THE LOCAL STRUCTURE THEOREM FOR REAL SPHERICAL VARIETIES

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Abstract. Let $G$ be an algebraic real reductive group and $Z$ a real spherical $G$-variety, that is, it admits an open orbit for a minimal parabolic subgroup $P$. We prove a local structure theorem for $Z$. In the simplest case where $Z$ is homogeneous, the theorem provides an isomorphism of the open $P$-orbit with a bundle $Q \times L/S$. Here $Q$ is a parabolic subgroup with Levi decomposition $L \rtimes U$, and $S$ is a homogeneous space for a quotient $D = L/L_n$ of $L$, where $L_n \subseteq L$ is normal, such that $D$ is compact modulo center.

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1. Introduction

Let $G_C$ be a complex reductive group and $B_C < G_C$ a fixed Borel subgroup. We recall that a normal $G_C$-variety $Z_C$ is called spherical provided that $B_C$ admits an open orbit. The local nature of a spherical variety is given in terms of the local structure theorem, [3], [4]. In its simplest form, namely applied to a homogeneous space $Z_C = G_C/H_C$ for which $B_C H_C$ is open, it asserts that there is a parabolic subgroup $Q_C > B_C$ with Levi-decomposition $Q_C = L_C \rtimes U_C$ such that the action of $Q_C$ on $Z_C$ induces an isomorphism of $(L_C/L_C \cap H_C) \times U_C$ onto $B_C H_C$.

The purpose of this paper is to continue the geometric study of real spherical varieties begun in [7]. We let $G$ be an algebraic real reductive group and $Z$ a normal real algebraic $G$-variety. Then $Z$ is called real spherical provided a minimal parabolic subgroup $P < G$ has at least one open orbit on $Z$. With this assumption on $Z$ we prove a local structure theorem analogous to the one above. In particular, when applied to a homogeneous real spherical space $Z = G/H$ with $PH$ open, it yields a parabolic subgroup $Q > P$ with Levi-decomposition $Q = L \times U$ such that

$$L_n < Q \cap H < L.$$ 

Here $L_n \triangleleft L$ denotes the product of all non-compact non-abelian normal factors of $L$. Furthermore, the action of $Q$ induces a diffeomorphism of $(L/L \cap H) \times U$ onto $PH$.

Our proof of the real local structure theorem is based on the symplectic approach of [4]. Our investigations also show the number of $G$-orbits on a real spherical variety is finite. Combined with the main result of [7] it implies that the number of $P$-orbits on a real spherical variety is finite.

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2. Homogeneous spherical spaces

Lie groups in this paper will be denoted by upper case Latin letters, $A, B \ldots$, and their associated Lie algebras with the corresponding lower case Gothic letter $a, b$ etc.

For a Lie group $G$ we denote by $G_0$ its connected component containing the identity, by $Z(G)$ the center of $G$ and by $[G, G]$ the commutator subgroup.

On a real reductive Lie algebra $\mathfrak{g}$ we fix a non-degenerate invariant bilinear form $\kappa(\cdot, \cdot)$, for example the Cartan-Killing form in case $\mathfrak{g}$ is semisimple.
A Lie group $G$ will be called real reductive provided that

- The Lie algebra $\mathfrak{g}$ is reductive.
- There exists a maximal compact subgroup $K < G$ such that we have a homeomorphism (polar decomposition)
  \[ K \times s \to G, \quad (k, X) \mapsto k \exp(X) \]

where $s := \mathfrak{k}^\perp$.

Observe that for a real reductive group the bilinear form $\kappa$ can (and will) be chosen $K$-invariant. A real reductive group is called algebraic if it is isomorphic to an open subgroup of the group of real points $G_C(\mathbb{R})$ where $G_C$ is a reductive algebraic group which is defined over $\mathbb{R}$.

Let now $G$ be a real reductive group, and let $P$ be a minimal parabolic subgroup. The unipotent part of $P$ is denoted $N$. If a maximal compact subgroup $K$ as above has been chosen, with associated Cartan involution $\theta$ of $G$, a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ can also be chosen. These choices then induce an Iwasawa decomposition $G = KAN$ of $G$ and a Langlands decomposition $P = MAN$ of $P$. Here $M = Z_K(\mathfrak{a})$. However, at present we do not fix $K$ and $\mathfrak{a}$.

Let $H$ be a closed subgroup of $G$ such that $H/H_0$ is finite. Recall that $Z = G/H$ is said to be real spherical, if the minimal parabolic subgroup $P$ admits an open orbit on $Z$. Furthermore, in this case $H$ is called a spherical subgroup. Note that $H$ is not necessarily reductive.

**Remark 2.1.** Here a remark on terminology is in order. Historically, spherical subgroups were first introduced by M. Krämer in the context of compact Lie groups, see [6]. However, as our focus is to investigate non-compact homogeneous spaces we allow a discrepancy between the original definition and the current one. In fact with our definition every closed subgroup of $G$ is spherical if $G$ is compact.

We denote by $z_0 \in Z$ the origin of the homogeneous space $Z = G/H$.

### 2.1. Semi-invariant functions and the local structure theorem.

Let $G$ be a real reductive Lie group.

**Definition 2.2.** Let $Z = G/H$ with $H \subseteq G$ a closed subgroup.

1. A finite dimensional real representation $(\pi, V)$ of $G$ is called $H$-semispherical provided there is a cyclic vector $v_H \in V$ and a character $\gamma : H \to \mathbb{R}^\times$ such that
   \[ \pi(h)v_H = \gamma(h)v_H, \quad \forall h \in H. \]

2. The homogeneous space $Z$ is called almost algebraic if there exists an $H$-semispherical representation $(\pi, V)$ such that the map
   \[ Z \to \mathbb{P}(V), \quad g \cdot z_0 \mapsto [\pi(g)v_H] \]
is injective.

According to a theorem of Chevalley (see [1] Thm. 5.1), \( Z = G/H \) is almost algebraic if \( G \) and \( H \) are algebraic. In the following we always assume that \( Z = G/H \) is almost algebraic.

For a reductive Lie algebra \( \mathfrak{g} \) we write \( \mathfrak{g}_n \) for the direct sum of the non-compact non-abelian ideals in \( [\mathfrak{g}, \mathfrak{g}] \). If \( \mathfrak{g} \) is the Lie algebra of \( G \), then \( G_n \) denotes the corresponding connected normal subgroup of \( [G, G] \).

**Theorem 2.3** (Local structure theorem, homogeneous case). Let \( Z = G/H \) be an almost algebraic real spherical space, and let \( P \subseteq G \) be a minimal parabolic subgroup such that \( PH \) is open. Then there is a parabolic subgroup \( Q \supseteq P \) with Levi-decomposition \( Q = LU \) such that:

1. The map \( Q \times_L (L/L \cap H) \to Z, \ [q, l(L \cap H)] \mapsto q l \cdot z_0 \) is a \( Q \)-equivariant diffeomorphism onto \( Q \cdot z_0 \subseteq Z \).
2. \( Q \cap H \subseteq L \).
3. \( L_n \subseteq H \).
4. \( (L \cap P)(L \cap H) = L \).
5. \( QH = PH \).

