SL-invariant entanglement measures in higher dimensions: the case of spin 1 and 3/2

Andreas Osterloh

Institut für Theoretische Physik, Universität Duisburg-Essen, D-47048 Duisburg, Germany

E-mail: andreas.osterloh@uni-due.de

Received 19 August 2014, revised 16 October 2014
Accepted for publication 11 December 2014
Published 21 January 2015

Abstract

An SL-invariant extension of the concurrence to higher local Hilbert-space dimension is due to its relation with the determinant of the matrix of a $d \times d$ two qudits state, which is the only $SL$-invariant of polynomial degree $d$. This determinant is written in terms of antilinear expectation values of the local $SL(d)$ operators. We use the permutation invariance of the comb-condition for creating further local antilinear operators which are orthogonal to the original operator. It means that the symmetric group acts transitively on the space of combs of a given order. This extends the mechanism for writing $SL(2)$-invariants for qubits to qudits. I outline the method, which in principle works for arbitrary dimension $d$, explicitly for spin 1 and spin 3/2. There is an odd–even discrepancy: whereas for half odd integer spin a situation similar to that observed for qubits is found, for integer spin the outcome is an asymmetric invariant of polynomial degree $2d$.

Keywords: entanglement, qudits, SL-invariance

1. Introduction

Since the development of quantum information theory, the importance of entanglement as a resource in physics has become clearer, and its quantification is hence an outstanding task. The minimal requirement for an entanglement measure is the symmetry under local $SU$ operations [1] and a lot of theoretical work has been devoted to its invariance group [2–7]. However, this group must be enlarged to the complexification of $SU$, the $SL$, which then encompasses the stochastic local operations and classical communication. The general linear $GL$ group of general local operations admits in its closure also projective measurements. The invariance group of $SL(2)$ has been explored in the mathematics and physics literature [7–23].
It has several nice properties, such as it leads automatically to entanglement monotones, it automatically contains the entangled states it measures [24], and it establishes a clue to what polynomial degree an invariant must have in order that it can possibly detect a given state [23–25]. Furthermore, first results for the SL(2) concerning optimal decompositions and entanglement of mixed states are also known [26–33] such that we are close to a breakthrough towards measuring the invariants of SL(2) experimentally. This astonishingly powerful tool of SL-invariance still awaits its extension on the local operator level to general spin, though formally it is known how to write down such invariants (see for example [20]). One of the main goals is to write invariants that permit control over their zeros as far as product states are concerned and hopefully to gain computational efficiency [23] when using an analogue of the procedure presented in [21, 22] for qubits, the method making use of the so-called combs, instead of the so-called Omega-process or the spinor contraction method as in [20].

Here, I will add to the invariant local antilinear operators presented in [21, 22] for qubits corresponding invariant local antilinear operators for higher spins 1 and 3/2, hence dimensions 3 and 4. With them, at least certain SL-invariant entanglement measures for an arbitrary number of qutrits (spin 1) and ququads (spin 3/2) can be written, having also control over the zeros of the corresponding invariants, as far as product states are concerned [21, 22]. I give two examples of how an analogue of the three-tangle for qubits would look like for qutrits and ququads. The method explained here is, however, much more general and admits to design entanglement measures for an arbitrary number of particles with spin 1 or 3/2.

The paper is laid out as follows. In section 2, I review the idea of combs (specific antilinear operators) for qubits, and devote the next section 3 to elaborate such combs for spin 1, and 3/2, together with two possible invariants for three constituents. The conclusions and an outlook are given in section 4.

2. The concept of a comb

The fundamental concept that represents the basis for the construction method of SL(2)-invariant operators, is that of a comb. It is a local antilinear operator \( A_i \) with zero expectation value for all states of the local Hilbert space \( \mathcal{H}_i \) [21, 24]. Every such operator can be written as a linear operator \( L_i \) times a complex conjugation \( C \). Here, I give a brief summary of this formalism. A condition for an operator to be a comb is hence

\[
\langle \psi | A_i | \psi \rangle = \langle \psi | L_i C | \psi \rangle = \langle \psi | L_i | \psi^* \rangle \equiv 0 \quad \forall \psi \in \mathcal{H}_i, \tag{1}
\]

where \( C \) is the complex conjugation in the computational basis

\[
| \psi^* \rangle := C | \psi \rangle \equiv C \sum_{j_1, \ldots, j_q=0}^1 \psi_{j_1 \cdots j_q} | j_1 \cdots j_q \rangle
\]

\[
= \sum_{j_1, \ldots, j_q=0}^1 \psi_{j_1 \cdots j_q}^* | j_1 \cdots j_q \rangle.
\]

In order to possibly vanish for every local state, the operator necessarily has to be antilinear (a linear operator with the above property is identically zero). The problem is, to identify a comb that is regular on the space it acts on, and with equal modulus of the corresponding eigenvalues due to the invariance property [24]. We will call such an antilinear...
operator $A^{(1)} : \mathfrak{h} \rightarrow \mathfrak{h}$ to be of order 1, where $\mathfrak{h}$ is the local Hilbert space. The expectation value of $A^{(1)}$ is a homogeneous polynomial of lowest possible degree 2.

In general we will call a (local) antilinear operator $A^{(n)} : \mathfrak{h}^{\otimes n} \rightarrow \mathfrak{h}^{\otimes n}$ to be of order $n$; its expectation value is a homogeneous polynomial of degree $2n$ in the coefficients of the state $|\psi\rangle$. It is worthwhile noticing that the $n$-fold tensor product $\mathfrak{h}^{\otimes n}$, a comb of order $n$ acts on, is made of $n$-fold copies of one single qubit (or qudit) state. In order to distinguish this merely technical introduction of a tensor product of copies of states from the physically motivated tensor product of different qubits we will denote the first tensor product of copies with the symbol $\otimes$, and hence write $A^{(n)} : \mathfrak{h}^{\otimes n} \rightarrow \mathfrak{h}^{\otimes n}$. I will sometimes use the abbreviation

$$\langle \psi | L_i C | \psi \rangle =: (L_i).$$

The term *comb* is used for the corresponding linear operator $L_i$ as for the antilinear operator $A_i$, identifying the two. In what follows, I will mainly deal with the corresponding linear operators $L_i$ instead of directly with the combs $A_i = L_i C$.

