A quantum algebraic description of D-branes on group manifolds

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Abstract: We propose an algebraic description of (untwisted) D-branes on compact group manifolds $G$ using quantum algebras related to $U_q(g)$. It reproduces the known characteristics of stable branes in the WZW models, in particular their configurations in $G$, energies as well as the set of harmonics. Both generic and degenerate branes are covered.

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1. Introduction

Recently the structure of $D$-branes in a $B$ field background has attracted much attention. The case of flat branes in a constant $B$ background has been studied extensively (see e.g. [1]), and leads to quantum spaces with a Moyal-Weyl star product. This was later generalized to non-constant, closed $B$ [2]. A rather different situation is given by $D$-branes on compact Lie groups $G$, which carry a (NSNS) $B$ field which is not closed. It has been shown, using CFT [3] and DBI (Dirac-Born-Infeld) [4] descriptions that stable branes can wrap certain conjugacy classes in the group manifold. On the other hand, the matrix model [5] and CFT calculations [6] led to a beautiful picture where, in a special limit, the macroscopic branes are formed as a bound state of $D0$-branes. Attempting to unify these various approaches, we proposed in a recent paper [7] a matrix description of $D$-branes on $SU(2)$. This led to a quantum algebra based on quantum group symmetries, which reproduced all static properties of stable $D$-branes on $SU(2)$.

In the present paper, we generalize the methods of [7] and propose a simple and compact description of all (untwisted) $D$-branes on group manifolds $G$, using quantum algebras related to $U_q(g)$. More specifically, we show that a simple algebra known for more than 10 years as reflection equation (RE) leads to precisely the same branes as the DBI approach or the WZW model. It not only reproduces their configurations in $G$, i.e. the positions of the corresponding conjugacy classes, but also a (quantized) algebra of functions on the branes which turns out to be essentially the same as given by CFT. Moreover, both generic and degenerate branes are predicted, again in agreement with the CFT results. In particular, we identify branes on $SU(N + 1)$ which are quantizations of $\mathbb{C}P^N$, and we show that they precisely correspond to the fuzzy $\mathbb{C}P^N$ constructed in [8, 9].

We do not attempt here to recover all known branes on $G$, such as twisted branes or “type B branes” [12] but concentrate on the untwisted branes. Given the success and simplicity of our description, it seems quite possible, however, that these other branes are described by RE as well. Our results can be briefly summarized as follows: $D$-branes on $G$ are described by the RE. A large class of irreps of RE corresponds to irreps of $U_q(g)$, and describes untwisted branes.

We should point out that all mathematical constructions are basically well-known. In spite of this, we tried to make the paper accessible to a wide audience, by giving the basic constructions and results in the main body of the paper while postponing many technical aspects to the Appendix. The paper can be read from a variety of viewpoints, starting from a string theorists perspective emphasizing the agreement with other approaches, but also from a more algebraic point of view given the simple and compact description of quantized adjoint orbits on $G$.

The paper is summarized as follows. Some basic facts about (untwisted) $D$-branes on compact Lie groups and their description in CFT are recalled in Section 2, with emphasis on those aspects which are useful in later considerations. We argue that the finite set of primaries of BCFT of a D-branes can be interpreted in terms of NCG, i.e. they provide a picture of
D-branes as quantum manifolds. Moreover, we claim that the appropriate symmetry algebra is a particular quantum group, replacing (in a sense) the chiral affine algebra \( \hat{\mathfrak{g}}_L \times \hat{\mathfrak{g}}_R \). The quantum manifold is defined as an associative algebra, generated by some elements subject to certain relations. In Section 3 we postulate these relations (the so-called reflection equations RE), and discuss their basic properties. In particular, we show that RE has all the required properties under the quantum group. The D-branes are then obtained from representations of RE. These are studied in Section 4, where we justify our claims. In particular, we calculate the positions of the D-branes on the group manifold, their energies, and perform the harmonic analysis on their quantized world-volumes. Our constructions are illustrated in some examples in Section 5, studying fuzzy \( \mathbb{C}P^N_q \) in more detail, and reviewing the \( SU(2) \) case in order to make clear the connection with our previous paper [7]. Finally, some technical discussions are collected in the Appendices.

2. CFT and classical description of untwisted D-branes

This section deals with the CFT description of branes in WZW models on \( G \), and their classical interpretation as certain sub-manifolds in the group manifold \( G \). All the results presented here are well known and serve only as inspiration to the algebraic considerations in the rest of the paper. The reader who is not familiar with CFT and string theory may skip this part of the paper and go directly to the Subsection 2.3.

2.1 Some Lie algebra notations

We collect some notations used throughout this paper. \( \mathfrak{g} \) denotes the (simple, finite-dimensional) Lie algebra of \( G \), with Cartan matrix \( A_{ij} = \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_j} \). Here \( \cdot \) is the Killing form which is defined for arbitrary weights, and \( \alpha_i \) are the simple roots. The set of dominant integral weights is denoted by

\[
P^+ = \{ \sum n_i \Lambda_i; \ n_i \in \mathbb{Z}_{\geq 0} \},
\]

where the fundamental weights \( \Lambda_i \) satisfy \( \alpha_i \cdot \Lambda_j = d_\alpha \delta_{ij} \), and the length of a root \( \alpha \) is \( d_\alpha = \alpha \cdot \alpha \). The Weyl vector is the sum over all positive roots, \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \). For a positive integer \( k \), one defines the “fundamental alcove” in weight space as

\[
P_k^+ = \{ \lambda \in P^+; \ \lambda \cdot \theta \leq k \}
\]

where \( \theta \) is the highest root. It is a finite set of dominant integral weights. For \( G = SU(N) \), this is explicitly \( P_k^+ = \{ \sum n_i \Lambda_i; \ \sum n_i \leq k \} \). We shall normalize the Killing form such that \( d_\theta = 1 \), so that the dual Coxeter number is given by \( g^\vee = (\rho + \frac{1}{2} \theta) \cdot \theta \), which is \( N \) for \( SU(N) \).

For any weight \( \lambda \), we define \( H_\lambda \in \mathfrak{g} \) to be the Cartan element which takes the value \( H_\lambda v_\mu = (\lambda \cdot \mu) \ v_\mu \) on vectors \( v_\mu \) with weight \( \mu \) in some representation. We shall consider only finite-dimensional representations (=modules) of \( \mathfrak{g} \). \( V_\lambda \) denotes the irreducible highest-weight module.
of $G$ with highest weight $\lambda \in P^+$, and $V_{\lambda^+}$ is the conjugate (=dual) module of $V_{\lambda}$. The defining representation of the classical matrix groups $SU(N)$, $SO(N)$, and $Sp(N)$ will be denoted by $V_N$, being $N$-dimensional.

2.2 WZW D-branes

The WZW model is specified by a group $G$ and a level $k$ \cite{10, 11}. We shall consider only simple, compact groups ($G$ will be $SU(N)$ mainly), so that the level $k$ must be a positive integer. The WZW branes can be described by boundary states $|B\rangle \in \mathcal{H}^{\text{closed}}$ respecting a set of boundary conditions. A large class of boundary conditions is of the form

$$\left( J_n + \tilde{\gamma}(\tilde{J}_{-n}) \right) |B\rangle = 0 \quad n \in \mathbb{Z} \quad (2.3)$$

where $\tilde{\gamma}$ is an auto-morphism of the affine Lie algebra $\hat{g}^1$. Here $J_n$ are the modes of the left-moving currents and $\tilde{J}_n$ are the modes of the right-moving currents. Boundary states with $\tilde{\gamma} = 1$ are called “symmetry-preserving branes” or ”untwisted branes”: these are the object of interest in this paper. The untwisted ($\tilde{\gamma} = 1$) boundary condition (2.3) breaks half of the symmetries of the WZW model $\hat{g}_L \times \hat{g}_R$ down to the vector part $\hat{g}_V$.

The condition (2.3) alone does not define a good boundary state: one must also impose open-closed string duality of the amplitude describing interactions of branes. This leads to so called Cardy (boundary) states. For the untwisted case they are labelled by $\lambda \in P_k^+$ corresponding to integrable irreps of $\hat{g}$, which are precisely the weights in the “fundamental alcove” (2.2). Therefore the untwisted branes are in one-to-one correspondence with $\lambda \in P_k^+$. The CFT description yields also an important formula for the energy of the brane $\lambda$,

$$E_\lambda = \prod_{\alpha > 0} \frac{\sin \left( \pi \frac{\alpha \cdot (\lambda + \rho)}{k + 9g^2} \right)}{\sin \left( \pi \frac{\alpha \cdot \rho}{k + 9g^2} \right)} \quad (2.4)$$

For $k \gg N$, one can expand the denominator in (2.4) to obtain a formula which compared with DBI \cite{12} shows that the leading $k$-dependence fits perfectly with the interpretation of a brane wrapping once a conjugacy class given by an element $t_\lambda$ of the maximal torus of $G$ (see the next subsection).

The CFT provides hints towards the description of branes as quantum manifolds. It is known that the dynamics of D-branes is given by open string excitations. The relevant operators, entering as building blocks of the string operators, are the primary fields of the BCFT with the symmetry algebra of the unbroken part $\hat{g}_L \times \hat{g}_R$, i.e. $\hat{g}_V$. The number of lowest conformal weight primaries is finite for any compact WZW model (in general for any RCFT). In the $k \to \infty$ limit, the primaries can be interpreted as corresponding to a (finite dimensional) algebra

\footnote{\hat{g} is the horizontal algebra of $\hat{g}$, and the Lie algebra of $G$.}
of functions on the brane (see [14, 13] and Section 4.2). For finite \( k \), the interpretation is not that clear because the candidate algebra as given in [14] is not associative. However as explained in [14, 7] for \( g = su(2) \), the algebra becomes associative after "twisting" (resulting in a modification of the product of the primary fields), so that it can be considered as an algebra of functions of a quantum manifold. Then the primaries become modules of the quantum group \( U_q(su(2)) \). We argued in [7] that the relations defining the algebra of functions on the quantum manifold is invariant under the full chiral counterpart of the chiral current algebra, i.e. under \( U_q(su(2)_L \times su(2)_R) \).

Here we shall follow the line of reasoning of [7] replacing \( g = su(2) \) by any compact, simple Lie algebra \( g \), noting that the technical arguments for twisting and associativity generalize. We shall therefore assume that one can modify the product of primary fields such that they form an associative algebra, and transform under a suitable quantum group \( U_q(\mathfrak{g}_L \times \mathfrak{g}_R) \) (or \( G_L \otimes^R G_R \)) as given below.

### 2.3 The classical description of D–branes on group manifolds.

The D-branes whose quantum description has been given in the previous subsection have a nice geometrical interpretation: they correspond to the conjugacy classes of the group manifold under the adjoint action. Here we describe some properties of those sub-manifolds. The results presented in the forthcoming sections can also be viewed as a quantization of those sub-manifolds.

