Decompactification near the horizon and non-vanishing entropy

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Abstract

Intersecting $D$-brane configurations are related to black holes in $D = 4$. Using the standard way of compactification only the Reissner-Nordstrøm black hole is non-singular. In this paper we argue, that also the other black holes are non-singular if i) we compactify over a periodic array and ii) we allow the string metric after reaching a critical curvature to choose the dual geometry. Effectively this means that near the horizon the solution completely decompactifies and chooses a non-singular $D$-brane configuration.

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If the 4d spacetime exhibits singularities one usually argues that physics breaks down there and if the singularity is not hidden by a horizon one should throw away this solution. On the other side in string theory there are some examples where these singularities are only a consequence of a too restrictive field theoretical description. We assume that if the string has the choice it always propagate in a non-singular geometry. Mainly there are two ways

i) Decompactification

ii) Transition into the dual geometry

The first possibility is based on the fact singularities are improved in higher dimensions. If the singularity disappears, it is not really a physical singularity, but a singular compactification. An example is the rotating BPS black hole in 4 dimensions that has a naked timelike singularity. In [1] it has been argued that near the core this solution decompactifies and the singularity becomes null. There are also other examples, where the solution becomes even non-singular. Below we will explain how this goes. The second possibility is a true stringy feature. String theory exhibits many additional symmetries that are not present in standard field theories. These are the dualities, which allow the string to live in different geometries. If the curvature reaches a certain level the string theory undergoes a phase transition (from the momentum to winding phase). From the string theory point of view both phases are not distinguishable. They are completely equivalent even though the singularity structure is different. This scenario has been discussed for cosmological solutions by many authors, see e.g. [2], [3] (pre-big-bang) or [4] for a recent discussion. In principle, there is yet a third possibility. Namely, that the singularity disappears on the quantum level. In this paper, however, we want to restrict ourselves on pure classical scenario (see e.g. [5]).

On the other side there are also arguments “in favour” of singularities. Any modification of general relativity which is completely non-singular cannot have a stable ground state [6]. Let us however discuss some examples for which we can give a non-singular description. These are the 4d supersymmetric static black holes and the non-singular geometry enables us to assign them a non-vanishing entropy density.

Let us now explain how the two possibilities works and what are the consequences for the black hole entropy.

1) Decompactification

Let us start with the example of the electric \( a = 1/\sqrt{3} \) black hole, which is given by

\[
\begin{align*}
\text{ds}^2 &= H^{-3/2} dt^2 - H^{3/2} dx^2, \\
&= e^{-2\phi/\sqrt{3}} = \sqrt{H}, \\
&= F_{0m} \sim \partial_m H^{-1},
\end{align*}
\]

with the harmonic function \( H(x) \). It can be obtained by compactification of the 5d Reissner-Nordström black-hole

\[
\begin{align*}
\text{ds}^2 &= \frac{1}{H^2} dt^2 - H(dx_5^2 + dx^2) \\
&= F_{0m} \sim \partial_m H^{-1},
\end{align*}
\]

where \( x_m = (x_5, \vec{x}) \) and

\[
H(x_5, \vec{x}) = 1 + \frac{r^2}{\rho^2},
\]
with \( \rho^2 = x^2_5 + r^2 \). After compactification of \( x_5 \) (with the dilaton related to \( g_{55} \)) we get the black hole \([1]\). For this, however, we have to assume that \( H \) is periodic: \( x_5 \sim x_5 + 2\pi R \). Then we can make the standard ansatz for \( H \) as a periodic array \([7]\)

\[
H = 1 + \sum_{n=-\infty}^{+\infty} \frac{r_h^2}{r^2 + (x_5 + 2\pi n R)^2} = 1 + \frac{r_h^2}{2 R r} + \mathcal{O}(e^{-r}) .
\]

(4)

