Lattice Tests of Supersymmetric Yang-Mills Theory?

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Abstract

Supersymmetric Yang Mills theory is directly accessible to lattice simulations using current methodology, and can provide a non-trivial check of recent exact results in SQCD. In order to tune the lattice simulation to the supersymmetric point it is necessary to understand the behavior of the theory with a small supersymmetry breaking gaugino mass. We introduce a soft breaking gaugino mass in a controlled fashion using a spurion analysis. We compute the gluino condensate, vacuum energy and bound-state masses as a function of the gaugino mass, providing more readily accessible predictions which still test the supersymmetric results. Finally we discuss diagnostics for obtaining the bare lattice parameters that correspond to the supersymmetric continuum limit.

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1 Introduction

The recently proposed exact solutions [1, 2, 3] of N=1 supersymmetric QCD (SQCD) satisfy striking tests of self-consistency and provide an extremely plausible picture of the rich low-energy dynamics of these models. Nevertheless, one may feel a little discomfort at the absence of direct non-perturbative tests of the results. An obvious possibility is that these models could be simulated directly on the lattice. Some initial work in these directions has already been performed in [4]. Unfortunately, as is well known, lattice regularization violates supersymmetry [5], and a special fine-tuning is required to recover the SUSY limit (this is analogous to the case of chiral symmetry in lattice QCD). Away from the SUSY point, the continuum limit of the lattice theory is described by a model with explicit SUSY violating interactions. In some cases, these violations may correspond only to soft breakings [6], although this is not guaranteed in general.

Softly broken SUSY models can be studied using spurion techniques, and “exact” results are possible [7, 8] (additional investigations of softly broken SQCD can be found in [9]). In this paper, we propose some tests of supersymmetric Yang-Mills theory (SYM – or SQCD with zero flavors of quark) which can be carried out using lattice techniques. SYM is a simple theory with only one parameter, the gauge coupling. The only low-dimension (renormalizable) SUSY violation allowed by gauge invariance is a gaugino mass, which is a soft violation. Therefore, the continuum limit of the lattice regularized version of SYM is simply SYM with a massive gaugino. The SUSY limit can be reached by fine-tuning the lattice parameter corresponding to a bare gaugino mass. In order to understand this limit as well as possible, we study continuum SYM with explicit gaugino mass, and derive some relations describing the approach to the SUSY limit. The spurion techniques used assume that there is not a phase transition to some totally new phase the moment that supersymmetry is broken, a fact that may be tested on the lattice (if such a transition did occur then testing the supersymmetric predictions would become very hard since that phase would constitute only a single point in parameter space and there would be no understanding of the approach to that point). Assuming no such transition, several non-trivial predictions can be made regarding the vacuum energy and of the behavior of the gaugino condensate. A less rigorously derived description of the lightest bound states of SYM theory has also been proposed in the literature [10, 11] from which predictions for the masses of the gluino-gluino and glue-gluino bound states and their splittings away from the supersymmetric point may be obtained. We also discuss aspects of tuning toward the SUSY limit.

A lattice test of the pure glue theory would also provide a test of the tower of SQCD theories with fundamental matter flavors. The numerical coefficients in the effective superpotential of SQCD theory with \( N_f = N_c - 1 \) have been determined analytically [12] (this is
the first value of $N_f$ where the gauge group can be broken completely and the theory studied in the perturbative regime). The pure glue theory predictions resulting from integrating out the quark flavors in that effective theory are closely related to these coefficients as we review below.

The outline of the paper is as follows. In section 2 we review the holomorphic analysis of SYM. In sections 3 and 4 we introduce soft breakings into the analysis, deriving predictions for the lattice simulations. In section 5 we discuss diagnostics for tuning towards the SUSY point. In section 6 we summarize our results. Finally, in the appendix, we discuss the rescaling anomaly and the holomorphic vs. canonical field normalizations, which are relevant to the comparison between exact and lattice results.

## 2 SUSY Yang-Mills

The bare Lagrangian of SYM with $SU(N_c)$ gauge group is

$$\mathcal{L} = \frac{1}{g_0^2} \left[ -\frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a + i \lambda_\alpha \gamma_5 D^{\alpha \beta} \lambda_\beta \right]. \quad (1)$$

This model possesses a discrete global $Z_{2N_c}$ symmetry, a residual non-anomalous subgroup of the anomalous chiral $U(1)$. The theory is believed to generate a gaugino condensate breaking the $Z_{2N_c}$ symmetry to $Z_2$ and also to exhibit a mass gap due to confinement.

