REMARK ON A CONJECTURE OF SHAFAREVICH

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We will assume several facts from [T, Sections 1, 2] and generalize one proposition from that note [T, Proposition 2.3] which is a special case of Shafarevich’s conjecture (see [K, p. 9, (0.3.1)]).

Proposition. Let $X \subset \mathbb{P}^r$ be a nonsingular connected complex $n$-dimensional projective variety with large residually finite fundamental group $\pi_1(X)$. We will also assume that a general curvilinear section through any point of $X$ is not an elliptic curve. Then the universal covering of $X$, denoted by $U$, is a Stein manifold.

Proof. Let $\mathcal{O}_X(m)$ be a fixed sufficiently very ample line bundle, i.e. $m \gg 0$, and $\mathcal{L} := \mathcal{O}_U(m)$ its pullback. As in [T, (2.2)], we can introduce a Hermitian metric on $\mathcal{O}_X(m)$ as well as a $\pi_1(X)$-invariant Hermitian metric on $\mathcal{L}$. We obtain the corresponding section $B_{U,\mathcal{L}}(z, w)$ holomorphic in $z$ and antiholomorphic in $w$. Thus $B_{U,\mathcal{L}}(z, w)$ is a Hermitian kernel which is not necessary positive, i.e., it may not satisfy the positivity condition [T, 1.4(ii)]. Let $\beta : U \to X$ be the covering map.

1.1. Given $U$, $\mathcal{L}$ and $B_{U,\mathcal{L}}(z, w)$ as above, we will show that $U$ has a natural Kähler metric. Since $B(z, z) := B_{U,\mathcal{L}}(z, z) > 0$, we may consider $\log B(z, z)$. We will show that $\log B(z, z)$ is strictly plurisubharmonic, i.e.,

$$ds_U^2 = 2 \sum g_{jk} dz_j d\bar{z}_k \quad (g_{jk} := \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log B(z, z))$$

determines a Hermitian metric. For an arbitrary real tangent vector at $z \in U$,

$$\mathbf{v} = \partial_a + \bar{\partial}_a \quad (\partial_a := \sum_{t=1}^n a_t \frac{\partial}{\partial z_t}, \quad \bar{\partial}_a := \sum_{t=1}^n \bar{a}_t \frac{\partial}{\partial \bar{z}_t}),$$

we set $|\mathbf{v}|^2 := \sum g_{jk} a_j \bar{a}_k$ hence $|\mathbf{v}|^2 = \partial_a \bar{\partial}_a \log B(z, z)$. The $g_{jk}$’s determine a Hermitian metric iff $|\mathbf{v}|^2 > 0$ for $\mathbf{v} \neq 0$. With the above notation,

$$\partial_a \bar{\partial}_a \log B(z, z) = \frac{1}{B(z, z)^2} \left\{ \sum_k |\mathbf{v}\psi_k(z)|^2 \cdot \sum_k |\psi_k(z)|^2 - \sum_k \mathbf{v}\psi_k(z) \cdot \overline{\psi_k(z)} \right\}$$

$$\geq \frac{1}{B(z, z)^2} \left\{ \sum_k |\mathbf{v}\psi_k(z)|^2 \cdot \sum_k |\psi_k(z)|^2 - \left( \sum_k |\mathbf{v}\psi_k(z) \cdot \overline{\psi_k(z)} \right)^2 \right\}$$

where $B(z, z) = \sum \psi_k(z) \overline{\psi_k(z)}$ in a neighborhood of $z$ (the calculations are similar to [FK, pp. 13 –14, pp. 190 –191]). In the Lagrange identity

$$\sum_{k=1}^s A_k^2 \sum_{k=1}^s B_k^2 - \left( \sum_{k=1}^s A_k B_k \right)^2 = \frac{1}{2} \sum_{i=1}^s \sum_{j=1}^s (A_i B_j - B_i A_j)^2 \quad (\forall A_i, \forall B_i \in \mathbb{R}_+, \ s \in \mathbb{N}),$$

the right-hand side is strictly positive for $s = \infty$, provided it is strictly positive for some $s \in \mathbb{N}$ (and the series are convergent). It follows that $|\mathbf{v}|^2 > 0$ for $\mathbf{v} \neq 0$, and we get the desired metric.
1.2. Set \( \Phi := \log B \). Let \( p \) and \( q \) be two points of \( U \) with coordinates \((z(p))\) and \((z(q))\). We assume \( p \) and \( q \) belong to a small open subset \( V \subset U \). The functional element of diastasis of the above metric is defined as follows (Calabi [C, (5)]):

\[
D_U(p, q) := \Phi(z(p));(z(p)) + \Phi(z(q));(z(q)) - \Phi(z(p));(z(q)) - \Phi(z(q));(z(p)).
\]

