BACKWARD MULTIVALUED MCKEAN-VLASOV SDES AND ASSOCIATED VARIATIONAL INEQUALITIES

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ABSTRACT. The work concerns a type of backward multivalued McKean-Vlasov stochastic differential equations. First, we prove the existence and uniqueness of solutions for backward multivalued McKean-Vlasov stochastic differential equations. Then, it is presented that their solutions depend continuously on the terminal values. Finally, we give a probabilistic interpretation for viscosity solutions of nonlocal quasi-linear parabolic variational inequalities.

1. INTRODUCTION

McKean-Vlasov stochastic differential equations (SDEs for short), also called mean-field SDEs or distribution-dependent SDEs, can track back to Kac [16] in 1956. Then in 2009 Buckdahn, Djehiche, Li and Peng [4] investigated a type of backward McKean-Vlasov SDEs. Since this, the theory of backward McKean-Vlasov SDEs and forward-backward McKean-Vlasov SDEs, as well as that of the associated partial differential equations have been widely studied. For example, in [5] Buckdahn, Li and Peng not only proved the existence, uniqueness and a comparison theorem of the solutions for backward McKean-Vlasov SDEs but also obtained a probabilistic interpretation of related nonlocal partial differential equations. Later, Li and Luo [19] studied reflected backward McKean-Vlasov SDEs, and proved the existence and uniqueness for their solutions under the Lipschitz condition. Li [18] also observed reflected backward McKean-Vlasov SDEs in a purely probabilistic method, and gave a probabilistic interpretation for obstacle problems of nonlinear and nonlocal partial differential equations by means of reflected backward McKean-Vlasov SDEs. Besides, Lu, Ren and Hu [20] dealt with a class of backward McKean-Vlasov SDEs with subdifferential operators corresponding to lower semi-continuous convex functions. By means of the Yosida approximation, they established the existence and uniqueness of the solutions for backward McKean-Vlasov SDEs with subdifferential operators, and attained a probability interpretation for the viscosity solutions of a class of nonlocal parabolic variational inequalities. Here we emphasize that in [5, 18, 19, 20] the coefficients of backward McKean-Vlasov SDEs depend on the distributions of solution processes through their expectations.

On the other hand, there also exist backward McKean-Vlasov SDEs in which the coefficients straightly depend on the distributions of solution processes. Note that the type...
of backward McKean-Vlasov SDEs is more general than the mentioned type in the above paragraph. Let us list some works related with ours. In [7] Carmona and Delarue provided an existence result for the solutions of a fully coupled forward-backward McKean-Vlasov SDE under a very mild Lipschitz condition. Then in [2] Bensoussan, Yam and Zhang proposed a broad class of natural monotonicity conditions and established the well-posedness for a type of forward-backward McKean-Vlasov SDEs. Recently, Li [17] studied a type of backward McKean-Vlasov SDEs driven by Brownian motions and independent Poisson random measures. There she proved the well-posedness of their solutions, and also obtained the well-posedness of classical solutions for related nonlocal quasi-linear integral-partial differential equations under some regular assumptions.

In the paper, we concentrate on a type of more general backward McKean-Vlasov SDEs. Concretely speaking, we fix \( T > 0 \) and a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\), and consider the following backward multivalued McKean-Vlasov SDE on \( \mathbb{R}^d \):

\[
\begin{aligned}
\mathrm{d}Y_t &\in A(Y_t)\mathrm{d}t - G(t, Y_t, Z_t, \mathcal{L}(Y_t, Z_t))\mathrm{d}t + Z_t \mathrm{d}W_t, \\
Y_T &= \xi, \mathcal{L}(Y_t, Z_t) = \text{the probability distribution of } (Y_t, Z_t),
\end{aligned}
\]

(1)

where \( W = (W^1, W^2, \ldots, W^l) \) is a \((\mathcal{F}_t)_{t \in [0, T]}\)-adapted \( l \)-dimensional standard Brownian motion, \( A: \mathbb{R}^d \mapsto 2^{\mathbb{R}^d} \) is a maximal monotone operator, for the coefficient \( G: \Omega \times [0, T] \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}^d \), \( \forall (y, z, \varrho) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^d) \), \( G(\cdot, y, z, \varrho) \) is \((\mathcal{F}_t)_{t \in [0, T]}\)-predictable, and \( \xi \) is a \( \mathcal{F}_T \)-measurable random variable with values in \( D(\mathbb{R}^d) \) and \( \mathbb{E}|\xi|^2 < \infty \). If \( A = 0 \), Eq.(1) becomes a backward McKean-Vlasov SDE in [2, 7, 17]. And if \( A \neq 0 \), backward multivalued McKean-Vlasov SDEs like Eq.(1) include reflected backward McKean-Vlasov SDEs in [18, 19] and backward McKean-Vlasov SDEs with subdifferential operators in [20]. Hence, as far as we know, backward multivalued McKean-Vlasov SDEs are the most general backward McKean-Vlasov SDEs. Since backward multivalued McKean-Vlasov SDEs like Eq.(1) are widely applied in finance, the control theory and the game theory (cf. [11] and cited references there), we are devoted to studying Eq.(1).

As a whole, our contribution are two-folded:

- We prove the well-posedness of Eq.(1) under the Lipschitz condition, and also obtain that its solution depend continuously on the terminal value.
- Combining Eq.(1) with a forward McKean-Vlasov SDE, we give a probabilistic interpretation for viscosity solutions of nonlocal quasi-linear parabolic variational inequalities.

It is worthwhile to mentioning our results and methods. Notice that Li [18] and Lu, Ren, Hu [20] also showed the well-posedness and probabilistic interpretation. Since our equation is more general than that in [18, 20], our results are better. Moreover, we establish continuous dependence of solutions on the terminal values. This is important for application of backward multivalued McKean-Vlasov SDEs. Besides, we emphasize that the appearance of maximal monotone operators brings a lot of trouble, and our methods and techniques are more subtle.

The content of the paper is arranged as follows. In the next section, we introduce notations and concepts, such as maximal monotone operators and the derivative for functions on \( \mathcal{M}_2(\mathbb{R}^d) \). Moreover, a result about backward McKean-Vlasov SDEs is listed and some conclusions on backward multivalued stochastic differential equations are proved in the section. Then we place the existence, uniqueness and continuous dependence on the
terminal values of the solutions for Eq. (1) in Section 3. In Section 4, viscosity solutions of nonlocal quasi-linear parabolic variational inequalities are established.

The following convention will be used throughout the paper: $C$ with or without indices will denote different positive constants whose values may change from one place to another.

2. Preliminary

In the section, we introduce notations and concepts, and recall some results used in the sequel.

2.1. Notations. In the subsection, we introduce some notations.

For convenience, we shall use $| \cdot |$ and $\| \cdot \|$ for norms of vectors and matrices, respectively. Furthermore, let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^d$. Let $B^*$ denote the transpose of a matrix $B$.

Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel $\sigma$-algebra on $\mathbb{R}^d$ and $\mathcal{P}(\mathbb{R}^d)$ be the space of all probability measures defined on $\mathcal{B}(\mathbb{R}^d)$ carrying the usual topology of weak convergence. Let $\mathcal{M}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathcal{B}(\mathbb{R}^d)$ with finite second order moments. That is,

$$\mathcal{M}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \|\mu\|_2 := \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.$$ 

Define the following metric on $\mathcal{M}_2(\mathbb{R}^d)$:

$$\rho^2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy), \quad \mu_1, \mu_2 \in \mathcal{M}_2(\mathbb{R}^d),$$

where $\mathcal{C}(\mu_1, \mu_2)$ denotes the set of all the probability measures whose marginal distributions are $\mu_1, \mu_2$, respectively. Thus, $\left( \mathcal{M}_2(\mathbb{R}^d), \rho \right)$ is a Polish space.

Let $\mathcal{S}_F^2([0, T], \mathbb{R}^d)$ be the space of all $\mathbb{F}$-adapted processes $Y : \Omega \times [0, T] \mapsto \mathbb{R}^d$ with $\mathbb{E}\left[ \sup_{t \in [0, T]} |Y_t|^2 \right] < \infty$, where $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. Let $\mathcal{H}_F^2([0, T], \mathbb{R}^{d \times l})$ be the space of all $\mathbb{F}$-predictable processes $Z : \Omega \times [0, T] \mapsto \mathbb{R}^{d \times l}$ with $\mathbb{E}\left[ \int_0^T \|Z_t\|^2 dt \right] < \infty$.

Let $C(\mathbb{R}^d)$ be the collection of continuous functions on $\mathbb{R}^d$ and $C^2(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ which have continuous partial derivatives of order up to 2.

2.2. Maximal monotone operators. In the subsection, we introduce maximal monotone operators.

For a multivalued operator $A : \mathbb{R}^d \mapsto 2^{\mathbb{R}^d}$, where $2^{\mathbb{R}^d}$ stands for all the subsets of $\mathbb{R}^d$, set

$$\mathcal{D}(A) := \left\{ x \in \mathbb{R}^d : A(x) \neq \emptyset \right\},$$

$$Gr(A) := \left\{ (x, y) \in \mathbb{R}^{2d} : x \in \mathcal{D}(A), \ y \in A(x) \right\}.$$ 

We say that $A$ is monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for any $(x_1, y_1), (x_2, y_2) \in Gr(A)$, and $A$ is maximal monotone if

$$(x_1, y_1) \in Gr(A) \iff \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \forall (x_2, y_2) \in Gr(A).$$

In the following, we recall some properties of a maximal monotone operator $A$. (cf. [13])
(i) \(\text{Int}(\mathcal{D}(A))\) and \(\overline{\mathcal{D}(A)}\) are convex subsets of \(\mathbb{R}^d\) with \(\text{Int} (\overline{\mathcal{D}(A)}) = \text{Int} (\mathcal{D}(A))\), where \(\text{Int}(\mathcal{D}(A))\) denotes the interior of the set \(\mathcal{D}(A)\).

(ii) For every \(x \in \mathbb{R}^d\), \(A(x)\) is a closed and convex subset of \(\mathbb{R}^d\). Let \(A^\circ (x) := \text{proj}_{A^\circ (x)}(0)\) be the minimal section of \(A\), where \(\text{proj}^D\) is designated as the projection on every closed and convex subset \(D \subset \mathbb{R}^d\) and \(\text{proj}_0(0) = \infty\). Then
\[
x \in \mathcal{D}(A) \iff |A^\circ (x)| < \infty.
\]

(iii) For \(\varepsilon > 0\), the resolvent operator \(J_\varepsilon := (I + \varepsilon A)^{-1}\) is a single-valued and contractive operator defined on \(\mathbb{R}^d\) and takes values in \(\mathcal{D}(A)\), and
\[
\lim_{\varepsilon \downarrow 0} J_\varepsilon(x) = \text{proj}_{\mathcal{D}(A)}(x), \quad x \in \mathbb{R}^d.
\]

(iv) \(A_\varepsilon := \frac{1}{\varepsilon}(I - J_\varepsilon)\), called the Yosida approximation of \(A\), is also a single-valued, maximal monotone and Lipschitz continuous operator with the Lipschitz constant \(\frac{1}{\varepsilon}\).

(v) \(A_\varepsilon(x) \in A(J_\varepsilon(x)), \quad x \in \mathbb{R}^d\),

(vi) \(|A_\varepsilon(x)| \leq |A^\circ (x)|, \quad x \in \mathcal{D}(A)\).

(vii) For any \(x \in \mathcal{D}(A)\), \(\lim_{\varepsilon \downarrow 0} A_\varepsilon(x) = A^\circ (x)\), and
\[
\lim_{\varepsilon \downarrow 0} |A_\varepsilon(x)| = |A^\circ (x)|, \quad x \in \mathcal{D}(A),
\]
\[
\lim_{\varepsilon \downarrow 0} |A_\varepsilon(x)| = \infty, \quad x \notin \mathcal{D}(A).
\]

About the Yosida approximation \(A_\varepsilon\), we also mention the following property ([8, Lemma 5.4]).

