TRIMMING A GORENSTEIN IDEAL

LARS WINther CHRISTENSEN, OANA VELICHE, AND JERZY WEYMAN

Abstract. Let $Q$ be a regular local ring of dimension 3. We show how to trim a Gorenstein ideal in $Q$ to obtain an ideal that defines a quotient ring that is close to Gorenstein in the sense that its Koszul homology algebra is a Poincaré duality algebra $P$ padded with a non-zero graded vector space on which $P_{≥1}$ acts trivially. We explicitly construct an infinite family of such rings.

1. Introduction

Let $Q$ be a regular local ring with maximal ideal $n$. Quotient rings of $Q$ that have projective dimension at most 3 as $Q$-modules have been classified based on the multiplicative structure of their Koszul homology algebras. To be precise, let $a \subseteq n^2$ be an ideal such that the minimal free resolution of $R = Q/a$ over $Q$ has length at most 3. By a result of Buchsbaum and Eisenbud [4], the resolution carries a structure of an associative differential graded commutative algebra, and based on that structure Avramov, Kustin, and Miller [3] and Weyman [9] established a classification in terms of the induced multiplicative structure on $\operatorname{Tor}^Q_*(R, k)$, where $k$ is the residue field of $Q$. Finally, as graded $k$-algebras, the Koszul homology algebra of $R$ and $\operatorname{Tor}^Q_*(R, k)$ are isomorphic; see Avramov [1] for an in-depth treatment.

An ideal $a \subseteq Q$ is called Gorenstein if the quotient $R = Q/a$ is a Gorenstein ring. By a classic result of Avramov and Golod [2], a Gorenstein ring is characterized by the fact that its Koszul homology algebra $\tilde{A} = \operatorname{H}(K^R)$ has Poincaré duality. In the classification mentioned above, a Gorenstein ring that is not complete intersection belongs to a parametrized family $G(r)$, where $r$ is the rank of the canonical map

$$\delta : A_2 \rightarrow \operatorname{Hom}_k(A_1, A_3);$$

see [1, 1.4.2]. It was conjectured in [1] that all rings of class $G(r)$ are Gorenstein, but Christensen and Veliche [5] gave sporadic examples of rings of class $G(r)$ that are not Gorenstein. In this paper we present a systematic construction and achieve:

(1.1) Theorem. Let $Q$ be the power series algebra in three variables over a field. For every $r \geq 3$ there is quotient ring of $Q$ that is of class $G(r)$ and not Gorenstein.

The quotient rings in Theorem (1.1) are obtained as follows: Let $n$ be the maximal ideal of $Q$ and start with a graded Gorenstein ideal $g \subseteq n^2$ generated by $2m + 1$
elements. Trim \( \mathfrak{g} \) by replacing one minimal generator \( g \) by \(ng\): this removes a 1-dimensional subspace from \( \mathfrak{g} \). The quotient of \( Q \) by the resulting ideal is a ring of type 2; in particular, it is not Gorenstein, and for \( m \geq 3 \) it is of class \( G(r) \). Theorem 1.1 is consequence of Proposition 3.5, which builds on a more general but slightly less precise statement about local rings, Theorem 2.4.

2. Local rings

Let \( Q \) be a \( d \)-dimensional regular local ring with maximal ideal \( \mathfrak{n} \) and residue field \( \mathbb{k} \). For an ideal \( \mathfrak{a} \) in \( Q \), we denote by \( \mu(\mathfrak{a}) \) the minimal number of generators of \( \mathfrak{a} \). Let \( \mathfrak{a} \subseteq \mathfrak{n}^2 \) be an ideal and set \( R = Q/\mathfrak{a} \). We denote by \( K^R \) the Koszul complex on a minimal set of generators for the maximal ideal \( \mathfrak{n}/\mathfrak{a} \) of \( R \); one has \( K^R = R \otimes_Q K^Q \).

The Koszul complex is an exterior algebra, and the homology algebra \( A = H(K^R) \) is a graded-commutative \( \mathbb{k} \)-algebra. Denote by \( c \) the projective dimension of \( R \) as a \( Q \)-module; by the Auslander–Buchsbaum Formula and depth sensitivity of the Koszul complex one has \( c = \text{max}\{i \mid \mathcal{A}_i \neq 0\} \). The number \( \text{rank}_k(A_c) \) is called the type of \( R \). If the ideal \( \mathfrak{a} \) is \( \mathfrak{n} \)-primary, then one has \( c = d \) and the type of \( R \) is the socle rank, i.e. \( \text{type}(R) = \text{rank}_k(0 :_R \mathfrak{n}/\mathfrak{a}) \).

