Error Analysis of Cycloid Gear Based on Ellipse Fitting Based on Least Square Approach

Sai Lou1,*, Fucong Liu1 and Haoran Wang1
1 Tianjin University of Technology and Education, Tianjin, China

*Corresponding author email: lousai20160610@163.com

Abstract. In this paper, under the condition of the existing processing and detecting technology, we used coordinate measuring machine (CMM) to measure the contour of cycloid gear. In order to improve the accuracy of measured data, we used Euclidean distance and Laida criterion to preprocess measured data. After preprocessing the measured data of the cycloid gear, we used ellipse fitting based on least square approach to fit the contour data points of the cycloid gear, and used the determination coefficient method to evaluate the goodness of fitting. According to the result of the curve fitting of cycloid gear, we used Matlab to analyse and calculate the pitch errors of cycloid gear, which provides data for the subsequent matching of parts combination of the best RV reducer with genetic algorithm.

Keywords: Error analysis of cycloid gear; Preprocessing of measured data; Ellipse fitting based on least square approach.

1. Introduction

RV reducer is widely used in industrial robots, precision machine tools, aerospace, medical instruments and other fields because of its small volume, large torsional stiffness, compact structure, large transmission ratio, light weight and other advantages. As one of the key transmission parts of the RV reducer, the manufacturing precision of the cycloid gear is the main factor affecting the transmission performance and transmission precision of the RV reducer. In recent years, China has made some achievements in research and development and manufacturing of RV reducer, but it still lags behind the international advanced level in finishing and testing. Therefore, it is very necessary to study the error analysis of cycloid gear under the condition of the existing processing and testing technology.

In recent years, the method of improving the precision of cycloid gear has been studied deeply. Zhang Fengshou et al.\cite{1} from Henan University of Science and Technology used the least square method to fit the contour of cycloid gear by circular arc, and putted forward the combination modification method of "eccentricity" + "equidistance" + "shift distance". Under the condition that the short amplitude coefficient and eccentricity of the cycloid gear remain unchanged, Wang Jun et al.\cite{2} from Hubei University of Technology carried out rolling cutting and modification on the standard contour of the cycloid gear by using the ellipse, deduced the contour equation expression after modification, and obtained a relatively ideal tooth profile of the cycloid gear. Li Tianxing et al.\cite{3} from Henan University of Science and Technology putted forward a digital correction method for cycloid gear tooth surface machining by using the detection data of cycloid gear profile from a domestic gear measurement center. This paper combined with previous research methods on improving the accuracy of cycloid gear, using CMM to detect contour data points of cycloid gear, and using ellipse fitting based on least square approach to fit the contour data points of the cycloid gear, combined with the curve fitting result of cycloid gear to evaluate the the error of cycloid gear. It provides technical support for precision...
machining and error evaluation of key transmission parts of RV reducer in China, which has certain theoretical significance and practical value.

2. Measurement of Contour Data Points of Cycloid Gear

The measuring instrument used in this paper is Global Advantage 5.7.5 CMM produced by Hexagon Measurement Technology Co., Ltd., the measurement accuracy is 1.7 μm. During the measurement, the room temperature was always kept at 18℃-22℃ to ensure the requirements of the measurement environment. The cycloid gear of the RV-80E reducer was fixed on the measuring platform to ensure that the cycloid was perpendicular to the side head. Taking the geometric center of cycloid gear as the coordinate origin, measuring points were uniformly arranged and measured strictly in accordance with the operation manual. A total of 2824 contour points were measured.

![Figure 1. Contour data points of cycloid gear are measured by CMM.](image)

3. Preprocessing of Measurement Data

When the CMM is used to measure the contour data points of the cycloid gear of RV-80E reducer. The environment, noise and other factors will affect the result of measurement in the measurement process, which will cause some errors in the measurement data points. Therefore, it is necessary to preprocess the result of measurement after the measurement is completed. We use the elimination algorithm based on the combination of Euclidean distance and Laida criterion to eliminate the outliers of result of measurement. The principle as follows:

Euclidean distance refers to the real distance between two points in n-dimensional space\(^4\), and its mathematical expression as follows:

\[
d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}
\]  

(1)

Laida criterion is also known as 3 times standard deviation criterion\(^5\). Its principle is that, for a set of measured values \(d_1,d_2...d_n\) the arithmetic mean value of the set of values is:

\[
\bar{d} = \frac{\sum_{i=1}^{n}d_i}{n}
\]  

(2)

