ABELIAN VARIETIES IN BRILL–NOETHER LOCI

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Abstract. In this paper, improving on results in [1, 7], we give the full classification of curves $C$ of genus $g$ such that a Brill–Noether locus $W^s_d(C)$, strictly contained in the jacobian $J(C)$ of $C$, contains a variety $Z$ stable under translations by the elements of a positive dimensional abelian subvariety $A \subset J(C)$ and such that $\dim(Z) = d - \dim(A) - 2s$, i.e., the maximum possible for such a $Z$.

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1. Introduction

In [1] the authors posed the problem of studying, and possibly classifying, situations like this:

(*) $C$ is a smooth, projective, complex curve of genus $g$, $Z$ is an irreducible $r$–dimensional subvariety of a Brill–Noether locus $W^s_d(C) \subset J^d(C)$, and $Z$ is stable under translations by the elements of an abelian subvariety $A \subset J(C)$ of dimension $a > 0$ (if so, we will say that $Z$ is $A$–stable).

Actually in [1] the variety $Z$ is the translate of a positive dimensional proper abelian subvariety of $J(C)$, while the above slightly more general formulation was given in [7].

The motivation for studying (*) resides, among other things, in a theorem of Faltings (see [9]) to the effect that if $X$ is an abelian variety defined over a number field $K$, and $Z \subset X$ is a subvariety not containing any translate of a positive dimensional abelian subvariety of $X$, then the number of rational points of $Z$ over $K$ is finite. The idea in [1] was to apply Faltings' theorem to the $d$–fold symmetric product $C(d)$ of a curve $C$ defined over a number field $K$. If $C$ has no positive dimensional linear series of degree $d$, then $C(d)$ is isomorphic to its Abel–Jacobi image $W_d(C)$ in $J^d(C)$. Thus $C(d)$ has finitely many rational points over $K$ if $W_d(C)$ does not contain any translate of a positive dimensional abelian subvariety of $J(C)$. The suggestion in [1] is that, if, by contrast, $W_d(C)$ contains the translate of a positive dimensional abelian subvariety of $J(C)$, then $C$ should be quite special, e.g., it should admit a map to a curve of lower positive genus (curves of this kind clearly are in situation (*)). This idea was tested in [1], where a number of partial results were proven for low values of $d$.

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The problem was taken up in [7], see also [8], where, among other things, it is proven that if (*) holds, then \( r + a + 2s \leq d \), and, if in addition \( d + r \leq g - 1 \), then \( r + a + 2s = d \) if and only if:

(a) there is a degree 2 morphism \( \varphi : C \to C' \), with \( C' \) a smooth curve of genus \( a \), such that \( A = \varphi^*(J(C')) \) and \( Z = W_{d-2a-2s}(C) + \varphi^*(J^{a+s}(C')) \).

In [7] there is also the following example with \( (d, s) = (g - 1, 0) \):

(b) there is an (étale) degree 2 morphism \( \varphi : C \to C' \), with \( C' \) a smooth curve of genus \( g' = r + 1 \), \( A \) is the Prym variety of \( \varphi \) and \( Z \subset W_{g-1}(C) \) is the connected component of \( \varphi^*(K_{C'}) \) consisting of divisor classes \( D \) with \( h^0(\mathcal{O}_C(D)) \) odd, where \( \varphi_* : J^{g-1}(C) \to J^{g-1}(C') \) is the norm map. One has \( Z \cong A \), hence \( r = a \).

One more family of examples is the following (see Corollary 3.9 below):

(c) \( C \) is hyperelliptic, there is a degree 2 morphism \( \varphi : C \to C' \) with \( C' \) a smooth curve of genus \( a \) such that \( g \geq 2a + 1 \), \( A = \varphi^*(J(C')) \), \( 0 < s < g - 1 \) and \( Z = \varphi^*(J^a(C')) + W_{d-2a-2s}(C) + W_{2a}(C) \) (notice that \( W_{2a}(C) \) is a point).

The aforementioned result in [7] goes exactly in the direction indicated in [1]. The unfortunate how strong it is, consider the case \( (d, s) = (g - 1, 0) \), which is indeed the crucial one (see [7, Proposition 3.3] and [3.2 below]) and in which Debarre–Fahaloui’s theorem is void.

The aim of the present paper is to give the full classification of the cases in which (*) happens and \( d = r + a + 2s \). What we prove (see Theorem 3.1 and Corollary 3.9) is that if (*) holds then, with no further assumption, either (a) or (b) or (c) occurs.

The idea of the proof is not so different, in principle, from the one proposed in [7] in the restricted situation considered there. Indeed, one uses the \( A \)-stability of \( Z \) and its maximal dimension to produce linear series on \( C \) which are not birational, in fact composed with a degree 2 irrational involution. The main tool in [7], inspired by [1], is a Castelnuovo’s type of analysis for the growth of the dimension of certain linear series.

Our approach also consists in producing a non birational linear series on \( C \), but it is in a sense more direct. We consider (*) with \( (d, s) = (g - 1, 0) \) and \( a + r = g - 1 \), i.e., the basic case (the others follow from this), in which \( Z \) is contained in \( W_{g-1}(C) \), which is a translate of the theta divisor \( \Theta \subset J(C) \). This immediately produces, restricting to \( Z \) the Gauss map of \( \Theta \), a base point free sublinear series \( L \) of dimension \( r \) of the canonical series of \( C \). It turns out that \( Z \) is birational to an irreducible component of the variety \( C(g - 1, L) \subset C(g - 1) \) consisting of all effective divisors of \( C \) of degree \( g - 1 \) contained in some divisor of \( L \). The \( A \)-stability of \( Z \) implies that \( C(g - 1, L) \) has some other component besides the one birational to \( Z \), and this forces \( L \) to be non–birational. Once one knows this, a (rather subtle) analysis of the map determined by \( L \) and of its image leads to the conclusion.

As for the contents of the paper, [2] is devoted to a few general facts about the varieties \( C(k, L) \subset C(k) \) of effective divisors of degree \( k \) contained in a divisor of a linear series \( L \) on a curve \( C \). These varieties, as we said, play a crucial role in our analysis. Section 3 is devoted to the proof of our main result.

One final word about our own interest in this problem, which is quite different from the motivation of [1, 7]. It is in fact related to the study of irregular surfaces \( S \) of general type, where situation (*) presents itself in a rather natural way. For example, let \( C \subset S \) be a smooth, irreducible curve, and assume that \( C \) corresponds to the general point of an irreducible component \( \mathcal{C} \) of the Hilbert scheme of curves on \( S \) which dominates \( \text{Pic}^0(S) \). Recall that there is also only one irreducible component \( \mathcal{K} \) of the Hilbert scheme of curves algebraically equivalent to canonical curves on \( S \) which dominates \( \text{Pic}^0(S) \) (this is called the main paracanonical system). The curves of \( \mathcal{C} \) cut out on \( C \) divisors which are residual, with respect to \( |K_C| \), of divisors cut out by curves in \( \mathcal{K} \). Consider now the one of the two systems \( \mathcal{C} \) and \( \mathcal{K} \) whose curves cut on \( C \) divisors of minimal
degree $d = \min\{C^2, K_S \cdot C\}$, and denote by $s$ the dimension of the general fibre of this system over $\text{Pic}^0(S)$. Thus the image of the natural map $\text{Pic}^0(S) \to J^d(C)$ defined by $\eta \mapsto O_C(C + \eta)$ (or by $\eta \mapsto \mathcal{O}_C(K_S + \eta)$) is a $q$–dimensional abelian variety contained in $W^r_d(C)$, which is what happens in (*). Thus understanding (*) would provide us with the understanding of (most) curves on irregular surfaces.

