Recollapse of the Closed Tolman Spacetimes

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Abstract

The closed-universe recollapse conjecture is studied for the spherically symmetric spacetimes. It is proven that there exists an upper bound to the lengths of timelike curves in any Tolman spacetime that possesses $S^3$ Cauchy surfaces and whose energy density is positive. Furthermore, an explicit bound is constructed from the initial data for such a spacetime.

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I. INTRODUCTION

Must the Universe end? The *closed-universe recollapse conjecture* offers the intriguing possibility that the ultimate fate of the Universe may be a consequence of its spatial topology. The conjecture says, roughly, that a closed Universe with ordinary matter must expand from an initial singular state to a maximal size and then recollapse to a final singular state, thus giving the Universe a finite lifetime [1–4]. In order to investigate this conjecture, we formulate a precise version as follows.

First, a restriction on the matter content is necessary for otherwise counterexamples to any reasonable conjecture are easily constructed. Exactly what energy condition should be imposed is an open issue, but it has proved useful in prior investigations [5] to demand that the dominant-energy and non-negative-pressures conditions hold. The dominant-energy condition is the demand that $G_{ab}t^a u^b \geq 0$ for all future-directed timelike $t^a$ and $u^b$. The non-negative-pressures condition is the demand that $G_{ab}x^a x^b \geq 0$ for all spacelike $x^a$. Through Einstein’s equation and for a stress-energy tensor that possesses a timelike eigenvector with eigenvalue $-\rho$ and principal pressures $p_i$, together these conditions are equivalent to the inequalities $\rho \geq p_i \geq 0$. It is interesting to note that simply demanding that the more standard energy conditions of dominant-energy and timelike-convergence ($R_{ab}t^a t^b \geq 0$ for all timelike $t^a$) hold is not sufficient to guarantee the recollapse of even the $k = +1$ Friedmann-Robertson-Walker spacetimes. A simple counterexample is given by the Friedmann-Robertson-Walker spacetime with scale factor $a(t) = t$ [4].

Second, by a spacetime having a finite lifetime we shall mean that there is a finite upper bound to the lengths of timelike curves in that spacetime. (A closed universe with such a bound is known as a Wheeler Universe [4].) If the spacetime satisfies the timelike-convergence condition and a genericity requirement, then the existence of a maximal hypersurface is sufficient to guarantee that such an upper bound exists [3,4]. However, proving the existence of a maximal hypersurface seems to be a very difficult task as spacetimes satisfying our energy-condition requirement can be constructed that don’t possess such hypersurfaces.
For example, consider the spacetime obtained by taking the past of any expanding spatially homogeneous hypersurface of a $k = +1$ Friedmann-Robertson-Walker spacetime with positive energy density and pressure. Although in this case it is possible to continue the spacetime to the future, examples can be constructed where the pressure diverges in a finite time (thus preventing a future extension) with the spacetime always expanding \[1\]. Note that in these examples, although the development of a maximal Cauchy surface is prevented by the “halting” of the evolution of the spacetime, there is a finite upper bound to the lengths of timelike curves in the spacetime. In light of this, rather than attempting to impose conditions designed to guarantee that a maximal hypersurface can develop, we simply investigate the question of whether there exists a finite upper bound to the lengths of timelike curves in the spacetimes under consideration.

Lastly, although the spacetimes being studied may not contain a maximal hypersurface, the existence of such a surface should not be precluded because of the spacetime’s topology. For such a maximal hypersurface $\Sigma$ to exist, it is necessary that the scalar curvature associated with the induced metric on $\Sigma$ be non-negative. (This is easily seen using the initial-value constraint equation and the non-negative-energy condition.) However, there are few compact orientable three-manifolds that admit metrics with non-negative scalar curvature \[6\]. We further narrow the allowed topologies for $\Sigma$ by demanding that the induced metric be non-flat, thereby excluding such possible counterexamples as the identified Minkowski spacetime with spatial topology $S^1 \times S^1 \times S^1$. The resulting allowed Cauchy surface topologies are $S^3$, $S^1 \times S^2$, and those constructed from these by making certain identifications and connected summations \[2,3\].

