Criticality in the randomness-induced second-order phase transition of the triangular Ising antiferromagnet with nearest- and next-nearest-neighbor interactions

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Abstract

Using a Wang-Landau entropic sampling scheme, we investigate the effects of quenched bond randomness on a particular case of a triangular Ising model with nearest- ($J_{nn}$) and next-nearest-neighbor ($J_{nnn}$) antiferromagnetic interactions. We consider the case $R = J_{nnn}/J_{nn} = 1$, for which the pure model is known to have a columnar ground state where rows of nearest-neighbor spins up and down alternate and undergo a weak first-order phase transition from the ordered to the paramagnetic state. With the introduction of quenched bond randomness we observe the effects signaling the expected conversion of the first-order phase transition to a second-order phase transition and using the Lee-Kosterlitz method, we quantitatively verify this conversion. The emerging, under random bonds, continuous transition shows a strongly saturating specific heat behavior, corresponding to a negative exponent $\alpha$, and belongs to a new distinctive universality class with $\nu = 1.135(11)$, $\gamma/\nu = 1.744(9)$, and $\beta/\nu = 0.124(8)$. Thus, our results for the critical exponents support an extensive but weak universality and the emerged continuous transition has the same magnetic critical exponent (but a different thermal critical exponent) as a wide variety of two-dimensional (2d) systems without and with quenched disorder.

Key words: quenched bond randomness, weak universality, first-order transitions, triangular Ising model - superantiferromagnetism, entropic sampling

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1 Introduction

Understanding the role played by impurities on the nature of phase transitions is of major importance, both from experimental and theoretical perspectives. It has been known that quenched bond randomness may or may not modify the critical exponents of second-order phase transitions, based on the Harris criterion [1,2]. It was more recently established that quenched bond randomness always affects first-order phase transitions by conversion to second-order phase transitions, for infinitesimal randomness in \(d = 2\) [3,4] and after a threshold amount of randomness in \(d > 2\) [4], as also inferred by general arguments [5].

In the last 15 years, softening effects on first-order transitions have been studied and confirmed in several investigations [6,7,8,9,10,11,12,13,14,15,16,17,18,19]. In particular in 2d the behavior of the random-bond \((q\text{-state})\) Potts model (RBPM) has attracted special attention, since by varying \(q\) one can observe the expected softening effects on both second- and first-order \((q > 4)\) phase transitions. For this model the extensive numerical study of Cardy and Jacobsen [13] clearly showed the existence of a new critical behavior for each value of \(q\), independently of the disorder. This fact was illustrated by producing a convincing continuous behavior for the magnetic exponent \(\beta/\nu\) as function of \(q\), exhibiting no singularity at \(q = 4\). Their results are in excellent agreement (up to \(q = 4\)) with the theoretical predictions, in the vicinity of \(q = 2\), originally of Ludwig [19] and improved by Dotsenko et al. [16]. Thus, the study of Cardy and Jacobsen [13] has resolved and critically discussed the controversy related to the early simplifying prediction from the numerical simulations on the \(q = 8\) RBPM, that the emerging second-order phase transitions may exhibit critical exponents consistent with those of the pure Ising \((q = 2)\) model [11,12].

The above mentioned simplifying picture, that in a variety of situations the universality class of random-bond models is that of the Ising model [11,12,20] cannot be also valid in systems with competing interactions, where a strong saturating behavior of the specific heat has been observed recently [21,22]. In the present paper, we shall study a further such novel second-order phase transition induced by bond randomness. In the corresponding pure generalized Ising system, the competition of the microscopic nearest- and next-nearest-neighbor exchange interactions gives rise to a weak first-order phase transition. We find now a rather delicate situation, where the introduction of a rather weak bond randomness on the also weak first-order transition of the pure system produces a dramatic saturating behavior on the specific heat and a new critical behavior.