**Lemma 2.1.** There exist three constants \(a \in \mathbb{R}^d, M_1 > 0, M_2 > 0\) only dependent on \(A\) such that for any \(\varepsilon > 0\) and \(x \in \mathbb{R}^d\)
\[
\langle A_\varepsilon(x), x - a \rangle \geq M_1 |A_\varepsilon(x)| - M_2 |x - a| - M_1 M_2.
\]

Let \(\mathcal{Y}_0\) be the set of all continuous functions \(K : [0, T] \mapsto \mathbb{R}^d\) with finite variations and \(K_0 = 0\). For \(K \in \mathcal{Y}_0\) and \(s \in [0, T]\), we shall use \(|K|^s_0\) to denote the variation of \(K\) on \([0, s]\) and write \(|K|_{TV} := |K|^s_0\). Set
\[
\mathscr{A} := \left\{ (Y, K) : Y \in C([0, T], \overline{\mathcal{D}(A)}), K \in \mathcal{Y}_0, \right. \\
\left. \text{and } \langle Y_t - x, dK_t - ydt \rangle \geq 0 \text{ for any } (x, y) \in \text{Gr}(A) \right\}.
\]

And about \(\mathscr{A}\) we have two following results (cf.[9, 27]).

**Lemma 2.2.** For \(Y \in C([0, T], \overline{\mathcal{D}(A)})\) and \(K \in \mathcal{Y}_0\), the following statements are equivalent:

(i) \((Y, K) \in \mathscr{A}\).

(ii) For any \((x, y) \in C([0, T], \mathbb{R}^d)\) with \((x_t, y_t) \in \text{Gr}(A)\), it holds that
\[
\langle Y_t - x_t, dK_t - y_t dt \rangle \geq 0.
\]

(iii) For any \((Y', K') \in \mathscr{A}\), it holds that
\[
\langle Y_t - Y'_t, dK_t - dK'_t \rangle \geq 0.
\]
Lemma 2.3. Assume that \( \{K^n, n \in \mathbb{N}\} \subset \mathcal{V}_0 \) converges to some \( K \) in \( C([0,T]; \mathbb{R}^d) \) and \( \sup_{n \in \mathbb{N}} |K^n|_{TV} < \infty \). Then \( K \in \mathcal{V}_0 \), and

\[
\lim_{n \to \infty} \int_0^T \langle Y^n_s, dK^n_s \rangle = \int_0^T \langle Y_s, dK_s \rangle,
\]

where the sequence \( \{Y^n\} \subset C([0,T]; \mathbb{R}^d) \) converges to some \( Y \) in \( C([0,T]; \mathbb{R}^d) \).

2.3. Backward McKean-Vlasov SDEs. In the subsection, we recall a result about backward McKean-Vlasov SDEs. Now given a function \( \tilde{G} : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d \times l} \times \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^{d \times l}) \to \mathbb{R}^d \) satisfying that for \( \forall (y, z, \vartheta) \in \mathbb{R}^d \times \mathbb{R}^{d \times l} \times \mathcal{M}_2(\mathbb{R}^d \times \mathcal{M}(\mathbb{R}^{d \times l})) \), \( \tilde{G}(\cdot, y, z, \vartheta) \) is \( (\mathcal{F}_t)_{t \in [0,T]} \)-predictable, and a \( \mathcal{F}_T \)-measurable random variable \( \xi \) with \( \mathbb{E}|\xi|^2 < \infty \).

Assume:

\( (H_1^G) \) There exists a non-random constant \( L_\tilde{G} > 0 \) such that for a.s. \( \omega \in \Omega \), it holds that

\[
\left| \tilde{G}(\omega, t, y, z, \vartheta) - \tilde{G}(\omega, t, y', z', \vartheta') \right| \leq L_\tilde{G} \left( |y - y'| + \|z - z'\| + \rho(\vartheta, \vartheta') \right),
\]

\( t \in [0,T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times l}, \vartheta, \vartheta' \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^{d \times l}). \)

\( (H_2^G) \) \( \mathbb{E} \int_0^T \left| \tilde{G}(t, 0, 0, \delta_{(0,0)}) \right|^2 dt < \infty \), where \( \delta_{(0,0)} \) is the Dirac measure at \( (0,0) \).

The following result comes from [17, Theorem A.1.].

Theorem 2.4. Assume that \( (H_1^G)-(H_2^G) \) hold. Then the following backward McKean-Vlasov SDE

\[
\begin{aligned}
\frac{dY_t}{dt} &= -\tilde{G}\left(t, Y_t, Z_t, \mathcal{L}(Y_t, Z_t)\right) dt + Z_t dW_t, \quad 0 \leq t < T, \\
Y_T &= \xi,
\end{aligned}
\]

has a unique solution \( (Y, Z) \in \mathbb{S}^2((0,T], \mathbb{R}^d) \times \mathbb{H}^2([0,T], \mathbb{R}^{d \times l}) \).

2.4. Backward multivalued SDEs. In the subsection, we recall and prove some results about backward multivalued SDEs.

Consider the following backward multivalued SDE on \( \mathbb{R}^d \):

\[
\begin{aligned}
\frac{dY_t}{dt} &\in A(Y_t) dt - g(t, Y_t, Z_t) dt + Z_t dW_t, \\
Y_T &= \xi,
\end{aligned}
\tag{2}
\]

where the coefficient \( g : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d \times l} \to \mathbb{R}^d \) is Borel measurable, \( \forall (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times l}, g(\cdot, y, z) \) is \( (\mathcal{F}_t)_{t \in [0,T]} \)-progressively measurable, and \( \xi \) is a \( \mathcal{F}_T \)-measurable random variable with values in \( \mathcal{D}(A) \) and \( \mathbb{E}|\xi|^2 < \infty \). We define the existence and uniqueness of solutions for Eq.\( (2) \).

Definition 2.5. We say that Eq.\( (2) \) admits a solution with the terminal value \( \xi \) if there exists a triple \( \{(Y_t, K_t, Z_t) : t \in [0,T]\} \) which is a \( (\mathcal{F}_t)_{t \in [0,T]} \)-progressively measurable process and satisfies

(i) \( (Y, K) \in \mathcal{X}, \text{ dP } \times \text{ dt-a.e. on } \Omega \times [0,T], \)

(ii) \( \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T \|Z_s\|^2 ds \right) < \infty, \)
Theorem 2.9.

\[ Y_t = \xi - (K_T - K_t) + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \]

Definition 2.6. Suppose that \( \{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \{0, T\}}, \mathbb{P}), (Y^1, K^1, Z^1)\} \) and \( \{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \{0, T\}}, \mathbb{P}), (Y^2, K^2, Z^2)\} \) are two solutions for Eq. (2) with \( Y^1_T = Y^2_T = \xi, K^1_T = K^2_T \). If \( Y^1_t = Y^2_t, K^1_t = K^2_t, t \in [0, T], \mathbb{P}\)-a.s. and \( Z^1 = Z^2, d\mathbb{P} \times dt \)-a.e., we say that the uniqueness holds for Eq. (2).

For the existence and uniqueness of Eq. (2), we assume:

(i) For \( \delta > 0 \), there exists a unique solution for Eq. (2), \( \text{Assume that Theorem 2.8.} \)

(ii) \( \{Y^1_t \in \mathbb{R}^d, t \in [0, T] \} \) is \( \mathcal{F}_t \)-progressively measurable, and \( \xi \) is \( \mathcal{F}_\tau \)-measurable, \( \forall (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times l} \), \( g(.; y, z) \) is \( (\mathcal{F}_t)_{t \in [0, T]} \)-progressively measurable, and \( \xi \) is a \( \mathcal{F}_T \)-measurable random variable with values in \( \mathcal{D}(A) \) and \( \mathbb{E}|\xi|^2 < \infty \), \( i = 1, 2 \). We have the following result.

Theorem 2.8. Assume that (H) holds and the coefficient \( g \) satisfies (H\(_1^g\))-(H\(_2^g\)). Then there exists a unique solution for Eq. (2).

Next, suppose that \( (Y^i, K^i, Z^i) \) is the unique solution of the following backward multi-valued SDE:

\[
\begin{align*}
\left\{ \begin{array}{l}
dY^i_t &\in A(Y^i_t)dt - g^i(t, Y^i_t, Z^i_t)dt + Z^i_t dW_t, \\
Y^i_T &\equiv \xi^i,
\end{array} \right.
\end{align*}
\]

where the coefficient \( g^i : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times l} \mapsto \mathbb{R}^d \) is Borel measurable, \( \forall (y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times l} \), \( g^i(\cdot ; y, z) \) is \( (\mathcal{F}_t)_{t \in [0, T]} \)-progressively measurable, and \( \xi^i \) is a \( \mathcal{F}_T \)-measurable random variable with values in \( \mathcal{D}(A) \) and \( \mathbb{E}|\xi|^2 < \infty \), \( i = 1, 2 \). We have the following result.

Theorem 2.9. Assume that (H) holds, \( g^1, g^2 \) satisfy (H\(_1^g\))-(H\(_2^g\)), and there exists a constant \( c^* > 0 \) such that for a.s. \( \omega \in \Omega \), it holds that

\[ |g^i(\omega, t, y, z) - g^j(\omega, t, y', z')| \leq c^* \left( |y - y'| + |z - z'| \right), \]

\[ t \in [0, T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times l}, i = 1, 2. \]

Then for \( \bar{Y} := Y^1 - Y^2, \bar{Z} := Z^1 - Z^2, \bar{g} := g^1 - g^2, \bar{\xi} := \xi^1 - \xi^2 \), it holds that

(i) For \( \delta > 0 \), there exists a constant \( \alpha > 0 \) such that for any \( t \in [0, T] \)

