ASYMPTOTIC BEHAVIOR OF 2D WAVE–KLEIN-GORDON
COUPLED SYSTEM UNDER NULL CONDITION

SHIJIE DONG, YUE MA, AND XU YUAN

Abstract. We study the 2D coupled wave–Klein-Gordon systems with semi-linear null nonlinearities $Q_0$ and $Q_{\alpha\beta}$. The main result states that the solution to the 2D coupled systems exists globally provided that the initial data are small in some weighted Sobolev space, which do not necessarily have compact support, and we also show the optimal time decay of the solution.

The major difficulties lie in the slow decay nature of the wave and the Klein-Gordon components in two space dimensions, in addition, extra difficulties arise due to the presence of the null form $Q_0$ which is not of divergence form and is not compatible with the Klein-Gordon equations. To overcome the difficulties, a new observation for the structure of the null form $Q_0$ is required.

1. Introduction

1.1. The model problem. In this article, we consider the 2D coupled wave–Klein-Gordon system under null condition,

\begin{align*}
-\Box w &= C_1 Q_0(w, v) + C_1^{\alpha\beta} Q_{\alpha\beta}(w, v), \quad (t, x) \in [0, \infty) \times \mathbb{R}^2, \\
-\Box v + v &= C_2 Q_0(w, v) + C_2^{\alpha\beta} Q_{\alpha\beta}(w, v), \quad (t, x) \in [0, \infty) \times \mathbb{R}^2, 
\end{align*}

where $C_1, C_2, C_1^{\alpha\beta}, C_2^{\alpha\beta} \in \mathbb{R}$ and $Q_0, Q_{\alpha\beta}$ are the standard null forms:

$Q_0(w, v) = \partial_\alpha w \partial_\alpha v$ and $Q_{\alpha\beta}(w, v) = \partial_\alpha w \partial_\beta v - \partial_\alpha v \partial_\beta w$.

The initial data are prescribed at $t = 0$

\begin{align*}
(w, \partial_t w, v, \partial_t v)|_{t=0} = (w_0, w_1, v_0, v_1).
\end{align*}

We use $\Box = \partial_\alpha \partial^\alpha = -\partial_t^2 + \partial_i^2 + \partial_\beta^2$ to denote the d’Alembert operator. We use Latin letters $i, j, \cdots \in \{1, 2\}$ to represent space indices, and use Greek letters $\alpha, \beta, \cdots \in \{0, 1, 2\}$ to represent spacetime indices, and the Einstein summation convention for repeated upper and lower indices is adopted.

For various kinds of wave-type equations and systems under a suitable restriction of dimension and nonlinearity, the local well-posedness of the Cauchy problem is well-known. Now, one question of basic importance is that whether the Cauchy problem admits a global-in-time solution provided the initial data are sufficiently smooth and small. In the 3D case, this question was answered in the affirmative in the seminal works of Christodoulou [5] and Klainerman [23] on nonlinear wave equations with null nonlinearities, and of Klainerman [21] and Shatah [31] on nonlinear Klein-Gordon equations with general quadratic nonlinearities. Shortly after these fundamental results, the study on 3D coupled wave–Klein-Gordon systems started in [4, 14]. On the other hand, in the 2D case, such study is somewhat harder due to the slower decay nature of the wave and the Klein-Gordon components compared to the 3D case (see for instance [1, 30] and references therein). In the present article, we aim to demonstrate the global existence for the 2D coupled wave–Klein-Gordon systems in some important cases, i.e., the semilinear null nonlinearities.
1.2. Main result. The goal of the present paper is twofold. The first one is to
generalize the classical result [14] of Georgiev on coupled wave–Klein-Gordon sys-
tems from 3D to 2D with more types of quadratic nonlinearities and non-compactly
supported initial data, which is more challenging. The other one is that we expect
to provide more tools and ideas which can be used to tackle other related physical
models in 2D. More precisely, the main result of this article is the following:

**Theorem 1.1.** Let \( N \geq 14 \) be an integer. There exists \( 0 < \varepsilon_0 \ll 1 \) such that for
all initial data \((w_0, w_1, v_0, v_1)\) satisfying the smallness condition

\[
\sum_{k \leq N+1} \left( \| \langle |x| \rangle^k \nabla^k w_0 \|_{L^2} + \| \langle |x| \rangle^{k+1} \log (2 + |x|) \nabla^k v_0 \|_{L^2} \right)
+ \sum_{k \leq N} \left( \| \langle |x| \rangle^{k+1} \nabla^k w_1 \|_{L^2} + \| \langle |x| \rangle^{k+2} \log (2 + |x|) \nabla^k v_1 \|_{L^2} \right) \leq \varepsilon < \varepsilon_0,
\]

the Cauchy problem (1.1)–(1.2) admits a global-in-time solution \((w, v)\), which enjoys
the following pointwise decay estimates

\[
|w(t, x)| \lesssim \varepsilon(t)^{-\frac{1}{4}}, \quad |v(t, x)| \lesssim \varepsilon(t)^{-1}, \quad |\partial_t w(t, x)| \lesssim \varepsilon(t - |x|)^{-\frac{3}{4}}(t)^{-\frac{1}{4}}.
\]

**Remark 1.2.** In Theorem 1.1, we can treat non-compactly supported initial data.
Besides, the pointwise asymptotic behavior (1.4) of the solution \((w, v)\) to (1.1) is
optimal in time, due to the reason that even for the 2D linear wave and Klein-Gordon
a better decay in time is not expected.

**Remark 1.3.** Compared with the previous work [7], here we can treat addi-
tionally the nonlinear term \(Q_0(w, v)\), which is much harder to handle compared with
\(Q_{\alpha,\beta}(w, v)\) due to two reasons. First, the term \(Q_0(w, v)\) is not compatible with the
Klein-Gordon equations. This means that we need to use the scaling vector field,
which is not compatible with the Klein-Gordon equations, to obtain an extra \((t)^{-1}\)
decay from the null structure of \(Q_0(w, v)\). Second, the term \(Q_0(w, v)\) is not of di-
vergence form, which means we cannot express \(Q_0(w, v)\) in the form of \(\partial(\cdot)\),
which prevents us from using some techniques of [7].

**Remark 1.4.** In the remarkable work [33] of Stingo, a 2D quasilinear coupled wave
and Klein-Gordon system with null nonlinearities has been shown to admit small
global solution. In this work, the initial data do not need to be compactly sup-
ported, and the author used the microlocal analysis method to reduce the weights
requirement on the initial data to be very low. See also the recent work Ifrim-
Stingo [17] for systems of 2D quasilinear wave–Klein-Gordon system.

For the 3D coupled wave–Klein-Gordon systems, after the works [4, 14], fruitful
progress has been made in [18, 20, 26]. In the 2D case, we refer to [1, 2, 6, 8, 12,
13, 27, 30] for the works on wave and Klein-Gordon equation, and to [7, 11, 29, 33]
for the study of coupled wave–Klein-Gordon systems.

The study of the coupled wave–Klein-Gordon systems has large physical signif-
cance and is mathematically challenging. In the physical aspect, several fundamental
physical models are governed by coupled wave–Klein-Gordon systems, and such
models include the Dirac–Klein-Gordon model, the Klein–Gordon–Zakharov model,
the Maxwell–Klein-Gordon model, the Einstein–Klein-Gordon model, and so on. To
show these models are stable, one is required to study the relevant coupled wave–
Klein-Gordon systems. In the mathematical aspect, the most well-known obstacle
to study such coupled equations is that the scaling vector field \(S = t\partial_t + x^i\partial_i\)
is not compatible with the Klein-Gordon operator. Besides, the slow pointwise
decay of 2D wave and Klein-Gordon components, compared with the 3D scenario,
causes serious problems in the study. Thus new ideas are demanded in such study.
Lastly, we also would like to draw one’s attention to some of the relevant works in [4, 9, 10, 19, 24] for the above-mentioned models.

1.3. Comparison between $Q_0$ and $Q_{\alpha\beta}$. The difficulties in studying coupled wave–Klein-Gordon systems include the non-commutativity of the scaling vector field and the Klein-Gordon operator, the different decay properties of the wave and the Klein-Gordon components etc., and we lead one to [7, Introduction] for the discussion. Here we only discuss in detail about the extra difficulties caused by the term $Q_0(w, v)$ compared with the term $Q_{\alpha\beta}(w, v)$.

First, the term $Q_0(w, v)$ is not compatible with the Klein-Gordon equations. We recall the estimates (see for instance [32, Section 3])

$$|Q_{\alpha\beta}(w, v)| \lesssim \langle t \rangle^{-1} |\Gamma w| |\Gamma v|,$$

in which $\Gamma \in \{\partial_\alpha, H, \Omega\}$ (see the definition in Section 2) which are compatible with the Klein-Gordon equations. On the other hand, the estimates on the term $Q_0(w, v)$ demands the scaling vector field $S$, which is not compatible with the Klein-Gordon equations (and that is the reason why we call $Q_0$ non-compatible), i.e.,

$$|Q_0(w, v)| \lesssim \langle t \rangle^{-1} \left( |\Gamma w| + |S w| \right) |\Gamma v|.$$

Thus in this sense, it is harder to bound the term $Q_0(w, v)$ by using the vector field method.

Second, we recall that if the nonlinearities in the wave equation take divergence form, then in most cases we can derive better bounds on the wave component. For instance, we consider the following wave equation (we ignore the initial data for simplicity)

$$-\Box u = \partial_\alpha f$$

where $\alpha = 0, 1, 2$ and $f$ is a sufficiently regular function. For the solution $u$, we can decompose it into two parts $u = u_1 + \partial_\alpha u_2$, in which $u_1, u_2$ are solutions to the following wave equations

$$-\Box u_1 = 0, \quad -\Box u_2 = f.$$

In this case, we can get very good control of $\partial u$ by the relation

$$\partial u = \partial u_1 + \partial \partial_\alpha u_2,$$

as $u_1$ satisfies the linear homogeneous wave equation (see the estimates in (2.13)), and $\partial \partial_\alpha u_2$ is part of the Hessian of $u_2$ which roughly speaking enjoys an extra $\langle t - |x| \rangle^{-1}$ decay compared with $\partial u_2$ (see the estimates in Lemma 2.10). This decomposition can be used to treat the term $Q_{\alpha\beta}(w, v)$, as it is of divergence form, i.e.,

$$Q_{\alpha\beta}(w, v) = \partial_\beta (v \partial_\alpha w) - \partial_\alpha (v \partial_\beta w).$$

However, since the term $Q_0(w, v)$ is not of divergence form, the above argument cannot be directly applied, thus it requires a new decomposition to handle the term $Q_0(w, v)$.

1.4. New ingredients. To prove Theorem 1.1, one crucial step is to close the highest-order energy estimates for the solution. One natural choice is to apply the ghost weight energy estimates of Alinhac [1]. We note that it is necessary to first show (see (2.33))

$$|\partial w| \lesssim \langle t \rangle^{-1/2} \langle t - |x| \rangle^{-1/2 - \delta}$$

for some $\delta > 0$, so that the ghost weight energy estimates can be applied. Unlike the case of $Q_{\alpha\beta}(w, v)$ in the wave equation which is of divergence form (note the nonlinearities in the wave equation of [7, 33] are also of divergence form), and from which roughly speaking we can gain one more derivative and hence an extra
\((t - |x|)^{-1}\) decay on the wave component, we need to find a new strategy to treat the case of \(Q_0(w, v)\) which is not of divergence form.

The key to treat the nonlinear term \(Q_0(w, v)\) is one simple but useful observation

\[
Q_0(w, v) = \partial_\alpha w \partial^\alpha v = \partial^\alpha (v \partial_\alpha w) - v \partial_\alpha \partial^\alpha w,
\]

and we note that the first term in the right hand side is of divergence form. Then we note that the term \(\partial_\alpha \partial^\alpha w = \Box w\) contains only quadratic terms which makes \(v \partial_\alpha \partial^\alpha w\) into cubic terms. Furthermore, this way of decomposing the nonlinear term \(Q_0(w, v)\) can be carried on so that we can express \(Q_0(w, v)\) as the summation of quartic terms and terms of divergence form; see Lemma 2.3 for more details.

Compared with the proof in [7], we do not need to rely on an iteration procedure, and the key is to apply the Sobolev inequality (2.10) proved in [15] by Georgiev. This makes the presentation of the proof in the present paper neater.

### 1.5. Organisation

The article is organized as follows. In Section 2, we introduce the notation and some fundamental estimates and tools to be used in Section 3: the estimates on commutators and null forms, global Sobolev inequality, and \(L^\infty\) estimates for wave equations and Klein-Gordon equations. In Section 3, we provide the proof for Theorem 1.1 by Klainerman’s vector field method enhanced with Alinhac’s ghost weight method.

### Acknowledgement

The author S.D. was partially supported by the China Post-doctoral Science Foundation, with grant number 2021M690702.

## 2. Preliminaries

### 2.1. Notation

We work in the \((1+2)\) dimensional spacetime \(\mathbb{R}^{1+2}\) with Minkowski metric \(\eta = (-1, 1, 1)\) which is used to raise or lower indices. For a point \((x_0, x_1, x_2) = (t, x_1, x_2) \in \mathbb{R}^{1+2}\), we denote its spacial radius by \(r = \sqrt{x_1^2 + x_2^2}\). The space indices are denoted by Latin letters \(i, j \in \{1, 2\}\). The spacetime indices are denoted by Greek letters \(\alpha, \beta \in \{0, 1, 2\}\).

To state global Sobolev inequalities, we first introduce the following four groups of vector fields:

(i) Translations: \(\partial_\alpha = \partial_{x_\alpha}\), for \(\alpha = 0, 1, 2\).

(ii) Spatial rotations: \(\Omega = x_1 \partial_2 - x_2 \partial_1\).

(iii) Scaling vector field: \(S = t \partial_t + x_i \partial_i\).

(iv) Hyperbolic rotations: \(H_i = t \partial_i + x_i \partial_t\), for \(i = 1, 2\).

Excluding the scaling vector field \(S\), we consider a general vector field set

\[
V = \{\Omega; \partial_\alpha, \alpha = 0, 1, 2; H_i, i = 1, 2\}.
\]

For future notational convenience, we order these vector fields that belong to \(V\) in some arbitrary manner, and we label them as \(\Gamma_1, \Gamma_2, \ldots, \Gamma_6\). Moreover, for any multi-index \(I = (I_1, I_2, \ldots, I_6) \in \mathbb{N}^6\), we denote

\[
\Gamma^I = \prod_{k=1}^6 \Gamma_k^{I_k}, \quad \text{where} \quad \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_6).
\]

In addition, we also introduce the good derivatives

\[
G_i = \frac{1}{r} (x_i \partial_t + r \partial_i) \quad \text{for} \quad i = 1, 2.
\]

The Fourier transform is defined as

\[
\hat{h}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(x) e^{-ix \cdot \xi} dx, \quad \text{for} \quad h \in L^2_x.
\]
For \((x, \rho) \in \mathbb{R}^2 \times \mathbb{R}_+\), we denote by \(B(x, \rho)\) (respectively, \(\partial B(x, \rho)\)) the ball (respectively, the sphere) of \(\mathbb{R}^2\) of center \(x\) and of radius \(\rho\).

