1. Introduction and the main result

1.1. Let $G$ be a complex reductive algebraic group, $B$ a Borel subgroup of $G$, $\Phi$ the root system of $G$ and $W = W(\Phi)$ the Weyl group of $\Phi$. It is well-known that the Bruhat–Chevalley order on $W$ encodes the cell decomposition of the flag variety $G/B$ (see, e.g., [BL]). Denote by $I(W)$ the poset of involutions in $W$ (i.e., elements of $W$ of order 2). In [RS], R. Richardson and T. Springer showed that $I(A_{2n})$ encodes the incidences among the closed $B$-orbits on the symmetric variety $SL_{2n+1}(\mathbb{C})/SO_{2n+1}(\mathbb{C})$. In [BC], E. Bagno and Y. Chernavsky presented a geometrical interpretation of the poset $I(A_n)$, considering the action of the Borel subgroup of $GL_n(\mathbb{C})$ on symmetric matrices by congruence. F. Incitti studied the poset $I(\Phi)$ from a purely combinatorial point of view for the case of classical root system $\Phi$ (see [In1], [In2]). In particular, he proved that this poset is graded, calculated the rank function and described the covering relation.

In [Ig3], we presented another geometrical interpretation of $I(A_n)$ in terms of coadjoint $B$-orbits. Precisely, let $G = GL_n(\mathbb{C})$, the general linear group, then $\Phi = A_{n-1}$ and $W = S_n$, the symmetric group on $n$ letters. Let $B = B_n$ be the group of invertible upper-triangular matrices. Denote by $U = U_n$ the unipotent radical of $B$. Let $n = n_n$ be the space of upper-triangular matrices with zeroes on the diagonal, and $n^*$ the dual space. Since $B$ acts on $n$ by conjugation, one can consider the dual action of $B$ on $n^*$. To each involution $\sigma \in I(A_{n-1})$ one can assign the $B$-orbit $\Omega_\sigma \subseteq n^*$ (see Subsection 1.2 or [Ig3, Subsection 1.2] for precise definitions). By [Ig3, Theorem 1.1] (see also Theorem 1.3), $\Omega_\sigma$ is contained in the Zariski closure of $\Omega_\tau$ if and only if $\sigma \leq B \tau$ with respect to the Bruhat–Chevalley order. In some sense, these results are “dual” to A. Melnikov’s results [Me1], [Me2], [Me3].

In this paper, we find similar results for the case $\Phi = C_n$. Namely, let $G = Sp_{2n}(\mathbb{C})$, the symplectic group, then $\Phi = C_{n-1}$ and $W$ is the hyperoctahedral group. Let $U$ be the unipotent radical of $B$, $n$ its Lie algebra and $n^*$ the dual space. Since $n$ is invariant under the adjoint action of the Borel subgroup on its Lie algebra, one can consider the dual action of $B$ on $n^*$. To each involution $\sigma \in I(C_n)$ one can assign the $B$-orbit $\Omega_\sigma \subseteq n^*$ (see Definition 1.0). The main result of the paper is as follows.

**Theorem 1.1.** Let $\sigma$, $\tau$ be involutions in the Weyl group of $C_n$. The orbit $\Omega_\sigma$ is contained in the Zariski closure of $\Omega_\tau$ if and only if $\sigma$ is less or equal to $\tau$ with respect to the Bruhat–Chevalley order.
\( \leq_B \) denotes the Bruhat–Chevalley order and \( \overline{Z} \) denotes the Zariski closure of a subset \( Z \subseteq n^* \).

Then, in Subsection 2.2, we define a partial order \( \preceq^* \) on \( \mathcal{I}(C_n) \) in combinatorial terms and, using [153, Theorem 1.10], prove Proposition 2.3, which says that \( \sigma \preceq \tau \) is equivalent to \( \sigma \leq_B \tau \). Finally, in Subsection 2.3, we prove that if \( \Omega_\sigma \subseteq \Omega_\tau \), then \( \sigma \preceq^* \tau \), see Proposition 2.5. Thus, the conditions \( \sigma \leq_B \tau \), \( \Omega_\sigma \subseteq \Omega_\tau \), and \( \sigma \preceq^* \tau \) are equivalent. This concludes the proof of our main result. Section 3 contains some related facts and conjectures. In Subsection 3.1, we present a formula for the dimension of the orbit \( \Omega \) (see Theorem 3.1). In Subsection 3.2, a conjectural approach to orbits associated with involutions in terms of tangent cones to Schubert varieties is described.

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1.2. Let \( G = \text{GL}_n(\mathbb{C}) \) be the general linear group, \( B = B_n \) the subgroup of invertible upper-triangular matrices, and \( U = U_n \) the unipotent radical of \( B \). Let \( n = n_n \) be the space of upper-triangular matrices with zeroes on the diagonal, and \( n^* \) the dual space. Let \( \Phi^+ \) be the set of positive roots with respect to \( B \). We identify \( \Phi^+ \) with the set \( \{ \epsilon_i - \epsilon_j, 1 \leq i < j \leq n \} \), where \( \{ \epsilon_i \}_{i=1}^n \) is the standard basis of \( \mathbb{R}^n \) (see, e.g., [160]). Denote by \( e_{i,j} \) the usual \((i,j)\)th matrix unit, then \( \{ e_{\alpha, \alpha} \in \Phi^+ \} \) is a basis of \( n^+ \), where \( e_{\epsilon_i - \epsilon_j} = e_{i,j} \). One can consider the dual basis \( \{ e^*_\alpha, \alpha \in \Phi^+ \} \) of the dual space \( n^* \).

The group \( B \) acts on \( n \) by conjugation, so one can consider the dual action of \( B \) on \( n^* \). By definition,

\[
(g, \lambda, x) = (\lambda, g^{-1}xg), \quad g \in B, \quad x \in n, \quad \lambda \in n^*.
\]

For a given \( \lambda \in n^* \), let \( \Omega_\lambda \) denote its orbit under this action. A subset \( D \subseteq \Phi^+ \) is called orthogonal if it consists of pairwise orthogonal roots. To each orthogonal subset \( D \) one can assign the element

\[
f_D = \sum_{\alpha \in D} e^*_\alpha \in n^*.
\]

Let \( W = W(A_{n-1}) \cong S_n \) be the Weyl group of \( G \). An involution \( \sigma \in \mathcal{I}(A_{n-1}) \) can be uniquely expressed as product of disjoint 2-cycles \( \sigma = (i_1, j_1) \ldots (i_t, j_t) \), \( i_1 < j_1 \) and \( i_1 < \ldots < i_t \). If we identify a transposition \((i, j)\) in \( S_n \) with the reflection \( r_{\epsilon_i - \epsilon_j} \) in the hyperplane orthogonal to the root \( \epsilon_i - \epsilon_j \), then

\[
\sigma = \prod_{\alpha \in D} r_\alpha,
\]

where the orthogonal subset \( D = \{ \epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_t} - \epsilon_{j_t} \} \) is called the support of \( \sigma \). We say that the \( B \)-orbit \( \Omega_\sigma = \Omega_{f_D} \) is associated with \( \sigma \).

It is very convenient to identify \( n^* \) with the space \( n^t \) of lower-triangular matrices with zeroes on the diagonal by putting \( e^*_\alpha = e^t_{\alpha} \). Under this identification,

\[
(\lambda, x) = \text{tr} \lambda x, \quad \lambda \in n^t, \quad x \in n.
\]

For this reason, we will denote \( n^t \) by \( n^* \) and interpret it as the dual space of \( n \). Note that if \( g \in B \), \( \lambda \in n^* \), then

\[
g.\lambda = (g\lambda g^{-1})_{\text{low}},
\]

where \( A_{\text{low}} \) denotes the strictly lower-triangular part of a matrix \( A \). To each involution \( \sigma \in \mathcal{I}(A_{n-1}) \) one can assign the 0–1 matrix \( X_\sigma \) such that \((X_\sigma)_{i,j} = 1\) if and only if \( \sigma(i) = j \).

