Thomae’s function on a Lie group

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Abstract

Let \( g \) be a simple complex Lie algebra of finite dimension. This paper gives an inequality relating the order of an automorphism of \( g \) to the dimension of its fixed-point subalgebra, and characterizes those automorphisms of \( g \) for which equality occurs. This is amounts to an inequality/equality for Thomae’s function on \( \text{Aut}(g) \). The result has applications to characters of zero weight spaces, graded Lie algebras, and inequalities for adjoint Swan conductors.

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1 Introduction

Thomae’s function $\tau : \mathbb{R} \to \mathbb{R}$ is discontinuous precisely on the rational numbers. It is traditionally defined as $\tau(x) = 1/m$ if $x = n/m$ is rational in lowest terms with $m > 0$, and $\tau(x) = 0$ if $x$ is irrational. So $\tau(n) = 1$ for every integer $n$, and on each open interval $(n, n+1)$ the maximum value of $\tau$ is $1/2$, taken just at the midpoint of the interval. More succinctly, $\tau(x)$ is the reciprocal of the order of $x$ in the group $\mathbb{R}/\mathbb{Z}$, with the convention that $1/\infty = 0$.

Every group $G$ has an analogous function $\tau_G : G \to \mathbb{Q}$, whose value at $g \in G$ equal to the reciprocal of the order of $g$. 

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Consider the group $G = \text{SO}_3$ of rotations about a fixed point $O$ in three-dimensional Euclidean space. Here $\tau_G(g) = 1/m$ if $g$ rotates by a rational multiple $n/m$ (in lowest terms) of a full circle and $\tau_G(g) = 0$ otherwise. So $\tau_G(g) = 1$ if $g$ is the identity rotation and elsewhere $\tau_G$ has maximum value $1/2$ taken just on the conjugacy class of half-turns. Since every element of $G$ is conjugate to a rotation about a fixed axis through $O$, this example is essentially the same as Thomae’s original one, but now we observe that $1/2 = 1/h$, where $h$ is the Coxeter number of $G$.

Suppose $G$ is either a compact Lie group or a complex algebraic group. For such groups the function $\tau_G$ is discontinuous precisely on the set of torsion elements in $G$. The proof is the same as for $\tau = \tau_{\mathbb{R}/\mathbb{Z}}$, using the facts 1) that torsion elements can be approximated by elements of infinite order, 2) for every $\varepsilon > 0$ there are only finitely many conjugacy classes in $G$ whose elements have order $\leq 1/\varepsilon$, and 3) the conjugacy class of any torsion element is closed in $G$.

If $G$ is connected and simple as an abstract group then on the regular elements of $G$ we have $\tau_G(g) \leq 1/h$, where $h$ is the Coxeter number of $G$. Equality holds on just the conjugacy class of principal elements. These are the analogues of the half-turns in $\text{SO}_3$ and were studied by Kostant in [12].

The aim of this paper is to extend this inequality/equality for Thomae’s function to singular elements in the group $G = \text{Aut}(\mathfrak{g})$ of automorphisms of a simple complex Lie algebra $\mathfrak{g}$ of finite dimension. We also indicate some applications of the result.

We will measure the singularity of an element $\theta \in G$ by the dimension of the fixed-point subalgebra $\mathfrak{g}^\theta$. We will give an upper bound for $\tau_G(\theta)$ in terms of $\dim \mathfrak{g}^\theta$, along with precise conditions for equality.

To explain these conditions we need some preparation. We say that an element $\theta \in G$ is ell-reg if $\theta$ normalizes a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that (i) $\mathfrak{t}^\theta = 0$ and (ii) the cyclic group generated by $\theta$ permutes the roots of $\mathfrak{t}$ in $\mathfrak{g}$ freely. There are only finitely many ell-reg classes in $\text{Aut}(\mathfrak{g})$. Their classification was given in [20] and is recalled in the appendix to this paper.\footnote{Such automorphisms are called $\mathbb{Z}$-regular in [20], in deference to [21]. In general, ell-reg elements are not regular elements of $G$. The point of “ell-reg”, besides brevity, is to avoid conflict between these two meanings of the word “regular”.
}

For ell-reg automorphisms it is known that the automorphism of $\mathfrak{t}$ given by $\theta|_{\mathfrak{t}}$ (as in (i) and (ii)) has the same order as $\theta$. It follows that if $\theta \in G$ is ell-reg then

$$\tau_G(\theta) = \frac{\dim \mathfrak{g}^\theta}{\dim(\mathfrak{g}/\mathfrak{t})},$$

where $\mathfrak{t}$ is any Cartan subalgebra of $\mathfrak{g}$.

Fix a connected component $\Gamma$ of $G$ and let $e \in \{1, 2, 3\}$ be the order of $\Gamma$ in the group $\text{Out}(\mathfrak{g})$ of connected components of $G$. If $\theta \in \Gamma$, the rank of $\mathfrak{g}^\theta$ depends only on $e$; we
write

\[ n_e = \text{rank}(g^\theta). \]

In \( \Gamma \) there is a unique conjugacy class \( P_\Gamma \) of elements \( \theta \) of minimal order for which \( g^\theta \) is a Cartan subalgebra of \( g^\theta \). This order, denoted \( h_e \), is the \textit{twisted Coxeter number} of the coset \( \Gamma \) [18]. The elements of \( P_\Gamma \) are ell-reg and it is known that

\[ \frac{1}{h_e} = \frac{n_e}{\text{dim}(g/t)}, \quad \text{if} \quad \theta \in P_\Gamma. \quad (2) \]

It follows that if \( \theta \in \Gamma \) has order \( m > h_e \), then

\[ \tau_G = \frac{1}{m} < \frac{\text{dim} g^\theta}{\text{dim}(g/t)}, \quad (3) \]

Where \( \tau_G \) is Thomae’s function for the group \( G = \text{Aut}(g) \). In this paper we extend (3) to all \( \theta \in \text{Aut}(g) \) as follows.

\textbf{Theorem 1} Let \( g \) be a simple complex Lie algebra of finite dimension and let \( \tau_G \) be Thomae’s function for the group \( G = \text{Aut}(g) \). Then for all \( \theta \in G \) we have

\[ \tau_G(\theta) \leq \frac{\text{dim} g^\theta}{\text{dim}(g/t)}. \quad (4) \]

Equality holds in (4) if and only if \( \theta \) is ell-reg.

From (2) we have equality in (4) if \( \theta \in P_\Gamma \). Also (4) holds trivially, and is a strict inequality, if the order of \( \theta \) is larger than \( h_e \), by (3). Therefore, the content of Theorem 1 is (i) the inequality (4) for all \( \theta \in G \) whose order \( m \) lies in the range \( 1 < m < h_e \), and (ii) the assertion that only ell-reg elements attain equality.

The proof of Theorem 1 consists of computations with Kac diagrams. It is given in section 3.

\section{Applications}

First we give some applications of Theorem 1 and connections to other results.

\subsection{Characters of zero-weight spaces}

This section describes the role played by Theorem 1 in the computation of characters of zero weight spaces [19].

An earlier version of this paper was an appendix to an earlier version of [19].
Let \( G \) be a connected and simply connected complex Lie group. Let \( T \) be a maximal torus in \( G \), with normalizer \( N \) and Weyl group \( W = N/T \). In every finite-dimensional irreducible representation \( V \) of \( G \), the zero weight space \( V^T \) affords a representation of \( W \). The problem is to compute the character of \( V^T \) afforded by \( W \), as a function of the highest weight of \( V \).

For example, in [13], Kostant used his results on principle elements to calculate the trace \( \text{tr}(\text{cox}, V^T) \) of a Coxeter element \( \text{cox} \in W \). He showed that \( \text{tr}(\text{cox}, V^T) \) is 0 or \( \pm 1 \) and gave an explicit formula for this trace in terms of the highest weight of \( V \).

In [16], Kostant’s proof was reformulated in terms of the dual group \( \hat{G} \) of \( G \). Since \( G \) is simply-connected, \( \hat{G} \) is the group of inner automorphisms of the Lie algebra \( \hat{g} \) whose root system is dual to that of \( g \). In [19], Theorem 1 is used in \( \hat{G} \) to compute traces of other Weyl group elements on \( V^T \).

We call an element \( w \in W \) ell-reg if (i) \( t^w = 0 \) and (ii) the group \( \langle w \rangle \) generated by \( w \) acts freely on the roots of \( t \) in \( g \). Equivalently, \( w \) is ell-reg if \( \text{Ad}(n) \) is an ell-reg automorphism of \( g \) for one (every) lift of \( w \) in \( N \). The classification of ell-reg elements of \( W \) is given by the untwisted diagrams in the Appendix 5.

