QUASILINEAR ROUGH PARTIAL DIFFERENTIAL EQUATIONS
WITH TRANSPORT NOISE

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Abstract. We investigate the Cauchy problem for a quasilinear equation of the form
\[ du = \text{div}(A(u)\nabla u)dt + \sigma(t,x)\nabla u, \quad u_0 \in L^2 \text{ on the torus } T^d, \]
where \( X \) is two-step geometric rough path. Using an energy approach, we provide sufficient conditions guaranteeing existence and uniqueness.

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1. Introduction

In this work, we consider the Cauchy problem for a quasilinear non-degenerate parabolic rough partial differential equation of the form
\[ du = \text{div}(A(t,x,u)\nabla u)dt + \sigma(t,x)\cdot \nabla u \, dX, \quad [0,T] \times T^d, \]
where \( dX \) denotes integration with respect to a geometric, \( \alpha \)-Hölder rough path
\[ (X_1^{1,\mu}, X_2^{2,\nu})_{1 \leq \mu, \nu \leq m} \]
with \( \alpha > 1/3, m \geq 1, \) and \( \sigma \) belongs to the Hölder space \( C_\gamma(0,T; W^{3,\infty}(T^d)) \) with \( \gamma > 1 - \alpha. \) We will consider the case where the matrix \( A(t,x,u) \) is non-degenerate, in the sense that
\[ \lambda |\xi|^2 \leq \sum_{1 \leq i,j \leq d} A_{ij}(t,x,z)\xi^i \xi^j \leq \lambda^{-1}|\xi|^2, \]
for some \( \lambda > 0, \) independent of \( \xi \in \mathbb{R}^d \) and \( (t,x,z) \in [0,T] \times T^d \times \mathbb{R}. \)

Our main achievement is to obtain existence and uniqueness of solutions to the problem (1.1) — actually a generalized version thereof, see (1.3) — in the energy space \( L^\infty(L^2) \cap L^2(H^1) \), under various conditions on the rough forcing term. It is possible to generalize our results to quasilinear equations of the form
\[ du + \text{div}(F(u)) = \text{div}(A(t,x,u)\nabla u)dt + \sigma(t,x)\cdot \nabla u \, dX, \quad [0,T] \times T^d, \]
where the flux
\[ F \equiv (F_1, \ldots, F_d) : \mathbb{R} \to \mathbb{R}^d \]
is \( C^1 \) and bounded. However, in order to simplify our presentation, we restrain from doing so.

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Relevant bibliography. Nonlinear stochastic equations with gradient noise were studied in [DS*04], see also [CFH11] in the rough path setting. The associated stochastic problem, where the noise is assumed to be of the multiplicative form \( \Phi(u) dW \), has been treated in [DHV*16], and more recently in [HZ17]. The first work also deals with the degenerate case, which corresponds to a stochastic conservation law. This problem has been independently studied in the framework of rough paths in [LPS13], where similar to our settings the rough forcing belongs to the flux term (yielding space derivatives of \( u \) in the noise).

In order to ease notations, but also to state the most general results possible, it is convenient to generalize the equation (1.1) to the more abstract problem

\[
du = \text{div}(A(t, x, u) \nabla u) dt + dBu \quad \text{on} \quad [0, T] \times \mathbb{T}^d,
\]
\[
 u_0 \in L^2(\mathbb{T}^d),
\]  

where \( B \) is a geometric, unbounded rough driver, i.e. a geometric rough path with values in a space of unbounded operators. The unknown is a mapping \( u : [0, T] \to L^2 \) with finite energy \( \|u\|_{L^\infty(L^2)} + \|u\|_{L^2(L^1)} < \infty \), and we will assume that \( B \) is derivation-valued, see Assumption 1.2. Following our previous work [HN18], such drivers will be called “transport-like”. A generic example of transport-like driver is that of (1.1) with a geometric rough path \( X \), see Example 1.1.

1.1. Notation. By \( \mathbb{N} \), we denote the set of natural integers \( 1, 2, \ldots \), and we let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), while \( \mathbb{Z} := \mathbb{N}_0 \cup (-\mathbb{N}) \). Real numbers are denoted by \( \mathbb{R} \) and we also adopt the notation \( \mathbb{R}_+ := [0, \infty) \).

Throughout the paper we shall consider a finite, fixed time horizon \( T > 0 \).

Sobolev spaces. We will consider the usual Lebesgue and Sobolev spaces in the space-like variable: \( L^p(\mathbb{T}^d) \), \( W^{k,p}(\mathbb{T}^d) \), for \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty] \), and we distinguish the case \( p = 2 \) by writing \( H^k(\mathbb{T}^d) := W^{2k}(\mathbb{T}^d) \). Their norms will be denoted by \( \|\cdot\|_{L^p(\mathbb{T}^d)}, \|\cdot\|_{W^{k,p}(\mathbb{T}^d)}, \|\cdot\|_{H^k(\mathbb{T}^d)} \).

