RADON, COSINE AND SINE TRANSFORMS ON GRASSMANNIAN MANIFOLDS

GENKAI ZHANG

Abstract. Let $G_{n,r}(K)$ be the Grassmannian manifold of $k$-dimensional $K$-subspaces in $K^n$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is the field of real, complex or quaternionic numbers. We consider the Radon, cosine and sine transforms, $\mathcal{R}_{r',r}, \mathcal{C}_{r',r}$ and $\mathcal{S}_{r',r}$, from the $L^2$ space $L^2(G_{n,r}(K))$ to the space $L^2(G_{n,r'}(K))$, for $r, r' \leq n - 1$. The $L^2$ spaces are decomposed into irreducible representations of $G$ with multiplicity free. We compute the spectral symbols of the transforms under the decomposition. For that purpose we prove two Bernstein-Sato type formulas on general root systems of type BC for the sine and cosine type functions on the compact torus $\mathbb{R}^{r'/2\pi\mathbb{Q}}$ generalizing our recent results for the hyperbolic sine and cosine functions on the non-compact space $\mathbb{R}^r$. We find then also a characterization of the images of the transforms. Our results generalize those of Alesker-Bernstein and Grinberg. We prove further that the Knapp-Stein intertwining operator for certain induced representations is given by the sine transform and we give the unitary structure of the Stein’s complementary series in the compact picture.

1. Introduction

The present paper is a continuation of our earlier papers [26] and [22] on Radon and related transforms and their relation to the Bernstein-Sato type formulas. We shall give here a rather unified approach to the Radon, cosine and sine transforms on Grassmannian manifolds. Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex or quaternionic numbers, and $G_{n,r} = G_{n,r}(K)$ be the Grassmannian manifold of $k$-dimensional subspaces over $K$ in $K^n$. For $1 \leq r \leq r' \leq n - 1, \nu \geq 0$, the Radon, cosine and sine transforms $\mathcal{R} = \mathcal{R}_{r',r}, \mathcal{C} = \mathcal{C}_{r',r}^{(2\nu)}, \mathcal{S} = \mathcal{S}_{r',r}^{(2\nu)}: C^\infty(G_{n,r}) \to C^\infty(G_{n,r'})$ are defined, for $\eta \in G_{n,r'}$, by

$$\mathcal{R}f(\eta) = \int_{\xi \subset \eta} f(\xi)d_\eta\xi,$$

$$\mathcal{C}f(\eta) = \int_{G_{n,r}} |\cos(\xi,\eta)|^{2\nu} f(\xi)d_\eta\xi,$$

$$\mathcal{S}f(\eta) = \int_{G_{n,r}} |\sin(\xi,\eta)|^{2\nu} f(\xi)d_\eta\xi,$$

where $d_\eta\xi$ is certain probability measure on the set $\{\xi \in G_{n,r} : \xi \subset \eta\}$ invariant with respect to the group of $K$-unitary transformations of $\eta$. The definition of the

Key words and phrases. Radon transform, sine and cosine transforms, Knapp-Stein intertwining operator, Lie groups, unitary representations, Stein’s complementary series.

Research supported by the Swedish Science Council (VR).
sine and cosine functions is given in Definition 3.1. Now the Grassmannian manifold $G_{n,r}$ is a compact Riemannian symmetric space $G_{n,r} = G/K$, and the space $C^\infty(G_{n,r})$ is decomposed under $G$ into irreducible subspaces by the Cartan-Helgason theorem. The operators $R$, $C$ and $S$ are $G$-invariant and hence they act on the irreducible subspaces as scalars, which can also be interpreted as the spectral symbols of the transforms. The spectral symbol of the Radon transform has been found earlier in [6] by integral computations using explicit formulas for highest weight vectors. In the present paper we shall find the spectral symbols of the cosine and sine transforms and a characterization of their images. We shall use the spherical transform on root systems instead of explicit formulas for highest weight vectors as in [6]. The same technique can also be used to derive some of the results of Grinberg [6]. Our work generalizes the recent results of [1] on cosine transform where the case $\mathbb{K} = \mathbb{R}$ and $2\nu = 1$ is considered.

As application we find further the unitary structure of the Stein’s complementary series of the group $GL(2r, \mathbb{K})$ realizing on the Grassmannian $G_{2r,r}$. We give below a more precise description of our results.

Firstly there is a close relation between the three transforms. The cosine and sine transforms can be factorized in terms of the Radon transform; see Lemma 3.4. The integral kernels $|\cos(\xi, \eta_0)|^{2\nu}$ and $|\sin(\xi, \eta_0)|^{2\nu}$, for $r = r'$, at a base point $\eta_0$ are $K$-invariant functions and their restriction on the Cartan subspace $A \cdot \eta_0$ are the product of the usual cosine and respectively sine functions. Thus they are related roughly speaking by a rotation of $\frac{\pi}{2}$. On the other hand, the square $R_{r',r}^* R_{r',r}$ for $r' > r$ is an integral operator on $G_{n,r}$ and the integral kernel is actually the cosine kernel; see [22] for the case of non-compact symmetric matrix domains. Now there is a fourth transform, the Berezin transform which appears naturally in the branching rule of holomorphic representations and quantization; see [25], [26] and references therein. We prove also that the cosine transform on Grassmannians is the compact analogue of the Berezin transform [25] on non-compact symmetric spaces. As another application we find the branching rule of certain scalar holomorphic representations on the complex Grassmannian under isometric group of the real or quaternionic Grassmannian as a submanifold of the former; see Section 6.2. Thus the three transforms and the Berezin transform are related intrinsically and we can treat them with a unified approach.

The sine transform $S_{r,r}$ is related to the unitary representations of a large non-compact group $GL(2r, \mathbb{K})$. In [19] Stein proved that there is a family of unitary irreducible representations of $GL(2r, \mathbb{C})$ induced from non-unitary representations of a maximal parabolic subgroup, also called the Stein’s complementary series. The unitarity of those representation was proved in [19] by computing the Fourier transform of the kernel $|\det(x)|^\alpha$ of the Knapp-Stein intertwining operator; it gives then the unitary structure in the so-called non-compact picture of the induced representations. Vogan [20] proved that the Stein’s results can be extended to $GL(2r, \mathbb{K})$ and proved the corresponding existence of the Stein’s complementary series. We prove that the sine transform $S_{r,r}$ is the Knapp-Stein intertwining operator for the group $GL(2r, \mathbb{K})$. We characterize the unitary structure of the Stein’s complementary series in the compact picture; see Section 6.1. The unitarity and composition series for this family of
principal series representation of $GL(n, \mathbb{K})$ are of considerable interests (see \[11\], \[18\] and \[23\]), and they have also applications in the theory of valuations on convex sets \[1\].

Here is an outline of our idea and method of computing the spectral symbol. As just mentioned the squared Radon transform $R^*_{r',r} R_{r',r}$, on a non-compact symmetric matrix domain has an integral kernel being a product of hyperbolic sine functions whereas \[26\] the Berezin transform has a product of hyperbolic cosine functions, both in terms of the geodesic coordinates. The spectral symbol of the Berezin transform is then the Harish-Chandra spherical transform of the hyperbolic cosine functions $\cosh^{\nu} t$. We find in \[26\] this spherical transform on a root system of type BC with general root multiplicities. This is done by deriving first certain Bernstein-Sato type formulas for hyperbolic cosine functions $\cosh^{\nu} t$, performing spherical transform and then by using the observation that the functions $\cosh^{-\nu} t$, $\nu \to \infty$, tend to the delta function at $t = 0$ up to normalization constants. This idea has been used before and might be due to Berezin. Similar formulas for the hyperbolic sine functions is proved in \[22\], and is a crucial step in finding the inversion formula for the Radon transform on non-compact symmetric domains. Using the techniques developed there we derive in this paper some corresponding Bernstein-Sato type formulas for the sine and cosine functions on the compact torus of a general root system of type BC using the Cherednik operators, and we obtain the Cherednik-Opdam transform for the sine and cosine transforms. This method can also be applied Radon transform, giving a different proof of some of the main results of Grinberg \[6\], but we will not present the details here.

Our main results are stated in Theorems 4.1, 4.3, 5.4 and 6.2. In Theorem 4.1 we present Bernstein-Sato type formulas for the cosine and sine functions, Theorem 4.3 computes their spherical transforms and Theorem 5.4 gives a characterization of the images of the cosine and sine transforms. In Theorem 6.2 we give an independent proof for the existence the Stein’s complementary series and provides a formula for their unitary structure.

