Diversification quotients: Quantifying diversification via risk measures

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Abstract

We establish the first axiomatic theory for diversification indices using six intuitive axioms – non-negativity, location invariance, scale invariance, rationality, normalization, and continuity – together with risk measures. The unique class of indices satisfying these axioms, called the diversification quotients (DQs), are defined based on a parametric family of risk measures. DQ has many attractive properties, and it can address several theoretical and practical limitations of existing indices. In particular, for the popular risk measures Value-at-Risk and Expected Shortfall, DQ admits simple formulas, it is efficient to optimize in portfolio selection, and it can properly capture tail heaviness and common shocks which are neglected by traditional diversification indices. When illustrated with financial data, DQ is intuitive to interpret, and its performance is competitive against other diversification indices.

Keywords: Expected Shortfall, axiomatic framework, diversification benefit, portfolios, quasi-convexity

1 Introduction

Portfolio diversification refers to investment strategies which spread out among many assets, usually with the hope to reduce the volatility or risk of the resulting portfolio. A mathematical formalization of diversification in a portfolio selection context was made by Markowitz (1952), and some early literature on diversification includes Sharpe (1964), Samuelson (1967), Levy and Sarnat (1970) and Fama and Miller (1972), amongst others.

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Although diversification is conceptually simple, the question of how to measure diversification quantitatively is never well settled. An intuitive, but non-quantitative, approach is to simply count the number of distinct stocks or industries of substantial weight in the portfolio; see e.g., Green and Hollifield (1992), Denis et al. (2002) and DeMiguel et al. (2009) in different contexts. This approach is heuristic as it does not involve statistical or stochastic modeling. The second approach is to compute a quantitative index of the portfolio model, based on e.g., the volatility, variance, an expected utility or a risk measure; this idea is certainly along the direction of Markowitz (1952). In addition, one may empirically address diversification by combining both approaches; see e.g., Tu and Zhou (2011) for the performance of different diversified portfolio strategies and D’Acunto et al. (2019) in the context of robo-advising. Green and Hollifield (1992) studied conditions under which the two approaches are roughly in-line with each other.

In this paper, we take the second approach by assigning a quantifier, called a diversification index, to each modeled portfolio. Carrying the idea of Markowitz (1952), we start our journey with a simple index, the diversification ratio (DR) based on the standard deviation (SD), defined as

\[ \text{DR}^{SD}(X) = \frac{\text{SD} \left( \sum_{i=1}^{n} X_i \right)}{\sum_{i=1}^{n} \text{SD}(X_i)} \]  

for a random vector \( X = (X_1, \ldots, X_n) \) representing future random losses and profits of individual components in a portfolio in one period; one can also replace SD by the variance. Intuitively, with a smaller value indicating a stronger diversification, the index \( \text{DR}^{SD} \) quantifies the improvement of the portfolio SD over the sum of SD of its components, and it has several convenient properties. Nevertheless, it is well-known that SD is a coarse, non-monotone and symmetric measurement of risk, making it unsuitable for many risk management applications, especially in the presence of heavy-tailed and skewed loss distributions; see Embrechts et al. (2002) for thorough discussions.

Risk measures, in particular the Value-at-Risk (VaR) and the Expected Shortfall (ES), are more flexible quantitative tools, widely used in both financial institutions’ internal risk management and banking and insurance regulatory frameworks, such as Basel III/IV (BCBS (2019)) and Solvency II (EIOPA (2011)). ES has many nice theoretical properties and satisfies the four axioms of coherence (Artzner et al. (1999)), whereas VaR is not subadditive in general, but it enjoys other practically useful properties; see Embrechts et al. (2014, 2018), Emmer et al. (2015) and the references therein for more discussions on the issues of VaR versus ES.

Some indices of diversification based on various risk measures have been proposed in the literature. For a given risk measure \( \phi \), an example of a diversification index is DR in (1) with
SD replaced by $\phi$, that is,

$$\text{DR}^{\phi}(X) = \frac{\phi \left( \sum_{i=1}^{n} X_i \right)}{\sum_{i=1}^{n} \phi(X_i)};$$

see Tasche (2007). Other studies on DR can be found in e.g., Choueifaty and Coignard (2008), Bürgi et al. (2008), Mainik and Rüschendorf (2010) and Embrechts et al. (2015). For a review of diversification indices, see Koumou (2020).

We find several demerits of DR built on a general risk measure $\phi$ such as VaR or ES. First, DR is not location invariant, meaning that adding a risk-free asset changes the value of DR. Second, the value of $\text{DR}^{\phi}$ is not necessarily non-negative. Since the risk measure may take negative values, it would be difficult to interpret the case where either the numerator or denominator in DR is negative, and this makes optimization of DR troublesome. This is particularly relevant if some components in the portfolio are used to hedge against other components, possibly leading to a negative or zero risk value. Another example is a portfolio of credit default losses, where VaR of individual losses are often 0. Third, DR is not necessarily quasi-convex in portfolio weights; this point is more subtle and will be explained later in the paper. In addition to the above drawbacks, we find that DR has wrong incentives for some simple models; for instance, it suggests that an iid portfolio of $t$-distributed risks is less diversified than a portfolio with a common shock and the same marginals; see Section 5.2 for details.

Based on the above observations, a natural question is whether we can design a suitable index based on risk measures to quantify the magnitude of diversification, which avoids all the deficiencies above. Answering this and related questions is the main purpose of this paper.

We take an axiomatic approach to find our desirable diversification indices. Axiomatic approaches for risk and decision indices have been prolific in economic and statistical decision theories; see e.g., the recent discussions of Gilboa et al. (2019) and the axiomatization of preferences in Klibanoff et al. (2005) and Gilboa et al. (2010). Closely related to diversification indices, risk measures (Artzner et al. (1999)) and acceptability indices (Cherny and Madan (2009)) also admit sound axiomatic foundation; the particular cases of VaR and ES are studied by Chambers (2009) and Wang and Zitikis (2021). In Section 3, we establish the first axiomatic foundation of diversification indices. This axiomatic theory leads to the class of diversification quotients (DQs), the main object of this paper. For a portfolio loss vector $X = (X_1, \ldots, X_n)$, a DQ is

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2 A negative value of a risk measure has a concrete meaning as the amount of capital to be withdrawn from a portfolio position while keeping it acceptable; see Artzner et al. (1999).

3 A different list of desirable axioms for diversification indices is studied by Koumou and Dionne (2022). Their framework is mathematically different from ours as their diversification indices are mappings of portfolio weights, instead of mappings of portfolio random vectors. They did not provide axiomatic characterization results.
Figure 1. Comparing DQ and DR

\[
\rho_\beta \left( \sum_{i=1}^n X_i \right) \\
\sum_{i=1}^n \rho_\alpha (X_i) \\
\rho_\alpha \left( \sum_{i=1}^n X_i \right)
\]

\[
DQ_\alpha^\rho (X) = \alpha^*/\alpha
\]

defined as

\[
DQ_\alpha^\rho (X) = \frac{\alpha^*}{\alpha}, \quad \text{with } \alpha^* = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\alpha (X_i) \right\},
\]

where \( \rho = (\rho_\alpha)_{\alpha \in I} \) is a class of risk measures decreasing in \( \alpha \in I = (0, \infty) \). Our axiomatic theory starts with three simple axioms satisfied by DR based on SD in (1) – non-negativity, location invariance and scale invariance – that are arguably natural for diversification indices. We then put forward three other axioms of rationality, normalization and continuity, and discuss their interpretation and desirability. Our first main result (Theorem 1) establishes that the six axioms together uniquely identify DQ based on monetary and positive homogeneous risk measures, including VaR and ES as special cases. In the most relevant case \( \rho_\alpha \left( \sum_{i=1}^n X_i \right) < \sum_{i=1}^n \rho_\alpha (X_i) \), we see from Figure 1 a conceptual symmetry between DQ, which measures the improvement of risk by pooling in the horizontal direction, and DR, which measures an improvement in the vertical direction.

A detailed analysis of properties of DQ based on general risk measures is discussed in Section 4, which reveals that DQ has many appealing features, both theoretically and practically. In addition to standard operational properties (Proposition 1), DQ has intuitive behaviour for several benchmark portfolio scenarios (Theorem 2). Moreover, DQ allows for consistency with stochastic dominance (Proposition 2) and a fair comparison across portfolio dimensions (Proposition 3). We proceed to focus on VaR and ES in Section 5. Alternative formulations of \( DQ_\alpha^{\text{VaR}} \) and \( DQ_\alpha^{\text{ES}} \) are first derived (Theorem 3). It turns out that \( DQ_\alpha^{\text{VaR}} \) and \( DQ_\alpha^{\text{ES}} \) have a natural range of \([0, n]\) and \([0, 1]\), respectively (Proposition 4). We further find that DQs based on both VaR and ES report intuitive comparisons between normal and t-models and it has the nice feature that it can capture heavy tails and common shocks.

In Section 6, we formulate DQ as a function of portfolio weights, and portfolio optimization
problems are studied. It is shown that $DQ_{\alpha}^{ES}$ is quasi-convex in portfolio weights (Theorem 4), and efficient algorithms to optimize $DQ_{\alpha}^{Val}$ and $DQ_{\alpha}^{ES}$ based on empirical observations are obtained (Proposition 5). Our new diversification index is applied to financial data in Section 7, where several empirical observations highlight the advantages of DQ. We conclude the paper in Section 8 by discussing a number of implications and promising future directions for DQ. Some additional results, proofs, and some omitted numerical results are relegated to the technical appendices.

**Notation.** Throughout this paper, $(\Omega, \mathcal{F}, P)$ is an atomless probability space, on which almost surely equal random variables are treated as identical. A risk measure $\phi$ is a mapping from $X$ to $\mathbb{R}$, where $X$ is a convex cone of random variables on $(\Omega, \mathcal{F}, P)$ representing losses faced by a financial institution or an investor, and $X$ is assumed to include all constants (i.e., degenerate random variables). For $p \in (0, \infty)$, denote by $L^p = L^p(\Omega, \mathcal{F}, P)$ the set of all random variables $X$ with $\mathbb{E}[|X|^p] < \infty$ where $\mathbb{E}$ is the expectation under $P$. Furthermore, $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ is the space of all (essentially) bounded random variables, and $L^0 = L^0(\Omega, \mathcal{F}, P)$ is the space of all random variables. Write $X \sim F$ if the random variable $X$ has the distribution function $F$ under $P$, and $X \overset{d}{=} Y$ if two random variables $X$ and $Y$ have the same distribution. Further, denote by $\mathbb{R}_+ = [0, \infty)$ and $\overline{\mathbb{R}} = [-\infty, \infty]$. Terms such as increasing or decreasing functions are in the non-strict sense. For $X \in L^0$, ess-sup$(X)$ and ess-inf$(X)$ are the essential supremum and the essential infimum of $X$, respectively. Let $n$ be a fixed positive integer representing the number of assets in a portfolio, and let $[n] = \{1, \ldots, n\}$. It does not hurt to think about $n \geq 2$ although our results hold also (trivially) for $n = 1$. The vector $\mathbf{0}$ represents the $n$-vector of zeros, and we always write $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$.

## 2 Preliminaries and motivation

The most important object of the paper, a **diversification index** $D$ is a mapping from $\mathcal{X}^n$ to $\overline{\mathbb{R}}$, which is used to quantify the magnitude of diversification of a risk vector $X \in \mathcal{X}^n$ representing portfolio losses. Our convention is that a smaller value of $D(X)$ represents a stronger diversification in a sense specified by the design of $D$.

As the evaluation of diversification is closely related to that of risk, diversification indices in the literature are often defined through risk measures. An example of a diversification index is the diversification ratio (DR) mentioned in the Introduction based on measures of variability such as the standard deviation (SD) and the variance (var):

$$DR_{SD}(X) = \frac{SD(\sum_{i=1}^n X_i)}{\sum_{i=1}^n SD(X_i)} \quad \text{and} \quad DR_{var}(X) = \frac{var(\sum_{i=1}^n X_i)}{\sum_{i=1}^n var(X_i)},$$

with the convention $0/0 = 0$. We refer to Rockafellar et al. (2006), Furman et al. (2017) and
Bellini et al. (2022) for general measures of variability. DRs based on SD and var satisfy the three simple properties below, which can be easily checked.

[+] Non-negativity: \( D(X) \geq 0 \) for all \( X \in \mathcal{X}^n \).

[LI] Location invariance: \( D(X + c) = D(X) \) for all \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) and all \( X \in \mathcal{X}^n \).

[SI] Scale invariance: \( D(\lambda X) = D(X) \) for all \( \lambda > 0 \) and all \( X \in \mathcal{X}^n \).

The first property, [+] , simply means that diversification is measured in non-negative values, where 0 typically represents a fully diversified or hedged portfolio (in some sense). The property [LI] means that injecting constant losses or gains to components of a portfolio, or changing the initial price of assets in the portfolio,\(^4\) does not affect its diversification index. The property [SI] means that rescaling a portfolio does not affect its diversification index. The latter two properties are arguably natural, although they are not satisfied by some diversification indices used in the literature (see (3) below). A diversification index satisfying both [LI] and [SI] is called location-scale invariant.

Next, we define the two popular risk measures in banking and insurance practice. The VaR at level \( \alpha \in (0, 1) \) is defined as

\[
\text{VaR}_\alpha(X) = \inf \{ x \in \mathbb{R} : P(X \leq x) \geq 1 - \alpha \}, \quad X \in \mathcal{L}^0,
\]

and the ES (also called CVaR, TVaR or AVaR) at level \( \alpha \in (0, 1) \) is defined as

\[
\text{ES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) d\beta, \quad X \in \mathcal{L}^1,
\]

and \( \text{ES}_0(X) = \text{ess-sup}(X) = \text{VaR}_0(X) \) which may be \( \infty \). The probability level \( \alpha \) above is typically very small, e.g., 0.01 or 0.025 in BCBS (2019); note that we use the “small \( \alpha \)” convention.

Artzner et al. (1999) introduced coherent risk measures \( \phi : \mathcal{X} \rightarrow \mathbb{R} \) as those satisfying the following four properties.

[M] Monotonicity: \( \phi(X) \leq \phi(Y) \) for all \( X, Y \in \mathcal{X} \) with \( X \leq Y \).\(^5\)

[CA] Constant additivity: \( \phi(X + c) = \phi(X) + c \) for all \( c \in \mathbb{R} \) and \( X \in \mathcal{X} \).

[PH] Positive homogeneity: \( \phi(\lambda X) = \lambda \phi(X) \) for all \( \lambda \in (0, \infty) \) and \( X \in \mathcal{X} \).

[SA] Subadditivity: \( \phi(X + Y) \leq \phi(X) + \phi(Y) \) for all \( X, Y \in \mathcal{X} \).

\(^4\)Recall that \( X_i \) represents the loss from asset \( i \). Suppose that two agents purchased the same portfolio of assets but at different prices of each asset. Denote by \( X \) the portfolio loss vector of agent 1. The portfolio loss vector of agent 2 is \( X + c \), where \( c \) is the vector of differences between their purchase prices. The two agents should have the same level of diversification regardless of their purchase prices, as they hold the same portfolio.

\(^5\)The inequality \( X \leq Y \) between two random variables \( X \) and \( Y \) is point-wise.
ES satisfies all four properties above, whereas VaR does not satisfy [SA]. We say that a risk measure is monetary if it satisfies [CA] and [M], and it is MCP if it satisfies [M], [CA] and [PH]. For discussions and interpretations of these properties, we refer to Föllmer and Schied (2016).

Some diversification indices, such as DR and the diversification benefit (DB, e.g., Embrechts et al. (2009) and McNeil et al. (2015)), are defined via risk measures. For a risk measure \( \phi \), DR and DB based on \( \phi \) are defined as

\[
\text{DR}^\phi(X) = \frac{\phi\left(\sum_{i=1}^{n} X_i\right)}{\sum_{i=1}^{n} \phi(X_i)} \quad \text{and} \quad \text{DB}^\phi(X) = \sum_{i=1}^{n} \phi(X_i) - \phi\left(\sum_{i=1}^{n} X_i\right).
\]

In contrast to DR, a larger value of DB represents a stronger diversification, but this convention can be easily modified by flipping the sign. By definition, DR is the ratio of the pooled risk to the sum of the individual risks, and thus a measurement of how substantially pooling reduces risk; similarly, DB measures the difference instead of the ratio. In general, the value of DR\(^\phi\) is not necessarily non-negative, since \( \phi(X) \) may be negative for some \( X \in \mathcal{X} \). As such, we cannot interpret diversification from the value of DR if either the denominator or the numerator in (3) is negative or zero. In addition to violating [+] , DR does not necessarily satisfy [LI] and [SI] even if the risk measure \( \phi \) has nice properties. The index DB\(^\phi\) satisfies [LI] for \( \phi \) satisfying [CA], but it does not satisfy [SI] in general and it may take both positive and negative values.

In financial applications, the risk measures VaR and ES are specified in regulatory documents such as BCBS (2019) and EIOPA (2011), and therefore it is beneficial to stick to VaR or ES as the risk measure when assessing diversification. Both DR\(^\text{VaR}^\alpha\) and DR\(^\text{ES}^\alpha\) satisfy [SI], but they do not satisfy [+] or [LI].\(^7\) It remains unclear how one can define a diversification index based on VaR or ES satisfying these properties. In the remainder of the paper, we will introduce and study a new index of diversification to bridge this gap.

### 3 Diversification quotients: An axiomatic theory

In this section, we fix \( \mathcal{X} = L^\infty \) as the standard choice in the literature of axiomatic theory of risk measures. Six axioms are used to uniquely characterize a new diversification index, called diversification quotient (DQ). The first three of the six axioms [+] , [LI], [SI] have been introduced in Section 2, and we proceed to introduce the other three.

For a risk measure \( \phi \), we say that two vectors \( X, Y \in \mathcal{X}^n \) are \( \phi \)-marginally equivalent if \( \phi(X_i) = \phi(Y_i) \) for each \( i \in [n] \), and we denote this by \( X \overset{\phi}{\sim} Y \). In other words, if an agent

\(6\)If the denominator in the definition of DR\(^\phi\)(X) is 0, then we use the convention 0/0 = 0 and 1/0 = \( \infty \).

\(7\)DR and DB for a general \( \phi \) do not satisfy some of [+] , [LI] and [SI]. Moreover, an impossibility result (Proposition EC.2) is presented in Appendix C which suggests that it is not possible to construct non-trivial diversification indices like DR and DB satisfying [+] , [LI] and [SI].
evaluates risks using the risk measure \( \phi \), then she would view the individual components of \( \mathbf{X} \) and those of \( \mathbf{Y} \) as indifferent. Similarly, denote by \( \mathbf{X} \succeq \mathbf{Y} \) if \( \phi(X_i) \leq \phi(Y_i) \) for each \( i \in [n] \), and by \( \mathbf{X} \succ \mathbf{Y} \) if \( \phi(X_i) < \phi(Y_i) \) for each \( i \in [n] \). The other three desirable axioms are presented below, and they are built on a given risk measure \( \phi \), such as VaR or ES, typically specified exogenously by financial regulation.

\[ [R]_\phi \text{ Rationality: } D({\bf X}) \leq D({\bf Y}) \text{ for } X, Y \in \mathcal{X}^n \text{ satisfying } X \succeq Y \text{ and } \sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i. \]

To interpret the axiom \([R]_\phi\), consider two portfolios \( \mathbf{X} \) and \( \mathbf{Y} \) satisfying \( \mathbf{X} \succeq \mathbf{Y} \), meaning that their individual components are considered as equally risky. If further \( \sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i \) holds, then the total loss from portfolio \( \mathbf{X} \) is always less or equal to that from portfolio \( \mathbf{Y} \), making the portfolio \( \mathbf{X} \) safer than \( \mathbf{Y} \). Since the individual components in \( \mathbf{X} \) and those in \( \mathbf{Y} \) are equally risky, the fact that \( \mathbf{X} \) is safer in aggregation is a result of the different diversification effects in \( \mathbf{X} \) and \( \mathbf{Y} \), leading to the inequality \( D(\mathbf{X}) \leq D(\mathbf{Y}) \). This axiom is called rationality because a rational agent always prefers to have smaller losses.

