Abstract. Let $M$ be a Riemmanian manifold with bounded geometry. We consider a generalization of Paley-Wiener functions and Lagrangian splines on $M$. An analog of the Paley-Wiener theorem is given. We also show that every Paley-Wiener function on a manifold is uniquely determined by its values on some discrete sets of points.

The main result of the paper is a generalization of the Whittaker-Shannon formula for reconstruction of a Paley-Wiener function from its values on a discrete set. It is shown that every Paley-Wiener function on $M$ is a limit of some linear combinations of fundamental solutions of the powers of the Laplace-Beltrami operator.

The result is new even in the one-dimensional case.

1. We introduce an appropriate generalization of Paley-Wiener functions. Our goal is to show that the reconstruction of such functions is possible as long as the distance between points from a discrete subgroup is small enough. The reconstruction formula involves the notion of a spline. In classical case this approach was used by Schoenberg [14].

Remark that consideration in the present paper is subelliptic in the sense that central role belongs to a certain subelliptic operator. The case of corresponding elliptic theory on manifolds was considered by author in [12], [13].

The Heisenberg group $H_m$ is the Lie group whose underlying manifold is direct product of $R$ and $C^m$ and composition is given by the formula

$$(t, z)(t', z') = (t + t' + 21mz z', z + z')$$

where $t, t' \in R, z, z' \in C^m$. Dilations on $H_m$ are given by

$$\delta_s(t, z) = (s^2 t, s z)$$

and homogeneous norm by

$$|(t, z)| = (t^2 + |z|^4)^{1/4}.$$ 

The number $Q = 2m + 2$ is called the homogeneous dimention of $H_m$ and in analysis on $H_m$ plays the same role as usual dimention $d$ in analysis on $R^d$.

To introduce Fourier transform on $H_m$ we consider irreducible unitary representations of $H_m$ in the Bargmann space which consist of holomorphic functions $F$ on $C^m$ such that

$$\|F\|^2 = (2\lambda/\pi)^m \int_{C^m} |F(w)|^2 exp(-2\lambda|w|^2)dw$$

is finite. The monomials

$$F_{\alpha, \lambda}(w) = ((2\lambda)^{1/2} w)\alpha / (\alpha !)^{1/2}, \alpha \in N^m$$

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form an orthonormal basis in the Bargmann space. For positive real $\lambda$ the representation $\pi_\lambda$ is given by

$$(\pi_\lambda(t,z)F)(w) = F(w-z)exp(i\lambda t + 2\lambda wz - |z|^2/2)$$

and for negative $\lambda$ by $\pi_\lambda(t,z) = \pi_{-\lambda}(-t,-z)$. The Fourier transform on $L^1(H_m)$ is given by the formula

$$\hat{f}(\lambda) = \int_{H^m} f(x)\pi_\lambda(x)dx, f \in L^1(H_m)$$

and can be extended to an isomorphism between $L^2(H_m)$ and the space of operator valued functions $\hat{f}(\lambda)$ such that

$$\int_{R-0} \|\hat{f}(\lambda)\|^2_{HS}d\lambda$$

exists. Here $\|\|_{HS}$ is the Hilbert Schmidt norm. If we set $z_k = x_k + iy_k$ then $(t,x_1,...,x_m,y_1,...,y_m)$ form a coordinate system on $H_m$. In this coordinate system we define the following vector fields

$$X_k = \partial_{x_k} + 2y_k\partial_t, 1 \leq k \leq m$$

$$X_k = \partial_{y_k} - 2x_k\partial_t, m + 1 \leq k \leq 2m$$

$$T = \partial_t.$$

The fields $T, X_1, ..., X_{2m}$ form the basis for the left-invariant vector fields on $H_m$. Every element on $H_m$ has a unique representation

$$exp(a_0T + a_1X_1 + ... + a_{2m}X_{2m}), a_k \in R$$

where $exp$ is the exponential map from Lie algebra onto group. One can easily verify that for the fixed integer $j \in Z$ all elements of the form

$$exp(2^{2j}n_0T + 2^jn_1X_1 + ... + 2^{2m}n_{2m}X_{2m})$$

where $n_k$ are integers form a discrete subgroup $\Gamma_j = \delta_2j\Gamma_0, \Gamma_0 = \Gamma$.