In supersymmetric notation the Lagrangian \( (1) \) can be written as

$$\mathcal{L} = \int d^2 \theta \frac{1}{8\pi} Im \tau_0 W^\alpha W_\alpha , \quad (2)$$

where the gauge coupling is defined to be $\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}$.

In this notation, the strong coupling scale of SYM is defined through the relation

$$\Lambda \equiv e^{2\pi i \tau_0 / b_0} \Lambda_{UV} , \quad (3)$$

where $b_0 = 3N_c$ is the first coefficient of SYM $\beta$-function and $\Lambda_{UV}$ is the UV cut off of the theory at which the coupling takes the value $g_0$. The relation of this definition of the strong scale to the $\overline{\text{MS}}$ scheme has been investigated in \( [13] \). Note that $\Lambda^b_0$ is explicitly $2\pi$-periodic in the $\theta_0$-angle. For the purposes of this paper we set $\theta_0 = 0$.

To derive the low energy effective theory of SQCD we note that there are two anomalous symmetries of the theory, $U(1)_R$ and scale invariance. In fact their anomalies are related since their currents belong to the same supermultiplet. These symmetries can be restored
in an enlarged theory provided we allow the spurion couplings to transform:

\[
U(1)_R : \quad W(x, \theta) \rightarrow e^{i\xi}W(x, \theta e^{i\xi}) \quad \Lambda \rightarrow \Lambda e^{2\xi/3} \quad \text{Scale Invariance :} \quad W(x, \theta) \rightarrow e^{i\xi/2}W(x e^{\xi}, \theta e^{i\xi/2}) \quad \Lambda \rightarrow \Lambda e^{\xi}.
\]

We may now determine the general form of the partition function (assuming a mass gap) in the large volume limit as a function of \(\tau\) subject to these symmetries. The only possible terms are

\[
Z[\tau] = \exp iV \left[ \frac{9}{\alpha} \Lambda^\dagger \Lambda|_D - (\beta \Lambda^3)|_F + h.c. \right]
\]

(4)

The numerical coefficients \(\alpha\) and \(\beta\) remain undetermined from the above symmetry arguments. \(\beta\) may be determined from the results for SQCD with massive quarks. The \(U(1)_R\) symmetries of models with matter are sufficient to determine the form of the superpotential as

\[
W_{\text{massive}} = c \left[ \det(m)\Lambda^{b_0} \right]^{1/N_c}
\]

(5)

where \(\Lambda\) is defined similarly to (3) but with the appropriate \(\beta\)-function \(b_0 = 3N_c - N_f\) and \(c\) is a constant, undetermined by the symmetry. The scale transformation anomaly is not sufficient to uniquely determine the Kähler potential in these theories with two mass scales \(\Lambda\) and \(m\). The coefficient \(c\) may be explicitly calculated in theories in which \(N_f \geq N_c - 1\) since the squark vevs may be chosen to completely break the gauge symmetry in the perturbative regime. The superpotential term is then generated by calculable instanton effects. The calculation was performed in [12]. The mass terms may then be taken to infinity removing the quarks from the theory and leaving the pure glue theory in a controlled fashion. The resulting prediction for \(c\) in Minkowski space is \((N_f - N_c)\) and hence \(\beta = N_c\).

These strong arguments lead to two predictions for the condensates of the SYM theory. The source \(J\) for the gaugino correlator \(\lambda\lambda\) occurs in the same position as the F-component of \(\tau\) and is hence known. There are two independent correlators

\[
\langle \lambda\lambda \rangle = -32\pi^2 \Lambda^3 \quad \langle \bar{\lambda}\lambda\bar{\lambda}\lambda \rangle = \frac{-1024i\pi^4}{\alpha N_c^2} |\Lambda|^2 /V .
\]

(6)

The IR theory has a gaugino condensate \(\simeq \Lambda^3\), with phase \(2\pi i\tau/N_c\) and hence there are \(N_c\) degenerate vacua associated with the \(N_c\)th roots of unity. Below, therefore, \(\Lambda^3\) is an \(N_c\) valued constant with phases \(n2\pi i/N_c\) where \(n\) runs from 0... \(N_c - 1\).
3 Soft Supersymmetry Breaking

