Approximation of fractional Brownian motion by martingales

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Abstract We study the problem of optimal approximation of a fractional Brownian motion by martingales. We prove that there exist a unique martingale closest to fractional Brownian motion in a specific sense. It shown that this martingale has a specific form. Numerical results concerning the approximation problem are given.

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1 Introduction

Let $B^H = \{B^H_t, \mathcal{F}_t^B; t \in [0, 1]\}$ be a fractional Brownian motion with Hurst index $H \in (0, 1)$. It means that $B^H$ is a centered Gaussian process with a covariance function $E[B^H_tB^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$. It is well known that a fractional Brownian motion is neither a semimartingale nor a Markov process unless $H = 1/2$. So a simple and natural question is how far is Brownian motion from being a martingale? That is, in a sense, we look for the projection of fractional Brownian motion on the space of (square integrable) martingales. Thus, initially, the problem is formulated in such a way: we are looking for a square integrable $\mathcal{F}_B^B$-martingale $M$ that minimizes the value

$$d_H(M)^2 := \sup_{t \in [0, 1]} E(B^H_t - M_t)^2.$$ 

To proceed with the solution of this problem, we can use the representation of the fractional Brownian motion via the standard Brownian motion on the finite interval $[0, 1]$. Introduce the kernel

$$K(t, s) = C_\alpha \left(t^\alpha s^{-\alpha}(t - s)^\alpha - \alpha s^{-\alpha} \int_s^t u^{\alpha - 1}(u - s)^\alpha du\right)_{1 \leq s < t \leq 1},$$

where $C_\alpha = \alpha \left(\frac{2\alpha + 1}{\Gamma(\alpha + 1)\Gamma(1 - \alpha)}\right)^{1/2}$, $\Gamma$ is the Gamma function, $\alpha = H - 1/2$. Then there exists $\mathcal{F}_B^B$-Wiener process $W = \{W_t, \mathcal{F}_t^B; t \in [0, 1]\}$ such that $B^H$ admits the representation

$$B^H_t = \int_0^1 K(t, s)dW_s = \int_0^t K(t, s)dW_s$$

$$= C_\alpha \int_0^t \left(t^\alpha s^{-\alpha}(t - s)^\alpha - \alpha s^{-\alpha} \int_s^t u^{\alpha - 1}(u - s)^\alpha du\right)dW_s.$$ 

In what follows we consider fractional Brownian motion with $H \in (1/2, 1)$, and in this case the kernel $K(t, s)$ has a simpler form:

$$K(t, s) = C_\alpha s^{-\alpha} \int_s^t u^\alpha(u - s)^{-\alpha}dudW_s, \quad 0 \leq s < t \leq 1.$$ 

(2)

Turning back to our problem, we observe first that $B^H$ and $W$ generate the same filtration, so any square integrable $\mathcal{F}_B^B$-martingale $M$ admits a representation

$$M_t = \int_0^t \alpha dW_s,$$ 

(3)
where $\alpha$ is an $\mathcal{F}_H$-adapted square integrable process. Hence we can write
\[
\mathbb{E}((B_H^t - M_t)^2) = \mathbb{E}\left(\int_0^t (K(t,s) - \alpha_s) dW_s\right)^2 = \int_0^t \mathbb{E}(K(t,s) - \alpha_s)^2 ds
\]
\[
= \int_0^t (K(t,s) - \mathbb{E}\alpha_s)^2 ds + \int_0^t \text{Var}(\alpha_s) ds.
\]
Consequently, it is enough to minimize $d_H(M)$ over Gaussian martingales, i.e. those having representation (3) with a non-random $\alpha$.

So, the main problem reduces to the following one:

(A) Find
\[
\inf_{a \in L^2([0,1])} \sup_{t \in [0,1]} \left(\int_0^t (K(t,s) - a(s))^2 ds\right)
\]
and a minimizing element $a \in L^2([0,1])$ if the infimum is attained.

Note that the expression being minimized does not involve neither the fractional Brownian motion nor the Wiener process, so the problem becomes purely analytic.

The paper is organized as follows. Sections 2 and 3 are devoted to the general problem of minimization of the functional $f$ on $L^2([0,1])$ that has the following form
\[
f(x) = \sup_{t \in [0,1]} \left(\int_0^t (K(t,s) - x(s))^2 ds\right)^{1/2}
\]
with arbitrary kernel $K(t,s)$ satisfying condition

(B) for any $t \in [0,1]$ the kernel $K(t, \cdot) \in L^2([0,t])$ and
\[
\sup_{t \in [0,1]} \int_0^t K(t,s)^2 ds < \infty.
\]
We shall call this functional the principal functional. It is proved in Section 2 that the principal functional $f$ is convex, continuous and unbounded on infinity, consequently, the minimum is attained. Section 3 gives an example of kernel $K(t,s)$ where a minimizing function for principal functional is not unique (moreover, being convex, the set of minimizing functions is infinite). Sections 4–6 are devoted to the problem of minimization of principal functional $f$ with the kernel $K$ corresponding to fractional Brownian motion, i.e., with the kernel $K$ from (3). It is proved in Section 4 that in this case the minimizing function for the principal functional is unique. In Section 5 it is proved that the minimizing function has a special form. Section 6 contains some numerical results.

2 The existence of minimizing function for the principal functional

In this section we consider arbitrary kernel $K$ satisfying assumption (B), which implies that the functional $f$ is well defined for any $x \in L^2([0,1])$. 
Lemma 1 For any \( x, y \in L_2([0, 1]) \)
\[
|f(x) - f(y)| \leq \|x - y\|_{L_2([0, 1])}. \tag{6}
\]

Proof Evidently, for any \( x, y \in L_2([0, 1]) \) and \( 0 \leq t \leq 1 \)
\[
\left( \int_0^t (K(t, s) - x(s))^2 \, ds \right)^{1/2} \leq \left( \int_0^t (x(s) - y(s))^2 \, ds \right)^{1/2} + \left( \int_0^t (K(t, s) - y(s))^2 \, ds \right)^{1/2}.
\]
Therefore
\[
\sup_{t \in [0, 1]} \left( \int_0^t (K(t, s) - x(s))^2 \, ds \right)^{1/2}
\leq \sup_{t \in [0, 1]} \left( \int_0^t (x(s) - y(s))^2 \, ds \right)^{1/2} + \sup_{t \in [0, 1]} \left( \int_0^t (K(t, s) - y(s))^2 \, ds \right)^{1/2},
\]
which is clearly equivalent to the inequality
\[
f(x) \leq \|x - y\|_{L_2([0, 1])} + f(y).
\]
Swapping \( x \) and \( y \), we get the proof.