Thus, away from the origin the dependence on \( x_5 \) is exponentially suppressed, this direction is compactified on a circle with radius \( R \). On the other side near the horizon (\( \rho = 0 \)) the 5th coordinate becomes visible and the solution decompactifies to its 5d origin. The asymptotic behaviour of the metric near the horizon is given by

\[
ds^2 \to \left( \frac{\rho^2}{r_h^2} \right)^2 dt^2 - r_h^2 \left( \frac{d\rho}{\rho} \right)^2 - r_h^2 d\Omega_3^2 = e^{4\eta/r_h} dt^2 - d\eta^2 - r_h^2 d\Omega_3^2 ,
\]

(5)

where \( \rho/r_h = e^{\eta/r_h} \) and \( r_h \) is the radius of the \( S_3 \) sphere. Hence, the asymptotic geometry is: (de Sitter)\(_2 \times S_3 \) which is non-singular.

On the other side if we assume that the harmonic function \( H \) depends only on \( \vec{x} \) and not on \( x_5 \) the scalar field is again divergent indicating that the compactification radius becomes infinite. Now, however, the radius of the \( S_3 \) does not stay finite. Instead it shrinks to zero yielding a singularity. This singularity is a consequence that we have been too restrictive in the coordinate dependencies.

2) Dual geometries

We explained already at the beginning that special symmetries allows the string to choose different geometries. To be explicit let us consider an euclidian model of a spatially flat space which is compactified on a torus with radius \( R \). This space is the near-horizon geometry of extremal \( p \)-branes and the metric is given by

\[
ds^2 = d\eta^2 + e^{2c\eta} dz^2 \]

(6)

where \( z \) are some spatial coordinates (world-volume coordinates for the \( p \)-brane). In the cosmological setting the big-bang is given by \( \eta \to -\infty \), where \( e^{2c\eta} \) shrinks to zero. But what happens if we approach this point? We assumed that the spatial coordinates \( z \) are compactified with the radius \( e^{\eta} \). For any compactified string theory there are two different kind of modes. One are the standard momentum modes which rule in the regime of large compactification radii. The other are the winding modes that become important for small radii. If the compactification radius is of order one (\( \eta = 0 \)) a phase transition happen, the momentum modes drop out and the winding modes become important. The consequence is that the string feels a different geometry where the radii are inverted

\[
\tilde{d}s^2 = d\eta^2 + e^{-2\eta} dz^2 \]

(7)

Going now to negative values of \( \eta \) the spatial geometry now starts to expand, the compactification radii increase. This is the pre-big-bang region \([3]\). This scenario has been discussed
for many cases [2]. Some of these models have a direct conformal field theory description [8] which makes the non-singular nature obvious. This effect is a consequence that in string theory, below a certain length scale, the usual local field theory loses their sense. There is a lower bound for space time structure that can be resolved and many singularities are hidden. Expressed in terms of thermodynamics it means that that there is a upper bound for the temperature \( T \sim 1/L \), where \( L \) is the compactification radius) and the maximum temperature is known as the Hagedorn temperature [2].

3) Consequences for black hole entropy

The supersymmetric black holes can be seen as compactified intersection of branes. In this paragraph we discuss what happens with the black holes when we approaching the horizon. Let us start with a short survey of the known supersymmetric black holes.

A common way to classify four–dimensional Maxwell/scalar black hole (BH) solutions is to specify the coupling of the scalar fields to the gauge fields. In the simplest case of only one scalar field and one gauge field this coupling is characterized by a single parameter \( a \) and the action in the Einstein frame is given by

\[
S_{4d} = \frac{1}{16\pi G_4} \int d^4x \sqrt{|g|} \{-R + 2(\partial\phi)^2 + e^{-2a\phi}F^2\} ,
\]

where \( G_4 \) is the 4–dimensional Newton constant. There are four different types of extremal\(^2\) black hole solutions, which are defined in terms of a function \( H \) which is harmonic on the 3–dimensional transverse space. The metric of these solutions is given by

