Information fractal dimension of mass function

Chenhui Qiang
Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, 610054 Chengdu, China
Yingcai Honors College, University of Electronic Science and Technology of China, Chengdu, 610054, China
2019270101007@std.uestc.edu.cn

Yong Deng∗
Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, 610054 Chengdu, China
School of Education, Shaanxi Normal University, Xi'an, 710062, China
School of Knowledge Science, Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 925-1211, Japan
Department of Management, Technology, and Economics, ETH Zürich, Zurich, Switzerland
dengentropy@uestc.edu.cn

Kang Hao Cheong†
Science, Mathematics and Technology Cluster, Singapore, University of Technology and Design (SUTD), S$4$87372, Singapore
SUTD-Massachusetts Institute of Technology International Design Centre, Singapore
kanghao_cheong@sutd.edu.sg

Received (received date)
Revised (revised date)

Fractal plays an important role in nonlinear science. The most important parameter to model fractal is fractal dimension. Existing information dimension can calculate the dimension of probability distribution. However, given a mass function which is the generalization of probability distribution, how to determine its fractal dimension is still an open problem of immense interest. The main contribution of this work is to propose an information fractal dimension of mass function. Numerical examples are illustrated to show the effectiveness of our proposed dimension. We discover an important property in that the dimension of mass function with the maximum Deng entropy is $\ln 3/\ln 2 \approx 1.585$, which is the well-known fractal dimension of Sierpiński triangle.

Keywords: Fractal; Information dimension; Probability distribution; Mass function; Shannon entropy; Deng entropy; Sierpiński triangle

∗Corresponding author
†Corresponding author
1. Introduction

Fractal is very common in nature [1,2], first proposed to measure the length of the coast. As the field progresses, it is now widely used across many different fields, such as time fractal [3,4], the calculation and geometry [5,7], and chaotic systems [8,9]. Irrefutably, the fractal dimension plays an vital role in fractal theory. To date, a lot of measures [10] has been proposed to determine the fractal dimension, including Hausdorff dimension [11], information dimension [12,13], correlation dimension [14], and multi-fractal dimension [15,16].

The uncertainty measurement is a topic of immense interest because it can be applied to the uncertain environment. Many algorithms have been proposed, including probability theory [17], Dempster-Shafer evidence theory [18,20] and belief structure [21,22]. The information volume of a probability distribution can be measured by Shannon entropy [23]. The mass function in evidence theory is a generalization of probability set to describe the uncertainty environment which is also called basic probability assignment (BPA). The uncertainty of mass function can be measured by Deng entropy [24, 25]. Compared with probability distribution, mass function has been widely applied due to its ability in dealing with uncertain information. Many measures and parameters are developed about mass function, such as entropy [20,27], negation [28], correlation coefficient [29] and information quality [30]. The information volume of the mass function has been studied recently [31,32].

However, how to determine the fractal dimension of mass function is still an open problem. In this paper, we have proposed information dimension of mass function based on Deng entropy and fractal theory, which can be further applied in decision-making [33]. Importantly, we discover that the dimension of mass function with the maximum Deng entropy is 1.585, which is the same as the fractal dimension of Sierpinski triangle.

Entropy is very important in complex systems [34,36]. There are many different kinds of entropy function, such as Tsallis entropy [37,38] and Renyi entropy [39]. In information theory, Shannon entropy plays an important role [40,41]. As part of our literature review, we will introduce these concepts briefly.

1.1. Shannon entropy

Given a probability distribution \( P = \{p_1, p_2, ..., p_n\} \), Shannon entropy is defined as follows [42]

\[
H_S = - \sum_i p_i \log(p_i).
\]  \hspace{1cm} (1)

If and only if \( p_i = \frac{1}{n} \), Shannon entropy reaches maximum

\[
H_{maxS} = \log(n).
\]  \hspace{1cm} (2)
1.2. Fractal and information dimension

Fractal has been widely studied [43] and fractal dimension is also attract a lot of focus [44,45]. For example, the fractal dimension of Sierpinski triangle is $\frac{\ln 3}{\ln 2}$ [46], the fractal dimension of Koch curve is $\frac{\ln 4}{\ln 3}$ [47], and the fractal dimension of Cantor sets is $\frac{\ln 2}{\ln 3}$ [48]. Besides, information dimension, as a kind of fractal dimension, plays a critical role in dealing with probability distribution. Below is a brief introduction of information dimension.

