CARNOT RECTIFIABILITY OF SUB-RIEMANNIAN MANIFOLDS WITH CONSTANT TANGENT

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ABSTRACT. We show that if $M$ is a sub-Riemannian manifold and $N$ is a Carnot group such that the nilpotentization of $M$ at almost every point is isomorphic to $N$, then there are subsets of $N$ of positive measure that embed into $M$ by bilipschitz maps. Furthermore, $M$ is countably $N$-rectifiable, i.e., all of $M$ except for a null set can be covered by countably many such maps.

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Date: February 1, 2019.

E.L.D. was partially supported by the Academy of Finland (grant 288501 ‘Geometry of subRiemannian groups’) and by the European Research Council (ERC Starting Grant 713998 GeoMeG ‘Geometry of Metric Groups’). R.Y. was supported by NSF grant 1612061.
1. Introduction

Given two metric measure spaces $M = (M, \mu, d)$ and $N$, we say that $M$ is countably $N$-rectifiable if there exist countably many biLipschitz embeddings $f_n : U_n \to M$, with $U_n \subseteq N$ measurable, such that

$$\mu \left( M \setminus \bigcup_{n \in \mathbb{N}} f_n(U_n) \right) = 0.$$  

When $N$ is the Euclidean space $\mathbb{R}^k$, this is the usual notion of a rectifiable set, see [Fed69].

In this paper we will consider the rectifiability of an equiregular sub-Riemannian manifold $M$, with respect to a Carnot group $N$. These are metric measure spaces in a natural way; since $M$ is equiregular, the Hausdorff dimension of any nonempty open subset of $M$ is the same, say $Q$, and the $Q$-dimensional Hausdorff measure $\mathcal{H}^Q$ absolutely continuous with respect to any smooth volume form. We will thus equip all equiregular sub-Riemannian manifolds with Hausdorff measures of appropriate dimension. We will prove the following theorem relating rectifiability to the nilpotentizations of $M$.

**Theorem 1.** Let $M$ be an equiregular sub-Riemannian manifold and let $G$ be a Carnot group. Then $M$ is countably $G$-rectifiable if and only if for almost every point $p \in M$, the nilpotentization $\text{Nil}(M, p)$ is isomorphic to $G$ as a Lie group.

For every sub-Riemannian manifold $M$ and every $p \in M$, the nilpotentization $\text{Nil}(M, p)$ of $M$ at $p$ is an invariant related to the tangent cone that encodes the infinitesimal structure of $M$. Let $\text{Cone}(M, p)$ be the Gromov–Hausdorff tangent of $M$ at $p$, also known as the metric tangent or tangent cone. Mitchell and Bellaïche showed that $\text{Cone}(M, p)$ always exists and is a sub-Riemannian manifold and gave a way to calculate $\text{Cone}(M, p)$ by constructing nilpotent approximations of a frame of the horizontal distribution of $M$. The graded Lie algebra generated by these approximations turns out to depend only on $p$ and the horizontal distribution of $M$, and we call it the nilpotentization of $M$ at $p$, denoted $\text{nil}(M, p)$. (Details of this construction can be found in [Bel96], [lea14], and we will give a sketch in Section 2.) When $p$ is a regular point of $M$, the dimension of $\text{nil}(M, p)$ is equal to the topological dimension of $M$ and $\text{Cone}(M, p)$ is isometric to the stratified Lie group $\text{Nil}(M, p)$ with Lie algebra $\text{nil}(M, p)$. (See [LD17] for an introduction to stratified groups and Carnot groups.)

The fact that countable $G$–rectifiability implies that the nilpotentization $\text{Nil}(M, p)$ is almost everywhere isomorphic to $G$ follows from Pansu differentiability. Indeed, if $U \subseteq G$ is a subset of positive measure and $f : U \to M$ is bilipschitz, then $f$ induces bilipschitz maps from $G$ to $\text{Cone}(M, p)$ for a generic $p$. By work of Pansu, the existence of such a map implies that $\text{Nil}(M, p)$ is isomorphic to $G$ as a Lie group; see Section 5.2 for more details.

The main result of this paper is to show that $M$ is countably $G$–rectifiable if $\text{Nil}(M, p) \cong G$ for almost every $p \in M$, where the symbol $\cong$ denotes Lie group isomorphism. We remark that even if $M$ is equiregular, the nilpotentization
Nil\((M, p)\) may be isomorphic to \(G\) almost everywhere but not everywhere, see Example 6.2. To avoid this problem, we prove the following proposition.

**Proposition 2.** Let \(M\) be an equiregular sub-Riemannian manifold and let \(G\) be a Lie group. Let \(X = \{p \in M \mid \text{Nil}(M, p) \cong G\}\). Then \(X\) is locally closed.

In particular, if \(\text{Nil}(M, p) \cong G\) almost everywhere, then \(\text{Nil}(M, p) \cong G\) on an open subset of \(M\) whose complement is a null set. By restricting to this subset, we may suppose that \(\text{Nil}(M, p) \cong G\) for any \(p \in M\).

In order to prove that \(M\) is countably \(G\)-rectifiable, we construct a family of biLipschitz maps \(f_n : U_n \to M\), where each \(U_n\) is a measurable subset of \(G \cong \text{Cone}(M)\). One difficulty is that in general, there may not be any biLipschitz maps between an open subset of \(\text{Cone}(M, p)\) and an open subset of \(M\). The first examples of this were manifolds such that \(\text{Nil}(M, p)\) is not constant on any set of positive measure \([V\text{ar}81]\); we will give a 7-dimensional example in Section 6.1. This can also happen when \(\text{Nil}(M, p)\) is constant; in \([\text{LOW}14]\) the authors showed that there are sub-Riemannian nilpotent groups \(G\) and \(H\) with isomorphic tangent cones that are not locally biLipschitz equivalent. Therefore, any biLipschitz embedding of a subset \(U \subset G\) into \(H\) has nowhere dense image. We thus prove Theorem 1 by showing that if all tangents of \(M\) are isomorphic to \(G\), then there is a Cantor set \(K \subset G\) of positive measure and a countable collection of biLipschitz embeddings from \(K\) to \(M\) whose images cover almost all of \(M\). As a corollary, if \(G\) and \(H\) are sub-Riemannian groups with isomorphic tangent cones, then there are positive-measure subsets \(U \subset G\) and \(V \subset H\) that are bilipschitz equivalent.

1.1. **Outline of paper.** In Section 2 we recall some results of Bellaïche and Jean on privileged coordinates, nilpotent approximations, and nilpotentizations. In Section 3 we use these results to approximate distances in a neighborhood of a point \(p \in M\) in terms of distances in \(\text{Cone}(M, p)\). Section 4 also contains the proof of Proposition 2. Section 5 contains the proof of Theorem 1. We first use a set of Christ cubes for \(G\) to construct the Cantor set \(K \subset G\), then for each \(p \in M\) we construct biLipschitz embeddings \(H_p : K \to M\) such that the image \(H_p(K)\) has positive density at \(p\). We devote Section 6 to the proof of Theorem 1. In Section 5.1 we use these maps to prove that \(M\) is countably \(G\)-rectifiable if \(G\) is the tangent almost everywhere; while in Section 5.2 we show that the tangent almost everywhere \(G\) if \(M\) is countably \(G\)-rectifiable.

Finally, in Section 6 we give two examples: an example of an equiregular sub-Riemannian manifold on which the tangent is not constant on any set of positive measure and an example of an equiregular sub-Riemannian manifold on which the tangent is constant almost everywhere but not everywhere.

2. **Preliminaries: privileged coordinates, tangent cones, and Bellaïche’s estimates**

Let \(M\) be a connected manifold of dimension \(n\), let \(\Delta \subset TM\) be a sub-bundle of the tangent bundle, and let \(g\) be a positive-definite quadratic form defined
on $\Delta$. For each $p$, let $\Delta_1(p) \subset \Delta_2(p) \subset \cdots \subset T_p M$ be the subspaces spanned by iterated brackets, so that $\Delta_1(p) = \Delta(p)$ and $\Delta_i(p)$ is spanned by vectors of the form $[X_1, \ldots, [X_{k-1}, X_k], \ldots](p)$, where $1 \leq k \leq i$ and $X_i$ are vector fields tangent to $\Delta$. Let $\Delta_i$ be the corresponding “bundle.” If $p \in M$ and there is a neighborhood $U$ of $p$ such that $\dim \Delta_i(q)$ is constant on $U$ for each $i$, we say that $p$ is a regular point. If every point is regular, we say that $M$ is equiregular; in this case, the $\Delta_i$’s are sub-bundles of $TM$.

