Multipole expansions for time-dependent charge and current distributions in quasistatic approximation

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Abstract

We propose a consistent approach to the definition of electric, magnetic, and toroidal multipole moments. Electric and magnetic fields are split into potential, vortex, and radiative terms, with the latter ones dropped off in the quasistatic approximation. The potential part of the electric field, the vortex parts of the magnetic field and vector potential contain gradients of scalar functions. Formally introducing magnetic and toroidal analogs of the electric charge, we apply multipole expansions for those scalars. Closed-form expressions are derived in an arbitrary order for electric, magnetic, and toroidal multipoles, which constitute a full system for expansions of the electromagnetic field.

Key words: multipole expansion; higher multipole moments; toroidal multipoles; quasistatic approximation.

1 Introduction

Multipole expansions are efficient techniques for calculations of electromagnetic fields generated by complex distributions of charges and currents [1, 2]. In problems of electro- and magnetostatics, expansions of scalar and vector potentials generating electric and magnetic multipoles are sufficient. Time-dependent distributions, however, generate a third type of multipoles generally known as toroidal moments.

We present a rigorous and consistent approach to the definitions of these three types of multipoles, namely, electric, magnetic, and toroidal multipole moments. Upon obtaining the first two types, which are better known, we proceed in the similar fashion to derive the latter one, for which the definitions vary in the literature [3, 4, 5, 6, 7, 8].

Following [9], the authors of [4] provide the exact multipole expansions for the radiation intensity, angular momentum loss, and the recoil force via electric, magnetic, and toroidal mean square radii. The authors of [10] start from the Jefimenko’s equations and thus present multipole expansions of electric and magnetic fields rather than traditional analysis of scalar and vector potentials. Multipole expansion of the action was applied to obtain expressions for the electric and magnetic multipole moments in [6]. Exact formulas for multipoles of arbitrary order, including toroidal multipoles, were obtained in [11]. In recent papers [12, 8], the terms corresponding to the toroidal moments were included in the definitions of the electric multipoles.

Usually, toroidal moments are related to the radiation fields [3, 13, 4, 14]. The presented approach remains within the quasistatic approximation and differs in this regard from most works briefly analyzed above. Moreover, closed-form expressions derived in an
arbitrary order for three multipole families – electric, magnetic, and toroidal – provide a full system for expansions of the electromagnetic fields. Despite lengthy derivations, the basic mathematical techniques applied are mostly limited to vector calculus, which ensures a low-effort tracking of the whole procedure.

The paper is organized as follows. Sec. 2 contains details of the quasistatic approximation. Expressions for the scalar and vector potentials are given in Sec. 3. Multipole expansions for different parts of the electric and magnetic fields are given in Secs. 4–6, where magnetic and toroidal multipole moments are defined by using a notion of formally introduced analogs of the electric charge from the electric multipoles. Discussion in Sec. 7 concludes the paper.

2 Quasistatic electrodynamics

We will proceed from Maxwell’s equations written in the Gaussian units:

\[ \text{div} \mathbf{E} = 4\pi \rho, \]  
\[ \text{div} \mathbf{B} = 0, \]  
\[ \text{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \]  
\[ \text{rot} \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \]

Let us write the electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) fields using the following splitting into potential ("p" subscript), vortex ("v" subscript), and radiative ("r" subscript) terms:

\[ \mathbf{E} = \mathbf{E}_p + \mathbf{E}_v + \mathbf{E}_r, \]  
\[ \mathbf{B} = \mathbf{B}_v + \mathbf{B}_r. \]

Obviously, the potential term in the magnetic field equals zero since \( \text{div} \mathbf{B} = 0 \).

It is convenient to rewrite Maxwell’s equations in the following form:

\[ \text{div} \mathbf{E}_p = 4\pi \rho \quad \text{div} \mathbf{E}_v = 0 \quad \text{div} \mathbf{E}_r = 0 \]  
\[ \text{div} \mathbf{B}_v = 0 \quad \text{div} \mathbf{B}_r = 0 \]  
\[ \text{rot} \mathbf{E}_p = 0 \quad \text{rot} \mathbf{E}_v = -\frac{1}{c} \frac{\partial \mathbf{B}_v}{\partial t} \quad \text{rot} \mathbf{E}_r = -\frac{1}{c} \frac{\partial \mathbf{B}_r}{\partial t} \]  
\[ \text{rot} \mathbf{B}_v = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}_p}{\partial t} \quad \text{rot} \mathbf{B}_r = \frac{1}{c} \frac{\partial \mathbf{E}_v}{\partial t} + \frac{1}{c} \frac{\partial \mathbf{E}_r}{\partial t}. \]

