Formulation of a Triaxial Three-Layered Earth Rotation: Theory and Rotational Normal Mode Solutions

Zhiliang Guo¹ and WenBin Shen¹,²

¹Department of Geophysics, School of Geodesy and Geomatics/Key Laboratory of Geospace Environment and Geodesy, Wuhan University, Wuhan, China. ²State Key Laboratory of Information Engineering in Surveying, Mapping and Remote Sensing, Wuhan University, Wuhan, China

Abstract In this study, we formulated a triaxial three-layered anelastic Earth rotation theory considering various core-mantle couplings, including the pressure and gravitational couplings acting on the elastic inner core by the outer core and mantle, and the viscoelectromagnetic couplings between the outer core and mantle and between the outer and inner cores. With this formulation, using the eigenvalue method, we solved four rotational normal modes, including the Chandler Wobble (CW), Free Core Nutation (FCN), Free Inner Core Nutation (FICN), and the Inner Core Wobble (ICW). The triaxiality of the actual Earth has notable effects on rotational normal modes, that is, 0.005, 0.0, −0.071, and 1.174 d (mean solar days), respectively, for the CW, FCN, FICN, and ICW. In the case that the triaxiality is bigger, for instance, if the Earth’s equatorial dynamical ellipticity were $5.22 \times 10^{-5}$, which is 1 order of magnitude larger than the presently determined ellipticity $(2.2 \times 10^{-5})$, the CW and ICW periods will be lengthened by 3.579 and 94.912 d, and the FICN period will be decreased by 0.591 d. Based on our present knowledge, the numerical solutions of the four normal modes under the new rotation frame suggest that the periods of CW, FCN, FICN, and ICW are, respectively, 432.2, 429.9, 934.0, and 2,718.7 d. Our theory for Earth can be extended to other planets in our solar system.

1. Introduction

The study of Earth’s rotation is an interdisciplinary pursuit of geodesy, geophysics, and astronomy. Observations from space geodesy and astrometry promote improvements of Earth’s rotation theory in geodesy and geophysics. Conventionally, scientists focus on the rotationally symmetric Earth rotation theory, from one-layer to three-layer models (Dehant & Mathews, 2015; Lambeck, 1980; Moritz & Mueller, 1987; Munk & MacDonald, 1960) due to the fact that (1) observation accuracies have been historically insufficient and (2) these models are more easily dealt with. Nowadays, the polar motion and length of day variations are observed with very high accuracies, about 40 microarcseconds (μas) and 10 microseconds (μs) (Bizouard et al., 2019), respectively, which demand more general Earth rotation theory, to deal with the effect of new features on Earth rotation, like triaxiality, mantle anelasticity, tide dissipations, and core-mantle couplings.

Previously, some scientists have studied the triaxial effects based on rigid Earth or elastic Earth models. Seitz and Schmidt (2005) studied the atmospheric and oceanic excitations for a triaxial elastic Earth based on their Dynamic Model for Earth Rotation and Gravity. Bizouard and Zotov (2013) found that based on a triaxial elastic Earth model, the pole tide and the triaxiality will both lead to asymmetric polar motion, while the effect of triaxiality will partly cancel the effect of the pole tide. Solutions for the high-frequency nutations forced by lunisolar attraction were derived shortly after 2000 for two-layer triaxial Earth models, most of them expressed as polar motion according to the conventions (Getino et al., 2001; Escapa et al., 2002; Ferrández et al., 2003; Mathews & Bretagnon, 2003).

Rotational normal modes of Earth rotation theories can be inspections of Earth rotation theories. Van Hoolst and Dehant (2002) successfully considered the triaxial and second-order effects of geometric and dynamical flattenings on the normal modes Chandler Wobble (CW) and Free Core Nutation (FCN) of a two-layered Earth and Mars model. Chen and Shen (2010) (hereafter CS10) formulated a triaxial two-layered Earth rotation theory and provided CW and FCN normal modes solutions. Based on the complex compliances formulation of triaxial two-layered Earth rotation frame by Sun and Shen (2016), Shen et al. (2019) further considered the electromagnetic coupling between the mantle and fluid core and analyzed the sensibility of the compliances and electromagnetic coupling parameters on the rotational CW and FCN normal modes.
Mathews et al. (1991a) (hereafter MBHS1991) formulated a rotationally symmetric three-layered Earth rotation theory with pressure and gravitational couplings considered and stated that two more normal modes will be induced because of the inner core, namely, the Free Inner Core Nutation (FICN) and Inner Core Wobble (ICW). Then, Mathews et al. (1991b) calculated four rotational normal modes for Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson, 1981) and 1066A Earth model (Gilbert & Dziewonski, 1975) without considering mantle anelasticity and pole tides. Further, Mathews et al. (2002) (hereafter MHB2002) considered the effects of electromagnetic couplings, mantle anelasticity, and ocean tides, finding that the FICN period will be prolonged to 1,025 mean solar days (d). Later, Dumberry (2009) (hereafter D09) successfully extended the MBHS1991 theory to consider the inner core tilt-induced deformations to further confirm the 7.5 year period of ICW given by Rochester and Crossley (2009).

Here, based on MBHS1991, MHB2002, and D09, we formulated a triaxial three-layered Earth rotation theory to provide rotational normal modes solutions with triaxiality effect investigated and compliances constrained by CW/FCN observations. In section 2, we provided basic descriptions of four rotational normal modes in three-layered Earth rotation, namely, CW, FCN, FICN, and ICW. In section 3, we summarized current CW/FCN observations and FICN/ICW predictions. In section 4, assuming the triaxial Earth is composed of a triaxial anelastic mantle (including the crust, hereafter, the same meaning), a triaxial fluid outer core, and a triaxial elastic inner core, we generalized the studies of MBHS1991, MHB2002, and D09 by extending their symmetric theory into triaxial theory, that is, by triaxializing moments of inertia tensor, pressure and gravitational coupling torque, and the viscoelectromagnetic coupling torques near the core-mantle boundary (CMB) and inner core boundary (ICB), and by considering elastic deformations of the inner core. In section 5, we describe the needed input parameters, including the dynamical figure parameters of the Earth models, the diurnal and static compliances, and the core-mantle coupling parameters. In section 6, under the frame of the triaxial three-layered Earth rotation theory, we calculated the corresponding four rotational normal modes and compared the new results with those obtained by conventional theories. Then, we examined two cases, Cases I and II, to investigate the effect of triaxiality on rotational normal modes, by enhancing the triaxiality of the Earth to different extents in Case I and Case II. Since there are lots of observations of the CW and FCN periods, here we used the weighted means of the CW and FCN period observations to adjust the values of the compliances to best match the observations for the usage of further studies. In section 7 we summarized the study and discussed relevant problems.

2. Descriptions of Rotational Normal Modes

Generally, for a three-layered Earth rotation with mantle, fluid outer core and solid inner core, there are four rotational normal modes, namely, CW, FCN, FICN, and ICW, see Figure 1. CW is the main component of the polar motion, described by misalignment of the Earth’s rotation axis and its figure axis (Figure 1a). For a rigid Earth, the period of CW is about 305 d, and after considering the fluid outer core and solid inner core, the period of CW will shorten to about 270 d (Dumberry, 2009; Hough, 1895; Smith & Dahlen, 1981). For an elastic one-layered Earth, the period of CW is about 440 d, and after considering the effect of ocean pole tide and mantle anelasticity, the CW period will be lengthened to about 480 d (Bizouard & Zotov, 2013). Further, if considering the effects of fluid outer core and solid inner core, the CW period will become to about 435 d (Smith & Dahlen, 1981), very close to current CW observations. FCN is denoted by the misalignment of the fluid outer core’s rotation axis and the whole Earth’s rotational axis (Figure 1b), the period of which is determined by the ellipticity of irregular CMB. In the Terrestrial Reference System, due to strong resonance at retrograde diurnal tide band, the FCN is called Nearly Diurnal Free Wobble (NDFW) with period less than 1 d. In the space fixed system, the FCN is a long-period motion and not included in precession-nutation theory. Then, extending their symmetric theory into triaxial theory, that is, by triaxializing moments of inertia tensor, pressure and gravitational coupling torque, and the viscoelectromagnetic coupling torques near the core-mantle boundary (CMB) and inner core boundary (ICB), and by considering elastic deformations of the inner core.

