Online Matroid Intersection:
Beating Half for Random Arrival

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Abstract

We study the online matroid intersection problem, which is related to the well-studied online bipartite matching problem in the vertex arrival model. For two matroids $M_1$ and $M_2$ defined on the same ground set $E$, the problem is to design an algorithm that constructs a large common independent set in an online fashion. The algorithm is presented with the ground set elements one-by-one in a uniformly random order. At each step, the algorithm must irrevocably decide whether to pick the element, while always maintaining a common independent set. Since the greedy algorithm — pick the element whenever possible — has a competitive ratio of half, the natural question is whether we can beat half. This problem generalizes online bipartite matching in the edge arrival model where a random edge is presented at each step; nothing better than half-competitiveness was previously known.

In this paper, we present a simple randomized algorithm for online matroid intersection that has a $\frac{1}{2} + \delta$ competitive ratio in expectation, where $\delta > 0$ is a constant. The expectation is over the randomness of the input order and the coin tosses of the algorithm. As a corollary, we obtain the first algorithm that beats half-competitiveness in the bipartite matching setting. We also extend our result to intersection of $k$ matroids and to general graphs, cases not captured by intersection of two matroids.
1 Introduction

Online bipartite matching is a fundamental problem that was introduced in the “vertex arrival” model by Karp, Vazirani, and Vazirani [KVV90]. Despite tremendous progress made in the online vertex arrival model (see Section 1.2), nothing non-trivial was known in the “edge arrival” model where the edges arrive one-by-one. In this work, we tackle this and the more general problems of online matroid intersection and online matching in general graphs. The greedy algorithm achieves a competitive ratio of half. We present the first algorithms that perform better than greedy in the random arrival model.

The online matroid intersection problem in the random arrival model (OMI) consists of two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$, where the elements in $E$ are presented one-by-one to an online algorithm whose goal is to construct a large common independent set. As an element arrives, the algorithm must immediately and irrevocably decide whether to pick it, while ensuring that the picked elements always form a common independent set. We assume that the algorithm knows the size of $E$ and has access to independence oracles for the already arrived elements. The following is the main result of this paper.

**Theorem 1.1.** The online matroid intersection problem in the random arrival model has a \((1/2 + \delta)\)-competitive randomized algorithm, where $\delta > 0$ is a constant.

A special case of OMI where both the matroids are partition matroids already captures the following online bipartite matching problem in the random edge arrival (OBME) model. OBME consists of a fixed bipartite graph $G$ whose edges arrive one-by-one to an online algorithm in a uniformly random order. As an edge arrives, the algorithm must immediately and irrevocably decide whether to pick it into a matching. The algorithm knows the number of edges to arrive but must always maintain a matching. The objective is to maximize the size of the final matching. Nothing better than half was previously known for OBME. Besides being a natural theoretical question after [KVV90], it captures various online content systems such as online libraries where the participants are known to the matching agencies but the requests arrive in an online fashion. Since several of our OMI ideas greatly simplify in this special case, we also present a self contained proof of the following corollary in Appendix B.

**Corollary 1.2.** The online bipartite matching problem in the random edge arrival model has a \((1/2 + \delta)\)-competitive randomized algorithm, where $\delta > 0$ is a constant.

Our online algorithm in Theorem 1.1 is quite simple and only makes a linear number of calls to the independence oracles of both the matroids. Given recent interest in finding fast approximation algorithms for fundamental polynomial-time problems, this result might be of independent interest even in the offline setting. Previously known algorithms that performed better than the greedy algorithm took quadratic time (see Offline Matroid Intersection in Section 1.2).

**Corollary 1.3.** The matroid intersection problem has a linear time \((1/2 + \delta)\) approximation algorithm.

Finally, it is the simplicity of our OMI algorithm that allows us to extend our result to the problems of online matching in general graphs and to online $k$-matroid intersection, cases that are not captured by intersection of two matroids (proof in Appendix D and Appendix F, respectively).

**Theorem 1.4.** In the random edge arrival model, the online matching problem for general graphs has a \((1/2 + \delta')\)-competitive randomized algorithm, where $\delta' > 0$ is a constant.

**Theorem 1.5.** The online $k$-matroid intersection problem in the random arrival model has a \((1/k + \delta''/k)\)-competitive randomized algorithm, where $\delta'' > 0$ is a constant.
1.1 Our Techniques

In this section we present an overview of our techniques to prove Theorem 1.1. See Section 2.1 for notation. At the heart of our algorithm lies Sampling Lemma (see Lemma 2.8), which informally states that the greedy algorithm can perform poorly on an arbitrary OMI instance but not on a randomly generated OMI instance.

Let OPT denote a fixed maximum independent set in the intersection of matroids $M_1$ and $M_2$. For any independent sets $A$ in $M_1$ and $B$ in $M_2$, let $OPT(M_1/A, M_2/B)$ denote a maximum cardinality subset $I \subseteq OPT \setminus (A \cup B)$ that is independent in both contracted matroids $M_1/A$ and $M_2/B$. For simplicity (discussed later), let us assume that our algorithm is allowed two passes over the ground elements, where in the first pass it runs the greedy algorithm (pick an element whenever possible) to select a maximal common independent set $T$ in $M_1$ and $M_2$. Our goal in the second pass is to improve $T$ to ensure that the expected size of our final solution is at least $(\frac{1}{2} + \epsilon)|OPT|$ for some constant $\epsilon$. We therefore restrict our attention to only those graphs where expected $|T| < (\frac{1}{2} + \epsilon)|OPT|$. Since the greedy algorithm always picks a maximal independent set, we know $OPT(M_1/T, M_2/T) = \emptyset$. This means that for most elements in $T$ picking them eliminates the possibility of two OPT elements (one for each matroid) from being picked, and we cannot hope to add more elements to $T$ in the second pass. Hence, to do better than half, the idea of the algorithm is to drop some of these “bad” elements in $T$ and instead pick “two” OPT elements per dropped element in the second pass. In fact, to get a small $\epsilon$ improvement over half, it suffices to pick on average $(1 + \gamma)$ elements (for some constant $\gamma > 0$) per every dropped element. Since most of the elements in $T$ are bad, we drop a random $p$ fraction of them to get a subset $S \subseteq T$ of expected size $(1 - p)|T|$. Our main challenge is to improve $S$ in the second pass to get on average $\gamma$ gain per dropped element of $T$.

To achieve this, we split the problem into two disjoint subproblems as follows. Consider $OPT(M_1/S, M_2/T)$, which denotes the set of OPT elements that can now be potentially picked in $M_1$ in the second pass since we have dropped some of the bad elements in $T$. Similarly, $OPT(M_1/T, M_2/S)$ denotes the set of OPT elements that can now be potentially picked in $M_2$ in the second pass. We independently solve two distinct OMI problems, $(M_1/S, M_2/T)$ and $(M_1/T, M_2/S)$ in the second pass. For simplicity, let us assume (discussed later) that both these problems return disjoint sets and their union is an independent set in both contracted matroids $M_1/S$ and $M_2/S$, i.e. they do not create any conflicts. The expected optimum size of each of these subproblems will be roughly $|T| - |S| \approx p|T|$. If we run the greedy algorithm on these subproblems independently, a naive analysis based on greedy is half competitive for adversarial inputs shows that we get $\frac{1}{2}p|T|$ elements (per subproblem). Hence, in total we obtain $|S| + p|T| \approx |T|$ elements, which does not suffice as expected $|T| < (\frac{1}{2} + \epsilon)|OPT|$. The Sampling Lemma, which is the main technical contribution of this paper and forms the core of our analysis, shows that if we run the greedy algorithm on the matroids $M_1/S$ and $M_2/T$, where $S$ is a random subset of $T$, then in expectation we select $p|T|(\frac{1}{1+p})$. For $p < 1$, this is better than the naive bound of $\frac{1}{2}p|T|$. The little advantage over half suffices to obtain $\gamma$ gain per dropped element on average.

Besides proving the Sampling Lemma, we still need to overcome some challenges to apply it. Firstly, we need to obtain a maximal independent set $T$ without doing two passes over the input. Here we use a crucial hastiness property, first made by Konrad et al. [KMM12] for bipartite matching. It says that if the greedy algorithm outputs a solution of expected size $(\frac{1}{2} + \epsilon)|OPT|$ for some small constant $\epsilon$, it picks most of its elements very quickly. I.e., even if we run the greedy algorithm on a small fraction of elements, it already picks roughly $\frac{1}{2}|OPT|$ elements. In Lemma 2.1 we extend this property to matroids. We therefore construct a two phase MARKING-GREEDY algorithm where Phase (a) runs the greedy algorithm for the first $f$ fraction of the elements to obtain a common independent set $T$; however, each element selected by greedy is picked (into set $S$) only with probability $1 - p$ and marked with remaining probability $p$ (equivalent to dropping $p$ fraction of the elements of $T$). By choosing $f$ to be a small constant, we can assume that almost all the OPT elements appear in Phase (b) and the two pass analysis still works.

The second challenge is to ensure that there are not many conflicts in the two OMI problems $(M_1/S, M_2/T)$
and \((\mathcal{M}_1/T, \mathcal{M}_2/S)\). The idea here is to make our algorithm behave cautiously by ignoring all elements that could potentially create conflicts. In particular, we consider only those elements that are in the span of \(T\) in exactly one of the two matroids. Using linearity of expectation and some careful counting arguments, we argue that we do not lose many OPT elements.

While our results are a qualitative advance, the quantitative improvement is small (\(\delta > 10^{-4}\)). Nonetheless, the lack of assumptions in our algorithm allows us to tackle a broad class of problems. In particular, we tackle the more difficult task of matching in general graphs (nothing better than half is known here even for the vertex arrival model) and intersection of \(k\)-matroid intersection (an \(NP\)-hard problem). It remains an interesting challenge to improve the approximation factor \(\delta\). Perhaps a more interesting challenge is to remove the random order requirement.

1.2 Related Works and Applications

Online Matching in Vertex Arrival Model
Karp, Vazirani, and Vazirani [KVV90] presented the ranking algorithm for online bipartite matching in the vertex arrival model. The problem is to find a matching in a bipartite graph where one side of the bipartition is fixed, while the other side vertices arrive in an online fashion. Upon arrival of a vertex, its edges to the fixed vertices are revealed, and the algorithm must immediately and irrevocably decide where to match it. [KVV90] gives an optimal \((1 - \frac{1}{e})\)-competitive ranking algorithm for adversarial vertex arrival. Since their original proof was incorrect, new ways of analyzing the ranking algorithm have since been developed [BM08, DJK13]. Due to its many applications in the online ad-market, the vertex arrival model and its weighted generalizations have been studied thoroughly (see survey [Meh12]).

Goel and Mehta [GM08] introduced the random vertex-arrival model. In this model, the adversary may choose the worst instance of a graph, but the online vertices arrive in a random order. The greedy algorithm is already \((1 - \frac{1}{e})\)-competitive for this problem, as the analysis reduces to [KVV90]. Later works [MY11, KMT11] showed that the ranking algorithm has a competitive ratio of at least 0.69, beating the bounds for adversarial vertex arrival model. There is still a gap between known upper and lower bounds, and closing this gap remains an open problem.

Online Matching in Edge Arrival Model
In the edge arrival model, a fixed bipartite graph is chosen by an adversary and its edges are revealed one by one to an online algorithm that is trying to find a maximum matching. If the edge arrival is adversarial, this problem captures the adversarial vertex arrival model as a special case: constraint the edges incident to a vertex to appear together. The greedy algorithm has a competitive ratio of half and a natural open question is whether we can beat half. The current best hardness result for adversarial edge arrival is \(\sim 0.57\), even when the algorithm is allowed to drop edges (see [ELMS11]).

Matching in the edge arrival model has also been studied in the streaming community. In the streaming model, the matching algorithm can revoke decisions made earlier, but has only a bounded memory; in particular, it has \(\tilde{O}(1)\) memory in the streaming model and \(\tilde{O}(n)\) memory in the semi-streaming model (see [FKM+05]). The algorithm may make multiple passes over the input; usually trading off the number of passes with the quality of the solution. For bipartite matching in adversarial edge arrival, Kapralov [Kap13] showed that no semi-streaming matching algorithm can do better than \(1 - \frac{1}{e}\). Beating half remains a major open problem.

On the other hand, for uniformly random edge arrival Konrad, Magniez, and Mathieu [KMM12] gave the first single pass algorithm that obtains a 0.501-competitive ratio for bipartite matching in the semi-streaming setting. Their algorithm crucially used the ability to revoke earlier decisions. One of the contributions in this paper is to show that a variant of the greedy algorithm, which appears simple in hindsight, achieves a competitive ratio better than half in the more restrictive online model.

A weighted generalization of OBME is online bipartite matching for random edge arrival in an edge weighted bipartite graph. This problem has exactly the same setting as OBME; however, the goal is to maximize the
weight of the matching obtained. Since it is a generalization of the secretary problem, the greedy algorithm is no longer constant competitive. Korula and Pal [KP09] achieved a breakthrough and gave a constant competitive ratio algorithm for this problem. Kesselheim et al. [KRTV13] later improved their results.

**Randomized Greedy Matching Algorithms**

Our result for matching in general graphs follows a line of work analyzing variants of the greedy algorithm for matching in general graphs. Dyer and Frieze [DF91] showed that greedy on a uniformly random permutation of the edges cannot achieve a competitive ratio better than half for general graphs; however, it performs well for some classes of sparse graphs. Aaronson et al. [ADFS95] proposed the Modified Randomized Greedy (MRG) algorithm and showed that it has a competitive ratio better than half for general graphs. Poloczek and Szegedy [PS12] provided an argument to improve the bounds on the competitive ratio of this algorithm; however, a gap has emerged in their contrast lemma. A ranking based randomized greedy algorithm has been also shown to have a competitive ratio better than half for general graphs (see [CCWZ14]). Neither MRG nor the ranking algorithm can be implemented in the original setting of [DF91] where the edges arrive in random order and the algorithm is only allowed a single pass. To prove Theorem 1.4, we give an algorithm that beats greedy for general graphs with a much simpler analysis and also works in the original setting of [DF91].

