Ideal Gases in Time-Dependent Traps

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We investigate theoretically the properties of an ideal trapped gas in a time-dependent harmonic potential. Using a scaling formalism, we are able to present simple analytical results for two important classes of experiments: free expansion of the gas upon release of the trap; and the response of the gas to a harmonic modulation of the trapping potential is investigated. We present specific results relevant to current experiments on trapped Fermions.

I. INTRODUCTION

Recently, impressive experimental results concerning the quantum degenerate regime of a dilute gas of trapped Fermionic atoms have been presented [1]. By cooling \( \sim 7 \times 10^5 \) \(^{40}\)K atoms to 0.5 of the Fermi temperature, the onset of Fermi degeneracy was observed in the thermodynamic and in the scattering properties of the gas. Also, progress towards achieving fermi degeneracy for trapped \(^6\)Li \([2]\) has been presented. Such a weakly-interacting, degenerate Fermi-Dirac gas provides a new platform for exploring fundamental quantum many-body physics. Several theoretical results dealing with the equilibrium properties of such a gas have been presented \([3–5]\). For example, it has been shown that a two-component gas of spin-polarized \(^6\)Li undergoes a Bardeen-Cooper-Schrieffer (BCS) transition to superfluidity at experimentally obtainable densities and temperatures \([6,7]\).

One class of experiments that is compatible with trapping protocols, and which has gleaned valuable information on the dynamics of dilute Bose-Einstein condensates (BECs), involves monitoring the response of the gas to a change in trapping potential. For example, transient modulation of the trapping potential induces free ringing of the gas, which in the case of BECs led to a direct determination of low-lying regions of the quasi-particle spectrum \([8]\). Complete release of the trapping potential enables one to view the free expansion of the gas; this established early on the essential validity of the time-dependent Gross-Pitaevski equation for describing the dynamics of latest generation of BECs \([9]\), which was later put to stringent quantitative tests \([10]\).

A trapped, single-component gas of ultracold Fermionic atoms can, for experiments of current interest, be considered to be ideal (noninteracting) as a reasonable approximation, since atomic collisions are strongly suppressed \([11,12]\). In this paper, we examine theoretically the dynamics of such an ideal gas in a time-dependent trap. Using a scaling formalism similar to that which has been successfully applied to trapped BECs \([13,14]\), we derive analytical results for the class of experiments mentioned above. The essential Fermionic character of the systems with which we are concerned is established by the initial distribution of particles in the trap; the time evolution of this distribution, under changes of the trapping potential that preserve its harmonicity, are rigorously equivalent to that of an ensemble of noninteracting particles, independent of statistics. We present a simple formula that describes the free expansion of such an ideal gas, and our results suggest a new approach to the problem of quantitative thermometry in the nanokelvin regime. Due to their weak pair interactions, single-component Fermi-Dirac gases are attractive candidates for nanokelvin thermometry; with an improved theoretical understanding of finite-temperature properties of BECs \([17]\), which are much more robust candidates for experiment at present, we can envisage a direct comparison of temperatures of ultracold Bose-Einstein and Fermi-Dirac gases. We also examine in detail the (non-linear) response of the gas to a harmonic oscillation of the trapping frequency. This reveals a domain of driving frequencies and amplitudes which generates a resonant response of the gas.

The non-interacting limit considered here should provide a good approximation for degenerate, single-component cold Fermi-Dirac gases. Since interactions in general play a smaller role for Fermi-Dirac vs. Bose-Einstein systems, the present approach may also provide a useful starting point for consideration of the dynamics of multiple-component Fermi gases.

II. FORMALISM

We start with a derivation of the equations describing the scaling properties of an ideal gas trapped in a time-dependent harmonic potential. Consider a classical particle of mass \( m \) trapped in a potential \( V(r, t) = m \sum_j \omega_j^2(t)^2 r_j^2 / 2 \) with \( j = x, y, z \) denoting the 3 spatial dimensions. We set \( \omega_j(t) = \omega_0 \) for \( t \leq 0 \), i.e. the trap potential is constant prior to \( t = 0 \). Newton’s law is then expressed by \( p_j = -m \omega_j(t)^2 r_j \) and \( m \dot{x}_j = k_j \).