**Example 1.2.** It is convenient to draw a 0–1 matrix \( A \) as a rook placement on the \( n \times n \) board: by definition, there is a rook in the \((i,j)\)th box if and only if \( A_{i,j} = 1 \). For instance, let \( n = 6 \), \( \text{Supp}(\sigma) = \{ \epsilon_1 - \epsilon_4, \epsilon_3 - \epsilon_5 \} \). On the picture below we draw \( X_\sigma \) (rooks are marked by \( \times \)'s).
To each 0–1 matrix $A$ one can assign the matrix $R(A)$ by putting

$$R(A)_{i,j} = \text{rk} \pi_{i,j}(A),$$

where $\pi_{i,j}$ denotes the lower-left $(n - i + 1) \times j$ submatrix of $A$. (In other words, $R(A)_{i,j}$ is just the number of rooks located non-strictly to the South-West of the $(i,j)$th box.) In particular, we set $R_\sigma = R(X_\sigma)$ and $R^*_\sigma = (R_\sigma)^{\text{low}}$, the lower-triangular part of $R_\sigma$. Suppose $A$ and $B$ are matrices with integer entries. We write $A \leq B$ if $A_{i,j} \leq B_{i,j}$ for all $i, j$. Denote by $\leq B$ the usual Bruhat–Chevalley order on $S_n$. Denote also by $\bar{Z}$ the closure of a subset $Z \subseteq n^*$ with respect to Zariski topology. We have the following description of the incidences among the closures of $B$-orbits associated with involutions.

**Theorem 1.3.** Let $\sigma, \tau \in I(A_{n-1})$. The following conditions are equivalent:

i) $\sigma \leq B \tau$;

ii) $R_\sigma \leq R_\tau$;

iii) $R^*_\sigma \leq R^*_\tau$;

iv) $\Omega_\sigma \subseteq \overline{\Omega_\tau}$.

**Proof.** i) $\Leftrightarrow$ ii). See, e.g., [In1, Theorem 1.6.4].

ii) $\Rightarrow$ iii) is evident. The converse follows from [Ig3, Theorem 1.10].

iii) $\Leftrightarrow$ iv). See [Ig3, Theorem 1.7].

1.3. From now on, let $G = \text{Sp}_{2n}(\mathbb{C})$ be the symplectic group, i.e.,

$$G = \{X \in \text{GL}_{2n}(\mathbb{C}) \mid X^t J X = J\}, \text{ where }$$

$$J = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}.$$ 

Here $s$ denotes $n \times n$ matrix with 1’s on the antidiagonal and zeroes elsewhere. Let $B = B_n \cap G$ be the Borel subgroup of $G$ consisting of all upper-triangular matrices from $G$. The unipotent radical of $B$ coincides with the group $U = U_n \cap G$ of all upper-triangular matrices from $G$ with 1’s on the diagonal.

The Lie algebra $\mathfrak{g}$ of $G$ is the symplectic algebra $\mathfrak{sp}_{2n}(\mathbb{C}) = \{X \in \text{Mat}_{2n}(\mathbb{C}) \mid X^t J + J X = 0\}$. The Lie algebra of $B$ coincides with the space $\mathfrak{b}$ of all upper-triangular matrices from $\mathfrak{g}$. The space $\mathfrak{n}$ of matrices from $\mathfrak{b}$ with zeroes on the diagonal is the Lie algebra of $U$. Let $\Phi^+$ be set of positive roots with respect to $B$. We identify $\Phi^+$ with the set

$$\{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n\} \cup \{2\epsilon_i, 1 \leq i \leq n\},$$

where $\{\epsilon_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$ (see, e.g., [Bo]).

For a given $\alpha \in \Phi^+$, put

$$e_\alpha = \begin{cases} 
\epsilon_{i,j} - \epsilon_{-j,-i}, & \text{if } \alpha = \epsilon_i - \epsilon_j, \\
\epsilon_{i,-j} + \epsilon_{j,-i}, & \text{if } \alpha = \epsilon_i + \epsilon_j, \\
\epsilon_{i,-i}, & \text{if } \alpha = 2\epsilon_i.
\end{cases}$$
Here we index rows and columns of any $2n \times 2n$ matrix by the numbers $1, 2, \ldots, n, -n, -n+1, \ldots, -1$ and denote by $e_{ij}$ the $(i,j)$th matrix unit. The set $\{e_\alpha, \alpha \in \Phi^+\}$ is a basis of $\mathfrak{n}$, so one can consider the dual basis $\{e_\alpha^*, \alpha \in \Phi^+\}$ of the dual space $\mathfrak{n}^*$.

Since $\mathfrak{n}$ is invariant under the adjoint action of $B$ on $\mathfrak{b}$, one can consider the dual action of $B$ on $\mathfrak{n}^*$. By definition, if $g \in B$, $\lambda \in \mathfrak{n}^*$ and $x \in \mathfrak{n}$, then

$$\langle g, \lambda, x \rangle = \langle \lambda, Ad_g^{-1}(x) \rangle,$$

where Ad denotes the adjoint action. (In fact, $Ad_g(x) = gxg^{-1}$.) For a given $\lambda \in \mathfrak{n}^*$, by $\Theta_\lambda$ (resp. by $\Theta_\lambda$) we denote the $B$-orbit (resp. the $U$-orbit) of $\lambda$.

A subset $D \subset \Phi^+$ is called orthogonal if it consists of pairwise orthogonal roots. To each orthogonal subset $D \subset \Phi$ and each map $\xi$: $D \to \mathbb{C}^\times$ one can assign the elements of $\mathfrak{n}^*$ of the form

$$f_D = \sum_{\alpha \in D} e_\alpha^*, \quad f_{D,\xi} = \sum_{\alpha \in D} \xi(\alpha)e_\alpha^*.$$ 

If $D = \emptyset$, then $f_D, f_{D,\xi} = 0$. Evidently, $f_D = f_D, f_{D,\xi}$, where $f_{\emptyset}$ sends all roots from $D$ to 1. Put $\Omega_D = \Omega_{f_D}$, $\Theta_D = \Theta_{f_D}$ and $\Theta_{D,\xi} = \Theta_{f_D,\xi}$. Clearly, $\Theta_D \subseteq \Omega_D$; Lemma [1.8] shows that in fact $\Omega_D = \bigcup \Theta_{D,\xi}$, where the union is taken over all maps $\xi$: $D \to \mathbb{C}^\times$.

Remark 1.4. Orbits associated with orthogonal subsets play an important role in representation theory of $U$. They were studied by the author in [Ig1]. (See also [Ig2] for further examples and generalizations to other unipotent algebraic groups.)

Let $\sigma \in I(\Phi)$ be an involution from $W$, the Weyl group of $\Phi$. An orthogonal subset $D \subset \Phi^+$ is called a support of $\sigma$, if $\sigma = \prod_{\alpha \in D} r_\alpha$ and there are no $\alpha, \beta \in D$ such that $\alpha - \beta \in \Phi^+$. Here $r_\alpha \in W$ denotes the reflection in the hyperplane orthogonal to a given root $\alpha$, and the product is taken in any fixed order. One can easily check that there exists exactly one support of $\sigma$ among all orthogonal subsets of $\Phi^+$. We denote it by $\text{Supp}(\sigma)$. We put also $\Omega_\sigma = \Omega_{\text{Supp}(\sigma)}$ and $f_\sigma = f_{\text{Supp}(\sigma)}$.

Example 1.5. Let $D = \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\}$, $D' = \{2\epsilon_1, 2\epsilon_2\}$. Then $\sigma = \prod_{\alpha \in D} r_\alpha = \prod_{\alpha \in D'} r_\alpha$, but $(\epsilon_1 + \epsilon_2) - (\epsilon_1 - \epsilon_2) = 2\epsilon_2 \in \Phi^+$, so $\text{Supp}(\sigma) = D'$, not $D$.

