Optimizing the centrally compressed step-variable stiffness rods

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Abstract. The article deals with the optimization parameters of centrally compressed step-variable stiffness rods, with the material that is able to work both within and out of proportionality. The following algorithms are developed: the maximization algorithm of loading critical parameter at the given volume of the rod material, and the minimization algorithm of the rod material volume observing the restrictions on durability and stability, taking into account the own weight of the rod. The analysis of the optimization results is carried out and the recommendations on calculation and rod design are offered.

1. Introduction
Step-variable stiffness rods of are widely used in construction practice, first of all, they are used as framework columns of industrial buildings. These rods often operate in the conditions close to the central compression. In addition, even at transverse loading, the calculation on stability of the centrally compressed rod is necessary within the approximate method limits of its calculation at longitudinal-transverse bending.

Many works are devoted to the issue of stability loss of the rods and rod systems [1-11]. The majority of the works, devoted to the analysis of rod systems stability, are limited by the use of linearly elastic model of systems deformation. The exceedance of proportionality in the calculation of individual rods was taken into account using the table of longitudinal bending coefficients. The iterative method of calculation on the stability of step stiffness rods with use of the differential equations solutions that describe the quasi-elastic bifurcation form of stability loss at the integration section is offered at work [11]. If voltage output was beyond the proportionality, the elastic modulus at iteration was replaced by a tangent modulus that was calculated by a piecewise approximation function $E'\left(\sigma\right)$ on the diagram $\delta-\varepsilon$ for the material used. The method of stability calculation, suggested in work [11], is extended in the present paper in order to take into account the own weight of the rod.

In view of the particular importance of stability requirements at the design of the most loaded bearing columns of the buildings, the tasks are set and solved in the present paper to optimize the parameters of the centrally compressed step-variable stiffness rods at the chosen criteria of optimality and taking into account the stability restrictions. Many works are devoted to the consideration of stability restrictions at rod systems optimization [12-19 and others]. The present work suggests the algorithms to solve the tasks to optimize the parameters of the centrally compressed step-
variable stiffness rods. They illustrate the efficiency of the developed calculation methodology at the stability restrictions and give the best (by the chosen criterion) projects.

2. Determination of the maximum critical load parameter $F_{cr}$ when the rod material volume is constant

Centrally compressed step-variable stiffness rod is considered. The calculation scheme of it is shown in Figure 1. The sectional view of sections 1, 2 is also shown in Figure 1. The task of parameters optimization of the rod is set as a task of nonlinear mathematical programming [12-16 and others]. It is required to find such values of the rod parameters $X_1, ..., X_n$, to be

$$F_{cr}(X_1',...,X_n') = F_{cr\max}$$

(1)

and restrictions are performed

$$X_{imin} \leq X_i \leq X_{imax}, i = 1, ..., n$$

(2)

$$V(X_1, ..., X_n) = V - \text{specified value},$$

(3)

$$X_i / (k_1X_2) - 1 \leq 0, X_i / (k_3X_4) - 1 \leq 0,$$

$$X_s / (k_5X_6) - 1 \leq 0, X_s / (k_7X_8) - 1 \leq 0,$$

(4)

where $V(X_1, ..., X_n)$ - the volume of the rod material, and equations (2), (4) correspond to parametric and constructive constraints. Since the previously developed program of constraint minimization was used to solve this problem [12], the objective function (1) was transformed to the form,

$$\text{min} f(x) = \text{min} 1/F_{cr}(X_1, ..., X_n)$$

The coefficients in constructive constraints (4) are accepted as equals: $k_1 = k_3 = k_5 = k_7 = 80$.

