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Polynilpotent Multipliers of Some Nilpotent Products of Cyclic Groups

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Abstract In this article, we present an explicit formula for the $c$th nilpotent multiplier (the Baer invariant with respect to the variety of nilpotent groups of class at most $c \geq 1$) of the $n$th nilpotent product of some cyclic groups $G = \mathbb{Z}^n \ast \cdots \ast \mathbb{Z}^n \ast \mathbb{Z}_{r_1} \ast \cdots \ast \mathbb{Z}_{r_t}$, $(m$-copies of $\mathbb{Z})$, where $r_{i+1} | r_i$ for $1 \leq i \leq t-1$ and $c \geq n$ such that $(p, r_1) = 1$ for all primes $p$ less than or equal to $n$. Also, we compute the polynilpotent multiplier of the group $G$ with respect to the polynilpotent variety $N_{c_1, c_2, \ldots, c_t}$, where $c_1 \geq n$.

Keywords Polynilpotent multiplier · Nilpotent product · Cyclic group

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1 Introduction and Motivation

Let $G$ be any group with a free presentation $G \cong F/R$, where $F$ is a free group. Then the Baer invariant of $G$ with respect to the variety of groups $\mathcal{V}$, denoted by $VM(G)$, is defined to be

$$VM(G) = \frac{R \cap V(F)}{|RV^*F|};$$

where $V$ is the set of laws of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of $F$ and

$$[RV^*F] = \langle v(f_1, \ldots, f_{i-1}, f_ir, f_{i+1}, \ldots, f_k)v(f_1, \ldots, f_i, \ldots, f_k)^{-1} | r \in R, f_i \in F, v \in V, 1 \leq i \leq k, k \in \mathbb{N} \rangle.$$

For example, if $\mathcal{V}$ is the variety of abelian groups $\mathcal{A}$, then the Baer invariant of the group $G$ will be $(R \cap F')/[R, F]$, which is isomorphic to $M(G)$, the Schur multiplier of $G$ (see [5]). If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1, c_2, \ldots, c_t}$, then the Baer invariant of a group $G$ with respect to this variety, which we call a polynilpotent multiplier, is as follows:

$$\mathcal{N}_{c_1, c_2, \ldots, c_t}M(G) = \frac{R \cap \gamma_{c_1+1} \circ \cdots \circ \gamma_{c_t+1}(F)}{[R, c_1F, c_2\gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_1+1} \circ \cdots \circ \gamma_{c_t+1}(F)]},$$

where $\gamma_{c_1+1} \circ \cdots \circ \gamma_{c_t+1}(F) = \gamma_{c_1+1}(\gamma_{c_1+1}(\cdots(\gamma_{c_t+1}(F)\cdots)))$ is the group which is attained from the iterated terms of the lower central series of $F$. See [4] for the equality

$$[RN^*_{c_1, c_2, \ldots, c_t}F] = [R, c_1F, c_2\gamma_{c_1+1}(F), \ldots, c_t \gamma_{c_1+1} \circ \cdots \circ \gamma_{c_t+1}(F)].$$

Note that the Baer invariant of $G$ is always abelian and independent of the choice of the free presentation for $G$ with respect to a variety $\mathcal{V}$ (see [5]). In particular, if $t = 1$ and $c_1 = c$, then the Baer invariant of $G$ with respect to the variety $\mathcal{N}_c$ is called the $c$-nilpotent multiplier and given by

$$\mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}.$$

Determining these Baer invariants of groups is known to be very useful for classification of groups into isologism classes with respect to the chosen varieties (see [5]). In 1979, Moghaddam [8] gave a formula for the $c$-nilpotent multiplier of a direct product of two groups, where $c + 1$ is a prime number or 4. Also, in 1998, Ellis [1] presented the formula for all $c \geq 1$. In 1997, Moghaddam and Mashayekhy [7] presented an explicit formula for the $c$-nilpotent multiplier of a finite abelian group for every $c \geq 1$.