**Online Matroid Problems**

The OMI problem studied in this paper is much more general than online matching and has many other applications, such as the following online network design problem. Consider a central depot that stores different types of commodities and is connected to different cities by rail-links. At various points cities order one of the commodities from the depot and the central manager must immediately and irrevocably decide whether to fulfill the order. If the central manager chooses to fulfill the order, it needs to find a path of rail-links from the depot to that city. Moreover, any rail-link can be used to fulfill at most one order as it can only run a single train. The question is to maximize the number of accepted requests given that there is only a finite amount of each commodity at the depot. This is a matroid intersection problem between a gammoid and a partition matroid. Our result implies an algorithm that beats half for this problem if the orders arrive uniformly at random. The intersection of two graphic matroids, with applications to electrical networks [Rec05], is another special case of matroid intersection that has received attention in the past [GX96].

The uniformly random order assumption in OMI is motivated from the work on the secretary problem. In 2007, Babaioff, Immorlica, and Kleinberg [BIK07] introduced the matroid secretary problem, which generalized the classical secretary problem. For a matroid with weighted elements arriving in a uniformly random order, the online algorithm needs to select an independent set of large weight. Despite recent breakthroughs (see [Lac14, FSZ15]), their question on the existence of a constant-competitive algorithm remains unanswered. This problem becomes trivial in the unweighted setting as the greedy algorithm finds the optimum solution. However, beating greedy remained challenging for intersection of matroids. Our Theorem 1.1 resolves this problem. For weighted online matroid intersection, constant factor competitive algorithms are known in the streaming model where the algorithm always maintains an independent set in the intersection but is allowed to drop elements (see [Var11]).

**Offline Matroid Intersection**

Until recently, the fastest unweighted offline matroids intersection algorithm was a variant of Hopcraft-Karp bipartite matching algorithm due to Cunningham [Cun86] taking \(O(mk^{3/2}Q)\) time — \(m, k, Q\) refer to the number of ground elements, the rank of matroid intersection, and to the independence oracle query time, respectively. In 2015, Lee, Sidford, and Wong [LSW15] improved this to \(\tilde{O}(m^2Q + m^3)\), both for weighted and unweighted matroid intersection. When looking for a \((1 - \epsilon)\) approximate weighted matroid intersection, recent works have improved the running time to \(\tilde{O}(mkQ/\epsilon^2)\) [CQ16, HKK16]. Our Theorem 1.1 gives the first algorithm that achieves an approximation factor greater than half with only a linear number of calls to the independence oracles, i.e., in \(O(mQ)\) time.

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1They also obtain similar results for hypergraphs and call it the “Hypergraph Edge-at-a-time Matching” problem.
2 Online Matroid Intersection

2.1 Definitions and Notation

An instance of the online matroid intersection problem \((M_1, M_2, E, \pi, m)\) consists of matroids \(M_1\) and \(M_2\) defined on ground set \(E\) of size \(m\), and where the elements in \(E\) arrive according to the order defined by \(\pi\). For any \(1 \leq i \leq j \leq m\), let \(E^\pi[i, j]\) denote the ordered set of elements of \(E\) that arrive in positions \(i\) through \(j\) according to \(\pi\). For any matroid \(M\) on ground set \(E\), we use \(T \in M\) to denote \(T \subseteq E\) is an independent set in matroid \(M\). We use the terminology of matroid restriction and matroid contraction as defined in Oxley [Oxl06]. To avoid clutter, for any \(e \in E\) we will abbreviate \(A \cup \{e\}\) to \(A \cup e\) and \(A \setminus \{e\}\) to \(A \setminus e\).

\[ \text{Algorithm \textsc{Greedy} (}M_1, M_2, E, \pi\) \]

1: Initialize set \(T\) to \(\emptyset\)
2: for each element \(e \in E^\pi[1, |E|]\) do
3: \hspace{1em} if \(T \cup e \in M_1 \cap M_2\) then
4: \hspace{2em} \(T \leftarrow T \cup e\)
5: return \(T\)

We note that \textsc{Greedy} is well defined even when matroids \(M_1\) and \(M_2\) are defined on larger ground sets as long as they contain \(E\). This notation will be useful when we run \textsc{Greedy} on matroids after contracting different sets in the two matroids. Since \textsc{Greedy} always produces a maximal independent set, it has a competitive ratio of at least half (see Theorem 13.8 in [KV08]). This is true because addition of an “incorrect” element to \textsc{OPT} can create at most two circuits, one for each matroid. In Appendix C we show that even for the special case of bipartite matching, \textsc{Greedy} cannot beat half in the random arrival model.

Let \(\textsc{OPT}\) denote a fixed maximum offline independent set in the intersection of both the matroids. For \(f \in [0, 1]\), let \(T_f^\pi\) denote the independent set that \textsc{Greedy} produces after seeing the first \(f\) fraction of the edges according to order \(\pi\). When clear from context, we will often abbreviate \(T_f^\pi\) with \(T_f\). Let \(G(f) := \frac{E_{\pi}(|T_f|)}{|\textsc{OPT}|}\), where \(\pi\) is a uniformly random chosen order.

For \(i \in \{1, 2\}\), let \(\text{span}_i(T) := \{e \mid (e \in E) \wedge (\text{rank}_{M_i}(T \cup e) = \text{rank}_{M_i}(T))\}\) denote the span of set \(T \subseteq E\) in matroid \(M_i\). Suppose we have \(T \in M_i\) and \(e \in \text{span}_i(T)\), then we denote the unique circuit of \(T \cup e\) in matroid \(M_i\) by \(C_i(T \cup e)\). If \(i = 1\), we use \(7\) to denote \(2\), and vice versa.

We provide a table of all notation used in Appendix A.

2.2 Hastiness Property

Before describing our algorithm \textsc{Marking-Greedy}, we need an important hastiness property of \textsc{Greedy} in the random arrival model. Intuitively, it states that if \textsc{Greedy}’s performance is bad then it makes most of its decisions quickly and incorrectly. This observation was first made by Konrad et al. [KMM12] in the special case of bipartite matching. We extend this property to matroids in Lemma 2.1 (proof in Appendix E.3). We are interested in the regime where \(0 < \epsilon \ll f \ll 1\).

\textbf{Lemma 2.1 (Hastiness Lemma).} For any two matroids \(M_1\) and \(M_2\) on the same ground set \(E\), let \(T_f^\pi\) denote the set selected by \textsc{Greedy} after running for the first \(f\) fraction of elements \(E\) appearing in order \(\pi\). Also, for \(i \in \{1, 2\}\), let \(\Phi_i(T_f^\pi) := \text{span}_i(T_f^\pi) \cap \textsc{OPT}\). Now for any \(0 < f, \epsilon \leq \frac{1}{2}\), if \(E_{\pi}(|T_f^\pi|) \leq \left(\frac{1}{2} + \epsilon\right)|\textsc{OPT}|\) then

\[ E_{\pi} [\Phi_1(T_f^\pi) \cap \Phi_2(T_f^\pi)] \leq 2\epsilon |\textsc{OPT}| \quad \text{and} \]

\[ E_{\pi} [\Phi_1(T_f^\pi) \cup \Phi_2(T_f^\pi)] \geq \left(1 - \frac{2\epsilon}{f} + 2\epsilon\right)|\textsc{OPT}|. \]
This implies $G(f) := \frac{\mathbb{E}[\|T_f\|]}{\|\text{OPT}\|} \geq \left(\frac{1}{2} - \left(\frac{1}{2} - 2\right)\epsilon\right)$.

### 2.3 Beating Half for Online Matroid Intersection

The following Lemma 2.2 shows that we can restrict our attention to the case when the expected size of GREEDY is small (proof in Appendix E.1). In Theorem 2.3, we give an algorithm that beats half for this restricted case, which when combined with Lemma 2.2 finishes the proof of Theorem 1.1.

**Lemma 2.2.** Suppose there exists an Algorithm $A$ that achieves a competitive ratio of $\frac{1}{2} + \gamma$ when $G(1) \leq \left(\frac{1}{2} + \epsilon\right)$ for some $\epsilon, \gamma > 0$. Then there exists an algorithm with competitive ratio at least $\frac{1}{2} + \delta$, where $\delta = \frac{\epsilon \gamma}{\frac{1}{2} + \epsilon + \gamma}$.

**Theorem 2.3.** For any two matroids $M_1$ and $M_2$ on the same ground set $E$, there exist constants $\epsilon, \gamma > 0$ and a randomized online algorithm MARKING-GREEDY such that if $G(1) \leq \left(\frac{1}{2} + \epsilon\right)$ then MARKING-GREEDY outputs an independent set in the intersection of both the matroids of expected size at least $\left(\frac{1}{2} + \gamma\right) |\text{OPT}|$.

#### 2.3.1 The MARKING-GREEDY Algorithm

**Algorithm** MARKING-GREEDY $(M_1, M_2, E, \pi, m, \Psi)$

**Phase (a)**
1: Initialize $S, T \rightarrow \emptyset$
2: for each element $e \in E^\pi[1, fm]$ do
3: if $T \cup e \in M_1 \cap M_2$ then
4: $T \leftarrow T \cup e$
5: if $\psi(e) = 1$ then
6: $S \leftarrow S \cup e$

**Phase (b)**
7: Fix $T_f$ to $T$ and initialize sets $N_1, N_2 \rightarrow \emptyset$
8: for each element $e \in E^\pi[fm, m]$ do
9: for $i \in \{1, 2\}$ do
10: if $e \in \text{span}_i(T_f)$ and $e \notin \text{span}_\pi(T_f)$ then
11: if $(S \cup N_i \cup e \in M_i)$ and $(T_f \cup N_i \cup e \in M_\pi)$ then
12: $N_i \leftarrow N_i \cup e$
13: return $(S \cup N_1 \cup N_2)$

MARKING-GREEDY consists of two phases (see notation in Appendix A). In Phase (a), it runs GREEDY for the first $f$ fraction of the elements, but picks each element selected by GREEDY into the final solution only with probability $(1 - p)$, where $p > 0$ is a constant. With the remaining probability $p$, it marks the element $e$, and behaves as if it had been selected. The idea of marking some elements in Phase (a) is that we hope to “augment” them in Phase (b), i.e. pick more than one element per marked element. To distinguish if an element is marked or picked, the algorithm uses auxiliary random bits $\Psi$ that are unknown to the adversary. We assume that $\Psi(e) \sim \text{Bern}(1 - p)$ i.i.d. for all $e \in E$.

In Phase (b), one needs to ensure that the augmentations of the marked elements do not conflict with each other. The crucial idea is to use the span of the elements selected by GREEDY in Phase (a) as a proxy to find two random disjoint OMI subproblems. The following Fact 2.4 (proof in Appendix E.2) underlies this intuition. It states that given any independent set $S$, we can substitute it by any other independent set contained in the span of $S$. In Lemma 2.5 we use it to prove the correctness of MARKING-GREEDY.
Fact 2.4. Consider any matroid $M$ and independent sets $A, B, C \in M$ such that $A \subseteq \text{span}_M(B)$ and $B \cup C \in M$. Then, $A \cup C \in M$.

Lemma 2.5. MARKING-GREEDY outputs sets $S, N_1$, and $N_2$ such that

$$(S \cup N_1 \cup N_2) \in M_1 \cap M_2.$$  

Proof. Observe that the outputs sets $S, N_1$, and $N_2$ of MARKING-GREEDY satisfy the following for $i \in \{1,2\}$:

1. $N_i \in M_i/S \cap M_i/f_j$ (due to Line 11)
2. $N_i \subseteq \text{span}_{M_i/S}(T_j \setminus S)$ (due to Line 10)

From Property (1) above we know that $N_i \cup (T_j \setminus S) \in M_i/S$. Also, Property (2) implies $N_i \subseteq \text{span}_{M_i/S}(T_j \setminus S)$. Using Fact 2.4, we have $(N_1 \cup N_2) \in M_i/S$. 

\[\square\]

2.3.2 Proof of Theorem 2.3

We know from Lemma 2.1 that $G(f)$ is close to half for $\epsilon \ll f \ll 1$. In the following Lemma 2.7, we show that MARKING-GREEDY (which returns $S \cup N_1 \cup N_2$ by Lemma 2.5) gets an improvement over GREEDY. This completes the proof Theorem 2.3 to get $\gamma \ge 0.03$ for $\epsilon = 0.001$, $f = 0.05$, and $p = 0.33$ in both these lemmas.

Definition 2.6 (Sets $\tilde{E}_i$). For $i \in \{1,2\}$, we define $\tilde{E}_i$ to be the set of elements $e$ that arrive in Phase (b) and satisfy $e \in \text{span}_i(T_f)$ and $e \not\in \text{span}_i(T_f)$.

Lemma 2.7. MARKING-GREEDY outputs sets $S, N_1$, and $N_2$ such that

$$E_{\pi,\psi}[[S \cup N_1 \cup N_2]] \ge (1 - p) G(f) |OPT| + \frac{2p}{1 + p} \left( 1 - \frac{2\epsilon}{f} - 2\epsilon - f - G(f) \right) |OPT|$$

Proof of Lemma 2.7. We treat the sets $S \subseteq T_f, N_1$, and $N_2$ as random sets depending on $\pi$ and $\Psi$. Since MARKING-GREEDY ensures the sets are disjoint, we have

$$E_{\pi,\psi}[[S \cup N_1 \cup N_2]] = E_{\pi,\psi}[[S]] + E_{\pi,\psi}[[N_1]] + E_{\pi,\psi}[[N_2]] \ge (1 - p) G(f) |OPT| + E_{\pi,\psi}[[N_1]] + E_{\pi,\psi}[[N_2]]$$

Next, we lower bound $E_{\pi,\psi}[[N_1]] + E_{\pi,\psi}[[N_2]]$. Observe, for $i \in \{1,2\}$, $N_i$ is the result of running GREEDY on the matroids $M_i/S$ and $M_{\pi}/T_f$ with respect to the elements in $\tilde{E}_i$. In other words, $N_i = \text{GREEDY}(M_i/S, M_{\pi}/T_f, \tilde{E}_i)$ and we wish to lower bound this in expectation (with respect to $\pi$ and $\Psi$). We use the following Sampling Lemma (proved in Section 3) that forms the core of our technical analysis. Intuitively, it says that if $S$ is a random subset of $T$ then for the obtained random OMI instance, with optimal solution of expected size $p |\tilde{T}|$, GREEDY performs better than half-competitiveness even for adversarial arrival order of ground elements.