Let \(\{\psi_j\}_{j=0}^\infty\) be a Littlewood-Paley partition of unity, i.e.
\[
1 = \sum_{j=0}^\infty \psi_j(s), \quad s \geq 0, \quad \psi_j \in C_0^\infty(\mathbb{R}^2), \quad \psi_j \geq 0 \quad \text{for all } j \geq 0,
\]
as well as
\[
\text{supp} \psi_0 \cap [0, \infty) = [0, 2], \quad \text{supp} \psi_j \subset [2^{j-1}, 2^{j+1}] \quad \text{for all } j \geq 1.
\]
We will use \(Q\) to denote a general null form in \(\{Q_0; Q_{\alpha\beta}, 0 \leq \alpha \neq \beta \leq 2\}\).

For simplicity of notation, we denote the initial data \((w_0, w_1, v_0, v_1)\) by
\[\vec{w}_0 = (w_0, w_1), \quad \vec{v}_0 = (v_0, v_1), \quad (\vec{w}_0, \vec{v}_0) = (w_0, w_1, v_0, v_1).\]

We write \(A \lesssim B\) to indicate \(A \leq CB\) with \(C\) a universal constant, and we use the notation \((s) = \sqrt{1 + |s|^2}\) for \(s \in \mathbb{R}\).

### 2.2. Estimates on commutators and null forms

In this subsection, we state some preliminary estimates related to commutators and null forms \(Q\). We first recall the well-known relations
\[
[\Box, \Gamma_k] = [(\Box - 1), \Gamma_k] = 0, \quad \text{for } k = 1, 2, \ldots, 6.
\]

Second, we introduce the following estimates related to the vector fields.

**Lemma 2.1.** For any smooth function \(m = m(t, x)\), the following estimates hold.

(i) Estimate on commutators. For all \(I \in \mathbb{N}^6\), we have
\[
\sum_{\alpha=0}^{2} \left| [\partial_\alpha, \Gamma^I] m \right| + \left| [S, \Gamma^I] m \right| \lesssim \sum_{|J| < |I|} \sum_{\beta=0}^{2} \left| \partial_\beta \Gamma^J m \right|. \tag{2.1}
\]

(ii) Estimates on \(\partial m\) and \(G_i m\). We have
\[
(t - r) \left| \partial m \right| + (t + r) \left| G_i m \right| \lesssim \sum_{|I| = 1} \left( |S m| + |\Gamma^I m| \right). \tag{2.2}
\]

**Proof.** Proof of (i). Note that, for \(k = 1, 2, \ldots, 6\) and \(\alpha = 0, 1, 2\), we have
\[
[\partial_\alpha, \Gamma_k] m, \quad [S, \Gamma_k] m \in \text{Span} \{\partial_1 m, \partial_2 m, \partial_3 m\},
\]
which implies (2.1) for \(I \in \mathbb{N}^6\) with \(|I| = 1\). Then by an induction argument, we obtain (2.1) for all \(I \in \mathbb{N}^6\).

Proof of (ii). By an elementary computation, for \(i = 1, 2\),
\[
\partial_i m = (t^2 - r^2)^{-1} \left( tS m - x_i H_j m \right),
\]
\[
\partial_i m = (t^2 - r^2)^{-1} \left( tH_i m - x_i S m - (-1)^i x_3 \cdot \Omega m \right),
\]
\[
G_i m = \frac{1}{r} \left( H_i m + (r - t) \partial_i m \right) = \frac{1}{r} \left( H_i m - \frac{x_i}{r} (r - t) \partial_i m \right).
\]
Based on the above identities, we obtain (2.2). \(\square\)

Third, we recall the following estimates related to the null form \(Q\) from [32].
Lemma 2.2 ([32]). For any $I \in \mathbb{N}^6$, and smooth functions $m$ and $n$, we have
\begin{equation}
|Q(m, n)| \lesssim \sum_{i=1,2} \left(|G_i m| |\partial n| + |G_i n| |\partial m|\right), \tag{2.3}
\end{equation}
\begin{equation}
|Q(m, n)| \lesssim (t + r)^{-1} \sum_{|J|=1} \left(|Sm| + \sum_{|J|=1} |\Gamma^J m|\right) |\Gamma^{I} n|, \tag{2.4}
\end{equation}
\begin{equation}
|\Gamma^{I} Q(m, n)| \lesssim \sum_{|\alpha|\neq |\beta| \leq |I|} \left(|Q_0 (\Gamma^I_{I_1} m, \Gamma^I_{I_2} n)| + |Q_{\alpha \beta} (\Gamma^I_{I_1} m, \Gamma^I_{I_2} n)|\right). \tag{2.5}
\end{equation}

Proof. The proof of (2.3) is based on the identity $\partial_t = G_i - \frac{n^i}{n^2} \partial_t$ and the structure of null form $Q$, and we omit it. For (2.4) and (2.5), we refer to [32, Lemma 3.3] and [32, Page 58] for the complete proofs. \hfill \square

Note that, from (2.4) and (2.5), for all $I \in \mathbb{N}^6$, we have the following pointwise estimate for $\Gamma^{I} Q$ where $Q \in \{Q_0, Q_{\alpha \beta}, 0 \leq \alpha \neq \beta \leq 2\}$,
\begin{equation}
| (t + r) \Gamma^{I} Q(w, v) | \lesssim \sum_{|I_1| + |I_2| \leq |I|} \left(|\Sigma^{I_1} w| + |\Gamma^{I_1} \Gamma^{I_2} w|\right) |\Gamma^{I_2} \Gamma^{I_2} v|. \tag{2.6}
\end{equation}

Last, we expand the nonlinear term $C_1 Q_0 + C_1^{\alpha \beta} Q_{\alpha \beta}$ by an elementary computation.

Lemma 2.3. Let $(m, n)$ be a solution of (1.1), then we have
\begin{equation}
C_1 Q_0 (m, n) + C_1^{\alpha \beta} Q_{\alpha \beta} (m, n) = \partial^\alpha F_\alpha(m, n) + \partial_\beta H^\alpha (m, n) + G(m, n), \tag{2.7}
\end{equation}
where
\begin{align*}
G(m, n) &= \frac{C_1^2}{2} n^2 \left(C_1 Q_0 (m, n) + C_1^{\alpha \beta} Q_{\alpha \beta} (m, n)\right), \\
F_\alpha(m, n) &= \sum_{k=1,2} \frac{C_1^k}{k!} (n^k \partial_\alpha m), \quad H^\alpha (m, n) = \sum_{k=1,2} \frac{C_1^{k-1}}{k!} \left(C_1^{\alpha \beta} - C_1^{\alpha \beta}\right) (n^k \partial_\beta m).
\end{align*}

Proof. First, we observe that
\begin{equation*}
C_1^{\alpha \beta} Q_{\alpha \beta} (m, n) = \left(C_1^{\alpha \beta} - C_1^{\alpha \beta}\right) \partial_\alpha (n \partial_\beta m).
\end{equation*}

Then, from (1.1), we have
\begin{equation*}
C_1 Q_0 (m, n) = C_1 \partial^\alpha (n \partial_\alpha m) + C_1^2 n Q_0 (m, n) + C_1 C_1^{\alpha \beta} n Q_{\alpha \beta} (m, n).
\end{equation*}

By an elementary computation,
\begin{align*}
C_1^2 n Q_0 (m, n) &= \frac{C_1^2}{2} \partial^\alpha (n^2 \partial_\alpha m) + G(m, n), \\
C_1 C_1^{\alpha \beta} n Q_{\alpha \beta} (m, n) &= \frac{C_1}{2} \left(C_1^{\alpha \beta} - C_1^{\alpha \beta}\right) \partial_\alpha (n^2 \partial_\beta m).
\end{align*}

Based on the above identities, we obtain (2.7). \hfill \square

Remark 2.4. Note that, the terms $\partial^\alpha F_\alpha$ and $\partial_\beta H^\alpha$ take the divergence form, and $G$ is a quartic term which decays fast enough as $t \to \infty$. This way of expressing the nonlinear terms can be used to establish very good $L^2$-norm estimate of wave component $w$, and hence a good estimate of the $L^2$-norm of $Sw$ by the aid of the conformal energy estimate (see more details in Lemma 2.9 and §3.3).
2.3. **Global Sobolev inequality.** In this subsection, we recall some Sobolev type inequalities associated with the vector field set $V$. These inequalities can be used to obtain the pointwise decay estimates of wave and Klein-Gordon equation from the weighted energy bounds.

**Lemma 2.5** ([15, 22]). Let $u = u(t, x)$ be a sufficiently regular function. Then the following estimates hold.

(i) **Standard Sobolev inequality.** We have

$$
|u(t, x)| \lesssim \langle r \rangle^{-\frac{1}{2}} \sum_{|I| \leq 2} \| \Gamma^I u(t, x) \|_{L^2_x}.
$$

(ii) **Estimate inside of a cone.** For $|x| \leq \frac{t}{2}$, we have

$$
|u(t, x)| \lesssim \langle t \rangle^{-\frac{1}{2}} \sum_{|I| \leq 3} \| \Gamma^I u(t, x) \|_{L^2_x}.
$$

(iii) **Global Sobolev inequality.** We have

$$
|u(t, x)| \lesssim \langle t \rangle^{-\frac{1}{2}} \sum_{|I| \leq 3} \| \Gamma^I u(t, x) \|_{L^2_x}.
$$

**Proof.** For (2.8) and (2.9), we refer to [22, Proposition 1] and [15, Lemma 2.4] for the details of the proof respectively. Then, the inequality (2.10) is a consequence of (2.8) and (2.9).

**Remark 2.6.** The inequality (2.10) by Georgiev can be used to get the pointwise decay of a nice function with $t^{-\frac{1}{2}}$ rate which is the same as in the Klainerman-Sobolev inequality in 2D, but no bound on $\| S \Gamma^I u \|_{L^2_x}$ is required, which is more compatible with the coupled wave–Klein-Gordon equations. However, these two kinds of inequalities, only in the 2D case, yield the same decay rate of a function; and for higher dimensional case, the decay rate derived from Georgiev [15] is slower than the rate derived from the Klainerman-Sobolev inequality (see more details in [15, Lemma 2.3 and Lemma 2.4] and [22, Proposition 3]).

2.4. **Estimates on 2D linear wave equation.** In this subsection, we recall several technical estimates for 2D linear wave equation. We start with the $L^2$ and $L^\infty$ estimates for 2D homogeneous wave equation. The proofs are similar to [28, Theorem 4.3.1 and Theorem 4.6.1], but they are given for the sake of completeness and for the readers’ convenience.

**Lemma 2.7** ([28]). Let $u$ be the solution to the Cauchy problem

$$
\begin{aligned}
-\Box u(t, x) &= 0, \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1).
\end{aligned}
$$

Then the following estimates hold.

(i) **$L^2$ estimate on solution.** We have

$$
\| u(t, x) \|_{L^2_x} \lesssim \| u_0 \|_{L^2_x} + \log^\delta (2 + t) \left( \| u_1 \|_{L^1_x} + \| u_1 \|_{L^2_x} \right).
$$

(ii) **$L^\infty$ estimate on solution.** We have

$$
\| u(t, x) \|_{L^\infty_x} \lesssim (t)^{-\frac{\delta}{2}} \left( \| u_0 \|_{W^{2,1}} + \| u_1 \|_{W^{1,1}} \right).
$$

**Proof.** Proof of (i). Taking the Fourier transform in the Cauchy problem with respect to the argument $x$, we have

$$
\begin{aligned}
\partial_t^2 \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) &= 0, \\
\hat{u}(0, \xi) &= \hat{u}_0(\xi), \\
\partial_t \hat{u}(0, \xi) &= \hat{u}_1(\xi).
\end{aligned}
$$
We solve the above second-order ODE in $t$ to arrive at the expression of the solution $u$ in Fourier space

$$
\hat{u}(t, \xi) = \cos(t|\xi|)\hat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \hat{u}_1(\xi), \quad \text{for } (t, \xi) \in [0, \infty) \times \mathbb{R}^2.
$$

Note that, from the Newton-Leibniz formula and change of variable, we have

$$
\int_0^t \left| \int_0^{t-s} \frac{\sin(|\xi|s)}{|\xi|} ds \right| dy \lesssim \int_0^t \left| \int_0^{t-s} \frac{\sin(|\xi|s)}{|\xi|} ds \right| dy \lesssim 1,
$$

Combining the above estimates, we obtain

$$
\left\| \hat{u}(t, \xi) \right\|_{L^1_{\xi}L^2_t} \lesssim \left\| \cos(t|\xi|)\hat{u}_0(\xi) \right\|_{L^1_{\xi}L^2_t} + \left\| \frac{\sin(t|\xi|)}{|\xi|} \hat{u}_1(\xi) \right\|_{L^1_{\xi}L^2_t},
$$

which implies (2.12).

