Magnetic Field Effects on the
Superconducting and Quantum Critical
Properties of Layered Systems with Dirac
Electrons

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Abstract

We study the effects of an external magnetic field on the superconducting properties of a quasi-two-dimensional system of Dirac electrons at an arbitrary temperature. An explicit expression for the superconducting gap is obtained as a function of temperature, magnetic field and coupling parameter ($\lambda R$). From this, we extract the $B \times \lambda R$, $T \times \lambda R$ and $B \times T$ phase diagrams. The last one shows a linear decay of the critical field for small values thereof, which is similar to the behavior observed experimentally in the copper doped dichalcogenide $Cu_xTiSe_2$ and also in intercalated graphite. The second one, presents a coupling dependent critical temperature $T_c$ that resembles the one observed in high-$T_c$ cuprates in the underdoped region and also in $Cu_xTiSe_2$. The first one, exhibits a quantum phase transition connecting a normal and a superconducting phase, occurring at a critical line that corresponds to a magnetic field dependent critical coupling parameter. This should be observed in planar materials containing Dirac electrons, such as $Cu_xTiSe_2$.

Key words: Dirac electrons, superconductivity, quantum criticality

1 Introduction

A lot of attention has been devoted recently to quasi-two-dimensional condensed matter systems presenting a band structure such that the dispersion relation of the active electrons corresponds to the one of a relativistic massless particle. The kinematics of such electrons is described by a Dirac instead of a Schrödinger term in the hamiltonian [1,2]. This fact has a profound impact on the physical properties of the system. The Fermi surface reduces to a point, the Fermi point, where the density of states vanishes. This will drastically...
affect the physical properties of such materials. In a previous work [3], we investigated the superconducting properties of a quasi-two-dimensional system of Dirac electrons and showed that they are completely different from the ones presented by usual Schrödinger electrons. There is, in particular, a quantum phase transition connecting the normal and superconducting phases, which is controlled by the magnitude of the effective superconducting interaction coupling parameter. The Cooper theorem, therefore, is no longer valid, as one should expect in the absence of a Fermi surface.

Among the materials presenting Dirac electrons as their elementary excitations, there are a few, which have been intensely focused lately. These are high-Tc cuprates [4,5,6,7], graphene [8,9,10], carbon nanotubes [11] and transition metal dichalcogenides [12,13].

The interplay between superconductivity and magnetism is a subject of central interest in any research involving superconducting materials. In particular, a key issue, both from the basic and applied physics points of view is the analysis of the effects of an external magnetic field on the superconducting properties of a system. In the present work, we study the effects of an applied constant magnetic field, perpendicular to a quasi-two-dimensional superconducting system containing Dirac electrons. As investigation method, we use the effective potential for the superconducting order parameter, which we evaluate both at $T = 0$ and $T \neq 0$, as a function of the applied field ($B$).

We firstly consider the zero temperature case and explicitly obtain the superconducting gap as a function of the magnetic field and of the renormalized effective physical coupling parameter ($\lambda_R$) that controls the superconducting interaction. This allows us to obtain the ($T = 0$) ($B \times \lambda_R$) phase diagram, which presents a quantum critical line separating the normal and superconducting phases. A renormalization group analysis is then performed, demonstrating that the physical results do not depend on the renormalization point.

We then study the nonzero temperature regime and explicitly obtain the superconducting gap as a function of the temperature and of the applied field. From this, we extract an implicit equation relating the critical temperature with the critical magnetic field, $T_c(T_c, B_c)$. This allows us to obtain the $T \times \lambda_R$ phase diagram, which shows the critical temperature line as a function of the coupling, for different values of the magnetic field. This line resembles the one observed in the underdoped regime of high-$T_c$ cuprates and also in transition metal dichalcogenides [14].

Also from $T_c(T_c, B_c)$, we can derive the $B \times T$ phase diagram, for different values of the coupling parameter $\lambda_R$. The critical line separating the normal and superconducting phases, in this case, shows a linear decay of the critical field, for small values of this field. This type of behavior has been reported recently
in the experimental study of the copper doped transition metal dichalcogenide $Cu_xTiSe_2$ [14] and also in intercalated graphite compounds [15]. Since both systems potentially possess Dirac electrons as elementary excitations, it is conceivable that the peculiar properties of such electrons would be responsible for this common behavior of the critical field. In connection to this point, it would be extremely interesting to measure the critical magnetic field as a function of doping in $Cu_xTiSe_2$, in order to compare with the result derived from our $B \times \lambda_R$ phase diagram.