Lemma 2.8 (Sampling Lemma). Given matroids $M_1, M_2$ on ground set $E$, a set $T \in M_1 \cap M_2$, and $\Psi(e) \sim \text{Bern}(1 - p)$ i.i.d. for all $e \in T$, we define set $S := \{e \mid e \in T$ and $\Psi(e) = 1\}$. I.e., $S$ is a set achieved by dropping each element in $T$ independently with probability $p$. For $i \in \{1,2\}$, consider a set $\tilde{E} \subseteq \text{span}_i(T)$ and a set $\tilde{T} \subseteq \tilde{E}$ satisfying $\tilde{T} \in M_i \cap (M_{\pi}/T)$. Then for any arrival order of the elements of $\tilde{E}$, we have

$$E_{\psi}[	ext{GREEDY}(M_i/S, M_{\pi}/T, \tilde{E})] \ge \frac{1}{1 + p} \left( p |\tilde{T}| \right).$$

To apply the above lemma, in the following Claim 2.9 we argue that in expectation there exist disjoint sets $\tilde{T}_i \subseteq \tilde{E}_i$ of “large” size that satisfy the preconditions of the Sampling Lemma.

Claim 2.9. If $G(1) \le \left( \frac{1}{2} + \epsilon \right)$ then for $i \in \{1,2\}$ there exist disjoint sets $\tilde{T}_i \subseteq \tilde{E}_i$ such that
Finally, to ensure that $\tilde{I}_i \in \mathcal{M}_i \cap (\mathcal{M}_\pi / T_f)$.

Proof. Let $\Phi_i(T_f) := \text{span}_i(T_f) \cap \text{OPT}$. Let $I_i$ denote $\Phi_i(T_f) \setminus \Phi_{i+1}(T_f)$. We construct sets $\tilde{I}_i$ by removing some elements from $I_i$, which implies $\tilde{I}_i \in \mathcal{M}_i$ because $I_i \in \mathcal{M}_i$. We first show that $|I_1| + |I_2|$ is large. From the Hastiness Lemma 2.1, we have

$$E_\pi[|I_1| + |I_2|] = E_\pi[|\Phi_1(T_f) \cup \Phi_2(T_f)|] - E_\pi[|\Phi_1(T_f) \cap \Phi_2(T_f)|] \geq \left(1 - \frac{2e}{f}\right) |\text{OPT}|. \quad (4)$$

Next, we ensure that $\tilde{I}_i \in \mathcal{M}_\pi / T_f$. Note that $I_\pi \subseteq \text{span}_\pi(T_f)$. Let $X_\pi$ denote a minimum subset of elements of $T_f$ such that $\text{span}_\pi(X_\pi \cup I_\pi) = \text{span}_\pi(T_f)$. Since $I_\pi$ and $T_f$ are independent in $\mathcal{M}_\pi$, we have $|X_\pi| = |T_f| - |I_\pi|$. Now starting with $(I_i \cup I_\pi) \in \mathcal{M}_\pi$, we add elements of $X_\pi$ into it. We will remove all elements from $I_i$ to get a set $I'_i$ such that $(I'_i \cup X_\pi \cup I_\pi) \in \mathcal{M}_\pi$ as $(X_\pi \cup I_\pi) \in \mathcal{M}_\pi$. Using Fact 2.4 and $\text{span}_\pi(X_\pi \cup I_\pi) = \text{span}_\pi(T_f)$, we also have $I'_i \cap T_f \in \mathcal{M}_\pi$. One can use a similar argument to obtain set $I'_i$ and $X_\pi$ such that $I'_i \cap T_f \in \mathcal{M}_i$. Since $E_\pi[|X_\pi|] = E_\pi[|T_f| - |I_i|]$,

$$E_\pi[|I'_1| + |I'_2|] \geq E_\pi[|I_1| + |I_2| - |X_1| - |X_2|] = 2E_\pi[|I_1| + |I_2| - |T_f|] \quad (5)$$

Finally, in expectation over $\pi$, at most $f |\text{OPT}|$ of these elements can appear in Phase (b). The remaining elements appear in Phase (b). Thus, combining the following equation with Eq. (4) and Eq. (5) completes the proof.

$$E_\pi[|\tilde{I}_1| + |\tilde{I}_2|] \geq E_\pi[|I'_1| + |I'_2|] - f |\text{OPT}| \quad \square$$

Finally, to finish the proof of Lemma 2.7, we use the sets $\tilde{I}_i$ from the above Claim 2.9 as $\tilde{I}$ and sets $\tilde{E}_i$ as $\tilde{E}$ in the Sampling Lemma 2.8. From Eq. (3) and Claim 2.9, we get

$$E_\pi, \Psi[|S \cup N_1 \cup N_2|] \geq (1 - p) G(f) |\text{OPT}| + \frac{p}{1 + p} E_\pi[|\tilde{I}_1| + |\tilde{I}_2|]$$

$$\geq (1 - p) G(f) |\text{OPT}| + \frac{2p}{1 + p} \left(1 - \frac{2e}{f} - G(f) \right) |\text{OPT}|. \quad \square$$

## 3 Sampling Lemma

We prove the lemma for $i = 1$ as the other case is analogous.

### 3.1 Alternate View of the Sampling Lemma

We prove the Sampling Lemma by first showing that Greedy($\mathcal{M}_1/S, \mathcal{M}_2/T, \tilde{E}$) produces the same output as the algorithm SAMP-ALG (proof in Appendix E.4).

**Lemma 3.1.** Given a fixed $\Psi$ and assuming the elements of $\tilde{E}$ are presented in the same order, the output of SAMP-ALG is the same as the output of Greedy($\mathcal{M}_1/S, \mathcal{M}_2/T, \tilde{E}$).

The idea behind SAMP-ALG is to run Greedy, but postpone distinguishing between the elements that are selected by Greedy (set $T$) and picked by our algorithm (set $S$). This limits what an adversary can do while ordering the elements of $\tilde{E}$. Intuitively, the sets in SAMP-ALG denote the following:
• $N'$ denotes the new elements to be added to the independent set.

• $T'$ are the elements of $T$ for which we still haven’t read the random bit $\Psi$.

• $S'$ are the elements $e \in T$ for which we have read $\Psi$ and they turn out to be picked, i.e., $\Psi(e) = 1$.

**Algorithm SAMP-ALG**

Input: $M_1, M_2, T$, and random bits $\Psi \in \{0, 1\}^{|T|}$.

1: Initialize: $N', S'$ to $\emptyset$, and $T' = T$

2: for each element $e \in \tilde{E}$ do

3: if $T \cup N' \cup e \in M_2$ then

4: Let $C \leftarrow C_1(S' \cup N' \cup T', e) \cap T'$

5: for each element $f \in C$ do

6: $T' \leftarrow T' \setminus f$

7: if $\Psi(f) = 1$ then

8: $S' \leftarrow S' \cup f$

9: else

10: $N' \leftarrow N' \cup e$

11: Break

12: return $N'$

3.2 Proof of the Sampling Lemma

By Lemma 3.1, it suffices to prove that given the preconditions of the Sampling Lemma, SAMP-ALG produces an output of expected size at least $\frac{p + 1}{2} |\tilde{I}|$. More precisely, we need to show that if $\Psi$ in SAMP-ALG is chosen as $\Psi(e) \sim \text{Bern}(1 - p)$ i.i.d. for all $e \in T$, we have $E[|N'|] \geq \frac{p}{1 + p} |\tilde{I}|$.

The main idea of the proof is to argue that before every iteration of the for-loop in Line 2, there are “sufficient” number of elements that are still to arrive and can be added to our solution. To achieve this, we define a set $I'$, which intuitively denotes the set of OPT elements that are still to arrive and can be added to the current solution. The properties of $I'$ are rigorously captured in Invariant 3.2, where $\tilde{E}_r$ denotes the remaining elements of $\tilde{E}$ that are still to be considered in the for-loop. Due to Lemma 3.1, this also denotes the elements of $\tilde{E}$ that are still to arrive for GREEDY. Starting with $I' = \tilde{I}$ at the beginning of SAMP-ALG, we wish to maintain the following.

**Invariant 3.2.** For given sets $S', N', T$, and $\tilde{E}_r \subseteq \tilde{E}$, we have set $I'$ satisfying this invariant if

\[
S' \cup N' \cup I' \in M_1 \\
T \cup N' \cup I' \in M_2 \\
I' \subseteq \tilde{E}_r
\]

As the algorithm SAMP-ALG progresses, set $I'$ has to drop some of its elements so that it continues to satisfy Invariant 3.2. These drops from $I'$ are rigorously captured in Updates 3.3. Note that set $I'$ and Updates 3.3 are just for analysis purposes, and never appear in the actual algorithm. Starting with $I' = \tilde{I}$ at the beginning of SAMP-ALG and satisfying Invariant 3.2, in Claim 3.4 we prove that Updates 3.3 to $I'$ ensure that the invariant is always satisfied. This lets us use induction to prove in Claim 3.5 that Updates 3.3 never drop too many elements from $I'$ and SAMP-ALG returns an independent set of large size.

**Updates 3.3.** We perform the following updates to $I'$ whenever SAMP-ALG reaches Line 8 or Line 10. Claim 3.4 shows that these updates are well-defined.
\begin{itemize}
  \item Line 8: If circuit \(C_1(S' \cup N' \cup I' \cup f)\) is non-empty then remove an element from \(I'\) belonging to \(C_1(S' \cup N' \cup I' \cup f)\) to break the circuit.
  \item Line 10: If circuit \(C_1(S' \cup N' \cup I' \cup e)\) is non-empty then remove an element from \(I'\) belonging to \(C_1(S' \cup N' \cup I' \cup e)\) to break the circuit. If \(C_2(T \cup N' \cup I' \cup e)\) is non-empty then remove another element from \(I'\) belonging to \(C_2(T \cup N' \cup I' \cup e)\) to break the circuit. In the special case where \(e \in I'\), we remove \(e\) from \(I'\).
\end{itemize}

The following claim (proof in Appendix E.5) shows that Updates 3.3 maintain Invariant 3.2.

**Claim 3.4.** Given matroids \(M_1, M_2\), a set \(T \in M_1 \cap M_2\), a set \(\tilde{E}_r \subseteq \text{span}_1(T)\) (denoting the set of remaining elements), and \(\Psi(e) \sim \text{Bern}(1 - p)\) i.i.d. for all \(e \in T\), suppose there exists a set \(I'\) satisfying Invariant 3.2 at the beginning of some iteration of the for-loop in Line 2 of SAMP-ALG. Then

(i) Updates 3.3 are well-defined.

(ii) Updates 3.3 ensure that Invariant 3.2 hold at the end of the iteration.

Finally, we use Invariant 3.2 to prove the main claim.

**Claim 3.5.** Given matroids \(M_1, M_2\), a set \(T \in M_1 \cap M_2\), a set \(\tilde{E}_r \subseteq \text{span}_1(T)\) (denoting the set of remaining elements), and \(\Psi(e) \sim \text{Bern}(1 - p)\) i.i.d. for all \(e \in T\), suppose there exists a set \(I'\) satisfying Invariant 3.2 at the beginning of some iteration of the for-loop of Line 2 in SAMP-ALG. Then the value of \(N'\) at the end of SAMP-ALG satisfies

\[
\mathbb{E}_{\Psi}[|N'|] \geq \frac{p}{1 + p}|I'|
\]

**Proof.** To prove the claim we use induction on \(|I'|\) where \(I' \subseteq \tilde{E}\). WLOG we can assume that \(e\) is the first element such that \(C\) in Line 4 is non-empty. Let \(C = \{t_1, \ldots, t_l\}\) where \(l \geq 1\). For \(j \in \{0, \ldots, l - 1\}\), define event \(B_j\) as \(\Psi(t_1) = \Psi(t_2) = \cdots = \Psi(t_j) = 1\) and \(\Psi(t_{j+1}) = 0\). Also, define \(\bar{B}\) as \(\Psi(t_1) = \Psi(t_2) = \cdots = \Psi(t_l) = 1\).

**Base Case:** Since \(C\) is a non-empty circuit, we can assume that any element \(f \in C\) satisfies the condition \(\Psi(f) = 0\) with probability \(p\). Hence, \(|N'| \geq 1\) with probability at least \(p\), proving the required claim.

**Induction Step:** The events \(B_0, \ldots, B_{l-1}\), and \(\bar{B}\) partition the entire probability space.

**Case 1** (Event \(B_j\)): Since applying the Updates 3.3 preserves Invariant 3.2 by Claim 3.4, we can apply the induction hypothesis to the updated set \(I'\). Moreover, Updates 3.3 remove at most \(j + 2\) elements from \(I'\) in the event \(B_j\). Applying the Induction hypothesis, we can conclude that \(\mathbb{E}_{\Psi}[|N'| \mid B_j] \geq 1 + \frac{p}{1 + p}(|I'|-j-2)\).

**Case 2** (Event \(\bar{B}\)): Since applying the Updates 3.3 preserves Invariant 3.2 by Claim 3.4, we can apply the induction hypothesis to the updated set \(I'\). Moreover, Updates 3.3 remove \(l\) elements from \(I'\) in the event \(\bar{B}\). Conditioned on the event \(\bar{B}\) and applying the induction hypothesis to the updated set \(I'\), we can conclude \(\mathbb{E}_{\Psi}[|N'|] \geq \frac{p}{1 + p}(|I'|-l)\).

Combining both the cases, we have \(\mathbb{E}_{\Psi}[|N'|]\) is at least

\[
\sum_{j=0}^{l-1} \Pr[B_j] \cdot \mathbb{E}_{\Psi}[|N'| \mid B_j] + \Pr[\bar{B}] \cdot \mathbb{E}_{\bar{B}}[|N'| \mid \bar{B}]
\]

\[
\geq \sum_{j=0}^{l-1} (1 - p)^j p \left(1 + \frac{p}{1 + p}(|I'|-2-j)\right) + (1 - p)^l \left(\frac{p}{1 + p}(|I'|-l)\right)
\]

\[
= \frac{p}{1 + p} |I'|
\]

where the last equality uses \(\sum_{j=0}^{l-1} j (1 - p)^j = -\frac{1}{p} \frac{(1-p)^{l+1}}{p^l} - \frac{(1-p)^{l+1}}{p^{l+1}}\). \(\square\)
To finish the proof of Lemma 2.8, we start with \( I' := \tilde{I}, T' := T, N' := \emptyset \), and \( S' := \emptyset \) in Claim 3.5. The preconditions hold true because \( T \cup I \in \mathcal{M}_2, T \in \mathcal{M}_1 \), and \( I \in \mathcal{M}_1 \).