Under the action of buoyancy force in fluid outer core, the solid inner core will drift in fluid outer core to some extent, and the figure axis of solid inner core will keep departures from the figure axis of the mantle, that is, about 1° (Greiner-mai & Barthelmes, 2001; Zhang & Huang, 2019), still large compared to the extent of polar motion, that is, less than 1”. Due to huge peripheral pressure force and rotation inertia of solid inner core, the rotation axis of the solid inner core is very close to the rotation axis of the mantle (D09). Since the magnitude of the polar motion is less than 1”, the rotation axis of the solid inner core is very close to the figure axis of the mantle. The precession of the rotation axis of the solid inner core with respect to the rotation axis of the mantle is referred to as FICN (Figure 1c). The ICW, namely, the wobble of the figure axis
Figure 1. Four rotational normal modes of three-layered Earth rotation. (a) Chandler Wobble (CW), \( \Omega \) axis moving around \( \Omega_0 \) axis in prograde direction; (b) Free Core Nutation (FCN), \( \Omega_f \) axis moving around \( \Omega \) axis in retrograde direction; (c) Free Inner Core Nutation (FICN), \( \Omega_s \) axis moving around \( \Omega \) axis in prograde direction; (d) Inner Core Wobble (ICW), \( \iota'_3 \) axis moving around \( \Omega_s \) axis in prograde direction.

of solid inner core around its rotation axis (Figure 1d), is in essence the same with the prograde precession of the figure axis of the solid inner core with respect to the figure axis of the mantle, due to small deviation of the rotation axis of the solid inner core from the figure axis of the mantle.

3. Observed and Predicted Rotational Normal Mode Values

There are quite a few observational results for the CW and FCN, by analyzing different kinds of data sets, including Earth orientation parameter (EOP) data (Bizouard & Gambis, 2009; Bizouard et al., 2019), very long baseline interferometry (VLBI) data, superconducting gravity (SG) data, and others. The period and quality factor of CW are often determined using polar motion data in EOP data set, SG data (Ding & Chao, 2017), and polar motion excitation data (Furuya & Chao, 1996; Nastula & Gross, 2015). Generally, the period and Q of the FCN are determined by VLBI data or gravity data using different methods (Chao & Hsieh, 2015; Lambert & Dehant, 2007; Krásná et al., 2013; Rosat et al., 2017; Zhou et al., 2016). On the one hand, the FCN parameters are obtained through VLBI data analysis, which is usually called the nutation method (Krásná et al., 2013; Rosat et al., 2017; Zhou et al., 2016). On the other hand, observed from the Terrestrial Reference System, the FCN will become NDFW, and due to strong resonance with the solid Earth tides in the diurnal band, the NDFW can be observed using gravity or SG data, which can be termed as gravity method (Ducarme et al., 2007; Rosat et al., 2009, 2017).

Nastula and Gross (2015) estimated the CW period as 430.9 d and the Q value as 127 by minimizing the modeled and observed polar motion excitation function, which are derived from polar motion data and second-degree gravitational potential coefficients based on observations of satellite laser range (SLR) (e.g., Cheng et al., 2011) and Gravity Recovery and Climate Experiment (GRACE) (e.g., Bettadpur, 2012). Rosat et al. (2009) analyzed the SG data using the gravity method and obtained the FCN period as 428 d with a quality factor of 7, 762 < Q < 31, 989. Here we choose CW and FCN observations with their accuracies being smaller than 8 d from various observations as summarized by Nastula and Gross (2015) and Rosat.
are defined as rotating with a mean angular velocity about a mean rotational vector $\mathbf{\Omega}_L$ along the $z$ axis of the reference frame $\mathbf{R}_L$. The kinematic Earth rotation theory by determining FICN and ICW periods in the time domain. After considering mantle anelasticity, ocean tides, and electromagnetic couplings near CMB and ICB, MHB2002 provided the accuracies of the CW, FCN, and ICW periods being 0.04 d, neither the FCN observations of Vondrák and Ron (2006), Lambert and Dehant (2007), and Krásná et al. (2013), which suggest the accuracies of the CW and FCN periods being 0.07, 0.4, and 0.1 d.

The first study of the FICN and ICW can be dated back to Smylie et al. (1984) and Szeto and Smylie (1984, 1984), and the related motions of the inner core were later confirmed as FICN and ICW by Xu and Szeto (1998). de Vries and Wahr (1991) evaluated the FCN and FICN periods based on displacement field (DF) approach by investigating the FICN effect on forced nutations and solid Earth tides, and the nonhydrostatic structure effect on FICN period. Mathews et al. (1991b) first estimated the FICN and ICW period with 475.5 d and 6.6 years, while Dehant et al. (1993) built a homogeneous three-layered Earth rotation theory and obtained similar results with Mathews et al. (1991b) under the condition that the mean density of the fluid outer core is taken as the density of the fluid outer core at ICB. Xu and Szeto (1998) formulated an Euler kinematic Earth rotation theory by determining FICN and ICW periods in the time domain. After considering mantle anelasticity, ocean tides, and electromagnetic couplings near CMB and ICB, MBHS2002 provided the FICN and ICW periods and quality factors. Rochester and Crossley (2009) found that the ICW period is 7.35 years if properly modeling elastic deformations of the inner core using Lagrangean description of the Euler-Liouville equations. D09 obtained FICN and ICW periods by considering the inner core tilt induced elastic deformations in the Euler-Liouville system of MBHS1991 and MBHB2002, and confirmed the result of Rochester and Crossley (2009). Crossley and Rochester (2014) obtained FICN and ICW periods with DF approach. Recently, Chao (2017) formulated the dynamics of the mantle-inner core gravitational interaction and provided an ICW model in their independent Euler-Liouville system, providing a quite different value of -15.6 years due to the wrong sign of gravitational coupling torque between inner core and the mantle. Later, Rochester et al. (2018) corrected the gravitational coupling torque and obtained the ICW period of 7 years. By using AR-z spectrum analysis (Ding & Chao, 2015, 2018), Ding et al. (2019) detected a signal with period of 8.7 years that is supposed from the ICW, which needs further verification and comparison with theoretical investigation. The relevant theoretical predictions of FICN and ICW are showed in Table 2.

## 4. Theoretical Formulation

### 4.1. Theory

In this study we adopted the basic assumptions deployed by MBHS1991, MHB2002, and D09 theory. The standard Earth, composed of triaxial elastic mantle, triaxial fluid outer core, and triaxial elastic inner core, are defined as rotating with a mean angular velocity $\mathbf{\Omega}_L$ about a mean rotational vector $\mathbf{I}_L$ along the $z$ axis of

| CW period/Q | Reference | FCN period/Q | Type | Reference |
|-------------|-----------|--------------|------|-----------|
| 433.2 ± 2.2/63 | Jeffreys (1972) | 435 ± 1/22, 000 ~ 10³ | Nutation | Herring et al. (1986) |
| 434.0 ± 2.6/100 | Wilson and Haubrich (1976) | 431 ± 6/2, 800 | Gravity | Neuberg et al. (1987) |
| 434.8 ± 2.0/96 | Ooe (1978) | 432 ± 4/ > 15, 000 | Nutation | Defraigne et al. (1994) |
| 433.3 ± 2.1/70 | Wilson and Vicente (1980) | 431 ± 1/1, 700 ~ 2, 500 | Gravity | Florsch et al. (1994) |
| 433.0 ± 1.1/179 | Wilson and Vicente (1990) | 430 ± 4/5, 500 ~ 10, 000 | Gravity | Merriam (1994) |
| 433.7 ± 1.8/49 | Furuya and Chao (1996) | 429.7 ± 1.4/9, 350 ~ 10, 835 | Gravity | Sato et al. (2004) |
| 439.5 ± 2.1/72 | Kuehne et al. (1996) | 429.7 ± 2.4/– | Gravity | Ducarme et al. (2007) |
| 433.1 ± 1.7/– | Vicente and Wilson (1997) | 430.00 ± 2/15, 000 | Gravity | Ducarme et al. (2009) |
| 434.0 ± 0.5/– | Gibert and Le Mouël (2008) | 428 ± 3/7, 762 ~ 31, 989 | Gravity | Rosat et al. (2009) |
| 430.9 ± 0.7/127 | Nastula and Gross (2015) | 434.0 ± 1.9/– | Nutation | Zhou et al. (2016) |
a geocentric quasi-inertial reference system called the geocentric celestial reference system. As discussed in MBHS1991 theory, the displacement field of the real Earth with respect to the standard Earth can be divided into a rigid rotation component and a deformation field. We applied the rigid rotation to the mantle and fluid outer core of the standard Earth to define the mantle fixed $i$ system ($i_1, i_2, i_3$); such Earth is called equilibrium Earth. These two coordinate systems are designed to separate the rigid rotation induced principal moments of inertia and the deformations induced moments and products of inertia, where the rigid rotation induced principal moments of inertia constitute two independent quasi-principal axis systems, and the deformations induced inertial moments and products caused by tides and nonuniform rotation are considered later as deviations. The tilt of the inner core is thus $n_0 = i'_1 - i_3 = (n'_1, n'_2, n'_3)$. The $i$ system is close to, but not, the Tisserand mean axial system (TMAS) (Munk & MacDonald, 1960), as stated by MBHS1991 theory. Thus, in geocentric celestial reference system, the angular velocities of the mantle, fluid outer core, and elastic inner core can be expressed as (Mathews et al., 1991a)

$$\Omega = \Omega_0 + \omega = \Omega_0(i_1 + m)$$

$$\Omega_f = \Omega + \omega_f = \Omega_0(i_3 + m + m_f)$$

$$\Omega_s = \Omega + \omega_s = \Omega_0(i_1 + m + m_s),$$

where $\Omega_0 = 7.292115 \times 10^{-5}$ rad/s is the mean rotation rate of equilibrium Earth, $m$ is the deviation of mantle rotation from steady rotation of equilibrium Earth, with $m_f$ and $m_s$ the differential rotation of fluid outer core and elastic inner core with respect to the mantle, expressed by the direction cosines of the individual rotation axes with respect to the mantle rotation axis (see Figure 2).