The results in this paper, even if restricted to the very special case of (*) in which $Z$ has maximal dimension, turn out to be useful in surface theory. For example, if $S$ is a minimal irregular surface of general type, then $K^2_S \geq 2p_g$ (see [6]). Using the results in this paper we are able to classify irregular minimal surfaces for which $K^2_S = 2p_g$. This classification is given in [5].

2. Linear series on curves and related subvarieties of symmetric products

2.1. Generalities. Let $C$ be a smooth, projective, irreducible curve of genus $g$. For an integer $k \geq 0$, we denote by $C(k)$ the $k$-th symmetric product of $C$ (by convention, $C(0)$ is a point).

Let $L$ be a base point free $g^r_k$ on $C$. We denote the corresponding morphism by $\phi_L : C \to C \subset \mathbb{P}^r$ and by $C'$ the normalization of $C$. We let $f : C \to C'$ be the induced morphism and $L'$ the linear series on $C'$ such that $L = f^*(L')$. We set $\deg(f) = \nu \geq 1$, so that $d = \delta \nu$, with $\delta = \deg(C)$. We say that $L$ is birational if $\phi_L : C \to C'$ is birational.

Let $k \leq d$ be a positive integer. We consider the incidence correspondence

$$\mathcal{C}(k, L) = \{(D, H) \in C(k) \times L \mid D \subseteq H\}$$

with projections $p_i$ (with $1 \leq i \leq 2$) to the first and second factor. Set $C(k, L) = p_1(\mathcal{C}(k, L))$, which has a natural scheme structure (see [1] or [2] p. 341]). Note the isomorphism

$$s : (D, H) \in \mathcal{C}(k, L) \mapsto (H - D, H) \in \mathcal{C}(d - k, L).$$

Lemma 2.1. (i) If $k \leq r$, then $C(k, L) = C(k)$.

(ii) If $k \geq r$, then $C(k, L)$ has pure dimension $r$. If $V$ is an irreducible component of $C(k, L)$, there is a unique component $V'$ of $C(k, L)$ dominating $V$ via $p_1$ and $p_1$ induces a birational morphism of $V$ onto $V'$. Finally $V$ dominates $L$ via $p_2$.

(iii) If $\min\{k, d - k\} \geq r$ the isomorphism $s$ induces a map $\tau : C(k, L) \dashrightarrow C(d - k, L)$, which is componentwise birational.

(iv) If $L$ is birational, then $C(k, L)$ is irreducible.

Proof. Part (i) is clear. The dimensionality assertion in (ii) follows from [1 § 1] or [2] Lemma (3.2), p. 342]. The rest of (ii) follows from these facts: the fibres of $p_1$ are isomorphic to linear subseries of $L$ and $p_2$ is finite, so no component of $C(k, L)$ has dimension larger than $r$. Part (iii) follows from (ii). Part (iv) follows from the Uniform Position Theorem (see [2] p. 112)).

Next we look at the case $L$ non–birational and $k \geq r$. Consider the induced finite morphism $f_s : C(k, L) \to C'(k)$. For each partition $m = (m_1, \ldots, m_s)$ of $k$ with $\delta \geq s \geq r$ and $1 \leq m_1 \leq m_2 \leq \cdots \leq m_s \leq \nu$, we denote by $C(m, L)$ the closure in $C(k, L)$ of the inverse image via $f_s$ of the set of divisors of the form $m_1y_1 + \cdots + m_s y_s$, with $y_1 + \cdots + y_s$ a reduced divisor in $C'(s, L')$. To denote a partition $m$ as above we may use the exponential notation $m = (1^{\mu_1}, \ldots, \nu^{\mu_\nu})$, meaning that $i$ is repeated $\mu_i$ times, with $1 \leq i \leq \nu$, and we may omit $i^{\mu_i}$ if $\mu_i = 0$. Note that $\sum_{i=1}^{\nu} \mu_i = s \leq \delta$. Set $\mu_0 := \delta - s$. Then $m^c = (1^{\mu_0-1}, \ldots, (\nu - 1)^{\mu_1}, \nu^{\mu_\nu})$ is a partition of $d - k$ which we call the complementary partition of $m$.

Lemma 2.2. In the above set up each irreducible component of $C(m, L)$ has dimension $r$, hence it is an irreducible component of $C(k, L)$ and all irreducible components of $C(k, L)$ are of this type.
Proof. Let $V$ be an irreducible component of $C(k, L)$: by Lemma 2.1 there is a unique irreducible component $\mathcal{V}$ of $C(k, L)$ dominating it and $p_1|_{\mathcal{V}}: \mathcal{V} \to V$ is birational. Let $D \in V$ be a general point and let $(D, H)$ be the unique point of $\mathcal{V}$ mapping to $D$ via $p_1$, so that $H$ is a general divisor in $L$ (cf. Lemma 2.1 (ii)). Hence $H$ consists of $\delta$ distinct fibres $F_1, \ldots, F_\delta$ of $f$, each being a reduced divisor of degree $\nu$ on $C$. Then $D$ consists of $m_i$ points in $F_i$, for $1 \leq i \leq \delta$, where we may assume $1 \leq m_1 \leq m_2 \leq \cdots \leq m_\delta \leq \nu$. Moreover, since $\dim(V) = \dim(\mathcal{V}) = r$ and $p_2$ is finite, one has $s \geq r$. Hence $D \in C(m, L)$, with $m = (m_1, \ldots, m_\delta)$, i.e. $V \subseteq C(m, L) \subseteq C(k, L)$ hence $V$ is a component of $C(m, L)$.

The above considerations and Lemma 2.1 (ii), applied to $C'(s, L')$, imply that the image of $C(m, L)$ in $C'(k)$ has dimension $r$, so each component of $C(m, L)$ has dimension $r$. □

2.2. Abel–Jacobi images. We assume from now on that $C$ has genus $g > 0$. For an integer $k$, we denote by $J^k(C) \subseteq \text{Pic}(C)$ the set of linear equivalence classes of divisors of degree $k$ on $C$. So $J(C) := J^0(C)$ is the Jacobian of $C$, which is a principally polarised abelian variety whose theta divisor class we denote by $\Theta_C$, or simply by $\Theta$.

The abelian variety $J(C)$ acts via translation on $J^k(C)$ for all $k$. If $X \subseteq J^k(C)$ and $Y \subseteq J(C)$, we say that $X$ is $Y$–stable, if for all $x \in X$ and for all $y \in Y$, one has $x + y \in X$.

For all integers $k$, fixing the class of a divisor of degree $k$ determines an isomorphism $J^k(C) \cong J(C)$. Given a subvariety $V$ of $J^k(C)$, one says that it generates $J^k(C)$ if the image of $V$ via one of the above isomorphisms generates $J(C)$ as an abelian variety. This definition does not depend on the choice of the isomorphism $J^k(C) \cong J(C)$.

For every $k \geq 1$, we denote by $j_k: C(k) \to J^k(C) \cong J(C)$ (or simply by $j$) the Abel–Jacobi map. We denote by $W^s_k(C)$ the subscheme of $J^k(C)$ corresponding to classes of divisors $D$ such that $h^0(\mathcal{O}_C(D)) \geq s + 1$ (these are the so–called Brill–Noether loci). One sets $W^s_k(C) := W^0_k(C) = \text{Im}(j_k)$ and $W^s_{g-1}(C)$ maps to a theta divisor of $J(C)$, so we may abuse notation and write $W^s_{g-1}(C) = \Theta_C$.

