The reality conditions for the new canonical variables of General Relativity

GIORGIO IMMIRZI

*Dipartimento di Fisica, Universitá di Perugia, and I.N.F.N., sez. di Perugia.*

ABSTRACT

We examine the constraints and the reality conditions that have to be imposed in the canonical theory of 4-d gravity formulated in terms of Ashtekar variables. We find that the polynomial reality conditions are consistent with the constraints, and make the theory equivalent to Einstein’s, as long as the inverse metric is not degenerate; when it is degenerate, reality conditions cannot be consistently imposed in general, and the theory describes complex general relativity.

*e-mail: immirzi@perugia.infn.it*
1. Introduction.

In 1986, A. Ashtekar found a new canonical formulation of Einstein’s theory of gravity [1], quite different and in many ways much more appealing than the one that had been evolved by P.A.M. Dirac, P. Bergmann and R. Arnowitt, S. Deser and C.W. Misner [2](amongst others) many years before. This remarkable work has given a new impulse to the field of canonical gravity, and has generated a totally new approach to the problem of formulating a quantum theory [3].

In this new formulation the basic canonical variables are a (densitized) inverse dreibein $\tilde{e}^{a\mu}$, and a complex gauge field $A^a_\mu$, and the constraints of the theory are polynomial in these new variables. In particular, this implies that the equations of the theory are regular regardless of whether $\tilde{e}^{a\mu}$ is invertible or not. This extension of Einstein’s theory has attracted a lot of attention, especially because it is expected to play a key role in quantum gravity [3-7].

A degenerate $\tilde{e}^{a\mu}$ signals a change of signature of the metric, which may be associated with a change in topology; we can hardly expect our interpretation of the variables of the theory to survive unscathed. In fact, besides constraints and equations of motion, there are reality conditions that must be imposed on the variables, i.e. a real section has to be identified on the constraint surface, for the theory to describe real space–time. In this paper we investigate whether these reality conditions can be consistently imposed on configurations with degenerate $\tilde{e}^{a\mu}$, and conclude that in general they cannot.

The reality conditions do not descend from the action, like constraints (be they first or second class), but have to be imposed "by hand" on the initial condition; all is well if the time evolution of the system preserves them, i.e. if the Poisson bracket of the Hamiltonian with a quantity stated to be real can be seen to be real.

What we find is that although constraints and reality conditions are polynomial in the basic variables, the expression of the time derivative of a reality condition in terms of constraints and of real quantities is not polynomial, and requires the
3–metric to be non degenerate to be valid. Therefore, when the metric is non degenerate we find that the theory expressed in terms of the new variables with polynomial reality conditions is completely equivalent to Einstein’s theory. When degeneracy occurs, we cannot consistently impose the reality conditions, and we have a complex general relativity theory.

In my opinion this situation is satisfactory, and the problems of interpretation which are left open are best investigated looking first at particular examples.

Before discussing the reality conditions in §4, I review the use of selfdual variables and the derivation the Ashtekar action in §2, and the structure of the constraints in §3. I limit myself to the case of pure gravity; the inclusion of matter and of a cosmological term, along the lines of [8], would complicate the argument, but presents no new difficulty.

2. The Ashtekar action.

One obtains the action expressed in terms of Ashtekar variables by projecting a Palatini-like action, that depends only on the self-dual part of the connection and on the vierbein forms \( e^i = e^i_\mu dx^\mu \star \), on a space slice \( \Sigma_t \) [9].

Let us begin with a digression on selfdual tensors. Given a real antisymmetric tensor \( A^{ij} \), the complex tensor:

\[
A^{+ij} := \frac{1}{2}(A^{ij} - i\frac{1}{2} \epsilon^{ij}_{kl} A^{kl})
\]  

(1)

is selfdual, i.e. \( \epsilon^{ij}_{kl} A^{+kl} = iA^{+ij} \). A selfdual tensor has 3 (complex) independent components, which transform under the (1, 0) representation of the Lorentz group. This can be made explicit using an appropriate Clebsch-Gordon coefficient to go

