Transition Functions of Diffusion Processes on the Thoma Simplex

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Abstract. The paper deals with a three-dimensional family of diffusion processes on an infinite-dimensional simplex. These processes were constructed by Borodin and Olshanski in 2009 and 2010, and they include, as limit objects, the infinitely-many-neutral-allels diffusion model constructed by Ethier and Kurtz in 1981 and its extension found by Petrov in 2009.

Each process $X$ in our family possesses a unique symmetrizing measure $M$, called the $z$-measure.

Our main result is that the transition function of $X$ has a continuous density with respect to $M$. This is a generalization of earlier results due to Ethier (1992) and to Feng, Sun, Wang, and Xu (2011). Our proof substantially uses a special basis in the algebra of symmetric functions, which is related to the Laguerre polynomials.

Key words: diffusion process, transition density, Thoma simplex, z-measure, symmetric functions.

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1. Introduction

Ethier and Kurtz [8] constructed a family of diffusion processes $X_\tau$ parameterized by $\tau > 0$ on the Kingman simplex

$$\nabla_\infty = \left\{ \alpha = (\alpha_n) \left| \alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \sum_i \alpha_i \leq 1 \right. \right\},$$

in which every $X_\tau$ has a unique invariant symmetrizing probability measure, being the Poisson–Dirichlet measure $PD(\tau)$ [13] (the Poisson–Dirichlet parameter is usually denoted by $\theta$, but we use this symbol to parameterize the Jack functions). Moreover, Ethier [6] showed that the transition function of $X_\tau$ has a continuous density with respect to $PD(\tau)$.

Petrov [19] discovered a broader family of diffusions on $\nabla_\infty$, which depend on two parameters $(a, \tau)$ and are related to the two-parameter Poisson–Dirichlet distributions $PD(a, \tau)$ [20].

The continuity of transition density for Petrov’s family was proved in the work [9] by Feng, Sun, Wang, and Xu.

The present paper deals with an even broader family of infinite-dimensional diffusion processes, which was constructed by Olshanski in [16] and even earlier (in a special case) by Borodin and Olshanski in [3]. These processes depend on a triplet $(z, z', \theta)$ of parameters. Their state space is the Thoma simplex

$$\Omega := \left\{ (\alpha, \beta) \in \mathbb{R}_{\geq 0}^\infty \times \mathbb{R}_{\geq 0}^\infty \left| \alpha_1 \geq \alpha_2 \geq \cdots; \beta_1 \geq \beta_2 \geq \cdots; \sum \alpha_i + \sum \beta_i \leq 1 \right. \right\},$$

with the topology of pointwise convergence induced by $\mathbb{R}_{\geq 0}^\infty \times \mathbb{R}_{\geq 0}^\infty$. Consider the algebra $\Lambda^0 := \mathbb{R}[p_2^0, p_3^0, \ldots]$ of continuous functions on $\Omega$, where

$$p_k^0 = \sum_{i=1}^\infty \alpha_i^k + (-\theta)^{k-1} \sum_{j=1}^\infty \beta_j^k, \quad k \geq 2. \quad (1)$$

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Let $A_{z,z',\theta} : \Lambda^0 \to \Lambda^0$ be the (formal) second-order differential operator defined by

$$
A_{z,z',\theta} = \sum_{i,j \geq 2} ij(p_i^0 p_{i-j-1}^0 - p_i^0 p_j^0) \frac{\partial^2}{\partial p_i^0 \partial p_j^0} + \sum_{i \geq 2} [(1 - \theta)i(i - 1)p_i^0 - (z + z')ip_{i-1}^0 - i(i - 1)p_i^0 - i\theta^{-1}zz'p_i^0] \frac{\partial}{\partial p_i^0} + \theta \sum_{i,j \geq 1} (i + j + 1)p_i^0 p_j^0 \frac{\partial}{\partial p_{i+j+1}^0}.
$$

(2)

It was shown in [16] that the operator $A_{z,z',\theta}$ is closable in $C(\Omega)$ and its closure generates a diffusion process $X_{z,z',\theta}$ on $\Omega$. In the limit as $\theta \to 0$, the $\beta$-coordinates disappear from the expression (1) for the functions $p_k^0$, so that the Thoma simplex turns into the Kingman simplex. Moreover, if the parameters $(z, z')$ vary together with $\theta$ in an appropriate way, then the operator (2) degenerates into the generator of the Petrov diffusion. In particular, in this way one can obtain the Ethier–Kurtz generator.

Our main result is Theorem 17. It asserts the existence of a transition density for the diffusion process $X_{z,z',\theta}$ with respect to the unique invariant symmetrizing measure (which is called the $z$-measure with Jack parameter) and, thereby, provides an extension of Ethier’s theorem for the processes $X_\tau$.

We derive an explicit expression for a transition density, from which it is seen that the density is continuous. In a recent work by Olshanski [18] our result was applied to prove that the topological support of the $z$-measures is the whole Thoma simplex, which leads to possible applications in other areas (see, e.g., [10]).

Our approach uses ideas of [6] and [9], as well as some new arguments. The central role is played by the Laguerre symmetric functions with Jack parameter $\theta$, which form bases in the algebra of symmetric functions.

The structure of this paper is as follows. At the beginning we recall several facts about the Jack symmetric functions and the process $X_{z,z',\theta}$, which are needed for what follows. In Section 4 we introduce the Laguerre symmetric functions, which are used to analyze $z$-measures. In Section 5 we describe the expansion of a transition density of $X_{z,z',\theta}$ in eigenfunctions of the operator $A_{z,z',\theta}$ (Theorem 17). At the end we give some corollaries of the main result.

### 2. The Jack Symmetric Functions

Here we will give a short reminder about the Jack symmetric functions; see [14] for further details. Let $\Lambda^{(N)} = \mathbb{R}[x_1, \ldots, x_N]^{S_N}$ denote the graded algebra of symmetric polynomials in $x_1, \ldots, x_N$, and let $\iota_N : \Lambda_N \to \Lambda_{N-1}$ be the projection map which acts by setting $x_N = 0$. The **graded algebra $\Lambda$ of symmetric functions** is the projective limit of the graded algebras $\Lambda^{(N)}$ with projection maps $\iota_N$. In other words, $\Lambda$ consists of symmetric formal power series in $x_1, x_2, \ldots$ of finite degree. The component of $\Lambda$ of graded degree $k$ is denoted by $\Lambda_k$. Note that we have the natural truncation map $\pi_N : \Lambda \to \Lambda^{(N)}$ setting $x_{N+1} = x_{N+2} = \cdots = 0$.

Let $\mathcal{Y}_n$ denote the set of partitions $\lambda$ of $n$, that is, the set of sequences of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $|\lambda| := \sum \lambda_i = n$, and let $\mathcal{Y} = \bigsqcup_{n \geq 0} \mathcal{Y}_n$ denote the set of all partitions of nonnegative integers. To each partition $\lambda$ we can assign a collection of boxes in the plane, called the **Young diagram**, consisting of rows of lengths $\lambda_i$. Given a pair of partitions $\mu$ and $\lambda$, we write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for every $i$; in this case, $\lambda/\mu$ denotes the **skew Young diagram**, which is obtained by removing the boxes corresponding to the partition $\mu$ from the Young diagram corresponding to $\lambda$ (see Fig. 1).
The Young diagram corresponding to \((5,3,2)\) (left) and the skew Young diagram corresponding to \((5,3,2)/(2,1)\) (right).