\[
\begin{align*}
|\bar{Y}_t|^2 &+ \frac{1}{2} \mathbb{E} \left( \int_t^T e^{\alpha(s-t)} (|\bar{Y}_s|^2 + ||\bar{Z}_s||^2)ds \right|_{\mathcal{F}_t} \\
&\leq \mathbb{E}e^{\alpha(T-t)} |\bar{\xi}|^2 \left|_{\mathcal{F}_t} \right| + c^* \delta \mathbb{E} \left( \int_t^T e^{\alpha(s-t)} |\bar{g}(s, Y^1_s, Z^1_s)|^2ds \right|_{\mathcal{F}_t}, \text{a.s.} \tag{3}
\end{align*}
\]
where $\delta > H \leq (ii) There exists a constant $\alpha t$ is a constant. Taking $
abla
abla \alpha s$ 1
$\tilde{\xi}^2 + \int_t^T \|\tilde{Z}_s\|^2 ds |\mathcal{F}_t \rangle - 2 \int_t^T e^{\alpha s} \tilde{Y}_s \tilde{d}W_s
\leq e^{\alpha T} \tilde{\xi}^2 + 2 \int_t^T e^{\alpha s} \tilde{Y}_s \langle \tilde{g} (s, Y_{t,s}, Z_{t,s}) + g^2 (s, Y_{t,s}, Z_{t,s}) - g^2 (s, Y_{t,s}, Z_{t,s}) \rangle ds
- 2 \int_t^T e^{\alpha s} \tilde{Y}_s \tilde{d}K_s
\leq e^{\alpha T} \tilde{\xi}^2 + 2 \int_t^T e^{\alpha s} \tilde{Y}_s \langle \tilde{g} (s, Y_{t,s}, Z_{t,s}) + g^2 (s, Y_{t,s}, Z_{t,s}) - g^2 (s, Y_{t,s}, Z_{t,s}) \rangle ds
- 2 \int_t^T e^{\alpha s} \tilde{Y}_s \tilde{d}W_s
Taking the conditional expectation with respect to $\mathcal{F}_t$ on two sides of the above inequality, we have that
\[ e^{\alpha T} \bar{Y}_t^2 + \mathbb{E} \left[ \int_t^T e^{\alpha s} \|\bar{Z}_s\|^2 ds + \alpha \int_t^T e^{\alpha s} \bar{Y}_s^2 ds |\mathcal{F}_t \right] \leq \mathbb{E} \left[ e^{\alpha T} \bar{\xi}^2 |\mathcal{F}_t \right] + 2 \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s \langle \tilde{g} (s, Y_{t,s}, Z_{t,s}) + g^2 (s, Y_{t,s}, Z_{t,s}) - g^2 (s, Y_{t,s}, Z_{t,s}) \rangle ds |\mathcal{F}_t \right].
By (H$^1$) and the Young inequality, it holds that
\[ e^{\alpha T} \bar{Y}_t^2 + \mathbb{E} \left[ \int_t^T e^{\alpha s} \|\bar{Z}_s\|^2 ds + \alpha \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right] \leq \mathbb{E} \left[ e^{\alpha T} \bar{\xi}^2 |\mathcal{F}_t \right] + 2 \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s \|\tilde{g} (s, Y_{t,s}, Z_{t,s}) \| ds |\mathcal{F}_t \right]
+ 2 \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s \| g^2 (s, Y_{t,s}, Z_{t,s}) - g^2 (s, Y_{t,s}, Z_{t,s}) \| ds |\mathcal{F}_t \right]
\leq \mathbb{E} \left[ e^{\alpha T} \bar{\xi}^2 |\mathcal{F}_t \right] + c^* \delta \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right] + \frac{1}{c^* \delta} \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right]
+ 2 c^* \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T e^{\alpha s} \|\tilde{Z}_s\|^2 ds |\mathcal{F}_t \right] + 2 c^* \mathbb{E} \left[ \int_t^T e^{\alpha s} \|\tilde{Z}_s\|^2 ds |\mathcal{F}_t \right]
\leq \mathbb{E} \left[ e^{\alpha T} \bar{\xi}^2 |\mathcal{F}_t \right] + c^* \delta \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T e^{\alpha s} \|\tilde{Z}_s\|^2 ds |\mathcal{F}_t \right]
+ \left( \frac{1}{c^* \delta} + 2 c^* + 2 c^2 \right) \mathbb{E} \left[ \int_t^T e^{\alpha s} \tilde{Y}_s^2 ds |\mathcal{F}_t \right],
where $\delta > 0$ is a constant. Taking $\alpha > \left( \frac{1}{c_0} + 2 c^* + 2 c^2 + \frac{1}{2} \right)$, we obtain (3).
Next, we prove (ii). For any $t \in [0, T]$, making use of the Itô formula and Lemma \ref{lemma2}, we attain that for $s \in [t, T]$

$$\begin{align*}
\left|\dot{Y}_s\right|^2 + \int_s^T \left|\dot{Z}_r\right|^2 dr &= \left|\hat{\xi}\right|^2 + 2 \int_s^T \left\langle \dot{Y}_r, \dot{g}(r, Y_r^1, Z_r^1) + g^2(r, Y_r^1, Z_r^1) - g^2(r, Y_r^2, Z_r^2) \right\rangle dr \\
- 2 \int_s^T \left\langle \dot{Y}_r, \dot{Z}_r dW_r \right\rangle - 2 \int_s^T \left\langle \dot{Y}_r, d\hat{K}_r \right\rangle \\
&\leq \left|\hat{\xi}\right|^2 + 2 \int_s^T \left\langle \dot{Y}_r, \dot{g}(r, Y_r^1, Z_r^1) + g^2(r, Y_r^1, Z_r^1) - g^2(r, Y_r^2, Z_r^2) \right\rangle dr \\
- 2 \int_s^T \left\langle \dot{Y}_r, \dot{Z}_r dW_r \right\rangle.
\end{align*}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$, by $(\mathbf{H}_3)$, we get

$$\begin{align*}
\mathbb{E}\left[\left|\dot{Y}_s\right|^2 | \mathcal{F}_t\right] + \mathbb{E}\left[\int_s^T \left|\dot{Z}_r\right|^2 dr | \mathcal{F}_t\right] &\leq \mathbb{E}\left[\left|\hat{\xi}\right|^2 | \mathcal{F}_t\right] + 2 \mathbb{E}\left[\int_s^T \left\langle \dot{Y}_r, \dot{g}(r, Y_r^1, Z_r^1) \right\rangle dr | \mathcal{F}_t\right] \\
+ 2 \mathbb{E}\left[\int_s^T \left\langle \dot{Y}_r, g^2(r, Y_r^1, Z_r^1) - g^2(r, Y_r^2, Z_r^2) \right\rangle dr | \mathcal{F}_t\right] \\
+ 2c^* \mathbb{E}\left[\int_s^T \left|\dot{Y}_r\right|^2 dr | \mathcal{F}_t\right] + 2c^2 \mathbb{E}\left[\int_s^T \left|\dot{Y}_r\right|^2 dr | \mathcal{F}_t\right] + \frac{1}{2} \mathbb{E}\left[\int_s^T \left|\dot{Z}_r\right|^2 dr | \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\left|\hat{\xi}\right|^2 | \mathcal{F}_t\right] + \mathbb{E}\left[\int_s^T \left|\dot{g}(r, Y_r^1, Z_r^1) \right|^2 dr | \mathcal{F}_t\right] + (1 + 2c^* + 2c^2) \int_s^T \mathbb{E}\left[\left|\dot{Y}_r\right|^2 | \mathcal{F}_t\right] dr \\
+ \frac{1}{2} \mathbb{E}\left[\int_s^T \left|\dot{Z}_r\right|^2 dr | \mathcal{F}_t\right],
\end{align*}$$

which together with the Gronwall inequality yields that

$$\sup_{s \in [t, T]} \mathbb{E}\left[\left|\dot{Y}_s\right|^2 | \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \left|\dot{Z}_r\right|^2 dr | \mathcal{F}_t\right] \leq C \mathbb{E}\left[\left|\hat{\xi}\right|^2 | \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \left|\dot{g}(r, Y_r^1, Z_r^1) \right|^2 dr | \mathcal{F}_t\right].$$

(6)

Now, we again observe (5). By the Young inequality, the BDG inequality and $(\mathbf{H}_3)$, it holds that

$$\begin{align*}
\mathbb{E}\left[\sup_{s \in [t, T]} \left|\dot{Y}_s\right|^2 \right] + \mathbb{E}\left[\int_t^T \left|\dot{Z}_r\right|^2 dr \right] &\leq \mathbb{E}\left[\left|\hat{\xi}\right|^2 \right] + 2 \mathbb{E}\left[\int_t^T \left\langle \dot{Y}_r, \dot{g}(r, Y_r^1, Z_r^1) + g^2(r, Y_r^1, Z_r^1) - g^2(r, Y_r^2, Z_r^2) \right\rangle dr \right]
\end{align*}$$
Combining with (6), we obtain that
\[
\text{The proof is complete. \hfill \Box}
\]

2.5. The derivative for functions on \( \mathcal{M}_2(\mathbb{R}^d) \). In the subsection, we recall the definition of the derivative for functions on \( \mathcal{M}_2(\mathbb{R}^d) \) (6).

A function \( f : \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R} \) is differentiable at \( \mu \in \mathcal{M}_2(\mathbb{R}^d) \), if for \( \tilde{f}(\gamma) := f(\mathcal{L}_\gamma) \), \( \gamma \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \), there exists some \( \zeta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) with \( \mathcal{L}_\zeta = \mu \) such that \( \tilde{f} \) is Fréchet differentiable at \( \zeta \), that is, there exists a linear continuous mapping \( D\tilde{f}(\zeta) : L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R} \) such that for any \( \eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \)
\[
\tilde{f}(\zeta + \eta) - \tilde{f}(\zeta) = D\tilde{f}(\zeta)(\eta) + o(\|\eta\|_{L^2}), \quad \|\eta\|_{L^2} \to 0.
\]

Since \( D\tilde{f}(\zeta) \in L(L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d), \mathbb{R}) \), it follows from the Riesz representation theorem that there exists a \( \mathbb{P}\)-a.s. unique variable \( \vartheta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) such that for all \( \eta \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \)
\[
D\tilde{f}(\vartheta)(\eta) = \langle \vartheta, \eta \rangle_{L^2} = \mathbb{E}[\vartheta \cdot \eta].
\]

**Definition 2.10.** We say that \( f \in C^1(\mathcal{M}_2(\mathbb{R}^d)) \), if there exists for all \( \gamma \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) a \( \mathcal{L}_\gamma \)-modification of \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \), again denoted by \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \), such that \( \partial_\mu f : \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \) is continuous, and we identify this continuous function \( \partial_\mu f \) as the derivative of \( f \).

**Definition 2.11.** We say that \( f \in C^2(\mathcal{M}_2(\mathbb{R}^d)) \), if for any \( \mu, \nu \in \mathcal{M}_2(\mathbb{R}^d) \), \( f \in C^1(\mathcal{M}_2(\mathbb{R}^d)) \) and \( \partial_\mu f(\mathcal{L}_\gamma)(\cdot) \) is differentiable, and its derivative \( \partial_\nu \partial_\mu f : \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) is continuous, and for any \( y \in \mathbb{R}^d \), \( \partial_\mu f(y)(\cdot) \) is differentiable, and its derivative \( \partial_\mu^2 f : \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) is continuous.
Definition 2.12. A function $F : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is said to be in $C^{2,2}(\mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d))$, if

(i) $F$ is $C^2$ in $x \in \mathbb{R}^d$ and $\mu \in \mathcal{M}_2(\mathbb{R}^d)$ respectively;
(ii) for any $\mu \in \mathcal{M}_2(\mathbb{R}^d)$, its derivatives

$$\partial_x F(x, \mu), \partial_x^2 F(x, \mu), \partial_\mu F(x, \mu)(y), \partial_y \partial_\mu F(x, \mu)(y), \partial_\mu^2 F(x, \mu)(y, y')$$

are jointly continuous in the variable family $(x, \mu), (x, \mu, y)$ and $(x, \mu, y, y')$ respectively.

Definition 2.13. A function $F : \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is said to be in $C^{2,2}_{b}(\mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d))$, if $F \in C^{2,2}_{b}(\mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d))$, and itself and all its derivatives are uniformly bounded on $\mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d)$.

Definition 2.14. A function $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is said to be in $C^{1,2}_{b}(\mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d))$, if

(i) $\Psi(\cdot, \cdot, \mu) \in C^{1,2}([0, T] \times \mathbb{R}^d)$, for all $\mu \in \mathcal{M}_2(\mathbb{R}^d)$;
(ii) $\Psi(t, x, \cdot) \in C^{2}(\mathcal{M}_2(\mathbb{R}^d))$, for all $(t, x) \in [0, T] \times \mathbb{R}^d$;
(iii) All derivatives of order 1 and 2 are continuous on $[0, T] \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d$, $\partial_\mu \Psi$ and $\partial_y \partial_\mu \Psi$ are bounded over $[0, T] \times \mathbb{R}^d \times \mathcal{M}_2(\mathbb{R}^d) \times \mathbb{R}^d$.

3. The existence and uniqueness of solutions to backward multivalued McKean-Vlasov SDEs

In this section, we study the existence and uniqueness of solutions for Eq. (1).

Definition 3.1. We say that Eq. (1) admits a solution with the terminal value $\xi$ if there exists a triple $\{(Y_t, K_t, Z_t) : t \in [0, T]\}$ which satisfies

(i) $(Y, K) \in \mathcal{A}$, $dP \times dt$-a.e. on $\Omega \times [0, T]$,
(ii) $(Y, Z) \in \mathcal{S}_F^2([0, T], \mathbb{R}^d) \times \mathbb{H}_F^2([0, T], \mathbb{R}^{d \times t})$,
(iii) $Y_t = \xi - (K_T - K_t) + \int_t^T G(s, Y_s, Z_s, \mathcal{L}_{(Y_s, Z_s)})ds - \int_t^T Z_s dW_s$.

