Hodge-to-de Rham degeneration for stacks

Dmitry Kubrak, Artem Prikhodko

Abstract

We introduce a notion of a Hodge-proper stack and extend the method of Deligne-Illusie to prove the Hodge-to-de Rham degeneration in this setting. In order to reduce the statement in characteristic 0 to characteristic p, we need to find a good integral model of a stack (a so-called spreading), which, unlike in the case of schemes, need not to exist in general. To address this problem we investigate the property of spreadability in more detail by generalizing standard spreading out results for schemes to higher Artin stacks and showing that all proper and some global quotient stacks are Hodge-properly spreadable. As a corollary we deduce a (non-canonical) Hodge decomposition of the equivariant cohomology for certain classes of varieties with an algebraic group action.

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0 Introduction

0.1 Deligne-Illusie method for schemes

Let X be a smooth scheme over $\mathbb{C}$ and let $X(\mathbb{C})$ be the topological space of its complex points. Grothendieck has shown that there is a formula for the singular cohomology of $X(\mathbb{C})$ in purely algebraic terms, namely

$$H^n_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq H^n_{\text{dR}}(X/\mathbb{C}),$$

where the de Rham cohomology $H^n_{\text{dR}}(X/\mathbb{C})$ is defined as the $n$-th hypercohomology of the algebraic de Rham complex of $X$. If, moreover, $X$ is projective, using Hodge theory one obtains the Hodge decomposition

$$H^n_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X, \Omega^p_X).$$
As was shown by Deligne and Illusie \([57x458]\) graded pieces are \(R\) called “stupid”) filtration \(\Omega_{\geq}^{p}\) whose associated graded is given by the sum above. Namely, the de Rham complex has a natural cellular (also called “stupid”) filtration \(\Omega_{\geq}^{p}\) given by subcomplexes

\[
\Omega_{X,\text{dR}}^{\geq p} := \ldots \rightarrow 0 \rightarrow \Omega_{X}^{p} \rightarrow \Omega_{X}^{p+1} \rightarrow \ldots \rightarrow \Omega_{X}^{\dim X}.
\]

This filtration induces a filtration on the complex of global sections \(R\Gamma(\text{dR})(X/\mathbb{C}) := R\Gamma(X, \Omega_{X,\text{dR}}^{p})\) whose associated graded pieces are \(R\Gamma(X, \Omega_{X}^{p}[-p])\). As a consequence one gets the so-called \(\text{Hodge-to-de Rham spectral sequence}\):

\[
E_{1}^{p,q} = H^{q}(X, \Omega_{X/k}^{p}) \Rightarrow H_{\text{dR}}^{p+q}(X/\mathbb{C}).
\]

As was shown by Deligne and Illusie \([D\text{I}87]\) there is a purely algebraic proof of the degeneration of the spectral sequence above, thus the induced filtration \(F^{*}H_{\text{dR}}^{n}(X/\mathbb{C})\) on the de Rham cohomology has the associated graded

\[
\text{gr} F^{*}H_{\text{dR}}^{n}(X/\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{q}(X, \Omega_{X}^{p}).
\]

The strategy of Deligne-Illusie is to reduce the statement in characteristic 0 to an analogous question in big enough positive characteristic. Let \(k\) be a perfect field of characteristic \(p\) and let \(Y\) be a smooth scheme over \(k\). Then we have:

**Theorem 0.1.1** (Cartier). Let \(Y^{(1)}\) denote the Frobenius twist of \(Y\) and let \(\varphi_{Y} : Y \rightarrow Y^{(1)}\) be the relative Frobenius morphism. Then there exists a unique isomorphism of sheaves of \(\mathcal{O}_{Y^{(1)}}\)-algebras on \(Y\)_{\text{zar}}

\[
C_{Y}^{-1} : \bigoplus_{i} \Omega_{Y^{(1)}}^{i} \rightarrow \bigoplus_{i} H^{i}(\varphi_{Y*} \Omega_{Y,\text{dR}}^{i}),
\]

determined by the property that for any local section \(f\) of \(\mathcal{O}_{Y}\)

\[
C_{Y}^{-1}(df) = "d f^p/p" := f^{p-1}df.
\]

The map \(C_{Y}^{-1}\) is called the inverse Cartier isomorphism.

This way we see that the Postnikov (also called “canonical”) filtration on \(\varphi_{Y*} \Omega_{Y,\text{dR}}^{*}\) induces another filtration on \(R\Gamma(Y, \Omega_{Y,\text{dR}}^{*}) \simeq R\Gamma(Y^{(1)}, \varphi_{Y} \Omega_{Y,\text{dR}}^{*})\) whose associated graded pieces are \(R\Gamma(Y^{(1)}, \Omega_{Y^{(1)}}^{p}[-p])\). Taking the corresponding spectral sequence induced by this filtration we obtain the so-called conjugate spectral sequence

\[
E_{2}^{p,q} = H^{q}(Y^{(1)}, \Omega_{Y^{(1)}}^{p}) \Rightarrow H_{\text{dR}}^{p+q}(Y/k).
\]

Note that for any spectral sequence the \(E_{\infty}\)-page is always a subfactor of the \(E_{r}\)-page \((r \geq 0)\), hence \(\dim_{k} E_{r}^{*,*} \leq \dim_{k} E_{\infty}^{*,*}\). If all vector spaces \(E_{r}^{*,*}\) are finite-dimensional, equality holds if and only if all differentials starting from the \(r\)-th page vanish. It follows that for \(Y\) proper, the conjugate spectral sequence degenerates if and only if \(\dim_{k} H_{\text{dR}}^{p,0}(Y) = \sum_{p+q=n} \dim_{k} H_{\text{dR}}^{p,q}(Y^{(1)})\). Since \(\dim_{k} H_{\text{dR}}^{p,q}(Y^{(1)}) = \dim_{k} H_{\text{dR}}^{p,q}(Y)\) this happens if and only if the Hodge-to-de Rham spectral sequence degenerates as well.

The differentials in the conjugate spectral sequence are induced by the connecting homomorphisms for the Postnikov filtration on \(\varphi_{Y*} \Omega_{Y,\text{dR}}^{*}\). In particular, if \(\varphi_{Y*} \Omega_{Y,\text{dR}}^{*}\) is formal (i.e. quasi-isomorphic to the sum of its cohomology), then the conjugate spectral sequence degenerates. While in general this is hard to guarantee, the formality of the truncation \(\tau_{\leq p-1} \varphi_{Y*} \Omega_{Y,\text{dR}}^{*}\) turns out to be equivalent to the existence of a lift to the second Witt vectors \(W_{2}(k)\):

**Theorem 0.1.2** (Deligne-Illusie). A smooth scheme \(Y\) over \(k\) admits a lift to \(W_{2}(k)\) if and only if there exists an equivalence

\[
\bigoplus_{i=0}^{p-1} \Omega_{Y^{(1)}}^{i} \rightarrow \tau_{\leq p-1} \varphi_{Y*} \Omega_{Y,\text{dR}}^{*}
\]

inducing the inverse Cartier isomorphism \(C_{Y}^{-1}\) on \(\mathcal{H}^{*}\). In particular if \(Y\) admits a lift to \(W_{2}(k)\) and \(\dim Y < p\), then the complex \(\varphi_{Y*} \Omega_{Y,\text{dR}}^{*}\) is formal, and hence the Hodge-to-de Rham spectral sequence degenerates at the first page.

The proof of the degeneration in characteristic 0 is then accomplished by choosing a smooth proper model (the so-called spreading) \(X_{R}\) of \(X\) over some finitely generated \(\mathbb{Z}\)-subalgebra \(R\) of \(F\). Enlarging \(R\) if needed, one can assume that the \(R\)-modules \(H^{q}(X_{R}, \Omega_{X_{R}}^{p})\) and \(H_{\text{dR}}^{p}(X_{R}/R)\) are free of finite rank. Picking a closed point of residue characteristic \(p > \dim X\) one reduces to Theorem 0.1.2.
0.2 Generalization to stacks

In this work we extend the results of Deligne-Illusie to the case of Artin stacks. For a smooth proper Deligne-Mumford stack one can proceed with the original arguments (see e.g. [Sat12, Corollary 1.7]), but they do not seem to work for a general smooth Artin stack (see Remark 0.2.3). Instead we use another approach relying on quasi-syntomic descent for the derived de Rham cohomology.

As in the case of schemes, to establish Hodge-to-de Rham degeneration, we need to impose some properness assumptions. However, the standard notion of a proper stack is too restrictive for our purposes. For example, the quotient stack \([X/G]\) of a proper scheme \(X\) by an action of a linear algebraic group \(G\) is proper if and only if the stabilizers of all points of \(X\) are finite group schemes. On the other hand, as we will see in Section 2.2.3, the Hodge-to-de Rham spectral sequence for \([X/G]\) always degenerates provided \(G\) is a reductive group (or even a parabolic subgroup in one, see Example 2.2.19). So, we consider the following more general notion instead:

**Definition 0.2.1.** Let \(R\) be a Noetherian ring. A smooth Artin stack \(X\) over \(R\) is called Hodge-proper if \(H^q(X, \wedge^p L_{X/R})\) is finitely generated for all \(p\) and \(q\), where \(L_{X/R}\) is the cotangent complex of \(X\) over \(R\).

\(R\Gamma(X, \wedge^p L_{X/R})\) is a natural analogue of \(R\Gamma(X, \Omega^p_{X})\) and, similarly to the scheme case, the de Rham cohomology complex \(R\Gamma_{\text{dR}}(X/R)\) has a natural (Hodge) filtration whose associated graded pieces are \(R\Gamma(X, \wedge^p L_{X/R}[-p])\); see Section 1.1 for more details. In this way one obtains a spectral sequence

\[
E_1^{p,q} = H^q(X, \wedge^p L_{X/R}) \Rightarrow H^{p+q}_{\text{dR}}(X/R),
\]

In the case \(R = F\) is a field this spectral sequence degenerates if and only if

\[
\dim F H^{p}_{\text{dR}}(X/F) = \sum_{p+q=n} \dim F H^q(X, \wedge^p L_{X/F}). \tag{1}
\]

**Remark 0.2.2.** By smooth descent for the cotangent complex, \(R\Gamma(X, \wedge^p L_{X/F})\) produces the same answer as a more common definition of the Hodge cohomology via the lisse-étale site of \(X\) (see Proposition 1.1.4).

We will now explain the strategy of our proof of the equality (1) above. The first step is to extend Theorem 0.1.2 to the setting of stacks:

**Theorem (1.3.18).** Let \(\mathcal{Y}\) be a smooth Artin stack over a perfect field \(k\) of characteristic \(p\) admitting a smooth lift to the ring of the second Witt vectors \(W_2(k)\). Then there is a canonical equivalence

\[
R\Gamma(\mathcal{Y}, \tau^{\leq p-1} \Omega^\bullet_{\mathcal{Y}, \text{dR}}) \simeq R\Gamma \left( \mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i L_{\mathcal{Y}^{(1)}}[-i] \right).
\]

In particular for \(n \leq p - 1\) we have \(H^n_{\text{dR}}(\mathcal{Y}) \simeq H^n(\mathcal{Y}^{(1)})\).

Since the de Rham cohomology for Artin stacks are defined as the right Kan extension from smooth affine schemes (Definition 1.1.3) one can more or less formally deduce the theorem above from the following very functorial form of Deligne-Illusie splitting for affine schemes:

**Theorem (1.3.16).** Let \(\text{Aff}^n_{W_2(k)}\) be the category of smooth affine schemes over \(W_2(k)\). Then there is a natural \(\sigma\)-linear equivalence of functors

\[
\bigoplus_{i=0}^{p-1} \Omega^i_{B} : B \mapsto \bigoplus_{i=0}^{p-1} \Omega^i_{(B^{(1)}/p)/k}[-i] \quad \text{and} \quad \tau^{\leq p-1} \Omega^\bullet_{-,-,\text{dR}} : B \mapsto \tau^{\leq p-1} \Omega^\bullet_{(B/p)/k, \text{dR}}
\]

from \(\text{Aff}^n_{W_2(k)}\) to \(\infty\)-category \(D(\text{Mod}_k)\) which induces the Cartier isomorphism on the level of the individual cohomology functors.

The splitting in Theorem 0.1.2 is already functorial with respect to liftings to \(W_2(k)\), but only on the level of the underlying homotopy category and not the \(\infty\)-category of complexes \(D(\text{Mod}_k)\) itself. To get this higher functoriality we follow [FM87, Section II] using a more convenient language of [BMS19].

The idea is to extend the de Rham (and crystalline) cohomology functors to a larger category of quasisyntomic algebras (Definition 1.3.1). This category, endowed with quasisyntomic topology, has a basis consisting of quasiregular semiperfectoid \(W_n(k)\)-algebras (Definition 1.3.2), on which the values of \(R\Gamma_{\text{dR}}\) (and \(R\Gamma_{\text{cryst}}\)) become ordinary.
rings. Additionally, the Frobenius morphism, the Hodge filtration and the conjugate filtration can be described explicitly. This way, using quasi-syntomic descent, the question reduces to a certain computation in commutative algebra.

More concretely, for a quasi-regular semiperfect $k$-algebra $S$ one can prove that $R\Gamma_{\mathrm{crys}}(S/W_n(k)) \simeq \mathcal{A}_{\mathrm{crys}}(S)/p^n$, where $\mathcal{A}_{\mathrm{crys}}(S)$ is the divided power envelope of the kernel of the natural surjection $W((S)^p) \to S$ (see Construction 1.3.10). Under this identification the Hodge filtration on $R\Gamma_{\mathrm{crys}}(-/k) \simeq R\Gamma_{\mathrm{dR}}(-/k)$ corresponds to the filtration by powers of the pd-ideal $I \triangleleft \mathcal{A}_{\mathrm{crys}}(S)/p$. The conjugate filtration $\mathcal{F}_{\mathrm{conj}}$ admits an explicit description as well (see Definition 1.3.12). Given a lifting $\tilde{S}$ of $S$ to $W_2(k)$ there is a natural morphism $\theta: \mathcal{A}_{\mathrm{crys}}(S)/p^2 \to \tilde{S}$. The image of $K := \ker \theta$ under the first divided Frobenius map $\varphi_1$ then provides a splitting of $\mathcal{F}_{\mathrm{conj}}$ into $\mathcal{F}_{\mathrm{conj}}^1 \oplus \mathcal{F}_{\mathrm{conj}}^1 / \mathcal{F}_{\mathrm{conj}}^0 \cong S^p/I \oplus I/I^2$ (Proposition 1.3.17). By multiplicativity this extends to the splitting of $\mathcal{F}_{\mathrm{conj}}^p$ whose descent to smooth schemes gives Theorem 1.3.16.

**Remark 0.2.3.** The original approach of Deligne-Illusie (at least applied literally) does not seem to work for a general Artin stack; the key result of [DI87] is the equivalence of two gerbes on the étale site of $Y(1)/k$ for a smooth $k$-scheme $Y$: the one of splittings of $\tau_{-1}^1 \varphi_Y \Omega_{Y,\mathrm{dR}}^n$ in $Q\mathrm{Coh}(Y(1))$ and the one of liftings of $Y(1)$ to $W_2(k)$. A general smooth Artin stack $\mathcal{Y}$ can be covered by an affine scheme only **smooth locally**, so one needs to replace the étale site of $\mathcal{Y}$ by the smooth one. But both the space of splittings of $\tau_{-1}^1 \varphi_Y \Omega_{Y,\mathrm{dR}}^n$ and the space of liftings to $W_2(k)$ are not even presheaves there. Nevertheless, it would be still interesting to have an explicit description of the space of liftings to $W_2(k)$ for an arbitrary smooth $n$-Artin stack $\mathcal{Y}$. We do not discuss this question here.

**Spreads.** Let now $X$ be a smooth Hodge-proper stack over a field $F$ of characteristic $0$. If there exists a finitely generated $\mathbb{Z}$-subalgebra $R \subset F$ and a Hodge-proper stack $X_R$ over $R$ such that $X_R \otimes_R F \simeq X$ (a so-called **spreading of $X$**), then one can deduce the equality (1) for the $n$-th cohomology from Theorem 1.3.18 by taking a suitable closed point $\mathrm{Spec} \ k \to \mathrm{Spec} \ R$ of characteristic $p > n$ and considering the fiber $X_k$. This way we obtain

**Theorem (1.4.2).** Let $X$ be a smooth Hodge-properly spreadable Artin stack over a field $F$ of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $X$ degenerates at the first page. In particular for each $n \geq 0$ there exists a (non canonical) isomorphism

$$H^n_{\mathrm{dR}}(X) \simeq \bigoplus_{p+q=n} H^{p,q}(X).$$

In order to address the question of the existence of spreadings we first extend the standard spreading out results for finitely presentable schemes to the case of Artin stacks:

**Theorem (2.1.12).** Let $\{S_i\}$ be a filtered diagram of affine schemes with limit $S$. For a class of morphism $\mathcal{P} = \text{smooth, flat, surjective or any other satisfying conditions of Definition 2.1.8}$ and an affine scheme $T$ let $\mathcal{Stk}^{n,\text{Art},fp,\mathcal{P}}_{/T}$ denote the category of finitely presentable $n$-Artin stacks over $T$ and morphism in $\mathcal{P}$ between them. Then the natural functor (induced by the base-change)

$$\lim_i \mathcal{Stk}^{n,\text{Art},fp,\mathcal{P}}_{/S_i} \longrightarrow \mathcal{Stk}^{n,\text{Art},fp,\mathcal{P}}_{/S}$$

is an equivalence.

As a corollary we deduce that any smooth $n$-Artin stack $X$ over $F$ admits a smooth spreading $X_R$ over some finitely generated $\mathbb{Z}$-algebra $R \subset F$ and that any such two spreadings become equivalent after enlarging $R$. Since all smooth proper stacks are Hodge-proper (see Proposition 2.2.6), we deduce Hodge-to-de Rham degeneration in this case. Note that this includes smooth proper Deligne-Mumford stacks as a special case.