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A Notation

Table 1: Table of Notation

| General Notation |
|------------------|
| \(\mathcal{M}_i\) | Matroid indexed by \(i\) |
| \(A \in \mathcal{M}\) | Subset \(A\) is an independent set in the matroid \(\mathcal{M}\) |
| \(T \cup e\) | Short form for notation \(T \cup \{e\}\) |
| \(\text{rank}_{\mathcal{M}}\) | The rank function defined by matroid \(\mathcal{M}\) |
| \(\mathcal{T}\) | Denotes the index \(3 - i\) |
| \(\mathcal{M}_1 \cap \mathcal{M}_2\) | The set of subsets that are independent in both matroids \(\mathcal{M}_1\) and \(\mathcal{M}_2\) |
| \(\mathcal{M}/T\) | The matroid resulting from contracting subset \(T\) in matroid \(\mathcal{M}\) |
| \(\text{span}_i(T)\) | \(\{e \mid e \in E \land \text{rank}_{\mathcal{M}_i}(T \cup \{e\}) = \text{rank}_{\mathcal{M}_i}(T)\}\) |
| \(\mathcal{C}_i(T \cup e)\) | The unique circuit formed by \(T \cup \{e\}\) in matroid \(\mathcal{M}_i\). This is undefined when \(T\) is not an independent set and \(e \notin \text{span}_i(T)\). |
| \(E\) | The set of ground elements common to the matroids \(\mathcal{M}_1\) and \(\mathcal{M}_2\) |
| \(\pi\) | A permutation on the set \(E\) |
| \(\text{OPT}\) | A fixed maximum independent set in the intersection of \(\mathcal{M}_1 \cap \mathcal{M}_2\) |
| \(G(f)\) | \(\mathbb{E}_\pi[|T_f|]/|\text{OPT}|\) |

Notation used by MARKING-GREEDY in Section 2.3.1

- \(\Psi\): The set of random bits used in the algorithm. For each \(e \in E\), we have \(\Psi(e) \sim \text{Bern}(1 - p)\)
- \(\text{selecting}\): The element is chosen by GREEDY in Phase (a)
- \(\text{picking}\): The element is chosen by MARKING-GREEDY in the final solution
- \(\text{marking}\): The element is chosen by GREEDY in Phase (a) but the algorithm does not pick it
- \(T_f\): The set of elements selected by GREEDY in Phase (a)
- \(S\): The set of elements picked by MARKING-GREEDY in Phase (a)
- \(N_i\): The set of elements belonging to \(\mathcal{M}_i/S \cap \mathcal{M}_i/T\) picked by MARKING-GREEDY in Phase (b)

B Online Bipartite Matching

In this section, we consider a special case of online matroid intersection, namely online bipartite matching in the random edge arrival model. Although, this is a special case of the general Theorem 1.1, we present it for the following two reasons. Firstly, this is a natural model to study and nothing non-trivial was known before (see Section 1.2). Secondly, several of our ideas simplify in this case (in particular the Sampling Lemma) and hence we can lay the framework for our ideas.

B.1 Definitions and Notation

An instance of the online bipartite matching problem \((G, E, \pi, m)\) consists a bipartite graph \(G = (U \cup V, E)\) with \(m = |E|\), and where the edges in \(E\) arrive according to the order defined by \(\pi\). We assume that the algorithm knows \(m\) and vertices of \(G\) but does not know \(E\) or \(\pi\). We use \(E^\pi[i, j]\) where \(1 \leq i \leq j \leq m\) to denote the set of edges that arrive inbetween positions \(i\) through \(j\) according to \(\pi\). When the permutation \(\pi\) is implicit, we will

\(^2\) We emphasize that our definition also works when \(i\) and \(j\) are non-integral
Figure 1: $U = X_1 \cup Y_2$ and $V = X_2 \cup Y_1$, where $X_1$ and $X_2$ denote the set of vertices matched by Greedy in Phase (a). Here thick-edges are picked and diagonal-dashed-edges are marked. Horizontal-dashed-edges show augmentations for the marked edges.

Abbreviate this to $E[i, j]$.

Greedy denotes the algorithm that picks an edge into the matching whenever possible. Let OPT denote a fixed maximum offline matching of graph $G$. For $f \in [0, 1]$, let $T^g_f$ denote the matching produced by Greedy after seeing the first $f$-fraction of the edges according to order $\pi$. For a uniformly random chosen order $\pi$, we define

$$G(f) := \frac{E_x[T^g_f]}{|OPT|}.$$

Hence, $G(1)|OPT|$ is the expected output size of Greedy and $G\left(\frac{1}{2}\right)|OPT|$ is the expected output size of Greedy after seeing half of the edges. Since Greedy always produces a maximal matching and any maximal matching has size at least half of OPT, its competitive ratio is at-least half. In Appendix C, we show that this ratio is tight for worst case input graphs.\(^3\)

### B.2 Beating Half

The following Lemma B.1 shows that we can restrict our attention to the case when the expected Greedy size is small (proof in Appendix E.1). In Theorem B.2 we give an algorithm that beats half for this restricted case.

**Lemma B.1.** Suppose there exists an Algorithm $A$ that achieves a competitive ratio of $\frac{1}{2} + \gamma$ when $G(1) \leq (\frac{1}{2} + \epsilon)$ for some $\epsilon, \gamma > 0$. Then there exists an algorithm with competitive ratio at least $\frac{1}{2} + \delta$, where $\delta = \frac{\epsilon \gamma}{\frac{1}{2} + \epsilon + \gamma}$.

**Theorem B.2.** If $G(1) \leq \left(\frac{1}{2} + \epsilon\right)$ for some constant $\epsilon > 0$ then the MARKING-GREEDY algorithm outputs a matching of size at least $(\frac{1}{2} + \gamma)|OPT|$ in expectation, where $\gamma > 0$ is a constant.

Before describing Marking-Greedy, we need an important property about the performance of Greedy in the random arrival model — if Greedy is bad then it makes most of its decisions quickly and incorrectly. This observation was first observed by Konrad et al. \([KMM12]\) and is crucial to our algorithm. We will be interested in the regime where $0 < \epsilon \ll f \ll 1$.

**Lemma B.3 (Hastiness property: Lemma 2 in \([KMM12]\)).** For any graph $G$ if $G(1) \leq \left(\frac{1}{2} + \epsilon\right)$ for some $0 < \epsilon < \frac{1}{2}$, then for any $0 < f < 1/2$

$$G(f) \geq \frac{1}{2} - \left(\frac{1}{f} - 2\right) \epsilon$$
B.4 shows that this suffices to pick in expectation
B.5 sampling each vertex matched by $G$
deciding the order of the edges in Phase (b). Finally, setting

can give a ratio better than $\frac{1}{2}$.

B.2.2 Proof of Theorem B.2

For a fixed order $\pi$ of the edges, graphs $G_i$ in MARKING-GREEDY are independent of the randomness $\Psi$. Since the algorithm uses $\Psi$ to pick a random subset of the GREEDY solution, this can be viewed as independently sampling each vertex matched by GREEDY in $G_i$. Lemma B.4 shows that this suffices to pick in expectation more than the number of marked edges. In essence, we use the randomness $\Psi$ to limit the power of an adversary deciding the order of the edges in Phase (b). Finally, setting $f = 0.07$, $p = 0.36$, and $\epsilon = 0.001$ in Lemma B.5, the theorem immediately follows and we get $\gamma > 0.05$.

Lemma B.4 (Sampling Lemma). Consider a bipartite graph $H = (X \cup Y, \tilde{E})$ containing a matching $\tilde{I}$. Let $\Psi(x) \sim \text{Bern}(1 - p)$ i.i.d. for all $x \in X$, and define $X' = \{x \mid x \in X \text{ and } \Psi(x) = 0\}$. I.e., the vertices of $X'$ are obtained by independently sampling each vertex in $X$ with probability $p$. Let $H'$ denote the subgraph induced on $X'$ and $Y$. Then for any arrival order of the edges in $H'$,

$$\mathbb{E}_\Psi[\text{GREEDY}(H', \tilde{E})] \geq \frac{p}{1 + p} |\tilde{I}|.$$}

\footnote{We also show that for regular graphs the competitive ratio of GREEDY is at least $\left(1 - \frac{1}{\epsilon^2}\right)$, and that no online algorithm for OBME can give a ratio better than $\frac{5}{6}$.}
**Proof.** We prove this statement by induction on $|\tilde{I}|$. Consider the base case $|\tilde{I}| = 1$. Whenever GREEDY does not select any edge, the vertex adjacent to $\tilde{I}$ in $X$ is not sampled. This happens with probability $1 - p$. Hence, the expected size of the matching is at least $p \geq \frac{p}{1 + p}$, which implies the statement is true when $|\tilde{I}| = 1$.

From the induction hypothesis (I.H.) we can assume the statement is true when the matching size is at most $|\tilde{I}| - 1$. We prove the induction step by contradiction and consider the smallest graph in terms of $|\tilde{I}|$ that does not satisfy the statement. Note that $|X| \geq |\tilde{I}|$. Consider the first edge $e = (x, y)$ that arrives. The first case is when $x \not\in X'$ and it happens with probability $1 - p$. Here any edge incident to $x$ does not matter for the remaining algorithm. We use I.H. on the subgraph induced on $(X \setminus x, Y)$ as $|X \setminus x| = (|X| - 1)$. Since this subgraph has a matching of size at least $|\tilde{I}| - 1$, I.H. gives a matching of expected size at least $\frac{p}{1 + p}(|\tilde{I}| - 1)$.

The second case is when $x \in X'$ and it happens with probability $p$. Now edge $(x, y)$ is included in the GREEDY matching for the induced graph on $(X', Y)$. Vertices $x$ and $y$, along with the edges incident to them, do not participate in the remaining algorithm. We apply I.H. on the subgraph induced on the vertices $(X \setminus x, Y \setminus y)$. Noting that this graph has a matching of size at least $|\tilde{I} - 1|$, I.H. gives a matching of expected size at least $\frac{p}{1 + p}(|\tilde{I} - 2|)$. Combining both cases, the expected matching size is at least

$$(1 - p) \left( \frac{p}{1 + p}(|\tilde{I} - 1|) + p \left( 1 + \frac{p}{1 + p}(|\tilde{I} - 2|) \right) \right) = \frac{p}{1 + p} |\tilde{I}|.$$  

This is a contradiction as we assumed that the graph did not satisfy the induction statement. \qed

We next prove the main lemma needed to prove Theorem B.2.

**Lemma B.5.** For any $0 < f < 1/2$ and bipartite graph $G$, MARKING-GREEDY outputs a matching of expected size at least

$$\left( 1 - \frac{2\epsilon}{f} \right) |\text{OPT}|$$

**Proof.** We remind the reader that for any $f \in [0, 1]$ and any permutation $\pi$ of the edges, $T^\pi_f$ denotes the matching that GREEDY produces on $E[1, fm]$. For $i \in \{1, 2\}$, let $H_i$ denote the subgraph of $G_i$ containing all its edges that appear in Phase (b). Let $I_i$ denote the set of edges of $\text{OPT}$ that appear in graph $G_i$.

**Claim B.6.**

$$\mathbb{E}_\pi [|I_1| + |I_2|] \geq \left( 1 - \frac{2\epsilon}{f} \right) |\text{OPT}|$$

**Proof.** We use the following two simple properties (Fact E.1 in Appendix E.2).

$$|T^\pi_1| \geq \frac{1}{2} \left( |\text{OPT}| + \sum_{e \in \text{OPT}} 1[\text{Both ends of } e \text{ matched in } T^\pi_1] \right)$$

(9)

$$|T^\pi_1| \geq |T^\pi_f| + \frac{1}{2} \sum_{e \in \text{OPT}} 1[\text{Both ends of } e \text{ unmatched in } T^\pi_f]$$

(10)

Moreover, $\mathbb{E}_\pi [|I_1| + |I_2|]$ is equal to

$$\mathbb{E}_\pi \left[ |\text{OPT}| - \sum_{e \in \text{OPT}} 1[\text{Both ends of } e \text{ matched in } T^\pi_f] - \sum_{e \in \text{OPT}} 1[\text{Both ends of } e \text{ unmatched in } T^\pi_f] \right]$$

$$\geq |\text{OPT}| - \mathbb{E}_\pi \left[ 2 |T^\pi_1| - |\text{OPT}| \right] - \mathbb{E}_\pi \left[ 2(|T^\pi_1| - |T^\pi_f|) \right]$$

(using Eq. (9) and Eq. (10))

$$\geq |\text{OPT}| - 2\epsilon |\text{OPT}| - 2 \left( \epsilon + \frac{1}{2} - 2 \right) |\text{OPT}|$$

(using $G(1) \leq \frac{1}{2} + \epsilon$ and Lemma B.3)

$$= \left( 1 - \frac{2\epsilon}{f} \right) |\text{OPT}|.$$  \qed

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For \(i \in \{1, 2\} \), let \(\bar{I}_i \subseteq I_i\) denote the set of edges of \(\text{OPT}\) that appear in Phase (b) of \textsc{Marking-Greedy}, i.e., they appear in graph \(H_i\). In expectation over uniform permutation \(\pi\), at most \(f|\text{OPT}|\) elements of \(\text{OPT}\) can appear in Phase (a). Hence,

\[
E_\pi \left[|\bar{I}_1| + |\bar{I}_2|\right] \geq E_\pi [|I_1| + |I_2|] - f|\text{OPT}| \geq \left(1 - \frac{2\epsilon}{f} - f\right)|\text{OPT}|.
\]

Marking a random subset of \(T^p\) independently is equivalent to marking a random subset of vertices independently. Thus, we can apply Lemma B.4 to both \(H_1\) and \(H_2\). The expected number of edges in \(N_1 \cup N_2\) is at least \(\frac{p}{1+p}(|\bar{I}_1| + |\bar{I}_2|)\), where the expectation is over the auxiliary bits \(\Psi\) that distinguishes the random set of edges marked. Taking expectations over \(\pi\) and noting that Phase (a) picks \((1 - p)G(f)|\text{OPT}|\) edges, we have

\[
E_{\Psi,\pi} [|S \cup N_1 \cup N_2|] = E_{\Psi,\pi} [|S|] + E_{\Psi,\pi} [|N_1| + |N_2|] \\
\geq G(f)(1-p)|\text{OPT}| + \frac{p}{1+p}E_\pi |\bar{I}_1| + |\bar{I}_2| \\
\geq \left[(1-p)\left(\frac{1}{2} - \frac{1}{1 - 2\epsilon}ight)\right] + \frac{p}{1+p} \left(1 - \frac{2\epsilon}{f} - f\right)|\text{OPT}| \quad \text{(using Lemma B.3)}
\]

\section{Miscellaneous Results}

\subsection{Greedy Beats Half on Almost Regular Graphs}

\textbf{Theorem C.1.} For online matching in random edge arrival model, \textsc{Greedy} has a competitive ratio of at least \(1 - \frac{1}{d}\) on any \(d\)-regular graph.