The absolute error of the measured data points is \(\Delta d_i = d_i - \bar{d}\) (i=1,2,3......n). The standard deviation of the measured value is:

\[
\sigma = \sqrt{\frac{\sum_{i=1}^{n}(\Delta d_i)^2}{n-1}}
\]  

(3)

If the absolute error of the measured data points satisfy \(|\Delta d_i| > 3\sigma\). It is considered that those points are outliers with a large error and should be eliminated. Based on the above principles, using Matlab to remove outliers. Firstly, calculating the Euclidean distance between two adjacent points in the measurement data points, and then calculating the mean value and standard deviation of the Euclidean distance between two adjacent points. If the absolute error of the measured data points (i.e. the difference between the Euclidean distance of two adjacent points and its mean value) is greater than 3 times the standard deviation, the Euclidean distance between this point and its adjacent points is considered to be too large or too small. A total of 320 outliers from the result of measurement were found by using the
above algorithm of outliers elimination. The effect of outliers elimination by using Matlab is shown in Figure 2.

![Figure 2. Outliers elimination.](image)

4. Curve Fitting of Contour Data Points of the Cycloid Gear

After preprocessing the measured data of the cycloid gear, curve fitting can be performed on the cycloid gear data. Firstly, we need to divide the dedendum and addendum of the cycloid gear. In this paper, we use the circle fitting based on least square approach to fit the measured contour data of cycloid gear. The fitting circle is used as the pitch circle of the cycloid gear to divide the addendum and dedendum of the cycloid gear. The principle of circle fitting based on least square approach as follows:

The standard equation of a circle can be written as:

\[
\frac{(x-x_0)^2}{r^2} + \frac{(y-y_0)^2}{r^2} = 1
\]  

(4)

It can be rewritten as:

\[
(x-x_0)^2 + (y-y_0)^2 = r^2
\]

(5)

\[
A = 2x_0, \quad B = 2y_0, \quad C = r^2 - x_0^2 - y_0^2
\]

(6)

The polynomial equation of the circle can be written:

\[
Ax + By + C - x^2 - y^2 = 0
\]

(7)

The coordinates of N data points to be fitted are \((x_N, y_N)\), the fitting error of each point is defined as:

\[
e_n = Ax + By + C - x^2 - y^2
\]

(8)

Take the first partial derivative of it, and set it as 0:

\[
2\begin{bmatrix}
    x_1 & y_1 & 1 \\
    x_2 & y_2 & 1 \\
    \vdots & \vdots & \vdots \\
    x_N & y_N & 1
\end{bmatrix}^T
\begin{bmatrix}
    x_1 & y_1 & 1 \\
    x_2 & y_2 & 1 \\
    \vdots & \vdots & \vdots \\
    x_N & y_N & 1
\end{bmatrix}
\begin{bmatrix}
    A \\
    B \\
    C
\end{bmatrix} + 2\begin{bmatrix}
    x_1 & y_1 \\
    x_2 & y_2 \\
    \vdots & \vdots \\
    x_N & y_N
\end{bmatrix}^T
\begin{bmatrix}
    x_1^2 + y_1^2 \\
    x_2^2 + y_2^2 \\
    \vdots \\
    x_N^2 + y_N^2
\end{bmatrix} = 0
\]

(10)

The least square solution of the vector \([A, B, C]^T\) can be obtained from the above equation as follows:

\[
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix} = \begin{bmatrix}
    x_1 & y_1 & 1 \\
    x_2 & y_2 & 1 \\
    \vdots & \vdots & \vdots \\
    x_N & y_N & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
    x_1 & y_1 & 1 \\
    x_2 & y_2 & 1 \\
    \vdots & \vdots & \vdots \\
    x_N & y_N & 1
\end{bmatrix}^T
\begin{bmatrix}
    x_1^2 + y_1^2 \\
    x_2^2 + y_2^2 \\
    \vdots \\
    x_N^2 + y_N^2
\end{bmatrix}
\]

(11)

The circle parameters can be obtained as follows:
The fitting equation of the circle can be obtained by substituting the parameters \([A \quad B \quad C]^T\) into the standard equation of the circle \((x-x_0)^2+(y-y_0)^2=r^2\). The data point matrix to be fitted can be defined in Matlab, and each parameter of the fitting circle can be solved by the above formula. According to each parameter of the fitting circle, the fitting circle is drawn in the form of parameter equation, as shown in Figure 3.

