Free Hilbert Transforms

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September 6, 2016

Abstract

We study analogues of classical Hilbert transforms as Fourier multipliers on free groups. We prove their complete boundedness on non-commutative $L^p$ spaces associated with the free group von Neumann algebras for all $1 < p < \infty$. This implies that the decomposition of the free group $F_\infty$ into reduced words starting with distinct free generators is completely unconditional in $L^p$. We study the case of Voiculescu’s amalgamated free products of von Neumann algebras as well. As by-products, we obtain a positive answer to a compactness-problem posed by Ozawa, a length independent estimate for Junge-Parcet-Xu’s free Rosenthal inequality, a Littlewood-Paley-Stein type inequality for geodesic paths of free groups, and a length reduction formula for $L^p$-norms of free group von Neumann algebras.

1 Introduction

Hilbert transform is a fundamental and influential object in the mathematical analysis and signal processing. It was originally defined for periodic functions. Given a trigonometric polynomial $f(z) = \sum_{k=-N}^{N} c_k z^k$, let $P_+ f = \sum_{k=1}^{N} c_k z^k$ be its analytic part and $P_- f = \sum_{k=-N}^{-1} c_k z^k$ be its anti-analytic part. The Hilbert transform is formally defined as

$$H = -iP_+ + iP_-$$

and clearly extends to an unitary on $L^2(T)$. The case of $L^p$, $1 < p < \infty$ is more subtle. M. Riesz first proved that $H$ extends to a bounded operator on $L^p(T)$ for all $1 < p < \infty$. It is also well known that $H$ is unbounded on $L^p(T)$ at the end point $p = 1, \infty$ but is of weak type $(1,1)$. In modern harmonic analysis, the Hilbert transform is considered as a basic example of Calderón-Zygmund singular integral. Its analogues have been studied in much more general situations with connections to $L^p$-approximation, the Hardy/BMO spaces, and more applied subjects.

*Research partially supported by the NSF grant DMS-1266042.
The Hilbert transforms appear also as the key tool to define conjugate functions in abstract settings such as for Dirichlet algebras. In operator algebras, they show up through Arveson’s concept of maximal subdiagonal algebra of a von Neumann algebra $\mathcal{M}$. Results about $L^p$-boundedness and weak-type $(1,1)$-estimates in this situation were obtained by N. Randrianantoanina in [Ran98].

The object of this article is a natural analogue of the Hilbert transform in the context of amalgamated free products of von Neumann algebras. The study is from a different viewpoint to Arveson’s and is motivated from questions in the theory of $L^p$-Herz-Schur multipliers on free groups.

Our model case is the von Neumann algebra $(\mathcal{L}(F_\infty), \tau)$ of free group with a countable set of generators $g_1, g_2, \ldots$. The associated $L^p$-spaces $L^p(F_\infty)$ is a non commutative analogue of $L^p(\mathbb{Z}) = L^p(T)$. Let $\mathcal{L}_{g_1}, \mathcal{L}_{g_1}^{-1}$ be the subsets of $F_\infty$ of reduced words starting respectively with $g_1, g_1^{-1}$. One can naturally associate to them projections; given a finitely supported function $\hat{x}$ on $F_\infty$, $\hat{x} = \sum_{g \in F_\infty} c_g \delta_g$, $c_g \in \mathbb{C}$, define

$$L_{g_1}^+ \hat{x} = \sum_{g \in \mathcal{L}_{g_1}} c_g \delta_g$$

and $L_{g_1}^- \hat{x}$ similarly. All of them obviously extend to norm 1 projections on $\ell_2(F_\infty) = L^2(F_\infty)$. A natural question is whether these projections are bounded on $L^p(F_\infty)$ and whether the decomposition $F_\infty = \{e\} \cup_{i \in \mathbb{N}, e \in \pm} \mathcal{L}_{g_i^k}$ is unconditional in $L^p(F_\infty)$. To that purpose, we define a free analogue of the classical Hilbert transform as the following map

$$H_\varepsilon = \varepsilon_1 L_{g_1}^+ + \varepsilon_{-1} L_{g_1}^- + \varepsilon_2 L_{g_2}^+ + \varepsilon_{-2} L_{g_2}^- + \ldots$$

(1)

for $\varepsilon_i = \pm 1$. We are interested in the (complete) boundedness of $H_\varepsilon$ on $L^p(F_\infty)$ as well as possible connections to semigroup-Hardy/BMO spaces and the $L^p$-approximation property in the non commutative setting. The question of the $L^p(F_\infty)$-boundedness of $H_\varepsilon$ has been around for some time. The authors learned from G. Pisier that P. Biane asked this question and discussed with him around 2000. N. Ozawa pointed out that the $L^4(F_\infty)$-boundedness of $H_\varepsilon$ answers positively the problem he posed at the end of [O10]. Junge-Parcet-Xu obtained some length dependent results for related questions in their work of Rosenthal’s inequality for amalgamated free products ([JPX07]).

The first result (Theorem 3.5) of this article is a positive answer to the $L^p$-boundedness question of $H_\varepsilon$ in the general case of Voiculescu’s amalgamated free products, which includes the free group of countable many generators as a particular case (Theorem 4.1).

One can also consider two similar Hilbert transforms. One is

$$H_\varepsilon^{Ld} = \varepsilon_c P_{d-1} + \sum_{h, |h| = d} \varepsilon_h L_h$$
with \( P_d \) the projection onto reduced words with length \( \leq d \) and \( L_h \)'s the projections onto reduced words starting with \( h \). Another is
\[
H^{(d)}_\varepsilon = \varepsilon_P d_{d-1} + \sum_{g, |g|=1} \varepsilon_g L^{(d)}_g
\]
with \( L^{(d)}_g \)'s the projections onto reduced words having \( g \) as its \( d \)-th letter. Their (complete) boundedness on \( L^p(\hat{F}_n) \) can be easily deduced from that of \( H_\varepsilon \) with constants depending on \( n \). The main result of this article (Theorem 4.8) says that \( H^{(d)}_\varepsilon \)'s are completely bounded on \( L^p(\hat{F}_\infty) \) for any \( d \geq 1 \). While \( H^{(d)}_{L_d} \)'s are bounded for all \( 1 < p < \infty \) but not completely bounded on \( L^p(\hat{F}_\infty) \), for any \( p \neq 2, d \geq 2 \). The authors also prove a length reduction formula to compute \( L^p \)-norms and a Rosenthal inequality with length independent constants.

A classical argument, in proving the \( L^p \)-boundedness of the Hilbert transform \( H \) is to use the following Cotlar’s identity
\[
|H(f)|^2 = |f|^2 + H(\overline{f}Hf + Hff), \tag{2}
\]
that allows to get the result for \( L^2 \) from that of \( L^p \) and implies optimal estimates. This identity holds in a general setting, if one can identify a suitable “analytic” algebra and defines the corresponding Hilbert transform as the subtraction of two projections on this algebra and its adjoint. This is the case of non commutative Hilbert transforms associated with Arveson’s maximal subdiagonal algebras (see Lemma 8.5 of [PX03]). After obtaining an initial proof of Theorem 4.1, we observed that a free version of Cotlar’s identity (see (3)) holds in the context of amalgamated free products for \( H_\varepsilon \) with \( |\varepsilon_k| \leq 1 \). We were slightly surprised when this observation came out, given that \( H_\varepsilon \), defined in (1), is associated to subsets instead of subalgebras. On the other hand, once it draws our attention, the proof of the identity and Theorem 4.1 are not hard. It is a surprise that this identity was not noticed earlier.

We will introduce the notations and necessary preliminaries in Section 2. The Cotlar’s formula for amalgamated free products and Theorem 3.3 are proved in Section 3.1. Section 3.2 includes a few immediate consequences. Section 3.3 obtains a length independent Rosenthal inequality, which was initially proved by Junge/Parcet/Xu (Theorem A, [JPX07]) restricted to a fixed length. Section 4.1 proves our main result Theorem 4.8. Corollary 4.7 of that section gives a length reduction formula and generalizes the main result of [PP05]. Corollary 4.11 (iii) answers positively the problem that Ozawa posed at the end of [O10]. Section 4.3 studies Littlewood-Paley-Stein type inequalities. Corollary 4.16 shows that the projection onto a geodesic path of the free group is completely bounded on \( L^p \) for \( 1 < p < \infty \). Theorem 4.19 is a dyadic Littlewood-Paley-Stein inequality for geodesic paths of free groups.

1Arveson’s “analytic” subalgebras do not seem available for amalgamated free products of von Neumann algebras in general. They are available for free group von Neumann algebras but the corresponding Hilbert transforms are different from ours and their formulations as Herz-Schur multipliers are difficult to determine.

2The classical Cotlar’s formula fails for \( H = -iP_+ + \varepsilon_iP_- \) if \( \varepsilon \neq \pm 1 \).
2 Notations and preliminaries

We refer the reader to [VDN92] and [JPX07] for the definition of amalgamated free products, and to [PX03] and the references therein for a formal definition and basic properties on non commutative $L^p$ spaces. For simplicity, we will restrict to the case of finite von Neumann algebras but all the arguments should be easily adapted to type III algebras with n.f. states.

About noncommutative $L^p$-spaces associated to a finite von Neumann algebra $(\mathcal{A},\tau)$, we will mainly need duality, interpolation and the non commutative Khintchine inequality (L86, LP91) in $L^p(\mathcal{A})$ as well as $p$-row and $p$-column spaces. As usual we denote by $c_k = e_{k,1}$ and $r_k = e_{1,k}$, $d_k = e_{k,k}$ the canonical basis of the column, row and diagonal subspaces of the Schatten $p$-class $S_p(\ell_2(\mathbb{N}))$.

We will use the duality $(x,y)_{L^p,L^q} = \tau(xy)$ to identify $L^q(\mathcal{A})$ with $L^p(\mathcal{A})^{*}$ isometrically for $1 \leq p < \infty$. At operator space level this gives a complete isometry $L^p(\mathcal{A})^* = (L^q(\mathcal{A})^{\circ})^\circ$, see [P98].