Hodge-proper spreadings need not to exist in general: one can show that the classifying stack $BG$ is Hodge-proper for any finite-type group scheme $G$ over $F$ (see Proposition 2.2.9) but it is not necessarily Hodge-properly spreadable. Indeed, the classifying stack $BG_a$ of the additive group has nontrivial Hodge cohomology but is de Rham contractible (i.e. has de Rham cohomology of a point), so the Hodge-to-de Rham spectral sequence is clearly nondegenerate. By Theorem 1.4.2 it follows that it is not Hodge-properly spreadable and this forces the Hodge cohomology of $BG_a,\mathbb{Z}$ to have infinitely generated $p$-torsion for a dense set of primes $p$, which one can also see from the explicit description (see Example 2.2.10). This illustrates the general phenomenon: the non-degeneracy of the Hodge-to-de Rham spectral sequence in characteristic 0 is often reflected arithmetically, namely the Hodge cohomology of any spreading must be infinitely generated.

In the main case of our interest, namely the quotient stacks $X = [X/G]$, we exhibit some sufficient conditions for Hodge-proper spreadability purely in terms of the geometry of $X, G$ and the action $G \acts X$. In this case
Hodge-proper spreadability is not automatic except for the case when $G$ is a torus (or an extension of one by a finite group). Nevertheless, using some cohomological finiteness results from [FvdK10] we prove

**Theorem (2.2.13).** Let $F$ be an algebraically closed field of characteristic 0. Let $X$ be a smooth scheme and let Spec $A$ be a finite-type affine scheme over $F$, both endowed with an action of a reductive group $G$. Assume that

- There is a proper $G$-equivariant map $\pi: X \to \text{Spec } A$.
- $\dim_F A^G < \infty$.

Then the quotient stack $[X/G]$ is Hodge-properly spreadable.

Note that we do not need any projectivity assumptions on the map $\pi$. We also prove a version of Theorem 2.2.13 where we drop the reductivity assumption on $G$ but impose an additional conicality assumption on the action, see Theorem 2.2.16. We expect that the Hodge-proper spreadability should also hold for a generalized version of KN-complete varieties (see [Tel00]) where one replaces projectivity with properness on each step in the definition.

Finally, for any Hodge-properly spreadable quotient stack $[X/G]$, we deduce an equivariant Hodge-to-de Rham degeneration:

**Corollary (2.3.2).** Let $X$ be a smooth scheme over $\mathbb{C}$ of characteristic 0 endowed with an action of an algebraic group $G$ such that the quotient stack $[X/G]$ is Hodge-properly spreadable. Then there is a (non-canonical) decomposition

$$H^*_G(\mathbb{C})(X(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q([X/G], \wedge^p \mathcal{L}[X/G]).$$

0.3 Relation to previous work

Our definition of Hodge-proper stacks is partially motivated by the work [HLP19], where various variants of the notion of properness for stacks are thoroughly studied. The splitting of the $(p−1)$-st truncation of the de Rham complex for a smooth tame 1-Artin stack over a perfect field $k$ of characteristic $p$ was established (among other things) in [Sat12]. The key observation in [Sat12] is that a smooth tame stack admits a smooth lift together with a lift of Frobenius étale-locally on its coarse moduli space, which enables to follow the original argument of Deligne-Illusie. In this work we give an alternative proof which works for an arbitrary smooth $n$-Artin stack.

The equivariant Hodge-to-de Rham degeneration for a reductive group $G$ acting on a scheme $X$ under Kempf-Ness-completeness assumption was treated (among other things) in [Tel00] by completely different methods. One advantage of our approach is that unlike in [Tel00] we do not need to assume the existence of a good $G$-equivariant line bundle on $X$. We also expect that all KN-complete quotient stacks are in fact Hodge-properly spreadable. If this is true, the present work covers all examples (at least to our knowledge) of the Hodge-to-de Rham degeneration in the case of Artin stacks.

Another approach to the equivariant Hodge theory was introduced in [HLP15], where the authors deduce (among other things) the noncommutative Hodge-to-de Rham degeneration for QCoh($[X/G]$)perf (under the KN-completeness assumption) and some purely non-commutative examples (like categories of matrix factorizations) by exploiting methods of non-commutative geometry. Note that the result of Kaledin (see [Kal08] and [Kal17]) does not apply in this situation, since the DG-category QCoh($[X/G]$)perf is usually not smooth. It is natural to ask whether the commutative degeneration implies the noncommutative one in this case. This is not immediately clear, since the relation between the Hochschild/periodic cyclic homology and the Hodge/de Rham cohomology for Artin stacks is more subtle than in the case of schemes.

0.4 Plan of the paper

Section 1 is devoted to a proof of the degeneration of the Hodge-to-de Rham spectral sequence for Hodge-properly spreadable stacks. In Subsections 1.1 and 1.2 we review Hodge and de Rham cohomology of stacks, define Hodge-proper stacks and prove some technical lemmas about them. In Section 1.3 we prove (a truncated version of) the Hodge-to-de Rham degeneration in positive characteristic for stacks admitting a lift to $W_2(k)$. Finally, in Section 1.4 we deduce from this the Hodge-to-de Rham degeneration in characteristic 0 for Hodge-properly spreadable stacks.

In Section 2 we study in more detail the notion of spreadability. In Subsection 2.1 we extend the standard spreading out results for finitely presented schemes and their morphisms to the case of Artin stacks (see Theorem 2.1.12). In Section 2.2 we give several examples of spreadable Hodge-proper stacks: in Section 2.2.1 we cover
the case of smooth proper stacks, in Section 2.2.2 we discuss for which algebraic groups $G$ the classifying stack $BG$ is spreadable, in Section 2.2.3 we discuss the case of global quotients by reductive groups, while Section 2.2.4 deals with slightly more general global quotients. Finally, in Section 2.3 we also deduce a (non-canonical) Hodge decomposition for the equivariant singular cohomology of some algebraic varieties with an algebraic group action.

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**Notations and conventions.**

1. We will freely use the language of higher categories, modeled e.g. by quasi-categories of $\text{Lur09}$. If not explicitly stated otherwise all categories are assumed to be $(\infty,1)$ and all (co-)limits are homotopy ones. The $(\infty,1)$-category of Kan complexes will be denoted by $\mathcal{S}$ and we will call it the category of spaces. By Lan, $F$ and Ran, $F$ we will denote left and right Kan extensions of a functor $F$ along $i$ (see e.g. $\text{Lur09}$, Definition 4.3.2.2 for more details).

2. For a commutative ring $R$ by $D(\text{Mod}_R)$ we will denote the canonical $(\infty,1)$-enhancement of the triangulated unbounded derived category of the abelian category of $A$-modules $\text{Mod}_A$. All tensor product, pullback and push-forward functors are implicitly derived.

3. Our conventions on (higher) Artin stacks are the standard ones: see e.g. $\text{TV08}$, Chapter 3.2 or $\text{GR17}$, Chapter 3.4 (in the definition we use fpf-topology). We stress that we work with classical (as opposed to derived) higher Artin stacks, i.e. they are defined on the category of ordinary commutative rings. We will denote the category of $n$-Artin stacks over a base scheme $S$ by $\text{St}_{/S}^{n}$.

4. For a stack $X$ we will denote by $\text{QCo}(X)$ the category of quasi-coherent sheaves on $X$ defined as the limit

\[ \lim_{\text{Spec } A \to X} D(\text{Mod}_A) \]

over all affine schemes $\text{Spec } A$ mapping to $X$ (see $\text{GR17}$, Chapter 3.1 for more details). Note that $\text{QCo}(X)$ admits a natural $t$-structure such that $\mathcal{F} \in \text{QCo}(X)^{\leq 0}$ if and only if $x^*(\mathcal{F}) \in D(\text{Mod}_A)^{\leq 0}$ for any $A$-point $x \in X(A)$. Moreover, by $\text{GR17}$, Chapter 3, Corollary 1.5.7 if $X$ is Artin stack, then $\text{QCo}(X)$ is left- and right-complete (i.e. Postnikov’s and Whitehead’s towers converge) and the truncation functors commute with filtered colimits.

5. For a locally Noetherian scheme $S$ we will denote by $\text{Coh}(S)$ the full subcategory of $\text{QCo}(S)$ consisting of bounded complexes of sheaves with coherent cohomology. For a finite presentable Artin stack $X$ over a locally Noetherian base scheme $S$ we will denote by $\text{Coh}(X)$ the full subcategory of $\mathcal{Y}$ consisting of sheaves $\mathcal{F}$ such that the restriction of $\mathcal{F}$ to some (equivalently to any) smooth finitely presentable atlas is coherent.

6. For an affine group scheme $G$ over a ring $R$, given a representation $M$ (i.e. a comodule over the corresponding Hopf algebra $R[G]$) we denote by $R\Gamma(G, M) \in D(\text{Mod}_A)$ the rational cohomology complex of $G$, namely the derived functor of $G$-invariants $M \mapsto M^G$. By flat descent, for $G$ flat over $R$, the abelian category $\text{Rep}(G)$ is identified with $\text{QCo}(BG)^{\otimes}$ and $R\Gamma(G, M) \simeq R\Gamma(BG, M)$.

## 1 Degeneration of the Hodge-to-de Rham spectral sequence

### 1.1 Hodge and de Rham cohomology

In this section we set up Hodge-to-de Rham spectral sequence for $n$-Artin stacks and prove some technical results needed in subsequent sections of the paper. For the rest of this section fix a base ring $R$. We refer the reader to $\text{TV08}$ for an introduction to the theory of Artin stacks and cotangent complexes.

**Definition 1.1.1** (Hodge cohomology). Let $X$ be an Artin stack over $R$. Define Hodge cohomology $R\Gamma_H(X)$ of $X$ to be

\[ R\Gamma_H(X) := \bigoplus_{p \geq 0} R\Gamma(X, \wedge^p L_X \otimes [p]) , \]
where $L_{X/R}$ is the cotangent complex of $X$ over $R$ and $\wedge^p L_{X/R}$ is its $p$-th derived exterior power (see [Ill71, Chapitre I.4] or [BM19, Section 3]). For a fixed $n \in \mathbb{Z}$ we will also denote

$$H^n_{dR}(X) := H^n \Gamma_{dR}(X) \simeq \bigoplus_{p+q=n} H^{p,q}(X), \quad \text{where } H^{p,q}(X) := H^q(X, \wedge^p L_{X/R}).$$

**Notation 1.1.2.** Let $S := \text{Spec } A$ be an affine smooth $R$-algebra. The algebraic de Rham complex of $S$ over $R$

$$A \xrightarrow{d} \Omega^1_{A/R} \xrightarrow{d} \Omega^2_{A/R} \xrightarrow{d} \ldots$$

will be denoted by $\Omega^\bullet_{S/R, dR}$. We define $\Gamma_{dR}(S) := \Omega^\bullet_{S/R, dR} \in D(\text{Mod}_R)$.

**Definition 1.1.3** (de Rham cohomology). Let $X$ be a smooth Artin stack over $R$. Define the *(Hodge-completed)* de Rham cohomology $\Gamma_{dR}(X)$ of $X$ to be

$$\Gamma_{dR}(X) := \lim_{\longrightarrow} \Gamma(S, \wedge^p L_{S/R}),$$

where $\text{Aff}^\text{sm}_X$ is the full subcategory of stacks over $X$ consisting of affine schemes that are smooth over $X$. We will also denote $H^\ast \Gamma_{dR}(X)$ by $H^\ast_{dR}(X)$.

In fact the Hodge cohomology complex admits a description similar to our definition of the de Rham cohomology:

**Proposition 1.1.4.** For any $p \in \mathbb{Z}_{\geq 0}$ the natural map

$$\Gamma(X, \wedge^p L_{X/R}) \to \lim_{\longrightarrow} \Gamma(S, \wedge^p L_{S/R}) \quad \text{(2)}$$

is an equivalence.

**Proof.** Since the relative cotangent complex of an étale morphism vanishes, the cotangent complex satisfies étale descent. Moreover, since étale locally any smooth morphism admits a section, the cotangent complex satisfies smooth descent as well. It follows that both sides of (2) satisfy smooth descent. Since $n$-Artin stacks are by definition iterated smooth quotients of schemes, we reduce by induction $n$ to the case when $X$ is a smooth affine scheme and the assertion of the proposition is true since $S \in \text{Aff}^\text{sm}_X$ has a final object given by $X$. 

**Corollary 1.1.5.** Let $X$ be a smooth Artin stack. Then there exists a complete (decreasing) Hodge filtration $F^\ast \Gamma_{dR}(X)$ such that $\text{gr } F^\ast \Gamma_{dR}(X) \simeq \Gamma_{H}(X)$.

**Proof.** Since $X$ is smooth, all schemes $Y \in \text{Sm}_X$ are smooth, hence $\Gamma_{dR}(X)$ admits a complete decreasing filtration (since complete filtered complexes are closed under limits). Moreover, by construction we have

$$\text{gr } F^\ast \Gamma_{dR}(X) \simeq \lim_{\longrightarrow} \Gamma(S, \wedge^p L_{S/R}) \simeq \lim_{\longrightarrow} \Gamma_{H}(S) \simeq \Gamma_{H}(X),$$

where the last equivalence follows from the previous proposition.

The following simple observation will be quite useful in what follows:

**Remark 1.1.6.** Let $X$ be a smooth Artin stack. Then the cotangent complex $L_{X/R}$ (and its exterior powers) is concentrated in nonnegative cohomological degrees (with respect to the natural $t$-structure on $\text{QCoh}(X)$). Since the global section functor $\Gamma$ is left $t$-exact, it follows that the natural map $\Gamma_{dR}(X) \to \Gamma_{dR}(X)/F^p \Gamma_{dR}(X)$ induces an isomorphism on $H^{<p}$.

Finally, we will need the following

**Proposition 1.1.7** (Base-change). Let $X$ be a smooth Artin stack over $R$ and let $R \to R'$ be a ring homomorphism such that either

- the map $R \to R'$ is flat, or
- $R'$ considered as an $R$-module is perfect.
Then for \( X' := X \otimes_R R' \) the natural map \( R\Gamma_{dR}(X/R) \otimes_R R' \to R\Gamma_{dR}(X'/R') \) is a filtered equivalence. In particular, for each \( p \in \mathbb{Z}_{\geq 0} \) the natural map \( R\Gamma(X, \wedge^p L_{X/R}) \otimes_R R' \to R\Gamma(X', \wedge^p L_{X'/R'}) \) is an equivalence.

Proof. By the smoothness assumption on \( X \) the fiber product \( X \otimes_R R' \) coincides with the derived fiber product. It follows by \cite[Lemma 1.4.1.16 (2)]{TV08} that \( L_{X/R} \otimes_R R' \simeq L_{X'/R'} \). By the base change for QCoh (see \cite[Chapter 3., Proposition 2.2.2 (b)]{GR17}) we deduce that the natural map \( R\Gamma_H(X/R) \otimes_R R' \to R\Gamma_H(X'/R') \) is an equivalence.

Next, note that each of the conditions on the morphism \( R \to R' \) guarantees that the natural map \( R\Gamma_{dR}(X) \otimes_R R' \to \lim_{\leftarrow p} (R\Gamma_{dR}(X)/F^p R\Gamma_{dR}(X) \otimes_R R') \) is an equivalence. Since both sides are complete with respect to the Hodge filtration, and, since by the above the induced map on the associated graded pieces

\[
R\Gamma_H(X) \otimes_R R' \simeq gr R\Gamma_{dR}(X/R) \otimes_R R' \to gr R\Gamma_{dR}(X'/R') \simeq R\Gamma_H(X'/R')
\]

is an equivalence, we deduce that the base-change map for de Rham cohomology is an equivalence as well. \( \square \)

### 1.2 Hodge-proper stacks

Fix a Noetherian base ring \( R \). In this section we will introduce a reasonable substitute for the notion of properness for stacks.

**Definition 1.2.1.** A complex of \( R \)-modules \( X \) is called almost coherent if it is cohomologically bounded below and for any \( i \in \mathbb{Z} \) the cohomology module \( H^i(X) \) is finitely generated over \( R \). We will denote the full subcategory of \( D(\text{Mod}_R) \) consisting of almost coherent \( R \)-modules by \( D(\text{Mod}_R)_{\text{acoh}} \).

**Remark 1.2.2.** One often encounters a dual variant of the finiteness condition above: a complex of \( R \)-modules \( X \) is called almost perfect if it is bounded from above and for all \( n \in \mathbb{Z} \) the cohomology module \( H^n(X) \) is finitely generated over \( R \). See \cite[Section 7.2.4]{Lur17} for a thorough discussion of this concept.

We have the following basic properties of the notion above:

**Proposition 1.2.3.** Let \( R \) be a Noetherian ring. Then:

1. The category \( D(\text{Mod}_R)_{\text{acoh}} \) is closed under finite (co-)limits and retracts. In particular \( D(\text{Mod}_R)_{\text{acoh}} \) is a stable subcategory of \( D(\text{Mod}_R) \).

2. For each \( n \in \mathbb{Z} \) the category \( D(\text{Mod}_R)_{\text{acoh}, \geq n} := D(\text{Mod}_R)_{\text{acoh}} \cap D(\text{Mod}_R)^{\geq n} \) is closed under totalizations.

**Proof.** 1. This follows from the fact, for a Noetherian \( R \) the abelian category of finitely generated \( R \)-modules is closed under (co)kernels, extensions and direct summands.

