Variation of the local topological structure of graph embeddings

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Abstract

The 2-cell embeddings of graphs on closed surfaces have been widely studied. It is well known that (2-cell) embedding a given graph $G$ on a closed orientable surface is equivalent to cyclically ordering the edges incident to each vertex of $G$. In this paper, we study the following problem: given a genus $g$ embedding $E$ of the graph $G$, if we randomly rearrange the edges around a vertex, i.e., re-embedding, what is the probability of the resulting embedding $E'$ having genus $g + \Delta g$? We give a formula to compute this probability. Meanwhile, some other known and unknown results are also obtained. For example, we show that the probability of preserving the genus is at least \[ \frac{2}{\deg(v)+2} \] for re-embedding any vertex $v$ of degree $\deg(v)$ in a one-face embedding; and we obtain a necessary condition for a given embedding of $G$ to be an embedding with the minimum genus.

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1 Introduction

Graph embedding is one of the most important topics in topological graph theory. In particular, 2-cell embeddings of graphs (loops and multiple edges allowed) have been widely studied. A 2-cell embedding of a given graph $G$ on a closed surface of genus $g$, $S_g$, is an embedding on $S_g$ such that every face is homeomorphic to an open disk. A 2-cell embedding is also called a map. The closed surfaces could be either orientable or unorientable. In this paper, we restrict ourselves to orientable case. Besides, by embedding we always mean 2-cell embedding. Note, by the classification theorem, any orientable closed surface of genus $g$ is homeomorphic to the connected sum of $g$ tori.

There are many interesting topics on graph embedding in the literature. For instance, given a graph $G$, what is the minimum (resp., maximum) genus $g$ such that there exists a 2-cell embedding of $G$ on $S_g$? For studies in detail, we refer the readers to [8, 13, 15, 18, 20, 21, 23, 26, 30, 31]
and references therein. Let \( g_{\text{min}}(G) \) and \( g_{\text{max}}(G) \) denote the minimum and the maximum genus \( g \) of the embeddings of \( G \), respectively. In Duke [6], an “interpolation” theorem showed that for any \( g_{\text{min}}(G) \leq g \leq g_{\text{max}}(G) \), there exists an embedding of \( G \) on \( S_g \). Assume \( G \) has \( e \) edges and \( v \) vertices, and embedded on \( S_g \) via the embedding \( \mathbb{E} \). The number \( \beta(G) = e - v + 1 \) is called the betti number of \( G \). According to Euler’s characteristic formula, there holds

\[
v - e + f = 2 - 2g \quad \iff \quad 2g = \beta(G) + 1 - f,
\]

where \( f \geq 1 \) is the number of faces of \( \mathbb{E} \). Thus, the largest possible value of \( g \) is \( \lfloor \frac{\beta(G)}{2} \rfloor \). If \( g_{\text{max}}(G) = \lfloor \frac{\beta(G)}{2} \rfloor \), \( G \) is called upper embeddable. When is \( G \) upper embeddable? See studies in [8, 13, 15, 18, 20, 21, 30, 31].

It is well known that an embedding of \( G \) on a closed orientable surface can be equivalently represented by \( G \) with a specified cyclic order of edges around (i.e., incident to) each vertex of \( G \), i.e., the topological structure of the embedding is implied in these cyclic orderings of edges [7, 23]. Any variation of the local topological structure around a vertex, i.e., the cyclic order of edges around the vertex, may change the topological properties of the embedding, e.g., the genus of the embedding.

Plane permutations were recently used to study hypermaps in Chen and Reidys [3]. It proved to be quite effective to enumerate hypermaps with one face. Besides, plane permutations allow to study the transposition and block-interchange distance of permutations as well as the reversal distance of signed permutations in a unified simple framework [4]. Since maps are specific class of hypermaps, it is natural to study graph embeddings using plane permutations as well.

This paper is organized as follows: in Section 2, we recall some basics of plane permutations [3] for later use. In Section 3, we study embeddings with one face. These objects have been studied in many fields [1, 2, 10, 12, 14, 19, 22, 29, 32] where the enumeration aspect is the main interest. Our interest in this paper is to understand the following problems: assume there exists a one-face embedding \( \mathbb{E} \) for the graph \( G \). How many different ways are there of changing the local embedding (re-embedding) around a vertex without changing the genus? By changing the local embedding, we mean changing the cyclic order of edges around the vertex. For a given vertex, is there another local embedding around it to preserve the genus? As results, we show that the probability of preserving the genus is at least \( \frac{2}{\deg(v)+2} \) for re-embedding any vertex \( v \) of degree (i.e., valence) \( \deg(v) \). Also, there is at least one alternative way to re-embed a vertex \( v \) preserving the genus if \( \deg(v) \geq 4 \). In Section 4, in order to study embeddings with more than one face, we generalize plane permutations into \( k \)-cyc plane permutations. We study more general questions, e.g., given an embedding \( \mathbb{E} \) for the graph \( G \), what is the maximum (resp., minimum) genus can be achieved by changing the local embedding of one of the vertices of \( G \)? For a vertex with larger degree, there are more alternatives to rearrange the edges around it. Is it true that re-embedding a vertex with larger degree always achieve a higher (resp., lower) genus than re-embedding a vertex with smaller degree? and so on. As results, we obtain a local version of the interpolation theorem which also provides an easy approach to roughly estimate the range \([g_{\text{min}}(G), g_{\text{max}}(G)]\), as well as a necessary condition for an embedding of \( G \) to be an embedding with the minimum genus.
2 Plane permutations

Let $\mathcal{S}_n$ denote the group of permutations, i.e. the group of bijections from $[n] = \{1, \ldots, n\}$ to $[n]$, where the multiplication is the composition of maps. We shall discuss the following three representations of a permutation $\pi$:

**two-line form:** the top line lists all elements in $[n]$, following the natural order. The bottom line lists the corresponding images of elements on the top line, i.e.

$$
\pi = \begin{pmatrix}
1 & 2 & 3 & \cdots & n-2 & n-1 & n \\
\pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n-2) & \pi(n-1) & \pi(n)
\end{pmatrix}.
$$

**cycle form:** regarding $\langle \pi \rangle$ as a cyclic group, we represent $\pi$ by its collection of orbits (cycles). The set consisting of the lengths of these disjoint cycles is called the cycle-type of $\pi$. We can encode this set into a non-increasing integer sequence $\lambda = \lambda_1 \lambda_2 \cdots$, where $\sum \lambda_i = n$, or as $1^{a_1} 2^{a_2} \cdots n^{a_n}$, where we have $a_i$ cycles of length $i$. A cycle of length $k$ will be called a $k$-cycle. A cycle of odd and even length will be called an odd and even cycle, respectively. It is well known that all permutations of a same cycle-type forms a conjugacy class of $\mathcal{S}_n$.

**Definition 2.1 (Plane permutation).** A plane permutation on $[n]$ is a pair $p = (s, \pi)$ where $s = (s_i)_{i=0}^{n-1}$ is an $n$-cycle and $\pi$ is an arbitrary permutation on $[n]$. The permutation $D_p = s \circ \pi^{-1}$ is called the diagonal of $p$.

Given $s = (s_0 s_1 \cdots s_{n-1})$, a plane permutation $p = (s, \pi)$ can be represented by two aligned rows:

$$
(s, \pi) = \left(\begin{array}{cccc}
s_0 & s_1 & \cdots & s_{n-2} \\
\pi(s_0) & \pi(s_1) & \cdots & \pi(s_{n-2})
\end{array}\right) \quad (2)
$$

Indeed, $D_p$ is determined by the diagonal-pairs (cyclically) in the two-line representation here, i.e., $D_p(\pi(s_{i-1})) = s_i$ for $0 < i < n$, and $D_p(\pi(s_{n-1})) = s_0$. For convenience, we always assume $s_0 = 1$ in the following and we mean by “the cycles of $p = (s, \pi)$” the cycles of $\pi$.