If there is an $i$ such that $TM = \Delta_i$ for some $i$, we say that $\Delta$ is bracket-generating and call the triple $M = (M, \Delta, g)$ a sub-Riemannian manifold with rank $d$ where $d$ is the dimension of the fiber $\Delta(p)$ of the bundle $\Delta$, i.e., $d := \dim \Delta(p)$ for any $p \in M$. We define the step of $M$ to be the minimal $i$ such that $TM = \Delta_i$. We can use the quadratic form $g$ to equip $M$ with the path metric called the Carnot–Carathéodory metric or the sub-Riemannian metric on $M$.

If $\dim \Delta_i(p)$ is independent of $p$ for all $i$, we say that $M$ is equiregular and define $n_i := \dim \Delta_i(p)$. By convention, we define $n_0 := 0$. Then the Hausdorff dimension of $M$, with respect to the sub-Riemannian metric, is $Q := \sum_i (n_i - n_{i-1})$. The $Q$–dimensional Hausdorff measure $\mathcal{H}^Q$ is doubling, Ahlfors regular, and it is absolutely continuous with respect to any smooth volume form. In particular, a set is null with respect to a Riemannian metric on $M$ if and only if it is null with respect to $\mathcal{H}^Q$.

Mitchell and Bellaïche associated a nilpotent Lie algebra, called the tangent Lie algebra, to each point of an equiregular manifold $M$. Let $p \in M$ and let $X = (X_1, \ldots, X_d)$ be a local frame for $\Delta$ defined on a neighborhood $U$ of $p$. There are a class of coordinate systems for $M$, called privileged coordinate systems, which induce a grading of the differential operators on $M$; we refer to [Bel96, Jea14] for definitions. Let $\phi: U \rightarrow \mathbb{R}^n$ be a coordinate system that is privileged at $p$. The corresponding grading decomposes smooth functions and vector fields on $U$ into functions and fields that are homogeneous with respect to a scaling associated to $\phi$. The grading respects the bracket operation and if $V$ is a homogeneous vector field, then the coefficients of $\phi(V)$ are polynomials. Each field $X_1, \ldots, X_d$ has weight $-1$, and we use the grading to write the $X_i$ as a sum of homogeneous vector fields

$$X_i = X_i^{(-1)} + X_i^{(0)} + \ldots$$

such that $X_i(p) = X_i^{(-1)}(p)$. We define the nilpotent approximation of $X_i$ with respect to $\phi$ as $\hat{X}_i^{\phi,p} := X_i^{(-1)}$. We write $\hat{X}^{\phi,p} = (\hat{X}_1^{\phi,p}, \ldots, \hat{X}_d^{\phi,p})$.

The nilpotent approximations of the $X_i$ generate a Lie algebra $\text{Lie}(\hat{X}^{\phi,p})$. This can be equipped with the stratification

$$\text{Lie}(\hat{X}^{\phi,p}) = V_1(\text{Lie}(\hat{X}^{\phi,p})) \oplus V_2(\text{Lie}(\hat{X}^{\phi,p})) \oplus \ldots,$$

where

$$V_i(\text{Lie}(\hat{X}^{\phi,p})) = \text{span}([\hat{X}^{\phi,p}_{a_1}, \ldots, [\hat{X}^{\phi,p}_{a_i}, \ldots]) \ | \ a_j \in \{1, \ldots, d\} \}.$$
can construct a basis for \( \text{Lie}(\hat{X}^{\phi,p}) \) by extending the \( X_i \)'s to an adapted frame \((Y_1, \ldots, Y_n)\) (possibly defined on a smaller neighborhood) such that \( X_i = Y_i \) for \( i = 1, \ldots, d \) and each \( Y_i \) is a \( w_i \)-iterated bracket of the \( X_i \)'s. Each vector field \( Y_i \) has weight \(-w_i\) and can be decomposed as a sum of homogeneous vector fields

\[
Y_i = Y_i^{(-w_i)} + Y_i^{(-w_i+1)} + \ldots.
\]

The nilpotent approximations of the \( Y_i \), defined as \( \hat{Y}_i^{\phi,p} := Y_i^{(-w_i)} \), form a basis of \( \text{Lie}(\hat{X}^{\phi,p}) \).

This Lie algebra depends \textit{a priori} on the choice of \( \phi \) and \( X \), but if \( p \) is a regular point and \( Z \) is another frame for \( \Delta \), then the fields \( \hat{Z}_i^{\phi,p} \) are linear combinations of the \( \hat{X}_i^{\phi,p} \)'s, so

\[
\text{Lie}(\hat{X}^{\phi,p}) = \text{Lie}(\hat{Z}^{\phi,p}).
\]

Furthermore, if \( \phi' \) is another privileged coordinate system, then by Proposition 5.20 of [Bel96], there is a canonical isomorphism

\[
i_{\phi,\phi'}: \text{Lie}(\hat{X}^{\phi,p}) \to \text{Lie}(\hat{X}^{\phi',p})
\]

such that

\[
i_{\phi,\phi'}(\hat{X}^{\phi,p}) = \hat{X}^{\phi',p}.
\]

That is, \( i_{\phi,\phi'} \) is the unique isomorphism such that \( i_{\phi,\phi'}(V)(p) = V(p) \) for every vector field \( V \in V_1(\text{Lie}(\hat{X}^{\phi,p})) \). We can thus define the tangent Lie algebra, also called the symbol, of \( M \) at \( p \) by

\[
\text{nil}(M, p) := \text{Lie}(\hat{X}^{\phi,p}).
\]

This depends on \( \phi \), but we suppress \( \phi \) in the notation because different choices of \( \phi \) lead to canonically isomorphic Lie algebras. Let \( \text{Nil}(M, p) \) be the simply connected stratified Lie group with Lie algebra \( \text{nil}(M, p) \). We call this the nilpotentization of \( M \) at \( p \).

Though \( \dim \text{nil}(M, p) = \dim M = n \), there is no canonical map from \( T_p M \) to \( \text{nil}(M, p) \). Regardless, the arguments above show that if \( \tau_{\phi,p}: \Delta(p) \to \text{Lie}(\hat{X}^{\phi,p}) \) is the linear map such that \( \tau_{\phi,p}(X_i(p)) = \hat{X}_i^{\phi,p} \), then for any privileged coordinate system \( \phi' \), we have \( \tau_{\phi',p} = i_{\phi,\phi'} \circ \tau_{\phi,p} \). That is, \( \tau_{\phi,p} \) induces an injective linear map \( \tau_p: \Delta(p) \to \text{nil}(M, p) \) whose image is the first stratum \( V_1(\text{nil}(M, p)) \) of \( \text{nil}(M, p) \).

Mitchell and Bellaïche proved the following theorem.

**Theorem 3** ([Bel96], Prop. 5.20, Thm. 7.36]). Let \( M = (\Delta, g) \) be an equiregular sub-Riemannian manifold. The tangent cone \( \text{Cone}(M, p) \) of \( M \) at \( p \) exists and is isometric to the sub-Riemannian metric on \( \text{Nil}(M, p) \) with horizontal bundle \( V_1(\text{nil}(M, p)) \) and quadratic form \( (\tau_p)_*(g_p) \).

We can also construct \( \text{Cone}(M, p) \) directly. Let \( X \) be an orthonormal frame and let \( \hat{X}^{\phi,p} \) be its nilpotent approximation with respect to a privileged coordinate system \( \phi \). The vector fields \( \phi_*(\hat{X}_i^{\phi,p}) \) have polynomial coefficients, so we can extend them to all of \( \mathbb{R}^n \). There is a product structure on \( \mathbb{R}^n \) that makes \( \mathbb{R}^n \) into a Lie group isomorphic to \( \text{Nil}(M, p) \) such that each vector field \( \phi_*(\hat{X}_i^{\phi,p}) \) is left-invariant. If we equip \( \mathbb{R}^n \) with the sub-Riemannian structure such that
φ∗(Xφ, p1), . . . , φ∗(Xφ, pd) are an orthonormal basis for the horizontal distribution, we obtain a left-invariant sub-Riemannian structure which is isometric to Cone(M, p).

