We construct an exact quantum gravitational state describing the collapse of an inhomogeneous spherical dust cloud using a lattice regularization of the Wheeler-DeWitt equation. In the semiclassical approximation around a black hole, this state describes Hawking radiation. We show that the leading quantum gravitational correction to Hawking radiation renders the spectrum non-thermal.

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I. INTRODUCTION

The end state of the gravitational collapse of a compact object is an important problem for which a full understanding has not yet been achieved. It is not completely clear what changes a quantum theory of gravity will bring about in our understanding of Hawking radiation, the entropy of a black hole, the information loss paradox and the nature of the gravitational singularity. This is because it has not been easy to develop dynamical models of quantum gravitational collapse in the various candidate theories of quantum gravity.

Over the last few years, useful and interesting developments in formulating the quantum gravitational collapse problem have taken place, within the framework of canonical quantum general relativity. Kastrup and Thiemann [1] and Kuchař [2, 3, 4] have pioneered the technique of midisuperspace quantization, a canonical quantization of a gravitational system with symmetries, and yet possessing infinitely many degrees of freedom. A prime example of this is the canonical quantization of the Schwarzschild geometry associated with an eternal black hole [2].

We have applied these earlier works to develop a canonical description of the collapse of a spherical time-like dust cloud, which is described by the LeMaître-Tolman-Bondi family of metrics [5]. Quantization of the classical dust system leads to the Wheeler-DeWitt equation for the wave-functional describing the quantum collapse. The quantum Schwarzschild black hole then becomes a special case of our model. We have successfully employed this equation to derive the Bekenstein mass spectrum and statistical entropy of the charged and uncharged black hole [6] and, in a recent work [7], we showed that the semiclassical (WKB) treatment of the Schwarzschild black hole in this canonical picture describes Hawking radiation. The midisuperspace program has also been used to describe the collapse of null dust, with interesting conclusions [8]. All these works demonstrate the overall consistency of the program.

Apart from the usual operator ordering ambiguities, a problem with the midisuperspace quantization program is that the Wheeler-DeWitt equation involves two functional derivatives taken at the same point and requires regularization in all but the effectively minisuperspace models. Any attempt at regularization, however, leads to ambiguities that are not easy to resolve. Our purpose, in the present article, is to address this problem by applying a lattice regularization scheme to the canonical quantization of LeMaître-Tolman-Bondi (LTB) models.

We first construct a stationary state solution of the diffeomorphism constraint. Upon discretization, the wave-functional reduces to a tensor product over shell wave-functions, one at each lattice point. The Wheeler-DeWitt equation is well defined on the lattice as an infinite set of Schroedinger equations, one for each shell. The results are independent of how the lattice is discretized when the size of all lattice cells approaches zero. Even though the regularization is on a lattice, diffeomorphism invariance is recovered when the lattice spacing approaches zero. The shell wave-functions reproduce Hawking’s thermal radiation near the horizon in the WKB limit and the first quantum gravitational correction to Hawking’s formula renders the spectrum non-thermal. We believe that our construction opens up a useful avenue for investigating some of the important unresolved issues related to black hole physics mentioned above.

The paper is organized as follows: in section II we review our pervious quantization of the LeMaître-Tolman-Bondi collapse models. In section III we construct a lattice to regularize the Wheeler-DeWitt equation and discuss its properties. Solutions of the Wheeler-DeWitt equation are discussed in section IV and Hawking radiation is reexamined in section V. We conclude with a few comments in section VI.
II. CANONICAL QUANTIZATION OF LTB COLLAPSE

The LeMaitre-Tolman-Bondi models \[ \text{constitute a complete solution of the Einstein equations for a matter continuum of inhomogeneous dust, } i.e., \text{they are solutions of the spherically symmetric Einstein's field equations, } G_{\mu\nu} = -8\pi G T_{\mu\nu}, \text{with vanishing cosmological constant and with stress-energy describing inhomogeneous, pressureless dust given by } T_{\mu\nu} = e u_{\mu} u_{\nu}. \text{The solutions are determined by two arbitrary functions, } F(r) \text{ and } E(r), \text{of a spatial label coordinate, } r, \text{and given in co-moving coordinates as}

\[ ds^2 = dr^2 - \frac{R^2}{1 + E} dr^2 - R^2 d\Omega^2, \]

\[ \epsilon = \frac{F}{R^2 R^*}, \]