Next, we formulate our idea about normalizing representative values of the diversification index. First, we assign the zero portfolio \( \mathbf{0} \) the value \( D(\mathbf{0}) = 0 \), as it carries no risk in every sense.\(^8\) A natural benchmark of a non-diversified portfolio is one in which all components are the same. Such a portfolio \( \mathbf{X}^{du} = (X, \ldots, X) \) will be called a duplicate portfolio, and we may, ideally, assign the value \( D(\mathbf{X}^{du}) = 1 \). However, since the zero portfolio \( \mathbf{0} \) is also duplicate but \( D(\mathbf{0}) = 0 \), we will require the weaker assumption \( D(\mathbf{X}^{du}) \leq 1 \) for duplicate portfolios. Lastly, we should understand for what portfolios \( D(\mathbf{X}) \geq 1 \) needs to occur. We say that a portfolio \( \mathbf{X}^{wd} = (X_1, \ldots, X_n) \) is worse than duplicate, if there exists a duplicate portfolio \( \mathbf{X}^{du} = (X, \ldots, X) \) such that \( \mathbf{X}^{wd} \succeq \mathbf{X}^{du} \) and \( \sum_{i=1}^n x_i \geq nX \). Intuitively, this means that each component of \( \mathbf{X}^{wd} \) is strictly less risky than \( X \), but putting them together always incurs a larger loss than \( nX \); in this case, diversification creates nothing but a penalty to the risk manager, and we assign \( D(\mathbf{X}^{wd}) \geq 1.\(^9\) Putting all of the considerations above, we propose the following normalization axiom.

\[ [N]_\phi \text{ Normalization: } D(\mathbf{0}) = 0 \text{ and } D(\mathbf{X}^{du}) \leq 1 \leq D(\mathbf{X}^{wd}) \text{ if } \mathbf{X}^{du} \text{ is duplicate and } \mathbf{X}^{wd} \text{ is worse than duplicate.} \]

Finally, we propose a continuity axiom which is mainly for technical convenience.

\[ [C]_\phi \text{ Continuity: For } \{Y^k\}_{k \in \mathbb{N}} \subseteq \mathcal{X}^n \text{ and } \mathbf{X} \in \mathcal{X}^n \text{ satisfying } Y^k \succeq \mathbf{X} \text{ for each } k, \text{ if } (\sum_{i=1}^n x_i - \sum_{i=1}^n y^k_i) \xrightarrow{L_\infty} 0 \text{ as } k \to \infty, \text{ then } (D(\mathbf{X}) - D(Y^k))_+ \to 0. \]

\(^8\)Indeed, the value of \( D(\mathbf{0}) \) may be rather arbitrary; this is the case for DR where \( 0/0 \) occurs.

\(^9\)Such situations may be regarded as diversification disasters; see Ibragimov et al. (2011).
The axiom \([C] \phi\) is a special form of semi-continuity. To interpret it, consider portfolios \(X\) and \(Y\) which are marginally equivalent. If the sum of components of \(X\) is not much worse than that of \(Y\) in \(L^\infty\), then the axiom says that the diversification of \(X\) is not much worse than the diversification of \(Y\). This property allows for a special form of stability or robustness\(^{10}\) with respect to statistical errors when estimating the distributions of portfolio losses.

One can check that the axioms \([R] \phi\), \([N] \phi\) and \([C] \phi\) are satisfied \(\text{DR} \text{VaR}_\alpha\) and \(\text{DR} \text{ES}_\alpha\) if we only consider positive portfolio vectors. The axioms are not satisfied by \(\text{DR} \text{SD}\) because SD is not monotone and hence the inequalities \(\sum_{i=1}^n X_i \leq \sum_{i=1}^n Y_i\) and \(\sum_{i=1}^n X_i \geq nX\) used in \([R] \phi\) and \([N] \phi\) are not relevant for SD.

We now formally introduce the diversification index \(DQ\) relying on a parametric class of risk measures.

**Definition 1.** Let \(\rho = (\rho_\alpha)_{\alpha \in I}\) be a class of risk measures indexed by \(\alpha \in I = (0, \overline{\alpha})\) with \(\overline{\alpha} \in (0, \infty]\) such that \(\rho_\alpha\) is decreasing in \(\alpha\). For \(X \in X^n\), the diversification quotient based on the class \(\rho\) at level \(\alpha \in I\) is defined by

\[
DQ^\rho_\alpha(X) = \frac{\alpha^*}{\alpha}, \quad \text{where } \alpha^* = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\alpha(X_i) \right\},
\]

with the convention \(\inf(\emptyset) = \alpha\).

\(DQ\) can be constructed from any monotonic parametric family of risk measures. All commonly used risk measures belong to a monotonic parametric family, as this includes VaR, ES, expectiles (e.g., Bellini et al. (2014)), mean-variance (e.g., Markowitz (1952) and Maccheroni et al. (2009)), and entropic risk measures (e.g., Föllmer and Schied (2016)). In addition, there are multiple ways to construct \(DQ\) from a single risk measure; see Section 4.4.

In the following result, we characterize \(DQ\) based on MCP risk measures by Axioms \([+]\), \([L]I\), \([S]I\), \([R] \phi\), \([N] \phi\) and \([C] \phi\).

**Theorem 1.** A diversification index \(D : X^n \to \mathbb{R}\) satisfies \([+]\), \([L]I\), \([S]I\), \([R] \phi\), \([N] \phi\) and \([C] \phi\) for some MCP risk measure \(\phi\) if and only if \(D\) is \(DQ^\rho_\alpha\) for some decreasing families \(\rho\) of MCP risk measures. Moreover, in both directions of the above equivalence, it can be required that \(\rho_\alpha = \phi\).

Theorem 1 gives the first axiomatic characterization of diversification indices, to the best of our knowledge. The diversification index \(DQ\) turns out to enjoy many intuitive and practically relevant properties, which we will study in the next few sections.

The axioms \([R] \phi\), \([N] \phi\) and \([C] \phi\) are formulated based on an exogenously specified risk measure \(\phi\), usually by financial regulation. This choice can also be endogenized in the context of

\(^{10}\)In the literature of statistical robustness, often a different metric than the \(L^\infty\) metric is used; see Huber and Ronchetti (2009) for a general treatment. Our choice of formulating continuity via the \(L^\infty\) metric is standard in the axiomatic theory of risk mappings on \(L^\infty\).
internal decision making. We present in Appendix A an axiomatization of DQ without specifying a risk measure via a few additional axioms on the preference of a decision maker.

4 Properties of DQ

In this section, we study properties of DQ defined in Definition 1. For the greatest generality, we do not impose any properties of risk measures in the decreasing family $\rho = (\rho_\alpha)_{\alpha \in I}$, i.e., the family $\rho$ is not limited to a class of MCP risk measures, so that our results can be applied to more flexible contexts in which some of the six axioms are relaxed.

4.1 Basic properties

We first make a few immediate observations by the definition of DQ. From (4), we can see that computing $\text{DQ}_\alpha$ is to invert the decreasing function $\beta \mapsto (\rho_\beta(\sum_{i=1}^n X_i))$ at $\sum_{i=1}^n \rho_\alpha(X_i)$. For the cases of VaR and ES, $I = (0,1)$, $\alpha^* \in [0,1]$, and DQ has simple formulas; see Theorem 3 in Section 5. For a fixed value of $\sum_{i=1}^n \rho_\alpha(X_i)$, DQ is larger if the curve $\beta \mapsto \rho_\beta(\sum_{i=1}^n X_i)$ is larger, and DQ is smaller if the curve $\beta \mapsto \rho_\beta(\sum_{i=1}^n X_i)$ is smaller. This is consistent with our intuition that a diversification index is large if there is little or no diversification, thus a large value of the portfolio risk, and a diversification index is small if there is strong diversification.

In Theorem 1, we have seen that DQ satisfies [SI] and [LI] if $\rho$ is a class of MCP risk measures. These properties of DQ can be obtained based on a more general version of properties [CA] and [PH] of risk measures, allowing us to include SD and the variance. The results are summarized in Proposition 1, which are straightforward to check by definition.

[CA] $m$ Constant additivity with $m \in \mathbb{R}$: $\phi(X + c) = \phi(X) + mc$ for all $c \in \mathbb{R}$ and $X \in \mathcal{X}$.

[PH] $\gamma$ Positive homogeneity with $\gamma \in \mathbb{R}$: $\phi(\lambda X) = \lambda^\gamma \phi(X)$ for all $\lambda \in (0, \infty)$ and $X \in \mathcal{X}$.

Proposition 1. Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a class of risk measures decreasing in $\alpha$. For each $\alpha \in I$,

(i) if $\rho_\alpha$ satisfies [PH] $\gamma$ with the same $\gamma$, then $\text{DQ}_\alpha^\rho$ satisfies [SI].

(ii) if $\rho_\alpha$ satisfies [CA] $m$ with the same $m$, then $\text{DQ}_\alpha^\rho$ satisfies [LI].

(iii) if $\rho_\alpha$ satisfies [SA], then $\text{DQ}_\alpha^\rho$ takes value in $[0,1]$.

It is clear that [CA] is [CA] $m$ with $m = 1$ and [PH] is [PH] $\gamma$ with $\gamma = 1$. Proposition 1 implies that if $\rho$ is a class of risk measures satisfying [CA], [PH] and [SA] such as ES, then $\text{DQ}_\alpha^\rho$ is location-scale invariant and takes value in $[0,1]$. More properties of DQs on the important families of VaR and ES will be discussed in Section 5. In particular, we will see that the ranges of $\text{DQ}_\alpha^{\text{VaR}}$ and $\text{DQ}_\alpha^{\text{ES}}$ are $[0, n]$ and $[0,1]$, respectively.
Example 1 (Liquidity and temporal consistency). In risk management practice, liquidity and time-horizon of potential losses need to be taken into account; see BCBS (2019, p.89). If liquidity risk is of concern, one may use a risk measure with \( \gamma > 1 \) to penalize large exposures of losses. For such risk measures, \( \text{DQ}_\alpha^\beta \) remains scale invariant, as shown by Proposition 1. On the other hand, if the risk associated to the loss \( X(t) \) at different time spots \( t > 0 \) are scalable by a function \( f > 0 \) (usually of the order \( f(t) = \sqrt{t} \) in standard models such as the Black-Scholes), then DQ is consistent across different horizons in the sense that \( \text{DQ}_\alpha^\beta (X(t)) = \text{DQ}_\alpha^\beta (X(s)) \) for two time spots \( s, t > 0 \), given that \( \rho_\beta (X_i(t)) = f(t) \rho_\beta (X_i(1)) \) for \( i \in [n], t > 0 \) and \( \beta \in I \).

Remark 1. Cherny and Madan (2009) proposed acceptability indices defined via a class of risk measures. More precisely, an acceptability index is defined by \( \text{AI}_\rho (X) = \sup \{ \gamma \in \mathbb{R}^+ : \rho_1/\gamma (X) \leq 0 \} \) for a decreasing class of coherent risk measures \( (\rho_\gamma)_{\gamma \in \mathbb{R}^+} \), which has visible similarity to \( \alpha^* \) in (4); see Kováčová et al. (2020) for optimization of acceptability indices. If \( \rho \) is a class of risk measures satisfying \([\text{CA}]\), then

\[
\text{DQ}_\alpha^\beta (X) = \frac{1}{\alpha} \left( \text{AI}_\rho \left( \sum_{i=1}^n (X_i - \rho_\alpha (X_i)) \right) \right)^{-1}.
\]

For Dhaene et al. (2012) studied several methods for capital allocation, among which the quantile allocation principle computes a capital allocation \( (C_1, \ldots, C_n) \) such that \( \sum_{i=1}^n C_i = \text{VaR}_\alpha (\sum_{i=1}^n X_i) \) and \( C_i = \text{VaR}_{\alpha_i} (X_i) \) for some \( c > 0 \). The constant \( c \) appearing as a nuisance parameter in the above rule has a visible mathematical similarity to \( \text{DQ}_\alpha^{\text{VaR}} \). Mafusalov and Uryasev (2018) studied the so-called buffered probability of exceedance, which is the inverse of the ES curve \( \beta \mapsto \text{ES}_\beta (X) \) at a specific point \( x \in \mathbb{R} \); note that \( \alpha^* \) in (4) is obtained by inverting the ES curve \( \beta \mapsto \text{ES}_\beta (\sum_{i=1}^n X_i) \) at \( \sum_{i=1}^n \text{ES}_\alpha (X_i) \).

4.2 Interpreting DQ from portfolio models

To use DQ as a diversification index, we need to make sure that it coincides with our usual perceptions of portfolio diversification. For a given risk measure \( \phi \) and a portfolio risk vector \( X \), we consider the following three situations which yield intuitive values of DQ.

(i) There is no insolvency risk with pooled individual capital, i.e., \( \sum_{i=1}^n X_i \leq \sum_{i=1}^n \phi (X_i) \) a.s.;

(ii) diversification benefit exists, i.e., \( \phi (\sum_{i=1}^n X_i) < \sum_{i=1}^n \phi (X_i) \);

(iii) the portfolio relies on a single asset, i.e., \( X = (\lambda_1 X, \ldots, \lambda_n X) \) for some \( X \in \mathcal{X} \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}^+ \).

The above three situations receive special attention because they intuitively correspond to very strong diversification, some diversification, and no diversification, respectively. Naturally, we
would expect $DQ$ to be very small for (i), $DQ$ to be smaller than 1 for (ii), and $DQ$ to be 1 for (iii). It turns out that the above intuitions all check out under very weak conditions that are satisfied by commonly used classes of risk measures.

Before presenting this result, we fix some technical terms. For a class $\rho$ of risk measures $\rho_\alpha$ decreasing in $\alpha$, we say that $\rho$ is non-flat from the left at $(\alpha, X)$ if $\rho_\beta(X) > \rho_\alpha(X)$ for all $\beta \in (0, \alpha)$, and $\rho$ is left continuous at $(\alpha, X)$ if $\alpha \mapsto \rho_\alpha(X)$ is left continuous. A random vector $(X_1, \ldots, X_n)$ is comonotonic if there exists a random variable $Z$ and increasing functions $f_1, \ldots, f_n$ on $\mathbb{R}$ such that $X_i = f_i(Z)$ a.s. for every $i \in [n]$. A risk measure is comonotonic-additive if $\phi(X + Y) = \phi(X) + \phi(Y)$ for comonotonic $(X, Y)$. Each of ES and VaR satisfies comonotonic-additivity, as well as any distortion risk measures (Yaari (1987), Kusuoka (2001)) and signed Choquet integrals (Wang et al. (2020)). We denote by $\rho_0 = \lim_{\alpha \uparrow 0} \rho_\alpha$. Note that $\rho_0 = \text{ess-sup}$ for common classes $\rho$ such as VaR, ES, expectiles, and entropic risk measures.

**Theorem 2.** For given $X \in \mathcal{X}^n$ and $\alpha \in I$, if $\rho$ is left continuous and non-flat from the left at $(\alpha, \sum_{i=1}^n X_i)$, the following hold.

(i) Suppose that $\rho_0 \leq \text{ess-sup}$. If for $\rho_\alpha$ there is no insolvency risk with pooled individual capital, then $DQ_\alpha^\rho(X) = 0$. The converse holds true if $\rho_0 = \text{ess-sup}$.

(ii) Diversification benefit exists if and only if $DQ_\alpha^\rho(X) < 1$.

(iii) If $\rho_\alpha$ satisfies [PH] and $X$ relies on a single asset, then $DQ_\alpha^\rho(X) = 1$.

(iv) If $\rho_\alpha$ is comonotonic-additive and $X$ is comonotonic, then $DQ_\alpha^\rho(X) = 1$.

In (i), we see that if there is no insolvency risk with pooled individual capital, then $DQ_\alpha^\rho(X) = 0$. In typical models, such as some elliptical models in Section 5.2, $\sum_{i=1}^n X_i$ is unbounded from above unless it is a constant. Hence, for such models and $\rho$ satisfying $\rho_0 = \text{ess-sup}$, $DQ_\alpha^\rho(X) = 0$ if and only if $\sum_{i=1}^n X_i$ is a constant, thus full hedging is achieved. This is also consistent with our intuition of full hedging as the strongest form of diversification.

**Remark 2.** We require $\rho$ to be left continuous and non-flat from the left to make the inequality in (ii) holds strictly. This requirement excludes, in particular, trivial cases like $X = c \in \mathbb{R}^n$ which gives $DQ_\alpha^{\text{VaR}}(X) = 0$ by definition. In case the conditions fail to hold, $DQ_\alpha^\rho(X) < 1$ may not guarantee $\rho_\alpha(\sum_{i=1}^n X_i) < \sum_{i=1}^n \rho_\alpha(X_i)$, but it implies the non-strict inequality $\rho_\alpha(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n \rho_\alpha(X_i)$, and thus the portfolio risk is not worse than the sum of the individual risks.

With the help of Theorem 2 (ii), we can show that $DQ$ and DR are equivalent when it comes to the existence of the diversification benefit. Under the conditions of Theorem 2, for a given $\alpha \in (0, 1)$, if $\sum_{i=1}^n \rho_\alpha(X_i)$ and $\rho_\alpha(\sum_{i=1}^n X_i)$ are positive, then it is straightforward to check that the following three statements are equivalent: (i) Diversification benefit exists, i.e.,
\( \rho_\alpha (\sum_{i=1}^{n} X_i) < \sum_{i=1}^{n} \rho_\alpha (X_i) \); (ii) \( \text{DR}^{\rho_\alpha} (X) < 1 \); (iii) \( \text{DQ}_\rho^{\phi} (X) < 1 \). The above equivalence only says that \( \text{DQ}_\rho^{\phi} \) and \( \text{DR}^{\rho_\alpha} \) agree on whether they are smaller than 1, but they do not have to agree in other situations, and they are not meant to be compared on the same scale.

4.3 Constructing DQ from a single risk measure

In this section, we discuss how to construct DQ from only a single risk measure \( \phi \). For commonly used risk measures like VaR and ES, a natural family \( \rho \) with \( \rho_\alpha = \phi \) exists. If in some applications one needs to use a different \( \phi \) which does not belong to an existing family, we will need to construct a family of risk measures for \( \phi \).

First, suppose that \( \phi \) is MCP. A simple approach is to take \( \rho_\alpha = (1 - \alpha)\text{ess-sup} + \alpha \phi \) for \( \alpha \in (0, 1) \). Clearly, \( \rho_1 = \phi \). As \( \phi \) is MCP, we have \( \phi(X) \leq \phi(\text{ess-sup}(X)) = \text{ess-sup}(X) \) for all \( X \in L^\infty \). Hence, \( \rho \) is a decreasing class of MCP risk measures. Therefore, \( \text{DQ}_\rho^\phi \) satisfies the six axioms in Theorem 1. Moreover, by checking the definition, this DQ has an explicit formula

\[
\text{DQ}_\rho^\phi (X) = \left( \frac{\text{ess-sup} (\sum_{i=1}^{n} X_i) - \sum_{i=1}^{n} \phi(X_i)}{\text{ess-sup} (\sum_{i=1}^{n} X_i) - \phi (\sum_{i=1}^{n} X_i)} \right)_{+}.
\]

If \( \sum_{i=1}^{n} X_i \leq \sum_{i=1}^{n} \phi(X_i) \), we have \( \text{ess-sup}(\sum_{i=1}^{n} X_i) \leq \sum_{i=1}^{n} \phi(X_i) \) and \( \text{DQ}_\rho^\phi (X) = 0 \); this is also reflected by Theorem 2 when \( \text{ess-sup}(\sum_{i=1}^{n} X_i) > \phi (\sum_{i=1}^{n} X_i) \).