Corollary 1 The functional \( f \) is continuous on \( L_2([0, 1]) \).

Lemma 2 The following inequalities hold for any function \( x \in L_2([0, 1]) \):
\[
\|x\|_{L_2([0, 1])} - \|K(1, \cdot)\|_{L_2([0, 1])} \leq f(x) \leq \|x\|_{L_2([0, 1])} + f(0). \tag{7}
\]

Proof The left-hand side of (7) immediately follows from the inequalities
\[
f(x) \geq \int_0^1 (K(1, s) - x(s))^2 \, ds \right)^{1/2} = \|K(1, \cdot) - x\|_{L_2([0, 1])}
\geq \|x\|_{L_2([0, 1])} - \|K(1, \cdot)\|_{L_2([0, 1])},
\]
and the right-hand side follows from (6).

Lemma 3 Functional \( f \) is convex on \( L_2([0, 1]) \).

Proof We have to prove that for any \( x, y \in L_2([0, 1]) \) and any \( \alpha \in [0, 1] \)
\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).
\]
applying the triangle inequality, we have for any \( t \in [0, 1] \)
\[
\left( \int_0^t (\alpha x(s) + (1 - \alpha)y(s) - K(t, s))^2 \, ds \right)^{1/2}
\leq \left( \int_0^t (\alpha (K(t, s) - x(s))^2 \, ds \right)^{1/2} + \left( \int_0^t ((1 - \alpha)(K(t, s) - y(s))^2 \, ds \right)^{1/2},
\]
whence
\[
\sup_{t \in [0, 1]} \left( \int_0^t (\alpha x(s) + (1 - \alpha)y(s) - K(t, s))^2 \, ds \right)^{1/2}
\leq \alpha \sup_{t \in [0, 1]} \left( \int_0^t (K(t, s) - x(s))^2 \, ds \right)^{1/2} + (1 - \alpha) \sup_{t \in [0, 1]} \left( \int_0^t (K(t, s) - y(s))^2 \, ds \right)^{1/2},
\]
and the proof follows.
**Theorem 1** Functional $f$ attains its minimal value on $L_2([0, 1])$.

**Proof** By Corollary 1 and Lemma 3 the functional $f$ is continuous and convex. By Lemma 2, $f(x)$ tends to $+\infty$ as $\|x\| \to \infty$. Hence it follows from Proposition 2.3 that $f$ attains its minimal value.

3 An example of the principal functional with infinite set of minimizing functions

Note that the set $\mathfrak{M}_f$ of minimizing functions for functional $f$ is convex. In this section we consider an example of kernel $K$ for which $\mathfrak{M}_f$ contains more than one point, consequently, is infinite. At first, establish the following lower bound for functional $f$.

**Lemma 4** 1. Let the kernel $K$ of functional $f$ defined by (4) satisfy assumption (B). Then for any $a \in L_2([0, 1])$ and $0 \leq t_1 < t_2 \leq 1$ the following inequality holds

$$\sup_{t \in [0,1]} \int_0^t (K(t,s) - a(s))^2 \, ds \geq \frac{1}{4} \int_0^{t_1} (K(t_2,s) - K(t_1,s))^2 \, ds. \quad (8)$$

2. The equality in (8) implies that

$$a(s) = 1/2(K(t_1,s) + K(t_2,s)) \quad \text{a.e. on } [0,t_1],$$

$$a(s) = K(t_2,s) \quad \text{a.e. on } [t_1,t_2]. \quad (9, 10)$$

**Proof** 1. Following inequalities are evident:

$$\sup_{t \in [0,1]} \int_0^t (K(t,s) - a(s))^2 \, ds$$

$$\geq \max \left\{ \int_0^{t_1} (K(t_1,t) - a(s))^2 \, ds, \int_0^{t_2} (K(t_2,t) - a(s))^2 \, ds \right\}$$

$$\geq \max \left\{ \int_0^{t_1} (K(t_1,s) - a(s))^2 \, ds, \int_0^{t_1} (K(t_2,s) - a(s))^2 \, ds \right\}$$

$$\geq \frac{1}{2} \int_0^{t_1} ((K(t_1,s) - a(s))^2 + (K(t_2,s) - a(s))^2) \, ds. \quad (11)$$

From $(P + Q - 2r)^2 \geq 0$ we immediately get

$$2 \left( \frac{P - r}{2} \right)^2 + 2 \left( \frac{Q - r}{2} \right)^2 \geq \frac{(P - Q)^2}{4}. \quad (12)$$

Setting $P = K(t_1,s)$, $Q = K(t_2,s)$ and $r = a(s)$ in this inequality, we get from (11)

$$\sup_{t \in [0,1]} \int_0^t (K(t,s) - a(s))^2 \, ds \geq \frac{1}{2} \int_0^{t_1} ((K(t_1,s) - a(s))^2 + (K(t_2,s) - a(s))^2) \, ds$$

$$\geq \frac{1}{4} \int_0^{t_1} (K(t_2,s) - K(t_1,s))^2 \, ds. \quad (13)$$
Thus, inequality (8) is proved.

2. We now show that equality in (8) implies (9) and (10). Indeed, equality in (12)
holds if and only if \( P + Q - 2r = 0 \). Equality in (13) has a form
\[
1/2 \int_0^{t_1} (K(t_1, s) - a(s))^2 + (K(t_2, s) - a(s))^2 ds = 1/4 \int_0^{t_1} (K(t_1, s) - K(t_2, s))^2 ds
\]
and holds if and only if
\[
K(t_1, s) + K(t_2, s) - 2a(s) = 0 \quad \text{a.e. on } [0, t_1),
\]
i.e. it holds if and only if condition (9) holds.

If (9) holds, then
\[
\int_0^{t_1} (K(t_1, s) - a(s))^2 ds = \frac{1}{4} \int_0^{t_1} (K(t_1, s) - K(t_2, s))^2 ds,
\]
and
\[
\int_0^{t_2} (K(t_2, s) - a(s))^2 ds = \frac{1}{4} \int_0^{t_1} (K(t_2, s) - K(t_1, s))^2 ds + \int_0^{t_2} (K(t_2, s) - a(s))^2 ds.
\]
It means that under condition (9) equality (8) holds only if
\[
\int_{t_1}^{t_2} (K(t_2, s) - a(s))^2 ds = 0,
\]
i.e. only if (10) holds.