The algebra \(\Lambda\) has several distinguished bases indexed by partitions. The simplest example is the basis of monomial symmetric functions

\[ m_\lambda := \sum_\alpha \prod_{i=1}^\infty x_\alpha^i, \]

where the sum is over all distinct permutations \(\alpha = (\alpha_1, \alpha_2, \ldots)\) of \((\lambda_1, \lambda_2, \ldots)\). Another basis, \(\{p_\lambda\}_{\lambda \in \mathcal{P}}\), is generated by the power sums \(p_k = \sum_i x_i^k\); it is defined by

\[ p_\lambda := \prod_{i=1}^\infty p_{\lambda_i}. \]

Here we set \(p_0 = 1\). Note that \(p_1, p_2, \ldots\) are algebraically independent generators of \(\Lambda\), and hence \(\Lambda \cong \mathbb{R}[p_1, p_2, \ldots]\) (see [14; Chap. I]).

In this work we will use a more complicated basis, or, more precisely, a family \(\{P_\lambda(x; \theta)\}_{\lambda \in \mathcal{P}}\) of bases depending on a positive real parameter \(\theta\). The functions in this family are called the Jack symmetric functions; they are uniquely characterized by the following properties:

(i) \(P_\lambda(x; \theta) = m_\lambda(x) + \sum_{\mu \leq \lambda} a_{\lambda \mu} m_\mu(x)\), where the sum is over the partitions \(\mu\) that are less than \(\lambda\) in the lexicographical order;

(ii) \(P_\lambda(x; \theta)\) are orthogonal with respect to the inner product \(\langle \cdot, \cdot \rangle_{\theta}\) on \(\Lambda\) defined on the basis \(\{p_\lambda\}_{\lambda \in \mathcal{P}}\) by

\[ \langle p_\lambda, p_\mu \rangle_{\theta} := \delta_{\lambda,\mu} z_\lambda \theta^{-l(\lambda)}. \]

Here we use the notation of [14], where the length \(l(\lambda)\) of a partition \(\lambda\) is defined as the number of nonzero parts in the partition \(\lambda\), \(m_i(\lambda)\) denotes the number of parts equal to \(i\) in \(\lambda\), and

\[ z_\lambda := \prod_i (m_i(\lambda) \cdot m_i(\lambda)!) \cdot \delta_{\lambda,\mu}. \]

These properties uniquely define a system of homogeneous symmetric functions in \(\Lambda\). Moreover, for fixed \(\theta\), the functions \(P_\lambda\) with \(|\lambda| = n\) form a basis of \(\Lambda_n\).

We set \(b_\lambda^{(\theta^{-1})} := \langle P_\lambda(x; \theta), P_\lambda(x; \theta) \rangle_{\theta}^{-1}\), \(Q_\lambda(x; \theta) := b_\mu^{(\theta^{-1})} P_\lambda(x; \theta)\); thus, \(\{Q_\lambda\}\) is the basis dual to \(\{P_\lambda\}\), that is,

\[ \langle P_\lambda(x; \theta), Q_\mu(x; \theta) \rangle_{\theta}^{-1} = \delta_{\lambda,\mu}. \]

The \(\theta\)-dimension \(\dim_\theta(\mu, \lambda)\) is defined as the coefficient in the expansion

\[ p_1^{n-|\mu|} P_\mu(x; \theta) = \sum_{|\lambda|=n} \dim_\theta(\mu, \lambda) P_\lambda(x; \theta). \tag{3} \]

By duality this is equivalent to

\[ \dim_\theta(\mu, \lambda) = \langle p_1^{n-|\mu|} P_\mu, Q_\lambda \rangle_{\theta}. \tag{4} \]
When $\mu = \emptyset$, we write simply $\dim \theta(\lambda)$.

In the sequel, we also use the following combinatorial notation. The *Pochhammer symbol* is denoted by $(t)_n$ and given by

$$(t)_n = t(t + 1) \cdots (t + n - 1).$$

The *$\theta$-content* of a box $(i, j)$ is defined by

$$c_\theta(i, j) := (j - 1) - \theta(i - 1).$$

We generalize the Pochhammer symbol, defining it for a skew diagram $\lambda/\mu$ by

$$(z)_{\lambda/\mu, \theta} := \prod_{(i, j) \in \lambda/\mu} (z + c_\theta(i, j)),$$

where the product is over all boxes $(i, j) \in \lambda/\mu$ (that is, $\mu_i < j \leq \lambda_i$). For $\mu = \emptyset$, we use the notation $(z)_{\lambda, \theta} := (z)_{\lambda/\emptyset, \theta}$.

**Remark.** The parameter $\theta$ is equal to $\alpha - 1$ in Macdonald’s notation; see [14; Chap. VI] for further details on the Jack symmetric functions.

### 3. Diffusion Processes on the Thoma Simplex

The *Thoma simplex* $\Omega$ is defined as the subspace of $\mathbb{R}_{\geq 0}^\infty \times \mathbb{R}_{\geq 0}^\infty$ (with the product topology) formed by pairs $\omega = (\alpha, \beta)$ such that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum_{i=1}^\infty \alpha_i + \sum_{j=1}^\infty \beta_j \leq 1.$$  

For the rest of this section, we fix $\theta > 0$. Let $C(\Omega)$ denote the space of continuous functions on $\Omega$ with the supremum norm. We define an algebra morphism $\Lambda \to C(\Omega)$ by setting $p_1 \mapsto 1$ and

$$p_k \mapsto \sum_{i=1}^\infty \alpha_i^k + (-\theta)^{k-1} \sum_{j=1}^\infty \beta_j^k, \quad k \geq 2.$$  

We denote the image of $f \in \Lambda$ in $C(\Omega)$ by $f^\circ$ and the image of the whole algebra $\Lambda$ by $\Lambda^\circ$. The functions $p_k^{n}$ for $n \geq 2$ are algebraically independent (see [16; 9.3]); hence $\Lambda^\circ = \mathbb{C}[p_2^2, \ldots] \cong \Lambda/(p_1 - 1)$. Note that the grading $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ induces a filtration on $\Lambda^\circ$:

$$\Lambda^\circ_{\leq n} = \left( \bigoplus_{i=1}^n \Lambda_i \right)^\circ.$$  

We consider a family of Markov processes on $\Omega$ parameterized by the triples $(z, z', \theta)$ satisfying one of the following conditions:

(i) $z \in \mathbb{C} \setminus \mathbb{R}$ and $z = z'$;

(ii) $\theta$ is rational and both $z$ and $z'$ are real numbers lying in one of the open intervals between two consecutive numbers in the lattice $\mathbb{Z} + \theta \mathbb{Z}$.

In the first (second) case, we say that $(z, z')$ belongs to the *principal series* (respectively, to the *complementary series*). Later on we will also need the *degenerate series* formed by the pairs $(z, z') = (N\theta, c + (N - 1)\theta)$ for $N \in \mathbb{Z}_{>0}$ and $c > 0$.

Consider the operator $A_{z, z', \theta}$ on $\Lambda^\circ$ defined by (2), where $\frac{\partial}{\partial p_i}$ denotes formal differentiation in $\Lambda^\circ \cong \mathbb{R}[p_2^2, p_3^2, \ldots]$. This operator was introduced and studied in [16]; here we will recall its key properties.
Theorem 1 [16; Theorems 9.6 and 9.10].
(i) The operator $A_{z,z'}$ is closable, and its closure is a generator of a Feller Markov process $X_{z,z'}$ on $\Omega$.
(ii) There exists a unique invariant probability measure for $X_{z,z'}$.
(iii) Moreover, this measure is the symmetrizing measure for $X_{z,z'}$.

