ILL-POSEDNESS FOR SEMILINEAR WAVE EQUATIONS WITH VERY LOW REGULARITY

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Abstract. In this paper, we study the ill-posedness of the Cauchy problem for semilinear wave equation with very low regularity, where the nonlinear term depends on \( u \) and \( \partial_t u \). We prove a ill-posedness result for the “defocusing” case, and give an alternative proof for the supercritical “focusing” case, which improves our previous result in Chin. Ann. Math. Ser. B 26(3), 361–378, 2005.

1. Introduction

This paper is mainly concerned with a low regular ill-posedness(ILP) of the Cauchy problem for the “defocusing” semilinear wave equation with nonlinear term depending on \( u \) and \( \partial_t u \). For the “focusing” case, it has been studied in [4], but here we give an alternative proof of the supercritical ill-posed result in the final section, which can improve the original result slightly.

Recently, the study of ill posed issues for nonlinear evolution equations is very active. For wave equations, one can refer to [1]-[12]. In [1], [2], [6], [7], [9] and [12], these authors give the ill-posed results for both focusing and defocusing type equations with nonlinear term depending only on \( u \) itself.

For nonlinear term depends also on the derivatives of \( u \), only a few results can be found but for the wave map type equations, which one can refer to [5] and references therein. In [8] and [4], the authors deal with the “focusing” type semilinear equations, where “focusing” means that the corresponding time-ODE equation admits finite time blow up solution. In [10] and [11], the author dealt with some particular “focusing” semilinear and quasilinear equations.

Let \( \Box := \partial^2_t - \Delta_x \), \( x \in \mathbb{R}^n \), be the usual d’Alembertian, we consider the Cauchy problem for the following “defocusing” equations \( (k + l > 1, \ k \in \mathbb{N} \ \text{and} \ \ 1 \leq l \in \mathbb{R}) \)

(1.1) \[ \Box u = -|u|^k |\partial_t u|^{l-1} \partial_t u, \ \text{for even} \ k \]

and

(1.2) \[ \Box u = -|u|^{k-1} |\partial_t u|^l, \ \text{for odd} \ k. \]

If we denote the right hand side of (1.1) or (1.2) by \( F(u, \partial_t u) \), then the corresponding ordinary differential equation(ODE) in \( t \) is

(1.3) \[ \partial^2_t u = F(u, \partial_t u). \]

Heuristically, for “focusing” equations of these type, there are two obstructions for well-posedness, one is scaling, which yields the scaling index \( s_c(k, l) = \frac{n}{2} + \frac{l-2}{k+l-1} \).

2000 Mathematics Subject Classification. 35L05, 35R25.
Key words and phrases. semilinear wave equation, ill-posedness, low regularity.
The authors were partially supported by NSF of China 10571158.
another is concentration along light cone, which yields the concentrative index $\dot{s}(k,l) = \frac{2}{n+4} + \frac{4}{n+4} - \frac{2}{n+1}$. For “defocusing” equations, the mechanism for ill posedness are currently not very clear.

In contrast, for both “focusing” and “defocusing” equations, if $k,l \in \mathbb{N}$ and $k + l > 1$, we have local well-posedness (LWP) in $C_t H^s$ for

$$s > \begin{cases} \max(\dot{s}(0,l), s_c(0,l)), & l \geq 2 \\ \max(\frac{2}{n+4}, s_c(k,1)), & l = 1 \text{ and } n \geq 3 \end{cases}$$

In particular, if $l - 1 \geq \frac{4}{n+4}$ and $s > s_c(0,l)$, or $k \geq 2$, $n \geq 3$ and $s > s_c(k,1)$, we have LWP in $C_t H^s$ (see [4]). Moreover, for $k = 0$ and $l - 1 > \max(2, \frac{4}{n+4})$, we have global well-posedness in $C_t H^s$ with small data for $s \geq s_c(0,l)$ (see [5]).

In this paper we will mainly prove the following ill posedness result:

**Theorem 1.** If $k + l$ is odd or $l \geq \max(k + 1, \left\lfloor \frac{2}{n+4} \right\rfloor)$, the problem (1.1) or (1.2) is ill posed in $H^s$ for $s \in (-\infty, \frac{2}{n+4}) \cap (-\infty, \frac{2}{n+4} + \frac{2}{n+4})$, in the sense that the solution map is not continuous from $H^s \times H^{s-1}$ to $C_t H^s \times C_t^1 H^{s-1}$. Precisely, we get that for any $\epsilon$, there exists a solution $u \in C_t H^s \times C_t^1 H^{s-1}$ and some $t \in (0, \epsilon)$ such that

$$u(0) = 0, \quad \partial_t u(0) = \varphi \in C_0^\infty, \quad \|\varphi\|_{H^s} < \epsilon \quad \text{and} \quad \|\partial_t u(t)\|_{H^{s-1}} > \frac{1}{\epsilon}.$$  

Inspired by the paper [2] of Christ, Colliander and Tao, we first use small dispersion analysis and scaling argument to get a well-controlled two-parameter solution. By choosing the parameters properly, one can get the desired estimate. Thus we have the following result,

**Proposition 1.** The problem (1.1) or (1.2) is ill posed in $H^s$ for $s \in (-\infty, \frac{2}{n+4}) \cap (-\infty, \frac{2}{n+4} + \frac{2}{n+4})$ if $k + l$ is odd or $l \geq k + 1$.

