MODULAR IDEMPOTENTS FOR THE DESCENT ALGEBRAS OF TYPE A AND HIGHER LIE MODULES

KAY JIN LIM

Abstract. The article focuses on three objects related to the descent algebras of type $A$. They are the modular idempotents, higher Lie modules and right ideals of the symmetric group algebras generated by the Solomon’s descent elements. We give a construction for the modular idempotents and study the structures of the higher Lie modules and the right ideals both in the ordinary and modular cases.

1. Introduction

Let $V$ be a finite dimensional vector space over a field $F$ of characteristic $p$ (either $p = 0$ or $p > 0$) and $T(V) = \bigoplus_{n \in \mathbb{N}_0} T^n(V)$ be the tensor algebra which is an associative $F$-algebra with unit. It can be made into a Lie algebra by means of the Lie bracket

$$[v, w] = v \otimes w - w \otimes v$$

for all $v, w \in T(V)$. The Lie subalgebra of $T(V)$ generated by $V$ is the free Lie algebra $L(V)$. The subspace of $L(V)$ containing the homogeneous elements of degree $n$ is $L^n(V)$ and it is called the $n$th Lie power of $V$. Moreover, we have $L(V) = \bigoplus_{n \in \mathbb{N}_0} L^n(V)$. When $V$ is a right $FG$-module, $T^n(V)$ is naturally an $FG$-module and $L^n(V)$ is a submodule of $T^n(V)$. For an example, if $G = \text{GL}(V)$ acts naturally on $V$, the dimension of $V$ is at least $n$ and $F$ is infinite, applying the Schur functor $f$, we obtain the Lie module $\text{Lie}_F(n) := f(L^n(V))$. The study of the Lie powers and Lie modules both in the zero and positive characteristic cases has drawn great attention. In the classical case when $p = 0$ and $G = \text{GL}(V)$, the structure of $L^n(V)$ is well-studied by the earlier work of Thrall [43], Brandt [13], Wever [44], Klyachko [31] and Kraskiewicz-Weyman [32]. In the case when $p > 0$, the study of these two objects has become significantly more difficult and relatively unknown. Many authors have contributed to this case and we name a few which is by no means complete. When $G = \text{GL}(V)$, the cases $p \nmid n$, $n = p$ and $n = pk$ with $p \nmid k$ have been studied by Donkin-Erdmann [20], Bryant-Stöhr [17] and Erdmann-Schocker [22] respectively. We should also like to mention the decomposition of the Lie powers obtained by Bryant-Schocker [16]. The motivation of the study of Lie modules in the modular case also arises from algebraic topology in the work of Selick-Wu [41]. For instance, they are interested in the maximal projective submodule of $\text{Lie}_F(n)$.

In the celebrated paper of Solomon [42], he showed that the integral group algebra of a Coxeter group $G$ has a subalgebra now widely known as the Solomon’s descent...
Lie algebra \( \mathcal{D}_{G,Z} \). In that paper, he gave a \( \mathbb{Z} \)-basis for the descent algebra and studied its structure, especially its radical over the rational field \( \mathbb{Q} \), via a ring epimorphism. The results in [12] have been extended to the modular case by Atkinson-van Willigenburg [6] for \( G \) is a symmetric group and Atkinson-Pfeiffer-van Willigenburg [9] for \( G \) an arbitrary Coxeter group. Let \( \Lambda(n) \), \( \Lambda^+(n) \) and \( \Lambda_p^+(n) \) be the sets consisting of all compositions, partitions and \( p \)-regular partitions of \( n \) respectively. In the symmetric group \( \mathfrak{S}_n \) case, we have the descent algebra \( \mathcal{D}_{n,F} \) of type \( A \) and the \( \mathbb{Z} \)-basis introduced by Solomon is \( \{ \Xi^q : q \in \Lambda(n) \} \). The descent algebra \( \mathcal{D}_{n,F} \) has deep connection with the Lie powers and Lie modules as the symmetric group \( \mathfrak{S}_n \) acts on both \( T^n(V) \) and \( L^n(V) \) on the left via the Pólya action and therefore they can be viewed as \( (F\mathfrak{S}_n,F\mathfrak{G}) \)-bimodules. Indeed, \( L^n(V) = \omega_n \cdot T^n(V) \) and therefore \( \text{Lie}_F(q) \cong \omega_n F\mathfrak{G}_n \) where \( \omega_n \) is the Dynkin-Specht-Wever element. The element \( \omega_n \) belongs to \( \mathcal{D}_{n,Z} \) and is almost an idempotent in the sense that \( \omega_n^2 = n\omega_n \) and called a Lie idempotent. Blessenohl-Laue [10] generalized it to obtain a set \( \{ \omega_q : q \in \Lambda(n) \} \) and it forms a basis for \( \mathcal{D}_{n,F} \) if \( p = 0 \). These elements are called higher Lie idempotents. In the case of \( p = 0 \), Garsia-Reutenauer [21] also constructed another basis \( \{ I_q : q \in \Lambda(n) \} \) for \( \mathcal{D}_{n,F} \) and obtained a complete set of primitive orthogonal idempotents \( \{ E_\lambda : \lambda \in \Lambda^+(n) \} \) for \( \mathcal{D}_{n,F} \) such that \( \sum_{\lambda \in \Lambda^+(n)} E_\lambda = 1 \). In the case of \( p > 0 \), Erdmann-Schocker [22] showed that there is a complete set of primitive orthogonal idempotents \( \{ e_{\lambda,F} : \lambda \in \Lambda_p^+(n) \} \) for \( \mathcal{D}_{n,F} \) such that \( \sum_{\lambda \in \Lambda_p^+(n)} e_{\lambda,F} = 1 \) and the image of \( e_{\lambda,F} \) under the Solomon’s epimorphism in the modular case is the characteristic function on the \( p \)-equivalent conjugacy class labelled by \( \lambda \). However, the modular idempotents of \( \mathcal{D}_{n,F} \) are relatively unknown nor how one can construct them.

Let \( q \in \Lambda(n) \). We define \( L^q(V) := \omega_q \cdot T^n(V) \) as the higher Lie power so that \( L^{(n)}(V) = L^n(V) \). The higher Lie module is defined as \( \text{Lie}_F(q) := \omega_q F\mathfrak{G}_n \) so that, whenever \( \text{GL}(V) \) acts naturally on \( V \) and \( F \) is infinite, under the Schur functor \( f \), we have \( \text{Lie}_F(q) \cong f(L^q(V)) \). In this paper, we focus on three objects related to \( \mathcal{D}_{n,F} \). They are the modular idempotents, higher Lie modules and right ideals \( \Xi^q F\mathfrak{G}_n \)'s where \( q \in \Lambda(n) \). In the study of the latter two objects, we consider both the ordinary and modular cases. These three objects are related in intricate ways. We select some for illustration purpose. In the case when \( p = 0 \), the right module \( \Xi^q F\mathfrak{G}_n \) can be written as a direct sum of some of the higher Lie modules. In the case when \( p > 0 \), if \( \lambda \) is coprime to \( p \) (that is, every part of \( \lambda \) is not divisible by \( p \)) and \( p \)-regular, the projective module \( e_{\lambda,F} F\mathfrak{G}_n \) is isomorphic to both the higher Lie module \( \text{Lie}_F(\lambda) \) and \( \Xi^\lambda e_{\lambda,F} F\mathfrak{G}_n \), which is a direct summand of \( \Xi^\lambda F\mathfrak{G}_n \).

In the next section, we collate together the necessary background material and prove some preliminary results along the way. In Section 3 we give a construction for the modular idempotents \( \{ e_{\lambda,F} : \lambda \in \Lambda_p^+(n) \} \) as mentioned earlier. The main result is Theorem 3.7 which shows that \( e_{\lambda,F} \) has the ‘leading term’ \( \frac{1}{n!} \Xi^\lambda \) (see Subsection 2.1 for the notation \( \mu \)). In Section 4 we turn our attention to higher Lie modules. Under the assumption that \( q \) is coprime to \( p \) (that is, every part of \( \lambda \) is not divisible by \( p \)) and \( p \)-regular, the projective module \( e_{\lambda,F} F\mathfrak{G}_n \) is isomorphic to both the higher Lie module \( \text{Lie}_F(\lambda) \) and \( \Xi^\lambda e_{\lambda,F} F\mathfrak{G}_n \), which is a direct summand of \( \Xi^\lambda F\mathfrak{G}_n \).
few corollaries; namely, when $q$ is coprime to $p$, $\text{Lie}_F(q)$ is a $p$-permutation module, its ordinary character can be computed using a result Schocker [39], its support variety and complexity are known and, when $\lambda \in \Lambda^+(n)$ is coprime to $p$, $\text{Lie}_F(\lambda)$ is isomorphic to $\epsilon_{\lambda,F}\mathfrak{S}_n$. Section 5 is devoted to the combinatorics to set the scene for Section 6. More precisely, in Section 5, we study the permutations in $\mathfrak{S}_n$ as words and, for each $q \in \Lambda(n)$, define a subset $\mathcal{B}_q$ of $\mathfrak{S}_n$. In Section 6, we show that $\{\Xi w : w \in \mathcal{B}_q\}$ is a free basis for $\Xi^n R\mathfrak{S}_n$ where $R$ is any arbitrary commutative ring with 1 (see Theorem 6.2) and study the right ideal $\Xi^n F\mathfrak{S}_n$ both in the cases when $p = 0$ and $p > 0$. When $p > 0$ and $q$ is coprime to $p$, if $\lambda$ is the partition obtained from $q$ by rearrangement, in Theorem 6.9, we show that $\Xi^n F\mathfrak{S}_n$ is a direct sum of the higher Lie modules $\text{Lie}_F(\lambda)$’s one for each partition $\lambda$ of $n$ such that $\lambda$ is a weak refinement of $q$. When $p > 0$ and $q$ is coprime to $p$, if $\lambda$ is the partition obtained from $q$ by rearrangement, in Theorem 6.13, we show that $\Xi^n F\mathfrak{S}_n$ has a projective summand $\Xi^n \epsilon_{\lambda,F}\mathfrak{S}_n$ which is also isomorphic to both $\text{Lie}_F(q)$ and $\epsilon_{\lambda,F}\mathfrak{S}_n$. Section 7 consists of some questions and conjectures arise and supported by the examples we have presented in the paper.

2. Preliminary

Throughout this article, let $\mathbb{N}_0$ and $\mathbb{N}$ be the sets of non-negative and positive integers respectively. Furthermore, $R$ is a commutative ring with 1 and $F$ is a field of characteristic $p$ (either $p = 0$ or $p > 0$). We will consider both the ordinary $p = 0$ and modular $p > 0$ cases. The reader will be reminded when the condition on $p$ is assumed.

2.1. Generalities. For any integers $a \leq b$, the set $\{a, a + 1, \ldots, b\}$ of consecutive integers is denoted as $[a, b]$. For a totally order set $S$, we write $S_1 \sqcup \cdots \sqcup S_k = S$ if $S$ is a disjoint union of the ordered subsets $S_1, \ldots, S_k$. In this case, we denote $S_i \subseteq S$. Sometimes, we view $[a, b]$ as an ordered set, in the obvious order, and consider its ordered subsets.

A composition $q$ of $n$ is a finite sequence $(q_1, \ldots, q_k)$ in $\mathbb{N}$ such that $\sum_{i=1}^{k} q_i = n$. In this case, we write $q \vdash n$ and $\ell(q) = k$. In the case $r$ is a finite sequence in $\mathbb{N}_0$, we write $r^*$ for the composition obtained from $r$ by deleting all the zero entries. We say that $q$ is coprime to $p$ if $p \nmid q_i$ for all $i \in [1, k]$ or simply write $(q, p) = 1$. For instance, any composition is coprime to 0. Furthermore, for each $j \in [1, k]$, we write

$$q_j^+ = \sum_{i=1}^{j} q_i,$$

i.e., the sum of the first $j$ parts of $q$. By convention, $q_0^+ = 0$. The composition $q$ is a partition if $q_1 \geq \cdots \geq q_k$ and we write $q \vdash n$. We denote the set of all compositions and partitions of $n$ by $\Lambda(n)$ and $\Lambda^+(n)$ respectively. The concatenation of two compositions $r, q$ is denoted as $r \# q$, i.e.,

$$r \# q = (r_1, \ldots, r_{\ell(r)}, q_1, \ldots, q_{\ell(q)}).$$

Let $q, r \in \Lambda(n)$. If the parts of $q$ can be rearranged to $r$ then we write $q \approx r$. Clearly, this is an equivalence relation and the equivalence classes are represented by $\Lambda^+(n)$. As such, we write $\lambda(q)$ for the partition such that $q \approx \lambda(q)$. A composition $r$ is a (strong)
refinement of \( q \) if there are integers \( 0 = i_0 < i_1 < \cdots < i_k \) where \( k = \ell(q) \) such that, for each \( j \in [1, k] \),
\[
 r^{(j)} := (r_{i_{j-1}+1}, r_{i_{j-1}+2}, \ldots, r_{i_j}) \| q_j,
\]
and we denote it as \( r \preceq q \). In this case, we define
\[
\ell(r, q) = \prod_{j=1}^{k} \ell(r^{(j)}), \quad \ell!(r, q) = \prod_{j=1}^{k} \ell(r^{(j)})!, \quad F_q(r) = \prod_{j=1}^{k} r_{i_j}.
\]
On the other hand, the composition \( r \) is a weak refinement of \( q \) if there is a rearrangement of \( r \) which is a refinement of \( q \), i.e., \( r \approx s \preceq q \) for some \( s \) and we denote this as \( r \preceq q \). In the case \( r \preceq q \) but \( r \not\preceq q \), we write \( r \prec q \).

Let \( q = (q_1, \ldots, q_k) \| n \). For each \( i \in \mathbb{N} \), we denote \( m_i(q) = |\{j : q_j = i\}| \), i.e., \( m_i(q) \) is the number of parts of \( q \) of size \( i \). Since \( m_i(q) = 0 \) for all \( i > n \), we shall omit them. Furthermore, we let
\[
m(q) = (m_1(q), \ldots, m_1(q))^*.
\]
In the case when \( \lambda \vdash n \), we often write \( \lambda = (\ldots, 2^{m_2(\lambda)}, 1^{m_1(\lambda)}) \). The partition \( \lambda \) is called \( p \)-regular if \( m_i(\lambda) < p \) for all \( i \in \mathbb{N} \). We denote the set of \( p \)-regular partitions by \( \Lambda_p^+(n) \). Notice that \( \Lambda_0^+(n) = \Lambda^+(n) \). Furthermore, let \( q_i = \prod_{i \geq 1} m_i(q)! \) and
\[
q^? = \prod_{i \geq 1} i^{m_i(q)} m_i(q)! = q_i \prod_{j=1}^{k} q_j.
\]
For examples, \((n)i = 1\) and \((1^n)i = n\). Notice that \( \lambda i \neq 0 \) in the field \( F \) if and only if \( \lambda \in \Lambda_p^+(n) \). Also, if \( q \approx r \), then we have both \( q_i = r_i \) and \( q^? = r^? \).

For a matrix \( A \), we denote the \( i \)th row and \( j \)th column of \( A \) by \( r_i(A) \) and \( c_j(A) \) respectively, i.e.,
\[
r_i(A) = (A_{i1}, \ldots, A_{in}),
\]
\[
c_j(A) = (A_{1j}, \ldots, A_{nj}),
\]
if \( A \) is an \((m \times n)\)-matrix. Let \( q, r, s \in \Lambda(n) \). We shall now define the numbers \(|N_{r,q}^s|\) and \(|\overline{N}_{r,q}^s|\) which play important roles in the descent algebras of type \( A \). Let \( N_{r,q}^s \) be the set consisting of all the \((\ell(r) \times \ell(q))\)-matrices \( A \) with entries in \( \mathbb{N}_0 \) such that
\begin{enumerate}[(a)]  
  \item for each \( i \in [1, \ell(r)] \), \( r_i(A)^* \) is a composition of \( r_i \),  
  \item for each \( j \in [1, \ell(q)] \), \( c_j(A)^* \) is a composition of \( q_j \), and  
  \item \( s = (r_1(A)^* \# \cdots \# r_{\ell(q)}(A))^* \).
\end{enumerate}
Furthermore, we let \( \overline{N}_{r,q}^s \) be the subset consisting of \( A \in N_{r,q}^s \) such that each column of \( A \) contains exactly one nonzero entry.

**Example 2.1.** Let \( q = (2, 1) \), \( r = (1, 2) \) and \( s = (1, 1, 1) \). We have
\[
N_{q,r}^q = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \right\}, \quad N_{r,q}^r = \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}, \quad N_{q,r}^s = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}, \quad N_{r,q}^s = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.
\]
Therefore \( \overline{N}_{q,r}^q = N_{q,r}^q \), \( \overline{N}_{r,q}^r = N_{r,q}^r \) but \( \overline{N}_{q,r}^s = \emptyset = \overline{N}_{r,q}^s \).
Lemma 2.2. Let $q, r \in \Lambda(n)$ such that $q \approx r$. We have $|N_{r,q}^r| = r! = q!$.

Proof. Since $q \approx r$, the set $N_{r,q}^r$ consists of ‘permutation matrices’ with the nonzero entries are precisely the components of $q$. Notice that, for each $i \in [1, n]$, any permutation of the $j$th rows of $A \in N_{r,q}^r$ such that $r_j(A)^* = (i)$ yields another element in $N_{r,q}^r$. □

Let $\mathfrak{S}_A$ be the symmetric group acting on a finite subset $A$ of $\mathbb{N}$. By convention, we let $\mathfrak{S}_0$ be the trivial group. For $n \in \mathbb{N}$, we write $\mathfrak{S}_n$ for $\mathfrak{S}_{[1,n]}$. In this article, the composition of the permutations is read from left to right. For an example, in terms of cycles, $(1, 2)(2, 3) = (1, 3, 2)$. The conjugacy class of $\mathfrak{S}_n$ labelled by $\lambda \in \Lambda^+(n)$ is denoted by $\mathcal{C}_\lambda$. Indeed, we have

$$|\mathcal{C}_\lambda| = \frac{n!}{\lambda!}.$$ 

Any partitions $\lambda, \mu \in \Lambda^+(n)$ are $p$-equivalent if the $p'$-parts of any $\sigma \in \mathcal{C}_\lambda$ and $\tau \in \mathcal{C}_\mu$ are conjugate in $\mathfrak{S}_n$. In this case, we write $\lambda \sim_p \mu$ for the equivalent relation. Notice that $\lambda \sim_0 \mu$ if and only if $\lambda = \mu$. The $p$-equivalent classes are represented by $p$-regular partitions $\Lambda^+_{p}(n)$. For $\lambda \in \Lambda^+_{p}(n)$, we write $\mathcal{C}_{\lambda,p}$ for the union of the conjugacy classes $\mathcal{C}_{\mu}$ such that $\mu \sim_p \lambda$. We record an easy lemma which we will need later.

Lemma 2.3. Suppose that $\lambda \in \Lambda^+_{p}(n)$ and $(\lambda, p) = 1$. We have $|\mathcal{C}_{\lambda,p}| = |\mathcal{C}_\lambda|$.

Proof. We only need to check $\mathcal{C}_{\lambda,p} \subseteq \mathcal{C}_\lambda$. Let $\sigma \in \mathcal{C}_\mu$ such that $\mu \sim_p \lambda$. By definition, given that $p \nmid \lambda_i$ for all $i \in [1, \ell(\lambda)]$, the $p'$-part of $\sigma$ is conjugate to $\lambda$. As noted in [22, §2], the cycle type of the $p'$-part of an element in $C_\mu$ obtained by replacing each entry $\mu_i = kp^m$ where $p \nmid k$ by $(k, \ldots, k) = (k^{p^m})^{k^{p^m}}$. Since $\lambda$ is $p$-regular, the cycle type of the $p'$-part of $\sigma$ is $\mu$. Therefore, $\mu = \lambda$. □

For any $r \in \mathbb{N}_0$ and $\sigma \in \mathfrak{S}_n$, we write $\sigma^{+r}$ for the permutation such that $(i)\sigma^{+r} = (i)\sigma + r$ if $i \in [1, n]$ and fixes the remaining numbers. Let $q \vdash n$ with $k = \ell(q)$. The Young subgroup of $\mathfrak{S}_n$ with respect to $q$ is

$$\mathfrak{S}_q = \coprod_{j=1}^k \mathfrak{S}_{[1+q^+_j, q^+_j]}.$$ 

In this article, we will also use another presentation for permutations, the word or one-line notation, which we shall now describe. Let $A$ be a set consisting of distinct elements called the alphabets. A word in $A$ is $w = w_1 w_2 \ldots w_n$ where $w_1, w_2, \ldots, w_n \in A$. In this case, we write $|w| = n$. A subword of $w$ is of the form $w_{i_1} \ldots w_{i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq n$ which is not necessarily contiguous. The group $\mathfrak{S}_n$ acts on the words of length $n$ via the Pólya action; namely,

$$\tau \cdot w_1 \ldots , w_n = w_{1\tau} \ldots w_{n\tau}. $$

In the case $A = \mathbb{N}$, we can identify a word $w = w_1 \ldots w_n$ such that $\{w_1, \ldots, w_n\} = [1, n]$ with the permutation $\sigma$ such that $(i)\sigma = w_i$ for all $i \in [1, n]$. As such, notice that

$$\tau \sigma = \tau w_1 \ldots w_n = w_{1\tau} \cdots w_{n\tau} \tau = \tau \cdot w$$
for another element \( \tau \in \mathfrak{S}_n \). In other words, the multiplication in \( \mathfrak{S}_n \) coincides with the Pólya action. With this identification, \( \mathfrak{S}_n \) can be totally order by the reversed colexicographic order \( \leq_{\text{clx}} \); namely \( w \leq_{\text{clx}} v \), if \( j \) is the least positive integer such that \( w_{j+1} = v_{j+1}, \ldots, w_n = v_n \), we have \( w_j > v_j \). In the case \( w \leq_{\text{clx}} v \) and \( w \neq v \), we write \( w <_{\text{clx}} v \). For an example, in \( \mathfrak{S}_3 \), omitting the superscript clx, we have

\[
123 < 213 < 132 < 312 < 231 < 321.
\]

Let \( G \) be a group. We denote the group algebra by \( RG \). The trivial \( RG \)-module is denoted as \( R \). Let \( H \) be a subgroup of \( G \), \( M \) be an \( RG \)-module and \( N \) be an \( RH \)-module. We denote the induction and restriction of modules by \( \text{ind}_H^G M \) and \( \text{res}_H^G M \) respectively. For any right \( RG \)-module \( M \) and non-negative integer \( n \), we write \( M^n \) for the \( R[G \wr \mathfrak{S}_n] \)-module which is the \( R \)-module \( M^\otimes n \) such that \( G \wr \mathfrak{S}_n \) acts via

\[
(m_1 \otimes \cdots \otimes m_n)(\tau; g_1, \ldots, g_n) = m_{(1)\tau^{-1}}g_1 \otimes \cdots \otimes m_{(n)\tau^{-1}}g_n
\]

where \( m_1, \ldots, m_n \in M \), \( \tau \in \mathfrak{S}_n \) and \( g_1, \ldots, g_n \in G \). By convention, \( M^0 = R \). For another \( RG' \)-module \( M' \), we write \( M \boxtimes M' \) for the outer tensor product of \( M \) and \( M' \) which is an \( R[G \times G'] \)-module.