We may induce a bare gaugino mass through a non zero $F$-component of the bare gauge coupling $\tau = \tau_0 + F_\tau \theta$ \[7\]

\[-\frac{1}{8\pi} Im[F_\tau \Lambda \lambda] \] (7)

To make the gaugino mass canonical we take $F_\tau = i \frac{8}{\pi} m\lambda$. In the IR theory $\tau$ enters through the spurion, $\Lambda_s$, the lowest component of which is the strong interaction scale $\Lambda$

$$\Lambda_s^{bo} = \Lambda^{bo} (1 - 16\pi^2 m\lambda \theta)$$ (8)

$\Lambda$ occurs linearly in the superpotential of the theory. Thus there will be a correction to the potential of the form:

$$\Delta V = -32\pi^2 Re(m\lambda \Lambda^3) - \frac{256\pi^4}{\alpha N_c^2} |m\lambda \Lambda|^2$$ (9)

Terms with superderivatives acting on the spurion field can also give rise to contributions to the potential but these are higher order in an expansion in $m\lambda / \Lambda$. The shift in the potential energy of the $N_c$ degenerate vacua of the SYM theory at linear order in $m\lambda$ is known and we may determine the vacuum structure

$$\Delta V = -32\pi^2 |m\lambda \Lambda^3| \cos \left( \frac{2\pi n}{N_c} + \theta m\lambda \right)$$ (10)

For small soft breakings, $m\lambda \ll \Lambda$, where the linear term dominates, the degeneracy between the SYM vacua is broken favoring one vacuum dependent on the phase of the gaugino mass. The coefficient in the energy shift is a test of the exact superpotential in Eq.(4).

We may also determine the leading shift in the gaugino condensate

$$\langle \lambda \lambda \rangle = -32\pi^2 \Lambda^3 + \frac{512\pi^4}{\alpha N_c^2} m^* \Lambda^2$$, (11)

which depends on the unknown parameter $\alpha$. Strictly speaking there are also divergent contributions to this quantity which are proportional to $m\lambda$ times the cut-off squared. We will have more to say about these divergences in section 5.

Reinserting the bare $\theta_0$ angle into the expression for the shift in vacuum energy we find

$$\Delta V = -32\pi^2 |m\lambda \Lambda^3| \cos \left[ \frac{2\pi n}{N_c} + \theta m\lambda + \theta_0 \frac{N_c}{N} \right]$$ (12)

As $\theta_0$ is changed first order phase transitions occur at $\theta_0 = (\text{odd}) \pi$ where two of the $N_c$ SYM vacua interchange as the minimum of the softly broken theory.
4 The Lightest Bound States

An alternative description of the low energy behaviour of SYM theory has been presented by Veneziano and Yankelowicz [10] which attempts to describe the lightest bound states of the theory. The form of their effective action can be rigorously obtained from the discussion above. Since the source \( J \) for \( WW \) occurs in the same places as the coupling \( \tau \) we also know the source dependence of \( Z \). If we wish we may Legendre transform \( Z[\tau, J] \) to obtain the effective potential for the classical field

\[
S \equiv \frac{1}{32\pi^2} \text{Tr} \langle W^2 \rangle .
\]  

(13)

We find

\[
\Gamma[\tau, S] = \frac{9}{\alpha} \left( SS \right)^{1/3} \bigg|_D - N_c \left( S - S \ln(S/\Lambda^3) \right) \bigg|_F + \text{h.c.} .
\]  

(14)

So derived, this effective action contains no more information than Eq.(4) simply being a classical potential whose minimum determines the vev of \( S \) and we find, by construction, Eq.(3).

A stronger interpretation can also be given to the VY action. We can attempt to use it to describe the physical gluino-gluino and gluino-glue bound states, if we assert that they interpolate in the perturbative regime to the field \( WW \). Under this assumption the symmetry properties of the bound states would be those of the \( S \) field, reproducing the VY action for those states, up to additional scale-invariant Kahler terms. (The latter only affect our predictions at higher order in the soft breaking.) To obtain the physical states one performs an appropriate rescaling of the \( S \)-field

\[
\Phi = \frac{3}{\sqrt{\alpha}} S^{1/3}
\]  

