A NEWTON POLYGON RULE FOR FORMALLY-REAL VALUED FIELDS AND MULTIPlicITIES OVER THE SIGNED TROPICAL HYPERFIELD

TREVOR GUNN

ABSTRACT. By defining multiplicities for zeros of polynomials over hyperfields, Baker and Lorscheid were able to provide a unifying perspective on Descartes’s rule and the Newton polygon rule for polynomials over a formally-real and valued field respectively. In this paper, we apply their multiplicity formula to the hyperfield associated with formally-real, valued fields to prove a Newton polygon rule which combines Descartes’s rule of signs with the classical Newton polygon rule.

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1. INTRODUCTION

Hyperfields are similar to fields except that their addition is set-valued. The notion of multi-valued groups goes back at least to Frédéric Marty in the 1930’s and hyperrings and hyperfields to Marc Krasner in the 1950’s. In the context of valued fields and tropical geometry, Oleg Viro described how tropical varieties arise as equations over hyperfields, replacing the standard “minimum occurs twice” formulation with an algebraic formulation [Vir10; Vir11]. We direct the reader to Viro’s paper [Vir10] for more on the history of multi-valued algebras.

Recently, Matthew Baker and Nathan Bowler have used hyperfields as a unifying framework to study matroids, oriented matroids and valuated matroids [BB19]. Also, Baker and Oliver Lorscheid have shown how Descartes’s rule and the Newton polygon rule can both be obtained from a single definition of multiplicity over hyperfields [BL18].

Baker and Lorscheid (loc. cit.) define multiplicities of roots of polynomials over hyperfields via the recursive formula

$$\text{mult}_a(p) = \begin{cases} 0 & \text{if } p(a) \not\in 0 \\ 1 + \max\{\text{mult}_a(q) : p \in (x-a)q\} & \text{if } p(a) \ni 0. \end{cases}$$

Baker and Lorscheid prove the following two theorems.

Theorem (Baker, Lorscheid). For the hyperfield of signs, \(\text{mult}_{+1}(p) = \Delta(p)\), where \(\Delta(p)\) is the number of sign changes in the coefficients of \(p\).
and is equal (modulo pairs of complex roots) to the number of positive roots of any lifting of \( p \) to a real-closed field.

**Theorem** (Baker, Lorscheid). For the tropical hyperfield, \( \text{mult}_a(p) \) is the horizontal length of the edge of the Newton polygon of \( f \) with slope \(-a\). This multiplicity is equal to the number of roots of \( F \) with valuation \( a \) for any lifting \( P \) of \( p \) to an algebraically closed, non-Archimedean valued field.

In this paper, we prove a similar result for the signed tropical hyperfield, \( \mathbb{T} \) which combines features of both the tropical and signed hyperfields.

**Theorem A.** For an element \( a = (1, r) \) of the signed tropical hyperfield, \( \text{mult}_a(p) \) equals \( \Delta(p_\sigma) \) where \( p_\sigma \) is the initial form of \( p \) corresponding to the edge of \( \text{Newt}(p) \) with slope \(-r\). This multiplicity is equal (modulo pairs of complex conjugate roots) to the number of roots with valuation \( r \) and sign +1 of any lifting of \( p \) to a real-closed, non-Archimedean-ordered valued field.

**Remark 1.1.** For \( \text{mult}_0(p) \), one verifies easily that \( \text{mult}_0(p) = k \) if and only if \( p = c_kx^k + c_{k+1}x^{k+1} + \cdots + c_nx^n \) with \( c_k \neq 0 \). For \( \text{mult}_a(p) \) with \( a < 0 \), we have \( \text{mult}_a(p(x)) = \text{mult}_{-a}(p(-x)) \).

Therefore Theorem A tells us exactly what the roots of \( p \) are, along with their multiplicities.

We also give a proof of a related theorem in the language of polynomials over a real-closed, non-Archimedean-ordered valued field.

**Theorem B.** Let \( K \) be a real-closed, non-Archimedean-ordered valued field. Let \( P(x) \) be a polynomial over \( K \). Then the number of real roots of \( P \) which are positive and have valuation \( a \) is congruent mod 2 to the number of sign changes of the monomials lying on the edge of \( \text{Newt}(P) \) with slope \( \log(a) \).

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2. **Preliminaries**

2.1. **Real fields.** A **real field** (or formally-real field) is a field in which \(-1\) is not a sum of squares. Every real field admits a not-necessarily-unique ordering that makes it into an ordered field.

A real field which admits no proper, real algebraic extensions is called **real-closed**. Every real-closed field has a unique ordering that makes it an
ordered field and that ordering is defined by \( x \geq 0 \) if \( x = y^2 \) for some \( y \).

By a theorem of Artin and Schreier, \( K \) is real-closed if and only if \( K \) is not algebraically closed and the algebraic closure of \( K \) is \( K[\sqrt{-1}] \).