In the MBHS1991 theory, only terms with a magnitude of $O(m)$ or $O(mc)$ are considered, and the second-order terms with magnitudes of $O(m^2)$ and $O(me^2)$ or higher-order terms are neglected. The pressure and gravitational coupling torques acting on the fluid outer core are neglected in the fluid outer core rotational equation (see equation (3)) due to the higher-order effect of $O(me^2)$. This is referred to as SOS approximation, that was first proposed by Sasao et al. (1980) and later used by MBHS1991. Similarly, in our study, we also considered only the first-order terms of $O(m)$ or $O(me)$, and neglected the higher-order terms for simplicity. We note that considering higher-order terms in triaxial three-layered rotation theory is much more complicated. Hence for example, the second-order dynamical and geometric flattening effects of a triaxial three-layered Earth with a magnitude of $O(me^2)$, or the second-order rotation effects of a triaxial three-layered Earth with a magnitude of $O(m^2)$ were not considered here. In addition, Mathews and Guo (2005) extended the electromagnetic coupling models used in MHB2002 theory to include viscous couplings; thus, the new viscoelectromagnetic coupling torques were generalized and applied to the triaxial case.

Based on the above assumptions, the coupled rotational equation for a triaxial three-layered Earth can be formulated in the $i$ ($i_1, i_2, i_3$) quasi-principal axis system (D09; Koot et al., 2008; MBHS1991; MHB2002):

$$\frac{dH}{dt} + \Omega \times H = 0$$

(2)

---

**Table 2**

Theoretical Predictions for the FICN and ICW Periods and $Q$'s Based on PREM Earth Model According to Euler (EL) Method or Displacement Field (DF) Approach (Unit: Mean Solar Days (d))

| FICN period/Q | ICW period/Q | Type    | Reference                        |
|---------------|--------------|---------|----------------------------------|
| 469.7         | —            | DF      | de Vries and Wahr (1991)         |
| 475.5         | 2,409        | EL      | Mathews et al. (1991b)           |
| 477.7         | 2,447.3      | EL      | Dehant et al. (1993)             |
| 502.6         | 2,427        | EL      | Xu and Szeto (1998)              |
| 1025/677      | 2,412        | EL      | MHB2002                          |
| —             | 2,743.9      | EL+DF   | Rochester and Crossley (2009)     |
| 478.6         | 2,715        | EL      | D09                              |
| 466.7         | 2,834.24     | DF      | Crossley and Rochester (2014)     |
| —             | —15.6 year   | EL      | Chao (2017)                      |
| —             | 7 year       | EL      | Rochester et al. (2018)          |
Figure 2. Triaxial three-layered Earth rotation model, including the triaxial elastic mantle (Earth-yellow color), triaxial fluid outer core (yellow), and triaxial elastic inner core (red). $i_3$ is the instantaneous figure axis of the mantle, $i_3'$ is the instantaneous figure axis of the elastic inner core, and $n_s = i_3' - i_3$ is the tilt of figure axis of inner core with respect to figure axis of mantle. $\Omega_0$ is the mean rotation rate of the Earth, $\Omega$ denotes the temporal angular velocity of the whole Earth, $\Omega_f$ the temporal angular velocity of the fluid outer core, and $\Omega_s$ the temporal angular velocity of the elastic inner core.

$$\frac{dH_f}{dt} - \omega_f \times H_f = \Gamma_{CMB} - \Gamma_{ICB}$$  

$$\frac{dH_s}{dt} + \Omega \times H_s = \Gamma_s + \Gamma_{ICB}$$  

$$\frac{dn_s}{dt} = \Omega_0 m_s \times i_3.$$  

These equations are essentially a generalization of the Liouville equations (Lambeck, 1980; Moritz & Mueller, 1987; Munk & MacDonald, 1960) adapted to whole Earth dynamics as opposed to the normal mode displacement formulation used for example in seismology. In equations (2)–(5), the variables are absolute angular momentum $H$ and angular velocity $\Omega$ for the whole Earth, absolute angular momentum $H_f$ of the fluid outer core, angular velocity $\omega_f$ of the outer core with respect to the mantle, viscoelectromagnetic coupling torque $\Gamma_{CMB}$ acting on the fluid outer core by the mantle, viscoelectromagnetic coupling torque $\Gamma_{ICB}$ acting on the elastic inner core by the fluid outer core, absolute angular momentum $H_s$ of the elastic inner core, and pressure and gravitational coupling torque $\Gamma_s$ acting on the elastic inner core by the fluid outer core and mantle.

Since our present study focuses on rotational normal modes, that is, free wobble and free nutation, including CW, FCN, FICN, ICW, we did not consider the tesseral tidal torque, neither the z components of the core-mantle coupling torques, which include the z components of the viscoelectromagnetic coupling torques near the CMB and ICB, and the z-components of the pressure and gravitational coupling torque.

The absolute angular momenta of the elastic inner core, the fluid outer core, and whole Earth can be written as (Mathews et al., 1991a)

$$H_s = |C_s| \cdot \Omega_s$$
\[ H_f = [C_f] \cdot \Omega_f \] (7)

\[ H = [C] \cdot \Omega + [C_f] \cdot (\Omega_f - \Omega) + [C_s] \cdot (\Omega_s - \Omega) + H^{(R)}, \] (8)

where the TMAS is used for the fluid outer core and elastic inner core, and the coordinate system fixed with the mantle is not, but close to, the TMAS, as discussed above. This is the reason why the relative angular momentum \( H^{(R)} \) variable exists in the equation for the whole Earth and can be neglected in the following derivations in our first-order approximation. Following MBHS1991 theory, in the triaxial case, the moments of inertia tensor can be generalized as follows:

\[
[C_s] = A_s i_1 i_1 + B_s i_2 i_2 + C_s i_3 i_3 + \left( C_s - \frac{A_s + B_s}{2} \right) (i'_1 i'_1 - i_s i_s) + \sum_{ij} c'_{ij} i_i i_j \] (9)

\[
[C_f] = A_f i_1 i_1 + B_f i_2 i_2 + C_f i_3 i_3 + \left( C_f - \frac{A'_f + B'_f}{2} \right) (i'_1 i'_1 - i_f i_f) + \sum_{ij} c'_{ij} i_i i_j \] (10)

\[
[C] = A i_1 i_1 + B i_2 i_2 + C i_3 i_3 + \left[ \left( C_s + \frac{A_s + B_s}{2} \right) - \left( C'_s - \frac{A'_s + B'_s}{2} \right) \right] (i'_1 i'_1 - i_s i_s) + \sum_{ij} c_{ij} i_i i_j . \] (11)

where \( A_s, B_s, \) and \( C_s \) are the principal moments of inertia for the elastic inner core; \( A_f, B_f, \) and \( C_f \) are the principal moments of inertia for the fluid outer core; \( A'_s, B'_s, \) and \( C'_s \) are the corresponding moments of inertia for the inner core that are replaced by an inner core with density \( \rho_f \) (1.2166 \times 10^4 kg/m^3) that is also the density of the fluid outer core at the ICB; \( A, B, \) and \( C \) are the principal moments of inertia for the whole Earth; and \( c'_{ij}, c'_{ij}, \) and \( c_{ij} \) are the moments and products of inertia caused by deformation of the elastic inner core, fluid outer core, and the whole Earth due to centrifugal potential, respectively.