The invariant measure mentioned in the theorem is denoted by $M_{z,z'}$ and called a $z$-measure on the Thoma simplex. It satisfies the relation
\[
\int_{\Omega} Q_\lambda^z(\omega; \theta) M_{z,z'}(d\omega) = \frac{\dim_\theta(\lambda)(z) \lambda, \theta(z', \lambda, \theta)}{|\lambda|! (\theta^{-1} z z') |\lambda|} \tag{5}
\]
for each $\lambda$. Moreover, this relation uniquely determines $M_{z,z'}$.

Theorem 2 [12; Theorem B]. The relation
\[
M(\lambda) = \dim_\theta(\lambda) \int_{\Omega} P^\alpha_\lambda(\omega; \theta) M(d\omega)
\]
gives a one-to-one correspondence between the probability measures $M$ on $\Omega$ and the nonnegative functions $M(\lambda)$ on $\mathbb{Y}$ such that $M(\emptyset) = 1$ and
\[
M(\lambda) = \sum_{\lambda \subseteq \mu} M(\mu) \frac{\dim_\theta(\lambda) \dim_\theta(\lambda, \mu)}{\dim_\theta(\mu)}.
\]
In particular, for every probability measure $M$ on $\Omega$, we have
\[
\int_{\Omega} P^\alpha_\lambda(\omega; \theta) M(d\omega) \geq 0,
\]
because $\dim_\theta(\lambda) > 0$. Hence the functions $P^\alpha_\lambda$ are nonnegative on $\Omega$.

4. The Laguerre Symmetric Functions

In the previous section we introduced the measures $M_{z,z'}$, which play an important role in our work. In this section we will describe our main tool for working with these measures.

The classical Laguerre polynomials can be defined as the eigenfunctions of the differential operator
\[
D^{(L)} = x \frac{d^2}{dx^2} + (c - x) \frac{d}{dx}.
\]
These polynomials are orthogonal in $L^2(\mathbb{R}_{\geq 0}, \gamma_c)$, where $\gamma_c$ is the gamma distribution, that is,
\[
\gamma_c(dr) = \frac{1}{\Gamma(c)} r^{c-1} e^{-r} dr.
\]
There is an explicit formula for the Laguerre polynomials:
\[
L^n_c = \frac{c}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{(-x)^j}{(c)_j}.
\]

Fix $\theta > 0$. The generalized Laguerre polynomials $L^{c,N,\theta}_n(x_1, \ldots, x_N)$ are the symmetric polynomial eigenfunctions of the $N$-variable generalization of $D^{(L)}$ given by
\[
D^{(L)}_N := \sum_{j=1}^{N} \left( x_j \frac{\partial^2}{\partial x_j^2} + (c - x_j) \frac{\partial}{\partial x_j} + 2\theta \sum_{k \neq j} \frac{x_j - x_k}{x_j - x_k} \frac{\partial}{\partial x_j} \right).
\]

They were described in the paper [1], in which some of their properties needed for what follows were proved. Recall that $P_\lambda$ and $Q_\lambda$ are the Jack symmetric functions, which can be viewed as symmetric polynomials in $N$ variables via the truncation map $\pi_N$. 

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**Proposition 3** [1; Proposition 4.3]. For all partitions \( \lambda \),

\[
L^{c,N,\theta}_\lambda(x) = \sum_{\mu \leq \lambda} (-1)^{|\lambda| - |\mu|} \frac{\dim_\theta(\mu, \lambda)}{(|\lambda| - |\mu|)!} (N\theta)e^{(N-1)\theta}Q_{\mu}(x; \theta),
\]

\[
Q_\lambda(x; \theta) = \sum_{\mu \leq \lambda} \frac{\dim_\theta(\mu, \lambda)}{(|\lambda| - |\mu|)!} (N\theta)e^{(N-1)\theta}L^{c,N,\theta}_\mu(x),
\]

where \( x \) stands for \((x_1, \ldots, x_N)\).

We define a probability measure \( \mu_{c,N} \) on the space \( \mathbb{R}^N_{\text{ord}} = \{x_1 \geq \cdots \geq x_N \mid x_i \in \mathbb{R}_{\geq 0}\} \) by

\[
\mu_{c,N}(dx) := \text{const} \prod_{i=1}^N x_i^{c-1} e^{-x_i} \prod_{1 \leq j < k \leq N} |x_j - x_k|^{2\theta} dx_1 \cdots dx_n. \tag{6}
\]

**Proposition 4** [1; Proposition 4.10]. The Laguerre symmetric polynomials are orthogonal in \( L^2(\mathbb{R}^N_{\text{ord}}, \mu_{c,N}) \). More precisely,

\[
\langle L^{c,N,\theta}_\lambda, L^{c,N,\theta}_\mu \rangle = \delta_{\lambda,\mu} \delta \theta b^{(\theta^{-1})}.
\]

**Remark.** The notation used in [1] is different from ours; see Appendix A, in which the notations are matched.

Now, using ideas of [5] and [17], we will build an “analytic continuation” of the Laguerre symmetric polynomials, treating \( L^{c,N,\theta}_\lambda \) as a degeneration of some symmetric function with coefficients in \( \mathbb{C}[z, z'] \). Recall that \( \pi_N \) denotes the \( N \)th truncation map \( \Lambda \rightarrow \Lambda^{(N)} \).

**Theorem 5.** For any partition \( \lambda \), there is a unique symmetric function \( \Sigma_\lambda \) in \( \Lambda \otimes \mathbb{C}[z, z'] \) such that

\[
\pi_N(\Sigma_\lambda|_{z=N\theta, z'=c+(N-1)\theta}) = L^{c,N,\theta}_\lambda,
\]

for any \( N \geq l(\lambda) \) and \( c > 0 \).

**Definition 6.** The symmetric function \( \Sigma_\lambda \) is called the Laguerre symmetric function.

**Proof of Theorem 5.** For every \( \lambda \) such that \( l(\lambda) \leq N \), we have

\[
\pi_N(\Sigma_\lambda(x; \theta)) = Q_\lambda(x_1, \ldots, x_N; \theta) \neq 0.
\]

Thus, the existence of the Laguerre functions follows from Proposition 3, because we can produce an expression satisfying (7):

\[
\Sigma_\lambda(x) = \sum_{\mu \leq \lambda} (-1)^{|\lambda| - |\mu|} \frac{\dim_\theta(\mu, \lambda)}{(|\lambda| - |\mu|)!} (N\theta)e^{(N-1)\theta}Q_{\mu}(x; \theta).
\]

To prove uniqueness, it is enough to show that the conditions

\[
\pi_N(f|_{z=N\theta, z'=c+(N-1)\theta}) = 0 \tag{8}
\]

force \( f = 0 \). Take \( f \) satisfying (8) and let \( a_\lambda(z, z') = \langle f, Q_\lambda(x; \theta) \rangle_{\theta} \in \mathbb{C}[z, z'] \) denote the coefficients in the expansion of \( f \) in the basis of Jack functions. Then, for any \( N \geq l(\lambda) \) and \( c > 0 \), we have

\[
a_\lambda(N\theta, c + (N-1)\theta) = 0.
\]

Therefore, \( a_\lambda(z, z') \equiv 0 \), because

\[
\{(z, z') = (N\theta, c + (N-1)\theta) \mid N \geq l(\lambda), c > 0\} \subset \mathbb{C}^2
\]

is a uniqueness set for polynomials in two variables. Hence \( f = 0 \), which proves uniqueness. \( \square \)
Corollary 7. For any partition \( \lambda \),
\[
\mathcal{L}_\lambda(x) = \sum_{\mu \subseteq \lambda} (-1)^{|\lambda|-|\mu|} \dim_\theta(\mu, \lambda) / (|\lambda| - |\mu|)! (z)_{\lambda/\mu, \theta}(z')_{\lambda/\mu, \theta}Q_\mu(x; \theta). \tag{9}
\]

Note that the functions \( \mathcal{L}_\lambda(x) \) are nonhomogeneous symmetric functions with highest-degree term equal to \( Q_\lambda(x; \theta) \). We introduce the following natural filtration of the graded algebra \( \Lambda \otimes \mathbb{C}[z, z'] \):
\[
\Lambda_{\leq n} \otimes \mathbb{C}[z, z'] := \bigoplus_{i=0}^n \Lambda_i \otimes \mathbb{C}[z, z'].
\]

Proposition 8. The Laguerre functions \( \mathcal{L}_\lambda \) with \( |\lambda| \leq n \) form a \( \mathbb{C}[z, z'] \)-basis of \( \Lambda_{\leq n} \otimes \mathbb{C}[z, z'] \).

