Topological Lattice Gravity
Using Self-Dual Variables

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Abstract

Topological gravity is the reduction of general relativity to flat space-times. A lattice model describing topological gravity is developed starting from a Hamiltonian lattice version of $B \wedge F$ theory. The extra symmetries not present in gravity that kill the local degrees of freedom in $B \wedge F$ theory are removed. The remaining symmetries preserve the geometrical character of the lattice. Using self-dual variables, the conditions that guarantee the geometricity of the lattice become reality conditions. The local part of the remaining symmetry generators, that respect the geometricity-reality conditions, has the form of Ashtekar’s constraints for GR. Only after constraining the initial data to flat lattices and considering the non-local (plus local) part of the constraints does the algebra of the symmetry generators close. A strategy to extend the model for non-flat connections and quantization are discussed.

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I. INTRODUCTION

Regularization appears as an inescapable step in the quantization of most interacting field theories. Ashtekar’s non-perturbative quantization program for gravity [1] is no exception; suitable regularization is needed to construct geometric operators and the constraints selecting the physical states. The standard expressions for the geometric operators to measure areas and volumes [2,3], and the scalar constraint [4,7] were derived through point-splitting regularization; on the other hand, the pioneering work of Renteln and Smolin [6] on lattice regularization remained quite distant from the main stream.

Paradoxically, ideas from lattice gauge theory have played a dominant role in the connectiondynamics approach to quantum gravity. Lattice gauge theory is used not to replace the continuum by a lattice as a regularization step, but to rigorously define the quantum configuration space of connectiondynamics $A/G$. The spaces of connections (modulo gauge transformations) of all the graphs embedded in the manifold are linked by a projective structure to define $A/G$; for a review of the field see [3,8].

Apart from the mentioned work intensive research in other approaches to quantum gravity and related areas like the dynamical triangulation approach and the Turaev-Viro model provide strong motivations to further develop lattice regularization within Ashtekar’s quantization program.

The recent work by Loll [9] and Immirzi [10], and the not so recent by Waelbroeck [13] and Katsymovsky [12] share some of the mentioned motivations; this research on lattice regularizations for Ashtekar’s quantization program has a geometrodynamical relative: the original Regge calculus [14]. One must recall that all the different attempts to construct an initial value formalism of Regge calculus as a fundamental theory have failed; their constraints are first-class only in the continuum limit.

This article presents a classical connectiondynamics model of 3 + 1 Regge calculus. From this classical toy model one can get hints to solve some problems that may occur in future lattice regularization of the constraints. The model describes flat space-times as the evolution of a three-dimensional simplicial lattice. It is based on a $SO(3,1)$ lattice gauge theory where every cell has four neighbors. In addition, the variables of the theory are required to satisfy some “geometricity conditions”. Once the geometricity conditions are fulfilled, the variables of the lattice gauge theory specify the geometry of a three-dimensional piecewise linear space that generates space-time as it evolves. In terms of self-dual variables the model for Regge calculus acquires an Ashtekar-like description. After changing to self-dual variables the geometricity conditions take the form of the reality conditions of Ashtekar’s $\text{GR}$. Spatial and time-like translations that preserve the geometricity conditions are generated by constraints; the local part of these symmetry generators resembles the constraints of general relativity written in terms of self-dual variables. The phase space variables of the model and the geometricity conditions of the lattice are closely related to the ones given by Immirzi in [10]; see also [12].

The origin of this model is a 2 + 1 lattice theory formulated by Waelbroeck [13], but a closer relative is the extension of the 2 + 1 theory: a lattice $B \wedge F$ theory in 3 + 1 dimensions (Waelbroeck and Zapata [16]). In the lattice $B \wedge F$ (LBF) case the geometricity conditions fixed a “geometrical gauge”, a precise statement of what the geometricity conditions mean in the LBF case is stated in section [V]. Once in the geometrical gauge the $B \wedge F$ constraints
take the form of the 4d-translation generators of the vertices of the lattice; these constraints together with the geometricity conditions render the lattice’s connection flat, making an extension of the theory to lattices with curvature not viable. Another manifestation of the same issue is the fact that the lattice $B \wedge F$ theory has no degrees of freedom associated with the lattice vertices or cells. The only possible degrees of freedom are topological [17].

In this article I introduce two results that help in the construction of a bridge between lattice $B \wedge F$ (LBF) and lattice gravity. Firstly, self-dual variables in the model allow the geometricity conditions, that in LBF were ordinary second-class constraints, to be treated as reality conditions. Secondly, the model’s symmetry group is effectively smaller than that of LBF and the symmetry generators do not restrict the lattice to be flat making viable an extension of the model to a theory with local degrees of freedom. A vector and a scalar constraint (per lattice cell) replace the four dimensional covariant generators of vertex translations of LBF. Remarkably, the local part of these new constraints has the form of the Ashtekar’s constraints which, as explained in sec. [V], implies that the continuum limit of the model is Ashtekar’s formulation of GR. An important aspect of this article is the notation. The mentioned relation between geometricity and reality conditions and the indication that the local part of the constraints are Ashtekar-like, are highly clarified after the introduction of an “affine” notation for the lattice [20]. The affine notation indicates “space directions” in a manner natural for the discreteness of the lattice, while resembling the notation used in the continuum. The “affine” notation simplifies the difficult task of translating physically meaningful expressions from the continuum to the lattice [21], thus, providing a useful tool for $3 + 1$ Regge calculus.

Once the relation between the model and canonical continuum gravity has been realized, it is natural to ask about quantization. One would like to adapt the original version of Ashtekar’s quantization program [1] to lattice gravity. To this end one would select the physical Hilbert space following Dirac’s prescription and then fix the inner product to make former real quantities Hermitian operators. However, in a study on the quantization of Regge calculus [11] Immirzi pointed out that the plan of choosing the inner product according to the quantum reality conditions fails when following the conventions derived from canonical gravity. The plan’s failure is nothing but another manifestation of the parallelism between the particular lattice approach to quantum gravity followed in this article and the approach of Ashtekar and collaborators for continuum gravity. In the context of continuum gravity Thiemann [18,19] introduced a generalized Wick transform to implement the quantum reality conditions. Some implications of importing Thiemann’s strategy to lattice gravity are discussed in the concluding section.

The organization of the article is the following. Section [II] describes the framework of the lattice theory. It presents the affine notation, a review of the lattice $B \wedge F$ theory, and introduces self-dual variables for the lattice. Section [III] contains a derivation of the geometricity conditions and its expression as reality conditions. In section [IV] the constraints of the model are introduced, as well as the induced symmetries and their algebra. The possibility of extending this model to space-times with curvature is thoroughly discussed in the concluding section.
II. FRAMEWORK

A. Affine Notation

A convenient tool for translating expressions from the continuum to a lattice framework is the affine notation \[2\]. In a n-dimensional simplicial lattice \(\Sigma\), the analog of its tangent bundle is a collection of n-dimensional vector spaces (one attached to every cell of the lattice). In each of these vector spaces, the \(n+1\) intrinsically defined bivectors related to the boundary faces of the simplex and the natural volume element select \(n+1\) one-forms. These intrinsically defined one-forms \((\omega^j)_a\) \((j = 1, ..., n+1\) label the one-forms, and \(a\) is an abstract Minkowski index) can play the role of an affine basis. For any one-form \(\sigma_a\)

\[
\sigma_a = (\omega^j)_a \sigma_j
\]

(2.1)

where

\[
\sum_j (\omega^j)_a = 0 \quad , \quad \sum_j \sigma_j = 0
\]

(2.2)

the first condition holds because the lattice is formed by closed cells and the second condition guarantees the uniqueness of the affine components \(\sigma_j\). One can construct a dual basis of vectors \((e_j)^a\) at each cell from the condition

\[
(\omega^j)_a (e_k)^a = \delta^j_k
\]

(2.3)

where

\[
\hat{\delta}^j_k = \delta^j_k - \hat{n}^j \hat{n}_k, \quad \hat{n}^j = \hat{n}_j = \frac{1}{\sqrt{n + 1}}
\]

(2.4)

in particular, for a three dimensional space

\[
\hat{n}^j = \hat{n}_j = \frac{1}{2}, \quad \hat{\delta}^j_k = \frac{3}{4}, \quad \hat{\delta}^j_{k\neq j} = -\frac{1}{4}
\]

(2.5)

One can see that the projector \(\hat{\delta}\) satisfies \(\hat{\delta}^j_k \hat{\delta}^k_l = \delta^j_l = \delta^j_j = n\). A basis for tensors of higher order can be constructed directly from these two. If one labels the lattice cells by Greek letters \(\alpha, \beta, \ldots\) in such a way that cell \(\alpha\) has as neighbors \(\beta, \gamma, \ldots\), then the directions of the affine basis of the vector space of cell \(\alpha\) can be labeled by its neighbors \(j = \beta, \gamma, \ldots\).

