RATIONAL HOMOTOPY AND SIMPLY-CONNECTED 8-MANIFOLDS

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Abstract. We introduce a rational homotopy invariant $\mathcal{P}$ of a topological space, which is a quintic tensor on the cohomology. For $n \geq 2$, formality of a closed $(n-1)$-connected manifold of dimension up to $5n - 2$ is equivalent to vanishing of $\mathcal{P}$ and the Bianchi-Massey tensor introduced by Crowley and the second author. We show also that elements of the group $\Theta_8(V^r S^2)$ of closed simply-connected spin 8-manifolds with the cohomology of an $r$-fold connected sum of $S^2 \times S^6$ are determined up to torsion by the value of $\mathcal{P}$.

1. Introduction

We introduce and study a new rational homotopy invariant $\mathcal{P}$ of a topological space $X$, which we call the pentagonal Massey tensor. It is a linear map of degree $-2$ to $H^*(X)$ from a subspace of the fifth tensor power of $H^*(X)$, and captures information similar to 4-fold Massey products. It is similar in style to the Bianchi-Massey tensor introduced by Crowley and the second author (see [4,2]). The definition of $\mathcal{P}$ will be given in Sections 2 and 4, and we summarise our main results below.

Throughout the paper all homology and cohomology will be with rational coefficients unless otherwise indicated.

1.1. 8-manifolds. Our main motivation for defining the pentagonal Massey tensor $\mathcal{P}$ is to study simply-connected 8-manifolds. The simplest closed simply-connected manifolds where $\mathcal{P}$ can be non-trivial are 8-manifolds with the homology of a connected sum of some copies of $P$. such a manifold amounts to a linear functional $S$ $\sim$, and it was computed by the first author [6] to be $8$ manifolds. The simplest closed simply-connected manifolds where $\mathcal{P}$ can be non-trivial are 8-manifolds with the homology of a connected sum of some copies of $S^2 \times S^6$. For such a manifold $M$ the pentagonal Massey tensor is a well-defined linear map $R(H^2(M)) \to H^8(M)$ (described in [2,2]), where $R$ is defined as follows.

Definition 1.1. For a vector space $V$ let $P^kV$ and $\Lambda^kV$ denote the degree-$k$ components of the polynomial and exterior algebra respectively, i.e. the quotients of $V^\otimes k$ by the relation of symmetry or antisymmetry. We define $R(V)$ to be the kernel of

$$m : V \otimes \Lambda^2 P^2 V \to P^2 V \otimes P^3 V, \quad q \otimes (xy \wedge zw) \mapsto xy \otimes zwq - zw \otimes xyq \quad (1)$$

Note that $R$ is a functor. By Lemma 2.1, if $\dim V = r$, then $\dim R(V) = 6(r^2 + 2)$.

We will denote by $\Theta_8(V^r S^2)$ the set of diffeomorphism classes of spin (polarized) 8-manifolds with the homology of $\#^r S^2 \times S^6$ (see Definition 3.4). It is a group under “connected sum along the 2-skeleton”, and it was computed by the first author [6] to be

$$\Theta_8(V^r S^2) \cong \mathbb{Z}^{a} \oplus (\mathbb{Z}/2)^{b}, \quad \text{where } a = 6(r^2 + 2) \text{ and } b = 2(r+3) - (r-1) + 2.$$ 

For a closed oriented 8-manifold $M$ the fundamental class determines an identification $H^8(M) \cong \mathbb{Q}$, so $\mathcal{P}$ amounts to a linear functional

$$\mathcal{P}(M) \in R(H^2(M))^\vee = \text{Hom}(R(H^2(M)), \mathbb{Q}).$$

Further, a manifold $M$ in $\Theta_8(V^r S^2)$ by definition comes with an identification $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^r$ (so $H_2(M) \cong \mathbb{Q}^r$ and $H^2(M) \cong (\mathbb{Q}^r)^\vee$), hence in this setting $\mathcal{P}(M)$ can be regarded as an element of $R((\mathbb{Q}^r)^\vee)^\vee \cong R(\mathbb{Q}^r)$. The first aim of this paper is to compute $\mathcal{P}$ (or, equivalently, $\mathcal{P}$) for elements of $\Theta_8(V^r S^2)$. We find that $\mathcal{P}$ captures all the non-torsion information, so in particular it determines elements of $\Theta_8(V^r S^2)$ up to finite ambiguity. More precisely, we prove in §3.3 that

Theorem 1.2. $\mathcal{P}$ defines an isomorphism $\Theta_8(V^r S^2) \otimes \mathbb{Q} \cong R(\mathbb{Q}^r)$.
1.2. Relation to fourfold Massey products. Much of the theory still works nicely even if we go from manifolds in $\Theta_8(\mathbb{V}^\perp S^2)$ to allowing $H_4$ to be non-trivial.

**Definition 1.3.** A closed manifold $M$ is called an E-manifold if it is simply-connected and its homology is concentrated in even dimensions, that is, $H_{2k+1}(M;\mathbb{Z}) \cong 0$ for every $k$.

**Example 1.4.** Examples of E-manifolds include homotopy spheres (in even dimensions), complex and quaternionic projective spaces, Bott-manifolds and complex projective complete intersections (in even complex dimensions).

In particular, $\mathcal{P}$ is still uniquely defined for an 8-dimensional E-manifold, although the domain $R(H^2)$ is replaced by $D := (H^2 \otimes \Lambda^2 E) \cap R(H^2)$, where $E$ is the kernel of the cup product map $P^2 H^2 \rightarrow H^4$. Moreover, the ambiguities in the definition of fourfold Massey products also remain more manageable than in general.

**Definition 1.5.** For a vector space $V$ we define the multilinear map $\star : V^5 \rightarrow V \otimes \Lambda^2 P^2 V$ by

$$\star(x_1, x_2, x_3, x_4, x_5) = \sum_{cyc} x_1 \otimes (x_2x_3 \wedge x_4x_5).$$

Note that $\star(x_1, x_2, x_3, x_4, x_5) \in R(V)$.

Note also that if $M$ is an 8-dimensional E-manifold and the elements $x_1, x_2, x_3, x_4, x_5 \in H^2(M)$ satisfy $x_1x_2 = x_2x_3 = x_3x_4 = x_4x_5 = x_5x_1 = 0 \in H^4(M)$, then $\star(x_1, x_2, x_3, x_4, x_5) \in D$. In \S 3.4 we prove the following:

**Proposition 1.6.** For any 8-dimensional E-manifold $M$ the fourfold Massey products are completely determined by $\mathcal{P}$ via the formula

$$\langle x_1, x_2, x_3, x_4, x_5 \rangle \in D \quad \text{if} \quad H_4(M) = 0 \quad \text{then also} \quad \mathcal{P} \quad \text{is determined by the fourfold Massey products.}$$

We can thus also say that elements of $\Theta_8(\mathbb{V}^\perp S^2)$ are determined up to $2^2(\kappa^2)^{-1} + 2$ possibilities by their fourfold Massey products.

1.3. The role of $\mathcal{P}$ in rational homotopy classification. Now we move on to considering $\mathcal{P}$ for more general closed manifolds. The general version of $\mathcal{P}$ is defined in \S 4.4. In \[2\] it was shown that for $n \geq 2$, the rational homotopy type of a closed $(n-1)$-connected manifold of dimension up to $5n - 3$ is determined by its cohomology algebra and Bianchi-Massey tensor $\mathcal{F}$. In particular, such a manifold is formal if and only if $\mathcal{F} = 0$. For an $(n-1)$-connected manifold of dimension $\geq 5n - 2$, fourfold Massey products and the pentagonal Massey tensor can be non-trivial. In the borderline case, we show in \S 4.2 that $\mathcal{P}$ can be used to characterise formality.

**Theorem 1.7.** For $n \geq 2$, a closed $(n-1)$-connected manifold of dimension $5n - 2$ is formal if and only if $\mathcal{F} = 0$ and $\mathcal{P} = 0$.

While the Bianchi-Massey tensor is always uniquely defined, $\mathcal{P}$ has a slightly complicated dependence on cochain choices in general (described in Theorem 4.4.10). However, if $\mathcal{F} = 0$ then $\mathcal{P}$ is uniquely defined too (Remark 5.3), so the hypothesis in Theorem 1.7 is unambiguous.

**Remark 1.8.** Combining Theorems 1.2 and 1.7 gives that the subset of formal elements of $\Theta_8(\mathbb{V}^\perp S^2)$ is precisely the torsion subgroup.

The term “formality” derives from the rational homotopy type of a formal space being a “formal consequence” of the cohomology algebra. In particular, Theorem 1.7 means that if $\mathcal{F}$ and $\mathcal{P}$ of a $(n-1)$-connected $(5n - 2)$-manifold vanish then its rational homotopy type is determined by the cohomology algebra. We expect that if we weaken the hypotheses of Theorem 1.7 to allow non-zero $\mathcal{P}$, then the rational homotopy type is determined by the cohomology algebra together with $\mathcal{P}$.
Conjecture 1.9. Let $F : H^*(X) \to H^*(Y)$ be an isomorphism of the cohomology algebras of closed $(n-1)$-connected manifolds of dimension $\leq 5n-2$ with trivial Bianchi-Massey tensor. Then $F$ is realised by a rational homotopy equivalence if and only if $F$ intertwines the pentagonal Massey tensors.

If the Bianchi-Massey tensor is allowed to be non-zero then it is a little more intricate to formulate the correct statement because of the ambiguities in $\mathcal{P}$, see Conjecture 5.11. Since a closed $(n-1)$-connected manifold of dimension $\leq 6n-4$ cannot have non-trivial Massey products of order greater than 4 we expect that the situation is largely the same in that dimension range, but it may be necessary to add a slight generalisation of the Bianchi-Massey tensor as discussed in Remark 5.10.

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2. The degree 8 component of $\mathcal{P}$

We describe the pentagonal Massey tensor $\mathcal{P}$ in general in §4. Let us now describe the component relevant for closed simply-connected 8-manifolds. In this section we will work solely in the algebraic setting.

2.1. The representation $R(V)$. Let us first make some brief remarks about the domain of $\mathcal{P}$. Given a vector space $V$, we defined $R(V) = \ker \mathfrak{m}$ in Definition 1.1 which can be thought of as a representation of $GL(V)$. Recall that any irreducible representation of $GL(V)$ is a Weyl module $S(\lambda)$ indexed by a partition $\lambda$ of an integer $n$ (for which the representation embeds in $V^{\otimes n}$).

Lemma 2.1. (i) If $V$ is a rational vector space of dimension $r$, then $\dim R(V) = 6(r^2 + 2)$. (ii) $R(V)$ is the irreducible representation $S(3,1,1)$ of $GL(V)$ (i.e. the Weyl module corresponding to the partition $5 = 3 + 1 + 1$).