Proof of (ii). From the expression of solutions for 2D linear wave equation, we decompose

$$
u(t, x) = \mathcal{I}_1(t, x) + \mathcal{I}_2(t, x) \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^2,
$$

where

$$
\mathcal{I}_1(t, x) = \frac{1}{2\pi} \int_{B(t, x)} \frac{u_1(y)dy}{\sqrt{t^2 - |x - y|^2}}, \quad \mathcal{I}_2(t, x) = \frac{1}{2\pi} \int_{B(t, x)} \frac{u_0(y)dy}{\sqrt{t^2 - |x - y|^2}}.
$$

Estimate on $\mathcal{I}_1$. We claim that

$$
|\mathcal{I}_1(t, x)| \lesssim (t)^{-\frac{1}{2}} \left\| u_1 \right\|_{W^{1,1}}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^2. \tag{2.14}
$$

Indeed, taking the change of variable $y \mapsto (x - y)$, we have

$$
\mathcal{I}_1(t, x) = \frac{1}{2\pi} \int_{B(0, t)} \frac{u_1(x - y)dy}{\sqrt{t^2 - |y|^2}}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^2. \tag{2.15}
$$

Case I: Let $(t, x) \in [0, 2] \times \mathbb{R}^2$. We rewrite (2.15) as

$$
\mathcal{I}_1(t, x) = \frac{1}{2\pi} \int_{-t}^t \int_{-\sqrt{t^2 - y_1^2}}^{\sqrt{t^2 - y_1^2}} \frac{u_1(x - (y_1, y_2))dy_2dy_1}{\sqrt{t^2 - y_1^2}}.
$$

Note that, from the Newton-Leibniz formula and change of variable, we have

$$
\int_{-t}^t \left\| u(x - (y_1, y_2)) \right\|_{L^\infty_{y_2}} dy_1 \lesssim \int_{-t}^t \left\| \partial_{x_1} u_1(x - (y_1, s)) \right\| ds dy_1 \lesssim \left\| u_1 \right\|_{W^{1,1}},
$$

$$
\max_{y_1 \in [-t, t]} \int_{-\sqrt{t^2 - y_1^2}}^{\sqrt{t^2 - y_1^2}} \frac{dy_2}{\sqrt{t^2 - y_2^2}} = \max_{y_1 \in [-t, t]} \int_{-1}^1 \frac{ds}{\sqrt{1 - s^2}} \lesssim 2 \int_0^1 \frac{ds}{\sqrt{\sqrt{2} - s}} \lesssim 1.
$$

Combining the above estimates, we obtain

$$
|\mathcal{I}_1(t, x)| \lesssim \left( \int_{-t}^t \left\| u(x - (y_1, y_2)) \right\|^2_{L^\infty_{y_2}} dy_1 \right)^{1/2} \cdot \left( \max_{y_1 \in [-t, t]} \int_{-\sqrt{t^2 - y_1^2}}^{\sqrt{t^2 - y_1^2}} \frac{dy_2}{\sqrt{t^2 - y_2^2}} \right)^{1/2} \lesssim \left\| u_1 \right\|_{W^{1,1}},
$$

which implies (2.14) for this case.
Case II: Let \((t, x) \in (2, \infty) \times \mathbb{R}^2\). We decompose
\[ I_2(t, x) = I_{11}(t, x) + I_{12}(t, x) \quad \text{for} \quad (t, x) \in (2, \infty) \times \mathbb{R}^2, \]
where
\[ I_{11}(t, x) = \frac{1}{2\pi} \int_{B(0, t-1)} \frac{u_1(x - y)dy}{\sqrt{t^2 - |y|^2}}, \quad I_{12}(t, x) = \frac{1}{2\pi} \int_{B(0, t) \setminus B(0, t-1)} \frac{u_1(x - y)dy}{\sqrt{t^2 - |y|^2}}. \]
First, we know that
\[ \max_{y \in B(0, t-1)} \frac{1}{\sqrt{t^2 - |y|^2}} = \max_{y \in B(0, t-1)} \frac{1}{\sqrt{t + |y|/\sqrt{t - |y|}}} \lesssim \langle t \rangle^{-\frac{3}{4}}, \]
which implies
\[ |I_{11}(t, x)| \lesssim \int_{\mathbb{R}^2} |u(y)|dy \left( \max_{y \in B(0, t-1)} \frac{1}{\sqrt{t^2 - |y|^2}} \right) \lesssim \langle t \rangle^{-\frac{3}{4}} \|u_1\|_{L^1}. \tag{2.16} \]
Second, by the polar coordinate transformation \(y = \rho \omega\) with \(dy = \rho d\rho d\omega\),
\[ I_{12}(t, x) = \frac{1}{2\pi} \int_{\partial B(0,1)} \int_{t-1}^t \frac{\rho u_1(x - \rho \omega)}{\sqrt{t^2 - \rho^2}} d\rho d\omega \]
\[ = -\frac{1}{\pi} \int_{\partial B(0,1)} \int_{t-1}^t \frac{\rho u_1(x - \rho \omega)}{\sqrt{t + \rho}} d\rho d\omega. \]
Based on the above identity and integration by parts, we decompose
\[ I_{12}(t, x) = I_{12}^1(t, x) + I_{12}^2(t, x), \]
where
\[ I_{12}^1(t, x) = \frac{1}{\pi} \int_{\partial B(0,1)} \frac{(t-1)u_1(x - (t-1)\omega)}{\sqrt{t^2 - 1}} d\omega, \]
\[ I_{12}^2(t, x) = \frac{1}{\pi} \int_{\partial B(0,1)} \int_{t-1}^t \rho d\rho \left( \frac{\rho u_1(x - \rho \omega)}{\sqrt{t + \rho}} \right) d\rho d\omega. \]
Using again the Newton-Leibniz formula, we have
\[ (t-1)u_1(x - (t-1)\omega) = -\int_{t-1}^\infty (u_1(x - \rho \omega) - \rho \omega \cdot \nabla u_1(x - \rho \omega)) d\rho, \]
which implies
\[ |I_{12}^1(t, x)| \lesssim (2t-1)^{-\frac{3}{4}} \int_{\partial B(0,1)} \int_{t-1}^\infty \left( |u_1(x - \rho \omega)| + |\rho \omega \cdot \nabla u_1(x - \rho \omega)| \right) d\rho d\omega \]
\[ \lesssim \langle t \rangle^{-\frac{3}{4}} \int_{\mathbb{R}^2 \setminus B(t-1)} \frac{|u_1(y)| |x-y| + |\nabla u_1(y)|}{|x-y|} dy \lesssim \langle t \rangle^{-\frac{3}{4}} \|u_1\|_{W^{1,1}}. \]
On the other hand, by direct computation, we have
\[ \partial_\rho \left( \frac{\rho u_1(x - \rho \omega)}{\sqrt{t + \rho}} \right) = \frac{u_1(x - \rho \omega)}{\sqrt{t + \rho}} - \rho \omega \cdot \nabla u_1(x - \rho \omega) \right) \frac{1}{\sqrt{t + \rho}}, \]
which implies
\[ |I_{12}^2(t, x)| \lesssim (2t-1)^{-\frac{3}{4}} \int_{\partial B(0,1)} \int_{t-1}^\infty \left( |u_1(x - \rho \omega)| + \rho |\nabla u_1(x - \rho \omega)| \right) d\rho d\omega \]
\[ \lesssim \langle t \rangle^{-\frac{3}{4}} \int_{\mathbb{R}^2 \setminus B(t-1)} \frac{|u_1(y)| |x-y| + |\nabla u_1(y)|}{|x-y|} dy \lesssim \langle t \rangle^{-\frac{3}{4}} \|u_1\|_{W^{1,1}}. \]
Combining the above two estimates, we obtain
\[ |I_{12}(t, x)| \lesssim |I_{12}^1(t, x)| + |I_{12}^2(t, x)| \lesssim \langle t \rangle^{-\frac{3}{4}} \|u_1\|_{W^{1,1}}. \tag{2.17} \]
We see that (2.14) for this case follows from (2.16) and (2.17).
Estimate on $I_2$. We claim that
\[ |I_2(t, x)| \lesssim (t)^{-\frac{3}{2}} \|u_0\|_{W^{2,1}}, \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^2. \] (2.18)
Indeed, taking the change of variable $y \mapsto x - ty$, we have
\[ I_2(t, x) = \frac{1}{2\pi} \int_{B(0, 1)} \frac{tu_0(x - ty)}{1 - |y|^2} \, dy = I_{21}(t, x) + I_{22}(t, x), \]
where
\[ I_{21}(t, x) = \frac{1}{2\pi} \int_{B(0, 1)} \frac{u_0(x - ty)}{1 - |y|^2} \, dy, \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^2, \]
\[ I_{22}(t, x) = -\frac{1}{2\pi} \int_{B(0, t)} \frac{y \cdot \nabla u_0(x - y)}{t\sqrt{t^2 - |y|^2}} \, dy, \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^2. \]

Using a similar argument as in the proof of (2.14), one can obtain (2.18). Combining (2.14) and (2.18), we complete the proof of (2.13).

Second, we introduce the $L^2$ and $L^\infty$ estimates of solutions for the 2D inhomogeneous wave equation with zero initial data. These estimates will be used to control a part of wave component $w$ which is related to quartic term $G(w, v)$ (recall Lemma 2.3 for the expression of $G$).

Lemma 2.8. Let $u$ be the solution to the Cauchy problem
\[
\begin{cases}
-\Box u(t, x) = f(t, x), \\
(u, \partial_t u)|_{t=0} = (0, 0),
\end{cases}
\]
with $f(t, x)$ a sufficiently regular function. Then the following estimates hold.
(i) $L^2$ estimate on solution. We have
\[ \|u(t, x)\|_{L^2_x} \lesssim \log^\frac{1}{2} (2 + t) \int_0^t \left( \|f(s, x)\|_{L^1_x} + \|f(s, x)\|_{L^2_x} \right) \, ds. \] (2.19)
(ii) $L^\infty$ estimate on solution. We have
\[ \|u(t, x)\|_{L^\infty_x} \lesssim (t)^{-\frac{1}{2}} \int_0^t (1 + s)^{-\frac{1}{2}} \|f(s, x)\|_{L^1_x} \, ds \]
\[ + (t)^{-\frac{1}{2}} \sum_{|\ell| \leq 1} \int_0^t (1 + s)^{-\frac{1}{2}} \|\Gamma^\ell f(s, x)\|_{L^1_x} \, ds. \] (2.20)

Proof. Proof of (i). Taking the Fourier transform for the Cauchy problem and using the Duhamel’s principle, \[ \hat{u}(t, \xi) = I_3(t, \xi) + I_4(t, \xi), \quad t \in [0, \infty), \]
where
\[ I_3(t, \xi) = 1_{\{\xi \geq 1\}} \int_0^t \frac{\sin((t - s)|\xi|)}{|\xi|} \hat{f}(s, \xi) \, ds, \quad t \in [0, \infty), \]
\[ I_4(t, \xi) = 1_{\{\xi \leq 1\}} \int_0^t \frac{\sin((t - s)|\xi|)}{|\xi|} \hat{f}(s, \xi) \, ds, \quad t \in [0, \infty). \]

Note that, from the Plancherel theorem
\[ \|I_3(t, \xi)\|_{L^2_x} \lesssim \int_0^t \|\hat{f}(s, \xi)\|_{L^2_x} \left( \int_0^t \frac{\sin^2 r}{r} \, dr \right)^{\frac{1}{2}} \, ds \lesssim \log^\frac{1}{2} (2 + t) \int_0^t \|f(s, x)\|_{L^2_x} \, ds, \]
\[ \|I_4(t, \xi)\|_{L^2_x} \lesssim \int_0^t \|\hat{f}(s, \xi)\|_{L^\infty_x} \left( \int_0^t \frac{\sin^2 r}{r} \, dr \right)^{\frac{1}{2}} \, ds \lesssim \log^\frac{1}{2} (2 + t) \int_0^t \|f(s, x)\|_{L^2_x} \, ds. \]
Therefore, using again the Plancherel theorem, we have
\[
\|u(t,x)\|_{L^2_x} = \|\hat{u}(t,\xi)\|_{L^2_\xi} \lesssim \|I_3(t,\xi)\|_{L^2_\xi} + \|I_4(t,\xi)\|_{L^2_\xi} \lesssim \int_0^t \|f(s,x)\|_{L^2_x} ds + \log^\frac{5}{2}(2+t) \int_0^t \|f(s,x)\|_{L^1_x} ds,
\]
which implies (2.19).

Proof of (ii). Estimate (2.20) is due to Hörmander [16]. We also refer to [28, Theorem 4.6.2] for details of the proof.

To establish the energy estimates of the wave equation for future reference, we first introduce the standard energy \(E\) and the conformal energy \(G\) for the 2D wave equation,
\[
E(t,u) = \int_{\mathbb{R}^2} \left( (\partial_t u)^2 + (\partial_1 u)^2 + (\partial_2 u)^2 \right) (t,x) dx,
\]
\[
G(t,u) = \int_{\mathbb{R}^2} \left( (Su + u)^2 + (\Omega u)^2 + \sum_{i=1,2} (H_i u)^2 \right) (t,x) dx.
\]

Now we recall the energy estimates for 2D wave equation from [3, 32].

**Lemma 2.9** ([3, 32]). Let \(u\) be the solution to the Cauchy problem
\[
\begin{cases}
-\Box u(t,x) = f(t,x), \\
(u, \partial_t u)|_{t=0} = (u_0, u_1),
\end{cases}
\]
with \(f(t,x)\) a sufficiently regular function. Then the following estimates hold.

(i) Standard energy estimate. We have
\[
E(t,u)^\frac{1}{2} \lesssim E(0,u)^\frac{1}{2} + \int_0^t \|f(s,x)\|_{L^2_x} ds. \tag{2.21}
\]

(ii) Conformal energy estimate. We have
\[
G(t,u)^\frac{1}{2} \lesssim G(0,u)^\frac{1}{2} + \int_0^t \|(s+r)f(s,x)\|_{L^2_x} ds. \tag{2.22}
\]

**Proof.** Proof of (i). First, we can rewrite the product \((-\Box u) \partial_t u\) as the following divergence form,
\[
(-\Box u) \partial_t u = \frac{1}{2} \sum_{\alpha=0}^2 \partial_\alpha (\partial_\alpha u)^2 - \partial^i (\partial_i u \partial_t u).
\]
Integrating the above identity on \([0,t] \times \mathbb{R}^2\) for any \(t > 0\), and using \(-\Box u = f\)
\[
E(t,u) \leq E(0,u) + 2 \int_0^t \|f(s,x)\|_{L^2_x} \|\partial_t u(s,x)\|_{L^2_x} ds \\
\leq E(0,u) + 2 \int_0^t \|f(s,x)\|_{L^2_x} E(s,u)^\frac{1}{2} ds.
\]
Based on the above inequality, we obtain (2.21).

Proof of (ii). Here, we briefly sketch the proof and refer to [3, Theorem 6.11] for the complete proof. Consider the nonspacelike multiplier
\[
K_0 = (r^2 + t^2) \partial_t + 2rt \partial_r \quad \text{where} \quad \partial_r = \frac{x^i}{r} \partial_i.
\]
By an elementary computation, we have
\[-\Box u(K_0 u + tu) = \frac{1}{2} \partial_t \left( (Su + u)^2 + (\Omega u)^2 + (H_1 u)^2 + (H_2 u)^2 \right) + \partial^i \left( t x_i \left( - (\partial_i u)^2 + (\partial_j u)^2 + (\partial_2 u)^2 \right) - tu \partial_i u \right) - \partial^i \left( (r^2 + t^2)(\partial_i u)(\partial_i u) + 2rt(\partial_i u)(\partial_i u) + \frac{1}{2} \partial_i(x_i u^2) \right).\]

Integrating the above identity on \([0, t] \times \mathbb{R}^2\) and then using the Cauchy-Schwarz inequality, we see that
\[G(t, u) - G(0, u) \lesssim \int_0^t \| (s + r) f(s, x) \|_{L^2} \| (s + r)^{-1} (K_0 u + tu) \|_{L^2} \, ds.\]

From the definition of \(K_0\), we observe that
\[K_0 u = t Su + r(\partial_r u + r \partial_t u) \Rightarrow |K_0 u + tu| \lesssim (t + r) (|Su + u| + |H_1 u| + |H_2 u|).\]
Combining the above inequalities, we obtain (2.22).