The group algebra \( FG \) can be identified with \( \mathbb{Z}G \otimes_{\mathbb{Z}} F \). For convenience, for any \( x \in \mathbb{Z}G \), we also write \( x \in FG \) for its ‘reduction modulo \( p \)’ under this identification. We have the following basic lemma in which \( \mathbb{Q} \) is the rational field.

**Lemma 2.4.** Let \( G \) be a group and \( x \in \mathbb{Z}G \). Then, taking reduced modulo \( p \), we have

\[
\dim_F xFG \leq \dim_{\mathbb{Q}} x\mathbb{Q}G.
\]

**Proof.** The \( \mathbb{Z} \)-module \( x\mathbb{Z}G \) is torsion free and therefore it has a \( \mathbb{Z} \)-basis \( B \). The dimension \( d := \dim_{\mathbb{Q}} x\mathbb{Q}G \) is obviously the cardinality of \( B \). Upon taking reduction modulo \( p \), the set \( B \) clearly spans \( xFG \) but could be linearly dependent. Therefore \( \dim_F xFG \) is not larger than \( d \). \( \square \)

For the remainder of this subsection, assume that \( p > 0 \). Let \( M \) be an indecomposable \( FG \)-module. A vertex of \( M \) is a subgroup \( H \) such that \( M \mid \text{ind}_H^G N \) for some \( FH \)-module \( N \). In this case, if \( N \) is indecomposable, it is called an \( FH \)-source of \( M \). By a result of Green [26], a vertex must be a \( p \)-subgroup, say \( P \), and all \( FP \)-sources are \( N_G(P) \)-conjugate. An \( FG \)-module \( M \) has trivial source if every indecomposable summand of \( M \) has trivial module as its source. An \( FG \)-module is projective if it is a direct summand of a direct sum of regular modules. An \( FG \)-module \( M \) is a \( p \)-permutation module if, for every Sylow \( p \)-subgroup \( P \) of \( G \), there exists a \( F \)-basis of \( M \) that is permuted by \( P \). The following theorem gives a characterization of \( p \)-permutation modules.

**Theorem 2.5 ([15] (0.3)).** An indecomposable \( FG \)-module \( M \) is a \( p \)-permutation module if and only if there exists a \( p \)-subgroup \( P \) of \( G \) such that \( M \mid \text{ind}_P^G F \); equivalently, \( M \) has trivial source.

Consider a \( p \)-modular system \((K, \mathcal{O}, F)\). That is, \( \mathcal{O} \) is a complete discrete valuation ring with quotient field \( K \) of characteristic 0 and with residue field \( F \). It is well-known that every trivial source \( FG \)-module \( M \) lifts uniquely to a trivial source \( \mathcal{O}G \)-module.
$M_O$ such that $M_O \otimes F \cong M$ (see, for example, [11 Corollary 2.6.3]). In this case, the ordinary character $\text{ch}(M)$ of $M$ is defined to be the character of $M_O \otimes K$. In particular, according to Theorem 2.5, trivial source modules have ordinary characters.

2.2. The descent algebras. We now consider particularly the symmetric group algebra $R\mathfrak{S}_n$. Notice that the operation $\sigma^+ r$ for $\sigma \in \mathfrak{S}_n$ and $r \in \mathbb{N}_0$, we introduced earlier, can be extended linearly to $R\mathfrak{S}_n$. A permutation $\sigma$ is said to have a descent at $i \in [1, n - 1]$ if $(i)\sigma > (i + 1)\sigma$. We write

$$\text{Des}(\sigma) = \{i \in [1, n - 1] : (i)\sigma > (i + 1)\sigma\}.$$

For each $q \in \Lambda(n)$, we define the Solomon’s descent element

$$\Xi^q = \sum_{\text{Des}(\sigma) \subseteq \{q^+_i : i \in [1, \ell(q)]\}} \sigma \in R\mathfrak{S}_n.$$

The element $\Xi^q$ is easy to write down as a sum of words associated to the row standard $q$-tableaux. For examples, $\Xi^{(n)} = 1$, $\Xi^{(1^n)} = \sum_{\sigma \in \mathfrak{S}_n} \sigma$ and

$$\Xi^{(2,2)} = 1234 + 1324 + 1423 + 2314 + 2413 + 3412.$$

Let $r \in \Lambda(n)$ and $\tau \in \mathfrak{S}_k$ where $k = \ell(r)$ and $q = (r_{(1)r}, \ldots, r_{(k)r})$. We write, in two-line form,

$$\tau_r = \begin{pmatrix} 1 & \cdots & q_1 & \cdots & q_{k-1}^+ + 1 & \cdots & q_k^+ \\ r_{(1)r-1}^+ + 1 & \cdots & r_{(1)r-1}^+ + q_1 & \cdots & r_{(k)r-1}^+ + 1 & \cdots & r_{(k)r-1}^+ + q_k \end{pmatrix}.$$

Notice that $\Xi^q = \tau_r \Xi^r$. For an example, if $r = (2,1,1)$, $q = (1,2,1)$, we could take $\tau = 213$ (swapping the first and second components) and

$$\tau_r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = 3124.$$

Furthermore, if $s \leq q$, we have $\Xi^s = \kappa \Xi^q$ where $\kappa = \sum_{\pi \in \mathfrak{S}_q/\mathfrak{S}_s} \pi$. Therefore, we have the following lemma.

**Lemma 2.6.** Let $r \leq q$ be compositions of $n$. Then there is a surjection $\Xi^q R\mathfrak{S}_n \rightarrow \Xi^r R\mathfrak{S}_n$ given by the left multiplication of $s_r \kappa$ where $(s_{(1)})_{\kappa}, \ldots, s_{(\ell(s))})_{\kappa} = r$ for some $s \leq q$ and $\kappa = \sum_{\pi \in \mathfrak{S}_q/\mathfrak{S}_s} \pi$.

Let $\mathcal{D}_{n,R}$ be the $R$-linear span of $\{\Xi^q : q \in \Lambda(n)\}$. Solomon proved the following far-reaching theorem in the case $R = \mathbb{Z}$ but the general case is obtained by tensoring with the general ring $R$.

**Theorem 2.7 ([12]).** For $q, r \in \Lambda(n)$, we have

$$\Xi^r \Xi^q = \sum_{s \in \Lambda(n)} |N^s_{r,q}| \Xi^s.$$

Furthermore, the set $\{\Xi^q : q \in \Lambda(n)\}$ is an $R$-basis for the ring $\mathcal{D}_{n,R}$.
For \( q \in \Lambda(n) \), let \( Y_q, Y^o_q \) be the subsets of \( D_{n,R} \) which are \( R \)-spanned by
\[
B_q := \{ \Xi : \Lambda(n) \ni \xi \prec q \}, \\
B^o_q := \{ \Xi : \Lambda(n) \ni \xi \preceq q \},
\]
respectively. As a consequence of the theorem, we have the following corollary.

**Corollary 2.8.** Let \( q, r \in \Lambda(n) \).

(i) The product \( \Xi r \Xi q \) is a linear combination of some \( \Xi s \) such that \( s \) is both a strong and weak refinements of \( r \) and \( q \) respectively, i.e., \( r \triangleright s \preceq q \). In particular, both \( Y_q \) and \( Y^o_q \) are two-sided ideals of \( D_{n,R} \).

(ii) The coefficient of \( \Xi r \) in \( (\Xi r)^2 \) is \( r! \).

**Proof.** For part (ii), notice that each row and column of a matrix in \( N_{r,r} \) has exactly one nonzero entry. Now use Lemma 2.2. \( \Box \)

The Dynkin-Specht-Wever element \( \omega_n \) for the group algebra \( R\mathfrak{S}_n \) is defined as
\[
\omega_n = (1 - c_n)(1 - c_{n-1}) \cdots (1 - c_2) \in R\mathfrak{S}_n
\]
where, for each \( i \in [2, n] \), \( c_i = (i, i-1, \ldots, 1) \) is the descending cycle of length \( i \). Notice that
\[
\omega_n = \sum (-1)^{|j|} s_j
\]
where the sum is taken over all subsets \( j = \{j_1 < \cdots < j_t\} \subseteq [2, n] \) where \( \{1 = k_1 < \cdots < k_{n-t}\} \sqcup j = [1, n] \) and
\[
s_j = j_t \cdots j_1k_1 \cdots k_{n-t} = \begin{pmatrix} 1 & \cdots & t & t+1 & \cdots & n \\ j_t & \cdots & j_1 & k_1 & \cdots & k_{n-t} \end{pmatrix}.
\]
For any word \( w = w_1 \cdots w_n \) or ordered set \( S = \{w_1, \ldots, w_n\} \), we denote
\[
Q_S = Q_w = \omega_n \cdot w = \sum_{j \in [2, n]} (-1)^{|j|} s_j \cdot w.
\]
For an example, \( \omega_3 \cdot abc = abc - bac - cab + cba \). It is well-known that \( \omega_n \in D_{n,R} \) and \( \omega_n^2 = n \omega_n \). Therefore, if \( n \) is a unit in \( R \), then \( \frac{1}{n} \omega_n \) is an idempotent in \( D_{n,R} \).

The Lie module for the group algebra \( R\mathfrak{S}_n \) is defined as the right ideal
\[
\text{Lie}_R(n) := \omega_n R\mathfrak{S}_n.
\]

The following proposition is well-known.

**Proposition 2.9.** The set \( \{ \omega_n \sigma : (1) \sigma = 1 \} \) is an \( R \)-basis for \( \text{Lie}_R(n) \).

2.3. **The higher Lie modules.** In the case \( p = 0 \), there is another basis for the descent algebra \( D_{n,F} \) found by Blessenohl-Laue [10]. In fact, in their paper, some results and proofs hold over arbitrary ring \( R \). We have summarized the results we shall need in Theorem 2.10 below and refer the readers to the proofs in that paper.

For each \( q \in \Lambda(n) \), we define
\[
\omega_q = \omega^q \Xi q \in D_{n,R}
\]
where \( \omega^q := \omega_{q_1}^{q_1^0} \cdots \omega_{q_k}^{q_k^k-1} \in R \mathfrak{S}_q \) (see [10] Proposition 1.1]). Clearly, \( \omega_{(n)} = \omega_n \). The element \( \omega_q \) can also be defined using the convolution product for the symmetric group but we have chosen this presentation which suits us best. By [10] Proposition 1.2, when \( p = 0 \), the set \( \{ \omega_q : q \in \Lambda(n) \} \) is a basis for \( D_{n,F} \). This is not true when \( p > 0 \) (see Example 2.13 below).

The following theorem summarizes a few properties which we will be using throughout.

**Theorem 2.10** ([10]). Let \( r, q \in \Lambda(n) \). In \( D_{n,R} \), we have

(i) \( \omega_q = \sum_{s \leq q} (-1)^{\ell(s)-\ell(q)} F_q(s) \Xi_s \),

(ii) \( \Xi^r \omega_q = \sum_{q \leq s \leq r} |N_{r,q}| \omega_s \),

(iii) \( \omega_q \omega_r = q? \omega_q \) if \( q \approx r \),

(iv) \( \omega_q \omega_r = 0 \) unless \( q \leq r \).

When \( q? \) is a unit in \( R \), the element \( \nu_q = \frac{1}{q?} \omega_q \) in \( D_{n,R} \) is called a higher Lie idempotent due to Theorem 2.10 (iii). In particular, when \( p = 0 \), any idempotent \( e \in D_{n,F} \) is called a higher Lie idempotent if \( eF \mathfrak{S}_n = \nu_q F \mathfrak{S}_n \) for some \( q \equiv n \).

For any \( q \in \Lambda(n) \), we define the higher Lie module with respect to \( q \) as the right ideal

\[ \text{Lie}_R(q) = \omega_q R \mathfrak{S}_n. \]

We have the following basic property.

**Lemma 2.11.** Suppose that \( q, r \) are compositions of \( n \) and \( q \approx r \). We have \( \sigma \omega_q = \omega_r \) for some \( \sigma \in \mathfrak{S}_n \). In particular, \( \text{Lie}_R(q) \cong \text{Lie}_R(r) \).

**Proof.** Let \( \sigma \Xi^q = \Xi^r \). By definition,

\[ \sigma \omega_q = \sigma \omega^q \sigma^{-1} \sigma \Xi^q = \omega^r \Xi^r = \omega_r. \]

The isomorphism is therefore given by the left multiplication by \( \sigma \). \( \Box \)

In the case when \( p = 0 \), it is well-known that \( \dim_F \omega_q F \mathfrak{S}_n = \frac{n!}{q!} \) (see, for an example, [37] Theorem 8.24]). Therefore, using Lemma 2.1 we obtain the following inequality.

**Proposition 2.12.** For any \( q \in \Lambda(n) \), we have \( \dim_F \omega_q F \mathfrak{S}_n \leq \frac{n!}{q!} \).

The inequality in Proposition 2.12 is not an equality in general. We give an example below and also refer the reader to Appendix C for the computational data using Magma [12].

**Example 2.13.** Let \( p = 2 \). Using Theorem 2.10 (i), in \( D_{3,F} \), we have

\[ \omega_{(2,1)} = 2 \Xi^{(2,1)} - \Xi^{(1,1,1)} = \Xi^{(1,1,1)} = \sum_{\sigma \in \mathfrak{S}_3} \sigma. \]

Therefore \( \omega_{(2,1)} F \mathfrak{S}_3 \cong F \) and the inequality in Proposition 2.12 is strict in this case. Similarly, we have \( \omega_{(1,2)} = \omega_{(2,1)} = \omega_{(1,1,1)} = \Xi^{(1,1,1)} \) and \( \omega_{(3)} = \Xi^{(3)} + \Xi^{(2,1)} + \Xi^{(1,1,1)} \). Therefore, \( \{ \omega_q : q \in \Lambda(3) \} \) cannot be a basis for \( D_{3,F} \).
2.4. The Solomon’s epimorphism. For each \( q \models n \), the Young permutation module is \( M^q_R = \text{ind}_{S_q}^S R \). If \( q \approx q' \) then \( M^q_R \cong M^{q'}_R \). The Young character \( \varphi^q \) is defined as the character of \( M^q_S \) where, for each \( \mu \in \Lambda^+(n) \), \( \varphi^q(\mu) \) is the number of right cosets (or \( q \)-tabloids) of \( S_q \) in \( S_n \) fixed by a permutation with cycle type \( \mu \), where we have identified \( \mu \) with the conjugacy class \( C_\mu \) of \( S_n \). Therefore, \( \varphi^q = \varphi^{q'} \) if \( q \approx q' \). We have the following lemma.

Lemma 2.14. If \( \varphi^q(\mu) \neq 0 \) then \( \mu \preceq q \). Furthermore, \( \varphi^q(q) = q_i \).

We denote \( \varphi^{q,R} \) for the \( R \)-valued Young character, i.e., for any \( \mu \in \Lambda^+(n) \),

\[
\varphi^{q,R}(\mu) = \varphi^q(\mu) \cdot 1_R \in R.
\]

Let \( C_{n,R} \) be the \( R \)-linear span of the \( R \)-valued Young characters. The ordinary irreducible characters of \( F S_n \) are labelled by \( \Lambda^+(n) \) and we label them as \( \zeta^\lambda \), one for each \( \lambda \in \Lambda^+(n) \), such that

\[
\varphi^\lambda,F = \zeta^\lambda + \sum_{\mu \succ \lambda} K_{\lambda,\mu} \zeta^\mu
\]

where \( K_{\lambda,\mu} \)'s are the Kostka numbers and \( \succ \) is the usual dominance order on \( \Lambda^+(n) \) (see, for example, [35, §II]); namely, \( \zeta^\lambda \) is the ordinary character of the Specht module \( S^\lambda \) (see [29]).

Suppose that \( p = 0 \). The ordinary irreducible constituents of \( \text{Lie}_F(q) \) has been computed in [39] (see also [11, Theorem 5.11]). The computation is done by reducing the general case to the case when \( q = (d^k) \) for some \( d, k \in \mathbb{N} \). Since we do not need the exact combinatorial description of the multiplicity, we simply denote the multiplicity of the irreducible character \( \zeta^\lambda \) in \( \text{Lie}_F((d^k)) \) given in [39] Main Theorem 3.1] as \( \text{s}_{d,k}^\lambda \). Furthermore, for partitions \( \mu(1), \ldots, \mu(k) \) such that \( \sum_{i=1}^k |\mu(i)| = |\lambda| = n \), let \( c^\lambda_{\mu(1),\ldots,\mu(k)} \) be the multiplicity of the irreducible character \( \zeta^\lambda \) in \( \text{ind}_{S_q}^{S_n}(S^{\mu(1)} \boxtimes \cdots \boxtimes S^{\mu(k)}) \) where \( r = (|\mu(1)|, \ldots, |\mu(k)|) \); namely, it is just the number obtained using Littlewood-Richardson Rule repeatedly (see [29]).

Theorem 2.15 ([39 Lemma 2.1 and Theorem 3.1]). Suppose that \( p = 0 \). For any \( \lambda \in \Lambda^+(n) \) and \( q \in \Lambda(n) \), the multiplicity of the irreducible character \( \zeta^\lambda \) in \( \text{Lie}_F(q) \) is

\[
C_q^\lambda = \sum_{\mu(1),\ldots,\mu(n)} c^\lambda_{\mu(1),\ldots,\mu(n)} \prod_{i=1}^n S_{i,m_i(q)^{\mu(i)}}.
\]

Recall the set \( N_{q,r}^s \) defined in Subsection 2.1. We have the following well-known identity, analogous to the Mackey’s formula,

\[
\varphi^q \varphi^r = \sum_{s \in \Lambda(n)} |N_{q,r}^s| \varphi^s
\]

which in turn gives rise to the following theorem.

Theorem 2.16 ([6, 42]). The \( F \)-linear map

\[
c_{n,F} : \mathcal{D}_{n,F} \rightarrow C_{n,F}
\]
sending $\Xi$ to $\varphi^{q,F}$ is a surjective $F$-algebra homomorphism. Furthermore, $\ker(c_{n,F}) = \text{rad}\mathcal{D}_{n,F}$ has an $F$-basis consisting of $\Xi$ such that $\lambda(q) \not\in \Lambda^+_p(n)$ and $\Xi - \Xi'$ such that $q \approx r$ with $q \neq r$.

For $\lambda \in \Lambda^+_p(n)$, let $\text{char}_{\lambda,F}$ be the characteristic function on the $p$-equivalent class $\mathcal{C}_{\lambda,p}$. The set $\{\text{char}_{\lambda,F} : \lambda \in \Lambda^+_p(n)\}$ forms a basis and complete set of primitive orthogonal idempotents of $\mathcal{C}_{n,F}$ (see [22, Proposition 5]). Furthermore, when $p = 0$, by [30, Proposition 1], under the Solomon’s epimorphism,

$$c_{n,F}(\nu_q) = c_{n,F}\left(\frac{1}{q}\omega_q\right) = \text{char}_{\lambda,F}$$

where $\lambda = \lambda(q)$.

In the $p = 0$ case, there are different sets of primitive orthogonal idempotents of $\mathcal{D}_{n,F}$ in the literature. Notably, those given by Blessenohl-Laue [10, 11] (see Proposition 2.17 below) and Garsia-Reutenauer [24] (see Subsection 2.5). In the $p > 0$ case, Erdmann-Schöker [22, Corollary 6] showed that there exists a complete set of primitive orthogonal idempotents $\{e_{\lambda,F} : \lambda \in \Lambda^+_p(n)\}$ for $\mathcal{D}_{n,F}$ such that $c_{n,F}(e_{\lambda,F}) = \text{char}_{\lambda,F}.$

**Proposition 2.17.** Suppose that $p = 0$. Using the orthogonalization procedure (see [36, §4] and also [30, Proposition 2.5]) for the set of higher Lie idempotents $\{\nu_\lambda : \lambda \in \Lambda^+(n)\}$, we obtain a complete set $\{e_\lambda : \lambda \in \Lambda^+(n)\}$ of primitive orthogonal idempotents of $\mathcal{D}_{n,F}$ such that $\sum_{\lambda \in \Lambda^+(n)} e_\lambda = 1$ and they satisfy the ‘triangularity property’

$$e_\lambda = \frac{1}{\lambda} E^\lambda + e_\lambda$$

where $e_\lambda$ is the sum of $E^\xi$ such that $\lambda \not\leq \xi \leq \lambda$.