*Proof.* The proof consists of an iterative procedure, in which we construct a strictly decreasing sequence of parabolic subgroups

\[ Q_0 \supset Q_1 \supset \cdots \supset P \]

and corresponding Levi subgroups \( L_0 \supset L_1 \supset \ldots \), all satisfying (1). Note that (2) is an immediate consequence of (1). After a finite number of steps a parabolic subgroup is reached which also satisfies (3)–(5).

Let \( Q_0 = G \). It clearly satisfies (1). If \( G_n \subseteq H \) then \( PH = G \) since \( P \) contains both the center of \( G \) and every compact normal subgroup of \( [G, G] \). Hence in this case \( Q = Q_0 \) solves (1)–(5). Note also that since \( L \cap P \) is a minimal parabolic subgroup of \( L \), the argument just given, but applied to \( L \), shows that (4) and (5) are consequences of (3).

Assume now that \( G_n \not\subseteq H \). By our general assumption on \( Z \) there is a finite dimensional representation \( (\pi, V) \) of \( G \) and a vector \( v_H \in V \) satisfying all the properties of Definition 2.2. The assumption on \( G_n \) implies that \( \pi(g)v_H \not\in Rv_H \) for some \( g \in G_n \), hence \( \pi \) does not restrict to a multiple of the trivial representation of \( G_n \).

Choose a Cartan involution for \( G \) and a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{s} \), but note that these choices may be valid only for the current step of the iteration. Let \( v^* \in V^* \setminus \{0\} \) be an extremal weight vector
so that the line $\mathbb{R}v^*$ is fixed by $AN$, say $\pi^*(g)v^* = \chi(g)v^*$ for $g \in AN$ and some character $\chi : AN \to \mathbb{R}^\times$. Now we need the following

**Lemma 2.4.** Let $G$ be a connected semisimple Lie group without compact factors, and with minimal parabolic $P = MAN \subseteq G$. Let $V$ be a non-trivial finite dimensional irreducible real representation of $G$. Then $V^{AN} = \{0\}$.

**Proof.** Let $\bar{N} = \theta(N)$ be the unipotent part of the parabolic subgroup $\theta(P)$ opposite to $P$. It follows from the representation theory of $\mathfrak{sl}(2, \mathbb{R})$ that vectors in $V^{AN}$ are also fixed by $\bar{N}$. Since $G$ has no compact factors it is generated by $\bar{N}$ and $AN$, hence $V^{AN} = V^G = \{0\}$. \hfill \square

By this lemma and what was just seen, we can choose $v^*$ such that $\chi$ is nontrivial on $G \cap A$. The matrix coefficient

$$f(g) := v^*(\pi(g)v_H) \quad (g \in G)$$

satisfies $f(anh) = \chi(a)^{-1}\gamma(h)f(g)$ for all $g \in G$, $an \in AN$ and $h \in H$. As $v_H$ is cyclic and $v^*$ non-zero, and as $PH$ is open, $f$ is not identically zero on $M$.

We construct a new function:

$$F(g) := \int_M f(mg)^2 \, dm \quad (g \in G).$$

This function is smooth, $G$-finite, non-negative valued, and satisfies

$$(2.1) \quad F(mangh) = \chi(a)^{-2}\gamma^2(h)F(g)$$

for all $g \in G$, $man \in P$ and $h \in H$. Furthermore, $F(e) > 0$.

It follows from the $G$-finiteness together with (2.1) that $F$ is a matrix coefficient

$$F(g) = w^*(\rho(g)w_H)$$

of a finite dimensional representation $(\rho, W)$ of $G$, with non-zero vectors $w_H \in W$ and $w^* \in W^*$ such that

$$\rho(h)w_H = \gamma(h)^2w_H, \quad \rho^*(man)w^* = \chi(a)^2w^*$$

for all $h \in H$ and $man \in P = MAN$. Here $W^*$ can be chosen to be the span of all left translates of $F$. Since $F$ is a highest weight vector, $W^*$ and hence $W$ is irreducible. Define $\nu \in a^*$ by

$$e^{\nu(X)} = \chi(\exp X)^2,$$

then $\nu$ is the highest $a$-weight of $\rho^*$, and it is dominant with respect to the set $\Sigma(a, n)$ of $a$-roots in $\mathfrak{n}$.

Now define a subgroup $Q_1 = Q \subseteq G$ to be the stabilizer of $\mathbb{R}w^*$,

$$Q = \{g \in G \mid \rho^*(g)w^* \in \mathbb{R}w^*\}.$$
and define a character \( \psi : Q \to \mathbb{R}^\times \) by
\[
\rho^*(g)w^* = \psi(g)w^*.
\]
In particular, we see that \( Q \) is a parabolic subgroup that contains \( P \). Moreover \( \psi : Q \to \mathbb{R} \) extends \( \chi^2 : AN \to \mathbb{R}^+ \). Let \( U \subseteq Q \) be the unipotent radical of \( Q \), its Lie algebra is spanned by the root spaces of the roots \( \alpha \in \Sigma(\mathfrak{a}, \mathfrak{n}) \) for which \( \langle \alpha, \nu \rangle > 0 \).

Note that since \( w_H \) is cyclic, \( \rho^*(g)w^* = cw^* \) if and only if \( F(g^{-1}x) = cF(x) \) for all \( x \in G \). Hence
\[
Q = \{ g \in G \mid F(g \cdot) \text{ is a multiple of } F \}
\]
and \( F(q \cdot) = \psi(q)F \) for all \( q \in Q \). (We use the symbol \( F(g \cdot) \) for the function \( x \mapsto F(gx) \) on \( G \).)

We note that \( Q \cap G_n \) is a proper subgroup of \( G_n \), for otherwise \( \rho^* \) would be one-dimensional spanned by \( w^* \), and this would contradict the non-triviality of its highest weight \( \chi^2 = \chi^2 \) on \( G_n \cap A \).

Set \( Z_0 := QH \subseteq Z \), then \( Z_0 \) is open since \( qPH \) is open for each \( q \in Q \). Following [4], Th. 2.3, we define a moment-type map:
\[
\mu : Z_0 \to \mathfrak{g}^*, \quad \mu(z)(X) := \frac{dF(q)(X)}{F(q)} = \frac{d}{dt} \bigg|_{t=0} \frac{F(\exp(tX)q)}{F(q)}
\]
for \( q \in Q \) such that \( z = qH \in Z_0 \) and \( X \in \mathfrak{g} \). Note that this map is well-defined: \( F(q) \neq 0 \) for \( q \in Q \), and if \( q \cdot z_0 = q' \cdot z_0 \) then \( q = q' h \) for some \( h \in H \).

We let \( G \) act on \( \mathfrak{g}^* \) via the co-adjoint action and record:

**Lemma 2.5.** The map \( \mu \) is \( Q \)-equivariant.

**Proof.** Let \( z \in Z_0 \), \( q \in Q \) and \( Y \in \mathfrak{g} \). Then
\[
\mu(qz)(Y) = \frac{d}{dt} \bigg|_{t=0} \frac{F(\exp(tY)qz)}{F(qz)} = \frac{d}{dt} \bigg|_{t=0} \frac{F(qq^{-1} \exp(tY)qz)}{\psi(q)F(z)} = \frac{d}{dt} \bigg|_{t=0} \frac{F(\exp(t \text{Ad}(q^{-1})Y)z)}{F(z)} = (\text{Ad}^*(q)\mu(z))(Y). \]

Note that
\[
(2.2) \quad \mu(z)(X) = d\psi(X), \quad (X \in \mathfrak{q})
\]
for all \( z \in Z_0 \). In particular, \( \mu(z_1) - \mu(z_2) \in \mathfrak{q}^\perp \subseteq \mathfrak{g}^* \) for \( z_1, z_2 \in Z_0 \). Moreover, \( \mu(z)(X + Y) = -\nu(X) \) for \( X \in \mathfrak{a} \) and \( Y \in \mathfrak{m} + \mathfrak{n} \).