Proof. Both claims follow from the observation that the image of $\mathfrak{m}$ is precisely the kernel of the symmetrisation map $P^3V \otimes P^2V \to P^5V$, which is easy to establish by picking a basis for $V$: modulo the image of $\mathfrak{m}$, a monomial $(qxy)(zw)$ in basis elements is equal to $(qzw)(xy)$, and it is clear that two monomials are related by a sequence of such swaps if and only if they involve the same 5 basis elements, i.e. the images in $P^5V$ are the same monomial.

(i) The exactness of $0 \to R(V) \to V \otimes \Lambda^2 P^2V \to P^3V \otimes P^2V \to P^5V \to 0$ allows us to evaluate

$$\dim R(V) = r \left( \frac{(r+1)}{2} \right) - \left( \frac{r+2}{3} \right) \left( \frac{r+1}{2} \right) + \left( \frac{r+4}{5} \right).$$

(ii) The other terms in the exact sequence are easy to decompose into irreducibles, see Fulton and Harris [5 §6]. $P^3V = S(5)$, and by Pieri’s formula $P^3V \otimes P^2V = S(5) \oplus S(4,1) \oplus S(3,2)$, so $\text{Im} \mathfrak{m}$ must be $S(4,1) \oplus S(3,2)$. On the other hand Pieri’s formula also gives $\Lambda^2 P^2V = S(3,1)$ (see [5 Exercise 6.16]) and hence $V \otimes \Lambda^2 P^2V = S(4,1) \oplus S(3,2) \oplus S(3,1,1)$, so $R(V)$ is $S(3,1,1)$. Having established (ii), one could of course alternatively deduce (i) by applying a standard formula [5 Theorem 6.3] for the dimension of a Weyl module. \hfill \Box

2.2. Definition of the degree 8 component of $\mathcal{P}$. Let $A$ be a DGA over $\mathbb{Q}$ (e.g. $\Omega^*_PL(X)$ for a topological space $X$), with cohomology $H^*$. Let $E^4 = E_A \subseteq P^2 H^2$ be the kernel of the product map $P^2 H^2 \to H^4$, and let $\mathcal{D} = \mathcal{D}_A \subseteq H^2 \otimes \Lambda^2 E$ be the kernel of the restriction of $\mathcal{D}$; equivalently, $\mathcal{D}_A = (H^2 \otimes \Lambda^2 E) \cap R(H^4)$.

Remark 2.2. In terms of Definition 4.6 in the more general setting, $\mathcal{D}_A$ coincides with the degree 10 component of the graded vector space $\mathfrak{D}^*(H^*(A))$ associated to the graded algebra $H^*(A)$. 

Let $Z \subseteq A$ be the subalgebra of closed elements. Pick a right inverse $\alpha : H^2 \to Z^2$ for the projection to cohomology. Then the restriction of $\alpha^2 : P^2H^2 \to Z^4$ to $E$ takes exact values, so one may pick $\gamma : E \to A^3$ such that $\alpha^2E = d\gamma$.

**Proposition 2.3.** The restriction of $\alpha^2 : H^2 \otimes \Lambda^2E \to A^8$

$$\text{to } D \text{ takes closed values.}$$

We will clarify the notation and prove Proposition 2.3 in §2.3

**Definition 2.4.** The pentagonal Massey tensor is the linear map $\mathcal{P} : D \to H^8$

induced by the restriction of $\alpha^2$ to $D$.

In general $\mathcal{P}$ has a slightly subtle dependence on the choices of $\alpha$ and $\gamma$, as set out in Theorem 4.10. We find it useful to prove separately in the next subsection the following special case.

**Theorem 2.5.** Let $A$ be a DGA with $H^3 = 0$. Then $\mathcal{P}$ induces a well-defined linear map $\mathcal{P} : D \to H^8$, independent of the choices of $\alpha$ and $\gamma$.

**Example 2.6.** Suppose $X$ is a simply-connected 8-manifold with $H^3(X) = 0$, and $a, x_1, x_2, x_3 \in H^2(X)$ such that $ax_i = 0 \in H^4(X)$. Then $x_1 \otimes (ax_2 \wedge ax_3) + x_2 \otimes (ax_3 \wedge ax_1) + x_3 \otimes (ax_1 \wedge ax_2) \in D$. If $\alpha, \beta_i \in \Omega^2(X)$ represent $a$ and $x_i$ respectively and $\gamma_i \in \Omega^3(X)$ satisfy $d\gamma_i = \alpha\beta_i$, then the result of evaluating $\mathcal{P}$ on $x_1(ax_2 \wedge ax_3) + x_2(ax_3 \wedge ax_1) + x_3(ax_1 \wedge ax_2)$ is the cohomology class

$$[\beta_1\gamma_2\gamma_3 + \beta_2\gamma_3\gamma_1 + \beta_3\gamma_1\gamma_2] \in H^8(X).$$

As a special case of Theorem 2.5 (and Remark 2.9) we thus we recover the claim from Fernández–Muñoz [4 Lemma 3.1] that the class (3) is independent of the choices, and if it is non-zero then $X$ cannot be formal.

**Example 2.7.** Suppose $H^2 \cong \mathbb{Q}^3(x, y, z)$, $H^4 \cong \mathbb{Q}$ and the product $P^2H^2 \to H^4$ is given by the quadratic form $ax + by + cz \mapsto a^2 + b^2 + c^2$. An 8-manifold with such a cohomology algebra has the “hard Lefschetz property” in the sense that there is a class $\varphi \in H^4$ such that multiplication by $\varphi$ defines an isomorphism $H^2 \to H^6$.

By considering $E$ as the $SO(3)$-representation $\text{Sym}_2^2\mathbb{Q}^3$ and decomposing $H^2 \otimes \Lambda^2E$ into irreducible $SO(3)$-representations one finds that $D$ is 1-dimensional. One can take

$$\sum_{xyc} x \otimes (yz \wedge (y^2 - z^2) - xy \wedge xz)$$

as an explicit generator. One can use [2 Proposition 3.4(i)] to produce minimal DGAs where $\mathcal{P}$ is non-trivial on this generator and whose cohomology algebra satisfies 8-dimensional Poincaré duality, and Sullivan [7 Theorem 13.2] to realise such a DGA as the minimal model of a closed simply-connected 8-manifold. This shows that the result of Cavalcanti [4 Theorem 4] that $(n-1)$-connected 4n-manifolds with the hard Lefschetz property and $b_n \leq 2$ are formal cannot be extended to $b_n = 3$.

**2.3. Well-definedness of the degree 8 component of $\mathcal{P}$ when $H^3 = 0$.** We now prove Proposition 2.3 and Theorem 2.5. Let us first set up some notation to keep the calculations manageable.

For linear maps $\alpha, \beta : H^2 \to A^*$, let $\alpha \beta : P^2H^2 \to A^*$ denote the linear map induced by the symmetric bilinear map $H^2 \times H^2 \to A^*, (x, y) \mapsto \alpha(x)\beta(y) + \alpha(y)\beta(x)$. This product is graded symmetric in the sense that if $\alpha$ and $\beta$ take values in $A^r$ and $A^s$ respectively, then $\beta\alpha = (-1)^{rs}\beta\alpha$. If $\alpha$ takes values in the even degree part of $A$, then we can also define $\alpha^2 : P^2H^2 \to A^*$ to be induced by $(x, y) \mapsto \alpha(x)\alpha(y)$. Note that $\alpha\alpha = 2\alpha^2$. We will not notionally distinguish the restrictions of these maps to $E \subseteq P^2H^2$.

For $\gamma, \delta : E \to A^*$ (and $\gamma$ of odd degree $r$), define $\gamma \wedge \delta : \Lambda^2E \to A^*$ (and $\gamma^2$) analogously; this is graded antisymmetric in the sense that $\delta \wedge \gamma = (-1)^{1+r}\gamma \wedge \delta$. 


The right-hand side is again of the form (4) so that its restriction to $D$.

Thus the restriction of any map of the form (4) to $D$ will vanish.

**Proof of Proposition 2.3.** We now verify the claim that the map $αγ^2$ takes closed values on $D$. As maps $H^2 ⊗ Λ^3 E → A^6$,

$$d(αγ^2) = αdγ ∧ γ = αα^2 ∧ γ.$$  

The right-hand side is of the form (4) so that its restriction to $D$ vanishes. Thus $αγ^2$ maps $D → Z^8$ as claimed.

**Proof of Theorem 2.3.** We start by checking that, for any fixed choice of $α, P$ is independent of the choice of $γ$. If $γ'$ is a different choice, then $γ' - γ : E → A^1$ takes closed values. The hypothesis that $H^3 = 0$ forces the difference to be exact, i.e.

$$γ' = γ + dη$$

for some $η : E → A^2$. Then

$$α(γ')^2 - αγ^2 + d(α(γ ∧ η - 1/2 η ∧ dη))$$

$$= αγ ∧ dη + α(ηdη) + α(αdγ ∧ η) + (d(γ ∧ η) - (dη)^2)$$

$$= αdγ ∧ η = αα^2 ∧ η.$$  

The right-hand side is again of the form (4) so that its restriction to $D$ vanishes. Hence the restriction of $α(γ')^2 - αγ^2$ to $D$ vanishes as required.

It remains to show that given $α, γ$, and another choice $α'$, there is also a choice of $γ'$ so that the restrictions to $D$ of $αγ^2$ and $α'(γ')^2$ differ by exact values. We can write $α' = α + dβ$ for some $β : H^2 → A^1$, and then take $γ' = γ + σ_1 E$ for $σ = αβ + 1/2 βdβ : P^2 H^2 → A^3$. Then, as functions $H^2 ⊗ Λ^3 E → A^6$,

$$α'(γ')^2 - αγ^2 = d(β(α')^2) + d((1/2 αβ + 1/2 βdβ) ∧ γ)$$

$$= (αγ ∧ σ + αα^2 + β(α')^2 ∧ γ') + d((1/2 αβ + 1/2 βdβ)) ∧ γ + (1/2 σ + 1/12 βdβ) ∧ α^2$$

$$(5) = (ασ + α(α')^2 + d((1/2 αβ + 1/2 βdβ))) ∧ γ + (1/2 ασ + β(α')^2 - 1/2 βα^2) ∧ σ + 1/12 βdβ ∧ α^2.$$  

Expanding a map $H^2 ⊗ E → A^5$:

$$ασ + β(α')^2 + d((1/2 αβ + 1/2 βdβ))$$

$$= ααβ + 1/2 αβdβ + β(α^2 + αdβ + (dβ)^2) + dβ((1/2 αβ + 1/2 βdβ) - β(1/2 αdβ + 1/2 (dβ)^2)$$

$$= ααβ + βα^2 + 1/2(αβdβ + βαdβ + dβαβ) + 1/3 (βdβ)^2 + dβdβ.$$  

The right hand side factors through $P^3 H^2$. Hence the first term on the right hand side of (5) factors through $m$, so that its restriction to $D$ vanishes.