Given a plane permutation $p = (s, \pi)$ on $[n]$ and a sequence $h = h_1 h_2 \cdots h_{n-1}$ on $[n-1]$, let $s^h = (s_0 s_1 h s_2 \cdots s_{h_{n-1}})$ and $\pi^h = D_p \circ \chi_h$. We write $(s^h, \pi^h) = \chi_h \circ (s, \pi)$. In particular, if $h = 12 \cdots (i-1)(j+1) \cdots l, i \cdots j(l+1) \cdots (n-1)$ where $0 < i < j < l < n$, we have

$$
s^h = (s_0, s_1, \ldots, \underline{s_{i-1}}, s_{i+1}, \ldots, \underline{s_{j-1}}, s_j, s_{j+1}, \ldots, \underline{s_{l+1}}, \ldots, s_{n-1}),
$$

i.e. the $n$-cycle obtained by transposing the blocks $[s_i, s_j]$ and $[s_{j+1}, s_l]$. Then, $(s^h, \pi^h)$ can be represented as

$$
(\cdots \pi(s) \pi(s_{j+1}) \cdots \pi(s_{i+1}) \pi(s_i) \cdots \pi(s_{j-1}) \pi(s_{j+1}) \cdots \pi(s_{l+1}) \pi(s_i) \cdots \pi(s_{j-1}) \pi(s_{l+1}) \cdots)
$$

Note that the bottom row of the two-row representation of $(s^h, \pi^h)$ is obtained by transposing the blocks $[\pi(s_{i-1}), \pi(s_{j-1})]$ and $[\pi(s_j), \pi(s_{l-1})]$ of the bottom row of $(s, \pi)$. In this particular case, we denote the sequence $h$ as $(i, j, j+1, l)$ for short and refer to $\chi_h$ a transpose. For general $h$, we observe that the two-row form of $(s^h, \pi^h)$ is obtained by rearranging the diagonal-pairs of $(s, \pi)$. As a result, we observe
Lemma 2.2. [3] Given a plane permutation $(s, \pi)$ on $[n]$ and a transpose $\chi_h$, where $h = (i, j, j + 1, l)$ and $0 < i \leq j < l < n$. Then $\pi(s_r) = \pi^h(s_r)$ if $r \in \{0, 1, \ldots, n - 1\} \setminus \{i - 1, j, l\}$, and

$$\pi^h(s_{i-1}) = \pi(s_j), \quad \pi^h(s_j) = \pi(s_l), \quad \pi^h(s_l) = \pi(s_{i-1}).$$

We shall proceed by analyzing the induced changes of the $\pi$-cycles when passing to $\pi^h$. By Lemma 2.2, only the $\pi$-cycles containing $s_{i-1}, s_j, s_l$ will be affected.

Lemma 2.3. [3] Given $(s, \pi)$ and a transpose $\chi_h$ where $h = (i, j, j + 1, l)$ and $0 < i \leq j < l < n$, then there exist the following six scenarios for the pairs $(\pi, \pi^h)$:

| Case 1 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}), (s_j, v^j_1, \ldots, v^j_{m_j}), (s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}), (s_j, v^j_1, \ldots, v^j_{m_j}), (s_l, v^l_1, \ldots, v^l_{m_l})$ |

| Case 2 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ |

| Case 3 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ |

| Case 4 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ |

| Case 5 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ |

| Case 6 | $\pi$ | $\pi^h$ |
|--------|-------|--------|
|        | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ | $(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l})$ |

Proof. We shall only prove Case 1 and Case 2, the remaining four cases can be shown analogously. For Case 1, the $\pi$-cycles containing $s_{i-1}, s_j, s_l$ are

$$(s_{i-1}, v^i_1, \ldots, v^i_{m_i}), (s_j, v^j_1, \ldots, v^j_{m_j}), (s_l, v^l_1, \ldots, v^l_{m_l}).$$

Lemma 2.2 allows us to identify the new cycle structure by inspecting the critical points $s_{i-1}, s_j$ and $s_l$. Here we observe that all three cycles merge and form a single $\pi^h$-cycle

$$(s_{i-1}, \pi^h(s_{i-1}), (\pi^h)^2(s_{i-1}), \ldots) = (s_{i-1}, \pi(s_j), \pi^2(s_j), \ldots) = (s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l}).$$

For Case 2, the $\pi$-cycle containing $s_{i-1}, s_j, s_l$ is

$$(s_{i-1}, v^i_1, \ldots, v^i_{m_i}, s_j, v^j_1, \ldots, v^j_{m_j}, s_l, v^l_1, \ldots, v^l_{m_l}).$$

We compute the $\pi^h$-cycles containing $s_{i-1}, s_j$ and $s_l$ in $\pi^h$ as

$$(s_{i-1}, \pi^h(s_{i-1}), (\pi^h)^2(s_{i-1}), \ldots) = (s_{i-1}, \pi(s_j), \pi^2(s_j), \ldots) = (s_{i-1}, v^i_1, \ldots, v^i_{m_i})$$

$$(s_j, \pi^h(s_j), (\pi^h)^2(s_j), \ldots) = (s_j, \pi(s_l), \pi^2(s_l), \ldots) = (s_j, v^j_1, \ldots, v^j_{m_j})$$

$$(s_l, \pi^h(s_l), (\pi^h)^2(s_l), \ldots) = (s_l, \pi(s_{i-1}), \pi^2(s_{i-1}), \ldots) = (s_l, v^l_1, \ldots, v^l_{m_l})$$
whence the lemma.

\[ \square \]

**Definition 2.4.** Two plane permutations \((s, \pi)\) and \((s', \pi')\) are equivalent if there exists a permutation \(\alpha\) such that
\[
 s = \alpha s' \alpha^{-1}, \quad \pi = \alpha \pi' \alpha^{-1}, \quad \alpha(1) = 1.
\]

For two equivalent plane permutations \(p = (s, \pi)\) and \(p' = (s', \pi')\), we have \(s = s'\) if and only if \(\pi = \pi'\). Clearly, the equation \(\alpha s' \alpha^{-1} = s'\) restricts \(\alpha\) to be a shift within the \(n\)-cycle \(s'\) and the latter has to be trivial due to \(\alpha(1) = 1\).

Let \(U_D\) denote the set of plane permutations having \(D\) as diagonals for some fixed permutation \(D\) on \([n]\). Note \(p = (s, \pi) \in U_D\) iff \(D = D_p = s \circ \pi^{-1}\). Then, the number \(|U_D|\) enumerates the ways to write \(D\) as a product of an \(n\)-cycle with another permutation. Or equivalently, assuming \(D\) is of cycle-type \(\lambda\), in view of
\[
 D = s\pi^{-1} \iff (12\cdots n) = \gamma \gamma^{-1} = (\gamma D \gamma^{-1})(\gamma \pi \gamma^{-1}),
\]
where \(\gamma\) is unique if \(\gamma(1) = 1\), \(|U_D|\) is also the number of factorizations of \((12\cdots n)\) into a permutation of cycle-type \(\lambda\) and another permutation, i.e., rooted hypermaps having one face. A rooted hypermap is a triple of permutations \((\alpha, \beta_1, \beta_2)\), such that \(\alpha = \beta_1 \beta_2\). The cycles in \(\alpha\) are called faces, the cycles in \(\beta_1\) are called (hyper)edges, and the cycles in \(\beta_2\) are called vertices. If \(\beta_1\) is an involution without fixed points, the rooted hypermap is an ordinary rooted map. We refer to \([1, 2, 5, 10, 12, 14, 19, 22, 29, 32]\) and references therein for an in-depth study of hypermaps and maps.

Let \(\mu, \eta\) be partitions of \(n\). We write \(\mu \triangleright_{2i+1} \eta\) if \(\mu\) can be obtained by splitting one \(\eta\)-block into \((2i+1)\) non-zero parts. Let furthermore \(\kappa_{\mu, \eta}\) denote the number of different ways to obtain \(\eta\) from \(\mu\) by merging \(\ell(\mu) - \ell(\eta) + 1\) \(\mu\)-blocks into one, where \(\ell(\mu)\) and \(\ell(\eta)\) denote the number of blocks in the partitions \(\mu\) and \(\eta\), respectively.

Let \(U_{\lambda}^D\) denote the set of plane permutations, \(p = (s, \pi) \in U_D\), where \(D\) has cycle-type \(\lambda\) and \(\pi\) has cycle-type \(\eta\).