![Figure 1. Calculation scheme of centrally compressed step stiffness rod and sectional view of sections.](image-url)
the sections (1), (2). At the internal level, the optimization task (1)÷(4) formed at iteration is solved with the use of the chosen search algorithm [13]. In this case, the critical parameter of the load is determined in the Quasi-Eulerian equation, the found values of tangent modules are used as conventional modulus of elasticity of. At the first iteration, the tangent modules of materials of the sections (1), (2) are accepted equal to their modulus of elasticity:

\[ E_{pr}^{(1)} = E_{pr}^{(2)}. \]  

The criterion to end the search in this task is the following condition:

\[ |F_{cr, max} (X_{1}^{n+1}, X_{n}^{n+1}) - F_{cr, max} (X_{1}^{n}, X_{n}^{n})| / F_{cr, max} (X_{1}^{n}, X_{n}^{n}) \leq \delta, \]  

where \( \delta > 0 \) is a small number given ahead of the task (e.g. 0.03).

To estimate the effect of a step stiffness creation at the load transfer point \( F \cdot m \) (see Figure 1), it was first considered the task to determine \( \max F_{cr} (X_{1}, X_{2}, X_{3}, X_{4}) \) for a constant stiffness rod with a fixed volume of the material. The restriction-equation of this task in the volume of the material is:

\[ (2X_{1}X_{2} + X_{3}X_{4})H / \overline{V} - 1 = 0, \]  

and the number of constructive constraints is reduced to two.

If there is a step stiffness at the point of load application \( F \cdot m \), the restriction (3) will be recorded as:

\[ (2X_{1}X_{2} + X_{3}X_{4})l_{1} + (2X_{3}X_{5} + X_{7}X_{8})l_{2} / \overline{V} - 1 = 0 \]  

Table 1 shows the results of the optimization task solution at the following initial data: \( H=12 \text{ m} \); \( \overline{V} = 0.072 \text{ m}^3 \); the material is steel 09G2S; \( E=2\times10^{5}\text{ MPa}; \sigma_{pr}=200\text{ MPa}; \sigma_{0,2}=305\text{ MPa}; \sigma_{u}=460\text{ MPa}; \) The position of the load transfer location \( F \cdot m \) varied with the purpose of the value \( l_{1} \) (Figure 1).

Table 1. Magnitudes \( F_{cr, max} (X_{1},...,X_{4}); F_{cr, max} (X_{1},...,X_{8}) \). Optimal values for variable parameters \( X_{i}^{*} \) (m)… \( X_{8}^{*} \) (m).

| \( l_{1} \) (m) | 3.0 | 6.0 | 9.0 |
| --- | --- | --- | --- |
| \( F_{cr, max} (X_{1},...,X_{4}) \) (kN) | 103.2 | 140.1 | 212.4 |
| \( X_{1}^{*} \) (m) | 0.458 | 0.457 | 0.456 |
| \( X_{2}^{*} \) (m) | 0.0057 | 0.00572 | 0.00571 |
| \( X_{3}^{*} \) (m) | 0.253 | 0.252 | 0.251 |
| \( X_{4}^{*} \) (m) | 0.00315 | 0.00312 | 0.00314 |
| \( F_{cr, max} (X_{1},...,X_{8}) \) (kN) | 144 | 227 | 317.41 |
| \( X_{1}^{*} \) (m) | 0.275 | 0.371 | 0.4407 |
| \( X_{2}^{*} \) (m) | 0.00344 | 0.00464 | 0.00551 |
| \( X_{3}^{*} \) (m) | 0.05 | 0.05 | 0.05 |
| \( X_{4}^{*} \) (m) | 0.001 | 0.001 | 0.001 |
| \( X_{5}^{*} \) (m) | 0.502 | 0.0543 | 0.566 |
| \( X_{6}^{*} \) (m) | 0.00636 | 0.0068 | 0.00708 |
| \( X_{7}^{*} \) (m) | 0.277 | 0.3 | 0.312 |
| \( X_{8}^{*} \) (m) | 0.00345 | 0.00375 | 0.0039 |
The results of the calculations showed that volume restriction-equation is very active and performed with high accuracy, while constructive constraints are performed by a wide margin. The increase of critical values of the load parameter at the increase of $l_1$ is explained by the approaching of load transfer section $F\cdot m$ to pinching. The choice of the load transfer point $F\cdot m$ (e.g. the crane one) is usually determined by the technological requirements. A more significant result is a significant increase (from 40% to 62%) of the critical value of the load parameter at the step stiffness device in comparison with a constant stiffness rod preserving the rod material volume. The critical stresses at the section (2) exceed the limit of proportionality.

