From $sl(2)$ Kirby weight systems
to the asymptotic 3-manifold invariant

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Abstract

We give a construction of Kirby weight systems associated to $sl(2)$ and valued into the finite field $\mathbb{Z}/p\mathbb{Z}$. We show that it is possible to apply this sequence of weight systems on the universal invariant of framed link. We also show that the corresponding sequence admits a Fermat limit, which defines an asymptotic rational homology 3-sphere quantum invariant. Moreover, this asymptotic invariant coincides with the Ohtsuki invariant.

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1 Introduction

The universal invariant for oriented framed link is now understood to be the most fundamental object in the study of quantum invariant of knots and links. One of the main features of this invariant is that it contains, at least, the same information as all the Witten-Reshetikhin-Turaev quantum invariants associated with semi-simple quantum groups [22]. There exist by now three constructive definition of this invariant: The first one was given by Kontsevich using the Khniznik-Zamolodchikov equation [11], the second was given by Piumikine [21] and Cartier [7] using a combinatorial definition inspired by the quasi-Hopf algebras construction of Drinfeld [8] and the third was given by the perturbative expansion of Chern-Simons theory ([1, 4]). The first two have been proved to be equal [1, 12] and are called canonical, but to my knowledge it is still a open question to show that the third one is also canonical.

Using the surgery presentation of a 3-dimensional manifold a framed link invariant can be promoted to a 3-d manifold invariant if it proves to be invariant under Kirby moves. Using the representation theory of quantum groups at roots of unity, Reshetikhin and Turaev constructed a quantum invariant of 3-manifolds as a weighted sum of framed colored Jones polynomials (in the case of SU(2)) over finite dimensional irreducible highest weight representations [23]. This formula was conjectured by Witten to be the partition function of Chern-Simons theory [26]. This construction uses the fact that representation theory of quantum groups at roots of unity is truncated, so the weighted sum involves only a finite number of terms. The restriction to the case of roots of unity appears as a peculiar regularisation of Chern-Simons theory. But it is also known from general grounds that behind every regularisation of a sensible quantum field theory there exists a renormalized theory which is regularisation independent. In this context this means that it should be possible to define an asymptotic partition function for Chern-Simons as a formal power series in $\hbar$.

The quest for such a theory has been concentrated on the value of the Chern-Simons theory for homology sphere expanded around the trivial connection, which is an isolated critical point of the Chern-Simons action. The oldest technique at disposal is the usual perturbative definition of quantum field theory [3] but it was only very recently that there appeared the possibility of defining Chern-Simons theory as a perturbative expansion around the trivial connection [8]. Roszansky developed techniques in order to study the asymptotic behaviour of W-R-T invariants in the limit where $k$ (the order of the root of unity) goes to infinity. In this context Ohtsuki [18] showed that the $su(2)$ W-R-T for integral and rational homology spheres admits a ”Fermat limit” (we use the terminology of [17]) and thus gives rise to $su(2)$ homology sphere invariants for generic $q$. The breakthrough in this quest was achieved in three seminal papers: [13, 14, 15], where T.Q. Le, H. Murakami, J. Murakami, T. Ohtsuki,
first showed that it was possible to construct a 3-manifold invariant from the universal framed link invariant if one could find a weight system satisfying the so-called Kirby relations (we call such a weight system a Kirby weight system). Then they constructed a universal Kirby weight system which maps the space of chord diagrams to the space of trivalent graphs satisfying Jacobi (or so-called I-H-X) relations. Moreover, T.Q. Le showed that the resulting object obtained by applying this Kirby weight system is the universal invariant of rational homology spheres, i.e., the graded isomorphisms between Jacobi trivalent graphs and finite type homology spheres invariants. This result is the 3-manifold analogue of the celebrated theorem of Kontsevich for knots. One can think that this is the end of the story. But a lot of questions were remaining like the correspondence between this universal invariants and the numerical ones obtained from the W-R-T or Ohtsuki constructions. Of course, Ohtsuki proved that by applying the su(2) trivalent graphs weight system on the universal Homology sphere invariant one obtains the Ohtsuki invariant, see also theorem 8.5 of [19].

Our aim in this paper is to give a completely different construction of a Kirby weight system in the case of su(2) and to evaluate the corresponding 3-manifold invariant. Our construction is very reminiscent of the Reshetikhin Turaev construction in the sense that the su(2) Kirby weight system on chord diagrams we propose is obtained as a weighted sum of weight systems over irreducible representations. In order to give a meaning to this sum, we were forced to work in non-zero characteristic fields. This restriction appears analogous to roots of unity regularisation at the level of weight systems. Using results of Le on integrality properties of the Universal framed link invariants we apply this weight system on this invariant. We show that the corresponding object admits a "Fermat limit" which is the Ohtsuki invariant given in terms of Gaussian integrals. This fact appears as a specific realization in the case of su(2) of the conjectural proposal given in [5]. Moreover this limit is the same for su(2) and so(3) invariants. The first part of this paper was presented in Knots 96 Tokyo conference. The study of the general Lie algebra case is under construction in collaboration with D. Altschuler.

In section 2 we present the construction of skein modules in non-zero characteristic fields. The material of this part is largely borrowed from [23] (see also [10]), but I couldn’t find in the literature any references concerning the finite field characteristic case (even if it is very similar to the root of unity case). I also present in this section some recoupling theory theorem that I uses in the following. The third section is devoted to the construction of weight system from skein modules, the link between zero and non-zero characteristic field weight system and the link with the usual definition of su(2) weight system. The section 4
presents the construction of a $so(3)$ and $su(2)$ weight system, the proof that they satisfy the Kirby relations and the study of some of the important properties that these weight systems satisfy. In the section 5 we take advantage of these properties in order to apply the Kirby weight systems the Universal Framed link invariant. We then show that the resulting object admits a Fermat limit expressed in terms of Gaussian integrals. This fermat limit is the same for the $su(2)$ and $so(3)$ case and moreover they coincide with Ohtsuki construction.

2 Skein Module

Preliminaries
In this article $p$ will denote a prime odd integer. $\mathbb{Z}/p\mathbb{Z}$ is identified with the set $\{0, \ldots, p-1\}$ and the set of invertible element of $\mathbb{Z}/p\mathbb{Z}$ with $\{1, \ldots, p-1\}$. If $k \in \mathbb{Z}$ we denote by $\psi_p(k)$ its image in $\mathbb{Z}/p\mathbb{Z}$; moreover, if $k$ is not divisible by $p$ then $\psi_p(k)$ is invertible in $\mathbb{Z}/p\mathbb{Z}$ and we denotes its inverse by $\psi_p(\frac{1}{k})$. Letting $\mathbb{Q}_p = \{ \frac{a}{b} \in \mathbb{Q}, \gcd(a, b) = 1, \gcd(p, b) = 1 \}$, we define the homomorphism $\psi_p : \mathbb{Q}_p \to \mathbb{Z}/p\mathbb{Z}$ by $\psi_p(\frac{a}{b}) = \psi_p(a)\psi_p(\frac{1}{b})$. If $x, y \in \mathbb{Q}_p$, $x \equiv y \pmod{p}$ means $\psi_p(x) = \psi_p(y)$.

A $(k,l)$ Tangle $T$ is an immersion of a one-dimensional compact submanifold, considered modulo ambient isotopy of $\mathbb{R} \times [0,1]$, into $\mathbb{R} \times [0,1]$, such that $\partial T = T \cap [\mathbb{R} \times \{0\} \cup \{1\}]$, and $\text{Card}(T \cap [\mathbb{R} \times \{0\}]) = k$, $\text{Card}(T \cap [\mathbb{R} \times \{1\}]) = l$. Denote $\mathcal{T}(k,l)$ the $\mathbb{Z}$-linear span of all $(k,l)$ tangles.