**Theorem 2.5.** \([3]\) Let \(f_{\eta, \lambda}(n) = |U_{\lambda}^D|\). Then, we have
\[
 f_{\eta, \lambda}(n) = \frac{\sum_{i=1}^{\lfloor n/\ell(\eta) \rfloor} \sum_{\mu \triangleright_{2i+1} \eta} \kappa_{\mu, \eta} f_{\mu, \lambda}(n) + \sum_{i=1}^{\lfloor n/\ell(\lambda) \rfloor} \sum_{\mu \triangleright_{2i+1} \lambda} \kappa_{\mu, \lambda} f_{\mu, \eta}(n)}{n + 1 - \ell(\eta) - \ell(\lambda)}.
\]  

**Corollary 2.6.** \([3]\) Let \(p_k^\lambda(n)\) denote the number of \(p \in U_D\) having \(k\) cycles, where \(D\) is of cycle-type \(\lambda\).
\[
 p_k^\lambda(n) = \frac{\sum_{i=1}^{\lfloor n/k \rfloor} (k+2i) p_{k+2i}^\lambda(n) + \sum_{i=1}^{\lfloor n/\ell(\lambda) \rfloor} \sum_{\mu \triangleright_{2i+1} \lambda} \kappa_{\mu, \lambda} p_k^\mu(n)}{n + 1 - k - \ell(\lambda)}.
\]

**Proposition 2.7.** \([3, 33]\)
\[
 \arg \max_k \{p_k^\lambda(n) \neq 0\} = n + 1 - \ell(\lambda).
\]
3 Local variation of embeddings with one face

As already mentioned, an embedding of the given graph $G$ can be combinatorially encoded into $G$ with a specified cyclic order of edges incident to each vertex of $G$. Such a cyclic ordering is also called a rotation system. A graph with a rotation system on it is called a fatgraph.

Conventionally, a fatgraph of $n$ edges is encoded into a triple of permutations $(\alpha, \beta, \gamma)$ on $[2n]$. This can be obtained as follows: given a fatgraph $F$, we firstly call the two ends of an edge as two half edges. Label all half edges using the labels from the set $[2n]$ so that each label appears exactly once. Then we immediately obtain two permutations $\alpha$ and $\beta$ where $\alpha$ is an involution without fixed points and each cycle consisting of the labels of the two half edges of a same edge and each cycle in $\beta$ is the counterclockwise cyclic arrangement of all half edges incident to a same vertex. The third permutation $\gamma = \alpha \beta$, which can be interpreted as the set of counterclockwise boundaries of the fatgraph. A boundary of the fatgraph is obtained as follows: start from some half edge, and every time when we meet a half edge we next go to the half edge paired with the counterclockwise neighbor of the present half edge until we meet the starting half edge again, the obtained cycle is a boundary of the fatgraph which corresponds to a cycle in $\gamma$. Starting from one half edge which does not appear in the former obtained boundary (or boundaries) and continuing the traveling process, we obtain all the boundaries of the fatgraph. If $\gamma$ has $k$ cycles, the fatgraph has $k$ boundaries, i.e., the corresponding embedding has $k$ faces. Obviously, a different triple of permutations can be obtained by relabeling the half edges of the fatgraph.

For a given fatgraph $(\alpha, \beta, \gamma)$, it is well known that $(\alpha, \alpha \beta, \alpha \gamma) = (\alpha, \gamma, \beta)$ is its Poincaré dual which transforms a face into a vertex and vice versa.

In this section, we will focus on embeddings with one face, i.e., in the triple $(\alpha, \beta, \gamma)$, $\gamma$ has only one cycle. These maps are also called unicellular maps [1,2]. At this point, we observe: Observation: A unicellular maps $(\alpha, \beta, \gamma)$ can be encoded into a plane permutation $(s, \pi)$ as well, i.e., $s = \gamma$ and $\pi = \beta$.

See Figure 1 for an example of a fatgraph with one boundary. The two drawings there are the same unicellular map. However, in the drawing on the righthand side the edges are drawn as ribbons so that it is sometimes called ribbon graphs. The corresponding plane permutation is

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 6 & 7 & 8 & 3 & 4 & 5 & 2 \end{pmatrix}.$$  

These objects have been considered in many different contexts, e.g., the computation of matrix integral [32], moduli space of curves [12], factorization of permutations [10,11,14], topological RNA and protein structure [1,22], etc.

In the following, we study embeddings (fatgraphs) in the framework of plane permutations. Let $p = (s, \pi)$ encode a one-face embedding of the graph $G$. The elements in the set on which the plane permutation is defined are called half edges. A cycle of $\pi$ is also called a vertex and $s$ is also called the boundary (i.e., face). The genus of the plane permutation refers to the genus of the corresponding unicellular maps. We focus on the local structure of unicellular maps, which is motivated by Case 3 transpose in Lemma 2.3 as follows: given a plane permutation, if we apply a Case 3 transpose, the set of half edges in each cycle of $\pi$ is not changed, which
means the underlying graphs before and after the transpose are the same. Namely, we may obtain different unicellular maps by rearranging the half edges around the vertices of a given unicellular map. Thus we could ask, given a unicellular map, if we randomly rearrange half edges around a vertex, what is the probability of the obtained fatgraph is still a unicellular map? For a given vertex in a unicellular map, whether it is the unique way of arranging all edges incident to the vertex to achieve the genus in the present map? and so on.

3.1 Variation of the embedding around one vertex

At first, we consider the case where only half edges around one vertex are rearranged. Let \( p = (s, \pi) \) correspond to a one-face embedding of the graph \( G \). A vertex \( v \) of \( G \) can be represented as a cycle of \( \pi \), which can be also naturally encoded into a plane permutation

\[
v = (s_v, \pi_v) = \left( \begin{array}{ccccccc} s_{i_0} & s_{i_1} & s_{i_2} & s_{i_3} & \cdots & s_{i_{k-1}} \\
\pi(s_{i_0}) & \pi(s_{i_1}) & \pi(s_{i_2}) & \pi(s_{i_3}) & \cdots & \pi(s_{i_{k-1}}) \end{array} \right),
\]

where \( s_v \) is a subsequence of \( s \) consisting of the half edges around \( v \) and \( \pi_v \) is equal to \( \pi \) with restriction to the half edges around \( v \), i.e., the set \( H(v) \).

Let \( X \) denote the set of \( k \)-cycles \( \theta \) on \( H(v) \) such that the resulting embedding has one face after rearranging half edges around \( v \) according to \( \theta \), \( Y \) denote the set of sequences \( h \) on \([k - 1] \) such that \( \chi_h \circ (s_v, \pi_v) \) has only one cycle.

Theorem 3.1. \( |X| = |Y| \).

Proof. Let

\[
p = \left( \begin{array}{cccccccccc}
\cdots & s_{i_0} & s_{i_0+1} & \cdots & s_{i_1} & \cdots & s_{i_2} & \cdots & s_{i_{k-2}} & \cdots & s_{i_{k-1}} & \cdots \\
\cdots & \pi(s_{i_0}) & \cdots & \pi(s_{i_1-1}) & \pi(s_{i_1}) & \cdots & \pi(s_{i_2}) & \cdots & \pi(s_{i_{k-2}}) & \cdots & \pi(s_{i_{k-1}}) & \cdots 
\end{array} \right).
\]

\( Y \rightarrow X \): each \( h = h_1 h_2 \cdots h_{k-1} \in Y \) uniquely induces a rearrangement of the following diagonal blocks of \( p \):

\[
[s_{i_0+1}, s_{i_1}], \quad [s_{i_1+1}, s_{i_2}], \quad \cdots \quad [s_{i_{k-2}+1}, s_{i_{k-1}}],
\]

where each segment (i.e., interval), e.g., \([s_{i_0+1}, s_{i_1}]\), refers to the diagonal block with the segment as the top row (or called top boundary). The resulting plane permutation is still an embedding with one face. Note in this operation, similar to transposes in Lemma 2.3, we only change the image of the elements in the set \( H(v) \) into the elements in \( H(v) \), all other cycles of \( p \) are not
changed. If after the rearrangement of the diagonal blocks according to $h$, the permutation on $H(v)$ forms only one cycle $\theta$, then the resulting unicellular map has the same underlying graph $G$. Namely, the resulting embedding with one face is obtained by rearranging the half edges around $v$ according to $\theta$. By construction, we have

$$\theta(s_{i_0}) = \pi(s_{i_{h_1-1}}), \ldots, \theta(s_{i_{h_j}}) = \pi(s_{i_{h_{j+1}-1}}), \ldots, \theta(s_{i_{h_k-1}}) = \pi(s_{i_{k-1}}).$$