As $\mathcal{A} = L^\infty(\mathcal{A})$ is finite, the obvious embedding $L^\infty(\mathcal{A}) \subset L^1(\mathcal{A})$ makes $(L^\infty(\mathcal{A}),L^1(\mathcal{A}))$ a compatible couple of Banach spaces. For $1 < p < \infty$, the complex interpolation spaces between $\mathcal{A}$ and $L^1(\mathcal{A})$ with index $\frac{1}{p}$ is isometric to $L^p(\mathcal{A})$ :

$$(L^\infty(\mathcal{A}),L^1(\mathcal{A}))_{\frac{1}{p}} = L^p(\mathcal{A}).$$

For a sequence $(x_k)$ in $L^p(\mathcal{A})$, we use the classical notation

$$\|x_k\|_{L^p(\mathcal{A},\ell^2_k)} = \left(\sum_k |x_k|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|x_k\|_{L^p(\mathcal{A},\ell^2_k)} = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}},$$

and

$$\|x_k\|_{L^p(\mathcal{A},\ell^{2*}_k)} = \left\{ \begin{array}{ll} \max \left\{ \|x_k\|_{L^p(\mathcal{A},\ell^2_k)}, \|x_k^*\|_{L^p(\mathcal{A},\ell^2_k)} \right\} & \text{if } 2 \leq p \leq \infty \\ \inf_y \left\{ \|y_k\|_{L^p(\mathcal{A},\ell^2_k)} + \|y_k\|_{L^p(\mathcal{A},\ell^2_k)} \right\} & \text{if } 0 < p < 2 \end{array} \right.$$

We refer to [P98] for non commutative vector-valued $L^p$-spaces. The above definition is justified by the non commutative Khintchine inequalities:

Lemma 2.1. ([L86], [LP91], [HM07]) Let $\varepsilon_k$ be independent Rademacher random variables, then for $1 \leq p < \infty$,

$$\alpha_p E_\varepsilon \| \sum_k \varepsilon_k \otimes x_k \|_p \leq \| (x_k)_{L^p(\mathcal{A},\ell^{2*}_k)} \| \leq \beta_p E_\varepsilon \| \sum_k \varepsilon_k \otimes x_k \|_p.$$

Here $\varepsilon_k$ can also be replaced by other orthonormal sequences of some $L^2(\Omega,\mu)$, e.g. $z^{2k}$ on the unit circle or standard Gaussian. For $z^{2k}$ on the unit circle or standard Gaussian, the best constant $\beta_p$ is $\sqrt{2}$ for $p = 1$ and is $1$ for $p \geq 2$ (see [HM07]). $\alpha_p$ is $1$ for $1 \leq p \leq 2$ and is of order $\sqrt{p}$ as $p \to \infty$. [P] was pushed further to the case of $0 < p < 1$ by Pisier and the second author recently (see [PRL]).
If \((A_k, \tau_k), k \geq 1\) are finite von Neumann algebras with a common sub-von Neumann algebra \((B, \tau)\) with conditional expectation \(E\) so that \(\tau_k E = \tau\), we denote by \((A, \tau) = (*_B A_k, \tau_k)\) the amalgamated free product of \((A_k, \tau_k)\)'s over \(B\). We will briefly recall the construction to fix notation.

For any \(x \in A_k\), we denote by \(\hat{x} = x - Ex\) and \(\hat{A}_k = \{\hat{x}; x \in A_k\}\); there is a natural decomposition \(A_k = B \oplus \hat{A}_k\).

The space

\[
\mathcal{W} = B \oplus_{n \geq 1} \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \neq i_2 \ldots \neq i_n} \hat{A}_{i_1} \otimes \cdots \otimes B \hat{A}_{i_n} = \bigoplus_{n \geq 0} \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \neq i_2 \ldots \neq i_n} \mathcal{W}_{i_1} \cdots \mathcal{W}_{i_n}
\]

is a \(*\)-algebra using concatenation and centering with respect to \(B\). The natural projection \(E\) onto \(B\) is a conditional expectation and \(\tau E\) is a trace on \(\mathcal{W}\) still denoted by \(\tau\). Then \((A, \tau)\) is the finite von Neumann algebra obtained by the GNS construction from \((\mathcal{W}, \tau)\). Thus \(\mathcal{W}\) is weak-\(*\) dense in \(A\) and dense in \(L^p(A)\) for \(p < \infty\).

For multi-indices, we write \((i_1, \ldots, i_n) \leq^L j = (j_1, \ldots, j_m)\) if \(m \geq n\) and \(i_k = j_k\) for \(k \leq n\) and \(i_{n+1} \leq^R j = (j_1, \ldots, j_m)\) if \(m \geq n\) and \(i_{m+1} = j_{m+1}\) for \(1 \leq k \leq n\). We also put \(i \leq^L j\) if \(i \leq^L j\) and \(n < m\). We extend those relations for non-zero elementary tensors \(g \in \mathcal{W}_{i_1} \cdots \mathcal{W}_{i_n}\) and \(h \in \mathcal{W}_{j_1} \cdots \mathcal{W}_{j_m}\), we write \(g \prec^L h\) if \(i \prec^L j\) and \(g \prec^R h\) if \(i \prec^R j\).

For \(k \in \mathbb{N}\), put

\[
\mathcal{L}_k = \oplus_{i \leq^L k} \mathcal{W}_i, \quad \text{and} \quad \mathcal{R}_k = \oplus_{i \leq^R k} \mathcal{W}_i.
\]

We denote the associated orthogonal projections on \(\mathcal{W}\) by \(L_k\) and \(R_k\). We use the convention \(L_0 = E\).

Given a sequence of \(\varepsilon_k \in \mathcal{B}, k \in \mathbb{N}\), and \(x \in \mathcal{W}\), we let

\[
H_\varepsilon(x) = \varepsilon_0 E(x) + \sum_{k \in \mathbb{N}} \varepsilon_k L_k(x); \quad H_\varepsilon^{op} = E(x) \varepsilon_0^* + \sum_{k \in \mathbb{N}} R_k(x) \varepsilon_k^*.
\]

The main theorem is that, for \(1 < p < \infty\), \(H_\varepsilon\) extends to \(L^p\) and for any \(x \in L^p(A)\),

\[
\|H_\varepsilon x\|_p \preceq \|x\|_p,
\]

for any choice of unitaries \(\varepsilon_k \in Z(B)\) in the center of \(\mathcal{B}\) and \(1 < p < \infty\).

3 Amalgamated Free products

3.1 The Cotlar formula for free products

We start with very basic observations, recall that \(\hat{x} = x - Ex\) for \(x \in A\).

**Proposition 3.1.** For \(g \in \mathcal{W}\), and \(\varepsilon, \varepsilon'\) sequences in \(\mathcal{B}\)

(i) \(H_\varepsilon(g^*) = (H_\varepsilon^{op}(g))^*\).
Proposition 3.2. For elementary tensors $g, h \in W$,

\begin{enumerate}[(i)]
  \item $H_c(\hat{g}) = H_c(g)$.
  \item $H_c(\hat{g}^* h) = H_c(g^*) h$ if $g \not\in L h$.
  \item $H_c^\text{op}(\hat{g}^* h) = g^* H_c^\text{op}(h)$ if $h \not\in R g$.
  \item $H_c(g^* h) = H_c(g^*) h$ if $g \not\in L h$.
  \item $H_c(g^* h) = H_c(g^*) h = \varepsilon_{i_n} g^* h$.
\end{enumerate}

Proof. This is clear on elementary tensors.

We now give the free version of Cotlar’s identity.

Proposition 3.2. For elementary tensors $g, h \in W$,

\[ H_c(g^* h) = H_c(g^*) h \text{ if } g \not\in L h \]

\[ H_c^\text{op}(g^* h) = g^* H_c^\text{op}(h) \text{ if } h \not\in R g. \]

And for any $g, h \in W$,

\[ H_c(g^* h) = H_c(g^*) h = \varepsilon_{i_n} g^* h. \]

Proof. Let $g = g_1 \otimes \ldots \otimes g_n \in W_2$ and $h = h_1 \otimes \ldots \otimes h_m \in W_2$ with $\hat{i} = (i_1, \ldots, i_n)$ and $\hat{j} = (j_1, \ldots, j_m)$, $n, m \geq 0$. We start by proving $\text{(iv)}$ by induction on $n + m$.

If $n + m = 0$, this is clear as $H_c(g^* h) = H_c(g^*) h = 0$.

Assume $n + m \geq 1$ and $g \not\in L h$. Note that necessarily $n \geq 1$.

First case: $i_1 \neq j_1$ or $m = 0$,

\[ g^* h = g_n^* \otimes \ldots \otimes g_2^* \otimes g_1^* \otimes h_1 \otimes h_2 \otimes \ldots \otimes h_m, \]

and $H_c(g^* h) = H_c(g^*) h = \varepsilon_{i_n} g^* h$.

Second case: $i_1 = j_1$,

\[ g^* h = g_n^* \otimes \ldots \otimes g_2^* \otimes (g_1^* h_1) \otimes h_2 \otimes \ldots \otimes h_m + (g_n^* \otimes \ldots \otimes g_2^*) ((E g_1^* h_1) h_2 \otimes \ldots \otimes h_m) \]

Put $\hat{g} = h_1^* g_1 \otimes \ldots \otimes g_n$, $\hat{h} = h_2 \otimes \ldots \otimes h_m$ and $\hat{g} = g_2 \otimes \ldots \otimes g_n$, $\hat{h} = (E g_1^* h_1) h_2 \otimes \ldots \otimes h_m$ (if $n = 1$, $\hat{g} = 1$). Note that $\hat{g} \not\in L \hat{h}$ (or $\hat{g} = 0$) and $\hat{g} \not\in L \hat{h}$ and the sum of their length is strictly smaller than $n + m$. We can apply the formula to them to get $H_c(\hat{g}^* \hat{h}) = H_c(\hat{g}^*) \hat{h} = \varepsilon_{i_n} \hat{g}^* \hat{h}$ and $H_c(\hat{g}^* \hat{h}) = H_c(\hat{g}^*) \hat{h} = \varepsilon_{i_n} \hat{g}^* \hat{h}$

(this holds if $n = 1$ because then $m = 1$ and $\hat{g}^* \hat{h} = 0$). Finally

\[ H_c(g^* h) = \varepsilon_{i_n} (\hat{g}^* \hat{h} + \hat{g}^* \hat{h}) = \varepsilon_{i_n} \hat{g}^* \hat{h} = H_c(g^*) h. \]

$(v)$ follows from $(iv)$ by taking adjoints.
To get (vi) it suffices to do it for elementary tensors by linearity. Assume first that \( g \not\approx^L h \), then obviously \( g \not\approx^L H^g_{\varepsilon}^p(h) \), so by (iv)
\[
H^g_{\varepsilon}(g^*H^g_{\varepsilon}^p(h)) = H^g_{\varepsilon}(g^*)H^g_{\varepsilon}^p(h), \quad H^g_{\varepsilon}(g^*)h = H^g_{\varepsilon}(g^*h).
\]
Since the centering operation commutes with \( H^g_{\varepsilon}^p \) by Proposition 3.1, we get (vi).

If \( g \not\approx^L h \) then \( h \not\approx^R g \) and we can use (v) and Proposition 3.1 (ii) as above and (iii) to get (vi) as
\[
H^g_{\varepsilon}(g^*)H^g_{\varepsilon}^p(h) = H^g_{\varepsilon}(g^*)H^g_{\varepsilon}^p(h) = H^g_{\varepsilon}(g^*)H^g_{\varepsilon}^p(h).
\]

Remark 3.3. Removing the centering, we have obtained a Cotlar’s formula for any \( x = \sum_i g_i, y = \sum_j h_j, g_i, h_j \in W \) as follows,
\[
H^g_{\varepsilon}x(H^g_{\varepsilon}y)^* - E[(H^g_{\varepsilon}x - \varepsilon_0x)(H^g_{\varepsilon}y - \varepsilon_0y)^*] = H^g_{\varepsilon}(xH^g_{\varepsilon}^p(y^*)) + H^g_{\varepsilon}^p(H^g_{\varepsilon}(x)y^*) - H^g_{\varepsilon}^pH^g_{\varepsilon}(xy^*). \tag{5}
\]
Note the justified Cotlar’s identity (5) holds for all \( \|\varepsilon_k\| \leq 1 \) while in the commutative setting the Cotlar’s formula holds for \( \varepsilon_k = \pm 1 \) only.