(2.1) Classification. Let \( Q \) be as above, and let \( \mathfrak{a} \subseteq \mathfrak{n}^2 \) be an ideal such that \( R = Q/\mathfrak{a} \) has projective dimension 3 as a \( Q \)-module. The possible multiplicative structures on the graded-commutative \( \mathbb{k} \)-algebra \( A = H(K^R) \cong \text{Tor}^Q_2(R, \mathbb{k}) \) were identified in [3]. By assumption one has \( A_{\geq 4} = 0 \), and the possible structures are described by the invariants

\[
p = \text{rank}_k(A_1 \cdot A_1), \quad q = \text{rank}_k(A_1 \cdot A_2), \quad \text{and} \quad r = \text{rank}_k(A_2 \xrightarrow{\delta} \text{Hom}_\mathbb{k}(A_1, A_3)).
\]

From [1] thm. 3.1 one extracts the following description of all the possible classes of rings that are not Gorenstein.

| Class | \( p \) | \( q \) | \( r \) | Restrictions |
|-------|-------|-------|-------|-------------|
| B     | 1     | 1     | 2     | 2 \leq r \leq \mu(\mathfrak{a}) - 2 |
| G(r)  | 0     | 1     | \( r \) | \( q \leq \text{type}(R) \) |
| H(p, q)| \( p \) | \( q \) | \( q \) | \( q \leq \text{type}(R) \) |
| T     | 3     | 0     | 0     | \( p = \text{rank}_k(A_1 \cdot A_1) \), \( q = \text{rank}_k(A_1 \cdot A_2) \), \( r = \text{rank}_k(A_2 \xrightarrow{\delta} \text{Hom}_\mathbb{k}(A_1, A_3)) \).

In [3] the multiplication tables for the different structures are given. In particular, if \( R = Q/\mathfrak{a} \) is a ring of class \( G(r) \), then with \( m = \mu(\mathfrak{a}) \) and \( t = \text{type}(R) \) there exist bases for \( A_1, A_2, \) and \( A_3 \):

\[
e_1, \ldots, e_m, \quad f_1, \ldots, f_{m+t-1}, \quad \text{and} \quad g_1, \ldots, g_t
\]

such that the only non-zero products are \( e_i f_i = g_1 = -f_i e_i \) for \( 1 \leq i \leq r \). That is, the subalgebra \( P \) of \( A \) spanned by \( e_1, \ldots, e_r, f_1, \ldots, f_r, \) and \( g_1 \) is a pure Poincaré duality algebra, in the sense that the only non-trivial products are those from the perfect pairing. Moreover, \( P_{\geq 1} \) acts trivially on the rest of \( A \).

The next result is proved in [6]; the argument is based on linkage theory and cannot be reproduced here without significant overhead.

(2.2) Proposition. Let \((Q, \mathfrak{n})\) be a regular local ring and let \( \mathfrak{a} \subseteq \mathfrak{n}^2 \) be a perfect ideal of grade 3 that is minimally generated by 5 elements and not Gorenstein. If, with the notation above, the ring \( Q/\mathfrak{a} \) has \( p = 0 \), then it has \( r \leq 1 \).
(2.3) Lemma. Let \((Q, n)\) be a regular local ring and consider an \(n\)-primary ideal \(g \subseteq n^2\), minimally generated by elements \(g_0, \ldots, g_k\). Let \(s_1, \ldots, s_t\) be elements of \(Q\) whose classes in \(Q/g\) form a basis for the socle. The ideal \(a = ng_0 + (g_1, \ldots, g_k)\) is \(n\)-primary, and if \(ns_i \subseteq a\) holds for all \(i = 1, \ldots, t\), then the classes of \(g_0, s_1, \ldots, s_t\) in \(Q/a\) form a basis for the socle; in particular one has \(\text{type}(Q/a) = \text{type}(Q/g) + 1\).

Proof. As \(g\) is \(n\)-primary, it follows from the containment \(ng \subseteq a\) that \(a\) is \(n\)-primary. Consider the rings \(R = Q/a\) and \(S = Q/g\); there is an exact sequence

\[0 \rightarrow g/a \rightarrow R \rightarrow S \rightarrow 0,\]

and an isomorphism of \(Q\)-modules \(g/a \cong k\), where \(k\) is the residue field of \(Q\). Tensoring with the Koszul complex \(K^Q\) one gets an exact sequence of \(Q\)-complexes,

\[0 \rightarrow k \otimes_Q K^Q \xrightarrow{\alpha} K^R \xrightarrow{\beta} K^S \rightarrow 0.
\]

Let \(d\) be the dimension of \(Q\). From the sequence in homology associated to \((\ast)\) one gets the following exact sequence

\[0 \rightarrow k \xrightarrow{H_d(\alpha)} H_d(K^R) \xrightarrow{H_d(\beta)} H_d(K^S).
\]

The rings \(R\) and \(S\) are artinian, and a rank count yields

\[\text{type}(R) = \text{rank}_k(H_d(K^R)) \leq \text{rank}_k(H_d(K^S)) + 1 = \text{type}(S) + 1.
\]

It is clear that the residue classes \([g_0]\) and \([s_1], \ldots, [s_t]\) in \(R\) are non-zero socle elements. Moreover, they are \(k\)-linearly independent. Indeed, the elements \([s_1], \ldots, [s_t]\) are \(k\)-linearly independent, because of the inclusion \(a \subseteq g\). Further, suppose one has \([g_0] = \sum_{i=1}^t [u_i][s_i]\) where the elements \(u_i\) are units in \(Q\). It follows that \(g_0 - \sum_{i=1}^t u_is_i\) is in \(a \subseteq g\), and as \(g_0 \in g\) one gets \(\sum_{i=1}^t u_is_i \in g\), a contradiction. Thus, there are \(t + 1\) \(k\)-linearly independent elements in the socle of \(R\).