After this paper was finished I was informed by Professor Semyon Alesker that Theorem 5.4 (1) was partly proved by him in an unpublished preprint \[2\] (under the assumption that $\nu$ is not a negative half-integer). The method there is quite different, it uses the localization theorem of Beilinson-Bernstein combined with some results by Braden-Grinberg on perverse sheaves and by Howe-Lee on the K-type structure of degenerate principal series representations. It is my pleasure to thank him for his comments on an earlier version of this paper, and for sending a copy of his preprint. I thank also Professor B. Rubin for informing me that a special case of Proposition 5.1 (on the real Grassmannian when all $m_j$ are equal) was also obtained in \[17\] as a consequence of Hecke equality and Gindikin Gamma functions. The expert comments on an earlier version of this paper by the referee are greatly acknowledged.
2. Grassmannian manifolds $G_{n,r}$ and the irreducible decomposition of $L^2(G_{n,r})$

2.1. Grassmannian manifolds $\mathcal{X} = G_{n,r}$ as symmetric spaces. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex or quaternionic numbers with the standard involution (conjugation) $x \to \bar{x}$ and let $a = \dim_{\mathbb{K}} \mathbb{K} = 1, 2, 4$. Let $M_{n,m} := M_{n,m}(\mathbb{K})$ be the space of all $n \times m$-matrices, also viewed as $\mathbb{K}$-linear transformations from $\mathbb{K}^m$ to $\mathbb{K}^n$, where $\mathbb{K}$ acts on the right by scalar multiplication. Denote $x^* = \bar{x}^T$, the conjugated transpose, for $x \in M_{n,m}$. Let

$$G := U(n, \mathbb{K}) = \{g \in M_{n,n}; g^* g = I_n\} = O(n), U(n), Sp(n)$$

be the orthogonal, unitary, and symplectic groups accordingly. Denote, for $x \in M_{n,n}$, $\det_{\mathbb{K}}(x)$ the determinant of $x$ as a real linear transformation on $\mathbb{K}^n = \mathbb{R}^n$. (For $\mathbb{K} = \mathbb{C}$ $\det_{\mathbb{K}}(x) = |\det_{\mathbb{C}}(x)|^2$ and for $\mathbb{K} = \mathbb{H}$, $\det_{\mathbb{K}}(x) = \det_{\mathbb{H}}(x)^4$ where $\det_{\mathbb{H}}(x)$ is the so-called Dieudonné determinant.)

Consider, for $r \leq n$, the Stiefel manifold $S_{n,r}$ of all orthonormal $r$-frames in $\mathbb{K}^n$, and

$$\mathcal{X} = G_{n,r}$$

the Grassmannian manifold of all $r$-dimensional subspaces over $\mathbb{K}$. $S_{n,r}$ can be realized as the set of all matrices $x \in M_{n,r}$ such that $x^* x = I$, namely $\mathbb{K}$-linear isometric transformations $x \in \mathbb{K}^r \to \mathbb{K}^n$. Each $x \in S_{n,r}$ defines uniquely a $r$-dimensional subspace $\xi$ over $\mathbb{K}$,

$$\xi = \{x\} := x \mathbb{K}^k \subset \mathbb{K}^n \in G_{n,r}.$$ 

Thus $\mathcal{X}$ is identified with the space of orbits in $S_{n,r}$ under the action of the unitary group $U(r, \mathbb{K})$ on $\mathbb{K}^r$,

$$\mathcal{X} = G_{n,r} = S_{n,r}/U(r, \mathbb{K}).$$

Let

$$\xi_0 = \mathbb{K}^r \oplus 0 = \{x_0\} \in G_{n,r}, \quad x_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in S_{n,r}.$$ 

We will fix $\xi_0$ and $x_0$ as reference points of $\mathcal{X}$ and $S_{n,r}$. The group $G$ acts on $S_{n,r}$ and $\mathcal{X}$ by the defining action; they are then realized as a homogeneous and respectively symmetric space,

$$S_{n,r} = G/U(n-r, \mathbb{K}), \quad \mathcal{X} = G/K = U(n, \mathbb{K})/U(r, \mathbb{K}) \times U(n-r, \mathbb{K}).$$

Here $K = U(r, \mathbb{K}) \times U(n-r, \mathbb{K})$ is the isotropic subgroup of $\xi_0$ consisting of block diagonal matrices of the form

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad A \in U(r, \mathbb{K}), \quad D \in U(n-r, \mathbb{K}).$$ 

Due to the $G$-isometry between $G_{n,r}$ and $G_{n,n-r}$ we can and will assume in this paper, if nothing else is stated, that

$$2r \leq n.$$
2.2. Irreducible decomposition of $L^2(\mathcal{X})$. To describe the irreducible decomposition of $L^2(\mathcal{X})$ under $G$ we let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of $K$. The linear subspace $\mathfrak{p}$ consists of $n \times n$ matrices of the block form
\[
p_X = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}, \quad X \in M_{r,n-r},
\]
which will be identified with $M_{r,n-r}$ via the mapping $X \mapsto p_X$. We let $\mathfrak{a}$ be the linear subspace of $\mathfrak{p} = M_{r,n-r}$ consisting of matrices of the form
\[
X = [\text{diag}\{t_1, \cdots, t_r\} \ 0_{r,n-2r}] = t_1E_1 + \cdots + t_rE_r, \quad t_1, \cdots, t_r \in \mathbb{R}
\]
with $E_j$ being the matrix having 1 on the $(j, j)$ position and 0 on the rest positions, $j = 1, \cdots, r$. The maximal torus $A = \exp(\mathfrak{a})$ is then the group of $\exp(X)$, with
\[
\exp(X) = \begin{bmatrix} \text{diag}(\cos t, \cdots, \cos t) & \text{diag}(\sin t, \cdots, \sin t) & 0 \\ \text{diag}(-\sin t, \cdots, -\sin t) & \text{diag}(\cos t, \cdots, \cos t) & 0 \\ 0 & 0 & I \end{bmatrix},
\]
written as block matrix under the decomposition of $\mathbb{K}^n = \mathbb{K}^r \oplus \mathbb{K}^r \oplus \mathbb{K}^{n-2r}$. Let $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$, the non-compact dual of $\mathfrak{g}$. The root system of $R(\mathfrak{g}^*, i\mathfrak{a})$ is of the form
\[
R(\mathfrak{g}^*, i\mathfrak{a}) = \{\pm \varepsilon_j \pm \varepsilon_k\} \cup \{\pm \varepsilon_j\} \cup \{\pm 2\varepsilon_j\}
\]
with respective multiplicities $a$, $a - 1$ and $a(n - 2r)$. Here \{\varepsilon_j\} is the dual basis of \{iE_j\}. It is of type C (if $n = 2r$, $a > 1$), type BC (if $n > 2r$, $a > 1$), type B (if $n > 2r$, $a = 1$), or type D (if $n = 2r$, $a = 1$). It is understood here and in Section 4 that roots with zero multiplicity will not appear, so that all types are considered as special case of type BC. We fix an ordering so that $\varepsilon_1 > \cdots > \varepsilon_r > 0$ (and the condition $\varepsilon_r > 0$ is dropped for type D).

The following result follows from the Cartan-Helgason theorem \cite{10}. (Strictly speaking the theorem is for $G/K$ with simply connected and semisimple Lie group $G$. The group $G = U(n, \mathbb{K})$ for $K = \mathbb{C}$ or $\mathbb{H}$ is not semisimple, but the result can be reduced to the semisimple group $SU(n, \mathbb{K})$. However for $\mathbb{K} = \mathbb{R}$ the group $G = O(n)$ is not connected nor $SO(n)$ is simple connected. This imposes more conditions on the highest weights than those given by that Theorem.)

**Proposition 2.1.** Under the action of $G = U(n, \mathbb{K})$ the space $L^2(\mathcal{X})$ decomposes as
\[
L^2(\mathcal{X}) = \bigoplus \mathfrak{m} V^\mathfrak{m}
\]
with multiplicity free, where each $V^\mathfrak{m}$ has highest weight $\mathfrak{m} = m_1\varepsilon_1 + \cdots m_r\varepsilon_r$ with $m_i$ being integers and
\[
m_1 \geq m_2 \cdots \geq m_r \geq 0
\]
for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ or $\mathbb{K} = \mathbb{R}$ with $n - 2r > 0$ and
\[
m_1 \geq m_2 \cdots \geq |m_r|
\]
for $K = \mathbb{R}$ and $n = 2r$. Each $V^m$ has a unique $K$-invariant function $\phi_m$, called spherical polynomial, normalized so that $\phi_m(\xi_0) = 1$.

We will view the spherical polynomial $\phi_m$ on $X$ as defined on $a = \mathbb{R}^r$ via the exponential mapping $a \to G_{n,r}$, $X \to \exp(X) \cdot \xi_0$, namely

$$\phi_m(t_1, \ldots, t_r) := \phi_m(\exp(t_1E_1 + \cdots + t_rE_r) \cdot \xi_0).$$

Finally we recall the polar decomposition of $G_{n,r}$. For any function $f$ on $G_{n,r}$ we have the following formula (which in turn fixes a normalization of the invariant measure)

$$(2.5) \quad \int_{G_{n,r}} f(\xi) d\xi = \int_K \int_A f(ka) \cdot \xi_0) dkd\mu(a)$$

where

$$d\mu(a) = d\mu_R(t) = \prod_{\alpha \in R_+} |2 \sin \alpha(it)|^{m_{\alpha}} d\alpha dt_1 \cdots dt_r, \quad a = \exp(t_1E_1 + \cdots + t_rE_r)$$

with $m_\alpha$ the root multiplicity of $\alpha$ and $dk$ the normalized measure on $K$; see [10, Chapter I, Theorem 5.10].

### 3. Radon, cosine and sine transforms

#### 3.1. Cosine and sine of angles between subspaces.

We fix the standard Euclidean norm on $K^n = \mathbb{R}^{an}$. For any convex subset $S$ in a $d$-dimensional real subspace of $K^n$ we let $vol_d(S)$ be the corresponding Euclidean volume. The following definition is given in [1] for the real Grassmannians.