The information dimension is defined as follows [49]

$$D = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N} P_i(\varepsilon) \ln P_i(\varepsilon)}{\ln(\varepsilon)},$$

(3)

where the numerator is Shannon entropy, $\varepsilon$ is the side length of the measured box, $P_i(\varepsilon)$ is the probability of the measured object falling into the $i$th box. $N$ represents $N$ measured boxes.

1.3. Mass function

Mass function is an important aspect of evidence theory, which is an extension of probability theory. $\Theta$ denotes the framework of discernment in evidence theory [19,20].

$$\Theta = \{\omega_1, \cdots, \omega_N\},$$

(4)

the power set of $\Theta$ is $2^\Theta$,

$$2^\Theta = \{A_1, A_2, \ldots, A_{2^N}\} = \{\emptyset, \{\omega_1\}, \cdots, \{\omega_N\}, \{\omega_1, \omega_i\}, \cdots, \{\Theta\}\},$$

(5)

The mass function is defined as follows [19,20]

$$m : 2^\Theta \to [0,1].$$

(6)

This mapping satisfies

$$m(\emptyset) = 0,$$

(7)

$$\sum_{A \in 2^\Theta} m(A) = 1.$$

(8)

where $A_i$ is called focal element when $m(A_i) > 0$. Mass function can be seen as a generalization of probability distribution, which is more efficient in dealing with uncertainty [32,50].
1.4. Deng entropy and maximum Deng entropy

Recently, a new entropy called Deng entropy has been proposed to measure uncertainty of mass function. Given a mass function: $\{A_1, ..., A_{2^N}\} : \{m(A_i), i = 1, ..., 2^N\}$, its Deng entropy is obtained as

$$H_D = - \sum_{A \in 2^\Theta} m(A) \log\left(\frac{m(A)}{2^{|A|} - 1}\right), \quad (9)$$

where $|A|$ is the cardinal of focal element $A$.

When mass function is Bayesian structure [20], Deng entropy degenerates to Shannon entropy.

When the mass function satisfies the condition

$$m(A_i) = \frac{2^{|A_i|} - 1}{\sum_i 2^{|A_i|} - 1}, \quad (10)$$

where $m(A_i)$ is the mass function for $A_i$ and $i = 1, 2, ..., 2^N - 1$, Deng entropy reaches the maximum, which is shown as follows [25]

$$H_{maxD} = - \sum_i m(A_i) \log\left(\frac{m(A_i)}{2^{|A_i|} - 1}\right) = \log \sum_i (2^{|A_i|} - 1). \quad (11)$$

In probability theory, Shannon entropy reaches maximum when all results have equal probability. However, in evidence theory, Deng entropy postulates that when the uncertainty of mass function reaches the maximum, multiple subsets should have more assignment.

The rest of the paper is organized as follows. Section 2 presents information dimension of mass function. Some numerical examples are given in Section 3 to illustrate the effectiveness of our proposed dimension. Finally, Section 4 concludes the paper.

2. A new proposal for information dimension of mass function

In this section, the new dimension is being proposed and we begin with some fundamental definitions. For convenience of the reader, we also provide the proof to the properties that are being discussed.

**Definition 1.** For a framework of discernment $\Theta$, the power set is $2^\Theta = \{A_1, A_2, ..., A_{2^N}\}$, a mass function is $m(A)$. Its information dimension is defined as follows,

$$D_m = \frac{H_D}{\log \sum_i (2^{|A_i|} - 1) m(A_i)}, \quad (12)$$

where $H_D$ is Deng entropy, $(2^{|A_i|} - 1)$ is the size of power set of the focal element $A_i$.

**Property 1:** When $m(A_i) = 1, |A_i| = 1$, both 0 in numerator and denominator of Equation (12), we defined $D_m$ as follows,
\[ D_m = 0 \ (m(A_i) = 1, \ |A_i| = 1). \] 

**Proof:**

Given a framework of discernment \( \Theta = \{\omega_1, \cdots, \omega_N\} \), a mass function is \( m(A_i) = a, \ m(A_1) = \ldots = m(A_{i-1}) = m(A_{i+1}) = \ldots = m(A_{2N-2}) = b \), where \( 0 < a < 1, \ 0 < b < 1, \ a + (2^N - 1) \cdot b = 1, \ |A_i| = 1 \).

