Leibniz Cohomology and the Calculus of Variations

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1 Introduction

In a previous publication [6] a definition of Loday’s Leibniz cohomology, $H L^*$, was proposed for differentiable manifolds. In particular every $k$-tensor $\omega$ (from classical differential geometry) is a cochain in the Leibniz complex. Although the Leibniz coboundary, $d\omega$, is not necessarily a $(k + 1)$-tensor, $d\omega$ remains a local operator on vector fields with the value of $d\omega$ at a point $p$ in the manifold $M$ determined by the values of $\omega$ in an arbitrary open neighborhood containing $p$. With this writing we offer an explicit formula for $d\omega$ in a local coordinate chart, and provide a geometric interpretation of $d\omega$ in terms of the calculus of variations. If $\omega$ is the metric two-tensor on a Riemannian manifold, then the local expression for $d\omega$ involves the Christoffel symbols, while the global definition of $d\omega$ reduces to the first variation formula for arc length. More generally the Leibniz coboundary of any two tensor $\omega$ can be written in terms of the necessary conditions to achieve a minimum (or maximum) value of $\int \omega$ over a locally immersed curve or surface. The paper closes with the computation of the Leibniz coboundary of the Riemann curvature tensor $R$ in terms of its covariant derivative $\nabla R$.

Section two of the paper begins with a brief recollection of $H L^*$ for a differentiable manifold $M$, and proceeds with the foundational material needed to prove that $d\omega$ is a local operator. Section three contains the results for two tensors and the calculus of variations. The final section provides the local
coboundary formula for arbitrary $k$-tensors as well as the global coboundary of the Riemann curvature tensor. For more background material about $HL^*(M)$ and in particular calculations of $HL^*$ for Euclidean $n$-space (which are highly non-trivial), see [6]. For more information about Leibniz homology and cohomology, see [3] [4] [5].

2 The Leibniz Coboundary as a Local Operator

We begin by reviewing the definition of Leibniz cohomology for differentiable manifolds [3], and show that the Leibniz coboundary of a $k$-tensor, $d\omega$, is a local operator, i.e. $d\omega$ at $p \in M$ is determined by the value of $\omega$ on an arbitrary open neighborhood of $p \in M$. This permits (in later sections) the formulation of $d\omega$ in terms of a local coordinate chart. Let $M$ be a differentiable ($C^\infty$) manifold of dimension $n$, $\chi(M)$ the Lie algebra of $C^\infty$ vector fields on $M$, and $C^\infty(M)$ the algebra of $C^\infty$ real-valued functions $f : M \to \mathbb{R}$. Recall that $C^\infty(M)$ is a left representation of $\chi(M)$ via

$$\chi(M) \otimes_{\mathbb{R}} C^\infty(M) \to C^\infty(M)$$

$$[X, f] \mapsto X(f),$$

where $X(f)$ is the Lie derivative of $f \in C^\infty(M)$ in the direction $X \in \chi(M)$. Let

$$C^k(M) = \text{Hom}_{\mathbb{R}}^c(\chi(M)^{\otimes k}, C^\infty(M)), \quad k \geq 0,$$

denote the $\mathbb{R}$-vector space of continuous homomorphisms

$$\alpha : \chi(M)^{\otimes k} \to C^\infty(M)$$

in the strong $C^\infty$ topology. See [2] for a discussion of this topology. Then the Leibniz cohomology of $M$ with coefficients in $C^\infty(M)$, written

$$HL^*(\chi(M); C^\infty(M)),$$

is the homology of the cochain complex

$$C^0(M) \to C^1(M) \to \ldots \to C^k(M) \xrightarrow{d} C^{k+1}(M) \to \ldots ,$$
where

\[ d\alpha(X_1 \otimes X_2 \otimes \ldots \otimes X_{k+1}) = \]
\[ \sum_{i=1}^{k+1} (-1)^{i+1} X_i (\alpha(X_1 \otimes \ldots \hat{X}_i \ldots \otimes X_{k+1})) + \]
\[ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j+1} \alpha(X_1 \otimes \ldots \otimes X_{i-1} \otimes [X_i, X_j] \otimes X_{i+1} \otimes \ldots \hat{X}_j \ldots \otimes X_{k+1}). \]

(2.1)

Let \( \omega \) be a \( k \)-tensor on \( M \), i.e.

\[ \omega : M \to T^*(M)^\otimes k \]

is a \( C^\infty \) section of the \( k \)-fold tensor product of the cotangent bundle. Then \( \omega \) determines an element of

\[ \text{Hom}^c_R(\chi(M)^\otimes k, C^\infty(M)) \]

via \( \omega(X_1 \otimes X_2 \otimes \ldots \otimes X_k) : M \to \mathbb{R} \)

\[ \omega(X_1 \otimes X_2 \otimes \ldots \otimes X_k)(p) = \omega(X_1(p) \otimes X_2(p) \otimes \ldots X_k(p)). \]

Although \( d\omega \) is not necessarily a \( (k+1) \)-tensor, the following local result remains valid.