**Proposition 3.4.** For any \( x \in W \), and any \( p \geq 1 \), and \( \varepsilon_k \in \mathcal{Z}(B), \|\varepsilon_k\| \leq 1 \)
\[
\max\{\|E(H^g_{\varepsilon}x(H^g_{\varepsilon}y^*))\|_p, \|E(H^g_{\varepsilon}(xH^g_{\varepsilon}^p(x^*))\|_p, \|E(H^g_{\varepsilon}^p(H^g_{\varepsilon}(x)y^*))\|_p, \|E(H^g_{\varepsilon}(xx^*))\|_p\} \leq \|E(xx^*)\|_p.
\]

**Proof.** Write \( g = \sum_k g_k \) with \( g_k \in W_k \). Then, by orthogonality of the \( W_i \) over \( B \), all the 4 elements on the left hand side are of the form \( \sum_k y_k E(g_k g_k^*)z_k^* \) with \( y_k, z_k \in \{1, \varepsilon_i, \varepsilon_j\} \). But \( \sum_k y_k E(g_k g_k^*)z_k^* = \sum_k a_k y_k z_k^* a_k \) with \( a_k = E(g_k g_k^*)^{1/2} \) so that the inequality follows by the Hölder inequality as \( \sum_k a_k^2 \leq E(xx^*) \).

We can prove the main result

**Theorem 3.5.** For \( 1 < p < \infty \), there is a constant \( c_p \) so that for \( \varepsilon_k \in \mathcal{Z}(B), \|\varepsilon_k\| \leq 1 \) and \( x \in W \)
\[
\|H^g_{\varepsilon}x\|_p \leq c_p \|x\|_p, \quad \|H^g_{\varepsilon}^p x\|_p \leq c_p \|x\|_p.
\]

Moreover the equivalence holds with constant \( c_p \) in both directions if \( \varepsilon_k \)'s are further assumed to be unitaries.

**Proof.** Assume \( \|H^g_{\varepsilon}\|_{L^p(A) \rightarrow L^p(A)} \leq c_p \). We will show that \( \|H^g_{\varepsilon}x\|_{2p} \leq (c_p + \sqrt{2c_p^2 + 4})\|x\|_{2p} \) for all \( x \in W \), and similarly for \( H^g_{\varepsilon}^p \) using the \( * \)-operation. Once this is proved, we get the upper desired estimate for all \( p = 2^n, n \in \mathbb{N} \), by
induction and the fact that \(\|H_\varepsilon x\|_2 = \|H_\varepsilon^{op} x\|_2 = \|x\|_2\). Applying interpolation and duality, we then get the result for all \(1 < p < \infty\) (note that the adjoint of \(H_\varepsilon\) is \(H_\varepsilon^{op}\)). The equivalence holds for unitary \(\varepsilon\) since \(H_\varepsilon H_\varepsilon^* = \text{id}_A\) in this case. In fact, Cotlar’s formula (5) implies that for \(x, y \in \mathcal{W}\)

\[
\widehat{H_\varepsilon x}(H_\varepsilon^* y) = H_\varepsilon(x H_\varepsilon^{op}(y^*)) + H_\varepsilon^{op}(H_\varepsilon(x)y^*) - H_\varepsilon^{op} H_\varepsilon(x y^*). \tag{7}
\]

Apply Hölder’s inequality and Proposition 3.4 to this identity for \(x = y, \varepsilon = \varepsilon'\), we get

\[
\|H_\varepsilon x\|_{2p}^2 \leq 2c_p \|x\|_{2p} \|H_\varepsilon x\|_{2p} + (4 + c_p^2) \|x\|_{2p}^2.
\]

That is \(\|H_\varepsilon x\|_{2p} \leq (c_p + \sqrt{2c_p^2 + 4}) \|x\|_{2p}\).

\[\square\]

**Remark 3.6.** As \(\prod_{n=0}^{\infty} \frac{1+\sqrt{2+4/c_p^2}}{1+\sqrt{2}} < \infty\), one gets that for \(p \geq 2\), \(c_p \leq C p^\gamma\) with \(\gamma = \frac{\ln(1+\sqrt{2})}{\ln 2}\).

**Remark 3.7.** By the usual trick to replace \(B, A_k\) by \(B \otimes M_n\) and \(A_k \otimes M_n\), one get that the maps \(H_\varepsilon\) are completely bounded on \(L^p\) for \(1 < p < \infty\).

**Remark 3.8.** We can use a slighter general definition for \(H_\varepsilon\) by taking \(\varepsilon_k \in \mathcal{B} \otimes \mathcal{M}\) where \(\mathcal{M}\) is a finite von Neumann algebra, then \(E(x)\) and \(L_k(x)\) have to be understood as \(E(x) \otimes 1\) and \(L_k(x) \otimes 1\). Theorem 3.3 remains valid with the assumption that \(\varepsilon \in \mathcal{Z}(\mathcal{B}) \otimes \mathcal{M}\).

### 3.2 Corollaries

In this section, we derive a few direct consequences of Theorem 3.3.

For any \(k_0 \in \mathbb{N}\), let \(\varepsilon_{k_0} = -1\) and \(\varepsilon_k = 1\) for \(k \neq k_0\). Then \(L_{k_0} = \frac{id_A - H_\varepsilon}{2}\).

**Corollary 3.9.** For any \(1 < p < \infty\),

\[
\|L_k x\|_p \leq \frac{1 + c_p}{2} \|x\|_p.
\]

**Corollary 3.10.** For \(1 < p < \infty\), we have

\[
\|(L_k x)\|_{L^p(\varepsilon_k^\infty)}^\infty \sim \sqrt{2c_p} \|x\|_p, \tag{8}
\]

\[
\|(R_k x)\|_{L^p(\varepsilon_k^\infty)}^\infty \sim \sqrt{2c_p} \|x\|_p. \tag{9}
\]

**Proof.** By duality we may only consider \(1 < p < 2\). For any \(x \in L^p\)

\[
\frac{1}{c_p} E_x H_\varepsilon(x) \|_p \leq \|x\|_p \leq c_p E_x H_\varepsilon(x) \|_p = c_p E_x \| \varepsilon_k L_k x\|_p.
\]

We conclude by the non commutative Khintchine inequality (4) for \(\varepsilon_k = z^{2^k}\). \[\square\]
We will prove a variant of Corollary 3.10 in the next subsection as Theorem 3.17.

Corollary 3.12. For any $1 < p < \infty$, any sequences $(i_k) \in \mathbb{N}^\mathbb{N}$ and $(x_k) \in \ell^2$, we have

$$\left\| \sum_{k=1}^{\infty} |L_{i_k}x_k|^2 \right\|^p_p \leq c_p \left\| \sum_{k=1}^{\infty} |x_k|^2 \right\|^p_p.$$  \hfill (10)

$$\left\| \sum_{k=1}^{\infty} |R_{i_k}x_k|^2 \right\|^p_p \leq c_p \left\| \sum_{k=1}^{\infty} |x_k|^2 \right\|^p_p.$$  \hfill (11)

Proof. Fix a sequence $\varepsilon_k = \pm 1$ and apply Theorem 3.5 to $x = \sum_l \varepsilon_l x_l \otimes c_l \in L^p(A \otimes B(\ell_2))$. We have

$$\left\| \sum_{k,l} \varepsilon_k \varepsilon_l L_k(x_l) \otimes c_l \right\|_p \leq c_p \left\| \sum_{l} \varepsilon_l x_l \otimes c_l \right\|_p = c_p \left\| \sum_{l=1}^{\infty} |x_l|^2 \right\|^\frac{1}{2}_p.$$  

Let $\varepsilon_k$ to be Rademacher variables, we have

$$\left\| \sum_{l=1}^{\infty} |L_k(x_l)|^2 \right\|^\frac{1}{2}_p = \left\| \mathbb{E} \sum_{k,l} \varepsilon_k \varepsilon_l L_k(x_l) \otimes c_l \right\|_p \leq c_p \left\| \sum_{l=1}^{\infty} |x_l|^2 \right\|^\frac{1}{2}_p.$$  

The proof of the second inequality is similar. \hfill \Box

Remark 3.13. Lemma 3.12 was proved in [JPX07] (Lemma 2.5, Corollary 2.9) for $x_k$’s supported on reduced words with length $= d$ with constants depending on $d$, independent on $p$.

3.3 Length independent estimates for Rosenthal’s inequality

We will apply Theorem 3.5 to obtain a length free estimate for the Rosenthal’s inequality proved in [JPX07] (Theorem A). In this subsection, we restrict $\varepsilon \in \{\pm 1\}^\mathbb{N}$ and $\varepsilon_0 = 0$ in the form of $H_\varepsilon = \sum_{k \in \mathbb{N}} \varepsilon_k L_k$ and $H_\varepsilon^{op}$. When no confusion can occur, we use the notation $T$ instead of $T \otimes Id$ for its ampliation.

Thanks to the previous results, we can define the following paraproduct for $x \in L^p(A) \otimes L^p(M) \ (1 < p < \infty)$ and $y \in L^q(A) \otimes L^q(M)$ with $\frac{1}{p} + \frac{1}{q} > 1$

$$x \overset{\varepsilon}{\star} y = E_\varepsilon H_\varepsilon(H_\varepsilon(x)y) = \sum_{k \in \mathbb{N}} L_k((L_kx)y),$$

with $E_\varepsilon$ the expectation with respect to the Haar measure on $\{\pm 1\}^\mathbb{N}$. We also set

$$x \overset{\varepsilon}{\star} y = xy - x \overset{\varepsilon}{\star} y - E(xy) = \sum_{k=0}^{\infty} L_k((L_kx)y).$$
Here \( \tilde{L}^k_j = \sum_{j \neq k, j \in \mathbb{N}} L_j \) for any \( k \geq 0 \).

If \( x \) and \( y \) are elementary tensors \( (x \notin \mathcal{B}) \), \( x \tilde{y} \) collects in the reduced form of \( xy \) all elements whose first letter is in the same algebra as \( x \) while \( x \hat{y} \) collects the rest in the reduced form of \( xy \).

**Proposition 3.14.** We have the following, for \( 1 < p < \infty \), \( 1 < q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} > 1 \)

1. \( \| H_r(x \hat{y}) \|_r \leq c_r c_p \|x\|_p \|y\|_q \), \( \| x \tilde{y} \|_r \leq 2(1+c_r c_p)\|x\|_p \|y\|_q \).
2. \( H_r(x \hat{y}) = H_r(x) \hat{y} \), \( x \hat{y} H_{r^p}(y) = H_{r^p}(x \hat{y}) \).

In particular \( x \hat{y} \in \mathcal{L}_k \) if \( x \in \mathcal{L}_k \) and \( x \hat{y} \in \mathcal{R}_k \) if \( y \in \mathcal{R}_k \).

**Proof.** (i) simply follows from Theorem 3.5 and the definitions. We now prove (ii). For \( \frac{1}{r} \), this follows from its definition.