Let \( G \) be the classical group manifold (we will consider mainly \( SU(N) \), but all constructions can be used for other groups such as \( SO(N) \), \( USp(N) \) as well). At the classical level, the D–branes under consideration are described by (twisted) conjugacy classes of the form

\[
C(t) = \{gt\gamma(g)^{-1}; \ g \in G\}.
\]

Here \( \gamma \) is an auto-morphism of \( G \), which is related to that of \([2,3]\). In this paper we shall consider only trivial \( \gamma \), leaving the \( \gamma \neq id \) case to a future publication. One can take \( t \) belonging to a maximal torus \( T \) of \( G \), i.e. \( t \) is a diagonal matrix for \( G = SU(N) \). Then \( C(t) \) can be viewed as homogeneous spaces (see Appendix [A.3]):

\[
C(t) \cong G/K_t.
\]

Here \( K_t = \{g \in G : [g,t] = 0\} \) is the stabilizer of \( t \in T \). “Regular” conjugacy classes are those with \( K_t = T \), and they are isomorphic to \( G/T \). In particular, their dimension is \( \text{dim}(C(t)) = \text{dim}(G) - \text{rank}(G) \). “Degenerate” conjugacy classes have a larger stability group \( K_t \), hence their dimension is smaller; e.g. at the extremal case \( C(t = 1) \) is a point. These conjugacy classes are invariant under the adjoint action

\[
G_V^{-1}C(t)G_V = C(t)
\]
of the vector subgroup $G_V \hookrightarrow G_L \times G_R$, which is diagonally embedded in the group of (left and right) motions on $G$. This reflects the breaking $\hat{g}_L \times \hat{g}_R \to \hat{g}_V$. We want to preserve this symmetry pattern in the quantum case, in a suitable sense.

The space of harmonics on $C(t)$. A lot of information about the spaces $C(t)$ can be obtained from the harmonic analysis, i.e. by decomposing scalar fields on $C(t)$ into harmonics under the action of the (vector) symmetry $G_V$. This is particularly useful here, because quantized spaces are described in terms of their algebra of functions. The decomposition of this space of functions $\mathcal{F}(C(t))$ into harmonics can be calculated explicitly using (2.6), and it must be preserved after quantization, at least up to some cutoff. Otherwise, the quantization would not be admissible. One finds (see Appendix A.3 and [13])

$$\mathcal{F}(C(t)) \cong \bigoplus_{\lambda \in P^+} \text{mult}^{(K_i)}_{\lambda^+} V_\lambda.$$  \hspace{1cm} (2.8)

Here $\lambda$ runs over all dominant integral weights $P^+$, $V_\lambda$ is the corresponding highest-weight $G$-module, and $\text{mult}^{(K_i)}_{\lambda^+}$ is the dimension of the subspace of $V_{\lambda^+}$ which is invariant under $K_i$.

Characterization of the stable $D$–branes. From the CFT [3, 13] and DBI considerations [4, 12], one finds that there is only a finite set of stable $D$–branes on $G$ (up to global motions), one for each integral weight $\lambda \in P^+$. They are given by $C(t_\lambda)$ for

$$t_\lambda = \exp\left(2\pi i \frac{H_\lambda + H_\rho}{k + g^\vee}\right).$$ \hspace{1cm} (2.9)

The restriction to $\lambda \in P^+_k$ follows from the fact that in general, different integral $\lambda$ may label the same conjugacy class. Because the exponential in (2.9) is periodic, this happens precisely if the weights are related by the affine Weyl group, which is generated by the ordinary Weyl group together with translations of the form $\lambda \to \lambda + (k + g^\vee)\frac{2a_i}{\alpha_i}$. Hence one should restrict the weights to the fundamental domain of this affine Weyl group, which is the fundamental alcove $P^+_k$ (2.2) but with $k \to k + g^\vee$.

Information about the location of these (untwisted) branes in $G$ is provided by the quantities

$$s_n = \text{tr}(g^n) = \text{tr}(t^n), \quad g \in C(t)$$ \hspace{1cm} (2.10)

which are invariant under the adjoint action (2.7). The trace is over the defining representation $V_N$ ($= V_{\Lambda_1}$ in the case of $SU(N)$, where $\Lambda_1$ is the fundamental weight) of the matrix group $G$, of dimension $N$. For the classes $C(t_\lambda)$, they can be easily calculated:

$$s_n = \text{tr}_{V_N} \left(q^{2n(H_\rho + H_\lambda)}\right) = \sum_{\nu \in V_N} e^{2\pi i n \frac{(\nu^+ + \lambda^+)_\mu}{k + g^\vee}}$$ \hspace{1cm} (2.11)

where

$$q = e^{\frac{2\pi i}{k + g^\vee}}.$$
The $s_n$ are independent functions of the weight $\lambda$ for all $n = 1, 2, ..., \text{rank}(G)$, which completely characterize the class $\mathcal{C}(t_{\lambda})$. These functions have the great advantage that their quantum analogs (3.11) can be calculated exactly.

An equivalent characterization of these conjugacy classes is provided by a characteristic equation: for any $g \in \mathcal{C}(t_{\lambda})$, the relation $P_{\lambda}(g) = 0$ holds in $\text{Mat}(V_N, \mathbb{C})$, where $P_{\lambda}$ is the polynomial

$$P_{\lambda}(x) = \prod_{\nu \in V_N} (x - q^{2(\lambda+\rho)\cdot \nu}).$$

This follows immediately from (2.9): $t_{\lambda}$ has the eigenvalues $q^{2(\lambda+\rho)\cdot \nu}$ on the weights $\nu$ of the defining representation $V_N$. Again, we will find analogous characteristic equations in the quantum case.

3. Quantum algebras and symmetries for branes

We expect that the relevant quantum spaces are described by quantum algebras $\mathcal{M}$ which transform appropriately under a quantum symmetry. To find $\mathcal{M}$ we shall make an “educated guess” based on the considerations in Section 2.2, and justify it by comparing its predictions with the results listed above. Thus first we postulate the form of the relations between generators of the quantum algebra. We expect the relations to be at most quadratic in generators, and to have appropriate covariance under the action of a quantum group which should correspond to the chiral $\hat{g}_L \times \hat{g}_R$. This quantum group will be $U_q(\hat{g}_L \times \hat{g}_R)$. Moreover we require the central terms of the algebra to be invariant under the "vector" subalgebra of this quantum symmetry. Thus our constructions mimic the symmetry pattern and its breaking by the D-branes in CFT. This is discussed in Section 3.2.

3.1 The module algebra

The discussion invoked in the end of Section 2.2 suggests that $\mathcal{M}$ should be a module algebra under some quantum group. Moreover, it suggests that this quantum group is a version of $U_q(\hat{g})$, the representations of which are parallel to those of $\hat{g}$ of the WZW model. Since we are considering matrix groups $G$, we assume that the appropriate quantum (module) algebra $\mathcal{M}$ is generated by elements $M^i_j$ with indices $i, j$ in the defining representation $V_N$ of $G$, subject to some commutation relations and constraints. With hindsight, we claim that these relations are given by the so-called reflection equation (RE) [16], which in a short notation reads

$$R_{21}M_1R_{12}M_2 = M_2R_{21}M_1R_{12}. \quad (3.1)$$

Here $R$ is the $\mathcal{R}$ matrix of $U_q(\mathfrak{g})$ in the defining representation. Displaying the indices explicitly, this means

$$(\text{RE})^k_{\ i \ j \ l} : \quad R^k_{\ a \ \ b}^{\ \ c} M^b_c R_{\ j \ d}^a M^d_l = M^k_{\ a \ c} R^a_{\ b \ \ c} M^b_{\ d} R^d_{\ j \ l}. \quad (3.2)$$

$^2$see Appendix A.2 for the mathematical definition
The indices \( \{i, j\}, \{k, l\} \) correspond to the first (1) and the second (2) vectors space \( V_N \) in (3.1). Some examples of algebras generated by RE relations are present in Section 5. For \( q = 1 \), this reduces to \([M^j_i, M^k_l] = 0\). Because \( \mathcal{M} \) should describe the quantized group manifold \( G \), we need to impose constraints which ensure that the branes are indeed embedded in such a quantum group manifold. In the case \( G = SU(N) \), these are \( \det_q(M) = 1 \) where \( \det_q \) is the so-called quantum determinant (3.14), and suitable reality conditions imposed on the generators \( M^j_i \). Both will be discussed below.

Following [7], the \( M^j_i \)'s can also be thought of as some matrices (as in Myers model [5]) out of which we can form an action invariant under the relevant quantum groups. The action has the structure \( S = \text{tr}_q(1 + ...), \) where dots represent some expressions in the \( M \)'s (the quantum trace is defined in (4.4)). The point of [7] was that for some equations of motion, the "dots"-terms vanish on classical configurations. We postulate that the equations of motion for \( M \) are given by RE (3.1). If so, then their energy is equal to

\[
E = \text{tr}_q(1). \tag{3.3}
\]

As we shall see this energy is not just a constant (as might be suggested by the notation), but it depends on the representations of the algebra, where it becomes the quantum dimension (4.4).

We should mention here that RE appeared more then 10 years ago in the context of the boundary integrable models, and is sometimes called boundary YBE [16]. Hence one might also think of (3.1) as being analogs of the boundary condition (2.3). As we shall see, RE has indeed similar symmetry properties. This is the subject of the following subsection.

### 3.2 Quantum symmetries of RE

Since \( \mathcal{M} \) is supposed to be a module algebra, we have to specify under which quantum group it transforms. The construction of the quantum symmetry algebra is a straightforward generalization of the approach of [5], replacing \( su(2) \) by \( g \). However we found it more convenient to work with a dual version of this symmetry, which leads directly to the desired results. We shall present here a simple practical version and postpone the precise mathematical definitions to Appendix A.2.

There are 2 equivalent ways to look at the symmetry of RE, involving the Hopf algebras \( G_L \otimes^R G_R \) and \( U_q(\mathfrak{g}_L \times \mathfrak{g}_R)_R \) respectively, which are dual to each other. We first assume that the matrix \( M \) transforms as

\[
M^j_i \rightarrow (s^{-1}Mt)^i_j \tag{3.4}
\]

where \( s^j_i \) and \( t^j_i \) generate the algebras \( G_L \) and \( G_R \) respectively, which both coincide with the well-known quantum groups \( \text{Fun}_q(G) \) as defined in [7] so e.g. \( s_2s_1R = Rs_1s_2, \ t_2t_1R = Rt_1t_2. \)

In (3.4) matrix multiplication is understood. This is a symmetry of RE if we impose that (the

\[3\]\( R \) with suppressed indices means \( R_{12} \).
matrix elements of) $s$ and $t$ commute with $M$, and additionally satisfy $s_2 t_1 R = R t_1 s_2$. Notice that (3.4) is a quantum analog of the action of the classical isometry group $G_L \times G_R$ on classical group element $g$ as in Section (2.3).