**Definition 3.1.** If $r \leq r'$ the cosine of the angle between two subspaces $\xi \in G_{n,r}$ and $\eta \in G_{n,r'}$ is defined by

$$(3.1) \quad |\cos(\xi, \eta)| = \left( \frac{\text{vol}_r P_\eta(A)}{\text{vol}_r(A)} \right)^{\frac{1}{n}},$$

where $P_\eta$ is the orthogonal projection from $K^n$ onto $\eta \subset K^n$ and $A \subset \xi$ is any convex set of non-zero volume. If $r \leq n - r'$ the sine of the angle between two planes $\xi \in G_{n,r}$ and $\eta \in G_{n,r'}$ is defined by

$$|\sin(\xi, \eta)| = |\cos(\xi, \eta^\perp)|.$$

For general $r$ and $r'$ we define $|\cos(\xi, \eta)|$ and $|\sin(\xi, \eta)|$ by the symmetry condition

$$|\cos(\xi, \eta)| = |\cos(\eta, \xi)|, \quad |\sin(\xi, \eta)| = |\sin(\eta, \xi)|.$$

In [5] Grinberg and Rubin introduce also a cosine of angle, $\text{COS}^2(y, x)$, between two elements $x \in S_{n,r}$ and $y \in S_{n,r'}$, for $r \leq r'$, defined to be the $r \times r$-semi-positive definite matrix

$$(3.2) \quad \text{COS}^2(y, x) = x^*yy^*x.$$

The following lemma computes the cosine (3.1) in terms of (3.2)
Lemma 3.2. (1) Let \( 1 \leq r \leq r' \) and the notations be as above. The two cosine functions (3.1) and (3.2) are related by,

\[
|\cos(\eta, \xi)|^a = (\det \cos^2(y, x))^\frac{1}{2}
\]

where \( \eta = y\mathbb{K}^r \subseteq G_{n,r}, \xi = y\mathbb{K}^r \subseteq G_{n,r} \).

(2) Let \( 1 \leq r = r' \leq n - r \). Write \( \xi \in G_{n,r} \) as \( \xi = k \exp(t_1 E_1 + \cdots + t_r E_r) \cdot \xi_0, \)

\( k \in K \). Then

\[
|\cos(\xi, \xi_0)|^\delta = \prod_{j=1}^r |\cos t_j|^\delta.
\]

\[
|\sin(\xi, \xi_0)|^\delta = \prod_{j=1}^r |\sin t_j|^\delta.
\]

Proof. We recall first that if \( T : \mathbb{R}^p \to \mathbb{R}^q, p \leq q \), is a linear transformation and \( B \subset \mathbb{R}^p \)

is any convex set of positive volume, then \( \text{vol}_p(T(B)) = \det(T^t T)^\frac{1}{2} \text{vol}_p(B) \), where \( T^t \)

is the transpose of \( T \) with respect to the Euclidean metrics in \( \mathbb{R}^p \) and \( \mathbb{R}^q \). Indeed, let

\( T = U(T^t)^\frac{1}{2} \) be the polar decomposition of \( T \) with \( U \) being a partial isometry. If \( \text{rank}(T) < p \), then both \( \text{vol}_p(T(B)) \) and \( \det(T^t T)^\frac{1}{2} \) are zero and the formula is trivially true. If \( \text{rank}(T) = p \) then \( U^t U = I \) and \( U \) is an isometry. We have then

\[
\text{vol}_p(T(B)) = \text{vol}_p(U(T^t)^\frac{1}{2}(B)) = \text{vol}_p((T^t)^\frac{1}{2}(B)) = \det((T^t)^\frac{1}{2}) \text{vol}_p(B),
\]

proving the identity.

Now if \( \xi = \{x\} = x\mathbb{K}^r \) and \( \eta = \{y\} = y\mathbb{K}^r \), with isometries \( x \in S_{n,r} \) and \( y \in S_{n,r'} \),

we have \( P_\eta = yy^* \). Let \( A \subset \xi = x\mathbb{K}^r \) be any convex set of nonzero volume. We write \( A = x(B) \) with \( B \subset \mathbb{K}^r \) a convex set, and \( \text{vol}_{ra}(B) = \text{vol}_{ra}(A) \) since \( x \) is an

isometry. Its image under \( P_\eta \) is \( P_\eta(A) = yy^*x(B) \). We apply the previous formula with \( T = P_\gamma x = yy^*x \), noticing that \( T^t T = x^*P_\eta^2 x = x^*x = yy^*x \),

\[
\text{vol}_{ra} P_\gamma(A) = (\det \cos(x^*y^*x))^\frac{1}{2} \text{vol}_{ra}(A) = \det \cos^2(x, y)^\frac{1}{2} \text{vol}_{ra}(A).
\]

This proves the first part, and the second part is then a special case by using the formula (2.3).

3.2. Factorization and diagonalization of the cosine and sine transforms. We define the Radon transform \( \mathcal{R}_{r',r} : C^\infty(G_{n,r}) \to C^\infty(G_{n,r'}) \) by

\[
(\mathcal{R}_{r',r} f)(\eta) = \int_{\xi \in G_{n,r} : \xi \subseteq \eta} f(\xi) d_\eta \xi,
\]

if \( r < r' \) and

\[
(\mathcal{R}_{r',r}^* f)(\eta) = \mathcal{R}_{r',r}^* f(\eta) = \int_{\xi \in G_{n,r} : \xi \supseteq \eta} f(\xi) d_\eta \xi.
\]

if \( r > r' \). Here \( d_\eta \xi \) is the unique Riemannian measure on the subset induced from the Riemannian measure on \( G_{n,r} \); see [5] for an expression of \( \mathcal{R}_{r',r} \) and the measure in terms of the realization (2.1). In particular the Radon transform \( \mathcal{R}_{r',r} \) commutes with the actions of \( G \) on \( C^\infty(G_{n,r}) \) and \( C^\infty(G_{n,r'}) \).
Definition 3.3. Let $1 \leq r, r' < n$ and $\nu \geq 0$. We define the sine and cosine transforms from $C^\infty(G_{n,r})$ to $C(G_{n,r'})$, by
\[
S_{r',r}^{(\nu)} f(\eta) = \int_{G_{n,r}} |\sin(\eta, \xi)|^{2\nu} f(\xi) d\xi, \quad C_{r',r}^{(2\nu)} f(\eta) = \int_{G_{n,r}} |\cos(\eta, \xi)|^{2\nu} f(\xi) d\xi.
\]

The sine and cosine transforms are related to the Radon transform via the following Lemma. This Lemma in the case when $2\nu = 1$ and $\mathbb{K} = \mathbb{R}$ is known and is proved in Lemma 1.7 in [1]. The same method can be applied to the present case for general $\nu \geq 0$, by using a variant of the Cauchy-Kubota formula
\[
\text{vol}_{ar}(B)^{\delta} = c_{\delta} \int_{\xi \in G_{n,r'}} \text{vol}_{ar}(\text{Pr}_{\xi} B)^{\delta} d\xi,
\]
for any convex polygon in $B \subset \xi_0$, which follows easily by the invariance of both sides under translations and linear actions on $\xi_0$. We skip the elementary proof.

Lemma 3.4. Let $1 \leq r \leq r' < n$, and $\nu \geq 0$. The cosine and sine transforms can be factorized as
\[
C_{r,r'}^{(\nu)} = c_1 C_{r,r'}^{(\nu)} R_{r,r'}, \quad S_{r,r'}^{(\nu)} = c_2 S_{r,r'}^{(\nu)} R_{r,r'},
\]
where $c_1 = c_1(\nu)$ and $c_2 = c_2(\nu)$ are some positive constants.

The constant $c_1$ and $c_2$ can be computed by using the integral formula in [21] but we will not need it here.

By taking conjugate of the above formulas we get factorizations for any $1 \leq r, r' < n$, noticing that $(C_{r,r'}^{(\nu)})^* = C_{r',r}^{(\nu)}$, $(S_{r,r'}^{(\nu)})^* = S_{r',r}^{(\nu)}$.

The transforms $C_{r,r'}^{(\nu)}$ and $S_{r,r'}^{(\nu)}$ clearly intertwine the action of $G$. Consider the corresponding decomposition of $L^2(G_{n,r'})$ according to Proposition 2.1,
\[
L^2(G_{n,r'}) = \bigoplus_{\mathbf{m}} W^{\mathbf{m}}, \quad (3.7)
\]
Thus $C_{r,r'}^{(\nu)}$ and $S_{r,r'}^{(\nu)}$ are diagonal operators up to a normalization. The eigenvalue of $R_{r,r'}$ has been found earlier by Grinberg [3]; see Theorem 5.2 below. In view of Lemma 3.4 above, to find the eigenvalues of $C_{r,r'}^{(2\nu)}$ and $S_{r,r'}^{(2\nu)}$ we need only consider the case when $r' = r$.

Denote $c_{\nu}(\mathbf{m}) = c_{\nu,r}(\mathbf{m}) = \text{ and } s_{\nu}(\mathbf{m})$ the eigenvalue of $C_{r,r}^{(\nu)}$ and respectively $S_{r,r}^{(\nu)}$ on $V^{\mathbf{m}}$. They can be evaluated by
\[
c_{\nu}(\mathbf{m}) = |\cos(\cdot, \xi_0)|^{2\nu} \phi_{\mathbf{m}}(\xi_0) = c_{\nu}(\mathbf{m}). \quad (3.8)
\]
Namely
\[
c_{\nu}(\mathbf{m}) = |\cos(\cdot, \xi_0)|^{2\nu} = \int_{G_{n,r}} |\cos(\xi, \xi_0)|^{2\nu} \phi_{\mathbf{m}}(\xi) d\xi. \quad (3.9)
\]
is the spherical transform of $|\cos(\cdot, \xi_0)|^{2\nu}$. Similarly for $s_{\nu}(\mathbf{m})$. They are integrals of $K$-invariant functions. We use the polar coordinates $(2.3)$, which further can be written as an integral on the quotient $\mathbb{R}^r/2\pi Q^r$ of $\mathbb{R}^r$ by the (spherical) coroots lattice.
$Q^\lor$, namely the lattice generated by $\alpha^\lor = \frac{\hat{\alpha}}{(\alpha, \alpha)}$ for $\alpha \in R$, where $\hat{\alpha}$ is the dual of $\alpha$, $\lambda(\hat{\alpha}) = (\lambda, \alpha)$; see [10, Chapter I, Theorem 5.10].