2. Let \( X^\bullet \) be a co-simplicial object of \( D(\text{Mod}_R)_{\text{acoh}, \geq n} \). By shifting if necessary, we can assume that all \( X^i \) are connective. Since coconnective modules are closed under limits, \( \text{Tot}(X^\bullet) \in D(\text{Mod}_R)^{\geq 0} \); hence it is enough to prove that \( H^i \text{Tot}(X^\bullet) \) is finitely generated \( R \)-module for all \( i \in \mathbb{Z}_{\geq 0} \). Since all \( X^i \) are connective, the natural map \( \text{Tot}(X^\bullet) \to \text{Tot}^\leq(X^\bullet) \) induces an isomorphism on \( H^\leq \). But since \( \text{Tot}^\leq \) is a finite limit, each \( H^i \text{Tot}^\leq(X^\bullet) \) is a finitely generated \( R \)-module. \( \square \)

**Remark 1.2.4.** Recall that the category of perfect \( R \)-modules \( D(\text{Mod}_R)^{\text{perf}} \) is defined as the smallest full subcategory of \( D(\text{Mod}_R) \) containing \( R \) and closed under finite (co-)limits and direct summands. Since \( R \in D(\text{Mod}_R)_{\text{acoh}} \) it follows from Proposition 1.2.3, that \( D(\text{Mod}_R)^{\text{perf}} \subseteq D(\text{Mod}_R)_{\text{acoh}} \).

After this technical digression we are ready to introduce the notion of a Hodge-proper stack:

**Definition 1.2.5 (Hodge-proper stacks).** A smooth Artin stack \( X \) over \( R \) is called Hodge-proper if for every \( p \in \mathbb{Z}_{\geq 0} \) the complex \( R\Gamma(X, \wedge^p L_{X/R}) \) is almost coherent.

For us the most important implication of Hodge-properness is the almost coherence of the de Rham cohomology:

**Proposition 1.2.6.** Let \( X \) be a smooth Hodge-proper Artin stack. Then \( R\Gamma_{dR}(X) \) is almost coherent.

**Proof.** By smoothness \( R\Gamma_{dR}(X) \) is bounded below by 0, hence it is enough to prove that for each \( n \in \mathbb{Z}_{\geq 0} \) the cohomology module \( H^n_{dR}(X) \) is finitely generated over \( R \). By Remark 1.1.6 the natural map \( R\Gamma_{dR}(X) \to R\Gamma_{dR}(X)/F^{n+1}R\Gamma_{dR}(X) \) induces an isomorphism on \( H^0 \). We conclude, since \( R\Gamma_{dR}(X)/F^{n+1}R\Gamma_{dR}(X) \), being a finite extension of almost coherent complexes \( R\Gamma(X, \wedge^i L_{X/R}[-i]), 0 \leq i \leq n \), is almost coherent. \( \square \)
1.3 Hodge-to-de Rham degeneration in positive characteristic

Let \( \mathcal{Y} \) be a smooth Artin stack over a perfect field \( k \) of characteristic \( p \) admitting a smooth lift to the ring of the second Witt vectors \( W_2(k) \). In this section we will prove that the Hodge-to-de Rham spectral sequence \( H^i(\mathcal{Y}, \Lambda^jL\mathcal{Y}/k) \Rightarrow H^i_{dR}(\mathcal{Y}/k) \) degenerates at the first page for \( i + j < p \). Our strategy is to interpret both Hodge and de Rham cohomology in terms of crystalline cohomology and then, following Fontaine-Messing [FM87] (and Bhattacharya-Morrow-Scholze [BMS19]), use (quasi-)syntomic descent for the crystalline cohomology to get a very functorial form of the Deligne-Illusie splitting.

We denote by \( \sigma: k \xrightarrow{x \rightarrow x^p} k \) the absolute Frobenius morphism of \( k \). We denote by the same letter \( \sigma \) the induced automorphisms \( W(k) \rightarrow W(k) \) and \( W_n(k) \rightarrow W_n(k) \) for any \( n \in \mathbb{N} \). For a \( W(k) \)-algebra (e.g. a \( W_n(k) \)-algebra for some \( s \)) \( A \) we denote by \( A^{(1)} := A \otimes_{W(k)} W(k) \) its Frobenius twist and by \( A^{(-1)} := A \otimes_{W(k)} W(k) \) its Frobenius untwist. For each \( n \in \mathbb{Z} \) we have the relative Frobenius map \( \varphi: A(n) \rightarrow A(n - 1) \).

**Definition 1.3.1.** A morphism \( A \rightarrow B \) of \( W_n(k) \)-algebras is called *quasisyntomic* if it is flat and \( L_{B/A} \) has cohomological Tor amplitude \([-1,0]\]. A morphism \( A \rightarrow B \) is a *quasisyntomic cover* if it is quasisyntomic and faithfully flat. We will denote by \( \text{QSyn}_n \) the site consisting of quasisyntomic \( W_n(k) \)-algebras with the topology generated by quasisyntomic covers.

The notion of a quasisyntomic morphism is a generalization of more classical notion of a *syntomic* morphism: a flat map \( A \rightarrow B \) that is locally a complete intersection in smooth one. Syntomic morphisms include smooth morphisms, and (in the case we are over \( k \)) the relative Frobenius morphism \( \varphi: A^{(1)} \rightarrow A \). The advantage of quasisyntomic morphisms is that they also include some natural non-finite-type maps, most importantly the direct limit perfection \( A \rightarrow A_{\text{perf}} := \lim_{\longleftarrow \varphi \in \mathbb{N}} A^{(-n)} \) and its tensor powers \( A \rightarrow A_{\text{perf}} \otimes_A \cdots \otimes_A A_{\text{perf}} \) for a smooth \( k \)-algebra \( A \). Using standard properties of the cotangent complex it is not hard to show that quasisyntomic morphisms are stable under composition and pushouts along arbitrary morphisms of algebras (and same for quasisyntomic covers). We refer to Section 4 of [BMS19] for more details.

Recall that an \( F_\rho \)-algebra \( S \) is called *semiperfect* if \( \varphi: S \rightarrow S \) is surjective.

**Definition 1.3.2.** An \( k \)-algebra \( S \) is called *quasiregular semiperfect* if \( S \) is quasisyntomic and the relative Frobenius homomorphism \( \varphi: S^{(1)} \rightarrow S \) is surjective. We call a \( W_n(k) \)-algebra \( \tilde{S} \) *quasiregular semiperfectoid* if it is flat over \( W_n(k) \) and \( \tilde{S}/p \) is quasiregular semiperfect. We will denote by \( \text{QRSPerf}_n \) the site consisting of quasiregular semiperfect \( W_n(k) \)-algebras with the topology generated by faithfully flat covers.

For any \( k \)-algebra \( S \), \( H^0(L_{S/k}) \) is identified with the Kahler differentials \( \Omega^1_{S/k} \). Since \( d(x^p) = 0 \), we get that \( H^0(L_{S/k}) = 0 \) for \( S \) semiperfect, and that \( L_{S/k} \) is concentrated in a single cohomological degree \(-1\) for \( S \) quasiregular semiperfectoid. The same is true for \( L_{\tilde{S}/W_n(k)} \) for a quasiregular semiperfectoid \( W_n(k) \)-algebra \( \tilde{S} \). Moreover, any flat map \( \tilde{S}_1 \rightarrow \tilde{S}_2 \) between quasiregular semiperfectoids over \( W_n(k) \) is quasisyntomic. This gives a map of sites \( \text{QRSPerf}_n \rightarrow \text{QSyn}_n \).

In fact quasiregular semiperfectoid algebras form a basis of topology in \( \text{QSyn}_n \). This leads to an equivalence between the corresponding categories of sheaves:

**Proposition 1.3.3.** The restriction along the natural embedding \( u: \text{QRSPerf}_n \rightarrow \text{QSyn}_n \) induces an equivalence

\[
\text{Shv}(\text{QSyn}_n, \mathcal{C}) \xrightarrow{u^{-1}} \text{Shv}(\text{QRSPerf}_n, \mathcal{C})
\]

of the categories of sheaves with values in any presentable \( \infty \)-category \( \mathcal{C} \).

**Proof.** Similar to [BMS19, Proposition 4.31]. \( \square \)

**Remark 1.3.4.** For a sheaf \( \mathcal{F} \) on \( \text{QRSPerf}_n \) we will denote its image under the inverse equivalence in Proposition 1.3.3 by \( \mathcal{F} \) as well.

**Example 1.3.5.** Let \( B \) be a smooth algebra over \( W_n(k) \). By smoothness there is an étale map \( P \rightarrow B \) from the polynomial algebra \( P = P_\rho := W_n(k)[x_1, \ldots, x_d] \) for some \( d \). Let \( P_{\text{perf}} = W_n(k)[x_1^{1/p^n}, \ldots, x_d^{1/p^n}] \) and let \( B_{\text{perf}} := B \otimes_B P_{\text{perf}} \); it is a quasiregular\(^4\) perfectoid \( W_n(k) \)-algebra and the natural map \( B \rightarrow B_{\text{perf}} \) is a quasisyntomic cover.

\(^4\)In fact it is even quasismooth, \( L_{B/W_n(k)} = 0 \).
Moreover all terms \((B\text{perf} \otimes B \ldots \otimes B\text{perf})_n\) in the corresponding Cech object are also quasiregular semiperfectoids. Given any sheaf \(\mathcal{F}\) on \(Q\text{R}^\text{SPerf}_n\) its value on \(B \in \text{QSyn}_n\) (via Proposition 1.3.3) can be computed as “the unfolding”:

\[
\text{R} \Gamma_{\text{QSyn}_n}(B, \mathcal{F}) \sim \text{Tot} \left( \mathcal{F}(B\text{perf}) \otimes B \text{perf} \rightarrow \mathcal{F}(B\text{perf} \otimes B \text{perf}) \otimes B \text{perf} \rightarrow \cdots \right).
\]

For a ring \(R\) let \(\text{Poly}_R \subset \text{CAlg}_R\) denote the full subcategory of finitely generated polynomial \(R\)-algebras. Recall that one of the ways to define the cotangent complex \(L_{A/R}\) for an \(R\)-algebra \(A\) is to consider the left Kan extension of the functor \(B \mapsto \Omega^1_{B/R}\) from the category of polynomial \(R\)-algebras, namely

\[
L_{A/R} \simeq \text{colim}_{\text{Poly}_R/A} \Omega^1_{B/R}.
\]

One can extend de Rham and crystalline cohomology functors in a similar way:

**Construction 1.3.6.** Let \(k\) be a perfect field.

- The derived de Rham cohomology functor
  \[
  \text{R} \Gamma_{\text{LdR}}(-/W_n(k)) : \text{CAlg}_{W_n(k)/} \rightarrow D(\text{Mod}_{W_n(k)})
  \]
  is defined as the left Kan extension of the functor \(B \mapsto \Omega^\bullet_{B/W_n(k),\text{dR}}\) on \(\text{Poly}_{W_n(k)}\).

- The derived crystalline cohomology functor
  \[
  \text{R} \Gamma_{\text{Lcrys}}(-/W(k)) : \text{CAlg}_{W_n(k)/} \rightarrow D(\text{Mod}_W(k))
  \]
  is defined as the (derived) \(p\)-adic completion of the left Kan extension of the functor \(B \mapsto \text{R} \Gamma_{\text{crys}}((B/p)/W(k))\) on \(\text{Poly}_{W_n(k)}\).

Similarly, we can extend the functors \(B \mapsto \Omega^m_{B/W_n(k),\text{dR}}\) and \(B \mapsto \tau^m \Omega^\bullet_{B/W_n(k),\text{dR}}\) to get filtered objects \((\text{R} \Gamma_{\text{LdR}}(-/W_n(k)), F^m_n)\) (Hodge filtration) and \((\text{R} \Gamma_{\text{LdR}}(-/W_n(k)), \text{Fil}^m_n)\) (conjugate filtration). The derived de Rham cohomology is complete with respect to its conjugate filtration since colimits commute. For \(\Gamma_{\text{LdR}}(-/k)\) the Cartier isomorphism identifies the corresponding associated graded with \(\text{Fil}^0_0 L_{B(1)/k}[-1]\). Thus \(\Gamma_{\text{LdR}}(-/k)\) satisfies flat descent (by flat descent for the cotangent complex [BMS19, Theorem 3.1]); for \(\Gamma_{\text{LdR}}(-/W_n(k))\) and \(\Gamma_{\text{Lcrys}}(-/W(k))\) the same holds since they are \(p\)-adically complete and their reduction mod \(p\) is \(\text{R} \Gamma_{\text{LdR}}(-/k)\). In particular all these functors define sheaves on \(\text{QS}\).

**Remark 1.3.7.** Let \(B\) be a smooth \(W_n(k)\)-algebra. We claim that the natural morphism \(\text{R} \Gamma_{\text{LdR}}(B) \rightarrow \text{R} \Gamma_{\text{dR}}(B)\) is an equivalence. To see this it is enough to prove that the induced map on the associated graded of the conjugate filtration is an equivalence. But since \(B\) is smooth \(L^\bullet_{B(1)/W_n(k)} \rightarrow \Omega^1_{B(1)/W_n(k)}\).

**Remark 1.3.8.** For any \(W_n(k)\)-algebra \(B\) the complexes \(\text{R} \Gamma_{\text{Tcrys}}(B/W(k)) \otimes_{\Lambda_{W(k)}} W_n(k)\) and \(\text{R} \Gamma_{\text{LdR}}(B/W_n(k))\) are canonically equivalent. Indeed, by construction both functors commute with geometric realizations, hence it is enough to prove the statement for \(B\) being a smooth \(W_n(k)\)-algebra. In this case this is a basic result in the crystalline cohomology theory, see e.g. [BO78, Corollary 7.4].

**Remark 1.3.9.** Since the absolute Frobenius \(\sigma : k \rightarrow k\) is an automorphism, the cotangent complex \(L_{k/\mathbb{F}_p}\) (and all its wedge powers) vanishes. It follows that \(\text{R} \Gamma_{\text{dR}}(k/\mathbb{F}_p) \simeq k\). Given any \(k\)-algebra \(B\) we have a natural morphism of \(E_{\infty}\)-algebras \(\text{R} \Gamma_{\text{dR}}(k/\mathbb{F}_p) \rightarrow \text{R} \Gamma_{\text{dR}}(B/\mathbb{F}_p)\). This endows \(\text{R} \Gamma_{\text{dR}}(B/\mathbb{F}_p)\) with a natural \(k\)-linear structure. Similarly, for any \(k\)-algebra \(A\) the complex \(\text{R} \Gamma_{\text{Lcrys}}(A/\mathbb{Z}_p)\) has a natural \(W(k)\)-linear structure. Moreover, the natural morphism

\[
\text{R} \Gamma_{\text{Lcrys}}(B/\mathbb{Z}_p) \rightarrow \text{R} \Gamma_{\text{Lcrys}}(B/W(k))
\]

is \(W(k)\)-linear. We claim that (3) is an equivalence. Since both sides are \(p\)-adically complete it is enough to show for the equivalence for the derived de Rham cohomology of the reduction \(B/p\). On the associated graded of the conjugate filtration \(\text{Fil}^m_n\) the induced map is an equivalence, since in the transitivity triangle

\[
\text{L}_{k/\mathbb{F}_p} \otimes_{\mathbb{F}_p} B \rightarrow \text{L}_{B/\mathbb{F}_p} \rightarrow \text{L}_{B/k}
\]

the term \(\text{L}_{k/\mathbb{F}_p}\) is equivalent to 0. Thus (3) is an equivalence.
Recall that the cotangent complex $L_{\tilde{S}/W_n(k)}$ of $\tilde{S}$ in $\mathrm{QRSPerf}_n$ is supported in cohomological degree $-1$, thus $\bigoplus_r \Lambda^r L_{\tilde{S}/W_n(k)}[-r]$ is supported in cohomological degree $0$. The same holds for $R_{\mathrm{LdR}}(\tilde{S}/W_n(k))$; in other words, it is a classical commutative ring. It has a description in terms of one of the Fontaine’s period rings $\mathcal{A}_{\text{crys}}$.

**Construction 1.3.10.** Let $S$ be a semiperfect $k$-algebra and let $S^0$ be the inverse limit perfection $S^0 := \lim_{\longleftarrow n \geq 0} S^{(n)}$. We have a natural map $S^0 \to S$ which is surjective. The ring $\mathcal{A}_{\text{crys}}(S)$ is defined as the $p$-adic completion of the divided power envelope of the kernel of the natural composite surjection $\theta_1 : W(S^0) \to S^0 \to S$ (where the divided power structure agrees with the one on $W(k)$). Note that $\mathcal{A}_{\text{crys}}(S)/p$ is identified with the $PD$-completion $D_{PD}^P(S^0)^{(1)}$ along the ideal $I \subset S^0$ defined as the kernel of the natural map $S^0 \to S$.

Theorem 8.14(3) of [BMS19] (together with Remark 1.3.9) identifies $R_{\mathrm{dR}}(S/W(k))$ with $\mathcal{A}_{\text{crys}}(S)$. The ring $\mathcal{A}_{\text{crys}}(S)$ comes with a natural ring morphism $\varphi : \mathcal{A}_{\text{crys}}(S)^{(1)} \to \mathcal{A}_{\text{crys}}(S)$ induced by the relative Frobenius $\varphi : S^{(1)} \to S$. It is identified with the natural Frobenius $\varphi : R_{\mathrm{dR}}(S/W(k))^{(1)} \to R_{\mathrm{dR}}(S/W(k))$ on the crystalline cohomology. For each $n$ we define a presheaf of rings $\mathcal{A}_{\text{crys}}$ on $\mathrm{QRSPerf}_n$ by sending $\tilde{S}$ to $\mathcal{A}_{\text{crys}}(\tilde{S}/p)$. By the above identification it is in fact a sheaf. Note that by the universal property of the PD-envelope there is a natural map $^2 \theta_n : \mathcal{A}_{\text{crys}}(S/p) \to S$ which quotients through $\mathcal{A}_{\text{crys}}(\tilde{S}/p)/p^n$.

