IRREDUCIBILITY OF THETA LIFTING FOR UNITARY GROUPS

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ABSTRACT. This article shows that for unitary dual reductive pairs the first occurrence of theta lift of an irreducible cuspidal automorphic representation is irreducible. It also proves a refined tower property for theta lifts and the involutive property for twisted theta lifts.

INTRODUCTION

This article studies the theta lifts between unitary groups. The main result is that the ‘first occurrence’ of theta lift is irreducible. Let $k$ be a number field and $E$ a quadratic field extension of $k$. Let $\mathbb{A}$ denote the adele ring of $k$. Fix a nontrivial additive character $\psi$ of $k \setminus \mathbb{A}$. Let $X$ be a skew-Hermitian vector space over $E$ and $Y$ a Hermitian vector space over $E$. Let $G(X)$ (resp. $G(Y)$) denote the isometry group of $X$ (resp. $Y$). Fix a pair of characters $\chi_1$ (resp. $\chi_2$) of $E^\times \setminus \mathbb{A}_E^\times$ for the splitting of metaplectic cover over $G(X)$ (resp. $G(Y)$) (c.f. Sec. 2.2). Let $\pi$ be an irreducible cuspidal automorphic representation of $G(X)$. Suppose that the theta lift $\theta^X_{X,\psi}(\pi)$ of $\pi$ from $G(X)$ to $G(Y)$ with respect to $\psi$ and $(\chi_1, \chi_2)$ is nonzero and cuspidal. Then we show that it is irreducible (Thm. 5.3). By the tower property, here cuspidality is equivalent to saying that $\theta^Y_{X,\psi}(\pi)$ is the first occurrence of the theta lift in the Witt tower associated to $Y$. We also show an involution result which is used in the proof of Thm. 5.3. Keep the above assumption. Then the twisted theta lift $\chi_1^{-1}\theta^X_{Y,\psi^{-1}}(\chi_2^{-1}\theta^Y_{X,\psi}(\pi))$ from $G(Y)$ back to $G(X)$ with respect to $\psi^{-1}$ and $(\chi_1, \chi_2)$ is equal to $\pi$ (Thm. 5.1). Analogous results hold for twisted theta lifts in the other direction.

This article is an extension of results of Mœglin[7, 6] and Jiang and Soudry[3] to the case of unitary dual reductive pairs. Mœglin[7, 6] showed for the dual reductive pairs $\text{Sp}_{2n}$ and $\text{O}(2m)$ the irreducibility of first occurrence of theta lift in either direction. She also proved the involutive property which is a key step towards the proof of irreducibility. The odd orthogonal case was treated by Jiang and Soudry[3]. The groups involved are the double cover $\tilde{\text{Sp}}(2n)$ of $\text{Sp}(2n)$ and $\text{O}(2m+1)$. Since in these cases the embedding of the dual reductive pairs into the metaplectic group is canonical ‘non-twisted’ theta lifts are used.

Here we treat the dual reductive pair $G(X)$ and $G(Y)$ which are unitary groups. The method of proof essentially follows Mœglin and Jiang and Soudry. However for unitary groups we need to treat the case where neither $G(X)$ nor $G(Y)$ is quasi-split. Also because of the non-uniqueness of splitting of metaplectic group over

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$G(Y) \times G(X)$, we have to keep track of the characters used to determine a splitting and this results in the twist in theta lifts in our formula.

We sketch the idea of the proof and point out the difficulties. For a non-negative integer $a$, form the skew-hermitian space $X_a$ by adjoining a hyperbolic planes $\ell_a = \ell_a^+ \oplus \ell_a^-$ to $X$. First we show the involutive property. Via the regularised Siegel-Weil formula for unitary groups due to Ichino[2] we show that for $a$ large enough the theta lift space $\theta_{X_a}^{Y,\varphi,-1}(\chi_2^{-1}\theta_{X_a}^Y(\pi))$ is contained in a certain space of residues of Eisenstein series (Prop. 5.9). Here we need to be extra careful with the choice of characters which determine the Weil representations involved in constructing theta series. We use $(\chi_1, \chi_2)$ both ways and $\varphi$ and $\varphi^{-1}$ for different directions of theta lift. We want to remove the requirement that $A$ is large enough. Let $a < A$. Let $Q_{A-a}$ be the parabolic subgroup of $G(X_A)$ stabilising an $(A-a)$-dimensional isotropic subspace of $X_A$. Then we take constant terms along $Q_{A-a}$ on the space of theta lift and the space of Eisenstein series. On the theta side we expect to get $\theta_{X_a}^{Y,\varphi,-1}(\chi_2^{-1}\theta_{X_a}^Y(\pi))$ for $a < A$. However we need a stronger tower property which is not known for unitary groups. The result of Rallis[11] on tower property does not readily apply since it deals with symplectic and orthogonal groups only and uses the property that symplectic group is always split. To work around it does not need to completely linearise the Weil representation. Our global computation is inspired by the local computation in [8]. We are able to generalise the tower property (Prop. 3.1) and the proof is simpler and more uniform than in [11]. Thus we show that taking constant term indeed gives the space $\theta_{X_a}^{Y,\varphi,-1}(\chi_2^{-1}\theta_{X_a}^Y(\pi))$. On the other hand it can be shown that the constant term on the Eisenstein side gives for $a > 0$ residues of Eisenstein series, for $a = 0$ exactly $\chi_1\pi$ and for $a < 0$ zero. It should be mentioned that we get a $\chi_1$-twist of $\pi$ because of the contribution from determinants of unitary groups. Then the involutive property combined with the tower property forces the irreducibility of $\theta_{X_a}^Y(\pi)$.

The fundamental result of irreducibility of theta lifting for the symplectic and orthogonal groups is used by Bergeron, Millson, and Mœglin in [9] that gives Hodge type theorems on Shimura varieties of orthogonal type. Thus our result has potential application to the Shimura varieties of unitary type along the same line.

1. Notation

Let $k$ be a number field and $E$ a quadratic extension of $k$. Fix an element $\delta \in E$ such that $\delta = -\overline{\delta}$. Let $\Delta = \delta^2$. Let $\mathcal{A}$ be the adeles of $k$. Let $X$ be a skew-Hermitian vector space of dimension $n$ over $E$ with form $\langle , \rangle_X$ and $Y$ a Hermitian vector space of dimension $m$ over $E$ with form $\langle , \rangle_Y$. Note that we assume that $\langle , \rangle_X$ is linear in the first variable and conjugate linear in the second variable whereas $\langle , \rangle_Y$ is conjugate linear in the first variable and linear in the second variable. Let $G(X)$ (resp. $G(Y)$) be the isometry group for $X$ (resp. $Y$). We let $G(X)$ act on the right and $G(Y)$ on the left. Fix an additive character $\varphi$ of $k \setminus \mathcal{A}$ and let $\psi_E = \psi \circ \frac{1}{2} \text{tr}_{E/k}$. The $k$-vector space $W = \text{Res}_{E/k}(Y \otimes_E X)$ endowed with the form

$$\langle y_1 \otimes x_1, y_2 \otimes x_2 \rangle := \text{tr}_{E/k}\langle y_1, y_2 \rangle_Y \overline{\langle x_1, x_2 \rangle}_X$$

is symplectic. Sometimes we will drop $\text{Res}_{E/k}$ to simplify notation. Let $\chi_1$ and $\chi_2$ be two characters of $E^\times \setminus \mathcal{A}_E^\times$ such that $\chi_1|_{\mathcal{A}} = \varepsilon_{E/k}^m$ and $\chi_2|_{\mathcal{A}} = \varepsilon_{E/k}^n$ where $\varepsilon_{E/k}$ is the quadratic character of $k^\times \setminus \mathcal{A}^\times$ associated to $E/k$ via Class Field Theory. This
is necessary if we need to determine a splitting of the metaplectic group over the unitary dual reductive pairs. Please see [5] for more details. It should be pointed out that $\chi_1$ (resp. $\chi_2$) is used to determine an embedding of $G(X)$ (resp. $G(Y)$) into the metaplectic group $\text{Mp}(W)$ but the choice of $\chi_1$ (resp. $\chi_2$) is restricted by the parity of the dimension of $G(Y)$ (resp. $G(X)$).

Let $X_a$ be the space formed by adjoining $a$ hyperbolic planes $\ell_a = \ell_a^+ \oplus \ell_a^-$ to $X$. It is in the same Witt tower as $X$. We may also define $X_{-b}$ if we can remove $b$ hyperbolic planes from $X$, so $X = \ell_b^+ \oplus X_{-b} \oplus \ell_b^-$. Define similarly $Y_a$ and $Y_{-b}$. Note that $\ell_a^\pm$ should conform to the type of the original space. We hope this choice of notation does not cause confusion. We will add in subscripts to indicate in which space the $\ell_a^\pm$’s lie.

### 2. Weil Representation

In this section we work in the local case, so temporarily we let $k$ denote a local field.

#### 2.1. Representation of Metaplectic Group.

We recall some results on the Weil representation. Because of the need to describe the mixed model which figures prominently in the proof of Rallis tower property for unitary groups, it is necessary that we start with representations of the Heisenberg group. Then we describe various models of Weil representation. References are [4] and [8].

Let $W = W^+ \oplus W^-$ be a symplectic space over $k$ with given complete polarisation. One model of the representation of the Heisenberg group $H(W)$ with central character $\psi$ is realised on the space $S_{W^-} := \text{Ind}^H_{H(W^+)} \psi$ where Schwartz induction is used. We also let $\psi$ denote the character of $H(W^-) := W^- \oplus k$ defined by $(w,t) \mapsto \psi(t)$. Let $\rho$ denote this representation. For $g \in \text{Sp}(W)$ define the representation $\rho^g$ by $\rho^g(h) = \rho(h^g)$ for $h \in H(W)$ where if $h = (w,t)$ then $h^g = (wg,t)$. It also acts on $S_{W^-}$ with central character $\psi$. By the Stone-von Neumann theorem, there is an isomorphism between the representations $\rho$ and $\rho^g$, which we now make explicit.

Consider the map $A^0(g) : S_{W^-} \to S_{W^-g^{-1}}$ given by $(A^0(g)f)(h) := f(h^g)$. We have $\rho(h)A^0(g) = A^0(g)\rho(h^g)$. In addition there exists an $H(W)$-intertwining isomorphism $I_{W^{-g^{-1}},W^-}$ between $S_{W^{-g^{-1}}}$ and $S_{W^-}$ given by

$$I_{W^{-g^{-1}},W^-}(f)(\cdot) = \int_{W^-g^{-1} \cap W^- \setminus W^-} f((w^-,0)\cdot)dw^-$$

where the choice of Haar measure is specified below. Define then $A(g) = I_{W^{-g^{-1}},W^-} \circ A^0(g)$. We use the unique choice of Haar measure such that $A(g)$ is unitary.