Then, by the argument of finite speed of propagation, Theorem 1 can be reduced to the $n = 1$ case of Proposition 1.

**Remark 1.** Our original purpose is to get the ill posedness result for any $s < s_c$. However, because of the lack of the property of blow up at infinity for the ODE solution, we couldn’t get ill posed result by exploring the continuous dependence. But we believe that there must be some other mechanism to develop ill posed result.

As a complementarity, for the supercritical ill-posed result of “focusing” equation in [4], we give an alternative proof, which can improve the original result slightly. Combined with the result of Theorem 1.2 in [4], we’ll prove the following result in section 3.

**Theorem 2.** Let $n \geq 1$, $k + l > 1$, $k,l \geq 0$ and $k,l \in \mathbb{R}$. Consider the model equation

$$(1.5) \quad \Box u = |u|^{l+1}|\partial_t u|^{-1}\partial_t u \quad \text{if} \quad l \neq 0$$

$$(1.6) \quad \Box u = |u|^{k-1}u \quad \text{if} \quad l = 0$$

they are s-ILP in $\dot{H}^s$ for

$$s \in \begin{cases} (1 - n/2, s_c) & l \geq 2 \\ (-n/2, s_c) & l < 2 \end{cases}$$

And, if $s_c > 1$, then it is s-ILP in $H^s$ for any $s < s_c$. Moreover, it’s also w-ILP in $H^s$ for $s < s_c$ if $s_c \geq 1$.  

Here s-ILP means that there is a sequence of data $f_j, g_j \in C_0^\infty(B_{R_j})$, for which the lifespan of the solutions $u_j$ tends to zero as the data’s norm and $R_j$ goes to 0, under the condition that the solutions obey finite speed of propagation; and w-ILP to express that the lifespan goes to zero and the data’s norm stay bounded. In fact, in the case of s-ILP, one also can find that the solution does not depend continuously on the data.

The main differences for Theorem 2 from Theorem 1.2 in [4] are in two cases, the first is that there isn’t the technical restriction $k = 0$ for $l = 2$, the second is that we have ILP in $H^s$ even for $s < -n/2$.

In this paper, we will use the following conventions. We use the notation $(f, g)$ to stand for the specification of the data $u(0) = f, \partial_t u(0) = g$. Let $\alpha = \frac{l-2}{k+l-1}$, $s_c = \frac{n}{2} + \alpha$ and $B_r = \{|x| < r\}$. Moreover, we use $\hat{f} = F(f)$ to denote the Fourier transform of the function $f$.

2. Proof of Proposition 1

In this section, we use the so-called “small dispersion analysis” in [2] to prove the ill posed result Proposition 1 based on the knowledge of the properties of the solutions of ODE (1.3).

2.1. ODE Solution. In this subsection, we study the asymptotic properties of the solutions of ODE (1.3), which is the basis of the small dispersion analysis.

Note that there is a “conserved quantity” for ODE (1.3) (for $u, \partial_t u \geq 0$),

\begin{equation}
I = \begin{cases}
\frac{|\partial_t u|^{2-l}}{2-l} + \frac{|u|^{k+1}}{k+1}, & l \neq 2, \\
\log(|\partial_t u|) + \frac{|u|^{k+1}}{k+1}, & l = 2, \partial_t u > 0.
\end{cases}
\end{equation}

Combine it with the equation, it is easy to get the following global asymptotic properties of the solution $u_1(t)$ of (1.3) with data (0,1).

