Dynamic Graph Algorithms with Batch Updates in the Massively Parallel Computation Model

Krzysztof Nowicki* 
Univ. of Wroclaw 
knowicki@cs.uni.wroc.pl

Krzysztof Onak 
IBM Research 
konak@us.ibm.com

Abstract

We study dynamic graph algorithms in the Massively Parallel Computation model, inspired by practical data processing systems. Our goal is to provide algorithms that can efficiently handle large batches of edge insertions and deletions.

We show algorithms that require fewer rounds to update a solution to problems such as Minimum Spanning Forest and Maximal Matching than would be required by their static counterparts to compute it from scratch. They work in the most restrictive memory regime, in which local memory per machine is strongly sublinear in the number of graph vertices. Improving on the size of the batch they can handle would improve on the round complexity of known static algorithms on sparse graphs.

Our algorithms can process batches of updates of size $\Theta(S)$ and $\Theta(S^{1-\epsilon})$ for Minimum Spanning Forest and Maximal Matching, respectively, in $O(1)$ rounds, where $S$ is the local memory of a single machine.

*Krzysztof Nowicki's research is supported by the Polish National Science Centre, under projects number 2017/25/B/ST6/02010 and 2019/32/T/ST6/00566. This work was partially done during an internship at the IBM Watson Research Center.
1 Introduction

The size of the data sets we want to process grows faster than the computational power of a single processor – due to this trend, classical sequential algorithms become not efficient enough. There are several ways of approaching this problem, and in this paper we study one of them: the Batch Dynamic MPC model, which is a model that is a mix of parallel computing with dynamic algorithms.

1.1 Computing in parallel: MPC model

There are several frameworks that allow to implement parallel algorithms in practice, the most recognizable being MapReduce, Hadoop and Spark. Even though the implementation details in those frameworks could be different, all of them share some properties, captured by a theoretical model called MPC (Massively Parallel Computation) model, proposed by Karloff et al. [KSV10].

In a nutshell, in the MPC model we have a set of $M$ machines, each with local memory $S$. Initially, the input is evenly partitioned between the machines, i.e. if an input consists of $N$ words, each machine knows part of the input of size roughly $N/M$.

The computation is performed in synchronous rounds, each consisting of a phase of local computation and a phase of communication. During the local computation phase, each machine performs computation based on the information stored in local memory. During the communication phase, each machine can exchange some number of $O(\log N)$ bit messages with any other machine, as long as each machine is a source and destination of $O(S)$ messages. Representation of the result of the algorithm depends on the problem: for some problems the result is small, and we can require that there is a single machine that knows whole result (e.g. for graph connectivity the size of the answer is one bit); for some we allow that the result is distributed (i.e. for Minimum Spanning Tree problem we can mark all the edges that are part of the MST).

The goal is to provide the algorithms that for $S \in O(N^{1-\varepsilon})$ and $S \cdot M \in O(N)$ require some small number of round of computation. Quite often, we consider slightly relaxed variant in which the memory bound is relaxed (i.e. $S \cdot M \in \omega(N)$), especially if it provides some significant improvement in the round complexity of the algorithm.

1.2 Slowly evolving data sets: dynamic algorithms

Let us consider a situation in which we want to process some sequence of data sets, the $(i+1)$th data set obtained by slightly changing the $i$th set. Then, it could be the case, that the result for the $i$th data set can be useful for computing the result for the $(i+1)$th data sets, and give an algorithm that can process the update from the $i$th to $(i+1)$th data set faster than processing $(i+1)$th data set from scratch.

In this paper, we focus on a variant of Dynamic Graph Algorithms [EGI98], in which the data set is a graph, the result we want to maintain is a solution to some graph problem (e.g. Minimum Spanning Tree [HdLT01, NSW17], Maximal Independent Set [AOS19, BDH19, CZ19], Maximal / Maximum Matching [ACC+18, BGS18, BFH19, CS18, Sol16, NS16], ...). More precisely, we consider a variant in which the change from the $i$th to $(i+1)$th graph is described by a batch of edge insert and edge delete operations [FL94, AABD19].

1.3 Graph problems in the MPC model and the need for dynamic algorithms

For graph problems in the MPC model, rather than expressing the memory of a single machine with respect to the size of the input (i.e. number of edges), we can express it as a function of

\[ \text{memory } S \text{ means that a machine can store } S \text{ words of size } O(\log n) \text{ bits} \]

1
the number of vertices. For an $n$ vertex $m$ edge graph, we can we usually consider three variants of the MPC model, depending on the relation between the local memory of a single machine and the number of vertices of a graph: $S \in \mathcal{O}(n^\alpha)$, for constant $\alpha < 1$, $S \in \tilde{\Theta}(n)$ and $S \in \mathcal{O}(n^{1+\alpha})$, for constant $\alpha > 0$. Usually, the goal is to provide efficient algorithms with the local memory as small as possible: the reason is that physical infrastructure simulates all machines of the MPC model on some cluster, and the smaller local memory allows for splitting work between physical machines more equally. There are no provable separation between those three variants, but there are problems, for which the best known upper bounds in each of those variants is different.

**Results related to the Minimum Spanning Forest problem in the MPC model:** Lattanzi et al. [LSV11] show that if we have $\mathcal{O}(n^{1+\alpha})$ local memory, it is possible to solve the Minimum Spanning Forest problem in $\mathcal{O}(1/\varepsilon)$ rounds.

The first sublogarithmic algorithm for this problem, for the MPC with nearly linear memory, is a deterministic $\mathcal{O}(\log \log n)$ round algorithm proposed by Lotker et al. [LPSPP05]. Even though it is phrased as an algorithm for a different model of computation, the technique easily translates to the MPC model with $\mathcal{O}(n)$ memory limit $^2$. This is still the best known deterministic algorithm, as all known faster algorithms heavily rely on sampling and sketching techniques.

The first $o(\log \log n)$ round algorithm was proposed by Hegeman et al. [HPP+15], where the authors show that application of Karger-Klein-Tarjan random sampling approach [KKT95], together with connectivity algorithm from the streaming model [AGM12], is enough to give an algorithm that needs only $\tilde{\mathcal{O}}(n)$ local memory to compute Minimum Spanning Forest in $\mathcal{O}(1)$ rounds or $\mathcal{O}(\log \log \log n)$ rounds with $\mathcal{O}(n)$ local memory limit. Further improvements were done, by providing better connectivity algorithms, that needed smaller number of rounds with $\mathcal{O}(n)$ local memory limit – firstly, Ghaffari and Parter presented a $\mathcal{O}(\log^* n)$ round algorithm [GP16], which was extended to an $\mathcal{O}(1)$ round algorithm by Jurdziński and Nowicki [JN18].

For the sublinear memory regime, it is known that one can solve the Minimum Spanning Forest problem in $\mathcal{O}(\log n)$ rounds, using some slightly modified variants of Boruvka’s algorithm [NMM01], but there are no known $o(\log n)$ algorithms. Moreover, it is conjectured that a problem as simple as deciding whether the input graph is a single cycle, or two disjoint cycles (so this problem is a very special case of the Connectivity problem) is conjectured to need $\Omega(\log n)$ rounds of computation, if we have $\mathcal{O}(n^\alpha)$ local memory for some constant $\alpha < 1$ and any polynomial number of machines.

**Results related to the Maximal Matching problem:** The first static algorithm for the Maximal Matching problem required $\mathcal{O}(n^{1+\epsilon})$ local memory, and $\mathcal{O}(1/\varepsilon)$ rounds to finish computation [LSV11]. For the linear memory regime, some algorithms for Maximum Matching were known since 2018 [CLM+18, GGK+18], but a poly($\log \log n$) round algorithm for the Maximal Matching was not known up until the recent breakthrough of Behnezhad et al. [BHH19]. For the sublinear memory regime the first sublogarithmic algorithm for the maximal matching was proposed in [GU19, Ona18], and for a while it was also the best algorithm for the linear memory regime.

**Need of Algorithms for Dynamic Data Sets:** In the strongly sublinear memory regime we don’t know any efficient $^3$ algorithms for neither the Maximal Matching problem or the Minimum Spanning Forest problem. It could be the case that the existing algorithms are optimal and obtaining any better bounds is impossible. Therefore, considering dynamic variants of those problems could be the only way of obtaining algorithms that can process the large datasets with provably lower round complexity, at least as long as the data we need to process can be modeled as a batch dynamic data set. In this paper show that there are efficient algorithms for both problems in their

---

$^2$that’s also the case for remaining algorithms for MST in Congested Clique

$^3$here by efficient we mean the algorithms with $\mathcal{O}(1)$ round complexity, or close to that; at very least the aim is to match the complexity of the variants of the algorithms from the linear memory regime MPC
1.4 Dynamic Graph Algorithms in the MPC model

The first paper that proposes studies of dynamic variant of the MPC algorithms is [ILMP19]: the authors propose the algorithms that after one update to the data set need only $O(1)$ rounds of computation to maintain the result for the Connectivity problem, Approximate MST problem and Maximal Matching problem. In that paper, the emphasis was mainly on minimizing the communication, and number of machines that are used by the protocol, but the authors didn’t address one issue: that if the number of changes to our data set is $\Theta(\log n)$, in order to apply all those changes we again need $\Theta(\log n)$ rounds.