Definition 1.6. We say that the $B$-orbit $\Omega_\sigma$ is associated with the involution $\sigma$.

Remark 1.7. i) If $D$ is an orthogonal subset of $C_\mathfrak{n}^+$ such that $\alpha - \beta \in \Phi^+$ for some $\alpha, \beta \in D$, then, by [Ig1] Proposition 2.1, $\Theta_{D,\xi} = \Theta_{D_1,\xi_1}$ for any $\xi$: $D \to \mathbb{C}^\times$, where $D_1 = D \setminus \{\beta\}$ and $\xi_1 = \xi|_{D_1}$. Applying Lemma [1.8], we see that $\Omega_D = \Omega_{D_1}$, while $\sigma \neq \sigma_1$, where $\sigma = \prod_{\alpha \in D} r_\alpha$, $\sigma_1 = \prod_{\alpha \in D_1} r_\alpha$.

Note, however, that if $\alpha - \beta \in \Phi^+$, then $\alpha = \epsilon_i + \epsilon_j$, $\beta = \epsilon_i - \epsilon_j$ for some $i, j$. Thus, $\text{Supp}(\sigma) = D'$, where $D'$ is obtained from $D$ by replacing each pair $\{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j\}$ by $\{2\epsilon_i, 2\epsilon_j\}$.

ii) In fact, we do not know how to define the support of an involution for other root systems. For instance, let $\Phi = D_n, n \geq 4$, then $\Phi^+$ can be identified with $\{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n\}$. Put

$$D = \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4\},$$

$$D' = \{\epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_2 + \epsilon_4\}.$$ 

Then $\sigma = \prod_{\alpha \in D} r_\alpha = \prod_{\alpha \in D'} r_\alpha \in I(D_n)$, so we have two candidates for the role of $\text{Supp}(\sigma)$ and we do not know how to choose one of them.

It is very convenient to identify $\mathfrak{n}^*$ with the space $n'$ of lower-triangular matrices from $\mathfrak{g}$ by putting $e_\alpha^* = e_\alpha'$. Under this identification,

$$\langle \lambda, x \rangle = t r \lambda' x, \ \lambda \in \mathfrak{n}', \ x \in \mathfrak{n}.$$ 

Here we put $\Phi_0^+ = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq n\}, \Phi_1^+ = \{2\epsilon_i, 1 \leq i \leq n\}$; evidently, $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, so $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$ as vector spaces, where $\mathfrak{n}_0$ (resp. $\mathfrak{n}_1$) is spanned by $e_\alpha, \alpha \in \Phi_0^+$ (resp. $\alpha \in \Phi_1^+$); if
\[ \lambda = \lambda_0 + \lambda_1, \lambda_0 \in \mathfrak{n}_0, \lambda_1 \in \mathfrak{n}_1, \] then, by definition, \( \lambda' = 1/2\lambda_0 + \lambda_1. \) For this reason, we will denote \( \mathfrak{n}' \) by \( \mathfrak{n}^* \) and interpret it as the dual space of \( \mathfrak{n}. \) Note that if \( g \in B, \lambda \in \mathfrak{n}^*, \) then

\[
g_{\lambda} = (g_{\lambda}g^{-1})_{\text{low}},
\]

where \( A_{\text{low}} \) denotes the strictly lower-triangular part of a matrix \( A \) (see, e.g., [AN p. 410]).

**Lemma 1.8.** Let \( D \) be an orthogonal subset of \( C_n^+ \). Then \( \bigcup_{D, \xi} \Omega_D = \bigcup_{D, \xi} \Theta_{D, \xi}, \) where the union is taken over all maps \( \xi : D \rightarrow C^\times \).

**Proof.** It is well-known (see, f.e., [Hu, Subsection 15.1]) that the exponential map

\[ \exp : \mathfrak{n} \rightarrow U : x \mapsto \sum_{i=0}^{\infty} \frac{x^i}{i!} \]

is well-defined; in fact, it is an isomorphism of affine varieties. For a given \( \alpha \in \Phi^+, s \in C^\times, \) put

\[
x_{\alpha}(s) = \exp(\theta e_\alpha) = 1 + \theta e_\alpha, \quad x_{-\alpha}(s) = x_{\alpha}(s)^t, \]
\[
w_{\alpha}(s) = x_{\alpha}(s)x_{-\alpha}(-s^{-1})x_{\alpha}(s), \quad h_{\alpha}(s) = w_{\alpha}(s)w_{\alpha}(1)^{-1}.
\]

Then \( h_{\alpha}(s) \) is a diagonal matrix from \( B. \)

Let \( \xi : D \rightarrow C^\times \) be a map. Suppose \( \alpha \in D. \) Pick a number \( s \in C^\times \) and put

\[
s' = \begin{cases} \sqrt{s^{-1}} & \text{if } \alpha = \epsilon_i + \epsilon_j, \\ s & \text{if } \alpha = \epsilon_i - \epsilon_j, \\ s^{-1} & \text{if } \alpha = 2\epsilon_i, \end{cases}
\]

and \( \alpha' = 2\epsilon_i \) in all cases above. (Here \( \sqrt{s^{-1}} \) is a number such that \( \sqrt{s^{-1}}^2 = s^{-1}. \) One can easily check by straightforward matrix calculations that

\[
h_{\alpha'}(s').f_{D, \xi} = \sum_{\beta \in D, \beta \neq \alpha} \xi(\beta)e_{\beta}^s + s\xi(\alpha)e_{\alpha}^s.
\]

Hence

\[ \left( \prod_{\alpha \in D} h_{\alpha}(\xi(\alpha)s') \right) f_D = f_{D, \xi}, \]

so \( \Theta_{D, \xi} \subseteq \Omega_D. \)

On the other hand, let \( h \in H, \) where \( H \) is the group of diagonal matrices from \( G. \) We claim that \( h.f_{D, \xi} = f_{D, \xi'} \) for some \( \xi'. \) Indeed, since \( H \) is generated by \( h_{\alpha}(s)'s, \alpha \in \Phi^+, s \in C^\times, \) we can assume without loss of generality that \( h = h_{\alpha}(s) \) for some \( \alpha \) and \( s. \) But in this case the statement follows immediately from the above.

The group \( B \) is isomorphic as an algebraic group to the semi-direct product \( U \rtimes H \) [Hu Subsection 19.1]. In particular, for a given \( g \in B, \) there exist unique \( u \in U, h \in H \) such that \( g = uh. \) If \( \xi : D \rightarrow C^\times \) is the map such that \( h.f_D = f_{D, \xi}, \) then \( g.f_D = u.f_{D, \xi} \in \Theta_{D, \xi} \). This concludes the proof.

**2. Proof of the main theorem**

\[ \text{2.1.} \] In this subsection, we will prove that if \( \sigma, \tau \in \mathcal{Z}(C_n) \) and \( \sigma \) is less or equal to \( \tau \) with respect to the Bruhat–Chevalley order, then \( \Omega_{\sigma} \) is contained in \( \Omega_{\tau}, \) the Zariski closure of \( \Omega_{\tau}. \) We will denote the set of fundamental roots \( \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\} \) by \( \Pi. \) A reflection \( r_{\alpha} \) is called fundamental
if \( \alpha \in \Pi \). An expression of \( w \in W \) as a product of fundamental reflections is called reduced if it has the minimal length among all such expressions. The length \( l(w) \) of a reduced expression is called the length of \( w \). Let \( w = r_{\alpha_1} \ldots r_{\alpha_l} \) be a reduced expression. By definition of the Bruhat–Chevalley order \( \leq_B \) on \( W \),

\[
\{w' \in W \mid w' \leq_B w\} = \{r_{\alpha_1} \ldots r_{\alpha_l}, \ t \leq l, \ i_1 < \ldots < i_t\}.
\]

Let \( \sigma, \tau \in \mathcal{I}(\Phi) \). We say that \( \tau \) covers \( \sigma \) and write \( \sigma <_B \tau \) if \( \tau <_B w <_B \sigma \) and there are no \( w \in \mathcal{I}(\Phi) \) such that \( \tau <_B w <_B \sigma \). In [In1], F. Incitti studied the restriction of \( \leq_B \) to \( \mathcal{I}(\Phi) \) from a combinatorial point of view. In particular, for a given involution \( \tau \), he described the set \( \mathcal{L}(\tau) = \{\sigma \in \mathcal{I}(\Phi) \mid \sigma <_B \tau\} \) (see [In1] pp. 75–81). We reformulate his description in our terms in Appendix.