Given an orthonormal frame X := (X1, . . . , Xn) for a sub-Riemannian manifold, we say that an absolutely continuous curve γ: I → M has controls u = (u1, . . . , un) with respect to X if

\[ \dot{\gamma} = u_1 X_1 \circ \gamma + \ldots + u_n X_n \circ \gamma \]

almost everywhere. Moreover, we say that the controls of such a γ are subunit if

\[ u_1^2 + \ldots + u_n^2 \leq 1 \]

almost everywhere. Let

\[ \|u\|_{L^1(L^2)} := \int_I \sqrt{u_1^2(t) + \ldots + u_n^2(t)} \, dt. \]

Since the X_i’s are orthonormal, we have \( \ell(\gamma) = \|u\|_{L^1(L^2)} \).

**Definition 4.** Let M1, M2 be two manifolds equipped with two frames X and Y, respectively. Assume that the ranks of the frames are the same. If α: I → M1 and β: I → M2 are two curves, we say that they have the same controls with respect to X and Y, respectively (or simply the same controls when the frames are clear) if the controls of α with respect to X and the controls of β with respect to Y are equal.

We can compare the geometry of the space and its tangent cone by comparing curves with the same controls in M and in Nil(M, p). Hereafter, we shall equip Nil(M, p) with the sub-Riemannian metric defined in Theorem 3, so that it is isometric to Cone(M, p), and view elements of nil(M, p) as left-invariant vector fields on Nil(M, p). We don’t claim originality in the following lemma, which follows from the work of Bellaïche-Jean.

**Lemma 5.** Let M be an equiregular sub-Riemannian n–manifold of step s equipped with an adapted orthonormal frame X = (X1, . . . , Xd). For p ∈ M, the images (τp(X1(p)), . . . , τp(Xd(p))) form a left-invariant frame for the horizontal bundle of Nil(M, p). We denote this frame by τp(X(p)). Let \( \hat{p} \in M \). There are \( C_0, L_0 > 0 \), and a compact neighborhood \( B_0 \) of \( \hat{p} \) with the following property. Let p ∈ B0. Let γ: [0, 1] → M and λ: [0, 1] → Nil(M, p) be two horizontal curves with the same control u with respect to X and τp(X(p)), respectively. Suppose that \( \|u\|_{L^1(L^2)} \leq L_0 \) and that γ(0) = p.

If λ is a closed curve, then

\[ d_M(\gamma(0), \gamma(1)) \leq C_0 \|u\|_{L^1(L^2)}^{1 + \frac{1}{s}}. \]

If γ is a closed curve, then

\[ d_{Nil(M,p)}(\lambda(0), \lambda(1)) \leq C_0 \|u\|_{L^1(L^2)}^{1 + \frac{1}{s}}. \]
Proof. By [Jea14, Theorem 2.3], there are $C_0, L_0 > 0$ and a compact set $B_0 \subset M$ containing a neighborhood of $\bar{p}$ with the following property. Let $p \in B_0$. There is a system of privileged coordinates $\phi: U \to \mathbb{R}^n$ defined on a neighborhood $U$ of $p$ such that $\phi(p) = 0$ and if $\| \cdot \|_p$ is the pseudo-norm
\[
\| (x_1, \ldots, x_n) \|_p = \sum_i |x_i|^\frac{1}{\alpha_i},
\]
then for any $q$ such that $d(p, q) \leq L_0$, we have
\[
(1) \quad \frac{1}{C_0} \| \phi(q) \|_p \leq d_M(p, q) \leq C_0 \| \phi(q) \|_p.
\]

We identify $\text{Nil}(M, p)$ with $\mathbb{R}^n$, equipped with the sub-Riemannian structure defined by the vector fields $\hat{X}_1^{\phi(p)}, \ldots, \hat{X}_d^{\phi(p)}$. By left-invariance, we may suppose that $\lambda(0) = 0$. By [Jea14, Theorem 2.2], we can choose $C_0, L_0, p_0$, and $B_0$ so that
\[
(2) \quad \frac{1}{C_0} \| u \|_p \leq d_{\text{Nil}(M, p_0)}(0, u) \leq C_0 \| u \|_p, \quad \text{for all } u \in \mathbb{R}^n.
\]

By [Jea14 (2.14)], (taking $q = p$), there is a $C > 0$ such that
\[
(3) \quad \| \phi(\gamma(1)) - \lambda(1) \|_p \leq C \| u \|_{L^1(L_2)}^{1 + \frac{1}{\alpha_i}}.
\]

Suppose that $\lambda$ is a closed curve, so that $\lambda(1) = 0$. By (1) and (3),
\[
d_M(\gamma(0), \gamma(1)) \leq C_0 \| \phi(\gamma(1)) \|_p \leq C \cdot C_0 \| u \|_{L^1(L_2)}^{1 + \frac{1}{\alpha_i}}.
\]
Likewise, if $\gamma$ is a closed curve, then $\phi(\gamma(1)) = 0$, so
\[
d_{\text{Nil}(M, p_0)}(\lambda(0), \lambda(1)) \leq C_0 \| \lambda(1) \|_p \leq C \cdot C_0 \| u \|_{L^1(L_2)}^{1 + \frac{1}{\alpha_i}}.
\]

\[\square\]

3. Manifolds with constant tangent

In this section, we prove an approximation result for sub-Riemannian manifolds with constant tangent.

If $M$ is a sub-Riemannian manifold with horizontal distribution $\Delta$ and $X$ is a frame for $\Delta$, we define $d_X$ to be the sub-Riemannian distance function on $M$ for which $X$ is an orthonormal frame.

Lemma 6. Let $(M, \Delta)$ be an equiregular sub-Riemannian manifold and let $G$ be a Carnot group with Lie algebra $\mathfrak{g}$ such that for every $p \in M$, there is an isomorphism $a_p: \mathfrak{g} \to \text{nil}(M, p)$. Suppose that the $a_p$’s vary smoothly in the sense that there is a basis $\mathcal{Y} = (Y_1, \ldots, Y_d) \in \mathcal{V}_1(\mathfrak{g})$ (i.e., a left-invariant frame of the horizontal bundle) such that if
\[
X_i(p) := \tau_p^{-1}(a_p(y_i)),
\]
then $X = (X_1, \ldots, X_d)$ is a smooth frame for $\Delta$, where $\tau_p: \Delta(p) \to \text{nil}(M, p)$ is the map defined in Section 2.

For every $p \in M$, there are $C, L > 0$, and a compact neighborhood $B$ of $p$ with the following property. Let $q \in B$ and for $i = 1, 2$ let $\gamma_i: [0, 1] \to M$ be horizontal
curves with controls $u_i$ such that $\gamma_i(0) = q$ and $\|u_i\|_{L_1(L_2)} \leq L$. Let $\lambda_i : [0, 1] \to G$ be the curves in $G$ with $\lambda_i(0) = 0$ and with controls $u_i$ with respect to $\gamma$. Then

$$d_X(\gamma_1(1), \gamma_2(1)) - d_Y(\lambda_1(1), \lambda_2(1)) \leq C\|u_1\|_1 + \|u_2\|_1)^{1+\frac{1}{2}}.$$  

**(Proof.)** Let $B_0$, $C_0$, and $L_0$ be as in Lemma 5. Let $0 < L \leq \frac{\|u\|}{2}$ be small enough that the sub-Riemannian ball $B_{1}(p)$ is contained in $B_0$ and let $B = \overline{B_{1}(p)}$. This choice ensures that if $q$, $\gamma_1$, and $\gamma_2$ satisfy the hypotheses, then $\gamma_1(1), \gamma_2(1) \in B_0$.

