Non-Hermitian boundary state engineering in anomalous Floquet topological insulators

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In Hermitian topological systems, the bulk-boundary correspondence strictly constrains boundary transport to values determined by the topological properties of the bulk. We demonstrate that this constraint can be lifted in non-Hermitian Floquet insulators. Provided that the insulator supports an anomalous topological phase, non-Hermiticity allows us to modify the boundary states independently. This new freedom is exploited in BSE.

Topological states of matter have proven to be a research topic where fundamental theoretical insights lead almost inevitably to state-of-the-art practical applications [1–3]. A key feature of topological systems is directional transport via chiral boundary states, which is protected by topological invariants and thus impervious to the imperfections of real-world implementations [4–8]. In combination with fundamental symmetries [9–11], especially time-reversal symmetry (TRS) in topological insulators [12, 13], topology even protects bidirectional helical transport via counterpropagating boundary states [6, 14]. Such symmetry-protected topological phases emerge in a variety of physical systems [15–23], where they give rise to a wide spectrum of experimentally observable phenomena [24–28]. The recent discovery of anomalous topological phases in periodically driven (i.e., Floquet) insulators demonstrates the singular relevance of topological concepts also in systems far from equilibrium [29–34]. Only very recently, the notion of topological phases has been extended to non-Hermitian systems [35–37]. The perception of the role of topology in this context is still changing rapidly through theoretical investigation and classification [38–43] as well as experimental exploration [27, 44, 45] of non-Hermitian topological phases. Intriguingly, topology is expected to protect transport even against energy losses and damping [42].

In the present work we introduce the topological concept of boundary state engineering (BSE) that combines the specific aspects of non-Hermitian and anomalous Floquet topological phases, and has no counterpart in systems with a Hermitian or static Hamiltonian. The concept underlying BSE is illustrated in Figure 1, where we sketch the spectrum of the Floquet propagator $U \equiv U(T)$, obtained as the solution of the Schrödinger equation $\imath \partial_t U(t) = H(t)U(t)$ after one period of a time-periodic Hamiltonian $H(t) = H(t + T)$.

For a Hermitian system, with real Floquet quasienergies $\varepsilon$, the spectrum $\{ e^{-\imath \varepsilon} \}$ of $U$ lies on the unit circle [46]. In regular topological phases, as they appear in systems with a static Hamiltonian, any boundary state, viewed as a continuous curve $k \mapsto e^{-\imath \varepsilon(k)}$ parametrized by momentum $k$, connects two different bulk bands. Anomalous Floquet topological phases, in contrast, possess boundary states that wind around the unit circle [30]. Thinking in terms of the quasienergy $\varepsilon$, this possibility results from the periodicity $\varepsilon \mapsto \varepsilon + 2\pi$.

In a non-Hermitian system, the spectrum of $U$ can move away from the unit circle. Regular boundary states have to remain attached to the bulk bands, since otherwise the continuous dependence on momentum would be violated. Anomalous boundary states, however, can detach from the bulk bands and thus be manipulated independently. This new freedom is exploited in BSE.

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BSE is indeed a topological concept: Since the propagator $U$ is invertible, its spectrum cannot move through the origin. Therefore, an anomalous boundary state that winds around the origin retains this property during BSE, and thus remains topologically protected.

From our discussion it is evident that BSE requires the combination of non-Hermiticity with anomalous Floquet topological phases. Non-Hermiticity arises naturally in optical settings such as photonic waveguide systems [27], since coupling involves losses due to the bending of the waveguides (see Fig. 2). The idealized Hamiltonian for the symmetric coupling of two lossy waveguides, $H_B = \begin{pmatrix} -i\gamma & J \\ J & -i\gamma \end{pmatrix}$, involves a coupling parameter $J$ and damping $\gamma$. The associated propagator $U_B = \exp[-iH_B]$, over a time step $\delta t \equiv 1$, is an SU(2) rotation, modified by the attenuation ($\gamma > 0$) or amplification ($\gamma < 0$) factor $e^{-\gamma}$. At perfect coupling $J = \pi/2$ we have $U_B = -ie^{-\gamma}\hat{\sigma}_x$, with the Pauli matrix $\hat{\sigma}_x$. Amplitude is swapped between the two coupled sites, but changes as $e^{-\gamma}$.

Regarding the experimental relevance of our theoretical considerations, it is useful to allow for a shift $\sigma(t) \in C$ of the Hamiltonian, where we map $H(t) \to H(t) + \sigma(t)$, and thus $U \to \Gamma U$ with $\Gamma = \exp(-i\int_0^T \sigma(t)dt)$. Through the shift, loss and gain become relative terms, and weak loss can be interpreted as (pseudo-) gain relative to strong loss. The physical content of $H(t)$ or $U$ remains unchanged: Measuring normalized intensities, of the form $I(r) = |\psi(r)|^2/\max_r|\psi(r')|^2$, the factor $\Gamma$ in $U$ cancels.