$X \rightarrow Y$: given $\theta \in X$, since the resulting embedding after rearrangement according to $\theta$ is still one-face embedding, then the corresponding plane permutation $(s', \pi')$ must have the form

$$\begin{pmatrix}
\cdots & s'_{i_0} & s'_{i_0+1} & \cdots & s'_{i_1} & s'_{i_2} & \cdots & s'_{i_k-2} & \cdots & s'_{i_k-1} & \cdots \\
\cdots & \pi'(s'_{i_0}) & \pi'(s'_{i_1}) & \pi'(s'_{i_2}) & \cdots & \pi'(s'_{i_k-2}) & \cdots & \pi'(s'_{i_k-1}) & \cdots
\end{pmatrix}$$

where we assume $s'_{i_0} = s_0$. Since by construction the local structures are not changed except for around $v$, $s_j = s'_j$ for $0 \leq j \leq i_0$. Assume

$$H(v) = \{s'_{i_0}, s'_{i_1}, \ldots, s'_{i_k-1}\} = \{\pi'(s'_{i_0}), \pi'(s'_{i_1}), \ldots, \pi'(s'_{i_k-1})\}.$$

Then, we have $\pi'(s'_{i_j}) = \theta(s'_{i_j})$ for $0 \leq j \leq k-1$. It suffices to show that each diagonal block $[s'_{i_j+1}, s'_{i_{j+1}}]$ for some $j$ is the same as the diagonal block $[s_{i_l+1}, s_{i_{l+1}}]$ for some $l$, i.e.,

$$\begin{pmatrix}
\cdots & s'_{i_j} & s'_{i_{j+1}} & \cdots & s'_{i_{j+2}} & \cdots & s'_{i_{j+1}} & \cdots \\
\pi'(s'_{i_j}) & \pi'(s'_{i_{j+1}}) & \cdots & \pi'(s'_{i_{j+1}}) & \cdots
\end{pmatrix} = \begin{pmatrix}
\pi(s_{i_l}) & s_{i_l+1} & s_{i_{l+2}} & \cdots & s_{i_{l+1}} \\
\pi(s_{i_l+1}) & \pi(s_{i_{l+2}}) & \cdots & \pi(s_{i_{l+1}}) & \pi(s_{i_{l+1}})
\end{pmatrix}$$

**Claim.** If $\pi'(s'_{i_j}) = \pi(s_{i_l})$, the diagonal block $[s'_{i_j+1}, s'_{i_{j+1}}]$ and the diagonal block $[s_{i_l+1}, s_{i_{l+1}}]$ are equal.

Note, if $\pi'(s'_{i_j}) = \pi(s_{i_l})$, then

$$s'_{i_{j+1}} = D_p \circ \pi'(s'_{i_j}) = D_p \circ \pi(s_{i_l}) = s_{i_{l+1}}.$$  

Since $s'_{i_{j+1}}$ is not in $H(v)$, $\pi'(s'_{i_{j+1}}) = \pi(s'_{i_{j+1}}) = \pi(s_{i_{l+1}})$. Continuing the analysis, we have

$$s'_{i_{j+2}} = D_p \circ \pi'(s'_{i_{j+1}}) = D_p \circ \pi(s_{i_{l+1}}) = s_{i_{l+2}},$$

and so on, finally we come to $s'_{i_{j+1}} = s_{i_{l+1}}$. This affirms the claim.

Therefore, the sequence of the diagonal blocks $[s'_{i_{j+1}}, s'_{i_{j+1}}]$ are rearrangement of the sequence of the diagonal blocks $[s_{i_{l+1}}, s_{i_{l+1}}]$. It is obvious that each rearrangement of the diagonal blocks $[s_{i_{l+1}}, s_{i_{l+1}}]$ uniquely induces a rearrangement of the diagonal-pairs of $(s_v, \pi_v)$ according to $h \in [k-1]$. This completes the proof. $\square$
Figure 2: Circular arrangement of diagonal blocks determined by the vertex $v$

From the discussion above, we observe that given any vertex $v$, the half edges in $H(v)$ will segment the plane permutation into $|H(v)|$ diagonal blocks. Each diagonal block is completely determined by its left lower corner $π(s_i)$ and its right upper corner $s_{i+1} = s_ν(s_i)$. We can view these diagonal blocks as arranged in a circular manner, as shown in Figure 2. To rearrange the half edges around $v$ is to rearrange these diagonal blocks circularly.

Given a plane permutation $p = (s, π)$ with only one cycle, how many different $h$ such that $χ_h ∘ (s, π)$ has only one cycle? From the discussion in Section 2, we know that it is equivalent to factorizing $D_ν$. Let $R_ν = |X| = |Y|$. Then we have

**Corollary 3.2.** Let $p = (s, π)$ correspond to a one-face embedding of the graph $G$, $v$ is a vertex of $G$ and

$$v = (s_ν, π_ν) = \left( \begin{array}{c} s_{i_0} \\ π(s_{i_0}) \\ s_{i_1} \\ π(s_{i_1}) \\ s_{i_2} \\ π(s_{i_2}) \\ ... \\ s_{i_{k-1}} \\ π(s_{i_{k-1}}) \end{array} \right).$$

Assume $D_ν = s_ν ∘ π_ν^{-1}$ is of cycle-type $λ$. Then, we have

$$R_ν = p_1^{λ}(k).$$

Furthermore, if $λ = (1^{a_1}, 2^{a_2}, ..., k^{a_k})$, then

$$R_ν = \sum_{i=0}^{k-1} \frac{i!(k-1-i)!}{k} \sum_{<r_1,...,r_i>} \left( \frac{a_1 - 1}{r_1} \right) \left( \frac{a_2}{r_2} \right) ... \left( \frac{a_i}{r_i} \right) (-1)^{r_2+r_4+r_6+...},$$

where $<r_1,...,r_i>$ ranges over all non-negative integer solutions to the equation $\sum j r_j = i$.

**Proof.** The number $R_ν$ is equal to the number of different ways to factorize $D_ν$ into a permutation with one cycle (e.g., $s_ν$) and the other permutation with one cycle (e.g., $π_ν^{-1}$). Then, Eq. (6) follows from Corollary 2.6. The explicit formula Eq. (7) follows from Stanley [25]. This completes the proof.

**Example 3.3.** Given a plane permutation

$$\begin{pmatrix} 1 & 2 & ... & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 1 & 7 & ... & 17 & 14 & 5 & 20 & 16 & 6 & 9 & 19 & 12 & 8 & 4 & 15 & 11 & 2 \end{pmatrix},$$

the corresponding unicellular map of which is shown on the left in Figure 3. Consider the vertex

$$v = \begin{pmatrix} 8 & 11 & 14 & 16 & 19 \\ 14 & 16 & 19 & 8 & 11 \end{pmatrix},$$
\[ D_v = \begin{pmatrix} 8 & 19 & 16 & 14 & 11 \\ 8 & 11 & 14 & 16 & 19 \\ 8 & 14 & 19 & 11 & 16 \end{pmatrix} \]

Rearranging the half edges around the vertex \( v \) following the second factorization of \( D_v \), we obtain another unicellular map as shown on the right hand side in Figure 3.

![Figure 3: A unicellular map with 10 edges (left) and rearranging half edges around one of its vertices (right) where after relabeling the boundary is \((1', 2', \ldots, 20')\).](image)

We can see that given a one-face embedding of the graph \( G \), randomly rearranging the half edges around the vertex \( v \), the probability of the resulting map to be unicellular is exactly \( R_n / (|H| - 1)! \). Furthermore, we have

**Theorem 3.4.** Let \( \mathcal{E} \) be a one-face embedding of \( G \), and \( v \) is a vertex of \( G \) with degree \( \deg(v) = k \). Assume \( v = (s_v, \pi_v) \) and \( D_v \) is of cycle-type \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, k^{a_k}) \). Then, the probability \( \text{prob}_1(v) \) of the resulting embedding to be unicellular after rearranging the half edges around \( v \) satisfies

\[
\frac{2}{\deg(v) - a_1 + 2} \leq \text{prob}_1(v) \leq \frac{2}{\deg(v) - a_1 + \frac{19}{29}}. \tag{8}
\]

In particular, for any vertex \( v \), \( \text{prob}_1(v) \geq \frac{2}{\deg(v) + 2} \).