Durfee et al. address this issue [DDK+20], and consider a model in which the updates come in batches, i.e. there is a set of updates, and all of them should be applied at once. They propose an algorithm that maintains the result for the Connectivity problem, and can process in $O(1)$ rounds a batch of size $\Theta(S^{1-\varepsilon})$.

Remark 1.1. Both of those papers [ILMP19, DDK+20] also try to minimize the overall number of messages that are sent in each round. In this paper, our goal is slightly different – it is to maximize the size of the batch that we can process, while minimizing the number of rounds we need for processing. As for communication complexity, we allow $\Theta(m)$ global communication during each round, which is usual in the static variant of the MPC model.

1.5 Our results

In this paper we further explore the direction suggested by Durfee et al. Firstly, we observe that there is some upper bound on the size of a batch, for which we can hope that dynamic algorithms can be better than static algorithms.

Besides some discussion regarding the maximum size of a batch, we show efficient algorithms for the Batch Dynamic Minimum Spanning Forest problem and Batch Dynamic Maximal Matching problem, that can process a batch of maximal size efficiently: any further improvements are either not possible or require some progress in developing the algorithms for the static variant of the MPC model.

Minimum Spanning Forest problem in the Batch Dynamic MPC model: In this problem we have to maintain a Minimum Spanning Forest of dynamically changing graph. If before the updates we have a graph $G$ with a Minimum Spanning Forest $F$, then for a sequence of updates that applied to $G$, give a graph $G'$ we have to compute a sequence of updates to $F$ such that applying this sequence of updates to $F$ give a minimum spanning forest of $G'$.

Actually, we show an algorithm that solves slightly more demanding variant of the problem, defined as follows. Let:

- $G$ be a graph before applying the batch of updates,
- $U = (u_1, u_2, \ldots, u_k)$ be a sequence of updates, i.e. sequence of edge insertions and edge deletions
- $F$ be a minimum spanning forest of $G$
- $F_x$ be a minimum spanning forest of $G$ after application of first $x$ updates $(u_1, u_2, \ldots, u_x)$.

The goal is to return a sequence of updates $U'$ to the minimum spanning forest, such that in order to obtain $F_x$, it is enough to execute some prefix of $U'$ on $F$. More precisely, as a result of processing a batch of updates, the algorithm has to return a sequence of updates $U' = (u'_1, u'_2, \ldots, u'_{k'})$, and
the sequence of indices \((y_1, y_2, \ldots, y_k)\), such that forest \(F_x\) is a forest \(F\) on which we apply a prefix of returned updates of length \(y_x\), i.e. some sequence of updates \((u_1', u_2', \ldots, u_{y_x}')\).

Our algorithm can process a single batch of updates in \(O\left(\frac{1}{\alpha}\right)\) rounds and the size of the batch can be any \(k \in O(n^\alpha)\).

In order to give an algorithm for Batch Dynamic Minimum Spanning Tree problem, we propose a generalization of the Top Tree data structure [AHdLT05], suited for the MPC model, and a generalization of the standard cycle property of Minimum Spanning Trees [Tar83, also known as red rule]. Those components, combined with a black box application of \(O(1)\) round algorithm for MPC model with \(O(n)\) memory limit of a single machine [JN18], give \(O(1)\) round algorithm that can process a batch of updates of size \(O(S)\).

**Maximal Matching problem in the Batch Dynamic MPC model:** In this problem, we have to maintain a Maximal Matching of dynamically changing graph. If before the updates we have a graph \(G\) with a Maximal Matching \(M\), then for a sequence of updates that applied to \(G\), gives a graph \(G'\) we have to compute a sequence of updates to \(M\) such that applying this sequence of updates to \(M\) gives a maximal matching of \(G'\).

In order to give an algorithm for the Batch Dynamic Maximal Matching problem we make few observations. Firstly, we observe that one can reduce the problem of processing a batch of updates to an instance of Maximal Matching problem on a graph that has a small vertex cover. Then, we observe that the algorithm for MPC model with \(O(n)\) memory [BHH19] actually can solve this kind of instance using local memory proportional to the size of this cover, which gives that we can process a batch of updates of size \(O(S^{1-\delta})\), in \(O(\log 1/\delta)\) rounds.

1.6 Comparison with other results

There is a set of papers concerning problems we address in this paper, developed independently by Acar et al. [AAB+20], Anderson et al. [ABT20], and Gilbert and Li [GL20]. In this section we briefly address some similarities and differences between our result and those papers.

**Batch Dynamic Trees by Acar et al. [AAB+20]:** This paper introduces a general technique for adapting PRAM algorithms to the batch dynamic MPC model. The authors, as an application of this general technique, show that one can implement Rake-compress trees, that can maintain a spanning tree of a graph undergoing batch updates. This paper is somewhat relevant to the part of our paper in which we present the top tree adaptation to the MPC model that supports batch link and cut operations, presented in Section 3.

The upside of the approach presented in [AAB+20] is that it gives a work efficient data structure, that can handle a batch of any size that is \(O(k)\). The downside is that they analysis fails to deliver any bound on the number of \(O(1)\) rounds, that is \(o(\log n)\). In comparison, our adaptation of top trees allows us to provide the same set of operations, in \(O(1)\) rounds, but it requires that the size of the batch of updates is \(O(n^\alpha)\).

**Batch Dynamic MST, by Anderson et al. [ABT20], and Gilbert and Li [GL20]:** The papers by Anderson et al. [ABT20], and Gilbert and Li [GL20] provide some algorithms for the MST problem undergoing batch updates, which are relevant to Section 4, where we discuss maintaining MST in the batch dynamic MPC model. Even though the paper by Anderson et al. considers only batch incremental model of computing MST and aims for work efficient PRAM implementation of the algorithm, the undergoing graph property seems to be akin to one used for the algorithm we develop in Section 4 and the algorithm developed by Gilbert and Li.

More precisely, all those algorithms for a set \(S\) of \(k\) edges to be inserted identify a set of \(O(k)\) edges \(R\) that if removed, create a partition into \(O(k)\) components \(C\), such that computing MST
on the graph where the set of vertices are components from \( C \) and the edges are edges of \( S \) and \( R \) finds all replacement edges for the maintained MST. As for the edge removal, we, as well as Gilbert and Li observe that it creates a set of \( \mathcal{O}(k) \) relevant connected components, for which we can use a MST algorithm with a linear limit on the memory of a single machine [JN18, Now19].

2 Maximum Batch Size

In this section we briefly discuss an upper bound on the size of a batch, for which we can hope that dynamic algorithms can be better than static algorithms. Let us consider a batch of updates of size \( k \), that we want to process using the MPC model with the local memory limit \( \mathcal{O}(k/f(k)) \), for some increasing function \( f \). This is equivalent to solving the static variant of the problem on a graph with \( \mathcal{O}(k) \) vertices and \( \mathcal{O}(k) \) edges using only sublinear local memory. It is because using those \( k \) updates, we can create a graph on \( \mathcal{O}(k) \) vertices, while having only \( \mathcal{O}(k/f(k)) \) memory limit. Therefore, any algorithm that can process such update, can also compute the result from scratch, for a \( \mathcal{O}(k) \) vertex, \( \mathcal{O}(k) \) edge input graph, using sublinear memory. Therefore, unless we make a progress in solving a static problem for some relatively sparse graphs, we can not expect to solve the batch dynamic problem faster.

**Batch size for Minimum Spanning Forest problem:** For the Minimum Spanning Forest problem, we consider \( k \in \mathcal{O}(S) \), for which we get \( \mathcal{O}(1) \) round algorithm. For \( k \in \Omega(S^{1+\varepsilon}) \), under two cycle conjecture, we can not get any \( o(\log n) \) round algorithms, so in some sense our result is as good as we could expect. Perhaps the algorithm extends to some slightly super-linear batch sizes, although the time complexity of possible algorithms is tied to the time complexity of algorithms that use slightly sublinear space in the static variant of the MPC model, and covering this gap seems not to be crucial to the content of this paper.

**Batch Size for the Maximal Matching problem:** For the Maximal Matching problem, we provide an algorithm, based on the static algorithm for the Maximal Matching problem, and its round complexity depends on the batch size in the same way as the round complexity depends on the relation between the local memory of a single machine and the number of vertices in the graph: for \( k \in \mathcal{O}(S^{1-\delta}) \) it requires \( \mathcal{O}(\log 1/\delta) \) rounds, which \( k \in \Theta(S) \) gives \( \mathcal{O}(\log \log n) \) rounds, and for \( k \in \mathcal{O}(S^{1-\varepsilon}) \), for a constant \( \varepsilon > 0 \) gives \( \mathcal{O}(1) \) rounds. For the Maximal Matching problem, we don’t have a presumably difficult sparse instance, therefore there is slight chance of getting batch dynamic algorithms that can efficiently process a batch of updates of size \( \mathcal{O}(S^{1+\varepsilon}) \), without actually solving the problem for a static variant of the MPC model with strongly sublinear memory.

