Solution to the Stieltjes moment problem
in Gelfand–Shilov spaces

by

Andreas Debrouwere (Gent)

Abstract. We characterize the surjectivity and the existence of a continuous linear
right inverse of the Stieltjes moment mapping on Gelfand–Shilov spaces, both of Beurling
and Roumieu type, in terms of their defining weight sequence. As a corollary, we obtain
some new results about the Borel–Ritt problem in spaces of ultraholomorphic functions
on the upper half-plane.

1. Introduction. In 1939, Boas [1] and Pólya [17] independently showed
that for every sequence \((a_p)_{p \in \mathbb{N}}\) of complex numbers there is a function \(F\) of
bounded variation such that
\[
\int_0^\infty x^p \, dF(x) = a_p, \quad p \in \mathbb{N}.
\]
A. J. Durán [7] (see also [8]) improved this result in 1989 by showing that
for every sequence \((a_p)_{p \in \mathbb{N}}\) of complex numbers the infinite system of linear
equations
\[
(1.1) \quad \int_0^\infty x^p \varphi(x) \, dx = a_p, \quad p \in \mathbb{N},
\]
ads a solution \(\varphi \in S(0, \infty) = \) the space of rapidly decreasing smooth
functions with support in \([0, \infty)\). Over the past 20 years, various authors
studied the (unrestricted) Stieltjes moment problem \((1.1)\) in the context of
Gelfand–Shilov spaces [10] (see [4, 3, 13, 14, 5]). In this article, we provide
a complete solution to this problem.

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In order to be able to discuss our results, we need to introduce some notation; see Section 2 for unexplained notions concerning weight sequences. Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence. We define $S_{(M_p)}(0, \infty)$ as the space consisting of all $\varphi \in S(0, \infty)$ such that

$$\sup_{p \in \mathbb{N}} \sup_{x \geq 0} \frac{h^p x^p |\varphi^{(n)}(x)|}{M_p} < \infty$$

for all $h > 0$ and $n \in \mathbb{N}$. Similarly, we define $S_{\{M_p\}}(0, \infty)$ as the space consisting of all $\varphi \in S(0, \infty)$ such that there is $h > 0$ for which (1.2) holds for all $n \in \mathbb{N}$. $S_{(M_p)}(0, \infty)$ and $S_{\{M_p\}}(0, \infty)$ are endowed with their natural Fréchet space and $(LF)$-space topology, respectively. Next, we define $\Lambda_{(M_p)}$ and $\Lambda_{\{M_p\}}$ as the sequence spaces consisting of all $(a_p)_{p \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{p \in \mathbb{N}} \frac{h^p |a_p|}{M_p} < \infty$$

for all $h > 0$ and some $h > 0$, respectively. $\Lambda_{(M_p)}$ and $\Lambda_{\{M_p\}}$ are endowed with their natural Fréchet space and $(LB)$-space topology, respectively. If $(M_p)_{p \in \mathbb{N}}$ satisfies (dc), the Stieltjes moment mapping

$$M : S_*(0, \infty) \to \Lambda_*, \quad \varphi \mapsto \left( \int_0^\infty x^p \varphi(x) \, dx \right)_{p \in \mathbb{N}},$$

is well-defined and continuous, where $*$ stands for either $(M_p)$ or $\{M_p\}$. Jiménez-Garrido, Sanz and the author [5] characterized the surjectivity of $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ in the following way (see [14] for earlier work in this direction).

**Theorem 1.1 ([5], Thm. 3.5]).** Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying (slc) (= $(M_p/p!)_{p \in \mathbb{N}}$ satisfies (lc)) and (dc). If $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective, then $(M_p)_{p \in \mathbb{N}}$ satisfies

$$(\gamma_2) \quad \sup_{p \in \mathbb{Z}^+_+} \frac{(M_p/M_{p-1})^{1/2}}{p} \sum_{q=p}^\infty \frac{1}{(M_q/M_{q-1})^{1/2}} < \infty.$$  

If in addition $(M_p)_{p \in \mathbb{N}}$ satisfies (mg), then $\gamma_2$ implies that $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective.

Condition $(\gamma_2)$ means that $(M_p^{1/2})_{p \in \mathbb{N}}$ is strongly non-quasianalytic [12]. The main goal of this article is to improve and complete Theorem 1.1 in the following three ways: consider the Beurling case as well; replace (slc) and (mg) by the weaker conditions (lc) and (dc); characterize the existence of a continuous linear right inverse of $M : S_*(0, \infty) \to \Lambda_*$. More precisely, we show the following result (see Theorem 6.1).
Theorem 1.2. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc) and (dc).

(a) The following statements are equivalent:

(i) \(M : S_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}(0, \infty)\) is surjective.

(ii) \(M : S_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}(0, \infty)\) has a continuous linear right inverse.

(iii) \((M_p)_{p \in \mathbb{N}}\) satisfies \((\gamma_2)\).

(b) \(M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) is surjective if and only if \((M_p)_{p \in \mathbb{N}}\) satisfies \((\gamma_2)\).

(c) \(M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) has a continuous linear right inverse if and only if \((M_p)_{p \in \mathbb{N}}\) satisfies

\[ (\beta_2) \quad \forall \varepsilon > 0 \ \exists n \in \mathbb{N}, \ n > 1 : \limsup_{p \to \infty} \left( \frac{M_{np}}{M_p} \right)^{\frac{1}{n-1}} \frac{M_{np-1}}{M_{np}} \leq \varepsilon. \]

Condition \((\beta_2)\) is due to Petzsche [16] and appears in his characterization of the existence of a continuous linear right inverse of the Borel mapping on spaces of ultradifferentiable functions of Roumieu type. We also give an analogue of Theorem 1.2 for Gelfand–Shilov spaces of type \(S^\dagger(0, \infty)\) (cf. [5, Thm. 3.5]); see Theorem 7.2.

In [5], Theorem 1.1 is shown by reducing it to the Borel–Ritt problem [18, 19, 20, 11] in spaces of ultraholomorphic functions on the upper half-plane and then using solutions to this problem from [20, 11], a technique that goes back to A. L. Durán and Estrada [8]. Up until now, this seems to be the only known method to study the Stieltjes moment problem in Gelfand–Shilov spaces. It also explains why we had to assume \((slc)\) and \((mg)\) in Theorem 1.1; these conditions are needed to solve the Borel–Ritt problem in spaces of ultraholomorphic functions [20, Thm. 3.2.1]. We develop here a completely new approach: we show Theorem 1.2 by reducing it to the Borel problem in spaces of ultradifferentiable functions of class \((N_p)_{p \in \mathbb{N}}\), where \((N_p)_{p \in \mathbb{N}}\) denotes the 2-interpolating sequence associated to \((M_p)_{p \in \mathbb{N}}\) [19], and then using Petzsche’s classical solution to this problem [16].

As a corollary, we obtain an analogue of Theorem 1.2 for the Borel–Ritt problem in spaces of ultraholomorphic functions on the upper half-plane, thereby improving some results of Schmets and Valdivia [19] and Thilliez [20] in the particular case of the upper half-plane (see Theorem 7.4 and Remark 7.5). Of course, due to the distinct geometry of the upper half-plane, this special case is much simpler to handle than the Borel–Ritt problem in spaces of ultraholomorphic functions on general sectors.

The plan of this article is as follows. In Section 2, we fix the notation, introduce weight sequences and recall Petzsche’s solution to the Borel problem in spaces of ultradifferentiable functions. In Section 3, we define Gelfand–Shilov spaces of type \(S^\dagger\) and collect several properties of these spaces that will be used later on. Next, in the auxiliary Sections 4 and 5, we present an
abstract result about the existence of a continuous linear right inverse and prove a Borel type theorem. These results are used in the proof of Theorem 1, which is given in Section 6. Finally, in Section 7 we consider the Stieltjes moment problem in Gelfand–Shilov spaces of type $S^*(0, \infty)$ and the Borel–Ritt problem in spaces of ultraholomorphic functions on the upper half-plane.

2. Preliminaries

2.1. Notation. We set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_+ = \{1, 2, \ldots\}$. The Fréchet space of rapidly decreasing smooth functions on $\mathbb{R}$ is denoted by $S(\mathbb{R})$. We fix the constants in the Fourier transform as follows

$$F(\varphi)(\xi) := \hat{\varphi}(\xi) := \int_{-\infty}^{\infty} \varphi(x)e^{ix\xi}dx, \quad \varphi \in S(\mathbb{R}).$$

The $p$th moment, $p \in \mathbb{N}$, of an element $\varphi \in S(\mathbb{R})$ is given by

$$\mu_p(\varphi) := \int_{-\infty}^{\infty} x^p \varphi(x)dx.$$

Notice that $\hat{\varphi}^{(p)}(0) = i^p \mu_p(\varphi)$ for all $p \in \mathbb{N}$.

We define the Borel mapping as

$$B : C^\infty(\mathbb{R}) \to \mathbb{C}^N, \quad \varphi \mapsto (\varphi^{(p)}(0))_{p \in \mathbb{N}},$$

and the Stieltjes moment mapping as

$$M : S(\mathbb{R}) \to \mathbb{C}^N, \quad \varphi \mapsto (\mu_p(\varphi))_{p \in \mathbb{N}}.$$

We will also use the notation $B$ and $M$ for the restriction of the Borel mapping and the Stieltjes moment mapping to various subspaces of $C^\infty(\mathbb{R})$ and $S(\mathbb{R})$, respectively. To avoid confusion, we will always clearly state the domain and range of these mappings.

A lcHs (= locally convex Hausdorff space) $E$ is said to be an $(LF)$-space if there is a sequence $(E_n)_{n \in \mathbb{N}}$ of Fréchet spaces with $E_n \subseteq E_{n+1}$ and continuous inclusion mappings for all $n \in \mathbb{N}$ such that $E = \bigcup_{n \in \mathbb{N}} E_n$ and the topology of $E$ coincides with the finest locally convex topology such that all the inclusion mappings $E_n \to E, \ n \in \mathbb{N}$, are continuous. We write $E = \lim_{\rightarrow} E_n$. If the sequence $(E_n)_{n \in \mathbb{N}}$ consists of Banach spaces, $E$ is called an $(LB)$-space. Finally, a lcHs is said to be a $(PLB)$-space if it can be written as the projective limit of a countable spectrum of $(LB)$-spaces.