**Proof.** Let $m = |\Lambda^+(n)|$ and $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be the partitions ordered so that, if $\lambda^{(j)} \lessdot \lambda^{(i)}$, then $j \leq i$. Notice that $\lambda^{(m)} = (n)$ and $\lambda^{(1)} = (1^n)$. Let $\nu_i = \nu_{\lambda^{(i)}}$ and $\text{char}_i = \text{char}_{\lambda^{(i)},F}$. By Theorem 2.16(iv), $\nu_i \nu_j = 0$ whenever $i > j$. Therefore the set $\{e_i : i \in [1, m]\}$ forms a set of primitive orthogonal idempotents such that $\sum_{i=1}^m e_i = 1$ where

$$e_i = \nu_i(1 - \nu_{i+1}) \cdots (1 - \nu_m).$$

It is a complete set due to Theorem 2.16. Since, by Theorem 2.10(i), $\nu_i$ is a sum of the $E^\xi$ such that $\xi \leq \lambda^{(i)}$, we have, by Corollary 2.8(i),

$$e_i = \sum_{s \leq \lambda^{(i)}} d_s E^s$$

for some $d_s \in F$. On the other hand, we have $c_{n,F}(\nu_i) = \text{char}_{\lambda^{(i)},F}$. Since $c_{n,F}$ is an $F$-algebra homomorphism, we have

$$\sum_{s \leq \lambda^{(i)}} d_s \varphi^{s,F} = c_{n,F}(\sum_{s \leq \lambda^{(i)}} d_s E^s) = c_{n,F}(e_i) = \text{char}_i(1 - \text{char}_{i+1}) \cdots (1 - \text{char}_m) = \text{char}_i.$$

(2.1)

For $\lambda^{(i)} \lessdot s \leq \lambda^{(i)}$, we must have $s = \lambda^{(i)}$. Evaluate Equation 2.1 at $\lambda^{(i)}$, by Lemma 2.14, we get $\varphi^{s,F}(\lambda^{(i)}) = 0$ unless $s = \lambda^{(i)}$ and hence $1 = \lambda^{(i)} d_{\lambda^{(i)}}$. \qed
2.5. **The higher Lie powers.** Let $V$ be an $R$-module with finite rank and $T^n(V) = V^\otimes n$ be the $n$-fold tensor product of $V$ over $R$. By convention, $T^0(V) = R$. Then $T(V) = \bigoplus_{n \in \mathbb{N}_0} T^n(V)$ is naturally an associative $R$-algebra with unit. It can be made into a Lie algebra by means of the Lie bracket

$$[v, w] = v \otimes w - w \otimes v$$

for all $v, w \in T(V)$. We denote by $L(V)$ the free Lie subalgebra of $T(V)$ generated by $V$. For each $n \in \mathbb{N}_0$, the $n$th Lie power of $V$ is defined as $L^n(V) = T^n(V) \cap L(V)$. Every element in $L^n(V)$ is called a Lie element and a Lie monomial is $P_1 \otimes P_2 \otimes \cdots \otimes P_m$ (simply written as $P_1 P_2 \cdots P_m$) such that each $P_i$ is a Lie element in $L^{n_i}(V)$ for some $n_i \in \mathbb{N}$. In this case, we denote $n_i = |P_i|$ and the composition $(n_1, \ldots, n_m)$ is called the type of the Lie monomial.

The symmetric group acts on $T^n(V)$ via the Pólya action; namely,

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{1\sigma} \otimes \cdots \otimes v_{n\sigma}.$$

It is well-known that $L^n(V) = \omega_n \cdot T^n(V)$. Hence, if $A$ is a finite set of distinct alphabets, the $R$-module $R[A]$ with an $R$-basis consisting of all words in $A$ can be identified with $T(V)$ where $V$ has $R$-basis $A$ and the multiplication in $R[A]$ induced by the concatenation of words is the same as the multiplication in $T(V)$. As such, the actions of the symmetric groups on $n$th homogeneous components are also identified.

If $V$ is a right $RG$-module then it is readily checked that both $T^n(V)$ and $L^n(V)$ are $(RG_n-RG)$-bimodules. In particular, when $V$ is an $F$-vector space with dimension at least $n$, $G = GL(V)$ acts naturally on $V$ and $F$ is infinite, under the Schur functor $f$ (see [25]), we have

$$f(L^n(V)) \cong \text{Lie}_F(n).$$

For each $q \in \Lambda(n)$, we define the higher Lie power $L^q(V) = \omega_q \cdot T^n(V)$. If $q \approx r$ then, by Lemma [2.4], we have an isomorphism $L^q(V) \cong L^r(V)$ of $RG$-modules. As before, under the similar assumption for the application of the Schur functor, we have

$$f(L^q(V)) \cong \text{Lie}_F(q).$$

Unlike the $p = 0$ case in which $T^n(V)$ decomposes as a direct sum of $L^\lambda(V)$ such that $\lambda \vdash n$, the $p > 0$ case does not.

**Example 2.18.** Let $p = 2$ and $V$ has an $F$-basis $\{v_1, v_2\}$. Since $\omega_2 = \Xi^{(1,1)} = \omega_{(1,1)}$, we have both $L^{(2)}(V)$ and $L^{(1,1)}(V)$ are identical with the $F$-basis $\{v_1 \otimes v_2 + v_2 \otimes v_1\}$.

The rest of this subsection is devoted to the description of the results in [24]. One of the main adaptations is by reversing the order of multiplication according to the conventions. Similar to [10], some of the results and proofs in [24] hold over $R$ and we shall refer the reader to the paper. In particular, we have:

**Theorem 2.19 ([24] Theorem 2.1]).** If $q \in \Lambda(n)$, $k = \ell(q)$ and $P_1 P_2 \cdots P_m$ is a Lie monomial of type $r \in \Lambda(n)$ then, over $R$,

$$\Xi^q \cdot P_1 P_2 \cdots P_m = \sum P_{S_1} P_{S_2} \cdots P_{S_k}$$
where the sum is taken over all $S_1 \sqcup \cdots \sqcup S_k = [1,m]$ such that, for each $i \in [1,k]$, $(|P_{t_1}|, \ldots, |P_{t_{c_i}}|) = q_i$, and $P_{S_i} = P_{t_1} \cdots P_{t_{c_i}}$ if $S_i = \{t_1 < \cdots < t_{c_i}\}$ for some $c_i$. In particular, if $r \not\approx q$, we have $\Xi^q \cdot P_1 P_2 \cdots P_m = 0$.

For an example, if $r, q$ are compositions such that $r \not\approx q$, then, by Theorem 2.19, $\Xi^q \omega^r = 0$ as $\omega^r \in R\mathfrak{S}_r$ can be viewed as a Lie monomial of type $r$.

For the rest of this section, suppose that $p = 0$.

Recall the notations $\ell(r, q)$ and $\ell!(r, q)$ for a (strong) refinement $r$ of $q$ in Subsection 2.1. For any $q \in \Lambda(n)$ and $\lambda \in \Lambda^+(n)$, we write

$$I_q = \sum_{r \leq q} \frac{(-1)^{\ell(r)-\ell(q)}}{\ell(r, q)} \Xi^r,$$

$$E_\lambda = \frac{1}{\ell(\lambda)!} \sum_{\lambda(q) = \lambda} I_q.$$

The set $\{E_\lambda : \lambda \in \Lambda^+(n)\}$ is a complete set of primitive orthogonal idempotents such that $\sum_{\lambda \in \Lambda^+(n)} E_\lambda = 1$. Each of the element $\Xi^q$ can be written explicitly in terms of the $I_r$’s as follows.

**Theorem 2.20 (24, Theorem 3.4).** The set $\{I_q : q \in \Lambda(n)\}$ is a basis for $\mathcal{D}_{n,F}$. Furthermore, for $q \vdash n$, we have

$$\Xi^q = \sum_{r \leq q} \frac{1}{\ell!(r, q)} I_r.$$

Next, we need to examine how the set of elements $\Xi^q$’s, $I_r$’s and $E_\lambda$’s interact with each others. We give a quick summary of the results we need and draw easy observations following them.

**Theorem 2.21 (24 Theorem 4.1).** Let $q, r \in \Lambda(n)$, $k = \ell(q)$ and $m = \ell(r)$. Then

$$\Xi^q I_r = \sum_{(r_j)_{j \in S_1} \#(r_j)_{j \in S_k}} I_{(r_j)_{j \in S_i} \#(r_j)_{j \in S_k}}$$

where the sum is taken over all $S_1 \sqcup \cdots \sqcup S_k = [1,m]$ such that, for each $i \in [1,k]$, we have $(r_j)_{j \in S_i} = q_i$. In particular, if $r \not\approx q$, then

(i) $\Xi^q I_r = 0$,

(ii) $I_q I_r = 0$, and

(iii) $E_\lambda I_r = 0$ where $\lambda = \lambda(q)$.

**Proof.** The equation $I_q I_r = 0$ follows since $I_q$ is a linear combination of $\Xi^s$ such that $s \leq q$ and $\Xi^s I_r = 0$. This implies $E_\lambda I_r = 0$. \qed

**Theorem 2.22 (24 Theorem 4.2(1,2,4)).** Let $q, r \in \Lambda(n)$ such that $q \approx r$ and let $\lambda = \lambda(q) \in \Lambda^+(n)$. We have

(i) $I_q I_r = \Xi^q I_r = \lambda I_q$,

(ii) $I_q E_\lambda = I_q$, and

(iii) $E_\lambda I_q = \lambda I E_\lambda$. 

To conclude the section, we give an example illustrating the above results.

**Example 2.23.** By definition, we have

\[ I_{(2,1)} = \Xi^{(2,1,1)} - \frac{1}{2} \Xi^{(1^4)}, \quad I_{(1,2)} = \Xi^{(1,2,1)} - \frac{1}{2} \Xi^{(1^4)}, \quad I_{(1,1,2)} = \Xi^{(1,1,2)} - \frac{1}{2} \Xi^{(1^4)}. \]

By Theorem 2.21, we have

\[ \Xi^{(3,1)} I_{(1,2)} = 2 I_{(1,2,1)} \]

where \( \{1, 3\} \sqcup \{2\} \) and \( \{2, 3\} \sqcup \{1\} \) are the only possibilities for such \( S_1 \sqcup S_2 \). Also, by definition,

\[ E_{(2,1)} = \frac{1}{6} (I_{(2,1,1)} + I_{(1,2,1)} + I_{(1,1,2)}) = \frac{1}{6} (\Xi^{(2,1,1)} + \Xi^{(1,2,1)} + \Xi^{(1,1,2)} - \frac{3}{2} \Xi^{(1^4)}). \]

Using Theorem 2.22(i), we have, for instance,

\[ E_{(2,1)} I_{(1,2,1)} = \frac{1}{6} (2 I_{(2,1,1)} + 2 I_{(1,2,1)} + 2 I_{(1,1,2)}) = 2 E_{(2,1,1)}. \]

One could of course also check the equations above using Theorem 2.7 as well.

### 3. Modular Idempotents of \( \mathcal{D}_{n,F} \)

Throughout this section, we assume that \( p > 0 \) and, by Theorem 2.16, we identify \( \mathcal{D}_{n,F} / \text{rad}(\mathcal{D}_{n,F}) \) with \( C_{n,F} \) through the Solomon’s epimorphism \( c_{n,F} \). We shall give a construction for the modular idempotents \( e_{\mu,F} \)'s for the descent algebra \( \mathcal{D}_{n,F} \) which satisfy the property as in [22, Corollary 6]. The main result is Theorem 3.7. A comparison with Proposition 2.17 shows that these two sets of idempotents enjoy the similar ‘triangularity property’.

We begin with some notations. Recall the Young characters \( \varphi^\lambda \) in Subsection 2.4. Fix a total order \( < \) on \( \Lambda^+(n) \) refining the weak refinement \( \preceq \) such that

\[ \Lambda^+(n) = \{(n) > \cdots > (1^n)\} \]

and let \( \Phi^F = (\varphi^{\lambda,F}(\mu))_{\lambda,\mu \in \Lambda^+_p(n)} \) where the last row and column are labelled by \( (n) \).

**Lemma 3.1.** The matrix \( \Phi^F \) is lower triangular with the diagonal entries \( \varphi^{\lambda,F}(\lambda) = \lambda_i \neq 0 \).

**Proof.** This follows from Lemma 2.14 and our choice of the total order. Since \( \lambda \) is \( p \)-regular, we have \( m_i(\lambda) < p \) for all \( i \) and therefore \( \lambda_i \neq 0 \) in \( F \). \( \square \)

By Lemma 3.1, the matrix \( \Phi^F \) is invertible. Let \( \Psi^F = (b_{\lambda,\mu}) \) be the inverse matrix of \( \Phi^F \). For each \( \lambda \in \Lambda^+_p(n) \), define

\[ f_{\lambda,F} = \sum_{\mu \in \Lambda^+_p(n)} b_{\lambda,\mu} \Xi^\mu \in \mathcal{D}_{n,F}. \]

Recall the ideals \( Y_q \) and \( Y_q^o \) of \( \mathcal{D}_{n,F} \) as in Subsection 2.2.

**Lemma 3.2.** Let \( \lambda \in \Lambda^+_p(n) \).

(i) \( c_{n,F}(f_{\lambda,F}) = \text{char}_{\lambda,F} \)
(ii) For any $\Lambda_p^+(n) \ni \mu \not\preceq \lambda$, we have $b_{\lambda,\mu} = 0$. In particular,

$$f_{\lambda,F} = \frac{1}{\lambda_1} \Xi^\lambda + \sum_{\Lambda_p^+(n) \ni \mu \prec \lambda} b_{\lambda,\mu} \Xi^\mu.$$ 

(iii) For any positive integer $r$, we have $(f_{\lambda,F})^r = \frac{1}{\lambda_1} \Xi^\lambda + \epsilon_\lambda$ for some $\epsilon_\lambda \in Y_\lambda^\circ$.

$$(\sum_{\lambda \in \Lambda_p^+(n)} f_{\lambda,F}) = 1$$

Proof. For part (i), for each $\gamma \in \Lambda_p^+(n)$, since $c_{n,F}$ is an $F$-algebra homomorphism, we have

$$c_{n,F}(f_{\lambda,F})(\gamma) = \sum_{\mu \in \Lambda_p^+(n)} b_{\lambda,\mu} \varphi_{\mu,F}(\gamma) = \delta_{\lambda,\gamma}.$$ 

For the first assertion in part (ii), suppose that $b_{\lambda,\mu} \neq 0$ for some largest (with respect to the total order $>$ we have fixed on $\Lambda_p^+(n)$) $\mu$ such that $\mu \not\preceq \lambda$. Evaluate part (i) at $\mu$, we get

$$0 = \sum_{\xi \in \Lambda_p^+(n)} b_{\lambda,\xi} \varphi_{\xi,F}(\mu) = \sum_{\Lambda_p^+(n)} b_{\lambda,\xi} \varphi_{\xi,F}(\mu) + \sum_{\Lambda_p^+(n) \ni \mu \prec \lambda} b_{\lambda,\mu} \varphi_{\mu,F}(\mu).$$

If $\xi \not\preceq \lambda$, then $\varphi_{\xi,F}(\mu) = 0$ by Lemma 2.14 (else, $\mu \not\preceq \xi \not\preceq \lambda$ and hence $\mu \not\preceq \lambda$). Therefore the first summand of the right-hand side of Equation 3.1 is 0. Suppose now that $\xi \not\preceq \lambda$. If $\mu > \xi$ then, in particular, $\mu \not\preceq \xi$ and, again by Lemma 2.14, $\varphi_{\xi,F}(\mu) = 0$. If $\xi > \mu$, by our choice of $\mu$, we have $b_{\lambda,\xi} = 0$. Therefore, using Lemma 2.14, Equation 3.1 reduces to

$$0 = b_{\lambda,\mu} \varphi_{\mu,F}(\mu) = b_{\lambda,\mu} \mu \neq 0.$$ 

This is a contradiction. Therefore

$$f_{\lambda,F} = \sum_{\Lambda_p^+(n) \ni \mu \prec \lambda} b_{\lambda,\mu} \Xi^\mu.$$ 

Apply $c_{n,F}$ to the equation and evaluation at $\lambda$, we get, by part (i) and Lemma 2.14

$$b_{\lambda,\lambda} \lambda = b_{\lambda,\lambda} \varphi_{\lambda,F}(\lambda) = c_{n,F}(f_{\lambda,F})(\lambda) = \text{char}_{\lambda,F}(\lambda) = 1.$$ 

We now prove part (iii). Notice that

$$(f_{\lambda,F})^2 = \left(\frac{1}{\lambda_1} \Xi^\lambda + \sum_{\Lambda_p^+(n) \ni \mu \prec \lambda} b_{\lambda,\mu} \Xi^\mu\right)^2 = \frac{1}{\lambda_1^2} \Xi^{2\lambda} + Z.$$ 

where $Z$ is the sum of the products of the form $\Xi^\zeta \Xi^\gamma$ where $\zeta, \gamma \in \Lambda(n)$ such that either $\zeta \not\preceq \lambda$ or $\gamma \not\preceq \lambda$. By Corollary 2.8, $Z \in Y_\lambda^\circ$ and $(\Xi^{2\lambda}) = \lambda_1 \Xi^{\lambda} + Y$ where $Y \in Y_\lambda^\circ$. Substitute these into Equation 3.2, we get the case for $r = 2$. Now the general case follows inductively.

For part (iv), let $z = \sum_{\lambda \in \Lambda_p^+(n)} f_{\lambda,F}$. Notice that, for any $\gamma \in \Lambda_p^+(n)$, we have

$$c_{n,F}(z)(\gamma) = \sum_{\lambda,\mu \in \Lambda_p^+(n)} b_{\lambda,\mu} \varphi_{\mu,F}(\gamma) = \sum_{\lambda \in \Lambda_p^+(n)} \delta_{\gamma = \lambda} = 1.$$
Therefore $c_{n,F}(z) = 1$ in $C_{n,F}$. Since $c_{n,F}(\Xi(n)) = 1$, we have $z - \Xi(n) \in \ker(c_{n,F})$. Since $z - \Xi(n)$ is a sum involving $\Xi^{(n)}$ such that $(n) \notin \Lambda_+^+(n)$, by Theorem 2.16 we must have $z - \Xi(n) = 0$.

The elements $\{f_{\lambda,F} : \lambda \in \Lambda_+^+(n)\}$ are generally not idempotents and far from being orthogonal to each other. We want to lift $f_{\lambda,F} + \text{rad}(\mathcal{D}_{n,F}) = \text{char}_{\lambda,F}$ and orthogonalize them. To do so, we use the idea in the proofs of Idempotent Lifting Theorem (see, for example, [7, Theorem 1.7.3]) and [7, Corollary 1.7.4] as follow.

**Proposition 3.3.** Let $A$ be a finite-dimensional algebra over $F$ and $N$ be a nilpotent ideal in $A$.

(i) Suppose that $c \in A/N$ is an idempotent and $c = a + N$ for some $a \in A$. Then there is a large enough $k$ (depending just on the nilpotency index of $N$) such that $a^k$ is an idempotent lifting $c$.

(ii) Let $c_1, \ldots, c_n$ be primitive orthogonal idempotents in $A/N$ such that $\sum_{i=1}^n c_i = 1 + N$. Suppose that $e_i' = 1$ and, for $i > 1$, $e_i'$ is a lift of $\sum_{j \geq i} c_j$ to an idempotent in $e_i' - 1 + Ae_{i-1}'$. Then $\{e_i = e_i' \cdot 1 : i \in [1, n]\}$ are primitive orthogonal idempotents in $A$ such that $\sum_{i=1}^n e_i = 1$ and $e_i + N = c_i$.

We write $a^{p^\infty}$ for the idempotent in Proposition 3.3(i). In the case of the descent algebra $\mathcal{D}_{n,F}$, it is sufficient to choose $k_n$ with $p^{k_n} \geq n - 1$ (see [6, Theorem 3]) such that, for each idempotent $e \in C_{n,F}$ and $x \in \mathcal{D}_{n,F}$ such that $c_{n,F}(x) = e$, we have $x^{p^\infty} = x^{p^{k_n}}$ is an idempotent lifting $e$.

We remark that, instead of repeatedly using $3a^2 - 2a^3$ for the idempotent-lifting procedure, the ‘advantage’ here allows us to lift $e$ immediately to an idempotent by taking enough high power of $p$ for $a$ and is easier to present in the subsequent proofs.

In the sequel, suppose that $m = |\Lambda_+^+(n)|$, with respect to the total order $>$, and we write

$$(n) = \lambda^{(m)} > \cdots > \lambda^{(1)}$$

for the elements in $\Lambda_+^+(n)$. Furthermore, to simplify notations, we denote

$$f_i = f_{\lambda^{(i)},F}, \quad f_{\geq i} = \sum_{j \geq i} f_j, \quad \text{char}_i = \text{char}_{\lambda^{(i)},F}, \quad \text{char}_{\geq i} = \sum_{j \geq i} \text{char}_j,$$

so that, by Lemma 3.2(i), $c_{n,F}(f_{\geq i}) = \text{char}_{\geq i}$. Furthermore, for each $i \in [1, m]$, let $Y_{\leq i} = \sum_{j \in [1, m]} Y_{\lambda^{(j)}}$ and $Y_{\geq i}^0 = Y_{\lambda^{(i)}} + \sum_{j \in [1, i-1]} Y_{\lambda^{(j)}},$ i.e., $Y_{\leq i}$ and $Y_{\geq i}^0$ have the following respective $F$-bases

$$\{\Xi^{\xi} : \Lambda(n) \ni \xi \preceq \lambda^{(j)}, j \in [1, i]\},$$

$$\{\Xi^{\xi} : \Lambda(n) \ni \xi \preceq \lambda^{(j)}, j \in [1, i], \xi \not\preceq \lambda^{(i)}\}.$$

The following follows from Corollary 2.8(ii).

**Lemma 3.4.** We have a chain of ideals of $\mathcal{D}_{n,F}$ given by

$$Y_{\leq 1} \subsetneq Y_{\leq 1} \subsetneq Y_{\leq 2} \subsetneq Y_{\leq 2} \subsetneq \cdots \subsetneq Y_{\leq m} \subsetneq Y_{\leq m} = \mathcal{D}_{n,F}.$$
We now define the elements \( e_{\lambda,F}'s \)'s which will form a complete set of primitive orthogonal idempotents for \( \mathcal{D}_{n,F} \).

**Definition 3.5.** Let \( f'_1 = 1 \) and, inductively, for \( i \in [2, m] \), we define \( f'_i = (f'_{i-1}f_{i}f'_{i-1})^{p^\infty} \) and \( f'_{m+1} = 0 \). For each \( i \in [1, m] \), define
\[
e_{\lambda(i),F} = f'_i - f'_{i+1}.
\]

In view of Definition 3.5, we want to show that the elements \( f'_i's \) satisfy the hypothesis stated in Proposition 3.3(ii).

**Lemma 3.6.**

(i) For each \( i \in [1, m] \), the element \( f'_i \) is an idempotent such that \( c_{n,F}(f'_i) = \text{char}_{\geq i} \) and, for \( i > 1 \), \( f'_i \in f'_{i-1}\mathcal{D}_{n,F}f'_{i-1} \).

(ii) For \( i \in [2, m] \),
\[
f'_i = 1 - \left( \frac{1}{\lambda^{(i-1)}} \Xi^{\lambda^{(i-1)}} + \epsilon_{i-1} \right)
\]
for some \( \epsilon_{i-1} \in Y_{\leq i-1}^2 \).