Note that $A$ is not a representation of $\text{Sp}(W)$, but $A$ lifts to a representation, called the Weil representation, of the metaplectic group $\text{Mp}(W)$ which is a nontrivial $\mathbb{C}^1$-extension of $\text{Sp}(W)$. However if we restrict to a standard unipotent subgroup of $\text{Sp}(W)$, $A$ gives a representation (c.f. [9]).

The Schrödinger model of the Weil representation is realised on $S(W^+)$ which denotes the Schwartz space of functions on $W^+$. There is an isomorphism of representations of $H(W)$

$$S_{W^-} \to S(W^+)$$

$$f \mapsto \phi$$
where $\phi$ is given by $\phi(w^+)=f(w^+,0)$ for $w^+ \in W^+$. To go back, given $\phi$ then its preimage $f$ is given by

\[ f(w^+ + w^-, t) = f((w^-, t + \frac{1}{2}(w^+, w^-))_W(w^+, 0)) = \psi(t + \frac{1}{2}(w^+, w^-)_W)\phi(w^+). \]

The action of $H(W)$ on $S(W^+)$ is given by [4]

\[ \rho((w^+ + w^-), t)\phi(w^+_0) = \psi(t + \langle w^+_0, w^- \rangle_W + \frac{1}{2}(w^+, w^-)_W)\phi(w^+_0 + w^-). \]

Let $g \in \text{Sp}(W)$ and write $g$ as $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ with respect to the polarisation. Note that $\text{Sp}(W)$ acts on $W$ from the right, so $a \in \text{End}(W^+)$, $b \in \text{Hom}(W^+, W^-)$, $c \in \text{Hom}(W^-, W^+)$ and $d \in \text{End}(W^-)$. Then we transfer the operator $A(g)$ to $S(W^+)$ and denote it by $r(g)$. Then $r(g)$ is given by [4]

\[ (r(g)\phi)(w^+) = \int_{\ker(c) \setminus W^-} \psi \left( \frac{1}{2}\langle w^+a, w^+b \rangle + \langle w^-c, w^+b \rangle + \frac{1}{2}\langle w^-c, w^-d \rangle \right) \times \phi(w^+a + w^-c)\text{d}\mu_dw^- \]

where we take the Haar measure on $\ker(c) \setminus W^-$ so that $r(g)$ becomes unitary. The obstruction to $r$ being a representation is given by the cocycle $c_{W^-}(g_1, g_2)$ which is equal to $\gamma_k(\psi \circ L(W^-, W^- g_2^{-1}, W^+_g))$, where $\gamma_k$ is the Weil index which takes values in 8-th roots of unity and $L$ gives the Leray invariant. Please see [12] for the definitions of these. We lift $r$ to a representation of the metaplectic group $\text{Mp}(W)$ on $S(W^+)$ and denote it by $\omega_{W,\psi}$. This is the Schrödinger model of the Weil representation.

More precisely, $\text{Mp}(W)$ is set-theoretically $\text{Sp}(W) \times \mathbb{C}^1$ with multiplication given by

\[ (g_1, z_1)(g_2, z_2) = (g_1g_2, z_1z_2c_{W^-}(g_1, g_2)) \]

and $r$ lifts to a representation of $\text{Mp}(W)$ on $S(W^+)$ as follows

\[ \omega_{W,\psi}(g, z)\phi = z \cdot r(g)\phi. \]

This definition of $\text{Mp}(W)$ actually depends on $\psi$.

Now we note some properties [8, pp. 36–37] of the Weil representation. Let $W'$ be the vector space with the same underlying vector space as $W$ but with form $\langle \cdot, \cdot \rangle_{W'} = -\langle \cdot, \cdot \rangle_W$. We consider the impact of changing $W$ to $W'$ and $\psi$ to $\psi_{-1}$ simultaneously.

**Proposition 2.1.** Let $j : \text{Sp}(W) \rightarrow \text{Sp}(W')$ be the natural identification and extend it to

\[ j : \text{Mp}(W) \rightarrow \text{Mp}(W') \]

\[ (g, z) \mapsto (j(g), z). \]

Then $\omega_{W,\psi} \cong \omega_{W',\psi_{-1}} \circ j$.

Another property that we will frequently invoke is the following

**Proposition 2.2.** Let $W_1$ and $W_2$ be two symplectic spaces and let $W = W_1 \oplus W_2$. Let $j$ denote the natural embedding $\text{Sp}(W_1) \times \text{Sp}(W_2) \rightarrow \text{Sp}(W)$ and extend it to
the homomorphism
\[ j : \text{Mp}(W_1) \times \text{Mp}(W_2) \to \text{Mp}(W) \]
\[ ((g_1, z_1), (g_2, z_2)) \mapsto (j(g_1, g_2), z_1z_2). \]
Then \( \omega_{W, \psi} \circ j \cong \omega_{W_1, \psi} \otimes \omega_{W_2, \psi} \) where \( \otimes \) is replaced by completed tensor \( \hat{\otimes} \) in the archimedean case.

2.2. Splitting of Metaplectic Cover. Let \( \eta = \psi_{1/2} \) as we use it often in Weil index. The metaplectic cover splits over the dual reductive pair \( G(Y) \times G(X) \) since we are dealing with unitary groups. Given a character \( \chi_1 \) of \( E^1 \) such that \( \chi_1|_{k^\times} = \epsilon_{E/k}^m \), there corresponds a splitting
\[
\begin{align*}
&\text{Mp}(Y \otimes X) \\
\gamma_{X_1} &\downarrow \\
G(X) &\longrightarrow \text{Sp}(Y \otimes X)
\end{align*}
\]
as in [5, 1]. Of course \( \gamma_{X_1} \) also depends on \( Y \).

In the case where \( X \) is split the splitting has an explicit description[5, Theorem 3.1]:
\[
\begin{align*}
G(X) &\longrightarrow \text{Mp}(Y \otimes X) \\
\gamma_{X_1} &\downarrow \\
g &\mapsto (g, \beta_Y(g))
\end{align*}
\]
where \( \beta_Y(g) = \chi_1(x(g)) \gamma_k(\eta \circ RY)^{j(g)} \). The definitions of \( x(g) \) and \( j(g) \) can be found in [5, pp 370-371] and \( RY \) is the underlying \( k \)-vector space of \( Y \) equipped with the symmetric bilinear form \( \frac{1}{2} \text{tr}_{E/k} \langle \cdot, \cdot \rangle_Y \).

By [1, Cor. A. 3] the splitting is compatible with taking direct sum. More precisely, suppose that \( X = X_1 \oplus X_2 \) is a direct sum of two skew-Hermitian spaces and use \( \chi_1 \) to determine embeddings of \( G(X) \), \( G(X_1) \) and \( G(X_2) \) into the corresponding metaplectic groups. Then we have a commutative diagram:
\[
\begin{align*}
&\text{Mp}(Y \otimes X) \\
\gamma_{X_1} &\downarrow \\
G(X_1) \times G(X_2) &\longrightarrow \text{Mp}(Y \otimes X_1) \times \text{Mp}(Y \otimes X_2)
\end{align*}
\]
where in the \( \mathbb{C}^1 \)-part of the metaplectic groups \( \hat{j} \) sends \((z_1, z_2)\) to \( z_1z_2 \).

Consider changing \( \chi_1 \) to some other character \( \mu_1 \) which satisfies \( \mu_1|_{k^\times} = \epsilon_{E/k}^m \). Let \( \mu_1 = \nu \chi_1 \). Then \( \nu \) is trivial on \( k^\times \) and we construct a character \( \nu' \) of \( E_1 \) by setting \( \nu'(a/\overline{a}) = \nu(a) \). Then (cf. [5])
\[
\nu_{\mu_1}(g) = \nu'(\text{det } g) \cdot \gamma_{X_1}(g).
\]
It is useful to note that \( \nu'^2 = \nu|_{E_1} \).

There is also an analogous version on the \( Y \) side. Let \( Y^4 \) be the skew-Hermitian space with the same underlying space as \( Y \) but with form \( \langle y_1, y_2 \rangle_{Y^4} = \delta(y_2, y_1) \) and let \( X^{d+1} \) be the Hermitian space with the same underlying space as \( X \) but with form \( \langle x_1, x_2 \rangle_{X^{d+1}} = \delta^{-1}(x_2, x_1) \). We have natural identifications \( G(X^{d+1}) = G(X) \), \( G(Y^4) = G(Y) \) and \( \text{Mp}(X^{d+1} \otimes Y^4) = \text{Mp}(Y \otimes X) \). Then the roles of \( G(X) \) and \( G(Y) \) are completely symmetric.
We also note when $Y$ is split using $\chi_2$ such that $\chi_2|_{k^*} = e^{\eta}_{E/k}$ we have explicit splitting
\begin{align}
G(Y) &\xrightarrow{\chi_2} \text{Mp}(Y \otimes X) \tag{2.10} \\
h &\mapsto (h, \beta_X(h)) \tag{2.11}
\end{align}
where $\beta_X(h) = \chi_2(x(h))\chi_2(\delta^{j(h)}\gamma_k(\eta \circ RX^{\delta^{-1}})^i(h))$ by [5, Theorem 3.1].

2.3. Representation of Dual Reductive Pair. We want to describe the Weil representation for dual reductive pairs more explicitly, especially in the case where the skew-Hermitian space is split. Suppose that $X$ is split with $\dim X = 2n$, so $G(X) = U(n, n)$. Choose a basis of $X$ such that the skew-Hermitian form is given by $(-1, 0)$. Let $m = \dim Y$.

We take $\chi_2$ to be the trivial character. This is allowed since $\dim X$ is even. We claim that the corresponding splitting is given by sending $h \in G(Y)$ to $(h, 1) \in \text{Mp}(Y \otimes_k X)$. Assume first that $Y$ is split. Then we have explicit description of splitting (2.10). We need to show that the quantity $\beta_X(h)$ for $h \in G(Y)$ is 1. Let $V = X^{\delta^{-1}}$ to simplify notation. Then as given in [5, Theorem 3.1] we have
\begin{equation}
\beta_X(h) = \gamma_k(\eta \circ RV)^j = (\Delta, \det V)_{k}\gamma_k(-\Delta, \eta)^{2n}\gamma_k(-1, \eta)^{-2n} \tag{2.12}
\end{equation}
where $(\ , \ )_k$ denotes the Hilbert symbol. Using properties of Weil index [4, Lemma 4.1] we find (2.12) is equal to
\begin{equation}
(\Delta, \det V)_{k}(-\Delta, -1)^n(-1, -1)^{-n} = (\Delta, \delta^{-2n})_{k}(\Delta, -1)^n = (\Delta, -\Delta)^n_{k} = 1. \tag{2.13}
\end{equation}
When $Y$ is non-split then splitting is determined via doubling. More precisely first we determine a splitting for $G(Y \oplus Y')$ and then restrict it to the subgroup $G(Y)$. Thus our claim is also true for non-split $Y$.

