From accelerating and Poincaré coordinates to black holes in spacelike warped AdS$_3$, and back

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Abstract
We first review spacelike stretched warped AdS$_3$ and we describe its black hole quotients by using accelerating and Poincaré coordinates. We then describe the maximal analytic extension of the black holes and present their causal diagrams. Finally, we calculate spacetime limits of the black hole phase space $(TR, TL)$. This is done by requiring that the identification vector $\partial \theta$ has a finite non-zero limit. The limits we obtain are the self-dual solution in accelerating or Poincaré coordinates, depending respectively on whether the limiting spacetimes are non-extremal or extremal, and warped AdS$_3$ with a periodic proper time identification.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A promising approach to describe the microscopic degrees of freedom in gravity lies in the conjectured AdS/CFT correspondence. The correspondence asserts that quantum gravity in its classical limit is dual to a lower-dimensional CFT in its strong regime. One can test the validity of the correspondence in different scenarios. Of particular interest is gravity in three dimensions because there the theory simplifies considerably. Pure and cosmological Einstein gravities in particular are trivial in the bulk and the solutions of the latter are described solely by their global properties [1, 2]. Topological massive gravity (TMG) is a three-dimensional third-order gravity theory that when expanded to linear order contains a single massive mode [3–6]. TMG presents a next-to-simplest model to explore quantum gravity.
The action of TMG contains an Einstein–Hilbert term with negative cosmological constant $-\frac{1}{\ell^2}$ plus a gravitational Chern–Simons term
\[
16\pi G S[g] = \int d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right) + \frac{\ell}{6\nu} \int d^3x \sqrt{-g} \epsilon^{\lambda\mu\nu} \Gamma_\lambda^{\nu\sigma} \left( \partial_\mu \Gamma_\sigma_{\rho\nu} + \frac{2}{3} \Gamma_\mu^{\sigma\tau} \Gamma_\nu_{\rho\tau} \right)
\].

In three dimensions, the gravitational constant $G$ has dimension of length and $\nu$ is a dimensionless positive constant that we shall take $\nu > 1$. The equations of motion are
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = \frac{\ell}{3\nu} \epsilon_{\rho\sigma} \left( R_{\rho\sigma} - \frac{1}{4} g_{\rho\sigma} R \right) \equiv -\ell \frac{1}{3\nu} C_{\mu\nu},
\]
where the Cotton tensor $C_{\mu\nu}$ is a measure of conformal flatness. A solution of TMG is given by a metric along with a preferred orientation of the Levi-Civita tensor $\epsilon_{\mu\nu\rho}$. However, its solution space is more relevant to four-dimensional physics than what one might expect from such a simplification. The near-horizon geometry of the extremal Kerr black hole [7], at a fixed polar angle, is a particular solution of TMG, the self-dual warped AdS3 space in Poincaré coordinates; see also [8]. The geometry of warped AdS3 plays a pivotal role here.

The last couple of years have seen a flurry of activity in TMG, due to the conjecture that the black hole solutions obtained by the quotients of spacelike warped AdS3 are dual to a CFT with separate left and right central charges [9]. More recently, real-time correlators were obtained for the self-dual geometry in accelerating coordinates that were chiral [10]. This motivates us to take a tour of the relevant geometries, starting from the quotient construction and arriving at the self-dual warped AdS3 as a spacetime limit of the black holes.

We begin in section 2 by describing the warped AdS3 geometry in three coordinate systems: the (global) warped AdS3 coordinates, accelerating coordinates and Poincaré coordinates. The use of the last two greatly simplifies the quotient construction of the black hole metric in section 3.

Although the quotient construction is well known from [9], we pay particular attention to the case when causal singularities do exist behind the Killing horizons. As customary, it is for this case that we speak of a 3D black hole [2]. We show that the phase space is such that the ratio of left to right temperatures $T_L/T_R$ has a lower bound, and there is a critical value of the ratio when the inner horizon coincides with the causal singularity. We also point out a subtlety in the parametrization with respect to the radii $(r_+ , r_-)$ of [9], namely that two separate regions of the ratio $r_+/r_-$ are in one-to-one isometric correspondence.

In section 4 we pause to describe the causal structure of the black holes. Following the previous discussion about the ratio $T_L/T_R$, we accordingly find that the causal diagrams fall into three different classes. These are similar to those of the non-extremal charged Reissner–Nordström (RN) 4D black hole for a generic ratio $T_L/T_R$, the extremal RN when $T_R = 0$ and the uncharged RN when the ratio is at its critical value.

Finally, in section 5 we describe the various spacetime limits that one can take in the black hole phase space. We describe the regular extremal limit, the near-horizon limit of the extremal black holes, a near-extremal limit $T_R \to 0$ for the non-extremal black holes and the limit when both temperatures $T_R$ and $T_L$ go to zero while keeping the Hawking temperature fixed. The near-horizon and near-extremal limits give the self-dual warped AdS3 geometry in accelerating or Poincaré coordinates, with which we were familiarized in section 2. The limit when both temperatures go to zero while keeping the Hawking temperature fixed gives the vacuum solution and is universal for all ratios $T_L/T_R$. The analysis is a systematic procedure and no other limits are obtained. We conclude in section 6 with a summary and comments on our results.

The appendices contain material that is not needed in the development of the text but may complement it. Appendix A describes the hyperbolic fibration of SL(2, $\mathbb{R}$), which is
only briefly mentioned in section 2. In appendix B we give explicit diffeomorphisms between the coordinate systems of section 2. Although the explicit diffeomorphisms are not used in proving any of the main text’s results, we present them for completeness. In appendix C we translate the bound on the ratio \( T_L/T_R \) to an inequality on the ADT mass and angular momentum and the parameters of [11].

2. Spacelike warped AdS

In this section we review the geometry of spacelike warped AdS\(_3\). This will prepare us for a clear understanding of the quotient construction in section 3. We describe the metric in warped, accelerating and Poincaré coordinates. In summary, the metric will be written in the form

\[
g_{\ell,\nu} = \frac{\ell^2}{\nu^2 + 3} (-f(x) \, d\tau^2 + \frac{dx^2}{f(x)} + \frac{4\nu^2}{\nu^2 + 3} (du + x \, d\tau)^2),
\]

(1)

where

\[
f(x) = \begin{cases} 
  x^2 + 1 & \text{for warped coordinates,} \\
  x^2 - 1 & \text{for accelerating coordinates,} \\
  x^2 & \text{for Poincaré coordinates.}
\end{cases}
\]

Metric (1) satisfies the TMG equations of motion with \( \epsilon_{\tau xu} = +\sqrt{-g} \). We will use the same labels \((\tau, x, u)\) for accelerating and Poincaré coordinates, hoping this will not cause confusion. For the warped coordinates we will use instead the coordinate labels \((\tilde{t}, \sigma, \tilde{u})\), where we replace \( x \rightarrow \sinh \sigma \), \( u \rightarrow \tilde{u} \) and \( \tau \rightarrow \tilde{t} \).

2.1. Warped coordinates

Let us start by expressing AdS\(_3\) as the universal cover of the special linear group SL(2, \( \mathbb{R} \)):

\[
\text{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} T_1 + X_1 & T_2 + X_2 \\
 X_2 - T_2 & T_1 - X_1 \end{pmatrix} : \quad T_1^2 + T_2^2 - X_1^2 - X_2^2 = 1 \right\}.
\]

As a group, SL(2, \( \mathbb{R} \)) acts on the left and right of the group manifold. We write the action as \( \text{SL}(2, \mathbb{R})_L \times \text{SL}(2, \mathbb{R})_R \). We choose a basis of the right- and left-invariant vector fields, respectively, \( l_a \) and \( r_b \):

\[
\begin{align*}
  l_1(r_2) &= \frac{1}{2} \begin{pmatrix} -X_2 \frac{\partial}{\partial T_1} - T_1 \frac{\partial}{\partial X_2} \pm T_2 \frac{\partial}{\partial X_1} \mp X_1 \frac{\partial}{\partial T_2} \\
  X_2 - T_2 & T_1 - X_1 \end{pmatrix} \\
  &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\
  -1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})_L \\
  l_0(r_0) &= \frac{1}{2} \begin{pmatrix} -T_1 \frac{\partial}{\partial T_1} + T_2 \frac{\partial}{\partial T_2} \pm X_1 \frac{\partial}{\partial X_2} \mp X_2 \frac{\partial}{\partial X_1} \\
  X_2 - T_2 & T_1 - X_1 \end{pmatrix} \\
  &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\
  +1 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})_L \\
  l_2(r_1) &= \frac{1}{2} \begin{pmatrix} -X_1 \frac{\partial}{\partial T_1} - T_1 \frac{\partial}{\partial X_2} \pm T_2 \frac{\partial}{\partial X_1} \mp X_2 \frac{\partial}{\partial T_2} \\
  X_2 - T_2 & T_1 - X_1 \end{pmatrix} \\
  &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\
  0 & +1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})_L \\
\end{align*}
\]

The non-zero commutators of the generators are \([l_a, l_b] = \epsilon_{ab}^c l_c \) and \([r_a, r_b] = \epsilon_{ab}^c r_c \), where the indices \( a = 0, 1, 2 \) are raised with a mostly-plus Lorentzian signature metric and \( \epsilon_{012} = +1 \).
We associate with the bases $l_a$ and $r_a$ the dual left- and right-invariant one-forms $\theta^a$ and $\bar{\theta}^a$, respectively, so that $\theta^a(l_b) = \delta^a_b$ and $\bar{\theta}^a(r_b) = \delta^a_b$. The Lie derivative therefore acts as $\mathcal{L}_l \theta^b = \epsilon_{abc} \theta^c$ and $\mathcal{L}_r \bar{\theta}^b = \epsilon_{abc} \bar{\theta}^c$. The left-invariant one-forms allow us to write metrics on $SL(2, \mathbb{R})$ with symmetry of rank 3, 4 and 6.

Let us introduce the parametrization

$$
T_1 = \cosh \frac{\sigma}{2} \cosh \frac{\tilde{u}}{2} \cos \frac{\tilde{t}}{2} + \sinh \frac{\sigma}{2} \sin \frac{\tilde{u}}{2} \sin \frac{\tilde{t}}{2},
$$

$$
T_2 = \cosh \frac{\sigma}{2} \cosh \frac{\tilde{u}}{2} \sin \frac{\tilde{t}}{2} - \sinh \frac{\sigma}{2} \sin \frac{\tilde{u}}{2} \cos \frac{\tilde{t}}{2},
$$

$$
X_1 = \cosh \frac{\sigma}{2} \sinh \frac{\tilde{u}}{2} \cos \frac{\tilde{t}}{2} + \sinh \frac{\sigma}{2} \cosh \frac{\tilde{u}}{2} \sin \frac{\tilde{t}}{2},
$$

$$
X_2 = \cosh \frac{\sigma}{2} \sinh \frac{\tilde{u}}{2} \sin \frac{\tilde{t}}{2} - \sinh \frac{\sigma}{2} \cosh \frac{\tilde{u}}{2} \cos \frac{\tilde{t}}{2},
$$

(3)

which was shown in [12] to cover the whole of $SL(2, \mathbb{R})$ with $\tilde{u}, \sigma \in \mathbb{R}$ and $\tilde{t} \sim \tilde{t} + 4\pi$. These are the hyperbolic asymmetric coordinates of [13]. With the above parametrization the $\theta^a$ are

$$
\theta^0 = - d\tilde{t} \cosh \tilde{u} \cosh \sigma + d\sigma \sinh \tilde{u},
$$

$$
\theta^1 = - d\sigma \cosh \tilde{u} + d\tilde{t} \cosh \sigma \sinh \tilde{u},
$$

$$
\theta^2 = d\tilde{u} + d\tilde{t} \sinh \sigma,
$$

the left-invariant vectors are

$$
r_0 = - \partial_{\tilde{t}},
$$

$$
r_1 = \sin \tilde{t} \partial_{\sigma} + \cos \tilde{t} \tanh \sigma \partial_{\tilde{u}} + \cos \sigma \partial_{\tilde{t}},
$$

$$
r_2 = - \cos \tilde{t} \partial_{\sigma} + \sin \tilde{t} \tanh \sigma \partial_{\tilde{u}} + \sinh \sigma \sin \tilde{t} \partial_{\tilde{u}}
$$

(9)

and the right-invariant vectors are

$$
l_0 = - \sinh \tilde{u} \partial_{\sigma} - \cosh \tilde{u} \tanh \sigma \partial_{\tilde{u}} + \cosh \sigma \partial_{\tilde{t}},
$$

$$
l_1 = - \cosh \tilde{u} \partial_{\sigma} - \sinh \tilde{u} \tanh \sigma \partial_{\tilde{u}} + \sin \sigma \partial_{\tilde{t}},
$$

$$
l_2 = \partial_{\tilde{u}}.
$$

(12)

The Killing form, or ‘round’ metric,

$$
g_\ell = \frac{\ell^2}{4} (-\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2),
$$

in the warped coordinates (3) becomes

$$
g_\ell = \frac{\ell^2}{4} [-\cosh^2 \sigma \, d\tilde{t}^2 + d\sigma^2 + (d\tilde{u} + \sinh \sigma \, d\tilde{t})^2].
$$

(13)

The isometry group of $SL(2, \mathbb{R})$ with the round metric is $SO(2, 2) = (SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R)/\mathbb{Z}_2$, where we take into account that $-I$ acts similarly on each side. Unwrapping $\tilde{t} \in \mathbb{R}$ gives the AdS$_3$ metric in warped coordinates [12]. The isometry group becomes a universal diagonal cover of $(SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R)/\mathbb{Z}_2$.