**Proposition 2.1.** Let \( \sigma, \tau \) be involutions in \( W(C_n) \). If \( \sigma \leq_B \tau \), then \( \Omega_\sigma \subseteq \Omega_\tau \).

**Proof.** There exist \( \tau_1, \ldots, \tau_r \in \mathcal{I}(\Phi) \) such that \( \tau_1 = \sigma, \tau_r = \tau \) and \( \tau_i < \tau_{i+1} \) for all \( 1 \leq i < r \), so we can assume without loss of generality that \( \sigma < \tau \). It is well-known that \( \overline{\Omega}_\tau \) coincides with \( \overline{\Omega}_\tau' \), the closure of \( \Omega_\tau \) with respect to the complex topology, so it suffice to construct \( g(s)f_\sigma \to f_\tau \) as \( s \to 0 \).

The proof is case-by-case. For example, let

\[
\text{Supp}(\sigma) \setminus \text{Supp}(\tau) = \{\epsilon_i + \epsilon_j, 2\epsilon_k\},
\]

\[
\text{Supp}(\tau) \setminus \text{Supp}(\sigma) = \{2\epsilon_i, \epsilon_k + \epsilon_j\}
\]

for some \( 1 \leq i < k < j \leq n \) (see Case 5 in Appendix). Put

\[
g(s) = h_{\epsilon_i - \epsilon_k}(s^{-1}) \cdot x_{\epsilon_i - \epsilon_j}(s) \cdot x_{\epsilon_k - \epsilon_j}((2s^2)^{-1}) \cdot x_{\epsilon_i - \epsilon_j}(-s^{-1})
\]

and \( f = g(s)f_\tau \). One can easily check by straightforward matrix calculations that

\[
f(e_\alpha) = \begin{cases} 1, & \text{if either } \alpha = \epsilon_i + \epsilon_j \text{ or } \alpha = 2\epsilon_k, \\ 0, & \text{if } \alpha = \epsilon_k + \epsilon_j, \\ -s, & \text{if } \alpha = \epsilon_i + \epsilon_k, \\ s^2, & \text{if } \alpha = 2\epsilon_i, \\ f_\tau(e_\alpha) & \text{otherwise.} \end{cases}
\]

Thus, \( f \to f_\sigma \) as \( s \to 0 \).

All other cases can be considered similarly, see Appendix. \( \square \)

**2.2.** Let \( S_{\pm n} \) be the symmetric group on \( 2n \) letters \( \{1, 2, \ldots, n, n, -n, -n + 1, \ldots, -1\} \). Put

\[
W' = \{w \in S_{\pm n} \mid w(-i) = -w(i) \text{ for all } 1 \leq i \leq n\}.
\]

It is known that the map \( W \to W' : w \mapsto w' \) defined by

\[
r'_{\epsilon_i - \epsilon_j} = (i, j)(-i, -j),
\]

\[
r'_{\epsilon_i + \epsilon_j} = (i, -j)(-i, j),
\]

\[
r'_{2\epsilon_i} = (i, -i)
\]

is an isomorphism of groups. Note that if \( \sigma \in \mathcal{I}(C_n) \), then \( \sigma' \) is an involution in \( S_{\pm n} \).

To each involution \( \sigma \in \mathcal{I}(C_n) \) one can assign the 0–1 matrix \( X_\sigma \) such that \( (X_\sigma)_{i,j} = 1 \) if and only if \( \sigma'(i) = j \) (recall that we index rows and columns of a \( 2n \times 2n \) matrix by the numbers \( 1, \ldots, n, -n, \ldots, -1 \)).

**Example 2.2.** It is convenient to draw a 0–1 matrix \( A \) as a rook placement on the \( 2n \times 2n \) board: by definition, there is a rook in the \( (i, j) \)th box if and only if \( A_{i,j} = 1 \). For instance, let \( n = 4 \), \( \text{Supp}(\sigma) = \{\epsilon_1 - \epsilon_4, 2\epsilon_2\} \). On the picture below we draw \( X_\sigma \) (rooks are marked by \( \otimes \)'s).
As above, to each 0–1 matrix \( A \) we assign the matrix \( R(A) \) by putting

\[
R(A)_{i,j} = \text{rk } \pi_{i,j}(A),
\]

where \( \pi_{i,j} \) sends \( A \) to its submatrix with rows \( i, \ldots, -1 \) and columns \( 1, \ldots, j \). (In other words, \( R(A)_{i,j} \) is just the number of rooks located non-strictly to the South-West of the \((i,j)\)th box.) In particular, we set \( R_\sigma = R(X_\sigma) \) and \( R_\sigma^* = (R_\sigma)_{\text{low}} \), the lower-triangular part of \( R_\sigma \).

Suppose \( A \) and \( B \) are matrices with integer entries. As above, we write \( A \leq B \) if \( A_{i,j} \leq B_{i,j} \) for all \( i, j \). Let \( \sigma, \tau \in \mathcal{I}(C_n) \). By \[\text{in1}\] Theorem 1.6.7,

\[
\sigma \leq_B \tau \text{ if and only if } R_\sigma \leq R_\tau.
\]

(Note that Incitti use another order of fundamental roots.) Let us define another partial order on \( \mathcal{I}(C_n) \). Namely, we put

\[
\sigma \leq^* \tau \text{ if } R_\sigma^* \leq R_\tau^*.
\]

Clearly, \( \sigma \leq_B \tau \) implies \( \sigma \leq^* \tau \). But the converse follows immediately from Theorem 1.3 since \( \sigma' \) and \( \tau' \) are involutions in \( S_{\pm n} \). This proves

**Proposition 2.3.** Let \( \sigma, \tau \) be involutions in \( W(C_n) \). Then \( \sigma \leq_B \tau \) if and only if \( \sigma \leq^* \tau \). \( \square \)

2.3. Suppose \( \sigma, \tau \in \mathcal{I}(C_n) \). To conclude the proof of Theorem 1.1 it remains to check that if \( \Omega_\sigma \subseteq \overline{\Omega}_\tau \), then \( \sigma \leq^* \tau \). To do this, we need the following

**Lemma 2.4.** Let \( \sigma \in \mathcal{I}(C_n) \). Then \( \text{rk } \pi_{i,j}(\lambda) = (R_\sigma^*)_{i,j} \) for all \( \lambda \in \Omega_\sigma \).