2.2. Weight sequences. A sequence $(M_p)_{p \in \mathbb{N}}$ of positive numbers is called a weight sequence if $M_0 = 1$ and $m_p := M_p/M_{p-1} \to \infty$ as $p \to \infty$. The associated function of a weight sequence $(M_p)_{p \in \mathbb{N}}$ is defined as $M(0) := 0$.
and

\[ M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p}{M_p}, \quad t > 0. \]

We will make use of the following conditions on weight sequences:

(lc) **Log-convexity:**

\[ M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{Z}_+. \]

(dc) **Derivation-closedness:** There are \( C_0, H \geq 1 \) such that

\[ M_{p+1} \leq C_0 H^{p+1} M_p, \quad p \in \mathbb{N}. \]

(mg) **Moderate growth:** There are \( C_0, H \geq 1 \) such that

\[ M_{p+q} \leq C_0 H^{p+q} M_p M_q, \quad p, q \in \mathbb{N}. \]

(\( \gamma \)) **Non-quasianalyticity:**

\[ \sum_{p=1}^{\infty} \frac{1}{m_p} < \infty. \]

(\( \gamma_1 \)) **Strong non-quasianalyticity:**

\[ \sup_{p \in \mathbb{Z}_+} \frac{m_p}{p} \sum_{q=p}^{\infty} \frac{1}{m_q} < \infty. \]

(\( \gamma_2 \))

\[ \sup_{p \in \mathbb{Z}_+} \frac{m_p^{1/2}}{p} \sum_{q=p}^{\infty} \frac{1}{m_q^{1/2}} < \infty. \]

(\( \beta_2 \))

\[ \forall \varepsilon > 0 \ \exists n \in \mathbb{N}, \ n > 1 : \ \limsup_{p \to \infty} \left( \frac{M_{np}}{M_p} \right)^{p(n-1)} 1 \frac{1}{m_{np}} \leq \varepsilon. \]

Clearly, (mg)\( \Rightarrow \) (dc) and (\( \gamma_1 \))\( \Rightarrow \) (\( \gamma \)). The conditions (lc), (dc), (mg), (\( \gamma \)) and (\( \gamma_1 \)) are standard in the theory of ultradifferentiable functions and their meaning is well explained in the classical work of Komatsu \[12\]. Conditions (\( \gamma_1 \)) and (\( \gamma_2 \)) are particular instances of

(\( \gamma_r \))

\[ \sup_{p \in \mathbb{Z}_+} \frac{m_p^{1/r}}{p} \sum_{q=p}^{\infty} \frac{1}{m_q^{1/r}} < \infty, \quad r > 0. \]

Condition (\( \gamma_r \)) means that \((M_p^{1/r})_{p \in \mathbb{N}}\) satisfies (\( \gamma_1 \)). If \((M_p)_{p \in \mathbb{N}}\) satisfies (lc), then (\( \gamma_r \))\( \Rightarrow \) (\( \gamma_s \)) for all \( r > s > 0 \). These conditions, which were introduced by Schmets and Valdivia \[19\] for \( r \in \mathbb{N} \) and by Thilliez \[20\] for arbitrary \( r > 0 \), play an important role in the study of the Borel–Ritt problem in spaces of ultraholomorphic functions \[18, 19, 20, 11\]. Condition (\( \beta_2 \)) is due to Petzsche \[16\] and appears in his characterization of the existence of a continuous linear right inverse of the Borel mapping in spaces of ultradifferentiable functions of Roumieu type (see Theorem 2.4 below). Finally, we
warn the reader that in [16] the notation \((\gamma_2)\) is used for a condition different from the one here. However, since we will never use the condition denoted by \((\gamma_2)\) in [16], this should not cause any confusion.

**Remark 2.1.** Consider the following conditions:

\[(\beta_2^0) \quad \exists n \in \mathbb{N}, n > 1 : \lim_{p \to \infty} \frac{m_{np}}{m_p} = \infty.\]

\[(\beta_2^1) \quad \lim_{p \to \infty} \frac{M_p^{1/p}}{m_p} = 0.\]

Petzsche has shown that \((\beta_2^0) \Rightarrow (\beta_2) \Rightarrow (\beta_2^1)\) [16, Prop. 1.5(b) and Prop. 1.6(a)] and that the converse implications are false in general [16, Example 1.8]. However, \((\beta_2^0)\) and \((\beta_2)\) are equivalent within the class of weight sequences \((M_p)_{p \in \mathbb{N}}\) satisfying the following mild regularity condition: There is \(n \in \mathbb{Z}_+\) such that the set of finite limit points of the set \(\{m_n^l/m_{n^l-1} | l \in \mathbb{Z}_+\}\) is bounded [16, Prop. 1.6(b)].

**Example 2.2.** (a) The Gevrey sequence \((p!^\alpha)_{p \in \mathbb{N}}, \alpha > 0\) satisfies (lc) and (mg); it satisfies \((\gamma_r)\) if and only if \(\alpha > r\); it does not satisfy \((\beta_2^1)\), and thus not \((\beta_2)\) either.

(b) The \(q\)-Gevrey sequence \((q^r)_{p \in \mathbb{N}}, q > 1\) satisfies (lc) and (dc) but not (mg); it satisfies \((\gamma_r) > 0\) for all \(r > 0\); it satisfies \((\beta_2^0)\), and thus also \((\beta_2)\).

Following [19], we define the 2-interpolating sequence \((N_p)_{p \in \mathbb{N}}\) associated to a weight sequence \((M_p)_{p \in \mathbb{N}}\) as

\[N_p := \begin{cases} M_q, & p = 2q, q \in \mathbb{N}, \\ (M_q M_{q+1})^{1/2}, & p = 2q + 1, q \in \mathbb{N}. \end{cases}\]

**Lemma 2.3.** Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc). Denote by \((N_p)_{p \in \mathbb{N}}\) its 2-interpolating sequence. Then \((N_p)_{p \in \mathbb{N}}\) is a weight sequence satisfying (lc). Moreover, the following statements hold:

(a) \((M_p)_{p \in \mathbb{N}}\) satisfies (dc) if and only if \((N_p)_{p \in \mathbb{N}}\) does.

(b) \((M_p)_{p \in \mathbb{N}}\) satisfies \((\gamma_2)\) if and only if \((N_p)_{p \in \mathbb{N}}\) satisfies \((\gamma_1)\).

(c) \((M_p)_{p \in \mathbb{N}}\) satisfies \((\beta_2)\) if and only if \((N_p)_{p \in \mathbb{N}}\) does.

**Proof.** (a) follows from a direct computation, while all other statements are shown in [19, Lemma 2.3].

**2.3. The Borel problem in spaces of ultradifferentiable functions.** Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence. For \(h > 0\) we define \(D^{M_p,h}_{[-1,1]}\) as the Banach space consisting of all \(\varphi \in C^\infty(\mathbb{R})\) with \(\supp \varphi \subseteq [-1,1]\) such that

\[\|\varphi\|_{D^{M_p,h}_{[-1,1]}} := \sup_{p \in \mathbb{N}} \max_{x \in [-1,1]} \frac{h^p |\varphi^{(p)}(x)|}{M_p} < \infty.\]
We set
\[ D^{(M_p)}_{[-1,1]} := \lim_{h \to \infty} D^{M_p,h}_{[-1,1]}, \quad D^{\{M_p\}}_{[-1,1]} := \lim_{h \to 0^+} D^{M_p,h}_{[-1,1]]. \]

Then \( D^{(M_p)}_{[-1,1]} \) is a Fréchet space, while \( D^{\{M_p\}}_{[-1,1]} \) is an (LB)-space. If \( (M_p)_{p \in \N} \) satisfies (lc), the spaces \( D^{(M_p)}_{[-1,1]} \) and \( D^{\{M_p\}}_{[-1,1]} \) are non-trivial if and only if \( (M_p)_{p \in \N} \) satisfies \( (\gamma) \), as follows from the Denjoy–Carleman theorem.

For \( h > 0 \) we define \( \Lambda_{M_p,h} \) as the Banach space consisting of all sequences \( a = (a_p)_{p \in \N} \in \C^n \) such that
\[ \|a\|\Lambda_{M_p,h} := \sup_{p \in \N} \frac{h^p|a_p|}{M_p} < \infty. \]

We set
\[ \Lambda_{(M_p)} := \lim_{h \to \infty} \Lambda_{M_p,h}, \quad \Lambda_{\{M_p\}} := \lim_{h \to 0^+} \Lambda_{M_p,h}. \]
\( \Lambda_{(M_p)} \) is a Fréchet space, while \( \Lambda_{\{M_p\}} \) is an (LB)-space. The mappings
\[ B : D^{(M_p)}_{[-1,1]} \to \Lambda_{(M_p)}, \quad B : D^{\{M_p\}}_{[-1,1]} \to \Lambda_{\{M_p\}} \]
are well-defined and continuous. Petzsche characterized the surjectivity and the existence of a continuous linear right inverse of these mappings in the following way.

**Theorem 2.4.** Let \( (M_p)_{p \in \N} \) be a weight sequence satisfying (lc) and (\( \gamma) \).

(a) ([16 Thm. 3.4]) The following statements are equivalent:

(i) \( (M_p)_{p \in \N} \) satisfies \( (\gamma_1) \).

(ii) \( B : D^{(M_p)}_{[-1,1]} \to \Lambda_{(M_p)} \) has a continuous linear right inverse.

(iii) \( B : D^{\{M_p\}}_{[-1,1]} \to \Lambda_{\{M_p\}} \) is surjective.

(b) ([16 Thm. 3.5]) \( (M_p)_{p \in \N} \) satisfies \( (\gamma_1) \) if and only if \( B : D^{\{M_p\}}_{[-1,1]} \to \Lambda_{\{M_p\}} \) is surjective.

(c) ([16 Thm. 3.1(a)]) \( (M_p)_{p \in \N} \) satisfies \( (\gamma_1) \) and \( (\beta_2) \) if and only if \( B : D^{\{M_p\}}_{[-1,1]} \to \Lambda_{\{M_p\}} \) has a continuous linear right inverse.

3. Gelfand–Shilov spaces of type \( S_\ast \). Let \( (M_p)_{p \in \N} \) be a weight sequence. For \( n \in \N \) and \( h > 0 \) we write \( S^n_{M_p,h}(\R) \) for the Banach space consisting of all \( \varphi \in C^n(\R) \) such that
\[ \|\varphi\|S^n_{M_p,h} := \max_{m\leq n} \sup_{p \in \N} \sup_{x \in \R} \frac{h^p|x^p\varphi^{(m)}(x)|}{M_p} < \infty. \]

(\(^1\)) As pointed out in [19, p. 223], the statement of [16 Thm. 3.1(a)] contains a mistake, namely, one should read “(\( \gamma_1) \) and \( (\beta_2) \)” instead of “(\( \beta_2) \)”.

Notice that
\[ \|\varphi\|_{\mathcal{S}_{M_p,h}^n} = \max_{m \leq n} \sup_{x \in \mathbb{R}} |\varphi^{(m)}(x)| e^{M(h|x|)}, \quad \varphi \in \mathcal{S}_{M_p,h}^n(\mathbb{R}). \]

We set
\[ \mathcal{S}_{(M_p)}(\mathbb{R}) := \lim_{n \to \infty} \mathcal{S}_{M_p,n}(\mathbb{R}), \]
\[ \mathcal{S}_{\{M_p\}}(\mathbb{R}) := \lim_{h \to 0^+} \lim_{n \to \infty} \mathcal{S}_{M_p,h}^n(\mathbb{R}), \]
\[ \mathcal{S}_{\{M_p\}}(\mathbb{R}) := \lim_{n \to \infty} \lim_{h \to 0^+} \mathcal{S}_{M_p,h}^n(\mathbb{R}). \]

Then \( \mathcal{S}_{(M_p)}(\mathbb{R}) \) is a Fréchet space, \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) \) is an \((LF)\)-space, while \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) \) is a \((PLB)\)-space. In what follows, we shall sometimes use \( \mathcal{S}_*(\mathbb{R}) \) as a common notation for \( \mathcal{S}_{(M_p)}(\mathbb{R}) \), \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) \) and \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) \); a similar convention will be used for other spaces. If \( (M_p)_{p \in \mathbb{N}} \) satisfies \((dc)\), the mapping
\[ \mathcal{M} : \mathcal{S}_*(\mathbb{R}) \to \Lambda_* \]
is well-defined and continuous. The following results will be used later on.

**Lemma 3.1.** Let \( (M_p)_{p \in \mathbb{N}} \) be a weight sequence satisfying \((lc)\). Then \( \mathcal{S}_{(M_p)}(\mathbb{R}) \) is an \((FS)\)-space (= Fréchet–Schwartz space).

**Proof.** We have \( M(kt) - M(ht) \to \infty \) as \( t \to \infty \) for all \( k > h > 0 \), as follows from the representation \([12], (3.11)\) of the associated function \( M \). By combining this property with the Arzelà–Ascoli theorem, one can show that the inclusion mapping \( \mathcal{S}_{M_p,k}^{n+1}(\mathbb{R}) \to \mathcal{S}_{M_p,h}^n(\mathbb{R}) \) is compact for all \( n \in \mathbb{N} \) and \( k > h > 0 \), which implies the result. \( \blacksquare \)

**Lemma 3.2.** Let \( (M_p)_{p \in \mathbb{N}} \) be a weight sequence. Then the \((LF)\)-space \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) \) is complete.

