Strong entanglement causes low gate fidelity in inaccurate one-way quantum computation

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We study how entanglement among the register qubits affects the gate fidelity in the one-way quantum computation if a measurement is inaccurate. We derive an inequality which shows that the mean gate fidelity is upper bounded by a decreasing function of the magnitude of the error of the measurement and the amount of the entanglement between the measured qubit and other register qubits. The consequence of this inequality is that, for a given amount of entanglement, which is theoretically calculated once the algorithm is fixed, we can estimate from this inequality how small the magnitude of the error should be in order not to make the gate fidelity below a threshold, which is specified by a technical requirement in a particular experimental setup or by the threshold theorem of the fault-tolerant quantum computation.

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I. INTRODUCTION

The one-way quantum computation \cite{1} is a novel scheme of quantum computation often contrasted with the traditional circuit model of quantum computation \cite{2}. It is believed to be one of the most promising approaches to the realization of scalable quantum computers, and indeed, small-size cluster states have already been created in laboratories \cite{3}. Some important quantum algorithms, such as Deutsch’s algorithm \cite{4} and Grover’s search algorithm \cite{5}, have also been demonstrated experimentally.

One great advantage of the one-way quantum computation over the circuit model is that the preparation of the resource (=entanglement) and the consumption of it are clearly separated with each other. This fact has prompted many researchers to explore lower-bounds or upper-bounds for the proper amount of resource entanglement for the one-way quantum computation \cite{6,7}. For example, it was shown \cite{6} that a certain amount of entanglement is necessary for any universal resource state for the one-way quantum computation. On the other hand, it was shown \cite{7,8} that a state having too much entanglement is useless for the one-way quantum computation. These important results and further research based on them will ultimately enable us to pin down the exact amount of resource entanglement which is neither too small nor too large for the one-way quantum computation.

If the proper amount of resource entanglement for the one-way quantum computation is determined, the next goal is to clarify how such proper entanglement affects the gate fidelity of the one-way quantum computation. Because a highly-entangled state is often fragile \cite{10,13}, we cannot make the most of the power of entanglement if the one-way quantum computation itself is unstable. Of course, a one-way quantum computer is, like the circuit model of a quantum computer, finally stabilized to some extent by embedding a quantum error-correcting code as shown in Ref. \cite{14}. However, it is still very important to investigate the stability of a bare one-way quantum computer for several reasons \cite{15}. First, it gives valuable feedback for the study of general fault-tolerant schemes. Second, it helps the development of “made-to-measure” error-correcting codes. Third, what experimentalists are now interested in is not the gigantic fully-fledged quantum computer but a bare elementary gate between a couple of qubits. Finally, and most importantly, although the stability of the final result of the computation is guaranteed by the threshold theorem, we must verify the stability of each gate independently, because the crucial assumption of the threshold theorem is that the fidelity of each gate is larger than a certain threshold \cite{14}.

In this Rapid Communication, we study how the gate fidelity of the one-way quantum computation is affected by the amount of entanglement between the measured qubit and other register qubits if the measurement is inaccurate in the sense that the direction to which the qubit is projected is slightly deviated from the ideal one. As the resource state having a proper amount of entanglement, we adopt the cluster state \cite{16}. Our main result is

$$F \leq 1 - S \sin^2 \frac{\epsilon}{2},$$

which shows that the mean gate fidelity $F$ ($0 \leq F \leq 1$) is upper bounded by the decreasing function of the amount $S$ ($0 \leq S \leq 1$) of entanglement and the magnitude $\epsilon$ ($0 \leq \epsilon \ll 1$) of the deviation. The main consequence of this inequality is that, for a given amount $S$ of entanglement, which is theoretically calculated once the algorithm is fixed and is often very large (see Refs. \cite{11,17,19} and Sec. \ref{sec:threshold}), we can estimate from this inequality how small $\epsilon$ should be in order not to make the gate fidelity $F$ below a threshold, which is specified by an experimentalist implementing the one-way quantum computation on his/her particular experimental instruments or by the
threshold theorem of the fault-tolerant quantum computation.