Symmetries become powerful only because they have a group-like structure, i.e. they can be iterated. In the above language this means that we can define a Hopf algebra (called from now on $G_L \otimes^R G_R$):

$$s_2 s_1 R = R s_1 s_2, \quad t_2 t_1 R = R t_1 t_2, \quad s_2 t_1 R = R t_1 s_2 \quad (3.5)$$

$$\Delta s = s \otimes s, \quad \Delta t = t \otimes t, \quad (3.6)$$

$$S(s) = s^{-1}, \quad \epsilon(s^i_j) = \delta^i_j, \quad S(t) = t^{-1}, \quad \epsilon(t^i_j) = \delta^i_j \quad (3.7)$$

(here $S$ is the antipode, and $\epsilon$ the counit). The inverse matrices $s^{-1}$ and $t^{-1}$ are defined after suitable further (determinant-like) constraints on $s$ and $t$ are imposed, as in [17]. Formally, $M$ is a right $G_L \otimes^R G_R$-comodule algebra; see Appendix A.2 for further details.

Furthermore, $G_L \otimes^R G_R$ can be mapped to a vector Hopf algebra $G_V$ with generators $r$, by $s^i_j \otimes 1 \rightarrow r^i_j$ and $1 \otimes t^i_j \rightarrow r^i_j$ (thus basically identifying $s = t = r$ on the rhs). The (co)action of $G_V$ on the $M$'s is then

$$M^i_j \rightarrow (r^{-1} M r)^i_j. \quad (3.8)$$

Equivalently, we can consider the Hopf algebra $U_q(g_L \times g_R)_R$ which is dual to $G_L \otimes^R G_R$. For the details we refer to Appendix A.2; we only state here that it is generated by 2 copies $U_q(g_L), U_q(g_L)$ of $U_q(g)$, which act on the the generators of $M$ as

$$(u_L \otimes u_R) \triangleright M^i_j = \pi^i_k(Su_L)M^k_l \pi^l_j(u_R) \quad (3.9)$$

where $\pi()$ is the defining representation $V_N$ of $U_q(g)$. This is a symmetry of $M$ in the usual sense, because the rhs is again an element in $M$. The “vector” part of this symmetry is obtained using the Hopf algebra map $u \in U_q(g_V) \rightarrow \Delta(u) \in U_q(g_L \times g_R)_R$. It acts on $M$ as

$$u \triangleright M^i_j = \pi^i_k(Su_1)M^k_l \pi^l_j(u_2) \quad (3.10)$$

where $u_1 \otimes u_2 = \Delta(u)$ is the standard coproduct of $U_q(g)$.

We would like to stress here two crucial points in our construction: the first is the existence of a vector sub-algebra $U_q(g_V)$ of $U_q(g_L \times g_R)_R$ (and the analogous notion for the dual $G_L \otimes^R G_R$). This is important because the central terms of $M$ which characterize its representations will be invariant only with respect to that $U_q(g_V)$ (and $G_V$). This will allow to interpret these sub-algebras as isometries of the quantum D-branes. The second point is the fact that the RE imposes very similar conditions on the symmetries and their breaking as the original BCFT WZW model described in section 2.2 does.
3.3 Central elements of RE

Below we discuss some general properties of the algebra defined by (3.1). We need to find the central elements, which are expected to characterize its irreps. This problem was solved in the second paper of [18]. The (generic) central elements of the algebra (3.1) are

\[ c_n = \text{tr}_q(M^n) \equiv \text{tr}_{V_N}(M^n v) \in \mathcal{M}, \]  

(3.11)

where the trace is taken over the defining representation \( V_N \), and

\[ v = \pi(q^{-2H_\rho}) \]  

(3.12)

is a numerical matrix which satisfies \( S^2(r) = v^{-1}rv \) for the generator \( r \) of \( G_V \). These elements \( c_n \) are independent for \( n = 1, 2, ..., \) rank \( G_V \). A proof of centrality can be found e.g. in the book [19], Section 10.3; see also Appendix A.5. Here we check only invariance under \( G_V \) (see (3.8)):

\[ c_n \rightarrow \text{tr}_q(r^{-1}M^n r) = (r^{-1})^I_J (M^n)^I_K r^I_K v^I_J \]

\[ = S(r^I_J)(M^n)^I_K v^I_J S^2(r^I_J) = (M^n)^I_K v^I_J S(S(r^I_J)r^I_J) = (M^n)^I_K v^I_J = c_n \]  

(3.13)

as required. As we shall see, the \( c_n \)'s for \( n = 1, \ldots, \text{rank}(G) - 1 \) fix the position of the brane configuration on the group manifold i.e. they are quantum analogs of the \( s_n \)'s (2.11).

There should be another central term which is the quantum analog of the ordinary determinant, which is necessary to define quantum \( SU(N) \). It is known as the quantum determinant, denoted by \( \det_q(M) \). While it can be expressed as a polynomial in \( c_n \)'s (\( n = 1, \ldots, \text{rank}(G) \)), \( \det_q(M) \) is invariant under the full chiral quantum algebra. Hence we impose the constraint

\[ 1 = \det_q(M) \]  

(3.14)

For other groups such as \( SO(N) \) and \( SP(N) \), additional constraints (which are also invariant under the full chiral quantum algebra) must be imposed. These are known and can be found in the literature [20], but their explicit form is not needed for the forthcoming considerations. Appendix A.4 contains details about how to calculate \( \det_q(M) \) and provides some explicit expressions.

3.4 Realizations of RE

In this section we find realizations (algebra homomorphisms) of the RE algebra (3.1) in terms of some other algebras. This can be viewed as an intermediate step towards finding representations. We use a technique generating new solutions out of constant solutions (i.e. trivial representations). Thus first we consider

\[ R_{21}M_1^{(0)}R_{12}M_2^{(0)} = M_2^{(0)}R_{21}M_1^{(0)}R_{12}, \]  

(3.15)

where the entries of the matrices \( M^{(0)} \) are c-numbers. Then one checks that

\[ M = L^+ M^{(0)} S(L^-) \]  

(3.16)
satisfies (3.1), if the matrices $L^\pm$ respect (see also [21], p.285)

$$RL^+_2 L^+_1 = L^+_1 L^+_2 R, \quad RL^+_2 L^-_1 = L^-_1 L^+_2 R.$$  \hspace{1cm} (3.17)

Notice that $\det_q(M) = \det_q(M^{(0)})$ due to chiral invariance of the $q$-determinant. Clearly the form of (3.16) is closely related to the $GL \otimes GR$ invariance of the RE. Thus we have trade our original problem to the problem of finding matrices $L^\pm$ respecting (3.17). Luckily this is known for a long time due to the famous work of Faddeev, Reshetikhin and Takhtajan [17], who noted that (3.1) together with the determinant–condition (and others for groups other than $SU(N)$) provides one possible definition of the quantized universal enveloping algebra $U_q(g)$.

More precisely, they showed that $L^\pm$ can be expressed in terms of generators of the $U_q(g)$ algebra as follows

$$L^+ = (id \otimes \pi)(\mathcal{R}), \quad L^- = (\pi \otimes id)(\mathcal{R}^{-1})$$  \hspace{1cm} (3.18)

where $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ is the universal R-matrix for $U_q(g)$, and $\pi$ is the defining representation of $U_q(g)$\footnote{The R-matrix of (3.2) is $R_{ij}^{kl} = \pi_j^i(\mathcal{R}_1)\pi_k^l(\mathcal{R}_2)$.}. In order to be more transparent, we write the component form of (3.18): $L^+_j = \mathcal{R}_1 \pi(\mathcal{R}_2)_j^i$, $L^-_j = \mathcal{R}^{-1}_2 \pi(\mathcal{R}^{-1}_1)_{-j}^i$. One can also show that $SL^-$ (which we shall need later) is

$$SL^- = (\pi \otimes id)(\mathcal{R}).$$  \hspace{1cm} (3.19)

The reason why (3.18) respect (3.17) is the YBE equation for $\mathcal{R}$, written in several equivalent forms

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$  \hspace{1cm} (3.20)

$$\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}^{-1}_{12} = \mathcal{R}^{-1}_{12}\mathcal{R}_{23}\mathcal{R}_{13}$$  \hspace{1cm} (3.21)

$$\mathcal{R}^{-1}_{13}\mathcal{R}^{-1}_{23}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}^{-1}_{23}\mathcal{R}^{-1}_{13}.$$  \hspace{1cm} (3.22)

The action of $1 \otimes \pi \otimes \pi$ in the first line, $\pi \otimes 1 \otimes \pi$ in the second line, and $\pi \otimes \pi \otimes 1$ in the third line immediately produces (3.17). It is useful to realize that $L^+$ are lower triangular matrices with $X^+\alpha$'s below the diagonal, and $L^-$ are upper triangular matrices with $X^-\alpha$'s above the diagonal. Explicitly, for $sl_2$ one has

$$L^+ = \begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2}\lambda X^+ & q^{-H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H/2} & -q^{1/2}\lambda X^- \\ 0 & q^{H/2} \end{pmatrix}$$  \hspace{1cm} (3.23)

The form of the solution (3.18) shows that $M$ generates a sub-algebra of $U_q(g)$. The sub-algebra depends on $M^{(0)}$. We will not discuss the most general $M^{(0)}$ here (see e.g. [18]), but consider only the most obvious solution, which is a diagonal matrix. The specific values of the diagonal entries do not change the algebra generated by the elements of $M$, because they simply multiply some entries of $M$. The other, non-diagonal $M^{(0)}$'s presumably also correspond to some
branes: we hope to come back to this issue in a future paper. Using \[17\], we conclude that the algebra generated by the elements of $M$ is essentially $U_q(g)$. As we will see, choosing a definite representation of $U_q(g)$ then corresponds to choosing a brane configuration, and determines the algebra of function on the brane. To be explicit, we give the solution for $g = sl_2$ and $M^{(0)} = \text{diag}(1,1)$:

$$M = L^+ M^{(0)} S(L^-) = \begin{pmatrix} q^H & q^{-\frac{1}{2}} \lambda q^{H/2} X_- \\ q^{-\frac{1}{2}} \lambda X_+ & q^{-H} + q^{-1} \lambda^2 X_+ X_- \end{pmatrix}$$ \hspace{1cm} (3.24)

One can verify that $\det_q(M) = 1$, according to \[A.17\].