Applying rotors to Eqs. (3c)–(3d) and taking into account Eqs. (3a)–(3b) one immediately arrives at the following set:

\[ \Delta \mathbf{E}_p = 4\pi \nabla \rho \quad \Delta \mathbf{E}_v = \frac{4\pi}{c^2} \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_p}{\partial t^2} \quad \Box \mathbf{E}_r = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_v}{\partial t^2} \]  
\[ \Delta \mathbf{B}_v = -\frac{4\pi}{c} \text{rot} \mathbf{j} \quad \Box \mathbf{B}_r = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}_v}{\partial t^2}, \]

where the d’Alembertian

\[ \Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \]
Note that for charges and current densities the charge conservation law holds:

\[ \text{div} \, j + \frac{\partial \rho}{\partial t} = 0. \tag{6} \]

The first two columns in (4) correspond to the quasistatic approximation. The third column involving the radiative parts \( E_r \) and \( B_r \) represents higher-order corrections. Note that retardation effects are contained only in this third column, so the quasistatic approximation can be seen as working on the plane of some fixed time \( t \). These considerations correspond to typical scales of the near and intermediate zones \( (r \ll \lambda) \), see Fig. 1.

Figure 1: Typical scales. The system of charges and currents is localized to the volume \( V \) with a characteristic linear size of order \( a \). In the intermediate zone, the distance of the observer from the origin \( r \) is much smaller than the wavelength \( \lambda \) and, at the same time, much larger that the system size \( a \).

3 Scalar and vector potentials

According to the above splitting of fields, the scalar and vector potentials are written as follows:

\[ \varphi = \varphi_p \tag{7a} \]
\[ \mathbf{A} = \mathbf{A}_v + \mathbf{A}_r \tag{7b} \]

so that:

\[ \mathbf{E}_p = -\nabla \varphi_p \quad \mathbf{E}_v = -\frac{1}{c} \frac{\partial \mathbf{A}_v}{\partial t} \quad \mathbf{E}_r = -\frac{1}{c} \frac{\partial \mathbf{A}_r}{\partial t} \tag{8a} \]
\[ \mathbf{B}_v = \text{rot} \, \mathbf{A}_v \quad \mathbf{B}_r = \text{rot} \, \mathbf{A}_r \tag{8b} \]

For convenience, we will drop the index “\( p \)” of the scalar potential as \( \varphi \) does not contain other terms.
Equations for the potentials are:

\[ \Delta \varphi = -4\pi \rho \]
\[ \Delta A_v = -\frac{4\pi}{c} j - \frac{1}{c} \frac{\partial E_p}{\partial t} \]
\[ \square A = \frac{1}{c^2} \frac{\partial^2 A_v}{\partial t^2} \]  
\[ \text{div } A_v = 0 \]
\[ \text{div } A_r = 0 \]  

(9a)

Note the recursive nature of these equations as well as of equations (4). First, the potential part of the field is obtained. Being inserted into the second column, it yields the vortex part, which is sufficient for the quasistatic approximation. Finally, equations for the radiative part of fields involve the vortex part.

The solutions for the potential and vortex parts are given by:

\[ \varphi = \int d^3r' \frac{\rho'}{|r - r'|} \]
\[ A_v = \int d^3r' \left\{ \frac{j'}{|r - r'|} + \frac{1}{4\pi} \frac{\partial E_p'}{\partial t} \frac{1}{|r - r'|} \right\} = A_{v1} + A_{v2}, \]

where the primes at \( \rho, j \), and \( E \) denote the dependence of the integration variable \( r' \). This convention is also used in further derivations.

The first term is thus

\[ A_{v1} = \frac{1}{c} \int d^3r' \frac{j'}{|r - r'|}. \]

(10a)

The second term \( A_{v2} \) of the vector potential can be written via currents after some transformations. First of all, it is straightforward to show that

\[ A_{v2} = \frac{1}{c} \int d^3r' \frac{1}{4\pi} \frac{\partial E_p'}{\partial t} \frac{1}{|r - r'|} = -\frac{1}{c} \int d^3r' \frac{1}{4\pi} \frac{\partial E_p'}{\partial t} \text{div}' \frac{(r - r')}{2|r - r'|}. \]

(10b)