According to MHB2002 theory and Mathews and Guo (2005), the viscoelectromagnetic coupling torque acting on the fluid outer core by the mantle, \( \Gamma_{CMB} \), can be expressed in the triaxial form as

\[
\Gamma_{CMB} = K_{CMB} \Omega_0^2 \begin{pmatrix} B_f m^2_f & -A_f m^1_f \\ -A_f m^1_f & 0 \end{pmatrix} \] (12)

where \( K_{CMB} \) is the dimensionless viscoelectromagnetic coupling parameter between the mantle and fluid outer core, \( A_f \) and \( B_f \) the equatorial principal moments of inertia for the fluid outer core, and \( m^1_f \) and \( m^2_f \) the equatorial components of the rotation vector of fluid outer core with respect to mantle. And the viscoelectromagnetic coupling torque acting on the elastic inner core by the fluid outer core \( \Gamma_{ICB} \) can be expressed in the triaxial form as

\[
\Gamma_{ICB} = K_{ICB} \Omega_0^2 \begin{pmatrix} B_s (m^1_s - m^1_f) & -A_s (m^1_s - m^1_f) \\ -A_s (m^1_s - m^1_f) & 0 \end{pmatrix} \] (13)

where \( K_{ICB} \) is the dimensionless viscoelectromagnetic coupling parameter between the fluid outer core and elastic inner core, \( A_s \) and \( B_s \) are the equatorial principal moments of inertia for the elastic inner core, and \( m^1_s \) and \( m^1_s \) are the equatorial components of rotation vector of elastic inner core with respect to mantle.

D09 theory extended the MBHS1991 and MHB2002 theories by modeling elastic deformations of the inner core properly, by considering the elastic deformations caused by gravitational coupling and centrifugal effect. Here we generalize the D09 pressure and gravitational coupling model into triaxial, and the new pressure and gravitational coupling torque \( \Gamma_s \) can be expressed as

\[
\Gamma_s = \Omega_0^2 \begin{pmatrix} (C_s - B_s)[a_1 (m^1_s + m^1_s) - a^2 s^2 - a_s a^2 s^2] \\ -(C_s - A_s)[a_1 (m^1_s + m^1_s) - a^2 s^2 - a_s a^2 s^2] \end{pmatrix} + \Omega_0^2 \begin{pmatrix} -a^2 s^2 \frac{a_2}{c_3} \\ a^2 s^2 \frac{a_2}{c_3} \end{pmatrix} \] (14)
where

\[ a_1 = 1 - \alpha_3 = \frac{C'_2 - (A'_1 + B'_1)/2}{C_1 - (A_1 + B_1)/2} \]

\[ a^{(1,2)}_g = \frac{3}{a_1^2\Omega_0^2} \left[ (C_1 - B_1, C_2 - A_2) \right] \left( (1 + \frac{5}{3} \frac{\tilde{\rho}_f}{\rho_f})a_1 - 1 \right) - 1 \]

\[ a^{(1,2)}_2 = a_1 - \alpha_3 a^{(1,2)}_g \]

\[ \alpha_1 \text{ and } \alpha_3 \text{ are pressure coupling parameters with } \alpha_3 \text{ a damping factor of pressure coupling. } \tilde{\rho}_f (1.2895 \times 10^4 \text{ kg/m}^3) \text{ is the mean density of elastic inner core as determined inversely from rotational symmetric version of equation (15) according to parameter values provided in Mathews et al. (1991b). } a^{(1,2)}_g \text{ represents the strength of the gravitational coupling in triaxial case, } (n'_1, n'_2) \text{ is the inner core tilt caused by relative motions of the fluid outer core and the mantle with respect to elastic inner core as explained in D09, } \rho \text{ the radius of the elastic inner core, } G = 6.67259 \times 10^{-11} \text{ kg}^{-1} \text{m}^3 \text{s}^{-2} \text{ the universal gravitational constant, and other variables keep the same meanings as described in previous text.} \]

The new products of inertia of the whole Earth, the fluid outer core, and the elastic inner core, along with the inner core tilt caused by relative motions of the fluid outer core and the mantle with respect to the elastic inner core, can be expressed as

\begin{align*}
  c_{13} &= A\{\kappa m_1 + \zeta m'_1 + \zeta m'_2 + S_{14}^6 n'_1 + S_{14}^6 (n'_1 - m'_1)\} \\
  c_{23} &= B\{\kappa m_2 + \zeta m'_2 + \zeta m'_1 + S_{14}^6 n'_2 + S_{14}^6 (n'_2 - m'_2)\} \\
  c'_{13} &= A_f\{\gamma m_1 + \beta m'_1 + \beta m'_2 + S_{14}^6 n'_1 + S_{14}^6 (n'_1 - m'_1)\} \\
  c'_{23} &= B_f\{\gamma m_2 + \beta m'_2 + \beta m'_1 + S_{14}^6 n'_2 + S_{14}^6 (n'_2 - m'_2)\} \\
  n'_1 &= S_{12} m_1 + S_{13} m'_1 + S_{13} m'_2 + S_{14}^6 n'_1 + S_{14}^6 (n'_1 - m'_1) \\
  n'_2 &= S_{12} m_2 + S_{13} m'_2 + S_{13} m'_1 + S_{14}^6 n'_2 + S_{14}^6 (n'_2 - m'_2),
\end{align*}

where the nine compliances \( \kappa, \zeta, \gamma, \beta, \theta, \chi, \nu \) and \( \nu \) are the classic compliances in MBHS1991 and MHB2002, and the new 11 compliances \( S_{14}^6, S_{12}, S_{13}, S_{14}^6, S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6 \) and \( S_{24}^6 \) that appear in D09 are additional compliances. The nine classic compliances \( \{\kappa, \zeta, \gamma, \beta, \theta, \chi, \nu\} \) characterize the deformations of whole Earth, fluid outer core, and elastic inner core due to centrifugal effects. The additional compliances \( S_{14}^6, S_{12}, S_{13} \) reflect the changes of inner core tilt caused by centrifugal effects of whole Earth, fluid outer core and elastic inner core. \( S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6, \) and \( S_{24}^6 \) are compliances related to the gravitational coupling between elastic inner core and the rest of the Earth, and \( S_{24}^6, S_{24}^6, S_{24}^6, S_{24}^6 \) are compliances related to the pressure coupling between elastic inner core and fluid outer core.

Noticeably, the dynamical ellipticities for anelastic whole Earth, compressible fluid outer core and elastic inner core are expressed respectively as

\begin{align*}
  e &= (C - (A + B)/2)/(A + B)/2 \\
  e_f &= (C_f - (A_f + B_f)/2)/(A_f + B_f)/2 \\
  e_i &= (C_i - (A_i + B_i)/2)/(A_i + B_i)/2.
\end{align*}

Hence, by treating \( m, m'_j, m'_m, n, c'_j, c'_m \) and \( c'_{m_i} \) as first-order small quantities and neglecting higher-order terms, the new triaxial three-layered Earth rotation theory with properly inner core elastic deformations modeling can be formulated as

\[ \mathbf{M}\frac{d\mathbf{y}}{dt} = \Omega_0 \mathbf{N}\mathbf{y}, \quad \mathbf{y} = \left[ m_1, m'_2, m'_3, m'_4, n'_1, n'_2, n'_3, n'_4 \right]^T. \]
Table 3
Dynamical Figure Parameters in Models 1, 2, and 3, Including the Principal Moments of Inertia (Unit: 10^{34} kg m^2) and the Dynamical Ellipticities of the Whole Earth, the Fluid Outer Core, and the Elastic Inner Core

| Parameter | Model 1 value | Model 2 value | Model 3 value |
|-----------|---------------|---------------|---------------|
| A         | 8,010.0085    | 8,010.0967    | 8,011.5       |
| B         | 8,010.1849    | 8,010.0967    | 8,011.5       |
| C         | 8,036.4063    | 8,036.4062    | 8,037.51      |
| A_1       | 905.6630      | 905.6713      | 905.83        |
| B_1       | 905.6793      | 905.6713      | 905.83        |
| C_1       | 908.0673      | 908.0673      | 908.138       |
| A_s       | 5.8520272     | 5.8520747     | 5.8531        |
| B_s       | 5.8521223     | 5.8520747     | 5.8531        |
| C_s       | 5.8662485     | 5.8662485     | 5.86728       |
| e         | ±0.003285479  | ±0.003285479  | ±0.003247     |
| e_1       | ±0.0026456    | ±0.0026456    | ±0.002548     |

Note: Model 1 = new triaxial three-layered Earth rotation theory; Model 2 = rotational symmetric case of Model 1; Model 3 = the parameters deploying from D09 under new triaxial three-layered Earth rotation theory.

*a In rotational symmetric case, the equatorial principal moments of inertia are averaged to get the symmetric values \( \overline{A}, \overline{A}_1, \) and \( \overline{A}_s, \) and polar principal moments of inertia \( C, C_1, \) and \( C_s \) are obtained from \( C = \overline{A}(1+e), C_1 = \overline{A}_1(1+e), C_s = \overline{A}_s(1+e). \) *b In Chen et al. (2015), only the uncertainties of principal moments of inertia for whole Earth are estimated simply, and in Mathews et al. (2002), the uncertainties for dynamical ellipticities of the whole Earth and fluid outer are, respectively, 0.0000000012 and 0.0000020, while the dynamical ellipticity of the elastic inner core is a theoretical value based on PREM Earth model, and the corresponding uncertainty is not provided.

where the matrices \( \mathbf{M} \) and \( \mathbf{N} \) are presented in Appendix A.

