Quantum Theory of Electrodynamics in Linear Media Subject to Boundary Conditions

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We show that the material dependencies of macroscopically quantized fields in linear media are not consistent with the classical electromagnetic boundary conditions. We then phenomenologically construct macroscopic quantized fields that satisfy quantum–classical correspondence with the result indicating that the canonical momentum in a linear medium is modified as a consequence of the reduced speed of light. We re-derive D’Alembert’s principle and Lagrange’s equations for an arbitrarily large region of space in which light signals travel slower than in the vacuum and show that the resulting modifications to Lagrangian dynamics, including the canonical momenta, repair the violation of the correspondence principle for macroscopically quantized fields.

I. INTRODUCTION

Quantum electrodynamics describes the quantum mechanical properties of electromagnetic fields in the vacuum. The analogous theory for the quantum optical properties of electric and magnetic fields in a dielectric or similar linear medium can be obtained by quantizing the macroscopic fields in the continuous medium [1, 2, 3, 4, 5, 6], as well as by applying quantum electrodynamics to a microscopic model of the medium [7, 8, 9, 10]. Adopting the former approach in 1940, Ginzburg [1] quantized the macroscopic electromagnetic field in a simple transparent linear dielectric. The idealized dielectric studied by Ginzburg is a model for a more complex material that exhibits dispersion, absorption, nonlinearity, and other extended effects to varying degrees. The difficulties of treating dispersion within the macroscopic quantization procedure have been addressed by Jauch and Watson [2], Drummond [3], and Milonni [4]. Huttner and Barnett [9] treated absorptive dielectrics using a macroscopic quantization procedure based on damped Hopfield polaritons. Quantization of the macroscopic field in nonlinear dielectrics [11], magnetodielectric matter [12], and negative-index materials [13] has also been investigated. In contrast to reports of extended aspects of quantized fields in matter, the current work challenges the validity of the macroscopic quantization procedure itself. Specifically, we use the classical electromagnetic boundary conditions to show that the quantized field inside a linear medium, as it is currently derived [1, 2, 3, 4, 5, 6], violates quantum–classical correspondence.

Materials are of finite extent and we are usually interested in how the dynamics of a field change with the properties of the medium. In the limit of large numbers, the quantum and classical relations must have the same form. We find, however, that the material dependencies of macroscopically quantized fields in linear media are not consistent with the classical electromagnetic boundary conditions. Subsequently, the boundary conditions are used to phenomenologically construct quantized fields in matter that are expressed in terms of material-independent creation and annihilation operators so that quantized fields in different media are related by the overall normalization that coincides with the classical boundary conditions. The generalized position and momentum operators of the conforming theory do not obey standard commutation relations. We re-derive D’Alembert’s principle and Lagrange’s equations based on the relativity of dynamics in a linear medium [14, 15] and show that the reduced speed of light changes the definition of the canonically conjugate momentum. Then we incorporate the modified conjugate momentum into the quantization procedure [6] and derive quantized fields that are the same as the phenomenological fields and likewise satisfy the correspondence principle.

II. QUANTIZATION PROCEDURE

A typical Ginzburg macroscopic quantization procedure [6] is based on an expansion of the vector potential in terms of modes

\[ \mathbf{A} = e \sum_{l\lambda} q_{l\lambda}(t) \mathbf{u}_{l\lambda}(\mathbf{r}) \] (2.1)

in a dielectric subject to the usual continuum assumption of an arbitrarily large, linear, isotropic, homogeneous, transparent medium with refractive index \( n \). All fields, variables, and operators are macroscopic in the sense of representing quantities as continuum averages over a volume larger than a cubic wavelength. The \( \mathbf{u}_{l\lambda} \) are orthonormal functions that satisfy periodic boundary conditions on the quantization volume. Substituting Eq. (2.1) into the electromagnetic Lagrangian