**Proof.** In the notation of \([6]\), we have \( \mathcal{S}_{\{M_p\}}(\mathbb{R}) = \mathcal{B}_{\mathcal{V}}(\mathbb{R}) \), where \( \mathcal{V} = (v_N)_{N \in \mathbb{N}} \) with \( v_N = e^{M(|\cdot|/N)} \) for \( N \in \mathbb{N} \). Notice that \( \mathcal{V} \) satisfies condition (3.2) from \([6]\) because \( M \) is increasing. Hence, by \([6], \text{Thm. 3.4}\) it suffices to show that \( \mathcal{V} \) satisfies \((\Omega)\), that is,
\[ \forall N \exists L \geq N \forall K \geq L \exists \theta \in (0, 1) \exists C > 0 \forall x \in \mathbb{R} : v_L(x) \leq C(v_N(x))^{1-\theta}(v_K(x))^{\theta}. \]
The latter follows from the fact that the function \( t \mapsto M(e^t) \) is increasing and convex on \([0, \infty)\). \( \blacksquare \)

Next, we discuss the Fourier transform on \( \mathcal{S}_*(\mathbb{R}) \). For \( n \in \mathbb{N} \) and \( h > 0 \) we write \( \mathcal{S}_{nM_p,h}^n(\mathbb{R}) \) for the Banach space consisting of all \( \varphi \in C^\infty(\mathbb{R}) \) such that
\[ \|\varphi\|_{\mathcal{S}_{nM_p,h}^n} := \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{h^p |x^m \varphi^{(p)}(x)|}{M_p} < \infty. \]
We set
\[ S^{(M_p)}(\mathbb{R}) := \lim_{n \to \infty} S^{M_p,n}_n(\mathbb{R}), \]
\[ \overrightarrow{S}^{(M_p)}(\mathbb{R}) := \lim_{h \to 0^+} \lim_{n \to \infty} S^{M_p,h}_n(\mathbb{R}), \]
\[ \overrightarrow{S}_{\{M_p\}}(\mathbb{R}) := \lim_{n \to \infty} \lim_{h \to 0^+} S^{M_p,h}_n(\mathbb{R}). \]

Then \( S^{(M_p)}(\mathbb{R}) \) is a Fréchet space, \( \overrightarrow{S}^{\{M_p\}}(\mathbb{R}) \) is an \((LF)\)-space, while \( \overrightarrow{S}_{\{M_p\}}(\mathbb{R}) \) is a \((PLB)\)-space. If \((M_p)_{p \in \mathbb{N}}\) satisfies (lc) and (dc), the Fourier transform is a topological isomorphism from \( S_*(\mathbb{R}) \) onto \( S^*(\mathbb{R}) \) (cf. [10 Sect. IV.6]).

We now introduce Gelfand–Shilov spaces of type \( S_*(0, \infty) \). Let \( n \in \mathbb{N} \) and \( h > 0 \). We define the following closed subspaces of \( S^{n}_{M_p,h}(\mathbb{R}) \):
\[ S^n_{M_p,h}(0, \infty) := \{ \varphi \in S^n_{M_p,h}(\mathbb{R}) \mid \text{supp } \varphi \subseteq [0, \infty) \}, \]
\[ S^{n,0}_{M_p,h}(\mathbb{R}) := \{ \varphi \in S^n_{M_p,h}(\mathbb{R}) \mid \varphi^{(m)}(0) = 0 \text{ for all } m = 0, \ldots, n \}, \]
and endow them with the norm \( \| \cdot \|_{S^{n}_{M_p,h}} \). Hence, they become Banach spaces.

We set
\[ S_{(M_p)}(0, \infty) := \lim_{n \to \infty} S^n_{M_p,n}(0, \infty), \quad S^0_{(M_p)}(\mathbb{R}) := \lim_{n \to \infty} S^{n,0}_{M_p,n}(\mathbb{R}), \]
\[ \overrightarrow{S}_{\{M_p\}}(0, \infty) := \lim_{h \to 0^+} \lim_{n \to \infty} S^n_{M_p,h}(0, \infty), \quad \overrightarrow{S}^0_{\{M_p\}}(\mathbb{R}) := \lim_{h \to 0^+} \lim_{n \to \infty} S^{n,0}_{M_p,h}(\mathbb{R}), \]
\[ \overrightarrow{S}_{\{M_p\}}(0, \infty) := \lim_{n \to \infty} \lim_{h \to 0^+} S^n_{M_p,h}(0, \infty), \quad \overrightarrow{S}^0_{\{M_p\}}(\mathbb{R}) := \lim_{n \to \infty} \lim_{h \to 0^+} S^{n,0}_{M_p,h}(\mathbb{R}). \]

Note that \( \overrightarrow{S}_{\{M_p\}}(0, \infty) \) was denoted by \( S_{\{M_p\}}(0, \infty) \) in the introduction. Now \( S_{(M_p)}(0, \infty) \) and \( S^0_{(M_p)}(\mathbb{R}) \) are Fréchet spaces, \( \overrightarrow{S}_{\{M_p\}}(0, \infty) \) and \( \overrightarrow{S}^0_{\{M_p\}}(\mathbb{R}) \) are \((LF)\)-spaces, while \( \overrightarrow{S}_{\{M_p\}}(0, \infty) \) and \( \overrightarrow{S}^0_{\{M_p\}}(\mathbb{R}) \) are \((PLB)\)-spaces. We have
\[ \begin{align*}
S_*(0, \infty) &= \{ \varphi \in S_*(\mathbb{R}) \mid \text{supp } \varphi \subseteq [0, \infty) \}, \\
S^0_*(\mathbb{R}) &= \{ \varphi \in S_*(\mathbb{R}) \mid \varphi^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N} \},
\end{align*} \]
as sets.

**Lemma 3.3.** Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence and let \( S_*(\mathbb{R}) = S_{(M_p)}(\mathbb{R}) \) or \( S_*(\mathbb{R}) = \overrightarrow{S}_{\{M_p\}}(\mathbb{R}) \). Then the equalities (3.1) and (3.2) hold topologically if the spaces on the right-hand side are endowed with the relative topology induced by \( S_*(\mathbb{R}) \).
We need some preparation for the proof of Lemma 3.3. Let \( E = \lim_{n \in \mathbb{N}} E_n \) be an \((LF)\)-space. A subspace \( L \) of \( E \) is called a limit subspace if \( L = \lim_{n \in \mathbb{N}} L \cap E_n \) topologically, where \( L \) is endowed with the relative topology induced by \( E \) and \( L \cap E_n, n \in \mathbb{N} \), is endowed with the relative topology induced by \( E_n \). The following result is a consequence of \([21, \text{Prop. 1.2}]\) and the fact that every Fréchet space is an acyclic \((LF)\)-space; we refer to \([21]\) for the definition of an acyclic \((LF)\)-space.

**Lemma 3.4 (cf. [21, Prop. 1.2]).** Let \( E \) be an \((LF)\)-space, let \( F \) be a Fréchet space and let \( T : E \to F \) be a surjective continuous linear mapping. Then \( \ker T \) is a limit subspace of \( E \).

**Lemma 3.5.** Let \( E \) be an \((LF)\)-space. Then every complemented subspace of \( E \) is a limit subspace of \( E \).

**Proof.** Since \((LF)\)-spaces are webbed and ultrabornological \([15, \text{Remark 24.36}]\) and the class of ultrabornological \(lCHs\) is closed under taking complemented subspaces, this follows from De Wilde’s open mapping theorem \([15, \text{Thm. 24.30}]\). ■

**Proof of Lemma 3.3.** \( S_* (\mathbb{R}) = S_{(M_p)} (\mathbb{R}) \): This is obvious.

\( S_* (\mathbb{R}) = \overrightarrow{S}_{(M_p)} (\mathbb{R}) \): It suffices to show that \( \overrightarrow{S}_{(M_p)} (0, \infty) \) and \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) \) are limit subspaces of \( \overrightarrow{S}_{(M_p)} (\mathbb{R}) \). We first consider \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) \). We claim that the continuous linear mapping \( B : \overrightarrow{S}_{(M_p)} (\mathbb{R}) \to \mathbb{C}^\mathbb{N} \) is surjective. Let \( a \in \mathbb{C}^\mathbb{N} \) be arbitrary. By Borel’s theorem, there is \( \varphi \in C^\infty (\mathbb{R}) \) such that \( B(\varphi) = a \). Pick \( \psi \in D(\mathbb{R}) \) such that \( \psi \equiv 1 \) in a neighbourhood of 0. Then, \( \varphi \psi \in D(\mathbb{R}) \subset \overrightarrow{S}_{(M_p)} (\mathbb{R}) \) and \( B(\varphi \psi) = a \). Notice that \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) = \ker B \). Hence, the result follows from Lemma 3.4. Next, we deal with \( \overrightarrow{S}_{(M_p)} (0, \infty) \). Since \( \overrightarrow{S}_{(M_p)} (0, \infty) \) is a complemented subspace of \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) \), by Lemma 3.5 \( \overrightarrow{S}_{(M_p)} (0, \infty) \) is a limit subspace of \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) \). Consequently, as we have already shown that \( \overrightarrow{S}_{(M_p)}^0 (\mathbb{R}) \) is a limit subspace of \( \overrightarrow{S}_{(M_p)} (\mathbb{R}) \), \( \overrightarrow{S}_{(M_p)} (0, \infty) \) is a limit subspace of \( \overrightarrow{S}_{(M_p)} (\mathbb{R}) \). ■

Finally, we present two technical lemmas that will play an important role later on.

**Lemma 3.6.** Let \( (M_p)_{p \in \mathbb{N}} \) be a weight sequence satisfying \((lc)\) and \((dc)\).

(a) \( T : S_{(M_p)} (0, \infty) \to S_{(M_p)} (0, \infty) \) and \( T : \overrightarrow{S}_{(M_p)} (0, \infty) \to \overrightarrow{S}_{(M_p)} (0, \infty) \) are well-defined continuous mappings, where

\[
T(\varphi)(x) = \begin{cases} \varphi(x)/x, & x > 0, \\ 0, & x \leq 0. \end{cases}
\]
(b) $T : S_{(M_p)}(0, \infty) \to S_{(M_p)}(0, \infty)$ and $T : \overline{S}_{(M_p)}(0, \infty) \to \overline{S}_{(M_p)}(0, \infty)$ are well-defined continuous mappings, where

$$T(\varphi)(x) = \begin{cases} x\varphi(x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

**Proof.** We start by recalling the following consequence of Taylor’s theorem: Let $\varphi \in C^n([0, 1]), n \in \mathbb{N},$ be such that $\varphi^{(m)}(0) = 0$ for all $m = 0, \ldots, n.$ Then

$$|\varphi^{(j)}(x)| \leq \frac{\|\varphi^{(j+k)}\|_{L^\infty([0,1])}}{k!} x^k, \quad x \in [0, 1],$$

for all $j, k \in \mathbb{N}$ with $j + k \leq n.$

(a) It suffices to show that $T : S_{M_p,H}^{n+2}(0, \infty) \to S_{M_p,H}^n(0, \infty)$ is well-defined and continuous for all $n \in \mathbb{N}$ and $h > 0.$ Let $\varphi \in S_{M_p,H}^{n+2}(0, \infty).$ Then

$$T(\varphi)^{(m)}(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j j! \frac{\varphi^{(m-j)}(x)}{x^{j+1}}, \quad x > 0,$$

for all $m \leq n.$ Hence, (3.3) yields $T(\varphi) \in C^n(\mathbb{R})$ with $\text{supp} \varphi \subseteq [0, \infty)$ and

$$\|T(\varphi)\|_{S_{M_p,H}^n} \leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \frac{h^p x^p}{M_p} \sum_{j=0}^{m} \binom{m}{j} \frac{|\varphi^{(m-j)}(x)|}{x^{j+1}}$$

$$\leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \frac{h^p}{M_p} \sum_{j=0}^{m} \binom{m}{j} \frac{|\varphi^{(m-j)}(x)|}{x^{j+1}}$$

$$+ \max_{m \leq n} \sup_{p \in \mathbb{N}} \frac{1}{M_p} \sum_{j=0}^{m} \binom{m}{j} \frac{h^p x^p |\varphi^{(m-j)}(x)|}{x^{j+1}}$$

$$\leq 2^n e^{M(h)} \max_{m \leq n} \|\varphi^{(m+1)}\|_{L^\infty([0,1])} + 2^n n! \|\varphi\|_{S_{M_p,H}^n} \leq C \|\varphi\|_{S_{M_p,H}^{n+2}},$$

where the last inequality follows from the fact that $\| \cdot \|_{L^\infty([0,1])} \leq \| \cdot \|_{S_{M_p,H}^0}.$