**Proof.** We first prove part (i). We argue by induction on \( i \). It is clearly true when \( i = 1 \) as \( \text{char}_{\geq 1} = 1 \) and \( f'_1 = 1 \). Let \( i > 1 \). Since \( c_{n,F} \) is an \( F \)-algebra homomorphism, we have
\[
c_{n,F}(f'_{i-1}f_{i}f'_{i-1}) = \text{char}_{\geq i-1} \text{char}_{\geq i} \text{char}_{\geq i-1} = \text{char}_{\geq i}.
\]
Therefore, \( c_{n,F}(f'_i) = \text{char}_{\geq i} \). By Proposition 3.3(i), \( f'_i = (f'_{i-1}f_{i}f'_{i-1})^{p^\infty} \) is an idempotent. Finally, since \( f'_{i-1} \) is an idempotent, we have
\[
f'_i = f'_{i-1}(f'_{i-1}f_{i}f'_{i-1})^{p^\infty} f'_{i-1} \in f'_{i-1}\mathcal{D}_{n,F}f'_{i-1}.
\]

For part (ii), we again argue by induction on \( i \). For \( i = 2 \), notice that
\[
f'_2 = (f_2)^{p^\infty} = (1 - f_1)^{p^\infty} = 1 - f_1^{p^\infty} = 1 - \left( \frac{1}{\lambda^{(1)}} \Xi^{\lambda^{(1)}} + \epsilon_1 \right),
\]
for some \( \epsilon_1 \in Y_{\leq 1}^2 \) where the second and last equalities are obtained using parts (iv) and (iii) of Lemma 3.2 respectively. Suppose inductively now that \( f'_i = 1 + \epsilon'_{i-1} \) where \( \epsilon'_{i-1} \in Y_{\leq i-1} \). Therefore, again using Lemma 3.2(iv),
\[
f'_{i+1} = (f'_i f_{i+1} f'_i)^{p^\infty}
\]
\[
= ((1 + \epsilon'_{i-1})(1 - f_i - \cdots - f_i)(1 + \epsilon'_{i-1}))^{p^\infty}
\]
\[
= ((1 + \epsilon'_{i-1})^2 - (1 + \epsilon'_{i-1})f_i(1 + \epsilon'_{i-1}) - \cdots - (1 + \epsilon'_{i-1})f_i(1 + \epsilon'_{i-1}))^{p^\infty}
\]
\[
= (1 - f_i + E)^{p^\infty}
\]
\[
= (1 - f_i)^{p^\infty} + E'
\]
\[
= 1 - f_i^{p^\infty} + E'
\]
\[
= 1 - \frac{1}{\lambda^{(i)}} \Xi^{\lambda^{(i)}} - \epsilon_i.
\]
where $E$ is the sum of products involving at least one of the $f_1, \ldots, f_{i-1}, e'_i$ and $E'$ is the sum of products involving at least one $E$. Since $f_1, \ldots, f_{i-1}, e'_i \in Y_{\leq i-1}$, by Lemma 3.4 $E \in Y_{\leq i-1}$ and consequently $E' \in Y_{\leq i-1}$. Lastly, by Lemmas 3.2(iii) and 3.4 we have $e_i = f_i' - 1/\lambda(\xi) \Xi^{\lambda(i)} - E' \in Y_{\leq i}$.

We are now ready to state and prove our main result in this section regarding the modular idempotents for the descent algebra $D_{n,F}$.

**Theorem 3.7.** The set $\{e_{\lambda,F} : \lambda \in \Lambda^+_F(n)\}$ is a complete set of primitive orthogonal idempotents of $D_{n,F}$ such that $c_{n,F}(e_{\lambda,F}) = \text{char}_{\lambda,F}$ and $\sum_{\lambda \in \Lambda^+_F(n)} e_{\lambda,F} = 1$. Furthermore,

$$e_{\lambda,F} = \frac{1}{\lambda(\xi)} \Xi^{\lambda} + \epsilon_{\lambda}$$

where $\epsilon_{\lambda}$ is a linear combination of some $\Xi^\xi$ such that $\xi < \lambda$, i.e., $\lambda \not\equiv \xi \not\equiv \lambda$.

**Proof.** Using Proposition 3.3(ii) and Lemma 3.6(i), we get the first sentence of our theorem. When $i = 1$, by Lemma 3.6(i), we have

$$e_{\lambda(1),F} = 1 - f_2 = \frac{1}{\lambda(1)} \Xi^{\lambda(1)} + \epsilon_1$$

where $\epsilon_{\lambda(1)} = e_1 \in Y_{\leq 1} = Y_{\lambda(1)}$. For $i \in [2, m]$, again by Lemma 3.6(ii), we have

$$e_{\lambda(i),F} = -\left(\frac{1}{\lambda(i-1)} \Xi^{\lambda(i-1)} + \epsilon_{i-1}\right) + \left(\frac{1}{\lambda(\xi)} \Xi^{\lambda(i)} + \epsilon_i\right) = \frac{1}{\lambda(\xi)} \Xi^{\lambda(i)} + \epsilon_{\lambda(i)}$$

where $\epsilon_{\lambda(i)} = \epsilon_i - \frac{1}{\lambda(i-1)} \Xi^{\lambda(i-1)} - \epsilon_{i-1} \in Y_{\leq i}$. To complete the proof, we claim that $e_{\lambda(i),F} \in Y_{\lambda(i)}$, so that $e_{\lambda(i)} = e_{\lambda(i),F} - \frac{1}{\lambda(\xi)} \Xi^{\lambda(i)} \in Y_{\leq i} \cap Y_{\lambda(i)} = Y_{\lambda(i)}$. The case $i = 1$ is done. Suppose that $i \in [2, m]$. We have

$$f_{i+1}' = (f_i' f_{i+1} f_i')^{p^\infty}$$

$$= (f_i' f_{i+1} f_i' - f_i' f_i f_i')^{p^\infty}$$

$$= (f_i' f_{i+1} f_i')^{p^\infty} + E$$

$$= ((f_i' f_{i+1} f_i')^{p^\infty} f_i' (f_i' f_{i+1} f_i')^{p^\infty})^{p^\infty} + E$$

$$= ((f_i' f_{i+1} f_i')^{p^\infty} f_i' f_{i+1} f_i' (f_i' f_{i+1} f_i')^{p^\infty})^{p^\infty} + E$$

$$= f_i' + E,$$

where $E$ is a sum of product involving the term $f_i' f_{i+1} f_i'$ and the fifth equality is due to the fact that $f_{i-1}'$ is an idempotent. Since $f_i' f_{i+1} f_i' \in Y_{\lambda(i)}$ by Lemmas 3.2(iii) and 3.4 we have $E \in Y_{\lambda(i)}$ and therefore $e_{\lambda(i),F} = f_i' - f_{i+1}' \in Y_{\lambda(i)}$.

We demonstrate our construction of the idempotents with an example.
Example 3.8. Let $p = 3$, $n = 4$ and $(4) > (3,1) > (2,2) > (2,1,1)$ be the total order on $\Lambda^+_3(4)$. It can be easily checked that we have

\[
\Phi^F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \Psi^F = (\Phi^F)^{-1} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
\]

Therefore we have

\[
f_1 = 2\Xi^{(2,1,1)}, \quad f_2 = 2\Xi^{(2,2)} + \Xi^{(2,1,1)}, \quad f_3 = \Xi^{(3,1)} + 2\Xi^{(2,1,1)}, \quad f_4 = \Xi^{(4)} + 2\Xi^{(3,1)} + \Xi^{(2,2)} + \Xi^{(2,1,1)}.
\]

As we mentioned earlier, it suffices to take power $p^n \geq n - 1$ (see the paragraph right after Proposition 3.3). In this case, it is enough to take the third power. We get

\[
f_1' = 1, \quad f_2' = (\Xi^{(4)} + \Xi^{(2,1,1)})^3 = 1 - (2\Xi^{(2,1,1)} + 2\Xi^{(1,1,1)}),
\]

\[
f_3' = (f_2'\Xi^{(4)} + \Xi^{(2,2)})^3 = 1 - (2\Xi^{(2,2)} + \Xi^{(2,1,1)} + 2\Xi^{(1,1,2)} + \Xi^{(1,1,1)}),
\]

\[
f_4' = (f_2'f_3f_4)^3 = 1 - (\Xi^{(3,1)} + 2\Xi^{(2,2)} + 2\Xi^{(1,1,2)} + \Xi^{(1,1,1)}),
\]

and $f_5' = 0$. Therefore,

\[
e_{(2,1,1),F} = 2\Xi^{(2,1,1)} + 2\Xi^{(1,1,1)}, \quad e_{(2,2),F} = 2\Xi^{(2,2)} + 2\Xi^{(2,1,1)} + 2\Xi^{(1,1,2)} + 2\Xi^{(1,1,1)},
\]

\[
e_{(3,1),F} = \Xi^{(3,1)} + 2\Xi^{(2,1,1)}, \quad e_{(4),F} = \Xi^{(4)} + 2\Xi^{(3,1)} + \Xi^{(2,2)} + \Xi^{(1,1,2)} + 2\Xi^{(1,1,1)}.
\]

We have presented the computation results of some modular idempotents in Appendices A and B using Magma [12]. We now state a few corollaries following our theorem.

Corollary 3.9. Let $\lambda \in \Lambda^+_p(n)$ and $q \in \Lambda(n)$. Then $e_{\lambda,F}\omega_q = 0$ unless $q \preceq \lambda$. When $q \approx \lambda$, we have $e_{\lambda,F}\omega_q = \omega_\lambda$.

Proof. By Theorem 2.10(ii), $\Xi^s\omega_q = 0$ unless $q \preceq s$. Therefore, by Theorem 3.7 we get the first assertion. For the second assertion, similarly, using Theorem 3.7 we have

\[
e_{\lambda,F}\omega_q = \frac{1}{\lambda!}\Xi^\lambda\omega_q + e_\lambda\omega_q = \frac{1}{\lambda!}\Xi^\lambda\omega_q = \frac{1}{\lambda!}|N_{\lambda,q}|\omega_\lambda = \frac{1}{\lambda!}\lambda!\omega_\lambda = \omega_\lambda
\]

where the third and fourth equalities follow from Theorem 2.10(ii) and Lemma 2.2 respectively.

By Lemma 2.11 if $\lambda \approx q$, there exists $\sigma$ such that $\sigma\omega_q = \omega_\lambda$. Therefore, Corollary 3.9 implies the following result.

Corollary 3.10. Let $q \in \Lambda(n)$ such that $q \approx \lambda \in \Lambda^+_p(n)$. The map $\alpha : \omega_qF\mathfrak{S}_n \rightarrow e_{\lambda,F}F\mathfrak{S}_n$ defined by the left multiplication by $e_{\lambda,F}\sigma$ is an injection, where $\sigma\omega_q = \omega_\lambda$.

Remark 3.11. The construction of the modular idempotents in this section can be extended for the descent algebras of Coxeter groups. We shall present them in a forthcoming paper.
As we have mentioned earlier, in the $p = 0$ case, we also could construct a list of primitive orthogonal idempotents for $D_{n,F}$ using the method in this section by replacing the idempotent lifting $a^\infty$ with performing $3a^2 - 2a^3$ repeatedly. To conclude the section, we give a simple example and compare it with the orthogonalization of the Lie idempotents $\nu_\lambda$’s as in Proposition 2.17.

**Example 3.12.** Let $n = 3$, $p = 0$ (or $p > 3$) and $(3) > (2, 1) > (1^3)$. We have

$$\Phi^0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix}, \quad \Psi^0 = \begin{pmatrix} 1 & -1 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{pmatrix}.$$

Therefore,

$$f_3 = \frac{1}{6} \Xi^{(1^3)}, \quad f_2 = \Xi^{(2, 1)} - \frac{1}{2} \Xi^{(1^3)}, \quad f_1 = \Xi^{(3)} - \Xi^{(2, 1)} + \frac{1}{3} \Xi^{(1^3)}.$$

It is readily checked that $\{f_1, f_2, f_3\}$ is already a complete set of primitive orthogonal idempotents and therefore it is the orthogonalization of itself; namely, $e_{(3)} = f_1$, $e_{(2, 1)} = f_2$ and $e_{(1,1,1)} = f_3$.

On the other hand, if we orthogonalize $\{\nu_{(3)}, \nu_{(2, 1)}, \nu_{(1^3)}\}$ as in Proposition 2.17 where

$$\nu_{(3)} = \Xi^{(3)} - \frac{1}{3} \Xi^{(2, 1)} - \frac{2}{3} \Xi^{(1, 2)} + \frac{1}{3} \Xi^{(1^3)}, \quad \nu_{(2, 1)} = \Xi^{(2, 1)} - \frac{1}{2} \Xi^{(1^3)}, \quad \nu_{(1^3)} = \frac{1}{6} \Xi^{(1^3)},$$

we get

$$e'_{(3)} = \nu_{(3)} = \Xi^{(3)} - \frac{1}{3} \Xi^{(2, 1)} - \frac{2}{3} \Xi^{(1, 2)} + \frac{1}{3} \Xi^{(1^3)},$$

$$e'_{(2, 1)} = \nu_{(2, 1)}(1 - \nu_{(3)}) = \Xi^{(2, 1)} + \frac{1}{2} \Xi^{(1^3)},$$

$$e'_{(1,1,1)} = \nu_{(1,1,1)}(1 - \nu_{(2, 1)})(1 - \nu_{(3)}) = \frac{1}{6} \Xi^{(1^3)}.$$

The set $\{e_\lambda : \lambda \in \Lambda^+(3)\}$ appears to be simpler than $\{e'_\lambda : \lambda \in \Lambda^+(3)\}$ in terms of the $\Xi^\psi$’s. But, when $p = 0$, in general, the computation requires the computation of the inverse matrix $\Psi^0$ which is not obvious at all. Indeed, in this case, $KC = \Phi^0$ where $K$ and $C$ are the Kostka matrix and character table for the symmetric group. Therefore the knowledge of the inverse of $\Phi^0$ would give us knowledge about the inverse of the Kostka matrix or character table because $K^{-1} = C\Psi^0$ and $C^{-1} = \Psi^0K$.

4. Higher Lie powers and higher Lie modules

In this section, we study the higher Lie power $L^q(V)$ and higher Lie module $\text{Lie}_F(q)$ when $q$ is coprime to $p$. Our main results in this section are Theorems 4.2 and 4.4 which basically establish a basis and describe the structure for $\text{Lie}_F(q)$ when $(q, p) = 1$. Both results work for arbitrary $p$. The extra assumption of $p > 0$ only applies after Theorem 4.4. We begin with some notations.
Let \( \lambda \in \Lambda^+(n) \) and \( k = \ell(\lambda) \). Recall the notations \( m_i(\lambda) \) and \( m(\lambda) \) in Section 2. Let \( V \) be a finite dimensional vector space over \( F \) and, for each \( i \in [1, k] \), let \( P_i \in \omega_i \cdot T^\lambda_i(V) = L^\lambda_i(V) \). We write
\[
(P_1, \ldots, P_k)_\lambda = \sum_{\pi \in \mathfrak{S}_{m(\lambda)}} P_{1\pi} \cdots P_{k\pi} = \Xi^\lambda \cdot P_1 \cdots P_k
\]
where the last equality follows from Theorem 2.19.

**Definition 4.1.** Let \( \Gamma(\lambda) \) be the set consisting of sequences
\[
(T_{n,1}, \ldots, T_{n,m_n(\lambda)}, \ldots, T_{1,1}, \ldots, T_{1,m_1(\lambda)})
\]
such that
(a) \( T_{i,j} \) are sequences of positive integers of length \( i \) such that \( \bigcup_{i,j} \{(T_{i,j})_k : k \in [1, i]\} = [1, n] \),
(b) \( \min(T_{i,j}) = (T_{i,j})_1 \) for all admissible \( i, j \), and
(c) \( \min(T_{i,1}) < \min(T_{i,2}) < \cdots < \min(T_{i,m_i(\lambda)}) \) for all \( i \in [1, n] \).

Notice that \( |\Gamma(\lambda)| = |\mathfrak{S}_\lambda| = \frac{n!}{\gamma(\lambda)} \) because \( \Gamma(\lambda) \) is in one-to-one correspondence with the set of permutations of cycle type \( \lambda \) in \( \mathfrak{S}_n \) and also the set of standard right coset representatives \( \prod_{i=1}^n (\mathfrak{S}_i \setminus \mathfrak{S}_{m_i}) \setminus \mathfrak{S}_n \). The second correspondence is obtained by reading, from left to right and top to bottom, the entries of \( \lambda \)-tableaux \( t \) such that \( t_{i,j} < t_{i,j+1} \) if \( \lambda_j = \lambda_{j+1} \).

Recall the notation \( Q_S \) in Subsection 2.2. For each \( T \in \Gamma \) with the presentation as before, we write
\[
Q_T = Q_{T_{n,1}} \cdots Q_{T_{n,m_n(\lambda)}} \cdots Q_{T_{1,1}} \cdots Q_{T_{1,m_1(\lambda)}},
\]
\[
(Q_T)_\lambda = (Q_{T_{n,1}}, \ldots, Q_{T_{n,m_n(\lambda)}}, \ldots, Q_{T_{1,1}}, \ldots, Q_{T_{1,m_1(\lambda)})}_\lambda.
\]
They are viewed as elements in \( R\mathfrak{S}_n \). For an example,
\[
Q_{(1,2),(3,4)} = Q_{(1,2)}Q_{(3,4)} = 1234 - 2134 - 1243 + 2143.
\]

Similar as before, we have \( (Q_T)_\lambda = \Xi^\lambda Q_T \). Notice that \( \omega_i \cdot Q_{T_{i,j}} = iQ_{T_{i,j}} \) since \( \omega_i^2 = i\omega_i \). Therefore, \( \omega^\lambda Q_T = \prod_{i=1}^n i^{m_i(\lambda)} Q_T \) and likewise for \( (Q_T)_\lambda \). With respect to \( \lambda \), for each pair of admissible numbers \( i, j \), denote \( d_{i,j} = i(j - 1) + \sum_{t \in [i+1, n]} tm_t(\lambda) \) and call the subword
\[
w_{d+1}w_{d+2} \cdots w_{d+i},
\]
the \( (i, j) \)-section of a word \( w \in \mathfrak{S}_n \).

Our first result in this section establishes a basis for the higher Lie module \( \text{Lie}_F(q) \) when \( q \) is coprime to \( p \) (cf. Appendix C).

**Theorem 4.2.** Let \( q \mid m \) and \( (q, p) = 1 \). The set \( \{\omega_i Q_T : T \in \Gamma(\lambda)\} \) is a basis for \( \text{Lie}_F(q) \). In particular, \( \dim_F \text{Lie}_F(q) = |\mathfrak{S}_\lambda| = \frac{n!}{\gamma(\lambda)} \) where \( \lambda = \lambda(q) \).

**Proof.** By Lemma 2.11 we may assume that \( q = \lambda(q) = \lambda \in \Lambda^+(n) \). By definition,
\[
\omega^\lambda Q_T = \omega^\lambda \Xi^\lambda Q_T = \omega^\lambda (Q_T)_\lambda = \left( \prod_{i=1}^n i^{m_i(\lambda)} \right) (Q_T)_\lambda.
\]
Since $\lambda$ is coprime to $p$, we have $(\prod_{i=1}^{n}t_{i}^{m_{i}(\lambda)}) \neq 0$. By Proposition 2.12, it suffices to show that the set $\{(Q_{T})_{\lambda} : T \in \Gamma(\lambda)\}$ is linearly independent.

Notice that, for a word $w$ appearing in $(Q_{T})_{\lambda}$, by the definition, the alphabets appearing in $(i, j)$-section of $w$ must be the set $\{t : t \in T_{ij}\}$ for some $j' \in [1, m_{i}(\lambda)]$. Fix an arbitrary $T \in \Gamma(\lambda)$. Therefore, it suffices to consider

$$\sum a_{S}(Q_{S})_{\lambda} = 0$$

where the sum is taken over all $S \in \Gamma(\lambda)$ such that $\{s : s \in S_{ij}\} = \{t : t \in T_{ij}\}$ for all admissible $i, j$. Fix an $S' \in \Gamma(\lambda)$ satisfying such property. We want to show that $a_{S'} = 0$. For each admissible $i, j$, we focus on the $(i, j)$-section in the sum. By Proposition 2.9, we have $\sum a_{S}(Q_{S})_{\lambda} = 0$ where the sum now is taken over all such $S$ with the extra condition $S'_{ij} = S'_{ij}$. Repeat this inductively for each admissible $i, j$, we have $a_{S}(Q_{S})_{\lambda} = 0$ where $S = S'$. Since $(Q_{S'})_{\lambda} \neq 0$, we must have $a_{S'} = 0$. The proof is now complete.

We could now prove the isomorphism

$$\text{Lie}_{F}(\lambda) \cong \text{ind}_{\prod\mathfrak{S}_{m_{i}(\lambda)}(\mathfrak{S}_{i})_{\lambda}}^{\mathfrak{S}_{n}}(\text{Lie}_{F}(1)^{m_{1}} \otimes \cdots \otimes \text{Lie}_{F}(n)^{m_{n}})$$

where $m_{i} = m_{i}(\lambda)$ as in Theorem 4.3 directly. But we could also, with the extra assumption that $F$ is infinite, prove this using the polynomial representation of $G_{F}$ and apply Schur functor. By doing this, it reveals that, in general, the higher Lie power $L^{q}(V)$ when $(q, p) = 1$ is a quotient of the tensor product of some symmetric powers.

In the next lemma, we denote the $n$th symmetric power of a vector space $V$ by $S^{n}(V)$, i.e., $S^{n}(V) \cong F \otimes_{F\mathfrak{S}_{n}} T^{n}(V)$ which has a basis represented by monomials $v_{1} \cdots v_{i_{n}}$ such that $1 \leq i_{1} \leq \cdots \leq i_{n} \leq m$ given that $\{v_{1}, \ldots, v_{m}\}$ is a basis for $V$.