* the Lorentz metric is \( \eta_{ij} = (-+++), \) the space-time metric is \( g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu, \) \( E := \det(e^i_\mu), \) \( \epsilon_{0123} = 1, \) and I have set \( 8\pi G = c = 1 \)
to a $(1,0)$ basis:

$$A^{+a} := C^a_{ij} A^{ij} = -\frac{1}{2} \epsilon_{abc} A^{bc} + i A^0 a = C^a_{ij} A^{ij}$$

with $a,b,\ldots = 1,2,3$. Lorentz transformations are represented by complex orthogonal matrices $D(1,0)(\Lambda)$, unless $\Lambda$ is an ordinary rotation, in which case $D_{ab}^{(1,0)}(\Lambda) = \Lambda_{ab}$.

It is important to stress that $A^{+a}$ contains the same information of $A^{ij}$; for example, for the electromagnetic field tensor $F^{ij}$ we have $F^{+a} = -B_a + i E_a$.

Applied to the connection form $\omega^{ij}$ this projection gives the forms $\omega^{+a}$ which, under infinitesimal local Lorentz transformations $\Lambda^i_j = \delta^i_j + \lambda^i_j + \ldots$, transform like:

$$\omega^{+a}_\mu \rightarrow \omega^{+a}_\mu - \frac{1}{2} \epsilon_{abc} \omega^{+a}_\mu \lambda^{ij} + \ldots := \omega^{+a}_\mu - D_\mu(C^a_{ij} \lambda^{ij}) + \ldots$$

A little more algebra shows that the corresponding curvature form satisfies:

$$F^{+a} := d\omega^{+a} + \frac{1}{2} \epsilon_{abc} \omega^{+a} \wedge \omega^{+c} = \frac{1}{2} (\partial_\mu \omega^{+a}_\nu - \partial_\nu \omega^{+a}_\mu + \epsilon_{abc} \omega^{+b}_\mu \omega^{+c}_\nu) dx^\mu \wedge dx^\nu = C^a_{ij} F^{ij}$$

The same arguments apply to the antiselfdual connection $\omega^{-a} = C^a_{ij} \ast \omega^{ij}$, and one finds:

$$\omega^{ij} = C^a_{ij} \omega^{+a} + C^a_{ij} \ast \omega^{-a} ; \quad F^{ij} = C^a_{ij} F^{+a} + C^a_{ij} \ast F^{-a}$$

An action that does use exclusively the self dual components has been introduced in [10][11][8]; it can be written in a variety of ways:

$$S_A := \int_\mathcal{M} \frac{1}{2} \epsilon_{ijkl} e^i \wedge e^j \wedge F^{+kl} = i \int_\mathcal{M} C^a_{ij} e^i \wedge e^j \wedge F^{+a} = \int_\mathcal{M} \frac{1}{4} \epsilon_{ijkl} e^i \wedge e^j \wedge F^{kl} + \frac{i}{2} e^i \wedge e^j \wedge F_{ij}$$

The real part of $S_A$ is the Palatini action [12], which is stationary for $\omega^{ij} = \Omega^{ij}$, the Levi–Civita connection†, for which it reproduces the Hilbert action.

---

† The Levi–Civita connection is related to the metric compatible derivative by $\Omega^{ij} = e^i_v \nabla_v e^j_\nu$, its curvature $R^{ij}_{\mu\nu}$ to the Riemann tensor by $R^{ij}_{\mu\nu\rho\sigma} = R^{ij}_{\mu\nu} e^e_{i,\rho} e^j_{j,\sigma}$. 
But, in spite of its appearance, $S_A$ is completely equivalent to the Palatini action; in fact, setting $\omega^+ = C^a_{ij} \Omega^{ij} + \phi^a$, one finds:

$$iC^a_{ij} e^i \wedge e^j \wedge F^a_{\mu} \big|_{\omega^+ = C^a_{ij} \Omega^{ij}} + \phi^a e^i \wedge e^j \wedge \phi^c + \text{id}(C^a_{ij} e^i \wedge e^j \wedge \phi^a)$$

(7)

Hence varying with respect to $\omega^+$, one finds that $S_A$ is stationary at $\omega^+ = C^a_{ij} \Omega^{ij}$, where it is real, and therefore reproduces the Hilbert action.