When \( a \to 1, \ b \to 0 \), the mass function degenerates into \( m(A_i) = 1, \ |A_i| = 1 \). According to Equation (9),

\[ \lim_{a \to 1} -m(A_i) \log \frac{m(A_i)}{2^{|A_i|} - 1} = \lim_{a \to 1} -a \cdot \log(a) \to 0 \] 

\[ \lim_{b \to 0^+} -m(A_j) \log \frac{m(A_j)}{2^{|A_j|} - 1} = \lim_{b \to 0^+} -b \cdot \log\left(\frac{b}{2^{|A_j|} - 1}\right) \to 0 \] 

\[ \lim_{a \to 1, b \to 0} H_D = -\sum_{A \in 2^\Theta} m(A) \log\left(\frac{m(A)}{2^{|A|} - 1}\right) = (-a \cdot \log(a) + \sum_j -b \cdot \log\left(\frac{b}{2^{|A_j|} - 1}\right)) \to 0 \] 

where \( 1 \leq j \leq 2^N - 1 \) and \( j \neq i \), according to Equation (12), we have

\[ D_m \to 0 \] 

The special case with the condition \( (m(A_i) = 1, \ |A_i| = 1) \) in Property 1 indicates that the information is deterministic, so its information dimension is 0.

**Property 2:** When mass function degenerates into probability distribution: \((\{A_1, \ldots, A_N\})\): \((\{P_i > 0, \ i = 1, \ldots, N\})\), where \( |A_i| = 1 \). Equation (12) can be rewritten as follows.

\[ D_m = \frac{H_D}{\log \sum (2^{|A_i|} - 1)^m(A_i)} = \sum_i -P_i \log(P_i) = \frac{H_S}{\log(N)} = D_p, \] 

where \( H_S \) is Shannon entropy. For denominator, due to the elementary event is no longer split in probability theory, which is represented by the value of any exponent of 1 is 1. In other words, the split of each singleton is itself.

In the case of one mass function or probability distribution, only a number is obtained by Equation (12). However, in the next section when the mass function changes in a certain regularity, the number either stays the same or converges gradually to a constant, indicating that there is a scale invariance between Deng entropy and the splitting of mass function. Therefore, the number is used to represent this property and named as information fractal dimension.
Authors’ Names

3. Numerical examples

In this section, we provide some numerical examples to better illustrate the definition of the proposed dimension. In order to verify the results easily, all examples below use $\log_2$ for the purpose of calculation. Nevertheless, base two or base e (or others) does not affect the calculation. In Equation (12), the numerator and the denominator will be different for different bases. However, we have proposed dimension as a ratio reflecting scale invariance. As long as the logarithm of the numerator and denominator has the same base, the result will not change. Therefore we can just use $\log$ for ease of convenience.

**Example 1:** A framework of discernment is $\Theta = \{\omega_1, \omega_2\}$, a mass function is

$m(\omega_1) = \frac{5}{6}, m_1(\omega_1, \omega_2) = \frac{1}{6}$

According to Equation (9) and Equation (12),

$$H_D = -\frac{5}{6}\log\left(\frac{5}{6}\right) - \frac{1}{6}\log\left(\frac{1}{2^2 - 1}\right) = 0.9142$$

$$D_m = \frac{0.9142}{\log((2^1 - 1)^\frac{5}{6} + (2^2 - 1)^\frac{1}{6})} = \frac{0.9142}{1.1381} = 0.8033$$

**Example 2:** Given a framework of discernment $\Theta, (|\Theta| = 1, 2, ..., 20)$. A mass function is $m(\Theta) = 1$. The result is show in Table 1. In Fig. 1, x axis is Deng entropy and y axis is $\log(2^{\Theta} - 1)^1$.