**Remark 1** Let the kernel \( K \) of functional \( f \) from (4) satisfy assumption (A). Then for any \( a \in L_2([0, 1]) \) and \( 0 \leq t_1 < t_2 \leq 1 \)
\[
\max_{t \in \{t_1, t_2\}} \int_0^t (K(t, s) - a(s))^2 ds \geq \frac{1}{4} \int_0^{t_1} (K(t_2, s) - K(t_1, s))^2 ds. \tag{14}
\]
Equality in (14) holds if and only if (9) and (10) hold.

**Example 1 (Functional \( f \) with infinite set \( \mathcal{M}_f \))** Take the kernel \( K(t, s) \) of the form
\[
K(t, s) = g(t)h(s), t, s \in [0, 1],
\]
where
\[
g(t) = (6t - 2)1_{\frac{1}{4} \leq t \leq \frac{1}{2}} + (4 - 6t)1_{\frac{1}{2} \leq t \leq \frac{3}{4}} + (6t - 6)1_{\frac{3}{4} \leq t \leq 1}
\]
and
\[
h(s) = 4s1_{0 \leq s \leq \frac{1}{4}} + (2 - 4s)1_{\frac{1}{4} \leq s \leq \frac{1}{2}}.
\]

Then
\[
\min_{a \in L_2([0, 1])} \max_{t \in [0, 1]} \int_0^t (K(t, s) - a(s))^2 ds = 1/6, \tag{15}
\]
and \( \mathcal{M}_f \) consists of functions \( a(s) \) satisfying the conditions
\[
a(s) = 0 \quad \text{a.e. on } [0, 5/6] \tag{16}
\]
and
\[
\int_{5/6}^t a(s)^2 ds \leq 1/6 - 6(1 - t)^2, \quad 5/6 \leq t \leq 1. \tag{17}
\]
Remark 2. Since $K \in C([0, 1]^2)$ and $a \in L_2([0, 1])$, we have that $\int_0^1 (K(t, s) - a(s))^2 ds$ is continuous in $t$, therefore we can replace $\sup_{t \in [0, 1]}$ with $\max_{t \in [0, 1]}$ in inequality (15).

2. Some examples of functions satisfying (16) and (17): $a(s) = 0, s \in [0, 1]; a(s) = (12(1 - s))^{1/2}1_{5/6<s\leq 1}; a(s) = \sqrt{3(6s - 5)}1_{5/6<s\leq 1}$.

To establish a lower bound on the left-hand side of (15), note that $\int_0^t h(s)^2 ds = 1/6$ for $1/2 \leq t \leq 1$. Therefore, applying Lemma 4 with $t_1 = 1/2$ and $t_2 = 5/6$ we obtain that

$$\sup_{t \in [0, 1]} \int_0^t (K(t, s) - a(s))^2 ds \geq \frac{1}{4} \int_0^{1/2} (K(5/6, s) - K(1/2, s))^2 ds$$

$$= \frac{1}{4} \int_0^{1/2} (g(5/6)h(s) - g(1/2)h(s))^2 ds = \frac{1}{4} \int_0^{1/2} 4h(s)^2 ds = 1/6. \quad (18)$$

Moreover, functions $a(s)$ satisfying (16) and (17) transform (18) into equality.

To establish an upper bound of the left-hand side of (15), consider functions satisfying conditions (16) and (17). Then for $0 \leq t \leq 5/6$ we have that

$$\int_0^t (K(t, s) - a(s))^2 ds = \int_0^t K(t, s)^2 ds = \int_0^t g(t)^2 h(s)^2 ds$$

$$= g(t)^2 \int_0^t h(s)^2 ds \leq \int_0^{5/6} h(s)^2 ds = 1/6,$n

since $a(s) = 0$ on $[0, 5/6]$ and $g(t)^2 \leq 1$. For $5/6 < t \leq 1$, we take into account the values of $a, h$ and $g$ on this interval and obtain that

$$\int_0^t (K(t, s) - a(s))^2 ds = \int_0^{5/6} (g(t)h(s) - a(s))^2 ds + \int_{5/6}^t (g(t)h(s) - a(s))^2 ds$$

$$= \int_0^{5/6} g(t)^2 h(s)^2 ds + \int_0^{5/6} a(s)^2 ds = g(t)^2 \int_0^{5/6} h(s)^2 ds + \int_0^{5/6} a(s)^2 ds$$

$$\leq (6t - 6)^2 \cdot 1/6 + 1/6 - 6(1 - t)^2 = 1/6. \quad (19)$$

Hence, if function $a$ satisfies (16) and (17), we have that

$$\sup_{t \in [0, 1]} \int_0^t (K(t, s) - a(s))^2 ds \leq 1/6.$$

Summing up, we obtain (15).

Now we prove that any minimizing function $a$ satisfies (16) and (17).

Indeed, let

$$\sup_{t \in [0, 1]} \int_0^t (K(t, s) - a(s))^2 ds = 1/6.$$

Then inequality (15) is transformed into equality, therefore

$$\sup_{t \in [0, 1]} \int_0^t (K(t, s) - a(s))^2 ds = \frac{1}{4} \int_0^{1/2} (K(5/6, s) - K(1/2, s))^2 ds. \quad (20)$$
It follows from (20) and from the 2nd part of Lemma 4 that

$$a(s) = \frac{1}{2} (K(5/6, s) + K(1/2, s)) = \frac{1}{2} (h(5/6) + h(1/2)) = 0$$

a.e. on $[0, 1/2]$ because $g(1/2) = 1$, $g(5/6) = -1$; we obtain also the equality

$$a(s) = K(5/6, s) = g(5/6)h(s) = 0$$

a.e. on $[1/2, 5/6]$ because $h(s) = 0$ for $s \geq 1/2$. Therefore, function $a$ satisfies condition (16). Then we can get similarly to (19) that

$$\int_0^t (K(t, s) - a(s))^2 ds = \frac{(6t - 6)^2}{6} + \int_{5/6}^t a(s)^2 ds$$

and it follows from inequality $\int_0^t (K(t, s) - a(s))^2 ds \leq 1/6$ that

$$\int_{5/6}^t a(s)^2 ds \leq 1/6 - \frac{(6t - 6)^2}{6} = 1/6 - 6(1-t)^2$$

for $5/6 < t \leq 1$.

It means that function $a$ satisfies condition (17).

4 Uniqueness of the minimizing function for the kernel connected to fractional Brownian motion

Now we return to the main problem (A) of approximation of fractional Brownian motion by martingales.

First we prove some simple but useful properties of the fractional Brownian kernel $K$ defined by (2).