**Proof.** In Zagier [33], it was proved that

\[
\frac{2(k - 1)!}{k - a_1 + 2} \leq p_1^\lambda(k) \leq \frac{2(k - 1)!}{k - a_1 + \frac{19}{29}}.
\]

However, there are \((k - 1)!\) different ways to arrange the half edges around \( v \). Then, according to Corollary 3.2, the probability \( \text{prob}_1(v) \) of the resulting embedding to be unicellular after
rearranging the half edges around \( v \) satisfies

\[
\frac{2}{\deg(v) - a_1 + 2} \leq \text{prob}_1(v) \leq \frac{2}{\deg(v) - a_1 + \frac{19}{29}}.
\]

This completes the proof of the former part. Since it always holds that \( \frac{2}{\deg(v) - a_1 + 2} \geq \frac{2}{\deg(v) + 2} \), the latter part follows.

Now we come to study the second question: given a cellular map and a vertex there, whether it is the unique way of arranging all half edges around the vertex to achieve the genus in the present map, i.e., keep one-face?

**Theorem 3.5.** Let \( \mathbb{E} \) be a one-face embedding of \( G \), and \( v \) is a vertex of \( G \) with \( \deg(v) \geq 4 \). Then there is at least one another way to arrange the half edges around the vertex \( v \) such that the obtained embedding \( \mathbb{E}' \) has the same genus as \( \mathbb{E} \).

**Proof.** Assume \( d \geq 4 \) and

\[
v = (s_v, \pi_v) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_{i,1} & v_{i,2} & \cdots & v_{i,d-1} & v_{i,d} \end{pmatrix},
\]

where \( \pi_v = (v_1, V_2, \ldots, V_{d-1}, V_d) \). Firstly, from Lemma 2.3 we know that if there is \( V_l = v_p, V_m = v_q \) and \( 1 < l < m \leq d, 1 < p < q \leq d \) (i.e., Case 3), then there is at least one another way to arrange all half edges around the vertex \( v \) to keep the genus and we are done. If this is not true, then we must have \( \pi_v = (v_1, v_d, v_{d-1}, \ldots, v_2) \). In this case, we have

\[
v = (s_v, \pi_v) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_d & v_1 & \cdots & v_{d-2} & v_{d-1} \end{pmatrix},
\]

so that

\[
D_v = \begin{cases} (v_1, v_3, \ldots, v_d, v_2, v_4, \ldots, v_{d-1}), & d \in \text{odd}, \\ (1, 3, \ldots, d-1)(2, 4, \ldots, d), & d \in \text{even}. \end{cases}
\]

Next, we only need to show that if \( d \geq 4 \) we have \( R_v \geq 2 \) in all cases. Applying the formula to compute \( R_v \), if \( d \in \text{odd} \) we have

\[
R_v = \frac{(d - 1)!}{d} \sum_{i=0}^{d-1} (-1)^i \binom{d - 1}{i}^{-1} = \frac{2(d - 1)!}{d + 1}.
\]

The simplification of the summation is from the following formula [24]

\[
\sum_{i=0}^{n} (-1)^i \binom{x}{i}^{-1} = \frac{x + 1}{x + 2} (1 + (-1)^n \binom{x + 1}{n + 1}^{-1}).
\]

It is not hard to see that \( R_v \geq 2 \) if \( d \geq 4 \). Similarly, if \( 4 \mid d \) and \( d \geq 4 \), we have

\[
R_v = \sum_{i=0}^{d-1} (-1)^i \frac{i!(d - 1 - i)!}{d} + \sum_{i=\frac{d}{2}}^{d-1} (-1)^i \frac{i!(d - 1 - i)!}{d} \left[ (-1)^i + (-1)^i \binom{2}{1} \binom{d}{\frac{d}{2}} \right]
\]

\[
= \frac{2(d - 1)!}{d + 1} \left( 1 - \binom{d}{\frac{d}{2}} \right).
\]
If \( d \in \text{even} \) and \( 4 \nmid d \), we have
\[
R_v = \frac{d-1}{d+1}(1 + \left(\frac{d}{d+1}\right) - 1).
\]

In both cases, if \( d \geq 4 \), it is not hard to show \( R_v \geq 2 \) since both \( \frac{2(d-1)!}{d+1} \) and \( 1 - \left(\frac{d}{d+1}\right) \) are increasing functions of \( d \). Therefore, in all cases, if \( d \geq 4 \), then \( R_v \geq 2 \). This completes the proof.

Note if the number of half edges around a vertex is 1 or 2, it is trivial. The only special case is when the number of half edges around a vertex is 3. For such a vertex, it may be the unique arrangement of half edges around it to achieve the genus of the present map. Along the discussion, we actually have the following corollary

**Corollary 3.6.** Any even permutation on \([n]\) with \( n \geq 4 \) has at least two different factorizations into two \( n \)-cycles.

**Proof.** Since \( D_v = s_v \circ \pi_v^{-1} \) and both \( s_v \) as well as \( \pi_v \) have only one cycle, \( D_v \) is an even permutation. The proof for Theorem 3.5 just implies that \( D_v \) has at least 2 factorizations into two \( n \)-cycles.

**3.2 Variation of the embedding around more vertices**

Next, we slightly generalize above results by considering changing the local structure around more vertices of the underlying graph and their local embeddings.

Firstly, we study rearrangement of half edges around \( m \geq 1 \) vertices simultaneously and independently, i.e., the underlying graph is not changed. Given a plane permutation \((s, \pi)\) and \( m \) vertices \( V_1, \ldots, V_m \) in \( \pi \). Similar as the case of single vertex above, we can represent all these vertices by the plane permutation \( V_{1-m} = (s_{1-m}, \pi_{1-m}) \), where \( s_{1-m} \) is the subsequence obtained from \( s \) by keeping only half edges in \( V_1, \ldots, V_m \) and \( \pi_{1-m} \) is the restriction of \( \pi \) to these half edges.

Denote \( Dsh_{1-m} \) the number of different ways of simultaneous rearrangement of half edges around \( V_i \), \( (1 \leq i \leq m) \), respectively, and keep the unicellular property.

**Theorem 3.7.** Given a one-face embedding of \( G \) and \( m \) vertices \( V_1, \ldots, V_m \) there, \( Dsh_{1-m} \) is equal to the number of different ways to factor \( D_{V_{1-m}} \) into \( \gamma \sigma \), where \( \gamma \) has one cycle while \( \sigma \) has \( m \) disjoint cycles and each cycle is on the set of half edges of \( V_i \), respectively.

**Proof.** Applying the same idea of diagonal blocks rearrangement as in the case of single vertex completes the proof.
Now for a plane permutation \((s, \pi)\) and \(m\) vertices \(V_1, \ldots, V_m\) in \(\pi\), if the half edges belonging to one of these vertices are allowed to attach to another vertex among these \(m\) vertices, i.e., change the incident relation of these vertices and half edges around them, how many different ways to keep one-face? Assume the degree distribution of these \(m\) vertices is encoded by the partition \(\mu\). Let \(Le(V_1, \ldots, V_m; \mu)\) denote the number of different variations (including both local incident relation and local embedding) of these vertices to preserve the degree distribution and preserve one-face. Note the degree of a single vertex may change, but as a whole the degree distribution will not change. Let \(Le(V_1, \ldots, V_m)\) denote the number of different variations (including both local incident relation and local embedding) of these vertices to keep the number of vertices and keep one-face. Then, we have

**Theorem 3.8.** Assume the cycle-type of \(D_{V_1 \cdots m}\) is \(\lambda\) and the total number of half edges around these \(m\) vertices are \(q\). Then we have

\[
Le(V_1, \ldots, V_m; \mu) = f_{\mu, \lambda}(q),
\]

(9)

\[
Le(V_1, \ldots, V_m) = p^\lambda_m(q).
\]

(10)

**Remark 3.9.** The method to study local variation of maps in this section can be easily employed to study local variation of hypermaps.