The conditions we are searching for, can then be written as

$$\langle \psi | A^{(1)} | \psi \rangle = 0,$$

$$\langle \psi | \cdot \langle \psi | A^{(2)} | \psi \rangle \cdot | \psi \rangle = 0,$$

$$\langle \psi | \cdot \langle \psi | A^{(3)} | \psi \rangle \cdot | \psi \rangle \cdot | \psi \rangle = 0,$$

$$\vdots$$

Before we go to higher spin, we restrict our focus on multipartite states of qubits or spins 1/2. There, the local Hilbert space is two-dimensional $\mathcal{H}_i = \mathbb{C}^2 =: \mathfrak{h}$ for all $i$. We will need the Pauli matrices

$$\sigma_0 := 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_i := \sigma_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_2 := \sigma_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 := \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and highlight that any tensor product $f (\{ \sigma_j \}) := \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}$ with an odd number $N_j$ of $\sigma_j$ is a comb. In particular, $L_{1/2}^{(2)} := \sigma_1$ is the unique comb$^2$ of order 1. Throughout the work the Einstein sum convention is used. The unique comb$^3$ of order 2, which is orthogonal with respect to the Hilbert–Schmidt scalar product to the trivial one $\sigma_i \cdot \sigma_j$, is then

$$L_{1/2}^{(2)} := \sigma_1 \cdot \sigma_2 = \sum_{\mu, \nu=0}^3 g^{\mu \nu} \sigma_\mu \cdot \sigma_\nu,$$

with

$$g^{\mu \nu} = g_{\mu \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$  

Finally, the comb$^4$ of third order is

$$L_{1/2}^{(3)} := \epsilon_{ijk} \tau_i \cdot \tau_j \cdot \tau_k,$$

---

2 Actually, this would be the corresponding linear operator for the comb $\sigma_0 C$.

3 Also here it is the corresponding linear operator to the comb $\sigma_0 \cdot \sigma_0 C$ of order 2.

4 Again, this is the corresponding linear operator of the comb $\epsilon_{ijk} \tau_i \cdot \tau_j \cdot \tau_k C$ of third order.
where $\tau_1 = \sigma_0$, $\tau_2 = \sigma_1$, $\tau_3 = \sigma_i$. This comb completes the set of local invariant operators for spin $1/2$ in that it closes the algebra of antilinear operators [23].

Filter invariants for $n$ qubits are then obtained as antilinear expectation values of filter operators; the latter are constructed from combs as to have vanishing expectation value for arbitrary product states (hence, they are zero for any bipartite state). The filter invariant $F$ is hence a $\mathbb{C}$-number that is intimately linked to its filter operator $\mathcal{F}$ by $F = \left( \prod \langle \psi_n \rangle \right) \left( \prod \langle \psi_\bar{n} \rangle \right)$, where $\psi_n$ is a pure state of $n$ constituents. Therefore, the word filter is often used for both the filter operator and its filter invariant in the literature. Filters for qubits have been constructed in [21–23, 26].

3. Local antilinear operators for higher spin

An elementary polynomial $SL$-invariant is the determinant for two qudits (see [34] for spin 1)

$$\left|\psi_{AB}\right| = \sum_{i,j=0}^{S} \psi_{ij} |ij\rangle,$$

(8)

$$\det M_{\psi_{AB}} = \begin{vmatrix} \psi_{00} & \cdots & \psi_{0S} \\ \vdots & \ddots & \vdots \\ \psi_{S0} & \cdots & \psi_{SS} \end{vmatrix}. \quad (9)$$

Its polynomial degree is $2S + 1$ and so the lowest polynomial degree we can expect for an invariant operator to exist, is $2S + 1$ for spin $S$. Because any expectation value of every operator leads to an even degree in the wave function, there are two cases to distinguish. First, the case of even dimension of the local Hilbert space, where the determinant could be expressible in terms of local invariant operators, as is the case for each odd half-integer spin as e.g. qubits. Second, there is the case of odd dimension of the local Hilbert space, which is given for each integer spin. The first occurrence of this latter scenario is for spin 1, which we will discuss next.

3.1. The case of spin 1

Spin 1 is the first example for an odd dimension of the local Hilbert space. The invariance group for this case is the $SL(3)$. The Gell-Mann matrices are generators on the algebra level

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(10)

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

(11)

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

(12)

including the identity matrix $\lambda_0$. The $SL$-invariant of lowest polynomial degree is the determinant [34, 35], which is of degree 3. In order to obtain an even polynomial degree, we take its square, which is of degree 6, and therefore is achievable as an expectation value of some local invariant operator. The invariant operator we obtain is therefore a comb of order 3,
and is given as follows

$$L_1^{(3)} := i\varepsilon_{ijk} \tau_i \cdot \tau_j \cdot \tau_k,$$

where

$$\tau_1 = \lambda_2; \quad \tau_2 = \lambda_5; \quad \tau_3 = \lambda_7.$$  \hfill (14)