**Proposition 2.** Let $l \geq 1, k \geq 0, k + l > 1$ and $M$ be the prescribed large numbers. The solution $u_1(t)$ of (1.3) with data $u(0) = 0, \partial_t u(0) = 1$ has the following properties:

1. $u_1(t)$ exists globally;
2. $\lim_{t \to \infty} u_1(t) = \begin{cases}
\infty, & l \geq 2, \\
\frac{k+1}{2-l} \frac{1}{\epsilon^{k+l-1}}, & l < 2;
\end{cases}$
3. $u_1(t) \in C^{m_0+2}$ with $m_0 = \begin{cases}
M \min(|k-1|, |l-1|), & k, l \in \mathbb{Z}, k + l \text{ odd}; \\
\min(|k-1|, |l-1|), & \text{else};
\end{cases}$
4. $\lim_{t \to \infty} \partial_t^m u_1(t) = 0, \forall 0 < m \leq m_0 + 2.$

**Proof.** Since we have the “conserved quantity” (2.1) for the solution, there exists a $T > 0$ such that

$\partial_t u_1 \leq 0, u_1 \not\nearrow a := \begin{cases}
\infty, & l \geq 2, \\
\frac{k+1}{2-l} \frac{1}{\epsilon^{k+l-1}}, & l < 2.
\end{cases}$
as \( t \to T \). Moreover, we claim that for \( l \geq 1 \), we have \( T = \infty \). In fact, by (2.2), we get that for \( t \in [0, T) \),

\[
\partial_t u_1 = f(u_1) := \begin{cases} 
\left(1 + \frac{l-2}{k+1}u_1^{k+1}\right)^{\frac{1}{k+1}}, & l \neq 2, \\
\exp\left(-\frac{a}{u_1^{k+1}}\right), & l = 2.
\end{cases}
\]

Then, note that \( f(u_1) \leq 1 \) for \( l \geq 2 \), we have \( \partial_t u_1 \leq 1 \). Thus \( u_1(t) \leq t \), for any \( t \), implies \( T = \infty \). For \( 1 \leq l < 2 \), note that \( 0 \leq a^{k+1} - u_1^{k+1} \leq (k+1)\alpha(a - u_1) \) and \( \frac{1}{(l-2)^{k+1}} \leq -1 \), we have

\[
T = \int_0^T dt = c \int_0^a (a^{k+1} - u_1^{k+1}) \frac{1}{1-l} du_1 \geq c \int_0^a (a - u_1) \frac{1}{1-l} du_1 = \infty.
\]

From the fact that \( F(a, b) \in C^{m_0,1}([0, u_1(T)] \times (0, 1]) \) for any \( T \in (0, \infty) \), we have \( u_1(t) \in C^{m_0+2}([0, \infty)) \).

At last, to prove the property of \( u_1^{(m)} := \partial^m_t u_1 \), we divide it into three cases \( l < 2, l > 2 \) and \( l = 2 \). For \( l < 2 \), it is obvious since \( u_1^{(m)} = P((u_1^{(j)}))_{j<m}) \), where \( P \) is polynomial and \( P|_{(u_1^{(j)})} = 0 \). For \( l > 2 \) and \( t \) large,

\[
\partial_t u_1 = \left(1 + \frac{l-2}{k+1}u_1^{k+1}\right)^{\frac{1}{k+1}} \approx u_1^{\frac{1}{k+1}} \Rightarrow u_1 \sim t^{\frac{1}{l-2}},
\]

thus we have, for any integer \( m > 0 \), \( u_1^{(m)} \approx t^{\frac{m-1}{l-2}} \to 0 \) as \( t \to \infty \). For \( l = 2 \) and the large \( t \), we have

\[
u_1(t) \geq c(\ln t)^{\frac{1}{l-2}}
\]

and \( u_1^{(m)}(t) \leq t^{-\delta} \) for some \( \delta > 0 \). This completes the proof. \( \blacksquare \)

For any \( \phi > 0 \), we can get the solution \( \tilde{u}_\phi(t) \) of (1.5) with data \( (0, \phi) \) by rescaling

(2.2) \( \tilde{u}_\phi(t) = u_1(t\phi^{-\frac{1}{k+1}})\phi^{-\frac{1}{k+1}}. \)

For the continuity in parameter, we choose \( N \) large and let \( \phi = \psi^{N(k+1)} \),

(2.3) \( u_\psi(t) = \tilde{u}_\phi(t) = u_1(t\psi^{N(k+1)})\psi^{-N(l-2)}. \)

Note that \( \lim_{\psi \to 0^+} u_\psi(t) = 0 \), we use the convention that \( u_0 = 0 \). Then we claim that

(2.4) \( u_\psi(t) \in C^{m_0+2}_{t,\psi}([0, \infty) \times [0, \infty)) \) if \( N(k+1) > m_0 + 2 \),

and consequently \( u_\psi(x) \in C^{m_0+2}_{t,x} \) for \( \psi(x) \in C^0_\infty(\mathbb{R}^n) \) and \( \psi \geq 0 \).