For any arbitrary risk measure \( \phi \), we can always define the decreasing family \( \{ \phi_+/\alpha : \alpha \in I \} \) for constructing DQ; here \( \phi_+ \) is the positive part of \( \phi \). This approach leads to DQs which are also DRs. Assuming that \( \phi \) is non-negative (such as SD), in Proposition EC.3 in Appendix D.1, we will see that DQ based on the class \( \{ \phi/\alpha : \alpha \in I \} \) is precisely DR based on \( \phi \); thus, DR with a non-negative \( \phi \) is a special case of DQ. For the interested reader, we further show that any DR\( ^\phi \) satisfying [+], [LI] and [SI] belongs to the DQ class; see Proposition EC.4 in Appendix D.1.

4.4 Stochastic dominance and dependence

In this section, we discuss the consistency of DQ with respect to stochastic dominance, as well as the best and worst cases for DQ among all dependence structures with given marginal distributions of the risk vector.

For a diversification index, monotonicity with respect to stochastic dominance yields consistency with common decision making criteria such as the expected utility model and the rank-dependent utility model. A random variable \( X \) (representing random loss) is dominated by a random variable \( Y \) in second-order stochastic dominance (SSD) if \( \mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)] \) for all decreasing concave functions \( f: \mathbb{R} \rightarrow \mathbb{R} \) provided that the expectations exist, and we denote this by \( X \leq_{\text{SSD}} Y \).

A risk measure \( \phi \) is \textit{SSD-consistent} if \( \phi(X) \geq \phi(Y) \) for all \( X, Y \in \mathcal{X} \) whenever \( X \leq_{\text{SSD}} Y \).  

\[11\] If \( X \) and \( Y \) represents gains instead of losses, then SSD is typically defined via increasing concave functions.
$X \leq_{SSD} Y$. SSD consistency is known as strong risk aversion in the classic decision theory literature (Rothschild and Stiglitz (1970)). SSD-consistent monetary risk measures, which include all law-invariant convex risk measures such as ES, admit an ES-based characterization (Mao and Wang (2020)).

**Proposition 2.** Assume that $\rho = (\rho_\alpha)_{\alpha \in I}$ is a decreasing class of SSD-consistent risk measures. For $X, Y \in \mathcal{X}^n$ and $\alpha \in I$, if $\sum_{i=1}^n \rho_\alpha(X_i) \leq \sum_{i=1}^n \rho_\alpha(Y_i)$ and $\sum_{i=1}^n X_i \leq_{SSD} \sum_{i=1}^n Y_i$, then $DQ^\rho_\alpha(X) \geq DQ^\rho_\alpha(Y)$.

Proposition 2 follows from the simple observation that
\[
\left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\alpha(X_i) \right\} \subseteq \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n Y_i \right) \leq \sum_{i=1}^n \rho_\alpha(Y_i) \right\},
\]
and we omit the proof.

Assume $\rho$ is a class of SSD-consistent risk measures (e.g., law-invariant convex risk measures). Proposition 2 implies that if the sum of marginal risks is the same for $X$ and $Y$ (this holds in particular if $X$ and $Y$ have the same marginal distributions), then DQ is decreasing in SSD of the total risk. The dependence structures which maximize or minimize DQ for $X$ with specified marginal distributions are discussed in Appendix D.2. For instance, a comonotonic portfolio has the largest DQ (thus the smallest diversification) among all portfolios with the same marginal distributions; this observation is related to Proposition 1 (iii) and Theorem 2 (iv).

### 4.5 Consistency across dimensions

All properties in the previous sections are discussed under the assumption that the dimension $n \in \mathbb{N}$ is fixed. Letting $n$ vary allows for comparison of diversification between portfolios with different dimensions. In this section, we slightly generalize our framework by considering a diversification index $D$ as a mapping on $\bigcup_{n \in \mathbb{N}} \mathcal{X}^n$; note that the input vector $X$ of DQ and DR can naturally have any dimension $n$. We present two more useful properties of DQ in this setting. For $X \in \mathcal{X}^n$ and $c \in \mathbb{R}$, $(X, c)$ is the $(n + 1)$-dimensional random vector obtained by pasting $X$ and $c$, and $(X, X)$ is the $(2n)$-dimensional random vector obtained by pasting $X$ and $X$.

[RI] **Riskless invariance:** $D(X, c) = D(X)$ for all $n \in \mathbb{N}$, $X \in \mathcal{X}^n$ and $c \in \mathbb{R}$.

[RC] **Replication consistency:** $D(X, X) = D(X)$ for all $n \in \mathbb{N}$ and $X \in \mathcal{X}^n$.

Riskless invariance means that adding a risk-free asset into the portfolio $X$ does not affect its diversification. For instance, the Sharpe ratio of the portfolio does not change under such an operation. Replication consistency means that replicating the same portfolio composition does
not affect $D$. Both properties are arguably desirable in most applications due to their natural interpretations.

**Proposition 3.** Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a class of risk measures decreasing in $\alpha$ and $\phi : L^p \to \mathbb{R}$ be a continuous and law-invariant risk measure. For $\alpha \in I$,

(i) If $\rho_\alpha$ satisfies $[CA]_m$ with $m \in \mathbb{R}$ and $\rho_\alpha(0) = 0$ then $DQ_\alpha^\rho$ satisfies $[RI]$.

(ii) If $\rho_\alpha$ satisfies $[PH]$, then $DQ_\alpha^\rho$ satisfies $[RC]$.

We further show in Proposition EC.5 that if $[RI]$ is assumed, then the only option for DR is to use a non-negative $\phi$. Thus, if $[RI]$ is considered as desirable, then DQ becomes useful compared to DR as it offers more choices, and in particular, it works for any classes $\rho$ of monetary risk measures with $\rho_\alpha(0) = 0$ including VaR and ES. Both DQ and DR satisfy $[RC]$ and $[RI]$ for MCP risk measures.

**Example 2.** Let $\phi$ be a risk measure satisfying $[CA]$, such as ES$_\alpha$ or VaR$_\alpha$. Suppose that $\phi(\sum_{i=1}^n X_i) = 1$ and $\sum_{i=1}^n \phi(X_i) = 2$, and thus $DR\phi(X) = 1/2$. If a non-random payoff of $c > 0$ is added to the portfolio, then $DR\phi(X, -c) = (1 - c)/(2 - c)$, which turns to 0 as $c \uparrow 1$, and it becomes negative as soon as $c > 1$. Hence, $DR\phi$ is improved or made negative by including constant payoffs (either as a new asset or added to an existing asset). This creates problematic incentives in optimization. On the other hand, DQ does not suffer from this problem due to $[LI]$ and $[RI]$.

## 5 DQ based on VaR and ES

Since VaR and ES are the two most common classes of risk measures in practice, we focus on the theoretical properties of $DQ^\text{VaR}_\alpha$ and $DQ^\text{ES}_\alpha$ in this section. We fix the parameter range $I = (0, 1)$, and we choose $X^n$ to be $(L^0)^n$ when we discuss $DQ^\text{VaR}_\alpha$ and $(L^1)^n$ when we discuss $DQ^\text{ES}_\alpha$, but all results hold true if we fix $X = L^1$.

### 5.1 General properties

We first provide alternative formulations of $DQ^\text{VaR}_\alpha$ and $DQ^\text{ES}_\alpha$. The formulations offer clear interpretations and simple ways to compute the values of DQs. The formula (7) below can be derived from the optimization formulation for the buffered probability of exceedance in Proposition 2.2 of Mafusalov and Uryasev (2018).

**Theorem 3.** For a given $\alpha \in (0, 1)$, $DQ^\text{VaR}_\alpha$ and $DQ^\text{ES}_\alpha$ have the alternative formulas

$$
DQ^\text{VaR}_\alpha(X) = \frac{1}{\alpha} \mathbb{P} \left( \sum_{i=1}^n X_i > \sum_{i=1}^n \text{VaR}_\alpha(X_i) \right), \quad X \in \mathcal{X}^n, \tag{5}
$$
\[ DQ_{\alpha}^{ES}(X) = \frac{1}{\alpha} \mathbb{P} \left( Y > \sum_{i=1}^{n} ES_{\alpha}(X_i) \right), \quad X \in \mathcal{X}^n, \]  

(6)

where \( Y = ES_U \left( \sum_{i=1}^{n} X_i \right) \) and \( U \sim U[0,1] \). Furthermore, if \( \mathbb{P}(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} ES_{\alpha}(X_i)) > 0 \),

\[ DQ_{\alpha}^{ES}(X) = \frac{1}{\alpha} \min_{r \in (0,\infty)} \mathbb{E} \left[ \left( r \sum_{i=1}^{n} (X_i - ES_{\alpha}(X_i)) + 1 \right) \right], \]  

(7)

and otherwise \( DQ_{\alpha}^{ES}(X) = 0 \).

As a first observation from Theorem 3, it is straightforward to compute \( DQ_{\alpha}^{VaR} \) and \( DQ_{\alpha}^{ES} \) on real or simulated data by applying (5) and (6) to the empirical distribution of the data.

Theorem 3 also gives \( DQ_{\alpha}^{VaR} \) a clear economic interpretation as the improvement of insolvency probability when risks are pooled. Suppose that \( X_1, \ldots, X_n \) are continuously distributed and they represent losses from \( n \) assets. The total pooled capital is \( s_{\alpha} = \sum_{i=1}^{n} VaR_{\alpha}(X_i) \), which is determined by the marginals of \( X \) but not the dependence structure. An agent investing only in asset \( X_i \) with capital computed by \( VaR_{\alpha} \) has an insolvency probability \( \alpha = \mathbb{P}(X_i > VaR_{\alpha}(X_i)) \).

On the other hand, by Theorem 3, \( \alpha^* \) is the probability that the pooled loss \( \sum_{i=1}^{n} X_i \) exceeds the pooled capital \( s_{\alpha} \). The improvement from \( \alpha \) to \( \alpha^* \), computed by \( \alpha^*/\alpha \), is precisely \( DQ_{\alpha}^{VaR}(X) \).

From here, it is also clear that \( DQ_{\alpha}^{VaR}(X) < 1 \) is equivalent to \( \mathbb{P}(\sum_{i=1}^{n} X_i > s_{\alpha}) < \alpha \).

To compare \( DQ_{\alpha}^{VaR} \) with \( DR_{\alpha}^{VaR} \), recall that the two diversification indices can be rewritten as

\[ DQ_{\alpha}^{VaR}(X) = \frac{\mathbb{P}(\sum_{i=1}^{n} X_i > s_{\alpha})}{\alpha} \quad \text{and} \quad DR_{\alpha}^{VaR}(X) = \frac{VaR_{\alpha}(\sum_{i=1}^{n} X_i)}{s_{\alpha}}. \]  

(8)

From (8), we can see a clear symmetry between DQ, which measures the probability improvement, and DR, which measures the quantile improvement. DQ and DR based on ES have a similar comparison.

Next, we compare the range of DQs based on VaR and ES.

**Proposition 4.** For \( \alpha \in (0,1) \) and \( n \geq 2 \), \( \{DQ_{\alpha}^{VaR}(X) : X \in \mathcal{X}^n\} = [0, \min\{n, 1/\alpha\}] \) and \( \{DQ_{\alpha}^{ES}(X) : X \in \mathcal{X}^n\} = [0, 1] \).

By Proposition 4, both \( DQ_{\alpha}^{VaR} \) and \( DQ_{\alpha}^{ES} \) take values on a bounded interval. In contrast, the diversification ratio \( DR_{\alpha}^{VaR} \) is unbounded, and \( DR_{\alpha}^{ES} \) is bounded above by 1 only when the ES of the total risk is non-negative. Moreover, similarly to the continuity axiom of preferences (e.g., Föllmer and Schied (2016)), a bounded interval can provide mathematical convenience for applications.

**Remark 3.** It is a coincidence that \( DQ_{\alpha}^{VaR} \) for \( \alpha < 1/n \) and \( DR_{\alpha}^{VaR} \) both have a maximum value \( n \). The latter maximum value is attained by a risk vector \((X/n, \ldots, X/n)\) for any \( X \in L^2 \).
5.2 Capturing heavy tails and common shocks

In this section, we analyze three simple normal and t-models to illustrate some features of DQ regarding heavy tails and common shocks in the portfolio models. A separate paper Han et al. (2023) contains a detailed study of DQs based on VaR and ES for elliptical distributions and multivariate regularly varying models, including explicit formulas to compute DQ for these models. Here, we only present some key observations.

Let \( Z = (Z_1, \ldots, Z_n) \) be an \( n \)-dimensional standard normal random vector, and let \( \xi^2 \) have an inverse gamma distribution independent of \( Z \). Denote by it \( n(\nu) \) the joint distribution with \( n \) independent t-marginals \( t(\nu, 0, 1) \), where the parameter \( \nu \) represents the degrees of freedom; see McNeil et al. (2015) for t-models. The model \( Y = (Y_1, \ldots, Y_n) \sim it_n(\nu) \) can be stochastically represented by

\[
Y_i = \xi_i Z_i, \quad \text{for } i \in [n],
\]

where \( \xi_1, \ldots, \xi_n \) are iid following the same distribution as \( \xi \), and independent of \( Z \). In contrast, a joint t-distributed random vector \( Y' = (Y'_1, \ldots, Y'_n) \sim t(\nu, 0, I_n) \) has a stochastic representation

\[
Y'_i = \xi Z_i, \quad \text{for } i \in [n].
\]

In other words, \( Y' \) is a standard normal random vector multiplied by a heavy-tailed common shock \( \xi \). All three models \( Z, Y, Y' \) have the same correlation matrix, the identity matrix \( I_n \).

Because of the common shock \( \xi \) in (10), large losses from components of \( Y' \) are more likely to occur simultaneously, compared to \( Y \) in (9) which does not have a common shock. Indeed, \( Y' \) is tail dependent (Example 7.39 of McNeil et al. (2015)) whereas \( Y \) is tail independent. As such, at least intuitively (if not rigorously), diversification for portfolio \( Y' \) should be considered as weaker than \( Y \), although both models are uncorrelated and have the same marginals.\(^\text{12}\) By the central limit theorem, for \( \nu > 2 \), the component-wise average of \( Y \) (scaled by its variance) is asymptotically normal as \( n \) increases, whereas the component-wise average of \( Y' \) is always t-distributed. Hence, one may intuitively expect the order \( D(Z) < D(Y) < D(Y') \) to hold.

In Tables 1 and 2, we present DQ and DR for a few different models based on \( N(0, I_n) \), \( t(\nu, 0, I_n) \), and \( it_n(\nu) \). We choose \( n = 10 \) and \( \nu = 3 \) or 4,\(^\text{13}\) and thus we have five models in total. As we see from Tables 1 and 2, DQs based on both VaR and ES report a lower value for \( it_n(\nu) \) and a larger value for \( t(\nu, 0, I_n) \), meaning that diversification is weaker for the common shock t-model (10) than the iid t-model (9). For the iid normal model, the diversification is

\(^{12}\) On a related note, as discussed by Embrechts et al. (2002), correlation is not a good measure of diversification in the presence of heavy-tailed and skewed distributions.

\(^{13}\) Most financial asset log-loss data have a tail-index between [3, 5], which corresponds to \( \nu \in [3, 5] \); see e.g., Jansen and De Vries (1991).
Table 1. DQs/DRs based on VaR, ES, SD and var, where $\alpha = 0.05$, $n = 10$ and $\nu = 3$; numbers in bold indicate the most diversified among $Z, Y, Y'$ according to the index $D$

| $D$ | $DQ_{\text{VaR}}$ | $DQ_{\text{ES}}$ | $DR_{\text{VaR}}$ | $DR_{\text{ES}}$ | $DR_{\text{SD}}$ | $DR_{\text{var}}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $Z \sim N(0, I_n)$ | $2.0 \times 10^{-6}$ | $1.9 \times 10^{-9}$ | $0.3162$ | $0.3162$ | $0.3162$ | $1$ |
| $Y \sim \text{i}n(3)$ | $0.0235$ | $0.0124$ | $0.3569$ | $0.2903$ | $0.3162$ | $1$ |
| $Y' \sim \text{t}(3, 0, I_n)$ | $0.0502$ | $0.0340$ | $0.3162$ | $0.3162$ | $0.3162$ | $1$ |
| $D(Z) < D(Y)$ | Yes | Yes | Yes | No | No | No |
| $D(Y) < D(Y')$ | Yes | Yes | No | Yes | No | No |

Table 2. DQs/DRs based on VaR, ES, SD and var, where $\alpha = 0.05$, $n = 10$ and $\nu = 4$; numbers in bold indicate the most diversified among $Z, Y, Y'$ according to the index $D$

| $D$ | $DQ_{\text{VaR}}$ | $DQ_{\text{ES}}$ | $DR_{\text{VaR}}$ | $DR_{\text{ES}}$ | $DR_{\text{SD}}$ | $DR_{\text{var}}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $Z \sim N(0, I_n)$ | $2.0 \times 10^{-6}$ | $1.9 \times 10^{-9}$ | $0.3162$ | $0.3162$ | $0.3162$ | $1$ |
| $Y \sim \text{i}n(4)$ | $0.0050$ | $0.0017$ | $0.3415$ | $0.2828$ | $0.3162$ | $1$ |
| $Y' \sim \text{t}(4, 0, I_n)$ | $0.0252$ | $0.0138$ | $0.3162$ | $0.3162$ | $0.3162$ | $1$ |
| $D(Z) < D(Y)$ | Yes | Yes | Yes | No | No | No |
| $D(Y) < D(Y')$ | Yes | Yes | No | Yes | No | No |

the strongest according to DQ. In contrast, DR sometimes reports that the iid $t$-model has a larger diversification than the common shock $t$-model, which is counter-intuitive. In the setting of both Tables 1 and 2, a risk manager governed by $DQ_{\text{VaR}}$ would prefer the iid portfolio over the common shock portfolio, but the preference is flipped if the risk manager uses $DR_{\text{VaR}}$. A more detailed analysis on this phenomenon for varying $\alpha \in (0, 0.1]$ is presented in Figure EC.1 in Appendix E, and consistent results are observed.

6 DQ as a function of the portfolio weight

In this section, we analyze portfolio diversification for a random vector $X \in \mathcal{X}^n$ representing losses from $n$ assets and a vector $w = (w_1, \ldots, w_n) \in \Delta_n$ of portfolio weights, where

$$\Delta_n := \{ x \in [0,1]^n : x_1 + \cdots + x_n = 1 \}.$$ 

The total loss of the portfolio is $w^\top X$. We write $w \odot X = (w_1 X_1, \ldots, w_n X_n)$ which is the portfolio loss vector with the weight $w$. For a portfolio selection problem, we need to treat
$DQ^n_\rho(w \odot X)$ as a function of the portfolio weight $w$. Denote by $x_\rho^n = (\rho_\alpha(X_1), \ldots, \rho_\alpha(X_n))$ which is a known vector that does not depend on the decision variable $w$.

6.1 Convexity and quasi-convexity

We first analyze convexity and quasi-convexity of the mapping $w \mapsto DQ^n_\rho(w \odot X)$ on $\Delta_n$. Recall that for any real-valued mapping $\phi$ on a space $\mathcal{Y}$, $\phi$ is convex (resp. quasi-convex) if

$$\phi(\lambda X + (1 - \lambda)Y) \leq \lambda \phi(X) + (1 - \lambda)\phi(Y) \quad (\text{resp. } \phi(\lambda X + (1 - \lambda)Y) \leq \phi(X) \vee \phi(Y))$$

for all $X, Y \in \mathcal{Y}$ and $\lambda \in [0, 1]$, where $\vee$ is the maximum operator. When formulated on monetary risk measures, convexity naturally represents the idea that diversification reduces the risk; see Föllmer and Schied (2016). For risk measures that are not constant additive, Cerreia-Vioglio et al. (2011) argued that quasi-convexity is more suitable than convexity to reflect the consideration of diversification; moreover, convexity and quasi-convexity are equivalent if $[\text{CA}]_m$ holds for $m \neq 0$.

We first note that, for a diversification index $D$, convexity or quasi-convexity on $\mathcal{X}^n$ should not hold, as illustrated in Example 3 below.