The \text{div}' operator acts on the \( r' \) variable. Upon applying the following vector identity,

\[ [[\nabla a]b] + [[\nabla b]a] + \nabla(ab) = [\text{rot } a, b] + [\text{rot } b, a] + a \text{ div } b + b \text{ div } a, \]

one can integrate the expression for \( A_{v2} \) by parts. Let \( a = E_p \) and \( b = \frac{(r - r')}{2|r - r'|} \), so that \( \text{rot } a = \text{rot } b = 0 \). The terms with \( \nabla \) become surface integrals vanishing – here and throughout further derivations – at large distances. So, making some manipulations,

\[ A_{v2} = \frac{1}{c} \int d^3r' \frac{1}{4\pi} \frac{\partial}{\partial t} \frac{E_p'}{2|r - r'|} = \frac{1}{c} \int d^3r' \frac{\partial}{\partial t} \frac{r - r'}{2|r - r'|} = \]
\[ = \frac{1}{c} \int d^3r' \left( - \text{div}' j' \right) \frac{r - r'}{2|r - r'|} = \frac{1}{c} \int d^3r' \left\{ j', \nabla \right\} \frac{r - r'}{2|r - r'|} = \]
\[ = \frac{1}{c} \int d^3r' \left\{ - \frac{j'}{2|r - r'|} + \frac{(j', r - r')(r - r')}{2|r - r'|^3} \right\} \]



finally we arrive at the following expression for \( A_v \):

\[ A_v = \frac{1}{c} \int d^3r' \left\{ \frac{j'}{2|r - r'|} + \frac{(j', r - r')(r - r')}{2|r - r'|^3} \right\} \]

(14)
4 Multipole expansion for $E_p$

Recall the following relations:

$$\frac{1}{|r - r'|} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (r', \nabla)^n \frac{1}{r}$$

(15)

$$\Delta \frac{1}{r} = 0, \quad \Delta \frac{r}{2} = \frac{1}{r}.$$ 

(16)

Note that we are interested in large $r$, so there is no need in writing the delta-function term in the first Laplacian.

The potential part of the field is:

$$E_p = -\nabla \varphi$$

(17)

The exact formula for the scalar potential

$$\varphi = \int d^3r' \frac{\rho'}{|r - r'|}$$

(18)

yields the series:

$$\varphi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3r' (r', \nabla)^n \frac{1}{r} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1=1}^{3} \cdots \sum_{i_n=1}^{3} Q_{i_1 \ldots i_n} \left( \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} \frac{1}{r} \right),$$

(19)

where electric multipole moments are:

$$Q_{i_1 \ldots i_n} = \int d^3r' \rho' x_{i_1}' \cdots x_{i_n}'. $$

(20)

Obviously, the zero-rank tensor $Q$ is just the total electric charge and the first-rank tensor $Q_i$ is nothing but the dipole moment of the system. For simplicity, we do not detract the $Q_{i_1 \ldots i_n}$ tensors for $n > 1$. That is, for instance, the quadrupole moment is just

$$Q_{ik} = \int d^3r' \rho' x_i' x_k',$$

(21)

but not the traceless

$$\bar{Q}_{ik} = \int d^3r' \rho' \left( x_i' x_k' - \frac{1}{3} \delta_{ik} r'^2 \right),$$

(22)

where $\delta_{ij}$ is Kornecker’s delta. The scalar potential $\varphi$ is independent of the multipole moment definition: one can equivalently use both the above form (20) and its traceless version due to the condition $\Delta \frac{1}{r} = 0$.

5 Multipole expansion for $B_v$

For a real physical system, charges and currents are localized. So, in the intermediate zone (see Fig. 1) $j = 0$ and the equations for the “vortex” part of the magnetic field become:

$$\text{div } B_v = 0$$

(23)

$$\text{rot } B_v = -\frac{1}{c} \frac{\partial}{\partial t} \nabla \varphi$$

(24)

$$\Delta B_v = 0.$$ 

(25)
Vector $\mathbf{B}_v$ can be represented in the following form:

$$
\mathbf{B}_v = -\nabla \psi + \frac{1}{c} \frac{\partial}{\partial t} \text{rot} \mathbf{C}.
$$

(26)

Here, the rot $\mathbf{C}$ field should depend on electric moments and cause nontrivial rot $\mathbf{B}$. As we will see further, this field will depend on the scalar potential $\varphi$ rather than on the electric moments specifically. The $\psi$ field is of purely Laplacian nature, it is known as the scalar magnetic potential. Its multipole coefficients are called magnetic multipole moments (by analogy with the electric moments of $\varphi$).