Let $a : [0, 1] \to G$ be a geodesic from $\lambda_1(1)$ to $\lambda_2(1)$. We have $\ell(a) \leq \|u_1\|_1 + \|u_2\|_1 \leq 2L$. The curves $\lambda_1, \lambda_2$, and $a$ form a triangle in $G$, and we define $a : [0, 3] \to G$ to be the closed curve that traces the triangle starting at $\lambda_2(1)$, i.e.,

$$a(t) = \begin{cases} 
\hat{\lambda}_2(1-t) & t \in [0, 1] \\
\lambda_1(t-1) & t \in [1, 2] \\
\mathbb{a}(t-2) & t \in [2, 3]
\end{cases}.$$  

Let $b : [0, 3] \to M$ be the curve with the same controls such that $b(0) = \gamma_2(1) \in B_0$. Since $b$ has the same controls as $a$, we have

$$d_X(b(2), b(3)) \leq \ell(a) = d_Y(\lambda_1(1), \lambda_2(1)).$$  

Furthermore, $b|_{[0, 1]}$ is the reverse of $\gamma_2$ and $b|_{[1, 2]}$ is $\gamma_1$, so $b(2) = \gamma_1(1)$. Thus, by Lemma 5,

$$d_X(\gamma_1(1), \gamma_2(1)) \leq d_X(b(2), b(3)) + d_X(b(3), \gamma_2(1)) \leq d_Y(\lambda_1(1), \lambda_2(1)) + C_0 \|b\|^{1+\frac{1}{2}} \leq d_Y(\lambda_1(1), \lambda_2(1)) + C_0 2(\|u_1\|_1 + \|u_2\|_1)^{1+\frac{1}{2}}.$$  

This proves one inequality. To prove the other inequality, we apply the same procedure with $\gamma$ and $\hat{\lambda}$ switched. That is, we connect $\gamma_1(1)$ and $\gamma_2(1)$ by a geodesic to construct a curve $a : [0, 3] \to M$ such that $c(0) = c(3) = \gamma_2(1), c(1) = q$, and $c(2) = \gamma_1(1)$. Let $b : [0, 3] \to M$ be the curve with the same controls such that $b(0) = \lambda_2(1)$, so that $b(1) = 0$ and $b(2) = \lambda_1(1)$. As above,

$$d_G(b(2), b(3)) \leq d_X(\gamma_1(1), \gamma_2(1)),$$

and by Lemma 5

$$d_Y(\lambda_1(1), \lambda_2(1)) \leq d_X(\gamma_1(1), \gamma_2(1)) + C_0 \|b\|^{1+\frac{1}{2}} \leq d_X(\gamma_1(1), \gamma_2(1)) + C_0 2(\|u_1\|_1 + \|u_2\|_1)^{1+\frac{1}{2}}.$$  

Next, in Lemma 8 we shall prove that a sub-Riemannian manifold $M$ has local frames satisfying Lemma 6 if and only if its tangent Lie algebra is constant. We first need some notation and the following Lemma 7.

Let $V$ be a finite-dimensional vector space and let $F_V := \text{Hom}(V \wedge V, V)$, seen as the set of alternating bilinear maps. This is a finite-dimensional vector space
and thus an algebraic variety. Thus, if \( \Psi \in F_V \) satisfies the Jacobi identity, then \((V, \Psi)\) is a Lie algebra. For any Lie algebra \( g \) of the same dimension, we define 

\[
E_\mathfrak{g} := \{ \Psi \in F_V \mid (V, \Psi) \cong g \},
\]

where the symbol \( \cong \) denotes Lie algebra isomorphism.

**Lemma 7.** Let \( g \) and \( E_\mathfrak{g} \) be as above. Then \( E_\mathfrak{g} \) is a smooth submanifold of \( F_V \).

**Proof.** Let \( V, W \) be vector spaces, let \( A : V \rightarrow W \) be a linear isomorphism, and let \( \Psi \in \text{Hom}(V \wedge V, V) \). We define the push-forward of \( \Psi \) by \( A \) to be the map \( A_* \Psi \in \text{Hom}(W \wedge W, W) \),

\[
(A_* \Psi)(w_1, w_2) = A\Psi(A^{-1}w_1, A^{-1}w_2).
\]

If \( B : W \rightarrow X \) is a linear isomorphism, then \( A_* B_* \Psi = (AB)_* \Psi \). That is, this defines a left action of \( GL(V) \) on \( F_V \). If \( (V, \Psi) \) is a Lie algebra with bracket \( \Psi \), then \((W, A_* \Psi)\) is a Lie algebra and \( A \) is an isomorphism from \((V, \Psi)\) to \((W, A_* \Psi)\).

Fix some \( \Psi_0 \in E_\mathfrak{g} \) and let \( f : GL(V) \rightarrow F_V \) be the orbit map \( f(A) := A_* \Psi_0 \). If \( \Psi \in E_\mathfrak{g} \), then there is a Lie algebra isomorphism \( A : (V, \Psi_0) \rightarrow (V, \Psi) \), so \( A_* \Psi_0 = \Psi \). It follows that \( E_\mathfrak{g} = GL(V)_* \Psi_0 = f(GL(V)) \).

The map \( f \) has polynomial coefficients, so it is a regular map (in the sense of algebraic geometry) on an affine variety. Consequently, its image is locally closed in the Zariski topology (see for instance [Har95, 3.16]). In particular, \( E_\mathfrak{g} \) is a smooth submanifold except possibly on a singular set of lower dimension. In particular the singular set is a proper subset of \( E_\mathfrak{g} \). Since \( GL(V) \) acts transitively on \( E_\mathfrak{g} \), the space \( E_\mathfrak{g} \) is homogeneous, so the singular set is empty. Thus \( E_\mathfrak{g} \) is a smooth submanifold.

If \( g \) is a stratified Lie algebra and \( V = V_1(V) \oplus V_2(V) \oplus \ldots \) is a graded vector space with \( \dim V_i(V) = \dim V_i(g) \), it likewise holds that

\[
E^{s}_\mathfrak{g} = \{ \Psi \in F_V \mid (V, \Psi) \cong_s g \}
\]

is a smooth submanifold of \( F_V \). Here the symbol \( \cong_s \) denotes a stratified Lie algebra isomorphism, i.e., an isomorphism \( A : (V, \Psi) \rightarrow g \) such that \( A(V_i(V)) = V_i(g) \).

**Lemma 7** allows us to prove the following.

**Lemma 8.** Let \((M, \Delta)\) be an equiregular sub-Riemannian manifold. Suppose that there is a Lie algebra \( g \) such that \( g \cong \text{nil}(M, p) \) for every point \( p \in M \). For any \( p \in M \), there are a neighborhood \( U \) of \( p \), a family of isomorphisms \( a_q : g \rightarrow \text{nil}(M, q) \), and a basis \( Y_1, \ldots, Y_d \in V_1(g) \) satisfying the assumptions of **Lemma 6** for \( M = U \).

**Proof.** We first construct a grading of \( \mathbb{R}^n \) and a family of smoothly varying forms \( \Psi_q \in F_{\mathbb{R}^n} \) such that \( (\mathbb{R}^n, \Psi_q) \cong_s \text{nil}(M, q) \); i.e., \( \Psi_q \in E^{s}_\mathfrak{g} \). By **Lemma 7** \( E^{s}_\mathfrak{g} \) is a smooth manifold, so we may apply the Implicit Function Theorem to produce the desired isomorphisms.

Let \( U \) be a neighborhood of \( p \) such that there is an adapted frame \((W_1, \ldots, W_n)\) defined on \( U \). As in [Jea14, 2.2.2], for every \( q \in U \), we can use the \( W_i \)'s to produce a system \( \phi_q \) of exponential coordinates such that \( \phi_q \) is privileged at \( q \) and varies smoothly with \( q \). Let \( \tilde{W}_i^q := \tilde{W}_i^{\phi_q, q} \) be the nilpotent approximation of the \( W_i \)'s at
$q$. By the results of Section 2, the span of the $\hat{W}_i^q$ is a Lie algebra of vector fields which is canonically isomorphic to $\text{nil}(M, q)$; let

$$g_q := \langle \hat{W}_1^q, \ldots, \hat{W}_n^q \rangle$$

and let $\iota_q: g_q \to \text{nil}(M, q)$ be the canonical isomorphism. This is the unique isomorphism such that $\iota_q(V) = \tau_q(V(q))$ for all $V \in V_1(g_q)$.