Now we give the proof of the claim. For the simplicity of notation, we denote here that \( f = u_1, b = N(k + l - 1), a = N(l - 2) \) and \( u(t, \psi) = f(t\psi^b)\psi^{-a} \). We extend the definition of \( u(t, \psi) \) from \( \psi \geq 0 \) to \( \psi \in \mathbb{R} \) by using the zero extension at first. Then we check that such extension lies in \( C^{m_0+2}_{t,\psi} \) and show that the only case to be examined is the case \( \psi = 0 \). We finish the proof by computing the right limit at \( \psi = 0 \). Note that if \( t > 0 \) and \( b - d > 0 \),

\[
\lim_{\psi \to 0^+} f(t\psi^b)\psi^{-d} = 0.
\]

Since we have

\[
\partial_\psi^a u(t, \psi) = C f(t\psi^b)\psi^{-a-j} + \sum_{1 \leq h \leq j} C_{h, t^h} (\partial^h f)(t\psi^b)\psi^{bh-a-j}
\]
then if \( b - a - j > 0 \), we have
\[
(2.5) \quad \lim_{\psi \to 0^+} \partial_j^\psi u(t, \psi) = 0 .
\]
Thus we have (2.5) for any \( 0 \leq j \leq m_0 + 2 \) if \( N(k + 1) = b - a > m_0 + 2 \). This complete the proof of the claim.

2.2. Small Dispersion Analysis. Based on Proposition 2 and (2.4), we make the small dispersion analysis in this subsection.

Consider the problem
\[
(2.6) \quad \left\{ \begin{array}{ll}
\Box_{\gamma} u := (\partial_t^2 - \gamma^2 \Delta) u = F(u, \partial_t u) \\
u(0) = 0, \quad \partial_t u(0) = \phi(x) = \psi(x)^N(k+1)
\end{array} \right.
\]
By the usual energy argument, we can compare the solution \( \phi^{(\gamma)} \) with the corresponding solution \( \phi^{(0)} = u_\psi \) of (1.3). The result is as following

**Proposition 3.** Let \( n \geq 1, \ k \geq 0, \ l \geq 1, \ k + l > 1, \ m \in \mathbb{N}, \ \left( \frac{2l}{2k+3} \right) \leq m \leq m_0, \ N \) such that \( N(k + 1) > m + 2, \) \( \phi = \psi^N(k+1) \) with \( \psi \in C^{\infty}_0(\mathbb{R}^n) \) and \( \psi \geq 0 \). Then there exist \( C \gg 1 \gg c > 0 \) such that \( \forall \gamma \in (0, c] \), there exist a solution \( u = \phi^{(\gamma)} \in C([0, T], H^{m+1}) \) \( \cap C^1([0, T], H^m) \) of (2.6) with \( T = c \log(\gamma)^c \). Moreover, for any \( t \in [0, T] \),
\[
(2.7) \quad \|\phi^{(\gamma)} - \phi^{(0)}\|_{H^{m+1}} + \|\partial_t \phi^{(\gamma)} - \partial_t \phi^{(0)}\|_{H^m} \leq C\gamma^{1/2}.
\]

**Proof.** Let \( w = u - \phi^{(0)} \), then we have
\[
(2.8) \quad \Box_{\gamma} w = F(u, \partial_t u) - F(\phi^{(0)}, \partial_t \phi^{(0)}) + \gamma^2 \Delta \phi^{(0)} = G(w)
\]
for \( w \) with data \((0, 0)\).

The energy method shows that this problem is local well-posed in \( H^{m+1} \times H^m \), the solution of (2.8) exists as long as the \( H^{m+1} \times H^m \) norm of it stays bounded.

We define the \( \gamma \)-energy of \( w \) by
\[
E_\gamma(w(t)) := \int \frac{1}{2} |w_t(t, y)|^2 + \frac{\gamma^2}{2} |\nabla_y w(t, y)|^2 \, dy ,
\]
and
\[
E_{\gamma, m}(w(t)) := \sum_{j=0}^m E_\gamma(\partial_j^w w(t)) .
\]
Then the standard energy inequality gives that
\[
(2.9) \quad |\partial_t E_{\gamma, m}^{1/2}(w(t))| \leq C\|G(t)\|_{H^m} .
\]

Let \( e(t) \) be the non-decreasing function \( e(t) := \sup_{0 \leq s \leq t} E_{\gamma, m}^{1/2}(w(s)) \), then
\[
(2.10) \quad \|w(t)\|_{H^m} \leq \int_0^t \|w_t(s)\|_{H^m} \, ds \leq C \int_0^t E_{\gamma, m}^{1/2}(w(s)) \, ds \leq Cte(t),
\]
Also, by the smoothness of \( \phi^{(0)} \) and \( F \), we can easily obtain the bounds
\[
\|\gamma^2 \Delta \phi^{(0)}\|_{H^m} \leq C\gamma^{2}(1 + |t|)
\]
and
\[
\|\phi^{(0)}\|_{H^{m+1}} + \|\partial_t \phi^{(0)}\|_{H^m} + \|\phi^{(0)}\|_{C^m} + \|\partial_t \phi^{(0)}\|_{C^m} \leq C(1 + |t|) .
\]