(b) It suffices to show that $T : S_{M_p,hH}^n(0, \infty) \to S_{M_p,hH}^n(0, \infty)$ is well-defined and continuous for all $n \in \mathbb{N}$ and $h > 0,$ where $H$ denotes the constant occurring in (dc). Let $\varphi \in S_{M_p,hH}^n(0, \infty).$ Then $T(\varphi) \in C^n(\mathbb{R})$ with $\text{supp} \varphi \subseteq [0, \infty)$ and

$$\|T(\varphi)\|_{S_{M_p,hH}^n} \leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \frac{h^p x^{p+1} |\varphi^{(m)}(x)|}{M_p}$$

$$+ \max_{m \leq n} \sup_{p \in \mathbb{N}} \frac{h^p x^p |\varphi^{(m-1)}(x)|}{M_p} \leq C \|\varphi\|_{S_{M_p,hH}^n}.$$
(a) \( T : S_{(N_p)}(0, \infty) \rightarrow S_{(M_p)}(0, \infty) \) and \( T : \overrightarrow{S}_{(N_p)}(0, \infty) \rightarrow \overrightarrow{S}_{(M_p)}(0, \infty) \) are well-defined continuous mappings, where
\[
T(\varphi)(x) = \begin{cases} \varphi(x^{1/2}), & x > 0, \\ 0, & x \leq 0. \end{cases}
\]

(b) \( T : S_{(M_p)}(0, \infty) \rightarrow S_{(N_p)}(0, \infty) \) and \( T : \overrightarrow{S}_{(M_p)}(0, \infty) \rightarrow \overrightarrow{S}_{(N_p)}(0, \infty) \) are well-defined continuous mappings, where
\[
T(\varphi)(x) = \begin{cases} \varphi(x^2), & x > 0, \\ 0, & x \leq 0. \end{cases}
\]

Proof. (a) It suffices to show that \( T : S_{N_p,h}^{2n+1}(0, \infty) \rightarrow S_{M_p,h}^n(0, \infty) \) is well-defined and continuous for all \( n \in \mathbb{N} \) and \( h > 0 \). Let \( \varphi \in S_{N_p,h}^{2n+1}(0, \infty) \) be arbitrary. We set \( I_0 = \{0\} \) and
\[
I_m = \left\{ \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \mid \sum_{j=1}^m j \alpha_j = m \right\}, \quad m \in \mathbb{Z}_+.
\]
Faà di Bruno’s formula implies that
\[
T(\varphi)^{(m)}(x) = \sum_{\alpha \in I_m} a_\alpha \frac{\varphi^{(|\alpha|)}(x^{1/2})}{x^{m-|\alpha|/2}}, \quad x > 0,
\]
for all \( m \leq n \), where \( a_\alpha \) are real constants. Hence, (3.3) yields \( T(\varphi) \in C^m([0, \infty)) \) with \( \text{supp} \varphi \subseteq [0, \infty) \). Since \( M_p = N_{2p} \) for all \( p \in \mathbb{N} \), we have
\[
\|T(\varphi)\|_{S_{M_p,h}^n} \leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{x > 0} \frac{h^p x^p}{M_p} \sum_{\alpha \in I_m} |a_\alpha| \frac{|\varphi^{(|\alpha|)}(x^{1/2})|}{x^{m-|\alpha|/2}}
\]
\[
\leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{0 < x < 1} \frac{h^p}{M_p} \sum_{\alpha \in I_m} |a_\alpha| \frac{|\varphi^{(|\alpha|)}(x^{1/2})|}{x^{m-|\alpha|/2}}
\]
\[
+ \max_{m \leq n} \sup_{p \in \mathbb{N}} \sum_{x \geq 1} \sum_{\alpha \in I_m} |a_\alpha| \frac{h^p x^p |\varphi^{(|\alpha|)}(x^{1/2})|}{M_p}
\]
\[
\leq e^{M(h)} \max_{m \leq n} \left( \sum_{\alpha \in I_m} |a_\alpha| \right) \|\varphi^{(2m)}\|_{L_\infty([0,1])}
\]
\[
+ \left( \max_{m \leq n} \sum_{\alpha \in I_m} |a_\alpha| \right) \|\varphi\|_{S_{N_p,h}^{2n+1}/2} \leq C \|\varphi\|_{S_{N_p,h}^{2n+1}/2},
\]
where the last inequality follows from the fact that \( \|\cdot\|_{L_\infty([0,1])} \leq \|\cdot\|_{S_{N_p,h}^{2n+1}/2} \).

(b) By Lemma 2.3(a), the sequence \((N_p)_{p \in \mathbb{N}}\) satisfies (dc). We may assume without loss of generality that the constants \( C_0 \) and \( H \) occurring in (dc) are the same for \((M_p)_{p \in \mathbb{N}}\) and \((N_p)_{p \in \mathbb{N}}\). It is enough to show that
that $S^{n}_{M_{p},H^{2n+1}h^{2}}(0,\infty) \rightarrow \mathbb{S}^{n}_{N_{p},h}(0,\infty)$ is well-defined and continuous for all $n \in \mathbb{N}$ and $h > 0$. Let $l = H^{2n+1}h^{2}$. Let $\varphi \in \mathbb{S}^{n}_{M_{p},l}(0,\infty)$. Clearly, $T(\varphi) \in C^{n}(\mathbb{R})$ with $\sup \varphi \subseteq [0,\infty)$. Faà di Bruno’s formula implies that

$$T(\varphi)(m)(x) = \sum_{j=0}^{[m/2]} a_{j}\varphi^{(m-j)}(x^{2})x^{m-2j}, \quad x > 0,$$

for all $m \leq n$, where the $a_{j}$ are positive constants. We have $N_{p+q} \leq C_{0}^{q}H^{q+1/2}H^{pq}N_{p}$ for all $p, q \in \mathbb{N}$. Moreover, there is $C_{1} > 0$ such that $M_{[p/2]} \leq C_{1}H^{[p/2]}N_{p}$ for all $p \in \mathbb{N}$. Therefore,

$$\|T(\varphi)\|_{\mathbb{S}^{n}_{N_{p},h}} \leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{x > 0} \frac{h^{p}x^{p}}{N_{p}} \sum_{j=0}^{[m/2]} a_{j}\varphi^{(m-j)}(x^{2})|x|^{m-2j}$$

$$\leq \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{0 < x < 1} \frac{h^{p}}{N_{p}} \sum_{j=0}^{[m/2]} a_{j}\varphi^{(m-j)}(x^{2})|$$

$$+ \max_{m \leq n} \sup_{p \in \mathbb{N}} \sup_{x \geq 1} \sum_{j=0}^{[m/2]} \frac{h^{p}x^{p+m}|\varphi^{(m-j)}(x^{2})|}{N_{p}}$$

$$\leq e^{N(h)} \left( \max_{m \leq n} \sum_{j=0}^{[m/2]} a_{j} \right) \|\varphi\|_{\mathbb{S}^{n}_{M_{p},l}}$$

$$+ \max_{m \leq n} \left( \sum_{j=0}^{[m/2]} a_{j} \right) \|\varphi\|_{\mathbb{S}^{n}_{M_{p},l}} \sup_{p \in \mathbb{N}} \frac{h^{p}M_{[p+m/2]}^{l}}{[(p+m)/2]}N_{p} \leq C\|\varphi\|_{\mathbb{S}^{n}_{M_{p},l}}.$$ 

**Remark 3.8.** An inspection of the proofs of Lemmas [3.6](a) and [3.7](a) shows that these results hold without assuming that $(M_{p})_{p \in \mathbb{N}}$ satisfies (dc).

**4. A functional analytic tool.** In this section, we show an abstract result about the existence of a continuous linear right inverse that is tailormade to prove Proposition [6.3](a) below. We start with the following simple observation.

**Lemma 4.1.** Let $E$, $F$, and $G$ be vector spaces and let $T : E \rightarrow F$ and $S : E \rightarrow G$ be linear mappings. If both $T : E \rightarrow F$ and $S_{|\ker T} : \ker T \rightarrow G$ are surjective, then $T_{|\ker S} : \ker S \rightarrow F$ is also surjective.

**Proof.** Let $x \in F$. Choose $y \in E$ such that $T(y) = x$ and $z \in \ker T$ such that $S(z) = S(y)$. Then, $y - z \in \ker S$ and $T(y - z) = T(y) = x$. ■

Now suppose that $E$, $F$, and $G$ are lcHs and that $T$ and $S$ are continuous linear mappings. If both $T$ and $S_{|\ker T}$ have a continuous linear right inverse, it is clear from the proof of Lemma [4.1](a) that $T_{|\ker S}$ also has a continuous linear
right inverse. Our goal is to show that, under suitable extra conditions on $F$ and $T$, $T_{|\ker S}$ has a continuous linear right inverse if one merely assumes that $S_{|\ker T}$ lifts bounded sets. We need some preparation to formulate and prove this result.

Let $E$ and $F$ be lcHs. We denote by $\text{csn}(E)$ the set of all continuous semi-norms on $E$ and by $L(E,F)$ the space of all continuous linear mappings from $E$ to $F$. Let $(x_n)_{n\in\mathbb{N}} \subset F$ and $(x'_n)_{n\in\mathbb{N}} \subset F'$. The pair $((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}})$ is said to be a Schauder frame (in $F$) [2, Def. 1.1] if

$$x = \sum_{n=0}^{\infty} \langle x'_n, x \rangle x_n$$

for all $x \in F$. A Schauder frame $((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}})$ is called absolute if for all $p \in \text{csn}(F)$ there is $q \in \text{csn}(F)$ such that

$$\sum_{n=0}^{\infty} |\langle x'_n, x \rangle| p(x_n) \leq q(x)$$

for all $x \in F$. Every absolute Schauder basis [15, p. 340] canonically determines a Schauder frame. We need the following lemma.

**Lemma 4.2.** Let $E$ and $F$ be lcHs and let $T \in L(E,F)$. Suppose that $E$ is sequentially complete and that $F$ possesses an absolute Schauder frame $((x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}})$. If there is a sequence $(y_n)_{n\in\mathbb{N}} \subset E$ such that $T(y_n) = x_n$ for all $n \in \mathbb{N}$ and for all $p \in \text{csn}(E)$ there is $q \in \text{csn}(F)$ such that $p(y_n) \leq q(x_n)$ for all $n \in \mathbb{N}$, then $T$ has a continuous linear right inverse.

**Proof.** For each $x \in F$ the sequence $(\sum_{n=0}^{N} \langle x'_n, x \rangle y_n)_{N\in\mathbb{N}}$ is Cauchy in $E$. Since $E$ is sequentially complete, we have

$$R(x) = \sum_{n=0}^{\infty} \langle x'_n, x \rangle y_n \in E.$$ 

Then $R : F \to E$ is a continuous linear right inverse of $T$. □

**Proposition 4.3.** Let $E$, $F$ and $G$ be lcHs and let $T \in L(E,F)$ and $S \in L(E,G)$. Endow $\ker T$ and $\ker S$ with the relative topology induced by $E$. Suppose that the following conditions are satisfied:

1. $E$ is sequentially complete.
2. $F$ possesses an absolute Schauder frame.
3. $S_{|\ker T} : \ker T \to G$ lifts bounded sets, that is, for every $B \subset G$ bounded there is $A \subset \ker T$ bounded such that $S(A) = B$.
4. There is a lcHs $E_0$ with the following properties:
   1. $E_0 \subset E$ with continuous inclusion mapping.
   2. $T_{|E_0} : E_0 \to F$ has a continuous linear right inverse.
Lemma 26.13. Suppose that \((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}}\) is an absolute Schauder frame in \(F\). Since \(\ker S\) is sequentially complete (as a closed subspace of \(E\)), it suffices to construct a sequence \((y_n)_{n \in \mathbb{N}} \subset \ker S\) satisfying the assumptions of Lemma \([4.2]\). By (4.2), there is a sequence \((y_{0,n})_{n \in \mathbb{N}} \subset E_0\) such that \(T(y_{0,n}) = x_n\) for all \(n \in \mathbb{N}\) and for all \(p \in \text{csn}(E_0)\) there is \(q \in \text{csn}(F)\) such that \(p(y_{0,n}) \leq q(x_n)\) for all \(n \in \mathbb{N}\). The property (4.3) means that there is \(p_0 \in \text{csn}(E_0)\) such that \(S_{|E_0} : (E_0, p_0) \to G\) is continuous, where \((E_0, p_0)\) stands for the vector space \(E_0\) endowed with the topology generated by the single seminorm \(p_0\). In particular, \(S(y) = 0\) for all \(y \in E_0\) with \(p_0(y) = 0\).