**Lemma 5 (Properties of the fractional Brownian kernel)** 1. Kernel $K$ satisfies condition (B).

2. Kernel $K$ increases in the first argument and decreases in the second argument.

3. Kernel $K$ is continuous on the set $[0, 1] \times (0, 1]$.

4. For any $c > 0$ and $0 < s \leq t$ we have that $K(ct, cs) = c^{\alpha} K(t, s)$ with $\alpha = H - 1/2$.

**Proof** 1. Since $K$ is the kernel of fractional Brownian motion, we have that

$$t^{2H} = E(B_t^H)^2 = E \left( \int_0^t K(t, s)dW_s \right)^2 = \int_0^t K(t, s)^2 ds.$$ 

Therefore, $\sup_{t \in [0,1]} \int_0^t K(t, s)^2 ds = 1$, and (5). Other statements follow directly from (2).

**Theorem 2** For any function $a \in \mathcal{M}_f$ there exists such function $\phi : [0, 1] \to \mathbb{R}$ that

$s \leq \phi(s) \leq 1, s \in [0, 1]$ and $a(s) = K(\phi(s), s)$ a.e.
Proof. Let \( a \in \mathcal{M}_f \). Consider the function \( b(s) = \min(K(1,s), \max(0, a(s)), s \in [0,1] \). Since the kernel \( K \) is nonnegative, then

\[
(a(s) - K(t,s))^2 \geq (\max(0, a(s)) - K(t,s))^2, t, s \in [0,1]
\]

and this inequality is strict on a set of positive Lebesgue measure if \( a(s) < 0 \) on a set of positive Lebesgue measure. Moreover, since the kernel \( K \) is increasing in the first argument, we have that

\[
(a(s) - K(t,s))^2 \geq (\min(K(1,s), a(s)) - K(t,s))^2, t, s \in [0,1]
\]

and this inequality is strict on the set of positive Lebesgue measure if \( a(s) > K(1,s) \) on a set of positive Lebesgue measure. Therefore, \( f(b) \leq f(a) \) and this inequality is strict if \( a(s) < 0 \) or \( a(s) > K(1,s) \) on a set of positive Lebesgue measure. Therefore,

\[
0 = K(s,s) \leq a(s) \leq K(1,s), s \in [0,1].
\]

Since the kernel \( K \) is continuous in the first argument, there exists a function \( s \leq \phi(s) \leq 1, s \in [0,1] \), such that \( a(s) = K(\phi(s), s) \).

Corollary 2 Functions in the set \( \mathcal{M}_f \) are nonnegative.

Now we are in position to establish the uniqueness of minimizing function for the principal functional corresponding to the kernel of fractional Brownian motion. In order to do this, prove at first the auxiliary statement concerning any minimizing function for this functional. For \( x \in L_2([0,1]) \), denote

\[
g_a(t) = \left( \int_0^t (K(t,s) - x(s))^2 \, ds \right)^{1/2}.
\]

Then we have from the definition of the principal functional \( f \) that \( f(x) = \sup_{c \in [0,1]} g_a(t) \). It follows from Lemma 5 that \( g_a \in C[0,T] \) for any \( x \in L_2[0,T] \). Using self-similarity property 4) of the kernel \( K \), it is easy to see that

\[
g_a(t) = c^{\alpha+1/2} g_{c^{-\alpha}a(a)}(t/c).
\]

Lemma 6 Let \( a \in \mathcal{M}_f \). Then the maximal value of \( g_a \) is attained at the point 1, i.e. \( f(a) = g_a(1) \).

Proof. Set \( a(t) = 0 \) for \( t > 1 \). Suppose that \( g_a(1) < f(a) \). Since \( g_a(t) \) is continuous in \( t \), there exists such \( c > 1 \) that \( g_a(t) < f_a \) for \( t \in [1, c] \). It means that \( \max_{x \in [0,c]} g_a(t) = f_a \). Set \( b(t) = c^{-\alpha} a(tc) \). It follows from equation (21) that \( g_b(t) = c^{-1/2-\alpha} g_a(tc) \), \( t \in [0,1] \). We get immediately that \( f(b) = c^{-\alpha-1/2} f(a) < f(a) \), which leads to a contradiction.

Theorem 3 (Uniqueness of minimizing function) For the principal functional \( f \) defined by (4) with fractional Brownian kernel \( K \) from (2), there is a unique minimizing function.
Proof Denote $M_f$ the minimal value of functional $f$. Recall that the set $\mathcal{M}_f$ is nonempty and convex. Let $\hat{K}(s) = K(1,s), s \in [0,1]$. It follows from Lemma 6 that for any function $x \in \mathcal{M}_f$ the following equality holds:

$$f(x) = \left( \int_0^1 (x(s) - K(1,s))^2 ds \right)^{1/2} = \|x - \hat{K}\|_{L^2([0,1])}.$$ 

For any $x, y \in \mathcal{M}_f, \alpha \in (0,1)$ we have that

$$M_f = f(\alpha x + (1-\alpha)y) = \|\alpha x + (1-\alpha)y - L\|_{L^2([0,1])} \leq \alpha \|x - L\|_{L^2([0,1])} + (1-\alpha)\|y - L\|_{L^2([0,1])} = \alpha f(x) + (1-\alpha)f(y) = M_f.$$ 

For arbitrary vectors $x$ and $y$ in a Hilbert space the equality $\|x + y\| = \|x\| + \|y\|$ implies that $x$ and $y$ differ by a non-negative multiple. Therefore, the functions $\hat{K} - x$ and $\hat{K} - y$ differ by a non-negative multiple, but since $\|\hat{K} - x\|_{L^2([0,1])} = \|\hat{K} - y\|_{L^2([0,1])}$, we have $\hat{K} - x = \hat{K} - y$. Therefore, $x = y$, as required.

5 Representation of the minimizing function

In this section we consider principal functional $f$ corresponding to fractional Brownian motion and establish that the minimizing function has some special form. We start by proving several auxiliary results of the fractional Brownian kernel and the minimizing function.

5.1 Auxiliary results

Lemma 7 The fractional Brownian kernel for any $0 \leq t \leq 1$ satisfies

$$\int_0^t (K(1,s) - K(t,s))^2 ds + \int_t^1 K(1,s)^2 ds = (1 - t)^{2H}. \tag{22}$$

Proof It follows from (1) that the left-hand side of (22) is equal to $E(B_t^H - B_s^H)^2 = (1 - t)^{2H}$.

The following statement will be essentially generalized in what follows. However, we prove it because its proof clarifies the main ideas and, moreover, it has the interesting consequences concerning the properties of the minimizing function. In the remainder of this section $a = a(s), s \in [0,1]$ denotes the minimizing function, i.e. the unique element of $\mathcal{M}_f$.