**Lemma 3.5.** Denote

$$|\cos(t)| := |\prod_{j=1}^r \cos t_j|, \quad |\sin(t)| := |\prod_{j=1}^r \sin t_j|,$$

with

$$t = t_1 E_1 + \cdots + t_r E_r = (t_1, \cdots, t_r) \in \mathbb{R}^r/2\pi Q^\lor.$$

Let $\nu \geq 0$. The eigenvalues $c_\nu(\mathbf{m})$ and $s_\nu(\mathbf{m})$ of the cosine respectively sine transforms are given by the spherical transforms

$$c_\nu(\mathbf{m}) = |\hat{\cos(t)}|^{2\nu}(\mathbf{m}) = \int_{\mathbb{R}^r/2\pi Q^\lor} |\cos(t)|^{2\nu} \phi(\mathbf{m}, t_1, \cdots, t_r) d\mu_R(t),$$

$$s_\nu(\mathbf{m}) = |\hat{\sin(t)}|^{2\nu}(\mathbf{m}) = \int_{\mathbb{R}^r/2\pi Q^\lor} |\sin(t)|^{2\nu} \phi(\mathbf{m}, t_1, \cdots, t_r) d\mu_R(t).$$

4. **Spherical transform on compact torus associated with general root system of Type BC**

4.1. **Bernstein-Sato type formulas for cosine and sine functions.** We consider the root system $R$ on a Euclidean space $i\alpha = i\mathbb{R}^r$ ($i = \sqrt{-1}$) of type BC

$$R = \{\pm \varepsilon_j \pm \varepsilon_k, j \neq k\} \cup \{\pm \varepsilon_k\} \cup \{\pm 2\varepsilon_k\}$$

with general non-negative multiplicities $a$, $2b$ and $\iota$ for the respective sets of roots. Here $\{\varepsilon_j\}$ is the dual basis in $(i\alpha)^\lor$ of a fixed orthonormal basis $\{iE_j\}$ of $i\alpha$; the notation $\{E_j\}$ here coincides with that in Section 2.2. We use the same convention there, so that type D and type B will be considered as special cases.

Denote by $W$ the Weyl group. We will compute the spherical transform of certain Weyl group invariant sine and cosine functions, by using the harmonic analysis of the Cherednik operators developed by Opdam [16]. We follow the presentation there loc. cit., however with our roots being twice of the roots there and our multiplicities half of the ones there.

We let $D_j = D_j E_j$, $j = 1, \cdots, r$, be the trigonometric Cherednik operators acting on functions on $\mathbb{R}^r/2\pi Q^\lor$,

$$D_j = \partial_j - ia \sum_{k < j} \frac{1}{1 - e^{-2i(t_k - t_j)}} (1 - s_{kj}) + ia \sum_{j < k} \frac{1}{1 - e^{-2i(t_j - t_k)}} (1 - s_{jk}) +$$

$$+ ia \sum_{k \neq j} \frac{1}{1 - e^{-2i(t_j + t_k)}} (1 - \sigma_{jk}) + 2ib \frac{1}{1 - e^{-4it_j}} (1 - \sigma_j) +$$

$$+ 2ib \frac{1}{1 - e^{-4it_j}} (1 - \sigma_j - i\rho_j),$$
where \( s_{kj}, \sigma_{kj} \) and \( \sigma_j \) are the reflections corresponding to the roots \( \varepsilon_j - \varepsilon_k, \varepsilon_j + \varepsilon_k \), and \( \varepsilon_j \). Here
\[
\rho = \frac{1}{2} \sum_{\alpha \in R^+} m_\alpha \alpha = \sum_{j=1}^r \rho_j \varepsilon_j, \quad \rho_j = \iota + b + \frac{a}{2}(r - j).
\]
is the half sum of positive roots, \( m_\alpha \) being the root multiplicities. Let \( \phi_m \) be the Heckman-Opdam Jacobi polynomials on the root system ([8], [7], and [16]), and
\[
\hat{f}(\mathbf{m}) := \int_{\alpha/2\pi Q^\vee} f(t) \phi_m(t) d\mu_R(t)
\]
be the spherical (or Jacobi) transform. Here \( Q^\vee \) is as before the (spherical) coroot lattice. Our objective is to find the spherical transform of the functions \(|\text{Cos}(t)|^{2\nu} \) and \(|\text{Sin}(t)|^{2\nu} \) in Lemma 3.5. We establish first certain Bernstein-Sato type formulas, more exactly we will find certain Weyl group invariant polynomials of the Cherednik operators mapping \(|\text{Cos}|^\delta \) to \(|\text{Cos}|^{\delta - 2}\).

**Theorem 4.1.** Let \( \delta \geq 0 \) and \( \mathcal{M}_\delta \) be the operator
\[
\mathcal{M}_\delta := \prod_{j=1}^r (D_j^2 + (\delta + \rho_1)^2).
\]
Then the following Bernstein-Sato type formulas hold,
\[
\mathcal{M}_\delta |\text{Cos}(t)|^\delta = \prod_{j=1}^r (\delta + a(j - 1)) (\delta + \iota + 1 + a(r - j)) |\text{Cos}(t)|^{\delta - 2}
\]
and
\[
\mathcal{M}_\delta |\text{Sin}(t)|^\delta = \prod_{j=1}^r (\delta + a(j - 1)) (\delta + \iota + 2b + 1 + a(r - j)) |\text{Sin}(t)|^{\delta - 2}.
\]

**Proof.** In [26, Theorem 2.1] and [22, Theorem 3.1] the following formulas are proved for the hyperbolic sine and cosine functions, defined on \( i\mathfrak{a} \),
\[
\prod_{j=1}^r ((iD_j)^2 - (\delta + \rho(\xi_1))^2) (\prod_{j=1}^r \cosh x_j)^\delta
\]
\[
= \prod_{j=1}^r (\delta + a(j - 1))(1 - \delta - \iota - a(r - j)) (\prod_{j=1}^r \cosh x_j)^{\delta - 2}, \quad x = (x_1, \ldots, x_r) \in \mathbb{R}^r,
\]
\[
\prod_{j=1}^r ((iD_j)^2 - (\delta + \rho_1)^2) (\prod_{j=1}^r \sinh x_j)^\delta
\]
\[
= \prod_{j=1}^r (\delta + a(j - 1))(\delta - 1 + \iota + 2b + a(r - j))(\prod_{j=1}^r \sinh x_j)^{\delta - 2}, \quad x = (x_1, \ldots, x_r) \in \mathbb{R}^r_+,
\]
where \( iD_j \) is acting on the variable \( x \). The function \((\sinh x)^\delta \) and \((\cosh x)^\delta \) are analytic functions on the right hand plane \( \Re x > 0 \), so are the products \( \prod_{j=1}^r (\sinh x_j)^\delta \) and \( \prod_{j=1}^r (\cosh x_j)^\delta \) on the product of the right half plane \( \{ x; \Re x_j > 0 \} \). All the identities
has a limit at the points $x = it$, $t \in \mathbb{R}$, $t_j \neq 0, \frac{\pi}{2}$. Our result follows then by taking the limit, observing also that $\sinh^2(it_j) = (-1)\sin^2 t$. \hfill \square

**Remark 4.2.** The above theorem can also be proved directly by a straightforward but long computations. First, we can choose a dense open subset of the orbit $W \setminus \mathbb{R}^r / 2\pi Q^\vee$ of the Weyl group so that the functions $\sin t_j > 0, \cos t_j > 0$. Furthermore the operator $\mathcal{M}_\delta$ can be factorized as follows,

$$ (4.2) \quad \mathcal{M}_\delta = \prod_{j=1}^r (D_j^2 + (\delta + \rho_1)^2) = \prod_{j=1}^r (-iD_j + (\delta + \rho_1))(iD_j + (\delta + \rho_1)) $$

and we may compute successively the action of each factors. We have, for each fixed $j$,

$$ \prod_{l=1}^j (iD_l + (\delta + \rho_1))\cos(t)\delta = \prod_{l=1}^j (\delta + a(l - 1)) \left( \cos(t)\delta \prod_{l=1}^j e^{-it_l} \right), $$

$$ \prod_{l=j}^r (-iD_l + (\delta + \rho_1)) \left( \cos(t)\delta \prod_{l=1}^j \frac{e^{-it_l}}{\cos t_l} \right) $$

$$ = \prod_{l=j}^r (\delta - 1 + \iota + a(r - l)) \left( \cos(t)\delta \prod_{l=1}^j \frac{e^{-it_l}}{\cos t_l} \right) \prod_{l=j}^r \frac{e^{it_l}}{\cos t_l} $$

$$ \prod_{l=j}^r (-iD_l + (\delta + \rho_1)) \left( \sin(t)\delta \prod_{l=1}^j \frac{e^{-it_l}}{\sin t_l} \right) $$

$$ = (-i)^{r-j+1} \prod_{l=j}^r (\delta - 1 + \iota + 2b + a(r - l)) \left( \sin(t)\delta \prod_{l=1}^j \frac{e^{-it_l}}{\sin t_l} \right) \prod_{l=j}^r \frac{e^{it_l}}{\sin t_l}, $$

by similar computations as in [26] and [22], which together with the factorization (4.2) implies our theorem. (The pattern of the identities is roughly that each action of $(iD_j + (\delta + \rho_1))$ on the functions $\cos t$ produces an extra factor $\frac{e^{-it_j}}{\cos t_j}$, further action by $(iD_j + (\delta + \rho_1))$ produces an extra factor $\frac{e^{it_j}}{\cos t_j}$; the $e^{\pm it_j}$ cancels and we get a singel factor of $\cos^{-2t}$, namely our formula.)