**Proof.** First, note that \( (R_\sigma^*)_{i,j} = \text{rk } \pi_{i,j}(f_\sigma) \). By Lemma 1.3, \( \Omega_\sigma = \bigcup_{D \subseteq \mathbb{C}^\times} D \cdot \xi \cdot \Omega_{D, \xi} \), where \( D = \text{Supp}(\sigma) \). Let \( \xi \colon D \to \mathbb{C}^\times \) be a map. Since \( \text{rk } \pi_{i,j}(f_{D, \xi}) = \text{rk } \pi_{i,j}(f_\sigma) = (R_\sigma^*)_{i,j} \) for all \( i, j \), it suffice to check that \( \text{rk } \pi_{i,j}(\lambda) = \text{rk } \pi_{i,j}(u.\lambda) \) for all \( u \in U, \lambda \in \mathfrak{n}^* \). But this follows immediately from the proof of \[\text{in3}\] Lemma 2.2], because \( u \) is an upper-triangular matrix with 1’s on the diagonal, \( \lambda \) is a lower-triangular matrix with zeroes on the diagonal, and \( u.\lambda = (u\lambda u^{-1})_{\text{low}} \). \( \square \)

Things now are ready to prove

**Proposition 2.5.** Let \( \sigma, \tau \) be involutions in \( W(C_n) \). If \( \Omega_\sigma \subseteq \overline{\Omega}_\tau \), then \( \sigma \leq^* \tau \)

**Proof.** Suppose \( \sigma \not\leq^* \tau \). This means that there exist \( i, j \) such that \( (R_\sigma^*)_{i,j} < (R_\tau^*)_{i,j} \). Denote

\[
Z = \{ f \in \mathfrak{n}^* \mid \text{rk } \pi_{r,s}(f) \leq (R^*_\tau)_{r,s} \text{ for all } r, s \}.
\]

Clearly, \( Z \) is closed with respect to the Zariski topology. Lemma 2.4 shows that \( \Omega_\tau \subseteq Z \), so \( \overline{\Omega}_\tau \subseteq Z \). But \( f_\sigma \not\in Z \), hence \( \Omega_\sigma \not\subseteq Z \), a contradiction. \( \square \)

The proof of Theorem 1.1 is complete.

---

\(^2\)Cf. \[\text{in3}\] Lemma 2.2

\(^3\)Cf. \[\text{in3}\] Proposition 2.3
3. Concluding remarks

3.1. Let $\sigma \in \mathcal{I}(C_n)$. Being an orbit of a connected unipotent group on an affine variety $\mathfrak{n}^*$, $\Omega_\sigma$ is a closed subvariety of $\mathfrak{n}^*$. In this subsection, we present a formula for the dimension of $\Omega_\sigma$. Recall the definition of the length $l(w)$ of an element $w \in W$.

**Theorem 3.1.** Let $\sigma \in W(C_n)$ be an involution. Then

\[ \text{dim } \Omega_\sigma = l(\sigma). \]

**Proof.** Put $D = \text{Supp}(\sigma)$. We claim that if $\xi_1, \xi_2$ are two distinct maps from $D$ to $\mathbb{C}^\times$, then $\Theta_{D,\xi_1} \neq \Theta_{D,\xi_2}$, Indeed, let $\tilde{U} = U_n$ be the uniprincipal group (i.e., the group of all upper-triangular matrices with 1’s on the diagonal). Since $\sigma'$ is an involution in $S_{\pm n}$, [Pa] Theorem 1.4] implies that $\Theta_{D,\xi_1} \neq \Theta_{D,\xi_2}$, where $\Theta_{D,\xi_1}$ (resp. $\Theta_{D,\xi_2}$) denotes the $\tilde{U}$-orbit of $f_{D,\xi_1}$ (resp. of $f_{D,\xi_2}$) under the action of $\tilde{U}$ on the space of all lower-triangular matrices with zeroes on the diagonal defined by the formula

\[ u.\lambda = (u\lambda u^{-1})_{\text{low}}, \ u \in \tilde{U}, \ \lambda \in \mathfrak{n}^*. \]

Since $U \subseteq \tilde{U}$, one has $\Theta_{D,\xi_1} \subseteq \tilde{\Theta}_{D,\xi_1}$ and $\Theta_{D,\xi_2} \subseteq \tilde{\Theta}_{D,\xi_2}$, hence $\Theta_{D,\xi_1} \neq \Theta_{D,\xi_2}$, as required.

Let $Z_B = \text{Stab}_B f_\sigma$ be the stabilizer of $f_\sigma$ in $B$. One has

\[ \text{dim } \Omega_\sigma = \text{dim } B - \text{dim } Z_B. \]

Recall that $B \cong U \rtimes H$ as algebraic groups. It was shown in the proof of Lemma [L8] that if $h \in H$, then there exists $\xi : D \to \mathbb{C}^\times$ such that $h.f_\sigma = f_{D,\xi}$. Hence if $g = uh \in Z_B$, then

\[ f_\sigma = (uh).f_\sigma = u.f_{D,\xi}, \]

so $f_\sigma \in \Theta_{D,\xi}$. In follows from the first paragraph of the proof that $f_\sigma = f_{D,\xi}$. This means that the map

\[ Z_U \times Z_H \to Z_B : (u, h) \mapsto uh \]

is an isomorphism of algebraic varieties, where $Z_U = \text{Stab}_U f_\sigma$ (resp. $Z_H = \text{Stab}_H f_\sigma$) is the stabilizer of $f_\sigma$ in $U$ (resp. in $H$). Hence

\[ \text{dim } Z_B = \text{dim } Z_U + \text{dim } Z_H. \]

By [Ig1] Theorem 1.2], $\text{dim } \Theta_D = l(\sigma) - |D|$, so

\[ \text{dim } Z_U = \text{dim } U - \text{dim } \Theta_D = \text{dim } U - l(\sigma) + |D|. \]

On the other hand, put $X = \bigcup_{\xi : D \to \mathbb{C}^\times} \{ f_{D,\xi} \}$. In follows from Lemma [L8] and the first paragraph of the proof that $X = \{ h.f_\sigma, h \in H \}$, the $H$-orbit of $f_\sigma$. Consequently,

\[ \text{dim } Z_H = \text{dim } H - \text{dim } X = \text{dim } H - |D|, \]

because $X$ is isomorphic as affine variety to the product of $|D|$ copies of $\mathbb{C}^\times$. Thus,

\[ \text{dim } \Omega_\sigma = \text{dim } B - \text{dim } Z_B = (\text{dim } U + \text{dim } H) - (\text{dim } Z_U + \text{dim } Z_H) \]

\[ = \text{dim } U + \text{dim } H - (\text{dim } U - l(\sigma) - |D|) - (\text{dim } H - |D|) = l(\sigma). \]

The proof is complete. \( \square \)

3.2. In the remainder of the paper, we briefly discuss a conjectural geometrical approach to orbits associated with involutions in terms of tangent cones to Schubert varieties. Recall that $W$ is isomorphic to $N_G(H)/H$, where $N_G(H)$ is the normalizer of $H$ in $G$. The flag variety $\mathcal{F} = G/B$ can be decomposed into the union $\mathcal{F} = \bigcup_{w \in W} X^\circ_w$, where $X^\circ_w = BwB/B$ is called the Schubert cell. (Here

\[ \text{Cf. [Ig2] Proposition 4.1} \]
\(w\) is a representative of \(w\) in \(N_G(H)\). By definition, the Schubert variety \(X_w\) is the closure of \(X_w^o\) in \(\mathcal{F}\) with respect to Zariski topology. Note that \(p = X_{\text{id}} = B/B\) is contained in \(X_w\) for all \(w \in W\). One has \(X_w \subseteq X_w^o\) if and only if \(w \leq B w'\). Let \(T_w\) be the tangent space and \(C_w\) the tangent cone to \(X_w\) at the point \(p\) (see [BL] for detailed constructions); by definition, \(C_w \subseteq T_w\), and if \(p\) is a regular point of \(X_w\), then \(C_w = T_w\). Of course, if \(w \leq B w'\), then \(C_w \subseteq C_w^o\).

Let \(T = T_p \mathcal{F}\) be the tangent space to \(\mathcal{F}\) at \(p\). It can be naturally identified with \(n^*\) by the following way: since \(\mathcal{F} = G/B\), \(T\) is isomorphic to the factor \(g/b \cong n^*\). Next, \(B\) acts on \(\mathcal{F}\) by conjugation. Since \(p\) is invariant under this action, the action on \(T = n^*\) is induced. One can check that this action coincides with the action of \(B\) on \(n^*\) defined above. The tangent cone \(C_w \subseteq T_w \subseteq T = n^*\) is \(B\)-invariant, so it splits into a union of \(B\)-orbits. Furthermore, \(\Omega_\sigma \subseteq C_\sigma\) for all \(\sigma \in I(C_n)\).