The remaining terms on the right-hand side of (5) expand to

$$(1/2 ααβ + 1/2 αβdβ + 1/2 βα^2 + βαdβ + β(dβ)^2) ∧ σ + 1/12 βdβ ∧ α^2$$

$$= (1/2 ααβ + 1/2 βα^2) ∧ σ + (1/2 αβdβ + βαdβ) ∧ αβ + 1/12 βdβ ∧ α^2$$

$$+ 1/4 α(βdβ)^2 + 1/4 βαdβ ∧ βdβ + β(dβ)^2 ∧ αβ + 1/2 (βdβ)^2 ∧ βdβ.$$
The first term on the right-hand side factors through $\mathfrak{m}$ as it is. The remaining terms can be arranged into 3 groups, each of which factors through $\mathfrak{m}$ after addition of an exact term.

$$\frac{1}{12}(\alpha\beta \delta + 3\alpha \beta \delta) \wedge \alpha \beta = \frac{1}{12}(\alpha\beta \delta + 3\alpha \beta \delta) \wedge \alpha \beta + \frac{1}{12}d(\beta \alpha \delta)$$

$$= -\frac{1}{12}(\alpha\beta \delta + 3\alpha \beta \delta) \wedge \alpha \beta + \frac{1}{2}d(\beta \alpha \delta) - \frac{1}{2}d(\beta \alpha \delta) \wedge \alpha \beta$$

and

$$\frac{1}{4}(\beta \delta \beta \beta + \beta \delta \alpha \delta + \beta \delta \alpha \delta) \wedge \alpha \beta = \frac{1}{2}(\beta \delta \beta \beta + \beta \delta \alpha \delta + \beta \delta \alpha \delta) \wedge \alpha \beta + \frac{1}{4}d(\beta \delta \beta \beta)$$

$$= \frac{1}{4}(\beta \delta \beta \beta + \beta \delta \alpha \delta + \beta \delta \alpha \delta) \wedge \alpha \beta + \frac{1}{4}(d\beta \delta \beta \beta + \beta \delta \alpha \delta) \wedge \alpha \beta$$

and

$$\frac{1}{4}(\beta \delta \beta \beta + \beta \delta \alpha \delta + \beta \delta \alpha \delta) \wedge \beta \delta \alpha \beta + \frac{1}{8}d(\beta \delta \beta \beta) \wedge \beta \delta \alpha \beta$$

In summary, $\alpha'(\gamma')^2 - \alpha_2$ is a sum of exact terms and terms that factor through $\mathfrak{m}$, so the restriction to $D$ is exact as required. This completes the proof of Theorem 2.3.

2.4. Naturality. Suppose that $f : A \to B$ is a morphism of DGAs, it induces a homomorphism

$$H^2(f) \otimes \Lambda^2 P^2 H^2(f) : H^2(A) \otimes \Lambda^2 P^2 H^2(A) \to H^2(B) \otimes \Lambda^2 P^2 H^2(B),$$

which restricts to a map $H^2(A) \otimes \Lambda^2 E_A \to H^2(B) \otimes \Lambda^2 E_B$. Another restriction of the same map is $R(H^2(f)) : R(H^2(A)) \to R(H^2(B))$. Hence by restricting $H^2(f) \otimes \Lambda^2 P^2 H^2(f)$ (or $R(H^2(f)))$ to $D_A$ we get a homomorphism $D_A = (H^2(A) \otimes \Lambda^2 E_A) \cap R(H^2(A)) \to D_B = (H^2(B) \otimes \Lambda^2 E_B) \cap R(H^2(B))$.

**Proposition 2.8.** If $H^2(f) : H^2(A) \to H^2(B)$ is an isomorphism, then for any choice of $P_A$ there is a choice of $P_B$ such that the diagram

$$\begin{array}{ccc}
D_A & \xrightarrow{P_A} & H^2(A) \\
\downarrow R(H^2(f)) & & \downarrow R(H^2(f)) \\
D_B & \xrightarrow{P_B} & H^2(B)
\end{array}$$

commutes.

**Proof.** The map $P_A$ is determined by the choice of $\alpha_A$ and $\gamma_A$ and $P_B$ is determined by the choice of $\alpha_B$ and $\gamma_B$. So it is enough to show that for any choice of $\alpha_A$ and $\gamma_A$ we can choose $\alpha_B$ and $\gamma_B$ such that the diagram

$$\begin{array}{ccc}
H^2(A) \otimes \Lambda^2 E_A & \xrightarrow{\alpha_A \gamma_A} & A^3 \\
\downarrow H^2(f) \otimes \Lambda^2 P^2 H^2(f) & & \downarrow f \\
H^2(B) \otimes \Lambda^2 E_B & \xrightarrow{\alpha_B \gamma_B} & B^3
\end{array}$$

commutes, which follows from the commutativity of the diagrams

$$\begin{array}{ccc}
H^2(A) & \xrightarrow{\alpha_A} & Z^2(A) \\
\downarrow H^2(f) & & \downarrow f \\
H^2(B) & \xrightarrow{\alpha_B} & Z^2(B)
\end{array} \quad \text{and} \quad \begin{array}{ccc}
E_A & \xrightarrow{\gamma_A} & A^3 \\
\downarrow f & & \downarrow f \\
E_B & \xrightarrow{\gamma_B} & B^3
\end{array}$$

Since $H^2(f)$ is an isomorphism, for any section $\alpha_A$ of the projection $Z^2(A) \to H^2(A)$ we can choose $\alpha_B = f \circ \alpha_A \circ H^2(f)^{-1}$. Then $\alpha_B$ is a section of $Z^2(B) \to H^2(B)$ and the first diagram commutes.

Since $P^2 H^2(f)$ is also an isomorphism, its restriction is an injective map $P^2 H^2(f)|_{E_A} : E_A \to E_B$, and $E_B = P^2 H^2(f)(E_A) \otimes E'$ for some subspace $E' \leq E_B$. Given a map $\gamma_A : E_A \to A^3$ such that $d\gamma_A = \alpha_A|_{E_A}$ we define $\gamma_B$ by $\gamma_B| P^2 H^2(f)(E_A) = f \circ \gamma_A \circ P^2 H^2(f)^{-1}$ and setting $\gamma_B|_{E'}$ to be any map such that $d\gamma_B|_{E'} = \alpha_B|_{E'}$. Then $d\gamma_B = \alpha_B|_{E_B}$ and the second diagram commutes too. \qed
Remark 2.9. If \( H^3(A) \cong H^3(B) \cong 0 \), then \( \mathcal{P}_A \) and \( \mathcal{P}_B \) are independent of choices by Theorem 2.5 and any morphism \( f : A \to B \) that induces an isomorphism \( H^2(f) : H^2(A) \to H^2(B) \) intertwines \( \mathcal{P}_A \) and \( \mathcal{P}_B \). It follows that in this case \( \mathcal{P} \) is invariant under quasi-isomorphisms, in particular it is an obstruction to formality.

3. The pentagonal Massey tensor of closed 8-manifolds

3.1. \( \mathcal{P} \) of spaces. First we define \( \mathcal{P}_X \) for a space \( X \).

Definition 3.1. Let \( X \) be a simply-connected space. Let \( E_X \leq P^2 H^2(X) \) denote the kernel of the cup product \( P^2 H^2(X) \to H^4(X) \), and let \( \mathcal{D}_X \leq H^2(X) \otimes \Lambda^2 E_X \) be the kernel of the restriction of \( \mathcal{D} \). We define \( \mathcal{P}_X : \mathcal{D}_X \to H^8(X) \) to be \( \mathcal{P}_{\Omega^* (\mathcal{X})} \), using the canonical identification between \( H^*(X) \) and \( H^*(\Omega^* (\mathcal{X})) \).

By Theorem 2.5 \( \mathcal{P}_X \) is well-defined if \( H^3(X) \cong 0 \). If \( H^3(X) \cong 0 \) and \( f : A \to \Omega^* (\mathcal{X}) \) is a quasi-isomorphism for some DGA \( A \), then by Proposition 2.8 the (well-defined) maps \( \mathcal{P}_A \) and \( \mathcal{P}_{\Omega^* (\mathcal{X})} \) fit into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_A & \xrightarrow{\mathcal{P}_A} & H^8(A) \\
\downarrow r(H^2(f)) & & \downarrow H^8(f) \\
\mathcal{D}_{\Omega^* (\mathcal{X})} & \xrightarrow{\mathcal{P}_{\Omega^* (\mathcal{X})}} & H^8(\Omega^* (\mathcal{X}))
\end{array}
\]

where the vertical maps are isomorphisms. This means that \( \mathcal{P}_X \) can be computed from any model of \( X \).

Proposition 3.2. Let \( f : X \to Y \) be a continuous map between simply-connected spaces such that \( H^2(f) : H^2(Y) \to H^2(X) \) is an isomorphism. Then for any choice of \( \mathcal{P}_Y \) there is a choice of \( \mathcal{P}_X \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{D}_Y & \xrightarrow{\mathcal{P}_Y} & H^8(Y) \\
\downarrow r(H^2(f)) & & \downarrow H^8(f) \\
\mathcal{D}_X & \xrightarrow{\mathcal{P}_X} & H^8(X)
\end{array}
\]

commutes.

Proof. Apply Proposition 2.8 to the induced map \( \Omega^* (\mathcal{X}) \to \Omega^* (\mathcal{Y}) \). \( \square \)

Definition 3.3. Let \( M \) be a simply-connected closed oriented 8-manifold with \( H^3(M) \cong 0 \). We define the canonical element \( \overline{\mathcal{P}}(M) \in \mathcal{D}_M \) by \( \overline{\mathcal{P}}(M) = \langle [M], \cdot \rangle \circ \mathcal{P}_M : \mathcal{D}_M \to \mathbb{Q} \), where \( \langle [M], \cdot \rangle : H^8(M) \to \mathbb{Q} \) denotes evaluation on the fundamental class \( [M] \in H_8(M) \).

3.2. The group \( \Theta_8(\sqrt[n]{S^2}) \). Recall the definition of E-manifolds (Definition 1.3). We also recall some definitions and results from [8], starting with the definition of \( \Theta_8(\sqrt[n]{S^2}) \):

Definition 3.4. For an integer \( r \geq 0 \) let

\[
\Theta_8(\sqrt[n]{S^2}) = \left\{ \langle M, \varphi \rangle \mid M \text{ is a spin } 8\text{-dimensional E-manifold} \right\} / \sim
\]

where \( \langle M, \varphi \rangle \sim \langle M', \varphi' \rangle \) if there is an orientation-preserving diffeomorphism \( F : M \to M' \) such that \( \varphi' = H_2(F; \mathbb{Z}) \circ \varphi \). The equivalence class of \( \langle M, \varphi \rangle \) will be denoted by [\( [M, \varphi] \)].

Remark 3.5. If we identify \( \mathbb{Z} \) with \( H_2(\sqrt[n]{S^2}; \mathbb{Z}) \), then the choice of an isomorphism \( \varphi : \mathbb{Z} \to H_2(M; \mathbb{Z}) \) determines a 4-connected map \( \varphi : \sqrt[n]{S^2} \to M \) (up to homotopy). Conversely, if \( M \) is a closed 8-manifold, then the existence and choice of a 4-connected map \( \varphi : \sqrt[n]{S^2} \to M \) ensures that \( M \) is an E-manifold with \( H_4 \cong 0 \) and determines an isomorphism \( \varphi : \mathbb{Z} \to H_2(M; \mathbb{Z}) \) respectively.