For \( \hat{y} \), we check the following formula from which the identity because of the translation invariance of the Haar measure on \( \{-1, 1\}^N \):

\[
x \hat{y} = E_{\varepsilon'}(H_{r^p}(x \hat{y} H_{r^p}(y))).
\] (12)

We first notice that the identity holds if \( x \in \mathcal{B} \) as \( x \hat{y} = 0 \) and \( x \hat{y} = x(y - E(y)) \). Similarly if \( y \in \mathcal{B} \), \( x \hat{y} = (x - E(x))y \) and \( x \hat{y} = 0 \). Thus we can assume \( E(x) = 0 \). Apply the Cotlar identity \( 5 \) to \( H_r(x) \) and \( H_{r^p}(y^*) \) and note \( H_r^2(x) = x \) and \( H_{r^p}^2(y^*) = y^* \), we get

\[
xy - E_{\varepsilon}H_{r}(x \hat{y}) = H_{r^p}(x \hat{y} H_{r^p}(y)) - H_{r^p}H_{r}(x \hat{y} H_{r^p}(y)).
\]

Taking expectations with respect to \( \varepsilon \) and \( \varepsilon' \) gives \( 12 \). One can also verify directly the identity for \( \hat{y} \) by its bilinearity and looking at elementary tensors \( x, y \in \mathcal{W} \) and using Proposition 3.2 (iv)-(v). \( \square \)

**Remark 3.15.** There are situations for which one can slightly improve those inequalities. For instance if \( r = 2 \), then \( \| x \hat{y} \|_r \leq (1+c_r)\|x\|_p \|y\|_q \). Or in general \( \| x \hat{y} \|_r \leq c_r \sup_\varepsilon \| H_r(x) \|_p \|y\|_q \) and \( \| xy \|_r \leq 2(1+c_r)\sup_\varepsilon \| H_r(x) \|_p \|y\|_q \).

**Lemma 3.16.** For \( 2 \leq p < \infty \) and \( x \in L^p(A) \)

\[
\| \sum_{k \in \mathbb{N}} \overline{L_k(x) L_k(x)^*} \|_q^\frac{1}{2} \leq \alpha_p \| \sum_{k \in \mathbb{N}} |(L_k x)^*|^2 \|_q^\frac{1}{2} \left( \sum_{k \in \mathbb{N}} \|L_k x\|_p^2 \right)^{\frac{1}{2}} (13)
\]

\[
\| \sum_{k \in \mathbb{N}} \overline{R_k(x) R_k(x)^*} \|_q^\frac{1}{2} \leq \alpha_p \| \sum_{k \in \mathbb{N}} |R_k(x)|^2 \|_q^\frac{1}{2} \left( \sum_{k \in \mathbb{N}} \|R_k x\|_p^2 \right)^{\frac{1}{2}} (14)
\]

with \( \alpha_p \leq 3c_4^2 \) for \( 2 < p < 4 \) and \( \alpha_p \leq 2\sqrt{2}(c_4^2 + c_4^2) \) for \( p \geq 4 \).
Combining these two estimates we get
\[ \| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \|_2^2 \leq \sqrt{2} c_{\sharp} \max \left[ \| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \otimes c_k \|_2^2, \| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \otimes r_k \|_2^2 \right]. \]

Using the bilinearity of \( \dagger \), we have
\[
\sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \otimes r_k = \sum_{k \in \mathbb{N}} (L_k(x) \otimes r_k)^\dagger (L_k(x)^* \otimes d_k)
= E_x \left( \sum_{k} \varepsilon_k L_k(x) \otimes r_k \right)^\dagger \left( \sum_{k} \varepsilon_k L_k(x)^* \otimes d_k \right)
= E_x \left[ H_x \left( \sum_{k} L_k(x) \otimes r_k \right)^\dagger H_x^{op} \left( \sum_{k} L_k(x)^* \otimes d_k \right) \right]
\]
So we can conclude from Theorem 5.5 and Remark 5.17 that
\[
\| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \otimes r_k \|_2^2 \leq c_{\sharp} \sup_{z, z'}\| H_x \left( L_k(x) \otimes r_k \right) \|_p \| H_x^{op} \left( L_k(x)^* \otimes d_k \right) \|_p
\leq c_{\sharp} \left( \sum_{k} \| L_k(x) \otimes r_k \|_p \right) \left( \sum_{k} \| L_k(x)^\dagger \|_p \right)^{\frac{1}{2}}.
\] (15)

Similarly we have
\[
\| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \otimes c_k \|_2^2 = E_x \left[ H_x \left( \sum_{k} L_k(x) \otimes d_k \right)^\dagger H_x^{op} \left( \sum_{k} L_k(x)^* \otimes c_k \right) \right]
\leq c_{\sharp} \left( \sum_{k} \| L_k(x) \|_p \right)^{\frac{1}{2}} \left( \sum_{k} \| L_k(x)^\dagger \|_p \right)^{\frac{1}{2}}.
\] (16)

Combining these two estimates we get
\[
\| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \|_2^2 \leq \sqrt{2} c_{\sharp} \left( \sum_{k \in \mathbb{N}} \| L_k(x) \otimes r_k \|_p \left( \sum_{k \in \mathbb{N}} \| L_k(x)^\dagger \|_p \right)^{\frac{1}{2}} \right),
\]
for \( p \geq 4 \). We can treat the \( \dagger \) term similarly since \( L_k(x)^\dagger (L_k x)^* \in \mathcal{R}_k \) and get
\[
\| \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(x)^* \|_2^2 \leq \sqrt{2} c_{\sharp} (2 + c_{\sharp}) \left( \sum_{k \in \mathbb{N}} \| L_k(x) \otimes r_k \|_p \left( \sum_{k \in \mathbb{N}} \| L_k(x)^\dagger \|_p \right)^{\frac{1}{2}} \right).
\]
We then get (13) for \( p \geq 4 \) with constant \( 2\sqrt{2} (c_{\sharp}^2 + c_{\sharp}) \).

To deal with the remaining cases, we will use interpolation by proving a better bilinear inequality for \( 2 \leq p \leq 4 \).
\[
\| \sum_{k \in \mathbb{N}} \overline{\sum_{\mathcal{R}_k}} \sum_{\mathcal{R}_k} \sum_{k \in \mathbb{N}} L_k(x)^\dagger L_k(y)^* \|_2^2 \leq 3 c_{\sharp} \left( \sum_{k \in \mathbb{N}} \| L_k(x) \otimes r_k \|_p \left( \sum_{k \in \mathbb{N}} \| L_k(x)^\dagger \|_p \right)^{\frac{1}{2}} \right).
\]
The spaces consisting of elements of the form $\sum_k L_kx \otimes r_k$ and $\sum_k R_ky \otimes d_k$ are $c_p$-complemented in $L^p(A) \otimes S_p$ by Theorem 3.16. Hence the norms on the right hand side interpolate for $2 \leq p \leq 4$ (both with constant $c^2_4(1-2/p)$).

We just need to justify the endpoint inequalities. For $p = 2$, we have by Hölder’s inequality that

$$\| \sum_{k \in \mathbb{N}} \overline{\alpha} L_k(x)R_k(y) \|_1 \leq 2 \| \sum_k L_k(x) \otimes r_k \|_2 \| \sum_k R_k(y) \otimes c_k \|_2 = 2 \| \sum_k L_k(x) \otimes r_k \|_2 (\sum_k \| L_ky \|_2^2)^{1/2}.$$  

For $p = 4$, by orthogonality and as in (13)

$$\| \sum_{k \in \mathbb{N}} L_k(x) \frac{i}{4} R_k(y) \|_2 = \| \sum_{k \in \mathbb{N}} L_k(x) \frac{i}{4} R_k(y) \otimes r_k \|_2 \leq \| \sum_k L_kx \otimes r_k \|_4 (\sum_k \| R_ky \|_4^4)^{1/4}.$$  

Similarly we get $\| \sum_{k \in \mathbb{N}} L_k(x) \frac{i}{4} R_k(y) \|_2 \leq 2 \| \sum_k L_kx \otimes r_k \|_4 (\sum_k \| R_ky \|_4^4)^{1/4}$.

Thus by interpolation we get (13) for $2 < p < 4$ with a constant $3c^4_4(1-2/p)$.

**Theorem 3.17.** For $2 \leq p < \infty$ and $x \in L^p$

$$\beta_p^{-1}\|x\|_p \leq \max \left\{ \left\| \sum_{k=0}^\infty \| L_k(x) \|^2 \right\|_p, \| E(xx^*) \|_p \right\} \leq \sqrt{c_p}\|x\|_p$$  

and

$$\beta_p^{-1}\|x\|_p \leq \max \left\{ \left\| \sum_{j=0}^\infty \| R_j(x^*) \|^2 \right\|_p, \| E(xx^*) \|_p \right\} \leq \sqrt{c_p}\|x\|_p$$  

with $\beta_p \leq \sqrt{c_p}(1 + \alpha_p) \leq c^2_p$.

**Proof.** For the first equivalence, the upper inequality follows from Corollary 3.10. For the lower bound, by Lemma 3.10 as $E(xx^*) = \sum_{k \geq 0} E(L_k(x)L_k(x)^*)$ and $L_0(x)L_0(x)^* = 0$, we have

$$\| \sum_{k \geq 0} L_k(x) \otimes r_k \|_p \leq \alpha_p \| \sum_{k \in \mathbb{N}} L_k(x) \otimes d_k \|_p \| E(xx^*) \|_p^{1/2}.$$  

But as $p \geq 2$ the map $c_k \mapsto d_k$ is a contraction on $L^p$, so we deduce

$$\| \sum_{k \geq 0} L_k(x) \otimes r_k \|_p \leq \alpha_p \| \sum_{k \geq 0} L_k(x) \otimes c_k \|_p \| E(xx^*) \|_p^{1/2},$$  

and we conclude the lower bound by Corollary 3.10 again. The other inequality follows by taking adjoints. □
We get the following Rosenthal type inequality as a direct application.

**Corollary 3.18.** Let $2 < p < \infty$

(i) For $x = \sum_{k=0}^{\infty} a_k$ with $a_k \in \mathcal{L}_k$, we have
\[
\beta_p^{-2} \|x\|_p \leq \|E(xx^*)\|^{\frac{1}{2}}_p + \|E(x^*x)\|^{\frac{1}{2}}_p + \sum_{k,j} R_j(a_k) \otimes e_{k,j} \|_p \leq (2c_2^2 + 2) \|x\|_p.
\]

(ii) For $x = \sum_{k=0}^{\infty} a_k$ with $a_k \in \mathcal{L}_k \cap \mathcal{R}_k$, we have
\[
\beta_p^{-2} \|x\|_p \leq \|E(xx^*)\|^{\frac{1}{2}}_p + \|E(x^*x)\|^{\frac{1}{2}}_p + \left( \sum_k \|a_k\|^{\frac{1}{p}}_p \right)^p \leq (2c_2^2 + 2) \|x\|_p.
\]

**Proof.** Apply Theorem 3.17 twice and notice that $(r_k \otimes c_k)$ generate the canonical basis of $\ell_p$ in $S_p$. We get (i) and (ii) follows immediately.

