GAUSSIAN COMPLEX ZEROES ARE NOT ALWAYS NORMAL:
LIMIT THEOREMS ON THE DISC

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Abstract. We study the zeroes of a family of random holomorphic functions on the unit disc, distinguished by their invariance with respect to the hyperbolic geometry. Our main finding is a transition in the limiting behaviour of the number of zeroes in a large hyperbolic disc. We find a normal distribution if the covariance decays faster than a certain critical value. In contrast, in the regime of ‘long-range dependence’ when the covariance decays slowly, the limiting distribution is skewed. For a closely related model we emphasise a link with Gaussian multiplicative chaos.

1. Introduction

1.1. Statement of results. We are interested in the zeroes of the random holomorphic functions

\[ f_0(z) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \zeta_m z^m \quad \text{and} \quad f_L(z) = \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(L+m)}{\Gamma(L)m!}} \zeta_m z^m \quad \text{for } L > 0, \]

where \( \{\zeta_m\} \) is a sequence of iid \( \mathcal{N}_\mathbb{C}(0,1) \) standard complex Gaussians and \( z \) belongs to the unit disc \( \mathbb{D} \). The distribution of \( f_L \) as a Gaussian analytic function (GAF) on \( \mathbb{D} \) is determined by its covariance kernel

\[ K_L(z, w) = \mathbb{E} \left[ f_L(z) f_L(w) \right] = \begin{cases} (1 - zw)^{-L}, & \text{if } L > 0; \\ \log \frac{1}{1 - zw}, & \text{if } L = 0. \end{cases} \]

(1)

A short computation of covariance kernels shows that if \( \psi: \mathbb{D} \rightarrow \mathbb{D} \) is a disc automorphism then

\[ f_0 \circ \psi - (f_0 \circ \psi)(0) \overset{d}{=} f_0 \quad \text{and} \quad f_L \circ \psi \cdot (\psi')^{L/2} \overset{d}{=} f_L \quad \text{for } L > 0, \]

where \( \overset{d}{=} \) denotes equality in distribution as Gaussian processes. Since \( \psi' \) is a deterministic non-vanishing function, this means that for \( L > 0 \) the zeroes of \( f_L \) form a stationary point process in \( \mathbb{D} \). Furthermore, if we fix the intensity of the zero process (equivalently \( L \)), that is, the mean number of zeroes per unit hyperbolic area, then \( f_L \) is essentially the only GAF with this property. For further details see [17, Chapter 2].

The functions \( f_L \) have arisen in different contexts. Diaconis and Evans [10, Example 5.6] showed that the function \( f_2 \) (up to normalisation) arises as the limit of the logarithmic derivative of the characteristic polynomial of a random \( n \times n \) unitary matrix, for large \( n \). Peres and Virág [30] showed that the zeroes of \( f_1 \) form a determinantal process, and used this to describe statistical properties of the zero set. Chhaibi and Najnudel [8] recently showed a relation between the ‘boundary values’ of the function \( f_0 \) and a certain limit of the circular \( \beta \) ensemble.

Let \( n_L(r) \) be the number of zeroes of \( f_L \) in the disc \( D(0, r) \) for \( 0 < r < 1 \). In this article we will describe the fluctuations of \( n_L(r) \) about its mean as \( r \to 1 \). This mean can be computed via the Edelman-Kostlan formula, see [17, Section 2.4]. For \( L > 0 \), the asymptotic growth of the variance was studied in [6] and one of the interesting features is a transition at the value \( L = \frac{1}{2} \). In this

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Theorem 2. This means that the ‘left’ tail of $X \leq L$ shows the limiting distribution is Gumbel, while for the normalised version of $X$, we have

$$\frac{X}{\sqrt{\text{Var}[X]}} \xrightarrow{r \to 0} \mathcal{N}(0,1)$$

In order to state our results we write

$$\hat{n}_L(r) = \frac{n_L(r) - \mathbb{E}[n_L(r)]}{\sqrt{\text{Var}[n_L(r)]}}$$

for the normalised version of $n_L(r)$, put

$$a_{m,0} = \begin{cases} 0 & m = 0, \\ \frac{1}{m} & m \geq 1, \end{cases} \quad \text{and} \quad a_{m,L} = \frac{\Gamma(L + m)}{\Gamma(L)m!} \quad \text{for } L > 0,$$

and introduce the random variable $X_L = \sum_{m=0}^{\infty} a_{m,L} (\zeta_m^2 - 1)$ for $0 \leq L < \frac{1}{2}$. By Stirling’s approximation we have $\frac{\Gamma(L+m)}{m!} \sim m^{-\frac{2}{3}} m(L-1)^{1/2}$ so that $\sum_{m=0}^{\infty} a_{m,L}^2 < \infty$ for such $L$ and so the sum defining $X_L$ converges almost surely.

**Theorem 1.**

(i) If $L \geq \frac{1}{2}$ is fixed, then we have $\hat{n}_L(r) \to \mathcal{N}(0,1)$ (the standard Gaussian) in law, as $r \to 1$.

(ii) If $L \to \frac{1}{2}$ and $r \to 1$ simultaneously, then $\hat{n}_L(r) \to \mathcal{N}(0,1)$ in law.

(iii) If $0 \leq L < \frac{1}{2}$ is fixed, then we have $\hat{n}_L(r) \to -c_L X_L$ in $L^2$ as $r \to 1$ where

$$c_L^2 = \left( \sum_{m=0}^{\infty} a_{m,L}^2 \right)^{-1} = \begin{cases} \frac{\pi^2}{6}, & L = 0; \\ \frac{\Gamma((L-1)^2)}{\Gamma(L^2)}, & 0 < L < \frac{1}{2}. \end{cases}$$

**Remarks.**

(1) The case $L = 1$ of this theorem is [30, Corollary 3 (iii)]. It was proved using the determinantal structure, and so the methods do not apply to other values of $L$.

(2) For $L < \frac{1}{2}$ the limit $X_L$ is determined by the ‘boundary values’ of the process $f_L$, we shall elaborate on this remark in Section 4.

(3) Using Lyapunov’s criterion, one can check that $\sum_{m=0}^{N} a_{m,L} (\zeta_m^2 - 1)$ obeys a CLT when $N \to \infty$, for $L \geq \frac{1}{2}$. This is essentially the reason for Gaussian behaviour when $L \to \frac{1}{2}$.

It is clear that $X_L$ is non-Gaussian, e.g., since $\mathbb{E}[X_L^3] = 2 \sum_{m=0}^{\infty} a_{m,L}^3 \neq 0$. In the case $L = 0$ a direct computation using characteristic functions shows the limiting distribution is Gumbel, while for $0 < L < \frac{1}{2}$ we give sharp estimates on the decay of the tail probability. Since $\hat{n}_L(r) \to -c_L X_L$, this means that the ‘left’ tail of $X_L$ corresponds to the ‘right’ tail of $\hat{n}_L$ and vice-versa.

**Theorem 2.**

(i) $X_0$ is a Gumbel distributed random variable with mean $0$ and variance $\frac{\pi^2}{6}$.

(ii) If $0 < L < \frac{1}{2}$ then $\mathbb{P}[X_L > x] = (\kappa_L + o(1)) e^{-x}$ and $\log \mathbb{P}[X_L < -x] = -L \Gamma(L)(-\text{sinc} \left( \frac{\pi}{1-L} \right))^{\frac{1}{2}}$ as $x \to \infty$, where

$$\kappa_L = \frac{1}{e} \prod_{m=1}^{\infty} \frac{e^{-a_{m,L}}}{1-a_{m,L}} \quad \text{and} \quad \lambda_L = L \Gamma(L)^{1/2} \left( -\text{sinc} \left( \frac{\pi}{1-L} \right) \right)^{\frac{1}{2}}.$$
Remarks. 

1. We recall that the Gumbel CDF (with our normalisation) is \( \exp(-e^{-x-\gamma}) \) where \( \gamma \) is Euler’s constant. This means that \( \mathbb{P}[X_0 > x] \sim e^{-\gamma e^{-x}} \) as \( x \to \infty \) and \( \log \mathbb{P}[X_0 < -x] = -e^{x-\gamma} \). The right tails of \( X_L \) therefore have an exponential profile for all \( L \), while the left tails are quite different. See Figure 1 for an illustration of the PDF of \( c_L X_L \) (that is, normalised to have mean 0 and variance 1).

2. In Section 1.4 we give a heuristic explanation for the appearance of the Gumbel distribution, using the theory of Gaussian multiplicative chaos.

1.2. Background and motivation. The function \( f_L \) was (to the best of our knowledge) first introduced by Lebœuf in [23] where it is referred to as an “analytic chaotic eigenstate”. It is viewed as a coherent state representation of a random quantum state. The unit disc is interpreted as the phase space of the corresponding quantum mechanical system, which is assumed to exhibit SU(1, 1) symmetry. Furthermore, the fact that the coefficients \( \zeta_m \) are complex valued reflects the absence of time-reversal symmetry. A different point of view, motivated by signal processing, is to consider \( f_L \) as a Daubechies-Paul wavelet transform of white noise, see, e.g., [2, Theorem 2.3] and [21, Section 3].

Lebœuf views the invariance of the zeroes of \( f_L \) as a manifestation of ergodicity in phase space. The zero set is sometimes referred to as the ‘stellar representation’ of the state (or ‘Majorana representation’ see, e.g., [3, Chapter 7]) and is physically expected to determine the ‘Husimi function’ \( \frac{|f_L(z)|^2}{\mathbb{E}|f_L(z)|^2} \), which gives the probability of finding a particle in a small neighbourhood of \( z \). Interestingly [30, Theorem 6] gives an explicit formula to reconstruct \( |f_L| \) from the random zeroes.

Another motivation for studying the zeroes of random holomorphic functions is to view the resulting point process as a system of interacting particles that exhibit local repulsion [17, Chapter 1]. One fruitful approach is to compare and contrast the properties of different processes. It is particularly interesting to contrast \( f_L \) with the ‘flat GAF’ which is entire and invariant with respect to the Euclidean geometry [17, Section 2.3]. Roughly speaking, the zeroes of \( f_L \) for large \( L \) behave like the flat zeroes, in contrast for small \( L \) one expects to see ‘genuine hyperbolic phenomena’. Asymptotic normality of the zeroes of the flat GAF was described by Sodin-Tsirleson and Nazarov-Sodin [27,28,37].
In the Euclidean setting there are many similarities between the zeroes of the ‘flat GAF’ and the infinite Ginibre ensemble, see for example [15]. It is therefore also natural to compare the behaviour of the zeroes of $f_L$ with the determinantal process $X_L$ with kernel

$$K_L(z, w) = \frac{1}{\pi} \frac{1}{(1 - z w)^{L+1}}$$

and reference measure $d\mu_L(z) = (1 - |z|^2)\frac{1}{2}(L-1)dm(z)$; here $m$ is the Lebesgue measure. Krishnapur [22, Theorem 3.0.5] showed that $X_L$ are the only determinantal processes on the disc with analytic kernel that are invariant with respect to the automorphisms, and the intensity of the point process $X_L$ and the zeroes of $f_L$ is the same. For $L = 1$ the processes are the same [30]. Moreover, it is also shown there that if $L \neq 1$ then the zeroes of $f_L$ do not have a determinantal structure. Kartick Adhikari (private communication) has shown that there is no transition in the behaviour of the variance for the determinantal models and by [17, Theorem 4.6.1] a CLT holds for all $L$. See also the recent work of Fenzl-Lambert [12, Section 2.3].

1.3. Related work. Consider a real-valued stationary Gaussian sequence $(X_n)_{n \in \mathbb{Z}}$ with covariance kernel $r(n) = \mathbb{E}[X_nX_0]$ which decays like $n^{-\alpha}$ for large $n$. Let $H$ be a function of Hermite rank $k$, that is, we can expand $H = \sum_{n \geq k} c_n H_n$ in terms of the Hermite polynomials in an appropriate sense. Consider the random variable

$$Y_N = \sum_{n=1}^{N} H(X_n).$$

By results of Breuer-Major and Dobrushin-Major [5, Theorem 1; 5, Theorem 1'; 11, Theorem 1]:

- If $\alpha > \frac{1}{k}$ then the variance of $Y_N$ grows linearly with $N$ and a CLT holds.
- If $\alpha = \frac{1}{k}$ then the variance of $Y_N$ grows at the rate $N \log N$, but a CLT still holds.
- If $\alpha < \frac{1}{k}$ then the variance of $Y_N$ grows at the rate $N^{2-\alpha k}$, and a non-CLT holds.

