Abstract

Sample efficiency is critical in solving real-world reinforcement learning problems, where agent-environment interactions can be costly. Imitation learning from expert advice has proved to be an effective strategy for reducing the number of interactions required to train a policy. Online imitation learning, which interleaves policy evaluation and policy optimization, is a particularly effective technique with provable performance guarantees. In this work, we seek to further accelerate the convergence rate of online imitation learning, thereby making it more sample efficient.

We propose two model-based algorithms inspired by Follow-the-Leader (FTL) with prediction: MoBIL-VI based on solving variational inequalities and MoBIL-Prox based on stochastic first-order updates. These two methods leverage a model to predict future gradients to speed up policy learning. When the model oracle is learned online, these algorithms can provably accelerate the best known convergence rate up to an order. Our algorithms can be viewed as a generalization of stochastic Mirror-Prox (Juditsky et al., 2011), and admit a simple constructive FTL-style analysis of performance.

1 INTRODUCTION

Imitation learning (IL) has recently received attention for its ability to speed up policy learning when solving reinforcement learning problems (RL) [1, 2, 3, 4, 5, 6]. Unlike pure RL techniques, which rely on uniformed random exploration to locally improve a policy, IL leverages prior knowledge about a problem in terms of expert demonstrations. At a high level, this additional information provides policy learning with an informed search direction toward the expert policy.

The goal of IL is to quickly learn a policy that can perform at least as well as the expert policy. Because the expert policy may be suboptimal with respect to the RL problem of interest, performing IL is often used to provide a good warm start to the RL problem, so that the number of interactions with the environment can be minimized. Sample efficiency is especially critical when learning is deployed in applications like robotics, where every interaction incurs real-world costs.

By reducing IL to an online learning problem, online IL [2] provides a framework for convergence analysis and mitigates the covariate shift problem encountered in batch IL [7, 8]. In particular, under proper assumptions, the performance of a policy sequence updated by Follow-the-Leader (FTL) can converge on average to the performance of the expert policy [2]. Recently, it was shown that this rate is sufficient to make IL more efficient than solving an RL problem from scratch [9].

In this work, we further accelerate the convergence rate of online IL. Inspired by the observation of Cheng and Boots [10] that the online learning problem of IL is not truly adversarial, we propose two MOdel-Based IL (MoBIL) algorithms, MoBIL-VI and MoBIL-Prox, that can achieve a fast rate of convergence. Under the same assumptions of Ross et al. [2], these algorithms improve on-average convergence to $O(1/N^2)$, e.g., when a dynamics model is learned online, where $N$ is the number of iterations of policy update.

The improved speed of our algorithms is attributed to using a model oracle to predict the gradient of the next per-round cost in online learning. This model can be realized, e.g., using a simulator based on a (learned) dynamics model, or using past demonstrations. We first conceptually show that this idea can be realized as a variational inequality problem in MoBIL-VI. Next, we propose a practical first-order stochastic algorithm MoBIL-Prox, which alternates between the steps of taking the true gradient and of taking the model gradient. MoBIL-Prox is a generalization of stochastic Mirror-Prox proposed by Juditsky et al. [11] to the
case where the problem is weighted and the vector field is unknown but learned online. In theory, we show that having a weighting scheme is pivotal to speeding up convergence, and this generalization is made possible by a new constructive FTL-style regret analysis, which greatly simplifies the original algebraic proof [11]. The performance of MoBiL-Prox is also empirically validated in simulation.

2 PRELIMINARIES

2.1 Problem Setup: RL and IL

Let $\mathcal{S}$ and $\mathcal{A}$ be the state and the action spaces, respectively. The objective of RL is to search for a stationary policy $\pi$ inside a policy class $\Pi$ with good performance. This can be characterized by the stochastic optimization problem with expected cost\(^1\) $J(\pi)$ defined below:

$$\min_{\pi \in \Pi} J(\pi), \quad J(\pi) := \mathbb{E}_{(s,t) \sim d_\pi} [c_l(s,a)], \quad (1)$$

in which $s \in \mathcal{S}$, $a \in \mathcal{A}$, $c_l$ is the instantaneous cost at time $t$, $d_\pi$ is a generalized stationary distribution induced by executing policy $\pi$, and $\pi_s$ is the distribution of action $a$ given state $s$ of $\pi$. The policies here are assumed to be parametric. To make the writing compact, we will abuse the notation $\pi$ to also denote its parameter, and assume $\Pi$ is a compact convex subset of parameters in some normed space with norm $\| \cdot \|$. Based on the abstracted distribution $d_\pi$, the formulation in (1) subsumes multiple discrete-time RL problems. For example, a $\gamma$-discounted infinite-horizon problem can be considered by setting $c_l = c$ as a time-invariant cost and defining the joint distribution $d_\pi(s,t) = (1 - \gamma)^t d_\pi(s)$, in which $d_\pi(s)$ denotes the probability (density) of state $s$ at time $t$ under policy $\pi$. Similarly, a $T$-horizon RL problem can be considered by setting $d_\pi(s,t) = \frac{1}{T} d_\pi(s)$. Note that while we use the notation $\mathbb{E}_{a \sim \pi_s}$, the policy is allowed to be deterministic; in this case, the notation means evaluation. For notational compactness, we will often omit the random variable inside the expectation (e.g. we shorten (1) to $\mathbb{E}_{d_\pi} \mathbb{E}_\pi [c_l]$). In addition, we denote $Q_{\pi,t}$ as the Q-function\(^2\) at time $t$ with respect to $\pi$.

In this paper, we consider IL, which is an indirect approach to solving the RL problem. We assume there is a black-box oracle $\pi^*$, called the expert policy, from which demonstration $a^* \sim \pi^*$ can be queried for any state $s \in \mathcal{S}$. To satisfy the querying requirement, usually the expert policy is an algorithm; for example, it can represent a planning algorithm which solves a simplified version of (1), or some engineered, hard-coded policy (see e.g. [12]).

The purpose of incorporating the expert policy into solving (1) is to quickly obtain a policy $\pi$ that has reasonable performance. Toward this end, we consider solving a surrogate problem of (1),

$$\min_{\pi \in \Pi} \mathbb{E}_{(s,t) \sim d_\pi} [D(\pi^* || \pi)], \quad (2)$$

where $D$ is a function that measures the difference between two distributions over actions (e.g. KL divergence; see Appendix B). Importantly, the objective in (2) has the property that $D(\pi||\pi^*) = 0$ and there is constant $C_{\pi^*} \geq 0$ such that $\forall t \in \mathbb{N}, s \in \mathcal{S}, \pi \in \Pi$, it satisfies $\mathbb{E}_{a \sim \pi} [Q_{\pi^*,t}(s,a)] - \mathbb{E}_{a \sim \pi} [Q_{\pi^*,t}(s,a^*)] \leq C_{\pi^*} D(\pi||\pi^*_s)$, in which $\mathbb{N}$ denotes the set of natural numbers. By the Performance Difference Lemma [13], it can be shown that the inequality above implies [10],

$$J(\pi) - J(\pi^*) \leq C_{\pi^*} \mathbb{E}_{d_\pi} [D(\pi^* || \pi)]. \quad (3)$$

Therefore, solving (2) can lead to a policy that performs similarly to the expert policy $\pi^*$.

2.2 Imitation Learning as Online Learning

The surrogate problem in (2) is more structured than the original RL problem in (1). In particular, when the distance-like function $D$ is given, and we know that $D(\pi^* || \pi)$ is close to zero when $\pi$ is close to $\pi^*$. On the contrary, $\mathbb{E}_{a \sim \pi} [c_l(s,a)]$ in (1) generally can still be large, even if $\pi$ is a good policy (since it also depends on the state). This normalization property is crucial for the reduction from IL to online learning [10].

The reduction is based on observing that, with the normalization property, the expressiveness of the policy class $\Pi$ can be described with a constant $\epsilon_1$ defined as,

$$\epsilon_1 \geq \max_{\pi_n \in \Pi} \min_{\pi \in \Pi} \frac{1}{N} \sum_{n=1}^N \mathbb{E}_{d_{\pi_n}} [D(\pi^* || \pi)], \quad (4)$$

for all $N \in \mathbb{N}$, which measures the average difference between $\Pi$ and $\pi^*$ with respect to $D$ and the state distributions visited by a worst possible policy sequence. Ross et al. [2] make use of this property and reduce (2) into an online learning problem by distinguishing the influence of $\pi$ on $d_\pi$ and on $D(\pi^* || \pi)$ in (2). To make this transparent, we define a bivariate function

$$F(\pi', \pi) := \mathbb{E}_{d_{\pi'}} [D(\pi^* || \pi)]. \quad (5)$$

Using this bivariate function $F$, the online learning setup can be described as follows: in round $n$, the learner applies a policy $\pi_n \in \Pi$ and then the environment reveals a per-round cost

$$f_n(\pi) := F(\pi_n, \pi) = \mathbb{E}_{d_{\pi_n}} [D(\pi^* || \pi)]. \quad (6)$$
Ross et al. [2] show that if the sequence \( \{\pi_n\} \) is selected by a no-regret algorithm, then it will have good performance in terms of (2). For example, DAGGER updates the policy by FTL, \( \pi_{n+1} = \arg \min_{\pi \in \Pi} f_{1:n}(\pi) \) and has the following guarantee (cf. [10]), where we define the shorthand \( f_{1:n} = \sum_{m=1}^{n} f_m \).

**Theorem 2.1.** Let \( \mu_f > 0 \). If each \( f_n \) is \( \mu_f \)-strongly convex and \( \|\nabla f_n(\pi)\| \leq G, \forall \pi \in \Pi \), then DAGGER has performance on average satisfying

\[
\frac{1}{N} \sum_{n=1}^{N} J(\pi_n) \leq J(\pi^*) + C_{\pi^*} \left( \frac{G^2}{2\mu_f} \ln \frac{N+1}{N} + \epsilon_N \right). \tag{7}
\]

First-order variants of DAGGER based on Follow-the-Regularized-Leader (FTRL) have also been proposed by Sun et al. [5] and Cheng et al. [9], which have the same performance but only require taking a stochastic gradient step in each iteration without keeping all the previous cost functions (i.e. data) as in the original FTL formulation. The bound in Theorem 2.1 also applies to the expected performance of a policy randomly picked out of the sequence \( \{\pi_n\}_{n=1}^{N} \), although it does not necessarily translate into the performance of the last policy \( \pi_{N+1} \) [10].

### 3 ACCELERATING IL WITH PREDICTIVE MODELS

The reduction-based approach to solving IL has demonstrated success in speeding up policy learning. However, because interactions with the environment are necessary to approximately evaluate the per-round cost, it is interesting to determine if the convergence rate of IL can be further improved. A faster convergence rate will be valuable in practical applications where data collection is expensive.

We answer this question affirmatively. We show that, by modeling the \( \nabla F \) the convergence rate of IL can potentially be improved by up to an order, where \( \nabla F \) denotes the derivative to the second argument. The improvement comes through leveraging the fact that the per-round cost \( f_n \) defined in (6) is not completely unknown or adversarial as it is assumed in the most general online learning setting. Because the same function \( F \) is used in (6) over different rounds, the online component actually comes from the reduction made by Ross et al. [2], which ignores information about how \( F \) changes with the left argument; in other words, it omits the variations of \( d_n \) when \( \pi \) changes [10]. Therefore, we argue that the original reduction proposed by Ross et al. [2], while allowing the use of (4) to characterize the performance, loses one critical piece of information present in the original RL problem: both the system dynamics and the expert are the same across different rounds of online learning.

We propose two model-based algorithms (MoBIL-VI and MoBIL-Prox) to accelerate IL. The first algorithm, MoBIL-VI, is conceptual in nature and updates policies by solving variational inequality (VI) problems [14]. This algorithm is used to illustrate how modeling \( \nabla F \) through a predictive model \( \nabla F \) can help to speed up IL, where \( F \) is a model bivariate function. The second algorithm, MoBIL-Prox, is a first-order method. It alternates between taking stochastic gradients by interacting with the environment and querying the model \( \nabla F \). We will prove that this simple yet practical approach has the same performance as the conceptual one: when \( \nabla F \) is learned online and \( \nabla F \) is realizable, e.g. both algorithms can converge in \( O \left( \frac{1}{\sqrt{N}} \right) \), in contrast to DAGGER’s \( O \left( \frac{\ln N}{N} \right) \) convergence. In addition, we show the convergence results of MoBIL under relaxed assumptions, e.g. allowing stochasticity, and provide several examples of constructing predictive models. (See Appendix A for a summary of notation.)

#### 3.1 Performance and Average Regret

Before presenting the two algorithms, we first summarize the core idea of the reduction from IL to online learning in a simple lemma, which builds the foundation of our algorithms (proved in Appendix C.1).

**Lemma 3.1.** For arbitrary sequences \( \{\pi_n \in \Pi\}_{n=1}^{N} \) and \( \{w_n > 0\}_{n=1}^{N} \), it holds that

\[
E \left[ \sum_{n=1}^{N} \frac{w_n \hat{f}(\pi_n)}{w_1:N} \right] \leq J(\pi^*) + C_{\pi^*} \left( \epsilon_{\Pi}^w + E \left[ \text{regret}^w(\Pi) \right] \right)
\]

where \( \hat{f}_n \) is an unbiased estimate of \( f_n \), regret\(^w\)(\(\Pi\)) := max_\pi E \sum_{n=1}^{N} w_n \hat{f}_n(\pi_n) - w_n f_n(\pi), \epsilon_{\Pi}^w \) is given in Definition 4.1, and the expectation is due to sampling \( f_n \).

In other words, the on-average performance convergence of an online IL algorithm is determined by the rate of the expected weighted average regret \( E [\text{regret}^w(\Pi)/w_1:N] \). For example, in DAGGER, the weighting is uniform and \( E [\text{regret}^w(\Pi)] \) is in \( O(\log N) \); by Lemma 3.1 this rate directly proves Theorem 2.1.

#### 3.2 Algorithms

From Lemma 3.1, we know that improving the regret bound implies a faster convergence of IL. This leads to the main idea of MoBIL-VI and MoBIL-Prox: use model information to approximately play Be-the-Leader (BTL) [15], i.e. \( \pi_{n+1} \approx \arg \min_{\pi \in \Pi} f_{1:n+1}(\pi) \). To understand why playing BTL can minimize the regret, we recall a classical regret bound of online learning.

While we only concern predicting the vector field \( \nabla F \), we adopt the notation \( \hat{F} \) to better build up the intuition, especially of MoBIL-VI; we will discuss other approximations that are not based on bivariate functions in Section 3.3.

We use notation \( x_n \) and \( l_n \) to distinguish general online learning problems from online IL problems.
Lemma 3.2 (Strong FTL Lemma [16]). For any sequence of decisions \(\{x_n \in X\}\) and loss functions \(\{l_n\}\), regret(\(X\)) \(\leq \sum_{n=1}^{N} l_n(x_n^*)\), where \(x_n^* = \arg\min_{x \in X} l_n(x)\), where \(X\) is the decision set.

Namely, if the decision \(\pi_{n+1}\) made in round \(n\) in IL is close to the best decision in round \(n+1\) after the new per-round cost \(f_{n+1}\) is revealed (which depends on \(\pi_{n+1}\)), then the regret will be small.

The two algorithms are summarized in Algorithm 1, which mainly differs in the policy update rule (line 5). Like DAGGER, they both learn the policy in an interactive manner. In round \(n\), both algorithms execute the current policy \(\pi_n\) in the real environment to collect data to define the per-round cost functions (line 3): \(f_n\) is an unbiased estimate of \(f_n\) in (6) for policy learning, and \(\bar{h}_n\) is an unbiased estimate of the per-round cost \(h_n\) for model learning. Given the current per-round costs, the two algorithms then update the model (line 4) and the policy (line 5) using the respective rules. Here we use the set \(F\), abstractly, to denote the family of predictive models to estimate \(\nabla F\), and \(\bar{h}_n\) is defined as an upper bound of the prediction error. For example, \(F\) can be a family of dynamics models that are used to simulate the predicted gradients, and \(\bar{h}_n\) is the empirical loss function used to train the dynamics models (e.g. the KL divergence of prediction).