To any \(p\)-form in a manifold, with components \(\sigma(x)_{j_1, \ldots, j_p}\), one assigns a lattice counterpart \(\sigma(\alpha)_{j_1, \ldots, j_p}\) (where \(\alpha\) plays the role of the base point \((x)\)). The lattice \(p\)-form can be regarded as a \(p\)-cochain, that is, a linear function that assigns real numbers to the \(p\)-chains of the lattice (e.g. a 2-cochain assigns numbers to the faces of the lattice). I should emphasize that \(\sigma(\alpha)_{j_1, \ldots, j_p} \equiv \sigma(\alpha)_{\beta}\) should be regarded as the lattice counterpart of e.g. \(\sigma(x)_{j_1, \ldots, j_p} \equiv \sigma(x)_{1}\); therefore, the Einstein summation convention should not be applied for Greek indices appearing in the right. Using the affine notation, one can easily find the discrete counterpart of the \(B \wedge F\) action.
B. Lattice $B \wedge F$ Theory

This subsection mirrors part of [10]; however, it is reviewed here using the affine notation for the convenience of the reader (the notation and conventions of this paper follow Peldán in [15], where he gives a formulation of continuum GR using $SO(3, 1)$ as internal group).

The $B \wedge F$ theory, as Horowitz formulated it [22], starts from a modified Palatini action

$$S_{B^\wedge F} = \int_{\mathcal{M}} B_A \wedge F^A$$

(2.6)

where the internal group is $SO(3, 1)$ and the indices $A$ can be written as $A = [ab], a, b = 0, 1, 2, 3$. In this modified Palatini action the constraint on the $so(3, 1)$ valued two-form

$$B_A := B_{[ab]} = \varepsilon_{abcd} e^c \wedge e^d$$

(2.7)

has been dropped, making gravity and $B \wedge F$ theory different theories.

A space-like discrete version of $B \wedge F$ theory can be formulated from the discrete counterpart of the $3 + 1$ split of the $B \wedge F$ action [22]:

$$S[B, A]_{B^\wedge F} = 3 \int dx^0 \int_{\Sigma} (\dot{A}^A_{[ij]} B_{jk]}A - F^A_{[ij]} B_{k]0A} + A^A_{\alpha} D^i_{[ij]} B_{jk]}A) dx^i dx^j dx^k$$

(2.8)

The lattice counterpart of the last expression should be considered as the starting point of this lattice formalism [16]

$$S[E, M] = 3 \int dx^0 \sum_{\alpha} (4 E_{(\alpha)j} \cdot \dot{A}_{(\alpha)j} - P_{(\alpha)jk} \cdot E_{(\alpha)jk} + A_{(\alpha)\alpha} \cdot J_{(\alpha)})$$

(2.9)

where $E_{(\alpha)A} := \frac{1}{8} B_{(\alpha)klA} \varepsilon^{jkl}$, $\varepsilon^{jkl} := \varepsilon^{jklm} \hat{n}_m$, and the action is a functional of the variables $E_{(\alpha)A}, M_{(\alpha)\beta A}$ attached to every face $(\alpha, \beta)$ of the lattice. In the action, the curvature form was replaced by $P_{(\alpha)jk} = F^A_{[ij]} + O(E^3)$, where $P_{(\alpha)jk} := \frac{1}{4} f^ABW_{(\alpha)\beta A}$, and $W_{(\alpha)\beta B} := (M_{(\alpha)\beta} M_{(\beta)\mu} \ldots M_{(\nu)\gamma} M_{(\gamma)\alpha} C)_{B}$ is the holonomy around the lattice link of cell $\alpha$ where faces $\beta$ and $\gamma$ intersect. The three-form of the Gauss law term in the continuum action is replaced by the integral of $B$ over the boundary of a lattice cell $J_{(\alpha)}$. In the $3 + 1$ split $A_{(\alpha)A}$, and $E_{(\alpha)jk} := B_{(\alpha)ab} \varepsilon^{jkA}$ are Lagrange multipliers.

In a discrete scenario, the role of a connection is better played by matrices that define parallel transport along non-infinitesimal paths. Thus, the connection $A_{(\alpha)j}^C$ that appears in the Lagrangian is regarded as a secondary quantity defined in terms of the matrix that parallel transports to the reference frame at cell $\alpha$, from its neighbor in direction $(j)$, by

$$\exp(A_{(\alpha)j}^C f_{CA}) := M_{(\alpha)jA}^B$$

(2.10)

In the adjoint representation the structure constants of $so(3, 1)$ and the generators of the group are related by $(T_A)_{B}^{C} = f_{AB}^{C} = f_{[ab][cd]}^{[ef]} = -\delta_{\alpha[a] t}^{a} \delta_{\alpha[b]}^{e} \delta_{\alpha[c]}^{f} \delta_{\alpha[d]}^{t}$ and the Lie algebra indices are raised and lowered with the Cartan metric $g_{AB} = -\frac{1}{4} f_{AC}^{D} f_{BD}^{C}$.
To write the Lagrangian explicitly in terms of the parallel transport matrices, one manipulates formally the kinetic term

\[ fA = \ln M \]  
\[ \dot{A} = \frac{1}{4} f M^{-1} \]  
\[ L = \sum_{\alpha} (4E_{(\alpha)}^\beta \cdot \dot{A}_{(\alpha)} - P_{(\alpha)j} \cdot E_{(\alpha)jk} + A_{(\alpha)0} \cdot J_{(\alpha)}) \]  
\[ = \sum_{\alpha\beta} E_{(\alpha)}^{\beta A} f_{B}^{C} M_{(\beta)}^{\alpha B} \dot{M}_{(\beta)}^{\alpha D} + \sum_{\alpha} (-P_{(\alpha)j} \cdot E_{(\alpha)jk} + A_{(\alpha)0} \cdot J_{(\alpha)}) \]  

where in the \((\alpha\beta)\) sum there is a term for \((\alpha, \beta)\) and a term for \((\beta, \alpha)\) if cells \(\alpha, \beta\) share a face. Notice that for this Lagrangian (2.14) the variables \(E_{(\alpha)}^{\beta A}, E_{(\beta)}^{\alpha A}, M_{(\alpha)}^{\beta},\) and \(M_{(\beta)}^{\alpha}\) are all independent. To relate these variables to a lattice, one has to impose the relations

\[ E_{(\alpha)}^{\beta A} = -M_{(\alpha)}^{\beta A} E_{(\beta)}^{\alpha B} \]  
\[ M_{(\alpha)}^{\beta A} M_{(\beta)}^{\alpha B} = \delta_{A}^{B} \]  
\[ M_{(\alpha)}^{\beta A} M_{(\beta)}^{\alpha C} = \delta_{A}^{B} \]  

which form a second-class set with the momentum constraints coming from the action. Through the Dirac procedure, one gets \[13\] the result first derived in the context of lattice gravity by Renteln and Smolin \[6\]

\[ \{E_{(\alpha)}^{\beta A}, E_{(\beta)}^{\alpha A}\} = f_{AB}^{D} E_{(\alpha)}^{\beta D} \]  
\[ \{E_{(\alpha)}^{\beta A}, M_{(\beta)}^{\alpha B}\} = f_{AB}^{D} M_{(\beta)}^{\alpha D} \]  
\[ \{E_{(\alpha)}^{\beta A}, M_{(\beta)}^{\alpha C}\} = f_{AB}^{D} M_{(\beta)}^{\alpha D} . \]  

Now \(M_{(\alpha)}^{\beta j}\) can be considered a parallel transport matrix and \(E_{(\alpha)}^{\beta j=\beta}\) a variable related to the boundary between cells \(\alpha, \beta\), because the relations (2.15)-(2.17) are identities for the Poisson brackets (2.18)-(2.20).