3 Top trees for the MPC model

In this section we show a data structure that allows us to maintain a dynamically changing forest in the MPC model. The basic variant of the data structure allows to process edge insertion and edge deletion updates, in batches of size \( k \in \mathcal{O}(n^{\alpha}) \), in \( \mathcal{O}(\frac{1}{\alpha}) \) rounds, and its internal structure allows for efficient computation of various queries to the tree, some of them are used in our Batch Dynamic MSF algorithm.

On the top level, the data structure is a \( \Theta(n^{\alpha/2}) \)-ary variant of top trees [AHdLT05], that supports executing updates in batches. As in the paper that introduced top trees, whenever we want to address the input graph we say underlying tree, and when we address the tree that is the data structure, we use term top tree.

The top tree data structure is a balanced binary tree, such that each node of this tree corresponds
to a connected part of the underlying tree. In this paper, we propose a generalization of this data structure, that is somehow corresponds to generalizing balanced binary search trees to B-trees [Com79], but here we also add a way of executing operations in batches. Since our data structure is balanced, and it is $\Theta(n^{\alpha/2})$-ary tree, its depth is only $O(1/\alpha)$.

**Description:** Our variant of the top tree could be summarized as follows:

- each node of the top tree represents a connected set of edges of underlying tree
- each edge of a graph is a top tree node of rank 0, each node of the top tree of rank $r > 0$, that does not represent a whole underlying tree, contains $\Theta(n^{\alpha/2})$ nodes of rank $r - 1$
- two nodes of a top tree are sibling nodes if they are children of the same top tree node and share a single vertex of the underlying graph, we call such vertex a boundary vertex
- each node of the top tree remembers its own boundary vertices as well as all boundary vertices of its children
- each node of the top tree has a single vertex called the root of the node
- the root of a top tree node that does not represent whole underlying tree is its border vertex on the path to the root of its top tree parent node

Additionally, each vertex of the underlying graph maintains a set of top-tree nodes that contain this vertex, single top tree node per rank. We call this set a reference set. If a top tree node stores a vertex of underlying graph, it stores also its reference set.

**Memory requirements for storing a single top-tree node:** The total size of the description of a top tree node is proportional to the number of its children and the number of all boundary vertices (and their reference sets) stored by this node.

**Lemma 3.1.** The number of boundary vertices of children (which we also call sub-nodes) of a single top tree node is $O(n^{\alpha/2})$

**Proof.** Firstly, let observe that there exists a sub-node with exactly one boundary vertex. If that would not be the case, it would be possible to find a tour, which visits each boundary node only once, and contains cycle.

Assume that there are no sub-nodes with only one boundary vertex. Let consider a tour that starts in some sub-node and can not step on a boundary vertex twice. Whenever tour goes into a sub-node with at least two boundary vertices, stepping on a vertex $x$, it can leave this sub-node via some vertex $y \neq x$. This means that such tour either ends in the starting sub-node, or in a sub-node with at least 3 boundary vertices.

The part of the tour that left from this sub-node, and came back without stepping twice on any boundary vertex is a cycle in the underlying graph, which is impossible, because the underlying graph is a tree.

Knowing that there exist at least one sub-node with one boundary vertex, we can remove from the graph all edges contained in this sub-node, and all non boundary vertices. For the boundary vertex, if it is not shared between two sub-nodes it is not a boundary vertex anymore. Then, we can repeat this reasoning, since removing a single sub-node reduces the number of boundary vertices at most by 1, and at the end there are no boundary vertices, the initial number of boundary vertices has to be no larger than the number of sub-nodes.

By definition, each node stores only $O(n^{\alpha/2})$ sub-nodes, their boundary vertices and own boundary vertices. By Lemma 3.1 the number of boundary vertices of all children is $O(n^{\alpha/2})$, hence the total number of boundary vertices stored by a single node of the top tree is $O(n^{\alpha/2})$. Since for each
of those vertices the size of the reference set is $O\left(\frac{1}{2}^\alpha\right)$, the total space requirement for storing the whole top tree node is $O\left(\frac{1}{2}^\alpha n^{\alpha/2}\right)$.

**Top Trees – operations:** In the remaining part of this section, we give an implementation of the top tree operations. First, we propose protocols for edge insertion and edge deletion operations: in Section 3.1 we show a protocol that given a top tree that is slightly out of balance transforms it into a top tree that meets the size constraints on all nodes. Then, in Section 3.2 we show a reduction from the problem of executing edge insert and edge delete operations to the problem of rebalancing an almost balanced top tree. Finally, in Section 3.3, we present the algorithm that process a batch of path queries.

### 3.1 Edit operations – rebalancing the tree

The top tree data structure allows us to modify the underlying forest by deleting and inserting the edges. During the executions of the protocol, we allow the nodes of the top tree to slightly violate the size constraints: rather than requiring the size to be between $n^{\alpha/2}$ and $cn^{\alpha/2}$, we only require it to be smaller than $c'n^{\alpha/2}$, for some $c' > c$. We call the nodes larger than $cn^{\alpha/2}$ overloaded, and smaller than $n^{\alpha}$ underloaded, and the top tree meeting this relaxed size constraint almost-balanced.

The common part of edge insertion and edge deletion operations is a protocol that restores the size invariant for the nodes of the top tree. More precisely, if the size constraint on the size of an internal node is $n^{\alpha/2} \leq \text{size} \leq cn^{\alpha/2}$, for some constant $c > 1$, then given a top tree $T$ that has nodes that are of size smaller than $c'n^{\alpha/2}$, we can reorganize $T$ into $T'$ which has nodes preserving the size constraints in $O\left(\frac{1}{2}^\alpha\right)$ rounds. In the remaining part of this subsection, we present the algorithm rebalancing an almost balanced top tree.

To restore the size invariants on the nodes, we process the tree in two phases. In the first phase, we reorganize the top tree to provide that it does not contain nodes that are underloaded, but we possibly introduce some vertices that are overloaded. Then, in the second phase, we split and rearrange all overloaded nodes, so that in the resulting trees all internal nodes meet the size requirements.

The first phase is executed in a bottom - top fashion. While processing nodes of rank $r$, we assume that each node contains sub-nodes that are proper instances of a top tree, but possibly with a rank that is smaller than $r - 1$. Let $v$ be the node we process, and $r'$ be the highest rank among the sub nodes of $v$. To perform rebalancing, we merge all sub-nodes in such a way, that as a result we get a set of sub-nodes of rank $r'$, each having no underloaded internal nodes. To do so, rather than executing merges right away, we compute a set of merge operations that have to be executed, and execute them later, simultaneously, on all levels of the tree. In order to compute which merges have to be done, we can simulate the following process: while there exists a node $X$ that has rank $r_1$ that has a sibling node $Y$ of rank $r_2 > r_1$, then virtually merge $X$ into $Y$, (for the sake of this process replace nodes $X$ and $Y$ by $Y$).

At this point, we have a set of sub-nodes of rank $r'$, some of them could be underloaded. Since, as a result, we can not have that $v$ has underloaded sub-nodes, we have to fix it. To do so, we can look at all sub-nodes and their children (including some sub-nodes of rank $r' - 1$ that just got virtually merged into those nodes). The total number of children and grandchildren of $v$ is at most $O(n^\alpha)$, hence it fits into memory of a single machine, which can then split grandchildren into groups of size that meets the requirement. If as a result we get only one node, the rank of $v$ becomes $r'$, otherwise it becomes $r' + 1$.

**Merge:** The merge operation is defined for a top tree $T$ of rank $r$ and a set of top trees $T_1, T_2, \ldots, T_k$ of ranks $r_1, \ldots, r_k$, and set of edges $e_1, e_2, \ldots, e_k$ such that for all $i$ $r_i < r$, and $e_i$ is in $T_i$ and has
one endpoint in $T$. For each edge, we identify a node $v_i$ of top tree $T$ that has rank $r_i$ and contains endpoint of $e_i$ that is in $T$. We merge $T_i$ with $v_i$, which is executed by the parent node of $v_i$.

Let consider a node $v$, to which children we link trees $T_1, \ldots, T_{k'}$. Since $k' \leq k \in \mathcal{O}(n^{\alpha/2})$, and the any node of a top tree has $\mathcal{O}(n^{\alpha/2})$ direct sub-nodes, we can look into all children of root nodes of $T_1, T_2, \ldots, T_k$ and all grand-children of $v$, and establish partition into sub-nodes on this structure in such a way, that it does not have underloaded nodes. As a result, we get a set of nodes that is larger by at most $\mathcal{O}(n^{\alpha/2})$ nodes.

