Zero range (contact) interactions conspire to produce Efimov trimers and quadruply
April 19, 2018

G.F. Dell’Antonio
Math. Dept. University Sapienza (Roma)
and
Mathematics Area, Sissa (Trieste)

Summary
We introduce contact (zero range) interactions, a special class of self-adjoint extensions of
the N-body Schrödinger free hamiltonian $H_0$ restricted to functions with support away from
the contact manyfold $\Gamma \equiv \cup \Gamma_{i,j} \quad \Gamma_{i,j} \equiv \{x_i = x_j \ i \neq j\} \ x_i \in \mathbb{R}^3$.
These extensions are defined by boundary conditions at $\Gamma$.
We discuss the spectral properties as function of the masses and the statistics
The (Efimov) spectrum is entirely due “conspiracy” of the contact interactions of two pairs.
These states are called in Theoretical Physics trimers if the two pairs have an element in
common, quadruply otherwise.
The analysis can be extended to the case in which there is a regular two-body potential, but
then the spectral properties cannot be given explicitly.
We prove that these interactions are limit (in strong resolvent sense) when $\epsilon \to 0$ of N-body
hamiltonians $H_\epsilon = H_0 + \sum_{i,j} \frac{1}{\epsilon} V_{i,j} (\frac{|x_i - x_j|}{\epsilon}), \ V_{i,j} \in L^1$.
The result is independent of the shape of the potential
Strong resolvent convergence implies convergence of spectra. This makes contact interaction
a valuable tool in the study of the spectrum of a system of $N$ particles interacting through
potentials of very short range.

1 Introduction

We introduce contact interactions as a special type of self-adjoint extensions of the operator
$H_0$ defined as the free Schrödinger operator $H_0$ with masses $m_i$ restricted to functions that
have support away from $\Gamma \equiv \cup \Gamma_{i,j} \quad \Gamma_{i,j} \equiv \{x_i = x_j \ i \neq j\} \ x_i \in \mathbb{R}^3$.
The prime indicates that only some pairs of indices are present.
The self-adjoint extensions we consider are characterized by boundary conditions at $\Gamma$.
At $\Gamma_{i,j}$ we admit a singularity $\frac{C_{i,j}}{|x_i - x_j|}$ for functions in the domain (some of the $C_{i,j}$ can be zero).
Imposing this behavior at $\Gamma$ is formally equivalent to placing a (negative) distributional potential at the boundary; this is seen by taking the scalar product with functions in the range of $H_0$ and integrating by parts twice (the boundary terms vanish).

Notice that functions in the domain of $H_0$ are absolutely continuous; therefore they are in the form domain of $H_0$ and of the boundary potential.

Conditions of this type were imposed by H.Bethe and R.Peirels [B,P] in 1935 in their analysis of the proton-neutron interaction; later they were used for the case $N = 3$ by G. Skorniakov and K.Ter Martirosian [S,T] in their study of the three body problem in Nuclear Physics in the Faddaev formalism.

For every $N$ we shall denote them Ter Martirosian -Skorniakov (T-S) boundary conditions.

The analysis we make can be extended to any Schrödinger hamiltonian bounded below with regular potentials. Here we consider only the case of the free hamiltonian $H_0$

We prove that hamiltonians with contact interactions are limits in strong resolvent sense when $\epsilon \to 0$ of hamiltonians

$$H_\epsilon = H_0 + \sum_{i,j} V_{i,j}^\epsilon (|x_j - x_i|)$$

with two-body negative potentials which scale according to $V^\epsilon(|x|) = \epsilon^{-3} V(|x|/\epsilon)$ where $V(|x|) \in L^1$.

Notice that the $L^1$ norm of $V^\epsilon$ does not depend on $\epsilon$ and that the result is independent of the shape of the potential.

This makes contact interactions a valuable tool in the the study of the spectrum of a system of $N$ particles of mass $m_i$ interacting through potentials of very short range.

Remark 1

Conspiracy of contact interaction is the counterpart of conspiracy of zero energy resonances [K,S].

Resonances may be viewed as contacts in a point at infinity (direction of slow decrease in the difference of the coordinates).

Since the Laplace operator in dimension three is conformally covariant, a pair of particles contact interaction with a zero energy resonance can be considered to be equivalent to contact interaction between two pairs of particles (one pair at infinity).

Recall that zero energy resonances between two particles correspond to a $\frac{1}{|x-x_j|}$ behavior of the wave function at infinity; this is the same behavior (at $\Gamma_{i,j}$) that we have required in case of contact interactions.

Conspiracy may occur between resonances [K,S]; this leads to the production of an Efimov set of bound states.

The analogous bound states for contact interactions are called trimers and quadrimers in the Physics literature.

Remark 2

A different hamiltonian of ”zero range interaction” was introduced [A] and called point interaction.

The system is composed of one particle in interaction with a fixed point trough a potential that has a zero-energy resonance.

In an appendix we comment on the relation between contact and point interactions.

In particular we prove that point interactions are limit of a contact interaction for a system of three identical bosons (two at infinity) of mass one and a particle of mass $M$, when $M \to \infty$.
(so that this particle in the limit is a fixed point) and the scaling factor of the potential is that given in [A] i.e $V(\epsilon |y|) = \frac{1}{\epsilon^2} V(|y|)$.

If the spectrum is positive, the spectral measure has a singularity at zero.

If the spectrum is not positive we show that the entire negative part of the spectrum of the point interaction hamiltonian is singular continuous.

It is the limit of a four-body Efimov point spectrum, when the distance between adjacent points of the spectrum goes to zero.

In the limit each point of the negative part of the spectrum is an accumulation point of the spectrum of the "point interaction" hamiltonian for $M < \infty$.

This "explains" the mapping properties of the Wave operator for point interactions [D].

We show also that a "simpler explanation" is obtained by a change of time scale.

In the appendix we also mention briefly the relation of our approach to the "heat kernel renormalization" for the three-body problem [E,T].

2 Auxiliary space

Contact interactions are a special class of self-adjoint extensions of the symmetric operator $\hat{H}_0$ defined to be the free hamiltonian restricted to functions that vanish in a neighborhood of $\tilde{\Gamma}$.

It will convenient to introduce an auxiliary space, suggested (as we will see) by the special class of self-adjoint extensions we consider.

For the three-body case our approach follows the lines of the approach of R.Minlos ([M1][M2] , see also [C1][C2] ).

For this reason we call the auxiliary space Minlos space and denote it by the symbol $\mathcal{M}$.

The "physical space" $L^2(\mathbb{R}^3)$ is compactly embedded in $\mathcal{M}$ by a map that we will give explicitly.

We call this map Krein map and we denote it by the symbol $K$.

The extensions of positive symmetric operators have been studied in great detail by Birman, Visik and Krein [B][K]. We use a quadratic form version described in [A,S].