We denote by $\Gamma_C(k, L)$ [resp. $\Gamma_C(m, L)$] the image in $W^s_k(C)$ of $C(k, L)$ [resp. $C(m, L)$] (we may drop the subscript $C$ if there is no danger of confusion). The expected dimension of $\Gamma(k, L)$ is $\min\{r, g\}$ (by dimension of a scheme we mean the maximum of the dimensions of its components).

We set $\rho_C(k, L) := \dim(\Gamma_C(k, L))$ (simply denoted by $\rho(k, L)$ or by $\rho$ if no confusion arises). By Lemma 2.1 (i), one has:

$$\tag{2.1}$$
if $k \leq r$ then $\Gamma(k, L) = W_k(C)$, hence $\rho = \min\{k, g\}$

So we will consider next the case $k > r$, in which $\rho \leq \dim(C(k, L)) = r$, by Lemma 2.1 (ii). Then the class $c(k, L)$ of $C(k, L)$ in the Chow ring of $C(k)$ is computed in [2, Lemma VIII.3.2]. If $x$ is the class of $C(k-1) \subseteq C(k)$ and $\theta := j^*(\Theta)$, one has

$$\tag{2.2}$$
c(k, L) = \sum_{s=0}^{k-r} \binom{d-g-r}{s} x^s \delta^{k-r-s} \theta^{k-r-s} / (k-r-s)!.
$$

Lemma 2.3. Assume $k > r$ and $d - g - r \geq 0$. Then:

(i) if $k - g \leq \min\{k-r, d-g-r\}$ one has $\rho = r \leq g$;

(ii) if $k - g \geq \min\{k-r, d-g-r\} = k-r$, one has $g \leq r$;

(iii) if $k - g \geq \min\{k-r, d-g-r\} = d-g-r$, one has $\rho = d-k \leq \min\{r, g\}$.

Proof. Note that $x^s$ is the class of $C(k-s) \subseteq C(k)$, for $1 \leq s \leq k$. Applying the projection formula (cf. [10, Example 8.1.7]) to (2.2), we find the class $\gamma(k, L)$ of $\Gamma(k, L)$

$$\tag{2.3}$$
$$\gamma(k, L) = \sum_{s=0}^{k-r} \binom{d-g-r}{s} w_{k-s} \delta^{k-r-s} \theta^{k-r-s} / (k-r-s)!. \quad \Box$$
where $w_i$ is the class of $W_i(C)$ for any $i \geq 0$. By Poincaré’s formula (cf. [3] §11.2] one has

$$w_{k-s} \Theta^{k-r-s} = \begin{cases} \Theta^{k-r-s}, & \text{if } k - g \geq s > 0, \\ \Theta^{g-r}, & \text{if } \max\{0, k - g\} \leq s \leq \min\{k - r, d - g - r\}, \end{cases}$$

whence the assertion follows. \qed

**Lemma 2.4.** (i) If $\rho = g$ then $g \leq \min\{k, r\}$;
(ii) if $r \geq k \geq g$, then $\rho = g$;
(iii) if $k > r \geq g$ and $d \geq k + g$ then $\rho = g$;
(iv) $\rho = 0$ if and only if $k = d$.

**Proof.** Parts (i) and (ii) follow from Lemma 2.1

(iii) In (2.3) one has the summand corresponding to the index $s = k - r > 0$, which is $\Theta^0$ with the positive coefficient $(d_{k-r})$, and no other summand in (2.3) cancels it.

(iv) If $k = d$ then $C(k, L) = L$ and clearly $\rho = 0$. Conversely, if $\rho = 0$ then in (2.3) the term $\Theta^0$ has to appear with non-zero coefficient and no other term $\Theta^i$ with $0 \leq i < g$ appears with non-zero coefficient. By looking at the proof of Lemma 2.3 we see that the summand $\Theta^0$ appears in (2.3) only if $0 \leq s = k - r - g$. Then $d \geq k \geq r + g$. So we may apply Lemma 2.3 and conclude that $\rho = 0$ occurs only in case (iii), if $k = d$. \qed

**Lemma 2.5.** Let $A \subseteq J(C)$ be an abelian subvariety of dimension $a$ and let $p: J^k(C) \to J' := J(C)/A$ be the map obtained by composing an isomorphism $J^k(C) \cong J(C)$ with the quotient map $J(C) \to J'$. Then

$$\dim(p(\Gamma(k, L))) = \min\{g - a, \rho\}.$$

**Proof.** If $\rho = g$ the statement is obvious, hence we assume $\rho < g$.

Consider first the case $k > r$. Assume by contradiction that $\dim(p(\Gamma(k, L))) < \min\{\rho, g - a\}$. Let $\xi$ be the class of the pull back to $J(C)$ of an ample line bundle of $J'$. We have $\overline{\gamma}(k, L)\xi^a = 0,$ where $\overline{\gamma}(k, L)$ is the $\rho$-dimensional part of $\gamma(k, L)$. By (2.3) one has $\overline{\gamma}(k, L) = \alpha \Theta^{g-\rho}$, where $\alpha \in \mathbb{Q}$ is positive because $\Gamma(k, L)$ is an effective non-zero cycle of dimension $\rho$. Hence $\overline{\gamma}(k, L)\xi^a = \alpha \Theta^{g-\rho} \xi^a > 0$, because $\Theta$ is ample. Thus we have a contradiction.

If $k \leq r$, then $\Gamma(k, L) = W_k(C)$, $\rho = k$ and $\gamma(k, L)$ is again a rational multiple of $\Theta^{g-k}$ (by Poincaré’s formula), so the proof proceeds as above. \qed

**Corollary 2.6.** If $A \subseteq J(C)$ is an abelian subvariety of dimension $a > 0$ and $\Gamma(k, L)$ is $A$-stable, then the restriction of $p$ to $\Gamma(k, L)$ is surjective onto $J' = J(C)/A$, hence $\Gamma(k, L) = J^k(C) = W_k(C)$, i.e., $\rho = g$.

**Corollary 2.7.** If $d - 1 \geq k \geq 1$, then $\Gamma(k, L)$ generates $J^k(C)$.

**Proof.** If $k \leq r$ then $\Gamma(k, L) = W_k(C)$ and the assertion is clear. Assume $k > r$. By Lemma 2.5, $\Gamma(k, L)$ generates $J^k(C)$ as soon as $\rho > 0$, which is the case by Lemma 2.4 (iv). \qed

### 2.3. A useful lemma

Let $L$ be a base point free $g^1_d$, let $\phi_L: C \to \mathbb{P}^1$ be the corresponding map and denote by $G_L$ the Galois group of $\phi_L$.

**Lemma 2.8.** If $L$ is a base point free $g^1_d$, then one of the following occurs:

(a) $C(2, L)$ is irreducible.

(b) $C(2, L)$ has two components. This occurs if and only if $G_L \cong \mathbb{Z}_2, \mathbb{Z}_4$.

(c) $C(2, L)$ has 3 components. This occurs if and only if $G_L \cong \mathbb{Z}_2^3$.