Concerning the second aspect of BSE, the anomalous Floquet topological phases, we resort to the idea of a driving protocol [30]. The protocol used here concatenates $n = 6$ steps, with the Hamiltonians $H_1, \ldots, H_6$ as given in Fig. 3 in an intuitive pictorial representation. The protocol takes place on a (finite or infinite) square lattice, which is composed of a “red” and “blue” sublattice. This type of driving protocol has been comprehensively investigated in Ref. [23], where its universality for symmetry-protected topological phases is established.

By construction, the driving protocol incorporates fermionic TRS, where the Hamiltonian obeys the relation $SH(t)S^{-1} = H(T-t)^*$, with a unitary symmetry operator $S$ that fulfills $SS^* = -1$ [10]. Here, it is $S = \hat{\sigma}_x \otimes \hat{\sigma}_y$ if we identify the “red” and “blue” sublattice with the up and down component of a pseudo-spin $1/2$.

Fermionic TRS is essential for the $\mathbb{Z}_2$ phases of topological insulators [12]. As can be seen in Fig. 3, the driving protocol indeed supports a symmetry-protected $\mathbb{Z}_2$ topological phase, with counterpropagating boundary states whose intersection at momentum $k = 0$ is protected by Kramers degeneracy. This phase is an anomalous Floquet phase, since the boundary states connect the same bulk band at quasienergies separated by $2\pi$. This phase has been explored experimentally in Ref. [14].

In general, the driving protocol supports topological phases on a continuous parameter manifold. Here, we consider a minimal parameter set, with the two parameters $J$ (for diagonal couplings in steps 1, 3, 4, 6) and $J'$ (for horizontal couplings in step 2, 5). We use $J = 1.5$, $J' = 0.4$ in all plots. The minus sign $\pm J'$ between steps 2, 5 shown in Fig. 3 is required for fermionic TRS, but negative couplings can be replaced by positive couplings to facilitate the experimental implementation [14].

Since the $\mathbb{Z}_2$ phase in Fig. 3 is an anomalous topological phase, it is a candidate for BSE. To understand the possibilities arising in this situation we first address the analytically tractable case of perfect coupling ($J = \pi/2$, $J' = 0$), before returning to general parameters.

At perfect coupling, the “red” and “blue” sublattice are decoupled (since $J' = 0$). As depicted in Fig. 4, states in the bulk move in a circular clockwise (counterclockwise) orbit on the “red” ("blue") sublattice. At a boundary, which in Fig. 4 is oriented horizontally with respect to Fig. 3 and lies at the bottom of the (half-infinite) lattice, states propagate either to the right (red sublattice) or the left (blue sublattice). The propagation direction does not depend on the precise position of the boundary, but is prescribed by the bulk-boundary correspondence. Here, with counterpropagating boundary states, transport is protected by topology and fermionic TRS.

According to these patterns of motion, perfect coupling gives rise to a four-fold degenerate dispersionless bulk band at quasienergy $\varepsilon = 0$, and two counterpropagating

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Idealized Hamiltonians for an uncoupled (A) and two coupled (B) lossy waveguides, as they might appear in the experimental realization of a non-Hermitian driving protocol.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Driving protocol for a Floquet insulator with TRS: Six steps of alternating interactions (left panel) between neighboring lattice sites (representing, e.g., waveguides as in Fig. 2) lead to a $\mathbb{Z}_2$ topological phase with counterpropagating boundary states (bottom right panel), which is protected by the fermionic TRS mediated by the symmetry operator $S \equiv \hat{\sigma}_x \otimes \hat{\sigma}_y$ (top right panel).}
\end{figure}
boundary states with linear dispersion $\varepsilon_r(k) = \pi + k$, $\varepsilon_b(k) = \pi - k$. As has been seen in Fig. 3, these features survive qualitatively for general parameters.

Non-Hermiticity is now introduced into the driving protocol in the following way. In the bulk, we assume uniform losses (as in the type B configuration in Fig. 2) for the diagonal couplings of the red (losses with $\gamma_r$) or blue (losses with $\gamma_b$) sublattice in steps 1, 3, 4, 6. The bulk bands thus acquire imaginary quasienergies

$$\bar{\varepsilon}_r = -4\gamma_r, \quad \bar{\varepsilon}_b = -4\gamma_b.$$  \hfill (2)

Here and in the following, a $\bar{\varepsilon}$ variable denotes the imaginary part $\bar{\varepsilon} \equiv \text{Im} \varepsilon$ of some quasienergy. Positive (negative) $\bar{\varepsilon}$ implies gain (loss) according to $e^{-i\bar{\varepsilon}t} = e^{t \bar{\varepsilon}} \times e^{-i dt \text{Re} \varepsilon}$. For the horizontal couplings in step 2, 5 we may assume identical losses, but they can be absorbed into the shift of the Hamiltonian and will not be listed explicitly.