It is worth noticing that the operators $\lambda_i$ for $i = 2, 5, 7$ are precisely the matrices $\sigma_y^{(1,2)}, \sigma_y^{(1,3)},$ and $\sigma_y^{(2,3)}$, respectively. Hence, they are combs of order 1 that however transform to a nontrivial orbit under $SL(3)$ operations [24]. The square of the determinant

$$T_2^{(1)}[\psi_{12}] = \det^2 M_{\psi_{12}} = \left| \begin{array}{ccc} \psi_{1,1} & \psi_{1,0} & \psi_{1,-1} \\ \psi_{0,1} & \psi_{0,0} & \psi_{0,-1} \\ \psi_{-1,1} & \psi_{-1,0} & \psi_{-1,-1} \end{array} \right|^2$$



$$= \left( \psi_{1,1}\psi_{0,0}\psi_{-1,-1} + \psi_{1,0}\psi_{0,-1}\psi_{-1,1} + \psi_{1,-1}\psi_{0,1}\psi_{-1,0} \\
- \psi_{-1,1}\psi_{0,0}\psi_{1,-1} - \psi_{-1,0}\psi_{0,-1}\psi_{1,1} - \psi_{-1,-1}\psi_{0,1}\psi_{1,0} \right)^2$$

is obtained by

$$-\frac{1}{48} \sum_{i,j,k,l,m,n} \varepsilon_{ijkl} \epsilon_{ijlm} \left( \tau_i \otimes \tau_l \right) \left( \tau_j \otimes \tau_m \right) \left( \tau_k \otimes \tau_n \right) \psi_{12} \cdot \psi_{12} \cdot \psi_{12} \cdot \psi_{12} \cdot \psi_{12} \cdot \psi_{12}.$$

Since the procedure to obtain higher order combs is not described in detail in [23], it is worth to make some remarks here. We consider the comb property of order $n$

$$\langle \psi | \cdot \langle \psi | \cdot \cdots \cdot \langle \psi \mid A^{(n)} \psi \rangle \cdot \mid \psi \rangle \cdot \cdots \cdot \mid \psi \rangle = 0.$$  \hfill (17)

This condition surely does not depend on how we reorder the local states $\mid \psi \rangle$ on either side. Therefore, it is invariant under the full symmetric group, $S_n$, which means that also

$$\langle \psi | \cdot \langle \psi | \cdot \cdots \cdot \langle \psi \mid \Pi A^{(n)} \Pi \mid \psi \rangle \cdot \mid \psi \rangle \cdot \cdots \cdot \mid \psi \rangle = 0$$  \hfill (18)

for $\Pi, \Pi' \in S_n$. This means that if $A^{(n)}$ is a comb of order $n$ (satisfying hence the property (17)), also the operator $\Pi A^{(n)} \Pi$ satisfies the comb relation (equation (18)). It therefore is a comb of order $n$ as well. In what follows, we introduce an additional symbol $\sigma$, which is used in the same way as $\cdot$ for highlighting and keeping track of the different nature of tensor products in the sequel. It symbolizes tensor products of the same local state but it indicates where the permutation operator from above acts on in a nontrivial way. In order to give a simple example of this notion, we look at the comb of first order for qubits, $\sigma_y$. A corresponding comb of second order is $\sigma_y \circ \sigma_y$, and hence another comb of the same order 2 is

$$\sigma_y \circ \sigma_y \Pi_2 = -\frac{1}{2} \sum_{\mu=0}^1 (\sigma_\mu \circ \sigma_\mu - \sigma_\mu \circ \sigma_\mu) \Pi,$$

as reported in [23], where $\Pi_2 = \frac{1}{2} \sum_{\mu=0}^3 \sigma_\mu \circ \sigma_\mu.$

Next, the procedure is to render both operators orthogonal in the trace norm and we have an orthogonal comb of order 2. This is a straightforward method for constructing combs of any order for $d$-dimensional qudits. I will apply this line of thought to spin 1.

The second comb of order six will emerge from a multiplication of the trivial comb

$$\varepsilon_{ijk} \sum_{l,m,n} \left( \tau_i \circ \tau_j \right) \left( \tau_j \circ \tau_m \right) \left( \tau_k \circ \tau_n \right)$$
with the permutation operator

\[
P_3 = \frac{1}{3} + \frac{1}{2} \sum_{i=1}^{7} \lambda_i \circ \lambda_i + \frac{1}{6} \lambda_8 \circ \lambda_8
\]  

(19)

as

\[
L_1^{(6)} := -\epsilon_{ijk} \epsilon_{lmn} (\tau_i \circ \tau_l) \ast (\tau_j \circ \tau_m) \ast (\tau_k \circ \tau_n) P_3 \ast P_3 \ast P_3 .
\]  

(20)

We at first have to determine the nine operators \( O_{ij} = (\tau_i \circ \tau_j)P_3 \), which are obtained in the following form

\[
O_{11} = \frac{1}{2} \left( \lambda_2 \circ \lambda_2 - \lambda_1 \circ \lambda_1 - \lambda_3 \circ \lambda_3 \right) + \frac{1}{18} \left( 2\lambda_0 + \lambda_8 \right) \circ \left( 2\lambda_0 + \lambda_8 \right) =: \xi_{11,\mu} \circ \xi_{11,\mu},
\]  

(21)

\[
O_{12} = \frac{1}{12} \left( \left( 2\lambda_0 + \lambda_8 + 3\lambda_3 \right) \circ \left( \lambda_6 - i\lambda_7 \right) + \left( \lambda_6 + i\lambda_7 \right) \circ \left( 2\lambda_0 + \lambda_8 + 3\lambda_3 \right) \right)
- \frac{1}{4} \left( \left( \lambda_1 - i\lambda_2 \right) \circ \left( \lambda_4 - i\lambda_5 \right) + \left( \lambda_4 + i\lambda_5 \right) \circ \left( \lambda_2 + i\lambda_1 \right) \right) =: \xi_{12,\mu} \circ \xi_{12,\mu},
\]  