It is well-known that \(C_w\) is a subvariety of \(T_w\) of dimension \(\dim C_w = l(w)\) [BL, Chapter 2, Section 2.6]. Let \(\sigma \in I(C_n)\). Since \(\Omega_\sigma\) is irreducible, \(\Omega_\sigma\) is. Theorem 3.4 implies \(\dim \Omega_\sigma = \dim \Omega_\sigma^o = l(\sigma)\), so \(\Omega_\sigma\) is an irreducible component of \(C_\sigma\) of maximal dimension.

\textbf{Conjecture 3.2.} Let \(\sigma \in W(C_n)\) be an involution. Then the closure of the \(B\)-orbit \(\Omega_\sigma\) coincides with the tangent cone \(C_w\) to the Schubert variety \(X_w\) at the point \(p = B/B\).

Note that this conjecture implies that if \(\sigma \leq B \tau\), then \(\Omega_\sigma \subseteq \Omega_\tau\).

\textbf{Appendix}

Let \(\tau \in I(C_n)\), \(L(\tau) = \{\sigma \in I(C_n) \mid \sigma \triangleleft \tau\}\). Here we reformulate Incitti’s description of \(L(\tau)\) used in the proof of Proposition 2.1. Precisely, for each pair \((\sigma, \tau)\), \(\sigma \in L(\tau)\), we construct the matrix \(g(s) \in B, s \in \mathbb{C}^n\), such that \(f = g(s), f_\tau \rightarrow f_\sigma\) as \(s \rightarrow 0\). By [In1] p. 96, \(\sigma \in L(\tau)\) if and only of \((\sigma, \tau)\) is one of the pairs described below. This concludes the proof of Proposition 2.1. In the second column of the table below, we indicate the number of a case referred to Incitti’s paper [In1]. (Note, however, that Incitti use another order of fundamental roots.) Note that there are some additional conditions to be satisfied by \((\sigma, \tau), \sigma \in L(\tau)\) (see [In1] pp. 76–81), but \(g(s)\) does not depend on them. In particular, all \(\epsilon_i\) occurring in the third or in the fourth column of the table must satisfy \((\beta, \epsilon_i) = 0\) for all \(\beta \in D_\sigma \cap D_\tau\). We put here \(D_\sigma = \text{Supp}(\sigma)\) and \(D_\tau = \text{Supp}(\tau)\). We denote by \(I\) a fixed complex number such that \(I^2 = -1\). For simplicity, if \(f(e_\alpha) = f_\tau(e_\alpha)\), then we omit \(\alpha\) in the last column of the table.

| \(D_\sigma \setminus D_\tau\) | \(D_\tau \setminus D_\sigma\) | \(g(s)\) | \(f(e_\alpha), \alpha \in \Phi^+\) |
|-----------------|-----------------|-----------------|-----------------|
| \(A_1, M1\)\(\emptyset\) | \(2\epsilon_i\) | \(h_{2\epsilon_i}(s^{-1})\) | \(s^2\) if \(\alpha = 2\epsilon_i\). |
| \(A_1, M5\) | \(\epsilon_i + \epsilon_j, i < j\) | \(2\epsilon_i, 2\epsilon_j\) | \(h_{2\epsilon_i}(s^{-1}) \times h_{2\epsilon_j}(-Is) \times x_{\epsilon_i - \epsilon_j}(I)\) | \(s^2\) if \(\alpha = 2\epsilon_i\), \(0\) if \(\alpha = 2\epsilon_j\), \(1\) if \(\alpha = \epsilon_i + \epsilon_j\). |
| \(A_1, M6\) | \(\epsilon_i - \epsilon_j, i < j\) | \(2\epsilon_i\) | \(h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i + \epsilon_j}(s^{-1})\) | \(s^2\) if \(\alpha = 2\epsilon_i\), \(1\) if \(\alpha = \epsilon_i - \epsilon_j\). |
| \(A_2, M2\) | \(2\epsilon_j\) | \(2\epsilon_i, i < j\) | \(h_{2\epsilon_j}(s^{-1}) \times x_{\epsilon_i - \epsilon_j}(1)\) | \(s^2\) if \(\alpha = 2\epsilon_i\), \(1\) if \(\alpha = \epsilon_i + \epsilon_j\). |
| \(A_2, M5\) | \(\epsilon_i + \epsilon_j, 2\epsilon_k, i < k < j\) | \(2\epsilon_i, \epsilon_k + \epsilon_j\) | \(h_{\epsilon_i - \epsilon_k}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(s) \times x_{\epsilon_k - \epsilon_j}(2s^2)^{-1} \times x_{\epsilon_i - \epsilon_j}(-s^{-1})\) | \(s^2\) if \(\alpha = 2\epsilon_i\), \(1\) if \(\alpha = \epsilon_i + \epsilon_j\). |