### 3.5 Covariance

We show in Appendix \[A.2\] that for any solutions of the form $M = L^+ M^{(0)} S(L^-)$ where $M^{(0)}$ is a constant solution of the RE, the “vector” rotations (3.10) can be realized as quantum adjoint action:

$$u \triangleright M^i_j = \pi_k^i (Su_1) M^k_l \pi_j^l (u_2) = u_1 M^i_j S u_2 \in \mathcal{M}$$ \hspace{1cm} (3.25)

for $u \in \mathcal{M}$, where $\pi()$ is the defining representation $V_N$ of $U_q(g)$. Here we consider $\mathcal{M} \subset U_q(g)$ so that $\Delta(u) = u_1 \otimes u_2$ is defined in $U_q(g) \otimes U_q(g)$, nevertheless the rhs is in $\mathcal{M}$. This is as it should be in a quantum theory: the action of a symmetry is implemented by a conjugation in the algebra of operators. It will be essential later to do the harmonic analysis on the branes.

### 3.6 Reality structure

An algebra $\mathcal{M}$ can be considered as a quantized (algebra of complex-valued functions on a) space only if it is equipped with a $\ast$-structure, i.e. an anti-linear (anti)-involution. For classical unitary matrices, the condition would be $M^\dagger = M^{-1}$. To find the correct quantum version is a bit tricky; we determine it by requiring that on finite-dimensional representations of $M = L^+ SL^-$ (i.e. on the branes, see below), the $\ast$ will become the usual matrix adjoint. In term of the generators of $U_q(g)$, this means that $(X_i^\pm)^\ast = X_i^\mp$, $H_i^\ast = H_i$. In the $SU(2)$ case, this leads to

$$\begin{pmatrix} a^\ast & b^\ast \\ c^\ast & d^\ast \end{pmatrix} = \begin{pmatrix} a^{-1} & -qca^{-1} \\ -qca^{-1} & q_d a^{-2} + a - q^2 a^{-1} \end{pmatrix};$$ \hspace{1cm} (3.26)

$a^{-1}$ indeed exists on the irreps of $\mathcal{M}$ considered here. A closed form for this star structure for general $g$ could also be given, but shall be omitted here.

### 4. Representations of $\mathcal{M}$ and quantum $D$–branes

By construction, the $M^i_j$ can be considered as quantized coordinate functions on $G$, defining some kind of quantization of the manifold $G$. However, we are interested here in the quantization of the orbits $C(t_\lambda)$, which are submanifolds of $G$. We claim that they are described by irreps (fixed
by the set of Casimirs) \( \pi_\lambda : M \rightarrow Mat(V_\lambda, \mathbb{C}) \) of \( M \). Indeed, the map \( \pi_\lambda \) can be considered as the dual of the embedding map \( C(t_\lambda) \hookrightarrow G \). This will allow us to make statements on the location of the branes in \( G \).

Consider an irreducible representation of \( M \). The Casimirs \( c_n \) (3.11) then take distinct values which can be calculated. Moreover, they are invariant under (vector) rotations as shown in (3.13). In view of their form (3.11), this suggests that an irrep of \( M \) should be considered as quantization of (the algebra of functions on) some conjugacy class \( C(t_\lambda) \), the position of which is determined by the values of the Casimirs \( c_n \).

We will show that the irreps of \( M \) describe indeed precisely the stable D-branes. Since the algebra \( M \) is\(^5\) the direct sum of the corresponding representations, the whole group manifold is recovered in the limit \( k \rightarrow \infty \) where the branes become dense. To confirm this interpretation, we will calculate the position of the branes on the group manifold, and study their geometry by performing the harmonic analysis on the branes, i.e. by determining the set of harmonics.

Here we shall consider only those representations of \( M \) which arise from \( M^{(0)} = 1 \) i.e. \( M = L^+SL^- \) (3.10). Then the representations of the algebra \( M \) coincide with those of \( U_q(g) \), which are largely understood, although quite complicated at roots of unity. The fact relevant for us is that representations \( V_\lambda \) of \( U_q(g) \) with \( \lambda \in P^+_k \) have the following properties:

- they are unitary, i.e. \( \ast \) reps of \( M \) with respect to the \( \ast \) structure of Section 3.6 (see [22])
- their quantum-dimension \( \dim_q(V_\lambda) = \text{tr}_{V_\lambda}(q^{2H_\rho}) \) given in (4.4) is positive [15]
- \( \lambda \) corresponds precisely to the integrable modules of the affine Lie algebra \( \hat{g} \) which governs the CFT.

The representations belonging to the boundary of \( P^+_k \) will correspond to the degenerate branes.

Having characterized the admissible representations \( V_\lambda \), we propose that the representation of \( M \) on \( V_\lambda \) for \( \lambda \in P^+_k \) is a quantized or “fuzzy” D–brane, denoted by \( D_\lambda \). It is an algebra of maps from \( V_\lambda \) to \( V_\lambda \) which transforms under the quantum adjoint action (3.25) of \( U_q(g) \). For “small” weights\(^6\) \( \lambda \), this algebra coincides with \( Mat(V_\lambda) \). There are some modifications for “large” weights \( \lambda \) because \( q \) is a root of unity, which will be discussed in Section 4.2. The reason is that \( Mat(V_\lambda) \) then contains unphysical degrees of freedom which should be truncated.

A first justification is that there is indeed a one–to–one correspondence between the (untwisted) branes in string theory and these quantum branes, since both are labeled by \( \lambda \in P^+_k \). To give a more detailed comparison, we calculate the traces (3.11), derive a characteristic equation, and then perform the harmonic analysis on \( D_\lambda \). Furthermore, the energy (2.4) of the branes in string theory will be recovered precisely in terms of the quantum dimension.

\(^5\)more precisely the semi-simple quotient of \( M \), see Section 4.2
\(^6\)see Section 4.2
4.1 Value of the central terms

The values of the Casimirs \( c_n \) on \( D_\lambda \) are calculated in Appendix A.6:

\[
c_0 = \text{tr}_{V_N} \left( q^{-2H_\rho} \right) = \dim_q(V_N), \tag{4.1}
\]

\[
c_1(\lambda) = \text{tr}_{V_N} \left( q^{2(H_\nu + H_\lambda)} \right), \tag{4.2}
\]

\[
c_n(\lambda) = \sum_{\nu \in V_N; \lambda + \nu \in P_k^+} q^{2n((\lambda + \rho) \cdot \nu - \lambda_N \cdot \rho)} \frac{\dim_q(V_{\lambda + \nu})}{\dim_q(V_\lambda)}, \quad n \geq 1. \tag{4.3}
\]

Here \( \lambda_N \) is the highest weight of the defining representation \( V_N \), and the sum in (4.3) goes over all \( \nu \in V_N \) such that \( \lambda + \nu \) lies in \( P_k^+ \). \( c_0 \) is \( \lambda \)-independent uninteresting number.

The value of \( c_1(\lambda) \) agrees with the corresponding value (2.11) of \( s_1 \) on the classical conjugacy classes \( C(t_\lambda) \). For \( n \geq 2 \), the values of \( c_n(\lambda) \) agree only approximately with \( s_n \) on \( C(t_\lambda) \), more precisely they agree if \( \frac{\dim(V_{\lambda + \nu})}{q^{\dim(V_\lambda)}} \approx 1 \), which holds provided \( \lambda \) is large (hence \( k \) must be large too). In particular, this holds for branes which are not “too close” to the unit element. This discrepancy for small \( \lambda \) is perhaps not too surprising, since the higher–order Casimirs are defined in terms of non-commutative coordinates and are hence subject to operator–ordering ambiguities.

We should emphasize here that the agreement of the values of \( c_n \) with their classical counterparts (2.11) shows that the \( M \)'s are indeed very reasonable variables to describe the branes.

Hence we see that the positions and the “size” of the branes essentially agree with the results from string theory. In particular, their size shrinks to zero if \( \lambda \) approaches a corner of \( P_k^+ \), as can be seen easily in the \( SU(2) \) case [7]: as \( \lambda \) goes from 0 to \( k \), the branes start at the identity \( e \), grow up to the equator, and then shrink again around \( -e \). We will see that the algebra of functions on \( D_\lambda \) precisely reflects this behavior; however this is more subtle and will be discussed below. All of this is fundamentally tied to the fact that \( q \) is a root of unity.

Furthermore, the quantum dimension of the representation space \( V_\lambda \) is

\[
\dim_q(V_\lambda) = \text{tr}_q(1) = \text{tr}_{V_\lambda} \left( q^{2H_\rho} \right) = \prod_{\alpha > 0} \sin(\pi \frac{\alpha(\lambda + \rho)}{k + g}) = \frac{\sin(\pi \frac{\alpha^2}{k + g})}{\sin(\pi \frac{\alpha}{k + g})}. \tag{4.4}
\]

The last equality above follows from Weyl’s character formula. According to the interpretation (1.3) it should be the energy of the D-brane, and this is indeed the case (see (2.4)).

Finally, we show in Appendix A.7 that the generators of \( \mathcal{M} \) satisfy the following characteristic equation on \( D_\lambda \):

\[
P_\lambda(M) = \prod_{\nu \in V_N} \left( M - q^{2(\lambda + \rho) \cdot \nu - 2\lambda_N \cdot \rho} \right) = 0. \tag{4.5}
\]

Here the usual matrix multiplication of the \( M_j^i \) is understood. Again, this (almost) matches with the classical version (2.12).
4.2 The space of harmonics on $D_\lambda$.

As discussed in Section 3, we must finally match the space of functions or harmonics on $D_\lambda$ with the ones on $C(t_\lambda)$, up to some cutoff. Using covariance (3.22), this amounts to calculating the decomposition of $\mathcal{M}$ generated by (3.16) characterized by $\lambda \in P_k^+$ under the quantum adjoint action of $U_q(\mathfrak{g})$ (3.10). i.e. decomposing $V_\lambda \otimes V_\lambda^*$ under $U_q(\mathfrak{g})$. To simplify the analysis, we assume first that $\lambda$ is not too large, so that this tensor product is completely reducible. Then $D_\lambda$ coincides with the matrix algebra acting on $V_\lambda$,

$$D_\lambda \cong \text{Mat}(V_\lambda) = V_\lambda \otimes V_\lambda^* \cong \bigoplus \mu N^{\mu}_{\lambda \lambda^+} V_\mu,$$

(4.6)

where $N^{\mu}_{\lambda \lambda^+}$ are the usual fusion rules of $\mathfrak{g}$ which can be calculated explicitly using formula (A.10). Here $\lambda^+$ is the conjugate weight to $\lambda$, so that $V_\lambda^* \cong V_{\lambda^+}$. This has a simple geometrical meaning if $\mu$ is small enough (smaller than all nonzero Dynkin labels of $\lambda$, roughly speaking; see Appendix A.3 for details): then

$$N^{\mu}_{\lambda \lambda^+} = \text{mult}^{(K_\lambda)}_{\mu^+},$$

(4.7)

where $K_\lambda \subset G$ is the stabilizer group of $\lambda$, and $\text{mult}^{(K_\lambda)}_{\mu^+}$ is the dimension of the subspace of $V_{\mu^+}$ which is invariant under $K_\lambda$. This is proved in Appendix A.3. Note in particular that the mode structure (for small $\mu$) does not depend on the particular value of $\lambda$, only on its stabilizer $K_\lambda$. Comparing this with the decomposition (2.8) of $\mathcal{F}(C(t_\lambda))$, we see that indeed

$$D_\lambda \cong \mathcal{F}(C(t'_\lambda))$$

(4.8)

up to some cutoff in $\mu$, where $t'_\lambda = \exp(2\pi i \frac{H_\lambda}{k + g^\vee})$. This differs slightly from (2.9), by a shift $\lambda \to \lambda + \rho$. It implies that degenerate branes do occur in the our quantum algebraic description, because $\lambda$ may be invariant under a nontrivial subgroup $K_\lambda \neq T$. These degenerate branes have smaller dimensions than the regular ones. An example for this is fuzzy $\mathbb{C}P^N$, which will be discussed in some detail below.