2 Model

We investigate here the effect of applying a constant magnetic field $B\hat{z}$ along the c-axis of a quasi-two-dimensional superconducting electronic system containing two Dirac points. Assuming that the active electrons correspond to these points, it follows that the electron kinematics will be described by a Dirac equation [1]. The electron creation operator will be $\psi^\dagger_{i,\sigma,a}$, where $i = 1, 2$ denotes the Dirac point and $\sigma = \uparrow, \downarrow$, the z-component of the spin. $a = 1, ..., N$ is an extra label, identifying the plane to which the electron belongs. Alternatively, in a multi-band system, $a$ could be used to specify the electron band.

The Lagrangian describing the system, in the presence of the external magnetic field is given by

$$L = i \bar{\psi}_{\sigma a} \left[ \hbar \partial_0 + v_F \gamma^i \left( \hbar \partial_i + \frac{e}{c} A_i \right) \right] \psi_{\sigma a} - \psi^\dagger_{\sigma a} \left( \mu_B \vec{B} \cdot \vec{\sigma} \right) \psi^\dagger_{\sigma a} + g \left( \psi^\dagger_{1\uparrow a} \psi^\dagger_{2\downarrow a} + \psi^\dagger_{1\downarrow a} \psi^\dagger_{2\uparrow a} \right) (\psi_{2\downarrow b} \psi_{1\uparrow b} + \psi_{1\downarrow b} \psi_{2\uparrow b})$$

(1)

where $A_i$ is the vector potential corresponding to $\vec{B}$, $\vec{\sigma}$ are Pauli matrices and $\mu_B$ is the Bohr magneton. The second and third terms, respectively, contain the coupling of the magnetic field to orbital and spin degrees of freedom. As in [3], we assume there is an effective superconducting interaction whose origin will not influence the results of this work. $g$ is the superconducting coupling constant, which is supposed to depend on some external control parameter. In order to make the lagrangian smooth, we define $g \equiv \lambda/N$. We use the same convention for the Dirac matrices as in [3].

Performing a Hubbard-Stratonovich transformation we arrive at

$$L [\Psi, \sigma] = -\frac{1}{g} \sigma^* \sigma + \Psi^\dagger_a A_a \Psi_a \quad (2)$$
where
\[
\mathcal{A} = \begin{pmatrix}
\tilde{\partial}_0 & -\tilde{\partial}_- & 0 & \sigma \\
-\tilde{\partial}_+ & \tilde{\partial}_0 & \sigma & 0 \\
0 & \sigma^* & \tilde{\partial}_0 & \tilde{\partial}_+ \\
\sigma^* & 0 & -\tilde{\partial}_- & \tilde{\partial}_0,
\end{pmatrix}
\] (3)

with \( \tilde{\partial}_0 \equiv i (\hbar \partial_0 + \mu_B B) \), \( \tilde{\partial}_\pm \equiv i v_F (\hbar \partial_\pm + i (e/c) A_\pm) \) and \( \partial_\pm = \partial_2 \pm i \partial_1 \). The fermions are in the form of a Nambu field \( \Psi^+_a = (\psi_{1\uparrow}^a, \psi_{2\uparrow}^a, \psi_{1\downarrow}^a, \psi_{2\downarrow}^a) \) and the auxiliary Hubbard-Stratonovitch field \( \sigma \) satisfies the equation,
\[
\sigma = -g (\psi_{2\downarrow}^a \psi_{1\uparrow}^a + \psi_{1\downarrow}^a \psi_{2\uparrow}^a).
\] (4)

This shows that \( \sigma^\dagger \) is a Cooper pair creation operator.

