Bayesian Model Updating Based on Kriging Surrogate Model and Simulated Annealing Algorithm

Zenghui Wang, Hong Yin and Zhenrui Peng

School of Mechanical Engineering, Lanzhou Jiaotong University, Lanzhou 730070, Gansu Province, China
Email: pzrui@163.com

Abstract. Aiming at the problem of difficulty in selecting the proposal distribution and low computational efficiency in the traditional Markov chain Monte Carlo algorithm, a Bayesian model updating method using surrogate model technology and simulated annealing algorithm is proposed. Firstly, the Kriging surrogate model is used to mine the implicit relationship between the structural parameters to be updated and the corresponding dynamic responses, and the Kriging model that meets the accuracy requirement is used to replace the complex finite element model to participate in the iterative calculation to improve the model updating efficiency. Then, the simulated annealing algorithm is introduced to reorganize the Markov chains from different proposal distributions to obtain high-quality posterior samples, which are used to estimate the parameters posterior distributions. Finally, a space truss structure is used to verify the effectiveness of the proposed method.

Keywords. Bayesian inference; model updating; Markov chain Monte Carlo algorithm; Kriging model; simulated annealing algorithm

1. Introduction

In recent years, finite element model updating (FEMU) technology has gradually become a research hotspot at the structural dynamics domain, and is widely used in structural damage identification [1]. Most of the current model updating methods are deterministic methods, however, the application of deterministic methods is limited due to uncertainty factors in the structural materials, boundary conditions and the test process [2].

The uncertainty model updating methods based on Bayesian statistical theory comprehensively considers historical data and experts’ experience to suppose the prior distributions, and combines the statistical information of the measured responses to continuously update the prior distribution to make it close to the real posterior probability distribution, which is widely used in uncertainty model updating [3-4]. Beck et al. [5] introduced Bayesian theory into model updating for the first time. In order to estimate the parameters posterior distributions, he also proposed a Markov chain Monte Carlo (MCMC) algorithm [6-7], and verified the effectiveness of the MCMC algorithm through a two-degree-of-freedom steel frame structure, and then the Bayesian method has been developed rapidly. Ching et al. [8] proposed the Transition MCMC (TMCMC) method, which samples from a string of intermediate probability density functions (pdf) to estimate the posterior distributions, which can avoid sampling directly from the complex posterior pdf, which improved sampling efficiency. Cheung et al. [9] proposed a Hybrid MCMC (HMCMC) method, which shows great potential in the Bayesian model updating with high-dimensional uncertainty parameters. However, most of the above methods are based on a single Markov chain, which means that these methods will rely too much on the selection
of the proposal distribution variance, the sampling efficiency of single-chain MCMC method is low when meets high parameter dimension, the quality of the obtained posterior samples is poor.

To address these problems, a Bayesian FEMU method using Kriging surrogate model technology and simulated annealing algorithm is proposed in this paper. Firstly, select the parameters to be updated; use the uniform experimental design method to construct the training set samples; use the training set samples as the inputs of the Kriging model, and the corresponding dynamic responses as the outputs to construct the accurate Kriging model; then, use simulated annealing mechanism and Metropolis criteria to reorganize Markov chains from different proposal distributions to obtain high-quality posterior samples. The parameter posterior pdf is obtained by statistical analysis of the Markov chain’s stationary phase. Lastly, the viability of the method is demonstrated through a space truss structure.

2. Basic Theory and Enhancements

2.1. FEMU theory based on Bayesian Inference

Bayesian FEMU methods combine prior information and test data, the posterior pdf of the parameters is inferred by the MCMC algorithm. This process can be explained as [10-11]:

\[
p(\theta|x) = \frac{p(x|\theta)\pi(\theta)}{\int_\theta p(x|\theta)\pi(\theta)d\theta} = c \cdot p(x|\theta)\pi(\theta)
\]

where \(x\) is test data; \(\theta\) is the vector composed of all the parameters to be updated; \(\pi(\theta)\) is the prior distribution, usually taken as the generalized unbiased uniform distribution \(\pi(\theta) = 1\); \(p(x|\theta)\) is the given conditional distribution, usually called the likelihood function; \(c\) is a constant factor, usually called a regularization constant. Assuming \(N\) independent experiments, the likelihood function in the Bayesian equation can be expressed as:

\[
\begin{align*}
L(\theta) &= \sum_{i=1}^{N} (y_i - y(\theta_i))^T \text{cov}_{y}^{-1} (y_i - y(\theta_i)) \\
p(x|\theta) &= \frac{1}{(2\pi\text{cov}_y)^{n/2}} \exp\left(-\frac{1}{2} (y_i - y(\theta_i))^T \text{cov}_{y}^{-1} (y_i - y(\theta_i))\right)
\end{align*}
\]

