The effect of algorithm parameters on the number of iterations of the dual fat boundary method

Kui Liu¹, Zhendong Hu¹

¹School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai, 200092, PR China

Corresponding author and e-mail: Zhendong Hu, zdhu@tongji.edu.cn

Abstract. The Dual Fat Boundary Method (DFBM) is an improvement of the Fat Boundary Method (FBM), which is proposed to solve elasticity problems defined on a perforated domain. The DFBM does not need an analytical solution in the holes, so it is suitable for any reasonable hole shapes and Dirichlet boundary conditions on the holes. However, as an iteration method, the relationship between iteration times and algorithmic parameters has not been studied. Here we give a numerical study by taking the perforated infinite plate problem as an example. It reveals that the relaxation parameters for two local domains should be equal to obtain the fastest convergence. Furthermore, the number of iterations for DFBM is smaller than FBM in the larger range of relaxation parameters despite of the decreasing range of convergence.

1. Introduction

Mesh generation techniques are needed for the traditional Finite Element Method (FEM). However, it is difficult to completely solve the mesh generation problem [1]. Ted Blacker [2] reports that it takes about 80% of the overall analysis time for mesh generation and only 20% for analysis. Thus, it is time-consuming for mesh generation and it is necessary to develop some methods to tackle this problem. Such methods include isogeometric analysis [3], meshless method [4], and fictitious domain method [5].

Fictitious domain methods put the physical domain in a frame which is larger and more regular and can be meshed by a structured mesh. The fat boundary method [6] is one type of fictitious domain methods which has the capacity of computing more accurately around the holes. At the same time, it still uses a relatively coarse structured mesh for the global domain. This method has been applied to fluid dynamics [7]. However, this method has some restrictions. For example, it needs an analytical solution in the hole before the numerical solution. Although some methods have been proposed to avoid solving the analytical solution, a harmonic extension is also necessary. This restricts the application of this method, such as into elasticity.

Liu [8] applied the FBM to elasticity and proposed the Dual Fat Boundary Method (DFBM) which breaks these restrictions. It reveals that the accuracy of DFBM is the same with FBM and it can abandon the analytical solution totally. The analytical solution in the holes can be solved numerically at the same time. However, it introduces some algorithmic parameters, such as the relaxation parameter of the second local domain. How these parameters influence the convergence of the method needs to be studied. This paper studied this by taking the perforated infinite plate problem as an example.
2. A review of the FBM and DFBM

For the Poisson problem in a domain with holes:
\[
\begin{align*}
-\Delta u &= f \text{ in } \Omega \\
\gamma_1 \left|_{\Gamma_0} & = 0 \right. \\
\gamma_1 \left|_{\Gamma_1} & = \gamma_\Gamma \right. \\
\gamma_0 \left|_{\Gamma_0} & = 0
\end{align*}
\]
(1)

Which is defined on the perforated domain \( \Omega \) (see Fig.1). \( \Gamma \) and \( \Gamma_0 \) are outer and inner boundaries of \( \Omega \) respectively.

Maury [6] divided the domain into two subdomains: the local domain \( \Omega_1 \) and and the global fictitious domain \( \Omega_{fict} \). The local domain is bounded by \( \Gamma_0 \) and \( \Gamma_1 \). The global domain is the union of \( \Omega \) and \( \Omega_0 \) (see Fig.1). By this way, the Poisson problem (1) can be stated as follows:

\[
\begin{align*}
I. \quad & \begin{cases}
\Delta u_1 = f & \text{in } \Omega_1 \\
u_1 |_{\Gamma_0} = 0, \ u_1 |_{\Gamma_1} = \bar{u} |_{\Gamma_1}
\end{cases} \\
II. & \begin{cases}
\Delta \bar{u} = \bar{f} + \frac{\partial u_1}{\partial n} \delta r_0 \text{ in } \Omega_{fict} \\
\bar{u} |_{\Gamma} = 0
\end{cases}
\end{align*}
\]
(2)

Where \( u_1 \) is the variable \( u \) while defined in part of domain \( \Omega_1 \). \( \bar{u} \) and \( \bar{f} \) are defined in \( \Omega_{fict} \) and vanishes in \( \Omega_0 \), which are the extension of \( u \) and \( f \) respectively. The derivative \( \partial u_1 / \partial n \) is taken outward with respect to \( \Omega \). \( \delta r_0 \) is a function which is unity on \( \Gamma_0 \) and is zero elsewhere.