Integrating on the fermion fields, we obtain the effective action for \( \sigma \), namely
\[
S_{\text{eff}} (|\sigma|, B) = \int d^3x \left( -\frac{N}{\lambda} |\sigma|^2 \right) - i N \ln \det \left[ \mathcal{A} [\sigma, B] \right].
\] (5)

3 Effect of a magnetic field on the superconducting quantum phase transition at \( T = 0 \)

The effective potential per \( N \) corresponding to (5) may be obtained by a saddle point procedure, in which the field configurations assume their ground state average values that correspond to the classic lowest energy configurations.

Observeing that the determinant of the matrix \( \mathcal{A} \) is given by
\[
\det \mathcal{A} = \left[ \tilde{\partial}_0^2 - \tilde{\partial}_+ \tilde{\partial}_- + |\sigma|^2 \right]^2
\] (6)

and choosing the asymmetric gauge, in which \( \vec{A} = B(0, x) \), we infer from the effective action (5) that the effective potential will be
\[
V_{\text{eff}} (|\sigma|, B) = \frac{|\sigma|^2}{\lambda} - \int \frac{d^2k}{(2\pi)^2} \int \frac{d\omega}{2\pi} \ln \left\{ \frac{\det \mathcal{A} [\sigma, B]}{\det \mathcal{A} [\sigma = 0, B = 0]} \right\},
\] (7)
where (6), in momentum-frequency space, is given by
\[
\det A[\sigma, B] = \left( (\hbar \omega + \mu_B B)^2 + \hbar^2 v_F^2 \left[ k_x^2 + \left( k_y + \frac{e}{c} B \langle x \rangle \right)^2 \right] + |\sigma|^2 + B\kappa \right)^2,
\]
with \( \kappa = v_F^2 \hbar (e/c) \).

In this expression, by \( \sigma \), we mean \( \langle 0 | \sigma | 0 \rangle \). Analogously \( \langle x \rangle \) is the average of the \( x \)-coordinate in the lowest energy state of the relativistic Landau problem [16].

The effective potential in the presence of a magnetic field may be obtained by decomposing the logarithm in two parts and performing the shift of integration variables \( k_y + (e/c) B \langle x \rangle \rightarrow k_y \) and \( \hbar \omega + \mu_B B \rightarrow \hbar \omega \), in the first term. Introducing the momentum cutoff \( \Lambda/v_F \), we obtain, up to a constant,
\[
V_{\text{eff}} \left( |\sigma|, B \right) = |\sigma|^2 \lambda_R - f(B, \sigma_0) \lambda_c - \frac{2}{3} \alpha \left( |\sigma|^2 + B \kappa \right)^{\frac{3}{2}},
\]
where \( \alpha = 2\pi v_F^2 \).

The divergence can be eliminated as usual by the renormalization
\[
\left. \frac{\partial^2 V_{\text{eff}}}{\partial \sigma \partial \sigma^*} \right|_{|\sigma| = \sigma_0} \equiv \frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{\Lambda}{\alpha} + \frac{f(B, \sigma_0)}{\lambda_c},
\]
where \( \lambda_R \) is the renormalized coupling parameter, \( \lambda_c = 2\alpha/3\sigma_0 \) and
\[
f(B, \sigma_0) = \frac{1 + \frac{2B\kappa}{3\sigma_0^2}}{\sqrt{1 + \frac{B\kappa}{\sigma_0^2}}}. \tag{11}
\]

In the above equations, \( \sigma_0 \) is an arbitrary scale, the renormalization point. In the next section, we perform a renormalization group analysis, which shows that physical quantities do not depend on \( \sigma_0 \).

Using (10) in (9), we obtain the renormalized effective potential
\[
V_{\text{eff}, R} \left( |\sigma|, B \right) = \frac{|\sigma|^2}{\lambda_R} - \frac{f(B, \sigma_0)}{\lambda_c} |\sigma|^2 + \frac{2}{3} \alpha \left( |\sigma|^2 + B \kappa \right)^{\frac{3}{2}} \tag{12}
\]
We may now determine the \( T = 0 \) phase diagram as a function of the magnetic field by analyzing the minima of the effective potential above. For this purpose, we consider the derivatives of \( V_{\text{eff}, R} \) with respect to \( |\sigma| \), namely
\[
V'_{\text{eff}, R} \left( |\sigma|, B \right) = 2\sigma \left( \frac{1}{\lambda_R} - \frac{f(B, \sigma_0)}{\lambda_c} + \frac{1}{\alpha} \sqrt{|\sigma|^2 + B \kappa} \right). \tag{13}
\]
and

