On the asymptotic Plateau problem for area minimizing surfaces in $\mathbb{E}(-1, \tau)$.

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Abstract

We prove some existence and non-existence results for complete area minimizing surfaces in the homogeneous space $\mathbb{E}(-1, \tau)$. As one of our main results, we present sufficient conditions for a curve $\Gamma$ in $\partial_\infty \mathbb{E}(-1, \tau)$ to admit a solution to the asymptotic Plateau problem, in the sense that there exists a complete area minimizing surface in $\mathbb{E}(-1, \tau)$ having $\Gamma$ as its asymptotic boundary.

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1 Introduction.

In the last few years the asymptotic Plateau problem in the homogeneous space $\mathbb{H}^2 \times \mathbb{R}$ has been actively studied. For instance, Nelli and Rosenberg [7] proved that for any given Jordan curve $\Gamma \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ that is a graph over $\partial_\infty \mathbb{H}^2$ there exists an entire minimal graph $\Sigma$ with $\Gamma$ as its asymptotic boundary; in particular, $\Sigma$ is area minimizing. Sa Earp and Toubiana [10] also considered the asymptotic Plateau problem in $\mathbb{H}^2 \times \mathbb{R}$ and they showed a general non existence result (see Theorem 2.1 in [10]) and got as a consequence that there is no complete properly immersed minimal surface whose asymptotic boundary is a Jordan curve homologous to zero in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ contained in an open slab between two horizontal circles of $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ with height equal to $\pi$.

Kloeckner and Mazzeo [5] worked with a more general class of curves in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ (considering different compactifications of the space) and got a good characterization of curves $\Gamma$ for which there exists a minimal surface that has $\Gamma$ as its asymptotic boundary (see, for instance, Proposition 4.4 and Theorem 4.5 in [5]).

For the Plateau problem involving two closed curves (not homotopically trivial) in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$, Ferrer, Martín, Mazzeo and Rodríguez [3] proved some existence and non existence results for minimal annuli having these two curves as the asymptotic boundary (see Theorem 1.2 and Theorem 5.1 in [3]).

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In addition to the aforementioned results, Coskunuzer [1] showed that for any tall curve (i.e., a curve with height greater than \( \pi \), see Definition 1.1 below) in \( \partial_\infty \mathbb{H}^2 \times \mathbb{R} \equiv \mathbb{S}^1 \times \mathbb{R} \), there exists an area minimizing surface with that curve as the asymptotic boundary. He also showed a non existence result for certain curves that are not tall. Here, we obtain similar results to the ones in [1] in the ambient space \( \mathbb{E}(-1, \tau) \), which is the total space of a fibration over \( \mathbb{H}^2 \) with bundle curvature \( \tau \). In particular, when \( \tau = 0 \), \( \mathbb{E}(-1, 0) \) is isometric to the Riemannian product \( \mathbb{H}^2 \times \mathbb{R} \), which allows us to reobtain and extend some of the results of [1].

Throughout this work, unless specified otherwise, we use the cylinder model for \( \mathbb{E}(-1, \tau) \). Specifically, let \( \mathbb{D} \) denote the unitary open disk in the complex plane and let, for \( \tau \in \mathbb{R} \), 
\[
\mathbb{E}(-1, \tau) = (\mathbb{D} \times \mathbb{R}, ds^2_\tau),
\]
for \( \lambda = \frac{2}{1-x^2-y^2} \). We consider the asymptotic boundary of \( \mathbb{E}(-1, \tau) \) as being induced by the product topology of \( \mathbb{D} \times \mathbb{R}, \partial_\infty \mathbb{E}(-1, \tau) = (\partial \mathbb{D}) \times \mathbb{R} = \mathbb{S}^1 \times \mathbb{R} \). Moreover, if \( \Sigma \) is a complete surface immersed in \( \mathbb{E}(-1, \tau) \) we define the asymptotic boundary of \( \Sigma \) as the set
\[
\partial_\infty \Sigma = \{(p, t) \in \mathbb{S}^1 \times \mathbb{R} \mid \exists (p_n, t_n)_{n \in \mathbb{N}} \subset \Sigma \text{ s.t. } (p_n, t_n) \to (p, t)\}.
\]

In order to state our main results, we next give the definition of height of a curve in \( \partial_\infty \mathbb{E}(-1, \tau) \). We notice that throughout the paper, curves will be assumed to be piecewise smooth and non-degenerate.

**Definition 1.1** (Height of a curve). Let \( \Gamma \) be a finite collection of pairwise disjoint simple closed curves in \( \partial_\infty \mathbb{E}(-1, \tau) \) and \( \Omega = \partial_\infty \mathbb{E}(-1, \tau) \setminus \Gamma \). For each \( p \in \mathbb{S}^1 \), let \( \ell_p = \{p\} \times \mathbb{R} \) denote the vertical line over \( p \) in \( \partial_\infty \mathbb{E}(-1, \tau) \) and let \( \ell_p^1, \ell_p^2, \ldots, \ell_p^{n_p} \) be the connected components of \( \Omega \cap \ell_p \). For \( i \in \{1, 2, \ldots, n_p\} \), let \( |\ell_p^i| \) denote the (possibly infinite) euclidean length of \( \ell_p^i \). Then, the height of \( \Gamma \) at \( p \) is 
\[
h_\Gamma(p) = \min_{i \in \{1, \ldots, n_p\}} |\ell_p^i|,
\]
and the height of \( \Gamma \) is
\[
h(\Gamma) = \inf_{p \in \mathbb{S}^1} h_\Gamma(p).
\]

**Remark 1.2.** As in \( \mathbb{H}^2 \times \mathbb{R} \), an isometry of \( \mathbb{H}^2 \) induces an isometry in \( \mathbb{E}(-1, \tau) \). Nevertheless, for \( \tau > 0 \), the induced isometry changes the \( t \)-coordinate, as observed in Proposition 2.1. Since this change is constant along any fiber, the vertical distance between two points in the same fiber is invariant under isometries. In particular, the definition of the height of a curve is well posed. Furthermore, differently from \( \mathbb{H}^2 \times \mathbb{R} \), there is no intrinsic notion of a height function in \( \mathbb{E}(-1, \tau) \), \( \tau > 0 \). An example that shows this dependence is the horizontal slice \( \{t = 0\} \subset \mathbb{E}(-1, \tau) \) in the half-plane model, which becomes (see (7)) a piece of a helicoid in the disk model: its height should be constant and equal to zero in the half plane model but it is not constant in the disk model. To avoid this ambiguity, throughout the paper the height of a curve in \( \partial_\infty \mathbb{E}(-1, \tau) \) is here defined for the cylinder model of \( \mathbb{E}(-1, \tau) \).

We next make precise the notion of a tall curve in \( \mathbb{E}(-1, \tau) \). Note that our definition differs slightly from the one introduced by [1, Definition 2.4], which allows us to treat a broader class of curves.
Definition 1.3. Let $\Gamma$ be a finite, pairwise disjoint collection of simple closed curves in $\partial_\infty \mathbb{E}(-1, \tau)$. We say that $\Gamma$ is a tall curve if $h_\Gamma(p) > \sqrt{1 + 4\tau^2}\pi$ for all $p \in \Gamma$. Otherwise, we say that $\Gamma$ is a short curve.

Our first main result is the following.

Theorem 1.4. Let $\Gamma \subset \partial_\infty \mathbb{E}(-1, \tau)$ be a finite collection of pairwise disjoint simple closed curves. If $\Gamma$ is tall, there exists a complete, possibly disconnected, area minimizing surface $\Sigma$ in $\mathbb{E}(-1, \tau)$ with $\partial_\infty \Sigma = \Gamma$.