Then the Weil representation of $G(Y) \times G(X)$ is characterised by [2]
\begin{align}
\omega\left(\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\right) \phi(x) = &\chi_1(\det A)\det A|^{\dim Y/2} \phi(xA) \\
\omega\left(\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\right) \phi(x) = &\phi(x)\psi_E\left(\frac{1}{2} \text{tr}(x, x)YB)\right) \\
\omega\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \phi(x) = &\gamma_k(\psi_2^{\frac{1}{2}} \circ RV)^{-n} \int_{Y^n(k)} \phi(y)\psi_E(\text{tr}(y, -x)Y)dy \\
\omega(h)\phi(x) = &\phi(h^{-1}x). \tag{2.14}
\end{align}

Note that we regard $X$ and $Y$ as schemes defined over $k$. In particular $Y(k) \cong E^n$.

2.4. Structure of Parabolic Subgroups. To prepare for the discussion of the mixed model we describe the structure of parabolic subgroups. We do this in a coordinate free way and we deviate a little from previous notation.

Let $E$ be a quadratic extension of $k$ or just $k$ itself. Now let $W$ be an $\epsilon$-Hermitian space with $\epsilon = \pm 1$ to signify whether the form is Hermitian or skew-Hermitian. Suppose $W$ can be written as $\ell^+_a \oplus W_0 \oplus \ell^-_a$. If the isometry group acts on the right then we consider the parabolic subgroup stabilising $\ell^+_a$. If the isometry group acts on the left then we consider the parabolic subgroup stabilising $\ell^-_a$. Then the
parabolic subgroup of $G(W)$ has Levi part consisting of elements

\[ m(\alpha, \delta) := \begin{pmatrix} \alpha & \delta \\ \delta & (\alpha^*)^{-1} \end{pmatrix} \]

where $\alpha \in \text{GL}(\ell_+^\ast)$ and $\alpha^* \in \text{GL}(\ell_a^-)$ is the adjoint of $\alpha$ and $\delta \in G(W_0)$. Adjoints are taken with respect to the pairing $(\cdot, \cdot)_W$. If the action is on the right then it has unipotent subgroup consisting of elements of the form

\[ n(\mu, \beta) := \begin{pmatrix} 1 & \mu \beta - \mu^* \mu/2 \\ 1 & -\mu^* \end{pmatrix} \]

where $\mu \in \text{Hom}_E(\ell_+^\ast, W_0)$ and $\beta \in \text{Hom}_E(\ell_a^+, \ell_a^-)$ such that $\beta + \beta^* = 0$. If the action is on the left then it has unipotent subgroup consisting of elements of the form

\[ n(\mu, \beta) := \begin{pmatrix} 1 & \mu \beta - \mu^* \mu/2 \\ 1 & -\mu^* \end{pmatrix} \]

where $\mu \in \text{Hom}_E(W_0, \ell_+^\ast)$ and $\beta \in \text{Hom}_E(\ell_a^-, \ell_a^+)$ such that $\beta + \beta^* = 0$.

In particular after taking dual standard basis if $W$ is symplectic then $\beta$ can be regarded as in $\text{Sym}_a$; if $W$ is Hermitian then $\beta$ can be regarded as in $\text{sHer}_a$; if $W$ is skew-Hermitian then $\beta$ can be regarded as in $\text{Her}_a$.

2.5. Mixed Model. We are more concerned with the action of the unipotent subgroups on the mixed model. Note that the splitting is trivial over our unipotent subgroups. For all other expressions that we will need we will invariably reduce via Prop. 2.2 to the split case. This is key in the demonstration of tower property.

Let $V = V^+ \oplus V^-$ be a symplectic space. Let $W_0$ be another symplectic space and consider the symplectic space $W = V \oplus W_0$. Thus we are adding hyperbolic planes to $W_0$. Assume we have polarisation $W_0 = W_0^+ \oplus W_0^-$. Suppose the representation of the Heisenberg group $H(W)$ is realised on $S_{W_0^+ \oplus V^-}$. Recall the element $n(\mu, \beta)$ defined in (2.16). We compute the action of the operator $A(n(\mu, 0))$ on $f_0 \otimes f \in S_{W_0^- \otimes S_V^-}$. Note that $n(\mu, 0)$ preserves $W_0^- \oplus V^-$ and thus there is no need to apply $\pi_W^{-1} \pi^{W^-}$. We compute:

\[
A(n(\mu, 0))(f_0 \otimes f)((w_0, t), (v^+, 0))
= f_0((w_0 + \mu(v^+), t))f((v^+ - \mu^*(w_0) - \frac{1}{2}\mu^* \mu(v^+), 0))
= f_0((w_0, t)(\mu(v^+), -\frac{1}{2}(w_0, \mu(v^+))_{W_0}))
\quad \times f((-\mu^*(w_0) - \frac{1}{2}\mu^* \mu(v^+), \frac{1}{2}\mu^*(w_0) + \frac{1}{2}\mu^* \mu(v^+), v^+)_V)(v^+, 0))
= f_0((w_0, t)(\mu(v^+), 0))
\quad \times f((-\mu^*(w_0) - \frac{1}{2}\mu^* \mu(v^+), \frac{1}{2}\mu^*(w_0) + \frac{1}{2}\mu^* \mu(v^+), v^+)_V - \frac{1}{2}(w_0, \mu(v^+))_{W_0})(v^+, 0))
= f_0((w_0, t)(\mu(v^+), 0))f((-\mu^*(w_0) - \frac{1}{2}\mu^* \mu(v^+), 0)(v^+, 0))
= f_0((\mu(v^+), 0))f_0((w_0, t)) \cdot f(v^+, 0)
\]
where $\rho_0$ is the representation of $H(W_0)$ on $S_{W_0}$. We transfer the action to the model $S_{W_0} \otimes S(V^+)$ to get

$$r(n(\mu, 0))(f_0 \otimes \phi)((w_0, t), v^+) = \rho_0((\mu(v^+), 0))f_0((w_0, t)) \cdot \phi(v^+).$$

(2.18)

This is unitary.

Now we describe the Weil representation $\omega_{X, Y}$ of $G(Y) \times G(X)$ on the mixed model for certain unipotent elements. We do not assume that $X$ or $Y$ is split. Suppose that $X$ can be written as $X = \ell^+_a \oplus X_0 \oplus \ell^-_a$. Here $X_0$ is not necessarily anisotropic. Let $S_0$ be any model for the Weil representation $G(Y) \times G(X)$. Let the Weil representation of $G(Y) \times G(\ell^+_a \oplus \ell^-_a)$ be realised on $S(Y \otimes \ell^+_a)$ as in Sec. 2.3. Note that $Y \otimes \ell^+_a \cong Y^n$. We describe the action of $n(\mu, 0) \in G(X)$ on the model $S_0 \otimes S(Y \otimes \ell^+_a)$, where $\mu \in \text{Hom}_E(\ell^+_a, X_0)$. In the archimedean version we replace tensor by completed tensor. This is the special case where $W = \text{Res}_{E/k}(Y \otimes X)$ and thus $V^\pm = \text{Res}_{E/k}(Y \otimes \ell^\pm_a)$ and $W_0 = \text{Res}_{E/k}(Y \otimes X_0)$. Thus by (2.18) we find for $\phi_0 \otimes \phi \in S_0 \otimes S(Y \otimes \ell^+_a)$

$$\omega_{X, Y}(n(\mu, 0), 1_{G(Y)})(\phi_0 \otimes \phi)(., y) = \rho_0((1_Y \otimes \mu(y), 0))\phi_0(\cdot) \times \phi(y)$$

(2.19)

where $\rho_0$ is the representation of $H(Y \otimes X_0)$ on $S_0$.

3. Tower Property

Let $\pi$ be an irreducible cuspidal automorphic representation of $G(Y)$. Let $W^+$ be some maximal isotropic subspace of $W = \text{Res}_{E/k} Y \otimes E X$. Let the Weil representation $\omega$ of $G(Y) \times G(X)$ be realised on the Schwartz space $S(W^+)$ with respect to $\psi$ and $(\chi_1, \chi_2)$. For $f \in \pi$ and $\phi \in S(W^+)$ we define

$$\theta(g, \phi, f) = \int_{[G(Y)]} \theta_{X, Y}(g, h, \phi)f(h)dh,$$

where $\theta_{X, Y}(g, h, \phi) = \sum_{w \in W_{+, k}} \omega(g, h)\phi(w)$. When there is no need to emphasise the chosen maximal isotropic subspace $W^+$ we write $(Y \otimes X)$ for some maximal isotropic subspace of $\text{Res}_{E/k} Y \otimes E X$ and the space $S(W^+)$ will be denoted as $S_{X, Y}$. The global theta lift $\theta^Y_X(\pi)$ of $\pi$ from $G(Y)$ to $G(X)$ is defined to be the space generated by all such functions $\theta(g, \phi, f)$. We consider the tower of theta lifts to $G(X_a)$ for varying $a$’s. We can also lift from $G(X)$ to $G(Y_a)$ for $\pi$ an irreducible cuspidal automorphic representation of $G(Y)$. Note that as the Weil representation depends on the additive character $\psi$, the theta lifts depend on $\psi$, but we suppress it from notation in this section. We have also suppressed the dependency on $\chi_1$ and $\chi_2$. Since the dimensions of $X_a$ and $X$ are of the same parity we can use the same $\chi_2$ for the splittings over $G(Y)$. Implicitly we also need an embedding of $G(Y) \to \text{Mp}(Y \otimes \ell_a)$ and the trivial character is used to determine it.

We will extend the computation in [11] to the unitary case. Our computation is simpler and more uniform. It follows the spirit of the local computation of covariants in [8, Chap. 3. V]. Since the isometry group of a Hermitian space can also be regarded as the isometry group of a skew-Hermitian space via a (non-canonical) isomorphism which depends on $\delta \in E$ (c.f. Sec. 2.2), we just need to demonstrate the tower property in one direction. Let $Q_a$ be the parabolic subgroup of $G(X_a)$ stabilising $\ell^-_a$ which is an isotropic subspace of $X_a$. Similarly define $R_a$ to be the parabolic subgroup of $G(Y_a)$ stabilising $\ell^+_a$, an isotropic subspace of $Y_a$. 

