Hyperskewness of $(1 + 1)$-dimensional KPZ height fluctuations

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Abstract. We evaluate the fifth-order normalised cumulant, known as hyperskewness, of height fluctuations dictated by the $(1 + 1)$-dimensional KPZ equation for the stochastic growth of a surface on a flat geometry in the stationary state. We follow a diagrammatic approach and invoke a renormalisation scheme to calculate the fifth cumulant given by a connected loop diagram. This, together with the result for the second cumulant, leads to the hyperskewness value $\tilde{S} = 0.0835$.

Keywords: kinetic roughening (theory), self-affine roughness (theory), dynamical processes (theory), stochastic processes (theory)
1. Introduction

The dynamics of surface growth has extensively been studied in nonequilibrium statistical physics in the last few decades [1–4]. To describe a local surface growth, Kardar, Parisi and Zhang [5] (KPZ) first proposed a prototypical nonlinear equation for the fluctuating height field \( h(\mathbf{x}, t) \), expressed as

\[
\frac{\partial}{\partial t} h(\mathbf{x}, t) = \nu_0 \nabla^2 h + \frac{\lambda_0}{2} (\nabla h)^2 + \eta(\mathbf{x}, t),
\]

(1)

where \( \nu_0 \) is the surface tension that smoothens the surface curvature and minimises the surface area and \( \lambda_0 \) is a coupling constant representing the strength of the non-linear term. The stochastic term \( \eta(\mathbf{x}, t) \) is modelled as a Gaussian white noise of zero average, \( \langle \eta(\mathbf{x}, t) \rangle = 0 \), and its covariance

\[
\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle = 2D_0 \delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t')
\]

(2)

where \( d \) is the substrate dimension. We will consider the stationary state for the growth of a surface on a flat substrate of dimension \( d = 1 \).

There are various systems that are mathematically equivalent to the KPZ dynamics. A few examples are: the vorticity free noisy Burgers equation [6], the stochastic heat equation (SHE), the diffusion equation governed by random source and sink, directed polymer in random potentials [7], in random media [8, 9], the sequence alignment of a gene or protein [10, 11] etc. Various growth phenomena belong to the \((1 + 1)\)-dimensional KPZ universality class (on the basis of scaling exponents [1]). For example, the growth of a bacteria colony [12, 13], fluid flow in a porous media [14], turbulent liquid crystal [15, 16], slow combustion of a sheet of paper [17, 18] etc.

The \((1 + 1)\)-dimensional KPZ equation satisfies the fluctuation dissipation theorem. The standard RG treatment of the \((1 + 1)\)-dimensional KPZ equation leads to the same scaling exponents for the renormalised noise amplitude \( D(k) \) and the renormalised...
surface tension $\nu(k)$. The resulting roughness and dynamic exponents are $\chi = \frac{1}{2}$ and $z = \frac{3}{2}$, respectively [5, 8], which satisfies the scaling relation $\chi + z = 2$. There have been various numerical models, namely, ballistic deposition (BD) [4, 19], the Eden model [20–23], the restricted solid on solid model (RSOS) [24], the single step model (SSM) [19, 25] and polynuclear growth (PNG) [26–29], which have the same scaling exponents, and consequently belong to the universality class of the $(1 + 1)$-dimensional KPZ equation.