For all $q \in U$, we define the basis $\tilde{\Psi}^q := (\hat{W}_1^q, \ldots, \hat{W}_n^q)$, and for all $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$, let

$$\tilde{\Psi}^q \nu := \nu_1 \hat{W}_1^q + \cdots + \nu_n \hat{W}_n^q \in g_q.$$ 

This induces an linear isomorphism from $\mathbb{R}^n$ to $g_q$. Let $\Psi_q := ((\tilde{\Psi}^q)^{-1})_{\ast} \cdot \cdot \cdot q \in F_{\mathbb{R}^n}$, i.e.,

$$\Psi_q(v, w) = (\tilde{\Psi}^q)^{-1}[\tilde{\Psi}^q v, \tilde{\Psi}^q w]_q,$$

so that $(\mathbb{R}^n, \Psi_q)$ is a stratified Lie algebra isomorphic to $g_q$, with strata

$$V_i(\mathbb{R}^n) := (\tilde{\Psi}^q)^{-1}V_i(g_q).$$

That is, $V_i(\mathbb{R}^n)$ is the subspace spanned by the $n_i$th through $(n_{i+1} - 1)$th coordinate vectors. The coordinates of $\Psi_q$ in $F_{\mathbb{R}^n}$ are the structure coefficients of $g_q$ with respect to $\tilde{\Psi}^q$, so $\Psi_q$ varies smoothly with $q$.

Let $S \subset \text{GL}_n(\mathbb{R})$ be the subgroup of block-diagonal matrices that preserve the grading of $\mathbb{R}^n$ and let $E^s_{\mathbb{R}}$ be as in (3). For every $q$, we have $(\mathbb{R}^n, \Psi_q) \cong g$, so $\Psi_q \in E^s_{\mathbb{R}}; \text{ indeed, } E^s_{\mathbb{R}} = S_{\ast} \Psi_q$. By the remark after Lemma 4, $E^s_{\mathbb{R}}$ is a smooth submanifold of $F_{\mathbb{R}^n}$. Let $k = \dim E^s_{\mathbb{R}}$.

Let $f: S \to E^s_{\mathbb{R}}$ be the map $f(A) := A_{\ast} \Psi_p$. This map $f$ is smooth and surjective, and its derivative $Df$ has constant rank, so by Sard’s theorem, $Df$ has rank $k$ everywhere. By the Implicit Function Theorem, there is a neighborhood $T \subset E^s_{\mathbb{R}}$ of $\Psi_p$ and a smooth section $\alpha: T \to S$ such that $\alpha(\Psi_p) = I$ and $f(\alpha(\Psi)) = \alpha(\Psi)_{\ast} \Psi_p = \Psi$ for all $\Psi \in T$. That is, $\alpha(\Psi)$ is a stratified isomorphism from $(\mathbb{R}^n, \Psi_p)$ to $(\mathbb{R}^n, \alpha(\Psi)_{\ast} \Psi_p) = (\mathbb{R}^n, \Psi)$. We fix a stratified isomorphism $\beta: g \to (\mathbb{R}^n, \Psi_p)$, and define

$$a_q := \iota_q \circ \tilde{\Psi}^q \circ \alpha(\Psi_q) \circ \beta$$

for all $q \in U \cap \psi^{-1}(T)$. This is a family of stratified isomorphisms from $g$ to $g_q$, and for any $Y \in V_1(g)$,

$$\tau_q^{-1} \circ a_q(Y) = \tau_q^{-1} \left( \iota_q \left[ (\tilde{\Psi}^q \alpha(\Psi_q) \beta(Y)) \right] \right) = (\tilde{\Psi}^q \alpha(\Psi_q) \beta(Y))(q).$$

This depends smoothly on $q$, so any basis $Y_1, \ldots, Y_d \in V_1(g)$ satisfies the assumptions of Lemma 6 as desired.

The following lemma is a further consequence of Lemma 7. We say that a subset $X$ is locally closed if, for all $p \in X$, there is a neighborhood $V$ of $p$ such that $X \cap V$ is relatively closed in $V$.

**Proposition 9.** Let $M$ be an equiregular sub-Riemannian manifold and let $g$ be a Lie algebra. Let $X = \{ p \in M \mid \text{nil}(M, p) \cong g \}$. Then $X$ is locally closed.
Proof. Let \( p \in X \). As in Lemma 8, let \( U \) be a neighborhood of \( p \) equipped with an adapted frame \((W_1, \ldots, W_n)\) and a family of privileged coordinate systems \( \phi_q \) at \( q \) that varies smoothly with \( q \). This induces a map \( \psi: U \to F_{\mathbb{R}^n} \) such that \( X \cap U = \psi^{-1}(E_{\overline{q}}) \), and the preimage of a locally closed set is locally closed. \( \square \)

In particular, if \( X \) is dense in \( M \), then for all \( p \in X \), there is a neighborhood \( V \) of \( p \) such that \( X \cap V \) is relatively closed in \( V \) and thus \( V \subset X \); i.e., \( X \) is an open subset of \( M \). Note that \( X \) need not be all of \( M \). See Section 6.2 for an example where \( M \) is equiregular and \( X \) is almost all of \( M \) but \( X \neq M \).

4. Bilipschitz maps with images of positive measure

To prove Theorem 1, we will first construct a family of bilipschitz maps from a subset of \( G \) to \( M \) whose images have positive measure. In Section 5, we will cover almost all of \( M \) by countably many such images.

We will show the following proposition.

**Proposition 10.** Let \( G \) be a Carnot group with Lie algebra \( \mathfrak{g} \). Let \((M, \Delta)\) be a sub-Riemannian manifold such that \( \mathfrak{g} \cong \text{nil}(M, p) \) for every point \( p \in M \). For every \( p \in M \) there is a subset \( K \subset G \) and a bilipschitz embedding \( H: K \to M \) such that \( H(K) \) has positive density at \( p \).

Let \( Y \) be a left-invariant frame for \( V_1(\mathfrak{g}) \). By Lemma 8, there is a local frame \( X \) for \( \Delta \) defined on a neighborhood \( U \) of \( p \) that satisfies the hypothesis of Lemma 6.

We equip \( G \) with the metric \( d_Y \) and equip \( U \) with the metric \( d_X \). There is no loss of generality here because any two Carnot metrics on \( G \) are bilipschitz equivalent and, after possibly passing to a smaller neighborhood, \( d_X \) is bilipschitz equivalent to the sub-Riemannian metric on \( U \).

We divide the proof of the novel direction of Theorem 1 into three parts. We will first construct a Cantor set \( K \) by using a set of Christ cubes for \( G \), then construct the map \( H \) that embeds \( K \) into \( M \). This proves Proposition 10. Finally, we will use the maps produced by Proposition 10 to show that \( M \) is countably \( G \)-rectifiable.

4.1. Constructing \( K \). If \( S_i \) is a collection of sets, we denote the disjoint union of the \( S_i \)'s by

\[
\sqcup_i S_i = \{(s, i) \mid i \in \mathbb{Z}, s \in S_i\}.
\]

We will often refer to elements of a disjoint union and elements of the constituent sets interchangeably.

Let \( X \) be a metric space and let \( \mu \) be a measure on \( X \) that is Ahlfors regular. A *cubical patchwork* or *set of Christ cubes* for \( X \) is a collection of nested partitions of \( X \), analogous to the tilings of \( \mathbb{R}^n \) by dyadic cubes. That is, for each \( i \), there is a partition \( \Delta_i \) of \( X \) (a set of subsets of \( X \) that are pairwise disjoint and whose union is all of \( X \)) that satisfies the following properties. There are \( \sigma \in (0, 1), a_0 > 0, \eta > 0, \) and \( 0 < C_1 < C_2 < \infty \) such that:

1. If \( Q \in \Delta_k \) and \( Q' \in \Delta_l \), with \( k \leq l \), then either \( Q' \subset Q \) or \( Q \cap Q' = \emptyset \).
2. Every \( Q \in \Delta_k \) has a *parent* \( P(Q) \in \Delta_{k-1} \) such that \( Q \subset P(Q) \).
3. Every \( Q \in \Delta_k \) contains a ball of radius \( C_1 \sigma^k \).
4. \( \text{diam} \ Q \leq C_2 \sigma^k \), for every \( Q \in \Delta_k \).
5. For all \( t > 0 \) and all \( Q \in \Delta_k \), let

\[ \partial_t Q := \{ x \in Q \mid d(x, X \setminus Q) < t\sigma^k \} \cup \{ x \in X \setminus Q \mid d(x, Q) < t\sigma^k \}. \]

Then for any \( 0 < t \leq 1 \),

(6) \[ \mu(\partial_t Q) \leq a_0 t^\eta \mu(Q). \]

We call the elements of \( \Delta_i \) cubes, and we let \( \Delta := \bigcup_i \Delta_i \) be the disjoint union of the \( \Delta_i \)’s. Every Ahlfors regular metric space admits a cubical patchwork [Dav88, Chr90] for each \( \sigma \in (0, 1) \), and if \( X \) is Ahlfors \( d \)–regular, then \( \mu(Q) \approx (\sigma^k)^d \) for all \( Q \in \Delta_k \).