Invoking a scaling transformation \( q_j = r_j / \gamma_j(t) \) and \( p_j = \gamma_j(t) k_j - \gamma_j(t)m \dot{r}_j \), we obtain \( \partial_t p_j = -m \omega_0^2 q_j^2 \), with \( \tau_j(t) = \int_0^t dt' \gamma_j(t')^{-2} \), if the scaling parameters \( \gamma_j(t) \) satisfy the equations

\[
\dot{\gamma}_j(t) = \frac{\omega_0^2}{\gamma_j(t)^3} - \omega_j(t)^2 \gamma_j(t).
\]
In quantum mechanics, the Heisenberg equation for a non-interacting gas in a time-dependent harmonic potential takes the form:

\[ i\hbar \partial_t \hat{\psi}(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right] \hat{\psi}(\mathbf{r}, t) \]  

(2)

where \( \hat{\psi}(\mathbf{r}) \) is the field operator for an atom in a single hyperfine state at position \( \mathbf{r} \), which obeys the usual Fermi-Dirac anticommutation relations. The quantum mechanical analogue of the rescaling outlined for the classical case above corresponds to writing \( \hat{\phi}_j(\mathbf{r}, t) \) as

\[ \hat{\phi}_j(\mathbf{r}, t) = \frac{\hat{\phi}(\mathbf{q}(t))}{\sqrt{\gamma x \gamma y \gamma z}} e^{i m \sum \gamma_j^2 q_j / 2 \hbar \gamma_j} \]  

(3)

with \( q_j(t) = r_j / \gamma_j(t) \), as defined above. If each \( \gamma_j \) satisfies Eq. (1), by writing \( \hat{\phi}[\mathbf{q}(t)] = \Pi_j \hat{\phi}[q_j(t)] \), we obtain

\[ i\hbar \partial_t \bar{\phi}[q_j(t)] = \left[ -\frac{\hbar^2}{2m} \partial^2_{q_j} + \frac{1}{2} m \omega^2 \bar{q}_j \right] \hat{\phi}[q_j(t)] \]  

(4)

Thus, the time-dependent problem has been reduced to the trivial case of evolution in a time-independent harmonic trap in the rescaled variables \( (\tau_j, q_j) \). Determination of these variables requires only the solution of the three ordinary differential equations expressed by Eq. (6); the three spatial dimensions can be treated independently. An appealing qualitative description of the solution of these equations, cast in the context of one-dimensional scattering theory, has been given by Kagan et al. [14]. From Eq. (6), it follows that the density \( \rho(\mathbf{r}, t) \) of the gas for a given time \( t \) is given by

\[ \rho(\mathbf{r}, t) \equiv \langle \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \rangle = \frac{1}{\gamma x(t) \gamma y(t) \gamma z(t)} \rho_0[\mathbf{q}(t)] \]  

(5)

where \( \rho_0(\mathbf{r}) \) is the particle density for \( t = 0 \). Thus, we can calculate \( \rho(\mathbf{r}, t) \) for any time \( t > 0 \) for modulated frequencies \( \omega_j(t) \), by solving Eq. (6), subject to the boundary conditions \( \gamma_j(0) = 1 \) and \( \gamma_j = 0 \). Equation (6) also describes the two-dimensional, non-ideal BEC subject to isotropic variations of the trap potential, with Eq. (6) also being applicable if \( \rho_0 \) is obtained by a solution of the Gross-Pitaevskii equation [14].

We now analyze the solution of Eq. (6) for two cases of experimental relevance: free expansion; and harmonic modulation of the trapping potential.