The above family of identities is to be compared with the trivial trigonometric formulas

$$ (i \frac{d}{dt} + \delta) \cos(t)\delta = \delta \cos(t)\delta \frac{e^{-it}}{\cos t}, $$

$$ (-i \frac{d}{dt} + \delta) \left( \cos(t)\delta \frac{e^{-it}}{\cos t} \right) = (\delta - 1) \left( \cos(t)\delta \frac{e^{-it}}{\cos t} \right) \frac{e^{it}}{\cos t}, $$

$$ (i \frac{d}{dt} + \delta) \sin(t)\delta = i\delta \left( \sin(t)\delta \frac{e^{-it}}{\sin t} \right), $$

$$ (-i \frac{d}{dt} + \delta) \left( \sin(t)\delta \frac{e^{-it}}{\sin t} \right) = (-i)(\delta - 1) \left( \sin(t)\delta \frac{e^{-it}}{\sin t} \right) \frac{e^{it}}{\sin t}. $$
It would be interesting to reformulate the identities systematically in terms of Hecke algebras [3].

4.2. Spherical transform for cosine and sine functions. We let $N_{\nu}$ and $N'_{\nu}$ be the following normalization constants,

$$N_{\nu} = \int_{\mathbb{R}^r/2\pi Q} |\cos|^{2\nu}(t)d\mu(t)$$

and

$$N'_{\nu} = \int_{\mathbb{R}^r/2\pi Q} |\sin|^{2\nu}(t)d\mu(t).$$

Their exact values can be evaluated by using the Macdonald formula for generalized Beta-integrals (see [15, Ex. 7, Sect. 10, Chapt. VII]),

$$N_{\nu} = 2^{nr(r-1)+2rb+2r_{\nu}}! \prod_{1\leq i<j\leq r} \frac{\Gamma\left(\frac{a}{2}(j-i)\right)}{\Gamma\left(\frac{a}{2}(j)\right)} \times \frac{\Gamma_{a}(1+b+\frac{r-1}{2}+\frac{r}{2}(r-1))\Gamma_{a}(\nu+1+\frac{r-1}{2}+\frac{r}{2}(r-1))}{\Gamma_{a}(\nu+1+b+i+a(r-1))},$$

$$N'_{\nu} = 2^{nr(r-1)+2rb+2r_{\nu}}! \prod_{1\leq i<j\leq r} \frac{\Gamma\left(\frac{a}{2}(j-i)\right)}{\Gamma\left(\frac{a}{2}(j)\right)} \times \frac{\Gamma_{a}(1+\frac{r-1}{2}+\frac{r}{2}(r-1))\Gamma_{a}(\nu+1+\frac{r-1}{2}+\frac{r}{2}(r-1))}{\Gamma(\nu+1+b+i+a(r-1) - \frac{a}{2}(j-1))}. $$

Here $\Gamma_{a}(\alpha)$ is the Gindikin’s Gamma function

$$\Gamma_{a}(\alpha) = \prod_{j=1}^{r} \Gamma(\alpha - \frac{a}{2}(j-1)).$$

We recall also the Pochammer symbol $(\nu)_{k} = (\nu)(\nu+1)\cdots(\nu+k-1)$.

**Theorem 4.3.** The spherical transforms of the functions $|\cos(t)|^{2\nu}$ and $|\sin(t)|^{2\nu}$ are given by

$$c_{\nu,r}(\mathbf{m}) := \widehat{\cos|^{2\nu}(\mathbf{m})} = N_{\nu} \prod_{j=1}^{r} \frac{(\nu+1+\frac{a}{2}(j-1) - \frac{m_{j}}{2})^{m_{j}}}{(\nu+1+b+a(r-1) - \frac{a}{2}(j-1))^{m_{j}}}$$

and

$$s_{\nu,r}(\mathbf{m}) := \widehat{|\sin|^{2\nu}(\mathbf{m})} = N'_{\nu} \prod_{j=1}^{r} \frac{(\nu+1+\frac{a}{2}(j-1) - \frac{m_{j}}{2})^{m_{j}}}{(\nu+1+b+a(r-1) - \frac{a}{2}(j-1))^{m_{j}}} \phi_{\mathbf{m}}(\frac{\pi}{2}, \cdots, \frac{\pi}{2}).$$

For root systems of Type C or Type D we have $\phi_{\mathbf{m}}(\frac{\pi}{2}, \cdots, \frac{\pi}{2}) = 1$ and thus

$$s_{\nu,r}(\mathbf{m}) := \widehat{|\sin|^{2\nu}(\mathbf{m})} = N'_{\nu} \prod_{j=1}^{r} \frac{(-\nu-\frac{a}{2}(j-1))^{m_{j}}}{(\nu+1+b+a(r-1) - \frac{a}{2}(j-1))^{m_{j}}}.$$
We need the following elementary result, which states simply that the normalized integration of the cosine functions \(\cos^n t\) and \(\sin^n t\) tends to the \(\delta\)-function at \(t = 0\) respectively \(t = \frac{\pi}{2}\).

**Lemma 4.4.** Suppose \(\phi\) be a bounded and continuous function on \(\mathbb{R}^r/2\pi Q^v\). Then

\[
\lim_{\nu \to \infty} \frac{1}{N\nu} \int_{\mathbb{R}^r/2\pi Q^v} |\cos|^{2\nu}(s)\phi(s)\,d\mu(s) = \phi(0)
\]

and

\[
\lim_{\nu \to \infty} \frac{1}{N\nu} \int_{\mathbb{R}^r/2\pi Q^v} |\sin|^{2\nu}(s)\,d\mu(s) = \phi\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right).
\]

The following lemma asserts that \(\phi_m\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right)\) is always nonzero, which is needed in Theorem 5.4.

**Lemma 4.5.**

1. Suppose \(R\) is a root system of type C, or D with general non-negative root multiplicity. Then

\[
\phi_m\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right) = \prod_{j=1}^r (-1)^{m_j}
\]

2. Suppose \(R\) is the root system of the Grassmannian manifold \(G_{n,r}\). Then

\[
\phi_m\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right) \neq 0.
\]

**Proof.** (1) From the formula for \(D_j\) we see that, if \(R\) is a root system of type C, or D, then \(\{D_j\}\) are invariant under the map \((t_1, \ldots, t_r) \mapsto (t_1 + \frac{\pi}{2}, \ldots, t_r + \frac{\pi}{2})\). Moreover it is easy to prove that the polynomial \(f(t_1, \ldots, t_r) := \phi_m(t_1 + \frac{\pi}{2}, \ldots, t_r + \frac{\pi}{2})\) is also invariant under the Weyl group. We prove this for type C, the type D is the same. The Weyl group is generated by the simple reflections \(\sigma_r\) and \(s_{jj+1}\), so we need only to check the invariance for those elements. Notice that by definition \(\phi_m\) is invariant under the mapping \((t_1, \ldots, t_r) \mapsto (t_1, \ldots, t_r + \pi)\). We have

\[
(\sigma_r f)(t_1, \ldots, t_r) = f(t_1, \ldots, -t_r) = \phi_m(t_1 + \frac{\pi}{2}, \ldots, -t_r + \frac{\pi}{2})
\]

and that \(s_{jj+1}f = f\) is trivially true. Thus \(f\) is an eigenfunction of the Weyl group invariant polynomials of the operators \(\{D_j\}\) with the eigenvalues being the same as that of \(\phi_m\). Namely \(f(t) = c\phi_m(t)\) for some constant \(c\), by the uniqueness of the spherical eigenfunctions \([16]\). Comparing the leading coefficients we find the constant \(c\), and the evaluation of \(f\) at \(t = 0\) proves our result.
(2) Consider the case of Grassmannian manifolds. We need only to treat the case when the corresponding root system $R$ is of type B or BC, namely $G_{n,r}(K)$ for $n-r > r$. Write

$$\xi_1 := \exp\left(\frac{\pi}{2} E\right) : \xi_0 = 0 \oplus K^r \oplus 0 \in G_{n,r},$$

where $\exp\left(\frac{\pi}{2} E\right)$ is the short-hand notation $\exp\left(\frac{\pi}{2} E\right) = \exp\left(\frac{\pi}{2} E_1 + \cdots + \frac{\pi}{2} E_r\right)$. We have, by the integral formula for spherical polynomials, [10, Chapter IV, Proposition 2.2], that

$$\phi_m(\xi_1)^2 = \int_K \phi_m(\exp\left(\frac{\pi}{2} E\right) k \exp\left(\frac{\pi}{2} E\right) \cdot \xi_0) dk,$$

which is further an integration on a subset of $G_{n,r}$. We claim the set of this integration, namely

$$S := \{\xi = \exp\left(\frac{\pi}{2} E\right) k \exp\left(\frac{\pi}{2} E\right) \cdot \xi_0; k \in K\},$$

is given by

$$S = \{\xi \in G_{n,r}; \xi \in \eta_1\}$$

where $\eta_1 \in G_{n,n-r}$ is the element

$$\eta_1 := K^r \oplus 0 \oplus K^{n-2r} \subset K^r \oplus K^r \oplus K^{n-2r} = K^n.$$

Indeed let $\xi = \exp\left(\frac{\pi}{2} E\right) k \exp\left(\frac{\pi}{2} E\right) \cdot \xi_0$ be any element in the set $S$ for some $k \in K$. Let $k = \text{diag}(A, D) \in U(r) \times U(n-r)$ and write $D$ as a $2 \times 2$-block matrix under the decomposition of $K^{n-r} = K^r \oplus K^{n-2r}$,