At the boundaries, additional losses occur at isolated uncoupled sites (as in the type A configuration in Fig. 2), namely $\gamma_r$ or $\gamma_b$ on a red or blue site. Note that the designation as an isolated site changes during the protocol, as visible in Fig. 4. For example, a state starting on a “filled red” site moves by two sites in steps 1, 3 (incurring losses $-2\gamma_r$), and remains at an isolated site during steps 4, 6 (incurring losses $-2\gamma_r$). Therefore, the boundary states acquire the imaginary quasienergies

$$\bar{\varepsilon}_r = \bar{\varepsilon}_r/2 - 2\gamma_r, \quad \bar{\varepsilon}_b = \bar{\varepsilon}_b/2 - 2\gamma_b.$$  \hfill (3)

Free choice of the four loss parameters $\gamma_r, \gamma_b, \gamma_r^*, \gamma_b^*$ allows for free placement of boundary states relative to the bulk, as we had anticipated with the concept of BSE introduced in Fig. 1. In particular, the boundary states are detached from the bulk bands if $\bar{\varepsilon}_r \neq \bar{\varepsilon}_r^*$ or $\bar{\varepsilon}_b \neq \bar{\varepsilon}_b^*$.

The free assignment of losses $\gamma$ (or imaginary parts $\bar{\varepsilon}$) is not compatible with TRS. To restore TRS, we can now impose two independent conditions in extension of the Hermitian case, reading

$$(\text{TRS}^*) : \quad SH(t)S^{-1} = H(T - t)^* + \xi_r(t),$$ \hfill (4a)

$$(\text{TRS}^t) : \quad SH(t)S^{-1} = H(T - t)^t + \xi_r(t).$$ \hfill (4b)

Note that these conditions have been modified by inclusion of the shift of the Hamiltonian, such that the functions $\xi_r(t) = \sigma(t) - \sigma(T - t)^*, \xi_r(t) = \sigma(t) - \sigma(T - t)$ appear in comparison to the standard relations [37]. For a constant shift $\sigma(t) \equiv \sigma$, we have $\xi_r \equiv \text{i} \text{Im} \sigma$ and $\xi_r \equiv 0$. Both conditions collapse into the previous TRS relation in the Hermitian case, where $H^I = H^*$ and $\sigma \equiv 0$.

Similar to the coupling parameters $J,J'$, where fermionic TRS requires a minus sign between steps 2, 5, non-Hermitian TRS imposes constraints

$$(\text{TRS}^*) : \quad \gamma_r + \gamma_b = \gamma_b^* + \gamma_r, \quad \gamma_r = \gamma_b, \quad \gamma_b = \gamma_r^*,$$ \hfill (5a)

$$(\text{TRS}^t) : \quad \bar{\varepsilon}_r + \bar{\varepsilon}_b = \bar{\varepsilon}_b^* + \bar{\varepsilon}_r, \quad \bar{\varepsilon}_r = \bar{\varepsilon}_b, \quad \bar{\varepsilon}_b = \bar{\varepsilon}_r^*.$$ \hfill (5b)

on the loss parameters, which are equivalent to

$$(\text{TRS}^*) : \quad \bar{\varepsilon}_r + \bar{\varepsilon}_b = \bar{\varepsilon}_b^* + \bar{\varepsilon}_r, \quad \bar{\varepsilon}_r = \bar{\varepsilon}_b, \quad \bar{\varepsilon}_b = \bar{\varepsilon}_r^*.$$ \hfill (6a)

$$(\text{TRS}^t) : \quad \bar{\varepsilon}_r + \bar{\varepsilon}_b = \bar{\varepsilon}_b^* + \bar{\varepsilon}_r, \quad \bar{\varepsilon}_r = \bar{\varepsilon}_b, \quad \bar{\varepsilon}_b = \bar{\varepsilon}_r^*.$$ \hfill (6b)

To interpret these relations, note that TRS$^t$, but not TRS$^*$, implies equal damping of counterpropagating boundary states.