(15)

in the Lagrangian (14) to make the kinetic term canonical

\[
\mathcal{L} = (\bar{\Phi} \Phi) \bigg|_D - \frac{\alpha^{3/2} N_c}{9} \left( \frac{1}{3} \Phi^3 - \Phi^3 \ln \left( \frac{\alpha^{1/2} \Phi}{3 \Lambda} \right) \right) \bigg|_F + \text{h.c.} .
\]  

(16)

In fact, as discussed in [16], this effective Lagrangian is not complete since it does not possess the full \( Z_{2N_c} \) symmetry of the quantum theory. To restore that symmetry the extra term

\[
\Delta \mathcal{L} = \frac{2\pi i m}{3} (S - \bar{S})
\]  

(17)

where \( m \) is an integer valued Lagrange multiplier must be added. For the \( n = 0 \) vacuum with vanishing phase this extra term vanishes and the VY model above is recovered. We shall concentrate on that vacuum and real, positive mass perturbations which make that vacuum the true vacuum of the perturbed theory.
In addition to the particle states associated with the $\phi$ field (gluino–glunio or gluino–glue balls), one might expect to find states associated with the familiar glueballs of QCD, with interpolating fields $F^2$ or $F\tilde{F}$, and their superpartners. A chiral supermultiplet can be constructed out of $D^2W^2 = D^2S$ which contains the appropriate fields. We have no reason to expect that the glueball states are parametrically heavier than the $\phi$ field states, and so there is no a priori reason to integrate them out of the effective Lagrangian for $\phi$. Furthermore, any glueballs are likely to be strongly coupled to $\phi$ and have a non-negligible effect on its dynamics.

On the other hand, any couplings to glueballs are still constrained by scale invariance and anomalous $U(1)_R$-invariance. Let us postulate the existence of some effective glueball chiral superfield $\phi'(x)$, which is dimension one and has a canonical kinetic energy term. Since $\phi'(x)$ is constructed from $D^2W^2$ it must have R-charge zero. A superpotential term describing the interaction of $\phi'$ with $S$ would be of the form:

$$\int d^2\theta \, \Lambda^{3-3a-b} S^a \phi'^b .$$

(18)

The constraint from anomalous R-invariance (under which $\Lambda$ has charge 2/3), requires that

$$2 = 2a - 2/3(3a + b - 3) ,$$

(19)

or $b = 0$. Therefore there are no superpotential terms of the correct type. All interactions between the potential glueball excitations and $S$ (including mass or wavefunction mixing) must appear as Kähler terms in the effective Lagrangian. There are already unknown Kähler terms that enter into the physical masses. These are higher derivative terms which, while suppressed by powers of $\Lambda$, contribute to the wave function renormalization, $Z(p^2)$, which must be evaluated at $p^2 = m^2 \simeq \Lambda^2$ to obtain the physical masses. The Kähler corrections from the $\phi'$ may be subsumed into these unknown terms.

Of course, it is possible that the glueball fields do not enter the effective Lagrangian as $\phi'(x)$, but rather as some other effective field, or even the auxiliary field of $S$ (which contains $G^2_{\mu\nu}$ and $G\tilde{G}$). In this case the description advocated here may not be correct. We will discuss this possibility (more specifically, the status of glueballs in SYM) in more detail below, but for now we simply adopt the VY Lagrangian and discuss its mass predictions. A lattice simulation will hopefully test these predictions and shed light on whether the action is indeed the correct description.

The straightforward evaluation of bosonic ($\lambda\lambda$) and fermionic ($g\lambda$) excitation masses around the minimum from Eq(16) gives

$$m_{\lambda\lambda} = m_{g\lambda} = N_c \alpha \Lambda .$$

(20)
It is important to stress again that these masses are not the physical masses of the bound states. Rather, they are zero-momentum quantities, which are related to the physical ones by wave function factors \( Z(p^2 = m^2_{\text{phys}}) \). These wave function factors result from higher-derivative Kähler terms in \( \mathcal{L} \), and are unknown.