The notations \( L^p, W^{k,p} \) and \( H^k \) will be sometimes used as abbreviations of the above spaces. When \( k \) is negative, we define \( W^{k,p} \) to be the range of the linear mapping \( T : (L^p)^{N_k} \ni h \mapsto Th := (D^\gamma h_\gamma)_{|\gamma| \leq -k} \) where \( |\gamma| := \gamma_1 + \cdots + \gamma_d \) and \( N_k := \sum_{|\gamma| \leq -k} |\gamma| \). The space \( W^{k,p} \) is endowed with the norm corresponding to the infimum over the \( L^p \) norms of its antiderivatives, namely

\[
|f|_{W^{k,p}} := \inf_{h \in (L^p)^{N_k}} \sum_{|\alpha| \leq |k|} |h^{\alpha}|_{L^p}.
\]  

(1.4)

As is well known, for \( p \neq 1 \), the space \( W^{k,p} \) is isomorphic to the dual space of \( W^{-k,\frac{p}{p-1}} \). Note that with our convention (1.4), for \( p = 1 \), \( W^{-k,1} \) is a proper subspace of \( (W^{k,\infty})^* \).

For functions \( f \) also depending on the time-like variable, we use the notation

\[
\|f\|_{L^r(s,t;L^s)} := \left( \int_s^t \left( \int_{\mathbb{R}^d} |f_r(x)|^q dx \right)^{r/q} dr \right)^{1/r}, \quad \text{for} \quad 0 \leq s \leq t \leq T,
\]

and we will also write \( \|f\|_{L^r(L^s)} \) instead of \( \|f\|_{L^r(0,T;L^s)} \). By \( C(0,T;E) \), we denote the space of continuous function with values in a Banach space \( E \). It is endowed with the usual supremum norm. Given another space \( F \), we will denote by \( \mathcal{L}(E,F) \) the space of linear, continuous maps from \( E \) to \( F \), endowed with the operator norm. For \( f \in E^* := \mathcal{L}(E,\mathbb{R}) \), we will denote the dual pairing by \( \langle f, g \rangle \) (i.e. the evaluation of \( f \) at \( g \in E \)).
Rough paths. We now introduce some notation related to controlled paths theory [Gub04] and rough paths. We will denote by $\Delta, \Delta_2$ the simplices

$$
\Delta := \{(s, t) \in [0, T]^2, s \leq t\}, \quad \Delta_2 := \{(s, \theta, t) \in [0, T]^3, s \leq \theta \leq t\}.
$$

(1.5)

If $E$ is a vector space and $g : [0, T] \to E$, we define a two-parameter element $\delta g$ as

$$
\delta g_{st} := g_t - g_s, \quad \text{for all } (s, t) \in \Delta.
$$

Similarly, we define another operation $\delta'$ by letting, for any $g : \Delta \to E$, $\delta' g$ be the quantity

$$
\delta' g_{s\theta t} := g_{st} - g_{s\theta} - g_{\theta t}, \quad \text{for every } (s, \theta, t) \in \Delta_2,
$$

and we recall that $\text{Ker} \delta' = \text{Im} \delta$. As usual in the framework of controlled paths, we will omit the “′” on the second operation, by writing abusively $\delta$ instead of $\delta'$.

If $E$ is equipped with a norm $| \cdot |_E$, we shall denote by $C^\alpha_2(0, T; E)$ the set of 2-index maps $g : \Delta \to E$ such that $g_{tt} = 0$ for every $t \in [0, T]$ and

$$
|g|_{\alpha, E} \overset{\text{def}}{=} \sup_{(s, t) \in \Delta} \frac{|g_{st}|_E}{(t - s)^{\alpha}} < \infty.
$$

(1.6)

We also denote by $C^\alpha_1(0, T; E)$ the space of $g : [0, T] \to E$, such that $\delta g$ belongs to $C^\alpha_2(0, T; E)$ (which corresponds to the usual Hölder space) and for such $g$ we shall abusively write $|g|_{\alpha, E}$ instead of $|\delta g|_{\alpha, E}$.

