simple game induced manifolds

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Abstract. Starting by a simple game \(Q\) as a combinatorial data, we build up a cell complex \(M(Q)\), whose construction resembles combinatorics of the permutohedron. The cell complex proves to be a combinatorial manifold; we call it the \textit{simple game induced manifold}. By some motivations coming from polygonal linkages, we think of \(Q\) and of \(M(Q)\) as of a \textit{quasilinkage} and the \textit{moduli space of the quasilinkage} respectively. We present some examples of quasilinkages and show that the moduli space retains many properties of moduli space of polygonal linkages. In particular, we show that the moduli space \(M(Q)\) is homeomorphic to the space of stable point configurations on \(S^1\), for an associated with a quasilinkage notion of stability.

Polygonal linkage, simple game, permutohedron, cell complex, configuration space

1. Introduction

It is a usual praxis that some combinatorial data produce a geometric object. Classical examples are \textit{permutohedron}, \textit{associahedron} (see [13]), other “famous” polytopes, and their generalizations \textit{graph-associahedra} and \textit{nestohedra} (see [9]). In the paper, we act in a somewhat similar way starting by a \textit{simple game} \(M(Q)\) (in the usual sense of game theory) as a combinatorial data. We build up a cell complex \(M(Q)\), whose construction although resembles very much the combinatorics of the permutohedron, yet depends on the simple game. The cell complex proves to be a combinatorial manifold, which we call the \textit{simple game induced manifold}.

The idea is borrowed from the cell decomposition of the moduli space of polygonal linkages (see [8]). This motivates us to treat a simple game \(Q\) as a \textit{quasilinkage} since it provides a natural generalization of polygonal linkages. By the same reason, we call the cell complex \(M(Q)\) the \textit{moduli space of the quasilinkage}. The paper presents the basic study of the simple game generated manifolds.
**Polygonal linkages: definitions and overview of the results.** Given a vector \( L = (l_1, ..., l_n) \in \mathbb{R}^n_+ \) of \( n \) positive real numbers, consider \( n \) rigid bars of lengths \( l_1, ..., l_n \) joined in a closed chain. Such a construction is called a polygonal linkage. By \( M(L) \) we denote its moduli space, or the space of planar configurations:

\[
M(L) := \{ z_1, ..., z_n \in \mathbb{R}^2 : |z_i| = 1, \sum l_i z_i = 0 \}/SO(2)
\]

\[= \{ z_1, ..., z_n \in \mathbb{R}^2 : |z_i| = 1, \sum l_i z_i = 0, z_1 = 1 \}.
\]

Denote by \([n]\) the set \( \{1, ..., n\} \).

**Definition 1.** The length vector \( L \) is called **generic**, if there is no subset \( J \subset [n] \) such that

\[
\sum_{i \in J} l_i = \sum_{i \notin J} l_i.
\]

Throughout the paper, we consider only generic length vectors \( L \).

The hyperplanes

\[
\sum_{i \in J} l_i = \sum_{i \notin J} l_i
\]

called **walls** subdivide \( \mathbb{R}^n_+ \) into a collection of **chambers**.

Here is a (far from complete) summary of facts about \( M(L) \):

- For a generic length vector, \( M(L) \) is a smooth manifold [6].
- The topological type of \( M(L) \) depends only on the chamber of \( L \) [6].
- As it was shown in [8], \( M(L) \) admits a structure of a regular cell complex. The combinatorics is very much related (but not equal) to the combinatorics of the permutahedron. The construction will be explained in details in Section 3.

**Definition 2.** For a generic length vector \( L \), a subset \( J \subset [n] \) is called **long**, if

\[
\sum_{i \in J} l_i > \sum_{i \notin J} l_i.
\]

Otherwise, \( J \) is called **short**. The set of all short sets we denote by \( S(L) \).

- Homology groups of \( M(L) \) are free abelian groups. For a generic length vector \( L \), the rank of the homology group \( H_k(M(L)) \) equals \( a_k + a_{n-3-k} \), where \( a_i \) is the number of short subsets of size \( i + 1 \) containing the longest edge (see [5]).

We stress that the manifold \( M(L) \) (considered either as a topological manifold, or as a cell complex) is uniquely defined by the collection of short subsets of \([n]\).
Quasilinkages. The following definition generalizes Definition 2.

**Definition 3.** A family $Q$ of subsets of $[n]$ is called a *quasilinkage*, if it satisfies the following properties:

1. **Q contains all singletons**: for any $i \in [n]$, $\{i\} \in Q$.
2. **Monotonicity**: if $S \in Q$, and $T \subset S$ then $T \in Q$.
3. **Strong complementarity**: if $S \in Q$ then $([n] \setminus S) \notin Q$, and, conversely, if $S \notin Q$, then $([n] \setminus S) \in Q$.