Proposition 3.1. Let \( \pi \) be an irreducible cuspidal automorphic representation of \( G(Y) \). Let \( \phi \in S((Y \otimes X)^+ \oplus Y^a)(\A) \). Then for \( f \in \pi \) and \( g \in G(X)(\A) \), the constant term along \( Q_a \) of the theta lift is equal to

\[
[\theta_{X_a,Y}(g, \phi, f)]_{Q_a} = \theta_{X,Y}(g, \phi(\cdot, 0), f).
\]

Hence as \( G(X) \)-representations

\[
(3.3) \quad \Res_{G(X)}(\theta_{X_a}^X(\pi)_{Q_a}) = \theta_Y^X(\pi).
\]

The action of \( m(\GL_n(\A_E), 1) \) on \( \theta_{X_a}^X(\pi)_{Q_a} \) is given by the character

\[
(3.4) \quad \chi_1 \circ \det | \det \|_{\A_E}^{\dim Y/2}.
\]

There is also a symmetric version for lifts from \( G(X) \) to \( G(Y_a) \).

Proof. To make notation less cluttered we omit writing \((k)\) for taking rational points. We need to compute

\[
(3.5) \quad \int_{[U_{Q_a}]} \int_{[G(Y)]} \theta_{X_a,Y}(ng, h, \phi)f(h)dhdn.
\]

Step 1. Consider first

\[
(3.6) \quad \int_{[U_{Q_a}]} \theta_{X_a,Y}(ng, h, \phi)dn.
\]

The exchange of order of integration will be justified at the end of the computation. Recalling the structure of \( U_{Q_a} \) (c.f. (2.16)) we find that (3.6) is equal to

\[
(3.7) \quad \int_{[\text{Hom}_E(\ell_a^+, X)]} \int_{\beta \in [\text{Her}_a]} \theta_{X_a,Y}(n(\mu, \beta)g, h, \phi)d\beta d\mu
\]

\[
= \int_{[\text{Hom}_E(\ell_a^+, X)]} \int_{\beta \in [\text{Her}_a]} \sum_{z \in (Y \otimes X)^+} \sum_{y \in Y^a} \omega_{X_a,Y}(n(\mu, \beta)g, h)\phi(z, y)d\beta d\mu.
\]

Without loss of generality we assume \( \phi = \phi_0 \otimes \phi_1 \) with \( \phi_0 \in S_{X,Y} \) and \( \phi_1 \in S(Y \otimes \ell_a^+) \). By (2.14) we write out the action of \( \beta \) to get

\[
(3.9) \quad \int_{[\text{Hom}_E(\ell_a^+, X)]} \int_{\beta \in [\text{Her}_a]} \sum_{z \in (Y \otimes X)^+} \sum_{y \in Y^a} \omega_{X_a,Y}(n(\mu, 0)g, h)\phi_0 \otimes \phi_1(z, y)\psi_E(\tr(\langle y, y \rangle_Y \beta))d\beta d\mu
\]

so the integration vanishes unless \( \langle y, y \rangle_Y = 0 \) and we get

\[
(3.10) \quad \int_{[\text{Hom}_E(\ell_a^+, X)]} \sum_{y \in Y^a} \omega_{X_a,Y}(n(\mu, 0)g, h)\phi_0 \otimes \phi_1(z, y)d\mu.
\]

The condition \( \langle y, y \rangle_Y = 0 \) means that the columns of \( y \in Y^a \) span an isotropic subspace of \( Y \). We have identified \( Y \otimes \ell_a^+ \) with \( Y^a \).

Step 2. We will show that only \( y = 0 \) contributes to the above integral. Note that if \( Y \) is anisotropic we are done now. Decompose \( Y \) as \( \ell_a^+ \oplus Y_0 \oplus \ell_a^- \) with \( Y_0 \) anisotropic. Choose dual bases for \( \ell_a^+ \) and \( \ell_a^- \) and denote the elements by \( e_1, e_2, \ldots, e_b \) and \( e_{-1}, e_{-2}, \ldots, e_{-b} \) respectively. We consider the orbits of \( y \) under the action of \( G(Y)(k) \times \GL_n(E) \). To alleviate notation \((k)\) and \((E)\) are dropped afterwards. We will split the integral (3.10) according to the orbits and analyse
them one at a time. The orbits are parametrised by the rank $r$ of $y$ and we can choose
\begin{equation}
\label{eq:3.11}
y_r := (e_{1}, e_{2}, \ldots, e_{r}, 0, \ldots, 0)
\end{equation}
as representatives of the orbits for $r$ running from 0 to $\min(a, b)$.

Then \eqref{eq:3.10} is equal to
\begin{equation}
\label{eq:3.12}
\sum_{r=0}^{\min(a, b)} \int_{\Hom_{\mathbb{R}}(\ell_{r}^{+}, X)} \sum_{z \in (Y \otimes X)^{+}} \sum_{(\gamma, \nu)} \omega_{X_{a}, Y}(n(\mu, 0)g, h)\phi_{0} \otimes \phi_{1}(z, \gamma^{-1}y_{r}\nu)d\mu
\end{equation}
where $(\gamma, \nu)$ runs over $\text{Stab}(G(Y) \times \GL_{b}) \backslash (G(Y) \times \GL_{a})$. By the explicit description of Weil representation the above is equal to
\begin{equation}
\label{eq:3.13}
\sum_{r=0}^{\min(a, b)} \int_{\Hom_{\mathbb{R}}(\ell_{r}^{+}, X)} \sum_{z \in (Y \otimes X)^{+}} \sum_{(\gamma, \nu)} \omega_{X_{a}, Y}(m(\nu, 1)n(\mu, 0)g, \gamma h)\phi_{0} \otimes \phi_{1}(z, y_{r})d\mu
\end{equation}
\begin{equation}
\label{eq:3.14}
= \sum_{r=0}^{\min(a, b)} \int_{\Hom_{\mathbb{R}}(\ell_{r}^{+}, X)} \sum_{z \in (Y \otimes X)^{+}} \sum_{(\gamma, \nu)} \omega_{X_{a}, Y}(n(\mu, 0)m(\nu, 1)g, \gamma h)\phi_{0} \otimes \phi_{1}(z, y_{r})d\mu.
\end{equation}
The above equality holds because
\begin{equation}
\sum_{z \in (Y \otimes X)^{+}} \omega_{X_{a}, Y}(g, \gamma h)\phi_{0}(z) = \sum_{z \in (Y \otimes X)^{+}} \omega_{X_{a}, Y}(g, h)\phi_{0}(z).
\end{equation}
and because exchanging $m(\nu, 1)$ and $n(\mu, 0)$ does not change the value of the integral.

Step 3. Now we focus on the subintegral of \eqref{eq:3.14} for a fixed $r \geq 1$. Using \eqref{eq:2.19}, for each $r$ we get:
\begin{equation}
\int_{\Hom_{\mathbb{R}}(\ell_{r}^{+}, X)} \sum_{z \in (Y \otimes X)^{+}} \sum_{(\gamma, \nu)} \omega_{X_{a}, Y}(n(\mu, 0)m(\nu, 1)g, \gamma h)(\phi_{0} \otimes \phi_{1})(z, y_{r})d\mu
\end{equation}
\begin{equation}
= \int_{\Hom_{\mathbb{R}}(\ell_{r}^{+}, X)} \sum_{z \in (Y \otimes X)^{+}} \sum_{(\gamma, \nu)} \rho_{0}(1Y \otimes \mu(y_{r}), 0)\omega_{X_{a}, Y}(m(\nu, 1)g, \gamma h)(\phi_{0} \otimes \phi_{1})(z, y_{r})d\mu.
\end{equation}
The representation $\rho_{0}$ of the Heisenberg group $H(Y \otimes X)$ acts only on the $\mathcal{S}(Y \otimes X)^{+}$-part. Let $\ell_{r}^{\pm}$ denote the span of columns of $y_{r}$. These are subspaces of $\ell_{b}^{\pm}$. Let $Y_{r}$ denote the orthogonal complement of $\ell_{r}^{\pm}$ in $Y$. Now we use a more concrete model $\mathcal{S}(Y_{r} \otimes X)^{+} \otimes \ell_{r}^{\pm} \otimes X$ for $\rho_{0}$. For $\phi_{00} \in \mathcal{S}(Y_{r} \otimes X)^{+}$ and $\phi_{r} \in \mathcal{S}(\ell_{r}^{\pm} \otimes X)$, $\rho_{0}$ acts as
\begin{equation}
\rho_{0}(1Y \otimes \mu(y_{r}), 0)(\phi_{00} \otimes \phi_{r})(z_{0}, x) = \rho_{r}(1_{\ell_{r}} \otimes \mu(y_{r}), 0)\phi_{r}(x) \cdot \phi_{00}(z_{0})
\end{equation}
\begin{equation}
= \psi_{\mathcal{E}}(\text{tr}(\langle x, \mu(y_{r}) \rangle_{X}))(\phi_{r}(x) \cdot \phi_{00}(z_{0})
\end{equation}
where $\rho_{r}$ is the representation of the Heisenberg group $H(\ell_{r} \otimes X)$. The above equation holds because the action of $1Y \otimes \mu(y_{r})$ concentrates in the $\phi_{r}$-part as
1_ℓ_x \otimes \mu(y_r) \) and in the final line we let \( \mu \) acts on \( y_r \) row-wise by abuse of notation. Applying this the integral (3.16) becomes

\[
\int_{[\text{Hom}_E(\ell^+_a,X)]} \sum_{y_0 \in (Y_-,\otimes X)^+} \sum_{x \in \ell^+_a \otimes X} \sum_{(\gamma,\nu)} \psi_E(\text{tr}(x,\mu(y_r))) \omega_{X_a,Y}(m(\nu,1)g,\gamma h)(\phi_0 \otimes \phi_1)(z_0, x, y_r) d\mu.
\]

Thus for a fixed \( x \) the integration against \( \mu \) vanishes unless the space spanned by the rows of \( x \) is orthogonal to the space spanned by the rows of \( \mu(y_r) \) for all \( \mu \). Note that \( y_r \in \ell^+_a \otimes \ell^+_a \) and thus \( \mu(y_r) \in \ell^+_a \otimes X \). Since we are in the case \( r \geq 1 \) the space spanned by the rows of \( \mu(y_r) \) for all \( \mu \) is the whole of \( X \). Thus only \( x = 0 \) contributes and (3.19) becomes

\[
\sum_{y_0 \in (Y_-,\otimes X)^+} \sum_{(\gamma,\nu)} \omega_{X_a,Y}(m(\nu,1)g,\gamma h)(\phi_0 \otimes \phi_1)(z_0, 0, y_r).
\]

