Domain decomposition solvers for operators with fractional interface perturbations

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Abstract Operators with fractional perturbations are crucial components for robust preconditioning of interface-coupled multiphysics systems. However, in case the perturbation is strong, standard approaches can fail to provide scalable approximation of the inverse, thus compromising efficiency of the entire multiphysics solver. In this work, we develop efficient and parameter-robust algorithms for interface-perturbed operators based on the non-overlapping domain decomposition method. As preconditioners for the resulting Schur complement problems we utilize (inverses of) weighted sums of fractional powers of the interfacial Laplacian. Realization of the preconditioner in terms of rational approximation is discussed. We demonstrate performance of the solvers by numerical examples including application to coupled Darcy-Stokes problem.

1 Introduction

Mathematical models featuring interaction of physical systems across a common interface describe numerous phenomena in engineering, environmental sciences and medicine. Here the large variations in material coefficients or wide ranges of temporal/spatial scales at which the phenomena can be studied demand parameter-robust solution algorithms. In [4, 3] such algorithms were recently developed for Darcy-Stokes and Biot-Stokes models by establishing uniform stability of the respective problems in (non-standard) parameter-dependent norms. In particular, the authors show that in order to attain robustness, mass conservation at the interface $\Gamma$ of the porous domain $\Omega$ must be accounted for in the functional setting, leading to control of the porous pressure $p$ in the norm $\|p\|_\Omega$ such that

$$\|p\|_{\Omega}^2 = \|K^{1/2} \nabla p\|_{0,\Omega}^2 + \|\mu^{-1/2} p\|_{1/2,\Gamma}^2.$$  (1)

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Fig. 1 (Left) Sparsity pattern of the operator in (2) on $\Omega = (0, 1)^3$ with $\Gamma \subset \partial \Omega$. Interface perturbation leads to dense block (in blue) which is challenging for sparse LU solvers. (Right) Number of PCG iterations under mesh refinement when solving (2) on $\Omega = (0, 1)^2$ with $\Gamma \subset \partial \Omega$ and AMG [8] preconditioner. In both cases $K = 1$ and the problems are discretized by continuous linear Lagrange ($P_1$) elements.

Here $\| \cdot \|_{k,D}$ denotes the standard norm of Sobolev space $H^k(D)$ on domain $D$. The coefficients $K, \mu > 0$ are due to material properties, namely the permeability of the porous medium and the fluid viscosity.

By operator preconditioning, the choice of norm (1) yields a Riesz map preconditioner $b \mapsto x$ defined by solving the problem

$$- K\Delta x + \mu^{-1} (-\Delta)^{-1/2} x = b. \tag{2}$$

Note that the operator in (2) contains a bulk part $-\Delta$ and an interface part $(-\Delta)^{-1/2}$, which, from the point of view topological dimension of the underlying domains, can be viewed as a lower order perturbation.

Efficiency of the block-diagonal Darcy/Biot-Stokes preconditioners [4, 3] hinges on performant solvers for (2). However, the problem might not be amenable to standard (generic, black-box) approaches in particular in case when the fractional interface perturbation becomes dominant. We illustrate this behaviour in Figure 1 where (2) is solved by preconditioned conjugate gradient (PCG) method with algebraic multigrid (AMG) preconditioner. Indeed, the number of iterations increases with the weight of the perturbation term and, worryingly, for large enough values mesh-independence is lost.

Non-overlapping domain decomposition (DD) is a solution methodology which has been successfully applied to number of challenging problems including coupled multiphysics systems e.g. [5, 7, 12, 9]. A key component of the method are then the algorithms for the problems arising at the interface which can be broadly divided into two categories. In FETI or BDDC variants (see e.g. [1] and references therein) the solvers utilize suitable auxiliary problems on the subdomains. To develop tailored solvers for operators with fractional interface perturbation we here follow an alternative approach [1] and address the problem directly at the interface. In particular, we shall construct preconditioners for the resulting Steklov-Poincaré operators using sums of fractional order interfacial operators which include contribution due to the DD and the perturbation (which is only localized at the interface).
2 Domain decomposition solvers