| $|\Theta|$ | $H_D$ | $\log \sum_i (2^{|A_i|} - 1)^{|m(A_i)|}$ | $D_m$ |
|----------|-------|---------------------------------|--------|
| 1        | 0     | 0                               | 0      |
| 2        | 1.5850| 1.5850                          | 1      |
| 3        | 2.8074| 2.8074                          | 1      |
| 4        | 3.9069| 3.9069                          | 1      |
| 5        | 4.9542| 4.9542                          | 1      |
| 6        | 5.9773| 5.9773                          | 1      |
| 7        | 6.9887| 6.9887                          | 1      |
| 8        | 7.9944| 7.9944                          | 1      |
| ...      | ...   | ...                            | ...    |
| 19       | 18.9999| 18.9999                        | 1      |
| 20       | 20.0000| 20.0000                        | 1      |
As can be seen from Table 1, the dimension of $|\Theta| = 1$ is 0. It means the complexity of a definite information is 0. Fig. 1 indicates a linear relationship between Deng entropy and the size of split of mass function when $|\Theta| = 2, 3, ..., 20$. The value of the slope is 1 and this means that the information dimension of the total uncertainty case is 1.
Example 3: Consider a framework of discernment $\Theta$, ($|\Theta| = 1, 2, \ldots, 10$). A mass function is $(\{A_1, A_2, \ldots, A_N\}, m(A_i) = \frac{1}{N})$, where $|A_i| = 1$. The information dimension of this mass function is 1. The results of $D_p$ are listed in Table 2 and Fig. 2.

| $|\Theta|$ | $H_S$ | $\log \sum_i (2^{|A_i|} - 1)^{P_i}$ | $D_p$ |
|-----------|-------|----------------------------------|-------|
| 1         | 0     | 0                                | 0     |
| 2         | 1     | 1                                | 1     |
| 3         | 1.5850 | 1.5850                           | 1     |
| 4         | 2     | 2                                | 1     |
| 5         | 2.3219 | 2.3219                           | 1     |
| 6         | 2.5850 | 2.5850                           | 1     |
| 7         | 2.8074 | 2.8074                           | 1     |
| 8         | 3     | 3                                | 1     |
| 9         | 3.1699 | 3.1699                           | 1     |
| 10        | 3.3219 | 3.3219                           | 1     |

Compared with Example 2 and Example 3, $m(\Theta) = 1$ means total uncertainty in evidence theory and average distribution in probability theory $m(A_i) = \frac{1}{N}$ has equal information dimension. According to Equation (18), the calculation of Example 3 is

$$D_p = \frac{\sum_i -p_i \log(p_i)}{\log \sum_i (2^{|A_i|} - 1)^{P_i}} = \frac{-\frac{1}{N} \log(\frac{1}{N}) \times N}{\log(1 \times N)}.$$  

(19)

The calculation of Example 2 according Equation (12) is

$$D_m = \frac{H_D}{\log \sum_i (2^{|A_i|} - 1)^{m(A_i)}} = \frac{-1 \times \log(\frac{1}{2^{N-1}})}{\log(2^N - 1)^1}.$$  

(20)

Equation (20) can be rewritten as

$$D_m = \frac{-\frac{1}{2^{N-1}} \log(\frac{1}{2^{N-1}}) \times (2^N - 1)}{\log(1 \times (2^N - 1))}.$$  

(21)

From above Equation (19) and Equation (21), the case of $m(\Theta) = 1$, $|\Theta| = N$ and the case of average distribution in probability, where the number of elementary events are $2^N - 1$, are equivalent in expressing the complexity of information.

Example 4: Given a framework of discernment $\Theta$, $|\Theta| = N = 1, 2, \ldots, 25$. Its power set is $2^\Theta = \{\emptyset, A_1, A_2, \ldots, A_{2^N - 1}\}$. A mass function with average assignment in power set is $m(A_i) = \frac{1}{2^{N-1}}$. As can be seen from Table 3, with the increase of the size of $\Theta$, $D_m$ is changed but eventually goes to 1.5. Different from Example 2 and Example 3, which is a constant from the beginning, we assume that for this
example the convergent value is the final information fractal dimension of mass function with average distribution in power set.

Table 3. The convergence process of Example 4 ($m(A_i) = \frac{1}{2^{i-1}}$)

| $|\Theta|$ | $H_D$ | $\log \sum_i (2^{|A_i|} - 1)^{m(A_i)}$ | $D_m$ |
|----------|-------|---------------------------------|-------|
| 1        | 0     | 0                               | 0     |
| 2        | 2.1133| 1.7834                          | 1.1850|
| 3        | 3.8877| 2.9691                          | 1.3094|
| 4        | 5.5500| 4.0186                          | 1.3811|
| 5        | 7.1610| 5.0260                          | 1.4248|
| 6        | 8.7428| 6.0214                          | 1.4520|
| 7        | 10.3048| 7.0418                          | 1.4690|
| 8        | 11.8523| 8.0095                          | 1.4798|
| 9        | 13.3886| 9.0058                          | 1.4867|
| 10       | 14.9162| 10.0034                         | 1.4911|
| ...      | ...   | ...                             | ...   |
| 21       | 31.4965| 21.0000                          | 1.4998|
| 22       | 32.9974| 22.0000                          | 1.4999|
| 23       | 34.4981| 23.0000                          | 1.4999|
| 24       | 35.9985| 24.0000                          | 1.4999|
| 25       | 37.4989| 25.0000                          | 1.5000|