(we have set \( 8\pi G = 1 = c \)) where \( R(\tau, r) \) is the physical radius of a spherical shell labeled by \( r \). A prime represents a derivative w.r.t. \( r \) and a star (*) represents a derivative with respect to the dust proper time, \( \tau \). The arbitrary functions, \( E(r) \) and \( F(r) \), are interpreted respectively as the energy and mass functions. The energy density of the collapsing matter is \( \epsilon(\tau, r) \), and the negative sign in the third equation above is required to describe a collapsing cloud. Its general solution is given up to an arbitrary function \( \psi(r) \) of the shell label coordinate. This arbitrariness reflects only a freedom in our choice of units i.e., at any given time, say \( \tau_0 \), the function \( R(\tau_0, r) \) can be chosen to be an arbitrary function of \( r \).

The mass function, \( F(r) \), represents the weighted mass (weighted by the factor \( \sqrt{1 + E} \)) contained within the matter shell labeled by \( r \). If a scaling is chosen so that the physical radius coincides with the shell label coordinate, \( r \), at \( \tau = 0 \), then it can be expressed in terms of the energy density at \( \tau = 0 \) according to

\[ F(r) = \int \epsilon(0, r) r^2 dr, \]

while the energy function, \( E(r) \), can be expressed in terms of the initial velocity profile, \( v(r) = R'(0, r) \), according to

\[ E(r) = v^2(r) - \frac{1}{r} \int \epsilon(0, r) r^2 dr, \]

showing that it represents the total energy (gravitational plus potential) of the matter-gravity system. The marginally bound models, which we will consider in this paper, are defined by \( E(r) = 0 \). For the scaling referred to above, we must choose \( \psi(r) = r^{\frac{3}{2}} \), whence the solution of \[ \text{(1)} \] can be written as

\[ R^{\frac{3}{2}}(\tau, r) = \frac{r^3}{3} - \frac{3}{2} \sqrt{F(r)} \tau. \]

The epoch \( R = 0 \) describes a physical singularity, whose singularity curve

\[ \tau(r) = \frac{2r^2}{3\sqrt{F(r)}}, \]

gives the proper time when successive shells meet the central physical singularity. Various models are obtained from choices of the mass function, \( F(r) \). For example, the Schwarzschild black hole is the marginally bound solution with \( F(r) = 2M \), a constant.

Reinserting the constants \( c, \hbar \) and \( G \), and defining the dimensionless variable \( x = r/l_p \) (where \( l_p = \sqrt{\hbar G/c^3} \) is the Planck length) and the dimensionless mass function \( f(x) = F/l_p \), the wave-functional of the quantized Einstein-Dust system can be shown \[ \text{to satisfy the Wheeler-DeWitt equation,}

\[ \left[ \frac{1}{c^2} \frac{\delta^2}{\delta \tau^2} \pm \frac{\delta^2}{\delta R^2} + \frac{f^2}{4l_p^2 F} \right] \Psi[\tau, R, f] = 0, \]

developed so that the DeWitt configuration space metric is manifestly flat. Invariance under spatial diffeomorphisms is implemented by the momentum constraint \[ \text{[}] \]

\[ \left[ \frac{r'}{\delta \tau} + R' \frac{\delta}{\delta R} + f \frac{\delta}{\delta f} \right] \Psi[\tau, R, f] = 0. \]

(8)

To complete the quantum theory, one must define an inner product on the Hilbert space of wave-functionals. In \[ \text{[}] \], we defined it in a natural way, by exploiting the fact that the DeWitt super-metric \[ \text{[}] \] is manifestly flat in the configuration space \( (\tau, R_*) \), in terms of the functional integral

\[ \langle \Psi_1 | \Psi_2 \rangle = \int_{R_*(0)}^{\infty} DR_* \Psi_1^* \Psi_2. \]

Equations \[ \text{[}] \)–\( \text{[}] \) clearly imply a specific choice, albeit a natural one, of operator ordering, for which no justification deeper than the fact that they were shown to reproduce the Bekenstein mass spectrum \[ \text{[}] \)–\( \text{[}] \] of the eternal black hole and its Hawking radiation in the WKB approximation \[ \text{[}] \] can be given at this time. This defect is unfortunately endemic to current theories of quantum gravity.

