Amplitude death in coupled slow and fast dynamical systems

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We study how mismatch between dynamical time scales of interacting identical systems can result in the suppression of collective dynamics leading to amplitude death. We find that the inability of the interacting systems to fall in step leads to difference in phase as well as change in amplitude. If the mismatch is small, the systems settle to a frequency synchronised state with constant phase difference. But as mismatch in time scale increases, the systems have to compromise to a state of no oscillations. We establish that this regime of amplitude death exists in a net work of identical systems also for sufficient number of slow systems. For standard nonlinear systems, the regions of quenched dynamics in the parameter plane and the transition curves are studied analytically and confirmed by numerical simulations.

I. INTRODUCTION

The complexity of several dynamical phenomena that occur in many physical, chemical, biological, geophysical and social systems arise due to the interaction of dynamical processes at differing time scales[1–7]. Thus fast and slow dynamical processes occur in modulated lasers and chemical reactions[8, 9]. It is known that in biological systems, dynamics with time scales of days coexist and interact with dynamics of biochemical reactions in sub second time scales[10]. There are several intracellular processes of differing time scales that directly or indirectly influence the electrical activity of neurons[11]. The weather and climate systems of earth contain subsystems spanning over widely differing time scales. Some of the subsystems are basically nonlinear and are strongly coupled like tropical atmospheric ocean system[12, 13]. In this context some of the relevant questions are how the fast dynamics can affect the predictability of the slow dynamics and how the slow and fast modes can be separated[14]. In general, coupled slow and fast systems occur in engineering design where issues related to regulation and optimal control are relevant topics for study[15].

The method of analysis mostly followed in such contexts is adiabatic elimination of fast variables[16] which is applicable when the time scales are widely different. Recent studies clearly indicate that fast time scales can affect the slow dynamics in systems of different time scales[17, 18]. Therefore, a detailed study on the interplay of dynamics occurring at all ranges of time scales is highly relevant and will have several applications. If such cases involve many interacting sub units, the dynamical phenomena can be understood by considering the sub systems as evolving at different time scales while being coupled to each other.

In the context of coupled dynamical systems, some of the well studied emergent phenomena are synchronisa-

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In this work, we investigate the dynamics of identical systems that evolve with different time scales and interact among themselves. We start by considering the coupled dynamics of two identical systems in which one is slower than the other. Our results indicate that for sufficient mismatch in time scale two such systems go into a synchronized state of amplitude death. If the mismatch is small, the two systems get locked into a state of same frequency with constant phase difference. The resultant frequency of the emergent state decreases while the phase difference increases with increase in time scale mismatch. We analyse the stability of the amplitude death state for differing coupling strengths and time scale mismatch and the nature of transition to this state as parameters are tuned. We present the results for standard nonlinear systems like coupled Landau-Stuart, Rössler and Lorenz systems.
We further extend this to a network of $N$ such systems in which $m$ number of systems evolve on a slow time scale. We find as the fraction of slow systems increases the whole network collapses to a state of amplitude death.

II. COUPLED SLOW AND FAST SYSTEMS

We construct a simple model of interacting slow and fast dynamics by considering two identical systems evolving at different time scales. For this a time scale parameter is attached to the intrinsic equations of the dynamical systems. The system having relatively large time scale parameter is considered as the fast system whereas system with relatively small time scale parameter is considered to be the slow system. These systems are then coupled to each other and the equations governing their dynamics will then be as given below.

$$
\begin{align*}
\dot{X}_1 &= \tau_1 F(X_1) + \epsilon G(X_1, X_2) \\
\dot{X}_2 &= \tau_2 F(X_2) + \epsilon G(X_2, X_1)
\end{align*}
$$

Here $X_{1,2} \in \mathbb{R}^n$, $F$ is the intrinsic dynamics of the system, $G$ denotes the coupling function, $\epsilon$ the coupling strength and $G$ is an $n \times n$ matrix which decides the variables to be coupled. The parameters $\tau_1$ and $\tau_2$ decide the difference in time scales. Without loss of generality, we can take $\tau_1 = \tau$ and $\tau_2 = 1$ as the tunable time scale parameter, by tuning which, the time scale mismatch between the two systems can be varied.

