Determinantal ideals and monomial curves in the three-dimensional space

Margherita Barile
Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy

Abstract We show that the defining ideal of every monomial curve in the affine or projective three-dimensional space can be set-theoretically defined by three binomial equations, two of which set-theoretically define a determinantal ideal generated by the 2-minors of a $2 \times 3$ matrix with monomial entries.

Keywords: Determinantal ideals, monomial curves, binomial ideals, complete intersections.

Introduction

Let $K$ be an algebraically closed field, and let $R$ be a polynomial ring in $n$ indeterminates over $K$. If $I$ is a proper ideal of $R$, then $I$ is called a set-theoretic complete intersection if the corresponding variety $V(I)$ in the affine space $K^n$ can be defined by a system containing the least possible number $s$ of equations: in this case $s = \text{height}(I)$. More generally, if we can take $s \leq \text{height}(I) + 1$, then $I$ is called an almost set-theoretic complete intersection. We recall that an ideal is a(n ideal-theoretic) complete intersection if it is generated by height $(I)$ elements.

The set-theoretic complete intersection property has been intensively studied for ideals generated by binomials, such as the determinantal ideals (see, e.g., [1], [7], and [9]), and the ideals defining toric varieties (see, e.g., [3]), which include the special class of monomial curves (see, e.g., [4], [11], [12], [13], and [15]). Monomial curves in $K^3$ are known to always be set-theoretic complete intersections (see [4] and [11]), and the same is true for all monomial curves in $\mathbb{P}^3$ (see [12]) if char $K > 0$: the extension to the characteristic zero case is a longstanding open problem, besides some special cases (see, e.g., [14]). What makes the problem so hard is the fact that, in characteristic zero, a monomial curve (and, more generally, a toric variety) is not a set-theoretic complete intersection on “simple” (i.e., on binomial) equations, except in the trivial case where it is a complete intersection (see [3], Theorem 4). Even for determinantal varieties, the minimum number of defining equations is in general provided by non-binomial ones (see [4], and also [7], Theorem 2 together with [9], Section 5.E or with [6], Section 2): this is also true for monomial curves in $K^3$ and for arithmetically

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Cohen-Macaulay curves in $\mathbf{P}^3$, which are defined by the vanishing of the 2-minors of a $2 \times 3$ matrix (see [4] or [11], and [5] or [14]).

In this paper we first describe a class of ideals, generated by the maximal minors of a two-row matrix, whose variety is defined by the vanishing of a proper subset of the generating minors; then we show that the defining ideal of any monomial curve in $K^3$ or $\mathbf{P}^3$ contains the ideal of 2-minors of a $2 \times 3$ matrix that is a set-theoretic complete intersection on two of these minors. Moreover, these two minors and an additional binomial define the curve set-theoretically. This is an attempt to give a unifying feature of affine and projective monomial curves, and could help us to shed some light on the conjecture according to which every projective curve in $\mathbf{P}^3$ is a set-theoretic complete intersection. This also provides a new proof to the following well-known result: the defining ideal of every monomial curve in the three-dimensional space contains a complete intersection ideal of height 2 generated by binomials. The problem of finding complete intersection ideals in toric ideals has been recently treated in [8].

1 A result on determinantal ideals

Let $A = (\alpha_{ij})$ be a $2 \times r$ matrix with entries in $R$, where $r \geq 3$. Moreover, let $J$ be the ideal of $R$ generated by the 2-minors of $A$. For all distinct indices $i, j$ let $\Delta_{ij}$ denote the minor of $A$ formed by the $i$-th and the $j$-th column. Finally, for all $k = 1, \ldots, r$, let $J_k$ be the ideal of $R$ generated by the set $\{\Delta_{ik} | 1 \leq i \leq r, i \neq k\}$; in other words, $J_k$ is generated by all 2-minors of $A$ which involve the $k$-th column.

Proposition 1 $\sqrt{J} = \sqrt{J_1}$ if and only if $J \subset \sqrt{(\alpha_{11}, \alpha_{21})}$.

Proof. Since $J_k \subset (\alpha_{1k}, \alpha_{2k})$, the only if part is trivial. We prove the if part.

For the sake of simplicity, and without loss of generality, we assume that $k = 1$.

