QUASICRYSTALS AND POISSON’S SUMMATION FORMULA

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ABSTRACT. We characterize the measures on \( \mathbb{R} \) which have both their support and spectrum uniformly discrete. A similar result is obtained in \( \mathbb{R}^n \) for positive measures.

1. Introduction

The subject of this paper is the analysis of measures in \( \mathbb{R}^n \) with discrete support and spectrum. This subject is often discussed in the framework of so-called Fourier quasicrystals, see J. C. Lagarias’ survey [12] and the references therein. The name “quasicrystals” was inspired by an experimental discovery in the middle of 80’s of non-periodic atomic structures with diffraction patterns consisting of spots.

Sometimes a Fourier quasicrystal is defined as a countable set \( \Lambda \) which supports an (infinite) pure point measure \( \mu \), such that its Fourier transform \( \hat{\mu} \) is also a pure point measure, see [6]. This definition is too wide, though, and includes examples where the support and spectrum are both everywhere dense sets. Usually the support \( \Lambda \) is assumed to be a uniformly discrete set (see e.g. [2], [3]).

The subject goes back to the classical Poisson summation formula: if \( f \) is a function on \( \mathbb{R} \) (satisfying some mild smoothness and decay conditions) and \( \hat{f} \) is its Fourier transform, then

\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).
\]

In other words, the measure

\[
\mu = \sum_{n \in \mathbb{Z}} \delta_n
\]

satisfies the equality

\[
\hat{\mu} = \mu.
\]

There is also a multi-dimensional version of Poisson’s formula. Let \( L \) be a (full-rank) lattice in \( \mathbb{R}^n \), and \( L^* \) be the dual lattice. Then

\[
(\sum_{\lambda \in L} \delta_\lambda) = \frac{1}{\det(L)} \sum_{s \in L^*} \delta_s.
\]

By simple procedures – shifts, multiplication on exponentials, and taking linear combinations – one may get different forms of this result. In particular (for \( n = 1 \)) it includes the Cauchy-Ramanujan formulas and more general ones due to V. Lin (see [8, pp. 283–289]).

However, there are Poisson-type formulas which cannot be obtained this way. In the one-dimensional case, the problem of which other discrete summation formulas may exist was studied by J.-P. Kahane and S. Mandelbrojt [9].

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An interesting example can be found in [7, p. 265], which involves weighted sums of \( f \) and \( \hat{f} \) at the nodes \( \{\pm(n + \frac{1}{3})^{1/2}\} \) \((n = 0, 1, 2, \ldots)\). This summation formula is also deduced from Poisson’s one, but in a more tricky way. Notice that in contrast to the classical case, the nodes in this example do not lie in a uniformly discrete set.

The cut-and-project method, applied to lattices in a generic position, leads to an important class of quasicrystals – the “model sets”. Y. Meyer [16, 17] discovered fundamental connections of these non-periodic sets to harmonic analysis.

On the other hand, if \( \mu \) is the sum of equal atoms along a discrete set \( \Lambda \) and \( \hat{\mu} \) is a positive pure point measure, then \( \Lambda \) is just a lattice. A simple proof of this fact was given by A. Córdoba [4]. A more general situation, when the atoms take finitely many different values, was considered in [15, p. 25], [5], [10]. These results are based on the Helson-Cohen characterization of idempotent measures in locally compact abelian groups.

There is a conjecture (see e.g. [12, p. 79]) that if the support and spectrum of a measure are both uniformly discrete sets, then the measure has a periodic structure, and the corresponding summation formula can be obtained from Poisson’s one by the procedures mentioned above.

The main goal of this paper is to prove this conjecture. In the one-dimensional case this is done in full generality, while in several dimensions – for positive (or positive-definite) measures. Our results were outlined in [14].

2. Results

A set \( \Lambda \subset \mathbb{R}^n \) is called uniformly discrete (u.d.) if

\[
\text{d}(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.
\]

We consider a (complex) measure \( \mu \) on \( \mathbb{R}^n \) supported on a u.d. set \( \Lambda \):

\[
\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_{\lambda}, \quad \mu(\lambda) \neq 0, \quad \text{d}(\Lambda) > 0.
\]

Assume that \( \mu \) is a temperate distribution, and that its Fourier transform

\[
\hat{\mu}(x) := \sum_{\lambda \in \Lambda} \mu(\lambda) e^{-2\pi i \langle \lambda, x \rangle}
\]

(in the sense of distributions) is also a measure, supported by a u.d. set \( S \):

\[
\hat{\mu} = \sum_{s \in S} \hat{\mu}(s) \delta_s, \quad \hat{\mu}(s) \neq 0, \quad \text{d}(S) > 0.
\]

The set \( S \) is the spectrum of the measure \( \mu \).

**Theorem 1.** Let \( \mu \) be a measure in \( \mathbb{R} \) satisfying (2) and (3). Then the support \( \Lambda \) is contained in a finite union of translates of a certain lattice. The same is true for \( S \) (with the dual lattice).

**Theorem 2.** Let \( \mu \) be a positive measure in \( \mathbb{R}^n, n > 1 \), satisfying (2) and (3). Then the conclusion of Theorem 1 holds.

The following proposition completes the results, describing the explicit form of \( \mu \).
Theorem 3. Let $\mu$ be a measure in $\mathbb{R}^n$, $n \geq 1$, satisfying (2) and (3), and such that $\Lambda$ is contained in a finite union of translates of a lattice $L$. Then $\mu$ is of the form

$$\mu = \sum_{j=1}^N P_j \sum_{\lambda \in L+\theta_j} \delta_\lambda$$

where $\theta_j$ is a vector in $\mathbb{R}^n$, and $P_j(x)$ is a trigonometric polynomial ($1 \leq j \leq N$).

By a trigonometric polynomial $P(x)$ on $\mathbb{R}^n$ we mean a finite linear combination of exponentials $\exp 2\pi i \langle \omega, x \rangle$.

The conclusion of Theorem 3 shows that $\mu$ can be obtained from the measure $\sum_{\lambda \in L} \delta_\lambda$ in Poisson’s summation formula by a finite number of shifts, multiplication on exponentials, and taking linear combinations.

Conversely, one can easily see that every measure $\mu$ of the form (4) satisfies both (2) and (3), since $\hat{\mu}$ is of the same form (with the dual lattice).

3. Preliminaries

3.1. Notation. By $\langle \cdot, \cdot \rangle$ and $| \cdot |$ we denote the Euclidean scalar product and norm in $\mathbb{R}^n$. The open ball of radius $r$ centered at the origin is denoted $B_r := \{ x \in \mathbb{R}^n : |x| < r \}$.

A set $\Lambda \subset \mathbb{R}^n$ is uniformly discrete (u.d.) if it satisfies (1). The set $\Lambda$ is relatively dense if there is $R > 0$ such that every ball of radius $R$ intersects $\Lambda$.

By a “distribution” we shall mean a temperate distribution on $\mathbb{R}^n$ (see [23]). By a “measure” we mean a complex, locally finite measure (usually infinite) which is also a temperate distribution. As usual $\delta_\lambda$ is the Dirac measure at the point $\lambda$.