We now identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) via the invariant non-degenerate form \( \kappa(\cdot, \cdot) \), then \( \mathfrak{q}^\perp \) is identified with \( \mathfrak{u} \), and \( (\mathfrak{m} + \mathfrak{n})^\perp \) with \( \mathfrak{a} + \mathfrak{n} \). Let
\[
X_0 = \mu(z_0) \in \mathfrak{a} + \mathfrak{n},
\]
then $X_0 \not\in \mathfrak{n}$ since $\nu \neq 0$ and hence $X_0$ is a semisimple element. Write $X_s$ for the $\mathfrak{a}$-part of $X_0$, then the eigenvalues of $\text{ad}(X_0)$ on $\mathfrak{n}$ are the $\alpha(X_s)$ where $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$. By the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ these are the inner products $\langle -\nu, \alpha \rangle$, in particular they are all $\leq 0$ and on $\mathfrak{u}$ they are $< 0$.

We conclude from the above that $\text{im} \mu \subseteq X_0 + \mathfrak{u}$. We claim equality:

(2.3) $\text{im} \mu = X_0 + \mathfrak{u}$.

As $\mu$ is $Q$-equivariant we have $\text{im} \mu = \text{Ad}(Q)X_0$. The lemma below (with $X = -\text{ad}(X_0)$) implies $\text{Ad}(U)X_0 = X_0 + \mathfrak{u}$, and then (2.3) follows.

**Lemma 2.6.** Let $\mathfrak{u}$ be a nilpotent Lie algebra and $X : \mathfrak{u} \rightarrow \mathfrak{u}$ a derivation which is diagonalizable with non-negative eigenvalues. Then in the solvable Lie algebra $\mathfrak{g} := \mathbb{R}X \ltimes \mathfrak{u}$ the following identity holds:

(2.4) $e^{\text{ad}u}X = X + [X, u]$.

**Proof.** Note that $[X, u] = u$ if all eigenvalues are positive. The inclusion $\subseteq$ in (2.4) is easy. The proof of the opposite inclusion is by induction on $\text{dim} \mathfrak{u}$, and the case $\text{dim} \mathfrak{u} = 0$ is trivial. Assume $\text{dim} \mathfrak{u} > 0$ and let $u = \sum_{\lambda \geq 0} u(X, \lambda)$ be the eigenspace decomposition of the operator $X : \mathfrak{u} \rightarrow \mathfrak{u}$. Let $\lambda_1 \geq 0$ be the smallest eigenvalue and set $u_1 := u(X, \lambda_1)$ and $u_2 := \sum_{\lambda > \lambda_1} u(X, \lambda)$. Note that $u_2$ is an ideal in $\mathfrak{u}$, and $\mathfrak{u} = u_1 + u_2$ as vector spaces.

By induction we have $e^{\text{ad}u_2}X = X + u_2$. If $\lambda_1 = 0$ then $[X, u] = u_2$, and we are done. Otherwise $[u_1, u_1] \subseteq u_2$ and hence $e^{\text{ad}u}X = X + \lambda_1 U + u_2.$

for $U \in u_1$. Note that $e^{\text{ad}u}$ is a group as $u$ is nilpotent. It follows that

$e^{\text{ad}u}X \supseteq e^{\text{ad}u_1} e^{\text{ad}u_2} X = e^{\text{ad}u_1} (X + u_2) = \bigcup_{U \in u_1} e^{\text{ad}U}(X + u_2) = \bigcup_{U \in u_1} (X + \lambda_1 U + u_2) = X + \mathfrak{u}$.

Continuing with the proof of Theorem 2.3, we conclude that the stabilizer $L \subseteq Q$ of $X_0 \in \mathfrak{q}$ is a reductive Levi-subgroup. Let

$S := \mu^{-1}(X_0) = \{z \in Z_0 \mid \mu(z) = X_0\},$

then for $q \in Q$ we have

(2.5) $qz_0 \in S \iff \mu(qz_0) = X_0 \iff qX_0 = X_0 \iff q \in L.$
Hence $L$ acts transitively on $S$. As $\mu$ is submersive, $S$ is a submanifold of $Z_0$ and we obtain with
\begin{equation}
Q \times_L S \rightarrow Z_0.
\end{equation}
a $Q$-equivariant diffeomorphism. As $L$-homogeneous space $S$ is isomorphic to $L/L \cap H$. Hence (1) is valid.

Note that (2.5) implies that $(L \cap P)H = S \cap (PH)$, which is open in $S$. Thus $L/L \cap H$ is a real spherical space.

If (3) is valid, we are done. Otherwise we let $Q_1 = Q$ and consider the real spherical space $Z_1 = L_1/L_1 \cap H$ for $L_1 = L$. Iterating the procedure of before yields a proper parabolic subgroup $R$ of $L_1$ containing $L_1 \cap P$ and with a Levi subgroup $L_2 \subseteq L_1$ such that
\begin{equation}
(R \cap N) \times L_2/(L_2 \cap H) \rightarrow R \cdot z_0
\end{equation}
is a diffeomorphism. We let $Q_2 = RP = RU_1$, which is a subgroup since $R$ normalizes $U_1$. Note that (2.7) together with the property (1) for $Q_1$ implies that this property is valid also for $Q_2$. We continue iterations until $H$ contains the non-compact semisimple part of some $L_i$. This will happen eventually since the non-compact semisimple part of a Levi subgroup of $P$ is trivial.

\section*{2.2. $Z$-adapted parabolics.}

\textbf{Definition 2.7.} Let $Z = G/H$ be a real spherical space. A parabolic subgroup $Q \subseteq G$ will be called $Z$-\textit{adapted} provided that
\begin{enumerate}
\item There is a minimal parabolic subgroup $P \subseteq Q$ with $PH$ open.
\item There is a Levi decomposition $Q = LU$ such that $Q \cap H \subseteq L$.
\item $l_n \subseteq h$.
\end{enumerate}
A parabolic subalgebra $q$ of $g$ is called $Z$-adapted if it is the Lie algebra of a $Z$-adapted parabolic subgroup $Q$.

\textbf{Theorem 2.8.} Let $Z = G/H$ be an almost algebraic real spherical space and $P$ a minimal parabolic subgroup such that $PH$ is open. Then there exists a unique parabolic subgroup $Q \supseteq P$ with unipotent radical $U$ such that $u$ is complementary to $n \cap h$ in $n$. Moreover, this parabolic subgroup $Q$ is $Z$-adapted, and it is the unique parabolic subgroup above $P$ with that property.