Along these lines we approach the spacelike warped metric

$$
g_{\ell,v} = \frac{\ell^2}{v^2 + 3} \left(-\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \frac{4v^2}{v^2 + 3} \theta^2 \otimes \theta^2\right),
$$

(14)
so that for $\nu > 1$ or $\nu < 1$ we have a stretching or squashing, respectively, of the fibre in the direction of $l_2$ [13–15]. The isometry group is broken to that generated by the $l_2$ and the $r_\nu$. In the warped coordinates ($t$, $\sigma$, $\tilde{u}$), the warped metric is

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2 + 3} \left( - \cosh^2 \sigma \, d\tilde{u}^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} \left( d\tilde{u} + \sinh \sigma \, d\tilde{f} \right)^2 \right).$$

(15)

The metric is compatible with a double cover over a quadric base space, a point we elaborate on in appendix A. As before, we unwrap the time coordinate to run over $\tilde{t} \in \mathbb{R}$. This is the metric of warped AdS$_3$ in (the global) warped coordinates, which was given in (1) for $f(x) = x^2 + 1$. The isometry group is the universal cover $SL(2, \mathbb{R}) \times \mathbb{R}$.

If we compactify spacelike warped AdS$_3$ along $l_2$, that is $\tilde{u} \sim \tilde{u} + 2\pi \alpha$, we obtain the so-called self-dual solution of TMG:

$$g_{\ell,\nu,\alpha} = \frac{\ell^2}{\nu^2 + 3} \left( - \cosh^2 \sigma \, d\tilde{t}^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2 + 3} \left( \alpha \, d\tilde{\phi} + \sinh \sigma \, d\tilde{f} \right)^2 \right),$$

with $\tilde{t}, \sigma \in \mathbb{R}$ and $\tilde{\phi} \sim \tilde{\phi} + 2\pi$. The isometry group of the self-dual geometry becomes $SL(2, \mathbb{R}) \times U(1)$.

### 2.2. Accelerating coordinates

Let us ask how we would write the warped AdS$_3$ metric in a coordinate system $(\tau, x, u)$ where $\partial_\tau$ is a linear combination of the $r_\nu$ and $l_2$. Since $l_2$ acts freely we can choose $u$ to be such that $\partial_\nu = l_2$. The translations in $\tau$ and $u$ will be manifest symmetries of the metric. We still need to make an appropriate choice for the coordinate $x$, which should be invariant under $\partial_\nu$ and $\partial_\nu$. That is, we require $\partial_\nu x = \partial_\nu u = 0$. We choose $x = \frac{\nu^2 + 3}{\nu^2 + 1} g_{\ell,\nu} (\partial_\nu, \partial_\nu)$, which is indeed invariant because $\partial_\nu$ and $\partial_\nu$ are the Killing vectors and they commute. The coordinate system $(\tau, x, u)$ is thus described by the surfaces $(u, \tau)$ generated by the flows of two Killing vectors, and a coordinate $x$ which smoothly labels them.

Under an $SL(2, \mathbb{R})_R$ rotation on the $r_\nu$ and an $GL(2, \mathbb{R})$ transformation on $(u, \tau)$ we can bring $\partial_\nu$ to one of the following forms: $r_\nu$, $-r_\nu$, or $r_\nu \pm r_2$. We also keep $\partial_\nu = l_2$ as before. The case $\partial_\nu = -r_2$ corresponds to the warped coordinates, see (7) and (12). In this subsection we consider the second case, $\partial_\nu = r_2$, and in the next subsection we will consider the third case. We thus have a set of coordinates defined by the action of the Killing vectors $r_2$ and $l_2$ and their metric product. Using the metric in (15) and the present data$^5$, we can write the metric

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2 + 3} \left( -(x^2 - 1) \, d\tau^2 + \frac{dx^2}{x^2 - 1} + \frac{4\nu^2}{\nu^2 + 3} \left( d\tilde{u} + x \, d\tau \right)^2 \right),$$

(16)

where we fixed $dx$ to be orthogonal to the $(u, \tau)$ hypersurfaces. This is precisely metric (1), with $f(x) = x^2 - 1$. We have derived the metric form without an explicit diffeomorphism between these coordinates and warped coordinates. An explicit diffeomorphism can be found in appendix B, any other being related to it by the symmetries of the metric.

We call this set of coordinates accelerating. Accelerating coordinates have a lot in common with those of the Rindler spacetime. They describe observers with proper velocity $v = \frac{\nu^2 - 1}{2\nu \sqrt{\nu^2 + 1}}$, whose acceleration $V_{\nu} v$ is position dependent. In contrast to Rindler coordinates though, where $\partial_\nu$ is a Lorentz boost in Minkowski space, here $\partial_\nu$ is never timelike with respect to the metric. Nevertheless the $\tau$-constant surfaces are spacelike. If we compactify along $l_2$,

$^5$ That is, $\partial_\nu = r_2$, $\partial_\nu = l_2$ and $x = \frac{\nu^2 + 3}{2\nu \sqrt{\nu^2 + 1}} g_{\ell,\nu} (\partial_\nu, \partial_\nu) = \cosh \sigma \sin \tilde{t}$. 

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that is replace \( u = \alpha \phi \) in (16) with \( \phi \sim \phi + 2\pi \), we obtain the self-dual solution in accelerating coordinates:

\[
g_{\ell,\nu,a} = \frac{\ell^2}{\nu^2 + 3} \left( -(x^2 - 1) \, d\tau^2 + \frac{dx^2}{x^2 - 1} + \frac{4\nu^2}{\nu^2 + 3} (\alpha \, d\phi + x \, d\tau)^2 \right). \tag{17}
\]

As expected for Rindler-like coordinates, there are apparent Killing horizons appearing at \( x = \pm 1 \). On the Killing horizons the flow of \( r_2 \) takes us to a line where \( r_2 \) becomes collinear to \( l_2 \). Thus the coordinates are valid only away from the Killing horizon. The warped AdS3 spacetime has an infinite number of such regions. Figure 1(a) gives a visualization of the situation\(^6\). The value of the level \( x \) tells us where we are with respect to the Killing horizons in each region, for each of which there is an appropriate isometric embedding of \((\tau, x, u)\) in warped AdS3. Figure 1(b) is precisely a choice of patches that the accelerating coordinates will cover.

2.3. Poincaré coordinates

We can go through the same construction as above, but this time choosing \( \partial_t = -r_0 + r_2 \). We define as before \( \partial_u = l_2 \) and \( x = \frac{(\nu^2 + 3)^2}{4\nu^2} g_{\ell,\nu}(\partial_u, \partial_t) \). We also use the freedom to make \( x \) hypersurface orthogonal. The metric is

\[
g_{\ell,\nu} = \frac{\ell^2}{\nu^2 + 3} \left( -x^2 \, d\tau^2 + \frac{dx^2}{x^2} + \frac{4\nu^2}{\nu^2 + 3} (du + x \, d\tau)^2 \right), \tag{18}
\]

in what have been called Poincaré coordinates of warped AdS for obvious reasons. This is the metric in (1) with \( f(x) = x^2 \). Similar to the accelerating coordinates, the surface \( x = 0 \) is a Killing horizon. Figure 2 shows a Poincaré patch of warped AdS3. An explicit diffeomorphism can be found in appendix B.

\(^6\) In this figure we take \( \tilde{u} = \text{const} \), which is possible because \( \tilde{u} \) is defined globally.
The case $\partial_\tau = r_0 + r_2$ is similar to the above, simply by the warped AdS$_3$ discrete symmetry\((\tilde{t}, \tilde{u}) \mapsto (-\tilde{t}, -\tilde{u})\) that flips the sign of $r_0$ while preserving that of $r_2$. Compactifying along $l_2$, that is $u \sim u + 2\pi\alpha$, gives us the self-dual solution in Poincaré coordinates:

$$g_{\ell,\nu,\alpha} = \frac{\ell^2}{\nu^2 + 3} \left( -\nu^2 \partial^2 + \frac{d\nu^2}{\nu^2 + 3} + \frac{4\nu^2}{\nu^2 + 3} (\alpha d\phi + x \partial r)^2 \right). \quad (19)$$

3. Black hole quotients

Here we will follow the construction of [9], and find the quotients of spacelike warped AdS$_3$ that have causal singularities hidden behind Killing horizons. We quotient spacelike warped AdS$_3$ by $\exp(2\pi\partial_\theta)$ with $\partial_\theta$ given by

$$\partial_\theta = \begin{cases} 2\pi\ell T_R r_2 + 2\pi\ell T_L l_2 & \text{non-extremal black holes} \\ (r_2 \pm r_0) + 2\pi\ell T_L l_2 & \text{extremal black holes}. \end{cases} \quad (20)$$

The timelike case $\partial_\theta = Ar_0 + Bl_2$ yields naked closed timelike curves (CTCs) and so we do not consider it. Up to an $\text{SL}(2,\mathbb{R})_R$ rotation, which is an isometry of warped AdS, these three cases cover all choices of $\partial_\theta$. The quotient by (20) defines the left and right temperatures $T_L$ and $T_R$ in analogy to the Banados–Teitelboim–Zanelli (BTZ) case.

We pay attention to two points of interest. The first is that singular regions of a non-extremal quotient can be hidden behind a Killing horizon only when $T_L / T_R$ is bigger than a critical value. The second is that the ansatz for $T_L$ and $T_R$ as a function of $r_+$ and $r_-$ in [9] is not one-to-one for $T_L / T_R$ smaller than a second (different) critical value.

The method we employ is to describe the quotient in accelerating or, for the case of extremal black holes, Poincaré coordinates. The reason is quite simple: other than $\partial_\theta$ we would like a metric where the translation $\partial_t$ is the remaining and manifest isometry. The coordinates $(t, \theta)$ should then be given by a $\text{GL}(2,\mathbb{R})$ transformation on the accelerating, respectively Poincaré, coordinates $(\tau, u)$, see (20). The remaining radial coordinate is then any function of $x$ that labels the integral flows of $(\partial_t, \partial_\theta)$. The non-extremal black hole horizons are none other than the Killing horizons of warped AdS$_3$ at $x = \pm 1$, while the extremal black hole horizon lies on the Poincaré horizon $x = 0$. 

![Figure 2. Isometric embedding of the Poincaré patch.](image-url)
3.1. Non-extremal black holes

Assume the accelerating coordinates \((\tau, x, u)\) and the quotient defined by

\[
\begin{pmatrix}
t \\
\theta
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
\tau \\
u
\end{pmatrix}.
\]  

(21)

The periodicity \(\theta \sim \theta + 2\pi\) is preserved under the coordinate transformation

\[
\begin{pmatrix}
t' \\
\theta'
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
B & 1
\end{pmatrix} \begin{pmatrix}
t \\
\theta
\end{pmatrix},
\]  

(22)

That is, the quotient matrix in (21) is equivalent under

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \approx \begin{pmatrix}
aA & Ab \\
acB & b + d
\end{pmatrix}.
\]

When \(b = 0\) we bring the matrix to the form

\[
\begin{pmatrix}
t \\
\theta
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{a}
\end{pmatrix} \begin{pmatrix}
\tau \\
u
\end{pmatrix}.
\]  

(23)

This quotient is the self-dual solution, albeit in accelerating coordinates, where the parameter \(\alpha\) is the one in (17). When \(b \neq 0\) we bring the matrix to the form

\[
\begin{pmatrix}
t \\
\theta
\end{pmatrix} = \frac{2\nu}{\nu^2 + 3} \begin{pmatrix}
a & 1 \\
c & 0
\end{pmatrix} \begin{pmatrix}
\tau \\
u
\end{pmatrix},
\]  

(24)

\[
\Leftrightarrow \begin{pmatrix}
\tau \\
u
\end{pmatrix} = \frac{\nu^2 + 3}{2\nu} \begin{pmatrix}
0 & 1/c \\
1 & -a/c
\end{pmatrix} \begin{pmatrix}
t \\
\theta
\end{pmatrix},
\]  

(25)

\[
\Leftrightarrow \left( \frac{\partial}{\partial \theta} \right) = \frac{\nu^2 + 3}{2\nu} \begin{pmatrix}
0 & 1/c \\
1 & -a/c
\end{pmatrix} \left( \frac{\partial}{\partial \tau} \right),
\]  

(26)

where our choice is to normalize the length \(|\partial| = \ell^2\). In this case, we identify

\[
2\pi \ell T_R = \frac{\nu^2 + 3}{2\nu} \frac{1}{c}
\]  

(27)

\[
2\pi \ell T_L = -\frac{\nu^2 + 3}{2\nu} \frac{a}{c}.
\]  

(28)

By reflecting \(\theta \mapsto -\theta\) if necessary, we choose \(c > 0\). Note that \(1/c \neq 0\) and so we cannot describe the extremal case \(T_R = 0\) regularly.