If $\alpha$ is a temperate distribution, and $\varphi$ is a Schwartz function on $\mathbb{R}^n$, then $\langle \alpha, \varphi \rangle$ will denote the action of $\alpha$ on $\varphi$.

The Fourier transform in $\mathbb{R}^n$ will be normalized as follows:

$$\hat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i \langle t, x \rangle} dx.$$ 

If $\alpha$ is a temperate distribution then its Fourier transform $\hat{\alpha}$ is defined by $\langle \hat{\alpha}, \varphi \rangle = \langle \alpha, \hat{\varphi} \rangle$.

We denote by $\text{supp}(\alpha)$ the support of the distribution $\alpha$, and by $\text{spec}(\alpha)$ the support of its Fourier transform $\hat{\alpha}$.

By a (full-rank) lattice $L \subset \mathbb{R}^n$ we mean the image of $\mathbb{Z}^n$ under some invertible linear transformation $T$. The determinant $\det(L)$ is equal to $| \det(T) |$. The dual lattice $L^*$ is the set of all vectors $\lambda^*$ such that $\langle \lambda, \lambda^* \rangle \in \mathbb{Z}$, $\lambda \in L$.

If $A$ is a set in $\mathbb{R}^n$ then $\#A$ is the number of elements in $A$, $\text{mes}(A)$ or $|A|$ denote the Lebesgue measure of $A$, $\text{diam}(A)$ is the diameter of $A$, and $1_A$ is the indicator function of $A$. By $A+B$ and $A-B$ we denote the set of sums and set of differences of two sets $A, B \subset \mathbb{R}^n$.

3.2. Measures. We will need a few simple facts about measure in $\mathbb{R}^n$.

Lemma 1. Let $\mu$ be a measure in $\mathbb{R}^n$ supported by a u.d. set $\Lambda$. Then $\mu$ is a temperate distribution if and only if

$$|\mu(\lambda)| \leq C(1 + |\lambda|^N), \quad \lambda \in \Lambda,$$

for some positive constants $C$ and $N$.

This can be proved using standard arguments.
Lemma 2. Let $\mu$ be a measure in $\mathbb{R}^n$ satisfying (2) and (3). Then
$$\sup_{\lambda \in \Lambda} |\mu(\lambda)| < \infty.$$ \hfill (5)

Proof. Fix a Schwartz function $\varphi$ such that $\hat{\varphi}(0) = 1$ and $\text{supp}(\hat{\varphi}) \subset B_\delta$, where $\delta := d(\Lambda) > 0$. Then
$$|\mu(\lambda)| = \left| \int \hat{\varphi}(\cdot - \lambda) \, d\mu(x) \right| = \left| \int \varphi(t) e^{2\pi i \langle \lambda, t \rangle} \, d\hat{\mu}(t) \right| \leq \sum_{s \in S} |\varphi(s)| |\hat{\mu}(s)|.$$ \hfill (6)

By Lemma 4 there are constants $C, N$ such that $|\hat{\mu}(s)| \leq C(1 + |s|^N)$. Thus the sum on the right-hand side of (6) converges, and this establishes (5).

Lemma 3. Let $\mu$ be a non-zero, positive measure in $\mathbb{R}^n$. Then $0 \in \text{spec}(\mu)$.

Proof. If not, there is $\delta > 0$ such that the support of the distribution $\hat{\mu}$ is disjoint from $B_\delta$. Choose a Schwartz function $\varphi$ such that $\text{supp}(\varphi) \subset B_\delta$ and $\hat{\varphi} > 0$. Then
$$\int \hat{\varphi} \, d\mu = \langle \hat{\mu}, \varphi \rangle = 0.$$ Hence $\hat{\varphi} \mu$ is a non-zero positive measure with zero total mass, a contradiction. \hfill \square

3.3. Densities. We will use the classical concepts of lower and upper uniform density of a set $\Lambda$. The first one plays a central role in Beurling’s sampling theory for entire functions of exponential type. The second one was used by Kahane and Beurling in the interpolation problem. Here are their definitions:

$$D^{-}(\Lambda) := \liminf_{R \to \infty} \inf_{x \in \mathbb{R}^n} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|},$$
$$D^{+}(\Lambda) := \limsup_{R \to \infty} \sup_{x \in \mathbb{R}^n} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|}.$$ We also need the following version of density:

$$D_\#(\Lambda) = \liminf_{R \to \infty} \frac{\#(\Lambda \cap B_R)}{|B_R|}.$$ Clearly we have $D^{-}(\Lambda) \leq D_\#(\Lambda) \leq D^{+}(\Lambda)$.

Notice that if $\Lambda$ is a u.d. set then the densities above are finite, and that their values are invariant under translation of $\Lambda$. The last claim is obvious for $D^{-}$ and $D^{+}$, and is easy to check for $D_\#$.

3.4. Sampling and interpolation. Let $\Omega$ be a compact set in $\mathbb{R}^n$, whose boundary has Lebesgue measure zero. We denote by $\mathfrak{B}(\Omega)$ the Bernstein space consisting of all bounded, continuous functions $f$ on $\mathbb{R}^n$ such that the distribution $\hat{f}$ is supported by $\Omega$.

Let $\Lambda$ be a u.d. set in $\mathbb{R}^n$. One says that

(i) $\Lambda$ is a sampling set for $\mathfrak{B}(\Omega)$ if there is a constant $C = C(\Lambda, \Omega)$ such that
$$\sup_{x \in \mathbb{R}^n} |f(x)| \leq C \sup_{\lambda \in \Lambda} |f(\lambda)|, \quad f \in \mathfrak{B}(\Omega);$$

(ii) $\Lambda$ is an interpolation set for $\mathfrak{B}(\Omega)$ if for any bounded sequence of complex numbers $\{c_\lambda\}_{\lambda \in \Lambda}$, there exists some $f \in \mathfrak{B}(\Omega)$ satisfying $f(\lambda) = c_\lambda$ ($\lambda \in \Lambda$).
Landau proved in [13] that the classical density conditions for sampling and interpolation remain to be necessary in the more general situation:

(i) If $\Lambda$ is a sampling set for $B(\Omega)$, then $D^-(\Lambda) \geq \text{mes}(\Omega)$;
(ii) If $\Lambda$ is an interpolation set for $B(\Omega)$, then $D^+(\Lambda) \leq \text{mes}(\Omega)$.

Actually, Landau considered $L^2$ versions of the sampling and interpolation problems (a simple proof can be found in [20]). The above results for the Bernstein space can be deduced e.g. as in [22, Theorem 2.1].

4. Spectral gaps

4.1. A measure (or a distribution) $\mu$ is said to have a spectral gap of size $a > 0$ if the Fourier transform $\hat{\mu}$ vanishes on a ball of radius $a$.

In dimension one, there is a simple condition which is necessary for a u.d. set $\Lambda$ to support a measure with a spectral gap.

Proposition 4. Let $\Lambda \subset \mathbb{R}$ be a u.d. set, $d(\Lambda) \geq \delta > 0$. Assume that $\Lambda$ supports a non-zero measure $\mu$, such that $\hat{\mu}$ vanishes on the open interval $(0, a)$ for some $a > 0$. Then

$$D\#(\Lambda) \geq c(a, \delta),$$

where $c(a, \delta) > 0$ depends on $a$ and $\delta$ only.

