WEIGHTED BERGMAN SPACES AND THE $\bar{\partial}$–EQUATION

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Dedicated to Professor Jinhao Zhang on the occasion of his seventieth birthday

Abstract. We give a Hörmander type $L^2$–estimate for the $\bar{\partial}$–equation with respect to the measure $\delta_{\Omega}^{-\alpha}dV$, $\alpha < 1$, on any bounded pseudoconvex domain with $C^2$–boundary. Several applications to the function theory of weighed Bergman spaces $A^2_{\alpha}(\Omega)$ are given, including a corona type theorem, a Gleason type theorem, together with a density theorem. We investigate in particular the boundary behavior of functions in $A^2_{\alpha}(\Omega)$ by proving an analogue of the Levi problem for $A^2_{\alpha}(\Omega)$ and giving an optimal Gehring type estimate for functions in $A^2_{\alpha}(\Omega)$. A vanishing theorem for $A^2_{1}(\Omega)$ is established for arbitrary bounded domains. Relations between the weighted Bergman kernel and the Szegö kernel are also discussed.

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1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let $\varphi$ be a $C^2$ plurisubharmonic (psh) function on $\Omega$. A fundamental theorem of Hörmander (cf. [23, 26], see also [1, 13]) states that for any $\bar{\partial}$–closed $(0,1)$–form $v$, there exists a solution $u$ to the equation $\bar{\partial}u = v$ such that

$$
\int_{\Omega} |u|^2 e^{-\varphi} dV \leq \int_{\Omega} |v|^2 i\partial \bar{\partial} \varphi e^{-\varphi} dV
$$

(1.1)

provided the right-hand side is finite.

In 1983, Donnelly-Fefferman [14] made a striking discovery that under certain condition, the $\bar{\partial}$–equation may have solutions of finite $L^2$–norm with some non-psh weight. Such a discovery was extended and simplified substantially by a number of mathematicians (see e.g. [17, 14, 6, 33, 9]), now may be formulated as follows: if $\psi$ is another $C^2$ psh function on $\Omega$ satisfying $i\alpha \partial \bar{\partial} \psi \geq i \partial \psi \wedge \bar{\partial} \psi$ for some $0 < \alpha < 1$, then the $L^2(\Omega, \varphi)$–minimal solution of the $\bar{\partial}$–equation enjoys the estimate

$$
\int_{\Omega} |u|^2 e^{-\varphi} dV \leq \text{const}_\alpha \int_{\Omega} |v|^2 i\partial \bar{\partial} (\varphi + \psi) e^{-\varphi} dV
$$

(1.2)

provided the right-hand side is finite. In particular, if we take $\psi = -\frac{\alpha}{\alpha_0} \log(-\rho)$, where $\rho$ is a negative $C^2$ psh function verifying $-\rho \asymp \delta_{\Omega}^{\alpha_0}$, $\alpha_0 > \alpha > 0$ and $\delta_{\Omega}$ is the boundary distance function, then (1.2) implies

$$
\int_{\Omega} |u|^2 e^{-\varphi} \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \int_{\Omega} |v|^2 i\partial \bar{\partial} \varphi e^{-\varphi} \delta_{\Omega}^{-\alpha} dV,
$$

(1.3)

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Theorem 1.1. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$-boundary and $\varphi$ a $C^2$ psh function on $\Omega$. Then for each $\alpha < 1$ and each $\partial$-closed $(0,1)$-form $v$ with $\int_{\Omega} |v|_0^2 e^{-\varphi} \delta^{-\alpha} dV < \infty$, there is a solution $u$ to the equation $\partial u = v$ such that (1.3) holds.

We shall give various applications of this result to the function theory of the weighted Bergman space $A^2_\alpha(\Omega)$, that is, the Hilbert space of holomorphic functions $f$ on $\Omega$ with

$$||f||^2_\alpha := \int_{\Omega} |f|^2 \delta^{-\alpha} dV < \infty.$$  

The spaces $A^2_\alpha(\Omega)$ coincide with the usual Sobolev spaces of holomorphic functions for $\alpha < 1$, i.e.,

$$A^2_\alpha(\Omega) = \mathcal{O}(\Omega) \cap W^\alpha(\Omega)$$

(see Ligocka [32]). Despite of deep results achieved for strongly pseudoconvex domains (see e.g., [2, 18]), few progress has been made in the case of weakly pseudoconvex domains.

Theorem 1.2. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$-boundary. Let $f_1, f_2 \in \mathcal{O}(\Omega)$ and $\delta > 0$ be such that

$$\delta^2 \leq |f_1|^2 + |f_2|^2 \leq 1.$$  

Then for each $h \in A^2_\alpha(\Omega)$, $\alpha < 1$, there are functions $g_1, g_2 \in A^2_\alpha(\Omega)$ satisfying

$$f_1 g_1 + f_2 g_2 = h.$$  

Theorem 1.3. Let $\Omega \subset \subset \mathbb{C}^2$ be a pseudoconvex domain with $C^2$-boundary. If $w \in \Omega$ and $h \in A^2_\alpha(\Omega)$, $\alpha < 1$, then there are functions $g_1, g_2 \in A^2_\alpha(\Omega)$ satisfying

$$h(z) - h(w) = (z_1 - w_1) g_1(z) + (z_2 - w_2) g_2(z), \quad \forall z \in \Omega.$$  

Theorem 1.4. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$-boundary.

1. For each $\alpha < 1$, $A^2_\alpha(\Omega)$ is dense in the space $\mathcal{O}(\Omega)$, equipped with the topology of uniform convergence on compact subsets.
2. For any $\alpha_1 < \alpha_2 < 1$, $A^2_{\alpha_1}(\Omega)$ is dense in $A^2_{\alpha_2}(\Omega)$.

The following result is an analogue of the Levi problem for $A^2_\alpha(\Omega)$, which also generalizes an old result of Pfug (cf. [38]):

Theorem 1.5. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$-boundary. Then for each $\alpha < 1$, there are $\beta > 0$ and $f \in A^2_\alpha(\Omega)$ such that for all $\zeta \in \partial \Omega$,

$$\limsup_{z \to \zeta} |f(z)| \delta(\zeta)^{1 - \frac{\alpha}{2}} |\log \delta(\zeta)|^\beta = \infty.$$  

It should be pointed out that each bounded pseudoconvex domain with $C^\infty$-boundary is the domain of existence of a function in $A^\infty(\Omega) := \mathcal{O}(\Omega) \cap C^\infty(\Omega)$ (cf. [10], see also [22]).

On the other side, we have the following Gehring type estimate:
Theorem 1.6. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$-boundary and let $f \in A^2_\alpha(\Omega)$, $\alpha < 1$. Then for almost all $\zeta \in \partial \Omega$

$$|f(z)| = o(\delta_\zeta(z)^{\frac{1-\alpha}{2}})$$

uniformly, as $z$ approaches $\zeta$ admissibly. Here $\delta_\zeta(z) =$ minimum of $\delta_\Omega(z)$ and the distance from $z$ to the tangent space at $\zeta$, and $A = o(B)$ means $\lim A/B = 0$.

The concept of admissible approach was introduced by Stein [11] in his far-reaching generalization of Fatou’s theorem for holomorphic functions in a bounded domain with $C^2$-boundary.

It turns out that the above bound is optimal for the case of the unit ball:

Theorem 1.7. Let $B^n$ be the unit ball in $\mathbb{C}^n$ and $S^n$ the unit sphere. For each $\alpha < 1$, there is a number $t_\alpha > 1$ such that for each $\epsilon > 0$, there exists a function $f \in A^2_\alpha(B^n)$ so that for each $\zeta \in S^n$,

$$\limsup |f(z)|(1-|z|)^{\frac{1-\alpha}{2}} \log(1-|z|)^{\frac{1+\epsilon}{2}} > 0$$

as $z \to \zeta$ from the inside of the Koranyi region $\mathcal{A}_{t_\alpha}(\zeta)$ defined by

$$\mathcal{A}_{t_\alpha}(\zeta) = \{ z \in B^n : |1 - z \cdot \overline{\zeta}| < t_\alpha(1-|z|) \}.$$

Stein [11] suggested to study the relation between the Bergman and Szegö kernels. In [12], Chen-Fu obtained a comparison of the Szegö and Bergman kernels for so-called $\delta$-regular domains including domains of finite type and domains with psh defining functions. Here we shall prove the following natural connection between the weighted Bergman kernels $K_\alpha$ and the Szegö kernel $S$,

which seems not to have been noticed in the literature:

Theorem 1.8. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$-boundary. Then

$$(1 - \alpha)^{-1} K_\alpha(z,w) \to S(z,w)$$

locally uniformly in $z,w$ as $\alpha \to 1^-$. In particular,

$$\frac{\partial K_\alpha(z,w)}{\partial \alpha} \bigg|_{\alpha = 1^-} := \lim_{\alpha \to 1^-} \frac{K_\alpha(z,w) - K_1(z,w)}{\alpha - 1} = -S(z,w).$$

For general bounded domains, a fundamental question immediately arises:

When is $A^2_\alpha(\Omega)$ trivial or nontrivial?

Clearly, $A^2_\alpha(\Omega)$ is always nontrivial for $\alpha \leq 0$. On the other side, we have the following vanishing theorem:

Theorem 1.9. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$.

(1) For each $f \in \mathcal{O}(\Omega)$ with $\int_{\Omega} |f|^2 \delta_\Omega^{-1} (1 + |\log \delta_\Omega|)^{-1} \, dV < \infty$, we have $f = 0$. In particular, $A^2_\alpha(\Omega) = \{0\}$ for each $\alpha \geq 1$.

(2) Let $\Omega_\epsilon = \{ z \in \Omega : \delta_\Omega(z) > \epsilon \}$ and let $c(\epsilon) := \text{cap}(\overline{\Omega_\epsilon}, \Omega)$ denote the capacity of $\overline{\Omega_\epsilon}$ in $\Omega$. Suppose there is a sequence $\epsilon_j \to 0^+$, so that $c(\epsilon_j) = O(\epsilon_j^{-\alpha})$, then $A^2_\alpha(\Omega) = \{0\}$.

