The Exact Rate Memory Tradeoff for Small Caches with Coded Placement

Vijith Kumar K P, Brijesh Kumar Rai and Tony Jacob

Abstract

The idea of coded caching was introduced by Maddah-Ali and Niesen who demonstrated the advantages of coding in caching problems. To capture the essence of the problem, they introduced the \( (N, K) \) canonical cache network in which \( K \) users with independent caches of size \( M \) request files from a server that has \( N \) files. Among other results, the caching scheme and lower bounds proposed by them led to a characterization of the exact rate memory tradeoff when \( M \geq \frac{N}{K}(K - 1) \). These lower bounds along with the caching scheme proposed by Chen et al. led to a characterization of the exact rate memory tradeoff when \( M \leq \frac{1}{K} \). In this paper we focus on small caches where \( M \in \left[ 0, \frac{N}{K} \right] \) and derive new lower bounds. For the case when \( \left\lfloor \frac{K+1}{2} \right\rfloor \leq N \leq K \) and \( M \in \left[ \frac{1}{K}, \frac{N}{K(N-1)} \right] \), our lower bounds demonstrate that the caching scheme introduced by Gómez-Vilardebó is optimal and thus extend the characterization of the exact rate memory tradeoff. For the case \( 1 \leq N \leq \left\lfloor \frac{K+1}{2} \right\rfloor \), we show that the new lower bounds improve upon the previously known lower bounds.

Index Terms

Coded caching, coded placement, exact rate memory tradeoff, lower bounds.

I. INTRODUCTION

Content distribution networks use memory distributed across the network, known as caches, to reduce the peak time data traffic by keeping copies of file fragments near the end-users. These techniques, known as caching techniques, generally operate in two phases. In the first phase, called the placement phase, the server fills the caches with fragments of files available in the server. In the second phase, called the delivery phase, the server broadcasts a set of packets to meet each user’s requests, aided by the caches available near to the user. Maddah-Ali and Niesen, in their seminal work \([1]\), noted that traditional caching techniques fail to exploit the multicast opportunity available in such networks. To address this limitation, they introduced the notion of coded caching and proposed a scheme to demonstrate that coding reduces the peak
data traffic load over traditional uncoded caching schemes. They introduced the canonical $(N, K)$ cache network where the server has $N$ files $\{W_1, \ldots, W_N\}$ and is communicating with $K$ users $\{U_1, \ldots, U_K\}$ through a common shared error-free link. Here, each user $U_k$ is equipped with an isolated cache $Z_k$ of size $M \in [0, N]$ as shown in Fig. [1]. During the placement phase, the server fills each user’s cache without knowing their future demands. Let the users’ requests be represented by $d = (W_{d_1}, \ldots, W_{d_K})$, where $W_{d_l}$ is the file requested by user $U_l$. During the delivery phase, when demand $d$ is revealed, the server broadcasts a set of packets $X_d$ of size $R_d(M)$ so that users can obtain their requested file, aided by their cache contents. The quantity $R_d(M)$ is called the load experienced by the network. The fundamental issue in a caching scheme is to decide what to store in each user’s cache and, accordingly, what to broadcast to fulfill each user’s demand such that the load experienced by the shared link is minimized. Variations of this problem have been introduced for decentralized networks [2], hierarchical networks [3], multiple servers [4], heterogeneous networks [6], D2D networks [8], and shared cache networks [7]. Issues of privacy [5] and data shuffling [9] have also been studied in this setup.

As a result of the inherent symmetry of the problem of coded caching for the canonical $(N, K)$ cache network, all demands related to each other through a permutation are naturally grouped together. This leads to the consideration of symmetric caching schemes which were shown by Tian [10] to operate at the same rate. Thus, we consider only the class of symmetric caching schemes in this paper. Consider a demand $d$, where the user $U_l$ requires the file $W_{d_l}$,

$$d = (W_{d_1}, \ldots, W_{d_K})$$

Fig. 1: The $(N, K)$ cache network
Let \( \pi(.) \) be a permutation operation defined over the set \( \{1, \ldots, K\} \) and \( \pi^{-1}(.) \) be its inverse. Now consider another demand \( \pi d \), which is obtained by permuting the files requested by the users,

\[
\pi d = (W_{d_{\pi^{-1}(1)}}, \ldots, W_{d_{\pi^{-1}(K)}}).
\]

In the demand \( \pi d \), the user \( U_{\pi(l)} \) requires the file \( W_{d_l} \). In response to the demand \( \pi d \), the server broadcasts a set of packets \( X_{\pi d} \). For a symmetric caching scheme, we have \([10]\)

\[
H(W_{d_l}, Z_{\pi(l)}, X_{\pi d}) = H(W_{d_l}, Z_l, X_d)
\]

Consider the demands where each of the \( N \) files is required by at least one user (and hence \( N \leq K \)). The set of all such demands is denoted by \( \mathcal{D} \) and the corresponding rate is denoted by \( R(M) \), where

\[
R(M) = \max\{R_d(M) \mid d \in \mathcal{D}\}.
\]

For the \( (N, K) \) cache network with cache size \( M \), the memory rate pair \((M, R)\) is said to be achievable if there is a scheme with \( R(M) \leq R \). For a such a scheme, we have

\[
H(Z_l) \leq M
\]

\[
H(X_d) \leq R
\]

\[
H(Z_l, X_d) = H(W_{d_l}, Z_l, X_d),
\]

\[
H(W_1, \ldots, W_N, Z_l, X_d) = H(W_1, \ldots, W_N),
\]

where \([5]\) follows from the fact that size of each cache is \( M \), \([6]\) follows from the fact that for any demand in \( \mathcal{D} \) the size of \( X_d \) is at most \( R(M) \leq R \), \([7]\) follows from the fact that the file \( W_{d_l} \) can be computed from \( X_d \) and \( Z_l \) by the user \( U_l \), and \([8]\) follows from the fact that \( Z_l \) and \( X_d \) are functions of files \( \{W_1, \ldots, W_N\} \). For a given cache size \( M \), the smallest \( R \) such that \((M, R)\) is achievable is called the exact rate memory tradeoff denoted by

\[
R^*(M) = \min\{R : (M, R) \text{ is achievable}\}
\]

