Wave-Particle Complementarity and Reciprocity of Gauss Sums on Talbot Effects

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Summary

Berry and Klein (J. Mod. Opt. (1997) 43 2139-2164) showed that the Talbot effects in classical optics are naturally expressed by Gauss sums in number theory. Their result was obtained by a computation of Helmholtz equation. In this article, we calculate the effects using Fresnel integral and show that the result is also represented by Gauss sums. However function forms of these two computational results are apparently different. We show that the reciprocity law of Gauss sums connects these results and both completely agree with. The Helmholtz equation can be regarded as an equation based upon wavy nature in optics whereas the Fresnel integral is defined by a sum over the paths based upon a particle picture in optics. Thus the agreement of these two computational results could be interpreted in terms of the concept of the wave-particle complementarity, though the concept is for quantum mechanical phenomenon. This interpretation leads us to a relation between the reciprocity of Gauss sums in number theory and the wave-particle complementarity in wave physics. We discuss it in detail.

Keywords

Talbot Effects, Gauss Sums, Complementarity, Reciprocity

§1. Introduction

The Talbot effects are diffraction grating phenomenon in classical optics, discovered by Talbot (1838), which are known as self-image (Patorski 1989); due to the effects, same patterns arrayed on one or two dimensional grating sharply recover on the screen for a certain condition without lens system. It is remarkable that such the effects occurs due to wavy properties.

Recently Berry and Klein (1996) discovered that behind the phenomenon there is an arithmetic structure. They considered optical wave with wavelength $\lambda$ which stems from a one-dimensional grating with a period $a$. The grating is set up so that its plane agrees
with the wave front and the screen is set parallel to the grating with distance \( z \). They investigated Talbot effects when \( z \) was a fractional number times the Talbot distance \( z_T := a^2/\lambda \). They found a beautiful connection between wave physics and number theory in the investigation of the Talbot effects, which is based upon a deep insight by Hannay and Berry (1980). Berry and Klein investigated the Talbot effects by solving the Helmholtz equation (Noponen & Turunen 1993) and showed that the Talbot effects were explained by Gauss sums and the quadratic reciprocity law (see for example Ireland & Rosen 1990, Chap. 6).

In this article, we present another connection of the problem with Gauss sums based upon the arguments of Berry and Klein. We will also neglect the polarization effect (Noponen & Turunen 1993) and partially review investigation of Berry and Klein by solving the Helmholtz equation in $\S$2. In $\S$3, we compute the same system by means of Fresnel integral (Winthrop & Worthington 1965), whose apparent function form differs from the results in $\S$2. However in terms of the reciprocity of Gauss sums we will show the agreement of both results in $\S$4. We should note that whereas Helmholtz equation is a wave equation, the Fresnel integral is an integration over the optical paths with a natural integration weight. The former one should be regarded as an expression of the wavy properties and later one should be interpreted as a transformation to wave expression from the particle nature. Thus the agreement of the both results reminds us of wave-particle complementarity, i.e., the wave-particle duality of nature in the quantum mechanics (Bohr 1928). We also argue that the complementarity is related to the reciprocity of Gauss sums (Hecke 1981 Chap.8) in the Talbot effects, by using the analogy between the quantum mechanics and the optics in $\S$4.

In order to make the agreements between two computations from Helmholtz equation and Fresnel integral easier, we sometimes employ different expressions from those in the papers of Berry and Klein and Hannay and Berry (loc.cit.). Thus the difference between our formula and theirs sometimes occurs in this paper.

Interestingly in the same period as Talbot’s discovery, Gauss lived and studied optics, number theory including Gauss sums and quadratic residues and so on (Klein 1926).

$\S$2. Wavy Expressions

As in the paper of Berry and Klein (1996), we will start with an incident plane wave of wavelength \( \lambda \) coming through periodic \( \delta \) functions transparency whose period is \( a \). Such a grating is called \( \delta \)-comb. Let the transverse direction denote \( x \) while \( z \) denotes along the optical axis.