The sub-Laplacian $D = -X_1^2 - ... - X_{2m}^2$ is a second order self-adjoint and positive definite hypoelliptic operator in $L^2(H_m)$ which is homogeneous with respect to the above dilations.

Using sub-Laplacian $D$ one can introduce the Sobolev scale of spaces with the norm $\|f\|_{S^\sigma(H_m)} = \|(I+D)^{\sigma/2}f\|, \sigma \geq 0$. As was shown by Folland [3] (see also [2], [5], [9], [10], [11]) this norm is equivalent to the norm $\|f\| + \|D^{\sigma/2}f\|$ and if $\sigma = r$ is an integer to the norm

$$\|f\| + \sum_{1 \leq i_1,...,i_r \leq 2m} \|X_{i_1}...X_{i_r}f\|.$$ 

For negative $\sigma$ spaces $S^\sigma(H_m)$ can be introduced using duality. The full scale $S^\sigma(H_m), -\infty < \sigma < \infty$ serves the sub-Laplacian $D$ in the same way as standard Sobolev spaces $H^\sigma(R^d), -\infty < \sigma < \infty$ serve standard Laplacian $\Delta$.

2. First of all we introduce an abstract definition of Paley-Wiener functions.

Let $E$ be a Hilbert space with the norm $\|\|$ and $D$ a self-adjoint positive definite operator in $E$. According to the spectral theory [7] there exist a direct integral of Hilbert spaces $X = \int X(\lambda)dm(\lambda)$ and a unitary operator $F$ from $E$ onto $X$, which transforms domain of $D^k$ onto $X^k = \{x \in X | \lambda^k x \in X\}$ with norm

$$\|x\|_k = \|F(D^k x)\|_E.$$
besides $F(D^kf) = \lambda^k(Ff)$, if $f$ belongs to the domain of $D^k$. As known, $X$ is the set of all $m$-measurable functions $\lambda \rightarrow x(\lambda) \in X(\lambda)$, for which the norm

$$\|x\|_X = \left( \int_0^\infty \|x(\lambda)\|^2_{X(\lambda)} dm(\lambda) \right)^{1/2}$$

is finite.

We will say that a vector $f$ from $E$ belongs to $PW_\omega(D)$ if its "Fourier transform" $Ff$ has support in $[0, \omega]$. The next theorem can be considered as an abstract version of Paley-Wiener theorem.

**THEOREM 1.** The following conditions are equivalent:

a) a vector $f$ belongs to $PW_\omega(D)$;

b) a vector satisfies Bernstein inequality

$$\|D^k f\| \leq \omega^k \|f\|$$

for every natural $k$.

**PROOF.** Let $f$ belongs to the space $PW_\omega(D)$ and $Ff = x \in X$. Then

$$\left( \int_0^\infty \lambda^{2k} \|x(\lambda)\|^2_{X(\lambda)} dm(\lambda) \right)^{1/2} = \left( \int_0^\omega \lambda^{2k} \|x(\lambda)\|^2_{X(\lambda)} dm(\lambda) \right)^{1/2} \leq \omega^k \|x\|_X, \quad k \in \mathbb{N},$$

which gives Bernstein inequality for $f$.

Conversely, if $f$ satisfies Bernstein inequality then $x = Ff$ satisfies $\|x\|_{X^*} \leq \omega^k \|x\|_X$. Suppose that there exists a set $\sigma \subset [0, \infty] \setminus [0, \omega]$ whose $m$-measure is not zero and $x|_\sigma \neq 0$. We can assume that $\sigma \subset [\omega + \epsilon, \infty)$ for some $\epsilon > 0$. Then for any $k \in \mathbb{N}$ we have

$$\int_\sigma \|x(\lambda)\|^2_{X(\lambda)} dm(\lambda) \leq \int_{\omega + \epsilon}^\infty \lambda^{-2k} \lambda^{2k} \|x(\lambda)\|^2_{X(\lambda)} d\mu \leq \|x\|^2_{X} (\omega/\omega + \epsilon)^{2k}$$

which shows that or $x(\lambda)$ is zero on $\sigma$ or $\sigma$ has measure zero.