Maddah-Ali and Niesen in \([1]\) proposed a coding scheme with an uncoded placement phase and a coded delivery phase for the demands in \( \mathcal{D} \) and demonstrated that the rate achieved by the proposed scheme is within a multiplicative gap of 12 from the optimal rate using cut set arguments. Several caching schemes were proposed in \([11]\)–\([19]\) to improve upon the rate achieved by the scheme proposed in \([1]\). Despite several lower bounds on the achievable rates
Caching Scheme | Cache Size ($M$) | Rate Memory Tradeoff | Condition
--- | --- | --- | ---
Chen et al. [11] | $[0, \frac{1}{K}]$ | $R^*(M) = N - NM$ | $N \leq K$ 
Gómez-Vilardebó [14] | $\left[ \frac{1}{N}, \frac{1}{N-1} \right]$ | $R^*(M) = \frac{N^2 - 1}{N} - (N-1)M$ | $N = K$ 
Vijith et al. [16], [17] | $\left[ N - 1 - \frac{1}{K}, N - 1 \right]$ | $R^*(M) = \frac{N+1}{N^2} - \frac{1}{N-1}M$ | $N = K$ 
Maddah-Ali & Niesen [1] | $\left[ \frac{N(K-1)}{K}, N \right]$ | $R^*(M) = 1 - \frac{1}{N}M$ | - 
Yu et al. [13] | $[0, N]$ | $R(M) = R_r + (R_r - R_{r+1}) \left( r - \frac{N}{K} \right)$ | Optimal among 
and $r \in \{1, \ldots, K\}$ uncoded prefetching 
schemes
This paper | $\left[ \frac{1}{K} \times \frac{N}{K(N-1)} \right]$ | $R^*(M) = \frac{KN - 1}{K} - (N-1)M$ | $\left[ \frac{K+1}{2} \right] \leq N \leq K$

TABLE I: Previous work in coded caching

being proposed in [20]–[24], the nature of the exact rate memory tradeoff is still elusive, except for the $(N, 2)$ cache network. The schemes proposed in [1], [11] provide a characterization of the exact rate memory tradeoff when $M \in \left[ 0, \frac{1}{K} \right] \cup \left[ \frac{N(K-1)}{K}, N \right]$. For the special case of $N = K$, the schemes proposed in [14], [17] provide a characterization of the exact rate memory tradeoff when $M \in \left[ \frac{1}{N}, \frac{1}{N(N-1)} \right] \cup \left[ N - 1 - \frac{1}{N}, N - 1 \right]$. In a surprising result, Yu et al. [13] showed the existence of a universal scheme among caching schemes with an uncoded placement phase. These results are summarised in TABLE I. In this paper, we consider the $(N, K)$ cache network where $N \leq K$ and $M \in \left[ \frac{1}{K} \times \frac{N}{K(N-1)} \right]$ and derive a set of new lower bounds for the demands in $D$.

The contributions of this paper are as follows:

- For the case $\left[ \frac{K+1}{2} \right] \leq N \leq K$, we derive a new lower bound and obtain a characterization of the exact rate memory tradeoff when $M \leq \frac{N}{K(N-1)}$.
- For the case $1 \leq N \leq \left[ \frac{K+1}{2} \right]$, we derive a new lower bound which improves upon the previously known lower bounds.

Throughout this paper we use $[L]$ to represent the set $\{1, 2, \ldots, L\}$, and $W_{[L]}$ to represent the set $\{W_1, W_2, \ldots, W_L\}$.

II. EXAMPLE NETWORKS

In this section, we consider two examples to motivate the results we present in the paper. The $(3, 4)$ network is an example for the case $\left[ \frac{K+1}{2} \right] \leq N \leq K$ and the $(2, 4)$ network is an example
for the case $1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil$.

A. Case I: The (3, 4) cache network

Here, users $\{U_1, U_2, U_3, U_4\}$ are connected to a server with three files $\{A, B, C\}$ (each of size $F$ bits). Each user $U_k$ has a cache $Z_k$ of size $MF$ bits. For a demand $d$, we have:

**Lemma 1.** For the (3, 4) cache network, achievable memory rate pairs $(M, R)$ must satisfy the constraint

$$8M + 4R \geq 11$$

**Proof.** We have,

$$8M + 4R \geq \begin{equation}
\begin{aligned}
&2H(Z_1) + 2H(Z_2) + H(Z_3) + 3H(Z_4) + 2H(X_{\{A,B,C,A\}}) + H(X_{\{B,C,A,A\}}) + H(X_{\{C,A,A,B\}}) \\
&\geq H(Z_2, X_{\{A,B,C,A\}}) + H(Z_4, X_{\{A,B,C,A\}}) + H(Z_1, Z_2, X_{\{A,B,C,A\}}) + H(Z_3, Z_4, X_{\{A,B,C,A\}}) \\
&= H(A, B, Z_1, Z_2, X_{\{A,B,C,A\}}) + H(A, B, Z_2, Z_4, X_{\{A,B,C,A\}}) + H(A, B, Z_1, Z_4, X_{\{B,C,A,A\}}) \\
&+ H(A, B, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq H(A, B, Z_1, Z_2, Z_4) + H(A, B, Z_1, Z_4, X_{\{B,C,A,A\}}) + H(A, B, Z_2, X_{\{A,B,C,A\}}) \\
&+ H(A, B, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq H(A, B, Z_1, Z_2, X_{\{B,C,A,A\}}) + H(A, B, Z_1, Z_4) + H(A, B, Z_2, X_{\{A,B,C,A\}}) \\
&+ H(A, B, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq H(A, B, Z_1, Z_2, Z_4) + H(A, B, Z_1, Z_4) + H(A, B, Z_2, X_{\{A,B,C,A\}}) \\
&+ H(A, B, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq H(A, B, Z_1, Z_2, Z_4) + H(A, B, Z_1, Z_4) + H(A, B, Z_2, X_{\{A,B,C,A\}}) \\
&+ H(A, B, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq 2H(A, B, Z_2, X_{\{A,B,C,A\}}) + H(A, B, Z_4) + H(A, B, C, Z_1, Z_3, Z_4, X_{\{C,A,A,B\}}) \\
&\geq 2H(A, B, C) + H(A, B, Z_2, X_{\{A,B,C,A\}}) + H(A, B, Z_4) \\
&\geq 2H(A, B, C) + H(A, B, X_{\{A,B,C,A\}}) + H(A, B, Z_4)
\end{aligned}
\end{equation}$$
\[(e) 2H(A, B, C) + H(A, B, X_{(A,A,B,C)}) + H(A, B, Z_4)\]
\[(b) \geq 2H(A, B, C) + H(A, B) + H(A, B, Z_4, X_{(A,A,B,C)})\]
\[(c) = 2H(A, B, C) + H(A, B) + H(A, B, C, Z_4, X_{(A,A,B,C)})\]
\[(c) = 3H(A, B, C) + H(A, B) \geq 11,\]

where

(a) follows from (6) and (5),
(b) follows from the submodularity property of entropy,
(c) follows from (7),
(d) follows from (8),
(e) follows from (3).