**Example 3** (Convexity or quasi-convexity on $\mathcal{X}^n$ is not desirable). Let $(X, Y) \in \mathcal{X}^2$ represent any diversified portfolio (e.g., with iid normal components), and assume that $Z := (X + Y)/2$ is not a constant. Since the portfolio $(Z, Z)$ relies only on one asset and has no diversification benefit, for a good diversification index $D$ we naturally want $D(Z, Z)$ to be larger than both $D(X, Y)$ and $D(Y, X)$; recall that $D(Z, Z) = 1$ in the setting of Theorem 2 (iii). This argument shows that it is unnatural to require $D$ to be convex or quasi-convex on $\mathcal{X}^2$; the case of $\mathcal{X}^n$ is similar. Indeed, if a real-valued $D$ satisfies $[\text{SI}]$ and convexity on $\mathcal{X}^n$, then it is a constant (see Proposition EC.6 in Appendix E).

Despite that quasi-convexity of $D$ is unnatural on $\mathcal{X}^n$, quasi-convexity may hold for $w \mapsto D(w \odot X)$ for each given $X$; this property will be called quasi-convexity in $w$ for short. Quasi-convexity in $w$ means that combining a portfolio with a better-diversified one on the same set of assets does not reduce the diversification of the original portfolio; this interpretation is different from quasi-convexity on $\mathcal{X}^n$ which means that combining a portfolio with an arbitrary better-diversified portfolio does not reduce diversification (this is not desirable as discussed in Example 3).

In the next result, we see that $DQ$ is quasi-convex in $w$ for convex risk measures satisfying $[\text{PH}]$, such as ES. In contrast, DR may not be quasi-convex in $w$ for a convex risk measure since the denominator in (3) may be negative.
Theorem 4. Let $\rho = (\rho_\alpha)_{\alpha \in I}$ be a class of convex risk measures satisfying [PH] and decreasing in $\alpha$. For every $X \in \mathcal{X}^n$ and $\alpha \in I$, $w \mapsto \text{DQ}_\alpha^\rho(w \otimes X)$ is quasi-convex.

Theorem 4 implies, in particular, that DQ based on SD, ES, or other coherent risk measures is quasi-convex in $w$. In contrast, the stronger condition of convexity in $w$ generally fails to hold. Indeed, we discuss in the next example that convexity in $w$ is not desirable for a good diversification index.

Example 4 (Convexity in $w$ is not desirable). Consider a risk vector $X = ((1 - \varepsilon)Z, -\varepsilon Z)$ where $Z$ is standard normal and $\varepsilon > 0$ is a small constant. Let $w = (1, 0)$ and $v = (\varepsilon, 1 - \varepsilon)$. Note that $w \otimes X = (1 - \varepsilon)(Z, 0)$ and $v \otimes X = (\varepsilon - \varepsilon^2)(Z, -Z)$. The portfolio $w \otimes X$ is not diversified since it has only one non-zero component, and the portfolio $v \otimes X$ is perfectly hedged since the sum of its components is 0. Hence, for a good diversification index $D$, it should hold that $D(w \otimes X) = 1$ and $D(v \otimes X) = 0$: Theorem 2 confirms this. On the other hand, the portfolio

$$\left(\frac{1}{2}w + \frac{1}{2}v\right) \otimes X = \frac{1}{2} ((1 - \varepsilon^2)Z, -(\varepsilon - \varepsilon^2)Z)$$

is not well diversified since its second component is very small compared to its first component. Intuitively, for $\varepsilon \approx 0$, we expect $D((w/2 + v/2) \otimes X) \approx 1 > D(w \otimes X)/2 + D(v \otimes X)/2$. This shows that $w \mapsto D(w \otimes X)$ is not convex. One can verify that this is indeed true if $D$ is DQ or DR based on commonly used risk measures such as SD, VaR ($\alpha < 1/2$) and ES.

To summarize, as we see from Examples 3 and 4, convexity and quasi-convexity on $\mathcal{X}^n$ and convexity in $w$ are not desirable for a diversification index. In contrast, quasi-convexity in $w$ is desirable, and it is satisfied by DQ based on coherent risk measures by Theorem 4.

6.2 Portfolio optimization

Next, we focus on the following optimal diversification problem

$$\min_{w \in \Delta_n} \text{DQ}^\text{VaR}_\alpha(w \otimes X) \quad \text{and} \quad \min_{w \in \Delta_n} \text{DQ}^\text{ES}_\alpha(w \otimes X); \quad (11)$$

recall that a smaller value of DQ means better diversification.\(^{14}\) We do not say that optimizing a diversification index has a decision-theoretic benefit; here we simply illustrate the advantage of DQ in computation and optimization. Whether optimizing diversification is desirable for individual or institutional investors is an open-ended question which goes beyond the current paper; we refer to Van Nieuwerburgh and Veldkamp (2010), Boyle et al. (2012) and Choi et al. (2017) for relevant discussions.

\(^{14}\)A possible alternative formulation to (11) is to use DQ as a constraint instead of an objective in the optimization. This is mathematically similar to a risk measure constraint (e.g., Rockafellar and Uryasev (2002) and Mafusalov and Uryasev (2018)), but it is perhaps less intuitive, as DQ is not designed to measure or control risk.
For the portfolio weight \( \mathbf{w} \), DQ based on VaR at level \( \alpha \in (0, 1) \) is given by

\[
\text{DQ}_\alpha^{\text{VaR}}(\mathbf{w} \otimes \mathbf{X}) = \frac{1}{\alpha} \inf \left\{ \beta \in (0, 1) : \text{VaR}_\beta \left( \sum_{i=1}^{n} w_i X_i \right) \leq \sum_{i=1}^{n} w_i \text{VaR}_\alpha(X_i) \right\},
\]

and DQ based on ES is similar. In what follows, we fix \( \alpha \in (0, 1) \) and \( \mathbf{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n \), where \( \mathcal{X} \) is \( L^0 \) for VaR and \( L^1 \) for ES, as in Section 5. Write \( \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n \).

**Proposition 5.** Fix \( \alpha \in (0, 1) \) and \( \mathbf{X} \in \mathcal{X}^n \). The optimization of \( \text{DQ}_\alpha^{\text{VaR}}(\mathbf{X}) \) in (11) can be solved by

\[
\min_{\mathbf{w} \in \Delta_n} \mathbb{P} \left( \mathbf{w}^\top (\mathbf{X} - \mathbf{x}_\alpha^{\text{VaR}}) > 0 \right).
\]

Assuming \( \mathbb{P}(X_i > \text{ES}_\alpha(X_i)) > 0 \) for each \( i \in [n] \), the optimization of \( \text{DQ}_\alpha^{\text{ES}}(\mathbf{X}) \) in (11) can be solved by the convex program

\[
\min_{\mathbf{v} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}} \mathbb{E} \left[ (\mathbf{v}^\top (\mathbf{X} - \mathbf{x}_\alpha^{\text{ES}}) + 1)_+ \right],
\]

and the optimal \( \mathbf{w}^\star \) is given by \( \mathbf{v}/\|\mathbf{v}\|_1 \).

Proposition 5 offers efficient algorithms to optimize \( \text{DQ}_\alpha^{\text{VaR}} \) and \( \text{DQ}_\alpha^{\text{ES}} \) in real-data applications. The values of \( \mathbf{x}_\alpha^{\text{VaR}} \) and \( \mathbf{x}_\alpha^{\text{ES}} \) can be computed by many existing estimators of the individual losses (see e.g., McNeil et al. (2015)). In particular, a popular way to estimate these risk measures is to use an empirical estimator. More specifically, if we have data \( \mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)} \) sampled from \( \mathbf{X} \) satisfying some ergodicity condition (being iid would be sufficient), then the empirical version of the problem (12) is

\[
\minimize_{\mathbf{w} \in \Delta_n} \sum_{j=1}^{N} \mathbf{1}_{\{\mathbf{w}^\top (\mathbf{X}^{(j)} - \hat{\mathbf{x}}_{\alpha}^{\text{VaR}}) > 0\}} \text{ over } \mathbf{w} \in \Delta_n,
\]

where \( \hat{\mathbf{x}}_{\alpha}^{\text{VaR}} \) is the empirical estimator of \( \mathbf{x}_\alpha^{\text{VaR}} \) based on sample \( \mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)} \); see McNeil et al. (2015). Write \( \mathbf{y}^{(j)} = \mathbf{X}^{(j)} - \hat{\mathbf{x}}_{\alpha}^{\text{VaR}} \) and \( z_j = \mathbf{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)} > 0\}} \) for \( j \in [n] \). Problem (14) involves a chance constraint (see e.g., Luedtke (2014) and Liu et al. (2016)). By using the big-M method (see e.g., Shen et al. (2010)) via choosing a sufficient large \( M \) (e.g., it is sufficient if \( M \) is larger than the components of \( \mathbf{y}^{(j)} \) for all \( j \)), (14) can be converted into the following linear integer program:

\[
\begin{align*}
\minimize_{\mathbf{z}, \mathbf{w}} & \quad \sum_{j=1}^{N} z_j \\
\text{subject to} & \quad \mathbf{w}^\top \mathbf{y}^{(j)} - M z_j \leq 0, & \sum_{j=1}^{n} w_j = 1, \\
& \quad z_j \in \{0, 1\}, & w_j \geq 0 \text{ for all } j \in [n].
\end{align*}
\]

Similarly, the optimization problem (13) for \( \text{DQ}_\alpha^{\text{ES}} \) can be solved the empirical version of the problem (13), which is a convex program:

\[
\minimize_{\mathbf{v}} \sum_{j=1}^{N} \max \left\{ \mathbf{v}^\top (\mathbf{X}^{(j)} - \hat{\mathbf{x}}_{\alpha}^{\text{ES}}) + 1, 0 \right\} \text{ over } \mathbf{v} \in \mathbb{R}_+,
\]
where $\hat{x}_{\alpha}^{ES}$ is the empirical estimator of $x_{\alpha}^{ES}$ based on sample $X^{(1)}, \ldots, X^{(N)}$. Both problems (15) and (16) can be efficiently solved by modern optimization programs, such as CVX programming (see e.g., Matmoura and Penev (2013)).

Additional linear constraints, such as those on budget or expected return, can be easily included in (12)-(16), and the corresponding optimization problems can be solved similarly.

Tie-breaking needs to be addressed when working with (14) since its objective function takes integer values. In dynamic portfolio selection, it is desirable to avoid adjusting positions too drastically or frequently. Therefore, in the real-data analysis in Section 7.3, among tied optimizers, we pick the closest one (in $L^1$-norm $\| \cdot \|_1$ on $\mathbb{R}^n$) to a given benchmark $w_0$, the portfolio weight of the previous trading period. With this tie-breaking rule, we solve

$$
\text{minimize} \quad \| w - w_0 \|_1 \quad \text{over} \quad w \in \Delta_n \quad \text{subject to} \quad \sum_{j=1}^{N} I_{\{w^\top y^{(j)} > 0\}} \leq n^*,
$$

where $n^*$ is the optimum of (14). A tie-breaking for (16) may need to be addressed similarly since (16) is not strictly convex.

7 Numerical illustrations

To illustrate the performance of DQ, we collect the stocks' historical data from Yahoo Finance and conduct three sets of numerical experiments based on financial data. We use the period from January 3, 2012, to December 31, 2021, with a total of 2518 observations of daily losses and 500 trading days for the initial training. In Section 7.1, we first compare DQs and DRs based on VaR and ES. In Section 7.2, we calculate the values of $DQ_{\alpha}^{VaR}$ and $DQ_{\alpha}^{ES}$ under different selections of stocks. Finally, we construct portfolios by minimizing $DQ_{\alpha}^{VaR}$, $DQ_{\alpha}^{ES}$ and $DR^{SD}$ and by the mean-variance criterion in Section 7.3.

7.1 Comparing DQ and DR

We first identify the largest stock in each of the S&P 500 sectors ranked by market cap in 2012. Among these stocks, we select the 5 largest stocks (XOM from ENR, AAPL from IT, BRK/B from FINL, WMT from CONS, and GE from INDU) to build our portfolio. We compute $DQ_{\alpha}^{VaR}$, $DQ_{\alpha}^{ES}$, $DR_{\alpha}^{VaR}$, and $DR_{\alpha}^{ES}$ on each day using the empirical distribution in a rolling window of 500 days, where we set $\alpha = 0.05$.

Figure 2 shows that the values of DQ and DR are between 0 and 1. This corresponds to the observation in Section 4.2 that $DQ_{\alpha} < 1$ is equivalent to $DR_{\alpha} < 1$. DQ has a similar temporal pattern to DR in the above period of time, with a large jump when COVID-19 exploded, which is more visible for DQ than for DR. We remind the reader that DQ and DR are not meant to be
compared on the same scale, and hence the fact that DQ has a larger range than DR should be taken lightly. We also note that the values of $DQ^\text{VaR}_0$ are in discrete grids. This is because the empirical distribution function takes value in multiples of $1/N$ there $N$ is the sample size (500 in this experiment) and hence $DQ^\text{VaR}_0$ takes the values $k/(N\alpha)$ for an integer $k$; see (5). If a smooth curve is preferred, then one can employ a smoothed VaR through linear interpolation. This is a standard technique for handling VaR; see McNeil et al. (2015, Section 9.2.6) and Li and Wang (2022, Remark 8 and Appendix B).

### 7.2 DQ for different portfolios

In this section, we fix $\alpha = 0.05$ and calculate the values of $DQ^\text{VaR}_0$ and $DQ^\text{ES}_0$ under different portfolio compositions of stocks. We consider portfolios with the following stock compositions:

(A) the two largest stocks from each of the 10 different sectors of S&P 500;

(B) the largest stock from each of 5 different sectors of S&P 500 (as in Section 7.1);

(C) the 5 largest stocks, AAPL, MSFT, IBM, GOOGL and ORCL, from the Information Technology (IT) sector;

(D) the 5 largest stocks, BRK/B, WFC, JPM, C and BAC, from the Financials (FINL) sector.

We make a few observations from Figure 3. Both $DQ^\text{VaR}_0$ and $DQ^\text{ES}_0$ provide similar comparative results. The order $(A) \leq (B) \leq (C) \leq (D)$ is consistent with our intuition.\(^{15}\) First, portfolio (A) of 20 stocks has the strongest diversification effect among the four compositions. Second,

\(^{15}\)The observations here are consistent with those from applying DR$^{SD}$ (which is also a DQ) in the same setting; see Appendix F.
portfolio (B) across 5 sectors has stronger diversification than (C) and (D) within one sector. Third, portfolio (C) of 5 stocks within the IT sector has a stronger diversification than portfolio (D) of 5 stocks within the FINL sector, consistent with the fact that the stocks in the IT sector are less correlated. Moreover, DQ$_{VaR}^\alpha$ for the FINL sector is larger than 1 during some period of time, which means that there is no diversification benefit if risk is evaluated by VaR. All DQ curves based on ES show a large up-ward jump around the COVID-19 outbreak; such a jump also exists for curves based on VaR but it is less pronounced.

7.3 Optimal diversified portfolios

In this section, we fix $\alpha = 0.1$ and build portfolios via DQ$_{VaR}^\alpha$, DQ$_{ES}^\alpha$, DR$^{SD}$, and the mean-variance criterion in the Markowitz (1952) model. The optimal portfolio problems using DR$^{SD}$ and the Markowitz model are well studied in literature; see e.g. Choueifaty and Coignard (2008). We compare these portfolio wealth with the equal weighted (EW) portfolio and the simple buy-and-hold (BH) portfolio. For an analysis on the EW strategy, see DeMiguel et al. (2009).

We apply the algorithms in Proposition 5 to optimize DQ$_{VaR}^\alpha$ and DQ$_{ES}^\alpha$, which are extremely fast. A tie-breaking is addressed for each objective as in (17). Minimization of DR$^{SD}$ and the Markowitz model can be solved by existing algorithms. The initial wealth is set to 1, and the risk-free rate is $r = 2.84\%$, which is the 10-year yield of the US treasury bill in Jan 2014. The target annual expected return for the Markowitz portfolio is set to 10%. We optimize the portfolio weights in each month with a rolling window of 500 days. That is, in each month,

16One may try other portfolio criteria other than mean-variance. For instance, Levy and Levy (2004) found that portfolio strategies based on prospect theory perform similarly to the mean-variance strategies.
roughly 21 trading days, starting from January 2, 2014, we use the preceding 500 trading days to compute the optimal portfolio weights using the method described above. The portfolio is rebalanced every month. We choose the 4 largest stocks from each of the 10 different sectors of S&P 500 ranked by market cap in 2012 as the portfolio compositions (40 stocks in total). The portfolio performance is reported in Figure 4 with some summary statistics in Table 3.

From these results, we can see that the portfolio optimization strategies based on minimizing DQ perform on par with other strategies such as the EW, BH strategies and those by minimizing DR$^{SD}$ or mean-variance. We remark that it is not our intention to analyze which diversification strategy performs the best, which is a challenging question that needs a separate study; also, we do not suggest diversification should or should not be optimized in practice. The empirical results here are presented to illustrate how our proposed diversification indices work in the context of portfolio selection. More empirical results with some other datasets and portfolio strategies are given in Appendix F, and the results show similar observations.

Figure 4. Wealth processes for portfolios, 40 stocks, Jan 2014 - Dec 2021

Table 3. Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021

|     | $\text{DQ}_{\alpha}^{\text{VaR}}$ | $\text{DQ}_{\alpha}^{\text{ES}}$ | $\text{DR}^{SD}$ | Markowitz | EW | BH |
|-----|---------------------------------|---------------------------------|------------------|-----------|----|----|
| AR  | 12.56                           | 14.59                           | 14.36            | 7.93      | 11.91 | 12.88 |
| AV  | 14.64                           | 15.74                           | 14.99            | 12.98     | 15.92 | 14.34 |
| SR  | 66.40                           | 74.66                           | 76.85            | 39.22     | 56.95 | 70.02 |
8 Concluding remarks

In this paper, we propose putting forward six axioms to jointly characterize a new index of diversification. The new diversification index DQ has favourable features both theoretically and practically, and it is contrasted with its competitors, in particular DR. At a high level, because of the symmetry in Figure 1 (see also (8)), we expect both DQ and DR to have advantages and disadvantages in different applications, and none should fully dominate the other. Nevertheless, we find many attractive features of DQ through the results in this paper, which suggest that DQ may be a better choice in many situations.

We summarize these features below. Certainly, some of these features are shared by DR, but many are not. (i) DQ defined on a class of MCP risk measures can be uniquely characterized by six intuitive axioms (Theorem 1), laying an axiomatic foundation for using DQ as a diversification index. (ii) DQ further satisfies many properties for common risk measures (Propositions 1-3). These properties are not shared by the corresponding DR. (iii) DQ is intuitive and interpretable with respect to dependence and common perceptions of diversification (Theorem 2). (iv) DQ can be applied to a wide range of risk measures, such as the regulatory risk measures VaR and ES, as well as entropic risk measures. In cases of VaR and ES, DQ has simple formulas and convenient properties (Theorem 3 and Proposition 4). (v) DQ based on a class of convex risk measures is quasi-convex in portfolio weights (Theorem 4). (vi) Portfolio optimization of DQs based on VaR and ES can be computed very efficiently (Proposition 5). (vii) DQ can be easily applied to real data and it produces results that are consistent with our usual perception of diversification (Section 7).

We also mention a few interesting questions on DQ, which call for thorough future study. (i) DQ is defined through a class of risk measures. It would be interesting to formulate DQ using expected utility or behavioral decision models, to analyze the decision-theoretic implications of DQ. For instance, DQ based on entropic risk measures can be equivalently formulated using exponential utility functions. Alternatively, one may also build DQ directly from acceptability indices (see Remark 1). (ii) To compute DQ, one needs to invert the decreasing function \( \beta \mapsto \rho_\beta(\sum_{i=1}^n X_i) \). In the case of VaR and ES, the formula for this inversion is simple (Theorem 3). For more complicated classes of risk measures, this computation may be complicated and requires detailed analysis. (iii) For general distributions and risk measures other than VaR and ES, finding analytical formulas or efficient algorithms for optimal diversification using either DQ or DR is a challenging task, and it is unclear which diversification index is easier to work within a specific application.