It is known that the nilpotent product is a generalization of the direct product. In 1992, Gupta and Moghaddam [2] calculated the $c$-nilpotent multiplier of the nilpotent dihedral group of class $n, G_n = \langle x, y | x^2, y^2, [x, y]^{2n-1} \rangle$. It is routine to verify that $G_n \cong \mathbb{Z}_n \ast \mathbb{Z}_2$. In 2003, Moghaddam et al. [9] extended the previous result and calculated the $c$-nilpotent multiplier of $nth$ nilpotent products of two cyclic groups for $n = 2, 3$ and 4 under some conditions. Also, the second author [6] gave an implicit formula for the $c$-nilpotent multiplier of a nilpotent product of cyclic groups.

In this paper, we first obtain an explicit formula for the $c$-nilpotent multiplier of the $nth$ nilpotent product of some cyclic groups $G = \mathbb{Z}_{r_1} \ast \cdots \ast \mathbb{Z}_{r_t} \ast \cdots \ast \mathbb{Z}_{r_1}$, where $r_i+1 \mid r_i$ for $1 \leq i \leq t-1$ and $c \geq n$ such that $(p, r_1) = 1$ for all primes $p$ less than or equal to $n$. This result extends the works of Moghaddam and Mashayekhy [7] and Moghaddam et al. [9]. Second, we present an explicit formula for the polynilpotent multiplier of such a group $G$ with respect to the polynilpotent variety $\mathcal{N}_{c_1, c_2, \ldots, c_t}$, where $c_1 \geq n$.

2 Notation and Preliminaries

Definition 2.1 ([3, §11.1 and §12.3]). The basic commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined as follows:

(i) The letters $x_1, x_2, \ldots, x_n, \ldots$ are basic commutators of weight one, ordered by setting $x_i < x_j$, if $i < j$. 

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(ii) Having defined the basic commutators of weight less than \( n \), the basic commutators of weight \( n \) are defined as \( c_k = [c_i, c_j] \), where

(a) \( c_i \) and \( c_j \) are basic commutators and \( w(c_i) + w(c_j) = n \), where \( w(c) \) is the weight of \( c \) and

(b) \( c_i > c_j \), and if \( c_i = [c_j, c_l] \), then \( c_j \geq c_l \).

(iii) The basic commutators of weight \( n \) follow those of weights less than \( n \). The basic commutators of weight \( n \) are ordered among themselves lexicographically; that is, if \( [b_1, a_1] \) and \( [b_2, a_2] \) are basic commutators of weight \( n \), then \( [b_1, a_1] < [b_2, a_2] \) if and only if \( b_1 < b_2 \) or \( b_1 = b_2 \) and \( a_1 < a_2 \).

Basic commutators are special cases of outer commutators. Outer commutators on the letters \( x_1, x_2, \ldots, x_n, \ldots \) are defined inductively as follows:

The letter \( x_i \) is an outer commutator word of weight one. If \( u = u(x_1, \ldots, x_s) \) and \( v = v(x_{s+1}, \ldots, x_{s+t}) \) are outer commutator words of weights \( s \) and \( t \), then \( w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})] \) is an outer commutator word of weight \( s + t \) and will be written \( w = [u, v] \).

**Theorem 2.2** ([3, §11.2]). Let \( F \) be the free group on \( x_1, x_2, \ldots, x_d \); then for all \( 1 \leq i \leq n \),

\[
\gamma_n(F) = \frac{1}{\gamma_{n+1}(F)}
\]

is the free abelian group, and freely generated by the basic commutators of weights \( n, n + 1, \ldots, n + i - 1 \) on \( d \) letters.

**Theorem 2.3** ([3, §11.2]). The number of basic commutators of weight \( n \) on \( d \) generators is given by the following formula:

\[
\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^\frac{n}{m},
\]

where \( \mu(m) \) is the Möbius function, which is defined to be

\[
\mu(m) = \begin{cases} 
1 & \text{if } m = 1, \\
0 & \text{if } m = p_1^{a_1} \cdots p_k^{a_k} \quad \exists a_i > 1, \\
(-1)^s & \text{if } m = p_1 \cdots p_s,
\end{cases}
\]

where \( p_i \)'s are distinct prime numbers.