The proposed notion exists in the literature; yet in completely different frameworks. It appeared as “simple game with constant sum” in game theory, see [12, 4] and also as “strongly complementary simplicial complex”, see [2, 3].

Following the aforementioned motivation by polygonal linkages, we call any $S \in Q$ a *Q-short set*, or simply a *short set*, and any $S \notin Q$ a *long set*.

**Remark 4.** Each polygonal linkage $L$ yields a quasilinkage by the above defined short sets family $S(L)$ (see Definition 1).

**Definition 5.** A quasilinkage $Q$ is called *real*, if there exists a length vector $L$ such that $Q = S(L)$ (see Definition 1). Otherwise, $Q$ is called *imaginary*.

Here we list some additional properties that are true for real quasilinkages, but in general may not hold for imaginary ones:

1. **Comparability**: For any $A, B \in 2^{[n]}$, and any $i, j \notin A \cup B$, if $A \cup i$ is long, $A \cup j$ is short and $B \cup i$ is short then $B \cup j$ is also short. The property means that the edge $i$ is in a sense “longer” than $j$.

2. **Trade robustness**: Given $k$ long subsets, there is no interchanging of the elements of these sets, which makes all of them short.

There arises a natural question: given a family of subsets $Q$, under what conditions there exists a length vector $L$ such that $Q = S(L)$? This question has been studied a lot in game theory. The family of subsets with the monotonicity property is called a *simple game*, and if there exists a corresponding length vector, then this family is called a *weighted majority game*. In [11] it was shown, that a simple game is a weighted majority game if and only if it satisfies the trade robustness condition. Other characterizations of weighted majority games are given in, for example, [12, 4].

In our terminology, the trade robustness condition guarantees that a given quasilinkage is real.

**Main results.** We start with small examples of imaginary quasilinkages. Next, we give two ways of cooking up quasilinkages: the *flip technique* and the *conflict-free family extensions* (Section 2). This implies that the class of all quasilinkages is much wider than the class of all linkages. Yet more examples arise via oriented matroid approach in Section 5.
In Section 3 we associate with a quasilinkage $Q$ a cell complex $CWM(Q)$ by applying the rules from [8]. We prove that $CWM(Q)$ is locally isomorphic to $CWM(L)$ for some real linkage $L$ (however, $L$ depends on the location, and there may be no real linkage associated to the entire complex). As a corollary, we immediately see that $CWM(Q)$ is a manifold of dimension $n - 3$.

In Section 4 we show that the manifold $CWM(Q)$ is homeomorphic to the moduli space of stable point configurations on $S^1$ for an appropriate definition of stability.

2. Imaginary quasilinkages

2.1. Small symmetric examples and non-examples. Elementary case analysis shows that for $n \leq 5$ there are no imaginary quasilinkages. However, for $n \geq 6$ there are many. We start with some symmetric examples of imaginary quasilinkages in low dimensions.

Definition 6. We say that a quasilinkage $Q$ is symmetric if for any $i, j \in [n]$ there exists an element $\sigma$ of the symmetric group $S_n$ such that:

1. $\sigma$ takes $i$ to $j$, and
2. $\sigma$ takes short sets to short sets. (Equivalently, if $\sigma$ takes long sets to long sets.)

Example 7. [12] Let $n = 6$. A symmetric quasilinkage is defined by the following rules:

1. All 2-element sets are short. (Equivalently, all 4-element sets are long.)
2. The only ten short 3-element subsets are:

   123, 124, 135, 146, 156, 236, 245, 256, 345, 346.

We give another example for $n = 7$, which is also symmetric:

Example 8. [12] A symmetric quasilinkage for $n = 7$ is defined as follows:

1. All 2-element subsets are short.
2. The only seven 3-element long subsets are:

   123, 145, 167, 257, 246, 347, 356.

Example 8 actually corresponds to Fano plane, and its automorphism group is known to be transitive, so this example is again symmetric.

Example 7 corresponds to the 6-vertex triangulation of projective 2-plane, and can be generalized as vertex-minimal triangulation of projective space only in dimensions 4, 8, 16, see [2, 3].
Lemma 9. (1) If \( n \) is odd, there exists exactly one symmetric real linkage. It assigns equal lengths to all the edges. Equivalently, a set is short whenever its size is smaller than \( n/2 \).

(2) If \( n \) is even, there exists no symmetric real linkage.