Last, we recall the extra decay for Hessian of 2D inhomogeneous wave equation. For the convenience of the reader, we revisit the complete proof in [25].

**Lemma 2.10** (Extra decay for Hessian). Let \(u\) be the solution of the Cauchy problem

\[
\begin{aligned}
&\Box u(t, x) = f(t, x), \\
&(u, \partial_t u)|_{t=0} = (u_0, u_1).
\end{aligned}
\]

Then we have
\[|\partial^r u| \lesssim \sum_{|I|=0, 1} \frac{1}{(t - r)} |\partial^I u| + \frac{t}{(t - r)} |f|, \quad \text{for } r \leq 2t. \tag{2.23}\]

**Proof.** Case I: Let \(t \in [0, 1]\). It is easily seen that
\[|\partial^r u| \lesssim \sum_{|I|=0, 1} |\partial^I u| \lesssim \sum_{|I|=0, 1} \frac{1}{(t - r)} |\partial^I u| + \frac{t}{(t - r)} |f|, \quad \text{for } r \leq 2t. \tag{2.24}\]

Case II: Let \(t \in (1, \infty)\). We first express the wave operator \(-\Box\) by \(\partial_{rr}\) and \(H_i\) to obtain
\[-\Box = \frac{(t - r)(t + r)}{t^2} \partial_t \partial_r + \frac{x^i}{t^2} \partial_i \partial_r H_i - \frac{1}{t} \partial^i H_i + \frac{2}{t} \partial_t - \frac{x^i}{t^2} \partial_i.\]

Based on the above identity and \(-\Box u = f\), we have
\[|\partial_t \partial_r u| \lesssim \sum_{|I|=0, 1} \frac{1}{(r - t)} |\partial^I u| + \frac{t^2}{(r - t)(r + t)} |f|, \quad \text{for } r \leq 2t. \tag{2.25}\]

On the other hand, for \(i, j \in \{1, 2\}\), we have the following two identities
\[
\begin{aligned}
\partial_i \partial_j &= \frac{x_i x_j}{t^2} \partial_i \partial_j + \frac{1}{t} \partial_i H_j + \frac{1}{t} \partial_j H_i - \frac{\delta_{ij}}{t} \partial_t + \frac{x_i}{t^2} \partial_j, \\
\partial_i \partial_j &= \frac{x_i x_j}{t^2} \partial_i \partial_j - \frac{x_i}{t^2} \partial_i H_j + \frac{x_i}{t^2} \partial_j H_i - \frac{\delta_{ij}}{t} \partial_t + \frac{x_i}{t^2} \partial_j.
\end{aligned}
\]

Based on the above two identities and (2.25), we have
\[|\partial_i \partial_j u| \lesssim \sum_{|I|=0, 1} \frac{1}{(r - t)} |\partial^I u| + \frac{t^2}{(r - t)(r + t)} |f|, \quad \text{for } r \leq 2t. \tag{2.26}\]

We see that (2.23) follows from (2.24), (2.25) and (2.26).
Remark 2.11. The above Lemma states that, for the solution $u$ of wave equation, the Hessian form $\partial^2 u$ has extra $(t-r)^{-1}$ decay than $\partial u$ in the spacetime region $\{(t, x) : r \leq 2t\}$ if the source term $f$ has sufficiently fast decay. Note that, this extra decay can be used to obtain the sharp pointwise decay for $\partial w$ thanks to the hidden divergence form structure in the wave equation of $w$ (see more details in §3.3).

2.5. Estimates on 2D linear Klein-Gordon equation. In this subsection, we recall some decay estimates and energy estimates for 2D linear Klein-Gordon equation. First, we recall the following decay estimates from [15].

Theorem 2.12 ([15]). Let $u$ be the solution of the Cauchy problem

$$
\begin{cases}
(\Box + 1) u(t, x) = f(t, x), \\
(u(0, x), \partial_t u(0, x)) = (u_0(x), u_1(x)),
\end{cases}
$$

with $f(t, x)$ a sufficiently regular function. Then we have

$$
\langle t + r \rangle|u(t, x)| \lesssim \sum_{j=0}^{\infty} \sum_{|I| \leq 5} \|\langle |x| \rangle \psi_j(\langle |x| \rangle) \Gamma^I u(0, x)\|_{L_2^5}^2 + \sum_{j=0}^{\infty} \sum_{|I| \leq 5} \max_{0 \leq s \leq t} \psi_j(s) \|\langle s + |x| \rangle \Gamma^I f(s, x)\|_{L_2^5}. 
$$

(2.27)

As a consequence, we have the following simplified version of Theorem 2.12.

Corollary 2.13. With the same settings as Theorem 2.12, let $\delta_0 > 0$ and assume

$$
\sum_{|I| \leq 4} \max_{0 \leq s \leq t} \langle s \rangle^\delta_0 \|\langle s + |x| \rangle \Gamma^I f(s, x)\|_{L_2^5} \leq C_f,
$$

(2.28)

then we have

$$
\langle t + r \rangle|u(t, x)| \lesssim \frac{C_f}{1 - 2^{-\delta_0}} + \sum_{|I| \leq 5} \|\langle |x| \rangle \log(2 + |x|) \Gamma^I u(0, x)\|_{L_2^5}. 
$$

(2.29)

Proof. From $\text{supp}\psi_0 \cap \mathbb{R}_+ = [0, 2]$ and $\text{supp}\psi_j = [2^{j-1}, 2^{j+1}]$ for $j \geq 1$, we infer

$$
\sum_{|I| \leq 5} \|\langle |x| \rangle \psi_j(\langle |x| \rangle) \Gamma^I u(0, x)\|_{L_2^5} \lesssim \sum_{|I| \leq 5} \|\langle |x| \rangle \log(2 + |x|) \Gamma^I u(0, x)\|_{L_2^5}
$$

$$
\lesssim \frac{1}{(j + 1)} \sum_{|I| \leq 5} \|\langle |x| \rangle \log(2 + |x|) \psi_j^\frac{1}{2}(\langle |x| \rangle) \Gamma^I u(0, x)\|_{L_2^5} 
$$

for $j \geq 0$.

Based on the above estimates, the Cauchy-Schwarz inequality and $\sum \psi_j = 1$,

$$
\sum_{j=0}^{\infty} \sum_{|I| \leq 5} \|\langle |x| \rangle \psi_j(\langle |x| \rangle) \Gamma^I u(0, x)\|_{L_2^5} \lesssim \sum_{|I| \leq 5} \|\langle |x| \rangle \log(2 + |x|) \Gamma^I u(0, x)\|_{L_2^5}. 
$$

(2.30)

On the other hand, using again the definition of $\psi_j$ and (2.28), for $j \geq 0$, we have

$$
\max_{0 \leq s \leq t} \psi_j(s) \|\langle s + |x| \rangle \Gamma^I f(s, x)\|_{L_2^5} \lesssim C_f \max_{0 \leq s \leq t} \psi_j(s) \langle s \rangle^{-\delta_0} \lesssim C_f 2^{-j\delta_0},
$$

which implies

$$
\sum_{j=0}^{\infty} \sum_{0 \leq s \leq t} \max_{0 \leq s \leq t} \psi_j(s) \|\langle s + |x| \rangle \Gamma^I f(s, x)\|_{L_2^5} \lesssim C_f \sum_{j=0}^{\infty} 2^{-j\delta_0} \lesssim \frac{C_f}{1 - 2^{-\delta_0}}.
$$

(2.31)

We see that (2.29) follows from (2.30) and (2.31).
Similar to the case of 2D linear wave equation, we introduce the following standard
energy \( E_1 \) for the 2D linear Klein-Gordon equation,
\[
E_1(t, u) = \int_{\mathbb{R}^2} \left( (\partial_t u)^2 + (\partial_1 u)^2 + (\partial_2 u)^2 + u^2 \right) (t, x) \text{d}x.
\]
Now we introduce the energy estimates for 2D Klein-Gordon equation.

**Lemma 2.14.** Let \( u \) be the solution to the Cauchy problem
\[
\begin{align*}
\begin{cases}
-\Box + 1 & u(t, x) = f(t, x), \\
(u(0, x), \partial_t u(0, x)) & = (u_0(x), u_1(x)),
\end{cases}
\end{align*}
\]
with \( f(t, x) \) a sufficiently regular function. Then the following estimates hold.
(i) Standard energy estimate. We have
\[
E_1(t, u) \lesssim \int_0^t \|f(s, x)\|_{L^2_x} \text{d}s. \tag{2.32}
\]
(ii) Ghost weight estimate (see also [1]). For all \( \delta_0 > 0 \), we have
\[
\sum_{i=1, 2} \int_0^t (s)^{-\delta_0} \int_{\mathbb{R}^2} \frac{u^2}{(r-s) ^2} + \frac{|G_i u|^2}{(r-s) ^2} \text{d}x \text{d}s \lesssim E_1(0, u) + \int_0^t (s)^{-\delta_0} \|f(s, x)\|_{L^2_x} \|\partial_t u(s, x)\|_{L^2_x} \text{d}s. \tag{2.33}
\]

**Proof.** Proof of (i). The proof is similar to Lemma 2.9 (i), and we omit it.
Proof of (ii). Set
\[
q(t, r) = \int_{-\infty}^{r-t} (s)^{-\frac{3}{2}} \text{d}s \quad \text{for } (t, r) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]
We apply the multiplier \((t)^{-\delta_0} e^{\delta u} \partial_t u\) and obtain:
\[
\begin{align*}
&(t)^{-\delta_0} e^{\delta u} \partial_t u (-\Box u + u) \\
&= \frac{1}{2} \partial_t \left( (t)^{-\delta_0} e^{\delta u} \left( \sum_{\alpha=0}^3 (\partial_\alpha u)^2 + u^2 \right) \right) - \partial^i \left( (t)^{-\delta_0} e^{\delta u} \partial_t u \partial_i u \right) \\
&\quad + \frac{1}{2} \frac{1}{(t-r) ^2} \left( (G_1 u)^2 + (G_2 u)^2 + u^2 \right) + \frac{\delta_0}{2} \frac{1}{t} \left( t \right)^{-\delta_0} (\delta_0 + 2) e^{3/2} \left( \sum_{\alpha=0}^3 (\partial_\alpha u)^2 + u^2 \right).
\end{align*}
\]
Integrating the above identity on \([0, t] \times \mathbb{R}^2\) and then using the Cauchy-Schwarz inequality, we obtain (2.33).

\[ \square \]

**3. Proof of Theorem 1.1**

In this section, we prove the existence of global-in-time solution \((w, v)\) of (1.1)
satisfying (1.4) in Theorem 1.1. The proof relies on a bootstrap argument of high-
order energy and pointwise decay of solutions.

**3.1. Bootstrap setting.** Fix \(0 < \delta \ll 1\). The proof of Theorem 1.1 is based on
the following bootstrap setting: for \(C_0 \gg 1\) and \(0 < \varepsilon \ll C_0^{-1}\) to be chosen later
\[
\begin{align}
\begin{align*}
&\left< t \right>^{\frac{\delta}{2}} \|w\|_{L_x^\infty} + \sum_{|I| \leq N-1} \mathcal{E}(t, \Gamma I^w) + \sum_{|I| \leq N} \left< t \right>^{-\delta} \mathcal{E}(t, \Gamma I^w) \leq C_0 \varepsilon, \\
&\sum_{|I| \leq N} \left< t \right>^{-\frac{\delta}{2} - 2\delta} \|w\|_{L_x^2} + \sum_{|I| \leq N} \left< t \right>^{-\delta} \|\text{ST}^I w\|_{L_x^2} \leq C_0 \varepsilon, \\
&\sum_{|I| \leq N} \left< t \right>^{-\delta} \|\Gamma I^w\|_{L_x^2} + \sup_{|I| \leq N-9} \sup_{x \in \mathbb{R}^2} \left< t - r \right>^{\frac{\delta}{2}} \left| \partial \Gamma I^w(t, x) \right| \leq C_0 \varepsilon,
\end{align*}
\end{align}
\]
\[ \tag{3.1} \]
For all initial data \((\vec{w}_0, \vec{v}_0)\) satisfying (3.1) and (3.2) on \([0, t]\), we have \(T_*(\vec{w}_0, \vec{v}_0) = \infty\).

Note that, combining the bootstrap setting (3.1) and (3.2), Theorem 1.1 is a consequence of Proposition 3.1. The rest of the section is devoted to the proof of Proposition 3.1. From now on, the implied constants in \(\lesssim\) do not depend on the constants \(C_0\) and \(\varepsilon\) appearing in the bootstrap assumption (3.1) and (3.2).

Note also that, from the estimate on commutator (2.1), the global Sobolev inequality (2.10) and the bootstrap assumption (3.1), for all \(t \in [0, T_*(\vec{w}_0, \vec{v}_0))\), we have the following \(L^\infty\) estimates related to the wave component \(w\),

\[
\sum_{|I| \leq N-3} \frac{\langle t \rangle^{-\delta}}{|t|} \| \Gamma^I w(t, x) \|_{L^\infty_x} + \sum_{|I| \leq N-9} \frac{(t)^{-\delta}}{|t|} \| \Sigma^I w(t, x) \|_{L^\infty_x} \lesssim C_0 \varepsilon \langle t \rangle^{-\frac{1}{4}},
\]

\[
\sum_{|I| \leq N-4} \| \partial \Gamma^I w(t, x) \|_{L^\infty_x} + \sum_{|I| \leq N-4} \langle t \rangle^{-\frac{1}{4} - 2\delta} \| \Sigma^I w(t, x) \|_{L^\infty_x} \lesssim C_0 \varepsilon \langle t \rangle^{-\frac{1}{4}}.
\]

From (3.2) and the Hölder inequality, for all \(t \in [0, T_*(\vec{w}_0, \vec{v}_0))\), we also have

\[
\int_0^t \left( \frac{\langle t \rangle^{-\delta}}{|t|} \left( \left\| \frac{\Gamma^I v}{(s-r)^{\frac{\delta}{2}}} \right\|_{L^2_x}^2 + \left\| \frac{G_i \Gamma^I v}{(s-r)^{\frac{\delta}{2}}} \right\|_{L^2_x}^2 \right) \right) \, ds \lesssim \left( \int_0^t \langle s \rangle^{1+\delta} \left( \int_0^t \langle s \rangle^{-\delta} \left( \left\| \frac{\Gamma^I v}{(s-r)^{\frac{\delta}{2}}} \right\|_{L^2_x}^2 + \left\| \frac{G_i \Gamma^I v}{(s-r)^{\frac{\delta}{2}}} \right\|_{L^2_x}^2 \right) \, ds \right)^{\frac{1}{2}} \leq C_0 \varepsilon \langle t \rangle^{\frac{1}{2}} \varepsilon \langle t \rangle^\delta,
\]

for \(I \in \mathbb{N}^6\) with \(|I| \leq N\) and \(i = 1, 2\).