**Example 5**: Given a framework of discernment $\Theta$, $|\Theta| = 1, 2, ..., 20$ and a mass function with maximum Deng entropy: $m(A) = \frac{1}{2^{|A|-1}}$, Fig. 4 show the result and Fig. 5 is a Sierpinski triangle.

From Table 4, with the size of $\Theta$ increasing, $D_m$ is a convergent sequence. The dimension of mass function with the maximum Deng entropy is 1.585, which is the well-known fractal dimension of Sierpiński triangle.

Finally, we conclude the results from Example 2, Example 4 and Example 5 with a picture. Fig. 6 shows that there is a linear relationship between Deng entropy and the size of split of mass function. However, given any mass function like Example 1, is there a special distribution form of mass function that has the same dimension? What does the calculated dimension actually mean?

There is still no common agreement on the interpretation of dimension, but one plausible explanation is related to the degree of freedom. In Euclidean Space, one-dimensional means that particle can only move in one direction. In two-dimensional space, particle can move in two orthogonal directions, and for three-dimensional, particle can move in three orthogonal directions. The higher the dimension, the more directions a particle can move; more variables are needed to measure it. However, we postulate here to express information fractal dimension as complexity.
That is to say, the larger the information dimension is, the more complex the information represented by mass function will be. With regards to the property of fractal, the proposed information dimension can be applied to pattern recognition and multicriteria decision making in highly uncertain environment, in which col-
Table 4. The convergence process of Example 5 \( (m(A) = \frac{2^{|A|} - 1}{\sum_{i=1}^{2^{|A|}} m(A_i)}) \)

| \(|\Theta|\) | \(H_{\text{max}}D\) | \(\log \sum_i (2^{|A_i|} - 1)^{m(A_i)}\) | \(D_m\) |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 2.3219 | 1.9757 | 1.1752 |
| 3 | 4.2479 | 3.1071 | 1.3672 |
| 4 | 6.0224 | 4.0970 | 1.4699 |
| 5 | 7.7211 | 5.0679 | 1.5235 |
| 6 | 9.3772 | 6.0434 | 1.5516 |
| 7 | 11.0077 | 7.0265 | 1.5666 |
| 8 | 12.6223 | 8.0157 | 1.5747 |
| 9 | 14.2266 | 9.0091 | 1.5791 |
| 10 | 15.8244 | 10.0052 | 1.5816 |
| 11 | 17.4178 | 11.0029 | 1.5830 |
| 12 | 19.0084 | 12.0016 | 1.5838 |
| 13 | 20.5971 | 13.0009 | 1.5843 |
| 14 | 22.1845 | 14.0005 | 1.5846 |
| 15 | 23.7711 | 15.0003 | 1.5847 |
| 16 | 25.3572 | 16.0001 | 1.5848 |
| 17 | 26.9429 | 17.0001 | 1.5849 |
| 18 | 28.5283 | 18.0000 | 1.5849 |
| 19 | 30.1136 | 19.0000 | 1.5849 |
| 20 | 31.6988 | 20.0000 | 1.5849 |

Acknowledgements

Yong Deng greatly appreciate China academician of the Academy of Engineering, Professor Shan Zhong and Professor You He, for their encouragement to do this research. Yong Deng greatly appreciates Professor Yugeng Xi to support this work. Ph.D student, Lipeng Pan, discussed the fractal dimension of Sierpiński tri-
Fig. 5. Sierpinski triangle, whose fractal dimension is $\frac{\ln 3}{\ln 2} \approx 1.585$. Interested readers can refer to Ref. [49] for the full construction steps.