This can be deduced from the results in the paper [18], where a complete characterization is given of u.d. sets in $\mathbb{R}$ which may support a finite measure with a spectral gap of given size, in terms of the lower Beurling-Malliavin density. It follows from this characterization that one may take $c(a, \delta) = a$.

Below we present an independent proof of Proposition 4 similar to the one used in [21, pp. 1044–1045].

Lemma 5. Let $\Lambda$ be a finite set contained in $(-R, R) \setminus (-\delta, \delta)$, where $d(\Lambda) \geq \delta > 0$, $R \geq 1$, and let $a > 0$. There is $c(a, \delta) > 0$ such that if $(\#\Lambda)/(2R) < c(a, \delta)$ then one can find a Schwartz function $\varphi$ with the following properties:

$$\varphi(0) = 1, \quad \varphi(\lambda) = 0 \quad (\lambda \in \Lambda), \quad \text{spec}(\varphi) \subset (0, a), \quad \sup_{|x| \geq R} |\varphi(x)| \leq 1.$$

Proof. It will be convenient to assume that the number of points in $\Lambda$ is even (if not, we may just add a point to $\Lambda$). Let $n := (\#\Lambda)/2$ and $\varepsilon := n/R$. Define the polynomial

$$P(z) := \prod_{\lambda \in \Lambda} \frac{z - e^{i\pi \lambda/R}}{1 - e^{i\pi \lambda/R}}.$$

Then $P(1) = 1$. We have

$$\max_{|z|=1} |P(z)| \leq \prod_{\lambda \in \Lambda} \frac{2}{2 \sin \left( \frac{\pi \lambda}{2R} \right)} \leq \prod_{\lambda \in \Lambda} \frac{R}{|\lambda|}.$$

The right-hand side is maximized when $\Lambda$ is the set $\{j\delta : 1 \leq |j| \leq n\}$. Hence

$$\max_{|z|=1} |P(z)| \leq \frac{R^{2n}}{\delta^{2n}(n!)^2} \leq \left( \frac{eR}{\delta \varepsilon} \right)^{2n} = \left( \frac{e}{\delta \varepsilon} \right)^{2\varepsilon R}.$$

Given $a > 0$, we choose a Schwartz function $\psi$ satisfying

$$\text{spec}(\psi) \subset (0, a/4), \quad \psi(0) = 1, \quad \gamma := \sup_{|x| \geq 1} |\psi(x)| < 1.$$
Set
\[ \varphi(x) := P(e^{i\pi x/R}) \cdot (\psi(x/R))^{[R]+1}. \] (7)

Then \( \varphi \) is a Schwartz function, \( \varphi(0) = 1, \varphi(\lambda) = 0 \) for \( \lambda \in \Lambda \). The spectrum of the first factor in (7) is contained in \([0, \varepsilon]\), while the spectrum of the second factor is contained in \((0, a/2)\). Hence, if \( \varepsilon < a/2 \) then \( \text{spec}(\varphi) \subset (0, a) \). Finally, we have
\[ \sup_{|x| \geq R} |\varphi(x)| \leq \left[ 2e^{\delta \varepsilon} \right]^{\frac{1}{R}}. \]

If \( \varepsilon \) is sufficiently small (depending on \( a, \delta \)) then the expression in square brackets is smaller than one. The lemma is therefore proved. \( \square \)

**Proof of Proposition 4.** It will be enough to prove the claim under the assumption that \( \mu \) is a finite measure. The general case may be easily reduced to this one by multiplying \( \mu \) on a Schwartz function \( \varphi \), such that \( |\varphi| > 0 \) and \( \text{spec}(\varphi) \subset (-a/2, 0) \). Then \( \varphi \mu \) is a non-zero, finite measure (by Lemma 1) supported by \( \Lambda \) and has a spectral gap \((0, a/2)\).

Assume that \( D\#(\Lambda) < c(a, \delta) \), where \( c(a, \delta) \) is given by Lemma 5. We will show that this implies \( \mu = 0 \). Observe that, by translating \( \mu \) and \( \Lambda \), and since \( D\#(\Lambda - \lambda) = D\#(\Lambda) \) for every \( \lambda \), it will be enough to consider the case when \( 0 \in \Lambda \) and to prove that \( \mu(0) \) must be zero.

Choose a sequence \( R_j \to \infty \) such that \( (#\Lambda_j)/(2R_j) < c(a, \delta) \), where
\[ \Lambda_j := \Lambda \cap (-R_j, R_j) \setminus \{0\}, \]
and let \( \varphi_j \) be the function given by Lemma 5 with \( \Lambda = \Lambda_j \) and \( R = R_j \). Since \( \hat{\mu} \) vanishes on \((0, a)\) we have
\[ \int_R \varphi_j(t) \hat{\mu}(t) \, dt = 0. \]

On the other hand,
\[ \int_R \varphi_j(t) \hat{\mu}(t) \, dt = \int_R \varphi_j(x) \, d\mu(x) = \mu(0) + \sum_{|\lambda| > R_j} \varphi_j(\lambda) \mu(\lambda). \]

It follows that
\[ |\mu(0)| \leq \sum_{|\lambda| > R_j} |\mu(\lambda)| \to 0 \quad (j \to \infty), \]
hence \( \mu(0) = 0 \). \( \square \)

4.2. The situation in the multi-dimensional case \((n > 1)\) is different, and the existence of a spectral gap is not sufficient to make a conclusion about the density of the support. As a simple example consider the set \( \Lambda = \mathbb{Z} \times \{0\} \) in \( \mathbb{R}^2 \), which has density zero, but which is the support of the measure
\[ \mu = \sum_{n \in \mathbb{Z}} (-1)^n \delta_{(n,0)} \]
having a spectral gap around the origin.

However, if a u.d. set \( \Lambda \) supports a measure which has not just a spectral gap, but an isolated atom in the spectrum, then the support must have positive density. More precisely, we have the following.
Lemma 6. Let \( \Lambda \) be a u.d. set in \( \mathbb{R}^n \). Assume that \( \Lambda \) supports a measure \( \mu \) satisfying (5), and such that \( \text{spec}(\mu) \cap B_a = \{0\} \) for some \( a > 0 \). Then
\[
D^- (\Lambda) \geq c(a,n),
\]
where \( c(a,n) > 0 \) depends on \( a \) and \( n \) only.

Proof. It is well-known that a distribution supported by the origin is a finite linear combination of derivatives of \( \delta_0 \). But condition (5) ensures that the distribution \( \hat{\mu} \) can only have order zero in a neighborhood of the origin. Hence there is a non-zero complex number \( w \) such that \( \hat{\mu} = w \delta_0 \) in \( B_a \). By multiplying \( \mu \) on \( 1/w \) we may suppose that \( w = 1 \).

Fix a Schwartz function \( \psi \), such that \( \text{supp}(\hat{\psi}) \subset B_{a/2} \) and \( \hat{\psi} = 1 \) in \( B_{a/3} \). For each \( x \in \mathbb{R}^n \) define a measure \( \nu_x \) by
\[
\nu_x := \hat{\psi}_x \mu, \quad \text{where} \quad \psi_x(y) := \psi(y-x).
\]
Then we have the following properties:
(i) \( \nu_x \) is supported by \( \Lambda \);
(ii) \( \hat{\nu}_x(t) = (\hat{\psi}_x * \hat{\mu})(t) = e^{-2\pi i(x,t)} \) in \( B_{a/3} \);
(iii) \( \nu_x \) is a finite measure, and \( \int |d\nu_x| \leq C \) for some constant \( C \) not depending on \( x \).