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Technical appendices

Outline of the appendices

We organize the technical appendices as follows. In Appendix A, we provide an axiomatization result of DQ via the relation of preference. The proofs of the main results, Theorems 1–4, are presented in Appendix B. Additional results, discussions, and proofs of propositions are presented in Appendices C (for Section 2), D (for Section 4), E (for Section 5), F (for Section 6). Finally, in Appendix G, we present other examples for the optimal portfolio problem which complement the empirical studies in Section 7.3.

A Axiomatization of DQ using preferences

In this section, we provide an axiomatization of DQ as in Theorem 1 without specifying a risk measure φ. We first define the preference of a decision maker over risks. A preference relation ≥ is defined by a non-trivial total preorder\(^{17}\) on \(\mathcal{X}\). As usual, \(>\) and \(\simeq\) correspond to the antisymmetric and equivalence relations, respectively. On the preference \(\geq\) of risk, the relation \(X \geq Y\) means the agent prefers \(X\) to \(Y\) for any \(X, Y \in \mathcal{X}\). We will use the following axioms.

[A1] \(X \leq Y \implies X \geq Y\).

[A2] \(X \geq Y \implies X + c \geq Y + c\) for any \(c \in \mathbb{R}\).

[A3] \(X \geq Y \implies \lambda X \geq \lambda Y\) for any \(\lambda > 0\).

[A4] For any \(X \in \mathcal{X}\), there exists \(c \in \mathbb{R}\) such that \(X \simeq c\).

The four axioms are rather standard and we only briefly explain them. The axiom [A1] means that the agent always prefers a smaller loss. The axioms [A2] and [A3] mean that if the agent prefers one random loss over another, then this is preserved under any strictly increasing linear transformations. The axiom [A4] implies that any random losses can be equally favourable as a constant loss which is commonly referred to as a certainty equivalence.

A numerical representation of a preference \(\geq\) is a mapping \(\phi : \mathcal{X} \to \mathbb{R}\), such that \(X \geq Y \iff \phi(X) \leq \phi(Y)\) for all \(X, Y \in \mathcal{X}\). In other words, \(\geq\) is the preference of an agent favouring less risk evaluated via \(\phi\). There is a simple relationship between preferences satisfying [A1]-[A4] and MCP risk measures.

\(^{17}\)A preorder is a binary relation on \(\mathcal{X}\), which is reflexive and transitive. A binary relation \(\geq\) is reflexive if \(X \geq X\) for all \(X \in \mathcal{X}\), and transitive if \(X \geq Y\) and \(Y \geq Z\) imply \(X \geq Z\). A non-trivial total preorder is a preorder which in addition is complete, that is, \(X \geq Y\) or \(Y \geq X\) for all \(X, Y \in \mathcal{X}\), and there exist at least two alternatives \(X, Y\) such that \(X\) is preferred over \(Y\) strictly.
Lemma EC.1. A preference satisfies [A1]–[A4] if and only if it can be represented by an MCP risk measure $\phi$.

Proof. The “if” statement is straightforward to check, and we will show the “only if” statement. The preference $\succeq$ can be represented by a risk measure $\phi$ through $X \succeq Y \iff \phi(X) \leq \phi(Y)$ for all $X, Y \in \mathcal{X}$ since $\succeq$ is separable by [A1] and [A4]; see Debreu (1954) and Drapeau and Kupper (2013). If $\phi(0) = \phi(1)$, then by using [A1]–[A3], the preference $\succeq$ is trivial, contradicting our assumption on $\succeq$. Hence, using [A1], $\phi(0) < \phi(1)$, we can further let $\phi(0) = 0$ and $\phi(1) = 1$. It is then straightforward to verify that $\phi$ is MCP from [A1]–[A3].

Similarly to Section 3, but with the preference $\succeq$ replacing the risk measure $\phi$, we denote by $X \supseteq Y$ if $X_i \simeq Y_i$ for each $i \in [n]$, by $X \succeq Y$ if $X_i \succeq Y_i$ for each $i \in [n]$, and by $X \supset Y$ if $X_i > Y_i$ for each $i \in [n]$. With this new formulation and everything else unchanged, the axioms of rationality, normalization and continuity are now denoted by $[R]_{\succeq}$, $[N]_{\succeq}$ and $[C]_{\succeq}$.

Proposition EC.1. A diversification index $D : \mathcal{X}^n \to \mathbb{R}$ satisfies $[+], [LI], [SI], [R]_{\succeq}, [N]_{\succeq}$ and $[C]_{\succeq}$ for some preference $\succeq$ satisfying [A1]–[A4] if and only if $D$ is $DQ^\phi_{\alpha}$ for some decreasing families $\rho$ of MCP risk measures. Moreover, in both directions of the above equivalence, it can be required that $\rho_\alpha$ represents $\succeq$.

Proof. The proof follows from Theorem 1 by noting that Lemma EC.1 allows us to convert between a preference $\succeq$ satisfying [A1]–[A4] and an MCP risk measure $\phi$.

B Proofs of Theorems 1–4

Proof of Theorem 1. For $X \in \mathcal{X}^n$ and a risk measure $\phi : \mathcal{X} \to \mathbb{R}$, denote by $S(X) = \sum_{i=1}^{n} X_i$ and $X_\phi = (X_1 - \phi(X_1), \ldots, X_n - \phi(X_n))$.

We first verify the “if” statement. By definition, it is straightforward to see that $DQ^\phi_{\alpha}$ is non-negative. For all $\alpha \in I$, if $\rho_\alpha$ satisfies [CA] and [PH], then the properties of [LI] and [SI] hold for $DQ^\phi_{\alpha}$. To show [LI], for $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we have

$$DQ^\phi_{\alpha}(X + c) = \frac{1}{\alpha} \inf_{\beta \in I} \left\{ \beta \in I : \rho_{\beta} \left( \sum_{i=1}^{n} (X_i + c_i) \right) \leq \sum_{i=1}^{n} \rho_\alpha(X_i + c_i) \right\}$$

$$= \frac{1}{\alpha} \inf_{\beta \in I} \left\{ \beta \in I : \rho_{\beta} \left( \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} c_i \right) \leq \sum_{i=1}^{n} \rho_\alpha(X_i) + \sum_{i=1}^{n} c_i \right\} = DQ^\phi_{\alpha}(X).$$

Hence, $DQ^\phi_{\alpha}$ satisfy [LI]. For $\lambda > 0$, we have

$$DQ^\phi_{\alpha}(\lambda X) = \frac{1}{\alpha} \inf_{\beta \in I} \left\{ \beta \in I : \rho_{\beta} \left( \sum_{i=1}^{n} \lambda X_i \right) \leq \sum_{i=1}^{n} \rho_\alpha(\lambda X_i) \right\}$$

$$= \frac{1}{\alpha} \inf_{\beta \in I} \left\{ \beta \in I : \lambda \rho_{\beta} \left( \sum_{i=1}^{n} X_i \right) \leq \lambda \sum_{i=1}^{n} \rho_\alpha(X_i) \right\} = DQ^\phi_{\alpha}(X).$$
Thus, $DQ^\phi_{\alpha}$ satisfies $[S\text{I}]$.

To show $[R]_{\phi}$, for $X, Y \in \mathcal{X}^n$ such that $X \overset{m}{=} Y$ and $\sum_{i=1}^n X_i \leq \sum_{i=1}^n Y_i$, we have $\sum_{i=1}^n \rho_\alpha(X_i) = \sum_{i=1}^n \rho_\alpha(Y_i)$ and $\rho_\beta(\sum_{i=1}^n X_i) \leq \rho_\beta(\sum_{i=1}^n Y_i)$ for all $\beta \in I$. Hence,

$$\begin{align*}
DQ^\phi_{\alpha}(X) &= \frac{1}{\alpha} \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\alpha(X_i) \right\} \\
&\leq \frac{1}{\alpha} \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n Y_i \right) \leq \sum_{i=1}^n \rho_\alpha(Y_i) \right\} = DQ^\phi_{\alpha}(Y).
\end{align*}$$

To show $[C]_{\phi}$, for $X \in \mathcal{X}^n$, we have $a_X^\phi = \inf \{ \beta \in I : \rho_\beta(S(X_{\rho_n})) \leq 0 \}$. If $a_X^\phi = 0$, it is clear that $DQ^\phi_{\alpha}(X) = 0$ and $[C]_{\phi}$ holds as $DQ^\phi_{\alpha}(Y) \geq 0$ for any $Y \in \mathcal{X}^n$. Now, we assume that $a_X^\phi > 0$. For any $0 \leq \beta < a_X^\phi$, we have $\rho_\beta(S(X_{\rho_n})) > 0$. Since $Y_k \overset{m}{=} X$ for each $k$ and $(S(X) - S(Y_k))_+ \overset{L^\infty}{\rightarrow} 0$ as $k \rightarrow \infty$, for any $\varepsilon > 0$, there exists $K$ such that $S(X_{\rho_n}) - S(Y_k_{\rho_n}) \leq \varepsilon$ for all $k \geq K$. For any $0 < \delta < a_X^\phi$, let $0 < \varepsilon < \rho_\beta(s(S(X_{\rho_n})))$. It is clear that $0 < \varepsilon < \rho_\beta(S(X_{\rho_n}))$ for all $0 < \beta < a_X^\phi - \delta$. Hence, for all $0 < \beta < a_X^\phi - \delta$, there exists $K$ such that $0 < \rho_\beta(S(X_{\rho_n})) \leq \rho_\beta(S(Y_k_{\rho_n}))$ for all $k \geq K$, which implies $a_Y^\phi \geq a_X^\phi - \delta$. Therefore, $(DQ^\phi_{\alpha}(X) - DQ^\phi_{\alpha}(Y_k))_+ \rightarrow 0$.

To show $[N]_{\phi}$, it is straightforward that $DQ^\phi_{\alpha}(0) = 0$. Let $X = (X, \ldots, X)$ for any $X \in \mathcal{X}$. We have

$$DQ^\phi_{\alpha}(X) = \frac{1}{\alpha} \inf \{ \beta \in I : \rho_\beta(nX) \leq n\rho_\alpha(X) \} \leq \frac{\alpha}{\alpha} = 1.$$

If $Y \overset{m}{=} (X, \ldots, X)$ and $\sum_{i=1}^n Y_i \geq nX$, then $\sum_{i=1}^n \rho_\alpha(Y_i) < n\rho_\alpha(X) \leq \rho_\alpha(\sum_{i=1}^n Y_i)$. Hence, $DQ^\phi_{\alpha}(Y) \geq 1$.

Next, we will show the “only if” statement. Assume that $D : \mathcal{X}^n \rightarrow \mathbb{R}$ satisfies $[+]$, $[L\text{I}]$, $[S\text{I}]$, $[R]_{\phi}$, $[C]_{\phi}$ and $[N]_{\phi}$. Note that $X_{\phi} \overset{m}{=} Y_{\phi}$ for all $X, Y \in \mathcal{X}^n$ since $\phi(X - \phi(X)) = 0$ for any $X \in \mathcal{X}$. Hence, by using $[R]_{\phi}$, we know that $S(X_{\phi}) = S(Y_{\phi})$ implies $D(X_{\phi}) = D(Y_{\phi})$. Using $[L\text{I}]$, we have $D(X) = D(Y)$ if $S(X_{\phi}) = S(Y_{\phi})$. This means that $D(X)$ is determined by $S(X_{\phi})$. Define the mapping

$$R : \mathcal{X} \rightarrow [0, \infty], \quad R(X) = \inf \{ D(X) : X \leq S(X_{\phi}), \ X \in \mathcal{X}^n \},$$

with the convention $\inf \emptyset = \infty$. Next, we will verify several properties of $R$.

(a) $R(S(X_{\phi})) = D(X)$ for $X \in \mathcal{X}^n$. The inequality $R(S(X_{\phi})) \leq D(X)$ follows directly from (EC.1). To see the opposite direction of the inequality, suppose $R(S(X_{\phi})) < D(X)$. By (EC.1), there exists $Y \in \mathcal{X}^n$ such that $D(Y) < D(X)$ and $S(X_{\phi}) \leq S(Y_{\phi})$. This conflicts with $[R]_{\phi}$ of $D$.

(b) $R(\lambda X) = R(X)$ for all $\lambda > 0$ and $X \in \mathcal{X}$. This follows directly from (EC.1), $[S\text{I}]$ of $D$ and positive homogeneity of $\phi$ which gives $(\lambda X)_{\phi} = \lambda X_{\phi}$.
(c) \( R(X) \leq R(Y) \) for all \( X, Y \in \mathcal{X} \) with \( X \leq Y \). This follows directly from (EC.1).

(d) \( R(0) = 0 \). This follows directly from (EC.1) and \( D(0) = 0 \) in \([N]_{\phi}\).

(e) \( \lim_{c \downarrow 0} R(S(X_\phi) - c) = R(S(X_\phi)) \) for \( X \in \mathcal{X}^n \). Let \( X = S(X_\phi) \). By (c), we have \( \lim_{c \downarrow 0} R(X-c) \leq R(X) \). Assume \( \lim_{c \downarrow 0} R(X-c) < R(X) \); that is, there exists \( \delta > 0 \) such that \( R(X-c) < R(X) - \delta \) for all \( c > 0 \). Let \( c_k = 1/k \) for \( k \in \mathbb{N} \). By (EC.1), there exists a sequence \( \{Y^k\}_{k \in \mathbb{N}} \) such that \( X - c_k \leq S(Y^k_\phi) \) and \( D(Y^k_\phi) < D(X_\phi) - \delta \). For \( \{Y^k_\phi\}_{k \in \mathbb{N}} \), we have \( 0 \leq (S(X_\phi) - S(Y^k_\phi))_+ \leq c_k \), which implies \( (S(X_\phi) - S(Y^k_\phi))_+ \xrightarrow{L^\infty} 0 \) as \( k \to \infty \). By \([C]_{\phi}\), we have \( (D(X_\phi) - D(Y^k_\phi))_+ \to 0 \); that is, for any \( \delta > 0 \), there exists \( K \in \mathbb{N} \) such that \( D(X_\phi) - \delta \leq D(Y^k_\phi) \) for all \( k > K \), which is a contradiction. Therefore, we have \( \lim_{c \downarrow 0} R(S(X_\phi) - c) = R(S(X_\phi)) \).

Let \( I = (0, \infty) \). For each \( \beta \in (0, \infty) \), let \( A_\beta = \{X \in \mathcal{X} : R(X) \leq \beta\} \). Since \( R \) is monotone, \( A_\beta \) is a decreasing set; i.e., \( X \in A_\beta \) implies \( Y \in A_\beta \) for all \( Y \leq X \). Moreover, \( A_\beta \) is conic; i.e., \( X \in A_\beta \) implies \( \lambda X \in A_\beta \) for all \( \lambda > 0 \). Moreover, we have \( A_\beta \subseteq A_\gamma \) for \( \beta \leq \gamma \), and \( A_\beta \neq \emptyset \) since \( 0 \in A_\phi \).

Let \( \rho_\beta(X) = \inf\{m \in \mathbb{R} : X - m \in A_\beta\} \) for \( \beta \in I \). Since \( \rho_\beta \) is defined via a conic acceptance set, \( (\rho_\beta)_{\beta \in I} \) is a class of MCP risk measures; see Föllmer and Schied (2016). It is also clear that \( \rho_\beta \) is decreasing in \( \beta \). Note that \( X \in A_\beta \) implies \( \rho_\beta(X) \leq 0 \). Hence,

\[
R(X) = \inf\{\beta \in I : R(X) \leq \beta\} = \inf\{\beta \in I : X \in A_\beta\} \geq \inf\{\beta \in I : \rho_\beta(X) \leq 0\}.
\]

For \( X \in \{S(X_\phi) : X \in \mathcal{X}^n\} \), using (e), we have \( R(X-m) \leq \beta \) for all \( m > 0 \) implies \( R(X) \leq \beta \). Then we have

\[
\inf\{\beta \in I : \rho_\beta(X) \leq 0\} = \inf\{\beta \in I : R(X-m) \leq \beta \text{ for all } m > 0\} \geq \inf\{\beta \in I : R(X) \leq \beta\}.
\]

Therefore, \( \inf\{\beta \in I : \rho_\beta(S(X_\phi)) \leq 0\} = \inf\{\beta \in I : R(S(X_\phi)) \leq \beta\} = R(S(X_\phi)) \) for all \( X \in \mathcal{X}^n \). Using (a), we get, for all \( X \in \mathcal{X}^n \),

\[
D(X) = R(S(X_\phi)) = \inf\left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n X_i\right) \leq \sum_{i=1}^n \phi(X_i)\right\}.
\]

Let \( X = (X_1, \ldots, X_n) \). Together with [PH] of \( \rho_\beta \), \( D(X) \leq 1 \) implies that

\[
D(X, \ldots, X) = \inf\{\beta \in I : \rho_\beta(X) \leq \phi(X)\} \leq 1
\]

and it is equivalent to \( \rho_\beta(X) \leq \phi(X) \) for \( \beta > 1 \). For any \( \varepsilon > 0 \) and \( X \in \mathcal{X} \), we have \( \rho_{1-\varepsilon}(X) = \inf\{m \in \mathbb{R} : R(X-m) \leq 1-\varepsilon\} \). For any \( m < \phi(X) \), let \( Z = (Y_1 - \phi(Y_1) + m/n, \ldots, Y_n - \phi(Y_n) + m/n) \) such that \( \sum_{i=1}^n Y_i - \sum_{i=1}^n \phi(Y_i) \geq X - m \). For \( Z \), we have \( \sum_{i=1}^n Z_i = \sum_{i=1}^n Y_i - \sum_{i=1}^n \phi(Y_i) + m \geq X \) and \((Z_1, \ldots, Z_n) \geq \geq (1/nX, \ldots, 1/nX) \). By [LI] and \([N]_{\phi} \), we have \( D(Y) = \)
\( D(\mathbf{Z}) \geq 1 \), which implies that \( R(X - m) \geq 1 \) and \( \rho_{1 - \varepsilon}(X) \geq \phi(X) \). Hence, the value of \( D \) does not change by replacing \( \rho_1 \) with \( \phi \). Therefore,

\[
D^\rho_\alpha(X) = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \phi(X_i) \right\} = D(X)
\]

for any \( X \in \mathcal{X}^n \). For any \( \alpha \in I \), let \( \pi_\beta = \rho_{\beta/\alpha} \). We can show that

\[
D^\rho_\alpha(X) = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\beta(X_i) \right\} = \inf \left\{ \beta \in I : \pi_\beta(X) \leq \sum_{i=1}^n \pi_\beta(X_i) \right\} = \frac{1}{\alpha} \inf \left\{ \gamma \in I : \pi_\gamma \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \pi_\gamma(X_i) \right\} = D^\rho_\alpha(X)
\]

and \( \pi_\alpha = \rho_1 = \phi \). □

**Proof of Theorem 2.** (i) As \( \sum_{i=1}^n X_i \leq \sum_{i=1}^n \rho_\alpha(X_i) \) a.s. and \( \rho_0 \leq \text{ess-sup} \), it is clear that \( \alpha^* = 0 \) in (4), which implies \( D^\rho_\alpha(X) = 0 \). Conversely, if \( D^\rho_\alpha(X) = 0 \), then \( \alpha^* = 0 \). By definition of \( \rho_0 \) and \( D^\rho_\alpha \), this implies \( \rho_0(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n \rho_\alpha(X_i) \), and hence \( \sum_{i=1}^n X_i \leq \sum_{i=1}^n \rho_\alpha(X_i) \) a.s.