Let \( G_i = \langle x_i \rangle_{x_i}^{k_i} \), for \( i \in I \), be the cyclic group of order \( k_i \) if \( k_i > 1 \), and the infinite cyclic group if \( k_i = 0 \). The \( n \)th nilpotent product of the family \( \{G_i\}_{i \in I} \) is defined as follows (see [10]):

\[
\prod_{i \in I}^n G_i = \frac{\prod_{i \in I} G_i}{\gamma_{n+1}(\prod_{i \in I} G_i)},
\]

where \( \prod_{i \in I} G_i \) is the free product of the family \( \{G_i\}_{i \in I} \).

Let

\[
1 \to R_i = \langle x_i \rangle \to F_i = \langle x_i \rangle \to G_i \to 1
\]

be a free presentation for \( G_i \). It is routine to check that a free presentation for the \( n \)th nilpotent product of \( \prod_{i \in I}^n G_i \) is as follows (see [9]):

\[
1 \to R = S\gamma_{n+1}(F) \to F = \prod_{i \in I}^n F_i \to \prod_{i \in I}^n G_i \to 1,
\]

where \( S = \langle x_i \rangle_{i \in I}^F \). Therefore, if \( c \geq n \), then the \( c \)-nilpotent multiplier of \( \prod_{i \in I}^n G_i \) is

\[
\mathcal{N}_c M \left( \prod_{i \in I}^n G_i \right) = \frac{R \cap \gamma_{c+1}(F)}{|R, c F|} = \frac{\gamma_{c+1}(F)}{[S, c F]\gamma_{c+n+1}(F)} = \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+n+1}(F)},
\]

where \( \rho_k(S) \) is defined inductively by \( \rho_1(S) = S \) and \( \rho_{c+1}(S) = [\rho_c(S), F] \).
Lemma 2.4 If $1 \leq i < r$ and $(p, r) = 1$ for all primes $p$ less than or equal to $i$, then $r$ divides $\binom{r}{i}$.

Proof Clearly $\binom{r}{i} = \frac{(r-1)\cdots(r-i+1)}{i! \cdot (r-i)!}$ is an integer. For any prime $p \leq i$, $p|(r-1)\cdots(r-i+1)$, since $p \not| r$. Thus, $1 \times 2 \times \cdots \times i|(r-1)\cdots(r-i+1)$ and, hence, the result holds. \hfill \Box

The following consequences of the collecting process are vital in the proof of our main result:

Lemma 2.5 ([10]). Let $x$, $y$ be any elements of a given group and let $c_1, c_2, \ldots$ be the sequence of basic commutators of weights at least two in $x$ and $[x, y]$, in ascending order. Then

$$[x^n, y] = [x, y]^n c_1^{f_1(n)} c_2^{f_2(n)} \cdots c_i^{f_i(n)} \cdots ,$$

where

$$f_i(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + \cdots + a_{w_i} \binom{n}{w_i} ,$$

with $a_i \in \mathbb{Z}$ and $w_i$ being the weight of $c_i$ in $x$ and $[x, y]$. If the group is nilpotent, then the expression in (2) gives an identity, and the sequence of commutators terminates.

Lemma 2.6 ([10]). Let $\alpha$ be a fixed integer and $G$ a nilpotent group of class at most $n$. If $b_j \in G$ and $r < n$, then

$$[b_1, \ldots, b_{l-1}, b_j^n, b_{l+1}, \ldots, b_r] = [b_1, \ldots, b_r]^n c_1^{f_1(n)} c_2^{f_2(n)} \cdots ,$$

where the $c_i$’s are commutators in $b_1, \ldots, b_r$ of weight strictly greater than $r$, and every $b_j$, $1 \leq j \leq r$ appears in each commutator $c_i$, the $c_i$’s listed in ascending order. The $f_i$’s are of the form (3), with $a_j \in \mathbb{Z}$ and $w_i = (\text{the weight of } c_i \text{ on the } b_i) - (r - 1)$.