![Figure 1. FBM: problem definition.](image)

To solve problems (2) iteratively, we introduce the operator \( \mathcal{T} \):

\[
\mathcal{T}: (\bar{u}, u_1) \rightarrow (\bar{u}', u_1')
\]
(3)

where \( u_1' \) and \( \bar{u}' \) are solutions of

\[
\begin{align*}
I. \quad & \begin{cases}
\Delta u_1' = f & \text{in } \Omega_1 \\
u_1'|_{\Gamma_0} = 0, \ u_1'|_{\Gamma_1} = (\theta u_1 + (1 - \theta)\bar{u}) |_{\Gamma_1}
\end{cases} \\
II. & \begin{cases}
\Delta \bar{u}' = \bar{f} + \frac{\partial u_1'}{\partial n} \delta r_0 \text{ in } \Omega_{fict} \\
\bar{u}' |_{\Gamma} = 0
\end{cases}
\end{align*}
\]
(4)

and

\[
\begin{align*}
I. \quad & \begin{cases}
\Delta u_1' = f & \text{in } \Omega_1 \\
u_1'|_{\Gamma_0} = 0, \ u_1'|_{\Gamma_1} = (\theta u_1 + (1 - \theta)\bar{u}) |_{\Gamma_1}
\end{cases} \\
II. & \begin{cases}
\Delta \bar{u}' = \bar{f} + \frac{\partial u_1'}{\partial n} \delta r_0 \text{ in } \Omega_{fict} \\
\bar{u}' |_{\Gamma} = 0
\end{cases}
\end{align*}
\]
(5)
respectively. The letters with prime denote variables of the next iteration. The relaxation parameter is in the range between 0 and 1, which is introduced to guarantee the convergence. Maury [6] proved that the operator $\mathcal{T}$ is convergent if $\theta \in ]1 - 2/(1 + C)^2, 1[$, where $C$ is a constant of the problem.

For problems with non-homogeneous boundary conditions defined on $\Gamma_0$, the FBM equations (2) are:

$$
\begin{cases}
I. & -\Delta u_1 = f \text{ in } \Omega_1 \\
 & u_{1|\Gamma_0} = g, u_{1|\Gamma_1} = \bar{u}|_{\Gamma_1} \\
 & \bar{u}|_{\Gamma_0} = 0
\end{cases}
$$

(6)

It can be seen from (6) that a harmonic extension $u_0$ is needed. However, it is not easy to obtain $u_0$ for the most general problem. Therefore, Liu etc. [8] applied the FBM to elasticity and proposed the DFBM. The DFBM can be stated as follows:

For the elasticity equations defined on the domain with holes (see Fig.2):

$$
\begin{cases}
-\nabla \cdot \sigma = f \text{ in } \Omega \\
u_{i|\Gamma_{Di}} = g_{1i}, u_{i|\Gamma_{oi}} = g_{2i} \\
(\sigma_{ij} n_j)_{\nu_{Ni}} = t_i 
\end{cases}
$$

(7)

where $u_i$ is displacements, $\sigma$ is the stress tensor and $f$ is the body force vector. $g_{1i}$ and $g_{2i}$ are prescribed boundary conditions defined on $\Gamma_{Di}$ and $\Gamma_{Oi}$ respectively. $t_i$ is the traction boundary conditions defined on $\Gamma_{Ni}$. $n_i$ is the components of outward normal unit vector.

The DFBM equations of the problem (7) are:

$$
\begin{cases}
I. & -\nabla \cdot \sigma(u_1) = f \text{ in } \Omega_1 \\
 & u_{1|\Gamma_{Di}} = g_{2i}, u_{1|\Gamma_{oi}} = \bar{u}_{i|\Gamma_{oi}} \\
 & \bar{u}|_{\Gamma_{0i}} = 0 \\
II. & -\nabla \cdot \sigma(\bar{u}) = \bar{f} + \left(\sigma(u_1) \cdot n - \sigma(u_0) \cdot n\right)\delta_{\Gamma_0} \text{ in } \Omega_{\text{fict}} \\
 & \bar{u}|_{\Gamma_{Di}} = g_{1i}, (\sigma_{ij} n_j)_{\nu_{Ni}} = t_i \\
III. & -\nabla \cdot \sigma(u_2) = 0 \text{ in } \Omega_2 \\
 & u_{2|\Gamma_{0}} = g_2, u_{2|\Gamma_{2}} = \bar{u}|_{\Gamma_{2}}
\end{cases}
$$

(8)

Figure 2. DFBM: problem definition.