**Lemma 4.3.** Let $\lambda \in \Lambda^{+}(n)$, $(\lambda, p) = 1$ and $m_{i} = m_{i}(\lambda)$ for each $i \in [1, n]$. There is a surjection $\psi : S^{m_{n}}(L^{1}(V)) \otimes \cdots \otimes S^{m_{1}}(L^{1}(V)) \to \omega_{\lambda} \cdot T^{n}(V)$ given by

$$\psi((P_{n,1} \cdots P_{n,m_{n}}) \otimes \cdots \otimes (P_{1,1} \cdots P_{1,m_{1}})) = \omega^{\lambda} \cdot (P_{n,1}, \ldots, P_{n,m_{n}}, \ldots, P_{1,1}, \ldots, P_{1,m_{1}})_{\lambda}$$

where $P_{i,j} \in L^{i}(V)$ for each admissible $i, j$. Furthermore, if $V$ is a right $FG$-module then $\psi$ is an $FG$-module homomorphism.

**Proof.** Notice that the image is fixed by the action of $\mathfrak{S}_{m(\lambda)}$ on the left. Also, as we have noted earlier,

$$\omega^{\lambda} \cdot (P_{n,1}, \ldots, P_{1,m_{1}})_{\lambda} = (\omega^{\lambda} \Xi^{\lambda}) \cdot (P_{n,1} \cdots P_{1,m_{1}}) = \omega^{\lambda} \cdot P_{n,1} \cdots P_{1,m_{1}}.$$ 

So $\psi$ is well-defined. Let $k = \ell(\lambda)$ and $N = \prod_{i=1}^{k} \lambda_{i}$ so that $N \neq 0$ by our assumption. Notice that

$$\Xi^{\lambda} = \sum_{S_{1}, \ldots, S_{k} \in [1, n], \mid S_{i} \mid = \lambda_{i}} S_{1} \cdots S_{k} = \sum_{\pi \in \mathfrak{S}_{m(\lambda)}} \sum_{(S_{1}, \ldots, S_{k}) \in \Lambda} S_{1^{\pi}} \cdots S_{k^{\pi}}$$
where \( \Lambda \) consists of \((S_1, \ldots, S_k)\) such that \( S_1 \sqcup \cdots \sqcup S_k = [1, n] \) such that \( |S_i| = \lambda_i \) and \( \min(S_i) < \min(S_{i'}) \) if \( |S_i| = |S_{i'}| \). Therefore,

\[
\omega_\lambda = \omega^\Lambda \Xi_\lambda = \sum_{\pi \in \mathfrak{S}_m(\lambda)} \sum_{(S_1, \ldots, S_k) \in \Lambda} (\omega_{\lambda_1} S_{1\pi}) \cdots (\omega_{\lambda_k} S_{k\pi})
\]

For any \( v_1, \ldots, v_n \in V \), we have

\[
\omega_\lambda \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_{\pi \in \mathfrak{S}_m(\lambda)} \sum_{(S_1, \ldots, S_k) \in \Lambda} (\omega_{\lambda_1} S_{1\pi}) \cdots (\omega_{\lambda_k} S_{k\pi}) \cdot (v_1 \otimes \cdots \otimes v_n)
\]

\[
= \sum_{\pi \in \mathfrak{S}_m(\lambda)} \sum_{(S_1, \ldots, S_k) \in \Lambda} P_{S_1\pi} \cdots P_{S_k\pi}
\]

\[
= \sum_{(S_1, \ldots, S_k) \in \Lambda} \left( \frac{1}{N} \omega^\Lambda \cdot (P_{S_1}, \ldots, P_{S_k})_\lambda \right)
\]

\[
= \sum_{(S_1, \ldots, S_k) \in \Lambda} \frac{1}{N} \psi((P_{S_1} \cdots P_{S_{mn}}) \otimes \cdots \otimes (P_{S_{k-m+1}} \cdots P_{S_k}))
\]

where, for each \( j \in [1, k] \), \( P_{S_j} = \omega_{\lambda_j} \cdot (v_{s_1} \otimes \cdots \otimes v_{s_{\lambda_j}}) \) if \( S_j = \{s_1 < \cdots < s_{\lambda_j}\} \). Therefore \( \psi \) is surjective. The fact that \( \psi \) is an \( FG \)-module homomorphism follows from the definitions of the tensor product and symmetric power. \( \square \)

In general, the map \( \psi \) in Lemma 4.3 is not an injection. For an example, let \( p = 2 \), \( V \) be 2-dimensional and \( \lambda = (1, 1) \). Then \( S^2(V) \) is 3-dimensional but \( \omega_{(1,1)} \cdot T^2(V) \) is 1-dimensional.

We can now state and prove the second main result for this section.

**Theorem 4.4.** Let \( q \in \Lambda(n) \) and suppose that \( (q, p) = 1 \). We have isomorphisms of right \( F \mathfrak{S}_n \)-modules

\[
\text{Lie}_F(q) \cong \text{ind}_{\prod_{i=1}^n (C_i) \rtimes \mathfrak{S}_m_i}^{\mathfrak{S}_n} (\text{Lie}_F(1)^{m_1} \boxtimes \cdots \boxtimes \text{Lie}_F(n)^{m_n}) \cong \text{ind}_{\prod_{i=1}^n (C_i) \rtimes \mathfrak{S}_m_i}^{\mathfrak{S}_n} F_\delta
\]

where \( m_i = m_i(q) \) and \( F_\delta \) is certain one-dimensional module for \( \prod_{i=1}^n (C_i \rtimes \mathfrak{S}_m_i) \).

**Proof.** Let \( \Lambda^+(n) \ni \lambda \approx q \), \( H = \prod_{i=1}^n (\mathfrak{S}_i \ltimes \mathfrak{S}_m_i) \) and \( Z_F \) denote the induced module \( \text{ind}_H^{\mathfrak{S}_n}(\text{Lie}_F(1)^{m_1} \boxtimes \cdots \boxtimes \text{Lie}_F(n)^{m_n}) \) defined over \( F \). Assuming first that \( F \) is infinite. Let \( V \) be the natural \( GL(V) \)-module and assume that \( \dim_F V \geq n \). By [20, §2.5 Lemma] and \[23\] Corollary 3.2(i), the Schur functor \( f \) maps the module \( S^{m_n}(L^n(V)) \otimes \cdots \otimes S^{m_1}(L^1(V)) \) isomorphically to \( Z_F \). Since \( f \) is exact, by Lemma 4.3, we have a surjection from \( Z_F \) to \( \text{Lie}_F(\lambda) \). Since \( \lambda \) is coprime to \( p \), by Theorem 4.2 \( \dim_F \text{Lie}_F(\lambda) = \frac{n!}{\lambda !} \). On the other hand, using Proposition 2.9 we have

\[
\dim_F(Z_F) = \frac{n!}{\prod_{i=1}^n (i!)^{m_i} m_i!} \prod_{i=1}^n ((i-1)!)^{m_i} = \frac{n!}{\prod_{i=1}^n (i!)^{m_i} m_i!} = \frac{n!}{\lambda !}.
\]
Comparing the dimensions, the surjection is indeed an isomorphism. Suppose now that $F$ is arbitrary and let $\overline{F}$ be the algebraic closure of $F$. Via extension of the ground field, since
\[
\text{Lie}_F(\lambda) \otimes_F \overline{F} \cong \text{Lie}_{\overline{F}}(\lambda) \cong Z_{\overline{F}} \cong Z_F \otimes_F \overline{F},
\]
by [27] Theorem 1.21, we have $\text{Lie}_F(\lambda) \cong Z_F$.

Alternatively, the isomorphism can also be proved by an explicit map. Let $W$ be the $F$-linear span of
\[
\{\omega_\lambda Q_T : T \in \Gamma(\lambda), \{t : t \in T_{i,j}\} = [d_{i,j} + 1, d_{i,j} + i]\}.
\]
As we pointed out, $\omega_\lambda Q_T = \omega^\lambda(Q_T)_\lambda$. By the characterization of the induced module as in [2, Section 8, Corollary 3], the fact that $W$ generates $\text{Lie}_F(\lambda)$ using Theorem 4.2 and our previous observation that $\dim_F \text{Lie}_F(\lambda) = |S_n : H| \dim_F W$, it suffices to check that $\text{Lie}_F(n)^{m_n} \boxtimes \cdots \boxtimes \text{Lie}_F(1)^{m_1}$ is isomorphic to $W$ as $FH'$-modules where $H' = (\mathfrak{S}_n \wr \mathfrak{S}_{m_n}) \times \cdots \times (\mathfrak{S}_1 \wr \mathfrak{S}_{m_1})$. Since it is a direct product, we may further assume that $\lambda = (d^k)$ and $H' = \mathfrak{S}_d \wr \mathfrak{S}_k$. We claim that the linear map $\phi$ sending $\omega_d \sigma_1 \otimes \cdots \otimes \omega_d \sigma_k$, where, for all $j \in [1, k]$, $\sigma_j \in \mathfrak{S}_d$ such that $1 \sigma_j = 1$, to $\omega_\lambda Q_T$ is an isomorphism, where
\[
Q_T = (\omega_d \sigma_1)(\omega_d \sigma_2)^+ \cdots (\omega_d \sigma_k)^{+(k-1)d}
\]
and $(\omega_d \sigma_j)^{+(j-1)d}$ means translation of the alphabets of $\omega_d \sigma_j$ by $(j-1)d$. Let $\tau$ be an element in the top group of $\mathfrak{S}_d \wr \mathfrak{S}_k$. We have
\[
\omega_\lambda Q_T \tau = \omega^\lambda(Q_T)_\lambda \tau = \omega^\lambda((\omega_d \sigma_1)^{+(1\tau-1)d}, \ldots, (\omega_d \sigma_k)^{(k\tau-1)d})
\]
\[
= \omega^\lambda(\omega_d \sigma_{1\tau-1}, \ldots, (\omega_d \sigma_{2\tau-1})^{+(k-1)d}, \ldots, (\omega_d \sigma_{k\tau-1})^{(k-1)d})
\]
\[
= \omega_\lambda(\omega_d \sigma_{1\tau-1} \odot \cdots \odot \omega_d \sigma_{k\tau-1}).
\]
For the action of the base group of $\mathfrak{S}_d \wr \mathfrak{S}_k$, it can be easily checked that it commutes with $\phi$. Therefore, $\phi$ is an isomorphism of $FH'$-modules.

We now prove the second isomorphism in our statement. For $p \nmid i$, by [34] Lemma 3.1, we have $\text{Lie}_F(i) \cong \text{ind}_{C_i}^{\mathfrak{S}_i} F_{\delta_i}$ where $C_i$ is generated by a cyclic permutation of length $i$ in $\mathfrak{S}_i$ and $F_{\delta_i}$ is one-dimensional. By [24] Lemma 2.6 (with $|I| = 1$ in that section), we have
\[
\text{Lie}_F(i)^{m_i} \cong \text{ind}_{C_i \wr \mathfrak{S}_{m_i}}^{\mathfrak{S}_i \wr \mathfrak{S}_{m_i}} F_{\delta_i}^{m_i}.
\]
Therefore, we obtain the second isomorphism where $F_{\delta_i} = (F_{\delta_i}^{m_i}) \boxtimes \cdots \boxtimes (F_{\delta_i}^{m_i})$. The proof is now complete since $\text{Lie}_F(q) \cong \text{Lie}_F(\lambda)$. \hfill \Box

For the rest of this section, we assume that $p > 0$. We now establish a few corollaries following our results.

**Corollary 4.5.** Let $q \in \Lambda(n)$, $(q, p) = 1$ and $m_i = m_i(q)$. Then $\text{Lie}_F(q)$ is a $p$-permutation module. Furthermore, any indecomposable summand of $\text{Lie}_F(q)$ is a trivial source module and has a vertex a $p$-subgroup of $T = T_{m_1} \times \cdots \times T_{m_n}$ where $T_{m_i}$ is the
top group of $C_i \wr \mathfrak{S}_{m_i}$. The multiplicity of the irreducible character $\zeta^\mu$ in the ordinary character of $\text{Lie}_F(q)$ is the number $C^\mu_q$ as given in Theorem 2.13.

Proof. The fact that $\text{Lie}_F(q)$ is a $p$-permutation module follows from Theorem 1.4 and Theorem 2.5. Let $N$ be an indecomposable summand of $\text{Lie}_F(q)$. By definition, $N$ has a vertex $Q$ a $p$-subgroup of $\prod_{i=1}^n (C_i \wr \mathfrak{S}_{m_i})$. Since $m_i = 0$ if $p \mid i$, we have that $Q$ is conjugate to a subgroup of $T$. Since $\text{Lie}_F(q)$ is induced from a one-dimensional module, we have that $N$ is a trivial source module. For the last assertion, the module $\omega_q \mathbb{Z} \mathfrak{S}_n$ clearly is the unique lift of $\omega_q F \mathfrak{S}_n$. As such, the ordinary character of $\omega_q F \mathfrak{S}_n$ is the ordinary character of $\nu_q \mathbb{Q} \mathfrak{S}_n$, which can be described as in Theorem 2.15. □

There are certain cohomology invariants of the $FG$-modules called the support variety and complexity (see [3 4]). For instance, a module is projective if and only if its complexity is 0. A module is non-projective periodic if and only if it has complexity one.

In the case for the Lie modules, $\text{Lie}_F(n)$ has complexity $c$ where $c \in \mathbb{N}_0$ is the largest such that $p^c \mid n$ (see [18 21]). In particular, $\text{Lie}_F(n)$ is projective if and only if $p \nmid n$ (see [20]) and $\text{Lie}_F(n)$ is non-projective periodic if and only if $p \mid n$ and $p^2 \nmid n$. For the notation we use in the next corollary, we refer the reader to [8 §5.7].

Corollary 4.6. Suppose that $q \in \Lambda(n)$, $(q, p) = 1$, $m_i = m_i(q)$ and $P$ be a Sylow $p$-subgroup of $\prod_{i=1}^n (C_i \wr \mathfrak{S}_{m_i})$. The support variety of $\text{Lie}_F(q)$ is $\text{res}^*_G \text{res}_F \text{V}_P(F)$ and its complexity is $\sum_{i=1}^n \lfloor \frac{m_i}{p} \rfloor$. In particular,

(i) $\text{Lie}_F(q)$ is projective if and only if $m_i < 0$ for all $i \in [1, n]$, i.e., $\lambda(q) \in \Lambda^+_{p}(n)$, and

(ii) $\text{Lie}_F(q)$ is non-projective periodic if and only if $p \leq m_j < 2p$ for some unique $j \in [1, n]$ and $m_i < p$ for $i \neq j$.

Proof. By [8 Proposition 5.7.5] and Corollary 1.5 the support variety of $\text{Lie}_F(q)$ is equal to $\text{res}^*_G \text{res}_F \text{V}_P(F)$ (see, for example, [19 Lemma 6.2]). Again, by [19 Lemma 6.2], the complexity of $\text{Lie}_F(q)$ is the $p$-rank of the group $\prod_{i=1}^n (C_i \wr \mathfrak{S}_{m_i})$ which is $\sum_{i=1}^n \lfloor \frac{m_i}{p} \rfloor$ since $m_i = 0$ if $p \mid i$. As such, the complexity is 0 if and only if $\lfloor \frac{m_i}{p} \rfloor = 0$ for all $i$ and is 1 if and only if $\lfloor \frac{m_i}{p} \rfloor = 1$ for precisely one such $j$ and $\lfloor \frac{m_i}{p} \rfloor = 0$ for $i \neq j$. □

Recall the set of primitive orthogonal idempotents $\{e_{\lambda,F} : \lambda \in \Lambda^+_{p}(n)\}$ of $\mathcal{D}_{n,F}$ we have constructed in Section 3. It follows that the regular module $FG \mathfrak{S}_n$ is isomorphic to $\bigoplus_{\lambda \in \Lambda^+_{p}(n)} e_{\lambda,F} FG \mathfrak{S}_n$ and, it has been proved in [22] that,

$$\dim_F e_{\lambda,F} FG \mathfrak{S}_n = |\mathcal{C}_{\lambda,p}|.$$ 

We now study the relation between the higher Lie modules and such projective modules.

Since, by Corollary 1.10 $\text{Lie}_F(q)$ is not projective in general, the map $\alpha$ in Corollary 3.10 may not split. However, when $\lambda$ is both $p$-regular and coprime to $p$, we get isomorphism as shown in the next corollary.

Corollary 4.7. Let $q \in \Lambda(n)$, $(q, p) = 1$ and $\lambda = \lambda(q) \in \Lambda^+_{p}(n)$. We have an isomorphism $\text{Lie}_F(q) \cong e_{\lambda,F} FG \mathfrak{S}_n$. 
Proof. By Corollary 3.10, we only need to check their dimensions. By [22 Proposition 7], Lemma 2.3 and Theorem 1.2, we have
\[
\dim_F e_{\lambda,F} F \Phi_n = |\Phi_{\lambda,p}| = |\Phi_{\lambda}| = \dim_F \text{Lie}_F(q).
\]
\[
\square
\]

To conclude this section, we give an example to illustrate how the modular twisted Foulkes modules are related to the next Lie modules.

**Example 4.8.** Let \( p \geq 3 \) and \( q = (2^a, 1^b) \) such that \( a, b \in [0, p-1] \). By Theorem 4.4,
\[
\text{Lie}_F((2^a, 1^b)) \cong \text{ind}_{\langle S_1 \rangle S_1 \times \langle S_2 \rangle \langle S_2 \rangle} (\text{Lie}_F(1)^b \boxtimes \text{Lie}_F(2)^a).
\]

Notice that \( \text{Lie}_F(1)^b = F \) is the trivial F\( \Phi \_n \)module with the identification \( S_1 \). On the other hand, \( \text{Lie}_F(2) = S_2 \) the signature representation for \( F \Phi \_n \)-module. Observe that \( \text{sgn}(2)^a = \text{sgn}(2) \) the signature representation for \( F \Phi \_n \)-module and
\[
(\text{ind}_{\langle S_2 \rangle \langle S_2 \rangle} \text{Lie}_F(2)^a) \otimes \text{sgn}(2a) \cong \text{ind}_{\langle S_2 \rangle \langle S_2 \rangle} (\text{sgn}(2)^a \otimes (\text{res}_{\langle S_2 \rangle \langle S_2 \rangle} \text{sgn}(2a)))
\]
\[
= \text{ind}_{\langle S_2 \rangle \langle S_2 \rangle} F = H^{(2^a)}
\]
which \( H^{(2^a)} \) is the Foulkes module (see [23]) for \( F \Phi \_n \) in the modular case. Therefore we obtain
\[
\text{Lie}_F((2^a, 1^b)) \otimes \text{sgn}(2a + b) \cong \text{ind}_{\langle S_2 \rangle \langle S_2 \rangle} (\text{sgn}(b) \boxtimes H^{(2^a)}).
\]
The module on the right hand side is called a twisted Foulkes module.

5. PIVOTS AND \( q \)-SEGMENTATIONS OF WORDS

Our next focus is the module \( \Xi^q R \Phi_n \). We aim to establish a free basis for the module and study its structure in the next section. This section sets up the necessary combinatorics. More precisely, for each \( n \in \mathbb{N} \) and \( q \mid n \), we define maps \( \Phi : \Phi_n \rightarrow \Phi_n \) and \( \Upsilon_q : \Phi_n \rightarrow \Phi_n \) (see Definitions 5.4 and 5.8), which will help us to partition \( \Phi_n \) for the construction of a free basis for \( \Xi^q R \Phi_n \).

Throughout this section, a word \( w \) is a word in \( \mathbb{N} \) with distinct alphabets. We begin with a definition.

**Definition 5.1.** Let \( w = w_1 \ldots w_n \) be a word in \( \mathbb{N} \) (with distinct alphabets) and \( W = \{w_1, \ldots, w_n\} \). Define the numbers \( \rho_0, \rho_1, \ldots, \rho_{\ell} \) inductively as follows. Let \( \rho_0 = 0 \). Suppose that \( \rho_i \neq n \) for some \( i \geq 0 \). Let \( \rho_{i+1} \) be the index such that \( w_{\rho_{i+1}} \) is the smallest number in the set \( W \setminus \{w_j : j \in [1, \rho_i]\} \). Continue in this manner until we get \( \rho_{\ell} = n \) for some \( \ell \).

(i) The pivots of \( w \) are the alphabets \( w_{\rho_1}, \ldots, w_{\rho_{\ell}} \).

(ii) For each \( i \in [1, \ell] \), the pivot word and pivot cycle of \( w \) are the subword \( w^{(i)} = w_{1+\rho_{i-1}} w_{2+\rho_{i-1}} \cdots w_{\rho_i} \) and the cycle \( (w_{1+\rho_{i-1}}, w_{2+\rho_{i-1}}, \ldots, w_{\rho_i}) \) respectively.

(iii) The pivot cycle type of \( w \) is the sequence
\[
(\rho_1, \rho_2 - \rho_1, \ldots, \rho_{\ell} - \rho_{\ell-1}) = (|w^{(1)}|, \ldots, |w^{(\ell)}|) \equiv n.
\]
Example 5.2. Consider
\[ w = 5613427 \in S_7. \]
The pivots of \( w \) are 1, 2, 7 as underlined. The pivot words of \( w \) are \( w^{(1)} = 561, w^{(2)} = 342 \) and \( w^{(3)} = 7 \), and its pivot cycle type is \( (3,3,1) \).

Recall the definition of reversed colexicographic order \( \leq_{clx} \) on \( S_n \) as words in Subsection 2.1.

Lemma 5.3. Let \( w(1), w(2), \ldots, w(k) \) be words such that \( v = w(1)w(2) \cdots w(k) \in S_n \), each \( w(i) \) has a unique pivot; namely, the least alphabet is at the last position of \( w(i) \), and the pivots are ordered in increasing order. Then \( v \) is, with respect to \( \leq_{clx} \), the least permutation containing all the subwords \( w(1), \ldots, w(k) \).

Proof. There is nothing to prove if \( k = 1 \). So we assume that \( k > 1 \). Let \( c_i = |w(i)| \) so that \( w(i)(c_i) \) is the pivot of \( w(i) \). Clearly \( v \) is a permutation containing all the subwords \( w(1), w(2), \ldots, w(k) \). Let \( u \) be another such permutation where \( u \leq_{clx} v \). By the definition of subword, for any \( j \in [1, k] \), for any \( i \in [1, c_j - 1] \), the alphabet \( w(j)_i \) in \( u \) must be in an earlier position than the alphabet \( w(j)_{i+1} \). Therefore, in particular, \( u_n = w(j)_{c_j} \) for some \( j \in [1, k] \). Since \( u \leq_{clx} v \), we have \( u_n \geq v_n = w(k)_{c_k} \) and thus \( j = k \). Let \( s \) be the largest index such that \( u_s = w(j')_{c_{j'}} \) for some \( j' \neq k \). In fact, \( s \) is the largest position in \( u \) such that \( u_s \) is not an alphabet in \( w(k) \). As such, \( |n - s| \leq |w(k)| \) and \( u_i = v_i \) for all \( i \in [s + 1, n] \), i.e.,
\[ u_{s+1} \cdots u_n = w(k)_{c_k - n + s + 1} \cdots w(k)_{c_k} = v_{s+1} \cdots v_n \]
and \( u_s < w(k) \) for any \( i \in [1, c_k] \). Since \( u \leq_{clx} v \), we have \( u_s \geq v_s \). If \( c_k - n + s + 1 > 1 \) then \( u_s \geq v_s = w(k)_{c_k - n + s} \), which is absurd. Therefore \( u = u'w(k) \) for some permutation \( u' \). The proof is now complete using induction.