In particular,
\[
P_\lambda(x; \theta) = (b_\mu^{(\theta-1)})^{-1} \sum_{\mu \subseteq \lambda} \dim_\theta(\mu, \lambda) / (|\lambda| - |\mu|)! (z)_{\lambda/\mu, \theta}(z')_{\lambda/\mu, \theta} \mathcal{L}_\mu(x). \tag{10}
\]

Proof. Note that (9) and (10) imply that the mutually inverse transition matrices between \( P_\lambda \) and \( \mathcal{L}_\lambda \) are nonsingular and upper-triangular (with respect to the order \( \lambda \subseteq \mu \)). Since the Jack functions \( P_\lambda(x; \theta) \) with \( |\lambda| \leq n \) form a basis of \( \Lambda_{\leq n} \), it follows that the functions \( \mathcal{L}_\lambda(x) \) with \( |\lambda| \leq n \) form a basis of \( \Lambda_{\leq n} \otimes \mathbb{C}[z, z'] \). Thus, it is enough to prove (10).

Using (9), we write the right-hand side of (10) as a linear combination of functions \( P_\mu \) with coefficients in \( \mathbb{C}[z, z'] \). Note that \( \deg \mathcal{L}_\lambda = |\lambda| \), and hence only functions \( P_\mu \) with \( |\mu| \leq |\lambda| \) will occur in the decomposition. By Proposition 3 (10) holds for \( (z, z') = (N \theta, c + (N - 1) \theta) \) with \( N \geq l(\lambda) \). Hence (10) holds in \( \Lambda[z, z'] \) by the same argument as at the end of the previous proof. \( \square \)

In order to describe the orthogonality of functions \( \mathcal{L}_\lambda \), we define the Thoma cone \( \tilde{\Omega} \) as the subspace of \( \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) consisting of triples \((\tilde{\alpha}, \tilde{\beta}, r)\), where \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots) \) and \( \tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2, \ldots) \), such that
\[
\tilde{\alpha}_1 \geq \tilde{\alpha}_2 \geq \cdots \geq 0, \quad \tilde{\beta}_1 \geq \tilde{\beta}_2 \geq \cdots \geq 0, \quad \sum_i \tilde{\alpha}_i + \sum_i \tilde{\beta}_i \leq r.
\]

We define an embedding of \( \Lambda \) in the space of continuous functions on \( \tilde{\Omega} \) by
\[
p_1 \mapsto r, \quad p_k \mapsto \sum_i \tilde{\alpha}_i^k + (-\theta)^{k-1} \sum_i \tilde{\beta}_i^k, \quad k \geq 2.
\]

Note that the Thoma cone is a cone with base \( \Omega \) and vertex \( \tilde{0} = (0, 0, 0) \), i.e., \( \tilde{\Omega} \setminus \{\tilde{0}\} \cong \Omega \times \mathbb{R}_{>0} \). The isomorphism sends each point \((\tilde{\alpha}, \tilde{\beta}, r)\) to \((\tilde{\alpha}_i/r, \tilde{\beta}_i/r, r) \in \Omega \times \mathbb{R}_{>0} \). The lifting of a measure \( M_{z, z', \theta} \) is defined as the probability measure \( \tilde{M}_{z, z', \theta} \) on \( \tilde{\Omega} \setminus \{\tilde{0}\} = \mathbb{R}_{>0} \times \Omega \) given by
\[
\tilde{M}_{z, z', \theta} = M_{z, z', \theta} \otimes \gamma_{zz'-1},
\]
where \( \gamma_{zz'-1} \) is the gamma distribution defined above.

It is readily seen that, for any \( f \in \Lambda_n \), we have
\[
\int_{\tilde{\Omega}} f(x) \tilde{M}_{z, z', \theta}(dx) = \int_{\mathbb{R}_{>0}} r^n \gamma_{\theta-1zz'}(dr) \int_{\Omega} f^\circ(\omega) M_{z, z', \theta}(d\omega) \frac{\Gamma(\theta^{-1}zz' + n)}{\Gamma(\theta^{-1}zz')} \int_{\Omega} f^\circ(\omega) M_{z, z', \theta}(d\omega)
\]
\[
= (\theta^{-1}zz')_n \int_{\Omega} f^\circ(\omega) M_{z, z', \theta}(d\omega). \tag{11}
\]

The following proposition shows that \( \tilde{M}_{z, z', \theta} \) is in fact an “analytic continuation” of the measures \( \mu_{c,N} \) defined by (6).
Proposition 9. For \((z, z') = (N\theta, c + (N - 1)\theta)\), the measure \(\tilde{M}_{z, z', \theta}\) degenerates to the measure \(\mu_{c, N}\) on

\[
\mathbb{R}^N_{\text{ord}} = \{ (\tilde{\alpha}, \tilde{\beta}, r) \in \tilde{\Omega} : \alpha_{N + 1} = \alpha_{N + 2} = \cdots = 0, \beta = 0, r = \alpha_1 + \cdots + \alpha_N \}.
\]

The proof of this proposition is based on the consideration of the measures \(M_n(\lambda)\) defined on the partitions of \(n\) by

\[
M_n(\lambda) := \frac{\text{dim}_\theta(\lambda)^2(z)_{\lambda, \theta}(z')_{\lambda, \theta}}{\lambda! (\theta^{-1}zz')_{\lambda, \theta}^{(\theta - 1)}}.
\]

The measure \(M_{z, z', \theta}\) can be represented as the limit of these measures, and the proposition follows by a direct computation of \(M_n(\lambda)\) for \((z, z') = (N\theta, c + (N - 1)\theta)\); see [2; Remark 1.10] and [11; Sec. 12] for a more detailed explanation of this fact.

Theorem 10. The functions \(\mathcal{L}_\lambda\) are orthogonal in \(L^2(\tilde{\Omega}, \tilde{M}_{z, z', \theta})\):

\[
\langle \mathcal{L}_\lambda, \mathcal{L}_\mu \rangle_{L^2(\tilde{\Omega}, \tilde{M}_{z, z', \theta})} = \delta_{\lambda, \mu}(z)_{\lambda, \theta}(z')_{\lambda, \theta}^{(\theta - 1)}.
\]

**Proof.** First, note that \(\int_{\tilde{\Omega}} P_\lambda(\omega) \tilde{M}_{z, z', \theta}(d\omega)\) is a rational function in \(z, z'\) (by (5)). Hence, for any \(f \in \Lambda[z, z']\),

\[
\int_{\tilde{\Omega}} f(\omega) \tilde{M}_{z, z', \theta}(d\omega) \in \mathbb{R}(z, z').
\]

In particular, \(\langle \mathcal{L}_\lambda, \mathcal{L}_\mu \rangle_{L^2(\tilde{\Omega}, \tilde{M}_{z, z', \theta})}\) is a rational function in \(z, z'\).

