ON CERTAIN ARITHMETIC FUNCTIONS INVOLVING EXPOENTIAL DIVISORS

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Abstract

The integer \( d \) is called an exponential divisor of \( n = \prod_{i=1}^{r} p_i^{a_i} \) if \( d = \prod_{i=1}^{r} p_i^{c_i} \), where \( c_i | a_i \) for every \( 1 \leq i \leq r \). The integers \( n = \prod_{i=1}^{r} p_i^{a_i}, m = \prod_{i=1}^{r} p_i^{b_i} > 1 \) having the same prime factors are called exponentially coprime if \( (a_i, b_i) = 1 \) for every \( 1 \leq i \leq r \).

In this paper we investigate asymptotic properties of certain arithmetic functions involving exponential divisors and exponentially coprime integers.

1. Introduction

Let \( n > 1 \) be an integer of canonical form \( n = \prod_{i=1}^{r} p_i^{a_i} \). The integer \( d \) is called an exponential divisor of \( n \) if \( d = \prod_{i=1}^{r} p_i^{c_i} \), where \( c_i | a_i \) for every \( 1 \leq i \leq r \), notation: \( d|_e n \). By convention \( 1|_e n \). This notion was introduced by M. V. Subbarao [9]. Note that 1 is not an exponential divisor of \( n > 1 \), the smallest exponential divisor of \( n > 1 \) is its squarefree kernel \( \kappa(n) = \prod_{i=1}^{r} p_i \).

Let \( \tau^{(e)}(n) = \sum_{d|_e n} 1 \) and \( \sigma^{(e)}(n) = \sum_{d|_e n} d \) denote the number and the sum of exponential divisors of \( n \), respectively. The integer \( n = \prod_{i=1}^{r} p_i^{a_i} \) is called exponentially squarefree if all the exponents \( a_i \) \( (1 \leq i \leq r) \) are squarefree. Let \( g^{(e)} \) denote the characteristic function of exponentially squarefree integers. Properties of these functions were investigated by several authors, see [1], [2], [3], [5], [8], [9], [12].

Two integers \( n, m > 1 \) have common exponential divisors if they have the same prime factors and in this case, i.e. for \( n = \prod_{i=1}^{r} p_i^{a_i}, m = \prod_{i=1}^{r} p_i^{b_i}, a_i, b_i \geq 1 \) \( (1 \leq i \leq r) \), the greatest common exponential divisor of \( n \) and \( m \) is

\[
(n, m)_{(e)} := \prod_{i=1}^{r} p_i^{(a_i, b_i)}.
\]

Here \( (1, 1)_{(e)} = 1 \) by convention and \( (1, m)_{(e)} \) does not exist for \( m > 1 \).

The integers \( n, m > 1 \) are called exponentially coprime, if they have the same prime factors and \( (a_i, b_i) = 1 \) for every \( 1 \leq i \leq r \), with the notation of above. In this case \( (n, m)_{(e)} = \kappa(n) = \kappa(m) \). 1 and 1 are considered to be exponentially coprime. 1 and \( m > 1 \) are not exponentially coprime.

For \( n = \prod_{i=1}^{r} p_i^{a_i} > 1, a_i \geq 1 \) \( (1 \leq i \leq r) \), denote by \( \phi^{(e)}(n) \) the number of integers \( \prod_{i=1}^{r} p_i^{c_i} \) such that \( 1 \leq c_i \leq a_i \) and \( (c_i, a_i) = 1 \) for \( 1 \leq i \leq r \), and let \( \phi^{(e)}(1) = 1 \). Thus \( \phi^{(e)}(n) \) counts the number of divisors \( d \) of \( n \) such that \( d \) and \( n \) are exponentially coprime.