The following equations hold for $\psi$ and $\mathbf{C}$:

$$
\Delta \psi = 0,
$$

(27)

$$
\Delta \mathbf{C} = 0,
$$

(28)

$$
\text{div} \mathbf{C} = -\varphi + \frac{Q}{r}.
$$

(29)

We can choose the form of $\mathbf{C}$ as follows:

$$
\mathbf{C} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' \rho' (r') \nabla (r')^{n-1} \frac{1}{r},
$$

(30)

$$
\text{rot} \mathbf{C} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' [\nabla, r'] (r')^{n-1} \frac{1}{r}.
$$

(31)

From Eq. (4b) we can obtain the following exact formula for $\mathbf{B}_v$:

$$
\mathbf{B}_v = \frac{1}{c} \int d^3 r' \frac{\text{rot} j'}{|r - r'|}.
$$

(32)

For the scalar magnetic potential $\psi$ one has:

$$
-\nabla \psi = \mathbf{B}_v - \frac{1}{c} \frac{\partial}{\partial t} \text{rot} \mathbf{C}.
$$

(33)

The terms in the right-hand side are expressed as series:

$$
\mathbf{B}_v = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' \text{rot} j'(r'), \nabla (r')^{n-1} \frac{1}{r}
$$

$$
= \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' [j', \nabla] (r')^{n} \frac{1}{r}
$$

(34)

$$
\frac{1}{c} \frac{\partial}{\partial t} \text{rot} \mathbf{C} = -\frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' \frac{\partial \rho'}{\partial t} [\nabla, r'] (r')^{n-1} \frac{1}{r}
$$

$$
= -\frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' [j', \nabla'] [\nabla, r'] (r')^{n-1} \frac{1}{r}
$$

(35)

The series for $-\nabla \psi$ becomes:

$$
-\nabla \psi = \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 r' \{ [j', \nabla'] (r', \nabla) + [j', \nabla'] [\nabla, r'] \} (r')^{n-1} \frac{1}{r}.
$$

(36)
We will normalize the operator in curly braces by moving the primed nablas to the rightmost positions:

\[
\{ \ldots \} = [j', \nabla] + (\nabla, r') [j', \nabla'] + [\nabla, j'] + [\nabla, r'][j', \nabla']
\]

\[
= (\nabla, r') [j', \nabla'] + [\nabla, r'][j', \nabla'].
\] (37)

It is now convenient to use the vector identity

\[
(ab)[cd] = (ac)[bd] + (ad)[cb] + a([bc]d)
\] (38)

following from two different ways of writing the expression \([b[a(cd)]]\), where \(a\) corresponds to \(\nabla\) in the first case and to \(j'\) in the second one:

\[
\{ \ldots \} = (\nabla, j')[r', \nabla'] + [j', r'][\nabla, \nabla'] + \nabla([r', j'], \nabla')
\]

\[
+ (j', r')[\nabla, \nabla'] - (\nabla, j')[r', \nabla'] + j'([\nabla, r'], \nabla')
\]

\[
= [j', r'][\nabla, \nabla'] + \nabla([r', j'], \nabla') + (j', r')[\nabla, \nabla'] - j'([\nabla, r'], \nabla')
\] (39)

One can easily check that \((\nabla, \nabla')\) and \([\nabla, \nabla']\) in Eq. (36) yield zero contribution, thus only the second term remains:

\[
-\nabla \psi = \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \nabla([r', j'], \nabla')(r', \nabla)^{n-1} \frac{1}{r}
\]

\[
= \frac{1}{c} \nabla \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3r' (\text{div}'[r', j']) (r', \nabla)^{n-1} \frac{1}{r}
\] (40)

Shifting the summation index by unity, from \(n\) to \((n + 1)\), we finally get:

\[
\psi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \left\{ -\frac{1}{c} \text{div}'[r', j'] \right\} (r', \nabla)^n \frac{1}{r}
\]

(41)

So, the magnetic scalar potential is now formally written in a form similar to the multipole expansion of the electric scalar potential, namely,

\[
\psi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \rho'_m(n)(r', \nabla)^n \frac{1}{r}
\] (42)

where a certain effective \(n\)-dependent “magnetic charge” appears:

\[
\rho'_m(n) = -\frac{1}{c} \frac{\text{div}'[r', j']}{n + 1}.
\] (43)

Consequently, magnetic multipole moments occur naturally as follows:

\[
M_{i_1 \ldots i_n} = \int d^3r' \rho'_m(n)x_{i_1}' \ldots x_{i_n}'.
\] (44)

The first three of them are

\[
M = 0,
\] (45)

\[
\mathbf{m} = \frac{1}{2c} \int d^3r' [r', j'],
\] (46)

\[
M_{ij} = \frac{1}{3c} \int d^3r' \{ [r', j'], x_k + [r', j']k x_i' \}
\] (47)

coinciding with the zero magnetic monopole, standard magnetic dipole moment, and magnetic quadrupole moment, respectively.
6 Multipole expansion of $E_v$