The following two filtrations on $\mathcal{A}_{\text{crys}}/p$ correspond to the Hodge and the conjugate filtrations:

**Definition 1.3.11.** Let $S$ be a quasiregular semiperfect $k$-algebra and let $I$ be the ideal of the natural projection $S^0 \to S$. The descending *Hodge filtration* on $\mathcal{A}_{\text{crys}}(S)/p \simeq D^P(S^0)^{(1)}$ is defined as the filtration by the divided powers of $I$: $\mathcal{A}_{\text{crys}}(S)/p \simeq I^0 \supset I \supset I^2 \supset I^3 \supset \cdots$. This filtration is functorial in $S$ and thus defines a filtration by presheaves $I^0 \supset I \supset I^2 \supset I^3 \supset \cdots$ on the sheaf $\mathcal{A}_{\text{crys}}(S)/p$ on $\mathrm{QRSPerf}_n$ for any $n$. Via Proposition 8.12 of [BMS19] it is identified with the Hodge filtration on $R_{\mathrm{LdR}}(S/k) \simeq \mathcal{A}_{\text{crys}}(S)/p$ and thus is in fact a filtration by sheaves.

**Definition 1.3.12.** The ascending *conjugate filtration* $\text{Fil}^{\text{conj}}_s$ on $\mathcal{A}_{\text{crys}}(S)/p \simeq D_{PD}^P(I)$ is defined by taking $F^{\text{conj}}_s$ to be the $S^0$-submodule generated by the elements of the form $s_1^{[t_1]}s_2^{[t_2]} \cdots s_m^{[t_m]}$ with $s_i \in I$ and $\sum_{i=1}^m t_i < (r+1)p$. This construction is functorial in $S$ and determines an (ascending) filtration $\text{Fil}^{\text{conj}}_s$ on the sheaf $\mathcal{A}_{\text{crys}}(S)/p$ on $\mathrm{QRSPerf}_n$ for any $n$. By Proposition 8.12 of [BMS19] it is identified with the conjugate filtration on $R_{\mathrm{LdR}}(S/k)$ and thus is also a filtration by sheaves.

Note that both filtrations are multiplicative and exhaustive.

The following is an analogue of the inverse Cartier isomorphism (see Theorem 0.1.1) between $(\mathcal{A}_{\text{crys}}/p, 
\text{Fil}^{\text{conj}}_s)$ and $(\mathcal{A}_{\text{crys}}/p, \text{Fil}^{\text{conj}}_s)$.

**Proposition 1.3.13** ([BMS19], Proposition 8.11). Let $S$ be a semiperfect $k$-algebra. There is a well-defined surjective homomorphism of $S^0$-algebras $\kappa_* : \Gamma_{S^0}(I/I^2)^{(1)} \otimes_{S^0} S^0 \to \text{gr}^{\text{conj}}_s(\mathcal{A}_{\text{crys}}(S)/p)^3$. If $S$ is quasiregular $\kappa_*$ is an isomorphism.

**Proof.** The map is defined as follows: for $s_i \in I$

$$\kappa_{k_1, \ldots, k_m} : (s_1^{[k_1]} \ldots s_m^{[k_m]})^{(1)} \otimes 1 \to \prod_{i=1}^m \binom{(p^{k_i})!}{p^{k_i}k_i!} s_1^{[p^{k_i}]} \ldots s_m^{[p^{k_m}]}.$$

We have $(s_1s_2)^{[p]} = p!(s_2^{[p]}s_2^{[p]}) = 0$ and $(s_1s_2)^{[l]} = F^{\text{conj}}_s$ for any $l < p$. This shows that for $s \in I^2$, $s^{[l]} \in F^{\text{conj}}_s$ for all $l$ and so the map is well-defined. Elements $\{s_1^{[p^{k_1}]} \ldots s_m^{[p^{k_m}]})_{k_1+\cdots+k_m < r+1}$ in fact generate $F^{\text{conj}}_s$ over $S^0$. Since the integer $\prod_{i=1}^m \binom{(p^{k_i})!}{p^{k_i}k_i!}$ is a $p$-adic unit the map $\kappa_*$ is surjective. The fact that $\kappa_*$ is an isomorphism for $S$ quasiregular semiperfect is a part of Proposition 8.12 of [BMS19].

**Remark 1.3.14.** In particular we get an isomorphism $\kappa_* : \Gamma_{S^0}(I/I^2) \overset{\sim}{\to} \text{gr}^{\text{conj}}_s(\mathcal{A}_{\text{crys}}(S)/p)$ of sheaves on $\mathrm{QRSPerf}_n$.

Now we descend everything back to the quasisyntomic site $\mathrm{QSyn}_n$. We record what the sheaves defined above give when computed on a smooth $W_n(k)$-algebra $B$.

**Proposition 1.3.15.** Let $B$ be a smooth $W_n(k)$-algebra considered as an object of $\mathrm{QSyn}_n$. Then:

---

2Here endow ideal $(p) \subset \tilde{S}$ with the standard PD-structure.

3Here $\Gamma^*$ denote the free commutative divided power algebra.
1. For any $0 \leq s \leq n$ there is a natural equivalence of $E_\infty$-algebras $R\Gamma_{Qsyn}(B, \mathcal{A}_{crys}/p^s) \simeq \Omega^\bullet_{(B/p^s)}/W_s(k)$. 

2. For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{Qsyn}(B, \mathbb{I}^r) \simeq \Omega^{\geq r}_{(B/p)}/k$. 

3. For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{Qsyn}(B, \mathbb{I}^r) \simeq \Omega^{\leq r}_{(B/p)}/k$. 

4. The natural map $\Gamma_S(I/I^2)^{(1)} \to \mathbb{I}/I^{r+1}$ induces an equivalence $R\Gamma_{Qsyn}(B, \Gamma_S(I/I^2)^{(1)}) \simeq R\Gamma_{Qsyn}(B, \mathbb{I}^r/I^{r+1})$ for any $r \geq 0$. 

5. The isomorphism $\kappa_*: \Gamma^r_{S^e}(I/I^2)^{(1)} \to \mathfrak{g}_{r}^{\text{conj}}(\mathcal{A}_{crys}/p)$ from Proposition 1.3.13 induces the inverse Cartier isomorphism $\bigoplus_{r=0}^\infty \Omega^r/(I/I^2)^{(1)} \cong \bigoplus_{r=0}^\infty H^r(\Omega^\bullet_{(B/p)}/k, dR)$ via the above equivalencies.

Proof. Using 1.3.7, Remark 1.3.9 and Remark 1.3.8 parts 1, 2, 3 of the proposition follow from Proposition 8.12 and Theorem 8.14(3) of [BMS19] and flat descent for the derived crystalline cohomology.

4. We use the notations of Example 1.3.5. We have

$$R\Gamma_{Qsyn}(B, \mathcal{F}) \simeq \text{Tot}
\begin{array}{ccc}
\mathcal{F}(B_{\text{perf}}) & \dashrightarrow & \mathcal{F}(B_{\text{perf}} \otimes_B B_{\text{perf}}) \\
\mathcal{F}(B_{\text{perf}} \otimes_B B_{\text{perf}}) & \dashrightarrow & \mathcal{F}(B_{\text{perf}} \otimes_B B_{\text{perf}} \otimes_B B_{\text{perf}}) \\
& \cdots & \cdots
\end{array}$$

for any quasisijective sheaf $\mathcal{F}$. Moreover all terms $(B_{\text{perf}} \otimes_B \ldots \otimes_B B_{\text{perf}})_n$ are in fact regular semiperfect, meaning $I \subset S^n$ is generated by a regular sequence. Thus for them $\Gamma^*_S(I/I^2) \dashrightarrow I^r/I^{r+1}$ and so $R\Gamma_{Qsyn}(B, \Gamma_S(I/I^2)^{(1)}) \simeq R\Gamma_{Qsyn}(B, \mathbb{I}^r/I^{r+1})$.

5. The inverse Cartier isomorphism $C^{-1}$ is uniquely defined by the property that it is multiplicative, $C^{-1}(f) = f^p$ and $C^{-1}(df) = f^{p-1}df$ for any $f \in B$. The map $\kappa_*$ is multiplicative, $\kappa_0$ is by definition given by Frobenius, so it remains to check the third assertion. By functoriality (considering the map $k[x] \xrightarrow{\beta} B$) it is enough to check this in the case $B = k[x]$. Recall that the map $f \mapsto C^{-1}(df)$ is given more functorially as $f \mapsto \beta(f)$ where $\beta: H^0_{dR}(k[x]) \to H^1_{dR}(k[x])$ is the Bockstein operator associated to the de Rham complex of the lifting $\tilde{B} := W_2(k)[x]$ (see e.g. Proposition 4 of [Fra]). Thus it is enough to check that $\kappa_1$ agrees with Bockstein (associated to $R\Gamma_{Qsyn}(B, \mathcal{A}_{crys}/p^2) \simeq \Omega^\bullet_{B/W_2(k, dR)}$) in the same way. Let $\tilde{B}_{\text{perf}} := W_2(k)[x^{1/p^\infty}]$, then $R\Gamma_{Qsyn}(B, \mathcal{A}_{crys}/p^2)$ can be computed by the “unfolding”, given in this case by

$$W_2(k[x^{1/p^\infty}]) \xrightarrow{d} W_2(k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]_{(y_1 - y_2)}) \xrightarrow{PD} W_2(k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]_{(y_1 - y_2)})_{\text{Ker} \theta_1} \rightarrow \cdots$$

By part 1 we know that the cohomology is non-trivial only in degree 0 and 1. We have an analogous complex for $R\Gamma_{Qsyn}(B, \mathcal{A}_{crys}/p)$:

$$k[x^{1/p^\infty}] \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]_{(y_1 - y_2)} \to \cdots$$

The first arrow in both is given by $x \mapsto y_1 - y_2$. For $\mathcal{A}_{crys}/p$ its kernel is $k[x^p]$. We see that the class of $\beta(x^p) \in H^1_{dR}(k[x])$ under the identification $R\Gamma_{Qsyn}(B, \mathcal{A}_{crys}/p) \simeq \Omega^\bullet_{B/k, dR}$ can be computed using the complex for $k_{crys}/p^2$ as

$$\frac{d([x^p])}{p} \in k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]_{(y_1 - y_2)} \xrightarrow{PD} W_2(k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]_{(y_1 - y_2)})_{\text{Ker} \theta_1} \otimes W_2(k)$$

since the Teichmuller lift $[x^p] \in W_2(k[x^{1/p^\infty}])$ is indeed a lifting of $x^p \in k[x^{1/p^\infty}]$. We have $d([x^p]) = [y_1 - y_2]^p = p! \cdot |y_1 - y_2|^p$ and so $\frac{d([x^p])}{p} = (p - 1)! \cdot |y_1 - y_2|^p$.

$$R\Gamma_{Qsyn}(B, \mathbb{I}/I^2)$$ is computed by

$$0 \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}] \cdot (y_1 - y_2) / (y_1 - y_2)^2 \to \cdots$$
and $R\Gamma_{\text{QSyn}}(\overline{B}, \text{Fil}_1^{\conj}/ \text{Fil}_0^{\conj})$ is computed by

$$0 \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}] \cdot [(y_1 - y_2)^p] \to \ldots$$

By a direct computation one can see that $1 \cdot (y_1 - y_2)$ generates $H^1_{\text{QSyn}}(\overline{B}, \mathbb{I}/I^2)$, and that it is sent exactly to $(p - 1)! \cdot (y_1 - y_2)^p$ under the map $\kappa_1$ from Proposition 1.3.13. On the other hand $y_1 - y_2$ is equal to $d(x) \in k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]$ in the complex for $\text{Fil}_0^{\conj}/p$ and so corresponds to the differential form $dx$ under the quasiisomorphism $R\Gamma_{\text{QSyn}}(\overline{B}, \text{Fil}_1^{\conj}/ \text{Fil}_0^{\conj}) \simeq \Omega^1_{B/k}[-1]$. Thus $\kappa_1$ agrees with $C^{-1}$.

Next we prove the following enhancement of the classical Deligne-Illusie theorem:

**Theorem 1.3.16.** Let $\text{Aff}^{\text{sm}}/W_2(k)$ be the category of smooth affine schemes over $W_2(k)$. Then there is a natural $k$-linear equivalence of functors

$$\bigoplus_{i=0}^{p-1} \Omega^i_{(B^i)/k} \overset{\sim}{\to} \bigoplus_{i=0}^{p-1} \Omega^i_{(B^{i+1})/k}$$

from $\text{Aff}^{\text{sm,op}}/W_2(k)$ to $D(\text{Mod}_k)$ which induces the Cartier isomorphism on the level of the individual cohomology functors.

By Proposition 1.3.3 and Proposition 1.3.15 to deduce the statement of the theorem it is enough to prove the following:

**Proposition 1.3.17.** There is a natural isomorphism $\varphi : \bigoplus_{r=0}^{p-1} \Gamma_0(I/\mathbb{I}^2) \simeq \text{Fil}^\conj_{p-1}$ of sheaves of abelian groups on $\text{QRSPerf}_2$ such that it agrees with $\kappa_{\leq p-1} : \Gamma_0^p(I/\mathbb{I}^2) \simeq \text{Fil}^\conj_{p-1}(\text{Fil}_0^{\conj}/p)$ after passing to the associated graded.

**Proof.** Given $\overline{S} \in \text{QRSPerf}_2$ we denote by $S$ the reduction of $\overline{S}$ modulo $p$. As before we denote the kernel of the natural map $S^p \to S$ by $I$. Note that $\Gamma_0^p(I/\mathbb{I}^2) \simeq \text{Sym}_0^p(I/\mathbb{I}^2)$ for $i \leq p - 1$ and so, extending the map by multiplicativity, it is enough to construct a splitting $f : S^p/I \oplus I/\mathbb{I}^2 \to \text{Fil}^\conj_{p-1}$. Recall that we have a natural endomorphism $\varphi : \text{Fil}_0^{\conj}(S) \to \text{Fil}_0^{\conj}$. We consider the Nygaard filtration (see Definition 8.9 of [BMS19])

$$N^{\geq i} \text{Fil}_0^{\conj}(S) := \{ x \in \text{Fil}_0^{\conj}(S) \mid \varphi(x) \in p^i \text{Fil}_0^{\conj}(S) \}.$$ 

In fact we will be interested only in the first two of its associated graded terms. We will construct $f$ by using the divided Frobenii, defined as follows. By Theorem 8.15(1) of [BMS19] $\text{Fil}_0^{\conj}(S)$ is $p$-torsion free and so for each $i \geq 0$ one has a well defined map

$$\varphi_i : \varphi/p^i : N^i \text{Fil}_0^{\conj}(S) \to \text{Fil}_0^{\conj}(S)/p$$

from the $i$-th graded piece $N^i \text{Fil}_0^{\conj}(S) := N^i \text{Fil}_0^{\conj}(S)/N^{i+1} \text{Fil}_0^{\conj}(S)$ of the Nygaard filtration.

Obviously $\text{Fil}_0^{\conj}(S) \subseteq N^{\geq 1} \text{Fil}_0^{\conj}(S)$ and by Theorem 8.14(4) of [BMS19] $N^{\geq 1} \text{Fil}_0^{\conj}(S)$ mod $p \cdot \text{Fil}_0^{\conj}$ is given by $I \subseteq \text{Fil}_0^{\conj}(S)/p$, thus $N^0 := N^{\geq 0}/N^{\geq 1} \simeq S^p/I$ and $\varphi_0$ induces an isomorphism $S^p/I \xrightarrow{\sim} \text{Fil}_0^{\conj}$. Since $\kappa_0 = \varphi$ this follows from Proposition 1.3.13). Then we also have a map $\varphi_1 : N^1 \text{Fil}_0^{\conj}(S) \to \text{Fil}_0^{\conj}(S)/p$ and by Theorem 8.14(2) of [BMS19] it is an isomorphism on $\text{Fil}_0^{\conj}$. Multiplication by $p$ induces a natural map $N^0 \text{Fil}_0^{\conj}(S) \to N^1 \text{Fil}_0^{\conj}(S)$ which after composing with $\varphi_1$ is identified with the embedding $\text{Fil}_0^{\conj} \subseteq \text{Fil}_1^{\conj}$. In fact by flatness of $\text{Fil}_0^{\conj}(S)$ we have $N^{\geq 1} \text{Fil}_0^{\conj}(S) \cap p \cdot \text{Fil}_0^{\conj}(S) \simeq N^{\geq 0} \text{Fil}_0^{\conj}(S)$ and so $\text{Fil}_0^{\conj}$ (under the isomorphism given by $\varphi_1$) is identified exactly with the subspace of those elements in $N^{\geq 1} \text{Fil}_0^{\conj}(S)$ which (or rather their liftings to $N^{\geq 1} \text{Fil}_0^{\conj}(S)$) are divisible by $p$.