Thus our assumption is

$$J \subset \sqrt{(\alpha_{11}, \alpha_{21})}, \quad (1)$$

and from this we want to deduce that $\sqrt{J} = \sqrt{J_1}$, or, equivalently, that $V(J) = V(J_1)$. Since $J_1 \subset J$, it suffices to show that $V(J) \supset V(J_1)$. Let $v \in K^n$ be a point where all elements of $J_1$ vanish, i.e., such that

$$\alpha_{11}(v)\alpha_{2j}(v) - \alpha_{1j}(v)\alpha_{21}(v) = \Delta_{1j}(v) = 0 \quad \text{for all } j = 2, \ldots, r. \quad (2)$$

Let

$$B = \begin{pmatrix} A \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \end{pmatrix}.$$ 

Since the first and the last row of $B$ are equal, all 3-minors of $B$ vanish. In particular, for all indices $i, j$ such that $1 < i < j \leq r$ we have

$$0 = \begin{vmatrix} \alpha_{11} & \alpha_{1i} & \alpha_{1j} \\ \alpha_{21} & \alpha_{2i} & \alpha_{2j} \\ \alpha_{11} & \alpha_{1i} & \alpha_{1j} \end{vmatrix} = \alpha_{11}\Delta_{ij} - \alpha_{1i}\Delta_{1j} + \alpha_{1j}\Delta_{1i},$$

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whence
\[ \alpha_{11} \Delta_{ij} \in J_1, \]
and, consequently, due to the choice of \( v \),
\[ \alpha_{11}(v) \Delta_{ij}(v) = 0. \]
We have to show that \( \Delta_{ij}(v) = 0 \) for all indices \( i, j \). This is certainly true if \( \alpha_{11}(v) \neq 0 \). So assume that \( \alpha_{11}(v) = 0 \). Then for all \( j = 2, \ldots, r \), (2) implies that \( \alpha_{1j}(v)\alpha_{21}(v) = 0 \). If \( \alpha_{21}(v) = 0 \), then (1) implies that
\[ \Delta_{ij}(v) = 0 \quad \text{for all } i, j. \] (3)
Otherwise \( \alpha_{1j}(v) = 0 \) for all \( j = 2, \ldots, r \), so that the whole first row of \( A \) vanishes at \( v \), and (3) holds in this case, too. This proves our claim.

2 Matrices with monomial entries

In this section we shall apply Proposition 1 to the case where the entries of \( A \) are monomials, i.e., products of nonnegative powers of indeterminates. A difference of two distinct monomials will be called a binomial. We shall say that a matrix with monomial entries is simple if none of its 2-minors has a nonconstant monomial factor.

Example 1 In the polynomial ring \( R = K[a, b, c, d] \) consider the matrix
\[
A = \begin{pmatrix}
  a^m d^n & b^p & c^q \\
  b^r & a^s & d^t
\end{pmatrix},
\]
where \( m, n, p, q, r, s, t \) are nonnegative integers. The ideal generated by the 2-minors of \( A \) is
\[
J = (a^{m+s} d^n - b^r c^q, a^{n+t} d^s - b^r c^q, b^p d^t - a^s c^q),
\]
so that
\[
J \subset (a, b) \subset \sqrt{(a^s b^r d^t - a^s c^q}).
\]
By Proposition 1 it follows that
\[
\sqrt{J} = \sqrt{J_2} = \sqrt{(a^{m+s} d^n - b^r c^q, b^p d^t - a^s c^q)}.
\]

Remark 1 Let \( J \) be the ideal of 2-minors of a \( 2 \times 3 \) matrix. Then \( J \) is generated by three elements of \( R \), and, according to Eagon and Northcott ([10], Theorem 3), height \( J \leq 2 \). Hence, if equality holds, \( J \) is an almost set-theoretic complete intersection. In general, it is not a set-theoretic complete intersection; for instance, it is not if the entries of the matrix are pairwise distinct indeterminates. In this case height \( J \) = 2, but \( J \) cannot be generated, up to radical, by less than three polynomials (see [7], Theorem 2). This shows the interest of Proposition 1 which describes a class of \( 2 \times 3 \) matrices with entries in \( R \) such that the ideal generated by its 2-minors is generated by two polynomials up to radical. A different class was characterized by Robbiano and Valla, who showed the following result.
Proposition 2 ([11], Theorem 2.2). Let $J$ be the ideal of $R$ generated by the 2-minors of the matrix

$$A = \begin{pmatrix} x & y^m & z \\ y^n & s & t \end{pmatrix},$$

where $m, n$ are nonnegative integers. Then there are $f, g \in R$ such that $\sqrt{J} = \sqrt{(f, g)}$.