Step 4. Still in the case \( r \geq 1 \), we multiply the above by \( f(h) \) and integrate over \( h \). Note that the sum over \( (\gamma,\nu) \in \text{Stab}_{G(Y) \times GL_a} y_r \setminus (G(Y) \times GL_a) \) can be written as the double sum over \( \nu \in S_1 := \text{Stab}_{GL_a} y_r \setminus GL_a \) and \( \gamma \) in the stabiliser \( S_2 \) in \( G(Y) \) of the \( GL_a \)-orbit of \( y_r \). Obviously this contains the unipotent subgroup \( U_{R_r} \) of \( G(Y) \) for \( r \geq 1 \). Thus we get

\[
\int_{[G(Y)]} \sum_{y_0 \in (Y_-,\otimes X)^+} \sum_{(\gamma,\nu) \in \text{Stab}_{GL_a} y_r \setminus GL_a} \omega_{X_a,Y}(m(\nu,1)g,\gamma h)(\phi_0 \otimes \phi_1)(z_0, 0, y_r) f(h) d\mu
\]

\[
= \int_{S_2(k) \backslash G(Y)(A)} \sum_{y_0 \in (Y_-,\otimes X)^+} \sum_{\nu \in \text{Stab}_{GL_a} y_r \setminus GL_a} \omega_{X_a,Y}(m(\nu,1)g,\gamma h)(\phi_0 \otimes \phi_1)(z_0, 0, y_r) f(h) d\mu.
\]

The expression

\[
\sum_{y_0 \in (Y_-,\otimes X)^+} \sum_{\nu \in \text{Stab}_{GL_a} y_r \setminus GL_a} \omega_{X_a,Y}(m(\nu,1)g,\gamma h)(\phi_0 \otimes \phi_1)(z_0, 0, y_r)
\]

is by the following lemma invariant under all \( n(\mu',\beta') \in U_{R_r}(A) \) for \( \mu' \in \text{Hom}_E(\ell^+_a, Y_-) \) and \( \beta' \in \text{Hom}_E(\ell^+_a, \ell^-_a) \) such that \( \beta'^* = -\beta' \). Thus we can decompose the integration over \( h \) to get an inner integral

\[
\int_{[U_{R_r}]} f(nh) d\mu
\]

and this vanishes because \( f \) is cuspidal. Thus only for \( r = 0 \) does the subintegral not necessarily vanish.

Step 5. Consider the case \( r = 0 \) i.e. the orbit containing the single element \( y = 0 \). Setting \( y = 0 \) in (3.10) and restricting to \( g \in G(X)(A) \) we are left with

\[
\int_{[\text{Hom}_E(\ell^+_a,X)]} \sum_{z \in (Y_\otimes X)^+} \omega_{X_a,Y}(n(\mu,0)g,\gamma h)(\phi_0 \otimes \phi_1)(z, 0) d\mu
\]

\[
= \sum_{z \in (Y_\otimes X)^+} \omega_{X_a,Y}(g,\gamma h)(\phi_0 \otimes \phi_1)(z, 0)
\]

\[
= \theta_{X_a,Y}(g,\gamma,h,\phi_0 \otimes \phi_1(\cdot,0)).
\]
Finally we integrate over $[G(Y)]$ against $f(h)$ to get $\theta_{X,Y}(g, \phi_0 \otimes \phi_1(\cdot,0), f)$ as required. The change of order in integration is justified by the absolute convergence of the integral.

The action of $m(GL_n(\mathbb{A}_E), 1)$ follows simply from the explicit formulae of Weil representation.

\textbf{Lemma 3.2.} Let $a \geq 1$ and $b \geq 1$. Assume $X = \ell^+_{a,X} \oplus X_{-a} \oplus \ell^-_{a,X}$ and $Y = \ell^+_{b,Y} \oplus Y_{-b} \oplus \ell^-_{b,Y}$. Suppose the Weil representation $\omega_{X,Y}$ of $G(Y) \times G(X)$ is realised on $S := S((Y_{-b} \otimes X_{-a})^+ \oplus (\ell^-_{b,Y} \otimes X_{-a}) \oplus (Y \otimes \ell^+_{a,X}))$. Then for $\phi \in S$

\begin{equation}
\omega_{X,Y}(1, n_Y(\mu, \beta))\phi(z_0, 0, y_b) = \phi(z_0, 0, y_b)
\end{equation}

where $y_b$ is as in (3.11).

\textbf{Proof.} We only need to show the equality for $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$ for $\phi_1 \in \hat{S}((Y_{-b} \otimes X_{-a})^+)$, $\phi_2 \in \hat{S}(\ell^+_{b,Y} \otimes X_{-a})$ and $\phi_3 \in (Y \otimes \ell^+_{a,X})$. We compute

\begin{align}
\omega_{X,Y}(1, n_Y(0, \beta))\phi_1 \otimes \phi_2 \otimes \phi_3(z_0, 0, y_b) \\
= \phi_1(z_0) \omega_{X_{a,Y}} (1, n_Y(0, \beta)) \phi_2(0) \omega_{\ell^+_{a,X}}(1, n_Y(0, \beta)) \phi_3(y_b).
\end{align}

By (2.14) we find the above is equal to $\phi_1 \otimes \phi_2 \otimes \phi_3(z_0, 0, y_b)$.

Next we compute

\begin{align}
\omega_{X,Y}(1, n_Y(\mu, 0))\phi_1 \otimes \phi_2 \otimes \phi_3(z_0, 0, y_b) \\
= \omega_{X_{a,Y}} (1, n_Y(\mu, 0)) \phi_1 \otimes \phi_2(0) \omega_{\ell^+_{a,X}}(1, n_Y(\mu, 0)) \phi_3(y_b)
\end{align}

By (2.19) and (2.14) we get also $\phi_1 \otimes \phi_2 \otimes \phi_3(z_0, 0, y_b)$.

\hfill $\Box$

\section{4. Regularised Siegel-Weil Formula}

To relate the theta lift space to Eisenstein series we recall the regularised Siegel-Weil Formula for unitary groups from [2]. We deviate from our usual notation.

Let $V$ be a Hermitian space of dimension $m$. Let $H = U(V)$ and $G = U(n, n)$. Suppose $n < m \leq 2n$ and $m - r \leq n$. Then we define the complementary space $V^c$ of $V$ as follows. Let $m^c = \dim V^c$. Then $m + m^c = 2n$ and $V^c$ is required to be in the same Witt tower as $V$. Fix $K := K_H$ a maximal compact subgroup of $H(\mathbb{A})$ and $K_G$ a maximal compact subgroup of $G(\mathbb{A})$. In this section we use the trivial character to split the metaplectic group over $H(\mathbb{A})$ and $\chi_1$ to split the metaplectic group over $G(\mathbb{A})$.

We consider the theta integral

\begin{equation}
I(g, \phi) := \int_{[H]} \theta(g, h, \phi)dh.
\end{equation}

This may not be absolutely convergent. Let $S(V^n(\mathbb{A}))_{abc}$ denote the subspace of $S(V^n(\mathbb{A}))$ consisting of function $\phi$ such that $I(g, \phi)$ is absolutely convergent for all $g$. This space is nonempty. Then $I$ defines an $H(\mathbb{A})$-invariant map

$I : S(V^n(\mathbb{A}))_{abc} \to \mathcal{A}^\infty(G)$

where $\mathcal{A}^\infty(G)$ is the space of smooth automorphic forms on $G(\mathbb{A})$ without the $K_G$-finiteness condition. Then Ichino[2] showed that there exists a canonical extension of $I$:
Proposition 4.1. Assume $m \leq n$. Then there exists a unique $H(\mathbb{A})$-invariant extension $I_{\text{REG}}$ of $I$ to $S(V^n(\mathbb{A}))$. More precisely, it is realised as

$$c_{\alpha}^{-1} \int_{[H]} \theta(g, h, \omega(\alpha)\phi) dh$$

where $\alpha$ is a suitable element in the Hecke algebra of $G$. It can also be taken to be an element in the Hecke algebra of $H$ which acts on the trivial representation of $H$ by the scalar $c_{\alpha}$.

Now to distinguish the regularised theta integral associated to different groups we add in subscripts, so $I_{V,\text{REG}}$ is what we call $I_{\text{REG}}$ above.

We also need the definition of the Siegel-Eisenstein series. First we define the Siegel-Weil section associated to $\Phi \in S(V^n(\mathbb{A}))$. Let $P$ be the Siegel parabolic subgroup of $G$ and $N$ the unipotent part. For $g \in G(\mathbb{A})$ decompose $g$ as $g = m(A)nk$ with $A \in \text{Res}_{E/k} \text{GL}_n(\mathbb{A})$, $n \in N(\mathbb{A})$ and $k \in K_G$. Set $a(g) = \det A$ in any such decomposition of $g$ and then the quantity $|a(g)|$ is well-defined. The Siegel-Weil section associated to $\Phi \in S(U^n(\mathbb{A}))$ is defined to be

$$F_{\Phi}(g, s) = |a(g)|^{s - s_0} \omega(g)\Phi(0),$$

where $s_0 = (m-n)/2$. This is a section in the induced representation $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi_1|^{s}$.

It is useful to also note the local version. We let $R_n(V_v)$ denote the set of sections $g \mapsto \omega(g)\Phi(0)$ inside $\text{Ind}_{P_v}^{G_v} \chi_1|^{s_0}$.

Returning to the global case, form the Siegel-Eisenstein series

$$E(g, s, F_\Phi) = \sum_{\gamma \in P(k) \setminus G(k)} F_\Phi(\gamma g, s).$$

It is absolutely convergent for $\text{Re } s > 0$. For a $K_G$-finite element $\Phi$ in $S(V^n(\mathbb{A}))$, $E(g, s, F_\Phi)$ has meromorphic continuation to the whole complex plane.

Then the regularised Siegel-Weil formula says

**Theorem 4.2.** [2, Thm. 4.1] Suppose $m > n$. Let $\Phi$ be a $K_G$-finite element in $S(V^n(\mathbb{A}))$. Then

$$\text{Res}_{s = \frac{m-n}{2}} E(g, s, F_\Phi) = c_K I_{V^c,\text{REG}}(g, \pi_{V^c}^{\pi_K} \Phi).$$

**Remark 4.3.** Here $c_K$ is a constant depending only on $K$. For the definition of $\pi_{V^c}^{\pi_K}$ please see (4.6). It sends $\Phi$ to a function in $S(V^{c,n}(\mathbb{A}))$.