## 4 Embeddings with \(k\) faces and \(k\)-cyc plane permutations

In this section, we generalize plane permutations \((s, \pi)\) to \(k\)-cyc plane permutations \((s, \pi)_k\) where \(s\) has \(k\) cycles, in order to study graph embeddings with \(k\) faces.

**Definition 4.1.** A \(k\)-cyc plane permutation on \([n]\) is a pair \(p = (s, \pi)\) where \(s\) is a permutation having \(k\) cycles and \(\pi\) is an arbitrary permutation. The permutation \(D_p = s \circ \pi^{-1}\) is called the diagonal of \(p\).

Assume \(s = (s_{11}, \ldots, s_{1m_1})(s_{21}, \ldots, s_{2m_2}) \cdots (s_{k1}, \ldots, s_{km_k})\), where \(\sum m_i = n\). A \(k\)-cyc plane permutation \((s, \pi)_k\) can be represented by two aligned rows:

\[
(s, \pi)_k = \begin{pmatrix}
S_{11} & s_{12} & \cdots & s_{1m_1} \\
\pi(s_{11}) & \pi(s_{12}) & \cdots & \pi(s_{1m_1}) \\
S_{21} & s_{2m_2} & \cdots & s_{km_k} \\
\pi(s_{21}) & \pi(s_{2m_2}) & \cdots & \pi(s_{km_k})
\end{pmatrix}
\]

where each adjacent pair of “boxed” elements (one is on the top row and the other is on the bottom row, e.g., \(s_{11}\) and \(\pi(s_{1m_1})\)) indicates a face. Then, \(D_p\) can be explicitly defined as follows:

- For \(1 \leq i \leq k\), \(D_p(\pi(s_{ij})) = s_{i(j+1)}\) if \(j \neq m_i\);  
- For \(1 \leq i \leq k\), \(D_p(\pi(s_{im_i})) = s_{11}\).

Since every embedding with \(k\) faces can be encoded into a triple \((\alpha, \beta, \gamma)\) where \(\gamma = \alpha \beta\) and \(\gamma\) has \(k\) cycles, every embedding can be encoded into a \(k\)-cyc plane permutation as well, i.e., \(s = \gamma\) \(\pi = \beta\) and \(D_p = \alpha\). A \(k\)-cyc plane permutation can be viewed as a concatenation of \(k\) “pseudo”
plane permutations induced by $k$ faces, where the “pseudo” plane permutation induced by the face $f_i$ is

$$\left( \begin{array}{cccc}
    s_{i1} & s_{i2} & \cdots & s_{im_i} \\
    \pi(s_{i1}) & \pi(s_{i2}) & \cdots & \pi(s_{im_i}) 
\end{array} \right).$$

We will denote $f_i$ this “pseudo” plane permutation if no confusions occur. Let $H(f)$ denote the set of half edges contained in the face $f$.

**Lemma 4.2.** Let $v$ be a vertex of the graph $G$ and $\mathbb{E}$ be an embedding of $G$, where $v$ is incident to $q$ faces, $f_i$, for $1 \leq i \leq q$. Let $\mathbb{E}'$ be another embedding which is obtained by rearranging the half edges around $v$ so that $v$ is incident to $q'$ faces, $f'_i$, for $1 \leq i \leq q'$. Then,

$$\bigcup_{i=1}^{q} H(f_i) = \bigcup_{i=1}^{q'} H(f'_i), \quad q \equiv q' \pmod{2}.$$

**Proof.** Firstly, every face $f$ of $\mathbb{E}$ can be expressed as $\{D_p \pi(z), (D_p \pi)^2(z), \ldots\}$ for any $z \in H(f)$. If $f \neq f_i$ for $1 \leq i \leq q$, then for any $z \in f$, $\pi'(z) = \pi(z)$ and $D_p(\pi(z)) = D_p(\pi'(z))$. Hence, the face $f$ in $\mathbb{E}$ is a face in $\mathbb{E}'$. Next, we will show that $\bigcup_i H(f_i)$ will reorganize into $q'$ faces, $f'_i$, for $1 \leq i \leq q'$, and all these $q'$ faces are incident to $v$. It suffices to show each $f'_i$ contains at least one half edge of $v$. For any half edge $u \in \bigcup_i H(f_i)$ that does not belong to $H(v)$, assume $u$ is contained in the face $f_j$ of $\mathbb{E}$ while in the face $f'_k$ of $\mathbb{E}'$. Then,

$$f_j = \{D_p \pi(u), (D_p \pi)^2(u), \ldots, v_i, D_p(\pi(v_i)), \ldots\}$$

$$f'_k = \{D_p \pi(u), (D_p \pi)^2(u), \ldots\}$$

where $v_i$ is the first half edge of $v$ appeared in $f_j$. We know that, if $\pi'(z) = \pi(z)$ then $D_p(\pi(z)) = D_p(\pi'(z))$. Thus, $v_i \in H(f'_k)$ and the segment from $D_p(\pi(u))$ to $v_i$ in $f_j$ is completely the same as the segment from $D_p(\pi(u))$ to $v_i$ in $f'_k$. Therefore, there is no face among $f'_i$ for $1 \leq i \leq q'$ which does not contain a half edge of $v$. Finally, the parity equivalence between $q$ and $q'$ comes from Euler characteristic formula. This completes the proof. \Halmos

**Corollary 4.3.** Let $\mathbb{E}$ be an embedding of the graph $G$ and $v$ be a vertex of $G$, where $v$ is of degree $\deg(v)$ and incident to $q$ faces in $\mathbb{E}$. Assume $\mathbb{E}'$ is another embedding which is obtained by rearranging the half edges around $v$. Then,

$$-\left\lfloor \frac{\deg(v) - q}{2} \right\rfloor \leq g(\mathbb{E}') - g(\mathbb{E}) \leq \left\lfloor \frac{q - 1}{2} \right\rfloor$$

(11)

**Proof.** According to Lemma 4.2, rearranging half edges around $v$ will at most increase the number of faces by $\deg(v) - q$ and at most decrease the number faces by $q - 1$ whence the corollary. \Halmos

Let $\mathbb{E}$ be an embedding of the graph $G$ and $v$ be a vertex of $G$. Assume $v$ is incident to $q$ faces. Then, $v$ can be naturally encoded into a $q$-cyc plane permutation $(s_v, \pi_v)_q$ obtained as follows: $s_v$ has $q$ cycles, where each cycle is obtained by deleting all half edges in a face.
incident to \( v \) except the elements in \( H(v) \), i.e., each cycle is induced from the cyclic order of a face incident to \( v \), \( \pi_v \) is the restriction of \( \pi \) to \( H(v) \).

Given an embedding \( \mathbb{E} \) and a vertex \( v \), where \( v \) is incident to the faces \( f_i \) for \( 1 \leq i \leq q \). Then, similar to one-face case, the half edges in \( H(v) \) will segment the \( q \) “pseudo” plane permutations corresponding to these \( q \) faces into diagonal blocks as well, where each diagonal block is determined by the left lower corner \( \pi_s(x) \) and the right upper corner \( s_v(x) \). We will show that rearranging the half edges around \( v \) is to rearrange these diagonal blocks.

**Theorem 4.4.** Let \( \mathbb{E} \) be an embedding of the graph \( G \) and \( v \) be a vertex of \( G \). Assume \( v \) is incident to \( q \) faces. Let \( D_v \) be the diagonal of \( (s_v, \pi_v)_q \), and denote \( R_v(\Delta g) \) the number of different ways to arrange the half edges around \( v \) such that the obtained embedding \( \mathbb{E}' \) has genus \( g(\mathbb{E}') = g(\mathbb{E}) + \Delta g \). Then, we have

\[
R_v(\Delta g) = p_{q+2\Delta g}^\lambda(D_v)(\deg(v)),
\]

where \( \lambda(D_v) \) is the cycle-type of \( D_v \).