\[ L = \frac{1}{2} \int \left( \frac{n^2}{c^2} \left( \frac{d\mathbf{A}}{dt} \right)^2 - (\nabla \times \mathbf{A})^2 \right) d^3 \mathbf{r} \] (2.2)

and the wave equation, applying separation of variables and the identity

\[ \int_V (\nabla \times \mathbf{u}_{l\lambda}) \cdot (\nabla \times \mathbf{u}_{\ell\nu\lambda}) d^3 \mathbf{r} = \int_V \mathbf{u}_{l\lambda} \cdot (\nabla \times (\nabla \times \mathbf{u}_{\ell\nu\lambda})) d^3 \mathbf{r}, \] (2.3)
one obtains the Lagrangian

\[ L = \frac{1}{2} \sum_{l\lambda} \left( n^2 \dot{q}_{l\lambda}^2 - n^2 \omega_l^2 q_{l\lambda}^2 \right) \quad (2.4) \]

using orthonormality of the mode functions. The conjugate momenta \[ p_{l\lambda} = \frac{\partial L}{\partial \dot{q}_{l\lambda}} = n^2 \dot{q}_{l\lambda} \quad (2.5) \]
are then used to construct the effective Hamiltonian

\[ H = \frac{1}{2} \sum_{l\lambda} \left( \frac{p_{l\lambda}^2}{n^2} + n^2 \omega_l^2 q_{l\lambda}^2 \right) . \quad (2.6) \]
The effective Hamiltonian takes the form

\[ H = \frac{1}{2} \sum_{l\lambda} \left( P_{l\lambda}^2 + \omega_l^2 Q_{l\lambda}^2 \right) \quad (2.7) \]
by application of the canonical transformation

\[ P_{l\lambda} = p_{l\lambda}/n \quad (2.8a) \]
\[ Q_{l\lambda} = nq_{l\lambda} . \quad (2.8b) \]
Quantization is accomplished by treating \( P_{l\lambda} \) and \( Q_{l\lambda} \) as operators satisfying the commutation relations

\[ [q_{l\lambda}, p_{l\lambda}] = i \hbar \delta_{l\lambda} \]. \quad (2.9) \]
One then introduces the creation operators \( \zeta_{l\lambda}^\dagger \) and the annihilation operators \( \zeta_{l\lambda} \) by

\[ P_{l\lambda} = i \sqrt{\frac{\hbar \omega_l}{2}} (\zeta_{l\lambda}^\dagger - \zeta_{l\lambda}) . \quad (2.10a) \]
\[ Q_{l\lambda} = \sqrt{\frac{\hbar \omega_l}{2 \omega_l}} (\zeta_{l\lambda}^\dagger + \zeta_{l\lambda}) , \quad (2.10b) \]
where the operators obey boson commutation relations

\[ [\zeta_{l\lambda}, \zeta_{l'\lambda'}^\dagger] = [\zeta_{l\lambda}^\dagger, \zeta_{l'\lambda'}] = 0 ; \ [\zeta_{l\lambda}, \zeta_{l'\lambda'}^\dagger] = \delta_{ll'} \delta_{\lambda\lambda'} . \quad (2.11) \]
The effective, or macroscopic, Hamiltonian is therefore

\[ H = \frac{1}{2} \sum_{l\lambda} \hbar \omega_l \left( \zeta_{l\lambda}^\dagger \zeta_{l\lambda} + \zeta_{l\lambda} \zeta_{l\lambda}^\dagger \right) . \quad (2.12) \]
It is then straightforward to derive the traveling-wave representation of the medium-assisted vector potential operator \[ A = c \sum_{l\lambda} \sqrt{\frac{\hbar}{2 n^2 \omega_l}} \left( \zeta_{l\lambda} e^{ik_l \cdot r} + \zeta_{l\lambda}^\dagger e^{-ik_l \cdot r} \right) \hat{e}_{k_{l\lambda}} . \quad (2.13) \]
The effective Hamiltonian \[ (2.12) \] and the macroscopic vector potential operator \[ (2.13) \] are the principle products of the quantization procedures that appear in the macroscopic quantum electrodynamics literature \[ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] \].