\( \Theta_8(\sqrt[n]{S^2}) \) is a group under “connected sum along the 2-skeleton”. If \( \langle M, \varphi \rangle \) represents an element of \( \Theta_8(\sqrt[n]{S^2}) \), then \( M \) has a unique String structure compatible with its orientation.
Definition 3.6. Let $P_3 = P_3(V^i S^2)$ denote the third Postnikov-stage of $V^i S^2$.

The identification $Z^r \cong H_2(V^i S^2; \mathbb{Z})$ induces identifications $Z^r \cong H_2(P_3; \mathbb{Z})$ and $(Q)^r \cong H^2(P_3)$.

Definition 3.7. Let $\Omega_8^{\text{string}}(P_3)$ denote the kernel of the homomorphism $\sigma : \Omega_8^{\text{string}}(P_3) \to \mathbb{Z}$ given by the signature.

Definition 3.8. The map $\eta : \Theta_8(V^i S^2) \to \Omega_8^{\text{string}}(P_3)$ is given by $\eta([M, \varphi]) = [f]$, where $f : M \to P_3$ is the composition of the natural map $M \to P_3(M)$ and the homotopy inverse of $P_3(\varphi) : P_3(V^i S^2) \to P_3(M)$, where $\varphi : V^i S^2 \to M$ is the map (well-defined up to homotopy) which induces the isomorphism $\varphi : Z^r = H_2(V^i S^2; \mathbb{Z}) \to H_2(M; \mathbb{Z})$.

Theorem 3.9 ([6 Proposition 2.2.34]). $\eta$ is a well-defined isomorphism.

Proposition 3.10. $\Omega_8^{\text{string}}(P_3) \otimes \mathbb{Q} \cong H_8(P_3)$.

Proof. Let $K_3$ denote the Eilenberg-MacLane space $K(\pi_3(V^i S^2), 3)$. The Postnikov-stage $P_3$ is the total space in a fibration $K_3 \to P_3 \to K(Z^r, 2)$, which allows us to compute $\Omega_8^{\text{string}}(P_3) \otimes \mathbb{Q}$ and $H_8(P_3)$ using Atiyah–Hirzebruch–Leray–Serre spectral sequences. The natural transformation $\Omega_8^{\text{string}}(-) \otimes \mathbb{Q} \to H_8$ of homology theories induces a morphism between these spectral sequences. We have $H^*(K_3) \cong (A(H^3(K_3)))^r$, hence $H_8(K_3) \cong (A(H^3(K_3)))^r$. From [6 Theorem 3.1.29] we see that the homomorphism $\Omega_8^{\text{string}}(K_3) \otimes \mathbb{Q} \to H_8(K_3)$ is an isomorphism for $i < 8$. Moreover, $\Omega_8^{\text{string}}(K_3) \otimes \mathbb{Q} \cong \Omega_8^{\text{string}} \otimes \mathbb{Q} \cong \mathbb{Q}$ (detected by the signature) and $H_8(K_3) \cong 0$. Therefore $\Omega_8^{\text{string}}(P_3) \otimes \mathbb{Q} \cong H_8(P_3) \oplus \mathbb{Q}$ and $\Omega_8^{\text{string}}(P_3) \otimes \mathbb{Q} \cong H_8(P_3)$.

By combining Theorem 3.9 and Proposition 3.10 we get the following:

Theorem 3.11. The map $\Theta_8(V^i S^2) \otimes \mathbb{Q} \to H_8(P_3)$, $[M, \varphi] \otimes 1 \mapsto H_8(f)([M])$ is an isomorphism, where $f : M \to P_3$ is the map from Definition 3.8 and $[M] \in H_8(P_3)$ is the fundamental class of $M$.

3.3 Computing $\mathcal{P}$ for $\Theta_8(V^i S^2)$. In order to understand $\mathcal{P}$ for elements of $\Theta_8(V^i S^2)$, we first consider it for their third Postnikov stage.

Theorem 3.12. We have $D_{P_3} = R(H^2(P_3))$ and the map $\mathcal{P}_{P_3} : D_{P_3} = R(H^2(P_3)) \to H^8(P_3)$ is an isomorphism.

Proposition 3.13. The Sullivan minimal model of $P_3$ is $(AV, d^\star)$, where $V = V^2 \oplus V^3$, $V^2 \cong H^2(P_3)$, $d^2 = 0$ and $d^3 : V^3 \to P^2 V^2$ is an isomorphism.

Proof. The Sullivan minimal model satisfies $V^i \cong \pi_i(P_3)^r \otimes \mathbb{Q}$, therefore $V^2 \cong \pi_2(V^i S^2)^r \otimes \mathbb{Q}$, $V^3 \cong \pi_3(V^i S^2)^r \otimes \mathbb{Q}$ and $V^2 \cong 0$ for $i \neq 2, 3$. Since $P_3$ is simply-connected, $H^2(AV, d^\star) \cong H^2(P_3) \cong \pi_2(V^i S^2)^r \otimes \mathbb{Q}$, hence $d^2 = 0$. We have $(AV)^4 \cong P^2 V^2$, so $d^3 = 0$, therefore $H^3(AV, d^\star) \cong \operatorname{Ker} d^3$ and $H^4(AV, d^\star) \cong \operatorname{Coker} d^3$. The Postnikov-stage $P_3$ can be constructed from $V^i S^2$ by adding cells of dimension 5 and above to kill homotopy groups in dimensions 4 and above, so $H^4(P_3) \cong H^4(P_3) \cong 0$. This implies that $d^3$ is an isomorphism.

Proof of Theorem 3.12: We will use the Sullivan minimal model $(AV, d^\star)$ of $P_3$ to compute $\mathcal{P}_{P_3}$. Since $H^4(P_3) \cong 0$, we have $E_{P_3} = P^2 H^2(P_3)$. This implies that $D_{P_3} = R(H^2(P_3))$. We have $Z^2(AV, d^\star) = V^2 \cong H^2(AV, d^\star)$, so $\alpha = \text{Id}_{P^2 V^2}$ is the only section of the projection $Z^2(AV, d^\star) \to H^2(AV, d^\star)$ (where $V^2$ is identified with $H^2(P_3)$). Since $d^4 = 0$, we have $Z^4(AV, d^\star) = (AV)^4 = P^2 V^2 = P^2 H^2(P_3)$, and $\alpha^2 = \text{Id}_{P^2 H^2(P_3)}$. We also saw that $d^3$ is an isomorphism, so $\gamma = (d^3)^{-1} : E_{P_3} = P^2 H^2(P_3) \to V^3$ is the only map satisfying $\alpha^2 = d^3 \gamma$.

We have

\begin{align*}
(\text{AV})^7 &= P^2 V^2 \otimes V^3 \\
(\text{AV})^8 &= P^4 V^2 \oplus (V^2 \otimes \Lambda^2 V^3) \\
(\text{AV})^9 &= (P^2 V^2 \otimes \Lambda^3 V^3)
\end{align*}
Corollary 3.14. We will show that this map is the composition of the isomorphisms in Theorem 3.11 on choices. We are therefore content to restrict attention to the context of \(x\) as an isomorphism.

\[\alpha^2 : H^2(P_3) \otimes \Lambda^2 P^2 H^2(P_3) \to (AV)^8\]

is the inclusion of the second component (using the identifications \(V^2 \cong H^2(P_3)\) and \(V^3 \cong P^2 H^2(P_3)\)). Therefore \(\mathcal{P}_{P_3} : R(H^2(P_3)) \to H^8(P_3)\), the map induced by the restriction of \(\alpha^2\), is an isomorphism.

Since \(H_8(P_3) \cong H^8(P_3)\), we immediately get the following:

Corollary 3.14. The map \(H_8(P_3) \to R(H^2(P_3)) \cong Q\) is an isomorphism.

Now we can prove the main result of this section:

Theorem 3.15. The map \(\Theta_8(V^1 S^2) \otimes Q \to R((Q\gamma^0))\) given by \([M, \varphi] \otimes 1 \mapsto ((\varphi^0\gamma^0)^{-1})^*(\mathcal{T}(M))\) is an isomorphism, where \(\mathcal{T}(M) \in D_M = R(H^2(M))\) is the canonical element defined in Definition 3.3 and \((\varphi^0\gamma^0)^{-1})^* : R(H^2(M)) \to R((Q\gamma^0))\) is the isomorphism induced by the inverse of \(\varphi^0 : H^2(M) \cong H_2(M) \to \gamma^0\). As the canonical element \(\mathcal{T}(M)\) is determined by \(\mathcal{P}_M\) and \((\varphi^0\gamma^0)^{-1})^*\) is just the identification of \(R(H^2(M))\) with the standard space \(R((Q\gamma^0))\) coming from \(\varphi\), this means that \(\mathcal{P}\) is a complete invariant of \(\Theta_8(V^1 S^2) \otimes Q\).

Proof. We will show that the composition of the isomorphisms in Theorem 3.11 and Corollary 3.14 (if \(H^2(P_3)\) is identified with \((Q\gamma^0))\). The composition maps \([M, \varphi] \otimes 1 \to (H_8(f)[(M)], \varphi) \otimes \mathcal{P}_{P_3}\). By the naturality of the cap product and of \(\mathcal{P}\) (see Proposition 3.2) we have

\[\Theta_8(f)[(M), \varphi) \otimes \mathcal{P}_{P_3} = (H_8(f)[(M), \varphi) \otimes \mathcal{P}_{P_3} = (H^2(f)[(M)]) \otimes \mathcal{T}(M) \circ R(H^2(f)) = \mathcal{T}(M) \circ R(H^2(f))\].

Since \(H_2(f[Z]) \otimes \varphi\) is the fixed identification \(Z^* \cong H_2(P_3; Z)\) (see Definition 3.8), this element is identified with \(\mathcal{T}(M) \circ R((\varphi^0\gamma^0)^{-1}) = ((\varphi^0\gamma^0)^{-1})^*(\mathcal{T}(M))\) when \(H^2(P_3)\) is identified with \((Q\gamma^0))\). □

4.4. Fourfold Massey products on 8-dimensional E-manifolds. We now consider the relation of \(\mathcal{P}\) to fourfold Massey products. Recall that more generally, for \(x_1, x_2, x_3, x_4 \in H^*(X)\) the fourfold Massey product is defined if \(x_1x_2 = x_2x_3 = x_3x_4 = 0\) and the triple products \((x_1, x_2, x_3)\) and \((x_2, x_3, x_4)\) vanish simultaneously, in the sense that it is possible to choose representatives of \(x_i \in Z^*\) and representative \(\gamma_{12}, \gamma_{23}, \gamma_{34} \in A^*\) for \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) such that \(\alpha_1\gamma_{23} + \alpha_2\gamma_{12}, \alpha_3\gamma_{23} + \alpha_4\gamma_{12}, \alpha_3\gamma_{34} + \alpha_4\gamma_{23}\) are both exact. Then the fourfold product \((x_1, x_2, x_3, x_4)\) is represented by

\[\alpha_1 \sigma_2 + \alpha_2 \gamma_{34} + \alpha_1 \alpha_4\]

whenever

\[d\sigma_1 = \alpha_1 \gamma_{23} + \alpha_2 \alpha_3, \quad d\sigma_2 = \alpha_2 \gamma_{34} + \alpha_2 \gamma_{23}\].