In $\mathcal{M}$ the boundary conditions at $\Gamma$ are represented by $L^1$ potentials and in this space the asymptotic convergence as $\epsilon \to 0$ is easily proved.

In $\mathcal{M}$ we study in some detail the case of three particles, in particular the interaction of two identical particles of mass one with a third one, paying attention to the dependence on the mass of the third particle and to the statistics.

We consider also in some detail the four-particle case, in particular the case of two pairs of identical particles, either fermions or bosons.

Finally we consider the case of $N$ pairs of identical particles, both in the case of fermions and in the case of bosons.

In $\mathcal{M}$ both the kinetic energy the potential are essentially self-adjoint unbounded operators; one is positive the other negative (we have assumed that the interaction is attractive).

Their sum is a priori only symmetric.

In the case we are considering the "boundary potential" has a special structure in $\mathcal{M}$: it is the sum of a negative Coulomb potential and a positive regular term.

The "kinetic term" is a positive (pseudo)-differential operator of order one (the square root of the free hamiltonian).

For some choices of masses and symmetries the sum of the kinetic and potential terms defines in $\mathcal{M}$ a unique (essentially) self-adjoint operator while for other choices there are continuous
families of self-adjoint extensions (this is an instance of Weyl limit circle property [D,R]). Depending on the masses and the symmetry properties, the negative point spectrum of each member of the family may be absent or contain a finite or infinite set of points. In the latter case the spectrum shows the Thomas effect (geometric divergence of the eigenvalues to $-\infty$).

One has now to come back to the "physical space" $L^2(R^{3N})$. If there are no bound states this is done by inverting the Krein map. It is then easy to prove in physical space convergence when $\epsilon \to 0$ in strong resolvent sense of the approximating hamiltonians (with shrinking potentials) to the hamiltonian with contact interactions.

This follows because for positive quadratic forms weak convergence implies strong convergence and this in turn implies resolvent convergence.

If there are bound states in $\mathcal{M}$ showing the Thomas effect, inverting the Krein map implies a difference in metric and the sequence of states is now an Efimov sequence: in physical space (the points in the negative spectrum converge to zero geometrically).

One verifies that, after reducing with respect to the continuous symmetries (our potentials are all rotationally invariant), the quadratic form of the potential in $\mathcal{M}$ is strictly convex.

A unique limit when $\epsilon \to 0$ is then obtained using convexity, compactness and lower semicontinuity (Gamma-convergence) [Dal].

Strong resolvent convergence follows from Gamma-convergence.

We prove that, depending on the masses and the symmetries, the conspiracy of two contact interactions can give rise to three-body bound states (called trimers in the Physics literature) and four-body bound states (quadrimmers).

The trimers’ wave functions have essential support in a neighborhood of the triple coincidence point.

The eigenfunctions are obtained (by duality) applying the the Krein map to the eigenfunctions found in $\mathcal{M}$ (the latter have a $\frac{1}{|x_i-x_j|}$ singularity at the triple coincidence point).

The quadrimer’s wave functions are centered in the quadruple coincidence point and are functions of the difference of the coordinates $X_i$ of the barycenters of the two pairs. In $\mathcal{M}$ they have a $\frac{1}{|X_i-X_j|}$ singularity. They are obtained in physical space by applying the Krein map.

We consider briefly the N-body problem, $N \geq 4$, in particular in the case of pairs of identical particles (either fermions or bosons).

We show that in the case of fermions the hamiltonian is positive (stability of the unitary Fermi gas).

In the case of identical bosons no new types of bound states occur (the bound states are due to conspiracy of two contact interactions).

The negative point spectrum is bounded by $-cN$ where the constant $c$ depends on the mass and on the strength of the contact.

Presumably the HKVZ theorem is valid for contact interactions as well as all threshold properties and Mourre estimates but we don’t give an explicit proof.

Since there are no singularities at the bottom of the continuous spectrum, the mapping properties of the wave operator are the usual ones [Y]

Remark 1

The strategy we use has much in common with the method of boundary triplets [B,M,N][D,M] which is used to find self-adjoint extensions of differential symmetric operators defined by
excluding boundaries; the triplet is composed of the operator itself, an operator at the boundary and the (generalized) Weyl function connecting the two.

A basic example in electrostatics is given by the potential and the distribution of charges at the boundary. The theory has been much generalized [D,H,M]

The Krein map (induced by a first order differential operator) has the structure of the abstract Weyl function [D,M].

We shall not elaborate here on this point.

**Remark 2**

For $N = 2$ the approach we use does not work, the reason being that in the center of mass frame the contact manyfold reduces to a point and there is no room for a compact embedding. The ”physical reason” is that for $N = 2$ there is only one contact interaction and ”conspiracy” cannot occur.

The same problem is encountered in electrostatics: the potential due to a ”point charge” is well defined, but in order to define a charge at a point one must consider the distribution of charges over a sphere of radius $\epsilon$ and then let $\epsilon \to 0$ (dropping a diverging term).

**3 Analytic formulation**

We study in the limit $\epsilon \to 0$ a system of $N \geq 3$ particles in $R^3$ which satisfy the Schrödinger equation with hamiltonian

$$H'(V) = -\sum_{k=1}^{N} \frac{1}{2m_k} \Delta_k + \sum_{i,j=1\ldots N} V_{i,j}^\epsilon(|x_i - x_j|)$$

where the potentials scale as $V_{i,j}^\epsilon(|y|) = \epsilon^{-3} V_{i,j}(\frac{|y|}{\epsilon})$ and $V_{i,j}(x)$ are (negative) $L^1$ functions.

The sum is in general over a subset of the indices.

Notice that the $L^1(R^3)$ norm of the potentials does not depend on $\epsilon$ and the potentials converges weakly (in distributional sense) when $\epsilon \to 0$ to a delta distribution at the boundary $\Gamma$.

In the limit $\epsilon \to 0$ the ”potentials” are therefore distributions supported by some of the contact hyper-planes $\Gamma_{i,j} \equiv \{x_i = x_j\}$.

We take the potential to be negative.

The limit is described equivalently by boundary conditions at $\Gamma$.

The equivalence can be seen by taking the scalar product with functions in the range of $\hat{H}_0$ and integrating by parts.

This step is crucial if one wants to see contact interactions as limits of interactions with smaller and smaller support.

This is the same procedure which provides different realizations e.g. of the laplacian on $(0, \infty)$.

We will investigate resolvent convergence and spectral properties.
With our scaling in the limit $\epsilon \to 0$ the potential is formally a distribution supported by some of the *coincidence* hyper-planes $\Gamma_{i,j} \equiv \{ x_i = x_j \}$.