**Lemma 2.1.** Let \( \omega \) be a \( k \)-tensor on \( M \) and \( O \subset M \) open. If

\[ X_1, X_2, \ldots, X_{k+1}, Y_1, \ldots, Y_{k+1} \in \chi(M) \]

with \( X_i = Y_i, i = 1, 2, \ldots k+1, \) on \( O \), then

\[ d\omega(X_1 \otimes \ldots \otimes X_{k+1})(p) = d\omega(Y_1 \otimes \ldots \otimes Y_{k+1})(p) \]

for all \( p \in O \).

**Proof.** Since the Lie bracket at \( p \)

\[ [X_i, X_j](p) \]

is determined by the values of \( X_i \) and \( X_j \) on an open set containing \( p \), we have

\[ [X_i, X_j](p) = [Y_i, Y_j](p), \quad p \in O. \]
Also, the (Lie) derivative of a function \( f : M \to \mathbb{R} \) at \( p \) is determined by the values of \( f \) on an open set containing \( p \). Thus,

\[
X_i(\omega(X_1 \otimes \ldots \hat{X}_i \ldots \otimes X_{k+1}))(p) = Y_i(\omega(Y_1 \otimes \ldots \hat{Y}_i \ldots \otimes Y_{k+1}))(p), \quad p \in O.
\]

Let \( x : U \to \mathbb{R}^n \) be a coordinate chart for \( M \), \( p \in U \) fixed with \( x(p) = 0 \in \mathbb{R}^n \). If \( X_1, \ldots, X_{k+1} \) are \( C^\infty \) vector fields on \( U \), we may define

\[
d\omega(X_1 \otimes \ldots \otimes X_{k+1})(p)
\]

for a \( k \)-tensor as follows. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a \( C^\infty \) function with

\[
g(v) = \begin{cases} 
1, & ||v|| \leq 1, \\
0, & ||v|| \geq 2.
\end{cases}
\]

Then \( g \circ x : U \to \mathbb{R} \) is \( C^\infty \) and may be extended to a \( C^\infty \) function \( \varphi : M \to \mathbb{R} \) via

\[
\varphi(q) = \begin{cases} 
0, & q \in M - U, \\
(g \circ x)(q), & q \in U.
\end{cases}
\]

Define \( C^\infty \) vector fields \( Y_i \) on \( M \) by setting \( Y_i = \varphi X_i \) on \( U \) and \( Y_i = 0 \) on \( M - U \). Clearly

\[
Y_i = X_i \quad \text{on} \quad O = x^{-1}\left( \{ v \in \mathbb{R}^n | ||v|| < 1 \} \right).
\]

Set

\[
d\omega(X_1 \otimes \ldots \otimes X_{k+1})(p) := d\omega(Y_1 \otimes \ldots \otimes Y_{k+1})(p). \tag{2.2}
\]

By lemma (2.1), the value of \( d\omega(X_1 \otimes \ldots \otimes X_{k+1})(p) \) is independent of the choice of \( g : \mathbb{R}^n \to \mathbb{R} \). The formula in equation (2.2) is useful for the construction of the Leibinz coboundary of a tensor in local coordinate chart.
3 Two Tensors and the Calculus of Variations

In this section we compute the Leibniz coboundary of a two tensor in terms of the local coordinate chart \((x, U)\), where \(U \subset M\) is open, and

\[
x : U \to \mathbb{R}^n
\]

is a homeomorphism belonging to the atlas of charts for the differentiable structure of \(M\). The coefficients of this coboundary can be identified with those which occur in the optimization process for the integral of a two tensor over an immersed curve or surface (within \(U\)). For example, the Leibniz coboundary of the metric two-tensor (for \(M\) Riemannian) can be expressed in terms of the Christoffel symbols.

For completeness we begin with a one-form (i.e. a one-tensor), which has a local expression on \(U\) as

\[
\omega = \sum_{i=1}^{n} a_i \, dx^i,
\]

where \(a_i : U \to \mathbb{R}\) are \(C^\infty\) functions. From equation (2.1), the Leibniz coboundary of \(\omega\) agrees with the de Rham coboundary of \(\omega\), and in local coordinates

\[
d\omega = \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial x^j} \, dx^j \wedge dx^i = \sum_{j<i} \left( \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^j \wedge dx^i.
\]

(3.1)

We now discuss in what sense the functions

\[
\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}
\]

arise from the calculus of variations. Let

\[
\gamma : [0, 1] \to U, \quad \gamma(0) = p, \ \gamma(1) = q
\]

be a \(C^\infty\) curve with a \(C^\infty\) variation

\[
\alpha : (-\epsilon, \epsilon) \times [0, 1] \to U
\]
satisfying

\[ \alpha(0, t) = \gamma(t) \]
\[ \alpha(s, 0) = p \quad \text{for} \quad -\epsilon < s < \epsilon \]
\[ \alpha(s, 1) = q \quad \text{for} \quad -\epsilon < s < \epsilon. \]

We wish to investigate to what extent

\[ J(\gamma) = \int_0^1 \omega \left( \frac{d\gamma}{dt} \right) dt \]

is an extreme value (as a function of \( s \)) of

\[ J(\alpha(s)) = \int_0^1 \omega \left( \frac{\partial \alpha}{\partial t}(s, t) \right) dt. \]

