Large deviation principle of occupation measures for non-linear monotone SPDEs

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Abstract Using the hyper-exponential recurrence criterion, we establish the occupation measures’ large deviation principle for a class of non-linear monotone stochastic partial differential equations (SPDEs) driven by Wiener noise, including the stochastic $p$-Laplace equation, the stochastic porous medium equation and the stochastic fast-diffusion equation. We also propose a framework for verifying hyper-exponential recurrence, and apply it to study the large deviation problems for strong dissipative SPDEs. These SPDEs can be stochastic systems driven by heavy-tailed α-stable process.

Keywords stochastic partial differential equation, large deviation principle, occupation measure, hyper-exponential recurrence

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1 Introduction

The ergodicity for stochastic partial differential equations (SPDEs) has been studied in abundant literature (see, e.g., [4,5,7,12,16,24,29,31,36]). An SPDE being ergodic means that the occupation measures of its solution converge to a unique invariant measure in some topology. It is natural to further ask whether the occupation measures satisfy a Donsker-Varadhan large deviation principle (LDP for short), which gives an estimate on the probability that the occupation measures are deviated from the invariant measure. Refer to [6,8–11] for an introduction to the large deviation theory of Markov processes. However, in the infinite-dimensional setting their assumptions are not satisfied.

Wu [41] gave a criterion of the LDP of occupation measures for strong Feller and irreducible Markov processes, in which one needs to check the hyper-exponential recurrence. Many techniques such as the Bismut-Elworthy-Li formula [13] and the Harnack inequality [34,35] have been developed for studying the strong Feller property and the irreducibility, while it is often difficult to verify the hyper-exponential recurrence. There seem to be only a few papers about the applications of Wu’s criterion (see [14,15] for

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stochastic Burgers and Navier-Stokes equations, [20] for some dissipative SPDEs and [39] for stochastic reaction-diffusion equations driven by subordinate Brownian motions).

Our contribution is twofold. One is to generalize the spirit of [14,15] to prove Donsker-Varadhan LDP for a family of monotone nonlinear SPDEs such as the stochastic porous medium equation, the stochastic p-Laplace equation and the stochastic fast-diffusion equation. All these equations are very important not only for modeling the phenomena in physics and biology but also for stimulating the development of stochastic analysis. For example, besides its numerous applications in physics [33], the porous medium equation is also the prototype that Otto used to put forward his famous “Otto calculus” [28] in optimal transport theory (see more detail about stochastic porous medium equations in [1–4] and the references therein). The other is that we build a framework for verifying the hyper-exponential recurrence condition by Wu [41], and apply it to a family of strong dissipative SPDEs driven by Wiener noise or heavy tailed noise. Furthermore, this framework also works for stochastic systems with heavy tailed noise. Note that the LDP result of the α-stable type system can be proved by following this framework.

The rest of the paper is organized as follows. In Section 2, we recall some general results on the LDP for strong Feller and irreducible Markov processes. In Section 3, we establish the LDP for a class of nonlinear monotone stochastic partial differential equations driven by Wiener noise. In Section 4, we first introduce a framework for verifying the hyper-exponential recurrence condition, and then give two examples. The appendix provides some regularities of PDEs and gives the detailed proofs of the log-Harnack inequality, strong Feller properties and the irreducibility of the first example in Section 4.

2 General results about large deviations

In this section, we recall some general results on the LDP for strong Feller and irreducible Markov processes from [40,41].

Let $E$ be a Polish space. Consider a general $E$-valued càdlàg Markov process

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t(\omega)\}_{t \geq 0}, \{P^x\}_{x \in E}),$$

where

- $\Omega = \mathcal{D}([0, +\infty); E)$, which is the space of all the càdlàg functions from $[0, +\infty)$ to $E$ equipped with the Skorokhod topology; for each $\omega \in \Omega$, $X_t(\omega) = \omega(t)$;
- $\mathcal{F}_t = \sigma\{X_s; 0 \leq s \leq t\}$ for any $t \geq 0$;
- $\mathcal{F} = \sigma\{X_t; t \geq 0\}$ and $P^x(X_0 = x) = 1$.

Hence, $P^x$ is the law of the Markov process with initial state $x \in E$. For any initial measure $\nu$ on $E$, let

$$P^\nu(d\omega) := \int_E P^x(d\omega)\nu(dx).$$

For each $f$, which belongs to $\mathcal{B}_b(E)$, the space of all bounded measurable functions on $E$, define

$$P_tf(x) := E^x[f(X_t)], \quad \text{for all } t \geq 0, \quad x \in E.$$ 

For any $t > 0$, $P_t$ is said to be strong Feller if for any $\varphi \in \mathcal{B}_b(E)$, $P_t\varphi \in C_b(E)$, the space of all bounded continuous functions on $E$; $P_t$ is irreducible in $E$ if $P_11_O(x) > 0$ for any $x \in E$ and any non-empty open subset $O$ of $E$. $\{P_t\}_{t \geq 0}$ is topological transitivity, if the resolvent $\{R_\lambda\}_{\lambda > 0}$ satisfies

$$R_\lambda(x, \mathcal{U}) := \int_0^\infty e^{-\lambda t} P_t(x, \mathcal{U})dt > 0, \quad \forall \lambda > 0$$

for all $x \in E$ and all neighborhoods $\mathcal{U}$ of $x$.

Let $L_t$ be the occupation measure of the system $\{X_t\}_{t \geq 0}$ given by

$$L_t(A) := \frac{1}{t} \int_0^t \delta_{X_s}(A)ds \quad \text{for any measurable set } A,$$

(2.1)
where \( \delta_a \) is the Dirac measure at \( a \in E \). Then \( \mathcal{L}_t \) is in \( \mathcal{M}_1(E) \), the space of probability measures on \( E \). On \( \mathcal{M}_1(E) \), let \( \sigma(\mathcal{M}_1(E), \mathcal{B}_b(E)) \) be the \( \tau \)-topology of convergence against measurable and bounded functions which is much stronger than the usual weak convergence topology \( \sigma(\mathcal{M}_1(E), C_b(E)) \) (see [6, 8–11]).

The level-3 entropy functional of Donsker-Varadhan \( H : \mathcal{M}_1(\Omega) \to [0, +\infty] \) is defined by

\[
H(Q) := \begin{cases} 
E^{\check{Q}} h_{\mathcal{F}_s^t}(\check{Q}_{\omega(-\infty, 0)}; \mathcal{P}_{\omega(0)}), & \text{if } Q \in \mathcal{M}_1^s(\Omega), \\
+\infty, & \text{otherwise},
\end{cases}
\]

where

- \( \mathcal{M}_1^s(\Omega) \) is the subspace of \( \mathcal{M}_1(\Omega) \), whose elements are moreover stationary;
- \( \check{Q} \) is the unique stationary extension of \( Q \in \mathcal{M}_1^s(\Omega) \) to

\[
\check{\Omega} := D(\mathbb{R}; E); \quad \mathcal{F}_s^t = \sigma\{X(u); s \leq u \leq t\}, \quad \forall s, t \in \mathbb{R}, \quad s \leq t;
\]

- \( \check{Q}_{\omega(-\infty, t]} \) is the regular conditional distribution of \( \check{Q} \) knowing \( \mathcal{F}_s^{-\infty} \);
- \( h_{\mathcal{G}}(\nu; \mu) \) is the usual relative entropy or Kullback information of \( \nu \) with respect to (w.r.t. for short) \( \mu \) restricted to the \( \sigma \)-field \( \mathcal{G} \), given by

\[
h_{\mathcal{G}}(\nu; \mu) := \begin{cases} 
\int \frac{d\nu}{d\mu} \log \left( \frac{d\nu}{d\mu} \right) d\mu, & \text{if } \nu \ll \mu \text{ on } \mathcal{G}, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

The level-2 entropy functional \( J : \mathcal{M}_1(E) \to [0, \infty] \) is

\[
J(\mu) := \inf \{ H(Q); Q \in \mathcal{M}_1^s(\Omega) \text{ and } Q_0 = \mu \}, \quad \forall \mu \in \mathcal{M}_1(E), \tag{2.2}
\]

where \( Q_0(\cdot) = Q(X_0 \in \cdot) \) is the marginal law at \( t = 0 \).

For any measurable set \( K \subset E \), let

\[
\tau_K := \inf \{ t \geq 0; X_t \in K \} \quad \text{and} \quad \tau_K^{(1)} := \inf \{ t \geq 1; X_t \in K \}. \tag{2.3}
\]

Recall the following hyper-exponential recurrence criterion established by Wu [41].

**Theorem 2.1** (See [41, Theorem 2.1]). Let \( \mathcal{A} \subset \mathcal{M}_1(E) \) and assume that

\[
\{P_t\}_{t \geq 0} \text{ is strong Feller and topologically transitive on } E.
\]

If for any \( \lambda > 0 \), there exists some compact set \( K \subset E \), such that

\[
\sup_{\nu \in \mathcal{A}} E^{\nu} e^{\lambda \tau_K} < \infty \quad \text{and} \quad \sup_{x \in K} E^x e^{\lambda \tau_K^{(1)}} < \infty, \tag{2.4}
\]

then the family \( P^\nu(\mathcal{L}_t \in \cdot) \) satisfies the LDP on \( \mathcal{M}_1(E) \) w.r.t. the \( \tau \)-topology with the rate function \( J \) defined by (2.2), uniformly for initial measures \( \nu \) in the subset \( \mathcal{A} \). More precisely, the following three properties hold:

1. (a1) for any \( a \geq 0 \), \( \{\mu \in \mathcal{M}_1(E); J(\mu) \leq a\} \) is compact in \( (\mathcal{M}_1(E), \tau) \);
2. (a2) (Lower bound) for any open set \( G \) in \( (\mathcal{M}_1(E), \tau) \),

\[
\liminf_{T \to \infty} \frac{1}{T} \log \inf_{\nu \in \mathcal{A}} P^\nu(\mathcal{L}_T \in G) \geq -\inf_G J;
\]

3. (a3) (Upper bound) for any closed set \( F \) in \( (\mathcal{M}_1(E), \tau) \),

\[
\limsup_{T \to \infty} \frac{1}{T} \log \sup_{\nu \in \mathcal{A}} P^\nu(\mathcal{L}_T \in F) \leq -\inf_F J.
\]
3 The LDP of occupation measures for nonlinear monotone SPDEs

3.1 Framework and examples

Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a separable Hilbert space and \((V, \| \cdot \|_V)\) a Banach space such that \(V \subset H\) continuously and densely. Let \(V^*\) be the dual space of \(V\). It is well known that

\[ V \subset H \subset V^* \]

continuously and densely. If \(V^*, \langle \cdot, \cdot \rangle_V\) denotes the dualization between \(V^*\) and \(V\), it follows that

\[ \langle v^*, z \rangle_V = \langle z, v \rangle_H \quad \text{for all } z \in H, \quad v \in V. \]

\((V, H, V^*)\) is called a Gelfand triple. In this paper, we always assume that \(V\) is compactly embedded in \(H\).