We shall consider solvers for (2) in a more general setting. To this end, let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) be a bounded domain with boundary \( \partial \Omega \) and \( \Gamma \subseteq \partial \Omega \). Moreover, let \( V = V(\Omega), Q = Q(\Gamma) \) be a pair of Hilbert spaces with \( V' \), \( Q' \) being their respective duals and let \( R : V \rightarrow Q' \) be a restriction operator. For \( b \in V' \) we are then interested in solving

\[
A\mathbf{x} = b \quad \text{in} \quad V' \quad \text{with} \quad A = A_\Omega + \gamma R^T B_T^{-1} R,
\]

where \( \gamma \geq 0 \) and \( A_\Omega : V \rightarrow V' \) is some symmetric operator coercive on \( V \) while \( B_T : Q \rightarrow Q' \) is assumed to induce an inner product on \( Q \). Note that the norm operator in (2) is a special case of (3) with \( V = H^1_0(\Omega), Q = H^{1/2}(\Gamma) \), \( R \) the trace operator and \( A_\Omega = -K \Delta_\Omega \) while \( B_T = (-\Delta_T)^{1/2} \).

To formulate our non-overlapping domain decomposition approach for (3), we follow [1] and decompose \( V = V_0 \oplus V_T \) where \( V_0 = \{ v \in V ; R v = 0 \} \). Assuming that \( V_T \) can be identified with \( Q \) we observe that the operator \( A \) takes a block structure

\[
A = \begin{pmatrix}
A^{00}_\Omega & A^{01}_\Omega \\
A^{10}_\Omega & A^{11}_\Omega
\end{pmatrix} + \gamma \begin{pmatrix}
0 & 0 \\
0 & B_T^{-1}
\end{pmatrix},
\]

which we exploit to design a preconditioner for \( A \). Specifically, under the assumption that \( A^{00}_\Omega \) is invertible, let us define the DD preconditioner

\[
B = \begin{pmatrix}
I^{00}_\Omega & (A^{00}_\Omega)^{-1} A^{01}_\Omega \\
0 & I^{11}_\Omega
\end{pmatrix} \begin{pmatrix}
A^{00}_\Omega & 0 \\
0 & S_T
\end{pmatrix}^{-1} \begin{pmatrix}
I^{00}_\Omega & 0 \\
A^{10}_\Omega & A^{11}_\Omega
\end{pmatrix} = \begin{pmatrix}
I^{00}_\Omega & 0 \\
A^{10}_\Omega (A^{00}_\Omega)^{-1} & I^{11}_\Omega
\end{pmatrix}. \tag{5}
\]

Here \( I^{00}_\Omega : V_0 \rightarrow V_0 \) and \( I^{11}_\Omega : V_T \rightarrow V_T \) are identity operators on the respective subspaces while \( S_T \) is spectrally equivalent to the DD Schur complement/Steklov-Poincaré operator. We note that preconditioner (5) preserves symmetry of the original problem (3) as we target PCG solvers. However, with Krylov methods which do not require symmetry a more efficient triangular variant of the preconditioner is sufficient.

Our main contribution is an observation that for problems with interface perturbations, the Schur complement preconditioner \( S_T \) in (5) takes the form

\[
S_T = L_A^s + \gamma L_B^t,
\]

for some constants \( s, t \in \mathbb{R} \) and symmetric, positive-definite operators \( L_A, L_B \) depending on the regularity of \( A \), the restriction operator and the perturbation. In particular, the structure of the preconditioner reflects the two contributions to the Schur complement; the decomposition \( V = V_0 \oplus V_T \) applied to operator \( A_\Omega \) yields \( L_A^s \) while \( L_B^t \) is due to the perturbation.

Motivated by the initial example (2) we shall in the following focus on problems for which \( -1 < s, t < 1 \) and \( L_A, L_B \) are spectrally equivalent to \( L = -\Delta_T + I_T \).
However, we highlight that the operators might in general differ by their boundary conditions (which for \(L_A\) reflect boundary conditions on \(\partial \Omega \setminus \Gamma\)).