Fix now $m \geq 1$. Recall that a two-step, $m$-dimensional geometric rough path is a pair

$$
X \equiv (X^{1, \mu}, X^{2, \mu\nu})_{1 \leq \mu, \nu \leq m}
$$

(1.7)

in the product space

$$
C^\beta_2(0, T; \mathbb{R}^m) \times C^{2\beta}_2(0, T; \mathbb{R}^{m \times m})
$$

(1.8)

such that Chen’s relations hold, namely:

$$
\delta X^\mu_{s\theta t} = 0, \quad \delta X^{2, \mu\nu}_{s\theta t} = X^\mu_{s\theta} X^\nu_{\theta t}, \quad \text{for } (s, \theta, t) \in \Delta_2, \quad 1 \leq \mu, \nu \leq m.
$$

(1.9)

Informally speaking, the first algebraic relation in (1.9) reflects the fact that $X^{1, \mu}_{s\theta t}$ is an increment $x_t - x_s$ for some $x : [0, T] \to \mathbb{R}^m$, while the second essentially means that the quantity $X^{2, \mu\nu}_{s\theta t}$ should be thought of as a prescribed value for the (a priori ill-defined) integral $\int_s^t (x_r - x_s) \otimes dx_r$.

We call control on $I$ any superadditive map $\omega : \Delta \to \mathbb{R}_+$, that is, for all $(s, \theta, t) \in \Delta_2$ there holds

$$
\omega(s, \theta) + \omega(\theta, t) \leq \omega(s, t).
$$

(1.10)

(Note that the property (1.10) implies in particular that $\omega(t, t) = 0$ for any $t \in [0, T]$.) We will call $\omega$ regular if in addition $\omega$ is continuous. Example of regular controls are given by $(t - s)^\alpha$ or $(\int_s^t f_r dr)^\alpha$ with $\alpha \geq 1$ and a non-negative $f \in L^1$.

We will denote by

$$
Z(0, T; E)
$$

(1.11)

the space consisting of $g \in C^\beta_2(0, T; E)$ such that there exists a constant $\ell > 0$, a regular control $\omega$, and a real number $z > 1$ so that $|g_{st}|_E \leq \omega(s, t)^z$, for every $(s, t) \in \Delta$ such that $|t - s| \leq \ell$. 

1.2. Unbounded rough drivers. As far as we are interested in a theory in the energy space \( L^\infty(L^2) \cap L^2(H^1) \) for (1.1), we do not need to work with the whole Sobolev scales \((W^{k,p})_{k \in \mathbb{Z}}\) for \( p \in [1, \infty] \), but only a finite sequence. In fact, we will only need the following indices
\[
S := \{-3, -2, -1, 0, 1, 2, 3\}. \tag{1.12}
\]

The following notion was introduced in [BG17].

**Definition 1.1** (unbounded rough driver). Let \( \alpha \in (1/3, 1/2] \) and fix \( p \in [1, \infty] \). A pair of 2-index maps \( B \equiv (B^1, B^2) \) is called an \( \alpha \)-Hölder unbounded rough driver in \( L^p \), if

- (RD1) \( B^i_{st} \) belongs to the space \( C^\alpha_2(0, T; \mathcal{L}(W^{k,p}, W^{k-i,p})) \) for \( i = 1, 2 \), and \(-3 \leq k-i \leq k \leq 3 \).

- (RD2) Chen’s relations hold true, namely, for every \((s, \theta, t) \in \Delta_2 \), we have
\[
\delta B^1_{st} = 0, \quad \delta B^2_{st} = B^1_{st} B^1_{st}, \tag{1.13}
\]
as linear operators on the scale \((W^{k,p})_{k \in S}\).

For \( i = 1, 2 \), we shall denote by
\[
\mathcal{T}_i := \cap_{k=-3+i}^3 \mathcal{L}(H^k, H^{k-i}), \tag{1.14}
\]
equipped with the sum of the corresponding operator norms. If \( b : [0, T] \to \mathcal{T}_1 \) is a path of finite-variation, one can define the canonical lift \( B \equiv (B^1, B^2) := S_2(b) \) which is the URD
\[
B^1_{st} := \delta b_{st} \in \mathcal{T}_1
\]
and
\[
B^2_{st} := \int_s^t (b_r - b_s)db_r \in \mathcal{T}_2.
\]
The latter integral is well-defined in the sense of Riemann-Stieltjes, in the space \( \mathcal{T}_2 \). We will assume in this paper that \( B \) can be approximated by sequences of such canonical lifts, more precisely:

**Assumption 1.1** (Geometricity). The unbounded rough driver \( B \) is geometric, namely: there exists a sequence of paths \( b(n) \in C^1([0, T]; \mathcal{T}_1), n \geq 0 \), such that letting \( B(n) := S_2(b(n)) \), it holds for each \( p \in [1, \infty] \), \( n \geq 0 \), and \( 1 \leq i \leq k \leq 2 \) :
\[
\rho\alpha(B(n), B) \overset{\text{def}}{=} \sum_{k=-2}^3 \|b(n) - B^1_k\|_{L^\infty([0,T]; \mathcal{L}(W^{k,p}, W^{k-1,p}))} + \sum_{i=1}^2 \sum_{k=-3+i}^3 \|B^i(n) - B^i\|_{\mathcal{L}(W^{k,p}, W^{k-1,p})} \to 0. \tag{1.15}
\]

Moreover, we will work under the standing assumption that \( B \) is “transport-like”, which in particular covers the case (1.1).