The generalized (pure) Ising Hamiltonian is described by the following

\[
H_p = J_{nn} \sum_{\langle i,j \rangle} S_i S_j + J_{nnn} \sum_{\langle i,j \rangle} S_i S_j,
\]  
(1)
and the Ising spin system is embedded on a 2d triangular lattice. In terms of the exchange interactions, the pure spin system on the triangular lattice obeys, depending on the interactions, a variety of interesting orderings [23,24], as illustrated by the ground-state phase diagram in figure 1. For the case of positive (antiferromagnetic) nearest- $J_{nn}$ and next-nearest-neighbor $J_{nnn}$ competing interactions, the illustrated layered ground state corresponds to a six-fold degenerate arrangement in which ferromagnetic lines alternate with opposite oriented spins in the three lattice directions [23,24]. We shall refer to the corresponding ordered phase as the superantiferromagnetic (SAF) phase and the corresponding triangular (Tr) model as the TrSAFM. For this case, Rastelli et al. [25] have shown via Monte Carlo (MC) simulations that the system undergoes a first-order phase transition from the SAF phase to a high-temperature paramagnetic phase by studying the ratio of interactions $R = J_{nnn}/J_{nn} = 0.1, 0.5$, and 1. For the particular case $R = 1$ our study [26] verified and improved the results of Rastelli et al. [25], showing that the first-order transition has characteristics that lie between those of the 5- and 6-states Potts model.

We now consider here the random-bond version of this last case by introducing the well-known bimodal distribution of bond disorder, applied on both nearest- and next-nearest-neighbor spins $i$ and $j$

$$P(J_{ij}) = \frac{1}{2}[\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)]; \quad \frac{J_1 + J_2}{2} = 1,$$  

(2)
where the new ratio \( r = J_2/J_1 \) of the weak to the strong bonds denotes the disorder strength and we fix \( 2k_B/(J_1 + J_2) = 1 \) to set the temperature scale. Thus, using the above distribution into Hamiltonian (1), the resulting random-bond version (RBTrSAFM) reads now as

\[
H = \sum_{<i,j>} J_{ij} S_i S_j + \sum_{(i,j)} J_{ij} S_i S_j.
\]  

(3)

In the present study, we will restrict ourselves only on the rather weak value of disorder strength \( r = 0.9/1.1 \), a value, however, that will produce a dramatic saturation of the originally pure model’s \( L^d \) divergence of the specific heat.

The rest of the paper is organized as follows: In the next Section we outline our numerical scheme and in Section 3 we present and discuss our results on the RBTrSAFM. Specifically, in Section 3.1 we illustrate the conversion, under the presence of quenched bond randomness, of the first-order transition of the pure model to second-order and using the Lee-Kosterlitz method [27] we quantitatively verify this conversion. We continue in Section 3.2 by performing a finite-size scaling (FSS) analysis on our data in order to estimate the critical exponents describing the induced continuous transition. Finally, we summarize our conclusions in Section 4.

2 Entropic simulation scheme

Resorting to large scale MC simulations is often necessary and useful [28], especially for the study of the critical behavior of disordered systems. It is also well-known [29] that for such complex systems traditional methods become very inefficient and that in the last few years several sophisticated algorithms, some of them based on entropic iterative schemes, have been proven to be very effective. The present numerical study of the RBTrSAFM will be carried out by applying our recent and efficient entropic scheme [21,30,31]. In this approach we follow a two-stage strategy of a restricted entropic sampling, which is described in our study of random-bond Ising models (RBIM) in 2d [21] and is very similar to the one applied also in our numerical approach to the 3d random-field Ising model (RFIM) [31]. In these papers, we have presented in detail the various sophisticated routes used for the restriction of the energy subspace and the implementation of the Wang-Landau (WL) algorithm [32]. Therefore, only a brief outline will be presented below.

