Weighted Gaussian entropy and determinant inequalities

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Abstract

We produce a series of results extending information-theoretical inequalities (discussed by Dembo–Cover–Thomas in 1989-1991) to a weighted version of entropy. The resulting inequalities involve the Gaussian weighted entropy; they imply a number of new relations for determinants of positive-definite matrices.

1 Introduction

The aim of this paper is to give a number of new bounds involving determinants of positive-definite matrices. These bounds can be considered as generalizations of inequalities discussed in \cite{2, 5}. A common feature of determinant inequalities (DIs) from \cite{2, 5} is that most of them have been previously known but often proven by individual arguments (see the bibliography in \cite{2, 5}). The unifying approach adopted in \cite{2, 5} emphasized their common nature connected with/through information-theoretical entropies.

The bounds presented in the current paper are also obtained by a unified method which is based on weighted entropies (WEs), more precisely, on Gaussian WEs. Hence, we speak here of weighted determinant bounds/inequalities. The weighted determinant inequalities (WDIs) offered in the present paper are novel, at least to the best of our knowledge. Moreover, when we choose the weight function to be a (positive) constant, a WDI become a ‘standard’ DI. In fact, the essence of this work is that we subsequently examined DIs from \cite{2, 5} for a possibility of a (direct) extension to non-constant weight functions; successful attempts formed the present paper. This reflects a particular feature of the present paper: a host of new inequalities are obtained by an old method while \cite{2, 5} re-establish old inequalities by using a new method.

As a primary example, consider the so-called Ky Fan inequality. (We follow the terminology used in \cite{2, 5, 3}.) This inequality asserts that \(\delta(C) := \log \det C\) is a concave function of a

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positive-definite $d \times d$ matrix $C$. In other words, for all strictly positive-definite $d \times d$ matrices $C_1, C_2$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$,

$$
\delta(\lambda_1 C_1 + \lambda_2 C_2) - \lambda_1 \delta(C_1) - \lambda_2 \delta(C_2) \geq 0; \text{ equality iff } \lambda_1 \lambda_2 = 0.
$$

For original ‘geometric’ proofs of (1.1) and other related inequalities, see Ref [8] and the bibliography therein. In [2, 5, 3] the derivation of (1.1) occupies few lines and is based on the fact that under a variance constraint, the differential entropy is maximized at a Gaussian density.

A weighted Ky Fan inequality (1.2) has been proposed in [9, Theorem 3.2]; the derivation is also short and based on a maximization property of the weighted entropy (cf. Theorem 3.1 below). Namely, given $C_1, C_2$ and $\lambda_1, \lambda_2$ as above and a nonnegative function $x \in \mathbb{R}^d \mapsto \phi(x)$, positive on an open domain in $\mathbb{R}^d$, assume condition (1.6). Then

$$
\sigma(\lambda_1 C_1 + \lambda_2 C_2) - \lambda_1 \sigma(C_1) - \lambda_2 \sigma(C_2) \geq 0; \text{ equality again iff } \lambda_1 \lambda_2 = 0.
$$

Here, for a strictly positive-definite $C$, the value $\sigma(C) = \sigma_\phi(C)$ is as follows:

$$
\sigma_\phi(C) = \frac{\alpha_\phi(C)}{2} \log \left[ (2\pi)^d (\det C) \right] + \frac{1}{2} \log \left[ \exp \left( \int C^{-1} \Phi_{C, \phi} := h_\phi^w(f_C No). \right. \right.
$$

Next, $\alpha_\phi(C) > 0$ and positive-definite matrix $\Phi_{C, \phi}$ are given by

$$
\alpha_\phi(C) = \int_{\mathbb{R}^d} \phi(x_1^T) f_C^{No}(x_1^T) dx_1^T, \quad \Phi_{C, \phi} = \int_{\mathbb{R}^d} x_1^T (x_1^T)^T \phi(x_1^T) f_C^{No}(x_1^T) dx_1^T,
$$

and $f_C^{No}$ stands for a normal probability density function (PDF) with mean $0$ and covariance matrix $C$:

$$
f_C^{No}(x) = \frac{1}{(2\pi)^{d/2}(\det C)^{1/2}} \exp \left( -\frac{1}{2} x^T C^{-1} x \right), \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^d.
$$

In terms of a multivariate normal random vector $X_1^d \sim f_C^{No}$: $\alpha_\phi(C) = \mathbb{E}(\phi(X_1^T))$ and $\Phi_{\phi, C} = \mathbb{E} \left( \phi(X_1^T) (X_1^T)^T \right)$. In (1.5) and below we routinely omit the indices in the notation like $x_1^T$ and $X_1^d$. The quantity $h_\phi^w(f_C^{No}) = -\int_{\mathbb{R}^d} \phi(x) f_C^{No}(x) \log f_C^{No}(x) dx$ is the weighted entropy of $f_C^{No}$ with weight function $\phi$, a concept analyzed in detail below. For $\phi(x) \equiv 1$, $h_\phi^w(f_C^{No})$ coincides with a ‘standard’ (differential) entropy of a normal PDF.

The assumption upon $C_1, C_2$ and $\lambda_1, \lambda_2$ consists of two bounds and reads

$$
\begin{align*}
\lambda_1 \alpha(C_1) + \lambda_2 \alpha(C_2) &- \alpha(\lambda_1 C_1 + \lambda_2 C_2) \geq 0, \\
\left[ \lambda_1 \alpha(C_1) + \lambda_2 \alpha(C_2) - \alpha(\lambda_1 C_1 + \lambda_2 C_2) \right] \times \log \left( (2\pi)^d \left[ \det (\lambda_1 C_1 + \lambda_2 C_2) \right] \right) + \text{tr} \left[ (\lambda_1 C_1 + \lambda_2 C_2)^{-1} \Delta \right] &\leq 0.
\end{align*}
$$

where matrix $\Delta = \lambda_1 \Phi_{C_1} + \lambda_2 \Phi_{C_2} - \Phi_{\lambda_1 C_1 + \lambda_2 C_2}$. Bounds (1.6) have opposite directions and stem from the weighted Gibbs inequality. Cf. Eqns (1.3), (3.3) from Ref [9] and (3.1), (3.5) from Section 3 below.
When $\phi(x) \equiv 1$, Eqn (1.6) is satisfied: we have equalities. In this case the weighted Ky Fan inequality (1.2) transforms into (1.1). In general, condition (1.6) is not trivial: in a simplified case of an exponential weight function $\phi(x) = \exp(t^T x)$, $t \in \mathbb{R}^d$, it has been analyzed, both analytically and numerically, in [10]. (Here, $\phi(x) \equiv 1$ means $t = 0$.) As was shown in [10], for given $C_1$, $C_2$, $\lambda_1$, $\lambda_2$ and $\phi$ (that is, for a given $t$), Eqn (1.6) may or may not be fulfilled. (And when (1.6) fails, (1.2) may still hold true.) Moreover, when (1.6) holds, it may or may not produce a strictly positive expression in the RHS of bound (1.1). (Thus, in some cases we can speak of an improvement in the Ky Fan inequality.) See Ref [10]. We believe that further studies in this direction should follow, focusing on specific forms of weight function $\phi$.

In our opinion, this paper paves way to a similar analysis of the whole host of newly established WDIs. These inequalities should be taken with a justified degree of caution: offered sufficient conditions (stated in the form of bounds involving various weight function) may fail for particular $C_1$, $C_2$, $\lambda_1$, $\lambda_2$, and $\phi$, and a given WDI may or may not yield an improvement compared to its ‘standard’ counterpart. For reader’s convenience we list the sufficient conditions figuring across the paper: Eqns (2.8), (2.20), (3.1), (3.5), (4.1), (4.4), (4.7), (4.10), (5.3), (5.12), (5.20), (5.24), (6.3), (6.11), (6.12) and (6.17).

The presented WDIs generalize what is sometimes called elementary information-theoretic inequalities. An opposite example is the entropy-power inequality; and related bounds. Here the intuition is more intricate; some initial results have been proposed in [11].

The paper is organized as follows. In Section 2 we work with a general setting, elaborating on properties of weighted entropies which have been established earlier in [9]. Section 3 summarizes some properties of Gaussian weighted entropies while Section 4 analyzes the behavior of weighted entropies under mappings; these sections also rely on Ref. [9]. The WDIs are presented in Sections 5 and 6 as a sequel to the material from Section 2 – 4. Again, for reader’s convenience we list them here as Eqns (5.4), (5.13), (5.14), (5.21), (5.25), (5.26), (5.3), (6.5), (6.14) and (6.18).

## 2 Random strings and reduced weight functions

The WE of a probability distribution was introduced in late 1960s – early 1970s; see, e.g., [1]. (Another term that can be used is a context-dependent or a preferential entropy.) The reader is referred to [9] where a number of notions and elementary inequalities were established for the WE, mirroring well-known facts about the standard (Shannon) entropy. We also use Refs [2, 5] as a source of standard inequalities which we extend to the case of the WE. To keep pre-emptiveness, we follow the system of notation from [2, 5, 9] with minor deviations.

Let us begin with general definitions. The WE of a random element $X$ taking values in a standard measure space (SMS) $(\mathcal{X}, \mathfrak{M}, \nu)$ with a weight function (WF) $x \in \mathcal{X} \mapsto \phi(x) \geq 0$ is defined by

$$h_{\phi}^\nu(X) = h_{\phi}^\nu(f) = \mathbb{E}(\phi(X) \log f(X)) = -\int_{\mathcal{X}} \phi(x) f(x) \log f(x) \nu(dx),$$

(2.1)

assuming that $\phi$ is measurable and the integral is absolutely convergent. Here $f = f_X$ is the probability mass/density function (PM/DF) of $X$ relative to measure $\nu$. Symbol $\mathbb{E}$ stands for the expected value (relative to a probability distribution that is explicitly specified or emerges from the context in an unambiguous manner).
A number of properties of the WE are related to a Cartesian product structure. Let random elements \(X_1, \ldots, X_n\) be given, taking values in SMSs \((\mathcal{X}_i, \mathbb{M}_i, \nu_i), 1 \leq i \leq n\). Set \(X^n := \{X_1, \ldots, X_n\}\) and assume that \(X_1, \ldots, X_n\) have a joint PM/DF \(f_{X^n}(x^n), x^n \in X^n := \times_{1 \leq i \leq n} \mathcal{X}_i\), relative to the measure \(\nu^n := \times_{1 \leq i \leq n} \nu_i\); for brevity we will sometimes set \(f_{X^n} = f\). The joint WE of string \(X^n\) is defined as

\[
h^n_w(X^n) = -E(\phi(X^n) \log f(X^n)) = -\int_{X^n} \phi(x^n) f(x^n) \log f(x^n) \nu^n(dx^n). \tag{2.2}
\]

Given a set \(S \subseteq I := \{1, 2, \ldots, n\}\), write

\[
\overline{X}(S), \overline{X}(S^c) \quad \text{for strings} \quad \{X_i : i \in S\}, \quad \{X_i : i \in S^c\}, \quad \text{respectively, where} \quad S^c = I \setminus S. \tag{2.3}
\]

Next, let \(\overline{x}(S)\) and \(\overline{x}(S^c)\) stand for

\[
\{x_i : i \in S\} \in \mathcal{X}(S) := \times_{i \in S} \mathcal{X}_i \quad \text{and} \quad \{x_i : i \in S^c\} \in \mathcal{X}(S^c) := \times_{i \in S^c} \mathcal{X}_i. \tag{2.4}
\]

Accordingly, the marginal PD/MF \(f_{X^i}(\overline{x}(S))\) emerges, for which we will often write \(f_S(\overline{x}(S))\) or even \(f(\overline{x}(S))\) for short. Furthermore, given a WF \(x^n \mapsto \phi(x^n) \geq 0\), we define the function \(\psi(S) : \overline{x}(S) \mapsto \psi(S; \overline{x}(S)) \geq 0\) involving the conditional PM/DF \(f_{\overline{X}(S^c)|\overline{X}(S)}(\overline{x}(S^c)|\overline{x}(S))\):

\[
\psi(S; \overline{x}(S)) = \int_{\mathcal{X}(S^c)} \phi(x^n) f_{\overline{X}(S^c)|\overline{X}(S)}(x(S^c)|\overline{x}(S)) \nu_{\overline{X}(S^c)}(dx(S^c)) \tag{2.5}
\]

where \(\nu_{\overline{X}(S^c)} := \times_{i \in S^c} \nu_i\). For brevity we again write sometimes \(f_{S|\overline{X}(S)}\) instead of \(f_{\overline{X}(S^c)|\overline{X}(S)}\) or omit subscripts altogether. We also write \(d_x(S)\) and \(d_x(S^c)\) instead of \(\nu_{\overline{X}(S)}(d_x(S))\) and \(\nu_{\overline{X}(S^c)}(d_x(S^c))\) and \(d_x\) instead of \(\nu^n(dx^n)\).