Let \( \Delta = \bigcup_{i \in \mathbb{Z}} \Delta_i \) be a cubical patchwork for \( G \) with \( \sigma = \frac{1}{2} \). For any \( Q \in \Delta \), let

\[ \Delta_i(Q) := \{ R \in \Delta_i \mid R \subset Q \} \]

and let \( \Delta(Q) := \bigcup_{i \in \mathbb{Z}} \Delta_i(Q) \).

We use \( \Delta \) to construct a Cantor set in \( G \). Let \( \tau > 0 \) be a small number to be determined later and let \( Q_0 \in \Delta_0 \). Let \( K_0 = Q_0 \), and for \( k \geq 0 \), let

(7) \[ K_{k+1} = K_k \setminus \bigcup_{Q \in \Delta_i(Q_0)} \partial_{\tau^2^{-\frac{k}{2}}} Q. \]

Note that \( \partial_t Q \) is open for any \( t \) and \( Q \) and that \( K_1 = Q_0 \setminus \partial_1 Q_0 \) is closed, so \( K_k \) is closed for all \( k \geq 1 \). Let \( K := \bigcap \limits_1 \Delta_i(Q_0) \). This is a compact, totally disconnected set.

**Lemma 11.** When \( \tau > 0 \) is sufficiently small, then \( \mu(K) > 0 \).

**Proof.** For any \( k \geq 0 \),

\[
\mu(K_k) - \mu(K_{k+1}) \leq \mu \left( \bigcup_{Q \in \Delta_i(Q_0)} \partial_{\tau^2^{-\frac{k}{2}}} Q \right) \leq \sum_{Q \in \Delta_i(Q_0)} a_0 \cdot (\tau^2^{-\frac{k}{2}})^\eta \mu(Q).
\]

Since \( \Delta_i(Q_0) \) is a partition of \( Q_0 \), we have

\[
\mu(K_k) - \mu(K_{k+1}) \leq a_0 \tau^\eta 2^{-\frac{kn}{2}} \mu(Q_0)
\]

and

\[
\mu(K_0) - \mu(K_k) \leq \sum_{i=0}^{k-1} a_0 \tau^\eta 2^{-\frac{in}{2}} \mu(Q_0) \leq a_0 \tau^\eta (1 - 2^{-\frac{n}{2}})^{-1} \mu(Q_0).
\]

As \( \tau \) goes to zero, this difference goes to zero, so \( \lim_{\tau \to 0} \mu(K) = \mu(K_0) = \mu(Q_0) > 0 \). \( \square \)
4.2. **Constructing $H$.** We construct $H$ by constructing a metric tree $\mathcal{T}$ and a map $A: \mathcal{T} \to G$ that sends the ends of $\mathcal{T}$ to the points of $K$ bijectively. Each point $q \in K$ then corresponds to a curve $\alpha_q$ in $\mathcal{T}$, and there is a map $F: \mathcal{T} \to M$ so that the controls of $F \circ \alpha_q$ with respect to $\%$ are a rescaling of the controls of $A \circ \alpha_q$ with respect to $\%$. We then use Lemma 6 to show that $H = F \circ A^{-1}$ satisfies the desired properties.

Let $\Lambda := \{Q \in \Delta(Q_0) \mid Q \cap K \neq \emptyset\}$ and let $\mathcal{T}$ be the rooted tree with one vertex $v_Q$ for every cube $Q \in \Lambda$. We let $v_0 = v_{Q_0}$ be the root of $\mathcal{T}$, and for each $Q \in \Delta(Q_0)$ with $Q \neq Q_0$, we connect $Q$ to its parent $P(Q)$. Let $V(\mathcal{T})$ be the vertex set of $\mathcal{T}$ and let $V_i(\mathcal{T}) \subset V(\mathcal{T})$ be the $i$th generation of the tree; i.e., the set of vertices corresponding to elements of $\Delta_i$. For every vertex $v \in V(\mathcal{T})$, we denote the corresponding cube by $[v] \in \Lambda$.

We equip $\mathcal{T}$ with a path metric so that the edges between $V_i(\mathcal{T})$ and $V_{i-1}(\mathcal{T})$ have length $2^{-i}$. Then $d(v_0, v) = 1 - 2^i$ for every $v \in V_i(\mathcal{T})$. Let $\overline{\mathcal{T}}$ be the metric completion of $\mathcal{T}$. This completion consists of the union of $\mathcal{T}$ with a Cantor set, which we denote by $J$.

Let $\rho_i: J \to V_i(\mathcal{T})$ be the map that sends $x \in J$ to the $i$th generation vertex that is closest to $x$. For every $x \in J$, the path starting at $\rho_0(x) = v_0$ that passes through $\rho_1(x), \rho_2(x)$, and so on is a geodesic of length 1 connecting $v_0$ to $x$. For each $i$, we have $\{\rho_{i+1}(x)\} \subset [\rho_i(x)]$.

Next, we define a Lipschitz map $A: \mathcal{T} \to G$. For each $Q$, we send $v_Q$ to a point $A(v_Q) \in Q$ and send the edge from $Q$ to $P(Q)$ to a geodesic in $G$. For every $i > 0$ and $Q \in V_i(\mathcal{T})$, we have $A(v_Q) \in Q \subset P(Q)$, so

$$d(A(v_Q), A(v_{P(Q)})) \leq \text{diam}(P(Q)) \leq C_22^{-i} = 2C_2d(v_Q, v_{P(Q)}).$$

Thus $A$ is Lipschitz. It extends to a map $\overline{A}: \overline{\mathcal{T}} \to G$, and for every $x \in J$,

$$\bigcap_{i \in \mathbb{N}} [\rho_i(x)] = [\overline{A}(x)].$$

The intersections $K \cap [\rho_i(x)]$ are all nonempty, so since $K$ is closed, $[\overline{A}(x)] \in K$.

For every $x, y \in J$, let

$$i(x, y) := \sup\{i \in \mathbb{Z} \mid \rho_i(x) = \rho_i(y)\}$$

and let $a(x, y) = \rho_{i(x,y)}(x) \in V(\mathcal{T})$, so that $[a(x, y)]$ is the minimal cube containing both $\overline{A}(x)$ and $\overline{A}(y)$. Then

$$d(x, y) = d(x, a(x, y)) + d(a(x, y), y) = 2^{-i(x,y)+1}.$$ 

This is an ultrametric on $J$.