Any functional that is a spatial scalar will obey the momentum constraint. In particular the functional
\[
\Psi[\tau, R, f] = \exp \left[ -\frac{ic}{2l_p} \int_0^\infty dx f'(x)W(\tau(x), R(x), f(x)) \right],
\]

is a spatial scalar if \(W(\tau, R, F)\) has no explicit dependence on \(x\) (equivalently, \(r\)) and we will use this as a solution ansatz. If we further require (10) to represent a stationary state then, noting that \(F'(r)/2\) is the proper energy density of the collapsing cloud, we choose

\[
W(\tau, R, f) = \tau + U(R, f).
\]

In a classical collapse problem, \(f(x)\) is generally monotonically increasing up to the boundary of the star, which is specified by some label \(x_b\). The exterior of the star is then matched to the Schwarzschild exterior and \(f(x)\) is taken to be constant for \(x > x_b\). While this is a reasonable condition to impose upon the mass function on the classical level, it is too strong in the quantum theory. To account for Hawking radiation, it is necessary for \(f(x)\) to be a monotonically increasing function of \(x\) over the entire range of \(x\), as we do not expect to have a sharp boundary between the collapsing matter and its exterior owing to the evaporation process.

\[
\Psi[\tau, R, f] = \prod_j \Psi_j(\tau_j, R_j, f_j) = \prod_j \exp \left[ -\frac{ic}{2l_p} (f_{j+1} - f_j) [\tau_j + U_j(R_j, f_j)] \right].
\]

The energy, \(\varepsilon_j\), contained between lattice points \(j\) and \(j + 1\) is related to \(f_{j+1} - f_j\) by

\[
f_{j+1} - f_j = \frac{2G}{l_p c^2} (M_{j+1} - M_j)c^2 = \frac{2G}{l_p c^2} \varepsilon_j = \frac{2l_p}{\hbar} \omega_j,
\]

implying that the wave-functional can be expressed in the form

\[
\Psi = \prod_j e^{-i\omega_j(\tau_j + U_j(R_j, F_j))}
\]

where we have defined \(\varepsilon_j = \hbar \omega_j = (M_{j+1} - M_j)c^2\). Our strategy is to solve the Wheeler-DeWitt equation (9) independently at each label, \(j\). The continuum limit would be defined by putting together the shell wave-functions and taking the limit as the lattice spacing approaches zero. We shall see that the wave functions are independent of the original choice of cell size, as it should be.

Before proceeding further it is necessary to define what is meant by a functional derivative when functions are defined on a lattice. The defining equations can be understood by analogy with the simplest properties of functional derivatives of the functions \(J(x)\).

\[
\frac{\delta J(\tau)}{\delta J(x)} = \delta(\tau - x),
\]

\[
\delta \frac{\delta J(x)}{\delta J(y)} = 1
\]

From these definitions follows

\[
\frac{\delta}{\delta J(x)} \int dy J(y) \phi(y) = \phi(x).
\]

On a lattice we define, for the lattice intervals \(x_i\) and \(x_j\),

\[
\frac{\delta J(x_i)}{\delta J(x_j)} = \Delta(x_i - x_j) = \lim_{\sigma_i \to 0, \forall \sigma_i} \frac{\delta_j}{\delta J(x_j)}
\]

where \(x_i\) labels the \(i^{th}\) lattice site and \(\delta_{ij}\) is just the Kronecker \(\delta\), equal to zero when the lattice sites \(x_i\) and \(x_j\) are different and one when they are the same. \(\sigma_i\) is the size of the \(i^{th}\) site, which we take to be arbitrarily given. Just as \(\delta(\tau - x)\) is only defined as an integrand in an integral, so \(\Delta(x_i - x_j)\) should also be considered
defined only as a summand in a sum over lattice sites. Hence

$$\lim_{\sigma_i \to 0, \forall i} \frac{\delta}{\delta J(x_j)} \sum_{i} \sigma_i J(x_i) = \lim_{\sigma_i \to 0, \forall i} \sum_{i} \sigma_i \frac{\delta J(x_i)}{\delta J(x_j)} = 1$$  \hfill (18)