3. Optimization of the centrally compressed rod volume with account for its own weight

The main objectives of this part of the research are as follows:

- the development of the calculation engineering method for stability of compressed rods of constant and step-variable cross-sections with account for their own weight;
- the consideration of the rational use of step-variable section from the point of material consumption;
- the formulation and solution of the rod parameters optimization problem using the criterion of the minimum volume of the rod material.

The structural design, presented in Figure 2, is considered as an example (to illustrate the developed calculation method of stability and optimization algorithm). With the purpose to reduce the calculations volume within the framework of the offered calculation methods on stability and optimization algorithm, the rod with one step of stiffness change the sections of which are set through one parameter is considered (if necessary, the number of steps and parameters of the section can be increased).

![Figure 2. Calculation scheme and rod sectional view of the sections.](image)

3.1. Development of the rod calculation engineering method for stability with account for its own weight

The account for its own weight complicates significantly the differential state equations describing the bifurcation form of rod stability loss at the sections. For example, at section (1) (Figure 3) it is:

$$E_i I_1 V_1^{(x_1)} = M_1(x_1) - F V_1^{(x_1)} x_1 - q_{01}^{(x_1)} V_1^{(x_1)} x_1 + \int_0^{x_1} V_1^{(z)dz}$$  \hspace{1cm} (10)

The solution of the system of equations of type (10), where the sought functions of the form $V_1^{(x_1)}, V_2^{(x_2)}$ at sections (1), (2) are included under the sign of integral, is hardly possible. Even if, simplifying the problem definition, we consider the own weight load directed along the axis $x_i$, we will come to differential equations with variable coefficients at sections (1), (2), which also complicates the search for a solution. Below we propose an engineering technique that allows to
reduce the problem of calculation of these rods to the case considered in [11], when the compressive load is expressed through a single parameter $F$ and is applied to the nodes (in this case to nodes 1,2).

Figure 3. Bifurcation form of stability loss at the section (1): z- auxiliary variable coordinate, $q_{ovq}$ - deadweight intensity (1).

The procedure for calculating the rod for stability, which is necessary in this case for building a stability constraint, is built (as noted above) into the algorithm for finding the optimal project $X_1^*,...,X_n^*$. Therefore, at any step of searching for this project we know the current values of variable parameters $X_{i1},...,X_{in}$. In order to use the methodology and program for determining the critical value of the load parameter [11], the nodal loads from the eigenstock weight must be expressed through the external load parameter $F$ (Figure 2). By distributing the eigenfant load of the sections (1), (2) equally in nodes 1, 2 and nodes 2, 3, we obtain the full values of loads in nodes 1, 2 at the $i$-th step of the search, expressed through the parameter $F$. The node loads (Figure 4) are from the equation:

$$F_{1i}(X_1^i,X_3^i) = F(1+k_1q_i); F_{1i} \cdot m_i = F(k_1q_i + k_2q_i); m_i = (k_1q_i + k_2q_i) / (1+k_1q_i)$$

where

$$k_1q_i = 0.5A^{(1)i} \gamma X_3^i; 2q_i = 0.5A^{(2)i} \gamma (H-X_3^i); A^{(1)i}, A^{(2)i} - cross-sectional areas,$$

$i$-th the step number. The same approach can be used for any number of intermediate nodes.

Figure 4. Approximate rod loading scheme.