From the Lagrangian (2.23), one also obtains the Gauss law and the flatness constraints

\[ J_{(\alpha) A} = \sum_{j} E_{(\alpha)}^{j A} \approx 0 \]  
\[ P_{(\alpha)j}^{A} = \frac{1}{4} f_{AB}^{C} W_{(\alpha)jkB}^{C} = \frac{1}{4} f_{AB}^{C} (M_{(\alpha)j} M_{(\beta)l} \ldots M_{(n)l})_{B}^{C} \approx 0 . \]  

If the geometricity conditions presented in the next section are satisfied, the previous conditions can be interpreted as the requirements that the lattice cells close and that the parallel

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1Equation (2.12) is strictly correct only for an Abelian group, since it neglects the ordering ambiguity of the two matrices. However, there are only two ways to write the kinetic term (2.14) considering that indices can be contracted only if they live in the same frame. One shows the equivalence between the other possibility and (2.14) integrating by parts.
transport around a lattice link is the identity map. The Gauss law constraint generates gauge transformations

\[ \{ E(\alpha)_A^j, J(\alpha)_B \} = f_{AB}^D E(\alpha)_D^j \]  
\[ \{ M(\alpha)_{jA}^C, J(\alpha)_{jB}^C \} = f_{AB}^D M(\alpha)_{jD}^C \]

and the flatness constraint generates “translations” of \( E \)

\[ \{ E(\alpha)_A^j, P(\alpha)_{jk}^B \} = \delta_A^B + O(P) \]  
where in \( O(P) \) I group a collection of terms of first and higher order in the curvature. Along the paper I am going to keep track of terms that vanish in this model where the lattice is flat, in order to be able to discuss the issue of extending the model to a theory for general lattices.

A remarkable feature of the Poisson algebra (2.18)-(2.20), and hence of the constraints (2.21), (2.22), is that under a decomposition of the variables into their self-dual and antiself-dual parts, the whole theory splits into two identical parts related by complex conjugation.

C. Self-Dual Variables

In \( SO(3,1) \) apart from the Cartan metric \( g_{AB} \), there is another invariant symmetric bilinear form \( g^*_AB = g_{[ab][cd]} := \varepsilon_{[ab][cd]} \). Its invariance follows directly from the invariance of the four-volume element under Lorentz transformations. This metric is used to define duality in the Lie algebra

\[ V_A^* = g^*_AV_B , \quad g^*_A := g^*_{AC}g^{CB} . \]  

The Lorentzian signature of the Cartan metric implies \( g^*_B g^*_C = -\delta_A^C \). Therefore, to split them real Lorentz Lie algebra into its self-dual \((\pm)\) and anti-self-dual \((-\pm)\) components, the projectors involve complex numbers.

\[ V_A^{(\pm)} := \delta_A^{(\pm)B}V_B = \frac{1}{2}(\delta_A^{B} \mp i g^*_B)V_B \]  
\[ g^*_A V_B^{(\pm)} = \pm i V_A^{(\pm)} . \]

The images of the self-dual and antiself-dual projectors are complementary orthogonal subspaces of \( so(3,1;C) \). Also, the following formulas containing the structure constants hold

\[ \delta_A^{(\pm)B} f_{BCD} = \delta_A^{(\pm)B} \delta_C^{(\pm)E} f_{BED} = \delta_A^{(\pm)B} \delta_C^{(\pm)E} \delta_D^{(\pm)F} f_{BEF} := f_{ACD}^{(\pm)} \]  
\[ \delta_A^{(\pm)B} \delta_C^{(-\pm)E} f_{BED} = 0 \]  
\[ f^{(+)}_{ABC} + f^{(-)}_{ABC} = f_{ABC} . \]

These are projectors of the complexified Lie algebra. Here, one first includes the real Lie algebra into the complex Lie algebra and then split it into its self and antiself-dual parts. The fact that the images of \( \delta^{(\pm)} \) lie out side of the image of the real Lie algebra does not prevent the “split”; the only objection could be to call \( \delta^{(\pm)} \) projectors.
In terms of self and antiself-dual variables

\[ E_{(\alpha)}^{(\pm)j}_A := \delta_{A}^{(\pm)C} E_{(\alpha)}^{j}_C \]
\[ M_{(\alpha)}^{(\pm)B}_j := \delta_{D}^{(\pm)C} \delta_{D}^{(\pm)B} M_{(\alpha)}^{jC} = \delta_{A}^{(\pm)C} M_{(\alpha)}^{(\pm)B}_C , \]

the Poisson algebra is

\[ \{ E_{(\alpha)}^{(\pm)\beta}_A , E_{(\alpha)}^{(\pm)\beta}_B \} = f^{D}_{AB} E_{(\alpha)}^{(\pm)\beta}_D = \pm f^{D}_{AB} E_{(\alpha)}^{(\pm)\beta}_D \]
\[ \{ E_{(\alpha)}^{(\pm)\beta}_A , M_{(\beta)}^{(\pm)C}_B \} = f^{D}_{AB} M_{(\beta)}^{(\pm)C}_D = \pm f^{D}_{AB} M_{(\beta)}^{(\pm)C}_D \]
\[ \{ E_{(\alpha)}^{(\pm)\beta}_A , M_{(\beta)}^{(\pm)C}_B \} = f^{D}_{AB} M_{(\beta)}^{(\pm)C}_D = \pm f^{D}_{AB} M_{(\beta)}^{(\pm)C}_D . \]

The Poisson brackets between self-dual and antiself-dual variables always vanish. Since the structure constants \( f^{D}_{AB} \) and \( f^{D}_{AB} \) are totally antisymmetric three tensors in three-dimensional (complex) spaces, they are proportional to the intrinsic volume element. In the basis suggested by the reality conditions of next section the proportionality constant for the self-dual part is \( i\sqrt{2} \) and for the antiself-dual is \( -i\sqrt{2} \). That is, the Lie algebra \( so(3;1) \) “splits” into two copies of the Lie algebra \( so(3;C) \). Each of these \( so(3;C) \) algebras contains all the information of \( so(3,1) \)

\[ V_A = V_A^{(+)} + V_A^{(-)} = V_A^{(+)} + c.c. \]

An immediate but important consequence is that the symmetry generators also split, yielding two parallel theories.

It would have been possible to start with self-dual variables in the action; however, I decided against it in order to preserve the direct geometric interpretation of the variables. On the other hand, using self-dual variables one learns that the geometricity conditions are the lattice counterpart of the reality conditions of Ashtekar’s GR.

### III. GEOMETRICITY-REALITY CONDITIONS

The motivation for demanding geometricity conditions on the variables is to guarantee the existence of a one-to-one mapping between the space of simplicial lattices and the space of variables \( E, M \) satisfying the geometricity conditions. Simultaneously, one gets a selection rule for the symmetry generators, ruling out the symmetries that do not preserve the geometricity conditions.

A set of variables \( E_{(\alpha)}^{j}_A \) related to a face of a lattice of simplices is of the form

\[ E_{(\alpha)}^{j}_A = E_{(\alpha)}^{j}_A = \frac{1}{2} \varepsilon_{abcd} l_{(\alpha,j_1)}^{e} l_{(\alpha,j_2)}^{d} \]
\[ = \frac{1}{2} \varepsilon_{abcd} l_{(\alpha,j_2)}^{e} l_{(\alpha,j_3)}^{d} = \frac{1}{2} \varepsilon_{abcd} l_{(\alpha,j_3)}^{e} l_{(\alpha,j_1)}^{d} \]

where the space-like Minkowski vectors \( I_{(\alpha,j)} \) are associated with the links of the face that is the frontier between the cell \( \alpha \) and its neighbor in direction \( (j) \). Clearly, the link vectors satisfy the condition \( I_{(\alpha,j_1)} + I_{(\alpha,j_2)} + I_{(\alpha,j_3)} = 0 \). Since every link of a tetrahedron is shared by two of its faces, a relation of the form \( I_{(\alpha,j_2)} = -I_{(\alpha,k_1)} \) holds for each link too.

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The geometricity conditions (3.1) are equivalent to the lattice analog of the condition $B = e \wedge e$ that distinguishes gravity from $B \wedge F$ theory. In this sense, discarding the symmetries that do not preserve the geometricity conditions bring us an step closer to gravity. To avoid confusion between the $B$ of $B \wedge F$ theory and the magnetic field of the curvature in the lattice I will write $b = e \wedge e$, more precisely,

$$E(\alpha)^{j}_{[ab]} \approx \frac{1}{8} \hat{\varepsilon}^{jkl} b(\alpha)_{kl[ab]} = \frac{1}{8} \hat{\varepsilon}^{jkl} \hat{\varepsilon}^{cde} e(\alpha)^{c}_{k} e(\alpha)^{d}_{l}$$

where $l(\alpha)^{jk}_{c} = \hat{\varepsilon}^{jkl} e(\alpha)^{c}_{l}$. The weak equivalence sign indicates that I have used the constraint $J(\alpha) = \sum_{j} E(\alpha)^{j}_{A} \approx 0$. I decided to write the “affine triads” $e(\alpha)^{a}_{j}$ that appear just as an intermediate step between $E$’s and $l$’s to make contact with other works, and because these affine triads are the ones that indicate directions naturally in the lattice, and are going to be very helpful to write the constraints.