The above result improves upon the previous results from [1], [14], [22] and is summarised in TABLE II and Fig. 2.

| Memory | Rate [14] | Lower Bound [1], [22] | New Lower Bound |
|--------|-----------|------------------------|-----------------|
| \[\frac{1}{4} \leq M \leq \frac{3}{8}\] | \[\frac{11}{4} - 2M\] | \[R \geq \max \left\{ (3 - 3M), \left( \frac{8}{3} - 2M \right) \right\}\] | \[R \geq \frac{11}{4} - 2M\] |

TABLE II: Rate memory tradeoff for the \((3, 4)\) cache network

\[\text{Rate Memory Tradeoff } [14, 11], \quad \text{New Rate Memory Tradeoff}\]

Fig. 2: Rate memory tradeoff for the \((3, 4)\) cache network
B. Case II: The (2, 4) cache network

Here, users \( \{U_1, U_2, U_3, U_4\} \) are connected to a server with files \( \{A, B\} \) (each of size \( F \) bits). Each user \( U_k \) has cache \( Z_k \) of size \( MF \) bits. For a demand \( d \), we have:

**Lemma 2.** For the (2, 4) cache network, achievable memory rate pairs \((M, R)\) must satisfy the constraint

\[
8M + 6R \geq 11.
\]

**Proof.** We have,

\[
8M + 6R \geq 2H(Z_1) + H(Z_2) + 2H(Z_3) + 3H(Z_4) + 3H(X_{(A,B,A,A)}) + 2H(X_{(B,A,A,A)}) + H(X_{(A,A,B,A)})
\]
\[
\geq H(Z_1, X_{(A,B,A,A)}) + H(Z_3, X_{(A,B,A,A)}) + H(Z_4, X_{(A,B,A,A)}) + H(Z_3, X_{(B,A,A,A)}) + H(Z_4, X_{(B,A,A,A)})
\]
\[
+ H(Z_4, X_{(A,A,B,A)}) + H(Z_2) + H(Z_1)
\]
\[
\geq H(A, Z_1, X_{(A,B,A,A)}) + H(A, Z_3, X_{(A,B,A,A)}) + H(A, Z_4, X_{(A,B,A,A)}) + H(A, Z_3, X_{(B,A,A,A)})
\]
\[
+ H(A, Z_4, X_{(B,A,A,A)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(Z_2) + H(Z_1)
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}) + 2H(A, X_{(A,B,A,A)}) + H(A, Z_3, Z_4, X_{(B,A,A,A)}) + H(A, X_{(B,A,A,A)})
\]
\[
+ H(A, Z_4, X_{(A,A,B,A)}) + H(Z_2) + H(Z_1)
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}) + H(A, Z_3, Z_4, X_{(B,A,A,A)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ H(A, X_{(A,B,A,A)}) + H(A, X_{(B,A,A,A)}) + H(Z_2) + H(Z_1)
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}) + H(A, Z_3, Z_4, X_{(B,A,A,A)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ H(A, Z_4, X_{(A,A,B,A)}) + H(A, Z_1, X_{(B,A,A,A)})
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}) + H(A, Z_3, Z_4, X_{(B,A,A,A)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ H(A, Z_2, X_{(A,B,A,A)}) + H(A, Z_1, X_{(B,A,A,A)})
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}) + H(A, Z_3, Z_4, X_{(B,A,A,A)}) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ 2H(A, B)
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}, X_{(B,A,A,A)}) + H(A, Z_3, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ 2H(A, B)
\]
\[
\geq H(A, Z_1, Z_3, Z_4, X_{(A,B,A,A)}, X_{(B,A,A,A)}) + H(A, Z_3, Z_4) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})
\]
\[
+ 2H(A, B)
\]
\((d)\) \(3H(A, B) + H(A, Z_3, Z_4) + H(A, Z_4, X_{(A,A,B,A)}) + H(A, X_{(A,A,A,B)})\)

\((a)\) \(\geq 3H(A, B) + H(A, Z_3, Z_4, X_{(A,A,B,A)}) + H(A, Z_4) + H(A, X_{(A,A,A,B)})\)

\((b)\) \(\geq 3H(A, B) + H(A, Z_3, Z_4, X_{(A,A,B,A)}) + H(A, Z_4) + H(A, X_{(A,A,A,B)})\)

\((d)\) \(= 4H(A, B) + H(A, Z_4) + H(A, X_{(A,A,A,B)})\)

\((a)\) \(\geq 4H(A, B) + H(A, Z_4, X_{(A,A,A,B)}) + H(A)\)

\((b)\) \(\geq 4H(A, B) + H(A, Z_4, X_{(A,A,A,B)}) + H(A)\)

\((d)\) \(\geq 5H(A, B) + H(A) \geq 11,\)

where

\((a)\) follows from the submodularity property of entropy,

\((b)\) follows from (7),

\((c)\) follows from (3),

\((d)\) follows from (8).

The above result improves upon the previous results from [1], [14], [22] and is summarised in TABLE III and Fig. 3.

| Memory | Rate [14] | Lower Bound [1], [22] | New Lower Bound |
|--------|-----------|------------------------|------------------|
| \(\frac{1}{2} \leq M \leq \frac{1}{2}\) | \(\frac{32}{18} - \frac{10}{9} M\) | \(R \geq 2 - 2M\) | \(R \geq \frac{11}{6} - \frac{4}{3} M\) |

TABLE III: Rate memory tradeoff for the (2, 4) cache network

**Remark 1.** It should be noted that, for the (2, 4) cache network, the bound \(8M + 6R \geq 11\) is already mentioned in [10]. We present the proof above, which can be extended to the \((N,K)\) cache network.

### III. New Lower Bounds

In this section, we derive new lower bounds on the rate memory tradeoff for the \((N,K)\) cache network where \(N \leq K\) and cache size \(M \in \left[\frac{1}{K}, \frac{N}{K}\right]\). The key ideas we employ are identities (7), (8) and the properties of symmetric caching schemes stated in (3). As in Section II, we consider two cases, namely \(\left\lfloor \frac{K+1}{2} \right\rfloor \leq N \leq K\) and \(1 \leq N \leq \left\lfloor \frac{K+1}{2} \right\rfloor\).
Fig. 3: Rate memory tradeoff for the (2, 4) cache network

A. Case I: \( \left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K \)

Consider the demand

\[ d_1 = (W_1, W_2, \ldots, W_N, W_1, W_2, \ldots, W_{K-N}) \]  \hspace{1cm} (11)

Demands \( \{d_l : 2 \leq l \leq K\} \), are obtained from the demand \( d_1 \) by cyclic left shifts as shown in TABLE IV. For the demand \( d_l \), let \( X_{d_l} \) denote the set of packets broadcast by the server. Consider the user index \( \bar{l} \) defined as

\[ \bar{l} = \begin{cases} 
N + 1 - l, \text{ for } 1 \leq l \leq N \\
K + N + 1 - l, \text{ for } N + 1 \leq l \leq K 
\end{cases} \]  \hspace{1cm} (12)

It can be noted that in demand \( d_l \), the user \( U_7 \) requires the file \( W_N \). For \( S \subseteq \{U_1, \ldots, U_K\} \), let \( Z_S \) denote the cache contents of all the users in set \( S \).