III. FREE EXPANSION

To model a free expansion experiment, we take \( \omega_j(t) = 0 \) for \( t > 0 \). Solving Eq. (6) with the requirement that the gas is in equilibrium for \( t \leq 0 \), we obtain \( \gamma_j(t) = 1 \) for \( t \leq 0 \) and

\[ \gamma_j(t) = \sqrt{1 + \omega^2_0 t^2} \]  

(6)

for \( t > 0 \). The time-dependent width of the cloud after the trap has been dropped is given by

\[ \sqrt{\langle r_j^2 \rangle(t)} = \frac{\gamma_j(t)}{\sqrt{\langle r_j^2 \rangle(0)}}. \]  

To describe the aspect ratio \( \alpha(t) \) of a cylindrically-symmetric cloud, we find

\[ \alpha(t) = \frac{\sqrt{(x^2(t))/(z^2(t))}}{\sqrt{(x^2(t))/(z^2(t))}} = \alpha_0 \frac{1 + \omega_0^2 t^2}{1 + \omega_0^2 z^2} \]  

(7)

We now apply these results to an ideal gas in two limiting cases.

We first treat the case of \( T = 0 \), with the chemical potential \( \mu_F(T = 0)/\hbar \omega_0 \gg 1 \) for \( j = x, y, z \). This corresponds to the semiclassical limit, appropriate to current experiments with more than a few hundred trapped atoms. In this limit, the initial density profile is well described by the Thomas-Fermi (TF) approximation [3]. This gives the integrated density \( \rho(x, z, t) = \int dy \rho(\mathbf{r}, t) \),

\[ \rho(x, z, t) = \frac{m \mu_F}{4 \pi \hbar^3 \omega_0^2 \gamma_x \gamma_z} \frac{1}{1 + \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} + \frac{z^2}{\lambda_z^2}} \]  

(8)

with \( \lambda_x \equiv \omega_x/\omega_0 \) and \( R_F = \sqrt{2 \mu_F / m \omega_0^2} \). We see that the cloud becomes isotropic for \( \omega_0 t \gg 1 \). This is the result one would get for a classical gas at finite temperature, in which the momentum distribution is isotropic, and is consistent with the initial isotropic momentum distribution implicit in the TF approximation (in BECs in anisotropic traps, on the other hand, the long-time expansion is anisotropic due to macroscopic population of an anisotropic initial state, a key result in their first observation [15]). The initial aspect ratio of \( \alpha(0) = \lambda_z \) and Eq. (6) evolves to \( \alpha(t) \to 1 \) as \( t \to \infty \). In Fig. 1(a), we plot the integrated density \( \rho(x, z, t) \) for \( T = 0 \), \( \mu_F(T = 0) = 20 \hbar \omega_0 \), and \( t = 0 \) (a) and \( t = 20/\omega_0 \) (b). We have taken \( \lambda_z = 19.5/137 \) and \( \lambda_y = \omega_y/\omega_0 = 1 \) corresponding to current experiments on trapped \(^{39}\)K atoms [16].

![FIG. 1. Contour images of the integrated density \( \rho(x, z, t) \) for a freely expanding gas. There are 9310 atoms in the cloud. We have defined \( l_h \equiv (\hbar/m \omega_0)^{1/2} \).](image)
Note that $\alpha(0) = \lambda_z$ for any initial density distribution of the form $\rho_0(\mathbf{r}) = f(\sum_j \omega_{0j}^2 r_j^2)$. Such initial distributions will become isotropic in an free expansion experiment. For instance, for $T > T_F = \mu_F(T = 0)/k_B$, the density is well described by a classical gaussian profile, i.e. $\rho_0(\mathbf{r}) \propto \exp[-\beta(\mu_F - \hbar \omega_{0j}/2)]$ and Fig. 2 thus describes a trapped gas of fermions for any $T$ within the TF approximation. This means that one cannot detect the onset of Fermi degeneracy by looking at the aspect ratio of the expanding gas.