Lemma 8 Let $t^* = \sup\{t \in (0,1) : a_t = f(a)\} \ (t^* = 0 \ if \ this \ set \ is \ empty). \ If \ t^* < 1, \ then \ a(t) = K(1,t) \ for \ a.e. \ t \in [t^*, 1].$
Proof Fix some $t_1 \in (t^*, 1]$ and prove that for any $h \in L_2([0, 1])$ the following equality holds:

$$\int_{t_1}^{1} h(s)(a(s) - K(1,s)) \, ds = 0.$$ 

Evidently, proof follows immediately from this statement.

Assume the contrary. Then, without loss of generality, there exists such $h \in L_2([0, 1])$ that

$$\int_{t_1}^{1} h(s)(a(s) - K(1,s)) \, ds =: \kappa > 0.$$ 

It follows from the continuity of the last integral w.r.t. upper bound that for some $t_2 \in (t_1, 1]$ we have

$$\int_{t_1}^{t_2} h(s)(a(s) - K(t,s)) \, ds \geq \kappa/2$$

for any $t \in [t_2, 1]$. Note also that our assumption implies that

$$m := \max_{s \in [t_1, t_2]} g_a(s) < f(a).$$

Consider now $h_\delta(t) = a(t) - \delta h(t)1_{[t_1, 1]}(t)$ for $\delta > 0$. We have that $g_{h_\delta}(t) = g_a(t)$ for $t \in [0, t_1]$, and

$$g_{h_\delta}(t)^2 = g_a(t)^2 - 2\delta \int_{t_1}^{t} h(s)(a(s) - K(t,s)) \, ds + \delta^2 \int_{t_1}^{t} h(s)^2 \, ds$$

for $t > t_1$. For $t \in (t_1, t_2]$ the following inequality holds,

$$g_{h_\delta}(t)^2 \leq m^2 - 2\delta \int_{t_1}^{t} h(s)(a(s) - K(t,s)) \, ds + \delta^2 \int_{t_1}^{t} h(s)^2 \, ds \leq m^2 + C\delta$$

with the constant $C$ that does not depend on $t, \delta$. Then for sufficiently small $\delta > 0$ we have that $g_{h_\delta}(t) < f(a)$ for any $t \in (t_1, t_2]$.

Furthermore, if $t \in (t_2, 1]$, then

$$g_{h_\delta}(t)^2 \leq f(a)^2 - 2\delta \int_{t_1}^{t} h(s)(a(s) - K(t,s)) \, ds + \delta^2 \int_{t_1}^{t} h(s)^2 \, ds \leq f(a)^2 - \kappa \delta + \delta^2 \int_{t_1}^{1} h(s)^2 \, ds.$$

Again, for sufficiently small $\delta > 0$ and any $t \in (t_2, 1]$ we have that $g_{h_\delta}(t) < f(a)$. Therefore, for sufficiently small $\delta > 0$ we get that $f(h_\delta) = f(a)$ and $g_{h_\delta}(1) < f(a) = f(h_\delta)$. We obtain the contradiction with Lemma 8 whence the proof follows.

Corollary 3 There exists such point $t \in (0, 1)$ that $g_a(t) = f(a)$.

Proof Assuming the contrary, we get from Lemma 8 that $a(t) = K(1,t)$ for a.a. $t \in [0, 1]$. However, in this case $g_a(1) = 0$, which contradicts Lemma 8.

Denote $\mathcal{E}_a = \{ t \in [0, 1] : g_a(t) = f(a) \}$, the set of the maximal points of the function $g_a$. 


Lemma 9 Let point $a \in [0, 1)$ is such that $g_a(u) < f(a)$. Then there does not exist function $h \in L_2([0, 1])$ such that for any $t \in \mathcal{G}_a \cap (u, 1]$ the inequality \( \int_t^u h(s)(a(s) - K(t,s))ds > 0 \) holds.

Proof Assume the contrary, i.e. let for some function $h \in L_2([0, 1])$ we have that \( \int_t^u h(s)(a(s) - K(t,s))ds > 0 \) for any $t \in \mathcal{G}_a \cap (u, 1]$. The set $\mathcal{G}_a \cap (u, 1]$ is closed because $g_a(u) < f(a)$. Therefore

$$\kappa := \min_{t \in \mathcal{G}_a \cap (u, 1]} \int_t^u h(s)(a(s) - K(t,s))ds > 0.$$ 

Denote

$$\mathcal{B}_\varepsilon = \{ t \in (u, 1] : \mathcal{G}_a \cap (u, 1] \cap (t - \varepsilon, t + \varepsilon) \neq \emptyset \}$$

the intersection of $\varepsilon$-neighborhood of the set $\mathcal{G}_a \cap (u, 1]$ with interval $(u, 1]$. Continuity argument implies that for some $\varepsilon > 0$ it holds that

$$\int_u^t h(s)(a(s) - K(t,s))ds > \kappa/2$$

for any $t \in \mathcal{B}_\varepsilon$. Similarly to the proof of Lemma 8 denote $b_\delta(t) = a(t) - \delta h(t)1_{(u,1]}(t)$ for any $\delta > 0$. Then we have that $g_{b_\delta}(t) = g_a(t)$ for any $t \in [0, u]$, and

$$g_{b_\delta}(t)^2 \leq f(a)^2 - 2\delta \int_u^t h(s)(a(s) - K(t,s))ds + \delta^2 \int_u^t h(s)^2ds \leq$$

$$\leq f(a)^2 - \kappa \delta + \delta^2 \int_0^1 h(s)^2ds$$

for any $t \in \mathcal{B}_\varepsilon$. It follows from the continuity of $g_a$ that $m = \max_{t \in [a, 1]} g_a(t) < f(a)$. Therefore we have for $t \in (u, 1] \setminus \mathcal{B}_\varepsilon$ that

$$g_{b_\delta}(t)^2 = g_a(t)^2 - 2\delta (a(s) - K(1,s)) + \delta^2 \int_{t_1}^t h(s)^2ds \leq$$

$$\leq m^2 - 2\delta \int_{t_1}^t h(s)(a(s) - K(t,s))ds + \delta^2 \int_{t_1}^t h(s)^2ds \leq m^2 + C\delta,$$

with the constant $C$ that does not depend on $t$ and $\delta$. It follows from the above bounds that for sufficiently small $\delta > 0$ and for any $t \in (u, 1]$ we have the inequality $g_{b_\delta}(t) < f(a)$. It means that for sufficiently small $\delta > 0$ we get the equality $f(b_\delta) = f(a)$, and moreover, $g_{b_\delta}(1) < f(a) = f(b_\delta)$, which contradicts Lemma 8.