Let \( \gamma^i = x^i(\gamma(t)) \in \mathbb{R} \) be the \( i \)-th component of the curve \( \gamma(t) \). Then

\[ \frac{d\gamma}{dt} = \sum_{i=1}^n \frac{d\gamma^i}{dt} \frac{\partial}{\partial x^i} \bigg|_{\gamma(t)} \]
\[ \omega \left( \frac{d\gamma}{dt} \right) = \sum_{i=1}^n a_i(\gamma(t)) \frac{d\gamma^i}{dt} \]
\[ \int_0^1 \omega \left( \frac{d\gamma}{dt} \right) dt = \int_0^1 \left( \sum_{i=1}^n a_i(\gamma(t)) \frac{d\gamma^i}{dt} \right) dt \]

Recalling the treatment of the calculus of variations, [9, p. 438], we define

\[ F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \]

by \( F(x, y) = \sum_{i=1}^n a_i(x) y^i \), where \( y = (y^1, y^2, \ldots y^n) \). In our case

\[ a_i(x) := (a_i x^{-1})(x(\gamma(t))) \quad \text{and} \quad y^i = \frac{d\gamma^i}{dt}. \]

A necessary condition that \( J(\gamma) \) be an extreme value is that the “Euler-Lagrange” equations hold [9, p. 438]

\[ \frac{\partial F}{\partial x^\ell} = \frac{d}{dt} \left( \frac{\partial F}{\partial y^\ell} \right), \quad \ell = 1, 2, \ldots n. \]
Now,

$$\frac{\partial F}{\partial x^\ell}(\gamma(t), \frac{d\gamma}{dt}) = \sum_{i=1}^{n} \frac{\partial a_i}{\partial x^\ell}(\gamma(t)) \frac{d\gamma^i}{dt}$$

$$\frac{\partial F}{\partial y^\ell}(\gamma(t), \frac{d\gamma}{dt}) = a_\ell(\gamma(t))$$

$$\frac{d}{dt}(a_\ell(\gamma(t))) = \sum_{j=1}^{n} \frac{\partial a_\ell}{\partial x^j}(\gamma(t)) \frac{d\gamma^j}{dt}$$

The condition in equation (3.2) may be rewritten as

$$\sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial x^\ell} - \frac{\partial a_\ell}{\partial x^i}\right)(\gamma(t)) \frac{d\gamma^i}{dt} = 0 \quad (3.3)$$

for each $\ell = 1, 2, 3, \ldots, n$. The coefficients arising in (3.3) also appear in (3.1). Of course,

$$d\omega\left(\sum_{i=1}^{n} \frac{d\gamma^i}{dt} \left(\frac{\partial}{\partial x^\ell}(\gamma(t)) \otimes \frac{\partial}{\partial x^i}(\gamma(t))\right)\right) = \sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial x^\ell} - \frac{\partial a_\ell}{\partial x^i}\right)(\gamma(t)) \frac{d\gamma^i}{dt}$$

We now prove an identical result for the Leibniz coboundary of a two-tensor (which is not necessarily a two-form).

Let $\omega$ be a two-tensor on $M$ with local expression

$$\omega = \sum_{i,j=1}^{n} a_{ij} \, dx^i \otimes dx^j$$

on $U$. Let $\gamma: I^2 \to U$ be an immersion ($C^\infty$ is sufficient), where

$$I^2 = \{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leq t_1 \leq 1, \ 0 \leq t_2 \leq 1\}.$$  

Although the value of the integral

$$J(\gamma) = \int_0^1 \int_0^1 \omega\left(\frac{\partial \gamma}{\partial t_1} \otimes \frac{\partial \gamma}{\partial t_2}\right) \, dt_1 \, dt_2 \quad (3.4)$$
generally depends on the parameterization \(\gamma\) (and not just the image of \(\gamma\)), necessary conditions for an extreme value of \(J(\gamma)\) can still be sought. Consider the \(C^\infty\) variation

\[
\alpha : (-\epsilon, \epsilon) \times I^2 \rightarrow U
\]
satisfying

\[
\begin{align*}
\alpha(0, t_1, t_2) &= \gamma(t_1, t_2) \\
\alpha(s, 1, t_2) &= \gamma(1, t_2), \quad \alpha(s, 0, t_2) = \gamma(0, t_2), \quad -\epsilon < s < \epsilon \\
\alpha(s, t_1, 1) &= \gamma(t_1, 1), \quad \alpha(s, t_1, 0) = \gamma(t_1, 0), \quad -\epsilon < s < \epsilon.
\end{align*}
\]