Function \(\psi(S; \cdot)\) will play the role of a reduced (or induced) WF when we pass from \(X^n\) to a sub-string \(X(S)\). More precisely, set

\[
h_{\psi(S)}^w(\overline{X}(S)) = -E(\psi(S; X(S) \log f_S(X; \overline{X}(S)))) = -\int_{\mathcal{X}(S)} \psi(S; \overline{x}(S)) f_S(\overline{x}(S)) \log f_S(\overline{x}(S)) d_x(S), \tag{2.6}
\]

with \(\nu_{\overline{X}(S)} := \times_{i \in S} \nu_i\). Cf. [9]. Next, for \(k = 1, \ldots, n\) define

\[
h^{w,n}_k = \binom{n}{k}^{-1} \sum_{S \subseteq I: \#(S) = k} h^{w}_\psi(\overline{X}(S)). \tag{2.7}
\]

(Here and below, \(#(S)\) and \(#(S^c)\) are the cardinalities of \(S\) and \(S^c\).) Here \(h^{w,n}_k\) renders the averaged WE (per string and per element) of a randomly drawn \(k\)-element sub-string in \(X^n\).
In what follows we use the concepts of the conditional and mutual WE and their properties; cf. [9]. These objects are used with a host of WFs, depending on the context. Consider the following condition:

\[ \forall i \in S \subseteq I, \text{ with } S_i^- = \{ j \in S : j < i \} \text{ and } S_i^+ = \{ j \in S : j > i \}, \]

\[ \int_{X(S)} \psi(S; x(S)) \left\{ f(x(S)) - f(x(S_i^-)) \times \left[ f(x_i|x(S_i^-)) f(x(S_i^+)|x(S_i^-)) \right] \right\} \, dx(S) \geq 0, \quad (2.8) \]

with standard agreements when one of the sets \( S_i^\pm = \emptyset \). Pictorially, Eqn (2.8) is an extension of bound (1.27) from [9]; it means that for all \( i \in S \subseteq I \), the induced WF \( \psi(S; \cdot) \) is correlated positively with the marginal PM/DF \( f_S(x(S)) \) than with the dependence-product \( f_{S_i^-}(x(S_i^-)) \times \left[ f(x_i|x(S_i^-)) f(x(S_i^+)|x(S_i^-)) \right] \). Another version of (essentially) the same property is Eqn (2.20) below.

**Remark 2.1** The special choice of sets \( S_i^\pm \) is not particularly important: it can be a general partition of \( S \setminus \{ i \} \) allowing us to use the chain rule for the conditional WE (see below).

**Theorem 2.2** (Cf. [2], Lemma 7 or [5], Theorem 1.) Let \( h_{k,n}^{w,n} \) be defined as in (2.7) and assume (2.8). Then

\[ h_{1,n}^{w,n} \geq h_{2,n}^{w,n} \geq \ldots \geq h_{n-1,n}^{w,n} \geq h_{n,n}^{w,n}. \quad (2.9) \]

**Proof.** Begin with the last inequality, \( h_{n-1,n}^{w,n} \geq h_{n,n}^{w,n} \). Let \( 1 \leq i \leq n \) and choose \( S = I \), \( S_i^- = I_i^- := \{ 1, \ldots, i-1 \} \) and \( S_i^+ = I_i^+ := \{ i+1, \ldots, n \} \), with \( \{ i \} \subseteq I_i^- \cup I_i^+ \) (cf. (2.3), (2.4)). Then the condition

\[ \int_{(X^n)^1} \phi(x) \left[ f(x) - f_{X_i-1}(x_{i-1}) f(x_i|x_{i-1}) f(x_{i+1}|x_{i-1}) \right] \, dx \geq 0 \quad (by \ virtue \ of \ (2.8)), \]

yields:

\[ h_{\phi}^w(X_i^1) = h_{\phi}^w(X_i|X(\{ i \})^C) + h_{\phi(\{ i \})^C}^w(X(\{ i \})^C) \quad \text{by the chain rule} \]

\[ \leq h_{\phi(\{ i \})^C}^w(X_i|X_{i-1}^{i-1}) + h_{\phi(\{ i \})^C}^w(X(\{ i \})^C) \quad \text{by Lemma 1.3 from [9].} \]

Here reduced WFs \( \psi(\{ i \})^C \) and \( \psi(I_i^-) \) are calculated according to the recipies in (2.5), (2.6).

Taking the sum, we obtain:

\[ n h_{\phi}^w(X^n) \leq \sum_{i=1}^{n} h_{\phi(\{ i \})^C}^w(X(\{ i \})^C) + \sum_{i=1}^{n} h_{\phi(I_i^-)}^w(X_i|X_{i-1}^{i-1}). \quad (2.10) \]

By using the chain rule, \( \sum_{i=1}^{n} h_{\phi(I_i^-)}^w(X_i|X_{i-1}^{i-1}) = h_{\phi}^w(X^n) \). Hence, Eqn (2.10) becomes

\[ (n-1)h_{\phi}^w(X^n) \leq \sum_{i=1}^{n} h_{\phi(\{ i \})^C}^w(X(\{ i \})^C). \]
Consequently,
\[ h_w^\phi(X^n_1) \leq \sum_{i=1}^{n} h_w^\phi(i) \frac{(X(i))}{n - 1}, \]  
(2.11)

which yields that \( h_{n-1}^{w,n-1} \geq h_n^{w,n} \).

This argument can be repeated if we restrict the WE and the PM/DF to a \( k \)-element subset \( S = \{i_1, \ldots, i_k\} \subset I \) listed in an increasing order of its points and perform a uniform choice over its \((k-1)\)-elements subsets. Condition \((2.8)\) yields the bound
\[
\frac{1}{k} h_w^\psi(S)(X(S)) \leq \frac{1}{k} \sum_{i \in S} h_w^\psi(S \setminus \{i\})(X(S \setminus \{i\})) k - 1.
\]

Hence for each \( k \)-element subset, \( h_w^{k,k} \geq h_k^{w,k} \). Therefore, the inequality remains true after taking the average over all \( k \)-element subsets drawn uniformly.

In Theorem 2.3 we extend the result of Theorem 2.2 to exponents of WEs for sub-strings in \( X^n \).

**Theorem 2.3** (Cf. [2], Corollary of Lemma 7 or [5], Corollary 1) Given \( r > 0 \), define:
\[
g_w^{\psi,n} = \left(\frac{n}{k}\right)^{-1} \sum_{S \subseteq I: \#(S) = k} \exp \left[ r \frac{h_w^\psi(X(S))}{k} \right]. \tag{2.12}
\]

Then, under assumption \((2.8)\),
\[
g_1^{w,n} \geq g_2^{w,n} \geq \cdots \geq g_{n-1}^{w,n} \geq g_n^{w,n}. \tag{2.13}
\]

**Proof.** Again, it is convenient to start with the last bound in \((2.13)\). As in [2], multiply Eqn \((2.11)\) by \( r \), exponentiate and apply the arithmetic–geometric mean inequality to obtain \( g_{n-1}^{w,n} \geq g_n^{w,n} \). The result is then completed with the help of same argument as in the proof of Theorem 2.2.

In Theorem 2.4 we analyse the averaged conditional WEs for sub-strings in \( X^n \).

**Theorem 2.4** (Cf. [5], Theorem 2.) Let \( p_k^{w,n} \) be defined as
\[
p_k^{w,n} = \left(\frac{n}{k}\right)^{-1} \sum_{S \subseteq I: \#(S) = k} h_w^\phi(X(S)|X(S^c)) \tag{2.14}
\]

Then under the assumption
\[
\int_{X^n_i} \phi(x) \left[ f(x) - \prod_{i=1}^{n} f(x_i) \right] dx \geq 0 \tag{2.15}
\]
we have that
\[
p_1^{w,n} \leq p_2^{w,n} \leq \cdots \leq p_{n-1}^{w,n} \leq p_n^{w,n}. \tag{2.16}
\]
Proof. Following the argument used in [9], Theorem 3.1, condition (2.15) yields
\[ h^w_\phi(X^n) \leq \sum_{i=1}^{n} h^w_\psi(i|x_i)(X_i). \]
Subtracting both sides from \( nh^w_\phi(X^n) \), we obtain:
\[ (n-1)h^w_\phi(X^n) \geq \sum_{i=1}^{n} \left[ h^w_\phi(X^n) - h^w_\psi(i|x_i)(X_i) \right], \]
By the conditional WE definition,
\[ h^w_\phi(X^n) = h^w_\phi(X^n|X_{i-1},X_{i+1}|X_i) + h^w_\psi(i|x_i)(X_i). \]
Hence,
\[ (n-1)h^w_\phi(X^n) \geq \sum_{i=1}^{n} h^w_\phi(X^n|X_{i-1},X_{i+1}|X_i). \]
Dividing (2.17) by \( n(n-1) \) yields that \( p^n_{w,n} \leq p^n_{w,n} \). Finally, applying the same argument as in Theorem 2.2 completes the proof.

The next step is to pass to mutual WEs.

Theorem 2.5 (Cf. [5], Corollary 2.) Consider the averaged mutual WE between a subset (or a sub-string) and its complement:
\[ q^w_{k,n} = \left( \frac{n}{k} \right)^{-1} \sum_{S\subseteq I: \#(S)=k} \frac{i^w_\phi(X(S):X(S^c))}{k}, \]
and assume (2.8). Then
\[ q^w_{1,n} \geq q^w_{2,n} \geq \ldots \geq q^w_{n-1,n} \geq q^w_{n,n}. \]

Proof. The result is straightforward, from Theorems 2.2 and 2.3 and the following relation between conditional and mutual WEs:
\[ i^w_\phi(X(S):X(S^c)) = h^w_\psi(S)(X(S)) - h^w_\phi(X(S)|X(S^c)). \]

In Theorem 2.6 we consider the following condition: for all set \( S \) with \( \#S \geq 2 \) and \( i,j \in S \) with \( i \neq j \),
\[ \int_{X^n_1} \phi(x) f(x(S^c)|x(S)) \left[ f(x(S)) - f(x(S \setminus \{i,j\})) f(x_i|x(S \setminus \{i,j\})) f(x_j|x(S \setminus \{i,j\})) \right] dx \geq 0. \]

The meaning of (2.20) is that for all \( S \) and \( i,j \) as above, the reduced WF \( \psi_S(x(S)) \) is correlated more positively with \( f(x(S)) \) than with the PM/DF \( f(x(S \setminus \{i,j\})) f(x_i|x(S \setminus \{i,j\})) f(x_j|x(S \setminus \{i,j\})) \) where the conditional dependence between \( X_i \) and \( X_j \) is broken, given \( X(S \setminus \{i,j\}) \).
Theorem 2.6 (Cf. [5], Theorem 3.) Define the average mutual WE as

\[ I_k^{w,n} = \left( \frac{n}{k} \right)^{-1} \sum_{S \subseteq I: \#(S) = k} i^w_\phi \left( X(S) : X(S^c) \right). \]  

(2.21)

By symmetry of the mutual WE, \( I_k^{w,n} = I_{n-k}^{w,n} \). Assume condition (2.20). Then

\[ I_1^{w,n} \leq I_2^{w,n} \leq \ldots \leq I_{\lfloor n/2 \rfloor}^{w,n}. \]  

(2.22)

Proof. Let \( k \leq \lfloor n/2 \rfloor \). If \( S \) is a subset of size \( k \) then \( S + \) has \( k \) subsets of size \( k - 1 \). Thus, we write:

\[
k i^w_\phi \left[ X(S) : X(S^c) \right] - \sum_{j \in S} i^w_\phi \left[ X(S_j) : X((S_j)^c) \right] \\
= \sum_{j \in S} \left\{ i^w_\phi \left[ X(S_j), X_j : X(S^c) \right] - i^w_\phi \left[ X(S_j) : (X(S^c), X_j) \right] \right\}.
\]

After direct computations, we obtain:

\[ i^w_\phi \left[ X(S_j), X_j : X(S^c) \right] = i^w_{\psi(S_j \cup S^c)} \left[ X(S_j) : X(S^c) \right] + i^w_\phi \left[ X_j : X(S^c) | X(S_j) \right], \]

and

\[ i^w_\phi \left[ X(S_j) : (X(S^c), X_j) \right] = i^w_{\psi(S_j \cup S^c)} \left[ X(S_j) : X(S^c) \right] + i^w_\phi \left[ X_j : X(S_j) | X(S^c) \right]. \]