We set

\[
S_{|E_0} : E_0 \to G\]

is locally bounded, that is, there is a neighbourhood \(U\) of 0 in \(E_0\) such that \(S(U)\) is bounded in \(G\).

Then \(T_{|\ker S} : \ker S \to F\) has a continuous linear right inverse.

**Proof.** Suppose that \(((x_n)_{n \in \mathbb{N}}, (x'_n)_{n \in \mathbb{N}})\) is an absolute Schauder frame in \(F\). Since \(\ker S\) is sequentially complete (as a closed subspace of \(E\)), it suffices to construct a sequence \((y_n)_{n \in \mathbb{N}} \subset \ker S\) satisfying the assumptions of Lemma \([4.2]\). By (4.2), there is a sequence \((y_{0,n})_{n \in \mathbb{N}} \subset E_0\) such that \(T(y_{0,n}) = x_n\) for all \(n \in \mathbb{N}\) and for all \(p \in \text{csn}(E_0)\) there is \(q \in \text{csn}(F)\) such that \(p(y_{0,n}) \leq q(x_n)\) for all \(n \in \mathbb{N}\). The property (4.3) means that there is \(p_0 \in \text{csn}(E_0)\) such that \(S_{|E_0} : (E_0, p_0) \to G\) is continuous, where \((E_0, p_0)\) stands for the vector space \(E_0\) endowed with the topology generated by the single seminorm \(p_0\). In particular, \(S(y) = 0\) for all \(y \in E_0\) with \(p_0(y) = 0\).

We set

\[
s_n = \begin{cases} 
\frac{S(y_{0,n})}{p_0(y_{0,n})}, & p_0(y_{0,n}) \neq 0, \\
0, & p_0(y_{0,n}) = 0,
\end{cases}
\]

for \(n \in \mathbb{N}\). Since the sequence \((s_n)_{n \in \mathbb{N}}\) is bounded in \(G\), by (3) there is a bounded sequence \((z_n)_{n \in \mathbb{N}} \subset \ker T\) such that \(S(z_n) = s_n\) for all \(n \in \mathbb{N}\). Set

\[
y_n = y_{0,n} - p_0(y_{0,n})z_n\]

for all \(n \in \mathbb{N}\). Then \(y_n \in \ker S\) and \(T(y_n) = T(y_{0,n}) = x_n\) for all \(n \in \mathbb{N}\). Finally, let \(p \in \text{csn}(E)\). Since \((z_n)_{n \in \mathbb{N}}\) is bounded in \(E\), there is \(C > 0\) such that

\[
p(y_n) \leq p(y_{0,n}) + p_0(y_{0,n})p(z_n) \leq p|_{E_0}(y_{0,n}) + Cp_0(y_{0,n}) \leq p'(y_{0,n})
\]

for all \(n \in \mathbb{N}\), where \(p' = (1 + C) \max\{p|_{E_0}, p_0\}\). By (4.1), \(p' \in \text{csn}(E_0)\). Hence, there is \(q \in \text{csn}(F)\) such that \(p(y_n) \leq p'(y_{0,n}) \leq q(x_n)\) for all \(n \in \mathbb{N}\).

**Remark 4.4.** If \(E\) is an \((FS)\)-space and \(G\) is a Fréchet space, then condition (3) in Proposition \([4.3]\) may be relaxed to “\(S_{|\ker T} : \ker T \to G\) is surjective”. Indeed, as a closed subspace of an \((FS)\)-space is again an \((FS)\)-space, \(\ker(S_{|\ker T})\) is an \((FS)\)-space. Since every \((FS)\)-space is quasinormable, the result follows from the fact that a surjective continuous linear mapping \(Q\) between Fréchet spaces lifts bounded sets if \(\ker Q\) is quasinormable \([15]\) Lemma 26.13].

5. **The Borel problem in \(\mathcal{S}(M_p,0)(\mathbb{R})\).** Given a weight sequence \((M_p)_{p \in \mathbb{N}}\), we define the following closed subspace of \(\mathcal{S}(M_p)(\mathbb{R})\):

\[
\mathcal{S}(M_p,0)(\mathbb{R}) := \{ \varphi \in \mathcal{S}(M_p)(\mathbb{R}) \mid \mu_n(\varphi) = 0 \text{ for all } n \in \mathbb{N} \}
\]

and endow it with the relative topology induced by \(\mathcal{S}(M_p)(\mathbb{R})\). Hence, it becomes a Fréchet space. The goal of this section is to show the following result, which will be used in the proof of Proposition \([6.3]\) below.
Proposition 5.1. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc), (dc) and \((\gamma_1)\). Then \(B : \mathcal{S}(M_p,0)(\mathbb{R}) \to \mathbb{C}^\mathbb{N}\) is surjective.

Since, by Lemma 3.1, \(\mathcal{S}(M_p)(\mathbb{R})\) is an \((FS)\)-space, Proposition 5.1 can be strengthened as follows (cf. Remark 4.4).

Proposition 5.2. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc), (dc) and \((\gamma_1)\). Then \(B : \mathcal{S}(M_p,0)(\mathbb{R}) \to \mathbb{C}^\mathbb{N}\) lifts bounded sets.

The proof of Proposition 5.1 is based on the following variant of Eidelheit’s theorem.

Proposition 5.3. Let \(E\) be a Fréchet space and let \((x'_n)_{n \in \mathbb{N}} \subset E'\). Let \(F\) be a closed subspace of \(E\) and set
\[
F^\perp = \{ x' \in E' \mid \langle x', x \rangle = 0 \text{ for all } x \in F \}.
\]
The mapping \(F \to \mathbb{C}^\mathbb{N}, x \mapsto (\langle x'_n, x \rangle)_{n \in \mathbb{N}}\), is surjective if and only if:

1. For all \(N \in \mathbb{N}\) and \(c_0, \ldots, c_N \in \mathbb{C}\),
\[
\sum_{n=0}^{N} c_n x'_n \in F^\perp
\]
implies that \(c_0 = \cdots = c_N = 0\).

2. For every \(B \subset E'\) equicontinuous there is \(\nu \in \mathbb{N}\) such that for all \(N \geq \nu\) and \(c_0, \ldots, c_N \in \mathbb{C}\),
\[
\sum_{n=0}^{N} c_n x'_n \in B + F^\perp
\]
implies that \(c_\nu = \cdots = c_N = 0\).

Proof. This is a consequence of the classical theorem of Eidelheit [15 Thm. 26.27] and the Hahn–Banach theorem.

We shall prove Proposition 5.1 via Proposition 5.3 with \(E = \mathcal{S}(M_p)(\mathbb{R})\), \(F = \mathcal{S}(M_p,0)(\mathbb{R})\) and \((x'_n)_{n \in \mathbb{N}} = ((-1)^n \delta^{(n)})_{n \in \mathbb{N}}\). To this end, we first give an explicit description of the space \((\mathcal{S}(M_p,0)(\mathbb{R}))^\perp\) and the equicontinuous subsets of \(\mathcal{S}'(M_p)(\mathbb{R})\). We need some preparation.

Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence. An entire function \(P(z) = \sum_{p=0}^{\infty} b_p z^p, b_p \in \mathbb{C}\), is said to be an ultrapolynomial of class \((M_p)\) [12 Def. 2.11] if
\[
(5.1) \quad \sup_{p \in \mathbb{N}} \frac{|b_p|M_p}{h^p} < \infty \quad \text{for some } h > 0.
\]

If \((M_p)_{p \in \mathbb{N}}\) satisfies (lc), an entire function \(P\) is an ultrapolynomial of class \((M_p)\) if and only if [12 Prop. 4.5]
\[
\sup_{z \in \mathbb{C}} |P(z)|e^{-M(h|z|)} < \infty \quad \text{for some } h > 0.
\]
Lemma 5.4. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc), (dc) and \((\gamma_1)\). Then, \(f \in S'_{(M_p)}(\mathbb{R})\) belongs to \((S_{(M_p),0}(\mathbb{R}))^\perp\) if and only if there is an ultrapolynomial \(P\) of class \((M_p)\) such that \(f = P\) in \(S'_{(M_p)}(\mathbb{R})\).

Proof. Since \(\mathcal{D}^{(M_p)}_{[-1,1]} \subseteq S^{(M_p)}(\mathbb{R})\), Theorem 2.4(a) implies that \(\mathcal{B} : S^{(M_p)}(\mathbb{R}) \to \Lambda_{(M_p)}\) is surjective. By taking the Fourier transform, we find that \(\mathcal{M} : S_{(M_p)}(\mathbb{R}) \to \Lambda_{(M_p)}\) is surjective. Consequently, the sequence

\[
0 \to S_{(M_p),0}(\mathbb{R}) \xrightarrow{\iota} S_{(M_p)}(\mathbb{R}) \xrightarrow{\mathcal{M}} \Lambda_{(M_p)} \to 0
\]

is exact, where \(\iota : S_{(M_p),0}(\mathbb{R}) \to S_{(M_p)}(\mathbb{R})\) denotes the inclusion mapping. Therefore, its dual sequence

\[
0 \to \Lambda'_{(M_p)} \xrightarrow{\mathcal{M}'} \Lambda_{(M_p)} \xrightarrow{\iota^t} S_{(M_p),0}(\mathbb{R}) \to 0
\]

is also exact [15, Prop. 26.4]. In particular, \(\text{im} \mathcal{M}^t = \ker \iota^t\). It is clear that \(\ker \iota^t = (S_{(M_p),0}(\mathbb{R}))^\perp\). On the other hand, \(\Lambda'_{(M_p)}\) may be identified with the space consisting of all sequences \(b = (b_p)_{p \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}\) satisfying (5.1) and, under this identification, the duality is given by

\[
\langle b, a \rangle = \sum_{p=0}^{\infty} b_p a_p, \quad a = (a_p)_{p \in \mathbb{N}} \in \Lambda_{(M_p)}, \quad b = (b_p)_{p \in \mathbb{N}} \in \Lambda'_{(M_p)}.
\]

Hence, \(\text{im} \mathcal{M}^t\) coincides with the subspace of \(S'_{(M_p)}(\mathbb{R})\) consisting of all ultrapolynomials of class \((M_p)\).

The next result follows from [12, Prop. 3.4] and the structural theorem for general Gelfand–Shilov spaces [10, p. 113].

Lemma 5.5. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc) and (dc). For every \(B \subset S'_{(M_p)}(\mathbb{R})\) equicontinuous there are \(\nu \in \mathbb{N}\) and \(C, h > 0\) such that for all \(f \in B\) there are measurable functions \(g_0, \ldots, g_\nu\) such that

\[
f = \sum_{n=0}^{\nu} g_n^{(n)} \quad \text{in} \ S'_{(M_p)}(\mathbb{R})
\]

and

\[
|g_n(x)| \leq C e^{M(h|x|)}, \quad x \in \mathbb{R}, \quad n = 0, \ldots, \nu.
\]

Finally, we will also use the following well-known fact from distribution theory.