**Assumption 1.2** (transport property). For every \((s, t) \in \Delta \), \( f, g \in C^\infty(\mathbb{T}^d) \), we assume that it holds
\[
B^1_{st}(fg) = (B^1_{st}f)g + f(B^1_{st}g) \quad \text{and} \quad B^2_{st}(fg) = (B^2_{st}f)g + (B^1_{st}f)(B^1_{st}g) + f(B^2_{st}g), \tag{1.17}
\]
a.e. on \( \mathbb{T}^d \).

**Example 1.1.** Fix a dimension \( m \in \mathbb{N} \), let \( X \equiv (X^1, X^2) \) be an \( \alpha \)-Hölder geometric rough path with \( \alpha > 1/3 \), and assume that we are given coefficients \( \sigma^{\mu,i}(t, x), 1 \leq \mu \leq m, 1 \leq i \leq d \) such that the mapping \( t \mapsto \sigma(t, \cdot) \) is \( \gamma \) Hölder with \( \gamma > 1 - \alpha \), as a path with values in \( W^{3,\infty}(\mathbb{T}^d; \mathbb{R}^{m \times d}) \).
Next, define in the Young sense the integral \( v^{1,ij}_{st} = \int_t^s \sigma^{\mu,ij}(r, x) \, dX^\mu_r \) (here and below we use a summation convention over repeated indices) and observe that the mapping \([0, T] \to W^{3,\infty}, t \mapsto v^1_t\) is controlled by \(X\) with Gubinelli derivative \(\sigma^i(t, \cdot)\). Hence one can define the iterated integrals
\[
v^{2,ij}_{st} = \int_{s<\tau<r<t} \sigma^{\mu,ij}(r, x) \partial_\tau \delta^j(\tau, x), \quad w^{2,ij}_{st} = \int_{s<\tau<r<t} \sigma^{\mu,ij}(r, x) \sigma^{\nu,j}(\tau, x) \, dX^\nu_r \, dX^\mu_r,
\]
in the sense of controlled paths, see [Gub04]. We then define \(B \equiv (B^1, B^2)\) as follows: for \(s, t \in \Delta\) we let
\[
B^1_{st} := v^{1,ij}_{st} \partial_i \in T_1 \quad \text{and} \quad B^2_{st} := v^{2,ij}_{st} \partial_j + w^{2,ij}_{st} \partial_{ij} \in T_2.
\]
It is easily seen that \(B\) defines an unbounded rough driver in \(L^p\), for every \(p \in [1, \infty]\).

Moreover, using the geometricity of \(X\), it can be shown that \(B\) is itself geometric, which is reflected by the algebraic relation
\[
w^{2,ij}_{st} = \frac{1}{2} v^{1,ij}_{st} v^{1,j}_{st}.
\] (1.19)

Furthermore, \(B\) is transport-like. Indeed, the first order Leibniz rule (1.17) is trivial, and the second follows from the fact that, thanks to (1.19), the “bracket” \(\mathbb{L} \equiv 2B^2 - \frac{1}{2}B^1B^1\) is first-order. More explicitly, we have
\[
\mathbb{L}_{st} := B^2_{st} - \frac{1}{2} B^1_{st} B^1_{st} = \left( v^{2,ij}_{st} - \frac{1}{2} v^{1,ij}_{st} \partial_i v^{1,j}_{st} \right) \partial_j, \quad (s, t) \in \Delta.
\] (1.20)

1.3. Notion of solution and main result. We can now proceed to the definition of a weak solution for an equation of the form
\[
dv = f dt + dB \nu\]
(1.21)
on \([0, T] \times \mathbb{R}^d\), where \(f\) is \(p\)-integrable as a mapping with values in \(W^{-1,p}\) for some given \(p \in [1, \infty]\), and the unknown \(v\) will always be assumed to be bounded as a path with values in \(L^p\).