**Corollary 4.4.** Suppose $m^c = \dim V^c < n$. For any $K_G$-finite element $\Phi^c$ in $S((V^c)^n(\mathbb{A}))$, there exists a $K_G$-finite element $\Phi$ in $S(V^n(\mathbb{A}))$ such that the following holds:

$$I_{V^c,\text{REG}}(g, \Phi^c) = \text{Res}_{s = \frac{m-n}{2}} E(g, s, F_\Phi).$$

**Proof.** We recall the definition of $\pi_{V^c}$ and $\pi_K$:

$$\pi_{V^c}(v^c) = \int_{M_{0,n}(E)} \Phi \begin{pmatrix} x & v^c \\ 0 & 0 \end{pmatrix} dx,$$

$$\pi_K\Phi(v) = \int_K \Phi(kv)dk.$$
Note that $\pi_V^{\mathfrak{c}}$ is $\pi_Q^{\mathfrak{c}}$ in [2]. To simplify notation let $\pi$ be the composite. We consider the local version. It is easy to see for almost all places $v$ of $k$ if we take $\Phi_v$ to be the characteristic function of the standard lattice in $V_v^{\mathfrak{c}}$, then $\pi_v, \Phi_v$ is the characteristic function of the standard lattice in $V_v^{\mathfrak{c}}$. Fix a place $v$ of $k$ and for all other places $w$ fix $\Phi_w, 0 \in S((V_v^{\mathfrak{c}})_0^n)$. Let $\Phi = \Phi_v \otimes (\otimes_{w \neq v} \Phi_w^0)$. Consider the functional

\[(4.7) \ell(\Phi_v^c) := I_{V^{\mathfrak{c}}, \text{REG}}(1, \Phi_v^c \otimes (\otimes_{w \neq v} \Phi_w^0)).\]

Here $1$ is the identity element of $G(\mathfrak{a})$. It is obviously $U(V_v^{\mathfrak{c}})$-invariant. Thus it factors through the $U(V_v^{\mathfrak{c}})$-coinvariant quotient of $S((V_v^{\mathfrak{c}})_0^n)$. Consider the commutative diagram

\[
\begin{array}{ccc}
S(V_v^{n}) & \longrightarrow & S((V_v^{\mathfrak{c}})_0^n) \\
\downarrow & & \downarrow \\
\ell & & \ell \\
\downarrow & & \downarrow \\
R_n(V_v^{\mathfrak{c}}) & \rightarrow & \text{Ind}_{G_v}^{G} \chi_{1,v} |_{\mathfrak{c}}
\end{array}
\]

By invariant distribution theorem, $\alpha$ is an isomorphism. It is shown in [2] that $F \circ \pi$ is $G_v$-equivariant. Also since we assume $m_\mathfrak{c} < n$, $R_n(V_v^{\mathfrak{c}})$ is irreducible. Thus $F \circ \pi$ is a surjection. This means for any $\Phi_v^c \in S((V_v^{\mathfrak{c}})_0^n)$ there exists a $\Phi_v \in S((V_v^{\mathfrak{c}})_0^n)$ such that $F_{\Phi_v^c} = F_{\Phi_v^0}$. Therefore for any factorisable $\Phi \in S((V_v^{\mathfrak{c}})_0^n(\mathfrak{a}))$ we can find a $\Phi \in S((V_v^{\mathfrak{c}})_0^n(\mathfrak{a}))$ such that $I_{V^{\mathfrak{c}}, \text{REG}}(g, \Phi^c) = I_{V^{\mathfrak{c}}, \text{REG}}(g, \pi \Phi)$. Then the corollary follows from the previous theorem. □

5. Theta Correspondence

5.1. Doubling Method. In this subsection we review the doubling method to prepare for the next subsection and set up some notation. Let $X$ be an $\epsilon$-Hermitian space. It may not be split. Let $X_\mathfrak{a}$ be as in Sec. 1 and let $X'$ be the vector space that has the same underlying vector space as $X$ but with the form $-\langle \cdot, \cdot \rangle_X$. We identify elements in $X$ and $X'$ naturally. Set $W = X \oplus X'$. Then there is a complete polarisation of $W$ given by $W^+ = X^\Delta$ and $W^- = X^\nabla$ where

\[
X^\Delta = \{(x, x) \mid x \in X\};
\]

\[
X^\nabla = \{(x, -x) \mid x \in X\}.
\]

Now consider the more general version. Let $W_\mathfrak{a} = X_\mathfrak{a} \oplus X'$. Then it has complete polarisation given by $W^+_\mathfrak{a} = \ell^+_\mathfrak{a} \oplus X^\Delta$ and $W^-_\mathfrak{a} = \ell^-_\mathfrak{a} \oplus X^\nabla$.

5.2. Main Theorems. Let $\pi$ be a cuspidal automorphic representation of $G(X)$. At this point the additive character that figures in the Weil representation becomes important, so it is put back in notation. We always use the character $\chi_1$ (resp. $\chi_2$) to determine the splitting of metaplectic group over $G(X_\mathfrak{a})(\mathfrak{a})$ (resp. $G(Y_\mathfrak{a})(\mathfrak{a})$). Let $\theta_{X, \psi}^Y(\pi)$ denote the theta lift of $\pi$. The main theorems are as follows.

**Theorem 5.1.** Let $\pi$ be an irreducible cuspidal automorphic representation of $G(X)$. Assume that $\theta_{X, \psi}^Y(\pi)$ is nonvanishing and cuspidal. Then
(1) \( \theta_{X,\psi}^Y(\chi_2^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) = \chi_1 \pi; \)

(2) \( \theta_{X,\psi}^Y(\chi_2^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) \) is orthogonal to all cusp forms on \( G(X_a) \) for \( a > 0; \)

(3) \( \theta_{X,\psi}^Y(\chi_2^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) = 0 \) for \( b > 0. \)

**Remark 5.2.** Note that the theta lifts in opposite directions use additive characters inverse to each other. Here \( \chi_1 \) (resp. \( \chi_2 \)) is regarded as character of \( G(X)(A) \) (resp. \( G(Y)(A) \)) via det. If we choose \( \chi_1 \) and \( \chi_2 \) to be quadratic characters then \( \chi_i|_{A^\times} \) is trivial and thus \( \chi_i \circ \det \) is trivial. Hence in this case we can leave these out of the formulae.

**Theorem 5.3.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \( G(X) \). Assume that \( \theta_{X,\psi}^Y(\pi) \) is nonvanishing and cuspidal. Then \( \theta_{X,\psi}^Y(\pi) \) is irreducible.

For the above we have also similar results in the opposite direction:

**Theorem 5.4.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \( G(Y) \). Assume that \( \theta_{X,\psi}^Y(\pi) \) is nonvanishing and cuspidal. Then

(1) \( \theta_{X,\psi}^Y(\chi_1^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) = \chi_2 \pi; \)

(2) \( \theta_{X,\psi}^Y(\chi_1^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) \) is orthogonal to all cusp forms on \( G(Y_a) \) for \( a > 0; \)

(3) \( \theta_{X,\psi}^Y(\chi_1^{-1} \cdot \theta_{X,\psi}^{-1}(\pi)) = 0 \) for \( b > 0. \)

**Theorem 5.5.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \( G(Y) \). Assume that \( \theta_{X,\psi}^Y(\pi) \) is nonvanishing and cuspidal. Then \( \theta_{X,\psi}^Y(\pi) \) is irreducible.

As pointed out in Sec. 2.2 the roles of \( G(X) \) and \( G(Y) \) are completely symmetric. Thus we only need to show the theorems in one direction. The proofs hinge on the following key computation. Essentially the following analysis shows that \( \theta_{Y,\psi}^X(\chi_2^{-1} \theta_{X,\psi}^{-1}(\pi)) \) is in a space of Eisenstein series.

By definition \( \theta_{Y,\psi}^X(\chi_2^{-1} \theta_{X,\psi}^{-1}(\pi)) \) consists of functions of the form

\[
(5.1) \quad g_a \mapsto \int_{[G(Y)]} \int_{[G(X)]} \chi_2^{-1}(\det h) \theta_{X_a,Y,\psi}(g_a, h, \phi_1) \theta_{X,Y,\psi}^{-1}(g, h, \phi_2) f(g) dg dh
\]

for \( \phi_1 \in S_{X,Y}, \phi_2 \in S_{X,Y} \) and \( f \in \pi. \)

We change \( X \) to \( X' \) which has the same underlying vector space as \( X \) but with the form \( \langle \cdot, \cdot \rangle_X \). We identify \( X' \) with \( X \) and \( G(X') \) with \( G(X) \) accordingly. Then by Prop. 2.1 to keep the action of Weil representation ‘unchanged’ we just need to change \( \psi \) to \( \psi^{-1}. \) By unfolding the definition of splitting we can check that the splittings are compatible with this move. Set \( W_a = X_a \oplus X'. \) Then the integral in (5.1) is equal to:

\[
(5.2) \quad \int_{[G(Y)]} \int_{[G(X)]} \chi_2^{-1}(\det h) \theta_{X_a,Y,\psi}(g_a, h, \phi_1) \theta_{X,Y,\psi}(g, h, \phi_2) f(g) dg dh
\]

\[
(5.3) \quad = \int_{[G(Y)]} \int_{[G(X)]} \chi_2^{-1}(\det h) \theta_{W_a,Y,\psi}((g_a, g), h, \phi_1 \otimes \phi_2) f(g) dg dh.
\]

where \( (g_a, g) \) denotes an element in \( G(W_a)(A) \) via obvious identification. Now the character used for splitting over \( G(Y)(A) \) is \( \chi_2 \) and the one for \( G(W_a)(A) \) is \( \chi_1. \)
By (2.9) to use the trivial splitting for $G(Y)(\mathbb{A})$ we just need to twist the action of $G(Y)(\mathbb{A})$ by the character $\chi_2^{-1}$. Thus we find the above is equal to

$$
\int_{[G(Y)]} \int_{[G(X')]} \theta_{W_a, Y, \psi}((g_a, g), h, \phi_1 \otimes \phi_2) f(g) dg dh
$$

(5.4)

where triv means that we are using the trivial character to determine the splitting for $G(Y)(\mathbb{A})$. From now on we will drop triv with the trivial splitting understood. We would like to exchange order of integration, so first consider the integral

$$
\int_{[G(Y)]} I_{\text{REG}}((g_a, g), \phi_1 \otimes \phi_2) f(g) dg
$$

(5.5)

$$
= \int_{[G(X')]} c_\alpha^{-1} \int_{[G(Y)]} \theta_{W_a, Y, \psi}((g_a, g), h, \omega(\alpha)(\phi_1 \otimes \phi_2)) f(g) dh dg.
$$

(5.6)

Since the inner integral, which is $I_{\text{REG}}((g_a, g), \phi_1 \otimes \phi_2)$, is absolutely convergent we can exchange order of integration to get

$$
\int_{[G(Y)]} \int_{[G(X')]} c_\alpha^{-1} \theta_{W_a, Y, \psi}((g_a, g), h, \omega(\alpha)(\phi_1 \otimes \phi_2)) f(g) dg dh.
$$

(5.7)

By the adjoint property of $\omega(\alpha)$ the above is equal to

$$
\int_{[G(Y)]} \int_{[G(X')]} \theta_{W_a, Y, \psi}((g_a, g), h, (\phi_1 \otimes \phi_2)) f(g) dg dh
$$

(5.8)

which is exactly (5.4). When $\dim Y < \dim X + a$ we can apply Cor. 4.4. Thus (5.4) is equal to

$$
\int_{[G(Y)]} \text{Res}_{s = \dim X + a - \dim Y} E((g_a, g), s, F_\varphi) f(g) dg
$$

(5.9)

for some $K_{G(W_a)}$-finite $\varphi \in S(Y^{\dim X + a}(\mathbb{A}))$ if we start with $\phi_1$ and $\phi_2$ that are $K_{G(X_a)}$- and $K_{G(X')}$-finite. Here $K_{G(W_a)} = K_{G(X_a)} \times K_{G(X')}$. 

