Derivative of a Conic Problem with a Unique Solution

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Abstract

We view a conic optimization problem that has a unique solution as a map from its data to its solution. If sufficient regularity conditions hold at a solution point, namely that the implicit function theorem applies to the normalized residual function of [BMD18], the problem solution map is differentiable. We obtain the derivative, in the form of an abstract linear operator. This applies to any convex optimization problem in conic form, while a previous result [AK17] studied strictly convex quadratic programs. Such differentiable problems can be used, for example, in machine learning, control, and related areas, as a layer in an end-to-end learning and control procedure, for backpropagation. We accompany this note with a lightweight Python implementation which can handle problems with the cone constraints commonly used in practice.

1 Introduction

Convex optimization problems, which can be always formulated as conic problems, are ubiquitous in many areas of applied mathematics. They are often embedded in procedures involving multiple computational layers, such as multi-steps estimations of statistical models [HTF09], learning and control policies [MBBW18], or iterative procedures that solve many problems in sequence [BBD+17]. In those cases it is interesting to analyze perturbations of the primal-dual solution in the numerical data of the problem. We do not consider, in this paper, changes in the shape of the cone of the problem. As an example, in model predictive control, one might be interested in knowing how the optimal action changes with changes in the forecasts. Whenever a cone problem has a unique solution, it can be viewed as a real-valued operator from the vector space of problem data to the vector space of primal-dual solutions. We obtain sufficient conditions for which this solution map is differentiable. In those cases we obtain the derivative linear operator and give efficient computational procedures to evaluate its matrix-vector products. We mirror the mathematical exposition with an open source Python implementation, with minimal dependencies.

Related work. This work is in line with a current trend of differentiable convex programming [Amo19, AWK18], which aims to bring together the computational power of modern machine learning techniques, such as end-to-end learning with automatic differentiation [ABC+16, PGC+17], and the maturity, clarity, and beauty of mathematical, and in particular convex, optimization.

2 Conic problem and homogeneous primal-dual embedding

We consider a conic optimization problem in its primal (P) and dual (D) forms (see, e.g., [BV04 §4.6.1] and [BTN01]):
\[(P) \text{ minimize } c^T x \quad \text{subject to } Ax + s = b \quad s \in K \]
\[(D) \text{ minimize } b^T y \quad \text{subject to } A^T y + c = 0 \quad y \in K^* \]

Here \(x \in \mathbb{R}^n\) is the primal variable, \(y \in \mathbb{R}^m\) is the dual variable, and \(s \in \mathbb{R}^m\) is the primal slack variable. The set \(K \subseteq \mathbb{R}^m\) is a nonempty closed convex cone and the set \(K^* \subseteq \mathbb{R}^m\) is its dual cone, \(K^* = \{y \in \mathbb{R}^m \mid \inf_{k \in K} y^T k \geq 0\}\). The problem data are the matrix \(A \in \mathbb{R}^{m \times n}\), the vectors \(b \in \mathbb{R}^m\), \(c \in \mathbb{R}^n\) and the cone \(K\).

**Solving a primal-dual conic program.** We call \((x,y,s)\) a solution of the primal-dual conic program \((1)\) if and only if it satisfies the conic optimality conditions
\[Ax + s = b, \quad A^T y + c = 0, \quad s \in K, \quad y \in K^*, \quad s^T y = 0. \quad (2)\]

**The solution map.** The goal of this paper is to understand how the solution \((x,y,s)\) changes when the data \((A,b,c)\) changes. For given data \((A,b,c)\) the corresponding primal-dual conic program may have no solution, a unique solution or multiple solutions. We focus on the case when \((1)\) has a unique solution \((x,y,s)\) for given data \((A,b,c)\). The problem solution map \(P : \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+2m}\) is defined by \(P(A,b,c) = (x,y,s)\). In this paper we obtain conditions sufficient to guarantee that \(P(A,b,c)\) is a differentiable function of \((A,b,c)\). Those are often, but not always, satisfied in practice.

