A Flow Formulation for Horizontal Coordinate Assignment with Prescribed Width

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Sugiyama framework

- (Cycle breaking)
- Layer assignment
- Crossing minimization
- Coordinate assignment
- (Edge routing)
Coordinate assignment

Input:
- DAG $G$
- Layering $\mathcal{L}$
- Vertex ordering $ord$

Output:
- Feasible $x$-coordinates, i.e.
  \[ x(u) + \delta \leq x(v), \]
  if $ord(u) < ord(v)$
Introduction

Coordinate assignment

Goals:

• Short horizontal edge lengths → [Gansner et al., 1993]
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- Vertical edges
Introduction

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What about the width?
• Width of the **layering**: maximum number of nodes in a layer \rightarrow [Coffman and Graham, 1972]
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  maximum number of nodes in a layer $\rightarrow$ [Coffman and Graham, 1972]
≠ Width of the drawing:
  maximum horizontal distance between any two nodes
• Width of the **layering**: maximum number of nodes in a layer $\rightarrow$ [Coffman and Graham, 1972]

≠ Width of the **drawing**: maximum horizontal distance between any two nodes

\[
\begin{align*}
\text{width} &= 1 \\
\text{edge length} &= k - 3 \\
\end{align*}
\]

\[
\begin{align*}
\text{width} &= k - 2 \\
\text{edge length} &= 0 \\
\end{align*}
\]
Our Contribution

Efficient algorithm for drawings with minimized total edge length under the restriction of bounded drawing width.
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In addition it realizes

• vertical inner segments
• minimum/maximum/exact distances between nodes
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Method: Minimum cost flow formulation for the coordinate assignment problem
Network Construction

Idea:

- Network with super source and super sink
- Send flow from top to bottom
Network Construction

Idea:

- Network with super source and super sink
- Send flow from top to bottom
- Flow = horizontal distance
- Cost = edge length
Network Construction

Minimum distance $\delta$ between nodes

+ Flow means distance

$\Rightarrow$ Network arcs between nodes with lower bound $\delta$
⇒ x-coordinate for node $v$:

$$x(v) = \sum_{a \text{ left of } v} f(a),$$

where $f(a)$ is the flow over network arc $a$. 

A Flow Formulation for Horizontal Coordinate Assignment with Prescribed Width
Network Construction

Send flow along layers and over some inter-layer connections without lower or upper bounds
Connections between layers:
- arc from leftmost upper node to leftmost lower node
Network Construction

Connections between layers:
- arc from leftmost upper node to leftmost lower node
- arc from rightmost upper node to rightmost lower node
Network Construction

Connections between layers:

- arc from leftmost upper node to leftmost lower node
- arc from rightmost upper node to rightmost lower node
- in *hug* situations: two network nodes enclosed by graph edges
Network Construction

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Network Construction

Cost corresponds to edge length and flow corresponds to distance
Network Construction

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Network Construction

Cost corresponds to edge length and flow corresponds to distance

⇒ cost = number of crossed edges
Network Construction

cost = 1
lower bound = 0
upper bound = ∞
Network Construction

cost = 2
lower bound = 0
upper bound = ∞
Network Construction

Cost = 3
Lower bound = 0
Upper bound = ∞
Network Construction

A Flow Formulation for Horizontal Coordinate Assignment with Prescribed Width

- Cost = 0
- Lower bound = 0
- Upper bound = ∞
Network Construction

![Diagram of network construction]

- Cost: $\text{cost} = 0$
- Lower bound: $\text{lower bound} = \delta$ (or 0)
- Upper bound: $\text{upper bound} = \infty$
Network Construction

super source and super sink

cost = 0
lower bound = 0
upper bound = ∞
Example Flows and Corresponding Drawings

\[ \delta = 1 \]

*flow in blue*

*arc cost in red*
Example Flows and Corresponding Drawings

\[ \sum \text{cost} \cdot \text{flow} = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 4 \]

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Example Flows and Corresponding Drawings

\[ \sum \text{cost} \cdot \text{flow} = 2 \cdot 1 + 1 \cdot 1 = 3 \]
Main Theorem

Every minimum cost flow in network described above defines feasible $x$-coordinates and minimum total horizontal edge length.
Results

Main Theorem

Every minimum cost flow in network described above defines feasible $x$-coordinates and minimum total horizontal edge length.

What about other constraints?
Further constraints

Width:

split super source and set upper bound of new arc to maximum width
Further constraints

Maximum/exact distances:
set bounds to appropriate values
Further constraints

Vertical edges:
delete appropriate arcs
Computational results

Test set: 1277 graphs, |V| = 10-100 (subset of AT&T)
Implementation: OGDF, network simplex algorithm

A Flow Formulation for Horizontal Coordinate Assignment with Prescribed Width
Conclusion

• Minimum cost flow formulation for the coordinate assignment problem
• Minimize edge length while respecting further constraints
• Comparable running time to state-of-the-art algorithms
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A Flow Formulation for Horizontal Coordinate Assignment with Prescribed Width
Brandes, U. and Köpf, B. (2001). Fast and simple horizontal coordinate assignment. In Mutzel, P., Jünger, M., and Leipert, S., editors, Graph Drawing, 9th International Symposium, GD 2001 Vienna, Austria, September 23-26, 2001, Revised Papers, volume 2265 of Lecture Notes in Computer Science, pages 31–44. Springer.

Buchheim, C., Jünger, M., and Leipert, S. (2000). A fast layout algorithm for $k$-level graphs. In Marks, J., editor, Graph Drawing, 8th International Symposium, GD 2000, Colonial Williamsburg, VA, USA, September 20-23, 2000, Proceedings, volume 1984 of Lecture Notes in Computer Science, pages 229–240. Springer.
Coffman, E. G. and Graham, R. L. (1972). Optimal scheduling for two-processor systems. *Acta Informatica*, 1(3):200–213.

Gansner, E. R., Koutsofios, E., North, S. C., and Vo, K.-P. (1993). A technique for drawing directed graphs. *Software Engineering*, 19(3):214–230.