Proof. Fix any symmetric real quasilinkage \( Q \) with length vector \( L \). For \( j \in [n] \) and \( k \in \mathbb{N} \), denote by \( a_k(j) \) the number of short subsets of size \( k + 1 \) containing \( j \). By symmetry assumption, \( a_k(j) \) does not depend on \( j \). Now assume that \( l_i < l_j \) for some \( i, j \in [n] \). Take a set \( A \subseteq [n] \) such that \( i, j \notin A \). If \( A \cup j \) is short, then \( A \cup i \) is also short. If \( A \cup i \) is short and \( A \cup j \) is long, then \( a_{|A|(i)} > a_{|A|(j)} \), which contradicts the symmetry assumption. Therefore \( A \cup j \) is short if and only if \( A \cup i \) is short for any \( i, j \in [n] \). This means that for any \( k \), all the \( k \)-element subsets of \( [n] \) are either simultaneously short or simultaneously long. This immediately implies the result of the lemma.

Lemma 10. (1) If \( n \) is odd, there exists exactly one symmetric real linkage. It assigns equal lengths to all the edges. Equivalently, a set is short whenever its size is smaller than \( n/2 \).

(2) If \( n \) is even, there exists no symmetric real linkage.

Proof. Fix any symmetric real quasilinkage \( Q \) with length vector \( L \). For \( j \in [n] \) and \( k \in \mathbb{N} \), denote by \( a_k(j) \) the number of short subsets of size \( k + 1 \) containing \( j \). By symmetry assumption, \( a_k(j) \) does not depend on \( j \). Now assume that \( l_i < l_j \) for some \( i, j \in [n] \). Take a set \( A \subseteq [n] \) such that \( i, j \notin A \). If \( A \cup j \) is short, then \( A \cup i \) is also short. If \( A \cup i \) is short and \( A \cup j \) is long, then \( a_{|A|(i)} > a_{|A|(j)} \), which contradicts the symmetry assumption. Therefore \( A \cup j \) is short if and only if \( A \cup i \) is short for any \( i, j \in [n] \). This means that for any \( k \), all the \( k \)-element subsets of \( [n] \) are either simultaneously short or simultaneously long. This immediately implies the result of the lemma.

Corollary 11. Examples 7 and 8 present imaginary quasilinkages.

Proposition 12. For \( n = 8 \), there is no symmetric quasilinkage (neither real, no imaginary).

Proof. There are \( \binom{8}{4} = 70 \) four-element subsets of \( [n] \). For any quasilinkage, exactly 35 of them are long, and 35 of them are short. By symmetry, any of the 8 elements of \( [n] \) should be contained in the same number of short 4-element subsets, therefore \( 35 \cdot 4 \) should be divisible by 8, but it is not.
Flips.

**Definition 13.** Let $Q$ be a quasilinkage, and let $T$ be a maximal (by inclusion) subset of $[n]$ such that $T \in Q$. Define the flip $F_T(Q)$ as follows:

$$F_T(Q) := (Q \setminus \{T\}) \cup \{([n] \setminus T)\}$$

In other words, a flip is an operation that makes the $Q$-short set $T$ long, and its complement short, leaving all the other sets unchanged.

**Proposition 14.** $F_T(Q)$ is again a quasilinkage.

*Proof.* The strong complementarity property obviously holds for $F_T(Q)$, so it remains to check monotonicity for $F_T(Q)$. Assume that $S \subset S' \subset [n]$, and $S' \in F_T(Q)$. We need to prove that $S \in F_T(Q)$. If $S' \neq T := ([n] \setminus T)$ then every proper subset of $S'$ is $Q$-short and is not equal to $T$ by maximality, so the only remaining case is $S' = T$. But every proper subset of $T$ is $Q$-short, again, by maximality of $T$, so the proposition is proven. □

**Example 15.** Take the length vector $L = (l_1, ..., l_6)$ with

$$l_1 = l_2 = l_3 = 1 + \varepsilon, \quad l_4 = l_5 = l_6 = 1.$$ 

It corresponds to a real quasilinkage $S(L)$. Now take the (maximal short) set $T = \{4, 5, 6\}$ and make a flip $Q := F_T(S(L))$. This quasilinkage is imaginary, because it violates the comparability condition: $\{4, 5, 6\}$ is $Q$-long, while $\{1, 5, 6\}$ is $Q$-short, so 4 must be longer than 1, but, from the other hand, $\{1, 3, 5\}$ is $Q$-long, while $\{4, 3, 5\}$ is $Q$-short.

This example differs from Example 7. One more example of an imaginary quasilinkage arises from the below proposition.