\[
\begin{align*}
\text{a} = 0 & : ds^2 = H^{-2}dt^2 - H^2 dx^2 , \quad e^{-2\phi} = 1 , \\
\text{a} = 1/\sqrt{3} & : ds^2 = H^{-3/2}dt^2 - H^{3/2} dx^2 , \quad e^{\pm 2\phi/\sqrt{3}} = \sqrt{H} , \\
\text{a} = 1 & : ds^2 = H^{-1}dt^2 - H dx^2 , \quad e^{\pm 2\phi} = H , \\
\text{a} = \sqrt{3} & : ds^2 = H^{-1/2}dt^2 - H^{1/2} dx^2 , \quad e^{\pm 2\phi/\sqrt{3}} = \sqrt{H} .
\end{align*}
\]

These solutions have been generalized to different harmonic functions in [9] (for \( a = 0 \) this generalization has been given in [10]). For a discussion of these solutions as bound states, see [13]. The four solutions (9) are also known as

\[
\begin{align*}
\text{a} = 0 & : 4d \text{ Reissner-Nordstrøm (RN) solution}, \\
\text{a} = 1/\sqrt{3} & : 5d \text{ RN (\( \phi \) is a modulus field)}, \\
\text{a} = 1 & : \text{dilaton black hole (\( \phi \) has standard dilaton coupling)}, \\
\text{a} = \sqrt{3} & : 5d \text{ KK black hole (\( \phi \) is a modulus field)}. 
\end{align*}
\]

For \( a \neq 0 \) the gauge fields can be electric or magnetic. The two possibilities correspond to different signs of the scalar field \( \phi \). In formula (9) the “+” sign corresponds to the magnetic case. On the other hand, the \( a = 0 \) RN solution is dyonic. It turns out that in

\(^2\)In this letter we consider only extremal solutions.
four dimensions only the $a = 0$ RN solution is non-singular at the horizon $r = 0$. However, following [14], also the $a \neq 0$ solutions can be understood in a non-singular way, in the sense that they follow from the dimensional reduction of the following higher-dimensional non-singular solutions [11]

\[ a = 1/\sqrt{3} : \text{5d RN electric black hole or magnetic string} , \]
\[ a = 1 : \text{6d self-dual string} , \]
\[ a = \sqrt{3} : \text{10d self-dual D-3-brane} . \]

In addition, these are just the brane solutions that exhibits supersymmetry restoration near the horizon (see [12] and refs. therein). So, the singularities in the solution (9) can be seen as a consequence of the compactification. For the case of the electric $a = 1/\sqrt{3}$ solution (11) we have seen already that it decompactified into a non-singular solution (2) when we allow the harmonic functions to depend also on $x^5$. A singularity appears only if we are to restrictive in the compactification. On the other side for the magnetic $a = 1/\sqrt{3}$ case things work differently. Here, the non-singular analogue is the magnetic string in 5 dimensions where we have wrapped the string around the 5th direction. Now, the transversal space stays 3-dimensional but if we approach $r = 0$ the radius of the 5th coordinate shrinks and would vanish at $r = 0$. As we argued in the last paragraph this situation will not happen in string theory. Instead, at a certain point a phase transition will happen (the momentum modes drop out and the winding modes become important). For the geometry this means that the compactification radius inverts and expand if we further approach the point $r = 0$ where finally the string is completely decompactified. For the other black holes ($a = 1$ and $a = \sqrt{3}$) we argue similar, but now both effects happen at the same time. The $a = 1$ black hole decompactifies completely to the 6d self-dual string and the $a = \sqrt{3}$ solution to the 10d self-dual 3-brane. The decompactification of the transversal space can really be seen in our field theoretical formulation. On the other side the inversion of the compactification radius is a dynamical process and related to a phase transition at the Hagedorn temperature. Even more, the inversion of the world volume radii makes them to transversal coordinates, i.e. the magnetic strings becomes a black hole. Basically, in string theory what we have to expect is that the object in the target space (e.g. 2-brane or 1-brane) is not well defined. The string theory chooses that object which is least singular. The confusion here comes from the point that we are looking on the object from two different sides, from the momentum modes and from the winding modes. But the main point is that the compactification radius becomes effectively infinite.