The definition of the uniqueness for solutions to Eq. (1) is the same to that for solutions to Eq. (2).

In the following, we give some assumptions to assure the existence and uniqueness of solutions for Eq. (1).

$(H_1^G)$ There exists a non-random constant $L_G > 0$ such that for a.s. $\omega \in \Omega$, it holds that

$$|G(\omega, t, y, z, \vartheta) - G(\omega, t, y', z', \vartheta')| \leq L_G \left(\|y - y'\| + \|z - z'\| + \rho(\vartheta, \vartheta')\right),$$

$t \in [0, T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times t}, \vartheta, \vartheta' \in \mathcal{M}_2(\mathbb{R}^d \times \mathbb{R}^{d \times t})$.

$(H_2^G)$ $\mathbb{E} \int_0^T |G(t, 0, 0, \delta_{(0, 0)})|^2 dt < \infty$, where $\delta_{(0, 0)}$ is the Dirac measure at $(0, 0)$.

$(H_A)$ $\mathcal{D}(A) = \mathbb{R}^d$ and $|A^c(x)| \leq C(1 + |x|), x \in \mathbb{R}^d$, where $C > 0$ is a constant.

Now, it is the position to state the main result in the section.

Theorem 3.2. Assume that $(H_A)$ holds and the coefficient $G$ satisfies $(H_1^G)$-$(H_2^G)$. Then there exists a unique solution for Eq. (1).
To prove the above theorem, we prepare some key lemmas. For any \( \varepsilon > 0 \), consider the penalized backward multivalued McKean-Vlasov SDE on \( \mathbb{R}^d \):

\[
\begin{aligned}
    dY_t^\varepsilon &= \left[ A_\varepsilon (Y_s^\varepsilon) - G \left( t, Y_t^\varepsilon, Z_t^\varepsilon, \mathcal{L}(Y_t^\varepsilon, Z_t^\varepsilon) \right) \right] dt + Z_t^\varepsilon dW_t, \quad 0 \leq t < T, \\
    Y_T^\varepsilon &= \xi,
\end{aligned}
\]

where \( A_\varepsilon \) is the Yosida approximation of \( A \). Note that \( A_\varepsilon \) is a single-valued, maximal monotone and Lipschitz continuous function (cf. Subsection 2.2). Thus, by Theorem 2.4 we know that under (\( H_G^1 \))-(\( H_G^2 \)) Eq.(7) has a unique solution denoted as \( (Y_s^\varepsilon, Z_t^\varepsilon) \) with

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 + \int_0^T \|Z_s^\varepsilon\|^2 ds \right) < \infty.
\]

Next, we give some uniform moment estimates about \( Y_s^\varepsilon, Z_t^\varepsilon \).

**Lemma 3.3.** Assume that \( \text{Int}(\mathcal{D}(A)) \neq \emptyset \) and the coefficient \( G \) satisfies (\( H_G^1 \))-(\( H_G^2 \)). Then there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 \right) + \mathbb{E} \int_0^T \|Z_s^\varepsilon\|^2 ds \leq C.
\]

**Proof.** Applying the Itô formula to \( |Y_t^\varepsilon - a|^2 \), where \( a \) is the same to that in Lemma 2.1, we obtain that

\[
\begin{aligned}
    |Y_t^\varepsilon - a|^2 + \int_t^T \|Z_s^\varepsilon\|^2 ds &= |\xi - a|^2 + 2 \int_t^T \left\langle Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right\rangle ds \\
    &\quad - 2 \int_t^T \left\langle Y_s^\varepsilon - a, Z_s^\varepsilon dW_s \right\rangle - 2 \int_t^T \left\langle Y_s^\varepsilon - a, A_\varepsilon (Y_s^\varepsilon) \right\rangle ds \\
    &\quad \leq |\xi - a|^2 + 2 \int_t^T \left\langle Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right\rangle ds \\
    &\quad - 2 \int_t^T \left\langle Y_s^\varepsilon - a, Z_s^\varepsilon dW_s \right\rangle - 2 M_1 \int_t^T |A_\varepsilon (Y_s^\varepsilon) | ds \\
    &\quad + 2 M_2 \int_t^T |Y_s^\varepsilon - a| ds + 2 M_1 M_2 T,
\end{aligned}
\]

where the last inequality is based on Lemma 2.1. Taking the expectation on two sides, by (S), one can have that

\[
\begin{aligned}
    \mathbb{E} |Y_t^\varepsilon - a|^2 + \mathbb{E} \int_t^T \|Z_s^\varepsilon\|^2 ds &\leq \mathbb{E} |\xi - a|^2 + 2 \mathbb{E} \int_t^T |Y_s^\varepsilon - a| \left| G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right| ds \\
    &\quad - 2 M_1 \mathbb{E} \int_t^T |A_\varepsilon (Y_s^\varepsilon) | ds + 2 M_2 \mathbb{E} \int_t^T |Y_s^\varepsilon - a| ds \\
    &\quad + 2 M_1 M_2 T.
\end{aligned}
\]

Let us estimate the second term in the right side of the above inequality. By (\( H_G^2 \)) it holds that

\[
\begin{aligned}
    \left| G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right| &\leq L_G \left( |Y_s^\varepsilon| + \|Z_s^\varepsilon\| + \rho (\mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon), \delta_0) \right) + |G \left( s, 0, \delta_0 \right) | \\
    &\leq L_G \left( |Y_s^\varepsilon| + \|Z_s^\varepsilon\| + (\mathbb{E}|Y_s^\varepsilon|^2)^{1/2} + (\mathbb{E}\|Z_s^\varepsilon\|^2)^{1/2} \right) + |G \left( s, 0, \delta_0 \right) | \\
    &\leq L_G \left( 3|a| + |Y_s^\varepsilon - a| + \|Z_s^\varepsilon\| + (\mathbb{E}|Y_s^\varepsilon - a|^2)^{1/2} + (\mathbb{E}\|Z_s^\varepsilon\|^2)^{1/2} \right)
\end{aligned}
\]
where we use the following fact that

\[
\rho^2 \left( \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon), \delta(0,0) \right) \leq \inf_{x \in \mathcal{C}(\mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon), \delta(0,0))} \int_{\mathbb{R}^d \times (\mathbb{R}^d \times t)} |(y_1, z_1) - (y_2, z_2)|^2 \pi(d(y_1, z_1), d(y_2, z_2))
\]

\[
\leq \mathbb{E}|(Y_s^\varepsilon, Z_s^\varepsilon) - (0,0)|^2 = \mathbb{E}|Y_s^\varepsilon|^2 + \mathbb{E}|Z_s^\varepsilon|^2.
\]

Thus, by the Young inequality and the inequality $|x| \leq 1 + |x|^2, x \in \mathbb{R}^d$, one can obtain that

\[
2|Y_s^\varepsilon - a| \left| G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right|
\leq 6|a_L|Y_s^\varepsilon - a| + 2L_G|Y_s^\varepsilon - a|^2 + 2L_G|Y_s^\varepsilon - a||Z_s^\varepsilon|| + 2L_G|Y_s^\varepsilon - a|\left| \mathbb{E}|Y_s^\varepsilon - a|^2 \right|^{1/2}
+ 2L_G|Y_s^\varepsilon - a|\left| \mathbb{E}|Z_s^\varepsilon|^2 \right|^{1/2} + 2|Y_s^\varepsilon - a| \left| G \left( s, 0, 0, \delta(0,0) \right) \right|
\leq 6|a_L| + (6|a_L| + 2L_G + 9L_G^2 + 1)|Y_s^\varepsilon - a|^2 + \mathbb{E}|Y_s^\varepsilon - a|^2
+ \frac{1}{4}||Z_s^\varepsilon||^2 + \frac{1}{4}\mathbb{E}||Z_s^\varepsilon||^2 + \left| G \left( s, 0, 0, \delta(0,0) \right) \right|^2.
\]

(12)

Combining (10) with (12), by (H_2^0) we get that

\[
\mathbb{E}|Y_t^\varepsilon - a|^2 + \frac{1}{2}\mathbb{E} \int_t^T ||Z_s^\varepsilon||^2 ds \leq C + C \int_t^T \mathbb{E}|Y_s^\varepsilon - a|^2 ds,
\]

(13)

where $C$ is independent of $\varepsilon$. By the Gronwall inequality, it holds that

\[
\sup_{0 \leq t \leq T} \mathbb{E}|Y_t^\varepsilon - a|^2 \leq Ce^{CT}.
\]

From this and (13), it follows that

\[
\mathbb{E} \int_0^T ||Z_s^\varepsilon||^2 ds \leq C.
\]

(14)

Next, for (5), by the BDG inequality and the Young inequality we have that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - a|^2 \right) \leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
+ 2\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_t^T \left< Y_s^\varepsilon - a, Z_s^\varepsilon dW_s \right> \right| \right) + 2M_2E \int_0^T |Y_s^\varepsilon - a|ds
+ 2M_1M_2T
\leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
+ 2\mathbb{E} \left( \int_0^T |Y_s^\varepsilon - a|^2 ||Z_s^\varepsilon||^2 ds \right)^{1/2} + 2M_2E \int_0^T |Y_s^\varepsilon - a|^2 ds
+ 2(M_1 + 1)M_2T
\leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
\]

\[
\leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
\]

\[
\leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
\]

\[
\leq \mathbb{E}|\xi - a|^2 + 2\mathbb{E} \int_0^T \left| \left< Y_s^\varepsilon - a, G \left( s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon) \right) \right> \right| ds
\]
\[ \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - a|^2 \right) + C \mathbb{E} \int_0^T \|Z_s^\varepsilon\|^2 ds + 2M_2 \mathbb{E} \int_0^T |Y_s^\varepsilon - a|^2 ds + 2(M_1 + 1)M_2 T. \]

From (12) and (14), it follows that
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2 \right) \leq 2 \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - a|^2 \right) + 2|a|^2 \leq C. \tag{15} \]

The proof is complete. \( \square \)

**Lemma 3.4.** Assume that \((H_A)\) holds and the coefficient \(G\) satisfies \((H_G^1)\)-(\(H_G^2\)). Then there exists a constant \(C > 0\) independent of \(\varepsilon\) such that
\[ \mathbb{E} \int_0^T |A_\varepsilon (Y_s^\varepsilon)| ds + \mathbb{E} \int_0^T |A_\varepsilon (Y_s^\varepsilon)|^2 ds \leq C. \]

**Proof.** By (9), it holds that
\[ 2M_1 \int_t^T |A_\varepsilon (Y_s^\varepsilon)| ds \leq |\xi - a|^2 + 2 \int_t^T \left\langle Y_s^\varepsilon - a, G\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon)\right) \right\rangle ds \]
\[ -2 \int_t^T (Y_s^\varepsilon - a, Z_s^\varepsilon dW_s) + 2M_2 \int_t^T |Y_s^\varepsilon - a| ds + 2M_1 M_2 T, \]
and furthermore
\[ 2M_1 \mathbb{E} \int_t^T |A_\varepsilon (Y_s^\varepsilon)| ds \leq \mathbb{E} |\xi - a|^2 + 2 \mathbb{E} \int_t^T \left\langle Y_s^\varepsilon - a, G\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon)\right) \right\rangle ds \]
\[ + 2M_2 \mathbb{E} \int_t^T |Y_s^\varepsilon - a| ds + 2M_1 M_2 T, \]
where (8) is used. Thus, by (12), (14) and (15), we know that
\[ \mathbb{E} \int_t^T |A_\varepsilon (Y_s^\varepsilon)| ds \leq C, \]
where the constant \(C\) is independent of \(\varepsilon\).