(ii) We first show the “only if” statement. As \( \rho \) is left continuous and non-flat from the left at \( (\alpha, \sum_{i=1}^n X_i) \) and \( \sum_{i=1}^n \alpha(X_i) - \alpha(\sum_{i=1}^n X_i) > 0 \), there exists \( \delta > 0 \) such that

\[
\rho_\beta \left( \sum_{i=1}^n X_i \right) - \rho_\alpha \left( \sum_{i=1}^n X_i \right) < \sum_{i=1}^n \rho_\alpha(X_i) - \rho_\alpha \left( \sum_{i=1}^n X_i \right)
\]

for all \( \beta \in (\alpha - \delta, \alpha) \). Hence, we have \( \alpha^* \leq \alpha - \delta < \alpha \), which leads to \( D^\rho_\alpha(X) < 1 \).

Next, we show the “if” statement. As \( D^\rho_\alpha(X) < 1 \), we have \( \alpha > \alpha^* \). By (4), there exists \( \beta \in (\alpha^*, \alpha) \) such that

\[
\sum_{i=1}^n \rho_\alpha(X_i) \geq \rho_\beta \left( \sum_{i=1}^n X_i \right).
\]

Because \( \rho \) is non-flat from the left at \( (\alpha, \sum_{i=1}^n X_i) \), we have

\[
\sum_{i=1}^n \rho_\alpha(X_i) \geq \rho_\beta \left( \sum_{i=1}^n X_i \right) > \rho_\alpha \left( \sum_{i=1}^n X_i \right).
\]

(iii) If \( \rho_\alpha \) satisfies [PH], for \( X = (\lambda_1 X, \ldots, \lambda_n X) \) where \( \lambda_1, \ldots, \lambda_n \geq 0 \), we have

\[
\alpha^* = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n \lambda_i X \right) \leq \sum_{i=1}^n \lambda_i \rho_\alpha(X) \right\}.
\]

It is clear that \( \rho_\alpha \left( \sum_{i=1}^n \lambda_i X \right) = \left( \sum_{i=1}^n \lambda_i \right) \rho_\alpha(X) \). Together with the non-flat condition and \( \rho_\beta \left( \sum_{i=1}^n \lambda_i X \right) > \sum_{i=1}^n \lambda_i \rho_\alpha(X) \) for all \( \beta < \alpha \), we have \( \alpha^* = \alpha \), and thus \( D^\rho_\alpha(X) = 1 \).

(iv) If \( \rho_\alpha \) is comonotonic-additive and \( X \) is comonotonic, then

\[
\alpha^* = \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^n X_i \right) \leq \sum_{i=1}^n \rho_\alpha(X_i) = \rho_\alpha \left( \sum_{i=1}^n X_i \right) \right\},
\]

which, together with the non-flat condition, implies that \( \alpha^* = \alpha \), and thus \( D^\rho_\alpha(X) = 1 \). □
Proof of Theorem 3. We first show (5). For any $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\alpha \in (0, 1)$, by Lemma 1 of Guan et al. (2022), $\mathbb{P}(X > t) \leq \alpha$ if and only if $\text{VaR}_\alpha(X) \leq t$. Hence,

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} \text{VaR}_\alpha(X_i)\right) = \inf \left\{ \beta \in (0, 1) : \mathbb{P}\left(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} \text{VaR}_\alpha(X_i)\right) \leq \beta \right\}$$

$$= \inf \left\{ \beta \in (0, 1) : \text{VaR}_\beta\left(\sum_{i=1}^{n} X_i\right) \leq \sum_{i=1}^{n} \text{VaR}_\alpha(X_i) \right\},$$

and (5) follows. The formula (6) for $\text{DQ}^{\text{ES}}_\alpha$ follows from a similar argument to (5) by noting that $Y$ is a random variable with $\text{VaR}_\alpha(Y) = \text{ES}_\alpha(\sum_{i=1}^{n} X_i)$.

Next, we show the last statement of the theorem. If $\mathbb{P}(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} \text{ES}_\alpha(X_i)) = 0$, then $\text{DQ}^{\text{ES}}_\alpha(X) = 0$ by Theorem 2 (i).

Below, we assume $\mathbb{P}(\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} \text{ES}_\alpha(X_i)) > 0$. The formula (7) is very similar to Proposition 2.2 of Mafusalov and Uryasev (2018), where we additionally show that the minimizer to (7) is not 0. Here we present a self-contained proof based on the well-known formula of ES (Rockafellar and Uryasev (2002)),

$$\text{ES}_\beta(X) = \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{\beta} \mathbb{E}\left[ (X - t)^+ \right]\right\}, \text{ for } X \in \mathcal{X} \text{ and } \beta \in (0, 1).$$

Using this formula, we obtain, by writing $X'_i = X_i - \text{ES}_\alpha(X_i)$ for $i \in [n]$,

$$\text{DQ}^{\text{ES}}_\alpha(X) = \frac{1}{\alpha} \inf \left\{ \beta \in (0, 1) : \text{ES}_\beta\left(\sum_{i=1}^{n} X_i\right) - \sum_{i=1}^{n} \text{ES}_\alpha(X_i) \leq 0 \right\}$$

$$= \frac{1}{\alpha} \inf \left\{ \beta \in (0, 1) : \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{\beta} \mathbb{E}\left[ \left(\sum_{i=1}^{n} X'_i - t \right)^+ \right]\right\} \leq 0 \right\}$$

$$= \frac{1}{\alpha} \inf \left\{ \beta \in (0, 1) : \frac{1}{\beta} \mathbb{E}\left[ \left(\sum_{i=1}^{n} X'_i - t \right)^+ \right] \leq -t \text{ for some } t \in \mathbb{R} \right\}$$

$$= \frac{1}{\alpha} \inf \left\{ \beta \in (0, 1) : \mathbb{E}\left[ \left( r \sum_{i=1}^{n} X'_i + 1 \right)^+ \right] \leq \beta \text{ for some } r \in (0, \infty) \right\}$$

$$= \frac{1}{\alpha} \inf_{r \in (0, \infty)} \mathbb{E}\left[ \left( r \sum_{i=1}^{n} X'_i + 1 \right)^+ \right].$$

Let $f : [0, \infty) \to [0, \infty)$, $r \mapsto \mathbb{E}[r \sum_{i=1}^{n} X'_i + 1]^+]$. It is clear that $f(0) = 1$. Moreover,

$$f(r) \geq r \mathbb{E}\left[(X'_i)^+\right] \to \infty \text{ as } r \to \infty.$$
of \( \rho \) implies convexity of \( w \mapsto r_\beta(w) \). Hence, for the portfolio weight \( \lambda w + (1 - \lambda)v \in \Delta_n \), \( DQ \) based on \( \rho \) at level \( \alpha \in (0, 1) \) is given by

\[
DQ_\alpha^\rho((\lambda w + (1 - \lambda)v) \odot X) = \frac{1}{\alpha} \inf \{ \beta \in I : r_\beta(\lambda w + (1 - \lambda)v) \leq 0 \}
\leq \frac{1}{\alpha} \inf \{ \beta \in I : \lambda r_\beta(w) + (1 - \lambda)r_\beta(v) \leq 0 \}
\leq \frac{1}{\alpha} \max \{ \inf \{ \beta \in I : r_\beta(w) \leq 0 \} ; \inf \{ \beta \in I : r_\beta(v) \leq 0 \} \}
= \max \{ DQ_\alpha^\rho(w \odot X) ; DQ_\alpha^\rho(v \odot X) \},
\]

which gives us quasi-convexity.

\[\square\]

C Additional results for Section 2

In this appendix, we present an impossibility result showing a conflicting nature of the three natural properties \([+], [LI] \) and \([SI] \) for diversification indices defined via risk measures. As mentioned in Section 2, the most commonly used diversification indices depend on \( X \) through its values assessed by some risk measure \( \phi \). That is, given a risk measures \( \phi \) and a portfolio \( X \), the diversification index can be written as

\[
D(X) = R\left(\phi\left(\sum_{i=1}^{n} X_i\right), \phi(X_1), \ldots, \phi(X_n)\right)
\]

for some function \( R : \mathbb{R}^{n+1} \to \mathbb{R} \). (EC.2)

We will say that \( D \) is \( \phi \)-determined if (EC.2) holds. Often, one may further choose \( R \) so that \( D(X) \) decreases in \( \phi(\sum_{i=1}^{n} X_i) \) and increases in \( \phi(X_i) \) for each \( i \in [n] \), for a proper interpretation of measuring diversification.

We show that a diversification index based on an MCP risk measure, such as VaR or ES satisfying all three properties \([+], [LI] \) and \([SI] \) can take at most 3 different values. In this case, we will say that the diversification index \( D \) is degenerate. In fact, this result can be extended to more general properties \([PH]_\gamma \) and \([CA]_m \) with \( \gamma \in \mathbb{R} \) and \( m \in \mathbb{R} \) of the risk measure \( \phi \), with definitions given at the beginning of Section 4.

**Proposition EC.2.** Fix \( n \geq 1 \). Suppose that a risk measure \( \phi \) satisfies \([PH]_\gamma \) and \([CA]_m \) with \( \gamma \in \mathbb{R} \) and \( m \neq 0 \). A diversification index \( D \) is \( \phi \)-determined and satisfies \([+], [LI] \) and \([SI] \) if and only if for all \( X \in \mathcal{X}^n \),

\[
D(X) = C_1 I_{\{d < 0\}} + C_2 I_{\{d = 0\}} + C_3 I_{\{d > 0\}},
\]

where \( d = DB^\phi(X) = \sum_{i=1}^{n} \phi(X_i) - \phi(\sum_{i=1}^{n} X_i) \) for some \( C_1, C_2, C_3 \in \mathbb{R}_+ \cup \{\infty\} \).

We first present a lemma to prepare for the proof of Proposition EC.2.
Lemma EC.2. A function $R : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies, for all $x_0 \in \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $\lambda > 0$, (i) $R(x_0 + \sum_{i=1}^n c_i, x + c) = R(x_0, x)$ and (ii) $R(\lambda x_0, \lambda x) = R(x_0, x)$, if and only if there exist $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$R(x_0, x) = C_1 \mathbb{1}_{\{r < 0\}} + C_2 \mathbb{1}_{\{r = 0\}} + C_3 \mathbb{1}_{\{r > 0\}},$$

where $r = \sum_{i=1}^n x_i - x_0$, for all $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Proof. First, we show that $R$ in (EC.4) satisfies (i) and (ii). Assume that $r < 0$. For any $c \in \mathbb{R}^n$ and $\lambda > 0$, it is clear that $x_0 + \sum_{i=1}^n c_i < \sum_{i=1}^n (x_i + c_i)$ and $\lambda x_0 < \sum_{i=1}^n \lambda x_i$. Therefore, (i) and (ii) are satisfied. The cases of $r = 0$ and $r > 0$ follow by the same argument.

Next, we verify the “only if” part. Given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ satisfying $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, let $c = y - x$. For any $x_0 \in \mathbb{R}$, we have $\sum_{i=1}^n c_i = \sum_{i=1}^n (y_i - x_i) = 0$. Therefore,

$$R(x_0, x) = R \left( x_0 + \sum_{i=1}^n c_i, x + c \right) = R(x_0, y).$$

Thus, the value of $R(x_0, x)$ only depends on $x_0$ and $\sum_{i=1}^n x_i$. Let $R : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\tilde{R}(x_0, \sum_{i=1}^n x_i) = R(x_0, x)$. From the properties of $R$, $\tilde{R}$ satisfies $\tilde{R}(a + c, b + c) = \tilde{R}(a, b)$ for any $c \in \mathbb{R}$, and $\tilde{R}(\lambda a, \lambda b) = \tilde{R}(a, b)$ for any $\lambda > 0$. Hence, we have

$$\tilde{R}(a, b) = \tilde{R}(a - b, 0) = \tilde{R}(1, 0) \quad \text{for } a > b,$$

$$\tilde{R}(a, b) = \tilde{R}(0, b - a) = \tilde{R}(0, 1) \quad \text{for } a < b,$$

and

$$\tilde{R}(a, b) = \tilde{R}(a - b, -b) = \tilde{R}(0, 0) \quad \text{for } a = b.$$

Let $C_1 = \tilde{R}(1, 0)$, $C_2 = \tilde{R}(0, 0)$ and $C_3 = \tilde{R}(0, 1)$. We have $R(x_0, x) = \tilde{R}(x_0, \sum_{i=1}^n x_i)$, which has the form in (EC.4).

Proof of Proposition EC.2. Let us first prove sufficiency. By definition, $D$ satisfies $[+]$ and $D$ is $\phi$-determined. Next, we prove $D$ satisfies $[LI]$ and $[SI]$. Similarly to Lemma EC.2, we only prove the case $d < 0$. It is straightforward that

$$\phi \left( \sum_{i=1}^n \lambda X_i \right) = \lambda^\gamma \phi \left( \sum_{i=1}^n X_i \right) < \lambda^\gamma \sum_{i=1}^n \phi(X_i) = \sum_{i=1}^n \phi(\lambda X_i),$$

and

$$\phi \left( \sum_{i=1}^n (X_i + c_i) \right) = \phi \left( \sum_{i=1}^n X_i \right) + m \sum_{i=1}^n c_i < \sum_{i=1}^n (\phi(X_i) + mc_i) = \sum_{i=1}^n \phi(X_i + c_i).$$

Thus, we have $D(\lambda X) = C_1$ and $D(X + c) = C_1$, which completes the proof of sufficiency.
Next, we show the necessity. Define the set
\[
\mathcal{A} = \left\{ \left( \phi \left( \sum_{i=1}^{n} X_i \right), \phi(X_1), \ldots, \phi(X_n) \right) : (X_1, \ldots, X_n) \in \mathcal{X}^n \right\}.
\]
Note that \( \phi \) satisfies [PH], with \( \gamma \neq 0 \) since \([CA]_{\gamma} m \) for \( m \neq 0 \) implies \( \rho(2) \neq \rho(1) \), which in turn implies \( \gamma \neq 0 \). We always write \( x = (x_1, \ldots, x_n) \) and \( c = (c_1, \ldots, c_n) \). Consider the two operations \((x_0, x) \mapsto (x_0 + \sum_{i=1}^{n} c_i, x + c)\) for some \( c \in \mathbb{R}^n \) and \((x_0, x) \mapsto (\lambda x_0, \lambda x)\) for some \( \lambda > 0 \). Let \( r(x_0, x) = \sum_{i=1}^{n} x_i - x_0 \). By using \([CA]_{\gamma} \) and [PH] of \( \phi \), we have that (see also the proof of Lemma EC.2) the regions \( \mathcal{A}_+ := \{(x_0, x) : r(x_0, x) > 0\} \), \( \mathcal{A}_0 := \{(x_0, x) : r(x_0, x) = 0\} \) and \( \mathcal{A}_- := \{(x_0, x) : r(x_0, x) < 0\} \) are closed under the above two operations, and each of them is connected via the above two operations. Therefore, \( \mathcal{A} \) is the union of some of \( \mathcal{A}_+, \mathcal{A}_0 \) and \( \mathcal{A}_- \).

We define a function \( R : \mathbb{R}^{n+1} \to \mathbb{R} \). For \((x_0, x) \in \mathcal{A} \), let \( R(x_0, x) = D(X_1, \ldots, X_n) \), where \((X_1, \ldots, X_n)\) is any random vector such that \( x_0 = \phi(\sum_{i=1}^{n} X_i) \) and \( x = (\phi(X_1), \ldots, \phi(X_n)) \). The choice of \((X_1, \ldots, X_n)\) is irrelevant since \( D \) is \( \phi \)-determined. For \((x_0, x) \in \mathbb{R}^{n+1} \setminus \mathcal{A} \), let \( R(x_0, x) = 0 \). We will verify that \( R \) satisfies conditions (i) and (ii) in Lemma EC.2.

For \((x_0, x) \in \mathcal{A} \), there exists \( X = (X_1, \ldots, X_n) \in \mathcal{X}^n \) such that \( x_0 = \phi(\sum_{i=1}^{n} X_i) \) and \( x = (\phi(X_1), \ldots, \phi(X_n)) \). For any \( c \in \mathbb{R}^n \), using \([CA]_{\gamma} \) with \( m \neq 0 \) of \( \phi \) and [LI] of \( D \), we obtain
\[
R \left( x_0 + \sum_{i=1}^{n} c_i, x + c \right) = R \left( \phi \left( \sum_{i=1}^{n} X_i \right), \phi(X_1) + c_1, \ldots, \phi(X_n) + c_n \right)
\]
\[
= R \left( \phi \left( \sum_{i=1}^{n} \frac{c_i}{m} \right), \phi(X_1 + \frac{c_1}{m}), \ldots, \phi(X_n + \frac{c_n}{m}) \right)
\]
\[
= D \left( X + \frac{c}{m} \right) = D(X) = R(x_0, x).
\]
Using [PH], with \( \gamma \neq 0 \) of \( \phi \) and [SI] of \( D \), for any \( \lambda > 0 \), we obtain
\[
R(\lambda x_0, \lambda x) = R \left( \lambda \phi \left( \sum_{i=1}^{n} X_i \right), \lambda \phi(X_1), \ldots, \lambda \phi(X_n) \right)
\]
\[
= R \left( \phi \left( \sum_{i=1}^{n} \lambda^{1/\gamma} X_i \right), \phi(\lambda^{1/\gamma} X_1), \ldots, \phi(\lambda^{1/\gamma} X_n) \right)
\]
\[
= D(\lambda^{1/\gamma} X) = D(X) = R(x_0, x).
\]
Hence, \( R \) satisfies (i) and (ii) in Lemma EC.2 on \( \mathcal{A} \). By definition, \( R \) satisfies (i) and (ii) also on \( \mathbb{R}^{n+1} \setminus \mathcal{A} \). Since \( \mathcal{A} \) and \( \mathbb{R}^{n+1} \setminus \mathcal{A} \) are both closed under the two operations, we know that \( R \) satisfies (i) and (ii) on \( \mathbb{R}^{n+1} \).

Using Lemma EC.2, we have \( R \) has the representation (EC.4), which gives
\[
D(X) = C_1 1_{\{d < 0\}} + C_2 1_{\{d = 0\}} + C_3 1_{\{d > 0\}}
\]
with \( d = \sum_{i=1}^{n} \phi(X_i) - \phi(\sum_{i=1}^{n} X_i) \) and \( C_1, C_2, C_3 \in \mathbb{R} \) for all \( X \in \mathcal{X}^n \). As \( D \) satisfying [+] we have \( C_1, C_2, C_3 \in \mathbb{R}_+ \cup \{\infty\} \). \( \square \)
D Additional results and proofs for Section 4

In this appendix, we present additional results, proofs, and discussions supplementing Sections 4.3, 4.4 and 4.5.

D.1 Connection between DQ and DR (Section 4.3)

For a single non-negative risk measure φ and its corresponding DR, we can construct a class ρ = (ρα)α∈I such that DQρα = DRφ which shows that DR is a subclass of DQ.

Proposition EC.3. For a given φ : X → ℜ+, let ρ = (φ/α)α∈(0,∞). For α ∈ (0, ∞), we have DQρα = DRφ. The same holds if ρ = (bE + cφ/α)α∈(0,∞) for some b ∈ ℜ and c > 0 and X = L1.

Proof of Proposition EC.3. First, we compute α∗ by the definition of DQρα. For any X ∈ (L1)n,

\[α^* = \inf \left\{ \beta \in (0, \infty) : b E \left[ \sum_{i=1}^{n} X_i \right] + \frac{c}{\beta} \phi \left( \sum_{i=1}^{n} X_i \right) \leq b \sum_{i=1}^{n} E [X_i] + \sum_{i=1}^{n} \frac{c}{\alpha} \phi (X_i) \right\} \]

= \inf \left\{ \beta \in (0, \infty) : \phi \left( \frac{\sum_{i=1}^{n} X_i}{\beta} \right) \leq \frac{\sum_{i=1}^{n} \phi (X_i)}{\alpha} \right\}.