where \(y_i\) is test response; \(y(\theta)\) is model calculated response; \(\text{cov}_{y}\) is test information covariance matrix. Substituting the above equations into equation (1), the pdf can be written as:

\[
p(\theta|x) = c' \cdot \exp\left[\sum_{i=1}^{N} (y_i - y(\theta_i))^T \text{cov}_{y}^{-1} (y_i - y(\theta_i))\right]
\]

where \(c'\) is a constant. In actual problems, there is usually no explicit response expression \(y(\theta)\) and the integration computation is more difficult. Therefore, the pdf is calculated using the MCMC to overcome complex integration computation problem.

2.2. Kriging Surrogate Model

Kriging model is a surrogate model, which is on the basis of stochastic theory. It has good fitting and prediction ability to nonlinear problems and error estimation function. Kriging model regards unknown function as the concrete realization of Gaussian process, which includes linear regression and non-parametric part [12]:

\[
y(x) = \sum_{i=1}^{N} \beta_i f_i(x) + z(x) = f(x)^T \beta + z(x)
\]
where \( f(x) \) is a polynomial function; \( \beta \) is a regression model coefficient matrix, and \( z(x) \) is a static stochastic process obeys normal distribution. The correlation between responses \( z(x) \) is expressed by a function with spatial distance between two sample points, and the Gaussian function which fits the actual engineering practice is selected as the correlation function, which is in the following form:

\[
R(x_i, x_j) = \exp(-\sum_{k=1}^{m} \theta_k |x_i^k - x_j^k|^2)
\]

where \( x_i^k \) and \( x_j^k \) are the \( k \) component of any two sample points; \( m \) is the design variables; \( \theta_k \) is correlation coefficient.

Maximum likelihood is introduced to estimate model parameters:

\[
\hat{\beta} = (F^T R^{-1} F)^{-1} F^T R^{-1} Y
\]

\[
\hat{\sigma}^2 = \frac{1}{n} (Y - F \hat{\beta})^T R^{-1} (Y - F \hat{\beta})
\]

where \( F \) is sample point matrix; \( Y \) is response column vector; \( R \) is the correlation matrix; the elements are \( R_{ij} = R(x_i, x_j) \) \((i, j = 1, 2, \ldots, n)\); \( n \) is test points number.

### 2.3. Simulated Annealing Algorithm

The goal of simulated annealing (SA) algorithm, a probabilistic algorithm, is to obtain an approximate solution of an energy function. The Metropolis sampling can be seen as the core of SA, which yields stochastic samples by a proposal pdf, the mathematical description of which is unknown, but whose value can be calculated [13]. A rapid annealing plan is adopted in SA, which temperature is set at \( \alpha \in (0, 1) \), build on this and decline at an exponential rate. The relationship between the temperature of the next anneal cycle \( c+1 \) and the temperature of the current anneal cycle \( c \) is as: \( T_{c+1} = \alpha T_c \).

When the initial and final temperatures are given, the cooling rate can be obtained by the relation \( \alpha = (T_f / T_i)^{1/(N_c - 1)} \), with \( N_c \) the number of annealing cycles. The new random candidate \( k^{c+1} \) are generated by the following equation:

\[
k^{c+1} = k^c + \omega \ast (U^{k^c} - L^{k^c})
\]

\[
\omega = \text{sgn}(u - \frac{1}{2}) T_c \left[ (1 + \frac{1}{T_c})^{2\mu - 1} - 1 \right]
\]

where \( U^{k^c} \) and \( L^{k^c} \) are the upper bound and the lower bound of parameters at the current anneal cycle \( c \); \( \omega \in [-1, 1] \) is generated by the uniform distribution \( u \in [0, 1] \).

### 2.4. Multi-Chains Simulated Annealing Reorganization Algorithm

Combining delay rejection and adaptive Metropolis (DRAM) strategies, Kriging surrogate model and simulated annealing algorithm, the specific steps of Bayesian model updating method are as follows.

1. Select the initial sample points \( \theta_i^0 \) of \( M \) Markov chains randomly by the prior information, and set the random sampling times is \( T \).