**Proposition 16.** Any flip of an imaginary quasilinkage $Q$ from Example 7 is again imaginary.

*Proof.* Because of the total symmetry of $Q$, it does not matter what set we will choose to be flipped, so we can choose $T := \{1, 2, 3\}$. But the quasilinkage $G := F_T(Q)$ still violates the comparability condition: the sets $\{1, 2, 4\}$ and $\{3, 4, 5\}$ are $G$-short while the sets $\{3, 2, 4\}$ and $\{1, 4, 5\}$ are $G$-long, so 1 and 3 are not comparable. □

**Proposition 17.** For a fixed $n$, any two $n$-quasilinkages are connected by a sequence of flips.

*Proof.* Take an arbitrary quasilinkage $Q$, and take any maximal short set $T \subset [n]$ such that $1 \in T$. Apply the flip $F_T(Q)$, take any other maximal short set containing 1, and make it long by another flip, and so on. After a finite number of steps we get a quasilinkage $Q'$ such that the set $S$ is $Q'$-long if and only if it contains 1. This quasilinkage corresponds to the real quasilinkage $S(L)$ for the length vector $L = (1, \varepsilon, \varepsilon, ..., \varepsilon)$. □
Conflict-free family extensions.

**Definition 18.** A family $\mathcal{G}$ of subsets of $[n]$ is called *conflict-free*, if for any $T, S \in \mathcal{G}$ it is true that $([n] \setminus T) \nsubseteq S$.

A conflict-free family represents our partial knowledge about which sets are short and which sets are long, and every short set doesn’t contain any long subsets.

A subset $S \subset [n]$ is called $\mathcal{G}$-*unknown* if neither $S$, nor its complement $\overline{S}$ is contained in an element of $\mathcal{G}$.

**Lemma 19.** Any conflict-free family of subsets $\mathcal{G}$ extends to a quasilinkage, i.e., there exists a quasilinkage $Q$ such that $\mathcal{G} \subset Q$.

**Proof.** Let $S$ be some $\mathcal{G}$-unknown subset. Then $\mathcal{G}' := \mathcal{G} \cup \{S\}$ is again conflict-free. This means that we can add $\mathcal{G}'$-unknown subsets one by one until unknown subsets exist. Finally, we arrive at a conflict-free family of subsets $\mathcal{G}'$, with no $\mathcal{G}'$-unknown subsets. It is a desired quasilinkage. □

So, now we have a way of constructing imaginary quasilinkages:

1. Start with some small conflict-free family $\mathcal{G}$, which cannot be incorporated into any real linkage. For instance, one can take a set violating the comparability property.
2. Add one by one all $\mathcal{G}$-unknowns.
3. The result will be automatically an imaginary quasilinkage.

For example, let $n = 6$, and let $\mathcal{G} = \{123, 356, 245, 146\}$. If these subsets are short, then the subsets 124 and 235 are long, whereas 123, 245 are short. It is a conflict-free family which doesn’t satisfy the comparability property (therefore, imaginary).

**Definition 20.** (Freezing for quasilinkages) Assume that $S_1, \ldots, S_k$ is a (non-ordered) partition of $[n]$ into $k$ non-empty short sets. We build a new quasilinkage $\text{FREEZE}(Q)$ on the set $[k]$ by the rule:

$$J \subset [k] \text{ is short iff } \bigcup_{i \in J} S_i \text{ is short.}$$

3. Moduli space of a quasilinkage

**Cell structure on the moduli space of a real linkage: a reminder.** Fix a generic length vector $L$. We remind that to describe a regular cell complex, it suffices to list all the (closed) cells ranged by dimension, and to describe incidence relations for closed cells.

**Definition 21.** A cyclically ordered partition $S_1, \ldots, S_k$ of $[n]$ into $k$ non-empty subsets is called *admissible*, if every $S_i$, $1 \leq i \leq k$, is short.
Theorem 22. The below described cell complex $CWM^*(L)$ is a combinatorial manifold homeomorphic to the moduli space $M(L)$.

1. The $k$-cell of the complex are labeled by (all possible) admissible cyclically ordered partition of $[n]$ into $(n-k)$ non-empty subsets. Given a cell $C$, its label is denoted by $\lambda(C)$.
2. A closed cell $C$ belongs to the boundary of another closed cell $C'$ whenever the label $\lambda(C')$ is finer than the label $\lambda(C)$.

We stress that the complex $CWM^*(L)$ depends only on the family of short subsets $S(L)$. This hints that this construction can be extended to quasilinkages.

Cell complex associated to a quasilinkage. Assume that a quasilinkage $Q$ is fixed. Although the notion of (planar) configurations has no sense, we can literally repeat the construction of the above cell complex.