Since \( H^m \) is an algebra, then
\[
\|F(u, \partial_t u)(t) - F(\phi^{(0)}, \partial_t \phi^{(0)})(t)\|_{H^m} \leq C(1 + |t|)^C(e(t) + e(t)^C) .
\]
The above estimates yield that
\[ \| G \|_{H^m} \leq C(1 + |t|)^C(\gamma^2 + e(s) + e(s)^C), \]
which by (2.9) gives the differential inequality
\[ \partial_t e(t) \leq C(1 + |t|)^C(\gamma^2 + e(t) + e(t)^C). \]
Since \( e(0) = 0 \), we can assume a priori that \( e(t) \leq \gamma \), then
\[ \partial_t e(t) \leq C(1 + |t|)^C(\gamma^2 + e(t)), \]
and hence
\[ e(t) \leq \gamma^2(\exp(C(1 + |t|)^C) - 1). \]
Thus if \( |t| \leq c \log \gamma \) for suitably chosen \( c \) and \( \gamma \) to be sufficiently small, we obtain
\[ e(t) \leq C\gamma^{3/2} \]
and furthermore we can recover the a priori assumption, which can then be removed by the usual continuity argument. The claim then follows from (2.10) if \( \gamma \) is sufficiently small. \( \blacksquare \)

2.3. Estimate for Solution. By Proposition 3 and rescaling \( (\lambda > 0) \), we get two-parameter solutions for the problem (2.6) with \( \gamma = 1, \)
\[ u^{(\gamma, \lambda)} := \lambda^\alpha \phi(\gamma)(\lambda^{-1}t, \lambda^{-1}\gamma x). \]
In particular, we have the initial data
\[ \left( 0, \lambda^{-\frac{k+1}{s+1}} \phi(\lambda^{-1}\gamma x) \right). \]

Proposition 4. Let \( 0 < \lambda \leq \gamma \ll 1, \)
\[ \| \partial_t u^{(\gamma, \lambda)}(0) \|_{H^{s-1}} = C \lambda^{s_c - s} \gamma^{-\frac{(s_c - s)}{2}} := C \epsilon \]
where \( \phi \in C^\infty_0(\mathbb{R}^n) \) such that \( \hat{\phi}(\xi) = O(|\xi|^k) \) as \( \xi \to 0 \) with \( k + s > 1 - n/2 \). Note that \( \lambda = c\gamma^\sigma \) with \( \sigma > 1 \) for fixed \( \epsilon \) and \( s < s_c. \)

Proof. Note that
\[ [\partial_t u^{(\gamma, \lambda)}(0)](\xi) = \lambda^{-\frac{k+1}{s+1}}(\lambda/\gamma)^n \hat{\phi}(\lambda/\gamma), \]
we have
\[ \| \partial_t u^{(\gamma, \lambda)}(0) \|_{H^{s-1}}^2 = \lambda^{-\frac{2(k+1)}{s+1}}(\lambda/\gamma)^{2n} \int |\hat{\phi}(\lambda^{-1}\xi)|^2 (1 + |\xi|^2)^{s-1} d\xi \]
\[ = \lambda^{-\frac{2(k+1)}{s+1}}(\lambda/\gamma)^{n} \int |\hat{\phi}(\eta)|^2 (1 + |\gamma^{-1}\eta|^2)^{s-1} d\eta. \]
\[ \sim \lambda^{-\frac{2(k+1)}{s+1}}(\lambda/\gamma)^{n-2(s-1)} \int_{|\eta| \geq \lambda^{-1}} |\hat{\phi}(\eta)|^2 |\eta|^{2(s-1)} d\eta \]
\[ + \lambda^{-\frac{2(k+1)}{s+1}}(\lambda/\gamma)^{n} \int_{|\eta| \leq \lambda^{-1}} |\hat{\phi}(\eta)|^2 d\eta \]
\[ = \lambda^{-\frac{2(k+1)}{s+1}}(\lambda/\gamma)^{n-2(s-1)} \left[ \int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2 |\eta|^{2(s-1)} d\eta \right] \]
\[ - \int_{|\eta| \leq \lambda^{-1}} |\hat{\phi}(\eta)|^2 ((\lambda/\gamma)^{2(s-1)} - |\eta|^{2(s-1)}) d\eta \]
Thus for any $s - 1 > -n/2$,

\begin{equation}
(2.13) \quad \|\partial_t u^{(\gamma, \lambda)}(0)\|_{H^{s-1}} = c\lambda^{-\frac{k+1}{s}}(\lambda/\gamma)^{n/2-(s-1)} \cdot (1 + O((\lambda^{-1})^{s-1}+n/2)),
\end{equation}

where $c \neq 0$ provided that $\phi$ is not identically zero. In particular,

\begin{equation}
(2.14) \quad \|\partial_t u^{(\gamma, \lambda)}(0)\|_{H^{s-1}} = c\lambda^{-\frac{k+1}{s}}(\lambda/\gamma)^{n/2-(s-1)}
\end{equation}

provided that $s - 1 > -n/2$ and $\lambda \ll \gamma$.