The identification of the universality class of experimental processes and numerical models have long been based on the numerical values of the scaling exponents. New insights have been gained by Prâhofer and Spohn [29] via the study of the PNG model by mapping the problem to the longest permutation of random Gaussian matrices [30]. Thus, they [29] estimated different probability distributions which are close to the Gaussian unitary ensemble (GUE) Tracy–Widom (TW), the Gaussian orthogonal ensemble (GOE) TW and the Baik-Rains $F_0$ distribution for the curved, flat and random initial conditions, respectively. The effects of these initial conditions on the growth have further been studied in various works. GUE TW $F_2$ distribution has been observed for the sharp-wedge initial condition [31, 32]. Calabrese and Doussal [33] mapped the one end free directed polymer to the flat initial condition KPZ equation and obtained GOE TW $F_1$ distribution. A universal crossover function from GOE TW $F_1$ to the Baik–Rains $F_0$ distribution has been studied in the experiment of turbulent liquid crystal (TLC) and the numerical simulation of the PNG model by Takeuchi [34]. Imamura and Sasamoto [35] considered a two-sided Brownian motion as an initial condition and obtained an exact solution of the $(1 + 1)$-dimensional KPZ equation as a function of time $t$, which goes to the Baik–Rains distribution asymptotically. Halpin–Healy and Lin [36] considered the numerical models (RSOS, BD, SHE with multiplicative noise, SSM) belonging to the $(1 + 1)$-dimensional KPZ universality class and studied the statistics in the stationary state corresponding to the $(1 + 1)$-dimensional KPZ height fluctuations. The estimated stationary state distribution functions from these numerical models yield a close comparison to the Baik–Rains $F_0$ distribution. Hence, they established that the stationary state $(1 + 1)$-dimensional KPZ-type height fluctuations obey the universal Baik–Rains $F_0$ distribution. Halpin–Healy and Takeuchi [37] performed Euler a numerical integration of $(1 + 1)$-dimensional KPZ the equation considering a large system size (for flat $L = 250\,000$ and stationary $L = 10^4$) and a large number of statistical realisations (for flat $4000$, curved $25\,000$ and stationary $10^5$). They obtained distributions that agree well with the TW GOE, TW-GUE and Baik–Rains $F_0$ distributions for flat, curved and stationary initial conditions, respectively. Thus, the probability distributions of the $(1 + 1)$-dimensional KPZ height fluctuations are governed by the nature of the initial conditions despite having the same scaling exponents [38]. In principle the probability distribution function can be obtained from the solution of the Fokker–Planck version [39, 40] of the KPZ equation, which is next to impossible due to the nonlinear term. A more viable way to obtain a partial information of the PDF is to calculate the higher order cumulants such as the third, fourth and fifth cumulants. These cumulants, when normalised with respect to the second cumulant, yield skewness, kurtosis and hyperskewness, respectively.
In this paper, we employ a diagrammatic approach and an RG scheme to calculate the fifth cumulant. We follow an RG approach without rescaling that was found to be successful for calculating the skewness [41] and kurtosis [42]. We evaluate the loop diagram for the fifth cumulant in the large-scale and long-time limits, which yields \( \tilde{S} = 0.0835 \).

The rest of the paper is organised as follows. In section 2, the definition of hyperskewness in terms of moments and cumulants are presented. In section 3 the calculations of the fifth cumulant and the resulting hyperskewness are presented. Finally, the discussion and conclusion are presented section 4.

2. Moments and cumulants

Relations between moments and cumulants can be obtained by means of generating functions. The moment-generating function \( Z(\beta) \) and cumulant-generating function \( F(\beta) \) are defined as

\[
Z(\beta) \equiv \langle e^{\beta h} \rangle = \sum_{n=0}^{\infty} \frac{\langle h^n \rangle}{n!} \beta^n \tag{3}
\]

\[
F(\beta) = \ln Z(\beta) = \sum_{n=1}^{\infty} \frac{\langle h^n \rangle_c}{n!} \beta^n, \tag{4}
\]

respectively, where \( \langle h^n \rangle \) is the \( n \)th moment and \( \langle h^n \rangle_c \) is the \( n \)th cumulant; the angular bracket denotes an average with respect to the probability distribution. The relations between the first few moments and cumulants are expressed as

\[
\langle h \rangle = \langle h \rangle_c
\]

\[
\langle h^2 \rangle = \langle h^2 \rangle_c + \langle h \rangle_c^2
\]

\[
\langle h^3 \rangle = \langle h^3 \rangle_c + 3\langle h \rangle_c \langle h^2 \rangle_c + \langle h \rangle_c^3
\]