While in a uniform system the parameters $\gamma_r, \gamma_b$ can be assumed to be identical throughout the bulk, it is essential for BSE that the losses $\gamma_r, \gamma_b$ may vary well depend on the boundary. For the schematic plot in Fig. 4, we consider a semi-infinite horizontal strip with different losses at the “top” ($\gamma_r^t, \gamma_b^t$) and “bottom” ($\gamma_r^b, \gamma_b^b$) boundary (this setup is also used in the remaining figures). The central part of this plot uses a parametrization with

$$(\text{TRS}^*) : \quad \gamma_r = -4\gamma_r^t, \quad \gamma_b = -4\gamma_b^b,$$ \hfill (7a)

$$(\text{TRS}^t) : \quad \bar{\varepsilon}_r^t = -2\bar{\varepsilon}_b^t, \quad \bar{\varepsilon}_r^b = -6\bar{\varepsilon}_b^b.$$ \hfill (7b)

The schematic plot in Fig. 4 illustrates a few particularly noteworthy features of BSE. First, without TRS, the damping of bulk and boundary states can be chosen entirely freely. Second, even with TRS, the damping
of boundary states relative to bulk states can be chosen freely, suppressing or enhancing either boundary transport or bulk motion. Third, the properties of boundary transport are not dictated by the properties of the bulk, as a strict bulk-boundary correspondence would demand. For example at the right axis of the plot, boundary transport occurs predominantly via “red” states, opposite to the predominant motion in the bulk via “blue” states.

In the general situation, away from perfect coupling, both the real and imaginary part of the quasienergies depend on momentum. TRS implies the constraints [47]

\begin{align}
(TRS^*): \quad & \text{Re} \{\varepsilon(k)\} = \text{Re} \{\varepsilon(-k)\}, \quad (8a) \\
(TRS): \quad & \text{Im} \{\varepsilon(k)\} = -\text{Im} \{\varepsilon(-k)\} + 2 \text{Im} \sigma, \quad (8b) \\
(TRS^t): \quad & \text{Im} \{\varepsilon(k)\} = \text{Im} \{\varepsilon(-k)\} \quad (8c)
\end{align}
on the quasienergy spectrum \{\varepsilon(k)\} at momentum \(k\), which generalize Eq. (6). Here, Eq. (8b) includes an imaginary shift, which drops out of Eq. (8c).

In Fig. 5, we observe the Kramers-like crossing of \(\text{Re} \varepsilon(k)\) according to Eq. (8a). For \(\text{TRS}^t\) the two boundary states to cross at the same \(\text{Im} \varepsilon(k)\) and are thus truly degenerate, while for \(\text{TRS}^*\) they are separated by their imaginary part. This difference suggests that in the first case \(\text{TRS}^t\) is required to protect the boundary states, while in the second case they are robust against breaking of \(\text{TRS}^*\). Indeed, if we break TRS by adding detunings [48], we observe an avoided crossing in Fig. 5 only for broken \(\text{TRS}^t\) but not for broken \(\text{TRS}^*\). Evidently, when counterpropagating boundary states have been separated via BSE, TRS is no longer required for their protection. However, now a preferred direction of transport exists due to the different damping of the boundary states. Only if the damping is equal, as in the \(\text{TRS}^t\) case in Fig. 5, true bidirectional transport without a preferred direction can be observed. In this scenario, TRS is still required for the protection of transport.

The new freedom introduced by BSE is thus two-fold: It allows us to modify boundary transport relative to the bulk motion, and to selectively modify transport on different boundaries. Fig. 6 provides a visual demonstration of the potential of such modifications. The point of reference is the Hermitian case in panel (B), with bidirectional helical transport via symmetry-protected counterpropagating boundary states (recall the quasienergy dispersion in Fig. 3). When protected by \(\text{TRS}^t\), the counterpropagating boundary states survive the transition into the non-Hermitian regime, but now BSE allows us to suppress bulk motion in favor of boundary transport (panel (C)). In the \(\text{TRS}^*\) case, we can use BSE to additionally suppress either one of the two states on each boundary, which we do in such a way that transport on opposite boundaries takes place in the same (and not opposite) direction (panel (D)). Such modifications are a unique feature of BSE. They are impossible in a Hermitian system, where they are prohibited by the bulk-boundary correspondence, as well as in a non-Floquet system, where boundary states remain attached to the bulk bands (cf. Fig. 1) and cannot be selectively amplified or suppressed.

To assess the full extent of topological protection in the present situation we have to examine the robustness of transport for imperfect boundaries and under the influence of disorder. In panels (E), (F) in Fig. 6 we use serrated boundaries, and include disorder [49] that pre-
serves (breaks) TRS in the lower (upper) half of each strip. In panel (E), TRS$^f$ indeed protects the scatter-free bidirectional boundary transport, while breaking TRS$^t$ leads to visible back-scattering. In panel (F), even disorder that breaks TRS$^*$ does not lead to appreciable back-scattering because of the suppression of one of the two boundary states. Effectively, this situation realizes chiral transport with a preferred direction, which, in contrast to helical bidirectional transport, is protected by topology but does no longer require TRS.