It follows that

$$\frac{\delta}{\delta J(x_j)} \sum_{i} \sigma_i J(x_i) \phi(x_i) = \lim_{\sigma_i \to 0, \forall i} \sum_{i} \sigma_i \Delta(x_i - x_j) \phi(x_i) = \phi(x_j)$$  \hfill (19)

Factoring out the term $1/\sigma_j^2$ the equation becomes independent of the choice of cell sizes,

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial R_j^2} + \frac{\omega_j^2}{c^2} \left( 1 \pm \frac{1}{|R_j|} \right) \right) \Psi_j(R_j, F_j) = 0.$$  \hfill (20)

We see that the limit of vanishing cell size may be taken and that the wave-functional is invariant under diffeomorphisms in this limit by construction. Upon making some obvious replacements,

$$\left[ \frac{\partial^2}{\partial R_j^2} + \frac{\omega_j^2}{c^2} \left( 1 \pm \frac{1}{|R_j|} \right) \right] \Psi_j(R_j, F_j) = 0, \hfill (23)$$

The upper sign refers to the exterior ($R > F$) and the lower to the interior ($R < F$). $\Psi_j$ is the time independent part of the wave function. Changing back from the coordinate $R_j$ to the original coordinate $R$, the equation has the form

$$\left| F_j \right| \frac{\partial^2 \Psi_j}{\partial R_j^2} + \frac{1}{2} \frac{\partial \left| F_j \right|}{\partial R_j} \frac{\partial \Psi_j}{\partial R_j} + \frac{\omega_j^2}{c^2} \left[ 1 \pm \frac{1}{|F_j|} \right] \Psi_j = 0 \hfill (24)$$

where the upper sign refers to the exterior ($R > F$) and the lower sign to the interior ($R < F$) and $\Psi_j$ refers to the time independent part of the wave-functional in the $j^{th}$ cell. Defining the dimensionless variables $z_j = R_j/F_j$ and $\gamma_j = F_j \omega_j/c$, redefining the time independent part of the wave function $\Psi_j \equiv y_j$, and suppressing the subscript $j$ throughout, the equation for each shell takes the form

$$z(z - 1) \frac{d^2 y}{dz^2} + z - 1 \frac{dy}{dz} + \gamma^2 z y = 0 \hfill (25)$$

both in the exterior ($z > 1$) and in the interior ($z < 1$).

IV. SOLUTIONS

Before examining the solutions of (25) we recall some results of the WKB approximation. The WKB solutions near spatial infinity were shown to be

$$\Psi_{\pm} \approx \exp \left[ \frac{-ic}{2F} \int d\tau' \left( \tau \pm \frac{2F}{c} \sqrt{z} \right) \right] \hfill (26)$$

In addition, finding the WKB approximation functional required no regularization. This suggests that, for consistency, our shell wave-functions should agree with the WKB approximation.

Writing (26) in terms of $u = 1/z$ and keeping only the lowest order term in $u$ we find

$$\frac{d^2 y}{du^2} + \frac{2}{u} \frac{dy}{du} + \gamma^2 y = 0, \hfill (27)$$
which has as a solution

\[ y_{\pm}(z) \approx A_{\pm} e^{ \pm 2i\gamma z} \sqrt{z + \frac{1}{2} \ln z}. \]  \tag{28}

The time dependent wave-function then has the form

\[ \Psi_{\pm}^\infty = e^{-i \sum_j \omega_j (\tau_j \pm 2F_j \sqrt{\tau_j + \frac{1}{4} \ln z_j})} \]

\[ = \prod_j e^{- \frac{i \epsilon_j}{\tau_j} (\tau_j \pm 2F_j \sqrt{\tau_j + \frac{1}{4} \ln z_j})} \]  \tag{29}

We see that the last term is of order \( \hbar \) and may be ignored when compared with the WKB approximation. Hence the wave function we obtain at infinity agrees with the results of the WKB approximation at infinity.