In this section, we will deal with the system in terms of the Helmholtz wave equation following Berry and Klein (1996) and Noponen and Turunen(1993). Let us introduce the dimensionless parameters with respect to Talbot distance \( z_T \),

\[
\xi := \frac{x}{a}, \quad \zeta := \frac{z}{z_T} = \frac{z\lambda}{a^2}.
\]  

(2-1)
On the $\delta$-comb grating plane $\zeta = 0$, the optical wave is expressed by the Poisson sum formula,
\[
\psi_{\text{comb}}(\xi, 0) = \sum_{n=-\infty}^{\infty} \exp(2\pi i \xi n) = \sum_{m=-\infty}^{\infty} \delta(\xi - m). \quad (2-2)
\]
As there is a discrete translation symmetry, we can set $\xi \in \mathbb{R}$ as $\xi = \xi_0 + n$ for $\xi_0 \in (-1/2, 1/2)$ and an integer $n$. By letting
\[
\psi_{\text{comb}}(\xi, \zeta) = \sum_{n=-\infty}^{\infty} \eta(\zeta) \exp(2\pi i \xi n), \quad (2-3)
\]
we substitute it into the Helmholtz equation,
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \left( \frac{2\pi}{\lambda} \right)^2 \right) \psi_{\text{comb}} = 0. \quad (2-4)
\]
Then we obtain the solution of (2-4) up to a constant factor $C$,
\[
\psi_{\text{comb}}(\xi, \zeta) = C \sum_{n=-\infty}^{\infty} \exp(2\pi i \xi n) \exp \left( 2\pi i \zeta \left( \frac{a}{\lambda} \right)^2 \left[ 1 - \left( \frac{n\lambda}{a} \right)^2 \right]^{1/2} \right). \quad (2-5)
\]
Here we note $\delta(cx) = \delta(x)/c$. For the case $n > a/\lambda$, the exponential factor becomes a damping function.

By employing the paraxial approximation, (2-5) is expressed by,
\[
\psi_{\text{comb}}(\xi, \zeta) \approx C \sum_{n=-\infty}^{\infty} \exp(2\pi i \xi n) \exp \left( -2\pi i \zeta \left( \frac{1}{2} \left( \frac{n\lambda}{a} \right)^2 \right) \right). \quad (2-6)
\]
Of course, large $n$ is out of the approximation but as it is expected that the contribution from the large $n$ is less than that from small $n$ in Fourier analysis, we will go on to consider (2-6) as in the paper of Berry and Klein (1996). Let the right hand side of (2-6) be denoted by $\psi_p(\xi, \zeta) \exp(ikz)$ following the notations in the paper, where $k := 2\pi/\lambda$. We obtain,
\[
\psi_p(\xi, \zeta) = C \sum_{n=-\infty}^{\infty} \exp(2\pi i \xi n) \exp \left( -2\pi i \zeta \left( \frac{1}{2} n^2 \right) \right). \quad (2-7)
\]
Let us consider the case that $\zeta$ is a rational number, i.e., for coprime positive numbers $p$ and $q$
\[
\zeta = \frac{p}{q}. \quad (2-8)
\]
By introducing the integers $l, s$ such that $n = lq + s$, the quantity of each term in (2-7) become,
\[
\xi n - \frac{1}{2} n^2 \zeta = \xi s - \frac{1}{2} \frac{p}{q} s^2 + \xi lq - \frac{1}{2} l^2 qp - lps. \quad (2-9)
\]
Further we note the relation,
\[ e^{-\pi i qpl^2} = (-1)^{q^2p} = e^{-\pi i qpl}, \] (2-10)
and it is the unity if \( pq \) is even. Then we have
\[ \psi_p(\xi, \zeta) = C \sum_{s=0}^{q-1} e^{\pi i (2\xi qs - ps^2)/q} \sum_{l=-\infty}^{\infty} e^{\pi i (2\xi q - pq)l}, \] (2-11)

Using the Poisson sum formula again, we obtain,
\[ \psi_p(\xi, \zeta) = C \sqrt{q} \sum_{n=-\infty}^{\infty} A(n; q, p) \delta(\xi - \frac{1}{2}e_{qp} - \frac{n}{q}), \] (2-12)
where
\[ e_{qp} := \begin{cases} 1, & \text{if } pq \text{ odd}, \\ 0, & \text{if } pq \text{ even}, \end{cases} \] (2-13)
\[ A(n; q, p) = \frac{1}{\sqrt{q}} \sum_{s=0}^{q-1} \exp \left( i\pi \left[ (2n + qe_{qp})s - ps^2 \right]/q \right). \] (2-14)