Then as a function of \(s\),

\[
J(\alpha(s)) = \int_0^1 \int_0^1 \omega \left( \frac{\partial \alpha}{\partial t_1}(s, t_1, t_2) \otimes \frac{\partial \alpha}{\partial t_2}(s, t_1, t_2) \right) \, dt_1 \, dt_2 = \int_0^1 \int_0^1 \sum_{i,j=1}^n a_{ij}(\alpha(s, t_1, t_2)) \frac{\partial \alpha^i}{\partial t_1}(s, t_1, t_2) \frac{\partial \alpha^j}{\partial t_2}(s, t_1, t_2) \, dt_1 \, dt_2,
\]

where \(\alpha^i(s, t_1, t_2) = x^i(\alpha(s, t_1, t_2)) \in \mathbb{R}\). Likewise, set

\[
\gamma^i = x^i(\gamma(t_1, t_2)) \in \mathbb{R}.
\]

**Lemma 3.1.** A necessary condition that \(J(\gamma)\) in equation (3.4) be an extreme value for the variation \(J(\alpha(s))\) is that

\[
\sum_{i,j=1}^n \left( -\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{ij}}{\partial x^l} + \frac{\partial a_{il}}{\partial x^j} \right)(\gamma(t_1, t_2)) \frac{\partial \gamma^i}{\partial t_1} \frac{\partial \gamma^j}{\partial t_2} + \sum_{j=1}^n (a_{\ell j} + a_{j \ell})(\gamma(t_1, t_2)) \frac{\partial^2 \gamma^j}{\partial t_1 \partial t_2} = 0
\]

for each \(\ell = 1, 2, 3, \ldots n\).

**Proof.** One computes \(\frac{d(J(\alpha(s)))}{ds}\) directly and equates

\[
\frac{d(J(\alpha(s)))}{ds} \bigg|_{s=0} = 0.
\]
In the following $a_{ij}$ and $\frac{\partial a_{ij}}{\partial x^\ell}$ are evaluated at $\alpha(s, t_1, t_2)$ while all partial derivatives of the $\alpha^i$'s are evaluated at $(s, t_1, t_2)$. Then

$$\frac{d(J\alpha(s))}{ds} = \int_0^1 \int_0^1 \left\{ \sum_{i,j=1}^n \left( \sum_{\ell=1}^n \frac{\partial a_{ij}}{\partial x^\ell} \frac{\partial \alpha^i}{\partial s} \frac{\partial \alpha^j}{\partial t_1} \right) \right. $$
$$+ \sum_{i,j=1}^n a_{ij} \frac{\partial^2 \alpha^i}{\partial s \partial t_1} \frac{\partial \alpha^j}{\partial t_2} + \sum_{i,j=1}^n a_{ij} \frac{\partial \alpha^i}{\partial t_1} \frac{\partial \alpha^j}{\partial s \partial t_2} \left. \right\} \, dt_1 \, dt_2,$$

which can be simplified using integration by parts and, in certain terms, the boundary values of $\alpha$. For example,

$$\int_0^1 \int_0^1 a_{ij} \frac{\partial^2 \alpha^i}{\partial s \partial t_1} \frac{\partial \alpha^j}{\partial t_2} \, dt_1 \, dt_2 = (I) + (II),$$

$$(I) = \int_0^1 \left. \left[ a_{ij} \frac{\partial \alpha^i}{\partial s} \frac{\partial \alpha^j}{\partial t_2} \right] \right|_{t_1=0}^{t_1=1} \, dt_2 = 0,$$

$$(II) = -\int_0^1 \int_0^1 \left\{ \left( \sum_{k=1}^n \frac{\partial a_{ij}}{\partial x^k} \frac{\partial \alpha^k}{\partial t_1} \frac{\partial \alpha^i}{\partial s} \frac{\partial \alpha^j}{\partial t_2} \right) + a_{ij} \frac{\partial \alpha^i}{\partial s} \frac{\partial \alpha^j}{\partial t_1 \partial t_2} \right\} \, dt_1 \, dt_2.$$  

After reindexing,

$$\left. \frac{d(J\alpha(s))}{ds} \right|_{s=0} = \sum_{\ell=1}^n \int_0^1 \int_0^1 \frac{\partial \gamma^\ell}{\partial s} \left\{ \sum_{i,j=1}^n \left( \frac{\partial a_{ij}}{\partial x^\ell} \frac{\partial \alpha^i}{\partial x^j} - \frac{\partial a_{ij}}{\partial x^j} \frac{\partial \alpha^i}{\partial x^\ell} \right) \frac{\partial \gamma^i}{\partial t_1} \frac{\partial \gamma^j}{\partial t_2} \right. $$
$$- \left. \sum_{j=1}^n (a_{ij} + a_{ji}) \frac{\partial^2 \gamma^j}{\partial t_1 \partial t_2} \right\} \, dt_1 \, dt_2,$$

where $a_{ij}$ and $\frac{\partial a_{ij}}{\partial x^\ell}$ are evaluated at $\gamma(t_1, t_2)$ and $\frac{\partial \alpha^i}{\partial s}$ is evaluated at $(0, t_1, t_2)$. The lemma now follows from the standard techniques of the calculus of variations, for example [9, p. 432–438], and in particular [9, p. 435].