Let \( w \in W \) be ell-reg of order \( m \) and let \( V = V_\lambda \) have highest weight \( \lambda \), with respect to a choice of positive roots \( R^+ \) for \( T \) in \( G \). Let \( P \) and \( Q \) be the weight and root-lattices of \( T \), so that \( P \) and \( Q \) are also the co-weight and co-root lattices of the dual maximal torus \( \hat{T} \) of \( \hat{G} \). Let \( \rho \in P \) be the half-sum of the roots in \( R^+ \). Let \( \theta \in \hat{T} \) be the value at \( e^{2\pi i/m} \) of the one-parameter subgroup \( \lambda + \rho \).

From Theorem 1 we get an inequality of centralizers
\[
\dim C_G(t) \leq \dim C_{\hat{G}}(\theta),
\] (5)
with equality if and only if \( (\lambda + \rho) + mQ \) is conjugate to \( \rho + mQ \) under the natural \( W \)-action on \( P/mQ \). From this inequality and the theory of \( W \)-harmonic polynomials we deduce that \( \text{tr}(w, V^T_\lambda) = 0 \) unless there exists \( v \in W \) such that \( v(\lambda + \rho) \in \rho + mQ \), in which case
\[
\text{tr}(w, V^T_\lambda) = \text{sgn}(v) \prod_{\tilde{\alpha} \in \hat{R}^+_m} \frac{\langle v(\lambda + \rho), \tilde{\alpha} \rangle}{\langle \rho, \tilde{\alpha} \rangle},
\]
where the product is over the positive co-roots \( \tilde{\alpha} \) of \( G \) for which \( \langle \rho, \tilde{\alpha} \rangle \in m\mathbb{Z} \). If \( m = h \) is the Coxeter number then \( \hat{R}^+_m = \emptyset \) and we recover Kostant’s result for \( \text{tr}(\text{cox}, V^T_\lambda) \).

### 2.2 Graded Lie algebras

Let \( \theta \in \text{Aut}(g) \) have order \( m \), and let \( \zeta = e^{2\pi i/m} \). Then \( \theta \) determines a \( \mathbb{Z}/m\mathbb{Z} \) grading
\[
g = \bigoplus_{k \in \mathbb{Z}/m\mathbb{Z}} g_k,
\] (6)
where \( g_k = \{ x \in g : \theta(x) = \zeta^k x \} \). Note that \( g_0 = g^\theta \).

From [20, Cor.14] it is known that the following are equivalent:

(i) There exists a semisimple element \( x \in g_1 \) for which \( \text{ad}(x) : g_0 \to g_1 \) is injective;

(ii) \( \theta \) is ell-reg.

Therefore we can also use (i) as the condition for equality in Theorem 1.

Theorem 1 makes no \textit{a priori} assumptions on the kinds of elements contained in \( g_1 \). But let us now assume that \( g_1 \) contains nonzero semisimple elements. Such gradings are said to have \textbf{positive rank}. Their classification is contained in the combined papers [22], [14] and [20].

In the case of positive rank gradings, Theorem 1 complements results of Panyushev. According to [15, Thm 2.1], if \( x \in g_1 \) is semisimple, then

\[
\frac{1}{m} = \frac{\dim[g_0, x]}{\dim[g, x]}. \tag{7}
\]

Since \( \dim[g_0, x] \leq \dim g_0 \) with equality exactly when (i) holds for \( x \), and since \( \dim[g, x] \leq \dim(g/t) \) with equality exactly when \( x \) is a regular element of \( g \), Theorem 1 combines with (7) to interpose \( \dim(g/t)/m \) in the inequality \( \dim[g_0, x] \leq \dim g_0 \). That is, we have

\textbf{Corollary 2} Assume there is a semisimple element \( x \in g_1 \). Then we have two inequalities

\[
\frac{\dim[g_0, x]}{1} \leq \frac{\dim(g/t)}{m} \leq \frac{2 \dim g_0}{2}
\]

\( \leq \frac{1}{1} \) is equality if and only if \( x \) is regular (semisimple), and \( \leq \frac{2}{2} \) is equality if and only if \( \theta \) is ell-reg.

Under the additional assumption that \( g_1 \) contains a regular semisimple element, Panyushev [15, 4.2] also showed that

\[
\dim g_0 = \frac{\dim[g/t]}{m} + k_0,
\]

where \( k_0 \geq 0 \) is an integer depending only on the orders \( m \) and \( e \) of \( \theta \) in \( \text{Aut}(g) \) and \( \text{Out}(g) \). For example, if \( e = 1 \) then \( k_0 \) is the number of exponents of \( g \) divisible by \( m \). This is a sharper form of Theorem 1 in the case that \( g_1 \) contains a regular semisimple element.

\subsection*{2.3 Adjoint Swan Conductors}

In the setting of section 2.1, sending a representation \( V \) to its highest weight \( \lambda \) is a simple case of the much broader and still partly conjectural local Langlands correspondence.
(LLC). In section 2.1 we saw that the inequalities/equalities of Theorem 1 appear on the
dual side of this LLC.

They also appear on the dual side of the LLC for reductive $p$-adic groups, now as mea-

sures of ramification.

We use notation parallel to that of section 2.1. Let $k$ be a $p$-adic field and let $G$ be the group
of $k$-rational points in a connected and simply-connected almost simple $k$-group $G$.

Let $\hat{g}$ be a simple complex Lie algebra whose root system is dual to that of $G$.

The LLC predicts the existence of a partition

$$\text{Irr}^2(G) = \bigsqcup_{\phi} \Pi_{\phi},$$

of the set $\text{Irr}^2(G)$ of irreducible discrete series representations of $G$ (up to equivalence)
into finite sets $\Pi_{\phi}$, where $\phi$ ranges over certain representations $\phi : \mathcal{W}_k \to \text{Aut}(\hat{g})$ of the
Weil group of $k$. (See [10] for more background on the LLC.)

It is of interest to find invariants relating the discrete series representation $\pi$ of $G$ to the
representation $\phi$ of $\mathcal{W}_k$ for which $\pi \in \Pi_{\phi}$.

One invariant of $\phi$ is its adjoint Swan conductor $sw(\phi, g)$. This is an integer depending only
on the image $I = \phi(I)$ of the inertia subgroup $I \subset \mathcal{W}_k$. We have a factorization $I = S \times P$,
where $P$ is a $p$-group and $S$ is a cyclic group of order prime to $p$. We have $sw(\phi, g) \geq 0$,
with equality if and only if $P$ is trivial.

Expected properties of the LLC imply certain inequalities for $sw(\phi, g)$. These inequalities
have been found to hold unconditionally. For example if $\phi$ is totally ramified (that is, if $g.I = 0$) then the LLC predicts that

$$\dim g^\theta \leq sw(\phi, g),$$

where $\theta$ is a generator of $S$. This inequality has been proved in [17] and [8].

Assume now that $p$ does not divide the order of $W$. Let $m$ be the order of $\theta$. Combining
(8) with Theorem 1 gives the inequality

$$\frac{\dim g/t}{m} \leq sw(\phi, g),$$

which is weaker than (8), but which depends only of the order $m$ of $S$, not on $S$ itself. Moreover, the two inequalities (8) and (9) coincide if and only if $\theta$ is ell-reg.

3 Proof of Theorem 1

The torsion automorphisms of $g$ are classified by Kac diagrams. We start with a summary
of Kac diagrams that I hope is sufficient for the reader to follow the computations. For
more background, see [11] and [18].

3.1 Kac Diagrams

Fix a divisor \( e \in \{1, 2, 3\} \) of the order of the component group \( \text{Out}(g) \) of \( \text{Aut}(g) \). Let \( \text{Aut}(g, e) \) be the set of elements in \( \text{Aut}(g) \) whose image in \( \text{Out}(g) \) has order \( e \). Then \( \text{Aut}(g, e) \) has one or two connected components, the latter only when \( g = \text{so}_8 \) and \( e = 3 \).

For any torsion automorphism \( \theta \in \text{Aut}(g, e) \), the rank of the fixed point subalgebra \( g^\theta \) depends only on \( e \); we denote this rank by \( n_e \). If \( e = 1 \) then \( G_1 := \text{Aut}(g, 1) \) is the identity component of \( \text{Aut}(g) \) and \( n_1 \) is the rank of \( g \).

To the pair \((g, e)\) one associates an affine Dynkin diagram \( D(g, e) \). As we vary over all pairs \((g, e)\), the diagrams \( D(g, e) \) range exactly over the affine Coxeter diagrams together with all possible orientations on the multiple edges. If \( e = 1 \), then \( D(g, 1) \) is the usual affine Dynkin diagram of \( g \).

The vertices in \( D(g, e) \) are indexed by a set \( I \) whose cardinality is \( n_e + 1 \), and these vertices are labelled by certain positive integers \( \{c_i : i \in I\} \), where \( 1 \leq c_i \leq 6 \).