II. SETUPS

Before showing our main result, some setups are necessary. As a universal set of quantum gates, we adopt the set of single-qubit rotations about $x$-axis and $z$-axis, and the controlled-NOT (C-NOT) gate between two qubits [2]. This is a universal gate set, since, according to the Euler decomposition, any single-qubit rotation can be written as a combination of these two types of rotations. We denote Pauli’s $x, y,$ and $z$ operators acting on $i$th qubit by $X_i, Y_i,$ and $Z_i,$ respectively. We also define eigenvectors of $X_i$ and $Z_i$ by $X_i|±⟩_i = ±|±⟩_i$ and $Z_i|z⟩_i = (−1)^z|z⟩_i \ (z = 0, 1),$ respectively. Let us remind that the single-qubit rotation $e^{−iX}u$ about $x$-axis, the single-qubit rotation $e^{−iZ}u$ by $u$ about $z$-axis, and the C-NOT gate are realized in the one-way scheme as (a), (b), and (c) in Fig. 1 respectively.

![FIG. 1: (Color online.) Circles represent qubits, bonds represent the controlled-Z (C-Z) interaction $|0⟩⟨0| ⊗ 1 + |1⟩⟨1| ⊗ Z$, and squares represent the input state. $X$ represents the measurement in $X$-basis. $±u$ represents the adaptive measurement in $(\cos uX ± \sin uY)$-basis according to the result of the previous measurement $±,$ respectively. $u$ represents the measurement in $(\cos uX − \sin uY)$-basis. Output states are modified according to the measurement history [20]. (a): The single-qubit rotation $e^{−iX}u$ by $u$ about $x$-axis. (b): The single-qubit rotation $e^{−iZ}u$ by $u$ about $z$-axis. (c): The C-NOT gate.](image)

Let us also remind that there are two possibilities for the implementation of the one-way quantum computation. One is that appeared in the original proposal [1] of the one-way quantum computation, where the whole cluster state is created before the onset of adaptive measurements. The other, which is called the “one-buffered implementation” [14], is the repetition of the addition of a single column of the cluster state to the register column and the measurement of register qubits (see Fig. 2). We will adopt the one-buffered implementation.

![FIG. 2: (Color online.) The one-buffered implementation [14] of the one-way quantum computation. The green ellipse represents the register state. The black solid arrow represents the measurement.](image)

Let $|ψ⟩$ be an $N$-qubit state, which is considered as the quantum register. We assume that one of the three operations, (a), (b), or (c), in Fig. 1 is applied to $|ψ⟩$ in the one-buffered implementation as is shown in (d), (e), and (f) in Fig. 3. We are interested in the fidelity of these operations assuming that a measurement is inaccurate.

![FIG. 3: (Color online.) (d): The register state $|ψ⟩$ is represented by the green ellipse. The rotation of the bottommost qubit of $|ψ⟩$ by $u$ about $x$-axis. (e): The rotation of the bottommost qubit of $|ψ⟩$ by $u$ about $z$-axis. (f): The C-NOT gate between the bottommost qubit and a qubit of $|ψ⟩$. (g): The red bond represents entanglement between the first qubit and other register qubits (which are in the blue ellipse). Processes (d) and (e) can be written as a combination of (g). (h): The process (f) can be written as a combination of (g) and (h). (i): The state after the measurement on the first qubit in (g).](image)

It is easy to see that we have only to consider the fidelity of the process (g) in Fig. 3 for the study of (d), (e), (f). First, any of three operations, (d), (e), and (f) in Fig. 3 is a combination of the two elementary processes (g) and (h) in Fig. 2 (see Fig. 4). Therefore, the study of the fidelity of (d), (e), and (f) are reduced to that of (g) and (h). Second, in the process (h), the measurement on the first qubit [which is labeled as “1”] commutes with the C-Z interaction between the second qubit and the third qubit [which are labeled as “2” and “3”, respectively]. Therefore, the study of (h) is reduced to that of (g). In summary, we have only to consider the fidelity of the process (g) for our purpose.
Let us therefore calculate the fidelity of the process (g) in Fig. 3. We assume that the measurement on the first qubit [which is labeled as “1” in (g)] is inaccurate in the sense that the direction to which the qubit is projected is slightly deviated from the ideal one. In other words, the measurement is not the ideal one \(|u_+\rangle, |u_-\rangle\), where