Propositions 1 and 2 can be summed up as follows.

Corollary 1 Let $A$ be a simple $2 \times 3$ matrix whose entries are monomials of $R = K[a, b, c, d]$. Let $J$ be the ideal generated by its 2-minors. Then there are $f, g \in R$ such that $\sqrt{J} = \sqrt{(f, g)}$.

Proof .-It suffices to show that $A$ fulfills the assumptions of Proposition 1 or Proposition 2. The assumption of Proposition 1 is certainly true if one of the entries of $A$ is equal to 1. So assume that this is not the case. Then note that, according to the simplicity condition, each entry is coprime with respect to the entries lying in the same row. Hence, up to permuting columns and renaming the indeterminates, the first row of matrix $A$ is of the following form

$$\begin{pmatrix} a^m & b^n & c^p d^q \\ b^r & c^s d^t \end{pmatrix},$$

where $m, n$ are positive integers and $p, q$ are nonnegative integers, not both zero. The entries of the second row are, of course, of the same form. Moreover, the simplicity condition implies that every entry is coprime with respect to the entries lying in the same column. Hence the possible forms of $A$ are in the list below (up to interchanging $c$ and $d$). We assume that each monomial contains at least one positive exponent.

(i) $$\begin{pmatrix} a^m & b^n & c^p d^q \\ b^r & c^s d^t & a^u \end{pmatrix}$$

(ii) $$\begin{pmatrix} a^m & b^n & c^p \\ b^r & c^s d^t & a^u \end{pmatrix}$$

(iii) $$\begin{pmatrix} a^m & b^n & c^p \\ b^r & c^s & a^t d^u \end{pmatrix}$$

(iv) $$\begin{pmatrix} a^m & b^n & c^p d^q \\ b^r c^s & d^t & a^u \end{pmatrix}$$

(v) $$\begin{pmatrix} a^m & b^n & c^p \\ b^r c^s & a^t & d^u \end{pmatrix}$$
Proposition 2 can be applied to (i), (ii), (iii), (iv), (v), (vi), (viii) and (ix). It can be applied in case (vii) if one of the exponents $p, q, t, u$ is zero. Otherwise we have $J \subset (a, c)$ (which is true if $t, p > 0$) and $J \subset (b, d)$ (which is true if $q, u > 0$), so that Proposition 1 applies. This completes the proof.

Remark 2 Note that (ii), (iv), (v) and (vi) may also fulfill the assumption of Proposition 1. This is true in case (ii) if $s > 0$, since then $J \subset (a, b)$; in case (iv) if $r = 0$ and $p > 0$, or $q = 0$ and $s > 0$, since then $J \subset (a, c)$; in case (v) if $r > 0$, since then $J \subset (a, b)$; in case (vi) if $s = 0$ and $p > 0$ and $t > 0$, since then $J \subset (b, d)$. In these special cases, if we are asked to explicitly exhibit the polynomials $f, g$ generating $J$ up to radical, Proposition 1 is in fact more convenient than Proposition 2: the former yields two binomials, the latter a binomial and polynomial of a more complicated form. We compare the two approaches in the next examples.

Example 2 Let

$$A = \begin{pmatrix} a^m & b^n & c^p \\ b^r & c^s & a^u \end{pmatrix},$$

with positive exponents, which is a matrix of type (i) for $t = q = 0$. Then $J = (a^m c^s - b^{n+r}, a^{m+u} - b^r c^p, a^u b^n - c^{p+s})$. Two polynomials $f, g \in K[a, b, c]$ generating $J$ up to radical are given by Valla through an explicit formula in [10], Section 3:

$$f = a^m c^s - b^{n+r},$$

$$g = \sum_{k=0}^{n+r} (-1)^{n+r-k} \binom{n+r}{k} a^{k u + \tau_k} b^{\sigma_k} c^{(n+r-k)(p+s)+\tau_k s - ns},$$

where $\tau_k$ and $\sigma_k$ are the quotient and the remainder of the Euclidean division of $kn$ by $n + r$. Here Proposition 1 cannot be applied. It only works in the
“degenerate” case, when one of the exponents is set equal to zero. If, e.g., \( n = 0 \), then \( f = a^nc^s - b^r \) and we can take \( g = a^u - c^p + s \). In this special case one can easily check that we even have \( J = (f, g) \).