Lemma 8 supplies the form of minimizing function on the part of the interval $[0, 1]$. All equalities below are considered a.s.

Lemma 10 Let $t_1 = \min\{ t \in (0, 1] : g_a(t) = f(a) \}$. Then there exist $t_2 \in (t_1, 1] \cap \mathcal{G}_a$ and random variable $\xi$ with the values in $[t_1, t_2] \cap \mathcal{G}_a$ such that for $t \in [0, t_2]$ we have that $P(\xi \geq t) > 0$, and the equality

$$a(t) = E[K(\xi,t)|\xi \geq t]$$

holds.
Proof Consider the set of functions

\[ \mathcal{K} = \{ k(t,s) = K(t,s)1_{s \leq t} + a(s)1_{s > t}, t \in \mathfrak{G}_a \} \]

and let

\[ \mathcal{C} = \left\{ \int_0^1 k_t(s)F(dt), F \text{ is the distribution function on } \mathfrak{G}_a \right\} \]

be the closure of the convex hull of \( \mathcal{K} \). According to Lemma 9 applied to \( u = 0 \), there does not exist \( h \in L^2([0,1]) \) such that \( (h,k) < (h,a) \) for any \( k \in \mathcal{K} \). Moreover, there is no \( h \in L^2([0,1]) \) such that \( (h,k) < (h,a) \) for any \( k \in \mathcal{C} \), i.e. the element \( a \) and the set \( \mathcal{K} \) can not be separated properly. Then, according to the proper separation theorem (see e.g. [2, Corollary 4.1.3]), \( a \in \mathcal{C} \), so there exists such distribution \( F \) on \( \mathfrak{G}_a \) that

\[ a(s) = \int_0^1 k_t(s)G(dt) = \int_{[s,1]} k_t(s)F(dt) + \int_{[0,s)} a(s)F(dt). \tag{23} \]

Hence

\[ a(s)F([s,1]) = \int_{[s,1]} k_t(s)F(dt). \tag{24} \]

Note that the equality \( \text{supp} F = \{ t_1 \} \) is impossible because otherwise it follows from equation (24) that \( a(s) = K(t_1,s) \) for \( s \leq t_1 \), therefore \( g_a(t_1) = 0 \) which contradicts the assumption \( g_a(t) = f(a) \).

Using the latter statement and (24), we get the statement of the theorem with \( t_2 = \max(\text{supp} F) \) and random variable \( \xi \) with the distribution \( F \).

Conditions on minimizing function from Lemma 10 are sufficient in the following sense.

Lemma 11 Let \( y \in L^2([0,1]) \). Define the kernel \( K_y(t,s) \) for \( s,t \in [0,1] \) as

\[ K_y(t,s) = \begin{cases} K(t,s) & \text{for } t \geq s, \\ y(s) & \text{for } t < s. \end{cases} \]

Function \( y \) is the minimizing function of the principal functional \( f \) if and only if there exists random variable \( \xi \) taking values in \([0,1]\) such that the following conditions hold:

\[ y(s) = \mathbb{E}K_y(\xi,s) \quad a.a. \ s \in [0,1], \tag{25} \]

\[ g_y(\xi) = f(y) \quad a.s. \tag{26} \]

Proof The necessity was proved in Lemma 10. Indeed, take \( \xi \) that was obtained in the course of the proof of Lemma 10. Then condition (25) follows from the equality (23), while condition (26) follows from the fact that \( \xi \in \mathfrak{G}_a \).

The sufficiency is proved basically by reversing a proper separation argument from Lemma 10: if a function belongs to the convex set \( \mathcal{C} \), then it cannot be properly separated from this set, which means that it is a minimizer. To make this idea rigorous, assume the contrary: let a function \( y \) satisfy (25) and (26), but \( y \notin \mathcal{M}_f \). Then there
exists function \(a \in L_2([0,1])\) such that \(f(y) > f(a)\) (for example, we can take \(a\) as the minimizing function). Functional \(f^2\) is convex, therefore

\[
f(y + \delta(a-y))^2 \leq f(y)^2 + \delta (f(a)^2 - f(y)^2), \quad 0 \leq \delta \leq 1.
\]

It is easy to see that for any function \(b \in L_2([0,1])\)

\[
\max_{r \in [0,1]} \|K_r(t, \cdot) - b\|^2 = \max_{r \in [0,1]} \left( \int_0^t (K(t,s) - b(s))^2 ds + \int_t^1 (y(s) - b(s))^2 ds \right) \leq \max_{r \in [0,1]} \int_0^t (K(t,s) - b(s))^2 ds + \int_t^1 (y(s) - b(s))^2 = f(b)^2 + \|y - b\|^2.
\]

Therefore for \(0 \leq \delta \leq 1\) we have that

\[
\max_{r \in [0,1]} \|K_r(t, \cdot) - y - \delta (a-y)\|^2 \leq f(y)^2 - \delta (f(y)^2 - f(a)^2) + \delta^2 \|a-y\|^2.
\]

It means that for sufficiently small \(\delta > 0\)

\[
\max_{r \in [0,1]} \|K_r(t, \cdot) - y - \delta (a-y)\|^2 < f(y)^2. \tag{27}
\]

On one hand, choose arbitrary \(\delta\) for which the inequality (27) holds, and set \(b = y + \delta (a-y)\). Then

\[
\max_{r \in [0,1]} \|K_r(t, \cdot) - b\|^2 < f(y)^2. \tag{28}
\]

On the other hand,

\[
\max_{r \in [0,1]} \|K_r(t, \cdot) - b\|^2 \geq \mathbb{E} \|K_r(\xi, \cdot) - b\|^2 \geq \mathbb{E} \|K_r(\xi, \cdot) - \mathbb{E} K_r(\xi, \cdot)\|^2 = \mathbb{E} \|K_r(\xi, \cdot) - \mathbb{E} g_\xi(\xi)\|^2 = \mathbb{E} g_\xi(\xi)^2 = f(y)^2. \tag{29}
\]

Inequalities (28) and (29) contradict each other. So, assuming that function \(y\) is not minimizing for principal functional \(f\), we get the contradiction. Therefore, \(f(y) = \min f\).

Now we are in position to prove that

\[
\text{ess sup } \xi := \min \{t : P(\xi \leq t) = 1\} = \max(\text{supp } \xi) = 1,
\]

which will imply that \(t_2 = 1\) in Lemma 10.

**Lemma 12**  Let \(a\) be the minimizing function for principal functional \(f\) and let \(\xi\) be random variable satisfying conditions (25) and (26) with \(x = a\). Then \(\text{ess sup } \xi = 1\).