In conclusion, BSE opens up new avenues to control topological transport in non-Hermitian Floquet systems. In conjunction with fermionic TRS to protect counter-propagating boundary states, BSE enables the selective enhancement of individual topological transport channels, and thus the effective manipulation of (bi-) directional boundary transport. The potential for applications is immediate, and photonic waveguides, which are intrinsically non-Hermitian systems, are a natural platform to explore this potential. Experiments should also investigate the robustness of the different transport phenomena described here, thereby extending our analysis of the influence of disorder and symmetry breaking. Not least, the surprising new possibilities of BSE highlight the importance of further theoretical research regarding the status of topological invariants and the bulk-boundary correspondence in non-Hermitian Floquet systems.

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[46] Note that we measure quasienergies normalized to time step unity.
[47] See the supplemental material for a detailed derivation.
[48] For specifics, see the supplemental material.
[49] See the supplemental material for specifics.
Supplemental material for:
Non-Hermitian boundary state engineering
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The supplemental material contains (i) a detailed derivation of Eq. (8) in the main text, (ii) the explicit specification of the non-Hermitian driving protocol, (iii) the explicit constraints on the protocol parameters from time-reversal symmetry, and (iv) the specification of detunings and disorder to break time-reversal symmetry in Figs. 5, 6 in the main text.

I. TRS AND ITS CONSEQUENCES

The standard relation for time-reversal symmetry (TRS) reads $SHS^{-1} = H^* = H^t$ for a static Hamiltonian, with a unitary operator $S$. One can distinguish bosonic TRS with $SS^* = 1$ from fermionic TRS with $SS^* = -1$. The TRS relations generalizes to

$$SH(t)S^{-1} = H(T-t)^* = H(T-t)^t$$

(1)

for a Floquet system with period $T$, where $H(t+T) = H(t)$. This relation generalizes even further for a non-Hermitian Floquet system, where we have the two separate relations [1]

$$\text{(TRS}^*) : \quad SH(t)S^{-1} = H(T-t)^*,$$  

(2a)

$$\text{(TRS}^F) : \quad SH(t)S^{-1} = H(T-t)^t.$$  

(2b)

Clearly, both relations agree for a Hermitian Hamiltonian with $H(t)^* = H(t)^t$.

TRS implies relations on the propagator $U(t)$ that lead, eventually, to the relations in Eq. (8) in the main text. To obtain these relations it is useful to consider the symmetrized propagator

$$U_*(t) = U \left( \frac{T+t}{2}, \frac{T-t}{2} \right).$$  

(3)

It is $U_*(0) = 1$ and $U_*(T) = U(T)$. By applying $S \cdots S^{-1}$ on both sides of the equation of motion

$$2i \partial_t U_*(t) = H \left( \frac{T+t}{2} \right) U_*(t) + U_*(t) H \left( \frac{T-t}{2} \right),$$  

(4)

replacing terms according to the symmetry relations (2), and using the additional equations of motion

$$-2i \partial_t U_*(t)^{-1} = H \left( \frac{T-t}{2} \right) U_*(t)^{-1}$$  

$$\quad + U_*(t)^{-1} H \left( \frac{T+t}{2} \right),$$  

(5a)

$$2i \partial_t U_*(t)^t = U_*(t)^t H \left( \frac{T+t}{2} \right)^t$$  

$$\quad + H \left( \frac{T-t}{2} \right)^t U_*(t)^t,$$  

(5b)

we see that

$$\text{(TRS}^*) : \quad SU_*(t)S^{-1} = (U_*(t)^{-1})^*,$$  

(6a)

$$\text{(TRS}^F) : \quad SU_*(t)S^{-1} = U_*(t)^t.$$  

(6b)

Therefore, we have

$$\text{(TRS}^*) : \quad SUS^{-1} = (U^{-1})^*,$$  

(7a)

$$\text{(TRS}^F) : \quad SU^*S^{-1} = U^t.$$  

(7b)

for the Floquet propagator $U \equiv U(T) = U_*(T)$.

The TRS relations (2) are somewhat too restrictive for non-Hermitian systems, where we want to be able to freely interpret the meaning of “gain” and “loss” in relative terms. A simple modification suffices to achieve that freedom, namely, we allow for a (time-dependent) shift $\sigma(t) \in \mathbb{C}$ of the Hamiltonian

$$H(t) \mapsto H(t) + \sigma(t)$$  

(8)

and demand that the modified TRS relations are invariant under such shifts. Replacing $H(t)$ by $H(t) + \sigma(t)$ in Eq. (2), these modified relations are obtained as

$$\text{(TRS}^*) : \quad SH(t)S^{-1} = H(T-t)^* + \xi_*(t),$$  

(9a)

$$\text{(TRS}^F) : \quad SH(t)S^{-1} = H(T-t)^t + \xi(t),$$  

(9b)

with arbitrary complex valued functions $\xi_*(t), \xi(t) \in \mathbb{C}$ that fulfill $\xi_*(T-t) = -\xi_*(t)^*$ and $\xi(T-t) = -\xi(t)$. These functions are related to the specific shift $\sigma(t)$ introduced in Eq. (8) through

$$\text{(TRS}^*) : \quad \xi_*(t) = \sigma(t) - \sigma(T-t)^*,$$  

(10a)

$$\text{(TRS}^F) : \quad \xi(t) = \sigma(t) - \sigma(T-t).$$  

(10b)