Here we differ from [13] who identify only regular $D$–branes in the CFT description, arguing that $\lambda + \rho$ is always regular. This is due to a particular limiting procedure for $k \to \infty$ which was chosen in [13]. We assume $k$ to be large but finite, and find that degenerate branes do occur. This is in agreement with the CFT description of harmonics on $D_\lambda$, as will be discussed below. Also, note that (4.8) reconciles the results (4.2), (2.11) on the position of the branes with their mode structure as found in CFT.

Now we consider the general case where the tensor product $\text{Mat}(V_\lambda) = V_\lambda \otimes V_\lambda^*$ may not be completely reducible. Then $\text{Mat}(V_\lambda) = V_\lambda \otimes V_\lambda^*$ contains non-classical representations with vanishing quantum dimension, which have no obvious interpretation. However, there is a well–known remedy: one can replace the full tensor product by the so-called “truncated tensor

7 roughly speaking if $\lambda = \sum n_i \Lambda_i$, then $\sum_i n_i < \frac{1}{2}(k + g^\vee)$.

8 which acts by the (co)adjoint action on weights
product”, which amounts to discarding the representations with \( \dim_q = 0 \). This gives a decomposition into irreps

\[
D_\lambda \cong V_\lambda \otimes V_\lambda^* \cong \bigoplus_{\mu \in P^+_k} N_{\lambda \lambda}^\mu V_\mu
\]

involving only modules \( V_\mu \) of positive quantum dimension. These \( N_{\lambda \lambda}^\mu \) are known to coincide with the fusion rules for integrable modules of the affine Lie algebra \( \hat{g} \) at level \( k \), and can be calculated explicitly. These fusion rules in turn coincide (see e.g. [13]) with the multiplicities of harmonics on the D-branes in the CFT description, i.e. primary (boundary) fields.

We conclude that the structure of harmonics on \( D_\lambda \), (4.9) is in complete agreement with the CFT results. Moreover, it is known (see also [13]) that the structure constants of the corresponding boundary operators are essentially given by the 6j symbols of \( U_q(\mathfrak{g}) \), which in turn are precisely the structure constants of the algebra of functions on \( D_\lambda \), as explained in [23]. Therefore our quantum algebraic description not only reproduces the correct set of boundary fields, but also essentially captures their algebra in (B)CFT.

Finally, it is interesting to note that branes \( D_\lambda \) which are “almost” degenerate (i.e. for \( \lambda \) near some boundary of \( P^+_k \)) have only few modes \( \mu \) in some directions and should therefore be interpreted as degenerated branes with “thin”, but finite walls. They interpolate between branes of different dimensions.

5. Examples

5.1 Fuzzy \( \mathbb{C}P^N_q \)

Particularly interesting examples of degenerate conjugacy classes are the complex projective spaces \( \mathbb{C}P^N_q \). We shall demonstrate the scope of our general results by extracting some explicit formulae for this special case. This gives a \( q \)-deformation of the fuzzy \( \mathbb{C}P^N_q \) discussed in [8, 9].

We first give a more explicit description of branes on \( SU(N) \). Let \( \lambda^a = (\lambda^a)^\alpha_\beta \) for \( a = 1, 2, ..., N^2 - 1 \) be the \( q \)-deformed Gell-Mann matrices, i.e. the intertwiners \( (N) \otimes (N) \to (N^2 - 1) \) for \( U_q(\mathfrak{su}(N)) \). We can then parameterize the matrix \( M (= L^+SL^- \text{ acting on } V_\lambda) \) as

\[
M = \sum_a \xi_a \lambda^a + \xi_0 \lambda^0
\]

where we set \( \lambda^0 \equiv 1 \). The \( \xi_a \) will be generators of a non-commutative algebra. The matrices \( \lambda_a \) satisfy

\[
\lambda^a \lambda^b = \frac{1}{\dim_q(V_N)} g^{ab} + (d^{ab} + f^{ab} c)\lambda
\]

\footnote{Note that the calculation of the Casimirs in Section [4.1] is still valid, because \( V_\lambda \) is always an irrep.}

\footnote{This is just the condition on \( \mu \) discussed before [4.3].}
where \( g^{ab}, d^{ab}_{c} \) and \( f^{ab}_{c} \) are invariant tensors in a suitable normalization, and \( \text{tr}_q(\lambda^a) = 0 \) (for \( a \neq 0 \)). We can now express the Casimirs \( c_n \) \((4.3)\) in terms of the new generators:

\[
c_1 = \text{tr}_q(M) = \xi_0 \dim_q(V_N),
\]

\[
c_2 = g^{ab} \xi_a \xi_b + \xi_0^2 \dim_q(V_N),
\]

\(\text{etc, which are numbers on each } D_\lambda. \text{ An immediate consequence of } (5.3) \text{ is}

\[
[\xi_0, \xi_a] = 0
\]

for all \( a \). One can show furthermore that the reflection equation \((3.1)\), which is equivalent to the statement that the \((q-)\)antisymmetric part of \( MM \) vanishes, implies that

\[
f^{ab}_{c} \xi_a \xi_b = \alpha \xi_0 \xi_c.
\]

On a given brane \( D_\lambda, \xi_0 \) is a number determined by \((5.3)\), while \( \alpha \) is a (universal) constant which can be determined explicitly, as indicated below.

\((5.6)\) and \((5.7)\) hold for all branes \( D_\lambda \). Now consider \( \mathbb{C}P^{N-1} \cong SU(N)/U(N-1) \), which is the conjugacy class through \( \lambda = n \Lambda_1 \) (or equivalently \( \lambda = n \Lambda_N \)) where \( \Lambda_i \) are the fundamental weights; indeed, the stabilizer group for \( n \Lambda_1 \) is \( U(N-1) \). The quantization of \( \mathbb{C}P^{N-1} \) is therefore the brane \( D_\lambda \). It is characterized by a further relation among the generators \( \xi_a \), which has the form

\[
d^{ab}_{c} \xi_a \xi_b = \beta_n \xi_c
\]

where the number \( \beta_n \) can be determined explicitly as indicated below. For \( q = 1 \), these relations reduce to the ones given in \([8]\). \((5.8)\) can be derived using the results in Section 4.2: It is easy to see using \((A.10)\) that

\[
D_{n \Lambda_1} \cong \oplus_n(n, 0, ..., 0, n)
\]

up to some cutoff, where \((k_1, ..., k_N)\) denotes the highest-weight representation with Dynkin labels \( k_1, ..., k_N \). In particular, all multiplicities are one. This implies that the function \( d^{ab}_{c} \xi_a \xi_b \) on \( D_{n \Lambda_1} \) must be proportional to \( \xi_c \), because it transforms as \((1, 0, ..., 0, 1)\) (which is the adjoint). Hence \((5.8)\) follows.

The constant \( \alpha \) in \((5.7)\) can be calculated either by working out RE explicitly, or by specializing \((5.7)\) for \( D_{\Lambda_1} \). We shall only indicate this here: On \( D_{\Lambda_1}, \xi_a = c \lambda_a \) for some \( c \in \mathbb{C} \). Plugging this into \((5.7)\), one finds \( c f^{ab}_{c} \lambda_a \lambda_b = \alpha \xi_0 \lambda_c \), and \( c^2 g^{ab}_{c} \lambda_a \lambda_b + \xi_0^2 \dim_q(V_N) = c_2 \). Calculating \( \xi_0 \) and the Casimirs explicitly on \( D_{\Lambda_1} \), one obtains \( \alpha \) which vanishes as \( q \to 1 \). Similarly using the explicit value of \( c_3 \) given in Section 4.1, one can also determine \( \beta_n \). Alternatively, they be calculated using creation - and annihilation operator techniques of \([24]\), \([23]\).

In any case, we recover the relations of fuzzy \( \mathbb{C}P^{N-1} \) as given in \([8]\) in the limit \( q \to 1 \). As an algebra, it is in fact identical to it, as long as \( k \) is sufficiently large.
5.2 $G = SU(2)$ model

In this section we shall show how one can recover the results of [7] from the general formalism we discussed so far. The representation of the RE given by $L^\pm$ operators and $M^{(0)} = \text{diag}(1,1)$ is

$$M = L^+ M^{(0)} S(L^-) = \begin{pmatrix} q^H & q^{-\frac{1}{2}} \lambda q^{H/2} X_- \\ q^{-\frac{1}{2}} \lambda X_+ & q^{-H} + q^{-1} \lambda^2 X_+ X_- \end{pmatrix}$$

(5.10)

Let us parameterize the $M$ matrix as

$$M = \begin{pmatrix} M^4 - iM^0 & -iq^{-3/2} \sqrt{2}[M^+] \\ iq^{-1/2} \sqrt{2}[M^-] & M^4 + iq^{-2}M^0 \end{pmatrix}$$

(5.11)

(c.p. (5.1)), then RE is equivalent to

$$[M^4, M^l] = 0, \quad c_{ij}^l M^i M^j = i(q - q^{-1})M^4 M^l$$

(5.12)

In order to calculate the central terms we need

$$v = \pi(q^{-2H_\nu}) = \pi(q^{-H}) = \text{diag}(q^{-1}, q)$$

(5.13)

so that (using (5.11), (A.17))

$$c_1 = \text{tr}_q(M) = q^{-a} + qd = [2] M^4$$

(5.14)

$$c_2 = \text{tr}_q(M^2) = [2] ((M^4)^2 - q^{-2}g_{ij} M^i M^j)$$

(5.15)

$$\det_q(M) = (M^4)^2 + (M^0)^2 - q^{-1} M^+ M^- - q M^- M^+ = (M^4)^2 + g_{ij} M^i M^j.$$ 

(5.16)

Only $\det_q(M)$ is invariant under $U_q(\mathfrak{g}_L \times \mathfrak{g}_R).$ The explicit value of $M^4 = c_1/[2]$ is obtained from

$$M^4 = \left[\frac{1}{2}\right](q^{-a} + qd) = \left[\frac{1}{2}\right](q^{H-1} + q^{-(H-1)} + \lambda^2 X_+ X_-)$$

(5.17)

which is proportional to the standard Casimir of $U_q(su(2)).$ On the n-th brane $D_n,$ $H$ takes the value $-n$ on the lowest weight vector, thus $M^4 = \cos\left(\frac{(n+1)\pi}{k+2}\right)/\cos\left(\frac{\pi}{k+2}\right).$ If the square of radius of the quantum $S^3$ is chosen to be $\det_q(M) = k$ (which is the value given by the supergravity solution for the background), $g_{ij} M^i M^j$ leads to the correct formulae for the square of the radius of the n-th branes.