Thus, there exists a constant \(\eta > 0\) such that

\[ \| x \|_V \geq \eta \| x \|_H \quad \text{for all } x \in V. \quad (3.1) \]

Consider the following stochastic differential equation on \(H\):

\[ dX_t = A(X_t)dt + B(X_t)dW_t, \quad (3.2) \]

where \(\{W_t\}_{t \geq 0}\) is a cylindrical \(Q\)-Wiener process with \(Q := I\) on another separable Hilbert space \((U, \langle \cdot, \cdot \rangle_U)\) and being defined on a complete probability space \((\Omega, \mathcal{F}, P)\) with the normal filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), and \(B\) belongs to the Hilbert-Schmidt space \(L_2(U, H)\). Assume the following:

(H1) (Hemicontinuity) For all \(v_1, v_2, v \in V\),

\[ \forall s \mapsto \langle v^*, A(v_1 + sv_2), v \rangle_V \]

is continuous.

(H2) (Weak monotonicity) There exists \(c_0 \in \mathbb{R}\) such that for all \(v_1, v_2 \in V\),

\[ 2\langle v^*, A(v_1) - A(v_2), v_1 - v_2 \rangle_V + \| B(v_1) - B(v_2) \|_{L_2(U, H)}^2 \leq c_0 \| v_1 - v_2 \|_H^2. \]

(H3) (Coercivity) There exist \(r > 0, c_1, c_3 \in \mathbb{R}\) and \(c_2 > 0\) such that for all \(v \in V\),

\[ 2\langle v^*, A(v), v \rangle_V + \| B(v) \|_{L_2(U, H)}^2 \leq c_1 - c_2 \| v \|_V + c_3 \| v \|_H^2. \]

(H4) (Boundedness) There exist \(c_4 > 0\) and \(c_5 > 0\) such that for all \(v \in V\),

\[ \| A(v) \|_{V^*} \leq c_4 + c_5 \| v \|_V, \]

where \(r\) is as in (H3).

**Definition 3.1.** A continuous \(H\)-valued \(\mathcal{F}_t\)-adapted process \(\{X_t\}_{t \geq 0}\) is called a solution of (3.2), if

\[ \mathbb{E}\left[ \int_0^t (\| X_s \|_V^{r+1} + \| X_s \|_H^2)ds \right] < \infty, \quad (3.3) \]

and \(P\)-a.s.

\[ X_t = X(0) + \int_0^t A(X_s)ds + \int_0^t B(X_s)dW_s, \quad \forall t \geq 0. \]

According to [21, 30], under Conditions (H1)--(H4), for any \(X_0 \in L^2(\Omega \rightarrow H; \mathcal{F}_0; P)\), (3.2) admits a unique solution \(\{X_t\}_{t \geq 0}\). Moreover, we have the following Itô’s formula:

\[ \| X_t \|_H^2 = \| X_0 \|_H^2 + \int_0^t (2\langle v^*, A(X_s), X_s \rangle_V + \| B(X_s) \|_{L_2(U, H)}^2)ds \]

\[ + 2\int_0^t \langle X_s, B(X_s)dW_s \rangle_H. \]

The formulation of (3.2) is standard, and it includes a lot of interesting examples of SPDEs (see [21, 30]). Here are three of them.
**Example 3.2** (Stochastic $p$-Laplace equation [23, 30]). Let $\Lambda$ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary. Consider the following Gelfand triple:

$$H_0^{1,p}(\Lambda) \cap L^q(\Lambda) \subset L^2(\Lambda) \subset (H_0^{1,p}(\Lambda) \cap L^q(\Lambda))^*,$$
and the stochastic $p$-Laplace equation

$$dX_t = [\text{div}(|\nabla X_t|^{p-2}\nabla X_t) - \gamma |X_t|^{q-2}X_t]dt + BdW_t, \quad X_0 = x,$$  \hspace{1cm} (3.4)

where $\max\{1, 2d/(d+2)\} < p \leq q$ and $\gamma > 0$, $B$ is a Hilbert-Schmidt operator on $L^2(\Lambda)$ and $W_t$ is a cylindrical Wiener process on $L^2(\Lambda)$ w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

On the Sobolev space $H_0^{1,p}(\Lambda)$, consider the norm

$$||u||_{1,p} := \left( \int_{\Lambda} |\nabla u|^{p} d\xi \right)^{1/p}.$$  

Denote the norm in $L^q(\Lambda)$ by $||\cdot||_q$. By the Rellich-Kondrachov theorem, the embedding $H_0^{1,p}(\Lambda) \subset L^2(\Lambda)$ is compact.

According to [30, Example 4.1.9], Conditions (H1)–(H4) hold for (3.4) with $r > 1$.

Assume that $B$ is non-degenerate, i.e., $Bx = 0$ implies that $x = 0$. For $x \in H_0^{1,p}(\Lambda)$, let

$$||x||_B := \begin{cases} \|y\|_2, & \text{if } y \in L^2(\Lambda), \quad By = x, \\ +\infty, & \text{otherwise}. \end{cases}$$

If there exist constants $\sigma \geq \frac{4}{p}$ and $\delta > 0$ such that

$$||x||_{1,p}^2 \cdot ||x||_2^{\sigma - 2} \geq \delta ||x||_B^2, \quad \forall x \in H_0^{1,p}(\Lambda),$$  \hspace{1cm} (3.5)

then the associated Markov semigroup $P_t$ is strong Feller and irreducible on $L^2(\Lambda)$ by [23, Theorems 1.2 and 1.3].

**Remark 3.3.** According to [22, Example 3.3], if $p > 2$ and $q \in [1, p]$ in (3.4), $P_t$ is strong Feller, and furthermore if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (1/4, 1/2]$, then $P_t$ is irreducible.

**Example 3.4** (Stochastic generalized porous media equation [4.32]). Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and $\mu$ be the normalized Lebesgue measure on $D$. Let $\Delta$ be the Dirichlet Laplace operator on $D$ and $L := (-\Delta)^{\gamma}$ for $\gamma > 0$. For $r > 1$, consider the Gelfand triple

$$L^{r+1}(D, \mu) \subset H^\gamma(D, \mu) \subset (L^{r+1}(D, \mu))^*,$$

where $H^\gamma(\mu)$ is the completion of $L^2(\mu)$ under the norm

$$||x||_{\gamma} := \left( \int_D |(-\Delta)^{-\gamma/2}x|^2 d\mu \right)^{1/2}, \quad x \in L^2(D, \mu).$$

Notice that $L^{r+1}(D, \mu)$ is compactly embedded in $H^\gamma(D, \mu)$.

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$$

be the eigenvalues of $-\Delta$ including multiplicities with unit eigenfunctions $\{e_j\}_{j \geq 1}$. For $r > 1, c \geq 0$ and $q > 1/2$,

$$\Psi(s) := s|s|^{r-1}, \quad \Phi(s) := cs, \quad B(x)e_j := b_j(x)j^{-q}e_j, \quad j \geq 1,$$  \hspace{1cm} (3.6)

where $\{b_j\}_{j \geq 1}$ satisfies that

$$\min \limits_{u \in H^\gamma(D, \mu)} \inf \limits_{j \geq 1} b_j(u) > 0.$$  \hspace{1cm} (3.7)
Consider the equation
\[ dX_t = (L\Psi(X_t) + \Phi(X_t))dt + B(X_t)dW_t, \]
where \( W \) is a cylindrical Brownian motion on \( H^\gamma(D, \mu) \) w.r.t. a complete filtered probability space \((\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)\). According to [30, Example 4.1.11], Conditions (H1)–(H4) hold for (3.8) and the constant \( r > 1 \) in (H3). If (3.7) holds and \( \gamma \geq dp \), the Markov semigroup \( P_t \) is strong Feller by [37, Example 3.3] or [44, Example 3.3], and \( P_t \) is irreducible by [44, Example 3.3].

**Example 3.5** (Stochastic fast-diffusion equation [27, 32]). Let \( D = (0, 1) \subset \mathbb{R} \) and let \( \mu, H^\gamma, L, \Psi, \Phi \) and \( B \) be as in Example 3.4 with \( 1/3 < r < 1, \gamma = 1 \) and \( c < 0 \). Let \( V := L^{r+1}(D, \mu) \cap H^1(D, \mu) \) with \( \|v\|_V = |v|_{L^{r+1}} + |v|, \quad v \in V \).

We consider the equation (3.8) under the triple
\[ V \subset H^1(D, \mu) \subset V^*. \]

According to [32, Theorem 3.9], (H1)–(H4) hold with \( 1/3 < r < 1 \) and \( c_3 < 0 \). Furthermore, for all \( \theta \in (\frac{4}{r+1}, \frac{6r+2}{r+1}) \) and \( q \in (\frac{1}{2}, \frac{3r+1}{2(r+1)}) \), the Markov semigroup \( P_t \) is strong Feller and irreducible by [44, Example 3.4].

### 3.2 LDP for the nonlinear monotone SPDEs

**Theorem 3.6.** Assume that (H1)–(H4) hold with \( r \in (0, 1] \) and \( c_3 < 0 \) or with \( r > 1 \), \( P_t \) is strong Feller and irreducible in \( H \), and
\[ C_B := \sup_{u \in H} \|B(u)\|_{L^2(U,H)}^2 < \infty. \]

For any \( \lambda_0 \in (0, -\frac{c_3}{2C_B}) \) (or \( \lambda_0 \in (0, \frac{c_3}{2C_B}) \), respectively) and \( L \geq 1 \), let
\[ \mathcal{M}_{\lambda_0, L} := \left\{ \nu \in \mathcal{M}_1(H) : \int_H e^{\lambda_0 \|x\|^2_H} \nu(dx) \leq L \right\}. \]

Then the family \( P^\nu(\mathcal{L}_t \in \cdot) \) as \( T \to +\infty \) satisfies the large deviation principle on \( (\mathcal{M}_1(H), \tau) \), with the speed \( T \) and the rate function \( J \) defined by (2.2), uniformly for any initial measure \( \nu \) in \( \mathcal{M}_{\lambda_0, L} \).