A soft breaking gaugino mass may again be introduced through the F-component of the spurion \( \Lambda_s \) (first investigated in \[11\]). The new scalar potential is

\[
V = \frac{\alpha^3 N_c^2}{9} |\phi|^2 \left| \ln \left( \frac{\alpha^{1/2} \phi}{3 \Lambda} \right) \right|^2 - \frac{32 \pi^2 \alpha^{3/2}}{27} \operatorname{Re} m_\lambda \phi^3
\]

from which we can calculate the shifts in the masses of the bound states. The two scalar fields and the fermionic field are all split in mass. The eigenvalues of the mass matrices are

\[
\begin{align*}
M_{\text{fermion}} &= N_c \alpha \Lambda + \frac{16 \pi^2 |m_\lambda|}{N_c} \\
M_{\text{scalar}} &= N_c \alpha \Lambda + \frac{56 \pi^2 |m_\lambda|}{3 N_c} \\
M_{\text{p–scalar}} &= N_c \alpha \Lambda + \frac{40 \pi^2 |m_\lambda|}{3 N_c}
\end{align*}
\]

these results have been derived for real, positive mass which favor the supersymmetric vacuum with vanishing phase. For even \( N_c \) there is a supersymmetric vacuum characterized by \( \langle S \rangle = -\Lambda^3 \) which is prefered by negative, real mass perturbations. It is easy to construct an effective lagrangian about that vacuum in a similar fashion to \([14]\). The bound state masses are again given by \([23]\).

The physical masses are again related to these quantities by unknown wave function renormalizations \( Z \) which arise from Kähler terms,

\[
M_{\text{physical}} \equiv Z M
\]

Fortunately, we know that in the SUSY limit the wavefunction factors are common within a given multiplet. This degeneracy holds even after the vev of the field is shifted by the soft breakings since a shift in the vev alone (without SUSY breaking) leaves the physical masses degenerate within a multiplet. We also know that the relative change in these Kähler terms is of order \( \mathcal{O}(f^2) \), and hence can be ignored at leading order in the soft breakings. Therefore, we may still obtain a prediction for the rate of change of the ratios of the physical masses,

\[
\bar{M}(m_\lambda) \equiv \frac{Z(m_\lambda) M(m_\lambda) - Z(0) M(0)}{Z(0) M(0)}
\]

\[
\simeq \frac{\partial M}{\partial m_\lambda} \left[ \frac{1}{M} + \frac{1}{Z \partial M} \right] m_\lambda
\]

\[\text{(23)}\]
near the SUSY limit. The factor in brackets is common within a given multiplet. Since
the quantity $Z(m_0)$ is unknown, we can only predict the ratios of $\tilde{M}$ at the SUSY point or
equivalently the ratios of $\partial M/\partial m_\lambda$
\[
\partial M_{ps}/\partial m_\lambda : \partial M_{term}/\partial m_\lambda : \partial M_s/\partial m_\lambda = 5 : 6 : 7
\] (24)

At this point it is worthwhile to return to the question of glueball masses and their
interaction with the S bound states. We make an observation based on some results of
West’s on glueball masses in QCD [14]. Using QCD inequalities, West has shown that the
mass of the lightest non-vacuum state coupling to the operator $G^{2}_{\mu\nu}$ is less than the mass of
the lightest non-vacuum state coupling to $G\tilde{G}$. In QCD, this implies that the lightest glueball
is a scalar, not a pseudoscalar. West’s results can be applied to SYM due to the positivity
of the gluino determinant [15]. In SYM the results are relevant both to the glueballs and
the gluino-gluino bound states, as they have the same quantum numbers and can mix even
at the perturbative level. Now, if (22) is correct, then very close to the SUSY point the
shift in S bound state masses is such that the pseudo-scalar S bound state is lighter
than the scalar S bound state. This is the case for sufficiently small $m_\lambda$ since the unkown Kähler
terms are higher order in $m_\lambda$, and the superpotential splitting dominates. Combining this
with West’s result suggests that in SYM the glueball states may actually be lighter than the
S states near the SUSY point! (In other words, the S states cannot be dominating West’s
inequalities.) If so, lattice measurements of properties of the $\lambda\lambda(x)$ correlation functions will
actually be dominated by those light glueballs (due to the mixing), and we will not obtain
any information on the VY model. Of course, another possibility is that the glueball–S
mixing is so strong that the VY model is not a very good description in any case.

Finally we note, as pointed out in [16], that the VY model apparently has an extra SUSY
vacuum corresponding to $\langle \phi \rangle = 0$. At this point the expectation value of $S$ is singular and
so it is not clear how to interpret this vacuum. Shifman and Kovner have proposed that the
vacuum is real and represents some conformal, $Z_{2N_c}$ preserving point of the theory. It would
be interesting to look for this vacuum in lattice simulations but unfortunately as can be seen from Eq.(21)
there is no value of soft breaking mass for which such a vacuum would be the
global minimum. This will make it difficult to observe in lattice simulations. We return to
a possible lattice signature of this vacuum in the following section.