The identification of the appropriate energy subspace \((E_1, E_2)\) for the entropic sampling of each random-bond realization is carried out by applying our critical minimum energy subspace restriction [30] and taking the union subspace...
at both pseudocritical temperatures of the specific heat and magnetic susceptibility. This subspace, extended by 10% from both low- and high-energy sides, is sufficient for an accurate estimation of all finite-size anomalies. Following references [21,31], the identification of the appropriate energy subspace is carried out in the first multi-range (multi-R) WL stage in a wide energy subspace. The WL refinement levels \( G(E) \rightarrow fG(E) \), where \( G(E) \) is the density of states (DOS); for more details see references [21,31]) used in this stage \( (j = 1, \ldots, j; f_{j+1} = \sqrt{f_j}) \) were \( j_i = 19 \) for \( L < 100 \) and \( j_i = 20 \) for \( L \geq 100 \). The same process was repeated several times, typically \( \sim 5 \) times, in the newly identified restricted energy subspace and the average DOS was determined. From our experience, this repeated application of the first multi-R WL approach greatly improves accuracy and then the resulting accurate DOS is used for a final redefinition of the restricted subspace, in which the final second stage is then applied.

In the second entropic stage we accumulate new and more accurate \((E, M)\) histogram data, by implementing now the refinement WL levels \( j = j_i + 1, \ldots, j_i + 4 \), improving also eventually the final DOS. Now, the implementation can be carried out in an one-range (one-R) or in a multi-R fashion. However, our comparative study of the first-order transition features of the 3d RFIM [31] has revealed the need for a careful one-R implementation of the final stage, in order to probe correctly the possible double-peaked (dp) structure of the energy probability density function (PDF). Thus, having to do with the RBTrSAFM and possible remaining characteristics of the dp structure of the pure model, we carried out the final stage of our approach in various ways and verified that both a demanding multi-R (with larger energy pieces) and also an one-R approach were very successful and gave comparable results. Furthermore, we carried out an alternative and very simple one-R approach, in which the WL modification factor was adjusted according to the rule \( \ln f \sim t^{-1} \) proposed recently by Belardinelli and Pereyra [33]. Our comparative tests showed that this last approach gave very good results, for all disorder realizations, and was therefore used for most of our subsequent simulations.

Thus, using this scheme, we performed extensive simulations for several lattice sizes in the range \( L = 20 - 160 \), over large ensembles \( \{1, \ldots, q, \ldots, Q\} \) of random realizations - \( Q_{L \leq 40} = 256 \), \( Q_{L \leq 80} = 512 \), and \( Q_{L \geq 100} = 768 \). It is well-known that, extensive disorder averaging is necessary when studying random systems, especially near ex-first-order transitions, where, as was shown by Fisher [34], extremely broad distributions are expected leading to a strong violation of self-averaging [35,36]. Figure 2 presents evidence that the above number of random realizations is sufficient in order to obtain the true average behavior and not a typical one. In particular, we plot in this figure (for \( L = 100 \)) the disorder distribution of the susceptibility maxima \( \chi_q^* \) and the corresponding running average, i.e. a series of averages of different subsets of the full data set - each of which is the average of the corresponding subset.
of a larger set of data points, over the samples for the simulated ensemble of $Q = 768$ disorder realizations. A first striking observation from this figure is the existence of very large variance of the values of $\chi_q^*$, indicating the expected violation of self-averaging for this quantity. This figure illustrates that the simulated number of random realizations is sufficient in order to probe correctly the average behavior of the system, since already for $Q \approx 400$ the average value of $\chi_q^*$ appears quite stable.

Closely related to the above issue of self-averaging in disordered systems is the manner of averaging over the disorder. This non-trivial process may be performed in two distinct ways when identifying the finite-size anomalies, such as the peaks of the magnetic susceptibility. The first way corresponds to the average over disorder configurations ($[\ldots]_{av}$) and then taking the maxima ($[\ldots]^*_{av}$), or taking the maxima in each individual configuration first, and then taking the average ($[\ldots]^*_{av}$). In the present paper we have undertaken our FSS analysis using both ways of averaging and have found comparable results for the values of the critical exponents, as will be discussed in more detail below.