If $H(t)$ is modified by the shift in Eq. (8), the Floquet propagator is modified as

$$U \mapsto U \Gamma$$  

(11)

with the scalar factor

$$\Gamma = \exp \left( -i \int_0^T \sigma(t) dt \right).$$  

(12)
With this modification, the TRS relations for the Floquet propagator read
\[(TRS^*) : \quad SU^{-1} = (\Gamma^* U^*)^{-1}, \quad (13a)\]
\[(TRS^t) : \quad SU^{-1} = U^t  \quad (13b)\]
in generalization of Eq. (7). Note that the scalar factor $\Gamma$ drops out of the TRS$^t$ relation.

Both the complex conjugation in TRS$^*$ and the transposition in TRS$^t$ map momentum $k \mapsto -k$. Therefore, the above relations give
\[(TRS^*) : \quad SU(k,T)S^{-1} = (\Gamma^* U(-k,T)^*)^{-1}, \quad (14a)\]
\[(TRS^t) : \quad SU(k,T)S^{-1} = U(-k,T)^t. \quad (14b)\]
for the Floquet-Bloch propagator $U(k,T)$ that depends also on momentum $k$.

Thinking in term of the Floquet quasienergy $\varepsilon = i \log \lambda$ to eigenvalue $\lambda = e^{-i\varepsilon}$ of $U$, we have that Re$\varepsilon$ is preserved but Im$\varepsilon$ changes sign under the mapping $\lambda \mapsto (\lambda^*)^{-1}$. From this, we immediately obtain the relations on the spectrum of the Floquet-Bloch propagator in Eq. (8) in the main text.

For a driving protocol with discrete steps $k = 1, \ldots, n$, where the propagator of each step is $U_k = \exp[-iH_k\delta t]$ for the time step $\delta t = T/n$, the above TRS relations can be stated more explicitly. With a shift
\[H_k \mapsto H_k + \sigma_k \quad (15)\]
in the $k$-th step, and the associated scalar factor $\Gamma_k = e^{-i\sigma_k\delta t}$, we have
\[(TRS^*) : \quad SU_k S^{-1} = (\Gamma_k \Gamma_{n-k+1}^* U_{n-k+1}^*)^{-1}, \quad (16a)\]
\[(TRS^t) : \quad SU_k S^{-1} = (\Gamma_{n-k+1}/\Gamma_k) U_{n-k+1}^t . \quad (16b)\]
for the propagators $U_k$ of each step. If we multiply these equations for all $n$ steps, we see again that the propagator $U \equiv U(T) = U_n \cdots U_1$ of one period of the driving protocol obeys the relations (13), now with $\Gamma = \Gamma_1 \cdots \Gamma_n$.

II. EXPLICIT FORM OF THE NON-HERMITIAN DRIVING PROTOCOL

In the general case, the six-step protocol has $6 \times 2 \times 2^2 = 48$ complex parameters. Hermiticity reduces the number to $24$ real and $12$ complex parameters, which have been tabulated in Ref. [2] together with the constraints resulting from (fermionic or bosonic) TRS.

For the present study, we choose a restricted set of parameters, with two coupling parameters ($J$ for diagonal couplings and $J'$ for horizontal couplings), two parameters for uniform losses in the bulk ($\gamma_r$ for red and $\gamma_b$ for blue sites) and individual losses for isolated boundary sites ($\dot{\gamma}_r, \dot{\gamma}_b$ for each boundary).

Specifically, the bulk Hamiltonian has the following form, using a graphic notation that agrees with Fig. 3 in the main text:

Step 1: diagonal couplings
\[\begin{align*}
\bullet & \quad \circ \quad 1 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\cdot & \quad \circ \quad 2 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]

Step 2: horizontal couplings
\[\begin{align*}
\circ & \quad \circ \quad 3 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\circ & \quad \circ \quad 4 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]

Step 3: diagonal couplings
\[\begin{align*}
\circ & \quad \circ \quad 3 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\circ & \quad \circ \quad 4 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]

Step 4: diagonal couplings
\[\begin{align*}
\bullet & \quad \circ \quad 5 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\circ & \quad \circ \quad 6 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]

Step 5: horizontal couplings
\[\begin{align*}
\circ & \quad \circ \quad 5 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\circ & \quad \circ \quad 6 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]

Step 6: diagonal couplings
\[\begin{align*}
\bullet & \quad \circ \quad 6 \quad = \quad \begin{pmatrix} -i\gamma_r & J \\ J & -i\gamma_r \end{pmatrix} \\
\bullet & \quad \circ \quad 6 \quad = \quad \begin{pmatrix} -i\gamma_b & J \\ J & -i\gamma_b \end{pmatrix}
\end{align*}\]
FIG. 1. Sketch of an imperfect “bottom boundary”.