Note that if $\Gamma \subset \partial_\infty \mathbb{E}(-1, \tau)$ is a short curve, then there exists a point $p \in \Gamma$ such that $h_\Gamma(p) \leq \sqrt{1 + 4\tau^2}\pi$. Concerning such curves, we expect that, at least for the case where there is an open arc $I \subset S^1$ such that $h_\Gamma(p) \leq \sqrt{1 + 4\tau^2}\pi$ for all $p \in I$, there is no area minimizing surface with asymptotic boundary $\Gamma$. However, this question is still open, even in the case of $\mathbb{H}^2 \times \mathbb{R}$. In the following result we are able to prove a special situation of this nonexistence result.

Theorem 1.5. Let $\Gamma$ be a short curve for which there exists an open arc $I \subset S^1$ where

$$h_\Gamma(p) < (\sqrt{1 + 4\tau^2} - 4|\tau|)\pi, \text{ for all } p \in I.$$  \hspace{1cm} (2)

Then, there is no area minimizing surface $\Sigma$ in $\mathbb{E}(-1, \tau)$ with $\partial_\infty \Sigma = \Gamma$.

Remark 1.6. In the case $\tau = 0$, Theorem 1.5 is equivalent to the nonexistence result of Coskunuzer [1]. When $\tau \neq 0$, it is not clear whether the bound assumed in (2) is sharp, and it only gives information for $|\tau| < \frac{1}{\sqrt{12}}$. This bound is necessary to our proof since isometries in $\mathbb{E}(-1, \tau)$ do not preserve the $t$-coordinate; see Proposition 2.1, Corollary 2.2 and Remark 2.3.

The organization of the paper is the following. In Section 2, we present some background material for the study of minimal surfaces in $\mathbb{E}(-1, \tau)$. In Section 3 we prove our main theorems; and in Section 4 we prove a technical fact used in the proof of Theorem 1.5.

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2 Preliminaries.

Let $\widetilde{SL}(2, \mathbb{R})$ denote the universal covering of the special linear group of $2 \times 2$ real matrices. For each $\tau \in \mathbb{R}$ there exists a left invariant metric $ds_\tau^2$ in $\widetilde{SL}(2, \mathbb{R})$ such that $(\widetilde{SL}(2, \mathbb{R}), ds_\tau^2) = \mathbb{E}(-1, \tau)$ becomes the total space of a Riemannian fibration over the hyperbolic plane $\mathbb{H}^2$ with bundle curvature $\tau$. Note that for any $\tau \in \mathbb{R}$ the group of isometries of $\mathbb{E}(-1, \tau)$ has dimension four (for a nice discussion about the $\mathbb{E}(\kappa, \tau)$ spaces, see Daniel [2]). A special case to be considered is when $\tau = 0$, where $\mathbb{E}(-1, 0)$ is isometric to the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$. In particular, all of our results also hold in $\mathbb{H}^2 \times \mathbb{R}$. We also note that for $\tau \neq 0$, the
spaces $\mathbb{E}(-1, \tau)$ and $\mathbb{E}(-1, -\tau)$ are isometric, hence it is without loss of generality that we assume that $\tau \geq 0$.

As stated in the Introduction, we use the cylinder model $(\mathbb{D} \times \mathbb{R}, ds^2)$ to $\mathbb{E}(-1, \tau)$, where $ds^2$ is given in (1). We also let $\pi_1 : \mathbb{E}(-1, \tau) \rightarrow \mathbb{D}$ and $\pi_2 : \mathbb{E}(-1, \tau) \rightarrow \mathbb{R}$ be the projections onto the first and the second coordinates, respectively.

The isometry group of $\mathbb{E}(-1, \tau)$ is generated by the lifts of the isometries of the disk model of $\mathbb{H}^2$, together with vertical translations along the fibers (see, for instance, Theorem 2.9 in [11]). Precisely, the following holds.

**Proposition 2.1.** The isometries of $\mathbb{E}(-1, \tau)$ are given by

$$F(z, t) = (f(z), t - 2\tau \arg f'(z) + c) \quad (3)$$

or

$$G(z, t) = (f(z), -t + 2\tau \arg f'(z) + c), \quad (4)$$

where $f$ is a positive isometry of the disk model of $\mathbb{H}^2$, $c \in \mathbb{R}$ and $\arg f' : \mathbb{H}^2 \rightarrow \mathbb{R}$ is a smooth angle function for $f'$.

One of the main difficulties that arises when working in the cylinder model of $\mathbb{E}(-1, \tau)$ when $\tau \neq 0$ is that isometries do not preserve the $t$-coordinate. The next result gives an upper bound to this gap on the $t$-coordinate for some isometries of $\mathbb{E}(-1, \tau)$; we make use of this bound in the proof of our non-existence result.

**Corollary 2.2.** For any positive isometry $f$ of the disk model of $\mathbb{H}^2$, there exists an isometry $F : \mathbb{E}(-1, \tau) \rightarrow \mathbb{E}(-1, \tau)$ such that the projections $\pi_1$ and $\pi_2$ satisfy, for all $z \in \mathbb{D}$ and $t \in \mathbb{R}$, that $\pi_1(F(z, t)) = f(z)$ and $|\pi_2(F(z, t)) - t| < 2\tau \pi$.

**Proof.** First, note that any positive isometry of the disk model of $\mathbb{H}^2$ can be represented by a M"obius transformation

$$f(z) = \frac{w_1 z - \overline{w_2}}{w_2 z - \overline{w_1}},$$

where $w_1, w_2 \in \mathbb{C}$ are such that $|w_1|^2 - |w_2|^2 = 1$. In particular, it holds that

$$f'(z) = \frac{-1}{(w_2 z - \overline{w_1})^2} = -\overline{w_2}^2 \frac{(\overline{z} - o)^2}{|w_2 z - \overline{w_1}|^4}, \quad (5)$$

where $o = \frac{w_1}{w_2}$. Note that $|o| = \frac{1}{|f'(0)|} > 1$, hence $f'(z) \neq 0$ for all $z \in \mathbb{D}$. For any $\theta_1 < \theta_2$, let

$$\Lambda_{\theta_1, \theta_2} = \{ re^{i\theta} \in \mathbb{C} \mid r > 0 \text{ and } \theta \in (\theta_1, \theta_2) \}.$$ 

We next analyze the image set $f'(\mathbb{D})$ to show that there exist $\theta_1 < \theta_2$ with $\theta_2 - \theta_1 < 2\pi$ such that $f'(\mathbb{D}) \subset \Lambda_{\theta_1, \theta_2}$.

Let $\tilde{\theta} \in [0, 2\pi)$ be such that $-\overline{w_2}^2 = |w_2|^2 e^{i\tilde{\theta}}$. Since multiplication by a positive constant does not change the argument of a complex number, it follows from (5) that

$$f'(z) \in \Lambda_{\theta_1, \theta_2} \iff (\overline{z} - o)^2 \in \Lambda_{\theta_1 - \tilde{\theta}, \theta_2 - \tilde{\theta}} \iff \overline{z} - o \in \Lambda_{\frac{\theta_1 - \tilde{\theta}}{2}, \frac{\theta_2 - \tilde{\theta}}{2}}.$$
Note that \( \{ \overline{z} - o \mid z \in \mathbb{D} \} \) is an open disk in \( \mathbb{C} \) with a positive distance \( |o| - 1 \) to the origin. Hence, there are \( \varphi_1 < \varphi_2 \) with \( \varphi_2 - \varphi_1 < \pi \) such that \( \{ \overline{z} - o \mid z \in \mathbb{D} \} \subset \Lambda_{\varphi_1, \varphi_2} \). After choosing, for \( i = 1, 2 \), \( \theta_i = 2\varphi_i + \theta \), it follows that \( f'(\mathbb{D}) \subset \Lambda_{\theta_1, \theta_2} \) with \( \theta_2 - 2\pi < \theta_1 < \theta_2 \).