\textbf{Proof.} Consider a vertex \(v\), and let \(u_1, u_2, \ldots, u_d\) be its neighbours. The probability that \((u_1, v)\) is the first to occur amongst all the edges of \(u_1\) is exactly \(\frac{1}{d}\). If this occurs, then we know that vertex \(v\) will be surely matched. Thus, the probability that \(v\) is not matched by the end of the algorithm is at most \((1 - \frac{1}{d})^d \leq \frac{1}{e}\). This means that each vertex is matched with probability at least \(1 - \frac{1}{e}\), leading to the stated theorem. \hfill \Box

The same analysis also extends to graphs that are almost regular, i.e., graphs with vertex degrees between \(d(1 \pm \epsilon)\), for any small constant \(\epsilon\).

\subsection{Greedy Cannot Always Beat Half}

\textbf{Definition C.2 (Thick-Z graph).} Let graph \(\text{Thick-Z} := ((U_1 \cup U_2) \cup (V_1 \cup V_2), E)\) be a bipartite graph with \(|U_1| = |V_1|\) and \(|U_2| = |V_2|\). The edge set \(E\) consists of the union of a perfect matching between \(U_i\) and \(V_i\) for \(i \in \{1, 2\}\) and a complete bipartite graph between \(U_2\) and \(V_1\). If additionally \(|U_1| = |V_2|\), we call the graph a balanced \(\text{Thick-Z}\).

\textbf{Lemma C.3.} When the edges of a balanced \(\text{Thick-Z}\) are revealed one-by-one in a random order to \textsc{Greedy} then in expectation it produces a matching of size \((\frac{1}{2} + o(1))|\text{OPT}|\).

\textbf{Proof.} We note that after an edge is picked by \textsc{Greedy}, both the end points of the edge do not participate later in the algorithm. Hence, at any instance during the execution of \textsc{Greedy}, the participating graph is still a \(\text{Thick-Z}\) graph \(((U'_1 \cup U'_2) \cup (V'_1 \cup V'_2), E')\), where \(U'_1 \subseteq U_1\) and \(V'_i \subseteq V_i\) for \(i \in \{1, 2\}\).

We can view the choices made by \textsc{Greedy} as being done in time steps, where \textsc{Greedy} chooses one edge at each time step. At each time step, at least one of \(U_1\) or \(U_2\) decrease by \(1\), and \textsc{Greedy} halts when \(|U'_1| = |U'_2| = 0\). Let \(t\) be the random variable indicating the first time step during the execution of \textsc{Greedy} when \(\min\{|U'_1|, |U'_2|\} = n^{2/3}\). Let \(a, b\) be the random variables denoting \(a := |U'_1| = |V'_1|\) and \(b := |U'_2| = |V'_2|\) at time \(t\). Let \(O_1\) denote the number of edges of \(\text{OPT}\) chosen by \textsc{Greedy} before time \(t\) and let \(O_2\) denote the number of edges of \(\text{OPT}\) chosen after time \(t\).
Figure 2: The above example is a conjunction of two Thick-Z graphs ($Z_1$ and $Z_2$) by a single edge (the red edge). Note that for a Thick-Z graph even knowing the degree 2 vertex does not allow any algorithm to achieve more than $\frac{5}{3}$ edges in expectation.

We observe that the matching produced by GREEDY is of size $\frac{n}{2} + |O_1| + |O_2|$. Observe $|(|U'_1| - |U'_2|)|$ changes only when GREEDY chooses an edge from OPT, implying that we can bound $|a - b| \leq |O_1|$. Since $O_2$ is bounded by $|U'_1| + |U'_2|$ at time $t$, we can say

$$|O_2| \leq a + b = 2 \min\{a, b\} + |a - b| \leq 2n^{2/3} + |O_1|.$$

Next, to bound $|O_1|$, we note that before time $t$ the probability of an edge picked by GREEDY being from OPT is at most $\frac{2n}{2^{2/3}n^{2/3}} = \frac{2}{n^{1/3}}$. Since GREEDY picks at most $n$ edges before time $t$, we have $E[|O_1|] \leq \frac{2n^2}{n^{1/3}} = 2n^{2/3}$. This proves that expected size of the matching chosen by GREEDY is $\frac{n}{2} + E[|O_1| + |O_2|] \leq \frac{n}{2} + 2n^{2/3} + 2E[|O_1|] \leq \frac{n}{2} + 6n^{2/3}.$

\[ \square \]

C.3 Limitations on any OBME Algorithm

Lemma C.4. No randomized algorithm can achieve a competitive ratio greater than $\frac{5}{6} \sim 0.833$ for online bipartite matching in random edge arrival model when the graph is a balanced Thick-Z with $n = 1$. This is true, even the adversary knows the graph and can identify one vertex which has degree 2.

Proof. The optimum offline matching size is two. However, no randomized online algorithm, (even one which knows the input graph), can obtain more than $\frac{5}{3}$ edges in expectation over the random edge order. To see this, let $p$ denote the probability that the algorithm picks the first edge it sees.

Case 1: The first edge is from the optimal matching (i.e. the first edge is of the form $(u_i, v_i)$ for $i \in \{1, 2\}$). In this case, the algorithm will achieve the optimal value 2 with probability $p$. If it skips one of these edges, it will retain at most 1 edge in the remaining graph.

Case 2: The first edge is not from the optimal matching (i.e. the first edge is $(u_1, v_2)$). In this case the algorithm will achieve a value of at most $1 \cdot p + 2 \cdot (1 - p)$.

Since Case 1 occurs with probability $\frac{2}{3}p$ and Case 2 occurs with probability $\frac{1}{3}$, the expected value of the algorithm is $\frac{2}{3}p + \frac{1}{3}(1 - p) \leq \frac{5}{6}.$

\[ \square \]

Lemma C.5. No randomized algorithm can achieve a competitive ratio greater than $\frac{69}{84} \sim 0.821$ for online bipartite matching in random edge arrival model.

Proof. Our instance corresponds to the case where we take two copies of balanced Thick-Z graph joined by a single edge (see Figure 2). The input is some permutation of the graphs (where the vertices or edges may be permuted and $U$ and $V$ may be swapped). We show by case analysis that no algorithm can achieve a competitive ratio better than $\frac{69}{84} < \frac{5}{6}$. Intuitively, the addition of the single edge only hurts any algorithm without compromising the independence between the two instances.

Let $p$ be the probability that the algorithm picks the first edge. Consider the following cases based on Figure 2:

Case 1: Suppose the first edge is the red edge. This occurs with probability $\frac{1}{3}$. If the algorithm picks this edge (which happens with probability $p$), then the optimal value in the remaining graph is 3. Otherwise, it can
get at most \(2 \cdot \frac{5}{3}\) as the two Thick-Z graphs are disjoint and we can use the previous lemma. Hence the expected outcome is \(\frac{1}{7}(p \cdot 3 + (1 - p) \cdot \frac{10}{3})\).

**Case 2:** Suppose the first edge is a blue edge, this occurs with probability \(\frac{2}{7}\). If the algorithm chooses this edge, then we can get value of 1. Since this affords no information about the second \(Z\), the best an algorithm can do is \(\frac{5}{7}\). Hence the expected solution is \(\frac{2}{7}(p(1 + \frac{5}{3}) + (1 - p)(2 + \frac{5}{3}))\).

**Case 3:** Suppose the first edge is a black edge. This occurs with probability \(\frac{1}{7}\). If the algorithm chooses the first edge, then we can get value of 2 in this copy of the Thick-Z. However, still the algorithms gets at most \(\frac{5}{7}\) in the remaining copy of Thick-Z. Hence the expected cost of the solution is \(\frac{1}{7}(p(2 + \frac{5}{3}) + (1 - p)(1 + \frac{5}{3}))\).

Adding these cases together, we get that expected solution has value at most \(\frac{64 + 5p}{21}\). Since the optimal solution is 4, this gives an upper bound of \(\frac{64}{74}\).

\[\square\]

### C.4 When Size of the Ground Set is Unknown

**Theorem C.6.** For any constant \(\epsilon > 0\), any randomized algorithm \(A\) that does not know the number of edges to arrive has a competitive ratio \(\alpha \leq \frac{2}{3} + \epsilon\) for online bipartite matching in random edge arrival model.

**Proof.** To prove this theorem, we show that for any \(\epsilon > 0\) there exists an instance where \(A\) is less than \(\frac{2}{3} + \epsilon\)-competitive.

Since \(A\) does not know the number of edges to arrive, it must maintain an \(\alpha\) approximation in expectation after the arrival of every edge. This is because \(A\) does not know if the current edge will be the last edge.

Consider the instance given by the graph balanced Thick-Z (see Definition C.2) where the size of the \(|U_1| = |V_1| = N\) will be set later. Consider a random permutation \(\pi\) on the set of all edges and note that each edge \(e\) appears in the first \(T\) edges with probability \(\frac{T}{N^2 + 2N}\), where \(T = 4(N + 2)\log N\). The previous probability is at least \(\frac{4\log N}{N}\). Let \(G_T\) denote the set of edges from the perfect matching between \(U_i\) and \(V_i\) that appear in the first \(T\) edges. Let \(B_T\) denote the set of edges from \(U_2\) to \(V_1\) that appear in the first \(T\) edges. By linearity of expectation, we can say \(\mathbb{E}[|G_T|] \leq 8\log N\) and \(\mathbb{E}[|B_T|] \leq 4N\log N\).

Let \(OPT_T\) denote the expected size of the maximum matching on the graph induced by the first \(T\) edges.

**Claim C.7.** \(N(1 - \epsilon) \leq \mathbb{E}_\pi[|OPT_T|]\)

**Proof.** Consider the graph induced between \(U_2\) and \(V_1\) in the first \(T\) edges. Since any particular edge occurs with probability \(\frac{4\log N}{N}\) and the edges are negatively correlated, we can conclude that

\[
\Pr[\exists a \text{ perfect matching between } U_2 \text{ and } V_1 \text{ in the first } T \text{ edges}] \geq \Pr[\exists a \text{ perfect matching in } \mathcal{G}_{N,N,\frac{4\log N}{N}}]
\]

By a result of Erdos and Renyi (see [ER64]), we know that

\[
\lim_{N \to \infty} \Pr[\exists a \text{ perfect matching in } \mathcal{G}_{N,N,\frac{4\log N}{N}}] = 1
\]

Hence, we can choose an \(N\) such that the above probability is at least \(1 - \epsilon\). Thus we can conclude that \(\mathbb{E}[OPT_T] \geq N(1 - \epsilon)\).

Let \(M_{OPT}\) denote the expected number of edges picked by \(A\) that belong to the perfect matching between \(U_i\) and \(V_i\) (for \(i = 1, 2\)) at time \(T\). Similarly, let \(M_{Rest}\) denote the expected number of edges between \(U_2\) and \(V_1\) chosen by \(A\).

Since \(A\) must maintain an \(\alpha\) approximation, we can say \(M_{OPT} + M_{Rest} \geq \alpha(1 - \epsilon)N\). Since \(M_{OPT} \leq \mathbb{E}[|G_T|] = 8\log N \leq \alpha\epsilon N\), we can say

\[
M_{Rest} \geq (\alpha - 2\epsilon)N \quad (11)
\]
Figure 3: $U$ denotes the set of vertices matched by GREEDY in Phase (a) and $V$ denotes the remaining vertices of $G$. Solid edges within $U$ denote the picked edges and dashed edges within $U$ denote the marked ones. Dashed edges from $U$ to $V$ denote the OPT edges. However, every edge chosen from $M_{Rest}$ decreases the value of the optimal algorithm by one. Let $F$ be the expected size of the matching chosen by the algorithm. We know that $\alpha \cdot 2N \leq F \leq 2N - M_{Rest}$. Substituting into Eq. (11) and dividing by $2N$, we get $\alpha \leq \frac{2}{3} + \epsilon$. \qed

D Beating Half for General Graphs

**Theorem 1.4.** In the random edge arrival model, the online matching problem for general graphs has a $(\frac{1}{2} + \delta')$-competitive randomized algorithm, where $\delta' > 0$ is a constant.

**Proof overview.** Let $G$ be the arrival graph with edge set $E$. Using the same idea as Lemma B.1, we can again focus on graphs where GREEDY has a competitive ratio of at most $(\frac{1}{2} + \epsilon)$ for any constant $\epsilon > 0$. We construct a two-phase algorithm that uses the algorithm from Theorem 1.2 as a subroutine. In Phase (a), we run GREEDY; however, each edge selected by GREEDY is picked only with probability $(1 - p)$. With probability $p$, we mark it along with its vertices and behave as if it has been matched for the rest of Phase (a). Since the hastiness property (Lemma B.3) is also true for general graphs, in expectation we pick $(1 - p) \left( \frac{1}{2} - O(\frac{\epsilon}{f}) \right)$ $|OPT|$ edges and mark $p \left( \frac{1}{2} - O(\frac{\epsilon}{f}) \right)$ $|OPT|$ edges in Phase (a). Now we need to ensure that in expectation $(1 + \gamma)$ edges, for some constant $\gamma > 0$, are picked per marked edge in Phase (b).

Let $T_f$ denote the set of edges selected by GREEDY in Phase (a), i.e., both picked and marked edges. Let $U$ denote the set of vertices matched in $T_f$ and $V$ denote the remaining set of vertices of $G$. Using the following simple Fact D.1 and Lemma B.3, we can argue that $\left( 1 - O(\frac{\epsilon}{f}) \right)$ $|OPT|$ edges go from a vertex in $U$ to a vertex in $V$ in graph $G$.

**Fact D.1 (Lemma 1 in [KMM12]).** Consider a maximal matching $T$ of graph $G$ such that $|T| \leq \left( \frac{1}{2} + \epsilon \right) |OPT|$ for some $\epsilon \geq 0$. Then $G$ contains at least $\left( \frac{1}{2} - 3\epsilon \right) |OPT|$ vertex disjoint 3-augmenting paths with respect to $T$.