For $s - 1 \leq -n/2$, (2.14) still holds, under the supplementary hypothesis that

\begin{equation}
(2.15) \quad \hat{\phi}(\xi) = O(|\xi|^k) \text{ as } \xi \to 0, \text{ for some } k > -(s - 1) - n/2.
\end{equation}

Then, if $\lambda \ll \gamma$, we have $\int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2|\eta|^{2(s-1)} \, d\eta < \infty$ and

\[
\int_{|\eta| \leq \lambda^{-1}} |\hat{\phi}(\eta)|^2 \left( (\lambda/\gamma)^{2(s-1)} - |\eta|^{2(s-1)} \right) \, d\eta \leq C(\lambda^{-1})^{n+2(s-1)+2k} \leq C < \infty.
\]

To complete the proof of Proposition 1, we need to choose the data $\varphi$ appropriately.

At first glance, for $s \leq 1 - n/2$, the condition of the data in Proposition 3 couldn’t be fulfilled since we assume the data to be nonnegative in Proposition 3. However, noting that $-u$ is also a solution of (1.1) or (1.2) whenever $u$ is, and the solution exhibit uniformly finite speed of propagation so long as $|\gamma| \leq 1$, such condition can be easily fulfilled by taking $\varphi$ to be an appropriate linear combination of nonnegative $C^\infty_0$ functions with widely spaced supports as in [2].

Moreover, by the conditions about $k$ and $l$ in Proposition 1, we can choose $\varphi$ such that $\varphi^{(0)}$ has the following property: there is a $t_0 > 0$ such that

\begin{equation}
(2.16) \quad F(\partial_t \varphi^{(0)}(t_0))(0) \neq 0
\end{equation}

even if

\[
F(\partial_t \varphi^{(0)})(0) = 0 .
\]

In fact, we consider, for the case that $k$ is even for example, the quantity $A(t) = \int \partial_t \varphi^{(0)}(t, x) dx$. Then from the equation (1.13), we have

\[
\partial_t A(t) = - \int |\varphi^{(0)}|^k |\partial_t \varphi^{(0)}|^{l-1} \partial_t \varphi^{(0)} \, dx ,
\]

which is vanishing at $t = 0$ for $k > 0$ since $\varphi^{(0)}$ is zero at $t = 0$. Similarly, for any $i \leq k$, $\partial_t^i \varphi^{(0)}$ is also null at $t = 0$. Since $k$ is even, we have

\begin{equation}
(2.17) \quad \partial_t^{k+1} A(t) = k! \int |\partial_t \varphi^{(0)}(0)|^{k+l-1} \partial_t \varphi^{(0)}(0) dx = k! \int |\varphi|^{k+l-1} \varphi dx
\end{equation}

at $t = 0$. Then we may require that $\varphi$ satisfies an additional condition that the right hand side of (2.17) is nonzero. Hence we get (2.16) immediately. Here, if $l$ isn’t odd, we use the condition $l \geq k + 1$ to ensure $\varphi^{(0)} \in C^{k+2}$.

Then, from (2.16), we get for some $c > 0$

\[
|F(\partial_t \varphi^{(0)}(t_0))(\xi)| \geq c \text{ when } |\xi| \leq c .
\]
However, by Proposition 3 and Sobolev embedding,

\[ |\mathcal{F}(\partial_t(\varphi^\gamma - \varphi^0))(\xi)| \lesssim \|\mathcal{F}(\partial_t(\varphi^\gamma - \varphi^0))(\xi)\|_{H^m} \lesssim \sum_{j=0}^m \|x|^j \partial_t(\varphi^\gamma - \varphi^0)\|_{L^2} \]

\[ \lesssim \|\partial_t(\varphi^\gamma - \varphi^0)\|_{L^2} \lesssim C\gamma , \]

where we have used the fact that \(\partial_t(\varphi^\gamma - \varphi^0)\) are supported in a fixed compact set. Thus we get that

\[ |\mathcal{F}(\partial_t(\varphi^\gamma(t_0))(\xi)| \geq c \quad \text{for} \quad |\xi| \leq c \]

(2.18) \[ |\mathcal{F}(\partial_t^{(\gamma,\lambda)}(\lambda_0))(\xi)| \geq c\lambda^{\alpha - 1} \left(\frac{\gamma}{\lambda}\right)^{-n} \quad \text{for} \quad |\xi| \leq \frac{c\gamma}{\lambda}. \]