Furthermore, all top trees that are being merged into $v$ used to be sibling nodes of some ancestor of $v$. Since there are only $\mathcal{O}(\frac{1}{\alpha})$ ancestors of $v$, and each of them has up to $\mathcal{O}(n^{\alpha/2})$ sibling nodes, the total number of new nodes in $v$ is $\mathcal{O}(\frac{1}{\alpha} n^{\alpha/2})$. Therefore, at the end of rebalancing on all levels, the number of children of $v$ could increase by $\mathcal{O}(\frac{1}{\alpha} n^{\alpha/2})$, which for constant $\alpha$ means that the size constraint on $v$ is violated at most by a constant factor.

**Clean-up phase**: As mentioned before, after first phase of rebalancing, we remove all nodes that are too small, but create some that could be too large by a constant factor $x$. To clean this up, we use following pipelined splitting protocol. On all levels of the tree in parallel we do the following: simultaneously, each node that is too large splits itself into up to $x$ parts and notifies a parent node about this. Then, each old-node passes the notification to proper newly created vertices, i.e. if a notification come from a sub-node $X$ that is now a part of newly created node $Y$, then notification is sent to $Y$. Since each newly created node has size within the constraints, and each notification splits a vertex into up to $x$ parts, the size of each newly created node is at most $x$ times too large. Furthermore, in order to create overloaded node, the level lower also had to contain an overloaded node. Therefore, in each step the minimum rank of the overloaded node increases by 1, hence rebalancing whole tree will end in time proportional to the depth of the tree, which is $\mathcal{O}(\frac{1}{\alpha})$.

### 3.2 Edit operations – split and link

In this subsection we propose protocols for linking and splitting the top trees. We show how to execute those operation in such a way that the resulting top trees can be balanced by the protocol described in Section 3.1.

**Edit operations – split (delete edge)**: To execute the splitting, for each edge $e$ we identify the splitting node: the node $v$ of the top tree that contains $e$ and the root-endpoint of $e$ is a boundary vertex of $v$. Then, the parent node of $v$ splits the set of sub-nodes into parts that are reachable from the root without going through $v$, and those that are not. The parts that are not reachable from the root, become new instances of the top tree, for the connected components created by the edge delete operation. All those operations can be executed simultaneously, for all edges. As a result, we get a set of data structures, one per connected component, but those structures do not necessarily meet the size constraints of the internal nodes.

**Edit operations – link (add edge)**: On the top level, the tree link operation can be executed as follows: firstly, we treat each edge as a bag of rank 0 and merge it into a bag that contains one of its endpoint. Then, we merge all trees that share a vertex. Both operations can be understood as a variant of a merge that was executed during rebalancing after the delete operation, but the number of data structures that are being merged into one vertex could be $\Theta(n^\alpha)$, which prevents us from direct application of merge protocol from the rebalancing algorithm.

To bypass this issue, as a first step, a vertex $v$ of rank $r$ that performs the merging treats the nodes to be merged as each of the nodes does not violate the size constraints. Thus, we can partition $v$ into $\mathcal{O}(n^{\alpha/2})$ nodes of the same rank $r$. Then, such $v$ notifies the parent node about the fact that in place of $v$ we now have $\mathcal{O}(n^{\alpha/2})$ nodes of rank $r$. After this operation, we have
some node that possibly are underloaded (trees that were merged in), and some that are possibly overloaded (a parent node of $v$), but only by a constant factor – this is because in order to increase the parent node size, a child node has to split. In order to split a child into at least three parts, we have to add at least $\Theta(n^{\alpha/2})$ nodes to this a child. therefore computing the value of $f$ therefore, each parent node increases size by at most $O(n^{\alpha/2})$ from children splitted into two parts, and $O(n^{\alpha})/\Theta(n^{\alpha/2}) = O(n^{\alpha/2})$ nodes from child that were splitted into at least three parts. From this point, we can execute exactly the same rebalancing protocol as in case of rebalancing after edge deletion - firstly we get rid of underloaded vertices, then we fix all overloaded vertices.

3.3 Query operations – batch $(u, v)$-path query operation

In this section we propose a protocol that recovers a value of a distributive aggregative function $f$ on a given path. The path is defined by the two endpoints, and the queries can be executed in batches. A single query is executed in a recursive manner. We use that a path from $u$ to $v$ can be splitted into parts that are fully contained by the sub-node of the node answering the query, therefore computing the value of $f$ on the parts of the path connecting $u$ and $v$ contained in sub-nodes is sufficient to compute the heaviest edge on the whole path. In this paragraph we discuss this idea in more details and provide that this approach gives an $O(\frac{1}{\alpha})$ round protocol, that uses only $O(n)$ global memory, while respecting the $O(n^\alpha)$ bound on a memory of a single machine.

**Execution of batch $(u, v)$-path query operation**: The execution of the protocol in a single node of the top tree is following. For nodes of rank 2 we can gather whole subtree represented by this node and answer the queries. For nodes of larger rank, for each query pair, we compute a sub-node path between the given vertices, i.e. set of sub-nodes that covers the whole path between the query vertices. If $u$ and $v$ are in the same sub-node, the node recursively asks it for the answer (we call it direct sub-query). Otherwise, for each internal subnode on the $(u, v)$ path, it generates a subquery $(u', v')$, where $u'$ and $v'$ are boundary vertices of this structure on the $u, v$ path (we call it internal sub-query), while for the structures that hold the endpoints of the the node asks for $(u, v')$ and $(u', v)$, where $u'$ is a boundary node from node holding $v$ towards node holding $u$, and $v'$ is a boundary node from node holding $u$ towards node holding $v'$ (we call it endpoint sub-query). The result is maximum of the answers gathered from recursive sub-queries.

**Communication complexity – local bounds**: Executing naively all subqueries in parallel could result in large number of subqueries, i.e. we could generate $\Theta(n^{\alpha/2})$ subqueries for a single query, hence $\Theta(n^{3\alpha/2})$ in total. to bypass this issue, we argue that the number of distinct subqueries we have to ask is $O(n^{\alpha/2})$, and each sub-node does not receive more subqueries than the parent node. Obviously, the number of direct subqueries can not be larger than the number of original queries, hence it is $O(n^\alpha)$. Let us consider queries that have endpoints in different sub-nodes: firstly, each query generates two endpoint sub-queries, so those contribute $O(n^\alpha)$ to the total number of sub-queries. Finally, each sub-query is defined by a pair of boundary nodes of a sub-node. Since by Lemma 3.1, there only $O(n^{\alpha/2})$ boundary vertices in sub nodes, the total number of possible subqueries is at most $O(n^{\alpha/2})^2 = O(n^\alpha)$. This mans that all those subqueries can be sent to sub-nodes in $O(1)$ rounds. Furthermore, for each query, we generate at most one sub-query per sub-node, hence the total number of subqueries is at most constant factor larger than the number of original queries, and each sub-node gets no more than the number of original queries.

**Communication complexity – global bound**: Additionally, we can give a bound on the total number of queries that are generated during the execution. Since the query is answered in the top tree nodes of rank 2 and there are only $O(n^{1-\alpha})$ such nodes, even if all those nodes get a sub-query, for each of all $O(n^\alpha)$ queries, the total number of sub-queries is $O(n^{1-\alpha}) \cdot O(n^\alpha) = O(n)$. 

9
4 Batch Dynamic MST

In this section we propose an algorithm for Batch Dynamic Minimum Spanning Forest problem, in the MPC model.

**Theorem 4.1.** Given an $n$ node, $m$ vertex graph, it is possible to solve Dynamic Minimum Spanning Forest Problem in the MPC model with $O(n^\alpha)$ memory limit on a single machine using

- $O(\frac{1}{\alpha})$ round to process a batch of updates of length $O(n^\alpha)$,
- $O(\frac{1}{\alpha^2} \log n)$ rounds for preprocessing.

**Proof.** **Building the initial Top Tree data structure:** To compute Minimum Spanning Forest in the MPC model with $O(n^\alpha)$ local memory limit, we use a variant of Boruvka’s algorithm [NMM01]. The algorithm maintains a forest that is a subgraph of Minimum Spanning Forest of the input graph, and gradually, in phases, grows this forest. To do so, the algorithm for each vertex chooses a minimum weight outgoing edge, and for each component tosses a coin. To grow the forest, we use all edges that were selected by components that tossed a head and have the other endpoint in a component that tossed a tail.

This variant of the Boruvka’s algorithm with high probability requires only $O(\log n)$ rounds to finish computation, and in a single phase we merge only star shaped graphs, which can be done efficiently in the MPC model. This algorithm allows us to build the top tree data structure along the way. Whenever we execute a merge in the Boruvka’s algorithm, we also merge top trees for all merged underlying trees.

For a single star, we can sort all the subtrees that have to be merged, and each machine holding a block of subtrees to be merged initiates a sequence of merges of corresponding top trees – as a result we get some set of partially merged trees, of size $\Theta(n^\alpha)$ times smaller (but at least of size one), and corresponding top trees. Iterating this idea $\Theta(1/\alpha)$ times ends the merging phase, computing corresponding top trees along the way. Each phase of this merging algorithm requires to run merge operation on top trees, which takes $O(1/\alpha)$ rounds. In total, a single phase of Boruvka’s algorithm takes $O(1/\alpha^2)$ rounds, and whole algorithm $O(\frac{1}{\alpha^2} \log n)$ rounds, which for a constant $\alpha$ is $O(\log n)$.