The normalization of the expansion of the vector potential in Eq. \[ (2.1) \] is arbitrary. It is sometimes customary to use a normalization of \( 1/n \) or \( 1/\sqrt{\varepsilon} \) in Eq. \[ (2.1) \]. This is equivalent to the canonical transformation \[ (2.8) \] and eliminates a step in the derivation. No normalization was used here in order to facilitate comparison with a similar quantization procedure that is presented later.

## III. BOUNDARY CONDITIONS

In the limit of large numbers, quantum and classical relations must have the same form. The nature of macroscopic fields as averages means that the limit will be satisfied. Classically, the spatial extent of a field inside a material is reduced in width by a factor of \( n \) compared to the width in vacuum due to the reduced speed at which light travels in the medium. The refracted field is then reduced in amplitude by \( \sqrt{n} \) compared to the incident field from vacuum as a requirement of conservation of electromagnetic energy in the limit that reflection is negligible or suppressed by an anti-reflection coating.

We consider an electromagnetic field normally incident on a minimally reflective interface between the vacuum and a linear medium. Forward propagating plane-wave solutions of the wave equation can be represented by the vector potential

\[ A_f = A_f \cos (-\omega t + k z + \phi) \hat{e}_i . \quad (3.1) \]
Here, \( \hat{e}_i \) is a unit vector transverse to the direction of propagation and \( k = n \omega / c \). The amplitudes of the vector potential for the incident and refracted fields are respectively denoted as \( A_i \) and \( A_f \).

The Poynting–Umov vector \( c \mathbf{E} \times \mathbf{B} \) is the continuous energy flux vector that is associated with energy conservation. In terms of vector potential amplitudes,

\[ A_i^2 = n A_f^2 \quad (3.2) \]
is obtained from continuity of the Poynting–Umov vector in the plane-wave cw limit. We write

\[ A_i = \frac{A_f}{\sqrt{n}} \quad (3.3) \]
for the relation between the refracted field \( A_i \) and the field \( A_f = A_v \) that is incident from the vacuum. Because fields can be transferred from one material to another through index-matching or other anti-reflection techniques,

\[ \sqrt{n_1} A_1 = \sqrt{n_2} A_2 \quad (3.4) \]
is the relation between vector potential amplitudes in different materials and is equivalent to the Fresnel relation.
in the limit that the change in refractive index is sufficiently small that reflections are negligible.

We write the vector potential operator (2.13) as

$$A = c \sum_{l\lambda} \sqrt{\frac{\hbar}{2n\omega_l V}} \left( \frac{\zeta_{l\lambda}}{\sqrt{n}} e^{ik_{l\lambda}\cdot r} + \frac{\zeta_{l\lambda}^\dagger}{\sqrt{n}} e^{-ik_{l\lambda}\cdot r} \right) \hat{e}_{\lambda\lambda},$$

and define macroscopic annihilation and creation operators

$$a_{l\lambda} = \frac{\zeta_{l\lambda}}{\sqrt{n}}, \quad a_{l\lambda}^\dagger = \frac{\zeta_{l\lambda}^\dagger}{\sqrt{n}}.$$  (3.5)

By substitution from Eqs. (3.6), the macroscopic vector potential operator (3.5) becomes

$$A = c \sum_{l\lambda} \sqrt{\frac{\hbar}{2n\omega_l V}} \left( a_{l\lambda} e^{ik_{l\lambda}\cdot r} + a_{l\lambda}^\dagger e^{-ik_{l\lambda}\cdot r} \right) \hat{e}_{\lambda\lambda}. \quad (3.7)$$