The limit hamiltonian is therefore a self-adjoint extension of $\hat{H}_0$, the free hamiltonian defined on functions that have support away from the coincidence manyfold

$$\Gamma \equiv \bigcup'_{i,j} \Gamma_{i,j} \quad \Gamma_{i,j} \equiv \{ x_i - x_j = 0 \}$$

The prime over the sum indicates that some of the hyper-planes do not contribute to the interaction.

Functions in the domain of the extension may have a singularity $\frac{1}{|x_i - x_j|}$ at $\Gamma_{i,j}$.

We have remarked that contact interactions are extensions of the symmetric operator $\hat{H}_0$ defined as the free hamiltonian restricted to functions with support away from $\Gamma$ [P]. Since this operator is positive, the possible extensions are classified by the theory of Birman, Krein, Visik [B][K].

To analyze the specific extension we consider we use a quadratic-form version of the theory [A,S] and introduce the auxiliary space $\mathcal{M}$ in which the "physical space" is compactly embedded. The space $\mathcal{M}$ is obtained acting on $L^2(\mathbb{R}^{3N})$ with $(\hat{H}_0 + \lambda)^{-\frac{1}{2}}$ where $\hat{H}_0$ is the free hamiltonian. Here $\lambda$ is arbitrary *but strictly positive*. We have called $\mathcal{K}$ this map.

Since to go back to physical space one inverts the map $\mathcal{K}$, the actual value of the parameter $\lambda > 0$ is irrelevant.

The embedding has the advantage that the distributional "boundary potentials" are now less singular (they are $L^1$ functions) and convergence when $\epsilon \to 0$ is easier to prove.

In $\mathcal{M}$ the kernels of the quadratic forms are continuous in $\lambda$ at $\lambda = 0$ and one can set $\lambda = 0$; this simplifies considerably the analysis because the kernels are now homogeneous of order $-1$ in the coordinates.

This helps greatly in keeping the formulae simpler and does not alter the small distance behavior.

We denote $H_M$ the operator in $\mathcal{M}$ which is image of the hamiltonian under the map $\mathcal{K}$.

For the case $N = 3$ the operator $H_M$ has been introduced by R. Minlos [M1][M2] (see also [C1]) in his analysis of contact interactions.

In an unpublished manuscript (private communication) R.Minlos attempted to treat in this way the case of two pairs of identical particles but the formulation in momentum space (typical of the Fadeev formalism) prevented him to reach definite conclusions.

The operator $H_M$ is the difference of two unbounded positive self-adjoint operators associated respectively to the kinetic energy and to a negative potential that has only Coulomb-type singularities.

Depending on the masses and the strength of the potential, the resulting operator may be essentially self-adjoint and positive ("regular" case) or only symmetric and not positive ("singular" case).

In $[M_1][M_2]$ the multiplicity of self-adjoint is derived from the multiplicity of solutions of the equation $(\hat{H}_0^* + 1)u = 0$.

Here we work in configuration space and the multiplicity is seen as a Weyl limit circle property.

In the regular case the proof of convergence in physical space is done by using the compactness of the map $\mathcal{K}$ and the fact that weakly closed positive quadratic forms are strongly closed.

In the singular case some of the components represent quadratic forms in a space of three particle, some in the space of four particles.
Each component can be decomposed using invariance under rotations; after this decomposition the quadratic form is \textit{strictly convex and lower semicontinuous}.

The same properties hold for the quadratic forms of the approximating hamiltonian.

This allows to use Gamma-convergence [Dal] when \( \epsilon \to 0 \); strong resolvent converge follows.

This procedure selects \textit{uniquely} in each channel the limit operator: it is the one with the lowest spectrum among the possible extensions.

The resulting operator \textit{in physical space} has a negative point spectrum; it may consists of infinitely many points, accumulating geometrically at zero (this difference between the spectral properties in \( \mathcal{M} \) and in physical space is due to the difference in metric).

This "Efimov effect" is due to conspiracy of contact interactions. Compare with the fact that in the case of two body smooth interactions with a zero energy resonance this effect is due to conspiracy of zero-energy resonances [K,S].

This correspondence should not come as a surprise since the two effects are both due to a \( \frac{1}{|x_i - x_j|} \) behavior of functions at the boundary (i.e. at \( \Gamma \) for contact interactions, at the sphere at infinity in the case of resonances).

The spectrum of the N-body system with contact interactions depends on the masses and on the statistics.

We will prove that for identical particles the (negative) point spectrum is \textit{completely determined} by the three- and four-particles sub-sectors; this should be expected since the bound states are due to "conspiracy of two two-body contact interactions".

In order to exemplify the method, we analyze in detail the following cases:

1) a pair of identical particles of mass one interacting with a third particle of mass \( m \); we consider both the fermionic and the boson case.

2) two pairs of identical particles of mass 1, either fermions or bosons.

3) \( N \) pairs of identical particles either fermions or bosons.

In the last case we prove that in the case of fermions (unitary gas) the hamiltonian is positive for any value of \( N \).

In the case of bosons the (negative) lower bound of the spectrum is linear in \( N \).

\textit{Remark 1}

In the space \( \mathcal{M} \) the singularity at \( \Gamma_{i,j} \) of the wave functions associated to the continuous part of the spectrum depends on the position, masses and symmetries of all the particles.

They are given now as

\[
\phi(X) = \sum_{i,j} \frac{a_{i,j}(Y)}{|x_i - x_j|} + \sum_{i,j} b_{i,j}(Y) + o\left(\frac{1}{|x_i - x_j|}\right)
\]  

(3)

where \( Y \) represents the other differences of variables and \( A \equiv \{a_{i,j}\}, \ B \equiv \{b_{i,j}\} \) are suitable functions at the boundary.

The functions \( a_{i,j}(Y), b_{i,j}(Y) \) depend on the masses of all the particles; this dependence is entirely due to the Krein map.

There are choices of the masses for which the matrix \( A \equiv a_{i,j}(Y) \) is singular (in some cases as singular as \( \frac{1}{|x_i - x_j| \log(|x_i - x_j|)} \) in some variable).

Notice that these singular boundary conditions occur in \( \mathcal{M} \); \textit{in physical space one has the T-S boundary conditions on function in the continuous spectrum}.

\textit{Remark 2}
A stronger modification of the hamiltonian occurs if one considers simultaneous contact of three identical particles.

Functions in the domain have now in physical space a singularity $\frac{1}{|x_i-x_j||x_i-x_k|}$ at $\Gamma$, for different values of the indices.

In [M,F] the authors consider this case for $N = 3$. It describes in physical space three particles in simultaneous contact interaction.

Also in this case there is a family of dynamics (self-adjoint extensions).

Each self-adjoint extension is unbounded below in physical space and has infinitely many eigenvalues $\mu_i$, $i \in \mathbb{Z}$, which diverge geometrically to $-\infty$.