We note that if \( pq \) is odd, \((2n + qe_{qp})\) is also odd. Hence if we can use the formula given by (A-18) in the Appendix which is based on the Appendix of the paper of Hannay and Berry (1980), then (2-14) is expressed as follows (Berry & Klein 1996);
\[ A(n; q, p) = \begin{cases} \left( \frac{p}{q} \right) \exp \left( i\pi \left[ \frac{1}{4}(q - 1) + \frac{p}{q} \left( \left[ \frac{1}{p} \right]_q \right)^2 n^2 \right] \right), & \text{p even, q odd,} \\ \left( \frac{q}{p} \right) \exp \left( -i\pi \left[ \frac{1}{4}p - \frac{p}{q} \left( \left[ \frac{1}{p} \right]_q \right)^2 n^2 \right] \right), & \text{p odd, q even,} \\ \left( \frac{p}{q} \right) \exp \left( i\pi \left[ \frac{1}{4}(q - 1) + \frac{2p}{q} \left[ \frac{1}{2} \right]_q \left( \left[ \frac{1}{2p} \right]_q \right)^2 (2n + q)^2 \right] \right), & \text{p odd, q odd.} \end{cases} \] (2-15)

where \( \left[ \frac{1}{p} \right]_q \) is a unique positive integer smaller than \( q \) satisfying
\[ p \left[ \frac{1}{p} \right]_q \equiv 1 \mod q, \] (2-16)
and \( \left( \frac{p}{q} \right) \) is the Jacobi symbol which is a product of the Legendre symbol \( \left( \frac{p}{s} \right) \) for the prime factors \( s \) of \( q \) (Ireland & Rosen 1990, Chap. 5),
\[ \left( \frac{p}{s} \right) := \begin{cases} +1, & \text{if there is an integer } m \text{ such that } m^2 = p \mod s, \\ -1, & \text{otherwise.} \end{cases} \] (2-17)
3. Particle Expressions

In this section, we will consider the system in terms of the Fresnel integration following
the study by Winthrop and Worthington (1965). Here we note that the Fresnel integration
should be considered as a simple path integration (Schulman 1981, Chap.20; Sánchez &
Wolf 1985) over the shortest optical paths obeying the minimal principle of Fermat and
these paths should be regarded as a particle picture in optics.

Using the Fresnel integral for $\delta$-comb wave function (2-2), we obtain the $\psi_{comb}$ at the
point $(\xi, \zeta)$,

$$
\tilde{\psi}_{comb}(\xi, \zeta) = \int \frac{ad\xi'}{\sqrt{(\xi - \xi')^2a^2 + z^2}} \exp \left( \frac{2\pi i}{\lambda} \sqrt{(\xi - \xi')^2a^2 + z^2} + \frac{\xi^2}{\zeta} \right) \psi_{comb}(\xi',0) \cos \theta + 1,
$$

where $\cos \theta = z/\sqrt{(\xi - \xi')^2a^2 + z^2}$. Here we note that (3-1) is a solution of (2-4) in the
sense of $O(\lambda/ \sqrt{(\xi - \xi')^2a^2 + z^2})$ as the function is an approximate kernel of the Helmholtz
differential operator, called Fresnel approximation [BW, Chap.8].

The optical distance between a point $(\xi, \zeta)$ and the $n$-th split $(-n, 0)$ is given by

$$
\frac{1}{\lambda} \sqrt{(\xi + n)^2a^2 + z^2},
$$

(3-2)

and in the paraxial approximation, it is approximated by

$$
\frac{1}{\lambda} \frac{1}{z} + \frac{1}{2} n \frac{1}{2} \zeta + \frac{\xi}{\zeta} n + \frac{1}{2} \frac{\xi^2}{\zeta}.
$$

(3-3)

In this approximation, the denominator of the first factor is approximated by $z$ and $\cos \theta$
can be regarded as the unity, (3-1) becomes

$$
\tilde{\psi}_{comb}(\xi, \zeta) \approx C' \sum_{n=-\infty}^{\infty} \exp\left(2\pi iz/\lambda\right) \exp \left( \pi i \left( \frac{2\xi n}{\zeta} + \frac{n^2}{\zeta} + \frac{\xi^2}{\zeta} \right) \right),
$$

(3-4)

using a certain constant factor $C'$. Of course, this approximation is not contradict with
the Fresnel approximation. By letting the right hand side of (3-4) denoted by $\tilde{\psi}_p \exp(i\xi z)$, we have

$$
\tilde{\psi}_p(\xi, \zeta) = C' \sum_{n=-\infty}^{\infty} \exp \left( \pi i \left( \frac{2\xi n}{\zeta} + \frac{n^2}{\zeta} + \frac{\xi^2}{\zeta} \right) \right).
$$

(3-5)

This formula is essentially obtained by Winthrop and Worthington (1965). In the formula,
we have also assumed that contribution from large $n$ is less than that from small $n$ and
have applied the paraxial approximation to large $n$ case. (This application can be justified
by the considerations that for the case that either $\zeta$ or $\xi$ is an irrational number, the phase
at large $n$ becomes random phase for each $n$ and sum of each terms with the random phase
damps its amplitude and for the case that both $\zeta$ and $\xi$ are rational numbers, the net effect
from (3-1), in which the larger \( n \) contributes less, repeats to appear in (3-5) but rescaling \( C' \) like (A-1) in the Appendix, (3-5) brings us the data of the net effect.)