As an example where $\rho_0(\mathbf{r}) \neq f(\sum_j \omega_{0j}^2 r_j^2)$, we now consider a case with $\mu_F/\hbar \omega_{0z} < 3/2$ such that only one level is occupied in the $x$-direction and $\mu_F/\hbar \omega_{0z} \gg 1$ for $j = y, z$. The gas is initially strongly confined in the $x$-direction. The integrated density profile is then

$$
\rho(x, y, t) \propto e^{x^2/\bar{r}_0^2 \gamma_z^2}
\left(1 - \frac{\mu \omega_{0y}^2 y^2 / \gamma_y^2}{2\mu_F - \hbar \omega_{0x}}\right)^{3/2}
$$

for $t > 0$, yielding $\alpha(0) = \sqrt{3}\hbar \omega_{0z} \lambda_z \sqrt{2\mu_F - \hbar \omega_{0x}}$. In Fig. 3, we plot the aspect ratio for a free expansion for $\mu_F = \hbar \omega_{0z}$ and $\lambda_y = \lambda_z = 1/50$ using Eq. (8). We have $\alpha(0) = \sqrt{3}/50$ and $\alpha(t) \rightarrow \sqrt{3}$ for $t \rightarrow \infty$. The gas, which initially is strongly confined in the $x$-direction will for $\omega_{0z}/\tau \gg 1$ become most confined in the $y$- and $z$-directions. This is, of course, a direct reflection of the uncertainty relation giving higher average momentum in the $x$-direction. However, observation of such a quantum effect requires a highly anisotropic trap. For the case of $10^4$ trapped atoms we would require $\lambda_y = \lambda_z \approx 1/250$. Thus, for Fermi-Dirac particles the anisotropy of the expanded cloud due to the Heisenberg uncertainty principle is considerably less than that encountered in the Bose-Einstein case.

From a measurement of the density at any time $t$ under free expansion it is straightforward to determine the initial density distribution $\rho_0(\mathbf{r})$ using Eq. (3)–(5). This suggests that a trapped, single-component Fermi-Dirac gas could serve as a low $T$ thermometer. Assuming the gas is in thermodynamic equilibrium for $t \leq 0$, one could infer the temperature of the gas from a knowledge of its density distribution, the calculation of which is a trivial problem of summing over fractionally occupied trap levels. Hence, a determination of $T$ is simply a matter of fitting a calculated density to the measured $\rho(\mathbf{r})$. Density profiles for an ideal Fermi-Dirac gas for various temperatures have already been given elsewhere [3]. However, if only one hyperfine level is present at all times, the atomic collisions required for thermalization are suppressed at low temperatures, and evaporative cooling of the gas will be inhibited. This can be remedied by initially trapping the gas in two hyperfine states, to allow s-wave collisions between atoms in different states. One can then deplete the trap of one hyperfine state by applying a microwave field that depletes one of the states. If this depletion occurs on a time scale that is long compared to collision times in the sample, the remaining atoms will be in thermodynamic equilibrium, and their expansion can be described by the results in this section. In fact, a procedure of this type is being used in present experiments on trapped $^{40}$K atoms [4].

Alternatively, one could drop the trap with both hyperfine levels present. The initial density profile and the free expansion of the gas will then be perturbed away from the ideal gas result by the interactions. When the effect of interactions is small, the initial equilibrium density and the subsequent expansion can be calculated relatively easily. A calculation of the effect of the interactions on the equilibrium distribution has been presented elsewhere [3] and the effect of the interactions on the free expansion can be calculated within the mean-field approximation, starting from the simple uncoupled solutions given by Eq. (1) for each spatial direction [10].