6. Conclusion

In this paper we propose a simple and compact description of all (untwisted) D-branes on group manifolds $G$ based on the reflection equation RE. The model can be viewed as a finite matrix model in the spirit of the non-abelian DBI model of D0-branes [8], but contrary to the latter it yields results well beyond the $1/k$ approximation. In fact, the model properly describes all branes on the group manifold regardless of their positions. This covers an astonishing wealth
of data on the configurations and properties of branes such as their positions and spaces of functions, which are shown to be in very good agreement with the CFT data. It also shows that \( M \) is a very reasonable variable to describe the branes. Our construction also sheds light on the fact that the energies of these branes are given by so-called quantum dimensions.

The branes are uniquely given by certain “canonical” irreps of the \( \mathcal{RE} \) algebra, and their world-volume can be interpreted as quantum manifolds. The characteristic feature of our construction is the covariance of \( \mathcal{RE} \) under a quantum analog of the group of isometries \( G_L \times G_R \) of \( G \). A given brane configuration breaks it to the diagonal (quantum) \( G_V \), an analog of the classical vector symmetry \( G_V \).

Let us also mention that the methods worked out in this paper might also serve as tools describing branes in RR background. E.g. it is known [28] that for \( G = SU(2) = S^3 \) the dynamics of branes is very similar for both NSNS and RR backgrounds.

It should be clear to the reader that the present paper does not cover all aspects of branes physics on group manifolds. For example, we did not study all representations of \( \mathcal{RE} \), only the most obvious ones which are induced by the algebra map \( \mathcal{RE} \to U_q(g) \). There exist other representations of \( \mathcal{RE} \), some of which can be investigated using the technique in Section 3.4, some of which may be entirely different. One may hope that all of the known \( D \)-branes on groups, including those not discussed here such as twisted branes or “type B”-branes, can be described in this way. We plan to investigate this further in a future publication. Moreover, we did not touch here the dynamical aspects of \( D \)-branes, such as their excitations and interactions. For this it may be necessary to extend the algebraic content presented here, and the well-developed theory of quantum groups may become very useful.

The paper has also an interesting mathematical side. The general construction of quantized branes presented here yields immediately a variety of specify examples of finite (“fuzzy”) quantum spaces, including \( \mathbb{CP}^N_q \). They may serve as useful testing grounds for noncommutative field theories, which can be defined in a very clean way on fuzzy spaces, being finite. This should lead to further insights into the problems encountered recently on other spaces with star-products. Some work in that direction can be found for example in [24, 23, 26].

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A. Appendix: technical details

A.1 Some properties of $U_q(g)$

We collect here some definitions, in order to establish the notations. We basically follow the conventions of [19]. $g$ is a simple Lie algebra, with Cartan matrix $A_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j^2}$. The generators $X_{i}^{\pm}, H_i$ of $U_q(g)$ satisfy the relations

$$[H_i, H_j] = 0, \quad [H_i, X_{j}^{\pm}] = \pm A_{ij} X_{j}^{\pm}, \quad (A.1)$$

$$[X_{i}^{+}, X_{j}^{-}] = \delta_{ij} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{ij} [H_i] q_i \quad (A.2)$$

where $q_i = q^{d_i}$. Comultiplication and antipode are defined by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_{i}^{\pm}) = X_{i}^{\pm} \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_{i}^{\pm},$$

$$S(H_i) = -H_i, \quad S(X_{i}^{\pm}) = -q^{\pm d_i} X_{i}^{\pm} \quad (A.3)$$

The coproduct is conveniently written in Sweedler-notation as $\Delta(u) = u_1 \otimes u_2$, for $u \in U_q(g)$, where a summation is implied. It is easy to verify that $S^2(u) = q^{2H_\rho} u q^{-2H_\rho}$ for all $u \in U_q(g)$, where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is the Weyl vector. This is used in the definition of the quantum traces $\langle \lambda | \langle \xi |$, (4.4).

A.2 The dual symmetries of the reflection equation.

Let $G$ and $U$ be Hopf algebras. An algebra $M$ is a (left) $U$-module algebra if there is an action $\triangleright : U \times M \rightarrow M$ such that $u \triangleright (mn) = (u_1 \triangleright m)(u_2 \triangleright n)$ and $(uv) \triangleright m = u \triangleright (v \triangleright m)$ for $m, n \in M$ and $u, v \in U$, where $\Delta(u) = u_1 \otimes u_2$. $M$ is a (right) $G$-comodule algebra if there is a coaction $\nabla : M \rightarrow M \otimes G$ which is an algebra map and satisfies $(id \otimes \Delta) \nabla(m) = (\nabla \otimes id) \nabla(m)$. These are dual concepts: if $G$ and $U$ are dually paired Hopf algebras (see e.g. [18]), then a (right) $G$ - comodule algebra $M$ with coaction $\nabla$ is automatically a (left) $U$ – module algebra by

$$u \triangleright M = \langle \nabla(M), u \rangle \quad (A.4)$$

where $\langle m \triangleright a, u \rangle = m \langle a, u \rangle$, and vice versa.

This is exactly our situation: The Hopf algebra $G_L \otimes^R G_R$ (3.3) is dual to $U_q(g_L \times g_R)_R$, which as an algebra is the usual tensor product $U_q(g_L) \otimes U_q(g_R)$, but has the twisted Hopf structure

$$\Delta_R : U_q^L \otimes U_q^R \rightarrow (U_q^L \otimes U_q^R) \otimes (U_q^L \otimes U_q^R),$$

$$u^L \otimes u^R \mapsto \mathcal{F} (u_1^L \otimes u_1^R) \otimes (u_2^L \otimes u_2^R) \mathcal{F}^{-1} \quad (A.5)$$

with $\mathcal{F} = 1 \otimes R^{-1} \otimes 1$. This is a special case of a “Drinfeld twist”, which provides also a corresponding antipode and counit (and $R$ - matrix; for the general theory of twisting we refer
to [13], and [19, Section 2.3). The dual evaluation $\langle , \rangle$ between $G_L \otimes G_R$ and $U_q(g_L \times g_R)$ is defined componentwise, using the standard dualities of $G_L,R$ with $U_q(g_L,R)$.

The action of $U_q(g_L \times g_R)$ on $\mathcal{M}$ which is dual to (3.4) then comes out as

$$(u_l \otimes u_R) \triangleright M_{ij}^k = M_{ij}^k (S(s^j_k) t_j^i, u_l \otimes u_R) = \pi(Su_L) \pi(u_R), \quad (A.6)$$

as in (3.9). Moreover, there is a Hopf-algebra map $u \in U_q(g_V) \rightarrow \Delta(u) \in U_q(g_L \times g_R)$ where $\Delta$ is the usual coproduct. This defines the vector sub-algebra $U_q(g_V)$. It induces on $\mathcal{M}$ the action (3.10), which is again dual to the coaction (3.8).

At roots of unity, these dualities are somewhat more subtle. We will not worry about this, because covariance of the reflection equation under $U_q(g_L \times g_R)$ can also be verified directly.

### A.3 The harmonics on the branes

#### The modes on $C(t)$ (2.8)

Consider the map

$$G/K_t \rightarrow C(t),$$

$$gK_t \mapsto gtg^{-1}$$

which is clearly well-defined and bijective. It is also compatible with the group actions, in the sense that the adjoint action of $G$ on $C(t)$ translates into the left action on $G/K_t$. Hence we want to decompose functions on $G/K_t$ under the left action of $G$.

Functions on $G/K_t$ can be considered as functions on $G$ which are invariant under the right action of $K_t$, and this correspondence is one-to-one (because this action is free). Now the Peter-Weyl theorem states that the space of functions on $G$ is isomorphic as a bimodule to

$$\mathcal{F}(G) \cong \bigoplus_{\lambda \in P^+} V_\lambda \otimes V_\lambda^*.$$  \hspace{1cm} (A.7)

Here $\lambda$ runs over all dominant integral weights, and $V_\lambda$ is the corresponding highest-weight module. Let $\text{mult}_{\lambda^+}^{(K_t)}$ be the dimension of the subspace of $V_\lambda^* \equiv V_\lambda^+$ which is invariant under the action of $K_t$. Then

$$\mathcal{F}(C(t)) \cong \bigoplus_{\lambda \in P^+} \text{mult}_{\lambda^+}^{(K_t)} V_\lambda$$

(A.8)

follows.

#### The modes on $D_\lambda$ and proof of (4.7)

We are looking for the Littlewood–Richardson coefficients $N^\mu_{\lambda\lambda^+}$ in the decomposition

$$V_\lambda \otimes V_\lambda^* \cong \bigoplus_\mu N^\mu_{\lambda\lambda^+} V_\mu \quad (A.9)$$

of $g$-modules. Now we use $N^\mu_{\lambda\lambda^+} = N^{\lambda^+}_{\lambda\mu^+}$ (because $N^\mu_{\lambda\lambda^+}$ is given by the multiplicity of the trivial component in $V_\lambda \otimes V_\lambda^+ \otimes V_{\mu^+}$, and so is $N^{\lambda^+}_{\lambda\mu^+}$). But $N^{\lambda^+}_{\lambda\mu^+}$ can be calculated using the
formula \[10\]

\[ N^\lambda_{\mu^+} = \sum_{\sigma \in W} (-1)^\sigma \text{mult}_{\mu^+}(\sigma \ast \lambda - \lambda), \tag{A.10} \]

where \( W \) is the Weyl group of \( g \). Here \( \text{mult}_{\mu^+}(\nu) \) is the multiplicity of the weight space \( \nu \) in \( V_{\mu^+} \), and \( \sigma \ast \lambda = \sigma(\lambda + \rho) - \rho \) denotes the action of \( \sigma \) with reflection center \( -\rho \). Now one can see already that for large, generic \( \lambda \) (so that \( \sigma \ast \lambda - \lambda \) is not a weight of \( V_{\mu^+} \) unless \( \sigma = 1 \)), it follows that \( N^\lambda_{\mu^+} = \text{mult}_{\mu^+}(0) = \text{mult}^{(T)}_{\mu^+} \), which proves (4.7) for the generic case. To cover all possible \( \lambda \), we proceed as follows:

Let \( \mathfrak{k} \) be the Lie algebra of \( K_\lambda \), and \( W_\mathfrak{k} \) its Weyl group; it is the subgroup of \( W \) which leaves \( \lambda \) invariant, generated by those reflections which preserve \( \lambda \) (the \( u(1) \) factors in \( \mathfrak{k} \) do not contribute to \( W_\mathfrak{k} \)). If \( \mu \) is “small enough”, then the sum in (A.10) can be restricted to \( \sigma \in W_\mathfrak{k} \), because otherwise \( \sigma \ast \lambda - \lambda \) is too large to be in \( V_{\mu^+} \); this defines the cutoff in \( \mu \). It holds for any given \( \mu \) if \( \lambda \) has the form \( \lambda = n\lambda_0 \) for large \( n \in \mathbb{N} \) and fixed \( \lambda_0 \).\(^{11}\) We will show below that

\[ \text{mult}^{(K_\lambda)}_{\mu^+} = \sum_{\sigma \in W_\mathfrak{k}} (-1)^\sigma \text{mult}_{\mu^+}(\sigma \ast \lambda - \lambda) \tag{A.11} \]

for all \( \mu \), which implies (4.7). Recall that the lhs is the dimension of the subspace of \( V_{\mu^+} \) which is invariant under \( K_\lambda \).

