An extended LINMAP method for multi-attribute group decision making under interval-valued intuitionistic fuzzy environment

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Abstract
The linear programming technique for multidimensional analysis of preference (LINMAP) is one of the well-known methods for multiple attribute group decision making (MAGDM). In this paper, we extend the LINMAP method to solve MAGDM problems in interval-valued intuitionistic fuzzy environment in which all the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices and the preference information about the alternatives is completely unknown. We introduce two definitions of distances between interval-valued intuitionistic fuzzy sets. To calculate the weights of attributes we develop a new linear programming model based on the group consistency and inconsistency index defined on the basis of pairwise comparison preference relations on alternatives given by the decision makers, the weighted distance of each alternative to the interval-valued intuitionistic fuzzy positive ideal solution can be calculated to determine the ranking order of all alternatives for the decision makers, and the ranking order of alternatives and the best alternative(s) for the group are generated by using the Borda’s function.

Keywords: Interval-valued intuitionistic fuzzy sets (IVIFSs), Multi-attribute group decision making (MAGDM), Linear programming technique for multidimensional analysis of preference (LINMAP).

1. Introduction
Atanassov introduce the concept of intuitionistic fuzzy set (AIFS) [1], which is a generalization of the concept of fuzzy set (FS) [2]. Gau and Buehrer [3] introduced the concept of vague set, but Bustince and Burillo [4] showed that vague sets are AIFSs. Atanassov and Gargov [5] further generalized the AIFSs in the spirit of ordinary interval-valued fuzzy sets (IVFSs) and defined the notion of an interval-valued intuitionistic fuzzy set (IVIFS), which is characterized by a membership function and a non-membership function whose values are intervals rather than exact numbers. Current research mainly focuses on basic operations and relations of IVIFSs as well as their properties [6]. The correlation coefficients of IVIFSs were systematically investigated from different points of view [7–9]. Other aspects of IVIFSs were also investigated, such as topological properties [10], relationships between AIFSs, L-fuzzy sets, IVFSs and IVIFSs [11–13], and the distance of IVIFSs [14–16].

Another active research topic is the investigation of multi-attribute decision making (MADM) by introducing interval-valued intuitionistic fuzzy aggregation operators [17–21], where the weights are known completely.

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Some researchers have proposed programming models to focus attention on the issue of MADM with IVIFSs and incomplete information [22–28].

The linear programming technique for multidimensional analysis of preference (LINMAP), developed by Srinivasan and Shocker [29], is one of the existing well-known methods based on distance measure for MADM problems in crisp environments. The classical LINMAP methodology was developed to solve MAGDM problems in fuzzy environments [30–32], and further extended to develop a new methodology for solving MAGDM problems using AIFSs [33]. Intuitively, extending from AIFSs to IVIFSs furnishes additional capability to handle vague information because the membership and non-membership degrees are only needed to be expressed as ranges of values rather than exact values. When the uncertainty in an IVIFS’s membership and non-membership degrees diminishes to zero, the IVIFS is reduced to an IFS. Therefore, it motivates to further extend the LINMAP to develop a new methodology for solving MAGDM problems with IVIFSs, where each alternative is assessed on the basis of its distance to a positive ideal solution, which is known a priori. The weights of attributes are calculated by constructing a new linear programming model based on the group consistency and inconsistency index defined on the basis of pairwise comparison preference relations on alternatives given by the decision makers. The weighted distance of each alternative to the interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS) can be calculated to determine the ranking order of all alternatives for the decision makers, and the ranking order of alternatives and the best alternative(s) for the group are generated by using the Borda’s function.

The remainder of this paper is organized as follows: In Section 2 some basic concepts related to AIFSs, IVIFSs and MAGDM are reviewed. Section 3 extends the classical LINMAP to develop a new methodology for solving MAGDM problems with IVIFSs, where each alternative is assessed on the basis of pairwise comparison preference relations on alternatives given by the decision makers. The weighted distance of each alternative to the interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS) can be calculated to determine the ranking order of all alternatives for the decision makers, and the ranking order of alternatives and the best alternative(s) for the group are generated by using the Borda’s function.