\[
\langle h^4 \rangle = \langle h^4 \rangle_c + 4\langle h \rangle_c \langle h^3 \rangle_c + 3\langle h^2 \rangle_c \langle h^2 \rangle_c + 6\langle h \rangle_c^2 \langle h^2 \rangle_c + \langle h \rangle_c^4
\]

\[
\langle h^5 \rangle = \langle h^5 \rangle_c + 5\langle h \rangle_c \langle h^4 \rangle_c + 10\langle h^2 \rangle_c \langle h^3 \rangle_c + 10\langle h \rangle_c^2 \langle h^3 \rangle_c + 15\langle h \rangle_c^3 \langle h^2 \rangle_c + \langle h \rangle_c^5. \tag{5}
\]

Since \( h \) represents height fluctuations with respect to the mean height, \( \langle h(\mathbf{x}, t) \rangle = 0 \) in the stationary state. Consequently, the relevant quantities for hyperskewness are

\[
\langle h^2 \rangle_c = \langle h^2 \rangle
\]

and

\[
\langle h^5 \rangle_c = \langle h^5 \rangle - 10\langle h^2 \rangle \langle h^3 \rangle. \tag{7}
\]
Hyperskewness $\tilde{S}$ is defined as

$$\tilde{S} = \frac{\langle h^5 \rangle_c}{\langle h^3 \rangle_c^{5/2}} = \frac{\langle h^5 \rangle}{\langle h^3 \rangle^{5/2}} - 10 \frac{\langle h^3 \rangle}{\langle h^3 \rangle^{5/2}}. \quad (8)$$

The contribution to $n$th cumulant $\langle h^n(x, t) \rangle_c$ comes from the connected loop diagrams with $n$ external legs.

3. The fifth cumulant

In this section, we calculate the fifth-order cumulant via a renormalisation scheme at one-loop order. The Fourier transformation of $h(x, t)$ is expressed as

$$h(x, t) = \int \frac{d^d k}{(2\pi)^{d+1}} h(k, \omega) e^{i(k \cdot x - \omega t)} \quad (9)$$

The Fourier transformation of equation (1) leads one to obtain the following form as

$$(-i\omega + \nu_0 k^2)h(k, \omega) = \eta(k, \omega) - \frac{\lambda_0}{2} \int \frac{d^d q d\Omega}{(2\pi)^{d+1}} [q \cdot (k - q)]h(q, \Omega)h(k - q, \omega - \Omega) \quad (10)$$

We will treat $[-i\omega + \nu_0 k^2]^{-1} = G_0(k, \omega)$ as the bare propagator. The Fourier transformation of the fifth cumulant is expressed as

$$\langle h^5(x, t) \rangle_c = \int \frac{d^{d+1} k_1}{(2\pi)^{d+1}} \int \frac{d^{d+1} k_2}{(2\pi)^{d+1}} \int \frac{d^{d+1} k_3}{(2\pi)^{d+1}} \int \frac{d^{d+1} k_4}{(2\pi)^{d+1}} \int \frac{d^{d+1} k_5}{(2\pi)^{d+1}} \ e^{i(k_1 + k_2 + k_3 + k_4 + k_5) \cdot x} \langle h(k_1)h(k_2)h(k_3)h(k_4)h(k_5) \rangle_c \quad (11)$$

where $\hat{x} \equiv (x, t)$ and $\hat{k} \equiv (k_1, \omega)$. The connected loop diagram corresponding to the fifth-order cumulant is shown in figure 1.