It is evident that the set $\bigcup_{\omega > 0} PW_\omega(D)$ is dense in $E$ and that the $PW_\omega(D)$ is a linear closed subspace in $E$.

In the case of a stratified group $H_m$ we use sub-Laplacian $D$ in the space $L_2(H_m)$. It is a self-adjoint positive definite operator. We apply the above construction to the operator $D$ and it gives us the notion of the space $PW_\omega(D)$ on the group $H_m$. Using results from [4] one can show that $f$ belongs to $PW_\omega(D)$ if and only if its Fourier transform has compact support in the following sense:

$$\hat{f}(\lambda)_{\alpha,\beta} = 0, (2|\beta| + m)|\lambda| > \omega^2$$

where $\hat{f}(\lambda)_{\alpha,\beta} = (\hat{f}(\lambda)F_{\alpha,\lambda}, F_{\beta,\lambda})$ and inner product is taken in the Bargmann space.

Let $B(x, r)$ be a ball in homogeneous metric $\rho(g, h) = |X - Y|, g = expX, h = expY$ with center $x \in H_m$ and radius $r$. Suppose that $\{B(x_\gamma, r)\}_\gamma \subset F$ is a cover of $H_m$. It is clear that this cover has a finite multiplicity $M$ in the sense that every ball from this family has non-empty intersections with no more than $M$ other
balls from the same family. Since metric $\rho(x, y)$ is homogeneous the family of balls $\{B(x_\gamma, 2^j r)\}_{x_\gamma \in \Gamma_j}$ will also be a cover of $H_m$ of the same multiplicity $M$.

Given a subgroup $\Gamma_j$ and a sequence $\{s_\gamma\} \in l_2$ we will be interested to find a function $s_{k,j} \in S^{2k}(H_m), k > Q/4$ such that

- a) function $s_{k,j}(x_\gamma) = s_\gamma, x_\gamma \in \Gamma_j$;
- b) function $s_{k,j}$ minimizes functional $u \rightarrow \|D^k u\|_2$.

Let us remark first that the same problem for functional $u \rightarrow \|u\|_{S^{2k}(H_m)}$, $u \in S^{2k}(H_m), k > Q/4$ can be solved easily.

Pick a ball $B(0, r)$ of very small radius $r$ and then by translations construct the family of pair ways disjoint balls $B(x_\gamma, r), x_\gamma \in \Gamma_j$. In the ball $B(0, r)$ we consider any function $\varphi_0 \in C_0^\infty(B(0, r))$ such that $\varphi(0) = 1$. Using translations we construct similar functions $\varphi_\gamma$ in balls $B(x_\gamma, r)$. Because of invariance all this functions have the same Sobolev norm

$$\|\varphi_\gamma\|_{S^k(H_m)} = \|\varphi\| + \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq 2^m} \|X_{i_1}X_{i_2}\ldots X_{i_k}\varphi\|, \gamma = 1, 2, \ldots$$

It is clear that for any sequence $\{s_\gamma\} \in l_2$ the formula

$$f = \sum s_\gamma \varphi_\gamma$$

defines a function from $S^k(H_m)$. Let $Pf$ will denote the orthogonal projection of this function $f$ (in the Hilbert space $S^{2k}(H_m)$ with natural inner product) on the subspace $U^{2k}(\Gamma_j) = \{f \in S^{2k}(H_m) | f(x_\gamma) = 0\}$ with $S^{2k}(H_m)$-norm. Then the function $g = f - Pf$ will be a unique solution of the above minimization problem for the functional $u \rightarrow \|u\|_{S^{2k}(H_m)}, k > Q/4$.