We note that MBHS1991, MHB2002, and D09 theories can be derived from our triaxial three-layered Earth rotation theory by simplifying or replacing relevant parameters. For instance, by adopting D09’s rotationally symmetric principal moments of inertia, dynamical ellipticities and compliances without considering the mantle anelasticity and ocean tides, we obtain D09 theory, and further setting viscoelectromagnetic coupling parameters and additional compliances to zero, we obtain MBHS1991 theory; by adopting rotationally symmetric principal moments of inertia, dynamical ellipticities of MHB2000 and setting the additional compliances to zero, we obtain MHB2002 theory. In addition, by neglecting the elastic inner core, and setting the viscoelectromagnetic coupling parameters and additional compliances to zero, we obtain triaxial two-layered Earth rotation theory of CS10.

4.2. Rotational Normal Modes Solutions

Concerning the triaxial three-layered Earth rotation, it is more complex to solve for the rotational normal modes. In fact, there are two methods to solve the rotational normal modes for the triaxial three-layered Earth rotation equations. One was proposed by Van Hoolst and Dehant (2002) and later used by CS10, which is referred to as the trigonometric function method. The other was proposed by MHB2002 theory and later was used by Sun and Shen (2015), which is referred to as the eigenvalue method. The eigenvalue method originated from the resonant frequency study of MHB2002, who considered the electromagnetic couplings near the CMB and ICB, mantle anelasticity, and ocean tides to study the forced nutations, and was developed by Sun and Shen (2015) to study the CW and FCN under the frame of the triaxial two-layered Earth rotation of CS10. Here, the eigenvalue method will be used for rotational normal modes solutions.

Suppose \( \sigma_j \) is the eigenvalue of \( \mathbf{M}^{-1}\mathbf{N} \), the rotational normal modes solutions will be \(-i\sigma_j\). According to the theory of ordinary differential equations, the solutions of the equation system (18) have the form \( \mathbf{r}_e e^{\sigma_j t}, \) where \( \sigma_j \) are the eigenvalues of \( \mathbf{M}^{-1}\mathbf{N} \), because the motions of \( \mathbf{y} \) are in fact sinusoidal and hence \(-i\sigma_j \) are the rotational normal mode solutions in cycles per sidereal day (cpsd) used in this study due to the fact that \( \sigma_j = \text{Im}(-i\sigma_j) \). Here, one cpsd means the frequency of the mean diurnal rotation of the Earth with angular velocity \( \Omega_0 \).
The Core-Mantle Coupling Parameters in Models 1, 2, and 3, Including the Viscoelectromagnetic Coupling Parameters

| Parameter | Value |
|-----------|-------|
| $K_{CMB}$ | $(2.97 \pm 0.02) - (1.78 \pm 0.02)i \times 10^{-5}$ |
| $K_{ICB}$ | $(1.01 \pm 0.02) - (1.09 \pm 0.03)i \times 10^{-3}$ |
| $a_1$ | 0.9463 |
| $a_3$ | 0.0537 |
| $a_2^g$ | 2.1640 |
| $a_2^g$ | 2.1853 |
| $a_2^g$ | 0.830093 |
| $a_2^g$ | 0.828949 |
| $a_2^g$ | 2.1752 |
| $a_2^g$ | 0.829492 |

Note: In Model 1, we deploy triaxial pressure and gravitational coupling parameters, that is, $a_1^g, a_2^g, a_2^g$, and $a_2^g$, while in Models 2 and 3, we deploy rotational symmetric pressure and gravitational coupling parameters, that is, $a_2^g$ and $a_2^g$. In Model 3, the viscoelectromagnetic coupling parameters are set to zero according to Dumberry (2009).

Theoretically, according to equation (18), solving the eigenvalues of $M^{-1}N$, we obtain eight original solutions, given as follows:

$$\sigma_1 = -\sigma_2, \quad \sigma_3 = -\sigma_4, \quad \sigma_5 = -\sigma_6, \quad \sigma_7 = -\sigma_8. \quad (19)$$

where $\sigma_1$ and $\sigma_2$ are solutions of $CW$; $\sigma_3$ and $\sigma_4$, $\sigma_5$ and $\sigma_6$, and $\sigma_7$ and $\sigma_8$ are solutions of $FCN$, $FICN$, and $ICW$, respectively. Suppose $\sigma_{2i-1}$ ($i = 1, 2, 3, 4$) denotes a prograde direction (or retrograde direction) solution, then $\sigma_{2i}$ ($i = 1, 2, 3, 4$) denotes a retrograde direction (or prograde direction) solution. It may be a bit surprising that the number of solutions for the free frequencies is eight, since there are only four physical oscillations associated to this Earth model. The reason is that free motions are elliptical in the triaxial case, and thus they are approximated by two circular oscillations with unbalanced amplitudes, one prograde and the other retrograde. However, we do not need to perform any mathematical derivation to find out the actual sense of the physical motion, since the comparison with observations (or two-layer solutions as a limit case) is simpler. Observations show that CW is prograde and FCN is retrograde; thus, we choose the prograde solution $\sigma_1$ for CW and the retrograde solution $\sigma_4$ for $FCN$. For $FICN$, because the elastic inner core is subjected to pressure and gravitational coupling torques, and visco-electromagnetic coupling torque, the direction of $FICN$ is determined by the relative magnitude of these coupling torques. According to the study of Xu and Szeto (1998), FICN should be retrograde if there is no pressure coupling.

Later, Dumberry and Wieczorek (2016) stated that for the planet Earth, since the pressure coupling dominates over the gravitational coupling, the FICN is prograde, just as indicated in the analysis of Rosat et al. (2017), although they only provide some weak constraints. Hence, we choose the prograde one, denoted by $\sigma_3$ as the FICN solution. MBHS1991 provided theoretical analysis to show that the ICW is prograde, and later various studies (Crossley & Rochester, 2014; Dehant et al., 1993; D09; MHB2002; Rochester & Crossley, 2009; Rochester et al., 2018) further demonstrate that ICW is prograde. Hence, we choose the prograde one, denoted by $\sigma_7$ as the ICW solution. We expect that future observations may further determine and confirm the motion status of the FICN and ICW.

5. Model Input Parameters

The model input parameters involve three kinds of data, namely, dynamical figure parameters, core-mantle coupling parameters, and compliances. We designed three special models for rotational normal modes solutions and comparisons. Model 1 is an Earth model consisting of triaxial anelastic mantle, triaxial fluid outer core, and triaxial elastic inner core. Model 2 is the rotational symmetrical model derived from Model 1. Model 3 follows the D09 to deploy the dynamical figure parameters of Mathews et al. (1991b) and uses the compliances calculated by D09. All three Earth models will be evaluated under our new triaxial three-layered Earth rotation theory.