It is immediately, that \( \phi^{(e)} \) is a prime independent multiplicative function and for \( n > 1 \),

\[
\phi^{(e)}(n) = \prod_{i=1}^{r} \phi(a_i),
\]
where \( \phi \) is the Euler-function. Exponentially coprime integers and function \( \phi^{(c)} \) were introduced by J. SÁNDOR [6]. He showed that

\[
\limsup_{n \to \infty} \frac{\log \phi^{(c)}(n) \log \log n}{\log n} = \frac{\log 4}{5}.
\]

We consider the functions \( \hat{\sigma} \) and \( \hat{P} \) defined as follows. Let \( \hat{\sigma}(n) \) be the sum of those divisors \( d \) of \( n \) such that \( d \) and \( n \) are exponentially coprime. Function \( \hat{\sigma} \) is multiplicative and for every prime power \( p^a \),

\[
\hat{\sigma}(p^a) = \sum_{1 \leq i \leq a, (c, a) = 1} p^i.
\]

Here \( \hat{\sigma}(p) = \hat{\sigma}(p^2) = p, \hat{\sigma}(p^3) = p + p^2, \hat{\sigma}(p^4) = p + p^3, \) etc.

Furthermore let \( \hat{P}(n) \) be given by

\[
\hat{P}(n) = \sum_{1 \leq i \leq n, (j, n) = 1} (j, n)^{(c)},
\]

representing an analogue of Pillai’s function \( P(n) = \sum_{j=1}^n (j, n) \).

Function \( \hat{P} \) is also multiplicative and for every prime power \( p^a \),

\[
\hat{P}(p^a) = \sum_{1 \leq i \leq a} p^{(c,a)} = \sum_{d|a} p^d \phi(a/d),
\]

here \( \hat{P}(p) = p, \hat{P}(p^2) = p + p^2, \hat{P}(p^3) = 2p + p^2, \hat{P}(p^4) = 2p + p^2 + p^4, \) etc.

We call an integer \( n = \prod_{i=1}^r p_i^{a_i} \) exponentially \( k \)-free if all the exponents \( a_i \) (\( 1 \leq i \leq r \) are \( k \)-free, i.e. are not divisible by the \( k \)-th power of any prime \( k \geq 2 \). Let \( q^{(c)}_k \) denote the characteristic function of exponentially \( k \)-free integers.

The aim of this paper is to investigate the functions \( \phi^{(c)}(n), \hat{\sigma}(n), \hat{P}(n) \) and \( q^{(c)}_k(n) \).

The estimate given for the sum \( \sum_{n \leq x} q^{(e)}_k(n) \) generalizes the result of J. WU [12] concerning exponentially squarefull integers. Our main results are formulated in Section 2, their proofs are given in Section 3.

Our estimates for \( \sum_{n \leq x} (\hat{\sigma}(n))^n \) and \( \sum_{n \leq x} q^{(c)}_k(n) \) are consequences of a general result due to V. SITA RAMAIAH and D. SURYANARAYANA [7], the proof of which uses the estimate of A. WALFISZ [11] concerning \( k \)-free integers and is simpler than the proof given by J. WU [12].

A. SMATI and J. WU [8] deduced some interesting analogues of known results on the divisor function \( \tau(n) \) in case of \( \tau^{(c)}(n) \). They remarked that their results can be stated also for certain other prime independent multiplicative functions \( f \) if \( f(n) \) depends only on the squarefull kernel of \( n \).

We point out two such results in case of \( \phi^{(c)}(n) \). Note that, since \( \phi(1) = \phi(2) = 1, \phi^{(c)}(n) \) depends only on the cubfull kernel of \( n \). These results are contained in Section 4. Here some open problems are also stated.

2. Main results

Regarding the average orders of the functions \( \phi^{(c)}(n), \hat{\sigma}(n) \) and \( \hat{P}(n) \) we prove the following results.