We write equation (11) in more compact form where the amputated part of the loop and the external legs are presented distinctly as

$$\langle h^5(x, t) \rangle_c = \int \frac{d^{d+1} \hat{k}_1}{(2\pi)^{d+1}} \int \frac{d^{d+1} \hat{k}_2}{(2\pi)^{d+1}} \int \frac{d^{d+1} \hat{k}_3}{(2\pi)^{d+1}} \int \frac{d^{d+1} \hat{k}_4}{(2\pi)^{d+1}} G(\hat{k}_1)G(\hat{k}_2)G(\hat{k}_3) \ G(\hat{k}_4)G(-\hat{k}_1 - \hat{k}_2 - \hat{k}_3 - \hat{k}_4)L(\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4). \quad (12)$$

where $L$ is the amputated part of the loop diagram (excluding the external legs). The unrenormalised (bare) form of the loop is written as
We will find a renormalised equivalent of this bare quantity by means of a renormalisation scheme employed earlier for the calculation of skewness and kurtosis \cite{41, 42}. We perform the internal frequency integration in equation (13) and carry out the integration over the internal momentum restricted in the shell $\Lambda_0 e^{-r} \ll q \ll \Lambda_0$, leading to

$$L^{(0)}(\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4) = \frac{32}{2} (2D_0)^5 \int \frac{d^{d+1} \hat{q}_1}{(2\pi)^{d+1}} [q_1 \cdot (q_1 - k_1)][q_1 \cdot (q_1 + k_2)]$$

$$\times [\{q_1 + k_2\} \cdot (q_1 + k_2 + k_3)][\{(q_1 + k_2 + k_3) \cdot (q_1 + k_2 + k_3 + k_4)]$$

$$[(q_4 - k_4) \cdot (k_4 + k_2 + k_3 + k_4 + k_3)]G_0(q_1)G_0(\hat{k}_1 - \hat{q}_1)$$

$$\times G_0(-\hat{q}_1)G_0(\hat{k}_2 + \hat{q}_1)G_0(-\hat{q}_1 - \hat{k}_2)G_0(\hat{k}_3 + \hat{q}_1 + \hat{k}_2)G_0(-\hat{q}_1 - \hat{k}_2 - \hat{k}_3)$$

$$G_0(\hat{k}_4 + \hat{q}_1 + \hat{k}_2 + \hat{k}_3)G_0(-\hat{q}_1 - \hat{k}_2 - \hat{k}_3 - \hat{k}_4)G_0(\hat{q}_1 + \hat{k}_3 + \hat{k}_2 + \hat{k}_3 + \hat{k}_4)$$

(13)

Considering the iterative nature of the momentum shell elimination in the RG scheme, we obtain the differential equation

$$\frac{dL}{dr} = \frac{35}{8} \frac{\lambda_0^5 D_0^5(r)}{\pi \nu^5(r) \Lambda_0^7} \left[ e^{2r} - 1 \right].$$

(14)