All the geometricity requirements (3.1) for cell $(i)$ can be written purely in terms of the variables $E(\alpha)^{j}_{A}$

$$q(\alpha)^{\ast jk} := g^{AB} E(\alpha)^{j}_{A} E(\alpha)^{k}_{B} = 0$$

or in terms of self-dual variables

$$i(E(\alpha)^{(+)j}_{A} E(\alpha)^{(+)k}_{A} - c.c.) = -2 \text{Im}(E(\alpha)^{(+)j}_{A} E(\alpha)^{(+)k}_{A} ) = 0 .$$

This first set of conditions guarantees the geometricity of each separate cell: for $j = k = \beta$ it requires that $E(\alpha)^{j}_{A}$ represent the dual of the area bivector of the face $(\alpha, \beta)$ between cells $\alpha$ and $\beta$. In addition, for $j \neq k$, the condition is satisfied if the faces $(\alpha, j)$ and $(\alpha, k)$ of cell $\alpha$ intersect. Once conditions (3.4) are satisfied, the variables $E(\alpha)^{j}_{A}$ characterize a tetrahedron that is contained in a space-like three-dimensional subspace of Minkowski space-time if

$$g^{AB} E(\alpha)^{j}_{A} E(\alpha)^{j}_{B} = 2 \text{Re}(g^{AB} E(\alpha)^{(+)j}_{A} E(\alpha)^{(+)j}_{B}) < 0 .$$

The similarity between (3.4) and the condition which requires the spatial metric of Ashtekar’s formulation of gravity to be real is remarkable considering that the geometricity conditions (3.3) were first proposed [16] in a context not related to Ashtekar’s formulation of general relativity. Furthermore, inequality (3.5) has a continuum analog that demands the metric to be Lorentzian.

One also wants a covariant description in which parallel transport between neighboring faces is described by Lorentz matrices $M(\alpha)^{b}_{a}$. The variables $l$ and $M$ are called the geometrical variables. After enforcing the first set of geometricity conditions (3.4), the

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3I will use the same notation $M(\alpha)^{b}_{a} = \exp(A(\alpha)^{cb}_{a} \eta_{ca})$ for these matrices, which act on Minkowski vectors, as for the previously defined $M(\alpha)^{B}_{A} = \exp(A(\alpha)^{Cc}_{f} f_{CA}^{B})$ in the bivector representation. Both matrices are different representations of the same $SO(3, 1)$ element.
geometrical variables are completely determined by the fundamental variables $E, M$. The complete set of geometricity conditions for the fundamental variables must imply that the geometrical variables of cell $\alpha$ and its neighbor $\beta$ satisfy the compatibility conditions

$$I_{(\alpha)}^{\beta \gamma} = -M_{(\alpha)\beta} I_{(\beta)}^{\alpha \phi} \quad (3.6)$$

where the faces defining $I_{(\alpha)}^{\beta \gamma}$ are $E_{(\alpha)}^{\beta}, E_{(\alpha)}^{\gamma}$, and the ones defining $I_{(\beta)}^{\alpha \phi}$ are $E_{(\beta)}^{\alpha}, E_{(\beta)}^{\phi}$. Some of these requirements are contained in the identity $E_{(\alpha)}^{\beta} = -M_{(\alpha)\beta} E_{(\beta)}^{\alpha}$; however, other conditions exist. These new restrictions relate to the different ways in which $E^*$ can be written as $I \wedge I$; in particular, there is one degree of freedom corresponding to the rotations within the plane defined by $E^*$. The constraint on the connection matrices, which freezes this degree of freedom, imposes zero torsion on the lattice.

An expression of condition (3.6) in terms of the fundamental variables $E, M$ follows from its geometrical meaning. The condition requires that links on the boundary between cells $\alpha$ and $\beta$ be the same when defined using variables $E$ from either cell. Consider $I_{(\alpha)}^{\beta \gamma}$ lying in the intersection of faces $E_{(\alpha)}^{*\beta}$ and $E_{(\alpha)}^{*\gamma}$ of cell $\alpha$ and $I_{(\beta)}^{\alpha \phi}$ lying in the intersection of two faces $E_{(\beta)}^{*\alpha}$ and $E_{(\beta)}^{*\phi}$ of cell $\beta$. Then equation (3.6) holds if the planes defined by $E_{(\alpha)}^{*\beta}$, $E_{(\alpha)}^{*\gamma}$ and $M_{(\alpha)\beta} E_{(\beta)}^{*\alpha}$ all intersect (3.7), (3.8), (3.9) and intersect along the same line (3.10):

$$g^{AB} E_{(\alpha)}^{\beta} E_{(\alpha)}^{\gamma} = -2\text{Im}(E_{(\alpha)}^{(+)\beta B} E_{(\alpha)}^{(+)\gamma}) = 0 \quad (3.7)$$

$$g^{AB} E_{(\alpha)}^{\beta} M_{(\alpha)\beta} C E_{(\beta)}^{\phi} = -2\text{Im}(E_{(\alpha)}^{(+)\beta B} M_{(\alpha)\beta} C E_{(\beta)}^{(+)\phi}) = 0 \quad (3.8)$$

$$g^{AB} E_{(\alpha)}^{\gamma} M_{(\alpha)\beta} C E_{(\beta)}^{\phi} = -2\text{Im}(E_{(\alpha)}^{(+)\gamma B} M_{(\alpha)\beta} C E_{(\beta)}^{(+)\phi}) = 0 \quad (3.9)$$

$$E_{(\alpha)}^{*\beta} f^{ABC} E_{(\alpha)}^{*\gamma} M_{(\alpha)\beta} D E_{(\beta)}^{*\phi} = 2\text{Im}(E_{(\alpha)}^{(+)\beta A} f^{(+)ABC} E_{(\alpha)}^{(+)\gamma} M_{(\alpha)\beta} D E_{(\beta)}^{(+)\phi}) = 0 \quad (3.10)$$

where there is no summation over the underlined index $\beta$ of the last equation. Condition (3.7) was already present in the first set of geometricity conditions on cell $\alpha$ (3.4), and condition (3.8) is a consequence of the geometricity conditions for cell $\beta$ and the identity (2.13). The second set of geometricity conditions is: one condition of the kind (3.3) and one condition of the kind (3.10) for the relation $I_{(\alpha)}^{\beta \gamma} = -M_{(\alpha)\beta} I_{(\beta)}^{\alpha \phi}$ and other set of conditions for the relation $I_{(\alpha)}^{\beta \gamma} = -M_{(\alpha)\gamma} I_{(\gamma)\beta \alpha}$. Clearly, the relations arising from $I_{(\alpha)}^{\beta \gamma} = -M_{(\alpha)\beta} I_{(\beta)}^{\alpha \phi}$ and the ones coming from $I_{(\beta)}^{\alpha \phi} = -M_{(\beta)\alpha} I_{(\alpha)}^{\beta \gamma}$ are equivalent; then an straightforward exercise in linear algebra shows that the “symmetrized” relations

$$\text{Im}(E_{(\alpha)}^{(+)\gamma B} M_{(\alpha)\beta}^{(+)C} E_{(\beta)}^{(+)\phi}) + \text{Im}(E_{(\alpha)}^{(+)\beta B} M_{(\alpha)\gamma B}^{(+)C} E_{(\gamma)}^{(+)\phi}) = 0 \quad (3.11)$$

$$\text{Im}(E_{(\alpha)}^{(+)\beta A} f^{(+)ABC} E_{(\alpha)}^{(+)\gamma} M_{(\alpha)\beta} D E_{(\beta)}^{(+)\phi})$$

$$+ \text{Im}(E_{(\alpha)}^{(+)\gamma A} f^{(+)ABC} E_{(\alpha)}^{(+)\beta} M_{(\alpha)\beta} D E_{(\beta)}^{(+)\phi}) = 0 \quad (3.12)$$

imposed for every link of each cell are equivalent to relations (3.3), (3.10) imposed once for each of the three links of each face of every cell.

In a recent paper, Immirzi wrote the reality conditions (3.4) for the lattice; the motivation of his work is to obtain a consistent Ashtekar-like framework for the lattice [10]. This
coincidence with Immirzi’s formulas, that come from a rather different approach, is the lattice manifestation of a well known result of Capovilla, et al [23]. The mentioned result arrives to the Ashtekar formalism as the Hamiltonian version of a formulation of gravity based on two forms.