As a consequence Theorem 1.9 we have

Theorem 1.10. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. For each $\epsilon > 0$, there does not exist a continuous psh function $\rho < 0$ on $\Omega$ such that

$$-\rho \leq \text{const} \delta_\Omega (1 + |\log \delta_\Omega|)^{-\epsilon}.$$ 

In particular, the order of hyperconvexity of $\Omega$ is no larger than 1. In case $\partial \Omega$ is of class $C^2$, this result is a direct consequence of the Hopf lemma.
2. Proof of Theorem 2.1

Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$-boundary. Let $\varphi$ be a real-valued $C^2$-smooth function on $\Omega$. Let $L^p_{(2)}(\Omega, \varphi)$ denote the space of $(p, q)$-forms $u$ on $\Omega$ satisfying

$$||u||^2_{\varphi} := \int_{\Omega} |u|^2 e^{-\varphi} dV < \infty.$$ 

Let $\bar{\partial}^{\ast}_\varphi$ denote the adjoint of the operator $\bar{\partial}$ with respect to the corresponding inner product $(\cdot, \cdot)_\varphi$. We recall the following twisted Morrey-Kohn-Hörmander formula, which goes back to Ohsawa-Takegoshi (cf. [36, 4, 40, 33, 37, 9]):

**Proposition 2.1.** Let $\rho$ be a $C^2$-defining function of $\Omega$. Let $u$ be a $(0, 1)$-form that is continuously differentiable on $\overline{\Omega}$ and satisfies the $\bar{\partial}$--Neumann boundary conditions on $\partial\Omega$, $\partial\rho \cdot u = 0$, and let $\eta$ and $\varphi$ be real-valued functions that are twice continuously differentiable on $\overline{\Omega}$ with $\eta \geq 0$. Then

$$||\sqrt{\eta} \bar{\partial} u||^2_\varphi + ||\sqrt{\eta} \bar{\partial}^{\ast}_\varphi u||^2_\varphi = \sum_{j,k=1}^n \int_{\Omega} \eta \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} u_{j|k} e^{-\varphi} \frac{d\sigma}{|\nabla \rho|} + \sum_{j=1}^{n} \int_{\Omega} |\eta \frac{\partial u_j}{\partial \bar{z}_j}|^2 e^{-\varphi} dV$$

$$+ \sum_{j,k=1}^n \int_{\Omega} \left( \eta \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 \eta}{\partial z_j \partial \bar{z}_k} \right) u_{j|k} e^{-\varphi} dV$$

$$+ 2\text{Re} \int_{\Omega} (\partial \eta \cdot u) \bar{\partial}^{\ast}_\varphi u e^{-\varphi} dV.$$

Now we prove Theorem 2.1. It is well-known that locally the Diederich-Fornæss exponents can be arbitrarily close to 1 (cf. [15], Remark b), p. 133). Thus for any given $\alpha < 1$, there exists a cover $\{U_j\}_{1 \leq j \leq m_\alpha}$ of $\partial\Omega$ and $C^2$ psh functions $\rho_j < 0$ on $\Omega \cap U_j$ such that

$$C^{-1} \delta_\Omega(z)^{\frac{\alpha+1}{2}} \leq -\rho_j(z) \leq C \delta_\Omega(z)^{\frac{\alpha+1}{2}}, \quad z \in \Omega \cap U_j, \ 1 \leq j \leq m_\alpha$$

(Throughout this section, $C$ denotes a generic positive constant depending only on $\alpha$ and $\Omega$). Take an open subset $U_0 \subset \subset \Omega$ such that $\{U_j\}_{0 \leq j \leq m_\alpha}$ forms a cover of $\overline{\Omega}$. Clearly, we can take a negative $C^2$ psh function $\rho_0$ on $U_0$ such that

$$C^{-1} \delta_\Omega(z)^{\frac{\alpha+1}{2}} \leq -\rho_0(z) \leq C \delta_\Omega(z)^{\frac{\alpha+1}{2}}, \quad z \in U_0$$

(for example, $\rho_0(z) = |z|^2 - \sup_{\Omega} |z|^2 - 1$).

Put $\varphi_\tau(z) = \varphi(z) + \tau |z|^2$, $\tau > 0$, and $\Omega_\varepsilon := \{z \in \Omega : \delta_\Omega(z) > \varepsilon\}$, $\varepsilon \ll 1$. By Proposition 2.1, we have

$$\int_{\Omega_\varepsilon} (\eta + c(\eta)^{-1})|\bar{\partial}^{\ast}_\varphi w|^2 e^{-\varphi_\tau} dV + \int_{\Omega_\varepsilon} \eta |\bar{\partial} w|^2 e^{-\varphi_\tau} dV$$

$$(2.1) \geq \sum_{k,l} \int_{\Omega_\varepsilon} \left( \eta \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} - \frac{\partial^2 \eta}{\partial z_k \partial \bar{z}_l} \right) w_k w_l e^{-\varphi_\tau} dV - \int_{\Omega_\varepsilon} c(\eta) \left| \sum_k \frac{\partial \eta}{\partial z_k} w_k e^{-\varphi_\tau} dV$$

where $w = \sum_k w_k dz_k$ lies in $\text{Dom} \bar{\partial}^{\ast}_\varphi$ and is continuously differentiable on $\overline{\Omega}_\varepsilon$ (i.e., it satisfies the $\bar{\partial}$--Neumann boundary condition on $\partial\Omega_\varepsilon$), $\eta \geq 0$, $\eta \in C^2(\Omega)$ and $c$ is a positive continuous function on $\mathbb{R}^+$. 

Let $\{\chi_j\}_{0 \leq j \leq m_\alpha}$ be a partition of unity subordinate to the cover $\{U_j\}_{0 \leq j \leq m_\alpha}$ of $\overline{\Omega}$. The point is that $w^j = \chi_j w$ still lies in $\text{Dom} \bar{\partial}^{\ast}_\varphi$. Now we choose a real-valued function $\tilde{\chi}_j \in$
\(C_0^\infty(U_j)\) so that \(\tilde{\chi}_j = 1\) on \(\text{supp}\, \chi_j\). Put \(\psi_j = -\frac{2\alpha}{\alpha+1} \log(-\rho_j)\). Applying (2.1) to each \(w^j\) with \(\eta = e^{-\tilde{\chi}_j\psi_j}\) and \(c(\eta) = \frac{1-\alpha}{2\alpha} e^{-\tilde{\chi}_j\psi_j}\), we get
\[
\sum_{k,l} \int_{\Omega_{\epsilon} \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \ e^{-\varphi_\tau - \psi_j} dV
\leq \int_{\Omega_{\epsilon} \cap U_j} |\bar{\partial}(\chi_j w)|^2 e^{-\varphi_\tau - \psi_j} dV + \frac{1 + \alpha}{1 - \alpha} \int_{\Omega_{\epsilon} \cap U_j} |\bar{\partial}_{\varphi_\tau} (\chi_j w)|^2 e^{-\varphi_\tau - \psi_j} dV
\]
because
\[-i(\bar{\partial} \eta + c(\eta) \partial \eta \wedge \bar{\partial} \eta) = i e^{-\psi_j} \left( \bar{\partial} \bar{\partial} \psi_j - \frac{\alpha + 1}{2\alpha} \bar{\partial} \psi_j \wedge \bar{\partial} \psi_j \right) \geq 0\]
holds on \(\Omega \cap \text{supp}\, \chi_j\). Since \(e^{-\psi_j} \asymp \delta_\Omega^\alpha\) on \(\Omega \cap U_j\), we get
\[(2.2) \sum_{k,l} \int_{\Omega_{\epsilon} \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \ e^{-\varphi_\tau} \delta_\Omega^\alpha dV \leq C \int_{\Omega_{\epsilon} \cap U_j} (|\bar{\partial}(\chi_j w)|^2 + |\bar{\partial}_{\varphi_\tau} (\chi_j w)|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha dV.\]
Thus
\[
\sum_{k,l} \int_{\Omega_{\epsilon}} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l \ e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
= \sum_{k,l} \int_{\Omega_{\epsilon}} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} \left( \sum_{j=0}^{m_\alpha} \chi_j \right)^2 w_k \bar{w}_l \ e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
\leq (m_\alpha + 1) \sum_{j=0}^{m_\alpha} \sum_{k,l} \int_{\Omega_{\epsilon} \cap U_j} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} |\chi_j|^2 w_k \bar{w}_l \ e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
\leq (m_\alpha + 1) \sum_{j=0}^{m_\alpha} \int_{\Omega_{\epsilon} \cap U_j} (|\bar{\partial}(\chi_j w)|^2 + |\bar{\partial}_{\varphi_\tau} (\chi_j w)|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
by (2.2). Since
\[
\bar{\partial}(\chi_j w) = \chi_j \bar{\partial} w + \bar{\partial} \chi_j \wedge w, \quad \bar{\partial}_{\varphi_\tau} (\chi_j w) = \chi_j \bar{\partial}_{\varphi_\tau} w - \bar{\partial} \chi_j w,
\]
thus by Schwarz’s inequality,
\[
\sum_{k,l} \int_{\Omega_{\epsilon}} \frac{\partial^2 \varphi_\tau}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l \ e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
\leq 2(m_\alpha + 1) \sum_{j=0}^{m_\alpha} \int_{\Omega_{\epsilon} \cap U_j} (|\bar{\partial} w|^2 + |\bar{\partial}_{\varphi_\tau} w|^2 + 2|w|^2 |\bar{\partial} \chi_j|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
\leq 2(m_\alpha + 1)^2 C \int_{\Omega_{\epsilon}} (|\bar{\partial} w|^2 + |\bar{\partial}_{\varphi_\tau} w|^2) e^{-\varphi_\tau} \delta_\Omega^\alpha dV
\]
\[
(2.3) + 4(m_\alpha + 1) C \int_{\Omega_{\epsilon}} |w|^2 \sum_j |\bar{\partial} \chi_j|^2 e^{-\varphi_\tau} \delta_\Omega^\alpha dV.
\]
Since $\bar{\partial}\bar{\partial} \varphi_r = \partial \bar{\partial} \varphi + \tau \partial \bar{\partial}|z|^2$, thus when $\tau = \tau(\alpha, \Omega)$ is sufficiently large, the term in (2.3) may be absorbed by the left-hand side and we get the following basic inequality

\[
\sum_{k,l} \int_{\Omega_\varepsilon} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l e^{-\varphi_r} \delta^\varepsilon_\Omega dV \leq C \int_{\Omega_\varepsilon} (|\bar{\partial}w|^2 + |\bar{\partial}^* w|^2) e^{-\varphi_r} \delta^\varepsilon_\Omega dV.
\]

(2.4)

The remaining argument is standard. By Hörmander [23, Proposition 2.1.1], the same inequality holds for any $w \in L^0_{(2)}(\Omega_\varepsilon, \varphi_r) \cap \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}^*$, (Note that $C^{-1}_\varepsilon \leq \delta^\varepsilon_\Omega \leq C_\varepsilon$ on $\Omega_\varepsilon$). In particular, if $\bar{\partial}w = 0$, then

\[
\sum_{k,l} \int_{\Omega_\varepsilon} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l e^{-\varphi_r} \delta^\varepsilon_\Omega dV \leq C \int_{\Omega_\varepsilon} |\bar{\partial}^* w|^2 e^{-\varphi_r} \delta^\varepsilon_\Omega dV.
\]

By Schwarz’s inequality,

\[
\left| \int_{\Omega_\varepsilon} \langle v, w \rangle e^{-\varphi_r} dV \right|^2 \leq \int_{\Omega_\varepsilon} |v|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi_r} \delta^\varepsilon_\Omega dV \sum_{k,l} \int_{\Omega_\varepsilon} \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l e^{-\varphi_r} \delta^\varepsilon_\Omega dV \leq C \int_{\Omega_\varepsilon} |v|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi_r} \delta^\varepsilon_\Omega dV \int_{\Omega_\varepsilon} |\bar{\partial}^* w|^2 e^{-\varphi_r} \delta^\varepsilon_\Omega dV.
\]

For general $w \in \text{Dom } \bar{\partial}^*$, one has the orthogonal decomposition $w = w_1 + w_2$ where $w_1 \in \text{Ker } \bar{\partial}$ and $w_2 \in (\text{Ker } \bar{\partial})^\perp \subset \text{Ker } \bar{\partial}^*$. Thus

\[
\left| \int_{\Omega_\varepsilon} \langle v, w \rangle e^{-\varphi_r} dV \right|^2 = \left| \int_{\Omega_\varepsilon} \langle v, w_1 \rangle e^{-\varphi_r} dV \right|^2 \leq C \int_{\Omega_\varepsilon} |v|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi_r} \delta^\varepsilon_\Omega dV \int_{\Omega_\varepsilon} |\bar{\partial}^* w_1|^2 e^{-\varphi_r} \delta^\varepsilon_\Omega dV = C \int_{\Omega_\varepsilon} |v|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi_r} \delta^\varepsilon_\Omega dV \int_{\Omega_\varepsilon} |\bar{\partial}^* w|^2 e^{-\varphi_r} \delta^\varepsilon_\Omega dV.
\]

Applying the Hahn-Banach theorem to the anti-linear map

\[
\delta^\varepsilon_\Omega \bar{\partial}^* w \mapsto \int_{\Omega_\varepsilon} \langle v, w \rangle e^{-\varphi_r} dV
\]

together with the Riesz representation theorem, we get a solution $u_\varepsilon$ of the equation $\bar{\partial}(\delta^\varepsilon_\Omega u_\varepsilon) = v$ on $\Omega_\varepsilon$ with the estimate

\[
\int_{\Omega_\varepsilon} |u_\varepsilon|^2 e^{-\varphi_r} dV \leq C \int_{\Omega_\varepsilon} |v|^2_{i \partial \bar{\partial} \varphi} e^{-\varphi_r} \delta^\varepsilon_\Omega dV.
\]

Taking a weak limit of $\delta^\varepsilon_\Omega u_\varepsilon$ as $\varepsilon \to 0+$, we immediately obtain the desired solution. Q.E.D.