2. Preliminaries

Deschrijver and Kerre [12] have shown that IFSs are equivalent to IVFSs (also called vague sets [3]) and both can be regarded as L-fuzzy sets in the sense of Goguen [34]. In reality, it may not be easy to identify exact values for the membership and non-membership degrees of an element to a set. In this case, a range of values may be a more appropriate measurement to accommodate the vagueness. As such, Atanassov and Gargov [5] introduce the notion of IVIFS:

**Definition 2.1.** Let \( X \) be a non-empty set of the universe, and \( D[0, 1] \) be the set of all closed subintervals of the interval \([0, 1]\), an IVIFS \( \tilde{A} \) in \( X \) is an expression given by

\[
\tilde{A} = \{(x, \tilde{\mu}_\tilde{A}(x), \tilde{\nu}_\tilde{A}(x)) \mid x \in X\}
\]

where \( \tilde{\mu}_\tilde{A} : X \rightarrow D[0, 1] \), \( \tilde{\nu}_\tilde{A} : X \rightarrow D[0, 1] \) with the condition \( 0 \leq \sup(\tilde{\mu}_\tilde{A}(x)) + \sup(\tilde{\nu}_\tilde{A}(x)) \leq 1 \) for every \( x \in X \).

The intervals \( \tilde{\mu}_\tilde{A}(x) \) and \( \tilde{\nu}_\tilde{A}(x) \) denote the degree of membership and non-membership of \( x \in X \) to \( \tilde{A} \), respectively. But, here, for each \( x \in X \), \( \tilde{\mu}_\tilde{A}(x) \) and \( \tilde{\nu}_\tilde{A}(x) \) are closed intervals rather than real numbers and their lower and upper boundaries are denoted by \( \tilde{\mu}_\tilde{A}^L(x) \), \( \tilde{\mu}_\tilde{A}^U(x) \), \( \tilde{\nu}_\tilde{A}^L(x) \) and \( \tilde{\nu}_\tilde{A}^U(x) \), respectively. Therefore, another equivalent way to express an IVIFS \( \tilde{A} \) is

\[
\tilde{A} = \{(x, [\tilde{\mu}_\tilde{A}^L(x), \tilde{\mu}_\tilde{A}^U(x)], [\tilde{\nu}_\tilde{A}^L(x), \tilde{\nu}_\tilde{A}^U(x)]) \mid x \in X\}
\]

where \( 0 \leq \tilde{\mu}_\tilde{A}^L(x) \leq \tilde{\mu}_\tilde{A}^U(x) \leq 1, 0 \leq \tilde{\nu}_\tilde{A}^L(x) \leq \tilde{\nu}_\tilde{A}^U(x) \leq 1 \) and \( \tilde{\mu}_\tilde{A}^L(x) + \tilde{\nu}_\tilde{A}^U(x) \leq 1 \).

For each element \( x \in X \) we can compute its hesitation interval relative to \( \tilde{A} \) as:

\[
\tilde{\pi}_\tilde{A}(x) = [\tilde{\pi}_\tilde{A}^L(x), \tilde{\pi}_\tilde{A}^U(x)] = [1 - \tilde{\mu}_\tilde{A}^U(x) - \tilde{\nu}_\tilde{A}^U(x), 1 - \tilde{\mu}_\tilde{A}^L(x) - \tilde{\nu}_\tilde{A}^L(x)]
\]

which is called the interval-valued intuitionistic fuzzy index of the element \( x \in X \) to the set \( \tilde{A} \), representing the degree of indeterminacy membership of the element \( x \in X \) to the set \( \tilde{A} \). It is obvious that \( \tilde{\pi}_\tilde{A}(x) \in D[0, 1] \) for every \( x \in X \).