The set of geometricity conditions (3.4), (3.11), (3.12) on the fundamental variables form a complete set in the sense that they imply the existence and uniqueness of the geometrical variables (3.1) and that condition (3.6) holds. An expression for the link $l(\alpha)_{\gamma \delta}^a$ of tetrahedron $\alpha$ where face $(\alpha \gamma) (E(\alpha)^\gamma_A = E(\alpha)^\gamma_B = 1/2 \varepsilon^a_{cd} I(\alpha)^{\gamma \delta}_d)$, and face $(\alpha \delta)$ $(E(\alpha)^\delta_A = E(\alpha)^\delta_B = 1/2 \varepsilon^a_{ab} I(\alpha)^{\delta \gamma}_a)$ intersect is

$$l(\alpha)^{\gamma \delta}_a = \frac{1}{\nu(\alpha)^2} \varepsilon(\alpha)_{abc} \varepsilon(\alpha)^{b \delta} (\nu(\alpha)^{\gamma \delta} E(\alpha)^h \varepsilon(\alpha)^{c h} E(\alpha)^a),$$

(3.13)

where the volume of the tetrahedron $\alpha$ is given by

$$\nu(\alpha)^2 = \frac{1}{9} \varepsilon_{j k l} f^{A B C} E(\alpha)^{j A} E(\alpha)^{k B} E(\alpha)^{l C}$$

(3.14)

and the volume element for cell $\alpha$

$$\varepsilon(\alpha)_{abc} = \frac{\phi(\alpha)_{abc}}{\sqrt{\frac{1}{2} \phi(\alpha)_{def} \phi(\alpha)_{def}}}$$

(3.15)

$$\phi(\alpha)_{abc} = \varepsilon_{j k l} \varepsilon_{def} f E(\alpha)^{j a} \varepsilon_{b} d E(\alpha)^{k e} \varepsilon_{c} f E(\alpha)^{l g} x(\alpha)^q$$

(3.16)

do not depend on the choice of the transverse vector $x(\alpha)^q$. The geometricity conditions (3.4), (3.9), (3.10) are also necessary: any set of variables $E$, that can be written as in (3.1) where the compatibility condition (3.6) holds, satisfies them.

(i) From (3.6) one can see that if $W(\alpha)^{\gamma \delta} := (M(\alpha)^{\gamma A} M(\beta)^{\delta}_{\mu} \ldots M(\omega)^{\gamma}_{\lambda} M(\gamma)^{\delta})$ is the holonomy around $l(\alpha)^{\gamma \delta}$, then a consequence of the geometricity conditions is

$$l(\alpha)^{\gamma \delta} = W(\alpha)^{\gamma \delta} l(\alpha)^{\beta \gamma}$$

(3.17)

Because the geometricity conditions imply that $W(\alpha)^{\gamma \delta}$ has an axis and that the axis cross the planes of $E(\alpha)^{\gamma \delta}$ and $E(\alpha)^{\delta \gamma}$

$$P(\alpha)^{1 A} (P(\alpha)^{\gamma A} P(\alpha)^{\gamma A} = -2 \text{Im}(P(\alpha)^{\gamma A} P(\alpha)^{\gamma A}) = 0$$

(3.18)

$$P(\alpha)^{1 A} (P(\alpha)^{\gamma A} P(\alpha)^{\gamma A} = -2 \text{Im}(P(\alpha)^{\gamma A} P(\alpha)^{\gamma A}) = 0$$

(3.19)

$$E(\alpha)^{1 A} f^{A B} f(\alpha)^{1 B} P(\alpha)^{1 C} = 2 \text{Im}(E(\alpha)^{\gamma A} f^{A B} f(\alpha)^{\gamma B} P(\alpha)^{\gamma C}) = 0$$

(3.20)

Evidently the geometricity conditions in the lattice are stronger than what one expected from experience in the continuum; in particular, the continuum counterpart of relations (3.19), (3.20) does not hold. This rigidity of the lattice makes some of the constraints trivial. I discuss this issue in the next two sections.
IV. THE CONSTRAINTS

Not all the symmetries generated by the constraints \( P \approx 0 \) of lattice \( B\wedge F \) theory (LBF) preserve the geometricity conditions. The largest subgroup that preserves them is the group of translation of vertices of the lattice. The generator of translations of vertex \((v)\) of cell \(\alpha\), in the direction of \(w(\alpha)^{(+)a}\), was introduced in [16]. It is easily written in self-dual variables using that \( \delta^{(+)a} = \frac{1}{2}(g - ig^*) \) and that \( w(\alpha)^{(+)a} \) is real

\[
2\text{Re}(w(\alpha)^{(+)a}T(\alpha, v)^{(+)a}) := \left[ w(\alpha)^{(+)a} \sum_{\{jk\} \rightarrow v} \frac{1}{2} \varepsilon_{abcd} l(\alpha)^{jk} \epsilon^+_a P^{(+)cd}_{jk} \right. \\
+ \left. (M(\beta)_\alpha w(\alpha)^{(+)a}) \sum_{\{jk\} \rightarrow v} \frac{1}{2} \varepsilon_{abcd} l(\beta)^{jk} \epsilon^+_b P^{(+)cd}_{jk} + \ldots \right] + \text{c.c.}
\]

The summation written above runs over the links \(I(\alpha)^{jk} a\) pointing in the direction of vertex \((v)\). The terms of the summation have been split for convenience, according to the cell \(\alpha, \beta, \ldots\) where the variables are expressed, but each link must be included only once in the summation. For this reason, the index of the second summation written is \(\{jk\}^-\) indicating to sum only over the links not included in the previous summation.

One can easily prove that the action of \(T(\alpha, v)_a := 2\text{Re}(T(\alpha, v)^{(+)a})\) on the variables \(E\), that determine the geometricity of the lattice, is to generate translations of the vertex \((v)\). First one sees that if the face \(\sigma\) of cell \(\rho\) does not contain vertex \((v)\) the Poisson bracket of \(T(\alpha, v)_a\) and \(E(\rho)^a\) is zero; and that in the case of a face that contains \((v)\), like face \(\beta\) of cell \(\alpha\), the action of the generator \(T(\alpha, v)_a\) on the variable \(E(\alpha)^{\beta\beta}_A = E(\alpha)^{\gamma\beta}_{[a]b} = l(\alpha)^{\beta\delta}_a l(\alpha)^{\beta\gamma}_b = l(\alpha)^{\beta\delta}_a l(\alpha)^{\beta\eta}_b\) has the same geometrical effect as a translation of the vertex \((v)\), where faces \(\beta, \gamma, \delta\) of cell \(\alpha\) intersect.

\[
\{E(\alpha)^{\beta\beta}_A, w(\alpha)^{(+)a}T(\alpha, v)^{(+)a}\} = \frac{1}{4} \varepsilon_{ab} \varepsilon_{gh} w(\alpha)^{(+)a}_A l(\alpha)^{\beta\gamma}_b E(\alpha)^{\gamma\beta}_g P(\alpha)^{cd}_j \\
+ \frac{1}{4} \varepsilon_{ab} \varepsilon_{gh} w(\alpha)^{(+)a}_A l(\alpha)^{\beta\delta}_b \{E(\alpha)^{\gamma\beta}_g, P(\alpha)^{cd}_j\} + O(P) \\
= w(\alpha)^{(+)a}_A l(\alpha)^{\beta\gamma}_j + O(P) \\
= w(T(\alpha, v)^{(+)a}_a) + O(P) .
\]

An immediate consequence is that the geometricity conditions are preserved by the translation generators

\[
\{q(\beta)^{+jk}, w(\alpha)^{(+)a}T(\alpha, v)_a\} = 0 + O(P) .
\]

Two remarks are in order. First, even though it seems natural to consider \(T(\alpha, v)_a\) as the symmetry generator but leave the full (complex) \(T(\alpha, v)^{(+)a} \approx 0\) as constraint, with the reality condition \(w(\alpha)^{(+)a} \epsilon R\) and a real Hamiltonian, the model would not make sense without the geometricity conditions. The strategy of forging a consistent theory for lattice gravity even without geometricity conditions is discussed in the last section. Second, \(\text{Im}(T(\alpha, v)^{(+)a}) = 0\) is automatically satisfied for lattices where the geometricity conditions are satisfied; thus,
writing the constraints as \(T(\alpha, v)_a^{(+)} \approx 0\) is correct, but only half of them are independent of the geometricity conditions (and the formula makes sense only for geometrical lattices).

The result of imposing the symmetries generated by \(T(\alpha, v)_a\) is a theory (GLBF) that describes the geometric sector of LBF \([16]\). More precisely:

- All the solutions of GLBF are solutions of LBF.
- The solutions of GLBF are the flat space-times \(\Sigma \times R\) generated by a geometrical lattice (a lattice made of vertices, links, faces and cells) \(\Sigma\) during its evolution.
- There are some solutions of LBF with “global torsion” that do not admit any geometric representation \([17]\).
- Both GLBF and LBF have zero local degrees of freedom. Both theories have only discrete, topological, degrees of freedom.
- GBF, the restriction of \(B \wedge F\) to torsion free connections is equivalent to GLBF. The proof follows the procedure used by Waelbroeck to prove the equivalence of the lattice 2 + 1 theory and the continuum theory \([24]\).