Definition 23. A cyclically ordered partition $S_1, \ldots, S_k$ of $[n]$ into $k$ non-empty subsets is called $Q$-admissible, if every $S_i, 1 \leq i \leq k$, is $Q$-short.

Definition 24. For a quasilinkage $Q$ it’s moduli space $M(Q)$ is the cell complex defined as follows:

1. The $k$-cell of the complex are labeled by (all possible) admissible cyclically ordered partition of $[n]$ into $(n-k)$ non-empty subsets. Given a cell $C$, its label is denoted by $\lambda(C)$.
2. A closed cell $C$ belongs to the boundary of another closed cell $C'$ whenever the label $\lambda(C')$ is finer than $\lambda(C)$.

The complex is a combinatorial manifold, which is locally isomorphic to the complex $CWM^*(L)$ of some real linkage.

Theorem 25. (1) For every vertex $v$ of cell complex $M(Q)$, there exists a length vector $L_v$ such that the star of the vertex $v$ is combinatorially isomorphic to the star of some vertex of $CWM^*(L)$.
(2) For every cell $\sigma$ of cell complex $M(Q)$, there exists a length vector $L_\sigma$ such that the star of the cell $\sigma$ is combinatorially isomorphic to the star of some vertex of $CWM^*(L_\sigma)$.
(3) For every quasilinkage $Q$, the complex $M(Q)$ is a combinatorial manifold.

Proof. (1) Fix a vertex $v$ of $M(Q)$. By construction, it is labeled by some $Q$-admissible cyclically ordered partition of $[n]$ into $n$ short non-empty subsets, that is, by a cyclic ordering on $[n]$. Without loss of generality we may assume that $v$ is labeled by the partition

$$\lambda(v) = \{1\}, \{2\}, \ldots, \{n\}.$$
The partition $p$ should be viewed as numbers $1, \ldots, n$ placed on the circle counterclockwise.

We need the following observation: let $\sigma$ be a $k$-cell of $M(Q)$ labeled by a partition $\lambda = S_1, \ldots, S_{n-k}$. Then $\sigma$ is incident to $v$ if and only if each of the sets $S_i$ is of the form $\{a, a+1, \ldots, a+b\}$ for some natural numbers $a$ and $b$ (the sums are taken modulo $n$). It is true because otherwise the partition $\lambda(v)$ would not be a refinement of $S$. Let us call the sets of the form $\{a, a+1, \ldots, a+b\}$ the segments of the partition $\lambda(v)$.

Now (1) follows from the lemma:

**Lemma 26.** In the above notation, there exists a length vector $L_v$ (depending on the vertex $v$) such that for any segment $T$ of the partition $\lambda(v)$, the set $T$ is $Q$-short if and only if $T$ is $L_v$-short.

**Proof of the lemma.**

To construct such a length vector, we will need some additional observations. Recall that $\lambda(v)$ is viewed as numbers $1, \ldots, n$ placed on the circle. There are $n$ ways to break the circle into a line: $1, 2, \ldots, n$, $2, 3, \ldots, n, 1, \ldots, n, 1, 2, \ldots, n-1$. We will call such way a separator position. Take a separator position $s$, for example, $2, \ldots, n, 1$. There exists a unique number $q = q(s) \in [n]$, such that the set $\{2, 3, \ldots, q-1\}$ is short, and the set $\{2, 3, \ldots, q\}$ is long. We analogously define $q(s)$ for all separators $s'$.

We are now ready to define the length vector. For any $j \in [n]$ put $l_j := 1 + |q^{-1}(j)|$. Equivalently speaking,

$$l_j := 1 + \frac{1}{2} |\{S \subset [n] : S \text{ is a short segment of } p; S \cup \{j\} \text{ is a long segment of } p\}|.$$  

It is clear that the total length of all edges is always equal to $2n$. We need to prove that the segment $S$ of $p$ is short iff $\sum_{j \in S} l_j < n$. Note that $\sum_{j \in S} l_j = |S| + |q^{-1}(S)|$.

Take arbitrary short segment $S$ of $p$. If $s$ is a separator position adjacent to some element of $S$ (there are $|S| + 1$ such separator positions), then it is obvious that $q(s) \notin S$. Therefore $|q^{-1}(S)| \leq n - |S| - 1$, because the total number of separator positions equals to $n$. So for short segment $S$ of $p$ we conclude that $\sum_{j \in S} l_j = |S| + |q^{-1}(S)| \leq n - 1$. Lemma is proven. □

(2) The star of a cell can be reduced to the case (1) by freezing technique. Indeed, for a cell $\sigma$ labeled by $\lambda(\sigma) = S_1, S_2, \ldots, S_k$, we freeze all the entries in each of the sets $S_i$, and arrive at a quasilinkage on the set $[k]$.