A. Coupled slow and fast periodic oscillators

In this section we consider the specific case of two Rössler systems with slow and fast time scales, coupled diffusively. The dynamics then evolves as

$$
\begin{align*}
\dot{x}_1 &= \tau(-y_1 - z_1) + \epsilon(x_2 - x_1) \\
\dot{y}_1 &= \tau(x_1 + ay_1) \\
\dot{z}_1 &= \tau(b + z_1(x_1 - c)) \\
\dot{x}_2 &= (-y_2 - z_2) + \epsilon(x_1 - x_2) \\
\dot{y}_2 &= (x_2 + ay_2) \\
\dot{z}_2 &= (b + z_2(x_2 - c))
\end{align*}
$$

The intrinsic dynamics is periodic with parameters chosen as $a=0.1$, $b=0.1$ and $c=4$. In this case, in addition to the coupling strength $\epsilon$, the time scale mismatch parameter $\tau$ also controls the asymptotic dynamics of the coupled systems. We find that, for sufficiently large value of $\epsilon$ and small value of $\tau$, the two systems go into a state of amplitude death. This is shown in Fig. 1 where the time series for the $x$- variable of both systems are plotted for $\tau = 0.5$ and $\epsilon = 0.8$.

We do a detailed numerical analysis of the system in equation (2) for different values of these parameters scanning the parameter plane $(\tau, \epsilon)$. To identify region of amplitude death in this plane, we compute an index $A$ by taking the difference between global maximum and global minimum of variable $x$ for each system after neglecting the transients. Then $A = 0$ would indicate regions of AD[28]. Using this we observe an island of amplitude death in the $(\tau, \epsilon)$ plane where both systems settle to a synchronised fixed point or state of amplitude death(Fig. 2).

When amplitude death happens in the systems in equation (2), the fixed point of the whole system stabilises. The parameters for which this happens can be obtained analytically by a linear stability analysis of the system about the fixed point. The synchronised fixed points of the systems in equation (2) are $(x^*, y^*, z^*)$ equal to $(\pm \sqrt{\frac{c^2-4ab}{2a}}, \pm c \sqrt{\frac{c^2-4ab}{2a}}, \pm \sqrt{\frac{c^2-4ab}{2a}})$. The Jacobian of

![Amplitude death region for coupled slow and fast systems](image1.png)

![Amplitude death region for coupled slow and fast systems](image2.png)
the coupled slow fast systems at the fixed points is

\[
J = \begin{pmatrix}
-\epsilon & -\tau & -\tau & \epsilon & 0 & 0 \\
\tau & \tau a & 0 & 0 & 0 & 0 \\
\tau z^* & 0 & \tau(x^* - c) & 0 & 0 & 0 \\
\epsilon & 0 & 0 & -\epsilon & -1 & -1 \\
0 & 0 & 0 & 1 & a & 0 \\
0 & 0 & 0 & z^* & 0 & (x^* - c)
\end{pmatrix}
\tag{3}
\]

The eigen values of J are calculated for different values of \(\tau\) and \(\epsilon\) and the values at which at least one of the eigen values changes from negative to positive are plotted to get the transition curves. This identifies the island of amplitude death in the parameter plane shown by the red line boundaries in Fig. 2. We clearly see that this boundary matches with that obtained from direct numerical simulations.

**B. Phase locked states and transition to amplitude death**

We now study the nature of dynamics for parameter values outside the island of AD. When the time scale parameter \(\tau\) is equal to one, i.e. with equal time scales, for sufficient strength of coupling both systems asymptotically reach complete synchronization. However as \(\tau\) decreases, depending upon the coupling strength the oscillators show different dynamical states. For very large time scale mismatch i.e. for very small \(\tau\) and large coupling strength, the system becomes unstable (corresponding to top left part of Fig 2). We discuss below the dynamics in the region below the AD island for very low coupling strength, and the region on its right for higher coupling strength (Fig 2) separately.

When the coupling strength is very small (\(\epsilon < 0.1\)), for a very small mismatch in time scales such as \(\tau = 0.9\) we see each system settles to a two frequency state of vibration as shown in Fig 3.