For the next result, recall from work of J. Watanabe that a grade 3 Gorenstein ideal in a regular ring is minimally generated by an odd number of elements.

(2.4) Theorem. Let \((Q, n)\) be a regular local ring of dimension 3 and let \(g \subseteq n^2\) be an \(n\)-primary Gorenstein ideal minimally generated by elements \(g_0, \ldots, g_{2m}\). The ideal \(a = ng_0 + (g_1, \ldots, g_{2m})\) is \(n\)-primary, one has \(\text{type}(Q/a) = 2\) and:

(a) If \(m = 1\), then \(\mu(a) = 5\) and \(Q/a\) is of class \(\text{B}\).
(b) If \(m = 2\), then one of the following holds:
   • \(\mu(a) = 4\) and \(Q/a\) is of class \(\text{H}(3, 2)\).
   • \(\mu(a) = 5\) and \(Q/a\) is of class \(\text{B}\).
   • \(\mu(a) \in \{6, 7\}\) and \(Q/a\) is of class \(\text{G}(r)\) with \(\mu(a) - 2 \geq r \geq \mu(a) - 3\).
(c) If \(m \geq 3\), then \(Q/a\) is of class \(\text{G}(r)\) with \(\mu(a) - 2 \geq r \geq \mu(a) - 3\).

Proof. As \(g\) defines a Gorenstein ring, one has \(g : (g : b) = b\) for every ideal \(b\) in \(Q\) that contains \(g\). Let \(s \in Q\) be a representative of the socle of \(Q/g\); in \(Q\) one has

\[g \subseteq (a : n) \subseteq (g : n) = g + (s).
\]

Forming colon ideals one gets \(g : (a : n) \supseteq g : (g : n) = n\) and hence \(g : (a : n) = n\). Forming colon ideals a second time now yields \((a : n) = (g : n) = g + (s)\); in particular, one has \(na \subseteq a\), so it follows from Lemma (2.3) that \(a\) is \(n\)-primary and \(R = Q/a\) has type 2; in particular, \(R\) is not Gorenstein.
Note that one has
\[ 2m \leq \mu(a) \leq 2m + 3. \]
Set \( S = Q/\mathfrak{g}; \) there is an exact sequence of \( Q \)-modules
\[ 0 \to \mathfrak{g}/\mathfrak{a} \to R \to S \to 0 \]
and an isomorphism \( \mathfrak{g}/\mathfrak{a} \cong \mathbb{k}. \) Tensor with the Koszul complex \( K^Q \) to get an exact sequence of \( Q \)-complexes,
\[ 0 \to k \otimes_Q K^Q \xrightarrow{\alpha} K^R \xrightarrow{\beta} K^S \to 0, \]
where \( \beta \) is a morphism of DG \( Q \)-algebras. Set \( A = H(K^R) \) and \( G = H(K^S) \), one has
\[
\begin{align*}
\text{rank}_k(G_0) &= 1 = \text{rank}_k(G_3) \quad \text{and} \quad \text{rank}_k(G_1) = 2m + 1 = \text{rank}_k(G_2) \\
\text{rank}_k(A_0) &= 1, \quad \text{rank}_k(A_1) = \mu(a), \quad \text{rank}_k(A_2) = \mu(a) + 1, \quad \text{and} \quad \text{rank}_k(A_3) = 2,
\end{align*}
\]
and consider the exact sequence in homology associated to \((*)\)
\[
\begin{array}{cccccccc}
& 0 & \to & k & \to & H_3(\alpha) & \to & A_3 & \to & H_3(\beta) & \to & G_3 & & \\
& & & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & k^3 & \to & H_2(\alpha) & \to & A_2 & \to & H_2(\beta) & \to & G_2 & & \\
& & & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & & k^3 & \to & H_1(\alpha) & \to & A_1 & \to & H_1(\beta) & \to & G_1 & & \\
& & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & & k & \to & H_0(\alpha) & \to & A_0 & \to & H_0(\beta) & \to & G_0 & \to & 0
\end{array}
\]
where the numbers below the arrows indicate the ranks of the maps. As \( H(\beta) \) is a homomorphism of graded \( k \)-algebras, there is a commutative diagram
\[
\begin{array}{ccc}
G_2 & \xrightarrow{\delta_G} & \text{Hom}_k(G_1, G_3) \\
\downarrow & & \downarrow \\
H_2(\beta) & \xrightarrow{\mu(a)-2} & \text{Hom}_k(H_1(\beta), G_3) \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{\varepsilon} & \text{Hom}_k(A_1, G_3) \\
\downarrow & & \downarrow \\
A_2 & \xrightarrow{\delta_A} & \text{Hom}_k(A_1, A_3)
\end{array}
\]
The rank of \( \delta_A \) is at least the rank of \( \varepsilon = \text{Hom}_k(A_1, H_3(\beta)) \circ \delta_A \). The rank of \( \varepsilon \) is at least \( (\mu(a) - 2) - 1 = \mu(a) \) as \( \delta_G \) is an isomorphism, see [1, 1.4.2], and the kernel of \( \text{Hom}_k(H_1(\beta), G_3) \) has rank 1. Thus, one has \( r = \text{rank}(\delta_A) \geq \mu(a) - 3 \).