5 Tuning to SUSY

In this section we discuss the problem of tuning a lattice simulation toward the SUSY limit.
As in the case of chiral symmetry in QCD, the lattice regularization explicitly violates SUSY
Gauge invariance only allows a single renormalizable operator that breaks SUSY, the gaugino mass. One must tune the bare gaugino mass to a special value in order to ensure that the symmetry (in this case SUSY) is restored, and even then it is only fully recovered in the continuum limit. It is important to identify a sensitive diagnostic with which to fine-tune the bare gaugino mass. (Of course the mass splittings described in the previous section are a possibility, but they are comparatively slowly varying with $m_{\lambda}$.) This is particularly true, as we will discuss below, because one of the most interesting quantities we wish to compute is the gaugino condensate $\langle \lambda \lambda \rangle$ which depends sensitively on small SUSY-breaking effects. We are tempted to propose the partition function itself as a diagnostic, since it must approach unity in the SUSY limit, where the vacuum energy, $\epsilon_0$, is zero,

$$ Z = \exp[-V \epsilon_0] \to 1 \quad . $$  

The behavior of the vacuum energy is extremely sensitive to the SUSY limit. In a continuum calculation, one obtains an expression of the form

$$ \epsilon_0 = 0 + m a^{-2} + m \Lambda^3 + \Lambda^4 O((\Lambda a)^k) \quad . $$

Here $m$ is the effective gaugino mass, which vanishes when the bare gaugino mass $m_{\lambda}$ is properly chosen. When the gaugino mass is non-zero, the vacuum energy receives quadratically divergent corrections, resulting from non-cancellation of gaugino and gauge vacuum loops. Since the (bare) gaugino condensate can be related to the derivative of $Z$ with respect to $m_{\lambda}$, it too is very sensitive to SUSY breaking corrections, which will appear multiplied by two powers of the UV cutoff. If $m$ is not tuned sufficiently close to zero, the resulting contamination will preclude a measurement of the non-zero condensate which remains in the SUSY limit.

Unfortunately, due to the nature of the lattice regularization, the vacuum energy $\epsilon_0$ does not actually approach zero, even in the SUSY limit. This is because SUSY can only be recovered in the IR and never in the UV part of the lattice model, which nevertheless contributes to $\epsilon_0$. This is easy to see, because the lattice dispersion relations of fermions and bosons differ significantly in higher orders of $(ka)$. SUSY cannot hold even approximately for modes with momentum $k \sim a^{-1}$.

A better method for identifying the point in parameter space corresponding to supersymmetry has been proposed by Montvay [17]. As can be seen from Eq.(10) the shift in energy of the $N_c$ SYM vacua depend on the phase of the gaugino mass in the softly broken theory. The lattice simulations are restricted to real mass terms in order to allow Monte Carlo techniques to be employed but the mass can be tuned through zero to negative values. For even values of $N_c$ the $n = 0$ and $n = N_c/2$ vacua interchange as the true minima when
the sign of $m_\lambda$ is flipped. For odd $N_c$ the $n = 0$ vacua is preferred for positive masses, while a negative real mass places the system on the edge of a first order phase transition between two vacua with conjugate phases.

We can use the phase transition between the different $Z_{N_c}$ vacua as a rough indicator of SUSY. This transition occurs near $m = 0$ (i.e. as the mass term switches sign), although there will be a slight overshoot due to supercooling. (For $|m|$ sufficiently small the critical bubble necessary for the transition will be larger than the lattice volume.) Note that the sign of the overshoot depends on the direction from which $m = 0$ is approached (hysteresis). We can define the corresponding phase transition points as $m_\lambda(\pm)$, and average them to obtain the true SUSY point:

$$m_\lambda^* = (m_\lambda(+) + m_\lambda(-))/2,$$

where $m(m_\lambda^*) = 0$.

In practice, $m_\lambda(\pm)$ should be defined as the points where some specified order parameter deviates by some specified amount from its behavior in the pure phase. A possible order parameter is the Wilson loop, $\langle W \rangle$, which is almost independent of $m$ for small $m$, and hence will only display jumps near the transition points.