### 3.2.1 A Conceptual Algorithm: MoBIL-VI

We first present our conceptual algorithm MoBIL-VI, which is simpler to explain. We assume that \(f_n\) and \(\bar{h}_n\) are given, as in Theorem 2.1. This assumption will be removed in MoBIL-PROX later. To realize the idea of BTL, in round \(n\), MoBIL-VI uses a newly learned predictive model \(\nabla F_{n+1}\) to estimate \(\nabla F\) in (5) and then updates the policy by solving the VI problem below: finding \(\pi_{n+1} \in \Pi\) such that \(\forall \pi' \in \Pi\),

\[
\langle \Phi_n(\pi_{n+1}), \pi' - \pi_{n+1} \rangle \geq 0, \tag{8}
\]

where the vector field \(\Phi_n\) is defined as

\[
\Phi_n(\pi) = \sum_{m=1}^{n} w_m \nabla f_m(\pi) + w_{n+1} \nabla F_{n+1}(\pi, \pi)
\]

Suppose \(\nabla F_{n+1}\) is the partial derivative of some bivariate function \(F_{n+1}\). If \(w_n = 1\), then the VI problem\(^6\) in (8) finds a fixed point \(\pi_{n+1}\) satisfying \(\pi_{n+1} = \arg\min_{\pi \in \Pi} \int_{1:n} (\pi) + \int_{n+1}^{n}(\pi_{n+1}, \pi)\). That is, if \(\hat{F}_{n+1} = F\) exactly, then \(\pi_{n+1}\) plays exactly BTL and by Lemma 3.2 the regret is non-positive. In general, we can show that, even with modeling errors, MoBIL-VI can still reach a faster convergence rate such as \(O\left(\frac{1}{n^2}\right)\), if a non-uniform weighting scheme is used, the model is updated online, and \(\nabla F\) is realizable within \(\tilde{F}\). The details will be presented in Section 4.2.

### 3.2.2 A Practical Algorithm: MoBIL-PROX

While the previous conceptual algorithm achieves a faster convergence, it requires solving a nontrivial VI problem in each iteration. In addition, it assumes \(f_n\) is given as a function and requires keeping all the past data to define \(f_{1:n}\). Here we relax these unrealistic assumptions and propose MoBIL-PROX. In round \(n\) of MoBIL-PROX, the policy is updated from \(\pi_n\) to \(\pi_{n+1}\) by taking two gradient steps:

\[
\begin{align*}
\hat{\pi}_{n+1} &= \arg\min_{\pi \in \Pi} \sum_{m=1}^{n} w_m \left(\langle \hat{g}_m, \pi \rangle + \hat{r}_m(\pi)\right), \\
\hat{\pi}_{n+1} &= \arg\min_{\pi \in \Pi} \sum_{m=1}^{n} w_{n+1} \left(\langle \hat{g}_{n+1}, \pi \rangle + \hat{r}_m(\pi)\right) \tag{9}
\end{align*}
\]

We define \(r_n\) as an \(\alpha_n\)-\(\mu_f\)-strongly convex function (with \(\alpha_n \in (0, 1]\); we recall \(\mu_f\) is the strongly convexity modulus of \(f_n\)) such that \(\pi_{n}\) is its global minimum and \(r_n(\pi_{n}) = 0\) (e.g. a Bregman divergence). And we define \(g_n\) and \(\hat{g}_{n+1}\) as estimates of \(\nabla f_n(\pi_n) = \nabla F(\pi_n, \pi_n)\) and \(\nabla F_{n+1}(\hat{\pi}_{n+1}, \hat{\pi}_{n+1})\), respectively. Here we only require \(g_n = \nabla f_n(\pi_{n})\) to be unbiased, whereas \(\hat{g}_{n+1}\) could be a biased estimate of \(\nabla F_{n+1}(\hat{\pi}_{n+1}, \hat{\pi}_{n+1})\).

MoBIL-PROX treats \(\hat{\pi}_{n+1}\), which plays BTL with \(g_n\) from the real environment, as a rough estimate of the next policy \(\pi_{n+1}\) and uses it to query an gradient estimate \(g_{n+1}\) from the model \(\nabla F_{n+1}\). Therefore, the learner’s decision \(\pi_{n+1}\) can approximately play BTL. If we compare the update rule of \(\pi_{n+1}\) and the VI problem in (8), we can see that MoBIL-PROX linearizes the problem and attempts to approximate \(\nabla F_{n+1}(\pi_{n+1}, \pi_{n+1})\) by \(g_{n+1}\). While the above approximation is crude, interestingly it is sufficient to speed up the convergence rate to be as fast as MoBIL-VI under mild assumptions, as shown later in Section 4.3.

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\[^{6}\text{Because }\Pi\text{ is compact, the VI problem in (8) has at least one solution [14]. If }f_n\text{ is strongly convex, the VI problem in line 6 of Algorithm 1 is strongly monotone for large enough }n\text{ and can be solved e.g. by basic projection method [14]. Therefore, for demonstration purpose, we assume the VI problem of MoBIL-VI can be exactly solved.}\]

\[^{7}\text{MoBIL-VI assumes }\hat{f}_n = f_n\text{ and }\hat{h}_n = h_n\]
3.3 Predictive Models

MoBIL uses $\nabla_2 \tilde{F}_{n+1}$ in the update rules (8) and (9) at round $n$ to predict the unseen gradient at round $n+1$ for speeding up policy learning. Ideally $\tilde{F}_{n+1}$ should approximate the unknown bivariate function $F$ so that $\nabla_2 F$ and $\nabla_2 \tilde{F}_{n+1}$ are close. This condition can be seen from (8) and (9), in which MoBIL concerns only $\nabla_2 \tilde{F}_{n+1}$ instead of $\tilde{F}_{n+1}$ directly. In other words, $\nabla_2 \tilde{F}_{n+1}$ is used in MoBIL as a first-order oracle, which leverages all the past information (up to the learner playing $\pi_n$ in the environment at round $n$) to predict the future gradient $\nabla_2 F_{n+1}(\pi_{n+1}, \pi_{n+1})$, which depends on the decision $\pi_{n+1}$ the learner is about to make. Hence, we call it a predictive model.

To make the idea concrete, we provide a few examples of these models. By definition of $F$ in (5), one way to construct the predictive model $\nabla_2 \tilde{F}_{n+1}$ is through a simulator with an (online learned) dynamics model, and define $\nabla_2 F_{n+1}$ as the simulated gradient (computed by querying the expert along the simulated trajectories visited by the learner). If the dynamics model is exact, then $\nabla_2 \tilde{F}_{n+1} = \nabla_2 F$. Note that a stochastic/biased estimate of $\nabla_2 \tilde{F}_{n+1}$ suffices to update the policies in MoBIL-Prox.

Another idea is to construct the predictive model through $\tilde{f}_n$ (the stochastic estimate of $f_n$) and indirectly define $\tilde{F}_{n+1}$ such that $\nabla_2 \tilde{F}_{n+1} = \nabla \tilde{f}_n$. This choice is possible, because the learner in IL collects samples from the environment, as opposed to, literally, gradients. Specifically, we can define $g_n = \nabla \tilde{f}_n(\pi_n)$ and $\hat{g}_{n+1} = \nabla \tilde{f}_n(\hat{\pi}_{n+1})$ in (9). The approximation error of setting $g_{n+1} = \nabla \tilde{f}_n(\hat{\pi}_{n+1})$ is determined by the convergence and the stability of the learner’s policy. If $\pi_n$ visits similar states as $\hat{\pi}_{n+1}$, then $\nabla \tilde{f}_n$ can approximate $\nabla_2 F$ well at $\hat{\pi}_{n+1}$. Note that this choice is different from using the previous gradient (i.e. $\hat{g}_{n+1} = g_n$) in optimistic mirror descent/FTL [17], which would have a larger approximation error due to additional linearization.

Finally, we note that while the concept of predictive models originates from estimating the partial derivatives $\nabla_2 F$, a predictive model does not necessarily have to be in the same form. A parameterized vector-valued function can also be directly learned to approximate $\nabla_2 F$. e.g., using a neural network and the sampled gradients $\{g_n\}$ in a supervised learning fashion.

4 THEORETICAL ANALYSIS

Now we prove that using predictive models in MoBIL can accelerate convergence, when proper conditions are met. Intuitively, MoBIL converges faster than the usual adversarial approach to IL (like DAgGER), when the predictive models have smaller errors than not predicting anything at all (i.e. setting $\hat{g}_{n+1} = 0$). In the following analyses, we will focus on bounding the expected weighted average regret, as it directly translates into the average performance bound by Lemma 3.1. We define, for $w_n = n^p$,

$$R(p) := \mathbb{E} \left[ \text{regret}^w(\Pi)/w_{1:N} \right] \quad (10)$$

Note that the results below assume that the predictive models are updated using FTL as outlined in Algorithm 1. This assumption applies, e.g., when a dynamics model is learned online in a simulator-oracle as discussed above. We provide full proofs in Appendix C and provide a summary of notation in Appendix A.

4.1 Assumptions

We first introduce several assumptions to more precisely characterize the online IL problem.

Predictive models Let $\tilde{F}$ be the class of predictive models. We assume these models are Lipschitz continuous in the following sense.

**Assumption 4.1.** There is $L \in [0, \infty)$ such that

$$\|\nabla_2 F(\pi, \pi) - \nabla_2 F(\pi', \pi')\|_\ast \leq L \|\pi - \pi'\|, \forall F \in \tilde{F} \text{ and } \forall \pi, \pi' \in \Pi.$$  

Per-round costs The per-round cost $f_n$ for policy learning is given in (6), and we define $h_n(\hat{F})$ as an upper bound of $\|\nabla_2 F(\pi_n, \pi_n) - \nabla_2 F(\pi_n, \pi_n)\|_\ast^2$ (see e.g. Appendix D). We make structural assumptions on $f_n$ and $h_n$, similar to the ones made by Ross et al. [2] (cf. Theorem 2.1).

**Assumption 4.2.** Let $\mu_f, \mu_h > 0$. With probability 1, $\tilde{f}_n$ is $\mu_f$-strongly convex, and $\|\nabla \tilde{f}_n(\pi)\|_\ast \leq G_{f}, \forall \pi \in \Pi$; $\tilde{h}_n$ is $\mu_h$-strongly convex, and $\|\nabla \tilde{h}_n(\tilde{F})\|_\ast \leq G_{h}, \forall \tilde{F} \in \tilde{F}$. By definition, these properties extend to $f_n$ and $h_n$. We note they can be relaxed to solely convexity and our algorithms still improve the best known convergence rate (see Table 1 and Appendix E).

Expressiveness of hypothesis classes We introduce two constants, $\epsilon_{\Pi}^w$ and $\epsilon_{\tilde{F}}^w$, to characterize the policy class $\Pi$ and model class $\tilde{F}$, which generalize the idea of (4) to stochastic and general weighting settings. When $\tilde{f}_n = f_n$ and $\theta_n$ is constant, Definition 4.1 agrees with (4). Similarly, we see that if $\pi^* \in \Pi$ and $\tilde{F} \in \tilde{F}$, then $\epsilon_{\Pi}^w$ and $\epsilon_{\tilde{F}}^w$ are zero.

\[8\]The rates here assume $\sigma_\ast, \sigma_\circ, \epsilon_{\Pi}^w = 0$. In general, the rate of MoBIL-Prox becomes the improved rate in the table plus the ordinary rate multiplied by $C = \sigma_\ast^2 + \sigma_\circ^2 + \epsilon_{\Pi}^w$. For example, when $\tilde{f}$ is convex and $\tilde{h}$ is strongly convex, MoBIL-Prox converges in $O(1/N + C/\sqrt{N})$, whereas DAgGER converges in $O(G_{f}^2/\sqrt{N})$. 

Definition 4.1. A policy class $\Pi$ is $\epsilon^w_\Pi$-close to $\pi^*$, if for all $N \in \mathbb{N}$ and weight sequence $\{\theta_n > 0\}_{n=1}^N$ with $\theta_{1:N} = 1$, $\mathbb{E}\left[\max\{\pi \in \Pi\} \min\{\pi \in \Pi\} \sum_{n=1}^N \theta_n f_n(\pi)\right] \leq \epsilon^w_\Pi$. Similarly, a model class $\hat{\mathcal{F}}$ is $\epsilon^w_{\hat{\mathcal{F}}}$-close to $\mathcal{F}$, if $\mathbb{E}\left[\max\{\pi \in \Pi\} \min_{\hat{F} \in \hat{\mathcal{F}}} \sum_{n=1}^N \theta_n \hat{h}_n(\hat{F})\right] \leq \epsilon^w_{\hat{\mathcal{F}}}$. The expectations above are due to sampling $\hat{f}_n$ and $\hat{h}_n$.

4.2 Performance of MoBIL-VI

Here we show the performance for MoBIL-VI when there is prediction error in $\nabla_2 \hat{F}_n$. The main idea is to treat MoBIL-VI as online learning with prediction [17] and take $\hat{F}_{n+1}(\pi_{n+1} \cdot)$ obtained after solving the VI problem (8) as an estimate of $f_{n+1}$.

Proposition 4.1. For MoBIL-VI with $p = 0$, $\mathcal{R}(0) \leq \frac{G^2_f}{2\mu_f \rho} \frac{1}{N} + \frac{c^\mathcal{F}}{p \mu_f} \ln N + 1$.

By Lemma 3.1, this means that if the model class is expressive enough (i.e. $c^\mathcal{F} = 0$), then by adapting the model online with FTL, we can improve the original convergence rate in $O(\ln N/N)$ of Ross et al. [2] to $O(1/N)$. While removing the $\ln N$ factor does not seem like much, we will show that running MoBIL-VI can improve the convergence rate to $O(1/N^2)$, when a non-uniform weighting is adopted.

Theorem 4.1. For MoBIL-VI with $p > 1$, $\mathcal{R}(p) \leq C_p \left(\frac{G^2_f}{2(p-1) \mu_f} \frac{1}{N} + \frac{c^\mathcal{F}}{p \mu_f} \right)$, where $C_p = \frac{p+1}{p} \frac{1}{\rho/N}$.

The key is that regret$^w(\Pi)$ can be upper bounded by the regret of the online learning for models, which has per-round cost $\frac{w_n}{n} h_n$. Therefore, if $c^\mathcal{F} = 0$, randomly picking a policy out of $\{\pi_n\}_{n=1}^N$ proportional to weights $\{w_n\}_{n=1}^N$ has expected convergence in $O\left(\frac{1}{N^2}\right)$ if $p > 1$.\footnote{If $p = 1$, it converges in $O\left(\frac{\ln N}{\sqrt{N}}\right)$; if $p \in [0,1)$, it converges in $O\left(\frac{1}{N^{1/2}}\right)$. See Appendix C.2.}

4.3 Performance of MoBIL-PROX

As MoBIL-PROX uses gradient estimates, we additionally define two constants $\sigma_f$ and $\sigma_g$ to characterize the estimation error, where $\sigma_g$ also entails potential bias.

Assumption 4.3. $\mathbb{E}||g_n - \nabla F(\pi_n, \pi_n)||^2_2 \leq \sigma^2_g$ and $\mathbb{E}||g_n - \nabla_2 \hat{F}_n(\pi_n, \hat{\pi}_n)||^2_2 \leq \sigma^2_g$.

We show this simple first-order algorithm achieves similar performance to MoBIL-VI. Toward this end, we introduce a stronger lemma than Lemma 3.2.

Lemma 4.1 (Stronger FTL Lemma). Let $x^*_n \in \arg\min_{x \in \mathcal{X}} l_{1:n}(x)$. For any sequence of decisions $\{x_n\}$ and losses $\{l_n\}$, regret($\mathcal{X}$) = $\sum_{n=1}^N l_{1:n}(x_n) - l_{1:n}(x^*_n) - \Delta_n$, where $\Delta_{n+1} := l_{1:n}(x_{n+1}) - l_{1:n}(x^*_n) \geq 0$.

The additional $-\Delta_n$ term in Lemma 4.1 is pivotal to prove the performance of MoBIL-PROX.

Theorem 2.4. For MoBIL-PROX with $p > 1$ and $\alpha_n = \alpha \in (0,1]$, it satisfies

$$\mathcal{R}(p) \leq \frac{(p+2)^2 \mathcal{F}}{\alpha \mu_f} \left(\frac{\sigma^2_g \mu_f - 1}{N} + \frac{2 \sigma^2_g + \sigma^2_f + \sigma^w_g}{N^2}\right) + \frac{(p+1)\mu_f}{Np+1},$$

where $\mu_f = O(1)$ and $\text{ncell} \leq \frac{2 \mathcal{F}(p+1) \mathcal{L} \mathcal{G} \mathcal{L}}{\alpha \mu_f}$.