Definition 1 We denote by $S_p(k,l)$ the $\mathbb{Z}/p\mathbb{Z}$-module generated by $(k,l)$ tangles with the following relations :

\begin{equation}
T \cup O = -2T \quad \text{where } O \text{ denote the circle and } T \text{ is an arbitrary tangle,}
\end{equation}

\begin{equation}
+ + = 0
\end{equation}

Figure 1:

\begin{equation}
\left(\begin{array}{c}
\cdot \\
\end{array}\right) + \left(\begin{array}{c}
\cdot \\
\end{array}\right) = 0
\end{equation}

Lets us call simple the tangle diagrams without any crossing and without any circle; using the skein relation, we have the following lemma :
Lemma 1 $S_p(k,l)$ is a free $K_p$ module with basis given by simple $(k,l)$ diagrams. In $S_p(k,l)$ the relations shown in figures 2, 3, 4 are satisfied:

Figure 2:

\[
\begin{align*}
\includegraphics[width=0.5\textwidth]{figure2}
\end{align*}
\]

Figure 3:

\[
\begin{align*}
\includegraphics[width=0.5\textwidth]{figure3}
\end{align*}
\]

Figure 4:

\[
\begin{align*}
\includegraphics[width=0.5\textwidth]{figure4}
\end{align*}
\]

Using the skein property any tangle can be decomposed uniquely into a sum of diagrams without any crossings. Using (2.1), this sum can be expanded into a sum of simple diagrams. Thus we get the first property of the lemma. The second part of the lemma is obtained by direct computations. In particular, the basis of $S_p(0,0)$ is given by the empty tangle, so this gives an identification of $S_p(0,0)$ with $\mathbb{Z}/p\mathbb{Z}$. If $x \in S_p(k,n)$ and $y \in S_p(l,k)$ we can define the composition $x \cdot y \in S_p(l,n)$ by stacking elements (see fig 5) and the tensor product $x \otimes y$ by juxtaposing elements (see fig 6). We also define a trace $tr$ to be the closure of $f$ as in figure 7. In $S_p(k,l)$ we consider the submodule of null elements $N_p(k,l)$ defined as follows:

\[
N_p(k,l) = \left\{ n \in S_p(k,l) | \forall f \in S_p(l,k), tr(fn) \equiv 0 \right\}
\]

(2.3)

Let $\sigma \in \Sigma(n)$ be an element of the symmetric group on $n$ elements, and denote by $T(\sigma)$ the associated tangle in $S_p(n,n)$. If $n \in \{0,1,\ldots,p-1\}$, then we define an element $f_n$ of $S_p(n,n)$ as:

\[
f_n = \psi_p(\frac{1}{n!}) \sum_{\sigma \in \Sigma(n)} (-1)^{\ell(\sigma)} T(\sigma).
\]

(2.4)
Figure 5:

\[
\begin{array}{c}
\text{f} \circ \text{g} = \\
\end{array}
\]

Figure 6:

\[
\begin{array}{c}
\text{f} \otimes \text{g} = \\
\end{array}
\]

Figure 7:

\[
\begin{array}{c}
\text{tr} \text{f} = \\
\end{array}
\]
In particular $f_0$ is the empty tangle. This element satisfies the following properties:

\[(f_k \otimes I_{n-k}) \circ f_n = f_n \circ (I_{n-k} \otimes f_k) = f_n,\]  

(2.5)

\[T(\sigma) \circ f_n = f_n \circ T(\sigma) = (-1)^{|\sigma|} f_n, \; \sigma \in \Sigma_n,\]  

(2.6)

\[g \circ f_n = 0 \text{ (resp } f_n \circ g = 0) \text{ if } g \text{ possess an arc connecting } \mathbb{R} \times 0 \text{ (resp } \mathbb{R} \times 1) \text{ with itself,}\]  

(2.7)

relation in fig 8 if $n \leq p - 2$,

\[trf_n = (-1)^n(n + 1).\]  

(2.8)

\[g \circ f_n = 0 \text{ (resp } f_n \circ g = 0) \text{ if } g \text{ possess an arc connecting } \mathbb{R} \times 0 \text{ (resp } \mathbb{R} \times 1) \text{ with itself,}\]  

(2.9)

\[\psi_p(1) = (n + 1) f_n \otimes I - \psi_p(\frac{n}{n+1}) T(t_{n,n+1}) f_n \otimes I \]  

(2.10)

Figure 8:

The first two properties are clear from the definition of $f_n$; for the third property we can suppose, using the behavior of $f_n$ under permutations, that $g$ connects the strands $i$ and $i + 1$, in which case proof is given in figure 9, while the first equality is obtained from 2.6 and the second using fig 4. From the definition of $f_n$ it is clear that, for $n \leq p - 2$

\[f_{n+1} = \psi_p(1) f_n \otimes I - \psi_p(\frac{n}{n+1}) T(t_{n,n+1}) f_n \otimes I \]  

(2.10)

where $t_{n,n+1}$ is the permutation of $n$ with $n + 1$. Then, using the skein relation and property 2.5 the conclusion of 2.8 is direct and 2.9 is shown by recurrence using 2.8. In particular the relation 2.9 implies that $f_{p-1} \in \mathcal{N}_p(n, n)$.

\[f = 0 \]  

Figure 9:

The following lemma is satisfied:
Lemma 2 Let $D_p(k,l) \subset S_p(k,l)$ be the set of elements of the form

$$f = n + \sum_s x_s f_i y_s, \quad (2.11)$$

where $s$ runs over a finite set of indices, $i_s \in \{0, 1, \cdots, p-2\}$, $n \in N_p(k,l)$. And $x_s$ (resp. $y_s$) is an element of $S_p(i_s,l)$ (resp. $S_p(k,i_s)$). Then $S_p(k,l) = D_p(k,l)$.

Proof

First, note that if $x, y \in D_p(k,l)$ then $x + y \in D_p(k,l)$ and if $x \in D_p(k,l) y \in S_p(l,n)$ (resp. $y \in S_p(n,k)$) then $y \cdot x \in D_p(k,l)$ (resp. $x \cdot y \in D_p(n,l)$). So it is enough to show this property for the identity tangle $I_n \in S_p(n,n)$. We will show it by recurrence : $I_0 = f_0, I_1 = f_1$, suppose that $I_n \in D_p(n,n)$, i-e $I_n = n + \sum_s x_s f_i y_s$ where $n \in N_p(n,n)$ and $x_s$ (resp. $y_s$) is an element of $S_p(i_s,n)$ (resp. $S_p(n,i_s)$). Then $I_{n+1} = I_n \otimes I_1 = n \otimes I_1 + \sum_s (x_s \otimes I_1) (f_i \otimes I_1) (y_s \otimes I_1)$. $n \otimes I_1 \in n \in N_p(n+1,n+1)$, moreover by relation (2.8) $f_i \otimes I_1 = f_{i+1} + x_{i+1} f_{i+1} y_{i+1}$ with $x_{i+1} \in S_p(i_s-1,i_s+1)$, $y_{i+1} \in S_p(i_s+1,i_s-1)$. Thus $f_i \otimes I_1 \in D_p(i+1,i+1)$ $(f_{p-1} \in N_p(p-1,p-1))$, thus $(x_s \otimes I_1) (f_i \otimes I_1) (y_s \otimes I_1) \in D_p(n+1,n+1)$ and so is $I_{n+1}$.

Define $I_p = \{0, 1, \cdots, p-2\}$ and $J = \{\vec{k} = (k_1, \cdots, k_n) n \in \mathbb{Z}, k_i \in I_p\}$. We denote $|\vec{k}| = k_1 + \cdots + k_n$ and $f_{\vec{k}} \in S_p(|\vec{k}|,|\vec{k}|)$ the element $f_{k_1} \otimes \cdots \otimes f_{k_n}$. And we define the invariant skein module $\mathcal{I}(\vec{k}, \vec{l}) \subset S_p(|\vec{k}|,|\vec{l}|)$ as follows:

$$\mathcal{I}(\vec{k}, \vec{l}) = \{x \in S_p(|\vec{k}|,|\vec{l}|) | f_{\vec{k}} x f_{\vec{k}} = x\}.$$  \hspace{1cm} (2.12)

We say that the triple $(i, j, k) \in I_p^3$ is admissible if and only if $i + j - k \geq 0$, $i - j + k \geq 0$, $-i + j + k \geq 0$ and $i + j + k$ is even. We say that the triple $(i, j, k)$ is $p$-admissible if and only if it is admissible and $i + j + k \leq 2(p - 2)$.

Lemma 3 The space $\mathcal{I}((i),(j,k))$, $(i, j, k) \in I_p^3$ is a zero dimensional space if $(i, j, k)$ is not admissible, is a one dimensional module if $(i, j, k)$ is admissible and in this case it is a null submodule $(i \cdot e \subset N_p(i, j + k))$ if $i + j + k \geq 2p - 2$.