The automorphism group \( \text{Aut}(D(g, e)) \) of the oriented and labelled diagram \( D(g, e) \) contains a (very small) subgroup \( \Omega \) with the following property: If \( e > 1 \) then \( \Omega = \text{Aut}(D(g, e)) \). If \( e = 1 \) then \( \Omega \simeq \pi_1(G) \).

We fix a connected component \( \Gamma \) of \( \text{Aut}(g, e) \). For any positive integer \( m \), let \( \Gamma_m \) be the set of elements of \( \Gamma \) having order \( m \). Then \( \Gamma_m \) is nonempty if and only if \( e \) divides \( m \). The \( G_1 \)-conjugacy classes in \( \Gamma_m \) are parametrized as follows. Let \( S_m \) be the set of \( I \)-tuples \( s = (s_i : i \in I) \) consisting of integers \( s_i \geq 0 \) such that gcd \( \{s_i : i \in I\} = 1 \) and

\[
m = e \cdot \sum_{i \in I} c_i s_i.
\]

There is a surjective mapping from \( S_m \) to the set of \( G_1 \)-conjugacy classes in \( \Gamma_m \) (Kac coordinates). Two elements \( s \) and \( s' \in S_m \) map to the same conjugacy class in \( \Gamma_m \) if and only if \( s \) and \( s' \) are conjugate under the group \( \Omega \).

For example in \( \Gamma \) there is a unique conjugacy class of automorphisms of minimal order having abelian fixed-point subalgebras. Such automorphisms are called principal. They are ell-reg and have Kac coordinates \( s = (s_i) \) where \( s_i = 1 \) for all \( i \). The order of a principal automorphism in \( \Gamma \), namely

\[
h_e := e \cdot \sum_{i \in I} c_i s_i
\]

is the Coxeter number of \( \text{Aut}(g, e) \). It is known [18] that equality holds in Theorem 1 for principal elements, namely we have

\[
\frac{1}{h_e} = \frac{n_e}{[g : t]}.
\]
For any subset $J \subset I$ we set

$$c_J = \sum_{j \in J} c_j, \quad c_I = \sum_{i \not\in J} c_i.$$  

If $J \neq I$ then the subgraph of $D(g, e)$ supported on $J$ is the finite Dynkin graph of a reductive subalgebra $g_J$ of $g$. Let $|R_J|$ be the number of roots of $g_J$.

Let $\theta \in \Gamma$ be a torsion automorphism with Kac-coordinates $s = (s_i)$ and let $J = \{j \in I : s_j = 0\}$. Then $J \neq I$ and we have $g^\theta = g_J$.

**Lemma 3** The inequality in Theorem 1 for all torsion elements in a component $\Gamma \subset Aut(g, e)$ is equivalent to the inequality

$$n_e \cdot c_J \leq c_I \cdot |R_J|$$  \hspace{1cm} (11)

for every subset $J \subset I, J \neq I$.

**Proof:** Let $\theta \in \Gamma_m$ have Kac coordinates $(s_i)$ and let $J = \{j \in I : s_j = 0\}$. Then $m \geq e \cdot c_I$ with equality if and only if $s_i = 1$ for all $i \in I - J$. Since

$$\dim g^\theta = \dim g_J = n_e + |R_J|, \quad \text{and} \quad \dim(g/t) = h_e n_e = e \cdot c_I \cdot n_e$$

It follows that

$$\frac{1}{m} \leq \frac{1}{e \cdot c_I} \quad \text{and} \quad \frac{\dim g^\theta}{\dim(g/t)} = \frac{n_e + |R_J|}{e \cdot c_I \cdot n_e},$$

so the inequality in Theorem 1 for every $\theta$ is equivalent to having

$$e \cdot c_I \cdot n_e \leq (n_e + |R_J|) \cdot e \cdot c_I$$

for every $J$. Since $c_I = c_I + c_J$, the result follows. 

If $J$ is empty then both sides of (11) are zero. We may assume from now on that $J$ is nonempty and that $s_i = 1$ for all $i \in I - J$. Thus $J$ is identified with a Kac diagram.

We will show that the integer $f(g, e, J)$ defined by

$$f(g, e, J) = c_I |R_J| - n_e c_J$$

satisfies $f(g, e, J) \geq 0$. Our analysis will also find those $J$ for which $f(g, e, J) = 0$.

On the other hand, the Kac diagrams of the ell-reg automorphisms of $g$ were tabulated in [20] section 7 and are recalled in the Appendix 5 of this paper. It turns out that the Kac diagrams of ell-reg automorphisms are exactly those for which $f(g, e, J) = 0$. 

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3.2 Type $A_n$

The case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\epsilon = 1$ is very simple but different from the other cases, so we treat it separately here. Fix a nonempty subset $J \subset I$. The root system $R_J$ has type

$$\prod_{i=1}^{a} A_{q_i}$$

for some positive integers $q_1, \ldots, q_a$. Let $q = \sum q_i$. Since all $c_i = 1$ we have $c_J = q$ and $c_J = n + 1 - q \geq a$. Now

$$f(\mathfrak{g}, 1, J) = c_J \sum q_i (q_i + 1) - (c_J + q - 1)q = c_J \sum q_i^2 - q^2 + q \geq a \sum q_i^2 - q^2 + q \geq q.$$

where the arithmetic-geometric inequality is used in the last step. Since $J \neq \emptyset$ we have $f(\mathfrak{g}, 1, J) \geq q > 0$.

3.3 Preliminary reductions

In this section, $(\mathfrak{g}, \epsilon)$ is of classical type not equal to $(\mathfrak{sl}_n, 1)$. We will write $n = n_\epsilon$ and $h = h_\epsilon$.

The relevant diagrams $D(\mathfrak{g}, \epsilon)$ for $n \geq 3$ are listed below. Each diagram has $n + 1$ nodes. They are grouped according to their underlying Coxeter diagram.
Let $X$ be the set of all triples $(g, e, J)$, where $(g, e)$ is one of the above classical types for $n \geq 3$ and $J$ is a nonempty proper subset of the set $I$ of vertices of $\mathcal{D}(g, e)$. Let $X_0 = \{(g, e, J) : f(e, g, J) = 0\}$. We must prove that $f \geq 0$ on $X$ and that $X_0$ consists precisely of the diagrams listed in the appendix 5 for classical $(g, e)$.

By induction on $n$, we may assume that $f(g', e, J') \geq 0$ for all $(g', e)$ of the same type as $(g, e)$, such that $\mathcal{D}(g', e)$ has at most $n$ nodes indexed by a set $I'$ and $J' \subset I'$.

**Definition.** If $Y' \subset Y$ are subsets of $X$, we say $Y'$ is a refinement of $Y$ if for every $(g, e, J) \in Y - Y'$, we have either

(i) $f(g, e, J) > 0$ or

(ii) there exists $(g', e', J') \in Y'$ such that $f(g, e, J) > f(g', e', J')$.

We observe the following.

(i) Refinement is transitive: if $Y''$ is a refinement of $Y'$ and $Y'$ is a refinement of $Y$ then $Y''$ is a refinement of $Y$.

(ii) If $Y$ is a refinement of $X$ and $f \geq 0$ on $Y$ then $f > 0$ on $X - Y$ and $X_0 = Y_0$.

| $(g, e)$ | $\mathcal{D}(g, e)$ | $h = e \cdot c_1$ |
|----------|---------------------|-------------------|
| $^{2A_{2n}} \underset{n\geq2}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2(2n + 1)$ |
| $^{C_n} \underset{n\geq2}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2n$ |
| $^{2D_{n+1}} \underset{n\geq2}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2(n + 1)$ |
| $^{2A_{2n-1}} \underset{n\geq3}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2(2n - 1)$ |
| $^{B_n} \underset{n\geq3}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2n$ |
| $^{D_n} \underset{n\geq4}{\longrightarrow}$ | $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \circ$ | $2n - 2$ |
From (ii), it suffices to find a refinement \( \mathcal{U} \) of \( \mathcal{X} \) such that \( f \geq 0 \) on \( \mathcal{U} \) and that \( \mathcal{U}_0 \) consists precisely of the ell-reg triples listed in the appendix.

Say that a vertex \( i \in I \) is interior if \( i \) is adjacent to at least two other vertices in \( D(g, e) \). In every pair of adjacent vertices, at least one vertex is interior. The table of diagrams shows that all interior \( i' \)'s have the same value \( c_i \) (\( c = 1 \) in type \( 2D_{2n+1} \) and \( c = 2 \) in the other classical diagrams), and \( c \geq c_i \) for all \( i \in I \).

**Lemma 4** Let \( \mathcal{U} \) be the set of \( (g, e, J) \) \( \in \mathcal{X} \) for which no two interior vertices of \( I - J \) are adjacent in \( D(g, e) \). Then \( \mathcal{U} \) is a refinement of \( \mathcal{X} \).