To prove (A.11), first observe the following fact: Let \( V_\lambda \) be the highest weight irrep of some simple Lie algebra \( \mathfrak{k} \) with highest weight \( \lambda \). Then

\[ \sum_{\sigma \in W_\mathfrak{k}} (-1)^\sigma \text{mult}_{V_\lambda}(\sigma \ast 0) = \delta_{\lambda,0} \tag{A.12} \]

i.e. the sum vanishes unless \( V_\lambda \) is the trivial representation; here \( \mathfrak{k} = u(1) \) is allowed as well. This follows again from (A.10), considering the decomposition of \( V_\lambda \otimes V_0 \). More generally, assume that \( \mathfrak{k} = \bigoplus_i \mathfrak{k}_i \) is a direct sum of simple Lie algebras \( \mathfrak{k}_i \), with corresponding Weyl group \( W_\mathfrak{k} = \prod_i W_i \). Its irreps have the form \( V = \bigotimes_i V_{\lambda_i} \), where \( V_{\lambda_i} \) denotes the highest weight module of \( \mathfrak{k}_i \) with highest weight \( \lambda_i \). We claim that the relation

\[ \sum_{\sigma \in W_\mathfrak{k}} (-1)^\sigma \text{mult}_{V}(\sigma \ast 0) = \prod_i \delta_{\lambda_i,0} \tag{A.13} \]

still holds. Indeed, assume that some \( \lambda_i \neq 0 \); then

\[ \sum_{\sigma \in W_\mathfrak{k}} (-1)^\sigma \text{mult}_{V}(\sigma \ast 0) = \left( \sum_{\sigma' \in W_\mathfrak{k}} (-1)^{\sigma'} \right) \left( \sum_{\sigma \in W_i} (-1)^{\sigma'} \text{mult}_{V_i}(\sigma' \ast (\sigma' \ast 0)) \right) = 0 \]

in self–explanatory notation. The last bracket vanishes by (A.12), since \( (\sigma' \ast 0) \) has weight 0 with respect to \( \mathfrak{k}_i \), while \( V \) contains no trivial component of \( \mathfrak{k}_i \) (notice that \( \rho = \sum \rho_i \), and the

\(^{11}\)This constitutes our definition of “classical limit”. For weights \( \lambda \) which do not satisfy this requirement, the corresponding \( D \)–brane \( D_\lambda \) cannot be interpreted as “almost–classical”. Here we differ from the approach in [13], which do not allow degenerate \( \lambda_0 \).
operation ⋆ is defined component-wise). Therefore for any (finite, but not necessarily irreducible) \( \mathfrak{g} \)-module \( V \), the number of trivial components in \( V \) is given by \( \sum_{\sigma \in W_\mathfrak{g}} (-1)^{\sigma} \text{mult}_V(\sigma \ast 0) \).

We now apply this to (A.11). Since the sum is over \( \sigma \in W_\mathfrak{g} \), we have \( \sigma(\lambda) = \lambda \) by definition, and \( \sigma \ast \lambda - \lambda = \sigma \ast 0 \). Hence the rhs can be replaced by \( \sum_{\sigma \in W_\mathfrak{g}} (-1)^{\sigma} \text{mult}_{\mu^+}(\sigma \ast 0) \). But this is precisely the number of vectors in \( V_{\mu^+} \) which are invariant under \( K_\lambda \), as we just proved. Notice that we use here the fact that \( \mathfrak{g} \) contains the Cartan sub-algebra of \( g \), so that the space of weights of \( \mathfrak{g} \) is the same as the space of weights of \( g \); therefore the multiplicities in (A.11) and (A.13) are defined consistently. This is why we had to include the case \( \mathfrak{g}_i = u(1) \) in the above discussion.

To calculate the decomposition (4.9) for all allowed \( \lambda \) (with \( \dim q(V_\lambda) > 0 \)), the ordinary multiplicities in (4.6) should be replaced with their truncated versions corresponding to \( U_q(g) \) at roots of unity. There exist generalizations of the formula (A.10) which allow to calculate \( N_{\lambda \mu}^{\lambda_1, \ldots, \lambda_N} \) efficiently; we refer here to the literature, e.g. [13].

A.4 The quantum determinant

Here we present a formula for the quantum determinant, following [20]. First we have to introduce \( q \)-deformed totally \((q)\)-antisymmetric tensors \( \varepsilon_{i_1 \ldots i_N} \) of \( U_q(sl(N)) \).

\[ \varepsilon_{\sigma(1) \ldots \sigma(N)} = (-q)^{-l(\sigma)} = \varepsilon_{\sigma(1) \ldots \sigma(N)} \] (A.14)

where \( l(\sigma) \) is the length of the permutation \( \sigma \). The important formula respected by \( \varepsilon_q \) is (in notation of (3.1))

\[ \varepsilon_{1 \ldots N} R'_i = -\frac{1}{q} q_{i}^{1 \ldots N} \] (A.15)

where \( R'_i = \hat{R}_{i(i+1)} \) and \( \hat{R}_{ij}^{ik} = R_{ij}^{ik} \). With this notation we define

\[ \varepsilon_{1 \ldots N}^{1 \ldots N} \det_q(M) = N(M_1 R'_1 R'_2 \ldots R'_{N-1})^{1 \ldots N} \varepsilon_q^{1 \ldots N} \] (A.16)

where \( N \) is an arbitrary normalization constant. One can show that this is invariant under the chiral symmetries \( G_L \otimes^R G_R \) as in (3.4), or equivalently under the action of \( U_q(g_L \times g_R) \).

For \( N = 2 \) we have \( (M_1 R'_1)(M_1 R'_1) \varepsilon_q^{12} = -\frac{1}{q} q M_1 R'_1 M_1 \varepsilon_q^{12} \). After using the RE relations, this becomes proportional to \( q^{-1} (M_1^2 M_2^2 - q^2 M_1^2 M_2^2) \) thus we choose the quantum determinant

\[ \det_q(M) = (M_1^2 M_2^2 - q^2 M_1^2 M_2^2) \] (A.17)

For other groups such as \( SO(N) \) and \( SP(N) \), the explicit form for \( \varepsilon_{1 \ldots N} \) is different, and additional constraints (which are also invariant under the chiral symmetries) must be imposed. These are known and can be found in the literature [20].
A.5 Covariance of $M$ and central elements

For any numerical matrix $M(0)$ (in the defining representation of $U_q(g)$), consider

$$M = L^+M(0)SL^- = (\pi \otimes 1)(R_{21}) M(0) (\pi \otimes 1)R_{12}. \quad (A.18)$$

Let $\mathcal{M} \subset U_q(g)$ be the sub-algebra generated by the entries of this matrix. First, we note that $\mathcal{M}$ is a (left) coideal sub-algebra, which means that $\Delta(\mathcal{M}) \in U_q(g) \otimes \mathcal{M}$. This is verified simply by calculating the coproduct of $M$,

$$\Delta(M_i^j) = L^+_i S L^-_i^s \otimes (M^s_i). \quad (A.19)$$

In particular if $M(0)$ is a constant solution of the reflection equation (3.1), it follows by taking the defining representation of (A.18) that $[\pi(M_i^j), M(0)] = 0$, and therefore $[\pi(M), M(0)] = 0$. Then for any $u \in \mathcal{M} \subset U_q(g)$,

$$(\pi \otimes 1)(\Delta(u))M = (\pi \otimes 1)(\Delta(u)R_{21}) M(0) SL^-$$
$$= (\pi \otimes 1)(R_{21} \Delta'(u)) M(0) SL^-$$
$$= L^+ M(0) (\pi \otimes 1)(\Delta'(u)R_{12})$$
$$= L^+ M(0) SL^- (\pi \otimes 1)\Delta(u) = M (\pi \otimes 1)\Delta(u). \quad (A.20)$$

In the second line we used $\Delta'(u) \equiv u_2 \otimes u_1 = R\Delta(u)R^{-1}$, in the third line, the coideal property (A.19). Using Hopf algebra identities (i.e. multiplying from left with $(\pi(Su_0) \otimes 1)$ and from the right with $(1 \otimes Su_3)$), this is equivalent to $(1 \otimes u_1)M(1 \otimes Su_2) = (\pi(Su_1) \otimes 1)M(\pi(u_2) \otimes 1)$, or

$$u_1 MSu_2 = \pi(Su_1)M\pi(u_2) \quad (A.21)$$

for any $u \in \mathcal{M}$, as desired. This implies immediately that

$$u_1 tr_q(M^n)Su_2 = tr_q(\pi(Su_1)M^n\pi(u_2)) = \varepsilon(u) tr_q(M^n), \quad (A.22)$$

or equivalently

$$[u, tr_q(M^n)] = 0 \quad (A.23)$$

for any $u \in \mathcal{M}$. This proves in particular that the Casimirs $c_n$ (3.1) are indeed central.