\[
V''_{\text{eff,R}}(|\sigma|, B) = 2\left(\frac{1}{\lambda_R} - \frac{f(B, \sigma_0)}{\lambda_c} + \frac{1}{\alpha \sqrt{|\sigma|^2 + B\kappa}}\right) + \left(\frac{2|\sigma|^2}{\alpha \sqrt{|\sigma|^2 + B\kappa}}\right).
\]

(14)

The minima of the effective potential will occur for the solutions of

\[
V''_{\text{eff,R}}(|\sigma|, B) = 0, \quad V'_{\text{eff,R}}(|\sigma|, B) > 0.
\]

We, henceforth, call these solutions \(\Delta\). Observe that

\[
\Delta = |\langle 0 | \sigma | 0 \rangle| \quad \text{and is, therefore, a superconducting order parameter.}
\]

The peculiarities of the superconducting transition for \(T \neq 0\) in 2D are well-known and have been discussed extensively in the literature [17].

From (14) we see that a solution \(\Delta = 0\) exists for

\[
\lambda_R < \lambda_c(B) = \lambda_c \frac{\sqrt{1 + \tilde{B}}}{1 + \frac{\alpha}{2} \tilde{B} - \frac{2}{3} \sqrt{\tilde{B}(1 + \tilde{B})}}.
\]

(15)

where \(\tilde{B} = B(\kappa/\sigma_0^2)\).

A solution with \(\Delta \neq 0\) will only occur for \(V'_{\text{eff,R}}(\Delta, B) = 0\). From (13), we find

\[
\Delta_0 = \sqrt{\alpha^2 \left(\frac{f(B, \sigma_0)}{\lambda_c} - \frac{1}{\lambda_R}\right)^2 - B\kappa}.
\]

(16)

Since the first term of (14) vanishes at this solution, we can readily see that \(V''_{\text{eff,R}}(\Delta_0, B) > 0\), provided \(\Delta_0\) is real, namely, for \(\lambda_R > \lambda_c(B)\), where \(\lambda_c(B)\) is given by (15). We see that for this range of the renormalized coupling parameter, the superconducting gap (16) is a true minimum of the effective potential.

The conclusion is that a quantum phase transition connecting a normal and a superconducting phase occurs at the magnetic field dependent quantum critical point \(\lambda_c(B)\), given by (15). Notice that in the limit \(B \to 0\) both the quantum critical point and the superconducting gap reduce to the ones found previously [3] in the absence of a magnetic field. Conversely, for each value of the physical coupling parameter \(\lambda_R\), there is a critical magnetic field above which superconductivity is destroyed. This observation allows us to infer the zero temperature phase diagram of the system, which is depicted in Fig. 1.

Recent advanced techniques of controlled intercalation of \(Cu\) in \(TiSe_2\) enabled the attainment of the experimental measure of important physical parameters as a function of doping in this layered dichalcogenide [14]. It would be interesting to compare this phase diagram with corresponding experimental results of the critical field as a function of doping in \(Cu_xTiSe_2\).
We have shown in [3], that the mean-field phase structure obtained at zero magnetic field is robust against quantum fluctuations. The same arguments apply here and, therefore, we reach the same conclusion in the presence of an applied magnetic field.

4 Renormalization Group Analysis

In order to eliminate the high-momentum divergence, we have renormalized the theory by performing the subtraction (10) at the arbitrary scale $\sigma_0$. In this section, we show that the physical quantities obey renormalization group equations which show that they are actually independent of $\sigma_0$.

By inserting the expressions of $\lambda_c$ and of $f(B, \sigma_0)$, in (12), we can show, for instance, that the effective potential satisfies the renormalization group equation

$$\left(\sigma_0 \frac{\partial}{\partial \sigma_0} + \beta \frac{\partial}{\partial \lambda_R}\right) V_{\text{eff,R}} = 0,$$

(17)

where the $\beta$-function, defined by

$$\beta \equiv \sigma_0 \frac{\partial \lambda_R}{\partial \sigma_0},$$

(18)

is given by

$$\beta = \frac{\lambda_c^2 \left(1 + \frac{4}{3} \tilde{B}\right)}{\lambda_c \left(1 + \tilde{B}\right)^{\frac{3}{2}},}$$

(19)

where $\tilde{B} = B(\kappa/\sigma_0^2)$.