\(\text{Cf. [In2] Conjecture 1.11}.\)
|   | $D_\alpha \setminus D_\tau$ | $D_\tau \setminus D_\sigma$ | $g(s)$ | $f(\epsilon_\alpha), \alpha \in \Phi^+$ |
|---|---|---|---|---|
| 6 | (A2, M5)$b$ | $\epsilon_i - \epsilon_j, 2\epsilon_k,$ $i < k < j$ | $2\epsilon_i, \epsilon_k - \epsilon_j$ | $h_{\epsilon_i+\epsilon_k}(s^{-1}) \times x_{\epsilon_i-\epsilon_k}(s^{-1}) \times x_{\epsilon_i+\epsilon_j}(s^{-1}) \times h_{\epsilon_k-\epsilon_j}(s)$ | $s^2$, if $\alpha = 2\epsilon_i,$ $-s$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i - \epsilon_j,$ $1$, if $\alpha = 2\epsilon_k,$ $0$, if $\alpha = \epsilon_k - \epsilon_j.$ |
| 7 | (A2, M6) | $\epsilon_i - \epsilon_k, 2\epsilon_j,$ $i < k < j$ | $2\epsilon_i$ | $h_{\epsilon_i-\epsilon_k}(s^{-1}) \times x_{\epsilon_i+\epsilon_k}(s^{-2}) \times x_{\epsilon_i-\epsilon_j}(1)$ | $s^2$, if $\alpha = 2\epsilon_i,$ $-s$, if $\alpha = \epsilon_i + \epsilon_j,$ $1$, if $\alpha = \epsilon_i - \epsilon_k,$ $1$, if $\alpha = 2\epsilon_j.$ |
| 8 | (A3, M4) | $\epsilon_i - \epsilon_k, 2\epsilon_j,$ $i < k < j$ | $\epsilon_i - \epsilon_j, 2\epsilon_k$ | $h_{\epsilon_k}(s^{-2}) \times h_{\epsilon_i-\epsilon_j}(s^{-1}) \times x_{\epsilon_k-\epsilon_j}(s)$ | $s^4$, if $\alpha = 2\epsilon_k,$ $s^2$, if $\alpha = \epsilon_i - \epsilon_j,$ $-s^2$, if $\alpha = \epsilon_k + \epsilon_j,$ $1$, if $\alpha = \epsilon_i - \epsilon_k,$ $1$, if $\alpha = 2\epsilon_j.$ |
| 9 | (A3, M5) | $\epsilon_i + \epsilon_j, 2\epsilon_k,$ $i < k < j$ | $\epsilon_i + \epsilon_k, 2\epsilon_j$ | $h_{\epsilon_k}(s^{-1}) \times h_{\epsilon_i-\epsilon_j}(s) \times x_{\epsilon_k-\epsilon_j}(-(2s^2)^{-1}) \times x_{\epsilon_i-\epsilon_j}(-2s^{-1}) \times x_{\epsilon_k-\epsilon_j}(1/2)$ | $s$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i + \epsilon_j,$ $1$, if $\alpha = 2\epsilon_k,$ $1$, if $\alpha = 2\epsilon_j.$ |
| 10 | (A3, M6) | $\epsilon_i - \epsilon_j, 2\epsilon_k,$ $i < k < j$ | $\epsilon_i + \epsilon_k$ | $x_{\epsilon_i-\epsilon_k}(-(2s^2)^{-1}) \times x_{\epsilon_k+\epsilon_j}(s^{-1}) \times h_{\epsilon_i-\epsilon_j}(s^{-1}) \times h_{\epsilon_k-\epsilon_j}(1/2)$ | $s$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i - \epsilon_j,$ $1$, if $\alpha = 2\epsilon_k.$ |
| 11 | (A4, M3) | $\epsilon_i + \epsilon_j$ | $\epsilon_i + \epsilon_k,$ $i < k < j$ | $h_{\epsilon_i-\epsilon_j}(s^{-1}) \times x_{\epsilon_k-\epsilon_j}(1)$ | $s$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i + \epsilon_j.$ |
| 12 | (A4, M4)$a$ | $\epsilon_k - \epsilon_j,$ $\epsilon_i + \epsilon_l,$ $i < k < j < l$ | $\epsilon_i + \epsilon_j,$ $\epsilon_k - \epsilon_l$ | $x_{\epsilon_i-\epsilon_l}(s^{-1}) \times h_{\epsilon_j-\epsilon_l}(s^{-1}) \times h_{2\epsilon_i}(1)$ | $s$, if $\alpha = \epsilon_k - \epsilon_j,$ $-s$, if $\alpha = \epsilon_i + \epsilon_l,$ $1$, if $\alpha = \epsilon_k - \epsilon_l,$ $1$, if $\alpha = \epsilon_i + \epsilon_j.$ |
| 13 | (A4, M4)$b$ | $\epsilon_i - \epsilon_j,$ $\epsilon_k + \epsilon_l,$ $i < k < j < l$ | $\epsilon_i - \epsilon_l,$ $\epsilon_k + \epsilon_j$ | $x_{\epsilon_j-\epsilon_l}(-(s^2)^{-1}) \times h_{\epsilon_j-\epsilon_l}(s^{-1}) \times h_{2\epsilon_i}(1)$ | $s$, if $\alpha = \epsilon_k - \epsilon_l,$ $-s$, if $\alpha = \epsilon_i - \epsilon_l,$ $1$, if $\alpha = \epsilon_i - \epsilon_j,$ $1$, if $\alpha = \epsilon_k + \epsilon_j.$ |
| 14 | (A4, M4)$c$ | $\epsilon_i + \epsilon_j,$ $i < j$ | $\epsilon_i + \epsilon_j$ | $h_{2\epsilon_j}(-s^{-1}) \times x_{2\epsilon_j}(-s)$ | $s$, if $\alpha = \epsilon_i + \epsilon_j,$ $1$, if $\alpha = \epsilon_i - \epsilon_j.$ |
| 15 | (A4, M5)$a$ | $\epsilon_i + \epsilon_j,$ $\epsilon_k + \epsilon_l,$ $i < k < j < l$ | $\epsilon_i + \epsilon_k,$ $\epsilon_j + \epsilon_l$ | $h_{2\epsilon_i}(-1) \times h_{\epsilon_i-\epsilon_l}(s^{-1}) \times h_{\epsilon_k-\epsilon_j}(-(s^2)^{-1}) \times x_{\epsilon_k-\epsilon_j}(1/2) \times h_{\epsilon_i-\epsilon_l}(1)$ | $s^2$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i + \epsilon_j,$ $1$, if $\alpha = \epsilon_k + \epsilon_l,$ $0$, if $\alpha = \epsilon_j + \epsilon_l.$ |
| 16 | (A4, M5)$b$ | $\epsilon_i + \epsilon_j,$ $\epsilon_k - \epsilon_l,$ $i < k < j < l$ | $\epsilon_i + \epsilon_k,$ $\epsilon_j - \epsilon_l$ | $h_{2\epsilon_i}(-1) \times h_{\epsilon_k-\epsilon_l}(s^{-1}) \times x_{\epsilon_i+\epsilon_l}(s^{-1}) \times h_{\epsilon_k-\epsilon_l}(s)$ | $-s$, if $\alpha = \epsilon_i + \epsilon_k,$ $1$, if $\alpha = \epsilon_i + \epsilon_j,$ $1$, if $\alpha = \epsilon_k - \epsilon_l,$ $0$, if $\alpha = \epsilon_j - \epsilon_l.$ |
|   | $\mathcal{D}_\alpha \setminus \mathcal{D}_\tau$ | $\mathcal{D}_\tau \setminus \mathcal{D}_\alpha$ | $g(s)$ | $f(\alpha), \alpha \in \Phi^+$ |
|---|---|---|---|---|
|17 | (A4, M6) | $\epsilon_i + \epsilon_l, \epsilon_k - \epsilon_j, i < k < j < l$ | $\epsilon_i + \epsilon_k, \epsilon_i - \epsilon_j, i < k < j$ | $h_{\epsilon_k + \epsilon_i}(s^{-1}) \times x_{\epsilon_k - \epsilon_i}(-s^{-1}) \times x_{\epsilon_i + \epsilon_k}(s^{-1})$ | 1, if $\alpha = \epsilon_i + \epsilon_k$, 1, if $\alpha = \epsilon_i + \epsilon_l$, 1, if $\alpha = \epsilon_k - \epsilon_j$, s, if $\alpha = \epsilon_i - \epsilon_j$. |
|18 | (A5, M1) | $\emptyset$ | $\epsilon_i - \epsilon_j, i < j$ | $h_{\epsilon_i - \epsilon_j}(s^{-1})$ | 1, if $\alpha = \epsilon_i - \epsilon_j$. |
|19 | (A5, M2) | $\epsilon_k - \epsilon_j$ | $\epsilon_i - \epsilon_j, i < k < j$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(-1)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_k - \epsilon_j$. |
|20 | (A5, M3) | $\epsilon_i - \epsilon_k$ | $\epsilon_i - \epsilon_j, i < k < j$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(s^{-1})$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_i - \epsilon_k$. |
|21 | (A5, M4)a | $\epsilon_i - \epsilon_k, \epsilon_j - \epsilon_l, i < k < j < l$ | $\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s^{-1}) \times h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_k - \epsilon_i}(-1)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_i - \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_k$. |
|22 | (A5, M4)b | $\epsilon_i - \epsilon_k, \epsilon_j + \epsilon_l, i < k < j < l$ | $\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s^{-1}) \times h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_k - \epsilon_i}(-1)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_i + \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_k$. |