Proof Let $T$ be a simple tangle in $S(i, j + k)$, and denote by $IT = f_j \otimes f_k T f_i$ the associated invariant tangle. If $(i, j, k)$ is not admissible then $T$ possesses an arc connecting $i$ (resp. $j$, $k$) with itself. Thus the property (2.7) implies that $IT = 0$, so $\mathcal{I}((i),(j,k)) = 0$. If $(i, j, k)$ is admissible there is only one simple diagram $S^{(j,k)}_i$ in $S(i, j + k)$ which do not possess an arc.
connecting \(i\) (resp. \(j, k\)) with itself. \(S^{(j,k)}_{(i)}\) is the simple diagram with \(a = (-i + j + k)/2\) arcs connecting \(j\) with \(k\), \(b = (i - j + k)/2\) arcs connecting \(i\) with \(k\), \(c = (i + j - k)/2\) arcs connecting \(i\) with \(j\). Let \(Y^{(j,k)}_{(i)} = f_j \otimes f_k S^{(j,k)}_{i} f_i\) Then \(Y^{(j,k)}_{i}\) is a base element of \(\mathcal{I}(i, (j,k))\) (i.e \(\mathcal{I}(i, (j,k)) = K_p Y^{(j,k)}_{i}\)). Moreover, if \((i,j,k) \in I^3_p\), then
\[
tr(Y^{(i)}_{(j,k)} Y^{(j,k)}_{(i)}) = \psi_p(\Theta(i,j,k)),
\]
where
\[
\Theta(i,j,k) = (-1)^{\frac{i+j+k}{2}} \binom{i+j+k+1}{i,j,k} \binom{i+j-k}{i,j,k} \binom{i+j-k}{i,j,k}.
\]
This property is a consequence of \([3,1]\). Thus \(\psi_p(\Theta(i,j,k)) = 0\) if and only if \(i+j+k \geq 2(p-1)\), so in that case \(Y^{(j,k)}_{(i)} \in N_p(i, j + k)\) (resp. \(Y^{(i)}_{(j,k)} \in N_p(i, j + k)\)). Moreover we have the following identity :
\[
Y^{(l)}_{(j,k)} Y^{(j,k)}_{(i)} = \psi_p(\frac{\Theta(i,j,k)}{\Delta_i}) \delta_{i,l} f_i,
\]
if \((i,j,k) \in I^3_p\). Here \(\delta_{i,l}\) denotes the Kronecker symbol and \(\Delta_i = tr f_i = (-1)^i(i + 1)\). Since \(Y^{(l)}_{(j,k)} Y^{(j,k)}_{(i)} \in \mathcal{I}(i, l)\), by the preceding lemma it is equal to 0 if \(i \neq l\) and proportional to \(f_i\) if \(i = l\). The coefficient of proportionality is calculated by taking the trace.

\[\blacksquare\]

**Proposition 1** Let \(f \in \mathcal{I}_p((k_1, k_2), (l_1, l_2))\), \(k_i, l_i \in I_p\), then
\[
f = n + \sum_i \alpha_i Y^{(l_1,l_2)}_{i(k_1,k_2)},
\]
where \(n \in N_p(k_1 + k_2, l_1 + l_2)\), the sum is over all \(i\) such that \((i,l_1,l_2)\) and \((i,k_1,k_2)\) are admissible triples and
\[
\alpha_i = tr(Y^{(l_1,l_2)}_{i} f Y^{(k_1,k_2)}_{i}) \frac{\Delta_i}{\Theta(i,k_1,k_2) \Theta(i,l_1,l_2)}.
\]

**Proof**

Let \(f \in \mathcal{I}((k_1, k_2), (l_1, l_2))\); using theorem 1 we know that \(f = n + \sum_s x_s f_{i_s} y_s\) where \(s\) runs over a finite set of indices, \(i_s \in I_p\), \(n \in N_p\). And \(x_s\) (resp. \(y_s\)) is an element of \(S_p(i_s, l_1 + l_2)\) (resp. \(S_p(k_1 + k_2, i_s)\)). By composing on the left by \(f_{l_1} \otimes f_{l_2}\), on the right by \(f_{k_1} \otimes f_{k_2}\) we can assume that \(x_s \in \mathcal{I}((l_1, l_2), i_s)\), \(y_s \in \mathcal{I}(i_s, (k_1, k_2))\). So, by lemma \([3]\), \(x_s = \lambda_s Y^{(l_1,l_2)}_{i_s}\) if \((i_s, l_1, l_2)\) is \(p\)-admissible or \(x_s \in \mathcal{N}(i_s, l_1 + l_2)\) (resp. \(y_s = \mu_s Y^{(k_1,k_2)}_{(i_s,l_2)}\) if \((i_s, k_1, k_2)\) is \(p\)-admissible or \(y_s \in \mathcal{N}(l_1 + l_2, i_s)\))
\[
f = n + \sum_i \alpha_i Y^{(l_1,l_2)}_{i} Y^{(k_1,k_2)}_{i}.
\]
The sum is over all $i$ such that $(i, l_1, l_2)$ and $(i, k_1, k_2)$ are $p$-admissible triples and $\alpha_i \in K_p$. Moreover, multiplying both side by $Y_{i(l_1, l_2)}$ on the left and $Y_{i(k_1, k_2)}$ on the right and using \[2.13\], we get the desired result for $\alpha_i$.

\[\square\]

In particular,

\[f_i \otimes f_j = n + \sum_{(i, j, k)_{p-admissible}} \psi_p(\frac{\Delta_k}{\Theta(i, j, k)}) Y_{i(i, j)}^k Y_{k(i, j)}^k.\] (2.19)

### 3 Spin Networks, Chord diagrams and weight system

A $p$-spin network is a trivalent graph $\Gamma$ equipped with a cyclic orientation at each vertex and with a $p$-admissible coloring, which is a mapping from the edges of $\Gamma$ to $I_p$, such that every triple $(i, j, k) \in I_p^3$ surrounding a trivalent vertex is $p$-admissible. If $\Gamma$ is a $p$-spin network we can associate a closed invariant tangle, denoted $e_p(\Gamma) \in I_p(0, 0) = \mathbb{Z}/p\mathbb{Z}$, to any immersion of $\Gamma$ into $\mathbb{R} \times [0, 1]$. The correspondence is the following: To each edge colored by $i \in I_p$ of the immersed spin network we associate the tangle $f_i \in I(i, i)$, and to each trivalent vertex of the immersed spin network colored by the admissible triple $(i, j, k)$, with a positive orientation we associate the tangle $Y(i, j, k)$ (see figure 10). Moreover, the relations of lemma [1] imply

\[f_i \otimes f_j = n + \sum_{(i, j, k)_{p-admissible}} \psi_p(\frac{\Delta_k}{\Theta(i, j, k)}) Y_{i(i, j)}^k Y_{k(i, j)}^k.\] (2.19)

\[\text{Figure 10:}\]

that this correspondence does not depend on the particular chosen immersion of the spin network.

Until now we have worked in characteristic $p$, but of course it is much more standard to work in characteristic 0 [10, 25], in which case we just have to replace in what we said $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Q}$, $\psi_p$ by the identity map, $I_p$ by $\mathbb{N}$ and so on. Let $e$ be the standard spin network evaluation in characteristic 0, then:
**Proposition 2**  If $\Gamma$ is a $p$-spin network

$$e_p(\Gamma) = \psi_p(e(\Gamma))$$  \hspace{1cm} (3.1)

**Proof**

If $\Gamma$ is a $p$-spin network, using the definition of $f_i$ we can expand $e(\Gamma)$ into a finite sum $\sum_i a_i T^{(i)}$, where $T^{(i)}$ are closed tangles and $a_i \in \mathbb{Q}_p$. If we evaluate $e_p(\Gamma)$ we get the same result but with $a_i$ replaced by $\psi_p(a_i)$. Each closed tangle is evaluated as an integer using 2.1, 2.2, these relations don’t depend on the characteristic thus we get the proposition 2. $\Box$

Let $X$ be a one dimensional oriented compact manifold without boundary. A chord diagram (usually referred as Chinese character chord diagram) is the union $D = \bar{D} \cup X$ where $\bar{D}$ is a graph with univalent and trivalent vertices, together with a cyclic orientation of trivalent vertices such that univalent vertices lie in $X$. Trivalent vertices are referred to as internal vertices, and the degree of $D$, denoted $d^\circ(D)$, is half the number of vertices of the graph $\bar{D}$. Let $\mathcal{A}_n$ the $\mathbb{Z}$ module freely generated by chord diagrams of degree $n$. We define the $\mathbb{Z}$ module of chord diagrams of degree $n$, denoted $\bar{\mathcal{A}}_n$, as being the quotient of $\mathcal{A}_n$ by the relations (STU, IHX, AS) shown in figure 11.