For \((z, z') = (N\theta, c + (N - 1)\theta)\), the orthogonality relations hold by Proposition 4. Therefore, these relations hold for every \((z, z')\).

**Corollary 11.** For any partitions \(\lambda\) and \(\mu\),

\[
\langle P_\lambda, \mathcal{L}_\mu \rangle_{L^2(\tilde{\Omega}, \tilde{M}_{z, z', \theta})} = (\theta_{\lambda}^{(\theta - 1)})^{-1} b_{\mu}^{(\theta - 1)} \frac{\text{dim}_\theta(\mu, \lambda)}{|\lambda| - |\mu|} (z)_{\lambda, \theta}(z')_{\lambda, \theta}.
\]

**Proof.** This assertion follows directly from the orthogonality relations for \(\mathcal{L}_\lambda\) and Proposition 8.

**Remark.** The Laguerre symmetric functions for \(\theta = 1\) were described in [17]. They were also described from a different perspective in [5], where the Laguerre symmetric functions were defined as the eigenfunctions of the differential operator

\[
\mathcal{D} := \sum_{i=1}^{\infty} \left( -ip_i \frac{\partial}{\partial p_i} + (z + z')(i + 1)p_i \frac{\partial}{\partial p_{i+1}} + (1 - \theta)i(i + 1)p_i \frac{\partial}{\partial p_{i+1}} \right) + \frac{zz'}{\theta} \frac{\partial}{\partial p_1} + \sum_{i,j=1}^{\infty} \left( ij p_{i+j-1} \frac{\partial^2}{\partial p_i \partial p_j} + \theta(i + j + 1)p_ip_j \frac{\partial}{\partial p_{i+j+1}} \right).
\]

Similarly to the process on the Thoma simplex considered here, one can define a process on \(\tilde{\Omega}\) by using \(\mathcal{D}\) as a pregenerator. In this case, \(\tilde{M}_{z, z', \theta}\) turns out to be the symmetrizing measure of the process; see [4] for more details on the resulting process on \(\tilde{\Omega}\).
5. The Eigenfunction Expansion of the Transition Density

In this section, following the general scheme of proof in [6], we show the existence and the continuity of the transition density of \( X_{z,z',\theta} \) with respect to \( M_{z,z',\theta} \).

Fix \((z,z',\theta)\) in either the principal or the complementary series and let \( \langle \cdot, \cdot \rangle \) denote the inner product in \( L^2(\Omega, M_{z,z',\theta}) \). First, we will describe the spectral structure of \( A_{z,z',\theta} \) as an essentially self-adjoint operator on \( L^2(\Omega, M_{z,z',\theta}) \).

**Theorem 12.** (i) The spectrum of \( A_{z,z',\theta} \) is purely discrete and equals \( \{0, -\alpha_2, -\alpha_3, \ldots \} \), where \( \alpha_m = m(m - 1 + zz'\theta^{-1}) \). The multiplicity of the eigenvalue 0 is equal to 1, and the multiplicity \( d_m \) of \(-\alpha_m\) is equal to the number of partitions of the number \( m \) without parts equal to 1.

(ii) The space \( L^2(\Omega, M_{z,z',\theta}) \) decomposes into eigenspaces of \( A_{z,z',\theta} \):

\[
L^2(\Omega, M_{z,z',\theta}) = \bigoplus_m W_m,
\]

where \( m = 0, 2, 3, \ldots \). Moreover, \( \bigoplus_{m=0}^{\infty} W_m = \Lambda_0^\circ \).

The proof given below is due to G. Olshanski.

**Proof.** We will use the similar fact for the operator \( A_{z,z',\theta} \) acting on \( \Lambda^\circ \). By Theorem 9.9 in [16] the action of \( A_{z,z',\theta} \) on \( \Lambda^\circ \) is diagonalizable with eigenvalues \( 0, -\alpha_2, -\alpha_3, \ldots \) and the same multiplicities \( d_m \) as in the statement of the theorem.

Note that \( \Lambda^\circ \) can be regarded as a subspace of \( L^2(\Omega, M_{z,z',\theta}) \). Indeed, we have a natural mapping from \( \Lambda^\circ \) to \( L^2(\Omega, M_{z,z',\theta}) \), and it suffices to show that it is injective. Suppose that \( f^\circ = 0 \) in \( L^2(\Omega, M_{z,z',\theta}) \) for some \( f \in \Lambda \). Multiplying the homogeneous components of \( f \) by powers of \( p_1 \), we may assume that \( f \) is homogeneous of degree \( n \). Then \( f^\circ t^n = \tilde{f} = 0 \), where \( \tilde{f} \in L^2(\tilde{\Omega}, \tilde{M}_{z,z',\theta}) \) is the realization of the symmetric function \( f \) on the Thoma cone \( \tilde{\Omega} \). But the map \( f \mapsto \tilde{f} \) is injective, because \( \{\Sigma_\lambda\} \) is simultaneously a linear basis of \( \Lambda \) and an orthogonal basis of \( L^2(\tilde{\Omega}, \tilde{M}_{z,z',\theta}) \). Hence \( f = 0 \).

Recall that \( \Lambda^\circ \) is a dense subspace of \( C(\Omega) \). Therefore, \( \Lambda^\circ \) is a dense subspace of \( L^2(\Omega, M_{z,z',\theta}) \), and it decomposes into eigenspaces of the symmetric operator \( A_{z,z',\theta} \). Hence this decomposition extends to a decomposition of \( L^2(\Omega, M_{z,z',\theta}) \).

Thus, the pregenerator \( A_{z,z',\theta} \) has an orthonormal eigenbasis \( \{g_\lambda\} \), where \( \lambda \) runs over all partitions without parts equal to 1. Let \( T(t) \) denote the semigroup on \( C(\Omega) \) generated by \( A_{z,z',\theta} \). Then, for any \( f \in \Lambda^\circ \) and \( t > 0 \), we have

\[
T(t)f = \sum_\lambda e^{-\alpha_\lambda t} \langle f, g_\lambda \rangle g_\lambda.
\]

Note that since \( \{g_\lambda\} \) is a linear orthonormal basis in \( \Lambda^\circ \), it follows that only a finite number of inner products \( \langle f, g_\lambda \rangle \) are nonzero, and hence the sum on the right-hand side is finite. Writing the inner product as an integral, we obtain

\[
T(t)f(\omega) = \sum_\lambda \int_\Omega e^{-\alpha_\lambda t} g_\lambda(\omega) g_\lambda(\sigma) f(\sigma) M_{z,z',\theta}(d\sigma).
\]

Clearly, if the series

\[
p(t,\omega,\sigma) := \sum_\lambda e^{-\alpha_\lambda t} g_\lambda(\omega) g_\lambda(\sigma)
\]

absolutely converges, then

\[
T(t)f(\omega) = \int_\Omega p(t,\omega,\sigma)f(\sigma)M_{z,z',\theta}(d\sigma)
\]

(13)
for any \( f \in \Lambda^0 \) and \( t > 0 \). By continuity, (13) holds for any \( f \in C(\Omega) \), so that \( p(t, \omega; \sigma) \) is a transition density of the process \( X_{z,z',\theta} \) with respect to \( M_{z,z',\theta} \).

In order to prove the convergence of (12), we express \( p(t, \sigma, \omega) \) in terms of Jack symmetric functions. For \( m \geq 1 \), we define \( G_m \in C(\Omega \times \Omega) \) by

\[
G_m(\omega, \sigma) := \sum_{|\lambda|=m} g_{\lambda}(\omega)g_{\lambda}(\sigma).
\]

Note that \( G_1 = 0 \).