**Processing a batch of updates:** Similarly, as in [DDK+20] in order to process a dynamic sequence, we firstly focus on processing a batch of updates in an offline manner, in order to identify a set of components and edges, that later allow us to process a dynamic sequence of updates in the memory of a single machine.

Firstly, we process all delete operations. If an edge is deleted, a new edge could be added to the Minimum Spanning Forest, and it is either an edge from the original graph, or it is one of the edges that is a part of the update sequence. The goal of this part is to identify all edges that could be a good replacement edge after some delete, that come from the original graph.

Then, we process all insert operations, in order to identify all edges that could violate the cycle property after executing some insert operation. By the cycle property, in order to update the minimum spanning forest after an edge insert operation, we can add this edge to the minimum spanning forest, and if we created a cycle, remove the heaviest edge on this cycle. Here, we want to apply whole batch of edge insert and delete operations at once, using only the information provided by the top tree for unaltered variant of the underlying tree. This means, for the $j$th insert, we don’t really know what would be the spanning forest after execution of $(j - 1)$th update operation. Therefore, we want to identify a set of edges that contains all the edges of the original graph, that could violate the cycle property, which is formulated in Lemma 4.2.
Lemma 4.2. Let us consider a graph \( G \), its minimum spanning forest \( F = (V, E_F) \), and some sequence of updates \( U \) of length \( k \). Let

- \( D = d_1, d_2, \ldots, d_k \) be a sequence of edges to delete,
- \( R = r_1, r_2, \ldots, r_k \) be a set of edges that would be included to the minimum spanning forest, if we execute deletes from \( D \),
- \( I = (i_1, i_2, \ldots, i_k) \) is a sequence of edges to be inserted to \( G \)
- \( V_U \) be a set of vertices incident to the edges from \( D, R, I \).
- \( S \) be a set of edges of \( F \) that for each pair of vertices \( u, v \in V_U \) contains the heaviest edge on the path between \( u \) and \( v \) in \( F \).
- \( F' = (V, E_F \setminus (S \cup D)) \).

Then, we have that \( |S| \leq 4k - 1 \) and the Minimum Spanning Forest after each update consists of components of \( F' \) connected by some of the edges from \( U, R, \) and \( S \).

Proof. To prove this lemma, we show the following:

1. \( |R| \leq |D| \)
2. \( |S| \leq 2(|U| + |R|) - 1 \)
3. after any delete operation, a replacement edge (if exists) is in the set of edges from \( U \) or \( R \)
4. after each insert operation, an edge violating the cycle property (if exists) is in the set of edges from \( U \) or in \( R \cup S \).

Bound on size of \( R \) and \( S \) (points 1 and 2): The first statement simply says that for each deleted edge, we can have at most one replacement edge. The set \( S \) is defined as set of edges, such that for each pair of vertices from \( (u, v) \in V_U \) it contains the heaviest edge on the path from \( u \) to \( v \) over the edges of forest \( F \). Let consider the edges of \( F \) from heaviest to the lightest. If the considered edge separates a pair of vertices from \( V_U \), we include it to \( S \) and remove it from \( F \). Since each edge included to \( S \) splits a set of vertices into two smaller sets. Since we can do at most \( |V_U| - 1 \) splittings, we have \( |S| \leq |V_U| - 1 \). Since \( |V_U| \leq 2(|D| + |R| + |I|) \) and \( |R| \leq |D| \leq |D| + |I| \), we have \( |S| \leq |V_U| - 1 \leq 4(|D| + |I|) - 1 = 4k - 1 \).

All relevant edges are in sets \( D, R, I \) or \( S \) (point 3, computing set \( R \)): Let consider a deleting of edge \( e \). If the replacement edge \( r \) exists, it is the lightest edge that connects two connected components of the spanning forest obtained by removing \( e \). If the edge \( r \) is either a member of \( I \) or \( D \), then we are done.

Let us assume that’s not the case and consider the graph induced by edges of the graph \( (V, E \setminus D) \) that are lighter than \( r \). If in such graph the endpoints of \( e \) are in a single connected component, then \( r \) is not the best replacement edge, as there exist path connecting endpoints of \( e \) using only lighter edges, and none of them got deleted. If the endpoints are in separate connected components, then \( r \) is the next edge that is to be considered by the Kruskal algorithm, hence it is part of a minimum spanning forest of \( (V, E \setminus D) \). Therefore, \( r \) is a part of set \( R \).

All relevant edges are in sets \( D, R, I \) or \( S \) (point 4, computing set \( S \)): Let consider an insert operation. The inserted edge could create a cycle in the spanning forest, and our goal is to remove the heaviest edge on such cycle. This cycle could go over several edges that were included during the execution of update sequence. No matter what the shape of such cycle is, it can be splitted into parts consisting of paths containing only edges from \( U \cup R \) and paths containing only
the edges of the original minimum spanning forest. For the first kind of path, we know explicitly all the edges, hence we know an edge with maximum weight on this part of the cycle. The paths of second kind connect the vertices of \( V_U \), hence by definition of \( S \) we include maximum weight on such path in \( S \). Therefore, no matter what is the shape of the cycle, the edge with maximal weight is in \( U \cup R \cup S \).

Therefore, if we can identify all edges of \( R \) and \( S \), and all components of \( F' \) incident to those edges, we can simulate the sequence of updates in the local memory of a single machine. In following two paragraphs, we briefly discuss implementation of the approach from Lemma 4.2 in the MPC model.

**Processing delete operations:** In order to find all replacement edges, we exploit the following:

**Fact 4.3.** Let us consider a graph \( G \) and its minimum spanning forest \( F = \{T_1, T_2, \ldots, T_x\} \), and the set \( E_d \) of \( k \) edges to delete. Let \( F' = \{T'_1, T'_2, \ldots, T'_y\} \) be a spanning forest of \( F \) without the edges from the set \( E_d \), and \( G' \) be a graph \( G \) without edge from \( E_d \). Then, all edges of the minimum spanning forest of graph \( G' \), that are not in \( F' \) are between \( T'_i \) and \( T'_j \), such that \( T'_i \notin F \) and \( T'_j \notin F \).

In other words, all edges that could be included in the new minimum spanning forest have to be between connected components of the maintained forest created by deleting the edges. Since we remove only \( \mathcal{O}(n^\alpha) \) edges, we create only \( \mathcal{O}(n^\alpha) \) connected components in \( F \) by removing the edges. Furthermore, having top tree data structure, we can identify all vertices inside of those connected components in \( \mathcal{O}(\frac{1}{\alpha}) \) rounds, hence we can identify all the edges that could be part of the new minimum spanning forest in \( \mathcal{O}(\frac{1}{\alpha}) \) rounds. Having those components and edges, we can treat them as an instance of the Minimum Spanning Forest problem, with \( \mathcal{O}(n^\alpha) \) vertices, and some number of edges \( m' \leq m \), hence we can solve it by \( \mathcal{O}(1) \) round algorithm \([JN18, Now19]\), with \( \mathcal{O}(m') \) global memory and \( \mathcal{O}(n^\alpha) \) memory limit of a single machine.

**Processing insert operations:** In order to compute set \( S \), we can use recursive approach: given a partition of the tree into disjoint connected sets of edges, we can compute such set for all the points that into each part, in a recursive manner, obtaining some sets \( S_1, S_2, \ldots, S_x \). Between each pair of parts, there can be at most single pair of vertices that don’t have the heaviest edge between them in \( \bigcup S_i \), if that would not be the case, there would be a pair of vertices without the heaviest edge in one of the subtrees. Therefore, if in addition to sets \( S_i \) we also compute the heaviest edge on the path between a vertex of \( V_U \) and a border vertex (vertex that is shared with some other subset), we have full information required for computing set \( S \).

Furthermore, the size of the result in a subproblem with \( y \) points, is at most \( y - 1 \), which also provides that the size of \( \bigcup S_i \) is at most \( y - x \), and we still have at most \( x \) points to split, which requires adding up to \( x - 1 \) edges. In total the size of \( S \) is at most \( y - 1 \), as required.

Since the nodes of the top tree data structure correspond to connected parts of the input graph, this data structure can be used to implement this approach. Our algorithm processes the nodes of the top tree in a bottom - top fashion, which requires only \( \mathcal{O}(1/\alpha) \) rounds.

**Generating a sequence of updates:** By Lemma 4.2 the minimum spanning forest at each moment consist of connected components of graph \( F' \) which is a spanning forest of the original graph without all edges from \( S, RandU \). In order to identify connected components of \( F' \), we can use the top tree data structure: it is enough to split the minimum spanning forest of the original graph using edges of \( S \), and edges of \( D \) that are part of \( F \). Then, the connected components that
are relevant to our computation are all connected components of $F'$ that are incident to the edges from $U$, $R$ and $S$.