The following lemma are easy to obtain:

**Lemma 3.** For \( S, T \subset \{U_1, \ldots, U_K\} \setminus \{U_7\} \), we have the identity

\[ H(W_{[N-1]}, Z_S, Z_7) + H(W_{[N-1]}, Z_T, X_{d_l}) \geq H(W_{[N-1]}, Z_{S \cap T}) + N, \]
Proof. We have,

\[
H(W_{[N-1]}, Z_S, Z_T) + H(W_{[N-1]}, Z_T, X_{d_l}) \overset{\text{(a)}}{=} H(W_{[N-1]}, Z_{S \cup T}) + H(W_{[N-1]}, Z_{S \cup T}, Z_T, X_{d_l}) \\
\overset{\text{(b)}}{=} H(W_{[N-1]}, Z_{S \cup T}) + H(W_{[N-1]}, W_N, Z_{S \cup T}, Z_T, X_{d_l}) \\
\overset{\text{(c)}}{=} H(W_{[N-1]}, Z_{S \cup T}) + H(W_{[N]}) \\
= H(W_{[N-1]}, Z_{S \cup T}) + N
\]

where

(a) follows from the submodularity property of entropy,

(b) follows from (7),

(c) follows from (8).

\[\square\]

**Lemma 4.** For a sequence of sets \(S_i \subset \{U_1, \ldots, U_K\} \setminus \{U_l\}\), such that \(S_i = S_{i+1} \cup \{U_{i+1}^{-}\}\), we
have

\[ H(W_{[N-1]}, Z_{S_j}) + \sum_{i=l+1}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \geq (j - l)N + H(W_{[N-1]}, Z_{S_j}) \]

**Proof.** We have,

\[
H(W_{[N-1]}, Z_{S_j}) + \sum_{i=l+1}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \\
= H(W_{[N-1]}, Z_{S_j}) + H(W_{[N-1]}, Z_{S_{i+1}}, X_{d_{i+1}}) + \sum_{i=l+2}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \\
\overset{(a)}{=} \left[ H(W_{[N-1]}, Z_{S_{i+1}}, Z_{r_{i+1}}) + H(W_{[N-1]}, Z_{S_{i+1}}, X_{d_{i+1}}) \right] + \sum_{i=l+2}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \\
\overset{(b)}{\geq} N + \left[ H(W_{[N-1]}, Z_{S_{i+1}}, X_{d_{i+1}}) \right] + \sum_{i=l+2}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \\
\overset{(c)}{\geq} 2N + H(W_{[N-1]}, Z_{S_{i+2}}) + H(W_{[N-1]}, Z_{S_{t+3}}, X_{d_{t+3}}) + \sum_{i=l+4}^{j} H(W_{[N-1]}, Z_{S_i}, X_{d_i}) \\
\overset{(d)}{\geq} (j - l)N + H(W_{[N-1]}, Z_{S_j})
\]

where

(a) follows from definition of set \( S_j \),

(b) follows from Lemma 3 with \( S \cup \{ U_7 \} = S_{i+1} \) and \( T = S_{i+2} \),

(c) follows from Lemma 3 with \( S \cup \{ U_7 \} = S_{i+2} \) and \( T = S_{i+3} \),

(d) follows from repeated use of Lemma 3 with \( S \cup \{ U_7 \} = S_i \) and \( T = S_{i+1} \)

for \( l + 3 \leq i \leq j \).

\[ \square \]

In a similar fashion, for a sequence of sets \( T_i \subset \{ U_1, \ldots, U_K \} \setminus \{ U_7 \} \), such that \( T_{i+1} = T_i \cup \{ U_7 \} \), we can obtain

\[
H(W_{[N-1]}, Z_{T_j}, Z_{T_j}) + \sum_{i=l}^{j} H(W_{[N-1]}, Z_{T_i}, X_{d_i}) \geq (j - l + 1)N + H(W_{[N-1]}, Z_{T_j})
\]

(13)

For \( 1 \leq i \leq N \), consider the sets of users as shown below:
These sets are also indicated in TABLE IV. Note that

\[ A_N = B_1 = C_N = \phi \]  \hspace{1cm} (14)

\[ A_{i+1} \cup \{U_{i+1}\} = A_i \]  \hspace{1cm} (15)

\[ B_i \cup \{U_{N+i}\} = B_{i+1} \]  \hspace{1cm} (16)

\[ B_{K-N} \cup \{U_{1}\} = B_{K-N+1} = E \]  \hspace{1cm} (17)

\[ A_i \cap C_i = C_i \]  \hspace{1cm} (18)

\[ B_i \cap E = \left\{ \begin{array}{ll} B_i & \text{when } 1 \leq i \leq K-N \\ E & \text{when } K-N+1 \leq i \leq N \end{array} \right. \]  \hspace{1cm} (19)

It can be noted that in the demands \(d_i\) and \(d_{N+i}\), the users in set \(B_i\) are requesting for the same set of files \(\{W_1, \ldots, W_{i-1}\}\) (for \(1 \leq i \leq N\)). Thus, from (3) we have

\[ H(W_{[i-1]}, Z_{B_i}, X_{d_i}) = H(W_{[i-1]}, Z_{B_i}, X_{d_{N+i}}) \]  \hspace{1cm} (20)

Note that \(|A_i \cup B_i| = |C_i \cup E| = N - 1\). Thus, we have

\[ (N - 1)M + R \geq H(Z_{A_i \cup B_i}) + H(X_{d_i}) \geq H(Z_{A_i \cup B_i}, X_{d_i}) \]  \hspace{1cm} (21)

Similarly,

\[ (N - 1)M + R \geq H(Z_{C_i \cup E}) + H(X_{d_i}) \geq H(Z_{C_i \cup E}, X_{d_i}) \]  \hspace{1cm} (22)

Now, we have the following result:

**Theorem 1.** For the \((N, K)\) cache network, when \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\), achievable memory rate pairs \((M, R)\) must satisfy the constraint

\[ K(N - 1)M + KR \geq KN - 1. \]
Proof. We have,

\[ K(N - 1)M + KR = \sum_{i=1}^{N} \left[ (N - 1)M + R + (N - 1)M + R \right] + \sum_{i=K-N+1}^{N} \left[ (N - 1)M + R \right] \]

\[ \geq \sum_{i=1}^{K-N} \left[ H(Z_{A_i \cup B_i}, X_{d_i}) + H(Z_{C_i \cup E}, X_{d_i}) \right] + \sum_{i=K-N+1}^{N} H(Z_{A_i \cup B_i}, X_{d_i}) \]

\[ \geq \sum_{i=1}^{K-N} \left[ H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) + H(W_{[N-1]}, Z_{C_i \cup E}, X_{d_i}) \right] + \sum_{i=K-N+1}^{N} H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) \]

\[ \geq \sum_{i=1}^{K-N} \left[ H(W_{[N-1]}, Z_{A_i \cup E}, X_{d_i}) + \sum_{i=2}^{K-N} H(W_{[N-1]}, Z_{A_i \cup E}, X_{d_i}) \right] + \sum_{i=K-N+1}^{N} H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) \]

\[ + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i \cup C_i}, X_{d_i}) \]

\[ \geq (K - N - 1)N + \left[ H(W_{[N-1]}, Z_{A_{K-N} \cup E}) + \sum_{i=K-N+1}^{N} H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) \right] \]

\[ + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i \cup C_i}, X_{d_i}) \]

\[ \geq (K - N - 1)N + \left[ H(W_{[N-1]}, Z_{A_{K-N} \cup B_{K-N+1}}) + \sum_{i=K-N+1}^{N} H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) \right] \]

\[ + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i \cup C_i}, X_{d_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, Z_{A_N \cup B_{K-N+1}}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i \cup C_i}, X_{d_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, Z_{B_{K-N+1}}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i}, X_{d_i}) \]

\[ \geq (N - 1)N + H(W_{[N-1]}, Z_{B_{K-N+1}}) + \sum_{i=1}^{K-N} H(W_{[N-1]}, Z_{B_i}, X_{d_{N+i}}) \]

\[ \geq (K - 1)N + H(W_{[N-1]}, Z_{B_1}) \]

\[ \geq (K - 1)N + H(W_{[N-1]}) \geq KN - 1 \]
where

(a) follows from (21) and (22),
(b) follows from (7) and definition of sets \(A_i, B_i, C_i\) and \(E\),
(c) follows from the submodularity property of entropy, and the facts that \(A_i \cap C_i = C_i\) and \(B_i \cap E = B_i\) for \(1 \leq i \leq K - N\) (refer (18) and (19)),
(d) follows from Lemma 4 with \(S_i = A_i \cup E\), \(l = 1\), \(j = K - N\) and (15),
(e) follows from (17),
(f) follows from Lemma 4 with \(S_i = A_i \cup B_i\), \(l = K - N\), \(j = N\) and (15),
(g) follows from (14),
(h) follows from (20),
(i) follows from (13) with \(T_i = B_i\), \(l = 1\), \(j = K - N\) and (16).

\[ \square \]

B. Case II: \(1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil \)

Consider the demand

\[ d_1 = (W_1, W_2, \ldots, W_N, W_1, W_2, \ldots, W_{N-1}, W_1, W_1, \ldots, W_1) \quad (23) \]

Demands \(\{d_l : 2 \leq l \leq K\}\), are obtained from the demand \(d_1\) by cyclic left shifts as shown in TABLE V.

Consider the user index \(\overline{l}\) defined as

\[ \overline{l} = \begin{cases} 
N + 1 - l, & \text{for } 1 \leq l \leq N \\
K + N + 1 - l, & \text{for } N + 1 \leq l \leq K 
\end{cases} \quad (24) \]

It can be noted that in demand \(d_l\), the user \(U_{\overline{l}}\) requires the file \(W_N\). The following lemma is easy to obtain.

**Lemma 5.** Let \(A, B, C \subset \{U_1, \ldots, U_K\}\) be such that in demand \(d_l\), every user in \(B\) requests the file \(W_1\) and users in \(C\) together request all the files in \(\{W_2, \ldots, W_N\}\). We have

\[ H(W_{[N-1]}, Z_A, X_{d_l}) + \sum_{i \in B} H(Z_i) + |B| H(X_{d_l}) + |B| H(Z_C) \geq H(W_{[N-1]}, Z_{A \cup B}, X_{d_l}) + |B| N \]
Proof. We have,

\[ H(W_{[N-1]}, Z_A, X_{d_1}) + \sum_{i \in B} H(Z_i) + |B| H(X_{d_1}) + |B| H(Z_C) \]

\[ = H(W_{[N-1]}, Z_A, X_{d_1}) + \sum_{i \in B} \left[ H(Z_i) + H(X_{d_1}) \right] + |B| H(Z_C) \]

\[ \geq H(W_{[N-1]}, Z_A, X_{d_1}) + \sum_{i \in B} H(Z_i, X_{d_1}) + |B| H(Z(C) \]

\[ \equiv H(W_{[N-1]}, Z_A, X_{d_1}) + \sum_{i \in B} H(W_1, Z_i, X_{d_1}) + |B| H(Z_C) \]

\[ \geq H(W_{[N-1]}, Z_A, X_{d_1}) + \left[ H(W_1, Z_B, X_{d_1}) + (|B| - 1)H(W_1, X_{d_1}) \right] + |B| H(Z_C) \]

TABLE V: Demand set \{d_l : 1 \leq l \leq K\}
\[ a \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + H(W_1, Z_{A \cup B}, X_{d_i}) + |B| H(W_1, X_{d_i}) + |B| H(Z_C) \]

\[ \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| \left[H(W_1, X_{d_i}) + H(Z_C)\right] \]

\[ (a) \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| \left[H(W_1, X_{d_i}) + H(Z_C)\right] \]

\[ (b) \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| H(W_{N}, Z_C, X_{d_i}) \]

\[ (c) \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| H(W_{N}), Z_C, X_{d_i}) \]

\[ (d) \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| H(W_{N}) \]

\[ \geq H(W_{N-1}, Z_{A \cup B}, X_{d_i}) + |B| N \]

where

(a) follows form the submodularity property of entropy,

(b) follows from (7) and the definition of \( B \),

(c) follows from (7) and the definition of \( C \),

(d) follows from (8).