**Definition 1.2** \((L^p\)-solution). Letting \(p \in [1, \infty]\), we will say that \(v\) is an \(L^p\)-solution of (1.21) if
\[
v \in L^\infty(0, T; L^p) \cap L^p(0, T; W^{1,p}).
\] (1.22)
and for every \(\phi\) in \(W^{3,\frac{p}{p-1}}\), (with the conventions \(1/0 = \infty\) and \(\infty/\infty = 1\)), it holds
\[
\langle \partial v_{st}, \phi \rangle = \int_s^t \langle f_r, \phi \rangle \, dr + \langle (B^1_{st} + B^2_{st}) v_s, \phi \rangle + \langle v^2_{st}, \phi \rangle, \quad (s, t) \in \Delta,
\] (1.23)
for some remainder term \(v^2\) in \(Z(0, T; W^{-3,\frac{p}{p-1}})\), see notation (1.11).

We can now state our main existence result.

**Theorem 1.1.** Given \(u_0 \in L^2\), and an unbounded rough driver \(B \equiv (B^1, B^2)\), satisfying assumptions 1.1 and 1.2, there exists an \(L^2\)-solution to the problem (1.1).

Concerning uniqueness of solutions, we have the following.

**Theorem 1.2.** Assume that the hypotheses of Theorem 1.1 are fulfilled, and assume in addition that \(B^1_{st}1 = B^1_{st} = 0\) (see (1.20)) for every \((s, t) \in \Delta\). Then, the solution constructed above is unique in the class of \(L^2\)-solutions.
The proof of the main theorems will be performed in sections 3 and 4. Existence will be performed by a compactness argument, using the fact that the unbounded rough driver is geometric. We will obtain convergence of a subsequence of approximate solutions, using energy estimates. This follows the lines of the variational approach for rough PDEs recently developed in the series of papers [BG17, DGHT16, HH18, HZ17]. Concerning uniqueness, the proof benefits from the results obtained in [HN18], but also makes use of ideas from the stochastic context [HZ17]. The core of the argument is the so-called renormalization property, namely Theorem 2.1, which states that Nemytskii operations of the form $\beta(u)$, $\beta \in C^2$ give rise, to new solutions of a similar problem. This, together with a suitable approximation argument, yields the possibility to estimate the $L^1$-norm of the difference of two solutions, and then conclude thanks to a Gronwall-type argument. Note that the additional assumption made in Theorem 1 essentially means that the vector field associated to $B$ is divergence-free. Generalization to the case where $B$ is transport-like is much more intricate and will be the object of a forthcoming work.

2. Preliminaries

We recall some useful notions and results that will be used in the proof of our main results. The proof of uniqueness appeals to the following result.

**Theorem 2.1 (Renormalization property).** Let $B$ such that the assumptions 1.1 and 1.2 hold. Let $u$ be an $L^2$-solution of the parabolic equation

$$du = (\text{div}(A(t,x)\nabla u) + f)dt + dBu, \quad \text{on } [0,T] \times \mathbb{T}^d,$$

where $f \in L^2(0,T; H^{-1}), A : [0,T] \times \mathbb{T} \to \mathbb{R}^{d \times d}$ is measurable and such that

$$\lambda |\xi|^2 \leq \sum_{1 \leq i,j \leq d} A^{ij}(t,x)\xi_i \xi_j \leq \lambda^{-1}|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d.$$

Then, for every $\beta \in C^2(\mathbb{R})$ with $|\beta'|_{L^\infty} + |\beta''|_{L^\infty} < \infty$, it holds

$$d\beta(u) = \beta'(u)(\text{div}(A(t,x)\nabla u + f))dt + dB\beta(u),$$

(2.2)

in the sense that the path $[0,T] \to \mathbb{L}^1$, $t \mapsto \beta(u_t)$ is an $L^1$-solution to the above equation. More explicitly, we have for any $\phi \in W^{3,\infty}$ and $(s,t) \in \Delta :$

$$\langle \delta \beta(u_{st}), \phi \rangle = \int_s^t \langle \beta'(ut) \text{div}(A\nabla ut), \phi \rangle dt + \langle (B^1_{st} + B^2_{st}) (\beta(ut)), \phi \rangle + \langle \beta^3_{st}, \phi \rangle$$

(2.3)

for some uniquely-determined remainder term $\beta^3 \in \mathcal{Z}(0,T; W^{-3,1})$.

**Proof.** This result is essentially contained in [HN18, Theorem 1]. The only difference is that here we replace the whole space $\mathbb{R}^d$ by the $d$-dimensional torus $\mathbb{T}^d$, however the proof adapts *mutatis mutandis*. Note that in contrast to the previous result, here we allow for functions $\beta \in C^2$ that do not necessarily vanish at the origin, which is permitted thanks to the fact that the torus has finite Lebesgue measure. Details are left to the reader.