For large $r$, $\Lambda(r) = \Lambda_0 e^{-r}$ is identified as the momentum $k$. Substituting the following functional forms
\[ \nu(k) = \lambda_0 \sqrt{\frac{D_0}{2\pi \nu_0}} k^{-1/2}, \]  
\text{and}  
\[ D(k) = \lambda_0 \sqrt{\frac{D_0}{2\pi \nu_0}} k^{-1/2}, \]
we integrate the differential equation equation (15) and obtain
\[ L(r) = \frac{7\pi}{2} \lambda_0 \frac{D_0^3}{\nu_0^3 \nu_0^3} e^{5r}. \]
This expression for the renormalised loop needs to be expressed as a symmetric combination of the external momentum \( \hat{k}_i \). Consequently, \( \Lambda_0 e^{-r} \) is expressed as the fully symmetric combination
\[ \pi \lambda \nu = \Lambda L D kkkk,0,0,0,0,0 \]  
for vanishing external frequencies. The frequency dependence is constructed by considering a scaling function of the form
\[ \omega \nu \omega = \left| k_f^{11/4} \nu^2(k_f) G(k_f, \omega) \right|^2. \]
Substituting from equation (21) in equation (12), and carrying out the frequency integrations over \( \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \), the fifth cumulant is obtained as
\[ \langle h^5(x, t) \rangle_c = \frac{7}{4} \left( \frac{D_0}{2\pi \nu_0} \right)^{5/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{X(k_1, k_2, k_3, k_4)}{Y(k_1, k_2, k_3, k_4)} \]
where
\[ J(k_1, k_2, k_3, k_4) = \frac{X(k_1, k_2, k_3, k_4)}{Y(k_1, k_2, k_3, k_4)} \]
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\[ X(k_1, k_2, k_3, k_4) = \left( -3|k_1|^6 - 3|k_2|^6 - 10|k_2|^{9/2} \left( 2|k_3|^{3/2} + |k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right. \]
\[ \left. - 10|k_1|^{9/2} \left( 2|k_2|^{3/2} + 2|k_3|^{3/2} + |k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right. \]
\[ \left. - \left( |k_3|^{3/2} + |k_4|^{3/2} + |k_1 + k_2 + k_3 + k_4|^{3/2} \right)^2 \right) \left( 3|k_3|^{3} + 3|k_4|^{3} \right. \]
\[ \left. + 4|k|^{3/2} |k_1 + k_2 + k_3 + k_4|^{3/2} + |k_1 + k_2 + k_3 + k_4|^{3} \right. \]
\[ \left. + 2|k_3|^{3/2} \left( 7|k_4|^{3/2} + 2|k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right) \]
\[ - 2|k_2|^{3/2} \left( 17|k_3|^{3} + 17|k_4|^{3} + 23|k_1|^{3/2} \right) \]
\[ \left. + 2|k_1 + k_2 + k_3 + k_4|^{3/2} \left( 62|k_4|^{3/2} + 23|k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right) \]
\[ - 2|k_1|^{3} \left( 17|k_2|^{3} + 17|k_3|^{3} + 23|k_4|^{3/2} \right) \]
\[ \left. + 6|k_1 + k_2 + k_3 + k_4|^{3/2} \left( 62|k_4|^{3/2} + 23|k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right) \]
\[ + |k_2|^{3/2} \left( 62|k_3|^{3/2} + 62|k_4|^{3/2} + 23|k_1 + k_2 + k_3 + k_4|^{3/2} \right) \right) \]

\[ (24) \]

and

\[ Y(k_1, k_2, k_3, k_4) = 256|k_1|^{5/4} |k_2|^{5/4} |k_3|^{5/4} |k_4|^{5/4} \left( |k_1|^{3/2} + |k - 2|^{3/2} + |k_3|^{3/2} \right) \]
\[ + |k_2|^{3/2} + |k_1 + k_2 + k_3 + k_4|^{3/2} \right) \]

Considering the symmetry of the function $J(k_1, k_2, k_3, k_4)$, the integrations in equation (22) can be written as

\[ \langle \hat{h}^5(x, t) \rangle_c = \frac{7}{4} \left( \frac{D_0}{2\pi v_0} \right)^{5/2} \int_{\mu}^{\infty} dk_1 \int_{\mu}^{\infty} dk_2 \int_{\mu}^{\infty} dk_3 \int_{\mu}^{\infty} dk_4 \]
\[ \left[ 2J(k_1, k_2, k_3, k_4) + 8J(-k_1, k_2, k_3, k_4) \right. \]
\[ + 6J(-k_1, -k_2, k_3, k_4) \right]. \] \[ (25) \]

where an infrared cut off $\mu$ has been set due to the infrared divergences in the integrations. We write the integrations as

\[ J_1(\mu) = \int_{\mu}^{\infty} dk_1 \int_{\mu}^{\infty} dk_2 \int_{\mu}^{\infty} dk_3 \int_{\mu}^{\infty} dk_4 J(k_1, k_2, k_3, k_4), \]
\[ (26) \]

\[ J_2(\mu) = \int_{\mu}^{\infty} dk_1 \int_{\mu}^{\infty} dk_2 \int_{\mu}^{\infty} dk_3 \int_{\mu}^{\infty} dk_4 J(-k_1, k_2, k_3, k_4) \]
\[ (27) \]

and

\[ J_3(\mu) = \int_{\mu}^{\infty} dk_1 \int_{\mu}^{\infty} dk_2 \int_{\mu}^{\infty} dk_3 \int_{\mu}^{\infty} dk_4 J(-k_1, -k_2, k_3, k_4) \]
\[ (28) \]