This implies that we may choose a branch of the argument function such that for all \( z \in \mathbb{D} \), \( \arg(f'(z)) \in (\theta_1, \theta_2) \). After letting \( c = 2\tau(\theta_1 - \pi) \) in (3), the result follows. \( \square \)

**Remark 2.3.** The bound \( 2\tau\pi \) on Corollary 2.2 cannot, in general, be improved. Indeed, \( \sup_{z \in \mathbb{D}} |\pi_2(F(z, t)) - t| \) depends uniquely on \( |f(0)| \), as shown in the proof of Corollary 2.2. Moreover, if \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of isometries such that \( \lim_{n \to \infty} |f_n(0)| = 1 \), then the respective isometries \( F_n \) satisfy \( \lim_{n \to \infty} \sup_{z \in \mathbb{D}} |\pi_2(F_n(z, t)) - t| = 2\tau\pi \).

In the cylinder model to \( \mathbb{E}(-1, \tau) \), both horizontal planes \( \{ t = t_0 \} \) and vertical planes (i.e. the inverse image of a geodesic of \( \mathbb{H}^2 \) by \( \tau_1 \)) are minimal (in fact, they are area minimizing) surfaces. We next describe some other families of minimal surfaces in \( \mathbb{E}(-1, \tau) \) that will be used as barriers throughout this paper.

### 2.1 Rotational Catenoids

We first describe a one-parameter family of complete (without boundary) minimal annuli in \( \mathbb{E}(-1, \tau) \), which plays a key role in the proof of Theorem 1.5. Such a family was first obtained by B. Nelli and H. Rosenberg [7] for the case \( \tau = 0 \) and extended to the case where \( \tau \neq 0 \) by C. Penafiel [9]. Each surface in such family is called a catenoid of \( \mathbb{E}(-1, \tau) \) and is invariant under the group of isometries corresponding to rotations about the \( t \)-axis of the cylinder model.

Following the notation of [9], for any \( d > 0 \) let \( u_d : (\arcsinh(d), \infty) \to (0, \infty) \) be defined by

\[
    u_d(s) = \int_{\arcsinh(d)}^{s} d \sqrt{\frac{1 + 4\tau^2 \tanh^2(\frac{s}{2})}{\sinh^2(r) - d^2}} \, dr.
\]

Then, \( u_d \) extends continuously to \( s = \arcsinh(d) \) by setting \( u_d(\arcsinh(d)) = 0 \) and is strictly increasing. Moreover, there exists an increasing function \( d > 0 \mapsto h(d) \in (0, \frac{\pi}{2}\sqrt{1 + 4\tau^2}) \) such that, for each \( d > 0 \), \( \lim_{s \to \infty} u_d(s) = h(d) \). It also holds that

\[
    \lim_{d \to 0^+} h(d) = 0, \quad \lim_{d \to \infty} h(d) = \frac{\pi}{2}\sqrt{1 + 4\tau^2}.
\]

Using this notation, for each \( d > 0 \) the catenoid \( M_d \) given by [9, Propositions 3.6 and 3.9] is \( M_d = M_d^+ \cup M_d^- \), where \( M_d^+ \) and \( M_d^- \) are the rotational surfaces parameterized by

\[
    M_d^\pm = \{ (\tanh(r/2) \cos(\theta), \tanh(r/2) \sin(\theta), \pm u_d(r)) \mid r \in [\arcsinh(d), \infty), \theta \in [0, 2\pi) \}.
\]

It follows directly from its definition that the asymptotic boundary of \( M_d \) is the union of the two horizontal circles \( S^1 \times \{-h(d)\} \) and \( S^1 \times \{h(d)\} \).
2.2 Tall Rectangles

Here we will present some key properties of complete minimal planes in $E(-1, \tau)$ that are invariant under a one-parameter group of hyperbolic isometries. These surfaces are the so-called tall rectangles and were first described in the $\tau = 0$ case by Sa Earp and Toubiana [10] and extended, when $\tau \neq 0$, to the halfspace model for $E(-1, \tau)$ by Folha and Peñafiel [4]. In what follows, we describe this family in the cylinder model and prove that they are in fact area minimizing surfaces.

For a fixed $\tau \geq 0$, let $h > \pi \sqrt{1 + 4\tau^2}$ and $r \in (0, \pi)$ be given and let

$$\gamma_0 = \left\{ \left( -\cos(\theta), -\sin(\theta), 4\tau \arctan \left( \frac{\sin(\theta)}{1 + \cos(\theta)} \right) \right) | \theta \in [-r, r] \right\},$$

$$\gamma_1 = \left\{ \left( -\cos(\theta), -\sin(\theta), h + 4\tau \arctan \left( \frac{\sin(\theta)}{1 + \cos(\theta)} \right) \right) | \theta \in [-r, r] \right\}.$$

Using this notation, we next prove the following.

**Proposition 2.4.** There exists an area minimizing plane $R_h(r) \subset E(-1, \tau)$, invariant under a one-parameter group of hyperbolic isometries of $E(-1, \tau)$ and with asymptotic boundary given by the union of $\gamma_0, \gamma_1$ and the two vertical segments joining their endpoints (see Figure 1).

**Proof.** When $\tau = 0$, the result follows immediately from Proposition 2.1 of [10], hence we next assume that $\tau > 0$. We follow the notation of Folha and Peñafiel [4], where such tall rectangles were described. For the purpose of simplifying the computations, we start our proof in the half space model for $E(-1, \tau)$, i.e.,

$$E(-1, \tau) = \left\{ (x, y, t) \in \mathbb{R}^3 | y > 0 \right\},$$

where

$$ds^2_\tau = \frac{1}{y^2}(dx^2 + dy^2) + \left( -\frac{2}{y^2}dx + dt \right)^2.$$

In this model, Corollary 5.1 of [4] implies that for $h > \pi \sqrt{1 + 4\tau^2}$ and $m > 0$, there exists a minimal plane $S_h(m)$ with asymptotic boundary given by the rectangle at $\{ y = 0 \}$ with the four vertices $(-m, 0, 0), (-m, 0, h), (m, 0, 0)$ and $(m, 0, h)$, see Figure 1. Furthermore, $S_h(m)$ is invariant under the one-parameter subgroup of isometries of the half space model which is generated by the hyperbolic isometries of $\mathbb{H}^2$ that fix the points at infinity corresponding to the vertical segments of $\partial_\infty S_h(m)$.

Note that the family of hyperbolic translations $\{f_s(x, y, t) = (sx, sy, t)\}_{s > 0}$ are isometries of $ds^2_\tau$. Hence, the image surfaces $\{f_s(S_h(m))\}_{s > 0}$ give a foliation of the open slab $\{(x, y, t) | y > 0, 0 < t < h \}$ by minimal surfaces. This was proved by Lima [6, Lemma 6] and follows from the fact that for any $t_0 \in (0, h)$, the intersection $S_h(m) \cap \{ t = t_0 \}$ is a graphical arc of circle with endpoints $(-m, 0, t_0)$ and $(m, 0, t_0)$. Hence, it follows that each $S_h(m)$ is area minimizing.
Figure 1: $\partial_\infty S_h(m)$ is the rectangle in $\{y = 0\}$ with vertices as above, and $\partial_\infty R_h(r)$ is the image of $\partial_\infty S_h(m)$ by $\psi$.

To prove the existence of $R_h(r)$ as claimed, we just use an isometry between models as we next present. Let $z = x + iy$ be a complex coordinate system for $\mathbb{R}_2^+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and let $\phi : \mathbb{R}_2^+ \to \mathbb{D}$ be the Möbius transformation given by $\phi(z) = \frac{z - i}{z + i}$. Then,$$
abla x, y, t) = \psi(z, t) = \left(\phi(z), t + 4\tau \arctan \left(\frac{x}{y + 1}\right)\right) \quad (7)$$is an isometry between the two models of $E(-1, \tau), (\mathbb{R}_2^+ \times \mathbb{R}, ds^2_\tau)$ and $D \times \mathbb{R}, ds^2_\tau)$. For a given $r \in (0, \pi)$, take $m = \frac{\sin(r)}{1 + \cos(r)}$ and let $R_h(r) = \psi(S_h(m))$. It is straightforward to see that $R_h(r)$ has the asymptotic boundary as claimed.