Let \( f \) be a function in the Bernstein space \( \mathfrak{B}(\Omega) \), where \( \Omega := \{x : |x| \leq a/4\} \). Let \( \varphi \) be a Schwartz function such that \( \varphi(0) = 1 \) and \( \text{spec}(\varphi) \) is contained in the open unit ball. Then \( f_\delta(x) := f(x)\varphi(\delta x) \) is a Schwartz function, and \( \text{spec}(f_\delta) \subset B_{a/4} \). Hence
\[
f_\delta(x) = \int f_\delta(x) e^{2\pi i(x,t)} dt = \int \hat{f}_\delta(t) \hat{\psi}_x(t) dt = \int \hat{f}_\delta \overline{\hat{\psi}_x}.
\]
Letting \( \delta \to 0 \) it follows (e.g. by the bounded convergence theorem) that
\[
f(x) = \int f \overline{\hat{\psi}_x},
\]
and hence
\[
|f(x)| \leq C \sup_{\lambda \in \Lambda} |f(\lambda)|.
\]
As this holds for any \( f \in \mathfrak{B}(\Omega) \), we get that \( \Lambda \) is a sampling set for \( \mathfrak{B}(\Omega) \). By Landau’s theorem we therefore have \( D^- (\Lambda) \geq \text{mes}(\Omega) = c(a,n) \), and this proves the claim.

4.3.

Lemma 7. Given \( a > 0 \) there is \( R = R(a,n) \) such that, if a measure \( \nu \) is supported by a u.d. set \( Q \) in \( \mathbb{R}^n \), \( d(Q) > a \), and if \( \hat{\nu} \) vanishes on a ball of radius \( R \), then \( \nu = 0 \).

Proof. This follows from Ingham type theorems used in interpolation theory in \( \mathbb{R}^n \). Given \( a > 0 \) there is \( R = R(a,n) \) such that if \( Q \) is any u.d. set in \( \mathbb{R}^n \), \( d(Q) > a \), then \( Q \) is an interpolation set for the Bernstein space \( \mathfrak{B}(\Omega) \), where \( \Omega := \{x : |x| \leq R/2\} \) (see for example [22]).

Let \( \nu \) be a measure supported by \( Q \) and such that the distribution \( \hat{\nu} \) vanishes on \( B_R \) (there is no loss of generality in assuming that the ball is centered at the origin). Given \( \lambda \in \Omega \) one can find \( f \in \mathfrak{B}(\Omega) \) such that \( f(\lambda) = 1 \) and \( f(\lambda') = 0 \) for any \( \lambda' \in Q \), \( \lambda' \neq \lambda \). Let \( \varphi(x) := f(x)\psi(x) \), where \( \psi \) is a Schwartz function such that \( \psi(\lambda) = 1 \) and \( \text{spec}(\psi) \subset B_{R/2} \). Then \( \varphi \) is a Schwartz function, satisfying
\[
\varphi(\lambda) = 1, \quad \varphi(\lambda') = 0 \quad (\lambda' \in Q, \lambda' \neq \lambda), \quad \text{spec}(\varphi) \subset B_R.
\]
It follows that
\[ \nu(\lambda) = \int \overline{\varphi} \, d\nu = \langle \widehat{\nu}, \overline{\varphi} \rangle = 0. \]
As this holds for any \( \lambda \in Q \), we obtain \( \nu = 0 \). \( \square \)

5. Delone and Meyer sets

5.1. We will need the following concepts of Delone and Meyer sets in \( \mathbb{R}^n \).

**Definition 1.** \( \Lambda \) is called a Delone set if \( \Lambda \) is both a u.d. and relatively dense set.

**Definition 2.** \( \Lambda \) is called a Meyer set if the following two conditions are satisfied:

(i) \( \Lambda \) is a Delone set;
(ii) There is a finite set \( F \) such that \( \Lambda - \Lambda \subset \Lambda + F \).

Meyer [16, 17] discovered important connections of this class of sets to certain problems in harmonic analysis. In particular, to the characterization of classes of almost-periodic functions with common almost-periods, and to the concepts of Pisot and Salem numbers in algebraic number theory.

5.2. Meyer observed that a Delone set \( \Lambda \) is a Meyer set if and only if \( \Lambda - \Lambda - \Lambda \) is u.d. (see [17]).

Lagarias [11] proved that if \( \Lambda \) is a Delone set and \( \Lambda - \Lambda \) is u.d. then \( \Lambda \) is a Meyer set. We need a stronger version of this result:

**Lemma 8.** Let \( \Lambda \subset \mathbb{R}^n \) be a Delone set, such that \( D^+(\Lambda - \Lambda) < \infty \). Then \( \Lambda \) is a Meyer set.

The proof below follows Lagarias’ argument, and simplifies it, basing also on [19].

**Proof of Lemma 8.** By translation we may assume that 0 \( \in \Lambda \). We fix \( R > 0 \) such that every ball of radius \( R \) intersects \( \Lambda \).

Let \( h \in \Lambda - \Lambda \). Then \( h = y - x \) for some \( x, y \in \Lambda \). Choose a sequence \( x_0, x_1, \ldots, x_s \) such that \( x_0 = x, \, x_s = y, \, |x_i - x_{i+1}| < R \). Define \( y_i = x_i + h \), then \( y_0 = y, \, y_s = h, \, |y_i - y_{i+1}| < R \). Choose \( p_i, q_i \in \Lambda \) such that \( |p_i - x_i| < R, \, |q_i - y_i| < R \) (0 \( \leq i \leq s \)), where \( p_0 = x, \, q_0 = y \) and \( p_s = 0 \) (recall that 0 \( \in \Lambda \)). It follows that \( p_i - p_{i+1} \) and \( q_i - q_{i+1} \) belong to the finite set \( F_1 := (\Lambda - \Lambda) \cap B_{3R} \).

Set \( h_i := q_i - p_i \). Then
\[ h_i - h_{i+1} = (q_i - q_{i+1}) - (p_i - p_{i+1}) \in F_2 := F_1 - F_1. \]
Also
\[ |h_i - h| = |(q_i - y_i) - (p_i - x_i)| < 2R, \]

hence
\[ h_i \in V(h) := (\Lambda - \Lambda) \cap (h + B_{2R}). \]

Since \( D^+(\Lambda - \Lambda) < \infty \), there is a constant \( M \) independent of \( h \) such that \( \#V(h) \leq M \). Thus in the sequence \( h_0, h_1, \ldots, h_s \) appear at most \( M \) distinct values. Write
\[ h_0 - h_s = (h_0 - h_1) + (h_1 - h_2) + \cdots + (h_{s-1} - h_s). \]

If some \( h_i \) and \( h_j \) (\( i < j \)) admit the same value, then we may remove from the sum above all the terms \( (h_k - h_{k+1}), \, i \leq k < j \). By removing all such “cycles” it follows that
\( h_0 - h_s \) belongs to the finite set \( F \) consisting of all vectors which may be expressed as the sum of at most \( M - 1 \) elements from \( F_2 \). Hence
\[ h = h_0 = h_0 + (q_s - h_s) = q_s + (h_0 - h_s) \in \Lambda + F. \]
This proves that \( \Lambda - \Lambda \subset \Lambda + F \), so \( \Lambda \) is a Meyer set. \( \square \)

5.3. Let \( \Gamma \) be a lattice in \( \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \) \( (m \geq 0) \), and let \( p_1 \) and \( p_2 \) denote the projections onto \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. We assume that the restriction of \( p_1 \) to \( \Gamma \) is injective, and that \( p_2(\Gamma) \) is dense in \( \mathbb{R}^m \). Let \( \Omega \) be a bounded set in \( \mathbb{R}^m \).