**Example 3** Let

\[
A = \begin{pmatrix}
a^{2u-1} & b^n & c^p \\
b^n & d^q & a^u
\end{pmatrix},
\]

so that \( J = (a^{2u-1}d^n - b^{n+r}c^s, a^u b^n - c^p d^q, a^{3u-1} - b^r c^p + s) \). Suppose that the exponents are all positive. Applying the general constructive method described in [14] requires more than one step for all \( u > 1 \), and one of the equations we obtain is non-binomial. Proposition [4] immediately gives \( \sqrt{J} = \sqrt{(a^u b^n - c^p d^q, a^{3u-1} - b^r c^p + s)} \).

3 Complete intersections and monomial curves

There is a well-known connection between the defining ideals of affine and projective monomial curves and determinantal ideals of simple matrices. Consider the affine curve of \( K^3 \) parametrized by

\[
x_1 = \xi^\alpha, \quad x_2 = \xi^\beta, \quad x_3 = \xi^\gamma.
\]

Its defining ideal in \( K[x_1, x_2, x_3] \) is the ideal of 2-minors of a matrix

\[
A = \begin{pmatrix}
x_1^m & x_2^n & x_3^p \\
x_2^m & x_3^n & x_1^p
\end{pmatrix},
\]

which has the form given in Example [2]. By Proposition [2] it is a set-theoretic complete intersection; see also [4] and [11]. This property does not extend to the projective case. Consider the projective curve of \( P^3 \) parametrized by

\[
C : x_0 = \xi^\delta, \quad x_1 = \xi^{\epsilon_1} \omega^{\delta - \epsilon_1}, \quad x_2 = \xi^{\epsilon_2} \omega^{\delta - \epsilon_2}, \quad x_3 = \omega^\delta,
\]

where \( \delta, \epsilon_1, \epsilon_2 \) are positive integers. The defining ideal \( I = I(C) \) of \( C \) in \( R = K[x_0, x_1, x_2, x_3] \) is not necessarily the determinantal ideal of a \( 2 \times 3 \) matrix. In fact it is generated by three elements if and only if it is arithmetically Cohen-Macaulay, in which case it is generated by the 2-minors of a matrix

\[
A = \begin{pmatrix}
x_1^m & x_2^n & x_3^p & x_0^q \\
x_2^m & x_3^n & x_1^p & x_0^q
\end{pmatrix},
\]

which is of type (i). This result was found by Bresinsky, Schenzel and Vogel in [3], see also [13], Theorem 2.1. According to Proposition [2] it follows that \( I \) is a set-theoretic complete intersection. It was proven by Moh [12] that the set-theoretic complete intersection property extends to all projective monomial curves \( C \) if the characteristic of \( K \) is positive, whereas the question is open in characteristic zero. Our aim is to show a general property that links all ideals \( I \) to determinantal ideals which are set-theoretic complete intersections.
A binomial will be called monic if one of its two monomials is a power of an indeterminate. Before stating the main result, we need to introduce some notation, which is referred to the projective curve $C$ defined by (4). Set $\phi_i = \delta - \epsilon_i$ for $i = 1,2$. Up to exchanging the parameters $\xi$ and $\omega$ we may assume that \( \epsilon_1 \geq \epsilon_2 \) (or, equivalently, $\phi_2 \geq \phi_1$). Then let

\[
\begin{align*}
\vartheta &= \gcd(\delta, \epsilon_1) = \gcd(\delta, \phi_1) \\
\varepsilon &= \gcd(\epsilon_1, \epsilon_2) \\
\varphi &= \gcd(\phi_1, \phi_2).
\end{align*}
\]

Further set

\[
\begin{align*}
\delta^* &= \frac{\delta}{\vartheta}; \quad \epsilon_1^* = \frac{\epsilon_1}{\vartheta}; \quad \phi_1^* = \frac{\phi_1}{\vartheta}; \\
m &= \frac{\phi_2}{\varphi}; \quad p = \frac{\phi_1}{\varphi}; \quad n = m - p; \\
u &= \frac{\epsilon_1}{\varepsilon}; \quad v = \frac{\epsilon_2}{\varepsilon}; \quad w = u - v.
\end{align*}
\]

Note that $n$ and $w$ are nonnegative integers. Consider the following three binomials

\[
\begin{align*}
f &= x_1^{\delta^*} - x_3^{\epsilon_1^*} x_3^{\phi_1^*}; \quad (5) \\
f_1 &= x_1^{m} - x_0^n x_2^p; \quad (6) \\
f_2 &= x_2^{u} - x_1^v x_3^w. \quad (7)
\end{align*}
\]