The problem with functional $u \rightarrow \|D^k u\|$ is that it is not a norm. But fortunatly we are able to show that for all natural $k > Q/4$ and all integer $j$ the norm

$$\|D^k f\| + \left(\sum_{x_\gamma \in \Gamma_j} |f(x_\gamma)|^2 \right)^{1/2}$$

is equivalent to the norm $\|f\|_{S^{2k}(H_m)}$. So, the above procedure can still be applied to the Hilbert space $S^{2k}(H_m)$ with inner product

$$<f, g> = \sum_{x_\gamma \in \Gamma_j} f(x_\gamma)g(x_\gamma) + <D^{k/2} f, D^{k/2} g>$$

and it clearly proves existance and uniqueness of the solution of our minimization problem for the functional $u \rightarrow \|D^k u\|, k > Q/4$.

3. We will need the following lemmas.

**Lemmas.** If $A$ is a self-adjoint operator in a Hilbert space and for some element $f$

$$\|f\| \leq b + a\|Af\|, a > 0,$$

then for all $m = 2^l, l = 0, 1, 2, \ldots$

$$\|f\| \leq mb + 8^{m-1}a^m\|A^m f\|$$

as long as $f$ belongs to the domain of $A^m$.

**Proof.** For any self-adjoint operator $B$ in a Hilbert space we have
\[ \|f\| \leq \|(I + \varepsilon iB)f\| \]
and the same for the operator \( I - \varepsilon iB \). It gives
\[ \varepsilon \|Bf\| \leq \|(I - \varepsilon iB)f\| + \|f\| \leq \|(I + \varepsilon^2 B^2)f\| + \|f\| \leq \varepsilon^2 \|B^2f\| + 2\|f\|. \]
So, for any \( f \) from the domain of \( B^2 \) we have the inequality
\[ \|Bf\| \leq \varepsilon \|B^2f\| + 2/\varepsilon \|f\|, \varepsilon > 0. \]
Our lemma is true for \( m = 1 \). If it is true for \( m \) then applying the last inequality for \( B = A^m \) we obtain
\[ \|f\| \leq mb + 8^{m-1} a^m (\varepsilon \|A^2mf\| + 2/\varepsilon \|f\|). \]
Setting \( \varepsilon = 8^{m-1}(a)^{2m} \), we obtain
\[ \|f\| \leq 2mb + 8^{2m-1}(a)^{2m} \|A^{2m}f\|. \]
The lemma is proved.

We consider Sobolev spaces \( S^\sigma(H_m) \) with the norm \( \|f\|_{S^\sigma(H_m)} = \|f\| + \|D^{\sigma/2}f\|, \sigma > 0 \) and for any open \( \Omega \) in \( H_m \) we define the space \( S^\sigma(\Omega) \) as the collection of all restrictions \( g_\Omega = g|_\Omega, g \in S^\sigma(H_m) \) with the norm \( \|g_\Omega\|_{S^\sigma(\Omega)} = \inf \|g\|_{S^\sigma(H_m)} \) where \( g \) runs over the set of all functions from \( S^\sigma(H_m) \) whose restriction to \( \Omega \) gives \( g_\Omega \).
Let \( B(\lambda, M) = B(x_\gamma, \lambda) \) be a cover of \( H_m \) of finite multiplicity \( M \). We introduce a map
\[ T_B(\lambda, M) : S^\sigma(H_m) \to l_2(S^\sigma(B_\gamma)), \sigma \geq 0, \]
where the Hilbert space on the right is defined as the set of all sequences \( g_\gamma, g_\gamma \in S^\sigma(B(x_\gamma, \lambda)) \) for which \( (\sum_\gamma \|g_\gamma\|^2_{S^\sigma(B(x_\gamma, \lambda))})^{1/2} < \infty \).