**Proof:** Consider a triple \( (g, e, J) \in \mathcal{X} \) and let \( i, j \in I - J \) be adjacent vertices in \( D(g, e) \).

First assume either that \( i \) has degree two, or that \( j \) is also interior. Let \( k \) be another vertex adjacent to \( i \). The possible configurations of \( i, j, k \) in the Kac diagram are

\[
\begin{array}{ccccccc}
\cdots & j & i & k & \cdots \\
\cdots & j & i & = & k \\
\cdots & k & i & j \\
\end{array}
\]

where \( *, \cdot \in \{0, 1\} \) are arbitrary.

Removing \( i \) and joining \( j \) to \( k \) with a bond of the appropriate type, we obtain a diagram \( D(g', e) \) of the the same type as \( D(g, e) \). The vertices of \( D(g', e) \) are indexed by \( I' = I - \{i\} \) and we have \( J \subset I' \). In this way, the diagram \( D(g, e, J) \) contracts by one vertex to the diagram \( D(g', e, J) \). The root system \( R'_j \) of \( g'_j \) is isomorphic to \( R_j \), we have \( \sum_{i' \in I' - J} c_{i'} = c^j - c \) and \( c_j \) is unchanged. It follows that

\[
f(g, e, J) - f(g', e, J) = [c^j |R_j| - nc_j] - [(c^j - c)|R_j| - (n - 1)c_j] \\
= c|R_j| - c_j \geq 2c|J| - c|J| = c|J| > 0.
\]

Since \( |I' - J| = |I - J| - 1 \), repeating this procedure will eventually produce a diagram \( D(g'', e, J) \in \mathcal{U} \) and we will have \( f(g, e, J) > f(g'', e, J) \).

Now assume \( i \) has degree three and \( j \) is one of the two boundary vertices adjacent to \( i \). Let \( k \) be the interior vertex adjacent to \( i \).

From the first case we may assume \( k \in J \). So \( (g, e, J) \) has the form

\[
J : \begin{array}{ccccccccc}
\vdots & j & i & k & \cdots & q \text{ vertices} \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots \\
\end{array}
\]

where \( s \in \{0, 1\} \) and \( q \geq 0 \). Switch \( s_j \) and \( s_k \) to obtain

\[
J' : \begin{array}{ccccccccc}
\vdots & j & i & k & \cdots & q \text{ vertices} \\
1 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots \\
\end{array}
\]

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If $q = 0$ then $f(g, e, J) = f(g, e, J') > 0$ from the previous case.

Assume $q \geq 1$. Since $c' = c$, $n' = n$ and $c_{J'} = c_J$, we find that

$$f(g, e, J) - f(g, e, J') = 2(q + s - 1)c \geq 0$$

with equality only if $q = 1$ and $s = 0$. In this case,

$$J': \begin{array}{cccc}
  j & i & k & \\
  1 & 0 & 1 & 0 \ldots \\
  0 &
\end{array} \quad (12)$$

In section 3.3.1 we will prove directly that for the case (12) we have $f(g, e, J') > 0$, so $f(g, e, J) > 0$. ■

Our next refinement heads toward equilibrium for the interior components of $R_J$.

Given a diagram $D(g, e, J) \in \mathcal{X}$, let $J^\circ$ be the set of interior vertices in $J$ and let $R_j^\circ$ be the union of those irreducible components of $R_j$ whose bases are contained in $J^\circ$. Let $R_1, R_2, \ldots, R_a$ be the components of $R_j^\circ$. Each $R_i$ has type $A_{q_i}$ for some integers $q_i \geq 1$. Let

$$d(J) = \max\{|q_i - q_j| : 1 \leq i \leq a\}.$$  

be the largest difference between the ranks of any two components of $R_j^\circ$.

Lemma 5 Let $\mathcal{V}$ be as in Lemma 4 and let $Z$ be the set of $(g, e, J) \in \mathcal{V}$ for which $d(J) \leq 1$. Then $Z$ is a refinement of $\mathcal{V}$.

Proof: The value of $f(g, e, J)$ is unchanged by permuting the components $R_1, \ldots, R_a$. If $d(J) \geq 2$, then we may choose such a permutation to arrange that $q_1 - q_2 \geq 2$ and there are three interior vertices $\{i, j, k\}$ such that $j \in R_1, i \in I - J, k \in R_2$, as shown.

\[ \cdots 0 - i - k \cdots. \]

Now switch $s_i$ and $s_j$ to obtain a diagram $(g, e, J')$:

\[ \cdots j - 1 - 0 \cdots. \]

Then $n, c_I$ and $c_J$ are unchanged, and one checks that

$$f(g, e, J) - f(g, e, J') = 2c_J(q_1 - q_2 - 1) > 0.$$

Repeating this process, we eventually find a subset $J'' \subset I$ with $f(g, e, J) > f(g, e, J'')$ and $d(J'') \leq 1$. ■

It suffices to calculate $f(g, e, J)$ only for diagrams $(g, e, J)$ in the set $Z$ of Lemma 5. Note that $Z$ consists of those diagrams for which no two interior vertices in $I - J$ are adjacent and whose components of $R_j^\circ$ have at most two types $A_{q_j-1}$ and $A_{q_j}$ occurring, say, $x$ and $y$ times respectively.
Lemma 6 For \((g, e, J) \in Z\), the integer \(f(g, e, J)\) has the form
\[
f(g, e, J) = cxy + ax + \beta y + \gamma,
\]
where \(c\) is the common value of \(c_i\) on the interior vertices of \(I\) and \(\alpha, \beta, \gamma\) are polynomial expressions in \(q\). Also, \(\beta\) is obtained from \(\alpha\) upon replacing \(q\) by \(q + 1\).

Proof: Let \(R_{\partial J}\) be the union of the components of \(R_J\) not in \(R_J^0\) and let \(\partial J\) be the subset of \(J\) supporting \(R_{\partial J}\). Then \(R_J = R_{\partial J} \cup R_J^0\) so we have
\[
|R_J| = |R_{\partial J}| + q(q + 1)x + q(q + 1)y \quad \text{and} \quad c_J = c_{\partial J} + (q - 1)x + c_qy,
\]
where
\[
c_{\partial J} = \sum_{j \in \partial J} c_j.
\]
Define integers \(a\) and \(b\) by
\[
c^I = a + c(x + y) \quad \text{and} \quad n = b + qx + (q + 1)y.
\]
Then
\[
f(g, e, J) = |R_J|c^I - nc_J
\]
\[
= [|R_{\partial J}| + q(q - 1)x + q(q + 1)y] \cdot [a + cx + cy] - [b + qx + (q + 1)y] \cdot c_{\partial J} + (q - 1)x + c_qy]
\]
\[
= cxy + ax + \beta y + \gamma,
\]
where
\[
\alpha = [c|R_{\partial J}| + aq(q - 1)] - [bc(q - 1) + q_{\partial J}]
\]
\[
\beta = [c|R_{\partial J}| + aq(q + 1)] - [bc(q + 1) + (q + 1)c_{\partial J}]
\]
\[
\gamma = a|R_{\partial J}| - bc_{\partial J},
\]
(13)
as claimed.

We will show that \(\alpha, \beta, \gamma\) are \(\geq 0\). This implies that \(f(g, e, J) \geq 0\), with equality if and only if \(0 = xy = \alpha = \gamma = \beta\). Without loss of generality, we may then assume \(y = 0\). Theorem 1 will follow by comparing with the tables of ell-reg automorphisms.

3.3.1 Types \(\mathbf{2A}_{2n}, \mathbf{C}_n, \mathbf{2D}_{n+1}\)

The underlying Coxeter diagram with indexing set \(I = [0, n]\) is
\[
0 = 1 - 2 - \cdots - (n - 1) = n
\]
The three types differ only in the labels \(c_i\) which do not affect \(|R_J|\). Let \((g, e)\) and \((g', e')\) be two of \(\mathbf{2A}_{2n}, \mathbf{C}_n, \mathbf{2D}_{n+1}\), with corresponding labellings \(c_i, c'_i\). For each subset \(A \subset I\) we set
\[
c_A = \sum_{i \in A} c_i \quad c'_A = \sum_{i \in A} c'_i.
\]
We set $K = I - J$.

We will compare
\[ f = f(g, e, J) = |R_J|c_K - nc_J \quad \text{and} \quad f' = f(g', e', J) = |R_J|c'_K - nc'_J. \]

Suppose $(g, e) = 2A_{2n}$ and $(g', e') = C_n$. If $n \in K$ then $c_K = c'_K + 1$ and $c_J = c'_J$, so $f > f'$. If $n \in J$ then $c_K = c'_K$ and $c_J = c'_J + 1$ so $f < f'$.