Assuming that \((A^{[\Omega]}_L)^{-1}\) can be efficiently computed, the main challenge for scalability of preconditioner \([5]\) is an efficient realization of (an approximate) inverse of \([6]\). Upon discretization, the operators \(L_A^x, L_B^y\) can be approximated by eigenvalue factorization\([7]\). However, this approach suffers from cubic scaling. For the specific case of \(L^{1/2}\) a more efficient strategy with improved scaling is applied in \([1]\) based on the Lanczos process while, more recently, \([10]\) prove that rational approximations (RA) lead to non-overlapping DD methods with linear scaling. Building on this observation to obtain order optimal solvers for the perturbed problem \([5]\) we follow \([6]\) where rational approximations\([8]\) were developed for Riesz maps with norms induced by sum operator \(\alpha L^x + \beta L^\Gamma\) with \(\alpha, \beta \geq 0\). In particular, this setting fits our Schur complement operator \([5]\) if constant material properties and suitable boundary conditions are prescribed on \(A\) in \([5]\).

3 Model problem

We shall illustrate performance of the domain decomposition preconditioner \([5]\) using a model interface-perturbed problem: Find \(x \in V = H^1(\Omega)\) such that

\[
K(-\Delta + I_L)x + \gamma(-\Delta_l + I_l)^t x = b \text{ in } V',
\]

where \(K > 0, \gamma \geq 0\) and \(-1 < t < 1\). Here \(\Omega = (0, 1)^d, d = 2, 3\) and \(\Gamma = \partial \Omega\). We note that this choice maximizes the size of the interface. At the same times, it enables the RA-favorable setting of \(L_A = L, L_B = L, L = -\Delta + I_\Gamma\) in \([5]\). Following

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1 For \(L : Q \to Q'\) let \(L_h\) be the matrix realization of the operator in the basis of some finite dimensional approximation space \(Q_h, n = \dim Q_h\). Moreover, let \(M_h\) be the mass matrix, i.e. matrix realization of the inner product of the Lebesgue space \(L^2\) on \(Q_h\). Assuming \(L\) is symmetric, positive definite, the factorization \(L_h U_h = M_h U_h \Lambda_h, U_h^T M_h U_h = \Id\) holds where \(\Lambda_h\) is a diagonal matrix of eigenvalues while the corresponding eigenvectors constitute the columns of matrix \(U_h\). We then define

\[
L_h^x = (M_h U_h) \Lambda_h^x (M_h U_h)^T.
\]

Note that for \(L = (-\Delta + I)\) and \(f \in Q\) represented in \(Q_h\) by interpolant with coefficient vector \(f_h \in \mathbb{R}^n\) the function \(f_h \mapsto f_h \cdot L_h^x f_h\) represents an approximation of the square of the Sobolev norm \(\|f\|_2^2\).

2 Referring to the definitions in \([7]\) the RA construct approximate solutions \(x \in Q\) satisfying \(\alpha L^x x + \beta L^\Gamma x = b, b \in Q'\) in the finite dimensional space \(Q_h\) via a solution operator

\[
c_0 M_h^{-1} + \sum_{k=1}^m c_i (L_h + p_k M_h)^{-1}.
\]

Here, \(c_i\) and \(p_i \geq 0\) are respectively the residues and the poles of the rational approximation \(f_{RA}\) to function \(f : x \mapsto (\alpha x^x + \beta x^\gamma)^{-1}\). Importantly, the number of poles \(m\) does not depend on the dimensionality of \(Q_h\) and is instead determined by the accuracy \(\varepsilon_{RA}\) of the RA, i.e. \(\|f - f_{RA}\| \leq \varepsilon_{RA}\). We refer to \([10, 6]\) and references therein for more details.
the DD Schur complement of the operator $A_\Omega = K(-\Delta_\Omega + I_\Omega)$ in $A$ is spectrally equivalent to fractional operator $KL^{1/2}$. In turn we apply preconditioner $S_\Gamma = KL^{1/2} + \gamma L'$. 