### 3.2. Estimates on the nonlinear terms

In this subsection, we establish the estimates on the nonlinear terms that will be needed in the proof of Proposition 3.1. We start with the technical estimates related to \(\Gamma^I Q\).

**Lemma 3.2.** For all \(t \in [0, T_*(\vec{w}_0, \vec{v}_0))\), the following estimates hold.

(i) Weight \(L^2\) estimates on \(\Gamma^I Q\). We have

\[
\sum_{|I| \leq N-1} \left\| (t+r)^I Q(w, v) \right\|_{L^2_x} \lesssim C_0^2 \varepsilon^2 \langle t \rangle^{-\frac{1}{2} + 2\delta}.
\]

(ii) \(L^1_tL^2_x\) estimates on \(\Gamma^I Q\). We have

\[
\sum_{|I| \leq N-1} \int_0^t \left\| \Gamma^I Q(w, v) \right\|_{L^2_x} \, ds \lesssim C_0^2 \varepsilon^2,
\]

\[
\sum_{|I| \leq N} \int_0^t \left\| \Gamma^I Q(w, v) \right\|_{L^2_x} \, ds \lesssim C_0^2 \varepsilon^2 \langle t \rangle^\delta.
\]
(iii) Weighted $L^1_t L^2_x$ estimates on $\Gamma^f Q$. We have
\[
\sum_{|I| \leq N-6} \int_0^t \| (s+r)\Gamma^f Q(w,v) \|_{L^2_x} \, ds \lesssim C_0^2 \varepsilon^2 (t)^{\delta},
\]
\[
\sum_{|I| \leq N-1} \int_0^t \| (s+r)\Gamma^f Q(w,v) \|_{L^2_x} \, ds \lesssim C_0^2 \varepsilon^2 (t)^{\frac{3}{4}+2\delta}.
\]

**Proof.** Proof of (i). From (2.6) and $N \geq 14$, we see that
\[
\sum_{|I| \leq N-1} \| (t+r)\Gamma^f Q(w,v) \|_{L^2_x} \lesssim J_{11}(t) + J_{12}(t) + J_{13}(t) + J_{14}(t),
\]
where
\[
J_{11} = \sum_{|I| \leq N-3} \sum_{|I_2| \leq N} \| \Gamma^{I_1} w \Gamma^{I_2} v \|_{L^2_x}, \quad J_{12} = \sum_{|I_1| \leq N} \sum_{|I_2| \leq N-5} \| \Gamma^{I_1} w \Gamma^{I_2} v \|_{L^2_x},
\]
\[
J_{13} = \sum_{|I| \leq N-9} \sum_{|I_2| \leq N} \| \Sigma^{I_1} w \Gamma^{I_2} v \|_{L^2_x}, \quad J_{14} = \sum_{|I_1| \leq N-1} \sum_{|I_2| \leq N-5} \| \Sigma^{I_1} w \Gamma^{I_2} v \|_{L^2_x}.
\]
Using (3.1), (3.2) and (3.4), we have
\[
J_{11}(t) \lesssim \sum_{|I| \leq N-3} \sum_{|I_2| \leq N} \| \Gamma^{I_1} w \|_{L^\infty_x} \| \Gamma^{I_2} v \|_{L^2_x} \lesssim C_0^2 \varepsilon^2 t^{-\frac{1}{4}+2\delta},
\]
\[
J_{12}(t) \lesssim \sum_{|I_1| \leq N} \sum_{|I_2| \leq N-5} \| \Gamma^{I_1} w \|_{L^2_x} \| \Gamma^{I_2} v \|_{L^\infty_x} \lesssim C_0^2 \varepsilon^2 t^{-1+\delta},
\]
\[
J_{13}(t) \lesssim \sum_{|I| \leq N-9} \sum_{|I_2| \leq N} \| \Sigma^{I_1} w \|_{L^\infty_x} \| \Gamma^{I_2} v \|_{L^2_x} \lesssim C_0^2 \varepsilon^2 t^{-\frac{1}{4}+2\delta},
\]
\[
J_{14}(t) \lesssim \sum_{|I_1| \leq N-1} \sum_{|I_2| \leq N-5} \| \Sigma^{I_1} w \|_{L^2_x} \| \Gamma^{I_2} v \|_{L^\infty_x} \lesssim C_0^2 \varepsilon^2 t^{-\frac{1}{4}+2\delta}.
\]
Combining the above estimates, we obtain (3.6).

Proof of (ii). First, from (3.6) and $0 < \delta \ll 1$, we have
\[
\sum_{|I| \leq N-1} \| \Gamma^f Q(w,v) \|_{L^2_x} \lesssim (t)^{-1} \sum_{|I| \leq N-1} \| (t+r)\Gamma^f Q(w,v) \|_{L^2_x} \lesssim C_0^2 \varepsilon^2 (t)^{-\frac{3}{4}},
\]
which implies (3.7). Second, from (2.3), (2.5) and $N \geq 14$, we infer
\[
\sum_{|I| \leq N} \int_0^t \| \Gamma^f Q(w,v) \|_{L^2_x} \, ds \lesssim J_{21}(t) + J_{22}(t) + J_{23}(t) + J_{24}(t),
\]
where
\[
J_{21}(t) = \sum_{i=1,2} \sum_{|I| \leq N-9} \sum_{|I_1|+|I_2| \leq N} \int_0^t \| (G_i \Gamma^{I_1} w) (\partial \Gamma^{I_2} v) \|_{L^2_x} \, ds,
\]
\[
J_{22}(t) = \sum_{i=1,2} \sum_{|I_1| \leq N-6} \sum_{|I|+|I_2| \leq N} \int_0^t \| (G_i \Gamma^{I_1} w) (\partial \Gamma^{I_2} v) \|_{L^2_x} \, ds,
\]
\[
J_{23}(t) = \sum_{i=1,2} \sum_{|I_2| \leq N-6} \sum_{|I|+|I_1| \leq N} \int_0^t \| (\partial \Gamma^{I_1} w) (G_i \Gamma^{I_2} v) \|_{L^2_x} \, ds,
\]
\[
J_{24}(t) = \sum_{i=1,2} \sum_{|I| \leq N-9} \sum_{|I_1|+|I_2| \leq N} \int_0^t \| (\partial \Gamma^{I_1} w) (G_i \Gamma^{I_2} v) \|_{L^2_x} \, ds.
\]
Using (2.2), (3.2), (3.4) and $0 < \delta \ll 1$, we have
\[
J_{21} \lesssim \sum_{|I| \leq N - 9 \atop |I_1| + |I_2| \leq N} \int_0^t (s)^{-1} \| \mathcal{S} \Gamma^I w \|_{L^\infty_x} \| \partial \Gamma^I v \|_{L^2_x} \, ds \\
+ \sum_{1 \leq |I| \leq N - 8 \atop |I_1| + |I_2| \leq N + 1} \int_0^t (s)^{-1} \| \Gamma^I w \|_{L^\infty_x} \| \partial \Gamma^I v \|_{L^2_x} \, ds \lesssim C^2_0 \varepsilon^2 \int_0^t (s)^{-1+\delta} \, ds \lesssim C^2_0 \varepsilon^2.
\]

Then, using again (3.1), (3.2), (3.4) and $|G_i| \lesssim |\partial|$, we infer
\[
J_{22} \lesssim \sum_{1 \leq |I| \leq N - 5 \atop |I_1| + |I_2| \leq N + 1} \int_0^t \| \partial \Gamma^I w \|_{L^2_x} \| \Gamma^I v \|_{L^\infty_x} \, ds \lesssim C^2_0 \varepsilon^2 \int_0^t (s)^{-1+\delta} \, ds \lesssim C^2_0 \varepsilon^2 (t)^{\delta},
\]
\[
J_{23} \lesssim \sum_{1 \leq |I| \leq N - 5 \atop |I_1| + |I_2| \leq N + 1} \int_0^t \| \partial \Gamma^I w \|_{L^2_x} \| \Gamma^I v \|_{L^\infty_x} \, ds \lesssim C^2_0 \varepsilon^2 \int_0^t (s)^{-1+\delta} \, ds \lesssim C^2_0 \varepsilon^2 (t)^{\delta}.
\]

Next, from (3.1) and (3.5), we see that
\[
J_{24} \lesssim \sum_{i=1,2} \sum_{|I| \leq N - 9 \atop |I_1| + |I_2| \leq N} \int_0^t \| \langle s - r \rangle^\frac{\delta}{4} (\partial \Gamma^I w) \|_{L^\infty_x} \| G_i \Gamma^I v \|_{L^2_x} \, ds \\
\lesssim C^2_0 \varepsilon \sum_{i=1,2} \int_0^t (s)^{-\frac{1}{2}} \| G_i \Gamma^I v \|_{L^2_x} \, ds \lesssim C^2_0 \varepsilon^2 (t)^{\delta}.
\]

Combining the above estimates, we obtain (3.8).

Proof of (iii). First, from (2.6), we have
\[
\sum_{|I| \leq N - 6} \int_0^t \langle s + \rho \rangle^\delta \mathcal{S} \Gamma Q(w, v) \|_{L^2_x} \lesssim J_{31}(t) + J_{32}(t),
\]
where
\[
J_{31}(t) = \sum_{|I| \leq N - 5 \atop |I_1| \leq N - 5} \sum_{|I_2| \leq N} \int_0^t \| \Gamma^I w \|_{L^2_x} \| \Gamma^I v \|_{L^\infty_x} \, ds,
\]
\[
J_{32}(t) = \sum_{|I| \leq N - 6 \atop |I_1| \leq N - 6} \sum_{|I_2| \leq N - 5} \int_0^t \| \mathcal{S} \Gamma^I w \|_{L^2_x} \| \Gamma^I v \|_{L^\infty_x} \, ds.
\]

Note that, using (3.1) and (3.2), we obtain
\[
J_{31}(t) + J_{32}(t) \lesssim C^2_0 \varepsilon^2 \int_0^t (s)^{-1+\delta} \, ds \lesssim C^2_0 \varepsilon^2 (t)^{\delta},
\]
which implies (3.9). Last, integrating (3.6) on $[0, t]$, we obtain (3.10).

Second, we introduce the $L^1_t L^2_x$ for the nonlinear terms that are related to hidden divergence type terms $F_\alpha$ and $H^\alpha$.

**Lemma 3.3.** For all $t \in [0, T, (\w_0, \v_0))$, the following estimates hold.

(i) $L^1_t L^2_x$ estimate for $\Gamma^I F_\alpha$ and $\Gamma^I H^\alpha$. We have
\[
\sum_{|I| \leq N} \int_0^t \left( \| \Gamma^I F_\alpha(w, v) \|_{L^2_x} + \| \Gamma^I H^\alpha(w, v) \|_{L^2_x} \right) \, ds \lesssim C^2_0 \varepsilon^2 (t)^{\delta}, \tag{3.11}
\]
(ii) $L^1_t L^2_x$ estimate for extra nonlinear terms. We have

\[
\sum_{|I| \leq N-5} \int_0^t \left( \| \Gamma^I Q (\partial w, v) \|_{L^2_x} + \| \Gamma^I (v^2 \partial w) \|_{L^2_x} \right) ds \lesssim C_0^2 \varepsilon^2,
\]

\[
\sum_{|I| \leq N-5} \int_0^t \left( \| \Gamma^I (v \partial Q (w, v)) \|_{L^2_x} + \| \Gamma^I (Q (w, v) \partial w) \|_{L^2_x} \right) ds \lesssim C_0^2 \varepsilon^2.
\]

(3.12)

Proof. Proof of (i). For all $\alpha = 0, 1, 2$, we claim that

\[
\sum_{|I| \leq N} \int_0^t \left( \| \Gamma^I (v \partial \alpha w) \|_{L^2_x} + \| \Gamma^I (v^2 \partial \alpha w) \|_{L^2_x} \right) ds \lesssim C_0^2 \varepsilon^2 \langle t \rangle^\delta.
\]

(3.13)

Indeed, from $N \geq 14$, we decompose

\[
\sum_{|I| \leq N} \int_0^t \left( \| \Gamma^I (v \partial \alpha w) \|_{L^2_x} + \| \Gamma^I (v^2 \partial \alpha w) \|_{L^2_x} \right) ds \lesssim J_{41} + J_{42} + J_{43} + J_{44},
\]

where

\[
J_{41} = \sum_{|I| \leq N} \sum_{|I_2| \leq N-5} \int_0^t \left( \| \Gamma^{I_1} (v) \right. \Gamma^{I_2} \partial \alpha w \left. \|_{L^2_x} \right) ds,
\]

\[
J_{42} = \sum_{|I| \leq N} \sum_{|I_2| \leq N-9} \int_0^t \left( \| \Gamma^{I_1} (v) \right. \Gamma^{I_2} \partial \alpha w \left. \|_{L^2_x} \right) ds,
\]

\[
J_{43} = \sum_{|I_1| + |I_2| \leq N-5} \int_0^t \left( \| \Gamma^{I_1} (v) \right. \Gamma^{I_2} \partial \alpha w \left. \|_{L^2_x} \right) ds,
\]

\[
J_{44} = \sum_{|I_1| \leq N-5, |I_2| \leq N} \int_0^t \left( \| \Gamma^{I_1} (v) \right. \Gamma^{I_2} \partial \alpha w \left. \|_{L^2_x} \right) ds.
\]

Then, using again (2.1), (3.1), (3.2), (3.4) and (3.5), we have

\[
J_{41} \lesssim \sum_{|I| \leq N-5} \int_0^t \| \Gamma^{I_1} v \|_{L^\infty_x} \| \partial \Gamma^{I_2} w \|_{L^2_x} ds \lesssim C_0^2 \varepsilon^2 \int_0^t \langle s \rangle^{-1+\delta} ds \lesssim C_0^2 \varepsilon^2 \langle t \rangle^\delta,
\]

\[
J_{42} \lesssim \sum_{|I| \leq N-9} \int_0^t \left\| \frac{\Gamma^{I_1} v}{\langle s-r \rangle^{\frac{1}{2}}} \right\|_{L^2_x} \left\| \langle s-r \rangle^{\frac{3}{2}} \partial \Gamma^{I_2} w \right\|_{L^\infty_x} ds \lesssim C_0 \varepsilon \int_0^t \langle s \rangle^{-\frac{1}{2}} \left\| \frac{\Gamma^{I_1} v}{\langle s-r \rangle^{\frac{3}{2}}} \right\|_{L^2_x} ds \lesssim C_0^2 \varepsilon^2 \langle t \rangle^\delta,
\]

\[
J_{43} \lesssim \sum_{|I_1| + |I_2| \leq N-5} \int_0^t \| \Gamma^{I_1} v \|_{L^\infty_x} \| \Gamma^{I_2} v \|_{L^\infty_x} \| \partial \Gamma^{I_2} w \|_{L^2_x} ds \lesssim C_0^3 \varepsilon^3 \int_0^t \langle s \rangle^{-1} \langle s \rangle^{-1} \langle s \rangle^{-\delta} ds \lesssim C_0^3 \varepsilon^3 \int_0^t \langle s \rangle^{-2+\delta} ds \lesssim C_0^3 \varepsilon^3,
\]

\[
J_{44} \lesssim \sum_{|I| \leq N-5, |I| \leq N-5} \int_0^t \| \Gamma^{I_1} v \|_{L^\infty_x} \| \Gamma^{I_2} v \|_{L^\infty_x} \| \partial \Gamma^{I_2} w \|_{L^\infty_x} ds \lesssim C_0^3 \varepsilon^3 \int_0^t \langle s \rangle^{-1} \langle s \rangle^{-\delta} ds \lesssim C_0^3 \varepsilon^3 \int_0^t \langle s \rangle^{-2+\delta} ds \lesssim C_0^3 \varepsilon^3.
\]
Combining the above estimates, we obtain (3.13). Last, we see that (3.11) follows from (3.13) and the definition of $F_\alpha$ and $H^n$ in Lemma 2.3.