Formally these extensions could be recovered as limit of hamiltonians with smooth potential $V_{i,j}^\epsilon(|x_i - x_j|)$ which belong uniformly to $L^1$, have support in a region of volume $\epsilon^3$ and are scaled so that the $L^1$ norm stays constant.

A rigorous proof is more difficult.

4 Two identical scalar fermions and a different particle

For three particles we will consider only the case in which a particle of mass $m$ is in contact interaction with two identical particles of mass 1 while these two particles do not interact.

We consider first the case of two identical fermions.

The analysis in $\mathcal{M}$ of the quadratic form of the operator has reported in [M1][M2] [Pa] (see also [C1][C2]).

Due to the symmetry one can consider a quadratic form on a space of functions on $R^3$.

Setting for simplicity $\lambda = 0$ in $\mathcal{M}$ the form is the sum of two terms

$$Q = Q_1 + Q_2$$

$$Q_1(\phi) = \frac{m}{m+1} (\phi, \sqrt{H_0}\phi)$$

$$Q_2(p,q) = -\frac{2}{1+m} (p,q) \frac{(p^2 + q^2)^2 - \frac{4}{(1+m)^2} (p,q)^2}{(p^2 + q^2)^2}$$

These are the quadratic forms in $\mathcal{M}$ that correspond respectively the kinetic energy and to the distributional potential.

These quadratic forms are invariant under rotations and therefore can be analyzed separately in each angular momentum sector.

The mass $m$ of the third particle is the only parameter.

Denote by $Q_l^t$ the restriction of $Q$ to the sector of angular momentum $l$.

This form can be diagonalized by a Mellin transform $[M_1]$.

In [M1][M2] the Author proves that there are constants $m_l^{**}$ and $m_l^*$ such that $Q_l$ s positive for $m > m_l^{**}$ and therefore corresponds to a self-adjoint operator.

If $m \leq m_l^{**}$ the form $Q_l$ ceases to be positive and for $m \leq m_l^*$ it is unbounded bellow.

Estimates for these constants are $[C_1] m_0^* \simeq (13.607)^{-1}$ and $m_0^{**} = (8.62)^{-1}$ if the mass of the fermions is one.
While the two components represent quadratic forms of self-adjoint operators their sum is the quadratic form of an operator that is a priori symmetric but need not be self-adjoint. If \( m \leq m^{**} \) the quadratic form is no longer closed. The symmetric operator associated to the form can be "disintegrated" using rotation invariance into a family of self-adjoint operators with negative point spectrum. For each extension the space of bound states is one-dimensional for \( m^*_i < m \leq m^{**}_i \) and infinite dimensional for \( 0 < m \leq m^*_i \). In this latter case the energy of the energies of the bound states accumulate (geometrically) at minus infinity (Thomas effect).

Recall that the Thomas effect \textit{takes place in} \( \mathcal{M} \).

The presence of infinitely many extensions was noticed first by Danilov [Da] (see also [Pa]). It is demonstrated in \([M_1][M_2]\) by finding in this space infinitely many solutions of an algebraic equation.

The analysis so far refers to the space \( \mathcal{M} \). \textit{To come back to the "physical space"} \( L^2(\mathbb{R}^9) \) one must undo the Krein map. This is not stressed in \([M_1],[M_2]\].

It is mentioned briefly that the map back to physical space changes the metric; due to the difference in metric now the eigenvalues of the three-body bound states accumulate at zero. The use of weak sequential convergence is not explicitly mentioned.

We now show how to obtain these thresholds in \( \mathcal{M} \) by comparing the form we have described with the form of the relativistic Coulomb model \[ \sqrt{-\Delta} - \frac{C(m)}{|x|} \] for a suitable function of the mass.

This hamiltonian has been thoroughly investigated [D,R], [lY]. In this model the plurality of extensions for \( m \leq m^{**} \) is clearly seen as a \textit{Weyl limit-circle effect}.

We have noticed that in \( \mathcal{M} \) all terms are continuous in \( \lambda \) at zero; the positivity of \( \lambda \) was essential only in making the embedding in \( \mathcal{M} \) compact.

Setting \( \lambda = 0 \) one has for the potential term if the two identical particles are (spin zero) fermions

\[
- \frac{1 + m}{m} \frac{1}{(p - q)^2} + \Xi(p, q, \lambda)
\]

where \( \Xi \) is a \textit{positive} smooth kernel with \( \Xi(p, p, 0) = 0 \). Therefore in position space the symmetric operator is

\[
2\pi^2 \frac{m}{m + 1} \sqrt{-\Delta} - \frac{1 + m}{8\pi(2m + m^2)} \frac{1}{|x|} + \tilde{\Xi}'
\]

where \( \tilde{\Xi}' \) is a \textit{positive} operator with locally bounded kernel vanishing on the diagonal. It follows that the form in \( \mathcal{M} \) should be compared with the quadratic form of the symmetric operator

\[
\sqrt{-\Delta} - C(m) \frac{1}{|x|}
\]

where \( C(m) \) is a suitable positive function of the parameter \( m \).

These symmetric operators have been studied extensively [lJ][B,R] as a function of \( C(m) \), originally in the context of the non relativistic hydrogen atom.
Since they are invariant under rotations, their domain can be decomposed into eigenstates of the angular momentum.

In [B,R] it is proven that for each eigenvalue \( l \) of the angular momentum there are threshold \( M_l^* \), \( M_{l}^{**} \) such that for \( m > M_l^{**} \) the spectrum is absolutely continuous and positive.

For \( M_l^* < m \leq M_l^{**} \) there is a continuous family of self-adjoint extensions, each with a negative eigenvalue, and for \( 0 < m \leq M_l^* \) the negative spectrum is pure point and accumulates geometrically to \(-\infty\) (a Weyl limit circle effect).

In the latter case the eigenfunctions concentrate at the origin (eigenvalues and eigenfunctions are known explicitly).

Of course one has \( M_l^* = m_l^* \), \( M_l^{**} = m_l^{**} \).

It is easy to verify that \( m_l^{**} < 1 \) for every \( l \).

For equal masses the hamiltonian is positive.

Recall that these statements hold true in \( \mathcal{M} \).

### 5 Two identical bosons and a third particle

We now consider briefly the case of two identical bosons of mass 1 and a third particle of mass \( m \).

In the case of two identical boson in \( \mathcal{M} \) the potential part of the quadratic form is

\[
Q_2(p,q)' = \frac{-2}{1+m}(p.q) \quad c_f(m) = \frac{4}{(1+m)^2}
\]

Now the quadratic form is the sum of a positive form \( \Xi'' \) with kernel that vanishes on the diagonal and of \( \sqrt{-\Delta} - \frac{C_b(m)}{|x|} \) where \( C_b(m) > C_f(m) \) (\( b \) for bosons).