Proof sketch. Here we give a proof sketch in big-O notation (see Appendix C.3 for the details). To bound $\mathcal{R}(p)$, recall the definition $\text{regret}^w(\Pi) = \sum_{n=1}^N w_n f_n(\pi_n) - \min_{\pi \in \Pi} \sum_{n=1}^N w_n f_n(\pi)$. Now define $\hat{f}_n(\pi) := \langle g_n, \pi \rangle + \rho_n(\pi)$. Since $\hat{f}_n$ is $\mu_f$-strongly convex, $\rho_n$ is $\alpha \mu_f$-strongly convex, and $r(\pi) = 0$, we know that $\hat{f}_n$ satisfies that $f_n(\pi_n) - \hat{f}_n(\pi) \leq \hat{f}_n(\pi_n) - \hat{f}_n(\pi), \forall \pi \in \Pi$. This implies $\mathcal{R}(p) \leq \mathbb{E}[\text{regret}^w(\Pi)/w_{1:n} \cdot \text{ncell}]$, where regret$^w(\Pi) := \sum_{n=1}^N w_n \hat{f}_n(\pi_n) - \min_{\pi \in \Pi} \sum_{n=1}^N w_n \hat{f}_n(\pi_n)$.

The following lemma upper bounds regret$^w(\Pi)$ by using Stronger FTL lemma (Lemma 4.1).

Lemma 4.2. regret$^w(\Pi) \leq \frac{p+1}{\alpha \mu_f} \sum_{n=1}^N n^{p-1} ||g_n - \hat{g}_n||^2_2 - \frac{\alpha \mu_f}{p} \sum_{n=1}^N (n-1)^{p-1} ||\pi_n - \hat{\pi}_n||^2_2$.

Since the second term in Lemma 4.2 is negative, we just need to upper bound the expectation of the first item. Using the triangle inequality, we bound the model’s prediction error of the next per-round cost.

Lemma 3.4. $\mathbb{E}[||g_n - \hat{g}_n||^2_2] \leq 4(\sigma^2_g + \sigma^2_f + L^2 \mathbb{E}[||\pi_n - \hat{\pi}_n||^2_2] + \mathbb{E}[\hat{h}_n(\hat{F}_n)])$.

With Lemma 4.3 and Lemma 4.2, it is now clear that $\mathbb{E}[\text{regret}^w(\Pi)] \leq \mathbb{E}[\sum_{n=1}^N \rho_n ||\pi_n - \hat{\pi}_n||^2_2] + O(N \mathbb{P}(\sigma^2_g + \sigma^2_f + O(\mathbb{E}[\sum_{n=1}^N n^{p-1} \hat{h}_n(\hat{F}_n)]))$, where $\rho_n = O(n^{p-1} - n^{p+1})$. When $n$ is large enough, $\rho_n \leq 0$, and hence the first term is $O(1)$. For the third term, because the model is learned online using, e.g., FTL with strongly convex cost $n^{p-1} h_n$, we can show that $\mathbb{E}[\sum_{n=1}^N n^{p-1} \hat{h}_n(\hat{F}_n)] = O(Nn^{p-1} + Np^2 \mathcal{F})$. Thus, $\mathbb{E}[\text{regret}^w(\Pi)] \leq O(1 + N^{p-1} + \ldots)$.


\( (\epsilon_y^2 + \sigma_g^2 + \sigma^2_z) N^p \). Substituting this bound into \( R(p) \leq \mathbb{E} \left( \text{regret} \right) / w_{1:N} \) and using that the fact \( w_{1:N} = \Omega(N^{p+1}) \) proves the theorem.

The main assumption in Theorem 4.2 is that \( \nabla_2 \hat{F} \) is \( L \)-Lipschitz continuous (Assumption 4.1). It does not depend on the continuity of \( \nabla_2 F \). Therefore, this condition is practical as we are free to choose \( \hat{F} \). Compared with Theorem 4.1, Theorem 4.2 considers the inexactness of \( f_n \) and \( h_n \) explicitly; hence the additional term due to \( \sigma_g^2 \) and \( \sigma_z^2 \). Under the same assumption of MoBIL-VI that \( f_n \) and \( h_n \) are directly available, we can actually show that the simple MoBIL-PROX has the same performance as MoBIL-VI, which is a corollary of Theorem 4.2.

**Corollary 4.1.** If \( \hat{f}_n = f_n \) and \( \hat{h}_n = h_n \), for MoBIL-PROX with \( p > 1 \), \( R(p) \leq \Omega \left( \frac{\sigma^2_g + \sigma^2_z}{\sqrt{n}} \right) \).

The proof of Theorem 4.1 and 4.2 are based on assuming the predictive models are updated by FTL (see Appendix D for a specific bound when online learned dynamics models are used as a simulator). However, we note that these results are essentially based on the property that model learning also has no regret; therefore, the FTL update rule (line 4) can be replaced by a no-regret first-order method without changing the result. This would make the algorithm even simpler to implement. The convergence of other types of predictive models (like using the previous cost function discussed in Section 3.3) can also be analyzed following the major steps in the proof of Theorem 4.2, leading to a performance bound in terms of prediction errors. Finally, it is interesting to note that the accelerated convergence is made possible when model learning puts more weight on costs in later rounds (because \( p > 1 \)).

### 4.4 Comparison

We compare the performance of MoBIL in Theorem 4.2 with that of DAgGER in Theorem 2.1 in terms of the constant on the \( 1/\epsilon_N \) factor. MoBIL has a constant in \( O(\sigma_g^2 + \sigma_z^2 + \epsilon_y^2) \), whereas DAgGER has a constant in \( G_2^2 = O(G^2 + \sigma^2_g) \), where we recall \( G_f \) and \( G \) are upper bounds of \( \| \nabla \hat{f}_n(\pi) \|_* \) and \( \| \nabla f_n(\pi) \|_* \), respectively. Therefore, in general, MoBIL-PROX has a better upper bound than DAgGER when the model class is expressive (i.e. \( \epsilon_y \approx 0 \)), because \( \sigma_g^2 \) (the variance of the sampled gradients) can be made small as we are free to design the model. Note that, however, the improvement of MoBIL may be smaller when the problem is noisy, such that the large \( \sigma_g^2 \) becomes the dominant term.

An interesting property that arises from Theorems 4.1 and 4.2 is that the convergence of MoBIL is not biased by using an imperfect model (i.e. \( \epsilon_y^2 > 0 \)). This is shown in the term \( \epsilon_y^2 / N \). In other words, in the worst case of using an extremely wrong predictive model, MoBIL would just converge more slowly but still to the performance of the expert policy.

MoBIL-PROX is closely related to stochastic Mirror-Prox [18, 11]. In particular, when the exact model is known (i.e. \( \nabla_2 F_n = \nabla_2 F \) and MoBIL-PROX is set to convex-mode (i.e. \( r_n = 0 \) for \( n > 1 \), and \( w_n = 1/\sqrt{n} \); see Appendix E), then MoBIL-PROX gives the same update rule as stochastic Mirror-Prox with step size \( O(1/\sqrt{n}) \) (See Appendix F for a thorough discussion). Therefore, MoBIL-PROX can be viewed as a generalization of Mirror-Prox: 1) it allows non-uniform weights; and 2) it allows the vector field \( \nabla_2 F \) to be estimated online by alternately taking stochastic gradients and predicted gradients. The design of MoBIL-PROX is made possible by our Stronger FTL lemma (Lemma 4.1), which greatly simplifies the original algebraic proof in [18, 11]. Using Lemma 4.1 reveals more closely the interactions between model updates and policy updates. In addition, it more clearly shows the effect of non-uniform weighting, which is essential to achieving \( O(1/\sqrt{n}) \) convergence. To the best of our knowledge, even the analysis of the original (stochastic) Mirror-Prox from the FTL perspective is new.

### 5 EXPERIMENTS

We experimented with MoBIL-PROX in simulation to study how weights \( w_n = n^p \) and the choice of model oracles affect the learning. We used two weight schedules: \( p = 0 \) as baseline, and \( p = 2 \) suggested by Theorem 4.2. And we considered several predictive models: (a) a simulator with the true dynamics (b) a simulator with online-learned dynamics (c) the last cost function (i.e. \( g_n+1 = \nabla \hat{f}_n(\hat{\pi}_{n+1}) \)) (d) no model (i.e. \( \hat{\pi}_{n+1} = 0 \); in this case MoBIL-PROX reduces to the first-order version of DAgGER [9], which is considered as a baseline here).

#### 5.1 Setup and Results

Two robot control tasks (CartPole and Reacher3D) powered by the DART physics engine [19] were used as the task environments. The learner was either a linear policy or a small neural network. For each IL problem, an expert policy that shares the same architecture as the learner was used, which was trained using policy gradients. While sharing the same architecture is not required in IL, here we adopted this constraint to remove the bias due to the mismatch between policy class and the expert policy to clarify the experimental results. For MoBIL-PROX, we set \( r_n(\pi) = \frac{C_n}{\sqrt{2}} \frac{\| \pi - \pi_n \|^2}{\eta_n} \) and set \( \alpha_n \), such that \( \sum w_n \alpha_n \mu_f = c \), where \( c = 0.1 \) and \( \eta_n \) was adaptive to the norm of the prediction error. This leads to an effective learning
Figure 1: Experimental results of MoBIL-PROX with neural network (1st row) and linear policies (2nd row). The shaded regions represent 0.5 standard deviation.

5.2 Discussions

We observe that, when $p = 0$, having model information does not improve the performance much over standard online IL (i.e., no model), as suggested in Proposition 4.1. By contrast, when $p = 2$ (as suggested by Theorem 4.2), MoBIL-PROX improves the convergence and performs better than not using models.\footnote{We note that the curves between $p = 0$ and $p = 2$ are not directly comparable; we should only compare methods within the same $p$ setting as the optimal step size varies with $p$. The multiplier on the step size was chosen such that MoBIL-PROX performs similarly in both settings.}

It is interesting to see that this trend also applies to neural network policies.

From Figure 1, we can also study how the choice of predictive models affects the convergence. As suggested in Theorem 4.2, MoBIL-PROX improves the convergence only when the model makes non-trivial predictions. If the model is very incorrect, then MoBIL-PROX can be slower. This can be seen from the performance of MoBIL-PROX with online learned dynamics models. In the low-dimensional case of CartPole, the simple neural network predicts the dynamics well, and MoBIL-PROX with the learned dynamics performs similarly as MoBIL-PROX with the true dynamics. However, in the high-dimensional Reacher3D problem, the learned dynamics model generalizes less well, creating a performance gap between MoBIL-PROX using the true dynamics and that using the learned dynamics. We note that MoBIL-PROX would still converge at the end despite the model error. Finally, we find that the performance of MoBIL with the last-cost predictive model is often similar to MoBIL-PROX with the simulated gradients computed through the true dynamics.

6 CONCLUSION

We propose two novel model-based IL algorithms MoBIL-PROX and MoBIL-VI with strong theoretical properties: they are provably up-to-and-order faster than the state-of-the-art IL algorithms and have unbiased performance even when using imperfect predictive models. Although we prove the performance under convexity assumptions, we empirically find that MoBIL-PROX improves the performance even when using neural networks. In general, MoBIL accelerates policy learning when having access to an predictive model that can predict future gradients non-trivially. While the focus of the current paper is theoretical in nature, the design of MoBIL leads to several interesting questions that are important to reliable application of MoBIL-PROX in practice, such as end-to-end learning of predictive models and designing adaptive regularizations for MoBIL-PROX.

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A Notation

Table 2: Summary of the symbols used in the paper

| Symbol | Definition |
|--------|------------|
| $N$ | the total number of rounds in online learning |
| $J(\pi)$ | the average accumulated cost, $\mathbb{E}_{d_x, x}[c_t]$ of RL in (1) |
| $d_\pi$ | the generalized stationary state distribution |
| $D(q||p)$ | the difference between distributions $p$ and $q$ |
| $\pi^*$ | the expert policy |
| $\Pi$ | the hypothesis class of policies |
| $\pi_n$ | the policy run in the environment at the $n$th online learning iteration |
| $\tilde{F}$ | the hypothesis class of models (elements denoted as $\tilde{F}$) |
| $F_n$ | the model used at the $n-1$ iteration to predict the future gradient of the $n$th iteration |
| $\epsilon_{\tilde{F}}$ | the policy class complexity (Definition 4.1) |
| $\epsilon^{\Pi}_n$ | the model class complexity (Definition 4.1) |
| $F(\pi', \pi)$ | the bivariate function $E_{d_x}[D(\pi' || \pi)]$ in (5) |
| $f_n(\pi)$ | $F(\pi_n, \pi)$ in (6) |
| $\hat{f}_n(\pi)$ | an unbiased estimate of $f_n(\pi)$ |
| $h_n(F)$ | an upper bound of $\|\nabla F(\pi_n, \pi_n) - \nabla \tilde{F}(\pi_n, \pi_n)\|^2$ |
| $\hat{h}_n(\tilde{F})$ | an unbiased estimate of $h_n(\tilde{F})$ |
| $\mu_f$ | the modulus of strongly convexity of $\hat{f}_n$ (Assumption 4.2) |
| $G_f$ | an upper bound of $\|\nabla f_n\|_*$ (Assumption 4.2) |
| $G$ | an upper bound of $\|\nabla f_n\|_*$ (Theorem 2.1) |
| $\mu_h$ | modulus of strongly convexity of $\hat{h}_n$ (Assumption 4.2) |
| $G_h$ | an upper bound of $\|\nabla \hat{h}_n\|_*$ (Assumption 4.2) |
| $L$ | the Lipschitz constant such that $\|\nabla F(\pi, \pi) - \nabla \tilde{F}(\pi', \pi')\|_* \leq L\|\pi - \pi'\|$ (Assumption 4.1) |
| $\mathbb{R}(p)$ | the expected weighted average regret, $\mathbb{E}_{w_1:N}[\text{regret}^w(\Pi)]$ in (10) |
| $\{w_n\}$ | the sequence of weights used to define $\text{regret}^w$; we set $w_n = n_p$ |

B Imitation Learning Objective Function and Choice of Distance

Here we provide a short introduction to the objective function of IL in (2). The idea of IL is based on the Performance Difference Lemma, whose proof can be found, e.g. in [13].

**Lemma B.1 (Performance Difference Lemma).** Let $\pi$ and $\pi'$ be two policies and $A_{\pi', t}(s, a) = Q_{\pi', t}(s, a) - V_{\pi', t}(s)$ be the (dis)advantage function with respect to running $\pi'$. Then it holds that

$$J(\pi) = J(\pi') + \mathbb{E}_{d_x, \pi}[A_{\pi', t}].$$  \hspace{1cm} (B.1)

Using Lemma B.1, we can relate the performance of the learner’s policy and the expert policy as

$$J(\pi) = J(\pi^*) + \mathbb{E}_{d_x, \pi}[A_{\pi^*, t}]$$

$$= J(\pi^*) + \mathbb{E}_{d_x} [(\mathbb{E}_\pi - \mathbb{E}_{\pi^*})[Q_{\pi^*, t}]]$$

where the last equality uses the definition of $A_{\pi^*, t}$ and that $V_{\pi, t} = \mathbb{E}_\pi [Q_{\pi, t}]$. Therefore, if the inequality below holds

$$\mathbb{E}_{a \sim \pi^*}[Q_{\pi^*, t}(s, a)] - \mathbb{E}_{a \sim \pi^*_t}[Q_{\pi^*, t}(s, a^*)] \leq C_{\pi^*} D(\pi^* || \pi), \quad \forall t \in \mathbb{N}, s \in S, \pi \in \Pi$$

then minimizing (2) would minimize the performance difference between the policies as in (3)

$$J(\pi) - J(\pi^*) \leq C_{\pi^*} \mathbb{E}_{d_x} [D(\pi^* || \pi)].$$
Intuitively, we can set \( D(\pi^* || \pi) = \mathbb{E}_\pi[A_{\pi^*, t}] \) and (3) becomes an equality with \( C_{\pi^*} = 1 \). This corresponds to the objective function used in AGGRESSVATE by Ross and Bagnell [3]. However, this choice requires \( A_{\pi^*, t} \) to be given as a function or to be estimated online, which may be inconvenient or complicated in some settings.

Therefore, \( D \) is usually used to construct a strict upper bound in (3). The choice of \( D \) and \( C_{\pi^*} \) is usually derived from some statistical distances, and it depends on the topology of the action space \( \mathcal{A} \) and the policy class \( \Pi \). For discrete action spaces, \( D \) can be selected as a convex upper bound of the total variational distance between \( \pi^* \) and \( \pi \) as a bound on the range of \( \sum_{t=1}^N \mathbb{E}_t \psi_t \pi^*_t \) (e.g., a hinge loss used by [2]). For continuous action spaces, \( D \) can be selected as an upper bound of the Wasserstein distance between \( \pi^* \) and \( \pi \) and \( C_{\pi^*} \) is the Lipschitz constant of \( \pi^*_t \) with respect to action [12]. More generally, for stochastic policies, we can simply set \( D \) to Kullback-Leibler (KL) divergence (e.g. by [9]), because it upper bounds both total variational distance and Wasserstein distance.