To develop a canonical formalism appropriate to $S_A$, let us assume that $\mathcal{M}$ is foliated into space-like 3-manifolds $\Sigma_t$, indexed by a global time function $t(x)$. Mimicking the procedure one adopts for gauge theories, one chooses the "time gauge", setting

$$e^0_\mu = -n_\mu := N \partial_\mu t$$

(8)

which still leaves the theory invariant under local $O(3)$ transformations. The metric induced on $\Sigma_t$ is given by:

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu = e^a_\mu e^a_\nu$$

(9)

The flow of time is represented by a time-like vector field $t^\mu$, such that $t^\mu \partial_\mu t = 1$, and

$$t^\mu = N n^\mu + N^\mu$$

(10)

The scalar function $N$ and the space-like vector field $N^\mu$ are the "lapse" and "shift", in the terminology of ADM[2]. In coordinates adapted to the foliation one would have:

$$t^\mu = (1, 0, 0, 0), \quad n_\mu = (-1, 0, 0, 0), \quad e^{0\mu} = (-\frac{1}{N}, \frac{N^\alpha}{N}), \quad e^{a\mu} = (0, e^{a\alpha}), \quad E = Ne$$

with $e = \det(e^a_\alpha)$. Replacing eq.s(8)(10) in eq.(6) one finds:

$$S_A = \int dt \int_{\Sigma_t} d^3x(-\frac{1}{2} N e_{abc} e^{a\mu} e^{b\nu} F^{c+}_{\mu\nu} + i N^\mu e^{a\nu} F^{+a}_{\mu\nu} - i e^\mu e^{a\nu} F^{+a}_{\mu\nu})$$

(11)

We now introduce the Ashtekar variables $A^a_\mu$ and $\tilde{e}^{a\mu}$, which will play the role of
"configuration" and "momentum" variables in the scheme:

\[ A^a_\mu := q^\nu \omega^{+a}_\nu; \quad \tilde{e}^{a\mu} := e e^{a\mu} \]  

(12)

\( \tilde{e}^{a\mu} \) is a vector density, which, because of the gauge choice, is tangent to \( \Sigma \), i.e. \( \tilde{e}^{a\mu} = q^{\mu} e^{a\nu} \), just like \( N^\mu \); the (complex) Ashtekar connection \( A^a_\mu \) is a gauge field on the 3-space \( \Sigma_t \). Under an infinitesimal local rotation, with \( \lambda^{ab} = \epsilon_{abc} \lambda^c \), the transformation laws will be:

\[ \tilde{e}^{a\mu} \rightarrow \tilde{e}^{a\mu} + \epsilon_{abc} \tilde{e}^{b\mu} \lambda^c + \ldots; \quad A^a_\mu \rightarrow A^a_\mu + 3D_\mu \lambda_a + \ldots \]  

(13)

Covariant 3-derivatives will be well defined for objects which belong to \((j, 0)\) representations of \( O(3,1) \), e.g.:

\[ 3D_\mu S_a = q^\nu \partial_\nu S_a + \epsilon_{abc} A^b_\mu S_c \]  

(14)

where \( S_a \) belongs to the \((1, 0)^*\), but also, since for rotations \( D_{ab}^{(1,0)}(\Lambda) = \Lambda_{ab} \), if by \( S_a \) we mean the 1, 2, 3 components of an \( S_i \) belonging to the \((\frac{1}{2}, \frac{1}{2})\) representation. It is because of this unusual situation that we loose control on the reality of \( \tilde{e}^{a\mu} \), and we have to impose reality conditions.