Recall that the symbols

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$$
may be interpreted as vector fields on $U$ as well as derivations of the ring $C^\infty(U)$. To state the next lemma, we introduce the composition operators

$$\frac{\partial}{\partial x^\ell} \circ dx^p : \chi(U) \to C^\infty(U)$$

given by

$$\left( \frac{\partial}{\partial x^\ell} \circ dx^p \right) \left( \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \right) = \frac{\partial a_p}{\partial x^\ell}.$$

Lemma 3.2. Let $\omega$ be a two-tensor on $M$ with local expression on $U$

$$\omega = \sum_{p,q=1}^n a_{pq} \, dx^p \otimes dx^q,$$

where each $a_{pq} : U \to \mathbb{R}$ is $C^\infty$. Then

$$d\omega = \sum_{p,q=1}^n \frac{\partial a_{pq}}{\partial x^\ell} \, dx^\ell \otimes dx^p \otimes dx^q$$

$$- \sum_{p,q=1}^n \frac{\partial a_{pq}}{\partial x^\ell} \, dx^\ell \otimes dx^q \otimes dx^p$$

$$+ \sum_{\ell=1}^n a_{pq} \, dx^\ell \otimes dx^p \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^q \right)$$

$$+ \sum_{\ell=1}^n a_{pq} \, dx^\ell \otimes dx^q \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^p \right).$$

Proof. Let $X_1, X_2, X_3 \in \chi(M)$ with local expressions

$$X_1 = \sum_{i_1=1}^n c_{i_1} \frac{\partial}{\partial x^{i_1}}, \quad X_2 = \sum_{i_2=1}^n c_{i_2} \frac{\partial}{\partial x^{i_2}}, \quad X_3 = \sum_{i_3=1}^n c_{i_3} \frac{\partial}{\partial x^{i_3}}.$$

Then

$$d\omega(X_1 \otimes X_2 \otimes X_3) = \sum_{i_1,i_2,i_3=1}^n d\omega \left( c_{i_1} \frac{\partial}{\partial x^{i_1}} \otimes c_{i_2} \frac{\partial}{\partial x^{i_2}} \otimes c_{i_3} \frac{\partial}{\partial x^{i_3}} \right).$$
and from equation (2.1)

\[ d\omega \left( c_{i_1} \frac{\partial}{\partial x^{i_1}} \otimes c_{i_2} \frac{\partial}{\partial x^{i_2}} \otimes c_{i_3} \frac{\partial}{\partial x^{i_3}} \right) = \\
\frac{1}{2} \left( a_{i_2 i_3} \frac{\partial}{\partial x^{i_1}} - a_{i_1 i_3} \frac{\partial}{\partial x^{i_2}} + a_{i_1 i_2} \frac{\partial}{\partial x^{i_3}} \right) + \left( a_{i_2 i_3} + a_{i_3 i_2} \right) c_{i_1} c_{i_2} c_{i_3} \frac{\partial^3}{\partial x^{i_1} \partial x^{i_2} \partial x^{i_3}}. \]

Applying the right-hand side of \( d\omega \) in the statement of the lemma to

\[ c_{i_1} \frac{\partial}{\partial x^{i_1}} \otimes c_{i_2} \frac{\partial}{\partial x^{i_2}} \otimes c_{i_3} \frac{\partial}{\partial x^{i_3}}, \]

the same result is obtained.

Thus, given \( \omega = \sum_{p,q=1}^n a_{pq} \, dx^p \otimes dx^q \), we have

\[ d\omega \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^j} \right) = \frac{\partial a_{ij}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^j} + \frac{\partial a_{ij}}{\partial x^j}. \]  

(3.6)

Although \( d\omega \) is \( C^\infty(M) \)-linear in the first two tensor factors, this is not the case for the third factor:

\[ d\omega \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes c \frac{\partial}{\partial x^j} \right) = \\
c \left( \frac{\partial a_{ij}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^j} + \frac{\partial a_{ij}}{\partial x^j} \right) + \left( a_{ij} + a_{ij} \right) \frac{\partial c}{\partial x^i}. \]  

(3.7)

The coefficients of \( d\omega \) appearing in equation (3.7) are the same as those in lemma 3.1.

**Lemma 3.3.** Let \( M \) be a Riemannian manifold with metric tensor

\[ \omega = \langle \ , \ \rangle = \sum_{p,q=1}^n g_{pq} \, dx^p \otimes dx^q, \quad g_{pq} = g_{qp}, \]

which is compatible with the Levi-Civita connection \( \nabla \). Then

(i) \( d\omega \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^j} \right) = 2[ij, \ell], \) (twice the Christoffel symbol),

(ii) \( d\omega (X \otimes Y \otimes Z) = 2\langle Y, \nabla_X Z \rangle \) for \( X, Y, Z \in \chi(M) \).
Proof. Part (i) follows from equation (3.6) and the definition of the Christoffel symbols of the first kind

\[
[ij, \ell] = \frac{1}{2} \left( \frac{\partial g_{\ell \ell}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^\ell} - \frac{\partial g_{\ell j}}{\partial x^\ell} \right).
\]

For part (ii), recall that since \( \nabla \) is compatible with the metric tensor [1, p. 54], we have

\[
X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.
\]

Since the Levi-Civita connection is symmetric (i.e. torsion-free) [1, p. 54–55] [10, p. 255–256], we have

\[
[X, Y] = \nabla_X (Y) - \nabla_Y (X).
\]