3 Main Results

Keeping the previous notation, let $k_i = 0$, for $1 \leq i \leq m$, and $k_{m+j} = r_j > 1$ such that $r_{j+1} | r_j$, for $1 \leq j \leq t$; then $\prod_{i \in I} G_i = \bigoplus_{m \text{-copies}} \mathbb{Z} * \cdots * \mathbb{Z} * \mathbb{Z} r_1 \cdots * \mathbb{Z} r_t$. In order to compute the $c$-nilpotent multiplier of the above group, we need two technical lemmas.

Lemma 3.1 With the above notation and assumption, if $(p, r_1) = 1$, for all primes $p$ less than or equal to $l - i$, then $\rho_{c+i}(S)\gamma_{c+i}(F)/\rho_{c+i+1}(S)\gamma_{c+i}(F)$ is the free abelian group with a basis $D_{i,j} \cup \cdots \cup D_{i,t}$, where

$$D_{i,j} = \{ b_j^{r_j} \rho_{c+i+1}(S)\gamma_{c+i}(F) \mid b \text{ is a basic commutator of weight } c+i \text{ on } x_1, \ldots, x_m, \ldots, x_{m+j} \text{ such that } x_{m+j} \text{ appears in } b \} ,$$

for $1 \leq i \leq l - 1$ and $1 \leq j \leq t$.

Proof Using the collecting process (see [3, §11.1]), one can easily check that $\rho_{c+i}(S)\gamma_{c+i}(F)/\rho_{c+i+1}(S)\gamma_{c+i}(F)$ is generated by all $b' \rho_{c+i+1}(S)\gamma_{c+i}(F)$, where $b'$ belongs to the set of basic commutators of weight $c+i$, $\ldots, c+i+l-1$ on letters $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that one of the $x_m^{r_j} \cdots x_{m+t}^{r_j}$ appears in them. It is easy to check that all the above commutators of weight greater than $c+i$ belong to $\rho_{c+i+1}(S)$. Now, we show that if $b'$ is one of the above commutators of weight $c+i$ such that $x_m^{r_j}$ appears in it, then

$$b' \equiv b^{r_j} \pmod{\rho_{c+i+1}(S)\gamma_{c+i}(F)} ,$$

where $b$ is a basic commutator of weight $c+i$ on $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that $x_{m+j}$ appears in it. (Note that $b$ is actually a basic commutator according to the definition, and $b'$ is the same as $b$, but the letter $x_{m+j}$ with exponent $r_j$.) In order to prove the above claim, first we use reverse induction on $k$ ($i+1 \leq k \leq l-1$) to show that if $u$ is an outer commutator of weight $c+k$ on $x_1, \ldots, x_m, \ldots, x_{m+t}$ such that $x_{m+j}$ appears in $u$, then

$$u^{r_j} \in \rho_{c+i+1}(S) \pmod{\gamma_{c+i}(F)} .$$
Let \( k = l - 1 \) and \( u = [\ldots, x_{m+j}, \ldots] \); then clearly \( u^{r_j} \equiv [\ldots, x_{m+j}^{r_j}, \ldots] \in \rho_{c+i+1}(S) \) (mod \( \gamma_{c+l}(F) \)).

Now, suppose the above property holds for every \( k > k' \). We will show that the property (5) holds for \( k' \).