With the help of eq.(8) (more properly, of Frobenius theorem) one can show that the curvature of \( A^a_\mu \) is the pull back to \( \Sigma_t \) of \( F^{+a}_{\mu\nu} \):

\[ 3F^{+a}_{\mu\nu} := q^\rho q^\sigma (\partial_\nu A^a_\sigma - \partial_\sigma A^a_\nu + \epsilon_{abc} A^b_\rho A^c_\sigma) = q^\rho q^\sigma F^{+a}_{\rho\sigma} \]  

(15)

\* for a right-handed Dirac spinor \( \psi_R = \frac{1}{2}(1 + \gamma_5)\psi \), which belongs to the \((\frac{1}{2}, 0)\) representation, we would have

\[ q^{\nu} D_\nu \psi_R := q^{\nu}(\partial_\nu + \frac{1}{2} i \epsilon^{ijk} \Sigma_{ij}) \psi_R = (q^{\nu} \partial_\nu + \frac{1}{2} i C^a_{ij} \Sigma^{ij} A^a_\nu) \psi_R \]

because \( \Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j] \) and \( \frac{1}{2} i \epsilon_{ijkl} \Sigma^{kl} = i \gamma_5 \Sigma_{ij} \). This is why the Ashtekar variables were first introduced in a spinor representation.
and that:

\[ t^\mu e^{\alpha \nu} F_{\mu \nu}^{+a} = e^{\alpha \nu} \mathcal{L}_t \omega_\nu^{+a} - e^{\alpha \nu} [\partial_\nu (t^\mu \omega_\mu^{+a}) + \epsilon_{abc} A_b^\nu (t^\mu \omega_\mu^{+c})] = e^{\alpha \nu} \mathcal{L}_t A_\nu^a - e^{\alpha \nu} 3 D_\nu \omega_\nu^{+a} \]  

(16)

where \( \mathcal{L}_t \) indicates the Lie derivative with respect to \( t^\mu \), \( \omega_\nu^{+a} := t^\mu \omega_\mu^{+c} \). Putting these ingredients together, defining \( N := N/e \), and with a final partial integration, we may write the action (11) as a functional of quantities defined on 3-space:

\[ S_A = \int dt \int d^3 x \left\{ -i \tilde{e}_a^\mu \mathcal{L}_t A_\mu^a + i N^\mu \tilde{e}_a^{\alpha \nu} 3 F_{\mu \nu}^a - \frac{1}{2} N \epsilon_{abc} \tilde{e}_a^{\alpha \mu} \tilde{e}_b^{\beta \nu} 3 F_{\mu \nu}^c - i \omega_\nu^{+a} 3 D_\nu e^{\alpha \mu} \right\} \]  

(17)

This remarkable expression is the main point of the whole approach; one immediately notices that, as we anticipated, it is polynomial in the basic variables and their derivatives. Of course, it is complex, which presumably makes it useless for path integral quantization, and which might spoil the whole approach if we let spurious solutions creep in.

Notice that that all the indices that appear in (17) are space like, as will all indices from now on. So, if one interprets them as component indices, they range from 1 to 3.

3. The constraints of the theory.

The most plausible interpretation of the action (17) is that it refers to a phase space described by two sets of complex variables, with the basic Poisson bracket:

\[ \{ \tilde{e}_a^\mu (x), A_b^\nu (y) \} = i \delta^{ab} \delta^{\mu \nu} \delta^{(3)} (x - y) \]  

(18)

There are also Lagrange multipliers \( N, N^\mu \) (real), \( \omega_\nu^{+a} \) (complex); if we vary the action with respect to them we find:

\[ (i) \quad \tilde{H} := \epsilon_{abc} \tilde{e}_a^{\alpha \mu} \tilde{e}_b^{\beta \nu} 3 F_{\mu \nu}^c = 0 \]

\[ (ii) \quad \tilde{F}_{\mu} := i \tilde{e}_a^{\alpha \nu} 3 F_{\mu \nu}^a = 0 \]  

(19)

\[ (iii) \quad \tilde{G}_a := 3 D_\mu e^{a \mu} = 0 \]
respectively the "scalar", the "vector", and the "Gauss law" constraint. The constraints reflect the local invariance of the theory, and are all first class, i.e. their Poisson brackets turn out to be linear combinations of themselves. To see this we shall follow the procedure of ref.[8], using complex test functions \( \lambda_a(x) \), a real test 3-vector \( f^\mu(x) \), and a real test density \( n(x) \) to smear the constraints:

\[
G_{\lambda} := i \int_\Sigma d^3x \lambda_a 3D_\mu \tilde{e}^{a\mu} \\
F_f := i \int_\Sigma d^3x f^\mu (\tilde{c}^{a\nu} 3F^{a}_{\nu\mu} + A^a_\mu 3D_\nu \tilde{c}^{a\nu}) \\
H_n := \frac{1}{2} \int_\Sigma d^3x \epsilon_{abc} \tilde{e}^{b\mu} \tilde{e}^{c\mu}
\]