By considering the term of the interval-valued intuitionistic fuzzy index, we can represent Eq. (1) as follows:

\[
\tilde{A} = \{(x, \tilde{\mu}_\tilde{A}(x), \tilde{\nu}_\tilde{A}(x), \tilde{\pi}_\tilde{A}(x)) \mid x \in X\}
\]
this representation of the IVIFS will be a cuboid (i.e., \( V = \{(x, y, z) | x \leq \tilde{\mu}_A(x), \tilde{v}_A(x) \leq y \leq \tilde{\nu}_A(x), \tilde{\pi}_A(x) \leq z \leq \tilde{\pi}_A(x)\} \)) of departure for considering the another method in calculating distances between IVIFSs. As it shown in [35], if an IVIFS reduce an IFS, then Eq.(4) is three dimension representation of the intuitionistic fuzzy set, but Eq.(1) is two dimension representation which is the orthogonal projection of Eq.(4).

Given an element \( x \), the pair \((\tilde{\mu}_A(x), \tilde{\nu}_A(x))\) is called an interval-valued intuitionistic fuzzy number (IVFN) [18]. For taking into account all parameters describing IVIFS when calculating distances, an IVIFN is presented to a triplet \((\tilde{\mu}_A(x), \tilde{\nu}_A(x), \tilde{\pi}_A(x))\), and simply denoted by \( \tilde{a} = [(a, b), [c, d], [e, f]] \), where \([a, b] \in D \in [0, 1], [c, d] \in D \in [0, 1], [e, f] \in D \in [0, 1], b + d + e = 1 \) and \( a + c + f = 1 \).

3. Linear programming models and methods for MAGDM using IVIFSs

A MAGDM problem is to find a best compromise solution from all feasible alternatives assessed on multiple attributes [36–39]. Assume that there is a group consisting of \( V \) decision makers (or experts) \( D_v (v = 1, 2, \ldots, V) \) who have to choose one(s) of (or rank) \( n \) alternatives on \( m \) attributes given by the decision makers. Namely, ratings of alternatives on attributes are given using IVIFNs through judgments of the decision makers. Usually, ratings of alternatives \( o_i \in O \) on attributes \( u_j \in U \) are expressed with IVIFNs \( \tilde{r}_{ij} = ([a_{ij}^1, b_{ij}^1], [c_{ij}^1, d_{ij}^1]) \), respectively. \([a_{ij}^1, b_{ij}^1] \in D \in [0, 1]\) and \([c_{ij}^1, d_{ij}^1] \in D \in [0, 1]\) are the degree of satisfaction (or membership) and the degree of non-satisfaction (or non-membership) of \( o_i \) on the qualitative attribute \( u_j \) with respect to the concept “excellence” given by the decision maker \( D_v \), and such that they satisfy the following condition: \( b_{ij}^1 + d_{ij}^1 \leq 1 \). The IVIFN index, \( [1 - b_{ij}^1 - d_{ij}^1, 1 - a_{ij}^1 - c_{ij}^1] \), reflects the fact that the decision maker \( D_v \) may not always be certain of membership and non-membership degrees.

Let \( \tilde{R}_v = (\tilde{r}_{ij})_{n \times m} = ((\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}) \in D \in [0, 1], \tilde{r}_{ij} = ([a_{ij}^1, b_{ij}^1], [c_{ij}^1, d_{ij}^1]), \ldots, ([a_{im}^1, b_{im}^1], [c_{im}^1, d_{im}^1]) \) denotes the vector of IVIFNs, which represents relative degrees of satisfaction and relative degrees of non-satisfaction for every alternative \( o_i \in O \) on all \( m \) attributes given by the decision maker \( D_v \). Usually \( \tilde{R}_v \) and \( o_i \in O \) may be interchangeably used for the decision maker \( D_v \). Thus, a MAGDM problem using IVIFNs can be concisely expressed in the matrix format as follows:

\[
\tilde{R}_v = (\tilde{r}_{ij})_{n \times m} = \begin{pmatrix}
\tilde{r}_{11}^v & \tilde{r}_{12}^v & \ldots & \tilde{r}_{1m}^v \\
\tilde{r}_{21}^v & \tilde{r}_{22}^v & \ldots & \tilde{r}_{2m}^v \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{r}_{n1}^v & \tilde{r}_{n2}^v & \ldots & \tilde{r}_{nm}^v
\end{pmatrix}, v = 1, 2, \ldots, V
\]

(5)

where \( \tilde{r}_{ij}^v = ([a_{ij}^1, b_{ij}^1], [c_{ij}^1, d_{ij}^1]) \).