\[
|u_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{-iu}|1\rangle),
\]

but the slightly deviated one \(|\tilde{u}_+\rangle, |\tilde{u}_-\rangle\), where

\[
|\tilde{u}_+\rangle = \cos \frac{\epsilon}{2}|u_+\rangle + e^{-i\delta} \sin \frac{\epsilon}{2}|u_-\rangle,
\]
\[
|\tilde{u}_-\rangle = \sin \frac{\epsilon}{2}|u_+\rangle - e^{-i\delta} \cos \frac{\epsilon}{2}|u_-\rangle.
\]

[If the measurement is done in the \(\tilde{X}\)-basis, we have only to put \(u = 0\) in Eq. (1).] It is easy to see that the degree of the deviation is parametrized by \(\epsilon\) and \(\delta\): \(|\tilde{u}_+\rangle\) \((|\tilde{u}_-\rangle)\) is the vector obtained by rotating \(|u_+\rangle\) \(|u_-\rangle\) by \(\epsilon\) about \(z\)-axis and by \(\frac{\epsilon}{2} - \delta\) about \(|u_+\rangle\)-axis. This kind of inaccuracy is ubiquitous in quantum physics. For example, in the measurement model of von Neumann [21], the direction to which the primary state is projected is deviated in this way if there is an inaccuracy in the control of the coupling constant or the coupling time between the primary system and the apparatus, or if the projection measurement on the apparatus is inaccurate.

After the measurement of the first qubit in (g), the entanglement between the first qubit and other qubits is broken. Then, (g) changes into (i) in Fig. 3. Let the register state after this measurement, i.e., the state of qubits in the green ellipse in (i), be \(|\phi_{e,\delta}\rangle\). If the measurement was accurate, this is \(|\phi_{0,0}\rangle\). Then, we can show that

\[
F \equiv \mathbb{E}[|\langle \phi_{0,0} | \phi_{e,\delta} \rangle|^2] \leq 1 - S \sin^2 \frac{\epsilon}{2}
\]

(2)

where \(\mathbb{E}[\cdot]\) means the average over all measurement histories,

\[
S \equiv 2(1 - \text{Tr}(\hat{\rho}_2^{2}))
\]

is the entanglement between the first qubit and other register qubits [which is indicated by the red bond in (g)], and \(\hat{\rho}_1 \equiv \text{Tr}_1(|\psi\rangle \langle \psi|)\) is the reduced density operator of the first qubit (\(\text{Tr}_1\) is the trace over all qubits except for the first qubit). If the first qubit and other register qubits are not entangled, \(S = 0\), whereas if they are maximally entangled, \(S = 1\). Equation (2) is our main result.

Proof of Eq. (2): Let us see (g) in Fig. 3. The register state \(|\psi\rangle\) (which is represented by the green ellipse) is written as

\[
|\psi\rangle = \alpha |0\rangle_1 \otimes |\eta_0\rangle_b + \beta |1\rangle_1 \otimes |\eta_1\rangle_b,
\]

where \(|0\rangle_1\) and \(|1\rangle_1\) are states of the first qubit [labeled as “1” in (g)], and \(|\eta_0\rangle_b\) and \(|\eta_1\rangle_b\) are states of other register qubits [represented by the blue ellipse in (g)]. \(|\eta_0\rangle_b\) and \(|\eta_1\rangle_b\) are not necessarily orthogonal with each other. Let us add the second qubit \(|+\rangle_2\) labeled as “2” in (g) to \(|\psi\rangle\) and perform the \(C-Z\) interaction between the first qubit and the second qubit:

\[
|\psi\rangle \otimes |+\rangle_2 \rightarrow \alpha |0\rangle_1 \otimes |\eta_0\rangle_b \otimes |+\rangle_2 + \beta |1\rangle_1 \otimes |\eta_1\rangle_b \otimes |+\rangle_2.
\]