Lemma 5.6. Let \(\nu \in \mathbb{N}\). For all \(N \geq \nu\), \(c_0, \ldots, c_N \in \mathbb{C}\) and \(g_0, \ldots, g_\nu \in L^1_{\text{loc}}(\mathbb{R})\),

\[
\sum_{n=0}^{N} c_n \delta^{(n)} = \sum_{n=0}^{\nu} g_n^{(n)} \quad \text{in} \ \mathcal{D}'(\mathbb{R})
\]

implies that \(c_\nu = \cdots = c_N = 0\).
Proof of Proposition 5.7. We use Proposition 5.3 with $E = S_{(M_p)}(\mathbb{R})$, $F = S_{(M_p),0}(\mathbb{R})$ and $(x'_n)_{n \in \mathbb{N}} = ((-1)^n \delta(m))_{n \in \mathbb{N}}$. In view of Lemma 5.4 condition (1) of Proposition 5.3 follows from Lemma 5.6 (with $\nu = 0$), while condition (2) follows from Lemmas 5.5 and 5.6.

6. The Stieltjes moment problem in $S_*(0, \infty)$. We are ready to characterize the surjectivity and the existence of a continuous linear right inverse of $M : S_* (0, \infty) \to \Lambda_*$ (cf. Theorem 1.2).

Theorem 6.1. Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying $(lc)$ and $(dc)$.

(a) The following statements are equivalent:

(i) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$.

(ii) $M : S_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}$ has a continuous linear right inverse.

(iii) $M : S_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}$ is surjective.

(b) The following statements are equivalent:

(i) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$.

(ii) $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective.

(iii) $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective.

(c) The following statements are equivalent:

(i) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$ and $(\beta_2)$.

(ii) $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ has a continuous linear right inverse.

(iii) $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ has a continuous linear right inverse.

In view of Lemma 2.3 Theorem 6.1 is a consequence of the following two results.

Proposition 6.2. Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying conditions $(lc)$ and $(dc)$. Denote by $(N_p)_{p \in \mathbb{N}}$ its 2-interpolating sequence.

(a) $M : S_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}$ is surjective [has a continuous linear right inverse] if and only if $M : S_{(N_p)}(0, \infty) \to \Lambda_{(N_p)}$ is surjective [has a continuous linear right inverse].

(b) If $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective [has a continuous linear right inverse], then $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective [has a continuous linear right inverse] as well.

(c) If $M : S_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}$ is surjective [has a continuous linear right inverse], then $M : S_{\{N_p\}}(0, \infty) \to \Lambda_{\{N_p\}}$ is surjective [has a continuous linear right inverse] as well.

Proposition 6.3. Let $(N_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying conditions $(lc)$ and $(dc)$. 

(a) The following statements are equivalent:

(i) \((N_p)_{p\in\mathbb{N}}\) satisfies \((\gamma_1)\).
(ii) \(\mathcal{M} : \mathcal{S}^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)}\) has a continuous linear right inverse.
(iii) \(\mathcal{M} : \mathcal{S}^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)}\) is surjective.

(b) The following statements are equivalent:

(i) \((N_p)_{p\in\mathbb{N}}\) satisfies \((\gamma_1)\).
(ii) \(\mathcal{M} : \mathcal{S}^0_{(N_p)}(\mathbb{R}) \to \Lambda_{\{N_p\}}\) is surjective.
(iii) \(\mathcal{M} : \mathcal{S}^0_{(N_p)}(\mathbb{R}) \to \Lambda_{\{N_p\}}\) is surjective.

(c) The following statements are equivalent:

(i) \((N_p)_{p\in\mathbb{N}}\) satisfies \((\gamma_1)\) and \((\beta_2)\).
(ii) \(\mathcal{M} : \mathcal{S}^0_{\{N_p\}}(\mathbb{R}) \to \Lambda_{\{N_p\}}\) has a continuous linear right inverse.
(iii) \(\mathcal{M} : \mathcal{S}^0_{\{N_p\}}(\mathbb{R}) \to \Lambda_{\{N_p\}}\) has a continuous linear right inverse.

The rest of this section is devoted to the proofs of the above two results.

Proof of Proposition 6.2. We only show that \(\mathcal{M} : \mathcal{S}_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}\) has a continuous linear right inverse if and only if \(\mathcal{M} : \mathcal{S}^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)}\) does so; all other statements follow from a similar argument. We start with the direct implication. The proof is divided into two steps.

**Step I.** The mappings \(\mathcal{M}_e : \mathcal{S}_{(N_p)}(0, \infty) \to \Lambda_{(M_p)}, \varphi \mapsto (\mu_{2p}(\varphi))_{p\in\mathbb{N}},\) and \(\mathcal{M}_o : \mathcal{S}_{(N_p)}(0, \infty) \to \Lambda_{(M_p)}, \varphi \mapsto (\mu_{2p+1}(\varphi))_{p\in\mathbb{N}},\) have continuous linear right inverses. Let \(R : \Lambda_{(M_p)} \to \mathcal{S}_{(M_p)}(0, \infty)\) be a continuous linear right inverse of \(\mathcal{M} : \mathcal{S}_{(M_p)}(0, \infty) \to \Lambda_{(M_p)}\). Lemmas 3.6(b) and 3.7(b) imply that the mapping

\[
T_e : \mathcal{S}_{(M_p)}(0, \infty) \to \mathcal{S}_{(N_p)}(0, \infty), \quad T_e(\varphi)(x) = \begin{cases} 
2x\varphi(x^2), & x > 0, \\
0, & x \leq 0,
\end{cases}
\]

is well-defined and continuous. We claim that \(T_e \circ R : \Lambda_{(M_p)} \to \mathcal{S}_{(N_p)}(0, \infty)\) is a continuous linear right inverse of \(\mathcal{M}_e : \mathcal{S}_{(N_p)}(0, \infty) \to \Lambda_{(M_p)}\). Let \(a = (a_p)_{p\in\mathbb{N}} \in \Lambda_{(M_p)}\). Then

\[
\mu_{2p}((T_e \circ R)(a)) = \int_0^\infty x^{2p}[(T_e \circ R)(a)](x) \, dx = 2 \int_0^\infty x^{2p}[R(a)](x^2) \, dx
\]

\[
= \int_0^\infty x^{p}[R(a)](x) \, dx = \mu_p(R(a)) = a_p
\]
for all \( p \in \mathbb{N} \). Likewise, by Lemma 3.7(b) the mapping

\[
T_o : S_{(M_p)}(0, \infty) \to S_{(N_p)}(0, \infty), \quad T_o(\varphi)(x) = \begin{cases} 2\varphi(x^2), & x > 0, \\ 0, & x \leq 0, \end{cases}
\]

is well-defined and continuous. We claim that \( T_o \circ R : \Lambda_{(M_p)} \to S_{(N_p)}(0, \infty) \) is a continuous linear right inverse of \( M_o : S_{(N_p)}(0, \infty) \to \Lambda_{(M_p)} \). Let \( a = (a_p)_{p \in \mathbb{N}} \in \Lambda_{(M_p)}. \) Then

\[
\mu_{2p+1}((T_o \circ R)(a)) = \int_0^\infty x^{2p+1}([T_o \circ R](a))(x) \, dx
\]

\[
= 2 \int_0^\infty x^{2p}[R(a)](x^2) \, dx = \int_0^\infty x^p[R(a)](x) \, dx = \mu_p(R(a)) = a_p
\]

for all \( p \in \mathbb{N} \).

**Step II.** The mapping \( M : S^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)} \) has a continuous linear right inverse. Let \( R_e : \Lambda_{(M_p)} \to S_{(N_p)}(0, \infty) \) and \( R_o : \Lambda_{(M_p)} \to S_{(N_p)}(0, \infty) \) be continuous linear right inverses of \( M_e : S_{(N_p)}(0, \infty) \to \Lambda_{(M_p)} \) and \( M_o : S_{(N_p)}(0, \infty) \to \Lambda_{(M_p)}, \) respectively. Consider the continuous mappings

\[
T_e : \Lambda_{(N_p)} \to \Lambda_{(M_p)}, \quad (a_p)_{p \in \mathbb{N}} \mapsto (a_{2p})_{p \in \mathbb{N}},
\]

\[
T_o : \Lambda_{(N_p)} \to \Lambda_{(M_p)}, \quad (a_p)_{p \in \mathbb{N}} \mapsto (a_{2p+1})_{p \in \mathbb{N}},
\]

and

\[
S_e : S_{(N_p)}(0, \infty) \to S^0_{(N_p)}(\mathbb{R}), \quad \varphi \mapsto \frac{\varphi + \varphi(-\cdot)}{2},
\]

\[
S_o : S_{(N_p)}(0, \infty) \to S^0_{(N_p)}(\mathbb{R}), \quad \varphi \mapsto \frac{\varphi - \varphi(-\cdot)}{2}.
\]

Let \( a = (a_p)_{p \in \mathbb{N}} \in \Lambda_{(N_p)} \). Then

\[
\mu_{2p}((S_e \circ R_e \circ T_e)(a)) = a_{2p}, \quad \mu_{2p+1}((S_e \circ R_e \circ T_e)(a)) = 0,
\]

\[
\mu_{2p}((S_o \circ R_o \circ T_o)(a)) = 0, \quad \mu_{2p+1}((S_o \circ R_o \circ T_o)(a)) = a_{2p+1},
\]

for all \( p \in \mathbb{N} \). Hence, \( S_e \circ R_e \circ T_e + S_o \circ R_o \circ T_o \) is a continuous linear right inverse of \( M : S^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)}. \)

Next, we show the converse implication. Again, we divide the proof into two steps.

**Step I.** The mapping \( M_e : S_{(N_p)}(0, \infty) \to \Lambda_{(M_p)}, \varphi \mapsto (\mu_{2p}(\varphi))_{p \in \mathbb{N}}, \) has a continuous linear right inverse. Let \( R : \Lambda_{(N_p)} \to S^0_{(N_p)}(\mathbb{R}) \) be a continuous linear right inverse of \( M : S^0_{(N_p)}(\mathbb{R}) \to \Lambda_{(N_p)}. \) Consider the continuous
mappings $T : A_{(M_p)} \to A_{(N_p)}$ given by $T((a_p)_{p \in \mathbb{N}}) = (b_p)_{p \in \mathbb{N}}$, where

$$b_p = \begin{cases} a_q, & p = 2q, \, q \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$S : S_{(N_p)}^0(\mathbb{R}) \to S_{(N_p)}(0, \infty), \quad S(\varphi)(x) = \begin{cases} \varphi(x) + \varphi(-x), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

We claim that $S \circ R \circ T : A_{(M_p)} \to S_{(N_p)}(0, \infty)$ is a continuous linear right inverse of $\mathcal{M}_e : S_{(N_p)}(0, \infty) \to A_{(M_p)}$. Let $a = (a_p)_{p \in \mathbb{N}} \in A_{(M_p)}$. Then

$$\mu_{2p}((S \circ R \circ T)(a)) = \int_{0}^{\infty} x^{2p}[(S \circ R \circ T)(a)](x) \, dx$$

$$= \int_{0}^{\infty} x^{2p}[(R \circ T)(a)](x) \, dx + \int_{0}^{\infty} x^{2p}[(R \circ T)(a)](-x) \, dx$$

$$= \int_{-\infty}^{\infty} x^{2p}[(R \circ T)(a)](x) \, dx = \mu_{2p}(R(T(a))) = (T(a))_{2p} = a_p$$

for all $p \in \mathbb{N}$.