**Remark 3.19.** We point out that Corollary 3.18 (ii) was proved in [JPX07] (Theorem A) when $a_k$ are supported on reduced words with a fixed length with constants independent of $p$ but depends on the length. Noticing that by the Khintchine inequalities from [RX06], $H_\varepsilon$ and $H_{\varepsilon^p}$ are bounded on words of length at most $d$ with a constant that depends only on $d$ and $A$, thus by interpolation, the argument above also implies Corollary 3.18 (ii) with constants independent of $p$ but dependent on the length.

**Remark 3.20.** All the results of this section also hold in the completely bounded setting.

### 4 Free groups

We can apply the previous results to the free group as it is naturally a free product. Let $g_i, i \in \mathbb{N}$ be the set of generators of $F_\infty$. We let $\mathcal{L}_{g_i}$ and $\mathcal{E}_{g_i^{-1}}$ be the set of reduced word starting by $g_i$ and $g_i^{-1}$ respectively and $\mathcal{L}_{g_i^\pm} = \mathcal{L}_{g_i} \cup \mathcal{L}_{g_i^{-1}}$. We denote by $L_{g_i}, L_{g_i^{-1}}$ and $L_{g_i^\pm}$ the associated projections. We use the notation $R_{g_i}$ and $R_{g_i^\pm}$ for the right analogues. We will often use the convention $g_i = g_i^{-1}$ for $i < 0$ so that $L_{g_i^{-1}} = L_{g_i}$ for any $i \in \mathbb{Z}^*$. Finally $S$ will denote the set $\{g_i; i \in \mathbb{Z}^*\}$.

Let $\mathcal{M}$ be a finite von Neumann algebra. Theorem 3.5 immediately gives that, for any $x \in L_p(\widehat{F}_{\infty}) \otimes L_p(\mathcal{M}), 1 < p < \infty$ and sequences of unitaries $\varepsilon_i \in \mathcal{Z}(\mathcal{M}), \|\varepsilon_k\| \leq 1$,
\[
\|(Id \otimes \tau)x + \sum_i \varepsilon_i(Id \otimes L_{g_i^\pm})(x)\|_p \preceq \varepsilon_p \|x\|_p, \tag{17}
\]

We slightly extend it
Theorem 4.1. Let \((\varepsilon_k)_{k \in \mathbb{Z}}\) be a sequence in \(Z(\mathcal{M}), \|\varepsilon_k\| \leq 1\). Then for any \(x \in L^p(\hat{F}_\infty) \otimes L^p(\mathcal{M})\) and \(1 < p < \infty\)

\[
\|\varepsilon_0(Id \otimes \tau)x + \sum_{k \in \mathbb{Z}^*} \varepsilon_k(Id \otimes Lg_k)(x)\|_p \leq c_p \|x\|_p.
\]

The equivalence holds if we assume further that \(\varepsilon_k\) are unitaries in \(Z(\mathcal{M})\).

Proof. We may assume \(\varepsilon_0 = 1\). We consider the following group embedding \(\pi : F_\infty \to F_\infty \ast F_\infty\) defined by \(\pi(g_i) = g_i h_i\) where \((h_i)\) are the free generators of the second copy of \(F_\infty\). This extends to a complete isometry for \(L^p\)-spaces and one checks directly that

\[
\left(\sum_{k=0}^{\infty} \varepsilon_k L_h^\pm + \tau + \sum_{k=0}^{\infty} \varepsilon_k Lg_k^\pm\right) \circ \pi = \pi \circ \left(\sum_{k=0}^{\infty} \varepsilon_k Lg_k^{-1} + \tau + \sum_{k=0}^{\infty} \varepsilon_k Lg_k\right).
\]

The statement follows from the amalgamated version of \(17\). \(\square\)

The proof of Lemma 3.16 and Theorem 3.17 can easily be adapted to the free group where \(H_\epsilon = \varepsilon_L c + \sum_{h \in S} \varepsilon_h L_h\) with \(|\varepsilon_h| = 1\) and the convention \(L_h x = \tau x\). We simply give the result

Theorem 4.2. For \(2 < p < \infty\), \(x \in L_p(\hat{F}_\infty) \otimes L_p(\mathcal{M})\)

\[
\beta_p^{-1} \|x\|_p \leq \max \left\{\left\|\left(\sum_{|h| \leq 1} |L_h(x)|^2\right)^{1/2}\right\|_p, \left\|\left(\tau \otimes Id(x^*)\right)\right\|_{p/2}^{1/2}\right\} \lesssim \sqrt{2c_p} \|x\|_p
\]

and

\[
\beta_p^{-1} \|x\|_p \leq \max \left\{\left\|\left(\sum_{|h| \leq 1} |R_h(x^*)|^2\right)^{1/2}\right\|_p, \left\|\left(\tau \otimes Id(x^* x)\right)\right\|_{p/2}^{1/2}\right\} \leq \sqrt{2c_p} \|x\|_p.
\]

Corollary 4.3. Let \(2 < p < \infty\)

(i) For \(x = \sum_{k=-\infty}^{\infty} a_k\) with \(a_k \in L_k\), we have

\[
\beta_p^{-2} \|x\|_p \leq \|\left(\tau \otimes Id(x^* x)\right)\|_p^{1/2} + \|\left(\tau \otimes Id(x^* x)\right)\|_p^{1/2} + \sum_{k,j} R_j(a_k) \otimes e_{k,j}\|_p \leq (2c_p^2 + 2) \|x\|_p
\]

(ii) For \(x = \sum_{k=-\infty}^{\infty} a_k\) with \(a_k \in L_k \cap \mathcal{R}_{\phi(k)}\) and \(\phi : Z \to Z\) an one to one map, we have

\[
\beta_p^{-2} \|x\|_p \leq \|\left(\tau \otimes Id(x^* x)\right)\|_p^{1/2} + \|\left(\tau \otimes Id(x^* x)\right)\|_p^{1/2} + \sum_{k} \|a_k\|_p^{1/2} \leq (2c_p^2 + 2) \|x\|_p.
\]

Proof. Apply Theorem 4.2 twice we get (i). (ii) follows immediately because \(\phi\) is one to one. \(\square\)
Remark 4.4. All results before this subsection hold for free groups with \( L_k, R_k \) replaced by \( L_{g_k} \) (resp. \( L_{g_k^{-1}} \) or \( L_{g_k^±} \)) and \( R_{g_k} \) (resp. \( R_{g_k^{-1}} \) or \( R_{g_k^±} \)). We can strengthen some of them. These will be recorded in the following.

Given \( g, h \) reduced word of \( F_\infty \), we write \( g \leq h \) (or \( h \geq g \)) if \( h = gk \) with \( g, h, k \) reduced words, i.e. \( |g^{-1}h| = |h| - |g| \). We write \( g \nmid h \) otherwise. Let

\[
L_h := \{ g \in F_\infty, g \geq h \}
\]

and \( L_h \) the associated \( L^2 \)-projection, this is compatible with our previous notation.

Corollary 4.5. For any \( 1 < p < \infty \), \( h \in F_\infty \) and \( x \in L^p(\overline{F_\infty}) \otimes L^p(M) \),

\[
\|L_h x\|_p \leq \frac{c_p + 1}{2} \|x\|_{L^p}.
\]

Moreover, \( \lim_{|h| \to \infty} \|L_h x\|_p \to 0 \).

Proof. Without loss of generality, we may assume \( h \in R_{g_1} \) and \( h = h'g_1 \). Then \( L_h x = \lambda_{h'} L_{g_1}(\lambda_{h'} x) \). The \( L^p \) bounds follow from Theorem 4.1. Note the \( L^p \) space is defined as the closure of \( C_c(F_\infty) \), we get the convergence by the uniformly boundedness of \( L_h \) on \( L^p \).

Corollary 4.6. For any \( 1 < p < \infty \), any sequences \((h_k) \in F_\infty \setminus \{e\} \) and \((x_k) \in L^p(\ell_2^\infty) \), we have

\[
\|\sum_{k=1}^\infty |L_{h_k} x_k|^2\|_p \leq c_p \|\sum_{k=1}^\infty \lambda_{h_k} e_{h_{k-1}} x_k\|_p.
\]

(18)

Proof. Let us assume such that \( h_k \in R_{g_{i_k}}, i_k \in \mathbb{Z} \). Assume \( h_k = h'_{i_k} g_{i_k} \). Then

\[
L_{h_k} x_k = \lambda_{h'_{i_k}} L_{g_{i_k}} (\lambda_{h'_{i_k}^{-1}} x_k).
\]

So

\[
\sum_{k=1}^\infty |L_{h_k} x_k|^2 = \sum_{k=1}^\infty |L_{g_{i_k}} (\lambda_{h'_{i_k}^{-1}} x_k)|^2.
\]

We get the result by the free group version of Corollary [5.12].

4.1 A length reduction formula

Let \( W_{\geq d} \) be the set of word in \( F_\infty \) of length greater than \( d \), also denote by \( W_{\geq d} \) the subspace in \( L^p \) generated by \( \lambda w, w \in W_{\geq d} \). For \( w \in F_\infty \), we let \( w_l \) denote its \( l \)-th letter (if it exists) and \( \partial w = w^{-1} w \).

Take any \( x = \sum_{w \in W_{\geq 1}} x_w \lambda_w \in W_{\geq 1} \), we have

\[
\|\sum_{h \in S} |L_h(x)|^2\|_p = \|\sum_{w \in W_{\geq 1}} x_w \lambda_w \otimes c_{w_1}\|_p = \|\sum_{w \in W_{\geq 1}} x_w \lambda_{\partial w} \otimes c_{w_1}\|_p.
\]

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At the operator space level, Theorem 4.2 means that the map \( \iota : W_{\geq d} \to C_p \otimes W_{\geq d-1} \oplus R_p \) given by \( \iota(\lambda_w) = \lambda_{\partial w} \otimes c_w \oplus r_w \) is a complete isomorphism. Iterating, we obtain a complete isomorphism for \( 2 < p < \infty \)

\[
\iota_d : \begin{cases}
W_{\geq d} & \to C_p^{\otimes d} \otimes L_p(\hat{F}_\infty) \oplus C_p^{\otimes d-1} \otimes R_p \oplus \cdots \oplus C_p \otimes R_p \oplus R_p \\
\lambda_w & \mapsto c_{w_1, \ldots, w_d} \otimes \lambda_{\partial w} \oplus c_{w_1, \ldots, w_{d-1}} \otimes r_{\partial w} \oplus \cdots \oplus c_{w_1} \otimes r_{\partial w} + r_w
\end{cases}
\]

Let us state this as a Corollary, which generalizes the result of [PP05].

**Corollary 4.7.** (Length reduction formula) For any \( d \geq 1 \), \( \iota_d \) extends to a completely isomorphism such that for \( x \in W_{\geq d} \), \( 2 \leq p < \infty \),

\[
\beta_p^{-d} \|x\|_p \leq \|\iota_d x\| \leq (\sqrt{2}c_p)^d \|x\|_p,
\]

for all \( x \in L^p(\hat{F}_\infty) \).