One may also consider the number of zeroes in the interval $[0, T]$ of a real-valued stationary Gaussian process $f : \mathbb{R} \rightarrow \mathbb{R}$, as $T \rightarrow \infty$. For sufficiently fast decay of the covariance, combining results of Cuzick and Slud, one gets a CLT for the zeroes [9, Theorem 1; 35, Theorem 3]; Slud found a non-Gaussian limit for a family with long range dependence [36, Theorem 3.2]. In our setting the random variable $n_L(r)$ has Hermite rank 2, see Proposition 4 and large values of $L$ correspond to fast decay of the covariance, see [1].

A related problem in higher dimensions is the study of the nodal (i.e., zero) sets of random Laplace eigenfunctions; we refer the interested reader to the survey [34] and the references therein. For example, Marinucci, Rossi and Wigman found that a CLT holds on the sphere [26, Corollary 1.3] while, in contrast, the same authors with Peccati showed that the fluctuations on the torus are non-Gaussian [25, Theorem 1.1].

Curiously, when studying the pair correlations in the circular $\beta$ ensemble, Aguirre, Soshnikov and Sumpter [1, Theorem 2.1] discovered a non-Gaussian limit that is similar in form to the $X_L$ appearing in our Theorem 1.

1.4. Links with Gaussian multiplicative chaos (GMC) when $L = 0$. The random Fourier series

$$\sum_{m=0}^{\infty} \sqrt{a_{m,L}} \zeta_m e^{im\theta}$$

does not converge to a function, but can be understood mathematically as a random distribution (i.e., generalised function). Such an object is sometimes referred to as a (complex) $1/f^\alpha$ noise on the unit circle; here $\alpha = 1 - L$ since $a_{m,L} \sim m^{L-1}$ [24]. Heuristically we can think of this noise as representing the ‘boundary values’ of $f_L$ on the unit circle, or conversely we can regard $f_L$ as the Poisson extension of the noise on the unit circle to the interior of the disc (see [19, Theorem 4].
for a deterministic statement). The $\alpha = 1$ (aliter $L = 0$, so-called pink noise) case is particularly interesting and there is an extensive literature (in both mathematics and physics) on log-correlated processes. This theory, moreover, has links to random matrix theory and conjecturally with number theory [31, Section 4]. We shall only touch on a small part of the theory here.

For $0 \leq L < \frac{1}{2}$, a careful examination of the proof of Theorem 1 shows that $\hat{n}_L(r)$ can be approximated in $L^2$ by the random variable

$$-c_L r^2 \sum_{m=0}^{\infty} a_{m,L} \left( |\zeta_m|^2 - 1 \right) r^{2m}$$

which converges to $-c_L X_L$ in $L^2$, as $r \to 1$. On the other hand it is easy to compute

$$\int_{-\pi}^{\pi} \left( |f_L(re^{i\theta})|^2 - \mathbb{E} \left[ |f_L(re^{i\theta})|^2 \right] \right) \frac{d\theta}{2\pi} = \sum_{m=0}^{\infty} a_{m,L} \left( |\zeta_m|^2 - 1 \right) r^{2m}$$

and so we may think of $X_L$ as representing the integral

$$\int_{-\pi}^{\pi} \left( |f_L(e^{i\theta})|^2 - \mathbb{E} \left[ |f_L(e^{i\theta})|^2 \right] \right) \frac{d\theta}{2\pi},$$

bearing in mind that $f_L$ is properly a generalised function on the unit circle.

We now restrict to the case $L = 0$. Write $u_0 = \text{Re} (f_0)$ and notice that

$$\frac{1}{2} \mathbb{E} \left[ u_0^2(z) \right] = \mathbb{E} \left[ |f_0(z)|^2 \right].$$

Consider, for $0 < r < 1$, the measures on the unit circle defined by

$$d \text{GMC}_r^\gamma (\theta) = \exp \left( \gamma u_0(re^{i\theta}) - \frac{\gamma^2}{2} \mathbb{E} \left[ u_0^2(re^{i\theta}) \right] \right) \frac{d\theta}{2\pi}.$$

The weak limit (as $r \to 1$) of this sequence of measures, denoted by $\text{GMC}^\gamma$, is the Gaussian multiplicative chaos with coupling coefficient $0 < \gamma < 1$; it is a singular continuous random measure. For a comprehensive introduction to the theory we refer the reader to the survey [33].

Curiously it turns out that one can derive Theorem 2 (i) via the theory of $\text{GMC}$. We will only give a heuristic explanation. We are not aware of any way to extend this to $L > 0$.

**Proposition 3 (The Fyodorov-Bouchaud formula (Remy, Chhaibi-Najnudel) [8, Corollary 2.5; 13, 32, Theorem 1.1]).** For $\gamma \in (0,1)$ the law of the total mass of the $\text{GMC}$ is given by

$$\text{GMC}^\gamma (\mathbb{T}) \overset{d}{=} K_\gamma e^{-\gamma^2},$$

where $K_\gamma = \Gamma \left( 1 - \gamma^2 \right)^{-1}$ and $e$ is a standard exponential random variable.

We may think of $\text{GMC}^\gamma$ as a sort of generating function for $X_0$. Expanding $\text{GMC}^\gamma$ in powers of $\gamma$ we find

$$\text{GMC}^\gamma (\mathbb{T}) = 1 + \gamma \int_{-\pi}^{\pi} u_0(e^{i\theta}) \frac{d\theta}{2\pi} + \frac{\gamma^2}{2} \int_{-\pi}^{\pi} \left( u_0^2(e^{i\theta}) - \mathbb{E} \left[ u_0^2(e^{i\theta}) \right] \right) \frac{d\theta}{2\pi} + \ldots$$

and thus

$$X_0 \overset{d}{=} \lim_{\gamma \to 0} \frac{K_\gamma e^{-\gamma^2} - 1}{\gamma^2}.$$

\footnote{Fyodorov and Keating studied the integral defining $X_0$ in a different context [14 Section 3 (d)].}
To see how this leads to the Gumbel distribution we compute
\[ \Pr \left[ \frac{K_\gamma e^{-\gamma^2} - 1}{\gamma^2} \leq t \right] = \Pr \left[ e^{\gamma^2} \left( \frac{K_\gamma}{1 + \gamma^2 t} \right)^{1/\gamma^2} \right] = \exp \left( - \left( \frac{K_\gamma}{1 + \gamma^2 t} \right)^{1/\gamma^2} \right). \]
Taking the limit \( \gamma \to 0 \) we find (here \( \gamma_e \) is Euler’s constant)
\[ \Pr [X_0 \leq t] = \exp \left( -e^{-t - \gamma_e} \right) \]
which is the Gumbel CDF.

The paper is organised as follows. In Section 2 we give an outline of the method. In Section 3 we prove Theorem 1 for \( L > \frac{1}{2} \). In Section 4 we compute the asymptotic growth of the variance of \( n_0(r) \). In Section 5 we complete the proof of Theorem 1. In Section 6 we prove Theorem 2.

We conclude the introduction with a word on notation. We write \( A \subset B \) if there exists a constant \( C \), independent of the relevant variables, such that \( A \leq C B \). We write \( A \asymp B \) if \( A \lesssim B \) and \( B \lesssim A \). We write \( A = O(B) \) if \( |A| \lesssim B \). We write \( A \sim B \) if \( A / B \to 1 \) when we take an appropriate limit.

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2. OUTLINE OF THE METHOD

Our investigations centre on the Wiener chaos expansion (sometimes called the Hermite-Itô expansion) of the random variable \( n_L(r) \). This expansion is well-known to experts, and appears implicitly in the papers \([6, 27, 37]\). In order to state it we first introduce some notation.

Let \( d\mu(\zeta) = \frac{1}{\pi} e^{-|\zeta|^2} \, dm(\zeta) \) denote the Gaussian measure on the plane (here \( m \) is the planar Lebesgue measure) and write \( \mathcal{P}_q \) for the polynomials (in the variables \( \zeta \) and \( \bar{\zeta} \)) of degree at most \( q \) considered as subspace of \( L^2(\mu) \). Denote by \( \mathcal{H}^0 = \mathcal{P}_0 \) and \( \mathcal{H}^q = \mathcal{P}_q \ominus \mathcal{P}_{q-1} \) for \( q \geq 1 \) (here \( \ominus \) denotes orthogonal complement). Given a monomial \( \zeta^\alpha \bar{\zeta}^\beta \) with \( \alpha + \beta = q \) we write \( \zeta^\alpha \bar{\zeta}^\beta : \) to denote its projection to \( \mathcal{H}^q \), which is usually called a Wick product (a complex Hermite polynomial of degree \( \alpha + \beta \)).

We now state the expansion, for completeness we include more details and a proof in Appendix A.

Proposition 4. Write \( \hat{f}_L(z) \) \( \frac{f_L(z)}{K_L(z, \bar{z})^{1/2}} \) and define
\[ n_L(r; \alpha) = \frac{(-1)^{\alpha+1}}{\alpha (\alpha!)^{1/2}} \int_{\partial D_0(0, r)} \frac{\partial}{\partial z} |\hat{f}_L(z)|^{2\alpha} \, dz. \]
Then \( n_L(r; \alpha) \) belongs to the \( 2\alpha \)-th component of the Wiener chaos corresponding to \( f_L \) and
\[ n_L(r) - \mathbb{E}[n_L(r)] = \sum_{\alpha=1}^{\infty} n_L(r; \alpha) \]
where the sum converges in \( L^2 \).

Let us indicate a heuristic explanation of the expansion. A computation (see \([18, \text{Example 3.32}]\)) shows that the set of all Wick products \( \zeta^\alpha \bar{\zeta}^\beta : \) with \( \alpha + \beta = q \) is an orthogonal basis for \( \mathcal{H}^q \), and moreover \( ||\zeta^\alpha \bar{\zeta}^\beta:||^2 = \alpha! \beta! \) (the norm here is the norm inherited from \( L^2(\mu) \)). Furthermore \([18, \text{Theorem 2.6}]\)
\[ L^2(\mu) = \bigoplus_{q=0}^{\infty} \mathcal{H}^q. \]
We expand the logarithm with respect to this orthonormal basis and a calculation \cite[Lemma 2.1]{27} yields
\[
\log |\zeta|^2 = -\gamma_e + \sum_{\alpha=1}^{\infty} \frac{(-1)^{\alpha+1}}{\alpha!} (\zeta)^{2\alpha};
\]
where the equality holds in $L^2(\mu)$.

From the argument principle and direct computation we have
\[
n_L(r) = \frac{1}{2\pi i} \int_{\partial D(0,r)} f'_L(z) \frac{1}{f_L(z)} \frac{d}{dz} \log |f_L(z)|^2 \, dz
\]
and the Edelman-Kostlan formula \cite[Section 2.4]{17} gives
\[
n_L(r) - \mathbb{E}[n_L(r)] = \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{d}{dz} \log |\tilde{f}_L(z)|^2 \, dz.
\]
Inserting (4) into this expression and exchanging the sum with the derivative and the integral formally yields the expansion given in the proposition. Furthermore, the orthogonality of the Wick products yields the orthogonality of $n_L(r; \alpha)$ for different values of $\alpha$.

Let us now outline how we use the expansion to prove the main theorem. The orthogonality of the expansion allows us to compute
\[
\text{Var} \left[ n_L(r) \right] = \sum_{\alpha=1}^{\infty} \mathbb{E} \left[ n_L(r; \alpha)^2 \right]
\]
and we will show that if $0 \leq L \leq \frac{1}{2}$ then
\[
\text{Var} \left[ n_L(r) \right] \sim \mathbb{E} \left[ n_L(r; 1)^2 \right],
\]
that is, only the first (non-trivial) component of the chaos contributes. In contrast, if $L > \frac{1}{2}$ then all of the terms of the sum are of comparable size. In this latter case we show that each of the terms $n_L(r; \alpha)$ is asymptotically normal, through the method of moments. This idea goes back to \cite{37}, although the scheme developed there and modified in \cite{7} only works for $L > 1$, essentially due to the slower decay of the covariance kernel in the hyperbolic setting. Instead we use the Fourth Moment Theorem \cite{29}, a powerful method for proving a CLT for random variables that belong to a fixed component of the Wiener chaos.