If \(x_5 \in H^*(X)\) that also \(x_4x_5 = x_5x_1 = 0\), then \((x_1, x_2, x_3, x_4)\) is closely related to the evaluation of \(\mathcal{P}\) on certain elements of \(D\). Recall from Definition 1.5 that the multilinear map \(\mathcal{T} : H^2(X) \otimes 5 \to R(H^2(X)) \leq H^2(X) \otimes \Lambda^2 P^2 H^2(X)\) is given by the formula

\[\mathcal{T}(x_1, x_2, x_3, x_4, x_5) = \sum_{\text{cyc}} x_1 \otimes (x_2 x_3 \wedge x_4 x_5).\]

Definition 3.16. We will call an element of \(D_X \leq R(H^2(X))\) ordinary if it can be obtained as \(\mathcal{T}(x_1, x_2, x_3, x_4, x_5)\) for some \(x_1, x_2, x_3, x_4, x_5 \in H^2(X)\) with \(x_1 x_2 = x_2 x_3 = x_3 x_4 = x_4 x_5 = x_5 x_1 = 0 \in H^4(X)\).

In general, the relation between the fourfold Massey products and the evaluation of \(\mathcal{P}\) on ordinary elements of \(D\) is complicated by the dependence of both objects, and in particular the fourfold Massey product, on choices. We are therefore content to restrict attention to the context of 8-dimensional E-manifolds \(M\). Then for \(x_1, x_2, x_3, x_4 \in H^2(M)\), the only choices in the definition of \((x_1, x_2, x_3, x_4) \in H^6(M)\) that affect the result are those of \(\sigma_1\) and \(\sigma_2 \in A^4\). Adding a closed form
to either of those has the effect of changing \( \langle x_1, x_2, x_3, x_4 \rangle \) by an element of \( x_1 H^4(M) + x_4 H^4(M) \). In particular, if \( x_5 \in H^2(M) \) has \( x_5 x_1 = x_4 x_5 = 0 \), then
\[
\langle x_1, x_2, x_3, x_4 \rangle x_5 \in H^8(M) \cong \mathbb{Q}
\]
is independent of all choices. Moreover, by Poincaré duality the set of possible values of \( \langle x_1, x_2, x_3, x_4 \rangle \) is completely determined by knowing the values of \( \langle x_1, x_2, x_3, x_4 \rangle x_5 \in H^8(M) \) for all such \( x_5 \). Those are determined by evaluating \( P \) on the ordinary elements:

**Proposition 3.17.** For any space \( X \) with \( H^3 = H^5 = 0 \) and any \( x_1, x_2, x_3, x_4, x_5 \in H^2(X) \) such that \( x_1 x_{i+1} = 0 \in H^4(X) \),
\[
\langle x_1, x_2, x_3, x_4 \rangle x_5 = P(\bigstar(x_1, x_2, x_3, x_4)) \in H^8(X).
\]

**Proof.** Since the LHS is independent of the choices, it suffices to prove the claim if we pick \( \alpha : H^2 \to \mathbb{Z}^2 \) and \( \gamma : E \to A^3 \) as in the definition of \( P \) and take \( \alpha_1 = \alpha(x_i) \), \( \gamma_{i+1} = \gamma(x_i x_{i+1}) \) in the definition of \( \langle x_1, x_2, x_3, x_4 \rangle \) (while choosing \( \sigma_1, \sigma_2 \) arbitrarily). Now
\[
[a_1 \sigma_2 + \gamma_{12} \gamma_{34} + \sigma_1 a_4] x_5 = [(d \gamma_{51}) \sigma_2 + \gamma_{12} \gamma_{34} a_5 + \sigma_1 d \gamma_{45}]
= [-d \gamma_{51} \sigma_2 + \gamma_{12} \gamma_{34} a_5 + (d \sigma_1) \gamma_{45}]
= [(a_2 \gamma_{34} + \gamma_{23} a_4) \gamma_{51} + \gamma_{12} \gamma_{34} a_5 + (a_1 \gamma_{23} + \gamma_{12} a_3) \gamma_{45}]
= \sum_{\text{cyc}} a_1 \gamma_{23} \gamma_{45}
\]
which equals \( P \) evaluated on \( \bigstar \). \qed

In summary, for any 8-dimensional E-manifold the fourfold Massey products are completely determined by \( P \). The reverse holds too if \( D \) is generated by ordinary elements, which is in fact the case for elements of \( \Theta_3(\mathbb{V}^r S^2) \). More generally, if the cup product \( H^2 \times H^2 \to H^4 \) is trivial, then \( D \) is simply \( R(H^2) \) and every element in the image of \( \bigstar \) is ordinary in the sense of Definition 3.16, so the next lemma implies that \( P \) is determined by the fourfold Massey products.

**Lemma 3.18.** For any finite-dimensional vector space \( V \), \( R(V) \) is generated by the image of \( \bigstar : V^{\times 3} \to R(V) \).

**Proof.** This amounts to proving surjectivity of the linear map \( V^{\otimes 5} \to R(V) \) induced by \( \bigstar \). Since that is a non-zero \( GL(V) \)-equivariant map and \( R(V) \) is irreducible by Lemma 2.1, it is surjective by Schur’s lemma. \qed

### 4. General definition of \( P \)

The definition and proof of well-definedness of \( P \) in general works much the same as in the 8-dimensional case, once we have set up the appropriate notation. However, in the discussion of the 8-dimensional case we used the hypothesis \( H^3 = 0 \) to completely eliminate dependence of \( P \) on choices of cochains. Without such a simplifying hypothesis we will need to describe a transformation rule for how \( P \) depends on the choices. The transformation rule is described in terms of uniform triple Massey products, and we will therefore also need to describe those.

#### 4.1. Products of algebra-valued maps.

For a graded algebra \( A \), graded vector spaces \( V \) and \( W \), and linear maps \( \alpha : V \to A \) and \( \beta : W \to A \), we can define an obvious product \( \alpha \beta : V \otimes W \to A \). But we also wish to introduce some notions of products of graded symmetric and antisymmetric maps.

For a graded vector space \( V \), let \( G^p V \) and \( G^p V \) denote the quotients of \( V \otimes p \) by relations of graded symmetry and graded antisymmetry respectively. Thus a linear map \( \alpha : G^p V \to A \) is essentially the same as a multilinear map \( \alpha : V^p \to A \) such that
\[
\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) = (-1)^{\sum d_j} \alpha(x_1, \ldots, x_p)
\]
for any permutation $\sigma$ and any $x_1, \ldots, x_p \in V$ of degrees $d_1, \ldots, d_p$, where the sum is taken over all $i < j$ such that $\sigma(i) > \sigma(j)$. Meanwhile a linear map $\alpha : \mathcal{G}^p V \to \mathcal{A}$ is equivalent to an $\alpha : V^p \to \mathcal{A}$ such that

$$
\alpha(x_{\sigma(1)}, \ldots , x_{\sigma(p)}) = (\text{sign } \sigma)(-1)^{\sum d_i} \alpha(x_1, \ldots, x_p) = (-1)^{\sum (1+d_i d_j)} \alpha(x_1, \ldots, x_p).
$$

Given $\alpha : \mathcal{G}^p V \to \mathcal{A}$ and $\beta : \mathcal{G}^q V \to \mathcal{A}$, of degree $r$ and $s$ respectively (i.e. $\alpha$ maps the degree $d$ part of the graded vector space $\mathcal{G}^p V$ to $\mathcal{A}^d$), we define a product $\alpha \beta : \mathcal{G}^{p+q} V \to \mathcal{A}$ (of degree $r + s$) as follows.

$$(\alpha \beta)(x_1, \ldots , x_{p+q}) = \frac{1}{p! q!} \sum_{\sigma \in S_{p+q}} \text{sign } \sigma (-1)^{\sum d_i} \alpha(x_{\sigma(1)}, \ldots , x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, \ldots , x_{\sigma(p+q)}).$$

**Remark.** The notation is a bit ambiguous in that $\alpha \beta$ could be interpreted as being defined on $\mathcal{G}^p V \otimes \mathcal{G}^q V$ or $\mathcal{G}^{p+q} V$, or also on $\mathcal{G}^2 \mathcal{G}^p V$ if $p = q$. Rather than introducing more cumbersome notation for the products themselves, we aim to disambiguate by specifying the domains clearly.

The symmetric product satisfies convenient versions of associativity, graded commutativity and (if $\mathcal{A}$ is a DGA) the Leibniz rule:

$$
\alpha(\beta \gamma) = (\alpha \beta) \gamma : \mathcal{G}^{p+q+r} V \to \mathcal{A}
$$

$$
\alpha \beta = (-1)^{rs} \beta \alpha : \mathcal{G}^{p+q} V \to \mathcal{A}
$$

$$
d(\alpha \beta) = (d\alpha) \beta + (-1)^r \alpha (d\beta) : \mathcal{G}^{p+q} V \to \mathcal{A}
$$

For $\alpha : \mathcal{G}^p V \to \mathcal{A}$ and $\beta : \mathcal{G}^q V \to \mathcal{A}$, we define $\alpha \wedge \beta : \mathcal{G}^{p+q} V \to \mathcal{A}$ analogously, i.e.

$$(\alpha \wedge \beta)(x_1, \ldots , x_{p+q})
= \frac{1}{p! q!} \sum_{\sigma \in S_{p+q}} (\text{sign } \sigma) (-1)^{\sum d_i} \alpha(x_{\sigma(1)}, \ldots , x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, \ldots , x_{\sigma(p+q)}).$$

This product enjoys the same associativity property and Leibniz rule as the graded symmetric one, and it is bigraded commutative in the sense that

$$
\alpha \wedge \beta = (-1)^{pq+rs} \beta \wedge \alpha : \mathcal{G}^{p+q} V \to \mathcal{A}.
$$

Finally, given $\alpha : V \to \mathcal{A}$ of degree $r$, the map $(x_1, \ldots , x_p) \mapsto \alpha(x_1) \cdots \alpha(x_p)$ is itself graded symmetric or antisymmetric, depending on whether $r$ is even or odd. We denote the resulting map $\mathcal{G}^p V \to \mathcal{A}$ by $\alpha^p$. Note that with this convention e.g. $6\alpha^3$ is $\alpha \alpha \alpha$ or $\alpha \wedge \alpha \wedge \alpha$; this convention is the right one if one wants the option to apply the set-up to free abelian groups rather than vector spaces.

### 4.2. Uniform triple products and the Bianchi-Massey tensor

The dependence of $\mathcal{P}$ on choices will be expressed in terms of the uniform triple products in the sense of [2, §2.3], which we now recall.