**Definition 3.** Under the assumptions above, the set
\[ \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega) := \{ p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega \}, \]
(8)
is called the model set defined by \( \Gamma \) and \( \Omega \).

This construction is known as “cut-and-project”.

Remark that the case \( m = 0 \) is not excluded in the above definition. In this case one should understand \( \mathbb{R}^m \) to be \( \{0\} \), and the model set obtained is just a lattice in \( \mathbb{R}^n \).

The following theorem \[16\] Sections II.5, II.14] gives a characterization of Meyer sets in terms of model sets (see also \[19\]).

**Theorem M** (Meyer). Let \( \Lambda \) be a Delone set in \( \mathbb{R}^n \). Then the following are equivalent:

(i) \( \Lambda \) is a Meyer set;

(ii) There exists a model set \( M \) and a finite set \( F \) such that \( \Lambda \subset M + F \).

5.4.

**Lemma 9.** Let \( M = \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega) \) be a model set in \( \mathbb{R}^n \), and suppose that the boundary of \( \Omega \) is a set of Lebesgue measure zero in \( \mathbb{R}^m \). Then
\[ D^-(M) = D^+(M) = \frac{\text{mes}(\Omega)}{\det(\Gamma)}. \]

This fact is well-known, see for example \[16\] Section V.7.3]. We include the proof in a simple, self-contained form, which we were not able to find elsewhere.

**Proof of Lemma 9.** (i) We show that \( p_1 \) restricted to \( \Gamma^* \) is injective, where \( \Gamma^* \subset \mathbb{R}^n \times \mathbb{R}^m \) is the lattice dual to \( \Gamma \). Indeed, let \( \gamma^* \in \Gamma^* \), and suppose that \( p_1(\gamma^*) = 0 \). Then we have
\[ \langle p_2(\gamma), p_2(\gamma^*) \rangle = \langle \gamma, \gamma^* \rangle \in \mathbb{Z}, \quad \gamma \in \Gamma. \]
Recall that \( p_2(\Gamma) \) is a dense set in \( \mathbb{R}^m \). Hence, if \( p_2(\gamma^*) \neq 0 \), then the left-hand side goes through a dense set in \( \mathbb{R}^m \) when \( \gamma \) goes through all elements in \( \Gamma \), which yields a contradiction. Thus we must have \( p_2(\gamma^*) = 0 \), and so \( \gamma^* = 0 \), which proves the claim.

(ii) Let \( \varphi \) be a Schwartz function on \( \mathbb{R}^n \), and \( \psi \) a Schwartz function on \( \mathbb{R}^m \). Denote
\[ S_R(\varphi, \psi, x) := \frac{1}{R^n} \sum_{\gamma \in \Gamma} \varphi \left( \frac{p_1(\gamma) - x}{R} \right) \psi(p_2(\gamma)), \quad x \in \mathbb{R}^n. \]

We show that
\[ \lim_{R \to \infty} S_R(\varphi, \psi, x) = \frac{1}{\det(\Gamma)} \int_{\mathbb{R}^n} \varphi \cdot \int_{\mathbb{R}^m} \psi, \]
uniformly with respect to $x$. Indeed, by the Poisson summation formula,
\[ S_R(\varphi, \psi, x) = \frac{1}{\det(\Gamma)} \sum_{\gamma^* \in \Gamma^*} \exp(-2\pi i \langle x, p_1(\gamma^*) \rangle) \hat{\varphi}(R_p(\gamma^*)) \hat{\psi}(p_2(\gamma^*)) \]
\[ = \frac{1}{\det(\Gamma)} \int \varphi \cdot \int \psi + \frac{1}{\det(\Gamma)} \sum_{\gamma^* \neq 0} \cdots . \]

Since $p_1(\gamma^*) \neq 0$ for all non-zero $\gamma^* \in \Gamma^*$, and due to the fast decay of $\varphi$ and $\psi$, the last sum tends to zero as $R \to \infty$ uniformly with respect to $x$, establishing the claim.

(iii) Let $\varepsilon > 0$. We may find non-negative Schwartz functions $\varphi_1, \varphi_2$ on $\mathbb{R}^n$ such that
\[ \varphi_1 \leq \frac{1}{|B|} \mathbf{1}_B \leq \varphi_2, \quad \int (\varphi_2 - \varphi_1) < \varepsilon, \]
where $B$ denotes the open unit ball in $\mathbb{R}^n$. Similarly, since the boundary of $\Omega$ is a set of Lebesgue measure zero (which is equivalent to the Riemann integrability of $\mathbf{1}_\Omega$), we may find non-negative Schwartz functions $\psi_1, \psi_2$ on $\mathbb{R}^m$ such that
\[ \psi_1 \leq \mathbf{1}_\Omega \leq \psi_2, \quad \int (\psi_2 - \psi_1) < \varepsilon. \]

Observe that
\[ S_R(\varphi_1, \psi_1, x) \leq \frac{|\#(M \cap (x + B_R))|}{|B_R|} \leq S_R(\varphi_2, \psi_2, x), \quad x \in \mathbb{R}^n. \]

Letting $R \to \infty$, this implies
\[ \frac{1}{\det(\Gamma)} \int \varphi_1 \cdot \int \psi_1 \leq D^-(M) \leq D^+(M) \leq \frac{1}{\det(\Gamma)} \int \varphi_2 \cdot \int \psi_2. \]

If $\varepsilon$ is chosen sufficiently small, then both the left- and right-hand sides will be arbitrarily close to $\text{mes}(\Omega)/\det(\Gamma)$. This proves the lemma. \qed

5.5. For a set $A \subset \mathbb{R}^n$ we shall denote by $\mathbb{Z}[A]$ the additive group generated by the elements of $A$.

**Lemma 10.** Let $M = \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega)$ be a model set, and $F$ be a finite set in $\mathbb{R}^n$. Then there is another model set $M' = \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma', \Omega')$ and a finite set $F'$, such that
\[ M + F \subset M' + F', \quad p_1(\Gamma') \cap \mathbb{Z}[F'] = \{0\}, \quad \Gamma' \subset \Gamma'. \]

**Proof.** The elements of $F$ generate a finite-dimensional vector space over the rationals $\mathbb{Q}$, which we denote by $V = \mathbb{Q}[F]$. Let $U := V \cap \mathbb{Q}[p_1(\Gamma)]$, a linear subspace of $V$. Let $W$ be any linear subspace of $V$ such that $U \oplus W = V$.