![Figure 11](image)

We denote by $\bar{\lambda}$ a $p$-coloring of $X$ i-e a mapping from the set of connected components of $X$ to $I_p = \{1, \cdots, p - 2\}$. And we define a weight system $\bar{\omega}_T^\bar{\lambda}$ which associates a spin network to a chord diagram. The rules defining $\omega^T$ are given in the figures (12, 13) where $\Delta$ is the coproduct, as shown in figure 14.
\[ D = \vec{\Delta} \]

\[ \omega^T(D) = i \cdot e(\vec{\Delta}) = \vec{\omega}_T(f_i) \]

Figure 12:

\[ \bar{\omega}^T(\vec{\Delta}) = -i \cdot e(\vec{\Delta}) \]

Figure 13:

\[ \bar{\omega}^T(\vec{\gamma}) = 2 \]

Figure 14:
Using the tangle evaluation $e$ of spin networks we define the linear form:

$$\omega^T_\lambda = e \circ \omega^T_\lambda : \tilde{A} \rightarrow \mathbb{Q}_p$$ \hspace{1cm} (3.2)

We have the following proposition:

**Proposition 3** $\omega^T_\lambda$ is a weight system i.e it defines a linear form on $A$.

**Proof**
This is a direct consequence of the relation presented in figure 15 which is itself consequence of the relations of figures 16 and 17.

It is well known that given a Lie algebra $\mathcal{G}$ and an invariant scalar product on $\mathcal{G}$ we can associate a weight system to any coloring of Wilson lines by representations of $\mathcal{G}$. We denote
by \((e, h, f)\) the basis elements of \(sl(2)\) which satisfy the following commutation relations:

\[
[h, e] = 2e, \quad [h, f] = -2f \quad (3.3)
\]

\[
[e, f] = h. \quad (3.4)
\]

The quadratic Casimir is given by \(C = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h\). We denote by \(V_\lambda\) the irreducible highest weight \(sl(2)\) module of weight \(\lambda\) given in a basis \((v_i), i \in \{0, 1, \cdots, \lambda\}\) by:

\[
ev_i = (\lambda - i + 1)v_{i-1} \quad (3.5)
\]

\[
fv_i = (i + 1)v_{i+1} \quad (3.6)
\]

\[
hv_i = (\lambda - 2i)v_i \quad (3.7)
\]

\[
ev_0 = 0, \quad fv_\lambda = 0. \quad (3.8)
\]

The value of the quadratic Casimir in the representation \(V_\lambda\) is given by \(C_\lambda = \frac{\lambda(\lambda+2)}{2}\). Let \(\vec{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{N}^\times\) and define \(\omega^{sl(2)}_\vec{\lambda}\) the \(sl(2)\) weight system associated with the Lie algebra \(sl(2)\) and with a coloring of of the components of \((S^1)^L\) by finite dimensional irreducible representations \(V_{\lambda_i}\) of \(sl(2)\) and with the normalization of the quadratic casimir given by the trace in the fundamental representation. Then :

**Lemma 4**

\[
\omega^{sl(2)}_\vec{\lambda}(D) = (-1)^{\lambda_1+\cdots+\lambda_n}(-1)^{deg(D)}\omega^{T}_\vec{\lambda}(D). \quad (3.9)
\]

**Proof**

Denote by \(\mathcal{T}(k, l)\) the \(\mathbb{Z}\) linear span of all \((k, l)\) tangles and by \(Hom_{sl(2)}(V_1^{\otimes k}, V_1^{\otimes l})\) the space of homomorphism from \(V_1^{\otimes k}\) to \(V_1^{\otimes l}\) which commute with the action of \(sl(2)\), where the action of \(sl(2)\) on \(V_1^{\otimes k}\) is the usual diagonal action and the action of \(sl(2)\) on \(V_1\) is given by \(3.3\).

We define a linear map \(\tau : \mathcal{T}(k, l) \rightarrow Hom_{sl(2)}(V_1^{\otimes k}, V_1^{\otimes l})\) as follows : Let \(I_1\) be the \((1, 1)\) tangle consisting of one vertical line, \(X\) be the \((2, 2)\) tangle consisting in a simple crossing, \(\cap\) (resp. \(\cup\)) be the simplest \((2, 0)\) (resp. \((0, 2)\)) tangle consisting of one line and no crossing, and define

\[
\tau(I) = Id_{V_1} \quad (3.10)
\]

\[
\tau(X) = -P : v \otimes v' \rightarrow -v' \otimes v \quad (3.11)
\]

\[
\tau(\cap) : v \otimes v' \rightarrow (v|\epsilon(v')) \quad (3.12)
\]

\[
\tau(\cup) : \alpha \rightarrow \alpha(v_0 \otimes v_1 - v_1 \otimes v_0) \quad (3.13)
\]

where \((v_0, v_1)\) is the basis \(3.3\) of \(V_1\), \((v_i|v_j) = \delta_{i,j}\) and \(\epsilon(v_0) = v_1, \epsilon(v_1) = -v_0\). Conjugation by \(\epsilon\) of an element \(x \in sl(2)\) is the automorphism *, \(e^* = -f, f^* = -e, h^* = -h\). We extend \(\tau\) to
all \((k,l)\) tangles by asking that
\[ \tau(T\cdot T') = \tau(T)\tau(T') \] and
\[ \tau(T\otimes T') = \tau(T)\otimes\tau(T') . \]
By a direct computation we verify that
\[ \tau(O) = -2, \]
where \(O\) denote the circle and \(\tau(I\otimes I) + \tau(X) + \tau(\cup\cdot\cap) = 0\), so that \(\tau\) is well defined on the skein module \(S\). Let \(C_1 \in Hom(V_1 \otimes V_1)\) be action of the quadratic Casimir on \(V_1 \otimes V_1\), then \(C_1 = \frac{1}{2} \tau(\cup\cdot\cap) - \frac{1}{2} \tau(X)\). Now if \(t\) is the chord diagram consisting of one horizontal chord connecting 2 vertical strands then, by the definition of the \(sl(2)\)-tangle weight system, we have that
\[ \omega^{sl(2)}(t) = -\tau(\omega^T(t)) = C_1 . \]
Moreover \(\tau(f_i)\) corresponds to the symmetrisation operation and thus to the projection operator \(P_i\) from \(V_i^\otimes\) to the irreducible representation \(V_i\). If \(f \in \mathcal{I}(i,i)\) is an invariant tangle then using the definitions of the traces and of \(\cap,\cup\) we see that
\[ tr(f) = (-1)^i tr_{V_i}(\tau f) . \]
Moreover, \(\omega^{sl(2)}_{i,j}(t) = c_{i,j} = P_i \otimes P_j \Delta^{(i)} \otimes \Delta^{(j)} CP_i \otimes P_j\) which is equal to \(-\tau(\omega^{T}_{i,j}(t))\).

\[ \blacksquare \]

4 \(sl(2) - so(3)\) Kirby Weight systems

Let \(D\) be a chord diagram which is built on an oriented one-dimensional compact manifold \(X\), and denote \(\hat{C}\) the chord diagram obtained from \(C\) by reversing the orientation of one connected component \(X_i\) of \(X\). A weight system \(\omega\) is said independent of the orientation if
\[ \omega(D) = (-1)^{n_i(D)} \omega(\hat{D}) \]
where \(n_i(D)\) is the number of vertices lying on \(X_i\).

**Definition 2** A weight system is a Kirby weight system if it is independent of the orientation and if it takes the same value on any two chord diagrams which are related as in figure 18.

![Figure 18](image)

Figure 18:

We are now able to state one of the main theorems of this paper : Let \(D\) be a chord diagram with support \(X\), let \((\lambda_i)i = 1, \cdots, n\) be the coloring of the connected components of \(X\) and denote \(\Delta_\lambda = \prod_{i=1}^n (-1)^{\lambda_i}(\lambda_i + 1)\).
Theorem 1  The weight system $\omega_{sl(2)}^{(p)}$ (resp. $\omega_{so(3)}^{(p)}$) given by:

$$\omega^{(p)}(D) = \sum_{\lambda} \Delta_{\lambda} \omega_{\lambda}^{(p)}(D),$$

where the sum is over all $\lambda \in I_p$ (resp. $\lambda \in I_p \cap 2\mathbb{Z}$), is a Kirby weight system valued into the field $\mathbb{Z}/p\mathbb{Z}$.