Since

\[
\begin{align*}
\epsilon_1 \delta^* &= \epsilon_1^* \delta^* \quad \text{(equivalently: } \phi_1 \delta^* = \phi_1^* \delta) ; \\
\epsilon_1 m &= \delta n + \epsilon_2 p \quad \text{(equivalently: } \phi_1 m = \phi_2 p) ; \\
\epsilon_2 u &= \epsilon_1 v \quad \text{(equivalently: } \phi_2 u = \phi_1 v + \delta w),
\end{align*}
\]

we have that $f, f_1, f_2 \in I(C)$.

**Proposition 3** The monomial curve $C$ in $\mathbb{P}^3$ is set-theoretically defined by the binomials $f, f_1, f_2$.

**Proof.** It suffices to prove that $V(f, f_1, f_2) \subset C$. Let $x = (x_0, x_1, x_2, x_3) \in \mathbb{P}^3$ be a common zero of $f, f_1, f_2$. Set $x_0 = \xi$ and $x_3 = \omega$. We will show that, after a suitable change of parameters, $x$ fulfils parametrization (4). This is true if $x_0 = 0$ or $x_3 = 0$. If $x_0 = 0$, then $f(x) = 0$ implies that $x_1 = 0$, and then $f_2(x) = 0$ implies that $x_2 = 0$; we thus can take $\xi = 0$ and set $\omega$ equal to any $\delta$-th root of $x_3$. Similarly, if $x_3 = 0$, we conclude that $x_1 = x_2 = 0$, so that we can take $\omega = 0$ and set $\xi$ equal to any $\delta$-th root of $x_0$. So assume that $x_0 \neq 0$ or $x_3 \neq 0$. Since $f(x) = 0$, we have that

\[
x_1^{\delta^*} = x_0^{\epsilon_1^*} x_3^{\phi_1^*} = \xi^{\epsilon_1^*} \omega^{\phi_1^*}. \quad (11)
\]
Let $\bar{\xi}, \bar{\omega} \in K$ be such that $\bar{\xi}^\delta = \xi$, and $\bar{\omega}^\delta = \omega$, i.e.,

$$x_0 = \bar{\xi}^\delta, \quad \text{and} \quad x_3 = \bar{\omega}^\delta. \tag{12}$$

Then, by (11) and (8),

$$x^\delta_1 = \bar{\xi}^{\delta_1^\delta} = \bar{\xi}^{\delta_1^\delta},$$

whence

$$x_1 = \bar{\xi}^{\epsilon_1^{\delta_1^\delta}} r,$$  \tag{13}

for some $r \in K$ such that $r^\delta = 1$. \tag{14}

From $f_2(x) = 0$ it follows that

$$x^u_2 = x^v_1 x^w_3,$$

so that, in view of (13) and (12),

$$x^u_2 = \bar{\xi}^{\epsilon_1^{\delta_1^\delta}} v^{\phi_1^\delta} r = \bar{\xi}^{\epsilon_1^{\delta_1^\delta}} v^{\phi_1^\delta} r = \bar{\xi}^{\epsilon_1^{\delta_1^\delta}} v^{\phi_1^\delta} r,$$

where the last equality follows from (10). Hence

$$x_2 = \bar{\xi}^{\epsilon_1^{\delta_1^\delta}} s,$$ \tag{15}

for some $s \in K$ such that $s^u = r^v$. \tag{16}

From $f_1(x) = 0$ it follows that

$$x^m_1 = x^n_0 x^p_2.$$  \tag{17}

By (13), (12) and (15) this is equivalent to

$$\bar{\xi}^{\epsilon_1^{\delta_1^\delta} + \epsilon_2^{\delta_2^\delta} + \epsilon_3^{\delta_3^\delta}} m^p = \bar{\xi}^{\delta_1^{\delta_2^\delta}} + \epsilon_2^{\delta_2^\delta} + \epsilon_3^{\delta_3^\delta} p.$$  \tag{18}