## IV. Driven Oscillations of the Gas

As noted above, modulation of the trapping frequency has proven to be a useful technique for probing the low-lying collective modes of trapped BECs. We therefore examine theoretically the non-linear response of a trapped gas to such a modulation. The non-interacting case treated here is in some sense opposite to the hydrodynamic limit we have treated elsewhere [21]. To model a typical experiment, we assume that the trapping frequencies take the form $\omega_j(t)^2 = \omega_{0j}^2[1 - 2\eta \cos(\omega_D t)]$ for $t > 0$ with $\omega_D$ being the driving frequency and $\eta$ the driving amplitude. Instead of solving Eq. (11) with this form for $\omega_j(t)$, it turns out to be easier to go back to the original classical equation of motion $\ddot{r}_j = -\omega_j(t)^2 r_j$ by using $r_j(t) = \beta_j(t) / \gamma_j(t)$ and $q_j(t) = \exp(\pm i \omega_{0j} \tau(t))$. Thus, by writing $\xi_j(t) = \gamma_j(t) \exp(\pm i \omega_{0j} \tau(t))$, and $\chi = \omega_D t/2$, we obtain:

$$
\partial_t^2 \xi_j + [a - 2q \cos(2\chi)] \xi_j = 0
$$

with $a = 4/\tilde{\omega}^2$, $q = 4\eta/\tilde{\omega}^2$ and $\tilde{\omega} = \omega_F/\omega_{0j}$. Equation (16) is a variant of Mathieu’s equation [22], whose properties have been extensively studied. Its behavior is easy to explain in one case of experimental interest: that when the trap frequency is modulated during a finite interval $0 < t \leq t_D$, and then returned to its original value. The
subsequent motion of the cloud is described by a solution of the time-independent problem \( \omega_j(t) = \omega_0j \) with arbitrary initial conditions:

\[
\gamma_j(t) = \sqrt{E^2 - 1} \sin(2\omega_0j t + c) + E, \tag{11}
\]

where \( E = (\omega_o^{-2} \gamma_j^2 + \omega_j^2 + \gamma_j^{-2})/2 \geq 1 \) is a conserved quantity for \( t > t_D; c = \arcsin(\gamma_j^2 - E)/\sqrt{E^2 - 1} \); and \( \gamma_j = \gamma_j(t_D) \) is the value of \( \gamma_j(t) \) immediately after the modulation ceases. Equation (11) has a discrete frequency spectrum (frequencies of \( 2\omega_0j \) with \( n = 0, 1, 2 \ldots \)), as expected for a non-interacting gas in a harmonic trap. The linear response limit is recovered by letting \( E \to 1+ \) in Eq. (11), to obtain \( \gamma_j(t) = 1 + \delta \sin(2\omega_0j t) \).

In essence, the problem of predicting the response of the gas to a harmonic driving with frequency \( \omega_D \) and amplitude \( \eta \) is reduced to an analysis of the well-known solutions to Mathieu’s equation. In the parameter space \((a, q)\), there are regions where the solutions of Eq. (11) are unstable, i.e. their amplitude increases exponentially with time. Also, there are stable regions where the solutions remain bounded. The solutions on the boundaries between these regions are the Mathieu functions [2]. Using \( |\gamma_j(t)| = |\xi_j(t)|, a = 4/\omega^2, \) and \( q = 4\eta/\omega^2 \), this means that for certain regions in the \((\omega_D, \eta)\)-space, the response of the gas diverges as the driving time \( t_D \) increases, i.e. there is a resonant response, whereas in other regions the response of the gas remains finite for any value of \( t_D \). Specifically, for \( a = n^2 \) with \( n = 0, 1, 2 \ldots \), the solutions to Eq. (11) diverge in time for an arbitrary small \( q \) [2]. Thus, for

\[
\omega_D = 2\omega_0j/n, \quad n = 1, 2, 3 \ldots \tag{12}
\]