**Proof.** We argue as in the proof of [3, Lemma 12.7.1]. Let $\Delta \subseteq \mathbb{P}^1$ be the set of critical values of $\phi_L: C \to \mathbb{P}^1$, let $\rho: \pi_1(\mathbb{P}^1 \setminus \Delta) \to \mathbb{S}_4$ be the monodromy representation and let $\Sigma := \text{Im}(\rho)$. The group $\Sigma$ acts transitively on $I_4 := \{1, 2, 3, 4\}$ (identified with the general divisor $x_1 + \ldots +$
Assume $\Sigma$ is a 2–group. If $s = 8$ then $\Sigma$ is a 2-Sylow subgroup of $G_4$, hence $\Sigma$ is the dihedral group $D_4$. Then $C$ is obtained from a $\Sigma$–cover $C_0 \rightarrow \mathbb{P}^1$ by moding out by a reflection $\sigma \in \Sigma$. In the $\Sigma$–action on $I_4$ we may assume that an element of order 4 acts by sending $i$ to $i + 1$ modulo 4, for $1 \leq i \leq 4$. So the order 2 element in the center of $\Sigma$ induces the involution $\iota$ of $C$ that maps $i$ to $i + 2$ modulo 4, for $1 \leq i \leq 4$, and $\iota$ generates $G_L \cong \mathbb{Z}_2$. There are two orbits for the $\Sigma$–action on the set of order two subsets of $I_4$: one of order 2 given by $\{(1, 3), (2, 4)\}$, the other of order 4 given by $\{(i, i + 1), \text{ for } 1 \leq i \leq 4\}$ (here $i$ is taken modulo 4). These orbits respectively correspond to two components $E_1, E_2$ of $C(2, L)$ and we are in case (b).

Assume $s = 4$. Then $\phi_L$ is Galois with $G_L = \Sigma$. If $G_L \cong \mathbb{Z}_4$, then the $\Sigma$–orbits on the set of order two subsets of $I_4$ are as in the previous case. If $G_L \cong \mathbb{Z}_2^2$, then one has $G_L = \{(1d, (12)(34), (13)(24), (14)(32))\}$. There are then three orbits, corresponding to three components $E_1, E_2, E_3$.

One can be more precise about the components $E_i$ of $C(2, L)$ in Lemma 2.8 whose geometric genera we denote by $g_i$ (with $1 \leq i \leq 2 + \epsilon$, and $\epsilon = 0$ in case (b), $\epsilon = 1$ in case (c)).

**Lemma 2.9.** Same setting and notation as in Lemma 2.8 and its proof. Then:

(i) each component of $C(2, L)$ maps birationally to its image in $J^2(C)$ unless $C$ is hyperelliptic and $L$ is composed with the hyperelliptic involution $\mathcal{L}$: in this case one of the components of $\Gamma(2, L) = C(2, L)$, which is contracted to a point in $J^2(C)$;

(ii) in case (b) one has $E_1 \cong C/\iota$ (where $\iota$ is the non–trivial involution in $G_L$), which is hyperelliptic, and the abelian subvariety of $J^2(C) \cong J(C)$ generated by $j(E_1)$ is the $\iota$–invariant part of $J(C)$. Moreover $2g_2 \geq g$;

(iii) In case (c) one has $E_i := C/\iota_i$ where $\iota_i$ are the three nonzero elements of $G_L$, for $1 \leq i \leq 3$.

**Proof.** We prove the only non–trivial assertion, i.e., $2g_2 \geq g$ in part (ii).

First assume $G_L = \mathbb{Z}_4 = \langle \rho \rangle$. Consider in $C(2)$ the curves $E_1 = \{P + \rho^2(P) \mid P \in C\}$ and $E_2 = \{P + \rho(P) \mid P \in C\}$. One has $C(2, L) = E_1 \cup E_2$ and $E_1 \cong C/\rho^2$. The curve $E_2$ is the image in $C(2)$ of the graph of $\rho$, so $g_2 = g$.

Suppose now $G_L = \mathbb{Z}_2$ and $\Sigma = D_4$. Recall that $C$ is obtained from a $D_4$–Galois cover $f : C_0 \rightarrow \mathbb{P}^1$ by moding out by a reflection $\sigma \in D_4$ (see the proof of Lemma 2.8). Denote by $g_0$ the genus of $C_0$. Let $\rho \in D_4$ be an element of order 4, so that $D_4 = \langle \sigma, \rho \rangle$. Let $n$ be the number of points of $C_0$ fixed by $\sigma$, $n'$ the number of points fixed by $\sigma \rho$, $m$ the number of points fixed by $\rho$ and $m + \epsilon$ the number of points fixed by $\rho^2$. The Hurwitz formula, applied to $C_0 \rightarrow \mathbb{P}^1$ and to $C_0 \rightarrow C = C_0/\sigma$, gives

$$g_0 = \frac{3}{2}m + n + n' + \frac{\epsilon}{2} - 7, \text{ and } g = \frac{n'}{2} + \frac{n + \epsilon}{4} - \frac{3}{4}m - 3. \quad (2.4)$$

**Claim 2.10.** (i) $m, n, n'$ and $\epsilon$ are even.

(ii) $n + m \equiv n' + m \equiv n + n' \equiv 0 \mod 4$, and at most one among $m, n, n'$ can be 0.

**Proof of the Claim.** (i) The numbers $n, n'$ and $m + \epsilon$ are even because $\sigma, \sigma \rho$ and $\sigma \rho^2$ are involutions. If $P \in C_0$ is fixed by $\rho$, then $\sigma(P)$ is also fixed by $\rho$. Since the stabilizer of any point is cyclic, then $\sigma(P) \neq P$. This implies that $m$ is even.

(ii) Consider the $\mathbb{Z}_2^2$–cover $D := C_0/\rho^2 \rightarrow \mathbb{P}^1$. The cardinalities of the images in $\mathbb{P}^1$ of the fixed loci of the three involutions are $n/2, n'/2$ and $m/2$. Indeed, denote by $\gamma_1$ [resp. by $\gamma_2$] the image of $\sigma$ [resp. of $\rho$] in $D_4/\rho^2 \cong \mathbb{Z}_2^2$. Let $Q \in \mathbb{P}^1$ be a branch point whose preimage in $D$ is fixed by $\gamma_1$. Then the preimage of $Q$ in $C_0$ consists of 4 points, two of which fixed by $\sigma$ and two by
\[\sigma p^2, \text{ so the number of such points } Q \text{ is } (2n)/4 = n/2. \] Similarly, the image in \(\mathbb{P}^1\) of the set of points of \(D\) fixed by \(\gamma_2\) has cardinality \(m/2\) and the image of the set of points fixed by \(\gamma_1\gamma_2\) has cardinality \(n'/2\). Hurwitz formula for \(D_1 := D/\gamma_1 \rightarrow \mathbb{P}^1\) gives
\[
2g(D_1) - 2 = \frac{m}{2} + \frac{n'}{2} - 4,
\]
hence \(m + n' > 0\) is divisible by 4. Similarly \(n + n'\) and \(m + n\) are positive and divisible by 4. □

We compute now the ramification of \(f: C_0 \rightarrow \mathbb{P}^1\) and of \(L\). Write the \(D_4\)-orbit of \(P \in C_0\) general as
\[
(2.5) \quad P, \rho(P), \rho^2(P), \rho^3(P), \sigma(P), \sigma\rho(P), \sigma\rho^2(P), \sigma\rho^3(P).
\]
Denote by \(Q_1, \ldots, Q_4\) the images in \(C\) of the points in the first row (or, what is the same, in the second row) of (2.5). The singular fibers of \(f\) occur when \(P\) has non trivial stabilizer, i.e., when:

- \(P \in C_0\) is fixed by \(\rho\). The fiber of \(f\) is \(4(P + \sigma(P))\) and the corresponding divisor of \(L\) is \(4Q_1\).