![Fig. 3](image)

**FIG. 3.** (colour online) Time series of the two frequency state for coupled slow (red) and fast (green) Rössler systems for \(\tau = 0.9\) and \(\epsilon = 0.05\)

The average frequency can be calculated from the time series using the relation

\[
\omega = \frac{1}{K} \sum_{k=1}^{K} \frac{2\pi}{(t_{k+1} - t_k)}
\tag{4}
\]

where \(t_k\) is the time of the \(k^{th}\) zero crossing point in the time series of the oscillator and \(K\) is the total number of intervals used. Using this we calculate the fast frequency (\(\omega_1\)) of each oscillator from the time series data, while the slow frequency (\(\omega_2\)) is obtained by taking \(t_k\) as the time of \(k^{th}\) local maximum of all the maxima. We find that the fast frequencies (\(\omega_1\)) differ, but the slow frequencies (\(\omega_2\)) are the same for both the oscillators. As the coupling strength increases the slow frequency disappears and the two systems get locked into a state of equal frequency. The variation of \(\omega_1\) and \(\omega_2\) as \(\epsilon\) increases is shown in Fig 4.

![Fig. 4](image)

**FIG. 4.** (colour online) Variation of frequency with \(\epsilon\) at \(\tau = 0.9\). (a) Fast frequency of fast (green) and slow (red) systems. (b) Slow frequency of both systems.

As the time scale mismatch increases or \(\tau\) decreases, the two systems go through a state of irregular behaviour and settle to a state of periodic oscillation with widely differing amplitudes. Fig 5 illustrates this for two different values of \(\tau\).

![Fig. 5](image)

**FIG. 5.** (colour online) Time series of coupled slow (red) and fast (green) Rössler systems with \(\epsilon = 0.05\) (a) irregular behaviour for \(\tau = 0.6\) (b) regular phase locked state with amplitude of slow system reduced at \(\tau = 0.2\).

For larger values of \(\epsilon\) (\(\epsilon > 0.1\)), as the time scale mismatch between oscillators increases the two systems fall out of step and soon settle into a state of constant phase relation (Fig 6).
amplitude death. The region where $A$ increases corresponds to the state of frequency synchronisation (similar to the state in Fig. 6).

FIG. 6. (colour online) Time series of coupled slow (red) and fast (green) R"ossler systems with $\tau = 0.6$ and $\epsilon = 1$.

To understand the phase relation between the systems, we calculate the phase $\phi(t)$ of each system from the time series using [31]

$$\phi(t) = 2\pi \frac{t - t_k}{t_{k+1} - t_k} + 2\pi k, \quad t_k \leq t \leq t_{k+1} \quad (5)$$

FIG. 7. (colour online) (a) Average phase difference between two oscillators vs $\tau$ for $\epsilon = 1$ (green), $\epsilon = 1.3$ (red), $\epsilon = 1.5$ (black) (b) Frequency of uncoupled fast oscillator (green), slow uncoupled oscillator (red) and coupled oscillator (black) for $\epsilon = 1$ as $\tau$ is varied.

The phase difference between the two systems, $|\phi_2(t) - \phi_1(t)|$, is found to be constant in time indicating the systems get locked into a state of constant phase difference. We plot the average phase difference, $\psi$, for three different values of $\epsilon$ as $\tau$ is varied and find $\psi$ increases as $\tau$ decreases until AD is reached (Fig. 7(b)). For these parameter values we find both systems have the same average frequency $\omega$ (computed using equation (4)) which is intermediate to the fast and slow system frequencies. However, this decreases as $\tau$ decreases for a given $\epsilon$. This is clear from (Fig. 7(b)) where the intrinsic frequencies are plotted along with that of the coupled systems.

In addition to the changes in frequencies and phase locking described above, the amplitudes of the systems also get reorganised due to the coupling. In general the amplitude of the slow system is lower than that of the fast system. The difference in the averaged amplitudes of the two systems, $A_d = \langle A_2 \rangle - \langle A_1 \rangle$ varies as the parameters are tuned. This is shown in Fig. 8 for different values of $\epsilon$ as $\tau$ is varied. This difference increases as $\tau$ decreases and reaches a maximum just before the onset of AD. After reaching the maximum, the amplitudes of both oscillators start decreasing in time and finally reach

FIG. 8. (colour online) Average amplitude difference between the two oscillators vs $\tau$ for $\epsilon = 0.5$ (blue), $\epsilon = 0.7$ (black), $\epsilon = 0.9$ (green), $\epsilon = 1.5$ (red). (colour online)

The variation of the average amplitude difference with $\epsilon$ is shown in Fig. 8 for four different values of $\tau$. Here also just before AD is reached $A_d$ increases and reaches a maximum. However, this difference is much larger than that for higher $\epsilon$ values (similar to the state in Fig. 6(b)).