Dropping the positive form \( \Xi'' \) one obtains the "relativistic atom" hamiltonian.

The system is invariant under rotations and can be decomposed in angular momentum states.

Notice that the expectation value of \( \sqrt{-\Delta} \) depends on the angular momentum of the state.

One can again derive the properties of the spectrum from the spectrum of \( H_R \).

Denote by \( M_l^*(b) \), \( M_{l}^{**(b)} \) the thresholds in the case of bosons.

One verifies that \( 1 < M_0^*(b) \) whereas \( M_j^{**}(b) > 1 \) for all \( j \geq 1 \).

Therefore for two identical bosons of mass 1 which do not interact among themselves and are in contact interaction with a particle of the same mass, in \( \mathcal{M} \) the Thomas effect is present (there are infinitely many extensions in the zero angular momentum sector and their energies diverge geometrically to \(-\infty\)). [B, T].

In physical space this leads to the Efimov effect.

### 6 \( N=3 \); convergence in physical space

Recall once more that this analysis is done in \( \mathcal{M} \), and to draw conclusions relevant for physics one must come back to physical space inverting the map \( K \).

If the limit quadratic form is positive \( \mathcal{M} \), it is also positive in physical space (the map preserves positivity).

In this case the approximating forms in \( \mathcal{M} \) converge strongly when \( \epsilon \to 0 \) and the limit form is strongly closed.
In physical space the convergence is only weak, ad the form is weakly closed. But for positive forms weak closure implies strong closure therefore if the limit form is positive, strong resolvent convergence follows.

If the limit form is not positive, we have seen that there is a continuous family of self-adjoint extensions.

Also here one can consider separately the sectors of fixed angular momentum.

In physical space in each sector, due to the special form of the potential term, the union of the quadratic forms of the self-adjoint extensions is bounded below and strictly convex (the Krein map preserves convexity).

Also the quadratic form of the approximating potentials are bounded below and strictly convex.

The conditions for Gamma-convergence are satisfied.

Recall that the Gamma-limit of a sequence of strictly convex forms $F_n$ in a topological function space $Y$ is the unique form $F$ such that for any subsequence the following holds

$$\forall y \in Y, \ y_n \to y : \ F(y) \leq \liminf_{n \to \infty} F_n(y_n) \quad \limsup_{n \to \infty} F_n(y_n) \geq F(y) \quad (11)$$

In our case the sequence is parametrized by $\{\epsilon_n\}$ where is any sequence of positive numbers that converge to zero.

The limit is independent of the chosen sequence (we express this independence with the notation $\epsilon \to 0$).

Therefore in the singular case for any choice of the masses there is in $L^2(\mathbb{R}^3)$ a distinguished extension (the Gamma-limit) obtained by Gamma-convergence.

In each angular momentum sector this extension is characterized by having the lowest point spectrum among all possible extensions.

Due to the change in metric the "Thomas spectrum" in $\mathcal{M}$ is turned into the "Efimov spectrum" in the physical space $L^2(\mathbb{R}^3)$.

And in this space the eigenfunctions are obtained from those in $\mathcal{M}$ by convolution with the three-particle Green function.

We use now the fact that Gamma-convergence implies strong resolvent of the associated operators [Dal].

Therefore we have proved

**Theorem**

Contact interactions between two identical particles and a third one of the same mass are limits in the strong resolvent sense of hamiltonians with two body potentials that have support that shrinks to a point. If the identical particles are fermions, the limit hamiltonian is positive. If the particles are bosons, the limit hamiltonian is bounded below and its negative point spectrum is of Efimov type (the sequence of eigenvalues converges geometrically to zero. The eigenfunctions are centered on the triple coincidence point.

Resolvent convergence implies, among other things, that bound states converge to bound states.

Therefore contact interaction are a good tool to find (approximately) the location of the bound states for two body potentials which are sharply peaked.

**Remark 1**
From the analysis we have made one sees that if the case of bosons, if mass of the third particle converges to zero the three-body point Efimov spectrum diverges to $-\infty$ in $\mathcal{M}$.

Remark 2
The "tail" of the Efimov states (trimers in the Physics literature) is hardly present in a realistic model, which corresponds to $\epsilon$ very small but not zero. Therefore what can be seen in experiments are a few members of the "head" (low energy states) of the Efimov sequence; they should be recognized as Efimov states for the geometrical scaling property of the energies. This states are less affected by the other short range interactions and on them the effect of the zero-energy resonances might be more visible.

Three body Efimov states have probably be seen experimentally [Pe], [C,M,P]



\section{Two pairs of identical particles}

For generic values of the masses and generic symmetry the quadratic form has a complicated structure.

We consider here only the case of two pairs of particles of equal mass, either fermions or bosons; in this case the quadratic forms are defined on a space of functions of two variables. Again we start the analysis in $\mathcal{M}$.

The kinetic term is again $\sqrt{H_0}$, the potential term is the convolution of the four particle Green function with the distributional potential. The denominator in the convolution can be decomposed in different ways in the sum of squares corresponding to the cases in which a particle does or does not belong to a same pair. In spatial coordinates this allows to isolate first order polar singularity in different positions. Depending on the statistic (bosons or fermions) the different components can be either all negative (for the bosonic case; recall that the potential is negative) or have alternate signs. As a consequence the total quadratic has a kernel that may be positive (fermionic case) or it may have a negative unbounded component with Coulomb singularities located in different positions.

Consider first the system of two pairs of identical scalar fermions of mass one in contact interaction.

One can equivalently consider a system of identical spin $\frac{1}{2}$ fermions.

The generalization to $N$ identical spin $\frac{1}{2}$ fermions will describe the unitary gas.

In $\mathcal{M}$ the quadratic form is the sum a term $C_0$ which represents the kinetic part of form minus three forms $C_1$, $C_2$, $C_3$.

The explicit expressions of these forms in momentum space were known to R.Minlos (private communication).

$C_1$ and $C_2$ are the images in $\mathcal{M}$ of the convolution of the four-particle Green function with two delta singularities of the potential between two identical particles and a third one

\begin{equation}
(\phi, C_1 \phi) = (\phi, C_2 \phi) = \int dk ds dw \phi(k, w) \frac{\phi(s, w) + \phi(k, s)}{k^2 + s^2 + w^2 + (k, s) + (k, w) + (s, w)}
\end{equation}

\[12\]
As in the tree particles case, when written in space coordinates they have a negative Coulomb type singularity in different variables related to the possible triples. Since they refer to pairs of identical spin $\frac{1}{2}$ fermions the contribution are identical. Contributions $C_1$, $C_2$ refer to the three particle case, i.e. two identical fermions and another particle with the same mass, that we have already considered. They differ from the expressions found in the case of three particles because for the Krein map we have used here the resolvent of the free four particle system instead of the one for the three particle case. This difference disappears upon coming back to physical space. In physical space in this contribution the presence of a fourth particle is irrelevant. We stress this fact because it means that in the formation of trimers the other body (or bodies) plays no role. The same will be true in the case of $N$ particles, $N > 4$.