**Proof.** Since \( P_t \) is strong Feller and irreducible in \( H \), according to Theorem 2.1, to prove Theorem 3.6, we need to prove that the hyper-exponential recurrence condition (2.4) is fulfilled. The verification of this condition in the case of \( r \in (0, 1] \) and \( c_3 < 0 \) will be given in Lemma 3.8. The proof for the case of \( r > 1 \) is parallel and thus omitted. \( \square \)

First, we establish the following crucial exponential estimate for the solution.

**Lemma 3.7.** Assume that (H1)–(H4) hold with \( r \in (0, 1] \) and \( c_3 < 0 \) and
\[ C_B := \sup_{u \in H} \|B(u)\|_{L^2(U,H)}^2 < \infty. \]

For any
\[ \lambda_0 \in \left( 0, -\frac{c_3}{2C_B} \right), \]

it holds that
\[ \mathbb{E}^\nu \left[ \exp \left( \frac{\lambda_0 c_2}{2} \int_0^t \|X_s\|^{r+1}_V \, ds \right) \right] \leq e^{\lambda_0 c_1 t} \cdot e^{\lambda_0 \|x\|^2_H}. \] (3.9)

**Proof.** Let
\[ Y_t := \|X_t\|^2_H + \frac{c_2}{2} \int_0^t \|X_s\|^{r+1}_V \, ds. \]
By Itô’s formula and (H3), we have
\[
dY_t = (2V \cdot (A(X_t), X_t) + \|B(X_t)\|_{L^2(U,H)}^2) dt + 2\langle X_t, B(X_t) dW_t \rangle_H + \frac{c_2}{2} \|X_t\|_{V^1} dt
\]
\[
\leq \left( c_1 - \frac{c_2}{2} \|X_t\|_{V^1}^2 + c_3 \|X_t\|_H^2 \right) dt + 2\langle X_t, B(X_t) dW_t \rangle_H.
\]
Denoting by \(d[Y,Y]_t\) the quadratic variation process of a semimartingale \(Y\), we can also compute with Itô’s formula,
\[
d\lambda^0 Y_t = e^{\lambda^0 Y_t} \left\{ \lambda_0 dY_t + \frac{\lambda^2}{2} d[Y,Y]_t \right\}
\]
\[
\leq \lambda_0 e^{\lambda^0 Y_t} \left( c_1 - \frac{c_2}{2} \|X_t\|_{V^1}^2 + c_3 \|X_t\|_H^2 + 2\lambda_0 \|B(X_t)\|_{L^2(U,H)}^2 \cdot \|X_t\|_H^2 \right) dt
\]
\[
+ 2\lambda_0 e^{\lambda^0 Y_t} \langle X_t, B(X_t) dW_t \rangle_H.
\]
Let \(Z_t := e^{-\lambda_0 t} e^{\lambda^0 Y_t}\). By Itô’s formula again, we obtain that when \(\lambda_0 < -c_3/(2CB)\),
\[
dZ_t = e^{-\lambda_0 t} d\lambda^0 Y_t + e^{\lambda^0 Y_t} d\lambda^0 Y_t - \lambda_0 \lambda^0 e^{\lambda^0 Y_t} \langle X_t, B(X_t) dW_t \rangle_H
\]
\[
\leq \lambda_0 Z_t \left( -\frac{c_2}{2} \|X_t\|_{V^1}^2 + c_3 \|X_t\|_H^2 + 2\lambda_0 CB \|X_t\|_H^2 \right) dt + 2\lambda_0 Z_t \langle X_t, B(X_t) dW_t \rangle_H
\]
\[
\leq 2\lambda_0 Z_t \langle X_t, B(X_t) dW_t \rangle_H.
\]
Since \(Z_t \geq 0\), we obtain by Fatou’s lemma \(E^x[Z_t] \leq E^x[Z_0]\), which is stronger than (3.9).

For any measurable set \(K \subset H\), recall the stopping times \(\tau_K\) and \(\tau_K^{(1)}\) defined by (2.3). Now, we will verify the hyper-exponential recurrence condition (2.4) in the following lemma.

**Lemma 3.8.** Assume that (H1)–(H4) hold with \(r \in (0,1]\) and \(c_3 < 0\) and
\[
CB := \sup_{u \in H} \|B(u)\|_{L^2(U,H)}^2 < \infty.
\]

For any \(\lambda > 0\), there exists some compact set \(K \subset H\), such that
\[
\sup_{x \in K} E^x[e^{\lambda \tau_K}] < \infty \quad \text{and} \quad \sup_{\nu \in M_{\lambda_0,\nu}} E^\nu[e^{\lambda \tau_K}] < \infty. \tag{3.10}
\]

**Proof.** The proof is inspired by Gourcy [14,15]. Take
\[
K := \{ x \in V : \|x\|_V \leq M \}, \tag{3.11}
\]
where the constant \(M\) will be fixed later. Since the embedding \(V \subset H\) is compact, \(K\) is a compact subset in \(H\).

The definition of the occupation measure implies that
\[
P^\nu(\tau_K^{(1)} > n) \leq P^\nu \left( \mathcal{L}_n(K) \leq \frac{1}{n} \right) = P^\nu \left( \mathcal{L}_n(K^c) \geq 1 - \frac{1}{n} \right).
\]

With our choice for \(K\), we have \(\|x\|_V \geq M1_{\mathbb{R}^N}(x)\). Hence, for any fixed \(\lambda_0 \in (0, -\frac{c_3}{2CB})\), we obtain by Chebyshev’s inequality that
\[
P^\nu(\tau_K^{(1)} > n) \leq P^\nu \left( \mathcal{L}_n(\|x\|_{V^1}^2 + M1_{\mathbb{R}^N}(x)) \geq M^{1+r} \left( 1 - \frac{1}{n} \right) \right)
\]
\[
\leq \exp \left( - \frac{n\lambda_0 c_3 M^{1+r}}{2} \left( 1 - \frac{1}{n} \right) \right) \left( \exp \left( \frac{\lambda_0 c_2}{2} \int_0^n \|X_s\|_{V^1}^2 \right) \right).
\]

Integrating (3.9) w.r.t. \(\nu(dx)\), by Lemma 3.7, we have
\[
P^\nu(\tau_K^{(1)} > n) \leq \nu(e^{\lambda_0 \|x\|_{V^1}^2} \exp(-n\lambda_0 C), \quad \forall n \geq 2,
\]

\[
\]
where

$$C(M) := \frac{c_2}{4} M^{1+r} - c_1.$$  

For any fixed $\lambda > 0$, by the formula of the integration by parts, we have

$$E_{\nu}[e^{\lambda K^{(1)}(t)}] = 1 + \int_0^{+\infty} \lambda e^{\lambda t} P_{\nu}(\tau^{(1)}_K > t) dt$$

$$\leq e^{2\lambda} + \sum_{n \geq 2} \lambda e^{\lambda(n+1)} P_{\nu}(\tau^{(1)}_K > n)$$

$$\leq e^{2\lambda} \left( 1 + \lambda \nu(e^{\lambda \|\cdot\|_H^2}) \sum_{n \geq 2} e^{-n(\lambda_0 C(M) - \lambda)} \right).$$

Now, we can choose $M$ such that $\lambda_0 C(M) - \lambda > 0$ in the definition (3.11) of $K$. Then, taking the supremum over $\{\delta_x; x \in K\}$, we get

$$\sup_{x \in K} E_{\delta_x}[e^{\lambda \tau^{(1)}_K}] \leq e^{2\lambda} \left( 1 + \lambda \nu(e^{\lambda \|\cdot\|_H^2}) \sum_{n \geq 2} e^{-n(\lambda_0 C(M) - \lambda)} \right) < \infty,$$

where (3.1) is used. Thus, the first inequality in (3.10) holds true. We obtain the second inequality in (3.10) in the same way: Since $\tau_K \leq \tau^{(1)}_K$, we have

$$\sup_{\nu \in M_{\lambda_0,L}} E_{\nu}[e^{\lambda \tau_K}] \leq \sup_{\nu \in M_{\lambda_0,L}} E_{\nu}[e^{\lambda \tau^{(1)}_K}]$$

$$\leq e^{2\lambda} \left( 1 + \lambda L \sum_{n \geq 2} e^{-n(\lambda_0 C(M) - \lambda)} \right) < \infty.$$

The proof is completed.

**Remark 3.9.** For every $f : H \to \mathbb{R}$ measurable and bounded, as

$$\nu \mapsto \int_H \, f \, d\nu$$

is continuous w.r.t. the $\tau$-topology, then by Theorem 3.6 and the contraction principle (see [6]), the empirical mean

$$\frac{1}{T} \int_0^T f(X_s) ds$$

satisfies the LDP on $\mathbb{R}$, with the rate function given by

$$J^f(r) = \inf \left\{ J(\nu) < +\infty; \nu \in M_1(H) \text{ and } \int f d\nu = r \right\}, \quad \forall r \in \mathbb{R}.$$

For the unbounded function $f$, $\nu \mapsto \int_H f d\nu$ is not continuous w.r.t. the $\tau$-topology, and then the contraction principle does not work. By using the technique of functional inequalities, Wu and Yao [42] studied the LDP of the empirical mean for the unbounded functions.

### 3.3 Some comments on the entropy $J$ defined by (2.2)

In many situations, one can evaluate the rate function $J$ defined by (2.2). In [8–11], Donsker and Varadhan defined the rate function in terms of the Cramer functional and the infinitesimal generator acting on bounded measurable functions, and they identified the rate function in terms of the associate Dirichlet form when the Markov process $X_t$ is symmetric w.r.t. $\mu$ (see [6,19,40] for further developments). Unfortunately, for the nonlinear monotone stochastic partial differential equation $X_t$ defined by (3.2), the explicit form of its invariant measure $\mu$ is actually unknown and the Markov process $X_t$ is merely symmetric w.r.t. $\mu$. 
Under the Feller assumption below,
\[ P_t(C_b(H)) \subset C_b(H), \quad \forall t \geq 0, \]
we know that (see, for example, [40, Lemma B.7])
\[ J(\nu) = \sup \left\{ -\int \frac{L u}{u} dv; 1 \leq u \in \mathbb{D}_e(\mathcal{L}) \right\}, \quad \nu \in \mathcal{M}_1(H), \quad (3.12) \]
where \( \mathbb{D}_e(\mathcal{L}) \) is the extended domain of the generator \( \mathcal{L} \) of \( P_t \) in \( C_b(H) \), i.e., \( u \in \mathbb{D}_e(\mathcal{L}) \) if \( u \in C_b(H) \) and there exists \( v \in C_b(H) \) such that
\[ P_t u - u = \int_0^t P_s v ds, \quad \forall t \geq 0. \]
In that case, \( v := \mathcal{L} u \).