If \( K \) is a real-closed field, its sign function is defined by \( \text{sign}(x) = +1 \) if \( x > 0 \), \( \text{sign}(x) = 0 \) if \( x = 0 \) and \( \text{sign}(x) = -1 \) if \( x < 0 \). Additionally, \( K \) has an absolute value given by \( |x| = \text{sign}(x) |x| \).

\[ 2.2. \textbf{Valued fields.} A (rank-1) non-Archimedean, valued field } K \text{ is a field with a map } v : K \to K \cup \{\infty\} \text{ such that } v(0) = \infty \text{ and such that } v : (K^\times, \cdot) \to (\mathbb{R}, +) \text{ is a group homomorphism that satisfies the relation } v(xy) \geq \min\{v(x), v(y)\} \text{ for all } x, y \in K. \text{ For simplicity, we will refer to these as just “valued fields.” We call the map } v \text{ a (non-Archimedean) valuation.}

The valuation ring of \( K \) is the ring \( \mathcal{O}_K := \{x \in K : v(x) \geq 0\} \). This is a local ring with maximal ideal \( \mathfrak{m}_K := \{x \in K : v(x) > 0\} \). We call \( k := \mathcal{O}_K/\mathfrak{m}_K \) the residue field of \( K \). If \( K \) is algebraically closed then so is its residue field.

\[ 2.2.1. \textbf{Ordered valued fields.} \text{ An ordering on a valued field is called Archimedean if for every } x \in K, \text{ there is a natural number } n \text{ such that } x \leq n. \text{ A non-Archimedean order is an order which is not Archimedean.}

\begin{definition} \text{ If } \leq \text{ is an ordering on } K, \text{ then the } \leq\text{-convex-hull of a subring } R \text{ is the set } \mathcal{O}_R(\leq) = \{x \in K : x, -x \leq a \text{ for some } a \in R\}. \text{ } \end{definition}

\( R \) is \( \leq\text{-convex} \) (or simply convex) if \( R = \mathcal{O}_R(\leq) \).

When \( \mathcal{O}_K \) is convex, then the ordering on \( K \) is non-Archimedean [EP05, Corollary 2.2.6.] and we call \( K \) a non-Archimedean-ordered field. Going forward, when we say \( K \) is a real (or real-closed) valued field, we mean that \( K \) is real (or real-closed) and its residue field is real (or real-closed). We need this hypothesis in order to have a map from \( K \) to the signed tropical hyperfield (Example 2.9).

When \( K \) is real-closed, then \( k \) is real-closed, whence \( K \) is Henselian and the valuation on \( K \) extends uniquely to a valuation on the algebraic closure, \( K[\sqrt{-1}] \) [EP05, Theorem 4.1.3. and Theorem 4.3.7.]. Additionally, the residue field of \( K[\sqrt{-1}] \) is \( k[\sqrt{-1}] \). Furthermore, valuations on Henselian fields are Galois-invariant, in particular, two complex-conjugate roots of a polynomial must have the same valuation.

See [EP05, Chapter 4.3] for more on Henselian valued fields and Chapter 4.3 in particular for ordered (i.e. formally-real), Henselian valued fields.
2.2.2. *Hahn Series.* Let $k = \mathbb{R}$ or $\mathbb{C}$ (more generally any real or algebraically closed field) and let $\Gamma \subseteq \mathbb{R}$ be an ordered group. We call elements of

$$k[[t^\Gamma]] = \left\{ \sum_{\gamma \in I} a_\gamma t^\gamma : a_\gamma \in k, I \subset \Gamma \text{ is well-ordered} \right\}$$

*Hahn series.* When $k$ is algebraically closed (resp. real-closed), and $\Gamma$ is divisible, then the Hahn series form an algebraically closed (resp. real-closed) field [Mac39, Theorem 1]. The two Hahn series fields we are interested in are with $\Gamma = \mathbb{R}$.

There are surjective maps $v_\mathbb{C} : \mathbb{C}[[t^\mathbb{R}]] \to \mathbb{R} \cup \{\infty\}$ and $v_\mathbb{R} : \mathbb{R}[[t^\mathbb{R}]] \to ((\{\pm 1\} \times \mathbb{R}) \cup \{\infty\}$ given by $v_k(0) = \infty$ and otherwise

$$v_\mathbb{C} \left( \sum_{\gamma \in I} c_\gamma t^\gamma \right) = \gamma_0,$$

$$v_\mathbb{R} \left( \sum_{\gamma \in I} c_\gamma t^\gamma \right) = (\text{sign}(c_\gamma), \gamma_0),$$

where $\gamma_0 = \min I$.

$v_\mathbb{C}$ is a valuation on $\mathbb{C}[[t^\mathbb{R}]]$ that makes it (and $\mathbb{R}[[t^\mathbb{R}]]$) into a non-Archimedean valued field. The field $\mathbb{R}[[t^\mathbb{R}]]$ has a non-Archimedean ordering given by $0 < \sum_{\gamma \in I} c_\gamma t^\gamma$ if $0 < c_\min I$.