Fix some \( d \in \mathbb{N} \) and any reduced word \( w = w_1 \ldots w_n \) in the generators, we define:

\[
L_h^{(d)}(\lambda_w) = \delta_{w_1=\lambda_h} \lambda_w, \quad \text{and} \quad H_\varepsilon^{(d)} = \varepsilon^{\ell} P_{d-1} + \sum_{h \in S} \varepsilon_h L_h^{(d)},
\]

for any choice of \( \varepsilon_h, |h| \leq 1 \) with \( |\varepsilon_h| \leq 1 \). Recall that by [RX06] or [JPX07], \( P_{d-1} \) is completely bounded on \( L^p \) (this also follows from Theorem 4.2). Note that

\[
\|\iota_{d-1} H_\varepsilon^{(1)} H_\varepsilon^{(2)} \cdots H_\varepsilon^{(d)} x\| = \|H_\varepsilon^{(d)} \iota_{d-1} H_\varepsilon^{(d-1)} \iota \cdots \iota H_\varepsilon^{(1)} x\|.
\]

We get immediately

**Theorem 4.8.** For any \( d \geq 1 \) and \( x \in L^p(\hat{F}_\infty) \otimes L^p(\mathcal{M}) \), \( 1 < p < \infty \),

\[
\|H_\varepsilon^{(1)} H_\varepsilon^{(2)} \cdots H_\varepsilon^{(d)} x\| \leq c_{p,d} \|x\|_p \sim_{p,d} \|x\|_p
\]

with \( c_{p,d} \leq (\sqrt{2}c_p)^{2d-1} \beta_{p-1} \leq \xi_{p}^{5d-4} \) and \( \|H_\varepsilon^{(d)} x\|_p \sim (\sqrt{2}c_p)^{d-1} \|x\|_p \) for any choice of \( |\varepsilon_h| = 1 \).

We give a faster argument for the boundedness of \( H_\varepsilon^{(d)} \). Consider, \( \varepsilon_h = \pm 1 \) for \( h \in \mathbb{F}_\infty \), let

\[
H_\varepsilon^{Ld} = \varepsilon^{\ell} P_{d-1} + \sum_{h \in \mathbb{F}_\infty, |h|=d} \varepsilon_h L_h, \quad H_\varepsilon^{Rd} = \varepsilon^{\ell} P_{d-1} + \sum_{h \in \mathbb{F}_\infty, |h|=d} \varepsilon_{h^{-1}} R_h.
\]

Recall that \( L_h \) (resp. \( R_h \)) is defined as the projection onto the set of all reduced words starting (resp. ending) with \( h \). We get \( H_\varepsilon^{(d)} \) from \( H_\varepsilon^{Ld} \) if \( \varepsilon_h \) depends only on the \( d \)-th letter of \( h \).

**Corollary 4.9.** For any \( 1 < p < \infty \), we have for any \( x \in L^p(\hat{F}_\infty) \),

\[
\|x\|_p \sim \|H_\varepsilon^{Ld} x\|_p \sim \|\iota_{h} x\|_p \leq \|L_h x\|_p \leq L^p(\hat{F}_\infty, \xi_\varepsilon^r)
\]
Proof. Note that a similar identity to (5) holds for free groups with $H_{x}^{Ld}$ and any $g, h$ with $|g^{-1}h| \geq 2d - 1$. We then have that

\[ P_{2d-2} \left[ H_{x}^{Ld}x(H_{x}^{Ld}x)^{\ast} \right] = P_{2d-2} \left[ H_{x}^{Ld}(xH_{x}^{Ld}(x^{\ast})) + H_{x}^{Ld}(H_{x}^{Ld}(x)x^{\ast}) - H_{x}^{Ld}H_{x}^{Ld}(xx^{\ast}). \right] \]  

Let $c_{p,d} \geq 2$ be the best constant such that $\|H_{x}^{Ld}x\|_{p} \leq c_{p} \|x\|_{p}$. Recall that by the Haagerup inequality, the $L^{1}$ and $L^{p}$ norms are equivalent on the range of $P_{2d-2}$:

\[ \|P_{2d-2}[H_{x}^{Ld}(xH_{x}^{Ld}(x^{\ast}))]\|_{p} \leq (2d - 1)^{2-\frac{3}{d}} \|H_{x}^{Ld}(xH_{x}^{Ld}(x^{\ast}))\|_{1} \leq (2d - 1)^{2-\frac{3}{d}} \|x\|_{2p}^{2}, \]

\[ \|P_{2d-2}[H_{x}^{Ld}(xH_{x}^{Ld}(x^{\ast}))]\|_{p} \leq (2d - 1)^{1-\frac{3}{d}} \|H_{x}^{Ld}(xH_{x}^{Ld}(x^{\ast}))\|_{2} \leq (2d - 1)^{1-\frac{3}{d}} c_{p,d} \|x\|_{2p} \]

for any $p > 2$. Therefore,

\[ c_{2p,d}^{2} \leq 2(2d - 1)^{2-\frac{3}{d}} + 2c_{2p,d}(2d - 1)^{1-\frac{3}{d}} + 2c_{p,d}c_{2p,d} + c_{p,d}^{2}. \]

We then have

\[ c_{2p,d} \leq (2d - 1)^{1-\frac{3}{d}} + c_{p,d} + \sqrt{2}(c_{p,d} + 3(2d - 1)^{1-\frac{3}{d}}). \]

Asymptotically $c_{p,d} \simeq \rho \frac{\ln(1 + d)}{\sqrt{d}}$ for $d$ given and $c_{p,d} \simeq d^{1-\frac{3}{d}}$ for $p$ given. So

\[ \|H_{x}^{Ld}x\|_{p} \leq c_{p,d} \|x\|_{p}. \]

Since $H_{x}^{Ld}H_{x}^{Ld} = id$, we get the equivalence. The $1 < p < 2$ case follows by duality. \qed

Remark 4.10. A straightforward c.h. version of Corollary 4.9 is false for $F_{\infty}$ (true for $F_{n}$ with a constant depending on $n$ though). This is because the operator valued Haagerup inequality is an equivalence between the $L^{p}$ norm and the more complicated norm given by Corollary 4.7. For instance it yields that the set $\{\lambda(g, g_{1})\}$ is not completely unconditional, this would be a direct consequence of Corollary 4.9.

For any $x \in L(F_{n})$, $n < \infty$ and any choice of signs $H_{x}x$ can be viewed as an unbounded operator on $L^{2}(F_{n})$ with domain $C_{c}(F_{n})$. As usual $K$ stands for the compact operators. Ozawa asked in [O10] whether the commutator $[R_{h}, x]$ sends the unit ball of $L^{2}(F_{n})$ into a compact set of $L^{2}(F_{n})$ for any $h \in F_{n}$, and $x \in L(F_{n})$ and pointed out that the $L^{p}$-boundedness of $R_{h}$ implies a positive answer. We record a general result in the following corollary.

Corollary 4.11. We have for $d \in \mathbb{N}$ and any choice of signs $\varepsilon$

(i) $[H_{\varepsilon}^{Ld}, x] \in B(L^{2}(F_{n}))$ if $x = x_{1} + H_{\varepsilon}^{Ld}x_{2}$ for some $|\varepsilon'| \leq 1, x_{1}, x_{2} \in L(F_{n})$.

(ii) $[H_{\varepsilon}^{Ld}, x] \in K(L^{2}(F_{n}))$ for all $x = x_{1} + H_{\varepsilon}^{Ld}x_{2}$ for some $|\varepsilon'| \leq 1, x_{1}, x_{2} \in L^{2}(F_{n})$. 

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(iii) \([H^R_x, x]\) maps the closed unit ball of \(L(\mathcal{F}_n)\) into a compact subset of \(L^2(\hat{\mathcal{F}}_n)\) if \(x \in L_p(\hat{\mathcal{F}}_n)\) for some \(p > 2\) (in particular, if \(x \in L(\mathcal{F}_n)\)).

**Proof.** Similar to (20), we have

\[
P_{\frac{1}{2d-2}}[H^L_{e,x}(H^R_{e,y})] = P_{\frac{1}{2d-2}}[H^L_{e,x}(H^R_{e,y})] + H^R_{e, x}((H^L_{e,x})y - H^L_{e,y}H^R_{e}(xy)).
\]

So, up to a finite rank perturbation, for \(y \in L^2(\hat{\mathcal{F}}_n)\)

\[
[H^R_{e,x}, H^L_{e,x}]y = -H^L_{e,x}(x(H^R_{e,y}) + H^L_{e,x}H^R_{e}(xy) = H^L_{e,x}((H^R_{e,x}y).
\]

Therefore, for \(x = x_1 + H^L_{e,x}x_2\), up to a finite rank perturbation

\[
[H^R_{e,x}] = [H^R_{e,x}, x] + [H^L_{e,x}].
\]

This implies (i). Note that \([R_h, g]\) is finite rank for each \(h, g\). We have that \([H^R_{e,x}, x] \in \mathbb{K}(L^2(\mathcal{F}_n))\) for all \(x \in C^\infty(\mathcal{F}_n)\). So (ii) is true. For (iii), following the argument of Ozawa, we have, by Hölder’s inequality and Theorem 4.1

\[
\|H^R_{e,x}\|_L^2(\mathcal{F}_n) \leq \|x\|_L^p(\mathcal{F}_n) \|y\|_L^q(\mathcal{F}_n)
\]

for any \(y \in L^q(\mathcal{F}_n), \frac{1}{q} + \frac{1}{p} = \frac{1}{2}\). By density of \(C^\infty(\mathcal{F}_n)\) in \(L^p(\mathcal{F}_n), p < \infty\) and since \(L(\mathcal{F}_n) \subset L^p(\hat{\mathcal{F}}_n)\) contractively, we get the desired result. □

**Remark 4.12.** When \(n = 1\), the space of functions \(x \in L^2(\mathcal{F}_n)\) (resp. \(C^\infty(\mathcal{F}_n)\)) is called BMO (resp. VMO). It characterises the class of functions \(x\) such that the commutator \([H, x]\) is bounded (resp. compact).