A lattice theory for gravity must be geometrical (at least when restricted to flat space-times), and must possess local degrees of freedom to reproduce a theory with two degrees of freedom per point in its macroscopic limit. The facts that GLBF has the largest symmetry group that preserves the geometricity of the lattice and has zero local degrees of freedom means that GLBF has too many symmetries. The next step in obtaining a lattice theory for gravity is therefore to select the correct subgroup of the symmetry group of GLBF. The result of choosing the correct subgroup of the symmetry group of GLBF is the model presented in this article (a precise definition of what I mean by the model will be given shortly). As the symmetry group is smaller than that of GLBF one can interpret the model as the result of restricting a theory of lattice gravity to act on flat initial data. This hypothetical theory of lattice gravity would be a theory with local degrees of freedom, and part of the motivation of this article is to learn as much as possible about the hypothetical theory from its restriction to flat initial data (for an extended discussion see the concluding section).

A suggestion of which subgroup to consider comes from the affine notation employed in this article. For the affine notation, cells are the lattice counterpart of points in the continuum. I will show that the translations of lattice cells constitute a proper subgroup of GLBF’s symmetry group (the group of vertex translations). The generator of translations of cell \(\alpha\) in the direction of \(w(\alpha)^{(+)}a\) is simply the one that moves the four vertices of \(\alpha\) by \(w(\alpha)^{(+)}a\).

\[
w(\alpha)^{(+)}aT(\alpha)_a := w(\alpha)^{(+)}aT(\alpha, v_1)_a + w(\alpha)^{(+)}aT(\alpha, v_2)_a + w(\alpha)^{(+)}aT(\alpha, v_3)_a + w(\alpha)^{(+)}aT(\alpha, v_4)_a
\]

(4.4)

The commutativity of cell translations inside the group of vertex translations is a simple consequence of the commutativity of vector addition. Hence the symmetry group of this model has dimension \(4N_3\) (four times the number of lattice cells). In a lattice that admits local deformations that make it flat the number of vertices is (locally) bigger than the
number of cells (see the appendix). As a result, the symmetry group of this model is a proper subgroup of that of GLBF.

In the lattice the links have been described using internal Minkowski vector spaces that are related by parallel transport matrices; the same internal space has been used to point the directions of translation for the translation generators. These internal Minkowski spaces are just side products of the geometricity conditions and are not the internal spaces where the dynamical variables live. Because of this, it would be desirable to label the translation generators of the lattice using notions that are more compatible with the continuum. A general translation of a cell $\alpha$ can be specified by a real “lapse” $N(\alpha)^{(+)\beta} j = \beta, \gamma, \delta, \eta$ (using the affine notation described in sec. 3) $2 \text{Re}(N(\alpha)^{(+)\beta}\mathcal{H}(\alpha)^{(+)\beta}) + N(\alpha)^{(+)\beta}\mathcal{H}(\alpha)^{(+)\beta})$. The generator $\mathcal{H}(\alpha)^{(+)\beta} := 2 \text{Re}(\mathcal{H}(\alpha)^{(+)\beta})$ moves cell $\alpha$ in the direction of its neighbor $\beta$, and $\mathcal{H}(\alpha) := 2 \text{Re}(\mathcal{H}(\alpha)^{(+)\beta})$ translates cell $\alpha$ in the direction orthogonal to the three dimensional space that contains it

\[
\mathcal{H}(\alpha)^{(+)\beta} := \frac{1}{16}(l(\alpha)^{\eta\pi} + l(\alpha)^{\gamma\delta} + l(\alpha)^{\delta\eta})T(\alpha)^{(+)\beta} \approx 0 \tag{4.5}
\]

\[
\mathcal{H}(\alpha)^{(+)\beta} := \frac{1}{2}l(\alpha)^{\eta\pi}l(\alpha)^{\gamma\delta}l(\alpha)^{\delta\eta}T(\alpha)^{(+)\beta} \approx 0 \tag{4.6}
\]

\[
J(\alpha)^{(+)\beta} = \sum_j E(\alpha)^{(+)\beta} \approx 0 \tag{4.7}
\]

\[
P(\alpha)^{(+)\beta}_j (\text{initial}) = 0 \tag{4.8}
\]

I wrote the lattice Gauss law, and the condition restricting to flat lattices as initial data in order to precise what I mean by the model. The model’s phase space is that of LBF described by variables $E^{(+)}$, $M^{(+)}$ and Poisson bracket structure \cite{2.37}, \cite{2.38}, \cite{2.39}. It is subject to the geometricity conditions \cite{3.4}, \cite{3.11}, \cite{3.12}, and to constraints (translation and gauge generators) \cite{4.3}, \cite{4.6}, \cite{4.7}. Also the model’s initial data is restricted to flat lattices \cite{4.8}. The initial data condition can also be written as $T(\alpha)^{(+)\beta}(\text{initial}) = 0$, in a geometrical lattice both relations are equivalent. This is what makes GLBF a genuine theory for the geometric sector of LBF. For the model presented in this article $\mathcal{H}(\alpha)^{(+)\beta} = 0$, $\mathcal{H}(\alpha)^{(+)\beta} = 0$ do not imply $P(\alpha)^{(+)\beta}_j = 0$ even for a geometrical lattice; this opens the possibility of considering the model as the restriction of a hypothetical theory for non flat lattices to the case of initial data satisfying $P(\alpha)^{(+)\beta}_j (\text{initial}) = 0$. The translation generators $\mathcal{H}(\alpha)^{(+)\beta} = 0$, $\mathcal{H}(\alpha)^{(+)\beta} = 0$ of the of the hypothetical lattice theory when restricted to flat lattices have the form \cite{4.3}, \cite{4.6}, i.e. $\mathcal{H}(\alpha)^{(+)\beta} = \mathcal{H}(\alpha)^{(+)\beta} + O(P^2)$, $\mathcal{H}(\alpha)^{(+)\beta} = \mathcal{H}(\alpha)^{(+)\beta} + O(P^2)$. The fact that the symmetry group of the model is smaller than that of GLBF means that the hypothetical lattice theory that reduces to the model for flat initial data is a theory with local degrees of freedom.

As in the case of GLBF’s constraints, one should regard only $\mathcal{H}(\alpha)_j$, $\mathcal{H}(\alpha)$ as symmetry generators and consider all the $\mathcal{H}(\alpha)^{(+)\beta} \approx 0$, $\mathcal{H}(\alpha)^{(+)\beta} \approx 0$ as constraints. However, they are not independent within themselves\cite{4}, and their imaginary part is a direct consequence of the

\footnote{In a three dimensional simplicial lattice there are $6(N_1 - N_0) = 6N_3$ independent curvature}.
non-locality of the translation constraints

Although it is difficult to calculate the algebra of the symmetry generators, there is a simple way to prove that their Poisson bracket vanishes weakly. The translation generators \( \mathcal{H}(\alpha)^j \) and \( \mathcal{H}(\alpha) \) are scalars and, therefore, their Poisson brackets with the gauge transformation generator \( \alpha J^{(+)}_A \) vanish. These translation generators, \( (\mathcal{H}(\alpha)^j \) and \( \mathcal{H}(\alpha)) \), span the same space as the former translation generators \( T(\alpha)_a \). Introducing a bit of notation, \( N(\alpha)^{(+)}_a \mathcal{H}(\alpha)_{\mu} = N(\alpha)^{(+)}_a \mathcal{H}(\alpha)^j_\mu + N(\alpha)^{(+)}_a \mathcal{H}(\alpha) \), the previous statement signifies that \( T(\alpha)_a = C(a)^{(+)}_a \mathcal{H}(\alpha)_{\mu} \) where the matrix \( C(a)^{(+)}_a \) has rank four. Then the Poisson brackets of \( T \)'s and those of \( \mathcal{H} \)'s are related by

\[
\{ T(\alpha)_a, T(\beta)_b \} = C(a)^{(+)}_a C(\beta)^{(+)}_b \{ \mathcal{H}(\alpha)_{\mu}, \mathcal{H}(\beta)_{\nu} \}
\]

+ \( C(a)^{(+)}_a \{ \mathcal{H}(\alpha)_{\mu}, C(\beta)^{(+)}_b \mathcal{H}(\beta)_{\nu} + \mathcal{H}(\alpha)_{\mu} \{ C(a)^{(+)}_a, \mathcal{H}(\beta)_{\nu} \} C(\beta)^{(+)}_b \}
\approx C(a)^{(+)}_a C(\beta)^{(+)}_b \{ \mathcal{H}(\alpha)_{\mu}, \mathcal{H}(\beta)_{\nu} \} .
\] (4.9)