**LEMMA 3.** For any natural \( M \) and any \( \sigma \geq 0 \) there exists a \( C = C(M, \sigma) \) such that for every cover \( B(\lambda, M), \lambda > 0, \)
\[ \|T_B(\lambda, M)\| \leq C(M, \sigma) max(1, \lambda^{-\sigma}). \]

**PROOF.** Let \( \theta \in C_0^\infty(R), \theta(t) = 1, |t| \leq 1, \text{supp } \theta \subset [-2, 2] \). We define \( \theta_{\lambda, \gamma}(x) = \theta(\rho(0, \delta_{\lambda^{-1}}(xx_\gamma^-)), x \in H_m, \lambda > 0 \). Then \( \theta_{\lambda} \in C_0^\infty(H_m), \theta_{\lambda}(x) = 1, x \in B(x_\gamma, \lambda), \text{supp } \theta_{\lambda} \subset B(x_\gamma, 2\lambda) \).
It is clear that \( |X_{i_1}...X_{i_r}\theta_{\lambda}(x)| \leq C(r, \theta) \lambda^{-r} \). Therefore if \( f \in S^k(H_m), k \geq 0 \) is an integer, then
\[ \|f|_{B(x_\gamma, \lambda)}\|_{S^k(B(x_\gamma, \lambda))} \leq \|f\theta_{\lambda}\|_{S^k(H_m)} \leq \]
\[ \sum_{|\gamma| \leq k} \int_{B(x_\gamma, 2\lambda)} |X_{i_1}...X_{i_r}(f\theta_{\lambda})(x)|^2 d\mu \leq \]
\[ C(k, \theta) max(1, \lambda^{-2k}) \sum_{|\gamma| \leq k} \int_{B(x_\gamma, 2\lambda)} |X_{i_1}...X_{i_r}f(x)|^2 d\mu \]
and then
\[ \sum_{\gamma} \|f|_{B(x_\gamma, \lambda)}\|_{S^k(B(x_\gamma, \lambda))} \leq \]
\[ C(k, \theta) max(1, \lambda^{-2k}) \sum_{\gamma} \sum_{|\gamma| \leq k} \int_{B(x_\gamma, 2\lambda)} |X_{i_1}...X_{i_r}f(x)|^2 d\mu \leq \]
\[ C(k, M, \theta) \max(1, \lambda^{-2k}) \|f\|^{2}_{S^{k}(H_{m})}. \]

Thus for natural \( s = k \) lemma is proved. General case can be obtained by interpolation since for the complex interpolation functor \([\cdot, \cdot]_{\theta}\)

\[ [l_2(L_2(B_{\gamma})), l_2(S^\sigma(B_{\gamma}))]_{\theta} = l_2(S^{\sigma}(B_{\gamma})), 0 < \theta < 1. \]

The proofs of all main results in the present paper are based on the following inequalities.

**Theorem 4.** There exist a \( j_0 \in \mathbb{Z} \) and a constant \( C_0 \geq 0 \) such that for \( j \leq j_0 \) and every \( f \in S^{2k}(H_{m}), k = 2^j Q, l = 1, 2, \ldots, \) the following inequality takes place

\[ \|f\| \leq 2^j C_0 \left( \sum_{x_{\gamma} \in \Gamma_{j}} |f(x_{\gamma})|^2 \right)^{1/2} + (C_0 2^{j/2Q})^k \|D^k f\|. \]

In particular for \( f \in U^k(\Gamma_{j}) \)

\[ \|f\| \leq (C_0 2^{j/2Q})^k \|D^k f\|. \]

**Proof.** Let \( \{B(x_{\gamma}, 1)\}_{x_{\gamma} \in \Gamma} \) be a cover of \( H_{m} \) of the finite multiplicity \( M \).

The cover \( \{B(x_{\gamma}, 2^l)\}_{x_{\gamma} \in \Gamma_j, \Gamma_j = \delta_2 \Gamma} \) also has the same multiplicity \( M \). Let \( \psi_{\gamma}, \text{supp}\psi_{\gamma} \subset B_{\gamma} \) be a corresponding partition of unity.