**Remark 4.13.** The content of this remark is communicated to the authors by N. Ozawa. Let \(M\) be a finite von Neumann algebra with a finite normal faithful trace \(\tau\). Let \(L^p(M), 1 \leq p < \infty\) be the associated non-commutative \(L^p\) spaces (see [PX03]). Recall that we set \(L^\infty(M) = M\). For the operators \(X \in B(L^p(M)), p \geq 2\), define a semi-norm

\[
\|X\|_{L^p \to L^2} = \sup\{\|Xy\|_{L^2(M)}; y \in L^p(M) \subset L^2(M), \|y\|_{L^p(M)} \leq 1\}.
\]

Note \(\|X\|_{L^p \to L^2}\) is just the operator norm \(\|X\|\). Identify \(M\) as sub algebra of \(B(L^2(M))\) by the left multiplication on \(L^2(M)\). Let \(M' \subset B(L^2(M))\) be the sub algebra of the right multiplication of \(M\) on \(L^2(M)\). For \(b \in M \cup M'\), we have by Hölder’s inequality that

\[
\|b\|_{L^p \to L^2} = \|b\|_{L^q}
\]

for \(\frac{1}{q} + \frac{1}{p} = \frac{1}{2}\). The lemma of [O10] Section 3 says that, for \(X \in B(L^2(M))\),

\[
\|X\|_{L^\infty \to L^2} \leq \inf\{\|Y\|\|b\|_{L^2(M)} + \|Z\|\|c\|_{L^2(M)}\} \leq 4\|X\|_{L^\infty \to L^2}.
\]

Here the infimum is taken over all possible decomposition \(X = Yb + Zc\) with \(Y, Z \in B(L^2(M)), b \in M, c \in M', \|b\|, \|c\| \leq \|X\|\). One can easily see that an analogue of the first inequality of (21) holds for all \(p > 2\), that is

\[
\|X\|_{L^p \to L^2} \leq \inf\{\|Y\|\|b\|_{L^p(M)} + \|Z\|\|c\|_{L^p(M)}\},
\]

\[18\]
Inequality (23) implies that \( M \) ball of Theorem 3.5 and Corollary 4.11 (iii) imply that \( H \) associated graph. The Gromov product on \( \bar{\gamma} \) generated by \( \rho \) suggested to study the

4.2 Connections to Carré du Champ

Suppose \( Y \in B(L^2(\mathcal{M})) \) satisfies that, for some \( p > 2 \),

\[
\|Y\|_{L^\infty \to L^p} = \sup \{\|Yx\|_{L^p(\mathcal{M})}; x \in L^\infty(\mathcal{M}) \subset L^2(\mathcal{M}), \|x\|_{\mathcal{M}} \leq 1\} < \infty.
\]

Inequality (23) implies that

\[
\|XY\|_{L^\infty \to L^2} \leq \|X\|_{L^p \to L^2} \|Y\|_{L^\infty \to L^p} \leq 4\|X\|_{L^\infty \to L^2} \|Y\|_{L^\infty \to L^p} \leq 4\|X\|_{L^\infty \to L^2} \|Y\|_{L^\infty \to L^p}. \tag{24}
\]

Let \( \mathbb{K}_{\mathcal{M}}^L \in B(L^2(\mathcal{M})) \) be the collection of all operators sending the unit ball of \( \mathcal{M} \) into a compact subset of \( L^2(\mathcal{M}) \). Let \( \mathbb{K}_{\mathcal{M}} = (\mathbb{K}_{\mathcal{M}}^L)^* \cap \mathbb{K}_{\mathcal{M}}^L \) be the associated \( C^* \)-algebra. Let \( M(\mathbb{K}_{\mathcal{M}}) \) be the multiplier algebra of \( \mathbb{K}_{\mathcal{M}} \), i.e., the algebra of all operators \( X \in B(L^2(\mathcal{M})) \) such that both \( X\mathbb{K}_{\mathcal{M}} \) and \( \mathbb{K}_{\mathcal{M}}X \) still belong to \( \mathbb{K}_{\mathcal{M}} \). Proposition of [O10] Section 2 says that \( X \in \mathbb{K}_{\mathcal{M}} \) iff for every sequence of finite rank projections \( Q_n \) strongly converging to the identity of \( B(L^2(\mathcal{M})) \), \( \|X - Q_nX\|_{L^\infty \to L_2} \to 0 \). Combining this with [24], we see that \( Y \) above belongs to \( M(\mathbb{K}_{\mathcal{M}}) \). This applies to the particular case when \( Y \) is the free Hilbert transform \( \mathcal{H}_\varepsilon \) or \( H^T_{\mathcal{M}} \). Let \( \mathcal{M} \) be an amalgamated free product. Ozawa suggested to study the \( C^* \)-algebra

\[
B_{\mathcal{M}} = \{X \in M(\mathbb{K}_{\mathcal{M}}); [X, y] \in \mathbb{K}_{\mathcal{M}}, \forall y \in \mathcal{M} \subset B(L^2(\mathcal{M}))\}.
\]

Theorem 5.4 and Corollary 4.11 (iii) imply that \( H_{\varepsilon}^{\mathcal{H}} \in B(\mathcal{F}_n) \) and similarly \( H_{\varepsilon}^{\mathcal{L}} \in B(\mathcal{F}_n) \). Here \( \mathcal{L}(\mathcal{F}_n) \) is the von Neumann algebra generated by the right regular representation \( \rho_y \)’s.

Let \( \mathcal{F}_n = F_n \cup \partial F_n \) and \( C(\mathcal{F}_n) \) be the \( C^* \)-algebra of continuous functions on \( \mathcal{F}_n \). Note that \( C(\mathcal{F}_n) \) is isomorphic to the sub \( C^* \)-algebra of \( B(\ell^2(\mathcal{F}_n)) \) generated by \( \rho_y L_h \rho_y^{-1}, g, h \in \mathcal{F}_n \). We then obtain

\[
C(\mathcal{F}_n) \subset B(\mathcal{F}_n).
\]

4.2 Connections to Carré du Champ

We use the same notation to denote elements of \( \mathcal{F}_\infty \) and points on its Cayley graph. The Gromov product for \( g^{-1}, g' \) (on the Cayley graph) is defined as

\[
\langle g, g' \rangle = \frac{|g| + |g'| - |gg'|}{2}.
\]

A closely related object is the so-called Carré du Champ of P. A. Meyer

\[
\Gamma(\lambda_g, \lambda_{g'}) = \frac{A(\lambda_{g}^*\lambda_{g'} + \lambda_{g}^*A(\lambda_{g'}) - A(\lambda_{g}^*\lambda_{g'}))}{2} = \langle g^{-1}, g' \rangle \lambda_{g^{-1}g'}
\]

associated to the conditionally negative operator \( A : \lambda_g \mapsto |g|\lambda_g \).
The following is a key connection to the operator $L_h$ studied in previous subsections, that
\[
2\Gamma(\lambda_g, \lambda_{g'}) = \sum_{h \in F_\infty} (L_h(\lambda_g))^* L_h(\lambda_{g'}). \tag{25}
\]
Let us extend these notations to $x = \sum_g c_g \lambda_g \in L^2(\hat{F}_\infty) \otimes L^2(M)$, and set
\[
A^r(x) = \sum_g c_g |g|^r \lambda_g
\]
\[
\Gamma(x, x) = \langle x, x \rangle = \sum c_g^* c_{g'} (g^{-1}, g') \lambda_{g^{-1}g'}.
\]
We then have
\[
2\langle H_\varepsilon x, H_\varepsilon x \rangle = \sum_{h \in F_\infty} |L_h x|^2 = A(x^*) x + x^* A(x) - A(|x|^2). \tag{26}
\]

The following square function estimate was proved in [JMP16]. One direction of the inequality had been proved in [JM10] and [JM12] in a more general setting.

**Lemma 4.14.** ([JMP16] Theorem A1, Example (c)) For any $2 \leq p < \infty, x \in L^p(\hat{F}_\infty) \otimes L^p(M)$,
\[
\|A_\frac{1}{2} x\|_p \simeq \left( \sum_{h \in F_\infty} |L_h x|^2 \right)^{\frac{1}{2}} p + \left( \sum_{h \in F_\infty} |L_h (x^*)|^2 \right)^{\frac{1}{2}} p.
\]

**Remark 4.15.** The equivalence above may fail if one replace $L_h(x^*)$ by $(L_h x)^{*}$ on the right hand side. Corollary 4.9 of [JM12] gives constants $\simeq p$ for the "$\leq" direction.

### 4.3 Littlewood-Paley inequalities

In the case of the free group we adapt the definition of the paraproducts studied in Section 3.3. Assume $x = \sum_g c_g \lambda_g \in L^p, y = \sum_h d_h \lambda_h \in L^q$. We then find that
\[
x \dot{\perp} y = \sum_{g^{-1} \not< h} c_g d_h \lambda_{gh}, \quad x \dot{\perp} y = \sum_{g^{-1} \not< h} c_g d_h \lambda_{gh}.
\]
Recall that we write $g \leq h$ (or $h \geq g$) if $h = gk$ with $g, h, k$ reduced words and $g < h$ if $g \leq h$ and $g \not= h$.

We consider a decomposition of $F_\infty$ into disjoint geodesic paths. To get one, first pick a (randomly decided) geodesic path $P_0$ starting at the unit element $e$. Then for any length 1 elements not in $P_0$ pick a (randomly decided) geodesic path starting at each of them. We then go to length 2 elements which are not contained in any of the previous picked paths, and pick a (randomly decided) geodesic path starting at each of them. We repeat this procedure and get countable many disjoint geodesic paths $P_n$ such that $\cup_n P_n = F_\infty$.

Let $T_n$ be the $L^2$-projection onto the span of $P_n$. Let $h_1(n)$ be the root of $P_n$, i.e. the first element in $P_n$. Let $S_n$ be the projection to the collection of words smaller than $h_1(n)$ (note that $S_0 = 0$).
Corollary 4.16. For any $1 < p < \infty$, the maps $T_n$ are completely bounded on $L^p$ with

$$\|T_n\|_{p \to p} \lesssim c_p^2. \quad (27)$$

Moreover, for any $p > 2$

$$\| \sum_n |T_n x + S_n x|^2 - |S_n x|^2 \|_p \lesssim c_p^2 \|x\|_p^2. \quad (28)$$

Proof. We write $x = \sum c_g \lambda_g$ and $T_n x = \sum_{g \in \mathbb{P}_n} c_g \lambda_g$. Then

$$(T_n x)^* T_n x - \sum_{g \in \mathbb{P}_n} |c_g|^2 \lambda_e = \sum_{g < h \in \mathbb{P}_n} c_g c_h \lambda_g \lambda_h + \sum_{h < g \in \mathbb{P}_n} c_g c_h \lambda_g \lambda_h = (T_n x)^* \lambda \lambda x + \lambda (T_n x)^* \lambda x.$$ 

Since $(T_n x + S_n x)^* \lambda x = x^* \lambda x$, we have that

$$(T_n x)^* \lambda \lambda x = x^* \lambda \lambda x - (S_n x)^* \lambda x.$$ 

Therefore, 

$$(T_n x)^* T_n x - \sum_{g \in \mathbb{P}_n} |c_g|^2 \lambda_e = x^* \lambda \lambda x + (x^* \lambda \lambda x)^* - (S_n x)^* \lambda x - (T_n x)^* \lambda x. \quad (29)$$

In particular for $n = 0$, we have actually

$$(T_0 x)^* T_0 x - \sum_{g \in \mathbb{P}_0} |c_g|^2 \lambda_e = x^* \lambda \lambda x + (x^* \lambda \lambda x)^*.$$ 

Assume $p > 2$, by Proposition 3.14, we have

$$\|T_0 (x)\|_p^2 \leq (4 + 2c_p c_p^2) \|x\|_p \|T_0 x\|_p + \|x\|_p^2.$$ 

So $\|T_0 (x)\|_p \leq (5 + 2c_p c_p^2) \|x\|_p$ for $p > 2$. One concludes that $T_0$ is (completely) bounded on $L^p$. One can improve the bound on $\|T_0\|_{p \to p}$ when $p$ is close to 2 by using interpolation. The case $p < 2$ follows by duality. Thus we have obtained (27) for an arbitrary $\mathbb{P}_0$ starting at $e$, for general $\mathbb{P}_n$ this follows by using translations.