In analogous fashion, we can show that the superconducting gap (16) also satisfies the renormalization group equation (17), being therefore independent of $\sigma_0$.

Finally, by integrating the $\beta$-function equation (18), we obtain the result

$$\frac{1}{\lambda_R(\sigma_0')} - \frac{f(\sigma_0', B)}{\lambda_c(\sigma_0', B)} = \frac{1}{\lambda_R(\sigma_0'') - \frac{f(\sigma_0'', B)}{\lambda_c(\sigma_0'', B)},}$$

(20)

for arbitrary scales $\sigma_0'$ and $\sigma_0''$. This explicitly shows that the combination

$$\frac{1}{\lambda_R(\sigma_0)} - \frac{f(\sigma_0, B)}{\lambda_c(\sigma_0, B)}$$

(21)

does not depend on the renormalization scale $\sigma_0$. From (16), then we explicitly see the scale independence of the superconducting gap $\Delta_0$. 

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We conclude that the physical properties of the system will be determined by the physical coupling parameter $\lambda_R$, which should be an experimental input. The value of the zero-magnetic-field quantum critical point $\lambda_c$ must also be determined experimentally. Subsequently, the magnetic field dependence of the quantum critical point may be obtained from (15).

5 Effect of a magnetic field on the superconducting phase transition at $T \neq 0$

Let us consider now the effect of an external magnetic field in the superconducting properties of a quasi-two-dimensional system of Dirac electrons for $T \neq 0$. In this case, the effective potential corresponding to (5) is given by

$$V_{\text{eff}} (|\sigma|, B) = \frac{||\sigma||^2}{\lambda} - 2 \int \frac{d^2k}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \left\{ \ln \left[ \left( \hbar \omega_n + \mu_B B \right)^2 + \hbar^2 v_F^2 |k|^2 + ||\sigma||^2 + B\kappa \right] - \ln \left[ \hbar \omega_n^2 + v_F^2 |k|^2 \right] \right\},$$

(22)

where $\omega_n$ are fermionic Matsubara frequencies, corresponding to the functional integration over the electron field. Observe that now we may no longer shift away the magnetic field coupled to the electron spin. This appears summed to the Matsubara frequencies.

The phase diagram at $T \neq 0$ is obtained from the minima of (22). For the purpose of determining these minima, we consider the necessary condition $V'_{\text{eff}} (|\sigma|, B) = 0$. Taking the derivative of (22) with respect to $|\sigma|$ and performing the Matsubara sum, we get

$$V'_{\text{eff}} (|\sigma|, B) = 2|\sigma| \left\{ \frac{1}{\lambda} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E} \left[ \frac{\sinh (\beta E)}{\cosh (\beta E) + \cosh (\beta \mu_B B)} \right] \right\} = 0,$$

(23)

where $E = \sqrt{v_F^2 k^2 + ||\sigma||^2 + B\kappa}$.

Let us look for a superconducting phase. In such a phase, a nonzero solution of (23) (which we call $\Delta$) is required. For this, the quantity between round brackets in (23) must vanish. After a change of variables, this condition leads to the equation for the superconducting gap $\Delta(T)$:

$$1 = \frac{\lambda}{\alpha} \int_{\Delta^2 + B\kappa}^{\Lambda} dE \frac{\sinh (\beta E)}{\cosh (\beta E) + \cosh (\beta \mu_B B)},$$

(24)

where $\Lambda/v_F$ is the high-momentum cutoff.
The second derivative of the effective potential evaluated at the solution of (24) is given by

$$V''_{\text{eff}}(\Delta, B) = \frac{2\Delta^2}{\alpha\beta\sqrt{\Delta^2 + \kappa B}} \left[ \frac{\sinh \beta \sqrt{\Delta^2 + \kappa B}}{\cosh \beta \sqrt{\Delta^2 + \kappa B} + \cosh \beta \mu B} \right] > 0. \quad (25)$$

This guarantees that the solution of (24) is indeed a minimum of the effective potential.