| Compliance | Diurnal value | Static value |
|------------|--------------|--------------|
| $\kappa$   | $1.243 \times 10^{-3} - 1.196 \times 10^{-5}i$ | $1.242 \times 10^{-3} - 1.195 \times 10^{-5}i$ |
| $\xi$      | $2.463 \times 10^{-4}$ | $2.460 \times 10^{-4}$ |
| $\zeta$    | $4.964 \times 10^{-9}$ | $5.134 \times 10^{-9}$ |
| $\gamma$   | $1.967 \times 10^{-3} - 1.075 \times 10^{-4}i$ | $1.964 \times 10^{-3} - 1.073 \times 10^{-4}i$ |
| $\beta$    | $6.271 \times 10^{-4}$ | $6.262 \times 10^{-4}$ |
| $\delta$   | $-4.869 \times 10^{-7}$ | $-4.865 \times 10^{-7}$ |
| $\theta$   | $6.794 \times 10^{-6}$ | $7.024 \times 10^{-6}$ |
| $\chi$     | $-7.536 \times 10^{-5}$ | $-7.529 \times 10^{-5}$ |
| $\nu$      | $7.984 \times 10^{-5}$ | $7.984 \times 10^{-5}$ |
The dynamical figure parameters are the principal moments of inertia and dynamical ellipticities of the whole Earth, the fluid outer core, and the elastic inner core, respectively. In Model 1, we use the triaxial three-layered dynamical figure parameters of Case III of Table 8 from Chen et al. (2015), who estimated these parameters based on new gravity field models EGM2008 (Pavlis et al., 2012), EIGEN-6C (Förste et al., 2012), and EIGEN-6C2 (Förste et al., 2012). In Model 2, the rotational symmetric equatorial principal moments of inertia $\tilde{A}_x$, $\tilde{A}_y$, and $\tilde{A}_z$ are obtained by averaging Model 1’s equatorial principal moments of inertia, and the polar principal moments of inertia are obtained by $C = \tilde{A}(1+e)$, $C_y = \tilde{A}_y(1+e)$, and $C_z = \tilde{A}_z(1+e)$, with the dynamical ellipticities deploying Model 1’s values. In Model 3, the dynamical figure parameters are from Mathews et al. (1991b). The dynamical figure parameters are shown in Table 3. The core-mantle coupling parameters include viscoelectromagnetic coupling parameters $K_{CMB}$ and $K_{ICB}$, and pressure and gravitational coupling parameters $a_1$, $a_3$, $a_2^{12}$, and $a_2^{12}$, and the electro-magnetic couplings are updated to viscoelectromagnetic couplings by Mathews and Guo (2005) and their values are provided by Koot et al. (2010). Especially, the $Re(K_{CMB})$ is obtained by subtracting from the estimated value $e_j + Re(K_{CMB})$ of Koot et al. (2010) with dynamical ellipticity of fluid outer core $e_j$ of Models 1 and 2 as listed in Table 1. In Models 1 and 2, we use the viscoelectromagnetic coupling parameters updated from Koot et al. (2010). In Model 1, we use new pressure and gravitational coupling parameters $a_1$, $a_3$, $a_2^{12}$, and $a_2^{12}$, while in Model 2, we use the rotational symmetric pressure and gravitational coupling parameters $a_1$, $a_3$, $a_2$, and $a_2$ that are formerly given in Mathews et al. (1991b). The new triaxial pressure and gravitational coupling parameters $a_2^{12}$ and $a_2^{12}$ are updated from Mathews et al. (1991b) according to equation (15). In Model 3, the viscoelectromagnetic coupling parameters are set to zero following the parameter settings of D09, and the rotational symmetric pressure and gravitational coupling parameters $a_1$, $a_3$, $a_2$, and $a_2$ are deployed. The core-mantle coupling parameters are shown in Table 4. In Models 1, 2, and 3, there are 9 compliances, as well as 11 additional compliances, which have diurnal ones and static ones. The diurnal ones are used for FCN and FICN, and the static ones are used for CW and ICW. For compliances in Models 1 and 2, the static values of $\kappa$, $\xi$, $\gamma$, and $\beta$ are from the Group I recommendations of Shen et al. (2019), and the other five static compliances and the corresponding diurnal compliances are from D09, with diurnal values of $\kappa$, $\xi$, $\gamma$, and $\beta$ scaled from static values according to D09. For additional compliances in Models 1, 2, and 3, we deploy the parameters from D09. The 9 compliances used in Models 1 and 2 and in Model 3 are listed in Table 5 and Table 6, respectively, and 11 additional compliances used in Models 1, 2, and 3 are listed in Table 7.

It is worthy to mention that although some parameters are fitted from observations, like $e$, $e_j$, and viscoelectromagnetic parameters $K_{CMB}$ and $K_{ICB}$ are fitted to nutation observations in MHB2002 and Koot et al. (2010), respectively, most parameters are determined theoretically based on hydrostatic equilibrium condition about the internal figures of the Earth corresponding to the PREM Earth model. So, it is inconvenient to provide the uncertainties of these parameters, but the values of these parameters are still credible for above reason.

### 6. Results of Rotational Normal Mode Solutions

#### 6.1. Basic Rotational Normal Modes Solutions

Now we provide rotational normal mode solutions for Models 1, 2, and 3 under our new triaxial three-layered Earth rotation frame that considers the elastic deformations induced by solid inner core. For comparisons, we list the results in Table 8, where the results obtained by Mathews et al. (1991b) and D09 are also listed. Comparing the Models 1 and 2, the triaxiality prolongs the CW period about 0.005 d, shortens the FICN period about 0.071 d, lengthens the ICW period about 0.174 d, but has no obvious effect on FCN, except for having slight effect on the quality factor of FCN. By comparisons, the Model 3 results and the results of D09 show that, if neglecting mantle anelasticity and ocean tides dissipations, the CW periods will be the same,
and the FCN period will be increased by 0.56 d; and if neglecting the viscoelectromagnetic couplings near CMB and ICB, the FICN period will be increased by 0.23 d, and the ICW period will be shortened by 0.25 d (see Table 8). If Model 3 has a rigid inner core, we need to set the additional compliances to zero for rotational normal modes solutions and deploy the diurnal ones of the compliances in Table 6 sourced from Mathews et al. (1991b). Especially, we adopt the same coupling parameter value with Mathews et al. (1991b), that is, 0.8294, rather than the more accurate result 0.849492 obtained in Table 4. We note the rigid inner core solutions with Model 3a, then we compare Model 3a solutions with the results of Mathews et al. (1991b). We find that the CW, FCN, and ICW periods of Model 3a are shortened, respectively, by 0.06, 0.13, and 0.06 d, while the FICN period is increased by 3.21 d.

### 6.2. The Effect of Triaxiality on Rotational Normal Modes

As discussed in section 6.1, concerning the Earth, the triaxiality can notably lengthen the CW and ICW periods and shorten the FICN period, but has no notable effect on FCN, since the triaxiality of the Earth is small. Suppose the triaxility of the Earth were enhanced to some extent, then there will be significant effects of triaxiality on rotational normal modes. We design two cases under Model 1, denoted as Case I and Case II, to examine the effect of triaxiality on rotational normal modes. In Cases I and II, due to the fact of hydrostatic equilibrium condition of the Earth that the geometrical and dynamical ellipticities of the Earth will be decreased from the surface to interior, the first and second equatorial principal moments of inertia of whole Earth, fluid outer core, and elastic inner core are decreased and increased by $0.0005 \times 10^{37}$, $0.0003 \times 10^{36}$, and $0.0001 \times 10^{34}$ kg m$^2$ in Case I and by $0.002 \times 10^{37}$, $0.001 \times 10^{36}$, and $0.0005 \times 10^{34}$ kg m$^2$ in Case II, from the model values as given in Model 1. In this way, the polar dynamical ellipticities of the whole Earth, fluid outer core, and elastic inner core in the above mentioned two cases are unchanged according to equation (17). The equatorial dynamical ellipticities of the whole Earth, fluid outer core, and elastic inner core in Cases I and II are calculated and displayed in Table 9. In Table 9, from Model 1 to Cases I and II, the equatorial dynamical ellipticities are increasing. The equatorial dynamical ellipticity of whole Earth in Cases I and II are larger than Model 1 by 1 order of magnitude, which varies from $2.20 \times 10^{-5}$ of Model 1, to $1.47 \times 10^{-4}$ of Case I, and to $5.22 \times 10^{-4}$ of Case II.

After considering the effect of triaxiality on gravitational coupling strength $a_{1,2}$ between elastic inner core and the rest through equation (15), the rotational normal modes solutions for Cases I and II are provided in Table 10. Since the first and second equatorial principal moments of inertia are decreased and increased

### Table 8

|                | CW   | FCN  | FICN | ICW  |
|----------------|------|------|------|------|
| Model 2        | 433.234 | 429.888 | 934.089 | 2717.499 |
| Q              | 85.43 | 22620.22 | 456.47 | 458.50 |
| Model 1–Model 2| 0.005 | 0.0 | −0.071 | 1.174 |
| Q              | 0.0 | −0.11 | 0.0 | 0.0 |
| Mathews et al. (1991b) | 400.7 | 455.8 | 475.5 | 2409 |
| Model 3a–Mathews et al. (1991b) | −0.06 | −0.13 | 3.21 | −0.06 |
| D09            | 400.5 | 455.2 | 478.6 | 2715 |
| Model 3–D09    | 0.0 | 0.56 | 0.23 | −0.25 |

**Note.** The viscoelectromagnetic coupling parameters are set to zero in Model 3, and the results of D09 and Mathews et al. (1991b) are provided for comparison with results of Model 3 and Model 3a.

### Table 9

|                | Model 1 | Case I | Case II |
|----------------|---------|--------|---------|
| $(B - A)/A$    | 0.02202245 | 0.14687543 | 0.52152792 |
| $(B_f - A_f)/A_f$ | 0.01774390 | 0.08399647 | 0.23860287 |
| $(B_s - A_s)/A_s$ | 0.01625078 | 0.05042783 | 0.18714772 |
by the same amount, Cases I and II will share the same rotational symmetric model, that is, Model 2. By comparing the rotational normal modes solutions of Cases I and II with the results of Model 2, we find that the CW and ICW periods will be lengthened by 0.280 and 7.027 d in Case I and by 3.579 and 94.912 d in Case II, while the FCN and FICN periods will be shortened by 0.0 and 0.105 d in Case I and by 0.002 and 0.591 d in Case II. Hence, the effect of triaxiality on CW and ICW in Case II is quite obvious. For CW and FCN in Case II, the quality factor difference is smaller by 1.4 and 0.72 respectively. In other cases, the deviations of quality factors of rotational normal modes due to triaxiality are too small to be considered. Cases I and II can be used on other planets in solar system, like Mars, which might have larger equatorial dynamical ellipticity than the Earth.

