ERMAKOV APPROACH FOR $Q = 0$ EMPTY FRW MINISUPERSPACE OSCILLATORS

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Summary. The Wheeler-DeWitt equation for empty FRW minisuperspace universes of Hartle-Hawking factor ordering parameter $Q = 0$ is mapped onto the dynamics of a unit mass classical oscillator. The latter is studied by the classical Ermakov invariant method. Angle quantities are presented in the same context.

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The formalism of Ermakov-type invariants \cite{1} can be a useful, alternative method of investigating evolutionary and chaotic dynamical problems in the “quantum” cosmological framework \cite{2}. Moreover, the Ermakov method is intimately related to geometrical angles and phases \cite{3}, so that one may think of cosmological Hannay’s angles as well as various types of topological phases as those of Berry and Pancharatnam \cite{4}.

Our purpose in the following is to apply the formal Ermakov scheme to the simplest cosmological oscillators, namely the empty Friedmann-Robertson-Walker (EFRW) “quantum” universes. When the Hartle-Hawking parameter for the factor ordering ambiguity is zero, $Q = 0$ \cite{5}, the EFRW Wheeler-DeWitt (WDW) minisuperspace equation reads \cite{6}

\begin{equation}
\frac{d^2 \Psi}{d\Omega^2} - \kappa e^{-4\Omega} \Psi(\Omega) = 0 ,
\end{equation}

where $\Omega$ is Misner’s time related to the volume of space $V$ at a given cosmological epoch through $\Omega = -\ln(V^{1/3})$ \cite{7}, $\kappa$ is the curvature index of the universe (1,0,-1 for closed, flat, open, respectively), and $\Psi$ is the wavefunction of the universe. The general solution is obtained as a superposition of modified Bessel functions of zero order, $\Psi(\Omega) = C_1 I_0\left(\frac{1}{2} e^{-2\Omega}\right) + C_2 K_0\left(\frac{1}{2} e^{-2\Omega}\right)$ in the $\kappa = 1$ case, and ordinary Bessel functions $\Psi(\Omega) = C_1 J_0\left(\frac{1}{2} e^{-2\Omega}\right) + C_2 Y_0\left(\frac{1}{2} e^{-2\Omega}\right)$ in the $\kappa = -1$ case, where $C_1$ and $C_2$ are two arbitrary superposition constants (we shall work with $C_1 = C_2 = 1$). Eq. (1) can be mapped...
onto the canonical equations for a classical point particle of mass $M = 1$, generalized coordinate $q = \Psi$, momentum $p = \Psi'$, evolving in Misner’s time considered as Newtonian time for which we shall keep the same notation. Thus, one is led to

$$\frac{dq}{d\Omega} = p \tag{2}$$

$$\frac{dp}{d\Omega} = \kappa e^{-4\Omega} q \, . \tag{3}$$

These equations describe the canonical motion for a classical point EFRW universe as derived from the time-dependent oscillator Hamiltonian of the inverted ($\kappa = 1$) and normal ($\kappa = -1$) type, respectively, \[8\]

$$H(\Omega) = \frac{p^2}{2} - \kappa e^{-4\Omega} q^2 \, . \tag{4}$$

For this EFRW Hamiltonian the triplet of phase-space functions $T_1 = \frac{p^2}{2}, T_2 = pq$, and $T_3 = \frac{q^2}{2}$ forms a dynamical Lie algebra (i.e., $H = \sum_n h_n(\Omega)T_n(p,q)$) which is closed with respect to the Poisson bracket, or more exactly $\{T_1, T_2\} = -2T_1, \{T_2, T_3\} = -2T_3, \{T_1, T_3\} = -T_2$. The EFRW Hamiltonian can be written down in the form

$$H = T_1 - e^{4\Omega} T_3 \, . \tag{5}$$

The Ermakov invariant $I$ belongs to the dynamical algebra

$$I = \sum_r \epsilon_r(\Omega)T_r \, , \tag{6}$$

and by means of the invariance condition

$$\frac{\partial I}{\partial \Omega} = -\{I, H\} \, , \tag{7}$$

one is led to the following equations for the unknown functions $\epsilon_r(\Omega)$

$$\dot{\epsilon}_r + \sum_n \left[ \sum_m C_{nm}^r h_m(\Omega) \right] \epsilon_n = 0 \, , \tag{8}$$

where $C_{nm}^r$ are the structure constants of the Lie algebra, that have been already given above. Thus, we get