Remark. (1) The additional weight $t|z|^2$ is somewhat inspired by Kohn [30].

(2) The following variation of Theorem 1.1 is more convenient for applications, which may be proved similarly, together with an additional approximation argument.

Theorem 2.2. Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with $C^2$ boundary and let $\tilde{\Omega} \subset \Omega$ be a pseudoconvex domain. Let $\varphi$ be a psh function on $\Omega$ such that $i \partial \bar{\partial} \varphi \geq i \partial \bar{\partial} \psi$ in the sense of distribution, where $\psi$ is a $C^2$ psh function on $\Omega$. Then for each $\alpha < 1$ and each
form $v$ with $\int_{\tilde{\Omega}} |v|^2 e^{-\varphi} \delta_{\Omega}^{-\alpha} dV < \infty$, there is a solution $u$ to the equation $\overline{\partial} u = v$ on $\tilde{\Omega}$ such that $\int_{\Omega} |u|^2 e^{-\varphi} \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \int_{\tilde{\Omega}} |v|^2 e^{-\varphi} \delta_{\Omega}^{-\alpha} dV$.

3. Some Consequences of Theorem 1.1

3.1. We first prove Theorem 1.2. Following Wolff’s approach to Carleson’s theorem (cf. [19], p. 315), we put

$$g_1 = h \frac{f_1}{|f|^2} - u f_2, \quad g_2 = h \frac{f_2}{|f|^2} + u f_1$$

where $|f|^2 = |f_1|^2 + |f_2|^2$. Clearly, $f_1 g_1 + f_2 g_2 = h$, so the problem is reduced to choose $u \in L^2_{\alpha}(\Omega)$, i.e., $\int_{\Omega} |u|^2 \delta^{-\alpha} dV < \infty$, so that $g_1, g_2$ are holomorphic. Thus it suffices to solve

$$\overline{\partial} u = h \frac{f_2 \partial f_1 - f_1 \partial f_2}{|f|^4} =: v$$

such that $u \in L^2_{\alpha}(\Omega)$. Applying Theorem 1.1 with $\varphi = \log |f|^2$, we get a solution $u$ satisfying

$$\int_{\Omega} |u|^2 |f|^{-2} \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \int_{\Omega} |v|^2 \delta_{\Omega}^{-\alpha} dV.$$ 

A straightforward calculation shows

$$\partial \overline{\partial} \varphi = \frac{(f_1 \partial f_2 - f_2 \partial f_1) \wedge (f_1 \partial f_2 - f_2 \partial f_1)}{|f|^4}$$

so that $|v|^2_{\overline{\partial} \varphi} \leq |h|^2/|f|^4 \leq |h|^2/\delta^4$. Thus

$$\int_{\Omega} |u|^2 \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \delta^{-6} \int_{\Omega} |h|^2 \delta_{\Omega}^{-\alpha} dV.$$ 

Q.E.D.

3.2. Next we prove Theorem 1.3. The argument is a slightly modification of 3.1. Without loss of generality, we assume $w = 0$, $h(0) = 0$, $|z|^2 < e^{-1}$ on $\Omega$. Put $f_k = z_k$, $k = 1, 2$ and $\varphi = -\log(-\log |f|^2)$. Then we have

$$\partial \overline{\partial} \varphi \geq \frac{(f_1 \partial f_2 - f_2 \partial f_1) \wedge (f_1 \partial f_2 - f_2 \partial f_1)}{|f|^4 (-\log |f|^2)}.$$ 

Let $g_k, v$ be defined as above and put $\hat{\Omega} = \Omega \setminus \{f_1 = 0\}$. By Theorem 2.2 we may solve the equation $\overline{\partial} u = v$ on $\hat{\Omega}$ such that

$$\int_{\hat{\Omega}} |u|^2 \delta_{\hat{\Omega}}^{-\alpha} dV \leq \int_{\hat{\Omega}} |u|^2 e^{-\varphi} \delta_{\hat{\Omega}}^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \int_{\hat{\Omega}} |v|^2 e^{-\varphi} \delta_{\hat{\Omega}}^{-\alpha} dV$$

since the last term is bounded by

$$\text{const}_{\alpha, \Omega} \int_{\hat{\Omega}} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_{\hat{\Omega}}^{-\alpha} dV$$

$$= \text{const}_{\alpha, \Omega} \int_{\Omega \cap \{|z| < \epsilon\}} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_{\hat{\Omega}}^{-\alpha} dV + \text{const}_{\alpha, \Omega} \int_{\Omega \setminus \{|z| < \epsilon\}} |h|^2 |f|^{-4} (\log |f|^2)^2 \delta_{\hat{\Omega}}^{-\alpha} dV$$

$$\leq \text{const}_{\alpha, \Omega} \int_{\{|z| < \epsilon\}} |z|^{-2} (\log |z|)^2 dV + \text{const}_{\alpha, \Omega} \int_{\Omega} |h|^2 \delta_{\hat{\Omega}}^{-\alpha} dV < \infty$$
where \( \varepsilon > 0 \) is so small that \( \{ |z| \leq \varepsilon \} \subset \Omega \). Thus \( g_1, g_2 \) are holomorphic on \( \hat{\Omega} \) such that

\[
\int_{\Omega} |g_k|^2 \delta_{\Omega}^{-\alpha} dV < \infty, \quad k = 1, 2.
\]

The assertion follows immediately from Riemann’s removable singularities theorem. Q.E.D.

**Remark.** It is possible to extend both the Corona and Gleason type theorems to general cases by using the Koszul complex technique introduced by Hörmander [24]. But the argument will be substantially longer and not very enlightening, so that we shall not treat here.

3.3. Finally, we prove Theorem 1.4. (a) Let \( K \) be a compact subset of \( \Omega \) and \( f \in \mathcal{O}(\Omega) \). We take a strictly psh exhaustion function \( \psi \in C^\infty(\Omega) \) such that \( K \subset \{ \psi < 0 \} \). Let \( \kappa \) be a \( C^\infty \) convex increasing function such that \( \kappa = 0 \) on \( (-\infty, 0] \) and \( \kappa' > 0, \kappa'' > 0 \) on \( (0, +\infty) \). Let \( \rho < 0 \) be a bounded strictly psh exhaustion function on \( \Omega \). Choose \( \varepsilon > 0 \) so small that \( \{ \psi \leq 0 \} \subset \{ \rho < -\varepsilon \} \). Let \( \chi \in C^\infty_0(\Omega) \) be a real-valued function satisfying \( \chi = 1 \) in a neighborhood of \( \{ \rho \leq -\varepsilon \} \). We construct a \( 2- \) parameter family of weight functions as follows

\[
\varphi_{t,s}(z) = |z|^2 + t\chi(z)\kappa(\psi(z)) + s\kappa(\rho(z) + \varepsilon), \quad t, s > 0.
\]

It is easy to see that for any \( t > 0 \) there is a sufficiently large number \( s = s(t) > 0 \) such that \( \partial \overline{\partial} \varphi_{t,s} \geq \partial \overline{\partial} |z|^2 \). Let \( \hat{\chi} \in C^\infty_0(\Omega) \) such that \( \hat{\chi} = 1 \) in a neighborhood of \( \{ \psi \leq 0 \} \) and \( \hat{\chi}(z) = 0 \) if \( \rho(z) \geq -\varepsilon \). By Theorem 1.1, we may solve the equation

\[
\partial u_t = f \partial \hat{\chi}
\]

such that

\[
\int_{\Omega} |u_t|^2 e^{-\varphi_{t,s}} \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{a,\Omega} \int_{\Omega} |f|^2 |\partial \hat{\chi}|^2 e^{-\varphi_{t,s}} \delta_{\Omega}^{-\alpha} dV \leq \text{const}_{a,\Omega} \int_{\text{supp} \partial \hat{\chi}} |f|^2 e^{-t\kappa_0 \psi} \delta_{\Omega}^{-\alpha} dV \to 0
\]

as \( t \to +\infty \). Since \( \varphi_{t,s}(z) = |z|^2 \) whenever \( \psi(z) \leq 0 \), we conclude that

\[
\int_{\{\psi \leq 0\}} |u_t|^2 dV \to 0
\]

as \( t \to +\infty \), so is the function \( f_t - f \) where \( f_t := \hat{\chi} f - u_t \). On the other hand, \( f_t \in A^2_\alpha(\Omega) \) because \( \varphi_{t,s} \) is a bounded function. Since \( f_t - f \) is holomorphic on \( \{ \psi < 0 \} \), a standard compactness argument yields

\[
\sup_K |f_t - f| \to 0
\]

as \( t \to +\infty \).

(b) We take a \( C^2 \) psh function \( \rho < 0 \) on \( \Omega \) such that \( -\rho \times \delta_{\Omega}^a \) for some \( a > 0 \). Let \( 0 \leq \hat{\chi} \leq 1 \) be a cut-off function on \( \mathbb{R} \) such that \( \hat{\chi}|_{(-\infty, -\log 2)} = 1 \) and \( \hat{\chi}|_{(0, \infty)} = 0 \). Let \( f \in A^2_{\alpha_1}(\Omega) \) be given. For each \( \varepsilon > 0 \), we define

\[
v_\varepsilon = f \partial \hat{\chi}( - \log(-\rho + \varepsilon) + \log 2\varepsilon), \quad \varphi_\varepsilon = -\frac{\alpha_2 - \alpha_1}{a} \log(-\rho + \varepsilon).
\]
By Theorem 1.1, we have a solution of \( \bar{\partial}u_\varepsilon = v_\varepsilon \) so that
\[
\int_\Omega |u_\varepsilon|^2 e^{-\varphi_\varepsilon} \delta_\Omega^{-\alpha_2} dV \leq \text{const.} \int_\Omega |v_\varepsilon|^2 e^{-\varphi_\varepsilon} \delta_\Omega^{-\alpha_2} dV \\
\leq \text{const.} \int_{\varepsilon \leq \rho \leq 3\varepsilon} |f|^2 \delta_\Omega^{-\alpha_1} dV
\]
for \( i\bar{\partial}\varphi_\varepsilon \geq \frac{\alpha_2 - \alpha_1}{\alpha} i\partial \log(-\rho + \varepsilon) \wedge \bar{\partial} \log(-\rho + \varepsilon) \). Put
\[
f_\varepsilon = f(\log(-\rho + \varepsilon) + \log 2\varepsilon) - u_\varepsilon.
\]
Since \( \varphi_\varepsilon \) is bounded and
\[
e^{-\varphi_\varepsilon} \geq e^{\frac{\alpha_2 - \alpha_1}{\alpha} \log(-\rho)} \delta_\Omega^{\alpha_2 - \alpha_1},
\]
we conclude that \( f_\varepsilon \in A^2_{\alpha_2}(\Omega) \) and
\[
\int_\Omega |f_\varepsilon - f|^2 \delta_\Omega^{-\alpha_1} dV \leq 2 \int_{\rho \leq 3\varepsilon} |f|^2 \delta_\Omega^{-\alpha_1} dV + 2 \int_\Omega |u_\varepsilon|^2 \delta_\Omega^{-\alpha_1} dV \\
\leq 2 \int_{\rho \leq 3\varepsilon} |f|^2 \delta_\Omega^{-\alpha_1} dV + \text{const.} \int_\Omega |u_\varepsilon|^2 e^{-\varphi_\varepsilon} \delta_\Omega^{-\alpha_2} dV \\
\leq 2 \int_{\rho \leq 3\varepsilon} |f|^2 \delta_\Omega^{-\alpha_1} dV + \text{const.} \int_{\varepsilon \leq \rho \leq 3\varepsilon} |f|^2 \delta_\Omega^{-\alpha_1} dV \\
\to 0
\]
as \( \varepsilon \to 0^+ \). Q.E.D.