Basing on the form (3.12) and using the Girsanov’s formula, Gourcy [15] gave a more explicit form of the rate function \( J \) for a family of measures associated with the perturbation equations. His method and result can be extended to the framework of the monotone SPDE (3.2). Taking the stochastic fast-diffusion equation for example, we give an explicit form of \( J \).

For the stochastic fast-diffusion equation (3.5), the generator \( \mathcal{L} \) is given by
\[ \mathcal{L} f(x) := \frac{1}{2} \text{Tr}(BB^* D^2 f)(x) + \langle -(-\Delta)^\gamma \Psi(x) + \Phi(x), \nabla^H f(x) \rangle, \quad (3.13) \]
at least for \( f \) cylindrical, i.e.,
\[ f(x) = g(\langle x, e_1 \rangle, \ldots, \langle x, e_n \rangle) \]
for some smooth function \( g \in C^\infty(\mathbb{R}^n; \mathbb{R}) \), with bounded gradients. Here, \( \nabla^H f(x) \) denotes the gradient in \( H \), and
\[ D^2(f) := (\partial_i \partial_j f)_{i,j \geq 1}. \]
Since \( f \) is cylindrical, the gradient \( \nabla^H f(x) \) is in \( H^\gamma := \mathcal{D}((-\Delta)^\gamma) \) for any \( \gamma \geq 0 \), and the right-hand side of (3.13) is well defined.

Let \( (X_t^x) \) be the solution of the fast-diffusion equation (3.5) with initial position \( x \), defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) and consider the Girsanov perturbation: For any \( T > 0, x \in H \),
\[ \frac{dQ^h_t}{dP} \bigg|_{\mathcal{F}_T} = \exp \left( \int_0^T \sqrt{BB^* \nabla^H h(X_s^x)} dW_s - \frac{1}{2} \int_0^T |\sqrt{BB^* \nabla^H h(X_s^x)}|^2 ds \right). \quad (3.14) \]
Here, \( h \) is in \( C^1(H) \) and satisfies that
\[ \langle BB^* \nabla^H h, \nabla^H h \rangle \leq C < \infty \]
so that
\[ \mathbb{E} \exp \left( \frac{1}{2} \int_0^T |\sqrt{BB^* \nabla^H h(X_s^x)}|^2 ds \right) < \infty, \quad \forall T \geq 0, \quad x \in H. \]
Under the above condition,
\[ L^x_t := \int_0^t \sqrt{BB^* \nabla^H h(X_s^x)} dW_s \]
is a continuous martingale under \( P \), and
\[ M^x_t := \exp \left( L^x_t - \frac{1}{2} \langle L \rangle_t \right) \]
is a martingale by Novikov’s criterion. Hence \( (Q^h_t)_{x \in H} \) defines a new Markov family with the transition semigroup
\[ Q^h_t f(x) = \mathbb{E}^{Q^h_t} f(X^x_t). \]
By Girsanov’s formula, the generator of $Q^h_t$ takes the form
\[ L_h u = L u + 2\Gamma(h, u), \]
where
\[ \Gamma(h, u) = \frac{1}{2} \langle BB^*\nabla^H h, \nabla^H h \rangle \]
is the carré du champ of $L$, and under $Q^h_t$ the process $(X^*_t)$ satisfies the following perturbation of the stochastic fast-diffusion equation:
\[
\begin{align*}
    dX^*_t &= \left(\frac{1}{2}(-\Delta)^\gamma \Psi(X^*_t) + \Phi(X^*_t)\right)dt \\
    &\quad + \sqrt{BB^*\nabla^H h}(X^*_t)dt + B(X^*_t)d\tilde{W}_t,
\end{align*}
\]
where $\tilde{W}$ is a cylindrical Wiener process under $Q^h_t$. The equation (3.15) also satisfies (H1)–(H4), and has a unique invariant measure $\mu^h \in \mathcal{M}_1(H)$.

Using the same method in the proof of Proposition 4.5 in [15], we can obtain the following explicit expression for $J(\mu^h)$, whose proof is omitted.

**Proposition 3.10.** For $h \in C^1(H)$ such that $\langle BB^*\nabla^H h, \nabla^H h \rangle \leq C < \infty$, we have
\[
    J(\mu^h) = \frac{1}{2} \int_H \langle BB^*\nabla^H h, \nabla^H h \rangle d\mu^h = \int_H \Gamma(h, h) d\mu^h.
\]

4 Framework for verifying hyper-exponential recurrence and its applications

To the best of our knowledge, [39] is the first paper reporting that the Donsker-Varadhan LDP holds for a heavy tailed stochastic system, and gives an example that strong dissipation overcomes the heavy tail effect to produce a long time LDP. In this section, we generalize the idea of verifying hyper-exponential recurrence in [39] to a framework, and apply it to study the other two examples. One is a strong dissipative SPDE driven by multiplicative Wiener noise, and the other is stochastic real Ginzburg-Landau equation driven by cylindrical $\alpha$-stable processes. Note that the SPDEs in [39] are the same type, but driven by a subordinate Brownian motion.

As will be seen in the two examples below, the advantage of our framework is twofold. For strong dissipative SPDEs driven by Wiener noise, we obtain an LDP, which is much stronger than that in Theorem 3.6 and not accessible by the method in the previous section. For the SPDEs driven by $\alpha$-stable noise, the approach to Theorem 3.6 does not work, but we can still use our framework.

4.1 Framework for verifying hyper-exponential recurrence

Let $(E, | \cdot |_E)$ and $(F, | \cdot |_F)$ be two metric spaces such that $F$ is compactly embedded in $E$, i.e., for any $M > 0$, the set $\{x \in E: |x|_F \leq M\}$ is compact in $E$. Let $(X_t)_{t \geq 0}$ be a time homogeneous Markov process valued on $E$, let $K$ be a compact set in $E$, and set
\[ \tau_K := \inf\{t \geq 0; X_t \in K\} \quad \text{and} \quad \tau^{(1)}_K := \inf\{t \geq 1; X_t \in K\}. \]
Recall that the hyper-exponential recurrence condition means that for any $\lambda > 0$, there exists some compact set $K$ depending on $\lambda$ so that
\[ \sup_{x \in A} E^x[e^{\lambda \tau_K}] < \infty \quad \text{and} \quad \sup_{x \in K} E^x[e^{\lambda \tau^{(1)}_K}] < \infty, \]
where $A \subset \mathcal{M}_1(E)$. 

4.1.1 Framework for verifying hyper-exponential recurrence

In order to prove (4.1), we propose a framework as follows:

(1) A sufficient condition for (4.1): Assume that the time homogeneous Markov process \((X_t)_{t \geq 0}\) admits the following uniform moment estimate.

**Condition 4.1.** There exist some constants \(T > 0\) and \(p > 0\) such that

\[
\sup_{x \in \mathbb{E}} \mathbb{E}^x[|X_T|^p] < C_{T,p}.
\]  

(2) Sampling a Markov chain from \((X_t)_{t \geq 0}\): Sample a Markov chain \((X_{kT})_{k \in \mathbb{Z}_+}\) with step size \(T\). Define the stopping times of Markov chain \((X_{kT})_{k \in \mathbb{Z}_+}\):

\[\sigma_K := \inf\{k \in \mathbb{Z}_+; X_{kT} \in K\}.\]

(3) Estimating \(\sigma_K\): Using Condition 4.1 and the Markov property recursively, we prove the following estimate.

**Lemma 4.2.** Under Condition 4.1, for any \(\lambda > 0\), there exists a compact set \(K \in \mathbb{E}\) depending on \(\lambda\) and \(T\) such that

\[
\sup_{\nu \in \mathcal{M}_1(\mathbb{E})} \mathbb{E}^\nu e^{\lambda \sigma_K} < \infty.
\]

(4) Hyper-exponential recurrence: By the easy comparison

\[
\sigma_K \geq \frac{\tau_K}{T}, \quad \sigma_K \geq \frac{\tau_K^{(1)}}{T},
\]

we immediately obtain the following theorem.

**Theorem 4.3.** Let \((X_t)_{t \geq 0}\) be a time homogeneous Markov process valued on \(\mathbb{E}\). Under Condition 4.1, \((X_t)_{t \geq 0}\) satisfies the hyper-exponential recurrence (4.1) with \(A = \mathcal{M}_1(\mathbb{E})\) therein.

Let us have a short discussion of this framework. It is often difficult to study the long time behavior of a continuous time stochastic process, due to the regularity of the process trajectories. A very useful technique is to sample an embedded Markov chain which captures the problems that we are interested in, for example, Hairer proved his famous recurrence theorem first for a discrete time Markov chain, and then extended it to continuous stochastic processes \([17, \text{Section 3}]\). Also see the recent work \([31]\) for a similar procedure.

From this framework, we know that, in order to verify the hyper-exponential recurrence condition, it suffices to verify Condition 4.1.

Now let us prove Lemma 4.2.