Here \( i^w_\phi \left[ X_j : X(S^c) | X(S_j) \right] \), \( i^w_\phi \left[ X_j : X(S_j) | X(S^c) \right] \) are mutual-conditional WEs emerging as in the proof of Theorem 3 from [3]:

\[
i^w_\phi \left[ X_j : X(S^c) | X(S_j) \right] = \mathbb{E} \left( \phi(X) \log \frac{f(X_j, X(S^c) | X(S_j))}{f(X_j | X(S_j)) f(X(S^c) | X(S_j))} \right) \\
= \int_{X_1^c} \phi(x) \frac{f(x_j, x(S^c) | x(S_j))}{f(x_j | x(S_j)) f(x(S^c) | x(S_j))} \, dx, \tag{2.23}
\]

and

\[
i^w_\phi \left[ X_j : X(S_j) | X(S^c) \right] = \mathbb{E} \left( \phi(X) \log \frac{f(X_j, X(S_j) | X(S^c))}{f(X_j | X(S^c)) f(X(S_j) | X(S^c))} \right) \\
= \int_{X_1^c} \phi(x) \frac{f(x_j, x(S_j) | x(S^c))}{f(x_j | x(S^c)) f(x(S_j) | x(S^c))} \, dx. \tag{2.24}
\]

In the remaining argument we will make an extensive use of definition (2.5), employing WF \( \psi(S) \) for a number of choices of set \( S \).
Using mutual-conditional WEs we can write:

\[
\begin{align*}
  k \sum_{i=1}^{n} i_{\phi}^{w} \left[ X(S) : X(S_{i}^{C}) \right] - \sum_{j \in S} \left( i_{\phi}^{w} \left[ X(S_{j}) : X((S_{j})^{C}) \right] \right) \\
  = \sum_{j \in S} \left\{ i_{\phi}^{w} \left[ X_{j} : X(S_{j}^{C}) \right] + i_{\phi}^{w} \left[ X_{j} : X(S_{j}) \right] \right\} \\
  = \sum_{j \in S} \left( h_{\psi(S)}(X_{j} | X(S_{j})) - h_{\psi(S)}(X_{j} | X(S_{j})^{C}, X(S_{j}^{C})) \right) \\
  \quad \quad - h_{\psi(j \cup S_{j})}(X_{j} | X(S_{j})) - h_{\psi(j \cup S_{j})}(X_{j} | X(S_{j})^{C}) \\
  = \sum_{j \in S} \left[ h_{\psi(S)}(X_{j} | X(S_{j})) - h_{\psi(j \cup S_{j})}(X_{j} | X(S_{j})) \right]. 
\end{align*}
\]

(2.25)

Summing over all subsets of size \( k \) and reversing the order of summation, we obtain:

\[
\sum_{S \subseteq I : \#(S) = k} \left\{ k \sum_{i=1}^{n} i_{\phi}^{w} \left[ X(S) : X(S_{i}^{C}) \right] - \sum_{j \in S} \left( i_{\phi}^{w} \left[ X(S_{j}) : X((S_{j})^{C}) \right] \right) \right\} \\
= \sum_{j=1}^{n} \sum_{S \subseteq I : \#(S) = k-1, j \notin S} \left[ h_{\psi(S \cup j)}(X_{j} | X(S')) - h_{\psi(S \cup j)}(X_{j} | X(S' \cup j)) \right], \\
\]

(2.26)

The RHS of (2.26) can be rewritten in the following way:

\[
\sum_{j=1}^{n} \sum_{S' : \#(S') = k-1, j \notin S} \left[ h_{\psi(S' \cup j)}(X_{j} | X(S')) - h_{\psi(S' \cup j)(S' \cup j)}(X_{j} | X((S' \cup j)^{C})) \right], \\
\]
or equivalently

\[
\sum_{j=1}^{n} \sum_{S' : \#(S') = k-1, S' \subseteq (j)^{C}} \left[ h_{\psi(S' \cup j)}(X_{j} | X(S')) - \sum_{S'' : \#(S'') = n-k, S'' \subseteq (j)^{C}} h_{\psi(S'' \cup j)}(X_{j} | X(S'')) \right]. \\
\]

Since \( k \leq \lfloor n/2 \rfloor \), then \( k-1 < n - k \). A set \( S'' \) with \( n - k \) elements has \( \binom{n-1}{k-1} \) subsets of size \( k-1 \). Owing to Lemma 1.3 from [9], for each such subset \( \tilde{S} \subset S'' \), under assumption (2.20) we have that

\[
\sum_{j=1}^{n} \sum_{S' : \#(S') = k-1, S' \subseteq (j)^{C}} \left[ h_{\psi(S' \cup j)}(X_{j} | X(S')) - \sum_{S'' : \#(S'') = n-k, S'' \subseteq (j)^{C}} h_{\psi(S'' \cup j)}(X_{j} | X(S'')) \right] \leq 0. \\
\]

(2.27)

With the same argument as in [5] we conclude from (2.27) that

\[
\sum_{S \subseteq I : \#(S) = k} \left\{ k \sum_{i=1}^{n} i_{\phi}^{w} \left[ X(S) : X(S_{i}^{C}) \right] - \sum_{j \in S} \left( i_{\phi}^{w} \left[ X(S_{j}) : X((S_{j})^{C}) \right] \right) \right\} \geq 0. \\
\]

Then, since each set of size \( k \) occurs \( n - k + 1 \) times in the second sum, we can write

\[
k \sum_{S \subseteq I : \#(S) = k} i_{\phi}^{w}(X(S) : X(S_{i}^{C})) \geq (n - k + 1) \sum_{S' \subseteq I : \#(S') = k-1} i_{\phi}^{w}(X(S') : X(S'_{i}^{C})). \\
\]

Dividing by \( k \binom{n}{k} \) concludes the proof. \( \square \)
3 Gaussian weighted entropies

As we said in the introduction, the WDIs are connected with the Gaussian WE $h_w^w(f^{\mathcal{N}}_C) := - \int_{\mathbb{R}^d} \phi(x)f_C^{\mathcal{N}}(x) \log f_C^{\mathcal{N}}(x)dx$; cf. (1.3), (1.5). Throughout the paper we use a number of properties established in [9]. One of them is maximization of the WE at $f = f_C^{\mathcal{N}}$. More precisely, consider the following inequalities

$$\int_{\mathbb{R}^d} \phi(x)[f(x) - f_C^{\mathcal{N}}(x)]dx \geq 0$$
$$\log [(2\pi)^d(\det C)] \int_{\mathbb{R}^d} \phi(x)[f(x) - f_C^{\mathcal{N}}(x)]dx + \text{tr} [C^{-1}(\Phi_C^{\mathcal{N}} - \Phi)] \leq 0. \quad (3.1)$$

**Theorem 3.1** Let $X = X_1^d \sim f(x)$, $x \in \mathbb{R}^d$, be a random vector with PDF $f$, mean zero and covariance matrix

$$C = \mathbb{E}_C \left( (X_1^d)(X_1^d)^T \right) = \int_{\mathbb{R}^d} xx^T f_C^{\mathcal{N}}(x)dx.$$

Set:

$$\Phi = \mathbb{E}_C \left( (X_1^d)(X_1^d)^T \phi(X_1^d) \right) = \int_{\mathbb{R}^d} xx^T \phi(x) f_C^{\mathcal{N}}(x)dx$$

and suppose that (3.1) is fulfilled. Then

$$h_w^w(f) \leq h_w^w(f_C^{\mathcal{N}}), \quad (3.2)$$

with equality iff $f = f_C^{\mathcal{N}}$ modulo $\phi$.

The proof of Theorem 3.1 follows the argument in Example 3.1 from [9] repeated verbatim in the multi-dimensional setting.

A conditional form of Theorem 3.1 is Theorem 3.2 below. The corresponding assertion for the standard entropy was noted in an earlier literature. See, e.g., Ref. [6, P. 1516]: the proof in the multi-dimensional setting.

The proof of Theorem 3.2 is essentially hinted in its statement (see Eqn (3.6)), and we omit it from the paper.

Given a $d \times d$ positive-definite matrix $C$ and $p = 1, \ldots, d - 1$, write $C$ in the block form:

$$C = \begin{pmatrix} C_{p}^{p} & C_{p}^{n-p} \\ C_{p}^{n-p} & C_{p+1}^{p} \end{pmatrix} \quad (3.3)$$

where $C_{n-p}$ and $C_{n-p}^{n-p}$ are mutually transposed $p \times (n - p)$ and $(n - p) \times p$ matrices. Given $x = \begin{pmatrix} x_1^d \\ x_{p+1}^d \end{pmatrix}$, set $Dx_{p+1}^d = C_{p}^{n-p}\left(C_{p+1}^{d}\right)^{-1}x_{p+1}^d$ and $K_1^p = C_{p+1}^{p} - C_{p}^{n-p}\left(C_{p+1}^{d}\right)^{-1}C_{p}^{n-p}$.

Correspondingly, if $X = X_1^d$ is a random vector (RV) with PDF $f_X$ and covariance matrix $C$ then $C_1^p$ represents the covariance matrix for vector $X_1^p$, with PDF $f_{X_1^p}(x_1^p)$. Let $X_{p+1}^d$ stand for the residual/remaining random vector and set $f_{X_{p+1}^d|X_1^p}(x_{p+1}^d|x_1^d) = \frac{f_X(x_{p+1}^d)}{f_X(x_1^d)}$. Also denote by $N_1^p$
and \( N_{p+1}^d \) the corresponding Gaussian vectors, with PDFs \( f_N(x) = f_{NC}^N(x), f_{N_t}^p(x_1^p) = f_{NC}^N(x_1^p) \) and \( f_{N_{p+1}}^N(x_{p+1}^d | x_1^p) \). Finally, for a given WF \( x \in \mathbb{R}^d \mapsto \phi(x) \) set:

\[
\psi(x_1^p) = \int_{\mathbb{R}^{d-p}} \phi(x)f_{N_{p+1}}^d(x_{p+1}^d | x_1^p)dx_{p+1}^d.
\]

Also, consider inequalities

\[
\begin{align*}
\int_{\mathbb{R}^d} \phi(x)f_{X_t}^p(\mathbf{x}_1^p) & \left[ f_{X_{p+1}}^d(x_{p+1}^d | x_1^p) - f_{N_{p+1}}^d(x_{p+1}^d | x_1^p) \right] dx \geq 0, \\
\int_{\mathbb{R}^d} \phi(x) \left[ f_X(x) - f_N(x) \right] \left\{ \log \left[ (2\pi)^p \det (K_t^T)^{-1} \right] \right. \\
& + \left. \log(e) \left[ (\mathbf{x}_1^p - \mathbf{D}x_{p+1}^d)^T (K_t^T)^{-1} (\mathbf{x}_1^p - \mathbf{D}x_{p+1}^d) \right] \right\} dx \leq 0.
\end{align*}
\]

**Theorem 3.2** Make an assumption that bounds (5.5) are satisfied. Then the following inequality holds true:

\[
h^w_\phi(x_{p+1}^d | x_1^p) := -\int_{\mathbb{R}^d} \phi(x)f_X(x) \log f_{X_{p+1}}^d(x_{p+1}^d | x_1^p)dx \\
\leq h^w_\phi(N_{p+1}^d | N_t^p) = h^w_\phi(N) - h^w_\psi(N_t^p) \\
= \frac{\alpha(C)}{2} \log \left[ (2\pi)^d \det C_t^T \right] + \frac{\log e}{2} \text{tr} [C_t^T^{-1} \Phi_N] \\
- \frac{\alpha(C_t^p)}{2} \log \left[ (2\pi)^d \det C_t^p \right] - \frac{\log e}{2} \text{tr} \left[ (C_t^p)^{-1} \Psi_{N_t} \right].
\]

**4 Weighted entropies under mappings**

In this section we give a series general theorems (Theorems 4.1–4.3 and Theorem 4.4) reflecting properties of the WEs under mappings of random variables (an example is a sum). Of a special importance for us is Theorem 4.3 used in Section 5. In essence, Theorems 4.1–4.3 are repetitions of their counterparts from [9], and we omit their proofs.