**Problem 1.** Is the Hardy space \( H^2(\Omega) \) dense in \( A^2_{\alpha}(\Omega) \) for each \( \alpha < 1 \)?

**Remark.** The referee of this paper pointed out the following

(1) Bell and Boas have proved a theorem related to Theorem 1.4 (cf. [3], Theorem 1).
(2) There is a standard argument as follows, which is perhaps more straightforward than the author’s proof. Choose a cover \( \{U_j\}_{j=1}^m \) of the boundary and vectors \( n_j \) such that \( z - \varepsilon n_j \in \Omega \) for \( 1 \leq j \leq m \), \( z \in U_j \), \( \varepsilon \leq \varepsilon_0 \). Choose \( \phi_0 \in C_\infty(\Omega) \) and \( \phi_j \in C_\infty^0(U_j) \), \( 1 \leq j \leq m \), with \( \sum \phi_j = 1 \) in a neighborhood of \( \Omega \). Set
\[
f_\varepsilon(z) = \phi_0(z) f(z) + \sum_{j=1}^m \phi_j(z) f(z - \varepsilon n_j).
\]

Then \( f_\varepsilon \to f \) in the norm with weight \( \delta_\Omega^{\alpha} \). The theorem now follows by correcting \( f_\varepsilon \) via
\[
\bar{\partial}f_\varepsilon = f \bar{\partial}\phi_0 + \sum_{j=1}^m f(z - \varepsilon n_j) \bar{\partial}\phi_j = \sum_{j=1}^m [f(z - \varepsilon n_j) - f(z)] \bar{\partial}\phi_j
\]
(because \( \sum_{j=0}^m \bar{\partial}\phi_j = 0 \) on \( \Omega \)). The norm of the right hand side tends to zero; so if we solve the \( \bar{\partial} \)–equation with the estimate that was shown, the corrections we make to the \( f_\varepsilon \) tend to zero as well in norm, and we are done.
4. Proof of Theorem 1.5

4.1. Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain. We define the pluricomplex Green function \( g_\Omega(\cdot, w) \) with pole at \( w \in \Omega \) as

\[
g_\Omega(z, w) = \sup \left\{ u(z) : u \in PSH(\Omega), u < 0, \limsup_{z \to w} (u(z) - \log |z - w|) < \infty \right\}.
\]

It is well-known that \( g_\Omega(\cdot, w) \in PSH(\Omega) \) for each fixed \( w \) and \( g_\Omega \in C(\overline{\Omega} \times \Omega \setminus \{z = w\}) \) when \( \Omega \) is hyperconvex (cf. [29]). We need the following estimate of \( g_\Omega \) due to Blocki [7]:

**Theorem 4.1.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a pseudoconvex domain. Suppose there is a negative psh function \( \rho \) on \( \Omega \) satisfying

\[
C_1 \delta^\alpha_0(z) \leq -\rho(z) \leq C_2 \delta^b_\Omega(z), \quad z \in \Omega
\]

where \( C_1, C_2 > 0 \) and \( a \geq b \geq 0 \) are constants. Then there are positive numbers \( \delta_0, C \) such that

\[
\{ g_\Omega(\cdot, w) \leq -1 \} \subset \{ C^{-1} \delta_\Omega(w)^{\frac{1}{2}} \log \delta_\Omega(w)^{-\frac{1}{2}} \leq \delta_\Omega \leq C \delta_\Omega(w)^{\frac{1}{2}} \log \delta_\Omega(w)^{\frac{1}{2}} \}
\]

holds for any \( w \in \Omega \) with \( \delta_\Omega(w) \leq \delta_0 \).

4.2. Let \( K_\alpha \) be the Bergman kernel of \( A^2_\alpha(\Omega) \).

**Proposition 4.2.** Suppose \( \lim_{z \to \partial \Omega} K_\alpha(z, \eta(z)) = \infty \) where \( \eta \) is a positive continuous function on \( \Omega \). Then there exists a function \( f \in A^2_\alpha(\Omega) \) such that

\[
\limsup_{z \to \zeta} |f(z)|/\sqrt{\eta(z)} = \infty, \quad \forall \zeta \in \partial \Omega.
\]

**Proof.** The argument is standard (see e.g. [27], p. 416–417). We claim that the following assertion holds:

For each \( \zeta \in \partial \Omega \) and each sequence of points in \( \Omega \) with \( z_j \to \zeta \), there exists a function \( f \in A^2_\alpha(\Omega) \) such that \( \sup_j |f(z_j)|/\sqrt{\eta(z_j)} = \infty \).

Suppose there is a point \( \zeta \in \partial \Omega \) and a sequence of points in \( \Omega \) such that \( z_j \to \zeta \) such that \( \sup_j |f(z_j)|/\sqrt{\eta(z_j)} < \infty \), \( \forall f \in A^2_\alpha(\Omega) \). Applying the Banach-Steinhaus theorem to the linear functional \( f \to f(z_j)/\sqrt{\eta(z_j)} \), we get

\[
\sup_j |f(z_j)|/\sqrt{\eta(z_j)} \leq \text{const.} \|f\|
\]

for all \( f \in A^2_\alpha(\Omega) \). Thus \( K_\alpha(z_j, \sqrt{\eta(z_j)}) \leq \text{const.} \), contradictory.

Now we construct the desired function \( f \). Pick a non-decreasing sequence of compact subsets \( \{K_j\} \) of \( \Omega \) such that \( D = \bigcup K_j \). Fix a dense sequence \( \{z_j\} \subset \Omega \). We reorder the points of the sequence as follows

\[
z_1, z_2, z_1, z_3, z_2, z_1, \ldots
\]

and denote the new sequence by \( \{w_j\} \). Let \( B_j = B(w_j, \delta_\Omega(w_j)) \) where \( B(z, r) \) is the euclidean ball with center \( z \) and radius \( r \). By the above claim, we may construct inductively sequences

\[
\{j_\nu\} \subset \mathbb{Z}^+, \quad \{\zeta_\nu\} \subset \Omega, \quad \{\theta_\nu\} \subset \mathbb{R}, \quad \{f_\nu\} \subset A^2_\alpha(\Omega)
\]

such that

\[
\zeta_\nu \in (B_{j_\nu} \setminus K_{j_\nu}) \cap K_{j_{\nu+1}}, \quad \|f_\nu\| = 1, \quad \left| \sum_{\mu=1}^{\nu} \frac{f_\mu(\zeta_\nu) e^{i\theta_\nu}}{\mu^2 (1 + \|f_\mu\| K_{j_\nu})} \right| \geq \frac{\nu}{\sqrt{\eta(\zeta_\nu)}}
\]
where \( \|f_\mu\|_{K_j} = \sup_{K_j} |f_\mu| \). It suffices to take \( f(z) = \sum_{\nu=1}^{\infty} \frac{f_\nu(z)e^{\theta_\nu}}{\nu!(1+\|f_\nu\|_{K_j})} \), Q.E.D.

Now we prove Theorem 1.5. The argument is essentially same as [12]. Fix first an arbitrary point \( w \) sufficiently close to \( \partial \Omega \). Put \( g_j = \max\{g_\Omega, \cdot, w, \cdot, -j\}, j = 1, 2, \ldots \). Since \( \Omega \) is hyperconvex, \( g_j \) is continuous on \( \Omega \) and \( g_j \downarrow g_\Omega, \cdot, w, \cdot, \) as \( j \to \infty \). By Richberg’s theorem (cf. [39]), there is a \( C^\infty \) strictly psh function \( \psi_j < 0 \) on \( \Omega \) such that \( |\psi_j(z) - g_j(z)| < 1/j \), \( z \in \Omega \). Put

\[
\varphi = 2n g_\Omega, \cdot, w, \cdot - \log(-g_\Omega, \cdot, w, \cdot + 1), \quad \varphi_j = 2n \psi_j - \log(-\psi_j + 1).
\]

Let \( \chi : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) cut-off function satisfying \( \chi((-\infty, -1]) = 1 \) and \( \chi([-\log 2, \infty)) = 0 \). Put

\[
v_j = \bar{\partial} \chi(-\log(-\psi_j)) \frac{K_\Omega(s, w)}{\sqrt{K_\Omega(w)}}
\]

where \( K_\Omega \) denotes the unweighted Bergman kernel of \( \Omega \). By Theorem 1.1, there is a solution of the equation \( \bar{\partial} u_j = v_j \) such that

\[
\int_{\Omega} |u_j|^2 e^{-\varphi} \delta_\Omega^\alpha dV \leq \text{const}_{\alpha, \Omega} \int_{\Omega} |v_j|^2 \bar{i} \partial \bar{\partial} \varphi_j e^{-\varphi} \delta_\Omega^\alpha dV \\
\leq \text{const}_{\alpha, \Omega} \int_{\sup \bar{\partial} \chi(s)} \frac{|K_\Omega(s, w)|^2}{K_\Omega(w)} \delta_\Omega^{-\alpha} dV
\]

where the second inequality follows from

\[
i \bar{\partial} \varphi_j \geq \frac{i \partial \psi_j \wedge \bar{\partial} \psi_j}{(-\psi_j + 1)^2}.
\]

By Blocki’s theorem, we have

\[
\sup \bar{\partial} \chi(s) \subset \{ \psi_j \leq -2 \} \subset \{ g_\Omega, \cdot, w, \cdot \leq -1 \} \subset \{ C^{-1} \delta_\Omega(w) | \log \delta_\Omega(w) |^{-\frac{\alpha}{\alpha}} \leq \delta_\Omega \}, \quad j \gg 1,
\]

where \( a \) is a Diederich-Fornaess exponent for \( \Omega \). Thus

\[
\int_{\Omega} |u_j|^2 e^{-\varphi} \delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \frac{|\log \delta_\Omega(w)|^{\alpha}}{\delta_\Omega(w)^{\alpha}}.
\]

Let \( u \) be a weak limit of a subsequence of \( \{u_j\} \). Thus

\[
f := \chi(-\log(-g_\Omega, \cdot, w, \cdot)) K_\Omega(s, w) / \sqrt{K_\Omega(w)} - u
\]

is holomorphic on \( \Omega \). Since \( u \) is holomorphic in a neighborhood of \( w \) and

\[
\int_{\Omega} |u|^2 e^{-\varphi} \delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \frac{|\log \delta_\Omega(w)|^{\alpha}}{\delta_\Omega(w)^{\alpha}},
\]

we conclude that \( u(w) = 0 \). Thus \( f(w) = \sqrt{K_\Omega(w)} \) and

\[
\int_{\Omega} |f|^2 \delta_\Omega^{-\alpha} dV \leq \text{const}_{\alpha, \Omega} \frac{|\log \delta_\Omega(w)|^{\alpha}}{\delta_\Omega(w)^{\alpha}}.
\]