(20)

In this notation we may write (17) in the form:

\[
S_A = \int dt \int_{\Sigma_t} (-i \tilde{e}^{a\mu} L_t A^a_{\mu} - \mathcal{H}) ; \quad \mathcal{H} := F_N + H_N + G_\omega , \text{ with } \omega^a := \omega^{+a} - N^\mu A^a_\mu
\]

(21)

For the Poisson bracket with the canonical variables we use eq.(18) to find:

\[
\{G_{\lambda}, \tilde{e}^{a\mu}\} = \epsilon_{abc} \tilde{e}^{b\mu} \lambda_c ; \quad \{G_{\lambda}, A^a_\mu\} = 3D_\mu \lambda_a \\
\{F_f, \tilde{e}^{a\mu}\} = f^\nu \partial_\nu \tilde{e}^{b\mu} - \tilde{c}^{a\nu} \partial_\nu f^\mu + \tilde{e}^{a\mu} \partial_\nu f^\nu = L_f \tilde{e}^{a\mu} \\
\{F_f, A^a_\mu\} = f^\nu \partial_\nu A^a_\mu + A^a_\nu \partial_\mu f^\nu = L_f A^a_\mu \\
\{H_n, \tilde{e}^{a\mu}\} = i 3D_\lambda (\eta \epsilon_{abc} \tilde{e}^{b\lambda} \tilde{e}^{c\mu}) ; \quad \{H_n, A^a_\mu\} = i \eta \epsilon_{abc} \tilde{e}^{b\nu} 3F^c_{\mu\nu}
\]

(22) (23) (24)

Comparing eq.(22) with eq.(13), we see that the smeared Gauss law is the generator of gauge rotations; and from eq.(23), that the peculiar combination of the vector and the Gauss law constraints was chosen because it generates diffeomorphisms on \( \Sigma \). On the contrary the scalar constraint, which is related to the arbitrariness of
the time parameter $t$, does not have an obvious geometric meaning in 3-space. With these relations we may derive the equations of motion:

$$\mathcal{L}_t A^a_\mu = \{\mathcal{H}, A^a_\mu\} = N^\nu 3F^a_{\nu \mu} + i N \epsilon_{abc} \tilde{e}^{b\nu} 3F^c_{\mu \nu} + 3D_\mu \omega^a_t$$

$$\mathcal{L}_t \tilde{e}^{a\mu} = \{\mathcal{H}, \tilde{e}^{a\mu}\} - 3D_\nu (N^\mu \tilde{e}^{a\nu} - N^\nu \tilde{e}^{a\mu} + i N \epsilon_{abc} \tilde{e}^{b\mu} \tilde{e}^{c\nu}) - \epsilon_{abc} \omega^a_t \tilde{e}^{c\mu}$$

We can also calculate all the Poisson brackets between constraints, and check that they are indeed all first class. The only one that requires some algebraic effort is the scalar-scalar one, which gives:

$$\{H_n, H_m\} = i \int_\Sigma d^3x (m_n \partial_\lambda m_m - m_m \partial_\lambda m_n) \tilde{e}^{b\nu} \tilde{e}^{b\lambda} \epsilon_{a\mu} 3F^{a\mu}_{\nu \mu}$$

This relation shows that the constraints do form an algebra under Poisson brackets, but the "structure constants" of this algebra depend on the canonical variables.