**Theorem 4.1** (Cf. Lemma 1.1 from [9].) Let \((\mathcal{X}, \mathcal{X}, \nu_X), (\mathcal{Y}, \mathcal{Y}, \nu_Y)\) be a pair of Lebesgue spaces and suppose \(X, Y\) are random elements in \((\mathcal{X}, \mathcal{X}), (\mathcal{Y}, \mathcal{Y})\) and PM/DFs \(f_X, f_Y\), relative to measures \(\nu_X, \nu_Y\), respectively. Suppose \(\eta : (\mathcal{X}, \mathcal{X}) \to (\mathcal{Y}, \mathcal{Y})\) is a measurable map onto, and that \(\nu_Y(B) = \nu_X(\eta^{-1}B), B \in \mathcal{Y}\). Consider the partition of \(\mathcal{X}\) with elements \(\mathcal{B}(y) := \{x \in \mathcal{X} : \)





\( \mathcal{X} : \, \eta x = y \) and let \( \nu_X(\cdot \mid y) \) be the family of induced measures on \( \mathcal{B}(y) \), \( y \in \mathcal{Y} \). Suppose that \( f_Y(y) = \int_{\mathcal{B}(y)} f_X(x)\nu(dx\mid y) \) and for \( x \in \mathcal{B}(y) \) let \( f_{X|Y}(x\mid y) := \frac{f_X(x)}{f_Y(y)} \) denote the PM/DF of \( X \) conditional on \( Y = y \). (Recall, \( f_{X|Y}(\cdot \mid y) \) is a family of PM/DFs defined for \( f_Y \)-a.a \( y \in \mathcal{Y} \) such that \( \int_{\mathcal{Y}} G(x)f_X(x)\nu_X(dx) = \int_{\mathcal{Y}} \int_{\mathcal{B}(y)} G(x)f_{X|Y}(x\mid y)\nu_X(dx\mid y)f_Y(y)\nu_Y(dy) \) for any non-negative measurable function \( G \).) Suppose that a WF \( x \in \mathcal{X} \mapsto \phi(x) \geq 0 \) obeys

\[
\int_{\mathcal{X}} \phi(x)f_X(x)\left[f_{X|Y}(x\mid \eta_x) - 1\right]\nu_X(dx) \leq 0
\]

(4.1)

and set

\[
\psi(y) = \int_{\mathcal{B}(y)} \phi(x)f_{X|Y}(x\mid y)\nu(dx\mid y), \quad y \in \mathcal{Y}.
\]

(4.2)

Then

\[
\begin{align*}
\psi^w(X) & \geq \psi^w(Y) := -\int_{\mathcal{Y}} \psi(y)f_Y(y)\log f_Y(y)\nu_Y(dy), \quad \text{or} \\
\psi^w(X|Y) & := -\int_{\mathcal{X}} \phi(x)f_X(x)\log f_{X|Y}(x\mid y(x))\nu_X(dx) \geq 0,
\end{align*}
\]

(4.3)

with equality iff \( \phi(x)[f_{X|Y}(x\mid \eta_x) - 1] = 0 \) for \( f\)-a.a. \( x \in \mathcal{X} \).

In particular, suppose that for \( f_Y \)-a.a. \( y \in \mathcal{Y} \) set \( \mathcal{B}(y) \) contains at most countably many values and \( \nu(\cdot \mid y) \) is a counting measure with \( \nu_1(x) = 1 \), \( x \in \mathcal{B}(y) \). Then the value \( f_{X|Y}(x\mid \eta_x) \) yields the conditional probability \( P(X = x\mid Y = \eta_x) \), which is \( \leq 1 \) for \( f_Y \)-a.a. \( y \in \mathcal{Y} \). Then \( \psi^w(X|Y) \geq 0 \) and the bound is strict unless, modulo \( \phi \), map \( \eta \) is 1 – 1.

**Theorem 4.2** (Cf. Lemma 1.2 from \[9\].) Let \((\mathcal{X}, \mathcal{X}, \nu_X), (\mathcal{Y}, \mathcal{Y}, \nu_Y), (\mathcal{Z}, \mathcal{Z}, \nu_Z)\) be a triple of SMSs and suppose \( X, Y, Z \) are random elements in \((\mathcal{X}, \mathcal{X}), (\mathcal{Y}, \mathcal{Y}), (\mathcal{Z}, \mathcal{Z})\). Let \( f_x \) be the PM/DF for \( X \) relative to measure \( \nu_X \) and \( f_{Y,Z} \) the joint PM/DF for \( Y, Z \) relative to measures \( \nu_Y \times \nu_Z \). Further, set \( f_Z(z) := \int_{\mathcal{Y}} f(y, z)\nu_Y(dy) \) and \( f_{Y|Z}(y\mid z) = \frac{f_{Y,Z}(y,z)}{f_Z(z)} \). Suppose that \( \eta : (\mathcal{X}, \mathcal{X}) \to (\mathcal{Y}, \mathcal{Y}), \quad \zeta : (\mathcal{X}, \mathcal{X}) \to (\mathcal{Z}, \mathcal{Z}) \) is a pair of measurable maps onto, and that

\[
\nu_Y(A) = \nu_X(\eta^{-1}A), \quad A \in \mathcal{Y}, \quad \nu_Z(B) = \nu_X(\zeta^{-1}B), \quad B \in \mathcal{Z}.
\]

Consider the partition of \( \mathcal{X} \) with elements \( \mathcal{B}(y, z) := \{ x \in \mathcal{X} : \eta x = y, \zeta x = z \} \) and let \( \nu_X(\cdot \mid y, z) \) be the family of induced measures on \( \mathcal{B}(y, z) \), \( (y, z) \in \mathcal{Y} \times \mathcal{Z} \). Suppose that

\[
f_{Y,Z}(y, z) = \int_{\mathcal{B}(y, z)} f_X(x)\nu_X(dx\mid y, z)
\]
and for $x \in \mathcal{B}(y,z)$ let $f_{X|Y,Z}(x|y,z) := \frac{f_X(x)}{f_{Y,Z}(y,z)}$ denote the PM/DF of $X$ conditional on $Y = y, Z = z$. (Recall, $f_{X|Y,Z} \cdot |y,z)$ is a family of PM/DFs defined for $f_{Y,Z}$-a.a. $(y,z) \in \mathcal{Y} \times \mathcal{Z}$ such that

$$\int_{\mathcal{X}} G(x)f_X(x)\nu_X(dx) = \int_{\mathcal{Y} \times \mathcal{Z} \mathcal{B}(y,z)} G(x)f_{X|Y,Z}(x|y,z)\nu_X(dx|y,z)f_{Y,Z}(y,z)\nu_Y(dy)\nu_Z(dz)$$

for any non-negative measurable function $G$. Assume that a WF $x \mapsto \phi(x) \geq 0$ obeys

$$\int_{\mathcal{X}} \phi(x)f(x)\left[f_{X|Y,Z}(x|\eta x, \zeta x) - 1 \right] \nu_X(dx) \leq 0 \quad (4.4)$$

and set

$$\psi(y,z) = \int_{\mathcal{B}(y,z)} \phi(x)f_{X|Y,Z}(x|y,z)\nu(dx|y,z). \quad (4.5)$$

Then

$$-\int_{\mathcal{Y} \times \mathcal{Z}} \psi(y,z)f_{Y,Z}(y,z)\log f_{Y,Z}(y,z)\nu_Y(dy)\nu_Z(dz)$$

$$=: h^w_\psi(Y|Z) \leq h^w_\psi(X|Z) := -\int_{\mathcal{X}} \phi(x)f_X(x)\log f_{X|Z}(x|\zeta x)\nu(dx); \quad (4.6)$$

equality iff $\phi(x)[f_{X|Y,Z}(x|\eta x, \zeta x) - 1] = 0$ for $f_X$-a.a. $x \in \mathcal{X}$.

As in Theorem [4.1] assume $\mathcal{B}(y,z)$ consists of at most countably many values and $\nu(x|y,z) = 1, x \in \mathcal{B}(y,z)$ for $f_{Y,Z}$-a.a. $(y,z) \in \mathcal{Y} \times \mathcal{Z}$. Then the value $f_{X|Y,Z}(x|y,z)$ yields the conditional probability $\mathbb{P}(X = x|Y = y, Z = z)$, for $f_{Y,Z}$-a.a. $y,z \in \mathcal{Y} \times \mathcal{Z}$. Then $h^w_\psi(X|Z) \geq h^w_\psi(Y|Z)$, with equality iff, modulo $\phi$, the map $x \mapsto (\eta x, \zeta x)$ is $1 - 1$.

**Theorem 4.3** (Cf. Lemma 1.3 from [9].) Let $(\mathcal{X}, \mathcal{X}, \nu_X), (\mathcal{Y}, \mathcal{Y}, \nu_Y), (\mathcal{Z}, \mathcal{Z}, \nu_Z)$ be a triple of SMSs and suppose $X, Y, Z$ are random elements in $(\mathcal{X}, \mathcal{X}), (\mathcal{Y}, \mathcal{Y}), (\mathcal{Z}, \mathcal{Z})$. Let $f_{X,Y}$ be the joint PM/DF for $X,Y$ relative to measure $\nu_X \times \nu_Y$ and set

$$f_Y(y) = \int_{\mathcal{X}} f_{X,Y}(x,y)\nu_X(dx), \quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$ 

Suppose that

$$\xi : (\mathcal{Y}, \mathcal{Y}) \rightarrow (\mathcal{Z}, \mathcal{Z})$$

is a measurable maps onto, and that

$$\nu_Z(C) = \nu_Y(\xi^{-1}C), \quad C \in \mathcal{Z}.$$
Consider a partition of $\mathcal{Y}$ with elements $\mathcal{C}(z) := \{ y \in \mathcal{Y} : \xi_y = z \}$ and let $\nu_y(\cdot | z)$ be the family of induced measures on $\mathcal{C}(z)$, $z \in \mathcal{Z}$. Given $(x, z) \in \mathcal{X} \times \mathcal{Z}$ and $y \in \mathcal{C}(z)$, let

$$f_{X,Z}(x, z) = \int_{\mathcal{C}(z)} f_{X,Y}(x, y) \nu_Y(dy | z), \quad f_Z(z) = \int_{\mathcal{X}} f_{X,Z}(x, z) \nu_X(dx),$$

and

$$f_{X|Z}(x | z) = \frac{f_{X,Z}(x, z)}{f_Z(z)}, \quad f_{Y|Z}(y | z) = \frac{f_Y(y)}{f_Z(z)}.$$  

Assume that a WF $(x, y) \mapsto \phi(x, y) \geq 0$ obeys

$$\int_{\mathcal{X} \times \mathcal{Y}} \phi(x, y) \left[ f_{X,Y}(x, y) - f_Z(\xi_y) f_{X|Z}(x | \xi_y) f_{Y|Z}(y | \xi_y) \right] \nu_X(dx) \nu_Y(dy) \geq 0 \quad (4.7)$$

and set

$$\psi(x, z) = \int_{\mathcal{C}(z)} \phi(x, y) f_{Y|Z}(y | z) \nu_Y(dy | z). \quad (4.8)$$

Then

$$- \int_{\mathcal{X} \times \mathcal{Z}} \psi(x, z) f_{X,Z}(x, z) \log f_{X|Z}(y | z) \nu_X(dx) \nu_Z(dz)$$

$$=: h^w_\phi(X|Z) \geq h^w_\phi(X|Y) := - \int_{\mathcal{X} \times \mathcal{Y}} \phi(x, y) f_X(x) \log f_{X|Y}(x | y) \nu_X(dx) \nu_Y(dy). \quad (4.9)$$

Furthermore, equality in $\mathclap{(4.9)}$ holds iff $X$ and $Y$ are conditionally independent given $Z$ modulo $\phi$, i.e. $\phi(x, y) \left[ f_{X,Y}(x, y) - f_Z(\xi_y) f_{X|Z}(x | \xi_y) f_{Y|Z}(y | \xi_y) \right] = 0$.

We will use an alternative notation $h^w_\phi(X) := h^w_\phi(f_X)$ where $X = X^d = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}$ is a $d$-dimensional random vector with PDF $f_X(x)$. In this context, we employ the notation $X \sim f_X$, $Y \sim f_Y$, $(X, Y) \sim f_{X,Y}$ and $(X|Y) \sim f_{X|Y}$ where $f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$.

Theorem 4.4 below mimics a result in [2], extending from the case of a standard entropy to that of the WE. A number of facts are related to the conditional WE

$$h^w_\phi(X|Y) := - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x, y) f_{X,Y}(x, y) \log f_{X|Y}(x | y) dx dy$$

or, more generally,

$$h^w_\phi(U|V) := - \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{\phi}(u, v) f_{U,V}(u, v) \log f_{U|V}(u | v) du dv,$$

Here a pair $(U, V)$ is a function of $(X, Y)$ with a joint PM/DF $f_{U,V}$, marginal PM/DFs $f_U$, $f_V$ and conditional PM/DF $f_{U|V}(u | v) := \frac{f_{U,V}(u, v)}{f_V(v)}$. (Viz., $U = Y$, $V = X + Y$.) WF $\tilde{\phi}$ may or may not be involved with the map $(X, Y) \mapsto (U, V)$.
Finally, Eqn (c) holds because $X$ from \cite{9}. Next, (b) is derived by applying the following equations:

Here bound (a) comes from the sub-additivity of the WE, see \cite{9}, Theorem 1.3 or Eqn (1.31) from \cite{9}. Assume that $WF(X,Y) \in \mathbb{R}^d \times \mathbb{R}^{d'} \to \phi(X,Y) \geq 0$ obeys

and set

\[
\theta(v) = \int_{\mathbb{R}^d} \phi(v-y,y)f_{Y|X}(y)dy, \quad \theta^*(x) = \int_{\mathbb{R}^d} \phi(x+y,y)f_{Y}(y)dy, \quad v,x \in \mathbb{R}^d.
\]

Then

\[
h_\theta^w(X + Y) \geq h_\theta^w(X),
\]

with equality iff $\phi(x,y)f_{Y}(y)f_{X|Y}(x|y) \log f_{X|Y}(x|y)dx dy = 0$ for Lebesgue-a.a. $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$.