**Theorem 1.**

\[
\sum_{n \leq x} \phi^{(c)}(n) = C_1 x + C_2 x^{1/3} + O(x^{1/5+\varepsilon}),
\]
for every \( \varepsilon > 0 \), where \( C_1, C_2 \) are constants given by

\[
C_1 = \prod_p \left( 1 + \sum_{a=3}^{\infty} \frac{\phi(a) - \phi(a-1)}{p^a} \right),
\]

\[
C_2 = \zeta(1/3) \prod_p \left( 1 + \sum_{a=5}^{\infty} \frac{\phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)}{p^{a/3}} \right).
\]

**Theorem 2.** Let \( u > 1/3 \) be a fixed real number. Then

\[
\sum_{n \leq x} (\tilde{\sigma}(n))^u = C_3 x^{u+1} + O(x^{u+1/2} \delta(x)),
\]

where \( C_3 \) is given by

\[
C_3 = \frac{1}{u+1} \prod_p \left( 1 + \sum_{a=2}^{\infty} \frac{(\tilde{\sigma}(p^a))^u - p^u(\tilde{\sigma}(p^{a-1}))^u}{p^{a(u+1)}} \right)
\]

and

\[
\delta(x) = \exp(-A (\log x)^{3/5} (\log \log x)^{-1/5}),
\]

\( A \) being a positive constant.

**Theorem 3.**

\[
\sum_{n \leq x} \tilde{P}(n) = C_4 x^2 + O(x (\log x)^{5/3}),
\]

where the constant \( C_4 \) is given by

\[
C_4 = \frac{1}{2} \prod_p \left( 1 + \sum_{a=2}^{\infty} \frac{\tilde{P}(p^a) - p\tilde{P}(p^{a-1})}{p^{2a}} \right).
\]

Concerning the maximal order of the function \( \tilde{P}(n) \) we have

**Theorem 4.**

\[
\limsup_{n \to \infty} \frac{\tilde{P}(n)}{n \log \log n} = \frac{6}{\pi^2} e^\gamma,
\]

where \( \gamma \) is Euler’s constant.

**Theorem 5.** If \( k \geq 2 \) is a fixed integer, then

\[
\sum_{n \leq x} q_k(x)(n) = D_k x + O(x^{1/2} \delta(x)),
\]

where

\[
D_k = \prod_p \left( 1 + \sum_{a=2^k}^{\infty} \frac{q_k(a) - q_k(a-1)}{p^a} \right),
\]

\( q_k(n) \) denoting the characteristic function of \( k \)-free integers.

In the special case \( k = 2 \) case this formula is due to J. Wu [12], improving an earlier result of M. V. Subbarao [9].
3. Proofs

The proof of Theorem 1 is based on the following lemma.

**Lemma 1.** The Dirichlet series of \( \phi^{(c)} \) is absolutely convergent for \( \Re s > 1 \) and it is of form

\[
\sum_{n=1}^{\infty} \frac{\phi^{(c)}(n)}{n^s} = \zeta(s)\zeta(3s)V(s),
\]

where the Dirichlet series \( V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s} \) is absolutely convergent for \( \Re s > 1/5 \).

**Proof of Lemma 1.** Let \( \mu_3(n) = \mu(m) \) or 0, according as \( n = m^3 \) or not, where \( \mu \) is the Möbius function, and let \( f = \mu_3 * \mu \) in terms of the Dirichlet convolution. Then we can formally obtain the desired expression by taking \( v = \phi^{(c)} * f \). Both \( f \) and \( v \) are multiplicative and easy computations show that \( f(p^a) = 0 \) for each \( a \geq 5 \), and \( v(p^a) = 0 \) for \( 1 \leq a \leq 4 \), \( v(p^a) = \phi(a) - \phi(a - 1) - \phi(a - 3) + \phi(a - 4) \) for \( a \geq 5 \).

Since \( |v(p^a)| < 4a \) for \( a \geq 5 \), we obtain that \( V(s) \) is absolutely convergent for \( \Re s > 1/5 \).