Combining these requirements, we arrive at the following

**Conjecture.** There exists an upper bound to the lengths of timelike curves in any spacetime that possesses $S^3$ or $S^1 \times S^2$ Cauchy surfaces and that satisfies the dominant-energy and non-negative-pressures conditions.

Currently, there is no known counterexample nor proof of this conjecture. However, there are results offering evidence for its truth. The simplest is the fact that the $S^3$ case of the
conjecture holds for all spatially homogeneous and isotropic spacetimes \[7,4\]. Lin and Wald \[8\] have generalized this last result by relaxing the isotropy assumption. That is, the above conjecture holds for the Bianchi type IX spacetimes.

Further, it is known that the $S^1 \times S^2$ case of the above conjecture holds for spacetimes that are spherically symmetric \[4\]. The basic idea behind the proof of this result is simple. Associated with the spherically symmetric spacetimes are two scalar fields $r$ and $m$. The field $r$ is simply the usual “size” associated with the spheres of symmetry while $m$ is a “quasi-local mass” associated with these spheres. (These are discussed in detail in Sec. II below.) In the $S^1 \times S^2$ case it turns out that $r$ is positive and globally bounded from above and $m$ is globally bounded from below by a positive constant. Further, as $r$ changes as a function of proper time $t$ along any timelike geodesic, it must obey the inequality $d^2 r/dt^2 \leq -m/r^2$. These facts together allow us to easily conclude that there is a global finite upper bound to the lengths of timelike curves in these spacetimes. (Further details of this argument can be found in Ref. \[3\].)

However, it is unknown whether the $S^3$ case of the above conjecture holds for spacetimes that are spherically symmetric. In the $S^3$ case we still find that $r$ is globally bounded from above but now we know only that $m$ is non-negative. Because of this we are unable to obtain in such a simple manner an upper bound to the lengths of timelike curves.

Herein, we show that the problems in proving the above conjecture for the spherically symmetric spacetimes with $S^3$ Cauchy surfaces can be overcome for the nowhere empty Tolman spacetimes. These are the spherically symmetric spacetimes whose matter content is dust (a perfect fluid with vanishing pressure) \[9\]. The method of proof can be understood roughly as follows. First, we introduce a “time-function” $t$ whose boundedness entails an upper bound to the lengths of timelike curves. Next, we divide the spacetime into three regions: a “northern” region, a “middle” region, and a “southern” region. Using the upper bound for $r$ and the positivity of $m$ in the “middle” region and on the boundaries of the “northern” and “southern” regions, the boundedness of $t$ in these regions follows in a straightforward manner. Lastly, we show how the boundedness of $t$ on the “northern” and
“southern” boundaries entails the boundedness of $t$ on all of “northern” and “southern” regions. Thus, $t$ is globally bounded which gives

*Theorem 1.* There exists an upper bound to the lengths of timelike curves in any Tolman spacetime that possesses $S^3$ Cauchy surfaces and whose energy density is positive.

This result is further strengthened by Theorem 5 where an explicit expression for an upper bound is given in terms of initial data for the spacetime on any spherically symmetric Cauchy surface.

The closed-universe recollapse conjecture has been investigated for the Tolman spacetimes (through a variety of methods) and either claimed to be true by argument [10][11] or false by counterexample [12]. However, none of these analyses provide either a true proof or counterexample of the closed-universe recollapse conjecture for the Tolman spacetimes (in any reasonable form).

Zel’dovich and Grishchuck showed that naive expectations regarding the unbounded expansion of a closed Tolman spacetime that is “locally open” are incorrect. However, their work does not show that there is indeed an upper bound to the lengths of timelike curves in such a spacetime. While their argument does show that the crossing of flow lines will develop somewhere in the spacetime, it does not necessarily mean that such an occurrence will be visible to all observers. Their work thus leaves open the possibility that some observer may forever be unaffected by the crossing of flow lines.