**Proof.** Set: $\phi^*(x,y) = \phi(x+y,y)$. The following relations (a)–(c) hold true:

\[
\begin{align*}
(a) & \quad h_\theta^w(X + Y) \geq h_\theta^w(X + Y|Y), \\
(b) & \quad h_\theta^w(X + Y|Y) = h_\theta^w(X|Y), \\
(c) & \quad h_\theta^w(X|Y) = h_\theta^w(X).
\end{align*}
\]

Here bound (a) comes from the sub-additivity of the WE, see \cite{9}, Theorem 1.3 or Eqn (1.31) from \cite{9}. Next, (b) is derived by applying the following equations:

\[
h_\theta^w (X + Y|Y) = \int_{\mathbb{R}^d} f_Y(y)h_\theta^w (X + Y|Y = y)dy \\
= - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x+y,y)f_Y(y)f_{X|Y}(x|y) \log f_{X|Y}(x|y)dx dy.
\]

Finally, Eqn (c) holds because $X$ and $Y$ are independent.

The proof of Theorem 4.4 is completed by observing that

\[
h_\theta^w (X|Y) = - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x+y,y)f_{X,Y}(x,y) \log f_{X|Y}(x|y)dx dy \\
= - \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \phi(x+y,y)f_Y(y)dy \right] f_X(x) \log f_X(x)dx.
\]

**Remark 4.5** The assertion of Theorem 4.4 remains valid, *mutatis mutandis*, when $X$ and $Y$ have different dimensions. Viz., we can assume that $Y$ has dimension $d' < d$ and append $Y$ and $y$ with zero entries when we sum $X + Y$ and $x + y$. 

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5  Miscellaneous weighted determinant inequalities

In this section we present a host of WDIs derived from properties of the WEs. As we said before, the proposed inequalities hold when WF $\phi \equiv 1$ (in this case the stated conditions are trivially fulfilled). To stress parallels with ‘standard’ DIs, we provide references to [2] or [5] in each case under consideration.

**Theorem 5.1** (Cf. [2] Theorem 2.) Let $X, Y$ be independent $d$-variate normal vectors with zero means and covariance matrices $C_1, C_2$, respectively: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, $x, y \in \mathbb{R}^d$, where $f_X = f_{C_1}^N$, $f_Y = f_{C_2}^N$. Given a WF $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \phi(x,y) \geq 0$, positive on an open domain in $\mathbb{R}^d \times \mathbb{R}^d$, consider a quantity $\beta$ and $d \times d$ matrices $\Theta, \Theta^*$:

$$\beta = \int_{\mathbb{R}^d} \theta(x)f_{C_1+C_2}^N(x)dx, \quad \Theta = \int_{\mathbb{R}^d} xx^T\theta(x)f_{C_1+C_2}^N(x)dx, \quad \Theta^* = \int_{\mathbb{R}^d} xx^T\theta^*(x)f_{C_1}^N(x)dx$$

(5.1)

where $\theta$ and $\theta^*$ are as in (4.11):

$$\theta(x) = \int_{\mathbb{R}^d} \phi(z,x-z)f_{Y|X+x}(x-z|x)dz, \quad \theta^*(x) = \int_{\mathbb{R}^d} \phi(x+y,y)f_{Y}(y)dy. \quad (5.2)$$

Assume the condition emulating (4.11):

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x,y)f_{C_2}^N(y)[f_{C_1}^N(x)-f_{C_1+C_2}^N(x+y)]dx dy \geq 0. \quad (5.3)$$

Then

$$\beta \log \left[ \frac{\det(C_1 + C_2)}{\det C_1} \right] + (\log e) \left\{ \text{tr} \left[ (C_1 + C_2)^{-1}\Theta \right] - \text{tr} \left( C_1^{-1}\Theta^* \right) \right\} \geq 0. \quad (5.4)$$

**Proof.** Using Theorem 4.1 and Eqn (1.3), we can write:

$$\frac{1}{2} \log \left[ (2\pi)^d(\det(C_1 + C_2)) \right] \int_{\mathbb{R}^d} \theta(x)f_{C_1+C_2}^N(x)dx + \frac{\log e}{2} \text{tr}(C_1 + C_2)^{-1}\Theta \geq \frac{1}{2} \log \left[ (2\pi)^d(\det C_1) \right] \int_{\mathbb{R}} \theta^*(x)f_{C_1}^N(x)dx + \frac{\log e}{2} \text{tr} C_1^{-1}\Theta^*.$$

The bound in (5.4) then follows.

**Remark 5.2.** It is instructive to observe that (5.4) is equivalent to:

$$\beta \log \left[ \det(I + C_1^{-1}C_2) \right] + (\log e)\text{tr} \left[ (C_1 + C_2)^{-1}\Theta^* - C_1^{-1}\Theta^* + (C_1 + C_2)^{-1}\bar{\Theta} \right] \geq 0 \quad (5.5)$$

where

$$\bar{\Theta} = \int_{\mathbb{R}^d \times \mathbb{R}^d} (xy^T + yx^T + yy^T) \phi(x+y,y)f_{C_2}^N(y)f_{C_1}^N(x)dy dx.$$

This claim is verified by observing that $\Theta = \Theta^* + \bar{\Theta}$.
**Remark 5.3** As above, we can assume that $C_2$ is a matrix of size $d' \times d'$, agreeing that in the sum $C_1+C_2$, matrix $C_2$ is identified as a top left block (say). This is possible because in Eqs (5.4) and (5.5) we do not use the inverse $C_2^{-1}$ or the determinant $\det C_2$.

To this end, recall the following theorem from [7]:

**Theorem 5.4** Let $G$ and $G+E$ be nonsingular matrices where $E$ is a matrix of rank one. Let $g = \text{tr} \left( EG^{-1} \right)$. Then $g \neq -1$ and

$$(G+E)^{-1} = G^{-1} - \frac{1}{1+g} G^{-1} E G^{-1}. $$

The above equation is essentially the Sherman-Morrison formula (see [4], p. 161).

Assuming that $C_2 = E$ has rank 1 and letting $g = \text{tr} \left( EC_1^{-1} \right)$, inequality (5.4) turns into the following bound:

$$\beta \log \left[ \frac{\det (C_1+E)}{\det C_1} \right] + (\log e) \left[ -\text{tr} \left( \frac{C_1^{-1}EC_1}{1+g} \Theta^* \right) + \text{tr} \left\{ (C_1+E)^{-1} \Theta \right\} \right] \geq 0. \quad (5.6)$$

The techniques developed so far allows us to prove Theorem 5.5 below rendering a weighted form of Szasz theorem. Suppose $C$ is a positive definite $d \times d$ matrix. Given $1 \leq k \leq d$ and a set $S \subseteq I^{(d)} := \{1, \ldots, d\}$ with $\#(S) = k$, denote by $C(S)$ be the $k \times k$ sub-matrix of $C$ formed by the rows and columns with indices $i \in S$. With every $S$ we associate a Gaussian random vector $X(S) \sim f_{C(S)}^N$ considered as a sub-collection of $X \sim f_C^N$. Accordingly, conditional PDFs emerge, $f_{S|S'}^N(x(S)|x(S'))$, for pairs of sets $S, S'$ with $S \cap S' = \emptyset$, where $x(S) \in \mathbb{R}^{\#(S)}$, $x(S') \in \mathbb{R}^{\#(S')}$. [The PDF $f_{S|S'}^N$ is expressed in terms of block sub-matrices forming the inverse matrix $C(S \cup S')^{-1}$.

Further, let a function $\phi(x) \geq 0, x \in \mathbb{R}^d$, be given, which is positive on an open domain in $\mathbb{R}^d$ and set, as in (2.5),

$$\psi(S; x(S)) = \int_{\mathbb{R}^{\#(S')}} \phi(x)f_{S|S'}^N(x(S)|x(S)) \, dx(S'). \quad (5.7)$$

Furthermore, define:

$$\tau(S) = \text{tr} \left[ C(S)^{-1} \Phi(S) \right], \quad T(k) = \sum_{S \subseteq I^{(d)}: \#(S) = k} \tau(S) \quad (5.8)$$

where matrix $\Phi(S)$ is given by

$$\Phi(S) = \Phi(C(S)) = \int_{\mathbb{R}^{\#(S)}} x(S) x(S)^T \psi(S; x(S)) f_{C(S)}^N(x(S)) \, dx(S). \quad (5.9)$$

(For $S = I^{(d)}$, we write simply $\Phi$; cf. (1.4).) Finally, set:

$$\alpha(S) = \alpha(C(S)) = \int_{\mathbb{R}^{\#(S)}} \psi(S; x(S)) f_{C(S)}^N(x(S)) \, dx(S), \quad A(k) = \sum_{S \subseteq I^{(d)}: \#(S) = k} \alpha(S) \quad (5.10)$$

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and
\[ \lambda(S) = \alpha(S) \log \det C(S), \quad \Lambda(k) := \sum_{S \subseteq \{1, \ldots, d\} : |S| = k} \lambda(S). \quad (5.11) \]

Consider the following condition invoking broken dependence and analogous to (2.8):
\[ \forall i \in S \subseteq I, \text{ with } S_i^- = \{ j \in S : j < i \} \text{ and } S_i^+ = \{ j \in S : j > i \}, \]
\[ \int_{(\mathbb{R}^{|S|})^d} \psi(S; x(S)) \left\{ f_{C(S)}^N(x(S)) - f_{C(S_i^-)}^N(x(S_i^-)) \times \left[ f_{C(S_i^+)}^N(x(S_i^+)) \right] \right\} \, dx(S) \geq 0, \quad (5.12) \]

**Theorem 5.5** (Cf. [2], Theorem 4 or [5], Theorem 31) Assume condition (5.12). Then the quantity \( m(k) = m(k, C, \phi) \) defined by
\[ m(k) := \left( \frac{d}{k} \right)^{-1} \left[ \log \Lambda(k) + \log \left( \frac{2\pi}{k} \right) \log \det C(S) + \log e \right] - \left[ \log \left( \frac{2\pi}{k} \right) \right] A(k) + \log e \Lambda(k) \]
is decreasing in \( k = 1, \ldots, d \):
\[ m(1) \geq \ldots \geq m(d). \quad (5.13) \]

**Proof.** For \( X(S) \sim f_{C(S)}^N \) we have, by using (1.3):
\[ \frac{h^w_{\psi(S)}(X(S))}{k} = \frac{\alpha(S)}{2k} \log \left[ (2\pi)^k \det C(S) \right] + \frac{\log e}{2k} - \text{tr} \left[ C(S)^{-1} \Phi(S) \right]. \]
Therefore,
\[ m(k) = \left( \frac{d}{k} \right)^{-1} \sum_{S \subseteq \{1, \ldots, d\} : |S| = k} \left\{ \frac{\alpha(S)}{2k} \log \left[ (2\pi)^k \det C(S) \right] + \frac{\log e}{2k} - \text{tr} \left( C(S)^{-1} \Phi(S) \right) \right\} \]
Invoking Theorem 2.2 completes the proof.

**Theorem 5.6** (Cf. [2], Theorem 5 or [5], Theorem 32) Assuming (5.12), for all \( r > 0 \) the values
\[ s(k) = s(k, C, \phi) := \left( \frac{d}{k} \right)^{-1} \sum_{S \subseteq \{1, \ldots, d\} : |S| = k} \Lambda(k)^{1/k} \exp \left\{ r \left[ \frac{\log (2\pi)}{2} A(k) + \frac{\log e}{2k} T(k) \right] \right\} \]

obey
\[ s(1) \geq \ldots \geq s(d). \quad (5.14) \]
\textbf{Proof.} The assertion follows readily from Theorem 2.3.\hfill \blacksquare

Our next goal is to establish bounds for Toeplitz determinants extending Theorem 6 from \cite{2} (or Theorem 27 from \cite{5}). It is said that $C = (C_{ij})$ is a $d \times d$ Toeplitz matrix if $C_{ij} = C_{kl}$ whenever $|i - j| = |k - l|$. A more restrictive property is cyclic Toeplitz where $C_{ij} = C_{kl}$ whenever $\text{dist}_{d}(i, j) = \text{dist}_{d}(k, l)$. Here, for $1 \leq i < j \leq d$ the cyclic distance $\text{dist}_{d}(i, j) = \min [j - i, d - j + 1]$; it is then extended to a metric with $\text{dist}_{d}(i, j) = \text{dist}_{d}(j, i)$ and $\text{dist}_{d}(i, i) = 0$. As before, we consider sub-matrices $C(S)$ where $S \subseteq I^{(d)} := \{1, \ldots, d\}$ and the Gaussian random vectors

$X(S) \sim f^{N_{0}}_{C(S)}$ as sub-collections in $X_{i}^{d} := \begin{pmatrix} X_{1}^{i} \\ \vdots \\ X_{d}^{i} \end{pmatrix} \sim f^{N_{0}}_{C}$. A special role is played by $S = I_{i,j}$ where $I_{i,j}$ stands for a segment of positive integers $\{i, i+1, \ldots, j\}$ of cardinality $j - i + 1$ where $1 \leq i < j \leq d$. In particular, for $S = I_{1,k}$, we set: $C(S) = C_{k}$ and deal with vectors $X_{i}^{k} \sim f^{N_{0}}_{C_{k}}$, $1 \leq k \leq d$, with $C_{d} = C$.