Definition 5.4. Fix \( n \in \mathbb{N} \). We define the function \( \Phi : S_n \to S_n \) as follows. Let \( \sigma \in S_n \). We write \( \sigma \) uniquely as a product of disjoint cycles
\[ \sigma = (w_1, \ldots, w_{\rho_1})(w_{\rho_1+1}, \ldots, w_{\rho_2}) \cdots (w_{\rho_{\ell-1}+1}, \ldots, w_{\rho_\ell}) \]
such that \( w_{\rho_1} < \cdots < w_{\rho_\ell} \) and, for each \( i \in [1, \ell] \), \( w_{\rho_i} \) is the smallest number in the cycle containing it. Define
\[ \Phi(\sigma) = w_1w_2 \cdots w_n \]
by removing all parenthesis and commas.

We give an example to illustrate the definition.

Example 5.5. Let \( \sigma = (5,6,1)(3,4,2)(7) \) so that \( w = \Phi(\sigma) = 5613427 \). Notice that \( w \) has cycle type \( (4,2,1) \) which is different from the cycle type of \( \sigma \) (or the pivot cycle type of \( w \) as in Example 5.2).

We have the following immediate lemma.

Lemma 5.6. The map \( \Phi : S_n \to S_n \) is bijective.

Proof. The product of the pivot cycles of \( w \in S_n \) gives rise to a permutation \( \sigma \) such that \( \Phi(\sigma) = w \). As such, we get the inverse map. \( \square \)
Following the definitions above, we are now ready to define the set which is central in our study for this and the next sections.

**Definition 5.7.** Let \( q \models n \). We define

\[
\mathcal{B}_q = \{ \Phi(\sigma) : \sigma \in S_n \text{ has cycle type a weak refinement of } q \}
\]

\[
= \{ w : w \in S_n \text{ has pivot cycle type a weak refinement of } q \}.
\]

It has cardinality the number permutations in \( S_n \) with cycle types refining \( q \) (cf. Lemma 5.6), i.e.,

\[
|\mathcal{B}_q| = \sum_{\lambda^+(n) \triangleright \lambda < q} n! \frac{1}{\lambda^!}.
\]

Next, for each \( q \models n \), we define a map \( \Upsilon_q \). The maps are crucial in this section in the sense that they give rise to another interpretation of the sets \( \mathcal{B}_q \) and partition \( S_n \) into disjoint union.

**Definition 5.8.** Let \( q \models n \) and \( w = w_1 \ldots w_n \in S_n \). The \( q \)-segments of \( w \) are the words

\[ w_{q,i} = w_{1+q_{i-1}}w_{2+q_{i-1}}\ldots w_{q_i} \]

one for each \( i \in [1, \ell(q)] \). For each \( q \)-segmentation \( w_{q,i} \), consider the pivot words \( w_{q,i}^{(1)}, \ldots, w_{q,i}^{(k_i)} \) of \( w_{q,i} \) for some \( k_i \). We define the map \( \Upsilon_q : S_n \to S_n \) where \( \Upsilon_q(w) \) is, with respect to \( \leq^{\text{clx}} \), the least permutation containing all the subwords \( w_{q,i}^{(j)} \)'s; equivalently, let \( k = \sum_{i=1}^{\ell(q)} k_i \) and \( w(1), w(2), \ldots, w(k) \) be the rearrangements of the subwords \( w_{q,1}^{(1)}, \ldots, w_{q,\ell(q)}^{(k_{\ell(q)})} \) in the increasing order with respect to their respective unique pivots, by Lemma 5.5 we have

\[ \Upsilon_q(w) = w(1)w(2)\ldots w(k). \]

By the construction, notice that \( \Upsilon_q(w) \) has pivot cycle type \( (|w(1)|, \ldots, |w(k)|) \) a weak refinement of \( q \). We should help the reader with a few examples below.

**Example 5.9.** Consider \( q = (4, 3) \) and \( w = 5613427 \) as in Example 5.2. The \( q \)-segmentations of \( w \) are \( w_{q,1} = 5613 \) and \( w_{q,2} = 427 \). The corresponding pivots words are \( w_{q,1}^{(1)} = 561, w_{q,1}^{(2)} = 3, w_{q,2}^{(1)} = 42 \) and \( w_{q,2}^{(2)} = 7 \). Therefore, \( \Upsilon_q(w) = w_{q,1}^{(1)}w_{q,1}^{(2)}w_{q,2}^{(1)}w_{q,2}^{(2)} = 5614237 \) and it has pivot cycle type \( (3, 2, 1, 1) \) which is a weak refinement of \( (4, 3) \). Notice that \( u = 5436127 \) is also a word with the subwords \( w_{q,i}^{(j)} \)'s but \( 5614237 <^{\text{clx}} 5436127 \).

**Example 5.10.** We get \( \Upsilon_{(1^n)}(w) = 12\ldots n \) for any \( w \in S_n \). Also, we get \( \Upsilon_q(12\ldots n) = 12\ldots n \) for any \( q \models n \).

**Example 5.11.** Let \( q = (2, 2) \). In the figure below, we draw an arrow \( u \to v \) if \( \Upsilon_q(u) = v \) for \( u, v \in S_4 \) but ignore the arrow if \( \Upsilon_q(u) = u \) and embolden all images of \( \Upsilon_q \). For instance, we ignore the arrow from 1234 to 1234 as the previous example shows.
Observe that $B_{(2,2)} = \text{im} \Upsilon_{(2,2)}$.

Indeed, the final equality we obtained in Example 5.11 is not a coincidence. We have the following lemma.

**Lemma 5.12.** Let $q \mid n$. Then $B_q = \text{im} \Upsilon_q$.

**Proof.** Let $w \in S_n$. By definition, $\Upsilon_q(w) = w(1)w(2) \cdots w(k)$ with pivot cycle type $c = ([w(1), w(2), \ldots, w(k)])$. Since $w(1), \ldots, w(k)$ are the rearrangements of the pivot words of the $q$-segments of $w$, we get that $c$ is a weak refinement of $q$. Therefore $\text{im} \Upsilon_q \subseteq B_q$.

Suppose now that $w$ has pivot cycle type a weak refinement of $q$. Let $w^{(1)}, \ldots, w^{(\ell)}$ be the pivot words of $w$. By assumption, there is a rearrangement of the pivot words of $w$, say $w(1), \ldots, w(\ell)$, and some $j_0 = 0 < j_1 < \cdots < j_{\ell(q)}$ such that, for each $i \in [1, \ell(q)]$,

$$q_i = |w(1 + j_{i-1})| + |w(2 + j_{i-1})| + \cdots + |w(j_i)|$$

and the pivots of $w(1 + j_{i-1}), w(2 + j_{i-1}), \ldots, w(j_i)$ are in increasing order. Let $u = w(1) \cdots w(\ell)$. By the construction of $u$, the $q$-segments of $u$ are the subwords $w(1 + j_{i-1})w(2 + j_{i-1}) \cdots w(j_i)$ one for each $i \in [1, \ell(q)]$ and the pivot words of the $q$-segments of $u$ are precisely $w^{(1)}, \ldots, w^{(\ell)}$ since the pivot words of $w^{(1)}, \ldots, w^{(\ell)}$ are increasing. Therefore $\Upsilon_q(u) = w$. The proof is now complete. \hfill \square

In view of the proof of the previous lemma, we define the following set.

**Definition 5.13.** Let $q \mid n$, $w \in B_q$ and $w^{(1)}, \ldots, w^{(\ell)}$ be the pivot words of $w$. A rearrangement $(t_1, \ldots, t_\ell)$ of $[1, \ell]$ is called $(q, w)$-compatible if there exist $j_0 = 0 < j_1 < \cdots < j_{\ell(q)} = \ell$ (a subsequence of $[1, \ell]$) such that, for each $i \in [1, \ell(q)]$, $w^{(t_1 + j_{i-1})}, w^{(t_2 + j_{i-1})}, \ldots, w^{(t_{j_i})}$ are precisely the pivot words of the word

$$w(i) = w^{(t_1 + j_{i-1})}w^{(t_2 + j_{i-1})}\cdots w^{(t_{j_i})}$$

(or equivalently, $t_1 + j_{i-1} < t_2 + j_{i-1} < \cdots < t_{j_i}$) and $q_i = |w(i)|$. We define the subset

$$\mathfrak{F}_q(w) = \{w^{(t_1)} \cdots w^{(t_\ell)} : (t_1, \ldots, t_\ell) \text{ is } (q, w)\text{-compatible}\}.$$

**Example 5.14.** Let $w = 416235$ and $q = (3, 2, 1)$. The pivot words are $w^{(1)} = 41$, $w^{(2)} = 62$, $w^{(3)} = 3$ and $w^{(4)} = 5$ and the $(q, w)$-compatible rearrangements are

$$(1, 3, 2, 4), (2, 3, 1, 4), (1, 4, 2, 3), (2, 4, 1, 3).$$

Therefore $\mathfrak{F}_q(w) = \{413625, 623415, 415623, 625413\}$. 
Example 5.15. In Example 5.11 we see that \( \mathcal{F}_{(2,2)}(1324) = \{1432, 3214\} = \Upsilon^{-1}(1324) \). Notice that 1324 \( \notin \mathcal{F}_{(2,2)}(1324) \).

As mentioned earlier, we can now partition \( \mathfrak{S}_n \) into disjoint unions.

Corollary 5.16. For each \( q \models n \) and \( v \in \mathcal{B}_q \), we have \( \mathcal{F}_q(v) = \Upsilon_q^{-1}(v) \). Furthermore, we have a disjoint union

\[
\mathfrak{S}_n = \bigsqcup_{v \in \mathcal{B}_q} \mathcal{F}_q(v).
\]

Proof. Fix a composition \( q \models n \). Suppose that \( \Upsilon_q(v) = v \). Let

\[
W_q = \{w_{q,i}^{(j)} : i \in [1, \ell(q)], j \in [1, \rho_i]\}
\]

be the pivot words as in Definition 5.8 so that \( W_q \) is a rearrangement of the sequence of pivot words \( v^{(1)}, \ldots, v^{(\rho)} \) of \( v \). This rearrangement \( (t_1, \ldots, t_{\rho}) \) of \( [1, \rho] \) is \( (q, v) \)-compatible where, in Definition 5.13, we have, for each \( \rho_i \), and the pivot words of \( w_{q,i} \) are precisely \( w_{q,i}^{(1)}, \ldots, w_{q,i}^{(\rho_i)} \) by definition. Therefore \( w \in \mathcal{F}_q(v) \).

Conversely, suppose that \( w \in \mathcal{F}_q(v) \), i.e., \( w = v(t_1) \cdots v(t_{\rho}) \) for some \( (q, v) \)-compatible rearrangement of \( [1, \ell] \) with the existence of the \( j_0 = 0 < j_1 < \cdots < j_{\ell(q)} = \ell \). By Definition 5.8 for each \( i \in [1, \ell(q)] \), we have \( \rho_i = j_i - j_{i-1} \)

\[
w_{q,i} = v^{(1+j_{i-1})}v^{(2+j_{i-1})} \cdots v^{(t_{j_i})}
\]

and the pivot words of \( w_{q,i} \) are precisely \( v^{(1+j_{i-1})}, v^{(2+j_{i-1})}, \ldots, v^{(t_{j_i})} \). It is clear that the least word containing the subwords \( v^{(1)}, \ldots, v^{(\ell)} \) is \( v \). So \( \Upsilon_q(w) = v \).

We now prove the second assertion. For any \( w \in \mathfrak{S}_n \), we have \( v = \Upsilon_q(w) \in \mathcal{B}_q \) by Lemma 5.12. So \( w \in \Upsilon_q^{-1}(v) \). For different \( v, v' \in \mathcal{B}_q \), the fibers \( \Upsilon_q^{-1}(v), \Upsilon_q^{-1}(v') \) are clearly disjoint.

To conclude the section, we give a proposition concerning the size of \( \mathcal{F}_q(v) \) as an independent interest.

Proposition 5.17. Let \( q \models n \) and suppose that \( v \in \mathcal{B}_q \) with pivot cycle type \( r \). Then \( |\mathcal{F}_q(v)| = |\Upsilon_q^{-1}(v)| \geq q_i \) and with equality if \( r \approx q \).

Proof. Let \( w \in \Upsilon_q^{-1}(v) \) and let \( w_{q,i} \)'s be the \( q \)-segments of \( w \) as in Definition 5.8. Notice that \( |w_{q,i}| = q_i \). Permute the \( q \)-segments of \( w \) of the same sizes yields \( q_i \) different words \( w \) with the same \( q \)-segments as \( w \) and therefore \( \Upsilon_q(w) = v \) by definition. So \( q_i \leq |\Upsilon_q^{-1}(v)| \).

Now assume that \( r \approx q \). If \( \Upsilon_q^{-1}(v) \) contains \( w \) such that \( w_{q,i} \) is not its own pivot word, then, definition, \( v = \Upsilon_q(w) \) has pivot cycle type \( r \prec q \), contradicting \( r \approx q \). Therefore, \( \Upsilon_q^{-1}(v) \) contains nothing else except the permutation of the \( q \)-segments we have mentioned earlier.

6. The module \( \Xi^q R \mathfrak{S}_n \) and its structure

Throughout this section, a word \( w \) is again a word in \( \mathbb{N} \) with distinct alphabets.

In this section, we study the right ideals of group algebra generated by the Solomon's descent elements \( \Xi^q \)'s, one for each composition \( q \). It turns out that the right ideals
are closely related to the projective modules generated by the modular idempotents of the descent algebra in Section 3 and the higher Lie modules in Section 4. As always expected, the modular case is significantly more difficult than the ordinary case and we could only take a glance at it.

We split this section into three subsections. Each subsection is devoted to the study of the right ideal generated by \( \Xi^q \) under different circumstances depending on the underlying ring. In the first subsection, we show that the right \( R\mathfrak{S}_n \)-module \( \Xi^q R\mathfrak{S}_n \) has a free \( R \)-basis as in Theorem 6.2. In the second part, we study the irreducible characters of \( \Xi^q F\mathfrak{S}_n \) when \( p = 0 \). For the third part, we study the case when \( p > 0 \).

We begin with the following basic observation.

**Lemma 6.1.** Let \( q, r \) be compositions of \( n \) such that \( r \ncong q \). Then \( \Xi^r R\mathfrak{S}_n \cong \Xi^q R\mathfrak{S}_n \) and there is a surjection \( \Xi^r R\mathfrak{S}_n \twoheadrightarrow \text{Lie}_R(q) \) given by the left multiplication by \( \omega^q \sigma \) where \( \sigma \Xi^r = \Xi^q \).

**Proof.** By Lemma 2.6, \( \sigma \Xi^r = \Xi^q \) for some \( \sigma \in \mathfrak{S}_n \) and we get the isomorphism. Now the lemma follows by the definition \( \omega^q = \omega^q \Xi^q \).

6.1. A free basis. We now introduce a few notations we shall be using throughout this subsection. For an element \( \alpha \in R\mathfrak{S}_n \), we write \( \text{supp}(\alpha) \) for the subset of \( \mathfrak{S}_n \) consisting of the words or permutations appearing in \( \alpha \) with nonzero coefficient. For each \( q \vdash n \), correspond to the set \( B_q \) in Definition 5.7, we let \( B_q = \{ \Xi^q w : w \in B_q \} \subset \Xi^q R\mathfrak{S}_n \).

The main result is the following free basis for the right \( R\mathfrak{S}_n \)-module \( \Xi^q R\mathfrak{S}_n \).

**Theorem 6.2.** The right ideal \( \Xi^q R\mathfrak{S}_n \) has an \( R \)-basis \( B_q \) where
\[
|B_q| = \sum_{\lambda \vdash n, \lambda \ncong q} \frac{n!}{\lambda!},
\]
i.e., the number of permutations of \( \mathfrak{S}_n \) with cycle types weakly refining \( q \).

To show the linearly-independence part for Theorem 6.2 we need the following straightforward lemma.

**Lemma 6.3.** Let \( v \in \mathfrak{S}_n \) and suppose that \( w \in \text{supp}(\Xi^q v) \). Then \( \Upsilon_q(w) \leq \text{clx} \ v \).

**Proof.** Let \( u = \Upsilon_q(w) \). By definition, we have \( w = u^{(i_1)} \cdots u^{(i_\ell)} \) where \( u^{(1)}, \ldots, u^{(\ell)} \) are the pivot words of \( u \) and \( \{i_1, \ldots, i_\ell\} = [1, \ell] \). On the other hand, since \( w \in \text{supp}(\Xi^q v) \), \( v \) is a permutation containing the subwords \( u^{(i_1)}, \ldots, u^{(i_\ell)} \). By definition (see Definition 5.8 and Lemma 5.3), \( u \) is the least of such permutations and therefore \( u \leq \text{clx} \ v \).

Recall the notations \( Q_S \) and \( s_j \) introduced in Subsection 2.2. To motivate the next lemma and the proof of the spanning part of Theorem 6.2 we give an example. Let \( w = 231 \). We reverse the alphabets in \( w \) and consider
\[
Q_{132} = Q_{\{1,3,2\}} = 132 - 312 - 213 + 231.
\]
Notice that $u \leq_{\text{clx}} 231$ for any word $u$ appearing in $Q_{132}$. Recall that, using Theorem 2.19, $\Xi^{(2,1)}Q_{132} = 0$ since $(3) \not\equiv (2,1)$. Therefore,

$$\Xi^{(2,1)}231 = \Xi^{(2,1)}312 + \Xi^{(2,1)}213 - \Xi^{(2,1)}132.$$ 

**Lemma 6.4.** Let $w^{(1)}, \ldots, w^{(\ell)}$ be the pivot words of $w \in S_n$. For each $i \in [1, \ell]$, let $S_i = \{w_i^{(0)}, \ldots, w_i^{(m_i)}\}$ be the ordered set with respect to the pivot word $w^{(i)} = w_1^{(i)} \cdots w_{c_i}^{(i)}$, i.e., $S_i$ reverses the alphabets in $w^{(i)}$. Then the largest word, with respect to the reversed colexicographic order $\leq_{\text{clx}}$, appearing in $Q_{S_1} \cdots Q_{S_\ell} \in R\Sigma_n$ is $w$ with coefficient $(-1)^{\ell+\sum_{i=1}^\ell c_i}$.

**Proof.** Let us use $w^{(i)} = w_{c_i}^{(i)} \cdots w_1^{(i)}$ and let $u$ be a word appearing in $Q_{S_1} \cdots Q_{S_\ell}$. Then it is of the form $u = u(1) \cdots u(\ell)$ where, for each $i \in [1, \ell]$, $u(i)$ is a word appearing in $Q_{S_i} = \sum (-1)^{|j|} s_j \cdot w^{(i)} = \sum (-1)^{|j|} w_{c_i-j_i+1}^{(i)} \cdots w_{c_i-k_i-1}^{(i)} w_{c_i-k_i-k_{n-s}}^{(i)}$ where both sums are taken over all $j = \{j_1 < \cdots < j_s\} \subseteq [2, c_i]$ where $1 = k_1 < \cdots < k_{n-s}$ and $j = [1, c_i]$. Suppose that $u \geq_{\text{clx}} w$. In particular, we have $S_\ell \ni u(\ell)_{c_\ell} = u_n \leq w_{c_\ell}^{(i)}$.

Since $w^{(i)}$ is a pivot word, we have $w_{c_i}^{(i)} = \min S_i$ and therefore $u(\ell)_{c_i} = w_{c_i}^{(i)}$. Consider $i = \ell$. The only $j \subseteq [2, c_\ell]$ such that $s_j \cdot w^{(i)}$ ends with $w_{c_\ell}^{(i)}$ is $j = [2, c_\ell]$. Therefore, $u(\ell) = s_{[2, c_\ell]} \cdot w^{(i)} = w^{(i)}$. We now work by reverse induction to conclude that $u(i) = w^{(i)}$ for all $i \in [1, \ell]$. Notice that, for each $i \in [1, \ell]$, the coefficient of $u(i) = s_{[2, c_i]} \cdot w^{(i)} = w^{(i)}$ in $Q_{S_i}$ is $(-1)^{c_i-1}$. \qed

We are now ready to prove the theorem.

**Proof of Theorem 6.2.** Let $T$ be the $R$-linear span of the set $B_q$. We argue by induction on the reversed colexicographic order to show that $\Xi^q w$ lies in $T$ for any word $w \in \Sigma_n$. If $w = 12 \ldots n$ then $w \in B_q$ because the pivot type of $w$ is $(1^n)$ and hence $\Xi^q w \in T$. If $w \in B_q$ then $\Xi^q w \in T$. Suppose that $w \not\in B_q$. Let $w = w^{(1)} \cdots w^{(\ell)}$ be written in the pivot words. For each $i \in [1, \ell]$, let $S_i$ be the ordered set with respect to $w^{(i)}$ as in Lemma 6.4. Since $w \not\in B_q$, i.e., the composition $(|S_1|, \ldots, |S_\ell|)$ is not a weak refinement of $q$, by Theorem 2.19 we have $\Xi^q Q_{S_1} \cdots Q_{S_\ell} = 0$.

By Lemma 6.4, we can rewrite

$$\Xi^q w = \Xi^q (w \pm Q_{S_1} \cdots Q_{S_\ell})$$

where the right hand side is a sum of elements of the form $\Xi^q u$ such that $u \leq_{\text{clx}} w$. By induction, $\Xi^q w \in T$. This shows that $T = \Xi^q R\Sigma_n$.