6.3. The CW/FCN Observational Constraints on Compliances
Since there are no evident observations for FICN and ICW, we will only use CW and FCN observations in Table 1 to constrain related compliances. Here, we used the weighted mean values of the CW and FCN observations in Table 1, that is, $433.249 \pm 0.344$ and $431.967 \pm 0.536$ d, respectively, to invert the compliances by matching the calculated normal mode solutions to these observations. Based on Model 1, when we invert compliances from CW observations, the static values of the compliances are used. Otherwise, if we invert compliances from FCN observations, the diurnal ones of compliances are used. The inverted static compliances $\kappa_{\text{static}}, \gamma_{\text{static}},$ respectively, have the values $1.242 \times 10^{-3} - 1.195 \times 10^{-5}i$ and $1.783 \times 10^{-3} - 1.073 \times 10^{-4}i$. The inverted diurnal compliances $\xi_{\text{diurnal}}, \rho_{\text{diurnal}},$ respectively, have the values $2.483 \times 10^{-4}$ and $6.370 \times 10^{-4}$. The constrained rotational normal modes solutions and inverted diurnal and static compliances, respectively, are listed in Table 11. These new obtained compliances can be used in polar motion excitation studies under triaxial three-layered Earth rotation frame.

7. Conclusions

As a generalization of both the rotationally symmetric three-layered Earth rotation theories (MBHS1991, MHB2002 and D09) and the triaxial two-layered Earth rotation theory (CS10), here we formulated a triaxial three-layered Earth rotation theory. In our theory, after considering an elastic inner core, we deploy the triaxial version of viscoelectromagnetic couplings, and the pressure and gravitational coupling is investigated to include the triaxial effect of gravitational coupling strength $\alpha_g$. We demonstrated that various presently existing theories, including MBHS1991, MHB2002, D09, and CS10 can be derived based on the new triaxial three-layered Earth rotation theory under different conditions.
Our theory provides the following rotational normal modes solutions, expressed as periods (quality factors): CW 433.239 (85.43), FCN 429.888 (22620.11), FICN 934.018 (456.47), ICW 2,718.673 d, or 7.44 years (458.50). When deploying the same parameters with D09, our theory provides very close results with them for CW, FCN, and ICW, except for FICN with a difference of 3 days. Concerning the effect of triaxiality on rotational normal modes, the triaxiality of the actual Earth has a notable effect on rotational normal modes, that is, 0.005, 0.0, −0.071, and 1.174 d, respectively, for the CW, FCN, FICN, and ICW. However, if increasing the triaxiality of the Earth, that is, equatorial dynamical ellipticities of the whole Earth, fluid outer core, elastic inner core, we find obvious effects of triaxiality on CW, FCN, FICN, and ICW. However, if increasing the triaxiality of the Earth, that is, equatorial dynamical ellipticities of the whole Earth, fluid outer core, elastic inner core, we find obvious effects of triaxiality on CW, FCN, FICN, and ICW. Hence, we conclude that, by enhancing the triaxiality, eventually the period shifts become more noticeable, and this could be a factor for other bodies in the solar system, even asteroids perhaps, which suggests that the formulation of this study could be applied to other planets in the solar system.

Appendix A: Matrixes M and N in Triaxial Three-Layered Earth Rotation Theory

Here the triaxial three-layered Earth rotation theory can be expressed in matrix form (equation (18)):

\[ \frac{dy}{dt} = \Omega_{\lambda} N y, \quad y = [m_1, m_2, m_1', m_2', n_1', n_1'']^T. \]  \hspace{1cm} (A1)

where the matrix \( M \) can be expressed as

\[ M = \begin{bmatrix} M_{ij} \end{bmatrix}, \quad i,j = 1,2 \]  \hspace{1cm} (A2)

with

\[
M_{11} = \begin{bmatrix}
(1 + \kappa - S_{14}^p)A & 0 & A_f + A_x & 0 \\
0 & (1 + \kappa - S_{14}^p)B & 0 & B_f + B_x \\
(1 + \gamma - S_{24}^p)A_f & 0 & A_f(1 + \beta) & 0 \\
0 & (1 + \gamma - S_{24}^p)B_f & 0 & B_f(1 + \beta)
\end{bmatrix},
\]

\[
M_{12} = \begin{bmatrix}
\frac{A_x + B_x}{2} \phi_1 & 0 & 0 & 0 \\
0 & B_x & B_f(\zeta - S_{14}^p) & 0 \\
A_f(\phi - S_{14}^p) & 0 & 0 & 0 \\
0 & 0 & B_f((\phi - S_{14}^p)) & 0
\end{bmatrix},
\]

\[
M_{21} = \begin{bmatrix}
A_f(\theta - S_{34}^p) & 0 & 0 & 0 \\
0 & B_f(\theta - S_{34}^p) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
M_{22} = \begin{bmatrix}
A_f(\nu - S_{34}^p) & 0 & \frac{A_x + B_x}{2} \phi_1 & 0 \\
0 & B_f(\nu - S_{34}^p) & 0 & \frac{A_x + B_x}{2} \phi_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and the matrix \( N \) is expressed as

\[ N = \begin{bmatrix} N_{ij} \end{bmatrix}, \quad i,j = 1,2 \]  \hspace{1cm} (A3)
with

\[ N_{11} = \begin{bmatrix} 0 & -C - (1 + \kappa - S_{14}^g)B \ 
0 & 0 \ 
0 & 0 \ 
-0 & B_J + B_{\zeta} \ 
-(A_J + \Lambda_J) & 0 \ 
0 & C_J + K_{CMB}B_J + K_{ICB}B_J \ 
-(C_J + K_{CMB}A_J + K_{ICB}A_J) & 0 \ 
-0 & 0 \ 
\end{bmatrix}, \]

\[ N_{12} = \begin{bmatrix} 0 & B_J + B_{\zeta} - S_{14}^g \ 
-0 & -K_{ICB}B_J \ 
K_{ICB}A_J & 0 \ 
0 & 0 \ 
-0 & 0 \ 
\end{bmatrix}, \]

\[ N_{21} = \begin{bmatrix} 0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
\end{bmatrix}, \]

\[ N_{22} = \begin{bmatrix} 0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
-(C_J - B_J)(1 - \alpha_1 + \alpha_3 \alpha_1^2 S_{c1} - S_{c4}^g) - (1 - \alpha_1^2)A_J(\theta - S_{s4}^g) \ 
0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
-(C_J - A_J)(1 - \alpha_1 + \alpha_3 \alpha_1^2 S_{c1} - S_{c4}^g) - (1 - \alpha_1^2)A_J(\theta - S_{s4}^g) \ 
0 & 0 \ 
0 & 0 \ 
0 & 0 \ 
\end{bmatrix}, \]
Then the eigenvalue method is used to solve the rotational normal modes solutions, that is, solving the eigenvalues $\sigma_j$ of matrix $M^{-1}N_i$, and the rotational normal modes solutions will be $-i\sigma_j$.

Acknowledgments

We would like to express our sincere thanks to W. Chen and H. S. Fok for extensive discussions, Z.Y. Shen for figure drawing, and thanks to José M. Fernández, Paul Tregoning, associate editor and additional three reviewers for their valuable comments and suggestions, which greatly improved this manuscript. All of data used in our work are from public publications as cited in the text, which can be accessed freely without any data limitation. This study was supported by the NSFC (grant Nos. 41631072, 41721003, 41874023, 41574007, and 41424901), the Discipline Innovative Engineering Plan of Modern Geodesy and Geodynamics (grant No. B17033), and the DAAD Thematic Network Project (grant No. 57173947).

References

Belda, S., Ferrándiz, J. M., Heinkelman, R., & Schuh, H. (2018). A new method to improve the prediction of the celestial pole offsets. *Scientific Reports*, 8, 13861. https://doi.org/10.1038/s41598-018-32082-1

Belda, S., Heinkelman, R., Ferrándiz, J. M., Karbon, M., Nilsson, T., & Schuh, H. (2017). An improved empirical harmonic model of the celestial intermediate pole offsets from a global VLBI solution. *The Astronomical Journal*, 154(4), 166. https://doi.org/10.3847/1538-3881/aab869

Bettadpur, S. (2012). CSR level-2 processing standards document for product release 05 grade 327–742.