Bonnor showed that his “physically acceptable cosmological models” must be everywhere “elliptic” and therefore must recollapse. The problem here is that the requirement for a Tolman spacetime to be “physically acceptable” is far too restrictive in that spacetimes where the dust lines eventually intersect are ruled out. Such spacetimes are physically acceptable as long as the regions where such crossing occur are excluded while keeping the spacetime globally hyperbolic. Even accepting the result, this still does not give a finite upper bound on the lengths of causal curves as the time to recollapse as one nears the “poles” may become infinite.

Finally, the counterexample to the closed-universe recollapse conjecture offered by
Hellaby and Lake is flawed in that it contains a surface layer of matter that has associated with it a negative pressure. (This was noted by Bonnor [11].) In fact, a simple example of the problem inherent with their spacetime is the following. Consider the static spherically symmetric spacetime with $S^3$ Cauchy surfaces with metric given by

$$g_{ab} = -(dt)_a (dt)_b + (dx)_a (dx)_b + r^2 \Omega_{ab}$$

where $r(t, x) = 1 - |1 - x|$, $\Omega_{ab}$ is the standard unit-metric on the 2-sphere, and $0 \leq x \leq 2$, $-\infty < t < \infty$. (The “poles” are at $x = 0, 2$.) This spacetime is flat everywhere except on the timelike surface $x = 1$ where there is a distributional stress-energy that possesses negative pressures. So, although this spacetime possesses $S^3$ Cauchy surfaces and has infinite length timelike curves, since the energy conditions are not met, it is not a counterexample to the closed-universe recollapse conjecture as formulated above.

In Sec. II the basics of the spherically symmetric spacetimes are presented. The Einstein equations are given in a form amenable to analysis, and the fields $r$ and $m$ are introduced and their basic properties established. Throughout this section the analysis is quite general in that it is independent of the Tolman matter assumption. In Sec. III the results obtained in Sec. II are applied to the Tolman spacetimes and Theorem 1 is proven. Lastly, in Sec. IV a few final remarks are made regarding possible extensions of this work.

The conventions used herein are those of Ref. [13]. In particular, our metrics are such that timelike vectors have negative norm and the Riemann and Ricci tensors are defined by

$$2\nabla_{[a} \nabla_{b]} \omega_c = R_{abc}^d \omega_d$$

and

$$R_{ab} = R_{amb}^m$$

respectively. All metrics are taken to be $C^2$. Our units are such that $G = c = 1$.

II. BASICS

In this section, the Einstein equations for the spherically symmetric spacetimes are presented and are used to prove

Theorem 2. Fix any spherically symmetric spacetime $(M, g_{ab})$ that possesses $S^3$ Cauchy surfaces and that satisfies the non-negative-pressures and dominant-energy conditions. Then
$m \geq 0$ and $r \leq \max_{\Sigma}(2m)$ where $\Sigma$ is any spherically symmetric Cauchy surface for this spacetime.

In other words, the spheres of symmetry can’t become arbitrarily large and their associated “quasi-local mass” is always non-negative.

A. Field Equations

Recall that a spacetime $(M, g_{ab})$ is said to be spherically symmetric if it admits the group $G \approx SO(3)$ of isometries, acting effectively on $M$, each of whose orbits is either a two-sphere or a point \[14\]. The value of the non-negative scalar field $r$ at each $p \in M$ is defined so that $4\pi r^2$ is the area associated with the orbit of $p$. So, in particular, $r(p)$ is zero if the orbit is a point, while $r(p)$ is positive if the orbit is a two-sphere.

For a spherically symmetric spacetime $(M, g_{ab})$ with $S^3$ Cauchy surfaces, the set of points $p$ for which $r(p) = 0$ (points whose orbits are themselves) consists of two disconnected components which we label $\gamma_n$ and $\gamma_s$. It follows from the spherical symmetry that these two sets must in fact be timelike geodesics. So, physically these curves are the world lines of the two privileged observers for whom the universe actually appears spherically symmetric.