2.3. *Hyperfields.* The *tropical* and *signed tropical hyperfield* are respectively $T = (\mathbb{R} \cup \infty, \cdot, \boxplus, \infty, 0)$ and $T_\mathbb{R} = ((\{\pm 1\} \times \mathbb{R}) \cup \{\infty\}, \cdot, \boxplus, \infty, (1, 0))$. One way to define $\cdot$ and $\boxplus$ is via the maps $v_k$ on $k[[t^\mathbb{R}]]$.

The multiplication on $T$ ($k = \mathbb{C}$) and on $T_\mathbb{R}$ ($k = \mathbb{R}$) can be described by the rule $a \cdot b = v_k(\alpha \cdot \beta)$ if $v_k(\alpha) = a$ and $v_k(\beta) = b$. The hyperaddition is defined by $a \boxplus b = \{v_k(\alpha + \beta) : v_k(\alpha) = a, v_k(\beta) = b\}$. Note: multiplication is well-defined.

**Remark 2.2.** Viro uses a different, but isomorphic, signed tropical hyperfield; he also calls it the *real tropical hyperfield* [Vir10; Vir11]. Viro’s hyperfield has $\mathbb{R}$ as its underlying set and is related to ours via the isomorphism $(s, r) \mapsto se^{-r}$ and $\infty \mapsto 0$.

Explicitly, the multiplication for $T$ is given by the usual addition in $\mathbb{R}$. Multiplication for $T_\mathbb{R}$ is given by $(s, r) \cdot (s', r') = (ss', r + r')$. Hyperaddition in $T$ is given by

$$a \boxplus b = \begin{cases} \{a\} & a < b, \\ \{b\} & b < a, \\ [a, \infty] & a = b. \end{cases}$$
Hyperaddition in $\mathbb{TR}$ is given by $(s, r) \boxplus \infty = \infty \boxplus (s, r) = \{(s, r)\}$ and otherwise

$$(s, r) \boxplus (s', r') = \begin{cases} \{(s, r)\} & r < r', \\ \{(s', r')\} & r' < r, \\ \{(s, r)\} & r = r' \text{ and } s = s', \\ \{(\pm 1, t) : t \geq r\} \cup \{\infty\} & r = r' \text{ and } s = -s'. \end{cases}$$

We define $a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n$ inductively by

$$a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n = \bigcup_{b \in a_1 \boxplus \cdots \boxplus a_{n-1}} b \boxplus a_n.$$ 

One can check that this is the same as $\{v_k(\alpha_1 + \cdots + \alpha_n) : v_k(\alpha_i) = a_i\}$. Addition in hyperrings and hyperfields is associative and commutative so we do not bother parenthesizing long sums.

The *sign hyperfield* (or *hyperfield of signs*) is $\mathbb{S} = (\{0, 1, -1\}, \cdot, 0, 1)$ with the usual multiplication and hyperaddition given by the following table

| $\boxplus$ | 0   | -1  | +1   |
|------------|-----|-----|------|
| 0          | $\{0\}$ | $\{-1\}$ | $\{+1\}$ |
| -1         | $\{-1\}$ | $\{-1\}$ | $\mathbb{S}$ |
| +1         | $\{+1\}$ | $\mathbb{S}$ | $\{+1\}$ |

We can identify $\mathbb{S}$ with the subset $\{\infty, (1, 0), (-1, 0)\} \subset \mathbb{TR}$ where $\infty \leftrightarrow 0$ and $(\pm 1, 0) \leftrightarrow \pm 1$. Thus $\mathbb{S}$ is a sub-hyperfield of $\mathbb{TR}$.

We will also use make use of the sub-hyperring $\mathbb{O}_{\mathbb{TR}} := (\{\pm 1\} \times \mathbb{R}_{\geq 0}) \cup \{\infty\} \subset \mathbb{TR}$ which is the image of $\mathbb{O}_{\mathbb{R}}[t]$ under $v_\mathbb{R}$.

**Remark 2.3.** The ordering on $\mathbb{R}[t][[t]]$ is given by $0 < \alpha$ if $\alpha \neq 0$ and $0 < (s, r)$ if $s = +1$.

2.3.1. **Axioms for hyperrings.** A *hyperring* $(H, \cdot, \oplus, 0, 1)$ consists of

- An Abelian monoid-with-zero $(H, \cdot, 0, 1)$ satisfying
  
  (M1) \ $\cdot$ is associative and commutative,
  
  (M2) \ $1 \cdot x = x$ for all $x$,
  
  (M3) \ $0 \cdot x = 0$ for all $x$.

  These are the axioms $(R, \cdot, 0, 1)$ satisfies when $R$ is a commutative ring.