What is this meaning for the entropy? There are two points of view. First, we should only consider the transversal space and define an entropy density (entropy per unit world-volume). In this picture we only have to take into account possible decompactifications of the transversal space, when we approach the horizon. We do not get into trouble with the phase transition. This procedure is well defined in the field theory. A more subtle point is the total entropy, where one integrates also over the world-volume coordinates. Here we do not have a standard field theory description, or if we simply integrate we get zero entropy (the radii vanish). This however is only a consequence that we have throw away possible
winding modes. For vanishing radii the momentum modes are completely suppressed but
the winding states have a high degeneracy. To count the degeneracy of winding states is its
own problem, but assuming that the number of momentum modes for large compactification
radius coincides with the number of winding modes at small radii we can simply count the
states by going on the brane and simultaneously making the compactification radius infinite
\((L \to \infty)\), where \(L\) is the periodicity of the coordinate). Thus we have the situation of “0 · ∞”
(infinite world volume but zero metric components), which could give a finite result for the
entropy. A further way to get a final total entropy is to calculate the expression not on the
horizon, but approach the horizon as far as possible. This means we calculate the entropy
just at the critical point, \(\eta = 0\) in (6), which is the \(T\)-self-dual radius. This defines us the
stretched horizon \([15]\).

To avoid this subtleties we discuss the entropy density for the black holes in (9). First, we
note that the radius of all horizons is given by \(r_h\), which can be expressed by the electric and
magnetic charges. Since the 4 dimensional solution can be electric as well as magnetic, let
us distinguish between both charges. Then, for \(a = 0, 1, \sqrt{3}\), after decompactification of the
transverse space and after integrating over the different spherical parts \(S_k\) (i.e. \(k = 2 + a^2\))
the entropy per unit world volume can be written as \([11]\)

\[
S = \frac{A}{4 G_k} = \frac{1}{4 G_k} \int_{S_k} (r_h)^k d\Omega_k = \pi (r_h)^k
\]

with:

\[
(r_h)^2 = 4 \sqrt{((\vec{n} + \frac{1}{2} \vec{Q}) L (\vec{n} + \frac{1}{2} \vec{Q})) ((\vec{p} + \frac{1}{2} \vec{P}) L (\vec{p} + \frac{1}{2} \vec{P}))}
\]

where \(L\) is the metric in the \(O(d, d)\) space\(^3\) and \(\vec{n}\) and \(\vec{p}\) are arbitrary unit vectors \((\vec{n} L \vec{n} = \vec{p} L \vec{p} = 1)\). With \(G_k\) we take into account that the Newton constant has to be rescaled when
one compares expressions in different dimensions. In our normalization in 4 dimensions we
have \(G_4 = 1\). This is a generalization of the entropy formula of Larsen and Wilczek \([16]\). It
reduces to their expression in the limit \(\vec{n} \cdot \vec{Q} = \vec{p} \cdot \vec{P} = 0\).

In our approach the case \(a = 1/\sqrt{3}\) is special. The electric case yields, integrating over
\(S_3\), an entropy density \(S \sim \sqrt{Q}^3\) whereas the magnetic case leads, after integrating over \(S_2\),
to \(S \sim P^2\). Many authors have investigated the electric case (see e.g. \([17]\)). However, it does
not fit into our entropy formula (we have 3 intersecting branes in this case). Note that the
\(a = 1/\sqrt{3}\) BH is also special in the sense that it cannot be expressed by \(D\)-3-branes only,
which are the only non-singular branes in 10 dimensions. In order to get a non-singular result
we need to include a boost or Taub-NUT soliton, i.e. KK modes. Thus, we can conclude that
the formula \([11]\) describes the entropy for all BH’s that can be expressed in a non-singular
way by \(D\)-3-branes only.

\(^3\)In this formula we have already included the different ways of embedding the intersection into the 10d
space, which causes this \(O(d, d)\) structure.
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