In case \( \mu(a) \geq 6 \), one has \( r \geq 3 \), and since the type of \( R \) is 2, this implies that \( R \) is of class \( \mathbb{G}(r); \) see [2, 2.1]. This proves part (c) and the last case of part (b). For \( m = 2 \) one has \( 4 \leq \mu(a) \leq 7 \). If \( \mu(a) = 5 \) it follows from Proposition [2, 2.2] and [2, 2.1] that \( R \) is of class \( \mathbb{B} \). If \( \mu(a) = 4 \), then \( R \) is of class \( \mathbb{H}(3, 2) \) by [1, 3.4.2.(a)]. Finally, part (a) is a result of Faucett [7]. ∎
3. A family of graded local rings of class $G(r)$

A grade 3 Gorenstein ideal of a local ring is by a result of Buchsbaum and Eisenbud [4, thm. 2.1] minimally generated by the sub-maximal Pfaffians of a $(2m + 1) \times (2m + 1)$ skew-symmetric matrix. Thus, skew-symmetric matrices are a source of Gorenstein rings and, via Theorem (2.4), also a source of rings of class $G(r)$ that are not Gorenstein. In this section, we construct an infinite family of such rings.

(3.1) Let $k$ be a field and set $Q = k[x, y, z]$; let $m$ be a positive integer.

Denote by $U_m$ the $m \times m$ matrix over $Q$ whose $i$th row has entries

\[ u_{i,m-i} = x, \quad u_{i,m-i+1} = z, \quad \text{and} \quad u_{i,m-i+2} = y \]

and 0 elsewhere; set $d_{-1} = 0, \ d_0 = 1,$ and $d_m = \det(U_m)$.

That is,

\[ U_1 = [z], \quad U_2 = \begin{bmatrix} x & z \\ z & y \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & x & z \\ x & z & y \\ z & y & 0 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 0 & 0 & x & z \\ 0 & x & z & y \\ x & z & y & 0 \\ z & y & 0 & 0 \end{bmatrix}, \quad \ldots \]

\[ d_1 = z, \quad d_2 = xy - z^2, \quad d_3 = 2xyz - z^3, \quad d_4 = -3xyz^2 + x^2y^2 + z^4, \quad \ldots \]

Notice that for every $i$ in the range $2, \ldots, m$ one has,

\[ U_m = \begin{bmatrix} O_x & U_{i-1} \\ U_{m-i+1} & yO \end{bmatrix}, \]

where $O_x$ is the appropriately sized matrix with $x$ in the lower right corner and 0 elsewhere, and $yO$ is the matrix with $y$ in the top left corner and 0 elsewhere.

Let $V_m$ be the $(2m + 1) \times (2m + 1)$ skew-symmetric matrix given by

\[ V_m = \begin{bmatrix} O & \begin{bmatrix} O_x \\ & 0 \\ & & yO \end{bmatrix} \\ \begin{bmatrix} -O_x & 0 \\ -O & -yO \end{bmatrix} & U_m \end{bmatrix}, \]

where $O$ is the $m \times m$ zero-matrix and, as above, $O_x$ and $yO$ are appropriately sized matrices with 0 everywhere but in the lower left and upper right corner, respectively. That is,

\[ V_1 = \begin{bmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 & 0 & x & z \\ 0 & 0 & x & z & y \\ 0 & -x & 0 & y & 0 \\ -x & -z & -y & 0 & 0 \\ -z & -y & 0 & 0 & 0 \end{bmatrix}, \quad \ldots \]

The sub-maximal Pfaffians of $V_m$ are determined (up to a sign) by minors,

\[ \operatorname{pf}_i(V_m)^2 = \det((V_m)_{ii}). \]

Consider the ideal of $Q$ generated by these Pfaffians,

\[ \mathfrak{g}_m = (\operatorname{pf}_1(V_m), \ldots, \operatorname{pf}_{2m+1}(V_m)) \].
(3.2) Lemma. In the notation from (3.1) the next equalities hold for every $m \geq 1$.

\[ d_m = (-1)^{m-1}zd_{m-1} + xyd_{m-2} \quad \text{and} \quad \]

\[ d_m = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} (-1)^{\frac{m-2j}{2}} xy^j z^{m-2j}. \]

Proof. Per (3.1.1) with $i = 2$, expansion of the determinant of $U_m$ along the first row yields

\[ d_m = (-1)^m x \det((U_m)_{1,m-1}) + (-1)^{m+1} z \det(U_{m-1}). \]

From (3.1.1) with $i = 3$ it follows that expansion along the last column yields

\[ \det((U_m)_{1,m-1}) = (-1)^m y \det(U_{m-2}). \]

Combining these two expressions, one gets the first equality. The second equality now follows by induction. \hfill \square

Evidently, the ideal $g_m$ from (3.1.4) is contained in $n^m$; in fact, one has $g_1 = n$. One can check that, though the generating matrices are different, the family of ideals $\{g_m\}_{m \geq 2}$ is the same as that provided by [4, prop. 6.2]. To understand what happens when one trims these ideals, we provide a more detailed description.