As in the $S^1 \times S^2$ case, we can construct a two-dimensional spacetime $(B, h_{ab})$ by setting $B = M/G$ (i.e. $B$ is the set of orbits) and $h^{ab} = (\pi^* g)^{ab}$ where $\pi$ is the natural projection map from $M$ to $M/G$. However, unlike the $S^1 \times S^2$ case, $(B, h_{ab})$ is a spacetime with boundary consisting of the two geodesics $\pi(\gamma_n)$ and $\pi(\gamma_s)$. (In other words, the boundary is timelike with zero extrinsic curvature. Further, it follows from the fact that $(M, g_{ab})$ possesses $S^3$ Cauchy surfaces that $B \approx R \times [0,1]$.) Because of this timelike boundary, the spacetime $(B, h_{ab})$ is not globally hyperbolic in the traditional sense, which makes working with the two-dimensional spacetime $(B, h_{ab})$ somewhat awkward. It is for this reason that the basic results concerning the spherically symmetric spacetimes shall be stated and proved using the four-dimensional spacetime $(M, g_{ab})$.

Denote the projection operator onto the tangent space of each sphere of symmetry by
\( q^a_b \). (So, e.g., if \( x^a \) is a vector tangent to a surface of symmetry then \( q^a_b x^b = x^a \), while if \( x^a \) is perpendicular to such a surface \( q^a_b x^b = 0 \).) From \( q^a_b \) we construct the projection operator onto the surfaces perpendicular to the spheres of symmetry: \( h^a_b = \delta^a_b - q^a_b \). Thus, using \( g^{ab} \) and \( g_{ab} \) to raise and lower the indices of these tensor fields, we have the following decomposition of the metric (where \( r > 0 \))

\[
  g_{ab} = h_{ab} + q_{ab}. \tag{2.1}
\]

We further decompose \( g_{ab} \) by decomposing \( q_{ab} \). Denote the Killing vector fields associated with the action of \( G \) on \( M \) by \( \xi^\alpha_a \) where lower-case Greek indices are Lie-algebra indices. With these we define the tensor field

\[
  \Omega^{ab} = \xi^a_\alpha \xi^b_\beta k^{\alpha\beta}, \tag{2.2}
\]

where \( k^{\alpha\beta} \) is the inverse of \( k_{\alpha\beta} = -\frac{1}{2} c^{\mu}_{\alpha\nu} c^{\nu}_{\beta\mu} \) (which is one-half of the Killing-Cartan metric) and \( c^{\nu}_{\alpha\beta} \) are the structure constants for the Lie algebra associated with \( G \) (so that, in particular, we have \( [\xi^\alpha, \xi^\beta]^a = c^{\gamma}_{\alpha\beta} \xi^a_\gamma \)). This tensor has the following properties: (1) It is tangent to the spheres of symmetry; (2) It is spherically symmetric, \( \mathcal{L}_{\xi} \Omega^{ab} = 0 \); (3) If \( \zeta^a \) is any spherically symmetric vector field, then \( \mathcal{L}_{\zeta} \Omega^{ab} = 0 \); (4) On the spheres of symmetry, \( \Omega^{ab} \) is positive definite; (5) The area of any sphere of symmetry computed using \( \Omega^{ab} \) is \( 4\pi \); (6) On \( \gamma_n \) and \( \gamma_s \), \( \Omega^{ab} = 0 \). These properties allow one to think of \( \Omega^{ab} \) as the preferred unit-metric on each sphere of symmetry.

Define \( \Omega_{ab} \) to be the inverse of \( \Omega^{ab} \) so that \( \Omega_{am} \Omega_{mb} = q^a_b \) and \( \Omega_{ab} = q^m_a q^b_m \Omega_{mn} \). We then have \( q_{ab} = r^2 \Omega_{ab} \). To show this, consider the linear map \( \Omega_{am} q_{mb} \). By spherical symmetry this map must be proportional to \( q^a_b \) for otherwise its preferred eigenvectors would violate the rotational symmetry about each point. Thus, \( \Omega^{ab} \) and \( q^{ab} \) must be proportional. Using the fact that the area computed using \( \Omega^{ab} \) is \( 4\pi \) and that the area computed using \( q^{ab} \) is \( 4\pi r^2 \) we find that \( q^{ab} = \frac{1}{r^2} \Omega^{ab} \) or more simply \( q_{ab} = r^2 \Omega_{ab} \).