Bizouard, C., & Gambis, D. (2009). The combined solution C04 for Earth orientation parameters consistent with international terrestrial reference frame 2005. In D. Hermann (Ed.), Geodetic reference frames: IAG Symposium Munich, Germany, 9-14 October 2006 (pp. 265–270). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-00860-3-41

Bizouard, C., Lambert, S., Gattano, C., Becker, O., & Richard, J.-Y. (2019). The IERS EOP 14C04 solution for Earth orientation parameters consistent with ITRF 2014. *Journal of Geodesy*, 93(5), 621–633. https://doi.org/10.1007/s00190-018-1186-3

Bizouard, C., & Zotov, L. (2013). Asymmetric effects on Earth’s polar motion. *Celestial Mechanics and Dynamical Astronomy*, 116(2), 195–212. https://doi.org/10.1007/s10569-013-9483-x

Chao, B. F. (2017). Dynamics of the inner core wobble under mantle inner core gravitational interactions. *Journal of Geophysical Research: Solid Earth*, 122, 7437–7448. https://doi.org/10.1002/2017JB014405

Chao, B. F., & Hsieh, Y. (2015). The Earth’s Free Core Nutation: Formulation of dynamics and estimation of eigenperiod from the very-long-baseline interferometry data. *Earth and Planetary Science Letters*, 432, 483–492. https://doi.org/10.1016/j.epsl.2015.10.010

Chen, W., Li, J. C., Ray, J., Shen, W. B., & Huang, C. L. (2015). Consistent estimates of the dynamic figure parameters of the Earth. *Journal of Geodesy*, 89(2), 179–188. https://doi.org/10.1007/s00190-014-0768-y

Chen, W., & Shen, W. (2010). New estimates of the inertia tensor and rotation of the triaxial nonrigid Earth. *Journal of Geophysical Research*, 115, B12419. https://doi.org/10.1029/2009JB007094

Cheng, M., Ries, J. C., & Tapley, B. D. (2011). Variations of the Earth’s figure axis from satellite laser ranging and gracing. *Journal of Geophysical Research*, 116, B01409. https://doi.org/10.1029/2010JB008850

Crossley, D. J., & Rochester, M. G. (2014). A new description of Earth’s wobble modes using Clairaut coordinates 2: Results and inferences on the core mode spectrum. *Geophysical Journal International*, 198, 1890–1905. https://doi.org/10.1093/gji/ggu232

de Vries, D., & Waltz, J. M. (1991). The effects of the solid inner core and nonhydrostatic structure on the Earth’s forced nutations and Earth tides. *Journal of Geophysical Research*, 96, 8275–8293. https://doi.org/10.1029/90JB01958

Defraigne, P., Dehant, V., & Hinderer, J. (1991). The effects of the solid inner core wobble and its implications for the density of Earth’s core. *Physica Scripta*, T26, 54–58. https://doi.org/10.1238/PhysicaScriptaT26a054

Dumberry, M. (2009). Influence of elastic deformations on the inner core wobble. *Geophysical Journal International*, 178, 57–64. https://doi.org/10.1111/j.1365-246X.2009.04140.x

Dumberry, M., & Wieczorek, M. A. (2016). The forced precession of the Moon’s inner core. *Journal of Geophysical Research: Planets*, 121, 1284–1292. https://doi.org/10.1002/2015JE004986

123

123

123

123
Rosat, S., Lambert, S. B., Gattano, C., & Calvo, M. (2017). Earth’s core and inner-core resonances from analysis of VLBI nutation and superconducting gravimeter data. *Geophysical Journal International*, 208, 211–220. https://doi.org/10.1093/gji/ggw378

Sasao, T., Okubo, S., & Saito, M. (1980). A simple theory on dynamical effects of stratified fluid core upon nutational motion of the Earth. In R. L. Duncombe (Ed.), *Nutation and the Earth's rotation* (Vol. 78, pp. 165). Dordrecht: Springer.

Sato, T., Tamura, Y., Matsumoto, K., Imanishi, Y., & McQueen, H. (2004). Parameters of the fluid core resonance inferred from superconducting gravimeter data. *Journal of Geodynamics*, 38, 375–389. https://doi.org/10.1016/j.jog.2004.07.016

Seitz, F., & Schmidt, M. (2005). Atmospheric and oceanic contributions to Chandler Wobble excitation determined by wavelet filtering. *Journal of Geophysical Research*, 110, B11406. https://doi.org/10.1029/2005JB003826

Shen, W., Yang, Z., Guo, Z., & Zhang, W. (2019). Numerical solutions of rotational normal modes of a triaxial two-layered anelastic Earth. *Advances in Space Research*, 59(2), 118–129. https://doi.org/10.1016/j.asr.2019.03.001 From Space Geodesy to Astro-Geodynamics.

Smith, M. L., & Dahlen, F. A. (1981). The period and Q of the Chandler Wobble. *Geophysical Journal of the Royal Astronomical Society*, 64, 223–281. https://doi.org/10.1111/j.1365-246X.1981.tb02667.x

Smylie, D. E., Szeto, A. M. K., & Rochester, M. G. (1984). The dynamics of the Earth’s inner and outer cores. *Reports on Progress in Physics*, 47, 855–906. https://doi.org/10.1088/0034-4885/47/7/002

Sun, R., & Shen, W.-B. (2016). Influence of dynamical equatorial flattening and orientation of a triaxial core on prograde diurnal polar motion (Abstract). In *Journées 2014 “systèmes de référence spatio-temporels”* (Z. Malkin, & N. Capitaine, Eds.)

Sun, R., & Shen, W.-B. (2015). Triaxial Earth’s rotation: Chandler Wobble, Free Core Nutation and diurnal polar motion (Abstract). In *From Space Geodesy to Astro-Geodynamics.*

Sun, R., & Shen, W.-B. (2016). Influence of dynamical equatorial flattening and orientation of a triaxial core on prograde diurnal polar motion of the Earth. *Journal of Geophysical Research: Solid Earth*, 121, 7570–7597. https://doi.org/10.1002/2016JB013278

Süse, A. M. K., & Smylie, D. E. (1984). Coupled motions of the inner core and possible geomagnetic implications. *Physics of the Earth and Planetary Interiors*, 36, 27–42. https://doi.org/10.1016/0031-9201(84)90096-7

Süse, A. M. K., & Smylie, D. E. (1984). The rotation of the Earth’s inner core. *Philosophical Transactions of the Royal Society of London Series A*, 313, 171–184. https://doi.org/10.1098/rsta.1984.0093

Van Hoolst, T., & Dehant, V. (2002). Influcence of triaxiality and second-order terms in flattening on the rotation of terrestrial planets. *Physics of the Earth and Planetary Interiors*, 134, 17–33. https://doi.org/10.1016/S0031-9201(02)00066-7

Visconti, RaimundoO., & Wilson, ClarkR. (1997). On the variability of the Chandler frequency. *Journal of Geophysical Research*, 102(B9), 20,439–20,445. https://doi.org/10.1029/97JB01275

Vondrák, J., & Ron, C. (2006). Resonant period of Free Core Nutation—Its observed changes and excitations. *Acta Geodynamica et Geomaterialia*, 2(2), 53–60.

Vondrák, J., & Ron, C. (2017). New method for determining Free Core Nutation parameters, considering geophysical effects. *Astronomy and Geophysics*, 60(4), A56. https://doi.org/10.1002/2017JG003635

Vondrák, J., Ron, C., & Chapany, Y. (2017). New determination of period and quality factor of Chandler wobble, considering geophysical excitations. *Advances in Space Research*, 59, 1395–1407. https://doi.org/10.1016/j.asr.2016.12.001

Wilson, C. R., & Haubrich, R. A. (1976). Meteorological excitation of the Earth’s wobble. *Geophysical Journal of the Royal Astronomical Society*, 46, 707–743.

Wilson, C. R., & Vicente, R. O. (1980). An analysis of the homogeneous ILS polar motion series. *Geophysical Journal of the Royal Astronomical Society*, 62, 605–616. https://doi.org/10.1111/j.1365-246X.1980.tb02594.x

Wilson, C. R., & Vicente, R. O. (1990). Maximum likelihood estimates of polar motion parameters. In D. D. McCarthy, & W. E. Carter (Eds.), *Variations in Earth rotation* (pp. 151–155). Washington, DC: AGU.

Xu, S., & Szeto, A. M. K. (1998). The coupled rotation of the inner core. *Geophysical Journal International*, 133, 279–297. https://doi.org/10.1046/j.1365-246X.1998.00495.x

Zhang, M., & Huang, C. (2019). The effect of the differential rotation of the Earth inner core on the Free Core Nutation. *Geodesy and Geodynamics*, 10(2), 146–149. https://doi.org/10.1046/j.ggeo.2018.11.003From Space Geodesy to Astro-Geodynamics.

Zhou, Y., Zhu, Q., Salstein, D. A., Xu, X., Shi, S., & Liao, X. (2016). Estimation of the Free Core Nutation period by the sliding-window complex least-squares fit method. *Advances in Space Research*, 57, 2136–2140. https://doi.org/10.1016/j.asr.2016.03.028