If $0 \leq L \leq \frac{1}{2}$ then, by the variance estimates just mentioned, we have $n_L(r) = n_L(r; 1) + o_P(1)$. We analyse the expression for $n_L(r; 1)$ in detail: for $L < \frac{1}{2}$ we show that it is asymptotic to $X_L$ (when normalised properly), in contrast for $L = \frac{1}{2}$ we show that it is asymptotically normal (the transition is essentially down to the summability of $a^2_{\mu_0, L}$).

3. \textsc{Proof of the CLT for} $L > \frac{1}{2}$

In this section we will prove Theorem 1(i) in the case that $L > \frac{1}{2}$. The method is fairly standard, relying on the fourth moment theorem, and we accordingly do not give all of the details. The idea is to replace $n_L(r)$ by the random variable
\[
n_L^{(M)}(r) = \sum_{\alpha=1}^{M} n_L(r; \alpha)
\]
and to prove that:

(I) There exists $r_0 < 1$ such that
\[
\mathbb{E} \left[ \left( n_L(r) - \mathbb{E}[n_L(r)] - n_L^{(M)}(r) \right)^2 \right] \leq \frac{C_L}{\sqrt{M}} \text{Var} \left[ n_L(r) \right]
\]
for all $r_0 \leq r < 1$ and $M \geq 1$.

(II) For each fixed $M$

$$\frac{n_L^{(M)}(r)}{\sqrt{\text{Var}(n_L^{(M)}(r))}} \to N_{\mathbb{R}}(0, 1)$$

in distribution, as $r \to 1$.

The result then follows; for completeness we prove this in Appendix B.

3.1. Some preliminary calculations. In order to implement the strategy outlined above, we will need the following lemma, which uses the notation $K_L(z, w) = \frac{K_L(z, w)}{\sqrt{K_L(z, z)K_L(w, w)}}$.

**Lemma 5.** If $\alpha \geq 1$ then

$$\mathbb{E} \left[ (n_L(r; \alpha))^2 \right] = \left( \frac{1}{2\pi i} \right)^2 \frac{1}{\alpha^2} \int \int_{\partial D(0, r)^2} \frac{\partial^2}{\partial z \partial \bar{w}} \left| \hat{K}_L(z, w) \right|^{2\alpha} dzdw$$

$$= \frac{L^2r^4}{2\pi (1 - r^2)^2} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - r^2 e^{i\theta}} \left| \frac{1 - e^{i\theta}}{1 - r^2 e^{i\theta}} \right|^2 d\theta$$

and

$$\mathbb{E} \left[ (n_L(r; \alpha))^4 \right] = \left( \frac{1}{2\pi i} \right)^4 \frac{1}{\alpha^4 (\alpha!)^2} \int \int \int \int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} \mathbb{E} \left[ \prod_{j=1}^{4} |\hat{f}(z_j)|^{2\alpha} \right] \prod_{j=1}^{4} dz_j.$$

We postpone the proof of the lemma to Appendix A, since it simply involves exchanging expectation with integrals and derivatives. We will also need the following estimate.

**Lemma 6.** If $L > \frac{1}{2}$ is fixed then

$$c_{L, \alpha} (1 - r) \leq \int_{-\pi}^{\pi} \left| \frac{1 - r^2}{1 - r^2 e^{i\theta}} \right|^{2\alpha L} \left| \frac{1 - e^{i\theta}}{1 - r^2 e^{i\theta}} \right|^2 d\theta \leq C_L \alpha^{-3/2} (1 - r) + (C(1 - r))^{2\alpha L}$$

for $r \geq r_0$ and $\alpha \geq 1$.

**Proof.** An easy computation yields $|1 - r^2 e^{i\theta}|^2 = (1 - r^2)^2 + 2r^2 (1 - \cos \theta)$ and so we get

$$\int_{-\pi}^{\pi} \left| \frac{1 - r^2}{1 - r^2 e^{i\theta}} \right|^{2\alpha L} \left| \frac{1 - e^{i\theta}}{1 - r^2 e^{i\theta}} \right|^2 d\theta = 4 \int_{0}^{\pi} \left( 1 + \frac{2r^2}{(1 - r^2)^2} (1 - \cos \theta) \right)^{-\alpha L + 1} \frac{1 - \cos \theta}{(1 - r^2)^2} d\theta.$$

We separate the ‘small’ and ‘big’ values of $\theta$. The small values contribute (below $B$ denotes the beta function and $\varepsilon > 0$ is small but fixed)

$$\int_{0}^{\varepsilon} \left( 1 + \frac{2r^2}{(1 - r^2)^2} (1 - \cos \theta) \right)^{-\alpha L + 1} \frac{1 - \cos \theta}{(1 - r^2)^2} d\theta \leq \int_{0}^{\varepsilon} \left( 1 + \frac{\varepsilon^2}{(1 - r^2)^2} \right)^{-\alpha L + 1} \frac{\varepsilon^2}{(1 - r^2)^2} d\theta \leq (1 - r) B \left( \frac{\Gamma(\alpha L - \frac{1}{2})}{\Gamma(\alpha L + 1)} \right) \leq C_L \alpha^{-3/2} (1 - r).$$

(5)

The remaining contribution is

$$\int_{\varepsilon}^{\pi} \left( 1 + \frac{2r^2}{(1 - r^2)^2} (1 - \cos \theta) \right)^{-\alpha L} \frac{2(1 - \cos \theta)}{(1 - r^2)^2 + 2r^2 (1 - \cos \theta)} d\theta \leq (C(1 - r))^{2\alpha L} \int_{\varepsilon}^{\pi} d\theta.$$
In the other direction, arguing similarly to (5), we get
\[
\int_{0}^{\epsilon} \left(1 + \frac{2r^2}{(1-r^2)^2} (1 - \cos \theta) \right)^{-\alpha(L+1)} \frac{1 - \cos \theta}{(1-r^2)^2} \, d\theta \gtrsim (1-r) \int_{\frac{1}{2}}^{1} y^{\alpha L - \frac{3}{2}} \sqrt{1-y} \, dy
\]
where the lower limit in the last integral arises from choosing \( r_0 \) appropriately. \( \square \)

3.2. Proof of (I). Using Proposition 4 and Lemma 5 we get
\[
\mathbb{E} \left[ \left( n_L (r) - \mathbb{E} [n_L (r)] - n_{L}^{(M)} (r) \right)^2 \right] = \sum_{\alpha > M} \mathbb{E} \left[ (n_L (r; \alpha))^2 \right]
\]
\[
= \frac{L^2 r^4}{2\pi (1-r^2)^2} \sum_{\alpha > M} \int_{-\pi}^{\pi} \left| \frac{1 - r^2}{1 - r^2 e^{i\theta}} \right|^2 \left| \frac{1 - e^{i\theta}}{1 - r^2 e^{i\theta}} \right|^2 d\theta.
\]
Using the estimates in Lemma 6 we conclude that
\[
\mathbb{E} \left[ \left( n_L (r) - \mathbb{E} [n_L (r)] - n_{L}^{(M)} (r) \right)^2 \right] \leq \frac{C_L}{(1-r)^2} \sum_{\alpha > M} \left[ \alpha^{-3/2} (1-r) + (C(1-r))^{2\alpha L} \right]
\]
\[
\leq C_L \left[ \frac{1}{\sqrt{M} (1-r)} + (C(1-r))^{2(M-1)L} \right]
\]
\[
\leq \frac{C_L}{\sqrt{M} (1-r)} \leq \frac{C_L}{\sqrt{M}} \mathbb{V} [n_L (r)]
\]
uniformly in \( M \), where the last bound follows from (2). This completes the proof of (I).

3.3. Proof of (II). We wish to show that the sum
\[
n_{L}^{(M)} (r) = \sum_{\alpha=1}^{M} n_L (r; \alpha)
\]
is asymptotically normal, and so it suffices to see that the vector
\[
\begin{pmatrix}
  n_L (r; 1) \\
  \vdots \\
  n_L (r; M)
\end{pmatrix}
\]
satisfies a multi-variate CLT. By the multi-dimensional fourth moment theorem [29, Theorem 1] it is enough to check that
\[
\frac{\mathbb{E} \left[ (n_L (r; \alpha))^4 \right]}{\mathbb{E} \left[ (n_L (r; \alpha))^2 \right]^2} \rightarrow 3
\]
as \( r \rightarrow 1 \), for each fixed \( \alpha \). We recall that, by Lemma 5, we have
\[
\mathbb{E} \left[ (n_L (r; \alpha))^4 \right] = \left( \frac{1}{2\pi i} \right)^4 \frac{1}{\alpha^4 (\alpha!)^4} \int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} \mathbb{E} \left[ \prod_{j=1}^{4} :|\hat{f} (z_j)|^{2\alpha} : \right] \prod_{j=1}^{4} dz_j.
\]
Let \( D = D(\alpha) \) denote the set of (bipartite) graphs with \( 8\alpha \) vertices such that:
- For each \( 1 \leq j \leq 4 \) there are \( \alpha \) vertices labelled \( j \) and \( \alpha \) vertices labelled \( \bar{j} \).
- Each vertex has degree exactly 1, i.e., every vertex is paired with exactly one other vertex.
- Each edge joins a vertex labelled \( j \) to a vertex labelled \( \bar{k} \) for \( j \neq k \).
Now if the edge $e$ joins a vertex labelled $j$ to a vertex labelled $k$ then we write $\hat{K}_L(e) = \hat{K}_L(z_j, z_k)$ and we define the value of a graph $\gamma \in \mathcal{D}$ to be

$$v(\gamma) = \prod_e \hat{K}_L(e).$$

By [18, Theorem 3.12]

$$E \left[ \prod_{j=1}^4 |z_j|^{2\alpha} \right] = \sum_{\gamma \in \mathcal{D}} v(\gamma)$$

and so

$$E \left[ (n_L(r; 2\alpha))^4 \right] = \left( \frac{1}{2\pi i} \right)^4 \frac{1}{\alpha^4(\alpha!)^4} \sum_{\gamma \in \mathcal{D}} \int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} v(\gamma) \prod_{j=1}^4 dz_j = 3 E \left[ (n_L(r; \alpha))^2 \right]^2$$

We say that a diagram is regular if the set $\{1, 2, 3, 4\}$ can be partitioned into pairs $\{j, k\}$ such that each edge of the diagram joins a vertex labelled $j$ to $k$ or $j$ to $k$, otherwise the diagram is said to be irregular; see Figure 2. Exactly as in [7, Pages 324–5] we have

$$\left( \frac{1}{2\pi i} \right)^4 \frac{1}{\alpha^4(\alpha!)^4} \sum_{\gamma \text{ regular}} \int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} v(\gamma) \prod_{j=1}^4 dz_j = 3 E \left[ (n_L(r; \alpha))^2 \right]^2$$

and so it is enough to show, for fixed $L \geq \frac{1}{2}$, $\alpha$ and irregular diagram $\gamma$ that

$$\int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} v(\gamma) \prod_{j=1}^4 dz_j = o \left( E \left[ (n_L(r; \alpha))^2 \right]^2 \right).$$

Combining Lemmas 5 and 6 gives

$$E \left[ (n_L(r; \alpha))^2 \right] \approx \frac{1}{1 - r}$$

and so it suffices to show that

$$\int_{\partial D(0, r)^4} \frac{\partial^4}{\partial z_1 \ldots \partial z_4} v(\gamma) \prod_{j=1}^4 dz_j = o \left( (1 - r)^{-2} \right).$$
We have $v(\gamma) = \prod_e \hat{K}_L(e)$ and we compute the logarithmic derivative of $v(\gamma)$ w.r.t. a fixed $z_j$. We get

$$
\frac{\partial}{\partial z_j} v(\gamma) = \sum_e \frac{\partial}{\partial z_j} \hat{K}_L(e)
$$

and note that $\frac{\partial}{\partial z_j} \hat{K}_L(e)$ vanishes unless the edge $e$ joins a vertex labelled $j$ to a vertex labelled $k$ for some $k$, or a vertex labelled $\overline{j}$ to a vertex labelled $k'$ for some $k'$. In the former case, using the explicit expression for $\hat{K}_L(e)$ and differentiating, we get

$$
\frac{\partial}{\partial z_j} \hat{K}_L(e) = -\frac{L}{2} \frac{z_j}{1 - |z_j|^2} + L \frac{\overline{z}_k}{1 - z_j \overline{z}_k}
$$

while in the latter case we have

$$
\frac{\partial}{\partial z_j} \hat{K}_L(e) = -\frac{L}{2} \frac{z_j}{1 - |z_j|^2}.
$$

Since the total number of each type of edge is the same (and equal to $\alpha$) we get

$$
\frac{\partial}{\partial z_j} v(\gamma) = \frac{L}{1 - r^2} v(\gamma) \left( \sum_{k \neq j} E_{j,k} \frac{\overline{z}_k - z_j}{1 - z_j \overline{z}_k} \right)
$$

where $E_{j,k}$ denotes the number of edges joining $j$ to $\overline{k}$, and we have used the fact that $|z_j| = r$.