Summing (29) over $n$, we get

$$\sum_{n \geq 0} |T_n x + S_n x|^2 - |S_n x|^2 = \sum_{n \geq 0} |(T_n x)^* T_n x + (S_n x)^* T_n x + (T_n x)^* S_n x|$$

$$= \sum_{n \geq 0} x^* \lambda \lambda x + (x^* \lambda \lambda x)^* + \sum_{g} |c_g|^2 \lambda_e$$

$$= x^* \lambda \lambda x + (x^* \lambda \lambda x)^* + \tau |x|^2 \lambda_e.$$ 

(28) then follows from Proposition 3.14. \qed

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We now consider a concrete partition given by geodesic paths. For any \( h_0 \not\in \mathcal{R}_{g^2} \) and \( g \in S \), let \( \mathbb{P}_{h_0,g} = \{ h_0 g^k ; k \in \mathbb{N} \} \), they form a countable partition of \( F_\infty \setminus \{ \varepsilon \} \), we may index it with \( \mathbb{Z}^* = \mathbb{Z} \setminus \{ 0 \} \). We still denote the root of \( h_1(n) \) and put \( h_0(n) = h_0 \), \( k_n = k \) if \( \mathbb{P}_n = \mathbb{P}_{h_0,g_k} \). By definition \( h_0(n) \in \mathcal{R}_{g_k}^\perp \) if \( h_1(n) \in \mathcal{R}_{g_k}^\pm \).

**Lemma 4.17.** Let \( T_n \) be the \( L^2 \)-projection onto \( \mathbb{P}_n \) described above, we have for any \( p \geq 2 \), \( x \in L^p \).

\[
\| (\sum_{n \in \mathbb{Z}^*} |T_n x|^2)^{\frac{1}{2}} \|_{L^p} \lesssim c_p \| x \|_p.
\] (30)

**Proof.** We may assume \( \tau x = 0 \). Let \( E_k \) be the projection from the group von Neumann algebra \( \mathcal{L}(F_\infty) \) onto the von Neumann algebra generated by \( \lambda_{g_k} \). We can easily verify that for \( k \in \mathbb{Z}^* \)

\[
E_{|k|} |R_{g_k} x|^2 = E_{|k|} \sum_{h_1(n) \in R_{g_k}} T_n x|^2 = \sum_{h_1(n) \in R_{g_k}} |T_n x|^2,
\]

because, if \( h_1(n), h_1(n') \in R_{g_k} \), then \( h_0(n), h_0(n') \in R_{g_k}^\perp \) and \( h_0^{-1}(n) h_0(n') \in E_{|k|}(F_\infty) \) iff \( n = n' \). Therefore,

\[
\sum_{n \in \mathbb{Z}^*} |T_n x|^2 = \sum_{k=1}^{\infty} E_k (|R_{g_k} x|^2 + |R_{g-k} x|^2) = \tau |x|^2 + \sum_{k=1}^{\infty} E_k (|R_{g_k} x|^2 + |R_{g-k} x|^2).
\]

By the free Rosenthal inequality (Theorem A in [JPX07]) for length one polynomials, we get for \( p \geq 4 \), with \( X_k = E_k (|R_{g_k} x|^2 + |R_{g-k} x|^2) \).

\[
\| (\sum_n |T_n x|^2)^{\frac{1}{2}} \|_p \lesssim \tau |x|^2 + \left( \sum_{k \in \mathbb{N}} \|X_k\|_{\ell^p_{\varepsilon}}^{\frac{2}{p}} \right)^{\frac{1}{2}} + \left( \sum_{k \in \mathbb{N}} \|X_k\|_{\ell^2}^{\frac{2}{2}} \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( \sum_{k \in \mathbb{Z}^*} \|R_{g_k} x\|_{\ell^p_{\varepsilon}}^{\frac{2}{p}} \right)^{\frac{1}{2}} + \left( \sum_{k \in \mathbb{Z}^*} \|R_{g_k} x\|_{\ell^2}^{\frac{2}{2}} \right)^{\frac{1}{2}}
\]

\[
\lesssim \| (\sum_{k \in \mathbb{Z}^*} |R_{g_k} x|^2)^{\frac{1}{2}} \|_p \lesssim c_p \| x \|_p.
\]

Where we used the obvious facts by interpolation that \( L^p(\ell^p_{\varepsilon}) \to \ell^p(L^p) \) and \( L^p(\ell^2_{\varepsilon}) \to \ell_4(L_4) \) are contractions. The case of \( p = 2 \) is obvious. We then get the estimate for all \( 2 \leq p < \infty \) by interpolation. \( \square \)

Let \( \mathbb{P}_j = \{ h_1(j) < h_2(j) < \cdots h_k(j) < \cdots \} \) be arbitrary geodesic paths of \( F_\infty \). For \( x_j = \sum_{k \in \mathbb{N}} c_k \lambda_{h_k(j)} \) supported on \( \mathbb{P}_j \), we consider its dyadic parts \( M_{n,j} x = \sum_{2^n \leq k < 2^{n+1}} c_k \lambda_{h_k(j)} \).

\[
M_{n,j} x = \sum_{2^n \leq k < 2^{n+1}} c_k \lambda_{h_k(j)}.
\] (31)
Dealing with $F_1 = Z$ with the $N \cap \{0\}$ and $-N$ as geodesic paths, the classical Littlewood-Paley theory says that

$$\left\| \left( \sum_{n=1}^{\infty} |M_{n,1}x|^2 + |M_{n,2}x|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim_{\epsilon_p} \|x\|_{L^p}$$

(32)

for all $1 < p < \infty$ and $x \in C_c(Z)$.

In [JMP16], the authors proved a “smooth” one sided version of (32) for $p > 2$ and for $x$ supported on a geodesic path. The following theorem is a “truncated” version of it and says a little more.

**Theorem 4.18.** For $x_j$ supported on geodesic paths $\mathbb{P}_j$, we have

$$\left\| \left( \sum_{n,j=1}^{\infty} |M_{n,j}x_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq Cp^2 \left\| \left( \sum_{j} |x_j|^2 \right)^{\frac{1}{2}} \right\|_p$$

(33)

for all $2 \leq p < \infty$.

**Proof.** As usual $g_1, g_2, \ldots$ are the free generators of $F_\infty$. We embed $F_\infty$ into the free product $F_\infty \ast F_\infty$ and denote by $g'_1, g'_2, \ldots$ the generators of the second copy of $F_\infty$. Let $y_j = \lambda_{g'_1(h_1(j))}x_j$. The $y_j$’s are supported on disjoint paths $\mathbb{P}_j \subset F_\infty \ast F_\infty$ with roots of distinct generators $g'_j$. Note $|x_j|^2 = |y_j|^2$ and $|M_{n,j}x_j|^2 = |M_{n,j}y_j|^2$. By considering $y_j$ instead, we may assume $\mathbb{P}_j = \{h_1(j) < h_2(j) < \cdots h_k(j) \cdots \}$ with $|h_k(j)| = k$ and $L_{h_k(j)}x_m = 0$ for $j \neq m$.

Let

$$M_{\varphi, n,j} = 2^{1-\frac{2}{p}} \sum_{2^{n-1} < k \leq 2^n} L_{h_k(j)}A^{-\frac{1}{2}} + \sum_{2^n < k \leq 2^{n+1}} (\sqrt{k} - \sqrt{k-1})L_{h_k(j)}A^{-\frac{1}{2}} - 2^{-\frac{2}{p}} \sum_{2^n < k \leq 2^{n+2}} L_{h_k(j)}A^{-\frac{1}{2}}.$$

Then $M_{\varphi, n,j}(\lambda_{h_i(m)}) = 0$ unless $m = j$ and $l \in (2^{n-1}, 2^{n+2})$, and one can check that

$$M_{\varphi, n,j}(\lambda_{h_i(j)}) = \varphi_n(l)\lambda_{h_i(j)},$$

for some $\varphi_n : N \rightarrow \mathbb{R}$ with $\chi[2^n, 2^{n+1}) \leq \varphi_n \leq \chi(2^{n-1}, 2^{n+2})$. Note

$$M_{\varphi(n), j} = \sum_{2^{n-1} < k \leq 2^{n+2}} a_{k,j}L_{h_k(j)}A^{-\frac{1}{2}}$$

with $\sum_k a_{k,j} \leq c$. By the convexity of the operator valued function $|\cdot|^2$, we have

$$|M_{\varphi, n,j}x_j|^2 \leq c \sum_{2^{n-1} < k \leq 2^{n+2}} |L_{h_k(j)}A^{-\frac{1}{2}}x_j|^2,$$

and $M_{\varphi, n,j}x_m = 0$ for $m \neq j$. Note $M_{\varphi, n,j}, M_{\varphi, n', j}$’s are disjoint for $|n - n'| \geq 2$. Applying Lemma 4.14 to $x = \sum j x_j \otimes c_j$, we obtain,

$$\left\| \left( \sum_{n=0}^{\infty} |M_{\varphi, n,j}x_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq c \left\| \left( \sum_{k,j} |L_{h_k(j)}A^{-\frac{1}{2}}x_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq c p^2 \left\| \left( \sum_{j} |x_j|^2 \right)^{\frac{1}{2}} \right\|_p.$$
Assume $h_{2n}(j) \in R_{g_{n,j}}, h_{2n+1}(j) \in R_{g_{n',j}}$, we have that
\[
\lambda h_{2n+1}(j) L_{g_{n',j}} \lambda h_{2n+1}(j) - 1 (\lambda h_{2n-1}(j) L_{g_{n,j}} \lambda h_{2n-1}(j) - 1 M_{\phi, x_j}) = M_n x_j, \tag{34}
\]
because $h_{2n+1+1}(j) \in R_{g_{n'+1}}$. By (10),
\[
\| (\sum_{n=1}^{\infty} |M_{n,j} x_j|^2)^{\frac{1}{2}} \|_p \leq c \| (\sum_{n=1}^{\infty} |M_{\phi,n,j} x_j|^2)^{\frac{1}{2}} \|_p \leq c p \| (\sum_{j} |x_j|^2)^{\frac{1}{2}} \|_p
\]
for all $2 \leq p < \infty$.

Let $M_{n,k} x = M_{n,k} T_k x$ for the $T_k$ in Lemma 4.17. We obtain the following from Theorem 4.18 and duality:

**Corollary 4.19.** For all $2 \leq p < \infty$, and $x \in L^p$
\[
\max \left\{ \| (\sum_{n,k} |M_{n,k} x|^2)^{\frac{1}{2}} \|_p, \| (\sum_{n,k} |M_{\phi,n,k} x|^2)^{\frac{1}{2}} \|_p \right\} \leq C p^2 c_p^3 \| x \|_p, \tag{35}
\]
for all $1 \leq p < 2$, and $x \in L^p$
\[
\| x \|_p \leq C p^2 c_p^3 \inf \left\{ \| (\sum_{n,k} |M_{n,k} y|^2)^{\frac{1}{2}} \|_p + \| (\sum_{n,k} |M_{n,k} z|^2)^{\frac{1}{2}} \|_p; x = y + z \right\}.
\]

**Acknowledgment.** The first author would like to thank Marius Junge for helpful discussions. An initial argument for Theorem 4.11 was obtained during a visit to him at Urbana-Champaign.

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