- A binary hyperoperation $\boxplus : H \times H \to \mathcal{P}(H) \setminus \{\emptyset\}$ which has the following axioms:
  
  (H1) \ $\boxplus$ is commutative and associative, meaning
  
  $a \boxplus b = b \boxplus a$ and $\bigcup_{d \in b \boxplus c} a \boxplus d = \bigcup_{d \in a \boxplus b} d \boxplus c$,

  (H2) \ $0 \boxplus a = \{a\}$ for all $a$,

  (H3) \ for every $a$ there exists a unique $-a$ such that $a \boxplus (-a) \ni 0$,

  (H4) \ $a \cdot (b \boxplus c) := \{ad : d \in b \boxplus c\} = ab \boxplus ac$,

  (H5) \ $a \in b \boxplus c$ if and only if $(-b) \in a \boxplus c$. 

(H,·,0,1) is a hyperfield if additionally 0 ≠ 1 and every non-zero element of H has a multiplicative inverse, in which case (H×,·,1) is an Abelian group.

**Remark 2.4.** (H3) implies that a ∈ b ⊕ c if and only if 0 ∈ (−a) ⊕ b ⊕ c (because the inverse is unique).

We leave the following lemma as an easy exercise.

**Lemma 2.5.** If R is any commutative ring and G ⊆ R× is any multiplicative subgroup, then H = R/G is a hyperring with the usual multiplication, 0 and 1 and whose hyperaddition is defined by

\[ [a] ⊕ [b] = \{ [a' + b'] : a' ∈ [a], b' ∈ [b] \} \]

**Remark 2.6.** This lemma implies that any field is a hyperfield (G = {1}) and that the hyperrings in Section 2.3 satisfy the above axioms. Here

- S ∼= R/R>0,
- T ∼= C[[t^R]]/ν_C^{-1}(0),
- tR ∼= R[[t^R]]/ν_R^{-1}(0),
- \( O_t^R \) ∼= O_R[[t^R]]/ν_R^{-1}(0).

### 2.4. Morphisms of hyperfields.

A map \( f: H_1 → H_2 \) between two hyperfields is a function satisfying \( f(0) = 0, f(1) = 1, f(ab) = f(a)f(b) \) and \( f(a ⊕ b) ⊆ f(a) ⊕ f(b) \).

In the following examples and proposition, let R be any commutative ring.

**Example 2.7.** Maps \( v: R → S \) correspond to a prime ideal \( p = v^{-1}(0) \) and an ordering on \( R/p \) given by 0 < x if \( v(x) = +1 \).

**Example 2.8.** Maps \( v: R → T \) correspond to a prime ideal \( p = v^{-1}(0) \) and a rank-1 valuation on Frac(R/p).

**Example 2.9.** Maps \( v: R → tR \) correspond to
- a prime ideal \( p = v^{-1}(0) \);
- a rank-1 valuation on Frac(R/p);
- an non-Archimedean-ordering on Frac(R/p).

For our purposes, we are interested in the following morphisms

\[
\begin{array}{cccc}
R & \xrightarrow{\text{sign}} & R[[t^R]] & \xrightarrow{v_R} \quad C[[t^R]] \\
|t=0| & \downarrow & |t=0| & \downarrow \\
S & \xrightarrow{\text{sign}} & O_t^R & \xrightarrow{|·|} \quad T
\end{array}
\]

The unnamed arrows are inclusions. The map sign : R → S is the usual sign function. The maps sign : tR → S and |·| : tR → T are the projection onto the first and second factor respectively, with sign(∞) = 0 and |∞| = ∞.
The map \( \cdot |_{t=0} \) is defined for \( \mathcal{O}_{\mathbb{R}[[t]]} \) as evaluation at \( t = 0 \) and for \( \mathcal{O}_{\mathbb{R}} \) as \( (s,0)|_{t=0} = s \) and \( a|_{t=0} = 0 \) otherwise.

2.5. Polynomials and Multiplicities over a Hyperfield. A polynomial over a hyperfield \( H_1 \) is a formal sum \( p(x) = \sum_{i=0}^{n} c_i x^i \) with \( c_i \in H_1 \). If \( f: H_1 \to H_2 \) is a morphism of hyperfields, then \( f(p) := \sum_{i=0}^{n} f(c_i) x^i \).

Lemma 2.10. If \( \sum a_i x^i \in \bigoplus_{i=j+k} b_j c_k \) for all \( i \) and \( f \) is any morphism of hyperfields, then \( \sum f(a_i) x^i \in \left( \sum f(b_j) x^j \right) \left( \sum f(c_k) x^k \right) \).

Proof. By Remark 2.4, \( a \in b \bigoplus c \) implies \( 0 \in (-a) \bigoplus b \bigoplus c \) implies \( 0 \in (-f(a)) \bigoplus f(b) \bigoplus f(c) \) implies \( f(a) \in f(b) \bigoplus f(c) \).

By induction on the number of summands, \( a_i \in \bigoplus_{i=j+k} b_j c_k \) implies

\[ f(a_i) \in \bigoplus_{i=j+k} f(b_j) f(c_k) \]

as required. \( \square \)

Baker and Lorscheid define zeroes and multiplicities over a hyperfield as follows [BL18, Lemma A, Definition 1.4].