\textit{Proof.} Note first that if $Q \supseteq P$ and $Q = LU$ is a Levi decomposition then $n = (n \cap l) \oplus u$. Assuming in addition (2) and (3) above then $n \cap h = n \cap l$, and hence $n \cap h$ is complementary to $u$. Hence every $Z$-adapted parabolic subgroup $Q \supseteq P$ has this property of complementarity. In particular, this holds then for the parabolic subgroup $Q$ constructed with Theorem 2.3.
It remains to prove that if $Q' \supseteq P$ is another parabolic for which the unipotent radical $u'$ is complementary to $n \cap h$, then $Q' = Q$. Since $l_n \subseteq h$ we find
\[ u' \cap l \subseteq u' \cap h = \{0\}. \]
The lemma below now implies $u \supseteq u'$. But then $u = u'$ since both spaces are complementary to $n \cap h$, and hence $Q = Q'$. \hfill \square

**Lemma 2.9.** Let $p$ be a minimal parabolic subalgebra, and let $q, q' \supseteq p$ be parabolic subalgebras with unipotent radicals $u, u'$. If there exists a Levi decomposition $q = l + u$ such that $l \cap u' = \{0\}$, then $q \subseteq q'$.

**Proof.** This follows easily from the standard description of the parabolic subalgebras containing $p$ by sets of simple roots. \hfill \square

### 2.3. The real rank of $Z$.

Let $Q$ be $Z$-adapted, with Levi decomposition $Q = LU$ as in Definition 2.7. From the local structure theorem we obtain an isomorphism
\[ Q \times_L L/L \cap H \to Q \cdot z_0 = P \cdot z_0. \]
Recall that $l_n \subseteq h$. We decompose
\[ l = \mathfrak{z}(l) \oplus \mathfrak{l} \cdot l = \mathfrak{z}(l) \oplus l_c \oplus l_n, \]
where $l_c$ denotes the sum of all compact simple ideals in $l$. Note that $D = L/L_n$ is a Lie group with the Lie algebra $\mathfrak{d} = \mathfrak{z}(l) + l_c$, which is compact, and that
\[ l \cap h = c \oplus l_n \]
with $c = \mathfrak{d} \cap h$. Let $C = (L \cap H)/L_n \subseteq D$, then $L/L \cap H = D/C$, and
\[ U \times D/C \to P \cdot z_0 \]
is an isomorphism.

Consider the refined version of (2.8),
\[ l = \mathfrak{z}(l)_{np} \oplus \mathfrak{z}(l)_{cp} \oplus l_c \oplus l_n \]
in which $\mathfrak{z}(l)_{np}$ and $\mathfrak{z}(l)_{cp}$ denote the non-compact and compact parts of $\mathfrak{z}(l)$. Let $L = K_L A_L (L \cap N)$ be an Iwasawa decomposition of $L$, and let $G = KAN$ be an Iwasawa decomposition of $G$ which is compatible, that is, $K \supseteq K_L$ and $A = A_L$. Then $\mathfrak{a} = \mathfrak{z}(l)_{np} \oplus (\mathfrak{a} \cap l_n)$.

Let $\mathfrak{a}_h \subset \mathfrak{z}(l)_{np}$ be the image of $c$ under the projection $l \to \mathfrak{z}(l)_{np}$ along (2.10), and let $\mathfrak{a}_Z$ be a subspace of $\mathfrak{z}(l)_{np}$, complementary to $\mathfrak{a}_h$. Then
\[ \mathfrak{a} = \mathfrak{a}_Z \oplus \mathfrak{a}_h \oplus (\mathfrak{a} \cap l_n) \]
The number $\dim \mathfrak{a}_Z$ will be called the real rank of $Z$ in Section 3, where we show (under an additional hypothesis) that it is an invariant of $Z$ (it is independent of the choices of $P$ and $L$). See Remark 3.5.
2.4. \textit{HP-factorizations of a semi-simple group.} Let $Z = G/H$ be real spherical. In general $G/P$ admits several $H$-orbits. Here we investigate the simplest case where there is just one orbit.

\textbf{Proposition 2.10.} Let $G$ be semi-simple. Assume that $Z = G/H$ is real spherical and that $\mathfrak{h}$ contains no non-zero ideal of $\mathfrak{g}$. Then $HP = G$ if and only if $H$ is compact.

\textit{Proof.} Assume that $HP = G$. Note that then $HgP = G$ for every $g \in G$ and hence

$$\mathfrak{h} + \text{Ad}(g)(\mathfrak{p}) = \mathfrak{g}$$

for every $g \in G$.

We first reduce to the case where $H$ is reductive in $G$. Otherwise there exists a non-zero ideal $\mathfrak{h}_u$ in $\mathfrak{h}$ which acts unipotently on $\mathfrak{g}$. By conjugating $P$ if necessary we may assume that $\mathfrak{h}_u \subseteq \mathfrak{n}$. It then follows from $G = PH$ that $\text{Ad}(g)(\mathfrak{h}_u) \subseteq \mathfrak{n}$ for all $g \in G$, which is absurd.

Assume now that $H$ is reductive and let $H = K_H A_H N_H$ be an Iwasawa decomposition. Let $X \in \mathfrak{a}_H$ be regular dominant with respect to $\mathfrak{n}_H$, and let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ which is spanned by the non-negative eigenspaces of $\text{ad} X$. It follows that $\mathfrak{q} \cap \mathfrak{h}$ is a minimal parabolic subalgebra of $\mathfrak{h}$, and that $\mathfrak{n}_H$ is contained in the unipotent part $\mathfrak{u}$ of $\mathfrak{q}$. As $Q$ contains a conjugate of $P$ we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and hence $\dim(\mathfrak{h}/(\mathfrak{q} \cap \mathfrak{h})) = \dim(\mathfrak{g}/\mathfrak{q})$, from which we deduce that $\mathfrak{n}_H = \mathfrak{u}$. From $\mathfrak{n}_H = \mathfrak{u}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ we deduce that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. Let $\mathfrak{h}_n$ be the subalgebra of $\mathfrak{h}$ generated by $\mathfrak{n}_H$ and its opposite $\mathfrak{n}_H$ with respect to the Cartan involution of $H$ associated with $H = K_H A_H N_H$. Then $\mathfrak{h}_n$ is $\mathfrak{l}$-invariant and an ideal in $\mathfrak{h}$. With $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ we now infer that $\mathfrak{h}_n$ is an ideal in $\mathfrak{g}$, and hence it is zero. It follows that $H = K_H A_H$, where $A_H$ is central in $H$. We may assume $K_H \subseteq K$ and $A_H \subseteq A$. Then $G = HP$ implies $K = K_H M$, and hence $K$ centralizes $A_H$. This is impossible unless $A_H = \{1\}$ and then $H$ is compact.

Conversely, if $H$ is compact then the open $H$-orbit on $G/P$ is closed, and since $G/P$ is connected it follows that $HP = G$. \hfill $\square$

3. \textbf{Real spherical varieties}

All complex varieties $Z_C$ in this section will be defined over $\mathbb{R}$. Typically we denote by $Z$ the set of real points of $Z_C$. If $Z$ is Zariski-dense in $Z_C$, then we call $Z$ a real (algebraic) variety.

We say that a subset $U \subset Z$ is (quasi)affine if there exists a (quasi)affine subset $U_C \subset Z_C$ such that $U = U_C \cap Z$. 