We thus arrive at the following decomposition of the metric \( g_{ab} \) (where \( r > 0 \))

\[
  g_{ab} = h_{ab} + r^2 \Omega_{ab}. \tag{2.3}
\]
Define \( D_a \) to be that (torsion-zero) derivative operator associated with the (unphysical) metric \( h_{ab} + \Omega_{ab} \). It then follows (from arguments involving spherical symmetry) that \( D_a h_{bc} = D_a \Omega_{bc} = 0 \). Further, if \( \omega_a \) is spherically symmetric then \( D_a \omega_b = h^m_a h^n_b \nabla_m \omega_n \), and if \( \mathcal{L}_\zeta \omega_a = 0 \) for all spherically symmetric vector fields \( \zeta^a \) then \( D_a \omega_b = q^m_a q^n_b \nabla_m \omega_n \). These properties allow one to think of \( D_a \) as the derivative operator associated with \( h_{ab} \) on the surfaces perpendicular to the spheres of symmetry (or on \( B \) if desired) and as the derivative operator associated with \( \Omega_{ab} \) on the spheres of symmetry.

We define the “quasi-local mass” \( m \) by

\[
2m = r(1 - D_m r D^m r). 
\] (2.4)

That this quantity deserves such a title is born out by its direct relation to the stress-energy content of the spacetime (given by Eq. (2.11)), and in the vacuum case its being the mass of that extended Schwarzschild spacetime to which this spacetime is locally isometric. Furthermore, \( m \) is non-negative (proven in Sec. [13]) and is of great utility in all that follows.

Denote the scalar curvature associated with the surfaces perpendicular to the spheres of symmetry by \( R[h] \), and define \( \epsilon^{ab} \) to be either of the two antisymmetric tensor fields such that \( \epsilon^{ab} \epsilon_{cd} = -2 h^{[c} h^{d]}_{b} \). With these definitions, the Riemann tensor for the four-dimensional spacetime \( (M, g_{ab}) \) is found to be

\[
R_{abcd} = R[h] h_{c[a} h_{b]d} - \frac{4}{r} q_{c[a} D_{b]d} D_d r + \frac{4m}{r^3} q_{c[a} q_{b]d} \] (2.5)

from which the Einstein tensor is computed to be

\[
G_{ab} = \frac{2}{r} \left[ D_m D^m r - \frac{m}{r^2} \right] h_{ab} - \frac{2}{r} D_a D_b r + \left[ \frac{1}{r} D_m D^m r - \frac{1}{2} R[h] \right] q_{ab}. \] (2.6)

From Eq. (2.5) we see that for the curvature to be finite on \( \gamma_n \) and \( \gamma_s \), then \( m/r^3 \) must be finite on each of these two curves. Thus, both \( m/r^2 \) and \( m/r \) must be zero on \( \gamma_n \) and \( \gamma_s \), from which it follows that \( D^a r \) is unit-spacelike on \( \gamma_n \) and \( \gamma_s \).
Using the above decomposition of the metric \( g_{ab} \), we decompose the Einstein equation as follows. Decompose the Einstein tensor \( G_{ab} \) (which through Einstein’s equation is proportional to the stress-energy tensor of the matter) into the pair of spherically symmetric fields \((\tau^{ab}, P)\) where \( \tau^{ab} \) is the purely “radial” part of the Einstein tensor \((\tau^{ab} = h^a_m h^b_n G^{mn})\) and \( P \) is the pressure associated with the spheres of symmetry \((P = \frac{1}{2} G^{mn} q_{mn})\). Then,

\[
G_{ab} = \tau_{ab} + P g_{ab}. \tag{2.7}
\]

The twice-contracted Bianchi identity \((\nabla_b G^{ab} = 0)\) requires that the pair \((\tau^{ab}, P)\) satisfy

\[
D_b (r^2 \tau^{ab}) = PD^a r^2. \tag{2.8}
\]

With this decomposition of \( G_{ab} \), Eq. (2.6) becomes the pair of equations

\[
D_a D_b \frac{r^2}{r^2} h_{ab} - \frac{r}{2} \tau^{mn} \epsilon_{ma} \epsilon_{nb} D^b r, \tag{2.9}
\]

\[
R[h] = 4m \frac{4m}{r^3} + (\tau_m^m - 2P). \tag{2.10}
\]

From Eq. (2.3) we arrive at the following simple and very useful equation relating the gradient of \( m \) algebraically to the stress-energy tensor

\[
D_a (2m) = r^2 \tau^{mn} \epsilon_{ma} \epsilon_{nb} D^b r. \tag{2.11}
\]

If desired, one could work with Eqs. (2.8, 2.11) as equations on the two-dimensional space-time \((B, h_{ab})\) as was done in the \( S^1 \times S^2 \) case.