(22)

\[
O_{13} = \frac{1}{4} \left( \left( \lambda_1 + i\lambda_2 \right) \circ \left( \lambda_6 - i\lambda_7 \right) + \left( \lambda_6 + i\lambda_7 \right) \circ \left( \lambda_1 - i\lambda_2 \right) \right)
- \frac{1}{12} \left( \left( 2\lambda_0 + \lambda_8 - 3\lambda_3 \right) \circ \left( \lambda_4 - i\lambda_5 \right) + \left( \lambda_4 + i\lambda_5 \right) \circ \left( 2\lambda_0 + \lambda_8 - 3\lambda_3 \right) \right)
=: \xi_{13,\mu} \circ \xi_{13,\mu},
\]  

(23)

\[
O_{21} = \frac{1}{12} \left( \left( \lambda_6 - i\lambda_7 \right) \circ \left( 2\lambda_0 + \lambda_8 + 3\lambda_3 \right) + \left( 2\lambda_0 + \lambda_8 + 3\lambda_3 \right) \circ \left( \lambda_6 + i\lambda_7 \right) \right)
- \frac{1}{4} \left( \left( \lambda_4 - i\lambda_5 \right) \circ \left( \lambda_1 - i\lambda_2 \right) + \left( \lambda_1 + i\lambda_2 \right) \circ \left( \lambda_4 + i\lambda_5 \right) \right) =: \xi_{21,\mu} \circ \xi_{21,\mu},
\]  

(24)

\[
O_{22} = \frac{1}{2} \left( \lambda_5 \circ \lambda_5 - \lambda_4 \circ \lambda_4 - \frac{1}{4} \left( \lambda_3 + \lambda_8 \right) \circ \left( \lambda_3 + \lambda_8 \right) \right)
+ \frac{1}{72} \left( 4\lambda_0 + 3\lambda_3 - \lambda_8 \right) \circ \left( 4\lambda_0 + 3\lambda_3 - \lambda_8 \right) =: \xi_{22,\mu} \circ \xi_{22,\mu},
\]  

(25)

\[
O_{23} = \frac{1}{4} \left( \left( \lambda_6 - i\lambda_7 \right) \circ \left( \lambda_4 - i\lambda_5 \right) + \left( \lambda_4 + i\lambda_5 \right) \circ \left( \lambda_6 + i\lambda_7 \right) \right)
+ \frac{1}{6} \left( \left( \lambda_0 - \lambda_8 \right) \circ \left( \lambda_1 - i\lambda_2 \right) + \left( \lambda_1 + i\lambda_2 \right) \circ \left( \lambda_0 - \lambda_8 \right) \right) =: \xi_{23,\mu} \circ \xi_{23,\mu},
\]  

(26)

\[
O_{31} = \frac{1}{4} \left( \left( \lambda_6 - i\lambda_7 \right) \circ \left( \lambda_1 + i\lambda_2 \right) + \left( \lambda_1 - i\lambda_2 \right) \circ \left( \lambda_6 + i\lambda_7 \right) \right)
- \frac{1}{12} \left( \left( \lambda_4 - i\lambda_5 \right) \circ \left( 2\lambda_0 + \lambda_8 - 3\lambda_3 \right) + \left( 2\lambda_0 + \lambda_8 - 3\lambda_3 \right) \circ \left( \lambda_4 + i\lambda_5 \right) \right)
=: \xi_{31,\mu} \circ \xi_{31,\mu},
\]  

(27)
\[ O_{32} = \frac{1}{4} \left( (\lambda_4 - i\lambda_5) \circ (\lambda_6 - i\lambda_7) + (\lambda_6 + i\lambda_7) \circ (\lambda_4 + i\lambda_5) \right) \\
+ \frac{1}{6} \left( (\lambda_1 - i\lambda_2) \circ (2\lambda_0 - \lambda_8) + (\lambda_0 - \lambda_8) \circ (\lambda_1 + i\lambda_2) \right) =: \xi_{32,\mu} \circ \xi_{32}^\mu, \quad (28) \]

\[ O_{33} = \frac{1}{2} \left( \lambda_7 \circ \lambda_7 - \lambda_6 \circ \lambda_6 - \frac{1}{4} (\lambda_3 - \lambda_8) \circ (\lambda_3 - \lambda_8) \right) \\
+ \frac{1}{12} (4\lambda_0 - 3\lambda_3 - \lambda_8) \circ (4\lambda_0 - 3\lambda_3 - \lambda_8) =: \xi_{33,\mu} \circ \xi_{33}^\mu. \quad (29) \]

It is furthermore to be mentioned that \( O_{jk} = O_{kj}^T \), where \( T \) means the transposition. This is due to the relation \( O_{jk} = (\tau_6 \circ \tau_7)p_3 = p_3(\tau_7 \circ \tau_6)p_3 = p_3(\tau_6 \circ \tau_7) \). Every single operator \( O_{ij} \) consists of precisely four contributions, each having a single entry in the matrix, two with contribution +1, and two with −1. The contraction, indicated with upper and lower greek indices, means to sum over these four elements in each of the \( O_{ij} \). In principle one could choose for example \( \xi_{11i} = \lambda_i / \sqrt{2} \), \( \xi_{111} = \lambda_j / \sqrt{2} \), \( \xi_{110} = \frac{1}{\sqrt{3}}(2\lambda_0 + \lambda_8) \), and \( \xi_{111} = -\lambda_j / \sqrt{2} \), \( \xi_{111} = \lambda_j / \sqrt{2} \), \( \xi_{110} = -\lambda_j / \sqrt{2} \), \( \xi_{111} = \frac{1}{3\sqrt{2}}(2\lambda_0 + \lambda_8) \), but any other representation would be as good, as long as it leads to the form of \( O_{11} \). It is interesting that besides the \( \sigma_{ij}^{(j)} \) appearing in the operators \( O_{jk} \), the remaining part is \( \sigma_{ij}^{(j)} \circ \sigma_{ik}^{(j)} = \sum_{\mu=0}^3 g_\mu \sigma_{ij}^{(j)} \circ \sigma_{ik}^{(j)} \) with \( g_\mu = (-1, 1, 0, 1) \), resembling the situation of the two-dimensional Hilbert space for spin 1/2.