Next, note that \(|A_\varepsilon(x)| \leq |A^o(x)|\) for \(x \in D(A)\). Thus, by \((H_A)\) and (15), it holds that
\[ \mathbb{E} \int_0^T |A_\varepsilon (Y_s^\varepsilon)|^2 ds \leq \mathbb{E} \int_0^T |A^o (Y_s^\varepsilon)|^2 ds \leq C \mathbb{E} \int_0^T (1 + |Y_s^\varepsilon|)^2 ds \leq C. \]

The proof is complete. \( \square \)

**Lemma 3.5.** Assume that \((H_A)\) holds and the coefficient \(G\) satisfies \((H_G^1)\)-(\(H_G^2\)). Then for any \(\varepsilon, \delta > 0\), there exists a constant \(C > 0\) independent of \(\varepsilon, \delta\) such that
\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 \right) + \mathbb{E} \int_0^T \|Z_s^\varepsilon - Z_s^\delta\|^2 ds \leq C(\varepsilon + \delta). \]

**Proof.** For any \(\varepsilon, \delta > 0\), it holds that
\[ Y_t^\varepsilon - Y_t^\delta = \int_t^T \left[ G(s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon)) - G(s, Y_s^\delta, Z_s^\delta, \mathcal{L}(Y_s^\delta, Z_s^\delta)) \right] ds \]
\[ - \int_t^T \left( A_\varepsilon(Y_s^\varepsilon) - A_\delta(Y_s^\delta) \right) ds - \int_t^T (Z_s^\varepsilon - Z_s^\delta) dW_s. \]
Applying the Itô formula to $|Y^\varepsilon_t - Y^\delta_t|^2$, we know that

$$
|Y^\varepsilon_t - Y^\delta_t|^2 + \int_t^T \|Z^\varepsilon_s - Z^\delta_s\|^2 ds
$$

$$
= 2 \int_t^T \langle Y^\varepsilon_s - Y^\delta_s, G(s, Y^\varepsilon_s, Z^\varepsilon_s, \mathcal{L}(Y^\varepsilon_s, Z^\varepsilon_s)) - G(s, Y^\delta_s, Z^\delta_s, \mathcal{L}(Y^\delta_s, Z^\delta_s)) \rangle ds
$$

$$
-2 \int_t^T \langle Y^\varepsilon_s - Y^\delta_s, A_\delta(Y^\varepsilon_s) - A_\delta(Y^\delta_s) \rangle ds - 2 \int_t^T \langle Y^\varepsilon_s - Y^\delta_s, (Z^\varepsilon_s - Z^\delta_s) dW_s \rangle,
$$

and

$$
\mathbb{E}|Y^\varepsilon_t - Y^\delta_t|^2 + \mathbb{E} \int_t^T \|Z^\varepsilon_s - Z^\delta_s\|^2 ds
$$

$$
= 2 \mathbb{E} \int_t^T \langle Y^\varepsilon_s - Y^\delta_s, G(s, Y^\varepsilon_s, Z^\varepsilon_s, \mathcal{L}(Y^\varepsilon_s, Z^\varepsilon_s)) - G(s, Y^\delta_s, Z^\delta_s, \mathcal{L}(Y^\delta_s, Z^\delta_s)) \rangle ds
$$

$$
-2 \mathbb{E} \int_t^T \langle Y^\varepsilon_s - Y^\delta_s, A_\varepsilon(Y^\varepsilon_s) - A_\delta(Y^\delta_s) \rangle ds.
$$

From (H.1) and (11), it follows that

$$
2|\langle Y^\varepsilon_s - Y^\delta_s, G(s, Y^\varepsilon_s, Z^\varepsilon_s, \mathcal{L}(Y^\varepsilon_s, Z^\varepsilon_s)) - G(s, Y^\delta_s, Z^\delta_s, \mathcal{L}(Y^\delta_s, Z^\delta_s)) \rangle|
$$

$$
\leq 2|Y^\varepsilon_s - Y^\delta_s||G(s, Y^\varepsilon_s, Z^\varepsilon_s, \mathcal{L}(Y^\varepsilon_s, Z^\varepsilon_s)) - G(s, Y^\delta_s, Z^\delta_s, \mathcal{L}(Y^\delta_s, Z^\delta_s))|
$$

$$
\leq 2|Y^\varepsilon_s - Y^\delta_s|L\mathcal{G}(|Y^\varepsilon_s - Y^\delta_s| + \|Z^\varepsilon_s - Z^\delta_s\| + \rho(\mathcal{L}(Y^\varepsilon_s, Z^\varepsilon_s), \mathcal{L}(Y^\delta_s, Z^\delta_s)))
$$

$$
\leq 2|Y^\varepsilon_s - Y^\delta_s|L\mathcal{G}(|Y^\varepsilon_s - Y^\delta_s| + \|Z^\varepsilon_s - Z^\delta_s\| + (\mathbb{E}|Y^\varepsilon_s - Y^\delta_s|^2)^{1/2} + (\mathbb{E}\|Z^\varepsilon_s - Z^\delta_s\|^2)^{1/2})
$$

$$
\leq (2L + 9L^2)|Y^\varepsilon_s - Y^\delta_s| + \frac{1}{4}\|Z^\varepsilon_s - Z^\delta_s\|^2 + \mathbb{E}|Y^\varepsilon_s - Y^\delta_s|^2
$$

Moreover, by the deduction in [22] Proposition 6, we know that

$$
-2\langle Y^\varepsilon_s - Y^\delta_s, A_\varepsilon(Y^\varepsilon_s) - A_\delta(Y^\delta_s) \rangle \leq 3\varepsilon|A_\varepsilon(Y^\varepsilon_s)|^2 + 3\delta|A_\delta(Y^\delta_s)|^2.
$$

Inserting (18) and (19) in (17), by Lemma 3.4 one can obtain that

$$
\mathbb{E}|Y^\varepsilon_t - Y^\delta_t|^2 + \frac{1}{2}\mathbb{E} \int_t^T \|Z^\varepsilon_s - Z^\delta_s\|^2 ds \leq (2L + 9L^2 + 1) \int_t^T \mathbb{E}|Y^\varepsilon_s - Y^\delta_s|^2 ds
$$

$$
+ \mathbb{E} \int_t^T (3\varepsilon|A_\varepsilon(Y^\varepsilon_s)|^2 + 3\delta|A_\delta(Y^\delta_s)|^2) ds
$$

$$
\leq (2L + 9L^2 + 1) \int_t^T \mathbb{E}|Y^\varepsilon_s - Y^\delta_s|^2 ds + C(\varepsilon + \delta).
$$

Thus, by the Gronwall inequality, it holds that

$$
\sup_{0 \leq t \leq T} \mathbb{E}|Y^\varepsilon_t - Y^\delta_t|^2 \leq C(\varepsilon + \delta),
$$

and furthermore

$$
\mathbb{E} \int_t^T \|Z^\varepsilon_s - Z^\delta_s\|^2 ds \leq C(\varepsilon + \delta).
$$
Next, from \((16)\) and \((18)-(21)\), it follows that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 \right)
\leq 2 \mathbb{E} \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, G(s, Y_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}(Y_s^\varepsilon, Z_s^\varepsilon)) \rangle \, ds + 2 \mathbb{E} \left( \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta) dW_s \rangle \right)
\]
\[
+ C \mathbb{E} \left( \int_0^T |Y_s^\varepsilon - Y_s^\delta|^2 |Z_s^\varepsilon - Z_s^\delta|^2 \, ds \right)^{1/2}
\leq C(\varepsilon + \delta) + \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 \right) + C \mathbb{E} \int_0^T |Z_s^\varepsilon - Z_s^\delta|^2 \, ds,
\]
which yields that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t^\delta|^2 \right) \leq C(\varepsilon + \delta).
\]
The proof is complete. \(\square\)

Now, we apply the above lemmas to prove Theorem 3.2.

**Proof of the existence for Theorem 3.2.**

We prove that the limit \((Y, K, Z)\) of the sequence \(\{Y^\varepsilon, K^\varepsilon, Z^\varepsilon\}\) is a solution of Eq.\((1)\), where \(K_t^\varepsilon := \int_0^t A_s(Y_s^\varepsilon) \, ds\).

From Lemma 3.5, it follows that \(\{Y^\varepsilon\}\) is a Cauchy sequence in \(L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0, T], \mathbb{R}^d))\) and \(\{Z^\varepsilon\}\) is a Cauchy sequence in \(L^2(\Omega \times [0, T], d\mathbb{P} \times dt)\). So, there exist two processes \(Y, Z\) such that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t|^2 \right) = 0, \tag{22}
\]
\[
\lim_{\varepsilon \to 0} \int_0^T |Z_s^\varepsilon - Z_s|^2 \, ds = 0. \tag{23}
\]
Moreover, by \((11)\) it holds that
\[
\lim_{\varepsilon \to 0} \rho \left( \mathcal{L}(Y_t^\varepsilon, Z_t^\varepsilon), \mathcal{L}(Y_t, Z_t) \right) \leq \lim_{\varepsilon \to 0} \left( (\mathbb{E}|Y_t^\varepsilon - Y_t|^2)^{1/2} + (\mathbb{E}|Z_t^\varepsilon - Z_t|^2)^{1/2} \right) = 0. \tag{24}
\]
Now, let us put
\[
-K_t := Y_0 - Y_t - \int_0^t G(s, Y_s, Z_s, \mathcal{L}(Y_s, Z_s)) \, ds + \int_0^t Z_s \, dW_s.
\]
\((22)-(24)\) imply that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} |K_t^\varepsilon - K_t|^2 \right) = 0.
\]
which together with Lemma 2.3 gives that
\[
\lim_{\epsilon \to 0} \int_0^t \langle Y_s^\epsilon, dK_s^\epsilon \rangle = \int_0^t \langle Y_s, dK_s \rangle.
\] (25)

Note that \((Y_s^\epsilon, K_s^\epsilon) \in \mathcal{A}\). Thus, by Lemma 2.2 it holds that for any \((Y', K') \in \mathcal{A}\),
\[
\left\langle Y_t^\epsilon - Y_t', dK_t^\epsilon - dK_t' \right\rangle \geq 0,
\]
which together with (25), yields \((Y, K) \in \mathcal{A}, a.s.\). That is, \((Y_t, K_t, Z_t)_{t \in [0,T]}\) is a solution of Eq. (1).

To show the uniqueness part in Theorem 3.2 we prepare the following proposition.