Thus

\[
K_\alpha(w) \geq \frac{|f(w)|^2}{\int_{\Omega} |f|^2 \delta_\Omega^{-\alpha} dV} \geq \text{const}_{\alpha, \Omega} K_\Omega(w) \frac{\delta_\Omega(w)^{\alpha}}{|\log \delta_\Omega(w)|^{\alpha}} \geq \frac{\text{const}_{\alpha, \Omega}}{\delta_\Omega(w)^{2-\alpha} |\log \delta_\Omega(w)|^{\frac{\alpha}{2}}}
\]

as \( w \to \partial \Omega \) where the last inequality follows from the Ohsawa-Takegoshi extension theorem (cf. [36]). Applying Proposition 4.2 with \( \eta(z) = \delta_\Omega(z)^{2-\alpha} |\log \delta_\Omega(z)|^{\frac{\alpha}{2}} \), we conclude the proof. Q.E.D.
5. Proof of Theorem 1.6

We follows closely along Stein’s book [41]. For each \( \zeta \in \partial \Omega \), let \( \nu_\zeta \) denote the unit outward normal at \( \zeta \) and \( T_\zeta \) the tangent plane at \( \zeta \). For each \( t > 0 \), we define an approach region \( \mathcal{A}_t(\zeta) \) with vertex \( \zeta \) by

\[
\mathcal{A}_t(\zeta) = \{ z \in \Omega : |(z - \zeta) \cdot \nu_\zeta| < (1 + t)\delta_\zeta(z), |z - \zeta|^2 < t\delta_\zeta(z) \}
\]

where \( \delta_\zeta(z) = \min\{\delta_\Omega(z), d(z, T_\zeta)\} \). We shall say that \( |f(z)| = o(\delta_\Omega(z)^{-\beta}) \) uniformly as \( z \to \zeta \) admissibly for some \( \beta \geq 0 \) if for each \( t > 0 \)

\[
\limsup \delta_\Omega(z)^\beta |f(z)| = 0
\]
as \( z \to \zeta \) from the inside of \( \mathcal{A}_t(\zeta) \). For each \( \zeta_0 \in \partial \Omega \) and \( r > 0 \), we put

\[
B_1(\zeta_0, r) = \{ \zeta \in \partial \Omega : |\zeta - \zeta_0| < r \}
\]

\[
B_2(\zeta_0, r) = \{ \zeta \in \partial \Omega : |(\zeta - \zeta_0) \cdot \nu_{\zeta_0}| < r, |\zeta - \zeta_0|^2 < r \}
\]

and

\[
f_j^*(\zeta_0) = \sup_{r > 0} \frac{1}{\sigma(B_j(\zeta_0, r))} \int_{B_j(\zeta_0, r)} |f(\zeta)| d\sigma(\zeta), \ j = 1, 2
\]

where \( f \in L^p(\partial \Omega) \) and \( d\sigma \) is the surface measure for \( \partial \Omega \). The maximal function is defined by

\[
(Mf)(\zeta) = (f_j^*)^*_{\gamma}(\zeta).
\]

Theorem 5.1. (cf. [41], see also [25]).

1. \( \|Mf\|_p \leq \text{const}_p \|f\|_p \), \( \forall f \in L^p(\partial \Omega) \), \( 1 < p < \infty \).

2. Let \( u \) be a psh function on \( \Omega \) which is continuous on \( \Omega \) and let \( f = u|_{\partial \Omega} \). Then

\[
\sup_{z \in \mathcal{A}_t(\zeta)} |u(z)| \leq \text{const}_p (Mf)(\zeta).
\]

Now choose a cover of \( \Omega \) by finitely many subdomains \( \Omega_0, \Omega_1, \ldots, \Omega_m \subset \Omega \) with the following properties:

(a) \( \partial \Omega_j \) is \( C^2 \).

(b) \( \partial \Omega_j \cap \partial \Omega \subset \Omega \).

(c) There exists a domain \( W_j \subset \partial \Omega_j \cap \partial \Omega \) such that \( \{W_j\}_{j=0}^m \) forms a cover of \( \partial \Omega \).

(d) There exists an outward unit normal \( \nu_j \) at a point in \( \partial \Omega_j \cap \partial \Omega \) such that

\[
\overline{\Omega}_j - \varepsilon \nu_j \subset \Omega, \quad \forall 0 < \varepsilon < 1.
\]

It suffices to work on a single subdomain, say \( \Omega_0 \). Let \( \varepsilon_0 \) be a sufficiently small number. In order to apply Gehring’s method (cf. [20]), we define for each \( t > 0 \), \( 0 < \varepsilon < \varepsilon_0/2 \), \( \zeta \in W_0 \),

\[
U_\varepsilon^{(t)}(\zeta) = \{ z \in \mathcal{A}_t(\zeta) : 2\varepsilon < \delta_\zeta(z) < \varepsilon_0 \}
\]

\[
V_\varepsilon^{(t)}(\zeta) = \left\{ z \in \mathcal{A}_t(\zeta) - \varepsilon \nu_0 : \delta_\zeta(z) < \frac{3}{2} \varepsilon_0 \right\}.
\]

Lemma 5.2. For each \( t > 0 \), we may choose \( \varepsilon_0 > 0 \) so that

\[
U_\varepsilon^{(t)}(\zeta) \subset V_\varepsilon^{(s)}(\zeta) \subset \Omega_0, \quad s := 2 + 4t,
\]

for all \( \varepsilon < \varepsilon_0/2 \) and \( \zeta \in W_0 \).
Proof. For each $z \in U_{t}^{(t)}(\zeta)$, we have $\delta_{\zeta}(z) > 2\varepsilon$. Thus
\[
\begin{align*}
\delta_{\zeta}(z + \varepsilon \nu_{0}) & \geq \delta_{\zeta}(z) - \varepsilon > \varepsilon \\
\delta_{\zeta}(z + \varepsilon \nu_{0}) & \leq \delta_{\zeta}(z) + \varepsilon < \frac{3}{2} \varepsilon_{0}
\end{align*}
\]
for all $\varepsilon < \varepsilon_{0}/2$. Since
\[
|(z - \zeta) \cdot \nu_{\zeta}| < (1 + t)\delta_{\zeta}(z), \quad |z - \zeta| < (t\delta_{\zeta}(z))^{1/2},
\]
we get
\[
\begin{align*}
|(z + \varepsilon \nu_{0} - \zeta) \cdot \nu_{\zeta}| & \leq |(z - \zeta) \cdot \nu_{\zeta}| + \varepsilon < (1 + t)\delta_{\zeta}(z) + \varepsilon \leq (3 + 2t)\delta_{\zeta}(z + \varepsilon \nu_{0}) \\
|z + \varepsilon \nu_{0} - \zeta|^{2} & \leq 2|z - \zeta|^{2} + 2\varepsilon^{2} < 2t\delta_{\zeta}(z) + 2\varepsilon \leq (2 + 4t)\delta_{\zeta}(z + \varepsilon \nu_{0}).
\end{align*}
\]
Thus $z + \varepsilon \nu_{0} \in V_{t}^{(s)}(\zeta)$ where $s = 2 + 4t$ and we get the first inclusion in the lemma.

On the other hand, for each $z \in V_{t}^{(s)}(\zeta)$, we have $|z - \zeta|^{2} < s\delta_{\zeta}(z) \leq \frac{3}{2} s \varepsilon_{0}$, hence $V_{t}^{(s)}(\zeta) \subset \Omega_{0}$ for all $\varepsilon < \varepsilon_{0}/2$, provided $\varepsilon_{0}$ small enough. Q.E.D.

For each $f \in A_{t}^{2}(\Omega)$, we define
\[
\begin{align*}
u_{t}^{(s)}(\zeta) = \sup_{z \in U_{t}^{(t)}(\zeta)} |f(z)| & \quad \text{and} \quad v_{t}^{(s)}(\zeta) = \sup_{z \in V_{t}^{(s)}(\zeta)} |f(z)|.
\end{align*}
\]
Put $f_{\varepsilon}(z) = f(z - \varepsilon \nu_{0})$, $z \in \Omega_{0}$. Clearly, $|f_{\varepsilon}|$ is psh in $\Omega_{0}$ and continuous on $\overline{\Omega}_{0}$. Let $M_{0} f_{\varepsilon}$ be the corresponding maximal function on $\partial\Omega_{0}$. Take $0 < c < 1$ so that
\[
\Omega_{0} - \varepsilon \nu_{0} =: \Omega_{0}^{c} \subset \Omega_{ce} := \{ z \in \Omega : \delta_{\Omega}(z) > c \varepsilon \}.
\]
Let $d\sigma_{0}$ and $d\sigma_{ce}$ denote the surface measures on $\partial\Omega_{0}$ and $\partial\Omega_{ce}$ respectively and let $C$ denote a generic constant which is independent of $\varepsilon$ but probably depends on $\alpha, t, s$. By Theorem 5.1 and Lemma 5.2, we have
\[
u_{t}^{(s)}(\zeta) \leq C(M_{0} f_{\varepsilon})(\zeta), \quad \forall \zeta \in W_{0},
\]
so that
\[
\begin{align*}
\int_{W_{0}} |u_{t}^{(t)}(\zeta)|^{2} d\sigma_{0}(\zeta) & \leq C \int_{\partial\Omega_{0}} |M_{0} f_{\varepsilon}|^{2} d\sigma_{0} \leq C \int_{\partial\Omega_{0}} |f_{\varepsilon}|^{2} d\sigma_{0} \\
& = C \int_{\partial\Omega_{0}} |f|^{2} d\sigma_{0} \leq C \int_{\partial\Omega_{ce}} |f|^{2} d\sigma_{ce}
\end{align*}
\]
because of the following

Lemma 5.3. There is a constant $C > 0$ independent of $\varepsilon$ and $f$ such that
\[
\int_{\partial\Omega_{0}} |f|^{2} d\sigma_{0} \leq C \int_{\partial\Omega_{ce}} |f|^{2} d\sigma_{ce}
\]
for all sufficiently small $\varepsilon > 0$.

Thus for suitable small number $c_{0} > 0$ we have
\[
\int_{0}^{c_{0}} \varepsilon^{-a} \int_{W_{0}} |u_{t}^{(t)}(\zeta)|^{2} d\sigma_{0}(\zeta) d\varepsilon \leq C \int_{0}^{c_{0}} \int_{\partial\Omega_{ce}} |f|^{2} \varepsilon^{-a} d\sigma_{ce} d\varepsilon \leq C \int_{\Omega} |f|^{2} \delta^{a \varepsilon} dV < \infty,
\]
so that for $\sigma_{0}$–almost every $\zeta \in W_{0}$,
\[
\int_{0}^{c_{0}} \varepsilon^{-a} |u_{t}^{(t)}(\zeta)|^{2} d\varepsilon < \infty.
\]
Hence
\[ \int_0^{\varepsilon'} \varepsilon^{-\alpha}|u_\varepsilon(t)(\zeta)|^2 d\varepsilon = o(1) \]
as \( \varepsilon' \to 0 \). Given \( z \in A_{\varepsilon}(\zeta) \), we let \( \varepsilon' = \delta_\varepsilon(z)/2 \). Since \( z \in U_\varepsilon^{(t)}(\zeta) \) for each \( \varepsilon < \varepsilon' \), we have \( u_\varepsilon^{(t)}(\zeta) \geq |f(z)| \), thus
\[ |f(z)| = o(\delta_\varepsilon(z)^{-\frac{1}{1+\alpha}}) \quad \text{uniformly} \]
as \( z \to \zeta \) from the inside of \( A_{\varepsilon}(\zeta) \). Q.E.D. □

Finally we prove Lemma 5.3. The argument is essentially implicit in [12]. Let \( P(z, w), P_\varepsilon(z, w), P_0(z, w) \) and \( P_{0,\varepsilon}(z, w) \) denote the Poisson kernels of \( \Omega, \Omega_{\varepsilon}, \Omega_0 \) and \( \Omega_{\varepsilon,0} \) respectively. Put
\[ g(z) = \int_{\partial \Omega_{\varepsilon}} P_\varepsilon(z, w)|f(w)|^2 d\sigma_\varepsilon(w). \]
Then \( g \) is a harmonic majorant of \( |f|^2 \) on \( \Omega_{\varepsilon,0} \). Fix a point \( z_0 \) in \( \Omega_0 \). Since \( P_\varepsilon(z_0, \pi_\varepsilon^{-1}(\zeta)) \) converges uniformly on \( \partial \Omega \) to \( P(z_0, \zeta) \) where \( \pi_\varepsilon \) is the normal projection from \( \partial \Omega_{\varepsilon,0} \) to \( \partial \Omega \),
\[ g(z_0) \leq 2C_1 \int_{\partial \Omega_{\varepsilon,0}} |f(w)|^2 d\sigma_\varepsilon(w) \]
for all sufficiently small \( \varepsilon > 0 \) where \( C_1 = \sup_{\zeta \in \partial \Omega} P(z_0, \zeta) \). On the other hand,
\[ g(z_0) = \int_{\partial \Omega_{\varepsilon,0}} P_{0,\varepsilon}(z_0, w)g(w)d\sigma_0 \]
\[ \geq \frac{C_2}{2} \int_{\partial \Omega_{\varepsilon,0}} g(w)d\sigma_0 \geq \frac{C_2}{2} \int_{\partial \Omega_{\varepsilon,0}} |f(w)|^2 d\sigma_0 \]
for all sufficiently small \( \varepsilon > 0 \) where \( C_2 = \inf_{\zeta \in \partial \Omega_0} P_0(z_0, \zeta) \). The proof is complete. Q.E.D.