The next result is a direct consequence of Proposition 2.4. We will make use of this result in the proof of Theorem 1.4.

**Corollary 2.5.** Given $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1 + \pi \sqrt{1 + 4\tau^2}$, there is $\delta > 0$ such that for any $\theta_1, \theta_2 \in S^1$ with $|\theta_1 - \theta_2| < \delta$ there exists an area minimizing surface $\mathcal{R} \subset \mathbb{D} \times (t_1, t_2)$ with $\partial_\infty \mathcal{R} \subset [\theta_1, \theta_2] \times (t_1, t_2)$.

**Proof.** This proof follows from the fact that rotations about the $t$-axis and vertical translations in $\mathbb{D} \times \mathbb{R}$ are isometries of $ds^2_\tau$, as we next explain. Let $t_1$ and $t_2$ be as stated and let$$h = \frac{t_2 - t_1 + \pi \sqrt{1 + 4\tau^2}}{2}, \quad \text{and} \quad \varepsilon = \frac{h - \pi \sqrt{1 + 4\tau^2}}{2}.$$Let $\delta > 0$ be such that for any $\theta \in (-\delta, \delta)$ it holds that $4\tau \left|\arctan \left(\frac{\sin(\theta)}{1 + \cos(\theta)}\right)\right| < \varepsilon$. Then, if $\theta_1, \theta_2 \in S^1$ are such that $|\theta_1 - \theta_2| < \delta$, we may vertically translate and rotate the surface $R_h(|\theta_2 - \theta_1|)$ to find $\mathcal{R}$ as claimed.

7
3 Existence and nonexistence results.

We next prove the main results of the paper. In Section 3.1 we prove that for any tall curve \( \Gamma \subset \partial_\infty \mathbb{E}(-1, \tau) \) there exists an area minimizing surface \( \Sigma \subset \mathbb{E}(-1, \tau) \) with \( \partial_\infty \Sigma = \Gamma \). In Section 3.2, we prove that for certain short curves \( \Gamma \) with \( h(\Gamma) < (\sqrt{1 + 4\tau^2} - 4\tau)\pi \) there is no area minimizing surface \( \Sigma \) with \( \partial_\infty \Sigma = \Gamma \).

Throughout this section, for any \( t \in \mathbb{R} \), we let \( P_t = \mathbb{D} \times \{t\} \) denote the horizontal plane at height \( t \) in \( \mathbb{E}(-1, \tau) \).

3.1 The proof of Theorem 1.4.

First, we prove the theorem when \( \Gamma \) is a finite union of disjoint parallel circles,

\[
\Gamma = \bigcup_{i=1,\ldots,n} \mathbb{S}^1 \times \{h_i\},
\]

where \( h_{i+1} - h_i > \pi \sqrt{1 + 4\tau^2} \) for all \( i \in \{1, \ldots, n-1\} \). Since each \( P_{h_i} \) separates and is area minimizing, it is enough to show that there is no connected minimal surface in \( \mathbb{E}(-1, \tau) \) with asymptotic boundary \( \Gamma \) as above when \( n \geq 2 \).

Suppose to the contrary that there is a connected minimal surface \( \Sigma \) in \( \mathbb{E}(-1, \tau) \) with \( \partial_\infty \Sigma = \Gamma \). For two consecutive \( h_i < h_{i+1} \), let \( \delta > 0 \) be such that \( (h_{i+1} - \delta) - (h_i + \delta) > \pi \sqrt{1 + 4\tau^2} \) and let \( \hat{H}_i = h_i + \delta \) and \( H_{i+1} = h_{i+1} - \delta \). Let \( U = \mathbb{D} \times [H_i, H_{i+1}] \) be the slab bounded by the planes \( P_{H_i} \) and \( P_{H_{i+1}} \). Then \( U \cap \Sigma \) is a compact minimal surface that admits a connected component \( \hat{\Sigma} \) with \( \partial \hat{\Sigma} \subset P_{H_i} \cup P_{H_{i+1}}, \partial \hat{\Sigma} \cap P_{H_i} \neq \emptyset \) and \( \partial \hat{\Sigma} \cap P_{H_{i+1}} \neq \emptyset \).

Let \( \{M_d\}_{d>0} \) be the family of rotational catenoids of \( \mathbb{E}(-1, \tau) \) given in Section 2.1, vertically translated so that for all \( d > 0 \), \( \partial_\infty M_d \subset \mathbb{S}^1 \times (H_i, H_{i+1}) \). To obtain a contradiction, we now just recall that when \( d \) goes to infinity, the surfaces \( M_d \) escape from any compact, and when \( d \) approaches zero, they converge (away from the origin, with multiplicity two) to a horizontal plane. In particular, there must be a first contact point between \( \hat{\Sigma} \) and some \( M_d \), which is a contradiction by the maximum principle.

Hence, we next proceed with the proof of Theorem 1.4 with the additional assumption that \( \Gamma \) is not a family of parallel circles.

Proof of Theorem 1.4. We start the proof by setting up the notation. For each \( n \in \mathbb{N} \) and \( h \in \mathbb{R} \), let \( D_n(h) \) be the disk in the horizontal plane \( P_h \) centered at the origin and with euclidean radius \( \tanh(n) \). In particular, the family \( \{D_n(h)\}_{n \in \mathbb{N}} \) gives an exhaustion of \( P_h \). Also, for a given \( T > 0 \), let \( \Delta_n(T) = \bigcup_{-T \leq h \leq T} D_n(h) \) be a compact solid cylinder in \( \mathbb{E}(-1, \tau) \). Since both horizontal planes and vertical planes over complete geodesics are minimal surfaces in the metric of \( \mathbb{E}(-1, \tau) \), \( \Delta_n(T) \) is mean convex for all \( n \in \mathbb{N} \) and \( T > 0 \).

Let \( \Gamma \) be a tall curve in \( \partial_\infty \mathbb{E}(-1, \tau) \) and let \( T > 0 \) be such that \( \Gamma \) is contained in the open slab \( \mathbb{S}^1 \times (-T, T) \) of \( \partial_\infty \mathbb{E}(-1, \tau) \). For each \( n \in \mathbb{N} \), let \( \Gamma_n \subset \partial \Delta_n(T) \) be the radial projection of \( \Gamma \) in \( \partial \Delta_n(T) \). Since \( \Delta_n(T) \) is mean convex and \( \Gamma_n \) is an embedded, piecewise smooth curve in \( \partial \Delta_n(T) \), there exists an embedded, possibly disconnected, area minimizing surface \( \Sigma_n \subset \Delta_n(T) \) with \( \partial \Sigma_n = \Gamma_n \). Our next argument is to show that when \( n \to \infty \), then,
up to a subsequence, \( \Sigma_n \) converges to a nonempty complete surface \( \Sigma \subset \mathbb{E}(-1, \tau) \) such that \( \partial_\infty \Sigma = \Gamma \).

Since each \( \Sigma_n \) is area minimizing, the number of connected components of \( \Sigma_n \) is uniformly bounded by the number of connected components of \( \Gamma_n \), which is equal to the number of components of \( \Gamma \). In particular, we may pass to a subsequence to assume that there exists some \( k \in \mathbb{N} \) such that the number of connected components of each \( \Sigma_n \) is \( k \), and we let \( \Sigma_1, \ldots, \Sigma_k \) denote such components, labeled in such a way that for each \( i \in \{1, 2, \ldots, k\} \) the radial projection of \( \partial \Sigma_i \) to \( \partial_\infty \mathbb{E}(-1, \tau) \) correspond to the same component of \( \Gamma \) for all \( n \in \mathbb{N} \). In particular, we just need to prove the result when \( k = 1 \), since the general case follows from a finite diagonal argument. Hence, from now on we will assume that \( \Sigma_n \) is connected, for all \( n \in \mathbb{N} \).