We arrange $B_q$ with respect to the reversed colexicographic order restricted to $B_q$. By Lemma 6.3, for any $v \in B_q$, any word $w \in \text{supp}(\Xi^q v)$ satisfies $\Upsilon_q(w) \leq_{\text{clx}} v$, i.e., $\text{supp}(\Xi^q v)$ involves words in $F_q(u)$ with $B_q \ni u \leq_{\text{clx}} v$. Clearly, $v \in \Xi^q v$. Therefore $B_q$ is linearly independent using Corollary 5.16. \qed
6.2. The ordinary case. We now turn our attention to the decomposition problem of $\Xi^q F\mathfrak{S}_n$ in the ordinary case. More precisely, we study the module $\Xi^q F\mathfrak{S}_n$ and write it as the direct sum of higher Lie modules (see Theorem 6.9 and cf. Theorem 6.2). As the irreducible constituents of the higher Lie modules have been described by Schocker in [39], we get the complete knowledge about $\Xi^q F\mathfrak{S}_n$ in this case.

For this subsection we assume that $p = 0$. We remind the readers about the elements $I_q$’s and $E \lambda$’s in Subsection 2.5.

**Lemma 6.5.** Let $\lambda$ be a partition of $n$ and $\alpha = \sum_{\lambda(q) = \lambda} c_q I_q$ for some $c_q \in F$ such that $\sum_{\lambda(q) = \lambda} c_q \neq 0$. Then $E_\lambda F\mathfrak{S}_n \cong \alpha F\mathfrak{S}_n$ as right $F\mathfrak{S}_n$-modules. In particular, $E_\lambda F\mathfrak{S}_n \cong I_q F\mathfrak{S}_n$ if $\lambda(q) = \lambda$.

**Proof.** Define $\phi : E_\lambda F\mathfrak{S}_n \to \alpha F\mathfrak{S}_n$ as $\phi(E_\lambda) = \alpha E_\lambda$ for any $\gamma \in F\mathfrak{S}_n$. Since the map $\phi$ is defined by multiplying $\alpha$ on the left, $\phi$ is a right $F\mathfrak{S}_n$-module homomorphism. By Theorem 2.22(ii),

$$
\alpha E_\lambda \gamma = \left( \sum_{\lambda(q) = \lambda} c_q I_q \right) E_\lambda \gamma = \left( \sum_{\lambda(q) = \lambda} c_q I_q \right) \gamma = \alpha \gamma.
$$

Therefore $\phi$ is surjective. Suppose that $\phi(E_\lambda \gamma) = 0$, i.e., $\alpha \gamma = 0$. Multiplying on the left by $E_\lambda$, by Theorem 2.22(iii), we have

$$
0 = E_\lambda \alpha \gamma = E_\lambda \left( \sum_{\lambda(q) = \lambda} c_q I_q \right) \gamma = \left( \lambda \sum_{\lambda(q) = \lambda} c_q \right) E_\lambda \gamma.
$$

So $E_\lambda \gamma = 0$. This shows that $\phi$ is injective. The last assertion now follows. \hfill \Box

**Lemma 6.6.** For each partition $\lambda$ of $n$, let $\alpha_\lambda = \sum_{\lambda(q) = \lambda} c_{\lambda,q} I_q$ for some $c_{\lambda,q} \in F$ such that $\sum_{\lambda(q) = \lambda} c_{\lambda,q} \neq 0$ whenever $\alpha_\lambda \neq 0$. The sum $\sum_{\lambda \vdash n} \alpha_\lambda F\mathfrak{S}_n$ in $F\mathfrak{S}_n$ is direct.

**Proof.** Suppose that

$$
\sum_{\lambda \vdash n} \alpha_\lambda \gamma_\lambda = 0 \quad (6.1)
$$

for some $\gamma_\lambda \in F\mathfrak{S}_n$, one for each $\lambda \vdash n$. We want to prove that $\alpha_\lambda \gamma_\lambda = 0$ for all partitions $\lambda$. We argue by reverse induction with respect to the weak refinement on $\Lambda^+(n)$. Let $\mu = (1^n)$. Multiply $E_\mu$ on the left of Equation 6.1 by Theorems 2.21(iii) and 2.22(iii), since $I_\mu = n! E_\mu = \Xi^\mu$, we have

$$
0 = E_\mu c_{\mu,\mu} I_\mu \gamma_\mu = c_{\mu,\mu} I_\mu \gamma_\mu = \alpha_\mu \gamma_\mu.
$$

Now fix an arbitrary $\mu \in \Lambda^+(n)$ and suppose that, for any partition $\xi \neq \mu$ such that $\xi$ is a weak refinement of $\mu$, we have $\alpha_\xi \gamma_\xi = 0$. Multiply $E_\mu$ on the left of Equation 6.1 by Theorems 2.21(iii) and 2.22(iii), we have

$$
0 = E_\mu \sum_{\lambda \vdash n} \alpha_\lambda \gamma_\lambda = \sum_{\lambda \subseteq \mu} E_\mu \alpha_\lambda \gamma_\lambda = E_\mu \alpha_\mu \gamma_\mu = \sum_{\lambda(q) = \mu} c_{\mu,q} E_\mu I_q \gamma_\mu = \left( \sum_{\lambda(q) = \mu} \mu^\alpha c_{\mu,q} \right) E_\mu \gamma_\mu.
$$

Therefore, since $\sum_{\lambda(q)=\mu} c_{\mu,q} \neq 0$, we have $E_{\mu} \gamma_{\mu} = 0$ and hence, by Theorem 2.22(ii),
$$\alpha_{\mu} \gamma_{\mu} = \sum_{\lambda(q)=\mu} c_{\mu,q} I_q \gamma_{\mu} = \sum_{\lambda(q)=\mu} c_{\mu,q} I_q E_{\mu} \gamma_{\mu} = 0.$$  

The proof is now complete. \hfill \Box

Recall the Lie idempotent $\nu_q = \frac{1}{q^2 q^q}$ one for each $q \mid n$. In the following lemma, it basically shows that $\frac{1}{q^2} I_q$ is also a Lie idempotent.

**Lemma 6.7.** Let $q \mid n$. We have $\omega_q I_q = q^{-1} I_q$ and $I_q \omega_q = q q^{-1} I_q$. In particular, $I_q F \mathcal{S}_n \cong \text{Lie}_F(q)$.

**Proof.** Recall from Subsection 2.1 that $F_q(q)$ is the product of all parts of $q$ so that $q^{-1} = q^{-1} F_q(q)$. By Theorems 2.10(i), 2.21(i) and 2.22(i), we have
$$\omega_q I_q = \left( (-1)^{[\ell(q)]} \sum_{r \leq q} (-1)^{[\ell(r)]} F_q(r) \omega_q^{r} \right) I_q = (-1)^{[\ell(q)]} (-1)^{[\ell(q)]} F_q(q) \omega_q I_q$$
$$= F_q(q) q I_q = q^{-1} I_q.$$  

For the second equation, using Lemma 2.2 and Theorem 2.10(ii) twice, we have
$$I_q \omega_q = \left( \sum_{r \leq q} \frac{(-1)^{[\ell(r)]}}{[\ell(r), q]} \omega_q^{r} \right) \omega_q = \frac{(-1)^{[\ell(q)]}}{[\ell(q), q]} \omega_q = \sum_{q \mid s \leq q} |N_q, q| \omega_s = q \omega_q.$$  

The last assertion now follows. \hfill \Box

Combine Lemmas 6.5 and 6.7 we get the following corollary.

**Corollary 6.8.** Let $q \mid n$ and $\lambda(q) = \lambda$. Then $E_{\lambda} F \mathcal{S}_n \cong \text{Lie}_F(q)$ as right $F \mathcal{S}_n$-modules.

We are now ready to state and prove the main result in this subsection.

**Theorem 6.9.** Let $q$ be a composition of $n$. We have $\Xi q F \mathcal{S}_n \cong \bigoplus_{\lambda+q(\lambda) \leq q} \text{Lie}_F(\lambda)$.

**Proof.** For each $n \mid \lambda \leq q$, let
$$\alpha_\lambda = \sum_{r \leq q, \lambda(r)=\lambda} \frac{1}{\ell(r; q)} I_r$$
and $W = \sum_{n-\lambda \leq q} \alpha_\lambda F \mathcal{S}_n$. Clearly, $\sum_{\ell(r; q) \geq q} \frac{1}{\ell(r; q)} \neq 0$. We first prove that $\Xi q F \mathcal{S}_n = W$. By Theorem 2.20, since $\Xi q = \sum_{n-\lambda \leq q} \alpha$, we have $\Xi q F \mathcal{S}_n \subseteq W$. We now check that $W \subseteq \Xi q F \mathcal{S}_n$. It suffices to show that $\alpha_\lambda \in \Xi q F \mathcal{S}_n$. Fix now $\lambda \leq q$. By Theorem 2.21 when $s \leq q$ and $\lambda(s) = \lambda$, we have $\Xi q I_s = \sum_{\ell(r; q) \leq q} d_{r, s q} I_r$ for some suitable integers $d_{r, s q}$. Therefore
$$\Xi q E_\lambda = \frac{1}{\ell(\lambda)!} \sum_{\ell(\lambda)=\lambda} \Xi q I_s = \sum_{\ell(\lambda)=\lambda} d_{\lambda, q} I_r.$$
for some suitable integers \(d_{\lambda, q}\). Furthermore, using Theorem 2.21(i) and the definition of \(E_\lambda\), we have

\[
\Xi^q = \Xi^q \cdot \sum_{\lambda \vdash n} E_\lambda = \sum_{n: n \lambda \leq q} \Xi^q E_\lambda = \sum_{n: n \lambda \leq q} \sum_{\lambda(r) = \lambda} d_{\lambda, q} I_r.
\]

By Theorem 2.20, \(\{I_r : r \models n\}\) forms a basis for \(\mathcal{D}_{n, F}\) and compare Equation 6.2 with \(\Xi^q = \sum_{n: n \lambda \leq q} \alpha_\lambda\), we conclude that

\[
\alpha_\lambda = \sum_{\lambda(r) = \lambda} d_{\lambda, q} I_r = \Xi^q E_\lambda \in \Xi^q F \mathfrak{S}_n.
\]

By Lemmas 6.5 and 6.6, the sum in \(W\) is direct and \(W \cong \bigoplus_{n: n \lambda \leq q} E_\lambda F \mathfrak{S}_n\). The isomorphism in the statement now follows from Corollary 6.8.

In the example below, we demonstrate how the various right ideals \(\Xi^\lambda F \mathfrak{S}_4\), where \(\lambda \in \Lambda^+(4)\), stack up one on top of the others.

**Example 6.10.** Let \(n = 4\). The numbers beside the modules \(\Xi^\lambda F \mathfrak{S}_n\) are their dimensions and \(\lambda_{q}\) denotes \(\text{Lie}_F(q)\).

\[
\begin{align*}
\Xi^{(4)} F \mathfrak{S}_4 &\quad 24 \\
\Xi^{(3, 1)} F \mathfrak{S}_4 &\quad 15 \\
\Xi^{(2, 1, 1)} F \mathfrak{S}_4 &\quad 7 \\
\Xi^{(1^4)} F \mathfrak{S}_4 &\quad 1 \\
\end{align*}
\]

As a consequence of Theorems 6.9 and 2.15, we have the following corollary.

**Corollary 6.11.** The multiplicity of the irreducible character \(\zeta^\mu\) in \(\Xi^q F \mathfrak{S}_n\) is

\[
\sum_{\Lambda^+(n) \supseteq \lambda \leq q} C_{\lambda}^\mu
\]

where \(C_{\lambda}^\mu\) is the number given as in Theorem 2.15.

6.3. The modular case. In this subsection, we assume that \(p > 0\).

As expected, since, for an example, \(\Xi^{(n)} F \mathfrak{S}_n\) is the regular module, the decomposition into indecomposable summands and composition factors of such modules \(\Xi^q F \mathfrak{S}_n\)'s are difficult to describe in general. In the main result in this subsection (see Theorem 6.13), we identify a projective summand of \(\Xi^\lambda F \mathfrak{S}_n\) under the assumption that \(\lambda\) is coprime to \(p\) and \(p\)-regular.
We refer to reader to the proof of [22, Proposition 7] for the simple modules \( \{M_{\lambda,F} : \lambda \in \Lambda_p^+(n)\} \) of the descent algebra \( \mathcal{D}_{n,F} \) in the argument below.

**Lemma 6.12.** Let \( \mu \in \Lambda_p^+(n) \) and suppose that \( \varphi^{q,F}(\mu) \neq 0 \). Then \( \lambda(q) \in \Lambda_p^+(n) \), \( p \gg q \), and \( \dim_F \Xi^q e_{\mu,F} \mathcal{G}_n = \mathcal{B}_{\mu,p} \) and \( \Xi^q e_{\mu,F} \mathcal{G}_n \cong e_{\mu,F} \mathcal{G}_n \).

**Proof.** Since \( \varphi^{q,F}(\mu) \neq 0 \), we have \( p \gg q \) by Lemma 2.14. By Theorem 2.16 we have

\[
(c_{n,\mathcal{F}}(\Xi^q e_{\mu,F}))(\lambda) = (\varphi^{q,F} \text{char}_{\mu,F})(\lambda) = \delta_{\lambda,\mu}\varphi^{q,F}(\mu).
\]

As such, we have \( \Xi^q e_{\mu,F} \not\in \ker c_{n,\mathcal{F}} \) and therefore \( \lambda(q) \in \Lambda_p^+(n) \). For any \( \lambda \in \Lambda_p^+(n) \), as left \( \mathcal{D}_{n,F} \)-module \( M_{\lambda,F} \), \( \Xi^q e_{\mu,F} \) acts on it with multiplication by \( \varphi^{q,F}(\mu) \) if \( \lambda = \mu \) and annihilates it if \( \lambda \neq \mu \). Hence \( \Xi^q e_{\mu,F} M_{\lambda,F} = \delta_{\lambda,\mu} M_{\lambda,F} \). Therefore, \( \Xi^q e_{\mu,F} \mathcal{G}_n \) has dimension the number of composition factors of \( M_{\mu,F} \) in \( \mathcal{G}_n \) which is the dimension of \( e_{\mu,F} \mathcal{G}_n \) as pointed out in the proof of [22, Proposition 7]. The final isomorphism is now clear since the map \( \psi : e_{\mu,F} \mathcal{G}_n \to \Xi^q e_{\mu,F} \mathcal{G}_n \) given by the left multiplication by \( \Xi^q \) is surjective and both of the right ideals have the same dimension. \( \square \)

We remark that, if \( r \) is \( p \)-equivalent to \( \mu \in \Lambda_p^+(n) \), then \( r \gg \mu \) (see [22, §2]). Therefore, under the hypothesis given in Lemma 6.12 we have that \( \dim_F \Xi^q e_{\mu,F} \mathcal{G}_n \) is also equal to the number of permutations with cycle types both \( p \)-equivalent to \( \mu \) and a refinement of \( q \) (with the second condition is superfluous in this case). We conjecture that the equality holds in general (see Conjecture 7.6) and the importance is illustrated in Conjecture 7.7.

We are now ready to state and prove the main result for this subsection.

**Theorem 6.13.** Suppose that \( (q,p) = 1 \) and \( \lambda = \lambda(q) \in \Lambda_p^+(n) \). Then we have isomorphism of modules

\[
\Xi^q e_{\lambda,F} \mathcal{G}_n \cong \text{Lie}_F(q) \cong e_{\lambda,F} \mathcal{G}_n.
\]

In particular, \( \Xi^q e_{\lambda,F} \mathcal{G}_n \) is a projective summand of \( \Xi^q \mathcal{G}_n \) of dimension \( |\mathcal{B}_{\lambda,p}| = |\mathcal{B}_\lambda| \).

**Proof.** The second isomorphism is given in Corollary 4.7. Let \( f : \Xi^q e_{\lambda,F} \mathcal{G}_n \to \omega_{\lambda} \mathcal{G}_n \) be defined by the left multiplication with \( \sigma \omega_q \) where \( \sigma \omega_q = \omega_{\lambda} \). It is well-defined because

\[
f(\Xi^q e_{\lambda,F}) = \sigma \omega_q \Xi^q e_{\lambda,F} = \sigma \omega_q e_{\lambda,F} = \omega_{\lambda} e_{\lambda,F}.
\]

Using Theorem 2.10(iii) and Corollary 3.9 we have

\[
\lambda?\omega_{\lambda} = \omega_{\lambda}\omega_{\lambda} = \omega_{\lambda} e_{\lambda,F}\omega_{\lambda}.
\]

Since \( \lambda \) is both \( p \)-regular and coprime to \( p \), we have \( \lambda? \neq 0 \) in \( F \) and therefore

\[
f(\frac{1}{\lambda^2} \Xi^q e_{\lambda,F}\omega_{\lambda}) = \omega_{\lambda}.
\]

Therefore, \( f \) is surjective and hence an isomorphism due to Lemma 6.12 and Theorem 6.2. The last assertion follows since \( \Xi^q e_{\lambda,F} \mathcal{G}_n \) is a projective (equivalently, injective) submodule of \( \Xi^q \mathcal{G}_n \). \( \square \)

We now demonstrate the decomposition of \( \Xi^q \mathcal{G}_n \) with a few examples below.
Example 6.14. The module $\Xi^{(2)} F\mathfrak{S}_2$ is the regular $F\mathfrak{S}_2$-module. In the case when $p = 2$, $\omega(2) = \omega(1,1) = 12 + 21$ and therefore $\text{Lie}_F((2)) \cong F \cong \text{Lie}_F((1,1))$. In the case when $p \neq 2$, we have $\text{Lie}_F((2)) \cong \text{sgn}(2)$ and $\text{Lie}_F((1,1)) \cong F$. Therefore,

$$\Xi^{(2)} F\mathfrak{S}_2 \cong \begin{cases} \text{Lie}_F((2)) \oplus \text{Lie}_F((1)) & p \neq 2, \\ \text{Lie}_F((2)) & p = 2. \end{cases}$$

In the $p = 2$ case above, it means $\Xi^{(2)} F\mathfrak{S}_2$ is indecomposable with its head and socle both isomorphic to the trivial module and the choice of $\text{Lie}_F((2))$ on its top is due to Lemma 6.1.

Example 6.15. Let $p = 2$. The regular module $\Xi^{(3)} F\mathfrak{S}_3$ has dimension strictly bigger than $\sum_{\lambda \vdash \lambda} \text{Lie}_F(\lambda)$ (see Appendix C). Therefore Theorem 6.9 would not hold without the assumption $p = 0$, even with the direct-sum condition relaxed and the extra assumption that $q$ is coprime to $p$.

Example 6.16. In this example, we use the labelling of the simple modules for symmetric groups as in $[29]$; namely, $D^\lambda$ is the head of the Specht module $S^\lambda$ whenever $\lambda \in \Lambda^+(n)$. Also, we use the labelling of the Young modules as in $[28]$; namely, $Y^\lambda$ is the unique summand of $M^\lambda$ containing $S^\lambda$ as a submodule up to isomorphism. Consider $V = \Xi^{(2,1)} F\mathfrak{S}_3$. Notice that $\text{dim}_F V = 4$ and, following Theorem 6.2, $V$ has a basis $B_{(2,1)} = \{\Xi^{(2,1)}, \Xi^{(2,1)}213, \Xi^{(2,1)}312, \Xi^{(2,1)}132\}$. By Lemmas 2.6 and 6.1 there are surjections of $V$ onto $\Xi^{(1,1,1)} F\mathfrak{S}_3 \cong F$ and $\text{Lie}_F((2,1))$ respectively.

Suppose first that $p = 2$. Under this assumption, $(2,1)$ is not coprime to 2 and $\omega(2,1) = 2\Xi^{(2,1)} - \Xi^{(1,1,1)} = \Xi^{(1,1,1)}$. Therefore we have $\Xi^{(1,1,1)} F\mathfrak{S}_3 \cong F \cong \text{Lie}_F((2,1))$. On the other hand, using the idea as in the proof of Theorem 6.2, we can compute the matrix representation of the element $(1,2)$ on $V$ with respect to the basis $B_{(2,1)}$, that is

$$[(1,2)]_{B_{(2,1)}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$  

Since the matrix $[(1,2)]_{B_{(2,1)}} - I_4$ has rank 2 and $\langle (1,2) \rangle$ is a Sylow 2-subgroup of $\mathfrak{S}_3$, the module $V$ is projective. Since $V$ is a submodule of the regular module $F\mathfrak{S}_3 \cong Y^{(2,1)} \oplus Y^{(2,1)} \oplus Y^{(1,1,1)}$, it must be a direct sum of some of the Young modules in the decomposition where $Y^{(2,1)} \cong D^{(2,1)}$ and $Y^{(1,1,1)} \cong \begin{bmatrix} F \\ F \end{bmatrix}$. Comparing the dimensions and using the fact $V$ surjects onto $F$, we must have

$$V \cong Y^{(2,1)} \oplus Y^{(1,1,1)} \cong D^{(2,1)} \oplus \begin{bmatrix} F \\ F \end{bmatrix}.$$  

We have seen that $\text{Lie}_F((2,1)) \cong F$. In fact, $\text{Lie}_F((1,1,1)) \cong F$ too. It is well-known that $\text{Lie}_F((3))$ is projective (see also Corollary 4.6) and has dimension 2. An easy
calculation shows that \( \text{Lie}_F((3)) \) does not admit a trivial submodule and therefore it is isomorphic to \( Y^{(2,1)} \cong D^{(2,1)} \). As such, the composition factors of \( V \) must involve the simple module \( \text{Lie}_F((3)) \) and not just \( \text{Lie}_F((2,1)) \) and \( \text{Lie}_F((1,1,1)) \). This example shows that, in some modular cases, \( \Xi^qF\mathfrak{S}_n \) may involve part, if not all, of \( \text{Lie}_F(\lambda) \) of some partition \( \lambda \neq q \) (cf. Theorem 6.9).