For \( 1 \leq i \leq N \), consider the sets of users as shown below:

| Set | Users | Number | Files Requested in Demand \( d_i \) |
|-----|-------|--------|-----------------------------------|
| \( A_i \) | \( U_1, \ldots, U_{N-i} \) | \( N-i \) | \( W_j, \ldots, W_{N-1} \) |
| \( B_i \) | \( U_{K-i+2}, \ldots, U_K \) | \( i-1 \) | \( W_1, \ldots, W_{i-1} \) |
| \( F_i \) | \( U_{N-i+1}, \ldots, U_{2N-i} \) | \( N-i \) | \( W_j, \ldots, W_{N-1} \) |
| \( G_i \) | \( U_{2N-i+1}, \ldots, U_{K-i+1} \) | \( K-2N+1 \) | \( W_1 \) |
| \( J_i \) | \( U_1, \ldots, U_{N-i+1} \) | \( N-i+1 \) | \( W_i, \ldots, W_N \) |
| \( K_i \) | \( U_{K-i+3} \ldots U_K \) | \( i-2 \) | \( W_2, \ldots, W_{i-1} \) |

These sets are also indicated in TABLE V. Let

\[ I_i = J_i \cup K_i \] \hspace{1cm} (25)

\[ L_i = A_i \cup B_i \cup F_i \cup G_i \] \hspace{1cm} (26)

Note that

\[ A_N = B_1 = F_N = K_1 = K_2 = \phi \] \hspace{1cm} (27)

\[ L_{i+1} \cup \{U_{i+1}\} = L_i \] \hspace{1cm} (28)

\[ B_i \cup \{U_{N+i}\} = B_{i+1} \] \hspace{1cm} (29)
It can be noted that in the demands $d_i$ and $d_{N+i}$, users in the set $B_i$ are requesting for the same set of files $\{W_1, \ldots, W_{i-1}\}$ (for $1 \leq i \leq N$). Thus, from (3) we have

$$H(W_{[i-1]}, Z_{B_i}, X_{d_i}) = H(W_{[i-1]}, Z_{B_i}, X_{d_{N+i}}) \quad (30)$$

Note that $|A_i \cup B_i| = |B_i \cup F_i| = N - 1$. Thus, we have

$$(N - 1)M + R \geq H(Z_{A_i \cup B_i}) + H(X_{d_i}) \geq H(Z_{A_i \cup B_i}, X_{d_i}) \quad (31)$$

Similarly,

$$(N - 1)M + R \geq H(Z_{B_i \cup F_i}) + H(X_{d_i}) \geq H(Z_{B_i \cup F_i}, X_{d_i}) \quad (32)$$

We can now obtain the following lemma:

**Lemma 6.** The sets $B_i$ and $L_i$, defined as above, satisfy

$$(N^2(K - 2N + 3) - 3N + 1)M + (N(K - 2N + 3) - 1)R$$

$$\geq H(W_{[N-1]}, Z_{L_N}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i}, X_{d_{N+i}}) + N((K - 2N + 2)N - 1)$$

**Proof.** We have,

$$(N^2(K - 2N + 3) - 3N + 1)M + (N(K - 2N + 3) - 1)R$$

$$= \sum_{i=1}^{N} \left[ (N - 1)M + R + (K - 2N + 1)M + (K - 2N + 1)(R + (N - 1)M) \right] + \sum_{i=1}^{N-1} [(N - 1)M + R]$$

$$\geq \sum_{i=1}^{N} \left[ H(Z_{A_i \cup B_i}, X_{d_i}) + \sum_{j \in G_i} H(Z_j) + |G_i| \cdot H(X_{d_i}) + |G_i| \cdot H(Z_i) \right] + \sum_{i=1}^{N-1} H(Z_{B_i \cup F_i}, X_{d_i})$$

$$= \sum_{i=1}^{N} \left[ H(W_{[N-1]}, Z_{A_i \cup B_i}, X_{d_i}) + \sum_{j \in G_i} H(Z_j) + |G_i| \cdot N \right] + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i \cup F_i}, X_{d_i})$$

$$= H(W_{[N-1]}, Z_{A_N \cup B_N \cup G_N}, X_{d_N}) + \sum_{i=1}^{N-1} \left[ H(W_{[N-1]}, Z_{A_i \cup B_i \cup G_i}, X_{d_i}) + H(W_{[N-1]}, Z_{B_i \cup F_i}, X_{d_i}) \right]$$

$$+ \sum_{i=1}^{N} |G_i| \cdot N$$
(d) \[ H(W_{[N-1]}, Z_{A_N \cup B_N \cup G_N \cup F_N}, X_{d_N}) + \sum_{i=1}^{N-1} \left[ H(W_{[N-1]}, Z_{A_i \cup B_i \cup G_i \cup F_i}, X_{d_i}) + H(W_{[N-1]}, Z_{B_i}, X_{d_i}) \right] \]
+(K - 2N + 1)N^2

(e) \[ \sum_{i=1}^{N} H(W_{[N-1]}, Z_{L_i}, X_{d_i}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i}, X_{d_i}) + (K - 2N + 1)N^2 \]
\geq \left[ H(W_{[N-1]}, Z_{L_1}) + \sum_{i=2}^{N} H(W_{[N-1]}, Z_{L_i}, X_{d_i}) \right] + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i}, X_{d_i}) + (K - 2N + 1)N^2

(f) \[ (N - 1)N + H(W_{[N-1]}, Z_{L_{N_I}}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i}, X_{d_i}) + (K - 2N + 1)N^2 \]
\geq H(W_{[N-1]}, Z_{L_N}) + \sum_{i=1}^{N-1} H(W_{[N-1]}, Z_{B_i}, X_{d_{N_i+1}}) + N((K - 2N + 2)N - 1)

where

(a) follows from \((31)\) and \((32)\),
(b) follows from \((7)\) and definition of set \(A_i, B_i,\) and \(F_i,\)
(c) follows from Lemma 5 with \(A = A_i \cup B_i, B = G_i\) and \(C = I_i,\)
(d) follows from \((27)\) the submodularity property of entropy,
(e) follows from the definition of \(L_i,\)
(f) follows from Lemma 4 with \(S_i = L_i, l = 1, j = N\) and \((28)\),
(g) follows from \((30)\).

\[ \square \]

Now, for \(2N \leq j \leq K,\) consider another sets of users as shown below:

| Set  | Users                                      | Number | Files Requested in Demand \(d_j\) |
|------|--------------------------------------------|--------|-----------------------------------|
| \(P_j\) | \(U_{K+N+2-j}, \ldots, U_{K+2N-j}\)   | \(N - 1\) | \(W_1, \ldots, W_{N-1}\) |
| \(Q_j\) | \(U_{K+2N+1-j}, \ldots, U_K\)          | \(j - 2N\) | \(W_1\) |
| \(S_j\) | \(U_{K-j+3}, \ldots, U_{K+N-j+2}\)     | \(N - 1\) | \(W_2, \ldots, W_N\) |

These sets are also indicated in TABLE V. Let
\[ T_j = P_j \cup Q_j \] (33)

Note that
\[ Q_{2N} = S_{2N} = \phi \] (34)
\[ T_{j+1} \cup \{U_{j+1}\} = T_j \] (35)
\[ T_K \cup \{ U_K \} = L_N \]  
\[ B_{N-1} \cup \{ U_{2N-1} \} = B_N = T_{2N} \]