**Proof.** We prove the theorem by showing that for every cyclic arrangement \( \theta \) of the half edges around \( v \) such that the obtained embedding \( \mathbb{E}' \) has genus \( g(\mathbb{E}') = g(\mathbb{E}) + \Delta g \) satisfies that \( D_v \circ \theta \) has \( q + 2\Delta g \) cycles, and each \( \theta \) that \( D_v \circ \theta \) has \( q + 2\Delta g \) cycles gives an embedding \( \mathbb{E}' \) has genus \( g(\mathbb{E}') = g(\mathbb{E}) + \Delta g \).

(\( \Rightarrow \)): Suppose \( v \) is incident to \( q \) faces in \( \mathbb{E}, f_1, \ldots, f_q \). Thus, \( H(v) = \bigcup_i H(f_i) \). Assume \( \theta \) on \( H(v) \) gives an embedding \( \mathbb{E}' \) with genus \( g(E) + \Delta g \). According to Lemma 4.2, \( H(v) \) will be reorganized into \( q + 2\Delta g \) faces, \( f'_1, \ldots, f'_{q+2\Delta g} \). The rest of faces in \( \mathbb{E} \) will not be impacted. We will show that \( D_v \circ \theta \) has \( q + 2\Delta g \) cycles, where each cycle is uniquely induced from one face \( f'_k \). Given the face \( f'_k \),

\[
f'_k = \left( v_{i_1}^{j_1}, \ldots, v_{i_2}^{j_1}, \ldots, v_{i_1}^{j_2}, \ldots, v_{i_2}^{j_2}, \ldots, v_{i_1}^{j_{t_1}}, \ldots, v_{i_2}^{j_{t_1}}, \ldots, y \right),
\]

where \( v_{ik}^{j_i}, v_{ij}^{j_i} \in H(v), v_{ij}^{j_i} = \theta(v_{ik}^{j_i}) \). By the same reasoning as the proof for Theorem 3.1, the diagonal block

\[
\begin{array}{cccc}
x_1 & & & x_2 \\
& v_{i_1}^{j_1} & & v_{i_2}^{j_1} \\
& \cdots & \cdots & \cdots \\
& x_1 & x_2 & \cdots \end{array}
\]

is also a diagonal block in \( \mathbb{E} \), which implies \( D_v(v_{ij}^{j_i}) = v_{ij}^{j_2} \). Therefore, \( D_v \circ \theta(v_{i_1}^{j_1}) = v_{i_2}^{j_2} \). Considering all other diagonal blocks, we have \( (v_{i_1}^{j_1}, v_{i_2}^{j_1}, \ldots, v_{i_t}^{j_t}) \) is a cycle of \( D_v \circ \theta \).

(\( \Leftarrow \)): given any \( \theta \) on \( H(v) \) such that \( D_v \circ \theta \) has \( q + 2\Delta g \) cycles, it will induce a \( (q + 2\Delta g) \)-cycle plane permutation \( v = (s_v', \pi_v')_{q+2\Delta g}, \)

\[
v = \left( \begin{array}{cccccc}
\theta(v_{11}^{j_1}) & \ldots & \theta(v_{1r_1}^{j_1}) & \theta(v_{21}^{j_1}) & \ldots & \theta(v_{2r_2}^{j_1}) \\
\theta(v_{q+2\Delta g}^{j_1}) & \ldots & \theta(v_{(q+2\Delta g)1}^{j_1}) & \theta(v_{2q+2\Delta g}^{j_1}) & \ldots & \theta(v_{(q+2\Delta g)q+2\Delta g}^{j_1})
\end{array} \right)
\]

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By extending the pair \((\theta(v_{ik}), s'_v(v_{ik}))\) into the diagonal block which has \(\theta(v_{ik})\) as the left lower corner and \(s'_v(v_{ik})\) as the right upper corner, and concatenating with the rest of faces in \(E\), we obtain an embedding \(E'\) with \(2\Delta g\) more faces than \(E\), i.e., \(g(E') = g(E) + \Delta g\). It can be shown that only the half edges around the vertex \(v\) are cyclically arranged in different manners in \(E\) and \(E'\), so each \(\theta\) uniquely induces an embedding \(E'\) which is obtained by rearranging the half edges around \(v\) and has genus \(g(E') = g(E) + \Delta g\). This completes the proof. 

In fact, by similar reasoning, we can obtain a more general result. Given an embedding \(E\) of the graph \(G\) and one vertex \(v\) of \(G\), let \(deg(v) = k\) and assume there are \(a_i\) faces of \(E\) where each contains \(i\) half edges in \(H(v)\). We call \(\mu = 1^{a_1}2^{a_2} \cdots k^{a_k}\) the \(f\)-incidence degree distribution of \(v\) w.r.t. \(E\). Now how many embeddings \(E'\) where the \(f\)-incidence degree distribution of \(v\) is \(\eta\) can be obtained by rearranging the half edges around \(v\)? We denote this number as \(R_v(\eta)\).

**Theorem 4.5.**

\[
R_v(\eta) = f_{\eta, \lambda(D_v)}(deg(v)).
\] (13)

As the first corollary of Theorem 4.4, we obtain the following (local) version of “interpolation” theorem.

**Corollary 4.6.** Let \(E\) be an embedding of the graph \(G\). If there is a vertex \(v = (s_v, \pi_v)_q\) then there exists an embedding \(E'\) of \(G\) such that \(g(E') = g(E) + \Delta g\) for any

\[
-\left[\frac{deg(v) + 1 - \ell(\lambda(D_v)) - q}{2}\right] \leq \Delta g \leq \left[\frac{q - 1}{2}\right].
\]

In particular, if there is a vertex \(v\) of \(G\) which is incident to every face of \(E\), then \(G\) is upper embeddable.

**Proof.** According to Corollary 2.6, \(p^k(\lambda(n)) \neq 0\) as long as \(p_{k+2i}(\lambda(n)) \neq 0\) for some \(i > 0\). And from Proposition 2.7, \(p^\lambda_{deg(v) + 1 - \ell(\lambda(D_v))}(deg(v)) \neq 0\). Therefore, for any

\[
-\left[\frac{deg(v) + 1 - \ell(\lambda(D_v)) - q}{2}\right] \leq \Delta g \leq \left[\frac{q - 1}{2}\right],
\]

\(p^\lambda_{q - 2\Delta g}(deg(v)) \neq 0\). Namely, rearranging the half edges around \(v\) can lead to an embedding \(E'\) such that \(E'\) has \(2\Delta g\) less faces. Hence, \(g(E') = g(E) + \Delta g\). If there is a vertex \(v\) of \(G\) which is incident to every face of \(E\), then rearranging the half edges around \(v\) can lead to an embedding with either 1 face or 2 faces, depending on the parity of the number of faces in \(E\). Thus, \(G\) is upper embeddable.

In fact, the result in Corollary 4.6 can be further optimized. Given two vertices, if there is no face of the embedding \(E\) incident to both vertices, or if there is only one face \(f_0\) of the embedding \(E\) incident to both vertices where all the half edges of one vertex contained in \(f_0\) are completely contained in a diagonal block determined by the other vertex, the two vertices are called \(E\)-facial disjoint. Applying Lemma 4.2 and diagonal blocks rearrangement argument, if two vertices are \(E\)-facial disjoint, re-embedding them simultaneously will not interfere with each other. Hence, we have
Corollary 4.7. Let $E$ be an embedding of the graph $G$. If vertices $v_i = (s_{v_i}, \pi_{v_i})_{q_i}, 1 \leq i \leq m$, are mutually $E$-facial disjoint, then there exists an embedding $E'$ of $G$ such that $g(E') = g(E) + \Delta g$ for any

$$\sum_{i=1}^{m} \left[ \frac{\deg(v_i) + 1 - \ell(\lambda(D_{v_i})) - q_i}{2} \right] \leq \Delta g \leq \sum_{i=1}^{m} \left[ \frac{q_i - 1}{2} \right].$$

Proof. Since $v_i$ are mutually $E$-facial disjoint, the range of genus difference achieved by reembedding $v_i$ do not interfere with each other. Therefore, the differences can be combined together whence the corollary.