Since the profile of a cycloidal gear machined is not a theoretical circle in the mathematical concept, but an approximate circular tooth profile curve composed of multiple discrete points, THEREFORE, the ellipse fitting based on least square approach is used to fit the addendum of cycloid gear \([6]\), and the ellipse fitting principle of the least-square method as follows:

The second order polynomials for elliptic equations is:

\[
Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \quad (13)
\]

\[
A=\frac{(\cos \theta)^2}{a^2} + \frac{(\sin \theta)^2}{b^2} \quad (14)
\]

\[
B=2\sin \theta \cos \theta \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (15)
\]

\[
C=\frac{(\sin \theta)^2}{a^2} + \frac{(\cos \theta)^2}{b^2} \quad (16)
\]

\[
D=-\frac{2h(\cos \theta)^2-2k \sin \theta \cos \theta}{a^2} + \frac{-2h(\sin \theta)^2+2k \sin \theta \cos \theta}{b^2} \quad (17)
\]

\[
E=-\frac{2k(\sin \theta)^2-2h \sin \theta \cos \theta}{a^2} + \frac{-2k(\cos \theta)^2+2h \sin \theta \cos \theta}{b^2} \quad (18)
\]

\[
F=\left(\frac{h \cos \theta + k \sin \theta}{a}\right)^2 + \left(\frac{h \sin \theta - k \cos \theta}{b}\right)^2 - 1 \quad (19)
\]

Combined with N discrete data points of the ellipse to be fitted, let the ellipse fitting error be:

\[
e_n=Ax_n^2+Bx_ny_n+Cy_n^2+Dx_n+Ey_n+F\quad (20)
\]

The parameters of the fitting ellipse are obtained as follows:

\[
\theta=\arctan\left(\frac{B}{A-C}\right)
\]

\[
\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix} = \begin{bmatrix}
  -2A \\
  -B \\
  -2C
\end{bmatrix}^{-1} \begin{bmatrix}
  D \\
  E
\end{bmatrix}
\]

\[
a=\sqrt{\frac{\frac{y_0}{B/2}}{A(\cos \theta)^2-B \sin \theta \cos \theta+C(\sin \theta)^2}} \quad (21)
\]

\[
b=\sqrt{\frac{\frac{y_0}{B/2}}{A(\cos \theta)^2-B \sin \theta \cos \theta+C(\sin \theta)^2}} \quad (21)
\]

Using Matlab to define the fitting matrix of cycloid gear addendum, and according to the ellipse fitting formula of the least-square approach, calculating each parameter of the fitting ellipse. According to each parameter of the fitting ellipse, the fitting ellipse is drawn in the form of parameter equation. The diagram of fitting the of cycloid gear addendum as shown in Figure 4.
Evaluating the Effect of Fitting by the Determination Coefficient Method

Determination coefficient method is used to verify the goodness of addendum of cycloid gear. The principle of the determination coefficient method is as follows:

For $m$ samples:

$$\begin{pmatrix} x_1, y_1 \\ x_2, y_2 \\ \vdots \\ x_m, y_m \end{pmatrix}$$

The estimated value of the model is:

$$\begin{pmatrix} \bar{x}_1, \bar{y}_1 \\ \bar{x}_2, \bar{y}_2 \\ \vdots \\ \bar{x}_m, \bar{y}_m \end{pmatrix}$$

The total sum of squares of the sample is:

$$\text{TSS} = \sum_{i=1}^{m} (y_i - \bar{y})^2$$

The sum of squares of the residuals is:

$$\text{RSS} = \sum_{i=1}^{m} (\bar{y}_i - y_i)^2$$

The determination coefficient:

$$R^2 = 1 - \frac{\text{RSS}}{\text{TSS}} = 1 - \frac{\sum_{i=1}^{m} (y_i - \bar{y}_i)^2}{\sum_{i=1}^{m} (y_i - \bar{y})^2}$$

The value of the determination coefficient is between 0 and 1, and the closer it is to 1, the better the fitting effect is. Generally speaking, models with a determination coefficient over 0.8 have a higher goodness of fitting. According to the above equation, the determination coefficient of ellipse fitting based on least square approach for cycloid gear is 0.9994. It can be seen that the goodness of fitting is very high. So it can be used as the basis for the cycloid error evaluation.