Iterating this, and using the trivial bound $|z-w|/|1-zw| \leq 1$, we see that we can bound

$$
\left| \frac{\partial^4}{\partial z_1 \ldots \partial z_4} v(\gamma) \right| \leq C(L, \gamma) \frac{|v(\gamma)|}{(1 - r)^4}
$$

and so it suffices to see that

$$
\int_{\partial D(0,r)^4} |v(\gamma)| |dz_1| \ldots |dz_4| = o \left( (1 - r)^2 \right).
$$

We now form a reduced diagram $\gamma^*$ by ‘gluing’ together all of the vertices labelled $j$ or $\overline{j}$, for each $1 \leq j \leq 4$; again see Figure 2. The edges of the resulting diagram have multiplicities, and it is not difficult to see that they must be arranged as shown on the left of Figure 3; $\ell$, $m$ and $n$ denote the multiplicity of the edges which satisfy $0 \leq \ell, m, n < 2\alpha$ and $\ell + m + n = 2\alpha$. Writing $E$ for the set of edges we have

$$
|v(\gamma)| \leq \prod_{(j,k) \in E} \left| \hat{K}_L(z_j, z_k) \right|.
$$
The fact that $\gamma$ is irregular implies that at most one of $\ell, m, n$ is zero. Now since $|\tilde{K}_L(z, w)| \leq 1$ we may delete some of the edges of $\gamma^*$ (and re-label the vertices if necessary) to get to the diagram $\gamma^{**}$ depicted on the right of Figure 3 where each edge has multiplicity 1. We therefore need to estimate

$$J_{L,r} = \int_{\partial D(0,r)^4} |\tilde{K}_L(z_1, z_2) \tilde{K}_L(z_2, z_3) \tilde{K}_L(z_3, z_4) \tilde{K}_L(z_4, z_1)| |dz_1| \ldots |dz_4|$$

$$= r^4 \int_{[-\pi,\pi]^4} |\tilde{K}_L(\theta_1 - \theta_1) \tilde{K}_L(\theta_3 - \theta_2) \tilde{K}_L(\theta_4 - \theta_3) \tilde{K}_L(\theta_1 - \theta_4)| d\theta_1 \ldots d\theta_4,$$

where $\tilde{K}_L(\theta) = \tilde{K}_{L, r}(\theta) = \tilde{K}_L(r, re^{i\theta})$.

**Claim 7.** Define $I_{L,r}(\theta) = 1 - r$ for $|\theta| \leq 1 - r$ and

$$I_{L,r}(\theta) = \begin{cases} \frac{(1-r)^{1+L}}{|\theta|^L}, & \text{if } L > 1, \\ \frac{(1-r)^2}{|\theta|^L} \left(1 + \log \frac{|\theta|}{1-r} \right), & \text{if } L = 1, \\ \frac{(1-r)^2}{|\theta|^{1-2L}}, & \text{if } \frac{1}{2} < L < 1, \end{cases}$$

for $1 - r \leq |\theta| \leq \pi$. Extend $I_{L,r}$ to be a $2\pi$-periodic function on $\mathbb{R}$. Then

$$\int_{-\pi}^{\pi} |\tilde{K}_L(\theta_4 - \theta_3) \tilde{K}_L(\theta_1 - \theta_4)| d\theta_4 \simeq I_{L,r}(\theta_3 - \theta_1).$$

By the claim we need to estimate

$$J_{L,r} \simeq \iint_{[-\pi,\pi]^2} I_{L,r}(\theta_3 - \theta_1)^2 d\theta_1 d\theta_3 = 2\pi \int_{-\pi}^{\pi} I_{L,r}(\theta)^2 d\theta$$

and doing the integration we get

$$J_{L,r} \simeq \begin{cases} (1-r)^3, & \text{if } L > \frac{3}{4}, \\ (1-r)^3 \log \frac{1}{1-r}, & \text{if } L = \frac{3}{4}, \\ (1-r)^4 L, & \text{if } \frac{1}{2} < L < \frac{3}{4}, \end{cases}$$

so that $J_{L,r} = o\left((1-r)^2\right)$. It remains only to prove the claim.

**Proof of Claim 7.** First notice that

$$|\tilde{K}_L(r, re^{i\theta})| = \left(\frac{(1-r)^2}{(1-r^2)^2 + 2r^2 (1 - \cos \theta)}\right)^{\frac{L}{2}} \simeq \begin{cases} 1, & \text{if } |\theta| \leq (1-r), \\ \left(\frac{1-r}{|\theta|}\right)^L, & \text{if } (1-r) \leq |\theta| \leq \pi. \end{cases}$$
Now we just need to do some tedious integration. By periodicity, we may assume that \( \theta_3 = 0 \). If \( |\theta_1| \geq 1 - r \) then
\[
\int_{-\pi}^{\pi} |\tilde{K}_L(\theta_4) \tilde{K}_L(\theta_1 - \theta_4)| \, d\theta_4 \simeq \int_{|\theta_4| \leq (1-r)/2} \left( \frac{1-r}{|\theta_1 - \theta_4|} \right)^L \, d\theta_4 + \int_{|\theta_4 - \theta_1| \leq (1-r)/2} \left( \frac{1-r}{|\theta_4|} \right)^L \, d\theta_4
\]
\[
+ \int_{|\theta_4|, |\theta_4 - \theta_1| \geq (1-r)/2} \left( \frac{1-r}{|\theta_4|} \right)^L \, d\theta_4
\]
\[
\simeq 2 \cdot \left( \frac{1-r}{|\theta_1|} \right)^L + \left( \frac{1-r}{|\theta_1|} \right)^L \int_{(1-r)/2 \leq |\theta_4 - \theta_1| \leq |\theta_1|/2} \frac{1}{|\theta_4 - \theta_1|^L} \, d\theta_4
\]
\[
+ \int_{|\theta_4|, |\theta_4 - \theta_1| \geq |\theta_1|/2} \left( \frac{1-r}{|\theta_4|} \right)^{2L} \, d\theta_4
\]
Performing the integrals we get
\[
\int_{-\pi}^{\pi} |\tilde{K}_L(\theta_4) \tilde{K}_L(\theta_1 - \theta_4)| \, d\theta_4 \simeq \begin{cases} 
(1-r)^{1+L}/|\theta_1|^L, & L > 1, \\
(1-r)^2/(|\theta_1|^L \log (1-r)), & L = 1, \\
(1-r)^2L/|\theta_1|^{1-2L}, & \frac{1}{2} < L < 1,
\end{cases}
\]
for \( |\theta_1| \geq 1 - r \). On the other hand, when \( |\theta_1| \leq 1 - r \) we have
\[
\int_{-\pi}^{\pi} |\tilde{K}_L(\theta_4) \tilde{K}_L(\theta_1 - \theta_4)| \, d\theta_4 \simeq \int_{|\theta_4|, |\theta_4 - \theta_1| \leq 2(1-r)} \, d\theta_4
\]
\[
+ \left| \int_{|\theta_4| \geq 2(1-r)} \right| \tilde{K}_L(\theta_4) \tilde{K}_L(\theta_1 - \theta_4) | \, d\theta_4
\]
Now note that if \( |\theta_4 - \theta_1| \geq 2(1-r) \) and \( |\theta_1| \leq (1-r) \) then \( |\theta_4| \geq (1-r) \). We get
\[
\int_{-\pi}^{\pi} |\tilde{K}_L(\theta_4) \tilde{K}_L(\theta_1 - \theta_4)| \, d\theta_4 \simeq 1 - r + \int_{|\theta_4| \geq (1-r) \geq |\theta_1|/2} \left( \frac{1-r}{|\theta_4|} \right)^{2L} \, d\theta_4 \simeq 1 - r.
\]

\section{Variance for \( L = 0 \)}

In order to implement the strategy we outlined in Section 2 we need sharp estimates for the asymptotic growth of \( \text{Var} \left[ n_L(r) \right] \). For \( L > 0 \) these were computed in \( \cite{6} \), and the corresponding result for \( L = 0 \) is as follows.

\begin{proposition} \label{prop8}
We have
\[
\text{Var} \left[ n_0 \left( r \right) \right] \simeq \frac{\pi^2}{24(1-r)^2 \left( \log \frac{1}{1-r} \right)^4}.
\]
\end{proposition}

We will actually give a proof that recovers the result from \( \cite{6} \) for \( L < \frac{1}{2} \) (with no regard for the error term). Consider a GAF (in the unit disc) of the form
\[
f(z) = \sum_{m \geq 0} b_m z^m
\]
where \( b_m \geq 0 \) and denote by
\[
G(z) = \sum_{m \geq 0} b_m^2 z^m
\]
Proposition 9. Put

In particular, the (radial) boundary values $H$ where we may take

It is not difficult to check that the GAFs

Using the above notation we have the following formula for the variance of the number of zeroes of $f$ in the disc of radius $r < 1$ (see [20, Appendix A])

We write $\zeta = t + i\theta$, with $t \leq 0$, and mention that $|H(t + i\theta)| \leq H(t)$. In order to derive an asymptotic expression for the variance we will make the following assumptions; they will allow us to show that the integrand above may be approximated by $A^2(t)|H(i\theta)|^2$ when $\theta$ does not belong to a small neighbourhood of 0 and apply the Dominated Convergence Theorem.

(A1) $\sum_{m \geq 0} b_m^4 < \infty$.

(A2) $H(t + i\theta) = o(H(t))$ as $t \to 0^-$ for every $\theta \in [-\pi, \pi] \setminus \{0\}$.

(A3) There exist $t_0 < 0$ and a constant $C \geq 1$ such that for $t_0 < t < 0$ we have $|A(t + i\theta)| \leq CA(t)$ for every $\theta \in [-\pi, \pi]$. Furthermore, for every $\theta \in [-\pi, \pi] \setminus \{0\}$ we have $A(t + i\theta) = o(A(t))$ as $t \to 0^-$.

(A4) There is a function $\Delta : (-\infty, 0) \to [0, \pi]$ such that $\Delta(t) \downarrow 0$, as $t \to 0^-$ and moreover if $t$ is sufficiently close to 0 then

$$|\theta| \geq \Delta(t) \implies |H(t + i\theta)| \leq \frac{1}{2} H(t).$$

(A5) Additionally $\Delta(t)A'(t) = o(A^2(t)H^{-2}(t))$ as $t \to 0^-.$

Remark. Assumption (A1) implies that $G$ belongs to the Hardy space $H^2(\mathbb{D})$. Thus, there is a function $M \in L^2([-\pi, \pi])$ (e.g., the radial or non-tangential maximal function) that satisfies, for $\theta \in [-\pi, \pi],$

$$\sup_{t < 0} |H(t + i\theta)| \leq M(\theta).$$

In particular, the (radial) boundary values $H(i\theta)$ exist for a.e. $\theta \in \mathbb{T}.$

Proposition 9. Put $e^t = r^2$ and let $f$ be a GAF whose covariance function $G$ satisfies the above assumptions. Then

$$\text{Var} \left[n_f(r)\right] = (1 + o(1)) \frac{A^2(t)}{H^2(t)} \sum_{m \geq 0} b_m^4 = (1 + o(1)) \frac{(G'(r^2))^2}{(G(r^2))^4} \sum_{m \geq 0} b_m^4, \text{ as } r \to 1.$$  

In our case we have

$$H_L(\zeta) = \begin{cases} -\log(1 - e^\zeta), & L = 0; \\ (1 - e^\zeta)^{-L}, & L \in (0, \frac{1}{2}), \end{cases} \quad A_L(\zeta) = \begin{cases} -e^\zeta \log(1 - e^\zeta), & L = 0; \\ \frac{Le^\zeta}{1 - e^\zeta}, & L \in (0, \frac{1}{2}), \end{cases}$$

and

$$A'_L(\zeta) = \begin{cases} -e^\zeta(1 - \log(1 - e^\zeta)) & L = 0; \\ (1 - e^\zeta)^2 \log^2(1 - e^\zeta) & L \in (0, \frac{1}{2}), \end{cases}$$

It is not difficult to check that the GAFs $f_L$ satisfy the assumptions of Proposition 9 for $L \in [0, \frac{1}{2})$, where we may take

$$\Delta_L(t) = \begin{cases} \frac{1}{\log(1 - e^t)} & L = 0; \\ (1 - e^t)L + \frac{1}{2} & L \in (0, \frac{1}{2}). \end{cases}$$
We conclude that, as \( r \to 1 \),
\[
\text{Var} [n_L(r)] \sim \begin{cases} \frac{\pi^2}{24} \cdot \frac{1}{(1-r)^2 \log^2(1-r)}, & L = 0; \\ \frac{1}{r^2 T(1-2r)}, & L \in (0, \frac{1}{2}). \end{cases}
\]

**Proof of Proposition 5.4.** We rewrite the integrand in (6) as follows
\[
I_H(t; \theta) = \frac{|H(t) H'(t + i\theta) - H(t + i\theta) H'(t)|^2}{H^2(t)(H^2(t) - |H(t + i\theta)|^2)} = \frac{|A(t) - A(t + i\theta)|^2 |H(t + i\theta)|^2}{H^2(t) - |H(t + i\theta)|^2},
\]
and split the integral
\[
\text{Var} [n_f(r)] = \frac{1}{2\pi} \int_{J_1 \cup J_2} I_H(t; \theta) \, d\theta,
\]
where \( J_1 = \{ \Delta(t) \leq |\theta| \leq \pi \} \) and \( J_2 = \{ |\theta| \leq \Delta(t) \} \).