**Remark.** In various studies of boundary behavior of functions in Hardy spaces, the approach region defined as above is only best possible for strongly pseudoconvex domains (see e.g., [35, 31]). It is probably same in the case of weighted Bergman spaces.

### 6. Proof of Theorem 1.8

Let \( \| \cdot \|_\alpha \) and \( \| \cdot \|_{\partial \Omega} \) denote the corresponding norms of the weighted Bergman space \( A_\alpha^2(\Omega) \) and the Hardy space \( H^2(\Omega) \) respectively. Note first that for each \( f \in H^2(\Omega) \), and any sufficiently small \( \varepsilon_0 > 0 \),
\[ (1 - \alpha) \int_\Omega |f|^2 \delta_\alpha dV = (1 - \alpha) \int_{\Omega_{\varepsilon_0}} |f|^2 \delta_\alpha^\alpha dV + (1 - \alpha) \int_{\Omega \setminus \Omega_{\varepsilon_0}} |f|^2 \delta_\alpha^\alpha dV \]
\[ \leq (1 - \alpha) \int_{\Omega_{\varepsilon_0}} |f|^2 \delta_\alpha^\alpha dV + \varepsilon_0^{1 - \alpha} \sup_{0 < \varepsilon < \varepsilon_0} \| f \|_{\partial \Omega_\varepsilon}^{2}. \]
Applying this inequality with \( f(z) = S(z, w) \) for fixed \( w \in \Omega \), we get
\[ \lim_{\alpha \to 1^-} \inf_{\alpha}(1 - \alpha)^{-1} K_\alpha(w) \geq \lim_{\alpha \to 1^-} \inf_{\alpha}(1 - \alpha)^{-1} \frac{|f(w)|^2}{\| f \|_\alpha^2} \leq \frac{S(w)^2}{\sup_{0 < \varepsilon < \varepsilon_0} \| S(\cdot, w) \|_{\partial \Omega_\varepsilon}^{2}} \]
locally uniformly in \( w \) and uniformly in \( \varepsilon_0 \). Let \( S_\varepsilon \) denote the Szegö kernel of \( \Omega_\varepsilon \). It was proved by Boas [8] that \( S_\varepsilon(z, w) \to S(z, w) \) locally uniformly in \( z, w \) and
\[ \| S_\varepsilon(\cdot, w) - S(\cdot, w) \|_{\partial \Omega_\varepsilon} \to 0 \]
locally uniformly in $w$ as $\varepsilon \to 0^+$. Thus
\[
\liminf_{\alpha \to 1^-} (1 - \alpha)^{-1} K_\alpha(w) \geq \lim_{\varepsilon_0 \to 0^+} \frac{S(w)^2}{\sup_{0 < \varepsilon < \varepsilon_0} |S_\varepsilon(\cdot, w)|^2_{\partial \Omega_\varepsilon}} = \lim_{\varepsilon_0 \to 0^+} \frac{S(w)^2}{\sup_{0 < \varepsilon < \varepsilon_0} S_\varepsilon(w)} = S(w)
\]locally uniformly in $w$. On the other side, for any sufficiently small $\varepsilon > 0$
\[
\int_{\partial \Omega_\varepsilon} |(1 - \alpha)^{-1} K_\alpha(z, w) - S_\varepsilon(z, w)|^2 d\sigma_\varepsilon(z) = (1 - \alpha)^{-2} \|K_\alpha(\cdot, w)\|_{\partial \Omega_\varepsilon}^2 + (S_\varepsilon(\cdot, w))^2_{\partial \Omega_\varepsilon} - 2(1 - \alpha)^{-1} \text{Re} \int_{\partial \Omega_\varepsilon} K_\alpha(z, w) S_\varepsilon(z, w) d\sigma_\varepsilon(z).
\]Put $f_\alpha(z) := (1 - \alpha)^{-1/2} K_\alpha(z, w)/\sqrt{K_\alpha(w)}$. Following [12], we introduce
\[
\lambda_\alpha(\varepsilon) := \|f_\alpha\|_{\partial \Omega_\varepsilon} = \int_{\partial \Omega_\varepsilon} |f_\alpha|^2 d\sigma_\varepsilon.
\]Clearly, $\lambda_\alpha$ is continuous on $(0, a]$ for some sufficiently small $a > 0$ (independent of $\alpha$). For any sufficiently small $0 < \varepsilon_1 < \varepsilon_2 < a$, $\lambda_\alpha$ assumes the minimum at some point $\varepsilon^* = \varepsilon^*(\varepsilon_1, \varepsilon_2, \alpha)$ in $[\varepsilon_1, \varepsilon_2]$. Thus
\[
1 = (1 - \alpha) \|f_\alpha\|_\alpha^2 \geq (1 - \alpha) \int_{\varepsilon_1 \leq \varepsilon_0 \leq \varepsilon_2} |f_\alpha|^2_{\partial \Omega_\varepsilon} dV \geq (\varepsilon_2^{1 - \alpha} - \varepsilon_1^{1 - \alpha}) \lambda_\alpha(\varepsilon^*),
\]so that
\[
\|K_\alpha(\cdot, w)\|_{\partial \Omega_{\varepsilon^*}}^2 \leq (1 - \alpha) (\varepsilon_2^{1 - \alpha} - \varepsilon_1^{1 - \alpha})^{-1} K_\alpha(w).
\]Thus
\[
\int_{\partial \Omega_{\varepsilon^*}} |(1 - \alpha)^{-1} K_\alpha(z, w) - S_{\varepsilon^*}(z, w)|^2 d\sigma_{\varepsilon^*}(z) \leq S_{\varepsilon^*}(w) - (1 - \alpha)^{-1} \left( 2 - (\varepsilon_2^{1 - \alpha} - \varepsilon_1^{1 - \alpha})^{-1} \right) K_\alpha(w) = \left( 2 - (\varepsilon_2^{1 - \alpha} - \varepsilon_1^{1 - \alpha})^{-1} \right) (S(w) - (1 - \alpha)^{-1} K_\alpha(w)) + \left( (\varepsilon_2^{1 - \alpha} - \varepsilon_1^{1 - \alpha})^{-1} - 1 \right) S(w) + S_{\varepsilon^*}(w) - S(w).
\]It follow that
\[
\limsup_{\varepsilon_2 \to 0^+} \limsup_{\alpha \to 1^-} \limsup_{\varepsilon_1 \to 0^+} \int_{\partial \Omega_{\varepsilon^*}} |(1 - \alpha)^{-1} K_\alpha(z, w) - S_{\varepsilon^*}(z, w)|^2 d\sigma_{\varepsilon^*}(z) = 0
\]locally uniformly in $w$. Let $P_\varepsilon(z, \zeta)$ denote the Poisson kernel of $\Omega_\varepsilon$. For each compact set $M$ in $\Omega$ and $z, w \in M$, we have
\[
|(1 - \alpha)^{-1} K_\alpha(z, w) - S_{\varepsilon^*}(z, w)|^2 \leq \int_{\partial \Omega_{\varepsilon^*}} P_{\varepsilon^*}(z, \zeta) \left| (1 - \alpha)^{-1} K_\alpha(\zeta, w) - S_{\varepsilon^*}(\zeta, w) \right|^2 d\sigma_{\varepsilon^*}(\zeta) \leq \text{const}_M \int_{\partial \Omega_{\varepsilon^*}} \left| (1 - \alpha)^{-1} K_\alpha(\zeta, w) - S_{\varepsilon^*}(\zeta, w) \right|^2 d\sigma_{\varepsilon^*}(\zeta)
\]provided $\varepsilon^*$ sufficiently small. Thus $(1 - \alpha)^{-1} K_\alpha(z, w) \to S(z, w)$ uniformly in $z, w \in M$ as $\alpha \to 1^-$. The second assertion follows immediately from this fact and Theorem 1.9 Q.E.D.
Problem 2. Does \((1 - \alpha)^{-1} K_\alpha(z, w)\) admit an asymptotic expansion in powers of \(1 - \alpha\) as \(\alpha \to 1\)?

7. Proof of Theorem 1.7

Let \(ds^2 = \partial \bar{\partial}(- \log(1 - |z|^2))\) be the Bergman metric of \(\mathbb{B}^n\) and \(d(z, w)\) the Bergman distance between two points \(z, w\). Here we omit the factor \(n + 1\) in the classical definition of the Bergman metric for the sake of convenience. For each \(w \in \mathbb{B}^n, \tau > 0\) and \(0 < r < 1\), we put

\[ B_\tau(w) = \{z \in \mathbb{B}^n : d(z, w) < \tau\}, \quad \mathcal{B}_r(w) = \{z \in \mathbb{B}^n : |z - w| < r\}. \]

Note that \(B_\tau(0) = \mathcal{B}_r(0) \iff \tau = \frac{1}{2} \log \frac{1 + r}{1 - r}\).

Let \(\text{vol}_B\) and \(\text{vol}_E\) denote the Bergman and Euclidean volumes respectively.

Proposition 7.1. The following conclusions hold:

1. For each \(\tau > 0\), there is a constant \(C_\tau > 1\) such that for each \(w \in \mathbb{B}^n\),
   \[ B_\tau(w) \subset \{z \in \mathbb{B}^n : C_\tau^{-1}(1 - |w|) < 1 - |z| < C_\tau(1 - |w|)\} . \]
   \[ C_\tau^{-1}(1 - |w|)^{n+1} \leq \text{vol}_E(B_\tau(w)) \leq C_\tau(1 - |w|)^{n+1}. \]

2. For each \(r < 1\),
   \[ \text{vol}_B(B_r(0)) \leq \text{const}_n (1 - r)^{-n}. \]

3. For each \(\tau > 0\), there is a constant \(t > 1\) such that for each \(\zeta \in \mathbb{S}^n\) and each \(w \in L_\zeta\), where \(L_\zeta\) is the segment determined by \(0, \zeta\), we have
   \[ B_\tau(w) \subset \mathcal{A}_t(\zeta). \]

Proof. (1) See [43], Lemma 2.20, Lemma 1.23.
   (2) The Bergman volume form is
   \[ \text{const}_n (1 - |z|^2)^{-n-1} dV. \]
   Thus
   \[ \text{vol}_B(B_r(0)) = \text{const}_n \int_0^r (1 - s^2)^{-n-1} s^{2n-1} ds, \]
   from which the assertion immediately follows.
   (3) By [43], Lemma 2.20, there is a constant \(C_\tau > 0\) such that
   \[ |1 - z \cdot \bar{w}| < C_\tau(1 - |w|), \quad \forall z \in B_\tau(w). \]
   Thus
   \[ |1 - z \cdot \bar{\zeta}| \leq |1 - z \cdot \bar{w}| + \left| z \cdot \overline{(w - \zeta)} \right| \leq (C_\tau + 1)(1 - |w|) \leq t(1 - |z|) \]
   for suitable \(t \gg 1\) by (i). Q.E.D.