For a function \( f \) from \( S^\sigma(H_{m}), \sigma > Q/2 \) we consider decomposition

\[ f(x) = \sum_{\gamma} f(x) \psi_{\gamma}(x) = \sum_{\gamma} f(x_{\gamma}) \psi_{\gamma}(x) + \sum_{\gamma} (f(x) - f(x_{\gamma})) \psi_{\gamma}(x) \]

and then

\[ \|f\|^2 \leq C \left\{ \sum_{\gamma} |f(x_{\gamma})|^2 + \sum_{\gamma} \int_{B(x_{\gamma}, \lambda)} |f(x) - f(x_{\gamma})|^2 d\mu \right\}, \]

where \( C \) depends only on multiplicity \( M \).

Since every vector field on the group \( H_{m} \) is a linear combination over \( C^\infty \) of the fields \( X_{i}, [X_{i}, X_{i2}], 1 \leq i, i_1, i_2 \leq 2m \), the Newton-Leibnitz formula gives

\[ |f(x) - f(x_{\gamma})|^2 \leq \]

\[ C4j \left( \sum_{1 \leq i_{1}, i_{2} \leq 2m} \left( \sup_{y \in B(x_{\gamma}, 2l)} |X_{i_{1}, X_{i_{2}} f(y)|} \right)^2 + \sum_{1 \leq i \leq 2m} \left( \sup_{y \in B(x_{\gamma}, 2l)} |X_{i} f(y)| \right)^2 \right). \]

Applying anisotropic version of Sobolev inequality [3] we obtain

\[ |f(x) - f(x_{\gamma})|^2 \leq \]

\[ C4j \left( \sum_{1 \leq i_{1}, i_{2} \leq 2m} \left( \sup_{y \in B(x_{\gamma}, 2l)} |X_{i_{1}, X_{i_{2}} f(y)|} \right)^2 + \sum_{1 \leq i \leq 2m} \left( \sup_{y \in B(x_{\gamma}, 2l)} |X_{i} f(y)| \right)^2 \right) \leq \]

\[ C4j \left( \sum_{1 \leq i_{1}, i_{2} \leq 2m} \|X_{i_{1}, X_{i_{2}} f\|}_{S^{Q/2+\varepsilon}(B(x_{\gamma}, 2l))}^2 + \sum_{1 \leq i \leq 2m} \|X_{i} f\|_{S^{Q/2+\varepsilon}(B(x_{\gamma}, 2l))}^2 \right), \]

where \( x \in B(x_{\gamma}, 2^j), \varepsilon > 0, C = C(X_{i}, ..., X_{2m}; \varepsilon). \)
An application of lemma 3 gives
\[ \sum_{\gamma} \int_{B(x_{\gamma},2^j)} |f(x) - f(x_{\gamma})|^2 d\mu \leq \]

\[ C(2^j)^{Q+2} \left( \sum_{1 \leq i_1, i_2 \leq 2m} \|X_{i_1}X_{i_2}f\|_{S^Q/2^p(H_m)}^2 + \sum_{1 \leq i \leq 2m} \|X_i f\|_{S^Q/2^p(H_m)}^2 \right) \leq \]

\[ C(2^j)^{2-2\varepsilon} \left( \sum_{1 \leq i_1, i_2 \leq m} \|X_{i_1}X_{i_2}f\|_{S^\sigma(H_m)}^2 + \sum_{1 \leq i \leq m} \|X_i f\|_{S^\sigma(H_m)}^2 \right), \]

where \( \sigma \geq Q/2 + \varepsilon \), \( C \) depends only on \( X_1, ..., X_m \) on \( \sigma \) and on multiplicity \( M \).

Since
\[ \|X_i f\|_{S^\sigma(H_m)} + \|X_{i_1}X_{i_2}f\|_{S^\sigma(H_m)} \leq C \left\{ \|f\| + \|D^{1+\sigma/2}f\| \right\}, \]
we have for particular choice of \( \varepsilon = 1/2, \sigma = 2Q - 2 \),
\[ \|f\| \leq C \left\{ \left( \sum_{\gamma} |f(x_{\gamma})|^2 \right)^{1/2} + 2^{j/2} \|D^Q f\| + 2j/2 \|f\| \right\}, \]

where \( C \) depends only on \( X_1, ..., X_m \) and multiplicity \( M \). Thus, if \( j \) is smaller than some \( j_0 = j_0(X_1, ..., X_m; M) \) it gives
\[ \|f\| \leq C \left\{ \left( \sum_{\gamma} |f(x_{\gamma})|^2 \right)^{1/2} + C^{2j/2} \|D^Q f\| \right\}, C = C(X_1, ..., X_m; M). \]