It is strictly plurisubharmonic in \((z(q))\) for a fixed \( p \), and inductive on holomorphic submanifolds of a manifold with analytic Kahler metric [C, Propositions 4 and 6].

For a fixed \( p \in U \), \( f(q) := D_U(p, q) \) is a unique primitive function, defined in some neighborhood of \( p \), that satisfies the vanishing of the corresponding partial derivatives [C, Chap. 2], [U, Appendix].

To conclude the proof of our proposition, it suffices to show that the \( f(q) \) generates a single-valued strictly plurisubharmonic function \( P_U(x) \) on \( U \) such that, for any real number \( c \), the set \( U_c := \{ x \in U | P_U(x) < c \} \) is relatively compact in \( U \). Then \( U \) will be Stein by the Oka-Grauert-Narasimhan solution of the Levi problem.

1.3. We will show that a functional element \( f(q) = D_U(p, q) \) can be prolongated to a single-valued function along a path in \( U \) originating in \( p \) so that we get the corresponding functional element at each point of the path.

First, any holomorphic bundle on a noncompact Riemann surface is trivial. Second, given a finite number of points on \( U \), as in [T, Remark 2.2.1] we can assume they lie on a connected noncompact Riemann surface \( Y \subset U \) which is a Galois covering of a curvilinear section of \( X_\gamma \) \((\gamma \gg 0, \gamma \text{ depends on the points})\) by Deligne’s theorem [K, Theorem 2.14.1] since \( \pi_1(X) \) is large. Thus, if a path \( \alpha \subset Y \) then the functional element \( f(q) \) can be prolonged along \( \alpha \).

Finally, for an arbitrary path \( \alpha \subset U \) with initial point \( p_0 \) and end point \( p_1 \), one can choose an arbitrary finite number of points on \( \alpha \). As above, we obtain a connected noncompact Riemann surface \( Y \subset U \) passing through \( p_0, p_1 \), and the chosen points. We get the desired prolongation along \( \alpha \). Since \( U \) is simply connected, the functional element \( f(q) \) generates a function \( P_U(x) \) on the whole \( U \).

1.4. Since \( X \) is compact, to establish the relative compactness of \( U_c \) (see 1.2), it suffices to show \( |U_c \cap \beta^{-1}(x)| < \infty \) for all \( x \in X \). The inequality is trivial if \( \dim X = 1 \). Suppose to the contrary \( |U_c \cap \beta^{-1}(x)| = \infty \) for some \( x \in X \) with \( \dim X > 1 \).

We take a general curvilinear section \( C_x \) through \( x \) and obtain the covering \( \nu : \Delta \to S \), where \( S := \beta^{-1}(C_x) \) and \( \Delta \) is a disk. The corresponding \( P_\Delta \) (arising from the Poincaré-Bergman metric) goes to infinity on any infinite discrete subset in \( \Delta \). Furthermore, the relative Poincaré map transforms \( P_\Delta \) into \( P_S \). We derive a contradiction at once from the classical (integral) lifting formula for automorphic functions, i.e., \( P_S \) lifts to \( P_\Delta \) and the relative Poincaré map transforms \( P_\Delta \) back into \( P_S \).

This concludes the proof of the proposition.

References

[C] E. Calabi, Isometric imbedding of complex manifolds, Ann. of Math. 58 (1953), 1–23.

[FK] J. Faraut, S. Koneyuki, A. Korányi, Q.-k. Lu, G. Roos, Analysis and geometry on complex homogeneous domains, Birkhäuser, Boston, 2000.

[K] J. Kollár, Shafarevich maps and automorphic forms, Princeton Univ. Press, Princeton, 1995.

[T] R. Treger, Uniformization, arXiv:math.AG/1001.1951.

[U] M. Umehara, Kaehler Submanifolds of Complex Space Forms, Tokyo J. Math. 10 (1987), 203–214.

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