(3.3) Proposition. Adopt the notation from (3.1) and let $n$ denote the maximal ideal of $Q$. For every $m \geq 2$ the ideal $g_m \subseteq n^2$ is an $n$-primary Gorenstein ideal minimally generated by the elements

\[ x^{m-i}d_i \text{ and } y^{m-i}d_i \text{ for } 0 \leq i \leq m-1 \quad \text{and} \quad d_m. \]

The ring $Q/g_m$ has socle generated by the class of $x^{m-1}y^{m-1}$ and Hilbert series

\[ \text{Hilb}_{Q/g_m}(t) = \sum_{i=0}^{m-2} \binom{i+2}{2} (t^i + t^{2m-2-i}) + \binom{m+1}{2} t^{m-1}. \]

Proof. Per (3.1.3) the Pfaffians of $V_1$ are, up to signs,

\[ \text{pf}_1(V_1) = y = yd_0, \quad \text{pf}_2(V_1) = z = d_1, \quad \text{and} \quad \text{pf}_3(V_1) = x = xd_0. \]

For $m \geq 2$ we argue that, up to signs, one has

\[ \text{pf}_i(V_m) = y^{m-i+1}d_{i-1} \text{ for } 1 \leq i \leq m, \]

\[ \text{pf}_{m+1}(V_m) = d_m, \quad \text{and} \quad \]

\[ \text{pf}_{2m+i-2}(V_m) = x^{m-i+1}d_{i-1} \text{ for } 1 \leq i \leq m. \]

First notice that the equality $\text{pf}_{m+1}(V_m) = d_m$ is immediate from (3.1.2). Further, note that by symmetry in $x$ and $y$ it is sufficient to prove that $\text{pf}_i(V_m) = y^{m-i+1}d_{i-1}$ holds for $1 \leq i \leq m$. To compute $\text{pf}_1(V_m)$ notice that the matrix $(V_m)_{11}$ is a $2m \times 2m$-matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, one has $\text{pf}_1(V_m) = y^m = y^{m}d_0$. Now, for $i$ in the range $2, \ldots, m$ consider the matrix $(V_m)_{ii}$ as $2 \times 2$ block matrix with blocks of size $m \times m$,

\[ (V_m)_{ii} = \begin{bmatrix} X & W_i \\ -W_i^T & O \end{bmatrix}, \]
where $O$ is as in (3.1.2), i.e. it is zero. Thus, one has

$$\det((V_m)_{ii}) = \begin{vmatrix} X & W_i \\ -W_i^T & O \end{vmatrix} = (-1)^m \begin{vmatrix} W_i & X \\ O & -W_i^T \end{vmatrix} = (\det(W_i))^2.$$  

Next, notice that $W_i$ is obtained from $U_m$ by removing row $i$ and adding a row $^yO$ at the bottom. Thus, per (3.1.1) it has the form

$$W_i = \begin{bmatrix} O_r & U_{i-1} \\ Y & O \end{bmatrix},$$

where $Y$ is the matrix obtained from $U_{m-i+1}$ by removing the first row and adding a row $^yO$ at the bottom. In particular, it is a $(m - i + 1) \times (m - i + 1)$-matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, computing the determinant of $W_i$ by successive expansion on the last $m - i + 1$ rows one gets, up to a sign, $pf_i(V_m) = y^{m-i+1}d_{i-1}$. It follows that $g_m$ is generated by the listed elements.

The elements $x^m, y^m, d_m$ form a $Q$-regular sequence in $g_m$, so it follows from (3.1.1) that $g_m$ is a Gorenstein ideal minimally generated by the listed elements. In particular, $g_m$ is $n$-primary. In fact, in this case it is elementary to see that the generating set is minimal: Notice from Lemma (3.1.2) that $d_i$ is a linear combination of monomials of the form $x^iy^jz^{i-j}$. Hence, each generator $x^{m-i}d_i$ is a linear combination of monomials of the form $x^{m-i+j}y^jz^{i-j}$ while the generators $y^{m-i}d_i$ are linear combinations of monomials $x^iy^{m-i+j}z^{i-j}$. Thus the generators are linear combinations of disjoint sets of degree $m$ monomials and hence linearly independent.

The Hilbert series of the power series ring $Q$ is $\text{Hilb}_Q(t) = \sum_{j=0}^\infty \left(\frac{t^j}{j!}\right)^2$. Since $g_m$ is Gorenstein and minimally generated by $2m + 1$ elements of degree $m$, the Hilbert series of the ring $S_m = Q/g_m$ is symmetric and given by

$$\text{Hilb}_{S_m}(t) = \sum_{i=0}^{m-2} \left(\frac{i+2}{2}\right) (t^i + t^{2m-2-i}) + \left(\frac{m+1}{2}\right) t^{m-1}.$$  

In particular, the socle degree of $S_m$ is $2m - 2$. Evidently, one has $(x^{m-1}y^{m-1})n \subseteq g_m$, so it is sufficient to show that the element $x^{m-1}y^{m-1}$ is not in $g_m$, i.e. that it yields a non-zero socle element in $S_m$. If it were in $g_m$, then one would have $x(x^{m-2}y^{m-1}) = \text{socle}(x) = y(x^{m-2}y^{m-1}) = z(x^{m-2}y^{m-1})$. Thus, $x^{m-2}y^{m-1}$ would yield a socle element in $S_m$ of degree $2m - 3$, whence it must be 0; i.e. one would have $x^{m-2}y^{m-1} \in g_m$. Reiterating this argument, one arrives at the conclusion that $y^{m-1}$ is in $g_m$, which is absurd as the generators of $g_m$ have degree $m$.