For consistency, we have included losses $\gamma_h$ also for the horizontal steps 2 and 5. Since these affect all sites equally, they can be absorbed in the shift $H(t) + \sigma(t)$ of the Hamiltonian, and are thus redundant.

Isolated boundary sites, i.e., lattice sites that are not coupled in one step of the driving protocol, incur individual losses specified by the parameters $\gamma_r$ and $\gamma_b$ for red and blue sites. Isolated sites occur only at boundaries, and the set of isolated sites changes during the six steps of the protocol. In the situation of Fig. 1, which shows an imperfect “bottom” boundary in generalization of Fig. 4 in the main text, the “filled red” site $A$ is isolated in steps 4 and 6, and the “hollow blue” site $B$ is isolated only in step 4. This statement about lattice sites should not be confused with a statement about states starting at the respective site. A state starting at site $A$ is a boundary state that (at perfect coupling) moves to site 1 and 2 in one and two cycles of the driving protocol. A state starting at site $B$ is a bulk state that (still at perfect coupling) returns to site $B$ in every cycle of the protocol.

Note that with TRS, a boundary has to be compatible with the symmetry operator $S = \sigma_y \otimes \sigma_y$ introduced in Fig. 3 in the main text. For example, if a “filled red” site is included also the “filled blue” site to the right must be included, as indicated by the grey ovals in Fig. 1. This implies that in steps 2 and 5, all boundary sites are coupled through the horizontal couplings $\pm J'$, such that individual losses do not occur.

In principle, the individual losses can differ for each isolated boundary site, restricted only by the corresponding TRS constraints. In the present study, we only consider the possibility of different losses on different boundaries.

III. PARAMETER CONSTRAINTS FROM TRS

The constraints on parameter values resulting from TRS can be obtained along the lines of Ref. [2], extended to the non-Hermitian case. For example, using the graphical notation of the previous section to denote the coupling parameters, we have with the symmetry operator $S = \sigma_y \otimes \sigma_y$ that

$$S \left( \bullet \leftrightarrow \circ \right)_1 S^{-1} = \left( \bullet \leftrightarrow \circ \right)_{\text{TRS}} \equiv \left( \bullet \leftrightarrow \circ \right)_6$$  \hspace{1cm} (29)$$

relating the parameters of “red” diagonal couplings in step 1 to the “blue” diagonal couplings in step 6, but

$$S \left( \rightarrow \bullet \leftrightarrow \circ \right)_{2} S^{-1} = \left( \rightarrow (\leftarrow \bullet \leftrightarrow \circ) \right)_{\text{TRS}} \equiv \left( \rightarrow \bullet \leftrightarrow \circ \right)_{5}$$  \hspace{1cm} (30)$$

which explains the minus sign between the horizontal couplings $\pm J'$ in step 2 and step 5.

From these transformations it is straightforward to obtain the constraints required for TRS. In the general case, say with

$$\left( \bullet \leftrightarrow \circ \right)_1 = \left( \begin{array}{cc} A_{r1} & B_{r1} \\ C_{r1} & D_{r1} \end{array} \right)$$  \hspace{1cm} (31)$$

for the diagonal coupling of red sites in step 1 and

$$\left( \bullet \leftrightarrow \circ \right)_6 = \left( \begin{array}{cc} A_{b6} & B_{b6} \\ C_{b6} & D_{b6} \end{array} \right)$$  \hspace{1cm} (32)$$

for the diagonal coupling of blue sites in step 6, the TRS constraints on the eight parameters $A_{r1}, \ldots, D_{b6}$ are

$$(\text{TRS}^*) : \quad A_{r1} = A_{b6}^* + \sigma_1 - \sigma_6^* , \quad B_{r1} = B_{b6}^* ,$$  \hspace{1cm} (33a)$$

$$C_{r1} = C_{b6}^* , \quad D_{r1} = D_{b6}^* + \sigma_1 - \sigma_6^* .$$  \hspace{1cm} (33b)$$

$$(\text{TRS}^\dagger) : \quad A_{r1} = A_{b6} + \sigma_1 - \sigma_6 , \quad B_{r1} = B_{b6},$$  \hspace{1cm} (33c)$$

$$C_{r1} = C_{b6} , \quad D_{r1} = D_{b6} + \sigma_1 - \sigma_6 .$$  \hspace{1cm} (33d)$$