For the asymptotic wave-functions of \( \Psi_{\pm}^\infty \) to describe collapse, we must choose shell wave-functions that correspond to incoming waves in the infinite past and at large distances from the center. These are the wave-functions with a positive sign in the exponent and they vanish along \( I_+ \). The time reversed situation, in which the wave-functions vanish on \( I_- \) but not on \( I_+ \), are outgoing modes and represent expanding matter. They are described by the wave-functions in \( \Psi_{\pm}^\infty \) with the negative sign in the exponent.

Near the horizon, \( |z - 1| \approx 0 \), we define \( u = z - 1 \). Again, retaining only the lowest orders in \( u \), we find the effective equation

\[ \frac{d^2 y}{du^2} + \frac{1}{2u} \frac{dy}{du} + \frac{\gamma^2 - \frac{1}{4}}{u^2} y \approx 0 \]  \tag{30}

and the two independent solutions

\[ \Psi_{\text{hor}}^\infty \approx A_{\pm}^\infty \prod_j e^{-i \omega_j \tau_j + \frac{i}{4} (1 \pm \sigma_j) \ln |z_j - 1|} \]  \tag{31}

where \( \sigma_j = \sqrt{1 - 16 \gamma_j^2} \). We consider only astrophysical black holes, for which \( M \) is large. Let \( \gamma_j \gg 1 \), so that \( \sigma_j \approx 4i\gamma_j \), then the solution may be put in the form

\[ \Psi_{\text{hor}}^\infty \approx A_{\pm}^\infty \prod_j e^{- \frac{i \omega_j}{\epsilon_j} (\tau_j \pm 2F_j \ln |z_j - 1| + \frac{1}{4} \ln |z_j - 1|)} \]  \tag{32}

As before, the last term in the exponent above is small compared with the first two and may be neglected to leading order. Then we recover the near-horizon WKB solutions of \( \Psi_{\pm}^\infty \), where it was shown that the logarithmic term, which represents the scattering of the wave-function in the background geometry, yields Hawking radiation at the Hawking temperature. In the infinite future, the logarithmic term approaches negative infinity in the approach to the horizon, therefore on \( I_+ \) only the component with the positive sign is relevant and we have

\[ \Psi_{\text{hor}} \approx A_{\pm}^\text{hor} \prod_j e^{- \frac{i \omega_j}{\epsilon_j} (\tau_j + \frac{1}{4} \ln |z_j - 1| + \frac{1}{4} \ln |z_j - 1|)} \]  \tag{33}

representing outgoing shells, scattered near their horizons.

Exact solutions to \( \Psi_{\text{hor}} \) may be obtained in a neighborhood of ordinary points or about non-essentially singular points by expanding in a Frobenius series. From the point of view of Hawking radiation, the horizon is the more interesting of the two non-essential singularities of \( \Psi_{\text{hor}} \). Expanding the horizon will yield at least one solution by Fuch’s theorem. Instead of \( z \) then, it is convenient to use the variable \( u = z - 1 \). Assuming a solution of the form

\[ y(u) = \sum_{n=0}^{\infty} a_n u^{k+n} \]  \tag{34}

we find that the roots

\[ k_{\pm} = \frac{1}{4} \left( 1 \pm \sqrt{1 - 16 \gamma^2} \right) \]  \tag{35}

of the indicial equation

\[ k \left( k - \frac{1}{2} \right) + \gamma^2 = 0 \]  \tag{36}

differ by a non-integral number and therefore two independent solutions are obtained, each defined by the following recursion relations,

\[
\begin{align*}
a_{n+2}^+ &= -\frac{\gamma^2 a_n^+ + \left(k_{\pm}^2 + 2k_{\pm} + 1\right) a_0^+}{\gamma^2 + (k_{\pm} + n + 2)(k_{\pm} + n + \frac{1}{2})}, \\
a_{n+2}^- &= -\frac{\gamma^2 a_n^- + \left(k_{\pm}^2 + 2k_{\pm} + 1\right) a_0^-}{\gamma^2 + (k_{\pm} + n + 2)(k_{\pm} + n + \frac{1}{2})}.
\end{align*}
\]  \tag{37}

Each series provides an exact solution for small \( |z - 1| \), and can be shown to be convergent for \( |u| = |z - 1| < 1 \). The general solution at each shell is therefore a linear combination of the two,

\[ \Psi_{j,\pm}[u_j] = \sum_{n=0}^{\infty} a_{j,n}^\pm u_j^{k_{j,\pm} + n}, \]  \tag{38}
where we have reinserted the shell label, $j$. The very first order in the expansion gives precisely \( \Psi_j \), and therefore Hawking’s formula. Because this is the exact quantum state of the LTB class of collapse models it also contains all the information about the quantum evolution of the collapse close to the horizon, up to the scale of validity of the canonical theory. We have shown that the first order approximation yields the standard Hawking radiation at the Hawking temperature \( \frac{\hbar c}{8\pi GM} \). The higher orders represent corrections to Hawking’s original formula.