**Lemma 12.** The restriction $\overline{A}|_J$ is a bijection from $J$ to $K$ so that for all $x, y \in J$,

$$\frac{1}{8}d(x, y)^{1+\frac{1}{n}} \leq d(\overline{A}(x), \overline{A}(y)) \leq 2C_2d(x, y).$$

**Proof.** First, we show that $\overline{A}(J) = K$. If $k \in K$, then for each $i \geq 0$, there is a cube $R_i \in \Lambda_i$ such that $k \in R_i$. Since $R_i$ and $R_{i+1}$ intersect, $R_{i+1} \subset R_i$, so the vertices
\( \nu_{R_0}, \nu_{R_1}, \ldots \) form a path starting at the root of \( \mathcal{F} \); this path converges to a point \( x \in J \), and
\[
\overline{A}(x) = \bigcap_i R_i = \{k\}.
\]

We previously showed that \( A \) is \( 2C_2 \)-Lipschitz, so \( d(\overline{A}(x), \overline{A}(y)) \leq 2C_2 d(x, y) \) for all \( x, y \in J \). Suppose that \( x_1, x_2 \in J \) and \( x_1 \neq x_2 \), so that \( d(x_1, x_2) < \infty \). Let \( i = i(x_1, x_2) \) and let \( Q_i = [\rho_{i+1}(x_j)] \in \Delta_{i+1} \) for \( j = 1, 2 \). Then \( P(Q_1) = P(Q_2) = [\rho_i(x_j)] \) for \( j = 1, 2 \), but \( Q_1 \) and \( Q_2 \) are disjoint. Since \( \overline{A}(x_j) \in K \), equation \( 7 \) implies that \( \overline{A}(x_1) \notin \partial_{\tau_{2^{-i+1}}} Q_i \), so
\[
d(\overline{A}(x_1), G \setminus Q_i) \geq \tau 2^{-(i+1)(1 + \frac{1}{\sqrt{C_2}})}.
\]
Thus
\[
d(\overline{A}(x_1), \overline{A}(x_2)) \geq \tau 2^{-(i+1)(1 + \frac{1}{\sqrt{C_2}})}.
\]
Since \( d(x_1, x_2) = 2^{-i+1} \),
\[
d(\overline{A}(x_1), \overline{A}(x_2)) \geq \frac{1}{8} \tau d(x, y)^{1 + \frac{1}{\sqrt{C_2}}}.
\]
\( \square \)

Let \( p \in M \). Let \( B, C, \) and \( L \) be as in Lemma \([5]\) and suppose that \( L \) is small enough that \( B_L(p) < B \). Let \( r > 0 \) be small enough that
\[
2r < L \quad \text{and} \quad C r^2 \leq \frac{\tau}{80C_2}
\]
and let \( E: \overline{\mathcal{F}} \to G \) be the map
\[
E := \delta_{\frac{r}{\sqrt{C_2}}} \circ \overline{A}.
\]
Then \( \text{Lip}(E) \leq r (2C_2)^{-1} \text{Lip}(A) \leq r \).

For every \( x \in J \), there is a unique unit-speed geodesic \( \lambda_x: [0, 1] \to \overline{\mathcal{F}} \) connecting \( \nu_0 \) to \( x \). The composition \( \alpha_x := E \circ \lambda_x \) is a horizontal curve in \( G \) of length at most \( r \). Let \( k_0 \in K \) be a density point of \( K \) and let \( j_0 \in J \) be such that \( \overline{A}(j_0) = k_0 \). Let \( \gamma: [0, 1] \to M \) be the unique horizontal curve with the same controls as \( \alpha_{j_0} \) such that \( \gamma(1) = p \), and let \( q_0 := \gamma(0) \). Note that \( d(p, q_0) \leq r \).

For each \( x \in J \), let \( \tilde{\alpha}_x: [0, 1] \to M \) be the unique horizontal curve with the same controls as \( \alpha_x \) such that \( \tilde{\alpha}_x(0) = q_0 \). Then \( \tilde{\alpha}_{j_0} = \gamma \).

Let \( F: \overline{\mathcal{F}} \to M \) be the map such that \( F(\lambda_x(t)) = \tilde{\alpha}_x(t) \) for all \( x \in J \) and \( t \in [0, 1] \). This is well defined, because if \( x, y \in J \) and \( \lambda_x(t) = \lambda_y(t) \), then \( \lambda_x \) and \( \lambda_y \) agree on the interval \( [0, t] \), so \( \tilde{\alpha}_x \) and \( \tilde{\alpha}_y \) agree on the same interval. By construction, \( \tilde{\alpha}_{j_0} = \gamma \), so
\[
F(j_0) = F(\lambda_{j_0}(1)) = \tilde{\alpha}_{j_0}(1) = \gamma(1) = p.
\]

We use \( F \) to prove Proposition \([10]\).

**Proof of Proposition \([10]\)** Let \( H: K \to M \) be the map \( H := F \circ (\overline{A}(j))^{-1} \). We claim that \( H \) is a biLipschitz embedding.
Let \( E := \delta \frac{Ax}{\alpha_2} \circ \overline{A} \) as in the definition of \( A \). Then \( H = F \circ (E|_J)^{-1} \circ \delta \frac{Ax}{\alpha_2} \), so \( H \) is biLipschitz if and only if \( F \circ (E|_J)^{-1} \) is biLipschitz. Since \( E \) is injective on \( J \), it suffices to show that for all \( x, y \in J \), we have

\[
\frac{1}{2} \, d_G(E(x), E(y)) \leq d_M(F(x), F(y)) \leq 2d_G(E(x), E(y)).
\]  

Let \( i = i(x, y) \) be as in (9). By (10), \( d_{\mathcal{F}}(x, y) = 2^{-i+1} \). By Lemma 12

\[
d_G(E(x), E(y)) = \frac{r}{2C_2} \, d_G(A(x), A(y)) \geq \frac{tr}{16C_2} d_{\mathcal{F}}(x, y)^{1 + \frac{1}{2i}}.
\]  

Let \( \alpha, \beta : [0, 1] \to \mathcal{F} \) be the curves \( \alpha(t) = \lambda_x(1 - 2^{-i} + 2^{-i} t), \beta(t) = \lambda_y(1 - 2^{-i} + 2^{-i} t) \). These are geodesics such that \( \alpha(0) = \beta(0) = a(x, y), \alpha(1) = x, \beta(1) = y \). The compositions \( E \circ \alpha \) and \( E \circ \beta \) are horizontal curves in \( G \) of length at most \( r2^{-i} \) that both start at \( E(a(x, y)) \), and \( F \circ \alpha \) and \( F \circ \beta \) are curves in \( M \) with the same controls that both start at \( q := F(a(x, y)) \). Since

\[
d(p, q) \leq d(p, q_0) + d(q_0, q) \leq r + (1 - 2^{-i})r \leq L,
\]  

we have \( q \in B \), and we can apply Lemma 6.

Let \( u_\alpha, u_\beta \) be the controls of \( E \circ \alpha \) and \( E \circ \beta \); we have

\[\|u_\alpha(t)\|_1, \|u_\beta(t)\|_1 \leq r2^{-i}.\]

By Lemma 6

\[
|d_G(E(x), E(y)) - d_M(F(x), F(y))| \leq C(\|u_\alpha\|_1 + \|u_\beta\|_1)^{1 + \frac{1}{2i}}
\]

\[
\leq C(trd_{\mathcal{F}}(x, y))^{1 + \frac{1}{2i}}
\]

\[
= Cr^{\frac{1}{2i}} \cdot r \cdot d_{\mathcal{F}}(x, y)^{1 + \frac{1}{2i}}
\]

\[
\leq \frac{\tau}{80C_2} \cdot r \cdot 2d_{\mathcal{F}}(x, y)^{1 + \frac{1}{2i}}
\]

\[
\leq \frac{1}{2}d_G(E(x), E(y)).
\]  

This implies (13), so \( H \) is biLipschitz.

Finally, since \( H \) is a biLipschitz map from \( K \subset G \) to \( M \), \( k_0 \) is a point of density of \( K \), and \( G \) and \( M \) have the same Hausdorff dimension, the image \( H(K) \) has positive density (with respect to top-dimensional Hausdorff measure) at \( H(k_0) = F(\overline{A}^{-1}(\Lambda(j_0))) = F(j_0) = p \).

\[\square\]

5. Proof of Theorem 1

We break the proof of Theorem 1 into two parts.
5.1. **G is the tangent almost everywhere ⇒ M is countably G-rectifiable.** First, we prove the reverse direction in Theorem 1. It suffices to show that any compact subset of M can be covered by the union of countably many bilipschitz images of subsets of G and a null set. Let $C \subset M$ be compact.

Let $L = \inf \{ \mu(C \setminus \bigcup_{i=1}^{\infty} X_{i}) \}$, where the infimum is taken over countable sequences of sets $X_{i} \subset M$ such that $X_{i}$ is the bilipschitz image of a subset of $G$ (henceforth, a bilipschitz image). We claim that $L = 0$. First, note that this infimum is achieved by some sequence of sets $X_{i}$; if $X_{i}, j \subset M$ are bilipschitz images such that for any $j$, $X_{i}$, $j \subset M$, then $\mu(C \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} X_{i}, j) = L$.