Remark 3.1. Even if $Z_C$ is irreducible it might happen that $Z$ has several connected components with respect to the Euclidean topology. However, by Whitney’s theorem, the number of connected components is always finite. Take for example $Z = \text{GL}(n, \mathbb{R})$ and $Z_C = \text{GL}(n, \mathbb{C})$. Here $Z$ breaks into two connected components $\text{GL}(n, \mathbb{R})_+$ and $\text{GL}(n, \mathbb{R})_-$ characterized by the sign of the determinant; certainly it would be meaningful to call $\text{GL}(n, \mathbb{R})_+$ a real algebraic variety as well.

Let $Z_1 \amalg \ldots \amalg Z_n$ be the decomposition of $Z$ into connected components (with respect to the Euclidean topology). A more general notion of real variety would be to allow arbitrary unions of those $Z_j$ which are Zariski dense in $Z_C$. In fact, all the statements derived in this section for real varieties are valid in this more general setup.

In this section we let $G$ be a real algebraic reductive group and $G_C \supseteq G$ its complexification. Furthermore, $P$ is a minimal parabolic subgroup of $G$ and $P = MAN$ a Langlands decomposition of it.

By a real $G$-variety $Z$ we understand a real variety $Z$ endowed with a real algebraic $G$-action. A real $G$-variety will be called linearizable provided there is a finite dimensional real $G$-module $V$ such that $Z$ is realized as real subvariety of $P(V_C)$.

An algebraic real reductive group $G$ is called elementary if $G \cong M \times A$ with $M$ compact and $A = (\mathbb{R}^+)^l$. This is equivalent to $G = P$. A real $G$-variety $Z$ will then be called elementary if $G/J$ is elementary where $J$ is the kernel of the action on $Z$.

Definition 3.2. A linearizable real $G$-variety $Z$ will be called real spherical provided that:

- $Z_C$ is irreducible,
- $Z$ admits an open $P$-orbit.

Remark 3.3. (a) In the definition of a (complex) spherical variety one requests in particular that the variety is normal. We now explain how this is related to our notion of real spherical.

Assume that $Z_C$ is normal. Then it follows from a theorem of Sumihiro (\cite{5} p. 64) that every every point $z \in Z_C$ has a $G_C$-invariant open neighborhood $U$ which can be equivariantly embedded into $\mathbb{P}(V_C)$ where $V_C$ is a finite dimensional representation of $G_C$. It follows that if $z \in Z$ then $U_0 := (U \cap \overline{U}) \cap Z$ is a linearizable open neighborhood of $z$. Observe that there is always a normalization map $\nu : \tilde{Z} \to Z$ where $\tilde{Z}$ is a normal $G$-variety and $\nu$ is proper, finite to one, and invertible over an open dense subset of $Z$. 

(b) If \( Z \) is a real spherical variety, then the number of open \( P \)-orbits is finite. As \( Z_\mathbb{C} \) is irreducible, there is exactly one open \( P_\mathbb{C} \)-orbit on \( Z_\mathbb{C} \) and the real points of this open \( P_\mathbb{C} \)-orbit decomposes into finitely many \( P \)-orbits. We conclude in particular that there are only finitely many open \( G \)-orbits in \( Z \). Let \( \mathcal{O} \simeq G/H \) be one of them. Then \( G/H \) is a real spherical algebraic homogeneous space which we considered before.

(c) Let \( Z \) be an elementary real spherical variety. If \( G = A \), then \( Z \) consists of the real points of a toric variety defined over \( \mathbb{R} \).

(d) Let \( G = M \times A \) be an elementary algebraic real reductive group and \( Z = G/H \) a homogeneous real spherical \( G \)-variety. Since there are no algebraic homomorphisms between a split torus and a compact group, the group \( H \) splits as \( H = M_0 \times A_0 \) with \( M_0 \subseteq M \) and \( A_0 \subseteq A \). Thus \( Z = M/M_0 \times A/A_0 \).

3.1. Some general facts about real \( G \)-varieties. Let \( Z \) be an irreducible real variety. We denote by \( \mathbb{C}[Z] \), resp. \( \mathbb{C}(Z) \), the ring of regular, resp. rational functions on \( Z \), that is \( \mathbb{C}[Z] \) consists of the restrictions of the regular functions on \( Z_{\mathbb{C}} \) to \( Z \) and likewise for \( \mathbb{C}(Z) \).

As \( Z \) is Zariski-dense we observe that the restriction mapping \( \text{Res} : \mathbb{C}(Z_{\mathbb{C}}) \to \mathbb{C}(Z) \) is bijective. Next we note that both \( \mathbb{C}(Z) \) and \( \mathbb{C}[Z] \) are invariant under complex conjugation \( f \mapsto \overline{f} \). In particular with \( f \in \mathbb{C}[Z] \), resp. \( \mathbb{C}(Z) \), we also have that \( \text{Re} f \) and \( \text{Im} f \) belong to \( \mathbb{C}[Z] \), resp. \( \mathbb{C}(Z) \).

If a compact real algebraic group \( M \) acts on \( Z \), then the \( M \)-average
\[
 f \mapsto f^M; \quad f^M(z) := \int_M f(m \cdot z) \, dm \quad (z \in Z)
\]
preserves \( \mathbb{C}[Z] \). This follows from the fact that the \( G \)-action on \( \mathbb{C}[Z] \) is locally finite. Put together we conclude
\[
 f \in \mathbb{C}[Z] \Rightarrow (|f|^2)^M \in \mathbb{C}[Z]^M \text{ with } f \neq 0 \Rightarrow (|f|^2)^M \neq 0.
\] (3.1)

Let us denote by \( \widehat{P} \) the set of real algebraic characters \( \chi : P \to \mathbb{R}^\times \) such that \( MN \subseteq \ker \chi \). Note that the subgroup \( MN \) of \( P \), and hence \( \widehat{P} \), is independent of the choice of a Langlands decomposition of \( P \). However, when that has been chosen, there is a natural identification of \( \widehat{P} \) with a lattice \( \Lambda \subseteq a^\ast \).

For the rest of this subsection we let \( Z \) be a real \( G \)-variety. We denote by \( \mathbb{C}(Z)^{(P)} \) the set of \( P \)-semi-invariant functions, i.e. the rational functions \( f \in \mathbb{C}(Z) \setminus \{0\} \) for which there is a \( \chi \in \widehat{P} \) such that \( f(p^{-1}z) = \chi(p)f(z) \) for all \( p \in P, z \in Z \) for which both sides are defined. We denote by \( \mathbb{C}(Z)^P \) the set of \( P \)-invariants in \( \mathbb{C}(Z) \). Likewise we define
\(\mathbb{C}[Z]^{(P)}\) and \(\mathbb{C}[Z]^P\). Further we denote by \(\mathbb{R}(Z)\) and \(\mathbb{R}[Z]\) the real valued functions in \(\mathbb{C}(Z)\) and \(\mathbb{C}[Z]\).

**Lemma 3.4.** Let \(Z\) be a quasi-affine real \(G\)-variety. Then for all non-zero \(f \in \mathbb{R}(Z)^P\) there exists \(f_1, f_2 \in \mathbb{R}[Z]^{(P)}\) such that \(f = \frac{f_1}{f_2}\).

**Proof.** Let \(f \in \mathbb{R}(Z)^P\). As \(Z\) is quasi-affine, we find regular functions \(h_1, h_2 \in \mathbb{C}[Z]\), \(h_2 \neq 0\) such that \(f = h_1 h_2\). Consider the ideal \(I := \{h \in \mathbb{C}[Z] \mid hf \in \mathbb{C}[Z]\}\).