There are \(m/2\) divisors of \(L\) of this type;

- \(P\) is fixed by \(\rho^2\) but not by \(\rho\). The fiber of \(f\) is \(2(P + \sigma(P) + \rho(P) + \sigma\rho(P))\). Then \(Q_1 = Q_3, Q_2 = Q_4, \) and the corresponding divisor of \(L\) is \(2(Q_1 + Q_2)\). There are \(\epsilon/4\) such divisors;

- \(P\) is fixed by \(\sigma\). The fibre of \(f\) is \(2(P + \rho(P) + \rho^2(P) + \rho^3(P))\), and \(Q_2 = Q_4\), while \(Q_1, Q_2\) and \(Q_3\) are distinct, so the corresponding divisor of \(L\) is \(Q_1 + 2Q_2 + Q_3\) and there are \(n'/2\) such fibers;

- \(P\) is fixed by \(\sigma\rho^2\). This is the same as the previous case;

- \(P\) is fixed by \(\sigma\rho\). The fibre of \(f\) is again \(2(P + \rho(P) + \rho^2(P) + \rho^3(P))\), and \(Q_1 = Q_2, Q_3 = Q_4\), so the corresponding divisor of \(L\) is \(2Q_1 + 2Q_3\) and there are \(n'/2\) such fibers;

- \(P\) is fixed by \(\sigma\rho^3\). This is the same as the previous case.

Denote by \(\iota\) the involution of \(C\) induced by \(\rho^2\). Then \(C(2, L)\) is the union of \(E_1 = \{P + \iota(P) \mid P \in C\} = C/\iota\) and of the irreducible curve \(E_2\). Keeping the above notation, if \(Q_1 + \cdots + Q_4\) is the general divisor of \(L\), then \(E_1\) is described by the divisors \(Q_1 + Q_3, Q_2 + Q_4\) and \(E_3\) by \(Q_1 + Q_2, Q_1 + Q_4, Q_2 + Q_3, Q_3 + Q_4\).

To compute \(g_2\), define a map \(\phi: C_0 \rightarrow E_2\) by sending \(P \in C_0\) to the image via \(C_0 \rightarrow C\) of the divisor \(P + \rho(P)\), i.e., \(Q_1 + Q_2 \in E_2\).

**Claim 2.11.** (i) \(\deg(\phi) = 2\) and \(E_2\) is birational to \(C_0/\sigma\rho\);
(ii) \(g_2 = \frac{3}{4}m + \frac{n}{2} + \frac{n'}{4} + \frac{\epsilon}{4} - 3\).

**Proof of the Claim.** (i) Let \(Q + Q' \in E_2\) be a general point and let \(P, \sigma(P)\) [resp. \(P', \sigma(P')\)] be the preimages of \(Q\) [resp. of \(Q'\)] on \(C_0\). Since \(Q + Q'\) is of the form \(Q_1 + Q_2, Q_1 + Q_4, Q_2 + Q_3,\) or \(Q_3 + Q_4\), we may assume that \(P' = \rho(P)\), so that \(\psi^{-1}(Q + Q') = \{P, \sigma\rho(P)\}\).

Part (ii) follows by applying Hurwitz formula. □

Finally, suppose by contradiction that \(2g_2 \leq g - 1\). Then, by (ii) of Claim 2.11 and by (2.4), we would have \(3(m + n) + \epsilon \leq 8\), hence \(m + n \leq 2\), contradicting (ii) of Claim 2.10. □

### 3. Abelian subvarieties of Brill–Noether loci

As in [1, 7], we consider \(Z \subseteq W^a_d(C) \subseteq J^d(C)\) an irreducible \(A\)-stable variety of dimension \(r\), with \(A \subseteq J(C)\) an abelian subvariety of dimension \(a > 0\). Note that \(r \geq a\), with equality if and only if \(Z \cong A\). Moreover, since \(W^a_d(C) \subseteq J^d(C)\), the general linear series \(L \in Z\) is special, thus \(s > d - g\) and \(d \geq 2s\) by Clifford’s theorem. From [7 Prop. 3.3], we have
\[
(3.1) \quad r + a + 2s \leq d.
\]
In this section, we classify the cases in which equality holds in (3.1), thus improving the partial results in [7] on this subject. Note that if equality holds in (3.1), then $Z \not\subset W_d^{s+1}(C)$ and $A$ is a maximal abelian subvariety of $J(C)$ such that $Z$ is $A$-stable.

### 3.1. The Theta divisor case

As in [7], we first consider the case $(d, s) = (g - 1, 0)$, i.e., $Z \subset W_{g-1}(C) = \Theta$ is an irreducible $A$-stable variety of dimension $r = g - 1 - a$. Then $Z \not\subset W_{g-1}^1 (C) = \text{Sing}(\Theta)$, i.e., $Z$ has a non-empty intersection with $\Theta_{\text{sm}} := \Theta - \text{Sing}(\Theta)$.

**Theorem 3.1.** Let $C$ be a curve of genus $g$. Let $A \subset J(C)$ be an abelian variety of dimension $a > 0$ and $Z \subset \Theta$ an irreducible, $A$-stable variety of dimension $r = g - 1 - a$. Then there is a degree 2 morphism $\varphi: C \to C'$, with $C'$ smooth of genus $g'$, such that one of the following occurs:

- (a) $g' = a$, $A = \varphi^*(J(C'))$ and $Z = W_{g-1-2a}(C) + \varphi^*(J_a(C'))$;
- (b) $g' = r+1$, $\varphi$ is étale, $A$ is the Prym variety of $\varphi$ and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes $D$ with $h^0(\mathcal{O}_C(D))$ odd, where $\varphi_*: J^{g-1}(C) \to J^{g-1}(C')$ is the norm map. In particular, $Z \cong A$ is an abelian variety if and only if either we are in case (a) and $g = 2a + 1$, or in case (b).

**Remark 3.2.** Cases (a) and (b) of Theorem 3.1 are not mutually exclusive. Indeed, if the curve $C$ in case (b) is hyperelliptic, then the Abel-Prym map $C \to A$ induces a 2-to-1 map $\psi: C \to D$, where $D$ is a smooth curve embedded into $A \cong J(D)$ by the Abel-Jacobi map. One can check that $A = \psi^*(J(D))$, namely this is also an instance of case (a) of Theorem 3.1.

The proof of Theorem 3.1 requires various preliminary lemmas. First, recall that the tangent space to $J(C)$ at 0 can be identified with $H^1(C, \mathcal{O}_C) \cong H^0(K_C)^*$. Denote by $T \subseteq H^0(K_C)^*$ the tangent space to $A$ at 0 and by $L$ the linear series $\mathbb{P}(T^\perp) \subseteq |K_C|$. One has $\dim(L) = g - 1 - a = r$.

Since $Z \cap \Theta_{\text{sm}} \neq \emptyset$, the Gauss map of $\Theta$ restricts to a rational map $\gamma: Z \dashrightarrow \mathbb{P} := |K_C|$.

**Lemma 3.3.** One has $\gamma(Z) = L$.

**Proof.** Since $Z$ is $A$-stable, one has $\gamma(Z) \subseteq L$. A point of $\Theta_{\text{sm}}$ can be identified with a divisor $D$ of degree $g - 1$ such that $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_C(K_C - D)) = 1$. The Gauss map sends $D \in \Theta_{\text{sm}}$ to the unique divisor of $|K_C|$ containing $D$. Then $\gamma^{-1}(\gamma(D))$ is finite. Hence $\dim(\gamma(Z)) = \dim(Z) = r = \dim(L)$. The assertion follows.