(3) follows directly from (1), (2), and Theorem 22 □

The below construction gives an analysis of the vertex stars of the complex $M(Q)$.

Assume that a quasilinkage $Q$ and a vertex $v$ of $M(Q)$ are fixed. Theorem 25 assigns to $v$ a length vector $L_v = (l_1, \ldots, l_n)$. Without loss of generality
we may assume that $l_1 + \ldots + l_n = 2\pi$ and that $v$ is labeled by the cyclical ordering $\lambda(v) = (1, 2, \ldots, n)$.

Decompose the (metric) circle $S^1$ centered at the origin $0$ into a union of arches of lengths $l_1, \ldots, l_n$. The endpoints of the arches give the Gale diagram (see [13]) of some convex polytope $K = K(F, v) \subset \mathbb{R}^{n-3}$.

**Proposition 27.** The star of the vertex $v$ is combinatorially dual to the above defined convex polytope $K$.

**Proof.** The vertices of $K$ correspond to partitions of $[n]$ into $n - 1$ short subsets, and, equivalently, to the short pairs of the form $(i, i + 1)$ (this pair is represented by the vector $u_i$. By a property of Gale diagrams, the vertices of the set $I \subset [n]$ form a facet if and only if the convex hull $\text{conv}(\{u_i | i \in ([n] \setminus I)\}$ contains the origin 0 the in its relative interior. This means that the angle between every two succeeding vectors of the set $([n] \setminus I)$ is smaller than $\pi$. Let the indices $i_1, i_2 \notin I$ be such that for any $i_1 < i < i_2$, we have $i \in I$. Then the angle between $u_i$ and $u_{i_2}$ is equal to the sum $\sum_{i_1 < i \leq i_2} l_i$. So the vertices of the set $I$ form a facet if and only if $I$ gives a refinement of partition $\lambda(v)$ into short subsets. This corresponds to the cell incident to $v$, which completes the proof of the proposition. □

**Theorem 28.** For any quasilinkage $Q$, the complex $M(Q)$ admits a PL structure.

**Proof.** We refer the reader to literally the same proof of the analogous theorem for real linkages from [8]. In short, each cell is combinatorially equivalent to a Cartesian product of permutohedra. We metrically realize each of the cells by the Cartesian product of standard permutohedra. Then the gluing map is an isometry. □

The next proposition gives us information about what happens to the moduli space of after a flip (see subsection 2.1).

**Proposition 29.** Let $Q$ be a quasilinkage, and let $T$ be any maximal $Q$-short subset of $[n]$. Then the moduli space of the flipped quasilinkage $M(F_T(Q))$ differs from $M(Q)$ by a Morse surgery of index $(n - |T| - 1)$.

**Proof.** Consider the cell complex $M(Q)$. The flip deletes from the complex some of the cells and adds some new cells. Assume that a cell labeled by some partition $S = (S_1, \ldots, S_k)$ gets deleted. This means that $T \subseteq S_i$ for some $i$. Since $T$ is a maximal $Q$-short set, we have $T = S_i$. Therefore, all the $(n-k)$-cells which are deleted during the flip are labeled by all possible partitions of type $(T, S_1, S_2, \ldots, S_{k-1})$. Thus we arrive at the cell structure of the boundary of the permutohedron (see [13]) $\Pi_{n-|T|} \subset \mathbb{R}^{n-|T|-1}$ multiplied by a disk. The cell structure of $M(Q)$ converts this disk to the permutohedron $\Pi_{|T|}$. So,
we cut out a cell subcomplex \((\partial \Pi_{n-|T|}) \times \Pi_{|T|}\) and then we patch instead the cell complex \(\Pi_{n-|T|} \times \partial \Pi_{|T|}\) along the identity mapping on their common boundary \(\partial \Pi_{n-|T|} \times \partial \Pi_{|T|}\). This operation is the Morse surgery of index \((n - |T| - 1)\).

\[\square\]

Remark 30. Propositions 29 and 17 give an alternative proof of Theorem 25.

4. Stable point configurations

There is an important relationship between moduli space of a polygonal linkage and moduli space of stable point configurations on \(S^1\). The relationship almost automatically extends to quasilinkages. We stress that the below is a combination of the classical construction borrowed from [6] with the cell decomposition approach from [8].

Assume that a quasilinkage \(Q\) is fixed.