Equations in the intermediate zone for the “vortex” part of the electric field are:

$$E_v = -\frac{1}{c} \frac{\partial A_v}{\partial t}$$  \hspace{1cm} (48)

$$\text{div} \ A_v = 0$$  \hspace{1cm} (49)

$$\text{rot} \ A_v = B_v = -\nabla \psi + \frac{1}{c} \frac{\partial}{\partial t} \text{rot} \ C$$  \hspace{1cm} (50)

$$\Delta A_v = \frac{1}{c} \frac{\partial}{\partial t} \nabla \varphi$$  \hspace{1cm} (51)

The vector potential is represented as:

$$A_v = -\nabla \xi + \text{rot} \ F + \frac{1}{c} \frac{\partial}{\partial t} (C + C' + C'')$$

The rot $F$ field should generate $(-\nabla \psi)$ in $B_v$, therefore, it depends on the magnetic moments and is invariant with respect to their detracing. The third term is purely electric. The $C$ part appears naturally, $C'$ is required to cancel the div $C$ term, while $C''$ ensures invariance with respect to the detracing of multipole moments. The $\nabla \xi$ term is a “defect” generating a third type of multipole moments, which are called toroidal moments.

Let the following conditions hold for the constituents of the $E_v$ field:

$$\Delta \xi = 0,$$  \hspace{1cm} (52)

$$\Delta F = 0,$$  \hspace{1cm} (53)

$$\text{div} \ F = -\psi,$$  \hspace{1cm} (54)

$$\text{div} \ C' = -\text{div} \ C,$$  \hspace{1cm} (55)

$$\text{rot} \ C' = 0,$$  \hspace{1cm} (56)

$$\text{div} \ C'' = 0,$$  \hspace{1cm} (57)

$$\text{rot} \ C'' = 0.$$  \hspace{1cm} (58)

To conform with previous considerations we choose $F, C, C', C''$ in the following forms:

$$F = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \rho'_m(n) r'(r', \nabla)^{n-1} \frac{1}{r},$$  \hspace{1cm} (59)

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3r' \rho'(r', \nabla)^n \frac{1}{r},$$  \hspace{1cm} (60)

$$C' = -\nabla \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3r' \rho'(r', \nabla)^{n+1} \frac{r}{2},$$  \hspace{1cm} (61)

$$C'' = \nabla \sum_{n=1}^{\infty} \alpha_n \frac{(-1)^n}{(n+1)!} \int d^3r' r'^2 \rho'(r', \nabla)^{n-1} \frac{1}{r}.$$  \hspace{1cm} (62)

The only remaining free parameters are $\alpha_n$. They can be fixed from the condition that the sum of fields $C + C' + C''$ is independent of the detracing procedure for the electric multipole moments. So, inserting into the expressions for $C, C', C''$ at given $n > 1$ instead of $\rho'(x'_{i_1} \ldots x'_{i_{n+1}})$
“detracers” given by

\[
\sum_{\{j_1 j_2\} \subset \{i_1 \ldots i_{n+1}\}} \rho r^2 \delta_{j_1 j_2} x'_{k_1} \ldots x'_{k_{n-1}}
\]

should yield zero. In the above expression, the summation runs over all the pairs of indices \(\{j_1, j_2\}\) within given \(\{i_1, \ldots, i_{n+1}\}\), while \(\{k_1, \ldots, k_{n-1}\}\) are the remaining indices; the total number of summands is \(n(n + 1)/2\).

We thus have an equation for \(\alpha_n\) as follows:

\[
0 = \nabla \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3 \rho' \rho'^2 \left\{ \frac{n}{2} + \alpha_n [3 + 2(n - 1)] \right\} (r', \nabla)^{n-1} \frac{1}{r} \tag{63}
\]

and

\[
\alpha_n = \frac{(n-1)n}{2(2n+1)}. \tag{64}
\]

The series for \(C'\) can be written in the following integral form:

\[
C' = \int d^3 \rho' \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} (r', \nabla)^{n+1} \frac{r}{2r} = \int d^3 \rho' \frac{r - r'}{2|\mathbf{r} - \mathbf{r}'|} - \frac{Q}{2r}. \tag{65}
\]

Therefore,

\[
\frac{1}{c} \frac{\partial}{\partial t} C' = \frac{1}{c} \int d^3 \rho' \frac{\partial}{\partial t} \frac{r - r'}{2|\mathbf{r} - \mathbf{r}'|} = \mathbf{A}_{v2} \tag{66}
\]

The equation for \(\xi\) becomes:

\[
- \nabla \xi = \mathbf{A}_{v1} - \text{rot} \mathbf{F} - \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{C} + \mathbf{C}''). \tag{67}
\]