The direction of KL divergence, i.e. \( \text{KL}(\pi || \pi^*) \) is available, \( \text{KL}(\pi || \pi^*) \) is a function or to be estimated online, which may be inconvenient or complicated in some settings.

### C Missing Proofs

#### C.1 Proof of Section 3.1

**Lemma 3.1.** For arbitrary sequences \( \{\pi_n \in \Pi\}_{n=1}^N \) and \( \{w_n > 0\}_{n=1}^N \), it holds that

\[
\mathbb{E} \left[ \sum_{n=1}^N w_n (J(\pi_n) - J(\pi^*)) \right] \leq C_{\pi^*} \mathbb{E} \left[ \sum_{n=1}^N w_n f_n(\pi_n) \right] = C_{\pi^*} \mathbb{E} \left[ \sum_{n=1}^N w_n \hat{f}_n(\pi_n) \right],
\]

where \( \hat{f}_n \) is an unbiased estimate of \( f_n \), \( \text{regret}(\Pi) := \max_{\pi \in \Pi} \sum_{n=1}^N w_n \hat{f}_n(\pi) - w_n f_n(\pi) \), \( c^w_\pi \) is given in Definition 4.1, and the expectation is due to sampling \( f_n \).

**Proof.** By inequality in (3) and definition of \( f_n \),

\[
\mathbb{E} \left[ \sum_{n=1}^N w_n (J(\pi_n) - J(\pi^*)) \right] \leq w_{1:N} J(\pi^*) + C_{\pi^*} \mathbb{E} \left[ \sum_{n=1}^N w_n \hat{f}_n(\pi_n) \right] = w_{1:N} J(\pi^*) + C_{\pi^*} \mathbb{E} \left[ \min_{\pi \in \Pi} \sum_{n=1}^N w_n \hat{f}_n(\pi) + \text{regret}(\Pi) \right]
\]

The statement is obtained by dividing both sides by \( w_{1:N} \) and by the definition of \( c^w_\pi \).

#### C.2 Proof of Section 4.2

**Theorem 4.1.** For MoBIL-VI with \( p > 1 \),

\[
R(p) \leq C_p \left( \frac{pG^2_\mu}{2(p-1)\mu \ln N} \frac{e^\frac{N}{p}}{N} + \frac{1}{2\mu} \right) \left( p+1 \right)^2 \frac{e^\frac{e^\frac{N}{p}}{p}}{N} \frac{1}{N} c^w_\pi,
\]

where \( C_p = \left( \frac{p+1}{2\mu} \right)^2 \frac{e^\frac{N}{p}}{N} \frac{1}{N} \).

**Proof.** We prove a more general version of Theorem 4.1 below.

**Theorem C.1.** For MoBIL-VI,

\[
\mathcal{R}(p) \leq \begin{cases} 
\frac{G^2_\mu}{4\mu^2 \mu_\mathcal{B}} \left( p+1 \right)^2 \frac{e^\frac{N}{p}}{N} \frac{1}{N} + \frac{1}{2\mu} \left( p+1 \right)^2 \frac{e^\frac{e^\frac{N}{p}}{p}}{N} \frac{1}{N} c^w_\pi, & \text{for } p > 1 \\
\frac{G^2_\mu}{4\mu^2 \mu_\mathcal{B} \ln(1+N)} + \frac{1}{2\mu} c^w_\pi, & \text{for } p = 1 \\
\frac{G^2_\mu}{4\mu^2 \mu_\mathcal{B} \left( p+1 \right)^2} \left( \frac{N}{p} \right)^{p+1} \frac{1}{\mu_\mathcal{B} \ln N} \left( p+1 \right)^2 e^\frac{e^\frac{N}{p}}{p} \frac{1}{N} c^w_\pi, & \text{for } 0 < p < 1 \\
\frac{G^2_\mu}{4\mu^2 \mu_\mathcal{B}} \ln N + \frac{1}{2\mu} \frac{N}{N} c^w_\pi, & \text{for } p = 0
\end{cases}
\]
Proof. The solution \( \pi_{n+1} \) of the VI problem (8) satisfies the optimality condition of

\[
\pi_{n+1} = \arg\min_{\pi \in \Pi} \sum_{m=1}^{n} w_m f_m(\pi_n) + w_{n+1} \hat{F}_{n+1}(\pi_{n+1}, \pi).
\]

Therefore, we can derive the bound of \( \mathcal{R}(p) \)\(^1\) as

\[
\mathcal{R}(p) = \frac{\text{regret}^w(\Pi)}{w_{1:N}}
\leq \frac{p + 1}{2 \mu_f w_{1:N}} \sum_{n=1}^{N} n^{p-1} \| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 \hat{F}_n(\pi_n, \pi_n) \|^2
\]

\[\text{(Lemma H.5)}\]

\[
\leq \frac{p + 1}{2 \mu_f w_{1:N}} \sum_{n=1}^{N} n^{p-1} h_n(\pi_n)
\]

\[\text{(Property of } h_n)\] \hspace{1cm} (C.1)

Next, we treat \( n^{p-1} h_n \) as the per-round cost for an online learning problem, and utilize Lemma H.6 to upper bound the accumulated cost. In particular, we set \( w_n \) in Lemma H.6 to \( n^{p-1} \) and \( l_n \) to \( h_n \). Finally, \( w_{1:N} = \sum_{n=1}^{N} n^p \) can be lower bounded using Lemma H.1. Hence, for \( p > 1 \), we have

\[
\mathcal{R}(p) \leq \frac{p + 1}{2 \mu_f w_{1:N}} \left( \frac{G^2_h}{2 \mu_h} \frac{p}{p - 1} \left( \frac{N + 1}{N} \right)^{p-1} \frac{1}{N^2} + \frac{1}{p} (p + 1) \left( \frac{N + 1}{N} \right)^{p-1} \frac{1}{N^{p-1}} \right).
\]

where in the last inequality we utilize the fact that \( 1 + x \leq e^x, \forall x \in \mathbb{R} \). Cases other than \( p > 1 \) follow from straightforward algebraic simplification.

\[\text{Proposition 4.1. For MoBIL-VI with } p = 0, \mathcal{R}(0) \leq \frac{G^2_h}{2 \mu_f \mu_h} \frac{1}{N} + \frac{e^w}{2 \mu_f} \frac{1}{N^{\frac{1}{2}}}.\]

Proof. Proved in Theorem C.1 by setting \( p = 0 \). \hspace{1cm} \[\Box\]

C.3 Proof of Section 4.3

Lemma 4.1 (Stronger FTL Lemma). Let \( x_n^* \in \arg\min_{x \in \mathcal{X}} l_{1:n}(x) \). For any sequence of decisions \( \{x_n\} \) and losses \( \{l_n\} \), \( \text{regret}(\mathcal{X}) = \sum_{n=1}^{N} l_{1:n}(x_n) - l_{1:n}(x_n^*) - \Delta_n \), where \( \Delta_{n+1} := l_{1:n}(x_{n+1}) - l_{1:n}(x_n^*) \geq 0 \).

Proof. The proof is based on observing \( l_n = l_{1:n} - l_{1:n-1} \) and \( l_{1:N} \) as a telescoping sum:

\[
\text{regret}(\mathcal{X}) = \sum_{n=1}^{N} l_n(x_n) - l_{1:N}(x_N^*)
\]

\[
= \sum_{n=1}^{N} (l_{1:n}(x_n) - l_{1:n-1}(x_n)) - \sum_{n=1}^{N} (l_{1:n}(x_n^*) - l_{1:n-1}(x_n^*))
\]

\[
= \sum_{n=1}^{N} (l_{1:n}(x_n) - l_{1:n}(x_n^*) - \Delta_n),
\]

where for notation simplicity we define \( l_{1:0} = 0 \). \hspace{1cm} \[\Box\]

Lemma 4.2. \[\text{regret}^w_{\text{path}}(\Pi) \leq \frac{p + 1}{2 \mu_f} \sum_{n=1}^{N} n^{p-1} \| g_n - \hat{g}_n \|^2 + \frac{\alpha \mu_f}{2 (p+1)} \sum_{n=1}^{N} (n - 1)^{p+1} \| \pi_n - \hat{\pi}_n \|^2.\]

\(^1\)The expectation of \( \mathcal{R}(p) \) is not required here because MoBIL-VI assumes the problem is deterministic.
Proof. We utilize our new Lemma 4.1. First, we bound \( \sum_{n=1}^{N} l_1(n) - l_1(n)^* \), where \( \pi_n^* = \arg \min_{\pi \in \Pi} l_1(n)(\pi) \). We achieve this by Lemma H.4. Let \( l_n = w_n f_n = w_n((g_n, \pi) + r_n(\pi)) \). To use Lemma H.4, we note that because \( r_n \) is centered at \( \pi_n, \pi_{n+1} \) satisfies

\[
\pi_{n+1} = \arg \min_{\pi \in \Pi} \sum_{m=1}^{n} w_m \tilde{f}(\pi) + w_{n+1}(\hat{g}_{n+1}, \pi)
\]

\[
= \arg \min_{\pi \in \Pi} \sum_{m=1}^{n} w_m \frac{\tilde{f}(\pi)}{l_n(\pi)} + w_{n+1}(\hat{g}_{n+1}, \pi) + w_{n+1} r_{n+1}(\pi_{n+1})
\]

Because by definition \( l_n \) is \( w_n \alpha\mu_f \)-strongly convex, it follows from Lemma H.4 and Lemma H.1 that

\[
\sum_{n=1}^{N} l_1(n)(\pi_n) - l_1(n)^* \leq \frac{1}{\alpha \mu_f} \sum_{n=1}^{N} \frac{w_n}{w_1} \| \hat{g}_n - g_n \|^2 \leq \frac{p+1}{2 \alpha \mu_f} \sum_{n=1}^{N} n^{p-1} \| g_n - \hat{g}_n \|^2.
\]

Next, we bound \( \Delta_{n+1} \) as follows

\[
\Delta_{n+1} = l_1(n)(\pi_{n+1}) - l_1(n)^*
\]

\[
\geq \langle \nabla l_1(n)^*(\pi^*_n), \pi_{n+1} - \pi^*_n \rangle + \frac{\alpha \mu_f w_1}{2} \| \pi_{n+1} - \pi^*_n \|^2 \quad \text{(Strong convexity)}
\]

\[
\geq \frac{\alpha \mu_f w_1}{2} \| \pi_{n+1} - \pi^*_n \|^2 \quad \text{(Optimality condition of } \pi^*_n \text{)}
\]

\[
= \frac{\alpha \mu_f w_1}{2} \| \pi_{n+1} - \hat{\pi}_{n+1} \|^2 \quad \text{(Definition of } \hat{\pi}_{n+1} \text{)}
\]

\[
\geq \frac{\alpha \mu_f n^{p+1}}{2(p+1)} \| \pi_{n+1} - \hat{\pi}_{n+1} \|^2. \quad \text{(Definition of } w_n \text{ and Lemma H.1)}
\]

Combining these results proves the bound.

\[\text{Lemma 4.3. } \mathbb{E}[\| g_n - \hat{g}_n \|^2] \leq 4(\sigma_g^2 + \sigma_g^2 + L^2 \mathbb{E}[\| \pi_n - \hat{\pi}_n \|^2] + \mathbb{E}[\hat{h}_n(F_n)]).\]

Proof. By Lemma H.3, we have

\[
\mathbb{E}[\| g_n - \hat{g}_n \|^2] \leq 4 \left( \mathbb{E}[\| g_n - \nabla_2 F(\pi_n, \pi_n) \|^2] + \mathbb{E}[\| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 F_n(\pi_n, \pi_n) \|^2] + \mathbb{E}[\| \nabla_2 F_n(\pi_n, \pi_n) - \hat{F}_n(\pi_n, \pi_n) \|^2] \right).
\]

Because the random quantities are generated in order \( \ldots, \pi_n, g_n, \hat{F}_{n+1}, \hat{\pi}_{n+1}, \hat{g}_{n+1}, \pi_{n+1}, g_{n+1}, \ldots \), by the variance assumption (Assumption 4.3), the first and fourth terms can be bounded by

\[
\mathbb{E}[\| g_n - \nabla_2 F(\pi_n, \pi_n) \|^2] = \mathbb{E}_{\pi_n} \left[ \mathbb{E}_{g_n}[\| g_n - \nabla_2 F(\pi_n, \pi_n) \|^2_{\pi_n}] \right] \leq \sigma_g^2,
\]

\[
\mathbb{E}[\| \nabla_2 F_n(\pi_n, \pi_n) - \hat{F}_n(\pi_n, \pi_n) \|^2]\leq \mathbb{E}[\hat{h}_n(F_n)] = \mathbb{E}[\hat{h}_n(F_n)]
\]

And, for the second term, we have

\[
\mathbb{E}[\| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 F_n(\pi_n, \pi_n) \|^2] \leq \mathbb{E}[\hat{h}_n(F_n)]
\]

Furthermore, due to the Lipschitz assumption of \( \nabla_2 F_{n+1} \) (Assumption 4.1), the third term is bounded by

\[
\mathbb{E}[\| \nabla_2 F_n(\pi_n, \pi_n) - \nabla_2 F_n(\pi_n, \pi_n) \|^2] \leq L^2 \mathbb{E}[\| \pi_n - \hat{\pi}_n \|^2].
\]

Combining the bounds above, we conclude the lemma.

\[\text{Theorem 4.2. } \text{For MoBIL-Prox with } p > 1 \text{ and } \alpha_n = \alpha \in (0, 1), \text{ it satisfies}
\]

\[
\mathcal{R}(p) \leq \frac{(p+1)^2 e^p}{\alpha \mu_f} \left( \frac{C_n^2}{\mu_n} \frac{p}{p-1} \frac{1}{N^p} + \frac{2 \sigma_g^2 + \sigma_g^2 + \sigma_g^2}{N^2} \right) + \frac{p+1}{N^{p-1}} \nu_p,
\]

where \( \nu_p = O(1) \) and \( n_{\text{ceil}} = \left\lfloor \frac{2e^2(p+1)LG}{\alpha \mu_f} \right\rfloor \).
Proof. We prove a more general version of Theorem 4.1 below.

**Theorem C.2.** For MoBIL-Prox,

\[
R(p) \leq \frac{4}{\alpha} R_{\text{MoBIL-VI}}(p) + e_{\Pi}^w + \sigma(p) \left( \sigma_f^2 + \sigma_g^2 \right) + \frac{(p + 1)\nu_p}{N^{p+1}},
\]

where \( R_{\text{MoBIL-VI}}(p) \) is the upper bound of the average regret \( R(p) \) in Theorem C.1, and the expectation is due to sampling \( f_n \) and \( h_n \).

**Proof.** Recall \( R(p) = \mathbb{E}[\text{regret}^w_{\Pi}] \), where

\[
\text{regret}^w_{\Pi} = \sum_{n=1}^{N} w_n f_n(\pi_n) - \min_{\pi \in \Pi} \sum_{n=1}^{N} w_n f_n(\pi).
\]

Define \( f_n(\pi) := (g_n, \pi) + r_n(\pi) \). Since \( f_n \) is \( \mu_f \)-strongly convex, \( r_n \) is \( \alpha \mu_f \)-strongly convex, and \( r(\pi_n) = 0 \), \( f_n \) satisfies

\[
f_n(\pi_n) - \bar{f}_n(\pi_n) \leq \bar{f}_n(\pi_n) - \bar{f}_n(\pi), \quad \forall \pi \in \Pi.
\]

which implies \( R(p) \leq \mathbb{E}[\text{regret}^w_{\text{path}}(\Pi)] \), where

\[
\text{regret}^w_{\text{path}}(\Pi) := \sum_{n=1}^{N} w_n f_n(\pi_n) - \min_{\pi \in \Pi} \sum_{n=1}^{N} w_n f_n(\pi)
\]

is regret of an online learning problem with per-round cost \( w_n f_n \).

Lemma 4.2 upper bounds \( \text{regret}^w_{\text{path}}(\Pi) \) by using Stronger FTL lemma (Lemma 4.1). Since the second term in Lemma 4.2 is negative, which is in our favor, we just need to upper bound the expectation of the first item. Using triangular inequality, we proceed to bound \( \mathbb{E}[\| g_n - \hat{g}_n \|^2 ] \), which measures how well we are able to predict the next per-round cost using the model.