In order to solve (24), we perform the integration in \(E\). This may be done exactly and the result depends on the cutoff \(\Lambda\). It is a well known fact that the inclusion of a finite temperature does not change the divergence structure of the theory. Indeed, the cutoff may be eliminated precisely by the same renormalization operation (10), which we used at \(T = 0\). After this, we obtain the following expression for the superconducting gap \(\Delta(T, B)\), as a function of the temperature and of the magnetic field,

$$\Delta^2(T, B) = \left\{ k_B T \cosh^{-1} \left[ \frac{e^{\frac{\sqrt{\Delta_0^2 + B \kappa}}{k_B T}}}{2} - \cosh \left( \frac{\mu B}{k_B T} \right) \right] \right\}^2 - B \kappa, \quad (26)$$

where \(k_B\) is the Boltzmann’s constant and \(\Delta_0 \equiv \Delta(T = 0, B)\) is the zero temperature gap, given by (16). It is not difficult to show from the above equation that indeed, for \(T \to 0\), we have \(\Delta(T) \to \Delta_0\), as it should.

Now, from the gap expression we may extract the critical temperature for the superconducting transition, \(T_c\). Indeed, using the fact that \(\Delta(T_c) = 0\) in (26), we get,

$$k_B T_c = \frac{\sqrt{\Delta_0^2 + B \kappa}}{\ln \left\{ 2 \left[ \cosh \left( \frac{\sqrt{\Delta_0^2 + B \kappa}}{k_B T_c} \right) + \cosh \left( \frac{\mu B}{k_B T_c} \right) \right] \right\}} \quad (27)$$

Notice that for \(B = 0\), we have \(\Delta_0(0)/k_B T_c(0) = 2 \ln 2\) in agreement with the result obtained in [3].

We may then re-express the gap as a function of \(T\) and \(T_c\) as

$$\Delta^2(T, B) = \left\{ k_B T \cosh^{-1} \left[ 2^{\frac{T_c}{T} - 1} \left[ \cosh \left( \frac{\sqrt{\Delta_0^2 + B \kappa}}{k_B T_c} \right) + \cosh \left( \frac{\mu B}{k_B T_c} \right) \right]^{T_c/T} - \cosh \left( \frac{\mu B}{k_B T} \right) \right] \right\}^2 - B \kappa, \quad (28)$$

from which we confirm that the gap vanishes at \(T = T_c\).

In order to display the superconducting gap as a function of the temperature for different values of the magnetic field, we insert the expression for \(\Delta_0\), given by (16), in (26). Our results for \(\lambda_R/\lambda_c = 2\) are shown in Fig. 2 (the numerical results we present were obtained using \(v_F^2 = 1.69 \times 10^8 \, \text{(m/s)}^2\) and
σ₀ = \( \frac{8}{3} \ln 2 k_B T_c(0) \), which yields \( T_c(0) = 2 \) K for \( \lambda_R/\lambda_c = 2 \) in the absence of magnetic field.

Moreover, by replacing (16) in (27) we may obtain \( T_c \) self-consistently as a function of the coupling parameter. Our results are depicted in Fig. 3, which is the \( T \times \lambda_R \) phase diagram of the system for different values of the magnetic field. As the magnetic field increases, the quantum critical point is shifted to the right. Our results suggest that this behavior may be experimentally verified for the copper-doped dichalcogenide \( Cu_xTiS_2e_2 \) [14].

From (27), we can also obtain the critical magnetic field as a function of the coupling parameter for different values of temperature, namely, the \( B \times \lambda_R \) phase diagram, which is shown in Fig. 4. We see that for very low temperatures, it reduces to the same phase diagram displayed in Fig. 1. However, as the temperature is increased, the superconducting region, which corresponds to the area below \( B_c \), becomes smaller.