**Proof of Lemma 4.2.** Without loss of generality, we assume that \(T = 1\). For any \(n \in \mathbb{N}\), let

\[
B_n := \{[X_j|_F > M; j = 1, \ldots, n\} = \{\tau_K > n\}.
\]

By the Markov property of \(\{X_n\}_{n \in \mathbb{N}}\) and Chebyshev’s inequality, we obtain that for any \(\nu \in \mathcal{M}_1(\mathbb{E})\),

\[
P^\nu(B_n) = \int_{[x_k|_F > M, k = 1, \ldots, n-1]} \mathbb{P}([X_n|_F > M | X_k = x_k, k = 1, \ldots, n-1]
\times \mathbb{P}^\nu(X_{n-1} \in dx_{n-1}, \ldots, X_1 \in dx_1)
= \int_{[x_k|_F > M, k = 1, \ldots, n-1]} \mathbb{P}([X_n|_F > M | X_{n-1} = x_{n-1}]\mathbb{P}^\nu(X_{n-1} \in dx_{n-1}, \ldots, X_1 \in dx_1)
\leq \int_{[x_k|_F > M, k = 1, \ldots, n-1]} \frac{1}{M^p} \mathbb{E}[[X_n|_F^p | X_{n-1} = x_{n-1}]\mathbb{P}^\nu(X_{n-1} \in dx_{n-1}, \ldots, X_1 \in dx_1)
\leq \int_{[x_k|_F > M, k = 1, \ldots, n-1]} \frac{C_p}{M^p} \mathbb{P}^\nu(X_{n-1} \in dx_{n-1}, \ldots, X_1 \in dx_1)
\[ = \frac{C_p}{M_p} p^\nu(B_{n-1}), \]

where Lemma 4.1 is used in the last inequality. Repeating this argument \(n\) times yields

\[ P^\nu(\tau_M > n) = P^\nu(B_n) \leq \left( \frac{C_p}{M_p} \right)^n. \]

This inequality, together with Fubini’s theorem, implies the desired estimate in Lemma 4.2 for any \(M > C_p^1 p^{\lambda/p}.\)

**4.1.2 Machinery of verifying Condition 4.1 for strong dissipative SPDEs**

Let \(T = \mathbb{R}/\mathbb{Z}\) be equipped with the usual Riemannian metric, and let \(d\xi\) denote the Lebesgue measure on \(T\). For any \(p > 1, \)

\[ L^p(T; \mathbb{R}) := \left\{ x: T \to \mathbb{R}; \|x\|_{L^p} := \left( \int_T |x(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty \right\}. \]

Denote

\[ H := \left\{ x \in L^2(T; \mathbb{R}); \int_T x(\xi) d\xi = 0 \right\}. \]

It is a separable real Hilbert space with the inner product

\[ \langle x, y \rangle_H := \int_T x(\xi)y(\xi) d\xi, \quad \forall x, y \in H. \]

For any \(x \in H\), let

\[ \|x\|_H := \|x\|_{L^2} = (\langle x, x \rangle_H)^{\frac{1}{2}}. \]

Let \(\Delta\) be the Laplace operator on \(H\). Denote \(Z_* := \mathbb{Z} \setminus \{0\}\). Let \(\{e_k\}_{k \in \mathbb{Z}_*}\) be the eigenfunctions of \(-\Delta\) associated with the eigenvalues \(\{\gamma_k\}_{k \in \mathbb{Z}_*}\). It is well known that for all \(k \in \mathbb{Z}_*\),

\[ \Delta e_k = -\gamma_k e_k \quad \text{with} \quad \gamma_k = 4\pi^2 |k|^2. \]

Denote \(A = -\Delta\) and \(A^\sigma\) with \(\sigma \in \mathbb{R}\) by

\[ A^\sigma x := \sum_{k \in \mathbb{Z}_*} \gamma_k^\sigma x_k e_k, \]

where \(x_k = \langle x, e_k \rangle\), and

\[ D(A^\sigma) := H_\sigma := \left\{ x = \sum_{k \in \mathbb{Z}_*} x_k e_k \in H, \|x\|_{H_\sigma} := \left( \sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma} |x_k|^2 \right)^{\frac{1}{2}} < \infty \right\}. \]

Then, \(H_\sigma\) is densely and compactly embedded in \(H\). Particularly, let \(V = D(A^{1/2})\).

Let us consider the following semi-linear SPDEs valued on \(H\):

\[ dX_t = [\Delta X_t + N(X_t)] dt + B(X_t) dL_t, \quad X_0 = x, \quad (4.3) \]

where \(L_t\) is Wiener noise or \(\alpha\)-stable type noise, the detailed conditions of \(B\) and \(L_t\) will be given in specific examples, and

\[ N(x) = -x^3 + x, \quad x \in V. \quad (4.4) \]

\(N\) can possibly take other odd order polynomials such as

\[ N(x) = -x^{2m+1} + \sum_{k=0}^{2m} a_k x^k \]
for some $m \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for $k = 0, \ldots, 2m$. Here, we only take the specific form (4.4) for simplicity to show the usage of our machinery below.

Denote

$$Z_t := \int_0^t e^{-(t-s)A}B(X_s)dL_s, \quad Y_t := X_t - Z_t, \quad t \geq 0. \quad (4.5)$$

In order to verify that (4.3) satisfies Condition 4.1, we propose the following machinery:

1. The estimate of the stochastic convolution $Z_t$: Find a $\delta > 0$ and an $r > 0$ such that for all $T > 0$,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \|Z_t\|_{H_4}^r\right] \leq C(T, \delta, r). \quad (4.6)$$

Using the estimate of (4.6), the Sobolev embedding theorem and the property of $N$, we have the following lemma.

**Lemma 4.4.** If (4.6) holds, we have

$$\sup_{x \in H} \mathbb{E}[\sup_{t \in [T/2, T]} \|Y_t\|_H^q] \leq C(T, r, q). \quad (4.7)$$

2. The estimate of $Y_t$ in $H_4$: Find a $q > 0$ satisfying that there exists some $q > 0$ such that

$$\sup_{x \in H} \mathbb{E}[\|Y_T\|_{H_4}^q] \leq C(T, \delta, q). \quad (4.8)$$

3. The combination of $Z_t$ and $Y_t$: Taking $p = r \wedge q$, we immediately obtain the desired inequality

$$\sup_{x \in H} \mathbb{E}[\|X_T\|_{H_4}^p] \leq C(T, \delta, p).$$

Thanks to the framework and machinery above, to prove the Donsker-Varadhan LDP for the SPDE (4.3), it suffices to verify (4.6) and (4.8). We hope our framework and machinery can be applied to study many other models.

**Proof of Lemma 4.4.** By the chain rule, we obtain that

$$\frac{d\|Y_t\|_H^2}{dt} + 2\|Y_t\|_V^2 = 2\langle Y_t, N(Y_t + Z_t) \rangle. \quad (4.9)$$

Using the Young inequality, Hölder’s inequality and the elementary inequality $2\sqrt{ab}$ for all $a, b > 0$, we obtain that there exists a constant $C \geq 1$ satisfying that

$$2\langle Y_t, N(Y_t + Z_t) \rangle \leq -\|Y_t\|_{L^4}^4 + C(1 + \|Z_t\|_{L^4}^4).$$

This inequality, together with (4.9), (A.2) and Hölder’s inequality, implies that

$$\frac{d\|Y_t\|_H^2}{dt} + 2\|Y_t\|_V^2 \leq -\|Y_t\|_H^2 + C(1 + \|Z_t\|_{L^4}^4). \quad (4.10)$$

For any $t \geq 0$, denote

$$h(t) := \|Y_t\|_H^2, \quad K_T := \sup_{0 \leq t \leq T} \sqrt{C(1 + \|Z_t\|_{L^4}^4)} \geq 1.$$

By (4.10), we have

$$\frac{dh(t)}{dt} \leq -h^2(t) + K_T^2, \quad \forall t \in [0, T],$$

with the initial value $h(0) = \|x\|_H^2 \geq 0$. By the comparison theorem, we obtain that

$$h(t) \leq g(t), \quad \forall t \in [0, T], \quad (4.11)$$
where the function $g$ solves the following equation:

$$\frac{dg(t)}{dt} = -g^2(t) + K_T^2, \quad \forall t \in [0, T], \text{ with } g(0) = h(0).$$

(4.12)

The solution of (4.12) is

$$g(t) = K_T + 2K_T \left( \frac{g(0) + K_T e^{2K_T t} - 1}{g(0) - K_T} \right)^{-1}, \quad \forall t \in [0, T],$$

where it is understood that $g(t) \equiv K_T$ when $g(0) = K_T$. It is easy to show that for any initial value $g(0)$, we have

$$g(t) \leq \frac{K_T(e^t + 1)}{e^t - 1}, \quad \forall t \in [T/2, T].$$

This inequality, together with (4.11) and the definition of $K_T$, implies that

$$\sup_{t \in [T/2, T]} \|Y_t\|_H \leq C(e^T + 1) e^{T/2} - 1 \left( 1 + \sup_{0 \leq t \leq T} \|Z_t\|_{L^4} \right).$$

(4.13)

Using the estimate of (4.6) and the Sobolev embedding theorem, we get the desired estimate (4.7). □

### 4.2 Semilinear stochastic equation driven by Wiener noise

Let $H, V$ and $(H_s)_{s \in \mathbb{R}}$ be the Hilbert spaces defined in Subsection 4.1.2. Let us assume that the SPDE (4.3) satisfies

$$dX_t = (-AX_t + N(X_t))dt + B(X_t)dW_t,$$

(4.14)

where $-A = \Delta$ is the Laplace operator on $H$, and $W = \sum_{k \in \mathbb{Z}} W_k e_k$ is a cylindrical Brownian motion on $H$ with $\{W_k\}_{k \in \mathbb{Z}}$ being i.i.d. one-dimensional standard Brownian motions.

The following hypotheses hold:

(C1) The nonlinear term $N$ is

$$N(x) = -x^3 + x, \quad \text{for } x \in V.$$

(C2) $B(x) = \sum_{k \in \mathbb{Z}} \beta_k \sigma_k(x)e_k$, where $\beta_k$ and $\sigma_k(x)$ satisfy that for some $1/4 < \alpha < 1/2$, $c, C, K > 0$,

$$c/\gamma_k^\alpha \leq \beta_k \leq C/\gamma_k^\alpha, \quad \forall k \in \mathbb{Z},$$

and

$$|\sigma_k(x)| \leq K, \quad |\sigma_k(x) - \sigma_k(y)| \leq K\|x - y\|_H, \quad \forall k \in \mathbb{Z}, \quad x, y \in H.$$

**Theorem 4.5** (See [25, Example 3.2] or [26, Example 5.1.8 and Appendix G]). Under (C1) and (C2), for any $X_0 \in L^6(\Omega, \mathbb{P}; H)$, (4.3) has a unique Markov solution $\{X_t\}_{t \in [0, T]}$ such that

$$E\left( \sup_{t \in [0, T]} \|X_t\|_H^6 + \int_0^T \|X_t\|_H^4 \cdot \|X_t\|_V^2 dt \right) < C(1 + E[\|X_0\|_H^6]),$$

(4.15)

and

$$X_t = e^{-At}X_0 + \int_0^t e^{-A(t-s)}N(X_s)ds + \int_0^t e^{-A(t-s)}B(X_s)dW_s.$$

(4.16)

Consequently, for any $X_0 \in L^6(\Omega, \mathbb{P}; H)$, the solution $\{X_t\}_{t \in [0, T]}$ belongs to $C([0, T]; H_\delta)$ for any $\delta \in (0, 1/2)$.