In the numerical experiments we consider $H^1$-conforming finite element spaces $V_h \subset V$ constructed in terms of $P_1$ elements. Consequently, the matrix realization of the fractional interface perturbation then reads $\gamma T_h^\Gamma L_h^\Gamma T_h$ where matrix $L_h^\Gamma$ is defined in (7) and $T_h$ is a discrete trace operator such that $T_h \phi = \sum_{j=1}^n l_j(\phi|_\Gamma) \psi_j$ for any $\phi \in V_h$ and $\psi_j, l_j, j = 1, \ldots, n$ being respectively the basis functions and degrees of freedom (point evaluations) of the discrete trace space $V_{\Gamma,h} = Q_h$ built likewise using $P_1$ elements.

The linear systems due to discretized (9) shall be solved by PCG solver using our DD preconditioner (5) which now requires inverse of the linear system due to $KL_h^{1/2} + \gamma L'$. Here we shall either apply the eigenvalue realization (7) (which allows for closed form evaluation of the exact inverse) or the approximate inverse due to RA, see (8). To put focus on the Schur complement the leading block $A_{00}$ in (5) will be computed exactly by LU factorization. For results with approximate inverse of $A_{00}$ we refer to Remark 1. Finally, the PCG solver is always started from 0 initial vector and terminates upon reducing the preconditioned residual norm by factor $10^10$.

We summarize performance of the DD preconditioner in Figure 2 and Figure 3 which consider (9) with $\Omega = (0,1)^2$ and $\Omega = (0,1)^3$ respectively. It can be seen that the PCG convergence is in general bounded in mesh size, fractionality $t$ and the perturbation strength $\Gamma$. Important for the scalability of (5) is the fact that iteration counts with RA realization of the Schur complement preconditioner practically match the exact inverse of $S_\Gamma$. We remark that the chosen tolerance of $\varepsilon_{RA} = 10^{-14}$ yields roughly $m = 20$ poles in (8). The computation setup in 3d then leads to linear systems with < 6200 unknowns at the interface.

Remark 1 (Evaluation of the operator in (9)) In numerical experiments shown in Figure 2, Figure 3 the operator $A$ in (9) utilized the eigenvalue decomposition (7) of $L_h^\Gamma$. This realization restricts the size of $\Gamma$ or $\dim Q_h$ that can be considered. However, action of the perturbation can instead be computed via RA leading to evaluation of $A$ with optimal complexity and enabling large scale problems. In Figure 4 we revisit (9) with $t = -1/2$, $\Omega = (0,1)^3$ and RA used both for the operator and the preconditioner.
\[ \log_{10} \gamma_t = -\frac{1}{2} \quad t = -\frac{1}{4} \quad t = \frac{1}{4} \quad t = \frac{1}{2} \]

Fig. 3 PCG iterations when solving (9) on \( \Omega = (0, 1)^3 \) and preconditioner (5) with \( S_F = KL^{1/2} + \gamma L' \). Problem is discretized by \( P_1 \) elements. Leading block of preconditioner is computed exactly. Results with realization of the Schur complement preconditioner by RA with tolerance \( \epsilon_{\text{RA}} = 10^{-14} \) are depicted by (○) markers while (×) markers correspond to definition via the eigenvalue problem (7).

Moreover, the leading block of the preconditioner is approximated by a single V-cycle of AMG [8]. The number of PCG iterations then appears to be bounded in the mesh size and the parameter \( \gamma \). As before, \( P_1 \) elements were used for discretization.

Fig. 4 PCG iterations for computing the inverse of fractional perturbed operators by preconditioner (5). (Left) The operator is (9) with \( t = -1/2 \) and \( \Omega = (0, 1)^3 \). Both the operator and preconditioner are evaluated by RA. On the finest refinement level \( \dim V_{\Gamma,h} = 24 \cdot 10^3 \). (Right) Operator (10) is considered with \( S_F = KL^{-1/2} + \gamma I \) in the Schur complement (6).