Corollary 4.8. Let $E$ be an embedding of the graph $G$ with genus $g_{\text{max}}(G)$. Then, every vertex is incident to at most 2 faces.

Proof. If there is a vertex $v$ with no less than 3 faces, according to Corollary 2.6, there exists an embedding with 2 less faces. Hence, $E$ can not be an embedding of the graph $G$ with the maximum genus $g_{\text{max}}(G)$.

The fact that if there exists a vertex incident to at least 3 faces in an embedding, an embedding with higher genus always exists has been well known in the literature, e.g., in [16, 30]. In particular, Corollary 4.8 is the same as a very recent result in [16] where locally maximal embedding is studied. In our context, a locally maximal embedding can be defined as an embedding from which a higher genus embedding can not be obtained by rearranging the half edges around one of the vertices. Then, we can actually restate that if an embedding is locally maximal, then every vertex is incident to at most 2 faces.

However, to the best of our knowledge, it seems there is no simple characterization to determine if there exists a lower genus embedding based on a given embedding. Thus, the lower bound of $\Delta g$ in Corollary 4.6 may be the first simple characterization. Furthermore, we obtain the following necessary condition for an embedding of $G$ to be an embedding with the minimum genus. We mention that there is a sufficient condition for an embedding of $G$ to be an embedding with the minimum genus in Thomassen [27]

Corollary 4.9. Let $E$ be an embedding of the graph $G$ with genus $g_{\text{min}}(G)$, and a vertex $v = (s_v, \pi_v)_{q_v}$. Then,

$$\ell(\lambda(D_v)) + q_v = \deg(v) + 1. \quad (14)$$

Proof. Otherwise, we can increase the number of faces by rearranging the half edges around $v$ so that we obtain an embedding with even lower genus, which contradict the fact that $E$ is an embedding of the graph $G$ with genus $g_{\text{min}}(G)$.

Surely, we can define locally minimal embedding analogously. So, we obtain that if an embedding is locally minimal, then, for every vertex $v$,

$$\ell(\lambda(D_v)) + q_v = \deg(v) + 1. \quad (15)$$

It is also obvious that if the embedding $E$ is locally minimal and all vertices of $G$ are mutually $E$-facial disjoint, then the genus of $E$ is equal to $g_{\text{min}}(G)$. 

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In addition, Corollary 4.6 provides an easy approach to estimate the range \([g_{\min}(G), g_{\max}(G)]\) for a given graph \(G\): we can randomly try several embeddings of \(G\), and from each embedding, we obtain an estimate for \([g_{\min}(G), g_{\max}(G)]\) using the following theorem, and finally combine these estimates.

**Theorem 4.10.** Let \(E\) be an embedding of the graph \(G\) and for a vertex \(v\) of \(G\), assume \(v = (s_v, \pi_v)_q\). Let

\[
T_1 = \min_v \{-\frac{\deg(v) + 1 - \ell(\lambda(D_v)) - q_v}{2} \}, \quad T_2 = \max_v \{\lfloor \frac{q_v - 1}{2} \rfloor \}.
\]

Then, we have

\[
g_{\min}(G) \leq g(E) + T_1 \leq g(E) + T_2 \leq g_{\max}(G).
\]

(16)

We remark that this approach to estimate the genus range can be optimized in the similar manner as in Corollary 4.7, i.e., the minimum (resp. maximum) can take over all \(E\)-facial disjoint covers, where a \(E\)-facial disjoint cover is a set of mutually \(E\)-facial disjoint vertices such that the union of their incident faces is the set of all faces of \(E\). Additionally, it may be possible to obtain a more efficient procedure to determine \([g_{\min}(G), g_{\max}(G)]\) if we combine our approach here and other algorithms to generate an embedding (if there is) of \(G\) on a given surface of genus \(g\), e.g., the linear algorithm in [17]. A rough idea could be as follows: suppose we know that \(0 \leq g_{\min}(G) \leq a \leq b \leq g_{\max}(G) \leq \frac{B(G)}{2}\). We can next choose a number in \([0, a]\) or \([b, \frac{B(G)}{2}\])], say \(k\). Then, apply the linear algorithm to generate an embedding of \(G\) on \(S_k\) and extend the range \([a, b]\) based on the obtained embedding by our approach. If there is no embedding on \(S_k\), we can update the outer bound, i.e., \([0, \frac{B(G)}{2}\])]. Iterating this procedure, we can eventually obtain \([g_{\min}(G), g_{\max}(G)]\).

In the following, we present an analogue of Case 3 (and Case 4, 5, 6), Case 1 and Case 2 in Lemma 2.3, which increases the genus by 0, 1 and \(-1\), respectively. In the derivation, a kind of local Poincaré dual is applied.

**Proposition 4.11.** Let \(E\) be an embedding of the graph \(G\) and a vertex \(v = (s_v, \pi_v)_q\), where

\[
\pi_v = (s_{i-1}, v_1^j, \ldots, v_{m_j}, s_j, v_1^j, \ldots, v_{m_j}, s_t, v_1^l, \ldots, v_{m_l}).
\]

If in \(E\), there exists a face of the form \((s_{i-1}, \ldots, s_{j}, \ldots, s_t, \ldots)\), or two faces of the form

\[
(s_{i-1}, \ldots, s_j, \ldots)(s_t, \ldots),
\]

then rearranging \(H(v)\) according to the cyclic order

\[
(s_{i-1}, v_1^j, \ldots, v_{m_j}, s_t, v_1^l, \ldots, v_{m_l}, s_j, v_1^j, \ldots, v_{m_j})
\]

will lead to the embedding \(E'\) with \(g(E') = g(E)\).
Proof. Since \( v = (s_v, \pi_v)_q \), we have \( s_v = D_v \circ \pi_v \), where \( s_v \) has \( q \) cycles and \( \pi_v \) has only one cycle. This is equivalent to \( \pi_v = D_v^{-1} \circ s_v \) which corresponds to a plane permutation \((\pi_v, s_v)\) with diagonal \( D_v^{-1} \), i.e., a kind of local Poincaré dual. Now the given conditions in the proposition either agree with Case 3 or one of \{Case 4, Case 5, Case 6\} in Lemma 2.3. Namely, if we transpose \( \pi_v \) into
\[
(s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l),
\]
we obtain a new plane permutation \((\pi_v', s_v')\) where the number of cycles in \( s_v' \) equals to the number of cycles in \( s_v \). That is, rearranging \( H(v) \) according to the cyclic order
\[
(s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l)
\]
will not change the number of faces of the embedding. Therefore, the resulting embedding \( \mathcal{E}' \) satisfies \( g(\mathcal{E}') = g(\mathcal{E}) \).

**Proposition 4.12.** Let \( \mathcal{E} \) be an embedding of the graph \( G \) and a vertex \( v = (s_v, \pi_v)_q \), where
\[
\pi_v = (s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l).
\]
If \( s_{i-1}, s_j \) and \( s_I \) are contained respectively in three faces in \( \mathcal{E} \), then rearranging \( H(v) \) according to the cyclic order
\[
(s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l)
\]
will lead to the embedding \( \mathcal{E}' \) with \( g(\mathcal{E}') = g(\mathcal{E}) + 1 \).

**Proof.** After applying the “local Poincaré dual”, the given conditions in the proposition agree with Case 1 in Lemma 2.3 whence the proposition.

**Proposition 4.13.** Let \( \mathcal{E} \) be an embedding of the graph \( G \) and a vertex \( v = (s_v, \pi_v)_q \), where
\[
\pi_v = (s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l).
\]
If in \( \mathcal{E} \), there exists a face of the form \((s_{i-1}, \ldots, s_I, \ldots, s_j, \ldots)\), then rearranging \( H(v) \) according to the cyclic order
\[
(s_{i-1}, v_{j_1}^1, \ldots, v_{m_j}^j, s_I, v_{l_1}^1, \ldots, v_{m_l}^l, s^1_j, v_{l_1}^1, \ldots, v_{m_l}^l)
\]
will lead to the embedding \( \mathcal{E}' \) with \( g(\mathcal{E}') = g(\mathcal{E}) - 1 \).

**Proof.** After applying the “local Poincaré dual”, the given conditions in the proposition agree with Case 2 in Lemma 2.3 whence the proposition.

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