In this form, the material dependence of the classical boundary condition (3.3) is satisfied by the normalization prefactor of the field in which the annihilation and creation operators obey material-independent boson commutation relations

$$[a_{l\lambda}, a_{l'\lambda'}] = [a_{l\lambda}^\dagger, a_{l'\lambda'}^\dagger] = 0; \quad [a_{l\lambda}, a_{l\lambda'}^\dagger] = \delta_{l\lambda} \delta_{l'\lambda'}.$$  (3.6a)

while the polariton operators obey material-dependent commutation relations

$$[\zeta_{l\lambda}, \zeta_{l'\lambda'}] = [\zeta_{l\lambda}^\dagger, \zeta_{l'\lambda'}^\dagger] = 0; \quad [\zeta_{l\lambda}, \zeta_{l\lambda'}^\dagger] = n \delta_{l\lambda} \delta_{l'\lambda'}.$$  (3.6b)

This result contradicts the condition (2.11) used in the macroscopic Ginzburg quantization procedure in Section II. Using the operators (3.6), one can show that the commutator (2.10) should be

$$[Q_{l\lambda}, P_{l'\lambda'}] = [q_{l\lambda}, p_{l'\lambda'}] = i\hbar \delta_{l\lambda} \delta_{l'\lambda'}.$$  (3.9)

based on the polariton commutation relations (3.9). The scaling of the commutator in Eq. (3.10) proves that the generalized momentum and position variables are not canonically conjugate in the usual sense.

**IV. LAGRANGE’S EQUATIONS IN FILLED SPACETIME**

In order to derive the generalized momentum variables that are canonically conjugate to the generalized coordinates in a linear medium, we treat space as being entirely filled with an isotropic homogeneous continuous medium that responds linearly to electromagnetic radiation. Starting from first principles, we then derive the characteristics of relativity and Lagrangian dynamics for the case in which the effective speed of light is $c/n$.

Consider an inertial reference frame $S(t, x, y, z)$ with orthogonal axes $x$, $y$, and $z$. Position vectors in $S$ are denoted by $x = (x, y, z)$. If a light pulse is emitted from the origin at time $t = 0$, then

$$x^2 + y^2 + z^2 - \left(\frac{c}{n}t\right)^2 = 0 \quad (4.1)$$
describes wavefronts in the $S$ system. Writing time as a spatial coordinate $ct/n$, the four-vector $[15]$ in which $S’$ translates at a constant velocity $v$ in the direction of the positive $x$ axis and the origins of the two systems coincide at time $t = t' = 0$. If a light pulse is emitted from the common origin at time $t = 0$, then

$$(x')^2 + (y')^2 + (z')^2 - \left(\frac{c}{n}t'\right)^2 = 0 \quad (4.3)$$
describes wavefronts in the $S’$ system and Eq. (4.1) holds for wavefronts in $S$. The material Lorentz transformation $[15]

$$x = \gamma(x' + vt') \quad (4.4a)$$

$$y = y' \quad (4.4b)$$

$$z = z' \quad (4.4c)$$

$$t = \gamma \left(t' + \frac{n^2v}{c^2}x'\right) \quad (4.4d)$$

derived by the usual methods [18, 19, 20], where

$$\gamma = \frac{1}{\sqrt{1 - \frac{n^2v^2}{c^2}}}. \quad (4.5)$$

The square of the invariant spatial interval $\Delta s$ is $[14]

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c/n)^2(\Delta t)^2. \quad (4.6)$$