Let \( \Omega = \partial_\infty \mathbb{E}(-1, \tau) \setminus \Gamma \). Since \( \Gamma \) is tall, Corollary 2.5 gives that for any \( q \in \Omega \) there exists a tall rectangle \( R_q \) such that \( \partial_\infty \mathcal{R}_q \) is disjoint from \( \Gamma \) and separates \( q \) from \( \Gamma \) in \( \partial_\infty \mathbb{E}(-1, \tau) \). Let \( U_q \subset \mathbb{E}(-1, \tau) \) be the region defined by \( \mathcal{R}_q \) in \( \mathbb{E}(-1, \tau) \) such that \( q \in \partial_\infty U_q \).

We claim that \( \Sigma_n \cap U_q = \emptyset \), for all \( n \) sufficiently large. In the topology of \( \overline{\mathcal{B}} \times \mathbb{R}, \overline{U_q} \) and \( \Gamma \) are two disjoint compact sets, and the sequence \( \Gamma_n \) converges to \( \Gamma \). Hence there is \( n(q) > 0 \) such that for all \( n \geq n(q) \), \( \Gamma_n \cap \overline{U_q} = \emptyset \), from where it follows that \( \Gamma_n \cap U_q = \emptyset \) in \( \mathbb{E}(-1, \tau) \).

To prove the claim, we argue by contradiction and assume that \( \Sigma_n \) intersects \( U_q \) for some \( n \geq n(q) \). Now, a standard replacement argument yields a contradiction. In fact, since \( \Gamma_n = \partial \Sigma_n \) does not intersect \( U_q \), then \( S = \Sigma_n \cap \overline{U_q} \) is a compact smooth surface with boundary in \( \partial U_q = \mathcal{R}_q \). Since \( \mathcal{R}_q \) is a topological plane, there exists a compact subdomain \( \hat{S} \subset \mathcal{R}_q \) with \( \partial \hat{S} = \partial S \). Then, from the fact that both \( \Sigma_n \) and \( \mathcal{R}_q \) are area minimizing, we obtain that \( \text{Area}(S) = \text{Area}(\hat{S}) \). In particular, the compact surface defined by

\[
\Sigma'_n = (\Sigma_n \setminus S) \cup \hat{S}
\]

is a nonsmooth area minimizing surface, a contradiction.

Next, we use the fact proved above to show that the sequence \( (\Sigma_n)_{n \in \mathbb{N}} \) admits a limit point in \( \mathbb{E}(-1, \tau) \); in other words, the surfaces \( \Sigma_n \) do not escape to infinity. Since \( \Gamma \) is not a finite collection of parallel circles, there exists a horizontal plane \( P_h \) in \( \mathbb{E}(-1, \tau) \) such that \( \partial_\infty P_h \) intersects \( \Gamma \) transversely at some point \( p \). Hence, we may choose points \( p_1, p_2 \in \partial_\infty P_h \) that bound a closed arc \( [p_1, p_2] \subset \partial_\infty P_h \) containing \( p \) in its interior and such that \( [p_1, p_2] \cap \Gamma = \{p\} \), see Figure 2 (a). Let \( \gamma \) be a complete arc in \( P_h \) with endpoints \( p_1, p_2 \) and let \( A \subset P_h \) be the region bounded by \( \gamma \) in \( P_h \) that contains \( p \) in its asymptotic boundary. Since \( \Gamma_n \) converges to \( \Gamma \), it follows that \( \Gamma_n \) intersects \( A \) transversely and only in one point, for all \( n \) sufficiently large.

The above argument shows that, when we consider \( [p_1, p_2] \cup \gamma \) as a simple closed curve in \( \overline{\mathcal{B}} \times \mathbb{R} \), the linking number between \( [p_1, p_2] \cup \gamma \) and \( \Gamma_n \) is one, for all \( n \) sufficiently large. In particular, since \( \partial \Sigma_n = \Gamma_n \), there must be a point \( q_n \in \gamma \cap \Sigma_n \).

Let \( U_1 = U_{p_1} \) and \( U_2 = U_{p_2} \) be the respective regions bounded by two tall rectangles \( \mathcal{R}_{p_1} \) and \( \mathcal{R}_{p_2} \) as before. Then, for \( n \) sufficiently large, \( \Sigma_n \cap (U_1 \cup U_2) = \emptyset \). Since \( \gamma \setminus (U_1 \cup U_2) \) is compact, the sequence \( \{q_n\}_{n \in \mathbb{N}} \) admits a convergent subsequence, and then the surfaces
Figure 2: (a) shows the plane $P_h$ intersecting $\Gamma$ transversely at $p$ and the arc $[p_1,p_2] \subset \partial_\infty P_h$. In (b) we have the arc $\gamma \subset P_h$ and, highlighted, the region $A$.

$\Sigma_n$ do not escape to infinity. In particular, after passing to a subsequence, it follows that $\Sigma_n$ converges (in the $C^{2,\alpha}$ topology on compacts of $\mathbb{E}(-1,\tau)$) to a complete, area minimizing surface $\Sigma \subset \mathbb{E}(-1,\tau)$.

It remains to prove that $\partial_\infty \Sigma = \Gamma$. First, note that the fact that for any $p \in \Omega$ there exists $n(p) \in \mathbb{N}$ such that $\Sigma_n \cap U_p = \emptyset$ for all $n \geq n(p)$ gives immediately that $\partial_\infty \Sigma \subset \Gamma$. Next, we show that given $p \in \Gamma$, then $p \in \partial_\infty \Sigma$. First, assume that there is a plane $P_h \subset \mathbb{E}(-1,\tau)$ such that $\partial_\infty P_h$ intersects $\Gamma$ transversely at $p$. Take a sequence of arcs $\gamma_n \subset P_h$ (each $\gamma_n$ resembles the arc $\gamma$ in Figure 2 (b)) such that the endpoints of $\gamma_n$ determine arcs in $\partial_\infty P_h$ that intersect $\Gamma$ uniquely at $p$ and such that the respective regions $A_n \subset P_h$ bounded by $\gamma_n$ satisfy that $A_{n+1} \subset A_n$ and that $\bigcap_{n \in \mathbb{N}} \overline{A_n} = \{p\}$. The same arguments as above give that for all $n \in \mathbb{N}$ there exists a point $q_n \in \Sigma \cap \gamma_n$, from where it follows that $p = \lim_{n \to \infty} q_n \in \partial_\infty \Sigma$. Since the above argument is purely topological, we notice that the general case when the $t$-coordinate of $\Gamma$ has a local extremal value at $p$ can be treated in a similar manner, by considering a vertical plane instead of a horizontal one, and this finishes the proof of Theorem 1.4.

### 3.2 The proof of Theorem 1.5.

In this section, we prove our nonexistence result stated as Theorem 1.5 in the Introduction. The proof follows the ideas contained in Step 2 of the proof of Theorem 2.13 in [1], with a few changes and necessary adaptations to the $\tau \neq 0$ setting. A key step in the proof of our result is, when $\tau \neq 0$, to show the existence of a compact, connected area minimizing surface in $\mathbb{E}(-1,\tau)$ with boundary contained in two parallel planes that are sufficiently far from each other. This is stated in Proposition 3.1 below and is proved in Section 4, since the arguments used in its proof are technical.
To what follows, for each $t \in \mathbb{R}$ and $r > 0$, we let

$$C_t(r) = \{(\tanh(r) \cos(u), \tanh(r) \sin(u), t) \mid u \in [0, 2\pi]\}$$

be the circle in the horizontal plane $P_t$ (with coordinates given by the open disk $\mathbb{D}$) centered at the origin with euclidean radius $\tanh(r) \in (0, 1)$.

**Proposition 3.1.** For any $h \in (0, \frac{\pi}{2}\sqrt{1+4\tau^2})$ there exist $R > 0$ and a compact, connected, area minimizing surface $S(h) \subset \mathbb{E}(-1, \tau)$ such that

$$\partial S(h) = C_h(R) \cup C_{-h}(R).$$

Assuming Proposition 3.1, we now proceed to the proof of Theorem 1.5.