the response of the gas is resonant for an arbitrarily small driving amplitude and the amplitude of its oscillations will diverge with the driving time. In terms of Eq. (11), if the gas remains trapped in the time-independent potential \( \omega_j(t) = \omega_0j \) after the driving \((t > t_D)\), we have \( E \gg 1 \) and the oscillations of the gas will be large and contain many harmonics. The resonance for \( \omega_D = 2\omega_0j \) is, of course, the usual excitation frequency for an even-parity perturbation for a non-interacting gas. However, the resonances for \( \omega_D = \omega_0j, 2\omega_0j/3, \omega_0j/2 \) etc. do not correspond to new modes. They simply reflect the fact that for these driving frequencies, the trapping potential is doing resonant work on the gas. Note that the present scaling formalism does not predict the higher-frequency modes at \( n2\omega_0 \) with \( n \geq 2 \), as they do not correspond to simple dilations/contractions of the cloud.

We will now examine the width of the unstable (resonant) regions around \( \omega_D = 2\omega_0j/n \) for \( \eta \to 0 \). For the \( \omega_D = 2\omega_0j \) resonance, the unstable region of the Mathieu equation is bounded by \( 1 - q < \eta < 1 + q \) for \( q \to 0 \) [2]. This gives, that for driving frequencies and amplitudes such that

\[
2 - \eta < \tilde{\omega} < 2 + \eta, \quad \eta \ll 1, \tag{13}
\]

the response of the gas is divergent. Hence, the resonance region of the gas for finite but small driving amplitude \( \eta \) has a reasonable width and should be relatively easy to access experimentally. Likewise, for the \( \omega_D = \omega_0j \) resonance, the unstable region is bounded by

\[
1 - 5\eta^2 / 6 < \tilde{\omega} < 1 + \eta^2 / 6, \quad \eta \ll 1 \tag{14}
\]

and it should be experimentally accessible. For the lower frequency resonances \( \omega_D = 2\omega_0j/n, n \geq 3 \), it turns out that the resonance regions are very narrow for \( \eta \to 0 \) as the boundary lines between the stable and unstable solutions of Eq. (11) only differ by terms of order \( q^3 \) or larger [2]. Thus, these resonances are difficult to find experimentally for \( \eta \to 0 \). Instead, one could increase the driving amplitude \( \eta \) for a given frequency \( \omega_D \); for large enough \( \eta \), an unstable region is reached and the response of the gas diverges. The above results are illustrated in Fig. (3), where we plot the response of the gas as a function of driving frequency \( \omega_D \) and amplitude \( \eta \).

![FIG. 3. The regions of stability/unstability for the response of a non-interacting gas to a modulation of the trapping potential with frequency \( \omega_D \) and amplitude \( \eta \).](image)
Mathematically, the transition region for $\eta \simeq 0.5$ comes from the fact, that for $a \sim 2q$, the even and odd periodic solutions (Mathieu functions) of Eq. (10) start to differ in "energy" (the Mathieu characteristic value $a_{21}$), as the tunneling between successive minima of the potential $\cos(2\chi)$ becomes significant.

Again, since a spin-polarized gas of cold fermionic atoms is effectively ideal, the results presented here should apply to the response of such a gas to a harmonic modulation of the trapping potential.

V. CONCLUSION

In conclusion, using a scaling formalism we have been able to derive simple analytical results for the dynamics of ideal gases trapped in time-dependent harmonic traps. We have concentrated on two important classes of experiments: For free expansion, we show how the initial density profile of the gas can easily be determined from a measurement of the density profile at any time $t$ after the trap has been dropped. We proposed a low $T$ thermometer based on these results. Also, we showed how the problem of determining non-linear response of the gas to a harmonic modulation of the trapping frequency can be mapped to an analysis of the well-known properties of the solutions to Mathieu's equation. We identified regions in $(\omega_D, \eta)$-space where the response of the gas was divergent with the modulation time. Especially, we were able to predict unstable regions for $\eta \to 0$ reflecting the fact, that the trap is doing resonant work on the gas. Since ultracold spin-polarized fermions are non-interacting to a very good approximation, our results should be directly relevant for current experiments in this very active field of research.

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