We now use the lifting $\overline{S}$ of $S$ to construct the splitting of $\text{Fil}_1^{\conj}$. Recall that we have a map $\theta_2^{\conj} : \text{Fil}_0^{\conj}(S)/p^2 \to \overline{S}$ and let $K := \ker \theta_2^{\conj}$. Since both $\overline{S}$ and $\text{Fil}_0^{\conj}(S)/p^2$ are flat over $W_2(k)$, we get that $K$ is also flat over $W_2(k)$ and that $K/pK \simeq I$:

$$0 \overset{K}{\longrightarrow} \text{Fil}_0^{\conj}(S)/p^2 \overset{\overline{S}}{\longrightarrow} 0 \quad 0 \overset{I}{\longrightarrow} \text{Fil}_0^{\conj}(S)/p \overset{\scriptscriptstyle S}{\longrightarrow} 0.$$
The splitting is then given by applying $\varphi_1$ to $K$. Namely, since $\varphi(I) = 0 \in \mathcal{A}_{\text{cris}}(S)/p$ it follows that $\varphi(K) \subset p \cdot \mathcal{A}_{\text{cris}}(S)/p^2$ and $K \subset N^{2,1} \mathcal{A}_{\text{cris}}(S) \mod p^2 \mathcal{A}_{\text{cris}}(S)$. The natural projection from $K$ to $N^{1,1} \mathcal{A}_{\text{cris}}(S)$ contains $p \cdot K + N^{2,2} \mathcal{A}_{\text{cris}}(S) \mod p^2 \mathcal{A}_{\text{cris}}(S)$ in its kernel. Since $K/pK = I$ and the image of $N^{2,2} \mathcal{A}_{\text{cris}}$ modulo $p$ is given by $I^2$ (e.g. by Theorem 8.14(4) of [BMS19]), we get that $\varphi_1$ (applied to $K$) gives a well-defined map $f : I/I^2 \to \text{Fil}_1^{\text{conj}}$. Moreover $K \cap (p \cdot \mathcal{A}_{\text{cris}}(S)/p^2) \subset p \cdot K$, since $K$ is flat over $W_2(k)$, and so the image of $f$ does not intersect with $\text{Fil}_0^{\text{conj}}$.

It remains to check that the constructed $f : I/I^2 \to \text{Fil}_1^{\text{conj}}$ coincides with $\kappa_1$ after the projection to $\text{Fil}_1^{\text{conj}} / \text{Fil}_0^{\text{conj}}$. Given $s \in I$ let $\bar{s} = [s] + p \cdot s' \in K \subset \mathcal{A}_{\text{cris}}/p^2$ be a lifting of $s$ to an element of $K$. Then

\[ \varphi(\bar{s}) = \varphi([s]) + p \cdot \varphi(s') = (p-1)! \cdot p \cdot [s]^p + p \cdot \varphi(s') \Rightarrow f(s) = (p-1)! \cdot s^p + \varphi(s'). \]

By the discussion above (see also Theorem 8.14(2) in [BMS19]) $\varphi(s') \in \text{Fil}_0^{\text{conj}}$ and $f(s) = (p-1)! \cdot s^p$ modulo $\text{Fil}_0^{\text{conj}}$.

Since the above splitting is clearly functorial in $\bar{s}$ we get the statement of the proposition.

As a corollary we deduce

**Theorem 1.3.18.** Let $\mathcal{Y}$ be a smooth Artin stack over a perfect field $k$ of characteristic $p$ admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. Then there is a canonical equivalence

\[ R\Gamma(\mathcal{Y}, \tau \leq p-1\Omega^*_\mathcal{Y},dR) \simeq R\Gamma \left( \mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \Lambda^1 L\mathcal{Y}^{(1)} \cdot [i] \right). \]

In particular for $n \leq p-1$ we have $H^n_{dR}(\mathcal{Y}) \simeq H^n_{\text{cris}}(\mathcal{Y}^{(1)})$.

**Proof.** Let $\pi : \text{Stk}_{k}^{n,\text{Art},\text{sm}} \to \text{Stk}_{k}^{n,\text{Art},\text{sm}}$ be the reduction functor, $\mathcal{Y} \to \mathcal{Y} \otimes_{W_2(k)} k$. By Theorem 1.3.16 it is enough to prove that the natural map (existing by the universal property of the right Kan extensions)

\[ R\Gamma_{dR}(-/k) \circ \pi \to \text{Ram}_{i_2}(R\Gamma_{dR}(-/k) \circ \pi|_{\text{Art}_{\text{sm}}^{n,W_2(k)}}) \]

(where $i_2$ denotes the inclusion functor $\text{Art}_{\text{sm}}^{n,W_2(k)} \to \text{Stk}_{k}^{n,\text{Art},\text{sm}}$) is an equivalence. Since both sides of (4) satisfy smooth descent, by induction on $n$ we reduce the statement to the case of smooth affine schemes over $W_2(k)$, where (4) is evidently an equivalence.

**Corollary 1.3.19.** Let $\mathcal{Y}$ be a smooth Hodge-proper stack over a perfect field $k$ of characteristic $p$ admitting a smooth lift to $W_2(k)$. Then the Hodge-to-de Rham spectral sequence $H^1(\mathcal{Y}, \Lambda^1 L\mathcal{Y}/k) \Rightarrow H^{1+i+j}_{dR}(\mathcal{Y}/k)$ degenerates at the first page for $i + j < p$.

**Proof.** This follows from Theorem 1.3.18 and the equality of dimensions $H^n_{\text{H}}(\mathcal{Y}) = H^n_{\text{H}}(\mathcal{Y}^{(1)})$.

### 1.4 Degeneration in characteristic zero

To reduce the statement in characteristic 0 to results of the previous section we introduce the following notion:

**Definition 1.4.1.** A smooth Hodge-proper Artin stack $X$ over a field $F$ of characteristic 0 is called *Hodge-properly spreadable* if there exists a finitely generated $\mathbb{Z}$-algebra $R \subset F$ and an Artin stack $X_R$ over $\text{Spec} R$ such that

- $X_R$ is smooth and $X \otimes_R F := X_R \times_{\text{Spec} R} \text{Spec} F \approx X$;
- $X_R$ is Hodge-proper, namely $R\Gamma(X_R, \Lambda^n L_{X_R}/R)$ is almost coherent over $R$ for any $p \geq 0$.

We defer a thorough discussion of spreadability of stacks till the next section. We only stress here again, that (unlike in the case of schemes) Hodge-proper spreadings do not exist in general (see Example 2.2.10).

Now we will deduce the promised Hodge-to-de Rham degeneration in characteristic 0:

**Theorem 1.4.2** (Hodge-to-de Rham degeneration for stacks). Let $X$ be a smooth Hodge-properly spreadable Artin stack over a field $F$ of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $X$ degenerates at the first page. In particular for each $n \geq 0$ there exists a (non canonical) isomorphism

\[ H^n_{dR}(X) \simeq \bigoplus_{p+q=n} H^{p,q}(X). \]
Proof. For the rest of the proof fix $n \in \mathbb{Z}_{\geq 0}$. By Hodge-properness of $X$ it is enough to prove

$$\dim_F H^n_{\text{dR}}(X) = \dim_F H^n_{\text{H}}(X).$$

Let $R$ and $X_R$ be as in Definition 1.4.1. Note that by the assumption on $X_R$ and Proposition 1.2.6 both $H^n_{\text{dR}}(X_R)$ and $H^n_{\text{H}}(X_R)$ are finitely generated $R$-modules. Localizing $R$ if necessary, we can assume that $R$ is regular and that the $n$-th cohomology groups $H^n_{\text{dR}}(X_R)$ and $H^n_{\text{H}}(X_R)$ are free $R$-modules of finite rank.\(^4\) By Proposition 1.1.7 (note that the map $s : R \to k$ satisfies conditions of Proposition 1.1.7 by the regularity assumption on $R$), for any point $s$: Spec $k \to$ Spec $R$ we have

$$R\Gamma_{\text{dR}}(X_k/k) \simeq R\Gamma_{\text{dR}}(X_R/R) \otimes_R k \quad \text{and} \quad R\Gamma_{\text{H}}(X_k/k) \simeq R\Gamma_{\text{H}}(X_R/R) \otimes_R k,$$

where $X_k := X_A \otimes_A k$. Since the $n$-th cohomology groups are free as $R$-modules we get $H^n_{\text{dR}}(X_k/k) \simeq H^n_{\text{dR}}(X_R/R) \otimes_R k$, so

$$\dim_F H^n_{\text{dR}}(X) = \text{rank}_R H^n_{\text{dR}}(X_R) = \dim_k H^n_{\text{dR}}(X_k)$$

and analogously for the Hodge cohomology. In particular to prove that $\dim_F H^n_{\text{dR}}(X) = \dim_F H^n_{\text{H}}(X)$ it is enough to show that $\dim_k H^n_{\text{dR}}(X_k/k) = \dim_k H^n_{\text{dR}}(X_k)$ for some point $s$: Spec $k \to$ Spec $R$.

To do so, consider first a point $t \in \text{Spec}(R \otimes \mathbb{Q})$ and its schematic closure $T \subset \text{Spec} R$. $T$ is a finite scheme over Spec $Z$, so if we take a closed point $s \in T$ of characteristic $p$ at which $T$ is étale over Spec $Z$, then the ($p$-adic) completion of the local ring $O_{T,s}$ is isomorphic to the Witt vectors $W(k(s))$ over the residue field $k(s)$. Moreover, without loss of generality we can assume $p > n$. Finally, the base change $X_{W(k(s))}$ is smooth and Hodge-proper over $W(k(s))$, so by Theorem 1.3.18 we have $\dim_{k(s)} H^n_{\text{dR}}(X_{k(s)}/k(s)) = \dim_{k(s)} H^n_{\text{H}}(X_{k(s)/k(s)})$ as desired. \(\Box\)

2 Spreadings

To apply Theorem 1.4.2 we need to find a good model of our stack over a finitely generated $\mathbb{Z}$-algebra: a so-called spreading. In Section 2.1 we prove a general result about the existence of spreadings for some natural classes of morphisms between Artin stacks (like smooth, flat, etc). Then some examples of Hodge-properly spreadable and nonspreadable stacks are given in Section 2.2. As a part of it we also discuss spreadings of proper morphisms in Section 2.2.1.

2.1 Spreadable classes

Definition 2.1.1. Let $\mathcal{P}$ be a class of morphisms of schemes (e.g. $\mathcal{P} =$ smooth, flat or proper morphisms) containing all isomorphisms and closed under compositions. For a scheme $S$ define $\text{Sch}_{/S}^{\text{fp}, \mathcal{P}}$ to be the (non-full) subcategory of schemes over $S$ consisting of finitely-presentable $S$-schemes and morphisms from $\mathcal{P}$ between them.

Theorem 2.1.2 ([Gro66, Theorems 8.10.5, 11.2.6] and [Gro67, Theorem 17.7.8]). Let $\{ S_i \}$ be a filtered diagram of affine schemes with limit $S$ and let $\mathcal{P}$ be one of the following classes of morphisms: isomorphisms, surjections, surjections of closed embeddings, flat, smooth or proper morphisms\(^5\). Then the natural functor

$$\lim_{\to i} \text{Sch}_{/S_i}^{\text{fp}, \mathcal{P}} \to \text{Sch}_{/S}^{\text{fp}, \mathcal{P}}$$

(induced by the base change $\text{Sch}_{/S_i}^{\text{fp}, \mathcal{P}} \ni X \mapsto X \times_{S_i} S$) is an equivalence.

We will say that a scheme $X$ is a $\mathcal{P}$-scheme over $S$ ($\mathcal{P}$-scheme/$S$) if $X$ is an $S$-scheme and the structure morphism $X \to S$ is in $\mathcal{P}$.

Corollary 2.1.3. Let $\{ S_i \}_{i \in I}, S$ and $\mathcal{P}$ be as above. Then if $X$ is a finitely presentable $\mathcal{P}$-scheme/$S$, then there exists $i \in I$ and a finitely presentable $\mathcal{P}$-scheme $X_i$ over $S_i$, such that $X \simeq X_i \times_{S_i} S$.

\(^4\)Note that there does not necessarily exist a localization $R[s^{-1}]$ such that for all $i$ the $R[s^{-1}]$-modules $H^n_{\text{dR}}(X_R)[s^{-1}]$ (or $H^n_{\text{H}}(X_R)[s^{-1}]$) are free, since there are infinitely many of them.

\(^5\)The list is not even nearly complete. See [Poo17, Appendix C.1] for a much more exhaustive list of classes of morphisms and their properties with precise references.
Proof. Let $\pi: X \to S$ be the structure morphism. By the previous theorem and description of objects in filtered colimits of categories (see e.g. [Roz12]) there exists a finitely presented scheme $\pi_j: X_j \to S_j$ such that $\pi_j \times_S S = \pi$. A morphism in a filtered colimit of categories is a filtered co-limit of morphisms, hence

$$\text{Hom}_{\text{Sch}_{/S}^{fp}}(X, S) \simeq \lim_{\overrightarrow{k}} \text{Hom}_{\text{Sch}_{/S_j}^{fp}}(X_j \times S_j, S_j).$$

Since the left hand side is non-empty by assumption, the right hand side also must also be nonempty for some $i$, i.e. there exists $i \in I$ such that $\pi_i: X_i \to S_i$ is in $P$. \qed

Our goal in this section is to extend the Theorem 2.1.2 to the setting of Artin stacks. First we recall how “finitely presentable” is defined in Artin setting:

**Definition 2.1.4 (Finitely presentable Artin stacks).** A $(-1)$-Artin stack over $R$, i.e. an affine scheme $\text{Spec} \, A$ over $R$, is called finitely presentable if $A$ is a finitely presentable $R$-algebra. An $n$-Artin stack $X$ over $R$ is called finitely presentable if there exists a smooth atlas $U \to X$ such that $U$ is a finitely presentable affine scheme and $U \times_X U$ is a finitely presentable $(n-1)$-Artin $R$-stack. We will denote the category of finitely presentable $n$-Artin stacks by $\text{Stk}^{n-\text{Art,fp}}$.

Our strategy for proving spreadability results is to inductively reduce to the case of finitely presentable schemes. For this we will need a description of an $n$-Artin stack as a quotient of a hypercover consisting of such schemes.

**Construction 2.1.5.** Let $X_\bullet: \Delta^{op} \to \mathcal{C}$ be a simplicial object in a category $\mathcal{C}$ admitting finite limits. Define $X(-): \text{SSet}^{\text{fin}, \text{op}} \to \mathcal{C}$ to be the right Kan extension of $X_\bullet$ along the inclusion of $\Delta^{op}$ into the opposite $\text{SSet}^{\text{fin}, \text{op}}$ of the category of finite simplicial sets (meaning simplicial sets with only finitely many non-degenerate simplices). More concretely, for a finite simplicial set $S$

$$X(S) \simeq \lim_{\Delta^n \in \Delta_{n,S}} X(\Delta^n).$$

In particular, we denote $M_n(X_\bullet) := X(\partial \Delta^n)$ and call it the $n$-th matching object of $X_\bullet$.

**Definition 2.1.6.** Let $\mathbb{H}$ be an $\infty$-topos. An augmented simplicial object $X_\bullet: \Delta^{op} \to \mathbb{H}$ is called a hypercover of $X_{-1}$ if for any $n \in \mathbb{Z}_{\geq 0}$ the natural map $X_n \to M_n(X_\bullet)$ is an effective epimorphism ($M_n$ is computed in the category $\mathbb{H}_{/X_{-1}}$). A hypercover $X_\bullet$ is called $n$-coskeletal if additionally for each $m > n$ the natural map $X_m \to M_n(X_\bullet)$ is an equivalence (equivalently $X_\bullet$ coincides with the right Kan extension of its restriction to $\Delta_{\leq n}$).

With this notation we have

**Theorem 2.1.7 ([Pri15, Theorem 4.15]).** Let $X$ be a finitely presented $n$-Artin stack. Then there exists an $(n-1)$-coskeletal hypercover $X_\bullet$ of $X$ such that all $X_k$ are finitely presentable affine schemes and for all $m, k, 0 \leq m \leq k$ the maps $X_k \to X(\Delta^n_{m,k})$ are smooth surjections. Conversely, given $X_\bullet$ as above, its geometric realization $|X_\bullet|$ is a finitely presentable $n$-Artin stack.

For convenience we introduce the following notation:

**Definition 2.1.8 (Spreadable class).** A class of morphism $P$ between Artin stacks is called spreadable if

- $P$ is closed under arbitrary base changes, compositions and contains all equivalences.
- (Locality on source and target) Let $f: X \to Y$ be a morphism of finitely presentable Artin stacks. If there exist smooth finitely presentable affine atlases $U \to Y$ and $V \to U \times_Y X$ such that the composite map $V \to U \times_Y X \to U$ is in $P$, then $f$ is also in $P$.
- (Affine spreadability) Let $\{S_i\}$ be a filtered diagram of affine schemes with the limit $S$. Let $f: X \to Y$ be a morphism in $P$ between affine finitely presentable $S$-schemes. Then for some $i$ there exists a map $f_i: X_i \to Y_i$ in $P$ of affine finitely presentable $S_i$-schemes, such that $f \simeq f_i \times_{S_i} S$.

**Example 2.1.9.** If $P$ and $Q$ is a pair of spreadable classes, then $P \cap Q$ and $P \cup Q$ are also spreadable. There exists the smallest spreadable class (consisting only of equivalences) and the largest one (consisting of all finitely presentable morphisms).

**Example 2.1.10.** Since surjective, smooth and flat morphisms of Artin stacks are by definition local on the source and the target for the flat topology, by Theorem 2.1.2 we get that these classes are spreadable.
**Theorem 2.1.12.** Let \( \{ S_i \} \) be a filtered diagram of affine schemes with limit \( S \). Then the natural functor

\[
\lim_{\rightarrow i} \text{Stk}^{n-\text{Art}, \text{fp}, P}_{/S_i} \to \text{Stk}^{n-\text{Art}, \text{fp}, P}_{/S}
\]

(induced by the base-change \( \text{Stk}^{n-\text{Art}, \text{fp}, P}_{/S_i} \to X_i \to X_i \times S_i, S \)) is an equivalence.

**Proof.** We will prove the statement by induction on \( n \). The base of the induction \( n = -1 \), i.e. the case of affine schemes, holds by the definition of spreadable class. To make the induction step, we first prove the statement for \( P = \text{all (finitely presented) morphisms} \) (using the induction assumption for smooth surjective morphisms) and then deduce the statement for a general spreadable class \( P \).