A.6 Evaluation of Casimirs

**Evaluation of $c_1$** Consider the fuzzy $D$–brane $D_\lambda$. Then $c_1$ acts on the highest–weight module $V_\lambda$, and has the form

$$c_1 = tr_q(L^+SL^-) = (tr_q \pi \otimes 1)(R_{21}R_{12}). \quad (A.24)$$

Because it is a Casimir, it is enough to evaluate it on the lowest–weight state $|\lambda-\rangle$ of $V_\lambda$, given by $\lambda- = \sigma_m(\lambda)$ where $\sigma_m$ denotes the longest element of the Weyl group. Now the universal $R$ has the form

$$R = q^{H_i(B^{-1})_{ij} \otimes H_j} (1 \otimes 1 + \sum U^+ \otimes U^-). \quad (A.25)$$
Here $B$ is the (symmetric) matrix $d_j^{-1} A_{ij}$ where $A$ is the Cartan Matrix, $d_i$ are the lengths of the simple roots ($d_i = 1$ for $g = su(N)$) and $U^+, U^-$ stands for terms in the Borel sub-algebras of rising respectively lowering operators. Hence only the diagonal elements of $(SL^-)^i_j$ are non-vanishing on a lowest-weight state, and due to the trace only the diagonal elements of $(L^+)^i_j$ enter. We can therefore write

$$c_1 |\lambda_-\rangle = (\text{tr}_q \pi \otimes 1)(q^2 H_i^1 \otimes H_j^1)|\lambda_-\rangle = (\text{tr}_q \pi \otimes 1)(q^{-2H_\rho} \otimes 1)(q^2 H_i^1 \otimes H_j^1)|\lambda_-\rangle$$

(A.26)

Here $H_\mu|\mu\rangle = (\lambda \cdot \mu)|\mu\rangle$ for any weight $\mu$. Therefore the eigenvalue of $c_1$ is

$$c_1 = \sum_{\mu \in V_N} q^{-2\mu \cdot \rho + 2\mu \cdot \lambda} = \sum_{\mu \in V_N} q^{2\mu \cdot (-\rho + \lambda)}.$$  

(A.27)

Using $\sigma_m(\rho) = -\rho$, this becomes

$$c_1 = \sum_{\mu \in V_N} q^{2(\sigma_m(\rho) \cdot (\rho + \lambda))} = \text{tr}_{V_N}(q^{2(\rho + \lambda)})$$

(A.28)

because the weights of $V_N$ are invariant under the Weyl group.

**Evaluation of $c_n$ in general** Since $c_n$ is proportional to the identity matrix on irreps, it is enough to calculate $\text{tr}_q(c_n) = \text{tr}_{V_\lambda}(c_n q^{-2H_\rho})$ on $V_\lambda$, noting that $\text{tr}_q(1) = \dim_q(V_\lambda)$ is known explicitly:

$$\text{tr}_q(c_n) = (\text{tr}_q \otimes \text{tr}_q)((\mathcal{R}_{21} \mathcal{R}_{12})^n)$$

(A.29)

where the traces are over $\text{Mat}(N)$ and $\text{Mat}(V_\lambda)$. Now we use the fact that $\mathcal{R}_{21} \mathcal{R}_{12}$ commutes with $\Delta(U_q(g))$, i.e. it is constant on the irreps of $V_N \otimes V_\lambda$, and observe that $\Delta(q^{-2H_\rho}) = q^{-2H_\rho} \otimes q^{-2H_\rho}$, which means that the quantum trace factorizes. Hence we can decompose the tensor product $V_N \otimes V_\lambda$ into irreps:

$$V_N \otimes V_\lambda = \bigoplus_{\mu \in P_N} V_\mu$$

(A.30)

where the sum goes over all $\mu$ which have the form $\mu = \lambda + \nu$ for $\nu$ a weight of $V_N$. The multiplicities are equal one because $V_N$ is the defining representation. The eigenvalues of $\mathcal{R}_{21} \mathcal{R}_{12}$ on $V_\mu$ are known [27] to be $q^{c_\mu - c_\lambda - c_\lambda N}$, where $\lambda_N$ denotes the highest weight of $V_N$ and $c_\lambda = \lambda \cdot (\lambda + 2\rho)$. Now for $\mu = \lambda + \nu$,

$$c_\mu - c_\lambda - c_\lambda N = 2(\lambda + \rho) \cdot \nu - 2\lambda \cdot \rho,$$

(A.31)

hence the set of eigenvalues of $\mathcal{R}_{21} \mathcal{R}_{12}$ is

$$\{ q^{2(\lambda + \rho) \cdot \nu - 2\lambda \cdot \rho}; \ \nu \in V_N \}.$$  

(A.32)

Putting this together, we obtain

$$\text{tr}_q(c_n) = c_n \text{tr}_{V_\lambda}(q^{-2H_\rho}) = \sum_{\mu} q^{2n(\lambda + \rho) \cdot \nu - 2\lambda \cdot \nu} \text{tr}_{V_\mu}(q^{-2H_\rho})$$

(A.33)

where the sum is as explained above. Then (1.3) follows, since $\text{tr}_{V_\mu}(q^{-2H_\rho}) = \dim_q(V_\mu)$. 


A.7 Characteristic equation for $M$.

\[(4.5)\] can be seen as follows: On $D_\lambda$, the quantum matrices $M^j_i$ become the operators

\[\left(\pi^j_i \otimes \pi_\lambda\right)(R_{21}R_{12})\]  \hspace{1cm} (A.34)

acting on $V_\lambda$. As above, the representation of $R_{21}R_{12}$ acting on $V_N \otimes V_\lambda$ has eigenvalues \[\{q^{c_\mu-c_\lambda-c_\lambda N} = q^{2(\lambda+N)-2\lambda N}\} \] on $V_\mu$ in the decomposition \[(A.30)\]. Here $\mu = \lambda + \nu$ for $\nu \in V_N$, and $\lambda_N$ is the highest weight of $V_N$. This proves (4.5). Note that if $\lambda$ is on the boundary of the fundamental Weyl chamber, not all of these $\nu$ actually occur in the decomposition; nevertheless, the characteristic equation holds.

References

[1] M. R. Douglas, N. A. Nekrasov, “Noncommutative Field Theory”, Rev.Mod.Phys. 73 (2002) 977-1029; hep-th/0106048.

[2] B. Jurco, P. Schupp, J. Wess, “Nonabelian noncommutative gauge theory via noncommutative extra dimensions”, Nucl.Phys. B604 (2001) 148-180.

[3] A. Yu. Alekseev, V. Schomerus, Phys. Rev. D60 (1999) 061901, hep-th/9812193.

[4] C. Bachas, M. Douglas, C. Schweigert, JHEP 0005 (2000) 048, hep-th/0003037; J. Pawelczyk, JHEP 08 (2000) 006, hep-th/0003057; P. Bordalo, S. Ribault, C. Schweigert, JHEP 0110 (2001) 036, hep-th/0108201.

[5] R. C. Myers, JHEP 9912 (1999) 022, hep-th/9910053; D. Kabat and W. Taylor, Adv. Theo. Math. Phys. 2 (1998) 181, hep-th/9711078; S.-J. Rey, hep-th/9711081.

[6] A. Yu. Alekseev, A. Recknagel, V. Schomerus, JHEP 0005 (2000) 010, hep-th/0003187.

[7] J. Pawelczyk, H. Steinacker, “Matrix description of D-branes on 3-spheres”, JHEP 0112 (2001) 018, hep-th/0107265.

[8] G. Alexanian, A.P. Balachandran, G. Immirzi, B. Ydri, “Fuzzy CP2”, hep-th/0103023; A.P. Balachandran, Brian P. Dolan, J. Lee, X. Martin, Denjoe O’Connor, “Fuzzy Complex Projective Spaces and their Star-products”, hep-th/0107099.

[9] H. Grosse, A. Strohmaier, “Towards a Nonperturbative Covariant Regularization in 4D Quantum Field Theory”, Lett.Math.Phys. 48 (1999) 163-179; hep-th/9902138.

[10] J. Fuchs, "Affine Lie Algebras and Quantum Groups", Cambridge University Press, 1992.

[11] P. Di Francesco, P. Mathieu and D. Senechal, “Conformal Field Theory”, Springer-Verlag, 1997.

[12] J. Maldacena, G. Moore, N. Seiberg, JHEP 0111 (2001) 062, hep-th/0108100.

[13] G. Felder, J. Fröhlich, J. Fuchs, C. Schweigert, “The geometry of WZW branes”, J.Geom.Phys. 34 (2000) 162-190.
[14] A. Yu. Alekseev, A. Recknagel, V. Schomerus, JHEP 9909 (1999) 023, hep-th/9908040.

[15] V. Chari and A. Pressley, ”A guide to quantum groups”, Cambridge University Press, 1994.

[16] E. Sklyanin, J. Phys. A 21 (1988) 2375; L. Mezincescu, R. Nepomeniche, Int. J. Mod. Phys. A6 (1991) 5231.

[17] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan. ”Quantization of Lie Groups and Lie Algebras” Algebra Anal. 1 178 (1989).

[18] P. P. Kulish, R. Sasaki, C. Schwiebert, J. Math. Phys. 34 (1993) 286, hep-th/9205039; P. P. Kulish, E. K. Sklyanin, J. Phys. A25 (1992) 5963-5976, hep-th/9209054; P. P. Kulish, R. Sasaki, Prog. Theor. Phys. 89 (93) 741, hep-th/9212007.

[19] S. Majid, “Foundations of Quantum Group Theory”, Cambridge University Press, 1995.

[20] P. Schupp, P. Watts, B. Zumino, ”Bicovariant Quantum Algebras and Quantum Lie Algebras”, Commun. Math. Phys. 157 (1993) 305-330; hep-th/9210150; L. D. Faddeev, P. N. Pyatov, ”The Differential Calculus on Quantum Linear Groups”, Am. Math. Soc. Transl. 175 (1996) 35-47; hep-th/9402070.

[21] A. Klimyk, K. Schmüdgen, “Quantum groups and their representations”, Springer-Verlag, 1997.

[22] H. Steinacker, “Unitary Representations of Noncompact Quantum Groups at Roots of Unity”, Rev. Math. Phys. 13, No. 8 (2001) 1035-1054; math-QA/9907021.

[23] H. Grosse, J. Madore, H. Steinacker, J. Geom. Phys. 38 (2001) 308-342, hep-th/0005273; H. Steinacker, Mod. Phys. Lett. A16 (2001) 361-366, hep-th/0102074.

[24] H. Grosse, C. Klimcik, P. Presnajder, ”Towards Finite Quantum Field Theory in Non-Commutative Geometry”, Int. J. Theor. Phys. 35 (1996) 231-244; hep-th/9505175.

[25] C.-S. Chu, J. Madore, H. Steinacker, “Scaling Limits of the Fuzzy Sphere at one Loop”, JHEP 0108 (2001) 038; hep-th/0106203.

[26] S. Vaidya, ”Scalar Multi-Solitons on the Fuzzy Sphere”, JHEP 0201 (2002) 011; hep-th/0109102.

[27] N. Reshetikhin, ”Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links I,II” LOMI preprint E-4-87, E-17-87.

[28] J. Pawelczyk, Soo-Jong Rey, ”Ramond-Ramond Flux Stabilization of D-Branes”, hep-th/0007154, Phys. Lett. B493 (2000) 395-401.