Suppose now that \( p \neq 2 \). We first examine \( \text{Lie}_F((2,1)) \). Since \((2,1)\) is coprime to \( p \) and \( p \)-regular, by Theorem 6.13 we have

\[
\Xi^{(2,1)}e_{(2,1),F}F\mathfrak{S}_3 \cong \text{Lie}_F((2,1))
\]

is a projective summand of \( \Xi^{(2,1)}F\mathfrak{S}_3 \) of dimension 3. By Theorem 1.4

\[
\text{Lie}_F((2,1)) \cong \text{ind}^\mathfrak{S}_3_{\mathfrak{S}_1 \times \mathfrak{S}_2}(\text{Lie}_F(1) \boxtimes \text{Lie}_F(2)) \cong \text{ind}^\mathfrak{S}_3_{\mathfrak{S}_1 \times \mathfrak{S}_2}(F \boxtimes \text{sgn}(2))
\]

with Specht factors \( S^{(2,1)} \) and \( S^{(1,1,1)} \). Using the result of Brauer and Robinson [14, 38]; namely, the Nakayama Conjecture, these two factors lie in the same \( p \)-block if and only if \( p = 3 \) (as \( p \neq 2 \) by our initial assumption). Therefore,

\[
\text{Lie}_F((2,1)) \cong \begin{cases} 
\begin{bmatrix} D^{(2,1)} \\ F \\ S^{(2,1)} \oplus S^{(1,1,1)} \end{bmatrix} & p = 3, \\
& p \neq 3,
\end{cases}
\]

where both \( S^{(2,1)} \) and \( S^{(1,1,1)} \) are simple in the case of \( p \neq 3 \) and all Young modules appearing above are projective. Since \( V \) also surjects onto \( \Xi^{(1,1,1)}F\mathfrak{S}_3 \cong F \) and \( \text{Hom}_{F\mathfrak{S}_3}(\text{Lie}_F((2,1)), F) = 0 \), we must have

\[
\Xi^{(2,1)}F\mathfrak{S}_3 \cong \text{Lie}_F((2,1)) \oplus \text{Lie}_F((1,1,1)).
\]

To conclude our example, in terms of decomposition into Young modules and composition factors, we have

\[
\Xi^{(2,1)}F\mathfrak{S}_3 \cong \left\{ \begin{array}{ll}
Y^{(2,1)} \oplus Y^{(1^3)} & p = 2, \\
Y^{(1^3)} \oplus Y^{(3)} & p = 3, \\
Y^{(2,1)} \oplus Y^{(1^3)} \oplus Y^{(3)} & p \geq 5,
\end{array} \right. \sim \begin{cases} 
D^{(2,1)} + 2D^{(3)} & p = 2, \\
2D^{(2,1)} + 2D^{(3)} & p = 3, \\
D^{(2,1)} + D^{(1^3)} + D^{(3)} & p \geq 5.
\end{cases}
\]

7. Further Questions and Conjectures

We end our article with the following list of questions and conjectures. They are organized based on the structure of this paper.

**Question 7.1.** Can one describe the modular idempotents in Section 3 explicitly in terms of the \( \Xi^q \)'s?

**Question 7.2.** What is the dimension of \( \text{Lie}_F(q) \) if \( q \) is not coprime to \( p \)?

**Question 7.3.** Write the projective module \( e_{\lambda,F}F\mathfrak{S}_n \) in terms of the projective Young modules.

**Question 7.4.** Are the modules \( \Xi^qF\mathfrak{S}_n \) self-dual or trivial source module? Can they be written as direct sums of Young modules?
As we have remarked right after Lemma 6.12, we give some computational data to further support our conjecture.

**Example 7.5.** Let $p = 2$. Using Magma [12], we obtain the following table.

| Partition $\lambda$ | $\dim \Xi^\lambda F \mathfrak{S}_5$ | $\dim \Xi^\lambda e_{(5)} F \mathfrak{S}_5$ | $\dim \Xi^\lambda e_{(4,1)} F \mathfrak{S}_5$ | $\dim \Xi^\lambda e_{(3,2)} F \mathfrak{S}_5$ |
|---------------------|-------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $(5)$               | 120                           | 24                              | 56                              | 40                              |
| $(4, 1)$            | 76                            | 0                               | 56                              | 20                              |
| $(3, 2)$            | 66                            | 0                               | 26                              | 40                              |
| $(3, 1, 1)$         | 31                            | 0                               | 11                              | 20                              |
| $(2, 2, 1)$         | 26                            | 0                               | 26                              | 0                               |
| $(2, 1, 1, 1)$      | 11                            | 0                               | 11                              | 0                               |
| $(1, 1, 1, 1, 1)$   | 1                             | 0                               | 1                               | 0                               |

Observe that $\dim \Xi^\lambda e_{\mu,F} F \mathfrak{S}_5$ is the number of permutations with cycle type a refinement of $\lambda$ and $2$-equivalent to $\mu$.

**Conjecture 7.6.** Let $\mu \in \Lambda^+_p(n)$ and $q \in \Lambda(n)$. The dimension of $\Xi^q e_{\mu,F} F \mathfrak{S}_n$ is the number of permutations with cycle types both a refinement of $q$ and $p$-equivalent to $\mu$.

Since

$$\Xi^q F \mathfrak{S}_n = \sum_{\mu \in \Lambda^+_p(n)} \Xi^q e_{\mu,F} F \mathfrak{S}_n,$$

Conjecture 7.6 implies the following conjectures.

**Conjecture 7.7.** Let $q \in \Lambda(n)$. Then we have a direct sum decomposition of right $F \mathfrak{S}_n$-modules

$$\Xi^q F \mathfrak{S}_n = \bigoplus_{\mu \in \Lambda^+_p(n)} \Xi^q e_{\mu,F} F \mathfrak{S}_n.$$

**References**

[1] C. Ahlbach and J. Swanson, Cyclic sieving, necklaces, and branching rules related to Thrall’s problem, Electron. J. Combin. 25 (2018), no. 4, Paper No. 4.4 2, 38 pp.
[2] J. L. Alperin, Local Representation Theory, Cambridge Stud. Adv. Math. 11, Cambridge University Press, Cambridge, 1986.
[3] J. L. Alperin and L. Evens, Representations, resolutions and Quillen’s dimension theorem, J. Pure Appl. Algebra 22 (1981), no. 1, 1–9.
[4] J. L. Alperin and L. Evens, Varieties and elementary abelian groups, J. Pure Appl. Algebra 26 (1982), no. 3, 221–227.
[5] M. D. Atkinson, G. Pfeiffer, and S. J. van Willigenburg, The $p$-modular descent algebras, Algebr. Represent. Theory 5 (2002), no. 1, 101–113.
[6] M. D. Atkinson and S. J. van Willigenburg, The $p$-modular descent algebra of the symmetric group, Bull. London Math. Soc. 29 (1997), no. 4, 407–414.
[7] D. J. Benson, Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, 1991, reprinted in paperback, 1998.
[8] D. J. Benson, Representations and Cohomology II: Cohomology of Groups and Modules, Cambridge Studies in Advanced Mathematics, vol. 31, Cambridge University Press, Cambridge, 1991.
[9] D. J. Benson, Modular Representation Theory, Lecture Notes in Math., vol. 1081, Springer-Verlag, Berlin, 2006, new trends and methods, second printing of the 1984 original.
[10] D. Blessenohl and H. Laue, On the descending Loewy series of Solomon’s descent algebra, J. Algebra 180 (1996), no. 3, 698–724.
[11] D. Blessenohl and H. Laue, The module structure of Solomon’s descent algebra, J. Aust. Math. Soc. 72 (2002), no. 3, 317–333.
[12] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., 24 (1997), 235–265.
[13] A. J. Brandt, The free Lie ring and Lie representations of the full linear group, Trans. Amer. Math. Soc. 56 (1944), 528–536.
[14] R. Brauer, On a conjecture by Nakayama, Trans. Roy. Soc. Canada Sect. III 41 (1947), 11–19.
[15] M. Broué, On Scott modules and p-permutation modules: An approach through the Brauer homomorphism, Proc. Amer. Math. Soc. 93 (1985), no. 3, 401–408.
[16] R. M. Bryant and M. Schocker, The decomposition of Lie powers, Proc. London Math. Soc. (3) 93 (2006), no. 1, 175–196.
[17] R. M. Bryant and R. Stöhr, Lie powers in prime degree, Q. J. Math. 56 (2005), no. 4, 473–489.
[18] F. R. Cohen, D. J. Hemmer, D. K. Nakano, The Lie module and its complexity, Bull. Lond. Math. Soc. 48 (2016), no. 1, 109–114.
[19] S. Danz and K. J. Lim, Signed Young modules and simple Specht modules, Adv. Math. 307 (2017), 369–416.
[20] S. Donkin and K. Erdmann, Tilting modules, symmetric functions, and the module structure of the free Lie algebra, J. Algebra 203 (1998), no. 1, 69–90.
[21] K. Erdmann, K. J. Lim and K. M. Tan, The complexity of the Lie module, Proc. Edinb. Math. Soc. (2) 57 (2014), no. 2, 393–404.
[22] K. Erdmann and M. Schocker, Modular Lie powers and the Solomon descent algebra, Math. Z. 253 (2006), no. 2, 295–313.
[23] H. O. Foulkes, Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form, J. Lond. Math. Soc. 25 (1950) 205–209.
[24] A. M. Garsia and C. Reutenauer, A decomposition of Solomon’s descent algebra. Adv. Math. 77 (1989), no. 2, 189–262.
[25] J. A. Green, Polynomial Representations of GLn, 2nd edn., Lecture Notes in Mathematics, vol. 830, Springer, Berlin, 2007.
[26] J. A. Green, On the indecomposable representations of a finite group, Math. Z. 70 (1959) 430–445.
[27] B. Huppert and N. Blackburn, Finite Groups II, Springer, Berlin, 1982.
[28] G. D. James, Trivial source modules for symmetric groups. Arch. Math. (Basel) 41 (1983), no. 4, 294–300.
[29] G. James and A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Mathematics and its Applications, Vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.
[30] A. Jöllenbeck and M. Schocker, Cyclic characters of symmetric groups, J. Algebraic Combin. 12 (2000), no. 2, 155–161.
[31] A. A. Klyachko, Lie elements in a tensor algebra, Sibirsk. Mat. Ž. 15 (1974), 1296–1304.
[32] W. Kraśkiewicz and J. Weyman, Algebra of coinvariants and the action of a Coxeter element, Bayreuth. Math. Schr. No. 63 (2001), 265–284.
[33] K. J. Lim and K. M. Tan, The Schur functor on tensor powers, Arch. Math. (Basel) 98 (2012), no. 2, 99–104.
[34] K. J. Lim and K. M. Tan, Periodic Lie modules. J. Algebra 445 (2016), 280–294.
[35] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edn., Oxford Science Publications.
[36] F. Patras and C. Reutenauer, Higher Lie idempotents, J. Algebra 222 (1999), no. 1, 51–64.
[37] C. Reutenauer, Free Lie Algebras, London Mathematical Society Monographs. New Series, vol. 7, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993.
[38] G. Robinson, On a conjecture by Nakayama, Trans. Roy. Soc. Canada Sect. III 41 (1947), 20–25.
[39] M. Schocker, Multiplicities of higher Lie characters, J. Aust. Math. Soc. 75 (2003), no. 1, 9–21.
[40] M. Schocker, Lie idempotent algebras, Adv. Math. 175 (2003), no. 2, 243–270.
[41] P. Selick and J. Wu, On natural coalgebra decompositions of tensor algebras and loop suspensions, Mem. Amer. Math. Soc. 148, no. 701 (2000) viii + 109 pp.
[42] L. Solomon, A Mackey formula in the group ring of a Coxeter group, J. Algebra 41 (1976) 255–268.
[43] R. M. Thrall, On symmetrized Kronecker powers and the structure of the free Lie ring, Amer. J. Math. 64 (1942), 371–388.
[44] F. Wever, Über Invarianten in Lie’schen Ringen, Math. Ann. 120 (1949) 563–580.
### Appendix A. Modular Idempotents when \( p = 2 \) and \( n = 2, 3, 4, 5, 6 \)

| \( n \) | 2-regular partitions \( \lambda \) | Modular Idempotents \( e_{\lambda,F} \) in terms of \( \Xi^q \)'s |
|-------|----------------------|-------------------------------------------------|
| 2     | (2) \( \Xi^{(2)} \) |                                                 |
| 3     | (3) \( \Xi^{(3)} + \Xi^{(2,1)} + \Xi^{(1,1,1)} \) |
|       | (2, 1) \( \Xi^{(2,1)} + \Xi^{(1,1,1)} \)       |
| 4     | (4) \( \Xi^{(4)} + \Xi^{(3,1)} + \Xi^{(2,1,1)} + \Xi^{(1,1,1,1)} \) |
|       | (3, 1) \( \Xi^{(3,1)} + \Xi^{(2,1,1)} + \Xi^{(1,1,1,1)} \) |
| 5     | (5) \( \Xi^{(5)} + \Xi^{(4,1)} + \Xi^{(3,2)} + \Xi^{(3,1,1)} + \Xi^{(2,2,1)} + \Xi^{(2,1,2)} + \Xi^{(1,2,1,1,1)} + \Xi^{(1,1,1,1,1)} \) |
|       | (4, 1) \( \Xi^{(4,1)} + \Xi^{(3,1,1)} + \Xi^{(2,2,1)} + \Xi^{(2,1,1,1)} + \Xi^{(2,1,2)} + \Xi^{(1,1,1,2)} + \Xi^{(1,1,1,1,1)} \) |
|       | (3, 2) \( \Xi^{(3,2)} + \Xi^{(1,2,2)} + \Xi^{(2,1,1,1)} + \Xi^{(1,2,1,1)} + \Xi^{(1,1,1,1,2)} \) |
| 6     | (6) \( \Xi^{(6)} + \Xi^{(1,1,2,1,1)} + \Xi^{(4,1,1)} + \Xi^{(4,2)} + \Xi^{(2,2,1,1)} + \Xi^{(5,1)} + \Xi^{(3,1,2)} + \Xi^{(1,1,1,2,1)} + \Xi^{(2,1,2,1)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(2,2,2)} + \Xi^{(1,2,1,2)} \) |
|       | (5, 1) \( \Xi^{(5,1)} + \Xi^{(4,1,1)} + \Xi^{(2,2,1,1)} + \Xi^{(1,2,1,1,1)} + \Xi^{(2,1,2,1)} + \Xi^{(3,1,1,1)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(1,2,2,1)} + \Xi^{(3,2,1)} \) |
|       | (4, 2) \( \Xi^{(4,2)} + \Xi^{(1,1,2,1,1)} + \Xi^{(3,1,2)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(2,2,2)} + \Xi^{(1,2,1,2)} \) |
|       | (3, 2, 1) \( \Xi^{(3,2,1)} + \Xi^{(1,2,1,1,1)} + \Xi^{(1,1,1,2,1)} + \Xi^{(3,1,1,1)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(1,2,2,1)} \) |
### Appendix B. Modular Idempotents when $p = 3$ and $n = 2, 3, 4, 5, 6$

| $n$ | 3-regular partitions $\lambda$ | Modular Idempotents $e_{\lambda,F}$ in terms of $\Xi^q$'s |
|-----|-------------------------------|--------------------------------------------------|
| 2   | $(2)$                         | $\Xi^{(2)} + \Xi^{(1,1)}$                        |
|     | $(1, 1)$                      | $2\Xi^{(1,1)}$                                   |
| 3   | $(3)$                         | $\Xi^{(3)} + 2\Xi^{(2,1)} + 2\Xi^{(1,1,1)}$     |
|     | $(2, 1)$                      | $\Xi^{(2,1)} + \Xi^{(1,1,1)}$                    |
| 4   | $(4)$                         | $\Xi^{(4)} + \Xi^{(2,2)} + 2\Xi^{(3,1)} + 2\Xi^{(1,1,1,1)} + \Xi^{(1,1,2)}$ |
|     | $(3, 1)$                      | $\Xi^{(3,1)} + 2\Xi^{(2,1,1)}$                   |
|     | $(2, 2)$                      | $2\Xi^{(2,2)} + 2\Xi^{(1,1,1,1)} + 2\Xi^{(2,1,1)} + 2\Xi^{(1,1,2)}$ |
|     | $(2, 1, 1)$                   | $2\Xi^{(2,1,1)} + 2\Xi^{(1,1,1,1)}$              |
| 5   | $(5)$                         | $\Xi^{(5)} + 2\Xi^{(4,1)} + \Xi^{(2,1,2)} + \Xi^{(2,1,1,1)} + \Xi^{(3,1,1)} + 2\Xi^{(3,2)} + 2\Xi^{(1,1,1,1,1)} + \Xi^{(1,1,1,2,1)} + 2\Xi^{(1,2,1,1,1)} + \Xi^{(1,1,2,1,1)}$ |
|     | $(4, 1)$                      | $\Xi^{(4,1)} + 2\Xi^{(2,1,1,1)} + 2\Xi^{(3,1,1)} + 2\Xi^{(1,1,1,1,1)} + \Xi^{(2,2,1)} + 2\Xi^{(1,2,1,1)}$ |
|     | $(3, 2)$                      | $\Xi^{(3,2)} + 2\Xi^{(2,2,1)} + 2\Xi^{(2,1,1,1)} + \Xi^{(3,1,1)} + 2\Xi^{(1,1,1,1,1)} + 2\Xi^{(1,1,1,2,1)} + 2\Xi^{(1,1,2,1,1)} + \Xi^{(1,2,1,1,1)}$ |
|     | $(3, 1, 1)$                   | $2\Xi^{(3,1,1)} + 2\Xi^{(2,1,1,1)} + \Xi^{(1,1,1,1,1)} + 2\Xi^{(1,1,2,1)}$ |
|     | $(2, 2, 1)$                   | $2\Xi^{(2,2,1)} + 2\Xi^{(2,1,1,1)} + 2\Xi^{(1,1,1,1,1)} + 2\Xi^{(1,1,2,1)}$ |
| 6   | $(6)$                         | $\Xi^{(6)} + \Xi^{(3,3)} + \Xi^{(1,1,2,1,1)} + \Xi^{(4,1,1)} + 2\Xi^{(1,1,1,1,1)} + 2\Xi^{(1,1,1,2,1)} + 2\Xi^{(2,1,2,1)} + 2\Xi^{(3,1,1,1,1)} + 2\Xi^{(2,1,1,3)} + 2\Xi^{(2,2,2)} + \Xi^{(1,1,3,1)} + 2\Xi^{(1,3,1,1)} + 2\Xi^{(3,2,1)} + \Xi^{(2,1,1,1,1)}$ |
|     | $(5, 1)$                      | $\Xi^{(5,1)} + 2\Xi^{(4,1,1)} + 2\Xi^{(2,1,1,1,1)} + \Xi^{(1,1,1,1,1)} + 2\Xi^{(1,1,1,2,1)} + \Xi^{(2,1,2,1)} + 2\Xi^{(1,1,1,1,1,1)} + 2\Xi^{(1,1,3,1,1)} + 2\Xi^{(3,2,1)} + \Xi^{(2,1,1,1,1)}$ |
|     | $(4, 2)$                      | $\Xi^{(4,2)} + \Xi^{(1,1,2,1,1)} + \Xi^{(4,1,1)} + 2\Xi^{(1,1,1,1,1)} + \Xi^{(1,1,1,2,1)} + \Xi^{(4,1,2)} + 2\Xi^{(3,2,1)} + 2\Xi^{(3,1,1,1,1)} + 2\Xi^{(1,1,1,1,1,1)} + \Xi^{(2,2,2)}$ |
|     | $(3, 3)$                      | $2\Xi^{(3,3)} + 2\Xi^{(1,1,1,1,1,1)} + \Xi^{(1,1,1,1,2,1)} + 2\Xi^{(2,1,2,1)} + 2\Xi^{(2,1,3)} + 2\Xi^{(3,1,1,1,1)} + 2\Xi^{(3,2,1)} + \Xi^{(2,1,1,1,1)}$ |
|     | $(4, 1, 1)$                   | $2\Xi^{(4,1,1)} + 2\Xi^{(2,2,1,1,1)} + \Xi^{(1,1,2,1,1)} + \Xi^{(3,1,1,1,1)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(2,1,1,1,1)}$ |
|     | $(3, 2, 1)$                   | $\Xi^{(3,2,1)} + \Xi^{(1,1,1,1,1,1)} + 2\Xi^{(2,1,2,1)} + \Xi^{(3,1,1,1,1)} + \Xi^{(2,1,1,1,1)}$ |
|     | $(2, 2, 1)$                   | $2\Xi^{(2,2,1,1,1)} + \Xi^{(1,1,2,1,1)} + \Xi^{(1,1,1,1,1,1)} + \Xi^{(2,1,1,1,1)}$ |
APPENDIX C. dim$_F$ Lie$_F(q)$ when $p = 0, 2, 3$ and $n = 2, 3, 4, 5, 6$

| $n$ | Partition $q$ | $p = 0$ | $p = 2$ | $p = 3$ |
|-----|---------------|---------|---------|---------|
| 2   | (2)           | 1       | 1       | 1       |
|     | (1, 1)        | 1       | 1       | 1       |
| 3   | (3)           | 2       | 2       | 2       |
|     | (2, 1)        | 3       | 1       | 3       |
|     | (1, 1, 1)     | 1       | 1       | 1       |
| 4   | (4)           | 6       | 6       | 6       |
|     | (3, 1)        | 8       | 8       | 7       |
|     | (2, 2)        | 3       | 1       | 3       |
|     | (2, 1, 1)     | 6       | 1       | 6       |
|     | (1, 1, 1, 1)  | 1       | 1       | 1       |
| 5   | (5)           | 24      | 24      | 24      |
|     | (4, 1)        | 30      | 21      | 30      |
|     | (3, 2)        | 20      | 20      | 20      |
|     | (3, 1, 1)     | 20      | 20      | 11      |
|     | (2, 2, 1)     | 15      | 1       | 15      |
|     | (2, 1, 1, 1)  | 10      | 1       | 10      |
|     | (1, 1, 1, 1, 1)| 1       | 1       | 1       |
| 6   | (6)           | 120     | 120     | 120     |
|     | (5, 1)        | 144     | 144     | 144     |
|     | (4, 2)        | 90      | 41      | 90      |
|     | (4, 1, 1)     | 90      | 41      | 90      |
|     | (3, 3)        | 40      | 40      | 40      |
|     | (3, 2, 1)     | 120     | 40      | 60      |
|     | (3, 1, 1, 1)  | 40      | 40      | 16      |
|     | (2, 2, 2)     | 15      | 1       | 15      |
|     | (2, 2, 1, 1)  | 45      | 1       | 45      |
|     | (2, 1, 1, 1, 1)| 15      | 1       | 15      |
|     | (1, 1, 1, 1, 1, 1)| 1       | 1       | 1       |

(K. J. Lim) Division of Mathematical Sciences, Nanyang Technological University, SPMS-04-01, 21 Nanyang Link, Singapore 637371.

Email address: limkj@ntu.edu.sg