**Proof of Theorem 1.5.** Arguing by contradiction, let us assume that $\Gamma$ is a curve as stated and that $\Sigma$ is a complete, connected, area minimizing surface in $\mathbb{E}(-1, \tau)$ such that $\partial_{\infty, \Sigma} = \Gamma$. Since $\Sigma$ is area minimizing, then $\Sigma$ is properly embedded. In particular, $\Sigma$ is orientable and the fact that $\Sigma$ is connected implies that it separates $\mathbb{E}(-1, \tau)$ into two connected open regions $E_1, E_2$.

The asymptotic boundaries of $E_1$ and $E_2$ intersect along $\Gamma$ and their union is the whole $\partial_{\infty, \mathbb{E}}(-1, \tau)$. In particular, if we let $\Omega_1 = \text{int}(\partial_{\infty} E_1)$ and $\Omega_2 = \text{int}(\partial_{\infty} E_2)$, it follows that $\partial_{\infty, \mathbb{E}}(-1, \tau) \setminus \Gamma = \Omega_1 \cup \Omega_2$.

After rotating $\Sigma$ about the $t$-axis and performing a vertical translation, the assumptions over $\Gamma$ imply that there exist $\delta > 0$ and $T \in (0, \frac{\pi}{2}\sqrt{1+4\tau^2} - 4\tau^2)$ such that, for all $\theta \in (-\delta, \delta)$ the vertical segment $\gamma_{\theta} = \{e^{i\theta} \times [-T, T]\}$ intersects $\Gamma$ transversely, in exactly two points, and both points are interior to $\gamma_{\theta}$. We may also reindex to assume, without loss of generality, that $I_\delta \times \{-T, T\} \subset \Omega_1$, where $I_\delta = \{e^{i\theta} \in \mathbb{S}^1 \mid \theta \in (-\delta, \delta)\}$; see Figure 3 (a).

Let $V_1, V_2$ be open sets in $\mathbb{E}(-1, \tau)$ such that $V_1, V_2 \subset \Omega_1$ and that $I_\delta \times \{-T\} \subset \text{int}(\partial_{\infty} V_1)$, $I_\delta \times \{-T\} \subset \text{int}(\partial_{\infty} V_2)$. Also, let $V_3$ be another open set in $\mathbb{E}(-1, \tau)$ such that $V_3 \subset E_2$ and that $\partial_{\infty} V_3 \subset \Omega_2$ with $(1, 0) \in \text{int}(\partial_{\infty} V_3)$. For instance, we could take $V_1, V_2, V_3$ as sufficiently small neighborhoods of $I_\delta \times \{-T\}$, $I_\delta \times \{-T\}$ and of $(1, 0)$, respectively (see Figure 3 (b)).

Let, for $i = 1, 2, 3$, $U_i = \pi_1(V_i) \subset \mathbb{D}$. Then $U = U_1 \cap U_2 \cap U_3$ is an open set of $\mathbb{D}$ that contains 1 in the interior of its asymptotic boundary. Let $V = U \times (-T, T) \subset \mathbb{E}(-1, \tau)$ (see Figure 3 (c)).

From equation (2), we may choose $h < \frac{\pi}{2}\sqrt{1+4\tau^2}$ such that $T < h - 2\tau \pi$, and then Proposition 3.1 implies the existence of $R > 0$ and of a connected, area minimizing surface $S = S(h)$ with boundary $C_h(R) \cup C_{-h}(R)$. As before, let $D_t(R)$ denote the disk in $P_t$ centered at the origin and with euclidean radius $\tanh(r)$. By the maximum principle using horizontal and vertical planes, we know that $S \subset \bigcup_{t \in [-h, h]} D_t(R)$. Furthermore, $\hat{S} = S \cup D_{-h}(R) \cup D_{-h}(R)$ is a connected, embedded, compact surface in $\mathbb{E}(-1, \tau)$; then $\hat{S}$ separates $\mathbb{E}(-1, \tau)$ and defines a unique bounded region $A \subset \mathbb{E}(-1, \tau)$ with $\partial A = \hat{S}$.

For $r > 0$, let $\varphi : \mathbb{D} \to \mathbb{D}$ be the hyperbolic isometry of $\mathbb{H}^2$ that translates along the geodesic given by the real axis and maps 0 to $\tanh(r)$, and let $\phi : \mathbb{E}(-1, \tau) \rightarrow \mathbb{E}(-1, \tau)$
be its related isometry of $\mathbb{E}(-1, \tau)$ given by Corollary 2.2. Then, for all $z \in \mathbb{D}$ and $t \in \mathbb{R}$, \( \pi_1(\phi_r(z, t)) = \varphi_r(z) \) and
\[
|\pi_2(\phi_r(z, t)) - t| < 2\tau_\pi. \tag{8}
\]

Let $S_r = \phi_r(S)$ and $A_r = \phi_r(A)$. Then $S_r$ is an area minimizing surface of $\mathbb{E}(-1, \tau)$ contained in the boundary of the region $A_r$. Let $D(R) = \pi_1(D_t(R))$ for $t \in \mathbb{R}$. Since, when $r \to \infty$, $\varphi_r(D(R))$ is collapsing into the point at infinity $z = 1$ and $U$ is an open set which contains 1 in the interior of its asymptotic boundary, there exists $r_0 > 0$ such that for all $r \geq r_0$, $\pi_1(A_r) \subset U$.

Note that (8), together with the fact that $h - 2\tau_\pi > T$, gives that $\partial A_{r_0}$ intersects both regions $\mathbb{D} \times (-\infty, -T)$ and $\mathbb{D} \times (T, +\infty)$. In particular, since $\pi_1(A_{r_0}) \subset U$ and $A_{r_0}$ is connected, there exists a connected component $A_0$ of $A_{r_0} \cap V$ with boundary intersecting both $P_T \cap V_1$ and $P_{-T} \cap V_2$. In particular, $A_0$ intersects $E_1$.

On the other hand, $V_3 \cap V$ separates $V$ into two connected components that intersect $A_0$, and then $V_3 \cap A_0 \neq \emptyset$, from where it follows that $A_0$ also intersects $E_2$ and that $A_0 \setminus \Sigma$ contains a compact connected component with boundary contained in $S_{r_0} \cup \Sigma$. Now, a standard replacement argument using that both $S_{r_0}$ and $\Sigma$ are area minimizing produces a nonsmooth area minimizing surface, which is a contradiction that proves Theorem 1.5. \( \Box \)

4 The proof of Proposition 3.1.

The proof of Proposition 3.1, which follows the ideas presented in [1, Lemma 7.1], will be carried out along this section. For $d > 0$, let $M_d$ be the rotational catenoid introduced in Section 2.1. The main idea here is to show that for any $h \in (0, \frac{\pi}{2} \sqrt{1 + 4\tau^2})$ and $d$ sufficiently
large, the surface

\[ M^h_d = M_d \cap (\mathbb{D} \times [-h, h]) \]

has less area than the union of the two disks in the parallel planes \( P_{-h}, P_h \) which share a boundary component with \( M^h_d \). Hence, the area minimizing surface with boundary \( \partial M^h_d \) is necessarily connected. Note that \( M^h_d \) is compact if and only if \( h < h(d) \), which is equivalent to the existence of \( R > \arcsinh(d) \) such that \( u_d(R) = h \). For convenience, for given \( d > 0 \) and \( R > \arcsinh(d) \), we define

\[ M_d(R) = M^u_d(R) = M_d \cap \{ -u_d(R) \leq t \leq u_d(R) \}. \]

Our first result is an area estimate for \( M_d(R) \) for large values of \( d \).