Closing this Section, let us comment on the statistical errors of our numerical data to be presented in the next section. The statistical errors of our WL scheme, on the observed averaged behavior, were found to be of relatively small magnitude (of the order of the symbol sizes) when compared to the relevant disorder-sampling errors due to the finite number of simulated realizations. Thus, the error bars shown in our figures below, used in the corresponding fitting attempts, reflect the disorder-sampling errors and have been estimated using groups of 32 or 64 realizations via the jackknife method [29].
3 Phase transition of the triangular random-bond SAF model

3.1 Nature of the transition

In the first part of our study we determine the order of the phase transition. This has traditionally been rather difficult for systems showing weak first-order transitions, but finite-size effects at first-order transitions are now much better understood [37,38]. As it is well known from the existing theories of first-order transitions, all finite-size contributions enter in the scaling equations in powers of the system size \( L^d [39] \). This holds for the general shift behavior (for various pseudotransition temperatures) and also for the FSS behavior of the peaks of various energy cumulants and of the magnetic susceptibility. It is also well known that the dp structure of the energy PDF, \( P_L(e) \), where \( e = H/L^d \), is signaling the emergence of the expected two delta-peak behavior in the thermodynamic limit, for a genuine first-order phase transition [40,41], and with increasing lattice size the barrier between the two peaks should steadily increase.

According to the arguments of Lee and Kosterlitz [27] the surface tension \( \Sigma(L) = \Delta F(L)/L^{d-1} = [k_B T \ln (P_{\text{max}}/P_{\text{min}})]/L \), where \( P_{\text{max}} \) and \( P_{\text{min}} \) are the maximum and minimum energy PDF values at the temperature \( T_h \) where the two peaks are of equal height (see figure 3), should tend to a non-zero value. Figure 3(a) shows this pronounced dp structure of the energy PDF of the pure and the random model (for a particular realization of bond disorder) at the corresponding temperatures where the two peaks are of equal height, for a lattice size \( L = 120 \). It appears that the dp structure of the PDF is partly maintained for the RBTrSAFM and relatively small lattices, but it is clear that the introduction of randomness has induced significant softening effects. With the help of figure 3(a) we may define the surface tension \( \Sigma(L) \) and also the width \( \Delta e(L) \) that represents, in the limit \( L \to \infty \), the latent heat of the transition, in the case of a first-order phase transition.

Following Chen et al. [11] we define the disorder averaged surface tension

\[
[\Sigma(L)]_{av} = \frac{1}{Q} \sum_{q=1}^{Q} \Sigma_i(L)
\] (4)

and the corresponding width of the transition

\[
[\Delta e(L)]_{av} = \frac{1}{Q} \sum_{q=1}^{Q} \Delta e_i(L),
\] (5)
3.2 Finite-size scaling - Critical exponents

We proceed now with a quantitative characterization of the induced second-order phase transition of the model, by providing estimates for the critical exponents. Let us start the presentation of our results with the most striking effect of the bond randomness on the specific heat of the TrSAFM. In figure 4(a) we present the average specific heat curves $[C]_{av}$ as a function of temperature for several lattice sizes in the range $L = 40 - 160$. The typical
average curve is a smooth function of temperature, as expected, and its maximum \( [C^*]_{av} \) increases for sizes \( L \leq 80 \) and clearly decreases for sizes \( L > 100 \). The remaining dp structure of the energy PDF [see figure 3(a)] becomes less important for large lattices (the two peaks come closer) and the limiting height will be finite, a behavior in agreement with an expected asymptotic cusp and a negative exponent \( \alpha \). In figure 4(b) we now contrast the size dependence of the specific heat maxima of both the pure and random-bond TrSAFM. For the random case we present two data set points corresponding to the two averaging processes, i.e. the upper curve corresponds to the average of the individual maxima \( [C^*]_{av} \) and the error bars shown are the sample-to-sample fluctuations of this quantity, whereas the lower curve corresponds to the maxima of the averaged curve \( [C^*]_{av} \). From figure 4 the suppression of the specific heat maxima is very clear and the asymptotic saturation of the specific heat seems to be unquestionable. Noteworthy here that, this behavior is similar to the one found in our recent studies of random-bond systems with competing interactions [21,22], but very different from that observed in the cases of the RBIM [21,42,43,44,45,46,47,48,49,50] and the \( q = 8 \) RBPM [11].