Using this, we have to consider

\[ L_1^{(6)} := -\varepsilon_{ijk} e_{lmi} O_{ij} \circ O_{jm} \circ O_{kn} = - \sum_{i,j,k=0}^3 \sum_{l,m,n=0}^3 \varepsilon_{ijk} e_{lmi} \times \left( \xi_{ik,j}^\mu \circ \xi_{ij}^\mu \right) \left( \xi_{jm,k}^c \circ \xi_{jm}^c \right) \left( \xi_{kn,r}^\rho \circ \xi_{kn}^\rho \right) \quad (30) \]

as a comb of order 6. This operator, however, still fails to be orthogonal to the original operator \( L_1^{(3)} \circ L_1^{(3)} \) in that we have \( \text{tr} \left( L_1^{(3)} \circ L_1^{(3)} \right) = 31104 \). Since \( \text{tr} \left( L_1^{(3)} \circ L_1^{(3)} \right) = 2304 \), this trace is removed in \( \left( L_1^{(3)} \circ L_1^{(3)} \right) \), such that both operators are orthogonal. I want to remind a fact that is also crucial here: as soon as operators as \( \sigma_{ij}^{(j)} \) occur an odd number of times, its corresponding antilinear expectation value \( \langle f(\sigma_{ij}^{(j)}) \rangle \) will vanish (see for example [21]). Therefore, the operator \( L_1^{(6)} \) can be directly inserted without resorting on the orthogonal version. In principle, with these two operators \( L_1^{(3)} \) and \( L_1^{(6)} \), we are in the position to write down SL-invariant entanglement measures for an arbitrary number of distinguishable particles of spin 1. The advantage of taking the two orthogonal operators is the better control on the zeros of corresponding filter invariants (see e.g. [22] for spin 1/2).

The square of the determinant \([34, 35]\), which leads to a degree 6 analogue of the concurrence for qutrits has been mentioned already. An analogue of the three-tangle for three qutrits would consequently be expressed by
It is a three-qutrit filter invariant. I have not checked for invariant operators existing besides this (see [23] for qubits). It is possible that one will even have to look for degree 6, where the permutations are set in a distinct way. This will be left for future work.

### 3.2. The case of spin 3/2

The next higher case of half integer spin is spin $\frac{3}{2}$, corresponding to an even dimension of the local Hilbert space, which is 4. Here, it is sufficient to prepare the determinant itself with expectation values of antilinear operators. The underlying group is $SL(4)$, whose generators, including the identity matrix, read

\[
\begin{align*}
\lambda_0 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
\lambda_1 &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_2 &= \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_8 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_9 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_{10} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
io & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_{11} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
\lambda_{12} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & io & 0 & 0
\end{pmatrix}, \\
\lambda_{13} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\lambda_{14} &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \\
\lambda_{15} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\end{align*}
\]

The comb of lowest order 2 is

\[
L^{(2)}_{3/2} := \sum_{i=1}^{6} (-1)^{\min(i, 7-i)} \tau_i \cdot \tau_{7-i},
\]

where we defined

\[
\tau_i = \lambda_{2i}; \quad \text{for } i = 1, \ldots, 6.
\]
It reproduces the determinant of two qudit states of spin 3/2 in the following way

\[
\frac{1}{24} \sum_{i,j=1}^{6} (-1)^{\min\{i,7-i\}+\min\{j,7-j\}} \left( (\tau_i \otimes \tau_j) \cdot \left( (\tau_{7-i} \otimes \tau_{7-j}) \right) \right)
\]

\[
= \begin{vmatrix}
\psi_{2,2} & \psi_{2,1} & \psi_{2,-1} & \psi_{2,-2} \\
\psi_{1,2} & \psi_{1,1} & \psi_{1,-1} & \psi_{1,-2} \\
\psi_{-1,2} & \psi_{-1,1} & \psi_{-1,-1} & \psi_{-1,-2} \\
\psi_{-2,2} & \psi_{-2,1} & \psi_{-2,-1} & \psi_{-2,-2} \\
\end{vmatrix}.
\]

(35)

Please notice that the operators \(\tau_i\) correspond to \(\sigma_{ij}^{(k,l)}\) which are the \(\sigma_y\) acting on the subspaces \(k\) and \(l\), where \((k, l)\) takes the values \((1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\) and \((3, 4)\) of the four-dimensional local space.

A second comb is of order 4 (hence, it is related to polynomial degree 8), and is obtained by means of the permutation operator

\[
P_4 = \frac{1}{4}I + \frac{1}{2} \sum_{i=1}^{14} \lambda_i \circ \lambda_i + \frac{1}{4} \lambda_{15} \circ \lambda_{15}
\]

(36)

in the following way

\[
L_{3/2}^{(4)} := \sum_{i,j=1}^{6} (-1)^{\min\{i,7-i\}+\min\{j,7-j\}} \left( \tau_i \circ \tau_j \right) \cdot \left( \tau_{7-i} \circ \tau_{7-j} \right) P_4 \cdot P_4.
\]

(37)