Proposition-Definition. Let \( p \) be a polynomial over a hyperfield \( H \). The following are equivalent:

1. \( 0 \in \bigoplus_{i=0}^{n} c_i a^i \)

2. There exist elements \( d_0, \ldots, d_{n-1} \in H \) such that

\[ c_0 = -ad_0, c_i \in (-ad_i) \bigoplus d_{i-1} \text{ for } i \in \{1, \ldots, n-1\}, c_n = d_{n-1}. \]

Baker and Lorscheid (after Viro) write \( 0 \in p(a) \) if (1) is satisfied. The classification (2) is new to Baker and Lorscheid’s work and they abbreviate it as \( p \in (x - a)^q \) where \( q = \sum d_i x^i \) —this is compatible with the notation introduced in Lemma 2.10. We say that \( a \) is a root (or zero) of \( p \) if these conditions hold.

Definition. For a polynomial \( p(x) = \sum c_i x^i \in H[x] \) and \( a \in H \), we define

\[ \text{mult}_a(p) = \begin{cases} 0 & p(a) \not\ni 0 \\ 1 + \max \{ \text{mult}_a(q) : p \in (x - a)q \} & p(a) \ni 0. \end{cases} \]

There are several pathologies to multiplicities over hyperfields. For instance, a polynomial can have infinitely many roots and even when there are finitely many roots, the sum of the multiplicities might exceed the degree of the polynomial. Additionally, the factorization \( p \in (x - a)q \) is generally not unique and \( \text{mult}_a(q) \) is generally not independent of \( q \). We point the reader to [BL18] for examples and further details.
2.6. Newton Polygons.

Definition. If \( p = \sum_{i=0}^{n} c_i x^i \) is a polynomial over a valued field \((K, v)\), its Newton polygon, \( \text{Newt}(p) \), is the lower convex hull of \( \{(i, v(c_i)) : i = 0, \ldots, n\} \). That is, it is the set defined by the lower inequalities of the convex hull where a lower inequality is an inequality \( \langle u, x \rangle \geq 0 \) with \( u \) in the upper half-plane.

The definition of \( \text{Newt}(p) \) only depends on the valuation of the coefficients, therefore it makes sense to define \( \text{Newt}(p) \) the same way when \( p \in T[x] \). That is, \( \text{Newt}(p) \) is the lower convex hull of \( \{(i, c_i) : i = 0, \ldots, n\} \) when \( p = \sum c_i x^i \in T[x] \). When \( p = \sum c_i x^i \in \mathbb{R}[x] \), we define \( \text{Newt}(p) = \text{Newt}(|p|) \) where \( |p| := \sum |c_i| x^i \in T[x] \).

An edge of \( \text{Newt}(p) \) will always mean a bounded edge. The horizontal length of such an edge is the length of its projection onto the \( x \)-axis.

Example 2.11. Consider \( p = (x - t^n)^n \in \mathbb{C}[[t^R]][x] \). The valuation of the \( i \)-th coefficient is
\[
v_{\mathbb{C}} \left( \binom{n}{i} t^{n-i}a \right) = (n-i)a.
\]
Therefore, the Newton polygon of \( p \) is the lower convex hull of \( \{(i, (n-i)a) : 0 \leq i \leq n\} \). This has a single edge of slope \(-a\) and horizontal length \( n \). ♦

Example 2.12. Let \( p = 2 + \infty x + 1 x^2 + (-\frac{1}{2}) x^3 + (-1) x^4 + \frac{1}{2} x^5 + 1 x^6 \in T[x] \). Then the Newton polygon of \( p \) is pictured in Figure 1. This Newton polygon

![Figure 1. Newton polygon of
\( 2 + \infty x + 1 x^2 + (-\frac{1}{2}) x^3 + (-1) x^4 + \frac{1}{2} x^5 + 1 x^6 \)](image)

has three edges: \( \sigma_1, \sigma_2, \sigma_3 \) with horizontal lengths 3, 1 and 2 respectively. The slopes of the edges are respectively \(-5/6, -1/2\) and 1. ♦

2.7. Sign changes. For a polynomial \( p \) over \( \mathbb{S} \) or over any ordered field, we define \( \Delta(p) \) to be the number of sign changes in the coefficients of \( p \).

Example 2.13. If \( p = 1 + x + 0 x^2 + 0 x^3 - x^4 + 0 x^5 - x^6 + x^7 \in \mathbb{S}[x] \), then \( \Delta(p) = 2 \): one sign change from \( x \) to \(-x^4\), one from \(-x^6\) to \( x^7 \). ♦
For polynomials over \(\mathbf{T}\mathbb{R}\), what we are interested in are the number of sign changes along an edge \(\sigma\) of \(\text{Newt}(p)\). That is, \(\Delta(p_\sigma)\) where

\[
p_\sigma = \sum \{ c_i x^i : (i, -\log |c_i|) \in \sigma \}.
\]

If \(a\) is a root of \(p\) and \(\sigma\) is the edge of \(\text{Newt}(p)\) corresponding to \(a\), we define \(\Delta_a(p) = \Delta_\sigma(p) := \Delta(p_\sigma)\). If \(a\) is not a root of \(p\), then we define \(\Delta_a(p) = \Delta_\sigma(p) = 0\).