**Proof**  Denote \(t_2 = \text{ess sup } \xi\). Evidently, \(\xi\) takes values from \([0, t_2]\).

Consider a function

\[
b(s) = t_2^{-\alpha} a(t_2 s), \quad s \in [0, 1].
\]

Then, in view of the self-similarity property (item 4 in Lemma 5),

\[
b(s) = \mathbb{E} K_\xi(\xi / t_2, s),
\]
where $K_b(t, s)$ is defined in the formulation of Lemma 11. Using (21), we get

$$g_b(t) = t_2^H g_a(t_2 t), \quad t \in [0, 1].$$

On one hand, since $a(s)$ satisfies (26), we have

$$f(b) = \max_{[0, 1]} g_b(\xi_{t_2}) = t_2^H g_a(\xi_{t_2}) = t_2^H f(a)$$

a.s.; on the other hand

$$f(b) = \max_{[0, 1]} g_b(\xi_{t_2}) = t_2^H \max_{[0, 1]} g_a = t_2^H f(a).$$

This implies

$$f(b) = g_b(\xi_{t_2}) = t_2^H f(a) \quad \text{a.e.}$$

Therefore, the function $b$ satisfies (25) and (26) and is therefore a minimizer of $f$. Hence

$$t_2^H f(a) = f(b) = \min_{L_2([0, 1])} f = f(a),$$

so $t_2 = 1$, as required.

5.2 Main properties of the minimizing function

We can refine Lemma 10 in view of Lemma 12. We remind that $a$ is the minimizing function for the principal functional $f$ and $G_a = \{ t \in [0, 1] : g_a(t) = f(a) \}$.

**Theorem 4** There exists a random variable $\xi_a$ assuming values in $G_a$ such that

$$P(\xi_a \geq s) > 0 \quad \text{for all } s \in [0, 1),$$

$$a(s) = E[K(\xi_a, s) \mid \xi_a \geq s] \quad \text{a.e. in } [0, 1]. \quad (30)$$

**Proof** This statement is a straightforward consequence of Lemma 12.

We will assume further (clearly, without loss of generality) that (30) holds for every $s \in [0, 1]$: $a(s) = E[K(\xi_a, s) \mid \xi_a \geq s]$ for any $s \in [0, 1]$. \hspace{1cm} (31)

**Corollary 4** 1. The minimizing function $a$ is left-continuous and has right limits.

2. For any $s \in [0, 1)$

$$0 < a(s) \leq K(1, s), \quad (32)$$

moreover,

$$a(s) < K(1, s)$$

on a set of positive Lebesgue measure.
Proof 1. Follows from (31), continuity of $K$ and the dominated convergence.

2. Taking into account statement 2 of Lemma 5 for $0 < s < t \leq 1$

$$0 < K(t, s) \leq K(1, s).$$

Now (32) follows from (31) and the fact that $P(\xi > s) > 0$ for $s < 1$. Further, if $a(s) = K(1, s)$ a.e., then $g_a(1) = 0$, which contradicts Lemma 5.

Further we investigate the distribution of $\xi$.

**Lemma 13** There exists $t^* \in (0, 1)$ such that

$$\forall t \in (t^*, 1): g_a(t) < f(a)$$

**Proof** Denote

$$h(t) = g_a(t)^2 = \int_0^t (K(t, s) - a(s))^2 ds.$$ 

The function $h$ is continuous on $[0, 1]$ and has left and right derivatives (except of $h'_-(0) = +\infty$):

$$h'_-(t) = a(t)^2 + 2 \int_0^t (K(t, s) - a(s)) K'_t(t, s) ds,$$

$$h'_+(t) = a(t)^2 + 2 \int_0^t (K(t, s) - a(s)) K'_t(t, s) ds,$$

where $K'_t(t, s) = \frac{\partial}{\partial t} K(t, s) = C_{\alpha s^{-\alpha} t^{\alpha}(t - s)^{\alpha-1}}$. Hence, by Corollary 4

$$h'_-(1) = 2 \int_0^1 (K(1, s) - a(s)) K'_t(1, s) ds > 0,$$

and the statement easily follows.

The lemma just proved means that 1 is an isolated point of $\mathcal{G}_a$.

As an immediate corollary, we have the following theorem.

**Theorem 5** There exists $t^*_a < 1$ such that $P(\xi_a \in (t^*_a, 1)) = 0$, and the distribution of $\xi_a$ has an atom at 1, i.e. $P(\xi = 1) > 0$. Consequently, $a(s) = K(1, s)$ for all $s \in [t^*_a, 1]$.

Further we prove that the distribution of $\xi_a$ has no other atoms.

**Theorem 6** For any $t \in (0, 1)$ $P(\xi_a = t) = 0$. Consequently, $a \in C[0, 1]$.

**Proof** We start by computing for $t \in (0, 1)$

$$a(t) - a(t) = E[K(\xi_a, t)|\xi_a > t] - E[K(\xi_a, t)|\xi_a \geq t]$$

$$= E[K(\xi_a, t)1_{\xi_a > t}] P(\xi_a > t) - E[K(\xi_a, t)1_{\xi_a \geq t}] P(\xi_a > t)$$

$$= E[K(\xi_a, t)1_{\xi_a > t}] P(\xi_a > t) - E[K(\xi_a, t)1_{\xi_a > t}] P(\xi_a > t)$$

$$= E[K(\xi_a, t)1_{\xi_a > t}] P(\xi_a = t) - E[K(\xi_a, t)1_{\xi_a = t}] P(\xi_a > t)$$

$$= \frac{E[K(\xi_a, t)1_{\xi_a > t}] P(\xi_a = t)}{P(\xi_a > t) P(\xi_a > t)} - \frac{E[K(\xi_a, t)1_{\xi_a = t}] P(\xi_a > t)}{P(\xi_a > t) P(\xi_a > t)}$$

$$= \frac{a(t+)}{P(\xi_a > t) P(\xi_a > t)} - \frac{a(t)}{P(\xi_a > t) P(\xi_a > t)} = \frac{a(t+)}{P(\xi_a > t) P(\xi_a > t)} - \frac{a(t)}{P(\xi_a > t) P(\xi_a > t)}.$$
Further, as in the proof of Lemma\[13\] denote $h = g_a^2$ and observe that it has left and right derivatives at $t$ equal to

$$h'_-(t) = a(t)^2 + 2\int_0^t (K(t,s) - a(s))K'_t(s)ds,$$

$$h'_+(t+) = a(t)^2 + 2\int_0^t (K(t,s) - a(s))K'_t(s)ds.$$  

But for any $t \in \mathcal{G}_a$, $h'_-(t) \geq 0$, $h'_+(t+) \leq 0$, so $a(t) \geq a(t+)$, whence from (53) we have that $a(t+) = a(t)$ and also $P(\xi_a = t) = 0$, as $a(t+) > 0$. For $t \notin \mathcal{G}_a$, $P(\xi_a = t) = 0$ (recall that $\xi_a$ takes values in $\mathcal{G}_a$) and $a(t+) = a(t)$.