**B. Non-negativity of \( m \)**

In the study of the spherically symmetric spacetimes with \( S^1 \times S^2 \) Cauchy surfaces, it was shown that (under certain energy conditions) the “quasi-local mass” \( m \) is globally bounded from below by a positive constant. That is, not only is \( m \) non-negative in that case, it can neither become zero nor get arbitrarily close to zero. In the \( S^3 \) case, however, \( m \) is zero on the curves \( \gamma_n \) and \( \gamma_s \). So, the best global bound one can hope for in this case is for \( m \) to be
non-negative. In fact, as we show below, not only is \( m \) non-negative, but in fact \( m \) can be zero only in flat (hence vacuum) regions about \( \gamma_n \) or \( \gamma_s \). It may be noted that the argument used here to establish the non-negativity of \( m \) is an “initial value” type of argument in that we show that \( m \) is non-negative on any Cauchy surface from which it follows that \( m \) is non-negative everywhere.

We begin with

**Lemma 1.** Fix any spherically symmetric globally hyperbolic spacetime \((M, g_{ab})\) that satisfies the dominant-energy condition. Fix a spherically symmetric Cauchy surface \( \Sigma \) therein and let \( C \) be any spherically symmetric compact subset of \( \Sigma \). Then

\[
\min_C(2m) \geq \min(\min_{\partial C}(2m), \min_C(r)).
\]  

(2.12)

**Proof.** Denote the right-hand side of Eq. (2.12) by \( 2\mu \) and consider the open proper subset \( U \) of \( C \) defined by \( U = \{ s \in C | 2m(s) < 2\mu \} \). On \( U \), \( 2m < r \) so \( D^a r \) is necessarily spacelike. Denote, by \( s^a \), the unit spherically symmetric vector field on each connected component of \( U \), tangent to the surface \( \Sigma \), such that \( s^a D_a r > 0 \). Then, by Eq. (2.11) and the fact that \( \tau^{ab} \) satisfies the dominant-energy condition, we have \( s^a D_a (2m) \geq 0 \) on \( U \). Using this fact and noting that \( 2m = 2\mu \) on the boundary of \( U \), we conclude that \( 2m = 2\mu \) on \( U \). This contradicts the definition of \( U \), so \( U \) must be empty. This establishes the above lower bound for \( m \) on \( C \). \( \square \)

The non-negativity of \( m \) in the \( S^3 \) case follows as a simple consequence of this result. For any point \( p \in M \), let \( \Sigma \) be any spherically symmetric Cauchy surface for \((M, g_{ab})\) with \( p \in \Sigma \). Take \( C = \Sigma \). Since \( \partial \Sigma \) is empty and \( \min_\Sigma (r) = 0 \), by Lemma 1, \( m \geq 0 \) on \( \Sigma \). So, \( m \) is non-negative at \( p \).

In fact, we can strengthen this last result in that if \( S \) is a sphere of symmetry for which \( m(S) = 0 \), then \( m = 0 \) for all points “inside” of \( S \). For a sphere of symmetry \( S \) with \( D^a r \) spacelike (i.e. the sphere is neither future/past trapped nor marginally trapped), we shall say that a point \( p \in M \) is inside of \( S \) if \( p \) lies in the component of \((J^+(S) \cup J^-(S))^c \) for which \( D^a r \) is outward pointing. (For \((M, g_{ab})\) globally hyperbolic and connected, the set
\((J^+(S) \cup J^-(S))^c\) can have at most two components and may have only one as in the \(S^1 \times S^2\) case.) With this definition, we have

**Theorem 3.** Fix any spherically symmetric globally hyperbolic spacetime \((M, g_{ab})\) that satisfies the dominant-energy condition. If \(m(S) = 0\) for some sphere of symmetry \(S\), then \(m = 0\) for all points inside of \(S\).