By way of contradiction, suppose that $L > 0$. Let $X_{1}, X_{2}, \ldots \subset M$ be bilipschitz images and let $S = C \setminus \bigcup_{i} X_{i}$ be such that $\mu(S) = L$. Let $p \in S$ be a point such that $S$ has density 1 at $p$. By Proposition 10 there is a bilipschitz image $Y$ that has positive density at $p$. Then

$$\mu(C \setminus (Y \cup \bigcup_{i} X_{i})) = \mu(S \setminus Y) < L,$$

which contradicts the minimality of $L$. Therefore, $L = 0$, and there is a sequence of bilipschitz images $X_{1}, X_{2}, \ldots \subset M$ such that

$$\mu(C \setminus \bigcup_{i=1}^{\infty} X_{i}) = 0,$$

as desired. Hence, one direction of Theorem 1 is proved.

5.2. **M is countably G-rectifiable ⇒ G is the tangent almost everywhere.** The forward direction of Theorem 1 follows from the work of Pansu.

Assume $M$ is countably G-rectifiable and of Hausdorff dimension $Q$. Since the measure $\mathcal{H}^{Q}$ is doubling then almost every point $p$ is a point of density 1 for the image $f(U)$ of a biLipschitz map $f : U \rightarrow M$, with $U \subset X$ measurable. Since $G$ and $M$ are doubling we can pass to tangents at $f^{-1}(p)$ and $p$ and the metric tangent of $M$ at $p$ equals the metric tangent of $F(U)$ at $p$, see [LD11]. Moreover, by Gromov compactness (or using ultralimits) we get some induced biLipschitz map from $G$ to Cone($M, p$), which may depends on rescalings of taking tangents. Since the space Cone($M, p$) is a Carnot group isomorphic to Nil($M, p$), Pansu's version [Pan89] of Rademacher differentiation implies that $G$ and Nil($M, p$) must be isomorphic as Lie groups.

6. **Examples**

6.1. **An example with no positive-measure set with constant tangent.** We give an example of an equiregular sub-Riemannian manifold on which the tangent is not constant on any set of positive measure. We thank Ben Warhurst for discussing this example with us.
Proposition 13. There exists a 7–dimensional equiregular sub-Riemannian manifold $M$ foliated by 6–dimensional manifolds with the property that if $p$ and $q$ are two points in different leaves, then the tangents $\text{Nil}(M, p)$ and $\text{Nil}(M, q)$ are not isomorphic as Lie groups.

Proof. Consider in $\mathbb{R}^7$ the rank-3 distribution spanned by

$$
\begin{align*}
X_1 &= \partial_1 \\
X_2 &= \partial_2 + x_1 \partial_4 - x_1 x_3 (-1 + x_1) \partial_7 \\
X_3 &= \partial_3 + x_2 \partial_5 - x_1 \partial_6 - x_1 x_2 x_1 \partial_7.
\end{align*}
$$

This is a bracket-generating equiregular distribution.

The nilpotentization at a point $(x_1, \ldots, x_7)$ is given by a stratified Lie algebra with basis $(E_1, \ldots, E_7)$ and non-trivial relations:

$$
\begin{align*}
[E_1, E_2] &= E_4 & [E_2, E_3] &= E_5 & [E_1, E_3] &= -E_6 \\
[E_1, E_5] &= -E_7 & [E_2, E_6] &= 2x_1 E_7 & [E_3, E_4] &= (1 - 2x_1)E_7 \\
[E_1, E_4] &= -2x_3 E_7 & [E_1, E_6] &= 2x_2 E_7
\end{align*}
$$

(15)

We consider the change of variables

$$
e_1 = E_1 + \alpha E_2 + \beta E_3, \quad e_2 = E_2, \quad e_3 = E_3,
$$

with $\alpha = -\frac{2x_2}{1 + 2x_1}$ and $\beta = \frac{x_3}{1 - 2x_1}$ so that

$$2x_2 + \alpha 2x_1 + \alpha = -2x_3 + \beta + \beta (1 - 2x_1) = 0.
$$

This change of variables proves that the nilpotentization (15) is isomorphic to the Lie algebra 147E, denoted by $g_{\xi}$, with parameter $\xi = 2x_1$. For $\xi \in (0, 1/2)$, such algebras are pairwise non-isomorphic, see Remark 14. Hence, the distribution above, restricted to the open set $\{0 < x_1 < 1/4\}$, has constant tangents on the $x_1$-hyperplanes, and on two such hyperplanes the tangent is different. □

Remark 14. In general, we know from the classification of nilpotent Lie algebras of dimension 7, see [Gon98], that the Lie algebras $g^\xi, \xi \in \mathbb{R}$, from the above proof are such that $g^\xi$ and $g^\eta$ are isomorphic if and only if

$$\eta \in \left\{ \xi, \frac{1}{\xi}, 1 - \xi, \frac{-1 + \xi}{\xi}, \frac{-1}{-1 + \xi}, \frac{\xi}{-1 + \xi} \right\}.
$$

Hence, one actually has that the distribution on $\mathbb{R}^7$ from the proof has that the tangent is not constant on any set of positive measure.

6.2. An example with constant tangent almost everywhere, but not everywhere.

We give an example of an equiregular sub-Riemannian manifold on which the tangent is constant everywhere, but it is not constant everywhere. We initially observe that a similar example that is not equiregular is given by the Martinet distribution in $\mathbb{R}^3$:

$$\partial_x, \quad \partial_y + x^2 \partial_z.$$
As a variation of this example, we shall consider the rank-4 distribution on $\mathbb{R}^5$, with coordinates $x_1, y_1, x_2, y_2, z$, defined by the vector fields
\[ X_1 := \partial_{x_1}, \quad Y_1 := \partial_{y_1} + x_1^2 \partial_z, \quad X_2 := \partial_{x_2}, \quad Y_2 := \partial_{y_2} + x_2 \partial_z. \]
The only non-trivial brackets of these four vector fields are
\[ [X_1, Y_1] = 2x_1 \partial_z, \quad [X_2, Y_2] = \partial_z. \]
This vector fields span the tangent at every point, so the structure is equiregular.

On a full-measure open set the metric tangent is the same Lie group. In fact, if $x_1 \neq 0$ then the nilpotentization is the second Heisenberg group, i.e., the basis $X_1, \frac{1}{x_1} Y_1, X_2, Y_2, \partial_z$ form a standard basis of such a group.

However, the nilpotentization is not the same at every point. Indeed, if $x_1 = 0$ then the nilpotentization is the direct product of $\mathbb{R}^2$ and the first Heisenberg group, i.e., the basis $X_1, Y_1, X_2, Y_2, \partial_z$ form a standard basis of such a group.

REFERENCES

[Bel96] André Bellaïche, *The tangent space in sub-Riemannian geometry*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 1–78.

[Chr90] Michael Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), no. 2, 601–628. MR 1096400

[Dav88] Guy David, *Morceaux de graphes lipschitziens et intégrales singulières sur une surface*, Rev. Mat. Iberoamericana 4 (1988), no. 1, 73–114. MR 1009120

[Fed69] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[Gon98] Ming-Peng Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and $\mathbb{R}$)*, ProQuest LLC, Ann Arbor, MI, 1998, Thesis (Ph.D.)–University of Waterloo (Canada). MR 2698220

[Har95] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR 1416564

[Jea14] Frédéric Jean, *Control of nonholonomic systems: from sub-Riemannian geometry to motion planning*, Springer Briefs in Mathematics, Springer, Cham, 2014.

[LD11] Enrico Le Donne, *Metric spaces with unique tangents*, Ann. Acad. Sci. Fenn. Math. 36 (2011), no. 2, 683–694.

[LD17] , *A Primer on Carnot Groups: Homogenous Groups, Carnot-Caratheodory Spaces, and Regularity of Their Isometries*, Anal. Geom. Metr. Spaces 5 (2017), 116–137.

[LOW14] Enrico Le Donne, Alessandro Ottazzi, and Ben Warhurst, *Ultrarigid tangents of sub-Riemannian nilpotent groups*, Ann. Inst. Fourier (Grenoble) 64 (2014), no. 6, 2265–2282.

[Pan89] Pierre Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) 129 (1989), no. 1, 1–60.

[Var81] A. N. Varčenko, *Obstructions to local equivalence of distributions*, Mat. Zametki 29 (1981), no. 6, 939–947, 957.

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