Analysis and Calculation of Cycloid Gear Error

Pitch error of a gear determines its working stability and is one of the important factors to evaluate its working performance. Better working stability can well reduce the noise, impact and vibration of the gear when it is working, and can well improve the service life of the gear. The single pitch error of a gear (also known as the adjacent pitch error) $\pm f_{pt}$ is defined as: On a circle concentric with the gear axis near the middle of the tooth height, the algebraic difference between the actual arc length and the nominal arc length (the average of all actual arc lengths) is shown in Figure 5.
As shown in Figure 5, the difference between the included angle on any adjacent ipsilateral tooth profile $\phi(i)$ and the nominal included angle $\phi$ (the average value of the actual included angle) is calculated by the following equation:

$$\Delta \phi = \phi(i) - \phi(i-1) - \phi $$  \hspace{1cm} (27)

The adjacent pitch errors (single pitch errors) of the cycloid gear and the accumulation pitch error can be calculated as follows:

$$f_{pt} = R \cdot \Delta \phi$$ \hspace{1cm} (28)

$$F_p = R \cdot \left\{ \frac{Z_a}{\max_{k=1}^{Z_a} \sum_{i=1}^{k} \Delta \phi(i)} \right\} - \left\{ \frac{Z_a}{\min_{k=1}^{Z_a} \sum_{i=1}^{k} \Delta \phi(i)} \right\}$$ \hspace{1cm} (29)

In the above formula, $Z_a$ is the number of teeth of the cycloid gear, and $R$ is the radius of the pitch circle of the cycloid gear.

### 6.1. Analysis and Calculation of Adjacent Pitch Error of Cycloid Gear

According to the tooth pitch error model and calculation formula of cycloid gear, and combined with the results of fitting of cycloid gear addendum. Using Matlab to analyze and calculate the tooth pitch error of cycloid gear. Connecting the center of each ellipse to the geometric center of the cycloid, and the included angle between the lines is the pitch angle of the cycloid, as shown in Figure 6.

According to the fitting results of the cycloid gear addendum. Matlab is used to analyze the adjacent pitch errors of the cycloid gear. The adjacent pitch error of the cycloid gear were shown in Figure 7, in which the x-axis was the tooth number of the cycloid gear, and the y-axis was the error value of the adjacent pitch of the cycloid gear.

Figure 5. Tooth pitch error model of cycloid gear.

Figure 6. Tooth pitch error analysis diagram of cycloid gear.

Figure 7. Diagram of adjacent pitch error of cycloid gear.
According to the analysis diagram of adjacent pitch error of cycloid gear and formula for calculating the adjacent pitch error of cycloid gear, the adjacent pitch error value of cycloid gear is calculated as shown in Table 1.

**Table 1. Error value of adjacent pitch of cycloid gear.**

| Tooth Number | Adjacent Pitch Error (mm) | Tooth Number | Adjacent Pitch Error (mm) | Tooth Number | Adjacent Pitch Error (mm) |
|--------------|----------------------------|--------------|----------------------------|--------------|----------------------------|
| 1            | 0.0023                     | 14           | 0.0001                     | 27           | -0.0005                    |
| 2            | -0.0013                    | 15           | -0.0036                    | 28           | 0.0032                     |
| 3            | -0.0042                    | 16           | 0.0028                     | 29           | -0.0006                    |
| 4            | 0.0026                     | 17           | 0.0013                     | 30           | -0.0010                    |
| 5            | -0.0034                    | 18           | -0.0022                    | 31           | 0.0028                     |
| 6            | -0.0019                    | 19           | 0.0025                     | 32           | -0.0022                    |
| 7            | 0.0030                     | 20           | 0.0006                     | 33           | -0.0018                    |
| 8            | 0.0016                     | 21           | -0.0032                    | 34           | 0.0022                     |
| 9            | -0.0014                    | 22           | -0.0009                    | 35           | -0.0009                    |
| 10           | 0.0031                     | 23           | 0.0011                     | 36           | -0.0016                    |
| 11           | -0.0016                    | 24           | -0.0033                    | 37           | 0.0057                     |
| 12           | -0.0015                    | 25           | 0.00027                    | 38           | 0.0006                     |
| 13           | 0.0006                     | 26           | -0.0001                    | 39           | -0.0015                    |

6.2. Analysis and Calculation of Accumulation Pitch Error of Cycloid Gear Pitch

The accumulation pitch error of the gear is defined as: On the pitch circle, the maximum absolute value of the difference between the actual and nominal arc lengths between any two ipsilateral tooth surfaces is shown in Fig. 5. According to the definition of accumulation pitch error of gear pitch, combined with the fitting results of cycloid gear addendum, the analysis results of error of cycloid gear are shown in Figure 8.