Finally, we apply the trimming procedure from Theorem (3.4) to the ideals $g_m$.

(3.4) Adopt the notation from (3.1.1). By Proposition (3.3) one has

$$g_2 = (x^2, xz, xy - z^2, yz, y^2).$$

Trimming the generators $xz$ and $yz$ one gets the following ideals of $Q$,

$$\begin{align*}
(x, y, z) &= (x^2, xy - z^2, yz, y^2) = (x^2, xy - z^2, yz, y^2) \quad \text{and} \\
(x, y, z) &= (x^2, xz, xy - z^2, y^2) = (x^2, xz, xy - z^2, y^2).
\end{align*}$$
They are both minimally generated by 4 elements, so they define quotient rings of class $\text{H}(3, 2)$; see Theorem (2.4) (b). Moreover, one has
\[
(x, y, z)x^2 + (x, xy - z^2, yz, y^2) = (x^3, xz, xy - z^2, yz, y^2),
\]
\[
(x, y, z)y^2 + (x^2, xz, xy - z^2, yz) = (x^2, xz, xy - z^2, yz, y^3),
\]
and
\[
(x, y, z)(xy - z^2) + (x^2, xz, yz, y^2) = (x^2, xz, z^3, yz, y^2),
\]
so by Theorem (2.4) (b) these ideals define rings of class $\text{B}$.

From the next result one immediately gets the statement of Theorem (1.1) about existence of infinite families of rings of class $\text{G}(r)$ that are not Gorenstein.

(3.5) **Proposition.** Adopt the notation from (3.1) and let $\mathfrak{n}$ denote the maximal ideal of $Q$. Let $\mathfrak{g}$ be one of the generators of $\mathfrak{g}_m$ listed in (3.3), let $\mathfrak{b}$ be the ideal generated by the remaining $2m$ generators of $\mathfrak{g}_m$, and set $\mathfrak{a} = \mathfrak{n} \mathfrak{g} + \mathfrak{b}$. For $m \geq 3$ the ring $R = Q/\mathfrak{a}$ has the following properties.

(a) $R$ is an artinian local ring of type 2 with socle generated by the classes of the elements $\mathfrak{g}$ and $x^{m-1}y^{m-1}$.

(b) If $\mathfrak{g}$ is $x^{m-i}d_i$ or $y^{m-i}d_i$ for some $i \in \{1, \ldots, m-1\}$, then $\mathfrak{a}$ is minimally generated by $2m$ elements and $R$ is of class $\text{G}(2m-3)$.

(c) If $\mathfrak{g}$ is $x^m$, $y^m$, or $d_m$, then $\mathfrak{a}$ is minimally generated by $2m + 1$ elements and $R$ is of class $\text{G}(2m-2)$.

**Proof.** Fix $m \geq 3$; for brevity the class in $R$ or $S = Q/\mathfrak{g}_m$ of an element $u$ in $Q$ is also written $u$.

Part (a) is immediate from Lemma (2.3). We prove parts (b) and (c) together. First we describe the generators of $\mathfrak{a}$ using the recurrence formula from Lemma (3.2).

For $1 \leq i \leq m$ one has
\[
x(x^{m-i}d_i) = x^{m-(i-1)}((-1)^{i-1}z_{d_{i-1}} + xyd_{i-2})
\]
\[
= (-1)^{i-1}z(x^{m-(i-1)}d_{i-1}) + y(x^{m-(i-2)}d_{i-2}).
\]

For $0 \leq i \leq m - 2$ one has
\[
y(x^{m-i}d_i) = x^{m-(i+1)}(xyd_i)
\]
\[
= x^{m-(i+1)}(d_{i+2} - (-1)^{i+1}z_{d_{i+1}})
\]
\[
= x(x^{m-(i+2)}d_{i+2}) + (-1)^iz(x^{m-(i+1)}d_{i+1})
\]
and moreover
\[
y(xd_{m-1}) = x(yd_{m-1}).
\]

For $0 \leq i \leq m - 1$ one has
\[
z(x^{m-i}d_i) = x^{m-i}(-1)^i(d_{i+1} - xyd_{i-1})
\]
\[
= (-1)^iz(x(x^{m-(i+1)}d_{i+1}) - (-1)^iy(x^{m-(i-1)}d_{i-1}).
\]

For $g = x^{m-i}d_i$ with $1 \leq i \leq m - 1$ it follows immediately from (1) – (3) that $\mathfrak{n} \mathfrak{g}$ is contained in $\mathfrak{a}$, so $\mathfrak{a}$ is minimally generated by $2m$ elements. By symmetry the same is true for $g = y^{m-i}d_i$ with $1 \leq i \leq m - 1$.