The lemma now follows from (2.1), i.e.

\[
d\omega(X \otimes Y \otimes Z) = X(\langle Y, Z \rangle) - Y(\langle X, Z \rangle) + Z(\langle X, Y \rangle) - \langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle X, [Y, Z] \rangle.
\]

Thus, the obstruction to \( C^\infty(M) \)-linearity of the Leibniz coboundary in lemma 3.3 is the failure of the connection \( \nabla \) to be \( C^\infty(M) \)-linear in its second argument. Also, the first variation of arc length of \( \gamma : I \to U \) can be recovered as

\[
\frac{1}{2} d\omega \left( \frac{d\gamma}{dt} \otimes Y \otimes \frac{d\gamma}{dt} \right) = \left\langle Y, \nabla_{\frac{d\gamma}{dt}} \left( \frac{d\gamma}{dt} \right) \right\rangle,
\]

where \( Y \) is a vector field which represents the variation of the curve \( \gamma \). A necessary condition that \( \gamma \) be a geodesic is that

\[
d\omega \left( \frac{d\gamma}{dt} \otimes Y \otimes \frac{d\gamma}{dt} \right) = 0
\]

for all \( Y \), since \( \nabla_{\frac{d\gamma}{dt}} \left( \frac{d\gamma}{dt} \right) \) must vanish along such curves.

The only two-tensors which are global cocycles, however, must be two-forms.

Lemma 3.4. Let \( \omega \) be a two-tensor on \( M \) with \( d\omega = 0 \) in the Leibniz cochain complex. Then \( \omega \) is a two-form.
Proof. Letting $\omega = \sum_{p,q=1}^n a_{pq} \, dx^p \otimes dx^q$ be a local expression for $\omega$, then
\[ d\omega \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^\ell} \otimes \frac{\partial}{\partial x^j} \right) = 0 \]
implies $\frac{\partial a_{ij}}{\partial x^\ell} - \frac{\partial a_{i\ell}}{\partial x^j} = 0$. Furthermore
\[ d\omega \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^\ell} \otimes x^i \frac{\partial}{\partial x^j} \right) = 0 \]
and (3.7) imply that $a_{j\ell} = -a_{\ell j}$. Thus, $\omega$ is skew-symmetric. \qed

The Leibniz coboundary of a $k$-tensor $\omega$ agrees with the terms occurring in the optimization of the integral of $\omega$ if $\omega$ is skew-symmetric in its last $(k-1)$-arguments. Such tensors naturally occur in the $E^2$ term of the Pirashvili spectral sequence for Leibniz cohomology \[7\] \[6\]. Suppose that $g$ is a Lie algebra over $\mathbb{R}$,
\[ g' = \text{Hom}(g, \mathbb{R}) \]
the coadjoint representation of $g$, and that
\[ H^*_\text{Lie}(g; g') \]
denotes the Lie algebra cohomology of $g$ with coefficients in $g'$. An element of $H^{k-1}_\text{Lie}(g; g')$ can be represented by a tensor
\[ \alpha : g^{\otimes k} \to \mathbb{R} \]
which is skew-symmetric in its last $(k-1)$-tensor factors \[6\]. Moreover, $H^*_\text{Lie}(g; g')$ occurs in the $E^2$ term of the Pirashvili spectral sequence for $HL^*(g)$.

4 The Local Coboundary Formula

Let $\omega$ be a $k$-tensor on $M$ with local expression
\[ \sum_I a_I \, dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k}, \quad (4.1) \]
where $I$ is the multi-index $(i_1, i_2, \ldots, i_k)$, and the summation ranges over
\[ 0 \leq i_1 \leq n, \quad 0 \leq i_2 \leq n, \quad \ldots, \quad 0 \leq i_k \leq n. \]
To state the coboundary formula, the following local operators are introduced. Let

\[
L(\omega) = \sum_I \left\{ \sum_{\ell=1}^n \frac{\partial a_I}{\partial x^\ell} dx^\ell \otimes dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k} - \sum_{\ell=1}^n \frac{\partial a_I}{\partial x^\ell} dx^{i_1} \otimes dx^\ell \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k} + \sum_{\ell=1}^n \frac{\partial a_I}{\partial x^\ell} dx^{i_1} \otimes dx^{i_2} \otimes dx^\ell \otimes \ldots \otimes dx^{i_k} + \ldots \right. \\
+ \left. (-1)^{k+2} \sum_{\ell=1}^n \frac{\partial a_I}{\partial x^\ell} dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k} \otimes dx^\ell \right\},
\]

