Uniform Approximation of Continuous Functions by Nontrivial Simple Functions

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Abstract

We prove that every nonnegative continuous real-valued function on a given compact metric space is the uniform limit of some increasing sequence of nonnegative simple functions being linear combinations of indicators of open sets; here the nontriviality is relative to the standard choice(s) of approximating simple functions for measurable functions, where one loses control over the indicated measurable sets. Thus the standard uniform approximation of bounded nonnegative measurable real-valued functions by increasing nonnegative simple functions may be improved for nonnegative continuous real-valued functions on compact metric spaces. There are also some interesting consequences regarding semi-continuous functions and smooth functions.

Keywords: compact metric spaces; nonstandard uniform approximation for continuous functions; semi-continuous functions; smooth functions

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1 Introduction

Let \( \mathbb{R}_+ := \{ x \in \mathbb{R} \mid x \geq 0 \} \). A continuous \( \mathbb{R}_+ \)-valued function on a compact metric space, being Borel and bounded then, is the uniform limit of some increasing sequence of nonnegative simple functions, the standard choice(s) (e.g. Theorem 2.10 in Folland [5]) of the approximating simple functions utilizing the preimages of half-open intervals under the given function. Such a classical approximation is silent on whether or not the involved indicator functions are indicators of open sets.

The present short communication furnishes an improvement of a uniform approximation for continuous \( \mathbb{R}_+ \)-valued functions on a compact metric space by an increasing sequence of nonnegative simple functions; the involved indicators may be chosen such

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that they are indicators of open sets. Since simple real-valued functions are in general
not continuous with respect to the standard topology of \( \mathbb{R} \), the relevant existing results
(such as Goodstein [6]) regarding uniform convergence of a monotonic sequence of con-
tinuous functions do not (immediately) apply. Our approximation thus also adds value
from this aspect.

Another facet of our result is supplying a new proof of the customary uniform ap-
proximation for a significant subclass of Borel functions.

Further, our main result and its proof imply some interesting results regarding semi-
continuous functions and smooth functions.

2 Results

With respect to the intended purpose, our main result is the slightly stronger

**Theorem 1.** If \( \Omega \) is a compact metric space, if \( f : \Omega \to \mathbb{R}_+ \) is continuous, and if
\((a_j)_{j \in \mathbb{N}}\) is a vanishing sequence of reals \( a_j > 0 \) such that \( \sum_{j \in \mathbb{N}} a_j \) diverges, then there
are some open subsets \( G_1, G_2, \ldots \) of \( \Omega \) such that \((\sum_{j=1}^n a_j \mathbb{1}_{G_j})_{n \in \mathbb{N}}\) converges uniformly
to \( f \).

**Proof.** We claim that there are some open subsets \( G_1, G_2, \ldots \) of \( \Omega \) such that
\( f(x) = \sum_{j \in \mathbb{N}} a_j \mathbb{1}_{G_j}(x) \)
for all \( x \in \Omega \), i.e. that \( f \) is the pointwise limit of the sequence \((\sum_{j=1}^n a_j \mathbb{1}_{G_j})_{n \in \mathbb{N}}\). Indeed,
Theorem 2.3.3 in Federer [4] gives in particular (with a one-line proof sket-
ch) a “harmonic” representation of \( f \) in the desired form with Borel \( G_j \). Theorem 1.12 in Evans
and Gariepy [3] provides a more detailed proof of the Federer’s result for the special
case where \((a_j)\) is the harmonic sequence \((j^{-1})\). This proof is readily generalizable to
cover our \((a_j)\), and a detailed proof of the generalization, adapted from the proof, may
be found in the proof of Theorem 1 in Chou [2].

To prove the claim, we modify the proof in Chou [2] as follows. If \( G_1 := \{ x \in \Omega \mid f(x) > a_1 \} \), let \( G_n := \{ x \in \Omega \mid f(x) > a_n + \sum_{j=1}^{n-1} a_j \mathbb{1}_{G_j} \} \) for all \( n \geq 2 \) by induction.
Then each \( G_n \) is open in \( \Omega \). It then can be shown that \( f \geq \sum_{j \in \mathbb{N}} a_j \mathbb{1}_{G_j} \) on \( \Omega \); the
assumed finiteness of \( f \) and the assumptions on \((a_j)\) in turn imply the desired pointwise
convergence, which proves the claim.

To obtain the desired uniform convergence, we remark that each \(-a_j \mathbb{1}_{G_j}\) is upper
semi-continuous. Moreover, a sum of finitely many upper semi-continuous functions
with values in \( \mathbb{R} \) is again upper semi-continuous: If \( g_1, g_2 \) are real-valued and upper
semi-continuous, and if \( b > 0 \), then for every \( x \) (in the domain) there is some open
neighborhood $V$ of $x$, e.g. $V$ being the intersection of some open neighborhoods of $x$ that are included respectively in $\{ y \mid g_1(y) < g_1(x) + b/2 \}$ and in $\{ y \mid g_2(y) < g_2(x) + b/2 \}$, such that $x \in V \subset \{ y \mid g_1(y) + g_2(y) < g_1(x) + g_2(x) + b \}$.