In those cases, we obtain the derivative \(D P\), a linear map from \(\mathbb{R}^{mn+3m+n}\) to \(\mathbb{R}^{2n+4m}\). We note that storing in a computer memory a matrix describing \(D P\) could be impractical even for moderately sized problems.

**Homogenous self-dual embedding.** We consider the homogenous self-dual embedding \([YTM94, OCPB16, BMB18]\) of \((1)\), which uses the optimization variable \(z \in \mathbb{R}^N\), for \(N = n + m + 1\), and the matrix
\[Q = Q(A,b,c) = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \in Q,\]
where \(Q\) is the vector space of \(N \times N\) skew-symmetric matrices. The function \(Q(A,b,c)\) is affine, and has trivial derivative \(D Q\). The variable \(z\) is partitioned as \(z = (u,v,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}\). The optimality conditions of the embedded problem are
\[w > 0, \quad Q\Pi z = \Pi z - z, \quad (3)\]
where \(\Pi\) is the projection onto \(\mathbb{R}^n \times K^* \times \mathbb{R}_+\). The embedding can also be used to obtain certificates of infeasibility or unboundedness, but we do not consider that possibility in this paper.

**Reconstructing a primal-dual solution form the embedding.** We then define \(\psi : \mathbb{R}^N \rightarrow \mathbb{R}^{n+2m}\) by
\[(x, y, s) = \psi(z) = (u/w, \Pi_{K^*}(v)/w, (\Pi_{K^*}(v) - v)/w).\]
If \(z\) satisfies \((3)\), then \((x,y,s) = \psi(z)\) satisfies \((2)\), i.e., it is a primal-dual solution of \((1)\). The converse also applies (see \([BMB18]\)). In the following we may refer to \(z\) as a solution of \((1)\), meaning
that $\psi(z)$ is. We note that $\psi$ is differentiable wherever $\Pi_{K^*}$ is, and the matrix $D\psi(z) \in \mathbb{R}^{(n+2m) \times N}$ is given by

$$D\psi(z) = \begin{bmatrix}
I/w & 0 & -u/w^2 \\
0 & D\Pi_{K^*}/w & -(\Pi_{K^*}(v))/w^2 \\
0 & (D\Pi_{K^*} - I)/w & -(\Pi_{K^*}(v) - v)/w^2
\end{bmatrix}.$$

**Normalized residual map.** The normalized residual map $N : \mathbb{R}^N \times Q \to \mathbb{R}^N$ is introduced in [BMB18] and defined as

$$N(z, Q) = ((Q - I)\Pi + I)(z/|z_N|).$$

For given data $Q(A, b, c)$, $z$ is a solution of the primal-dual pair (1) if and only if $N(z, Q) = 0$ and $z_N > 0$.

**Derivative mappings of $N$.** $N$ is an affine function of $Q$, hence $D_Q N$ always exists. In fact, $D_Q N(z, Q) : Q \to \mathbb{R}^N$ with

$$D_Q N(z, Q)(U) = U\Pi(z/|z_N|), \quad D_Q N(z, Q)^T(r) = \Pi(z/|z_N|)r^T,$$

for any skew-symmetric matrix $U \in Q$ and vector $r \in \mathbb{R}^N$. We now turn to $D_z N(z, Q)$. Let $z$ be such that $z_N \neq 0$, then if $\Pi$ is differentiable at $z$ also $N$ is differentiable at $z$, and

$$D_z N(z, Q) = ((Q - I)\Pi(z) + I)/|z_N| - \text{sign}(z_N)((Q - I)\Pi + I)(z/|z_N|)e^T,$$

where $e = (0, 0, \ldots, 1) \in \mathbb{R}^N$.

### 3 Derivative of the solution map

We give sufficient conditions for which the derivative of the problem solution map $D\mathcal{P}(A, b, c)$ exists, and provide methods to compute its matrix-vector products.