Besides the renormalization property, one of the core arguments that we shall use in this paper is a Gronwall-type lemma, well-adapted to incremental equations of the form (1.23).

**Lemma 2.1 (Rough Gronwall).** Let $E : [0,T] \to \mathbb{R}_+$ be a path such that there exist constants $\kappa, L > 0$, a regular control $\omega$, and a superadditive map $\varphi$ with:

$$\delta E_{st} \leq \left( \sup_{s \leq r \leq t} E_r \right) \omega(s,t)^\kappa + \varphi(s,t),$$

(2.4)
for every \((s, t) \in \Delta\) under the smallness condition \(\omega(s, t) \leq L\).

Then, there exists a constant \(C_{\kappa, L} > 0\) such that

\[
\sup_{0 \leq t \leq T} E_t \leq \exp \left(\frac{\omega(0, T)}{C_{\kappa, L}}\right) \left[ E_0 + \sup_{0 \leq t \leq T} |\varphi(0, t)| \right].
\]

(2.5)

Proof. See [DGHT16].

The following result was proven first in [DGHT16]. For an alternative proof we also refer to [HN18].

**Proposition 2.1** (Remainder estimates). Let \(p \in [1, \infty]\) and fix \(v\) such that (1.21) holds for some \(f \in L^p(0, T; W^{-2,p})\).

There are constants \(C, L > 0\) depending only on \(\alpha\) such that for each \((s, t) \in \Delta\) subject to the smallness assumption

\[|t - s| \leq L,\]

it holds the estimate

\[|v^k_{st}|_{W^{-3,p}} \leq C[B]_\alpha \left( (t - s)^3 \alpha \|v\|_{L^\infty(s,t;L^p)} + (t - s)^9 \int_s^t |f_r|_{W^{-2,p}} \, dr \right).\]

(2.6)

3. Existence

Let \(b(n) \in \mathcal{C}^1(0, T; T_1)\) such that the canonical lift \(B(n) \equiv S_2(b(n))\) converges to \(B\) for the unbounded rough driver metric, and assume without loss of generality that for each \(n \in \mathbb{N}\) and \(t \in [0, T]\), the operator \(b_t(n)\) is a derivation (so that \(B(n)\) is transport-like).

By classical results on quasilinear equations [LSU68, Chapter 5], there exists a unique solution \(u(n) \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)\) to the problem

\[
\begin{align*}
\frac{du(n)}{dt} - \text{div}\left( A(t, x, u(n)) \nabla u(n) \right) &= dB(n)u(n), \\
u_0(n) &= u_0.
\end{align*}
\]

(3.1)

Moreover, following [HN18, Section 4] we can compute the chain rule for the function \(\beta(z) = z^2\), yielding that for every \(\phi \in W^{3, \infty}\) and \((s, t) \in \Delta\):

\[
\begin{align*}
\int_{t \wedge T} \delta(u^2(n))_{st} \phi dx &+ 2 \int_{[s, t] \times T^d} \left[ \nabla u(n) \cdot A(t, x, u(n)) \nabla u(n) \phi + u(n)A(t, x, u(n)) \nabla u(n) \cdot \nabla \phi \right] dx dt \\
&= \langle u^2(n), (B_{st}^1(n) + B_{st}^2(n))^\phi \rangle + \langle u_{st}^2(n), \phi \rangle.
\end{align*}
\]

(3.2)

Testing (3.2) against \(\phi = 1\), and then estimating the remainder \(\langle u_{st}^2(n), 1 \rangle\) by its \(W^{-3,1}\) norm, we obtain thanks to Proposition 2.1 and (1.2):

\[
E_t(n) - E_s(n) \leq C(T, \lambda, [B]_\alpha)(t - s)^\alpha |u_s(n)|_{L^2}^2
\]

\[+ C(\lambda, [B]_\alpha)(t - s)^\alpha \int_s^t (|u(n)|_{L^2}^2 + |\nabla u(n)|^2) \, dr,
\]

(3.3)

where we let \(E_t(n) := |u_t|_{L^2}^2 + \int_0^t |\nabla u(n)|_{L^2}^2 \, dr\). For \((t - s)\) small (depending on \([B]_\alpha, \lambda\) but not on \(n\)) we can absorb the last term to the left, yielding (2.4) with \(\kappa = \alpha\) and \(\varphi = 0\). In particular we obtain that

\[
\sup_{n \in \mathbb{N}} \|u(n)\|_{L^\infty(L^2)} + \int_0^T |\nabla u(n)|_{L^2}^2 \, dt \leq C|u_0|_{L^2}^2.
\]