Since $|U| = k$, $|R| \leq k$ and $|S| \leq 4k - 1$, the total number of edges in $U$, $R$ and $S$ is $O(k)$. The number of connected components of $F'$ that are relevant is also $O(k)$. Therefore, the graph consisting of those components and edges can be stored in the memory of a single machine, where the algorithm can simulate the update sequence locally, and generate the output sequence. 

**Comparison with the previous result on the Batch Dynamic Connectivity in the MPC model:** Durfee et al. [DDK+20] show an $O(1)$ round algorithm that can process a batch of size $O(n^{\omega-\delta})$ using $\tilde{O}(n^\omega)$ communication. Even though our algorithm could use even $\Theta(m)$ messages, it solves more difficult problem, for polynomially larger batch sizes, which seems to be as large as possible (without giving any improvements for algorithms with sublinear space).

Apart from the Batch Dynamic Connectivity algorithm show that it can be applied to Adaptive Connectivity. In this problem, we ask to execute a sequence of pairs of commands, each consisting from a query (in their case query ask whether an edge is inside a single connected component) and a update, and executing the update may depend on the result to the query.

In our data structure, querying whether an edge connects different connected components is trivial, as for each endpoint, we know the topmost top tree node that contains it, which uniquely identifies the connected component containing this endpoint, furthermore our algorithm also generalizes to such adaptive variant of the problem, i.e. Adaptive Minimum Spanning Forest problem in quite similar way. During the execution of batch variant of the problem, we identified the set of edges that can be parts of the final MST in a way, that allows simulation in the local memory of a single machine, no matter what are the results of the queries.

As for edge deletions, the argument is quite similar as in [DDK+20]. Since we identified a set of edges that makes graph connected if all deletions would happen, we are prepared to handle any subset of the edge-delete operations. For edge insertion, the argument is slightly more complicated than in [DDK+20].

For the edges that connect separate connected components, we don’t need to compute any additional backup edges, but for the edges that are inside a single connected component, it could be that some edge from the Minimum Spanning Forest has to be removed. The way we computed a set of edges for the extended cycle property does not really consider a weight of an edge to be inserted, therefore we can think about the edges that have small weight, but according to the query result should not be inserted, as edges with infinite weight (hence not being included in the Minimum Spanning Forest).

Even though we can process in a constant number of rounds a batch of polynomially larger size, it does not give a huge improvement for circuit simulation [DDK+20, Corollary 1.2]. For a long sequence of updates, the algorithm of Durfee et al. can process a part of sequence of size $O(n^\omega)$ in $O(\log n)$ rounds, hence a sequence of $O(n)$ updates in $O(n^{1-\omega} \log n)$ rounds. Plugging in our result shaves off only the $\log n$ factor.

### 5 Maximal Matching in Batch Dynamic MPC

In this section, we propose an algorithm for the Maximal Matching problem in the Batch Dynamic MPC model. Our goal is to maintain a maximal matching of a graph undergoing changes, as described in Section 1.5.

**Theorem 5.1.** Let $G$ be a graph, $M$ be a maximal matching of $G$, and $U$ be a sequence of $k$ updates to $G$. Let $G_U$ be a graph $G$ on which we applied the updates. Given $G, M, \text{ and } U$, we can process
U and compute a batch of $O(k)$ updates $U'$, such that applying $U'$ on $M$ gives a maximal matching of $G_U$, in $O(\log 1/\delta)$ rounds, for $k \in \Theta(S^{1-\delta})$, which for constant $\delta$ gives an $O(1)$ round algorithm, and for $\delta = O(1/\log n)$, an $O(\log \log n)$ round algorithm.

**Proof.** Let $M$ be a maximal matching maintained by the algorithm before the updates are applied to the graph, and $M'$ be a matching $M$, from which we removed all edges according to a sequence of updates we process. Then, in order to extend $M'$ to a maximal matching of the graph after all the updates are applied, it is enough to solve a variant of the Maximal Matching problem, with a promise that the graph has a vertex cover of size $O(k)$.

**Claim 5.2.** The set of edges that can extend $M'$ to a maximal matching of the current graph has a vertex cover of size at most $2k$.

**Proof.** Applying $k$ updates to the graph can remove up to $k$ edges from maintained maximal matching and can insert up to $k$ new edges. Any edge that could extend $M'$ has to be either incident to an edge that got removed during the updates, or has to be a completely new edge. If $M' \subset M$ could be extended by some edge $e$ that is not incident to any edge from $M$, and is not new, then $e$ was present in the graph before the updates. Therefore, $e$ could be added to $M$, which contradicts maximality of $M$. The set of endpoints of $k$ edges (inserted and deleted) has size at most $2k$, hence all edges that can extend $M'$ are covered by at most $2k$ vertices. \( \square \)

To solve this special variant of the Maximal Matching problem, we use an $O(\log \log n)$ round Maximal Matching algorithm designed for the static variant of the MPC model, with a linear memory of a single machine [BHH19]. More precisely, we show that the memory requirement of a single machine is actually linear with respect to the size of the vertex cover of the input graph, which combined with Claim 5.2 and some small adjustments, is enough to provide an algorithm for Maximal Matching in Batch Dynamic MPC.

The static algorithm builds a maximal matching in phases. It starts with an empty matching, and input graph as a source of edges that can extend this matching. In a single phase it considers only a residual graph, i.e. a graph in which there are only the edges that still can extend the matching obtained at the end of the previous phase, and extends the matching by some set of edges from this residual graphs. In order to obtain $O(\log \log n)$ round complexity, the authors show that one phase can be implemented in $O(1)$ rounds, and that if $\Delta_i$ is the maximal degree of the residual graph after the $i$th phase, then $\Delta_i \in O(\Delta^{-1-\Omega(1)}_{i-1})$, with exponentially high probability.

Here, we show that a single phase of the static algorithm, assuming existence of a vertex cover of size $k$, can be executed in $O(1)$ rounds, using machines with local memory $O(k)$, and guarantees that degrees of vertices in the cover drop from $\Delta$ to $\Delta^{1-\Omega(1)}$, with exponentially high probability.

**Lemma 5.3.** Given a graph $G$ of maximal degree $\Delta$, and its vertex cover of size $k$, it is possible to compute a matching $M$, in $O(1)$ rounds of the MPC model, with the limit on local memory $S \in O(k)$, such that the maximum degree of a vertex in the cover of the residual graph is $O(\Delta^{0.999})$.

**Proof.** The proof is basically repetition of the analysis provided in [BHH19], in which we use that the number of edges in a random subgraph may be bounded by estimating the number of cover vertices in this subgraph and their degrees – as a result we get that the required memory is linear in $k$ rather than in $n$.

For the sake of completeness, we briefly recall analysis of the maximal matching algorithm [BHH19], slightly adjusted to the case, where the algorithm works on machines with $O(k)$ memory, and the input graph has a vertex cover of size $k$. Still, in order to get a more detailed version of the analysis, we recommend to look it up in [BHH19].
The algorithm from [BHH19] consists of phases, and in each phase the maximal degree drops significantly (form $\Delta$ to $\Delta^{1-\Omega(1)}$), which implies the $O(\log \log \Delta)$ round complexity. The key part of our variant of the analysis is that one can give a bound to the memory requirements of the algorithm by considering only the vertices from the given vertex cover. Furthermore, we show that the bound on the maximal degree in the cover set follows from analysis of [BHH19]. We also obtain similar bound on the degree in the whole graph, which is necessary if we want to use unaltered variant of the analysis. To do so, we use a simple observation: after executing a phase of the algorithm, the number of high degree vertices in the whole graph is $O(k)$. Therefore, running it twice, first time to reduce the degrees of the vertices in the given cover, and the second time to reduce the degrees in the high-degree residual graph, provides a guarantee that the maximal degree of whole graph is reduced.

In the remaining part of this section we focus on a single phase of the algorithm by [BHH19], and show that we can implement it in a way that requires only $O(1)$ rounds, and works with $O(k)$ memory limit on a single machine. A single phase of the algorithm consists of two stages.

**Stage 1:** The first stage of a phase is based on the following algorithm:

- partition vertices into $x = \Delta^{0.1}$ random sets $V_1, V_2, \ldots V_x$, and $V_i$ denotes intersection of $V_i$ with the cover set
- sample the edges with probability $1/\Delta^{0.85}$, let $G_i$ be a graph with the vertex set $V_i$ with edges with both endpoints in $V_i$ that survived the sampling
- compute greedy maximal matching $M_i$ of $G_i$, for a random ordering of the edges
- extend the result matching $M$ by $\bigcup M_i$, remove all edges incident to $\bigcup M_i$ from the graph

Our claim is that the set of edges of each $G_i$ is $O(k)$. Furthermore, the number of vertices of degree larger than $\Delta^{0.99}$ in the cover set is $O(k/\Delta^{0.03})$. The bound follows from the analysis provided [BHH19] slightly modified to take into account the fact that we want to give a bound dependent on the size of the cover rather than the number of all vertices.

**Analysis: memory requirement of Stage 1:** Firstly, we show that the algorithm can be executed using only machines with $O(k)$ local memory.