Let $\mathcal{A}$ be a DGA with homology $H^\ast$. As before, $Z^\ast \subseteq \mathcal{A}$ denotes the subalgebra of closed elements. Let $E^\ast \subseteq \mathcal{G}^2 H^\ast$ denote the kernel of the product map $\mathcal{G}^2 H^\ast \to H^\ast$.

**Definition 4.1.** A cochain choice is a pair $c = (\alpha, \gamma)$ where $\alpha : H^\ast \to Z^\ast$ is a right inverse for the projection to cohomology, and $\gamma : E^\ast \to A^{s-1}$ satisfies $d\gamma = \alpha^2_{\mathcal{E}}$.

Given two choices $c$ and $c'$, pick $\beta : H^\ast \to A^{s-1}$ such that $d\beta = \alpha' - \alpha$. Then $\gamma' - \gamma - \beta(\alpha + \frac{1}{2} d\beta)$ maps $E^\ast \to Z^{s-1}$, so induces a map $\delta(c', c) : E^\ast \to H^{s-1}$.

This $\delta$ can in turn depend on the choice of $\beta$. If we set $\beta' = \beta + \eta$ for some $\eta : H^\ast \to Z^{s-1}$, then $\beta'(\alpha + \frac{1}{2} d\beta') - \beta(\alpha + \frac{1}{2} d\beta) = \frac{1}{2} \eta(\alpha + \alpha')$, so induces the map $[\eta] : \mathcal{G}^2 H^\ast \to H^{s-1}$. But in any case, if we let

* $L_1 = \text{Hom}(H^\ast, H^{s-1})$, the space of degree $-1$ maps $H^\ast \to H^{s-1}$
* $L_2 = \text{Hom}(E^\ast, H^{s-1})$

and $\text{Id} L_1 \subseteq L_2$ the subspace of maps $E^\ast \to H^{s-1}$ that are restrictions of the symmetric product $\text{Id} n : \mathcal{G}^2 H^\ast \to H^{s-1}$ of $\text{Id} : H^\ast \to H^\ast$ and some $n : H^\ast \to H^{s-1}$, then $\delta$ is well-defined in $L_2 / \text{Id} L_1$. 

Definition 4.2. Say that two choices $c$ and $c'$ are equivalent if $\delta(c', c) = 0 \in L_2/\text{id}L_1$. The set of equivalence classes $\mathcal{C}$ then forms an affine vector space modelled on $L_2/\text{id}L_1$.

In particular, there is a cochain choice $c$ of $G$ with its domain. For any graded vector space $P$, let $F$ be a DGA with cohomology $H^*$ such that $F(\lfloor 2 \rfloor, \text{Lemma } 2.8)$.

Lemma 4.5. For an $m$-dimensional Poincaré DGA $A$, the top degree component $F : B^{m+1}(A) \to H^m(A)$ of the Bianchi-Massey tensor is equivalent to the uniform triple product.

Lemma 4.6. For a graded algebra $H^*$, if $E^* \subset \mathcal{G}^2 H^*$ denotes the kernel of the product map $\mathcal{G}^2 H^* \to H^*$, then we define $D^*(H^*)$ to be the kernel of the restriction of $m$ to $H^* \otimes \mathcal{G}^2 E^*$.

Lemma 4.7. Let $A$ be a DGA with cohomology $H^* = H^*(A)$. Given a choice $c = (\alpha, \gamma)$, the restriction of $\alpha \gamma^2 : H^* \otimes \mathcal{G}^2 \mathcal{G}^2 H^* \to A^{*-2}$ to $D = D(H^*)$ takes closed values.

Proof. Like in §2.3, we find that $d(\alpha \gamma^2) = \alpha(\alpha^2 \wedge \gamma + \frac{1}{2} \alpha^3 \gamma) \circ m : H^* \otimes \mathcal{G}^2 \mathcal{G}^2 H^* \to A^{*-1}$ and thus vanishes on $D$. \hfill \Box

Definition 4.8. Given a choice $c$, the pentagonal Massey tensor

$$\mathcal{P}_c : D^* \to H^{*-2}$$

is the degree $-2$ linear map induced by $\alpha \gamma^2$. 

\begin{align}
\mathcal{P}_c : D^* \to H^{*-2} \\
\alpha \gamma^2 : H^* \otimes \mathcal{G}^2 \mathcal{G}^2 H^* \to A^{*-2}
\end{align}
We now wish to understand the dependence of $P_c$ on $c$. Consider the natural inclusion
\[ j : G^2 H^* \to G^2 H^* \otimes G^2 H^*, \quad x \otimes y \mapsto xy - (-1)^{d_1d_2} yx. \] (11)

**Lemma 4.9.** The image of $D$ under $\text{Id} \, j : H^* \otimes G^2 H^* \to H^* \otimes G^2 H^* \otimes G^2 H^*$ is contained in $K[H^* \otimes E^*] \otimes E^*$ (where $K[H^* \otimes E^*]$ denotes the kernel of the full symmetrisation map $s$ from (3) like before).

**Proof.** $m$ is the composition of $\text{Id} \, j$ and $s \, \text{Id} : H^* \otimes G^2 H^* \otimes G^2 H^* \to G^2 H^* \otimes G^2 H^*$, so $\ker m$ is mapped to $(\ker s) \otimes G^2 H^*$.

For any linear map $\delta : E^* \to H^{*-1}$ Lemma 4.9 means that the composition
\[ (T_c \delta) \circ (\text{Id} \, j) : D^* \to H^{*-2} \]
of $\text{Id} \, j : D^* \to K[H^* \otimes E^*] \otimes E^*$ and $T_c \delta : K[H^* \otimes E^*] \otimes E^* \to H^{*-2}$ is well-defined.

**Theorem 4.10.** Let $c = (\alpha, \gamma)$ and $c' = (\alpha', \gamma')$ be two different choices, and let $\delta : E^* \to H^{*-1}$ be any representative of $\delta' - c$. Then
\[ P_{c'} - P_c = (T_c \delta) \circ (\text{Id} \, j) + \text{Id} \delta^2. \] (12)
(In particular the right-hand side is independent of the choice of representative $\delta$ of $c' - c$.)

As $P_{c+\delta+c} - P_{c+\delta} - P_{c+\epsilon} - P_c = \text{Id} \delta \wedge \epsilon$ is bilinear in $\delta$ and $\epsilon$ and independent of $c$, we can interpret the relation (12) to mean that $c \mapsto P_c$ is an "affine quadratic function" $C \to \text{Hom}(D^*, H^{*-2})$.

If $c$, $c'$ and $c''$ are three different choices, it is not immediately apparent that the expressions for $P_{c'} - P_c$ and $P_{c'} - P_c$ given by (12) add up to the expression for $P_{c''} - P_c$. As a sanity check, let us verify this directly. Suppose $c' - c$ is represented by $\epsilon : E^* \to H^{*-1}$. Then we want vanishing of
\[ T_c(\delta + \epsilon) \circ (\text{Id} \, j) + \text{Id} (\delta + \epsilon)^2 - T_c \epsilon \circ (\text{Id} \, j) - \text{Id} \epsilon^2 - T_c \delta \circ (\text{Id} \, j) - \text{Id} \delta^2 \]
\[ = (T_c - T_c) \epsilon \circ (\text{Id} \, j) + \text{Id} \delta \wedge \epsilon \]
\[ = (T_c - T_c) + \text{Id} \delta \wedge (\text{Id} \, j) \]
which does indeed vanish by (9).

**Proof of Theorem 4.10.** We prove the claim in three steps.

- If $\alpha' = \alpha$, then (12) holds for some representative $\delta : E^* \to H^{*-1}$ of $c' - c$.
- The right-hand side of (12) is independent of the choice of representative of $c' - c$.
- Given $\alpha$, $\gamma$ and $\alpha'$, there is some choice of $\gamma'$ such that $c' - c$ is represented by $\delta = 0$ and $P_{c'} = P_c$.

So let us first consider the case when $\alpha' = \alpha$. Then $\gamma' = \gamma + \eta$ for some $\eta : E^* \to Z^{*-1}$, and
\[ \alpha(\gamma')^2 - \alpha \gamma^2 = \alpha(\eta + \gamma + \eta^2) = (\alpha \eta) \circ (\text{Id} \, j) + \alpha \eta^2. \]

The restriction of this to $D$ induces precisely the map in (12), for $\delta : E^* \to H^{*-1}$ the map induced by $\eta$.

Next we show that both terms in (12) vanish when $\delta$ is the restriction of $n \text{Id} : G^2 H^* \to H^{*-1}$ for some $n : H^* \to H^{*-1}$. Writing $\text{Id}(n \text{Id})^2 : H^* \otimes G^2 H^* \to H^{*-2}$ as $\frac{1}{2}(\text{Id}(n \text{Id}) + \frac{1}{2}n \text{Id}^2) \wedge n \text{Id} - \frac{1}{4}n \text{Id}^2 \wedge n \text{Id}$, the restriction of the first term to $D = \ker m$ vanishes because $\text{Id}(n \text{Id}) + \frac{1}{2}n \text{Id}^2$ is fully symmetric, while the restriction to $D \subseteq H^* \otimes G^2 E^*$ of the second term vanishes because $\text{Id}(n \text{Id})^2 = 0$ by the definition of $E^*$.

Given a choice $(\alpha, \gamma)$, and some $\eta : H^* \to Z^{*-1}$ representing $n$, $(T_c(\text{Id} \, n \text{Id})) \circ (\text{Id} \, j) : D^* \to H^{*-2}$ is induced by the restriction of
\[ \alpha \gamma \wedge \eta : H^* \otimes G^2 E^* \to A^{*-2}. \]

Now
\[ \alpha \gamma \wedge \eta + d(\eta^2) = \alpha \eta \wedge \gamma + \eta \alpha^2 \wedge \gamma, \]
whose restriction to $D$ vanishes. Hence $(T_c(\text{Id} \, n \text{Id})) \circ (\text{Id} \, j)$ vanishes too.

Finally, given $\alpha$, $\gamma$ and $\alpha'$, write $\alpha' = \alpha + d\beta$ for some $\beta : H^2 \to A^1$, and take $\gamma' = \gamma + \alpha \beta + \frac{1}{2} \beta d \beta$. Then certainly $c' - c$ is represented by $\gamma = 0$. Meanwhile, with the notation we have set up we can show that the restriction of $\alpha'(\gamma')^2 - \alpha^2$ to $D$ is exact using exactly the same calculations as in (5) and the following equations. 

□
5. THE ROLE OF $\mathcal{P}$ IN RATIONAL HOMOTOPY CLASSIFICATION

We expect that for sufficiently highly connected closed manifolds, $\mathcal{P}$ should play a role in the rational homotopy classification alongside the Bianchi-Massey tensor, but even saying what it means for the pentagonal Massey tensors of two spaces to “be the same” is somewhat subtle because of the way that $\mathcal{P}$ depends on cochain choices.