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$ 

Using the formula (2.3) for the exponential we find that $\xi$ is of the form

$$\xi = \{D_{11}v \oplus 0 \oplus D_{21}v; v \in K^r\} \subset \eta_1.$$

Conversely, it is rather elementary to see that any $r$-dimensional subspace of $\eta_1$ is of the above form for some $D$. This proves our claim. Therefore, the integration (4.7) is precisely the Radon transform of $\phi_m$, namely

$$\phi_m(\xi_1)^2 = R_{r,n-r,\phi_m}(\eta_1).$$

Now the function $R_{n-r,r,\phi_m}$ is a $K$-invariant function on $G_{n,n-r}$, and its dual Radon transform evaluated at $\xi_0$, $R_{r,n-r,\phi_m}^*(R_{n-r,r,\phi_m})(\xi_0) = R_{r,n-r}(R_{n-r,r,\phi_m})(\xi_0)$ is an integration of $R_{r,n-r,\phi_m}$ over the set $\{\eta; \eta \supset \xi_0\}$ which is a $K$-orbit of $\eta_1$. Namely the integrand is constant and

$$R_{r,n-r}(R_{n-r,r,\phi_m})(\xi_0) = (R_{n-r,r,\phi_m})(\eta_1).$$

Putting those together we have

$$\phi_m(\xi_1)^2 = R_{n-r,r,\phi_m}(\eta_1) = R_{r,n-r,\phi_m}^*(R_{r,n-r,\phi_m})(\xi_0).$$

It follows from the main result in [5] (see Theorem 5.2 below) that $R_{n-r,r,\phi_m}(\xi_0)$ is a non-zero constant of $\phi_m(\xi_0) = 1$ since $r \leq n-r$. Thus $\phi_m(\xi_1) \neq 0$. \qed
Remark 4.6. One may follow the argument in the proof and find the constant \( \phi_m(\xi_1)^2 \) by using the result of Grinberg. It would be interesting to evaluate the constant for a general root system of positive multiplicities. For the real projective space \( G_{n,1} = P^{n-1}(\mathbb{R}) \) the second claim (2) is proved in [10, Chapter I, Lemma 4.9].

We prove now Theorem 4.3.

Proof. We perform the spherical transform on the Bernstein-Sato formula for the cosine function in Theorem 4.1 with \( \delta = 2\nu + 2 \). Using the self-adjoint property of the operator \( M^2_{2\nu} \) and that (see [16]),

\[
M^2_{2\nu} \phi_m = \prod_{j=1}^{r} ((2\nu + 2 + \rho_1)^2 - (m_j + \rho_j)^2) \phi_m,
\]

we find that (suppressing the subindex \( r \) in \( c_{\nu,r}(m) \))

\[
\frac{c_{\nu}(m)}{N_{\nu}} = \frac{N_{\nu+1}}{N_{\nu}} \prod_{j=1}^{r} \prod_{k=0}^{l} \left( 1 - \frac{m_j}{2} \right) \left( 1 + \frac{m_j}{2} \right) \frac{1}{N_{\nu+1}} c_{\nu+1}(m).
\]

After a simplification we get

\[
\frac{c_{\nu}(m)}{N_{\nu}} = \prod_{j=1}^{r} \prod_{k=0}^{l} \left( 1 - \frac{m_j}{2} \right) \left( 1 + \frac{m_j}{2} \right) \frac{1}{N_{\nu+1}} c_{\nu+1}(m).
\]

Iterating the result produces furthermore

\[
\frac{c_{\nu}(m)}{N_{\nu}} = \frac{c_{\nu+l+1}(m)}{N_{\nu+l+1}} \prod_{j=1}^{r} \prod_{k=0}^{l} \left( 1 - \frac{m_j}{2} \right) \left( 1 + \frac{m_j}{2} \right) \frac{1}{N_{\nu+1}} c_{\nu+1}(m).
\]

However \( \frac{c_{\nu+l+1}(m)}{N_{\nu+l+1}} \to \phi_m(0) = 1, l \to \infty \), according to Lemma 4.4. Therefore,

\[
\frac{c_{\nu}(m)}{N_{\nu}} = \prod_{j=1}^{r} \prod_{k=0}^{\infty} \left( 1 - \frac{m_j}{2} \right) \left( 1 + \frac{m_j}{2} \right) \frac{1}{N_{\nu+1}} c_{\nu+1}(m).
\]

which can also be written in terms of the Gamma function ([4, p.5])

\[
\prod_{j=1}^{r} \frac{\Gamma(\nu + 1 + \frac{a}{2}(j - 1)) \Gamma(\nu + 1 + b + \iota + a(r - 1) - \frac{a}{2}(j - 1))}{\Gamma(\nu + 1 + b + \iota + a(r - 1) - \frac{a}{2}(j - 1) + \frac{m_j}{2}) \Gamma(\nu + 1 + b + \iota + a(r - 1) - \frac{a}{2}(j - 1) + \frac{m_j}{2})}.
\]

This proves our formula for the cosine function. The sine function is done similarly. \( \square \)
5. Spectral symbols and range characterization

5.1. Diagonalization of the transforms. Theorem 4.3 applied to Lemma 3.5 gives then

**Proposition 5.1.** Let \( K = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), and \( 1 \leq r \leq n - 1 \). The eigenvalue of \( C_{r,r}^{(\nu)} \) and \( S_{r,r}^{(\nu)} \) are given by \( c_{\nu,r}(m) \) and \( s_{\nu,r}(m) \) in Theorem 4.3 with \( i = a - 1 \), \( b = a(n - 2r) \).

To state our result on the spectral symbol of \( C_{r,r}^{(\nu)} \) and \( S_{r,r}^{(\nu)} \) for different \( r \) and \( r' \) we recall first the following result of Grinberg [6], reformulated slightly differently here.

**Theorem 5.2.** Let \( K = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), and \( 1 \leq r < r' \leq n - 1 \). Then the operator \( R_{r',r} \) defines a bounded operator from \( L^2(G_{n,r}) \) into \( L^2(G_{n,r'}) \). The operator \( R_{r',r}^{\nu} R_{r',r} \) is a diagonal operator under the decomposition,

\[
R_{r',r}^{\nu} R_{r',r} f = c(m) f, \quad f \in V^m,
\]

with an explicit formula for the eigenvalue \( c(m) \). The closure in \( L^2(G_{n,r'}) \) of the image of the operator \( R_{r',r} \) on \( L^2(G_{n,r}) \) is

\[
\sum_{m \in L_{r,r'}} W^m
\]

where \( L_{r,r'} \) is the subset of those \( m \) for which \( m_j = 0 \) if \( j \geq \min\{r, r'\} \).

Note that the \( L^2 \)-bounded result was not stated in [6]. However it follows directly from the explicit formula for the eigenvalue \( c(m) \) there. We remark also that the explicit formula for \( c(m) \) found in [6] is under the condition \( r \leq r' \) and \( 2r' \leq n \). For general \( r \) and \( r' \) the formula for \( c(m) \) is given in [12, Theorem 6.4].

Using Lemma 3.4, Proposition 5.1 and Theorem 5.2 we get

**Corollary 5.3.** Suppose \( \nu > 0 \). Let \( 1 \leq r, r' \leq n - 1 \). The eigenvalue of \( C_{r,r}^{*}, C_{r,r'}^{*} \) and \( S_{r,r}^{*}, S_{r,r'}^{*} \) on the space \( V^m \) are given respectively by

\[
c(m)c_{\nu,r}(m)c_{\nu,r'}(m), \quad s(m)s_{\nu,r}(m)s_{\nu,r'}(m),
\]

where \( c_{\nu,r} \) is given in Proposition 5.1 and \( c(m) \) in Theorem 5.2 and [12]. In particular \( C_{r,r}^{*}, C_{r,r'}^{*} \) and \( S_{r,r}^{*}, S_{r,r'}^{*} \) are bounded operators from \( L^2(G_{n,r}) \) to \( L^2(G_{n,r'}) \).

5.2. Characterization of the image of the transforms. The following theorem follows immediately from Corollary 5.3 and Lemma 4.5.

**Theorem 5.4.** Let \( 1 \leq r, r' < n - 1 \) and \( \nu > 0 \).

1. Let \( K = \mathbb{R} \). If \( \nu \notin \frac{Z}{2} \). Then the closures of images of \( C_{r',r} \) and \( S_{r',r}, L^2(G_{n,r}) \to L^2(G_{n,r'}) \), are given by

\[
\sum_{m \in L_{r,r'}} W^m
\]

where \( L_{r,r'} \) is given in Theorem 5.2. In particular the images are dense if \( r' \leq r \).

If \( \nu \in \frac{Z}{2} \) then the closures of their images in \( L^2(G_{n,r'}) \) are given by

\[
\sum_{m \in L_{r,r'} \cap L_{\nu}} W^m,
\]
where $L_\nu$ is the subset of $\mathbf{m}$ such that
\begin{equation}
\frac{m_j}{2} < \nu + 1 + \frac{1}{2}(j - 1), \quad \text{if } \nu + 1 + \frac{1}{2}(j - 1) \text{ is an integer, } j = 1, \cdots, r.
\end{equation}

(2) Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. If $\nu \notin \mathbb{Z}$. Then the closure of the images are as in (5.1). If $\nu \in \mathbb{Z}$ then the closures are given by
$$
\sum_{\mathbf{m} \in L_{r,r} \cap L_\nu} W_{\mathbf{m}},
$$

where $L_\nu$ is the subset of $\mathbf{m}$ such that
\begin{equation}
\frac{m_j}{2} < \nu + 1 + \frac{a}{2}(j - 1), \quad j = 1, \cdots, r.
\end{equation}

6. The sine-transform as Knapp-Stein intertwining operator. Cosine transform and branching of holomorphic representations

In this section we give applications of our results to the existence of the Stein’s complementary series and on the branching rule of holomorphic representations.