Assume that weights of attributes \( u_j \in U \) are \( \omega_j (j = 1, 2, \ldots, m) \), respectively. Usually, the weights are required to satisfy the following normalization conditions: \( \omega_j \in [0, 1], j = 1, 2, \ldots, m \) and \( \sum_{j=1}^{m} \omega_j = 1 \). Denote the vector of the weights by \( \omega = (\omega_1, \omega_2, \ldots, \omega_m)^T \).

3.1. The normalized Euclidean distance between IVIFSs

The distance, as the measure of difference between IVIFSs, is an important measure in IVIFS theory. The concept of the distance between IVIFSs was firstly introduced by Xu [14], where taking into account only two parameters (membership degree and non-membership degree), and do not take into account the third parameter (uncertainty degree). In [35, 40, 41], researchers showed that omitting one of the three parameters may lead to incorrect results, and thus all the three parameters should be taken into account when calculating distance between two IFSs. Motivated by this idea, the distances of IVIFSs should be calculated by taking into account three parameters describing an IVIFS as Eq. (4). So the normalized Euclidean distance between two IVIFNs \( \tilde{a}_1 \) and \( \tilde{a}_2 \) can be defined as:

\[
d(\tilde{a}_1, \tilde{a}_2) = \left( \frac{1}{4} \left( (a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2 + (e_1 - e_2)^2 + (f_1 - f_2)^2 \right) \right)^{0.5}
\]

(6)
For convenience, from here we will assume that X is finite, i.e., \( X = \{x_1, x_2, \ldots, x_n\} \). Let \( \tilde{A} = \{(x_i, \tilde{\mu}_A(x_i), \tilde{\nu}_A(x_i), \tilde{\pi}_A(x_i)) | x_i \in X\} \) and \( \tilde{B} = \{(x_i, \tilde{\mu}_B(x_i), \tilde{\nu}_B(x_i), \tilde{\pi}_B(x_i)) | x_i \in X\} \) be two IVIFSs, where \( \tilde{\mu}_A(x_i) = [\tilde{\mu}^L_A(x_i), \tilde{\mu}^R_A(x_i)] \), \( \tilde{\nu}_A(x_i) = [\tilde{\nu}^L_A(x_i), \tilde{\nu}^R_A(x_i)] \), \( \tilde{\pi}_A(x_i) = [\tilde{\pi}^L_A(x_i), \tilde{\pi}^R_A(x_i)] \) and \( \tilde{\pi}_B(x_i) = [\tilde{\pi}^L_B(x_i), \tilde{\pi}^R_B(x_i)] \). Then, the normalized Euclidean distance between IVIFSs \( \tilde{A} \) and \( \tilde{B} \) can be defined as:

\[
d(A, B) = \left( \frac{1}{4n} \sum_{i=1}^{n} \left[ (\tilde{\mu}^L_A(x_i) - \tilde{\mu}^L_B(x_i))^2 + (\tilde{\mu}^R_A(x_i) - \tilde{\mu}^R_B(x_i))^2 + (\tilde{\nu}^L_A(x_i) - \tilde{\nu}^L_B(x_i))^2 + (\tilde{\nu}^R_A(x_i) - \tilde{\nu}^R_B(x_i))^2 \right]^{0.5} \right)
\]

Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of the elements \( x_i (i = 1, 2, \ldots, n) \), then, we define the weighted Euclidean distance as follows:

\[
d_{\omega}(A, B) = \left( \frac{1}{4} \sum_{i=1}^{n} \omega_i \left[ (\tilde{\mu}^L_A(x_i) - \tilde{\mu}^L_B(x_i))^2 + (\tilde{\mu}^R_A(x_i) - \tilde{\mu}^R_B(x_i))^2 + (\tilde{\nu}^L_A(x_i) - \tilde{\nu}^L_B(x_i))^2 + (\tilde{\nu}^R_A(x_i) - \tilde{\nu}^R_B(x_i))^2 \right]^{0.5} \right)
\]

where \( \omega_i \in [0, 1] (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} \omega_i = 1 \).