We also consider the same situation as (2-8) or \( \zeta = p/q \) for coprime positive numbers \( p \) and \( q \). By introducing the integers \( l, s \) such that \( n = lp + s \), the quantity of each terms in (3-5) becomes,

\[
(\xi n + \frac{1}{2} n^2) \frac{q}{p} = \xi s \frac{q}{p} + \frac{1}{2} q s^2 + \xi lq + \frac{1}{2} l^2 qp + lqs. \tag{3-6}
\]

Noting (2-10), (3-5) is expressed by

\[
\tilde{\psi}_p(\xi, \zeta) = C' \sum_{s=0}^{q-1} e^{\pi i (2q \xi s + qs^2 + q \xi^2) / p} \sum_{l=-\infty}^{\infty} e^{\pi i (2 \xi q + pq) l}. \tag{3-7}
\]

Corresponding to (2-12), we obtain the wave function of the system,

\[
\tilde{\psi}_p(\xi, \zeta) = C' \sqrt{p} \sum_{n=-\infty}^{\infty} \tilde{A}(n; q, p) \delta(\xi - \frac{1}{2} e_{qp} - \frac{n}{q}), \tag{3-8}
\]

where

\[
\tilde{A}(n; q, p) := \frac{1}{\sqrt{p}} \sum_{s=0}^{q-1} \exp \left( i \pi \left[ (2n + q e_{qp}) s + q s^2 \right] / p + (2n + q e_{qp})^2 / 4pq \right). \tag{3-9}
\]

As (2-15), we have

\[
\tilde{A}(n; q, p) = \begin{cases} 
(p \choose q) \exp \left( i \pi \left[ \frac{1}{4} q - \left( \frac{q}{p} \left[ \frac{1}{q} \right]_p \right)^2 - \frac{1}{4} \frac{1}{qp} \right] n^2 \right), & p \text{ even, } q \text{ odd,} \\
(q \choose p) \exp \left( -i \pi \left[ \frac{1}{4} (p-1) + \left( \frac{q}{p} \left[ \frac{1}{q} \right]_p \right)^2 - \frac{1}{4} \frac{1}{qp} \right] n^2 \right), & p \text{ odd, } q \text{ even,} \\
(q \choose p) \exp \left( -i \pi \left[ \frac{1}{4} (p-1) + \left( \frac{2q}{p} \left[ \frac{1}{2} \right]_p \left[ \frac{1}{2} \right]_p \right) - \frac{1}{4} \frac{1}{4qp} \right] (2n + q)^2 \right), & p \text{ odd, } q \text{ odd.} 
\end{cases} \tag{3-10}
\]

We note that the function form in (3-10) differs from that in (2-15). Especially the apparent form of the final case of odd \( pq \) is completely different from that in (2-15); the roles of \( q \) and \( p \) in both formulae look inverted.

\section*{§4. Discussion}

As we have two expressions of the same Talbot effect, we will discuss both expressions.
Due to Hecke (Hecke 1981 Chap.8), we have the reciprocity of Gauss sums, if $ab$ even and $c$ is even (or if $ab$ is odd and $c$ is odd) (Hannay & Berry 1980),

\[ \sum_{t=0}^{a-1} e^{\frac{\pi i}{a} (bt^2 + ct)} = \left[ \frac{a^2}{b} \right]^{1/2} e^{-i \frac{\pi a^2}{ab}} \sum_{t=0}^{b-1} e^{-i \frac{\pi t^2}{b}}. \] (4-1)

Here we use the convention $\sqrt{i} = i^{1/2} \equiv e^{\frac{i\pi}{4}}$. From the function forms in (2-15) and (3-10), we have the relation,

\[ A(n; q, p) = \sqrt{i} \tilde{A}(n; q, p). \] (4-2)

In other words, the result which we used the Helmholtz equation and that used by the Fresnel integration agree with after applying the paraxial approximation. Whereas we used the Fourier transformation on solving the Helmholtz equation, the computation of Fresnel integration is directly connected with the optical paths. As (4-1) is obtained by the Fourier transformation as mentioned in the Appendix (A-15) following Appendix in the paper of Hannay and Berry (1980), the correspondence is very natural. However as in both approaches, we used the essentially same but different paraxial approximations, (4-2) is not trivial. Further by considering the meanings of (4-2) as follows, it will turn out non-trivial.