The analysis of the results of the solution of the stability calculation tasks taking into account the own weight according to the proposed technique has shown that its use leads to an overestimation of the critical parameter $F_{cr}$ load, which, in turn, reduces the correctness of setting the optimization task of the rod volume. This especially affects at one intermediate node and in extreme cases ($X_3^i$), because then half of the eigenfant load goes to support 3. Therefore, in this optimization task, the variable parameter $X_3$, which is quite acceptable when increasing the number of intermediate nodes of the own load transfer, is excluded from the number of variable, and the value $X_3^i$ in all cases considered below is assumed to be $H/2$.

3.2. Setting and solving the task of rod volume optimization with account for its own weight and stability limitations

The task of optimizing the volume of the rod loaded in the upper section (Figure 2) by the external force $F$, as well as its own weight is set as a nonlinear mathematical programming task: it is necessary
to find the values of variable parameters of the rod \( X_1', ..., X_n' \), implementing the minimum function of the volume \( V(X_1', ..., X_n') \) of the rod, taking into account the limitations on stability and strength. In the annex to the rod, shown in Figure 2, and taking into account the proposed procedure for converting the own weight load into a nodal formulation of the optimization task has the following form: it is necessary to find

\[
\min V(X_1, X_2, X_3) = 99X_1^2 \cdot X_3 + 99X_2^2 (H - X_3),
\]

Subject to restrictions:

\[
(F1(X_1, X_3) \cdot k_a / F_{1,\text{cr}}(X_1, X_2, X_3)) - 1 \leq 0
\]

(14)

\[
(F(1 + k_1q) / (99X_1^2 \times R)) - 1 \leq 0
\]

(15)

\[
(F(1 + 2k_1q + 2k_2q) / (99X_1^2 \cdot R)) - 1 \leq 0
\]

(16)

where \( F1(X_1, X_3) \) - load parameter functions; \( F_{1,\text{cr}}(X_1, X_2, X_3) \) - load parameter critical value function; \( k_a \) - stability factor. When solving the optimization task, function values at each search step are calculated using the algorithm described in [11], and the function \( F1(X_1, X_3) \) possesses the form:

\[
F1(X_1, X_3) = F(1 + k_1q(X_1, X_3))
\]

(17)

In equations (16), (17):

\[
m = (k_1q + k_2q) / (1 + k_1q); k_1q = 0.5 \cdot 99X_1^2 \gamma X_3 / F; k_2q = 0.5 \cdot 99X_1^2 \gamma (H - X_3) / F
\]

(18)

The solution of the optimization task is carried out within the framework of the selected search method (the method of a mobile external penalty [13] in combination with the method of a deformable polyhedron in the solution of formed unconditionally extreme problems at iterations). Except for the internal iterative process connected with a search method of the decision of a mathematical programming problem, the external iterative process connected with updating of tangent modules at the sections for the changed project at search \( F_{1,\text{cr}}(X_1, X_2, X_3) \) is organized.

Tables 2, 3, 4, 5 present some results of the solution of the formed optimization problem for the rod presented in Fig.2, with the following initial data: \( F=10\text{kN} \); the material is steel 09G2S; \( \sigma_{pr} = 200\text{MPa}; \sigma_{u,z} = 305\text{MPa}; \sigma_u = 460\text{MPa} \). Length \( l_1 = X_3 \) taken equal \( H/2 \).

**Table 2.** Optimal parameters of constant stiffness rods not taking into account the influence of its own weight.

| H(m) | 10    | 20    | 30    | 40    | 50    | 60    |
|------|-------|-------|-------|-------|-------|-------|
| \( \min V_{0l} \) \( (m^3) \) | 0.0222 | 0.089 | 0.199 | 0.357 | 0.556 | 0.806 |
| \( X'_1 \) (m) | 0.00473 | 0.0067 | 0.00818 | 0.00949 | 0.0106 | 0.01165 |
| \( F_{1,\text{cr}} \) (kN) | 18.0 | 18.1 | 17.9 | 18.2 | 18 | 18.3 |