We now ask when singular regions \(|\partial| \leq 0\) exist and whether they are hidden behind the Killing horizon \(x = 1\). Observe that we have not yet restricted the parameter \(a = -T_L/T_R\) in (24). A simple calculation in accelerating coordinates reveals

\[
c^2 |\partial_x|^2 = \ell^2 \frac{\nu^2 + 3}{4\nu^2} \left( -\left( \frac{\nu^2 - 1}{\nu^2 + 3} (x - a)^2 \right) \right),
\]

with determinant

\[
\Delta_x = \ell^4 \frac{\nu^2 + 3}{\nu^3} \left( a^2 - 3 \frac{\nu^2 - 1}{\nu^2 + 3} \right)
\]

and

\[
\partial_x (c^2 |\partial_x|^2) = \ell^2 \frac{\nu^2 + 3}{2\nu^2} \left( 3 \frac{\nu^2 - 1}{\nu^2 + 3} x - \frac{4\nu^2}{\nu^2 + 3} a \right).
\]

8
It follows that for $|a| < \frac{\sqrt{3(\nu^2 - 1)}}{2\nu}$ there are no CTCs, for $a < -\frac{\sqrt{3(\nu^2 - 1)}}{2\nu}$ CTCs exist in $x < -1$ and for $a > \frac{\sqrt{3(\nu^2 - 1)}}{2\nu}$ there are CTCs after $x > 1$. This is summarized in figure 3.

In fact, the values $a > \frac{\sqrt{3(\nu^2 - 1)}}{2\nu}$ tell us that $x > 1$ is an accelerating patch where CTCs exist. One can then move by the discreet isometry $(x, u) \rightarrow (-x, -u)$ to the outer region of the black hole. This essentially flips the sign of $a$, so that the ratio $T_\text{L}/T_\text{R}$ is bounded from below by $\frac{\sqrt{3(\nu^2 - 1)}}{2\nu}$.

The quotient in [9] is parametrized by $(r_+, r_-)$, where the right and left temperatures are related to $T_\text{R}$ and $T_\text{L}$ by

$$T_\text{R} = \frac{(\nu^2 + 3)(r_+ - r_-)}{8\pi \ell},$$

$$T_\text{L} = \frac{\nu^2 + 3}{8\pi \ell} \left( r_+ + r_- - \frac{\sqrt{r_+ r_- (\nu^2 + 3)}}{\nu} \right),$$

respectively, and the radial coordinate is chosen to be

$$r = \frac{r_+ - r_-}{2} + \frac{r_+ + r_-}{2}.$$

The coordinate transformation from the accelerating coordinates $(\tau, x, u)$ to the black hole coordinates $(r, t, \theta)$ is given by (24) and (31). By using (16), we find the black hole metric in the ADM form

$$ds^2 = -N^2 dt^2 + \frac{\ell^4 dr^2}{4R^2 N^2} + \ell^2 R^2 (d\theta + N^\theta dt)^2,$$

where

$$R^2 = \frac{3(\nu^2 - 1)}{4} r(r - r_0),$$

$$N^2 = \frac{\ell^2 (\nu^2 + 3)}{4R^2} (r - r_+)(r - r_-) = \frac{\ell^2 (\nu^2 + 3)}{3(\nu^2 - 1)} \frac{(r - r_-)(r - r_+)}{r(r - r_0)}$$

$$N^\theta = \frac{2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}}{2R^2}$$

$$r_0 = \frac{4\nu \sqrt{r_+ r_- (\nu^2 + 3)} - (\nu^2 + 3)(r_+ + r_-)}{3(\nu^2 - 1)}.$$
Let us now point out a subtlety in the parametrization with respect to \( r^+ \) and \( r^- \) in (29) and (30). From these equations, it follows that the ratio of temperatures

\[
\frac{T_L}{T_R} = \frac{v(r^+ + r^-) - \sqrt{v^2 r_- (v^2 + 3)}}{v(r^+ - r^-)}
\]

(37)
can be written as a function of the ratio \( \lambda = r^+ / r^- > 1 \). At the same time, the right temperature \( T_R \) can remain arbitrary positive for any given \( \lambda \) by adjusting \( r^- > 0 \). It is instructive to draw the graph of the ratio \( T_L / T_R \) as a function of \( \lambda \), see figure 4. We find that \( T_L / T_R \) decreases from plus infinity until the minimum at

\[
\lambda_f = 1 + 6 \frac{v^2 - 1}{v^2 + 3} \left( 1 + \frac{\sqrt{3}}{3} \frac{2v}{\sqrt{v^2 - 1}} \right),
\]

(38)
for which

\[
\left. \frac{T_L}{T_R} \right|_{\lambda_f} = \sqrt{\frac{3(v^2 - 1)}{4v^2}} < 1.
\]

It then increases asymptotically to the value of 1. On the one hand, this agrees with the necessary bound on \( T_L / T_R \). On the other, we find that there is a hidden isometry between pairs \( (r^+, r^-) \) in the two regions \( (\lambda_c, \lambda_f) \) and \( (\lambda_f, \infty) \), where \( \lambda_c = \frac{4v^2}{v^2 + 3} \) gives \( \left. \frac{T_L}{T_R} \right|_{\lambda_c} = 1 \).

Indeed, two values of the ratio of radii, say \( r^+/r^- \) and \( \tilde{r}^+ / \tilde{r}^- \), can give the same value of \( T_L / T_R \) and so the two pairs \( (r^+, r^-) \) and \( (\tilde{r}^+, \tilde{r}^-) \) will result in the same two invariants \( T_L \) and \( T_R \), respectively, that define the quotient in (20).

More precisely, the isometry relates black hole metrics with

\[
r^+ = \frac{v^2 + 3}{3(v^2 - 1)} \left( \sqrt{r^-} - \frac{2v}{\sqrt{v^2 + 3}} \sqrt{r^+} \right)^2,
\]

(39)
\[
r^- = \frac{v^2 + 3}{3(v^2 - 1)} \left( \frac{2v}{\sqrt{v^2 + 3}} \sqrt{r^-} - \sqrt{r^+} \right)^2,
\]

(40)
while the radial coordinate transforms as \( r \mapsto \tilde{r} \) according to

\[
\frac{2r - r^+ - r^-}{r^+ - r^-} = \frac{2\tilde{r} - \tilde{r}^+ - \tilde{r}^-}{\tilde{r}^+ - \tilde{r}^-}.
\]

(41)
The last equation is true by its equality to the coordinate label \( x = \frac{(v^2 + 3)^2}{4v^2 + 3} g_{\partial_\nu}(\partial_\tau, \partial_\tau) \), see (31), which is fixed for given \( T_L \) and \( T_R \).
Note that $r_0$ in (36), as a function of the ratio $\lambda \equiv r_+/r_- \geq 1$ with $r_-$ fixed, presents a maximum $r_0(\lambda_c) = r_-$ and then decreases monotonically, as in figure 5. In particular, $r_0(\lambda_f) = 0$. As a result, the maximum root of $R(r)^2$, denoted $\bar{r}_0$ hereafter, is

$$\bar{r}_0 = \begin{cases} 0 & \text{if } r_0 < 0 \text{ i.e. } \lambda > \lambda_f \\ r_0 & \text{if } r_0 \in [0, r_-] \text{ i.e. } 1 \leq \lambda \leq \lambda_f, \end{cases}$$

and so $R(r)^2 > 0$ for $r > r_-$ and the CTCs are always hidden behind $r_-$. The equality $R(r_-)^2 = 0$ holds only for $r_0(\lambda_c) = r_-$, that is when the inner horizon coincides with the singularity. For later use, let us also define the shorthand $R^{\pm}_2$:

$$R^{\pm}_2 \equiv R(r^{\pm}_0)^2 = \frac{r^{\pm}_0}{4} (2\nu\sqrt{r^{\pm}_0} - (\nu^2 + 3)r^{\pm}_0)^2 \geq 0.$$  

We should stress that we arrive at global results using accelerating coordinates. This is because $\theta_0$ in (20) is a global identification and one can choose to cover any of the infinite regions discussed in section 2 using accelerating coordinates. The Killing horizons at $r = r_-, r_+$ are inherited from the accelerating horizon at $x = \pm 1$. The lower bound in $T_L/T_R$ was discussed in [9], section 6.1.1. The parametrization of $T_L$ and $T_R$ in terms of $r_-$ and $r_+$, respectively, in [9] is such that the lower bound is satisfied. A subtle feature of the parametrization is the isometry in parameter space for $r_+/r_- \geq \lambda_c$. Let us also comment that, by the above analysis, the parameter assignment $T_L = 0$ appears special and disconnected from the region $T_L/T_R > \frac{2\nu^2 - 1}{2\nu}$. We will nevertheless obtain it as the vacuum limit of the non-extremal black holes in section 5.

### 3.2. Extremal black holes

The quotient that gives the extremal black holes in terms of the second Killing vector in (20) does not present any particular point of interest. We can repeat the previous derivation mutatis mutandis, where now the coordinates $(\tau, u)$ in (21) are the Poincaré coordinates of warped AdS. The case $b = 0$, see (23), is the self-dual solution in Poincaré coordinates (19). The case $b \neq 0$ gives the black hole solution in ADM form (32) by setting $r_+ = r_-$ in (30) and using $x = r - r_-$. The singular regions are behind $r < r_-$ for all values of $T_L \neq 0$, which can be chosen positive by reflecting $\theta$ if necessary.

In the non-extremal quotient given by the matrix in (26), observe that the parameter $1/c \sim T_R$ is always positive. One can thus never reach the extremal black holes $T_R = 0$ from
a regular quotient of that type. It is clear though that the non-extremal black holes have an extremal limit given by setting $r_+ = r_-$ in the metric (32). We shall recover this result in section 5 as a limit of the non-extremal quotient (26).

However, let us first examine the causal structure of the black holes. We will see that the critical value $r_+/r_- = \lambda_c$ is special with respect to the causal structure. We thus have three cases to consider: generic non-extremal, the case $r_+/r_- = \lambda_c$ and the extremal case.

### 4. Causal structure

In this section we will examine the causal structure of the spacelike warped black holes in a manner similar to [1]. Although these geometries are ideal, they are likely to appear as the end state of physical processes where chronology is protected. We will show that the Penrose–Carter diagram of a generic non-extremal or extremal black hole is similar to the 4D non-extremal, respectively extremal, RN black hole. Recall that we uncovered a critical value $r_0 = r_-$ (for $r_+/r_- = \lambda_c$) that is isometric to $r_+ = 0$. We accordingly find that the $r_+ = r_0$ black hole has a causal diagram similar to that of the Schwarzschild black hole, that is the uncharged RN black hole.

In what follows we will work with the two-dimensional metric $g_2$:

$$g = -N^2 \text{d}t^2 + \frac{\ell^2 \text{d}r^2}{4R^2N^2} + \ell^2 R^2 (\text{d}\theta + N\theta \text{d}t)^2.$$ 

If a curve $\gamma : [0, 1] \rightarrow M$ has tangent vector $\dot{\gamma} \in \gamma^*TM$, then

$$g_2(\dot{\gamma}, \dot{\gamma}) > 0 \Rightarrow g(\dot{\gamma}, \dot{\gamma}) > 0;$$

thus a causal curve $\gamma$ must be non-positive on $g_2$:

$$g(\dot{\gamma}, \dot{\gamma}) \leq 0 \Rightarrow g_2(\dot{\gamma}, \dot{\gamma}) \leq 0.$$ 

On the other hand, any causal curve $g_2(\dot{\gamma}, \dot{\gamma}) \leq 0$ can be lifted to a causal curve on $g$, e.g. by choosing the horizontal lift

$$\dot{\theta} + N_0 \dot{t} = 0. \quad (42)$$

Let us note that the metric $g_2$ does not capture the behaviour of causal geodesics, see e.g. [16]. However null curves on $g$ such that (42) holds are geodesic on $g_2$. They correspond to zero angular momentum $p_\theta = g(\dot{\gamma}, \partial_\theta)$.