### 3. The Classical Multiplicity Formula

**Theorem B.** Let \(K\) be a real-closed, non-Archimedean-ordered valued field with residue field \(k\). Let \(P(x) = \sum_{i=0}^n c_i x^i \in K[x]\) be a polynomial. Suppose the Newton polygon of \(P\) has edges \(\sigma_1, \ldots, \sigma_r\) with slopes \(\lambda_1, \ldots, \lambda_r\) respectively. Let \(P_j = \sum \{ c_i x^i : (i, -\log |c_i|) \in \sigma_j \}\).

Factor \(P\) as \(P = c_0 Q(x) \prod_{i=1}^r (x - \alpha_i)\) where \(Q\) is irreducible, \(\deg Q > 1\) and \(\alpha_1, \ldots, \alpha_r\) are the roots of \(P\) in \(K\). Then

1. for all \(i\), there exists some \(j\) such that \(\log v(\alpha_i) = \lambda_j\)
2. \(#\{i : \log v(\alpha_i) = \lambda_j, \alpha_i > 0\} \equiv \Delta(P_j) \pmod{2}\).

**Proof.** Statement (1) is a result of the classical Newton polygon rule.

For statement (2), for any \(\gamma \in v(K^\times)\), let \(t^\gamma\) denote a fixed element of \(K\) with \(v(t^\gamma) = \gamma\) and \(t^\gamma > 0\).

Fix \(j \in \{1, \ldots, r\}\). Then the polynomial \(P(t^\lambda x)\) has roots \(\alpha_i t^{-\lambda_i}\) and the slopes of \(\text{Newt}(P(t^\lambda x))\) are \(\{\lambda_j - \lambda_i : i = 1, \ldots, r\}\). In particular, \(\sigma_j\) now has slope 0 after this transformation and hence \(\gamma := v(c_i)\) is constant for each monomial \(c_i x^i\) of \(P_j(t^\lambda x)\).

Consider the polynomial \(R(x) := t^{-\gamma} P(t^\lambda x)\). The corresponding edge \(\sigma_j\) of \(\text{Newt}(R)\) has slope 0 and lies along the \(x\)-axis. In particular, \(R \in \mathcal{O}_K[x]\). The number of real roots of \(R\) with valuation 0 is exactly the number of real roots of \(F\) with valuation \(\lambda_j\) and the signs of the roots match as well.

The residue of \(R\) mod \(m_K\) has the same support as \(P_j\), namely

\[
\bar{R}(x) = \sum \{ a_i t^{\lambda_j - \gamma} x^i : (i, -\log |a_i|) \in \sigma_j \}.
\]

Now factor \(\bar{R}\) in \(k[\sqrt{-1}]\) as \(cx^m \prod_{i=1}^s (x - \beta_i)\). The roots \(\beta_1, \ldots, \beta_s\) are the residues of the complex roots of \(R\) with valuation 0. These include residues of real roots plus pairs of residues of complex-conjugate non-real roots. For example, we might have a pair of complex-conjugate roots in \(K[\sqrt{-1}]\), whose residue in \(k[\sqrt{-1}]\) is real. They will have the same residue since the valuation on \(K[\sqrt{-1}]\) is Galois invariant.

Therefore, the number of positive, real roots of \(F\) with valuation \(\lambda_j\) is congruent, mod 2, to the number of positive, real roots of \(\bar{R}\). By Descartes’ rule, this is counted by the number of sign changes of \(P_j\). \(\square\)

**Remark 3.1.** Theorem B can fail if the ordering is incompatible with the valuation. For example, in \(\mathbb{Q}\) with the \(p\)-adic valuation and the usual
(Archimedean) order, the polynomial \(-p + (p - 1)x + x^2 = (x + p)(x - 1)\) does not have a positive root with valuation 1.

Corollary 3.2. Over \(K = \mathbb{R}[\mathbb{R}]\), the number of sign changes of \(P_j\) is also congruent, mod 2, to the number of complex roots of \(F\) with valuation \(\lambda_j\), whose leading term is real and positive.

The difference here is that there is a canonical choice of \(t^\gamma\) for every \(\gamma \in \mathbb{R}\) as well as a notion of “leading-term.”