An immediate consequence of the fact that the translation generators \( T(\alpha)^{(+)}_a \) commute (in flat space-time) \( \{ T(\alpha)_a, T(\beta)_b \} = 0 + O(P) \) is that the Poisson brackets of the new form of the translation generators weakly vanish up to second order in the curvature

\[
\{ \mathcal{H}(\alpha)_{\mu}, \mathcal{H}(\beta)_{\nu} \} \approx 0 + O(P^2) .
\] (4.10)

An extremely interesting feature of the spatial translation constraints \( \mathcal{H}(\alpha)^{(+)}_j \) and the time translation constraints \( \mathcal{H}(\alpha)^{(+)} \) is that their local parts are algebraically identical to the diffeomorphism and Hamiltonian constraints of the Ashtekar formulation of general relativity. To define the local parts one requires

\[
\{ E(\alpha)^{(+)}_j, \mathcal{H}(\alpha)^{(+)}_\mu \} = \{ E(\alpha)^{(+)}_j, \mathcal{H}(\alpha)^{(+)}_{\mu local} \}
\]

\[
\{ M(\alpha)^{(+)}_{jA}, \mathcal{H}(\alpha)^{(+)}_\mu \} = \{ M(\alpha)^{(+)}_{jA}, \mathcal{H}(\alpha)^{(+)}_{\mu local} \} .
\] (4.11 - 4.12)

and that in the expression for \( \mathcal{H}(\alpha)^{(+)}_{\mu local} \) only variables related to links of cell \( \alpha \) appear. The link variables \( l(\alpha)^{jk} \) occur in pairs that recombine in the form of the variables of the theory \( E(\alpha)^{(+)}_j \) to yield

\[
\mathcal{H}(\alpha)^{(+)}_{\mu local} \frac{1}{2} \bar{\epsilon}^{jkl} E(\alpha)^{(+)}_A B(\alpha)^{(+)}_B \left( \frac{1}{2} P(\alpha)^{(+)}_{jkl} \right)^k \bar{n}^l J(\alpha)^{(+)}_A
\]

\[
\mathcal{H}(\alpha)^{(+)}_{\mu local} \frac{1}{4} \bar{\epsilon}^{jkl} f^{AB} E(\alpha)^{(+)}_A E(\alpha)^{(+)}_B B(\alpha)^{(+)}_C
\] (4.13 - 4.14)

where the “magnetic field” is written as \( B(\alpha)^{(+)}_A := \bar{\epsilon}^{jkl} P(\alpha)^{(+)}_{jkl} \). The second term of \( \mathcal{H}(\alpha)^{(+)}_{\mu local} \) generates Lorentz transformations and vanishes for flat space-times. The non-locality of the translation constraints \( \mathcal{H}(\alpha)^{(+)} \) is very mild. The difference between

variables \( P(\alpha)^{A}_{jkl} \) because of the Bianchi identities; thus, the \( 8N_3 \) projections of them \( (\mathcal{H}(\alpha)^{(+)}_j, \mathcal{H}(\alpha)^{(+)}_\mu) \) cannot be independent.
$H(\alpha)^{(+)}_{\mu}\text{local}$ and $H(\alpha)^{(+)}_{\mu}$ is function only of variables of the lattice sharing vertices with cell $\alpha$. One of the first basic distinctions between the lattice and the continuum $^{3}$ appears: the concept of neighborhood fundamentally differs. Two points in the continuum $p$ and $q$ can either be the same or be separated by open neighborhoods. In contrast, two cells in the lattice can either be the same, be immediate neighbors, or be separated by other cells. Clearly the category of immediate neighbors disappears during any acceptable continuum limit; therefore, in any acceptable (see last section for a discussion) continuum limit, only the local parts of the expressions are going to remain. Hence, the continuum limit of the translation generators is, precisely, Ashtekar’s diffeomorphism and Hamiltonian constraints.

In order to summarize ideas and prepare the discussion, I am going to count the degrees of freedom of the hypothetical lattice theory resulting from an extension of the constraints of the model (4.3), (4.6) to first-class constraints $H^{h}(\alpha)_{j}$, $H^{h}(\alpha)$). Recall that the number of points, links, faces, and cells are denoted by $N_{0}$, $N_{1}$, $N_{2}$ and $N_{3}$ respectively, and also, for a lattice of tetrahedra, $N_{2} = 2N_{3}$. The phase space variables are given by the $12N_{2} = 24N_{3}$ numbers $E(\alpha)^{(+)i}_{A}$, $M(\alpha)^{(+)i}_{jA}$, that are subject to $6N_{3}$ closure constraints $J(\alpha)^{(+)i}_{A} \approx 0$, and $3N_{3}$ vector constraints $H^{h}(\alpha)_{j}$, and $N_{3}$ scalar constraints $H^{h}(\alpha)$. These constraints are first-class and generate Lorentz transformations, spatial translations, and time translations respectively. Thus, the dimension of the reduced phase space, without taking into account the geometricity conditions, is

$$24N_{3} - 2(6N_{3} + 3N_{3} + 1N_{3}) = 4N_{3} \quad (4.15)$$

And the geometricity conditions reduce the number of degrees of freedom to at least half. The reason is that all the scalar information in the variables $E(\alpha)^{(+)i}_{A}$ of cell $\alpha$ is captured in the symmetric matrix $q(\alpha)^{(+)ijk} = E(\alpha)^{(+)i}_{A} E(\alpha)^{(+)jk}_{B} g^{AB}$; thus, the first set of geometricity conditions ($\text{Im}(q(\alpha)^{(+)ijk}) = 0$) reduces the scalar information on $E(\alpha)^{(+)i}_{A}$ from 12 to 6 numbers. Some local degrees of freedom remain, because as proven in the appendix, the symmetry group of the theory is “locally smaller” than that of GLBF, a theory with zero local degrees of freedom. For a discrete, microscopic, theory of gravity local degrees of freedom are essential, but to get the expected $2N_{3}$ is not. A discussion of this point and related issues is the topic of the last section.

V. DISCUSSION

A classical lattice theory that describes space-time in 3+1 form must be geometrical, i.e. a unique three-dimensional piecewise linear space must be assigned to any set of variables of the theory (in the present case $\{E, M\}$ satisfying the geometricity conditions, $\{l, \pi_{l}\}$ in the usual formulation of Regge Calculus). Also, a lattice theory with no local degrees of freedom cannot be related to gravity simply because any sensible continuum limit cannot convert a set of configurations that are equivalent in the lattice description into inequivalent

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$^{5}$I am assuming that the continuum is a smooth manifold; obviously, excluding pathological topologies such as non-Hausdorff spaces.
geometries in the corresponding continuum theory. On one hand, the model presented in 
this article is geometrical; on the other hand, however, the model describes flat space-times. 
In this sense the model is not better than GLBF (geometric lattice $B \wedge F$ theory). There are 
two essential differences between the model and GLBF. First, GLBF is a theory with first-
class constraints that describes the geometric sector of $B \wedge F$ theory in the sense described in 
section IV. This is different from the model introduced in this article, that has a symmetry 
group whose flows commute only if the initial data is a flat lattice. Second, the constraints 
of the model do not force the lattice curvature to be zero; therefore one can consider the 
model as the restriction of a hypothetical theory, that includes non-flat lattices, restricted 
to the case of flat initial data. The reason to call the theory “hypothetical” is not that its 
existence is in doubt; for instance, the method to find symplectic coordinates \cite{25} can be 
used to get an extension of the translation generators $T(\alpha, v)_a$, in an open neighborhood of 
the submanifold $P = 0$ where their flows commute, to momentum coordinate functions that 
with some configuration coordinate functions form a set of symplectic coordinates. Then, 
the extensions $T^h(\alpha, v)_a$ of the translation generators of the model are the constraints of the 
hypothetical theory. The enormous freedom in the choice of coordinates makes this method 
more of an existence statement than a constructive process: that is the reason to call the 
extension h-theory. Along the article I presented four results concerning any $h$-lattice theory 
that reduces to the model when restricted to flat initial data.

(a) \textit{There are local degrees of freedom in the $h$- theory}. This follows from the fact 
that the symmetry group of the model is a proper subgroup of that of GLBF (see 
appendix).

(b) \textit{The continuum limit has to be a macroscopic limit}. If the continuum limit were 
taken by simply shrinking the lengths of the lattice links to zero, and by identifying 
the cells (or vertices) of the lattice with points of the continuum, the resulting 
continuum theory would have less degrees of freedom than gravity because the $h$-
theory has less than two degrees of freedom per lattice cell. Thus, to be consistent 
one must regard this theory as a microscopic theory.