Taking into consideration series (11), (59)–(62) for \(\mathbf{A}_{v1}, \text{rot} \mathbf{F}, \frac{1}{c} \frac{\partial}{\partial t} \mathbf{C}\), and \(\frac{1}{c} \frac{\partial}{\partial t} \mathbf{C}''\), expressed via currents:

\[
\mathbf{A}_{v1} = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3 \rho' \rho'^2 (r', \nabla)^n \frac{1}{r} = \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3 \rho' \rho'^2 (r', \nabla)^n \frac{1}{r} + \frac{1}{cr} \int d^3 \rho' \rho', \tag{68}
\]

rot \(\mathbf{F} = -\frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3 \rho' ([r', j', \nabla]) [\nabla, r'](r', \nabla)^{n-1} \frac{1}{r}, \tag{69}
\]

\[
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{C} = \frac{1}{c} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3 \rho' (j', \nabla') r'(r', \nabla)^n \frac{1}{r} = \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3 \rho' (j', \nabla') r'(r', \nabla)^n \frac{1}{r} + \frac{1}{cr} \int d^3 \rho' j', \tag{70}
\]

\[
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{C}'' = \nabla \mathbf{A} - \frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3 \rho' \frac{(n-1)n}{2(2n+1)} (j', \nabla') r'^2 (r', \nabla)^{n-1} \frac{1}{r}, \tag{71}
\]

\]
we obtain the series for \((-\nabla \xi)\):

\[
-\nabla \xi = \frac{1}{c^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^{3}r' \left\{ (n+1)j'(r', \nabla) + ([r', j'], \nabla')[\nabla, r'] - (j', \nabla')r'(r', \nabla) \right\} \\
\times (r', \nabla)^{n-1} \frac{1}{r} - \frac{1}{c} \frac{\partial}{\partial t} C''
\]

(72)

The \(C''\) vector is a gradient itself, so we will work with it later. The first term in curly braces together with the external factor of \((r', \nabla)^{n-1}\) might be written as follows in order to get rid of \(n\):

\[
(n+1)j'(r', \nabla)^n = nj'(r', \nabla)^n + j'(r', \nabla)^n = j'(r', \nabla')(r', \nabla)^n + j'(r', \nabla)^n
\]

(73)

The operator in curly braces becomes:

\[
\{ \ldots \} = j'(r', \nabla')(r', \nabla) + j'(r', \nabla) + ([r', j'], \nabla')[\nabla, r'] - (j', \nabla')r'(r', \nabla).
\]

(74)

As before, it can be normalized by moving the primed nablas to the rightmost positions:

\[
\{ \ldots \} = 2j'(r', \nabla) + j'(r', \nabla)(r', \nabla') + [\nabla, [r', j']][\nabla, r'] + [\nabla, r'][[r', j'], \nabla') -
\]

\[
- j'(r', \nabla) - r'(j', \nabla) - r'(r', \nabla)(j', \nabla') =
\]

\[
= j'(r', \nabla)(r', \nabla') + [\nabla, r'][[r', j'], \nabla') - r'(r', \nabla)(j', \nabla').
\]

Upon applying identity (38) to the second term choosing \(a\) as \([r', j']\) we obtain:

\[
[[\nabla, \nabla'], [r', j']] = ([r', j'], r')[\nabla, \nabla'] - ([r', j'], \nabla)[r', \nabla'] - [r', j'][r', [\nabla, \nabla']].
\]

(75)

Since the operator \([\nabla, \nabla']\) acts trivially, as we have seen before, we can again transform only the second term choosing \(a\) equal to \(\nabla\):

\[
-([r', j'], \nabla)[r', \nabla'] = -[r', [r', j']][\nabla, \nabla') - (r', \nabla)[[r', j'], \nabla'] - \nabla([r', j'], [r', \nabla']).
\]

(76)

The action of \((\nabla, \nabla')\) is also trivial, so only the second and third terms enter into \{\ldots\}:

\[
\{ \ldots \} = j'(r', \nabla)(r', \nabla') - (r', \nabla)[[r', j'], \nabla'] - \nabla([r', j'], [r', \nabla']) - r'(r', \nabla)(j', \nabla')
\]

\[
= -\nabla([r', j'], [r', \nabla']) = \nabla([r', [r', j']], \nabla').
\]

(77)

Only non-trivial contributions are preserved under transition to the last equality.