To solve problems (8) iteratively, we can also introduce the operator $\mathcal{T}^c$:

$$
\mathcal{T}^c: (d_1^m, d_2^m, d_3^m) \rightarrow (d_1^{m+1}, d_2^{m+1}, d_3^{m+1})
$$

(9)
The difference between DFBM and FBM is that we need to introduce two relaxation parameters $\theta$ and $\theta'$ for the two local domains. We will study the convergence of $T'$ with respect to these parameters in this article.

3. Convergence of DFBM with respect to the algorithmic parameters

The convergence of DFBM was proved in [8]. This section will study the influence of algorithmic parameters to the convergence of DFBM, such as the number of iterations. We consider the perforated infinite plate and the radial displacement of the hole is $\epsilon = 0.01$ (see Fig.3). The analytical solution for this problem is

$$u_r = \frac{\epsilon R}{r}, \quad u_0 = 0$$

(10)

![Figure 3. Infinite plates with expanded hole: problem definition.](image)

A harmonic extension is needed for the FBM because the conditions on $\Gamma_0$ is non-homogenous, while there is no need of this extension if we use the DFBM instead. The cost is one more local domain to be discretized (see Figure. 4).

![Figure 4. Meshes of DFBM.](image)

$r_1$ and $r_2$ denote the outer radius of the local mesh 1 and the inner radius of the local mesh 2 respectively. The relaxation parameters are $\theta$ and $\theta'$ respectively. The iteration is regarded as convergent when the energy norm of error between numerical and analytical solutions is no more than $10^{-2}$, i.e.

$$e^m \equiv \frac{\int_{\Omega} (\epsilon_m - \epsilon_a) (\sigma_m - \sigma_a)}{\int_{\Omega} \epsilon_a \sigma_a} \leq 10^{-2}$$

(11)

Firstly, the choice of relaxation parameter $\theta'$ is studied. The outer radius of local mesh 1 $r_1$ remains unchanged and different relaxation parameter $\theta$ is chosen. The number of iterations is plotted for different $r_2$ and $\theta'$ (see Figure. 5 – Figure.7).
Figure 5. The best choice of $\theta'$ is the same as $\theta$ ($\theta = 0.8$).

Figure 6. The best choice of $\theta'$ is the same as $\theta$ ($\theta = 0.85$).

Figure 7. The best choice of $\theta'$ is the same as $\theta$ ($\theta = 0.9$).

It can be seen from Fig.5 to Fig.7 that there is a sharp decrease of number of iterations at the point $\theta' = \theta$. This can also be observed when more sampling points are chosen in the vicinity of $\theta$, such as $\theta \pm 0.1, 0.01$. Therefore, the best choice of $\theta'$ is the same as $\theta$, i.e. $\theta' = \theta$.

Secondly, the influence of local domain size and a compare with FBM are studied. The same relaxation parameters are used according to the study made above. On the one hand, the size of local domain 1 increases by increasing the outer radius $r_1$ while remaining the inner radius $r_2$ unchanged. On the other hand, the size of local domain 2 increases by decreasing the inner radius $r_2$ while remaining the outer radius $r_1$ unchanged. The iteration curves are shown in Fig. 8 and Fig. 9.
It can be seen that there is a minimum number of iterations for each curve, which corresponds to the optimal value of $\theta$. For larger local domains (i.e. the larger $r_1$ or smaller $r_2$), the range of convergence increases and meanwhile, the optimal $\theta$ decreases. Comparing with FBM in Fig. 9, it can also be seen that the range of convergence of DFBM is smaller than FBM, while the number of iterations is smaller than FBM in the larger range of $\theta$ (for example, $\theta \geq 0.75$ for $r_2 = 0.35$).

4. Conclusions

The DFBM is an improved method based on FBM, which has great advantages for elasticity problems defined on a perforated domain. This article researches the convergence of DFBM with respect to the algorithmic parameters, such as the relaxation parameters and local domain sizes. It reveals that the relaxation parameters $\theta$ and $\theta'$ should be equal to obtain the fastest convergence. Increasing the local domains can increase the range of convergence. The number of iterations for DFBM is smaller than FBM in the larger range of relaxation parameters in spite of the decreasing range of convergence.

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