Multiplying the preceding equation by $-1$ and taking the square root of the result yields another invariant quantity $c\Delta \tau = \frac{c}{n} \sqrt{(\Delta t)^2 - (n/c)^2((\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2)} \quad (4.7)$
from which we obtain the integral of proper time

$$d\tau = \frac{dt}{\gamma n}. \quad (4.8)$$
Taking the derivative of the position four-vector with respect to the proper time, we obtain the four-velocity
\[
U = \frac{d\mathbf{x}}{d\tau} = \frac{dx}{dt}\frac{1}{d\tau} = \gamma c \left( 1, \frac{dx}{d(\tau/t)}, \frac{dy}{d(\tau/t)}, \frac{dz}{d(\tau/t)} \right)
\]
and the four-momentum
\[
\mathbf{P} = m_0 U = mc \left( 1, \frac{dx}{d(\tau/t)}, \frac{dy}{d(\tau/t)}, \frac{dz}{d(\tau/t)} \right),
\]
where \(m = \gamma m_0\) is the relativistic mass. The four-force \(\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = \frac{d\mathbf{P}}{dt}\frac{1}{d\tau}\) is derived in a similar manner.

In the nonrelativistic limit, \(\gamma = 1\), \(\tau = t/n\), we find the three-velocity
\[
\mathbf{u} = n\mathbf{x} = c \left( \frac{dx}{d(\tau/n)}, \frac{dy}{d(\tau/n)}, \frac{dz}{d(\tau/n)} \right)
\]
and the three-momentum
\[
\mathbf{p} = nm\mathbf{x} = mc \left( \frac{dx}{d(\tau/n)}, \frac{dy}{d(\tau/n)}, \frac{dz}{d(\tau/n)} \right)
\]
in a region of reduced light velocity.

For a system of particles, the transformation of the position vector of the \(i^{th}\) particle to \(J\) independent generalized coordinates is
\[
\mathbf{x}_i = \mathbf{x}_i(\tau; q_1, q_2, \ldots, q_J),
\]
where \(\tau = t/n\). Applying the chain rule, we obtain the virtual displacement
\[
\delta\mathbf{x}_i = \sum_{j=1}^{J} \frac{\partial\mathbf{x}_i}{\partial q_j} \delta q_j
\]
and the velocity
\[
\mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau} = \sum_{j=1}^{J} \frac{\partial\mathbf{x}_i}{\partial q_j} \frac{dq_j}{d\tau}
\]
of the \(i^{th}\) particle in the new coordinate system. The relation
\[
\frac{\partial\mathbf{u}_i}{\partial dq_j}/d\tau = \frac{\partial\mathbf{x}_i}{\partial dq_j}/d\tau
\]
comes from the derivative of Eq. (1.17). Substitution of Eq. (1.17) into the identity
\[
\frac{d}{d\tau} \left( \mathbf{m}_i \cdot \frac{\partial\mathbf{x}_i}{\partial q_j} \right) = \mathbf{m} \frac{d\mathbf{u}_i}{d\tau} \frac{\partial\mathbf{x}_i}{\partial q_j} - \mathbf{m}_i \frac{d}{d\tau} \left( \frac{\partial\mathbf{x}_i}{\partial q_j} \right)
\]
results in
\[
\frac{d\mathbf{p}_i}{d\tau} \cdot \frac{\partial\mathbf{x}_i}{\partial q_j} = \frac{d}{d\tau} \left( \frac{\partial}{\partial dq_j} \frac{1}{2} \mathbf{m}_i^2 \right) - \frac{\partial}{\partial q_j} \left( \frac{1}{2} \mathbf{m}_i^2 \right)
\]
by application of the calculus.

For a system of particles in equilibrium, the virtual work of the applied forces \(\mathbf{f}_i\) vanishes and the virtual work on each particle vanishes leading to the principle of virtual work
\[
\sum_i \mathbf{f}_i \cdot \delta\mathbf{x}_i = 0 \quad (4.20)
\]
and D’Alembert’s principle
\[
\sum_i \left( \mathbf{f}_i - \frac{d\mathbf{p}_i}{d\tau} \right) \cdot \delta\mathbf{x}_i = 0. \quad (4.21)
\]
Defining the kinetic energy of the \(i^{th}\) particle
\[
T_i = \frac{1}{2} m_i u_i^2,
\]
we can write D’Alembert’s principle as
\[
\sum_j \left[ \left( \frac{d}{d\tau} \left( \frac{\partial T}{\partial (dq_j/d\tau)} \right) - \frac{\partial T}{\partial q_j} \right) - Q_j \right] \delta q_j = 0 \quad (4.23)
\]
using Eqs. (4.15) and (4.19), where
\[
Q_j = \sum_i f_i \cdot \frac{\partial\mathbf{x}_i}{\partial q_j} \quad (4.24)
\]
If the generalized forces \(Q_j\) come from a generalized scalar potential function \(V\), then we can write the Lagrange equations of motion
\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial dq_j/d\tau} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (4.25)
\]
where \(L = T - V\) is the Lagrangian in a linear medium.