**Step II.** The mapping $\mathcal{M} : S_{(M_p)}(0, \infty) \to A_{(M_p)}$ has a continuous linear right inverse. Let $R_e : A_{(M_p)} \to S_{(N_p)}(0, \infty)$ be a continuous linear right inverse of $\mathcal{M}_e : S_{(N_p)}(0, \infty) \to A_{(M_p)}$. Lemmas 3.6(a) and 3.7(a) imply that the mapping

$$T : S_{(N_p)}(0, \infty) \to S_{(M_p)}(0, \infty), \quad T(\varphi)(x) = \begin{cases} \varphi(x^{1/2})/2x^{1/2}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

is well-defined and continuous. We claim that $T \circ R_e : A_{(M_p)} \to S_{(M_p)}(0, \infty)$ is a continuous linear right inverse of $\mathcal{M} : S_{(M_p)}(0, \infty) \to A_{(M_p)}$. Let $a = (a_p)_{p \in \mathbb{N}} \in A_{(M_p)}$. Then

$$\mu_p((T \circ R_e)(a)) = \int_{0}^{\infty} x^{p}[(T \circ R_e)(a)](x) \, dx = \int_{0}^{\infty} x^{p}[R_e(a)](x^{1/2})/2x^{1/2} \, dx$$

$$= \int_{0}^{\infty} x^{2p}[R_e(a)](x) \, dx = \mu_{2p}(R_e(a)) = a_p$$

for all $p \in \mathbb{N}$. □

**Proof of Proposition 6.3 (a) (i)⇒(ii).** We apply Proposition 4.3 with $E = S_{(N_p)}(\mathbb{R})$, $F = A_{(N_p)}$, $G = \mathbb{C}^\mathbb{N}$, $T = \mathcal{M} : S_{(N_p)}(\mathbb{R}) \to A_{(N_p)}$ and $S = \mathcal{B} : S_{(N_p)}(\mathbb{R}) \to \mathbb{C}^\mathbb{N}$. By Lemma 3.3, the topology induced by $E = S_{(N_p)}(\mathbb{R})$ on
ker \( S = S_{(N_p)}^0(\mathbb{R}) \) coincides with the original topology of \( S_{(N_p)}^0(\mathbb{R}) \). We now verify conditions (1)–(4) of Proposition 4.3: (1) is obvious. (2) The sequence of standard unit vectors is an absolute Schauder basis in \( A_{(N_p)} \). (3) has been shown in Proposition 5.2. (4) Set \( E_0 = \mathcal{F}^{-1}(\mathcal{D}_{[-1,1]}^{(N_p)}) \) and endow it with the topology generated by the system of seminorms \( \{ p \circ \mathcal{F} \mid p \in \text{csn}(\mathcal{D}_{[-1,1]}^{(N_p)}) \} \).

Notice that \( \mathcal{D}_{[-1,1]}^{(N_p)} \subset S^{(N_p)}(\mathbb{R}) \) with continuous inclusion mapping and recall that \( \mathcal{F} : S^{(N_p)}(\mathbb{R}) \to S^{(N_p)}(\mathbb{R}) \) is a topological isomorphism. Hence, (4.1) is clear, while (4.2) follows from Theorem 2.4(a). Finally,

\[
|\phi^{(p)}(0)| \leq \frac{1}{2\pi} \int_{-1}^{1} |x|^p |\hat{\phi}(x)| \, dx \leq \frac{1}{\pi} \|\hat{\phi}\|_{L^\infty([-1,1])}, \quad p \in \mathbb{N},
\]

for all \( \phi \in E_0 \), whence (4.3) holds.

(ii) \( \Rightarrow \) (iii). This is trivial.

(iii) \( \Rightarrow \) (i). In particular, \( M : S^{(N_p)}(\mathbb{R}) \to A^{(N_p)} \) is surjective. By taking the Fourier transform, we find that \( B : S^{(N_p)}(\mathbb{R}) \to A^{(N_p)} \) is surjective. Choose \( \varphi \in S^{(N_p)}(\mathbb{R}) \) such that \( \varphi^{(p)}(0) = \delta_{0,p} \) for all \( p \in \mathbb{N} \). Set \( \psi = \varphi - 1 \). Then

\[
\sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{h^p |\psi^{(p)}(x)|}{N_p} < \infty
\]

for all \( h > 0 \) and \( \psi^{(p)}(0) = 0 \) for all \( p \in \mathbb{N} \). Since \( \lim_{|x| \to \infty} \varphi(x) = 0 \), \( \psi \) is not identically zero. Hence, the Denjoy–Carleman theorem implies that \( (N_p)_{p \in \mathbb{N}} \) satisfies (γ). By Theorem 2.4(a), it therefore suffices to show that the mapping \( B : D_{[-1,1]}^{(N_p)} \to A^{(N_p)} \) is surjective. Let \( a \in A^{(N_p)} \) and choose \( \varphi \in S^{(N_p)}(\mathbb{R}) \) such that \( B(\varphi) = a \). Pick \( \psi \in D_{[-1,1]}^{(N_p)} \) such that \( \psi \equiv 1 \) in a neighbourhood of 0. Then \( \varphi \psi \in D_{[-1,1]}^{(N_p)} \) and \( B(\varphi \psi) = a \).

(b) (i) \( \Rightarrow \) (ii). We apply Lemma 4.1 with \( E = \mathcal{S}_{(N_p)}^0(\mathbb{R}), F = A_{(N_p)}, G = \mathbb{C}^N, T = M : \mathcal{S}_{(N_p)}^0(\mathbb{R}) \to A_{(N_p)} \) and \( S = B : \mathcal{S}_{(N_p)}^0(\mathbb{R}) \to \mathbb{C}^N \). Theorem 2.4(b) and the inclusion \( D_{[-1,1]}^{(N_p)} \subset \mathcal{S}_{(N_p)}^0(\mathbb{R}) \) imply that \( B : \mathcal{S}_{(N_p)}^0(\mathbb{R}) \to A_{(N_p)} \) is surjective. By taking the Fourier transform, we deduce that \( T \) is surjective. \( \mathcal{S}_{\ker T} \) is surjective because of Proposition 5.1 and the inclusion \( S^{(N_p)}(\mathbb{R}) \subset \mathcal{S}_{(N_p)}^0(\mathbb{R}) \).

(ii) \( \Rightarrow \) (iii). This follows from the inclusion \( \mathcal{S}_{(N_p)}^0(\mathbb{R}) \subset \mathcal{S}_{(N_p)}^0(\mathbb{R}) \).

(iii) \( \Rightarrow \) (i). This can be shown in a similar way to (iii) \( \Rightarrow \) (i) in (a).

(c) (i) \( \Rightarrow \) (ii). We apply Proposition 4.3 with \( E = \mathcal{S}_{(N_p)}^0(\mathbb{R}), F = A_{(N_p)}, G = \mathbb{C}^N, T = M : \mathcal{S}_{(N_p)}^0(\mathbb{R}) \to A_{(N_p)} \) and \( S = B : \mathcal{S}_{(N_p)}^0(\mathbb{R}) \to \mathbb{C}^N \). By Lemma 3.3, the topology induced by \( E = \mathcal{S}_{(N_p)}^0(\mathbb{R}) \) on \( \ker S = \mathcal{S}_{(N_p)}^0(\mathbb{R}) \)
Notice that $F$ call that half-plane $H$ the Borel–Ritt problem in spaces of ultraholomorphic functions on the upper half-plane.

In this final section, we show an analogue of Theorem 6.1 both for

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before, we will sometimes use $S_{\infty}^\dagger$ for all $\varphi \in E_0$, whence (4.3) holds.

(ii)⇒(iii) follows from the continuous inclusion $\hat{S}_{\{N_p\}}(\mathbb{R}) \subset S_{\{N_p\}}(\mathbb{R})$, and (iii)⇒(i) can be shown in a similar way to (iii)⇒(i) in (a). 

7. The Stieltjes moment problem in $S^\dagger_*(0, \infty)$ and the Borel–Ritt problem in spaces of ultraholomorphic functions on the upper half-plane. In this final section, we show an analogue of Theorem 6.1 both for the Stieltjes moment problem in Gelfand–Shilov spaces of type $S^\dagger_*(0, \infty)$ and the Borel–Ritt problem in spaces of ultraholomorphic functions on the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \Im z > 0 \}$.

We start by introducing Gelfand–Shilov spaces of type $S^\dagger_\ast$. Let $(M_p)p\in\mathbb{N}$ and $(A_p)p\in\mathbb{N}$ be weight sequences. For $h > 0$ we write $S^{A_p,h}_{M_p,h}(\mathbb{R})$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that

$$
\| \varphi \|_{S^{A_p,h}_{M_p,h}} := \sup_{p,q \in \mathbb{N}} \frac{h^{p+q}|x|^p\varphi(q)(x)}{A_q M_p} < \infty.
$$

We set

$$
S^{(A_p)}_{(M_p)}(\mathbb{R}) := \lim_{h \to \infty} S^{A_p,h}_{M_p,h}(\mathbb{R}), \quad S^{(A_p)}_{\{M_p\}}(\mathbb{R}) := \lim_{h \to 0^+} S^{A_p,h}_{M_p,h}(\mathbb{R}).
$$

Then $S^{(A_p)}_{(M_p)}(\mathbb{R})$ is a Fréchet space, while $S^{(A_p)}_{\{M_p\}}(\mathbb{R})$ is an $(LB)$-space. As before, we will sometimes use $S^\dagger_\ast(\mathbb{R})$ as a common notation for $S^{(A_p)}_{(M_p)}(\mathbb{R})$ and $S^{(A_p)}_{\{M_p\}}(\mathbb{R})$. If both $(M_p)p\in\mathbb{N}$ and $(A_p)p\in\mathbb{N}$ satisfy (lc), (dc) and (γ), the Fourier transform is a topological isomorphism from $S^\dagger_\ast(\mathbb{R})$ onto $S^\dagger_\ast(\mathbb{R})$ (cf. [10, Sect. IV.6] and [12, Lemma 4.1]).
Let $h > 0$. We define the following closed subspace of $S_{M_p,h}^{A_p}(\mathbb{R})$:

$$S_{M_p,h}^{A_p}(0, \infty) := \{ \varphi \in S_{M_p,h}^{A_p}(\mathbb{R}) \ | \ \text{supp} \varphi \subseteq [0, \infty) \}$$

and endow it with the norm $\| \cdot \|_{S_{M_p,h}^{A_p}}$. Hence, it becomes a Banach space.

We set

$$S_{(M_p)}^{A_p}(0, \infty) := \lim_{h \to \infty} S_{M_p,h}^{A_p}(0, \infty), \quad S_{\{M_p\}}^{A_p}(0, \infty) := \lim_{h \to 0^+} S_{M_p,h}^{A_p}(0, \infty).$$

Then $S_{(M_p)}^{A_p}(0, \infty)$ is a Fréchet space, while $S_{\{M_p\}}^{A_p}(0, \infty)$ is an $(LB)$-space.

Notice that

$$(7.1) \quad S_{1}^{\dagger}(0, \infty) = \{ \varphi \in S_{1}^{\dagger}(\mathbb{R}) \ | \ \text{supp} \varphi \subseteq [0, \infty) \}$$

as sets. In the Beurling case, it is clear that (7.1) also holds topologically if we endow the space on the right-hand side with the relative topology induced by $S_{(M_p)}^{A_p}(\mathbb{R})$. If $(M_p)_{p \in \mathbb{N}}$ satisfies (lc), the corresponding statement also holds in the Roumieu case. Indeed, in that case, an argument similar to the one in the proof of Lemma 3.1 shows that $S_{\{M_p\}}^{A_p}(\mathbb{R})$ is a $(DFS)$-space (= Schwartz (DF)-space). Therefore, the result follows from the fact that a closed subspace of a $(DFS)$-space is again a $(DFS)$-space and from De Wilde’s open mapping theorem.

The image of $S_{1}^{\dagger}(0, \infty)$ under the Fourier transform can be described as follows.

**Lemma 7.1 (cf. [3, Prop. 2.1]).** Let $(M_p)_{p \in \mathbb{N}}$ and $(A_p)_{p \in \mathbb{N}}$ be weight sequences satisfying (lc), (dc) and $(\gamma)$. Let $\psi \in S_{1}^{\dagger}(\mathbb{R})$. Then $\psi \in F(S_{1}^{\dagger}(0, \infty))$ if and only if there is $\Psi : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(i) $\Psi_{|\mathbb{R}} = \psi$.

(ii) $\Psi$ is continuous on $\mathbb{H}$ and holomorphic on $\mathbb{H}$.

(iii) $\lim_{z \in \mathbb{H}, z \to \infty} \Psi(z) = 0$.