As in [7], formula (3.1) for $(d, s) = (g - 1, 0)$ follows from the argument in the proof of Lemma 3.3 and the general case follows from this (see [3.2 below]).

Note the birational involution $\sigma: \Theta \dashrightarrow \Theta$, defined on $\Theta_{\text{sm}}$, sending a divisor $D \in \Theta_{\text{sm}}$ to the unique effective divisor $D' \in |K_C - D|$. Then $\sigma$ restricts on $Z$ to a birational map (still denoted by $\sigma$) onto its image $Z'$, which is also $A$-stable.

**Lemma 3.4.** The linear series $L$ is base point free.

**Proof.** Suppose $P \in C$ is a base point of $L$. If every $D \in Z$ contains $P$, then the map $D \to D-P$ defines an injection $Z \hookrightarrow W_{g-2}(C)$, contradicting (3.1). If the general $D \in Z$ does not contain $P$, then $\sigma(D)$ contains $P$ for $D \in Z$ general, and we can apply the previous argument to $Z'$.

**Lemma 3.5.** One has:

- (i) $Z$ is a component of $\Gamma(g-1, L)$.
- (ii) $Z \not\subseteq \Gamma(g-1, L)$.
- (iii) The linear series $L$ is not birational.

**Proof.** By Lemma 3.3 we have $Z \subseteq \Gamma(g-1, L)$ and the dimension of $\Gamma(g-1, L)$ is equal to $r$ (by Lemma 2.3). So (i) holds. If $Z = \Gamma(g-1, L)$, then $\Gamma(g-1, L)$ is $A$-stable. By Corollary 2.6, we have $r = g$, a contradiction. This proves (ii). Then (iii) holds by Lemma 2.1 (iv).
Recall the notation $\phi_L: C \to \bar{C} \subset \mathbb{P}^r$, $C'$ for the normalization of $\bar{C}$, $f: C \to C'$ the induced morphism and $\nu = \deg(f) > 1$. For any integer $h$ one has the morphism $f^*: J^h(C') \to J^{h\nu}(C)$. Let $g'$ be the genus of $C'$ and $L'$ the (birational) linear series on $C'$ of dimension $r$ such that $L = f^*(L')$, whose degree is $\delta = \frac{2g-2}{r}$.

**Lemma 3.6.** Assume $\nu \leq 3$ and let $m = (1^{\mu_1}, \ldots, \nu^{\mu_\nu})$ be the partition of $g - 1$ such that $Z \subseteq \Gamma(m, L)$. Set $\mu_1 = \mu, \mu_2 = \mu'$. Then $\nu = 2$ and there are the following possibilities:

(i) $\mu = g - 1$ (hence $\mu' = 0$);
(ii) $\mu = g - 2a, \mu' = a, A = f^*(J(C'))$ (hence $g' = a$) and $Z = W_{\mu}(C) + f^*(J^{\mu}(C'))$.

**Proof.** We first consider the case $\nu = 2$.

If $\mu = 0$, then $g - 1 = 2\mu'$ and $Z = f^*(\Gamma_{C'}(\mu', L'))$, since $\Gamma_{C'}(\mu', L')$ is irreducible by Lemma 2.1. The general point of $Z$ corresponds to a linearly isolated divisor, hence $\mu' \leq g'$. Since $f^*: J^\mu(C') \to J^{g-1}(C)$ is finite, one has $r = \rho_{C'}(\mu', L') \leq \mu' \leq g'$. Since $Z$ is $A$–stable, we have an isogeny $A \to \bar{A} \subseteq J(C')$ (so that $a \leq g'$) and $\Gamma_{C'}(\mu', L')$ is $A$–stable. Hence by Corollary 2.6 one has $\Gamma_{C'}(\mu', L') = J^\mu(C') = W_{\mu'}(C')$ and $g' \leq \min\{\mu', r\}$ (see Lemma 2.4 (i)). Then $Z$ is $f^*(J(C'))$–stable, and, since $a$ is the maximal dimension of an abelian subvariety of $J(C)$ for which $Z$ is stable and $a = \dim(\bar{A}) \leq g'$, it follows $a = g'$. In conclusion $\mu' = a = r = g'$, and we are in case (ii).

Assume now $\mu > 0$. The general point of $Z$ (which is a component $\Gamma(m, L)$) is smooth for $\Theta$, hence it corresponds to a linearly isolated, effective divisor $D$ of degree $g - 1$, which is reduced (see Lemma 2.1 (ii)) and can be written in a unique way as $D = M + f^*(N)$, where $M$ and $N$ are effective divisors, with $\deg(M) = \mu$, $\deg(N) = \mu'$ and $M' := f_\ast(M)$ reduced. So there is a rational map $h: Z \dashrightarrow J^\mu(C)$ defined by $D = M + f^*(N) \mapsto M$.

Assume $\mu \leq r$. By Lemma 2.1 the image of $h$ is $W_{\mu}(C)$. Identify $A$ with its general translate inside $Z$. Then we have a morphism $h|_A: A \to J(C)$ whose image we denote by $\bar{A}$. Then $W_{\mu}(C)$ is $\bar{A}$–stable. Since $W_{\mu}(C)$ is birational to $C(\mu)$, which is of general type because $\mu < g$, then $\bar{A} = \{0\}$. It follows that each component of the general fibre of $h$ is $A$–stable, in particular $r - \mu \geq a \geq 1$.

Take $M \in W_{\mu}(C)$ general and set $L'' := L'(\bar{M})$. Since $L'$ is birational and $M' \in W_{\mu}(C')$ is general, with $\mu < r = \dim(L')$, then $L''$ has dimension $r - \mu \geq 1$ and it is base point free. It is also birational as soon as $r - \mu \geq 2$.

Assume first $r - \mu \geq 2$. Then $C'(\mu', L'')$ is irreducible by Lemma 2.1 (iv), and there is a birational morphism $C'(\mu', L'') \to h^{-1}(M) \subset J^{g-1}(C)$ factoring through the map $C'(\mu', L'') \to J^\mu(C') \to J^{g-1}(C)$, where the last inclusion is translation by $M$. Namely, up to a translation, $h^{-1}(M) = \Gamma(\mu', L'')$. In particular, $\dim(\Gamma(\mu', L'')) = \dim(C(\mu', L'')) = r - \mu$, hence $\mu' \geq r - \mu$.

Remember that $h^{-1}(M) = \Gamma(\mu', L'')$ is $A$–stable for $M \in W_{\mu}(C)$ general. By Corollary 2.6 one has $\Gamma_{C'}(\mu', L'') = W_{\mu'}(C') = J^\mu(C')$, so $g' \leq \min\{\mu', r - \mu\} = r - \mu$. On the other hand, since $h^0(O_C(D)) = 1$, one has also $h^0(O_C(f^*(N))) = 1$, hence $\mu' \leq g'$ and we conclude that $\mu' = g' = r - \mu$. The same argument as above yields $a = g'$ and we are again in case (ii).

If $r - \mu = 1$, then $a = 1, r = g - 2, \mu = g - 3$ and $\mu' = 1$. On the other hand $L$ is cut out on the canonical image of $C$ by the hyperplanes through the point $p_A$ which is the projectivized tangent space to $A$ at the origin. Then $\phi_L: C \to \bar{C}$ is the projection from $p_A, \bar{C}$ it is a normal elliptic curve, and we are again in case (ii).