From (27), we may also obtain the \( B \times T \) phase diagram for the quasi-two-dimensional superconducting Dirac electronic system. This is represented in Fig. 5. Particularly interesting is the linear behavior of the critical magnetic field for \( B \gtrsim 0 \). We may derive explicitly from (27) the following expression for the critical field, in the small \( B \) region:

\[
B_c(T) \sim \frac{8 \ln 2 k_B^2}{A \kappa} T_c^2(0) \left( 1 - \frac{T}{T_c(0)} \right),
\]

where

\[
A = 1 - \frac{3}{4 \ln 2} \left( 1 - \frac{\lambda_c}{\lambda_R} \right),
\]

and

\[
k_B T_c(0) = \frac{3 \sigma_0}{4 \ln 2} \left( 1 - \frac{\lambda_c}{\lambda_R} \right).
\]

\( B_c(T) \) exhibits a linear behavior of the critical field, which is indicated by the dotted lines for different values of the dimensionless parameter \( x \equiv \lambda_R/\lambda_c \) in Fig. 5 (this is actually valid for \( \lambda_R < 13 \lambda_c \), when \( A \) is positive). This differs from the quadratic behavior predicted by BCS theory.

A linear decay of the critical field with the temperature similar to the one obtained here has been experimentally observed in intercalated graphite compounds [15] and also in the copper-doped dichalcogenide \( Cu_xTiS_2e_2 \) [14]. Since graphene and also the transition metal dichalcogenides are well-known to possess Dirac electrons in their spectrum of excitations, one is naturally led to wonder whether the presence of such electrons could explain such a behavior of the critical field.
It is interesting to observe that in expressions (26), (27) and (28) for the gap and $T_c$, we can trace back the contributions from the spin and orbital couplings of the external magnetic field. These are given, respectively by the $\mu_B$ and $\kappa$ proportional terms.

6 Conclusion

Starting from the explicit expression for the superconducting gap as a function of temperature, magnetic field and of the effective superconducting coupling parameter, we have derived three phase diagrams for a superconducting quasi-two-dimensional system of Dirac electrons.

The $T \times \lambda_R$ diagram displays a transition temperature with the same qualitative behavior as the critical temperature $T_c$ (as a function of doping) observed in high-$T_c$ cuprates in the underdoped region. It is conceivable that the effective coupling parameter $\lambda_R$, controlling the magnitude of the superconducting interaction, could be effectively determined by the amount of doping. In this case a direct comparison between our curve and the $T_c \times$ doping experimental results for the cuprates would be possible. Since the coupling magnitude should increase with doping, the curve of $T_c$ as a function of doping would have the same qualitative form as a function of the coupling $\lambda_R$, which by its turn would qualitatively agree with ours. The qualitative agreement with the cuprates data might be an indication that Dirac electrons play an important role in the mechanism of high-$T_c$ superconductivity. It would be interesting to investigate the magnetic field dependence of the superconducting dome in cuprates and especially of the $T = 0$ critical doping determining the onset of superconductivity.

The $B \times T$ phase diagram, by its turn shows a linear decay of the critical field as a function of temperature, for small fields, which differs from the corresponding quadratic behavior predicted by BCS theory. It is quite interesting that two materials that are supposed to have Dirac electrons as elementary excitations, namely, the copper-doped dichalcogenide $Cu_xTiSe_2$ [14] and intercalated graphite [15], both present the same type of linear behavior of the upper critical field as a function of temperature, in the small field regime. This might indicate that Dirac electrons are the common cause of this behavior in both materials. We intend to explore this point more profoundly in a next publication.

Finally, the $B \times \lambda_R$ phase diagram presents a quantum critical line, corresponding to a magnetic field dependent quantum critical parameter $\lambda_c(B)$, connecting the normal and superconducting phases. Again, we may assume a direct connection between the coupling parameter $\lambda_R$ and the doping parame-
ter, for instance, in the case of $Cu_xTiSe_2$. In this case, we would actually have in Fig. 4 a phase diagram $B \times x$, where $x$ is the doping parameter. It would be extremely interesting to have experimental curves of the critical magnetic field as a function of doping in dichalcogenides such as $Cu_xTiSe_2$, in order to compare with our theoretical results.