Any rational homotopy classification result will encompass as a special case a criterion for formality. In the present paper we are content to just prove such a formality criterion, in the context of closed $(n-1)$-connected manifolds of dimension $5n-2$. But even to state that we first need to at least clarify what it means for $\mathcal{P}$ to vanish.

5.1. Dependence of $\mathcal{P}$ on choices. While the transformation rule for how $\mathcal{P}$ depends on choices is given in Theorem 4.10 we would now like to clarify how the terms in the formula behave, and in particular in what sense $\mathcal{P}$ is independent of choices if the Bianchi-Massey tensor is trivial.

Lemma 5.1. Suppose that $\delta: E^* \to H^{*-1}$ is such that the restriction of $\text{Id}\delta: H^* \otimes E^* \to H^{*-1}$ to $K[H^* \otimes E^*]$ vanishes. Then the restriction of $\text{Id}\delta^2: H^* \otimes h^2E^* \to H^{*-2}$ to $\mathcal{D}$ vanishes, while $\mathcal{T}_c \circ (\text{Id} j): \mathcal{D}^* \to H^{*-2}$ is independent of the choice $c$.

Proof. $\text{Id}\delta^2$ can be factorised as $((\text{Id}\delta)\delta) \circ \text{Id} j$ for $j$ as in (11), so vanishes on $\mathcal{D}$ by Lemma 4.9.

Now suppose $c$ and $c'$ are two different choices. Then according to (12) $\mathcal{T}_c - \mathcal{T}_{c'}$ is the restriction to $K[H^* \otimes E^*]$ of $\text{Id}\delta': H^* \otimes E^* \to H^{*-1}$ for some $\delta': E^* \to H^{*-1}$. We thus need to prove that $((\text{Id}\delta')\delta) \circ \text{Id} j = 0$. But by the definition of $j$, $(\text{Id}\delta')\delta \circ (\text{Id} j) = ((\text{Id}\delta')\delta) \circ (\text{Id} j)$, whose restriction to $\mathcal{D}$ again vanishes by Lemma 4.9.

Corollary 5.2. If $\mathcal{T}_c = 0$ for some choice $c$, then $\mathcal{P}_c$ takes the same value for all such choices.

Proof. If $c$ and $c'$ are two different choices such that $\mathcal{T}_c = \mathcal{T}_{c'} = 0$ then the restriction of $\text{Id}\delta(c', c): H^* \otimes E^* \to H^{*-1}$ to $K[H^* \otimes E^*]$ vanishes by (9). Thus Lemma 5.1 implies that the second term in the transformation rule (12) for $\mathcal{P}$ vanishes, while the first term vanishes because $\mathcal{T}_c = 0$.

Remark 5.3. If the Bianchi-Massey tensor of a closed manifold vanishes, then there exists some choice $c$ such that $\mathcal{T}_c$ vanishes according to Lemma 4.5. Thus we can think of $\mathcal{P}$ as a single well-defined map $\mathcal{D}^* \to H^{*-2}$ in this situation.

This is the case in particular for 8-dimensional E-manifolds. Indeed, in this case the Bianchi-Massey tensor is vacuous, there is a single equivalence class $c$ of choice data, the degree 8 component of $\mathcal{P}_c$ reduces to the definition in (9) and Theorem 2.5 can be recovered as a special case of Theorem 4.10.

5.2. Formality criterion. For a closed manifold, asking that the Bianchi-Massey and pentagonal Massey tensors both vanish is a well-defined condition by Remark 5.3 and this is clearly a necessary condition for formality (Remark 2.9). We now prove Theorem 1.7 i.e. that this condition is in fact equivalent to formality for closed $(n-1)$-connected manifolds of dimension up to $5n-2$.

Theorem 5.4. Let $n \geq 2$ and let $M$ be a closed $(n-1)$-connected manifold of dimension $m \leq 5n-2$. Suppose that the Bianchi-Massey tensor $\mathcal{F}$ of $M$ vanishes, and that $\mathcal{P}_c = 0$ for the cochain choices $c$ such that $\mathcal{T}_c = 0$. Then $M$ is formal.

For $m \leq 5n-3$, it was already proved in [2] that vanishing of $\mathcal{F}$ on its own is equivalent to formality (and $\mathcal{P}$ vanishes for degree reasons), so really we are only interested in the case $m = 5n-2$. To prove formality it is convenient to employ the concept of $s$-formality of Fernández and Muñoz [3].

Definition 5.5. Let $AV$ be a minimal Sullivan-algebra, where $V = \bigoplus V^i$, $V^i$ denotes the degree $i$ component, and let $C^s \leq V^i$ be the subspace of closed elements. We say that $AV$ is $s$-formal if for each $i \leq s$ one can choose a direct complement $N^s \leq V^i$ to $C^s$ such that any closed element in $N^{s+1}AV^{s+1} \leq (\text{the ideal in } AV\leq s)$ is exact in $AV$.

A DGA $\mathcal{A}$ is $s$-formal if its minimal model is isomorphic to an $s$-formal minimal Sullivan-algebra, equivalently, if there is an $s$-formal minimal Sullivan-algebra $AV$ and a quasi-isomorphism $\psi: AV \to \mathcal{A}$.
Theorem 5.6 [3 Theorem 3.1]. Formality is equivalent to s-formality for closed manifolds of dimension $m \leq 2s + 1$.

For $n \geq 2$, it will thus suffice to show that a closed $(n-1)$-connected manifold of dimension up to $5n - 2$ with vanishing Bianchi-Massey tensor and $\mathcal{P}$ is $(3n-2)$-formal.

Remark 5.7. To verify s-formality of a closed manifold of dimension $m \leq 2s + 1$, it suffices to show that for some minimal model $(\Lambda V, \psi)$ there is a choice of complements $N^i \leq V^i$ such that the closed elements of degree $s$ or equal to $m$ in $\Lambda V \leq s$ are exact, by the following standard argument.

Let $\tilde{Z}^* \subseteq \Lambda V \leq s$ be the subalgebra of closed elements. If $w \in \tilde{Z}^*$ with degree $s < i < m$, and the image of $w$ in $H^i(A)$ is non-zero then by Poincaré duality there is a class $x \in H^{m-1}(A)$ such that $x[\psi(w)]$ is non-zero. This $x$ must be the image of some $u \in \Lambda V \leq s$. Then $uw \in \tilde{Z}^m$ and its image in $H^m$ is non-zero.

Proof of Theorem 5.4. By the assumptions (and Lemma 4.5) there is a cochain choice $c = (\alpha, \gamma)$ such that $T_c = 0$ (i.e. $\alpha \gamma$ takes exact values on $K[H^* \otimes E^*]$) and $P_c = 0$. We will fix such a $c$.

We build a minimal model $(\Lambda V, \psi)$ recursively as follows. We define $C^{i+1}$ and $N^i$ to be the cokernel and kernel of the map $\Lambda^{i+1}(\Lambda V < i) \to H^{i+1}(A)$ induced by $\psi$, and we set $V^i = C^i \oplus N^i$. The differential is given by $d_C^{i+1} = 0$ and $d_{N^i} = Id_{N^i} : N^i \to \Lambda^{i+1}(\Lambda V < i) \leq AV$. We choose $\psi_{(i+1)} : C^{i+1} \to \Lambda^{i+1}(A)$ such that the image of the composition $C^{i+1} \to \Lambda^{i+1}(A) \to H^{i+1}(A)$ is a complement to the image of $\Lambda^{i+1}(\Lambda V < i) \to H^{i+1}(A)$ and $\psi_{N^i} : N^i \to A^i$ such that the diagram

\[
\begin{array}{ccc}
N^i & \xrightarrow{d} & \Lambda^{i+1}(\Lambda V < i) \\
\downarrow{\psi} & & \downarrow{\psi} \\
A^i & \xrightarrow{d} & A^{i+1}
\end{array}
\]

commutes (such a $\psi$ exists, but in general it is not unique).

By construction the $(\Lambda V, \psi)$ we get is a minimal model of $A$. Next we give a more explicit description of the spaces $C^{i+1}$ and $N^i$ (in the relevant range), and, using an appropriate choice of $\psi$ and the assumptions on $c$, prove that $\Lambda V$ is s-formal (with the chosen complements $N^i$).

- We must in any case take $V^i = 0$ for $0 \leq i < n$.
- For $i \leq 2n - 2$ we must also have $N^i = 0$, because the degree $i + 1$ part of $\Lambda V < i$ is trivial.
- For $n \leq i \leq 2n - 1$ we take $C^i = H^i$, and on $C^i$ we define $\psi$ by $\alpha$.
- For $2n - 1 \leq i \leq 3n - 3$, the degree $i + 1$ part of $\Lambda V < i$ is simply $(G^2 C^{<i})^{i+1} \oplus (G^2 H^{<i})^{i+1}$, and the kernel of $(G^2 C^{<i})^{i+1} \to H^{i+1}$ corresponds to $E^{i+1}$. We can take $N^i = E^{i+1}$, and define $\psi$ on $N^i$ by $\gamma$.
- Similarly, for $2n - 1 \leq i \leq 3n - 2$ we can take $C^i$ to be a direct complement in $H^i$ to the image of $(G^2 H^{<i-2})^i \to H^i$, and define $\psi$ on $C^i$ by $\alpha$.

To complete the description of the part of the minimal model that is relevant for $(3n-2)$-formality, it remains only to describe $N^{3n-2}$. Now the degree $3n - 1$ part of $\Lambda V \leq 3n-3$ is a direct sum of $(G^2 C^{<3n-3})^{3n-1}$ and $C^i \otimes N^{2n-1} \cong H^n \otimes E^{2n}$. The closed subspace is the direct sum of $(G^2 C^{<3n-3})^{3n-1}$ and the closed subspace $K \leq C^{n} \otimes N^{2n-1}$, which corresponds to $K[H^n \otimes E^{2n}]$. On the first summand, the map to $H^{3n-1}$ is simply the cup product, so $N^{3n-3}$ should include a summand $\tilde{N}^{3n-3}$ isomorphic to $E^{3n-1}$. On the other summand $K$, it is induced by the restriction of $\alpha \gamma : H^n \otimes E^{2n} \to A^{3n-1}$, i.e. it corresponds to the uniform triple product $T_c : K[H^n \otimes E^{2n}] \to H^{3n-1}$ defined by the cochain choice $c = (\alpha, \gamma)$.

Recall that $(\alpha, \gamma)$ has been chosen such that $T_c = 0$ (i.e. $\alpha \gamma$ takes exact values on $K$). Thus we should take $N^{3n-2} = \tilde{N}^{3n-2} \oplus K$, defining $\psi$ by $\gamma$ on $\tilde{N}^{3n-2}$, and by some choice of predifferential of $\alpha \gamma$ on $K$.

Using that we also have $P_c = 0$, we now want to prove that it is possible to choose $\psi$ on $K$ such that $\Lambda V$ is a $(3n-2)$-formal minimal model. Let $\tilde{Z}^* \subseteq \Lambda V \leq 3n-2$ be the closed subalgebra. Every $w \in \tilde{Z}^*$ with $r \leq 3n - 2$ is exact in $\Lambda V$ by construction, so by Remark 5.7 it remains only to consider $\tilde{Z}^m$. 