Denote by $\theta_1, \ldots, \theta_s$ the elements of $F$. Then each $\theta_j$ admits a unique representation as $\theta_j = u_j + w_j$, where $u_j \in U$, $w_j \in W$. Since $U \subset \mathbb{Q}[p_1(\Gamma)]$ we may find a non-zero integer $q$ and elements $\gamma_1, \ldots, \gamma_s \in \Gamma$ such that $u_j = p_1(\gamma_j/q)$, $1 \leq j \leq s$. Define
\[ \Gamma' := (1/q)\Gamma, \quad \Omega' := \bigcup_{j=1}^s (\Omega + p_2(\gamma_j/q)), \quad F' := \{w_1, \ldots, w_s\}. \]

Then $\Gamma'$ is a lattice in $\mathbb{R}^n \times \mathbb{R}^m$, the restriction of $p_1$ to $\Gamma'$ is injective, and $p_2(\Gamma')$ is dense in $\mathbb{R}^m$. The set $\Omega'$ is a bounded set in $\mathbb{R}^m$, and $F'$ is a finite set in $\mathbb{R}^n$. \qed
Let $M'$ be the model set defined by $\Gamma'$ and $\Omega'$. We show that $M + F \subset M' + F'$. Indeed, an element $\lambda \in M + F$ is of the form $\lambda = p_1(\gamma) + \theta$, where $\gamma \in \Gamma$ and $p_2(\gamma) \in \Omega$. Set $\gamma' := \gamma + \gamma_j/q$, then $\gamma' \in \Gamma'$ and $p_2(\gamma') \in \Omega'$. Hence

$$\lambda = p_1(\gamma') + w_j \in M' + F'.$$

Finally, observe that the set $p_1(\Gamma') \cap \mathbb{Z}[F']$ must be equal to $\{0\}$, since it is contained in both $U$ and $W$. It is also clear that $\Gamma \subset \Gamma'$, and so the lemma is proved. \hfill \Box

Notice that in the special case when $m = 0$, Lemma [10] reduces to:

**Corollary 11.** Let $L$ be a lattice, and $F$ be a finite set in $\mathbb{R}^n$. Then there is another lattice $L'$ and a finite set $F'$, such that $L + F \subset L' + F'$, $L' \cap \mathbb{Z}[F'] = \{0\}$, $L \subset L'$.

**6. Proof of Theorems [1] and [2]**

6.1. We will use the following notation: for $h \in \Lambda - \Lambda$, denote

$$\Lambda_h := \Lambda \cap (\Lambda - h) = \{\lambda \in \Lambda : \lambda + h \in \Lambda\}.$$

Clearly $\Lambda_h$ is a non-empty subset of $\Lambda$.

Let $\mu$ be a measure in $\mathbb{R}^n$ satisfying (2) and (3). For each $h \in \Lambda - \Lambda$ we introduce a new measure

$$\mu_h := \sum_{\lambda \in \Lambda_h} \mu(\lambda) \overline{\mu(\lambda + h)} \delta_\lambda. \quad (9)$$

Clearly it is a non-zero measure with $\text{supp}(\mu_h) = \Lambda_h$ and with bounded atoms (by Lemma [2]), so it is a temperate distribution.

**Lemma 12.** Let $a := d(S) > 0$. Then we have $\text{spec}(\mu_h) \cap B_a \subset \{0\}$, that is, the punctured ball $B_a \setminus \{0\}$ is free from the spectrum of the measure $\mu_h$.

**Proof.** We fix a Schwartz function $\varphi$ on $\mathbb{R}^n$, such that $\varphi(0) = 1$, and whose spectrum is contained in the open unit ball. Denote $\varphi_\delta(x) := \varphi(\delta x)$.

Let $u \in \mathbb{R}^n$. Consider the measure

$$(\widehat{\varphi_\delta} \ast \widehat{\mu})(t + u) \cdot \overline{\mu(t)}. \quad (10)$$

It is a temperate distribution, supported by the set $S \cap (S - u + B_\delta)$. Hence, if

$$u \in U_\delta := \mathbb{R}^n \setminus [(S - S) + B_\delta],$$

then the measure in (10) vanishes identically.

Now consider the Fourier transform of the measure (10). It is the measure

$$[e^{2\pi i(u,x)} \varphi_\delta(-x) \mu(-x)] \ast \mu(x) = \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} e^{-2\pi i(u,\lambda')} \varphi_\delta(\lambda) \mu(\lambda) \overline{\mu(\lambda')} \delta_{\lambda - \lambda'}$$

$$= \sum_{h \in \Lambda - \Lambda} \left[ \sum_{\lambda \in \Lambda_h} e^{-2\pi i(u,\lambda)} \varphi_\delta(\lambda) \mu(\lambda) \overline{\mu(\lambda + h)} \right] \delta_h$$

$$= \sum_{h \in \Lambda - \Lambda} (\varphi_\delta \cdot \mu_h)(u) \cdot \delta_h.$$

It follows that for every $h \in \Lambda - \Lambda$ we have

$$(\varphi_\delta \cdot \mu_h)(u) = 0, \quad u \in U_\delta.$$
The finite measure \( \varphi_\delta \cdot \mu_h \) tends to \( \mu_h \) (in the sense of temperate distributions) as \( \delta \to 0 \). This implies that \( \text{spec}(\mu_h) \) is contained in the closure of the set \( S - S \), which is disjoint from \( B_a \setminus \{0\} \). The lemma is therefore proved. \( \square \)

**Remark.** If \( \mu \) is a positive measure, then so is \( \mu_h \). Hence in this case Lemmas 3 and 12 imply that the distribution \( \hat{\mu}_h \) has an isolated atom at the origin.

6.2.

**Lemma 13.** Let \( \Lambda \) be a u.d. set in \( \mathbb{R}^n \). Suppose there is \( c = c(\Lambda) > 0 \) such that \( D_\#(\Lambda_h) > c \) for every \( h \in \Lambda - \Lambda \). Then \( D^+(\Lambda - \Lambda) < \infty \).

**Proof.** Let \( x \in \mathbb{R}^n \). Suppose that \( h_1, \ldots, h_N \) are distinct vectors belonging to the set \( (\Lambda - \Lambda) \cap (x + B_\delta) \), where \( \delta := d(\Lambda)/2 > 0 \). If \( \lambda \in \Lambda_{h_i} \cap \Lambda_{h_j} \) (\( i \neq j \)) then

\[
h_i - h_j = (\lambda + h_i) - (\lambda + h_j) \in (\Lambda - \Lambda) \cap B_{2\delta} = \{0\},
\]

which is not possible. Hence \( \Lambda_{h_1}, \ldots, \Lambda_{h_N} \) are pairwise disjoint subsets of \( \Lambda \). Since the density \( D_\# \) is super-additive, it follows that

\[
D_\#(\Lambda) \geq \sum_{j=1}^{N} D_\#(\Lambda_{h_j}) \geq cN.
\]

This shows that the set \( \Lambda - \Lambda \) cannot have more than \( D_\#(\Lambda)/c \) elements in any ball of radius \( \delta \), thus \( D^+(\Lambda - \Lambda) < \infty \). \( \square \)

6.3.