Note that \(\gamma \gg \lambda\) for \(\gamma\) small and \(\epsilon\) fixed. For \(s < 1 - n/2\) and \(s < s_c\),

(2.19) \[ \|\partial_t^{(\gamma,\lambda)}(\lambda_0)\|_{H^{s-1}} \geq c\lambda^{\alpha - 1} \left(\frac{\gamma}{\lambda}\right)^{-n} = c\left(\frac{\gamma}{\lambda}\right)^{1-n/2-s} = 1/\epsilon \]

if \(\gamma\) is sufficiently small. For \(s = 1 - n/2 < s_c\),

\[ \|\partial_t^{(\gamma,\lambda)}(\lambda_0)\|_{H^{s-2}}^2 \geq c\lambda^{2\alpha - 2} \left(\frac{\gamma}{\lambda}\right)^{-2n} \int_{|\xi| \leq \frac{c\gamma}{2}} (1 + |\xi|)^{-n} d\xi \]

\[ \geq c\lambda^{2\alpha - 2} \left(\frac{\gamma}{\lambda}\right)^{-2n} \log(c\frac{\gamma}{\lambda}) = c^2 \log(c\frac{\gamma}{\lambda}) = e^2 \]

This complete the proof of Proposition 1.

3. Reduction to dimension \(n = 1\)

By the argument of finite speed of propagation, Theorem 1 can be reduced to the \(n = 1\) case of Proposition 1.

This reduction is worked by considering initial data of product form \(\eta(x')\varphi(x_n)\) where \(x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\), \(\varphi\) is the same as that in the proof of Proposition 1 and \(\eta\) is a fixed \(C_0^\infty\) function which equals 1 on a sufficiently large ball in \(\mathbb{R}^{n-1}\). By finite speed of propagation, the corresponding solutions, assuming existence and uniqueness, will likewise have product form for \(x'\) in a fixed smaller ball, so the norm estimate in \(\mathbb{R}^n\) follows from the estimate already established in \(\mathbb{R}^1\).

Precisely, we set \(\eta(x') \in C_0^\infty\) and \(\eta = 1\) on the ball \(B_R(\mathbb{R}^{n-1})\) with radius \(R \gg 1\). Then for some \(\nu \ll 1\), the problem (1.1) or (1.2) with data \((0, \eta(x')\varphi(x_n))\) are local well-posed in \(C([0, \nu], H^{m+1}) \cap L^1([0, \nu], H^m)\) with \(m = \left\lfloor \frac{n+2}{2} \right\rfloor\) (Note that this is where we need the extra condition that \(l \geq \left\lfloor \frac{n+2}{2} \right\rfloor\)). Thus the solution \(u(t)\) has the property of “finite speed of propagation”, and hence for \((x', x_n, t) \in B_{R-1}(\mathbb{R}^{n-1}) \times \mathbb{R} \times [0, \nu]\), the value of \(u(x, t)\) depends only on the data in \(B_R(\mathbb{R}^{n-1}) \times \mathbb{R}\), i.e., only on \(\varphi(x_n)\). Thus we have that

\[ u(x, t) = \tilde{u}_\varphi(x_n, t) \text{ in } B_{R-1}(\mathbb{R}^{n-1}) \times \mathbb{R} \times [0, \nu], \]

where \(\tilde{u}_\varphi\) denote the solution of (1.1) or (1.2) with data \((0, \varphi)\).

Now we give the corresponding estimate of such data and solution. Note that \(s - 1 < -n/2 < 0\) and \(|\xi| \geq |\xi_n|\), we have

\[ |||\eta(x')\varphi(x_n)||_H^{s-1} \leq C||\varphi||_H^{s-1}||\eta||_{L^2} \leq C\epsilon. \]
Let \( f(x) = g(x') h(x_n) \in C^\infty_0(\mathbb{R}^n) \) with \( h(x_n) \bar{u}_\varphi(x_n, t) = \bar{u}_\varphi(x_n, t) \), \( g \) supported in \( B_{R-1}(\mathbb{R}^{n-1}) \) and \( |\bar{g}(\xi')| \geq c > 0 \) for \( \xi' \leq c \). Since the generalized Leibniz rule (see e.g. Lemma 2.2 of [4]) yields that for any \( s \in \mathbb{R} \) and \( \varepsilon > 0 \),

\[
\|f u\|_{H^s} \leq C \|f\|_{H^{\max(|s|, \frac{\varepsilon}{\varepsilon} + 1)}} \|u\|_{H^s} = C_f \|u\|_{H^s},
\]

we have

\[
\|\partial_t u(t_0)\|_{H^{s-1}} \geq \frac{1}{C_f} \|f \partial_t u(t_0)\|_{H^{s-1}} = \frac{1}{C_f} \|g(x') h(x_n) \partial_t \bar{u}_\varphi(x_n, t_0)\|_{H^{s-1}} \geq c \|\partial_t \bar{u}_\varphi(x_n, t_0)\|_{H^{s-1}} \geq c \epsilon^{-1}
\]

This complete the reduction.