Suppose $(g, e) = 2A_{2n}$ and $(g', e') = 2D_{n+1}$. If $0 \in K$ then $1 + c_K = 2c'_K$ and $c_J = 2c'_J$, so $2f' > f$. If $0 \in J$ then $1 + c_J = 2c'_J$ and $c_K = 2c'_K$, so $f > 2f'$.

Suppose $(g, e) = C_n$ and $(g', e') = 2D_{n+1}$. If $\{0, n\} \subset J$ then $c_K = 2c'_K$ and $c_J = 2c'_J - 2$ so $f = 2f' + 2n > f'$. If $0 \in J$ and $1 \in K$ then $c_K + 1 = 2c'_K$ and $c_J + 1 = 2c'_J$ so $2f' = f + |R_J| - n$. Since no two vertices in $K$ are adjacent, it follows that $|R_J| \geq n + 1$, so $2f' > f$.

This discussion shows that we need only consider the following three cases.

1. $(g, e) = 2A_{2n}$ with $0 \in K$ and $n \in J$
2. $(g, e) = C_n$ with $\{0, n\} \subset K$
3. $(g, e) = 2D_{n+1}$ with $\{0, n\} \subset J$

Indeed, if $f(J, g, e) \geq 0$ in cases 1-3 then $f(J, g, e) \geq 0$ in all cases and we can only have $f(J, g, e) = 0$ in cases 1-3.

**Case 1:** $(g, e) = 2A_{2n}$ and $R_J = B_r + xA_{q-1} + yA_q, r \geq 1$.

\[
|R_J| = 2r^2 + q(q - 1)x + q(q + 1)y \\
n = r + xq + y(q + 1) \\
\gamma = 0 \\
c_K = 1 + 2x + 2y \\
c_J = 2(q - 1)x + 2qy + 2r \\
\alpha = (q - 2r)(q - 2r - 1)
\]

Thus we have $f(g, e, J) \geq 0$ with equality if and only if $q = 2r$ or $2r + 1$. These cases are the last two rows in the table of appendix 5.1 for $n \geq 2$.

**Case 2:** $(g, e) = C_n$ and $R_J = xA_{q-1} + yA_q$.

\[
|R_J| = q(q - 1)x + q(q + 1)y \\
n = qx + (q + 1)y \\
\gamma = 0 \\
c_K = 2x + 2y \\
c_J = 2(q - 1)x + 2qy \\
\alpha = 0
\]

Thus we have $f(g, e, J) \geq 0$ with equality if and only if $xy = 0$. These are the cases with $k = x$ in the table of appendix 5.4.

**Case 3:** $(g, e) = 2D_{n+1}$ and $R_J = B_p + xA_{q-1} + yA_q + B_r, r, p > 0, q > 1$. 

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Thus we have \( f(\mathfrak{g}, e, J) \geq 0 \) with equality if and only if \( xy = 0 \), \( p = r \) and \( q = 2p \) or \( q = 2p + 1 \). These are the cases with \( k = q \geq 2 \) in the appendix 5.6.

### 3.3.2 Types \( 2A_{2n-1}, B_n \)

The underlying Coxeter diagram with indexing set \( I = [0, n] \) is

```
1 — 2 — 3 — \cdots (n-1) — n
```

The two types differ only in the label \( c_n = 1 \) for \( 2A_{2n-1} \) and \( c_n = 2 \) for \( B_n \). Comparing as in the previous section we may assume \( n \in K \) for \( 2A_{2n-1} \) and \( n \in J \) for \( B_n \).

**Case A1:** \( \{0, 1, n\} \subset J \quad R_J = D_p + xA_{q-1} + yA_q, \quad p \geq 2. \)

\[
|R_J| = 2p^2 + 2r^2 + q(q-1)x + q(q+1)y \\
n = p + r + qx + (q+1)y \\
\gamma = (p-r)^2
\]

\[
c_K = 1 + x + y \\
c_J = p + r + (q-1)x + qy \\
\alpha = (p-r)^2(p + r - q)(p + r - q + 1)
\]

In this case we have \( f(\mathfrak{g}, e, J) \geq 0 \) with equality if and only if \( xy = 0 \) and \( q = 2p \) or \( q = 2p - 1 \). These are the cases with \( k = p \geq 2 \) in appendix 5.2.

**Case A2:** \( \{0, n\} \subset K,\ 1 \in J, \quad R_J = A_p + xA_{q-1} + yA_q \)

\[
|R_J| = p(p + 1) + q(q-1)x + q(q+1)y \\
n = 1 + p + qx + (q+1)y \\
\gamma = p + 1
\]

\[
c_K = 2 + 2x + 2y \\
c_J = 2p - 1 + 2(q-1)x + 2qy \\
\alpha = 2(p-q+1)^2 + q
\]

In this case we have \( f(\mathfrak{g}, e, J) > 0 \).

**Case A3:** \( \{0, 1, n\} \subset K, \quad R_J = xA_{q-1} + yA_q, \quad q \geq 2 \)

\[
|R_J| = q(q-1)x + q(q+1)y \\
n = 1 + qx + (q+1)y \\
\gamma = 0
\]

\[
c_K = 1 + 2x + 2y \\
c_J = 2(q-1)x + 2qy \\
\alpha = (q-1)(q-2)
\]

In this case we have \( f(\mathfrak{g}, e, J) \geq 0 \) with equality if and only if \( q = 2 \). This is the case \( k = n \) in appendix 5.2.
Case B1: \( \{0,1\} \subset J \), \( R_J = D_p + xA_{q-1} + yA_q + B_r \)

\(|R_J| = 2p(p - 1) + 2r^2 + q(q - 1)x + q(q + 1)y \quad c_K = 2(1 + x + y) \\
\quad n = p + r + qx + (q + 1)y \quad c_f = 2(p + r - 1) + 2(q - 1)x + 2qy \\
\quad \gamma = 2(p - r)(p - r - 1) \quad \alpha = 2(p - q)(p - q + 1) + (q - r)^2 + 3r^2 + 2p \)

In this case we have \( f(g, e, J) \geq 0 \) with equality if and only if \( p = r \) and \( q = 2r \), or \( p = r + 1 \) and \( q = 2r + 1 \). These are the cases in the last two rows of appendix 5.3 with \( k = q \).

Case B2: \( 0 \in K, 1 \in J \), \( R_J = A_p + xA_{q-1} + yA_q + B_r, p > 0 \)

\(|R_J| = p(p + 1) + 2r^2 + q(q - 1)x + q(q + 1)y \quad c_K = 2x + 2y \\
\quad n = p + r + 1 + qx + (q + 1)y \quad c_f = 2p + 2r - 1 + 2(q - 1)x + 2qy \\
\quad \gamma = (2r - p - 1)^2 + 3r \quad \alpha = 2(p - q)(p - q + 1) + (q - r)^2 + 3r^2 + 2p \)

In this case we have \( f(g, e, J) > 0 \).

Case B3: \( \{0,1\} \subset K \), \( R_J = xA_{q-1} + yA_q + B_r, r \geq 1 \).

\(|R_J| = 2r^2 + q(q - 1)x + q(q + 1)y \quad c_K = 2x + 2y \\
\quad n = r + 1 + qx + (q + 1)y \quad c_f = 2r + 2(q - 1)x + 2qy \\
\quad \gamma = 2r(r - 1) \quad \alpha = 2(q - r)^2 + 2r(r - 1) \)

In this case we have \( f(g, e, J) \geq 0 \) with equality if and only if \( r = 1 \) and \( q = 2 \). This is the case \( k = 2 \) in appendix 5.3.

3.3.3 Type \( D_n \)

We choose the indexing set \( I = \{0,1,\ldots,n\} \) as in [6], so that \( I_1 = \{0,1,n - 1,n\} \). Up to automorphisms of \( D \), there are five cases for \( J \cap I_1 \).

Case 1: \( \{0,1,n - 1,n\} \subset J \), \( R_J = D_p \times xA_{q-1} \times yA_q \times D_r, \quad p, q, r \geq 2. \)

\(|R_J| = 2p(p - 1) + 2r(r - 1) + q(q - 1)x + q(q + 1)y \quad c_K = 2x + 2y \\
\quad n = p + r + qx + (q + 1)y \quad c_f = 2(p + r - 2 + (q - 1)x + qy) \\
\quad \gamma = 2(p - r)^2 \quad \alpha = 2(p - r)^2 + 2(p - q + r)(p - q + r - 1) \)

In this case we have \( f(g, e, J) \geq 0 \) with equality holds if and only if \( p = r \) and \( q = 2p \) or \( q = 2p - 1 \). These are the cases \( 2 < k = q \) in appendix 5.5.