V. HAWKING RADIATION

We now demonstrate that the leading correction to Hawking radiation makes the spectrum non-thermal. First we show that the Hawking spectrum itself arises from retaining only the \( n = 0 \) term and dropping all terms of \( \mathcal{O}(\hbar) \) in the exponent of \( \Psi_\omega \). If one further assumes that \( \gamma \gg 1 \), then \( \Psi_\omega \) gives \( k_\pm \approx 1/4 \pm i\gamma \), and the above shell by shell wave-functions can be written as (dropping the shell labels)

\[
\Psi_{\text{hor}} \approx A^\text{hor} \exp \left[ -i\omega \left( \tau + \frac{F}{c} \ln |z - 1| \right) \right] \tag{39}
\]

and the asymptotic WKB wave-functions in \( \Psi_\omega \) are

\[
\Psi_{\pm} \approx A_\pm \exp \left[ -i\omega \left( \tau \pm \frac{2F}{c} \sqrt{z} \right) \right]. \tag{40}
\]

Taking the negative sign in the exponent, the function \( \Psi_\omega \) in (40) represents a free outgoing dust mode of frequency \( \omega \). We take a complete basis of outgoing modes to be given by

\[
\Psi_\omega = \exp \left[ -i\omega' \left( \tau - \frac{2F}{c} \sqrt{z} \right) \right], \tag{41}
\]

letting \( \omega' \) take all possible values between zero and infinity.

The states \( \Psi_{\text{hor}} \) are the scattered modes near the horizon. We are interested in computing the projection of \( \Psi_\omega \) on the negative frequency modes of the outgoing basis. For this purpose, we must consider the inner product of the states on a hypersurface of constant Schwarzschild time. As explained in \( \Psi_{\text{hor}} \), the Bogoliubov \( \beta \)-coefficient of interest is given by

\[
\beta(\omega, \omega') = \langle \Psi_{\omega'}^\dagger | \Psi_{\text{hor}}^\omega \rangle = F \int_1^\infty \frac{zd\tau}{z - 1} \Psi_{\omega'}^\dagger \Psi_{\omega}^\text{hor}. \tag{42}
\]

The measure in the above integral is determined by transforming to the Schwarzschild Killing time, \( T \), which is related to the Tolman-Bondi (proper) time by a standard transformation \( \tau = e^{\frac{\hbar c}{8\pi GM}} T \). The wave-functionals considered must also be expressed in terms of the Schwarzschild Killing time. Ignoring the unimportant \( T \)-dependent part (which will go away on squaring) the Bogoliubov coefficient turns out to be

\[
\beta(\omega, \omega') \approx F \int_1^\infty \frac{zd\tau}{z - 1} e^{\frac{\hbar c}{8\pi GM} \sqrt{z} (z - 1) - 2i\frac{\hbar c}{8\pi GM}}. \tag{43}
\]

This integral can be performed, after making the substitution \( s = \sqrt{z} - 1 \). Retaining only the lowest order in \( s \) one gets

\[
\beta(\omega, \omega') \approx 2Fe^{\frac{\hbar c}{8\pi GM}} \int_0^\infty ds e^{-s^2} [2s]^{-1 - 2i\frac{\hbar c}{8\pi GM}}. \tag{44}
\]

After squaring, one obtains the desired transition probability as

\[
|\beta(\omega, \omega')|^2 = 2\pi^2 F^2 kT_H = \frac{1}{e^{\frac{\hbar c}{8\pi GM} T} - 1}. \tag{45}
\]

which is the Hawking spectrum at the Hawking temperature, \( kT_H = \hbar c/8\pi GM \). Note that \( M \) is the total mass contained within the radiating shell.