(3.4)
Using (3.4) in (3.3), we also obtain uniform equicontinuity for $\delta E(n)$, in the sense that for any $\epsilon > 0$, there exists $\delta > 0$ such that for every $n \geq 0$:

$$|t - s| \leq \delta \Rightarrow |\delta E_{st}(n)| \leq \epsilon. \quad (3.5)$$

The same is true for the 2-parameter quantity $G_{st}(n) := |u_t - u_s|_{H^{-1}}$. Indeed, letting for $\eta > 0$, $\rho_\eta := \eta^{-d} \rho(\frac{\cdot}{\eta})$, where $\rho$ is a radially symmetric function integrating to one, one can write, using (3.1)

$$u_t(n) - u_s(n) = (\text{id} - \rho_\eta)\delta u_{st}(n) + \rho_\eta \left[ \int_s^t \text{div}(A(u(n)) \nabla u(n))dr ight] + (B^1 + B^2)(u_s(n)) + u^\beta_{st}(n) = I + II,$$

where “*” denotes the convolution operation. Next, making use of the inequalities

$$|\rho_\eta \ast v|_{H^k} \leq \frac{C_1(\rho)}{\eta^i}|v|_{H^{k-i}} \quad (3.6)$$

$$|/(\text{id} - \rho_\eta)\ast v|_{H^{k-i}} \leq C_2(\rho)\eta^j|v|_{H^k}, \quad (3.7)$$

for every $v \in H^{k-i}$, respectively $v \in H^k$, $-3 \leq k - i \leq k \leq 0$ (see e.g. [HH18]), we obtain that $|I|_{H^{-1}} \leq C_2\eta|u(n)|_{L^2} \leq C_2\eta$ while

$$|II|_{H^{-1}} \leq C(\lambda, [B]_\alpha, C_1) \left[ \delta E_{st}(n) + (t - s)^\alpha + \frac{(t - s)^2}{\eta^2} + \frac{(t - s)^3}{\eta^2} \right] |u(n)|_{L^\infty(s; t; L^2)} + \frac{(t - s)^\alpha}{\eta^2} \delta E_{st}(n).$$

By choosing $\eta = \theta((t - s)^\alpha + \delta E_{st}(n)^\alpha)$ for some sufficiently small but universal parameter $\theta > 0$, we obtain in particular that

$$G_{st}(n) \equiv |u_t(n) - u_s(n)|_{H^{-1}} \leq \delta E_{st}(n) + (t - s)^\alpha + \delta E_{st}(n)^{1-\alpha}, \quad (3.8)$$

where we have use the estimate $(t - s)^\alpha + \delta E_{st}(n)^\alpha \geq 2(t - s)^{\alpha/2}\delta E_{st}(n)^{\alpha/2}$. Using the uniform equicontinuity for $\delta E(n)$, we therefore obtain the claimed property for $G_{st}(n)$.

Now, from the Banach Alaoglu Theorem (3.4) and Ascoli, we obtain a limit point $u \in L^\infty(L^2) \cap L^2(H^1) \cap C(H^{-1})$ such that for a subsequence $u(n_k), k \geq 0$,

$$u(n_k) \rightharpoonup u \quad \text{weakly in } L^2(0, T; H^1), \quad (3.9)$$

$$u(n_k) \rightarrow u \quad \text{strongly in } C(0, T; H^{-1}), \quad (3.10)$$

and interpolating:

$$u(n_k) \rightarrow u \quad \text{strongly in } L^2([0, T] \times \mathbb{T}^d), \quad (3.11)$$

$$u(n_k) \rightharpoonup u \quad \text{almost everywhere}. \quad (3.12)$$

Because of (3.4), the drift term

$$\langle \mathcal{D}(n_k)_{st}, \phi \rangle := -\int_{|s,t|\times \mathbb{T}^d} A(r, x, u_r(n_k)) \nabla u_r(n_k) \nabla \phi dx dr$$

is uniformly bounded, and so is $|u(n_k)^\beta_{st}|_{H^{-3}}$ by Proposition 2.1. Therefore, there exists an element of $Z(0, T; H^{-3})$ denoted by $u^\ast$ so that up to another subsequence $u(n'_k)$:

$$\langle u^\ast(n'_k)_{st}, \phi \rangle \rightarrow \langle u^\ast_{st}, \phi \rangle. \quad (3.13)$$
Moreover, we have the estimates so that in particular \( \beta \)

Hence, using the renormalization property (Theorem 2.1) for

Thanks to (1.2), the equation (4.2) is strongly parabolic, i.e. of the form

This shows that \( u \) is a solution, thus proving existence and Theorem 1.1. ■

4. Uniqueness

We adapt the proof given in [HZ17]. Consider two solutions \( u^1 \) and \( u^2 \) and let \( v := u^1 - u^2 \). We cannot estimate the \( L^1 \)-norm of \( v \) directly because the map \( x \mapsto |x| \) is singular at \( x = 0 \), however we can define an approximation of it as follows.