The above remarks about the behavior of the specific heat of the RBTrSAFM implicitly suggest a new critical behavior. In order to estimate the critical exponents describing this disorder-induced continuous transition, we will now follow the standard FSS methods applying several alternatives for their estimation and use an appropriate procedure to evaluate the stability of our
Fig. 5. (a) Simultaneous fitting (equation (6) in the range $L = 20 - 160$) of two pseudocritical temperatures, $Z = C$ (filled squares) and $Z = \chi$ (open triangles), giving the values $T_c = 1.742(8)$ and $\nu = 1.135(11)$. The data are shown in a double logarithmic scale. (b) Simultaneous fitting (simple power-law for $L \geq 100$) of the average logarithmic derivatives for $n = 1, 2, 4$ giving the value $\nu = 1.141(9)$. The inset presents values of effective exponents $\nu_{\text{eff}}$ obtained from the data of the pseudocritical temperatures (open reversed triangles) and the logarithmic derivatives (open triangles) corresponding to panels (a) and (b). The solid line in the inset marks the proposed estimate $\nu = 1.135(11)$.

estimates. Such a stability test is useful in cases where one may expect finite-size problems. Such well-known problems may be due to possible cross-over effects, as those discussed by Picco [51] for the case of the RBPM. Also, they may appear in the presence of logarithmic corrections and in the past produced erroneous exponents, as explained by Ballesteros et al. [48] for the case of the 2d site-diluted Ising model. In particular, the determination of the correlation length exponent will be carried out by two alternatives. In the first, we observe and analyze the shift behavior of two pseudocritical temperatures corresponding to the maxima of the average specific heat and magnetic susceptibility $(T[Z]_{\text{av}}^*)$, according to the usual shift relation

$$T[Z]_{\text{av}}^* = T_c + bL^{-1/\nu}. \quad (6)$$

Furthermore, we report here the analogous behavior corresponding to the pseudocritical temperatures $[T[Z]_n^*]_{\text{av}}$ of the individual maxima. In the second alternative we study the divergences of the logarithmic derivatives of some powers of the order parameter with respect to the inverse temperature $K = 1/T$ [11]. The shift behavior of the pseudocritical temperatures corresponding to the
maxima of the average curves is shown in the main panel of figure 5(a). The solid lines in this panel are a simultaneous fitting using data from all lattice sizes \((L = 20 - 160)\), giving a convincing behavior and a value \(T_c = 1.742(8)\) for the critical temperature of the random system, well below the corresponding transition temperature \(T_p^* = 1.8084\) of the pure system [26]. The estimate of the correlation length exponent from the fitting in figure 5(a) is \(\nu = 1.135(11)\), in agreement with the inequality \(\nu \geq 2/d\) for disordered systems of Chayes et al. [52]. A similar simultaneous fitting attempt was performed, but not shown for brevity, to the numerical data for the pseudocritical temperatures \([T_Z^*]_{av}\) that corresponds to the averaging process of the individual maxima. The corresponding estimates are: \(T_c = 1.736(9)\) and \(\nu = 1.139(12)\).

The second alternative estimation of \(\nu\) is carried out by analyzing the divergence of the logarithmic derivatives of the order parameter defined as

\[
\frac{\partial \ln \langle M^n \rangle}{\partial K} = \frac{\langle M^n E \rangle}{\langle M^n \rangle} - \langle E \rangle.
\]  

As is well-known [11], the corresponding maxima scale with the system size as \(\sim L^{1/\nu}\). In panel (b) of figure 5 we illustrate the size dependence of the first- (filled squares), second- (open triangles), and fourth-order (filled stars) maxima of the average logarithmic derivatives. Now, the solid lines show a simultaneous power-law fitting using only the larger lattice sizes \((L \geq 100)\) and provide an estimate \(\nu = 1.141(9)\).