**Table 3.** Optimal parameters of constant stiffness rods with account for the influence of its own weight.

| H(m) | 10    | 20    | 30    | 40    | 50    | 60    |
|------|-------|-------|-------|-------|-------|-------|
| \( \min V_{0l} \) \( (m^3) \) | 0.0236 | 0.115 | 0.369 | 0.983 | 2.27 | 4.62 |
| \( X'_1 \) (m) | 0.00487 | 0.00762 | 0.0111 | 0.0157 | 0.0214 | 0.0278 |
| \( F_{1,\text{cr}} \) (kN) | 18.7 | 21.8 | 31.4 | 51.8 | 97.5 | 180 |

Analyzing the results of the calculations provided in Tables 2, 3 the following can be noted.
1. In all considered cases, when solving the optimization task, the stability limitation is active and the strength limitation is satisfied with a wide margin.

2. The criterion of optimality chosen at the decision of the given problem (a minimum volume of the rod), in a combination with an active restriction on stability leads to that the material of the structural design works within the limits of proportionality.

3. The account of forces of the own weight essentially affects all considered heights of the rod (from 10 m to 60 m), but their influence especially increases with height increase at a rather small external concentrated load $F=10$ kN in the top section. At $H=10$ m the optimal volume increases with account of its own weight by 10%, and at $H=60$ m the optimal volume increases by more than five times with account for its own weight. It suggests that the calculations without regard to the own weight at such heights do not make sense for rod structures.

Tables 4, 5 show the results of the rods volume optimization solution in the presence of one step of stiffness change.

Table 4. Optimal parameters for step stiffness rods not taking into account the influence of their own weight.

| H (m) | 10   | 20   | 30   | 40   | 50   | 60   |
|-------|------|------|------|------|------|------|
| $\min V_{0i}(m^3)$ | 0.0193 | 0.0770 | 0.173 | 0.307 | 0.480 | 0.691 |
| $X_i^1$ (m) | 0.0036 | 0.0052 | 0.0064 | 0.0073 | 0.0082 | 0.0098 |
| $X_i^2$ (m) | 0.0043 | 0.0071 | 0.0087 | 0.01 | 0.0126 | 0.0132 |
| $F_{1st} (kN)$ | 18.2 | 18.2 | 18.2 | 18 | 18.1 | 18.1 |

Table 5. Optimal parameters for step stiffness rods with account for the influence of their own weight.

| H (m) | 10   | 20   | 30   | 40   | 50   | 60   |
|-------|------|------|------|------|------|------|
| $\min V_{0i}(m^3)$ | 0.0199 | 0.0894 | 0.246 | 0.567 | 1.2 | 2.34 |
| $X_i^1$ (m) | 0.0036 | 0.0053 | 0.0068 | 0.0085 | 0.0107 | 0.0247 |
| $X_i^2$ (m) | 0.0051 | 0.0078 | 0.0109 | 0.0146 | 0.0193 | 0.0132 |
| $F_{1st} (kN)$ | 18.6 | 20.1 | 23.0 | 28.2 | 38.2 | 54.5 |

The analysis of the calculation results in Tables 4, 5 shows that the formation of a stiffness step and the increase of the number of variable parameters to two significantly improve the optimal projects. For example, at $F=10$ kN, $H=60$ m, even not taking into account the own weight, the presence of the step stiffness improves the minimum volume project by 14.3%. The introduction of one step stiffness at the height of $H=60$ m produces a greater effect in the projects with account for their own weight: the optimal volume is reduced from 4.62 m$^3$ to 2.34 m$^3$, that is 49.4%.

4. The main conclusions

1. The conducted researches have allowed to expand the previously developed [17] calculation method for stability of step-variable stiffness rods in the case with account for their own weight.

2. The tasks of step-variable stiffness rods optimization at use of various criteria of optimality and with account for stability restrictions are set and solved. The optimization algorithms with the use of the developed procedure of stability calculation with account for the own weight are offered.

3. It is shown that the introduction of step stiffness can significantly (from 10% to 50%) improve the optimal project with the use of various criteria of optimality and restrictions.

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