The metric $g_2$ then tells us about all causal relations by neglecting the angle $\theta$. One might wonder why we do not take a $\theta = \text{const}$ section. After disentangling the angle one can indeed find a Kruskal extension, as done generically in [17]. However, the angle is not defined globally on the different Kruskal patches, so our choice is simpler since the connection $d\theta + N_0 \text{d}r$ is global. Furthermore, a local $\theta$-section will not give us information on causal relations, nor can it be compatible with any geodesic. Indeed, observe that the restriction of the metric on a constant angle will always be positive definite far away from the horizon.

The similarities with the RN black holes are not coincidental. Our method involves reducing the causal properties to the two-dimensional quotient space under the angular isometry $\partial_\theta$. The difference to the RN solution then, other than the dimensionality of the sphere, is a non-trivial connection one-form $d\theta + N_0 \text{d}r$, compare e.g. with Carter’s extension in [18].

We will first describe the future horizon ingoing coordinates. This is done so as to immediately introduce the Regge–Wheeler tortoise coordinate $r_*$. We then write down the Kruskal–Szekeres extension. We shall also use the ingoing coordinates in section 5, in order to derive the near-horizon geometry of extremal black holes.
4.1. Ingoing Eddington–Finkelstein coordinates

To introduce Eddington–Finkelstein coordinates, one first solves for the Regge–Wheeler tortoise coordinate \( r_* \), which in our case satisfies

\[
\frac{dr_*}{dr} = \frac{r^2}{2RN^2} = \frac{\sqrt{3(v^2 - 1)}}{v^2 + 3} \frac{\sqrt{r(r - r_0)}}{(r - r_-)(r - r_+)}.
\]  

(43)

For \( r > r_0 \) and \( r_* \neq r_- \), the solution is branched as follows:

\[
r_* = \frac{\sqrt{3(v^2 - 1)}}{v^2 + 3} \left( \sqrt{r_+(r_+ - r_0) \ln \left( \frac{|r - r_+|}{\sqrt{r_+(r_+ - r_0) + \sqrt{r_0(r_+ - r_0)^3}}} \right)} - \sqrt{r_-(r_- - r_0) \ln \left( \frac{|r - r_-|}{\sqrt{r_-(r_- - r_0) + \sqrt{r_0(r_- - r_0)^3}}} \right)} \right) + 2 \ln(\sqrt{r} + \sqrt{r - r_0}).
\]  

(44)

For the critical value \( r_* = r_- \), the solution \( (44) \) is also well defined. For the extremal case \( r_* = r_- \), \( (43) \) becomes

\[
\frac{dr_*}{dr} = \frac{\sqrt{3(v^2 - 1)}}{v^2 + 3} \frac{\sqrt{r(r - r_0)}}{(r - r_-)^2}
\]  

(45)

and its solution is branched as

\[
r_* = \frac{\sqrt{3(v^2 - 1)}}{v^2 + 3} \left( -\sqrt{r(r - r_0)} + 2 \ln(\sqrt{r} + \sqrt{r - r_0}) + \frac{1}{\sqrt{r_-(r_0 - r_0)}} \ln \left( \frac{|r - r_-|}{\sqrt{r_-(r_0 - r_0) + \sqrt{r_0(r_- - r_0)^3}}} \right) \right).
\]  

(46)

The ingoing coordinate is defined as \( u = t + r_* \).

The coordinates \((u, r)\) are well defined on and past the future horizon. In contrast, the angle \( \theta \) is entangled; that is, it diverges for geodesics that cross the horizon. For \( r_* \neq r_- \) and \( r_* / r_- = 4v^2 / (v^2 + 3) \) we define the angle

\[
\theta_{in} = \theta - \frac{4\nu}{v^2 + 3} \frac{1}{v(r_* - r_-)} \left( -\frac{2vr_* + \sqrt{r_+(v^2 + 3)}}{2vr_* - \sqrt{r_+(v^2 + 3)}} \ln \left( \sqrt{r_+(r_0 - r_0) + \sqrt{r_0(r_0 - r_0)}} \right) + 2vr_* + \sqrt{r_+(v^2 + 3)} \right) - \frac{2vr_* - \sqrt{r_-(v^2 + 3)}}{2vr_* + \sqrt{r_-(v^2 + 3)}} \ln \left( \sqrt{r_-(r_0 - r_0) + \sqrt{r_0(r_0 - r_0)}} \right) + N_0(r_+)u.
\]  

while for \( r_* / r_- = 4v^2 / (v^2 + 3) \) we define

\[
\theta_{in} = \theta - \frac{4}{v^2 - 1} \ln \left( \sqrt{r} + \frac{2\nu}{\sqrt{3(v^2 + 1)}} \right) + N_0(r_+)u.
\]  

For the extremal black holes \( r_* = r_- \) we define

\[
\theta_{in} = \theta + N_0(r_-)u + \frac{4\nu}{\sqrt{3(v^2 + 1)(v^2 + 3)}} \left( -\sqrt{r(r - r_0)} + \frac{r - r_0}{2r - \sqrt{r_0(r_0 - r_0)}} \ln \left( \sqrt{r_0 + \sqrt{r_0(r - r_0)}} \right) \right).
\]
These definitions are such that, in \((u, r, \theta_m)\) coordinates, in all cases the metric becomes

\[
g = -N^2 \, du^2 + \frac{\ell^2}{R} \, dr^2 + \ell^2 R^2 \left( dt + N_0 \, du \right)^2,
\]

with \(N_0(r) = N_0(r) - N_0(r_+)\) being zero on the horizon. The coordinates \((u, r, \theta_m)\) are regular on the future horizon \(r = r_+\) and valid until \(r = r_{-}\). The Hamiltonian of a free-falling particle is

\[
\mathcal{H} = \frac{2}{\ell^2} \left( \ell^2 R p_r p_r + N^2 R^2 p_\theta^2 + \frac{\ell^2}{4 R^2} p_\theta^2 p_{\theta_m} - \ell^2 R N_0 p_r p_{\theta_m} \right)
\]

where \(p_{\theta_m}, p_r\) are the constants of motion. Null geodesics, \(\mathcal{H} = 0\), satisfy

\[
\dot{u} = \frac{2}{\ell^2} R p_r,
\]

and for \(p_{\theta_m} = 0\) the ingoing rays are those with \(p_r = 0\).

### 4.2. Kruskal extension of non-extremal black holes

We first describe the Kruskal extension across \(r = r_+\) for the case \(r_+ \neq r_{-}\). With

\[
b_+ = \frac{\nu^2 + 3 \, r_+ - r_{-}}{4 \, R_+} = \frac{1}{2} \frac{r_+ - r_{-}}{\sqrt{r_+(r_+ - r_0)}} \frac{\nu^2 + 3}{\sqrt{3(\nu^2 - 1)}}
\]

and \(\rho(r) = e^{h_+r_+}\), define

\[
\begin{align*}
U &= \rho(r) \, e^{h_+t} \\
V &= \rho(r) \, e^{-h_+t} \\
\theta_+ &= \theta - \frac{N_0(r_+)}{2b_+} \ln \frac{U}{V} \\
U &= \rho(r) \, e^{h_-t} \\
V &= -\rho(r) \, e^{-h_-t} \\
\theta_- &= \theta + \frac{N_0(r_+)}{2b_-} \ln \frac{U}{V}
\end{align*}
\]

for \(r > r_+\) and \(r_+ < r < r_+\). The transformation in \(r_{-} < r < r_+\) is given so that one can match the Kruskal patches using \((32)\). In these coordinates, the metric becomes

\[
ds^2 = \Omega^2 \, dU \, dV + \ell^2 R^2 \left( dt + N_{UV} (V \, dU - U \, dV) \right)^2,
\]

where

\[
\Omega^2 = \frac{4 \ell^2 \, r_+(r_+ - r_0) \, (r - r_0)}{\nu^2 + 3 \, (r_+ - r_{-})^2 \, r(r - r_0)} \times \left( \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^2
\]

\[
\times \left( \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^{2} \left( \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^{-2} \left( \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^{-2} \left( \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^{-2}
\]

is everywhere positive and \(N_{UV}\) can be shown to be regular at \(r = r_+\). The coordinate \(r\) is given implicitly by \(UV = \rho^2(r)\), which is monotonic in \(r > r_+\) and, separately, in \(r_{-} < r < r_+\). We have the limits \(\lim_{r \to r_+} UV = +\infty, \lim_{r \to r_{-}} UV = 0\) and \(\lim_{r \to r_+} UV = -\infty\). We can extend with the isometry \(V \mapsto -V\) and \(U \mapsto -U\) and the patch \(K_+ = \{ U, V \in \mathbb{R} \}\) is regular everywhere with a metric given by \((48)\).

We now build an extension across \(r_{-}\) for \(r_+ \neq r_{-}\) and \(r_{-} \neq r_0\). With

\[
b_+ = \frac{\nu^2 + 3 \, r_+ - r_{-}}{4 \, R_{-}} = \frac{1}{2} \frac{r_+ - r_{-}}{\sqrt{r_+(r_+ - r_0)}} \frac{\nu^2 + 3}{\sqrt{3(\nu^2 - 1)}}
\]
and \( \rho(r) = e^{b-r} \), define
\[
\begin{align*}
\bar{U} &= \tilde{\rho}(r) e^{b-\ell_r} \\
\bar{V} &= \tilde{\rho}(r) e^{-b-\ell_r} \\
\theta_- &= \theta - \frac{N_0(r_-)}{2b_-} \ln \frac{\bar{U}}{\bar{V}} \quad \text{for } \tilde{r}_0 < r < r_- \\
\bar{U} &= -\tilde{\rho}(r) e^{b-\ell_r} \\
\bar{V} &= \tilde{\rho}(r) e^{-b-\ell_r} \\
\theta_- &= \theta + \frac{N_0(r_-)}{2b_-} \ln \frac{\bar{U}}{\bar{V}} \quad \text{for } r_- < r < r_+.
\end{align*}
\]

The metric becomes
\[
\text{d}s^2 = \Omega^2_- \text{d}\bar{U} \text{d}\bar{V} + \ell^2 R^2 (\text{d}\theta_- + N_0 \tilde{\varphi}(\bar{V} \text{d}\bar{U} - \bar{U} \text{d}\bar{V}))^2
\]
with
\[
\Omega^2_- = \frac{4\ell^2}{v^2 + 3} \frac{r_- (r_- - r_0) (r_+ - r)}{(r_+ - r_0)^2 r (r - r_0)} \left( \sqrt{r} \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^2
\times \left( \sqrt{r} \sqrt{r_+ - r_0} + \sqrt{r} \sqrt{r - r_0} \right)^2
\times \left( \sqrt{r} \sqrt{r_+ - r_0} + \sqrt{r - r_0} \right)^2
\]
and \( r \) is given implicitly by \( \bar{U} \bar{V} \), which is again monotonic in \( r \). We have the limits \( \lim_{r \to r_0} \bar{U} \bar{V} = 0 \), \( \lim_{r \to r_-} \bar{U} \bar{V} = 0 \) and \( \lim_{r \to r_+} \bar{U} \bar{V} = -\infty \).

We similarly extend the coordinate range with the isometry \( U \leftrightarrow -U \), \( V \leftrightarrow -V \). The patch \( K_- = \{ \bar{U}, \bar{V} \in \mathbb{R} \} \) is defined regularly throughout with the metric given in (49).

By transforming into the finite-range coordinates \( \tan(u) = U \) and \( \tan(v) = V \), and similarly \( \tan(\bar{u}) = \rho_0 \bar{U} \) and \( \tan(\bar{v}) = \rho_0 \bar{V} \), we draw in figure 6 the Carter–Penrose diagrams for the two patches. Note that the conformal factor multiplying the connection one-form in the metric blows up as
\[
\frac{R^2}{U^2 V^2 \Omega^2_-} \sim \mathcal{O}(r^{1-\ell_r}).
\]

To circumvent any ambiguity, we compactify the manifold by using instead the coordinate system
\[
\begin{align*}
\hat{U} &= U^{z(U)+1} \\
\hat{V} &= V^{z(V)+1},
\end{align*}
\]
where the exponent \( z(x) \) is a function that is zero for small but positive \( x \) and grows smoothly within a finite range up to the constant value of \( \frac{r_- - r_0}{\sqrt{r_+ - r_0}} \). The factor multiplying the connection one-form then becomes finite and non-vanishing in the limit \( r \to \infty \). The maximal extension is obtained by concutting \( K_+ \) after \( K_- \) ad infinitum, as in figure 7.