Moreover, in expectation at most $f$ fraction of these $(U, V)$ $|OPT|$ edges can appear in Phase (a). Thus, setting $\epsilon \ll f \ll 1$ gives that most of the $|OPT|$ edges, i.e., $\left( 1 - O(\frac{\epsilon}{f}) - f \right)$ fraction, appear in Phase (b). This implies that most of the marked edges contain two 3-augmentation edges as shown in Figure 3.

Now consider a marked edge $(u_1, u_2)$ with $(u_1, v_1)$ and $(u_2, v_2)$ denoting its 3-augmentations. In comparison to bipartite graphs, the new concern in general graphs is that there might be an edge between $u_1$ and $v_2$ as
triangles are possible in non-bipartite graphs. Hence, the Sampling Lemma B.4 cannot be directly applied here. However, we are only interested in the bipartite graph between vertices $U$ and $V$. Therefore, in Phase (b), we run the algorithm from Theorem 1.2 for bipartite graphs restricted to $(U, V)$ edges. For sufficiently small values of constants $\epsilon$ and $f$, the constant $\delta$ gain in Theorem 1.2 is sufficient to obtain a constant $\delta'$ gain for this theorem.

E Missing Proofs

E.1 Restricting Attention to Graphs Where Greedy Performs Badly

Lemma B.1. Suppose there exists an Algorithm $\mathcal{A}$ that achieves a competitive ratio of $\frac{1}{2} + \gamma$ when $\mathcal{G}(1) \leq (\frac{1}{2} + \epsilon)$ for some $\epsilon, \gamma > 0$. Then there exists an algorithm with competitive ratio at least $\frac{1}{2} + \delta$, where $\delta = \frac{\epsilon \gamma}{\frac{1}{2} + \epsilon + \gamma}$.

Proof. Consider the algorithm that tosses a coin at the beginning and runs Greedy with probability $1 - r$ and Algorithm $\mathcal{A}$ with probability $r$, where $r > 0$ is some constant. This lemma follows from simple case analysis.

- Case 1: $\mathcal{G}(1) < \frac{1}{2} + \epsilon$
  Since Greedy is always $\frac{1}{2}$ competitive, we can say that in expectation, the competitive ratio will be at least
  $$(1 - r) \frac{1}{2} + r \left( \frac{1}{2} + \gamma \right) = \frac{1}{2} + r \gamma$$

- Case 2: $\mathcal{G}(1) \geq \frac{1}{2} + \epsilon$
  Since we have no guarantees on the performance of Algorithm $\mathcal{A}$ when Greedy performs well, we assume that it achieves a competitive ratio of 0. Our expected performance will be at least
  $$(1 - r) \left( \frac{1}{2} + \epsilon \right) + 0 = \frac{1}{2} + \epsilon - \frac{r}{2} - r \epsilon$$

Choosing $r = \frac{\epsilon}{\frac{1}{2} + \epsilon + \gamma}$, we get $\delta \geq \frac{\epsilon \gamma}{\frac{1}{2} + \epsilon + \gamma}$. □

E.2 Facts

Fact E.1.

$$|T^\pi_1| \geq \frac{1}{2} \left( |OPT| + \sum_{e \in OPT} 1[Both \ ends \ of \ e \ matched \ in \ T^\pi_f] \right)$$

$$|T^\pi_1| \geq |T^\pi_f| + \frac{1}{2} \sum_{e \in OPT} 1[Both \ ends \ of \ e \ unmatched \ in \ T^\pi_f]$$

Proof. We start by counting the vertices matched in $T^\pi_1$,

$$2 |T^\pi_1| \geq 2 \sum_{e \in OPT} 1[Both \ ends \ of \ e \ matched \ in \ T^\pi_1] + \sum_{e \in OPT} 1[Exactly \ one \ end \ of \ e \ matched \ in \ T^\pi_1]$$

Since $T^\pi_1$ is a maximal set,

$$|OPT| = \sum_{e \in OPT} 1[Exactly \ one \ end \ of \ e \ matched \ in \ T^\pi_1] + \sum_{e \in OPT} 1[Both \ ends \ of \ e \ matched \ in \ T^\pi_1]$$

Combining the previous two statements and the fact that $T^\pi_f \subseteq T^\pi_1$,

$$|T^\pi_1| \geq \frac{1}{2} \left( |OPT| + \sum_{e \in OPT} 1[Both \ ends \ of \ e \ matched \ in \ T^\pi_f] \right)$$
To prove the second part, observe that \( T_f^x \subseteq T_f^x \) and \( T_f^x \) is a maximal matching. For each edge of OPT that has both its end points unmatched in \( T_f^x \), at least one end point is adjacent to an edge \( T_f^x \). Since these edges must be part of \( T_f^x \setminus T_f^x \),
\[
|T_f^x| \geq |T_f^x| + \frac{1}{2} \sum_{e \in \text{OPT}} 1[\text{Both ends of } e \text{ unmatched in } T_f^x].
\]

\[\square\]

**Fact 2.4.** Consider any matroid \( M \) and independent sets \( A, B, C \in M \) such that \( A \subseteq \text{span}_M(B) \) and \( B \cup C \in M \). Then we also have \( A \cup C \in M \).

**Proof.** Suppose we start with \( B \in M \) and add elements of \( A = \{a_1, a_2, \ldots, a_k\} \) one by one. We show that one can ensure that the set remains independent in \( M \) by removing some elements from \( B \). First, note that \(|B| = \text{rank}(B) = \text{rank}(B \cup A)\). Our algorithm removes an element from \( B \) only if addition of every \( a_j \) creates a circuit. Hence the rank of the set is always \(|B|\) and addition of every \( a_j \) creates a unique circuit. Moreover, this circuit contains an element \( b_j \in B \) that can be removed as we know \( A \in M \).

Next we repeat the above procedure but by starting with \( B \cup C \in M \) and adding elements of \( A \). We know from before that addition of each element \( a_j \) creates a unique circuit that does not contain an element of \( C \). Hence we can remove element \( b_j \) while ensuring the set remains independent in \( M \). This will finally give \( A \cup C \in M \). \[\square\]

### E.3 Hastiness Lemma

The proof of the following lemma is similar to Lemma 2 in [KMM12].

**Lemma 2.1** (Hastiness Lemma). For any two matroids \( M_1 \) and \( M_2 \) on the same ground set \( E \), let \( T_f^x \) denote the set selected by GREEDY after running for the first \( f \) fraction of elements \( E \) appearing in order \( \pi \). Also, for \( i \in \{1, 2\} \), let \( \Phi_i(T_f^x) := \text{span}_i(T_f^x) \cap \text{OPT} \). Now for any \( 0 < f, \epsilon \leq \frac{1}{2} \) if \( E_\pi[|T_f^x|] \leq (\frac{3}{2} + \epsilon) |\text{OPT}| \) then
\[
E_\pi[|\Phi_1(T_f^x) \cap \Phi_2(T_f^x)|] \leq 2\epsilon |\text{OPT}| \quad \text{and} \quad (12)
\]
\[
E_\pi[|\Phi_1(T_f^x) \cup \Phi_2(T_f^x)|] \geq \left( 1 - \frac{2\epsilon}{f} + 2\epsilon \right) |\text{OPT}|. \quad (13)
\]

This implies \( G(f) := \frac{E_\pi[|T_f^x|]}{|\text{OPT}|} \geq \left( \frac{1}{2} - \left( \frac{1}{f} - 2 \right) \epsilon \right) \).

**Proof.** For ease of notation, we write \( T_f^x \) by \( T_f \). To prove Eq. (12),
\[
E_\pi[|\Phi_1(T_f) \cap \Phi_2(T_f)|] \leq E_\pi[|\Phi_1(T_1) \cap \Phi_2(T_1)|] \quad (\text{because } T_f \subseteq T_1)
\]
\[
= E_\pi[|\Phi_1(T_1)| + |\Phi_2(T_1)| - |\Phi_1(T_1) \cup \Phi_2(T_1)|] \quad (\text{because } T_1 \text{ is a maximal solution})
\]
\[
\leq 2 E_\pi[|T_1|] - |\text{OPT}| \quad (\text{because } |T_1| \geq |\Phi_1(T_1)|)
\]
\[
\leq 2\epsilon |\text{OPT}|.
\]

Now to prove Eq (13), we first bound \(|\Phi_1(T_f)| + |\Phi_2(T_f)|\). It is at least
\[
|\text{OPT}| + \sum_{e \in \text{OPT}} 1[e \in \text{span}_1(T_f) \cap \text{span}_2(T_f)] - \sum_{e \in \text{OPT}} 1[e \notin \text{span}_1(T_f) \cup \text{span}_2(T_f)]
\]
\[
\geq |\text{OPT}| + \sum_{e \in \text{T_f}} 1[e \notin \text{span}_1(T_f) \cup \text{span}_2(T_f)] \quad (\text{because } T_f \subseteq \text{span}_i(T_f))
\]

Taking expectations and using Claim E.2,
\[
E_\pi[|\Phi_1(T_f)| + |\Phi_2(T_f)|] \geq |\text{OPT}| - \left( \frac{1}{f} - 2 \right) E_\pi[|T_f \cap \text{OPT}|] \quad (14)
\]

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Since \( f \leq \frac{1}{2} \), we can use an upper bound on \( \mathbb{E}_\pi[|T_f \cap \text{OPT}|] \). Observe \( T_1 \supseteq T_f \) is a maximal solution implying \(|T_1| \geq |T_1 \cap \text{OPT}| + \frac{1}{2}(\text{OPT}-|T_1 \cap \text{OPT}|) \geq \frac{1}{2}(\text{OPT} + |T_f \cap \text{OPT}|) \). Taking expectations,

\[
\mathbb{E}_\pi[|T_f \cap \text{OPT}|] \leq 2 \mathbb{E}_\pi\left[|T_1| - \frac{1}{2}\text{OPT}\right] \leq 2 \varepsilon |\text{OPT}|.
\]

(because \( \mathbb{E}_\pi[|T_1|] \leq (\frac{1}{2} + \varepsilon) |\text{OPT}| \))

Combining this with Eq. (14) and Eq. (12) proves Eq. (13),

\[
\mathbb{E}_\pi\left[|\Phi_1(T_f^\pi) \cup \Phi_2(T_f^\pi)|\right] = \mathbb{E}_\pi[|\Phi_1(T_f)| + |\Phi_2(T_f)|] - \mathbb{E}_\pi\left[|\Phi_1(T_f^\pi) \cap \Phi_2(T_f^\pi)|\right] \geq \left(1 - \frac{2 \varepsilon}{f} + 2 \varepsilon \right) |\text{OPT}|.
\]

Finally, using Eq. (14) and \(|T_f| \geq |\Phi_i(T_f)|\), we also have \( \mathbb{E}_\pi[|T_f^\pi|] \geq \frac{1}{2} \mathbb{E}_\pi[|\Phi_1(T_f)| + |\Phi_2(T_f)|] \geq \left(\frac{1}{2} - \left(\frac{1}{2} - 2 \varepsilon\right)\right) |\text{OPT}|. \)

For intuition, imagine the following claim for \( f = \frac{1}{2} \), where it says that for a uniformly random order probability that \( e \) is in not in the span of \( T_f \) for either of the matroids is at most the probability \( e \) is selected by GREEDY into \( T_f \).

**Claim E.2.** Suppose \( G(1) \leq \left(\frac{1}{2} + \varepsilon\right) |\text{OPT}| \) for some \( \varepsilon < \frac{1}{2} \) and \( T_f \) is the output of GREEDY on \( E([1, mf]) \), then

\[
\forall e \in \text{OPT} \quad \mathbb{P}_\pi[e \notin \Phi_1(T_f) \land e \notin \Phi_2(T_f)] \leq \left(\frac{1}{f} - 1\right) \mathbb{P}_\pi[e \in T_f]
\]

**Proof.** Let us define the event \( X = \left(e \notin \Phi_1(T_f) \land e \notin \Phi_2(T_f)\right) \lor (e \in T_f) \). Consider the mapping \( g \) from permutations to permutations. If \( e \) occurs in the first \( f \) fraction of elements then \( g(\pi) = \pi \). If not, then remove \( e \) and insert it uniformly at random at a position in \([1, mf]\). This induces a mapping from the set of all permutations on the ground elements to the set of permutations that have \( e \) in the first \( f \) fraction of elements. The important observation is that the set of permutations satisfying the event \( X \) still satisfy the event under the mapping \( g \). We can conclude that \( \mathbb{P}[X] \leq \mathbb{P}[X \mid e \in [1, mf]] \). Conditioned on the event that \( e \in [1, fm] \), event \( X \) means \( e \in T_f \). This is because if \( e \notin \Phi_1(T_f) \land e \notin \Phi_2(T_f) \) and \( e \in [1, fm] \), then \( T_f \cup e \in M_1 \lor M_2 \). Thus, we can conclude that \( \mathbb{P}[X] \leq \mathbb{P}[e \in T_f \mid e \in [1, mf]] = \frac{1}{f} \mathbb{P}[e \in T_f] \). Moreover, since \( \left(e \notin \Phi_1(T_f) \land e \notin \Phi_2(T_f)\right) \) and \( (e \in T_f) \) are disjoint events, \( \mathbb{P}[X] = \mathbb{P}[\left(e \notin \Phi_1(T_f) \land e \notin \Phi_2(T_f)\right)] + \mathbb{P}[e \in T_f] \), which proves this claim. \( \square \)

### E.4 Proof of the Alternate View of Sampling Lemma

We restate the lemma for convenience.

**Lemma 3.1.** Given a fixed \( \Psi \) and assuming the elements of \( \tilde{E} \) are presented in the same order, the output of SAMP-ALG is the same as the output of GREEDY(\( M_1 / S, M_2 / T, \tilde{E} \)).

Starting with \( S' = \emptyset, N' = \emptyset \), and \( T' = T \), we make some simple observations and prove a small claim before proving Lemma 3.1.