Note that \(| P_j | = N - 1\). Thus, we have
\[(N - 1)M + R \geq H(Z_{P_j}) + H(X_{d_j}) \geq H(Z_{P_j}, X_{d_j}) \]  
(38)

The following lemma is easy to obtain:

**Lemma 7.** The set \( T_j \), as defined above, satisfy
\[
\frac{(K - 2N + 1)}{2} \left[ (N(K - 2N + 2) - 2)M + (K - 2N + 2)R \right] \geq \sum_{j=2N}^{K} \left( H(W_{[N-1]}, Z_{T_j}, X_{d_j}) \right) \\
+ \frac{(K - 2N + 1)(K - 2N)}{2} N
\]

**Proof.** We have,
\[
\frac{(K - 2N + 1)}{2} \left[ (N(K - 2N + 2) - 2)M + (K - 2N + 2)R \right] \\
= \sum_{j=2N}^{K} \left[ ((N - 1)M + R) + (j - 2N)M + (j - 2N)(R + (N - 1)M) \right] \\
\geq \sum_{j=2N}^{K} \left[ H(Z_{P_j}, X_{d_j}) + \sum_{i \in Q_j} H(Z_i) + | Q_j | H(X_{d_j}) + | Q_j | H(Z_{S_j}) \right] \\
\geq \sum_{j=2N}^{K} \left[ H(W_{[N-1]}, Z_{P_j}, X_{d_j}) + \sum_{i \in Q_j} H(Z_i) + | Q_j | H(X_{d_j}) + | Q_j | H(Z_{S_j}) \right] \\
\geq \sum_{j=2N}^{K} \left[ H(W_{[N-1]}, Z_{P_j} \cup Q_j, X_{d_j}) + | Q_j | N \right] \\
\geq \sum_{j=2N}^{K} \left[ H(W_{[N-1]}, Z_{T_j}, X_{d_j}) + \sum_{j=2N}^{K} (j - 2N)N \right] \\
= \sum_{j=2N}^{K} \left( H(W_{[N-1]}, Z_{T_j}, X_{d_j}) + \frac{(K - 2N + 1)(K - 2N)}{2} N \right)
\]

where

(a) follows from (38) and the fact that \( Q_{2N} = \phi \),
(b) follows from definition of sets \( P_j, Q_j \) and (7),
(c) follows from Lemma 5 with \( A = P_j, B = Q_j \) and \( C = S_j \),
(d) follows from the definition of \( T_j \).
Using the above lemma, we can obtain the following result:

**Theorem 2.** For the \((N, K)\) cache network, when \(1 \leq N < \left\lceil \frac{K+1}{2} \right\rceil\), achievable memory rate pairs \((M, R)\) must satisfy the constraint

\[
\frac{K(N(K + 3) - 2(N^2 + 1))}{2} M + \frac{K(K + 3 - 2N)}{2} R \geq \frac{NK(K - 2N + 3) - 2}{2}
\]

**Proof.** We have,

\[
\frac{K(N(K + 3) - 2(N^2 + 1))}{2} M + \frac{K(K + 3 - 2N)}{2} R
\]

\[
= \left[ (N^2(K - 2N + 3) - 3N + 1)M + (N(K - 2N + 3) - 1)R \right] + \left[ \frac{(K - 2N + 1)}{2} \right] \left( N(K - 2N + 2) - 2 \right) M + (K - 2N + 2) R \right]
\]

\[
\geq (a) \left[ H(W_{[N+1]}, Z_{L_{N}}) + \sum_{i=1}^{N-1} H(W_{[N+1]}, Z_{B_i}, X_{d_{(N+i)}}) + N((K - 2N + 2)N - 1) \right]
\]

\[
+ \left[ \sum_{j=2N}^{K} H(W_{[N+1]}, Z_{T_j}, X_{d_j}) + \frac{(K - 2N + 1)(K - 2N)}{2} N \right]
\]

\[
\geq (b) \left[ H(W_{[N+1]}, Z_{T_{K}}, Z_{K}) + \sum_{j=2N}^{K} H(W_{[N+1]}, Z_{T_j}, X_{d_j}) + \sum_{i=1}^{N-1} H(W_{[N+1]}, Z_{B_i}, X_{d_{(N+i)}}) \right]
\]

\[
+ \frac{(K(K - 2N + 1) + 2N - 2)}{2} N
\]

\[
\geq (c) \left[ (K - 2N + 1)N + H(W_{[N+1]}, Z_{T_{2N}}) \right] + \sum_{i=1}^{N-1} H(W_{[N+1]}, Z_{B_i}, X_{d_{(N+i)}}) + \frac{(K(K - 2N + 1) + 2N - 2)}{2} N
\]

\[
\geq (d) \left[ H(W_{[N+1]}, Z_{B_{N-1}}, Z_{2N-1}) + \sum_{i=1}^{N-1} H(W_{[N+1]}, Z_{B_i}, X_{d_{(N+i)}}) \right] + \frac{(K(K - 2N + 3) - 2N)}{2} N
\]

\[
\geq (e) \left[ (N - 1)N + H(W_{[N+1]}, Z_{B_1}) \right] + \frac{(K(K - 2N + 3) - 2N)}{2} N
\]

\[
\geq (f) H(W_{[N+1]}) + \frac{(K(K - 2N + 3) - 2N)}{2} N
\]

\[
\geq \frac{NK(K - 2N + 3) - 2}{2}
\]

where
(a) follows from Lemma 6 and Lemma 7
(b) follows from (36),
(c) follows from (13) with \(T_i = T_j, l = 2N, j = K\) and (35),
(d) follows from (37),
(e) follows from (13) with \(T_i = B_i, l = 1, j = N - 1\) and (29),
(f) follows from (27).

IV. COMPARISON WITH PREVIOUS BOUNDS

In [1], Maddah-Ali and Niesen derived a lower bound on achievable rates using cut set arguments, which was further improved in [20]–[24]. A comparison between these lower bounds and the new lower bounds in Section III, at cache size \(M = \frac{N}{K(N-1)}\), is given in TABLE VI. It can be noted that the new bounds improve upon the previous ones. For the \((N, K)\) cache network, the scheme proposed by Gómez-Vilardebó in [14] achieves memory rate pairs \((M, R_G) = \left(\frac{KN - 1}{K}, \frac{N}{K(N-1)} - (N-1)M\right)\), for \(M \in \left[\frac{1}{K}, \frac{N}{K(N-1)}\right]\). From Theorem 1, when \(\left[\frac{K+1}{2}\right] \leq N \leq K\), we have that all achievable memory rate pairs satisfy the constraint

\[
R \geq \frac{KN - 1}{K} - (N-1)M = R_G(M)
\]

Thus we have:
Theorem 3. For the \((N,K)\) cache network, when \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\), the exact rate memory tradeoff is given by

\[
R^*(M) = \frac{KN - 1}{K} - (N - 1)M
\]  

(39)

where \(M \in \left[ \frac{1}{K}, \frac{N}{K(N-1)} \right]\).