For $g = x^m$ one has $yg \in \mathfrak{b}$ and $zg \in \mathfrak{b}$ by (2) and (3), so $\mathfrak{a}$ is generated by the $2m$ generators of $\mathfrak{b}$ and $x^{m+1}$. To see that this is a minimal set of generators, note that the generators of $\mathfrak{b}$ have degree $m$ and none of them includes the term $x^m$.

The statement for $g = y^m$ follows by symmetry.
For $g = d_m$ one has $xg \in b$ by (1) and $yg \in b$ by symmetry. Thus $a$ is generated by the $2m$ generators of $b$ and $zd_m$. To see that this is a minimal set of generators, note from Lemma (3.2) that $zd_m$ has a $z^{m+1}$ term, while the generators of $b$ have degree $m$ and none of them has a $z^m$ term.

To determine the multiplicative structure on $A = H(K^R)$ we first describe a basis for $A_1$. The Koszul complex $K^R$ is the exterior algebra of the free $R$-module with basis $\{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$ endowed with the differential given by $\partial(\varepsilon_x) = x$, $\partial(\varepsilon_y) = y$, and $\partial(\varepsilon_z) = z$. We suppress the wedge in products on $K^R$ and adopt the following shorthands

$$\varepsilon_{xy} = \varepsilon_x \varepsilon_y, \quad \varepsilon_{xz} = \varepsilon_x \varepsilon_z, \quad \varepsilon_{yz} = \varepsilon_y \varepsilon_z, \quad \text{and} \quad \varepsilon_{xyz} = \varepsilon_x \varepsilon_y \varepsilon_z.$$ 

Because of the symmetry in $x$ and $y$ we only consider $g = x^{m-i}d_i$. Given the minimal generating set of $a$ described above, one gets:

If $g = x^m$ then the following cycles in $K_1^R$ yield a basis for $A_1$

$$x^m \varepsilon_x \quad \text{and} \quad x^{m-j-1}d_j \varepsilon_x \quad \text{for} \quad 1 \leq j \leq m - 1,$$

$$y^{m-j-1}d_j \varepsilon_y \quad \text{for} \quad 0 \leq j \leq m - 1, \quad \text{and} \quad (-1)^{m-1}z^{m-1}d_{m-1} \varepsilon_z + xd_{m-2} \varepsilon_y .$$

If $g = x^{m-i}d_i$ for some $i$ in the range $1, \ldots , m - 1$, then the following cycles in $K_1^R$ yield a basis for $A_1$

$$x^{m-j-1}d_j \varepsilon_x \quad \text{for} \quad 0 \leq j \leq m - 1, \quad j \neq i$$

$$y^{m-j-1}d_j \varepsilon_y \quad \text{for} \quad 0 \leq j \leq m - 1, \quad \text{and} \quad (-1)^{m-1}z^{m-1}d_{m-1} \varepsilon_z + xd_{m-2} \varepsilon_y .$$

If $g = d_m$ then the following cycles in $K_1^R$ yield a basis for $A_1$

$$x^{m-j-1}d_j \varepsilon_x \quad \text{for} \quad 0 \leq j \leq m - 1,$$

$$y^{m-j-1}d_j \varepsilon_y \quad \text{for} \quad 0 \leq j \leq m - 1, \quad \text{and} \quad d_m \varepsilon_z .$$

From Theorem (2.4) it is known that $R$ is of class $G(r)$ with $\mu(a) - 3 \leq r$. To prove that equality holds, which is the claim in (b) and (c), it suffices to show that the kernel of $\delta$ has rank at least $(\mu(a) + 1) - (\mu(a) - 3) = 4$; see (2.1). To this end we first notice that the cycles $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements of $A_2$. Assume towards a contradiction that they are not, then there exists an element $h\varepsilon_{xy}$ in $K_3^Q$ and elements $q_1$, $q_2$, and $q_3$ in $Q$ and not all in $n$ with

$$\partial(h\varepsilon_{xy}) - (q_1g\varepsilon_{xy} + q_2g\varepsilon_{xz} + q_3g\varepsilon_{yz}) \in aK_2^Q .$$

That is, one has $zh - q_1g \in a$, $yh + q_2g \in a$, and $xh - q_3g \in a$, and hence $h \notin n^m$ as $g \notin a + n^{m+1}$. Furthermore, the class of $h$ is a socle element in $S$ as one has $nh \subseteq a + Qg = g_m$. Thus, $h \in g_m$ or $h = qx^{m-1}y^{m-1}$ for some $q \in Q \setminus n$. In either case one has $h \in n^m$, which is a contradiction. Thus $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements in $A_2$ that clearly belong to the kernel of $\delta$.