and

\[
S(dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k}) = \sum_{\ell=1}^n dx^\ell \otimes dx^{i_1} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_2} \right) \otimes dx^{i_3} \otimes \ldots \otimes dx^{i_k} + \sum_{\ell=1}^n dx^\ell \otimes dx^{i_1} \otimes dx^{i_2} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_3} \right) \otimes \ldots \otimes dx^{i_k} + \ldots \\
+ \sum_{\ell=1}^n dx^\ell \otimes dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_{k-1}} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_k} \right) + (-1)^4 \sum_{\ell=1}^n dx^\ell \otimes dx^{i_2} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_1} \right) \otimes dx^{i_3} \otimes \ldots \otimes dx^{i_k} + (-1)^5 \sum_{\ell=1}^n dx^\ell \otimes dx^{i_2} \otimes dx^{i_3} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_1} \right) \otimes dx^{i_4} \otimes \ldots \otimes dx^{i_k} + \ldots \\
+ (-1)^{k+2} \sum_{\ell=1}^n dx^\ell \otimes dx^{i_2} \otimes dx^{i_3} \otimes \ldots \otimes dx^{i_k} \otimes \left( \frac{\partial}{\partial x^\ell} \circ dx^{i_1} \right).
\]
Theorem 4.1. If $\omega$ is a $k$-tensor on $M$, $k \geq 2$, with local expression given in equation (4.1), then locally the Leibniz coboundary of $\omega$ is

$$d(\omega) = L(\omega) + \sum_I a_I \left\{ S(dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k}) \\
- dx^{i_1} \otimes S(dx^{i_2} \otimes dx^{i_3} \otimes \ldots \otimes dx^{i_k}) \\
+ dx^{i_1} \otimes dx^{i_2} \otimes S(dx^{i_3} \otimes \ldots \otimes dx^{i_k}) \\
+ \ldots \\
+ (-1)^{k-2} dx^{i_1} \otimes \ldots \otimes dx^{i_{k-2}} \otimes S(dx^{i_{k-1}} \otimes dx^{i_k}) \right\}.$$ 

Proof. The proof proceeds by induction on $k$ with the case $k = 2$ proven in lemma 3.2. To streamline the inductive step, define a $(k-1)$-tensor $\beta$ (often called a contraction) by

$$\beta(v_2 \otimes \ldots \otimes v_k) = \omega \left( \frac{\partial}{\partial x^{j_1}} \otimes v_2 \otimes \ldots \otimes v_k \right),$$

where $\frac{\partial}{\partial x^{j_1}}$ is a fixed canonical vector field on a coordinate chart. Let

$$z = \left( c_{j_1} \frac{\partial}{\partial x^{j_1}} \otimes c_{j_2} \frac{\partial}{\partial x^{j_2}} \otimes c_{j_3} \frac{\partial}{\partial x^{j_3}} \otimes \ldots \otimes c_{j_{k+1}} \frac{\partial}{\partial x^{j_{k+1}}} \right)$$

Then

$$d\omega(z) =$$

$$= c_{j_1} c_{j_2} \ldots c_{j_{k+1}} \frac{\partial}{\partial x^{j_1}} \left( a_{j_2 j_3 \ldots j_{k+1}} \right)$$

$$+ \sum_{m=3}^{k+1} \left( a_{j_2 j_3 \ldots j_{k+1}} + (-1)^{m+1} a_{j_m j_2 j_3 \ldots j_m \ldots j_{k+1}} \right) c_{j_1} c_{j_2} \ldots \hat{c}_{j_m} \ldots c_{j_{k+1}} \frac{\partial}{\partial x^{j_i}}$$

$$+ \sum_{i_1=1}^n (-dx^{i_1} \otimes d\beta)(z)$$

$$= \sum_I \sum_{t=1}^n \left( \frac{\partial a_I}{\partial x^t} dx^t \otimes dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k} \right)(z)$$

$$+ \sum_I a_I S(dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_k})(z) + \sum_{i_1=1}^n -(dx^{i_1} \otimes d\beta)(z),$$

whence follows the result. \qed
Lemma 4.2. If $\omega$ is a $k$-tensor on an $n$-dimensional differentiable manifold $M$, $k \leq (n + 1)$, and $d\omega = 0$, then $\omega$ is a $k$-form.

Proof. Clearly $d\omega(\Xi) = 0$ for any $\Xi \in \chi(M)^{(k+1)}$. Since

$$d\omega\left(\frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_{k+1}}}\right) = 0,$$

we have (using the notation in equation (4.1)),

$$\frac{\partial}{\partial x^{j_1}}(a_{j_2 j_3 \ldots j_{k+1}}) - \frac{\partial}{\partial x^{j_2}}(a_{j_1 j_3 j_4 \ldots j_{k+1}}) + \ldots + \ldots + (-1)^{k+2} \frac{\partial}{\partial x^{j_{k+1}}}(a_{j_1 j_2 j_3 \ldots j_k}) = 0.$$

Choosing $j_1, j_2, \ldots, j_n$ to be distinct (which may be done since dim$(M) = n$), we also have

$$d\omega\left(\frac{\partial}{\partial x^{j_1}} \otimes \frac{\partial}{\partial x^{j_2}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_q}} \otimes x^{j_{q+1}} \frac{\partial}{\partial x^{j_p}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_{k+1}}}\right) = 0,$$

where $p \leq q - 2$. Thus,

$$a_{j_1 j_2 \ldots j_p \ldots j_{k+1}} + (-1)^{q-p} a_{j_1 j_2 \ldots j_{p-1} j_q j_{p+1} \ldots j_q \ldots j_{k+1}} = 0.$$