**Proposition 3.6.** Assume that \((H_A)\) holds. Let \(G^1, G^2\) be two functions satisfying the standard assumptions \((H_{A1}^1)-(H_{A2}^2)\). Let \(\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})\) and \((Y^i, K^i, Z^i), i = 1, 2,\) be the solutions of Eq. (1) with the terminal values \(\xi^1, \xi^2\), respectively. Set \(\overline{Y}_t := Y_t^1 - Y_t^2, \overline{Z}_t := Z_t^1 - Z_t^2, \overline{K}_t := K_t^1 - K_t^2, \overline{\xi} := \xi^1 - \xi^2, \overline{G} := G^1 - G^2\). Then it holds that
\[
\begin{align*}
\mathbb{E} \int_0^T (|\overline{Y}_t|^2 + |\overline{Z}_t|^2) dt & \leq CT, \\
\mathbb{E} \left( \sup_{t \in [0,T]} |\overline{Y}_t|^2 \right) & \leq CT,
\end{align*}
\] (26) (27)

where
\[
\Gamma := \mathbb{E}|\overline{\xi}|^2 + \mathbb{E} \int_0^T |\overline{G}(s, Y_{s}^1, Z_{s}^1, \mathcal{L}(Y_{s}^2, Z_{s}^2))|^2 ds.
\]

**Proof.** Since \((Y^i, K^i, Z^i), i = 1, 2,\) are the solutions of Eq. (1) with the terminal values \(\xi^1, \xi^2\), respectively, it holds that
\[
\begin{align*}
Y_t^1 &= \xi^1 - (K_t^1 - K_t^1) + \int_t^T G^1(s, Y_s^1, Z_s^1, \mathcal{L}(Y_s^2, Z_s^2)) ds - \int_t^T Z_s^1 dW_s, \\
Y_t^2 &= \xi^2 - (K_t^2 - K_t^2) + \int_t^T G^2(s, Y_s^2, Z_s^2, \mathcal{L}(Y_s^2, Z_s^2)) ds - \int_t^T Z_s^2 dW_s,
\end{align*}
\]
and
\[
\overline{Y}_t = \overline{\xi} - \int_t^T d\overline{K}_s + \int_t^T \left( G^1(s, Y_s^1, Z_s^1, \mathcal{L}(Y_s^2, Z_s^2)) - G^2(s, Y_s^2, Z_s^2, \mathcal{L}(Y_s^2, Z_s^2)) \right) ds \\
- \int_t^T \overline{Z}_s dW_s.
\]

Applying the Itô formula to \(\overline{Y}_t^2\), by Lemma 2.2 one can obtain that
\[
\begin{align*}
|\overline{Y}_t|^2 + \int_t^T |\overline{Z}_s|^2 ds &= |\overline{\xi}|^2 + 2 \int_t^T \langle \overline{Y}_s, G^1(s, Y_s^1, Z_s^1, \mathcal{L}(Y_s^2, Z_s^2)) - G^2(s, Y_s^2, Z_s^2, \mathcal{L}(Y_s^2, Z_s^2)) \rangle ds \\
& \quad - 2 \int_t^T \langle \overline{Y}_s, \overline{Z}_s dW_s \rangle - 2 \int_t^T \langle \overline{Y}_s, d\overline{K}_s \rangle
\end{align*}
\]
\[
\begin{align*}
\mathbb{E}|\overline{Y}_t|^2 & \leq |\xi|^2 + 2 \int_t^T \langle \overline{Y}_s, \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle ds - 2 \int_t^T \langle \overline{Y}_s, Z_s dW_s \rangle \\
& + 2 \int_t^T \langle \overline{Y}_s, \overline{g}^2(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) - G^2(s, Y_s^2, Z_s^2, \mathcal{L}_{(Y_s^2, Z_s^2)}) \rangle ds. \tag{28}
\end{align*}
\]

Taking the expectation on two sides of the above inequality, we get that

\[
\begin{align*}
\mathbb{E}|\overline{Y}_t|^2 & \leq \mathbb{E}|\overline{\xi}|^2 + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds \\
& + 2 \mathbb{E} \int_t^T \langle \overline{Y}_s, \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T \mathbb{E} \langle \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle^2 ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s| \rho \left( \overline{\mathcal{L}}_{(Y_s^1, Z_s^1), \mathcal{L}_{(Y_s^2, Z_s^2)}} \right) ds \\
& \leq \mathbb{E}|\overline{\xi}|^2 + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T \mathbb{E} \langle \overline{g}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle^2 ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s| \left( \left( \mathbb{E} |\overline{Y}_s|^2 \right)^{1/2} + \left( \mathbb{E} |\overline{Z}_s|^2 \right)^{1/2} \right) ds \\
& \leq \mathbb{E}|\overline{\xi}|^2 + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T \mathbb{E} \langle \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle^2 ds \\
& + 2 \mathbb{E} \int_t^T |\overline{Y}_s|^2 ds + 2 \mathbb{E} \int_t^T |\overline{Z}_s|^2 ds, \tag{29}
\end{align*}
\]

and

\[
\mathbb{E}|\overline{Y}_t|^2 \leq \mathbb{E}|\overline{\xi}|^2 + \mathbb{E} \int_t^T \mathbb{E}|\overline{Y}_s|^2 ds + \mathbb{E} \int_t^T \mathbb{E} \langle \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle^2 ds,
\]

which together with the Gronwall inequality yields that

\[
\sup_{t \in [0, T]} \mathbb{E}|\overline{Y}_t|^2 \leq C \left( \mathbb{E}|\overline{\xi}|^2 + \int_0^T \mathbb{E} \langle \overline{G}(s, Y_s^1, Z_s^1, \mathcal{L}_{(Y_s^1, Z_s^1)}) \rangle^2 ds \right). \tag{30}
\]

Inserting (30) in (29), we have (26).
Next, from (28) and the BDG inequality, it follows that
\[
\mathbb{E}\left( \sup_{t \in [0,T]} |\bar{Y}_t|^2 \right) 
\leq \mathbb{E}|\bar{\xi}|^2 + 2 \int_0^T |(Y_s, G(s, Y_s, Z_s, \mathcal{L}(Y_s, Z_s)))| ds + 2 \mathbb{E}\left( \sup_{t \in [0,T]} \int_t^T (\bar{Y}_s, Z_s dW_s) \right) 
+ 2 \mathbb{E}\int_0^T |(Y_s, G^2(s, Y_s, Z_s, \mathcal{L}(Y_s, Z_s))) - G^2(s, Y_s, Z_s, \mathcal{L}(Y_s, Z_s))| ds 
\leq \mathbb{E}|\bar{\xi}|^2 + \mathbb{E}\int_0^T |\bar{Y}_s|^2 ds + \mathbb{E}\int_0^T |\bar{X}_s|^2 ds + C\mathbb{E}\left( \int_0^T |\bar{Y}_t|^2 \|Z_t\|^2 dt \right)^{1/2} 
+ C \int_0^T \mathbb{E}|\bar{Y}_s|^2 ds + \frac{1}{2} \mathbb{E}\int_0^T \|\bar{Z}_s\|^2 ds 
\leq \mathbb{E}|\bar{\xi}|^2 + \mathbb{E}\int_0^T |\bar{Y}_s|^2 ds + \mathbb{E}\int_0^T |\bar{X}_s|^2 ds + \frac{1}{2} \mathbb{E}\int_0^T \|\bar{Z}_s\|^2 ds,
\]
which together with (26) gives (27). The proof is complete.

\textbf{The proof of the uniqueness for Theorem 3.2.} 
Assume that \((Y^1, K^1, Z^1)\) and \((Y^2, K^2, Z^2)\) are two solutions of Eq. (1) with \(Y^1_T = Y^2_T = \xi, K^1_T = K^2_T, \) i.e.
\[
Y^1_t = \xi - (K^1_T - K^1_t) + \int_t^T G(s, Y^1_s, Z^1_s, \mathcal{L}(Y^1_s, Z^1_s)) ds - \int_t^T Z^1_s dW_s, \\
Y^2_t = \xi - (K^2_T - K^2_t) + \int_t^T G(s, Y^2_s, Z^2_s, \mathcal{L}(Y^2_s, Z^2_s)) ds - \int_t^T Z^2_s dW_s.
\]
By Proposition 3.6 with \(\xi^1 = \xi^2, G^1 = G^2, \) it holds that
\[
\mathbb{E}\left( \sup_{t \in [0,T]} |\bar{Y}_t|^2 \right) = 0, \quad \mathbb{E}\int_0^T \|\bar{Z}_s\|^2 ds = 0,
\]
which gives that
\[
Y^1_t = Y^2_t, \quad t \in [0,T], \ \mathbb{P}-a.s., \quad Z^1_t = Z^2_t, \ \mathbb{P} \times dt - a.s.
\]
Finally, from the above deduction it follows that for any \(t \in [0,T], \)
\[
-K^1_t = \xi - Y^1_t + \int_t^T G(s, Y^1_s, Z^1_s, \mathcal{L}(Y^1_s, Z^1_s)) ds - \int_t^T Z^1_s dW_s - K^1_T 
= \xi - Y^2_t + \int_t^T G(s, Y^2_s, Z^2_s, \mathcal{L}(Y^2_s, Z^2_s)) ds - \int_t^T Z^2_s dW_s - K^2_T 
= -K^2_t.
\]
So, the fact that \(K_t\) is continuous in \(t\) assures that \(K^1_t = K^2_t, \ t \in [0,T] \ \mathbb{P}\text{-a.s.} \). The proof is complete.
By Proposition 3.6, we also have the following result.

**Corollary 3.7.** Assume that $(H_4)$ holds and $G$ satisfies $(H_{4}^1)$-$(H_{4}^2)$. Let $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and $(Y^i, K^i, Z^i), i = 1, 2$, be the solutions of Eq. (1) with the terminal values $\xi^1, \xi^2$, respectively. Then it holds that

$$E\left( \sup_{t \in [0,T]} |Y_t|^2 \right) \leq C E|\bar{\xi}|^2.$$

By the above corollary, we know that the solution of Eq. (1) depends continuously on the terminal value.

### 4. Connection with parabolic variational inequalities

In this section, we apply the result in the previous section to a type of parabolic variational inequalities and give a probabilistic representation of their solutions.

First of all, we construct a new filtration. Let $\\{\mathcal{F}_t\}_{t \in [0,T]}$ be the filtration generated by $(W_t)_{t \in [0,T]}$ and augmented by a $\sigma$-field $\mathcal{F}_0$, i.e.,

$$\mathcal{F}_t^W := \sigma\{W_s : 0 \leq s \leq t\}, \quad \mathcal{F}_t := \left( \bigcap_{s>t} \mathcal{F}_s \right) \vee \mathcal{F}_0, \quad t \in [0,T],$$

where $\mathcal{F}_0 \subset \mathcal{F}$ has the following properties:

(i) $(W_t)_{t \in [0,T]}$ is independent of $\mathcal{F}_0$;

(ii) $\mathcal{M}_2(\mathbb{R}^m) = \{\mathbb{P} \circ \eta^{-1}, \eta \in L^2(\mathcal{F}_0, \mathbb{R}^m)\}$;

(iii) $\mathcal{F}_0 \supset \mathcal{N}$ and $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets.

In addition, consider the following forward McKean-Vlasov SDEs on $\mathbb{R}^m$: for any $t \in [0,T]$

$$\begin{cases} dX^{t,\eta}_s = b\left(s, X^{t,\eta}_s, \mathcal{L}_{X^{t,\eta}} \right) ds + \sigma \left(s, X^{t,\eta}_s, \mathcal{L}_{X^{t,\eta}} \right) dW_s, t \leq s \leq T, \\
X^{t,\eta}_t = \eta, \end{cases} \quad (31)$$

and

$$\begin{cases} dX^{t,x,\eta}_s = b\left(s, X^{t,x,\eta}_s, \mathcal{L}_{X^{t,x,\eta}} \right) ds + \sigma \left(s, X^{t,x,\eta}_s, \mathcal{L}_{X^{t,x,\eta}} \right) dW_s, t \leq s \leq T, \\
X^{t,x,\eta}_t = x, \end{cases} \quad (32)$$

where $\eta$ is a $\mathcal{F}_t$-measurable random variable with $E|\eta|^2 < \infty$, and the coefficients $b : [0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \mapsto \mathbb{R}^m$, $\sigma : [0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \mapsto \mathbb{R}^{m \times l}$ are Borel measurable. Assume:

**(H_{b,\sigma}^1)** The functions $b, \sigma$ are continuous in $(t, x, \mu)$ and satisfy for $(t, x, \mu) \in [0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)$

$$|b(t, x, \mu)| + \|\sigma(t, x, \mu)\| \leq L_{b,\sigma}(t)(|x| + \|\mu||_2),$$

where $L_{b,\sigma} : [0, T] \mapsto (0, \infty)$ is an increasing function.