### 3.2. Consistency and inconsistency measurements

Let \( \tilde{\omega}^+ \) represent an interval-valued intuitionistic fuzzy positive ideal solution (IVIFPIS), whose vector of the IVIFSs is denoted by

\[
\tilde{\omega}^+ = (\tilde{\omega}^+_{1}, \tilde{\omega}^+_{2}, \ldots, \tilde{\omega}^+_{m})
\]

where \( \tilde{\omega}^+_{j} = ((a^+_{ij}, b^+_{ij}), (c^+_{ij}, d^+_{ij})) \) are IVIFSs on attributes \( u_j \in U \), respectively. Usually \( \tilde{\omega}^+ \) and \( \tilde{\omega}^+ \) may be interchangeably used. Furthermore, we can take the IVIFPIS \( \tilde{\omega}^+ = (\tilde{\omega}^+_{1}, \tilde{\omega}^+_{2}, \ldots, \tilde{\omega}^+_{m}) \) where \( \tilde{\omega}^+_{j} = ((1, 1), [0, 0]) \) are the \( m \) largest IVIFNs.

For the decision maker \( D_v \), using Eq. (8), the square of the weighted Euclidean distance between an alternative \( \tilde{\omega}^+_{j} \) and the IVIFPIS \( \tilde{\omega}^+ \) can be calculated as follows:

\[
S^v_j = \sum_{j=1}^{m} \omega_j \left[ d(\tilde{\omega}^+_{ij}, \tilde{\omega}^+) \right]^2 = \sum_{j=1}^{m} \omega_j d^v_{ij}
\]

where

\[
d^v_{ij} = \frac{1}{4} \left[ (a^+_{ij} - a^+_{ij})^2 + (b^+_{ij} - b^+_{ij})^2 + (c^+_{ij} - c^+_{ij})^2 + (d^+_{ij} - d^+_{ij})^2 + (e^+_{ij} - e^+_{ij})^2 + (f^+_{ij} - f^+_{ij})^2 \right]
\]

and

\[
e^+_{ij} = 1 - b^+_{ij} - d^+_{ij}, f^+_{ij} = 1 - a^+_{ij} - c^+_{ij}, e^+_{ij} = 1 - b^+_{ij} - d^+_{ij}, f^+_{ij} = 1 - a^+_{ij} - c^+_{ij}.
\]

Assume that the decision maker \( D_v \) (\( v = 1, 2, \ldots, V \)) expresses his/her comparison preference relations on alternatives with \( \Omega^v = \{k, i\} | \alpha_k \mathbb{R}_v \alpha_i, (i, k = 1, 2, \ldots, m) \} \) (\( v = 1, 2, \ldots, V \)) according to knowledge and experience, where the symbol “\( \mathbb{R}_v \)” is the preference relation of the decision maker \( D_v \). \( \alpha_k \mathbb{R}_v \alpha_i \) means that either the decision maker \( D_v \) prefers the alternative \( \alpha_k \) to \( \alpha_i \) or the decision maker \( D_v \) is indifferent between \( \alpha_k \) and \( \alpha_i \). If the weight vector \( \omega \) and the IVIFPIS \( \tilde{\omega}^+ \) are chosen by the group already, then using Eq. (10), the squares of the weighted Euclidean distances between each pair of alternatives \( (k, i) \in \Omega^v \) and the IVIFPIS \( \tilde{\omega}^+ \) are calculated as follows:

\[
S^v_k = \sum_{j=1}^{m} \omega_j \left[ d(\tilde{\omega}^+_{kj}, \tilde{\omega}^+) \right]^2 = \sum_{j=1}^{m} \omega_j d^v_{kj}
\]

and

\[
S^v_i = \sum_{j=1}^{m} \omega_j \left[ d(\tilde{\omega}^+_{ij}, \tilde{\omega}^+) \right]^2 = \sum_{j=1}^{m} \omega_j d^v_{ij}
\]
respectively.