2. Set the initial variance \( C_0 \) \((s=1,2,\ldots,M)\) of \( M \) proposal distributions randomly, and arrange the variance values of the proposal distributions sequentially, which are used to sample to produce Markov chains of different sources;

3. Obtain \( M \) Markov chains by parallel sampling according to the DRAM algorithm;

4. Take the initial sample point of the first Markov chain as the initial sample of simulated annealing algorithm, generate two sample points in the sample points of the remaining \((M-1)\) Markov
chains at the same iterative step randomly, which are used as the upper and lower boundary values \( U \) and \( L \) of the disturbance, respectively, and a new candidate sample point is obtained by equation (8):

(5) Determine whether to accept the new candidate sample by Metropolis criterion, and terminate the reorganization when the set sampling times \( T \) is reached;

(6) Analyse the stationary phase of the reorganized Markov chain statistically to obtain the updated parameter values.

3. Example Analysis

The structural model shown in figure 1 is a common space truss structure in engineering, which consists of 28 nodes, 66 rod units and 48 degrees of freedom. The constraint condition is that 4 supports (node number 1, 8, 9, 16) are fixed. When analyzing the structure, only the Y and Z directional DOF of each node are considered, which are numbered in turn. The truss is divided into three parts: the upper chords, the middle chords and the lower chords. In most cases, the damage will lead to significant changes in structural stiffness, mainly manifested in the uncertainty of elastic modulus. The elastic modulus of three groups are selected as the parameters to be updated, which are \( \theta_1 = E_i / E_n \), \( \theta_2 = E_2 / E_n \) and \( \theta_3 = E_3 / E_n \), respectively. The initial elastic modulus \( E_n \) is 190 GPa, and it is assumed that the measured mean value of each parameter obeys the normal distribution matrix \([1.5, 2, 1.2]\), and the standard deviation is all 0.1. Set the ranges of parameters are \([1, 2]\), \([1.5, 2.5]\) and \([0.7, 1.7]\), respectively.

![Figure 1. Space truss model.](image)

Kriging model is constructed using the method in Section 2.2, which meets the precision requirements, and the RMSE value of the first order frequency is selected to test the Kriging model accuracy, which are listed in table 1. It can be seen that Kriging model has high prediction accuracy and can replace the initial FE model for iterative calculation.

| Order | Minimum error | Maximum error | Average error |
|-------|---------------|---------------|---------------|
| 1     | 1.32×10^{-12} | 4.30×10^{-8}  | 1.70×10^{-9}  |
| 2     | 7.16×10^{-12} | 2.34×10^{-7}  | 9.25×10^{-9}  |
| 3     | 2.21×10^{-11} | 7.23×10^{-7}  | 2.86×10^{-8}  |

The Markov chains are obtained by standard MH algorithm and the proposed algorithm, respectively, set the sampling time is 20000. The Markov chains obtained by the two methods are shown in figure 2 and figure 3. It can be seen that the phenomenon of "sampling stagnation" is always occurs in Markov chains. However, when using the proposed algorithm to sample, the Markov chains
can always maintain a swing state, and the mixing performance of samples is significantly better than the standard MH algorithm.

The posterior samples in figure 3 are analysed using the normal probability test knowledge in probability statistics: because the samples in non-stationary period will seriously disrupt statistical analysis and affect estimation accuracy of parameter means, the first 10% samples of Markov chains in figure 3 are eliminated, and then the rest samples are tested for normal probability.

It can be seen from figure 4 that most posterior samples of the three parameters obtained by the proposed algorithm can fall on the assumed normal distribution line after removing inferior samples such as nonstationary samples. Therefore, the quality of samples is higher and the assumption of normal distribution holds.

| Parameter | True value | Initial value | Pre-updating error/% | Updated value | Updated error/% |
|-----------|------------|---------------|---------------------|---------------|-----------------|
| $\theta_1$ | 1.500      | 1.953         | 30.187              | 1.506         | 0.421           |
| $\theta_2$ | 2.000      | 2.399         | 19.930              | 2.009         | 0.443           |
| $\theta_3$ | 1.200      | 1.469         | 22.417              | 1.202         | 0.141           |

The mean value of parameters is estimated by the stable phases of corresponding Markov chains, and the updated results of the mean values are listed in table 2, where the updated errors of mean values are all less than 0.5%, and the updated errors of the three parameters are all better than the
standard MH algorithm, which indicates that the proposed algorithm achieves good results in the uncertainty model updating.

4. Conclusions
Compared with the traditional standard MH algorithm, "sampling stagnation" stage of Markov chains generated by the proposed algorithm is significantly reduced, which can better reflect the distribution characteristics of parameter space to be updated and get better updating accuracy, which is of great significance for the FEMU study of engineering structures.

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