Definition 7.1. (see e.g., [28]). A subset \(\Gamma = \{w_j\}_{j=1}^\infty\) of \(\mathbb{B}^n\) is said to be \(\tau\)-separated for \(\tau > 0\), if \(d(w_j, w_k) \geq \tau\) for all \(j \neq k\), and \(\tau\)-separated subset is called maximal if no more points can be added to \(\Gamma\) without breaking the condition.
Lemma 7.2. Let \( \Gamma = \{w_j\}_{j=1}^{\infty} \) be a \( \tau \)-separated sequence such that \( 0 \notin \Gamma \). For any \( \varepsilon > 0 \),
\[
\sum_{j=1}^{\infty} \frac{(1 - |w_j|)^n}{\left( \log \frac{1}{1 - |w_j|} \right)^{1+\varepsilon}} < \infty.
\]

Proof. The argument is standard (compare [22], Theorem XI.7 and Theorem XI.8). For each 0 < \( r < 1 \), let \( n_r \) denote the number of points \( w_j \) which are contained in the ball \( B_r(0) = B_{\frac{1}{2} \log \frac{1}{1 - r}}(0) \). Since \( \{B_{r/2}(w_j)\}_{j=1}^{\infty} \) do not overlap, we have
\[
n_r \text{vol}_B(B_{r}(0)) \leq \text{vol}_B(B_{\frac{1}{2} \log \frac{1}{1 - r}}(0)) = \text{vol}_B \left( \mathbb{B}_{e^{\frac{1}{2} \log \frac{1}{1 - r}} - \frac{1}{2}}(0) \right) \leq \text{const}_{n,\tau} (1 - r)^{-n}
\]
by Proposition 7.1/(2). Take \( r_0 > 0 \) such that \( |w_j| \geq r_0 \) for each \( j \). Thus
\[
\sum_{|w_j| < r < 1} \frac{(1 - |w_j|)^n}{\left( \log \frac{1}{1 - |w_j|} \right)^{1+\varepsilon}} = \int_{r_0}^{r} \frac{(1 - s)^n}{\left( \log \frac{1}{1 - s} \right)^{1+\varepsilon}} ds
\]
\[
\leq \frac{(1 - r)^n}{\left( \log \frac{1}{1 - r} \right)^{1+\varepsilon}} n_r + \int_{r_0}^{r} \frac{(1 - s)^n}{\left( \log \frac{1}{1 - s} \right)^{1+\varepsilon}} n_s ds
\]
\[
\leq \text{const}_{n,\tau} + \text{const}_{n,\tau,\varepsilon} \int_{r_0}^{r} \frac{1}{(1 - s)^{1+\varepsilon}} ds = O(1)
\]
as \( r \to 1^- \). Q.E.D.

Lemma 7.3. There is a constant \( C_n > 0 \) such that for each \( \alpha < 1 \), \( \varepsilon > 0 \) and each 2\( \tau \)-separated sequence \( \Gamma = \{w_j\}_{j=1}^{\infty} \) with \( 0 \notin \Gamma \) and \( \tau \geq \frac{C_n}{\sqrt{1 - \alpha}} \), there exists a function \( f \in A^2(\mathbb{B}^n) \) such that
\[
f(w_j) = (1 - |w_j|)^{-\frac{1}{2} - \varepsilon} \left( \log \frac{1}{1 - |w_j|} \right)^{-\frac{1 + \varepsilon}{2}}, \quad \forall j.
\]

Proof. Take a \( C^\infty \) cut-off function \( \chi : \mathbb{R} \to [0, 1] \) such that \( \chi|_{(-\infty, 1/4]} = 1 \), \( \chi|_{(1/2, \infty)} = 0 \) and \( \chi' \leq 0 \). Put \( d_j(z) = d(z, w_j) \) and
\[
\psi(z) = \sum_j \chi(d_j(z)/\tau) \log d_j(z)/\tau
\]
\[
\varphi(z) = -\frac{1 - \alpha}{2} \log(1 - |z|^2) + 2n\psi(z).
\]
A straightforward calculation shows
\[
\partial \bar{\partial} \psi = \sum_j \chi''(\cdot) \frac{\partial d_j \wedge \bar{\partial} d_j}{\tau^2} \log d_j/\tau + 2\chi'(\cdot) \frac{\partial d_j \wedge \bar{\partial} d_j}{\tau d_j}
\]
\[
+ \chi'(\cdot) \frac{\partial \bar{\partial} d_j}{\tau} \log d_j/\tau + \chi(\cdot) \partial \bar{\partial} \log d_j.
\]
\[
(7.1)
\]
Since \( ds^2_{\mathbb{B}^n} \) has negative Riemannian sectional curvature, it follows from [21] that \( \log d_j \) is psh (so is \( d_j \)) on \( \mathbb{B}^n \). Neglecting the last two semipositive terms in (8), we get
\[
\partial \bar{\partial} \psi \geq -\frac{C_n^2}{8n\tau^2} ds^2_{\mathbb{B}^n}
\]
for suitable constant $C_n > 0$. If $\tau \geq C_n/\sqrt{1 - \alpha}$, then

$$\partial \bar{\partial} \varphi \geq \frac{1 - \alpha}{4} ds_{\mathbb{B}^n}^2.$$  

By Theorem 1.1, we may solve the equation

$$\bar{\partial}u = \sum_j (1 - |w_j|)^{-\frac{1 + \alpha}{2}} \left( \log \frac{1}{1 - |w_j|} \right)^{-\frac{1 + \alpha}{2}} \bar{\partial} \chi(d_j/\tau) =: v$$  

such that

$$\int_{\mathbb{B}^n} |u|^2 e^{-\varphi}(1 - |z|)^{-1 - \alpha} dV \leq \text{const}_{n,\alpha} \int_{\mathbb{B}^n} |v|_{\bar{\partial} \varphi}^2 e^{-\varphi}(1 - |z|)^{-1 - \alpha} dV$$

$$\leq \text{const}_{n,\alpha,\tau} \sum_j (1 - |w_j|)^{-1 + \alpha} \left( \log \frac{1}{1 - |w_j|} \right)^{-1 + \varepsilon} \int_{B_{\tau}(w_j)} (1 - |z|)^{-\alpha} dV$$

$$\leq \text{const}_{n,\alpha,\tau} \sum_{j=1}^{\infty} \frac{(1 - |w_j|)^n}{(\log \frac{1}{1 - |w_j|})^{1 + \varepsilon}} < \infty$$

where the last inequality follows from Proposition 7.1/(1). To get the desired function, we only need to take

$$f := \sum_j \chi(d_j/\tau)(1 - |w_j|)^{-\frac{1 + \alpha}{2}} \left( \log \frac{1}{1 - |w_j|} \right)^{-\frac{1 + \alpha}{2}} - u.$$  

Q.E.D. \hfill $\Box$

Now we prove Theorem 1.9, 1.10. Take $\tau = C_n/\sqrt{1 - \alpha}$ as in Lemma 7.3. Pick a maximal $2\tau$–separated sequence $\Gamma = \{w_j\}_{j=1}^{\infty}$ with $0 \notin \Gamma$. It is easy to see that the geodesic balls $B_\tau(w_j)$ are disjoint and $\{B_{3\tau}(w_j)\}_{j=1}^{\infty}$ forms a cover of $\mathbb{B}^n$. In particular,

$$B_{4\tau}(w) \cap \Gamma \neq \emptyset, \quad \forall w \in \mathbb{B}^n.$$  

By Proposition 7.1/(3) and completeness of $ds_{\mathbb{B}^n}^2$, we conclude that there is a constant $t > 1$ such that for each $\zeta \in \mathbb{S}^n$, the set $A_t(\zeta)$ contains a sequence of disjoint geodesic balls of radius $4\tau$ whose centers approach $\zeta$. Consequently, this set contains a subsequence of $\Gamma$. On the other hand, there is a function $f \in A^2_t(\mathbb{B}^n)$ such that

$$f(w_j) = (1 - |w_j|)^{-\frac{1 + \alpha}{2}} \left( \log \frac{1}{1 - |w_j|} \right)^{-\frac{1 + \alpha}{2}}, \quad \forall j$$

by virtue of Lemma 7.3. Thus the proof is complete. Q.E.D.

8. PROOF OF THEOREM 1.9, 1.10

Let $dz = dz_1 \wedge \cdots \wedge dz_n$ and $\widehat{d\bar{z}}_j = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n$. The Bochner-Martinelli kernel is defined to be

$$K_{BM}(\zeta - z) = \frac{(n - 1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{(-1)^{j-1} (\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2n}} d\bar{\zeta}_j \wedge d\zeta.$$
Bochner-Martinelli Formula. Let $D \subset \mathbb{C}^n$ be a bounded domain with $C^1$-boundary. Let $f \in C^1(\overline{D})$. Then for each $z \in D$,

$$f(z) = \int_{\partial D} f(\zeta) K_{BM}(\zeta - z) - \frac{(n-1)!}{(2\pi i)^n} \int_D \sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_j) \frac{\partial f}{\partial \zeta_j} d\bar{\zeta} \wedge d\zeta.$$

First we prove Theorem 1.9. Without loss of generality, we assume that the diameter $d(\Omega)$ of $\Omega$ is less than $1/2$.

(a) Put $\delta(z) := d(z, \partial \Omega)$, $z \in \mathbb{C}^n$. Clearly, $|\delta(z) - \delta(w)| \leq |z - w|$ for all $z, w \in \mathbb{C}^n$.