**Lemma 4.1.** There exists \( d_0 > 0 \) such that for any \( d \geq d_0 \) and \( R > \arcsinh(d + 1) \) it holds that

\[ \text{Area}(M_d(R)) < 2\pi \sqrt{1 + 4 \tau^2} \left( \sqrt{e^{2R} - 2 - 4d^2 + 1} \right). \]

**Proof.** Let \( u : [a, b] \to \mathbb{R} \) be a smooth function and let \( \Sigma \) be the rotational surface in \( \mathbb{E}(-1, \tau) \) parameterized by

\[ \Sigma = \{ (\tanh(r/2) \cos(\theta), \tanh(r/2) \sin(\theta), u(r)) \mid r \in [a, b], \theta \in [0, 2\pi) \}. \]

Then, a straightforward computation implies that the area of \( \Sigma \) is given by

\[ \text{Area}(\Sigma) = 2\pi \int_a^b \sinh(s) \sqrt{(u'(s))^2 + 1 + 4\tau^2 \tanh^2(s/2)} ds. \]  

(9)

Hence, it follows from (9) and from (6) that

\[ \text{Area}(M_d(R)) = 4\pi \int_{\arcsinh(d)}^{R} \sinh^2(s) \sqrt{1 + 4\tau^2 \tanh^2(s/2) \cosh^2(s) - 1 - d^2} ds. \]

In particular, since \( |\tanh(x)| < 1 \) for all \( x \in \mathbb{R} \), we obtain that

\[ \text{Area}(M_d(R)) < 4\pi \sqrt{1 + 4\tau^2} \int_{\arcsinh(d)}^{R} \sinh^2(s) \sqrt{\cosh^2(s) - 1 - d^2} ds. \]  

(10)

The next argument presents an adequate estimate to the integral in (10), which we will denote by \( I \). Under the assumption that \( R > \arcsinh(d + 1) \), we may write \( I = I_1 + I_2 \) where

\[ I_1 = \int_{\arcsinh(d)}^{\arcsinh(d+1)} \frac{\sinh^2(s)}{\sqrt{\cosh^2(s) - 1 - d^2}} ds, \quad I_2 = \int_{\arcsinh(d+1)}^{R} \frac{\sinh^2(s)}{\sqrt{\cosh^2(s) - 1 - d^2}} ds. \]

To estimate \( I_1 \), we first use that \( s < \arcsinh(d + 1) \), obtaining

\[ I_1 \leq (d + 1) \int_{\arcsinh(d)}^{\arcsinh(d+1)} \frac{\sinh(s)}{\sqrt{\cosh^2(s) - 1 - d^2}} ds. \]
Next, we use the change of variables $u = \cosh(s)$, the identity $\cosh(\arcsinh(x)) = \sqrt{1 + x^2}$ and the fact that for any $a \in \mathbb{R}$ the function $\log(\sqrt{x^2 - a + x})$ is a primitive to $\frac{1}{\sqrt{x^2 - a}}$ to obtain that

$$I_1 \leq (d + 1) \log \left( \frac{\sqrt{1 + 2d + \sqrt{2 + 2d + d^2}}}{\sqrt{1 + d^2}} \right)$$

$$\leq (d + 1) \log \left( \frac{\sqrt{1 + 2d + \sqrt{2 + 2d + d^2}}}{d} \right).$$  \hspace{1cm} (11)

In order to estimate $I_2$, we use the inequalities $\sinh(x) < \frac{e^x}{2}$ and $\cosh^2(x) - 1 > \frac{e^{2x} - 2}{4}$, which hold for all $x \in \mathbb{R}$, so that

$$I_2 < \int_{\arcsinh(d+1)}^{R} \frac{e^{2s}}{2\sqrt{e^{2s} - 2 - 4d^2}} ds.$$

Since $\frac{e^{2s}}{2\sqrt{e^{2s} - 2 - 4d^2}}$ is a primitive to $\frac{\sqrt{e^{2s} - 2 - 4d^2}}{2}$ and $\arcsinh(x) = \log(\sqrt{x^2 + 1} + x)$, we obtain that

$$I_2 < \frac{\sqrt{e^{2R} - 2 - 4d^2} - \sqrt{(\sqrt{2 + 2d + d^2 + d + 1})^2 - 2 - 4d^2}}{2}$$

$$\leq \frac{\sqrt{e^{2R} - 2 - 4d^2} - \sqrt{2(d + 1)\sqrt{1 + 2d + d^2 + 1} + 4d - 2d^2}}{2}$$

$$= \frac{\sqrt{e^{2R} - 2 - 4d^2} - \sqrt{8d + 3}}{2}.$$  \hspace{1cm} (12)

Using (10), (11) and (12) we obtain that, for any $d > 0$ and $R > \arcsinh(d + 1)$,

$$\frac{\text{Area}(M_d(R))}{4\pi \sqrt{1 + 4\tau^2}} < (d + 1) \log \left( \frac{\sqrt{1 + 2d + \sqrt{2 + 2d + d^2}}}{d} \right)$$

$$+ \frac{\sqrt{e^{2R} - 2 - 4d^2} - \sqrt{8d + 3}}{2}.$$

Since

$$\lim_{d \to \infty} (d + 1) \log \left( \frac{\sqrt{1 + 2d + \sqrt{2 + 2d + d^2}}}{d} \right) - \frac{\sqrt{8d + 3}}{2} = 0,$$

the lemma follows.

Let $D_1(R)$ and $D_2(R)$ be the respective disks in the horizontal planes $P_{u_d(R)}$ and $P_{-u_d(R)}$ such that $\partial(D_1(R) \cup D_2(R)) = \partial M_d(R)$. Since vertical translations are isometries of $E(-1, \tau)$, it follows that $\text{Area}(D_1(R) \cup D_2(R)) = 2\text{Area}(D_1(R))$. Using this fact, we prove the next result, which compares the area of $M_d(R)$ with the area of $D_1(R) \cup D_2(R)$ for $d$ and $R$ sufficiently large.
**Lemma 4.2.** Let \( R(d) = \frac{3}{2} \log(d) \). Then, there exists \( d_1 > 0 \) such that for all \( d \geq d_1 \) it holds that
\[
2 \text{Area}(D_1(R(d))) > 2\pi \sqrt{1 + 4\tau^2} (\sqrt{d^3} - 4 - \sqrt{d}). \tag{13}
\]
In particular, there exists \( \tilde{d} > 0 \) such that for all \( d \geq \tilde{d} \) it holds that
\[
\text{Area}(D_1(R(d)) \cup D_2(R(d))) > \text{Area}(M_d(R)). \tag{14}
\]

**Proof.** For any \( R > 0 \), we have that
\[
2 \text{Area}(D_1(R)) = 4\pi \sqrt{1 + 4\tau^2} \int_0^R \sinh(s) \sqrt{\cosh(s) + \frac{1 - 4\tau^2}{1 + 4\tau^2}} ds.
\]
In order to see this, just apply (9) for the function \( u \equiv 0 \), after using the identity \( \tanh^2 \left( \frac{a}{2} \right) = \frac{\cosh(a) - 1}{\cosh(a) + 1} \). Thus, if we denote
\[
\tilde{I} = \int_0^R \sinh(s) \sqrt{\cosh(s) + \frac{1 - 4\tau^2}{1 + 4\tau^2}} ds,
\]
it holds that
\[
2 \text{Area}(D_1(R)) = 4\pi \sqrt{1 + 4\tau^2} \tilde{I}. \tag{15}
\]
Our next arguments estimate \( \tilde{I} \) from below.