**Lemma 14.** Let \( E \) be a bounded set in \( \mathbb{R}^m \), and let \( \xi \) be a vector in \( E - E \) such that

\[
|\xi|^2 > (\text{diam } E)^2 - \delta^2
\]

for some \( \delta > 0 \). Suppose that we are given two representations of \( \xi \) as the difference of two elements from \( E \):

\[
\xi = y_1 - x_1 = y_2 - x_2, \quad x_1, y_1, x_2, y_2 \in E.
\]

Then \( |x_1 - x_2| < \delta \).

**Proof.** By the parallelogram law we have

\[
|\xi|^2 + |x_1 - x_2|^2 = \frac{1}{2} (|y_1 - x_2|^2 + |y_2 - x_1|^2) \leq (\text{diam } E)^2,
\]

so the claim follows. \( \square \)

6.4.

**Lemma 15.** Let \( \Lambda \) be a Meyer set in \( \mathbb{R}^n \). Suppose there is \( c = c(\Lambda) > 0 \) such that

\[
D^+(\Lambda_h) > c
\]

for every \( h \in \Lambda - \Lambda \). Then \( \Lambda \) is contained in a finite union of translates of some lattice.

**Proof.** (i) By Theorem M there exists a model set \( M = \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega) \) and a finite set \( F \) such that \( \Lambda \subset M + F \). By Lemma 11 we may suppose that

\[
p_1(\Gamma) \cap \mathbb{Z}[F] = \{0\}.
\]

Thus each \( \lambda \in \Lambda \) admits a unique representation as

\[
\lambda = p_1(\gamma(\lambda)) + \theta(\lambda), \quad \gamma(\lambda) \in \Gamma, \quad p_2(\gamma(\lambda)) \in \Omega, \quad \theta(\lambda) \in F.
\]
The uniqueness follows from (12) and the fact that the restriction of \( p_1 \) to \( \Gamma \) is injective.

(ii) Let \( h \in \Lambda - \Lambda \), and suppose that \( \lambda_1, \lambda_2 \in \Lambda_h \). Denote
\[
\lambda_j' := \lambda_j + h, \quad j = 1, 2.
\]
Then from (13) we have
\[
h = \lambda_j' - \lambda_j = p_1(\gamma(\lambda_j') - \gamma(\lambda_j)) + (\theta(\lambda_j') - \theta(\lambda_j)), \quad j = 1, 2.
\]
The condition (12) implies that the representation of \( h \) as the sum of an element from \( p_1(\Gamma) \) and an element from \( F - F \) is unique. Hence, we must have
\[
p_1(\gamma(\lambda_j') - \gamma(\lambda_1)) = p_1(\gamma(\lambda_j') - \gamma(\lambda_2)).
\]
Since the restriction of \( p_1 \) to \( \Gamma \) is injective, this implies
\[
\gamma(\lambda_j') - \gamma(\lambda_1) = \gamma(\lambda_j') - \gamma(\lambda_2).
\]
We thus obtain the following: to each \( h \in \Lambda - \Lambda \) there corresponds an element \( H(h) \in \Gamma \) such that
\[
\gamma(\lambda + h) - \gamma(\lambda) = H(h), \quad \lambda \in \Lambda_h.
\]

(iii) Let \( E := \{ p_2(\gamma(\lambda)) : \lambda \in \Lambda \} \). Then \( E \) is a bounded set in \( \mathbb{R}^m \), \( E \subset \Omega \). Given \( \delta > 0 \), we may choose a vector \( \xi \in E - E \) such that \( |\xi|^2 > (\text{diam } E)^2 - \delta^2 \). Observe that
\[
E - E = \{ p_2(H(h)) : h \in \Lambda - \Lambda \},
\]
hence \( \xi = p_2(H(h)) \) for some \( h \in \Lambda - \Lambda \). Let us fix such an \( h \).

Now suppose that \( \lambda_1, \lambda_2 \in \Lambda_h \). Then by (14) we have
\[
H(h) = \gamma(\lambda_j + h) - \gamma(\lambda_j), \quad j = 1, 2.
\]
This yields two representations of \( \xi \) as the difference of two elements from \( E \):
\[
\xi = p_2(H(h)) = p_2(\gamma(\lambda_j + h)) - p_2(\gamma(\lambda_j)), \quad j = 1, 2.
\]
By Lemma 14 we must therefore have
\[
|p_2(\gamma(\lambda_2)) - p_2(\gamma(\lambda_1))| < \delta.
\]
Hence, we conclude the following: denote
\[
E(h) := \{ p_2(\gamma(\lambda)) : \lambda \in \Lambda_h \}.
\]
Then, given any \( \delta > 0 \) one can find \( h \in \Lambda - \Lambda \) such that \( \text{diam}(E(h)) < \delta \).

(iv) Let \( h \in \Lambda - \Lambda \), and suppose that \( \text{diam}(E(h)) < \delta \) for some \( \delta > 0 \). We may find an open ball \( \Omega' \) of radius \( \delta \) such that \( E(h) \subset \Omega' \). Consider the model set
\[
M' = \mathfrak{M}(\mathbb{R}^n \times \mathbb{R}^m, \Gamma, \Omega').
\]
Then by (8), (13) and (15) we have \( \Lambda_h \subset M' + F \). Since the density \( D^+ \) is sub-additive and invariant under translations, this implies
\[
D^+(\Lambda_h) \leq \# F \cdot D^+(M').
\]
Recall that \( D^+(M') = (\det \Gamma)^{-1}|\Omega'| \), according to Lemma 9. Hence
\[
D^+(\Lambda_h) \leq \# F \cdot \frac{c_m \delta^m}{\det \Gamma},
\]
where \( c_m \) denotes the volume of the unit ball in \( \mathbb{R}^m \).

(v) It follows from (iii),(iv) that if \( m \geq 1 \), then we may find elements \( h \in \Lambda - \Lambda \) with \( D^+(\Lambda_h) \) arbitrarily small, in contradiction to (11). Hence we must have \( m = 0 \), that
is, $M$ must be a lattice. Thus $M + F$ is a finite union of translates of a lattice. Since $\Lambda \subset M + F$, this concludes the proof. □

6.5. Now we can finish the proof of Theorems 1 and 2.

Proof of Theorems 1 and 2. For each $h \in \Lambda - \Lambda$, let $\mu_h$ be the measure defined by (9). Then $\mu_h$ is a non-zero measure, $\text{supp}(\mu_h) = \Lambda_h$, and $\sup_{\Lambda} |\mu_h(\lambda)| < \infty$ (by Lemma 2).

By Lemma 12 we have

$$\text{spec}(\mu_h) \cap B_a \subset \{0\},$$

(16)

where $a = d(S) > 0$.

In the one-dimensional case $n = 1$, observe that condition (16) implies that $\hat{\mu}_h$ vanishes on the open interval $(0, a)$. So we may use Proposition 4 which gives

$$D^-(\Lambda_h) \geq c, \quad h \in \Lambda - \Lambda,$$

(17)

where $c > 0$ is a constant which now depends on $d(S)$.