Finally, the inset of figure 5(a) illustrates our method to evaluate and discuss the stability of the estimation for the exponent \(\nu\). It shows values of effective exponents \(\nu_{eff}\) determined by imposing a lower cutoff \((L_{min})\) and applying simultaneous fittings in windows \((L_{min} - L_{max})\), where \(L_{max} = 160\) and \(L_{min} = 20, 40, 60, 80, 100,\) and 120 as a function of \(1/L_{min}\). The effective estimates, from the pseudocritical temperatures (inverse open triangles) are very stable around the value 1.135(11) and their behavior is a clear evidence in favor of the suggested new critical behavior. Their values don’t show any trend or tendency towards the value \(\nu = 1\) of the Ising universality class. On the other hand, the effective estimates from the logarithmic derivatives show a rather strong cross-over effect for small values of the lower cutoff. For larger values a trend for settlement to the value 1.135 (marked by the solid line) may be observed as \(L_{min} \rightarrow 120\). These observations explain also the reasons why we have used in the main panels (a) and (b) of figure 5 different windows for the proposed estimations. In conclusion, we feel that, the emerging from this figure picture rules out the possibility that, our main suggestion, of a new critical behavior, is a result of finite-size or cross-over effects. Of course, there is always some danger that, strong effects, persisting to very large lattices, may produce erroneous results and critical exponents. This fact has been pointed out in several papers dealing with disordered systems [48,51,53].
Fig. 6. (a) Log-log plots of the size dependence of the maxima of the average magnetic susceptibility $[\chi^*]_{av}$ and the average of the individual maxima $[\chi^*]_{av}$. The inset shows the limiting behavior of the ratio $R_{[\chi^*]_{av}}$ defined in the text. (b) Estimation of the magnetic exponent ratio $\beta/\nu$ from the average magnetization at the estimated critical temperature. Linear fittings are applied only on the data of sizes $L \geq 100$.

We turn now to the estimation of the magnetic exponents of the RBTrSAFM. In figure 6(a) we present the FSS behavior of the maxima of the average magnetic susceptibility $[\chi^*]_{av}$ and also the average of the individual maxima $[\chi^*]_{av}$. For the average $[\chi^*]_{av}$ the error bars indicate the statistical errors due to the finite number of the realizations, as discussed in Section 2. In view of this discussion, the relatively small error bars reflect the sufficiency of the disorder-averaging process and also the accuracy of our two-stage implementation of the WL scheme. For the average $[\chi^*]_{av}$ the errors bars shown reflect now the relatively very large sample-to-sample fluctuations. Using these sample-to-sample fluctuations, we construct the ratio $R_\chi = V_\chi/[\chi^2]_{av} = ([\chi^2]_{av} - [\chi^2]_{av}^2)/[\chi^2]_{av}$ and plot it as a function of the inverse linear size, as shown in the inset of figure 6(a). This ratio is a well-known measure of self-averaging [35,36] of the corresponding physical quantity and clearly for the present model the limiting value of $R_\chi$ is non-zero, indicating a strong violation of self-averaging, as mentioned already in Section 2. Following our earlier practice (panel (b) of figure 5) for the logarithmic derivatives, we present now by solid lines in figure 6(a) linear fittings using only the larger lattice sizes ($L \geq 100$), giving the estimates 1.744(9) and 1.740(11) for the ratio $\gamma/\nu$.

Finally, in figure 6(b), also in a double logarithmic scale, we plot the data of the average order parameter at the estimated critical temperature $T_c = 1.742$ of the model, $[M]_{av}(T = T_c)$. The solid line is a linear fitting for the larger lattice sizes giving the value 0.124(8) for the magnetic exponent ratio $\beta/\nu$. 