Case 2: \( \{0,n\} \subset J, \{1,n - 1\} \subset K \), \( R_J = A_{p-1} + xA_{q-1} + yA_q + A_{r-1}, p, r \geq 2. \)

\(|R_J| = p(p - 1) + r(r - 1) + q(q - 1)x + q(q + 1)y \quad c_K = 2(2 + x + y) \\
\quad n = p + r + qx + (q + 1)y \quad c_f = 2(p + r - 3 + (q - 1)x + qy) \\
\quad \gamma = 2(p - r)^2 + 2(p + r) \quad \alpha = 2(p - q)^2 + 2(p - r)^2 + 2q \)
In this case $f(g, e, J) > 0$.

**Case 3:** $\{0, 1\} \subset J, \{n - 1, n\} \subset K, R_J = D_p + xA_{q-1} + yA_q, p \geq 2$.

$$|R_J| = 2p(p - 1) + q(q-1)x + q(q+1)y \quad c_K = 2(1+x+y)$$
$$n = 1 + p + qx + (q+1)y \quad c_J = 2(p - 1 + (q-1)x + qy)$$
$$\gamma = 2(p-1)^2 \quad \alpha = 2[(p-q+1)^2 + (p-2)(p-1) + (q-2)]$$

In this case $f(g, e, J) > 0$.

**Case 4:** $0 \in J, \{1, n-1, n\} \subset K, R_J = A_{p-1} + xA_{q-1} + yA_q,$

$$|R_J| = p(p - 1) + q(q-1)x + q(q+1)y \quad c_K = 3 + 2x + 2y$$
$$n = 1 + p + qx + (q+1)y \quad c_J = 2p - 3 + 2(q-1)x + 2qy$$
$$\gamma = (p-1)^2 + 2 \quad \alpha = 2(p-q)^2 + (q-1)^2 + 1$$

In this case $f(g, e, J) > 0$.

**Case 5:** $\{0, 1, n-1, n\} \subset K, R_J = xA_{q-1} + yA_q, q \geq 2$.

$$|R_J| = q(q-1)x + q(q+1)y \quad c_K = 2 + 2x + 2y$$
$$n = 2 + qx + (q+1)y \quad c_J = 2(q-1)x + 2qy$$
$$\gamma = 0 \quad \alpha = 2(q-1)(q-2)$$

In this case $f(g, e, J) \geq 0$ with equality if and only if $q = 2$. This is the case $k = 2$ in appendix 5.5.

### 4 Exceptional Lie Algebras

In principal (or on a computer) one can verify Theorem 1 for the exceptional Lie algebras and $^3D_4$ by checking the theorem for each subset $J \subset I$. The aim of this section is to make this verification somewhat more illuminating.

Assume the diagram $\mathcal{D}(g, e)$, with labels $c_i$ has one of the types

- $G_2$
- $^3D_4$
- $F_4$
- $^2E_6$
4.1 Small $J$

We begin with cases where $|J| \leq 8$.

When $R_J = A_1$, Theorem 1 follows from an observation which applies uniformly to all exceptional cases. Namely, each coefficient $c_i$ is at most twice the average of the remaining coefficients, with equality just for the largest coefficient $c_i = c$ whose node is the target of the arrow or is the branch node. On the other hand, the Kac diagrams

\[
\begin{array}{c}
11 \Rightarrow 0 \\
10 \Leftrightarrow 1 \\
111 \Rightarrow 01 \\
110 \Leftrightarrow 11 \\
\end{array}
\]

are those of the ell-reg automorphisms of order $h - ec$.

Now suppose $R_J = 2A_1$. Then $J = \{i, j\}$ where $i, j$ are not adjacent in $\mathcal{D}(g, e)$. The maximum value of $c_i + c_j$ is $2c - 2$ (with $c$ as above), which gives

\[
|R_J|c^I - nc_J \geq 2(n - 2c + 4) \geq 0
\]

with equality only in $G_2, F_4, E_8$. On the other hand, the Kac diagrams

\[
\begin{array}{c}
01 \Rightarrow 0 \\
101 \Rightarrow 01 \\
\end{array}
\]

are those of ell-reg automorphisms of order $h - 2c + 2$.

If $R_J = A_2$, one finds similarly that

\[
|R_J|c^I - nc_J = 6c^I - (n + 6)(c_i + c_j) \geq 0,
\]

with equality only in $^3D_4$. The Kac diagram

\[
00 \Leftrightarrow 1
\]
Is the ell-reg automorphism of order $e = 3$.

If $R_J = B_2$ or $G_2$, one finds that $|R_J|c^J - nc_J > 0$.

At this point, the theorem is proved for $G_2$ and $^3D_4$, and we may assume $R_J$ has rank at least three in the remaining cases.

Assume that $R_J = 3A_1$. Then $f(g, e, J) = 6c^J - (n + 6)c_J$. The Kac diagrams with maximal $c_J$ are

$$
010 \Rightarrow 10 \quad 010 \Leftarrow 10 \quad 1010101 \quad 10101011 \quad 101010111.
$$

These all have $f(g, e, J) \geq 0$, with equality only in the $E_6$ case. This is the Kac diagram of the ell-reg inner automorphism of $g = E_6$ of order six.

Assume that $R_J = A_1 + A_2$. In the same manner we find $f(g, e, J) \geq 0$, with equality only in the cases

$$
101 \Rightarrow 00, \quad 100 \Leftarrow 10,
$$

which are the Kac diagrams for the ell-reg automorphisms of $F_4$ of order four and the outer ell-reg automorphism of $E_6$ of order six.

Assume that $R_J = 4A_1$. This only exists in type $E$ we find $f(g, e, J) \geq 0$, with equality only in the case

$$
0010101.
$$

This is the ell-reg automorphism of $E_8$ of order 15.

### 4.2 $F_4$ and $^2E_6$

We now complete the proof of Theorem 1 for $(g, e)$ of type $F_4$ and $^2E_6$, for which $\mathcal{D}(g, e)$ has the same underlying Coxeter diagram. By the previous section, we may assume $|R_J| > 8$.

Arguing as in section 3.3.1 we need only consider cases of the form

$$
* * * \Rightarrow 00 \quad * * * \Leftarrow 11 \quad * * * \Leftarrow 10 \quad * * * \Leftarrow 01,
$$

with $R_J > 8$. The possibilities are
\[ J \quad | \quad R_J \cdot c^l \quad | \quad 4 \cdot c^l \]

\begin{array}{ccc}
100 \Rightarrow 00 & 48 \cdot 1 & 4 \cdot 11 \\
010 \Rightarrow 00 & 20 \cdot 2 & 4 \cdot 10 \leftarrow \\
001 \Rightarrow 00 & 12 \cdot 3 & 4 \cdot 9 \leftarrow \\
110 \Rightarrow 00 & 18 \cdot 3 & 4 \cdot 9 \\
000 \Leftarrow 11 & 12 \cdot 3 & 4 \cdot 6 \\
000 \Leftarrow 10 & 14 \cdot 2 & 4 \cdot 7 \leftarrow \\
000 \Leftarrow 01 & 32 \cdot 1 & 4 \cdot 8 \leftarrow \\
100 \Leftarrow 01 & 18 \cdot 2 & 4 \cdot 7 \\
010 \Leftarrow 01 & 10 \cdot 3 & 4 \cdot 6 \\
\end{array}

We have \( f(g,e,J) \geq 0 \) with equality in the cases marked by \( \leftarrow \). These are the elliptic regular automorphisms of order 2 for \( F_4 \) and outer automorphisms of \( E_6 \) of orders 4 and 2. This completes the proof of Theorem 1 in the cases \( F_4 \) and \( ^2E_6 \).

### 4.3 Types \( E_6, E_7, E_8 \)

We consider the ends of the interval \( 1 < m < h \) in two steps.

**Step 1.** For each \( 1 < m < n \) we compute the minimum

\[ r(m) = \min \{|R_J| : c^l = m\}. \]

In the tables below we check that

\[ r(m) \geq \frac{|R|}{m} - n \quad (14) \]

for each \( m < n \) and we verify that equality holds in (14) for at most one \( J \) with \( c^l = m \). This will prove Theorem 1 when \( m < n \).