For any coordinate \(q_j\) that measures a linear displacement of a particle in a given direction, the partial derivative of the Lagrangian with respect to that coordinate vanishes. In that case, substitution of
\[
\frac{\partial L}{\partial q_j} = 0 \quad (4.26)
\]
into Eq. (4.25) produces
\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial dq_j/d\tau} \right) = 0. \quad (4.27)
\]
If the potential energy is velocity independent then we can define the canonical momentum
\[
\mathbf{p}_j = \frac{\partial L}{\partial dq_j/d\tau} = \frac{1}{c} \frac{\partial L}{\partial dq_j/d(\tau/n)} \quad (4.28)
\]
based on the proper temporal integration of Eq. (4.27).

The consequence that an effective speed of light has for the canonical momentum is easily illustrated in the
case of a nonrelativistic free particle. The kinetic energy of the particle, initially at rest in the local frame, is

\[ T = \frac{1}{2} m \mathbf{u} \cdot \mathbf{u}. \]  

Then the Lagrangian for the free particle can be written as

\[ L = \frac{m}{2} \sum_{j=1}^{3} \left( \frac{dx_j}{d\tau} \right)^2 \]  

in rectangular coordinates. Applying the definition of the canonical momentum \([4.28]\), one obtains

\[ p_j = \frac{\partial L}{\partial (dx_j/d\tau)} = m \frac{dx_j}{d\tau} = ma_j \frac{dx_j}{dt} \]  

in agreement with the linear momentum of a particle in a linear medium, Eq. \([4.13]\). Although macroscopic particles cannot travel unimpeded through a material in the continuum limit, light does at reduced speed. In the next section, the modified Lagrangian dynamics are applied to the derivation of macroscopic quantum electrodynamics principles.

**V. MACROSCOPIC QUANTIZATION**

Lagrangian dynamics is the basis for the procedure that was used in Sec. II to quantize the electromagnetic field in a linear medium. The macroscopically quantized field, Eq. \([2.13]\), was found to be inconsistent with the classical electromagnetic boundary conditions. In the preceding section, we derived a modification of Lagrangian dynamics in the context of a uniform \(c/n\) speed of light. Here we show that the quantized field that is derived using the modified dynamical theory is consistent with the electromagnetic boundary conditions.

The macroscopic quantization procedure is based on an expansion of the vector potential in terms of modes as

\[ \mathbf{A} = c \sum_{l\lambda} q_{l\lambda}(t) \mathbf{u}_{l\lambda}(\mathbf{r}). \]  

Substituting Eq. \([5.1]\) into the Lagrangian

\[ L = \frac{1}{2} \int \left( \frac{n^2}{c^2} \left( \frac{d\mathbf{A}}{dt} \right)^2 - (\nabla \times \mathbf{A})^2 \right) d^3\mathbf{r} \]  

and the wave equation, one obtains the Lagrangian

\[ L = \frac{1}{2} \sum_{l\lambda} \left( n^2 q_{l\lambda}^2 - n^2 \omega^2 q_{l\lambda}^2 \right), \]  

as in Section II. The point of departure for this derivation is the use of Eq. \([4.28]\), rather than Eq. \([2.5]\), for the conjugate momenta. Instead of \(p_{l\lambda} = n^2 q_{l\lambda}\), which leads to violation of quantum–classical correspondence, we obtain