**(H_{b,\sigma}^2)** The functions $b, \sigma$ satisfy for $t \in [0, T], (x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)$

$$|b(t, x_1, \mu_1) - b(t, x_2, \mu_2)| + \|\sigma(t, x_1, \mu_1) - \sigma(t, x_2, \mu_2)\| \leq L_{b,\sigma}(t)(|x_1 - x_2| + \rho(\mu_1, \mu_2)).$$

By Theorem 3.1 in [10], it holds that under $(H_{b,\sigma}^1)$-$(H_{b,\sigma}^2)$, Eq. (31) has a unique solution denoted as $X^{t,\eta}$. Thus, by inserting $\mathcal{L}_{X^{t,\eta}}$ in Eq. (32), it becomes a classical SDE. Based on [15] Theorem 19.3, we know that under $(H_{b,\sigma}^1)$-$(H_{b,\sigma}^2)$, Eq. (32) has a unique solution.
denoted as $X_{t,x,\eta}^s$. Moreover, about $X_{t,x,\eta}^s$, $X_{t,x,\eta}^1$ we have the following result ([17, Lemma 3.1]).

**Lemma 4.1.** Suppose that $(H_{1,\sigma}^1) - (H_{b,\sigma}^2)$ hold. Then there exists a constant $C > 0$ such that for any $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^m$, $\eta, \tilde{\eta} \in L^2(\mathcal{F}_t, \mathbb{R}^m)$,

(i) $\mathbb{E} \left[ \sup_{s \in [t, T]} |X_{s,x,\eta}^t - X_{s,\bar{x},\tilde{\eta}}^t|^2 \right] \leq C (|\eta - \tilde{\eta}|^2 + \rho^2(\mathcal{L}_x, \mathcal{L}_\eta))$,

(ii) $\mathbb{E} \left[ \sup_{s \in [t, T]} |X_{s,x,\eta}^t - X_{s,\bar{x},\tilde{\eta}}^t|^2 \right] \leq C (|x - \bar{x}|^2 + \rho^2(\mathcal{L}_x, \mathcal{L}_\eta))$,

(iii) $\sup_{s \in [t, T]} \rho(\mathcal{L}_{X_x}^t, \mathcal{L}_{X_{\bar{x}}}^t) \leq C \rho(\mathcal{L}_x, \mathcal{L}_\eta)$.

Next, consider the following backward multivalued McKean-Vlasov SDEs on $\mathbb{R}$:

$$
\begin{cases}
\frac{dY_{t,x,\eta}^s}{ds} \in A'(Y_{t,x,\eta}^s) ds - H\left( s, X_{s,x,\eta}^t, Y_{s,x,\eta}^t, Z_{s,x,\eta}^t, \mathcal{L}(X_{s,x,\eta}^t, Y_{s,x,\eta}^t, Z_{s,x,\eta}^t) \right) ds + Z_{s,x,\eta}^t dW_s, \\
Y_{T,x,\eta}^s = \Phi\left( X_{t,x,\eta}^t, \mathcal{L}_{X_{t,x,\eta}^t} \right),
\end{cases}
$$

(33)

and

$$
\begin{cases}
\frac{dY_{s,x,\eta}^t}{ds} \in A'(Y_{s,x,\eta}^t) ds - H\left( s, X_{s,x,\eta}^t, Y_{s,x,\eta}^t, Z_{s,x,\eta}^t, \mathcal{L}(X_{s,x,\eta}^t, Y_{s,x,\eta}^t, Z_{s,x,\eta}^t) \right) ds + Z_{s,x,\eta}^t dW_s, \\
Y_{T,x,\eta}^s = \Phi\left( X_{t,x,\eta}^t, \mathcal{L}_{X_{t,x,\eta}^t} \right),
\end{cases}
$$

(34)

where $A' : \mathbb{R} \mapsto 2^\mathbb{R}$ is a maximal monotone operator and $H : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l \times \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l) \mapsto \mathbb{R}$, $\Phi : \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \mapsto \mathcal{D}(A')$ are Borel measurable.

Assume:

(H$_{H,\Phi}$) The function $H$ is continuous in $(t, x, y, z, \vartheta)$, the function $\Phi$ is continuous in $(x, \mu)$ and they satisfy for $t \in [0, T], x \in \mathbb{R}^m, y \in \mathbb{R}, z \in \mathbb{R}^l, \vartheta \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l), \mu \in \mathcal{M}_2(\mathbb{R}^m)$

$$|H(t, x, y, z, \vartheta)| + |\Phi(x, \mu)| \leq L_{H,\Phi}(|x| + |y| + |z| + \|\vartheta\|_2 + \|\mu\|_2),$$

where $L_{H,\Phi} > 0$ is a constant.

(H$_{H,\Phi}^2$) The functions $H, \Phi$ satisfy for $t \in [0, T], x_1, x_2 \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^l, \vartheta_1, \vartheta_2 \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^l), \mu_1, \mu_2 \in \mathcal{M}_2(\mathbb{R}^m)$

$$|H(t, x_1, y_1, z_1, \vartheta_1) - H(t, x_2, y_2, z_2, \vartheta_2)| + |\Phi(x_1, \mu_1) - \Phi(x_2, \mu_2)| \leq L_{H,\Phi}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| + \rho(\vartheta_1, \vartheta_2) + \rho(\mu_1, \mu_2)).$$

(H$_{\vartheta}^\infty$) $\mathcal{D}(A') = \mathbb{R}$ and $|A'(\vartheta)| \leq C(1 + |x|), x \in \mathbb{R}$, where $C > 0$ is a constant.

By Theorem 3.2, we know that Eq. (33) has a unique solution denoted as $(Y_{t,x,\eta}^t, K_{t,x,\eta}^t, Z_{t,x,\eta}^t)$. So, Eq. (33) goes into a classical backward multivalued SDE. By Theorem 2.8, there exists a unique triple $(Y_{t,x,\eta}^t, K_{t,x,\eta}^t, Z_{t,x,\eta}^t)$ satisfying Eq. (34). About $Y_{t,x,\eta}^t$ and $Z_{t,x,\eta}^t$ we present the following estimate.

**Proposition 4.2.** Suppose that (H$_{H,\Phi}$) - (H$_{H,\Phi}^2$) - (H$_{\vartheta}^\infty$) hold. Then there exists a constant $C > 0$ such that

(i) for any $t \in [0, T], x \in \mathbb{R}^m, \eta \in L^2(\mathcal{F}_t, \mathbb{R}^m)$,

$$\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_{s,x,\eta}^t|^2 + \int_t^T |Z_{s,x,\eta}^t|^2 ds \right] \leq C,$$
(ii) for any $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^m$, $\eta_1, \eta_2 \in L^2(\tilde{\mathcal{F}}_t, \mathbb{R}^m)$,

$$
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s^{t, x_1, \eta_1} - Y_s^{t, x_2, \eta_2}|^2 + \int_t^T |Z_s^{t, x_1, \eta_1} - Z_s^{t, x_2, \eta_2}|^2 ds \right],
$$

$$
\leq C \left( |x_1 - x_2|^2 + \rho^2 (\mathcal{L}_{\eta_1}, \mathcal{L}_{\eta_2}) \right).
$$

Proof. By (ii) of Theorem 2.9 (i) we obtain (i).

Next, we prove (ii). The method is from [17] Proposition 4.1. First of all, $(X^{t, x, \eta}, Y^{t, x, \eta}, K^{t, x, \eta}, Z^{t, x, \eta})$ is independent of $\tilde{\mathcal{F}}_t$. Hence, it is independent of $\eta \in L^2 \left( \tilde{\mathcal{F}}_t, \mathbb{R}^m \right)$. We consider $(X^{t, x, \eta}, Y^{t, x, \eta}, K^{t, x, \eta}, Z^{t, x, \eta}) |_{x = \eta}$. Combining the uniqueness of solutions for Eq. (33) and Eq. (34), with $X_s^{t, \eta} = X_s^{x, \eta} |_{x = \eta} = X_s^{t, \eta}, s \in [t, T]$, we get $(X^{t, \eta}, Y^{t, \eta}, K^{t, \eta}, Z^{t, \eta}) = (X^{t, x, \eta}, Y^{t, x, \eta}, K^{t, x, \eta}, Z^{t, x, \eta}) |_{x = \eta}$. Besides, if $\eta \in L^2 \left( \tilde{\mathcal{F}}_t, \mathbb{R}^m \right)$, and the distribution of $\eta$ is the same to that of $\eta$, then

$$(X^{t, \eta}, Y^{t, \eta}, K^{t, \eta}, Z^{t, \eta}) : = (X^{t, x, \eta}, Y^{t, x, \eta}, K^{t, x, \eta}, Z^{t, x, \eta}) |_{x = \eta}$$

and $(X^{t, \eta}, Y^{t, \eta}, K^{t, \eta}, Z^{t, \eta})$ have the same law. So, given $\eta_1, \eta_2 \in L^2 \left( \tilde{\mathcal{F}}_t, \mathbb{R}^m \right)$ with $\mathcal{L}_{\eta_1} = \mathcal{L}_{\eta_2}, i = 1, 2$, we consider the following equation,

$$
\begin{align*}
\begin{cases}
\quad dY_s^{t, \eta_1, \eta_2} \in A^\prime \left( Y_s^{t, \eta_1, \eta_2} \right) ds - H \left( s, X_s^{t, \eta_1, \eta_2}, Y_s^{t, \eta_1, \eta_2}, Z_s^{t, \eta_1, \eta_2}, \mathcal{L}_{X^{t, \eta_1}, Y^{t, \eta_1}, Z^{t, \eta_1}} \right) ds + Z_s^{t, \eta_1, \eta_2} dW_s, \\
Y_s^{t, \eta_1, \eta_2} = \Phi \left( X_T^{t, \eta_1, \eta_2}, \mathcal{L}_{X_T^{t, \eta_1}, \eta_2} \right).
\end{cases}
\end{align*}
$$

By Theorem 2.9 (i) and (H'$_{H, \Phi}$), it holds that for any $\delta > 0$, there is an $\alpha > 0$ such that

$$
\mathbb{E} \left[ \int_t^T e^{\alpha(s-t)} \left( |Y_s^{t, \eta_1, \eta_2} - Y_s^{t, \eta_2, \eta_2}|^2 + |Z_s^{t, \eta_1, \eta_2} - Z_s^{t, \eta_2, \eta_2}|^2 \right) ds \right]
$$

$$
\leq C e^{\alpha T} \mathbb{E} \left[ |X_T^{t, \eta_1, \eta_2} - X_T^{t, \eta_2, \eta_2}|^2 + \rho^2 \left( \mathcal{L}_{X_T^{t, \eta_1}, \eta_2} \right) \right]
$$

$$
\leq C \delta \mathbb{E} \left[ \int_t^T e^{\alpha(s-t)} \left( |X_s^{t, \eta_1, \eta_2} - X_s^{t, \eta_2, \eta_2}|^2 + \rho^2 \left( \mathcal{L}_{X^t_{T}, \eta_1}, \mathcal{L}_{X^t_{T}, \eta_2} \right) \right) ds \right]
$$

$$
\leq C \delta \int_t^T e^{\alpha(s-t)} \rho^2 \left( \mathcal{L}_{X_s^{t, \eta_1, \eta_2}, X_s^{t, \eta_1, \eta_2}, Z_s^{t, \eta_1, \eta_2}}, \mathcal{L}_{X_s^{t, \eta_1, \eta_2}, X_s^{t, \eta_1, \eta_2}, Z_s^{t, \eta_1, \eta_2}} \right) ds.
$$