**Figure 8.** Diagram of accumulation pitch error of cycloidal gear.

According to the definition of accumulation pitch error and Figure 8, and using Matlab to calculate the accumulation pitch error of cycloid gear, as shown in Table 2.
Table 2. Accumulation pitch error value of cycloid gear.

| Tooth number | Accumulation pitch error (mm) | Tooth number | Accumulation pitch error (mm) | Tooth number | Accumulation pitch error (mm) |
|--------------|-------------------------------|--------------|-------------------------------|--------------|-------------------------------|
| 1            | 0.0059                        | 14           | 0.0049                        | 27           | 0.0058                        |
| 2            | 0.0092                        | 15           | 0.0049                        | 28           | 0.0063                        |
| 3            | 0.0079                        | 16           | 0.0071                        | 29           | 0.0032                        |
| 4            | 0.0046                        | 17           | 0.0043                        | 30           | 0.0037                        |
| 5            | 0.0064                        | 18           | 0.0054                        | 31           | 0.0047                        |
| 6            | 0.0054                        | 19           | 0.0052                        | 32           | 0.0044                        |
| 7            | 0.0074                        | 20           | 0.0057                        | 33           | 0.041                         |
| 8            | 0.0043                        | 21           | 0.0063                        | 34           | 0.0060                        |
| 9            | 0.0056                        | 22           | 0.0053                        | 35           | 0.0038                        |
| 10           | 0.0042                        | 23           | 0.0062                        | 36           | 0.0047                        |
| 11           | 0.0073                        | 24           | 0.0051                        | 37           | 0.0063                        |
| 12           | 0.0057                        | 25           | 0.0054                        | 38           | 0.0008                        |
| 13           | 0.0043                        | 26           | 0.0057                        | 39           | 0.0015                        |

6.3. Error Analysis and Calculation of Cycloid Gear Tooth Center Radius

In Figure 6, the center of the ellipse is the center of the cycloid gear teeth, the radius error of the center circle of the tooth is used to measure the degree of deviation between the tooth and the geometric center of the cycloid. This deviation will affect the radial assembly effect of cycloid gear, and then affect the transmission accuracy of RV reducer. Therefore, it is necessary to analyze the center radius error of cycloid gear teeth. According to the fitting results of cycloid gear, using Matlab to analyze the error of the center radius of cycloid gear tooth is shown in Figure 9 and Figure 10.

Figure 9. Error analysis diagram of cycloid gear tooth center radius.

Figure 10. Diagram of center radius error of cycloid gear teeth.

According to the results of tooth accumulation fitting of the cycloid gear, combined with the error analysis figure of the center radius of the cycloid gear tooth, the error value of the center radius of the cycloid gear 1 tooth is calculated as shown in Table 3.
Table 3. Error value of center radius of cycloid gear tooth.

| Tooth number | Radius error (mm) | Tooth number | Radius error (mm) | Tooth number | Radius error (mm) |
|--------------|-------------------|--------------|-------------------|--------------|-------------------|
| 1            | 0.0059            | 14           | -0.0132           | 27           | 0.0090            |
| 2            | 0.0019            | 15           | -0.0182           | 28           | 0.0232            |
| 3            | -0.0190           | 16           | 0.0005            | 29           | 0.0189            |
| 4            | -0.0212           | 17           | 0.0159            | 30           | -0.0062           |
| 5            | -0.0110           | 18           | 0.0099            | 31           | 0.0027            |
| 6            | -0.0068           | 19           | 0.0217            | 32           | -0.0122           |
| 7            | 0.0149            | 20           | -0.0091           | 33           | -0.0246           |
| 8            | 0.0240            | 21           | -0.0061           | 34           | -0.0105           |
| 9            | 0.0168            | 22           | -0.0085           | 35           | 0.0012            |
| 10           | 0.0152            | 23           | -0.0255           | 36           | -0.0008           |
| 11           | 0.0005            | 24           | -0.0271           | 37           | 0.0187            |
| 12           | -0.0132           | 25           | -0.0067           | 38           | 0.0218            |
| 13           | -0.0159           | 26           | 0.029             | 39           | 0.0089            |

7. Conclusion
In this paper, under the condition of the existing processing and detecting technology, the contour of cycloid gear were measured by coordinate measuring machine (CMM). The measured data points were preprocessed by using Euclidean distance and Laida criterion. The contour data points of the cycloid gear which were preprocessed were fitted by using ellipse fitting based on least square approach. The goodness of fitting was evaluated by using determination coefficient method. It is concluded that the fitting of cycloid gear addendum by least-square method has a higher goodness of fitting. Combined with result of the curve fitting of cycloid gear, and the tooth pitch error model of cycloid gear and the calculation formula of tooth pitch. Error of cycloid gear was analyzed and calculated by using Matlab, which provides data for the subsequent matching of parts combination of the best RV reducer with genetic algorithm.

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