Proof of (ii). The proof is similar to Lemma 3.2, and we omit it.

Last, we introduce the following technical estimates related to quartic term $\Gamma^4 G$.

**Lemma 3.4.** For all $t \in [0, T_\ast(\bar{w}_0, \bar{\nu}_0))$, the following estimates hold.

(i) $L_t^1 L_t^4$ estimate on $\Gamma^4 G$. We have

$$\sum_{|I| \leq N} \int_0^t \|\Gamma^4 G(w, v)\|_{L_t^4} \, ds \lesssim C_0^4 \varepsilon^4.$$

(ii) $L_t^2 L_t^4$ estimate on $\Gamma^4 G$. We have

$$\sum_{|I| \leq N} \int_0^t \|\Gamma^4 G(w, v)\|_{L_t^2} \, ds \lesssim C_0^4 \varepsilon^4.$$

(iii) Weighted $L_t^1 L_t^4$ estimate on ST$^4 G$. We have

$$\sum_{|I| \leq N-8} \int_0^t (1 + s)^{-\frac{1}{2}} \|\Gamma^4 G(w, v)\|_{L_t^4} \, ds \lesssim C_0^4 \varepsilon^4.$$

(iv) Weighted $L_t^1 L_t^4$ estimate on $S^2 \Gamma^4 G$. We have

$$\sum_{|I| \leq N-9} \int_0^t (1 + s)^{-\frac{1}{2}} \|S^2 \Gamma^4 G(w, v)\|_{L_t^1} \, ds \lesssim C_0^4 \varepsilon^4.$$

**Proof.** Proof of (i). For all $Q \in \{Q_0; Q_{\alpha \beta}, \alpha \neq \beta\}$, we claim that

$$\sum_{|I| \leq N} \int_0^t \|\Gamma^4 (v^2 Q(w, v))\|_{L_t^4} \, ds \lesssim C_0^4 \varepsilon^4.$$

Indeed, from the definition of $Q$ and $N \geq 14$, we infer

$$\sum_{|I| \leq N} \int_0^t \|\Gamma^4 (v^2 Q(w, v))\|_{L_t^4} \, ds \lesssim J_{51} + J_{52},$$

where

$$J_{51} = \sum_{|I| \geq N} \sum_{|I| \leq N} \int_0^t \int_{\mathbb{R}^2} \left| (\Gamma^{I_1} v) (\Gamma^{I_2} v) (\partial \Gamma^{I_3} w) (\partial \Gamma^{I_4} v) \right| \, dx \, ds,$$

$$J_{52} = \sum_{|I| \geq N} \sum_{|I| \leq N} \int_0^t \int_{\mathbb{R}^2} \left| (\Gamma^{I_1} v) (\Gamma^{I_2} v) (\partial \Gamma^{I_3} w) (\partial \Gamma^{I_4} v) \right| \, dx \, ds.$$

From (3.1), (3.2), (3.4) and the Hölder inequality, we have

$$J_{51} \lesssim \sum_{|I| \geq N} \sum_{|I| \leq N} \int_0^t \|\Gamma^{I_1} v\|_{L_t^4} \|\Gamma^{I_2} v\|_{L_t^{2\infty}} \|\partial \Gamma^{I_3} w\|_{L_t^2} \|\partial \Gamma^{I_4} v\|_{L_t^{2\infty}} \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-2+2\delta} \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-2+2\delta} \, ds \lesssim C_0^4 \varepsilon^4.$$
\[ J_{02} \lesssim \sum_{|I| \leq N} \sum_{|I_1| \leq N-5} \int_0^t \| \Gamma^{I_1} v \|_{L^2_x} \| \Gamma^{I_2} v \|_{L^\infty_x} \| \partial \Gamma^{I_3} w \|_{L^\infty_x} \| \partial \Gamma^{I_4} v \|_{L^2_x} \, ds \]

\[ \lesssim C_0^4 \varepsilon^4 \int_0^t \langle \delta(s) \rangle^{-1} \langle s \rangle^{-\frac{3}{2}} \langle s \rangle^\delta \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-\frac{3}{2} + 2\delta} \lesssim C_0^4 \varepsilon^4, \]

which implies (3.18). Last, we see that (3.14) follows from (3.18) and the definition of \( G(w, v) \) in Lemma 2.3.

Proof of (ii). For all \( Q \in \{ Q_0; Q_{\alpha \beta}, \alpha \neq \beta \} \), we claim that

\[ \sum_{|I| \leq N} \int_0^t \| \Gamma^{I} (v^2 Q(w, v)) \|_{L^2_x} \, ds \lesssim J_{01} + J_{02} + J_{03} + J_{04}, \]

Indeed, from the definition of \( Q \), we infer

\[ \sum_{|I| \leq N} \int_0^t \| \Gamma^{I} (v^2 Q(w, v)) \|_{L^2_x} \, ds \lesssim J_{01} + J_{02} + J_{03} + J_{04}, \]

where

\[ J_{01} = \sum_{0 \leq |I|, |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \, ds, \]

\[ J_{02} = \sum_{0 \leq |I|, |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \, ds, \]

\[ J_{03} = \sum_{0 \leq |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \, ds, \]

\[ J_{04} = \sum_{0 \leq |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \, ds. \]

From (3.1), (3.2) and (3.4), we see that

\[ J_{01} = \sum_{0 \leq |I|, |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \|_{L^2_x} \| \Gamma^{I_2} \|_{L^\infty_x} \| \partial \Gamma^{I_3} w \|_{L^\infty_x} \| \partial \Gamma^{I_4} v \|_{L^2_x} \, ds \]

\[ \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^\delta \langle s \rangle^{-1} \langle s \rangle^{-\frac{3}{2}} \langle s \rangle^{-1} \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{\frac{3}{2}} \, ds \lesssim C_0^4 \varepsilon^4, \]

\[ J_{02} = \sum_{0 \leq |I|, |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \| \partial \Gamma^{I_3} w \|_{L^2_x} \| \partial \Gamma^{I_4} v \|_{L^\infty_x} \, ds \]

\[ \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-1} \langle s \rangle^{-1} \langle s \rangle^\delta \langle s \rangle^{-1} \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-2} \, ds \lesssim C_0^4 \varepsilon^4, \]

\[ J_{03} = \sum_{0 \leq |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \| \Gamma^{I_1} \Gamma^{I_2} \Gamma^{I_3} \Gamma^{I_4} \|_{L^2_x} \| \partial \Gamma^{I_3} w \|_{L^\infty_x} \| \partial \Gamma^{I_4} v \|_{L^2_x} \, ds \]

\[ \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-1} \langle s \rangle^{-1} \langle s \rangle^{\frac{3}{2}} \langle s \rangle^{-1} \, ds \lesssim C_0^4 \varepsilon^4 \int_0^t \langle s \rangle^{-\frac{3}{2}} \, ds \lesssim C_0^4 \varepsilon^4, \]
\[ \mathcal{J}_{64} = \sum_{0 \leq |I_1|, |I_2|, |I_3|, |I_4| \leq N-6} \int_0^t \left\| \Gamma^{I_1}v \right\|_{L^\infty_x} \left\| \Gamma^{I_2}v \right\|_{L^\infty_x} \left\| \partial \Gamma^{I_3}w \right\|_{L^\infty_x} \left\| \partial \Gamma^{I_4}v \right\|_{L^2_x} ds \]
\[ \leq C_0^4 \varepsilon^4 \int_0^t (s)^{-1} (s)^{-1} (s)^{\frac{\delta}{2}} (s)^{\delta} ds \leq C_0^4 \varepsilon^4 \int_0^t (s)^{-\frac{3}{2}} (s)^{\delta} ds \leq C_0^4 \varepsilon^4. \]

Combining the above estimates, we obtain (3.19). Last, we see that (3.15) follows from (3.19) and the definition of \( G(w, v) \) in Lemma 2.3.

Proof of (iii). For all \( Q \in \{ Q_0; \ Q_{\alpha \beta}, \ \alpha \neq \beta \} \), we claim that
\[ \sum_{|I| \leq N-8} \left\| ST^I (v^2 Q(w, v)) \right\|_{L^2_x} \lesssim C_0^4 \varepsilon^4 (s)^{-1}, \quad \text{for all } s \in [0, T_*(\bar{w}_0, \bar{v}_0)). \] (3.20)

Indeed, from the inequality \(|S| \lesssim (s + r)|\partial|\), we infer
\[ \sum_{|I| \leq N-8} \left\| ST^I (v^2 Q(w, v)) \right\|_{L^1_x} \lesssim \sum_{|I| \leq N-7} \left\| (s + r)^I (v^2 Q(w, v)) \right\|_{L^1_x}. \]

Then, using again (2.6), we have
\[ \sum_{|I| \leq N-7} \left\| (s + r)^I (v^2 Q(w, v)) \right\|_{L^1_x} \lesssim \mathcal{J}_{71} + \mathcal{J}_{72}, \]
where
\[ \mathcal{J}_{71} = \sum_{0 \leq |I_1|, |I_2| \leq N-6} \left\| (\Gamma^{I_1}v) (\Gamma^{I_2}v) (\Gamma^{I_3}w) (\Gamma^{I_4}v) \right\|_{L^1_x}, \]
\[ \mathcal{J}_{72} = \sum_{0 \leq |I_1|, |I_2| \leq N-6} \left\| (\Gamma^{I_1}v) (\Gamma^{I_2}v) (ST^{I_3}w) (\Gamma^{I_4}v) \right\|_{L^1_x}. \]

Note that, from (3.1), (3.2) and the Hölder inequality, we have
\[ \mathcal{J}_{71} \lesssim \sum_{0 \leq |I_1|, |I_2| \leq N-6} \left\| \Gamma^{I_1}v \right\|_{L^\infty_x} \left\| \Gamma^{I_2}v \right\|_{L^\infty_x} \left\| \Gamma^{I_3}w \right\|_{L^2_x} \left\| \Gamma^{I_4}v \right\|_{L^2_x} \]
\[ \lesssim C_0^4 \varepsilon^4 (s)^{-1} (s)^{-1} (s)^{\delta} (s)^{\delta} \lesssim C_0^4 \varepsilon^4 (s)^{-2 + \frac{2\delta}{s}} \lesssim C_0^4 \varepsilon^4 (s)^{-1}, \]
\[ \mathcal{J}_{72} \lesssim \sum_{0 \leq |I_1|, |I_2| \leq N-6} \left\| \Gamma^{I_1}v \right\|_{L^\infty_x} \left\| \Gamma^{I_2}v \right\|_{L^\infty_x} \left\| ST^{I_3}w \right\|_{L^2_x} \left\| \Gamma^{I_4}v \right\|_{L^2_x} \]
\[ \lesssim C_0^4 \varepsilon^4 (s)^{-1} (s)^{\delta} (s)^{\delta} \lesssim C_0^4 \varepsilon^4 (s)^{-2 + \frac{2\delta}{s}} \lesssim C_0^4 \varepsilon^4 (s)^{-1}, \]
which implies (3.20). Last, multiplying both sides of (3.20) by \((1 + s)^{-\frac{\delta}{2}}\) and then integrating on \([0, t]\), we obtain (3.16).