**Essential surjectivity for** \( P = \text{all} \). Since all \( n \)-Artin stacks are \((n+1)\)-truncated, the Yoneda embedding \( \text{Stk}^{n-\text{Art}} \to \text{Fun}(\text{CAlg}, S)\) factors through a full subcategory \( \text{Fun}(\text{CAlg}, S_{n+1}) =: \text{PrStk} \). Let now \( X \) be a finitely presented \( n \)-Artin \( S \)-stack and let \( X_n \) be a simplicial diagram of finitely presented affine \( S \)-schemes, so that \( |X_n| \simeq X \) (see Theorem 2.1.7). Since for any simplicial diagram \( A_n \) in any \((n+1)\)-category the natural map \( |A_n| \to |A_n| \) is an equivalence, we see that \( X \simeq |X_n| \leq n \) in \( \text{PrStk} \). But \( X_n |_{\Delta \leq n+2} \) is a finite diagram of finitely presented affine schemes, hence there exists \( S_i \) and a diagram \( X_n |_{\Delta \leq n+2} \times S_i \) such that \( X_n |_{\Delta \leq n+2} \simeq X_n |_{\Delta \leq n+2} \times S_i \times S_i \). We set \( X_n := |X_n| \leq n+2, S_i| \). By applying the inductive assumption with \( P = \text{smooth surjective} \), we can assume that all maps \( X_n \to X_n \times S_i(A^h) \) are smooth and surjective for some \( S_i \); hence by Theorem 2.1.7 \( X_n \) is a finitely presented \( n \)-Artin spreading of \( X \).

**Fully-faithfulness for** \( P = \text{all} \). Let \( X_i, Y_i \) be a pair of \( n \)-Artin stacks of finite presentation over \( S_i \). Then have

\[
\lim_{\rightarrow i} \text{Hom}_{\text{PrStk}}(X_i \times S_i, S_j, Y_i \times S_j, S_j) \simeq \lim_{\rightarrow i} \text{Hom}_{\text{PrStk}}(X_i \times S_i, S_j, Y_i) \simeq \lim_{\rightarrow i} \text{Hom}_{\text{PrStk}}(X_i \times S_i, S_j, Y_i)_i
\]

where the second equivalence follows from the fact that filtered co-limits commutes with \( \pi_* \), hence preserve \((n+1)\)-truncated spaces. Let now \( X_n \to X \) be as in Theorem 2.1.7. Then

\[
\lim_{\rightarrow i} \text{Hom}_{\text{PrStk}}(X_n \times S_i, S_j, Y_n \times S_j, S_j) \simeq \text{Tot}_{\leq n+2} \lim_{\rightarrow i} \text{Hom}_{\text{PrStk}}(X_n \times S_i, S_j, Y_n)_i \simeq \text{Tot}_{\leq n+2} \lim_{\rightarrow i} \text{Hom}_{\text{Stk}}(X_n \times S_i, S_j, Y_n)_i
\]

where the second equivalence follows from the fact that since \( \Delta \leq n+2 \) is a finite diagram, limits along \( \Delta \leq n+2 \) commute with filtered co-limits. Similarly one shows that

\[
\text{Hom}_{\text{Stk}}(X_i \times S_i, S_j, Y_i \times S_i, S_j) \simeq \text{Tot}_{\leq n+2} \text{Hom}_{\text{Stk}}(X_i \times S_i, S_j, Y_i)_i
\]

Finally, since \( Y_i \) is finitely presentable, by [GR17, Chapter 2, Proposition 4.5.2]

\[
\lim_{\rightarrow i} \text{Hom}_{\text{Stk}}(X_i \times S_i, S_j, Y_i) \simeq \text{Hom}_{\text{Stk}}(X_i \times S_i, S_j, Y_i)_i
\]

**General** \( P \). Let \( f : X \to Y \) be a morphism in a spreadable class \( P \) over \( S \). It is enough to prove that there exists \( i \) and a map between finitely presentable \( n \)-Artin \( S_i \)-stacks \( f_i : X_i \to Y_i \) such that \( f_i \times S_i \simeq f \) and \( f_i \in P \). Choose affine finitely presentable atlases \( U \to Y \) and \( V \to U \times_S X \). The induced map \( g : V \to U \) belongs to \( P \) and is \((n-1)\)-representable, so the previous part and inductive assumption, the diagram

\[
\begin{array}{ccc}
V & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
U & \longrightarrow & Y
\end{array}
\]

can be spread out to some \( S_i \), such that \( g_{S_i} \) belongs to \( P \). It follows by the definition of spreadable class, that \( f_{S_i} \) is also in \( P \). \( \square \)
A stack $X$ will be called a $n$-Artin $\mathcal{P}$-stack over $S$ if the structure morphism $\pi : X \to S$ exhibits $X$ as an $n$-Artin stack and $\pi$ is in $\mathcal{P}$.

**Corollary 2.1.13** (Existence of spreading in predefined class). Let $\{S_i\}_{i \in I}$ be a filtered diagram of affine schemes, $S := \lim S_i$ and $\mathcal{P}$ be a spreadable class. Then if $X$ is a finitely presentable $n$-Artin $\mathcal{P}$-stack over $S$, then there exists $i \in I$ and a finitely presentable $n$-Artin $\mathcal{P}$-stack $X_i$ over $S_i$, such that $X \simeq X_i \times_{S_i} S$.

**Proof.** Apply Theorem 2.1.12 to the structure morphism $X \to S$ and argue as in the proof of Corollary 2.1.3. $\Box$

### 2.2 Examples of Hodge-properly spreadable stacks

In this subsection we study which Hodge-proper stacks in characteristic 0 admit a Hodge-proper spreading over some finitely generated $\mathbb{Z}$-algebra. We will make extensive use of Theorem 2.1.12 in the following situation: let $F$ be an algebraically closed field of characteristic 0, then $\text{Spec } F \simeq \lim R$ where $R \subset F$ runs through subrings of $F$ that are regular and finitely generated over $\mathbb{Z}$. In particular we have an equivalence

$$\lim_{R \subset F} \text{Stk}^{n,\text{Art},\text{fp},\mathcal{P}}_{/R} \xrightarrow{\sim} \text{Stk}^{n,\text{Art},\text{fp},\mathcal{P}}_{/F}$$

for any spreadable class $\mathcal{P}$. In what follows $F$ will always denote an algebraically closed field of characteristic 0 and $R$ will be a finitely generated regular $\mathbb{Z}$-subalgebra of $F$. We also freely use the standard spreading out results for schemes (Theorem 2.1.2) and their easy consequences (like spreading out group schemes, group actions, group homomorphisms, closed subgroups, etc.) without any additional reference.

The Hodge-proper spreadability for proper smooth Artin stacks is deduced from the spreadability of proper morphisms Theorem 2.1.12. This is done in Section 2.2.1. The non-proper case is much more interesting (and nontrivial); the following two important representation-theoretic results are used in the proofs.

**Theorem 2.2.1** (Theorem 9 and Proposition 57 of [FvdK10]). Let $G_\mathbb{Z}$ be a split reductive group over $\mathbb{Z}$ and $R$ be a finitely generated algebra over $\mathbb{Z}$. Let $A$ be a finitely generated $R$-algebra endowed with a (rational) action of $G_R$. Then the algebra $A^{G_R}$ of $G_R$-invariants is finitely generated over $R$ and $H^n(G_R, A)$ is a finitely generated $A^{G_R}$-module for any $n \geq 0$.

**Theorem 2.2.2** (Kempf’s theorem (Proposition II.4.5 of [Jan07])). Let $G_\mathbb{Z}$ be a split connected reductive group over $\mathbb{Z}$ and let $B_\mathbb{Z} \subset G_\mathbb{Z}$ be a Borel subgroup. Let $(G/B)_\mathbb{Z}$ be the corresponding flag variety. Then $R\Gamma((G/B)_\mathbb{Z}, \mathcal{O}_{(G/B)_\mathbb{Z}}) \simeq \mathbb{Z}$.

These two tools (together with more standard techniques, like Hochshild-Serre spectral sequence), allow to show in many cases that a quotient stack is Hodge-properly spreadable. We restricted ourselves to two examples we found the most illustrative: a proper-over-affine scheme with an action of a reductive group (Section 2.2.3) and a less general case of a conical resolution where the reductivity assumption on the group can be relaxed slightly (Section 2.2.4). We also discuss in great detail the question of Hodge-proper spreadability of $BG$ in Section 2.2.2.

#### 2.2.1 Proper stacks

In this subsection we show that all proper stacks are Hodge-proper and Hodge-properly spreadable.

**Definition 2.2.3** (Proper Artin stacks). Let $S$ be a scheme. A 0-Artin stack over $S$ is called *proper* if it is a proper $S$-scheme. An $n$-Artin stack $X$ over $S$ is called *proper* if there exists a smooth atlas $U \to X$, where $U$ is a proper $S$-scheme, such that $U \times_X U$ is a proper $(n-1)$-Artin $U$-stack.

**Remark 2.2.4.** A potentially more familiar definition of a (classical) proper algebraic stack $p : X \to S$ is that $p$ should be separated, finite type and universally closed. We note that such stacks over $S$ are proper 1-Artin stacks in the definition above. Indeed, by [Ols05] there exists a proper surjective map $U \to X$ from a proper scheme $U$. Then $U \times_X U$ is a proper scheme, so $X$ is a proper 1-Artin stack.

**Remark 2.2.5.** It follows from the definition that the full subcategory $\text{Stk}^{n,\text{Art},\text{pr}} \subset \text{Stk}$, consisting of proper $n$-Artin stacks, is closed under finite products.

**Proposition 2.2.6.** Let $X$ be a smooth proper stack over $R$. Then it is Hodge-proper.
Proof. Assume that \( X \) is \( n \)-Artin. We will prove the statement by induction on \( n \). Let \( U \to X \) be a smooth proper atlas. By the previous remark
\[
U_k := U \times_X U \times_X \ldots \times_X U \simeq (U \times_X U) \times_U (U \times_X U) \times_U \ldots (U \times_X U)
\]
is Hodge-proper \((k-1)\)-Artin stack over \( U \) (and hence over the base) for any \( n \). By the smooth descent for cotangent complex we have
\[
R \Gamma_H(X) \simeq \text{Tot} R \Gamma_H(U_k).
\]
By inductive assumption \( R \Gamma_H(U_n) \) is almost coherent for any \( n \) and connective since \( U_n \) is smooth. By Proposition 1.2.3 \( R \Gamma_H(X) \) is also almost coherent.

An inductive argument analogous to the one used in Theorem 2.1.12 gives the following:

**Proposition 2.2.7.** Let \( \{ S_i \} \) be a filtered diagram of affine schemes with a limit \( S \). Then the natural functor
\[
\lim_i \text{Stk}^{n,\text{Art,pr}}_{/S_i} \to \text{Stk}^{n,\text{Art,pr}}_{/S}
\]
is an equivalence.

**Corollary 2.2.8.** Let \( X \) be a smooth proper stack over a field \( F \) of characteristic 0. Then \( X \) is Hodge-proper and Hodge-properly spreadable.

### 2.2.2 Classifying stacks

Let’s first understand for which groups \( G \) in characteristic 0 the classifying stack \( BG \) is Hodge-proper. The answer is easy: more or less for all \( G \).

**Proposition 2.2.9.** Let \( G \) be a finite type group scheme over an algebraically closed field \( F \) of characteristic 0. Then \( BG \) is Hodge-proper.

**Proof.** In fact we will show a stronger statement, namely that for any perfect sheaf \( F \) on \( BG \), \( R \Gamma(BG,F) \) lies in \( D(\text{Mod}_F)^{\text{acoh}} \) in general and in \( D(\text{Mod}_F)^{\text{perf}} \) if \( G \) is linear. The exterior power \( \wedge^n_{LBG} \simeq \text{Sym}^i(g^*)[-i] \) is perfect, so this indeed will suffice.

Note that in characteristic 0 all finite-type group schemes are smooth, so \( BG \) is always a smooth 1-Artin stack and thus \( \text{Qcoh}(BG)^{\text{perf}} \) has a t-structure which coincides with the usual t-structure on \( D(\text{Mod}_F)^{\text{perf}} \) after applying the forgetful functor \( \text{Qcoh}(BG)^{\text{perf}} \to D(\text{Mod}_F)^{\text{perf}} \). In particular every perfect complex has a finite filtration with the associated graded given by the sum of its cohomology groups. Since \( R \Gamma \) is exact and both \( D(\text{Mod}_F)^{\text{perf}} \) and \( D(\text{Mod}_F)^{\text{acoh}} \) are closed under finite limits, it is enough to show the statement for \( F \) lying in the heart \( \text{Qcoh}(BG)^{\text{perf,\circ}} \). Note that such \( F \) is the same thing as a finite-dimensional algebraic representation of \( G \) over \( F \).

By Chevalley’s structure theorem there is an exact sequence \( 1 \to L \to G \to A \to 1 \) where \( L \) is a linear algebraic group and \( A \) is proper. Then for \( L \) we have another short exact sequence
\[
1 \to U \to L \to H \to 1,
\]
where \( U \) is the unipotent radical of \( L \) and \( H \simeq L/U \) is reductive.

Let \( j : BU \to BL \), \( f : BL \to BH \), \( i : BL \to BG \) and \( p : BG \to BA \) be the corresponding maps between classifying stacks. We will prove the statement by step considering different options for \( G \) and making \( G \) more and more general.

**Case 1.** \( G = U \) is unipotent. We assume \( F \in \text{Qcoh}(BU)^{\text{perf,\circ}} \). Since the characteristic of \( F \) is 0 and \( U \) is unipotent, \( R \Gamma(BU,F) \) can be computed as the cohomology of the Lie algebra \( u \). Explicitly this is given by the Chevalley complex:
\[
0 \to F \to F \otimes u^* \to F \otimes \wedge^2 u^* \to \ldots F \otimes \wedge^{\dim U} u^* \to 0.
\]
Since \( F \) is finite dimensional this complex is clearly perfect.

**Case 2.** \( G = H \) is reductive. This follows from the fact that (since char(\( F \)) = 0) the category \( \text{Rep}_H(Vect_F) \) is semisimple. Namely for \( F \in \text{Qcoh}(BH)^{\text{perf,\circ}} \), \( R \Gamma(BH,F) \) is equal to the \( H \)-invariants \( F^H \), since \( F \) is finite-dimensional we get \( R \Gamma(BH,F) \in D(\text{Mod}_F)^{\text{perf}} \).
Case 3. $G = A$ is proper. Let $\mathcal{F}$ be a perfect complex on $BA$. We can compute $R\Gamma(BA, \mathcal{F})$ using the Čech simplicial object associated to the cover $q : \text{pt} \to BA$. Let $p_n : A^n \to BA$ be the map from the $n$-th term of the object. We get a cosimplicial object

$$[n] \mapsto R\Gamma(A^n, p_n^*\mathcal{F}),$$

and

$$R\Gamma(BA, \mathcal{F}) \simeq \text{Tot } R\Gamma(A^n, p_n^*\mathcal{F}).$$

However, each term $R\Gamma(A^n, p_n^*\mathcal{F})$ lies in $D(\text{Mod}_{\mathcal{F}})^{\text{perf}}$ since $A^n$ is proper and has cohomology only in non-negative degrees. By Proposition 1.2.3 it follows that $R\Gamma(BA, \mathcal{F})$ lies in $D(\text{Mod}_{\mathcal{F}})^{\text{acoh}}$.

Now we will deduce the general case.

Case 4. $G = L$ is linear. We assume $\mathcal{F} \in \text{QCoh}(BU)^{\text{perf}}$ and consider $f_*\mathcal{F} \in \text{QCoh}(BH)$. We claim it actually lands in $\text{QCoh}(BH)^{\text{perf}}$. It is enough to check that after taking pull-back to the smooth cover $q : \text{pt} \to BH$. We have a fibered square

$$\begin{array}{ccc}
BU & \xrightarrow{j} & BL \\
\downarrow q & & \downarrow f \\
\text{pt} & \to & BH
\end{array}$$

and by the base change theorem we have $q^*f_*\mathcal{F} \simeq R\Gamma(j^*\mathcal{F})$. $j$ is flat so $j^*\mathcal{F}$ is perfect and thus $R\Gamma(j^*\mathcal{F})$ is perfect by Case 1. It follows that $f_*\mathcal{F}$ is perfect. But then $R\Gamma(BL, \mathcal{F}) \simeq R\Gamma(BH, f_*\mathcal{F})$ and we are done by Case 2. At this point we have the statement for $G$ linear.