Since $f$ is in particular upper semi-continuous by assumption, the function $f - \sum_{j=1}^{n} a_j I_{G_j}$ is upper semi-continuous for every $n \in \mathbb{N}$. The previously proved pointwise convergence claim implies that $f - \sum_{j=1}^{n} a_j I_{G_j}$ decreases to 0 pointwisely; it then follows from a variant of Dini’s theorem (e.g. the theorem in M8 in Appendix M of Billingsley [1]) that

$$f - \sum_{j=1}^{n} a_j I_{G_j} \to 0 \text{ uniformly on } \Omega$$

as $n \to \infty$. □

**Remark.** That part of proof of Theorem 1 for uniform convergence is in some sense a new proof; since the function $f$ in Theorem 1 is bounded, the uniform convergence also follows from the inequality $f \geq \sum_{j \in \mathbb{N}} a_j I_{G_j}$. But our new proof exploits further properties of the involved functions, and hence may be potentially useful for other purposes. □

Theorem 1 and its proof are also informative in another respect:

**Proposition 1.** Every continuous $\mathbb{R}^+\text{-valued}$ function on a compact metric space is the uniform limit of some increasing sequence of lower semi-continuous $\mathbb{R}^+\text{-valued}$ functions.

**Proof.** Every indicator function of an open set is lower semi-continuous. The sum of finitely many lower semi-continuous functions with values in $\mathbb{R}$ is lower semi-continuous; indeed, if $g_1, g_2$ are $\mathbb{R}$-valued and lower semi-continuous, and if $b > 0$, then for every $x$ (in the domain) there is some open neighborhood $V$ of $x$, e.g. $V$ being the intersection of some open neighborhoods of $x$ included respectively in $\{ y \mid g_1(y) > g_1(x) - b/2 \}$ and in $\{ y \mid g_2(y) > g_2(x) - b/2 \}$, such that $x \in V \subset \{ y \mid g_1(y) + g_2(y) > g_1(x) + g_2(x) - b \}$.

Let $\sum_{j \in \mathbb{N}} a_j I_{G_j}$ be a uniform representation of $f$ as given by Theorem 1. Since $f$ is in particular lower semi-continuous by assumption, and since each $a_j I_{G_j}$ and hence each $\sum_{j=1}^{n} a_j I_{G_j}$ is lower semi-continuous, the sequence $(\sum_{j=1}^{n} a_j I_{G_j})_{n \in \mathbb{N}}$ serves the purpose. □

A part of the proof of Theorem 1 and of that of Proposition 1 together suggest a lower bound for nonzero continuous functions $\mathbb{R}^n \to \mathbb{R}^+$ in terms of a series of nonzero smooth functions $\mathbb{R}^n \to \mathbb{R}^+$:

**Proposition 2.** For every nonzero continuous $f : \mathbb{R}^n \to \mathbb{R}^+$, there are some nonzero smooth $g_1, g_2, \cdots : \mathbb{R}^n \to \mathbb{R}^+$ such that i) $f(x) \geq \sum_{j \in \mathbb{N}} g_j(x)$ for all $x \in \mathbb{R}^n$ and ii) $f \geq \sum_{j \in \mathbb{N}} g_j$ uniformly on every compact $K \subset \mathbb{R}^n$.  

3
Proof. The proofs of Theorem 1 and Proposition 1 imply in particular that \( f \) is the pointwise limit of some increasing sequence of \( \mathbb{R}_+ \)-valued lower semi-continuous functions; the compactness assumption plays a role precisely in asserting uniform convergence.

Lemma 1.1.5 in Krantz and Parks [7] asserts in particular that every nonzero lower semi-continuous function \( \mathbb{R}^n \rightarrow \mathbb{R}_+ \) may be minorized by some nonzero smooth function \( \mathbb{R}^n \rightarrow \mathbb{R}_+ \) (pointwisely).

If we represent \( f \) as \( \sum_{j \in \mathbb{N}} a_j 1_{G_j} \) on \( \mathbb{R}^n \) by the proof of Theorem 1, then the lower semi-continuity of each \( a_j 1_{G_j} \) implies that for every \( j \in \mathbb{N} \) there is some nonzero smooth \( g_j : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) such that \( g_j \leq a_j 1_{G_j} \) on \( \mathbb{R}^n \); therefore, we have

\[
 f(x) \geq \sum_{j \in \mathbb{N}} g_j(x)
\]

for all \( x \in \mathbb{R}^n \).

The other desired property now follows either from the boundedness of a continuous function on a compact set or from the proof of the variant of Dini’s theorem cited in the proof of Theorem 1. \( \square \)

References

[1] Billingsley, P. (1999). *Convergence of Probability Measures*, second edition. Wiley.

[2] Chou, Y.-L. (2020). Tail probability and divergent series. arXiv:2004.13541v2 [math.PR].

[3] Evans, L. C. and Gariepy, R. F. (2015). *Measure Theory and Fine Properties of Functions*, first edition. Chapman & Hall.

[4] Federer, H. (1996). *Geometric Measure Theory*, reprint of the first edition. Springer.

[5] Folland, G. B. (2007). *Real Analysis: Modern Techniques and Their Applications*, second edition. Wiley.

[6] Goodstein, R. L. (1946). A theorem in uniform convergence. *The Mathematical Gazette* 30 287–290.

[7] Krantz, S.G. and Parks, H.R. (1999). *The Geometry of Domains in Space*. Birkhäuser.