Accordingly, we say that WF $x \in \mathbb{R}^{d} \mapsto \phi(x) \geq 0$ has a Toeplitz property if the value of the reduced WF $\psi(I_{i,j}; x_{i}^{j})$ coincides with $\psi(I_{i,k} + j,k; x_{i,k}^{j+k})$, provided that arguments $x_{i}^{j} = x(I_{i,j})$ and $x_{i,k}^{j+k} = x(I_{i,k} + j,k)$ are shifts of each other, where $1 \leq i < j \leq d$ and $1 \leq i + k < j + k \leq d$. An example is where $C$ is cyclic Toeplitz and $\phi$ has a product-form: $\phi(x) = \prod_{1 \leq i \leq d} \varphi(x_{i})$. Recall, the reduced WF in question involves the conditional PDF $f^{N_{0}}_{C_{i,j}}(x(I_{i,j})|x_{i}^{j})$:

$$
\psi(I_{i,j}; x_{i}^{j}) = \int_{\mathbb{R}^{d-j+i-1}} \phi(x)f^{N_{0}}_{C_{i,j}}(x(I_{i,j})|x_{i}^{j})dx(I_{i,j}) \quad \text{where} \quad I_{i,j}^{c} = I_{1,d} \setminus I_{i,j}.
$$

For $S = I_{1,k}$, $1 \leq k \leq d$, in accordance with (1.3),

$$
h_{\psi(k)}(X_{1}^{k}) = h_{\psi(I_{1,k})}(X_{1}^{k}) = \frac{\alpha(C_{k})}{2} \log [(2\pi)^{k} \det C_{k}] + \frac{\log e}{2} \text{tr} [C_{k}^{-1} \Psi_{k}] . \quad (5.15)
$$

Here the value $\alpha(C_{k}) = \alpha(C_{k}, C, \phi)$ and the $k \times k$ matrix $\Psi_{k} = \Psi_{k}(C_{k}, C, \phi)$ are given by

$$
\alpha(C_{k}) = \int_{\mathbb{R}^{k}} \psi(k; x_{k}^{k})f^{N_{0}}_{C_{k}}(x_{k}^{k})dx_{k}, \quad \Psi_{k} = \int_{\mathbb{R}^{k}} x_{k}^{k}(x_{k}^{k})^{T} \psi(k; x_{k}^{k})f^{N_{0}}_{C_{k}}(x_{k}^{k})dx_{k} \quad (5.16)
$$

and $\psi(k) = \psi(I_{1,k})$. (For $k = d$, the subscript $k$ will be omitted.)

\textbf{Theorem 5.7} (Cf. \cite{2}, Theorem 6 or \cite{5}, Theorem 27) Suppose $C_{n}$ is a positive definite $d \times d$ Toeplitz matrix and $\phi$ has the Toeplitz property. Consider the map $k \in \{1, \ldots, d\} \mapsto a(k) = a(k, C, \phi)$ where

$$
a(k) = \alpha(C_{k}) \left\{ \log(2\pi) + \log [(\det C_{k})^{1/k}] \right\} + \frac{\log e}{k} \text{tr} [C_{k}^{-1} \Psi_{k}] . \quad (5.17)
$$

Assuming condition (5.12), the value $a(k)$ is decreasing in $k$: $a(1) \geq \cdots \geq a(d)$.\hfill \(\blacksquare\)
\textbf{Proof.} By using the Toeplitz property of $C$ and $\phi$, we can write
\begin{equation}
 h_w^{\psi(I_{1,k})}(X_k|X_{1}^{k-1}) = h_w^{\psi(I_{2,k+1})}(X_{k+1}|X_{2}^{k}).
 \end{equation} 
(5.18)

Next, Theorem 4.3 yields:
\begin{equation}
 h_w^{\psi(I_{2,k+1})}(X_{k+1}|X_{2}^{k}) \geq h_w^{\psi(I_{1,k+1})}(X_{k+1}|X_{1}^{k}).
 \end{equation} 
(5.19)

From (5.18) and (5.19) we conclude that $h_w^{\psi(I_{1,k})}(X_k|X_{1}^{k-1})$ is decreasing in $k$. Thus the running average also decreases. On the other hand, by the chain rule
\begin{equation}
 \frac{1}{k} h_w^{\psi(I_{1,k})}(X_{1}^{k}) = \frac{1}{k} \sum_{i=1}^{k} h_w^{\psi(I_{1,i})}(X_{i}|X_{i-1}^{i-1}).
 \end{equation}

Consequently $\frac{1}{k} h_w^{\psi(I_{1,k})}(X_{1}^{k})$ too decreases in $k$. Referring to Eqns (5.16) and (5.15) leads directly to the result.

\textbf{Theorem 5.8} (Cf. [5], Theorem 33.) Given a WF $x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d \mapsto \phi(x)$, assume condition
\begin{equation}
 \int_{\mathbb{R}^d} \phi(x) \left[ f_{C}^N(x) - \prod_{i=1}^{n} f_{C^i}^N(x_i) \right] \, dx \geq 0.
 \end{equation} 
(5.20)

Then the quantity
\begin{equation}
 w(k) = w(k, C, \phi) = \left( \frac{d}{k} \right)^{-1} \frac{\alpha(C)}{2k} \log \left[ \prod_{S \subseteq I_n: \#(S) = k} \frac{(2\pi)^d (\det C)}{\frac{1}{2} \log e} \right]
 + \left( \frac{d}{k} \right)^{-1} \frac{\log e}{2k} \sum_{S \subseteq I_n: \#(S) = k} \left\{ \text{tr} \left[ C^{-1} \Phi \right] - \text{tr} \left[ C(S^C)^{-1} \Phi(S^C) \right] \right\}
\end{equation}

is increasing in $k$, with
\begin{equation}
 w(1) \leq \cdots \leq w(d).
 \end{equation} 
(5.21)

\textbf{Proof.} Using the conditional WE, we can write
\begin{align*}
 h_w^{\psi}(X(S)|X(S^C)) &= h_w^{\psi}(X(S), X(S^C)) - h_w^{\psi}(S^C)(X(S^C)) \\
 &= \frac{\alpha(C)}{2} \log \left[ (2\pi)^d (\det C) \right] + \frac{\log e}{2} \text{tr} \left[ C^{-1} \Phi \right] \\
 &\quad - \frac{\alpha(C)}{2} \log \left[ (2\pi)^{d-k} (\det C(S^C)) \right] + \frac{\log e}{2} \text{tr} \left[ C(S^C)^{-1} \Phi(S^C) \right].
\end{align*}
The proof of Theorems 5.10 and 5.11 is done with the help of Theorem 2.6, assuming that

\[
\psi(x(S^c)) f_{C^*}^N(x(S^c)) = \int_{\mathbb{R}^{d-N}} \psi(x(S^c)) f_{C^*}^N(x(S^c)) \, dx(S^c).
\]

Therefore,

\[
h_\phi^w(X(S) | X(S^c)) = \frac{\alpha(C)}{2} \log \left[ \frac{(2\pi)^d (\det C)^{-1}}{(2\pi)^{d-k} (\det C(S)^{-1})} \right] + \frac{\log e}{2} \left\{ \text{tr} [C^{-1} \Phi] - \text{tr} [C(S^c)^{-1} \Phi(S^c)] \right\}.
\]  

(5.22)

After that we apply Theorem 2.3 which completes the proof.

Remark 5.9 Note that the outermost inequality, \(w(1) \leq w(d)\), can be rewritten as

\[
\alpha(C) \log \left[ (2\pi)^d (\det C) \right] + \log e \, \text{tr} [C^{-1} \Phi] \geq \alpha(C) \log \left[ \prod_{i=1}^d \frac{2\pi(\det C)}{\det C(I_{i-1}^d \cup I_{i+1}^d)} \right] + \log e \sum_{i=1}^d \left\{ \text{tr} [C^{-1} \Phi] - \text{tr} [C(I_{i-1}^d \cup I_{i+1}^d)^{-1} \Phi(I_{i-1}^d \cup I_{i+1}^d)] \right\}.
\]  

(5.23)

Our next goal is to establish additional WDIs by using Theorem 2.6. For this purpose, we first analyse the mutual Gaussian WE, \(h_\phi^w(X(S) : X(S^c))\). According to the definition of the mutual WE in [9], we can write

\[
h_\phi^w(X(S) : X(S^c)) = h_\phi^w(X(S)) - h_\phi^w(X(S) | X(S^c)).
\]

Then, in accordance with (5.22), we have

\[
h_\phi^w(X(S) : X(S^c)) = \frac{\alpha(C)}{2} \log \left[ \frac{(\det C(S)) (\det C(S^c))}{(\det C)} \right] + \frac{\log e}{2} \left\{ \text{tr} [C(S)^{-1} \Phi(S)] + \text{tr} [C(S^c)^{-1} \Phi(S^c)] - \text{tr} [C^{-1} \Phi] \right\}.
\]

In Theorems 5.10 and 5.11 we consider the following condition (5.24) stemming from (2.20):

\[
\forall S \subseteq \{1, \ldots, n\} \text{ with } \#S \geq 2 \text{ and } i, j \in S \text{ with } i \neq j,
\]

\[
\int_{\mathbb{R}^d} \phi(x) f_{C^*}^N(x(S^c) | x(S)) \left[ f_{C^*}^N(x(S)) - f_{C^{(S)}}^N (x(S \setminus \{i, j\}) \big| f_{C(S \setminus \{i, j\})}^N (x(S \setminus \{i, j\})) f_{C^*}^N (x(S \setminus \{i, j\}) \big| x(S \setminus \{i, j\})) \right] \, dx(S) \geq 0.
\]  

(5.24)

The proof of Theorems 5.10 and 5.11 is done with the help of Theorem 2.6 assuming that \(X_1, X_2, \ldots, X_d\) are normally distributed with covariance matrix \(C\).

Theorem 5.10 (Cf. [9], Theorem 34.) Assume condition (5.24). Let

\[
u(k) = \left( \begin{array}{c} d \\ k \end{array} \right) \frac{\alpha(C)}{2k} \log \left[ \prod_{S \subseteq I^n: \#(S)=k} \frac{(\det C(S)) (\det C(S^c))}{(\det C)} \right]
\]

\[
+ \left( \begin{array}{c} d \\ k \end{array} \right) \frac{\log e}{2k} \sum_{S \subseteq I^n: \#(S)=k} \left\{ \text{tr} [C(S)^{-1} \Phi(S)] + \text{tr} [C(S^c)^{-1} \Phi(S^c)] - \text{tr} [C^{-1} \Phi] \right\}.
\]

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Then
\[ u(1) \geq u(2) \geq \cdots \geq u(d-1) \geq u(d). \] (5.25)

**Theorem 5.11** (Cf. [5], Theorem 35.) Under condition (5.24), let
\[
z(k) = \left( \frac{d}{k} \right)^{-1} \frac{\alpha(C)}{2} \log \left[ \prod_{S \subseteq I^{(d)}: \#(S) = k} \frac{(\det C(S)) (\det C(S^C))}{(\det C)} \right] \\
+ \left( \frac{d}{k} \right)^{-1} \frac{\log e}{2} \sum_{S \subseteq I^{(d)}: \#(S) = k} \left\{ \text{tr} \left[ C(S)^{-1} \Phi(S) \right] + \text{tr} \left[ C(S^C)^{-1} \Phi(S^C) \right] - \text{tr} \left[ C^{-1} \Phi \right] \right\}.
\]
Then
\[ z(1) \geq z(2) \geq \cdots \geq z(\lfloor d/2 \rfloor). \] (5.26)

6 Weighted Hadamard-type inequalities

In this section we group several results related to the weighted Hadamard inequality (WHI); cf. [9], Theorem 3.3. The WHI inequality asserts that for a \( d \times d \) positive definite matrix \( C \), under condition (5.20) we have:
\[
\alpha(C) \log \prod_i C_{ii} + (\log e) \sum_i C_{ii}^{-1} \Phi_{ii} - \alpha(C) \log \det C - (\log e) \text{tr} C^{-1} \Phi \geq 0,
\] (6.1)
with equality iff \( C \) is diagonal. Recall, \( \alpha(C) = \alpha_\phi(C) \) and \( \Phi = \Phi_C = \Phi_{C,\phi} \) are as in (1.4).