Let \( u = [u_1, u_2] \) be an outer commutator of weight \( c + k' \) on \( x_1, \ldots, x_{m+t} \), where \( x_{m+j} \) appears in \( u_1 \). Then, by Lemma 2.5, we have

\[
u^{r_j} \equiv [u_1^{r_j}, u_2](v_1^{f_1(r_j)} \cdots v_h^{f_h(r_j)})^{-1} \pmod {\gamma_{c+l}(F)},
\]

where \( v_s \) is a basic commutator of weight \( w_s \) in \( u_1 \) and \( [u_1, u_2] \), \( 1 \leq s \leq h \). Thus, \( v_s \) is an outer commutator of weight greater than \( c + k' \) and less than \( c + l \) on \( x_1, \ldots, x_{m+t} \) such that \( x_{m+j} \) appears in it. By the hypothesis, since \( r_j | r_1 \), we have \( (p, r_j) = 1 \) for all primes \( p \) less than or equal to \( l - 1 \). Also, it is easy to see that \( w_s \leq (c + l) - (c + k' - 1) = l - k' + 1 \leq l - 1 \). Therefore, by Lemma 2.4, \( r_j | f_i(r_j) \), and so, by induction hypothesis, \( v_s^{f_i(r_j)} \in \rho_{c+i+1}(S) \) (mod \( \gamma_{c+l}(F) \)). Hence, by repeating the above process, if \( u = [\ldots, x_{m+j}, \ldots] \), then \( u^{r_j} \equiv [\ldots, x_{m+j}^{r_j}, \ldots]v_1^{f_1(r_j)} \cdots v_h^{f_h(r_j)} \in \rho_{c+i+1}(S) \) (mod \( \gamma_{c+l}(F) \)). Now using (5), Lemma 2.6, and some commutator manipulations, the congruence (4) holds. Therefore, the set \( \bigcup_{j=1}^{l-1} D_{i,j} \) is a generating set for \( \rho_{c+i+1}(S) \gamma_{c+l}(F) / \rho_{c+i+1}(S) \gamma_{c+l}(F) \). On the other hand, by Theorem 2.2, distinct basic commutators are linearly independent, and hence, distinct powers of these basic commutators are also linearly independent. Therefore, the set \( \bigcup_{j=1}^{l-1} D_{i,j} \) is a basis. \( \square \)

**Lemma 3.2** With the notation and assumption of the previous lemma, if \( (p, r_1) = 1 \) for all primes \( p \) less than or equal to \( l - 1 \), then

\[
\rho_{c+i+1}(S) \gamma_{c+l}(F) / \gamma_{c+l}(F)
\]

is the free abelian group with a basis \( \bigcup_{i=1}^{l-1} (\bigcup_{j=1}^{l} D_{i,j}) \).

**Proof** Put

\[
A_j = \frac{\rho_{c+i}(S) \gamma_{c+l}(F)}{\rho_{c+i+1}(S) \gamma_{c+l}(F)}, \quad B_j = \frac{\rho_{c+i+1}(S) \gamma_{c+l}(F)}{\rho_{c+i+1}(S) \gamma_{c+l}(F)}.
\]

Then, clearly the following exact sequence exists for \( 1 \leq i \leq l - 1 \)

\[
0 \to A_i \to B_i \to B_{i-1} \to 0.
\]

By Lemma 3.1, \( B_1 \) is a free abelian group, so the following exact sequence

\[
0 \to A_2 \to B_2 \to B_1 \to 0
\]

splits, and hence, \( B_2 \cong A_2 \oplus B_1 \). Also, by Lemma 3.1 every \( A_i \) is free abelian, so by induction, every \( B_i \) is free abelian and

\[
\frac{\rho_{c+i}(S) \gamma_{c+l}(F)}{\gamma_{c+l}(F)} = B_{i-1} \cong A_{i-1} \oplus A_{i-2} \oplus \cdots \oplus A_2 \oplus A_1.
\]

Now, using the basis for \( A_i \) presented in Lemma 3.1, the result holds. \( \square \)

Now, we are in a position to state and prove the first main result of the paper.