Using lemma 2 for \( A = D^Q, b = \left( \sum_{\gamma} |f(x_{\gamma})|^2 \right)^{1/2} \) we obtain
\[ \|f\| \leq C^{2l} \left( \sum_{\gamma} |f(x_{\gamma})|^2 \right)^{1/2} + (C_0 2^{l/2})^2 \|D^Q f\|, \]

where \( l = 0, 1, 2, ..., C_0 = 8C \). After all, for \( f \in U^{2^l Q}(\Gamma_j), j < j_0 \)
\[ \|f\| \leq (C_0 2^{l/2})^2 \|D^Q f\|, l = 0, 1, 2, ... \]

Theorem 4 is proved.

**Lemma 5.** For any natural \( k > Q/4 \) and any \( \Gamma_j = \delta_{2^l} \Gamma, j \in Z \), the norm \( \|f\|_{S^2(k)} \) is equivalent to the norm
\[ \|D^k f\| + \left( \sum_{x_{\gamma} \in \Gamma_j} |f(x_{\gamma})|^2 \right)^{1/2}. \]
PROOF. The proof of the theorem 4 shows that for every natural \( k > Q/4 \) there exists a \( j(k) \) such that for every \( j \leq j(k) \) there is a \( C = C(k,j) \) for which

\[
\|f\| \leq C \left\{ \|D^k f\| + \left( \sum_{x_\gamma \in \Gamma_j} |f(x_\gamma)|^2 \right)^{1/2} \right\}.
\]

Now using homogeneity arguments one can easily show that for every natural \( k > Q/4 \) and every integer \( j \) there exists a \( C = C(k,j) \) for which the above inequality takes place.

In order to prove inverse inequality we consider \( C_0^\infty(H_m) \) functions \( \phi_\gamma \) with disjoint supports such that \( \phi_\gamma(x_\gamma) = 1 \). Using Sobolev embedding theorem we obtain for \( k > Q/4 \)

\[
\left( \sum_\gamma |f(x_\gamma)|^2 \right)^{1/2} \leq C_k \left( \sum_\gamma \|f\phi_\gamma\|_{C^2(H_m)}^2 \right)^{1/2} \leq C_k \|f\|_{C^2(H_m)}, \quad k > Q/4.
\]

The proof of the lemma 5 is finished.

Next, we consider the following minimization problem [6]. For the given \( f \in S^{2k}(G) \), \( k > Q \), the \( s_{k,j}(f) \in S^{2k}(G) \) will be the function that minimizes \( u \rightarrow \|D^k u\| \) and takes the same values on \( \Gamma_j \) i.e. \( s_{k,j}(f)|_{\Gamma_j} = f|_{\Gamma_j} \). Since \( D \) is invariant with respect to translations it is clear that \( s_{k,j}(f) = \sum_{x_\gamma \in \Gamma_j} f(x_\gamma)L_{k,j}(x x_\gamma^{-1}) \) where \( L_{k,j}(x) \in S^{2k}(G) \) is the function that minimizes the same functional and \( L_{k,j}(0) = 1 \), and is zero at all other points of \( \Gamma_j \). In classical case such functions are called Lagrangian splines.