We begin with the weighted version of the strong Hadamard inequality (WSHI). The inequality (and other bounds in this section) will involve determinants \( \det C(S) \) of sub-matrices \( C(S) \) in \( C \) where, as before, \( S \) is a subset of \( I^{(d)} := \{1, \ldots, d\} \) of a special type. Namely, we fix \( p \in \{1, \ldots, d-1\} \) and consider the segment \( I_{p+1,d} := \{p+1, \ldots, d\} \), segment \( I_{1,p} := \{1, \ldots, p\} \) and unions \( \{i\} \cup I_{p+1,d} \) and \( I_{1,i} \cup I_{p+1,d} = I_{I+1,p} \) where \( i \in I_{1,p} \). We deal with the related entry \( C_{ii} \) in \( C \) and sub-matrices
\[
C_{d+1}^{d+1} := C(I_{p+1,d}), C_{d-1}^{d-1} := C(I_{1,i-1}), C(\{i\} \cup I_{p+1,d}) \quad \text{and} \quad C(I_{1,i} \cup I_{p+1,d})
\]
and Gaussian random variables \( X_i \) and vectors \( X_i^{d+1} := X(I_{p+1,d}), X_i^{-1} := X(I_{1,i-1}), X_i \cup X_i^{d+1} := X(\{i\} \cup I_{p+1,d}) \) and symbols \( x_i, x_i^{d+1}, x_i^{-1} \), and \( x_i \cup x_i^{d+1} \) for their respective values. Thus, PDFs
\[
f_{X_i^{d+1} | X_i^{d+1}} (x_i^{d+1} | x_i^{p+1}) = f_{X_i^{d+1} | X_i^{d+1}} (x_i^{d+1} | x_i^{p+1})
\]
emerge, as well as conditional PDFs \( f_{X_i | X_i^{d+1}} (x_i | x_i^{p+1}) \) and \( f_{X_i^{-1} | X_i^{d+1}} (x_i^{-1} | x_i^{p+1}) \). Viz., \( X_i \cup X_i^{d+1} \) and \( x_i \cup x_i^{d+1} \) stand for the concatenated vectors
\[
\begin{pmatrix}
X_1 \\
\vdots \\
X_i \\
X_{p+1} \\
\vdots \\
X_d
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_i \\
x_{p+1} \\
\vdots \\
x_d
\end{pmatrix},
\]
each with
\( i + d - p \) entries. As above (see (1.4)), for a given WF \( \mathbf{x} \in \mathbb{R}^d \mapsto \phi(\mathbf{x}) \) we consider numbers
\[ \alpha(C^p) = \alpha_{\phi}(C^p) \] and matrices \( \Phi_{C^p} = \Phi_{C^p, C, \phi} \):
\[ \alpha(C^p) = \alpha_{\phi}(C^p) = \int_{\mathbb{R}^d} \phi(\mathbf{x}) f_{C^p}^\alpha(x^p_i) d\mathbf{x} \]
\[ \Phi_{C^p} = \Phi_{C^p, C, \phi} = \int_{\mathbb{R}^d} x^p_i (x^p_i)^T \phi(\mathbf{x}) f_{C^p}^\alpha(\mathbf{x}) d\mathbf{x} \].

(In Eqs (6.16) and (6.22) – (6.24) we will use variations of these formulas.) We also set
\[ \Phi_{p+1}^d = \int_{\mathbb{R}^p} x^d_{p+1} (x^d_{p+1})^T \psi(I_{p+1}; x^d_{p+1}) f_{x^d_{p+1}}(x^d_{p+1}) d\mathbf{x}^d_{p+1} \]
\[ \Phi(|\{i\} \cup I_{p+1,d}) = \int_{\mathbb{R}^{d-p}} (x_i \lor x^d_{p+1}) (x_i \lor x^d_{p+1})^T \psi(I_{p+1}; x^d_{p+1}) f_{x^d_{p+1}}(x^d_{p+1}) d\mathbf{x}^d_{p+1} \]
with reduced WFs \( \psi(I_{p+1,d}) \) and \( \psi(|\{i\} \cup I_{p+1,d}) \) calculated as in (2.5), for \( S = I_{p+1,d} \) and \( S = |\{i\} \cup I_{p+1,d} \).

Furthermore, we will assume in Theorem 6.1 that, \( \forall i = 1, \ldots, p \), the reduced WF \( \psi(S) \) with \( S = \{1, \ldots, i, p + 1, \ldots, d\} \) obeys
\[ \int_{\mathbb{R}^{i+d-p}} \psi(I_{i+1,p}; x^d_{i} \lor x^d_{p+1}) \left\{ f_{x^d_{i+1}}(x^d_{i} \lor x^d_{p+1}) - f_{x^d_{p+1}}(x^d_{i}) \right\} d\mathbf{x}^d_{i} \lor x^d_{p+1} \geq 0. \]

The ‘standard’ SHI is
\[ \frac{\det C}{\det C_{p+1}^d} \leq \prod_{1 \leq i \leq p} \frac{\det C(|\{i\} \cup I_{p+1,d}) \det C_{p+1}^d}{\det C_{p+1}^d} \]
or \[ \log \det C + (p - 1) \log \det C_{p+1}^d \leq \sum_{1 \leq i \leq p} \log \det C(|\{i\} \cup I_{p+1,d}) \].

The WE approach offers the following WSHI:

**Theorem 6.1** (Cf. [2], Theorem 8 or [5], Theorem 28.) Under condition (6.3), for \( 1 \leq p < d \),
\[ \alpha(C) \log \left( (2\pi)^d \det C \right) + (\log e) \text{tr} (C^{-1} \Phi) \]
\[ + (p - 1) \left\{ \alpha(C_{p+1}^d) \log \left( (2\pi)^{d-p} \det C_{p+1}^d \right) + (\log e) \text{tr} (C_{p+1}^d)^{-1} \Phi_{p+1}^d \right\} \]
\[ \leq \sum_{1 \leq i \leq p} \left\{ \alpha(C(|\{i\} \cup I_{p+1,d}) \log \left( (2\pi)^{d-p+1} \det C(|\{i\} \cup I_{p+1,d}) \right) \right. \]
\[ \left. + (\log e) \text{tr} (C(|\{i\} \cup I_{p+1,d})^{-1} \Phi(|\{i\} \cup I_{p+1,d}) \right\} \].

**Proof.** We use the same idea as in Theorem 3.3 from [9]. Recalling (6.3) we can write
\[ h^\phi_{\alpha}(X^d_{i}(X^d_{p+1}) = \frac{1}{2} \log \left( (2\pi)^d \det C \right) \alpha(C) + \log e \text{tr} (C^{-1} \Phi) \]
\[ - \frac{1}{2} \log \left( (2\pi)^{d-p} \det C_{p+1}^d \right) \alpha(C_{p+1}^d) - \frac{\log e}{2} \text{tr} (C_{p+1}^d)^{-1} \Phi_{p+1}^d \].

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Cf. Eqns [5.9], [5.10], [5.16]. Furthermore, by subadditivity of the conditional WE (see [9], Theorem 1.4), under assumption (6.3) we can write
\[
\frac{h_w^w}{\psi} (X_i | X_{p+1}) \leq \sum_{i=1}^p \frac{h_w^w}{\psi} (\{i\} \cup I_{p+1,d}) (X_i | X_{p+1}). \tag{6.6}
\]
Here for \( i = 1, \ldots, p \), again in agreement with (6.9),
\[
\frac{h_w^w}{\psi} (\{i\} \cup I_{p+1,d}) (X_i | X_{p+1}) = \frac{1}{2} \log \left( (2\pi)^{d-p} \det C(\{i\} \cup I_{p+1,d}) \right) \alpha(C(\{i\} \cup I_{p+1,d})
+ \frac{1}{2} \log \left( (2\pi)^{d-p} \det C^d_{p+1} \right) \alpha(C^d_{p+1}) - \frac{\log e}{2} \text{tr} \left( (C^d_{p+1})^{-1} \Phi^d_{p+1} \right).
\]
Substituting into (6.6) yields the assertion of the theorem.

Our next result, Theorem 6.2, gives an extension of Lemma 9 from [2] (or Lemma 8 from [5]). The latter asserts that an individual diagonal entry \( C_{ii} \) of a \( d \times d \) positive definite matrix equals the ratio of the relevant determinants, viz.,
\[
C_{dd} = \frac{\det C}{\det C^{d-1}_1}, \quad \text{or} \quad \log C_{dd} + \log \det C^{d-1}_1 - \log \det C = 0.
\]
Remarkably, Theorem 6.2 does not require assumption (6.3).

**Theorem 6.2** (Cf. [2], Lemma 9 or [5], Lemma 8.) The following equality holds true:
\[
\alpha(C_{dd}) \log [(2\pi)C_{dd}] + \alpha(C^{d-1}) \log [(2\pi)d^{-1} \det C^{d-1}_1] - \alpha(C) \log [(2\pi)^d \det C] = (\log e) \tr[C^{-1}\Phi] - (\log e) \tr\left( (C^{d-1}_1)^{-1} \Phi^{d-1}_1 \right) - (\log e) C^{d-1}_1 \Phi^{d-1}_1. \tag{6.7}
\]

**Proof.** Using the conditional normality of \( X_d \) given \( X^{d-1}_1 \), we can write
\[
\frac{h_w^w}{\psi} (X_d | X^{d-1}_1) = \frac{\alpha(C_{dd})}{2} \log [(2\pi)C_{dd}^2] + \frac{\log e}{2} C_{dd}^2 \Phi_{dd}.
\]
On the other hand,
\[
\frac{h_w^w}{\psi} (X_d | X^{d-1}_1) = \frac{h_w^w}{\psi} (X^{d-1}_1) + \frac{h_w^w}{\psi} (I_{1,d-1}) (X^{d-1}_1), \tag{6.8}
\]
and therefore
\[
\frac{\alpha(C_{dd})}{2} \log [(2\pi)C_{dd}^2] + \frac{\log e}{2} C_{dd}^2 \Phi_{dd}
= \frac{\alpha(C)}{2} \log [(2\pi)d \det C] + \frac{\log e}{2} \tr C \Phi
- \frac{\alpha(C^{d-1}_1)}{2} \log [(2\pi)^{d-1} \det C^{d-1}_1] - \frac{\log e}{2} \tr\left( (C^{d-1}_1)^{-1} \Phi^{d-1}_1 \right). \tag{6.9}
\]
The result then follows.

\[
\text{24}
\]
The next assertion, Theorem 6.3 extends the result of Theorem 9 from [2] (or Theorem 29 from [5]) that, \( \forall p = 1, \ldots, d \), \( C \mapsto \log \frac{\det C}{\det C_1^p} \) is a concave function of a positive definite \( d \times d \) matrix \( C \). We will write matrix \( C \) in the block form similar to (3.3):

\[
C = \begin{pmatrix}
C_1^p & C_{n-p}^p \\
C_{p}^{d-p} & C_{d-p}^p
\end{pmatrix}.
\] (6.10)

Set \( D\mathbf{x}_{p+1}^d = C_{p+1}^{d-p}(C_{p+1}^d)^{-1}\mathbf{x}_{p+1}^d \) and \( B_1^p = C_1^p - C_{p+1}^{d-p}(C_{p+1}^d)^{-1}C_{d-p}^p \). Consider the following inequalities

\[
\int_{\mathbb{R}^d} \phi(\mathbf{x})f_{\mathbf{x}_{p+1}^d}(\mathbf{x}_{p+1}^d) \left[ f_{\mathbf{x}_{p+1}^d|\mathbf{x}_{p}^d}(\mathbf{x}_{p}^d|\mathbf{x}_{p+1}^d) - f_{\mathbf{x}_{p+1}^d|\mathbf{x}_{p}^d}(\mathbf{x}_{p}^d|\mathbf{x}_{p+1}^d) \right] d\mathbf{x} \geq 0
\] (6.11)

and

\[
\int_{\mathbb{R}^d} \phi(\mathbf{x}) \left[ f_{\mathbf{x}}(\mathbf{x}) - f_{\mathbf{x}_{p+1}^d}(\mathbf{x}) \right] \left\{ \log \left[ (2\pi)^p \det (B_1^p) \right] \\
+ (\log e) \left[ (\mathbf{x}_{p+1}^d - D\mathbf{x}_{p+1}^d)^T (B_1^p)^{-1} (\mathbf{x}_{p+1}^d - D\mathbf{x}_{p+1}^d) \right] \right\} d\mathbf{x} \leq 0.
\] (6.12)