Therefore, the operators \(O_{ij} = (\tau_i \circ \tau_j)P_4\) are relevant, which are presented in the appendix. Again, we find \(O_{ik} = O_{kj}^T\). Also here it is striking that, similar to the previous case, the operators \(O_{jk}\) have only four contributions, corresponding to four entries (two with +1, and two with −1). The comb of order 4 is hence

\[
L_{3/2}^{(4)} = \sum_{i,j=1}^{6} (-1)^{\min\{i,7-i\}+\min\{j,7-j\}} O_{ij} \cdot O_{7-i,7-j}
\]

\[
= \sum_{i,j=1}^{6} (-1)^{\min\{i,7-i\}+\min\{j,7-j\}} \left( \xi_{ij,\mu} \circ \xi_{ij,\mu} \right) \cdot \left( \xi_{7-i,7-j,\mu} \circ \xi_{7-i,7-j,\mu} \right).
\]

(38)

This operator is again not orthogonal to the trivial one, \(L_{3/2}^{(2)} \circ L_{3/2}^{(2)}\), in that we find \(\text{tr} L_{3/2}^{(4)} (L_{3/2}^{(2)} \circ L_{3/2}^{(2)}) = \frac{3}{7}\). Together with \(\text{tr} L_{3/2}^{(2)} \circ L_{3/2}^{(2)} = 9\) we obtain the orthogonal operator \(L_{3/2}^{(4)} - \frac{3}{7} L_{3/2}^{(2)} \circ L_{3/2}^{(2)}\). These two orthogonal combs are the counterparts of \(\sigma_i\) and \(\sigma_y \circ \sigma_y\) for qubits. They are sufficient for the construction of many (most likely not of all) SL-invariants for spin 3/2. However, as for the spin-1 case, using the operator \(L_{3/2}^{(4)}\) an analogue of the three-tangle for spin 3/2 could be expressed as
The existence of further combs of higher order will be left for future work.

4. Conclusion

I have outlined a path towards the generalization of entanglement measures along the line pursued in [10, 22, 26] to systems with higher local dimensions i.e. for higher spin $S$. Therefore, the theory of local SL-invariant operators [21] has been considered and developed further. A comb, written in terms of expectation values, has an even polynomial degree. The lowest possible such degree would be $S^2 + 1$, due to the determinant of a $(2S + 1) \times (2S + 1)$ matrix. This lowest possible degree however can only be realized for an even dimension, hence for $S = m/2$ for odd $m$. In case of an odd dimension $2S + 1$, i.e. for integer spin $S$, the lowest dimension is therefore doubled as $4S + 2$. In order to have more than a single SL-invariant, we need at least one more and linearly independent local antilinear operator which is possibly (but not necessarily) orthogonal to the comb of lowest order or (multiple) tensor products of it. Two operators are sufficient for being able to construct measures for genuine multipartite entanglement which are related to an entanglement monotone [21–23, 36]. Therefore, it is crucial to observe that the comb condition for a local SL-invariant operator is invariant with respect to the symmetric group (a fact which has been observed and used excessively already in [23]). The symmetric group hence acts transitively on the space of combs of a certain order. Using this method, I give expressions for two local and orthogonal SL-invariant operators for each spin, 1 and 3/2, corresponding to odd and even dimensions of the local Hilbert space, respectively. I give explicit formulae for analogues to the concurrence and the three-tangle for qubits. Open questions remain. One of the next tasks would be to find out how this formalism is generalized to arbitrary dimension as a straightforward generalization of the formulae given in the text do not lead to an answer. This has to be left for future work. Also the question of how a complete set of local SL-invariant operators is obtained has to be analyzed later on.

Acknowledgments

This work was supported by the German Research Foundation within the SFB TR12.

Appendix. The operator for spin 3/2

The operators $O_{ij} = (\tau_i \circ \tau_j)P_{3}$ are given here explicitly for local dimension 4, or spin 3/2. We find them also being symmetric, $O_{ij} = O_{ji}$, and similar to spin 1 the operators $O_{i\bar{k}}$ have only four contributions, corresponding to four entries (two with +1, and two with −1).
\[ O_{11} = \frac{1}{2} (\lambda^2 \circ \lambda^2 - \lambda_1 \circ \lambda_1 - \lambda_{13} \circ \lambda_{13}) + \frac{1}{8} (\lambda_0 + \lambda_15) \circ (\lambda_0 + \lambda_{15}) = \xi_{11,\mu} \circ \xi''_{11}. \]  
(A.1)

\[ O_{12} = -\frac{1}{4} \left( (\lambda_1 - i\lambda_2) \circ (\lambda_3 - i\lambda_4) + (\lambda_3 + i\lambda_4) \circ (\lambda_1 + i\lambda_2) \right) \\
+ \frac{1}{8} \left( (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \circ (\lambda_{17} - i\lambda_8) + (\lambda_{17} + i\lambda_8) \circ (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \right) \\
=: \xi_{12,\mu} \circ \xi''_{12}. \]  
(A.2)

\[ O_{13} = -\frac{1}{4} \left( (\lambda_1 - i\lambda_2) \circ (\lambda_5 - i\lambda_6) + (\lambda_6 + i\lambda_5) \circ (\lambda_1 + i\lambda_2) \right) \\
+ \frac{1}{8} \left( (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \circ (\lambda_{19} - i\lambda_{10}) + (\lambda_{19} + i\lambda_{10}) \circ (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \right) \\
=: \xi_{13,\mu} \circ \xi''_{13}. \]  
(A.3)

\[ O_{14} = \frac{1}{4} \left( (\lambda_1 + i\lambda_2) \circ (\lambda_7 - i\lambda_8) + (\lambda_7 + i\lambda_8) \circ (\lambda_1 - i\lambda_2) \right) \\
- \frac{1}{8} \left( (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \circ (\lambda_{13} - i\lambda_4) + (\lambda_{13} + i\lambda_4) \circ (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \right) \\
=: \xi_{14,\mu} \circ \xi''_{14}. \]  
(A.4)