Case 5. $G$ is general. The argument in Step 1 works here as well after replacing $U$ with $L$ and $H$ with $A$. Namely $p_*\mathcal{F}$ is perfect and then by Case 3 we are done. \hfill \Box

Even though $BG$ is Hodge-proper practically for any $G$, there are definitely some algebraic groups $G$ for which $BG$ is non-spreadable. Indeed, if $G = \mathbb{G}_a$ were spreadable, then by Corollary 2.3.2 we would get a (non-canonical) decomposition

$$H^n_{\text{dR}}(BG_\mathbb{a}) \simeq \bigoplus_{p+q=n} H^q(BG_\mathbb{a}, \Lambda^p L_{BG_\mathbb{a}}).$$

By the $\mathbb{A}^1$-homotopy invariance of the de Rham cohomology in characteristic 0, the left hand side vanishes for $n > 0$. On the other hand $\Lambda^p L_{BG_\mathbb{a}} \simeq \mathcal{O}_{BG_\mathbb{a}}[-p]$, and $H^q(BG_\mathbb{a}, \mathcal{O}_{BG_\mathbb{a}})$ is non-zero for $i = 0, 1$. Thus the right hand side is non-zero for all $n$, a contradiction. From Theorem 1.4.2 it follows that the Hodge cohomology of any smooth spreading of $BG_\mathbb{a}$ has to be infinitely generated, which is confirmed by a direct computation:

Example 2.2.10. Let $X = BG_\mathbb{a}$ and let $X_R$ be a Hodge-proper spreading of $X$. Then, since $(BG_\mathbb{a})_R$ is a spreading of $BG_\mathbb{a}$, by Theorem 2.1.12, after enlarging $R$, we can actually assume $X_R \simeq (BG_\mathbb{a})_R$. The cohomology of $\mathcal{O}_{BG_\mathbb{a}}$ over $\mathbb{Z}$ is given by

$$H^\bullet(BG_\mathbb{a}, \mathcal{O}_{BG_\mathbb{a}}) \simeq \mathbb{Z}[v_1]/v_1^2 \otimes \mathbb{Z} \text{Sym}^\bullet \left( \bigoplus_p F_p(v_p, v_p^2, v_p^3, \ldots) \right),$$

where the sum is taken over all primes $p$, and $F_p(v_p, v_p^2, v_p^3, \ldots)$ denotes the free vector space spanned by $v_p, v_p^2, v_p^3, \ldots$, element $v_1$ has degree 1 and all other $v_i$ have degree 2. Since $R \subset F$ is torsion-free, by base change we also get that

$$H^\bullet(X_R, \mathcal{O}_{X_R}) \simeq R[v_1]/v_1^2 \otimes R \text{Sym}^\bullet \left( \bigoplus_p R/p(v_p, v_p^2, v_p^3, \ldots) \right).$$

Since $R$ is a finitely generated $\mathbb{Z}$-algebra, for $p$ big enough we have $R/p \neq 0$ and thus $H^i(X_R, \mathcal{O}_{X_R})$ is not finitely generated over $R$ for $i \geq 2$. A similar thing happens for any unipotent group $U$, namely $BU$ always has gigantic $p$-torsion in cohomology for almost all primes $p$.

So it is natural to ask when $BG$ is Hodge-properly spreadable. We provide a list of examples:

- $G$ is proper (=an extension of a finite group by an abelian variety). Then $BG$ is a proper stack and this is covered by Corollary 2.2.8;\footnote{The indexing of the generators is not arbitrary and corresponds to their $G_m$-weights, via the natural action of $G_m$ on $G_\mathbb{a}$ by rescaling, we will discuss that in more detail in Example 2.2.12.}
• $G$ is reductive. This is a particular case of Theorem 2.2.13;
• $G = P \subset H$ is some parabolic subgroup of some reductive group $H$. This is a particular case of Theorem 2.2.16.

**Remark 2.2.11.** By an argument similar to Proposition 2.2.9 it is also possible to show the spreadability of $BG$ for an extension of an abelian variety by a parabolic subgroup of some reductive group.

The fact that $BP$ is spreadable can look a little surprising and we would like to illustrate what happens by the simplest non-trivial example, a Borel subgroup $B \subset SL_2$:

**Example 2.2.12.** Let $G = B \subset SL_2$ be the standard Borel subgroup of $SL_2$, namely

$$B = \left\{ \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right\} \subset SL_2.$$  

Then $B \cong \mathbb{G}_a \times \mathbb{G}_m$ with $\mathbb{G}_m = \text{Spec} \mathbb{Z}[t, t^{-1}]$ acting on $\mathbb{G}_a = \text{Spec} \mathbb{Z}[x]$ by multiplication of $x$ by $t^2$. Consider the natural map $p: BB \to BG_a$ and take $p_*(\mathcal{O}_{BB})$. We have a fiber square

$$
\begin{array}{ccc}
BG_a & \longrightarrow & BB \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & BG_m.
\end{array}
$$

We have $j^*\mathcal{O}_{BB} \cong \mathcal{O}_{BG_a}$ and by base change the underlying complex $q^*p_*\mathcal{O}_{BB}$ is equal to $R\Gamma(BG_a, \mathcal{O}_{BG_a})$. It follows that

$$R\Gamma(BB, \mathcal{O}_{BB}) \cong R\Gamma(BG_a, p_*\mathcal{O}_{BB}) \cong R\Gamma(BG_a, \mathcal{O}_{BG_a})^\mathbb{G}_m,$$

since $\mathbb{G}_m$-invariants is an exact functor. So it left to understand the $\mathbb{G}_m$-action on $R\Gamma(BG_a, \mathcal{O}_{BG_a})$.

The description in Example 2.2.10 can be made more functorial. Namely let $V \cong \mathbb{Z}^n$ be a lattice and let $V \cong \mathbb{G}_a \otimes \mathbb{Z} V$ be the corresponding vector group scheme. Let $V F_p \cong V \otimes \mathbb{F}_p$ be the reduction and let $V F_p(i)$ denote the $i$-th Frobenius twist. Then

$$H^*(BV, \mathcal{O}_{BV}) \cong \Lambda^*_\mathbb{Z} V \otimes \mathbb{Z} \text{Sym}^\bullet \left( \bigoplus_p V F_p(i) \oplus V F_p(2) \oplus V F_p(3) \oplus \cdots \right),$$

as a representation of the algebraic group $GL(V)$. Here $V$ is in degree 1 and all $V F_p(i)$ are in degree 2. Applying this to our case and using the notation of Example 2.2.10 for the description of $H^*(BG_a, \mathcal{O}_{BG_a})$ we see that $\mathbb{G}_m$ acts on $v_1$ with the weight 2 and on $v_{p^k}$ with the weight $2p^k$. Since $H^*(BG_a, \mathcal{O}_{BG_a})$ is freely generated by $v_1$’s it follows that the $\mathbb{G}_m$-invariants are given by $H^0(BG_a, \mathcal{O}_{BG_a}) \cong \mathbb{Z}$. Consequently $R\Gamma(BB, \mathcal{O}_{BB}) = \mathbb{Z}$.

Summarizing, we see that even though the cohomology of $BG_a$ is enormous, the $\mathbb{G}_m$-action contracts it to something finitely generated and nice, ultimately making $BB$ Hodge-proper over $\mathbb{Z}$.

### 2.2.3 Global quotients by reductive groups

In [Tel00] Teleman proved the Hodge-to-de Rham degeneration for quotients of $KN^7$-complete schemes by an action of a reductive group (for the definitions of $KN$-stratification and $KN$-completeness see Section 1 [Tel00] or Section 2.1 of [HLP15]). We expect all these quotients to be Hodge-properly spreadable. We can prove this for the typical $KN$-complete example given by a smooth projective-over-affine scheme $X$ with the space of $G$-invariant functions being finite-dimensional. Moreover, we replace the projective-over-affine assumption by proper-over-affine almost for free:

**Theorem 2.2.13.** Let $F$ be an algebraically closed field of characteristic 0. Let $X$ be a smooth scheme and let $\text{Spec} \ A$ be a finite-type affine scheme over $F$, both endowed with an action of a reductive group $G$. Assume that

- There is a proper $G$-equivariant map $\pi: X \to \text{Spec} \ A$.
- $\dim_F A^G < \infty$.

Then the quotient stack $[X/G]$ is Hodge-properly spreadable.

---

7$KN$ stays for Kempf-Ness.
Proof. The group $G$ is split and has a Chevalley model $G_\mathbb{Z}$ over $\mathbb{Z}$; this defines a spreading of $G$ over any $R$, namely put $G_R := G_\mathbb{Z} \times_{\mathbb{Z}} R$. We can also spread $X$ with an action $\alpha : G \curvearrowright X$ to a smooth $X_R$ with an action $a_R : G_R \curvearrowright X_R$. Morphism $\pi : X \to \text{Spec } A$ can be spread out to a proper morphism $\pi_R : X_R \to \text{Spec } A_R$ for some finitely generated $R$-algebra $A_R$. We can also assume that $\pi_R$ is $G_R$-equivariant.

We claim that we can enlarge $R$ making the stack $[X_R/G_R]$ Hodge-proper over $R$. Indeed, $\mathbb{L}_{X_R/G_R}$ is represented by the 2-term complex $\Omega^{\cdot}_{X_R/R} \to \mathfrak{g}_R^0 \otimes_R \mathcal{O}_{X_R}$ of $G$-equivariant sheaves on $X$. It follows that the underlying complex of quasi-coherent sheaves on $X$ of

$$\wedge^p \mathbb{L}_{X_R/G_R} \cong \Omega^p_{X_R/R} \to \Omega^{p-1}_{X_R/R} \otimes \mathfrak{g}_R^0 \to \ldots \to \text{Sym}^p \mathfrak{g}_R^0$$

is coherent for all $p \geq 0$. Thus $\wedge^p \mathbb{L}_{X_R/G_R} \in \text{QCoh}([X_R/G_R])$ is coherent, and, since $\pi_R$ is proper, $\pi_R^*(\wedge^p \mathbb{L}_{X_R/G_R}) \in \text{QCoh}([\text{Spec } A_R/G_R])$ is also coherent for all $p \geq 0$. In particular, the underlying $A_R$-modules

$$H^p_{X_R} := R^p\pi_{R*}(\wedge^p \mathbb{L}_{X_R/G_R})$$

of its cohomology $R^{p,\pi_R*}(\wedge^p \mathbb{L}_{X_R/G_R})$ are finitely generated over $A_R$. Since $\text{Spec } A_R$ is affine, we have

$$R\Gamma([\text{Spec } A_R/G_R], H^p_{X_R}) \cong R\Gamma(BG_R, H^p_{X_R}).$$

We have $R\Gamma([X_R/G_R], \wedge^p \mathbb{L}_{X_R/G_R}) \cong R\Gamma([\text{Spec } A_R/G_R], R\pi_{R*}(\wedge^p \mathbb{L}_{X_R/G_R}))$ and $R\pi_{R*}(\wedge^p \mathbb{L}_{X_R/G_R})$ has a finite filtration whose associated graded terms are $H^p_{X_R}$ (considered as $G_R$-equivariant $A_R$-modules). We see that it is enough to show that for any $p, q$ we have $R\Gamma(BG_R, H^p_{X_R}) \in \text{Mod}^{\text{coh}}_R$ (after enlarging $R$). Note that $R\Gamma(BG_R, H^p_{X_R}) \cong R\Gamma(BG_R, H^p_{X_R})$ where the latter denotes the rational cohomology of $G_R$.

We first show that (after enlarging $R$) $A^G_R$ is a finitely-generated module over $R$. Since $X$ is smooth we can assume $A$ is reduced. In this case, $A^G$ is a finite-dimensional reduced algebra and so $A^G \cong F \cdot e_1 + \cdots + F \cdot e_d$, $d$ copies of $F$ with $e_i$ being basis idempotent elements. Since there is a finite number of $e_i$'s, enlarging $R$ we can assume $e_i \in A_R$ for all $i$. We have $A^G_R \subset A^G \cong F^{[d]}$ and also $R^{[d]} \subset A^G$ (since we assumed $e_i \in A_R$). Thus $\bigcup_{i=1}^{d}(\text{Spec } F)_i \subset \text{Spec } A^G_R \subset \bigcup_{i=1}^{d}(\text{Spec } R_i)$. Since $A_R$ is a finitely generated algebra over $R$, as well as over $R^{[d]}$, Chevalley’s constructibility theorem tells that there exists $U \subset \bigcup_{i=1}^{d}(\text{Spec } R_i)$, such that $U \subset p_{A_R}(\text{Spec } A_R)$ where $p : \text{Spec } A_R \to R$ is the projection. Moreover $p(\text{Spec } A_R) \subset \text{Spec } A^G_R$ and $\bigcup_{i=1}^{d}(\text{Spec } F)_i \subset p(\text{Spec } A_R)$, since fibers over all these points are non empty ($e_i \in A_R \otimes R F \cdot e_i$). Thus, there is a disjoint union $\bigcup_{i=1}^{d}U_i$ of nonempty opens $U_i \subset (\text{Spec } R_i)$ that is contained in $\text{Spec } A^G_R$. Enlarging $R$ so that $\text{Spec } R \subset U_i$ for all $i$ we can assume $A^G_R \cong R^d$.

Now, we use Theorem 2.2.1. Namely, given a finitely generated commutative $R$-algebra $B$, the cohomology $H^n(G_R, B)$ are finitely generated $B^G$-modules. Taking $B$ to be the square-zero extension $A_R \oplus H^p_{X_R}$ we get that $H^n(G_R, A_R \oplus H^p_{X_R})$ is finitely generated over the $G_R$-invariants $(A_R \oplus H^p_{X_R})^{G_R}$ for any $n$. It also says that $(A_R \oplus H^p_{X_R})^{G_R}$ is a finitely generated algebra over $R$, and thus $(H^p_{X_R})^{G_R}$ is a finitely generated module over $A^G_R$. Since Spec $A^G_R \cong R^d$ we get that $(A_R \oplus H^p_{X_R})^{G_R}$ is a finite $R$-algebra. The $n$-th cohomology $H^n(G_R, A_R \oplus H^p_{X_R})$ is a finitely generated module over $(A_R \oplus H^p_{X_R})^{G_R}$ for all $n$ and thus is finitely generated over $R$ as well. Since $H^n(G_R, H^p_{X_R})$ is a direct summand, it is also finitely generated. It follows that $R\Gamma(BG_R, H^p_{X_R}) \in \text{Mod}^{\text{coh}}_R$.

Remark 2.2.14. For a general KN-complete variety we expect that one can show the Hodge-proper spreadability by using cohomology with supports at KN-strata as Teleman did in [Te100]. Theorem 57 of [FvdK10], and a mixture of arguments of Theorem 2.2.13 and Theorem 2.2.16 could allow to drop the linear reducitivity of $G$ in the reduction to the projective case for each KN-stratum. We plan to return to this question in the future.

Remark 2.2.15. Note that the statement of Theorem 2.2.13 is no longer true after either the properness or affiness conditions are discarded. For example, the base affine space $G/U$ always is quasi-affine; however, the quotient stack $[(G/U)/G]$ is isomorphic to $BU$ which is not Hodge-properly spreadable.

2.2.4 Other global quotients

In this section we will prove a version of Theorem 2.2.13 with more strict conditions on the action but allowing quotients by groups that are not necessarily reductive.

Let $G$ be any linear algebraic group and let $B \subset G$ be a Borel subgroup. Let $U \subset B$ be the unipotent radical of $B$ and let $T \subset B$ be a maximal torus. Let $X^*(T) := \text{Hom}(T, \mathbb{G}_m)$ and $X_*(T) := \text{Hom}(\mathbb{G}_m, T)$ be the character

\footnote{Recall that a subgroup $B \subset G$ is called Borel if it is a maximal Zariski-closed solvable subgroup of $G$.}
Remark 2.2.17. Similarly to $A^\mathbb{G}$ the underlying complex of $-\mathbb{G}$-weights of the Lie algebra $h$ to the $G$-action on $\text{Spec} G/U$ we denote by $\Phi^\mathbb{G}$ and $\pi$ is Hodge-properly spreadable.

Let $G$ be connected then $A_{\mathbb{G}} = G$, and we denote by $\Phi_{\mathbb{G}}$ and $\pi$ is Hodge-properly spreadable.

Then the quotient stack $[X/G]$ is Hodge-properly spreadable.

Proof. Let’s first assume that $G$ is connected. Note that $B$ is a semidirect product $T \times U$ and can be spread out to a semidirect product $T_R \times U_R$ of a split torus $T_R$ and a unipotent group $U_R$ over $R$. Since $T_R$ is split $X^\mathbb{G}(T_R) \simeq X^\mathbb{G}(T)$. The subgroup $B \subset G$ can be spread out to a closed subgroup $B_R \subset G_R$. Let $U_G$ be the unipotent radical of $G$. Then $G/U_G$ is reductive and can be spread out to a split reductive group $(G/U_G)_R$. We then also have a spreading $p_R: G_R \to (G/U_G)_R$ of the projection $p: G \to G/U_G$ and the kernel $(U_G)_R := \text{Ker}(p_G)$ is a spreading of $U_G$ and thus can be assumed to be unipotent. Since $U_G$ is a closed subgroup of $B$, we can assume that $(U_G)_R$ is a closed subgroup of $B_R$. The image of $B_R$ under $p_R$ is a spreading of $B/U_G \subseteq G/U_G$ and thus can be assumed to be a Borel subgroup of the split reductive group $(G/U_G)_R$. Note that with all these assumptions $G_R/B_R \simeq (G/U_G)_R/p_R(B_R)$.

We can also spread $X$ with an action $a: G \times X \to X$ to a smooth $X_R$ with an action $a_R: G_R \times X_R \to X_R$. Morphism $\pi : X \to Spec A$ can be spread out to a proper morphism $\pi_R: X_R \to Spec A_R$ for some finitely generated $R$-algebra $A_R$. We can also assume that $\pi_R$ is $G_R$-equivariant. The induced $T_R$-action on $A_R$ gives a $X^\mathbb{G}(T)$-grading and, as before, we can apply the cocharacter $\delta$ to $X^\mathbb{G}(T)$ to get a $Z$-grading $A_R, h=\delta$. We can assume that the $T_R$-action on Spec $A$ is conical, namely that one has $A_R, h=\delta \leq 0$. Since $U_G$ is a closed subgroup of $B$, we can assume that $(U_G)_R$ is a closed subgroup of $B_R$. The image of $B_R$ under $p_R$ is a spreading of $B/U_G \subseteq G/U_G$ and thus can be assumed to be a Borel subgroup of the split reductive group $(G/U_G)_R$. Note that with all these assumptions $G_R/B_R \simeq (G/U_G)_R/p_R(B_R)$.