By substituting the result of Lemma 4.3 into Lemma 4.2, we see

\[
\mathbb{E}[\text{regret}^w_{\text{path}}(\Pi)] \leq \mathbb{E} \left[ \sum_{n=1}^{N} \rho_n \| \pi_n - \hat{\pi}_n \|^2 \right] + \left( \frac{2(p + 1)}{\alpha \mu_f} \sum_{n=1}^{N} n^{p-1} \right) \left( \sigma_f^2 + \sigma_g^2 \right) + \frac{2(p + 1)}{\alpha \mu_f} \mathbb{E} \left[ \sum_{n=1}^{N} n^{p-1} h_n(\hat{f}_n) \right]
\]

where \( \rho_n = \frac{2(p + 1)\mu_f^2}{\alpha \mu_f} n^{p-1} - \frac{\alpha \mu_f}{2(p + 1)} (n - 1)^{p+1} \). When \( n \) is large enough, \( \rho_n \leq 0 \), and hence the first term of (C.2) is \( O(1) \). To be more precise, \( \rho_n \leq 0 \) if

\[
\frac{2(p + 1)\mu_f^2}{\alpha \mu_f} n^{p-1} \leq \frac{\alpha \mu_f}{2(p + 1)} (n - 1)^{p+1}
\]

\[
\iff (n - 1)^2 \geq \left( \frac{2(p + 1)\mu_f}{\alpha \mu_f} \right)^2 \left( \frac{n}{n - 1} \right)^{p-1}
\]

\[
\iff (n - 1)^2 \geq \left( \frac{2(p + 1)\mu_f}{\alpha \mu_f} \right)^2 e^{\frac{p-1}{n-1}}
\]
\[
\begin{align*}
\Leftrightarrow (n-1)^2 &\geq \left( \frac{2(p+1)LG_f}{\alpha \mu_f} \right)^2 e \\
\Leftrightarrow n &\geq \frac{2e^{\frac{1}{2}}(p+1)LG_f}{\alpha \mu_f} + 1
\end{align*}
\]

(\text{Assume } n \geq p)

Therefore, we just need to bound the first \( n_{\text{ceil}} = \left\lceil \frac{2e^{\frac{1}{2}}(p+1)LG_f}{\alpha \mu_f} \right\rceil \) terms of \( \rho_n \| \pi_n - \hat{\pi}_n \|^2 \). Here we use a basic fact of convex analysis in order to bound \( \| \pi_n - \hat{\pi}_n \|^2 \)

**Lemma C.1.** Let \( X \) be a compact and convex set and let \( f, g \) be convex functions. Suppose \( f + g \) is \( \mu \)-strongly convex. Let \( x_1 \in \arg \min_{x \in X} f(x) \) and \( x_2 = \arg \min_{x \in X} (f(x) + g(x)) \). Then \( \| x_1 - x_2 \| \leq \| \nabla g(x_1) \|_{\mu} \).

\textbf{Proof of Lemma C.1.} Let \( h = f + g \). Because \( h \) is \( \mu \)-strongly convex and \( x_2 = \arg \min_{x \in X} h(x) \)
\[
\frac{\mu}{2} \| x_1 - x_2 \|^2 \leq h(x_1) - h(x_2) \leq \langle \nabla h(x_1), x_1 - x_2 \rangle - \frac{\mu}{2} \| x_1 - x_2 \|^2 \\
\leq \langle \nabla g(x_1), x_1 - x_2 \rangle - \frac{\mu}{2} \| x_1 - x_2 \|^2
\]

This implies \( \mu \| x_1 - x_2 \|^2 \leq \langle \nabla g(x_1), x_1 - x_2 \rangle \leq \| \nabla g(x_1) \|_{\mu} \| x_1 - x_2 \| \). Dividing both sides by \( \| x_1 - x_2 \| \) concludes the lemma. \hfill \Box

Utilizing Lemma C.1 and the definitions of \( \pi_n \) and \( \hat{\pi}_n \), we have, for \( n \geq 2 \),
\[
\| \pi_n - \hat{\pi}_n \|^2 \leq \frac{1}{\alpha \mu_f w_{1:n-1}} \| w_n \hat{\gamma} \|^2 \\
\leq \frac{(p+1)G_f^2}{\alpha \mu_f} n^{2p} (n-1)^{p+1} \quad \text{(Bounded } \hat{\gamma} \text{ and Lemma H.1)}
\leq \frac{(p+1)e^{\frac{1}{2}}G_f^2}{\alpha \mu_f} n^{p-1} \quad (1 + x \leq e^x)
\leq \frac{e(p+1)G_f^2}{\alpha \mu_f} n^{p-1} \quad \text{(Assume } n \geq p + 2)\]

and therefore, after assuming initialization \( \pi_1 = \hat{\pi}_1 \), we have the bound
\[
\sum_{n=2}^{n_{\text{ceil}}} \rho_n \| \pi_n - \hat{\pi}_n \|^2 \leq 2e \left( \frac{(p+1)LG_f}{\alpha \mu_f} \right)^2 \sum_{n=2}^{n_{\text{ceil}}} n^{2p-2} - \frac{eG_f^2}{2} \sum_{n=2}^{n_{\text{ceil}}} (n-1)^{p+1} n^{p-1} \quad \text{(C.3)}
\]

For the third term of (C.2), we can tie it back to the bound of \( R(p) \) of MoBIL-VI, which we denote \( R_{\text{MoBIL-VI}}(p) \).

More concretely, recall that for MoBIL-VI in (C.1), we have
\[
R(p) \leq \frac{p+1}{2\mu_f w_{1:N}} \sum_{n=1}^{N} n^{p-1} \hat{h}_n(\pi_n),
\]

and we derived the upper bound \( (R_{\text{MoBIL-VI}}(p)) \) for the RHS term. By observing that the third term of (C.2) after averaging is
\[
\begin{align*}
\frac{2(p+1)}{\alpha \mu_f w_{1:N}} \mathbb{E} \left[ \sum_{n=1}^{N} n^{p-1} \hat{h}_n(\tilde{F}_n) \right] &\leq \mathbb{E} \left[ \frac{4}{\alpha} \left( \frac{p+1}{2\mu_f w_{1:N}} \sum_{n=1}^{N} n^{p-1} \hat{h}_n(\tilde{F}_n) \right) \right] \\
&\leq \frac{4}{\alpha} \mathbb{E} \left[ R_{\text{MoBIL-VI}}(p) \right] \\
&= \frac{4}{\alpha} R_{\text{MoBIL-VI}}(p).
\end{align*}
\]

Dividing (C.2) by \( w_{1:N} \), and plugging in (C.3), (C.4), we see
\[
R(p) \leq \mathbb{E}[\text{regret}_{\text{path}}(\Pi)/w_{1:N}]
\]
\[
\leq \frac{4}{\alpha} R_{\text{MoBIL-VI}}(p) + \frac{1}{w_{1:N}} \left( \nu_p + \left( \frac{2(p+1)}{\alpha \mu_f} \sum_{n=1}^{N} n^{p-1} \right) \left( \frac{\sigma_2^2 + \sigma_3^2}{2} \right) \right)
\]

where \( \nu_p = 2e \left( \frac{(p+1)LG_f}{\alpha \mu_f} \right)^2 \sum_{n=1}^{\text{ceil}(n_{\text{cell}})} n^{2p-2} - \frac{eG_f^2}{2} \sum_{n=2}^{n_{\text{cell}}} (n-1)p^{p-1} + 1 \), \( n_{\text{cell}} = \left\lceil \frac{2\pi (p+1)LG_f}{\alpha \mu_f} \right\rceil \).

Finally, we consider the case \( p > 1 \) as stated in Theorem 4.2

\[
R(p) \leq \frac{4}{\alpha} \left( \frac{G_f^2}{4 \mu_f \mu_\pi} \right)^2 \frac{p+1}{p-1} \left( \frac{1}{N^2} + \frac{(p+1)^2 e^p}{p} \right) + \frac{2}{\alpha \mu_f} \frac{p+1}{N^{p+1}} \left( \nu_p + \left( \frac{2(p+1)}{\alpha \mu_f} \right) \left( \frac{\sigma_2^2 + \sigma_3^2}{2} \right) \right)
\]

where \( \nu_p = 2e \left( \frac{(p+1)LG_f}{\alpha \mu_f} \right)^2 \left( \frac{n_{\text{cell}}+1}{2} \right)^{p-1} - \frac{eG_f^2 (n_{\text{cell}}-1)^{p+1}}{2p+1} + 1 \), \( n_{\text{cell}} = \left\lceil \frac{2\pi (p+1)LG_f}{\alpha \mu_f} \right\rceil \).

\[\blacksquare\]

D Model Learning through Learning Dynamics Models

So far we have stated model learning rather abstractly, which only requires \( h_n(\hat{F}) \) to be an upper bound of \( \| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 \hat{F}(\pi_n, \pi_n) \|_2^2 \). Now we give a particular example of \( h_n \) and \( \hat{h} \) when the predictive model is constructed as a simulator with online learned dynamics models. Specifically, we consider learning a transition model \( M \in \mathcal{M} \) online that induces a bivariate function \( \hat{F} \), where \( \mathcal{M} \) is the class of transition models. Let \( D_{KL} \) denote the KL divergence and let \( d_{\pi_n}^M \) be the generalized stationary distribution (cf. (1)) generated by running policy \( \pi_n \) under transition model \( M \). We define, for \( M_n \in \mathcal{M} \), \( \hat{F}_n(\pi', \pi) := d_{\pi_n}^M [D(\pi' || \pi)] \). We show the error of \( \hat{F}_n \) can be bounded by the KL-divergence error of \( M_n \).

Lemma D.1. Assume \( \nabla D(\pi^* || \cdot) \) is \( L_D \)-Lipschitz continuous with respect to \( \| \cdot \|_* \). It holds that \( \| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 \hat{F}_n(\pi_n, \pi_n) \|_*^2 \leq 2^{-1} (L_D \text{Diam}([\pi]))^2 D_{KL}(d_{\pi_n}^M || d_{\pi_n}^M) \).

Directly minimizing the marginal KL-divergence \( D_{KL}(d_{\pi_n}^M || d_{\pi_n}^M) \) is a nonconvex problem and requires backpropagation through time. To make the problem simpler, we further upper bound it in terms of the KL divergence between the true and the modeled transition probabilities.

To make the problem concrete, here we consider \( T \)-horizon RL problems.

Proposition D.1. For a \( T \)-horizon problem with dynamics \( P \), let \( M_n \) be the modeled dynamics. Then \( \exists C > 0 \) s.t \( \| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 \hat{F}_n(\pi_n, \pi_n) \|_*^2 \leq C \sum_{t=0}^{T-1} (T-t) E_{d_{\pi_n}} E_{x} [D_{KL}(P \| M_n)] \).

Therefore, we can simply take \( h_n \) as the upper bound in Proposition D.1, and \( \hat{h} \) as its empirical approximation by sampling state-action transition triples through running policy \( \pi_n \) in the real environment. This construction agrees with the causal relationship assumed in the Section 3.2.1.

D.1 Proofs

Lemma D.1. Assume \( \nabla D(\pi^* || \cdot) \) is \( L_D \)-Lipschitz continuous with respect to \( \| \cdot \|_* \). It holds that \( \| \nabla_2 F(\pi_n, \pi_n) - \nabla_2 \hat{F}_n(\pi_n, \pi_n) \|_*^2 \leq 2^{-1} (L_D \text{Diam}([\pi]))^2 D_{KL}(d_{\pi_n}^M || d_{\pi_n}^M) \).

Proof. First, we use the definition of dual norm

\[
\| \nabla_2 \hat{F}(\pi_n, \pi_n) - \nabla_2 F(\pi_n, \pi_n) \|_* = \max_{\| \delta \| \leq 1} \langle \delta, \nabla D(\pi^* || \pi_n) \rangle
\]

and then we show that \( \langle \delta, \nabla D(\pi^* || \pi_n) \rangle \) is \( L_D \)-Lipschitz continuous: for \( \pi, \pi' \in \Pi \),

\[
\langle \delta, \nabla D(\pi^* || \pi_n) - \nabla D(\pi^* || \pi'_n) \rangle \leq \| \delta \| \| \nabla D(\pi^* || \pi_n) - \nabla D(\pi^* || \pi'_n) \|_\star \leq L_D \| \pi - \pi' \|_\star
\]

Note in the above equations \( \nabla \) is with respect to \( D(\pi^* || \cdot) \).

Next we bound the right hand side of (D.1) using Wasserstein distance \( D_W \), which is defined as follows [20]: for two probability distributions \( p \) and \( q \) defined on a metric space \( D_W(p, q) := \sup_{f \in \text{Lip}([f(\cdot)]) \leq 1} E_{x \sim p}[f(x)] - E_{x \sim q}[f(x)] \).
Using the property that $(\delta, \nabla D(\pi^*||\pi_n))$ is $L_D$-Lipschitz continuous, we can derive

$$ \| \nabla_2 \hat{F}(\pi_n, \pi_n) - \nabla_2 \hat{F}(\pi_n, \pi_n) \|_* \leq L_D \| d_{\pi_n}, \hat{d}_{\pi_n} \| \leq \frac{L_D \text{Diam}(S)}{\sqrt{2}} \sqrt{D_{KL}(d_{\pi_n}||d_{\pi_n})} $$

in which the last inequality is due to the relationship between $D_{KL}$ and $D_W$ [20].

**Proposition D.1.** For a $T$-horizon problem with dynamics $P$, let $M_n$ be the modeled dynamics. Then $\exists C > 0$ s.t $\| \nabla_2 \hat{F}(\pi_n, \pi_n) - \nabla_2 \hat{F}_n(\pi_n, \pi_n) \|_*^2 \leq C \sum_{t=0}^{T-1} (T-t) \mathbb{E}_{d_{\pi_n}, \mathbb{E}_\pi} [D_{KL}(P||M_n)]$.

**Proof.** Let $\rho_{\pi,t}$ be the state-action trajectory up to time $t$ generated by running policy $\pi$, and let $\hat{\rho}_{\pi,t}$ be that of the dynamics model. To prove the result, we use a simple fact:

**Lemma D.2.** Let $p$ and $q$ be two distributions.

$$ KL[p(x,y)||q(x,y)] = KL[p(x)||q(x)] + \mathbb{E}_{p(x)} KL[p(y|x)||q(y|x)] $$

Then the rest follows from Lemma D.1 and the following inequality.

$$ D_{KL}(d_{\pi_n}||d_{\pi_n}) \leq \frac{1}{T} \sum_{t=0}^{T-1} D_{KL}(\rho_{\pi_n,t}||\hat{\rho}_{\pi_n,t}) $$

$$ = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{\rho_{\pi_n,t}} \left[ \sum_{t=0}^{t-1} \ln \frac{p_M(s_{t+1}|s_t, a_t)}{p_{\hat{M}}(s_{t+1}|s_t, a_t)} \right] $$

$$ = \frac{1}{T} \sum_{t=0}^{T-1} (T-t) \mathbb{E}_{d_{\pi_n}, \mathbb{E}_\pi} [D_{KL}(p_M||p_{\hat{M}})] $$

### E Relaxation of Strong Convexity Assumption

The strong convexity assumption (Assumption 4.2) can be relaxed to just convexity. We focus on studying the effect of $\hat{f}_n$ and/or $\hat{h}_n$ being just convex on $R(p)$ in Theorem 2.1 and Theorem 4.2 in big-O notation. Suggested by Lemma 3.2, when strong convexity is not assumed, additional regularization has to be added in order to keep the stabilization terms $l_{1,n}(x_n) - l_{1,n}(x_n^*)$ small.

**Lemma E.1** (FTRL with prediction). Let $l_n$ be convex with bounded gradient and let $X$ be a compact set. In round $n$, let regularization $r_n$ be $\mu_n$-strongly convex for some $\mu_n \geq 0$ such that $r_n(x_n) = 0$ and $x_n \in \arg \min_{x \in X} r_n(x)$, and let $v_{n+1}$ be a (non) convex function such that $\sum_{m=1}^n w_m (l_n + r_n) + w_{n+1}v_{n+1}$ is convex. Suppose that learner plays Follow-The-Regularized-Leader (FTRL) with prediction, i.e. $x_{n+1} = \arg \min_{x \in X} \sum_{m=1}^n (w_m (l_n + r_n) + w_{n+1}v_{n+1})$, and suppose that $\sum_{m=1}^n w_m \mu_m = \Omega(n^k) > 0$ and $\sum_{m=1}^n w_m v_n(x)$ is convex for all $x \in \mathcal{X}$ and some $k \geq 0$. Then, for $w_n = n^p$,

$$ \text{regret}^w(\mathcal{X}) = O(N^k) + \sum_{n=1}^N O\left(n^{2p-k} - \| \nabla l_n(x_n) - \nabla v_n(x_n) \|_2 \right)^2 $$

**Proof.** The regret of the online learning problem with convex per-round cost $w_n l_n$ can be bounded by the regret of the online learning problem with strongly convex per-round cost $w_n (l_n + r_n)$ as follows. Let $x_n^* \in \arg \min_{x \in X} \sum_{m=1}^N w_m l_n(x)$.

$$ \text{regret}^w(\mathcal{X}) = \sum_{n=1}^N w_n l_n(x_n) - \min_{x \in \mathcal{X}} \sum_{n=1}^N w_n l_n(x) $$

$$ = \sum_{n=1}^N l_n(x_n) + r_n(x_n) - \sum_{n=1}^N w_n (l_n(x_n^*) + r_n(x_n^*)) + \sum_{n=1}^N w_n r_n(x_n^*) $$

$$ \leq \left( \sum_{n=1}^N l_n(x_n) + r_n(x_n) - \min_{x \in \mathcal{X}} \sum_{n=1}^N w_n (l_n(x) + r_n(x)) \right) + O(N^k). $$
Since the first term is the regret of the online learning problem with strongly convex per-round cost $w_n(l_n + r_n)$, and $x_{n+1} = \arg\min_{x'} \sum_{m=1}^n w_m (l_n + r_n) + w_{n+1} r_{n+1}$, we can bound the first term via Lemma H.5 by setting $w_n = n^p$ and $\sum_{m=1}^n w_m \mu_m = O(n^k)$.