First, use the substitution \( u = \cosh(s) \) to obtain that
\[
\tilde{I} = \int_1^{\cosh(R)} \sqrt{\frac{u + \frac{1 - 4\tau^2}{1 + 4\tau^2}}{u + 1}} du.
\]
Using that for any \( a \in \mathbb{R} \),
\[
\frac{d}{du} \left( \sqrt{(u + a)(u + 1)} + (a - 1) \log \left( \sqrt{u + a} + \sqrt{u + 1} \right) \right) = \frac{u + a}{u + 1},
\]
we may compute \( \tilde{I} \) in terms of \( R \) as follows
\[
\tilde{I} = \sqrt{(\cosh(R) + \frac{1 - 4\tau^2}{1 + 4\tau^2})(\cosh(R) + 1) - \frac{2}{\sqrt{1 + 4\tau^2}} - \frac{8\tau^2}{1 + 4\tau^2} \log \left( \frac{\sqrt{\cosh(R) + \frac{1 - 4\tau^2}{1 + 4\tau^2}} + \sqrt{\cosh(R) + 1}}{\sqrt{\frac{2}{1 + 4\tau^2}} + \sqrt{2}} \right)}.
\]
(15)

Since \( \frac{e^R}{2} < \cosh(R) < \frac{e^{R+1}}{2} \) and \(-1 < \frac{1 - 4\tau^2}{1 + 4\tau^2} < 1\), it follows from (15) that
\[
\tilde{I} > \frac{1}{2} \sqrt{e^{2R} - 4} - \frac{2}{\sqrt{1 + 4\tau^2}} - \frac{8\tau^2}{1 + 4\tau^2} \log \left( \frac{\sqrt{\frac{e^{R+1}}{2} + 1} + \sqrt{\frac{e^{R+1}}{2}} + 1}{\sqrt{\frac{2}{1 + 4\tau^2}} + \sqrt{2}} \right).
\]
Assuming that $R$ is large enough so $3 < e^R$ and setting
\[ c_1 = \frac{4\tau^2}{1 + 4\tau^2}, \quad c_2 = \frac{2}{\sqrt{1 + 4\tau^2}} + \frac{8\tau^2}{1 + 4\tau^2} \log \left( \frac{\sqrt{2}\sqrt{1 + 4\tau^2}}{1 + \sqrt{1 + 4\tau^2}} \right), \]
we obtain that
\[ \tilde{I} > \frac{1}{2} \sqrt{e^{2R}} - 4 - c_1 R - c_2. \]
In particular, for $R(d) = \frac{3}{2} \log(d)$, it holds that
\[ 2\text{Area}(D_1(R(d))) > 2\pi \sqrt{1 + 4\tau^2} \left( \sqrt{d^3} - 4 - 3c_1 \log(d) - 2c_2 \right), \]
for all $d$ sufficiently large. To conclude the proof that (13) holds, we just notice that there exists $d_1 > 0$ large enough so that for all $d \geq d_1$, it holds that $3c_1 \log(d) + 2c_2 < \sqrt{d}$.

Now, to finish the proof of the lemma it remains to show that there exists some $\tilde{d}$ such that (14) holds for all $d \geq \tilde{d}$. We first observe that $\lim_{d \to \infty} \frac{R(d)}{\text{arcsinh}(d+1)} = \frac{3}{2}$, hence there exists some $d_2 > 0$ such that for any $d \geq d_2$ it holds that $R(d) > \text{arcsinh}(d+1)$. Without loss of generality, we may assume that $d_2 \geq d_0$, where $d_0$ is defined in Lemma 4.1. In particular, for all $d \geq d_2$
\[ \text{Area}(M_d(R(d))) < 2\pi \sqrt{1 + 4\tau^2} \left( \sqrt{d^3} - 2 - 4d^2 + 1 \right). \tag{16} \]

Now, we just use the fact that
\[ \lim_{d \to \infty} \left( \sqrt{d^3} - 4 - \sqrt{\tilde{d}} - \sqrt{d^3} - 2 - 4d^2 - 1 \right) = \infty \]
to obtain some $\tilde{d} \geq \max\{d_0, d_1, d_2\}$ such that for all $d \geq \tilde{d}$
\[ \sqrt{d^3} - 2 - 4d^2 + 1 < \sqrt{d^3} - 4 - \sqrt{\tilde{d}} \]
and we can use (13) and (16) to finish the proof of the lemma.

At this point, we know that for $d$ sufficiently large and $R(d) = \frac{3}{2} \log(d)$, the area of $M_d(R)$ is less than the area of the two horizontal disks with the same boundary as $M_d(R)$. In particular, any area minimizing surface of $\mathbb{E}(-1, \tau)$ with boundary $\partial M_d(R)$ will be connected, since the unique disconnected minimal surface with such boundary is $D_1(R) \cup D_2(R)$.

Our next result shows that for any $h \in \left(0, \frac{\pi}{2} \sqrt{1 + 4\tau^2}\right)$ there exists $d \geq \tilde{d}$ so that $M_d(R(d)) = M^h_d$, thereby completing the proof of Proposition 3.1.

**Lemma 4.3.** For $R(d) = \frac{3}{2} \log(d)$, it holds that
\[ \lim_{d \to \infty} u_d(R(d)) = \frac{\pi}{2} \sqrt{1 + 4\tau^2}. \]
Proof. For a given $d > 0$ let $\lambda_d: (\text{arcsinh}(d), \infty) \to \mathbb{R}$ be defined by

$$\lambda_d(s) = \int_{\text{arcsinh}(d)}^{s} \frac{d}{\sqrt{\sinh^2(r) - d^2}} dr. \tag{17}$$

It is immediate to obtain that for all $R > \text{arcsinh}(d)$ we have the inequality

$$u_d(R) > \sqrt{1 + 4\tau^2 \tanh^2(\text{arcsinh}(d)/2)} \lambda_d(R).$$

Since the results from Peñaﬁel [9, Proposition 3.9] imply that $u_d(R) < \frac{\pi}{2} \sqrt{1 + 4\tau^2}$ and

$$\lim_{d \to \infty} \tanh^2(\text{arcsinh}(d)/2) = 1,$$

in order to prove the lemma it suffices to show that

$$\lim_{d \to \infty} \lambda_d(R(d)) = \frac{\pi}{2}. \tag{18}$$

Equation (18) is precisely Lemma 7.3 of [1], but, for the sake of completeness, we present its proof here.

We start with the change of variables used in Proposition 5.2 of [8], $t = \text{arccosh}\left(\frac{\cosh(r)}{\sqrt{1 + d^2}}\right)$, so that (17) becomes

$$\lambda_d(s) = \int_0^{\text{arccosh}\left(\frac{\cosh(s)}{\sqrt{1 + d^2}}\right)} \frac{d}{\sqrt{(1 + d^2) \cosh^2(t) - 1}} dt.$$

In particular, if we let $\rho(s) = \text{arccosh}(\sqrt{1 + d^2} \cosh(s))$, it follows that

$$\lambda_d(\rho(s)) = \int_0^{s} \frac{d}{\sqrt{(1 + d^2) \cosh^2(t) - 1}} dt.$$

Let $s(d)$ be defined by $\rho(s(d)) = R(d)$. Then

$$\lim_{d \to \infty} \lambda_d(R(d)) = \lim_{d \to \infty} \int_0^{s(d)} \frac{d}{\sqrt{(1 + d^2) \cosh^2(t) - 1}} dt. \tag{19}$$

For any $t > 0$ it holds that

$$\frac{d}{\sqrt{1 + d^2} \cosh(t)} \leq \frac{d}{\sqrt{(1 + d^2) \cosh^2(t) - 1}} \leq \frac{1}{\cosh(t)},$$

and then

$$\frac{d}{\sqrt{1 + d^2}} \int_0^{s(d)} \frac{1}{\cosh(t)} dt \leq \lambda_d(R(d)) \leq \int_0^{s(d)} \frac{1}{\cosh(t)} dt. \tag{20}$$
We note that \( s(d) = \text{arccosh} \left( \frac{d^3 + 1}{2\sqrt{d^2d^2 + 1}} \right) \); in particular, \( \lim_{d \to \infty} s(d) = +\infty \). On the other hand, \( \int_0^{s(d)} \frac{1}{\cosh(t)} \, dt = 2 \arctan \left( \tanh \left( \frac{s(d)}{2} \right) \right) \) from where it follows that both lower and upper bounds on (20) converge to \( \pi / 2 \), as \( d \to \infty \). This, together with (19), implies that

\[
\lim_{d \to \infty} \lambda_d(R(d)) = \frac{\pi}{2},
\]

which proves the lemma. As already explained, this also finishes the proof of Proposition 3.1.

\( \square \)

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