Proof  The proof is given in figure 15 where we use the formula 2.19 and the relation of figure 13. The proof is given for the $sl(2)$ case, for the $so(3)$ case it is exactly the same proof taking into account that if $(i, j, k)$ is an admissible triple and $i, j$ are even then $k$ is also even. \hfill $\square$

Proposition 4  Let $D \in A((S^1)^L)$, let $\lambda_i$ be the highest weight associated with the $i$th component of $(S^1)^L$ and let $n_i(D)$ be the number of univalent vertices on the $i$th component of $(S^1)^L$. Then expanding $\omega_{sl(2)}^{(2)}$ in powers of $(\lambda + 1)$ we get:

$$\omega_{\lambda}^{sl(2)}(D) = \sum_{k_1 \geq 0} \cdots \sum_{k_L \geq 0} \frac{2^{-d^o(D)}}{\prod_{i=1}^{k_L} (n_i(D) + 1)!} a_{k_1, \ldots, k_L}(D)(\lambda_1 + 1)^{2k_1 + 1} \cdots (\lambda_n + 1)^{2k_L + 1} \cdot \cdots$$

$$= \sum_{k_i \geq 0} (\lambda_1 + 1)^{2k_1 + 1} 2^{-d^o(D)} a_{\lambda,K_i}(D)$$

where $d^o(D)$ is the degree of the diagram $D$ and $[n_1/2]$ denotes the integral part of $n_1/2$. Moreover, the coefficients $a_{k_1, \ldots, k_L}(D)$ lies in $\mathbb{Z}$, and $a_{\lambda,K_i}(D)$ lies in $\mathbb{Z}$ if $\lambda_j \in \mathbb{Z}$.

Proof  Let $U(sl(2))$ the subring of $U(sl(2))$ generated over $\mathbb{Z}$ by the Chevalley basis elements $e, f, h$ (3.3). The structure constants appearing in the commutation relations of the Chevalley generators are integers; thus $U(sl(2))$ admits a $\mathbb{Z}$-basis consisting of all $e^a h^b f^c$, $a, b, c \in \mathbb{N}$.

Using the definition of the $sl(2)$ weight system and the fact that $2C \in G \otimes G$ where $G$ denotes the $\mathbb{Z}$-linear span of the Chevalley basis, it is clear that $2^{-d^o(D)} \omega_{\lambda}^{sl(2)}(D)$ is constructed as the trace over $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_i}$, $\lambda_i \in \mathbb{N}$ of an element $x(D) \in (U(sl(2)))_{\mathbb{Z}}^{\otimes L} G$, where the power of $G$ means that $x(D)$ commutes with the diagonal action of $G$. Moreover the degree of $x(D)$ with respect to the $i$th-component of $(U(sl(2)))_{\mathbb{Z}}^{\otimes L}$ is at most $n_i(D)$.

Let $Z(sl(2))$ be the center of $U(sl(2))$ and let $V_\lambda$ the irreducible highest weight module of (non necessarily integral) weight $\lambda$. If $z \in Z(sl(2))$, $z$ acts as a scalar on $V_\lambda$, denoted as $\lambda(z)$. From Harish-Chandra’s theorem we know that the infinitesimal character $\lambda(z)$ is an even polynomial in $\lambda + 1$ whose degree is at most the degree of $z$ with respect to
\[ \omega^{(p)}(\begin{array}{c}
\end{array}) = \sum_{i,j=0}^{p-2} (-1)^{i+j}(i+1)(j+1)i^n \]

\[ = \sum_{<i,j,k>_p} \frac{(-1)^{i+j+k}(i+1)(j+1)(k+1)}{\Theta(i,j,k)} i^n \]

\[ = \sum_{<i,j,k>_p} \frac{(-1)^{i+j+k}(i+1)(j+1)(k+1)}{\Theta(i,j,k)} (i^{n-1}j + i^{n-1}k) \]

\[ = \sum_{<i,j,k>_p} \frac{(-1)^{i+j+k}(i+1)(j+1)(k+1)}{\Theta(i,j,k)} \]

\[ = \ldots = \omega^{(p)}(\begin{array}{c}
\end{array}) \]

Figure 19:
the filtration of $\mathcal{U}(sl(2))$. In the case of $sl(2)$ the center $Z(sl(2))$ is generated by $C$ and 
$\chi_\lambda(C) = \frac{(\lambda+1)^2-1}{2}$. Let $D = [\mathcal{U}(sl(2)), \mathcal{U}(sl(2))]$ be the subspace of $\mathcal{U}(sl(2))$ generated by 
all commutators. We then have that $\mathcal{U}(sl(2)) = Z(sl(2)) \oplus D$ (H) so we can extend $\chi_\lambda$ 
to all of $\mathcal{U}(sl(2))$ by requiring it to be 0 on $D$. When $\lambda \in \mathbb{N}$, $V_\lambda$ is finite dimensional and 
$tr_{V_\lambda}(x) = (\lambda + 1)\chi_\lambda(x)$. Let $v_i$, $i = 0, \cdots, \lambda$ the basis of $V_\lambda$ given in (3.8) and denote $V_{\lambda Z}$ 
the $\mathbb{Z}$-span of $v_i$, if $x \in \mathcal{U}(sl(2))_\mathbb{Z}$ then $x \cdot v_i \in V_{\lambda Z}$ so $tr_{V_\lambda}(x) \in \mathbb{Z}$.

We can conclude from this analysis that : $2^{d_x(D)}\omega^{sl(2)}_\chi(D) = \prod_{i=1}^{L}(\lambda_i + 1)\chi_{\lambda_1} \cdots \chi_{\lambda_L} x(D)$
is an odd polynomial of degree $n_i(D) + 1$ in $\lambda_i + 1$ taking integer values when $\lambda_i, i = 1, \cdots, L$
are integers. It is a standard exercise to show by recurrence that such a polynomial is a 
$\mathbb{Z}$-linear combination of the polynomials

$$\prod_{i=1}^{L} \left( \frac{\lambda_i + 1 + k}{2k_i + 1} \right), \quad 2k_i \leq n_i(D),$$

where $\binom{x}{k} = \frac{x(x-1) \cdots (x-k+1)}{k!}$. Thus $2^{d_x(D)}\prod_{i=1}^{L}(n_i(D) + 1)!\omega^{sl(2)}_\chi(D)$ is a polynomial 
with integer coefficients.

In order to prove the second part of the theorem we consider the partial trace $x_1(D) = \prod_{i=2}^{L}(\lambda_i + 1)1 \otimes \chi_{\lambda_2} \otimes \cdots \otimes \chi_{\lambda_L} x(D)$ with $\lambda_2, \cdots, \lambda_L \in \mathbb{N}$. This is a well defined operation (i-e 
which do not depend on the particular choice of $x(D)$) moreover $x_1(D) \in \mathcal{U}(sl(2))_\mathbb{Z} \cap Z(sl(2))$
thus $x_1(D)$ is a polynomial of degree $n_i(D)/2$ in $2C$ with integral coefficients.

\[
\square
\]

**Preliminary**

Consider $D_{<2n}$ the subspace of $A(X)$ generated by all chord diagrams which possess less 
than $2n$ univalent vertices on one component of $X$, and denote $by \phi_n : A(X) \to A(X)/D_{<2n}$ 
the canonical projection. By definition,

$$\phi_n(D) = 0 \text{ if } n_i(D) < 2n,$$  \hfill (4.4)

where $n_i(D)$ denote the number of univalent vertices on the $i^{th}$ component of the diagram 
$D$. We define $N_p = \left(\frac{p-3}{2}\right)$ and we denote by $\mathbb{Z}_{(n)} \equiv \mathbb{Z}_{[\frac{1}{2}, \cdots, \frac{1}{n}]}$ the ring generated over $\mathbb{Z}$ by 
$\{\frac{1}{2}, \cdots, \frac{1}{n}\}$.

**Proposition 5** Let $D \in A((S^1)^L) \otimes \mathbb{Z}_{(p-1)}$, then :

$$\omega^{sl(2)}_{p_1}(D) = 2^{L} \omega^{sl(2)}_{p_2}(D), \text{ if } n_i(D) \leq 2N_p + 1$$

$$\omega^{sl(2)}_{p}(D) = \omega^{sl(2)}_{p} \circ \phi_{N_p}(D).$$
where * in $\omega_*$ means $sl(2)$ or $so(3)$. This proposition states that $sl(2)$ and $so(3)$ Kirby weight systems are essentially the same.