For the critical value \( r_+/r_- = 4v^2/(v^2 + 3) \) we define the patch \( K_+ \) as before. With the special value
\[
b_+ = \frac{v^2 + 3}{4v},
\]
we find
\[
\Omega^2_+ = \frac{4\ell^2}{3(v^2 - 1)} \frac{4v^2}{v^2 + 3} \frac{r_+}{r} \left( \sqrt{r} \sqrt{3(v^2 - 1)} + 2v \sqrt{r - r_-} \right)^2 \left( \sqrt{r} + \sqrt{r - r_-} \right)^2 \frac{\sqrt{r_+ - r_0}}{r_+ - r_0}.
\]
Figure 6. Penrose diagrams of Kruskal patches for $r_0 \neq r_-$ black holes.

Figure 7. The Penrose diagram of maximally extended $r_0 \neq r_-$ black holes.

However, here we do not extend beyond the inner horizon $r_-$ where $|\partial_\theta|^2 < 0$. The Kruskal coordinates have the limits $\lim_{r \to +\infty} UV = +\infty$, $\lim_{r \to r_+} UV = 0$ and

$$\lim_{r \to 0} UV = -\rho_0^2 = -\frac{1}{v^2 + 3} \frac{V \sin \frac{\pi l}{L}}{r}. $$

The Penrose diagram of the critical black hole is drawn in figure 8, where we use $U = \rho_0 \tan(u)$ and $V = \rho_0 \tan(v)$. 
4.3. Kruskal extension of extremal black holes

Finally, we describe the extremal case. We present the conformal compactification at once, by using a transformation similar to the one for the extremal RN in [18]. However, some care is needed to show that the connection one-form is also well defined. Using the tortoise coordinate, define for \( r > r_- \)

\[
\tan U = t + r_+ \tag{50}
\]

\[
\tan V = -t + r_+ \tag{51}
\]

\[
\theta_{UV} = \theta - N_0(r_-)t - C \left( 2 \tanh^{-1} \tan \frac{U}{2} - 2 \tanh^{-1} \tan \frac{V}{2} \right), \tag{52}
\]

with the constant

\[
C = -\frac{4\nu}{(\nu^2 + 3)\sqrt{3(\nu^2 - 1)}\sqrt{r_- - r_0}}.
\]

The metric takes the form

\[
g = \Omega^2 \, dU \, dV + \ell^2 R^2 (d\theta_{UV} + \tilde{N}_{UV}(dU - dV))^2,
\]

with

\[
\Omega^2 = \frac{N^2}{\cos^2 U \cos^2 V}
\]

and \( \tilde{N}_{UV} \) is zero on the horizon. We first observe that \( \Omega^2 \) is non-zero on the future and past horizon. Indeed, the dangerous factor \( \frac{(r - r_-)^2}{\cos^2 V} \) in the limit \( V \to 0 \) goes like

\[
\left( \frac{1}{\cos V} \right) \left( \frac{1}{r - r_-} \right)^{-1} \rightarrow \left( \frac{2 \sin V}{\cos^2 V} \right) \left( -\frac{2\nu}{\pi} \right)^{-1} \rightarrow 2 \frac{\sqrt{3(\nu^2 - 1)}}{\nu^2 + 3} \frac{1}{r_- - r_0},
\]

where the last equation uses the derivative of the tortoise co-ordinate in (45). It follows that \( \lim_{V \to 0} \Omega^2 \) is finite and non-vanishing on the future horizon, and similarly on the past.
We also defined $\theta_{UV}$ in (52) with the term linear in $C$ so that a potential pole of $g(\partial\theta, \partial U) \sim \tilde{N}_{UV}$ in $r - r_-$ vanishes.

Altogether, this means that we can use the same transformation on and behind the horizon but for a different domain of $U, V$, and by replacing $C \rightarrow -C$. The singular region is at $\tan U + \tan V = 2r_*$ which can be brought to zero by a suitable shift in $r_*$. The Penrose diagram of the extremal black hole is drawn in figure 9 and the maximal extension can be obtained with the isometry $U - V \mapsto U - V + 2\pi \mathbb{Z}$.

5. Spacetime limits

In the previous sections we explored the geometry of warped AdS, its black hole quotients and their causal properties. In particular, the extremal black holes are obtained from a different quotient than their non-extremal counterparts. At the same time, the extremal black holes are a regular limit of the non-extremal black holes, in the sense that we can set $r_- = r_+$ in the ADM form. In this section we explain this limit in more detail.

We also want to ask what other classical\footnote{That is, we consider $\ell$, $G$ and $\nu$ fixed.} limits we can obtain from the warped AdS$_3$ black holes. We will obtain the near-horizon geometry of extremal black holes and we will define several other spacetime limits, which give us the self-dual warped AdS, in either accelerating or Poincaré coordinates, and warped AdS$_3$ with a proper time identification.

We find it helpful to recall Geroch’s notion of a spacetime limit [19]. Here one collects a family of metric spacetimes $(M_L, g_L)$, where $L > 0$, and constructs the augmented manifold $\mathcal{M} = \{(M_L, g_L, L)\}$. A spacetime limit, $L \rightarrow 0$, is invariantly defined on the boundary
of \( \mathcal{C} / \mathcal{H} \). Spacetime limits are interesting for the properties of the family \((M_L, g_L)\) that are inherited in the limit, a typical example being that the rank of Killing vectors and Killing spinors [20] does not reduce. Naturally, the spacetime limit \((M_0, g_0)\) is of interest when its maximal extension is not included in the original phase space.

An instance of Geroch’s notion is when there is a local isometry \( f_L : M_L \rightarrow M_1 \), for \( L > 0 \), between the metrics \( g_L \) and \( g_1 \). The limit can then be said to be of the metric itself \( g_1 \) rather than a limit in the family of metrics \( g_L \). An example is the Penrose limit [20]. A metric limit typically involves blowing up a neighbourhood of the spacetime. Minkowski space is not only a spacetime limit of 4D black holes, where the mass \( M \equiv \lim_{L \rightarrow 0} \), but can also be written in terms of a metric limit [19]. In the latter case, one is translating in the limit to the asymptotically flat region while keeping the mass \( M \) fixed. In this paper, we call a metric limit the near-horizon geometry of \( g_1 \) when the isometry \( f_L \) fixes the outer horizon.

In our case, the non-extremal black hole metrics are parametrized by \((T_R, T_L)\) which we take as functions of \( L > 0 \). Each black hole in the phase space is given by the identification \( K \) as written in (20). Note, though, that the identification vector in (20) is unique up to \( SL(2, \mathbb{R}) \) rotations. The question we ask is, what are the limits of the non-extremal black holes as \( T_R \rightarrow 0 \).

In order to simplify our discussion, we do not ask what happens in the limit behind the outer horizon. We thus take \( M_L \) to cover only part of the maximally extended spacetime. In practice this means we can work with the accelerating, or Poincaré coordinates, and define the limits explicitly. The coordinates will thus depend explicitly on \( L \). This description is complementary to the previous not only for practical reasons, but also because it describes the relation of the coordinate range of the limit manifold \( M_0 \) to that of \( M_{L>0} \).

We first describe the near-horizon limit of the extremal black holes (section 5.1) using the coordinate description in the framework of [21–23]. We then consider spacetime limits of non-extremal black holes when \( T_R \rightarrow 0 \). There are two such limits. The first gives us a geometry similar to the near-horizon geometry of the extremal ones, but in accelerating coordinates (section 5.2). We call this the near-extremal limit. The second one gives the extremal black holes (section 5.3). We also describe the near-horizon geometry of extremal black holes in the invariant description (section 5.4). Finally, we consider the case when we send \( T_R \rightarrow 0 \) while keeping the Hawking temperature fixed (section 5.5).

5.1. Near-horizon limit

Let us erect Gaussian null coordinates on the future horizon of a spacelike warped black hole, as explained in [24]. The ingoing coordinates \((u, r, \theta_n)\) are such that \( \theta_n \) is a well-defined angle on a spacelike section of the horizon and \( u \) is the group parameter of \( \xi = \partial_u \). Recall that the metric in ingoing coordinates has the form (47):

\[
g = -N^2 \, du^2 + \frac{\ell^2}{R} \, dr \, du + \ell^2 R^2 (d\theta_n + N \theta_n \, du)^2.
\]

We are interested in defining a new coordinate \( \tilde{r} \) that is the affine parameter of a null geodesic congruence \( \gamma \) emanating from the horizon and parametrized by \((u, \theta_n)\). We fix its velocity \( \dot{\gamma} \) on the future horizon \( \partial \mathcal{R}^* \) to be the normalized null complement of \( \partial_u \) and \( \partial_{\theta_n} \), with respect to the metric: \( g(\dot{\gamma}_u, \partial_u) = 1/2 \) and \( g(\dot{\gamma}_{\theta_n}, \partial_{\theta_n}) = 0 \). The Hamiltonian of a free-falling particle and its geodesic equations are

\[
\mathcal{H} = \frac{2}{\ell^4} \left( \ell^2 R p_u + N^2 R^2 p_{\theta_n}^2 + \frac{\ell^2}{4 R^2} p_{\theta_n}^2 - \ell^2 R N p_u p_{\theta_n} \right),
\]

\[
\dot{r} = \frac{2}{\ell^4} \left( \ell^2 R p_u + 2N^2 R^2 p_{\theta_n} \right).
\]
\[ \dot{\theta} = \frac{2}{\ell^2} \left( \frac{p_{\theta_\infty}}{2R} - RN_{\theta_\infty}p_r \right) \quad \dot{u} = \frac{2}{\ell^2} Rp_r. \]

The equations can easily be solved. The constraint \( H = 0 \) implies \( p_r|_{\bar{r} = r} = p_{\theta_\infty} = 0 \) and with \( p_u = \frac{1}{2} \) we find \( p_\theta = 0, \theta = u = 0 \) and

\[ \frac{dr}{d\bar{r}} = R \ell^2, \]

where \( \bar{r} \) is the affine parameter. This equation is solved generically by

\[ r = r_0 \cosh^2 \left( \frac{\sqrt{3(v^2 - 1)}}{4\ell^2} \bar{r} - c \right). \]

The coordinate transformation (54) covers the region \( r \in (r_0, +\infty) \), which corresponds to

\[ \bar{r} \in \left( -\infty, \frac{4\ell^2}{3(v^2 - 1)} c \right). \]

The other coordinates remain \( u \in \mathbb{R} \) and \( \theta_\infty \) periodic.

For \( r_+ \neq r_- \) the metric takes the form

\[ g = -\bar{r} F(\bar{r}) \, d\bar{r}^2 + d\bar{r} \, du + \ell^2 R^2(r(\bar{r})) \left( d\theta_\infty + N_{\theta_\infty}(r(\bar{r})) \right) \, du^2, \]

where \( N^2 = \bar{r} F(\bar{r}) \) and \( F(\bar{r}) \) is regular non-vanishing on the horizon \( \bar{r} = 0 \). It follows that the near-horizon limit cannot be defined for non-extremal black holes. Indeed, if we assume a diffeomorphism \( \bar{r} \mapsto \bar{r}/L \) that zooms in on a neighbourhood of the horizon, then the component \( g_{ur} \) dictates an appropriate rescaling \( u \mapsto Lu \) so that \( \lim_{L \to 0} g_{ur} \) remains finite. However, this blows up the component \( g_{uu} \).