Let us now write the series for \((-\nabla \xi)\) taking into account \(C''\):

\[
-\nabla \xi = \frac{1}{c^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^{3}r' \left\{ ([r', [r', j']], \nabla') - \frac{(n-1)n}{2(2n+1)} (j', \nabla')r'^2 \right\} (r', \nabla)^{n-1} \frac{1}{r}.
\]

(78)

Normalization of the second term gives:

\[
(j', \nabla')r'^2 = r'^2(j', \nabla') + 2(j', r').
\]

(79)

The last term together with \((r', \nabla)^{n-1}\) can be transformed as follows (note that here \(n > 1\)):

\[
2(j', r')(r', \nabla)^{n-1} = \frac{2}{n-1}(j', r')(r', \nabla')(r', \nabla)^{n-1}.
\]

(80)
The expression for $\xi$ is now:

$$\xi = -\frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \int d^3r' \left\{ ([r', [r', j']], \nabla') - \frac{(n-1)n}{2(2n+1)} \left( r'^2 (j', \nabla') + \frac{2}{n-1} (j', r')(r', \nabla') \right) \right\} (r', \nabla')^{n-1} \frac{1}{r}. \quad (81)$$

The curly braces can be written in the form of $(\ldots, \nabla')$, where the left argument is:

$$\ldots = \frac{2(2n+1)(r', [r', j']) - n(n-1)r'^2 j' - 2n(j', r')r'}{2(2n+1)} = \frac{2(n+1)(j', r')r' - (n+1)(n+2)r'^2 j'}{2(n+1)}. \quad (82)$$

As a result we obtain:

$$\xi = -\frac{1}{c} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \rho'_i(n)(r', \nabla')^{n} \frac{1}{r}; \quad (83)$$

Shifting the summation index by unity, from $n$ to $n + 1$, we finally arrive at:

- toroidal scalar potential:
  $$\xi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^3r' \rho'_i(n)(r', \nabla')^{n} \frac{1}{r}; \quad (84)$$

- toroidal “charge”:
  $$\rho'_i(n) = -\frac{1}{c} \frac{\text{div}'\{2(j', r')r' - (n+3)r'^2 j'\}}{2(n+1)(2n+3)}; \quad (85)$$

- toroidal multipole moments (unreduced):
  $$T_{i_1...i_n} = \int d^3r' \rho'_i(n)x'_{i_1}...x'_{i_n}. \quad (86)$$

The first three toroidal moments are as follows:

$$T = 0, \quad (87)$$

$$T = \frac{1}{10c} \int d^3r' \{ (j', r')r' - 2r'^2 j' \}, \quad (88)$$

$$T_{ik} = \frac{1}{42c} \int d^3r' \{ 4(j', r')x'_i x'_k - 5r'^2 (j'_i x'_k + j'_k x'_i) \}. \quad (89)$$

The second expression, for vector $\mathbf{T}$, is nothing but the so called toroidicity. Note that, up to integer multipliers, the inverse factor of 42 coincides, e.g., with definitions in [15, 16, 8, 17] or differs by $\frac{3}{2}$ from 28 as given in [9]. As a relief after all the conducted derivations recall that “42” in the third moment is also the Answer to the Ultimate Question of Life, the Universe, and Everything [18].
7 Discussion

We have proposed a consistent approach to the definition of multipole moments occurring in expansions of the electromagnetic field generated by time-dependent distributions of charges and currents. Upon splitting the fields into potential, vortex, and radiative terms we have managed to derive general expressions for three types of moments: electric, magnetic, and toroidal. This became possible by means of formally introduced magnetic and toroidal analogs of the electric charge with subsequent application of techniques similar to the multipole expansion of the electrostatic field in terms of the scalar potential.

The proposed scheme is summarized in Table 1. Electric, magnetic, and toroidal multipole moments appear in expansions of the scalar fields $\varphi$, $\psi$, and $\xi$ under gradients in $E_p$, $B_v$, and $A_v$, respectively. The transitions from $A_v$ back to $E_p$ are made via consecutive applications of rotors. Vectors $\text{rot} \ F$ and $C, C', C''$s are expressed via multipole moments in the corresponding columns, i.e., magnetic and electric ones, respectively.

Table 1: General scheme for obtaining three types of multipoles.

| Toroidal moments | Magnetic moments | Electric moments |
|------------------|-----------------|-----------------|
| $\frac{1}{c} \frac{\partial E_p}{\partial t}$ | $1 \frac{\partial}{\partial t} (-\nabla \varphi)$ |
| rot: $\uparrow$ | $-\nabla \psi + \frac{1}{c} \frac{\partial}{\partial t} \text{rot} \ C$ |
| $A_v = -\nabla \xi + \text{rot} \ F + \frac{1}{c} \frac{\partial}{\partial t} (C + C' + C'')$ |

Some general observations regarding our approach can be formulated as follows:

- The splitting of $B_v$ into magnetic ($-\nabla \psi$) and electric ($\frac{1}{c} \frac{\partial}{\partial t} \text{rot} \ C$) parts is unique up to the choice of the radius-vector origin.
- The same holds true for the splitting of $A_v$ (and hence $E_v$) into toroidal ($-\nabla \xi$), magnetic ($\text{rot} \ F$), and electric ($\frac{1}{c} \frac{\partial}{\partial t} (C + C' + C'')$) parts.
- All the fields in the right-hand sides in Table 1 ($-\nabla \varphi$, $-\nabla \psi$, $-\nabla \xi$, $\text{rot} \ F$, $\text{rot} \ C$, $C + C' + C''$) do not depend on the types of moments, traceless or not.
- The series for electric, magnetic, and toroidal moments are generated by the large-$r$ expansions of quasistatic fields $E_p$, $B_v$, and $E_v$ only.
- The magnetic and toroidal moments together with time-derivatives of electric moments form a linearly independent and full system, i.e., they form a basis for current moments (while electric moments correspond to charge). So, there is no need to introduce other types of multipoles – they will appear as linear combinations of the three defined moments.
- The coefficients in expansions of the radiative fields $A_r$, $B_r$ and $E_r$ at large $r$ are unambiguously expressed via the found moments, their traces and time derivatives (the same holds true for the radiation intensity).

In summary, we expect that the proposed approach will take its proper place in the theory of multipole expansion of time-dependent charge and current distributions, even though this is a problem in classical electrodynamics with a long history and a number of definitions known so far.
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References

[1] David J. Griffiths. *Introduction to Electrodynamics*. Prentice Hall, Upper Saddle River, New Jersey, 3rd edition, 1999.

[2] John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, Inc., New York, 3rd edition, 1999.

[3] V. M. Dubovik and L. A. Tosunyan. Toroidal moments in the physics of electromagnetic and weak interactions. *Soviet Journal of Particles and Nuclei*, 14(5):504–519, 1983.

[4] E. E. Radescu and G. Vaman. Exact calculation of the angular momentum loss, recoil force, and radiation intensity for an arbitrary source in terms of electric, magnetic, and toroid multipoles. *Phys. Rev. E*, 65(4):046609, 2002.

[5] C. Vrejoiu and R. Zus. Singular behavior of the multipole electromagnetic field. *Journal of Physics A: Mathematical and Theoretical*, 43(40):405208, 2010.

[6] Andreas Ross. Multipole expansion at the level of the action. *Phys. Rev. D*, 85(12):125033, 2012.

[7] Stefan Nanz. *Toroidal Multipoles Moments in Classical Electrodynamics. An Analysis of their Emergence and Physical Significance*. Springer Spektrum, Wiesbaden, 2016.

[8] Rasoul Alaee, Carsten Rockstuhl, and I. Fernandez-Corbaton. An electromagnetic multipole expansion beyond the long-wavelength approximation. *Optics Communications*, 407:17–21, 2018.

[9] V. M. Dubovik and A. A. Cheshkov. Multipole expansion in classical and quantum field theory and radiation. *Fiz. Elem. Chastits At. Yadra*, 5(3):791–837, 1974. [Sov J. Part. Nucl. 5(3), 318–337].

[10] C. Vrejoiu and R. Zus. Explanation notes on the multipole expansions of the electromagnetic field. *ArXiv e-print*, 0910.1313, 2009.

[11] E. Radescu, Jr. and G. Vaman. Cartesian multipole expansions and tensorial identities. *Progress in Electromagnetics Research B*, 36:89–111, 2012.

[12] Ivan Fernandez-Corbaton, Stefan Nanz, and Carsten Rockstuhl. On the dynamic toroidal multipoles from localized electric current distributions. *Scientific Reports*, 7(1):7527, 2017.

[13] V. M. Dubovik and V. V. Tugushev. Toroid moments in electrodynamics and solid-state physics. *Physics Reports*, 187(4):145–202, 1990.

[14] Andrij Rovenchak and Yuri Krynytskyi. Radiation of the electromagnetic field beyond the dipole approximation. *ArXiv e-print*, 1803.07889, 2018.

[15] S. G. Porsev. Quadrupole toroidal moment of positronium. *Phys. Rev. A*, 49(6):5105–5107, 1994.

[16] C. Vrejoiu and Diana Nicmorus. On the multipole electromagnetic radiation. *Journal of Physics A: Mathematical and General*, 37(16):4671–4684, 2004.

[17] N. Talebi, S. Guo, and P. van Aken. Theory and applications of toroidal moments in electrodynamics: their emergence, characteristics, and technological relevance. *Nanophotonics*, 7(1):93–110, 2018.

[18] Douglas Adams. *The Hitchhiker’s Guide to the Galaxy*. Pocket Books, New York, 1979.