$C_3$ is a genuine four particle term which is not present in the three-particle sector. It represents an effective interaction between the barycenters of the two pairs. The corresponding quadratic form in $M$ is

$$ (\phi, C_3 \phi) = -\int dw ds dk \frac{\bar{\phi}(k,s)\phi(w - \frac{k+s}{2}, -w - \frac{k+s}{2})}{w^2 + \frac{3}{4}(k^2 + s^2) + \frac{1}{2}(k,s)} $$

(13)

It has a simpler when written as a function of the difference of coordinates of the barycenters of the two pairs. In these coordinates it is the image in the four particle sector of $M$ of an "effective" contact interaction between the barycenters of the two pairs.

Notice that only pairs of particles enter in this term. The problem is reduced to the three particle case.

The form can be decomposed into a symmetric and antisymmetric part under interchange of the two pairs.

Only the kinetic energy contributes to the antisymmetric part; this part is positive. The symmetric part can be decomposed as sum of a term which is symmetric and a term which is antisymmetric under interchange of the elements in one of the pairs.

For fermions the symmetric term is positive (only the kinetic energy contributes). From the analysis of the three particle problem it follows that also the antisymmetric term of is positive.

Therefore in the case of two pairs of fermions the quadratic form is positive. Since the Krein map is positivity preserving the same is true in physical space.

We have proved

**Theorem**

The operator associated to a system two pairs of identical fermions in contact interaction is a positive self-adjoint operator in $L^2(R^{12})$. Its hamiltonian is the limit, in the strong resolvent sense, of a sequence of approximating hamiltonians with potentials of decreasing support.

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In [M.P] positivity was conjectured with the aid of a computer.

Consider now the case of two pairs of identical bosons.
Also here there are four terms \( C'_k k = 0, 1, 2, 3 \).
As in the fermionic case, \( C'_1 \) and \( C'_2 \) describe a three-particle system.
\( C'_3 \) is negative.
It is different from zero only in the four-particle sector and has a first order pole in the barycenter of the two pairs.
It can be decomposed in a symmetric and antisymmetric part under interchange of the two pairs.
The antisymmetric part is positive; it represents the contact interaction of two fermions with a boson.
The symmetric part represents the (contact) interaction between two bosons (with reduced mass) with the barycenter.
From the results in the case of three bosons one derives that this term is not positive, convex and has a \(-\frac{1}{|x_i-x_b|}\) singularity.
It can be decomposed using rotation invariance. Each component is strictly convex.
It leads to a one-parameter family of self-adjoint extensions.
When lifted to physical space, under Gamma-convergence it represents a unique self-adjoint operator; its point spectrum represents an Efimov sequence of quadrimers.

Theorem

A system of two pairs of identical bosons in two body contact interaction has an infinite number of three body bound states (trimers) and of four-body bound states (quadrimers); both show the Efimov effect. The hamiltonian of the system is the limit in strong resolvent sense as \( \epsilon \to 0 \) of hamiltonians with potentials of radius \( \epsilon \) and \( L^1 \) norm independent of \( \epsilon \).

As for the three-body case, one expects to see experimentally only the head of the Efimov tail. Four-body Efimov states have been reported [C,M,P][Ba,P].

8 N-body systems

We have considered so far the cases \( N = 3 \) and \( N = 4 \).
Consider now the case of \( N \) identical spin \( \frac{1}{2} \) fermions or \( N \) identical bosons in contact interaction.
We have noticed that the negative part of the spectrum in \( \mathcal{M} \) is entirely due to the conspiracy between pairs of contact interactions.
This effect is independent from the presence of other particles.
From an analytic point of view, this is a consequence of the fact that in \( \mathcal{M} \) the ”kinetic part” is a first order differential operator and its Coulomb capacity is zero.
By additivity, the hamiltonian of a system of \( N \) identical spin \( \frac{1}{2} \) fermions is positive.
Therefore

Theorem

Under contact interaction a gas of \( N \) spin \( \frac{1}{2} \) fermions is stable i.e. its hamiltonian is a positive self-adjoint operator. It is the limit, in strong resolvent sense, of interactions through potentials \( V_{i,j}^\epsilon(|x_i - x_j|) \) that scale as \( V_{i,j}^\epsilon(|y|) = \frac{1}{\epsilon^3}V_{i,j}^\epsilon(|y|) \), \( V_{i,j} \in L^1(R^3) \).
For identical bosons in the space $\mathcal{M}$ the potential corresponding to a contact interaction of each pair has a $\frac{1}{|x_i-x_k|}$ singularity.

In $\mathcal{M}$ there are infinitely many extensions and for each of them there is an infinity of bound states for which the Thomas effect is present.

But also in this case the conspiracy between two contact interactions is independent from the presence of other particles.

Therefore in physical space the hamiltonian is bounded below by $-cN$ and shows the Efimov effect.

The positive constant $c$ depends on the masses and on the strength of the boundary potential.

**Theorem**
The hamiltonian of a system of $N$ identical bosons in contact interaction is bounded below by $-cN$ where the constant $c$ depends on the mass and on the strength of the interaction (the coefficient of the delta function at the boundary). The negative part of the spectrum is composed entirely of Efimov trimers and quadrimers. The hamiltonian is limit, in strong resolvent sense, of hamiltonians with (negative) two-body potentials $V_{ij}^\varepsilon(|x_i-x_j|) = \frac{\varepsilon}{|x_i-x_j|} V_{ij}(\frac{|x_i-x_j|}{\varepsilon})$.

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**Remark**
One can also consider a system composed of $N$ fermions and $M$ bosons, all with the the same mass, interacting through contact interactions.

Only interactions among tree or four of the particles are relevant.

It is then easy to see that for $M = 1$ the system is stable independently of $N \geq 2$; if $M = 2$ and $N \geq 2$ the system has an infinite number of bound states and the Efimov effect is present.

For an outlook on experimental and theoretical results on the Nbody problem one can consult [C,M,P] [C,T] [Pe].

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9 Krein resolvent formula for contact interactons

In quantum mechanics the Krein formula for the difference of two resolvents is used to transform a relation between unbounded operators into a relation between two families of bounded operators; it is mostly used to study convergence.

For regular two-body potentials if we set $H_\varepsilon = H_0 + V_\varepsilon$ the Krein map $K_\lambda$ is defined by duality on quadratic forms by

$$K_\lambda(W) = \left(\frac{1}{H_0 + \lambda}\right)^{\frac{\varepsilon}{2}} V_\varepsilon \left(\frac{1}{H_0 + \lambda}\right)^{\frac{\varepsilon}{2}}$$

where $V_\varepsilon$ is the quadratic form of the potential term.