Define

\[
\epsilon_p(i) \equiv (p) \begin{cases} 
-1, & \text{if } p - 1 \text{ divides } i, \\
0, & \text{if not } \end{cases}
\]  

(4.6)

We have the following lemma:

**Lemma 5**

\[
\sum_{k=0}^{p-2} (k + 1)^i \equiv \epsilon_p(i) \\
\sum_{k=0}^{p-2} (k + 1)^{2i} \equiv \frac{1}{(p)} \epsilon_p(2i)
\]

(4.7)

The first equality, which is the Von-Staudt theorem, follows directly from the fact that the group of invertible elements of $\mathbb{Z}/p\mathbb{Z}$ is the cyclic group of order $p - 1$ for $p$ prime. For the second equality consider:

\[
\epsilon_p(2i) \equiv \sum_{k=1}^{p-1} k^{2i} \equiv \frac{1}{(p)} \left( \sum_{k=1}^{p-1} k^{2i} \right) + \frac{1}{(p)} \epsilon_p(2i).
\]

Making the change of variables $j = p - k$ the second term in the LHS can be written as $\sum_{j=1}^{\frac{p-1}{2}} (p - k)^{2i} = \sum_{j=1}^{\frac{p-1}{2}} (k)^{2i}$. Thus $\sum_{k=1}^{p-1} k^{2i} \equiv \frac{1}{(p)} \epsilon_p(2i)$. Together with,

\[
\epsilon_p(2i) \equiv \sum_{k=1}^{p-1} (2k - 1)^{2i} + \sum_{k=1}^{p-1} (2k)^{2i},
\]

this imply

\[
\sum_{k=1}^{\frac{p-1}{2}} (2k - 1)^{2i} \equiv (1 - 2^{2i-1}) \epsilon_p(2i).
\]

We get the desired result using the fact that $2^{2i} \equiv 1$ if $p - 1$ divides $2i$.

**Proof of Proposition**

Let $\psi_p : \mathbb{Z}_{(p-1)} \to \mathbb{Z}/p\mathbb{Z}$ be the evaluation map. By definition we have for $D \in \mathcal{A}((S^1)^L)$.

\[
\omega_{sl(2)}(D) \equiv (p) \psi_p \left( \prod_{\lambda_1=0}^{p-2} \cdots \prod_{\lambda_L=0}^{p-2} \prod_{i=1}^L (\lambda_i + 1) \omega_{\lambda}(D) \right)
\]

(4.8)

This is well defined since we have seen in prop.4 that $\omega_{\lambda}(D) \in \mathbb{Z}(\lambda)$. If $D$ is such that $n_t(D) \leq 2N_p + 1$ this means that the coefficients of $\prod_{i=1}^L (\lambda_i + 1)^{2(k_i+1)}$ in the development
of $\omega_\lambda(D)$ belongs to $\mathbb{Z}_{(p-1)}$, thus we can distribute $\psi_p$ and apply the preceding lemma to get the first conclusion of the proposition.

If $D$ is such that $\phi_{N_p}(D) = 0$, for example $n_1(D) < 2N_p$, then using the results of prop.4 especially the fact that the coefficient of $(\lambda_1 + 1)^{2(k_1+1)}$ belong to $\mathbb{Z}_{(2)}$ we have

$$\omega_\lambda^{(p)}(D) = \sum_{k_1 \geq 0} \psi_p((\lambda_1 + 1)^{2(k_1+1)} \psi_p(a_{k_1}(D))) = 0$$

(4.9)

$k_1 < N_p$ imply $2(k_1 + 1) < p - 1$ thus $\epsilon_p(2(k_1 + 1)) = 0$.

\[\square\]

5 The asymptotic Invariant for rational homology 3-sphere

Let $L$ be a framed link and $\hat{Z}(L)$ be the canonical Vassiliev invariant \[1, 12, 7\]. Which is a formal power series in $\hbar : \hat{Z}(L) = \sum_{m=0}^{+\infty} \hbar^m Z_m(L)$ with $Z_m(L) \in \mathcal{A}_m(L) \otimes \mathbb{Q}$. Let $\nu$ be the value of $\hat{Z}$ for the un-framed trivial knot and define $\hat{Z}(L) = \nu \otimes \cdots \otimes \nu \cdot \hat{Z}(L)$, where the product is realised with the connected sum along each component of $L$. One of the main theorem of \[13\] states that if $\omega$ is a Kirby weight system then $\omega \circ \hat{Z}(L)$ is invariant under the second Kirby move (hand-slide) and under the change of orientation. Once we have a Kirby weight system we can construct, up to a normalisation problem, a framed link invariant, independent of the orientation and invariant under all Kirby moves, hence a 3-manifold invariant. The problem we face in our case comes from the fact that the weight systems we constructed are valued into a non-zero characteristic field. In order to apply our weight system to the universal invariant we need to know the integrality properties of its coefficients. T.Q. Le in a recent paper \[16\] completely study the denominators of the Kontsevich integrals. Combining the propositions 5.3 and 6.1 of \[16\], we can state :

**Lemma 6** If $L$ is an algebraically split (ASL) link then :

$$\phi_n(\hat{Z}_m(L)) \in \mathcal{A}_m(L) \otimes \mathbb{Z}_{(2(n+1))}, \text{ for } m \leq n(|L| + 1)$$

(5.1)

where $|L|$ denotes the number of connected component of the link $L$.

$L$ is called algebraically split if it is a framed link with diagonal linking matrix. It is known that if $M$ is rational homology 3-sphere then there exist Lens spaces such that the connected
sum of \( M \) with these lens spaces can be obtained by surgery along an algebraically split link possessing a non-degenerate linking matrix [13]. Using the multiplicative property of the invariant under connected sum this means that the computation of the asymptotic invariant we are going to make is applicable to at least the case of all rational homology 3-spheres.

**Definition 3** If \( L \) is an ASL we can define:

\[
F^p_\ast(L) = \sum_{m=0}^{N_p} h^m \omega^p_\ast(\hat{Z}_{N_pL+m})(L) = h^{-N_p|L|} \omega^p(\hat{Z}(L) \mod(h^{N_p(|L|+1)+1})),
\]

where \( p \) is a prime odd integer and \( N_p = (p-3)/2 \).

\( \omega^p \) here means \( \omega^p \circ \psi_{N_p} \), which are by proposition 3 identical on \( \mathcal{A}(L) \otimes \mathbb{Z}_{(p-1)} \), and \( * \) refers to \( sl(2) \) or \( so(3) \). \( \omega^p \) is well defined on \( \mathcal{A}(L) \otimes \mathbb{Z}_{(p-1)} \), thus from lemma 3 the definition of \( F^p \) suffers from no ambiguity.

Let \( U^\pm \) be the unknot with framing \( \pm 1 \), \( F_p(U^\pm) \) is an invertible element (i.e. the coefficient of \( h^0 \) is non-zero, see [5.10]). For \( L \) an ASL we can define:

\[
O^p_\ast(L) = \frac{F^p_\ast(L)}{F^p_\ast(U^+)^{\sigma_+} F^p_\ast(U^-)^{\sigma_-}}.
\]

where \( \sigma_+ \) (resp. \( \sigma_- \)) is the number of positives (resp. negatives) eigenvalues of the linking matrix of \( L \). \( O^p_\ast(L) \) is invariant under all Kirby moves.

### 5.1 Fermat limit

For a complete discussion of the notion of Fermat limit, see [17]. For the reader’s convenience we recall here some definitions that we are going to use.