Remark 2 (Application to \( H(\text{div}) \)-elliptic problem) Preconditioners (5) are not limited to \( H^1 \)-elliptic problems. To illustrate this fact we consider \( V = H(\text{div}, \Omega), \Omega = (0, 1)^2 \) and a variational problem induced by bilinear form due to operator \( A_hA_u - v = \int_{\Omega} K(u \cdot v + \nabla \cdot u \nabla \cdot v) + \gamma \int_{\partial \Omega} u \cdot \nabla v \cdot v \quad \forall u, v \in V \). (10)

We observe that \( A \) falls under the template problem (5). In order to apply the domain-decomposition preconditioner we then require a preconditioner for the DD Schur complement due to \( (A_{\Omega})^{-1} \) where \( A_{\Omega} \) is here the operator \( K(I - \nabla \nabla \cdot) \) on \( H_0(\text{div}, \Omega) \). Motivated by [2], we shall to this end consider the operator \( L_A = KL^{-1/2} \) such that \( S_F \) in (5) is defined as \( S_F = KL^{-1/2} + \gamma I \). For numerical experiments the system is discretized by lowest order Brezzi-Douglas-Marini elements which lead to the discrete trace space \( V_{\Gamma,h} = Q_h \) of discontinuous piecewise-linear functions on trace mesh \( \Gamma_h \). Robustness of the resulting preconditioner is shown in Figure 4.
4 Darcy-Stokes preconditioning

We finally apply the proposed non-overlapping DD solvers to realize preconditioners for the coupled Darcy-Stokes problem with Darcy problem in the primal form \(^7\). That is, assuming bounded domains \(\Omega_S, \Omega_D \subset \mathbb{R}^d, d = 2, 3\) sharing a common interface \(\Gamma\) (cf. Figure 5) we seek Stokes velocity and pressure \(u_S, p_S\) and Darcy pressure \(p_D\) such that

\[
\begin{align*}
-\nabla \cdot \sigma(u_S, p_S) &= f_S \quad \text{and} \quad \nabla \cdot u_S = 0 \quad \text{in} \ \Omega_S, \\
-\nabla \cdot K \nabla p_D &= f_D \quad \text{in} \ \Omega_D, \\
u_S \cdot v + K \nabla p_D \cdot v &= 0 \quad \text{on} \ \Gamma, \\
-v \cdot \sigma(u_S, p_S) \cdot v - p_D &= 0 \quad \text{on} \ \Gamma, \\
-P_y (\sigma(u_S, p_S) \cdot v) - \alpha \mu K^{-1/2} P_y u_S &= 0 \quad \text{on} \ \Gamma,
\end{align*}
\]

(11)

where \(P_y\) is the tangential trace operator \(P_y u = u - (u \cdot v)v\) and \(\sigma(u, p) = \mu \Delta u - p I_d\). In addition to the previously introduced coefficients \(K, \mu > 0\) the model also includes the Beavers-Joseph-Saffman parameter \(\alpha \geq 0\). We close the system by prescribing suitable boundary conditions to be discussed shortly.

We consider (11) with a parameter-robust block diagonal preconditioner \(^4\)

\[
\mathcal{B} = \text{diag} \left( -\mu \Delta + \alpha \mu K^{-1/2} P_y P_y, \mu^{-1} I, -K \Delta + \mu^{-1} (-\Delta)_{1/2}^{-1} \right)^{-1}.
\]

(12)

Observe that both the first and the final block in (12) are of the form of the interface-perturbed operators \(^3\). However, for simplicity we shall here set \(\alpha = 0\) and only focus on the pressure preconditioner. In particular, to efficiently approximate \(^2\) we shall perform few PCG iterations with the DD preconditioner (5) using \(S_I = KL^{1/2} + \mu^{-1} L^{-1/2}\). We note that the interface operator is thus identical to the one utilized in robust preconditioning of mixed Darcy-Stokes model \(^11\).

To illustrate performance of the preconditioner (12) we consider (11) in a 3d domain pictured in Figure 5 and set \(K = 10^{-2}, \mu = 10^{-4}\). Using discretization by \(P_2-P_1-P_2\) elements the linear system is solved by preconditioned Flexible GMRes (FGMRs) starting from 0 initial guess and relative tolerance \(10^{-10}\) on the preconditioned residual norm. The Darcy-Stokes preconditioner is then realized by applying single AMG V-cycle for the Stokes blocks while the Riesz map of the Darcy pressure \(^2\) is approximated by PCG solver using (5) and running with a relative tolerance of \(10^{-4}\). The DD preconditioner uses RA with tolerance \(\varepsilon_{RA} = 10^{-15}\) and AMG for the leading block of (5). With this setup the scalability study summarized in Figure 5 reveals that the proposed solver is order optimal. We note that the computations are run in serial.