For smooth potentials the Krein formula for the difference of two resolvent families is

$$\frac{1}{H_\varepsilon + \lambda} - \frac{1}{H_0 + \lambda} = \frac{1}{H_0 + \lambda} W_{\lambda,\varepsilon} \frac{1}{H_0 + \lambda}$$

where $W_{\lambda,\varepsilon}$ is the Krein kernel and $H_\varepsilon$ is the total hamiltonian.
For contact interactions elements \( \psi \) in the domain of \( H_\lambda \) can be written \([A,S]\) as \( \psi = \phi + \xi \) where \( \phi \) is an element the form domain of \( H_0 \) and \( \xi \) square integrable but more singular (it belongs to the space \( \mathcal{M} \)).

Under this decomposition one has

\[
(H_\epsilon + \lambda)\psi = (H^0 + \lambda)\phi_
\]

and the quadratic form of \( H \) can be written \([A,S]\) for \( \lambda = 0 \), as

\[
\Phi(\psi_1, \psi_2) = \Phi_0(\phi_1, \phi_2) + \Xi(\xi_1, \xi_2)
\]

where \( \Phi_0 \) is the quadratic form of \( H_0 \) and \( \Xi \) is a bilinear form in \( \mathcal{M} \).

For \( N \geq 3 \) the Krein kernel for the approximating hamiltonian converges in the limit \( \epsilon \to 0 \).

We can therefore study the resolvent before the limit \( \epsilon \to 0 \) by lifting it to \( \mathcal{M} \) where the "boundary potentials" have a finite \( L^1 \) norm.

In this space it is easy to verify convergence.

Also the Krein kernel converges.

In the regular case the convergence holds also in the physical space because the quadratic form is positive.

Also in the singular case by Gamma-convergence there is a unique limit operator.

Resolvent convergence occurs away from the spectrum of this operator.

Using \( K_\lambda \) for \( \lambda > 0 \) the Krein formula can be written

\[
\frac{1}{H + \lambda} - \frac{1}{H^0 + \lambda} = K_\lambda(K_\lambda(W_\lambda))
\]

where \( W_\lambda = \lim_{\epsilon \to 0} W_{\lambda,\epsilon} \) exists due to the smoothing properties of \( K_\lambda \).

For a general study of the Krein resolvent formula for self-adjoint extensions see [P,R].

**Remark**

For contact interactions there is a natural relation between the Minlos kernel and the quadratic form that characterizes the extension.

To see this, notice that the domain is the sum of a part that belongs to the domain of the free laplacian and a more singular part (it correspond in electrostatics to the contribution of the charges).

The relation is shown by writing in two different ways the energy form.

Let \( H \) be the self-adjoint extension that represent the contact interaction.

Choose \( \lambda \) in such a way that \( H_\lambda = H + \lambda I \) is invertible

Introducing the Krein kernel \( W_\lambda \) defined by

\[
\frac{1}{H + \lambda} = \frac{1}{H^0 + \lambda} + \frac{1}{H^0 + \lambda} W_\lambda \frac{1}{H^0 + \lambda}
\]

one has

\[
(H_\lambda \psi, \frac{1}{H_\lambda} H_\lambda \psi) = (H^0_\lambda \phi, \frac{1}{H_\lambda} H^0_\lambda \phi) = (\phi, H^0_\lambda \phi) + (K_\lambda \phi, K_\lambda \phi)
\]

This relation should be compared with (17) \([A,S]\).
10 Further comments

We conclude with an outline of possible developments.

Notice the similarity of the scaling of the potential with parameter $\epsilon$ (the inverse of the scattering length) with the scaling $V_N(x) = N^{-1}(N^3V(|x|/N))$ which is used in the study of the fluctuations of $N$-particle quantum systems, for $N$ very large, in the Gross-Pitaeuskii regime (see e.g. [B,C,S]).

The coefficient $N^{-1}$ is introduced in [B,C,S] to balance the kinetic and the potential terms.

In our case the extra factor $N^{-1}$ is introduced to take care of the lower bound in the spectrum of a bosonic $N$-body system with contact interactions.

This permits to see the landscape of the bound states.

If the gas of particles is very diluted, ”most of the non-free configurations” correspond to interaction of two particles.

A priori the parameters $\epsilon$ (inverse of the scattering length of the potential) and $1/N$ are independent parameters.

For a system with a large number $N$ of particles it is natural to take $\epsilon \simeq 1/N$. This allows to use both Fock space techniques [B,C,S] and techniques related to the Schrödinger equation for a many body system with regular potential of very small (contact interactions).

The choice of consider first the limit $\epsilon \to 0$ and then the limit of very large $N$, has some advantages.

It permits to prove that the only bound states present for any value of $N$ are three- and four-body bound states. Their total energy is bounded uniformly by $-cN$.

Recall that the explanation given in [B,C,S] is that this factor is needed to give equal weight to the kinetic and the potential term in the limit $N \to \infty$.

Consider now a system of $N$ identical bosons of mass one in contact interaction and restrict attention to the case in which two particles can be very close (in the limit, in contact) but the system is so diluted that the probability that find another particle nearby is negligible.

The equation for contact interaction is defined also for a system of two isolated particles.

If we assume that the probability of interaction of the two particles is proportional to the probability that they be simultaneously present, contact interaction leads in physical space to the non linear equations for the wave functions of the two particles

\[
\begin{align*}
    i\frac{\partial}{\partial t}\phi_1(t, x) &= \frac{1}{m_1}\Delta \phi_1(t, x) + c|\phi_2(t, x)|^2\phi_1(t, x) \\
    i\frac{\partial}{\partial t}\phi_2(t, x) &= \frac{1}{m_2}\Delta \phi_2(t, x) + c|\phi_1(t, x)|^2\phi_2(t, x)
\end{align*}
\]

(21)

where $c$ is a coupling constant.

These are the equations of two particles in contact interaction.

Since contact interactions are strong resolvent limit of interactions through short range potentials when the range goes to zero, the solutions of these effective equations, if unique, are strong limits, as $\epsilon \to 0$ of the solutions of the equation for the system of the two selected particles under the assumption on the initial data that the probability distributions are identical.

Setting $|\phi_1(x)|^2 = |\phi_2(x)|^2$ one can regard these equations as effective equations for a gas of identical particles if the gas is very diluted and the interaction is of range so small that one can omit the interactions of more than two particles.
The resulting equation for each particle in the pair is

\[ i \frac{\partial}{\partial t} \phi(t, x) = -\frac{1}{m} \Delta \phi(t, x) + c|\phi(t, x)|^2 \phi(t, x) \]  

(22)

The equation is meant to describe the dynamics of the one-particle marginals for a gas of identical particles if the gas is very diluted and the range of the interaction is so small that it can be described as a contact interaction.