Proof of (iv). For all \( Q \in \{ Q_0; \ Q_{\alpha \beta}, \ \alpha \neq \beta \} \), we claim that
\[ \sum_{|I| \leq N-9} \left\| S^2 \Gamma^I (v^2 Q(w, v)) \right\|_{L^2_x} \lesssim C_0^4 \varepsilon^4 (s)^{-1-\frac{2\delta}{s}}, \quad \text{for all } s \in [0, T_*). \] (3.21)

Indeed, from the inequality \(|S^2| \lesssim (s + r)^2|\partial|\), we infer
\[ \sum_{|I| \leq N-9} \left\| S^2 \Gamma^I (v^2 Q(w, v)) \right\|_{L^1_x} \lesssim \sum_{|I| \leq N-7} \left\| (s + r)^2 \Gamma^I (v^2 Q(w, v)) \right\|_{L^1_x}. \]

Then, using again (2.6), we have
\[ \sum_{|I| \leq N-7} \left\| (s + r)^2 \Gamma^I (v^2 Q(w, v)) \right\|_{L^1_x} \lesssim \mathcal{J}_{81} + \mathcal{J}_{82}, \]
where
\[ J_1 = \sum_{0 \leq |I_{1}|, |I_{2}| \leq N-6, 0 \leq |I_{3}|, |I_{4}| \leq N-6} \| (s + r) (\Gamma^{I_{1} v}) (\Gamma^{I_{2} w}) (\Gamma^{I_{3} v}) (\Gamma^{I_{4} v}) \|_{L^{2}_{x}}. \]
\[ J_2 = \sum_{0 \leq |I_{1}|, |I_{2}| \leq N-6, 0 \leq |I_{3}|, |I_{4}| \leq N-6} \| (s + r) (\Gamma^{I_{1} v}) (\Gamma^{I_{2} w}) (ST^{I_{3}} w) (\Gamma^{I_{4} v}) \|_{L^{2}_{x}}. \]

Note that, from (3.1), (3.2) and the Hölder inequality, we have
\[ J_1 \lesssim \sum_{0 \leq |I_{1}|, |I_{2}| \leq N-6, 0 \leq |I_{3}|, |I_{4}| \leq N-6} \| (s + r) \Gamma^{I_{1} v} \|_{L^{2}_{x}} \| \Gamma^{I_{2} w} \|_{L^{2}_{x}} \| \Gamma^{I_{3} w} \|_{L^{2}_{x}} \| \Gamma^{I_{4} v} \|_{L^{2}_{x}} \lesssim C_0 \varepsilon A(s)^{-1} \langle s \rangle^\delta \langle \delta \rangle^\delta \lesssim C_0 \varepsilon A(s)^{-1+2\delta}, \]
\[ J_2 \lesssim \sum_{0 \leq |I_{1}|, |I_{2}| \leq N-6, 0 \leq |I_{3}|, |I_{4}| \leq N-6} \| (s + r) \Gamma^{I_{1} v} \|_{L^{2}_{x}} \| \Gamma^{I_{2} w} \|_{L^{2}_{x}} \| ST^{I_{3}} w \|_{L^{2}_{x}} \| \Gamma^{I_{4} v} \|_{L^{2}_{x}} \lesssim C_0 \varepsilon A(s)^{-1} \langle s \rangle^\delta \langle \delta \rangle^\delta \lesssim C_0 \varepsilon A(s)^{-1+2\delta}, \]

which implies (3.21). Last, multiplying both sides of (3.21) by \((1 + s)^{-\frac{1}{2}}\) and then integrating on \([0, t]\), we obtain (3.17).

3.3. **End of the proof of Proposition 3.1.** We are in a position to complete the proof of Proposition 3.1.

**Proof of Proposition 3.1.** For all \((\bar{w}_0, \bar{v}_0)\) satisfying the smallness condition (1.3), we consider the corresponding solution \((w, v)\) of (1.1). From the definition of initial data, we have
\[ \sum_{|I| \leq N} \left( |E(0, \Gamma^I w)^{\frac{1}{2}} + G(0, \Gamma^I w)^{\frac{1}{2}} + E_1(0, \Gamma^I v)^{\frac{1}{2}} \right) \lesssim \varepsilon. \] (3.22)

**Step 1.** Closing the estimates in \(E(t, \Gamma^I w)\) and \(E_1(t, \Gamma^I v)\). Using (1.1), (2.21), (2.32), (3.7), (3.8) and (3.22), we deduce that
\[ \sum_{|I| \leq N-1} \left( |E(t, \Gamma^I w)^{\frac{1}{2}} + E_1(t, \Gamma^I v)^{\frac{1}{2}} \right) \lesssim \sum_{|I| \leq N-1} \left( |E(0, \Gamma^I w)^{\frac{1}{2}} + E_1(0, \Gamma^I v)^{\frac{1}{2}} \right) + \sum_{|I| \leq N-1} \int_0^t \| \Gamma^I Q_0(w, v) \|_{L^2_x} ds + \sum_{|I| \leq N-1} \sum_{|J| \leq N-1} \int_0^t \| \Gamma^I Q_{\alpha\beta}(w, v) \|_{L^2_x} ds \lesssim \varepsilon + C_0^2 \varepsilon^2 \text{ on } [0, T^*(\bar{w}_0, \bar{v}_0)), \]
and
\[ \sum_{|I| \leq N} \left( |E(t, \Gamma^I w)^{\frac{1}{2}} + E_1(t, \Gamma^I v)^{\frac{1}{2}} \right) \lesssim \sum_{|I| \leq N} \left( |E(0, \Gamma^I w)^{\frac{1}{2}} + E_1(0, \Gamma^I v)^{\frac{1}{2}} \right) + \sum_{|I| \leq N} \int_0^t \| \Gamma^I Q_0(w, v) \|_{L^2_x} ds + \sum_{|I| \leq N} \sum_{|J| \leq N} \int_0^t \| \Gamma^I Q_{\alpha\beta}(w, v) \|_{L^2_x} ds \lesssim \varepsilon + C_0^2 \varepsilon^2 (t)^\delta \text{ on } [0, T^*(\bar{w}_0, \bar{v}_0)). \]

These strictly improve the bootstrap estimates of \(E(t, \Gamma^I w)\) and \(E_1(t, \Gamma^I v)\) in (3.1) and (3.2) for \(C_0\) large enough and \(\varepsilon\) small enough (depending on \(C_0\)).
Step 2. Closing the estimates in $\|\Gamma^I w\|_{L^2_x}$. We decompose the wave component $w$ as (see also [20])

$$w = \Upsilon_0 + \Upsilon_1 + \partial^\alpha \Psi_\alpha + \partial^\alpha \Phi^\alpha,$$

where $\Upsilon_0$, $\Upsilon_1$, $\Psi_\alpha$ and $\Phi^\alpha$ are the solutions for the following 2D linear homogeneous or inhomogeneous wave equations,

$$\begin{cases}
-\Box \Upsilon_0 = 0, & \text{with } (\Upsilon_0, \partial_t \Upsilon_0)_{|t=0} = (w_0, w_1), \\
-\Box \Upsilon_1 = G(w, v), & \text{with } (\Upsilon_1, \partial_t \Upsilon_1)_{|t=0} = (0, 0), \\
-\Box \Psi_\alpha = F_\alpha(w, v), & \text{with } (\Psi_\alpha, \partial_t \Psi_\alpha)_{|t=0} = (0, 0), \\
-\Box \Phi^\alpha = H^\alpha(w, v), & \text{with } (\Phi^\alpha, \partial_t \Phi^\alpha)_{|t=0} = (0, 0).
\end{cases} \tag{3.23}$$

First, from the smallness condition (1.3),

$$\sum_{|I| \leq N} \left( \| \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} + \| \partial_t \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} + \| \partial_t \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} \right) \lesssim \varepsilon.$$

Thus, using (2.12), for all $I \in \mathbb{N}^6$ with $|I| \leq N$, we see that

$$\begin{align*}
\| \Gamma^I \Upsilon_0(t, x) \|_{L^2_x} & \lesssim \| \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} + \log^\frac{3}{2}(2 + t) \| \partial_t \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} \\
& \quad + \log^\frac{3}{2}(2 + t) \| \partial_t \Gamma^I \Upsilon_0(0, x) \|_{L^2_x} \lesssim \varepsilon \log^\frac{3}{2}(2 + t).
\end{align*} \tag{3.24}$$

Second, from (2.19), (3.14) and (3.15), for all $I \in \mathbb{N}^6$ with $|I| \leq N$, we see that

$$\begin{align*}
\| \Gamma^I \Upsilon_1 \|_{L^2_x} & \lesssim \log^\frac{3}{2}(2 + t) \int_0^t \| \Gamma^I G(w, v) \|_{L^2_x} ds \\
& \quad + \log^\frac{3}{2}(2 + t) \int_0^t \| \Gamma^I G(w, v) \|_{L^2_x} ds \lesssim C_0^4 \varepsilon^4 \log^\frac{3}{2}(2 + t). \tag{3.25}
\end{align*}$$

Then, from (2.21), (3.11), for all $I \in \mathbb{N}^6$ with $|I| \leq N$, we infer

$$\begin{align*}
\sum_{\alpha=0}^2 \mathcal{E}(t, \Gamma^I \Psi_\alpha)^\frac{3}{2} & \lesssim \int_0^t \| \Gamma^I F_\alpha(w, v) \|_{L^2_x} ds \lesssim C_0^4 \varepsilon^4 \langle t \rangle^\delta, \\
\sum_{\alpha=0}^2 \mathcal{E}(t, \Gamma^I \Phi^\alpha)^\frac{3}{2} & \lesssim \int_0^t \| \Gamma^I H^\alpha(w, v) \|_{L^2_x} ds \lesssim C_0^4 \varepsilon^4 \langle t \rangle^\delta.
\end{align*}$$

Based on the above estimates, for all $I \in \mathbb{N}^6$ with $|I| \leq N$, we see that

$$\begin{align*}
\sum_{\alpha=0}^2 \| \partial_t \Gamma^I \Psi_\alpha \|_{L^2_x} & \lesssim \sum_{\alpha=0}^2 \mathcal{E}(t, \Gamma^I \Psi_\alpha)^\frac{3}{2} \lesssim C_0^4 \varepsilon^4 \langle t \rangle^\delta, \\
\sum_{\alpha=0}^2 \| \partial_t \Gamma^I \Phi^\alpha \|_{L^2_x} & \lesssim \sum_{\alpha=0}^2 \mathcal{E}(t, \Gamma^I \Phi^\alpha)^\frac{3}{2} \lesssim C_0^4 \varepsilon^4 \langle t \rangle^\delta. \tag{3.26}
\end{align*}$$

Combining (2.1), (3.24), (3.25) and (3.26), we obtain

$$\begin{align*}
\sum_{|I| \leq N} \| \Gamma^I w \|_{L^2_x} & \lesssim \sum_{\alpha=0}^2 \sum_{k=0,1} \sum_{|I| \leq N} \left( \| \Gamma^I \Upsilon_k \|_{L^2_x} + \| \partial_t \Gamma^I \Psi_\alpha \|_{L^2_x} + \| \partial_t \Gamma^I \Phi^\alpha \|_{L^2_x} \right) \\
& \lesssim \left( \varepsilon + C_0^4 \varepsilon^4 \right) \log^\frac{3}{2}(2 + t) + C_0^4 \varepsilon^4 \langle t \rangle^\delta \lesssim \left( \varepsilon + C_0^4 \varepsilon^4 \right) \langle t \rangle^\delta.
\end{align*} \tag{3.27}$$

This strictly improves the bootstrap estimates of $\| \Gamma^I w \|_{L^2_x}$ in (3.1) for $C_0$ large enough and $\varepsilon$ small enough (depending on $C_0$).
Step 3. Closing the estimates in $\|\Sigma^I w\|_{L^2_\infty}$. From (2.22), (3.9), (3.10) and (3.22), we see that

$$
\sum_{|I| \leq N-6} G(t, \Gamma^I w)^{\frac{1}{2}} \lesssim \sum_{|I| \leq N-6} G(0, \Gamma^I w)^{\frac{1}{2}} + \sum_{|I| \leq N-6} \int_0^t \|\langle s + r \rangle \Gamma^I Q_0(w, v)\|_{L^2_\infty} ds + \sum_{|I| \leq N-6} \sum_{\alpha \neq \beta} \int_0^t \|\langle s + r \rangle \Gamma^I K_{\alpha\beta}(w, v)\|_{L^2_\infty} ds \lesssim (\varepsilon + C_0^2 \varepsilon^2) \langle t \rangle^\delta.
$$

Combining the above estimates with (3.27), we obtain

$$
\sum_{|I| \leq N-6} \|\Sigma^I w\|_{L^2_\infty} \lesssim \sum_{|I| \leq N-6} G(t, \Gamma^I w)^{\frac{1}{2}} + \sum_{|I| \leq N-6} \|\Gamma^I w\|_{L^2_\infty} \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^\delta.
$$

These strictly improve the bootstrap estimates of $\|\Sigma^I w\|_{L^2_\infty}$ in (3.1) for $C_0$ large enough and $\varepsilon$ small enough (depending on $C_0$).

Step 4. Closing the estimate in $|\partial \Gamma^I w|$. We split the proof into two parts according to different spacetime regions.

Case I: Let $(t, x) \in \{(t, x) \in \mathbb{R}^{1+2} : r \geq 2t\}$. From the Step 2 and Step 3, we have

$$
\sum_{|I| \leq N-10} \left(\|\Sigma^I w\|_{L^2_\infty} + \|\Gamma^I w\|_{L^2_\infty}\right) \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2} + \delta}.
$$

Therefore, from (2.1) and (2.10), we see that

$$
\sum_{|I| \leq N-9} \left(\|\Sigma^I w\|_{L^2_\infty} + \|\Gamma^I w\|_{L^2_\infty}\right) \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2} + \frac{\delta}{2}}.
$$

Based on (2.2), the above estimate and $r \geq 2t$, we have

$$
\sum_{|I| \leq N-9} |\partial \Gamma^I w(t, x)| \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2} + \delta} (t-r)^{-\frac{1}{4}} \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2}} (t-r)^{-\frac{3}{4}},
$$

which strictly improves the estimate of $|\partial \Gamma^I w|$ in (3.1) for $C_0$ large enough and $\varepsilon$ small enough (depend on $C_0$).