We claim that $[X_R/G_R]$ is a Hodge-properly spreadable of $[X/G]$. Since $\pi_R$ is proper the $G_R$-equivariant $A_R$-modules $H^p_{[X_R/G_R]} := R^p\pi_{R!}(\mathcal{L}_{[X_R/G_R]/R})$ are finitely generated (as in the proof of Theorem 2.2.1.3). Restricting the action to $T_R$ we get a $X^\mathbb{G}(T)$-grading $H^p_{[X_R/G_R]}$ making it a $X^\mathbb{G}(T)$-graded $A_R, h=\delta$-module. As before, we can apply $h$ to the $X^\mathbb{G}(T)$-grading getting a $Z$-graded $A_R, h=\delta$-module. Since $H^p_{[X_R/G_R]}$ is finitely generated over $A_R$ it follows (see Lemma 2.2.1.8 below) that $H^p_{[X_R/G_R], h=n}$ is a finitely generated $R$-module for any $n$ and that there exists $m$ such that $H^p_{[X_R/G_R], h=m} = 0$.

Let $j_R: BB_R \to BG_R$ be the natural morphism. Then by the projection formula, for any $G_R$-representation $M$

\[ R^\Gamma(BB_R, j_R^*M) \simeq R^\Gamma(BG_R, j_* j^*M) \simeq R^\Gamma(BG_R, M \otimes j_* \mathcal{O}_{BB_B}). \]

By base change, the underlying complex of $R$-modules of $j_* \mathcal{O}_{BB_B}$ is quasi-isomorphic to $R^\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R})$. But $G_R/B_R \simeq (G/U_G)_R/p_R(B_R)$, where $p_R(B_R)$ is a Borel subgroup and thus $R^\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R}) \simeq R$ by the theorem of Kempf (see Theorem 2.2.2). Consequently, $R^\Gamma(BB_R, j_R^*M) \simeq R^\Gamma(BG_R, M)$. In particular we get $R^\Gamma(BG_R, H^p_{[X_R/G_R]}) \simeq R^\Gamma(BB_R, H^p_{[X_R/G_R]})$ for all $p, q$.

We discarded $j^*$ from the notation because it just means restriction in terms of the corresponding representations of groups.
To compute $R\Gamma(BB_R,M)$ for a given $B_R$-representation $M$ we can use the Hochschild-Serre spectral sequence. Namely, $R\Gamma(BB_R,M) \simeq R\Gamma(BU_R,M)^{Tr}$ (note that a functor of $T_R$-invariants is $t$-exact) where $U_R$ is the unipotent radical of $B_R$. The complex $R\Gamma(BU_R,M)$ with the $T_R$-action can be computed by the explicit Hochschild-type complex

$$M \to M \otimes_R \mathcal{O}(U_R) \to M \otimes_R \mathcal{O}(U_R) \otimes_R \mathcal{O}(U_R) \to M \otimes_R \mathcal{O}(U_R) \otimes_R \mathcal{O}(U_R) \to \ldots$$

where the action of $T_R$ is given termwise by the tensor product of the action on $M$ and the action on $\mathcal{O}(U_R)$ induced by the adjoint action of $T_R$ on $U_R$. The underlying scheme of $U_R$ can be $T$-equivariantly identified with its Lie algebra $u_R$ (see II.1.7 in [Jan07]), so $\mathcal{O}(U_R) \simeq \text{Sym} u_R^*$ as $T_R$-representations. Since the weights of $u_R^*$ are exactly $\Phi^+$, the weights of $\mathcal{O}(U_R)$ are contained in $\mathbb{N} \cdot \Phi^+$ with the $T_R$-invariants given by $R \cdot 1 \subset \mathcal{O}(U_R)$. Since $h(\Phi^+) > 0$ it follows that the tensor products $\mathcal{O}(U_R) \otimes_R \ldots \otimes_R \mathcal{O}(U_R)$ (endowed with the grading $h = \bullet$) satisfies the conditions of Lemma 2.2.18. Thus, for any $n$ the $R$-module $(\mathcal{O}(U_R) \otimes_R \ldots \otimes_R \mathcal{O}(U_R))^{Tr}$ is finitely generated. Since taking the $T_R$-invariants is a $t$-exact functor, $H^n(BB_R,M) \simeq H^n(BU_R,M)^{Tr}$ is a subquotient of this module, so it is also finitely generated. We get that $R\Gamma(BB_R,M) \in \text{Mod}_{\mathcal{R}}^c$ is almost coherent. Substituting $M = H^{p,q}_{[X_R/G_R]}$ finishes the proof in the case $G$ is connected.

It remains to reduce to this case. We can write $[X/G] \simeq [[X/G^0]/\pi_0(G)]$ where $G^0$ is the connected component of $e \in G$ and $\pi_0(G)$ is the finite group of components. The homomorphism $p: G \to \pi_0(G) \simeq G/G^0$ can be spread out to $p_R: G_R \to \pi_0(G_R)$ where $G_R$ is some spreading out of $G$ and $\pi_0(G_R)$ is the constant group $R$-scheme associated to $\pi_0(G)$. Moreover the kernel $G^0_R$ of $p_R$ is a spreading of $G^0$. We have just shown that (after possibly enlarging $R$) $[X_R/G^0_R]$ is Hodge-proper over $R$. We also have $[X_R/G_R] \simeq [[X_R/G^0_R]/\pi_0(G_R)]$. The natural projection $q : [X_R/G^0_R] \to [X_R/G_R]$ is étale and $q^*\mathcal{L}_{[X_R/G_R]} \simeq \mathcal{L}_{[X_R/G^0_R]}$. From the projection formula it follows that

$$R\Gamma([X_R/G_R], \wedge^p\mathcal{L}_{[X_R/G_R]}) \simeq R\Gamma([X_R/G^0_R], \wedge^p\mathcal{L}_{[X_R/G^0_R]} \otimes \pi_0(G_R))$$

for all $p$. Let’s replace $R$ with $R[1/\pi_0(G)]$ so that $\pi_0(G)$ is invertible in $R$ so that the functor of $\pi_0(G)$-invariants is $t$-exact. Then we get $H^p([X_R/G^0_R], \mathcal{L}_{[X_R/G^0_R]} \otimes \pi_0(G_R)) \simeq H^p([X_R/G^0_R], \wedge^p\mathcal{L}_{[X_R/G^0_R]} \otimes (\pi_0(G_R)))$. So $H^p([X_R/G^0_R], \mathcal{L}_{[X_R/G^0_R]})$ is finitely generated over $R$ for all $p$ and $q$ and we are done. \qed

Along the way we have used the following simple fact from commutative algebra:

**Lemma 2.2.18.** Let $R$ be a Noetherian ring and let $A_\bullet$ be a positively $\mathbb{Z}$-graded finitely generated $R$-algebra, $A_\bullet \simeq A_{\geq 0}$ with $A_0 \simeq R^s$ for some $s$. Let $M$ be a finitely generated graded module over $A$. Then each graded component $A_n$ and $M_n$ are finitely generated over $R$. Also there exists $m$ such that $M_{<m} = 0$.

We end this subsection by giving some examples to which Theorem 2.2.16 does apply:

**Example 2.2.19.** 1. $X$ is proper. In this case $A \simeq F^0[\pi_0(X)]$ and the action of $T$ on $A$ is trivial. The only condition to check is on $G$: namely there should exist $h \in X_*(T)$ such that $h(\Phi^+) > 0$ (since all Borel subgroups of $G$ are conjugate to each other this does not depend on the choice of $B$). Here is the list of linear algebraic groups $G$ which satisfy this:

- $G$ reductive. Then one can take $h \in X_*(T)$ given by any dominant coweight. This case is also covered by Theorem 2.2.13;
- $G = P \subset H$ is a parabolic subgroup of a reductive group $H$. The same $h$ as above applies;
- More or less tautologically any $G$ with a 1-dimensional subtorus $\mathbb{G}_m \subset G$ such that the adjoint action of $\mathbb{G}_m$ on the dual $u_\mathbb{Z}$ to the Lie algebra of the unipotent radical $U_G \subset G$ has strictly positive weights and such that the projection of $\mathbb{G}_m$ to $G/U_G$ gives a regular coweight (meaning its centralizer is given by a maximal torus). Then one picks $B$ as the preimage of a Borel subgroup of $G/U_G$, with respect to which the $\mathbb{G}_m$-action gives a dominant coweight, under the projection $G \to G/U_G$ and take $h$ given by any lifting $\mathbb{G}_m \to B$. As a non-parabolic example of such $G$ one can take any semidirect product $\mathbb{G}_m \ltimes U$ where $U$ is unipotent and $\mathbb{G}_m$ acts on $u^\mathbb{C}$ with strictly positive weights.

2. $G = \mathbb{G}_m$. In this case we essentially get the definition of a conical resolution (for more details see e.g. [KT16]), except for the connectedness assumption on $X$. Indeed, $\Phi^+ = \{ \emptyset \}$, so there is no condition on $G$; $A$ is a finitely generated $\mathbb{Z}$-graded ring (note that in [KT16] $A$ is assumed to be $H^0(X, \mathcal{O}_X)$, $\pi : X \to \text{Spec} A$ is proper, $X$ is smooth and that the $\mathbb{G}_m$-action on $X$ agrees with the grading on $A$. Finally, there should exist $h \in X_*(\mathbb{G}_m) \simeq \mathbb{Z} \text{id}_{\mathbb{G}_m}$ such that $A_{h<0} = 0$ and $A_{h=0}$ is finite-dimensional. There are three options for $h$:
• $h = 0$. This is not the best choice, since then one should have $\dim A < \infty$ and consequently $X$ should be proper;  
• $h > 0$. Then $A_{<0} = 0$ (w.r.t to the $X^*(\mathbb{G}_m)$-grading, where $X^*(\mathbb{G}_m) \simeq \mathbb{Z} \cdot \text{id}_{\mathbb{G}_m}$) and $\dim A_0 < \infty$. By Remark 2.2.17 we have a decomposition $\text{Spec } A \simeq \text{Spec } (A \cdot e_1) \sqcup \cdots \sqcup \text{Spec } (A \cdot e_n)$. For all $i$, the algebra $A \cdot e_i$ is $\mathbb{N}$-graded and finitely generated with $(A \cdot e_i)_0 \simeq F$, so each $\text{Spec } (A \cdot e_i)$ is an affine cone over the projective variety $\text{Proj } (A \cdot e_i)$ with the vertex $v_i$ given by the ideal $(A \cdot e_i)_{\geq 0} \subset A \cdot e_i$. In particular the $\mathbb{G}_m$-action contracts each $\text{Spec } (A \cdot e_i)$ to its vertex $v_i$ as $t \to 0$. So $\text{Spec } A$ is a disjoint union of affine cones which are contracted to a finite set of points by the $\mathbb{G}_m$-action. The geometry of $X$ is the following: it is not proper itself, but it has a proper $\mathbb{G}_m$-equivariant map to $\text{Spec } A$ so that the $\mathbb{G}_m$-action contracts it to the disjoint union of $X_i := \pi^{-1}(v_i)$ which are proper over $F$. Note that even if $X$ is smooth, $X_i$ can be singular (for example take $X$ to be the minimal resolution of the $A_n$-singularity):  
• $h < 0$. In this case we have $A_{>0} = 0$ and $\dim A_0 < \infty$. The geometry is the same as above, but now $X$ is contracted to a proper scheme when $t \to \infty$. This case is equivalent to the previous one by twisting the action by the inverse map $t \to t^{-1}$ on $\mathbb{G}_m$.

3. $G$ reductive. In this case Theorem 2.2.13 is strictly stronger then Theorem 2.2.16.

4. $G$ general. In this case $h \in X_*(T)$ gives a map $h : \mathbb{G}_m \to G$ and the conditions of Theorem 2.2.16 imply that $X$ with the induced $\mathbb{G}_m$-action is a conical resolution (in the above sense). However there still remains an action of the whole group $G$. By case 2, $[X/\mathbb{G}_m]$ is already Hodge-properly spreadable, but Theorem 2.2.16 says that if, moreover, the subgroup given by $h$ acts with strictly positive weights on the unipotent radical of some Borel subgroup of $G$ then $[X/G]$ is Hodge-properly spreadable as well.

### 2.3 Equivariant Hodge degeneration

Let $X$ be a homotopy type with an action of a topological group $H$ (i.e. an $(\infty, 1)$-functor $X_\bullet : BH \to \mathcal{S}$). Recall that the $H$-equivariant cohomology $C^*_H(X, \Lambda)$ of $X$ with coefficients in a ring $\Lambda$ are defined as

$$C^*_H(X, \Lambda) := C^*(X_{hH}, \Lambda),$$

where $X_{hH}$ is the homotopy quotient of $X$ by $H$ (i.e. a colimit of the corresponding functor $X_\bullet$, or, more classically, $(X \times EH)/H$). Now, if $X$ is a smooth algebraic variety over a field $F \subseteq \mathbb{C}$ equipped with an action of a smooth algebraic group $G$, one has a de Rham model of $G(\mathbb{C})$-equivariant cohomology of $X(\mathbb{C})$:  

**Proposition 2.3.1.** Let $X$ and $G$ be as above. Then there is a canonical equivalence

$$C^*_G(X(\mathbb{C}), \mathbb{C}) \simeq R_{dR}(\{ X/G \}/F) \otimes_F \mathbb{C}.$$  

**Proof.** By definition we have

$$\begin{align*}
\cdots \longrightarrow G \times G \times X \longrightarrow G \times X \longrightarrow X \longrightarrow \cdots \\
\cdots \longrightarrow G(\mathbb{C}) \times G(\mathbb{C}) \times X(\mathbb{C}) \longrightarrow G(\mathbb{C}) \times X(\mathbb{C}) \longrightarrow X(\mathbb{C}) \longrightarrow \cdots
\end{align*}$$

Since the functor of cochains $C^*(-, \mathbb{C})$ sends colimits of homotopy types to limits of complexes and by smooth descent for $R_{dR}(-/F) \otimes_F \mathbb{C}$, the result follows from the analogous comparison between algebraic de Rham and Betti cohomology for ordinary smooth schemes $X \times G^n$.  

**Corollary 2.3.2** (Equivariant Hodge degeneration). Let $X$ be a scheme over $\mathbb{C}$ with an action of a group scheme $G$. Assume that $[X/G]$ is spreadable and Hodge-proper (e.g. $X$ and $G$ satisfy the conditions of Theorem 2.2.13 or Theorem 2.2.16). Then for all $n \in \mathbb{Z}_{\geq 0}$ there is an isomorphism

$$H^p_{G(\mathbb{C})}(X(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q([X/G], \wedge^p \mathcal{L}_{|X/\mathbb{C}}).$$

**Example 2.3.3.** Let $X = \text{Spec } \mathbb{C}$. Then $\wedge^n \mathcal{L}_{BG} \simeq \text{Sym}^n(\mathfrak{g}^\vee)[-n]$ and we get a standard isomorphism

$$H^n_{G(\mathbb{C})}(\text{pt}, \mathbb{C}) \simeq \left\{
\begin{array}{ll}
\text{Sym}^k(\mathfrak{g}^\vee)^G & \text{if } n = 2k, \\
0 & \text{if } n = 2k + 1.
\end{array}
\right.$$  

In particular,

$$H^*_G(\text{pt}, \mathbb{C}) \simeq \text{Sym}(\mathfrak{g}^\vee)^G, \quad \text{where } \deg(\mathfrak{g}^\vee) = 2.$$
Example 2.3.4. As another example one can take a conical resolution $\pi : X \to \text{Spec } A$ (see the $\mathbb{G}_m$-case of 2.2.19). By Theorem 2.2.16 $[X/\mathbb{G}_m]$ is Hodge-properly spreadable and we get a decomposition for $H^*_c(X(\mathbb{C}), \mathbb{C})$ as in Corollary 2.3.2. In particular, the first equivariant cohomology is isomorphic to the non-equivariant first cohomology (since $\tau_{\leq 1} BC^\times \simeq \text{pt}$) and we get a decomposition

$$H^1(X(\mathbb{C}), \mathbb{C}) \simeq H^0(X/\mathbb{G}_m, L_{X/\mathbb{G}_m}) \oplus H^1(X/\mathbb{G}_m, \mathcal{O}_{X/\mathbb{G}_m}).$$

We have $L_{X/\mathbb{G}_m} \simeq \Omega^1_X \xrightarrow{a^\ast} \mathcal{O}_X$ as a complex of $\mathbb{G}_m$-equivariant sheaves on $X$, where $a^\ast$ is the map dual to the derivative of the action $\text{Lie}(\mathbb{G}_m) \otimes_{\mathbb{C}} \mathcal{O}_X \to T_X$ (where $T_X$ denotes the tangent bundle). Then $H^0(X, \Omega^1_X \xrightarrow{a^\ast} \mathcal{O}_X) \simeq \ker \left( H^0(X, \Omega^1_X) \xrightarrow{a^\ast} H^0(X, \mathcal{O}_X) \right)$, which is identified with the invariants of the Lie algebra action, which also identifies with the group invariants $H^0(X, \Omega^1_X)^{\mathbb{G}_m}$. Finally we get

$$H^0(X/\mathbb{G}_m, L_{X/\mathbb{G}_m}) \simeq H^0(X, \Omega^1_X \xrightarrow{a^\ast} \mathcal{O}_X)^{\mathbb{G}_m} \cong H^0(X, \Omega^1_X)^{\mathbb{G}_m}$$

as well. The second summand is just $H^1(X, \mathcal{O}_X)^{\mathbb{G}_m}$. Thus for any conical resolution we get a formula

$$H^1(X(\mathbb{C}), \mathbb{C}) \simeq H^0(X, \Omega^1_X)^{\mathbb{G}_m} \oplus H^1(X, \mathcal{O}_X)^{\mathbb{G}_m}.$$ 

This is a partial generalization of results of Section 6 in [KT16] to the case when $R^1 \pi_* \mathcal{O}_X$ is not necessarily 0.

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Dmitry Kubrak, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, United States dmkubrak@mit.edu

Artem Prikhodko, Department of Mathematics, National Research University Higher School of Economics, Moscow; Center for Advanced Studies, Skoltech, Moscow, artem.n.prihodko@gmail.com