\[ p_{l\lambda} = \frac{\partial L}{\partial (dq_{l\lambda}/d\tau)} = n^2 q_{l\lambda} \]  

for the conjugate momenta. The effective Hamiltonian

\[ H = \sum_{l\lambda} \left( p_{l\lambda} \frac{dq_{l\lambda}}{d\tau} - L \right) = \frac{1}{2} \sum_{l\lambda} \left( p_{l\lambda}^2 + n^2 \omega^2 q_{l\lambda}^2 \right) \]  

is quantized in the usual way by taking \(P_{l\lambda} = \sqrt{n} q_{l\lambda}\) and \(Q_{l\lambda} = q_{l\lambda}/\sqrt{n}\) to be operators satisfying the material-independent commutation relations

\[ [q_{l\lambda}, p_{l'\lambda'}] = [Q_{l\lambda}, P_{l'\lambda'}] = i \hbar \delta_{l\lambda} \delta_{l'\lambda'}. \]  

Defining the usual material-independent annihilation and creation operators,

\[ a_{l\lambda} = \frac{1}{\sqrt{2\hbar \omega_l}} (\omega_l Q_{l\lambda} + i P_{l\lambda}) \]  

\[ a_{l\lambda}^\dagger = \frac{1}{\sqrt{2\hbar \omega_l}} (\omega_l Q_{l\lambda} - i P_{l\lambda}) \]  

yields the effective Hamiltonian

\[ H = \frac{1}{2} \sum_{l\lambda} n\hbar \omega_l \left( a_{l\lambda}^\dagger a_{l\lambda} + a_{l\lambda} a_{l\lambda}^\dagger \right). \]  

In the plane-wave cw limit, the classical energy density is

\[ \mathcal{H} = \frac{1}{2} \left( \frac{n \mathbf{dA}}{c dt} \right)^2 + (\nabla \times \mathbf{A})^2 = \frac{n^2 \omega^2}{c^2} \mathbf{A}^2. \]  

Using the classical boundary conditions \(A_t = A_t/c/\sqrt{n}\), the energy density inside the material is a factor of \(n\) greater than the energy density in the vacuum. The effective Hamiltonian \([5.8]\) exhibits the conforming enhancement of energy density in a medium. In contrast, the effective Hamiltonian \([2.12]\) in the material, derived by the original Ginzburg quantization procedure with the boson commutation relations, fails the test of quantum–classical correspondence.

The traveling wave representation of the macroscopic vector potential

\[ \mathbf{A} = c \sum_{l\lambda} \sqrt{\frac{\hbar}{2n\omega_l V}} \left( a_{l\lambda} e^{ik_l \cdot \mathbf{r}} + a_{l\lambda}^\dagger e^{-ik_l \cdot \mathbf{r}} \right) \mathbf{e}_{k_l \lambda} \]  

can be constructed from the relations \([5.7]\) between the canonical variables and the creation and annihilation operators using the canonical transformation. The field \([5.10]\), like the phenomenological field \([3.7]\), satisfies quantum–classical correspondence with the electromagnetic boundary condition \(A_t = A_t/c/\sqrt{n}\).
VI. SUMMARY

The quantized vector potential and the effective Hamiltonian for an electromagnetic field in a dielectric were derived from the Lagrangian by the conventional quantization procedure. This representation of the macroscopically quantized field was shown to violate quantum-classical correspondence. The principles of Lagrangian dynamics, including D’Alembert’s principle and the Lagrange equations of motion were re-derived using the thesis of continuum electrodynamics of a structureless medium that interacts with light through a constant. The resulting changes to the principles dynamics in the continuum, applied in the Ginzburg macroscopic quantization procedure, were found to be sufficient to repair the violation of quantum-classical correspondence for macroscopically quantized fields.

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