|23 | (A5, M5)a | $\epsilon_i + \epsilon_l, \epsilon_k + \epsilon_j, i < k < j < l$ | $\epsilon_i + \epsilon_j, \epsilon_k + \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s) \times h_{2\epsilon_i}(s) \times x_{\epsilon_j - \epsilon_i}(-s^{-1}) \times x_{\epsilon_i - \epsilon_k}(s)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 0, if $\alpha = \epsilon_k + \epsilon_l$, 1, if $\alpha = \epsilon_i + \epsilon_l$, 1, if $\alpha = \epsilon_k - \epsilon_l$. |
|24 | (A5, M5)b | $\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l, i < k < j < l$ | $\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s) \times h_{2\epsilon_i}(s) \times x_{\epsilon_j - \epsilon_i}(-s^{-1}) \times x_{\epsilon_i - \epsilon_k}(s)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 0, if $\alpha = \epsilon_k - \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_k + \epsilon_l$. |
|25 | (A5, M5)c | $\epsilon_i + \epsilon_l, \epsilon_k + \epsilon_j, i < k < j < l$ | $\epsilon_i + \epsilon_j, \epsilon_k - \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s) \times h_{2\epsilon_i}(s) \times x_{\epsilon_j + \epsilon_i}(s) \times x_{\epsilon_i - \epsilon_k}(s)$ | 1, if $\alpha = \epsilon_i + \epsilon_j$, 0, if $\alpha = \epsilon_k - \epsilon_l$, 1, if $\alpha = \epsilon_i + \epsilon_l$, 1, if $\alpha = \epsilon_k + \epsilon_j$. |
|26 | (A5, M5)d | $\epsilon_i - \epsilon_j, \epsilon_k + \epsilon_l, i < k < j < l$ | $\epsilon_i - \epsilon_j, \epsilon_k + \epsilon_l, i < k < j$ | $h_{2\epsilon_k}(-s) \times h_{2\epsilon_i}(s) \times x_{\epsilon_j + \epsilon_i}(s) \times x_{\epsilon_i - \epsilon_k}(s)$ | 1, if $\alpha = \epsilon_i - \epsilon_j$, 0, if $\alpha = \epsilon_k - \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_j$, 1, if $\alpha = \epsilon_k + \epsilon_j$. |
|27 | (A5, M6)a | $\epsilon_i - \epsilon_k, \epsilon_j - \epsilon_l, i < k < j < l$ | $\epsilon_i - \epsilon_l, i < k < j < l$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(s) \times x_{\epsilon_k - \epsilon_l}(s)$ | 1, if $\alpha = \epsilon_i - \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_k$, 1, if $\alpha = \epsilon_j - \epsilon_l$. |
|28 | (A5, M6)b | $\epsilon_i - \epsilon_k, \epsilon_j + \epsilon_l, i < k < j < l$ | $\epsilon_i + \epsilon_l, i < j < l$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i + \epsilon_k}(s^{-1}) \times x_{\epsilon_k - \epsilon_l}(-1)$ | 1, if $\alpha = \epsilon_i + \epsilon_l$, 1, if $\alpha = \epsilon_i - \epsilon_k$, 1, if $\alpha = \epsilon_j - \epsilon_l$. |
|29 | (A6, M1) | $2\epsilon_j$ | $\epsilon_i + \epsilon_j, i < j$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(-1/2)$ | 1, if $\alpha = \epsilon_i + \epsilon_j$, 1, if $\alpha = 2\epsilon_j$. |
|30 | (A6, M2) | $2\epsilon_k, 2\epsilon_j$ | $\epsilon_i + \epsilon_k, i < k < j$ | $h_{2\epsilon_i}(s^{-1}) \times x_{\epsilon_i - \epsilon_k}(-1/2) \times x_{\epsilon_k - \epsilon_j}(-1/2) \times x_{\epsilon_i - \epsilon_j}(I)$ | 1, if $\alpha = 2\epsilon_k$, 1, if $\alpha = 2\epsilon_j$. |
|   | $D_\alpha \setminus D_\tau$ | $D_\tau \setminus D_\alpha$ | $g(s)$ | $f(\alpha), \alpha \in \Phi^+$ |
|---|---|---|---|---|
| 31 | (A6, M3) | $\epsilon_i - \epsilon_k, 2\epsilon_j, \ i < k < j$ | $\epsilon_i + \epsilon_j$ | $
abla_2(\epsilon_2^{-1}) \times$ $\epsilon_i + \epsilon_j$ $x_{\epsilon_k + \epsilon_j}(s^{-1})$ $x_{\epsilon_i - \epsilon_j}(-1/2)$ $s$, if $\alpha = \epsilon_i + \epsilon_j$, $1$, if $\alpha = \epsilon_i - \epsilon_k$, $1$, if $\alpha = 2\epsilon_j$. |
| 32 | (A6, M4) | $\epsilon_i - \epsilon_k, 2\epsilon_j, 2\epsilon_l, \ i < k < j < l$ | $\epsilon_i - \epsilon_l, \ \epsilon_k + \epsilon_j$ | $\nabla_2(\epsilon_2^{-1}) \times$ $\epsilon_i - \epsilon_l$ $x_{\epsilon_k - \epsilon_j}(s^{-1})$ $x_{\epsilon_k - \epsilon_j}(-1/2)$ $x_{\epsilon_k - \epsilon_j}(I)$ $s$, if $\alpha = \epsilon_i - \epsilon_l$, $-s$, if $\alpha = \epsilon_k + \epsilon_l$, $I_s$, if $\alpha = \epsilon_i - \epsilon_j$, $-I_s$, if $\alpha = \epsilon_k + \epsilon_j$, $1$, if $\alpha = 2\epsilon_k$, $1$, if $\alpha = 2\epsilon_j$. |
| 33 | (A6, M5)$a$ | $\epsilon_i + \epsilon_l, 2\epsilon_k, 2\epsilon_j, \ i < k < j < l$ | $\epsilon_i + \epsilon_k, \ \epsilon_j + \epsilon_l$ | $\nabla_2(\epsilon_2^{-1}) \times$ $\epsilon_i + \epsilon_k$ $x_{\epsilon_l}(s^{-1})$ $x_{\epsilon_l}(-1/2)$ $x_{\epsilon_l}(I)$ $s$, if $\alpha = \epsilon_i + \epsilon_k$, $0$, if $\alpha = \epsilon_j + \epsilon_l$, $I_s$, if $\alpha = \epsilon_i + \epsilon_j$, $1$, if $\alpha = \epsilon_i + \epsilon_l$, $1$, if $\alpha = 2\epsilon_k$, $1$, if $\alpha = 2\epsilon_j$. |
| 34 | (A6, M5)$b$ | $\epsilon_i - \epsilon_l, 2\epsilon_k, 2\epsilon_j, \ i < k < j < l$ | $\epsilon_i + \epsilon_k, \ \epsilon_j - \epsilon_l$ | $\nabla_2(\epsilon_2^{-1}) \times$ $\epsilon_i - \epsilon_k$ $x_{\epsilon_l}(s^{-1})$ $x_{\epsilon_l}(-1/2)$ $x_{\epsilon_l}(I)$ $s$, if $\alpha = \epsilon_i + \epsilon_k$, $0$, if $\alpha = \epsilon_j - \epsilon_l$, $-I_s$, if $\alpha = \epsilon_i + \epsilon_j$, $1$, if $\alpha = \epsilon_i - \epsilon_l$, $1$, if $\alpha = 2\epsilon_k$, $1$, if $\alpha = 2\epsilon_j$. |
| 35 | (A6, M6)$a$ | $\epsilon_i - \epsilon_j, 2\epsilon_k, 2\epsilon_l, \ i < k < j < l$ | $\epsilon_i + \epsilon_k$ | $\nabla_2(\epsilon_2^{-1}) \times$ $\epsilon_i - \epsilon_k$ $x_{\epsilon_k}(s^{-1})$ $x_{\epsilon_k}(-1/2)$ $x_{\epsilon_k}(I)$ $s$, if $\alpha = \epsilon_i + \epsilon_k$, $2I_s$, if $\alpha = \epsilon_k + \epsilon_l$, $-I_s$, if $\alpha = \epsilon_i + \epsilon_l$, $1$, if $\alpha = \epsilon_i - \epsilon_k$, $1 - 2s$, if $\alpha = 2\epsilon_j$, $1 + 2s$, if $\alpha = 2\epsilon_l$. |
| 36 | (A6, M6)$b$ | $\epsilon_i - \epsilon_k, 2\epsilon_j, 2\epsilon_l, \ i < k < j < l$ | $\epsilon_i + \epsilon_j$ | $\nabla_2(\epsilon_2^{-1}) \times$ $\epsilon_i - \epsilon_j$ $x_{\epsilon_k}(s^{-1})$ $x_{\epsilon_k}(-1/2)$ $x_{\epsilon_k}(I)$ $s$, if $\alpha = \epsilon_i + \epsilon_j$, $2I_s$, if $\alpha = \epsilon_j + \epsilon_l$, $-I_s$, if $\alpha = \epsilon_i + \epsilon_l$, $1$, if $\alpha = \epsilon_i - \epsilon_j$, $1 - 2s$, if $\alpha = 2\epsilon_k$, $1 + 2s$, if $\alpha = 2\epsilon_l$. |
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