Remark 2. In \([14]\), with the help of the lower bounds derived in \([22]\) and \([24]\), Gómez-Vilardebó showed that when \(K = N\) and \(M = \frac{1}{N-1}\), his scheme is optimal. We extend his result to the case where \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\) in Theorem 3.

V. Conclusions

In this paper we considered the canonical \((N,K)\) cache network where \(N \leq K\) and \(M \in [0, \frac{N}{K}]\). We derived a new set of lower bounds on the achievable rate when each file in the server is requested by at least one user. Using these lower bounds, we showed that when \(\left\lceil \frac{K+1}{2} \right\rceil \leq N \leq K\) the scheme proposed in \([14]\) is optimal for \(M \in \left[ \frac{1}{K}, \frac{N}{K(N-1)} \right]\). For the case \(1 \leq N \leq \left\lceil \frac{K+1}{2} \right\rceil\), the new lower bound was shown to improve upon the previous lower bounds, but a matching scheme is still not known. The work presented forms another step in the attempt to find a characterization of the exact rate memory tradeoff for coded caching and is illustrated in Fig. 4.

![Rate Memory Tradeoff](image)

Fig. 4: Rate memory tradeoff for the \((N,K)\) cache network when \(\left\lceil \frac{K+1}{2} \right\rceil < N \leq K\)
REFERENCES

[1] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” IEEE Transactions on Information Theory, vol. 60, no. 5, pp. 2856–2867, 2014.
[2] M. A. Maddah-Ali and U. Niesen, “Decentralized coded caching attains order-optimal memory-rate tradeoff,” IEEE/ACM Transactions on Networking, vol. 23, no. 4, pp. 1029–1040, 2015.
[3] N. Karamchandani, U. Niesen, M. A. Maddah-Ali, and S. N. Diggavi, “Hierarchical coded caching,” IEEE Transactions on Information Theory, vol. 62, no. 6, pp. 3212–3229, 2016.
[4] S. P. Shariatpanahi, S. A. Motahari, and B. H. Khalaj, “Multi-server coded caching,” IEEE Transactions on Information Theory, vol. 62, no. 12, pp. 7253–7271, 2016.
[5] V. Ravindrakumar, P. Panda, N. Karamchandani, and V. M. Prabhakaran, “Private coded caching,” IEEE Transactions on Information Forensics and Security, vol. 13, no. 3, pp. 685–694, 2017.
[6] A. M. Daniel and W. Yu, “Optimization of heterogeneous coded caching,” IEEE Transactions on Information Theory, vol. 66, no. 3, pp. 1893–1919, 2019.
[7] E. Parrinello and P. Elia, “Coded caching with optimized shared-cache sizes,” in Information Theory Workshop, IEEE, 2019, pp. 1–5.
[8] Ç. Yapar, K. Wan, R. F. Schaefer, and G. Caire, “On the optimality of D2D coded caching with uncoded cache placement and one-shot delivery,” IEEE Transactions on Communications, vol. 67, no. 12, pp. 8179–8192, 2019.
[9] K. Wan, D. Tuninetti, M. Ji, and P. Piantanida, “Fundamental limits of distributed data shuffling,” in Allerton Conference on Communication, Control, and Computing, IEEE, 2018, pp. 662–669.
[10] C. Tian, “Symmetry, outer bounds, and code constructions: A computer-aided investigation on the fundamental limits of caching,” MDPI Entropy, vol. 20, no. 8, p. 603, 2018.
[11] Z. Chen, P. Fan, and K. B. Letaief, “Fundamental limits of caching: Improved bounds for users with small buffers,” IET Communications, vol. 10, no. 17, pp. 2315–2318, 2016.
[12] M. M. Amiri, Q. Yang, and D. Gündüz, “Coded caching for a large number of users,” in Information Theory Workshop, IEEE, 2016, pp. 171–175.
[13] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “The exact rate-memory tradeoff for caching with uncoded prefetching,” IEEE Transactions on Information Theory, vol. 64, no. 2, pp. 1281–1296, 2017.
[14] J. Gómez-Vilardebó, “Fundamental limits of caching: Improved rate-memory trade-off with coded prefetching,” IEEE Transactions on Communications, vol. 66, no. 10, pp. 4488–4497, 2018.
[15] C. Tian and J. Chen, “Caching and delivery via interference elimination,” in International Symposium on Information Theory, IEEE, 2016, pp. 830–834.
[16] K. P. Vijith Kumar, B. K. Rai, and T. Jacob, “Towards the exact rate memory tradeoff in coded caching,” in National Conference on Communications, IEEE, 2019, pp. 1–6.
[17] K. P. Vijith Kumar, B. K. Rai, and T. Jacob, “Fundamental limits of coded caching: The memory rate pair (K-1/(K-1)),” in International Symposium on Information Theory, IEEE, 2019, pp. 2624–2628.
[18] S. Shao, J. Gómez-Vilardebó, K. Zhang, and C. Tian, “On the fundamental limit of coded caching systems with a single demand type,” in Information Theory Workshop, IEEE, 2019, pp. 1–5.
[19] S. Shao, J. Gómez-Vilardebó, K. Zhang, and C. Tian, “On the fundamental limits of coded caching systems with restricted demand types,” arXiv preprint [arXiv:2006.16557], 2020.
[20] N. Ajaykrishnan, N. S. Prem, V. M. Prabhakaran, and R. Vaze, “Critical database size for effective caching,” in National Conference on Communications, IEEE, 2015, pp. 1–6.
[21] H. Ghasemi and A. Ramamoorthy, “Improved lower bounds for coded caching,” *IEEE Transactions on Information Theory*, vol. 63, no. 7, pp. 4388–4413, 2017.

[22] A. Sengupta, R. Tandon, and T. C. Clancy, “Improved approximation of storage-rate tradeoff for caching via new outer bounds.” in *International Symposium on Information Theory*, IEEE, 2015, pp. 1691–1695.

[23] C.-Y. Wang, S. S. Bidokhti, and M. Wigger, “Improved converses and gap results for coded caching,” *IEEE Transactions on Information Theory*, vol. 64, no. 11, pp. 7051–7062, 2018.

[24] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “Characterizing the rate-memory tradeoff in cache networks within a factor of 2,” *IEEE Transactions on Information Theory*, vol. 65, no. 1, pp. 647–663, 2018.