Next we consider \( |R_J| \) where \( c^l \geq n \). If \( |R_J| > h - n \), then since \( h = |R|/n \) and \( c^l \geq n \) we have

\[ |R_J| > h - n = \frac{|R|}{n} - n \geq \frac{|R|}{c^l} - n. \]

Hence we may also assume \( |R_J| \leq h - n \). Since we have already proved Theorem 1 for \( |R_J| \leq 8 \), we may in fact assume

\[ 10 \leq r < h - n \]

**Step 2.** For each even integer \( r \leq h - n \) we compute the minimum

\[ m(r) = \min \{c^l : |R_J| = r\}. \]

In the tables below we check that

\[ r \geq \frac{|R|}{m(r)} - n \quad (15) \]

for each \( r \leq h - n \) and we verify that equality holds in (15) for at most one \( J \) with \( |R_J| = r \). This will complete the proof of Theorem 1.
4.3.1 $E_6$

Step 1 for $E_6$ is tabulated as follows. The bold entries are those types of $R_j$ for which $r(m) = |R_j|$. 

| $m$ | types of $R_j$ with $c^l = m$ | $r(m)$ | $(|R|/m) - 6$ | $J$ |
|-----|--------------------------------|--------|----------------|-----|
| 2   | $A_1A_5$, $D_5$                | 32     | 30             | none|
| 3   | $3A_2$, $A_1A_4$, $D_4$, $A_5$ | 18     | 18             | 0 0 0 0 0 0 0 0 |
| 4   | $2A_2A_1$, $A_1A_3$, $2A_1A_3$, $A_4$ | 14     | 12             | none|
| 5   | $2A_1A_2$, $2A_2A_1$, $A_1A_3$, $A_3$ | 10     | 42/5           | none|

Since $h - n = 12 - 6 < 8$, the proof for $E_6$ is completed by Step 1 alone.

4.3.2 $E_7$

Step 1 for $E_7$ is tabulated as follows.

| $m$ | types of $R_j$ with $c^l = m$ | $r(m)$ | $(|R|/m) - 7$ | $J$ |
|-----|--------------------------------|--------|----------------|-----|
| 2   | $A_7$, $A_1D_6$, $E_6$        | 56     | 56             | none|
| 3   | $A_2A_5$, $A_1D_5$, $A_6$, $D_6$, | 36     | 35             | none|
| 4   | $2A_3A_1$, $A_2A_4$, $2A_1D_4$, $A_5$, $A_1A_5$, $D_5$ | 26     | 49/2           | none|
| 5   | $A_1A_2A_3$, $A_1A_4$, $A_2A_4$, $A_1D_4$, $A_5$, $A_1A_5$ | 20     | 91/5           | none|
| 6   | $2A_2A_1$, $2A_1A_3$, $A_2A_3$, $3A_2$, $3A_1A_3$, $A_4$, $A_1A_4$, $2A_3$, $D_4$, $A_5$ | 14     | 14             | 1 0 0 1 0 0 0 0 1 |

For step 2, we need only consider $r = 10$. The only root systems with 10 roots are $5A_1$ and $2A_1A_2$. All occurrences of these as $R_j$ in $E_7$ have $c^l = 8$. Since $|R|/8 - 7 < 10$, Theorem 1 is now proved for $E_7$.

4.3.3 $E_8$

In Step 1, we take $1 < m < 8$ and compute $r(m)$ in the following table. The types of $R_j$ for which $m_J = m$ are shown; those for which $|R_J| = r(m)$ are in bold face. The right column
gives the unique $J$ for which $r(m_J) = (240/m) - 8$, if it exists.

| $m$ | types of $R_J : m_J = m$          | $r(m)$ | $(240/m) - 8$ | $J$            |
|-----|---------------------------------|--------|---------------|---------------|
| 2   | $D_8, A_1 E_7$                  | 112    | 112           | 0 0 0 0 0 0 0 1 |
| 3   | $A_8, A_2 E_6, D_7, E_7$        | 72     | 72            | 0 0 0 1 0 0 0 0 |
| 4   | $A_3 D_5, A_7, A_1 A_7, A_1 D_6, A_1 A_6$ | 52     | 52            | 0 0 0 1 0 0 0 0 |
| 5   | $2 A_4, A_1 A_6, A_2 D_5, A_7, D_6, A_1 E_6$ | 40     | 40            | 0 0 0 1 0 0 0 0 |
| 6   | $A_3 A_4, A_1^2 A_5, A_3 D_4, A_2 A_5, A_1 A_2 A_5, A_1 D_5, A_6, A_1 A_6, A_2 D_5$ | 32     | 32            | 1 0 0 0 1 0 0 0 |
| 7   | $A_1 A_2 A_4, A_2 D_4, A_3 A_4, A_1 A_5$ | 28     | 184/7         | none          |

For Step 2, we take $r = 10, 12, \ldots, 22$ and compute $m(r)$ in the following table. The types of $R_J$ for which $|R_J| = r$ are shown; those for which $c^J = m(r)$ are in bold face and that $J$ for which $|R_J| = (240/m_J) - n$, if it exists, is shown in the right column.

| $r$ | types of $R_J$ with $|R_J| = r$ | $m(r)$ | $(240/m(r)) - 8$ | $J$            |
|-----|---------------------------------|--------|------------------|---------------|
| 10  | $A_1^2 A_2, A_1^2 A_2$          | 14     | 64/7             | none          |
| 12  | $A_1 A_2, A_2^2, A_3$           | 12     | 12              | 1 0 1 0 0 1 0 1 |
| 14  | $A_1^2 A_2, A_1 A_2^2, A_1 A_3$ | 12     | 12              | none          |
| 16  | $A_1^2 A_2, A_1^2 A_3$          | 10     | 16              | 1 0 1 0 0 1 0 0 |
| 18  | $A_2 A_3, A_3 A_3, A_2^3$       | 10     | 16              | none          |
| 20  | $A_1 A_2 A_3, A_1 A_3^2, A_4$   | 9      | 56/3            | none          |
| 22  | $A_1 A_2 A_3, A_1 A_4$          | 8      | 22              | none          |

In each case we have $r \geq [240/m(r)] - 8$ and equality is achieved by at most one $J$, as indicated in the rightmost column.

The proof of Theorem 1 for $E_8$ is now complete.
5 Appendix: The classification of ell-reg automorphisms

For reference in the proofs above, we recall the classification of ell-reg automorphisms given in [20]. There is only one inner ell-reg automorphism of \( sl_n \), namely the principle one, so we ignore this case.

5.1 Type \( ^2A_{2n} \)

The ell-reg outer automorphisms of \( sl_{2n+1} \) correspond to odd quotients \( d \) of \( 2n \) and \( 2n + 1 \). The graphs \( D(sl_{2n+1}, 2) \) are as shown:

The ell-reg outer automorphisms of \( sl_{2n+1} \) correspond to odd quotients \( d \) of \( 2n \) and \( 2n + 1 \). We write these quotients as

\[
 d = \frac{2n + 1}{2k + 1} \quad d = \frac{n}{k}
\]

respectively. The cases overlap only when \( d = 1 \). The corresponding ell-reg automorphism has order \( m = 2d \) in both cases.

\[
\begin{array}{cc}
 d = m/2 & s \\
 3 & 1 \equiv 1 \\
 2 & 1 \equiv 0
\end{array}
\]

\[
\begin{array}{cc}
 d = m/2 & s \\
 2n + 1 & 1 \Rightarrow 1 1 \cdots 1 1 \Rightarrow 1 \\
 1 & 1 \Rightarrow 0 0 \cdots 0 0 \Rightarrow 0 \\
 \frac{2n + 1}{2k + 1} & 1 \Rightarrow 0 \cdots 0 1 0 \cdots 0 1 \cdots 1 0 \cdots 0 \Rightarrow 0 \\
 \frac{n}{k} & 1 \Rightarrow 0 \cdots 0 1 0 \cdots 0 1 \cdots 1 0 \cdots 0 \Rightarrow 0
\end{array}
\]

In the two last rows we have \( 0 < k < n \) such that \( d \) is odd and the number of type-\( A \) factors is \( (d - 1)/2 \). The next-to-last row corrects an error in [20].

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5.2 Type $^2A_{2n-1}$

The graph $D(sl_{2n}, 2)$, with $n \geq 3$ and labels $c_0, c_1, \ldots, c_n$ is shown here, with $c_0 = c_n = 1$.

\[
\begin{array}{c}
1 \circ \overset{2}{\longrightarrow} \cdots \overset{2}{\longrightarrow} \overset{1}{\leftarrow} \circ \\
\end{array}
\]

The ell-reg outer automorphisms of $sl_{2n}$ correspond to odd quotients $d$ of $2n - 1$ and $2n$. We write these quotients as

\[
d = \frac{2n - 1}{2k - 1} \quad d = \frac{n}{k}
\]

respectively. The cases overlap only when $d = 1$. The corresponding ell-reg automorphism has order $m = 2d$ in both cases.

| $d = m/2$ | $s$ |
|-----------|-----|
| $2n - 1$  | 1 1 1 1 1 ... 1 1 $\iff$ 1 |
| 1         | 0 0 0 0 0 ... 0 0 $\iff$ 1 |
| $\frac{n}{k}$, $n$ odd    | 1 |
| $\frac{2n - 1}{2k - 1}$ | $D_k$ $A_{2k-2}$ $A_{2k-2}$ |
| 0         | 0 0 ... 0 1 0 ... 0 1 ... 1 0 ... 0 $\iff$ 1 |
| $\frac{n}{k}$ | $D_k$ $A_{2k-1}$ $A_{2k-1}$ |
| 0         | 0 0 ... 0 1 0 ... 0 1 ... 1 0 ... 0 $\iff$ 1 |

In the last two rows we have $1 < k < n$ such that $d$ is odd and there are $(d - 1)/2$ components of type $A$. 
5.3 Type $B_n$

The graph $D(so_{2n+1}, 1)$ with labels $c_0, c_1, \ldots, c_n$ is shown here, with $c_0 = c_n = 1$.