The section is closed with the computation of the Leibniz coboundary of one of most important tensors in differential geometry, the Riemann curvature tensor, $R$. Let $\nabla$ be the Levi-Civita connection on a Riemannian manifold $M$ with metric $\langle \ , \ \rangle$. Given $X, Y, Z, W \in \chi(M)$, then $R$ is the four-tensor defined by [10]

$$R(X \otimes Y \otimes Z \otimes W) = \langle \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) - \nabla_{[X,Y]}(Z), W \rangle.$$

The Leibniz coboundary $dR$ is expressed in terms of the covariant derivative $\nabla R$, which we briefly review. Let $\omega$ be a $k$-tensor on $M$ and

$$X_1, X_2, \ldots, X_k, Z \in \chi(M).$$
Then \[1, \text{p. 102}\] the covariant derivative \(\nabla\omega\) is the \((k + 1)\)-tensor given by
\[
\nabla\omega(X_1 \otimes X_2 \otimes \ldots \otimes X_k \otimes Z) = Z(\omega(X_1 \otimes X_2 \otimes \ldots \otimes X_k)) \\
- \omega(\nabla_Z(X_1) \otimes X_2 \otimes \ldots \otimes X_k) - \omega(X_1 \otimes \nabla_Z(X_2) \otimes \ldots \otimes X_k) \\
- \cdots - \omega(X_1 \otimes X_2 \otimes \ldots \otimes \nabla_Z(X_k)).
\]

Note that \(\nabla\omega(X_1 \otimes X_2 \otimes \ldots \otimes X_k \otimes Z)\) is often denoted as \((\nabla_Z\omega)(X_1 \otimes \ldots \otimes X_k)\). The following properties of \(R\) are useful in the computation of \(dR\), where \(X, Y, Z, W, T \in \chi(M)\):

(i) Bianchi’s second identity \([1, \text{p. 106}]\)
\[
(\nabla_T R)(X \otimes Y \otimes Z \otimes W) + (\nabla_Z R)(X \otimes Y \otimes W \otimes T) \\
+ (\nabla_W R)(X \otimes Y \otimes T \otimes Z) = 0,
\]

which may also be expressed as \([8, \text{p. 34}]\)
\[
(\nabla_X R)(Y \otimes Z \otimes W \otimes T) + (\nabla_Y R)(Z \otimes X \otimes W \otimes T) \\
+ (\nabla_Z R)(X \otimes Y \otimes W \otimes T) = 0,
\]

(ii) skew-symmetry in certain coordinates \([1, \text{p. 91}]\)
\[
R(X \otimes Y \otimes Z \otimes T) = -R(Y \otimes X \otimes Z \otimes T) \\
R(X \otimes Y \otimes Z \otimes T) = -R(X \otimes Y \otimes T \otimes Z) \\
R(X \otimes Y \otimes Z \otimes T) = +R(Z \otimes T \otimes X \otimes Y).
\]

**Lemma 4.3.** Let \(M\) be a Riemannian manifold with curvature tensor \(R\) and Levi-Civita connection \(\nabla\). Then for \(X, Y, Z, W, T \in \chi(M)\), one has
\[
dR(X \otimes Y \otimes Z \otimes W \otimes T) = -(\nabla_Z R)(X \otimes Y \otimes W \otimes T) \\
+ R(Z \otimes T \otimes Y \otimes \nabla_X W) - R(Y \otimes Z \otimes T \otimes \nabla_X W) \\
- R(Z \otimes W \otimes Y \otimes \nabla_X T) + R(Y \otimes Z \otimes W \otimes \nabla_X T) \\
- R(Z \otimes T \otimes X \otimes \nabla_Y W) + R(X \otimes Z \otimes T \otimes \nabla_Y W) \\
+ R(Z \otimes W \otimes X \otimes \nabla_Y T) - R(X \otimes Z \otimes W \otimes \nabla_Y T).
\]
Proof. After applying equations (2.1), (4.2), (4.4), and symmetry of the connection,

\[ [X, Y] = \nabla_X Y - \nabla_Y X, \]

one has

\[
dR(X \otimes Y \otimes Z \otimes W \otimes T) = \\
X(R(W \otimes T \otimes Y \otimes Z)) - Y(R(W \otimes T \otimes X \otimes Z)) \\
- R(W \otimes T \otimes \nabla_X Y \otimes Z) + R(W \otimes T \otimes \nabla_Y X \otimes Z) \\
+ R(W \otimes T \otimes \nabla_X Z \otimes Y) - R(Z \otimes T \otimes \nabla_X W \otimes Y) \\
+ R(Z \otimes W \otimes \nabla_X T \otimes Y) + R(W \otimes T \otimes X \otimes \nabla_Y Z) \\
- R(Z \otimes T \otimes X \otimes \nabla_Y W) + R(Z \otimes W \otimes X \otimes \nabla_Y T). \]

The lemma now follows from equations (4.3) and (4.4). \qed

In the statement of lemma 4.3, note that \( \nabla_Z R \) is the \( C^\infty(M) \)-linear term of \( dR \), while the remaining eight terms comprise the non-linear pieces.

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