Note that:
- \(I \neq \{0\}\) as \(h_2 \in I\),
- \(I = I\) as \(f\) is real,
- \(I\) is \(P\)-invariant as \(f\) is \(P\)-fixed.

The action of \(P\) on \(\mathbb{C}[Z]\) is algebraic, hence locally finite and thus we find an element \(0 \neq h \in I\) which is an eigenvector for the solvable group \(AN\). We use (3.1) to obtain with \(f_2 = (|h|^2)^M\) a non-zero element of \(I \cap \mathbb{R}[Z]^{(P)}\). Now we put \(f_1 = f_2f \in \mathbb{R}[Z]^{(P)}\). \(\square\)

For \(\chi \in \hat{P} = \Lambda\) we let
\[
\mathbb{C}[Z]_{\chi} := \{f \in \mathbb{C}[Z] \mid (\forall p \in P, z \in Z) f(p^{-1}z) = \chi(p)f(z)\}
\]
and likewise define \(\mathbb{C}(Z)_{\chi}\). We define a sub-lattice of \(\Lambda\) by
\[
\Lambda_Z := \{\chi \in \hat{P} \mid \mathbb{C}(Z)_{\chi} \neq \{0\}\}.
\]

With that we declare the **real rank** of \(Z\) by
\[
(3.2) \quad \text{rk}_\mathbb{R}(Z) := \dim_\mathbb{Q}(\Lambda_Z \otimes \mathbb{Q})\).
\]

It is easily seen that \(\text{rk}_\mathbb{R}(Z)\) is independent of the choice of minimal parabolic subgroup \(P\).

**Remark 3.5.** Let \(Z = G/H\) be homogeneous. Then \(\text{rk}_\mathbb{R}(Z) = \dim \mathfrak{a}_Z\) where \(\mathfrak{a}_Z\) is defined by (2.11). In fact, as a \(Q\)-variety, an open subset of \(Z\) is isomorphic to \(U \times L/L \cap H\). Thus \(\mathbb{R}(Z)^{(P)} = \mathbb{R}(L/L \cap H)^{(L \cap P)}\).

Since \(H\) contains \(L\), the variety \(L/L \cap H\) is elementary. By Remark 3.3(d), we have \(\mathbb{R}(L/L \cap H)^{(L \cap P)} = \mathbb{R}(A/A_0)^{(A)}\) which implies the claim, as \(A/A_0 \simeq \mathfrak{a}_Z\).

**Lemma 3.6.** Let \(Z\) be a linearizable irreducible real \(G\)-variety and \(Y \subseteq X\) a Zariski-closed \(G\)-invariant subvariety. Then there exists a \(P\)-stable affine open subset \(Z_0 \subseteq Z\) which meets \(Y\) and such that the restriction mapping:
\[
\mathbb{R}[Z_0]^{(P)} \to \mathbb{R}[Z_0 \cap Y]^{(P)}
\]
is onto.
Proof. If $G$ is complex, then this is the real points version of \cite{2}, Prop. 1.1. Further with $P$ replaced by $AN$ one can literally copy the proof of \cite{2}. Finally the additional $M$-invariance when moving from $AN$ to $P$ is obtained from (3.1). □

Denote by $\Lambda^+ \subseteq \Lambda$ the semigroup of elements dominant with respect to $P$. For all $\lambda \in \Lambda^+$ we set

$$m(\lambda) := \dim_\mathbb{C} \mathbb{C}[Z]_\lambda.$$ 

If we identify $\Lambda^+$ with a subset of the irreducible finite dimensional representations of $G$, then $m(\lambda)$ is the multiplicity of the irreducible representation $\lambda$ occurring in the locally finite $G$-module $\mathbb{C}[Z]$. The following is a real analogue of the Vinberg-Kimel’feld theorem \cite{9}:

**Proposition 3.7.** Let $Z$ be a quasi-affine irreducible $G$-variety. Then the following assertions are equivalent:

1. $Z$ is real spherical.
2. $m(\lambda) \leq 1$ for all $\lambda \in \Lambda^+$.

**Proof.** “(1) ⇒ (2)”: Let $z \in Z$ such that $P \cdot z$ is open in $Z$. Then two $P$-semi-invariant functions $f_1$ and $f_2$ with respect to the same character $\lambda \in \hat{P}$ satisfy $f_1|_{P \cdot z} = cf_2|_{P \cdot z}$ for some constant $c \in \mathbb{C}$. As $Z_\mathbb{C}$ is irreducible we conclude that $f_1 = cf_2$.

“(2) ⇒ (1)”: We recall that there is an open $P$-orbit on $Z$ if and only if $\mathbb{C}(Z)^P = \mathbb{C}1$. This follows from Rosenlicht’s theorem, \cite{8} p. 23, applied to $Z_\mathbb{C}$. Let now $f \in \mathbb{C}(Z)^P$. According to Lemma 3.4 there exists $f_1, f_2 \in \mathbb{C}[Z]^P$ such that $f = \frac{f_1}{f_2}$. Clearly $f_1$ and $f_2$ correspond to the same character $\lambda \in \hat{P}$. As $m(\lambda) \leq 1$, we conclude that $f_1$ is a multiple of $f_2$. □

**Corollary 3.8.** Let $Z$ be a real spherical variety and $Y \subseteq Z$ a closed $G$-invariant irreducible subvariety. Then $Y$ is real spherical.

**Proof.** If $Z$ is quasi-affine, then this is immediate from the previous proposition as the restriction mapping $\mathbb{C}[Z] \to \mathbb{C}[Y]$ is onto. The more general case is reduced to that by considering the affine cone over $Z$. Recall that $Z \subseteq \mathbb{P}(V)$. The preimage of $Z$ in $V \setminus \{0\}$ will be denoted by $\tilde{Z}$. Note that $\tilde{Z}$ is quasi-affine. Moreover $Z$ is real spherical if and only if $\tilde{Z}$ is real spherical for the enlarged reductive group $G_1 = G \times \mathbb{R}^\times$. □

**Corollary 3.9.** Let $Z$ be a real spherical variety. Then the number of $G$-orbits on $Z$ is finite and each $G$-orbit is spherical.

**Proof.** In view of the preceding corollary we only need to show that there are finitely many $G$-orbits. Suppose that there are infinitely many
G-orbits. We let $Y \subseteq Z$ be a closed irreducible $G$-subvariety of minimal dimension which admits infinitely many $G$-orbits. By Corollary 3.8, $Y$ is spherical. In particular $Y$ admits open $G$-orbits. After deleting the finitely many open $G$-orbits from $Y$ we obtain a $G$-invariant subvariety $Y_1 \subseteq Y$ with infinitely many $G$-orbits. As $\dim Y_1 < \dim Y$ we reach a contradiction. □

The main result of [7] was that every homogeneous real spherical space admits only finitely many $P$-orbits. With Corollary 3.9 we then conclude:

**Theorem 3.10.** Let $Z$ be a real spherical variety. Then the number of $P$-orbits on $Z$ is finite.

### 3.2. The local structure theorem.

Let $Z$ be a real spherical variety and $Y \subseteq Z$ a $G$-invariant closed subvariety. Our goal is to find a $P$-invariant coordinate chart $Z_0$ for $Z$ which meets $Y$. For that we may assume that $Z$ is Zariski-closed in $\mathbb{P}(V)$, where $V$ is a finite-dimensional $G$-module. Moreover we may assume that $Y \subseteq Z$ is a closed $G$-orbit. In particular $Y$ is real spherical by Corollary 3.9 and we let $Q_Y < G$ be a $Y$-adapted parabolic.