On the other hand, with

$$\left( \rightarrow \bullet \leftrightarrow \circ \right)_{2} = \left( \begin{array}{cc} A_{f2} & B_{f2} \\ C_{f2} & D_{f2} \end{array} \right)$$  \hspace{1cm} (34)$$

for the horizontal coupling of the filled sites in step 2 and

$$\left( \rightarrow \bullet \leftrightarrow \circ \right)_{5} = \left( \begin{array}{cc} A_{f5} & B_{f5} \\ C_{f5} & D_{f5} \end{array} \right)$$  \hspace{1cm} (35)$$

for the horizontal coupling of the filled sites in step 5, the TRS constraints are

$$(\text{TRS}^*) : \quad A_{f2} = A_{f5}^* + \sigma_2 - \sigma_5^* , \quad B_{f2} = -C_{f5} ,$$  \hspace{1cm} (36a)$$

$$C_{f2} = -B_{f6}^* , \quad D_{f2} = D_{f5} + \sigma_2 - \sigma_5^* .$$  \hspace{1cm} (36b)$$

$$(\text{TRS}^\dagger) : \quad A_{f2} = A_{f5} + \sigma_2 - \sigma_5 , \quad B_{f2} = -B_{f5} ,$$  \hspace{1cm} (36c)$$

$$C_{f2} = -C_{f5} , \quad D_{f2} = D_{f5} + \sigma_2 - \sigma_5 .$$  \hspace{1cm} (36d)$$

Analogous constraints are obtained for all other parameters. In total, TRS introduces 24 constraints on the 48 complex parameters of the general protocol. For the Hermitian protocol, this number reduces to exactly the $12 + 6$ constraints for the $24 + 12$ real and complex parameters that have been listed in Ref. [2].

For the restricted set of parameters in Eqs. (17)–(28) used in the present study, where $A_{r1} = -i\gamma_r$, $B_{r1} = J$ etc., we immediately identify the TRS constraints in Eq. (5) in the main text.
IV. DETUNINGS AND DISORDER

With the minimal set of parameters used for the non-Hermitian driving protocol in the present study, diagonal (i.e., on-site) terms in the Hamiltonian arise from losses and are purely imaginary, e.g., $-i\gamma_r, -i\gamma_b$ as listed in Eqs. (17)–(28). For Figs. 5, 6 we add real diagonal terms to the Hamiltonian that account for local potentials or fields. In a photonic waveguide system, these terms account for detunings that arise from variations in the optical path length of the different waveguides.

In the translational invariant situation, we have four parameters $\Delta, \ldots, \Delta''' \in \mathbb{R}$ corresponding to the four different types of sites. Graphically, we may write the detuning term as

\[
\begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} = \begin{pmatrix}
\Delta' & \Delta'''
\Delta & \Delta''
\end{pmatrix}
\]

(37)

In general, the detunings can differ in each step of the protocol, but for the purposes of Figs. 5, 6 it suffices to consider only constant detunings.

Constraints are required to preserve TRS. Here, the symmetry operator $S$ gives (cf. Fig. 3)

\[
S \begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} S^{-1} = \begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} \overset{\text{TRS}}{\equiv} \begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} \]  (38)

Since the detunings are real, $\text{TRS}^*$ and $\text{TRS}'$ give the same constraints $\Delta = \Delta''$ and $\Delta' = \Delta'''$. In other words, TRS preserving detunings have the form

\[
\begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} = \begin{pmatrix}
\Delta' & \Delta''
\Delta & \Delta
\end{pmatrix},
\]

(39)

with only two parameters $\Delta, \Delta' \in \mathbb{R}$.

In Fig. 5 in the main text, detunings

\[
\begin{pmatrix}
\bigcirc & \bigcirc \\
\bullet & \bullet
\end{pmatrix} = \begin{pmatrix}
\Delta & \Delta
\Delta & -\Delta
\end{pmatrix}
\]

(40)

with $\Delta = 0.5$ are used to break $\text{TRS}'$ or $\text{TRS}^*$.

For the real-space propagation in panels (E), (F) in Fig. 6 in the main text, disorder is included through detunings where the parameters $\Delta, \ldots, \Delta'''$ fluctuate randomly in dependence on the lattice site. We choose the parameters from a uniform probability distribution in the interval $[-\delta, \delta]$, with $\delta = 0.2$, and independently on each site. This gives the TRS-breaking disorder used in the upper half of panels (E), (F). To preserve TRS we assign the same detuning to the two sites in a pair ($\bigcirc \bigcirc$) or ($\bullet \bullet$) that is mapped onto itself by the symmetry operator $S$, thus respecting the constraints $\Delta = \Delta''$ and $\Delta' = \Delta'''$ locally. This gives the TRS-preserving disorder used in the lower half of panels (E), (F).

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[2] B. Höckendorf, A. Alvermann, and H. Fehske, Phys. Rev. B 99, 245102 (2019).