On \( J_1 \) we re-write the integrand as
\[
I_H(t; \theta) = \frac{A^2(t) |H(t + i\theta)|^2}{H^2(t)} R(t; \theta),
\]
where
\[
R(t; \theta) = \left| 1 - \frac{A(t + i\theta)}{A(t)} \right|^2 \left( 1 - \frac{|H(t + i\theta)|^2}{H^2(t)} \right)^{-1}.
\]
By Assumptions (A3) and (A4) we have that \( |R(t; \theta)| \leq C' \) for \( t \) sufficiently close to 0. Combining this with (7) we see that we may apply the Dominated Convergence Theorem, and using Assumptions (A2) and (A3) we get
\[
\lim_{t \to 0} \frac{1}{2\pi} \int_{J_1} I_H(t; \theta) \left( \frac{A^2(t)}{H^2(t)} \right)^{-1} \, d\theta = \frac{1}{2\pi} \int_{-\pi}^\pi |H(i\theta)|^2 \, d\theta = \sum_{m \geq 0} b_m^4.
\]

On \( J_2 \), we use the bound \(20 I_H(t; \theta) \leq A'(t) \) which is valid for all \( \theta \in [-\pi, \pi] \) and \( t < 0 \) (20 Corollary 5.3). Using Assumption (A5) we get
\[
\frac{1}{2\pi} \int_{J_2} I_H(t; \theta) \, d\theta \leq \frac{\Delta(t)}{\pi} A'(t) = o \left( \frac{A^2(t)}{H^2(t)} \right), \quad \text{as } t \to 0^-.
\]

5. **Non-CLT for \( 0 \leq L < \frac{1}{2} \) and CLT for \( L = \frac{1}{2} \)**

In this section we complete the proof of Theorem 5.1. As we outlined in Section 2 we will show that the main contribution comes from \( n_L(r; 1) \) and we begin by deriving another expression for it.

5.1. **An explicit formula for \( n_L(r; 1) \).** We begin with an elementary but useful lemma about deterministic power series.

**Lemma 10.** Suppose that \( f(z) = \sum b_m z^m \) has radius of convergence 1. Then, for \( 0 < r < 1 \),
- \( \frac{1}{2\pi r} \int_{|f(0, r)|^2} |f(z)|^2 \, dz = r^2 \sum |b_m|^2 r^{2m}, \) and
- \( \frac{1}{2\pi} \int_{|f(0, r)|^2} f'(z) \overline{f(z)} \, dz = \sum m |b_m|^2 r^{2m}. \)

\[20\] uses the notation \( B = A' \).
Proof. We have
\[
\frac{1}{2\pi i} \int_{\partial D(0, r)} z |f(z)|^2 \, dz = \frac{1}{2\pi i} \int_{-\pi}^{\pi} r e^{-i\theta} \sum_{m,m'=0}^{\infty} b_m \overline{b}_{m'} r^{m+m'} e^{i\theta(m-m')} r i e^{i\theta} \, d\theta
\]
\[
= r^2 \sum_{m,m'=0}^{\infty} b_m \overline{b}_{m'} r^{m+m'} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(m-m')} \, d\theta
\]
\[
= r^2 \sum_{m=0}^{\infty} |b_m|^2 r^{2m}
\]
since the Taylor series that defines \( f \) converges uniformly on compact subsets of \( \mathbb{D} \). Similarly
\[
\frac{1}{2\pi i} \int_{\partial D(0, r)} f'(z) \overline{f(z)} \, dz = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \sum_{m,m'=0}^{\infty} mb_m \overline{b}_{m'} r^{m-1+m'} e^{i\theta(m-1-m')} r i e^{i\theta} \, d\theta
\]
\[
= \sum_{m,m'=0}^{\infty} mb_m \overline{b}_{m'} r^{m+m'} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(m-m')} \, d\theta
\]
\[
= \sum_{m=0}^{\infty} m|b_m|^2 r^{2m}.
\]

Proposition 11. We have
\[
n_L (r; 1) = (1 - r^2)^{L-1} \sum_{m=0}^{\infty} a_{m,L} (m (1 - r^2) - L r^2) \left(|\zeta_m|^2 - 1\right) r^{2m}
\]
for \( L > 0 \) and
\[
n_0 (r; 1) = \frac{1}{(1 - r^2) \left(\log \frac{1}{1-r^2}\right)} \sum_{m=1}^{\infty} a_{m,0} \left(m (1 - r^2) \log \frac{1}{1-r^2} - r^2\right) \left(|\zeta_m|^2 - 1\right) r^{2m}.
\]

Proof. Recall that
\[
n_L (r; \alpha) = \frac{(-1)^{\alpha+1}}{\alpha (\alpha!)} \frac{1}{2\pi i} \int_{\partial D(0, r)} \frac{\partial}{\partial \zeta} |\hat{f}_L (z)|^{2\alpha} \, dz
\]
and that \(|\zeta|^2 := |\zeta|^2 - 1\). This yields, for \( L > 0 \),
\[
n_L (r; 1) = \frac{1}{2\pi i} \int_{\partial D(0, r)} \frac{\partial}{\partial \zeta} |\hat{f}_L (z)|^2 \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\partial D(0, r)} -L \zeta (1 - |\zeta|^2)^{L-1} |\hat{f}_L (z)|^2 + (1 - |\zeta|^2)^L f'_L (z) \overline{\hat{f}_L (z)} \, dz
\]
\[
= (1 - r^2)^{L-1} \sum_{m=0}^{\infty} a_{m,L} (m (1 - r^2) - L r^2) |\zeta_m|^2 r^{2m}
\]
where the last equality follows from the previous lemma. Since \( \mathbb{E} [n_L (r; 1)] = 0 \) and \( \mathbb{E} [|\zeta_m|^2] = 1 \) the result for \( L > 0 \) follows. The case \( L = 0 \) is similar and omitted. \[\Box\]

\(^3\)It is also possible to verify that \( \sum_{m=0}^{\infty} a_{m,L} (m(1 - r^2) - L r^2) r^{2m} = 0 \) directly.
5.2. **The second chaos dominates.** In this section we show that the main contribution to \( n_L(r) \) comes from \( n_L(r; 1) \), the projection to the second chaos. (Recall that the odd chaoses vanish and \( n_L(r; \alpha) \) denotes the \( 2\alpha \)-th chaos.)

**Proposition 12.** If \( 0 \leq L \leq \frac{1}{2} \) is fixed then

\[
E \left[ n_L(r; 1)^2 \right] \sim \text{Var} \left[ n_L(r) \right]
\]

as \( r \to 1 \). Moreover, this asymptotic also holds as \( L \to \frac{1}{2}, r \to 1 \) in an arbitrary way.

**Remarks.**

1. From \([6]\) we have

\[
\text{Var} \left[ n_L(r) \right] \sim \begin{cases} 
\frac{L^2 \Gamma \left( \frac{1}{2} - L \right)}{4 \sqrt{\pi} \Gamma \left( 1 - L \right)} \left( 1 - r \right)^{2L - 2}, & \ 0 < L < \frac{1}{2}; \\
\frac{1}{8\pi} \left( 1 - 2L \right)^{2L - 2} \left( 1 - (1 - x)^{1 - 2L} \right), & \ L \to \frac{1}{2}, L \neq \frac{1}{2}; \\
\frac{1}{8\pi \Gamma (1 - r)} \log \frac{1}{1 - r}, & \ L = \frac{1}{2}. 
\end{cases}
\]

The case \( L = 0 \) is given in Proposition \([8]\).

2. Indeed it is possible to mimic the proofs given in \([6]\) to prove Proposition \([12]\), but we give a different one for variety. Moreover this proof will boil down to proving estimates that will be necessary for the proof of Theorem \([4]\).

We begin with a useful lemma. We put

\[
\psi_L(x) = \begin{cases} 
\frac{1 - (1-x)^{1 - 2L}}{\pi (1 - 2L)}, & \text{for } L \neq \frac{1}{2}; \\
\frac{1}{\pi} \log \frac{1}{1 - x}, & \text{for } L = \frac{1}{2}.
\end{cases}
\]

**Lemma 13.** If \( 0 < L < \frac{1}{2} \) is fixed then

\[
\sum_{m=0}^{\infty} a_{m,L}^2 x^m \xrightarrow{x \to 1^-} \frac{\Gamma (1 - 2L)}{\Gamma (1 - L)^2} = \frac{\Gamma \left( \frac{1}{2} - L \right)}{4L \sqrt{\pi} \Gamma (1 - L)}.
\]

If \( x \to 1^- \) and \( L \to \frac{1}{2} \) then

\[
\sum_{m=0}^{\infty} a_{m,L}^2 x^m \sim \psi_L(x).
\]

**Proof.** For \( 0 < L < \frac{1}{2} \) and \( 0 < x < 1 \) we have\(^4\)

\[
\sum_{m=0}^{\infty} a_{m,L}^2 x^m = \frac{1}{\Gamma (L) \Gamma (1 - L)} \int_0^1 t^{L-1} (1 - t)^{-L} (1 - xt)^{-L} \, dt,
\]

which is easily verified by expanding the term \( (1 - xt)^{-L} \) as a power series in \( xt \). Since both sides are convergent when \( x = 1 \) we have

\[
\sum_{m=0}^{\infty} a_{m,L}^2 x^m \xrightarrow{x \to 1^-} \sum_{m=0}^{\infty} a_{m,L}^2 = \frac{1}{\Gamma (L) \Gamma (1 - L)} \int_0^1 t^{L-1} (1 - t)^{-2L} \, dt = \frac{B \left( L, 1 - 2L \right)}{\Gamma (L) \Gamma (1 - L)} = \frac{\Gamma (1 - 2L)}{\Gamma (1 - L)^2}
\]

where \( B \) is the beta function. Applying the identity

\[
\Gamma \left( z \right) \Gamma \left( z + \frac{1}{2} \right) = 2^{1 - 2z} \sqrt{\pi} \Gamma (2z)
\]

yields \([9]\).
Now suppose that $x \to 1^-$ and $L \to \frac{1}{2}^-$ simultaneously and write $\varepsilon = \varepsilon(x) = (1 - x) \log \frac{1}{1 - x}$. For $0 < t < 1 - \varepsilon$ we have
\[(1 - xt)^{-L} = (1 - t)^{-L} \left(1 + \frac{(1 - x)t}{1 - t}\right)^{-L} \sim (1 - t)^{-L}\]
uniformly, which yields
\[
\int_0^{1 - \varepsilon} t^{L-1}(1 - t)^{-L}(1 - xt)^{-L} \, dt \sim \int_0^{1 - \varepsilon} t^{L-1}(1 - t)^{-2L} \, dt
\]
\[= B(L, 1 - 2L) - \int_1^{1 - \varepsilon} t^{L-1}(1 - t)^{-2L} \, dt\]
\[\sim \frac{1 - \varepsilon^{1 - 2L}}{1 - 2L} \sim \frac{1 - (1 - x)^{1 - 2L}}{1 - 2L}.
\]
Further
\[
\int_1^{1 - \varepsilon} t^{L-1}(1 - t)^{-L}(1 - xt)^{-L} \, dt \leq (1 - x)^{-L} \int_1^{1 - \varepsilon} t^{L-1}(1 - t)^{-L} \, dt
\]
\[\lesssim (1 - x)^{-L} \int_0^\varepsilon s^{-L} \, ds
\]
\[= (1 - x)^{1 - 2L} \left(\log \frac{1}{1 - x}\right)^{1 - L} = o \left(\frac{1 - (1 - x)^{1 - 2L}}{1 - 2L}\right).
\]
Since $\Gamma \left(\frac{1}{2}\right) = \sqrt{\pi}$ we have $\Gamma(L) \Gamma(1 - L) \to \pi$ and the result follows in this case.