Fig. 6. Information dimension of Example 2, 4 and 5

This work has been continuously funded for the past 20 years by grants such as the National Natural Science Foundation of China, Grant Nos. 30400067, 60874105, 61174022, 61573290 and 61973332, Program for New Century Excellent Talents in University, Grant No. NCET-08-0345, Shanghai Rising-Star Program Grant No.09QA1402900, Chongqing Natural Science Foundation for distinguished scientist, Grant No. CSCT, 2010BA2003 and JSPS Invitational Fellowships for Research in Japan (Short-term).
References

[1] Mandelbrot and B. Benoit, “The fractal geometry of nature”, American Journal of Physics 51 (1998) 468 p.
[2] Y. Jin, X. Liu, H. Song, J. Zheng and J. Pan, “General fractal topography: an open mathematical framework to characterize and model mono-scale-invariances”, Nonlinear Dynamics 96 (2019) 2413–2436.
[3] K.-L. Wang and C.-H. He, “A remark on wang’s fractal variational principle”, Fractals 27 (2019) 1950134.
[4] O. Abu Arqub, “Application of residual power series method for the solution of time-fractional schrödinger equations in one-dimensional space”, Fundamenta Informaticae 166 (2019) 87–110.
[5] J.-H. He, “Fractal calculus and its geometrical explanation”, Results in Physics 10 (2018) 272–276.
[6] B. Stanojević, S. Dzitac, I. Dzitac et al., “Fuzzy numbers and fractional programming in making decisions”, International Journal of Information Technology & Decision Making (IJITDM) 19 (2020) 1123–1147.
[7] O. A. Arqub and N. Shawagfeh, “Application of reproducing kernel algorithm for solving dirichlet time-fractional diffusion-gordon types equations in porous media”, Journal of Porous Media 22.
[8] N. B. Slimane, K. Bouallegue and M. Machhout, “Designing a multi-scroll chaotic system by operating logistic map with fractal process”, Nonlinear Dynamics 88 (2017) 1655–1675.
[9] J. Lin and Z.-J. Wang, “Parameter identification for fractional-order chaotic systems using a hybrid stochastic fractal search algorithm”, Nonlinear Dynamics 90 (2017) 1243–1255.
[10] T. Wen and K. H. Cheong, “The fractal dimension of complex networks: A review”, Information Fusion 73 (2021) 87–102.
[11] R. Lopes and N. Betrouni, “Fractal and multifractal analysis: a review”, Medical image analysis 15 (2009) 634–649.
[12] R. Zhu, J. Chen and B. Kang, “Power law and dimension of the maximum value for belief distribution with the maximum deng entropy”, IEEE Access 8 (2020) 47713–47719.
[13] T. Bian and Y. Deng, “Identifying influential nodes in complex networks: A node information dimension approach”, Chaos: An Interdisciplinary Journal of Nonlinear Science 28 (2018) 043109.
[14] L. Lacasa and J. Gómez-Gardeneres, “Correlation dimension of complex networks”, Physical review letters 110 (2013) 168703.
[15] D. Wei, X. Chen and Y. Deng, “Multifractality of weighted complex networks”, Chinese Journal of Physics 54 (2016) 416–423.
[16] T. Wen, D. Pelusi and Y. Deng, “Vital spreaders identification in complex networks with multi-local dimension”, Knowledge-Based Systems 195 (2020) 105717.
[17] E. T. Jaynes, Probability theory: The logic of science (Cambridge university press, 2003).
[18] A. P. Dempster, “Upper and lower probabilities induced by a multivalued mapping”, in Classic works of the Dempster-Shafer theory of belief functions (Springer, 2008), pp. 57–72.
[19] A. P. Dempster, “Upper and lower probabilities induced by a multivalued mapping”, The annals of mathematical statistics (1967) 325–339.
[20] G. Shafer, A mathematical theory of evidence, volume 42 (Princeton university press, 1976).
[21] R. R. Yager and N. Alajlan, “Maxitive belief structures and imprecise possibility distributions”, IEEE Transactions on Fuzzy Systems 25 (2016) 768–774.
[22] R. R. Yager, “Fuzzy rule bases with generalized belief structure inputs”, Engineering Applications of Artificial Intelligence 72 (2018) 93–98.
[23] C. E. Shannon, “A mathematical theory of communication”, The Bell system technical journal 27 (1948) 379–423.