The following log-Harnack inequality, strong Feller property and irreducibility hold for (4.3); we defer their proofs to the appendix.
Theorem 4.6. Under (C1) and (C2), for any positive function \( f \in B_0(H) \),
\[
P_t \log f(x) \leq \log P_t f(y) + \frac{\tilde{K}\|A^{\alpha}(x-y)\|_H^2}{2c^2(1-e^{-Kt})}, \quad t > 0, \quad x, y \in H.
\]
(4.17)

Here, \( \tilde{K} = K + 2, \|A^{\alpha}(x-y)\|_H \in [0, \infty], \) which is finite if \( x - y \in H_\alpha \). Consequently, \( X_t \) is strong Feller and irreducible in \( H_\alpha \). Furthermore, \( X_t \) is strong Feller and irreducible in \( H \).

Our LDP result is the following.

Theorem 4.7. Assume that (C1) and (C2) hold. Then the family \( \mathbb{P}^\nu(\{\mathcal{L}_t \in \cdot\}) \) as \( T \to +\infty \) satisfies the large deviation principle with respect to the \( \tau \)-topology, with a good rate function \( J \) defined by (2.2), uniformly for any initial measure \( \nu \) in
\[
\left\{ \nu \in \mathcal{M}_1(H) : \int_H \|x\|_H^6 \nu(dx) < \infty \right\}.
\]

Proof. By Theorem 4.6, \( X \) is strong Feller and irreducible in \( H \). According to Theorem 2.1, in order to prove this theorem, we only need to prove that the hyper-exponential recurrence is fulfilled. By Theorem 4.3, it is sufficient to verify Condition 4.1, which can be done by the machinery proposed in Subsection 4.1.2.

(1) The estimate of stochastic convolution \( Z_t \): Recall that
\[
Z_t = \int_0^t e^{-(t-s)A}B(X_s)dW_s.
\]

By Burkholder’s inequality for stochastic convolutions (see [18]), we have for any \( p \geq 2 \) and \( \delta \in (0, \alpha-1/4) \),
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \|A^\delta Z_t\|_H^p \right] = \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left\| \int_0^T e^{-(t-s)A}A^\delta B(X_s)dW_s \right\|_H^2 \right]^{\frac{p}{2}} \\
\leq C \mathbb{E} \left[ \int_0^T \left\| A^\delta B(X_s) \right\|_H^2 ds \right]^{\frac{p}{2}} \\
\leq C(T, \delta, p).
\]
(4.18)

(2) The estimate of \( X_t \) in \( H_\delta \): We will prove below that for any \( \delta \in (0, 2\alpha - 1/2) \) and \( T > 0 \), there exists a constant \( C(\delta, T) \) such that
\[
\sup_{x \in H} \mathbb{E}^x \left[ \left\| X_T \right\|_H^2 \right] \leq C(\delta, T).
\]
(4.19)

By Lemma 4.4 and (4.18), we have for any \( p \geq 2 \),
\[
\sup_{x \in H} \mathbb{E}^x \left[ \sup_{t \in [T/2, T]} \left\| X_t \right\|_H^p \right] \leq C(T, P).
\]
(4.20)

For any \( x \in H \) and \( T > 0 \),
\[
X_T = e^{-T/2}X_{T/2} + \int_{T/2}^T e^{-(T-s)A}N(X_s)ds + \int_{T/2}^T e^{-(T-s)A}B(X_s)dW_s.
\]

By (4.20) and the Cauchy-Schwarz inequality, for any \( \delta \in (0, 2\alpha - 1/2) \), we have
\[
\mathbb{E}^x \left[ \left\| A^\delta X_T \right\|_H^2 \right] \leq C \mathbb{E}^x \left[ \left\| A^\delta e^{-T/2}X_{T/2} \right\|_H^2 \right] + C \int_{T/2}^T \left\| A^\delta e^{-(T-s)A}N(X_s)ds \right\|_H^2 \\
+ C \int_{T/2}^T \left\| A^\delta e^{-(T-s)A}B(X_s)dW_s \right\|_H^2 \\
\leq \frac{C}{T^{2\alpha}} \mathbb{E}^x \left[ \left\| X_{T/2} \right\|_H^2 \right] + C \int_{T/2}^T (T-s)^{-2\delta}ds \cdot \mathbb{E}^x \left[ \int_{T/2}^T \left\| N(X_s) \right\|_H^2 ds \right]
\]
+ CE \left[ \int_{T/2}^{T} (T-s)^{-2\delta}\|B(X_s)\|_{H^S}^2 ds \right] \\
\leq C(T, K, \delta) + CE \left[ \int_{T/2}^{T} \|N(X_s)\|_{H}^2 ds \right].

By (4.15) and the estimate
\[ \|x^3\|_{H}^2 \leq C(1 + \|x\|_{V}^2 \cdot \|x\|_{H}^4) \]
we obtain that
\[ E \left[ \int_{T/2}^{T} \|N(X_s)\|_{H}^2 ds \right] \leq E \left[ \int_{T/2}^{T} E[\|X_s\|_{H}^2 + \|X^3_s\|_{H}^2 | X_{T/2}|] ds \right] \]
\[ \leq E \left[ \int_{T/2}^{T} E[C(1 + \|X_s\|_{V}^2 \cdot \|X_s\|_{H}^4) | X_{T/2}|] ds \right] \]
\[ \leq CT(1 + E[\|X_{T/2}\|_{H}^6]). \]

Applying (4.20) again, we get the desired result. $\square$

4.3 Stochastic Ginzburg-Landau equation driven by cylindrical $\alpha$-stable processes

We shall study the stochastic Ginzburg-Landau equation on $\mathbb{T}$ as follows:
\[
\begin{aligned}
  dX_t &= -AX_t dt + N(X_t)dt + dL_t, \\
  X_0 &= x,
\end{aligned}
\tag{4.21}
\]
where
(i) the nonlinear term $N$ is defined by
\[ N(u) = u - u^3, \quad u \in H; \]
(ii) $L_t = \sum_{k \in \mathbb{Z}, \beta_k l_k(t)e_k}$ is an $\alpha$-stable process on $H$ with $\{l_k(t)\}_{k \in \mathbb{Z}}$, being i.i.d. one-dimensional symmetric $\alpha$-stable processes with $\alpha > 1$. Moreover, we assume that there exist some $C_1, C_2 > 0$ so that
\[ C_1 \gamma_k^{-\beta} \leq |\beta_k| \leq C_2 \gamma_k^{-\beta} \quad \text{with} \quad \beta > \frac{1}{2} + \frac{1}{2\alpha}. \]

Definition 4.8. We say that a predictable $H$-valued stochastic process $X = (X^x_t)_{t \geq 0}$ is a mild solution to (4.21) if, for any $t \geq 0$ and $x \in H$, it holds that
\[
X^x_t = e^{-At}x + \int_{0}^{t} e^{-A(t-s)}N(X^x_s)ds + \int_{0}^{t} e^{-A(t-s)}dL_s, \quad \text{P.-a.s.} \tag{4.22}
\]

The following properties for the solutions can be found in [38,43].

Proposition 4.9 (See [38,43]). Assume that $\alpha \in (3/2, 2)$ and
\[ \frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}. \]

For every $x \in H$, (4.21) admits a unique mild solution
\[ X = (X^x_t)_{t \geq 0} \in D([0, \infty); H) \cap D([0, \infty); V), \]
which is a strong Feller and irreducible Markov process in $H$, and admits a unique invariant measure $\pi$.

Moreover, we proved that the system converges to its invariant measure $\mu$ with an exponential rate under a topology stronger than the total variation, and the occupation measure $\mathcal{L}_t$ obeys the moderate deviation principle by constructing some Lyapunov test functions in our previous paper [38].

We shall establish the large deviation principle for the occupation measure $\mathcal{L}_t$ in the next theorem.
Theorem 4.10. Assume that $\alpha \in (3/2, 2)$ and
\[
\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}.
\]
Then the family $P^\nu(L_T \in \cdot)$ as $T \to +\infty$ satisfies the large deviation principle with respect to the $\tau$-topology, with a good rate function $J$ defined by (2.2), uniformly for any initial measure $\nu$ in $\mathcal{M}_1(H)$.

Proof. Since $X$ is strong Feller and irreducible in $H$, according to Theorem 2.1, to prove this theorem, we only need to prove that the hyper-exponential recurrence condition (2.4) is fulfilled. By Theorem 4.3, it is sufficient to establish the uniform estimate in Condition 4.1, which will be done by the machinery proposed in Subsection 4.1.2.

(1) The estimate of stochastic convolution $Z_t$: Recall that $Z_t$ is the Ornstein-Uhlenbeck process such that
\[
dZ_t + AZ_t dt = dL_t, \quad Z_0 = 0.
\]
(4.23)
By [43, Lemma 3.1], it holds that for any $T > 0, 0 \leq \theta < \beta - \frac{1}{2\alpha}$ and any $0 < p < \alpha$, we have
\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} \|A^p Z_t\|_H^p\right] \leq CT^{p/\alpha},
\]
(4.24)
where $C$ is independent of $T$.

(2) The estimate of $Y_t$ in $H_\delta$: Recall that $Y_t := X_t - Z_t$. We will prove that for all $T > 0$, $\delta \in (0, 1/2)$ and $p \in (0, \alpha/4)$, we have
\[
\mathbb{E}^\nu[\|Y_T\|_{H_\delta}^p] \leq C(\delta, p) \left(\frac{e^T + 1}{e^T - 1}\right)^4 (1 + T)(1 + T^{-\delta}),
\]
(4.25)
where the constant $C(\delta, p)$ does not depend on $Y_0 = x$ and $T$.