(c) \textit{Regarding the macroscopic limit of the $h$-theory and Ashtekar’s formulation of 
general relativity}. Although a serious study of the macroscopic limit is yet to 
be performed, two facts indicate that the macroscopic limit of the $h$-theory and 
general relativity formulated in terms of the new variables are related. First, the 
local part of the translation generators of the model is algebraically identical to 
Ashtekar’s constraints, and the reality conditions have the geometricity-reality 
conditions as lattice counterpart. Second, an appropriate procedure to take the 
macroscopic limit could be through refinements of the (dual) lattice and a pre-
scription for projecting down the the variables from the refined lattice to the 
original lattice. Along these refinements the phase space of the lattice becomes 
bigger and given a non-vanishing macroscopic curvature it is possible to reach it 
as the limit of lattices where the curvature in every link goes to zero. These special 
“smooth configurations” would be the ones that define the spatial manifold 
$\Sigma$ and for them the constraints of the $h$-theory become the translation generators 
of the model. Furthermore, in the macroscopic limit the concept of immediate
neighboring cells is lost and, hence, only the local part of the translation generators is relevant. This local part is the one that is algebraically identical to Ashtekar’s constraints.

(d) *The h-theory is condemned to remain classical.* Immirzi’s observation \[1\] that the quantum reality conditions and the algebra of the area bivectors is incompatible makes the theory well defined only classically (see discussion below).

Most of the effort behind this article was directed towards a slightly different goal than the one achieved. The relation between the geometricity conditions and the reality conditions (when using self-dual variables) was the motivation to structure a model for lattice gravity that precisely mirrored Ashtekar’s formulation of GR. In particular, a self-consistent model with symmetry generators whose flows commute for flat initial data \((P(\text{initial}) = 0)\), that resembled Ashtekar’s seemed feasible. A summary of the results, using as proposed constraints \(\mathcal{H}_{(\alpha)^{(+)}_{\mu} \text{local}} (4.13,4.14)\), is the following:

- \(\{\mathcal{H}_{(\alpha)^{(+)}_{\mu} \text{local}}, \mathcal{H}_{(\beta)^{(+)}_{\nu} \text{local}}\} \approx 0 + O(P)\) except in the case of \(\alpha\) and \(\beta\) being neighbors sharing only a link. In this case \(\{\mathcal{H}_{(\alpha)^{(+)}_{\mu} \text{local}}, \mathcal{H}_{(\beta)^{(+)}_{\nu} \text{local}}\} \approx 0 + O(P)\), which means that the symmetries generated by the constraints do not commute even for flat space-times.

- The “symmetry generators” \(\mathcal{H}_{(\alpha)^{+}_{\mu} \text{local}}\) do not respect the geometricity of neighboring cells even for flat space-times.

- On the other hand, the generators

\[
\mathcal{H}_{(\alpha)^{+}_{j} \text{local}} = 2\text{Re}\left(\frac{1}{2} \hat{\varepsilon}_{jkl} E_{(\alpha)(+)A} B_{(\alpha)(+)A} - \frac{1}{2} P_{(\alpha)(+)A} \hat{n}^{j} J_{(\alpha)(+)A} \right) = \hat{\delta}_{j}^{k} P_{(\alpha)(+)A} E_{(\alpha)(+)A} \\
\mathcal{H}_{(\alpha)^{+}_{j} \text{local}} = 2\text{Re}\left(\frac{1}{4} \hat{\varepsilon}_{jkl} f^{AB}_{C} E_{(\alpha)(+)A} E_{(\alpha)(+)B} B_{(\alpha)(+)C} \right) = f^{AB}_{C} E_{(\alpha)(+)A} E_{(\alpha)(+)B} P_{(\alpha)(+)C} 
\]

*do generate the translations expected for variables \(E_{(\alpha)(+)A}^{j}\) of the same cell.*

The conclusion is that the constraints \(\mathcal{H}_{(\alpha)^{(+)}_{\mu} \text{local}}\) are not correct even at first-order in the curvature; but they are the local part of constraints that are correct at first order. This attempt deviates from the path that uses the lattice \(B \wedge F\) theory as a firm ground from which one can extract a richer theory. Using only the local parts of the expressions written in this paper leads to a discrete theory closely related to a discretization of general relativity using Ashtekar’s new variables proposed recently by Boström, Miller and Smolin \[26\]. The translation generators introduced in this paper should be considered as a refinement of those proposed in \[26\], because they differ only in their non-local part that is necessary for them to have commuting flows (for initial data satisfying \(P = 0\)). In this sense, it is the first time that a connection oriented formulation of 3+1 lattice gravity has symmetry generators that are correct up to first order. Thus, this model is of potential interest to develop numerical codes. As shown in section \[\text{IV}\] these non-local terms have an entirely discrete origin and, therefore, their absentia is natural in a theory derived directly from a discrete analog of a continuum action. Unfortunately, only the local part is easily written in terms of the dynamical variables \(E, M\); the non-local terms are naturally written using the geometrical
variables $l$, $M$. The problem with straightforward substitution using the formulas giving the links $l_{\alpha}^{ijk}$ in terms of the variables $E_{\alpha}^{(\pm)}$ is that the 3-volume element $\varepsilon_{\alpha}^{abc}$ for each tetrahedron $\alpha$ is a complicated function of the $E$'s. A solution to this problem could be modifying the model to have $SO(3)$ as internal group and then inherit the natural volume element of the Lie algebra. The counterpart of this approach in the continuum has no complicated reality conditions and is linked to Lorentzian gravity via the generalized Wick transform \cite{19,28} while keeping the Hamiltonian constraint algebraically simple. To import this approach to the lattice means trading immediate geometrical interpretation for algebraic simplicity. Direct geometric interpretation of the lattice theory should not be regarded as a first priority, because a theory of classical gravity with the correct number of degrees of freedom can be recovered only as the macroscopic limit of the lattice theory anyway.

An extension of the model that is coherent even without geometricity conditions could be the first step in extending the model to a theory of lattice gravity that does not require $P(\text{initial}) = 0$. To pursue this option is particularly interesting, because the whole framework of lattices and projective techniques \cite{3,8} is tailored for quantization.

In an attempt to organize ideas for future investigations, a program towards a quantum theory based on this kind of lattice gravity has been outlined. The consequences up to now have been only to state several well known facts in the language used in this article. As mentioned earlier, the lattice theory must be regarded as a microscopic theory to achieve a continuum limit with the correct number of degrees of freedom. A particularly appealing strategy is to do both, the connection to continuum gravity and quantization, simultaneously; such a task is not an utopia, the projective techniques developed by Ashtekar et al \cite{3} and Baez \cite{8} were designed for this purpose. An adaptation of the mentioned strategy for quantization has already been used for abstract lattices by Loll \cite{9}. Immirzi showed that implementation of the quantum reality conditions that result from using $SL(2, C)$ or $SO(3, 1)$ as internal groups is inconsistent; rather than a final word this observation should be considered as an other factor in favor of adopting Thiemann’s strategy of solving the Lorentzian reality conditions via the generalized Wick transform. Work in this direction is in progress \cite{29}.

In the quantization program the issue of constructing a regularization of the constraints is a central one; provide hints to avoid future problems, like the presence of anomalies, was one of the motivations for working on this model.

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APPENDIX

The symmetry group of the model is a proper subgroup of the group of vertex translations

The symmetry group of the model (excluding $SO(3,1)$ symmetries) is the group of cell translations. It has dimension $4N_3$ and is a subgroup of the symmetry group of GLBF (the group of vertex translations) that has dimension $4N_0$. Thus, the fact that the symmetry group of the model is smaller than that of GLBF follows from the fact (to be shown below) that $N_0 > N_3$ in locally Euclidean simplicial lattices.

In three dimensions, the Euler number is zero, i.e., $N_0 - N_1 + N_2 - N_3 = \chi = 0$ where $N_i$ is the number of $i$-dimensional simplices (points, links, faces, and cells). Then the difference between $N_0$ and $N_3$ is the same as that between $N_1$ and $N_2$. In a simplicial lattice there are three links in each face, and each link is shared by three or more faces (excluding lattices with zero volume cells). Thus, $N_1 \geq N_2$, the equality holding only in a case where every link is shared by three faces. A simplicial lattice with this connectivity can not be deformed into a flat lattice as I explain now: It is easy to prove that in Euclidean three dimensional space one cannot draw a tetrahedron and its four neighbors in such a way that these five tetrahedra form a convex polyhedron. Start embedding one tetrahedron in $\mathbb{R}^3$, then because every link is shared by three faces the four neighbors of the tetrahedron must have their faces identified with each other. Thus, the five tetrahedron embedded in $\mathbb{R}^3$ form a convex polyhedron that has no boundary faces. The contradiction indicates that it is impossible to draw a three-dimensional simplicial lattice with more than five cells where three faces share each link or equivalently only simplicial lattices with $N_0 \geq N_3$ fit locally in three-dimensional Euclidean space.
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