\[ O_{15} = \frac{1}{4} \left( (\lambda_1 + i\lambda_2) \circ (\lambda_9 - i\lambda_{10}) + (\lambda_9 + i\lambda_{10}) \circ (\lambda_1 - i\lambda_2) \right) \\
- \frac{1}{8} \left( (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \circ (\lambda_5 - i\lambda_6) + (\lambda_5 + i\lambda_6) \circ (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \right) \\
=: \xi_{15,\mu} \circ \xi''_{15}. \]  
(A.5)

\[ O_{16} = \frac{1}{4} \left( (\lambda_0 + i\lambda_{10}) \circ (\lambda_3 - i\lambda_4) + (\lambda_3 + i\lambda_4) \circ (\lambda_9 - i\lambda_{10}) \right) \\
- \frac{1}{8} \left( (\lambda_5 + i\lambda_6) \circ (\lambda_7 - i\lambda_8) + (\lambda_7 + i\lambda_8) \circ (\lambda_5 - i\lambda_6) \right) =: \xi_{16,\mu} \circ \xi''_{16}. \]  
(A.6)

\[ O_{22} = \frac{1}{2} (\lambda_4 \circ \lambda_4 - \lambda_3 \circ \lambda_3) - \frac{1}{8} \left( (\lambda_{13} - \lambda_{14} + \lambda_{15}) \circ (\lambda_{13} - \lambda_{14} + \lambda_{15}) \right) \\
+ (\lambda_0 + \lambda_{13} + \lambda_{14}) \circ (\lambda_0 + \lambda_{13} + \lambda_{14}) \right) =: \xi_{22,\mu} \circ \xi''_{22}. \]  
(A.7)

\[ O_{23} = -\frac{1}{4} \left( (\lambda_3 - i\lambda_4) \circ (\lambda_5 - i\lambda_6) + (\lambda_5 + i\lambda_6) \circ (\lambda_3 + i\lambda_4) \right) \\
+ \frac{1}{8} \left( (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \circ (\lambda_{11} - i\lambda_{12}) + (\lambda_{11} + i\lambda_{12}) \circ (\lambda_0 + 2\lambda_{13} + \lambda_{15}) \right) \\
=: \xi_{23,\mu} \circ \xi''_{23}. \]  
(A.8)

\[ O_{24} = -\frac{1}{4} \left( (\lambda_3 + i\lambda_4) \circ (\lambda_7 + i\lambda_8) + (\lambda_7 - i\lambda_8) \circ (\lambda_3 - i\lambda_4) \right) \\
+ \frac{1}{8} \left( (\lambda_0 + 2\lambda_{14} - \lambda_{15}) \circ (\lambda_1 - i\lambda_2) + (\lambda_1 + i\lambda_2) \circ (\lambda_0 + 2\lambda_{14} - \lambda_{15}) \right) \\
=: \xi_{24,\mu} \circ \xi''_{24}. \]  
(A.9)
\[ O_{25} = -\frac{1}{4} \left( (\lambda_7 - i\lambda_8) \circ (\lambda_5 - i\lambda_6) + (\lambda_5 + i\lambda_6) \circ (\lambda_7 + i\lambda_8) \right) + \frac{1}{4} \left( (\lambda_{11} + i\lambda_{12}) \circ (\lambda_1 - i\lambda_2) + (\lambda_1 + i\lambda_2) \circ (\lambda_{11} - i\lambda_{12}) \right) =: \xi_{25,\mu} \circ \xi_{25}^\mu, \] (A.10)

\[ O_{26} = \frac{1}{4} \left( (\lambda_3 + \lambda_4) \circ (\lambda_{11} - i\lambda_{12}) + (\lambda_{11} + i\lambda_{12}) \circ (\lambda_3 - i\lambda_4) \right) - \frac{1}{8} \left( (\lambda_5 + i\lambda_6) \circ (\lambda_0 + 2\lambda_{14} - \lambda_{15}) + (\lambda_0 + 2\lambda_{14} - \lambda_{15}) \circ (\lambda_5 - i\lambda_6) \right) =: \xi_{26,\mu} \circ \xi_{26}^\mu, \] (A.11)

\[ O_{33} = \frac{1}{2} \left( (\lambda_6 \circ \lambda_6 - \lambda_5 \circ \lambda_5) - \frac{1}{8} \left( (\lambda_{13} + \lambda_{14} + \lambda_{15}) \circ (\lambda_{13} + \lambda_{14} + \lambda_{15}) - (\lambda_0 + \lambda_{13} - \lambda_{14}) \circ (\lambda_0 + \lambda_{13} - \lambda_{14}) \right) \right) =: \xi_{33,\mu} \circ \xi_{33}^\mu, \] (A.12)

\[ O_{34} = -\frac{1}{4} \left( (\lambda_9 - i\lambda_{10}) \circ (\lambda_3 - i\lambda_4) + (\lambda_3 + i\lambda_4) \circ (\lambda_9 + i\lambda_{10}) \right) + \frac{1}{4} \left( (\lambda_{11} - i\lambda_{12}) \circ (\lambda_1 - i\lambda_2) + (\lambda_1 + i\lambda_2) \circ (\lambda_{11} + i\lambda_{12}) \right) =: \xi_{34,\mu} \circ \xi_{34}^\mu, \] (A.13)

\[ O_{35} = -\frac{1}{4} \left( (\lambda_5 + i\lambda_6) \circ (\lambda_9 + i\lambda_{10}) + (\lambda_9 - i\lambda_{10}) \circ (\lambda_5 - i\lambda_6) \right) + \frac{1}{8} \left( (\lambda_1 + i\lambda_2) \circ (\lambda_0 - 2\lambda_{14} - \lambda_{15}) + (\lambda_0 - 2\lambda_{14} - \lambda_{15}) \circ (\lambda_1 - i\lambda_2) \right) =: \xi_{35,\mu} \circ \xi_{35}^\mu, \] (A.14)