Since
\[
Y_T = e^{-AT/2}Y_{T/2} + \int_{T/2}^T e^{-A(T-s)}N(Y_s + Z_s)ds,
\]
for any $\delta \in (0, 1/2)$, by the inequalities (A.1)–(A.3) and (4.13), there exists a constant $C = C(\delta)$ (whose value may be different from line to line by convention, but not dependent on $T$) satisfying that
\[
\|Y_T\|_{H_\delta} \leq CT^{-\delta}\|Y_{T/2}\|_H + C \int_{T/2}^T (T-s)^{-\delta} \|N(Y_s + Z_s)\|_H ds
\]
\[
\leq CT^{-\delta}\|Y_{T/2}\|_H + C \int_{T/2}^T (T-s)^{-\delta}(\|Y_s\|_H + \|Z_s\|_H + \|Y_s^2\|_H + \|Z_s^2\|_H) ds
\]
\[
\leq CT^{-\delta}\|Y_{T/2}\|_H + C \int_{T/2}^T (T-s)^{-\delta}(\|Y_s\|_H + \|Z_s\|_V + \|Y_s\|_V\|Y_s^2_H + \|Z_s\|_V^2) ds
\]
\[
\leq C T^{-\delta}\left(1 + \sup_{0 \leq t \leq T} |Z_t|_V^2\right) ds
\]
Next, we estimate the last term in the above inequality. By (4.10) and (4.13) again, we have
\[
\int_{T/2}^T (T-s)^{-\delta}\|Y_s\|_V\|Y_s^2_H ds
\]
\[
\leq C \left(\frac{e^T + 1}{e^T - 1}\right)^2 (1 + \sup_{0 \leq t \leq T} |Z_t|_V^2) \int_{T/2}^T (T-s)^{-\delta}\|Y_s\|_V ds
\]
\[
\leq C \left(\frac{e^T + 1}{e^T - 1}\right)^2 (1 + \sup_{0 \leq t \leq T} |Z_t|_V^2) \left(\int_{T/2}^T (T-s)^{-2\delta} ds\right)^\frac{1}{2} \left(\int_{T/2}^T \|Y_s\|_V^2 ds\right)^\frac{1}{2}
\]
\[
\leq C \left(\frac{e^T + 1}{e^T - 1}\right)^2 T^{\frac{1}{2}-\delta} \left(1 + \sup_{0 \leq t \leq T} |Z_t|_V^2\right) \left(\|Y_{T/2}\|_H^2 + \int_{T/2}^T (1 + \|Z_s\|_V^4) ds\right)^\frac{1}{2}
\]
\[ C \left( \frac{e^T + 1}{e^T - 1} \right)^4 (1 + T) \left( 1 + \sup_{0 \leq t \leq T} \| Z_t \|^4 \right). \]

Hence, by (4.24), we obtain that for any \( p \in (0, \alpha/4) \),
\[
E^x[\| Y_T \|_{H_p}^p] \leq C(\delta, p) \left( \frac{e^T + 1}{e^T - 1} \right)^4 (1 + T)(1 + T^{-\delta}).
\]

(3) The combination of \( Z_t \) and \( Y_t \): Putting (4.24) and (4.25) together, we obtain that for all \( T > 0 \), \( \delta \in (0, 1/2) \) and \( p \in (0, \alpha/4) \),
\[
E^x[\| X_T \|_{H_p}^p] \leq C(\delta, p) \left( \frac{e^T + 1}{e^T - 1} \right)^4 (1 + T)(1 + T^{-\delta}),
\]

where the constant \( C(\delta, p) \) does not depend on \( X_0 = x \) and \( T \).

Thus, we get the uniform estimate in Condition 4.1, which completes the proof.

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For the Laplace operator $\Delta = -A$ on $H$, recall the following inequalities (see [43]):

\begin{align}
\|A^\sigma e^{-At}\| &\leq C_\sigma t^{-\sigma}, \quad \forall \sigma > 0, \quad \forall t > 0, \quad \text{(A.1)} \\
\|x\|^2_{L^4} &\leq \|x\|^2_V \cdot \|x\|^2_{L^4} \leq \|x\|^2_V, \quad \forall x \in V, \quad \text{(A.2)} \\
\|x\|^2_H &\leq C\|A^{1/2}x\|^2_{L^4} \cdot \|x\|_H \leq C\|x\|^2_V \cdot \|x\|^2_{L^2}, \quad \forall x \in V, \quad \text{(A.3)} \\
\|N(x) - N(y)\|_H &\leq C(1 + \|A^{1/4}x\|^2_H + \|A^{1/4}y\|^2_H)\|x - y\|_H, \quad \forall x, y \in H. \quad \text{(A.4)}
\end{align}

Now let us prove Theorem 4.6. In order to make the proof easy to read, we split it into three pieces.
Theorem A.1. Under (C1) and (C2), for any positive function \( f \in B_b(H) \),
\[
P_t \log f(x) \leq \log P_t f(y) + \frac{\tilde{K} \|A^\alpha(x-y)\|_H^2}{2c_0^2(1-e^{-\tilde{K} t})}, \quad t > 0, \ x, y \in H. \tag{A.5}
\]
Here, \( \tilde{K} = K + 2 \), \( \|A^\alpha(x-y)\|_H \in [0, \infty) \), which is finite if \( x - y \in H_\alpha \).

Proof. The proof is divided into three steps.

**Step 1** (Galerkin approximation). Recall that \( \{e_n\}_{n \in \mathbb{Z}} \) are eigenfunctions of \( A \) in \( H \). Set
\[
H_n := \text{span}\{e_n, \ldots, e_n\}.
\]
Let \( P_n : V^* \to H_n \) be defined by
\[
P_n y := \sum_{i=-n}^n \langle y, e_i \rangle V \cdot e_i, \quad y \in V^*.
\]
Notice that \( P_n |_{H_n} \) is just the orthogonal projection onto \( H_n \) in \( H \). Set
\[
W_t^{(n)} := \sum_{i=-n}^n \langle W_t, e_i \rangle H \cdot e_i = P_n W_t.
\]
For each \( n \in \mathbb{N} \), consider the following stochastic equation on \( H_n \):
\[
dX_t^{(n)} = -P_n AX_t^{(n)} dt + P_n N(X_t^{(n)}) dt + P_n B(X_t^{(n)}) dW_t^{(n)}, \tag{A.6}
\]
with the initial \( X_0^{(n)} = P_n X_0 \). From the proof of Theorem 5.1.3 in [26], we know that
\( X^{(n)} \) converges weakly to \( X \) in \( L^2(\Omega; L^\infty([0,T]; H)) \).

Thus, to prove the log-Harnack inequality (A.5), it is enough to prove that for the finite-dimensional equation \( X^{(n)} \).

**Step 2** (The gradient estimate for \( X^{(n)} \)). For \( x^{(n)}, y^{(n)} \in \mathbb{R}^{2n} \), let \( X_t^{(n)} \) and \( Y_t^{(n)} \) be the solutions to (A.6) with \( X_0^{(n)} = x^{(n)} \) and \( Y_0^{(n)} = y^{(n)} \), respectively. By Itô’s formula, we obtain
\[
d\|X_t^{(n)} - Y_t^{(n)}\|^2 = 2 \langle P_n A(X_t^{(n)} - Y_t^{(n)}) + P_n [N(X_t^{(n)}) - N(Y_t^{(n)})] \rangle dt
+ \|P_n[B(X_t^{(n)}) - B(Y_t^{(n)})]\|_H^2 dt
+ 2 \langle X_t^{(n)} - Y_t^{(n)} \rangle P_n[B(X_t^{(n)}) - B(Y_t^{(n)})] dW_t^{(n)}
\leq \tilde{K} \|X_t^{(n)} - Y_t^{(n)}\|^2 dt + 2 \langle X_t^{(n)} - Y_t^{(n)} \rangle P_n[B(X_t^{(n)}) - B(Y_t^{(n)})] dW_t^{(n)}
\]
where \( \tilde{K} = K + 2 \). Since the solution \( X^{(n)} \) to (A.6) is non-explosive, this implies
\[
E[\|X_t^{(n)} - Y_t^{(n)}\|^2] \leq e^{\tilde{K} t} \|x^{(n)} - y^{(n)}\|^2.
\]
Denote
\[
P_t^{(n)} f := E[f(X_t^{(n)})],
\]
the Markov transition semigroup of \( X^{(n)} \). We have
\[
\|\nabla P_t^{(n)} f\|^2(x^{(n)}) = \limsup_{y^{(n)} \to x^{(n)}} \frac{|P_t^{(n)} f(y^{(n)}) - P_t^{(n)} f(x^{(n)})|^2}{\|x^{(n)} - y^{(n)}\|^2}
= \limsup_{y^{(n)} \to x^{(n)}} \left( \frac{E[f(Y_t^{(n)}) - f(X_t^{(n)})]}{\|x^{(n)} - y^{(n)}\|} \right)^2
\]
\[ \leq e^{\tilde{K}t} \mathbb{E}[\|\nabla f\|^2(X_t^{(n)})]. \quad (A.7) \]

**Step 3** (Log-Harnack inequality for \( X^{(n)} \)). For fixed \( f \in \mathcal{B}_b(\mathbb{R}^n) \) with \( f \geq 1, x^{(n)} \in \mathbb{R}^{2n}, \) let \( X_0^{(n)} = x^{(n)} \) and \( \mathcal{L}^{(n)} \) be the infinite generator operator of \( X^{(n)} \). By Itô’s formula, we have

\[
\begin{align*}
\frac{d}{dt} P_t^{(n)} f(x^{(n)}) & = \langle \nabla \log P_t^{(n)} f(x^{(n)}), \mathcal{L}^{(n)} f(x^{(n)}) \rangle + \mathcal{L}^{(n)} \log P_t^{(n)} f(X_t^{(n)}) \rangle ds - \frac{\mathcal{L}^{(n)} P_t^{(n)} f(x^{(n)})}{P_t^{(n)} f(x^{(n)})} (X_t^{(n)}) ds \\
& = \langle \nabla \log P_t^{(n)} f(x^{(n)}), \mathcal{L}^{(n)} f(x^{(n)}) \rangle - \frac{1}{2} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (X_t^{(n)}) ds.
\end{align*}
\]

Letting

\[ \tau_k^{(n)} = \inf \{ t \geq 0 : \| X_t^{(n)} \| \geq k \}, \quad k \in \mathbb{N}, \]

we have

\[
\mathbb{E}[\log P_{t-s \wedge \tau_k^{(n)}} (X_{s \wedge \tau_k}^{(n)})] - P_t^{(n)} f(x^{(n)}) = -\frac{1}{2} \mathbb{E}\left[ \int_0^{t \wedge \tau_k} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (X_t^{(n)}) ds \right].
\]

Since the process \( X^{(n)} \) is non-explosive, we have \( \tau_k^{(n)} \to \infty \) almost surely. By the dominated convergence theorem, as \( k \to \infty \), the left-hand side goes to

\[ P_s^{(n)} \log P_{t-s}^{(n)} f(x^{(n)}) - P_t^{(n)} f(x^{(n)}), \]

while by the monotone convergence theorem, the right-hand side goes to

\[ -\frac{1}{2} \int_0^t P_r^{(n)} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (x^{(n)}) dr. \]