It turns out that the functions \( G_m \) do not depend on the choice of an eigenbasis \( \{g_{\lambda}\} \) and that they can be explicitly computed in terms of symmetric functions. We set

\[
K_n^0(\omega, \sigma) := \sum_{|\lambda|=n} b_{\lambda}^{(\theta^{-1})} \frac{P_\lambda^0(\omega; \theta)P_\lambda^0(\sigma; \theta)}{(z)_{\lambda,\theta}(z')_{\lambda,\theta}}.
\]

Note that, for \((z, z', \theta)\) in the principal or the complementary series, the denominators in the definition of \( K_n^0 \) are positive real numbers.

**Lemma 13.** Let \( n \geq m \), and let \( f \in \Lambda^0_{\leq m} \). Then

\[
\int_{\Omega} K_n^0(\omega, \cdot) f(\omega) M_{z,z',\theta}(d\omega) - \frac{f(\cdot)}{(n-m)! (\frac{z}{\theta} z')^{m+n}} \in \Lambda^0_{\leq m-1}.
\]

This key lemma is similar to Lemma 3.2 in [6] and Lemma 3.2 in [9], but it is unknown whether the proofs given in [6] and [9] can be generalized to the case of a process with Jack parameter. Our proof is different and essentially uses the Laguerre symmetric functions introduced above.

**Proof.** First, we will prove a similar property for the Thoma cone. For \( n > 0 \), we define functions \( K_n(x, y) \) on \( \tilde{\Omega} \times \tilde{\Omega} \) by

\[
K_n(x, y) := \sum_{|\lambda|=n} b_{\lambda}^{(\theta^{-1})} \frac{P_\lambda(x; \theta)P_\lambda(y; \theta)}{(z)_{\lambda,\theta}(z')_{\lambda,\theta}}.
\]

According to Corollary 11, we have

\[
\int_{\tilde{\Omega}} K_n(x, y) \mathcal{L}_\mu(x) M_{z,z',\theta}(dx) = \sum_{|\lambda|=n} b_{\lambda}^{(\theta^{-1})} \frac{\langle P_\lambda, \mathcal{L}_\mu \rangle L^2}{(z)_{\lambda,\theta}(z')_{\lambda,\theta}} P_\lambda(y; \theta)
\]

\[
= b_{\mu}^{(\theta^{-1})} \sum_{|\lambda|=n} \dim(\mu, \lambda) \frac{1}{(|\lambda| - |\mu|)!} P_\lambda(y; \theta)
\]

\[
= b_{\mu}^{(\theta^{-1})} \frac{1}{(n - |\mu|)!} \sum_{|\lambda|=n} \dim(\mu, \lambda) P_\lambda(y; \theta).
\]

Hence by (3)

\[
\int_{\tilde{\Omega}} K_n(x, y) \mathcal{L}_\mu(x) M_{z,z',\theta}(dx) = b_{\mu}^{(\theta^{-1})} \frac{1}{(n - |\mu|)!} \sum_{|\lambda|=n} p_{1}^{n-|\mu|} P_{\mu}(y; \theta).
\]

Since the elements \( \mathcal{L}_\mu \) with \( |\mu| \leq m \) form a basis of \( \Lambda_{\leq m} \), it follows that, for every \( f \in \Lambda_{\leq m} \), we have

\[
\int_{\tilde{\Omega}} K_n(x, \cdot) f(x) M_{z,z',\theta}(dx) \in p_{1}^{n-m} \Lambda_m.
\]
Finally, the \( m \)th homogeneous component of \( \Sigma_\mu \) is \( b^{(\theta-1)}_\mu P_\mu \), whence
\[
\int_{\Omega} K_n(x, \cdot) P_\mu(x; \theta) \tilde{M}_{z, \theta}(dx) - \frac{1}{(n-|\mu|)!} p_1^{n-m} P_\mu(\cdot; \theta) \in p_1^{n-1-m} \Lambda_{m-1}.
\]
Since the elements \( P_\mu \) with \( |\mu| = m \) form a basis of \( \Lambda_m \), it follows that, for any \( f \in \Lambda_{\leq m} \),
\[
\int_{\Omega} K_n(x, y) f(x) \tilde{M}_{z, \theta}(dx) - \frac{1}{(n-m)!} p_1^{n-m}(y) f(y) \in p_1^{n-1-m} \Lambda_{m-1}.
\]
To deduce the assertion of the lemma from its analogue lifted to the cone, recall that every nonzero point \( x \in \Omega \setminus \{0\} \) can be identified with a pair \((r_x, \omega_x) \in \mathbb{R}_{>0} \times \Omega \). For any \( f \in \Lambda_k \), we have \( f(x) = f(r_x, \omega_x) = r_x^n f^0(\omega_x) \in C(\mathbb{R}_{>0} \times \Omega) \). Hence
\[
\int_{\Omega} K_n(x, y) f(x) \tilde{M}_{z, \theta}(dx) - \frac{1}{(n-k)!} p_1^{n-k}(y) f(y) = r_y^n \int_{\Omega} K_n^0(\omega_x, \omega_y) f^0(\omega_x) r_x^n (M_{z, \theta} \otimes \gamma_{\omega_y})(dx) - \frac{r_y^n f^0(\omega_y)}{(n-k)!}
\]
\[
= r_y^n (z z' \theta^{-1})_{n+k} \int_{\Omega} K_n^0(\omega, \cdot) f^0(\omega) M_{z, \theta} (d\omega) - \frac{f^0(\cdot)}{(n-k)!}) \in r_y^n \Lambda_{\leq k-1}^0.
\]
The required identity is obtained by setting \( r_y = 1 \).

**Lemma 14.** Let \( f_n \) and \( g_n \) be two sequences in a vector space over \( \mathbb{C} \). Assume that, for some \( c \in \mathbb{C} \) and any \( n \geq 0 \),
\[
f_n = \sum_{m=0}^{n} \frac{g_m}{(c)_{n+m} (n-m)!}.
\]
Then
\[
g_m = \sum_{n=0}^{m} (-1)^{m-n} \frac{(c+2m-1)(c)_{m+n-1}}{(n-m)!} f_n.
\]
(14)
The proof of this lemma is given in [6; Lemma 3.3]; it is similar to the inclusion-exclusion formula.

**Proposition 15.** For every \( m > 0 \),
\[
G_m = \sum_{n=0}^{m} (-1)^{m-n} \frac{z z' + 2m-1}{(z z')_{m+n-1}} K_n^0.
\]
(15)

**Proof.** Recall that \( \Lambda_{\leq m}^0 \) is spanned by the elements \( g_\mu \) with \( |\mu| \leq m \); hence if \( |\lambda| > m \), then \( g_\lambda \)
is orthogonal to \( \Lambda_{\leq m}^0 \). Thus, by Lemma 13
\[
\int_{\Omega \times \Omega} K_n^0(\omega, \sigma) g_\lambda(\omega) g_\mu(\sigma) M_{z, \theta}(d\omega) M_{z, \theta}(d\sigma) = \frac{\delta_{\lambda, \mu}}{(n-|\lambda|)! (z z')_{|\lambda|+n}}.
\]
Since \( \{g_\lambda(\omega) g_\mu(\sigma)\}_{\lambda, \mu} \), where \( \lambda \) and \( \mu \) run over partitions with no part equal to 1, is an orthonormal basis of \( L^2(\Omega \times \Omega, M_{z, \theta} \otimes M_{z, \theta}) \), we have
\[
K_n^0 = \frac{1}{n!(z z')_n} + \sum_{m>2} \frac{G_m}{(z z')_{n+m}(n-m)!}.
\]
Together with Lemma 14 this implies (15).
Recall that we are proving the convergence of the series
\[ p(t, \sigma, \omega) = 1 + \sum_{m \geq 2} e^{-t\alpha_m} G_m(\sigma, \omega). \]

Thus, we need an upper bound for \( G_m \).