If φ(∑i=1n X_i) = 0 and ∑i=1n φ(X_i) = 0, then α∗ = 0. If φ(∑i=1n X_i) > 0 and ∑i=1n φ(X_i) = 0, then α∗ = ∞ as the set on which the infimum is taken is empty. If φ(∑i=1n X_i) > 0 and ∑i=1n φ(X_i) > 0, then α∗ = αφ(∑i=1n X_i)/∑i=1n φ(X_i). Hence, DQρα(X) = DRφ(X) holds for all X ∈ (L1)n. By the same argument, for ρ = (φ/α)α∈(0,∞), we get DQρα(X) = DRφ(X) for all X ∈ Xn.

One may immediately observe that DQρα in Proposition EC.3 does not depend on α; this is not the case for DQ based on a general class ρ. Recall that DRvar and DRSD do not have a parameter. The form of bE + cφ/α in Proposition EC.3 represents a class of possibly non-monotone risk measures. Examples include mean-standard deviation, mean-variance, and mean-Gini; see Denneberg (1990). Furthermore, if φ satisfies [CA]0, then ρα = bE + cφ/α satisfies [CA]0. The fact that DRvar and DRSD are location-scale invariant can be seen as a special case of Proposition 1 since φ satisfies [PH] and [CA]0 are satisfied.

Next, we will show that if DRφ satisfies [+], [LI] and [SI], then φ satisfies the property [±]: φ is either non-negative or non-positive. We say that a risk measure on Lp is law invariant if φ(X) = φ(Y) whenever X = Y and it is continuous if Xn Lp → X implies ρ(Xn) → ρ(X), both as n → ∞. Law invariance and continuity are two technical conditions commonly satisfied by most risk measures. For instance, a VaR is continuous on L∞ whereas an ES is continuous on L1. The following characterization result is highly technical and relies on a recent result of Wang and Wu (2020).
**Proposition EC.4.** Fix $n \geq 3$ and $p \in [0, \infty)$. Assume that $\phi : L^p \to \mathbb{R}$ is law invariant and continuous, and $\text{DR}^\phi$ is not degenerate. Then, $\text{DR}^\phi$ satisfies $[+]$, $[LI]$ and $[SI]$ if and only if $\phi$ satisfies $[\pm]$, $[CA]_0$ and $[PH]_\gamma$ with $\gamma \in \mathbb{R}$. As a consequence, $\text{DR}^\phi$ satisfying $[+]$, $[LI]$ and $[SI]$ belongs to the class of DQs.

**Proof of Proposition EC.4.** We first show the “if” part. It is clear that if $\phi$ satisfies $[\pm]$, then $\text{DR}^\phi$ satisfies $[+]$. As $\phi$ satisfies $[CA]_0$, for any $c \in \mathbb{R}^n$ and $X \in (L^p)^n$,

$$
\text{DR}^\phi(X + c) = \frac{\phi\left(\sum_{i=1}^n X_i + \sum_{i=1}^n c_i\right)}{\sum_{i=1}^n \phi(X_i + c_i)} = \frac{\phi\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \phi(X_i)} = \text{DR}^\phi(X).
$$

For any $\lambda > 0$ and $X \in (L^p)^n$,

$$
\text{DR}^\phi(\lambda X) = \frac{\phi\left(\sum_{i=1}^n \lambda X_i\right)}{\sum_{i=1}^n \phi(\lambda X_i)} = \frac{\lambda^n \phi\left(\sum_{i=1}^n X_i\right)}{\lambda^n \sum_{i=1}^n \phi(X_i)} = \text{DR}^\phi(X).
$$

Hence, we have $\text{DR}^\phi$ satisfies $[+]$, $[LI]$ and $[SI]$.

Next, we show the “only if” part in 4 steps.

**Step 1.** Assume $\phi(0) \neq 0$. For any $X \in L^p$, let $c = \phi(X)/(\phi(X) + (n - 1)\phi(0))$. Note that the equation $y/(y + (n - 1)\phi(0)) = c$ has a unique solution $y = \phi(X)$. Hence, by using $[SI]$, for all $\lambda > 0$,

$$
\frac{\phi(\lambda X)}{\phi(\lambda X) + (n - 1)\phi(0)} = c = \frac{\phi(X)}{\phi(X) + (n - 1)\phi(0)}.
$$

Therefore, we obtain $\phi(\lambda X) = \phi(X)$, and thus $\phi$ satisfies $[PH]_0$. Using continuity of $\phi$ and $[PH]_0$, we have $\phi(X) = -\phi(0)$ as $\lambda \downarrow 0$, and hence $\phi(X) = \phi(0)$. This shows $\phi$ satisfies both $[CA]_0$ and $[\pm]$. Therefore, the desired statement holds if $\phi(0) \neq 0$ or $\phi$ satisfies $[PH]_0$. In what follows, we will assume $\phi(0) = 0$.

**Step 2.** Next, we show $[CA]_0$. Since $\phi(0) = 0$, if there exists $c_1 \in \mathbb{R}$ such that $\phi(c_1) \neq 0$, then by $[LI]$, we have

$$
\frac{\phi(c_1)}{\phi(c_1)} = 1 = \frac{\phi(c_1 + c_2)}{\phi(c_1) + \phi(c_2)}.
$$

Switching the roles of $c_1$ and $c_2$, we know that $\phi(c_1 + c_2) = \phi(c_1) + \phi(c_2)$ as long as $\phi(c_1)$ or $\phi(c_2)$ is not zero. If both of $\phi(c_1)$ and $\phi(c_2)$ are 0, then $\phi(c_1 + c_2) = 0$. To sum up, $\phi$ is additive on $\mathbb{R}$. Since $\phi$ is also continuous on $\mathbb{R}$, we know that $\phi$ is linear, that is, $\phi(c) = \beta c$ for some $\beta \in \mathbb{R}$.

Since $\text{DR}^\phi$ is not degenerate, we can take $X$ such that $\phi(X) \neq 0$. Using $[LI]$ and $\phi(0) = 0$, we have, for $c \in \mathbb{R}$,

$$
\text{DR}^\phi(X, c, 0, \ldots, 0) = \frac{\phi(X + c)}{\phi(X) + \phi(c)} = \text{DR}^\phi(X, 0, \ldots, 0) = 1,
$$

which implies $\phi(X + c) = \phi(X) + \phi(c) = \phi(X) + \beta c$.

Again, using the fact that $\text{DR}^\phi$ is not degenerate, there exists $X = (X_1, \ldots, X_n)$ such that $\text{DR}^\phi \in \mathbb{R} \setminus \{0, 1\}$. Applying $\text{DR}^\phi$ to $X$, we have $\phi(\sum_{i=1}^n X_i) \neq 0$, $\sum_{i=1}^n \phi(X_i) \neq 0$ and there
exists $X_i$ such that $\phi(X_i) \neq 0$. Without loss of generality, assume $\phi(X_1) \neq 0$. Note that

$$\text{DR}^\phi(X + (1, 0, \ldots, 0)) = \frac{\phi(\sum_{i=1}^n X_i + 1)}{\phi(X_1 + 1) + \sum_{i=2}^n \phi(X_i)} = \frac{\phi(\sum_{i=1}^n X_i) + \beta}{\sum_{i=1}^n \phi(X_i) + \beta} = \frac{\phi(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \phi(X_i)}.$$  

This implies $\beta = 0$, $\phi(c) = 0$ for all $c \in \mathbb{R}$ and $\phi(X + c) = \phi(X)$ for all $X \in L^p$ such that $\phi(X) \neq 0$. For any $X \in L^p$ such that $\phi(X) = 0$ and $c \in \mathbb{R}$, we have

$$\text{DR}^\phi(X, c, 0, \ldots, 0) = \frac{\phi(X + c)}{\phi(X) + \phi(c)} = \frac{\phi(X)}{\phi(X)} = \text{DR}^\phi(X, 0, 0, \ldots, 0),$$

which implies $\phi(X + c) = 0 = \phi(X)$. Therefore, $\phi$ satisfies $[\text{CA}]_0$.

Step 3. We show $[\text{PH}]_\gamma$ with $\gamma \in \mathbb{R}$.

For $Z \in L^p$, if $\phi(Z) \neq 0$, then $\phi(\lambda Z)/\phi(\lambda Z) = \phi(Z)/\phi(Z) = 1$ for all $\lambda > 0$ by [SI]. Hence, $\phi(\lambda Z) \neq 0$ for all $\lambda > 0$. If $\phi(Z) = 0$, then $\phi(\lambda Z)/\phi(\lambda Z) = \phi(Z)/\phi(Z) = 0$ for all $\lambda > 0$, which implies $\phi(\lambda Z) = 0$ for all $\lambda > 0$.

Take $Z \in L^p$ such that $\phi(Z) \neq 0$. Note that there exists $Y \in L^\infty$ such that $\phi(Y) \neq 0$; otherwise, by continuity of $\phi$, $\phi$ must be 0 on $L^p$, and $\text{DR}^\phi$ is degenerate, a contradiction. Together with $[\text{CA}]_0$, we know that there exists $Y \in L^\infty$ with $\mathbb{E}[Y] = 0$ and $\phi(Y) \neq 0$. In the following, we will show that $\phi(\lambda Z)/\phi(Z) = \phi(\lambda Y)/\phi(Y)$ for all $\lambda > 0$.

As $Y \in L^\infty$ and $\mathbb{E}[Y] = 0$, by Lemma 1 of Wang and Wu (2020), there exist $X, X' \in L^\infty$ such that $X \overset{d}{=} X'$ and $X - X' \overset{d}{=} Y$. Let $Y' = X - X'$. Using continuity of $\phi$ again, we have $\phi(\lambda Y' + Z) \to \phi(Z) \neq 0$ as $\lambda \to 0$ and we can replace $(Y, Y')$ by $(\lambda Y, \lambda Y')$ for a small $\lambda > 0$, which will satisfy $\phi(\lambda Y) \neq 0$ as we argued above. Hence, we can choose $Y$ such that $\phi(Y' + Z) \neq 0$ by scaling $Y$ down; this condition will be useful in (b) below.

(a) Suppose that $\phi(aX) + \phi(-aX) \neq 0$ for some $a > 0$. Let $X = (Z, X, -X, 0, \ldots, 0)$ and $X' = (X, -X', 0, \ldots, 0)$. Applying $\text{DR}^\phi$ to $\lambda X$ for $\lambda > 0$, by [SI] and $\phi(\lambda Z) \neq 0$, we have

$$\frac{1}{\text{DR}^\phi(\lambda X)} = \frac{\phi(\lambda X) + \phi(-\lambda X) + \phi(\lambda Z)}{\phi(\lambda Z)} = \frac{1}{\text{DR}^\phi(X)} := c_1 \in \mathbb{R}.$$  

Applying $\text{DR}^\phi$ to $\lambda X'$ for $\lambda > 0$, by the law invariance of $\phi$ and [SI], we have

$$\frac{1}{\text{DR}^\phi(\lambda X')} = \frac{\phi(\lambda X) + \phi(-\lambda X')}{\phi(\lambda(X - X'))} = \frac{\phi(\lambda X) + \phi(-\lambda X)}{\phi(\lambda Y')} = \frac{1}{\text{DR}^\phi(X')} := c_2 \in \mathbb{R}.$$  

Since $c_2$ does not depend on $\lambda$, $\phi(aY') \neq 0$ because $\phi(Y') \neq 0$, and $\phi(aX) + \phi(-aX) \neq 0$, we know $c_2 \neq 0$ by setting $\lambda = a$. Hence, for all $\lambda > 0$,

$$c_2\phi(\lambda Y') + \phi(\lambda Z) = c_1 \phi(\lambda Z)$$

and

$$\frac{\phi(\lambda Y')}{\phi(\lambda Z)} = \frac{c_1 - 1}{c_2} = \frac{\phi(Y')}{\phi(Z)},$$

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Finally, we show (b) Suppose that \( \phi(aX) + \phi(-aX) = 0 \) for all \( a > 0 \). Let \( X = (Z, X, -X', \ldots, 0) \) and \( X' = (Z, Y', \ldots, 0) \). Applying DR to \( \lambda X \), by the law invariance of \( \phi \) and \([SI]\), we have

\[
\frac{1}{\text{DR}^\phi(\lambda X)} = \frac{\phi(\lambda X) + \phi(-\lambda X') + \phi(\lambda Z)}{\phi(\lambda (X - X') + \lambda Z)} = \frac{\phi(\lambda Z)}{\phi(\lambda Y' + \lambda Z)} = \frac{1}{\text{DR}^\phi(X)} := c_3.
\]

Since \( c_3 \) does not depend on \( \lambda \), using \( \phi(Y' + Z) \neq 0 \), we have \( \phi(\lambda Y' + \lambda Z') \neq 0 \). Together with \( \phi(Z) \neq 0 \), we get \( c_3 \in \mathbb{R} \setminus \{0\} \). Applying DR to \( \lambda X' \) for \( \lambda > 0 \), by the law invariance of \( \phi \) and \([SI]\), we get

\[
\frac{1}{\text{DR}^\phi(\lambda X')} = \frac{\phi(\lambda Y') + \phi(\lambda Z)}{\phi(\lambda Y' + \lambda Z)} = \frac{\phi(\lambda Y) + \phi(\lambda Z)}{\phi(\lambda Y' + \lambda Z)} = \frac{1}{\text{DR}^\phi(X')} := c_4.
\]

Hence, for all \( \lambda > 0 \),

\[
\phi(\lambda Y) \phi(\lambda Z) = \frac{c_4}{c_3} - 1 = \frac{\phi(Y)}{\phi(Z)}.
\]

In both cases, we have \( \phi(\lambda Z)/\phi(Z) = \phi(\lambda Y)/\phi(Y) \). Therefore, for any \( Z, Z' \in L^p \) such that \( \phi(Z) \neq 0 \) and \( \phi(Z') \neq 0 \), by taking a common \( Y \) (by scaling down) with \( \phi(Y' + Z) \neq 0 \) and \( \phi(Y' + Z') \neq 0 \), we get

\[
\frac{\phi(\lambda Z)}{\phi(Z)} = \frac{\phi(\lambda Y)}{\phi(Y)} = \frac{\phi(\lambda Z')}{\phi(Z')},
\]

which implies that \( f(\lambda) := \phi(\lambda Z)/\phi(Z) \) does not depend on \( Z \). That is, \( \phi(\lambda Z) = f(\lambda)\phi(Z) \) for all \( \lambda > 0 \) and \( Z \in L^p \) such that \( \phi(Z) \neq 0 \). Note that

\[
\phi(\lambda_1 \lambda_2 Z) = f(\lambda_1) f(\lambda_2) \phi(Z)
\]

for all \( \lambda_1, \lambda_2 > 0 \). By the continuity of \( \phi \), \( f \) is continuous, which implies \( f(\lambda) = \lambda^\gamma \) for some \( \gamma \in \mathbb{R} \). Hence, \( \phi(\lambda Z) = \lambda^\gamma \phi(Z) \) if \( \phi(Z) \neq 0 \). If \( \phi(Z) = 0 \), we have \( \phi(\lambda Z) = 0 = \lambda^\gamma \phi(Z) \).

Therefore, \( \phi \) satisfies \([PH]_\gamma\) for some \( \gamma \in \mathbb{R} \).

Step 4. Finally, we show \( \phi \) is either non-negative or non-positive by considering the following three cases.

(i) Assume that there exists \( X \in L^p \) such that \( \phi(X) + \phi(-X) > 0 \). If there exists \( Y \in L^p \) such that \( \phi(Y) < 0 \), then by continuity of \( \phi \) and \( \phi(0) = 0 \), there exists \( m > 0 \) such that \( 0 < -\phi(mY) < \phi(X) + \phi(-X) \). We have

\[
\text{DR}^\phi(mY, X, -X, 0, \ldots, 0) = \frac{\phi(mY)}{\phi(mY) + \phi(X) + \phi(-X)} < 0,
\]

which contradicts the fact that \( \text{DR}^\phi \) is non-negative. Hence, \( \phi(Y) \geq 0 \) for all \( Y \in L^\infty \).

(ii) By the same argument, if there exists \( X \in L^p \) such that \( \phi(X) + \phi(-X) < 0 \), then \( \phi(Y) \leq 0 \) for all \( Y \in L^\infty \).
(iii) Assume \( \phi(X) + \phi(-X) = 0 \) for all \( X \in L^\infty \). Suppose that there exists \( Y \in L^\infty \) such that \( \phi(Y) < 0 \). Using Lemma 1 of Wang and Wu (2020) again, there exist \( Z, Z' \in L^\infty \) satisfying \( Z \overset{d}{=} Z' \) and \( Z - Z' \overset{d}{=} Y - E[Y] \). For \( Z = (Z, -Z', 0, \ldots, 0) \), using the law invariance of \( \phi \), we have

\[
\text{DR}^\phi(Z) = \frac{\phi(Z - Z')}{\phi(Z) + \phi(-Z')} = \frac{\phi(Y - E[Y])}{\phi(Z) + \phi(-Z')} = \frac{\phi(Y)}{0} = -\infty,
\]

which contradicts \( \text{DR}^\phi(Z) \geq 0 \). Hence, \( \phi(X) \geq 0 \) for all \( X \in L^\infty \). Together with \( \phi(X) + \phi(-X) = 0 \), we get \( \phi(X) = 0 \). To extend this to \( L^p \), we simply use continuity. For \( X \in L^p \), let \( Y_M = (X \wedge M) \vee (-M) \). Hence, \( Y_M \in L^\infty \) and \( Y_M \overset{L^p}{\to} X \) as \( M \to \infty \). As a result, we have \( \phi(X) = \lim_{M \to \infty} \phi(Y_M) = 0 \).

In conclusion, we have \( \phi(Y) \geq 0 \) or \( \phi(Y) \leq 0 \) for all \( X \in L^p \). Case (iii) is not possible because it contradicts that \( \text{DR}^\phi \) is not degenerate. Cases (i) and (ii) are possible, corresponding to, for instance, (i) \( \phi = \text{SD} \); (ii) \( \phi = -\text{SD} \).\( \square \)

The properties \([\pm], [\text{CA}]_0 \) and \([\text{PH}]\) are the three defining properties of variability measures in Bellini et al. (2022) (who additionally required \( \phi(0) = 0 \) and \( \gamma \geq 0 \)). By Proposition EC.4, in the setting where \([+], [\text{LI}] \) and \([\text{SI}]\) are imposed on \( \text{DR} \), one has to choose a variability measure instead of a monetary risk measure. There are two important implications. First, if VaR or ES is the risk measure of interest, then \( \text{DR} \) cannot be used in this setting. Second, all choices of DRs in this setting are in fact DQs.

### D.2 Worst-case and best-case dependence for DQ (Section 4.4)

We assume that two random vectors \( X \) and \( Y \) have the same marginal distributions, and we study the effect of the dependence structure. We will assume that a tuple of distributions \( F = (F_1, \ldots, F_n) \) is given and each component has a finite mean. Let

\[
\mathcal{Y}_F = \{(X_1, \ldots, X_n) : X_i \sim F_i \text{ for each } i = 1, \ldots, n \}.
\]

For \( X, Y \in \mathcal{Y}_F \), we say that \( X \) is smaller than \( Y \) in sum-convex order, denoted by \( X \preceq_{\text{scx}} Y \), if \( \sum_{i=1}^n X_i \geq_{\text{SSD}} \sum_{i=1}^n Y_i \); see Corbett and Rajaram (2006). We refer to Shaked and Shanthikumar (2007) for a general treatment of multivariate stochastic orders. With arbitrary dependence structures, the best-case value and worst-case value of \( \text{DQ}_\alpha^\phi \) are given by

\[
\inf_{X \in \mathcal{Y}_F} \text{DQ}_\alpha^\phi(X) \quad \text{and} \quad \sup_{X \in \mathcal{Y}_F} \text{DQ}_\alpha^\phi(X).
\]

For some mapping on \( \mathcal{X}^n \), finding the best-case and worst-case values and structures over \( \mathcal{Y}_F \) is known as a problem of risk aggregation under dependence uncertainty; see Bernard et al. (2014) and Embrechts et al. (2015).
If $\rho = (\rho_\alpha)_{\alpha \in I}$ is a class of SSD-consistent risk measures such as ES, then, by Proposition 2, $DQ^\rho_\alpha$ is consistent with the sum-convex order on $Y_F$. This leads to the following observations on the corresponding dependence structures.