**4. The reality conditions.**

So far we have really been dealing with complex general relativity, and we now have to face the problem of identifying a real section. The variable $A^a_\mu$ is complex, but we must demand that $\tilde{e}^{a\mu}$ be real; or rather, that if they are real initially, they remain so. Or we may be more tolerant, and let them wander off, choosing as reality condition:

$$(iv) \quad \tilde{Q}^{\mu \nu} := \tilde{e}^{a\mu} \tilde{e}^{a\nu} = \text{real}$$

$\tilde{Q}^{\mu \nu}$ is a gauge invariant tensor density of degree 2, related to the inverse 3-metric by $\tilde{Q}^{\mu \nu} = e^2 q^{\mu \nu}$. The point is that the equations of motion (25) are completely gauge-dependent, i.e. they depend on arbitrary Lagrange multipliers, so the motion might involve a complex gauge rotation, which in itself is harmless, since only gauge invariant quantities like $\tilde{Q}^{\mu \nu}$ matter.
A reality condition limits the possible initial values, but cannot be treated like the other constraints we have met, which descend from the action. Its consistency can be decided by checking that, once imposed on the initial data, it remains valid through the evolution of the system. By eq. (25), the time derivative of $\tilde{Q}^{\mu\nu}$ is:

$$
\mathcal{L}_t \tilde{Q}^{\mu\nu} = \mathcal{L}_N \tilde{Q}^{\mu\nu} - (\tilde{e}^{\alpha\mu} N^\nu + \tilde{e}^{\alpha\nu} N^\mu) 3 D_\lambda \tilde{e}^{\alpha\lambda} - i N \epsilon_{abc} \tilde{e}^{\alpha\lambda} (\tilde{e}^{\alpha\mu} 3 D_\lambda \tilde{e}^{b\nu} - \tilde{e}^{b\nu} 3 D_\lambda \tilde{e}^{a\mu})
$$

(28)

therefore we have to impose a "secondary" reality condition:

$$(v) \quad \tilde{P}^{\mu\nu} := i \epsilon_{abc} \tilde{e}^{\alpha\lambda} (\tilde{e}^{\alpha\mu} 3 D_\lambda \tilde{e}^{b\nu} - \tilde{e}^{b\nu} 3 D_\lambda \tilde{e}^{a\mu}) = \text{real}$$

(29)

Conditions (iv) and (v) together define the real section on the constraint surface in phase space. A useful check is to count the number of (real) degrees of freedom. Since $\tilde{e}^{\alpha\mu}$ and $A_a^\mu$ are in general complex, there are $2 \cdot 2 \cdot 3 \cdot 3 = 36$ real dynamical variables. We have $2 \cdot 3$ real constraints from (iii), and $1 + 3$ from (i) and (ii), if (as we expect) $\tilde{H}$ and $\tilde{F}_\mu$ are real, for a total of 10, and $6 + 6$ reality conditions. So:

$$2 \cdot \text{n. of degrees of freedom} = 36 - 2 \cdot 10 - 12 = 4$$

as it should be.

However we do have to prove that $\tilde{H}$ and $\tilde{F}_\mu$ are real, and that there are no "tertiary" reality conditions, namely that, once (iv) and (v) are imposed on the initial data, they remain valid through the evolution of the system.

As far as I can see neither proof goes through unless one assumes non degeneracy, namely:

$$(*) \quad \tilde{Q}^{\mu\nu} = \text{positive definite}$$

(30)

From this further assumption it follows that a complex $O(3)$ gauge transformation exists that makes $\tilde{e}^{\alpha\mu}$ real, with $e^2 = \det(\tilde{e}^{\alpha\mu}) > 0$. We can then reconstruct the local geometry of $\Sigma$, i.e. calculate $e_a^\mu$ from $\tilde{e}^{\alpha\mu}$ and the Ashtekar connection in
terms of $e^a_{\mu}$, $\tilde{G}^a$ and $\tilde{P}^{\mu\nu}$. In fact, using the definitions of $\tilde{G}^a$ and of $\tilde{P}^{\mu\nu}$, we find, after some algebra:

$$A^a_{\mu} = \frac{1}{2} \epsilon_{abc} e^{b\nu} (\partial_{\nu} e^c_{\mu} - \partial_{\mu} e^c_{\nu}) - e^{c\lambda} e^d_{\mu} e^d_{\lambda} + \frac{1}{2e} \epsilon_{abc} e^b_{\mu} \tilde{G}^c + i \frac{1}{4e^2} (2e^a q_{\mu\rho} - e^a q_{\lambda\rho}) \tilde{P}^{\lambda\rho}$$

(31)