**Proof of Theorem 1.** Lemma 1 shows that \( \phi^{(c)} = v * \tau(1,3,\cdot) \), where \( \tau(1,3,n) = \sum_{n \phi^3 = n} 1 \) for which

\[
\sum_{n \leq x} \tau(1,3,n) = \zeta(3)x + \zeta(1/3)x^{1/3} + O(x^{1/5}),
\]

cf. [4], p. 196-199. Therefore,

\[
\sum_{n \leq x} \phi^{(c)}(n) = \sum_{d \leq x} v(d) \sum_{e \leq x/d} \tau(1,3,e) =
\]

\[
= \zeta(3)x \sum_{d \leq x} \frac{v(d)}{d} + \zeta(1/3)x^{1/3} \sum_{d \leq x} \frac{v(d)}{d^{1/3}} + O \left( x^{1/5+\epsilon} \sum_{d \leq x} \frac{|v(d)|}{d^{1/3+\epsilon}} \right),
\]

and obtain the desired result by usual estimates.

For the proof of Theorem 2 we use the following general result due to V. Sita Ramaiah and D. Suryanarayana [7], Theorem 1.

**Lemma 2.** Let \( k \geq 2 \) be a fixed integer, \( \beta > (k+1)^{-1} \) be a fixed real number and \( g \) be a multiplicative arithmetic function such that \( |g(n)| \leq 1 \) for all \( n \geq 1 \). Suppose that either

(i) \( |g(p^j) - 1| \leq p^{-1} \) for \( 1 \leq j \leq k - 1 \), \( g(p^k) = 0 \) for all primes \( p \), or
(ii) \( g(p^j) = 1 \) for \( 1 \leq j \leq k - 1 \), \( g(p^k) = p^{-\beta} \) for all primes \( p \).

Then

\[
\sum_{n \leq x} g(n) = x \sum_{n=1}^{\infty} \frac{(g * \mu)(n)}{n} + O(x^{1/k} \delta(x)).
\]

**Proof of Theorem 2.** This is a direct consequence of Lemma 2 of above. Take \( g(n) = (\sigma(n)/n)^u \). Here \( g(p) = 1 \), \( g(p^2) = p^{-u} \), \( g(p^3) \leq p^{-u}(p + p^2 + \ldots + p^{a-1}) < (p - 1)^{-u} \leq 1 \) for every \( a \geq 3 \), hence \( 0 < g(n) \leq 1 \) for all \( n \geq 1 \). Choosing \( k = 2 \), \( \beta = u \), we obtain the given result by partial summation.
**Lemma 3.** The Dirichlet series of $\tilde{P}(n)$ is absolutely convergent for $Re s > 2$ and it is of form

$$\sum_{n=1}^{\infty} \frac{\tilde{P}(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)}W(s),$$

where the Dirichlet series $W(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s}$ is absolutely convergent for $Re s > 3/4$.

**Proof of Lemma 3.**

$$\sum_{n=1}^{\infty} \frac{\tilde{P}(n)}{n^s} = \prod_p \left(1 + \sum_{a=1}^{\infty} \sum_{d|a} \frac{p^d\phi(a/d)}{p^{as}} \right)$$

$$= \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi(j) \sum_{d=1}^{\infty} \frac{1}{p^{d(j-1)}}}{p^{js-1}} \right) = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi(j)}{p^{js-1} - 1} \right)$$

$$= \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)}W(s),$$

where

$$W(s) := \prod_p \left(1 + \frac{(p^{s-1} - 1)(p^{2s-1} - 1)}{p^{js-2} - 1} \sum_{j=3}^{\infty} \frac{\phi(j)}{p^{js-1} - 1} \right),$$

which is absolutely convergent for $Re s > 3/4$.

**Proof of Theorem 3.** By Lemma 3, $\tilde{P} = h \ast w$, where

$$h(n) = \sum_{ab^2c^3=n} abc^2 \mu(c),$$

and obtain the desired result, exactly like in proof of Theorem 2 of [5], using the estimate

$$\sum_{mn^2 \leq x} mn = \frac{1}{2} \zeta(3)x^2 + O(x(\log x)^{2/3})$$

due to Y. - F. S. PÉTERMANN and J. WU [5], Theorem 1.

Theorem 4 is a direct consequence of the following general result of L. TÓTH and E. WIRSing [10], Corollary 1.