When \( r_+ = r_- \), \( F(\bar{r}) = \bar{r} H(\bar{r}) \) where \( H(\bar{r}) \) is regular non-vanishing at \( \bar{r} = 0 \). Introducing the coordinate transformation

\[ \bar{r}' = \bar{r}/L \]
\[ u' = Lu, \]

and sending \( L \to 0 \) gives the metric limit

\[ g = \frac{\nu^2 + 3}{4\ell^2} \bar{r}'^2 \, du'^2 + d\bar{r}' \, du' + \ell^2 R^2 \left( d\theta_\infty + \frac{dN_{\theta_\infty}}{dr} \Bigg|_{r_\infty} \frac{R}{\ell^2} \bar{r}' \, du' \right)^2, \]

with

\[ \frac{dN_{\theta_\infty}}{dr} \Bigg|_{r_\infty} = \frac{4 \nu - 2\nu^2 r_\infty - \nu \sqrt{\nu^2 + 3} r_\infty}{(2\nu - \sqrt{\nu^2 + 3})^2}. \]

Observe that as \( L \to 0 \), any point \( \bar{r} \) close to \( \bar{r} = 0 \) is pushed away to infinity with respect to \( \bar{r}' \). The metric in (58) is the self-dual solution with \( \alpha = \frac{\nu^2 + 3}{2\nu} R_\infty \) in Poincaré coordinates. This can be verified by using the diffeomorphism

\[ u' = \tau - \frac{1}{\chi} \]
\[ \bar{r}' = \frac{2\ell^2}{\nu^2 + 3} \]
\[ \phi = \theta + \frac{2\nu}{\nu^2 + 3} R_\infty \ln \chi. \]
The above derivation zooms indefinitely into the future horizon of an extremal black hole along a geodesic congruence. Using the coordinate description we obtained the self-dual warped AdS3 in Poincaré coordinates. This result is universal. We would not have been able to arrive at the same geometry in, say, accelerating or warped coordinates. Since the horizon is non-bifurcate the same should be true for the limit spacetime. One could use equivalently the double null coordinates \((u, v)\), where \(u\) is the ingoing and \(v = -t + r_*\) is the outgoing coordinate. The description using \((u, v)\) serves to show that we are zooming in on the whole of the horizon. Finally, we could have used the ADM coordinates \((r, t)\). The limit is given by \(r - r_- = r' L\) and \(t = t'/L\). This description provides an equivalent explanation for why the limit is in Poincaré coordinates. This is the case because \(t\) is defined asymptotically by observers who wish to probe the horizon. As such, the near-horizon inherits a preferred time which is not related to the global warped time \(\tilde{t}\).

We can already ask what properties are inherited in the limit. It is clear that one such property is the nature of the horizon. The size of the radius of \(\theta\) on the horizon is also inherited, this being a consequence of definition (57) as an isometry that fixes the horizon. We will later describe the near-horizon geometry invariantly, using the identification vector \(\partial_\theta\), and see that this is related to the extremal black hole via \(\alpha = 2\pi \ell T_L\).

### 5.2. Near-extremal limit

Although a non-extremal black hole does not admit a near-horizon limit, we can consider a limit in the black hole phase space \((T_L, T_R)\) for \(T_R \to 0\). This limit cannot be considered a metric limit because \(T_R\) is continuously varied. Furthermore, there is more than one way to take the limit. Here we will consider the case when the limit gives us the self-dual solution in accelerating coordinates. We call the limit the near-extremal near-horizon limit, or near-extremal limit for short, and we stress it is a spacetime limit in the phase space of non-extremal black holes.

A black hole is described by \((T_R, T_L)\) that enter definition (20) of the Killing vector \(\partial_\theta\):  
\[
\partial_\theta = 2\pi \ell T_R r_2 + 2\pi \ell T_L l_2.
\]

There are however two gauge freedoms that we can use in its description. The first is an active \(\text{SL}(2, \mathbb{R})\) rotation that isometrically maps the outer region as embedded in warped AdS3 to a new region. The rotation transforms \(r_2 \mapsto A r_2 + B r_0\), with \(A^2 - B^2 = 1\), and we can use instead the vector 
\[
\partial_{\theta'} = 2\pi \ell T_R (A r_2 \pm B r_0) + 2\pi \ell T_L l_2.
\]  

Note that we are considering an active transformation in warped AdS. That is, the rotation \(\exp(\tanh^{-1}(\frac{r}{L}) r_1)\) is not an isometry of the metric.

The second gauge freedom is how we describe time \(t\). The \(\text{GL}(2, \mathbb{R})\) diffeomorphism in (22) keeps the identification vector \(\partial_\theta\) invariant. However, we are redefining \(\partial_t\) and so the metric form in the new coordinate system does change. It is this freedom that we shall use and fix here. Indeed, note that if we simply take \(T_R = 0\) in (26), that is send \(1/c \to 0\) and keep \(a/c\) fixed in (26), we end up with \(\partial_t\) collinear with \(\partial_\theta\). The coordinates \((t, \theta)\) are thus ill-defined in the limit. We use the transformation 
\[
\begin{pmatrix}
  t' \\
  \theta'
\end{pmatrix} =
\begin{pmatrix}
  -\frac{1}{b} \frac{v^2 + 3}{2\nu} \frac{T_R}{T_L} & 0 \\
  \frac{1}{2\nu} \left( \frac{v^2 + 3}{2\nu} \right) & 1
\end{pmatrix}
\begin{pmatrix}
  t \\
  \theta
\end{pmatrix},
\]

(61)
so that
\[
\begin{pmatrix}
\frac{\partial}{\partial t'} \\
\frac{\partial}{\partial \theta'}
\end{pmatrix} = \begin{pmatrix}
b & 0 \\
2\pi \ell T_R & 2\pi \ell T_L
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \theta}
\end{pmatrix}.
\] (62)
Here we have included an arbitrary \( b > 0 \) constant, which is equivalent to \( b = 1 \) by diffeomorphism invariance.

The near-extremal limit is now well defined in coordinates \( t' \) and \( \theta' \). By simply setting \( R_T = 0 \) in (62) we obtain
\[
\frac{\partial t'}{\partial \theta'} = \left( \frac{b}{2\pi \ell T_L} \right) \frac{\partial}{\partial \theta}.
\] (63)
This identification gives the self-dual geometry with \( \alpha = 2\pi \ell T_L \) in accelerating coordinates.

It is useful to describe the limit explicitly in coordinates. For this, we reuse the accelerating coordinate \( x \), which is related to \( r \) via (31). Recall that \( x \) is given linearly by \( g(\partial \tau, \partial u) \) and so it remains invariant under the transformation (61). We also use the coordinates \( (\theta', t') \) from (61). Altogether we have
\[
r = r_+ - r_-, \quad t = -\frac{2\nu}{v^2 + 3} \frac{b}{\ell T_R} t',
\] (63)
The ADM metric at fixed \( T_L \) and \( T_R > 0 \) in \( (t', x, \phi) \) coordinates is
\[
g = -\frac{\ell^2}{v^2 + 3} b^2 (x^2 - 1) \left( \frac{4\pi \nu \ell T_L}{R(r(x))(v^2 + 3)} \right)^2 \mathrm{d}'t'^2,
\]
\[
+ \frac{\ell^2}{v^2 + 3} \frac{\mathrm{d}x^2}{x^2 - 1} + \ell^2 R^2 (r(x)) (\mathrm{d}\phi + N_t (r(x)) \mathrm{d}'t')^2,
\] (64)
with
\[
N_t (r) = \frac{b}{2\pi \ell T_R} \left( 1 - \frac{2\nu}{v^2 + 3} \frac{2\pi \ell T_L}{2\nu R - \sqrt{r_+ r_- (v^2 + 3)}} \right).
\] (65)
Note that in the limit \( r_+ \rightarrow r_- \), \( r(x) \rightarrow \frac{r_+ + r_-}{2} \). We also have that
\[
R^2 \left( \frac{r_+ + r_-}{2} \right)^2 = \left( \frac{v^2 + 3}{(v^2 + 3)^2} \right)^2 \left( \frac{4\pi \nu \ell T_L}{r_+ r_- - \sqrt{r_+ r_- (v^2 + 3)}} \right)^2.
\]
By using the above, and the equations for \( T_L \) and \( T_R \) in (29) and (30), one sees that the term in parentheses in (65) is zero as \( r_+ \rightarrow r_- \). Therefore, it cancels the pole in \( T_R \). In order to find the limit we expand the function
\[
f \left( \frac{r_+ - r_-}{2} x + \frac{r_+ + r_-}{2} ; r_+, r_- \right) = \frac{2\nu r_+ - \sqrt{r_+ r_- (v^2 + 3)}}{2R^2 (r)}
\]
which is symmetric in its last two arguments, in powers of \( L \), with \( r_\pm = r_c \pm L \) and keeping \( r_c \) and \( x \) fixed:
\[
f (Lx + r_c ; r_c + L, r_c - L) = f (r_c ; r_c, r_c) + f^{(1,0,0)} (r_c ; r_c, r_c) L x + f^{(0,1,0)} (r_c ; r_c, r_c) L
\]
\[
- f^{(0,0,1)} (r_c ; r_c, r_c) L + \mathcal{O} (L^2)
\]
\[
eq \partial_r \left( \frac{2\nu r_+ - \sqrt{r_+ r_- (v^2 + 3)}}{2R^2 (r)} \right) \bigg|_{r = r_+ \rightarrow r_-} \frac{r_+ - r_- x + \mathcal{O} (T_R^2) }{2}.
\]
After some algebra, we find
\[
\lim_{r \to r_+} N_r(x) = \frac{1}{R_-} \frac{2\nu}{v^2 + 3} b x.
\]
With \( R_- = 4\pi \nu \ell T_L/(v^2 + 3) \), we confirm that metric (64) becomes at \( r_+ \to r_- \) the self-dual solution with \( \alpha = 2\pi \ell T_L \) and \( t' = b x \).

Observe that the bifurcate nature of the horizon is inherited in accelerating coordinates. Although this is not a metric limit, in the sense that we have not fixed a black hole geometry, we intuitively understand (63) as zooming in close to the outer horizon of non-extremal black holes with \( T_R \approx 0 \). Finally note that, in taking \( T_R \to 0 \), we can keep \( T_L \) or some other combination of \( T_L \) and \( T_R \) fixed. The interpretation of the near-horizon limit in the context of the 4D Planck scale limit \( L_p \to 0 \) for RN black holes has been discussed in [25, 26], see also [27].

5.3. Extremal limit

In the ADM form one can reach the extremal black holes by setting \( r_+ = r_- \) in (32). We can describe this by combining the limit \( T_R \to 0 \) with an \( \text{SL}(2, \mathbb{R})_R \) transformation,
\[
\partial \nu' = 2\pi \ell T_R (A r^2 \pm B r_0) + 2\pi \ell T_L l_2,
\]
where we set
\[
A = \frac{1}{\ell T_R}, \quad B = \sqrt{\left(\frac{1}{\ell T_R}\right)^2 - 1}.
\]
In the limit \( T_R \to 0 \) we have
\[
\partial \nu = \frac{\nu^2 + 3}{2\nu} \partial u, \quad \partial \nu' = 2\pi (r^2 \pm r_0) + 2\pi \ell T_L l_2,
\]
which describe precisely the extremal black holes. Here we do not need to use a \( \text{GL}(2, \mathbb{R}) \) transformation.

We claim that this limit is equivalent to setting \( r_- = r_+ \) in the ADM form. Indeed, in section 3 we only considered the case when \( \partial \theta \) is a linear combination of \( r^2 \) and \( l^2 \). Since \( e^{i\tau_1} \) is invertible, the identification along \( \partial \theta \) is equivalent to the identification along \( \partial \theta' \):
\[
e^{2\pi \nu_{\partial \theta}} p \sim p \iff e^{2\pi \partial \theta e^{i\tau_1}} p \sim e^{i\tau_1} p
\]
for every point \( p \) in warped AdS. We can define coordinates \((r', t', \theta')\) on the mapped region by using the \((r, t, \theta)\) coordinates of section 3, with \( r' = r, t' = t, \theta' = \theta \), see figure 10.

By using the invariant description of the identification vector, it is obvious that in sending \( T_R \to 0 \), and keeping \( T_L \) finite, non-extremal black holes can either limit to the near-extremal geometry with \( \alpha = 2\pi \ell T_L \), or to the extremal black hole with the same \( T_L \). That is, we can either try to keep the term in \( \partial \theta \) that is multiplied by \( T_R \) (the extremal limit) or not (the near-extremal limit).