So that equation (25) becomes

\[ \langle \hat{h}^5(x, t) \rangle_c = \frac{7}{4} \left( \frac{D_0}{2\pi v_0} \right)^{5/2} \left[ 2J_1(\mu) + 8J_2(\mu) + 6J_3(\mu) \right] \]
\[ (29) \]

The infrared cutoff dependent integrals are of the form

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\[ \tilde{J}_1(\mu) = a_2 \mu^{-5/2} \]
\[ \tilde{J}_2(\mu) = a_2 \mu^{-5/2} \]
\[ \tilde{J}_3(\mu) = a_3 \mu^{-5/2} \]  

where \( a_1, a_2 \) and \( a_3 \) are dimensionless constants. Numerical integrations yield the dimensionless constants as

\[ a_1 = \lim_{\mu \to 0^+} \mu^{5/2} J_1(\mu) = 0.00197 \]
\[ a_2 = \lim_{\mu \to 0^+} \mu^{5/2} J_2(\mu) = 0.00607 \]
\[ a_3 = \lim_{\mu \to 0^+} \mu^{5/2} J_3(\mu) = 0.00583 \]  

via numerical convergences. Substituting equations (26), (27), (28), (30) and (31) in equation (29), we finally obtain the expression for fifth cumulant as

\[ \langle h^5(x, t) \rangle_c = \frac{7}{4} [2a_1 + 8a_2 + 6a_3] \left( \frac{D_0}{2\pi \nu_0} \right)^{5/2} \frac{1}{\mu^{5/2}} \]  

4. Hyperskewness

To calculate the hyperskewness, the expression for the second cumulant is needed. The second cumulant in the frequency and momentum space is expressed as

\[ \langle h^2(x, t) \rangle_c = \langle h^2(x, t) \rangle = \int \frac{d^dk}{(2\pi)^d} \int \frac{d\omega}{(2\pi)} \int \frac{d^dk'}{(2\pi)^d} \int \frac{d\omega'}{2\pi} \langle h(k, \omega)h(k', \omega') \rangle_c \]
\[ e^{i(k-k') \cdot x - (\omega+\omega') t} \]  

The Feynman loop corresponding to the correlation function is shown in figure 2.

The frequency and momentum integrations in equation (33) can be evaluated [41, 42] and it can be obtained as

\[ \langle h^2(x, t) \rangle_c = \frac{4}{\pi} \left( \frac{D_0}{2\pi \nu_0} \right) \frac{1}{\mu} \]  

Using equations (32) and (34) in (8), we obtain
Substituting the values of $a_1$, $a_2$ and $a_3$ in equation (35) we obtain

\[ \mathcal{S} = 0.0835. \] (36)

5. Discussion and conclusion

In this paper, we considered the stochastic growth of a surface on a flat geometry (in the stationary state) governed by the $(1+1)$-dimensional KPZ equation. We followed a diagrammatic RG scheme to evaluate the Feynman diagram (figure 1) for the fifth cumulant at one-loop order corresponding to the $(1+1)$-dimensional KPZ dynamics. We started with the unrenormalised loop with bare parameters $(\nu_0, \lambda_0$ and $D_0)$ and invoked a shell elimination scheme (belonging to the shell $\Lambda q e r \Lambda$) to obtain a differential equation (equation (15)) representing the recursion relation for the successive elimination of momenta in thin shells. The solution of the differential equation yielded the renormalised expression (equation (14)) for the loop diagram. This facilitated the evaluation of the cumulant $\langle h^5(x, t) \rangle_c$ given by equation (12) involving renormalised quantities. The resulting integrals are found to be infrared divergent and hence $\langle h^5(x, t) \rangle_c$ depends on the infrared cutoff $\mu$. The momentum integrals are evaluated numerically to obtain $\langle h^5(x, t) \rangle_c$ as given by equation (33). Normalizing this value with respect to the $\langle h^2(x, t) \rangle_c^{5/2}$ resulted in the hyperskewness value as $\mathcal{S} = 0.0835$. It is interesting to note that all parameters of the KPZ dynamics $(\nu_0, \lambda_0, D_0)$ and the momentum cutoffs ($\Lambda$ and $\mu$) finally cancel out to yield this value, suggesting its universality.