For $L \geq \frac{1}{2}$ we use the asymptotic (which follows from Stirling’s bounds)
\[
\frac{\Gamma(L + m)}{m!} = m^{L-1} + O(m^{L-2})
\]
for $m \geq 1$, and the implicit constant is uniform for $L$ bounded away from $0$ and $\infty$. For $L = \frac{1}{2}$ we get
\[
\sum_{m=0}^{\infty} a_{m, L} x^m = 1 + \frac{1}{\pi} \sum_{m=1}^{\infty} \left(\frac{1}{m} + O \left(\frac{1}{m^2}\right)\right) x^m = \frac{1}{\pi} \log \frac{1}{1 - x} + O(1).
\]
For $L \to \frac{1}{2}^+$ we see that
\[
\sum_{m=0}^{\infty} a_{m, L} x^m = 1 + \frac{1}{\Gamma(L)^2} \sum_{m=1}^{\infty} \left(\frac{1}{m^{2 - 2L}} + O \left(\frac{1}{m^{3 - 2L}}\right)\right) x^m \sim \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{x^m}{m^{2 - 2L}}.
\]
On the other hand
\[
\frac{1 - (1 - x)^{1 - 2L}}{1 - 2L} = \frac{1}{(2L - 1) \Gamma(2L - 1)} \sum_{m=1}^{\infty} \frac{\Gamma(2L - 1 + m)}{m!} \frac{x^m}{m^{2 - 2L}} \sim \sum_{m=1}^{\infty} \frac{x^m}{m^{2 - 2L}}.
\]

\[\square\]

\textbf{Proof of Proposition} (12) Since the random variables $|\zeta_m|^2 - 1$ are orthonormal we get, for $L > 0$,
\[
\mathbb{E} \left[n_L \left(r; 1 \right)^2 \right] = (1 - r^2)^{2L - 2} \sum_{m=0}^{\infty} a_{m, L} \left(m \left(1 - r^2\right) - L r^2\right)^2 r^{4m}.
\]
We expand the term $(m \left(1 - r^2\right) - L r^2)^2$ and estimate, bearing in mind (8). From Lemma (13) we see that
\[
L^2 r^4 (1 - r^2)^{2L - 2} \sum_{m=0}^{\infty} a_{m, L} r^{4m} \sim \text{Var} \left[n_L \left(r\right)\right]
\]
for $0 < L \leq \frac{1}{2}$ and for $L \to \frac{1}{2}$. We also have, for $L > 0$,

$$(1 - r^2)^{2L} \sum_{m=0}^{\infty} a_{m,L} m^2 r^{4m} \simeq (1 - r)^{2L} \sum_{m=1}^{\infty} m^{2L} r^{4m} \simeq (1 - r)^{-1}$$

and this is all uniform when $L$ is close to $\frac{1}{2}$. Similarly

$$L r^2 (1 - r^2)^{2L-1} \sum_{m=0}^{\infty} a_{m,L} m r^{4m} \simeq (1 - r)^{2L-1} \sum_{m=1}^{\infty} m^{2L-1} r^{4m} \simeq (1 - r)^{-1}$$

and furthermore $(1 - r)^{-1} = o(\Var[n_L(r)])$ if $0 < L \leq \frac{1}{2}$ or $L \to \frac{1}{2}$.

Similarly.

$$\mathbb{E} \left[ n_0(r; 1)^2 \right] = \frac{1}{(1 - r^2)^2 \log \frac{1}{1 - r}} \sum_{m=1}^{\infty} \left( (1 - r^2) \log \frac{1}{1 - r^2} - \frac{r^2}{r^m} \right) \frac{1}{2} r^{4m}$$

as claimed. \hfill \Box

5.3. Completing the proof of Theorem 1

Proof of Theorem 1 [in]. We begin with the case $0 < L < \frac{1}{2}$. By Proposition 12 it is enough to see that $\frac{n_L(r; 1)}{\Var[n_L(r)]} \to -c_L X_L$ in $L^2$. Using the estimate [8], the alternative expression for $c_L$ given by [9] and Proposition 11, we see that it is enough to show that

$$\frac{1}{L} \sum_{m=0}^{\infty} a_{m,L} (m (1 - r^2) - L r^2) \left( |\zeta_m|^2 - 1 \right) r^{2m} \overset{L^2}{\to} -X_L.$$

But we have

$$\mathbb{E} \left[ \left( \sum_{m=0}^{\infty} a_{m,L} m (1 - r^2) \left( |\zeta_m|^2 - 1 \right) r^{2m} \right)^2 \right] = (1 - r^2)^2 \sum_{m=0}^{\infty} a_{m,L}^2 m^2 r^{4m} \simeq (1 - r)^{1 - 2L} = o(1)$$

as before, and so we need to show that

$$-r^2 \sum_{m=0}^{\infty} a_{m,L} \left( |\zeta_m|^2 - 1 \right) r^{2m} \overset{L^2}{\to} -X_L.$$

This will follow if we show that

$$\sum_{m=0}^{\infty} a_{m,L}^2 (r^{2m+2} - 1)^2 \to 0,$$

but this is obvious since $\sum_{m=0}^{\infty} a_{m,L}^2 < +\infty$.

Next we treat the case $L = 0$. We need to see that

$$\sum_{m=1}^{\infty} a_{m,0} \left( m (1 - r^2) \log \frac{1}{1 - r^2} - r^2 \right) \left( |\zeta_m|^2 - 1 \right) r^{2m} \overset{L^2}{\to} -X_0.$$

Similar to before we see that

$$\mathbb{E} \left[ \left( \sum_{m=1}^{\infty} a_{m,0} m (1 - r^2) \log \frac{1}{1 - r^2} \left( |\zeta_m|^2 - 1 \right) r^{2m} \right)^2 \right] = (1 - r^2)^2 \left( \log \frac{1}{1 - r^2} \right)^2 \sum_{m=1}^{\infty} r^{4m}

\simeq (1 - r) \left( \log \frac{1}{1 - r} \right)^2 = o(1)$$
and that
\[ \sum_{m=1}^{\infty} a_{m,0}^2 \left( r^{2m+2} - 1 \right)^2 \to 0. \]
\[ \square \]

Finally we treat the case \( L \to \frac{1}{2} \), which of course includes the case \( L = \frac{1}{2} \), and so completes the proof of Theorem 1. We first state a convenient lemma.

**Lemma 14.** Let \( \zeta_m \) be a sequence of iid \( \mathcal{N}_C(0,1) \) random variables as before and \( \alpha_m \) be square-summable real coefficients. Then
\[ S = \sum_{m=0}^{\infty} \alpha_m (|\zeta_m|^2 - 1) \]
is almost surely convergent, has mean 0, and satisfies
\[ \mathbb{E}[S^2] = \sum_{m=0}^{\infty} \alpha_m^2 \quad \text{and} \quad \mathbb{E}[S^4] = 3 \left( \sum_{m=0}^{\infty} \alpha_m^2 \right)^2 + 6 \sum_{m=0}^{\infty} \alpha_m^4. \]

**Remark.** The Gaussianity plays no role here, we may replace \( |\zeta_m|^2 - 1 \) by any iid real-valued random variables with the same mean, variance and fourth moment.

**Proof of Theorem 1 (ii) and the case \( L = \frac{1}{2} \).** As before, by Proposition 12 it is enough to see that
\[ \frac{n_L(r;1)}{\sqrt{\mathbb{E}[n_L(r;1)^2]}} \]
is asymptotically normal. By the Fourth Moment Theorem [29, Theorem 1] this is equivalent to
\[ \frac{\mathbb{E}[n_L(r;1)^4]}{\mathbb{E}[n_L(r;1)^2]^2} \to 3. \]
Combining Lemma 14, Proposition 11 and the estimate (8) we see that it is enough to show that
\[ \sum_{m=0}^{\infty} a_{m,L}^4 (m (1 - r^2) - Lr^2)^4 r^{8m} = o \left( \psi_L(r)^2 \right). \]
In fact we will show that
\[ \sum_{m=0}^{\infty} a_{m,L}^4 (m (1 - r^2) - Lr^2)^4 r^{8m} = O(1). \]

First note that for \( 1 \leq k \leq 4 \) we have
\[ (1 - r)^k \sum_{m=0}^{\infty} a_{m,L}^4 m^{k} r^{8m} \simeq (1 - r)^k \sum_{m=0}^{\infty} m^{4L-4+k} r^{8m} \simeq (1 - r)^{3-4L} = o(1) \]
and
\[ L^4 \sum_{m=0}^{\infty} a_{m,L}^4 r^{8m} \lesssim L^4 \sum_{m=0}^{\infty} m^{4L-4} = O(1) \]
for \( L < \frac{3}{4} \). This completes the proof. \( \square \)

**Proof of Lemma 14.** To shorten the expressions we write \( \hat{|\zeta|^2} = |\zeta|^2 - 1 \). The fact that \( S \) is convergent is an obvious consequence, e.g., of Kolmogorov’s Three Series Theorem, and since \( \mathbb{E}[\hat{|\zeta|^2}] = 0 \), it must have mean 0 also. Since the sequence \( \hat{|\zeta|^2} \) is orthogonal in \( L^2 \) we see that
\[ \mathbb{E}[S^2] = \sum_{m,m'=0}^{\infty} \alpha_m \alpha_{m'} \mathbb{E}[\hat{|\zeta|^2} \cdot \hat{|\zeta'|^2}] = \sum_{m=0}^{\infty} \alpha_m^2. \]
Finally note that a straightforward computation gives

\[
\mathbb{E} \left[ |\zeta_{m_1}|^2 : |\zeta_{m_2}|^2 : |\zeta_{m_3}|^2 : |\zeta_{m_4}|^2 \right] = \begin{cases} 
9, & \text{if } m_1 = m_2 = m_3 = m_4; \\
1, & \text{if } \{m_1, m_2, m_3, m_4\} = \{m, m'\} \text{ where } m \neq m'; \\
0, & \text{otherwise.}
\end{cases}
\]

This yields

\[
\mathbb{E} \left[ \mathcal{S}^4 \right] = \sum_{m_1, m_2, m_3, m_4 = 0}^{\infty} \alpha_{m_1} \alpha_{m_2} \alpha_{m_3} \alpha_{m_4} \mathbb{E} \left[ |\zeta_{m_1}|^2 : |\zeta_{m_2}|^2 : |\zeta_{m_3}|^2 : |\zeta_{m_4}|^2 \right]
= 6 \left( \sum_{m=0}^{\infty} \sum_{m' < m} \alpha_m^2 \alpha_{m'}^2 \right) + 9 \sum_{m=0}^{\infty} \alpha_m^4 = 3 \left( \sum_{m=0}^{\infty} \sum_{m' \neq m} \alpha_m^2 \alpha_{m'}^2 \right) + 9 \sum_{m=0}^{\infty} \alpha_m^4
= 3 \left( \sum_{m=0}^{\infty} \alpha_m^2 \right)^2 + 6 \sum_{m=0}^{\infty} \alpha_m^4.
\]

5.4. **A comment on the case** \( L \to 0^+ \). Theorem [1] does not cover the case \( L \to 0^+ \). We believe that the behaviour should be as follows, although the computations needed to prove it by our methods appear formidable. For \( x \geq 0 \) we define

\[
\Phi_\eta(x) = \begin{cases} 
1 - x^2, & \eta = 0, \\
\sqrt{1-\eta^2} \left( e^{-\frac{\eta}{\eta^2}} - 1 \right), & 0 < \eta < 1; \\
0 & \eta = 1.
\end{cases}
\]

If \( L \to 0, r \to 1, \frac{L}{1-r} \to \infty \) and \( (1-r)^L \to \eta \) then we should have \( \hat{n}_L(r) \to \Phi_\eta(|\zeta_0|) \) in \( L^2 \).