**Observation E.3.** The for-loop defined in Line 2 of SAMP-ALG maintains the following invariant

\[
S \subseteq S' \cup T' \subseteq T
\]

**Proof.** To show the first containment, observe that for each element if an \( \Psi(e) = 1 \) then it simply moves from \( T' \) to \( S' \). Hence, all the elements of \( S \subseteq S' \cup T' \). To observe the second containment, note that an element of \( T' \) either moves into \( S' \) or gets removed. Since \( T' \) was initialized to \( T \), the second containment follows. \( \square \)
Observation E.4. The for-loop defined in Line 2 of SAMP-ALG maintains the following invariant

\[ S' \cup N' \cup T' \in \mathcal{M}_1. \]

Proof. Since \( T \in \mathcal{M}_1 \) and \( S' = T' = \emptyset \) at the beginning, we can conclude that this is correct at the beginning of SAMP-ALG. Now consider an iteration of the for-loop defined in Line 2. When an element \( f \) is added to \( S' \) in Line 8, it must have belonged to \( T' \), implying that \( S' \cup N' \cup T' \) is unchanged. If an element \( e \) is added to \( N' \) (in Line 10) then we must remove an element \( f \) from \( T' \) (due to Line 6), which belonged to the unique circuit \( C_1(S' \cup T' \cup N', e) \). Hence, \( S' \cup N' \cup e \cup (T' \setminus f) \) is still an independent set in \( \mathcal{M}_1 \). \( \Box \)

Claim E.5. For an element \( e \in \tilde{E} \), if Line 4 of SAMP-ALG is reached then \( C_1(S' \cup N' \cup T', e) \) is non-empty.

Proof. We know \( \tilde{E} \subseteq \text{span}_1(T) \). Moreover, \( S' \cup T' \subseteq T \subseteq \text{span}_1(T) \) (using Observation E.3). Hence, \( S' \cup T' \cup \tilde{E} \subseteq \text{span}_1(T) \) implies

\[
\text{rank}_{\mathcal{M}_1}(S' \cup T' \cup \tilde{E}) \leq |T|.
\] (15)

We prove the lemma by contradiction and assume circuit \( C_1(S' \cup N' \cup T', e) \) is empty. Using Observation E.4, this implies \( (S' \cup N' \cup T' \cup e) \in \mathcal{M}_1 \). Now, \( \text{rank}_{\mathcal{M}_1}(S' \cup N' \cup T' \cup e) = |S' \cup N' \cup T'| + 1 \leq \text{rank}_{\mathcal{M}_1}(S' \cup T' \cup \tilde{E}) \leq |T| \) using Eq. (15). In the next paragraph, we show that the algorithm always maintains \( |S' \cup N' \cup T'| = |T| \), which gives the contradiction \( |T| + 1 \leq |T| \).

To prove \( |S' \cup N' \cup T'| = |T| \), we note that the only time \( T' \) decreases is in Line 6. In this case, we either add the dropped element to \( S' \) in Line 8 or a new element to \( N' \) in Line 10. Hence, the \( |S' \cup N' \cup T'| \) is unchanged in the for-loop of Line 2. Since we initialize \( S' = N' = \emptyset \) and \( T' = T \), we can conclude that this \( |S' \cup N' \cup T'| = |T| \) is maintained. \( \Box \)

We now have the tools to prove Lemma 3.1.

Proof of Lemma 3.1. Let us assume the elements of \( \tilde{E} \) are presented in order \( e_1, \ldots, e_t \) where \( t = |\tilde{E}| \). We will use induction on the following hypothesis.

**Induction Hypothesis (I.H.):** After both algorithms have seen the first \( k \) elements \( e_1, \ldots, e_k \), the set \( N' \) in SAMP-ALG is the same as the set of elements selected by GREEDY \((\mathcal{M}_1/S, \mathcal{M}_2/T, \tilde{E})\).

**Base Case:** Initially, both algorithms have not selected any element. Hence, \( N' = \emptyset \) is the set of all elements selected by GREEDY.

**Induction Step:** Suppose the I.H. is true for elements \( e_1, \ldots, e_{k-1} \) and we are considering element \( e_k \). If \( e_k \) does not satisfy \( T \cup N' \cup e_k \in \mathcal{M}_2 \), then it will also not satisfy the same condition for GREEDY because \( N' \) is the set selected by GREEDY (by I.H.) and \( N' \cup e \notin \mathcal{M}_2/T \). In this case we are done with the induction step. From now assume \( T \cup N' \cup e_k \in \mathcal{M}_2 \).

Suppose \( e_k \) is added to \( N' \) in SAMP-ALG, then we claim GREEDY \((\mathcal{M}_1/S, \mathcal{M}_2/T, \tilde{E})\) will also select this element. The only location where \( e_k \) could be added is Line 10. This occurs when we remove some appropriate element \( f \in T' \) to ensure \( S' \cup (T' \setminus f) \cup N' \cup e \in \mathcal{M}_1 \). Furthermore \( \Psi(f) = 0 \) implies \( f \notin S \). By Observation E.3, set \( S \subseteq S' \cup T' \setminus f \). Hence, \( S' \cup (T' \setminus f) \cup N' \cup e \in \mathcal{M}_1 \) implies \( S \cup N' \cup e \in \mathcal{M}_1 \) and GREEDY will also select this element.

Next, suppose \( e_k \) is not picked by the algorithm. By Claim E.5, we know that \( C_1(S' \cup N' \cup T', e) \) is non-empty. In this case, every element \( f \in C \) encountered in the for-loop of Line 5 must have had \( \Psi(f) = 1 \). This implies that at the end of the for-loop of Line 5, circuit \( C_1(S' \cup N' \cup T', e) \subseteq S' \cup N' \). Since \( S' \subseteq S \) (by Observation E.3), this gives \( N' \cup e \notin \mathcal{M}_1/S \). Hence, GREEDY cannot select element \( e_k \). \( \Box \)
E.5 Proof of Claim 3.4

Claim 3.4. Given matroids $\mathcal{M}_1, \mathcal{M}_2$, a set $T \in \mathcal{M}_1 \cap \mathcal{M}_2$, a set $\tilde{E}_r \subseteq \text{span}_1(T)$ (denoting the set of remaining elements), and $\Psi(e) \sim \text{Bern}(1 - p)$ i.i.d. for all $e \in T$, suppose there exists a set $I'$ satisfying Invariant 3.2 at the beginning of some iteration of the for-loop in Line 2 of SAMP-ALG. Then

(i) Updates 3.3 are well-defined.

(ii) Updates 3.3 ensure that Invariant 3.2 hold at the end of the iteration.

Proof. Since Invariant 3.2 holds before entering into the for-loop in Line 2, we prove this claim by showing that after one iteration of the for-loop, i.e., after arrival of an element $e$, Properties (i) and (ii) hold.

We first show that the properties hold if the set $C$ in Line 4 is empty. Since in this case we do not perform any updates to sets $S', N', I'$, and $T'$, Invariant 6, Invariant 7, and well-definedness trivially hold. To prove Invariant (8), we need to show $e \notin I'$. This is true because by Claim 3.5 element $e$ forms a circuit in $C_1(S' \cup N' \cup T', e)$, and by Invariant (6) we know $S' \cup N' \cup I' \in \mathcal{M}_1$. Hence, the circuit $C_1(S' \cup N' \cup T', e)$ contains some element of $T'$, which gives the contradiction that $C$ is non-empty.

Now WLOG, we can assume that element $e$ forms a non-empty set $C$ in Line 4. We prove Property (i), Invariant (6), and Invariant (7) by showing that they hold after any iteration of the for-loop of Line 5. Note that sets $S', N'$, and $I'$ can only change in Lines 8 or 10 of the for-loop. We prove the claim for both these cases.

Case 1 (Line 8): Since $f$ belonged to $T'$, from Observation E.4 we know $(S' \cup N' \cup f) \in \mathcal{M}_1$. Now using Invariant (6) (which holds before the iteration), we can deduce that $C_1(S' \cup N' \cup I', f) \cap I'$ is non-empty and the update is well-defined. Invariant (6) holds because the update breaks any circuit in $S' \cup N' \cup I'$ in $\mathcal{M}_1$. Since $T$ and $N'$ are unchanged and $I'$ only gets smaller, Invariant (7) holds trivially.

Case 2 (Line 10): Since we are adding $e$ to $N'$, it must be the case that $S' \cup N' \cup e \in \mathcal{M}_1$ (by Lemma 3.1). If $C_1(S' \cup N' \cup I' \cup e)$ is non-empty then $C_1(S' \cup N' \cup I' \cup e) \cap I'$ must be non-empty. Moreover, by Line 3, we know that $T \cup N' \cup e \in \mathcal{M}_2$. Hence, if $C_2(T \cup N' \cup I' \cup e)$ is non-empty then $C_2(T \cup N' \cup I' \cup e) \cap I'$ must be non-empty. Both of them together prove the the update is well-defined in this case. Invariant (6) and Invariant (7) hold because Updates 3.3 break any circuit $C_1(S' \cup N' \cup I' \cup e)$ and $C_2(T \cup N' \cup I' \cup e)$.

Finally, to finish the proof of this claim, we show that Invariant (8) also holds at the end of every iteration of the for-loop of Line 2. If $e \notin I'$ then Invariant (8) trivially holds as $\tilde{E}_r$ looses element $e$, and $I' \subseteq E_r \setminus e$. Now suppose $e \in I'$. Here we consider two cases.

Case 1 ($e$ is added to $N'$): Here SAMP-ALG reaches Line 10 and the special case of Update 3.3 ensures that $e$ is removed from $I'$. Hence $I' \subseteq E_r$.

Case 2 ($e$ is not added to $N'$): From the above proof, we know that Invariants (6) and (7) are preserved at the end of this iteration. We prove by contradiction and assume that $e \in I'$ at the end of this iteration. Since $e \notin N'$, all the elements of $C$ in Line 4 are added to $S'$ by the end of this iteration. Hence, the entire circuit in Line 4 (which is non-empty by Claim E.5) is contained in $S' \cup N' \cup e$ at the end of the iteration. Since $e \in I'$, this implies that $S' \cup N' \cup I'$ is not independent. This is a contradiction as Invariant (6) is violated.

F Generalization to the Intersection of $k$-matroids

The online $k$-matroid intersection problem in the random arrival model (OMI) consists of $k \geq 2$ matroids, $\mathcal{M}_i = (E_i, \mathcal{I}_i)$ for $i \in [k]$. The elements of $E$ are presented one-by-one to an online algorithm whose goal is to construct a large common independent set. As the elements arrive, the algorithm must immediately and irrevocably decide whether to pick them, while ensuring that the set of picked elements always form a common independent set. We assume that the algorithm knows the size of $E$ and has access to independence oracles of the $k$ matroids for the already arrived elements.
**Theorem F.1.** The online $k$-matroid intersection problem in the random arrival model has a $\left( \frac{1}{k} + \frac{\delta''}{k^2} \right)$-competitive randomized algorithm, where $\delta'' > 0$ is a constant.

The proof largely follows the proof of Theorem 1.1 for intersection of two matroids. We sketch the proof of the following lemma below (and make no effort in optimizing the parameters). When combined Lemma B.1, this proves Theorem F.1. As before, $\mathcal{G}(1) | \text{OPT}|$ denotes the expected size of the common independent produced by the greedy algorithm.

**Lemma F.2.** There exists constants $\epsilon, \gamma > 0$ and an online algorithm such that if $\mathcal{G}(1) \leq (\frac{1}{k} + \frac{\epsilon}{k^2})$ then the algorithm finds an independent set of expected size at least $\left( \frac{1}{k} + \frac{\gamma}{k} \right) | \text{OPT}|$.

## F.1 Hastiness Lemma

We need the following hastiness property for the proof of Lemma F.2.

**Lemma F.3 (Hastiness Lemma).** For any $k$ matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$ on the same ground set $E$, let $T^\pi_f$ denote the set selected by GREEDY after running for the first $f$ fraction of elements $E$ appearing in order $\pi$. Also, for $i \in [k]$, let $\Phi_i(T^\pi_f) := \text{span}_i(T^\pi_f) \cap \text{OPT}$. Now for any $0 < f \leq \frac{1}{k}$ and $0 \leq \epsilon < 1$, if $\mathbb{E}_\pi[|T^\pi_f|] \leq (\frac{1}{k} + \frac{\epsilon}{k}) | \text{OPT}|$ then

\[
\mathbb{E}_\pi \left[ |\Phi_i(T^\pi_f) \cap \Phi_j(T^\pi_f)| \right] \leq \frac{2\epsilon}{k^2} | \text{OPT}| \quad \text{for all } i \neq j \in [k] \tag{16}
\]

\[
\mathbb{E}_\pi \left[ \sum_{i=1}^{k} |\Phi_i(T^\pi_f)| \right] \geq \left( 1 - \frac{4\epsilon}{kf} + 4\epsilon \right) | \text{OPT}|. \tag{17}
\]

Hence, $\mathbb{E}_\pi[|T^\pi_f|] \geq \left( \frac{1}{k} - \left( \frac{4\epsilon}{kf} - 4\epsilon \right) \frac{1}{k} \right) | \text{OPT}|$.

**Proof Overview.** For ease of notation, we write $T^\pi_f$ by $T_f$. We prove Eq. (16) by contradiction and assume

\[
\mathbb{E}_\pi \left[ |\Phi_i(T_f) \cap \Phi_j(T_f)| \right] > \frac{2\epsilon}{k^2} | \text{OPT}| \implies \mathbb{E}_\pi \left[ |\Phi_i(T_1) \cap \Phi_j(T_1)| \right] > \frac{2\epsilon}{k^2} | \text{OPT}| \text{ because } T_f \subseteq T_1
\]

Let $S = (\Phi_i(T_1) \cup \Phi_j(T_1))$ and note that $T_1 \cup \text{OPT} \setminus S \in \mathcal{M}_i \cap \mathcal{M}_j$. Moreover,

\[
| \text{OPT} \setminus S | = | \text{OPT} | - (| (\Phi_i(T_1) \cup \Phi_j(T_1)) |)
\]

\[
= | \text{OPT} | - | \Phi_i(T_1) | - | \Phi_j(T_1) | + | \Phi_i(T_1) \cap \Phi_j(T_1) |
\]

\[
\geq | \text{OPT} | - 2 | T_1 | + | \Phi_i(T_1) \cap \Phi_j(T_1) |
\]

Since, $T_1 \in \bigcap_{i=1}^{k} \mathcal{M}_i$, we remove at most $(k - 2)|T_1|$ more elements, say $S'$, from set $\text{OPT} \setminus S$ such that $(T_1 \cup \text{OPT} \setminus (S \cup S')) \in \bigcap_{i=1}^{k} \mathcal{M}_i$. This gives

\[
| \text{OPT} \setminus (S \cup S') | \geq | \text{OPT} \setminus S | - (k - 2)|T_1| \geq | \text{OPT} | + | \Phi_i(T_1) \cap \Phi_j(T_1) | - k | T_1 |. \tag{18}
\]

However, as $T_1$ is a maximal independent set, $\text{OPT} \setminus (S \cup S')$ is an empty set. Taking expectations over $\pi$ in Eq. (18) gives $\mathbb{E}_\pi[| \text{OPT} \setminus (S \cup S') |] > | \text{OPT} | \left( 1 - k \left( \frac{1}{k} + \frac{\epsilon}{k^2} \right) + \frac{2\epsilon}{k^2} \right) > 0$. This is a contradiction.