The alternative \( o_k \) is closer to the IVIFPIS \( \bar{F}^+ \) than the alternative \( o_i \) if \( S_i^+ \geq S_k^+ \). So the ranking order of alternatives \( o_k \) and \( o_i \) determined by \( S_k^+ \) and \( S_i^+ \) based on \( (\omega, \bar{F}^+) \) is consistent with the preference relation \((k, i) \in \Omega^v \) given by the decision maker \( D_v \). Conversely, if \( S_i^+ < S_k^+ \) then \((\omega, \bar{F}^+) \) is not chosen properly since it results in that the ranking order of alternatives \( o_k \) and \( o_i \) determined by \( S_k^+ \) and \( S_i^+ \) based on \( (\omega, \bar{F}^+) \) is inconsistent with the preference relation \((k, i) \in \Omega^v \) given by the decision maker \( D_v \). Therefore, \((\omega, \bar{F}^+) \) should be chosen so that the ranking order of alternatives \( o_k \) and \( o_i \) determined by \( S_k^+ \) and \( S_i^+ \) is consistent with the preference relation \((k, i) \in \Omega^v \) provided by the decision maker \( D_v \).

An index \((S_i^+ - S_k^+)\) is defined as follows:

\[
(S_i^+ - S_k^+) = \begin{cases} 
S_i^+ - S_k^+ & \text{if } S_i^+ < S_k^+ \vspace{0.5em} \text{max} \{0, S_i^+ - S_k^+\} \\
0 & \text{if } S_i^+ \geq S_k^+ \vspace{0.5em} \text{max} \{0, S_i^+ - S_k^+\}
\end{cases}
\]

(15)

which can measure inconsistency between the ranking order of alternatives \( o_k \) and \( o_i \) determined by \( S_k^+ \) and \( S_i^+ \), and the preference relation \((k, i) \in \Omega^v \) given by the decision maker \( D_v \). Obviously, the index in Eq. (15) can be rewritten as follows:

\[
(S_i^+ - S_k^+) = \text{max} \{0, S_i^+ - S_k^+\}
\]

(16)

Let

\[
B^v = \sum_{(k, j) \in \Omega^v} (S_j^+ - S_k^+) = \sum_{(k, j) \in \Omega^v} \text{max} \{0, S_j^+ - S_k^+\}
\]

(17)

which is called the inconsistency index of the decision maker \( D_v \). Thus, the group inconsistency index is defined as follows:

\[
B = \sum_{v = 1}^{V} B^v = \sum_{v = 1}^{V} \sum_{(k, j) \in \Omega^v} \text{max} \{0, S_j^+ - S_k^+\}
\]

(18)

which is a sum of the inconsistency indices of all \( V \) decision makers in the group.

Similarly, an index \((S_i^+ - S_k^+)\) is defined as follows:

\[
(S_i^+ - S_k^+) = \begin{cases} 
S_i^+ - S_k^+ & \text{if } S_i^+ \geq S_k^+ \vspace{0.5em} \text{max} \{0, S_i^+ - S_k^+\} \\
0 & \text{if } S_i^+ < S_k^+ \vspace{0.5em} \text{max} \{0, S_i^+ - S_k^+\}
\end{cases}
\]

(19)

which can measure consistency between the ranking order of alternatives \( o_k \) and \( o_i \) determined by \( S_k^+ \) and \( S_i^+ \), and the preference relation \((k, i) \in \Omega^v \) given by the decision maker \( D_v \). Then, the index in Eq. (19) can be rewritten as follows:

\[
(S_i^+ - S_k^+) = \text{max} \{0, S_i^+ - S_k^+\}
\]

(20)

Hence, the consistency index of the decision maker \( D_v \) can be defined as follows:

\[
G^v = \sum_{(k, j) \in \Omega^v} (S_j^+ - S_k^+) = \sum_{(k, j) \in \Omega^v} \text{max} \{0, S_j^+ - S_k^+\}
\]

(21)

Hence, the group consistency index can be defined as follows:

\[
G = \sum_{v = 1}^{V} G^v = \sum_{v = 1}^{V} \sum_{(k, j) \in \Omega^v} \text{max} \{0, S_j^+ - S_k^+\}
\]