Let \( 1 > a_1 > a_2 > \cdots > a_n > \cdots > 0 \) be a decreasing sequence of numbers such that

For \( n \geq 1 \), we let \( \varrho_n(\theta) \) be a continuous function supported in \( (a_n, a_{n-1}) \) and such that

and integrating to one, i.e.

We then define

so that in particular \( \beta \in C^2(\mathbb{R}) \) and has bounded first and second order derivatives. Moreover, we have the estimates

Because \( u^1 \) and \( u^2 \) are solutions, we have in the \( L^2 \)-sense:

where for simplicity in the notations we abbreviate the term \( A(t, x, u^i) \) by \( A(u^i), i = 1, 2 \). Thanks to (1.2), the equation (4.2) is strongly parabolic, i.e. of the form

with \( A^{ij} \) bounded above and below, and \( f = \text{div} (A(u^1) - A(u^2)) \nabla u^2) \in L^2(H^{-1}) \).

Hence, using the renormalization property (Theorem 2.1) for \( v \), we have in the \( L^1 \)-sense:

\[
d\beta_n(v) = \beta_n(v) \text{div}(A(u^1)\nabla v) + \text{div}((A(u^1) - A(u^2))\nabla u^2)dt + dB\beta_n(v),
\]

where \( v_0 = 0 \).
Testing against $\phi \equiv 1 \in W^{3,\infty}(\mathbb{T}^d)$, we have
\[
\delta \left( \int_{\mathbb{T}^d} \beta_n(v) \, dx \right)_{st} = - \int_{[s,t] \times \mathbb{T}^d} \beta''_n(v) \nabla v \cdot A(u^1) \nabla v \, dxd\tau \\
- \int_{[s,t] \times \mathbb{T}^d} \beta''_n(v) \nabla v \cdot (A(u^1) - A(u^2)) \nabla v \, dxd\tau + \langle (B^1 + B^2) \beta_n(v), 1 \rangle + \langle \beta_n(v)^3, 1 \rangle.
\]
Using the bound below for $A$, as well as (4.1) and the fact that $A$ is Lipshitz with respect to the third variable, we get the estimate
\[
\delta \left( \int_{\mathbb{T}^d} \beta_n(v) \, dx \right)_{st} + \lambda \int \beta''_n(v) |\nabla v|^2 \, dxd\tau \\
\leq \frac{C}{n} \int |\nabla v||\nabla u^2| \, dxd\tau + C[B]_\alpha (t-s)\alpha |\beta_n(v)|_{L^1} + \langle \beta_n(v)^3, 1 \rangle. \quad (4.3)
\]
Next, we note that the remainder $R_{st} := \langle \beta_n(v)^3, 1 \rangle$ vanishes identically. Indeed one can write, using Chen’s relations (1.13):
\[
\delta R_{st} = \langle B^1 \delta (\beta_n(v))_{st} - B^1 \beta_n(v)|_{st}, 1 \rangle + \langle B^2 \delta (\beta_n(v))_{st}, 1 \rangle = 0
\]
because $B^1 = 0$ and $B^2 = B^1 + L_s 1 = 0$. Hence, $R$ is an increment, and it belongs to $Z(0,T; \mathbb{R})$, we have $R = 0$.

Going back to (4.3), this gives the estimate
\[
\delta \left( \int_{\mathbb{T}^d} \beta_n(v) \, dx \right)_{st} + \lambda \int \beta''_n(v) |\nabla v|^2 \, dxd\tau \\
\leq \frac{C'}{n} \int [\nabla u^1|^2 + |\nabla u^2|^2] \, dxd\tau + C'[B]_\alpha (t-s)\alpha \|\beta_n(v)\|_{L^\infty(s,t;L^1)}
\]
Next, the Rough Gronwall argument (Lemma 2.1) implies
\[
\sup_{r \in [0,T]} \int_{\mathbb{T}^d} \beta_n(v_r) \, dx \leq C \exp \left( \frac{[B]_\alpha T}{C_{L,\alpha}} \right) \int_0^T [\nabla u^1|^2 + |\nabla u^2|^2] \, dr
\]
As $n \to \infty$, the right hand side goes to 0, from which we infer by monotone convergence:
\[
\sup_{r \in [0,T]} \int_{\mathbb{T}^d} |v(x)| \, dx = 0,
\]
yielding uniqueness. This proves Theorem 1.2.

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