Case II: Let $(t, x) \in \{(t, x) : r \leq 2t\}$. Similar as in Step 2, we decompose the wave components $w$ as

$$
w = \tilde{Y}_0 + \tilde{Y}_1 + \partial^\alpha \Psi_\alpha + \partial_\alpha \Phi^\alpha.
$$

We claim that

$$
\sum_{|I| \leq N-9} \left(|\partial \Gamma^I \tilde{Y}_0(t, x)| + |\partial \Gamma^I \tilde{Y}_1(t, x)|\right) \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2}} (t-r)^{-\frac{3}{4}},
$$

$$
\sum_{|I| \leq N-9} \sum_{\alpha = 0, 1, 2} \left(|\partial \Gamma^I \Psi_\alpha(t, x)| + |\partial \Gamma^I \Phi^\alpha(t, x)|\right) \lesssim \left(\varepsilon + C_0^2 \varepsilon^2\right) \langle t \rangle^{-\frac{1}{2}} (t-r)^{-\frac{3}{4}}.
$$

(3.28)
Estimate on $\partial^I \Psi_0$. Indeed, from the smallness condition (1.3) and (3.23),
\begin{align*}
\sum_{|I| \leq N-8} (\|\Gamma^I \Psi_0(0, x)\|_{W^{2,1}} + \|\partial_1 \Gamma^I \Psi_0(0, x)\|_{W^{1,1}}) \lesssim \varepsilon,
\sum_{|I| \leq N-8} (\|\partial T^I \Psi_0(0, x)\|_{W^{2,1}} + \|\partial_1 \partial T^I \Psi_0(0, x)\|_{W^{1,1}}) \lesssim \varepsilon.
\end{align*}
Thus, from $[\Box, S] = 2\Box$, $[\Box, \Gamma_\alpha] = 0$, (2.13) and (3.23) for $\Psi_0$, we see that
\begin{equation}
\sum_{|I| \leq N-8} \left( \|\partial^I \Psi_0\|_{L^\infty} + \|\Gamma^I \Psi_0\|_{L^\infty} \right) \lesssim \varepsilon(t)^{-\frac{1}{2}}. \tag{3.29}
\end{equation}
Based on the above estimates and (2.2), we see that
\begin{align*}
\langle t - r \rangle \sum_{|I| \leq N-9} |\partial^I \Psi_0(t, x)| & \lesssim \sum_{|I| \leq N-8} \|\partial^I \Psi_0\|_{L^\infty} + \sum_{|I| \leq N-8} \|\Gamma^I \Psi_0\|_{L^\infty} \lesssim \varepsilon(t)^{-\frac{1}{2}}. \tag{3.30}
\end{align*}

Estimate on $\partial^I \Psi_1$. From (2.1), (2.20), (3.14), (3.16), (3.17) and (3.23), for all $I \in \mathbb{N}^0$ with $|I| \leq N - 8$, we have
\begin{align*}
\|\Gamma^I \Psi_1\|_{L^\infty} & \lesssim (t)^{-\frac{1}{4}} \int_0^t (1 + s)^{-\frac{7}{4}} \|\partial T^I G(w, v)\|_{L^1_s} ds \\
& + \langle t \rangle^{-\frac{1}{4}} \sum_{|I| \leq N-7} \int_0^t (1 + s)^{-\frac{1}{4}} \|\Gamma^I G(w, v)\|_{L^1_s} ds \lesssim C_0 \varepsilon^4 (t)^{-\frac{1}{2}}, \tag{3.31}
\end{align*}
and for all $I \in \mathbb{N}^0$ with $|I| \leq N - 9$, we have
\begin{align*}
\|\partial^I \Psi_1\|_{L^\infty} & \lesssim (t)^{-\frac{1}{4}} \int_0^t (1 + s)^{-\frac{7}{4}} \|\partial^2 T^I G(w, v)\|_{L^1_s} ds \\
& + \langle t \rangle^{-\frac{1}{4}} \sum_{|I| \leq N-8} \int_0^t (1 + s)^{-\frac{1}{4}} \|\partial T^I G(w, v)\|_{L^1_s} ds \\
& + \langle t \rangle^{-\frac{1}{4}} \sum_{|I| \leq N-8} \int_0^t \|\Gamma^I G(w, v)\|_{L^1_s} ds \lesssim C_0 \varepsilon^4 (t)^{-\frac{1}{2}}.
\end{align*}
From the above two estimates and (2.2), we see that
\begin{align*}
\langle t - r \rangle \sum_{|I| \leq N-9} |\partial^I \Psi_1(t, x)| & \lesssim \sum_{|I| \leq N-9} \|\partial^I \Psi_1\|_{L^\infty} + \sum_{|I| \leq N-9} \|\Gamma^I \Psi_1\|_{L^\infty} \lesssim C_0 \varepsilon^4 (t)^{-\frac{1}{2}}. \tag{3.32}
\end{align*}

Estimate on $\partial \partial^I \Psi_\alpha$. To obtain a sharp pointwise decay of $\partial \partial^I \Psi_\alpha$, we introduce a new variable by nonlinear transformation (see also [34]),
\[ \tilde{\Psi}_\alpha = \Psi_\alpha + C_1 v \partial_\alpha w, \quad \text{for } \alpha = 0, 1, 2. \]
Using (1.1), (3.23) and an elementary computation, we have
\[ -\Box \tilde{\Psi}_\alpha = \tilde{F}_\alpha(w, v), \quad \left( \tilde{\Psi}_\alpha, \partial_t \tilde{\Psi}_\alpha \right)_{|t=0} = C_1 \left( v \partial_\alpha w, \partial_t (v \partial_\alpha w) \right)_{|t=0}, \]
where
\begin{align*}
\tilde{F}_\alpha(w, v) & = \frac{C_2}{2} v^2 \partial_\alpha w + C_1 v \left( C_1 \partial_\alpha Q_0(w, v) + C_1^\beta \partial_\alpha Q_\beta(w, v) \right) \\
& - 2C_1 Q_0(\partial_\alpha w, v) + C_1 \partial_\alpha w \left( C_2 Q_0(w, v) + C_2^\alpha Q_\alpha w, v \right).
\end{align*}
First, using again the smallness condition \( (1.3) \),
\[
\sum_{|I| \leq N-5} \mathcal{E}(0, \Gamma^I \tilde{\varphi}_\alpha)^{1/2} \lesssim \sum_{|I| \leq N-4} \| \Gamma^I (v \partial w)(0, x) \|_{L^2_x} \lesssim \varepsilon.
\]

Then, from \((3.12)\) and the definition of \( \tilde{F}_\alpha \), we infer
\[
\sum_{|I| \leq N-5} \int_0^t \left\| \Gamma^I \tilde{F}_\alpha(w, v) \right\|_{L^2_x} ds \lesssim C_0^2 \varepsilon^2.
\]

Based on the above two estimates, we have
\[
\sum_{|I| \leq N-5} \mathcal{E}(t, \Gamma^I \tilde{\varphi}_\alpha)^{1/2} \lesssim \sum_{|I| \leq N-5} \left( \mathcal{E}(0, \Gamma^I \tilde{\varphi}_\alpha)^{1/2} + \int_0^t \left\| \Gamma^I \tilde{F}_\alpha(w, v) \right\|_{L^2_x} ds \right) \lesssim \varepsilon + C_0^2 \varepsilon^2.
\]

From the above estimates, \((2.1), (2.10), (3.2)\) and \((3.4)\),
\[
\sum_{|I| \leq N-8} |\partial \Gamma^I \Psi_\alpha| \lesssim \sum_{|I| \leq N-8} \left( |\partial \Gamma^I \tilde{\varphi}_\alpha| + |\Gamma^I (v \partial w)| \right) \lesssim (\varepsilon + C_0^2 \varepsilon^2) (t)^{-\frac{1}{2}}.
\]

Combining the above estimate with \((2.23), (3.2)\) and \((3.4)\), we obtain
\[
\sum_{|I| \leq N-9} \langle t-r \rangle |\partial \Gamma^I \Psi_\alpha(t, x)| \lesssim \sum_{|I| \leq N-8} \left( \left\| \partial \Gamma^I \Psi_\alpha \right\|_{L^2_x} + t \left\| \Gamma^I (v \partial w) \right\|_{L^2_x} \right) \lesssim (\varepsilon + C_0^2 \varepsilon^2) (t)^{-\frac{1}{2}}.
\]

**Estimate on \( \partial \Gamma^I \Phi_\alpha \).** Using a similar argument as above, we also have
\[
\sum_{|I| \leq N-8} |\partial \Gamma^I \Phi_\alpha| \lesssim (\varepsilon + C_0^2 \varepsilon^2) (t)^{-\frac{1}{2}},
\]
and so using again \((2.23), (3.2)\) and \((3.4)\), we obtain
\[
\sum_{|I| \leq N-9} \langle t-r \rangle |\partial \Gamma^I \Phi_\alpha(t, x)| \lesssim (\varepsilon + C_0^2 \varepsilon^2) (t)^{-\frac{1}{2}}.
\]

Combining \((3.30), (3.32), (3.34)\) and \((3.36)\), we obtain \((3.28)\) which strictly improves the estimate of \( |\partial \Gamma^I w| \) in \((3.1)\) for \( C_0 \) large enough and \( \varepsilon \) small enough (depending on \( C_0 \)).

**Step 5.** Closing the estimate in \( \| w \|_{L^\infty_x} \). Combining the estimates \((3.29), (3.31), (3.33)\) and \((3.35)\), we obtain
\[
\| w \|_{L^\infty_x} \lesssim \| \mathcal{Y}_0 \|_{L^\infty_x} + \| \mathcal{Y}_1 \|_{L^\infty_x} + \| \partial \Psi_\alpha \|_{L^\infty_x} + \| \partial \Phi_\alpha \|_{L^\infty_x} \lesssim (\varepsilon + C_0^2 \varepsilon^2) (t)^{-\frac{1}{2}},
\]

which strictly improves the estimate of \( \| w \|_{L^\infty_x} \) in \((3.1)\) for \( C_0 \) large enough and \( \varepsilon \) small enough (depending on \( C_0 \)).

**Step 6.** Closing the estimates of \( \Gamma^I v \) and \( G_1 \Gamma^I v \). First, from \((1.3), (3.6)\) and Corollary 2.13, we directly have
\[
\langle t+r \rangle \sum_{|I| \leq N-5} |\Gamma^I v(t, x)| \lesssim \varepsilon + C_0^2 \varepsilon^2,
\]

Then, from \((3.1)\) and \((3.8)\), we see that
\[
\sum_{|I| \leq N} \int_0^t \langle s \rangle^{-\frac{1}{2}} \| \Gamma^I Q(w, v) \|_{L^2} \| \partial_t \Gamma^I v \|_{L^2_x} ds \lesssim C_0 \varepsilon \sum_{|I| \leq N} \int_0^t \| \Gamma^I Q(w, v) \|_{L^2} ds \lesssim C_0^2 \varepsilon^2 (t)^{\delta}.
\]
Therefore, from (2.33) and (3.22), we obtain
\[
\sum_{|I| \leq N, i, j = 1, 2} \int_0^t (s)^{-\delta} \int_{\mathbb{R}^2} \left( \frac{|\Gamma^I v|^2}{(r-s)^{\frac{\delta}{2}}} + \frac{|G_i \Gamma^I v|^2}{(r-s)^{\frac{\delta}{2}}} \right) ds dx \lesssim \varepsilon^2 + C_0^2 \varepsilon^2 (t)^4. \tag{3.37}
\]
These strictly improve the estimates of $\Gamma^I v$ and $G_i \Gamma^I v$ in the bootstrap assumption (3.1) for $C_0$ large enough and $\varepsilon$ small enough (depending on $C_0$).

At this point, we have strictly improved all the bootstrap estimates of $(w, v)$ in (3.1) and (3.2). In conclusion, for all initial data $(\tilde{w}_0, \tilde{v}_0)$ satisfying (1.3), we show that $T_*(\tilde{w}_0, \tilde{v}_0) = \infty$ and thus the proof of Proposition 3.1 is complete. \qed

References

[1] S. Alinhac. The null condition for quasilinear wave equations in two space dimensions I. Invent. Math. 145 (2001), no. 3, 597–618.
[2] S. Alinhac. The null condition for quasilinear wave equations in two space dimensions II. Amer. J. Math. 123 (2001), no. 6, 1071–1101.
[3] S. Alinhac. Hyperbolic Partial Differential Equations. Universitext. Springer, Dordrecht, 2009. xii+150 pp.
[4] A. Bachelot. Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon. Ann. Inst. H. Poincaré Phys. Théor. 48 (1988), no. 4, 387–422.
[5] D. Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. Comm. Pure Appl. Math. 39 (1986), no. 2, 267–282.
[6] P. D. Ionescu and B. Pausader. On the global regularity for a wave-Klein-Gordon coupled system. Acta Math. Sin. (Engl. Ser.) 35 (2019), no. 6, 933–986.
[7] A. D. Ionescu and B. Pausader. The Einstein-Klein-Gordon coupled system: global stability of the Minkowski solution. Annals of Mathematics Studies, Vol 403, Princeton University Press (2022).
[8] M. Ifrim and A. Stingo. Almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems in two space dimensions. Preprint, arXiv:2010.08951. To appear in J. Math. Pures Appl.
[9] J.-M. Delort, D. Fang, and R. Xue. Global existence of small solutions for quadratic quasilinear strongly coupled wave-Klein-Gordon systems in two spatial dimensions. Preprint, arXiv:2111.00244.
[10] H. Fei and H. Yin. Global small data smooth solutions of 2-D null-form wave equations with strong couplings in divergence form. Preprint, arXiv:2010.08951. To appear in SIAM J. Math. Anal.
[11] S. Dong and Z. Wyatt. Hidden structure and sharp asymptotics for the Dirac–Klein-Gordon system in two space dimensions. Preprint, arXiv:2105.13780.
[12] S. Dong and Z. Wyatt. Global solutions of wave-Klein-Gordon system in two spatial dimensions with strong couplings in divergence form. Preprint, arXiv:2010.08951. To appear in SIAM J. Math. Anal.
[23] S. Klainerman. The null condition and global existence to nonlinear wave equations. Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326, Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986.

[24] S. Klainerman, Q. Wang, and S. Yang. Global solution for massive Maxwell-Klein-Gordon equations. Comm. Pure Appl. Math. 73 (2020), no. 1, 63–109.

[25] P. G. LeFloch and Y. Ma. The hyperboloidal foliation method. Series in Applied and Computational Mathematics, 2. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014. x+149 pp.

[26] P.G. LeFloch and Y. Ma. The global nonlinear stability of Minkowski space. Einstein equations, f(R)-modified gravity, and Klein-Gordon fields. Preprint, arXiv:1712.10045.

[27] D. Li. Uniform estimates for 2D quasilinear wave. Preprint, arXiv:2106.06419.

[28] T. Li and Y. Zhou. Nonlinear wave equations. Vol. 2. Translated from the Chinese by Yachun Li. Series in Contemporary Mathematics, 2. Shanghai Science and Technical Publishers, Shanghai; Springer-Verlag, Berlin, 2017. xiv+391 pp.

[29] Y. Ma. Global solutions of quasilinear wave-Klein-Gordon system in two-space dimension: completion of the proof. J. Hyperbolic Differ. Equ. 14 (2017), no. 4, 627–670.

[30] T. Ozawa, K. Tsutaya and Y. Tsutsumi, Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions. Math. Z. 222 (1996), no. 3, 341–362.

[31] J. Shatah. Normal forms and quadratic nonlinear Klein–Gordon equations. Comm. Pure Appl. Math. 38 (1985), no. 5, 685–696.

[32] C.D. Sogge. Lectures on non-linear wave equations. Second edition. International Press, Boston, MA, 2008. x+205 pp.

[33] A. Stingo. Global existence of small amplitude solutions for a model quadratic quasi-linear coupled wave-Klein-Gordon system in two space dimension, with mildly decaying Cauchy data. Preprint, arXiv:1810.10235. To appear in Mem. Amer. Math. Soc.

[34] Y. Tsutsumi. Global solutions for the Dirac-Proca equations with small initial data in 3 + 1 space time dimensions. J. Math. Anal. Appl. 278 (2003), no. 2, 485–499.