Assume now $\mu > r$ and keep the above notation. In this case the map $h: Z \dashrightarrow Z := h(Z)$ is generically finite, $\bar{A}$ is isogenous to $A$ and $\bar{Z}$ is $\bar{A}$–stable. By Lemma 3.1, we have $g - 1 \geq \mu \geq \dim(\bar{Z}) + \dim(\bar{A}) = r + a = g - 1$, thus $\mu = g - 1$, so we are in case (i).

Finally consider the case $\nu = 3$. Write the general $D \in \Gamma(m, L)$ as $D = M_1 + M_2 + f^*(N)$, where $M_1$ is reduced of degree $\mu$, $f_\ast(M_2) = 2M'_2$ with $M'_2$ reduced of degree $\mu'$ and, as above,
\[ \mu_3 = \deg(N) \leq g'. \] Set \( \tau = \mu + \mu' \) and consider the rational map \( h : Z \to W_\tau(C) \subseteq J'(C) \) defined by \( D \mapsto M_1 + f^*(M_2') - M_2 \).

If \( \tau \leq r \), arguing as above (and keeping a similar notation) one sees that \( h(Z) = W_\tau(C) \), the general fibre of \( h \) is \( A \)-stable, hence \( r - \tau \geq a \). However \( a = 1 \) and \( \phi_L \) non-birational, forces, as we have seen, \( \nu = 2 \), which is not the case here. Hence we have \( a \geq 2 \). We consider now \( L'' = L'(-f_2(M_1) - M_2') \), which has dimension \( r - \tau \geq 2 \) and is base point free and birational, so \( C'(\mu_3, L'') \) is irreducible. The general fiber \( h^{-1}(h(D)) \) of \( h \) is isomorphic to \( \Gamma_C'(\mu_3, L'') \) and is \( A \)-stable. In particular \( \dim(\Gamma_C(\mu_3, L'')) = r - \tau \), hence \( \mu_3 \geq r - \tau \). By Corollary 2.6, we have \( \Gamma_C'(\mu_3, L'') \cong W_{\mu_3}(C') \cong J'_{\mu_3}(C') \), so \( g' \leq \min\{\mu_3, r - \tau\} = r - \tau \), hence \( g' \leq \mu - r - \tau \leq \mu_3 \leq g' \), thus \( \mu_3 = r - \tau = g' \). In addition, as above, we have \( a = g' \), so that \( A = f^*(J(C')) \). Then \( g - 1 - a = r = \tau + \mu_3 = \mu + \mu' + \mu_3 \). On the other hand \( g - 1 = \mu + 2\mu' + 3\mu_3 \). This yields \( \mu + 2\mu_3 = a = \mu_3 \), hence \( \mu'_2 = \mu_3 = 0 \), which is not possible.

If \( \tau > r \) then \( h : Z \to \tilde{Z} := h(Z) \) is generically finite, \( A \) is isogenous to \( A \) and \( \tilde{Z} \) is \( \tilde{A} \)-stable. By [7, Lemma 3.1], we have \( g - 1 \geq \tau \geq \dim(\tilde{Z}) + \dim(A) = r + a = g - 1 \), thus \( \tau = g - 1 \), contradicting \( \tau \leq \deg(L') = \frac{r}{2}(g - 1) \).

**Lemma 3.7.** If \( \nu \geq 4 \) then \( \nu = 4 \) and either

(i) there is a degree 2 map \( \psi : C \to E_1 \) with \( E_1 \) a genus \( r \) hyperelliptic curve such that \( Z = A = \psi^*(J'(E_1)) \); or

(ii) there is a faithful \( \mathbb{Z}_2^2 \)-action on \( C \) with rational quotient; denoting by \( f_1 : C \to E_i \) (for \( 1 \leq i \leq 3 \)) the quotient map for the three non–trivial involutions of \( \mathbb{Z}_2^2 \), with \( E_i \) of genus \( g_i \), and \( g_1 \geq g_2 \geq g_3 \), then \( g_1 = r + 1 \), \( g_2 + g_3 = r \) and \( Z = A = f_2^*(J^{g_2}(E_2)) \times f_3^*(J^{g_3}(E_3)) \) is the Prym variety associated to \( f_1 \).

**Proof.** Since \( L \) is base point free by Lemma 3.4 and it is not birational by Lemma 3.5 we have \( \delta = \frac{g - 2}{\nu} \geq r \geq \frac{r - 1}{2} \), hence \( \nu \leq 4 \).

Assume \( \nu = 4 \). Then \( \tilde{C} \) is a curve of degree \( \frac{g - 1}{2} \) spanning a projective space of dimension \( r \geq \frac{g - 1}{2} \). Hence \( r = a = \frac{g - 1}{2} \), \( Z = A \), and \( \phi_L = f \) is the composition of a \( g_4 \) (that we denote by \( \mathcal{L} \)) with the degree \( r \) Veronese embedding \( \mathbb{P}^3 \to \mathbb{P}^r \).

Let \( m = (\mu_1, \ldots, \mu_4) \) be the partition of \( 2r \) such that \( Z \subseteq \Gamma(m, L) \).

**Claim 3.8.** One has \( m = (2^r) \).

**Proof of the Claim.** Assume by contradiction this is not the case, so that one among \( \mu_1, \mu_3, \mu_4 \) is non–zero. We have \( r = \dim(\Gamma(m, L)) = \mu_1 + \cdots + \mu_4 \), because \( \tilde{C} \) is a rational normal curve of degree \( r \) in \( \mathbb{P}^r \), and \( 2r = g - 1 = \mu_1 + 2\mu_2 + 3\mu_3 + 4\mu_4 \), hence \( \mu_1 = 2\mu_2 + 3\mu_3 + 4\mu_4 \). However, we may write the general divisor \( D \in Z \) as \( D = M + N \), where \( M' = f_4(M) \) is reduced of degree \( \mu := \mu_1 \). Then we proceed as in the proof of Lemma 3.6.

Consider the rational map \( h : Z = A \to J^\mu(C) \) defined by \( D = M + N \to M \). It extends to a morphism and \( A := h(A) \) is an abelian variety contained in \( W_{\mu}(C) \). Since \( \mu \leq r = \frac{g - 1}{2} \), one has \( \tilde{A} = W_{\mu}(C) \), which is impossible.

Let \( E := C(2, L) \). Assume first \( C \) is not hyperelliptic. Then there is a birational (dominant) morphism \( E(r) \to \Gamma(2^r, L) \) (see Lemma 2.9 (i)). If \( E \) is irreducible, then by Corollary 2.7, its image generates \( J^{2r}(C) \), hence also \( \Gamma(2^r, L) = Z \) generates \( J^{2r}(C) \) and we obtain a contradiction.

So \( E \) is reducible and we apply Lemma 2.8 and 2.9. Suppose we are in case (b) of Lemma 2.8 and (ii) of Lemma 2.9. Then \( A = Z = \Gamma(2^r, L) \) is birational to \( E_1(r_1) \times E_2(r_2) \), for non–negative integers \( r_1, r_2 \). However, if \( r_i > 0 \), then \( r_i = g_i \), for \( i \in \{1, 2\} \). Since \( 2g_2 \geq g \), one has \( r_2 = 0 \) because \( 2r = g - 1 < g \). Hence \( A \) is birational to \( E_1(g_1) \) and we are in case (i).