**Definition 4** Let \( (u_p)_{p \text{ prime}} \) be a sequence of \( \mathbb{Z}/p\mathbb{Z} \) numbers. We say that \( (u_p)_{p \text{ prime}} \) admits a Fermat-limit if there exists a rational number \( u \) and \( N \in \mathbb{N} \) such that \( \psi_p(u) \equiv u \) for all \( p > N \). Consider

\[
\mathcal{S}_p = \{ F_p = \sum_{n=0}^{N_p} h^n F_{n,p}, F_{n,p} \in \mathbb{Z}/p\mathbb{Z} \}.
\]

We say that \( (F_p)_{p \text{ prime}} \), \( F_p \in \mathcal{S}_p \) admits a strong Fermat limit if there exist \( F \in \mathbb{Q}[[h]] \), \( F = \sum_{n=0}^{+\infty} h^n F_n \) and \( N \in \mathbb{N} \), such that \( \psi_p(F \mod(h^{N_p+1})) \equiv \sum_{n=0}^{N_p} h^n \psi_p(F_n) \equiv F_p \) for all \( p > N \).
Definition 5  Given integers $f_1, \ldots, f_n$ we consider a formal integration procedure defined by:

$$I_{\vec{f}, \hbar} (\prod_{i=1}^{|L|} (h\alpha_i)^{k_i}) = \begin{cases} \prod_{i=1}^{|L|} \left(-\hbar \frac{k_i}{f_i}\right)^{\frac{1}{2^{k_i-1}}} k_i!, & \text{if } k_i \text{ is even for all } i \in \{1, \ldots, |L|\} \\ 0, & \text{otherwise} \end{cases}$$

(5.4)

We can extend by linearity $I_{\vec{f}, \hbar}$ to all formal power series of the form

$$\sum_n \vec{k} S_n \hbar^n (\prod_{i=1}^{|L|} \alpha_i^{2k_i})$$

if $\{S_n, k \neq 0 \mid n + \sum k_i = q\}$ is a finite set for all $q$.

Remark

If we consider $\hbar$ as an imaginary number whose value is $\frac{2i\pi}{K}$ then $I_{\vec{f}, \hbar}$ is really an integral:

$$I_{f, \hbar} (P(\alpha)) = e^{-\frac{i\pi}{4} \text{sgn}(f)} \left(\frac{2K}{|f|}\right)^{\frac{1}{2}} \lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} d\alpha e^{\frac{i\pi}{2K} f \alpha^2} P(\alpha) e^{\epsilon\alpha^2} \quad (5.5)$$

Theorem 2  Consider $N, n \in \mathbb{N}$ and $D \in A_n(L) \otimes \mathbb{Z}(N)$ Then,

$$\left(\frac{(N_p + 1)!}{\epsilon(\ast) h^{N_p}}\right)^{|L|} \frac{1}{\prod_{i=1}^{|L|} \left(\frac{L}{p}\right)} \omega(p)^{|L|} \prod_{i=1}^{|L|} e^{h \frac{f_i}{p} \theta_i} \cdot h^{d^p(D)} D \mod(h^{N_p(L+1)+1}), \in \mathcal{S}_p$$

(5.6)

admits a strong Fermat limit which is

$$\left(\prod_{i=1}^{|L|} e^{h \frac{f_i}{p} \theta_i}\right)^{|L|} h^{d^p(D)} I_{f, \hbar} \left(\prod_{i=1}^{|L|} \alpha_i \tilde{\omega}_\alpha(h^{d^p(D)} D)\right),$$

(5.7)

where $\ast$ denotes sl(2) or so(3), $\epsilon(\text{sl}(2)) = -1, \epsilon(\text{so}(3)) = -1/2$, $(\cdot)_p$ is the Legendre symbol and $\tilde{\omega}_\alpha = \omega(\alpha_1, \ldots, \alpha_{|L|-1})$.

Moreover, if $D$ is a chord diagram, $h^{d^p(D)} I_{f, \hbar} \left(\prod_{i=1}^{|L|} \alpha_i \tilde{\omega}_\alpha(h^{d^p(D)} D)\right)$ is a polynomial in $h$ which degree is lower than $d^p(D)$ and whose valuation is greater than half the number of internal vertices of $D$.

Note that the Fermat limit does not depend on the choice of sl(2) or so(3).

Theorem 3  Letting $F^p_n$ be the coefficient of $h^n$ in the developpement of $F^p$, then if $L$ is an ASL such that the framings $f_i$ of all its components are non zero then

$$\left(\frac{(N_p + 1)!}{\prod_{i=1}^{|L|} \left(\frac{L}{p}\right)}\right)^{|L|} F^p_n(L)$$

(5.8)
admits a Fermat limit denoted \( F_n \). Moreover, \( F = \sum_{n=0}^{+\infty} h^n F_n \) is equal to:

\[
F = \prod_{i=1}^{\lfloor L \rfloor} e^{-\frac{h_i}{2}} h^{\lfloor L \rfloor} I_{f,h} \left( \prod_{i=1}^{\lfloor L \rfloor} \alpha_i \tilde{\omega}_\alpha (\tilde{Z}(L)) \right) \tag{5.9}
\]

For example

\[
F(U^\pm) = \frac{\mp 2h}{e^\frac{\pm}{2} - e^\frac{\mp}{2}} e^{\pm \frac{2h}{4}} \tag{5.10}
\]

The asymptotic rational homology 3-sphere quantum invariant \( O \) is defined as the Fermat limit of \( O_p \). If we denote \( J_\alpha \) the colored Jones polynomial then:

\[
O(L) = e^{-\frac{h}{2} \sum_l (f_i + \frac{1}{f_i} - \frac{1}{4} \text{sgn}(f_i))} \prod_{i=1}^{\lfloor L \rfloor} \text{sgn}(f_i) I_{f,h}(J_\alpha + \frac{1}{4}(L)) \tag{5.11}
\]

This expression correspond to the Rozansky formulation of the Ohtsuki invariant \cite{24}.

### 5.2 Proofs

Let \( D \in \mathcal{A}_m(L) \otimes \mathbb{Z}_N \), \( N \in \mathbb{N} \) and denote by \( n_i(D) \) the number of chords arriving the \( i^{th} \) component of \( L \). Let \( \theta_i \in \mathcal{A}(L) \) be the diagram possessing only one chord on the \( i^{th} \) component of \( L \).

Using the definitions of \( \text{mod}(h^{N_p(L+1)+1}) \) and \( \phi_{N_p} \), we have the expansion,

\[
\phi_{N_p}(\prod_{i=1}^{\lfloor L \rfloor} e^{\frac{h_i}{2} \theta_i} D \text{ mod}(h^{N_p(L+1)+1})) = \sum_{\tilde{\alpha} \in \mathcal{J}_p(D)} \prod_{i=1}^{\lfloor L \rfloor} \left( \frac{\tilde{h}_i}{2} \right)^{n_i} (\theta_i)^{\frac{1}{2}} \cdot h^{d^c(D) D}, \tag{5.12}
\]

where \( \mathcal{J}_p(D) = \{ \tilde{\alpha} \in \mathbb{N}^{\lfloor L \rfloor}, d^c(D) + \sum_l n_l \leq N_p(\lfloor L \rfloor + 1), n_i(D) + 2l_i \geq 2N_p \} \). Remembering that \( 2d^c(D) \) is the number of vertices of the diagram \( D \) we have \( 2N_p \geq 2(d^c(D) - N_p |L|) + \sum_i 2l_i \geq \sum_i (n_i(D) + 2l_i - 2N_p) \), which implies that \( n_i(D) + 2l_i \leq 4N_p \), and a fortiori \( l_i \leq 2N_p \).

Thus \( \left( \frac{\theta_i}{l_i}, D \right) \in \mathcal{A}(L) \otimes \mathbb{Z}_{(\max(N,2N_p))} \) if \( \tilde{\alpha} \in \mathcal{J}_p(D) \), so we can apply \( \omega^{(p)} \) to this diagram for \( p > N \). Using the definition of \( \omega^{(p)}, p > N \) we have to compute:

\[
\sum_{\tilde{\alpha} \in \mathcal{J}_p(D)} \sum_{\tilde{\alpha} \in \mathcal{J}_p(D)} \psi_p \left( \prod_{i=1}^{\lfloor L \rfloor} \left( \frac{\tilde{h}_i}{2} \right)^{n_i} \alpha_i \tilde{\omega}_\alpha \left( \prod_{i=1}^{\lfloor L \rfloor} \theta_i^{\frac{1}{l_i}} \cdot h^{d^c(D) D} \right) \right). \tag{5.13}
\]

Using the notation of proposition \cite{4} and the fact that

\[
\tilde{\omega}_\alpha \left( \prod_{i=1}^{\lfloor L \rfloor} \theta_i^{\frac{1}{l_i}} \cdot D \right) = \prod_{i=1}^{\lfloor L \rfloor} \left( \frac{\alpha_i^2 - 1}{2} \right)^{n_i} \frac{1}{l_i} \tilde{\omega}_\alpha \left( h^{d^c(D) D} \right)
\]