Thus, by Lemma 4.11 and the definition of $\rho$, we get that

$$
\mathbb{E} \left[ \int_t^T e^{\alpha(s-t)} \left( |Y_s^{t, \eta_1, \eta_2} - Y_s^{t, \eta_2, \eta_2}|^2 + |Z_s^{t, \eta_1, \eta_2} - Z_s^{t, \eta_2, \eta_2}|^2 \right) ds \right]
$$

$$
\leq C_{\alpha, \delta} \left( \mathbb{E} \left[ |\eta_1 - \eta_2|^2 + \rho^2 (\mathcal{L}_{\eta_1}, \mathcal{L}_{\eta_2}) \right] \right).
The proof is complete. \qed
Based on the above proposition, we know that \( Y_{t,x}^{t,x,y} \) and \( Z_{t,x}^{t,x,y} \) depend on \( y \) only through its distribution. Therefore, to strengthen the impression set \( Y_{s,x}^{t,x,y} := Y_{s,x}^{t,x,y}, \quad Z_{s,x}^{t,x,y} := Z_{s,x}^{t,x,y}, \quad s \in [t,T], \)
and it holds that
\[
\begin{align*}
(i) \quad & (Y_{s,x}^{t,x,y}, Y_{s,x}^{t,x,y}) = (Y_{s,x}^{t,x,y}, Y_{s,x}^{t,x,y}), \quad r \in [s,T], \mathbb{P}.a.s., \\
(ii) \quad & (Z_{s,x}^{t,x,y}, Z_{s,x}^{t,x,y}) = (Z_{s,x}^{t,x,y}, Z_{s,x}^{t,x,y}), \quad r \in [s,T], \mathbb{P}.a.s..
\end{align*}
\]

Next, we introduce the following parabolic variation inequality (PVI for short) on \([0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)\):
\[
\left\{ \begin{array}{l}
\frac{\partial u(t,x,\mathcal{L}_n)}{\partial t} + Lu(t,x,\mathcal{L}_n) + H(t,x,u(t,x,\mathcal{L}_n),(\nabla u\sigma)(t,x,\mathcal{L}_n),\mathcal{L}_{(\mathcal{L}_n,u(t,x,\mathcal{L}_n),(\nabla u\sigma)(t,x,\mathcal{L}_n))}) \\
A(t,x,\mathcal{L}_n) = \Phi(t,\mathcal{L}_n), \quad (x,\mathcal{L}_n) \in \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m),
\end{array} \right.
\]
where
\[
Lu(t,x,\mathcal{L}_n) = (b^i \partial_{x^i} u)(t,x,\mathcal{L}_n) + \frac{1}{2} \left((\sigma^s)^{ij} \partial_{x^i,x^j}^2 u\right)(t,x,\mathcal{L}_n)
\]
\[
+ \int_{\mathbb{R}^m} \left(\partial_{u} u\right)_i(t,x,\mathcal{L}_n)(y) b^i(t,y,\mathcal{L}_n) \mathcal{L}_n(dy)
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^m} \partial_{y_j} \left(\partial_{u} u\right)_j(t,x,\mathcal{L}_n)(y) (\sigma^s)^{ij} (t,y,\mathcal{L}_n) \mathcal{L}_n(dy).
\]

Then we define viscosity solutions for PVI. \((36)\). For this, we introduce the following notations (cf.\([26]\)):
\[
A'_-(x) := \liminf_{x' \to x, x^* \in A'(x')} x^*, \quad A'_+(x) := \limsup_{x' \to x, x^* \in A'(x')} x^*.
\]

**Definition 4.3.** (i) we say that \( u \in C([0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)) \) is a viscosity subsolution of PVI.\((36)\) if \( u(T,\cdot,\cdot) = \Phi(\cdot,\cdot) \) on \( \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \), and
\[
\frac{\partial \Psi(t,x,\mathcal{L}_n)}{\partial t} + L \Psi(t,x,\mathcal{L}_n)
\]
\[
+ H(t,x,u(t,x,\mathcal{L}_n),(\nabla \Psi\sigma)(t,x,\mathcal{L}_n),\mathcal{L}_{(\mathcal{L}_n,u(t,x,\mathcal{L}_n),(\nabla \Psi\sigma)(t,x,\mathcal{L}_n))})
\]
\[
\geq A'_-(u(t,x,\mathcal{L}_n)),
\]
whenever \( \Psi \in C^{1,2,2}_b([0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)) \) and \( (t,x,\mathcal{L}_n) \in [0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \) is a local maximum point of \( u - \Psi \);
(ii) we say that \( u \in C([0,T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)) \) is a viscosity supersolution of PVI.\((36)\) if \( u(T,\cdot,\cdot) = \Phi(\cdot,\cdot) \) on \( \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m) \), and
\[
\frac{\partial \Psi(t,x,\mathcal{L}_n)}{\partial t} + L \Psi(t,x,\mathcal{L}_n)
\]
\[
\leq A'_+(u(t,x,\mathcal{L}_n)),
\]
+H(t, x, u(t, x, \mathcal{L}_\eta), \nabla \Psi \sigma)(t, x, \mathcal{L}_\eta), \mathcal{L}_{(\eta, u(t, \eta, \mathcal{L}_\eta), (\nabla \Psi \sigma)(t, \eta, \mathcal{L}_\eta))}

\leq A'_+(u(t, x, \mathcal{L}_\eta)),

whenever \( \Psi \in C^{1,2}_b([0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)) \) and \((t, x, \mathcal{L}_\eta) \in [0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)\) is a local minimum point of \( u - \Psi \);

(iii) \( u \) is a viscosity solution of PVI.\((36)\) if it is both a viscosity subsolution and a viscosity supersolution of PVI.\((36)\).

Now, we state the main result in this section.

**Theorem 4.4.** Assume that \((\mathcal{H}^\varepsilon_{H, \varepsilon}) - (\mathcal{H}^\varepsilon_{H, \varepsilon}), (\mathcal{H}^\varepsilon_{H, \varepsilon}) - (\mathcal{H}^\varepsilon_{H, \varepsilon}) (H_A')\) hold. Then the function \( u(t, x, \mathcal{L}_\eta) \) defined by \((35)\) is a viscosity solution of PVI.\((36)\).

To prove the above theorem, we make some preparation. First of all, for any \( \varepsilon > 0 \), consider the following penalized equations

\[
\begin{cases}
\frac{dY^t, s, \varepsilon}{ds} = \left[ A'_\varepsilon (Y^t, s, \varepsilon) - H(s, X^t, s, \varepsilon, Y^t, s, \varepsilon, Z^t, s, \varepsilon, \mathcal{L}(X^t, s, \varepsilon, Y^t, s, \varepsilon, Z^t, s, \varepsilon)) \right] ds + Z^t, s, \varepsilon dW_s, \\
Y^t, s, \varepsilon = \Phi(X^t, s, \varepsilon, \mathcal{L}X^t, s, \varepsilon),
\end{cases}
\]

and

\[
\begin{cases}
\frac{dY^t, x, \varepsilon}{dx} = \left[ A'_\varepsilon (Y^t, x, \varepsilon) - H(s, X^t, x, \varepsilon, Y^t, x, \varepsilon, Z^t, x, \varepsilon, \mathcal{L}(X^t, x, \varepsilon, Y^t, x, \varepsilon, Z^t, x, \varepsilon)) \right] ds + Z^t, x, \varepsilon dW_s, \\
Y^t, x, \varepsilon = \Phi(X^t, x, \varepsilon, \mathcal{L}X^t, x, \varepsilon),
\end{cases}
\]

where \( A'_\varepsilon \) is the Yosida approximation of \( A' \). So, by Theorem 2.4 we know that under \((\mathcal{H}^\varepsilon_{H, \varepsilon}) - (\mathcal{H}^\varepsilon_{H, \varepsilon})\) \((37)\) has a unique solution denoted as \((Y^t, s, \varepsilon, Z^t, s, \varepsilon)\). And \((38)\) is a classical backward SDE and has a unique solution denoted as \((Y^t, x, \varepsilon, Z^t, x, \varepsilon)\) (cf. \([24, \text{Theorem 4.1}]\)). Besides, set \( u_\varepsilon(t, x, \mathcal{L}_\eta) := Y^t, x, \varepsilon := Y^t, x, \varepsilon \), and by the similar deduction to that in the proof of Theorem 3.2, we obtain that \( u_\varepsilon \to u \) as \( \varepsilon \to 0 \). Moreover, by \([17, \text{Proposition 9.1 and Theorem 9.2}]\), it holds that \( u_\varepsilon(t, x, \mathcal{L}_\eta) \) is continuous with respect to \((t, x, \mathcal{L}_\eta) \in [0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)\) and a unique viscosity solution of the following parabolic partial differential equation:

\[
\begin{cases}
\frac{\partial u_\varepsilon(t, x, \mathcal{L}_\eta)}{\partial t} + \mathcal{L}(u_\varepsilon(t, x, \mathcal{L}_\eta)) + H(t, x, u_\varepsilon(t, x, \mathcal{L}_\eta), \nabla u_\varepsilon(t, x, \mathcal{L}_\eta), \mathcal{L}(\eta, u_\varepsilon(t, \eta, \mathcal{L}_\eta), (\nabla u_\varepsilon)(t, \eta, \mathcal{L}_\eta))) \\
\quad = A'_\varepsilon(u_\varepsilon(t, x, \mathcal{L}_\eta)), \\
u_\varepsilon(T, x, \mathcal{L}_\eta) = \Phi(x, \mathcal{L}_\eta), (x, \mathcal{L}_\eta) \in \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m).
\end{cases}
\]

**Proof of Theorem 4.4.**

First of all, we prove that \( u \) is a viscosity subsolution of PVI.\((36)\).

Let us take a \( \Psi \in C^{1,2}_b([0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)) \) such that \( u - \Psi \) attains a local maximum in \((t, x, \mathcal{L}_\eta) \in [0, T] \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)\). Thus, there exists \((t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)\) such that, at least along a subsequence,

(i) \((t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta) \to (t, x, \mathcal{L}_\eta)\) as \( \varepsilon \to 0 \);

(ii) \( u_\varepsilon - \Psi \leq u_\varepsilon(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta) - \Psi(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta) \) in a neighborhood of \((t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)\) for any \( \varepsilon > 0 \);

(iii) \( u_\varepsilon(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta) \to u(t, x, \mathcal{L}_\eta)\) as \( \varepsilon \to 0 \).

Since \( u_\varepsilon \) is a viscosity subsolution of Eq.\((39)\), it holds that

\[
\frac{\partial \Psi(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)}{\partial t} + \mathcal{L}(\Psi(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)) + H(t_\varepsilon, x_\varepsilon, u_\varepsilon(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta), \nabla \Psi(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta), (\nabla \Psi)(t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta), \partial \varepsilon)
\]
\[ A'_\varepsilon (u_\varepsilon (t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)) \geq \] (40)

where \( \vartheta := \mathcal{L}_{(t_\eta, u_\eta(t_\eta, \mathcal{L}_\eta), (\nabla \Psi)(t_\eta, \mathcal{L}_\eta))} \). Note that

\[ J_\varepsilon(u_\varepsilon (t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)) \to u(t, x, \mathcal{L}_\eta), \quad \varepsilon \to 0, \]

\[ A'_\varepsilon (u_\varepsilon (t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta)) \in A' (J_\varepsilon(u_\varepsilon (t_\varepsilon, x_\varepsilon, \mathcal{L}_\eta))). \]

Thus, taking the lower limit on two sides of (40) as \( \varepsilon \to 0 \), we obtain that

\[ \partial \Psi (t, x, \mathcal{L}_\eta) + \mathcal{L} \Psi (t, x, \mathcal{L}_\eta) + H (t, x, u(t, x, \mathcal{L}_\eta), (\nabla \Psi)(t, x, \mathcal{L}_\eta), \vartheta) \geq A'_\varepsilon (u(t, x, \mathcal{L}_\eta)), \]

where \( \vartheta := \mathcal{L}_{(t_\eta, u(t_\eta, \mathcal{L}_\eta), (\nabla \Psi)(t_\eta, \mathcal{L}_\eta))} \). So, by the definition, we know that \( u \) is a viscosity subsolution of PVI.\(^{(36)}\)

By the same deduction to that for viscosity subsolutions of PVI.\(^{(36)}\), one can show that \( u \) is a viscosity supersolution of PVI.\(^{(36)}\). The proof is complete.

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