Apart from neglecting all \( \mathcal{O}(\hbar) \) corrections to the wave-functionals, the Hawking formula is recovered by employing a “near horizon” approximation in which only the lowest order in \( s = \sqrt{z} - 1 \) is considered. The latter is justified by the fact that only the near horizon region contributes significantly to the integral in (43) through the pole at \( z = 1 \).

Consider the correction to this formula, which is obtained by taking into account the first \( \mathcal{O}(\hbar) \) correction. Keeping the asymptotic form of the wave-functional the same, as given in (41), but including the first \( \mathcal{O}(\hbar) \) correction to the near horizon wave-functional in (39), we evaluate the Bogoliubov coefficient in (42) to leading approximation in \( s \) (i.e., near the horizon). With \( s = \sqrt{z} - 1 \), this is

\[
\beta(\omega, \omega') \approx 2Fe^{\frac{\hbar c}{8\pi GM}} \int_0^\infty ds e^{\frac{\hbar c}{8\pi GM}s} [2s]^{-1 - 2i\frac{\hbar c}{8\pi GM}}. \tag{46}
\]

Although it appears that the r.h.s. of (46) has the same structure as the WKB expression, it is evident that \( |\beta(\omega, \omega')|^2 \) is no longer thermal. When \( \gamma \gg 1 \) we find

\[
|\beta(\omega, \omega')|^2 \approx 2\pi^2 F^2 kT_H = \frac{1}{e^{\frac{\hbar c}{8\pi GM} T} - 1} \left[ 1 + \frac{1}{2} \text{Re} \psi \left( -\frac{i\varepsilon}{2\pi kT_H} \right) \right]. \tag{47}
\]

where \( \psi(z) \) is the Polygamma function. As the first correction depends on the Hawking temperature (or the
mass $M$) and not just the emission frequency, it cannot be associated simply with a correction to the density of states. Furthermore, it cannot be obtained by modifying the Hawking temperature. When $\gamma \gg 1$, the real part of the Polygamma function can be approximated by $\text{Re}[\psi(-iy)] \approx \ln y$ and we find,

$$|\beta(\omega, \omega')|^2 \approx 2\pi^2 F^2 \left[ \frac{2\epsilon k T_H}{\omega^2} \frac{1}{\epsilon^{2\pi n}} - 1 \right] \left[ 1 - \frac{1}{2} \ln \left( \frac{\pi k T_H}{\epsilon} \right) \right],$$

(48)

which completes the first correction to the WKB approximation. The correction term is of order $O(\ln \hbar)$.

VI. DISCUSSION

The correction we have obtained cannot be accounted for by modifying the Hawking temperature and we conclude that it renders the radiation non-thermal. The breakdown of Hawking’s thermal spectrum, when quantum gravitational effects are accounted for i.e., when we go beyond the WKB approximation, is not equivalent to a correction to the black hole entropy. Corrections to the black hole entropy formula have been computed in various approaches before [17], but they can all be understood as relating to a thermal spectrum, simply modifying the Hawking temperature. Put in another way, we have shown that there are “grey body” factors whose origin is quantum gravitational and they are associated with the horizon. They are not to be confused with grey body factors that are associated with the radiation at spatial infinity, which originate in the backscattering of the radiation against the classical space-time geometry.

The thermal character of the Hawking radiation from collapsing matter is sometimes invoked to suggest that quantum gravitational evolution may not be unitary and that information is lost. Our result indicates that this is not necessarily the case, at least within the context and limitations of the canonical theory. A more complete picture will be obtained from a deeper understanding of the behavior of solutions to (28). We hope to come back to a more developed discussion of the properties of the solutions to this equation in a future publication. One of the issues we expect to discuss will be the nature of the singularity at infinity. It is an essential singularity so there will be an issue regarding normalizing the wave function. The collapsing system we study has been a marginally bound system - physically this means that the total vacuum energy is zero. From quantum considerations we expect there to be a non-vanishing zero point energy. A preliminary estimate of the result of allowing a non-zero vacuum energy is that the singularity at infinity will be softened but that the general features of the solution discussed so far will be unchanged.

It was especially gratifying to see that the results are completely independent of the choice of cell sizes in the lattice.

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