(i) It is well-known (e.g., Rüschendorf (2013)) that the $\leq_{scx}$-largest element of $Y_F$ is comonotonic, and thus a comonotonic random vector has the largest $DQ^\rho_\alpha$ in this case. Note that such $\rho$ does not include VaR. Indeed, as we have seen from Proposition 4, $DQ^{\text{VaR}}_\alpha(X) = 1$ for comonotonic $X$ under mild conditions, which is not equal to its largest value $n$.

(ii) In case $n = 2$, the $\leq_{scx}$-smallest element of $Y_F$ is counter-comonotonic, and thus a comonotonic random vector has the smallest $DQ^\rho_\alpha$.

(iii) For $n \geq 3$, the $\leq_{scx}$-smallest elements of $Y_F$ are generally hard to obtain. If each pair $(X_i, X_j)$ is counter-monotonic for $i \neq j$, then $X$ is a $\leq_{scx}$-smallest element of $Y_F$. Pairwise counter-monotonicity puts very strong restrictions on the marginal distributions. For instance, it rules out all continuous marginal distributions; see Puccetti and Wang (2015).

(iv) If a joint mix, i.e., a random vector with a constant component-wise sum, exists in $Y_F$, then any joint mix is a $\leq_{scx}$-smallest element of $Y_F$ by Jensen’s inequality. See Puccetti and Wang (2015) and Wang and Wang (2016) for results on the existence of joint mixes. In case a joint mix does not exist, the $\leq_{scx}$-smallest elements are obtained by Bernard et al. (2014) and Jakobsons et al. (2016) under some conditions on the marginal distributions such as monotonic densities.

In optimization problems over dependence structures (see e.g., Rüschendorf (2013) and Embrechts et al. (2015)), the above observations yield guidelines on where to look for the optimizing structures.

### D.3 Proofs and related discussions on RI and RC (Section 4.5)

Here we present the proof of Proposition 3 and an additional result (Proposition EC.5) on the properties RI and RC.

**Proof of Proposition 3.** (i) For any $n \in \mathbb{N}$, $X \in (L^p)^n$ and $c \in \mathbb{R}$, by [CA]$_m$ of $(\rho_\alpha)_{\alpha \in I}$,

$$DQ^\rho_\alpha(X, c) = \frac{1}{\alpha} \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^{n} X_i + c \right) \leq \sum_{i=1}^{n} \rho_\alpha(X_i) + \rho_\alpha(c) \right\}$$

$$= \frac{1}{\alpha} \inf \left\{ \beta \in I : \rho_\beta \left( \sum_{i=1}^{n} X_i \right) + mc \leq \sum_{i=1}^{n} \rho_\alpha(X_i) + mc \right\} = DQ^{\text{scx}}_\alpha(X),$$

and hence $DQ^\rho_\alpha$ satisfies [RI].
Proposition EC.5. Let $\phi : \ell^p \to \mathbb{R}$ be a continuous and law-invariant risk measure.

(i) Suppose that $\text{DR}^\phi$ is not degenerate for some input dimension. Then $\text{DR}^\phi$ satisfies [RI] and $[+]$ if and only if $\phi$ satisfies $[\text{CA}]_0$, $[\pm]$ and $\phi(0) = 0$.

(ii) If $\phi$ satisfies $[\text{PH}]$, then $\text{DR}^\phi$ satisfies $[\text{RC}]$.

Proof of Proposition EC.5. (i) We first show the “if” part. If $\phi$ satisfies $[\text{CA}]_0$ and $\phi(0) = 0$, then $\phi(c) = \phi(0) = 0$ for all $c \in \mathbb{R}$. For any $n \in \mathbb{N}$, $X \in (\ell^p)^n$ and $c \in \mathbb{R}$,

$$\text{DR}^\phi(X, c) = \frac{\phi\left(\sum^n_{i=1} X_i + c\right)}{\phi(X) + \phi(c)} = \frac{\phi\left(\sum^n_{i=1} X_i\right)}{\sum^n_{i=1} \phi(X_i)} = \text{DR}^\phi(X).$$

Thus, $\text{DR}^\phi$ satisfies [RI].

For the “only if” part, we first assume $\phi(0) \neq 0$. Since $\text{DR}^\phi$ satisfies [RI], for all $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $X = 0 \in \mathbb{R}^n$, we have

$$\text{DR}^\phi(X, c) = \frac{\phi(c)}{n\phi(0) + \phi(c)} = \text{DR}^\phi(X) = \frac{\phi(0)}{n\phi(0)} = \frac{1}{n}.$$ 

The above equality means that $\phi(c) = n\phi(0)/(n - 1)$ holds for any $n \in \mathbb{N}$ and $c \in \mathbb{R}$, and thus we have $\phi(0) = 0$, which violates the assumption $\phi(0) \neq 0$. Hence, $\phi(0) = 0$.

Similarly to Step 2 in the proof of Proposition EC.4, if there exists $c_1 \in \mathbb{R}$ such that $\phi(c_1) \neq 0$, then by [RI] and $\phi(0) = 0$, we have

$$\text{DR}^\phi(c_1, 0, 0, \ldots, 0) = \frac{\phi(c_1 + c)}{\phi(c_1) + \phi(c)} = \text{DR}^\phi(0, 0, \ldots, 0) = \frac{\phi(c_1)}{\phi(c_1)} = 1,$$

and thus $\phi(c_1 + c) = \phi(c_1) + \phi(c)$ as long as $\phi(c_1)$ or $\phi(c)$ is not zero. If both of $\phi(c_1)$ and $\phi(c)$ are zero, then $\phi(c_1 + c) = 0$. To sum up, $\phi$ is additive on $\mathbb{R}$. Since $\phi$ is also continuous on $\mathbb{R}$, we know that $\phi$ is linear, that is, $\phi(c) = \beta c$ for some $\beta \in \mathbb{R}$.

Suppose that there exists $X$ such that $\phi(X) \neq 0$; otherwise there is nothing to show. Using [RI] and $\phi(0) = 0$, we have, for $c \in \mathbb{R}$,

$$\text{DR}^\phi(X, 0, 0, \ldots, 0) = \frac{\phi(X + c)}{\phi(X) + \phi(c)} = \text{DR}^\phi(X, 0, \ldots, 0) = 1,$$

which implies $\phi(X + c) = \phi(X) + \phi(c) = \phi(X) + \beta c$. 

and hence $\text{DR}^\phi$ satisfies $[\text{RC}]$. \(\square\)
Using the fact that DR$^\phi$ is not degenerate for some dimension $n$, there exists $X = (X_1, \ldots, X_n)$ such that DR$^\phi(X) \in \mathbb{R} \setminus \{0, 1\}$. Note that $\phi(\sum_{i=1}^n X_i) \neq 0$ and $\sum_{i=1}^n \phi(X_i) \neq 0$. Hence,

$$\text{DR}^\phi(X, 1) = \frac{\phi(\sum_{i=1}^n X_i + 1)}{\sum_{i=1}^n \phi(X_i) + \phi(1)} = \frac{\phi(\sum_{i=1}^n X_i) + \beta}{\sum_{i=1}^n \phi(X_i) + \beta} = \text{DR}^\phi(X) = \frac{\phi(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \phi(X_i)}.$$ 

This implies $\beta = 0$, $\phi(c) = 0$ for all $c \in \mathbb{R}$ and $\phi(X + c) = \phi(X)$ for all $X \in L^p$ such that $\phi(X) \neq 0$. For any $X \in L^p$ such that $\phi(X) = 0$ and $c \in \mathbb{R}$, we have

$$
\text{DR}^\phi(X, 0, \ldots, 0, c) = \frac{\phi(X + c)}{\phi(X)} = \frac{\phi(X)}{\phi(X)} = \text{DR}^\phi(X, 0, \ldots, 0),
$$

which implies $\phi(X + c) = 0 = \phi(X)$. Therefore, $\phi$ satisfies [CA]$_0$.

The proof for $[\pm]$ follows from the same arguments in the proof (Step 4) of Proposition EC.4.

(ii) If $\phi$ satisfies [PH], then for any $n \in \mathbb{N}$ and $X \in (L^p)^n$,

$$\text{DR}^\phi(X, X) = \frac{\phi(2 \sum_{i=1}^n X_i)}{2 \sum_{i=1}^n \phi(X_i)} = \frac{\phi(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \phi(X_i)} = \text{DR}^\phi(X).$$

Hence, DR$^\phi$ satisfies [RC]. \qed

In Proposition EC.3, we show that if [RI] is assumed, then the only option for DR is to use a non-negative $\phi$ (we can use $-\phi$ if $\phi$ is non-positive) such as var or SD. Using Proposition EC.3, all such DRs belong to the class of DQs. The result in Proposition EC.5 has a similar implication to Proposition EC.4 with [RI] replacing [LI] and [SI].

E Additional result and proof for Section 5

In this appendix, we present the proof for Proposition 4 and one additional numerical result to complement those in Section 5.2.

We first look at the models $Y'$ and $Y$ in the setting of Tables 1 and 2. In Figure EC.1, we observe that the values of $D(Y')/D(Y)$ for $D = \text{DQ}^{\text{VaR}}_\alpha$ or DQ$^{\text{ES}}_\alpha$ are always smaller than 1 for $\alpha \in (0, 0.1]$, while the values of $D(Y')/D(Y)$ for $D = \text{DR}^{\text{VaR}}_\alpha$ are only smaller than 1 when $\alpha$ is relatively small. We always observe that, if the desired relation $D(Y')/D(Y) < 1$ holds for $D = \text{DR}^{\text{VaR}}_\alpha$ or DR$^{\text{ES}}_\alpha$ then it holds for $D = \text{DQ}^{\text{VaR}}_\alpha$ or DQ$^{\text{ES}}_\alpha$, but the converse does not hold. This means that if the iid model is preferred to the common shock model by DR, then it is also preferred by DQ, but in many situations, it is only preferred by DQ not by DR. Similarly to Tables 1 and 2, the iid normal model shows a stronger diversification according to DQ, and this is not the case for DR.

Proof of Proposition 4. This statement follows from Theorem 1 (i) of Han et al. (2023). \qed
Figure EC.1. $D(Y')/D(Y)$ based on VaR and ES for $\alpha \in (0, 0.1]$ with fixed $n = 10$

### F  Additional results and proofs for Section 6

The following result shows that there is a conflict between convexity and [SI]. This result is mentioned in Example 3.

**Proposition EC.6.** A mapping $D : X^n \to \mathbb{R}$ satisfies [SI] and convexity if and only if $D(X) = c$ for all $X \in X$ and some constant $c \in \mathbb{R}$.

**Proof.** If $D$ is a constant for all $X \in X^n$, it is clear that $D$ satisfies [SI] and convexity. Next we will show the “only if” part. Let $d_0 = D(0) \in \mathbb{R}$.

(i) If $d_0 \geq D(X)$ for all $X \in X^n$ and there exists $X_0$ such that $D(X_0) < d_0$, then

$$D \left( \frac{1}{2}X_0 + \frac{1}{2}(-X_0) \right) = D(0) > \frac{1}{2}D(X_0) + \frac{1}{2}D(-X_0),$$

which contradicts the convexity of $D$.

(ii) If there exists $X_0$ such that $d_0 < D(X_0)$, then, by [SI] of $D$,

$$D \left( \frac{1}{2}0 + \frac{1}{2}X_0 \right) = D(X_0) > \frac{1}{2}D(0) + \frac{1}{2}D(X_0),$$

which contradicts the convexity of $D$.

By (i) and (ii), we can conclude that $D$ only takes the value $d_0$. \hfill $\square$

From the proof of Proposition EC.6, we see that the conflict between convexity and [SI] holds for real-valued mappings on any closed convex cone, not necessarily on $X^n$.

**Proof of Proposition 5.** For the case of $DQ_{\alpha}^{VaR}(X)$, (5) in Theorem 3 gives that to minimize $DQ_{\alpha}^{VaR}(X)$ is equivalent to minimize

$$\mathbb{P} \left( w^\top X > w^\top x_{VaR}^{\alpha} \right) = \mathbb{P} \left( w^\top (X - x_{VaR}^{\alpha}) > 0 \right) \quad \text{over } w \in \Delta_n.$$
Next, we discuss the case of $DQ^E(X)$. Let $f(v) = \mathbb{E}[(v^T(X - x^E_\alpha) + 1)_+]$ for $v \in \mathbb{R}_+^n$. It is clear that $f$ is convex. Furthermore, for any $i \in [n]$, we have, for almost every $v \in \mathbb{R}_+^n$,

$$\frac{\partial f}{\partial v_i}(v) = \mathbb{E}[(X_i - ES_\alpha(X_i))\mathbb{I}_{\{v^T(X - x^E_\alpha) + 1 > 0\}]}$$

$$= \mathbb{E}[(X_i - ES_\alpha(X_i))\mathbb{I}_{\{(v^T(X - x^E_\alpha) + 1 > 0) \cap (X_i - ES_\alpha(X_i) > 0)\}]} + \mathbb{E}[(X_i - ES_\alpha(X_i))\mathbb{I}_{\{(v^T(X - x^E_\alpha) + 1 > 0) \cap (X_i - ES_\alpha(X_i) < 0)\}]}.$$

The set $\{(v^T(X - x^E_\alpha) + 1 > 0) \cap (X_i - ES_\alpha(X_i) > 0)\}$ increases in $v_i$ and the set $\{(v^T(X - x^E_\alpha) + 1 > 0) \cap (X_i - ES_\alpha(X_i) < 0)\}$ decreases in $v_i$. Hence, $v_i \mapsto \partial f/\partial v_i(v)$ is increasing. Furthermore, $\partial f/\partial v_i(v) \to \mathbb{E}[(X_i - ES_\alpha(X_i))\mathbb{I}_{(X_i - ES_\alpha(X_i) > 0)}] > 0$ as $v_i \to \infty$. Also, $\partial f/\partial v_i(v) \to \mathbb{E}[(X_i - ES_\alpha(X_i))] < 0$ as $v \downarrow 0$ component-wise. Hence, there exists a minimizer $v^*$ of the problem $\min_{v \in \mathbb{R}_+^n \setminus \{0\}} \mathbb{E}[(v^T(X - x^E_\alpha) + 1)_+]$.

Let $A = \{v \in \mathbb{R}_+^n \setminus \{0\} : \mathbb{P}(v(X - x^E_\alpha) > 0) > 0\}$ and $B = \{v \in \mathbb{R}_+^n \setminus \{0\} : \mathbb{P}(v(X - x^E_\alpha) > 0) = 0\}$. If $B$ is empty, it is clear that $\min_{w \in \Delta_n} DQ^E_\alpha(w \odot X) = \min_{v \in \mathbb{R}_+^n \setminus \{0\}} \mathbb{E}[(v^T(X - x^E_\alpha) + 1)_+]$ by Theorem 3.

If $B$ is not empty, assume $v^* \in A$. For any $v_A \in A$, $v_B \in B$ and $k > 0$, we have

$$\mathbb{E}[(v_A + kv_B)^T(X - x^E_\alpha) + 1)_+] \leq \mathbb{E}[(v_A^T(X - x^E_\alpha) + 1)_+]$$

This implies $f(v^* + kv_B) = f(v^*)$ for all $k > 0$, which contradicts $\partial f/\partial v_i(v) > 0$ as $v_i \to \infty$. Hence, we have $v^* \in B$. For $w^* = v^*/\|v^*\|$, we have $\mathbb{P}((w^*)^T(X - x^E_\alpha) > 0) = 0$ and $DQ^E_\alpha(w^* \odot X) = 0$ by Proposition 4, which means that $w^*$ is the minimizer of the problem $\min_{w \in \Delta_n} DQ^E_\alpha(w \odot X)$.

G Additional empirical results for Section 7

In this appendix, we present some omitted empirical results to complement those in Sections 7.2 and 7.3. In Section 7.2, the values of DQs based on VaR and ES are reported under different portfolio compositions of stocks during the period from 2014 to 2022. Using the same stock compositions in (A)-(D), we calculate the values of DRs based on SD and var (recall that they are also DQs), to see how they perform. The results are reported in Figure EC.2.

We can see that the same intuitive order $(A) \leq (B) \leq (C) \leq (D)$ as in Figure 3 in Section 7.2 holds for $DR^SD$, showing some consistency between DQs based on VaR and ES and $DR^SD$. The values of $DR^SD$ are between 0 and 1. On the other hand, the values of $DR^var$ are all larger than 1, and portfolio (A) of 20 stocks has the weakest diversification effect according to $DR^var$ among the four compositions. This is not in line with our intuition, but is to be expected since variance has a different scaling effect than SD, and more correlated stocks lead to a larger value.
of DR_{\text{var}} in general. For example, DR_{\text{var}} equals 1 even for an iid normal model of arbitrarily large dimension (which is often considered as quite well-diversified), and DR_{\text{var}} equals n if the portfolio has one single asset. These observations show that DR_{\text{var}} is difficult to interpret if it is used to measure diversification across dimensions.

In Section 7.3, we used the period from January 3, 2012, to December 31, 2021, to build up the portfolios. Next, we consider two different datasets from Section 7.3, first using the period 2002-2011 and second using 20 instead of 40 stocks, to see how the results vary.

For the first experiment, we choose the four largest stocks from each of the 10 different sectors of S&P 500 ranked by market cap in 2002 as the portfolio compositions and use the
Table EC.1. Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2004 to Dec 2011

|   | DQ\(^{VaR}_\alpha\) | DQ\(^{ES}_\alpha\) | DR\(^{SD}\) | Markowitz | EW | BH |
|---|----------------------|---------------------|-------------|------------|----|----|
| AR | 9.46                 | 8.13                | 9.10        | 7.98       | 5.30 | 6.23 |
| AV | 16.65                | 21.45               | 20.92       | 11.98      | 20.15 | 15.53 |
| SR | 30.48                | 17.47               | 22.58       | 30.06      | 4.57 | 11.94 |

period from January 3, 2002, to December 31, 2011, to build up the portfolio. The risk-free rate \(r = 4.38\%\), and the target annual expected return for the Markowitz portfolio is set to 5% due to infeasibility of setting 10%. The results are reported in Figure EC.3 and Table EC.1.

Figure EC.4. Wealth processes for portfolios, 20 stocks, Jan 2014 - Dec 2021

Table EC.2. Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021

|   | DQ\(^{VaR}_\alpha\) | DQ\(^{ES}_\alpha\) | DR\(^{VaR}_\alpha\) | DR\(^{ES}_\alpha\) | DR\(^{SD}\) | Markowitz | EW | BH |
|---|----------------------|---------------------|----------------------|---------------------|-------------|------------|----|----|
| AR | 13.54                | 14.79               | 12.77                | 13.85               | 14.37       | 8.59       | 12.74 | 14.22 |
| AV | 13.43                | 15.90               | 14.41                | 14.53               | 14.29       | 12.74      | 14.68 | 13.96 |
| SR | 79.69                | 75.17               | 68.89                | 75.79               | 80.67       | 45.14      | 67.40 | 81.54 |

For the second experiment, we choose the top two stocks from each sector to build the portfolios, and all other parameters are the same as in Section 7.3. The results including another two portfolios built by DR\(^{VaR}_\alpha\) and DR\(^{ES}_\alpha\) are reported in Figure EC.4 and Table EC.2. Since
we do not find an efficient algorithm for computing $DR^{VaR}_\alpha$ and $DR^{ES}_\alpha$, we use the preceding 500 trading days to compute the optimal portfolio weights using the random sampling method, which is relatively slow and not very stable. (If the previous month has an optimal weight $w^*_{t-1}$, then $10^5$ new weights are sampled from $\lambda w^*_{t-1} + (1-\lambda)\Delta_n$, where $\lambda$ is chosen as 0.9. Tie-breaking is done by picking the one that is closest to $w^*_{t-1}$. We set $w^*_0 = (1/n, \ldots, 1/n)$.) The results show similar observations to those in Section 7.3.