In the multi-dimensional case $n > 1$, we use the extra assumption that $\mu$ is a positive measure. It implies that $\mu_h$ is also positive, for every $h \in \Lambda - \Lambda$. By Lemma 3 we therefore have $0 \in \text{spec}(\mu_h)$, so $\text{spec}(\mu_h) \cap B_a = \{0\}$. This allows us to use Lemma 6, which gives that $D^-(\Lambda_h) \geq c$, where $c > 0$ is a constant which now depends on $d(S)$ only. Since $D^-(\Lambda_h) \geq D^-(\Lambda_h)$, we obtain (17) again.

With (17) established, we now proceed to apply Lemma 13 which gives

$$D^+(\Lambda - \Lambda) < \infty.$$  (18)

Also, using Lemma 7 with $Q = S$ and $\nu = \hat{\mu}$ gives that $\Lambda$ is a relatively dense set. Hence $\Lambda$ is a Delone set (see also Lemma 1 in [5]).

This together with (18) gives, by Lemma 8, that $\Lambda$ is a Meyer set.

Finally, we apply Lemma 15. Since from (17) we get $D^+(\Lambda_h) \geq c$ for every $h \in \Lambda - \Lambda$, the lemma gives that $\Lambda$ is contained in a finite union of translates of some lattice, and this completes the proof. □

7. Proof of Theorem 3

7.1. Lemma 16. Let $\theta \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Then the set

$$H(\theta) := \{m \in \mathbb{Z}^n : \langle \theta, m \rangle \in \mathbb{Z}\}$$

is contained in some $(n - 1)$-dimensional hyperplane.

Proof. Define $V(\theta) := \{x \in \mathbb{Q}^n : \langle \theta, x \rangle \in \mathbb{Q}\}$. It is a linear subspace of $\mathbb{Q}^n$ over the rationals. Since $\theta \notin \mathbb{Q}^n$, this subspace cannot contain all the standard basis vectors $e_1, \ldots, e_n$. Hence $V(\theta)$ is a proper subspace of $\mathbb{Q}^n$, and so it is necessarily contained in some $(n - 1)$-dimensional hyperplane. But $H(\theta) \subset V(\theta)$, so this proves the claim. □

Since the union of a finite number of hyperplanes cannot cover $\mathbb{Z}^n$, it follows that:

Corollary 17. Let $\theta_1, \ldots, \theta_s \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Then there is $m \in \mathbb{Z}^n$ such that

$$\langle \theta_j, m \rangle \notin \mathbb{Z}, \quad 1 \leq j \leq s.$$
7.2.

Proof of Theorem 3. We suppose that \( \mu \) is a measure in \( \mathbb{R}^n \) (\( n \geq 1 \)) satisfying (2) and (3), and that the support of \( \mu \) is contained in a finite union of translates of a lattice \( L \).

Using Corollary 11 we can find a larger lattice \( L' \supset L \) and a finite set \( F' \) such that \( L' \cap \mathbb{Z}[F'] = \{0\} \), and the support of \( \mu \) is contained in \( L' + F' \). We will show that \( \mu \) can be represented in the form (4) with the lattice \( L' \). The desired representation with the original lattice can be obtained by covering \( L' \) with a finite number of translates of \( L \).

It will be enough, by applying a linear transformation, to consider the case \( L' = \mathbb{Z}^n \).

Denote by \( \theta_1, \ldots, \theta_s \) the elements of \( F' \). For each \( j = 1, \ldots, s \) define a measure

\[
\mu_j := \sum_{k \in \mathbb{Z}^n} \mu(k + \theta_j) \delta_k.
\]

It is a temperate distribution (by Lemma 2) supported by \( \mathbb{Z}^n \), and we have

\[
\mu(x) = \sum_{j=1}^s \mu_j(x - \theta_j). \tag{19}
\]

The Fourier transform \( \hat{\mu}_j \) is a temperate distribution on \( \mathbb{R}^n \) which is \( \mathbb{Z}^n \)-periodic. Define a distribution

\[
\alpha_j(t) := e^{-2\pi i \langle \theta_j, t \rangle} \hat{\mu}_j(t). \tag{20}
\]

From (19), (20) and the periodicity of \( \hat{\mu}_j \) it follows that

\[
\hat{\mu}(t - k) = \sum_{j=1}^s e^{2\pi i \langle \theta_j, k \rangle} \alpha_j(t) \tag{21}
\]

for each \( k \in \mathbb{Z}^n \).

Since \( \mathbb{Z}^n \cap \mathbb{Z}[\theta_1, \ldots, \theta_s] = \{0\} \), we have

\[
\theta_j - \theta_\ell \notin \mathbb{Q}^n \quad (j \neq \ell).
\]

Using Corollary 17 we may therefore choose a vector \( m \in \mathbb{Z}^n \) such that

\[
\langle \theta_j - \theta_\ell, m \rangle \notin \mathbb{Z} \quad (j \neq \ell). \tag{22}
\]

Applying (21) with \( k = pm \) (\( p = 0, 1, 2, \ldots, s - 1 \)) yields a system of \( s \) linear equations, with a Vandermonde determinant that does not vanish due to (22). Hence this linear system may be inverted, and we obtain that

\[
\alpha_j(t) = \sum_{p=0}^{s-1} c_{jp} \hat{\mu}(t - pm)
\]

for appropriate coefficients \( \{c_{jp}\} \).

But now using (20) this implies that the distribution \( \hat{\mu}_j \) is a measure, supported by the closed, discrete set \( S + \{0, m, 2m, \ldots, (s - 1)m\} \). On the other hand, the measure \( \hat{\mu}_j \) is \( \mathbb{Z}^n \)-periodic. Hence it must be of the form

\[
\hat{\mu}_j = \nu_j \ast \sum_{k \in \mathbb{Z}^n} \delta_k,
\]

where \( \nu_j \) is a measure which is a finite sum of point masses. It follows that

\[
\mu_j(x) = P_j(x) \sum_{k \in \mathbb{Z}^n} \delta_k
\]
where $P_j$ is a trigonometric polynomial, $P_j(x) = \hat{\nu}_j(-x)$. By (19) this completes the proof of Theorem 3.

8. Remarks

1. Theorem 2 gives an affirmative answer to Problem 4.1(a) in [12, p. 79]. The Problem 4.1(b) from that paper, asking whether one can remove in Theorem 2 the uniformity requirement for discrete sets $\Lambda$ and $S$, remains open. For signed (not positive) measures, one may expect a counter-example due to the results in [7].

We also leave open the problem whether Theorem 2 holds for non-positive measures. In [14] we proved this under the additional assumption that $\Lambda - \Lambda$ is a u.d. set.

2. It is well-known that if one requires from $S$ in Theorems 1 and 2 to be just a countable (non-discrete) set, then the result fails. As an example one may take the model set defined by (8) (with $m \geq 1$). It is a u.d. set, which supports a positive measure $\mu$ whose Fourier transform is a sum of point masses (see [17]), but is not contained in a finite union of translates of a lattice.

3. Sometimes different approaches to mathematical models of quasicrystals are considered. In particular, inspired by the Fibonacci sequence, one may look at the “block complexity” of a u.d. sequence in $\mathbb{R}$, characterized by the number of distinct blocks of given length occurring in the sequence, see [1, 2] and the references therein. It seems to be interesting to investigate the spectral properties of measures supported by sequences with “low complexity”.

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