**Theorem 3.3** Let \( G = \mathbb{Z}_n \ast \cdots \ast \mathbb{Z}_n \ast \mathbb{Z}_{r_1} \ast \cdots \ast \mathbb{Z}_{r_t} \) be the \( m \)-th nilpotent product of some cyclic groups, where \( r_{i+1} \) divides \( r_i \) for \( 1 \leq i \leq t \). If \( c \geq n \) and \( (p, r_1) = 1 \) for all primes \( p \) less than or equal to \( n \), then the \( c \)-nilpotent multiplier of \( G \) is isomorphic to

\[
\mathbb{Z}^{(d_m)} \oplus \mathbb{Z}^{(d_{m+1}-d_m)}_{r_1} \oplus \cdots \oplus \mathbb{Z}^{(d_{m+t}-d_{m+t-1})}_{r_t},
\]

where \( d_m = \sum_{i=1}^{n} \chi_{c+i}(m) \) and \( \mathbb{Z}^{(d)}_{r_i} \) denotes the direct sum of \( d \) copies of the cyclic group \( \mathbb{Z}_{r_i} \).
Proof Using the previous notation and assumption, we have
\[ N_{c_1}M(G) = \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+n+1}(F)} \approx \frac{\gamma_{c+1}(F)/\gamma_{c+n+1}(F)}{\rho_{c+1}(S)/\gamma_{c+n+1}(F)}. \]

Also, by Theorem 2.2, \( \gamma_{c+1}(F)/\gamma_{c+n+1}(F) \) is a free abelian group with the basis consisting of all basic commutators of weight \( c + 1, \ldots, c + n \) on the letters \( x_1, \ldots, x_{m+t} \).

Now, by considering the basis presented for \( \rho_{c+1}(S)/\gamma_{c+n+1}(F)/\gamma_{c+n+1}(F) \) in Lemma 3.2 and note the points that \( D_{i,j} \)'s are mutually disjoint and the number of elements of \( D_{i,j} \) is equal to \( x_{c+i}(m+j) - x_{c+i}(m+j-1) \), the result holds. \( \square \)

Now the second main result of the paper, which is in turn an extension of the first one, is as follows:

Theorem 3.4 Let \( G = \mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n \oplus \mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n \) be the \( n \)th nilpotent product of some cyclic groups, where \( r_i+1 \) divides \( r_i \), for \( 1 \leq i \leq t \). If \( (p, r_1) = 1 \) for all primes \( p \) less than or equal to \( n \), then the poly-nilpotent multiplier with class row \( c_1, c_2, \ldots, c_\ell \) of \( G \) is
\[ N_{c_1,c_2,\ldots,c_\ell}M(G) = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_{r_1}} \oplus \cdots \oplus \mathbb{Z}_{d_{r_t}}, \]
where \( d_i = x_{c_i+1} \left( \cdots \left( x_{c_{2i}+1} \left( \sum_{n=1}^{n} x_{c_{n+i}(m)} \right) \right) \right), \) for \( c_1 \geq n \) and \( c_2, \ldots, c_\ell \geq 1 \) and \( 1 \leq i \leq t \).

Proof Let \( G \) be a nilpotent group of class \( n \leq c_1 \) with a free presentation \( G = F/R \). Since \( \gamma_{c+1}(F) \leq \gamma_{n+1}(F) \leq R \), it gives \( N_{c_1}M(G) = \gamma_{c+1}(F)/[R, c_1 F] \). Now, we can consider \( \gamma_{c+1}(F)/[R, c_1 F] \) as a free presentation for \( N_{c_1}M(G) \) and, hence,
\[ N_{c_1,c_2}M(N_{c_1}M(G)) = \frac{\gamma_{c+1}(\gamma_{c+1}(F))}{[R, c_1 F]}. \]

Therefore, by (1) we have
\[ N_{c_1,c_2}M(G) = N_{c_1,c_2}M(N_{c_1}M(G)). \]

By continuing the above process, we can show that
\[ N_{c_1,c_2,\ldots,c_\ell}M(G) = N_{c_1}M(\cdots N_{c_1}M(N_{c_1}M(G)) \cdots). \]