Hereafter let us consider the relation (4-2) primitively in order to reveal the arithmetic structure of (4-2). First we consider $n$-dependence up to constant factor. Since $p$ and $q$ are coprime,

\[ \left[ \frac{1}{q} \right]_p q + \left[ \frac{1}{p} \right]_q p = 1 + pq, \] (4-3)

and thus

\[ \left( \left[ \frac{1}{q} \right]_p q \right)^2 + \left( \left[ \frac{1}{p} \right]_q p \right)^2 \equiv 1 + (pq)^2 \mod 2pq. \] (4-4)

Hence the correspondence of the $n$-dependence parts of even $pq$ case is expressed by

\[ -\left( \frac{q}{p} \left( \left[ \frac{1}{q} \right]_p \right)^2 - \frac{1}{qp} \right) n^2 \equiv \frac{p}{q} \left( \left[ \frac{1}{p} \right]_q \right)^2 n^2 \mod 2. \] (4-5)

Similarly, for the case of $pq$ odd, we obtain,

\[ -\left( \frac{2q}{p} \left[ \frac{1}{2} \right]_p \left( \left[ \frac{1}{2} \right]_p \right)^2 - \frac{1}{4qp} \right) (2n + q)^2 \]

\[ \equiv \frac{2p}{q} \left[ \frac{1}{2} \right]_q \left( \left[ \frac{1}{2} \right]_q \right)^2 (2n + q)^2 + \left( \frac{1}{2} (p + q + 1) + \frac{pq}{4} \right) (2n + q)^2 \mod 2. \] (4-6)
The derivation of (4-6) needs heavy computations in the sense of primitive number theory due to the factor $1/4$. We used the relations for $pq$ odd case,

$$\left[\frac{1}{2}\right]_p = \frac{1+p}{2},$$

$$\frac{1}{4} \left( \left[\frac{1}{q}\right]_p + \left[\frac{1}{p}\right]_q \right) (1+pq) = \frac{p+q}{2} \mod 2,$$ (4-7)

which is proved by by expressing $p = 4\mu + \alpha$ and $q = 4\nu + \beta$ ($\mu, \nu$ are positive integers and $\alpha$ and $\beta$ are 1, or 3.) and by considering the possible cases.

With respect to the $n$-dependence, except constant phase and factor, we have the same dependence. It implies that they express the same intensity of light, square of wave function, in the screen.

Next we will consider the equality of (4-2) including the constant phase factor. The even $pq$ case is not difficult because (4-5) is essential for the equality. However for the odd $pq$ case, we need careful treatment.

Thus we will concentrate our attention on the odd $pq$ case. Let us consider the quantity $\tilde{A}(n, p, q)/A(n, p, q)$ and show that it is $\sqrt{i}$.

Further as for odd $pq$ case, we have

$$\exp\left(\pi i \left(\frac{1}{2} (p+q+1) + \frac{pq}{4}\right) (2n+q)^2\right) = \exp\left(\pi i \left(\frac{1}{2} (p+q+1) + \frac{pq}{4}\right)\right),$$ (4-8)

by considering possible cases. Noting (4-6) and (4-8), the terms $\exp\left(\pi i \left(\frac{pq}{4} + \frac{p+q+1}{2}\right)\right)$ multiplying with $\exp\left(\pi i \left(-\frac{p-1}{4}\right)\right)$ and $\exp\left(\pi i \left(-\frac{q-1}{4}\right)\right)$ in (2-15) and (3-10) becomes

$$\exp\left(\pi i \left(\frac{(p-1)}{2} \cdot \frac{(q-1)}{2} - \frac{1}{4}\right)\right).$$ (4-9)

As it is well-known, the Jacobi’s quadratic reciprocity is given by (Ireland & Rosen 1990, Chap.5)

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = \exp\left(i\pi \left(\frac{(p-1)}{2} \cdot \frac{(q-1)}{2}\right)\right),$$ (4-10)

It means that (4-9) expresses the Jacobi’s quadratic reciprocity. Hence $\tilde{A}(n, p, q)/A(n, p, q)$ is $\sqrt{i}$. In other words agreement in (4-2) for the odd $pq$ case is primitively proved.

We note that the agreement in (4-2) essentially comes from (4-3) and (4-10) and we emphasize that our correspondence including the sign of the phase of Gauss sums. It is known that determination of the sign of the phase in Gauss sums is a very subtle problem (Ireland & Rosen 1990, p.73).