(22)

which is a sum of the consistency indices of all \( V \) decision makers in the group. It is easily derived from Eqs. (15) and (19) that

\[
(S_i^+ - S_k^+) - (S_i^+ - S_k^+) = S_i^+ - S_k^+
\]

(23)

It is easily derived from Eqs. (13) and (14) that and

\[
S_i^+ - S_k^+ = \sum_{j = 1}^{m} \omega_j d_{ij}^v - \sum_{j = 1}^{m} \omega_j d_{kj}^v = \sum_{j = 1}^{m} \omega_j (d_{ij}^v - d_{kj}^v)
\]

(24)
Combining with Eqs. (18), (22) and (24), it directly follows that

\[
G - B = \sum_{v=1}^{V} \sum_{(k,j)\in\Omega} \left[ (S_i^v - S_k^v) + (S_k^v - S_j^v) - (S_i^v - S_j^v) \right]
\]

\[
\sum_{v=1}^{V} \sum_{i,j\in\Omega} (S_i^v - S_j^v)
\]

\[
\sum_{v=1}^{V} \sum_{i,j\in\Omega} m \omega (d_{ij}^v - d_{kj}^v)
\]

(25)

3.3. The linear programming models for MAGDM using IVIFSs

Since the IVIFPS \( \tilde{F}^+ = (\tilde{r}_1^+, \tilde{r}_2^+, \ldots, \tilde{r}_m^+) \) is given a priori, to determine \( \omega \), we construct the auxiliary mathematical programming model as follows:

\[
\min \{B\} \\
\begin{cases}
G - B \geq h, \\
\omega_j \geq \varepsilon \quad (j = 1, 2, \ldots, m)
\end{cases}
\]

(26)

which intends to minimize the inconsistency index \( B \) under the condition in which the consistency index \( G \) is not smaller than \( B \) by \( h \). Here, \( h \) is an arbitrary positive number given by the group a priori, and \( \varepsilon > 0 \) is a sufficiently small constant and can be chosen by the group a priori, such constraints ensure that the weights generated are not zero as it may be the case in the LINMAP [29]. Using Eqs. (18) and (25), the mathematical programming model (26) can be rewritten as follows:

\[
\min \left\{ \sum_{v=1}^{V} \sum_{(k,j)\in\Omega} \max \left\{ 0, S_k^v - S_j^v \right\} \right\} \\
\sum_{v=1}^{V} \sum_{i,j\in\Omega} (S_i^v - S_j^v) \geq h, \\
\omega_j \geq \varepsilon \quad (j = 1, 2, \ldots, m).
\]

(27)

For each pair of alternatives \((k, i) \in \Omega^v\), let

\[
\lambda_{ki}^v = \max \left\{ 0, S_k^v - S_i^v \right\}
\]

(28)

then \( \lambda_{ki}^v \geq S_k^v - S_i^v \), i.e., \( S_i^v - S_k^v + \lambda_{ki}^v \geq 0 \).

Combining with Eq. (13) and (14), it directly follows that

\[
\sum_{j=1}^{m} \omega_j (d_{ij}^v - d_{kj}^v) + \lambda_{ki}^v \geq 0
\]

(29)

According to Eqs. (12), (25), (28) and (29), the mathematical programming model (27) can be transformed into the linear programming model (30) as follows:

\[
\min \left\{ \sum_{v=1}^{V} \sum_{(k,j)\in\Omega} \lambda_{ki}^v \right\} \\
\sum_{v=1}^{V} \sum_{i,j\in\Omega} m \omega_j (d_{ij}^v - d_{kj}^v) \geq h, \\
\sum_{j=1}^{m} \omega_j (d_{ij}^v - d_{kj}^v) + \lambda_{ki}^v \geq 0 \quad (k, i) \in \Omega^v, (v = 1, 2, \ldots, V), \\
\omega_j \geq \varepsilon (j = 1, 2, \ldots, m), \\
\lambda_{ki}^v \geq 0 \quad (k, i) \in \Omega^v, (v = 1, 2, \ldots, V).
\]

(30)