Definition 31. A configuration of \(n\) (not necessarily distinct) marked points \(p_1, ..., p_n\) on the unit circle \(S^1\) is called \(Q\)-stable if the following holds:

If the points \(\{p_i\}_{i \in I}\) coincide, then the set \(I \subset [n]\) is \(Q\)-short.

We identify \(S^1\) with the real projective line \(\mathbb{R}P^1\), which enables us to speak of diagonal action of the group \(PSL(2, \mathbb{R})\) on the space of all stable configurations. We introduce the quotient space

\[M_{st}(Q) = \{\text{space of } Q\text{-stable configurations}\}/PSL(2, \mathbb{R}).\]

Theorem 32. Given a quasilinkage \(Q\),

1. \(M_{st}(Q)\) is a \((n - 3)\)-dimensional manifold.
2. \(M_{st}(Q)\) is homeomorphic to \(M(Q)\).
3. The stratification of the space \(M_{st}(Q)\) by combinatorial types is a regular cell complex dual to the cell complex \(M(Q)\).

Proof. We label each point configuration by its combinatorial type – the cyclically ordered partition of the set \([n]\). The labels do not change under the action of the group \(PSL(2, \mathbb{R})\). Equivalence classes are open balls of different dimensions, and can be considered as open cells of some cell decomposition.

We arrive at the cell complex on \(M_{st}(Q)\) defined as follows:

1. The \(k\)-cell of the complex are labeled by (all possible) admissible cyclically ordered partition of \([n]\) into \(k+3\) non-empty subsets. Given a cell \(C\), its label is denoted by \(\lambda(C)\).
2. A closed cell \(C\) belongs to the boundary of another closed cell \(C'\) whenever the label \(\lambda(C')\) is finer than \(\lambda(C)\).
This cell decomposition is obviously combinatorially dual to the cell complex $M(Q)$. \hfill \Box

5. A FAMILY OF QUASILINKAGES GENERATED BY AN ORIENTED MATROID

Oriented matroids: a short reminder. Let us start with some definitions.

Definition 33. A $(n-1)$-pseudosphere is a tame embedding of the oriented $(n-1)$-dimensional sphere $S^{n-1}$ into the $n$-dimensional sphere $S^n$. ”Tame” here means just ”not wild”, so it is sufficient to consider just piecewise linear embeddings.

Each $(n-1)$-pseudosphere $E$ divides $S^n$ into two parts, $E^+$ and $E^-$. We call them hemispheres related to $E$. Here ”plus” and ”minus” are assigned consistent with the orientation of $E$.

A pseudosphere arrangement on $S^n$ is a finite collection of $(n-1)$-pseudospheres that intersect along pseudospheres. That is,

1. Any number of pseudospheres from the arrangement intersect by some other pseudosphere.
2. Any number of (closed) hemispheres $E^+_i$ and $E^-_i$, where $E \in A$, intersect by a topological ball.

An oriented matroid is a concept which abstracts combinatorial properties of directed graphs, point configurations, vector configurations, sphere arrangements, etc. It is defined axiomatically, but we prefer not to present here the complete definition, referring the reader to [1]. The reason is that all what we need in the framework of the paper, is the following crucial feature of matroids, the Folkman-Lawrence topological representation theorem:

Oriented matroids of rank $n$ are in a one-to-one correspondence with arrangements of $(n-1)$-pseudospheres.

Here are some further facts about matroids:

1. Any configuration of spheres is automatically a pseudosphere arrangement, and therefore, represents some oriented matroid.
2. Some of pseudosphere arrangements can be straightened, that is, there exists a combinatorially equivalent arrangement of spheres. Such arrangements represent the realizable matroids.
3. However, there exist many non-realizable matroids. In other words, the class of pseudosphere arrangements is significantly wider than the class of sphere arrangements.
An oriented matroid with some extra properties generates a collection of quasilinkages. The below construction generalizes the walls-and-chambers stratification of the parameter space of polygonal linkages (see Section 1).

For the classical setting, there exists just one parameter space \( \mathbb{R}P^n_{>0} \) with a (unique) subdivision into chambers. However, for quasilinkages we have many different stratifications: as explained below, any matroid (with some extra properties) provides an analogue of "parameter space + chambers". The idea is to replace the walls \( \sum_I x_I = \sum_T x_T \) by appropriate pseudospheres. Besides, to single out the parameter space, we also need to replace coordinate hyperplanes by some pseudospheres.

Assume we have an arrangement \( \mathcal{A} \) of \( (n + 2^n - 2) \) pseudospheres on the sphere \( S^{n-1} \). Assume that \( \mathcal{A} \) contains

- \( n \) pseudospheres \( e_i \) labeled by the elements of \( [n] \), and
- \( (2^n - 2) \) pseudospheres \( E_I \) labeled by all proper non-empty subsets of the set \( [n] \).