**Claim 5.4.**

- for every $i \in [x]$, $|V_i^c| \in \Theta(k/\Delta^{0.1})$
- for every $i$, graph $G_i$ contains $O(k/\Delta^{0.05})$ edges

**Proof.** The analysis is more or less the same as in [BHH19], although we exploit the existence of a vertex cover of size $k$ $^4$, to show a bound that depends on $k$ rather than $n$.

The first property follows from Chernoff bound, as $E[|V_i^c|] \in O(k/\Delta^{0.1})$. For the second property, we consider two cases: $\Delta > n^{0.01}$ and $\Delta \leq n^{0.01}$.

For larger values of $\Delta$, it is enough to observe that the expected degree of a vertex is $O(\Delta^{0.05})$, and for $\Delta$ that is polynomial in $n$ it holds w.e.h.p. Since all edges have at least one endpoint in the cover set, we can bound the number of edges in $G_i$ by bounding the number of edges in $G_i$ that are incident to vertices of $V_i^c$. Since we already have that $|V_i^c| \in \Theta(k/\Delta^{0.1})$, w.e.h.p., the number of edges in $G_i$ is at most $O(\Delta^{0.05} \cdot \Theta(k/\Delta^{0.1})) = O(k/\Delta^{0.05})$.

For smaller values of $\Delta$, we still have the expected number of edges that is $O(k/\Delta^{0.1})$. Since the number of edges is a function of $O(k\Delta)$ independent random variables, and outcome of a single variable changes the outcome of hole function by $\Delta$ (number of edges is a $\Delta$-Lipschitz function), we can use the bounded differences inequality [Proposition 5.5].

---

$^4$to get exponentially high probability bounds, we consider $k \in \Omega(n^\epsilon)$
Proposition 5.5. [Bounded differences inequality, formulated in this useful way in [BHH19]]

Let \( f \) be a \( \lambda \)-Lipschitz function on \( y \) variables, and let \( X = (X_1, \ldots, X_y) \) be vector of \( y \) independent (not necessarily identically distributed) random variables. Then, w.e.h.p. (with respect to a parameter \( n \))

\[
f(X) \leq E[f(X)] + \lambda n^{0.01} \sqrt{y}
\]

Applying this inequality gives that the number of edges is \( O(k/\Delta^{0.05}) + \Delta k^{0.01} \sqrt{k\Delta} = O(k/\Delta^{0.05}) \) with probability \( 1 - \exp(k^{0.01}) \). As long as \( k \) is polynomial in \( n \), it is also exponentially high probability with respect to \( n \).

□

As a corollary from Claim 5.4 we have that the number of edges of each \( G_i \) is \( O(k) \), hence even though the number of vertices of the graph \( G_i \) could be \( \omega(k) \), we still can gather all relevant vertices and edges in the memory of a single machine and compute a matching of \( G_i \).

Analysis: degree reduction of Stage 1: Now we briefly sketch the part of the proof that shows that we can use this algorithm, to obtain a residual graph, in which the number of vertices in the cover, that have high degree larger than \( \Delta^{0.99} \), is \( O(k/\Delta^{0.03}) \), w.e.h.p.

Firstly, let consider single execution, for each vertex, the proof analyses what is the number of edges that go between this vertex and a single group of vertices (in random partition). By symmetry, it is enough to analyze this for a fixed group, e.g. \( Z_i \). The main idea is to splits vertices of the graph into two kinds: vertices that have small impact on the number of edges in the residual graph (variance of \( Z_{v,1} O(\Delta^{1.4}) \)) and all remaining vertices.

For the first kind of vertices, the authors show that the probability of having degree larger than \( \Delta^{0.99} \) is \( O(\Delta^{-0.03}) \), and the analysis does not depend on value of \( n \) or \( k \). This gives that if there are \( k' \) such vertices in the cover, the expected number of such vertices in the cover of residual graph with degree larger than \( \Delta^{0.99} \) is \( O(k'\Delta^{-0.03}) \).

For the second kind of vertices, the authors show that the expected sum of variances of all vertices in some set \( S \) is \( O(m/\Delta \cdot 2\Delta \cdot \Delta^{0.15} + |S|\Delta^{1.15}) \). If we take \( m = \Delta k \) and \( S \) to be a set of vertices in the cover, we get that the expected sum of variances of all vertices in the cover is \( O(k\Delta^{1.15}) \). From here, we have that in expectation, there can be at most \( k^{0.05} \) vertices that do not qualify as the first kind.

Therefore, the expected number of vertices in the cover of residual graph with degree larger than \( \Delta^{0.99} \) is \( O(k'\Delta^{-0.03}) + O(k\Delta^{-0.25}) = O(k\Delta^{-0.03}) \).

Reducing degree with exponentially high probability: If at the beginning, there are at most \( k/\Delta^{0.03} \) vertices of degree at least \( \Delta^{0.99} \), then we are done. Otherwise, we have that the number of edges in the graph is at least \( k\Delta^{0.96} \). There are two cases to consider.

If \( \Delta < k^{0.1} \), then we can use inequality from Proposition 5.5, which gives exponentially high probability of obtaining a residual graph with at most \( k/\Delta^{0.03} \) vertices of degree at least \( \Delta^{0.99} \), in the cover.

If \( \Delta > k^{0.1} \), then we can run \( k^{0.05} \) instances of the algorithm. A single instance, by Markov inequality, has a constant probability, that the number of vertices in the cover that have degree larger than \( \Delta^{0.99} \) is \( O(k/\Delta^{0.03}) \). Repeating this \( k^{0.05} \) times, in parallel, gives an exponentially high probability that in one of those instances we have \( O(k/\Delta^{0.03}) \) vertices of degree larger than \( \Delta^{0.99} \) in the cover of residual graph. Since one of the steps of the algorithm is to subsample the edges of the input graph, we can bound the total space required by all instances by \( k^{0.05} \cdot O(k\Delta \cdot \Delta^{-0.85}) \leq \Delta^{0.5} \cdot O(k\Delta \cdot \Delta^{-0.85}) = O(k\Delta^{0.65}) \), which is smaller than \( O(k\Delta^{0.96}) \) space required for just storing the

---

\(^5\) in the original analysis \( S \) was a set of all vertices, and \( m/\Delta \) is replaced by \( n \), as such bound was good enough for their purpose
edges of the input graph. Therefore, all instances of the algorithm can be run in parallel, without any additional space.

**Stage 2:** In the second phase of the algorithm, we sample each edge in the residual graph incident to a vertex with degree larger than $\Delta^{0.999}$ with probability $q = 1/\Delta^{0.999}$. Since after the first stage, we have at most $O(k/\Delta^{0.03})$ vertices of high degree, the total number of edges is $O(k\Delta^{0.999}) + O(k/\Delta^{0.03} \cdot \Delta) = O(k\Delta^{0.999})$. Sampling each edge with probability $q$ gives us a set of edges that has size $O(k)$, w.e.h.p. Thus, it can be gathered in the memory of a single machine, where we can compute a greedy maximal matching of this subsampled graph.

We compute a greedy maximal matching on the set of obtained edges, add it to the final solution and remove all edges incident to computed matching. By properties of greedy maximal matchings of uniformly random subgraphs we have that for each vertex, its degree in the residual graph is $O(\Delta^{0.991})$ with probability $1 - \exp(poly(\Delta))$. This implies that the expected number of high degree vertices in the cover of residual graph is $k \cdot \exp(poly(\Delta))$.

If $\Delta > k^{0.01}$, by Markov inequality the degree of all vertices in the cover of residual graph is $O(\Delta^{0.991})$, with exponentially high probability. For $\Delta \leq k^{0.01}$, we use that adding / removing a single edge changes the match status of $O(\Delta)$ vertices [BHH19], which means that the number of high degree vertices in residual graph is a $O(\Delta)$-Lipschitz function of $k\Delta$ random variables, which allows us to apply inequality from Proposition 5.5. This gives us that the number of vertices in the cover of residual graph with degree higher than $O(\Delta^{0.991})$ is $k \cdot \exp(poly(\Delta)) + \sqrt{k\Delta k^{0.01}} = O(k^{0.51} \Delta^{1.5} + k \cdot \exp(poly(\Delta)))$. By assumption $\Delta < k^{0.01}$, this is easily $O(k/\Delta^2)$.

This means, that the remaining number of high degree vertices in the cover of residual graph is so small, that we can gather all edges incident to those vertices in the memory of a single machine, and greedily match them, which eliminates all remaining high degree vertices from the cover. 

Some part of the proof of Lemma 5.3 require that degree of all vertices is small, therefore we have to somehow handle the fact that reducing degree of the vertices in the cover may be not enough. To handle this issue, we use the third observation, that the number of remaining high degree vertices is $O(k)$.

**Claim 5.6.** After executing a single phase of algorithm from [BHH19], on a graph $G$ with maximum degree $\Delta$, the number of vertices in the residual graph that have degree larger than $\Delta^{0.999}$ is $O(k)$.