This is a very useful equation; from it we see that, if the conditions (iii), (iv), (v) and (*) hold, the Ashtekar connection can be written in the form:

$$A^a_{\mu} = -\frac{1}{2} \epsilon_{abc} q^{\nu}_{\mu} \Omega^{bc}_{\nu} + i e^{a\nu} K_{\mu\nu}$$

(32)

where $\Omega^{ab}_{\mu}$ are components of the Levi-Civita connection, and the real, symmetric tensor

$$K_{\mu\nu} := \frac{1}{4e^2} (2q_{\mu\rho} q_{\nu\sigma} - q_{\mu\nu} q_{\rho\sigma}) \tilde{P}^{\rho\sigma}$$

(33)

can be interpreted as the extrinsic curvature of $\Sigma_t$, which is defined to be $\frac{1}{2} \mathcal{L}_n q_{\mu\nu}$. Eq.(32) is in fact the expression one gets if one calculates $A^a_{\mu}$ projecting the Levi-Civita connection, and is called the "Sen connection" [13].

From eq.(31) one can also prove directly that $\tilde{H}$ and $\tilde{F}_{\mu}$ are real, but it is more instructive to take for $A^a_{\mu}$ the Sen form $A^a_{\mu} = q_{\mu\nu} X_{ij}^{a} \Omega^{ij}_{\nu}$, for which one obtains [13][14]:

$$i e^{a\mu} F^a_{\mu\nu} = -e q^\rho_{\nu} n^\sigma (R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R)$$

$$\epsilon_{abc} e^{a\mu} e^{b\nu} F^c_{\mu\nu} = -2e^2 n^\mu n^\nu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$$

(34)

Thus, provided (iii), (iv) and (v) and (*) hold, the vector and the scalar constraints are just the constraint part of the Einstein equations.

However to assure the consistency of the scheme we still need to prove that the time derivative of $\tilde{P}^{\mu\nu}$ is real; which means that we have to express it in terms of objects which we know are real on the real section of the constraints surface. This turns out to be a fairly hard task, because from eq.(25) we have, with a few
simplifications:

\[ \mathcal{L}_t \tilde{P}^{\mu
u} = \{ \mathcal{H}, \tilde{P}^{\mu
u} \} = \mathcal{L}_N \tilde{P}^{\mu
u} - \int d^3x N^a \tilde{G}^a \frac{\delta \tilde{P}^{\mu\nu}}{\delta e^a_\rho} - 2 \, N \tilde{Q}^{\mu\nu} \tilde{H} - \]

\[ - 2 \epsilon_{abc} \tilde{e}^a(\mu 3D_\lambda \tilde{e}^{b\nu}) 3D_\rho (N \epsilon_{abc} \tilde{e}^{b\rho} \tilde{e}^{c\lambda}) + 2 \epsilon_{abc} \tilde{e}^{a\rho} 3D_\lambda \tilde{e}^{b\nu} 3D_\rho (N \epsilon_{abc} \tilde{e}^{b\lambda} \tilde{e}^{c\rho}) - \]

\[ - 2 \left[ 3D_\lambda 3D_\rho (N \epsilon_{abc} \tilde{e}^{b\nu} \tilde{e}^{c\lambda}) \tilde{e}_{ab'} \epsilon^{b'\lambda} \tilde{e}^{c'\rho} \right] + 2 \, N \tilde{Q}^{\lambda(\nu} \epsilon_{abc} \tilde{e}^{a\mu)} \tilde{e}^{b\rho} 3F_\lambda^c \]

(35)

in which the first three terms on the R.H.S. are real or zero, but little can be said about the rest; we cannot even be sure the expression is diffeomorphism invariant, written in this way. However if the metric is non degenerate we can substitute eq.(31) in it, and after a very long calculation obtain

\[ \mathcal{L}_t \tilde{P}^{\mu
u} = \mathcal{L}_N \tilde{P}^{\mu
u} - \int d^3x N^a \tilde{G}^a \frac{\delta \tilde{P}^{\mu
u}}{\delta e^a_\rho} - 2 \, N \tilde{Q}^{\mu\nu} \tilde{H} - \]