Next, to prove Eq. (17),

\[
\mathbb{E}[\sum_{i=1}^{k} | \Phi_i(T^\pi_f) |] = \sum_{e \in \text{OPT}} \sum_{i=1}^{k} \Pr[e \text{ is in the span of } T_f \text{ in exactly } i \text{ matroids ]}
\]

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Using Claim F.4 below and the fact that \( \Pr[e \text{ is in the span of } T_f \text{ in all the matroids}] \geq \Pr[e \in T_f] \), we have

\[
\geq |\text{OPT}| - \left( \frac{1 - \epsilon}{f} - k \right) \sum_{e \in \text{OPT}} \Pr[e \in T_f]
\]

\[
= |\text{OPT}| - \left( \frac{1}{f} - k \right) \mathbb{E}[|\text{OPT} \cap T_f|]
\]

Since \( f \leq \frac{1}{k} \), we use an upper bound \( \mathbb{E}[|\text{OPT} \cap T_f|] \leq \mathbb{E}[|\text{OPT} \cap T_1|] \leq \frac{2}{k - 1} \leq \frac{4}{k} \) to finish the proof.

To prove this bound, observe \( T_1 \) is a maximal set, and achieves a \( k \) approximation. We can conclude that \( |T_1| \geq |\text{OPT} \cap T_1| + \frac{1}{f}(|\text{OPT}| - |\text{OPT} \cap T_1|) \). Taking expectations and simplifying we get \( \mathbb{E}[|\text{OPT} \cap T_1|] \leq \frac{2k}{k - 1} \).

Since for all \( i \), we have \( |T_f| \geq |\Phi_i(T_f)| \), the hence part of the lemma follows because \( |T_f| \geq \frac{1}{f} \sum_i |\Phi_i(T_f)| \).

Finally, the proof of the following claim is similar to that of Claim E.2.

**Claim F.4.** Suppose we know that \( \mathcal{G}(1) \leq \left( \frac{1}{k} + \frac{e}{k^2} \right) |\text{OPT}| \) for some \( \epsilon > 0 \) and \( T_f \) is the output of Greedy on the \( E([1, m f]) \), then

\[
\forall e \in \text{OPT} \quad \Pr[\pi \neq \bigcup_{i=1}^{k} \Phi_i(T_f)] \leq \left( \frac{1}{f} - 1 \right) \Pr[e \in T_f]
\]

\[ \square \]

### F.2 Modifications to the Marking-Greedy Algorithm

Phase (a) of the algorithm remains the same; we will use the first \( f \) fraction to pick \( 1 - p \) fraction of the elements chosen by Greedy. Let \( T_f \) denote the elements chosen by Greedy and \( S \) to be the elements picked into the final solution.

In Phase (b), the algorithm is modified in a natural way; we pick elements that lie in \( M_1/T \cap M_2/T \cap \cdots \cap M_i/S \cap \cdots \cap M_k/T \) for each \( i \in \{1, \ldots, k\} \). When \( k = 2 \), this reduces to the algorithm given in Section 2. Let \( N_i \) denote the set of elements chosen by Greedy on the matroids \( \text{Greedy}(M_1/T_f, \ldots, M_i/S, \ldots, M_k/T_f) \). We will return \( S \cup \bigcup_{i=1}^{k} N_i \). Using Fact 2.4, we know that \( S \cup \bigcup_{i=1}^{k} N_i \) is an independent set in all the matroids \( M_1, \ldots, M_k \). To show that \( \mathbb{E}[\bigcup_{i=1}^{k} N_i] \) is large, we use the following modified sampling lemma.

**Lemma F.5.** (Sampling Lemma for k-matroids). Given matroids \( M_1, M_2, \ldots, M_k \) on ground set \( E \), a set \( T \in \bigcap_{j=1}^{k} M_j \), and \( \Psi(e) \sim \text{Bern}(1 - p) \) i.i.d. for all \( e \in T \), we define set \( S := \{ e \mid e \in T \text{ and } \Psi(e) = 1 \} \). I.e., \( S \) is a set achieved by dropping each element in \( T \) independently with probability \( p \). For \( i \in [k] \), consider a set \( \tilde{E} \subseteq \text{span}_i(T) \) and a set \( \tilde{T} \subseteq \tilde{E} \) satisfying \( \tilde{T} \in M_i \cap (\bigcap_{j \neq i} M_j/T) \). Then for any arrival order of the elements of \( \tilde{E} \), we have

\[
\mathbb{E}_{\Psi}[\text{Greedy}(M_1/T, \ldots, M_i/S, \ldots, M_k/T, \tilde{E})] \geq \frac{p}{1 + p(k - 1)} |\tilde{T}|.
\]
**Proof Overview.** We use the same proof outline as Lemma 2.8 and prove for \( i = 1 \) as the other cases are analogous. Once again, the performance of GREEDY can be mapped to a Sampling Algorithm like SAMP-ALG and we analyze its performance. The only difference is in Line 3 where we now ask that the new element \( e \) is independent in all the matroids \( \mathcal{M}_j \) for all \( j \neq 1 \) (instead of just \( \mathcal{M}_2 \)). Observations E.3 and E.4, and Claim E.5 remain the same for the new Sampling Algorithm.

We first show that at the end of each iteration we can maintain an invariant.

**Invariant F.6.** For given sets \( S', N', T, \) and \( \tilde{E}_r \subseteq \tilde{E} \), we have set \( I' \) satisfying this invariant if

\[
S' \cup N' \cup I' \in \mathcal{M}_1 \\
T \cup N' \cup I' \in \mathcal{M}_j \quad \text{for} \ j \in [2, k] \\
I' \subseteq \tilde{E}_r
\]

This invariant still contains Eq. (6) and Eq. (8) of Invariant 3.2; however, it contains one equation \( (T \cup N' \cup I') \in \mathcal{M}_j \) for each \( j \in [2, k] \) (instead of just \( \mathcal{M}_2 \)). The updates are also naturally extended. Whenever adding a new element violates any of these invariants, we simply remove some elements from \( I' \) to compensate.

**Updates F.7.** We perform the following updates to \( I' \) whenever SAMP-ALG reaches Line 8 or Line 10.

- Line 8: If circuit \( \mathcal{C}_1(S' \cup N' \cup I' \cup f) \) is non-empty then remove an element from \( I' \) belonging to \( \mathcal{C}_1(S' \cup N' \cup I' \cup f) \) to break the circuit.
- Line 10: If circuit \( \mathcal{C}_1(S' \cup N' \cup I' \cup e) \) is non-empty then remove an element from \( I' \) belonging to \( \mathcal{C}_1(S' \cup N' \cup I' \cup e) \) to break the circuit. For \( j \in [2, k] \), if \( \mathcal{C}_j(T \cup N' \cup I' \cup e) \) is non-empty then remove another element from \( I' \) belonging to \( \mathcal{C}_j(T \cup N' \cup I' \cup e) \) to break the circuit. In the special case where \( e \in I' \), we remove \( e \) from \( I' \).

A claim similar to Claim 3.4 shows that the above updates are well-defined and maintain the invariants. Now using the invariants, we prove that the expected number of elements picked is large. As before, we apply the principle of deferred decisions and define the events \( B_j \) and \( \overline{B} \) for \( j \in [0, l - 1] \), where \( l = |C| \). Let \( \alpha := \frac{p}{1 + p(k-1)} \). To prove the induction step, in the event \( B_j \), we use I.H. to get \( \mathbb{E}_\Phi [|N'| \mid B_j] \geq 1 + \alpha (|I'|-j-k) \) and in the event \( \overline{B} \), we use I.H. to get \( \mathbb{E}_\Psi [|N'| \mid \overline{B}] \geq \alpha (|I'|-l) \). A little algebra completes the proof. \( \square \)

### F.3 Putting Everything Together

**Definition F.8** (Sets \( \tilde{E}_i \)). For \( i \in [k] \), we define \( \tilde{E}_i \) to be the set of elements \( e \) that arrive in Phase (b), \( e \in \text{span}_i(T_f) \), and \( e \notin \text{span}_j(T_f) \) for \( j \neq i \).

To prove Lemma F.2 using the Sampling Lemma F.5, we show that there exist “large” disjoint subsets of OPT that are still to arrive in Phase (b). For \( i \in [k] \), let \( \Phi_i(T_f) := \text{span}_i(T_f) \cap \text{OPT} \) and \( I_i := \Phi_i(T_f) \setminus ( \bigcup_{j \neq i} \Phi_j(T_f) ) \).

**Claim F.9.** If \( G(1) \leq \left( \frac{1}{k} + \frac{\epsilon}{k^2} \right) \) then for \( i \in [k] \) there exist disjoint sets \( \tilde{I}_i \subseteq \tilde{E}_i \) such that

\[(i) \ \mathbb{E}_\pi \left[ \sum_{i} |\tilde{I}_i| \right] \geq \left( 1 + 4\epsilon(k - \frac{1}{k} - \frac{1}{k^2}) - \frac{1}{k} + \frac{\epsilon}{k^2} - \frac{1}{k^2} \right) |\text{OPT}| \\
(ii) \ \tilde{I}_i \in \mathcal{M}_i \cap \left( \bigcap_{j \neq i} \mathcal{M}_j/T_f \right) \]

**Proof Overview.** For any \( i \in [k] \), we first construct \( I''_i \subseteq I_i \) satisfying Eq. (ii). For a fixed \( i \) and each \( j \neq i \), let \( X_j' \subseteq T_f \) be a set of minimum size that ensures \( \text{span}_j(X_j' \cup \Phi_j(T_f)) = \text{span}_j(T_f) \). Here \(|X_j'| = |T_f| - |\Phi_j(T_f)|\) because \( \mathcal{M}_j \) is a matroid. Since \( I_i \cup \Phi_j(T_f) \in \mathcal{M}_j \) and \( I_i \cap \Phi_j(T_f) = \emptyset \), by eliminating at most \(|X_j'| \)
elements from \( I_i \), we can create \( I'_i \subseteq I_i \) that satisfies \( X_j^i \cup I'_i \cup \Phi_j(T_f) \in \mathcal{M}_j \). Now using using Fact 2.4 and \( \text{span}_j(X_j^i \cup \Phi_j(T_f)) = \text{span}_j(T_f) \), we conclude \( I'_i \in \mathcal{M}_j/T_f \). Hence, we can define \( I''_i := I'_i \setminus (\bigcup_{j \neq i} X_j^i) \) where \( I''_i \in \mathcal{M}_j/T_f \) for all \( j \neq i \). Observe,

\[
| \bigcup_{j \neq i} X_j^i | \leq \sum_{j \neq i} |X_j^i| = \sum_{j \neq i} (|T_f| - |\Phi_j(T_f)|)
\]

Combining this with \( |I_i| \geq |\Phi_i(T_f)| - \sum_{j \neq i} |\Phi_j(T_f) \cap \Phi_i(T_f)| \) (a fact from the definition of \( I_i \)), we get

\[
|I''_i| = |I_i| - | \bigcup_{j \neq i} X_j^i | \geq |\Phi_i(T_f)| - (k-1)|T_f| + \sum_{j \neq i} (|\Phi_j(T_f)| - |\Phi_j(T_f) \cap \Phi_i(T_f)|) = \sum_{j} |\Phi_j(T_f)| - (k-1)|T_f| - \sum_{j \neq i} |\Phi_j(T_f) \cap \Phi_i(T_f)|.
\]

Note that for all \( i \) sets \( I''_i \) are pairwise disjoint (as \( I_i \) are constructed to be disjoint). Furthermore, taking expectations over \( \pi \), we let \( \tilde{I}_i \) to be the elements of \( I''_i \) that are still to appear in Phase (b). Hence, \( \mathbb{E}_\pi[\sum_{i=1}^k |\tilde{I}_i|] \geq \mathbb{E}_\pi[\sum_{i=1}^k |I''_i|] - f|\text{OPT}| \). Using the bounds from Hastiness Lemma F.3, we conclude

\[
\mathbb{E}_\pi \left[ \sum_{i=1}^k |\tilde{I}_i| \right] \geq \sum_{i=1}^k \left( \sum_j (|\Phi_j(T_f)|) - (k-1)\mathbb{E}_\pi[|T_f|] - \sum_{j \neq i} \mathbb{E}_\pi[|\Phi_j(T_f) \cap \Phi_i(T_f)|] \right) - f|\text{OPT}|
\]

\[
\geq k \left( 1 - \frac{4\epsilon}{k^2} + 4\epsilon - (k-1)G(1) - k \cdot \frac{2\epsilon}{k^2} \right) |\text{OPT}| - f|\text{OPT}|
\]

\[
\geq \left( k + 4\epsilon(k - \frac{1}{f} - \frac{1}{2}) - k(k-1)(\frac{1}{k} + \frac{\epsilon}{k^2}) - f \right) |\text{OPT}|
\]

\[
= \left( 1 + 4\epsilon(k - \frac{1}{f} - \frac{1}{2}) - \frac{\epsilon}{k} + \frac{\epsilon}{k^2} - f \right) |\text{OPT}|
\]

Finally, to prove Lemma F.2, we use the disjoint sets \( \tilde{I}_i \) from Claim F.9 in the Sampling Lemma F.5 to say that the expected output size is at least

\[
(1-p)G(f) + \frac{p}{1 + p(k-1)} \left( 1 + 4\epsilon(k - \frac{1}{f} - \frac{1}{2}) - \frac{\epsilon}{k} + \frac{\epsilon}{k^2} - f \right) |\text{OPT}|
\]

We assume \( k \geq 3 \) (as the case \( k = 2 \) was presented in Section 2). By using Lemma F.3, setting \( f = \frac{k}{2} \), and choosing \( \epsilon \ll 1 \), we can conclude that the expected value is at least

\[
(1-p) \left( \frac{1}{k} - \frac{4\epsilon - 4\epsilon}{k} \right) + \frac{p}{1 + p(k-1)} \left( \frac{k-1}{k} - 3\epsilon \right).
\]

This value is at least \( \frac{1}{k} + \frac{\epsilon}{k} \) for some universal constant \( \gamma > 0 \) (e.g., when \( p = 0.2 \) and \( \epsilon = 10^{-10} \)).