As we have assumed, the first qubit is measured in \(|\tilde{u}_+\rangle, |\tilde{u}_-\rangle\}. Then, (g) changes into (i). Let the states of the green ellipse in (i) be \(|\phi_{e,\delta}^\pm\rangle\) if the result of the measurement is \(\pm\), respectively. By a straightforward calculation, the probabilities \(P_{\pm}\) of obtaining \(|\phi_{e,\delta}^\pm\rangle\) are

\[
P_{\pm} = \frac{1}{2}(1 \pm \xi \sin \epsilon \cos \delta),
\]

respectively, where \(\xi = \text{Tr}(\rho_1 \tilde{Z}_1)\). The fidelity for each output is also calculated as

\[
F_{\pm} \equiv |\langle \phi_{0,0} | \phi_{e,\delta}^\pm \rangle|^2 = \frac{1 \pm \xi \sin \epsilon \cos \delta - (1 - \xi^2) \sin^2 \frac{\epsilon}{2}}{2P_{\pm}},
\]

respectively, and the mean fidelity is therefore

\[
F_{+}P_{+} + F_{-}P_{-} = 1 - (1 - \xi^2) \sin^2 \frac{\epsilon}{2}.
\]

Our goal, Eq. (2), is obtained by applying the relation

\[
1 - \text{Tr}^2(\hat{\rho}_1 \tilde{Z}_1) \geq S,
\]

which is shown as follows. Let \(\rho_1 = \lambda_0 |\tau_0\rangle_1 \langle \tau_0| + \lambda_1 |\tau_1\rangle_1 \langle \tau_1|\), where \(\lambda_0 \geq 0, \lambda_1 \geq 0, \lambda_0 + \lambda_1 = 1\), and

\[
|\tau_0\rangle_1 = \cos \frac{\mu}{2}|0\rangle_1 + e^{-iu} \sin \frac{\mu}{2}|1\rangle_1,
\]
\[
|\tau_1\rangle_1 = \sin \frac{\mu}{2}|0\rangle_1 - e^{-iu} \cos \frac{\mu}{2}|1\rangle_1.
\]

Then, we obtain

\[
1 - \text{Tr}^2(\hat{\rho}_1 \tilde{Z}_1) = 1 - (\lambda_0 - \lambda_1)^2 \cos^2 \mu \geq 1 - (\lambda_0 - \lambda_1)^2 = S.
\]
IV. DISCUSSION

If $S$ was always 0 during any quantum computation, Eq. (2) would be of no use. However, in fact, $S$ often becomes very large during a quantum computation. For example, in Ref. [17], it was shown that if an $N$-qubit register state $|\psi\rangle$ is decomposed as the tensor product of inseparable states $|\psi\rangle = \otimes |\psi_i\rangle$, at least one of these inseparable states $\{ |\psi_i\rangle \}$ must have unboundedly increasing size during a quantum computation if the quantum computation offers an exponential speed-up over a classical one. This result is not changed even if a weak entanglement is established among $|\psi_i\rangle$'s. Therefore, there is a high probability that the measured qubit has sufficiently strong entanglement with other register qubits during a quantum computation. Moreover, in Ref. [18] [19], it was shown that the register state has a superposition of macroscopically distinct states during the execution of Shor’s factoring algorithm and Grover’s search algorithm. According to the result of Ref. [11], a randomly chosen single qubit is strongly entangling with other qubits with a high probability if the state has such a macroscopic superposition. In short, $S$ is often very large in a quantum computation, and therefore Eq. (2) offers a meaningful upper-bound for the gate fidelity of the inaccurate one-way quantum computation.

The error model studied here is not an atypical one. This type of error is indeed often considered in many studies of fault-tolerant quantum computations including Ref. [14], where the possibility of the fault-tolerant one-way quantum computation is shown. Therefore, the effect of our error is recoverable to some extent and the whole quantum computation can be performed successfully. However, as mentioned in Sec. I the study of the stability of a bare one-way quantum computation is very important. This is where our result can contribute.

In addition to the inaccurate measurement considered here, there are many other possibilities of errors in the one-way quantum computation. For example, if the one-way quantum computation is implemented with the discrete-variable linear-optics schemes [22], we must also consider the imperfection of the C-Z gate, since, in this case, the entangling operation is not deterministic. To consider other error models would lead to interesting generalizations of the present work. It is left for a future study.

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