**Proposition 16.** There exists a \( C > 0 \) and a \( d > 0 \) such that
\[ \|G_m\| \leq C m^d, \]
where \( \| \cdot \| \) is the supremum norm on \( C(\Omega \times \Omega) \).

**Proof.** Proposition 15 expresses \( G_m \) in terms of the kernels \( K_n^\circ \), which, in turn, are defined via the functions \( P^\circ \). Therefore, first we will give an upper bound for \( P^\circ \).

From (3) we have
\[ 1 = (p^n)^\circ = \left( \sum_{|\lambda|=n} \dim_\theta(\lambda) P^\circ(\omega; \theta) \right)^\circ = \sum_{|\lambda|=n} \dim_\theta(\lambda) P^\circ_\lambda(\omega; \theta). \]

As pointed out in Section 3, the functions \( P^\circ(\omega; \theta) \) are nonnegative; hence
\[ P^\circ_\lambda(\omega; \theta) \leq \dim_\theta(\lambda)^{-1}. \]

It is known (see [15; Sec. 5]) that
\[ \dim_\theta(\lambda) = \frac{|\lambda|!}{H(\lambda)}, \]
where
\[ H(\lambda) = \prod_{(i,j) \in \lambda} (\lambda_i - j + \theta(\lambda_j' - i) + 1). \]

Note that
\[ \lambda_i - j + \theta(\lambda_j' - i) + 1 \leq |\lambda| + \theta|\lambda| = |\lambda|(1 + \theta). \]

Therefore,
\[ P^\circ_\lambda(\omega; \theta) \leq \dim_\theta(\lambda)^{-1} = \frac{H(\lambda)}{|\lambda|!} \leq \frac{|\lambda|!(1 + \theta)^{|\lambda|}}{|\lambda|!}. \]

To estimate \( K_n^\circ \), we need bounds for \( b^\circ \) and \( (z)_\lambda^\theta \). Recall that the \( (z, z') \) are in the principal or the complementary series; hence there exists a \( \delta_z > 0 \) such that \( |z + k + \theta l| > \delta_z \) for any \( k, l \in \mathbb{Z} \), whence \( |(z)_\lambda^\theta| > \delta_z^{|\lambda|} \).

For \( b^\circ_\lambda^{(\theta^{-1})} \), we use a formula in [14; VI, 10.10], which gives
\[ b^\circ_\lambda^{(\theta^{-1})} = \prod_{(i,j) \in \lambda} \frac{\lambda_i - j + \theta(\lambda_j' - i) + 1}{\lambda_i - j + 1 + \theta(\lambda_j' - i)} \leq (\theta + 1)^{|\lambda|}. \]

This implies
\[ K_n^\circ(\omega, \sigma) \leq \rho(n) \frac{n^{2n}(1 + \theta)^{3n}}{(n!)^2 \delta_z^{|\lambda|}}, \]
where \( \rho(n) \) is the number of partitions of \( n \). Since \( \rho(n) \) equals the number of conjugacy classes in the symmetric group of order \( n \), we have \( \rho(n) \leq n! \). Hence there exists a \( C > 0 \) such that
\[ K_n^\circ(\omega, \sigma) \leq C n^{3n}. \]
Finally, for $G_m$, we have
\[ |G_m| \leq \sum_{n=0}^{m} \frac{\left| \frac{zz'}{\theta} + 2m - 1 \right|}{(m-n)!} |K_n^\theta| \leq C \sum_{n=0}^{m} (2m + \frac{zz'}{\theta})^{2m-n} n! \]
\[ \leq mC \left( 2m + \frac{zz'}{\theta} \right)^{2m-n} m^{3n} \leq Dm^{6m} \]
for some constant $D > 0$.

**Theorem 17.** (i) The process $X_{z, z', \theta}$ has a continuous transition density with respect to $M_{z, z', \theta}$, that is, there is a continuous function $p(t, \sigma, \omega)$ on $\mathbb{R}^+ \times \Omega \times \Omega$ such that, for any $f \in C(\Omega)$ and $t > 0$, the following identity holds:
\[ T(t)f(\sigma) = \int_{\Omega} p(t, \sigma, \omega) f(\omega) M_{z, z', \theta}(d\omega). \]

(ii) The transition density $p(t, \sigma, \omega)$ is given by the following series converging in the supremum norm:
\[ p(t, \sigma, \omega) = 1 + \sum_{n=0}^{\infty} \sum_{|\lambda|=n} C_n(t) \frac{P^\theta_\lambda(\sigma) P^\theta_\lambda(\omega)}{(z)_{\lambda, \theta}(z')_{\lambda, \theta}}, \]
where
\[ C_n(t) := \sum_{m \geq n} e^{-t\alpha_m} (-1)^{m-n} \frac{zz'}{\theta} + 2m - 1 \frac{(zz')^{m-n-1}}{(m-n)!}. \]

**Proof.** As noted above, it is enough to prove that, for any $\varepsilon > 0$, the series
\[ p(t, \sigma, \omega) = 1 + \sum_{m \geq 2} e^{-t\alpha_m} G_m(\sigma, \omega) \]
uniformly absolutely converges for $t \in (\varepsilon, \infty)$. From Proposition 16 we have
\[ \|e^{-t\alpha_m} G_m\| \leq Ce^{-\varepsilon(m-1)+dm\log(m)}. \]
Note that
\[ \log \left( \frac{\varepsilon m}{2d} \right) \leq \frac{\varepsilon m}{2d} - 1, \]
whence
\[ dm \log m \leq \frac{\varepsilon}{2} m^2 - dm + dm \log \left( \frac{\varepsilon}{2d} \right). \]
Thus, we have
\[ \|e^{-t\alpha_m} G_m\| \leq Ce^{-c_1 m^2 - c_2 m} \]
for $c_1 > 0$, and the desired convergence follows. \qed

6. An Ergodic Theorem for the Process with Jack Parameter

As shown in [16; Theorem 9.10(iii)], the diffusion with Jack parameter is ergodic in the sense that
\[ \lim_{t \to \infty} \left\| T(t)f - \int_{\Omega} f(\omega) M_{z, z', \theta}(d\omega) \right\|_{\sup} = 0. \]
Following [6; Remark 3.6], we use the expression for the transition density to strengthen the ergodic theorem.
Corollary 18. There is a constant $K > 0$ depending only on $z$, $z'$, and $\theta$ such that, for any $\sigma \in \Omega$,
\[ \| P(t, \sigma, \cdot) - M_{z,z',\theta} \|_{\text{var}} \leq Ke^{-t\alpha_2}, \]
where $P(t, \sigma, \cdot)$ is the transition function of the process $X_{z,z',\theta}$, $\| \cdot \|_{\text{var}}$ is the total variation, and $\alpha_m = m(m - 1 + zz'\theta^{-1})$.