The lemma below is a corollary of Lemma E.1.

**Lemma E.2 (FTRL).** Under the same condition in Lemma E.1, suppose that learner plays FTRL, i.e. $x_{n+1} = \arg\min_X \sum_{m=1}^n w_m (l_n + r_n)$. Then, for $w_n = n^p$ with $p > -\frac{1}{2}$, choose $\{r_n\}$ such that $\sum_{m=1}^n w_m \mu_m = \Omega(n^{p+1/2}) > 0$ and it achieves regret $\mathcal{R}(X) = O(N^{p + 1/2})$ and $\mathcal{R}_{H,N}(X) = O(N^{-1/2})$.

**Proof.** Let $\sum_{m=1}^n w_m \mu_m = \Theta(n^k) > 0$ for some $k \geq 0$. First, if $2p - k > -1$, then we have

$$\text{regret}(X) \leq O(N^k) + \sum_{n=1}^N O\left(\frac{n^{2p-k}}{2}\right) \|\nabla l_n(x_n)\|^2_w$$

(Lemma E.1)

$$\leq O(N^k) + \sum_{n=1}^N O\left(\frac{n^{2p-k}}{2}\right)$$

($l_n$ has bounded gradient)

$$\leq O(N^k) + O\left(\frac{n^{2p-k+1}}{2}\right)$$

(Lemma H.1)

In order to have the best rate, we balance the two terms $O(N^k)$ and $O\left(\frac{n^{2p-k+1}}{2}\right)$

$$k = 2p - k + 1 \implies k = p + \frac{1}{2},$$

That is, $p > -\frac{1}{2}$, because $2p - (p + \frac{1}{2}) > -1$. This setting achieves regret in $O(N^{p + \frac{1}{2}})$. Because $w_{1:N} = O(N^{p+1})$, the average regret is in $O(N^{-\frac{3}{2}})$.

With these lemmas, we are ready to derive the upper bounds of $\mathcal{R}(p)$ when either $\tilde{f}_n$ or $\hat{h}_n$ is just convex, with some minor modification of Algorithm 1. For example, when $\tilde{f}_n$ is only convex, $r_n$ will not be $\alpha f$ strongly; instead we will concern the strongly convexity of $\sum_{m=1}^n w_m r_n$. Similarly, if $\hat{h}_n$ is only convex, the model cannot be updated by FTL as in line 5 of Algorithm 1; instead it has to be updated by FTRL.

In the following, we will derive the rate for MoBIL-VI (i.e. $\tilde{f}_n = f_n$ and $\hat{h} = h$) and assume $c_{\nu}^p = 0$ for simplicity.

The same rate applies to the MoBIL-Prox when there is no noise. To see this, for example, if $\tilde{f}_n$ is only convex, we can treat $r_n$ as an additional regularization and we can see

$$\mathcal{R}(p) = \mathbb{E}\left[\frac{\text{regret}^w(\Pi)}{w_{1:N}}\right] \leq \frac{1}{w_{1:N}} \mathbb{E}\left[\sum_{n=1}^N w_n \tilde{f}_n(\pi_n) - \min_{\pi \in \Pi} \sum_{n=1}^N w_n \tilde{f}_n(\pi) + \sum_{n=1}^N w_n r_n(\pi_n)\right]$$

where $\pi_N = \arg\min_{\pi \in \Pi} \sum_{n=1}^N \tilde{f}_n(\pi)$. As in the proof of Theorem 4.2, regret$^w_{\text{path}}$ is decomposed into several terms: the $\hat{h}_n$ part in conjunction with $\sum_{n=1}^N w_n r_n(\pi_N^*)$ constitute the same $\mathcal{R}(p)$ part for MoBIL-VI, while other terms in regret$^w_{\text{path}}$ are kept the same.

**Strongly convex $\tilde{f}_n$ and convex $\hat{h}_n$** Here we assume $p > \frac{1}{2}$. Under this condition, we have

$$\text{regret}^w(\Pi) = \sum_{n=1}^N O(n^{p-1}) \hat{h}_n(\hat{F}_n)$$

(Lemma H.5)

$$= O\left(N^{p-\frac{1}{2}}\right)$$

(Lemma E.2)

Because $w_{1:N} = \Omega(N^{p+1})$, the average regret $\mathcal{R}(p) = O(N^{-3/2})$. 
Convex $\tilde{f}_n$ and strongly convex $\tilde{h}_n$ Here we assume $p > 0$. Suppose $r_{1:n}$ is $\Theta(n^k)$-strongly convex and $2p - k > 0$. Under this condition, we have

$$\text{regret}^w(\Pi) = O(N^k) + \sum_{n=1}^{N} O(n^{2p-k}) \tilde{h}_n(\hat{F}_{n+1})$$  \hspace{1cm} \text{(Lemma E.1)}

$$= O(N^k) + O(N^{2p-k}).$$  \hspace{1cm} \text{(Lemma H.6)}

We balance the two terms and arrive at

$$k = 2p - k \implies k = p,$$

which satisfies the condition $2p - k > 0$, if $p > 0$. Because $w_{1:N} = \Omega(N^{p+1})$, the average regret $R(p) = O(N^{-1})$.

Convex $\tilde{f}_n$ and convex $\tilde{h}_n$ Here we assume $p \geq 0$. Suppose $r_{1:n}$ is $\Theta(n^k)$-strongly convex and $2p - k > -\frac{1}{2}$. Under this condition, we have

$$\text{regret}^w(\Pi) = O(N^k) + \sum_{n=1}^{N} O(n^{2p-k}) \tilde{h}_n(\hat{F}_{n+1})$$  \hspace{1cm} \text{(Lemma E.1)}

$$= O(N^k) + O(N^{2p-k+\frac{1}{2}}).$$  \hspace{1cm} \text{(Lemma E.1)}

We balance the two terms and see

$$k = 2p - k + \frac{1}{2} \implies k = p + \frac{1}{4},$$

which satisfies the condition $2p - k > -\frac{1}{2}$, if $p \geq 0$. Because $w_{1:N} = \Omega(N^{p+1})$, the average regret $R(p) = O(N^{-3/4})$.

Convex $f_n$ without model Setting $p = 0$ in Lemma E.2, we have $\text{regret}(\Pi) = O(N^{\frac{1}{2}})$.

Therefore, the average regret becomes $O(N^{-\frac{1}{4}})$.

Stochastic problems The above rates assume that there is no noise in the gradient and the model is realizable. If the general case, it should be selected $k = p + 1$ for strongly convex $\tilde{f}_n$ and $k = p + \frac{1}{2}$ for convex $\tilde{f}_n$. The convergence rate will become $O(\frac{\epsilon^2 + \sigma^2 + \sigma^2}{N})$ and $O(\frac{\epsilon^2 + \sigma^2 + \sigma^2}{\sqrt{N}})$, respectively.

F Connection with Stochastic Mirror-Prox

In this section, we discuss how MoBIL-Prox generalizes stochastic MIRROR-PROX by Juditsky et al. [11], Nemirovski [18] and how the new Stronger FTL Lemma 4.1 provides more constructive and flexible directions to design new algorithms.

F.1 Variational Inequality Problems

MIRROR-PROX [18] was first proposed to solve VI problems with monotone operators, which is a unified framework of “convex-like” problems, including convex optimization, convex-concave saddle-point problems, convex multiplayer games, and equilibrium problems, etc (see [14] for a tutorial). Here we give the definition of VI problems and review some of its basic properties.

**Definition F.1.** Let $\mathcal{X}$ be a convex subset in an Euclidean space $\mathcal{E}$ and let $F: \mathcal{X} \to \mathcal{E}$ be an operator, the **VI problem**, denoted as $\text{VI}(\mathcal{X}, F)$, is to find a vector $x^* \in \mathcal{X}$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X}.$$

The set of solutions to this problem is denoted as $\text{SOL}(\mathcal{X}, F)$
It can be shown that, when $\mathcal{X}$ is also compact, then $\text{VI}(\mathcal{X}, F)$ admits at least one solution [14]. For example, if $F(x) = \nabla f(x)$ for some function $f$, then solving $\text{VI}(\mathcal{X}, F)$ is equivalent to finding stationary points.

VI problems are, in general, more difficult than optimization. To make the problem more structured, we will consider the problems equipped with some general convex structure, which we define below. When $F(x) = \nabla f(x)$ for some convex function $f$, the below definitions agree with their convex counterparts.

**Definition F.2.** An operator $F : \mathcal{X} \to \mathcal{E}$ is called

1. *pseudo-monotone* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,
   \[ \langle F(y), x - y \rangle \geq 0 \implies \langle F(x), x - y \rangle \geq 0 \]
2. *monotone* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,
   \[ \langle F(x) - F(y), x - y \rangle \geq 0 \]
3. *strictly monotone* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,
   \[ \langle F(x) - F(y), x - y \rangle > 0 \]
4. *$\mu$-strongly monotone* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$,
   \[ \langle F(x) - F(y), x - y \rangle \geq \mu \| x - y \|^2 \]

A VI problem is a special case of general equilibrium problems [21]. Therefore, for a VI problem, we can also define its dual VI problem.

**Definition F.3.** Given a VI problem $\text{VI}(\mathcal{X}, F)$, the dual VI problem, denoted as $\text{DVI}(\mathcal{X}, F)$, is to find a vector $x_D^* \in \mathcal{X}$ such that

\[ \langle F(x), x - x_D^* \rangle \geq 0, \quad \forall x \in \mathcal{X}. \]

The set of solutions to this problem is denoted as $\text{DSOL}(\mathcal{X}, F)$.

The solution sets of the primal and the dual VI problems are connected as given in next proposition, whose proof e.g. can be found in [22].

**Proposition F.1.**

1. If $F$ is pseudo-monotone, then $\text{SOL}(\mathcal{X}, F) \subseteq \text{DSOL}(\mathcal{X}, F)$.
2. If $F$ is continuous, then $\text{DSOL}(\mathcal{X}, F) \subseteq \text{SOL}(\mathcal{X}, F)$.

However, unlike primal VI problems, a dual VI problem does not always have a solution even if $\mathcal{X}$ is compact. To guarantee the existence of solution to $\text{DSOL}(\mathcal{X}, F)$ it needs stronger structure, such as pseudo-monotonicity as shown in Proposition F.1. Like solving primal VI problems is related to finding local stationary points in optimization, solving dual VI problems is related to finding global optima when $F(x) = \nabla f(x)$ for some function $f$ [23].

**F.2 Stochastic Mirror-Prox**

Stochastic MIRROR–PROX solves a monotone VI problem by indirectly finding a solution to its dual VI problem using stochastic first-order oracles. This is feasible because of Proposition F.1. The way it works is as follows: given an initial condition $x_1 \in \mathcal{X}$, it initializes $\hat{x}_1 = x_1$; at iteration $n$, it receives unbiased estimates $g_n$ and $\hat{g}_n$ satisfying $\mathbb{E}[g_n] = F(x_n)$ and $\mathbb{E}[\hat{g}_n] = F(\hat{x}_n)$ and then performs updates

\[
\begin{align*}
    x_{n+1} &= \text{Prox}_{\gamma_n} \left( \gamma_n \hat{g}_n \right) \\
    \hat{x}_{n+1} &= \text{Prox}_{\gamma_n} \left( \gamma_n g_{n+1} \right)
\end{align*}
\] (F.1)
where $\gamma_n > 0$ is the step size, and the proximal operator $\text{Prox}$ is defined as
\[
\text{Prox}_\gamma(y) = \arg \min_{x \in \mathcal{X}} \langle g, x \rangle + B_\omega(x\| y)
\]
and $B_\omega(x\| y) = \omega(x) - \omega(y) - \langle \nabla \omega(y), x - y \rangle$ is the Bregman divergence with respect to an $\alpha$-strongly convex function $\omega$. At the end, stochastic MIRROR-PROX outputs
\[
\bar{x}_N = \frac{\sum_{n=1}^{N} \gamma_n x_n}{\gamma_1 N}
\]
as the final decision.

For stochastic MIRROR-PROX, the accuracy of a candidate solution $x$ is based on the error
\[
\text{ERR}(x) := \max_{y \in \mathcal{X}} \langle F(y), x - y \rangle.
\]
This choice of error follows from the optimality criterion of the dual VI problem in Definition F.3. That is, $\text{ERR}(x) \leq 0$ if and only if $x \in \text{DSOL}(\mathcal{X}, F)$. From Proposition F.1, we know that if the problem is pseudo-monotone, a dual solution is also a primal solution. Furthermore, we can show an approximate dual solution is also an approximate primal solution.

Let $\Omega^2 = \max_{x,y \in \mathcal{X}} B_\omega(x\| y)$. Now we recap the main theorem of [11].

Theorem F.1. [11] Let $F$ be monotone. Assume $F$ is $L$-Lipschitz continuous, i.e.
\[
\| F(x) - F(y) \|_* \leq L \| x - y \| \quad \forall x, y \in \mathcal{X}
\]
and for all $n$, the sampled vectors are unbiased and have bounded variance, i.e.
\[
E[g_n] = F(x_n), \quad E[\hat{g}_n] = F(\hat{x}_n)
\]
\[
E[\| g_n - F(x_n) \|^2] \leq \sigma^2, \quad E[\| \hat{g}_n - F(\hat{x}_n) \|^2] \leq \sigma^2
\]
Then for $\gamma_n = \gamma$ with $0 < \gamma_n \leq \frac{\alpha}{\sqrt{3L}}$, it satisfies that
\[
E[\text{ERR}(\bar{x}_N)] \leq \frac{2\alpha \Omega^2}{N\gamma} + \frac{7\gamma \sigma^2}{\alpha}
\]
In particular, if $\gamma = \min\{\frac{\alpha}{\sqrt{3L}}, \alpha \Omega \sqrt{\frac{2}{7N\sigma^2}}\}$, then
\[
E[\text{ERR}(\bar{x}_N)] \leq \frac{7 \Omega^2 L}{2 \alpha N} + \Omega \sqrt{\frac{14\sigma^2}{N}}
\]
If the problem is deterministic, the original bound of Nemirovski [18] is as follows.

Theorem F.2. [18] Under the same assumption in Theorem F.1, suppose the problem is deterministic. For $\gamma \leq \frac{\alpha}{\sqrt{2L}}$, 
\[
\text{ERR}(\bar{x}_N) \leq \sqrt{2} \Omega^2 L \frac{1}{\alpha N}
\]
Unlike the uniform scheme above, a recent analysis by Ho-Nguyen and Kilinc-Karzan [24] also provides a performance bound the weighted average version of MIRROR-PROX when the problem is deterministic.

Theorem F.3. [24] Under the same assumption in Theorem F.1, suppose the problem is deterministic. Let $\{w_n \geq 0\}$ be a sequence of weights and let the step size to be $\gamma_n = \frac{\alpha}{E \max_m w_m} w_{1,n}$.
\[
\text{ERR}(\bar{x}_N) \leq \frac{\Omega^2 L \max_n w_n}{\alpha w_{1,N}}
\]

Theorem F.3 (with $w_n = w$) tightens Theorem F.1 and Theorem F.2 by a constant factor.