6.1. Stein’s complementary series. We fix $n = 2r$ in this subsection. Let $GL_n = GL(n, \mathbb{K})$ be the general linear group over $\mathbb{K}$ and $G = U(n, \mathbb{K})$ as before. Consider the parabolic subgroup $P$ of $GL_n$ consisting of block matrices of the form
$$
p = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}
$$
where $B, C, D \in M_{r,r}(\mathbb{K})$. The Langlands decomposition of $P$ is $P = LN = MAN$, with the nilpotent group $N$ and its opposite $\bar{N}$ consisting of upper respectively lower triangular matrices
\begin{equation}
n_X = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \in N, \quad n_Y = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \in \bar{N}, \quad X, Y \in M_{r,r},
\end{equation}
both being identified with $M_{r,r}$, the group $L = MA$ consists of diagonal matrices
$$
p = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}
$$
with $B, D \in GL_r$ and $A$ being a one-dimensional subgroup in the center of $L$.

Let $\delta_t$, be the one-dimensional representation
$$
\delta_t(p) = |\det_{\mathbb{R}}(BD^{-1})|^t,
$$
of $P$, for $t \in \mathbb{C}$. The special case $\delta_{\frac{1}{2}}$ corresponds the determinant of adjoint representation of $P$ on $N$ (written generally as $e^\rho$ where $\rho$ is half-sum of positive roots with respect to the one-dimensional Lie algebra of $A$).

Let $\text{Ind}^{GL_n}_{U(n,k)}(t)$ be the induced representation [13, Chapter VII] consisting of measurable functions $f$ on $G$ such that
$$
f(gp) = (\delta_{t+\frac{1}{2}}(p))^{-1}f(g)
$$
and that $f|_G \in L^2(G)$. When realized on $\tilde{N} = M_{r,r} = M_{r,r}(\mathbb{K})$ the Knapp-Stein intertwining operator from $\text{Ind}_{P}^{GL_n}(t)$ to $\text{Ind}_{P}^{GL_n}(-t)$ is given by \cite{19},

$$ \mathcal{I}_t F(\bar{n}_X) = \int_{M_{r,r}} |\det R(X - Y)|^{(-r+2t)} F(n_Y)dY. $$

We will use however the so-called compact realization of the induced representation, namely as acting the $L^2(GL_n/GL_n \cap L)$ of the compact manifold $GL_n/GL_n \cap L$. Now $X = GL_n/GL_n \cap L = G/K$ is precisely the Grassmannian manifold $X = G_{n,r}$. The relation between the two realizations is given by a change of variables,

\begin{equation}
(6.2) \quad f(\xi) \in L^2(X) \rightarrow F(\bar{n}) = \delta_{t+\frac{a}{2}}(p(\bar{n}))^{-1} f(\kappa(\bar{n})\xi_0)
\end{equation}

where $\bar{n} = k(\bar{n})p(\bar{n})$ is the Iwasawa $GL_n = GP$ decomposition of $\bar{n}$ in $GL_n$; see \cite[Chapter VII]{13} (our $GL_n$ and $G$ correspond to $G$ respectively $K$ there).

**Lemma 6.1.** In the compact picture the Knapp-Stein (formal) intertwining operator $\mathcal{I}_t$ is given by (up to a non-zero constant)

$$ \mathcal{J}_t f(\xi) = \int_{G_{n,r}} |\text{Sin}(\xi, \eta)|^{2\nu} f(\eta)d\eta, $$

where

\begin{equation}
(6.3) \quad \nu = -\frac{a}{2}(r - 2t).
\end{equation}

It is well-defined on the space $L^2(X)$ for $\nu \geq 0$.

**Proof.** Let $S(\xi, \eta)$ denote temporarily the kernel of the intertwining operator in the compact realization on $L^2(X)$, namely let

$$ \mathcal{J}_t f(\xi) = \int_X S(\xi, \eta)f(\eta)d\eta. $$

The kernel $S(\xi, \eta)$ is then uniquely determined by $S(\xi_0, \eta)$ (recalling that $\xi_0 = \mathbb{K}^r \oplus 0 \in \mathcal{X}$) by the transitivity of $G = U(n, \mathbb{K})$ on $\mathcal{X}$ and by the intertwining property. The evaluation $\mathcal{J}_t f(\xi_0)$ can also be computed via the intertwining operator $\mathcal{I}_t$ in the non-compact picture above,

\begin{equation}
(6.4) \quad \int_X S(\xi_0, \eta)f(\eta)d\eta = \mathcal{J}_t f(\xi_0) = \mathcal{I}_t F(0) = \int_{M_{r,r}} |\det R(Y)|^{(-r+2t)} F(n_Y)dY
\end{equation}

where the function $f$ on $X$ and $F$ on $\tilde{N}$ is given by $(6.2)$. Performing the change of variable $Y \in M_{r,r} \mapsto \eta := k(n_Y)\xi_0 \in X$ in the integral over $M_{r,r}$, according to $(6.2)$, we find that

$$ S(\xi_0, \eta) = |\det R(Y)|^{(-r+2t)} \delta_{t+\frac{a}{2}}(p(n_Y))^{-1}(\text{Jacy}_{\eta})^{-1}, $$

where Jac$Y \rightarrow \eta$ is the Jacobian of $Y \rightarrow \eta$, namely

$$ d\eta = \text{Jacy}_{\eta}dY. $$
We will find the two quantities \( \delta_{t+\frac{\pi}{2}}(p(n_Y)) \) and the Jacobian. Consider therefore the \( GL_n = G MAN \) decomposition (note that \( G \) is the maximal compact subgroup of \( GL_n \)) of \( n_Y \in \tilde{N} = M_{r,r}(\mathbb{K}) \) (under the identification (6.1))

\[
(6.5) \quad n_Y = \begin{bmatrix} I_r & 0 \\ Y & I_r \end{bmatrix} = k(n_Y)p(n_Y), \quad k(n_Y) \in G, \ p(n_Y) = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \in P = MAN.
\]

This is a factorization of the lower triangular matrix \( n_Y \) as a product of a unitary matrix \( k(n_Y) \) and an upper triangular matrix \( p(n_Y) \). Let the above element act on the base element \( \xi_0 \in \mathcal{X} \) via the defining action of \( GL_n \) acting on subspaces of \( \mathbb{K}^n \). Since \( p(n_Y) \) stabilize \( \xi_0 \) we see that the change of variables \( Y \to \eta = k(n_Y)\xi_0 \in \mathcal{X} \) is given by

\[
(6.6) \quad \eta = k(n_Y)\xi_0 = \begin{bmatrix} I_r & 0 \\ Y & I \end{bmatrix} \xi_0 = \{v \oplus Yv; v \in \mathbb{K}^r\} \in \mathcal{X}.
\]

An easy matrix computation then shows also that

\[
\delta_{t+\frac{\pi}{2}}(p(n_Y)) = \det_{\mathbb{R}}(1 + Y^*Y)^{t+\frac{\pi}{2}}.
\]

Indeed taking the determinant of (6.5) we find that \( \det_{\mathbb{R}} B \det_{\mathbb{R}} D = 1 \); computing the upper left entry of the block matrix \( n_Y^{-1}n_Y \) we get \( B^*B = I + Y^*Y \) and \( \delta_s(p(n_Y)) = \det_{\mathbb{R}}(BD^{-1})^s = \det_{\mathbb{R}}(B)^{2s} = \det_{\mathbb{R}}(B^*B)^s = \det_{\mathbb{R}}(I + Y^*Y)^s \) for any \( s \).

The measure \( d\eta \), and equivalently the Jacobian, is given by

\[
d\eta = \det_{\mathbb{R}}(I + Y^*Y)^{-r}dY,
\]

by direct computations (or by using Proposition 3.3 (ii) in [21]). Now the sine \( \text{Sin}(\eta, \xi_0) \) of the angle between \( \eta \) and \( \xi_0 \) is, by (6.6) and Lemma 3.2, given by

\[
|\text{Sin}(\xi_0, \eta)|^a = \frac{\det_{\mathbb{R}}(Y)}{\det_{\mathbb{R}}(1 + Y^*Y)^{t}}.
\]

It follows finally that

\[
S(\xi_0, \eta) = |\text{Sin}(\xi_0, \eta)|^{-a(r-2t)},
\]

as claimed. This completes the proof.

\[\square\]

**Remark 6.2.** Consider the simplest case when \( GL_n = GL_2(\mathbb{R}) \). The compact picture of the induced representation in question is on the half unit circle \( S^1/\mathbb{Z}_2 = \{e^{i\theta}, 0 \leq \theta < \pi \} \) representing lines \( \mathbb{R}(\cos \theta, \sin \theta) \) in \( \mathbb{R}^2 \). The intertwining operator in the non-compact picture is an integral operator defined by the convolution by \( |x|^s \) on \( \mathbb{R} \). The change of variables from non-compact to the compact is then \( x \to \mathbb{R}(1, x) \), namely \( x \to \mathbb{R}(\cos \theta, \sin \theta) \), with \( |\sin \theta| = \frac{|x|}{(1+x^2)^{\frac{1}{2}}} \). The above lemma states that the intertwining operator in the compact picture is then the convolution on the circle by \( |\sin \theta|^s = (\frac{1}{2})[1 - e^{2\theta}]^s \), namely it has the kernel \( |e^{2\phi} - e^{2\theta}|^s \), which is well-known.