\[ - 2(\tilde{Q}^{\mu\nu} \tilde{Q}^{\lambda\rho} - \tilde{Q}^{\mu\rho} \tilde{Q}^{\lambda\nu}) \tilde{\nabla}_\lambda \tilde{\nabla}_\rho N + 2 \tilde{Q}^{\rho(\mu} \tilde{e}^{b\nu)} \tilde{G}^b \tilde{\nabla}_\rho N - 2 \, N \tilde{Q}^{\mu\nu} \tilde{\nabla}_\lambda (\tilde{e}^{b\lambda} \tilde{G}^b) + \]

\[ + \frac{1}{2} \, N (\tilde{Q}^{\mu\nu} \delta_{ab} - \tilde{e}^{a\mu} \tilde{e}^{b\nu}) \tilde{G}^a \tilde{G}^b - i \, N \epsilon_{abc} \tilde{P}^{\lambda(\nu} \epsilon^{a\mu)} \tilde{G}^b \tilde{e}^{c\lambda} - 2 \, N \epsilon^2 \tilde{Q}^{\rho(\mu} 3R_\lambda^{\nu)} + \]

\[ + \frac{N}{2\epsilon^2} q^{\mu\nu}(q_{\lambda \nu'} q_{\rho \rho'} - \frac{1}{2} q_{\rho \lambda'} q_{\lambda \nu'}) \tilde{P}^{\lambda \rho'} \tilde{P}^{\rho'} + \frac{N}{\epsilon^2} q_{\lambda \rho} \tilde{P}^{\mu \lambda} \tilde{P}^{\nu \rho} \]

(36)

Here \( \tilde{\nabla}_\mu \) is the \( q_{\mu\nu} \)-compatible derivative on \( \Sigma_t \), \( 3R_{\mu\nu} \) the corresponding Ricci tensor. This unattractive expression has the merit of being manifestly diffeomorphism invariant, and of displaying explicitly that \( \mathcal{L}_t \tilde{P}^{\mu\nu} \) is indeed real if the conditions (i)...(v) and (*) are satisfied.

Thus, as long as \( \tilde{Q}^{\mu\nu} \) is positive definite, the theory is completely equivalent to Einstein’s theory. When \( \tilde{Q}^{\mu\nu} \) is not positive definite, we cannot, in general, impose consistently the reality conditions, and we have to make do with the complex theory.

Again one may count the degrees of freedom: there is no reason now to expect \( \tilde{H} \) and \( \tilde{F}_\mu \) to be real, so for 18 complex dynamical variables we have 3 + 1 + 3
complex constraints, and therefore 2 complex (4 real) degrees of freedom, and no sensible interpretation of the $\tilde{e}^{a\mu}$.

It is a pleasure to thank Abhay Ashtekar and Carlo Rovelli for some very enlightening conversations.

REFERENCES

1. A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244; Phys. Rev. D36 (1986) 1587; *New perspectives in canonical gravity* (with invited contributions), Bibliopolis, Napoli 1988.

2. R. Arnowitt, S. Deser, C.W. Misner, in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley 1962).

3. C. Rovelli, Class. and Quantum Gravity

4. R. D’Auria, T. Regge, Nucl. Phys. B195 (1982) 308

5. E. Witten, Nucl. Phys. B311 (1988) 46

6. G. T. Horowitz, Class. and Quantum Gravity 8 (1991) 587

7. I. Bengtsson, Class. and Quantum Gravity 8 (1991) 1847

8. A. Ashtekar, J. Romano, R. Tate, Phys. Rev. D40 (1989) 2572.

9. A. Ashtekar, A.P. Balachandran, S. Jo, Int.Jour. of Mod.Phys. A4 (1989) 1493

10. J. Samuel, Pramana J. of Phys. 28 (1987) L429.

11. T. Jacobson, L. Smolin, Class. Quantum Grav. 5 (1988) 583.

12. A. Palatini, Rend. Circ. Mat. Palermo 43 (1919) 203

13. A. Sen, J. Math. Phys. 22 (1981) 1718; Phys. Lett. 119B (1982) 89.

14. E. Witten, Commun. Math. Phys. 80 (1981) 725