23
this can be expressed as :

\[ \sum_{k \geq 0} \sum_{\vec{a} = 1}^{\vec{a}} \sum_{\vec{j} = 0}^{\vec{j}} \psi_p \left( \frac{h^2}{2} a_k(D) \prod_{i=1}^{\vec{l}} \left( \frac{h f_i}{2} \right) a_i^{2(k_i+1)} \left( \frac{\alpha_i^2 - 1}{2} \right) \frac{1}{l_i!} \right). \]  

We know from proposition 3 that \( a_k(D) \in \mathbb{Z}_{(\max(n_i(D)+1, N))} \), thus if \( p > \max(n_i(D) + 1, N) \) we can distribute \( \psi_p \) over all the factors appearing in 5.14. Using the result of Lemma 3 we get :

\[ \psi_p \left( \frac{\prod_{i=1}^{\vec{l}} \sum_{\alpha_i = 1}^{\alpha_i} a_i^{2(k_i+1)} \left( \frac{\alpha_i^2 - 1}{2} \right) \frac{1}{l_i!} \right) \equiv \sum_{\vec{q} + \vec{r} = \vec{i}} \frac{(-1)^{r_i}}{\vec{q}! \vec{r}!} \epsilon_p (2(k_i + q_i + 1)). \]  

But \( 2(k_i + q_i + 1) \leq n_i(D) + 2l_i + 2 \leq 4N_p + 2 = 2(p - 1) - 2 \), so \( \epsilon_p (2(k_i + q_i + 1)) \neq 0 \) if and only if \( k_i + q_i = N_p \). The computation for the case of \( \text{so}(3) \) is the same except that we have to replace \( \epsilon_p \) by \( \frac{1}{2} \epsilon_p \) (see proposition 3).

Making the change of variable \( j_i = l_i + k_i - N_p \), we can express 5.14 in the following way :

\[ \sum_{k, \vec{j}} \psi_p(a_k(h^2(D)D)) \psi_p \left( \frac{\prod_{i=1}^{\vec{l}} \left( \frac{h f_i}{4} \right) ^{j_i + N_p - k_i} \left( -1 \right) ^{j_i} \frac{1}{j_i! (N_p - k_i)!} \right), \]  

where the summation is over all \( k, \vec{j} \) satisfying \( 0 \leq 2k_i \leq n_i(D), j_i \geq 0 \) and \( d^2(D) + \sum_i (j_i - k_i) \leq N_p \) (recall that we have suppose that \( p > \max(n_i(D) + 1, N) \)).

Using the identity :

\[ \frac{1}{(N_p - k)!} \frac{1}{(N_p + 1)!} \left( \frac{1}{4} \right) ^{k+1} \frac{(2(k + 1))!}{(k + 1)!}, \]  

we get if \( \psi_p(f_i) \neq 0 \)

\[ \left( \sum_{k \geq 0, \vec{j} \geq 0} \psi_p(a_k(h^2(D)D)) \psi_p \left( \frac{\prod_{i=1}^{\vec{l}} a_i^{2(k_i+1)} \left( -1 \right) ^{j_i} \frac{1}{j_i! (N_p - k_i)!} \right) \right) \psi_p \left( \prod_{i=1}^{\vec{l}} e^{\frac{f_i - h}{4}} \right) \mod (h^{N_p (|L| + 1) + 1}), \]  

where we use the fact \( 4N_p + 1 = \frac{h}{p} \). This proves the first part of the theorem 2.

We know from 3 that \( \prod_{i=1}^{\vec{l}} a_i^{2(n_i(D)+1)} \) is a polynomial of total degree, with respect to \( \alpha_i \), smaller that \( \sum_{i=1}^{\vec{l}} (n_i(D) + 2) \), this means that the valuation of \( h^{\vec{l}} I_f \psi_p \left( \prod_{i=1}^{\vec{l}} a_i^{\vec{j}} \right) \) is greater than \( d^2(D) + L - (1/2) \sum_{i=1}^{\vec{l}} (n_i(D) + 2) = I(D)/2 \), where \( I(D) \) denote the number
Let $D$ be a chord diagram, denote $\tilde{D}$ the corresponding graph (the graph obtained by removing all Wilson circles from $D$). We say that $D \in I_1$ if each connected component of the graph $\tilde{D}$ contains at least one trivalent vertices. And an element $D \in A_n \otimes \mathbb{Q}$ is said to be in $I_1$ if it can be expressed as a sum of diagrams belonging to $I_1$.

**Lemma 7** If $D \in I_1$ then $I(D) \geq d^c(D)$.  

**Proof**

If $D \in I_1$ is a connected graph then $d^c(D) \geq 2$ thus, if $c(\tilde{D})$ denotes the number of connected components of $\tilde{D}$, then $d^c(D) \geq 2$. Therefore, if $\bar{D}$ denotes the number of connected components of $\tilde{D}$, then $d^c(D) \geq 2c(\bar{D})$. Using the Euler characteristic of $\bar{D}$ and the fact that the graph is trivalent we conclude that $I(D) + 2c(\bar{D}) \geq N(D)$, where $N(D) = \sum_{i=1}^{L} n_i(D)$ is the number of univalent vertices of $\tilde{D}$, thus $2d^c(D) = N(D) + I(D) \leq 2I(D) + 2c(\bar{D})$. 

Let $L$ be an ASL, and $L'$ is the associated link of framing zero, then $\hat{Z}(L) = \prod_{i=1}^{L} e^{\frac{h}{4}(\theta_i)} \tilde{Z}_n(L')$, where $\theta_i$ denotes the chord diagram possessing only one chord one the $i$-th-component and the product is realised as usual by the connected sum, moreover $\hat{Z}_n(L') \in I_1$ [10]. This implies, by the second part of theorem [2] and the preceding lemma, that $h^{-N_p(L)} \omega(p)(h^n \tilde{Z}_n(L'))$ has a valuation with respect to $h$ greater than $\frac{4}{l}$. We can conclude that if $n \leq N_p(L+1)$ then $F_{p,n}$ is equal to the coefficient of $h^n$ in the development of :

$$\frac{(N_p + 1)!}{\prod_{i=1}^{L} \left( \frac{1}{4} \right)} \sum_{k=0}^{4n} \omega(p) \left( \prod_{i=1}^{L} e^{\frac{h}{2}(\theta_i)} \tilde{Z}_k(L') \right).$$

From theorem [3] each term of this finite sum admits a Fermat limit; hence $F_{p,n}$ admits a Fermat limit which is the coefficient of $h^n$ in the development of :

$$F(L) = \prod_{i=1}^{L} e^{-\frac{h}{4} L} h^{|L|} I_j \prod_{i=1}^{L} \alpha_i \tilde{\omega}_a(\hat{Z}(L)).$$

Let us denote $U_\pm$ the unknot with framing $\pm$, $\nu$ the value of $\hat{Z}$ on the un-framed unknot and $J_{\nu}(L) = \omega_{\nu}(\hat{Z}(L'))$ the colored Jones polynomial.

$$\omega_\nu(\nu) = \frac{e^{\frac{h}{2} \nu} - e^{-\frac{h}{2} \nu}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}$$
and

\[ I_{f,h}(e^{q\hbar} P(\alpha)) = (e^{-\frac{\hbar^2}{f} J_{\frac{f}{2},h}(P(\alpha - \frac{2q}{f})))}. \quad (5.22) \]

Thus

\[ F(L) = \frac{-2\hbar}{e^\frac{h}{2} - e^{-\frac{h}{2}}} e^{-\frac{\hbar}{2} \sum_i (f_i + \frac{1}{f})} I_{f,h}(J_{\frac{f}{2},h}(L)), \quad (5.23) \]

\[ F(U^\pm) = -2 \frac{\hbar}{e^\frac{h}{2} - e^{-\frac{h}{2}}} e^{-\frac{h}{2} \pm}. \quad (5.24) \]

So, in conclusion :

\[ O(L) = e^{-\frac{\hbar}{2} \sum_i (f_i + \frac{1}{f_i})} \prod_{i=1}^{[L]} sgn(f_i) I_{f,h}(J_{\frac{f}{2},h}(L)), \quad (5.25) \]

and if, for example, we compute the invariant for the Lens space \( L(n,1) \), we get :

\[ O(L(n,1)) = e^{-\frac{\hbar}{2} (n + \frac{3}{2} - 3sgn(n)) \prod_{i=1}^{[L]} sgn(f_i)} \frac{e^\frac{\hbar}{2} - e^{-\frac{\hbar}{2}}}{e^\frac{\hbar}{2} - e^{-\frac{\hbar}{2}}}. \quad (5.26) \]

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