\[ O_{36} = -\frac{1}{4} \left( (\lambda_5 + i\lambda_6) \circ (\lambda_{11} + i\lambda_{12}) + (\lambda_{11} - i\lambda_{12}) \circ (\lambda_5 - i\lambda_6) \right) + \frac{1}{8} \left( (\lambda_3 + i\lambda_4) \circ (\lambda_0 - 2\lambda_{14} - \lambda_{15}) + (\lambda_0 - 2\lambda_{14} - \lambda_{15}) \circ (\lambda_3 - i\lambda_4) \right) =: \xi_{36,\mu} \circ \xi_{36}^\mu, \] (A.15)

\[ O_{44} = \frac{1}{2} \left( (\lambda_8 \circ \lambda_8 - \lambda_7 \circ \lambda_7) - \frac{1}{8} \left( (\lambda_{13} + \lambda_{14} - \lambda_{15}) \circ (\lambda_{13} + \lambda_{14} - \lambda_{15}) - (\lambda_0 - \lambda_{13} + \lambda_{14}) \circ (\lambda_0 - \lambda_{13} + \lambda_{14}) \right) \right) =: \xi_{44,\mu} \circ \xi_{44}^\mu, \] (A.16)

\[ O_{45} = -\frac{1}{4} \left( (\lambda_7 - i\lambda_8) \circ (\lambda_9 - i\lambda_{10}) + (\lambda_9 + i\lambda_{10}) \circ (\lambda_7 + i\lambda_8) \right) + \frac{1}{8} \left( (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \circ (\lambda_{11} - i\lambda_{12}) + (\lambda_{11} + i\lambda_{12}) \circ (\lambda_0 - 2\lambda_{13} + \lambda_{15}) \right) =: \xi_{45,\mu} \circ \xi_{45}^\mu, \] (A.17)
Also for spin 3/2 it is intriguing that $O_{kk}$ contains \( \sigma^{(i,j)} \circ \sigma^{(i,j)} = \sum_{\mu=0}^{3} g_{\mu} \sigma^{(i,j)}_{\mu} \circ \sigma^{(i,j)}_{\mu} \) with \( g_{\mu} = (-1, 1, 0, 1) \), besides an \( \sigma^{(i,j)} \). This is again resembling the situation of the two-dimensional Hilbert space for spin 1/2.

References

[1] Vidal G 2000 J. Mod. Opt. 47 355
[2] Linden N and Popescu S 1998 Fortschr. Phys.-Prog. Phys. 46 567
[3] Grassl M, Rötteler M and Beth T 1998 Phys. Rev. A 58 1833
[4] Carteret H, Linden N, Popescu S and Sudbery A 1999 Found. Phys. 29 527
[5] Meyer D A and Wallach N R 2002 J. Math. Phys. 43 4273
[6] Leifer M, Linden N and Winter A 2004 Phys. Rev. A 69 052304
[7] Wallach N 2005 Acta Appl. Math. 86 203
[8] Cayley A 1846 J. Reine Angew. Math. 30 1
[9] Hilbert D 1890 Math. Ann. 36 473
[10] Wong A and Christensen N 2001 Phys. Rev. A 63 044301
[11] Luque J-G and Thibon J-Y 2003 Phys. Rev. A 67 042303
[12] Brylinski J-L 2002 Mathematics of Quantum Computation (London/Boca Raton, FL: Chapman and Hall/CRC Press) ch 1
[13] Jaeger G, Sergienko A V, Saleh B E A and Teich M C 2003 Phys. Rev. A 68 022318
[14] Jaeger G, Teodorescu-Frumosu M, Sergienko A V, Saleh B E A and Teich M C 2003 Math. Model. Phys. Eng. Sci. 5 273
[15] Teodorescu-Frumosu M and Jaeger G 2003 Phys. Rev. A 67 052305
[16] Briand E, Luque J-G, Thibon J-Y and Verstraete F 2004 J. Math. Phys. 45 4855
[17] Lévy P 2005 J. Phys. A: Math. Gen. 38 9075
[18] Lévy P 2006 Phys. Rev. D 74 024030
[19] Luque J-G and Thibon J-Y 2005 J. Phys. A: Math. Gen. 39 371
[20] Verstraete F, Dehaene J and De Moor B 2003 Phys. Rev. A 68 012103
[21] Osterloh A and Siewert J 2005 Phys. Rev. A 72 012337
[22] Osterloh A and Siewert J 2006 Int. J. Quant. Inf. 4 531
[23] Doković D Z and Osterloh A 2009 J. Math. Phys. 50 033509
[24] Osterloh A and Siewert J 2010 New J. Phys. 12 075025
[25] Johansson M, Ericsson M, Sjöqvist E and Osterloh A 2014 Phys. Rev. A 89 012320
[26] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[27] Uhlmann A 2000 Phys. Rev. A 62 032307
[28] Lohmayer R, Osterloh A, Siewert J and Uhlmann A 2006 Phys. Rev. Lett. 97 260502
[29] Osterloh A, Siewert J and Uhlmann A 2008 Phys. Rev. A 77 032310
[30] Viehmann O, Eltschka C and Siewert J 2012 Appl. Phys. B 106 533
[31] Eltschka C, Osterloh A, Siewert J and Uhlmann A 2008 New J. Phys. 10 043014
[32] Eltschka C and Siewert J 2012 Phys. Rev. Lett. 108 020502
[33] Siewert J and Eltschka C 2012 Phys. Rev. Lett. 108 230502
[34] Cereceda J L 2003 arXiv:quant-ph/0305043
[35] Gour G 2005 Phys. Rev. A 71 012318
[36] Eltschka C, Bastin T, Osterloh A and Siewert J 2012 Phys. Rev. A 85 022301