To apply the B-M formula, we need to approximate $\delta(z)$ by $C^1$-smooth functions with uniformly bounded gradients by a standard argument as follows. Let $\kappa \geq 0$ be a $C^\infty$ function in $\mathbb{C}^n$ satisfying the following properties: $\kappa$ depends only on $|z|$, supp $\kappa \subset \mathbb{B}^n$ and $\int_{\mathbb{C}^n} \kappa(z) dV = 1$. For each $\varepsilon > 0$, we put $\kappa_\varepsilon(z) = \varepsilon^{-2n} \kappa(z/\varepsilon)$ and $\delta_\varepsilon = \delta * \kappa_\varepsilon$. Clearly, $\delta_\varepsilon$ converges uniformly on $\overline{\Omega}$ to $\delta$, and the gradient $\nabla \delta_\varepsilon$ of $\delta_\varepsilon$ verifies

$$\nabla \delta_\varepsilon(z) = \int_{\mathbb{C}^n} \delta_\varepsilon(r) \nabla \kappa_\varepsilon(z - r) dV_r = \int_{\mathbb{C}^n} \delta_\varepsilon(r) \nabla \kappa_\varepsilon(z - r) dV_r$$

because $\int_{\mathbb{C}^n} \kappa_\varepsilon(z - r) dV_r = 1$. Thus

$$|\nabla \delta_\varepsilon(z)| \leq \int_{\mathbb{C}^n} \left| \delta_\varepsilon(r) \right| \cdot \left| \nabla \kappa_\varepsilon(z - r) \right| dV_r \leq \text{const}_n.$$

Let $f \in \mathcal{O}(\Omega)$ and $z_0 \in \Omega$ arbitrarily fixed. For any sufficiently small $\varepsilon > 0$, there is a positive number $\varepsilon_1$ such that

$$\{ z \in \Omega : \varepsilon \leq \delta_{\varepsilon_1}(z) \leq \sqrt{\varepsilon} \} \subset \Omega_\varepsilon \setminus \Omega_{2 \sqrt{\varepsilon}}$$

and $\delta_{\varepsilon_1} \asymp \delta_\Omega$ holds on $\Omega_\varepsilon \setminus \Omega_{2 \sqrt{\varepsilon}}$ (with implicit constants independent of $\varepsilon, \varepsilon_1$). Now take a cut-off function $\chi$ on $\mathbb{R}$ such that $\chi_{(-\infty, -\log 2)} = 1$ and $\chi_{(0, \infty)} = 0$. Applying the B-M formula to the function

$$\chi(\log \log 1/\delta_{\varepsilon_1} - \log \log 1/\varepsilon) f^2$$

with $\varepsilon$ sufficiently small, we obtain

$$f^2(z_0) = -\frac{(n-1)!}{(2\pi i)^n} \int_{\Omega} f^2(\zeta) \chi'(\zeta) \delta_{\varepsilon_1}(\zeta) \int_{\mathbb{C}^n} \left( \sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_{0,j}) \frac{\partial \delta_{\varepsilon_1}}{\partial \zeta_j}(\zeta) \right) d\bar{\zeta} \wedge d\zeta.$$

Thus

$$|f(z_0)|^2 \leq \text{const}_{n, \varepsilon_1} \int_{\Omega_\varepsilon \setminus \Omega_{2 \sqrt{\varepsilon}}} |f^2 \delta_{\varepsilon_1}^2| \log \delta_\Omega |^{-1} dV \to 0 \quad (\varepsilon \to 0+)$$

provided

$$\int_{\Omega} |f^2 \delta_{\varepsilon_1}^2| \log \delta_\Omega |^{-1} dV < \infty.$$

(b) Recall first that for each compact set $M \subset \Omega$, the capacity of $M$ in $\Omega$ is defined by

$$\text{cap}(M, \Omega) = \inf \int_{\Omega} |\nabla \phi|^2 dV$$

where the infimum is taken over all $\phi \in C_c^\infty(\Omega)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ in a neighborhood of $M$. For each $j$, we may choose a function $\phi_j \in C_c^\infty(\Omega)$ with $0 \leq \phi_j \leq 1$, $\phi_j = 1$ in a neighborhood of $\overline{\Omega}_{\varepsilon_1}$, so that

$$\int_{\Omega} |\nabla \phi_j|^2 dV \leq 2c(\varepsilon_1).$$
Let $f \in A_2^\alpha(\Omega)$ and $z_0 \in \Omega$ arbitrarily fixed. Applying the B-M formula to the function $\phi_j f$ with $j$ sufficiently large, we get

$$f(z_0) = -\frac{(n-1)!}{(2\pi i)^n} \int_{\Omega} f(\zeta) \sum_{k=1}^{n} (\bar{\zeta}_k - \bar{z}_{0,k}) \frac{\partial \phi_j(\zeta)}{\partial \bar{\zeta}_k} \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta - z_0|^{2n}}$$

so that

$$|f(z_0)| \leq \text{const}_{n,z_0} \int_{\Omega} |\nabla \phi_j||f| dV$$

$$\leq \text{const}_{n,z_0} \left( \int_{\Omega \setminus \Omega_{x_j}} |\nabla \phi_j|^2 \delta^{-\alpha}_{\Omega} dV \right)^{1/2} \left( \int_{\Omega \setminus \Omega_{x_j}} |f|^2 \delta^{-\alpha}_{\Omega} dV \right)^{1/2}$$

$$\leq \text{const}_{n,z_0} c(\varepsilon_j)^{1/2} \varepsilon_j^{-\alpha/2} \left( \int_{\Omega \setminus \Omega_{x_j}} |f|^2 \delta^{-\alpha}_{\Omega} dV \right)^{1/2} \rightarrow 0$$

as $j \to \infty$. Q.E.D.

On the other side, we have

**Proposition 8.1.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and put $V(\varepsilon) = \text{vol}_E(\Omega \setminus \Omega_\varepsilon)$. If

$$\alpha < \liminf_{\varepsilon \to 0^+} \frac{\log V(\varepsilon)}{\log \varepsilon},$$

then $H^\infty(\Omega) \subset A_2^\alpha(\Omega)$.

**Proof.** It suffices to show that $1 \in A_2^\alpha(\Omega)$. Fix $\beta$ such that $\alpha < \beta < \liminf_{\varepsilon \to 0^+} \frac{\log V(\varepsilon)}{\log \varepsilon}$. Note that

$$\text{vol}_E(\Omega \setminus \Omega_\varepsilon) < \text{const}_{\beta} \varepsilon^\beta$$

for all $\varepsilon > 0$. Without loss of generality, we assume $\delta_{\Omega} < 1$ on $\Omega$ and $\alpha \geq 0$. Then we have

$$\int_{\Omega} \delta^{-\alpha}_{\Omega} dV \leq \sum_{j=0}^{\infty} \int_{\Omega_{2^{-j-1}} \setminus \Omega_{2^{-j}}} 2^{\alpha(j+1)} dV \leq \sum_{j=0}^{\infty} 2^{\alpha(j+1)} \text{vol}_E(\Omega \setminus \Omega_{2^{-j}})$$

$$\leq \text{const}_{\alpha,\beta} \sum_{j=0}^{\infty} 2^{-(\beta-\alpha)j} < \infty.$$ 

Q.E.D.

It is reasonable to introduce the following

**Definition 8.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. The critical exponent $\alpha(\Omega)$ of $\Omega$ for weighted Bergman spaces $A_2^\alpha(\Omega)$ is defined to be

$$\alpha(\Omega) := \sup \{ \alpha : A_2^\alpha(\Omega) \neq \{0\} \} = \inf \{ \alpha : A_2^\alpha(\Omega) = \{0\} \} .$$

From Proposition 8.1 and Theorem 1.9, we know that

$$\beta(\Omega) := \liminf_{\varepsilon \to 0^+} \frac{\log V(\varepsilon)}{\log \varepsilon} \leq \alpha(\Omega) \leq \min \left\{ 1, \liminf_{\varepsilon \to 0^+} \frac{\log c(\varepsilon)}{\log 1/\varepsilon} \right\} =: \gamma(\Omega).$$

Note that $2n - \beta(\Omega)$ is nothing but the classical Minkowski dimension of $\partial \Omega$. Thus $\alpha(\Omega) = 1$ in case $\partial \Omega$ is non-fractal, i.e., $\beta(\Omega) = 1$. This is the case for instance, when $\Omega$ is a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary or a domain in $\mathbb{C}$ whose boundary is a rectifiable Jordan curve. Unfortunately, the author is unable to find an example with $\alpha(\Omega) < 1$. 

Finally we prove Theorem 1.10. Without loss of generality, we may assume that \( \rho > -e^{-1} \) and \( d(\Omega) \leq 1/2 \). Suppose on the contrary there is a continuous psh function \( \rho < 0 \) on \( \Omega \) such that

\[
-\rho \leq \text{const}_\varepsilon \delta_\Omega |\log \delta_\Omega|^{-\varepsilon}.
\]

Then we have

\[
(8.1) \quad (-\rho)(- \log(-\rho))^{1+\varepsilon/2} \leq \text{const}_\varepsilon \delta_\Omega |\log \delta_\Omega|.
\]

By Richberg’s theorem, we may also assume that \( \rho \) is \( C^\infty \) and strictly psh on \( \Omega \). Fix \( z_0 \in \Omega \). Put \( \phi = -\log(-\rho) \) and

\[
\varphi(z) = 2n \log |z - z_0|, \quad \psi = \phi - \frac{\varepsilon}{2} \log \phi.
\]

Note that \( \bar{\partial} \varphi = \bar{\partial} \phi - \frac{\varepsilon}{2} \bar{\partial} \phi \) and

\[
i\bar{\partial} \bar{\partial} \psi = \left(1 - \frac{\varepsilon}{2\phi}\right)i\bar{\partial} \bar{\partial} \phi + \frac{\varepsilon}{2} i\bar{\partial} \bar{\partial} \phi \wedge \bar{\partial} \phi \geq \left(1 - \frac{\varepsilon}{2\phi} + \frac{\varepsilon}{2\phi^2}\right)i\bar{\partial} \phi \wedge \bar{\partial} \phi,
\]

so that

\[
(8.2) \quad |\bar{\partial} \psi|^2_{i\bar{\partial} \bar{\partial} \psi} \leq \frac{1 - \frac{\varepsilon}{\phi} + \frac{\varepsilon^2}{4\phi^2}}{1 - \frac{\varepsilon}{2\phi} + \frac{\varepsilon^2}{4\phi^2}}.
\]

Let \( \chi \) be as in the proof of Theorem 1.9 and put \( v = \bar{\partial} \chi(2|z - z_0|/\delta_\Omega(z_0) - 1) \). We need to solve the equation \( \bar{\partial} u = v \) on \( \Omega \) together with a Donnelly-Fefferman type estimate by using a trick from Berndtsson-Charpentier [6] essentially as [11]. Let \( m > 0 \) be sufficiently large and \( u_m \) the minimal solution of \( \bar{\partial} u = v \) in \( L^2(\Omega_{1/m}, \varphi + \psi) \). Then we have \( u_m e^\psi \perp \text{Ker} \bar{\partial} \) in \( L^2(\Omega_{1/m}, \varphi + \psi) \). Thus by Hörmander’s estimate (1.1),

\[
\int_{\Omega_{1/m}} |u_m|^2 e^{-\varphi + \psi} dV \leq \int_{\Omega_{1/m}} |\bar{\partial}(u_m e^\psi)|^2_{i\bar{\partial} \bar{\partial}(\varphi + \psi)} e^{-\varphi - \psi} dV
\]

\[
\leq \int_{\Omega_{1/m}} |v + \bar{\partial} \psi \wedge u_m|^2_{i\bar{\partial} \bar{\partial} \psi} e^{-\varphi + \psi} dV
\]

\[
\leq \int_{\Omega_{1/m}} \left(1 + \frac{4\phi}{\varepsilon}\right) |v|^2_{i\bar{\partial} \bar{\partial} \psi} e^{-\varphi + \psi} dV + \int_{\Omega_{1/m}} \left(1 + \frac{4\phi}{\varepsilon}\right) |\bar{\partial} \psi|^2_{i\bar{\partial} \bar{\partial} \psi} |u_m|^2 e^{-\varphi + \psi} dV.
\]

Together with (8.2), we get

\[
(8.3) \quad \int_{\Omega_{1/m}} |u_m|^2 \phi^{-1} e^{-\varphi + \psi} dV \leq \text{const}_\varepsilon \int_{\Omega} \left(1 + \frac{4\phi}{\varepsilon}\right) |v|^2_{i\bar{\partial} \bar{\partial} \psi} e^{-\varphi + \psi} dV < \infty,
\]

for we can make \( \phi \) sufficiently large if \( \rho \) is replaced by \( \rho/C \) with \( C \gg 1 \).

Now put \( f_m(z) := \chi(2|z - z_0|/\delta_\Omega(z_0) - 1) - u_m(z) \). Let \( f \) be a weak limit of \( \{f_m\}_{m=1}^\infty \). Clearly, \( f \in \mathcal{O}(\Omega) \), \( f(z_0) = 1 \) and by (8.1), (8.3),

\[
\int_{\Omega} |f|^2 \phi^{-1} |\log \delta_\Omega|^{-1} dV \leq \text{const}_\varepsilon \int_{\Omega} |f|^2 \phi^{-1} e^\psi dV < \infty.
\]

This contradicts with Theorem 1.9. Q.E.D.

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