Next, we define spaces of ultraholomorphic functions on $\mathbb{H}$. Given an open subset $\Omega \subseteq \mathbb{C}$, we denote by $O(\Omega)$ the space of holomorphic functions on $\Omega$. Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence. For $h > 0$ we write $A_{M_p,h}^{A}(\mathbb{H})$ for the Banach space consisting of all $f \in O(\mathbb{H})$ such that

$$\| f \|_{A_{M_p,h}^{A}} := \sup_{p \in \mathbb{N}} \sup_{z \in \mathbb{H}} \frac{h^p |f^{(p)}(z)|}{M_p} < \infty.$$ 

We set

$$A_{(M_p)}^{A}(\mathbb{H}) := \lim_{h \to \infty} A_{M_p,h}^{A}(\mathbb{H}), \quad A_{\{M_p\}}^{A}(\mathbb{H}) := \lim_{h \to 0^+} A_{M_p,h}^{A}(\mathbb{H}).$$
Then $\mathcal{A}^{(M_p)}(\mathbb{H})$ is a Fréchet space, while $\mathcal{A}^{\{M_p\}}(\mathbb{H})$ is an $(LB)$-space. Let $f \in \mathcal{A}^*(\mathbb{H})$. Since $f$ and all its derivatives are Lipschitz on $\mathbb{H}$, the limit
\[ f_p(x) = \lim_{z \to x, z \in \mathbb{H}} f^{(p)}(z) \in \mathbb{C}, \quad x \in \mathbb{R}, \]
exists for all $p \in \mathbb{N}$. Moreover, $f_0 \in C^\infty(\mathbb{R})$ and $f_0^{(p)} = f_p$ for all $p \in \mathbb{N}$. From now on, we simply write $f_0 = f$. The asymptotic Borel mapping
\[ \mathcal{B} : \mathcal{A}^*(\mathbb{H}) \to \Lambda_*, \quad f \mapsto (f^{(p)}(0))_{p \in \mathbb{N}}, \]
is well-defined and continuous.

We are ready to state the two main results of this section.

**Theorem 7.2.** Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying (lc) and (dc), and let $(A_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying (lc) and $(\gamma)$.

(a) The following statements are equivalent:

(i) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$.

(ii) $\mathcal{M} : S^{(A_p)}_{(M_p)}(0, \infty) \to A_{(M_p)}$ has a continuous linear right inverse.

(iii) $\mathcal{M} : S^{(A_p)}_{(M_p)}(0, \infty) \to A_{(M_p)}$ is surjective.

(b) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$ if and only if $\mathcal{M} : S^{\{A_p\}}_{\{M_p\}}(0, \infty) \to A_{\{M_p\}}$ is surjective.

(c) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$ and $(\beta_2)$ if and only if $\mathcal{M} : S^{\{A_p\}}_{\{M_p\}}(0, \infty) \to A_{\{M_p\}}$ has a continuous linear right inverse.

**Remark 7.3.** In [5] Thm. 3.5], the direct implication of Theorem 7.2(b) was shown under the assumptions (slc) $(= (M_p/p!)_{p \in \mathbb{N}}$ satisfies (lc)) and (mg), while the converse implication was shown under the assumptions (slc) and (dc). The reader should be aware that there is a difference in notation between [5] and the present article. Namely, given two weight sequences $M = (M_p)_{p \in \mathbb{N}}$ and $A = (A_p)_{p \in \mathbb{N}}$, the space denoted by $S^A_M(0, \infty)$ in [5] coincides with $S^{\{p!A_p\}}_{\{p!M_p\}}(0, \infty)$. Moreover, if $(p!M_p)_{p \in \mathbb{N}}$ satisfies (lc), then $(p!M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$ if and only if $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_1)$ [19] Lemma 2.1]. This shows that [5] Thm. 3.5] and Theorem 7.2 are consistent with one another.

**Theorem 7.4.** Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence satisfying (lc) and (dc).

(a) The following statements are equivalent:

(i) $M_p$ satisfies $(\gamma_2)$.

(ii) $\mathcal{B} : \mathcal{A}^{(M_p)}(\mathbb{H}) \to A_{(M_p)}$ has a continuous linear right inverse.

(iii) $\mathcal{B} : \mathcal{A}^{(M_p)}(\mathbb{H}) \to A_{(M_p)}$ is surjective.

(b) $(M_p)_{p \in \mathbb{N}}$ satisfies $(\gamma_2)$ if and only if $\mathcal{B} : \mathcal{A}^{\{M_p\}}(\mathbb{H}) \to A_{\{M_p\}}$ is surjective.
(c) \((M_p)_{p \in \mathbb{N}}\) satisfies \((\gamma_2)\) and \((\beta_2)\) if and only if \(B : A^{(M_p)}(\mathbb{H}) \to \Lambda_{\{M_p\}}\) has a continuous linear right inverse.

Remark 7.5. Theorem 7.4 improves various results from [19, 20] in the special case of the upper half-plane. The implication \((i) \Rightarrow (iii)\) from Theorem 7.4(a) and the direct implication of Theorem 7.4(b) were shown in [20] Cor. 3.4.1 and [20] Thm. 3.2.1, respectively, under the assumptions (slc) and (mg). The existence of a continuous linear right inverse of \(B : A^{(M_p)}(\mathbb{H}) \to \Lambda_{\{M_p\}}\), respectively) was shown in [19] Thm. 4.4 and Thm. 4.5 (Thm. 5.4 and Thm. 5.5), respectively) under the assumptions (lc) and \((\gamma_3)\) ((lc), \((\gamma_3)\) and \((\beta_2)\), respectively); recall from Subsection 2.2 that \((\gamma_3)\) is much more restrictive than \((\gamma_2)\). Finally, we point out that a stronger variant of the converse implication of Theorem 7.4(b) was recently shown in [11, Thm. 4.14] under the assumptions (slc) and (dc).

In view of Theorem 6.1, Theorems 7.2 and 7.4 are both consequences of the following result; it is essentially shown in [5, Sect. 3], but we repeat the argument here for the sake of completeness.

Proposition 7.6. Let \((M_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc) and (dc), and let \((A_p)_{p \in \mathbb{N}}\) be a weight sequence satisfying (lc) and \((\gamma)\).

(a) The following statements are equivalent:

(i) \(M : \mathcal{S}_{\{A_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

(ii) \(M : \mathcal{S}_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

(iii) \(B : A^{(M_p)}(\mathbb{H}) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

(b) The following statements are equivalent:

(i) \(M : \mathcal{S}_{\{A_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

(ii) \(M : \mathcal{S}_{\{M_p\}}(0, \infty) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

(iii) \(B : A^{(M_p)}(\mathbb{H}) \to \Lambda_{\{M_p\}}\) is surjective [has a continuous linear right inverse].

Proof. We only show the equivalences about the existence of a continuous linear right inverse stated in (a); all other cases follow from a similar argument.

(i) \(\Rightarrow (ii)\) follows from the continuous inclusion \(\mathcal{S}_{\{A_p\}}(0, \infty) \subset \mathcal{S}_{\{M_p\}}(0, \infty)\).
(ii)⇒(iii). We define the Laplace transform of \( \varphi \in S(M_p)(0, \infty) \) as

\[
\mathcal{L}(\varphi)(z) := \int_0^\infty \varphi(t) e^{itz} \, dt, \quad z \in \mathbb{H}.
\]

Then \( \mathcal{L} : S(M_p)(0, \infty) \rightarrow \mathcal{A}(M_p)(\mathbb{H}) \) is well-defined and continuous, and \( \mathcal{L}(\varphi)(p)(0) = ip \mu_p(\varphi) \) for all \( p \in \mathbb{N} \). Let \( R : A(M_p) \rightarrow S(M_p)(0, \infty) \) be a continuous linear right inverse of \( \mathcal{M} : S(M_p)(0, \infty) \rightarrow A(M_p) \). Consider the continuous mapping

\[
T : A(M_p) \rightarrow A(M_p), \quad T((a_p)_{p \in \mathbb{N}}) = ((-i)^p a_p)_{p \in \mathbb{N}}.
\]

Then \( \mathcal{L} \circ R \circ T : A(M_p) \rightarrow \mathcal{A}(M_p)(\mathbb{H}) \) is a continuous linear right inverse of \( \mathcal{B} : \mathcal{A}(M_p)(\mathbb{H}) \rightarrow A(M_p) \).

(iii)⇒(i). We start by making some preliminary observations about the weight sequences \( (M_p)_{p \in \mathbb{N}} \) and \( (A_p)_{p \in \mathbb{N}} \). By \([5, \text{Lemma 2.2}]\), we may assume without loss of generality that \( (A_p)_{p \in \mathbb{N}} \) satisfies \((\gamma)\). Next, choose \( f \in \mathcal{A}(M_p)(\mathbb{H}) \) such that \( f^{(p)}(0) = \delta_{1,p} \) for all \( p \in \mathbb{N} \). Set \( \varphi(x) = f(x) - x \) for \( x \in \mathbb{R} \). Then

\[
\sup_{p \in \mathbb{N}} \sup_{x \in [-R,R]} \frac{h^p|\varphi^{(p)}(x)|}{M_p} < \infty
\]

for all \( h > 0 \) and \( R > 0 \), and \( \varphi^{(p)}(0) = 0 \) for all \( p \in \mathbb{N} \). Since \( f \) is bounded, \( \varphi \) is not identically zero. Hence, the Denjoy–Carleman theorem implies that \( (M_p)_{p \in \mathbb{N}} \) satisfies \((\gamma)\). Consequently, \( \lim_{p \to \infty} (p! / M_p)^{1/p} = 0 \) \([12, \text{Lemma 4.1}]\).

We now turn to the actual proof. It is based on the following observation \([3, \text{Lemma 3.6}]\): Let \( (a_p)_{p \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \) and let \( G \in C^{\infty}((\delta, \delta)), \delta > 0, \) with \( G(0) \neq 0 \). Set

\[
b_p = \sum_{n=0}^p \binom{p}{n} a_n \left( \frac{1}{G} \right)^{(p-n)}(0), \quad p \in \mathbb{N}.
\]

Then

\[
\sum_{n=0}^p \binom{p}{n} b_n G^{(p-n)}(0) = a_p, \quad p \in \mathbb{N}.
\]

Set \( V = \{ z \in \mathbb{C} | \Im z > -1 \} \). By \([3, \text{Lemma 3.1}], [12, \text{Lemma 3.10}] \) and \([12, \text{Lemma 4.3}] \), there is \( G \in \mathcal{O}(V) \) with the following properties:

(i) \( G \) does not vanish on \( V \).

(ii) \( \sup_{z \in V} |G(z)|e^{A(|z|)} < \infty \) for all \( h > 0 \).

(iii) \( \sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|G^{(p)}(x)|e^{A(|x|)}}{2^{pp!}} < \infty \) for all \( h > 0 \).

Lemma \([7, \text{Lemma 7.1}]\) implies that the mapping \( T : \mathcal{A}(M_p)(\mathbb{H}) \rightarrow S^{(A_p)}(0, \infty), \ f \mapsto \mathcal{F}^{-1}((fG)|_\mathbb{R}) \), is well-defined. Since \( \mathcal{F} : S^{(A_p)}(\mathbb{R}) \rightarrow S^{(M_p)}(\mathbb{R}) \) is a topological
isomorphism and (7.1) holds topologically, $T$ is also continuous. Next, the Cauchy estimates yield
\[
\sup_{p \in \mathbb{N}} \frac{|(1/G)^{(p)}(0)|}{2^p p!} < \infty.
\]
Hence, the mapping
\[
S : A_{(M_p)} \to A_{(M_p)}: S((a_p)_{p \in \mathbb{N}}) = \left( \sum_{n=0}^{p} \binom{p}{n} a_n \left( \frac{1}{G} \right)^{(p-n)}(0) \right)_{p \in \mathbb{N}},
\]
is well-defined and continuous. Let $R : A_{(M_p)} \to A^{(M_p)}(\mathbb{H})$ be a continuous linear right inverse of $B : A^{(M_p)}(\mathbb{H}) \to A_{(M_p)}$. Then $T \circ R \circ S : A_{(M_p)} \to S_{(M_p)}^{(A_p)}(0, \infty)$ is a continuous linear right inverse of $M : S_{(M_p)}^{(A_p)}(0, \infty) \to A_{(M_p)}$.  

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Andreas Debrouwere
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University
Krijgslaan 281, 9000 Gent, Belgium
E-mail: Andreas.Debrouwere@UGent.be