4. We prove the following approximation theorem.

THEOREM 6. There exists \( c_0 > 0 \) such that for \( j \leq j_0 \) the following estimate takes place

\[
\|f - s_{k,j}(f)\| \leq (c_0 2^{j/2} Q)^k \|D^k f\|, \quad f \in S^{2k}(H_m), \quad k = 2^l Q, \quad l = 1, 2, \ldots.
\]

PROOF. If \( f \in S^{2k}(H_m), \quad k = 2^l Q \) then \( f - s_{k,j}(f) \in U^{2k}(\Gamma_j) \) and according to the theorem 4 we have

\[
\|f - s_{k,j}(f)|\| \leq (c_0 2^{j/2} Q)^k \|D^k (f - s_{k,j}(f))\|.
\]

Using minimization property of \( s_{k,j}(f) \) we obtain

\[
\|f - s_{k,j}(f)\| \leq (c_0 2^{j/2} Q)^k \|D^k f\|, \quad k = 2^l Q,
\]

where \( c_0 = 2C_0 \) and the constant \( C_0 \) is from theorem 4.

Using theorem 6 and Bernstein inequality we immediately come to the following uniqueness and reconstruction theorem.

THEOREM 7. For the same constant \( c_0 > 0 \) as above

a) every function \( f \in PW_\omega(D), \omega > 0 \) is uniquely determined by its values on any set \( \Gamma_j = \delta_2 \Gamma \) as long as \( j < -2Q \log_2(\omega) \);

b) for every such set \( \Gamma_j = \delta_2 \Gamma \) the sequence of splines

\[
s_{k,j}(f)(x) = \sum_{x_\gamma \in \Gamma_j} f(x_\gamma)L_{k,j}(x x_\gamma^{-1}), \quad k = 2^l Q, \quad l = 1, 2, \ldots,
\]

converges to \( f \in PW_\omega(D) \) in \( L^2(H_m) \)-norm.
5. As an concluding remark we will show that functions $s_{k,j}$ have the following remarkable property (see [7]).

$$D^{2k}s_{k,j} = \sum_{x, \gamma \in \Gamma_j} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure and $\{\alpha_\gamma\} \in l_2$.

Indeed, suppose that $s_{k,j} \in S^{2k}(H_m)$ is a solution to the minimization problem and $h \in U^{2k}(\Gamma_j)$. Then

$$\|D^k(s_{k,j} + \lambda h)\|^2 = \|D^k s_{k,j}\|^2 + 2 \text{Re}\lambda \int_{H_m} D^k s_{k,j} D^k h d\mu + |\lambda|^2 \|D^k h\|^2.$$

The function $s_{k,j}$ can be a minimizer only if for any $h \in U^{2k}(\Gamma_j)$

$$\int_{H_m} D^k s_{k,j} D^k h d\mu = 0.$$

So, the function $\Phi = D^k s_{k,j} \in L_2(H_m)$ is orthogonal to $D^k U^{2k}(\Gamma_j)$. Let $\varphi_\gamma$ be the same set of functions as above and $h \in C^\infty_0(H_m)$. Then the function $h - \sum h(x_\gamma) \varphi_\gamma$ belongs to the $U^{2k}(\Gamma_j) \cap C^\infty_0(H_m)$. Thus,

$$0 = \int_{H_m} \Phi D^k(h - \sum h_\gamma \varphi_\gamma) d\mu = \int_{H_m} \Phi D^k h d\mu - \sum h_\gamma \int_{H_m} \Phi D^k \varphi_\gamma d\mu.$$

In other words

$$D^k \Phi = \sum_{x, \gamma \in \Gamma_j} \alpha_\gamma \delta(x_\gamma),$$

or

$$D^{2k}s_{k,j} = \sum_{x, \gamma \in \Gamma_j} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure.

Moreover for any integer $r > 0$

$$\sum_{\gamma=1}^{r} |\alpha_\gamma|^2 \leq \sum_{\gamma=1}^{\infty} \alpha_\gamma \delta(x_\gamma), \sum_{\gamma=1}^{r} \alpha_\gamma \varphi_\gamma \leq C \sum_{\gamma=1}^{\infty} \alpha_\gamma \delta(x_\gamma) \|S^{-2k}(H_m)\| \left(\sum_{\gamma=1}^{r} |\alpha_\gamma|^2\right)^{1/2},$$

where $C$ is independent on $r$. It shows that the sequence $\{\alpha_\gamma\}$ belongs to $l_2$.

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