Proof. From the proof of Theorem B, the non-real roots came in pairs. One of those sets of pairs is the pairs of roots whose leading term is real but whose higher-order terms are non-real. When we add these pairs back into the count, we obtain this corollary. \(\square\)

4. The Baker-Lorscheid Multiplicity Formula

Proposition 4.1. A sum \(a_1 \oplus \cdots \oplus a_n\) in \(\mathbb{IR}\) contains \(\infty\) if and only if

1. The minimum of \(|a_1|, \ldots, |a_n|\) is achieved twice.
2. The minimum is achieved with opposite signs. That is, there exists \(i\) and \(j\) such that \(a_i = -a_j\) and \(|a_i| = |a_j| = \min\{|a_1|, \ldots, |a_n|\}\).

Proof. If \(\infty e a_1 \oplus \cdots \oplus a_n\) then \(\infty \in |a_1| \oplus \cdots \oplus |a_n|\). In analogy with valued fields, if the minimum of a sum over \(\mathbb{T}\) occurs just once, then the whole sum evaluates to that minimum. For let \(|a_1|\) be the unique minimum. Then \(|a_1| \oplus |a_2| = \{|a_1|\}\) and \(|a_1| \oplus |a_3| = \{|a_1|\}\) and so on. Therefore, for \(\infty\) to be in this sum, the minimum has to occur at least twice.

Next, suppose that after permuting the summands, \(\{i: |a_i| = \min_j |a_j|\} = \{1, \ldots, m\}\). Then by the same reasoning as above, \(a_1 \oplus \cdots \oplus a_n = a_1 \oplus \cdots \oplus a_m\).

Now applying the sign morphism, we get \(0 \in \text{sign}(a_1) \oplus \cdots \oplus \text{sign}(a_m)\) which is only possible if there is a pair of signs which is different.

Conversely, suppose the minimum is \(|a_1| = |a_2|\) and \(a_1 = -a_2\). Then one checks that \(a_1 \oplus \cdots \oplus a_n = a_1 \oplus a_2 \ni \infty\). \(\square\)

Corollary 4.2. If \(p(x) = \sum_{i=0}^n c_i x^i\) is a polynomial over \(\mathbb{IR}\), then a nonzero \(a\) is a root of \(p\) if and only if

1. \(\min\{|c_i a|^i\}\) is achieved twice.
2. \(\min\{|c_i a|^i\}\) is achieved with opposite signs.

Theorem A. Let \(p = \sum_{i=0}^n c_i x^i \in \mathbb{IR}[x]\). Let \(a\) be a positive root of \(p\). Then \(\text{mult}_a(p) = \Delta_a(p)\).

The “lifting” part of the Theorem A, as stated in the introduction, is the content of Theorem B.

Proof. The goal is to maximize \(\text{mult}_a(q)\) over all factorizations \(p \in (x-a)q\). That is, we claim that if \(p \in (x-a)q\) then:

1. \(\Delta_a(q) < \Delta_a(p)\).
2. There exists a \(q\) such that \(\Delta_a(q) = \Delta_a(p) - 1\).
The Theorem follows from these two claims by induction.

The proof of Claim (1) is identical to the corresponding proof for polynomials over \( S \) [BL18, proof of Theorem 3.1]. To make this connection clear, we apply the same transformations as in the proof of our Theorem B. That is: first consider \( p(ax) \) and then let \( r(x) = \gamma^{-1} p(ax) \) where \( \gamma = |c_i| \) is constant for each monomial \( c_i x^i \) of \( p_\sigma(ax) \).

Then, as before, \( \text{Newt}(r) \) lies above the \( x \)-axis with the edge corresponding to \( \sigma \) now having a slope of 0 and lying on the \( x \)-axis. So when we take \( r|_{t=0} \in S[x] \), we are recording just the signs of the monomials in \( p_\sigma \).

A factorization \( p \in (x - a)q \) corresponds to a factorization \( r \in (x - 1)s \) where \( s(x) = \gamma^{-1} q(ax) \). Applying the morphism \( : t_0 : O_{\mathbb{TR}} \to S \), we get a factorization \( r|_{t=0} \in (x - 1)s|_{t=0} \in S[x] \) by Lemma 2.10. Therefore, applying Baker and Lorscheid’s proof to this factorization, we have
\[
\Delta_\sigma(p) = \Delta(r|_{t=0}) > \Delta(s|_{t=0}) = \Delta_\sigma(q).
\]

For claim (2), it will be convenient to replace \( p(x) \) by \( r(x) \). That is, we may assume that \( a = (1,0) \) and that \( \sigma \) has slope 0 and lies on the \( x \)-axis and that \( \text{Newt}(p) \) lies on or above the \( x \)-axis.

First, let’s fix our notation. Write
\[
p = c_0 + c_1 x + \cdots + c_n x^n
\]
and suppose \( \sigma \) is the edge from \((n_1, |c_{n_1}|)\) to \((n_2, |c_{n_2}|)\). Suppose that \( c_r \) is the first sign change along \( \sigma \). That is, \( c_i = c_{n_1} \) or \( c_i > |c_{n_1}| \) for \( i = n_1, \ldots, r - 1 \) and \( c_r = -c_{n_1} \).

We will define a polynomial \( q = \sum d_i x^i \) as follows.