[24] Y. Deng, “Deng entropy”, Chaos, Solitons & Fractals 91 (2016) 549–553.
[25] Y. Deng, “Uncertainty measure in evidence theory”, SCIENCE CHINA Information Sciences 63 (2020) 210201.
[26] F. Xiao, “EFMCDM: Evidential fuzzy multicriteria decision making based on belief entropy”, IEEE Transactions on Fuzzy Systems 28 (2020) 1477–1491.
[27] Y. Xue and Y. Deng, “Interval-valued belief entropies for Dempster–Shafer structures”, Soft Computing 25 (2021) 8063–8071.
[28] F. Xiao, “On the maximum entropy negation of a complex-valued distribution”, IEEE Transactions on Fuzzy Systems (2020) DOI: 10.1109/TFUZZ.2020.3016723.
[29] W. Jiang, “A correlation coefficient for belief functions”, International Journal of Approximate Reasoning 103 (2018) 94–106.
[30] D. Li, Y. Deng and K. H. Cheong, “Multisource basic probability assignment fusion based on information quality”, International Journal of Intelligent Systems 36 (2021) 1851–1875.
[31] Y. Deng, “Information volume of mass function”, International Journal of Computers Communications & Control 15 (2020) 3983.
[32] J. Deng and Y. Deng, “Information volume of fuzzy membership function”, International Journal of Computers Communications & Control 16 (2021) 4106.
[33] H. Liao, Z. Ren and R. Fang, “A deng-entropy-based evidential reasoning approach for multi-expert multi-criterion decision-making with uncertainty”, International Journal of Computational Intelligence Systems 13 (2020) 1281–1294.
[34] S. J. Phillips, R. P. Anderson and R. E. Schapire, “Maximum entropy modeling of species geographic distributions”, Ecological modelling 190 (2006) 231–259.
[35] H. Zhang and Y. Deng, “Entropy Measure for Orderable Sets”, Information Sciences 561 (2021) 141–151.
[36] S. M. Pincus, “Approximate entropy as a measure of system complexity.”, Proceedings of the National Academy of Sciences 88 (1991) 2297–2301.
[37] C. Tsallis, R. Mendes and A. R. Plastino, “The role of constraints within generalized nonextensive statistics”, Physica A: Statistical Mechanics and its Applications 261 (1998) 534–554.
[38] Y. Xue and Y. Deng, “Tsallis extropy”, Communications in Statistics-Theory and Methods (2021) 10.1080/03610926.2021.1921804.
[39] T. Van Erven and P. Harremos, “Rényi divergence and kullback-leibler divergence”, IEEE Transactions on Information Theory 60 (2014) 3797–3820.
[40] C. Wang, Z. X. Tan, Y. Ye, L. Wang, K. H. Cheong and N.-g. Xie, “A rumor spreading model based on information entropy”, Scientific reports 7 (2017) 1–14.
[41] S. Babajanyan, A. Allahverdyan and K. H. Cheong, “Energy and entropy: Path from game theory to statistical mechanics”, Physical Review Research 2 (2020) 043055.
[42] C. Shannon, “A mathematical theory of communication bell syst.”, .
[43] S. Djennadi, N. Shawagfeh and O. A. Arqub, “A fractional tikhonov regularization method for an inverse backward and source problems in the time-space fractional diffusion equations”, Chaos, Solitons & Fractals 150 (2021) 111127.
[44] A. Yilmaz and G. Unal, “Multiscale higuchi’s fractal dimension method”, Nonlinear Dynamics 101 (2020) 1441–1455.
[45] P. Ziaukas and M. Ragulskis, “Fractal dimension and wada measure revisited: no straightforward relationships in ndds”, *Nonlinear Dynamics* **88** (2017) 871–882.

[46] Ettestad, David, Carbonara and Joaquin, “The sierpinski triangle plane”, *Fractals: An Interdisciplinary Journal on the Complex Geometry of Nature* .

[47] B. Jia, “Bounds of the hausdorff measure of the koch curve”, *Applied Mathematics & Computation* **190** (2007) 559–565.

[48] I. S. B.Ae.K, “Hausdorff dimension of perturbed cantor sets without some boundedness condition”, *Acta Mathematica Hungarica* **99** (2003) 279–283.

[49] H.-O. Peitgen, H. Jürgens and D. Saupe, *Chaos and fractals: new frontiers of science* (Springer Science & Business Media, 2006).

[50] F. Xiao, “GIQ: A generalized intelligent quality-based approach for fusing multi-source information”, *IEEE Transactions on Fuzzy Systems* (2020) DOI: 10.1109/TFUZZ.2020.2991296.