5.4. Near-horizon geometry, again

We are now able to describe the near-horizon geometry of the extremal black holes, which was given in section 5.1, in an invariant way. Let us accordingly switch to Poincaré coordinates \((x, \tau, u)\). From (20) and by using an \( \text{SL}(2, \mathbb{R})_R \) rotation generated by \( r_1 \), the identification vector is
\[
\partial \theta = 2\pi L (r^2 + r_0) + 2\pi \ell T_L l_2 \quad \text{with} \quad L > 0.
\]
It is also necessary to use a matrix transformation as in section 5.2, so that \( \partial_\nu \) is not collinear with \( \partial_\theta \) in the limit \( L \to 0 \). We use a matrix transformation identical in form to (61), but replace \( T_R \) with \( L \). In the limit \( L \to 0 \), we obtain the self-dual solution in Poincaré coordinates, with \( \alpha = 2 \pi \ell T_L \):

\[
\begin{align*}
\partial_t &= b \partial_\tau \\
\partial_\theta &= 2 \pi \ell T_L \partial_\nu.
\end{align*}
\]

One can use coordinates to describe the above limit. In fact, the coordinate transformation follows closely section 5.2, with some minor changes. In (63), the first equation should be replaced with

\[
x = L (r - r^-),
\]

and \( T_R \) should be replaced with \( L \) in the other two equations. The metric in \((r', t', \phi')\) coordinates, (64), becomes

\[
g = -\frac{\ell^2 v^2 + 3}{v^2 + 3} \left( \frac{4 \pi v \ell T_L}{R(r)(v^2 + 3)} \right)^2 \, dt'^2 + \frac{\ell^2}{v^2 + 3} \, dx^2 + \ell^2 R^2(r) (d\phi + N_t(r) \, dt')^2,
\]

and, in the limit \( L \to 0 \), the metric limits to the self-dual geometry in Poincaré coordinates, with \( \alpha = 2 \pi \ell T_L \) and \( t' = b \tau \).

It might seem surprising that this is the same limit as in section 5.1. Observe however that \( \partial_t - \partial_\phi \) is proportional to the Killing vector that is null on the horizon. In using the matrix transformation we are rescaling the ingoing coordinate as before. The radial coordinate is then rescaled appropriately so that the limit is finite.

### 5.5. Vacuum limit

We finally consider the limit \( T_R, T_L \to 0 \) with the ratio \( T_L/T_R \) kept constant. This is equivalent to keeping a fixed ratio \( r_e/r_- \) and sending \( r_- \to 0 \). In [9] this limit was called the vacuum solution. In order to keep \( \partial_\theta \) finite, we use the \( \text{SL}(2, \mathbb{R}) \) transformation in (66), with the same parameters (67), so that in the limit \( T_R \to 0 \) we obtain

\[
\begin{align*}
\partial_t &= \frac{v^2 + 3}{2v} \partial_u \\
\partial_\theta &= 2 \pi (r_2 \pm r_0).
\end{align*}
\]

Here we did not need the \( \text{GL}(2, \mathbb{R}) \) transformation. Observe that the Killing vectors \( \partial_t \) and \( \partial_\theta \) do not depend on \( r_e/r_- \). The limit is thus universal and there is no remnant of the ratio.
The geometry we obtain by (68) is warped AdS$_3$ in Poincaré coordinates with a periodic identification of the proper time. That is, metric (18)

$$g_{\ell,\nu} = \frac{\ell^2}{\nu^2 + 3} \left( -\nu^2 \, dr^2 + \frac{d\xi^2}{\nu^2 + 3} + \frac{4\nu^2}{\nu^2 + 3} (du + x \, d\tau)^2 \right),$$

with $\tau = \theta$ and $u = \frac{\nu^2 + 3}{2}r$; hence, $\tau \sim \tau + 2\pi$ and $x, u \in \mathbb{R}$.

We can see the same result by using coordinates. As in the extremal limit, we use the metric in ADM form, and we send the parameters $r_-$ and $r_+$ to zero keeping $r_+ / r_- \text{fixed}$. The metric becomes

$$\lim_{r_- \to 0, r_+ \to 0} \left( g = -\ell^2 \frac{\nu^2 + 3}{3(\nu^2 - 1)} \, dr^2 + \frac{\ell^2}{\nu^2 + 3} \, dr^2 + \ell^2 \frac{3(\nu^2 - 1)}{4} r^2 \left( d\theta + \frac{4\nu}{3(\nu^2 - 1)} \frac{1}{r} \, dr \right)^2 \right)$$

$$= -\ell^2 \frac{\nu^2 + 3}{4} \, r^2 \, d\theta^2 + \frac{\ell^2}{\nu^2 + 3} \, r^2 + \frac{4\nu^2 \ell^2}{(\nu^2 + 3)^2} \left( \frac{\nu^2 + 3}{2\nu} \, dr + \frac{\nu^2 + 3}{2} \frac{d\theta}{r} \right)^2.$$ (70)

The identification with Poincaré coordinates can be made with $x = \frac{\nu^2 + 3}{2}r$.

The limit corresponds to sending $M_{\text{ADT}}$ and $J_{\text{ADT}}$ to zero while keeping the Hawking temperature fixed, see appendix C. One can also interpret this as a limit to the far-away region. That is, the metric in (70) corresponds to keeping the leading-order components of the black hole metric when $r \gg r_+$. This agrees with the fixed components of the asymptotic boundary conditions of [11], which we recall in appendix C.

6. Discussion

In this paper we explored the geometry of black holes in spacelike stretched warped AdS. We elaborated on the construction of warped AdS$_3$ from first principles, described suitable coordinates and performed the quotient construction. We focused on the case when causal singularities do exist and are hidden behind a Killing horizon. The geometries are ideal, in the sense that they can be continued to regions that contain new singularities and new asymptotic regions. We found the causal structure and showed that the geometries fall into three classes that resemble the causal structure of the RN black hole.

We pointed out two features that are usually suppressed in the literature. The first is that the black hole metric parametrized by $r_+$ and $r_-$ presents a redundancy; in that for a certain region two sets of parameters $(r_+, r_-)$ describe the same geometry. The second is that the ratio of the left to right temperature is bounded from below, if the geometry is to describe a causal singularity that is hidden behind Killing horizons. In [11] care was taken to consistently define an asymptotically Killing algebra [28] that contains a centrally extended Virasoro algebra with generators $z_m$, so that $\mathcal{Z}_0$ has positive spectrum and a central extension that matches the AdS/CFT expectation [29]. The bound on the ratio of temperatures $T_L / T_R$ would then imply an upper bound on $\mathcal{Z}_0$.

We also described various spacetime limits that one can take in the black hole phase space. We do this by studying the behaviour of the identification vector $\hat{\partial}_0$ for different significant limits of the invariants $T_R, T_L$ and their ratio. We chose this exposition for the clarity of the geometric interpretation of the limits, and also to avoid the ambiguities that could come from a coordinate description. In this description, it is easy to see that the possible limits using this method are again quotients of warped AdS. Furthermore, the spacetime limits inherit suitable coordinates that are not global. In particular, we obtain the self-dual solution $T_L / T_R$. 

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in accelerating or Poincaré coordinates, and warped AdS$_3$ in Poincaré coordinates under a proper time identification.

The spacelike stretched black holes are a subset of the general black holes of cosmological Einstein–Maxwell theory with gravitational and gauge Chern–Simons couplings, which were presented in [30], with $\mu_E/\mu_G = 2/3$ and $\beta^2 = (v^2 + 3)/(4v^2)$. There, the causal structure of the general black holes was also first reported. We have here presented an explicit Kruskal extension, which underlies the Penrose diagram of the maximal extension. Our derivation focuses on the metric $g_2$ that is defined on the two-dimensional quotient space of a black hole by the global isometry $\partial_\theta$. One can successively remove detail from our presentation but retain the reduction on $g_2$, since this captures the essential causal relations of the 3D spacetime.

Also in [30], local coordinate transformations were given that relate the various black holes, the self-dual solution and the vacuum. Here, we only write the local coordinate transformations between the black holes, or the self-dual solution, and spacelike warped AdS, which precisely define the first as discrete quotients of the latter. Furthermore, the vacuum and self-dual solutions are obtained here as limits of the black holes. This was done invariantly using the identification vector, but also through well-defined coordinate transformations. We note this comparison so as to highlight the structure of this work. Let us also remark that the limits we consider are classical, that is $\nu$, $G$ and $\ell$ are kept fixed. This does not allow us to obtain, for instance, the black holes with vanishing cosmological constant [31].

Let us also compare to the construction to the BTZ black holes of Einstein gravity with a negative cosmological constant. The BTZ black holes are necessarily asymptotically conformally flat. Therefore, the conformal boundary is always timelike. The causal diagrams of the BTZ black holes fall into two classes, depending on whether the geometry is extremal or not [1, 2]. This is different to the warped AdS$_3$ case, which is not conformally flat.

One motivation for this work was to find a non-extremal spacetime limit where the acceleration coordinate $\partial_\tau$ would explicitly depend on a parameter $b$. This would imply that the limit inherits two parameters rather than the one in $u = 2\pi \ell T_L \phi$. Then one could approximate the chiral thermal Green functions of the near-extremal black holes with those computed in the self-dual warped AdS$_3$ space in accelerating coordinates, see [10, 32]. It is for this reason that we introduced the constant $b$ in (64). By diffeomorphism invariance though, we can set this constant equal to 1. We speculated on whether a suitable set of asymptotic conditions can break this freedom.

TMG is expected to have a rich spectrum and we believe that the solution space will present new insight in the AdS/CFT correspondence. Understanding better the relation of TMG and its solutions to four-dimensional reality is another direction we look forward to.

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Appendix A. Hyperbolic fibration

In section 2 we observed that the warped metric in warped coordinates

$$
\tilde{g}_{\ell,v} = \frac{\ell^2}{v^2 + 3} \left( -\cosh^2 \sigma \, d\tilde{t}^2 + d\sigma^2 + \frac{4v^2}{v^2 + 3} (d\tilde{u} + \sinh \sigma \, d\tilde{t})^2 \right)
$$
is compatible with a double cover of $\text{SL}(2, \mathbb{R})$. That is, the time identification on $\text{SL}(2, \mathbb{R})$ is $\tilde{t} \sim \tilde{t} + 4\pi$, whereas the base space describes a two-dimensional quadric with $\tilde{t} \sim \tilde{t} + 2\pi$.

Here we elaborate on this without using warped coordinates. The vector field $l_2$ in (2) defines a non-trivial real-line fibration of the quadric

$$T_1^2 + T_2^2 - X_1^2 - X_2^2 = 1$$

(A.1)

over the quadric

$$\tilde{T}_1^2 + \tilde{T}_2^2 - \tilde{X}_1^2 - \tilde{X}_2^2 = 1.$$  

(A.2)

Explicitly, the projection

$$\tilde{T}_1 - \tilde{X} = 2(X_1 + T_1)(X_2 - T_2)$$

$$\tilde{T}_1 + \tilde{X} = 2(X_1 - T_1)(X_2 + T_2)$$

$$\tilde{T}_2 = 2(T_1^2 - X_1^2) - 1$$

(A.3)

is invariant under $l_2$ and satisfies (A.2). Conversely, for every point $(\tilde{T}_1, \tilde{T}_2, \tilde{X}_1)$ that satisfies (A.2), there are two real-line orbits that satisfy (A.1) and (A.3). Indeed, (A.3) can be solved depending on the value of $\tilde{T}_2$: if $\tilde{T}_2 < -1$ the solutions will cross $\tilde{T}_2 = 0$ and the two orbits are distinguished by the sign of $X_2$; similarly, if $\tilde{T}_2 > -1$ the same happens, but with $\tilde{T}_2$ and $X_2$ exchanged; if $\tilde{T}_2 = -1$ the two orbits are given by $\tilde{T}_2 = \pm X_2$. This is shown in figure A1(b). Furthermore, the two orbits are connected by the action of $r_0$. On the base space, $r_0$ induces a rotation of period $2\pi$:

$$\mathcal{L}_{r_0}(\tilde{T}_2) = \tilde{T}_1$$

$$\mathcal{L}_{r_0}(\tilde{T}_1) = -\tilde{T}_2$$

$$\mathcal{L}_{r_0}(\tilde{X}) = 0,$$

while, from (3), in $\text{SL}(2, \mathbb{R})$ it has a period of $4\pi$. This is schematically depicted in figure A1(a).

Let us note that the hyperbolic fibration of $\text{SL}(2, \mathbb{R})$ is different to the Hopf fibration of the three-sphere; in that the latter covers the two-sphere once. If two complex numbers $z_1, z_2$ are used to describe the three-sphere as $|z_1|^2 + |z_2|^2 = 1$, then the projection of the Hopf fibration is $\pi(z_1, z_2) = (2z_1z_2^*, |z_1|^2 - |z_2|^2) \in S^2$. For every point in $S^2$ there is then precisely one orbit in $S^3$ given by the action $(z_1, z_2) \mapsto (e^{\theta}z_1, e^{\theta}z_2)$. 