**Implicit function theorem.** Suppose that $z$ is a solution of the primal-dual pair (1) for a given $Q$, and that $\Pi$ is differentiable at $z$. Then $N$ is differentiable at $z$, $N(z, Q) = 0$ and $z_N > 0$. Now, assume that $D_z N(z, Q)$ is invertible. It follows from the implicit function theorem (see [Dim07] and [DR09]) that there exists a set $V \subseteq Q$ with non-empty interior, $Q \in \text{int}(V)$, such that for any $Q \in V$ there always exists a unique $\bar{z}$ such that $N(\bar{z}, Q) = 0$, and $\bar{z}$ is given by a differentiable function $\phi : V \to \mathbb{R}^N$ such that $z = \phi(Q)$. Furthermore, $\phi$ is differentiable on $V$, and

$$D\phi(Q) = -(D_z N(\phi(Q), Q))^{-1}D_Q N(\phi(Q), Q).$$

**Chain rule.** The problem map is

$$\mathcal{P}(A, b, c) = \psi(\phi(Q(A, b, c))),$$

and we have, by the chain rule [Lei20],

$$D\mathcal{P}(A, b, c) = -D\psi(z)(D_z N(z, Q))^{-1}D_Q N(z, Q)DQ(A, b, c).$$

**LSQR.** It is possible to solve equations involving $D\mathcal{P}(A, b, c)$ by forming the matrix $D_z N(z, Q)$ and factorizing it, but for large problems this might be impractical. We instead resort to an iterative (or indirect) linear system solution technique, LSQR [PSS2], which does not require to store or factor the matrix $D\mathcal{P}(A, b, c)$. We provide methods to evaluate its right and left matrix-vector multiplications.
Evaluation of $D\mathcal{P}(A,b,c)(dA,db,dc)$. We use LSQR to solve the least squares problem

$$\text{minimize} \quad \|D_z\mathcal{N}(z,Q)dz + Q(dA,db,dc)\Pi(z/|z_N|)\|^2, \tag{4}$$

in the variable $dz \in \mathbb{R}^N$, and finally obtain $(dx,dy,dz) = D\psi(z)dz$.

Evaluation of $D\mathcal{P}(A,b,c)^T(dx,dy,ds)$. Similarly, we solve with LSQR the problem

$$\text{minimize} \quad \|D_z\mathcal{N}(z,Q)^Tdr + D\psi(z)^T(dx,dy,ds)\|^2, \tag{5}$$

in the variable $dr \in \mathbb{R}^N$, and obtain $(dA,db,dc) = DQ(A,b,c)^TDQ\mathcal{N}(z,Q)^Tdr$.

4 Implementation

CPSR. We provide a Python code called CPSR, available at [https://github.com/enzobusseti/cpsr](https://github.com/enzobusseti/cpsr), implementing the two multiplication procedures (4) and (5), for conic problems whose cone $K$ is the Cartesian product of a zero cone, a nonnegative cone, and any number of second-order cones, semidefinite cones, and exponential cones. The code can also be used for the refinement of solutions of conic problems, and is described in more detail in [BMB18]. It depends on the standard scientific libraries Numpy [Oli06] for vector arithmentics, Scipy [JOP+01] for sparse linear algebra, and Numba [Tea15] for just-in-time compilation. It also provides an interface to the convex optimization specification library Cvxpy [DB16].

Code example. We include a minimal example.

```python
import cpsr
A,b,c,cone_shape, _ = cpsr.generate_problem(mode='solvable')
DP = cpsr.derivative(A, b, c, cone_shape)
```

The object stored in DP is an abstract linear operator which represents $D\mathcal{P}(A,b,c)$, meaning that it implements the right and left matrix-vector products described above, as class methods, and it does not store the entries of $D\mathcal{P}(A,b,c)$.

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An earlier version of this paper credited Walaa Moursi and Stephen Boyd, who provided, respectively, mathematical and expert advices, as authors. Walaa Moursi also contributed some editing. They since asked to not be listed as authors, for disagreements over the scope of the project.

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