FIG. 9. (colour online) Average amplitude difference between two oscillators vs $\epsilon$ for different $\tau$, $\tau = 0.5$ (red), $\tau = 0.7$ (green), $\tau = 0.8$ (black), $\tau = 0.9$ (blue).

We repeat the analysis with another periodic non-linear oscillator called Landau-Stuart oscillator. The equations for two coupled slow and fast systems are given by

$$\begin{align*}
\dot{x}_1 &= \tau((\lambda - x_1^2 - y_1^2)x - \omega y) + \epsilon(x_2 - x_1) \\
\dot{y}_1 &= \tau((\lambda - x_1^2 - y_1^2)y + \omega x) \\
\dot{x}_2 &= (\lambda - x_1^2 - y_1^2)x - \omega y + \epsilon(x_1 - x_2) \\
\dot{y}_2 &= (\lambda - x_1^2 - y_1^2)y + \omega x
\end{align*} \quad (6)$$

where $\lambda$ is the bifurcation parameter. For $\lambda > 0$ the intrinsic oscillator gives a limit cycle behaviour, and for $\lambda < 0$, a fixed point behavior. With $\lambda = 0.1$ and $\omega = 2$ we study the system numerically using equation (6). In this case also similar phenomena of amplitude death,
phase locking, frequency synchronisation and two frequency states are observed with qualitatively similar results. The amplitude death region in \((\tau, \epsilon)\) obtained numerically is in good agreement with that from stability analysis for the synchronised fixed point \((0, 0)\).

### C. Coupled slow and fast chaotic systems

We repeat the study for the case of two coupled slow and fast chaotic Rössler systems, as in equation \((2)\), but with parameter values of each system chosen such that their intrinsic dynamics is chaotic \((a=0.2, b=0.2, c=5.7)\). Here also we find that with sufficient strength of coupling and time scale mismatch the systems settle to a state of synchronised fixed point or amplitude death. The region for which the coupled dynamics of chaotic Rössler systems goes to amplitude death in the plane \((\tau, \epsilon)\) obtained numerically shows good agreement with the stability analysis. The transition to AD is through reverse period doublings resulting in periodic dynamics before AD is reached. Even though the amplitudes are different, the bifurcations occur at the same parameter values in both systems. The bifurcation diagram corresponding to these transitions as \(\tau\) is varied for \(\epsilon = 0.9\), is given in Fig. 10. The average phase difference and average amplitude difference have qualitatively similar behaviour, as in the periodic case with change in parameters.

We do the same analysis with another standard chaotic system, viz. Lorenz system. The equations of two coupled slow and fast Lorenz systems are

\[
\begin{align*}
\dot{x}_1 &= \tau a(y_1 - x_1) + \epsilon(x_2 - x_1) \\
\dot{y}_1 &= \tau (x_1(b - z_1) - y_1) \\
\dot{z}_1 &= \tau (x_1y_1 - cz_1) \\
\dot{x}_2 &= a(y_2 - x_2) + \epsilon(x_1 - x_2) \\
\dot{y}_2 &= x_2(b - z_2) - y_2 \\
\dot{z}_2 &= x_2y_2 - cz_2
\end{align*}
\]

where \(a=10, b=28, c=8/3\).

In this case also an island of AD is observed in the plane \((\tau, \epsilon)\). However the transition to AD is through an intermittency behaviour where the duration of the small amplitude oscillations gets longer as \(\tau\) is decreased. The time series of the coupled Lorenz systems are plotted for increasing values of \(\tau\) with \(\epsilon = 4.0\) in Fig. 11 which indicates this intermittency route to AD.

![Fig. 10](image1)

**FIG. 10.** (colour online)Bifurcation diagram obtained by plotting the maximum values of the \(x\) variables of the two coupled chaotic Rössler systems for \(\epsilon = 0.9\) as \(\tau\) is varied.