Consider now case (c) of Lemma 2.8 and (iii) of Lemma 2.9. We claim that \( J(C) \) is isogenous to \( J(E_1) \times J(E_2) \times J(E_3) \). Indeed, consider the representation of \( \mathbb{Z}_2^2 \) on \( H^0(K_C) \). Since \( C/\mathbb{Z}_2^2 \) is
rational, we have $H^0(K_C) = V_{\chi_1} \oplus V_{\chi_2} \oplus V_{\chi_3}$, where for $i = 1, 2, 3$ the non-trivial character $\chi_i$ of $\mathbb{Z}_2^2$ is orthogonal to the involution $\iota_i$ such that $C/\iota_i \cong E_i$, and $\mathbb{Z}_2^2$ acts on $V_{\chi_i}$ by multiplication by $\chi_i$. Thus $V_{\chi_i}$ is the tangent space to $f_\nu^*(J(E_i))$.

Recall that there are three non-negative integers $r_1, r_2, r_3$ such that $A$ is birational to $E_1(r_1) \times E_2(r_2) \times E_3(r_3)$. If $r_1 > 0$, then $r_1 = g_1$ (for $1 \leq i \leq 3$). Since $g_1 + g_2 + g_3 = g = 2r + 1$, at least one of the integers $r_i$ is zero. If two of them are zero, we are again in case (i). So assume $r_1 = 0$ and $r_2, r_3$ non-zero. Then $r = g_2 + g_3$ and $g_1 = r + 1$. Moreover the involution $\iota_1$ acts on $A$ as multiplication by $-1$, hence $A$ is the Prym variety of $f_1: C \to E_1$, and we are in case (ii).

The case $C$ hyperelliptic can be treated similarly. In case (b) of Lemma 2.8 and (ii) of Lemma 2.9, $E_1$ is rational (hence it is contracted to a point by $j$), and $A = Z = \Gamma((2'), L)$ is birational to $E_2(r)$. Then $r = g_2$ and we reach a contradiction since $2g_2 \geq g$. In case (c) of Lemma 2.8 and (iii) of Lemma 2.9 one has $g_3 = 0$ and the above argument applies with no change.

Proof of Theorem 3.1. By Lemma 3.7, we may assume $\nu = 2$. Let $m$ be the partition of $2r$ such that $Z \subseteq \Gamma(m, L)$. By Lemma 3.6, it is enough to consider the case $\nu = 2$ and $m = (1^g)\frac{1}{g}$, i.e., case (i) of that lemma. Then $Z$ is contained in the kernel $P$ of $f_*: J^{g-1}(C) \to J^{g-1}(C')$ (namely $P$ is the generalized Prym variety associated with $f$). The space $H^0(K_{C'})$ decomposes under the involution $\iota$ associated with $f$ as $H^0(K_{C'}) \oplus V$, where $V$ is the space of antiinvariant 1-forms. Hence $V^*$ is the tangent space to $P$, the tangent space $T$ to $A$ is also contained in $V^*$ and the linear series $L$ is equal to $\mathbb{P}(T^*) \supseteq \mathbb{P}(H^0(K_{C'}))$. On the other hand, by construction $\iota$ acts trivially on $L$, hence $T = V^*$ and thus $A = P$. This implies that $Z = A = P$, $g = 2r + 1$ and $f$ is unramified with $g' = r + 1$.

3.2. General Brill–Noether loci. The proof of the general formula (3.1) in [7] uses an argument which is useful to briefly recall. Let $Z \subseteq W_d^g(C) \subseteq J^d(C)$ be $A$–stable and $L \in Z$ general, so $L$ is a special $g^d_s$, which we may assume to be complete, so $d \geq 2s$. Let $F_L$ be the fixed divisor of $L$ and, if $s > 0$, let $L'$ be the base point free residual linear series. Set $d' = \deg(L')$. Since $L'$ is also special, we have $d' \geq 2s$. Then, by Lemma 2.8 (iii), one has the birational map $\tau: C(d' - s, L') \dashrightarrow C(s, L') \cong C(s)$.

Consider the morphism $j: C(d' - s, L') \times C(g - 1 - d + s) \to J^{g-1}(C)$ such that $j(D, D')$ is the class of $D + F_L + D'$ (if $s = 0$, we define $j: C(g - 1 - d) \to J^{g-1}(C)$ by $D' \mapsto F_L + D'$). If $(D, D')$ is general in $C(d' - s, L') \times C(g - 1 - d + s)$, the divisor $D + F_L + D'$ is linearly isolated, hence $j$ is generically finite onto its image $Z_L$, which therefore has dimension $g - 1 - d + 2s$. Consider the closure $Z'$ of the union of the $Z_L$'s, with $L \in Z$ general, which is $A$–stable. One has $Z' \subseteq \Theta$ and the above discussion yields $r' := \dim(Z') = r + g - 1 - d + 2s$. Therefore $r' + a \leq g - 1$ if and only if (3.1) holds, with equality if and only if equality holds in (3.1).

Corollary 3.9. Let $C$ be a curve of genus $g$. Let $A \subseteq J(C)$ be an abelian variety of dimension $a > 0$ and $Z \subseteq W_d^g(C) \subseteq J^d(C)$ an irreducible, $A$–stable variety of dimension $r = d - 2s - a$. Assume $(d, s) \neq (g - 1, 0)$. Then there is a degree 2 morphism $\varphi: C \to C'$, with $C'$ a smooth curve of genus $a$ with $g > 2a + 1$, such that $A = \varphi^*(J(C'))$ and either

(i) $Z = W_{d-2a}^g(C) + \varphi^*(J^{a+1}(C'))$, or

(ii) $C$ is hyperelliptic and $Z = \varphi^*(J^g(C')) + W_{d-2s-2a}^g(C) + W_{2a}^g(C)$.

Proof. We keep the same notation as above.

We apply Theorem 3.1 to $Z'$. Since $d = r + a + 2s \geq 2(a + s)$ and $(d, s) \neq (g - 1, 0)$, then $2a < g - 1$, hence case (b) of Theorem 3.1 does not occur for $Z'$. So we have a degree 2 morphism $\varphi: C \to C'$, with $C'$ a smooth curve of genus $a$ with $g > 2a + 1$, such that $A = \varphi^*(J(C'))$ and $Z' = W_{g-1-2a}^g(C) + \varphi^*(J^a(C'))$. The case $s = 0$ follows right away and we are in case (i). So we assume $s > 0$ from now on.
If $L'$ is composed with the involution $\iota$ determined by $\varphi$, we are again in case (i). So assume $L'$ is not composed with $\iota$. The general $D \in C(d' - s, L')$ contains no fibre of $\varphi$ and the same happens for the general $D' \in C(g - 1 - d + s)$. Since $D + F_L + D'$ corresponds to the general point of $Z$, and contains exactly a general fibres of $\varphi$, then $F_L$ has to contain these a fibres, whose union we denote by $F$. Moreover, the description of $Z$ implies that $D + D' + (F_L - F)$ is a general divisor of degree $g - 1 - 2a$, in particular $D$ is a general divisor of degree $d' - s$. But $D$ is also general in $C(d' - s, L')$, and this implies $d' - s \leq s$. On the other hand $d' \geq 2s$, hence $d' = 2s$. Then, either $L'$ is the canonical series of $C$ or $C$ is hyperelliptic and $L'$ is the $s$–multiple of the $g^1_2$. However the former case does not occur since by construction $d' = \deg(L') \leq g - 1$, hence we are in case (ii).

\[\square\]

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