Now we compute

$$
\int_{[G(X')]} E^{P_a}(\varphi) f(g) dg,
$$

(5.10)

where $g \in G(X')$ and $h \in G(X_a)$. Note that we have added the superscript to indicate that $E$ is an Eisenstein series associated to the Siegel parabolic $P_a$ of $G(W_a)$ which stabilises $W_a^-$. Recall that $Q_a$ denotes the parabolic subgroup of $G(X_a)$ stabilising $E_a^-$. 

**Proposition 5.6.** Let $F$ be a $K_{G(W_a)}$-finite section of $\text{Ind}_{P_a(\mathbb{A})}^{G(W_a)(\mathbb{A})} \chi_1 \mid s$. Then

$$
\int_{[G(X')]} E^{P_a}(\varphi) f(g) dg = E^{Q_a}(h, s, F^f)
$$

(5.11)

where

$$
F^f(h, s) := \int_{G(X') \backslash G(X_a)} F((h, g), s) \pi(g) f dg.
$$

(5.12)

**Remark 5.7.** Here $F^f(h, s)$ comes from the computation below and as shown in the next lemma it is a section of $\text{Ind}^{G(X_a)(\mathbb{A})}_{Q_a(\mathbb{A})} \chi_1 \mid s \otimes \chi_1 \pi$ and thus the notation of Eisenstein series on the right handside of (5.11) is justified. We always use normalised induction.
Lemma 5.8. The function \( F^f(h, s) \) is absolutely convergent for \( \Re s > (\dim X + a)/2 \) and has meromorphic continuation to the whole complex plane. It is a \( K_{G(X_a)} \)-finite section in \( \text{Ind}^{G(X_a)}_{Q_a(\A)} \chi_1 |^* \otimes \chi_1 \pi \).

Proof. The equation (5.12) is equal to
\[
(5.13) \quad \int_{[G(X')]} \sum_{\gamma \in G(X')(k)} F((h, \gamma g), s) \pi(g) f dg.
\]
Since \( P_a(k) \cap G(X')(k) = \{1\} \), we have an embedding
\[
(5.14) \quad G(X')(k) \hookrightarrow P_a(k) \setminus G(W_a)(k)
\]
and thus the inner sum is a partial sum of the Eisenstein series \( E_{P_a}((h, g), s, F) \).
Since the Eisenstein series is absolutely convergent for \( \Re s > (\dim X + a)/2 \), we find that \( F^f \) is absolutely convergent for \( \Re s > (\dim X + a)/2 \).

Before checking meromorphic continuation we note how \( Q_a(\A) \) acts via left translation. Let \( p \in U_{Q_a}(\A) \). Then \( p \in P_a(\A) \) and also \( \det_{W_a^-}(\A) \) \( p = 1 \). The determinant is thus the action of \( p \) on \( W_a^-(\A) \). Thus \( F^f(ph, s) \) is equal to
\[
(5.15) \quad \int_{G(X')(\A)} F((ph, g), s) \pi(g) f dg = \int_{G(X')(\A)} F((h, g), s) \pi(g) f dg.
\]
Let \( p \in M_{Q_a}(\A) \). First suppose \( p \) acts trivially on \( X \). Then \( p \) is again in \( P_a(\A) \). Then \( F^f(ph, s) \) is equal to
\[
(5.16) \quad \int_{G(X')(\A)} F((ph, g), s) \pi(g) f dg = \chi_1(\det p) | \det p|^{\ell^+_a(\A)} \int_{G(X')(\A)} F((h, g), s) \pi(g) f dg.
\]
Also we can check that we have equalities for the modular characters: \( \delta_{Q_a}(p) = \delta_{P_a}(p) = | \det p |^{\dim X + a} \). Secondly suppose \( p |_{\ell^+_a(\A)} \) is trivial. Then \( p \in G(X)(\A) \). Thus
\[
(5.17) \quad \int_{G(X')(\A)} F((ph, g), s) \pi(g) f dg
\]
\[
(5.18) \quad = \int_{G(X')(\A)} \chi_1(\det p) F((h, p^{-1} g), s) \pi(g) f dg
\]
\[
(5.19) \quad = \int_{G(X')(\A)} \chi_1(\det p) F((h, g), s) \pi(pg) f dg.
\]
Note \((p, p)\) preserves \( X^\vee \) and as \( F \) is a section of \( \text{Ind}^{G(W_a)}_{P_a(\A)} \chi_1 |^* \) the first equality above holds.

Combining these we find that
\[
(5.20) \quad \int_{G(X')(\A)} F((h, g), s) \pi(g) f dg
\]
is an element in the \( \text{Ind}^{G(X_a)}_{Q_a(\A)}(\chi_1 |^* \otimes \chi_1 \pi) \). It is obviously \( K_{G(X_a)} \)-finite since \( F(s) \) is \( K_{G(W_a)} \)-finite.

Now we check that we have meromorphic continuation. Since \( G(X_a)(\A) = Q_a(\A)K_{G(X_a)} \) and \( F(s) \) is \( K_{G(W_a)} \)-finite, we may just assume that \( h \) is in \( Q_a(\A) \).
Since \( U_{Q_a}(\A) \) acts trivially on \( F \) by left translation and the \( GL(\ell^+_a(\A)E) \)-part of \( M_{Q_a}(\A) \) acts by \( \chi_1 |^* \) we may further assume that \( h \) is in the subgroup \( G(X)(\A) \).
of $G(X_a)(\mathbb{A})$. Then $F^f(s)$ restricted to $G(X)(\mathbb{A})$ is in the space of $\chi_1 \pi$. Consider for all cusp form $\xi \in \chi_1 \pi$ the $L^2$-inner product

$$\int_{[G(X')]\mathbb{A}} F^f(h, s)\overline{\xi(h)}dh.$$  \hspace{1cm} (5.21)

If it has meromorphic continuation to the whole complex plane for all cusp form $\xi \in \chi_1 \pi$, then $F^f(s)$ has meromorphic continuation. The equation (5.21) by definition is equal to

$$\int_{[G(X')]\mathbb{A}} F((h, g), s)f(g)\langle \xi(h), \xi(g) \rangle_{L^2}dg.$$  \hspace{1cm} (5.22)

$$= \int_{[G(X')]\mathbb{A}} F((1, h^{-1}g), s)f(g)\overline{\xi(h)}dgdh$$  \hspace{1cm} (5.23)

$$= \int_{[G(X')]\mathbb{A}} F((1, g), s)f(hg)\overline{\xi(h)}dgdh$$  \hspace{1cm} (5.24)

$$= \int_{G(X')(\mathbb{A})} F((1, g), s)f(g)\overline{\xi(g)}dg.$$  \hspace{1cm} (5.25)

This according to the basic identity in [10] is equal to

$$\int_{[G(W)]} E^P((g_1, g_2), s, F)f(g_2)\overline{\xi(g_1)}dg_1dg_2,$$  \hspace{1cm} (5.26)

where $P$ is the Siegel parabolic of $G(W)$ stabilising $W^-$ and we restrict $F$ to $G(W)(\mathbb{A})$ so it is in the induced representation $\text{Ind}_{P(\mathbb{A})}^{G(W)(\mathbb{A})} \chi_1|^{s+\frac{d}{2}}$. Thus (5.12) has meromorphic continuation. \hspace{1cm} $\square$

**proof of Prop. 5.6.** The method of proof is a generalisation of the basic identity in [10]. We unfold and study the double coset $P_a \setminus G(W_a)/G(X_a) \times G(X')$ and then identify the negligible orbits and the main orbit.

Consider the set $P_a \setminus G(W_a)$ that parametrises the maximal isotropic subspaces of $W_a$ over $E$. Let $L$ be a maximal isotropic subspace. Let $d = \dim(L \cap X')$. Then $\dim(L \cap X_a) = d + a$. The proof in [10, Lemma 2.1] goes through word for word. The $G(X_a) \times G(X')$-orbits of maximal isotropic subspaces are parametrised by the invariant $d$. When $d = 0$, one representative for the double coset is $W_a^+$. For each $d = 0, \ldots, \dim X$, take $g_d$ to be the element in $G(W_a)$ that conjugates to $W_a^+$ a representative of the orbit corresponding to $d$. For clarity of notation we omit taking $k$-points in the sums below. For Re $s$ large the left hand side of (5.11) is
equal to
\[
\int_{[G(X')]_\gamma} \sum_{g \in P_u \setminus G(W_u)} F(\gamma(h, g), s) f(g) dg
\]
\[
= \int_{[G(X')]_\gamma} \sum_{d=0}^{\dim X} \sum_{\gamma \in g_u^{-1} P_u g_d \cap (G(X_a) \times G(X)) \setminus G(X')} F(g_d \gamma(h, g), s) f(g) dg
\]
\[
= \sum_{d=0}^{\dim X} \int_{[G(X')]_\gamma} \sum_{\gamma \in g_u^{-1} P_u g_d \cap G(X')} \sum_{\delta \in g_u^{-1} P_u g_d \cap G(X')} F(g_d(\delta, \gamma)(h, g), s) f(g) dg
\]
\[
= \sum_{d=0}^{\dim X} \int_{[G(X')]_\gamma} \sum_{\delta \in g_u^{-1} P_u g_d \cap G(X')} \sum_{\gamma \in g_u^{-1} P_u g_d \cap G(X')} \int_{[g_u^{-1} P_u g_d \cap G(X')]_\gamma} F(g_d(\delta h, g), s) f(g g') dg dq\prime.
\]

We will verify as in [10] that only the integral corresponding to \( d = 0 \) contributes. Fix \( d \) and thus the corresponding maximal isotropic subspace is \( L = W_a + g_d \). Let \( Q_d \) be the parabolic subgroup of \( G(X') \) stabilising the isotropic subspace \( L_1 := L \cap X' \) and let \( U_d \) be its unipotent radical. Note that we view \( G(X') \) as a subgroup of \( G(W_u) \). Suppose \( d > 0 \). Then \( Q_d \) is a proper parabolic subgroup. Decompose \( X' \) as \( L_1 \oplus X'' \). Let \( n \in U_d \) and let \( x \in L \). Write \( x = x_1 + x_2 + x_3 \) where \( x_1 \in X_a, x_2 \in L_1 \) and \( x_3 \in X'' \). Then \( xn = x_1 + x_2 + x_3 + x_4 = x + x_4 \) for some \( x_4 \in L_1 \). Thus \( xn \in L \). We have shown that \( U_d \) stabilises \( L \), that \( U_d \subset g_u^{-1} P_u(k) g_d \cap (G(X')(k) \cap G(X')) \) and that \( \det_L n = 1 \). Thus \( F(g_d(\delta h, g), s) \) is invariant when \( g \) is changed to \( ng \).