![Fig. 11](image2)

**FIG. 11.** (colour online)Transition to amplitude death in two coupled chaotic Lorenz systems in \((\tau, \epsilon)\). Time series plotted for \(\tau = 0.952, 0.951\) and 0.95 and \(\epsilon = 4\).

### III. NETWORK OF SLOW AND FAST SYSTEMS

We extend the study to a network of \(N\) identical systems in which \(m\) evolve on a slower time scale. This subset of oscillators of lower timescale is defined as \(S\). The equations of \(n\) dimensional systems governing their dynamics are given by

\[
\dot{X}_i = \tau_i F(X_i) + G\epsilon \sum_{j=1}^{N} A_{ij} (X_j - X_i)
\]

where \(\tau_i = \tau\) if \(i \in S\), \(\tau_i = 1\) otherwise. \(G\) is an \(n \times n\) matrix which decides which variables are to be coupled. Here we take \(G = \text{diag}(1, 0, 0, \ldots)\) which means \(x\) variable of the \(i^{th}\) oscillator is coupled diffusively with the
x variable of $j^{th}$ oscillator. $A_{ij}$ is adjacency matrix of the network. For simplicity we consider an all to all connected network with $A_{ij} = 1$ for $i \neq j$ and $A_{ii} = 0$. The intrinsic dynamics is taken as the chaotic Rössler system. We study three cases of system size, $N=10$, 20 and 30, and analyse how the slowness of $m$ of the systems can affect the dynamics of the whole network. For small $m$, the network can separate into two clusters of slow and fast systems as indicated by time series in Fig 12 for $\tau = 0.35$, $\epsilon = 0.12$ and $m=3$.

![Cluster formation of slow and fast dynamics](image)

**FIG. 12.** (colour online)Cluster formation of slow and fast dynamics in a network of 10 systems with three of them slow for $\tau = 0.35$, $\epsilon = 0.12$. Here the time series of x variable of all the systems are plotted.

As $m$ increases, interaction between the slow and fast systems causes the suppression of dynamics in the whole network. We investigate this further by calculating the average amplitude $\langle A \rangle$ of all the systems for different values of the fraction of slow systems, $m/N$. For three values of $N$ the results are shown in the Fig. 13. It is interesting to note that there is a specific range of $m/N$ for which AD happens. As $m$ increases further, the network regains the dynamics and follows the slow time scale.

![The average amplitude $\langle A \rangle$](image)

**FIG. 13.** (colour online)The average amplitude $\langle A \rangle$ of $N$ coupled Rössler systems plotted with the fraction, $m/N$ of slow systems for different $N$. $\langle A \rangle = 0$ corresponds to AD. Here $N=10$ (red), $N=20$ (green), $N=30$ (blue).

For a value of $m/N$ chosen from the amplitude death region, we numerically obtain the region of amplitude death in $(\tau, \epsilon)$ plane for network size $N=10$ (red), $N=20$ (green), $N=30$ (blue) of coupled Rössler systems with $m/N=0.5$.

![Regions of AD in parameter plane](image)

**FIG. 14.** (colour online)Regions of AD in parameter plane $(\tau, \epsilon)$ for network size $N=10$ (red), $N=20$ (green), $N=30$ (blue) of coupled Rössler systems with $m/N=0.5$.

### IV. CONCLUSION

Our study illustrates that in general, mismatch in time scales of different interacting units can affect the group performance of any complex system leading to decay of overall outcomes and even suppression of all activity. The concepts introduced are quite general and can be applied to social networks, interacting species evolving at different time scales, climate systems etc. In all these cases how the connections can affect the performance and outcomes and how they can be controlled or prevented are important questions to be addressed.

In the context of coupled systems, we report this as another mechanism that can cause amplitude death. Taking two coupled periodic and chaotic systems like Rössler Landau-Stuart and Lorenz with differing time scales, we show that the systems get locked to a state of constant phase difference and undergo transition to amplitude death as the strength of interaction or mismatch in time scale increases. For many such systems coupled to form a network, we report the same phenomenon in the simplest case of an all to all connected network. However, we are aware that the topology of the connections and the nature of the slow nodes will have a non-trivial influence in the collective dynamics. Further studies in this direction are in progress and will be reported elsewhere.

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