4. An alternative proof of \( s < s_c \) ILP in [4]

In this section, combined with Theorem 1.2 in [4], we prove Theorem 2 for the following “focusing” equation:

\[
\square u = |u|^k |\partial_t u|^{l-1} \partial_t u \quad \text{if} \quad l \neq 0
\]

\[
\square u = |u|^{k-1} u \quad \text{if} \quad l = 0
\]

with \( k + l > 1, k, l \geq 0 \) and \( k, l \in \mathbb{R} \).

Several supercritical ill posed (ILP) results of above equations have been obtained in our previous paper [4], here we give an alternative proof and a slightly improvement for Theorem 1.2 in there. In [4], the starting point is the explicit blow-up solution in time-ODE, and here instead by the “conserved quantity” like (2.1).

For simplicity, we concentrate on the case \( l = 2 \) here. The proof of ILP in \( H^s \) with negative \( s \) and \( l \neq 2 \) directly follows from the following argument. In principle, Theorem 1.2 in [4] can also be covered by the argument here for the \( l \neq 2 \) cases and we’ll not exploit it further here.

Note that the ODE part in \( t \) for (4.1) is

\[
\partial_t^2 u = |u|^k |\partial_t u|^{l-1} \partial_t u,
\]

and we have the “conserved quantity” for (4.3) (with \( u \geq 0 \), and \( \partial_t u > 0 \))

\[
\ln \partial_t u - \frac{u^{k+1}}{k+1}.
\]

If we assign the data \((0, 1)\) for (4.3), then we get a solution \( u_T(t) \) defined on \( t \in [0, T) \) with \( 0 < T < \infty \) such that

\[
u, \partial_t u, \partial_t^2 u \not\to \infty \quad \text{as} \quad t \to T.
\]

Then for any \( a > 0 \), \( u_a(t) := u_T(\frac{t}{a}) \) is the solution of (4.3) with the data \((0, \frac{T}{a})\). Denote by \( T^a \) the lifespan of the solution of (4.1) with data \((0, g_a) := (0, \frac{2}{a} \phi(\frac{2}{a})) \) in \( H^s \) or \( H^a \), where \( \phi \in C^\infty_0 \) such that \( \phi = 1 \) on \( B(0, 1 + d) \) with \( d > 0 \).

If the solution space is \( H^s \) with \(|s| < n/2 \), we claim that

\[
\hat{T}^a \leq a.
\]
Since, otherwise, for $t \in [0, a]$ and $x \in B_{1+(d+1)a-t}$, the solution $u(t, x)$ equals $u_a(t)$. Note that for $|s| < n/2$, the generalized Leibnitz rule yields that
\begin{equation}
\|fu\|_{H^s} \leq C\|f\|_{H^{n/2}\cap L}\|u\|_{H^s}
\end{equation}
Thus we have
\begin{equation}
\|u(t, x)\|_{H^s} \geq c\|h\|_{H^{n/2}\cap L}\|h(\frac{x}{(1+d)a-t})u(t, x)\|_{H^s} = cu_a(t)((1+d)a-t)^\frac{n}{2-s} \to \infty
\end{equation}
for $h$ supported in the unit ball $B_1$ as $t \to a$ from below. On the other hand, for the data, we have(note here that we choose $\phi$ appropriately as in Section 2.3 such that $\|\phi\|_{H^{s-1}} < \infty$ for any prescribed $s$)
\begin{equation}
\|g_a\|_{H^{s-1}} = ca^{\frac{n}{2-s}}
\end{equation}
Thus by letting $a$ go to zero, we get the s-ILP of (4.1) in $\dot{H}^s$ for $|s| < n/2$.

For the case of $H^s$ with $s < n/2$, by the above result, we have
\begin{align*}
T^s_a &\leq a \\
\|g_a\|_{H^{s-1}} &\leq C(\|g_a\|_{L^2} + \|g_a\|_{H^{s-1}}) = C(a^{\frac{n}{2-s}} + a^{\frac{n}{2}-1})
\end{align*}
for any $s$. For the estimate of lifespan with $s < 0$, we substitute (4.6) by (3.1) and get the same estimate. This completes the proof of the $l = 2$ case of Theorem 2.

**Remark 2.** Note that the data we given in this section guarantee the derivatives of the solution $u_a$ for ODE is nonnegative, one can change the nonlinear term in (4.1) to any reasonable form such that we also have the “conserved quantity”, say, $|u|^{k-1}u\partial_t u^2$.

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