- For \( i \leq n_1 \), define \( d_i = -c_k \) where \( k \) is the smallest index where \(|c_k| = \min_{j \leq i} |c_j| \).
- For \( n_1 \leq i \leq r \), let \( d_i = -c_{n_1} \).
- For \( r \leq i \leq n \), let \( d_i = c_k \) where \( k \) is the smallest index \( i \) such that \(|c_k| = \min \{|c_j| : j > i\} \).

By construction, \( \Delta_\sigma(q) = \Delta_\sigma(p) - 1 \) because we have all the sign changes of \( p \) except for the one at \( i = r \). All that is left to do is to show that \( p \in (x - 1)q \).

For \( i = 0 \) and \( i = n \), we have \( d_0 = -c_0 \) and \( d_{n-1} = c_n \) which confirms the identity \( p \in (x - 1)q \) on the “boundary.” For all other \( i \), the relation \( p \in (x - 1)q \) says that
\[
c_i \in d_{i-1} \circledast (-d_i).
\]

For \( i \leq n_1 \), there are two cases. First, if \(|c_i| < \min_{j < i} |c_j| = |d_{i-1}| \) then \( d_i = -c_i \) in which case \( d_{i-1} \circledast (-d_i) = \{-d_i\} = \{c_i\} \). Second, if \(|c_i| \geq \min_{j < i} |c_j| \) then \( d_i = d_{i-1} \) and \( d_{i-1} \circledast (-d_i) \ni c_i \).

Next, for \( n_1 < i \leq r \), \( d_i \) is constant and \( d_{i-1} \circledast (-d_i) \ni c_i \) for any \( i \) in that range.

Finally, for \( r < i < n \), notice that \(|c_i| \geq |d_{i-1}| \). If
\[
|d_i| = \min \{|c_k| : k > i\} > |d_{i-1}| = \min \{|c_k| : k > i - 1\},
\]

then we must have \( d_{i-1} = c_i \) and hence \( c_i \in d_{i-1} \boxminus (-d_i) \). Otherwise, if \( d_{i-1} = d_i \) then \( c_i \in d_{i-1} \boxminus (-d_i) \) since \( |c_i| \geq |d_{i-1}| \). Lastly, if \( d_{i-1} = -d_i \) that can only happen if \( d_{i-1} = c_i \) and again \( c_i \in d_{i-1} \boxminus (-d_i) \). □

**Remark 4.3.** The \( q \) we define in this proof is the same as the corresponding \( q \) defined in Baker and Lorscheid’s proof [BL18, Theorem 3.1]. Also notice that Newt(\( q \)) is the same as Newt(\( p \)) except for shortening \( \sigma \) by 1.

The following Theorem is an analogue of Remark 1.14 in Baker and Lorscheid’s paper [BL18] concerning lifting polynomials in \( \mathbf{TR}[x] \) to polynomials in \( \mathbf{R}[[t^\mathbf{R}]][x] \) with the maximum number of real roots as allowed by the Theorem A.

**Theorem 4.4.** Let \( p \in \mathbf{TR}[x] \), then there exists a polynomial \( P \in \mathbf{R}[[t^\mathbf{R}]][x] \) such that \( v_\mathbf{R}(P) = p \) and for each edge \( \sigma \) of Newt(\( P \)) with slope \(-|a|\) (\( a \in \mathbf{TR}, a > 0 \)), \( P \) has exactly \( \text{mult}_a(p(x)) \) (resp. \( \text{mult}_{-a}(p(-x)) \)) roots of valuation \(|a|\) whose leading term is real and positive (resp. negative).

**Proof.** For each \( \sigma \), let \( r_\sigma(x) \) be the polynomial \( \gamma^{-1}p(ax) \) that we had in the proof of the Theorem A. Now we use the converse of Descartes’s rule (for example, [Gra99, Theorem 1]) to choose a lift \( R_\sigma(x) \in \mathbf{R}[x] \) of \( r_\sigma|_{t=0} \in \mathbf{S}[x] \) which has exactly \( \text{mult}_a(p(x)) \) (resp. \( \text{mult}_{-a}(p(-x)) \)) positive (resp. negative) real roots.

Now let

\[
\tilde{P}(x) = t^\delta \prod_\sigma R_\sigma(t^{-|a|}x)
\]

where \( \delta \) is chosen such that Newt(\( \tilde{P} \)) = Newt(\( p \)). From \( \tilde{P} \), we can take any choice of monomials not on the boundary of Newt(\( p \)) to a polynomial \( P \) such that \( v_\mathbf{R}(P) = p \).

Note that the leading terms of the roots of \( P_\sigma \) are exactly the roots of \( R_\sigma(x) \in \mathbf{R}[x] \). By construction, \( P \) has correct number of roots of valuation \(|a|\) whose leading term is real and positive/negative. □

It would be interesting to know if we could pick \( P \) such that the roots themselves are real, rather than just their leading terms.

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Email address: tgunn@gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, USA