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13Here simplify the condition they made by assuming $F$ is Lipschitz continuous and $g_n$ and $\hat{g}_n$ are unbiased.
F.3 Connection with MoBIL-Prox

To relate stochastic Mirror-Prox and MoBIL-Prox, we first rename the variables in (F.1) by setting $\hat{x}_{n+1} := \hat{x}_n$ and $\gamma_{n+1} := \gamma_n$

\[
x_{n+1} = \text{Prox}_{\hat{x}_n}(\gamma_n g_n) \quad \iff \quad x_{n+1} = \text{Prox}_{\hat{x}_{n+1}}(\gamma_{n+1} g_{n+1})
\]

and then reverse the order of updates and write them as

\[
\hat{x}_{n+1} = P_{\hat{x}_n}(\gamma_n g_n) \\
x_{n+1} = P_{\hat{x}_{n+1}}(\gamma_{n+1} g_{n+1})
\]  \hspace{1cm} (F.2)

Now we will show that the update in (F.2) is a special case of (9), which we recall below.

\[
\hat{\pi}_{n+1} = \arg\min_{\pi \in \Pi} \sum_{m=1}^{n} w_m \left( \langle g_m, \pi \rangle + r_m(\pi) \right), \\
\pi_{n+1} = \arg\min_{\pi \in \Pi} \sum_{m=1}^{n} w_m \left( \langle g_m, \pi \rangle + r_m(\pi) \right) + w_{n+1} \langle \hat{g}_{n+1}, \pi \rangle,
\]  \hspace{1cm} (9)

That is, we will show that $x_n = \pi_n$ and $\hat{x} = \hat{\pi}_n$ under certain setting.

**Proposition F.2.** Suppose $w_n = \gamma_n$, $\hat{F}_n = F$, $r_1(\pi) = B_\omega(\pi||\pi_1)$ and $r_n = 0$ for $n > 1$. If $\Pi = \mathcal{X}$ is unconstrained, then $x_n = \pi_n$ and $\hat{x}_n = \hat{\pi}_n$ as defined in (F.2) and (9).

**Proof.** We prove the assertion by induction. For $n = 1$, it is trivial, since $\pi_1 = \hat{\pi}_1 = x_1 = \hat{x}_1$. Suppose it is true for $n$. We show it also holds for $n + 1$.

We first show $\hat{x}_{n+1} = \hat{\pi}_{n+1}$. By the optimality condition of $\hat{\pi}_{n+1}$, it holds that

\[
0 = \sum_{m=1}^{n} w_m g_m + \nabla \omega(\hat{\pi}_{n+1}) - \nabla \omega(\pi_1)
\]

\[
= (w_n g_n + \nabla \omega(\hat{\pi}_{n+1}) - \nabla \omega(\hat{\pi}_n)) + \left( \sum_{m=1}^{n-1} w_m g_m + \nabla \omega(\hat{\pi}_n) - \nabla \omega(\pi_1) \right)
\]

\[
= w_n g_n + \nabla \omega(\hat{\pi}_{n+1}) - \nabla \omega(\hat{\pi}_n)
\]

where the last equality is by the optimality condition of $\hat{\pi}_n$. This is exactly the optimality condition of $\hat{x}_{n+1}$ given in (F.2), as $\hat{x}_n = \hat{\pi}_n$ by induction hypothesis and $w_n = \gamma_n$. Finally, because Prox is single-valued, it implies $\hat{x}_{n+1} = \hat{\pi}_{n+1}$.

Next we show that $\pi_{n+1} = x_{n+1}$. By optimality condition of $\pi_{n+1}$, it holds that

\[
0 = \sum_{m=1}^{n} w_{m+1} g_{m+1} + \nabla \omega(\pi_{n+1}) - \nabla \omega(\pi_1)
\]

\[
= (w_{n+1} g_{n+1} + \nabla \omega(\pi_{n+1}) - \nabla \omega(\pi_{n+1})) + \left( \sum_{m=1}^{n} w_m g_m + \nabla \omega(\hat{\pi}_n) - \nabla \omega(\pi_1) \right)
\]

\[
= w_{n+1} g_{n+1} + \nabla \omega(\pi_{n+1}) - \nabla \omega(\hat{\pi}_{n+1})
\]

This is the optimality condition also for $x_{n+1}$, since we have shown that $\hat{x}_{n+1} = \hat{\pi}_{n+1}$. The rest of the argument follows similarly as above.

In other words, stochastic Mirror-Prox is a special case of MoBIL-Prox, when $\hat{F}_n = F$ (i.e. the update of $\pi_n$ also queries the environment not the simulator) and the regularization is constant. The condition that $\mathcal{X}$ and $\Pi$ are unconstrained is necessary to establish the exact equivalence between Prox-based updates and FTL-based
updates. This is a known property in the previous studies on the equivalence between lazy mirror descent and FTRL [16]. Therefore, when \( \hat{F}_n = F \), we can view MoBIL-Prox as a lazy version of MIRROR-Prox. It has been empirically observed the FT(R)L version sometimes empirically perform better than the Prox version [16].

With the connection established by Proposition F.2, we can use a minor modification of the strategy used in Theorem 4.2 to prove the performance of MoBIL-Prox when solving VI problems. To show the simplicity of the FTL-style proof compared with the algebraic proof of Juditsky et al. [11], below we will prove from scratch but only using the new Stronger FTL Lemma (Lemma 4.1).

To do so, we introduce a lemma to relate expected regret and \( \text{ERR}(\bar{x}_N) \).

**Lemma F.1.** Let \( F \) be a monotone operator. For any \( \{x_n \in \mathcal{X}\}_{n=1}^N \) and \( \{w_n \geq 0\} \),

\[
\mathbb{E}[\text{ERR}(\bar{x}_N)] \leq \mathbb{E} \left[ \max_{x \in \mathcal{X}} \frac{1}{w_{1:N}} \sum_{n=1}^N w_n \langle F(x_n), x_n - x \rangle \right]
\]

where \( \bar{x}_N = \frac{\sum_{n=1}^N w_n x_n}{w_{1:n}} \).

**Proof.** Let \( x^* \in \arg \max_{x \in \mathcal{X}} \langle F(x), \bar{x}_N - x \rangle \). By monotonicity, for all \( x_n \), \( \langle F(x^*), x_n - x^* \rangle \leq \langle F(x_n), x_n - x^* \rangle \). and therefore

\[
\mathbb{E}[\text{ERR}(\bar{x}_N)] \leq \mathbb{E} \left[ \frac{1}{w_{1:N}} \sum_{n=1}^N w_n \langle F(x_n), x_n - x^* \rangle \right] 
\]

\[
\leq \mathbb{E} \left[ \max_{x \in \mathcal{X}} \frac{1}{w_{1:N}} \sum_{n=1}^N w_n \langle F(x_n), x_n - x \rangle \right] \]

\[\blacksquare\]

**Theorem F.4.** Under the same assumption as in Theorem F.1. Suppose \( w_n = n^p \) and \( r_n(x) = \beta_n B_\omega(x|x_n) \), where \( \beta_n \) is selected such that \( \sum_{n=1}^N w_n \beta_n = \frac{1}{\eta} n^k \) for some \( k \geq 0 \) and \( \eta > 0 \). If \( k > p \), then

\[
\mathbb{E}[\text{ERR}(\bar{x}_N)] \leq \frac{1}{w_{1:N}} \left( \frac{\alpha \Omega^2}{\eta} N^k + \frac{3 \sigma^2 \eta}{\alpha} \sum_{n=1}^N n^{2p-k} \right) + O(1)
\]

**Proof.** To simplify the notation, define \( l_n(x) = w_n(\langle F(x_n), x \rangle + r_n(x)) \) and let

\[
\text{regret}^w(\mathcal{X}) = \sum_{n=1}^N w_n \langle F(x_n), x_n \rangle - \min_{x \in \mathcal{X}} \sum_{n=1}^N w_n \langle F(x_n), x \rangle
\]

\[
\mathcal{R}^w(\mathcal{X}) = \sum_{n=1}^N l_n(x_n) - \min_{x \in \mathcal{X}} \sum_{n=1}^N l_n(x)
\]

By this definition, it holds that

\[
\text{regret}^w(\mathcal{X}) \leq \mathcal{R}^w(\mathcal{X}) + \max_{x \in \mathcal{X}} \sum_{n=1}^N w_n r_n(x)
\]

In the following, we bound the two terms in the upper bound above. First, by applying Stronger FTL Lemma (Lemma 4.1) with \( l_n \) and we can show that

\[
\mathcal{R}^w(\mathcal{X}) \leq \sum_{n=1}^N l_{1:n}(x_n) - l_{1:n}(x_n^*) - \Delta_n
\]

\[
\leq \sum_{n=1}^N \frac{\eta}{2\alpha} n^{2p-k} \| g_n - \hat{g}_n \|_*^2 - \frac{\alpha(n-1)^{k-1}}{2\eta} \| x_n - \hat{x}_n \|_*^2
\]
where $x^*_n := \arg\max_{x \in X} t_{i:n}(x)$. Because by Lemma H.3 and Lipschitz continuity of $F$, it holds

$$\|g_n - \hat{g}_n\|^2 \leq 3(L^2\|x_n - \hat{x}_n\|^2 + 2\sigma^2) \quad (F.3)$$

Therefore, we can bound

$$R_w(X) \leq \sum_{n=1}^{N} \left( \frac{3L^2\eta}{\alpha} n^{2p-k} - \frac{\alpha}{2\eta} (n-1)^k \right) \|x_n - \hat{x}_n\|^2 + \frac{3\sigma^2}{\alpha} \sum_{n=1}^{N} n^{2p-k} \quad (F.4)$$

If $k > p$, then the first term above is $O(1)$ independent of $N$. On the other hand,

$$\max_{x \in X} \sum_{n=1}^{N} w_n r_n(x) \leq \frac{\alpha \Omega^2}{\eta} \sum_{n=1}^{N} n^k \quad (F.5)$$

Combining the two bounds and Lemma F.1, i.e. $E[\text{ERR}(\bar{x}_N)] \leq E\left[\frac{\text{regret}_w(X)}{w_{1:N}}\right]$ concludes the proof. ■

Deterministic Problems For deterministic problems, we specialize the proof Theorem F.4 gives. We set $k = p = 0$, $x_1 = \arg\min_{x \in X} \omega(x)$, which removes the 2 factor in (F.5), and modify 3 to 1 in (F.3) (because the problem is deterministic). By recovering the constant in the proof, we can show that

$$E[\text{ERR}(\bar{x}_N)] \leq \frac{1}{N} \left( \frac{\alpha \Omega^2}{\eta} \sum_{n=1}^{N} \left( \frac{1}{2} \frac{L^2 \eta}{\alpha} - \frac{\alpha}{2\eta} \right) \|x_n - \hat{x}_n\|^2 \right)$$

Suppose . We choose $\eta$ to make the second term non-positive, i.e.

$$\frac{1}{2} \frac{L^2 \eta}{\alpha} - \frac{\alpha}{2\eta} \leq 0 \iff \eta \leq \frac{\alpha}{L}$$

and the error bound becomes

$$E[\text{ERR}(\bar{x}_N)] \leq \frac{L \Omega^2}{N}$$

This bound and the condition on $\eta$ matches that in [24].

Stochastic Problems For stochastic problems, we use the condition specified in Theorem F.4. Suppose $2p - k > -1$. To balance the second term in (F.4) and (F.5), we choose

$$2p - k + 1 = k \implies k = p + \frac{1}{2}$$

To satisfy the hypothesis $2p - k > -1$, it requires $p > -\frac{1}{2}$. Note with this choice, it satisfies the condition $k > p$ required in Theorem F.4. Therefore, the overall bound becomes

$$E[\text{ERR}(\bar{x}_N)] \leq \frac{1}{w_{1:N}} \left( \frac{\alpha \Omega^2}{\eta} N^{p+\frac{1}{2}} + \frac{3\sigma^2}{\alpha} \sum_{n=1}^{N} n^{p-\frac{3}{2}} \right) + \frac{O(1)}{w_{1:N}}$$

$$\leq \frac{p+1}{N^{p+1}} \left( \frac{\alpha \Omega^2}{\eta} + \frac{3\eta \sigma^2}{\alpha (p + \frac{1}{2})} \right) (N + 1)^{p+\frac{1}{2}} + \frac{O(1)}{N^{p+1}}$$

$$\leq e^{p+1/2} (p + 1) \left( \frac{\alpha \Omega^2}{\eta} + \frac{3\eta \sigma^2}{\alpha (p + \frac{1}{2})} \right) N^{-\frac{1}{2}} + \frac{O(1)}{N^{p+1}}$$

where we use Lemma H.1 and $(\frac{N+1}{N})^{p+1/2} \leq e^{p+1/2}$. If we set $\eta$ such that

$$\frac{\alpha \Omega^2}{\eta} = \frac{3\eta \sigma^2}{\alpha (p + \frac{1}{2})} \implies \eta = \frac{\alpha \Omega}{\sigma} \sqrt{\frac{p + \frac{1}{2}}{3}}$$
Then
\[
E[\text{ERR}(\bar{x}_N)] \leq 2e^{\frac{p+1}{N}}(p + 1)\Omega \sqrt{\frac{3}{p+\frac{1}{2}}} N^{-\frac{1}{2}} + O(1) N^{p+1}
\] (F.6)

For example, if \(p = 0\), then
\[
E[\text{ERR}(\bar{x}_N)] \leq O(1) N + 2\sqrt{6} \Omega e^{\frac{p+1}{N}} \frac{1}{\sqrt{N}}
\]
which matches the bound in by Juditsky et al. [11] with a slightly worse constant. We leave a complete study of tuning \(p\) as future work.

F.4 Comparison of stochastic Mirror-Prox and MoBIL-Prox in Imitation Learning

The major difference between stochastic Mirror-Prox and MoBIL-Prox is whether the gradient from the environment is used to also update the decision \(\pi_{n+1}\). It is used in the Mirror-Prox, whereas MoBIL-Prox uses the estimation from simulation. Therefore, for \(N\) iterations, MoBIL-Prox requires only \(N\) interactions, whereas Mirror-Prox requires \(2N\) interactions.

The price MoBIL-Prox pays extra when using the estimated gradient is that a secondary online learning problem has to be solved. This shows up in the term, for example of strongly convex problems,
\[
\frac{(p + 1)G_h^2}{2\mu_h} \frac{1}{N^2} + \frac{\epsilon_g^w + \sigma_g^2 + \sigma_g^2}{N}
\]
in Theorem 4.2. If both gradients are from the environment, then \(\epsilon_g^w = 0\) and \(\sigma_g^2 = \sigma_g^2\). Therefore, if we ignore the \(O(\frac{1}{N^2})\) term, using an estimated gradient to update \(\pi_{n+1}\) is preferred, if it requires less interactions to get to the magnitude of error, i.e.
\[
2 \times 2\sigma_g^2 \geq \epsilon_g^w + \sigma_g^2 + \sigma_g^2
\]
in which the multiplier of 2 on the left-hand side is due to MoBIL-Prox only requires one interaction per iterations, whereas stochastic Mirror-Prox requires two.

Because \(\sigma_g^2\) is usually large in real-world RL problems and \(\sigma_g^2\) can be made close to zero easily (by running more simulations), if our model class is reasonably expressive, then MoBIL-Prox is preferable. Essentially, this is because MoBIL-Prox can roughly cut the noise of gradient estimates by half.

The preference over MoBIL-Prox would be more significant for convex problems, because the error decays slower over iterations (e.g. \(\frac{1}{\sqrt{N}}\)) and therefore more iterations are required by the stochastic Mirror-Prox approach to counter balance the slowness due to using noisy gradient estimator.

G Experimental Details

G.1 Tasks

Two robot control tasks (Cartpole and Reacher3D) powered by the DART physics engine [19] were used as the task environments.

**Cartpole** The Cart-Pole Balancing task is a classic control problem, of which the goal is to keep the pole balanced in an upright posture with force only applied to the cart. The state and action spaces are both continuous, with dimension 4 and 1, respectively. The state includes the horizontal position and velocity of the cart, and the angle and angular velocity of the pole. The time-horizon of this task is 1000 steps. There is a small uniformly random perturbation injected to initial state, and the transition is deterministic. The agent receives +1 reward for every time step it stays in a predefined region, and a rollout terminates when the agent steps outside the region.
**Reacher3D** In this task, a 5-DOF (degrees-of-freedom) manipulator is controlled to reach a random target position in a 3D space. The reward is the sum of the negative distance to the target point from the finger tip and a control magnitude penalty. The actions correspond to the torques applied to the 5 joints. The time-horizon of this task is 500 steps. At the beginning of each rollout, the target point to reach is reset to a random location.

**G.2 Algorithms**

**Policies** We employed Gaussian policies in our experiments, i.e. for any state \( s \in S \), \( \pi_s \) is Gaussian distributed. The mean of \( \pi_s \) was modeled by either a linear function or a neural network that has 2 hidden layers of size 32 and tanh activation functions. The covariance matrix of \( \pi_s \) was restricted to be diagonal and independent of state. The expert policies in the IL experiments share the same architecture as the corresponding learners (e.g. a linear learner is paired with a linear expert) and were trained using actor-critic-based policy gradients.