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References

[1] G.Semenoff, Phys. Rev. Lett. 53 (1984) 2449; F.D.M.Haldane, Phys. Rev. Lett. 61 (1988) 2015; P.W.Anderson, cond-mat/9812063 (unpublished)

[2] S.G.Sharapov, V.P.Gusynin and H.Beck, Phys. Rev. B 69 (2004) 363; V.P.Gusynin and S.G.Sharapov, Phys. Rev. B 71 (2005) 125124; Phys. Rev. Lett. 95 (2005) 146801; Phys. Rev. B 73 (2006) 245411

[3] E.C.Marino and L.H.C.M. Nunes, Nucl. Phys. B741 [FS] (2006) 404

[4] S.H.Simon and P.A.Lee, Phys. Rev. Lett. 78 (1997) 1548; A.C.Durst and P.A.Lee, Phys. Rev. B 62 (2000) 1270;

[5] E.J.Ferrer, V.P.Gusynin and V. de la Incera, Mod. Phys. Lett. B 16 (2002) 107

[6] F.Herbut, Phys. Rev. Lett. 88 (2002) 047006

[7] M.Franz and Z.Tešanović, Phys. Rev. Lett. 84 (2000) 554, Phys. Rev. Lett. 87 (2001) 257003;

[8] Y.Zhang et al., Nature 438 (2005) 201; K.S.Novoselov et al., Nature 438 (2005) 197

[9] J.Gonzalez, F.Guinea and M.A.H.Vozmediano, Phys. Rev. Lett.69 (1992) 172; Nucl. Phys. B406 (1993) 771; Nucl. Phys. B424 (1994) 595; Phys. Rev. Lett. 77 (1996) 3589; Phys. Rev. B 59 (1999) 2474; Phys. Rev. B 63 (2001) 134421;

[10] N.M.R.Peres, F.Guinea and A.H.Castro Neto, Ann. Phys. 321 (2006) 1559; J.Nilsson et al., Phys. Rev. B73 (2006) 214418; N.M.R.Peres, A.H.Castro Neto and F.Guinea, Phys. Rev. B73 (2006) 239902; ibid. 241403; 245426; 195411; 205408; 125411; V.M.Pereira et al. Phys. Rev. Lett. 96 (2006) 036801
[11] L.Balents and M.P.A.Fisher, Phys. Rev. B 55 (1997) R 11973
[12] A.H.Castro Neto, Phys. Rev. Lett. 86 (2001) 4382
[13] B.Uchoa, A.H.Castro Neto and G.G.Cabrera, Phys. Rev. B 69 (2004) 144512; B.Uchoa, G.G.Cabrera and A.H.Castro Neto, Phys. Rev. B 71 (2005) 184509
[14] E.Morosan et al., cond-mat/0606529 (2006) (to appear in Nature Physics)
[15] N.Emery et al. Phys. Rev. Lett. 95 (2005) 087003; T.E.Eller et al. Nature Physics 1 (2005) 39; M.Ellerby et al., Physica B - Condensed Matter, 378-380 (2006) 378
[16] C.G.Beneventano and E.M.Santangelo, J. Phys. A: Mathematical and General 39 (2006) 6137, ibid. 7457.
[17] E.Witten, Nucl. Phys. B145 (1978) 110; E.Babaev, Phys. Lett. B497 (2001) 323; D.V.Khveshchenko and H.Leal, Nucl. Phys. B687 [FS] (2004) 323
Fig. 1. The zero temperature phase diagram of the system.
Fig. 2. The superconducting gap $\Delta / k_B$ as a function of the temperature for several values of the magnetic field.
Fig. 3. $T \times \lambda_R$ phase diagram. The superconducting critical temperature $T_c$ as a function of the dimensionless coupling parameter $\lambda_R/\lambda_c$ for several values of magnetic field.
Fig. 4. $B \times \lambda_R$ phase diagram. The critical magnetic field $B_c$ as a function of the dimensionless coupling parameter $\lambda_R/\lambda_c$ for several values of the temperature.
Fig. 5. $B \times T$ phase diagram. The critical magnetic field $B_c$ as a function of the temperature for several values of the dimensionless coupling parameter $x \equiv \lambda_R/\lambda_c$. The dotted lines in the figure indicate the linear behavior of $B$ given by (29) as $B \to 0$. 