In truth, we have a natural correspondence including $\pi/2$ phase in (4-2). The phase $\pi/2$ reminds us of the phase anomaly (Born & Wolf 1975 8.8.4) and caustic problem (Berry &
Ustill 1980). (In the paper of Berry and Ustill, we can find a beauty of another connection between classical optics and modern mathematics.) The $\pi/2$ phase shift was studied in context of the partial differential equations as Maslov connection (Maslov 1972), number theory as metaplectic representations (Weil 1964), phase anomaly in optics, connection formula in semi-classical method in quantum mechanics, and so on. The “phase” of the Gauss sums is a typical example of metaplectic system. In these systems, the phase is a delicate object. Since the phase usually appears in optics as a phase anomaly, it is expected that our phase also appears due to the effects. Further it is known that in path integral approach, (or in this case, the Fresnel integral), the treatment of the phase needs more carefulness (Schulman 1981 Chap. 17; Berry & Ustill 1980; Matsutani 1997). Hence the agreement between both approaches from Helmholtz equation and Fresnel integral is highly non-trivial.

However conversely, as we solve the same system by means of different methods, these physical investigations require the complete agreement between $\tilde{A}(n,p,q)$ and $A(n,p,q)$. It means that our computation gives a novel proof of the reciprocity of the Gauss sums from physical point of view.

Even though we deal only with the classical optics, we know a following correspondence between classical optics and quantum mechanics. The wave length $\lambda$ is translated into Planck constant $h$, the Helmholtz equation corresponds to the Schrödinger equation and the Fresnel integral is related to path integral (Guillemin & Sternberg 1984 I.12; Sánchez & Wolf 1985). In paraxial theory of geometrical optics, which is sometimes called Gaussian optics (Born & Wolf 1975, 4.4), we can find the symplectic structure in the angle and position of a ray (Guillemin & Sternberg 1984 Chap.I; Sánchez & Wolf 1985). We can define a Poisson bracket for the angle and the position as in the classical mechanics. In the sense, the wave optics is related to the quantum mechanics. In truth Hannay and Berry (1980) studied the Gauss sums in the quantum mechanical context, which play the same roles in Talbot effects in classical optics as mentioned above.

As mentioned in the introduction, we should regard the result from Helmholtz equation as a wavy property and one from the Fresnel integral as a particle property. The complementary principle leads us that complementary elements, such as configuration and momentum, are of dual and complementary to each other due to Planck constant $\hbar$ (Bohr 1928). Though the concept was introduced in order to explain quantum mechanical experiment (Bohr 1928), this concept might be applicable to an even purely theoretical phenomena e.g., one in number theory. In fact, in the wave expression (2-7), we find the term $-2\pi i \zeta \left(\frac{1}{2} n^2\right)$ which is proportional to $\lambda$ because $\zeta = z\lambda/a^2$, whereas in the particle expression (3-5), $\pi i \left(\frac{2\xi n}{\zeta} + \frac{n^2}{\zeta} + \frac{\xi^2}{\zeta}\right)$ is proportional to $1/\lambda$. When $\lambda$ vanishes, former one vanishes while the later case diverges. (In the later case, the operation is related to another connection between number theory and quantum system (Matsutani 2001a, 2001b).) They show the duality and complementarity. It implies that the agreement in (4-2) can be interpreted as wave-particle complementarity. Thus the complementarity in this system and reciprocity of Gauss sums should be regarded as double aspects of a thing.
The complementarity between wave and particle, for the quantum mechanics, is based upon the commutation relation,
\[ xp - px = \sqrt{-1}\hbar, \quad (4-11) \]
for position and momentum operators, \( x \) and \( p \). (4-11) is resembles to (4-3), i.e., for coprime integers \( p \) and \( q \), there exists integers \( \left\{ \frac{1}{q} \right\}_p \) and \( \left\{ \frac{-1}{p} \right\}_q \) such that
\[ \left\{ \frac{1}{q} \right\}_p q - \left\{ \frac{-1}{p} \right\}_q p = 1. \quad (4-12) \]
(4-12) is very essential and is the most important origin which generates a beauty of number theory.

Hence this analogy between (4-11) and (4-12) leads to our conclusion that the wave-particle complementarity (in this system) plays the same role as the reciprocity of coprime numbers in number theory.

In the studies of mathematical physics, we sometimes encounter the cases that we feel the resemblances between number theory and physics as mentioned in the papers (Matsutani 2001a, 2001b). The works of Hannay and Berry (1980) and Berry and Klein (1996) are the most typical cases. Further in the optical design, Gaussian optics in lens system can be expressed by a generalized bracket of the Gaussian bracket which represents continuous fraction (Tanaka 1986). In one-dimensional classical point-particle-system, the cyclomonic polynomial appears and expresses a sort of integrable condition (Ishiwata, Matsutani & ˆOnishi 1997). Further in quantum mechanical problems, we can find the several resemblances. We hope that this report might have some effects on these studies in future.