Under these assumption on $Y$ and $Z$ there is the following immediate generalization of Lemma 3.6.

**Lemma 3.11.** Let $Z$ be a real spherical variety, closed in $\mathbb{P}(V)$, and $Y \subseteq Z$ a closed $G$-orbit. Then there exists a $Q_Y$-stable affine open subset $Z_0 \subseteq Z$ which meets $Y$ and such that the restriction mapping:

$$\mathbb{R}[Z_0]^{(Q_Y)} \rightarrow \mathbb{R}[Z_0 \cap Y]^{(Q_Y)}$$

is onto.

**Proof.** The proof is analogous to the one of Lemma 3.6. We obtain $Z_0$ is the non-vanishing locus of a $Q_Y$-semi-invariant homogeneous polynomial function on $V$. □

**Corollary 3.12.** Let $Z \subset \mathbb{P}(V)$ be a closed real spherical variety and $Y$ an elementary closed subvariety. Then there exists a $G$-stable affine open subset $Z_0 \subset Z$ such that $Z_0 \cap Y \neq \emptyset$.

**Proof.** One has $Q_Y = G$. □

We now start with the construction of $Z_0$. In case $Y$ is elementary, $Z_0$ is given by Corollary 3.12. So let us assume that $Y$ is not elementary, i.e. $G_n$ does not act trivially on $Y$. Let $\bar{P} = MAN$ be opposite to $P$. As $Y \subseteq \mathbb{P}(V)$ is closed, we can find a vector $y_0 \in V$ such that $[y_0] \in Y$ is $A\bar{N}$-fixed, and such that $A$ acts by a non-trivial character on $y_0$. This
can be seen as follows. Assume for simplicity that \( V \) is irreducible. Then \( Y \) contains a vector \( y \) of which the \( A \)-weight decomposition has a non-trivial component \( y_0 \) in the lowest weight space of \( V \). Compression of \( y \) by \( A^+ \) then exhibits a non-zero multiple of \( y_0 \) as a limit of elements from \( Y \).

Next we choose \( v_0^* \in V^* \) such that \([v_0^*] \) is \( AN \)-fixed and \( v_0^*(y_0) = 1 \). Let \( \chi : A \to \mathbb{R}^+ \) be the character defined by \( a \cdot v_0^* = \chi(a)v_0^* \).

Consider the function
\[
F : V \to \mathbb{R}, \quad v \mapsto \int_M v_0^*(m \cdot v)^2 \, dm
\]
and note that
\[
F(man \cdot v) = \psi(a)F(v)
\]
for all \( man \in MAN \) and \( v \in V \), where \( \psi = \chi^{-2} \). Further \( F \) is real algebraic and homogeneous of degree 2. Thus \( \{[v] \in \mathbb{P}(V) \mid F(v) \neq 0\} \) defines an affine open set in \( \mathbb{P}(V) \) and the intersection with \( Z \) yields an affine open set \( Z_0 \). Note that \( F \) is not constant and hence \( Z_0 \) is a proper subvariety. We define \( Q \supseteq P \) to be the parabolic subgroup which fixes the line \( \mathbb{R}F|_{Z_0} \), that is \( Q = \{g \in G \mid gZ_0 = Z_0\} \).

As before we define on \( Z_0 \) a moment-type map:
\[
\mu : Z_0 \to g^*, \quad \mu(z)(X) := \frac{dF(v)(X)}{F(v)}
\]
for \( z = [v] \in Z \subseteq \mathbb{P}(V) \). This map is algebraic and \( Q \)-equivariant. Let \( U < Q \) be the unipotent radical.

We claim that \( \text{im} \, \mu \) is a \( Q \)-orbit. In fact for \( X \in \mathfrak{q} \) we have \( \mu(z)(X) = d\psi(X) \) for all \( z \in Z \), and after identifying \( g \) with \( g^* \) we obtain as in the previous section that
\[
\text{im} \, \mu = \text{Ad}(Q)X_0 = X_0 + u
\]
with \( X_0 = \mu([y_0]) \). The stabilizer of \( X_0 \) determines a Levi-subgroup \( L < Q \). Then \( S := \mu^{-1}(X_0) \) is an \( L \)-stable affine subvariety of \( Z_0 \) and we obtain an algebraic isomorphism
\[
Q \times_L S \to Z_0.
\]
The affine \( L \)-variety \( S \) is real spherical and meets \( Y \). We continue the procedure with \( (L, S, S \cap Y) \) instead of \( (G, Z, Y) \). The procedure will stop at the moment when \( S \cap Y \) is fixed under \( L_n \). We have thus shown:

**Theorem 3.13** (Local structure theorem, general case). Let \( Z \) be a real spherical variety and \( Y \subseteq Z \) a closed \( G \)-invariant subvariety. Then there is parabolic subgroup \( Q \supseteq P \) with Levi-decomposition \( Q = LU \) with the following properties: There is a \( Q \)-invariant affine open piece...
$Z_0 \subseteq Z$ meeting $Y$ and an $L$-invariant closed spherical subvariety $S \subseteq Z_0$ such that:

1. There is a $Q$-equivariant isomorphism $Q \times_L S \rightarrow Z_0$.
2. $S \cap Y$ is an elementary spherical $L$-variety.

4. The normalizer of a spherical subalgebra

As in the preceding section we assume that $G$ is algebraic and let $\mathfrak{h}$ be the Lie algebra of a spherical subgroup $H < G$. We denote by $\tilde{h} := n_\mathfrak{g}(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ and by $\tilde{H}$ the normalizer in $G$. Note that $\mathfrak{h} \trianglelefteq \tilde{h}$ is an ideal. Let $\mathfrak{p}$ be a minimal parabolic subalgebra such that $\mathfrak{p} + \mathfrak{h} = \mathfrak{g}$ and let $\mathfrak{q}$ denote the unique parabolic subalgebra above $\mathfrak{p}$, which is $Z$-adapted. Let $\tilde{Z} = G/\tilde{H}$. 

Lemma 4.1. The parabolic subalgebra $\mathfrak{q}$ is also $\tilde{Z}$-adapted.

Proof. We write $\tilde{\mathfrak{q}}$ for the unique $\tilde{Z}$-adapted parabolic above $\mathfrak{p}$ and $\tilde{u}$ for its unipotent radical. Then

$$n = (n \cap \mathfrak{h}) \oplus u = (n \cap \tilde{h}) \oplus \tilde{u}$$

It follows that $\tilde{u} \subseteq u$ and $\mathfrak{q} \subseteq \tilde{\mathfrak{q}}$. To obtain a contradiction we assume that $\mathfrak{q} \subsetneq \tilde{\mathfrak{q}}$. Then $\tilde{u} \subsetneq u$ and $n \cap \mathfrak{h} \subsetneq n \cap \tilde{h}$. In particular, the Lie algebra $\tilde{h}/\mathfrak{h}$ cannot be compact.