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To decompose $\hat{Z}^m$, set $\hat{N}^i = N^i$ for $i \leq 3n - 3$ (so that $\hat{N}^i \cong E^{i+1}$ for each $i \leq 3n - 2$), and $\hat{V} = \bigoplus_{i=0}^{3n-2} C^i \oplus \hat{N}^i \subseteq V^{\leq 3n-2}$ (so $V^{\leq 3n-2} = \hat{V} \oplus K$). Let $\hat{Z}^* \subset (\hat{N}A\hat{V})^*$ be the closed subalgebra.

**Lemma 5.8.** For $m = 5n - 2$, every closed element of $\hat{Z}^m$ is exact in $NAV$.

**Proof.** $\hat{A}V$ can be decomposed as a direct sum of the graded subspaces

$$L^*(a, b) = G^aC^* \oplus G^bN^*.$$

Because $m \leq 5n - 2$, the degree $m$ part of $\hat{N}A\hat{V}$ is a direct sum of $L^m(1, 1)$, $L^m(2, 1)$, $L^m(0, 2)$ and $L^m(1, 2)$. The differential maps $L^i(a, b) \to L^{i+1}(a+b-1, 1)$, so $\hat{Z}^m$ is a direct sum of the closed subspaces $Z^m(a, b) \leq L^m(a, b)$. It thus remains to check for each of these $Z^m(a, b)$ that $Z^m(a, b) \to H^m$ is trivial.

On $L^*(1, 1) \cong H^* \otimes E^*$, the differential corresponds to the symmetrisation map $H^* \otimes E^* \to G^3H^*$, so $Z^*(1, 1) \cong K[H^* \otimes E^*]$. $Z^*(1, 1) \to H^*$ is $T_c$, which is trivial by the choice we made for $c$.

On $L^*(2, 1) \cong G^2H^* \otimes E^*$, the differential corresponds to the symmetrisation $G^2H^* \otimes E^* \to G^4H^*$. By the Bianchi identity, its kernel $Z^*(2, 1)$ equals the image of the natural map from $H^* \otimes K[H^* \otimes E^*]$. The map $Z^*(2, 1) \to H^*$ corresponds to $aT_c$, so again vanishes because we ensured $T_c = 0$.

On $L^*(0, 2) \cong G^2E^*$ the differential is the composition $G^2E^* \to E^* \otimes E^* \to G^3H^* \otimes E^*$ of two injections, hence $Z^*(0, 2) \cong 0$.

On $L^*(1, 2) \cong H^* \otimes G^2E^*$ the differential corresponds to $m$ from (1), so the kernel is $Z^*(1, 2) \cong D$. The induced map $Z^*(1, 2) \to H^*$ is precisely $P_c$.

Now let $D^{2n} \leq G^2C^n \oplus C^{2n} \leq (A\hat{V})^{2n}$ be a direct complement to $E^{2n}$ (so that $D^{2n} \cong H^{2n}$).

**Lemma 5.9.** For $m = 5n - 2$, the closed subspace $\hat{Z}^m \subseteq (N^{\leq 3n-2}A\hat{V}^{\leq 3n-2})^m$ is contained in

$$K \otimes D^{2n} \oplus d(K \otimes N_{2n-1}) \oplus (\hat{N}A\hat{V})^m. \tag{13}$$

**Proof.** For degree reasons we have $(N^{\leq 3n-2}A\hat{V}^{\leq 3n-2})^m = (\hat{N}A\hat{V})^m \oplus K \otimes (A\hat{V})^{2n}$.

The differential maps $K \otimes N_{2n}$ to $K \otimes C^n \otimes C^{n+1} \oplus C^n \otimes N_{2n+1} \oplus N_{2n}$. The composition with the projection to the first summand is injective because the differential $N_{2n} \to C^n \otimes C^{n+1}$ is.

Therefore $\hat{Z}^m$ must be contained in $(\hat{N}A\hat{V})^m \oplus K \otimes (C^{2n} \oplus G^2C^n) = (\hat{N}A\hat{V})^m \oplus K \otimes (D^n \oplus E^{2n})$. It thus remains only to note that for any element $ke \in K \otimes E^{2n}$, there is a $y \in N_{2n-1}$ with $dy = c$, so $ke - d(\hat{y}k) = g(dy) \in (\hat{N}A\hat{V})^m$.

Finally we deduce that $\psi : K \to A^{m-2n}$ can be chosen so that every element in the closed subspace $\hat{Z}^m \subseteq (N^{\leq 3n-2}A\hat{V}^{\leq 3n-2})^m$ is exact in $A\hat{V}$. Let $W \subseteq \hat{Z}^m$ be a direct complement to $\hat{Z}^m \oplus d(K \otimes N^{2n-1})$. We already know that $\psi_* : \hat{Z}^m \oplus d(K \otimes N_{2n-1}) \to H^m$ is trivial. On the other hand, the projection to the $K \otimes D^{2n}$ summand in (13) is injective on $W$. Therefore by Poincaré duality we can adjust the values of $\psi_* : W \to H^*$ arbitrarily by changing the choice of $\psi$ on the $K$ summand by adding on some map $K \to Z^{m-2n}$. In particular we can choose it such that $\psi_*$ vanishes on all of $\hat{Z}^m$. This completes the proof of Theorem 5.4.

**Remark 5.10.** If one extended the dimension range to consider closed $(n-1)$-connected manifolds of dimension $m \leq 6n - 4$, the only part of the proof that would fail is that the degree $m$ part of $\hat{N}A\hat{V}$ could also have a contribution from $L^m(3, 1)$, so $\hat{Z}^m$ could have a non-trivial $Z^m(3, 1)$ summand. We expect that to prove formality in this case would require a further generalisation of the Bianchi-Massey tensor.

For any $r$, we could consider the kernel $E^*_r$ of the $r$-fold product $G^rH^* \to H^*$. Given a right inverse $\alpha : H^* \to Z^*$, one could then pick a predifferential $\gamma_r : E^*_r \to A^{*r-1}$ of the restriction to $E^*_r$ of $\alpha^r : G^rH^* \to A^*$. The restriction of the degree $-1$ map

$$\gamma_r \alpha^r : E^*_r \otimes E^*_s \to A^{*r+s-1}$$

to the kernel $K[E^*_r \otimes E^*_s]$ of full graded symmetrisation $E^*_r \otimes E^*_s \to G^{r+s}H^*$ takes closed values. The induced map $F_{r,s} : K[E^*_r \otimes E^*_s] \to H^{*r+s}$ is clearly independent of the choice of $\gamma$, and seems likely to be independent of the choice of $\alpha$ too.
F\_r,s\) amounts to the Bianchi-Massey tensor. Depending on the product structure on \(H^*\), other \(F_{r,s}\) may be determined by the Bianchi-Massey tensor too, but it seems it does not always need to be. Vanishing of \(F_{3,2}\) seems relevant for formality a \((n-1)\)-connected manifold of dimension \(5n - 2 < m \leq 6n - 4\).

Closed \((n-1)\)-connected manifolds of dimension \(> 6n - 4\) can have non-trivial 5-fold Massey products, so any criterion for formality of such manifolds would need to include some invariant that controls those products.

5.3. Using \(\mathcal{P}\) to distinguish and classify manifolds. For an isomorphism \(F : H^*(X) \to H^*(Y)\) to be realised by a rational homotopy equivalence, it is clearly necessary that there are some choices \(b\) and \(c\) on \(X\) and \(Y\) such that \(F\) intertwines the uniform triple products \(\mathcal{T}_b\) and \(\mathcal{T}_c\), and the pentagonal Massey tensors \(\mathcal{P}_b\) and \(\mathcal{P}_c\). Supposing that we can make the choices so that \(\mathcal{T}_b\) and \(\mathcal{T}_c\) are intertwined, let us now consider how to measure the failure of the pentagonal Massey tensors to agree.

Let \(\Delta \subseteq \text{Hom}(\mathcal{D}^*, H^{*-2})\) be the space of degree \(-2\) maps \(\mathcal{D}^* \to H^{*-2}\) generated by functions of the form \(\mathcal{T}_c \circ (\text{Id} j)\), for \(\delta : E^* \to H^{-1}\) such that the restriction of \(\delta \text{Id} \) to \(K[H^* \otimes E^*]\) vanishes. Then Lemma \[5.1\] (together with Theorem \[4.10\]) means that whenever two choices \(c'\) and \(c\) have \(\mathcal{T}_c = \mathcal{T}_{c'}\), then \(\mathcal{P}_{c'} - \mathcal{P}_c \in \Delta\). In consequence, if there are some cochain choices \(b\) on \(X\) and \(c\) on \(Y\) such that \(\mathcal{T}_b\) and \(\mathcal{T}_c\) are intertwined by \(F\), then

\[F^# \mathcal{P}_c - \mathcal{P}_b \in \text{Hom}(\mathcal{D}^*, H^{*-2})/\Delta\]

takes the same value for all such pairs of choices, and vanishes if and only if there is some pair for which \(F^# \mathcal{P}_c = \mathcal{P}_b\).

In the case of closed manifolds, the existence of choices such that \(\mathcal{T}_b\) and \(\mathcal{T}_c\) are intertwined by \(F\) is equivalent to the more convenient condition that the Bianchi-Massey tensors (which do not depend on choices at all) are intertwined by \(F\) [Lemma 2.8(ii)].

For closed \((n-1)\)-connected manifolds of dimension up to \(5n - 3\), Massey products of order 4 or higher (and analogous data like \(\mathcal{P}\)) are irrelevant for degree reasons. In this case, [2, Theorem 1.2] shows that \(F\) is realised by a rational homotopy equivalence if and only it intertwines the Bianchi-Massey tensors.

For closed \((n-1)\)-connected manifolds of dimension up to \(6n - 4\), Massey products of order 5 or greater are irrelevant. While we expect that \(\mathcal{P}\) captures all the “fourfold product like” information, Remark \[5.10\] indicates that some data may need to be added to the Bianchi-Massey and pentagonal Massey tensors to determine the rational homotopy type if the dimension is greater than \(5n - 2\). However, the following statement is reasonable in view of the formality criterion of Theorem \[1.2\].

**Conjecture 5.11.** Let \(F : H^*(X) \to H^*(Y)\) be an isomorphism of the cohomology algebras of closed \((n-1)\)-connected \(m\)-manifolds, for \(n > 1\) and \(m \leq 5n - 2\). Then \(F\) is realised by a rational homotopy equivalence if and only if

- \(F\) intertwines the Bianchi-Massey tensors, and
- for any choices \(b\) and \(c\) such that \(F\) intertwines \(\mathcal{T}_b\) and \(\mathcal{T}_c\), we have \(F^# \mathcal{P}_c - \mathcal{P}_b \in \Delta\).

In view of Remark \[5.3\], Conjecture \[1.9\] is a special case of Conjecture \[5.11\].

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