions of subsystems.

It is worth mentioning that this work is carried out within incomplete statistics by using the unnormalized $q$-expectation. It is found that, if one wants to establish relations between $\beta(A)$ and $\beta(B)$ when the subsystems have different $q$’s, the usual expectation $U = \sum_i p_i^q E_i$ for incomplete distribution cannot be used due to the factorization of joint probability $p_{ij}(A + B) = p_i(A)p_j(B)$. We would like to mention in passing that this constraint of thermodynamic splitting and of zeroth law establishing for inhomogeneous $q$-systems fundamentally changes the formulation of NSM with normalized distribution $\sum_i p_i = 1$ because in this case we have to use a general product rule $p_{ij}^q(A + B) = p_i^{q_A}(A)p_j^{q_B}(B)$ for “formally independent”\(^2\) subsystems $A$ and $B$ instead of the usual product probability $p_{ij}(A + B) = p_i(A)p_j(B)$ as used in this work. Detailed description of this approach can be found in [14].

\(^2\)Product joint probability means independence of the subsystems and no interaction in the conventional probability theory. This meaning does not apply for nonextensive systems having interacting and correlating subsystems due to which the nonextensivity of entropy and energy can arise. So by “formal independence” we mean a statistical situation where interacting systems verify the product rule of joint probability.
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