It is possible that the existence of the extra vacuum proposed in [16] and discussed above can be ascertained by the behavior of the Wilson loop near the phase transition. Consider the average of a large Wilson loop: $\langle W \rangle$. We expect that, due to confinement, this exhibits area law behavior with some string tension $\kappa$:

$$\langle W \rangle \sim \kappa A + \cdots,$$

where the ellipsis indicate subleading effects which scale like the perimeter of the loop. Note that since the gluinos are in the adjoint representation they, like the gluons, cannot fully screen sources in the fundamental representation, which must be connected by gluonic strings.

Now consider what happens to $\langle W \rangle$ at the phase transition, near $m = 0$. If the system only exhibits the two vacua with non-zero gluino condensate ( $\langle \lambda \lambda \rangle = \pm 32\pi^2\Lambda^3$ ), then the important configurations near the transition will consist of regions of each of these phases, with domain walls in between. In the bulk of each region the string tension will be essentially $\kappa$, assuming that the supercooling is negligible and $|m|$ is very small (this requires a large lattice). On each Euclidean time slice of the loop, one can imagine the gluonic string

\footnote{For the system to exhibit hysteresis, some “memory” of the path in $m$ is required. This would be the case if, as $m$ is varied, the previous configurations are used as the initial ones for the subsequent Monte Carlo update. This allows the system to be trapped in a metastable phase for $m_\lambda$ sufficiently close to $m_\lambda^*$.}
connecting the sources at either end, but passing through domain walls and regions of each phase in between. The area law part of \( \langle W \rangle \) will remain the same even at the transition, although the subleading perimeter effects may exhibit a discontinuity due to the sudden appearance of the domain walls. Therefore we do NOT expect a discontinuity in the leading behavior of the Wilson loop.

On the other hand, if a phase exists with \( \langle \lambda \lambda \rangle = 0 \), as has been suggested in [16], it would presumably have very distinct properties including a string tension \( \kappa' \neq \kappa \). In this case, at the phase transition the dominant configurations will contain all three phases, and the effective string tension will be altered to some value in between \( \kappa \) and \( \kappa' \). This would manifest itself as a very rapid change in the leading behavior of \( \langle W \rangle \). An alternative method of searching for the \( \langle \lambda \lambda \rangle = 0 \) vacuum is to use the “multicanonical” method, which induces transitions between the different vacua. If the simulation is tuned sufficiently close to the SUSY point, it will then spend a significant portion of its time in the exotic phase.

Also worth investigation is the difference in vacuum energy between the metastable and stable vacua, defined by

\[
\Delta V(m_\lambda) \equiv \epsilon_0^{m_\lambda}(m_\lambda) - \epsilon_0^s(m_\lambda).
\]  

\( \Delta V \) is finite, and vanishes at the exact SUSY point. The behavior of \( \Delta V(m_\lambda) \) is predicted by [10], which we emphasize follows directly from the (Seiberg) effective Lagrangian [1], independent of the further V-Y analysis concerning the bound states’ masses [10]. Furthermore, the slope of \( \Delta V \) is directly related to the gaugino condensate. Alternatively, one could also directly compute the quantity

\[
\lim_{m_\lambda \to 0} \left[ \langle \lambda \lambda \rangle^{m_\lambda}(m_\lambda) - \langle \lambda \lambda \rangle^s(m_\lambda) \right],
\]

in which the divergences also cancel. A sufficiently accurate measurement would also provide a determination of \( \alpha \), as can be seen from Eq.(11).

Finally, we mention the possibility of measuring the surface tension of domain walls separating two different phases. A domain wall configuration can be produced by splitting the lattice into two separate regions, each treated with different (say, opposite) values of \( m_\lambda \). Recently, Shifman and collaborators have given exact expressions for the profiles and energy densities of such configurations, using the \( N = 1 \) SUSY algebra with central extension [18].