**Proof.** By Lemma 5.3, we have that the degree of vertices in the cover of residual graph is $O(\Delta^{0.999})$, with high probability. Therefore, the number of edges in the whole graph is at most $O(k) \cdot O(\Delta^{0.999})$, with high probability. This means, that in the residual graph there are at most $O(k) \cdot O(\Delta^{0.999})/\Omega(\Delta^{0.999}) = O(k)$ vertices of degree $\Omega(\Delta^{0.999})$, with high probability. 

Having Lemma 5.3 and Claim 5.6 is enough to give an algorithm that guarantees that in $O(1)$ rounds the degree of whole graph drops significantly with high probability. Firstly, we run a phase of the algorithm from Lemma 5.3. Then we identify all vertices with degrees $\Omega(\Delta^{0.999})$, and add them to the cover. Since there are $O(k)$ such vertices, the size of the cover is still $O(k)$. Now we run the algorithm on the graph with this extended cover – by construction of this cover all vertices of degree $\Omega(\Delta^{0.999})$ are already in this cover, and running the algorithm from Lemma 5.3 reduces those degrees to $O(\Delta^{0.999})$. As a result we get a residual graph in which all vertices have degree $O(\Delta^{0.999})$

**Remark:** We use a single phase of the algorithm from [BHH19] in a black box manner (twice, to get reduction of degree in whole graph), showing that its memory requirements can be expressed as a function of the vertex cover. Similarly as in the paper [BHH19], if we are given larger memory,
we can use larger probabilities of sampling, which yields faster algorithms, more precisely for $k \in \Theta(S^{1-\varepsilon})$ the round complexity is $O(\log(1/\varepsilon))$.

References

[AAB+20] Umut A. Acar, Daniel Anderson, Guy E. Blelloch, Laxman Dhulipala, and Sam Westrick. Batch-dynamic algorithms via parallel change propagation and applications to dynamic trees, 2020.

[AABD19] Umut A. Acar, Daniel Anderson, Guy E. Blelloch, and Laxman Dhulipala. Parallel batch-dynamic graph connectivity. In Proc. of SPAA, pages 381–392, 2019.

[ABT20] Daniel Anderson, Guy E. Blelloch, and Kanat Tangwongsan. Work-efficient batch-incremental minimum spanning trees with applications to the sliding window model, 2020.

[ACC+18] Moab Arar, Shiri Chechik, Sarel Cohen, Cliff Stein, and David Wajc. Dynamic matching: Reducing integral algorithms to approximately-maximal fractional algorithms. In Proc. of ICALP, pages 7:1–7:16, 2018.

[AGM12] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Analyzing graph structure via linear measurements. In Proc. of SODA, pages 459–467. SIAM, 2012.

[AHdLT05] Stephen Alstrup, Jacob Holm, Kristian de Lichtenberg, and Mikkel Thorup. Maintaining information in fully dynamic trees with top trees. ACM Trans. Algorithms, 1(2):243–264, 2005.

[AOSS19] Sepehr Assadi, Krzysztof Onak, Baruch Schieber, and Shay Solomon. Fully dynamic maximal independent set with sublinear in n update time. In Proc. of SODA, pages 1919–1936, 2019.

[BDH+19] Soheil Behnezhad, Mahsa Derakhshan, MohammadTaghi Hajiaghayi, Cliff Stein, and Madhu Sudan. Fully dynamic maximal independent set with polylogarithmic update time. In Proc. of FOCS, 2019.

[BFH19] Aaron Bernstein, Sebastian Forster, and Monika Henzinger. A deamortization approach for dynamic spanner and dynamic maximal matching. In Proc. of SODA, pages 1899–1918, 2019.

[BGS18] Surender Baswana, Manoj Gupta, and Sandeep Sen. Fully dynamic maximal matching in $o(\log n)$ update time (corrected version). SIAM J. Comput., 47(3):617–650, 2018.

[BHH19] Soheil Behnezhad, MohammadTaghi Hajiaghayi, and David G. Harris. Exponentially faster massively parallel maximal matching. In Proc. of FOCS, 2019.

[CLM+18] Artur Czumaj, Jakub Lacki, Aleksander Madry, Slobodan Mitrovic, Krzysztof Onak, and Piotr Sankowski. Round compression for parallel matching algorithms. In Proc. of STOC, pages 471–484, 2018.

[Com79] Douglas Comer. The ubiquitous b-tree. ACM Comput. Surv., 11(2):121–137, 1979.
[CS18] Moses Charikar and Shay Solomon. Fully dynamic almost-maximal matching: Breaking the polynomial worst-case time barrier. In Proc. of ICALP, pages 33:1–33:14, 2018.

[CZ19] Shiri Chechik and Tianyi Zhang. Fully dynamic maximal independent set in expected poly-log update time. In Proc. of FOCS, 2019.

[DDK+20] David Durfee, Laxman Dhulipala, Janardhan Kulkarni, Richard Peng, Saurabh Sawlani, and Xiaorui Sun. Parallel batch-dynamic graphs: Algorithms and lower bounds. In Proc. of SODA, 2020.

[EGI98] David Eppstein, Zvi Galil, and Giuseppe F Italiano. Dynamic graph algorithms. In Algorithms and theory of computation handbook, pages 181–205. CRC Press, 1998.

[FL94] Paolo Ferragina and Fabrizio Luccio. Batch dynamic algorithms for two graph problems. In PARLE ’94: Parallel Architectures and Languages Europe, 6th International PARLE Conference, Athens, Greece, July 4-8, 1994, Proceedings, pages 713–724, 1994.

[GGK+18] Mohsen Ghaffari, Themis Gouleakis, Christian Konrad, Slobodan Mitrovic, and Ronitt Rubinfeld. Improved massively parallel computation algorithms for mim, matching, and vertex cover. In Proc. of PODC, pages 129–138, 2018.

[GL20] Seth Gilbert and Lawrence Li. How fast can you update your mst? (dynamic algorithms for cluster computing), 2020.

[GP16] Mohsen Ghaffari and Merav Parter. MST in Log-Star Rounds of Congested Clique. In Proc. of PODC, 2016.

[GU19] Mohsen Ghaffari and Jara Uitto. Sparsifying distributed algorithms with ramifications in massively parallel computation and centralized local computation. In Proc. of SODA, pages 1636–1653, 2019.

[HdLT01] Jacob Holm, Kristian de Lichtenberg, and Mikkel Thorup. Poly-logarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity. J. ACM, 48(4):723–760, 2001.

[HPP+15] James W. Hegeman, Gopal Pandurangan, Sriram V. Pemmaraju, Vivek B. Sardeshmukh, and Michele Scquizzato. Toward optimal bounds in the congested clique: Graph connectivity and mst. In Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, PODC ’15, pages 91–100, New York, NY, USA, 2015. ACM.

[ILMP19] Giuseppe F. Italiano, Silvio Lattanzi, Vahab S. Mirrokni, and Nikos Parotsidis. Dynamic algorithms for the massively parallel computation model. In Proc. of SPAA, pages 49–58, 2019.

[JN18] Tomasz Jurdzinski and Krzysztof Nowicki. MST in O(1) rounds of congested clique. In Proc. of SODA, pages 2620–2632, 2018.

[KKT95] David R. Karger, Philip N. Klein, and Robert E. Tarjan. A randomized linear-time algorithm to find minimum spanning trees. J. ACM, 42(2):321–328, March 1995.

[KSV10] Howard J. Karloff, Siddharth Suri, and Sergei Vassilvitskii. A model of computation for mapreduce. In Proc. of SODA, pages 938–948, 2010.
Silvio Lattanzi, Benjamin Moseley, Siddharth Suri, and Sergei Vassilvitskii. Filtering: a method for solving graph problems in mapreduce. In *Proc. of SPAA*, pages 85–94, 2011.

Zvi Lotker, Boaz Patt-Shamir, Elan Pavlov, and David Peleg. Minimum-weight spanning tree construction in $O(\log \log n)$ communication rounds. *SIAM Journal on Computing*, 35(1):120–131, 2005.

Jaroslav Nešetřil, Eva Milková, and Helena Nešetřilová. Otakar Borůvka on minimum spanning tree problem translation of both the 1926 papers, comments, history. *Discrete Mathematics*, 233(1):3–36, 2001.

Krzysztof Nowicki. A deterministic algorithm for the MST problem in constant rounds of congested clique. *CoRR*, 2019.

Ofer Neiman and Shay Solomon. Simple deterministic algorithms for fully dynamic maximal matching. *ACM Trans. Algorithms*, 12(1):7:1–7:15, 2016.

Danupon Nanongkai, Thatchaphol Saranurak, and Christian Wulff-Nilsen. Dynamic minimum spanning forest with subpolynomial worst-case update time. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 950–961, 2017.

Krzysztof Onak. Round compression for parallel graph algorithms in strongly sublinear space. *CoRR*, abs/1807.08745, 2018.

Shay Solomon. Fully dynamic maximal matching in constant update time. In *Proc. of FOCS*, pages 325–334, 2016.

Robert Endre Tarjan. *Data Structures and Network Algorithms*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1983.