Then the inner integral can be further split into two iterated integrals with an inner integral being over \( [U_d] \):
\[
(5.27) \quad \int_{[U_d]} f(ng) dn
\]
which is zero by cuspidality of \( f \). Thus when \( d > 0 \) these are indeed negligible orbits.

Now consider the main orbit corresponding to \( d = 0 \). Since \( P_u(k) \cap G(X')(k) = \{1\} \) the contribution is
\[
(5.28) \quad \int_{G(X')(k)} \sum_{\delta \in P_u G(X') \cap G(X_a) \setminus G(X_a)} F((\delta h, g), s) f(g) dg.
\]

We claim that \( P_u G(X') \cap G(X_a) = Q_a \). Here \( Q_a \) is from our old notation: the parabolic subgroup of \( G(X_a) \) stabilising \( \ell_a^+ \). It is easy to see \( Q_a \subset P_u G(X') \cap G(X_a) \). Now suppose \( h \in P_u G(X') \cap G(X_a) \). Then \( (h, 1) = p(1, g) \) for some \( p \in P_a \) and \( g \in G(X') \). We have \( (h, g^{-1}) \in P_u \) and thus \( (h, 1) \in P_u \) where we view \( g \in G(X') \) as an element in the subgroup \( G(X) \) of \( G(X_a) \) via the obvious embedding. In other words \( hg \) is contained in \( P_u \cap G(X_a) \) which is isomorphic to the group \( GL(\ell_a^+) \times U_{Q_a} \). Thus \( h \in GL(\ell_a^+) \times U_{Q_a} \times G(X') = Q_a \).
With this we find (5.28) is equal to

\[(5.29) \sum_{\gamma \in Q_\ast(k) \setminus G(X_\ast)(k)} F^f(\gamma h)\]

for \(h \in G(X_\ast)(A)\) as required. \(\square\)

It follows from the proposition above and the analysis before it (c.f. (5.9)) we have

**Proposition 5.9.** For \(\dim Y < \dim X + \alpha\), \(\theta_{Y,\psi}^{X_a} \theta_{X,\psi_1}^Y(\pi)\) is contained in the space

\[(5.30) \{\Res_{s=\frac{1}{2}(\dim X + a - \dim Y)} E^{Q_\ast}(r, s, F^f)|f \in \pi\}\]

where \(F^f\) is defined in (5.12).

Now we try to remove the condition on \(a\) in the previous proposition. The idea is to take constant terms on the Eisenstein series and on the theta lifts. Let \(Q_{A-a}\) denote the parabolic subgroup of \(G(X_A)\) stabilising \(\ell_{A-a}\) which is a subspace of \(\ell_A\) consisting of \(A - a\) hyperbolic planes. Temporarily let \(\chi_1\) be subsumed into \(\pi\). Then on the Eisenstein side we have

**Lemma 5.10.** Consider the constant term of \(E^{Q_\ast}(g, s, f)\) along \(Q_{A-a}\) for \(f \in \Ind_{Q_A(\A)}^{G(X_A)(\A)}(\chi_1)|^s \otimes \pi\). If we consider it as a function of \(g \in G(X_a)\) then its residue at \(s^0 = \frac{1}{2}(\dim X + A - \dim Y)\) is

1. orthogonal to all cusp forms on \(G(X_a)\) for \(a > 0\);
2. in \(\pi\) for \(a = 0\);
3. zero for \(a < 0\).

**Proof.** We compute the constant term. Since \(\chi_1| |^s \otimes \pi\) may be non-cuspidal, we go to the cuspidal support. Let \(Q'_A\) be the standard parabolic subgroup of \(G(X_A)\) whose Levi is isomorphic to \(\Res_{E/k} GL_1^A \times G(X)\) where \(\Res_{E/k}\) is Restriction of scalar of Weil. There exists a section \(F\) of

\[(5.31) \Ind_{Q_A(\A)}^{G(X_A)(\A)}(\chi_1)|^s_1 \times \cdots \times \chi_1| ^s_A \times \pi\]

such that

\[(5.32) E^{Q'_A}(s, f) = \prod_{i=1}^A (s_i - s - A - 2i + 1) E^{Q'_A}(s_1, \ldots, s_A, F)|_{s_i = s + A - 2i + 1}.\]

Thus we need to compute \([E^{Q'_A}(s_1, \ldots, s_A, F)]_{Q_{A-a}}\). We consider the double cosets \(Q'_A \setminus G(X_A)/Q_{A-a}\). Let \(\Omega\) be the set of Weyl elements \(w\) such that \(w\) is of minimal length in the double coset \(Q_{A-a}wQ'_A\). If we identify \(\Omega\) with the group of signed permutations which is a subset of maps from \(\{1, \ldots, A\}\) to \(\{\pm 1, \ldots, \pm A\}\), then \(\Omega\) is the set of maps

\[(5.33) \left\{ w \right\}_{w^{-1}(i)<w^{-1}(j) \text{ if } 1 \leq i < j \leq A-a \text{ or } A-a+1 \leq i < j \leq A \text{ \ and \ } w^{-1}(i)>0 \text{ if } A-a+1 \leq i \leq A} \\\n\text{w^{-1}(i)>0 \ if \ A-a+1 \leq i \leq A} \right\}.

Set \(\underline{s} = (s_1, \ldots, s_A)\) and \(w\underline{s} = (s_{w^{-1}(1)}, \ldots, s_{w^{-1}(A)})\). Let \(g\) be an element in the Levi \(M_{A-a}(\A)\) of \(Q_{A-a}(\A)\). Then \([E^{Q'_A}(g, \underline{s}, F)]_{Q_{A-a}}\) is equal to

\[(5.34) \sum_{w \in \Omega} E^{Q'_A w^{-1}\cap M_{A-a}}(g, w\underline{s}, M(w, \underline{s})F)\]
where $M(w, \mathfrak{s})$ is the intertwining operator associated to $w$. Here $E_{M_{A-a}}^{wQ_A w^{-1} \cap M_{A-a}}$ is an Eisenstein series on $M_{A-a}$ with respect to the parabolic subgroup $wQ_A w^{-1} \cap M_{A-a}$. We restrict to $g \in G(X_a)(\mathbb{A})$. Then in the cone of absolute convergence (5.34) is equal to

$$
\sum_{\gamma \in \text{w}Q_A w^{-1} \cap \text{Res}_{E/k} \text{GL}_{A-a} \setminus \text{Res}_{E/k} \text{GL}_A} E_{G(X_a)}^{wQ_A w^{-1} \cap G(X_a)}(\gamma g, w\mathfrak{s}, M(w, \mathfrak{s})F).
$$

If we integrate this against a cusp form on $G(X_a)$ the integral will vanish. This proves part (1).

Now let $\alpha = 0$. Then the residue in question becomes

$$
\prod_i (s_i - \frac{A - 2i + 1}{2}) \sum_{\gamma \in \text{w}Q_A w^{-1} \cap \text{Res}_{E/k} \text{GL}_A \setminus \text{Res}_{E/k} \text{GL}_A} M(w, \mathfrak{s})F(\gamma g, w\mathfrak{s}).
$$
evaluated first at $s_i = s + \frac{A - 2i + 1}{2}$ and then at $s = s^0$. If we restrict $g$ to the subgroup $G(X)(\mathbb{A})$ of $G(X_a)(\mathbb{A})$ then this is in the space of $\pi$. Thus we have proved part (2).

Let $b > 0$. For $[E^Q_A(g, s_1, \ldots, s_A, F)]_{Q_{A+b}}$ we take constant term in steps:

$$
[E^Q_A(g, s_1, \ldots, s_A, F)]_{Q_{A+b} \cap G(X)}.
$$
The residue of the first step falls in the space of $\pi$ which is assumed to be cuspidal. Thus the second step gives 0. Thus we have proved part (3).

With this preparation we are ready to begin.

**Proof of Theorem 5.1.** We take $A$ large so that $\dim X + A > \dim Y$ and thus we can apply Prop. 5.9. For the theta lift side, for any $a$ we have the tower property (Prop. 3.1)

$$
[\theta_{X,\psi}(\chi_2^{-1}) \theta_{Y,\psi}^{-1}(\pi)]_{Q_{A-a} \cap G(X_a)} = [\theta_{Y,\psi}(\chi_2^{-1}) \theta_{X,\psi}^{-1}(\pi)]_{Q_A \cap G(X_a)}.
$$

Suppose $0 < a < A$. We take constant terms along $Q_{A-a}$ on the Eisenstein side of the spaces and then restrict to $G(X_a)(\mathbb{A})$. By Lemma 5.10, the constant term is orthogonal to cusp forms on $G(X_a)$. Thus we have proved Part (2). We then take constant term along $Q_A$. The residue of Eisenstein series falls in the space of $\chi_1 \pi$. (We unsubsume the $\chi_1$.) Thus we get part (1). Now take constant term along $Q_{A+b}$ for $b > 0$. The residue of Eisenstein series becomes 0. Thus we get part (3).

Finally for the irreducibility result:

**Proof of Theorem 5.3.** Suppose $\sigma$ is an irreducible submodule of $\theta_{X,\psi}^{-1}(\pi)$. By Thm. 5.1 the subspace $\theta_{Y,\psi}(\chi_2^{-1}) \sigma$ of $\theta_{Y,\psi}(\chi_2^{-1}) \theta_{X,\psi}^{-1}(\pi)$ is orthogonal to cusp forms on $G(X_a)$ and also $\theta_{X,\psi}^{-1}(\chi_2^{-1}) \sigma = 0$. Thus by the cuspidality of first occurrence we must have that $\theta_{X,\psi}(\chi_2^{-1}) \sigma$ is nonvanishing and cuspidal. Since $\theta_{Y,\psi}(\chi_2^{-1}) \sigma \subset \theta_{X,\psi}(\chi_2^{-1}) \theta_{X,\psi}^{-1}(\pi) = \chi_1 \pi$ and $\pi$ is irreducible, we find $\theta_{Y,\psi}(\chi_2^{-1}) \sigma = \chi_1 \pi$. Then by theta lifting in the other direction we find $\sigma = \theta_{X,\psi}^{-1}(\pi)$. Thus $\theta_{X,\psi}^{-1}(\pi)$ is indeed irreducible.
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