In view of (14), cancelling equal powers of $\bar{\xi}$ and $\bar{\omega}$ yields

$$r^m = s^p.$$  \tag{19}

From the definition of $m, p, u, v$ we have

$$m \phi + v \epsilon = \phi_2 + \epsilon_2 = \delta$$

$$p \phi + u \epsilon = \epsilon_1 + \phi_1 = \delta.$$  \tag{20}

Hence, by (16) and (17), it holds that

$$s^\delta = s^{m \phi + v \epsilon} = r^{m \phi + v \epsilon} = r^\delta = 1,$$  \tag{21}
where the last equality is a consequence of (14). Thus both $s$ and $r$ are $\delta$-th roots of unity. Let $\eta$ be a primitive $\delta$-th root of unity. Then
\[ r = \eta^e, \quad \text{and} \quad s = \eta^f \]
for suitable integers $e, f$. Since, in view of (14), the order of $\eta^e$ as a root of unity is a divisor of $\delta^*$, it follows that $\vartheta$ divides $e$. Moreover, replacing (19) in (16) yields
\[ \eta^{uf} = \eta^{ve}, \]
whence
\[ \delta \text{ divides } uf - ve = \frac{\epsilon_1 f - \epsilon_2 e}{\epsilon}, \]
so that, in particular,
\[ \epsilon_1 f \equiv \epsilon_2 e \pmod{\delta} \] (20)
Let $\alpha = \frac{\epsilon}{\delta}$, where $\alpha$ is an integer such that
\[ \epsilon_1 \epsilon_1^* \equiv 1 \pmod{\delta} \] (21)
Such $\epsilon_1$ exists because $\gcd(\epsilon_1^*, \delta) = 1$. Then
\[ \eta^{\alpha r_1} = \eta^{\epsilon_1 r_1 / \vartheta} = \eta^{\epsilon_1 r_1^*} = \eta^e = r, \]
where the last two equalities follow from (21) and (19) respectively. Moreover
\[ \eta^{\alpha r_2} = \eta^{\epsilon_2 r_2 / \vartheta} = \eta^{\epsilon_2 r_2^*} = \eta^f = s, \]
where the second, the fourth and the fifth equality follow from (20), (21) and (19) respectively. Set
\[ \tilde{\xi} = \tilde{\omega} \eta^a. \] (24)
We then replace (24) in (12), (13) and (15). We obtain
\[ x_0 = \tilde{\xi}^\delta = \xi^\delta \eta^a\delta = \tilde{\xi}^\delta, \] (25)
by [26]
\[ x_1 = \xi^{\epsilon_1} \tilde{\omega}^{\delta_1} r = \xi^{\epsilon_1} \tilde{\omega}^{\delta_1} \eta^{\alpha r_1} = \xi^{\epsilon_1} \tilde{\omega}^{\delta_1}, \] (26)
and, by [27]
\[ x_2 = \xi^{\epsilon_2} \tilde{\omega}^{\delta_2} s = \xi^{\epsilon_2} \tilde{\omega}^{\delta_2} \eta^{\alpha r_2} = \xi^{\epsilon_2} \tilde{\omega}^{\delta_2}. \] (27)
Equalities (26)–(27), together with (12), show that $\xi$ and $\tilde{\omega}$ are the parameters which allow us to represent $x$ in the form (4). This completes the proof.

Now consider the following matrix with monomial entries:
\[ A = \begin{pmatrix} x_1^\min(\delta, m) & x_0^\max(0, n, 0) & x_3^\max(0, \epsilon, n, 0) & x_0^\max(0, m, 0) & x_2^p \\ x_0^\min(\delta, n) & x_3^\max(0, \epsilon, n, 0) & x_0^\max(0, m, 0) & x_1^\max(0, m, 0) & x_2^p \end{pmatrix}. \]
One can easily check that $A$ is simple. Let $J$ be the ideal generated by the 2-minors of $A$. In the sequel we will throughout refer to the projective curve $C$ given in (4).
Corollary 2 We have that $C = V(M_1, M_2, f_2)$, where $M_1, M_2$ are two minors of $A$ such that $V(M_1, M_2) = V(J)$.

Proof. We have to distinguish between different cases. In each case we will show that $A$ fulfills the assumption of Proposition 1 with respect to the second column ($k = 2$) or the third column ($k = 3$). We will choose $M_1, M_2$ accordingly, so as to have $V(M_1, M_2) = V(J)$.

Case (I): $\epsilon_1^* > n, \delta^* > m$. Then

$$A = \begin{pmatrix} x_1^m & x_0^{\epsilon_1^*-n} & x_3^{\phi_1^*} & x_2^p \\ x_0^n & x_1^{\delta^*-m} & 1 \end{pmatrix}.$$ 

In this case we can take $M_1 = \Delta_{13}$ and $M_2 = \Delta_{23}$.