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The high accuracy of our estimate for the critical temperature, as well as the sufficiency of the performed disorder-averaging are again reflected in the relatively small error bars, especially for the larger sizes where as may be seen the behavior appears to be quite smooth. The above estimations strongly indicate that the ratios $\gamma/\nu$ and $\beta/\nu$ of the RBTTrSAFM may share the values of the pure 2d Ising model. This property is supporting an extensive version of the generalized statement [54,55] of weak universality [56,57] and appears to be also obeyed in some cases of disordered models, including the corresponding random-bond version of the square SAF model [21], as well as the case of the induced second-order phase transition of the originally first-order transition of the 2d Blume-Capel (BC) model [22]. However, for the case of the RBIM the property of strong universality (logarithmic corrections) applies, as clearly shown in references [42,43,44,45,46,47,48,49,50]. Furthermore, as pointed out in the introduction for the corresponding induced second-order phase transitions in the case of the RBPM, this property is ruled out by the study of Cardy and Jacobsen [13], since in this paper a clear $q$ dependence has been shown for the magnetic exponent $\beta/\nu$.

We close this Section by determining the ratio $\alpha/\nu$ via the Rushbrooke relation: $\alpha/\nu = 2/\nu - 2\beta/\nu - \gamma/\nu = -0.23(4)$. As expected, this negative value reflects the saturation property of the specific heat, observed here but also in our similar recent studies [21,22]. Of course, the estimated exponents satisfy here also hyperscaling, although attention should be drawn for possible violations of hyperscaling in random systems [58]. From the above discussion and the corresponding referenced literature we may conclude that, there are still interesting and not settled questions for the origin of the universality properties of the disorder-induced continuous phase transitions and the problem remains of current interest.

4 Conclusions

We have illustrated and quantitatively verified, using the Lee-Kosterlitz method, that even a weak quenched bond randomness converts the first-order phase transition of the TrSAFM to a second-order transition. Thus, our finding strongly supports the theoretical prediction of Refs. [3,4] that quenched bond randomness always affects first-order phase transitions by conversion to second-order phase transitions, for infinitesimal randomness in $d = 2$. The emerging, under random bonds, continuous transition shows a strongly saturating specific heat behavior very similar to the one found in our investigation of the corresponding random-bond version of the square SAF model [21]. It also resembles the recently studied analogous case of the induced second-order phase transition of the originally first-order transition of the 2d BC model [22]. These similarities point toward the delicate nature of the corresponding pure sys-
tem’s phase transitions, due to the competition of the microscopic nearest- and next-nearest-neighbor exchange interactions, or the competition between nearest-neighbor exchange interactions and the crystal field in the case of the 2d BC model. It appears that such competitions are very sensitive to the introduction of quenched bond randomness.

We found that, the emerging second-order transition belongs to a new distinctive universality class with the following critical exponents $\nu = 1.135(11)$, $\gamma/\nu = 1.744(9) \approx 1.75$, and $\beta/\nu \approx 0.124(8) \approx 0.125$. The numerical evidence illustrated in figures 4 and 5 clearly show that the present model is out of the realm of a double logarithmic behavior predicted theoretically and numerically verified for the case of the marginal 2d RBIM [21,42,43,44,45,46,47,48,49,50]. Our conclusion, is in general agreement with the predictions of a distinctive universality class for the emerging second-order transitions [10,59], and the value $\nu = 1.135(11)$ of the correlation length’s exponent is in agreement with the inequality $\nu \geq 2/d$ for disordered systems of Chayes et al. [52].

The estimated magnetic critical exponents support an extensive but weak universality as a wide variety of 2d systems without [56,57] and with [21,22] quenched disorder. However, since in the case of the RBPM this property has been clearly ruled out by the study of Cardy and Jacobsen [13], it will be very interesting to undertake a more extensive numerical study of both square and triangular SAF models. Following Picco [51], such a study may be extended to several disorder strengths in order to observe the FSS behavior for the disorder strength regime which is close to the random fixed point.

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