To conclude the proof we now show that $\tilde{h}/\mathfrak{h}$ is compact. Suppose first that $Z$ is quasi-affine and let $\mathbb{C}[Z] = \bigoplus_{\pi \in \hat{G}} \mathbb{C}[Z]_{\pi}$ be the decomposition of the $G$-module $\mathbb{C}[Z]$ into $G$-isotypical components. For each $\pi$ we choose a model space $V_{\pi}$ and let $\mathcal{M}_{\pi} := \text{Hom}_G(V_{\pi}, \mathbb{C}[Z])$ be the corresponding multiplicity space. Note that $\mathcal{M}_{\pi}$ is finite dimensional as there is a natural identification of $\mathcal{M}_{\pi}$ with the space of $H$-fixed elements in $V_{\pi}^*$. Let $\mathcal{C} := \tilde{H}/H$. Note that $\mathcal{C}$ acts from the right on $\mathbb{C}[Z]$ and preserves each $\mathbb{C}[Z]_{\pi}$, thus inducing an action on $\mathcal{M}_{\pi}$. Since $Z$ is quasi-affine we can choose finitely many $\pi_1, \ldots, \pi_k$ so that we obtain a faithful representation of $\mathcal{C}$ on the sum $\mathcal{M} := \bigoplus_{j=1}^k \mathcal{M}_{\pi_j}$.

Let $B < G_C$ be a Borel subgroup contained in $P_C$. For every $\pi$ we let $v_{\pi}$ be a $B$-highest weight vector in $V_{\pi}$. To every $\eta \in \mathcal{M}_{\pi}$ we associate the function $f_{\eta}(g) = \eta(\pi(g^{-1})v_{\pi})$ and define an inner product on $\mathcal{M}_{\pi}$ by

$$\langle \eta, \eta \rangle_{\pi} := (|f_{\eta}|^2)^M(z_0)$$

with notation of [3.1]. As $(|f_{\eta}|^2)^M$ is a matrix coefficient of a representation in $\Lambda$, and as multiplicities for these are at most one by
Proposition 3.7, we obtain that there is a real character \( \chi : C \to \mathbb{R}^\times \)
such that
\[
\langle h \cdot \eta, h \cdot \eta \rangle = \chi(h) \langle \eta, \eta \rangle.
\]

The group \( C_1 := \bigcap_{j=1}^k \ker\chi_{\pi_j} \) acts unitarily and faithfully on \( \mathcal{M} \),
hence is compact. By definition \( C/C_1 < (\mathbb{R}^\times)^k \), hence the Lie algebra
of \( C \) is compact.

Finally we reduce to the quasi-affine case using the affine cone over
\( \mathbb{P}(V) \) as before, see the proof of Corollary 3.8.

Let \( Q = LU \) be a Levi decomposition as in Definition 2.7 and recall
the decomposition (2.10).

Proposition 4.2. The normalizer \( \tilde{h} \) of \( h \) is of the form

\[
\tilde{h} = h \oplus \tilde{c}
\]

with \( \tilde{c} \) a subalgebra of the form \( \tilde{c} = \tilde{a} \oplus \tilde{m} \) where \( \tilde{a} < Z(l)_{np} \) and \( \tilde{m} < Z(l)_{cp} + l_c \).

Proof. From Lemma 4.1 we conclude that \( \tilde{h} = h + \tilde{h} \cap l \), and we obtain
(4.1) with a subspace \( \tilde{c} \) of \( Z(l)_{cp} + l_c \). It is a subalgebra because \( Z(l) + l_c \)
is reductive and \( h \) is an ideal in \( \tilde{h} \).

Write \( \tilde{a} \) for the orthogonal projection of \( \tilde{c} \) to \( Z(l)_{np} \) and \( \tilde{m} \) for the
orthogonal projection of \( \tilde{c} \) to \( Z(l)_{cp} + l_c \). Then, \( \tilde{c} \subseteq \tilde{a} + \tilde{m} \) and it remains
to show equality. This will follow if we can show that both \( \tilde{a} \) and \( \tilde{m} \)
normalize \( h \). For that we decompose \( X \in \tilde{c} \) as \( X = X_a + X_m \) with
\( X_a \in \tilde{a} \) and \( X_m \in \tilde{m} \). Observe that \( \text{ad} X_a \) commutes with \( \text{ad} X_m \).
Both operators are diagonalizable with real, resp. imaginary, spectrum. As
\( \text{ad} X \) preserves \( h \) we therefore conclude that \( \text{ad} X_a \) and \( \text{ad} X_m \) preserve
\( h \) as well.

Corollary 4.3. Let \( H \subseteq G \) be real spherical. Then \( N_G(H)/H \) is an
elementary group.

Corollary 4.4. The normalizer \( \tilde{h} \) is its own normalizer: \( \tilde{h} = \tilde{h} \).

Proof. It suffices to show that the normalizer \( \tilde{h} \) of \( \tilde{h} \) normalizes \( h \), as well.
Let \( \tilde{H} = N_G(\tilde{h}) \). Observe that \( \tilde{H}/H \) is an elementary real algebraic group, it is in particular reductive. Thus, \( \tilde{h}_u = h_u \) for the
nilpotent radicals. This implies that \( \tilde{h} \) normalizes \( h_u \) and that \( \tilde{H}/H_u \)
is a reductive real algebraic group. A connected group, which acts by
algebraic automorphisms on a reductive Lie group, acts by inner auto-
morphisms, hence fixes every ideal. Thus \( h/h_u \subseteq h/h_u \) is normalized
by \( \tilde{h} \), as well.
Remark 4.5. On the group level, the statement is wrong. Let, e.g.,
\( G = GL(2, \mathbb{R}) \) and \( H = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \). Then \( N_G(H) = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \).
Thus \( N_G(N_G(H)) = N_G(T) \) is strictly larger than \( N_G(H) = T \).

References

[1] A. Borel, *Linear Algebraic Groups*, Springer-Verlag 1991.
[2] M. Brion, *Variétés Sphériques*, Notes de la session de la SMF, “Opérations hamiltoniennes et opérations de groupes algébriques” (Grenoble, 1997),
www-fourier.ujf-grenoble.fr/~mbrion/spheriques.pdf
[3] M. Brion, D. Luna and T. Vust, *Espaces homogènes sphériques*, Invent. Math. 84 (1986), 617–632.
[4] F. Knop, *The asymptotic behavior of invariant collective motion*, Invent. Math. 116 (1994), 309–328.
[5] F. Knop, H. Kraft, D. Luna and T. Vust, *Local properties of algebraic group actions*. Algebraische Transformationsgruppen und Invariantentheorie, 63–75, DMV Sem. 13, Birkhäuser, Basel, 1989.
[6] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Gruppen*, Compositio Math. 38 (1979), 129–153.
[7] B. Krötz and H. Schlichtkrull, *Finite orbit decomposition of real flag manifolds*, arXiv:1307.2375 To appear in J. Eur. Math. Soc. (JEMS)
[8] T. A. Springer, *Aktionen reductiver Gruppen auf Varietäten*. Algebraische Transformationsgruppen und Invariantentheorie, 3–39, DMV Sem. 13, Birkhäuser, Basel, 1989.
[9] E. Vinberg and B. Kimel’feld, *Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups*, Funktsional. Anal. i Prilozhen. 12 (1978), 12–19.