Case (II): $\epsilon_1^* > n, \delta^* \leq m$. Then

$$A = \begin{pmatrix} x_1^{\delta^*} & x_0^{\epsilon_1^*-n} & x_3^{\phi_1^*} & x_2^p \\ x_0^n & 1 & x_1^{m-\delta^*} \end{pmatrix}.$$ 

In this case we can take $M_1 = \Delta_{12}$ and $M_2 = \Delta_{23}$.

Case (III): $\epsilon_1^* \leq n, \delta^* > m$. Then

$$A = \begin{pmatrix} x_1^m & x_0^{\phi_1^*} & x_3^{n-\epsilon_1^*} & x_2^p \\ x_0^{\epsilon_1^*} & x_1^{\delta^*-m} & 1 \end{pmatrix}.$$ 

In this case we can take $M_1 = \Delta_{13}$ and $M_2 = \Delta_{23}$.

Case (IV): $\epsilon_1^* \leq n, \delta^* \leq m$. Then

$$A = \begin{pmatrix} x_1^{\delta^*} & x_0^{\phi_1^*} & x_3^{n-\epsilon_1^*} & x_2^p \\ x_0^{\epsilon_1^*} & x_1^{m-\delta^*} & 1 \end{pmatrix}.$$ 

In this case we can take $M_1 = \Delta_{12}$ and $M_2 = \Delta_{23}$.

Now, in each of the above cases,

$$\Delta_{12} = f$$

$$\Delta_{13} = f_1,$$

whence $f, f_1 \in J$, so that

$$V(J) = V(M_1, M_2) \subset V(f_1, f_2).$$

Moreover:

in case (I)

$$\Delta_{23} = x_0^{\epsilon_1^*-n} x_3^{\phi_1^*} - x_1^{\delta^*-m} x_2^p.$$
in case (II)
\[ \Delta_{23} = x_1^{m-\delta^*} x_0^{\epsilon_1^*} x_3^{\phi_1} - x_2, \]

in case (III)
\[ \Delta_{23} = x_1^{\phi_1} - x_0^{n-\epsilon_1^*} x_1^{\delta^*} x_2, \]

in case (IV)
\[ \Delta_{23} = x_1^{\phi_1} x_1^{m-\delta^*} - x_0^{n-\epsilon_1^*} x_2. \]

From (8) and (9) it follows that
\[ \epsilon_1 (\delta^* - m) = (\epsilon_1^* - n) \delta - \epsilon_2 p, \]

which implies that in all cases \( \Delta_{23} \in I(C) \). Thus \( M_2 \in I(C) \), and, in view of (28) and (29), \( M_1 \in \{ f, f_1 \} \subset I(C) \). Hence
\[ C \subset V(M_1, M_2, f_2) \subset V(f, f_1, f_2) = C, \]

where the second and the third inclusion follow from (28) and Proposition 3 respectively.

This completes the proof.

**Example 4** Consider the projective curve

\[ C : C : x_0 = \xi^4, \ x_1 = \xi^3 \omega, \ x_2 = \xi \omega^3, \ x_3 = \omega^4, \]

which is known as the rational quartic and is conjectured to be a set-theoretic complete intersection. Its defining ideal \( I(C) \) is minimally generated by \( x_1^2 x_3 - x_0 x_2^2, x_1^3 - x_0 x_2 x_3, x_2^3 - x_1 x_3, x_0 x_3 - x_1 x_2 \). Curve \( C \) fulfils case (I) in the proof of Corollary 2 and corresponds to the matrix

\[
A = \begin{pmatrix}
  x_1^3 & x_0 x_3 & x_2 \\
  x_0^2 & x_1 & 1 \\
\end{pmatrix}.
\]

Therefore curve \( C \) is set-theoretically defined by

\[
\Delta_{13} = x_1^3 - x_0^2 x_2, \\
\Delta_{23} = x_0 x_3 - x_1 x_2, \\
f_2 = x_3^3 - x_1 x_2^2.
\]

Note that matrix \( A \) satisfies the assumption of Proposition 1 with respect to the first and the third column. Hence both the pairs of minors \( \Delta_{12}, \Delta_{13} \) and \( \Delta_{13}, \Delta_{23} \) set-theoretically define the corresponding determinantal variety \( V(J) \),
$J$ being the ideal generated by the 2-minors of $A$. It follows that $C$ is also set-theoretically defined by

\[
\begin{align*}
    f &= \Delta_{12} = x_1^4 - x_0^3 x_3 \\
    f_1 &= \Delta_{13} = x_1^3 - x_0^2 x_2 \\
    f_2 &= x_2^3 - x_1 x_3^2.
\end{align*}
\]

This is a well-known result on the rational quartic, which is also recalled in [14], p. 391.