**Lemma 4.** Let $f$ be a nonnegative real-valued multiplicative function. Suppose that for all primes $p$ we have $\rho(p) := \sup_{a \geq 0} f(p^a) \leq (1 - 1/p)^{-1}$ and that for all primes $p$ there is an exponent $e_p = p^{\alpha(p)}$ such that $f(p^{e_p}) \geq 1 + 1/p$. Then

$$\limsup_{n \to \infty} \frac{f(n)}{\log \log n} = \gamma \prod_p \left(1 - \frac{1}{p} \right) \rho(p).$$

**Proof of Theorem 4.** Apply Lemma 4 for $f(n) = \tilde{P}(n)/n$, where $f(p^a) \leq (p + p^2 + \cdots + p^a)p^{-a} < (1 - 1/p)^{-1}$ for every $a \geq 1$ and $f(p^2) = 1 + 1/p$, hence we can choose $e_p = 2$ for all $p$. Moreover, $\rho(p) = 1 + 1/p$ for all $p$ and obtain the desired result.

**Proof of Theorem 5.** This follows from Lemma 2 by taking $2^k$ instead of $k$, where $q_k^{(c)}(p^a) = q_k^{(c)}(p^2) = \ldots = q_k^{(c)}(p^{2^{k-1}}) = 1$, $q_k^{(c)}(p^{2^k}) = 0$.

4. Further results and problems
The next result is an analogue of the exponential divisor problem of Titchmarsh, see Theorem 1 of [8]. The proof is the same using that \( \phi^{(e)}(n) \) is a prime independent multiplicative function depending only on the squarefull (cubfull) kernel of \( n \) and that \( \phi^{(e)}(p^a) = \phi(a) \leq a \) for every \( a \geq 1 \).

**Theorem 6.** For every fixed \( B > 0 \),

\[
\sum_{p \leq x} \phi^{(e)}(p - 1) = C_5 \text{li} x + O(x/(\log x)^B),
\]

where

\[
C_5 = \prod_p \left( 1 + \sum_{k=3}^{\infty} \frac{\phi(k) - 1}{p^k} \right).
\]

Let \( \omega(n) \) and \( \Omega(n) \) denote, as usual, the number of prime factors of \( n \) and the number of prime power factors of \( n \), respectively.

**Theorem 7.** A maximal order of \( \Omega(\phi^{(e)}(n)) \) is \( 2(\log n)/5 \log \log n \).

This can be obtained by the same arguments as those given in the proof of Theorem 3.(i) of [8]. Here the upper bound is attained for \( n_k = (p_1 \cdots p_k)^5 \), where \( p_k \) is the \( k \)-th prime.

**Problem 1.** Determine a maximal order of \( \omega(\phi^{(e)}(n)) \).

Since \( \sigma(n) \leq n \) for all \( n \geq 1 \) and \( \sigma(p) = p \) for all primes \( p \), it is clear that a maximal order of \( \sigma(n) \) is \( n \).

**Problem 2.** Determine a minimal order of \( \sigma(n) \).

J. Sándor [6] considered in fact the function \( \varphi_e(n) \) defined as the number of integers \( 1 < a < n \) for which \( a \) and \( n \) are exponentially coprime \( (n > 1) \) and \( \varphi_e(1) = 1 \). Although \( \varphi_e(p^a) = \phi^{(e)}(p^e) = \phi(a) \) for any prime power \( p^e \), functions \( \varphi_e \) and \( \phi^{(e)} \) are not the same. Take for example \( n = 2^3 \cdot 3^2 \), then numbers \( a < n \) exponentially coprime to \( n \) are \( a = 2 \cdot 3, 2^2 \cdot 3, 2^4 \cdot 3, \) hence \( \varphi_e(2^3 \cdot 3^2) = 3 \neq 2 \cdot 1 = \phi(3)\phi(2) = \varphi_e(2^3) \cdot \varphi_e(3^2) \).

Therefore, \( \varphi_e \) is not multiplicative and \( \varphi_e(n) \geq \phi^{(e)}(n) \) for every \( n \geq 1 \).

**Problem 3.** What can be said on the order of the function \( \varphi_e(n) \)?

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