A subtlety in the preceding discussion is that our expressions are given in terms of the bare quantities in the “Wilsonian” regularization scheme (e.g. \( \bar{\text{DR}} \)), whereas in the lattice simulations it is the bare lattice parameters which are varied. Fortunately, one can relate both quantities to the bare parameters in the usual MS scheme using perturbation theory.
6 Summary and Prospects

To summarize, we believe the following tests of the (softly broken) SUSY results will be feasible:

(1) The first essential step is to confirm the phase structure of the theory. As discussed in the introduction it is possible that the SUSY phase ceases to exist for any non-zero gaugino mass, although one expects that it will persist until some critical value of \( m \), possibly \( m \sim \Lambda \) if the gaugino plays an important role in the dynamics of the SUSY theory. Assuming that the SUSY phase persists, it should be described by the softly broken theory and one would hope to test the \( Z_{N_c} \) vacuum structure. This structure implies the existence of a phase transition near the SUSY point identified above. There is even the possibility of using the Wilson loop \( \langle W \rangle \) as a probe of Shifman’s putative vacuum at zero gaugino condensate \([16]\).

(2) Measurement of the gaugino condensate and fundamental scale \( \Lambda \). This would provide a direct test of Seiberg’s tower of SQCD superpotentials. The shift in the gaugino condensate with increasing \( m_\lambda \) could also be measured and would determine the parameter \( \alpha \).

(3) Measurement of mass-splitting ratios \( \tilde{m} \) \([23]\), which are independent of the relationship between different regularizations. This is a test of the V-Y Lagrangian itself. At the SUSY point, the masses should be degenerate and provide a measurement of the product \( \alpha \Lambda \).

(4) On a more speculative note, some of the recent results of Shifman and collaborators \([16, 18]\) are potentially testable, including the controversial existence of a \( Z_{2N_c} \)-preserving vacuum and the exact calculation of domain wall energy densities.

Finally, we mention a related theory which has been speculated to have very different behavior from normal QCD and which could be easily simulated using current lattice technology. The model is an SU(2) gauge theory with a Dirac spinor of adjoint fermions. This model was studied in \([19]\) with real Susskind fermions in the strong coupling expansion where it gives rise to a massless composite Dirac fermion. This theory can be reached at tree level by the inclusion of a large soft breaking scalar mass in N=2 SQCD, although in that case the spurion symmetries are not sufficient to determine the IR behavior, so the results of \([19]\) cannot be confirmed without further analysis.

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7 Appendix: Rescaling Anomaly

The results given in this paper implicitly assumed the holomorphic normalizations of fields in the bare Lagrangian. A rescaling is necessary in order to compare our predictions to lattice results which result from canonically normalized bare Lagrangians (i.e. unit kinetic energy coefficients for the gluino, as opposed to $1/g^2$). Due to the rescaling anomaly, some care is necessary in this rescaling [20].

The vector superfields in the holomorphic and canonical normalizations are related by

\[ V_h = g_c V_c, \quad (1) \]

which, in particular, implies $\lambda_h = g_c \lambda_c$. Naively, using classical rescaling, we would have the following relation between the condensates:

\[ \langle \lambda_h \lambda_h \rangle = g_c^2 \langle \lambda_c \lambda_c \rangle. \quad (2) \]

However, the rescaling anomaly introduces additional effects. In the holomorphic computation, we essentially computed

\[ \int D V_h \lambda_h \lambda_h \exp \left( -\frac{1}{16 g_h^2} \int d^4 x d^2 \theta W_h W_h \right). \quad (3) \]

Changing variables to $V_c$, we have

\[ \int D(g_c V_c) g_c^2 \lambda_c \lambda_c \exp \left( -\left( \frac{1}{g_h^2} - \frac{N_c}{4 \pi^2} \ln g_c \right) \frac{g_c^2}{16} \int d^4 x d^2 \theta W_c W_c \right), \quad (4) \]

where the shift in coupling by $\frac{N_c}{4 \pi^2} \ln g_c$ is due to the rescaling anomaly, arising from the functional measure. We see that if we take

\[ \frac{1}{g_h^2} = \frac{1}{g_c^2} + \frac{N_c}{8 \pi^2} \ln g_c^2, \quad (5) \]

the path integral in (4) is canonically normalized, with coupling constant $g_c$.

Our prediction for the canonically normalized lattice calculation is therefore

\[ \langle \lambda_c \lambda_c \rangle_{\text{lattice, } g_c} = -\frac{1}{g_c^2} 32 \pi^2 \Lambda^3, \quad (6) \]

where $\Lambda$ is the holomorphic strong coupling scale. $\Lambda$ is defined by the RGE evolution of $g_h$, governed by the exact one loop beta function, and is distinct from the strong coupling scale associated with $g_c$. 

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