Proof. Recall that if a measure $\nu$ has a density $f$ with respect to a measure $\mu$, then
\[ \| \mu - \nu \|_{\text{var}} = \int |1 - f| \, d\mu. \]
In our case, $P(t, \sigma, \cdot)$ has the density $p(t, \sigma, \cdot)$ with respect to $M_{z,z',\theta}$, and hence
\[ \| P(t, \sigma, \cdot) - M_{z,z',\theta} \|_{\text{var}} = \int |p(t, \sigma, \omega) - 1| M_{z,z',\theta}(d\omega). \]
Using the expression for $p(t, \sigma, \omega)$, we obtain
\[ |p(t, \sigma, \omega) - 1| \leq e^{-t\alpha_2} \sum_{m \geq 2} |G_m| e^{-t(\alpha_m - \alpha_2)} \]
for $t > 0$. Moreover, the sum on the right-hand side converges for any $t > 0$ and is decreasing in $t$. Therefore, for any $t_0 > 0$ and $t \geq t_0$, we have
\[ \| P(t, \sigma, \cdot) - M_{z,z',\theta} \|_{\text{var}} \leq K_0 e^{-t\alpha_2}, \]
where
\[ K_0 = \sum_{m \geq 2} \| G_m \|_{\sup} e^{-t(\alpha_m - \alpha_2)}. \]
Finally, note that the variance distance between any two measures is less than or equal to 2; hence, for $t \geq 0$ and $K = K_0 + 2e^{-t\alpha_2}$, we have
\[ \| P(t, \sigma, \cdot) - M_{z,z',\theta} \|_{\text{var}} \leq Ke^{-t\alpha_2}. \]

Remark. Having a bound for $\int_{\Omega} |G_2(\sigma, \omega)| M_{z,z',\theta}(d\omega)$, we can further improve the rate of convergence. Note that $K_0(\sigma, \omega) > 0$ for any $\sigma, \omega \in \Omega$, so that
\[ |G_2| = \left| \sum_{n=0}^{2} (-1)^{2-n} \frac{(zz' + 3)(zz')n+1}{(2-n)!} K_n^\circ \right| \leq \sum_{n=0}^{2} \frac{(zz' + 3)(zz')n+1}{(2-n)!} K_n^\circ. \]
By Lemma 13 we have
\[ \int_{\Omega} K_n^\circ(\sigma, \omega) M_{z,z',\theta}(d\omega) = \frac{1}{n!(\frac{zz'}{\theta})^n}. \]
Hence
\[ \int_{\Omega} |G_2(\sigma, \omega)| M_{z,z',\theta}(d\omega) \leq \sum_{n=0}^{2} \frac{(zz' + 3)(zz')n+1}{(2-n)!} \frac{1}{n!(\frac{zz'}{\theta})^n} \]
\[ = \sum_{n=0}^{2} \frac{(zz' + 3)(zz'+n)}{(2-n)!n!} = 2 \left( \frac{zz'}{\theta} + 3 \right) \left( \frac{zz'}{\theta} + 1 \right). \]
Thus, for some constant $C > 0$ depending only on $z$, $z'$, and $\theta$, we have
\[ \| P(t, \sigma, \cdot) - M_{z,z',\theta} \|_{\text{var}} \leq 2 \left( \frac{zz'}{\theta} + 1 \right) \left( \frac{zz'}{\theta} + 3 \right) e^{-t\alpha_2} + Ce^{-t\alpha_3}. \]
Appendix A: Matching the Notation with [1]

The purpose of this section is to match the notation in Section 4 of [1] with our notation.

As noted above, our parameter \( \theta \) is reciprocal to the conventional Jack parameter \( \alpha \), and the parameter \( a \) in [1] is equal to \( c - 1 \) used in our work. The operator \( \tilde{H}^{(L)} \) in [1] is equal to our operator \( D_N^{(L)} \).

The normalization of the Jack polynomials \( C_\kappa^{(\alpha)} \) used in [1] is different from ours; namely, it is defined by

\[
p_1^\alpha = \sum_{|\kappa|=n} C_\kappa^{(\alpha)}.
\]

Comparing this with (4), we obtain

\[
C_\kappa^{(\alpha)}(x) = \dim_\theta(\kappa) P_\kappa(x; \theta).
\]

Our Laguerre polynomials \( L_c^{\kappa, \alpha} \) differ from the polynomials \( L_k^{\alpha}(x_1, \ldots, x_N; \alpha) \) used in [1] by a multiplier not depending on \( x \):

\[
L_k^{\alpha}(x_1, \ldots, x_N; \alpha) = \frac{1}{\dim_\theta(\alpha)(N\theta)_\kappa} L_c^{\kappa, \alpha}.
\]

Finally, to obtain Propositions 3 and 4 from Propositions 4.3 and 4.10 of [1], we use a binomial formula in [15], which has the form

\[
\binom{\kappa}{\sigma} \frac{C_\kappa^{(\alpha)}(x_1, \ldots, x_N)}{C_\kappa^{(\alpha)}(1, \ldots, 1)} = \frac{|\kappa|!}{(|\kappa| - |\sigma|)!} \frac{\dim_\theta(\sigma, \kappa)}{\dim_\theta(\kappa)(N\theta)_\kappa} Q_\sigma(x_1, \ldots, x_N; \theta).
\]

Appendix B: Degeneration to the Petrov Diffusion

In the limit regime as \( \theta \to 0 \), \( z z' \to 0 \), \( \theta^{-1} z z' \to \tau \), \( z + z' \to -a \), the process with Jack parameter formally degenerates to Petrov’s generalization of the Ethier–Kurtz diffusion [19] (see [16; Remark 9.12]). In this regime the Thoma simplex degenerates to the Kingman simplex \( \nabla_\infty \) formed by sequences of real numbers \( x_1 \geq x_2 \geq \cdots \geq 0 \) such that

\[
\sum_{i \geq 1} x_i \leq 1.
\]

The functions \( p_\kappa^\theta \) degenerate to the moment coordinates \( q_k = \sum_i x_i^k \), the probability pregenerator is given by

\[
\sum_{i,j \geq 1} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} + \sum_{i \geq 1} (i+1)[(i-a)q_{i-1} - (i+\tau)q_i] \frac{\partial}{\partial q_i},
\]

and the invariant measure \( M_{z,z',\theta} \) degenerates in this regime to the Poisson–Dirichlet distribution \( PD(\alpha, \tau) \).

Recall that \( m_i(\mu) \) is the number of parts equal to \( i \) in the partition \( \mu \). The Laguerre functions degenerate to

\[
\mathcal{L}_\lambda^{PD(\alpha, \tau)} = \sum_{\mu \leq \lambda} (-1)^{|\lambda| - |\mu|} \dim_0(\mu, \lambda) \frac{\dim_0(\mu, \lambda)}{(|\lambda| - |\mu|)!} \left( \prod_{x \in \lambda/\mu} q_{\alpha x}(x) \right) r_{\mu} m_\mu(x; \theta).
\]
Here $\dim_0(\mu, \lambda)$ is the weighted sum of increasing paths from $\mu$ to $\lambda$ in the Kingman graph described in [12; Sec. 4], the coefficient $r_\mu$ is defined by $r_\mu = \prod_{i} m_i(\mu) / \prod_{i} m_i$, and $q_{\alpha\tau}(x)$ is defined by

$$q_{\alpha\tau}(i,j) = \begin{cases} (j - 1)(j - 1 - \alpha) & \text{if } j > 1, \\ \tau + \alpha(i - 1) & \text{if } j = 1. \end{cases}$$

The functions $L_{\lambda}^{PD(\alpha,\tau)}$ form an orthogonal system for the lifted Poisson–Dirichlet distribution $\tilde{PD}(\alpha, \tau) = PD(\alpha, \tau) \otimes \gamma_{\tau}$.

The proof of Lemma 13 also works in this regime, so that our approach can be used to obtain a new proof of the eigenfunction expansion of the transition density in the case of the Petrov diffusion.

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