Although there have been many studies on the lower order normalised moments such as skewness and kurtosis, the study of hyperskewness remains a rarity. However, there have been studies based on numerical models that focus on the probability distribution function belonging to the KPZ universality class [29, 34, 36]. In the steady state, this distribution has been shown to be identical with the Baik–Rains distribution [34, 36] with zero mean [29, 43].

The universal Baik–Rains $F_0$ distribution is a function of the solutions of the Painlevé-II equation, namely, $u''(x) = 2u^3(x) + xu(x)$. Hastings and McLeod [44] obtained a unique solution with the asymptotic boundary conditions $u(x) \sim -Ai(x)$ as $x \to \infty$ and $u(x) \sim -\sqrt{-x/2}$ as $x \to -\infty$ where $Ai(x)$ is the Airy function. To solve the Painlevé-II equation, Tracy and Widom performed a numerical integration [45]. Subsequently, Prăhofer and Spohn [46] obtained an arbitrary high precision solution by performing Taylor expansions. The Baik–Rains distribution is obtained from these solutions via known mathematical relations. On the other hand, the Fredholm determinant representation of the random matrix theory has been observed to be conceptually simpler and more numerically efficient than Painlevé-II to obtain the probability distributions [47] ($F_2, F_1, F_0,$ etc).
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It may, however, be noted that the Baik–Rains value for hyperskewness is 0.3092, which may be obtained from Fredholm determinant representation [47, 48] as well as from the numerical data [49] via the solution of the Painlevé-II. This Baik–Rains value is distinctly higher than our calculated value $\tilde{S} = 0.0835$. This underestimation via the RG calculation is due to the fact that the method is based on a perturbative approach where the calculation is carried out only at one-loop order. Since the dynamics is governed by a white noise following a Gaussian distribution, the lowest order approximation appears to be influenced rather strongly by the Gaussian noise. A similar trend was observed in the one-loop perturbative calculation for kurtosis, namely, $Q = 0.1523$ [42], which is lower than the Baik–Rains value 0.2892. However, the value for skewness via the perturbative scheme, namely, $S = 0.3237$ [41] is slightly lower than the Baik–Rains value 0.3594. Thus, it appears that the Gaussian white noise plays a more dominant role in the perturbative calculations for the higher order moments, namely, kurtosis and hyperskewness. More involved calculations with the incorporation of higher order contributions in the perturbative expansion are expected to yield better estimates for these higher order moments.

We note that it is next to impossible to obtain a closed-form analytical expression for the full probability distribution starting with the KPZ equation driven by the stochastic noise. Consequently, in an analytical approach, one usually evaluates a few lower and higher order moments (such as skewness, kurtosis and hyperskewness) from the governing dynamics. This outlook has motivated us to evaluate these numbers directly from the (1 + 1)-dimensional KPZ dynamics in the stationary state.