$$\dot{\epsilon}_1 = -2\epsilon_2$$

$$\dot{\epsilon}_2 = -\kappa e^{-4\Omega} \epsilon_1 - \epsilon_3$$

$$\dot{\epsilon}_3 = -2\kappa e^{-4\Omega} \epsilon_2 \, . \tag{9}$$

The solution of this system can be readily obtained by setting $\epsilon_1 = \rho^2$ giving $\epsilon_2 = -\rho \dot{\rho}$ and $\epsilon_3 = \rho^2 + \frac{1}{\rho^2}$, where $\rho$ is the solution of the Milne-Pinney equation \[9\]

$$\ddot{\rho} - \kappa e^{-4\Omega} \rho = \frac{1}{\rho^3} \, . \tag{10}$$

In terms of the function $\rho(\Omega)$ and using (6), the Ermakov invariant can be written as follows \[11\]

$$I = I_{\text{kin}} + I_{\text{pot}} = \frac{(\rho p - \dot{\rho}q)^2}{2} + \frac{q^2}{2\rho^2} = \rho^4 \left[ d\Omega \left( \frac{\Psi}{\rho} \right) \right]^2 + \frac{1}{2} \left( \frac{\Psi}{\rho} \right)^2 \, . \tag{11}$$
We have followed Pinney [9] and Eliezer and Gray [10] to calculate $\rho(\Omega)$ in terms of linear combinations of Bessel functions such that the initial conditions given by these authors were fulfilled. We worked with the set $A = 1$, $B = -1/W^2$ and $C = 0$ of Pinney’s constants, where $W$ is the Wronskian of the pair of Bessel functions. Moreover, we have chosen the angular momentum of the auxiliary Eliezer-Gray two-dimensional motion as unity ($\hbar = 1$).

Since $I = \hbar^2/2$ we must get a constant half-unity value for the Ermakov invariant. We have checked this by plotting $I(\Omega)$ for $\kappa = \pm 1$ in Fig. 1.

In order to get angle variables, we calculate the time-dependent generating function allowing one to pass to new canonical variables for which $I$ is chosen as the new “momentum” [11]

\[
S(q, I, \vec{c}(\Omega)) = \int q \, dq' p(q', I, \vec{c}(\Omega)) ,
\]
leading to

\[
S(q, I, \vec{c}(\Omega)) = \frac{q^2 \dot{\rho}}{2 \rho} + I \arcsin \left[ \frac{q}{\sqrt{2I\rho^2}} \right] + \frac{q\sqrt{2I\rho^2} - q^2}{2\rho^2} ,
\]
where we have put to zero the constant of integration. Thus,

\[
\theta = \frac{\partial S}{\partial I} = \arcsin \left( \frac{q}{\sqrt{2I\rho^2}} \right) .
\]

Moreover, the canonical variables are now

\[
q = \rho \sqrt{2I} \sin \theta , \quad p = \frac{\sqrt{2I}}{\rho} \left( \cos \theta + \dot{\rho} \rho \sin \theta \right) .
\]

The dynamical angle will be

\[
\Delta \theta^d = \int_{\Omega_0}^{\Omega} \left( \frac{\partial H_{\text{new}}}{\partial I} \right) d\Omega' = \int_{\Omega_0}^{\Omega} \left[ \frac{1}{\rho^2} - \frac{\rho^2}{2} \frac{d}{d\Omega} \left( \frac{\dot{\rho}}{\rho} \right) \right] d\Omega' ,
\]
whereas the geometrical (generalized Hannay) angle reads

\[
\Delta \theta^g = \frac{1}{2} \int_{\Omega_0}^{\Omega} \left[ \frac{d}{d\Omega'} \left( \dot{\rho} \rho \right) - 2\dot{\rho}^2 \right] d\Omega' .
\]

The sum of $\Delta \theta^d$ and $\Delta \theta^g$ is the total change of angle (Lewis angle)

\[
\Delta \theta^t = \int_{\Omega_0}^{\Omega} \frac{1}{\rho^2} d\Omega' .
\]

Plots of the angle quantities (16-18) for $\kappa = 1$ are presented in Figs. 2, 3, and 4, respectively. Similar plots have been obtained for the $\kappa = -1$ case.

In conclusion, a cosmological application of the classical Ermakov procedure has been presented here based on a classical point particle representation of the EFRW WDW equation. Finally, we notice that the Ermakov invariant is equivalent to the Courant-Snyder one in accelerator physics [12] allowing a beam physics analogy to cosmological evolution.

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Fig. 1: The Ermakov invariant as a function of $\Omega$ for the closed EFRW minisuperspace model. We got this plot for the open case as well.
Fig. 2: The dynamical angle for the closed EFRW model.
Fig. 3: The geometrical angle as a function of $\Omega$ for the same model.
Fig. 4: The total angle as a function of $\Omega$ for the same model.