Denote by \( \Delta \) the intersection of the hemispheres associated to all the \( e_i \) and to all pseudospheres labeled by one-element sets:

\[
\Delta = \bigcap_{i=1}^n e_i^+ \cap \bigcap_{i=1}^n E_{\{i\}}^+.
\]

**Definition 34.** In this notation, \( \mathcal{A} \) is called a \( Q \)-arrangement if the following holds:

1. Each subset \( I \) and its complement \( \overline{I} = [n] \setminus I \) label one and the same pseudosphere, but with different orientations. That is,

   \[
   E_I^+ = S^n \setminus E_{\overline{I}}^+.
   \]

2. All the pseudospheres are different: for each \( I \neq J \neq \overline{I} \), \( E_I^+ \neq E_J^+ \)

3. For any sets \( J \subseteq I \subset [n] \), we always have

   \[
   E_I^+ \cap \Delta \subseteq E_J^+.
   \]

Assume that a \( Q \)-arrangement \( \mathcal{A} \) is fixed. The pseudospheres from \( \mathcal{A} \) tile the domain \( \Delta \) into a number of (open) chambers separated by the intersections \( E_I \cap \Delta \) that are called walls. We say that two chambers \( C, C' \) are adjacent if there is exactly one wall separating them.

**Definition 35.** Given a \( Q \)-arrangement, we associate with each chamber \( C \) a collection of short subsets \( L(C) \) of the set \( [n] \) by the following rule:

A subset \( I \subset [n] \) is short whenever \( C \subset E_I^+ \).

The following theorem follows straightforwardly from the above constructions.
Theorem 36. Given a Q-arrangement,

(1) By the above rule, each chamber $C$ yields a quasilinkage $L(C)$, and, consequently, the PL manifold $M(L(C))$.

(2) The quasilinkages for two adjacent cameras differ by a flip.

(3) The manifolds $M(L(C))$ and $M(L(C'))$ for two adjacent cameras differ on a Morse surgery which is compatible to the cell structure.

Example 37. Consider the collection walls and cameras with $n = 6$ for the classical setting. Take the 10 walls of type $x_{i_1} + x_{i_2} + x_{i_3} = x_{i_4} + x_{i_5} + x_{i_6}$. They intersect at a single point $X = (1/2, 1/2, ..., 1/2)$, and no other wall contains the point $X$. We turn the walls to pseudospheres by a local perturbation in a neighborhood of $X$ in such a way that there arises a new camera corresponding to the symmetric quasilinkage from Example 7. Figure 1 gives an illustration of the idea (however, in the figure we present a smaller number of walls in the smaller dimension).

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References

[1] Björner, A., Las Vergnas, M., Sturmfels, B., White, N., and Ziegler, G. M.: Oriented Matroids. Cambridge Univ. Press (1993).
[2] Brehm, U., Wolfgang, K.: Combinatorial manifolds with few vertices. Topology 26.4, 465-473 (1987)
[3] Brehm, U., Wolfgang, K.: 15-vertex triangulations of an 8-manifold. Mathematische Annalen 294.1, 167-193 (1992)
[4] Elgot, Calvin C.: Truth functions realizable by single threshold organs. Switching Circuit Theory and Logical Design (1961)
[5] Farber, M., Schuetz, D.: Homology of planar polygon spaces. Geometriae Dedicata 125.1, 75-92 (2007)
[6] Kapovich, M., Millson, J.: On the moduli space of polygons in the Euclidean plane. J. Differential Geom 42.1, 133-164 (1995)
[7] Kapovich, M., Millson, J.: The symplectic geometry of polygons in Euclidean space. J. Differential Geom 44.3, 479-513 (1996)
[8] Panina, G.: Moduli space of planar polygonal linkage: a combinatorial description. arXiv:1209.3241 (2012).
[9] Postnikov, A.: Permutohedra, associahedra, and beyond. Int Math Res Notices, Vol. 2009, 1026-1106 (2009)
[10] Richardson, M.: On finite projective games. Proceedings of the American Mathematical Society 7.3, 458-465 (1956)
[11] Taylor, A., Zwicker, W.: A characterization of weighted voting. Proceedings of the American mathematical society 115.4, 1089-1094 (1992)
[12] Von Neumann, J., Morgenstern, O.: Theory of games and economic behavior. Bull. Amer. Math. Soc 51, 498-504 (1945)
[13] Ziegler, G. M.: Lectures on polytopes. Springer, Vol. 152 (1995)