---

Figure A1. Hyperbolic fibration.
Appendix B. Diffeomorphisms

In writing the warped metric in accelerating coordinates we did not need to use an explicit
diffeomorphism. Indeed, using the definition
\[ x = \left(\nu^2 + 3\right)^{\frac{1}{4}} g_{\ell,\nu}(\partial_u, \partial_{\tau}) = \cosh \sigma \sin \tilde{\tau}, \]
we were able to calculate the metric components
\[ g_{\tau\tau} = g(r_2, l_2), \quad g_{\tau u} = g(r_2, l_2), \quad g_{uu} = g(l_2, l_2) \]
and \( g^{xx} = g^{-1}(dx, dx) \) in terms of \( x \). Let us present here an explicit diffeomorphism of the
accelerating patch.

The region \( x > 1 \) with metric (16) isometrically embeds in warped \( \text{AdS}_3 \) under
\[
\begin{align*}
\sinh \sigma &= \sqrt{x^2 - 1} \cosh \tau \\
\cot \tilde{\tau} &= -\frac{\sqrt{x^2 - 1}}{x} \sinh \tau \\
\tilde{u} &= u + \tanh^{-1}\left(\frac{\tan \tau}{x}\right). 
\end{align*}
\]
This covers \( \tilde{u} \in \mathbb{R}, \sigma > 0 \) and \( \tilde{\tau} \in (0, \pi) \) with \( \cosh \sigma \sin \tilde{\tau} > 1 \). The inverse of (B.1) is
\[
\begin{align*}
x &= \cosh \sigma \sin \tilde{\tau} \\
\tanh \tau &= -\coth \sigma \cos \tilde{\tau} \\
u &= \tilde{u} + \tanh^{-1}\left(\frac{\cot \tilde{\tau}}{\sinh \sigma}\right),
\end{align*}
\]
which is well defined for \( \sigma > 0, \tilde{\tau} \in (0, \pi) \) and
\[
\frac{\cot \tilde{\tau}}{\sinh \sigma} < 1 \iff |\cosh \sigma \sin \tilde{\tau}| > 1.
\]
Similarly the region \( |x| < 1 \) can be embedded with
\[
\begin{align*}
\sinh \sigma &= \sqrt{1 - x^2} \sinh \tau \\
\tan \tilde{\tau} &= -\frac{x}{\sqrt{1 - x^2}} \cosh \tau \\
\tilde{u} &= u + \tanh^{-1}(x \tanh \tau),
\end{align*}
\]
whose inverse is
\[
\begin{align*}
x &= \cosh \sigma \sin \tilde{\tau} \\
\tanh \tau &= -\frac{\tanh \sigma}{\cos \tilde{\tau}} \\
u &= \tilde{u} + \tanh^{-1}(\sinh \sigma \tan \tilde{\tau}),
\end{align*}
\]
Here we cover \( \sigma \in \mathbb{R}, \tilde{u} \in \mathbb{R} \) and
\[
|\sinh \sigma \tan \tilde{\tau}| < 1 \iff |\cosh \sigma \sin \tilde{\tau}| < 1.
\]

Similarly, let us give an explicit diffeomorphism between the Poincaré and warped
coordinates and some detail on how we derived it. First we show that in Poincaré coordinates,
\( r_1 = x \partial_x - \tau \partial_{\tau} \) follows from
\[
\begin{align*}
r_1(x) &= \frac{(\nu^2 + 3)^2}{4v^2 \ell^2} \mathcal{L}_{\partial_{\tau}}(g_{\ell,\nu}(\partial_u, \partial_{\tau})) = \frac{(\nu^2 + 3)^2}{4v^2 \ell^2} g_{\ell,\nu}(\partial_u, [r_1, \partial_{\tau}]) = x \quad (B.3) \\
[r_1, \partial_{\tau}] &= \partial_{\tau} \Rightarrow \partial_{\tau}(r_1(\tau)) = -1 \quad \text{and} \quad \partial_{\tau}(r_1(u)) = 0 \\
r_1(\partial_u) = 0 \Rightarrow \partial_u(r_1(u)) = 0 \quad \text{and} \quad \partial_u(r_1(\tau)) = 0, \quad (B.5)
\end{align*}
\]
where we use that \( r_1 \) is Killing and the commutation relations. Indeed, note how rescaling \( x \mapsto e^{\zeta}x \) and \( \tau \mapsto e^{-\zeta} \tau \) is an isometry of the metric in Poincaré coordinates. Using this and the relation

\[
x = \frac{(v^2 + 3)^2}{4v^2 \ell^2} g_{t,v}(\partial_u, \partial_t) = \sinh \sigma + \sin i \cosh \sigma,
\]

we compute

\[
\partial_t x = -L_{r_0} x = \frac{(v^2 + 3)^2}{4v^2 \ell^2} g_{t,v}(\partial_u, r_1),
\]

or equivalently

\[
x \tau = -\cos i \cosh \sigma.
\]

Finally, integrating \( \sinh \sigma \, dt - x \, d\tau \) gives us

\[
u = \tilde{\nu} + \ln \left( \pm \frac{\cosh \sigma/2 \cos t/2 + \sinh \sigma/2 \sin t/2}{\cosh \sigma/2 \sin t/2 + \sinh \sigma/2 \cos t/2} \right).
\]

The explicit diffeomorphism is given by (B.6), (B.7) and (B.8). We confirm that they are well defined and \( x \not\equiv 0 \) is equivalent to

\[
\frac{\cosh \sigma/2 \cos t/2 + \sinh \sigma/2 \sin t/2}{\cosh \sigma/2 \sin t/2 + \sinh \sigma/2 \cos t/2} \not\equiv 0.
\]

The quotient construction of section 3 uses a linear relation between the black hole coordinates \((t, \theta)\) and the accelerating or Poincaré coordinates. Combining the linear relation with the above diffeomorphisms, one can write the diffeomorphism of the black hole coordinates with respect to the warped coordinates that define the quotient:

\[
\tilde{t} = \tan^{-1} \left( \frac{2\sqrt{(r-r_0)(r-r_-)}}{2r-r_+ - r_-} \sinh \left( \frac{1}{4} (r_+ - r_-)(v^2 + 3)\theta \right) \right)
\]

\[
\sigma = \sinh^{-1} \left( \frac{2\sqrt{(r-r_0)(r-r_-)}}{r_+ - r_-} \cosh \left( \frac{1}{4} (r_+ - r_-)(v^2 + 3)\theta \right) \right)
\]

\[
\tilde{\nu} = \frac{v^2 + 3}{4v} \left( 2\nu + \left( v(r_+ + r_-) - \sqrt{r_+ r_- (v^2 + 3)} \right) \theta \right)
\]

\[
+ \coth^{-1} \left( \frac{2r - r_+ - r_-}{r_+ - r_-} \cosh \left( \frac{1}{4} (r_+ - r_-)(v^2 + 3)\theta \right) \right).
\]

This is essentially the transformation in equations (5.3)–(5.5) of [9], although here they are defined in \( r > r_+ \) as opposed to \( r_- < r < r_+ \). Note that we have translated \( \tilde{t} \mapsto \tilde{t} + \frac{\pi}{2} \) with respect to (B.1).

**Appendix C. Thermodynamics**

We would like to recall here the thermodynamic quantities that were computed for the spacelike warped black holes in [9]. It is also noteworthy to translate the bound on the ratio of temperatures and the region \( r_+ / r_- < \lambda_f \) into conventions used in the literature. However, let us briefly comment on the ADM form. A general stationary, axisymmetric, asymptotically-flat black hole uniquely normalizes the Killing vector \( \xi = \partial_t - \Omega \partial_\theta \) that is null on its horizon, by using the asymptotically defined \( t \) and \( \theta \). Quantities like the surface gravity \( \kappa_0 = 2\pi T_H \) on its horizon \( \partial \mathscr{H} \) are unambiguously defined, e.g.

\[
\nabla_t \xi \equiv \kappa_0 \xi.
\]
Were we to use a different time and angle
\[ t' = \Lambda t \]
\[ \theta' = \theta + b t, \]
the Hawking temperature, angular velocity \( \Omega \) and ADT charges [33–37], here the mass \( M_{\text{ADT}} \) and angular momentum \( J_{\text{ADT}} \), would transform as
\[
T'_H = \frac{1}{\Lambda} T_H
\]
\[
\Omega' = \Omega + \frac{b}{\Lambda}
\]
\[
\delta M'_{\text{ADT}} = \frac{1}{\Lambda} \delta M_{\text{ADT}} - \frac{b}{\Lambda} \delta J_{\text{ADT}}
\]
\[
\delta J'_{\text{ADT}} = \delta J_{\text{ADT}}.
\]
On the other hand, the entropy variation in the first law, \( \delta S = \frac{1}{T_H} (\delta M_{\text{ADT}} - \Omega \delta J_{\text{ADT}}) \), is seen to be invariant under (C.1). The Wald formula for the entropy [38] as applied for TMG in [39] (see also [34], section 4.2) depends on the asymptotic orthonormal frame and its spin connection, and therefore is indeed invariant under (C.1).

We normalize the thermodynamic quantities with respect to the frame where
\[
g(\partial t, \partial t) = \ell^2.
\]
This is compatible to the asymptotically warped AdS3 conditions in [11]. In particular, it fixes both \( t \) and \( \theta \) coordinates as in the ADM form (32). From [9], we have
\[
T_H = \frac{v^2 + 3}{4\pi \ell v} \frac{T_R}{T_L + T_R}
\]
\[
\Omega = -\frac{v^2 + 3}{4\pi v} \frac{1}{T_R + T_L}
\]
\[
M_{\text{ADT}} = \frac{\pi}{3G} \ell T_L
\]
\[
J_{\text{ADT}} = \frac{v \ell}{3(v^2 + 3)G} \left( (2\pi \ell T_L)^2 - \frac{5v^2 + 3}{4v^2} (2\pi \ell T_R)^2 \right)
\]
\[
S = \frac{\pi^2 \ell}{3} \left( \frac{5v^2 + 3}{v(v^2 + 3)G} \ell T_R + \frac{4\nu}{(v^2 + 3)G} \ell T_L \right).
\]
The CFT correspondence conjecture in [9] allows one to write the entropy in the form of Cardy’s formula [29] with left/right central extension charges \( c_R = \frac{5v^2 + 3}{v^2 + 3} \ell \) and \( c_L = \frac{4\nu}{v^2 + 3} \ell \), respectively. The bounds in (38) and \( T_R \geq 0 \) become, respectively, the left-hand and right-hand sides of
\[
-\frac{8\nu \ell G}{v^2 - 1} M_{\text{ADT}}^2 \leq J_{\text{ADT}} \leq \frac{12\nu \ell G}{v^2 + 3} M_{\text{ADT}}^2.
\]
There is yet another form of the black hole metrics\(^8\) that is given in [40], [11] and [30]. The metric in [40] with parameters \( (v', J', a', L') \) is related to the one in [11], which we write here as
\[
\begin{align*}
\text{d}s^2 &= \text{d}T'^2 + \left( \frac{3}{\ell^2 (v^2 - 1)} R'^2 - \frac{4j \ell}{v} + 12m R' \right) \text{d}\theta^2 \\
&\quad - 4 \frac{v}{\ell} R' \text{d}T' \text{d}\theta + \frac{3m v^2}{\ell^2} R'^2 - 12m R' + \frac{4j \ell}{v},
\end{align*}
\]
\(^8\) The black hole metric was first written in [34].
by \( j = G' \), \( 6m = 4Gv', a' = -v/\ell \) and \( L' = \sqrt{2\ell/(3-v^2)} \). The metric in (C.2) is related to (32) under the transformation \( R' = \frac{e^2}{\ell} r - e^2 \sqrt{r_- r_+(v^2 + 3)} \) and \( T' = \ell t \) with

\[
6m = \frac{v^2 + 3}{4} (r_+ + r_- - \sqrt{r_+ r_-(v^2 + 3)} + \frac{\sqrt{r_+ r_-(v^2 + 3)}}{v}) = 2\pi \ell T_L \\
4l = \frac{5v^2 + 3}{16v} (v^2 + 3) \ell r_- r_+ - \frac{(v^2 + 3)^2}{8} \ell (r_+ + r_-) \sqrt{r_+ r_-}.
\]

The existence of a Killing horizon is given by the vanishing of \( g_{R'K} \) for some value of \( R' \). This is a quadratic equation in \( R' \) with determinant

\[
\Delta_R = (12m)^2 - 16 \frac{v^2 + 3}{v_l} = \left( \frac{v^2 + 3}{2} (r_+ - r_-) \right)^2 = (4\pi \ell T_R)^2 \geq 0.
\]

On the other hand, there are causal singularities hidden behind a Killing horizon when \( g_{00} \) vanishes. This is equivalent to \( r_+/r_- < \lambda_f \) and gives

\[
j \geq -3m^2v/\ell(v^2 - 1).
\]

For smaller values of \( j \) for fixed \( m \) we continue in the region where there are no CTCs.

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