By using Lemma 3.5 and Theorem 4.3 we obtain then the eigenvalues of the Knapp-Stein intertwining operator \( J_t \) on \( L^2(G_{n,r}) \) under the decomposition (2.4), and we can determine the range of \( t \) for which the eigenvalues are all non-negative (or all non-positive). This gives an independent proof of the existence proof Stein’s complementary series and with explicit formula for the corresponding unitary structure.
Theorem 6.3. Let $\nu = -\frac{a}{2}(r - 2t)$. The Knapp-Stein intertwining operator $J_t$, is well-defined for $\nu \geq 0$ on the algebraic span of the subspace $\mathfrak{m}$. It intertwines the actions of $\mathfrak{gl}(n, \mathbb{K})$ by the induced representations $\text{Ind}_{P}^{GL_n}(t)$ and $\text{Ind}_{P}^{GL_n}(-t)$, and has meromorphic continuation in $\nu$. Its eigenvalues are given by

$$
N'_\nu \prod_{j=1}^{r} \left( \frac{\left( \frac{a}{2} \right)(r + 1 - j - 2t)}{\left( \frac{a}{2} \right)(r + j - 1 + 2t)} \right)^{m_j}.
$$

The sesqui-linear form

$$(f, g)_\nu := \frac{1}{N'_{\nu}} (J_t f, g)$$

for

$$0 < t < \frac{1}{2},$$

is well-defined, and is a $\mathfrak{gl}(n, \mathbb{K})$-invariant positive definite Hermitian inner product. The completion of the pre-Hilbert space is a unitary representation of $GL(n, \mathbb{K})$.

For more details on unitarity and composition series of the whole family of the induced representations see also [11], [18] and [23].

6.2. Branching of holomorphic representations. Finally we give an application of our result to the branching of holomorphic representations on compact Hermitian symmetric spaces. The non-compact case has been studied intensively; see [25] and references therein. The result below can be deduced from the general theory [14] of (discrete) branching of highest weight representations. We only indicate here a concrete approach using the cosine transform. To keep the presentation of paper rather explicit we will only treat the case when the Hermitian symmetric spaces is the complex Grassmannian manifold with real form being the real or quaternionic Grassmannians.

We fix the complex Grassmannian manifold $X_1$ of $r_1$-dimensional complex subspace in $\mathbb{C}^{2r_1+b_1}$, $r_1 \geq 1, b_1 \geq 0$, namely

$$X_1 = U(2r_1 + b_1)/U(r_1) \times U(r_1 + b_1).$$

$X_1$ is equipped with the $U(2r_1 + b_1)$-invariant Hermitian metric. Consider the real Grassmannian $X$,

$$X = G/K = O(2r + b)/O(r) \times O(r + b), \quad r = r_1$$

or the quaternionic Grassmannian

$$X = G/K = Sp(r + b)/Sp(r) \times Sp(b), \quad r = \frac{r_1}{2}, \quad b = \frac{b_1}{2}$$

when $r_1$ and $b_1$ are even integers. Then $X$ can be realized as a totally geodesic real form of the Hermitian symmetric space $X_1$. We realize the space $V_1 = M_{r_1+b_1,r_1}(\mathbb{C})$ as a dense subset of $X_1$ by the identification

$$z \in V_1 \mapsto \{ y \oplus zy; y \in \mathbb{C}^{r_1} \} \in X_1.$$ 

similarly $V = M_{r+b,r}(\mathbb{K})$ can be realized as a dense subset of $X$ via

$$z \in V \mapsto \{ y \oplus zy; y \in \mathbb{K}^r \} \in X.$$
For any positive integer $\alpha$ there is a corresponding weighted Bergman space on the compact space $X_1$, denoted by $H_\alpha(X_1)$, with the reproducing kernel $\det(I + w^*z)^\alpha$, $z, w \in V \subset X_1$, which forms an irreducible unitary representation of $U(2r_1 + b_1)$. The elements in the space $H_\alpha(X_1)$ will be identified with holomorphic polynomials on $V_1$. Consider $H_\alpha(X_1)$ as a unitary representation of $G \subset U(2r_1 + b_1)$ and we will find the explicit irreducible decomposition.

We can first realize the $L^2(X)$ as a space of functions on the subset $V$. Indeed the $L^2(X)$ is $G$-equivalent to $L^2(V, \det(I + x^*x)^{-\frac{2r_1+b_1}{2}} dm(x))$ where $dm(x)$ is the Lebesgue measure.

The restriction mapping $T : H_\alpha(X_1) \to C^\infty(X)$, $f \mapsto f(x) \det(I + x^*x)^{-\frac{2r_1+b_1}{2}}$ defines then an intertwining map, where $G$ acts on $C^\infty(X)$ by the defining action. The operator $TT^*$ on $L^2(X) = L^2(V, \det(I + x^*x)^{-\frac{2r_1+b_1}{2}} dm(x))$ will be called the Berezin transform and is of the form

$$TT^* f(x) = \int_V \det(I + x^*x)^{-\frac{2r_1+b_1}{2}} \det(I + x^*y)^\nu f(y) \det(I + y^*y)^{-\frac{2r_1+b_1}{2}} dm(y);$$

see [25]. This is just the cosine transform, and its eigenvalues on the space $L^2(X)$ implies the branching rule of the representation of $U(2r_1 + b_1)$ under $G$. The first part of the next Proposition follows by similar computation as in the previous subsection, the second part follows from Theorem 5.4 and some abstract arguments.

**Proposition 6.4.**

1. The Berezin transform $TT^*$ on $X = G/K = G_{n,r}(\mathbb{K})$ is the cosine transform $C^{(\nu)}_{r_1}$ with $\nu = \frac{\alpha}{2}$ for $\mathbb{K} = \mathbb{R}$ and $\nu = \alpha$ for $\mathbb{K} = \mathbb{H}$.

2. The representation $H_\alpha(X_1)$ is decomposed under $G$ with multiplicity free as

$$H_\alpha(X_1) = \sum_{m \in L_\nu} V^m,$$

where $L_\nu$ is given in (5.2) and (5.3).

**References**

[1] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Adv. Math. 184 (2004), no. 2, 367–379. MR MR2054020 (2005b:22024)

[2] S. Alesker, *The $\alpha$-cosine transform and intertwining integrals on real Grassmannians*, preprint, 2003.

[3] I. Cherednik, *Inverse Harish-Chandra transform and difference operators*, Internat. Math. Res. Notices (1997), no. 15, 733–750. MR MR1470375 (99d:22018)

[4] A. Erdelyi et al, *Higher transcendental functions*, vol. 1, McGraw-Hill, New York - Toronto - London, 1953.

[5] E. Grinberg and B. Rubin, *Radon inversion on Grassmannians via Gårding-Gindikin fractional integrals*, Ann. of Math. (2) 159 (2004), no. 2, 783–817.

[6] E. L. Grinberg, *Radon transforms on higher rank Grassmannians*, J. Differential Geom. 24 (1986), no. 1, 53–68.

[7] G. J. Heckman, *Root systems and hypergeometric functions. II*, Compositio Math. 64 (1987), no. 3, 353–373.
[8] G. J. Heckman and E. M. Opdam, *Root systems and hypergeometric functions. I*, Compositio Math. 64 (1987), no. 3, 329–352.

[9] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, London, 1978.

[10] ———, *Groups and geometric analysis*, Academic Press, New York, London, 1984.

[11] Roger Howe and Soo Teck Lee, *Degenerate principal series representations of $\text{GL}_n(\mathbb{C})$ and $\text{GL}_n(\mathbb{R})$*, J. Funct. Anal. 166 (1999), no. 2, 244–309. MR MR1707754 (2000g:22023)

[12] T. Kakehi, *Integral geometry on Grassmann manifolds and calculus of invariant differential operators*, J. Funct. Anal. 168 (1999), 1–45.

[13] A. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, New Jersey, 1986.

[14] T. Kobayashi, *Multiplicity-free restrictions of unitary highest weight modules with respect to reductive symmetric pairs*, preprint, 2006.

[15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Clarendon Press, Oxford, 1995.

[16] E. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), no. 1, 75–121.

[17] E. Ournycheva and B. Rubin, *The composite cosine transform on the Stiefel manifold and generalized zeta integrals*, to appear.

[18] S. Sahi, *Jordan algebras and degenerate principal series*, J. Reine Angew. Math. 462 (1995), 1–18.

[19] E. M. Stein, *Analysis in matrix spaces and some new representations of $\text{SL}(n,c)$*, Ann. Math. 86 (1967), 461–490.

[20] D. Vogan, *The unitary dual of $\text{gl}(n)$ over an archimedean field*, Invent. Math. 83 (1986), 449–505.

[21] G. Zhang, *Radon transform on real, complex and quaternionic Grassmannians*, Duke Math. J. 138 (2007), no. 1, 137–160.

[22] ———, *Radon transform on symmetric matrix domains*, Trans. Amer. Math. Soc., to appear.

[23] ———, *Jordan algebras and generalized principal series representations*, Math. Ann. 302 (1995), 773–786.

[24] ———, *Berezin transform on compact hermitian symmetric spaces*, Manuscripta Math. 97 (1998), no. 3, 371–388.

[25] ———, *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc. 353 (2001), 3769–3787.

[26] ———, *Spherical transform and Jacobi polynomials on root systems of type BC*, Intern. Math. Res. Notices (2005), no. 51, 3169–3190.

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, Sweden

E-mail address: genkai@math.chalmers.se