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References

[1] Barabási A-L and Stanley H E 1995 Fractal Concepts in Surface Growth (Cambridge: Cambridge University Press)
[2] Krug J 1997 Adv. Phys. 46 139
[3] Halpin-Healy T and Zhang Y-C 1995 Phys. Rep. 254 215
[4] Family F and Vicsek T 1985 J. Phys. A: Math. Gen. 18 L75
[5] Kardar M, Parisi G and Zhang Y-C 1986 Phys. Rev. Lett. 56 889
[6] Forster D, Nelson D R and Stephen M J 1977 Phys. Rev. A 16 732
[7] Krug J, Meakin P and Halpin-Healy T 1992 Phys. Rev. A 45 638
[8] Kardar M and Zhang Y-C 1987 Phys. Rev. Lett. 58 2087
[9] Fisher D S and Huse D A 1991 Phys. Rev. B 43 10728
[10] Hwa T and Lässig M 1996 Phys. Rev. Lett. 76 2591
[11] Hwa T 1999 Nature 399 17
[12] Vicsek T, Cserz M and Horvth V K 1990 Physica A 167 315
[13] Huergo M A C, Pasquale M A, Bolzán A E, Arvia A J and González P H 2010 Phys. Rev. E 82 031903
[14] Rubio M A, Edwards C A, Dougherty A and Golübü J P 1989 Phys. Rev. Lett. 63 1685
[15] Takeuchi K A and Sano M 2010 Phys. Rev. Lett. 104 230601
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[16] Takeuchi K A, Sano M, Sasamoto T and Spohn H 2011 Sci. Rep. 1 34
[17] Myllys M, Maunukoski J, Alava M, Ala-Nissila T, Merikoski J and Timonen J 2001 Phys. Rev. E 64 036101
[18] Miettinen L, Myllys M, Merikoski J and Timonen J 2005 Eur. Phys. J. B 46 55
[19] Meakin P, Ramanal P, Sander L M and Ball R C 1986 Phys. Rev. A 34 5901
[20] Eden M 1961 A two-dimensional growth process Proc. Fourth Berkeley Symp. on Mathematical Statistics and Probability ed J Neyman (Berkeley, CA: University of California Press) pp 223–39
[21] Plischke M and Rácz Z 1984 Phys. Rev. Lett. 53 415
[22] Jullien R and Botet R 1985 Phys. Rev. Lett. 54 2055
[23] Plischke M and Rácz Z 1985 Phys. Rev. A 32 3825
[24] Meakin P 1993 Phys. Rep. 235 189
[25] Plischke M, Rácz Z and Liu D 1987 Phys. Rev. B 35 3485
[26] van Saarloos W and Gilmer G H 1986 Phys. Rev. B 33 4927
[27] Goldenfeld N 1984 J. Phys. A 17 2807
[28] Krug J and Spohn H 1988 Phys. Rev. A 38 4371
[29] Práhofer M and Spohn H 2000 Phys. Rev. Lett. 84 4882
[30] Baik J and Rains E M 2001 Random Matrix Models and Their Applications ed P M Bleher and A R Its (Cambridge: Cambridge University Press)
[31] Sasamoto T and Spohn H 2010 Phys. Rev. Lett. 104 230602
[32] Sasamoto T and Spohn H 2010 J. Stat. Mech. 13
[33] Calabrese P and Le Doussal P 2011 Phys. Rev. Lett. 106 250603
[34] Takeuchi K A 2013 Phys. Rev. Lett. 110 210604
[35] Imamura T and Sasamoto T 2012 Phys. Rev. Lett. 108 190603
[36] Halpin-Healy T and Lin Y 2014 Phys. Rev. E 89 010103
[37] Halpin-Healy T and Takeuchi K A 2015 J. Stat. Phys. 160 794
[38] Shim Y and Landau D P 2001 Phys. Rev. E 64 036110
[39] Huse D A, Henley C L and Fisher D S 1985 Phys. Rev. Lett. 55 2924
[40] Parisi G 1990 J. Phys. 51 1595
[41] Singha T and Nandy M K 2014 Phys. Rev. E 90 062402
[42] Singha T and Nandy M K 2015 J. Stat. Mech. P05020
[43] Baik J and Rains E M 2000 J. Stat. Phys. 100 523
[44] Hastings S P and McLeod J B 1980 Arch. Ration. Mech. Anal. 73 1
[45] Tracy C A and Widom H 1994 Commun. Math. Phys. 159 151
[46] Práhofer M and Spohn H 2004 J. Stat. Phys. 115 1
[47] Bornemann F 2010 Markov Processes Related Fields 16 803
[48] Bornemann F 2015 private communication
[49] Práhofer M and Spohn H www-m5.ma.tum.de/KPZ

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