**Theorem 6.3** (Cf. [2], Theorem 9 or [5], Theorem 29.) Assume that \( C = \lambda C' + (1 - \lambda)C'' \) where \( C, C' \) and \( C'' \) are positive definite \( d \times d \) matrices and \( \lambda \in [0,1] \). Given a WF \( \mathbf{x} \mapsto \phi(\mathbf{x}) \geq 0 \) and \( 1 \leq p \leq d \), define:

\[
\mu(C) = \alpha(C) \log \left[ (2\pi)^d \det C \right] + (\log e) \text{tr} \left[ C^{-1}\Phi_C \right] - \alpha(C_1^p) \log \left[ (2\pi)^d \det C_1^p \right] - (\log e) \text{tr} \left[ (C_1^p)^{-1}\Phi_{C_1^p} \right],
\] (6.13)

and similarly with \( \mu(C') \) and \( \mu(C'') \). Then

\[
\mu(C) \geq \lambda \mu(C') + (1 - \lambda)\mu(C'').
\] (6.14)

**Proof.** Again we essentially follow the method from [2] with modifications developed in [9]. Fix two \( d \times d \) positive definite matrices \( C' \) and \( C'' \) and set \( \mathbf{X}' \sim f_{C'}^{\Theta}, \mathbf{X}'' \sim f_{C''}^{\Theta} \). Given \( \lambda \in [0,1] \), consider a random variable \( \Theta \) taking values \( \bar{\Theta} = 1,2 \) with probabilities \( \lambda \) and \( 1 - \lambda \) independently of \( (\mathbf{X'}, \mathbf{X}'') \). Next, set

\[
\mathbf{X} = \begin{cases}
\mathbf{X}', & \text{when } \Theta = 1, \\
\mathbf{X}'', & \text{when } \Theta = 2.
\end{cases}
\]

Then \( \mathbf{X} \sim (\lambda f_{C'}^{\Theta} + (1 - \lambda) f_{C''}^{\Theta}) \) and the covariance matrix \( \text{Cov} \mathbf{X} = \lambda C' + (1 - \lambda)C'' =: C \).

With the WF \( \tilde{\phi}(\mathbf{x}_{1}^d, \bar{\Theta}) = \phi(\mathbf{x}_{1}^d) \), use Theorem 2.1 from [9] and Theorem 3.2 from Section 3 and write:

\[
h_{\tilde{\phi}}^w(\mathbf{X}_{p+1}^d|\mathbf{x}_{1}^p, \Theta) \leq h_{\tilde{\phi}}^w(\mathbf{X}_{p+1}^d|\mathbf{x}_{1}^p) \leq h_{\tilde{\phi}}^w(\mathbf{Y}_{p+1}^d|\mathbf{y}_{1}^p).
\] (6.15)
Here $\mathbf{Y}$ stands for the Gaussian random vector with the PDF $f_C^{No}(\mathbf{x}_1^d)$. The LHS in (6.15) coincides with $\lambda \mu(C') + (1 - \lambda) \mu(C'')$ and the RHS with $\mu(C)$. This completes the proof.

In a particular case $p = d - 1$, the function $C \mapsto \frac{\det C}{\det C_1^{d-1}}$ is also concave (see [2], Theorem 10). The weighted version of this property is encapsulated in the following result. For a positive definite $d \times d$ matrix $C$ and a WF $x \mapsto \psi(x)$, set:

$$\varpi_\psi(C) := \frac{\alpha_\psi(C)}{2} \log \left[ \frac{(2\pi)^d}{\det(C)} \right] + \frac{1}{2} \log e - \frac{1}{2} \text{tr} \left[ \Phi_{C,\psi}^{-1} \right],$$

(6.16)

Remark 6.4 When $\psi(\mathbf{x}_1^d) \equiv 1$, the expression for $\varpi_\psi(C)$ in (6.16) simplifies to $\log \frac{2\pi \det C}{\det C_1^{d-1}}$. The aforementioned concavity property from [2], Theorem 10 (or from [5], Theorem 30), is essentially equivalent to the following subadditivity-type property:

$$\log \frac{2\pi \det (A + B)}{\det (A_1^{d-1} + B_1^{d-1})} \geq \log \frac{2\pi \det A}{\det A_1^{d-1}} + \log \frac{2\pi \det B}{\det B_1^{d-1}}.$$  

The WE-version of this property is more involved: see Eqns (6.17) - (6.19). A crucial part is played by Lemma 4.3 with $X$ represented by the random variable $Z_d \sim f_{A_{dd} + B_{dd}}^{No}$ and $Y$ is associated with the independent Gaussian pair of vectors $(\mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1})$ having the joint PDF

$$f_{\mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}}(\mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}) = f_{\mathbf{A}_1^{d-1}}^{No}(\mathbf{x}_1^{d-1}) f_{\mathbf{B}_1^{d-1}}^{No}(\mathbf{y}_1^{d-1}).$$

The random element $Z$ from Theorem 4.3 is represented by $\mathbf{Z}_1^{d-1}$, and the map $\xi$ takes $(\mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}) \mapsto \mathbf{x}_1^{d-1} + \mathbf{y}_1^{d-1}$.

Theorem 6.5 (Cf. [2], Theorem 10 or [5], Theorem 30.) Let $A$, $B$ be two positive definite $d \times d$ matrices and $X \sim f_{\mathbf{A}}^{No}$, $Y \sim f_{\mathbf{B}}^{No}$ be the corresponding independent Gaussian vectors, with $\mathbf{Z} := \mathbf{X} + \mathbf{Y} \sim f_{\mathbf{A} + \mathbf{B}}^{No}$. Consider a WF

$$(\mathbf{z}_d, \mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \mapsto \phi(\mathbf{z}_d, \mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1})$$

and assume the following inequality involving conditional normal PDFs $f_{\mathbf{Z}_d|\mathbf{X}_1^{d-1}, \mathbf{Y}_1^{d-1}}$ and $f_{\mathbf{Z}_d|\mathbf{Z}_1^{d-1}}$:

$$\int_{\mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \phi(\mathbf{z}_d, \mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}) f_{\mathbf{A}_1^{d-1}}^{No}(\mathbf{x}_1^{d-1}) f_{\mathbf{B}_1^{d-1}}^{No}(\mathbf{y}_1^{d-1}) \left[ f_{\mathbf{Z}_d|\mathbf{X}_1^{d-1}, \mathbf{Y}_1^{d-1}}(\mathbf{z}_d|\mathbf{x}_1^{d-1}, \mathbf{y}_1^{d-1}) - f_{\mathbf{Z}_d|\mathbf{Z}_1^{d-1}}(\mathbf{z}_d|\mathbf{x}_1^{d-1} + \mathbf{y}_1^{d-1}) \right] \, d\mathbf{z}_d \, d\mathbf{x}_1^{d-1} \, d\mathbf{y}_1^{d-1} \geq 0.$$  

(6.17)

Then

$$\varpi_\psi(\mathbf{A} + \mathbf{B}) \geq \varpi_\chi(\mathbf{A}) + \varpi_\gamma(\mathbf{B}).$$  

(6.18)
We therefore obtain the property claimed in (6.18):

\[ \psi(z_1^d) = \int \phi(z_d - y_d, z_1^d - y_1^d) f_{A_1}^N(z_1^d - y_1^d) f_{B_1}^N(y_1^d) dy_1, \]

\[ \chi(x_1^d) = \int \psi(x_1^d + y_1^d) f_{A_1}^N(y_1^d) dy_1, \quad \gamma(x_1^d) = \int \psi(x_1^d + y_1^d) f_{A_1}^N(y_1^d) dy_1. \]  

(6.19)

**Proof.** As in [2], we use basic properties of Gaussian random variables. Assume \( X \sim f_{A_1}^N \) and \( Y \sim f_{B_1}^N \) are independent Gaussian random vectors and set \( Z = X + Y \sim f_{A_1 + B_1}^N \). By virtue of (6.6) and Theorem 4.3, we can write:

\[ h^w_{\psi}(Z_d|Z_1^{d-1}) = h^w_{\psi}(Z) - h^w_{\psi_{X,Y}}(Z_1^{d-1}) = \varpi(A + B) \geq h^w_{\psi}(Z_d|X_1^{d-1}, Y_1^{d-1}). \]  

(6.20)

Next, owing to independence of \( X \) and \( Y \), the conditional WE \( h^w_{\psi}(X_d + Y_d|X_1^{d-1}, Y_1^{d-1}) \) equals the sum

\[ \int_{\mathbb{R}^{d-1}} \chi_1^{-1}(x_1^{-1}) f_{X_1^{-1}}(x_1^{-1}) \left\{ \frac{1}{2} \log \left( \frac{2\pi}{A_1^{-1}} \right) \right\} \int_{\mathbb{R}} \gamma_1(x) f_{X_1|x_1^{-1}}(x|x_1^{-1}) dx \]

\[ + \frac{\log e}{2} A_1^{-1} \int_{\mathbb{R}} x^2 \gamma_1(x) f_{X_1|x_1^{-1}}(x|x_1^{-1}) dx \]

\[ + \int_{\mathbb{R}^{d-1}} \chi_1^{-1}(x_1^{-1}) f_{Y_1^{-1}}(x_1^{-1}) \left\{ \frac{1}{2} \log \left( \frac{2\pi}{B_1^{-1}} \right) \right\} \int_{\mathbb{R}} \gamma_1(x) f_{Y_1|x_1^{-1}}(x|x_1^{-1}) dx \]

\[ + \frac{\log e}{2} B_1^{-1} \int_{\mathbb{R}} x^2 \gamma_1(x) f_{Y_1|x_1^{-1}}(x|x_1^{-1}) dx. \]  

(6.21)

(The fact that \( X_d \) and \( Y_d \) are scalar Gaussian variables is crucial here.)

The first summand equals

\[ \frac{1}{2} \log \left( \frac{2\pi}{A_1^{-1}} \right) \int_{\mathbb{R}^{d-1}} \chi(x_1^{-1}) f_{X_1^{-1}}(x_1^{-1}) dx_1^d + \frac{\log e}{2} A_1^{-1} \int_{\mathbb{R}^{d-1}} \chi(x_1^{-1}) x_1^d f_{X_1^{-1}}(x_1^{-1}) dx_1^d \]  

(6.22)

and coincides with

\[ h^w_\chi(X_d|X_1^{d-1}) = \frac{\alpha_\chi(A)}{2} \log \left( (2\pi)^d \det A \right) + \frac{\log e}{2} \text{tr} A^{-1} \Phi_{A,\chi} \]

\[ - \frac{\alpha_\chi^{-1}(A_1^{d-1})}{2} \log \left( (2\pi)^{d-1} \det A_1^{d-1} \right) - \frac{\log e}{2} \text{tr} \left( (A_1^{d-1})^{-1} \Phi_{A_1^{d-1},\chi_1^{-1}} \right) =: \varpi_\chi(A). \]  

(6.23)

Similarly, the second summand coincides with

\[ h^w_\gamma(Y_d|Y_1^{d-1}) = \frac{\alpha_\gamma(B)}{2} \log \left( (2\pi)^d \det B \right) + \frac{\log e}{2} \text{tr} B^{-1} \Phi_{B,\gamma} \]

\[ - \frac{\alpha_\gamma^{-1}(B_1^{d-1})}{2} \log \left( (2\pi)^{d-1} \det B_1^{d-1} \right) - \frac{\log e}{2} \text{tr} \left( (B_1^{d-1})^{-1} \Psi_{B_1^{d-1},\gamma_1^{-1}} \right) =: \varpi_\gamma(B). \]  

(6.24)

We therefore obtain the property claimed in (6.18): \( \varpi_\psi(A + B) \geq \varpi_\chi(A) + \varpi_\gamma(B) \).

Finally, combining (5.23) and (6.1), we offer
Theorem 6.6 (Cf. [5], Corollary 4) Given a $d \times d$ positive definite matrix $C$, assume condition (5.20). Then

$$\alpha(C) \log \left( \prod_{i=1}^{d} \frac{2\pi \det(C)}{\det(C(I_{i-1}^{d-1} \cup I_{i+1}^{d}))} \right)$$

$$+ \log e \sum_{i=1}^{d} \left\{ \text{tr} \left[ C^{-1} \Phi \right] - \text{tr} \left[ C(I_{i-1}^{d-1} \cup I_{i+1}^{d})^{-1} \Phi(I_{i-1}^{d-1} \cup I_{i+1}^{d}) \right] \right\}$$

$$\leq \alpha(C) \log \det C - (\log e) \text{tr} C^{-1} \Phi \leq \alpha(C) \log \prod_{i} C_{ii} + (\log e) \sum_{i} C_{ii}^{-1} \Phi_{ii}.$$  \hfill (6.25)

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