**Remark 3** Due to monotonicity of $K$ in the first variable, the right-hand of inequality (8) is maximal for $t_2 = 1$, so we have that

$$f(a) \geq \frac{1}{4} \max_{t \in [0,1]} \int_0^t (K(1,s) - K(t,s))^2 ds. \quad (34)$$

Theorem\[6\] implies in particular that the inequality is strict, i.e. this lower bound is not attained. Indeed, if there were equality in (34), Lemma\[2\] would imply that the distribution of $\xi_a$ is $\frac{1}{2}(\delta_{b_0} + \delta_{b_1})$, where $t_0$ is the point where the minimum of the right-hand side of (34) is attained, which would contradict Theorem\[6\].

**Remark 4** From (33) it is easy to see that $a$ decreases on the complement of $\mathcal{G}_a$. The numerical experiments in the following section suggest that $a$ is decreasing on $[0,1]$ (the positive jumps in the graphs are due to atoms, which are, clearly, unavoidable in the discrete case, but there are no atoms in the continuous) case. It seems even that $a$ is constant on $\mathcal{G}_a \setminus \{1\}$, which would be a striking property to have. However, we did not manage to prove either of these facts.

### 6 Approximation of a discrete fBm by martingales

In this section we consider a problem of minimization of the principal functional, but in discrete time. This is an approximation to the original problem, so its solution can be considered as an approximate solution to the original problem.

Let $N$ be a natural number, and define $b_k = B^N_k, k = 0, \ldots, N$. The vector $b = (b_0, b_1, \ldots, b_N)$ will be called a discrete fBm. It generates a discrete filtration $\mathcal{F}_k = \sigma(b_0, \ldots, b_k), k = 0, \ldots, N$. For arbitrary random vector $\xi = (\xi_0, \xi_1, \ldots, \xi_N)$ with square integrable components denote

$$G(\xi) = \max_{k=0,\ldots,N} E(b_k - \xi_k)^2.$$  

Consider the problem of minimization of the functional $G(\xi)$, where $\xi$ is an $\mathcal{F}_k$-martingale.

Denote by $d_i = b_i - b_{i-1}, i = 1, \ldots, N$ the increments of the discrete fBm. Let $C$ be the covariance matrix of the vector $(d_i| i = 1, \ldots, N)$. Using the Cholesky decomposition, one can find a lower triangular real matrix $L = \{l_{ij}| i = 1, \ldots, N\}$ such that
$C = LL^T$. Then there exists a sequence $(\xi_1, \ldots, \xi_N)$ of independent standard Gaussian random variables such that $\xi_k$ is $\mathcal{F}_k$-measurable for $k = 1, \ldots, N$ and

$$
\begin{pmatrix}
    d_1 \\
    \vdots \\
    d_N
\end{pmatrix} = L
\begin{pmatrix}
    \xi_1 \\
    \vdots \\
    \xi_N
\end{pmatrix}.
$$

Define a matrix $K = (k_{ij})_{i, j = 1, \ldots, N}$ as follows:

$$
k_{ij} = \begin{cases} 
    0, & i < j \\
    \sum_{s=1}^{i} l_{ij}, & i \geq j.
\end{cases}
$$

It is clear that

$$
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_N
\end{pmatrix} = K
\begin{pmatrix}
    \xi_1 \\
    \vdots \\
    \xi_N
\end{pmatrix}.
$$

The matrix $K$ is therefore can be regarded as a discrete counterpart of a fractional Brownian kernel.

Further, we will show, as in the continuous case, that minimization of $G$ over martingales is equivalent to minimization over Gaussian martingales. Indeed, let $\xi = (\xi_0, 0, \xi_1, \ldots, \xi_N)$ be arbitrary square integrable $\mathcal{F}_k$-martingale. Owing to the fact that $\mathcal{F}_k = \sigma\{\xi_1, \ldots, \xi_k\}$, $k = 1, \ldots, N$, we have the following martingale representation:

$$
\xi_n = \sum_{k=1}^{n} \alpha_k \xi_k, \quad n = 1, \ldots, N,
$$

where $\alpha_k$ is a square integrable $\mathcal{F}_k$-measurable random variable, $k = 1, \ldots, N$. Thus,

$$
G(\xi) = \max_{j=0, \ldots, N} E(b_j - \xi_j)^2 = \max_{j=0, \ldots, N} \sum_{n=1}^{j} E(k_{jn} - \alpha_n)^2
$$

$$
= \max_{j=0, \ldots, N} \sum_{n=1}^{j} \left( E(k_{jn} - E \alpha_n)^2 + \text{Var}(\alpha_n) \right) \geq \max_{j=0, \ldots, N} \sum_{n=1}^{j} E(k_{jn} - E \alpha_n)^2.
$$

So we can assume that $\xi$ has a form $\xi_k = \sum_{j=1}^{k} a_j \xi_j$, $k = 1, \ldots, N$, with some non-random $a_1, \ldots, a_n$. Then

$$
G(\xi) = \max_{t=1, \ldots, N} \sum_{s=1}^{t} (k_{ts} - a_t)^2 =: F(a).
$$

Thus, we have arrived to the following optimization problem:

$$
\min F(a), \quad a \in \mathbb{R}^N.
$$

For fixed $N$ and $H$ we solve this problem numerically by using the MATLAB \texttt{fminimax} function.

The following table gives the values of the functional for different $H$ and $N = 200$. 
Figure 1 shows the values of $\min F$ for $H$ from 0.51 to 0.99 with a step 0.01 for $N = 200$. Figure 2 contain graphs of the minimizing vector (blue) and the scaled “distance” $R(t) = \sum_{s=1}^{t} (k_{ts} - a_{ts})^2$ (red), when $H = 0.75$ and $N = 500$. For other values of $H$ the picture is similar: $a$ is (mainly) decreasing and looks close to constant on the sets of maxima of $R$.

![Figure 1](image_url)

**Fig. 1** Values of $\min F$ for $H$ from 0.51 to 0.99.

### References

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3. I. Norros, E. Valkeila, J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*. Vol. 5, No. 4, 1999, 571-587.
Fig. 2 The minimizing vector (blue) and the scaled distance (red) for $H = 0.75$