\(^3\) Due to computational demands we did not perform parameter-robustness study for \(d = 3\). In the subsequent Appendix we demonstrate robustness using a 2d version of the geometry in Figure 5.
Fig. 5 (Left) Computation domain is obtained by extrusion of the pictured geometry. The interface \( \Gamma \), being part of a circle arc, is curved. No-slip and traction conditions are prescribed on \( \Gamma^w_S \) and \( \Gamma^o_S \) respectively. Darcy pressure is prescribed on \( \Gamma^p_D \). (Center) Error convergence study performed using the 3d setup. With \( P_2-P_1-P_2 \) elements, optimal quadratic rates are observed in all the variables. (Right) The solver time (including preconditioner setup and FGMRes runtime) scales linearly with the problem size.

Appendix

We verify parameter-robustness of Darcy-Stokes preconditioner (12) which utilizes (5)-preconditioned CG solver for the Darcy pressure block using the (2d) geometry and boundary condition setup depicted in Figure 5. Here, a relative tolerance of \( 10^{-5} \) is prescribed in PCG, while (i) either the RA in (5) is applied with \( \epsilon_{RA} = 10^{-12} \) and AMG for the leading block or (ii) the respective problems are solved with (7) and LU. For simplicity and to put focus on the presented DD approach the Stokes velocity and pressure blocks of (12) are computed exactly with LU. FGMRes settings are identical to Section 4.

In Figure 6 we observe that the number of FGMRes iterations is bounded in \( \mu, K \) and mesh size. Moreover, using RA for approximating the inverse of \( S_\Gamma = KL^{1/2} + \mu^{-1}L^{-1/2} \) leads to practically the same convergence as when the (exact) spectral realization is used. In Figure 7 we finally consider behaviour of PCG iterations in applications of the the Darcy-Stokes preconditioner. When (5) is computed using LU for \( (A_\Omega^{10})^{-1} \) and (7) in inverting \( S_\Gamma \), convergence of the Krylov solver is stable with respect to material and discretization parameters. In case (i) a slight increase of iterations for large values of \( \mu \) can be observed. We recall that for \( (A_\Omega^{10})^{-1} \) a single AMG V-cycle is applied and \( \epsilon_{RA} = 10^{-12} \) in approximating the Schur component inverse. This observation suggests that sufficient accuracy is needed when approximating the blocks in (5) in order to retain mesh-independent convergence. This effect will be investigated in future work.

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Domain decomposition solvers for operators with fractional interface perturbations

Fig. 6 FGMRes iterations for Darcy-Stokes problem (11) on 2d geometry in Figure 5. Beavers-Joseph-Saffman parameter is set to $\alpha = 3$. Discretization by $P_2-P_1-P_2$ elements. Darcy-Stokes preconditioner (12) uses LU for Stokes velocity and pressure blocks. (Top) Two different realizations of (2) are compared. Results depicted in ($\times$) markers are obtained with $\times$ computed by LU. With ($\bigcirc$) markers the Riesz map (2) is approximated by PCG using relative tolerance $10^{-5}$ and DD preconditioner (5). Within (5) ($A_0^{00} - 1$ is computed by LU and the Schur complement uses (7). (Bottom) DD preconditioner (5) within PCG solver for (2) using LU and (7) for the blocks ($\times$) is compared with a scalable version utilizing AMG, respectively RA ($\varepsilon_{RA} = 10^{-12}$) to approximate the blocks of (5).

Fig. 7 PCG convergence inside application of Darcy-Stokes preconditioner in Figure 6 (Top) DD preconditioner uses LU to realize $(A_0^{00})^{-1}$ and (7) for $S_1^{-1}$. (Bottom) The blocks of (5) are approximated by AMG V-cycle and RA ($\varepsilon_{RA} = 10^{-12}$) respectively. In both plots, values in ($\bigcirc$) markers represent the maximum number of PCG iterations needed for convergence when applying the outer preconditioner (12) while ($\times$) markers represent the (integer-rounded) mean iteration count.
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