**Appendix**

In this appendix, we will base upon the Appendix in the paper of Hannay and Berry (1980) and supply their arguments. For (2-15) and (3-10), we will show the derivations of (A-14) and (A-18), which slightly differ from those in the paper. The difference assures the equality in (4-2).

First we will define an infinite sum,
\[ G(a, b, c) := \lim_{N \to \infty} \frac{1}{2abN} \sum_{m=-Nab}^{Nab} \exp \left( \frac{i\pi}{b} \left[ am^2 + cm \right] \right), \quad (A-1) \]
and the Gauss sum,
\[ K(a, b, c) := \frac{1}{a} \sum_{m=0}^{a-1} \exp \left( \frac{i\pi}{b} \left[ am^2 + cm \right] \right). \quad (A-2) \]
By letting $m = bn + s$, each term in (A-1) becomes

$$
\exp \left( \frac{\pi i}{b} (am^2 + cm) \right) = \exp \left( \frac{\pi i}{b} (as^2 + cs) \right), \quad (A-3)
$$

if $ab$ and $c$ are even (or $ab$ and $c$ are odd). Hence if $ab$ and $c$ are even (or $ab$ and $c$ are odd), (A-1) and (A-2) coincide with,

$$G(a, b, c) \equiv K(a, b, c). \quad (A-4)$$

For the case of that $a$ and $c$ are even and $a$ and $b$ are coprime, we can compute $K(a, b, c)$ as,

$$K(a, b, c) = \frac{1}{b} \sum_{m=0}^{b-1} \exp \left( \frac{2\pi ia/2}{b} \left[ m + \frac{c}{2} \left[ \frac{1}{a} \right]_b \right]^2 \right) \exp \left( \frac{-\pi ia}{b} \left[ \frac{c}{2} \right] \left[ \frac{1}{a} \right]_b \right)$$

$$= \frac{1}{b} \sum_{n=0}^{b-1} \left[ 1 + \left( \frac{an/2}{b} \right) \right] \exp \left( \frac{2\pi i}{b} n \right) \exp \left( \frac{-\pi ia}{b} \left[ \frac{c}{2} \right] \left[ \frac{1}{a} \right]_b \right)$$

$$= \frac{1}{b} \left( \frac{a/2}{b} \right) \exp \left( \frac{-\pi i(a/2)}{b} \left[ \frac{c}{2} \right] \left[ \frac{1}{a} \right]_b \right) \sum_{n=0}^{b-1} \left( \frac{n}{b} \right) \exp \left( \frac{2\pi i}{b} n \right). \quad (A-5)$$

In the case for odd $b$ (Ireland & Rosen 1990, Chap.6), the Gauss sum is expressed by,

$$\sum_{n=0}^{a-1} \left( \frac{n}{b} \right) \exp \left( \frac{2\pi i}{b} n \right) = \begin{cases} \sqrt{b}, & \text{for } b = 1 \mod 4, \\ i\sqrt{b}, & \text{for } b = 3 \mod 4. \end{cases}$$

$$= \sqrt{b} \exp \left( \frac{i\pi}{8} (b-1)^2 \right). \quad (A-6)$$

Using the relations,

$$\left( \frac{2}{b} \right) = \left( \frac{1/2}{b} \right) = (-1)^{(b^2-1)/8}, \quad \left( -1 \right) = (-1)^{(b-1)/2}, \quad (A-7)$$

we have

$$\left( \frac{2}{b} \right) \sum_{n=0}^{a-1} \left( \frac{n}{b} \right) \exp \left( \frac{2\pi i}{b} n \right) = \sqrt{b} \exp \left( \frac{-i\pi}{4} (b-1) \right),$$

$$\left( -2 \right) \sum_{n=0}^{a-1} \left( \frac{n}{b} \right) \exp \left( \frac{2\pi i}{b} n \right) = \sqrt{b} \exp \left( \frac{i\pi}{4} (b-1) \right). \quad (A-8)$$

Next we will consider the case that $ab$ and $c$ are odd and $a$ and $b$ are coprime. If we deal with $\exp(2\pi im/b)$-type functions, we can use a technique mentioned in the paper of
Hannay and Berry (1980), i.e., $\sum_{m=1}^{b-1} \exp(2\pi iam/b) = \sum_{m=1}^{b-1} \exp(2\pi im/b)$ for coprime and odd $ab$. However we are treating $\exp(\pi im/b)$-type functions. Since the period of $\exp(\pi im/b)$ in $m$ is not $b$ but $2b$, we must independently consider this case, which is completely different from the even $a$ case.