Let us explain the importance of \( \eta \). If \( m \geq 1 \) is fixed then \( a_{m,L} \sim \frac{L}{m} \) as \( L \to 0 \), which means that, almost surely, \( \frac{1}{\sqrt{L}} \left( f_L - \zeta_0 \right) \to f_0 \) locally uniformly. Heuristically \( f_L(z) = 0 \) corresponds to \( f_0(z) = -\zeta_0/\sqrt{L} \). The typical size of \( |f_0(z)|^2 \) is \( \log \frac{1}{1-|z|} \) and \( \eta \) determines whether \( 1/L \) is much smaller than, roughly the same size as, or much larger than this value.

6. Tail asymptotics

We will finally prove Theorem [2]. Since the distribution of the random variable \( |\zeta_m|^2 \) is \( \text{Exp}(1) \) whose characteristic function is \( \frac{1}{1-t} \), we can calculate (recall [3])

\[
\varphi_{X_L}(t) = \mathbb{E} \left[ e^{i t X_L} \right] = \prod_{m=0}^{\infty} \mathbb{E} \left[ e^{i t a_{m,L} (|\zeta_m|^2 - 1)} \right] = \prod_{m=0}^{\infty} \frac{e^{-i t a_{m,L}}}{1 - i t a_{m,L}}.
\]

If \( L = 0 \) then we recognise Weierstrass’s definition of the Gamma function and get

\[
\varphi_{X_0}(t) = e^{-i \gamma t} \Gamma(1 - it)
\]

which yields Theorem [2] [6]. For the rest of this section we assume that \( 0 < L < \frac{1}{2} \) and write \( \varphi_{X_L}(t) = 1/\Psi_L(-it) \) where \( \Psi_L \) is the entire function

\[
\Psi_L(z) = \prod_{m=0}^{\infty} \left( 1 + \frac{z}{b_{m,L}} \right) e^{-z/b_{m,L}};
\]

here

\[
b_{m,L} = \frac{1}{a_{m,L}} = \frac{\Gamma(L)m!}{\Gamma(L+m)}
\]
which satisfies \( b_{m,L} \sim \Gamma(L)m^{1-L} \) as \( m \to \infty \). This allows us to explicitly describe the tail behaviour of \( X_L \).

Denote by \( n_L(R) \) the zero counting function of \( \Psi_L \) so that

\[
n_L(R) = \# \{ z \in \mathbb{C}: \Psi_L(z) = 0 \text{ and } |z| \leq R \} = \# \{ m \in \mathbb{N}: b_{m,L} \leq R \} \sim \left( \frac{R}{\Gamma(L)} \right)^{1/(1-L)}
\]

for fixed \( L \). We put \( \rho = \frac{1}{1-L} \in (1,2) \) and \( d_\rho = \Gamma(1-1/\rho)^{-\rho} = \Gamma(L)^{-1/(1-L)} \), and notice that \( \Psi_L \) is an entire function of order \( \rho \) and genus 1, which corresponds to the fact that

\[
\sum_{m=1}^\infty \frac{1}{b_{m,L}^2} < \infty.
\]

### 6.1. Preliminaries: growth bounds for \( \Psi_L \)

We will deduce Theorem (2) (ii) by standard techniques which require growth estimates for \( \Psi_L \). The estimates that we will use are given in the following proposition.

**Proposition 15.**

(i) As \( R \to \infty \)

\[
\log \Psi_L(R) = \frac{\pi d_\rho}{\sin(\pi \rho)} R^{\rho}(1 + o(1)).
\]

(ii) Fix \( x \leq 0 \). If \( |y| \geq 2|x| \) is sufficiently large then

\[
\log |\Psi_L(x + iy)| \geq c|y|^{\rho},
\]

where \( c > 0 \) is a constant that depends only on \( \rho \).

**Remark.** The estimate given in (ii) is rather crude, but suffices for our purposes.

**Proof.** Part (i) is immediate from \cite[Equation (7.2.3)]{4}. To see part (ii) we note that, with the definitions given above, we have

\[
\log |\Psi_L(z)| = -\int_0^\infty \Re \left( \frac{z^2}{t+z} \right) \frac{n_L(t)}{t^2} \, dt
\]

for \( \arg z \neq \pi \), by \cite[Theorem 7.2.1]{4}. Now fix \( x \leq 0 \) and note that if \( |y| \geq |x| \) then

\[
-\Re \left( \frac{z^2}{t+z} \right) = \frac{y^2(t-x) - x^2(t+x)}{(t+x)^2 + y^2} \geq 0.
\]

Further if \( t \leq |y| + |x| \) and \( |y| \geq 2|x| \) then we have the lower bound

\[
\frac{y^2(t-x) - x^2(t+x)}{(t+x)^2 + y^2} \geq \frac{t(y^2 - x^2)}{2y^2} \geq \frac{t}{4}.
\]

We conclude that if \( |y| \) is sufficiently large then there is a constant \( c = c_\rho > 0 \) such that

\[
\log |\Psi_L(z)| \geq c \int_1^{|y|+|x|} t^{\rho-1} \, dt \geq c' |y|^\rho.
\]

\( \square \)

### 6.2. The left tail

We will deduce the asymptotics for the left tail from a Tauberian theorem of Kasahara.

**Proposition 16 (\cite[Lemma 3]{16}).** Let \( \rho \in (1,2) \) and let \( Z \) be a random variable satisfying \( \mathbb{P}[Z < a] > 0 \) for any \( a \in \mathbb{R} \), then

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^\rho} \log \mathbb{E}\left[e^{-\lambda Z}\right] = A > 0
\]
if and only if
\[
\lim_{y \to \infty} \frac{1}{y^{\rho/(\rho-1)}} \log \mathbb{P} [Z < -y] = - \left( 1 - \frac{1}{\rho} \right) \left( \frac{1}{\rho A} \right)^{\frac{1}{\rho-1}}.
\]

**Remark.** Kasahara originally stated his result for the right tail of a random variable \(X\) that satisfies \(\mathbb{P} [X > a] > 0\) for every \(a > 0\) but, as noted in the remark after [16, Lemma 3], this assumption is easy to remove.

By Proposition [15][1]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^\rho} \log \mathbb{E} \left[ e^{-\lambda X_L} \right] = - \lim_{\lambda \to \infty} \frac{1}{\lambda^\rho} \log \Psi_L(\lambda) = - \frac{\pi d_\rho}{\sin(\pi \rho)}
\]
and so, by Proposition [16] we find that
\[
\log \mathbb{P} [X_L < -y] \sim - \left( 1 - \frac{1}{\rho} \right) \left( \frac{\pi d_\rho}{\sin(\pi \rho)} \right)^{\frac{1}{\rho-1}} y^{\rho/(\rho-1)}
\]
as \(y \to \infty\), where sinc \(x = \sin x/x\), which is one half of Theorem [2][ii].

### 6.3. The right tail.

For the right tail we first use the inversion formula to compute the density \(f_{X_L}\) of the random variable \(X_L\). We have
\[
f_{X_L}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{X_L}(t)e^{-ixt} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\Psi_L(-it)} e^{-ixt} dt,
\]
which holds by Proposition [15][ii]. By the same estimate, we can make a change of contour, and thus by the residue theorem,
\[
f_{X_L}(x) = \frac{1}{2\pi} \int_{\mathbb{R}-i\beta} \psi(w) dw - i \text{Res}(\psi, -ib_{0,L}),
\]
where \(\psi(w) = \frac{1}{\Psi_L(-iw)} e^{-ixw}\), and \(\beta \in (b_{0,L}, b_{1,L}) = (1, L^{-1})\) is some constant (the sign of the residue is negative since the contour runs clockwise). Again by Proposition [15][ii],
\[
\left| \frac{1}{2\pi} \int_{\mathbb{R}-i\beta} \psi(w) dw \right| \leq C e^{-\beta x} \int_{\mathbb{R}} e^{-c|t|^\rho} dt \leq C e^{-\beta x}.
\]

It remains to evaluate the residue. We have \(\text{Res}(\psi, -i) = e^{-x} \text{Res}(\frac{1}{\Psi_L(-i)}, -i)\) and we write
\[
\frac{1}{\Psi_L(-iw)} = \frac{e^{-iw}}{1 - iw} \prod_{m=1}^{\infty} \frac{e^{-iw/b_m,L}}{1 - \frac{iw}{b_m,L}}
\]
so that
\[
-i \text{Res}(\psi, -ib_{0,L}) = -i \text{Res}(\psi, -i) = -ie^{-x} \prod_{m=1}^{\infty} \frac{e^{-1/b_m,L}}{1 - \frac{1}{b_m,L}} \lim_{w \to -i} e^{-iw} \frac{w + i}{1 - iw} = \frac{1}{e} \prod_{m=1}^{\infty} \frac{e^{-a_m,L}}{1 - a_m,L} e^{-x} = \kappa_L e^{-x}.
\]
We conclude that, as \(x \to \infty\),
\[
f_{X_L}(x) \sim \kappa_L e^{-x}
\]
(the error term is actually exponentially small) and in particular, as \(y \to \infty\),
\[
\mathbb{P} [X_L > y] \sim \kappa_L e^{-y}.
\]
This completes the proof of Theorem 2 (iii).

APPENDIX A. WIENER CHAOS EXPANSION FOR NUMBER OF ZEROES

In this appendix we prove versions of Proposition 4 and Lemma 5. We return to the notation introduced in Section 4; we define a GAF of the form
\[ f(z) = \sum_{m \geq 0} b_m \zeta_m z^m \]
with \( b_m \geq 0 \) and covariance function
\[ G(z) = \sum_{m \geq 0} b_m^2 z^m \]
and write \( \hat{f}(z) = f(z) G(|z|^2)^{1/2} \). We also write \( n(r) \) for the number of zeroes of \( f \) in the disc \( D(0,r) \) for any \( r < R_0 \), where \( R_0 \) denotes the radius of convergence of \( G \) (which is a.s. the radius of convergence of \( f \), and we allow \( R_0 = \infty \)).

We next recall the notion of the Wiener chaos. We define the \( q \)-th component of the Wiener chaos to be
\[ \mathcal{W}^q = L^2 - \text{span}\{ \zeta_1^{\alpha_1} \zeta_2^{\beta_1} \cdots \zeta_k^{\alpha_k} \zeta_k^{\beta_k} : \alpha_1 + \beta_1 + \cdots + \alpha_k + \beta_k = q \} \]
It follows from [18, Theorem 3.12] that the \( \mathcal{W}^q \) define orthogonal subspaces of \( L^2 \), and from [18, Theorem 2.6] we get that \( L^2 = \bigoplus_{q=0}^{\infty} \mathcal{W}^q \). Given any random variable with finite second moment, we may therefore expand it in terms of its projection to each \( \mathcal{W}^q \), and this is known as the Wiener chaos expansion. We now state this expansion for \( n(r) \).

Proposition 17. Define
\[ n(r; \alpha) = (-1)^{\alpha+1} \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{\partial}{\partial z} |\hat{f}(z)|^{2\alpha} \, dz. \] (10)
Then \( n(r; \alpha) \) belongs to the \( 2\alpha \)-th component of the Wiener chaos corresponding to \( f \) and
\[ n(r) - \mathbb{E}[n(r)] = \sum_{\alpha=1}^{\infty} n(r; \alpha) \]
where the sum converges in \( L^2 \).

We will also need the following lemma, which generalises Lemma 5 above.