**Imitation learning loss** With regard to the IL loss, we set \( D(\pi_s^* || \pi_s) \) in (2) to be the KL-divergence between the two Gaussian distributions: \( D(\pi_s^* || \pi_s) = KL[\pi_s || \pi_s^*] \). (We observed that using \( KL[\pi_s || \pi_s^*] \) converges noticeably faster than using \( KL[\pi_s^* || \pi_s] \)).

**Implementation details of MoBIL-Prox** The regularization of MoBIL-Prox was set to \( r_n(\pi) = \frac{\mu^2}{2} || \pi - \pi_n ||^2 \) such that \( \sum w_n \alpha_n \mu F = (1 + cn^{p+1/2})/\eta_n \), where \( c = 0.1 \) and \( \eta_n \) was adaptive to the norm of the prediction error. Specifically, we used \( \eta_n = \eta \lambda_n : \eta > 0 \) and \( \lambda_n \) is a moving-average estimator of the norm of \( e_n = g_n - \hat{g}_n \) defined as

\[
\hat{\lambda}_n = \beta \hat{\lambda}_{n-1} + (1 - \beta) ||e_n||^2 \\
\lambda_n = \hat{\lambda}_n/(1 - \beta^n)
\]

where \( \beta \) was chosen to be 0.999. This parameterization is motivated by the form of the optimal step size of MoBIL-Prox in Theorem 4.2, and by the need of having adaptive step sizes so different algorithms are more comparable. The model-free version was implemented by setting \( \tilde{g}_n = 0 \) in MoBIL-Prox, and the same adaptation rule above was used (which in this case effectively adjusts the learning rate based on \( ||g_n|| \)). In the experiments, \( \eta \) was selected to be 0.1 and 0.01 for \( p = 0 \) and \( p = 2 \), respectively, so the areas under the effective learning rate \( \eta_n w^p/(1 + cn^{p+1/2}) \) for \( p = 0 \) and \( p = 2 \) are close, making MoBIL-Prox perform similarly in these two settings.

In addition to the update rule of MoBIL-Prox, a running normalizer, which estimates the upper and the lower bounds of the state space, was used to center the state before it was fed to the policies.

**Dynamics model learning** The dynamics model used in the experiments is deterministic (the true model is deterministic too). It is represented by a neural network with 2 hidden layers of size 64 and tanh activation functions. Given a batch of transition triples \( \{(s_{tk}, a_{tk}, s_{tk+1})\}_{k=1}^{K} \) collected by running \( \pi_n \) under the true dynamics in each round, we set the per-round cost for model learning as \( \frac{1}{K} \sum_{k=1}^{K} ||s_{tk+1} - M(s_{tk}, a_{tk})||^2_z \), where \( M \) is the neural network dynamics model. It can be shown that this loss is an upper bound of \( \|\nabla F(\pi_n, \pi_n) - \nabla \bar{F}(\pi_n, \pi_n)\|^2_z \) by applying a similar proof as in Appendix D. The minimization was achieved through gradient descent using ADAM [25] with a fixed number of iterations (2048) and fixed-sized mini-batches (128). The step size of ADAM was set to 0.001.
H Useful Lemmas

This section summarizes some useful properties of polynomial partial sum, sequence in Banach space, and variants of FTL in online learning. These results will be useful to the proofs in Appendix C.

H.1 Polynomial Partial Sum

Lemma H.1. This lemma provides estimates of $\sum_{n=1}^{N} n^p$.

1. For $p > 0$, $\frac{N+1}{p+1} = \int_{0}^{N} x^p dx \leq \sum_{n=1}^{N} n^p \leq \int_{1}^{N+1} x^p dx \leq \frac{(N+1)^{p+1}}{p+1}$.
2. For $p = 0$, $\sum_{n=1}^{N} n^p = N$.
3. For $-1 < p < 0$,

$$\frac{(N+1)^{p+1}}{p+1} = \int_{1}^{N+1} x^p dx \leq \sum_{n=1}^{N} n^p \leq 1 + \int_{1}^{N} x^p dx = \frac{N^{p+1}+1}{p+1} \leq \frac{(N+1)^{p+1}}{p+1}.$$ 

4. For $p = -1$, $\ln(N+1) \leq \sum_{n=1}^{N} n^p \leq \ln N + 1$.

5. For $p < -1$, $\sum_{n=1}^{N} n^p \leq \frac{N^{p+1}+1}{p+1} = O(1)$. For $p = -2$, $\sum_{n=1}^{N} n^p \leq \frac{N^{-1}-2}{-2+1} \leq 2$.

Lemma H.2. For $p \geq -1$, $N \in \mathbb{N}$,

$$S(p) = \sum_{n=1}^{N} \frac{n^{2p}}{\sum_{m=1}^{n} m^p} \leq \begin{cases} \frac{p+1}{p} (N+1)^p, & \text{for } p > 0 \\ \ln(N+1), & \text{for } p = 0 \\ O(1), & \text{for } -1 < p < 0 \\ 2, & \text{for } p = -1 \end{cases}.$$ 

Proof. If $p \geq 0$, by Lemma H.1,

$$S(p) = (p+1) \sum_{n=1}^{N} n^{p-1} \leq \begin{cases} \frac{p+1}{p} (N+1)^p, & \text{for } p > 0 \\ \ln(N+1), & \text{for } p = 0 \end{cases}.$$ 

If $-1 < p < 0$, by Lemma H.1, $S(p) \leq (p+1) \sum_{n=1}^{N} \frac{n^p}{(n+1)^{p+1}}$. Let $a_n = \frac{n^p}{(n+1)^{p+1}}$, and $b_n = n^{p-1}$. Since $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and by Lemma H.1 $\sum_{n=0}^{\infty} b_n$ converges, thus $\sum_{n=0}^{\infty} a_n$ converges too. Finally, if $p = -1$, by Lemma H.1, $S(-1) \leq \sum_{n=1}^{N} \frac{1}{n^{\ln(n+1)}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. □

H.2 Sequence in Banach Space

Lemma H.3. Let $\{a = x_0, x_1, \ldots, x_N = b\}$ be a sequence in a Banach space with norm $\| \cdot \|$. Then for any $N \in \mathbb{N}$, $\|a-b\| \leq N \sum_{n=1}^{N} \|x_{n-1} - x_n\|^2$.

Proof. First we note that by triangular inequality it satisfies that $\|a - b\| \leq \sum_{n=1}^{N} \|x_{n-1} - x_n\|$. Then we use the basic fact that $2ab \leq a^2 + b^2$ in the second inequality below and prove the result.

$$\|a - b\|^2 \leq \sum_{n=1}^{N} \|x_{n-1} - x_n\|^2 + \sum_{n=1}^{N} \sum_{m=1;m \neq n} \|x_{n-1} - x_n\| \|x_{m-1} - x_m\|$$

$$\leq \sum_{n=1}^{N} \|x_{n-1} - x_n\|^2 + \sum_{n=1}^{N} \sum_{m=1;m \neq n} \frac{1}{2} (\|x_{n-1} - x_n\|^2 + \|x_{m-1} - x_m\|^2)$$

$$= \sum_{n=1}^{N} \|x_{n-1} - x_n\|^2 + \frac{N-1}{2} \sum_{n=1}^{N} \|x_{n-1} - x_n\|^2 + \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1;m \neq n} \|x_{m-1} - x_m\|^2$$
where

\[ x = \arg\min_{x} \sum_{n=1}^{N} \| x_{n-1} - x_{n} \|^{2} + (N - 1) \sum_{n=1}^{N} \| x_{n-1} - x_{n} \|^{2} \]

\[ = N \sum_{n=1}^{N} \| x_{n-1} - x_{n} \|^{2} \]

**H.3 Basic Regret Bounds of Online Learning**

For the paper to be self-contained, we summarize some fundamental results of regret bound when the learner in an online problem updates the decisions by variants of FTL. Here we consider a general setup and therefore use a slightly different notation from the one used in the main paper for policy optimization.

**Online Learning Setup** Consider an online convex optimization problem. Let \( \mathcal{X} \) be a compact decision set in a normed space with norm \( \| \cdot \| \). In round \( n \), the learner plays \( x_{n} \in \mathcal{X} \) receives a convex loss \( l_{n} : \mathcal{X} \to \mathbb{R} \) satisfying \( \| \nabla l_{n}(x_{n}) \| \leq G \), and then make a new decision \( x_{n+1} \in \mathcal{X} \). The regret is defined as

\[ \text{regret}(\mathcal{X}) = \sum_{n=1}^{N} l_{n}(x_{n}) - \min_{x \in \mathcal{X}} \sum_{n=1}^{N} l_{n}(x) \]

More generally, let \( \{ w_{n} \in \mathbb{R}^{+} \}_{n=1}^{N} \) be a sequence of weights. The weighted regret is defined as

\[ \text{regret}^{w}(\mathcal{X}) = \sum_{n=1}^{N} w_{n} l_{n}(x_{n}) - \min_{x \in \mathcal{X}} \sum_{n=1}^{N} w_{n} l_{n}(x) \]

In addition, we define a constant \( \epsilon_{\mathcal{X}}^{w} \) (which can depend on \( \{ l_{n} \}_{n=1}^{N} \) ) such that

\[ \epsilon_{\mathcal{X}}^{w} \geq \min_{x \in \mathcal{X}} \frac{\sum_{n=1}^{N} w_{n} l_{n}(x)}{w_{1:N}}. \]

In the following, we prove some basic properties of FTL with prediction. At the end, we show the result of FTL as a special case. These results are based on the Strong FTL Lemma (Lemma 3.2), which can also be proven by Stronger FTL Lemma (Lemma 4.1).

**Lemma 3.2** (Strong FTL Lemma [16]). For any sequence of decisions \( \{ x_{n} \in \mathcal{X} \} \) and loss functions \( \{ l_{n} \} \),

\[ \text{regret}(\mathcal{X}) \leq \sum_{n=1}^{N} l_{1:n}(x_{n}) - l_{1:n}(x_{n}^{*}), \text{ where } x_{n}^{*} \in \arg\min_{x \in \mathcal{X}} l_{1:n}(x), \text{ where } \mathcal{X} \text{ is the decision set}. \]

To use Lemma 3.2, we first show an intermediate bound.

**Lemma H.4.** In round \( n \), let \( l_{1:n} \) be \( \mu_{1:n} \)-strongly convex for some \( \mu_{1:n} > 0 \), and let \( v_{n+1} \) be a (non)convex function such that \( l_{1:n} + v_{n+1} \) is convex. Suppose the learner plays FTL with prediction, i.e. \( x_{n+1} \in \arg\min_{x \in \mathcal{X}} (l_{1:n} + v_{n+1})(x) \). Then it holds

\[ \sum_{n=1}^{N} (l_{1:n}(x_{n}) - l_{1:n}(x_{n}^{*})) \leq \sum_{n=1}^{N} \frac{1}{2\mu_{1:n}} \| \nabla l_{n}(x_{n}) - \nabla v_{n}(x_{n}) \|_{*}^{2} \]

where \( x_{n}^{*} = \arg\min_{x \in \mathcal{X}} \sum_{n=1}^{N} l_{n}(x) \).

**Proof.** For any \( x \in \mathcal{X} \), since \( l_{1:n} \) is \( \mu_{1:n} \) strongly convex, we have

\[ l_{1:n}(x_{n}) - l_{1:n}(x) \leq \langle \nabla l_{1:n}(x_{n}), x_{n} - x \rangle - \frac{\mu_{1:n}}{2} \| x_{n} - x \|^{2}. \] (H.1)

And by the hypothesis \( x_{n} = \arg\min_{x \in \mathcal{X}} (l_{1:n-1} + v_{n})(x) \), it holds that

\[ \langle -\nabla l_{1:n-1}(x_{n}) - \nabla v_{n}(x_{n}), x_{n} - x \rangle \geq 0. \] (H.2)
Adding (H.1) and (H.2) yields
\[
l_{1:n}(x_n) - l_{1:n}(x) \leq (\nabla l_n(x_n) - \nabla v_n(x_n), x_n - x) - \frac{\mu_{1:n}}{2}\|x_n - x\|^2
\]
\[
\leq \max_d \langle \nabla l_n(x_n) - \nabla v_n(x_n), d \rangle - \frac{\mu_{1:n}}{2}\|d\|^2
\]
\[
= \frac{1}{2\mu_{1:n}} \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2,
\]
where the last equality is due to a property of dual norm (e.g. Exercise 3.27 of [26]). Substituting \(x_n^*\) for \(x\) and taking the summation over \(n\) prove the lemma. \hfill \blacksquare

Using Lemma 3.2 and Lemma H.4, we can prove the regret bound of FTL with prediction.

**Lemma H.5** (FTL with prediction). Let \(l_n\) be a \(\mu_n\)-strongly convex function such that \(\sum_{m=1}^n w_m l_m + w_{m+1} v_{n+1}\) is convex. Suppose the learner plays FTL with prediction, i.e. \(x_{n+1} = \arg\min_{x \in X} \sum_{m=1}^n (w_m l_m + w_{m+1} v_{n+1})(x)\) and suppose that \(\sum_{m=1}^n w_m \mu_m > 0\). Then
\[
\sum_{n=1}^N \mu_n \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2
\]
In particular, if \(\mu_n = \mu\), \(w_n = n^p\), \(p \geq 0\), \(\text{regret}^w(\mathcal{X}) \leq \frac{p+1}{2p} \sum_{n=1}^N n^{p-1} \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2\).

**Proof.** By Lemma 3.2 and Lemma H.4, we see
\[
\text{regret}^w(\mathcal{X}) \leq \sum_{n=1}^N \left( l_{1:n}(x_n) - l_{1:n}(x_n^*) \right) \leq \sum_{n=1}^N \frac{w_n^2}{\sum_{m=1}^{n-1} w_m \mu_m} \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2.
\]
If \(\mu_n = \mu\), \(w_n = n^p\), and \(p \geq 0\), then it follows from Lemma H.1
\[
\text{regret}^w(\mathcal{X}) \leq \frac{1}{2\mu} \sum_{n=1}^N \frac{n^p}{n^{p+1}} \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2 = \frac{p+1}{2\mu} \sum_{n=1}^N n^{p-1} \|\nabla l_n(x_n) - \nabla v_n(x_n)\|^2.
\]

The next lemma about the regret of FTL is a corollary of Lemma H.5.

**Lemma H.6** (FTL). Let \(l_n\) be \(\mu\)-strongly convex for some \(\mu > 0\). Suppose the learner play FTL, i.e. \(x_n = \arg\min_{x \in X} \sum_{m=1}^n w_m l_m(x)\). Then \(\text{regret}^w(\mathcal{X}) \leq \frac{G^2}{2\mu} \sum_{n=1}^N \frac{w_n^2}{\sum_{m=1}^n m^p}\). In particular, if \(w_n = n^p\), then
\[
\sum_{n=1}^N w_n l_n(x_n) \leq \begin{cases} \frac{G^2}{2\mu} (N + 1)^p + \frac{1}{p+1} (N + 1)^{p+1} \epsilon X, & \text{for } p > 0 \\ \frac{G^2}{2\mu} \ln(N + 1) + N \epsilon X, & \text{for } p = 0 \\ \frac{G^2}{2\mu} O(1) + \frac{1}{p+1} (N + 1)^{p+1} \epsilon X, & \text{for } -1 < p < 0 \\ \frac{G^2}{\mu} + (\ln N + 1) \epsilon X, & \text{for } p = -1 \end{cases}
\]

**Proof.** By definition of \(\text{regret}^w(\mathcal{X})\), the absolute cost satisfies \(\sum_{n=1}^N w_n l_n(x_n) = \text{regret}^w(\mathcal{X}) + \min_{x \in X} \sum_{n=1}^N w_n l_n(x)\). We bound the two terms separately. For \(\text{regret}^w(\mathcal{X})\), set \(v_n = 0\) in Lemma H.5 and we have
\[
\text{regret}^w(\mathcal{X}) \leq \frac{G^2}{2\mu} \sum_{n=1}^N \frac{w_n^2}{\sum_{m=1}^n m^p},
\]
\[
= \frac{G^2}{2\mu} \sum_{n=1}^N \frac{n^{2p}}{\sum_{m=1}^n m^p},
\]
in which \(\sum_{n=1}^N \frac{n^{2p}}{\sum_{m=1}^n m^p}\) is exactly what H.2 bounds. On the other hand, the definition of \(\epsilon_X\) implies that \(\min_{x \in X} \sum_{n=1}^N w_n l_n(x) \leq w_1 N \epsilon_X = \sum_{n=1}^N n^p \epsilon_X\), where \(\sum_{n=1}^N n^p\) is bounded by Lemma H.1. Combining these two bounds, we conclude the lemma. \hfill \blacksquare