So,

\[ \int_0^t P_r^{(n)} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (x^{(n)}) dr < \infty \]

and

\[ P_s^{(n)} \log P_{t-s}^{(n)} f(x^{(n)}) - P_t^{(n)} f(x^{(n)}) = -\frac{1}{2} \int_0^t P_r^{(n)} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (x^{(n)}) dr. \]

Now, for fixed \( x^{(n)}, y^{(n)} \in \mathbb{R}^{2n}, \) let

\[ x_s^{(n)} := (x^{(n)} - y^{(n)}) h_s + y^{(n)}, \quad s \in [0, t], \]

where \( h \in C^1([0, t]; \mathbb{R}) \) such that \( h_0 = 0 \) and \( h_t = 1. \) Since

\[ s \mapsto P_s^{(n)} \log P_{t-s}^{(n)} f(x_s^{(n)}) \]

is absolutely continuous, by (C2) and (A.7), we obtain that

\[
\begin{align*}
P_t^{(n)} \log f(y^{(n)}) - P_t^{(n)} f(x^{(n)}) & = \int_0^t \frac{d}{ds} P_s^{(n)} \log P_{t-s}^{(n)} f(X_s^{(n)}) ds \\
& = -\frac{1}{2} \int_0^t \left\{ P_s^{(n)} \| (\mathcal{P}_n B)^* \nabla \log P_t^{(n)} f \|^2 (x_s^{(n)}) + h'_s (x^{(n)} - y^{(n)}, \nabla P_t^{(n)} \log P_t^{(n)} f) (x_s^{(n)}) \right\} ds \\
& \leq -\frac{c_2}{2} \int_0^t \left\{ P_s^{(n)} A^{-\alpha} \nabla \log P_t^{(n)} f \|^2 (x_s^{(n)}) + h'_s (x^{(n)} - y^{(n)}, \nabla P_t^{(n)} \log P_t^{(n)} f) (x_s^{(n)}) \right\} ds \\
& \leq -\frac{c_2}{2} \int_0^t e^{-K_s} A^{-\alpha} P_s^{(n)} \log P_t^{(n)} f \|^2 (x_s^{(n)}) ds
\end{align*}
\]
Taking $\varepsilon$ which is impossible, because of the boundedness of the last term. Therefore, for any $x, y \in H_\alpha$, we get the log-Harnack inequality for $X^{(n)}$. Letting $n \to \infty$, we obtain (A.5).

**Corollary A.2.** Under (C1) and (C2), for any $t > 0$, $P_t$ is strong Feller and irreducible in $H_\alpha$.

**Proof.** These results are well known as the applications of the log-Harnack inequality (see [35]). For the completeness, we give those proofs.

1. (Strong Feller) It is sufficient to prove for nonnegative $f \in B_b(H_\alpha)$, $P_t$ is continuous in $H_\alpha$. Applying the log-Harnack inequality in Theorem A.1 for $1 + \varepsilon f$ in place of $f$, we obtain from the elementary inequality $r \leq \log(1 + r) + r^2$, $r \geq 0$, that for any $\varepsilon > 0$ and $x, y \in H_\alpha$,

$$P_t f(y) - \varepsilon \|f\|_\infty^2 \leq P_t \log(1 + \varepsilon f)(y) \leq \frac{1}{\varepsilon} \log(1 + \varepsilon P_t f(x)) + \frac{c_t}{\varepsilon} \|x - y\|_{H_\alpha}^2,$$

where

$$c_t := \frac{K}{2c^2(1 - e^{-Kt})}.$$

By Taylor’s formula,

$$\frac{1}{\varepsilon} \log(1 + \varepsilon P_t f(x)) = P_t f(x) + \varepsilon |P_t f(x)|^2 + o(\varepsilon).$$

Taking $\varepsilon = \|x - y\|_{H_\alpha}$ in (A.8) and (A.9), we have

$$P_t f(y) - P_t f(x) \leq \|x - y\|_{H_\alpha} (2\|f\|_\infty^2 + c_t) + o(\|x - y\|_{H_\alpha}).$$

Then, changing the positions of $x$ and $y$, we have

$$\|P_t f(y) - P_t f(x)\| \leq \|x - y\|_{H_\alpha} (2\|f\|_\infty^2 + c_t) + o(\|x - y\|_{H_\alpha}).$$

Therefore, $P_t f$ is continuous in $H_\alpha$.

2. (Irreducibility) It is enough to show that for any $x, y \in H_\alpha$, $\eta > 0, t > 0$,

$$P_t 1_{G_{y,\eta}}(x) > 0, \quad \text{for} \quad G_{y,\eta} := \{y' \in H_\alpha : \|y' - y\|_{H_\alpha} < \eta\}.$$

By the continuity of $X_t$ in $H_\alpha$, for any $y_0 \in H_\alpha$ and $\eta_0 > 0$, there exists $t_0 > 0$ such that for any $t \leq t_0$,

$$P_t 1_{G_{y_0,\eta_0}}(y_0) > \frac{1}{2}.$$

By contradiction, if there exist $x_0 \in H_\alpha$ and $t \leq t_0$ satisfying that $P_t 1_{G_{y_0,\eta_0}}(x_0) = 0$, applying Theorem A.1 to $1 + n 1_{G_{y_0,\eta_0}}$, $n \in \mathbb{N}$, we have

$$\log P_t(1 + n 1_{G_{y_0,\eta_0}})(x_0) = 0$$

and then

$$\frac{1}{2} \log(1 + n) \leq P_t \log(1 + n 1_{G_{y_0,\eta_0}}(y_0)) \leq \frac{K \|A^\alpha(x_0 - y_0)\|_{H_\alpha}^2}{2c^2(1 - e^{-Kt})},$$

which is impossible, because of the boundedness of the last term. Therefore, for any $t \leq t_0$, $P_t$ is irreducible in $H_\alpha$ and $P_t$ is irreducible in $H_\alpha$ for all $t > 0$ by the Markov property. \qed
Corollary A.3. Under (C1) and (C2), for any $t > 0$, $P_t$ is strong Feller and irreducible in $H$.

Proof. Since for any $t > 0$ the Markov process $X_t$ is in $H_3$ a.s., it is easy to observe that the irreducibility in $H_3$ implies the irreducibility in $H$. Now, we prove the strong Feller property in $H$.

Let $\{X_t^x\}_{t \geq 0}$ and $\{X_t^y\}_{t \geq 0}$ be the solutions of (4.14) with initial values $x, y \in H$, respectively. By Itô’s formula, we have

$$
\|X_t^x - X_t^y\|_H^2 + \int_0^t \|X_s^x - X_s^y\|_H^2 ds
= \|x - y\|_H^2 + 2 \int_0^t \langle X_s^x - X_s^y, N(X_s^x) - N(X_s^y) \rangle ds + \int_0^t \|B(X_s^x) - B(X_s^y)\|_{L^2}^2 ds
+ 2 \int_0^t \langle X_s^x - X_s^y, (B(X_s^x) - B(X_s^y)) dW_s \rangle
\leq \|x - y\|_H^2 + C \int_0^t \|X_s^x - X_s^y\|_H^2 ds + 2 \int_0^t \langle X_s^x - X_s^y, (B(X_s^x) - B(X_s^y)) dW_s \rangle.
$$

By Condition (C2) and (4.15), taking the expectation in the above equation, we have

$$
E[\|X_t^x - X_t^y\|_H^2] \leq \|x - y\|_H^2 + C \int_0^t E[\|X_s^x - X_s^y\|_H^2] ds.
$$

(A.11)

By Gronwall’s inequality, we obtain that

$$
E[\|X_t^x - X_t^y\|_H^2] \leq e^{Ct} \|x - y\|_H^2.
$$

(A.12)

By (4.15), (A.3) and (A.4), we have that

$$
E[\|A^\alpha(X_t^x - X_t^y)\|_H]
\leq C \|A^\alpha e^{tA}(x - y)\|_H + CE \left[ \int_0^t (t - s)^{-\alpha} \|B(X_s^x) - B(X_s^y)\|_{L^2} ds \right]
+ CE \left[ \int_0^t A^\alpha e^{(t-s)A} \|N(X_s^x) - N(X_s^y)\|_H ds \right]
\leq Ct^{-\alpha} \|x - y\|_H + C \left\{ E \int_0^t (t - s)^{-2\alpha} \|B(X_s^x) - B(X_s^y)\|_{H^2} ds \right\}^{1/2}
+ CE \left[ \int_0^t (t - s)^{-\alpha} (1 + \|X_s^y\|_V \|X_s^x\|_H + \|X_s^y\|_V \|X_s^y\|_H) \|X_s^x - X_s^y\|_H ds \right]
\leq Ct^{-\alpha} \|x - y\|_H + C \left\{ E \int_0^t (t - s)^{-2\alpha} \|X_s^x - X_s^y\|_H^2 ds \right\}^{1/2}
+ C \left\{ E \int_0^t (1 + \|X_s^x\|_V \|X_s^x\|_H + \|X_s^y\|_V \|X_s^y\|_H) ds \right\}^{1/2}
\times \left\{ E \int_0^t (t - s)^{-2\alpha} \|X_s^x - X_s^y\|_H^2 ds \right\}^{1/2}.
$$

(A.13)

Plugging (4.15) and (A.12) into (A.13), we have that for any $t > 0$,

$$
E[\|X_t - Y_t\|_{H_3}] \leq C_t \|x - y\|_H.
$$

For any $s \in (0, t)$, set $G := \{\|X_s^x - X_s^y\|_{H_3} \geq \|x - y\|_{H_3}^{1/2}\}$.

For any bounded function $f \in \mathcal{B}_b(H)$, we have

$$|P_tf(x) - P_tf(y)| = |E\{P_{t-s}f(X_s^x) - P_{t-s}f(X_s^y)\}| \leq I_1 + I_2,$$
where
\[
I_1 := |\mathbb{E}\{[P_{t-s}f(X_s^x) - P_{t-s}f(X_s^y)]1_G]\}|
\]
\[
I_2 := |\mathbb{E}\{[P_{t-s}f(X_s^x) - P_{t-s}f(X_s^y)]1_{G^c}\}|
\]

By Chebyshev’s inequality, we have
\[
I_1 \leq C\|f\|_\infty \cdot \|x - y\|_H^{1/2}.
\]

By the Markov property and (A.10), we have
\[
I_2 \leq \|x - y\|_H^{1/2} (2\|f\|_\infty^2 + C) + o(\|x - y\|_H^{1/2}).
\]

Combining the estimates of \(I_1\) and \(I_2\), we immediately get the strong Feller property in \(H\). 
\[\square\]