Lemma 18. If \( \alpha \geq 1 \) then
\[ \mathbb{E}\left[ (n(r; \alpha))^2 \right] = \left( \frac{1}{2\pi i} \right)^2 \frac{1}{\alpha^2} \int_{\partial D(0,r)^2} \frac{\partial^2}{\partial z \partial w} G(|z|^2)^{\alpha} G(|w|^2)^{\alpha} \, dz \, dw \]
and
\[ \mathbb{E}\left[ (n_L(r; \alpha))^4 \right] = \left( \frac{1}{2\pi i} \right)^4 \frac{1}{\alpha^4 (\alpha!)^4} \int_{\partial D(0,r)^4} \frac{\partial^4}{\partial z_1 \cdots \partial z_4} \mathbb{E}\left[ \prod_{j=1}^{4} |\hat{f}(z_j)|^{2\alpha} \right] \, \prod_{j=1}^{4} dz_j. \]
To prove the above results we require the following lemmas, which allow us to justify interchanging the order of some operations.
Lemma 19 (cf. [7] Lemma 7). Given a polynomial \( P, R < R_0 \) and \( 1 \leq p < +\infty \) we have
\[
\mathbb{E} \left[ \left| \nabla P \left( \left| \hat{f}(z) \right|^2 \right) \right|^p \right] \leq C(p, R, P, G)
\]
for \( z \in D(0, R) \).

Lemma 20 (cf. [7] Lemma 8). Let \( \psi_j : \mathbb{R}_+ \rightarrow \mathbb{R} \) be differentiable functions for \( 1 \leq j \leq N \) and let \( \Psi_j = \psi_j \circ \left| \hat{f} \right|^2 \). Suppose that
\[
\int_{\partial D(0, R)^N} \mathbb{E} \left[ \left| \prod_{j=1}^N \nabla \Psi_j(z_j) \right| \prod_{j=1}^N |dz_j| \right] < +\infty
\]
and that, for almost every tuple \( (z_1, \ldots, z_N) \) with respect to the measure \( \prod_{j=1}^N |dz_j| \), there exists \( \varepsilon_0 > 0 \) and \( 1 < p < 2 \) such that
\[
\sup_{\psi_j : w_j \in D(z_j, \varepsilon_0)} \mathbb{E} \left[ \left| \prod_{j=1}^N \nabla \Psi_j(w_j) \right|^p \right] < +\infty.
\]
Then
\[
\mathbb{E} \left[ \int_{\partial D(0, R)^N} \frac{\partial^N}{\partial z_1 \cdots \partial z_N} \prod_{j=1}^N \Psi_j(z_j) \prod_{j=1}^N |dz_j| \right] = \int_{\partial D(0, R)^N} \frac{\partial^N}{\partial z_1 \cdots \partial z_N} \mathbb{E} \left[ \prod_{j=1}^N \Psi_j(z_j) \right] \prod_{j=1}^N |dz_j|.
\]

Lemma 21 (cf. [7] Lemmas 9, 10 and 11). (a) Suppose that \( \psi_j \) are polynomials for \( 1 \leq j \leq N \). Then (11) and (12) hold.
(b) Suppose that \( N = 2, \psi_1 = \log \) and \( \psi_2 \) is a polynomial. Then (11) and (12) hold.
(c) Suppose that \( N = 2 \) and \( \psi_1 = \psi_2 = \log \). Then (11) holds and for every pair \( (z_1, z_2) \) with \( z_1 \neq z_2 \), (12) holds.

The proof of Lemma 19 is postponed until later. The proofs of Lemmas 20 and 21 are essentially identical to the proofs given in [7], and are accordingly omitted. Combining Lemmas 20 and 21 with [18] Theorem 3.9 immediately yields Lemma 18. We now proceed to prove Proposition 17.

Proof of Proposition 17. We first show that the random variable \( n(r; \alpha) \) defined in (10) belongs to \( \mathcal{W}^{2\alpha} \). Notice that Lemma 19 implies that
\[
\mathbb{E} \left[ \left| \nabla : \hat{f}(z) \right|^{2\alpha} \right] \leq C(R, \alpha, G)
\]
for \( z \in D(0, R) \). Standard arguments show that \( : \hat{f}(z) :^{2\alpha} \) is in \( \mathcal{W}^{2\alpha} \), and therefore so is \( :\hat{f}(z+h)^{2\alpha} : \) for any \( h \in \mathbb{C} \) such that \( |z+h| < R_0 \). Taking real \( h \rightarrow 0 \) and applying the mean value theorem and (13), we use dominated convergence to see that \( \frac{\partial}{\partial z} : \hat{f}(z) :^{2\alpha} \) is in \( \mathcal{W}^{2\alpha} \) for any fixed \( z \in D(0, R_0) \). Arguing similarly for imaginary \( h \) we see that \( \frac{\partial}{\partial \bar{z}} : \hat{f}(z) :^{2\alpha} \) is in \( \mathcal{W}^{2\alpha} \). Now write \( g_{\alpha}(z) = \frac{\partial}{\partial z} : \hat{f}(z) :^{2\alpha} \) and consider the Riemann sum \( \frac{1}{N} \sum_{j=1}^N g \left( re^{2\pi i \frac{j}{N}} \right) \) which is in \( \mathcal{W}^{2\alpha} \). Then
\[
\mathbb{E} \left[ \left| \frac{1}{N} \sum_{j=1}^N g \left( re^{2\pi i \frac{j}{N}} \right) \right|^2 \right] \leq \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[ \left| g \left( re^{2\pi i \frac{j}{N}} \right) \right|^2 \right] \leq C(R, \alpha, G)
\]
and once more applying dominated convergence we see that the Riemann sums converge to the integral
\[
\int_0^\pi g \left( re^{i\theta} \right) d\theta = \frac{1}{ir} \int_{\partial D(0, r)} \frac{\partial}{\partial z} : \hat{f}(z) :^{2\alpha} : dz
\]
in $L^2$ and so $n(r; \alpha)$ belongs to $W^{2\alpha}$. as claimed.

It remains to prove that $\sum_{\alpha=1}^{M} n(r; \alpha) \rightarrow n(r) - \mathbb{E}[n(r)]$ in $L^2$ as $M \rightarrow \infty$. The basic strategy is to implement the scheme outlined in Section 2, and to use the lemmas above to justify the steps. We will briefly outline the argument, which closely follows [7, Section 3.2].

We define

$$\log_M \|\zeta\|^2 = -\gamma_e + \sum_{\alpha=1}^{M} \frac{(-1)^{\alpha+1}}{\alpha(\alpha!)} :|\zeta|^{2\alpha}:$$

to be the truncation of the series in (4). Notice that

$$n(r) - \mathbb{E}[n(r)] - \sum_{\alpha=1}^{M} n(r; \alpha) = \frac{1}{2\pi i} \int_{\partial D(0, r)} \frac{\partial}{\partial z} \left( \log |\hat{f}_L(z)|^2 - \log_M |\hat{f}_L(z)|^2 \right) dz.$$

We square this expression and take its expectation. Appealing to Lemmas 20 and 21 we can exchange the order of operations to get

$$\mathcal{E}_M := \mathbb{E}\left[ \left( n(r) - \mathbb{E}[n(r)] - \sum_{\alpha=1}^{M} n(r; 2\alpha) \right)^2 \right]$$

$$= -\frac{1}{4\pi^2} \iint_{\partial D(0, r)^2} \frac{\partial^2}{\partial z \partial \bar{w}} \mathbb{E} \left[ \left( \log |\hat{f}_L(z)|^2 - \log_M |\hat{f}_L(z)|^2 \right) \left( \log |\hat{f}_L(w)|^2 - \log_M |\hat{f}_L(w)|^2 \right) \right] dz \, dw$$

$$= -\frac{1}{4\pi^2} \iint_{\partial D(0, r)^2} \frac{\partial^2}{\partial z \partial \bar{w}} \sum_{\alpha, \beta > M} \frac{(-1)^{\alpha+\beta}}{\alpha! \beta!} \mathbb{E} \left[ |\hat{f}_L(z)|^{2\alpha} : |\hat{f}_L(w)|^{2\beta} : \right] dz \, dw$$

$$= -\frac{1}{4\pi^2} \sum_{\alpha > M} \frac{1}{\alpha^2} \iint_{\partial D(0, r)^2} \frac{\partial^2}{\partial z \partial \bar{w}} \left| \frac{G(z \bar{w})}{\sqrt{G(|z|^2)G(|w|^2)}} \right|^{2\alpha} dz \, dw,$$

where the final equality follows from [18, Theorem 3.9]. Simplifying the integrand à la [20, Claim A.2] we get

$$\mathcal{E}_M = \mathcal{E}_M(r) = \frac{1}{2\pi} \sum_{\alpha > M} \int_{-\pi}^{\pi} \left| \frac{G(r^2 e^{i\theta})}{G(r^2)} \right|^{2\alpha} \left| \frac{G'(r^2 e^{i\theta}) r^2 e^{i\theta} - G'(r^2) r^2}{G(r^2)} \right|^2 d\theta$$

(14)

which is the tail of a convergent sum.

\[ \square \]

Remark. Putting $M = 0$ in (14) is a recasting of (6).

**Proof of Lemma 19** We mimic the proof of [7, Lemma 7]. It suffices to show that

$$\mathbb{E} \left[ \left| \nabla \left( |\hat{f}(z)|^2 \right) \right|^p \right] \leq C(p, R, G)$$

for $z \in D(0, R)$. Now

$$\left| \nabla |\hat{f}(z)|^2 \right| \lesssim |f'(z) f(z)| + |z| G' \left( |z|^2 \right) |\hat{f}(z)|^2$$

and since $\mathbb{E} \left[ |\hat{f}(z)|^{2p} \right]$ is independent of $z$ and $G$ is analytic and zero-free on $[0, 1)$ we see that

$$\mathbb{E} \left[ \left| z G' \left( |z|^2 \right) |\hat{f}(z)|^2 \right|^p \right] \leq C(R, p, G).$$
Similarly
\[
\mathbb{E}
\left[
\frac{|f'(z)f(z)|}{G\left(|z|^2\right)^p}\right]
\leq \mathbb{E}
\left[
\frac{|f'(z)|^{2p}}{G\left(|z|^2\right)^p}\right]^{\frac{1}{2}} \mathbb{E}
\left[
\frac{|f(z)|^{2p}}{G\left(|z|^2\right)^p}\right]^{\frac{1}{2}} \leq C(R,p,G) \mathbb{E}
\left[
\frac{|f'(z)|^{2p}}{G\left(|z|^2\right)^p}\right]^{\frac{1}{2}}.
\]

Now \(f'(z)\) is a Gaussian random variable with variance \(G'\left(|z|^2\right) + |z|^2 G''\left(|z|^2\right)\) and so
\[
\mathbb{E}
\left[
\frac{|f'(z)|^{2p}}{G\left(|z|^2\right)^p}\right] = \Gamma(1 + \frac{p}{2}) \left(\frac{G'\left(|z|^2\right) + |z|^2 G''\left(|z|^2\right)}{G\left(|z|^2\right)}\right)^p
\]
which is bounded as before. □

**Appendix B. Standard lemma to deduce CLT**

In this appendix we prove that the scheme outlined in Section 3 indeed implies a CLT.

**Lemma 22.** Let \(X_n\) and \(X_n(M)\), for \(n > 0\) and \(M > 0\), be real-valued random variables with mean 0 and variance 1. Suppose that the following holds:

- For each fixed \(M\),
  \[X_n(M) \xrightarrow{d} \mathcal{N}_\mathbb{R}(0,1)\] as \(n \to \infty\).

- We have
  \[\lim_{M \to \infty} \mathbb{E}\left[(X_n - X_n(M))^2\right] = 0,\]
  uniformly in \(n\).

Then
\[X_n \xrightarrow{d} \mathcal{N}_\mathbb{R}(0,1)\] as \(n \to \infty\).

We will give a short proof using characteristic functions, though one can also give an elementary direct proof.

**Proof.** We write \(\varphi(t) = e^{-t^2/2}\),
\[
\phi_n(t) = \mathbb{E}\left[e^{itX_n}\right] \quad \text{and} \quad \phi_{n,M}(t) = \mathbb{E}\left[e^{itX_n(M)}\right].
\]

We estimate, for fixed \(t\),
\[
|\phi_n(t) - \varphi(t)| \leq |\mathbb{E}\left[e^{it(X_n - X_n(M))}\right] - 1| + |\phi_{n,M}(t) - \varphi(t)|
\leq |t| |\mathbb{E}|X_n - X_n(M)| + |\phi_{n,M}(t) - \varphi(t)|.
\]

Applying the hypothesis we see that \(\phi_n\) converges pointwise to \(\varphi\), which implies the result. □

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