**Remark 3** The fact that every monomial curve in the three-dimensional space is set-theoretically defined by three monic binomials had already been proved in [2]. In that paper, however, the proof was not based on closed formulas like (5), (6) and (7), but on a constructive method that is less convenient from a practical point of view: the three binomials described there are

\[
\begin{align*}
    f &= x_1^{\delta} + x_0^{\epsilon_1} x_3^{\phi_1} \\
    f_1 &= x_2^{\delta^p} - x_0^{\epsilon_0} x_1^{\alpha_1} x_3^{\beta_1} \\
    f_2 &= x_2^{\delta^q} - x_0^{\beta_0} x_1^{\beta_1} x_3^{\beta_3},
\end{align*}
\]

(assuming, without loss of generality, that gcd $(\delta, \epsilon_1, \epsilon_2) = 1$), where $p$ and $q$ are arbitrary distinct primes, and $\mu, \nu$, and $\alpha_i, \beta_i$ are suitable nonnegative integers to be determined so as to have $f_1, f_2 \in I(C)$.

**Example 5** Consider the projective curve

\[
C : C : x_0 = \xi^{70}, \; x_1 = \xi^{66} \omega^4, \; x_2 = \xi^{15} \omega^{55}, \; x_3 = \omega^{70}.
\]

The parametrization fulfills case (IV) in the proof of Corollary [2]. The corresponding matrix is

\[
A = \begin{pmatrix}
    x_1^{35} & x_3^2 & x_0^{18} x_2^4 \\
    x_0^{33} & 1 & x_1^{20}
\end{pmatrix}.
\]

Hence, according to Proposition [3] $C$ is set-theoretically defined by

\[
\begin{align*}
    f &= \Delta_{12} = x_1^{35} - x_0^{33} x_3^2 \\
    f_1 &= \Delta_{13} = x_1^{55} - x_0^{51} x_2^4 \\
    f_2 &= x_2^{22} - x_1^{5} x_3^{17}.
\end{align*}
\]

The method in [2], however, for $p = 2$ and $q = 3$, yields binomials of higher degree, namely

\[
\begin{align*}
    f &= x_1^{35} - x_0^{34} x_3^2
\end{align*}
\]

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\[ f_1 = x_2^{64} - x_0^{9}x_1^5x_3^{50} \]
\[ f_2 = x_2^{162} - x_0^{30}x_1^5x_3^{127} \]

if one takes \( f_1 \) and \( f_2 \) of the least possible degree.

**Remark 4** We know from the work by Cattani, Curran and Dickenstein [8] that the defining ideal of every projective monomial curve in \( \mathbb{P}^3 \) contains a complete intersection ideal of height 2. Our Proposition 3 makes this result more precise: it establishes that this ideal can be chosen in such a way that it has the same radical as the determinantal ideal generated by the 2-minors of a \( 2 \times 3 \) simple matrix with monomial entries. In [8] the authors consider the curve known as the twisted cubic

\[ C : x_0 = \xi^3, \ x_1 = \xi^2\omega, \ x_2 = \xi\omega^2, \ x_3 = \omega^3, \]

and find that \( I(C) \) contains the ideal \( (x_1^2 - x_0x_2, \ x_2^3 - x_0x_3) \). Curve \( C \) fulfils case (I) in the proof of Corollary [2] hence it is associated with the matrix

\[ A = \begin{pmatrix} x_1^2 & x_0x_3 & x_2 \\ x_0 & x_1 & 1 \end{pmatrix}. \]

We deduce, as in Example [11] that \( (x_1^2 - x_0x_2, \ x_0x_3 - x_1x_2) \subset I(C) \), and \( (x_1^2 - x_0x_2, \ x_1^3 - x_0x_3) \subset I(C) \).

All the above can be applied to affine monomial curves in \( K^3 \); since these are the affine parts of projective monomial curves in \( \mathbb{P}^3 \), it suffices to set \( x_0 = \xi = 1 \).

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