The ell-reg automorphisms of $so_{2n+1}$ are of the form $\pi^k$, where $\pi$ is a principal automorphism and $k$ is a divisor of $n$. The order $m$ of $\pi^k$ is $m = 2n/k$ and the Kac coordinates of $\pi^k$ are given in the table below. We replace each node $i$ by the Kac coordinate $s_i \in \{0, 1\}$, and also omit the single bonds in the graph. Recall that $J = \{i \in I : s_i = 0\}$.

| $k$ | $n$ | $m$ | $s = (s_0, s_1, \ldots, s_n)$ |
|-----|-----|-----|-----------------------------|
| 1   | 2n  | 1 1 1 1 1 1 \ldots 1 1 \Rightarrow 1 |
| 2   | $n$ | 1 0 1 0 \ldots 1 1 \Rightarrow 0 |
| $k > 2$ | $\frac{2n}{k}$ | $D_{k/2}$ | $A_{k-1}$ | $A_{k-1}$ | $B_{k/2}$ |
| Even | $\frac{2n}{k}$ | 0 0 \ldots 0 1 0 \ldots 0 1 0 \ldots 0 1 \Rightarrow 0 |
| Odd  | $\frac{2n}{k}$ | $D_{(k+1)/2}$ | $A_{k-1}$ | $A_{k-1}$ | $B_{(k-1)/2}$ |
| $\frac{2n}{k}$ | 0 0 \ldots 0 1 0 \ldots 0 1 0 \ldots 0 1 \Rightarrow 0 |

The second line $m = n$ only occurs if $n$ is even. In the last two lines there are $\frac{n}{k} - 1$ factors of type $A_{k-1}$.

5.4 Type $C_n$

The graph $D(sp_{2n}, 1)$ with labels $c_0, c_1, \ldots, c_n$ is shown here, with $c_0 = c_n = 1$. The Coxeter number is $2n$.

As with $so_{2n+1}$, the ell-reg automorphisms of $sp_{2n}$ are powers $\pi^k$ of a principal automorphism $\pi$, where $k$ is a divisor of $n$. The order $m$ of $\pi^k$ is $m = 2n/k$ and the Kac coordinates of $\pi^k$ are given in the table below.
\[
\begin{array}{ccc}
  k | n & m & s = (s_0, s_1, \ldots, s_n) \\
  1 & 2n & 1 \Rightarrow 1 1 1 \cdots 1 1 \Leftarrow 1 \\
  k > 1 & \frac{2n}{k} & 1 \Rightarrow 0 \cdots 0 1 0 \cdots 0 1 \cdots 1 0 \cdots 0 \Leftarrow 1 \\
  A_{k-1} & A_{k-1} & A_{k-1}
\end{array}
\]

In the last line for \( k > 1 \) there are \( n/k \) factors of type \( A_{k-1} \).

### 5.5 Type \( D_n \)

The graph \( D(so_{2n}, 1) \) with labels \( c_0, c_1, \ldots, c_n \) is shown here, with \( c_0 = c_1 = c_{n-1} = c_n = 1 \).

```
1  2  2  \ldots  2  2  1  \\
   \circ  \circ  \circ  \ldots  \circ  \circ  \circ  \\
   1  1
```

The ell-reg congruacy classes in \( \text{Aut}(so_{2n}, 1) \) correspond to even divisors \( k \) of \( n \) (where \( m = 2n/k \)) and odd divisors \( k \) of \( n - 1 \) (where \( m = (2n - 2)/k \)), as shown in the table below.

\[
\begin{array}{ccc}
  k & m & s = (s_i) \\
  1 & 2n - 2 & 1 1 1 \cdots 1 1 1 \\
  2 & n & 1 0 1 0 1 \cdots 1 0 1 0 1 \\
  n & 2 & 0 0 \cdots 1 0 \cdots 0 0 \\
  \text{k even} & \frac{2n}{k} & D_{k/2} \quad A_{k-1} \quad A_{k-1} \quad D_{k/2} \\
  k \text{ divides } n & 0 0 \cdots 0 1 0 \cdots 0 1 0 \cdots 0 0 & 0 0 \\
  2 < k < n & 0 & 0 \\
  \text{k odd} & \frac{2n - 2}{k} & D_{(k+1)/2} \quad A_{k-1} \quad A_{k-1} \quad D_{(k+1)/2} \\
  k \text{ divides } n - 1 & 0 0 \cdots 0 1 0 \cdots 0 1 0 \cdots 0 0 & 0 \\
  1 < k < n - 1 & 0 & 0 
\end{array}
\]
In the last two rows, the number of type -A factors is one less than \( \frac{n}{k} \) and \( n - 1k \), respectively.

5.6 Type \( 2D_{n+1} \)

The graph \( D(so_{2n+2}, 2) \) (with \( n \geq 2 \)) with is shown here, with \( c_0 = c_1 = \cdots = c_n = 1 \).

\[
2D_{n+1}: \quad \circ \leftrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ 
\]

The ell-reg classes in \( Aut(so_{2n+2}, 2) \) correspond to even divisors \( k \) of \( n \) with order \( m = 2n/k \) and odd divisors \( k \) of \( n + 1 \) with order \( 2(n + 1)/k \).

\[
\begin{array}{ccc}
\hline
k & m & s = (s_0, s_1, \ldots, s_n) \\
\hline
1 & 2n + 2 & 1 \leftarrow 1 1 \cdots 1 1 \Rightarrow 1 \\
2 & n \text{ even} & 0 \leftarrow 1 0 1 0 \cdots 0 1 0 1 \Rightarrow 0 \\
\hline
k \text{ even} & 2n/k & B_{k/2} \quad A_{k-1} \quad A_{k-1} \quad B_{k/2} \\
& k \text{ divides } n & 0 \leftarrow 0 \cdots 0 1 0 \cdots 0 1 0 \cdots 0 \Rightarrow 0 \\
& 2 < k & \\
\hline
k \text{ odd} & 2n + 2/k & B_{(k-1)/2} \quad A_{k-1} \quad A_{k-1} \quad B_{(k-1)/2} \\
& k \text{ divides } n + 1 & 0 \leftarrow 0 \cdots 0 1 0 \cdots 0 1 0 \cdots 0 \Rightarrow 0 \\
& 1 < k & \\
\hline
\end{array}
\]

In the last two rows, the number of type -A factors is one less than \( \frac{n}{k} \) and \( \frac{n+1}{k} \), respectively.

5.7 Exceptional Lie Algebras
| $E_6$  | $^{2}E_6$ | $E_7$  | $E_8$  |
|-------|---------|-------|-------|
| $m$   | $s$     | $m$   | $s$   | $m$   | $s$     |
| 12    | 1 1 1 1 1 | 18    | 1 1 1 1 1 1 1 | 18    | 1 1 1 1 1 1 1 |
|       | 1       | 12    | 1 1 1 1 1 | 14    | 1 1 1 1 0 1 1 |
| 9     | 1 1 1 0 0 | 6     | 0 0 0 1 0 1 0 | 6     | 0 0 1 0 0 0 0 |
|       | 1       | 4     | 0 0 0 1 0 1 0 | 2     | 0 0 1 0 0 0 0 |
| 6     | 1 1 0 0 1 | 2     | 0 0 0 0 1 0 1 | 3     | 0 1 0 0 0 0 0 |
|       | 0       |       | 0 0 0 0 0 1 0 | 2     | 0 1 0 0 0 0 0 |
| 3     | 0       |       | 0 0 0 0 0 0 0 |       | 0 1 0 0 0 0 0 |

| $F_4$  | $G_2$  | $^{3}D_4$ |
|-------|-------|----------|
| $m$   | $s$     | $m$   | $s$   | $m$   | $s$ |
| 12    | 1 1 1 1 | 6     | 1 1 1 | 12    | 1 1 1 1 |
| 8     | 1 1 1 0 | 3     | 1 1 0 | 6     | 1 0 1 |
| 6     | 1 0 1 0 | 2     | 0 1 0 | 3     | 1 0 0 |
| 4     | 1 1 1 0 |       | 0     |       |       |
| 3     | 0 0 0 0 |       | 1     |       |       |
| 2     | 0 1 0 0 |       | 0     |       |       |

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