Finally we produce a fourth element in the kernel. For $g = x^n$ the element

$$f = y^{m-1}\varepsilon_{yz}$$

is clearly a cycle in $K_2^R$, and it is not a boundary. Indeed, if one had $f = \partial(h\varepsilon_{yz}) = hx\varepsilon_{yz} - hy\varepsilon_{xz} + hz\varepsilon_{xy}$ for some homogeneous element $h \in R$, then
it would have degree \( m - 2 \) and one would have \( hy = 0 = hz \) in \( R \), which is impossible as a has generators of degree at least \( m \). The products \((y^{m-j-1}d_j \varepsilon_y) \cdot f\) and \((-(1)^{m-1}z^{m-1}\varepsilon_z + xd_{m-2} \varepsilon_y) \cdot f\) in \( K^R \) vanish by graded commutativity. Moreover, one has

\[
(x^m \varepsilon_x) \cdot f = x(x^{m-1}y^{m-1})\varepsilon_{xyz} = 0 \quad \text{and} \quad (x^{m-j-1}d_j \varepsilon_x) \cdot f = x^{m-j-1}y^{j-1}(y^{m-j}) \varepsilon_{xyz} = 0.
\]

Thus the homology class of \( f \) annihilates \( A_1 \).

For \( g = x^{m-i}d_i \) and \( 1 \leq i \leq m - 1 \) the element

\[
f = y^{m-i}d_{i-1} \varepsilon_{xy} + (-1)^{i-1}y^{m-i-1}d_i \varepsilon_{yz}
\]

is a cycle in \( K^R_d \); indeed one has

\[
\partial(f) = xy^{m-i}d_{i-1} \varepsilon_y - y^{m-(i-1)}d_{i-1} \varepsilon_x + (-1)^{i-1}y^{m-i}d_i \varepsilon_z + (-1)^iy^{m-i-1}zd_i \varepsilon_y
\]

\[
= y^{m-i-1}((-1)^{i}zd_i + xyd_{i-1}) \varepsilon_y
\]

\[
= y^{m-(i+1)}d_{i+1}
\]

\[
= 0,
\]

where the third equality follows from Lemma \((2.3)\). An argument similar to the one above shows that \( f \) is not a boundary. The products \((y^{m-j-1}d_j \varepsilon_y) \cdot f\) in \( K^R \) vanish by graded commutativity. Moreover, one has

\[
(x^{m-j-1}d_j \varepsilon_x) \cdot f = (-1)^{i-1}x^{m-j-1}d_j y^{m-i-1}d_i \varepsilon_{xyz}.
\]

If \( i > j \) holds, then the element \( x^{m-j-1}d_j y^{m-i-1}d_i \) is 0 in \( R \) because it is divisible by \( g \), which is a socle element in \( R \). If one has \( i < j \), then the element \( x^{m-j-1}d_j y^{m-i-1}d_i \) is zero in \( R \) because it is divisible in \( Q \) by the generator \( y^{m-j}d_j \) of \( a \). Finally, one has

\[
((-1)^{m-1}z^{m-1}d_{m-1} \varepsilon_z + xd_{m-2} \varepsilon_y) \cdot f = (-1)^{m-1}y^{m-i}d_{i-1}z^{m-1}d_{m-1} \varepsilon_{xyz}
\]

\[
= (-1)^{m-1}y^{m-i-1}d_{i-1}z^{m-1}(yd_{m-1}) \varepsilon_{xyz}
\]

\[
= 0
\]

in \( K^R \), so the homology class of \( f \) annihilates \( A_1 \).

For \( g = d_m \) the element

\[
f = d_{m-1} \varepsilon_{xy}
\]

is evidently a cycle in \( K^R_d \), and as above it is not a boundary. The products \((x^{m-j-1}d_j \varepsilon_x) \cdot f\) and \((y^{m-j-1}d_j \varepsilon_y) \cdot f\) in \( K^R \) vanish by graded commutativity. Finally one has

\[
(d_m \varepsilon_z) \cdot f = d_{m-1}d_m \varepsilon_{xyz} = 0,
\]

as \( g = d_m \) is a socle element of \( R \).

\[\square\]

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References

1. Luchezar L. Avramov, *A cohomological study of local rings of embedding codepth 3*, J. Pure Appl. Algebra 216 (2012), no. 11, 2489–2506. MR2927181
2. Luchezar L. Avramov and Evgeniy S. Golod, *On the homology algebra of the Koszul complex of a local Gorenstein ring*, Mat. Zametki 9 (1971), 53–58. MR029157
3. Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra 118 (1988), no. 1, 162–204. MR09061334
4. David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. 99 (1977), no. 3, 447–485. MR0453723
5. Lars Winther Christensen and Oana Veliche, *Local rings of embedding codepth 3. Examples*, Algebr. Represent. Theory 17 (2014), no. 1, 121–135. MR3160716
6. Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Linkage classes of grade three perfect ideals*, in preparation.
7. Jessica Ann Faucett, *Expanding the socle of a codimension 3 complete intersection*, Rocky Mountain J. Math. 46 (2016), no. 5, 1489–1498. 1580796
8. Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. 50 (1973), 227–232. MR0319985
9. Jerzy Weyman, *On the structure of free resolutions of length 3*, J. Algebra 126 (1989), no. 1, 1–33. MR1023284

Texas Tech University, Lubbock, TX 79409, U.S.A.
E-mail address: lars.w.christensen@ttu.edu
URL: http://www.math.ttu.edu/~lchriste

Northeastern University, Boston, MA 02115, U.S.A.
E-mail address: o.veliche@neu.edu

University of Connecticut, Storrs, CT 06269, U.S.A.
E-mail address: jerzy.weyman@uconn.edu
URL: http://www.math.uconn.edu/~weyman