On the other hand, the solutions to this equation may be regarded as describing, in the contact interaction limit, the solutions of a linear equation for a pair of identically distributed particles. Notice that the resulting equation is linear for each member of the pair, but it is the pair structure that is relevant.

11 Conclusions

We have proved that for a Schrödinger system of three or more particles in contact interactions the Hamiltonians are self-adjoint operators which are constructed considering a natural extensions of \( \hat{H}_0 \), a symmetric operator defined on functions with support away from a subset of coincidence planes \( \Gamma_{i,j} \) (defined by \( x_i = x_j \)).

These operators are limits, in the strong resolvent sense, of Schrödinger Hamiltonians with two-body potentials with smaller and smaller support and constant \( L^1 \) norm.

If there are bound states this is proved by Gamma-convergence [Dal].

The negative (point-) spectrum of the Hamiltonian for contact interactions is completely determined by the structure of the three-body and four-body subsystems.

Functions in the continuous spectrum of the contact Hamiltonian satisfy T-S boundary conditions at \( \Gamma_{i,j} \).

The bound states are due to conspiracy of two-body contact interactions; three and four-body bound states are Efimov states.

Acknowledgments

Comments and constructive criticism by several friends at an early stage of the research were very helpful and are gratefully acknowledged.

I’m grateful to R.Minlos for valuable correspondence and to K.Yajima for the very warm hospitality at Gakushuin University.

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12 Appendix : contact interaction vs point interaction

If the two-body potential has a zero energy resonance a different hamiltonian of "zero range interaction" was introduced in [A]; the resulting system is called point interaction. The hamiltonian is obtained as limit of $H_0 + V_\epsilon$ where $V_\epsilon(|x|) = \epsilon^{-2}V(|x|/\epsilon)$ and the potential $V(|y|)$ has a zero energy resonance. Notice that in the limit the $L^1$ norm vanishes. It is proved in [A] that this Hamiltonian is self-adjoint.
The proof requires an accurate balancing in the Krein resolvent formula of the divergence due to the resonance and the vanishing of the $L^1$ norm of the potential.

In order to see the relation between contact interaction and point interaction, it is convenient to remark that zero energy resonances represent directions in which the wave function has a behavior.

This is the same behavior as that at $\Gamma$ for contact interactions.

Therefore in conformal coordinates (the laplacian in dimension 3 is conformal covariant), resonances can be seen as contact interaction at infinity between a pair of particles.

In this sense the Efimov effect for contact interactions has the same origin (conspiracy of contact interactions) as the effect with the same name in low energy nuclear physics [A.S] (conspiracy of zero energy resonances).

In the case of two particles of equal mass which are in contact interaction and have a zero energy (Feshbach) resonance, weakly bound pairs (Cooper pairs) are the counterpart of quadrimers.

A resonance between two particles is due to a two body potential with infinite scattering length.

Denoting by $\epsilon$ the inverse of the scattering length, the resonance between two particles provides a factor $\frac{1}{\epsilon}$ that makes the volume integral finite.

If one of the particle has very large mass $\frac{1}{\epsilon}$ the kinetic energy becomes small for a large spectrum of momenta and it is convenient to scale the masses by a factor $\epsilon^{-1}$.

This does not affect the resonances and changes the scaling of the potential; it now scales as $V'(|x|) = \frac{1}{\epsilon^2} V\left(\frac{|x|}{\epsilon}\right)$.

On the new scale he system consists of a particle of very small mass in contact interaction with a particle of mass one in presence of a zero energy resonance (zero energy resonances are scale invariant),

In the limit $\epsilon \to 0$ the particle of mass $\frac{1}{\epsilon}$ can be considered fixed at the origin and one recovers in the limit the hamiltonian of point interaction described in [A].

In physical space the hamiltonian obtained by Gamma-convergence is bounded below and has Efimov spectrum; but notice that before the limit the eigenvalues are spaced of order $\epsilon$ since the potential is reduced by a factor $\epsilon$.

When the spectrum has a negative part it is bounded below and each point is an accumulation point.

Therefore on the negative axis the spectral measure is continuos but not absolutely continuous and has a singularity at the the origin (the onset of the continuous spectrum).

If the hamiltonian is positive the spectral measure has a singularity at zero(this is due to the resonance).

This "explains" the the "anomalous" mapping properties of the Wave operator for point interactions [A].

A simpler way to find a relation between contact and point interactions is to notice that zero energy resonances have long time scale effects.

Consider a system composed of particle $A$ of mass one and particle $B$ of mass $\frac{1}{\epsilon}$ and particle interacting through a potential $V_\epsilon = \frac{1}{\epsilon^2} V\left(\frac{|x|}{\epsilon}\right)$, $V(x) \in L^1(R^3)$ that admits a zero energy resonance (this is true independently of $\epsilon$).

The dynamics is uniquely defined for arbitrary small values of $\epsilon$. 

We scale time setting $\tau = \epsilon t$ so that on the $\tau$ time scale the displacement of particle $A$ is of order $\epsilon^{-1}$.

In the limit $\epsilon \to 0$ particle $B$ can be regarded as fixed at the origin.

With this approximations \textit{and on the new time scale} the Schrödinger equation reads

$$i \frac{\partial}{\partial \tau} \phi = -\Delta \phi + \epsilon^{-2} V\left(\frac{|x|}{\epsilon}\right) \phi$$ (23)

In the absence of zero energy resonances the dynamics converges in the limit $\epsilon \to 0$ to free motion.

If a zero energy resonance is present, in the limit $\epsilon \to 0$ \textit{and in the new time scale} the dynamics is given the Schrödinger equation for a particle of unit mass interacting through a \textit{point interaction} with a particle fixed at the origin.

Therefore point interaction with a particle fixed at the origin can be regarded as an approximate description of the asymptotic (in time) dynamics of a particle of very small mass interacting with a particle of large mass through a potential of very short range that gives rise to a zero energy resonance.

Notice that after a time of order $\epsilon^{-1}$ the motion is almost free (ballistic) if the two-body potential has no zero energy resonance and is described by a point interactions if there is a zero energy resonance.

If one recalls that the spectrum of point interaction is singular at the origin this justifies the mapping properties of the Wave Operator for point interaction [DMSY].

\textbf{Remark}

We add here a comment on the relation between our procedure and an analysis described in [E,T] using "heat kernel regularization".

This is a procedure that regularizes by introducing in the expectation value of the operators a gaussian regularizing factor $e^{-\tau (H_0 + \lambda)}$ and taking the limit $\tau \to \infty$.

In a quadratic form analysis this map is precisely what we have called Krein map.

One has now to come back to the "physical space". This must be the meaning of the "renormalization procedure" in [E,T].