It is easily seen from linear programming theory that the optimal solution of the linear programming model (30) and its optimal value of the objective function are related to the parameter \( h \), denoted by \( \omega^*(h) \) and \( B^*(h) \), respectively. Hence, the ranking orders of the alternative set \( X = \{x_1, x_2, \ldots, x_n\} \) for the decision makers \( D_v \) \((v = 1, 2, \ldots, V)\) are generated according to the increasing orders of distances \( S_i^v(i = 1, 2, \ldots, n) \) calculated with Eq. (11), respectively.
3.4. Decision process and algorithm for MAGDM with IVIFSs and incomplete preference information

The algorithm and decision process for solving MADM problems using IVIFSs are summarized as follows:

Step 1: Identify all alternatives to be evaluated and evaluation attributes. Denote sets of all alternatives and attributes by \( O = \{o_1, o_2, \ldots, o_n\} \) and \( U = \{u_1, u_2, \ldots, u_m\} \), respectively.

Step 2: Pool the decision makers’ opinions to get appropriate IVIFSs (or ratings) of alternatives \( o_i \in O \) on attributes \( u_j \in U \) and construct the interval-valued intuitionistic fuzzy decision matrices \( \tilde{R}^i \) for the decision makers \( D_v \) \((v = 1, 2, \ldots, V)\) respectively. Furthermore, the decision makers \( D_v \) \((v = 1, 2, \ldots, V)\) give incomplete preference relations on alternatives with \( \Omega' = \{(k, i) | o_k \tilde{R}_v o_i \ (k, i = 1, 2, \ldots, m)\} \), or there are some contradiction relations in the sets \( \Omega' \) \((v = 1, 2, \ldots, V)\).

Step 3: For the given IVIFPIS \( \tilde{F}^+ = (\tilde{F}_{1}^+, \tilde{F}_{2}^+, \ldots, \tilde{F}_{m}^+) \), we construct the linear programming model (30), where \( \tilde{r}_{j}^+ = ([a_{j}^+, b_{j}^+], [c_{j}^+, d_{j}^+]) \) is the position ideal point on the attributes \( u_j \in U \) \((j = 1, 2, \ldots, m)\).

Step 4: Solve the linear programming model (30) using Simplex method for linear programming with a given parameters \( e > 0 \) and \( h > 0 \).

Step 5: Determine the weight vector \( \omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_m^*)^T \).

Step 6: Calculate the distances \( S_{j}^i(i = 1, 2, \ldots, n; v = 1, 2, \ldots, V) \) of alternatives \( o_i \in O \) \((i = 1, 2, \ldots, n)\) to the IVIFPIS \( \tilde{F}^+ = (\tilde{F}_{1}^+, \tilde{F}_{2}^+, \ldots, \tilde{F}_{m}^+) \) using Eq.(10).

Step 7: Rank the alternatives for the decision makers \( D_v \) \((v = 1, 2, \ldots, V)\) according to the increasing orders of the distances \( S_{j}^i(i = 1, 2, \ldots, n; v = 1, 2, \ldots, V) \), respectively.

Step 8: Rank the alternatives for the group using the Borda’s function [42] and determine the best alternative(s) from the alternative set \( O \).

4. Conclusions

We have extend the LINMAP method to solve MAGDM problems in interval-valued intuitionistic fuzzy environment in which all the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by IVIFN, and the preference information about the alternatives is completely unknown. Because each alternative would be assessed on the basis of its weighted distance to an IVIFPIS, we have proposed the normalized Euclidean distance and the weighted Euclidean distance between IVIFSs, and we have derived the weights of attributes by constructing a new linear programming model based on the group consistency and inconsistency index defined on the basis of preferences between alternatives given by the decision makers. Then we have calculated the weighted distance of each alternative to the IVIFPIS to determine the ranking order of all alternatives for the decision makers, and we have used the Borda’s function to generate the ranking order of alternatives and select the best alternative(s) for the group. The proposed methodology in this paper not only can satisfy the consistency of preferences between alternatives given by the decision makers, but also avoid losing or distorting the original decision objection (i.e., IVIFPIS) in the process of decision making.

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