Using Theorem 3.3, \( N_{c_1}M(G) \) is a finitely generated abelian group of the following form:
\[ \mathbb{Z}^n(\sum_{1}^{n} x_{c_{1+i}(m)}) \oplus \mathbb{Z}_{d_{r_1}}^{\sum_{1}^{n} x_{c_{1+(m+1)}}-x_{c_{1+i}(m)}} \oplus \cdots \oplus \mathbb{Z}_{d_{r_t}}^{\sum_{1}^{n} x_{c_{(m+t)}}-x_{c_{1+i}(m+t-1)}}. \]

Now applying Theorem 3.3 with \( n = 1 \), the result holds. \( \square \)

Remark 3.5 Let \( G = \mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n \oplus \mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n \) be the \( n \)th nilpotent product of some cyclic groups, where \( s_i \) are arbitrary natural numbers, for \( 1 \leq i \leq t \). If \( c \geq n \) and \( (p, s_i) = 1 \) for all primes \( p \) less than or equal to \( n \) and \( 1 \leq i \leq t \), then by a similar proof to Lemmas 3.1 and 3.2 and Theorem 3.3, one can compute the \( c \)-nilpotent multiplier of \( G \), but the formula is certainly more complicated than the one in Theorem 3.3. For example, if \( G = \mathbb{Z}_{s_1}^n \oplus \cdots \oplus \mathbb{Z}_{s_t}^n \oplus \mathbb{Z}_{s_t}^n \), then \( N_{c_1}M(G) \) is as follows:
\[ \mathbb{Z}^n(\sum_{1}^{n} x_{c_{1+i}(2)}) \oplus \mathbb{Z}_{d_1}^{\sum_{1}^{n} x_{c_{1+i}(2)}} \oplus \cdots \oplus \mathbb{Z}_{d_{r_t}}^{\sum_{1}^{n} x_{c_{1+i}(2)}} \oplus \mathbb{Z}^{\sum_{1}^{n} x_{c_{1+i}(3)-3} \sum_{1}^{n} x_{c_{1+i}(2)}}. \]

Moreover, using the proof of Theorem 3.4 and the above formula twice, we can compute the poly-nilpotent multiplier with class row \( c_1, c_2 \) of \( G \) as follows:
\[ N_{c_1,c_2}M(G) = \mathbb{Z}^{(e_1)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_2)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_3)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_4)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_5)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_6)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_7)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_8)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_9)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_10)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_11)}_{(a,\beta)} \oplus \mathbb{Z}^{(e_12)}_{(a,\beta)}. \]
where

\[ e_1 = x_{c_2+1} \left( \sum_{i=1}^{n} x_{c_1+i}(2) \right), \quad e_2 = x_{c_2+1} \left( \sum_{i=1}^{n} x_{c+i}(3) - 3 \sum_{i=1}^{n} x_{c+i}(2) \right), \]

\[ e_3 = x_{c_2+1} \left( 2 \sum_{i=1}^{n} x_{c_1+i}(2) \right) - 2e_1, \quad e_4 = x_{c_2+1} \left( \sum_{i=1}^{n} x_{c+i}(3) - 2 \sum_{i=1}^{n} x_{c+i}(2) \right) - e_1 - e_2, \]

\[ e_5 = x_{c_2+1} \left( 3 \sum_{i=1}^{n} x_{c_1+i}(2) \right) - 3x_{c_2+1} \left( 2 \sum_{i=1}^{n} x_{c_1+i}(2) \right), \]

\[ e_6 = x_{c_2+1} \left( \sum_{i=1}^{n} x_{c_1+i}(3) - \sum_{i=1}^{n} x_{c+i}(2) \right) - x_{c_2+1} \left( 2 \sum_{i=1}^{n} x_{c_1+i}(2) \right) \]

\[ -x_{c_2+1} \left( \sum_{i=1}^{n} x_{c+i}(3) - 2 \sum_{i=1}^{n} x_{c+i}(2) \right). \]

It seems that the general formula in this case is more complicated than to write!

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