Since $b + 1$ is even and $b + 1 \equiv 1$ modulo $b$, we have

$$\left[ \frac{1}{2} \right]_b = \frac{1 + b}{2}. \quad (A-9)$$

Hence we have

$$\exp \left( \frac{\pi i}{b} a \left( \left[ \frac{1}{2} \right]_b m^2 \right) \right) = \exp \left( 2\pi i \left[ \frac{1}{2} \right]_b a \left( \left[ \frac{1}{2} \right]_b m^2 - \pi iam^2 \right) \right), \quad (A-10)$$

and

$$\exp \left( \frac{\pi i}{b} c \left( \left[ \frac{1}{2} \right]_b m \right) \right) = \exp \left( 2\pi i \left[ \frac{1}{2} \right]_b c \left( \left[ \frac{1}{2} \right]_b m - \pi icm \right) \right). \quad (A-11)$$

Noting $(-1)^{am^2} \equiv (-1)^{am}$,

$$\exp(-\pi iam^2 + \pi icm) = 1. \quad (A-12)$$

Accordingly for odd $ab$ and odd $c$ case, we have,

$$K(a, b, c) = \frac{1}{b} \sum_{n=0}^{b-1} \exp \left( \frac{2\pi ia}{b} \left[ \frac{1}{2} \right]_b \left[ m + c \left[ \frac{1}{2} \right]_b \right]^2 \right) \exp \left( \frac{-2\pi i a c^2}{b} \left[ \frac{1}{2} \right]_b \left[ \frac{1}{2} \right]_b \right)$$

$$= \frac{1}{b} \left( \frac{a}{b} \right) \exp \left( \frac{-2\pi i a c^2}{b} \left[ \frac{1}{2} \right]_b \left[ \frac{1}{2} \right]_b \right) \sum_{n=0}^{b-1} \left( \frac{n}{b} \right) \exp \left( \frac{2\pi i n}{b} \right)$$

$$= \frac{1}{b} \left( \frac{\pi i}{4b} (b - 1) - \frac{2\pi i a c^2}{b} \left[ \frac{1}{2} \right]_b \left[ \frac{1}{2} \right]_b \right). \quad (A-13)$$

Here following the argument in the paper of Hannay and Berry (1980), we will prove the reciprocity of the infinite sums for any $a$, $b$ and $c$,

$$G(a, b, c) = G(-b, a, c) \sqrt{\frac{i a}{b}} \exp \left[ \frac{-i \pi c^2}{4ab} \right]. \quad (A-14)$$

We first note the relations;

$$\frac{1}{\sqrt{2\pi}} \int dx e^{-\beta x^2} e^{ikx} = \sqrt{\frac{1}{\beta}} e^{-k^2/4\beta}. \quad (A-15)$$
\[ \delta_c(x) = \sum_{m=-\infty}^{\infty} \delta(x - m) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dke^{ik(x-m)} = \sum_{m=-\infty}^{\infty} e^{2\pi i xm}. \] (A-16)

Secondly, we note that due to the paper of Hannay and Berry (1980), the left hand side in (A-14) is expressed by,

\[ G(a,b,c) = \lim_{\alpha \to +0} \int_{-\infty}^{\infty} \sqrt{\alpha} e^{-\pi \alpha x^2} \delta_c(x) \exp \left( \frac{i\pi}{b} (ax^2 + cx) \right) dx. \] (A-17)

Using (A-15) and (A-16), we compute (A-17) and then derive (A-14). Using (A-4), (A-14) becomes the reciprocity of the Gauss sums.

Hence the Gauss sums are expressed by,

\[ K(a,b,c) = \begin{cases} 
\frac{1}{\sqrt{b}} \binom{a}{b} \exp \left( -i\pi \left[ \frac{1}{4} (b-1) + \frac{a}{b} \left( \left[ \frac{1}{a} \right] \right) \left( \frac{c}{2} \right)^2 \right] \right), & a \text{ even, } b \text{ odd,} \\
\frac{1}{\sqrt{b}} \binom{b}{a} \exp \left( i\pi \left[ \frac{1}{4} \left( a - \frac{a}{b} \left[ \frac{1}{a} \right] \right) \left( \frac{c}{2} \right)^2 \right] \right), & b \text{ odd, } a \text{ even,} \\
\frac{1}{\sqrt{b}} \binom{a}{b} \exp \left( -i\pi \left[ \frac{1}{4} (b-1) + \frac{2a}{b} \left[ \frac{1}{2} \right] \left( \left[ \frac{1}{2a} \right] \right) \left( \frac{c}{2} \right)^2 \right] \right), & a \text{ odd, } b \text{ odd.}
\end{cases} \] (A-18)

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