**Proof.** Since \(m(S) = 0\), by Eq. (2.4) we have \(D_a r D^a r = 1\) on \(S\) showing that \(D^a r\) is spacelike on \(S\). For any point \(p\) inside of \(S\), let \(\sigma\) be a radial spacelike curve from \(S\) to \(p\) with tangent vector \(s^a\). (Such a curve always exists.) Let \(\sigma'\) be the maximal subset of \(\sigma\) connected to \(S\) on which \(D^a r\) is spacelike. Using the fact that \(s^a D_a r < 0\) on \(\sigma'\), the fact that \(\tau^{ab}\) satisfies the dominant energy condition, and Eq. (2.11), we find \(s^a D_a (2m) \leq 0\) on \(\sigma'\). Thus, for all points \(q \in \sigma'\) we have the inequalities \(0 \leq m(q) \leq m(S) = 0\) showing that \(m = 0\) on \(\sigma'\). However, this shows that \(D^a r D_a r = 1\) on \(\sigma'\) so in fact \(\sigma' = \sigma\). Thus, \(m(p) = 0\). \(\square\)

It is in this sense (a sense as strong as one could hope for) that \(m\) can vanish only “about” the poles. Further, such regions must be flat.

**Theorem 4.** Fix any spherically symmetric globally hyperbolic spacetime \((M, g_{ab})\) that satisfies the dominant-energy condition. If \(m(S) = 0\) for some sphere of symmetry \(S\) in \(M\), then the spacetime inside of \(S\) is flat \((R_{abcd} = 0)\).

**Proof.** Denote the spacetime inside of \(S\) by \(\mathcal{F}\). By Eq. (2.3), a spherically symmetric spacetime is flat on an open set \(\mathcal{F}\) iff \(m = 0\), \(D_a D_b r = 0\), and \(R[h] = 0\) on \(\mathcal{F}\). Since \(m(S) = 0\), by Theorem 3, \(m = 0\) on \(\mathcal{F}\). So, by Eq. (2.11), \(\tau^{mn}(\epsilon_{ma} D^a r)(\epsilon_{nb} D^b r) = 0\) on \(\mathcal{F}\). But, since \(\epsilon_{am} D^m r\) is timelike, by the dominant-energy condition, it must be the case that \(\tau^{ab} = 0\) and consequently \(P = 0\) on \(\mathcal{F}\). So, by Eq. (2.9), we have \(D_a D_b r = 0\) and by Eq. (2.10), \(R[h] = 0\). \(\square\)

Likewise, a region that is vacuum about a pole must be flat since, by Eq. (2.11), in that region \(m\) is zero and thus must be flat, by Theorem 4. However, it is interesting to note that a spherically symmetric spacetime with \(S^3\) Cauchy surfaces cannot be completely vacuum since, by Eq. (2.4), \(m\) is manifestly positive where \(r\) reaches its maximum value on
any Cauchy surface which, by Eq. \(2.11\), demands that \(G^{ab}\) be non-zero somewhere on that surface.

C. Upper bound for \(r\)

We now establish the upper bound for \(r\) given in Theorem 2. First, we have the following upper bound for \(r\) on a Cauchy surface.

**Lemma 2.** Fix any spherically symmetric globally hyperbolic spacetime \((M, g_{ab})\) that possesses compact Cauchy surfaces and that satisfies the dominant-energy condition. For any spherically symmetric Cauchy surface \(\Sigma\) of \((M, g_{ab})\) we have

\[
\max_{\Sigma}(r) \leq \max_{\Sigma}(2m) \tag{2.13}
\]

**Proof.** Consider a point \(p\) where \(r\) reaches its maximum value on \(\Sigma\). At such a point \(D^a r\) is necessarily timelike or zero. Hence

\[
\max_{\Sigma}(r) = r(p) \leq 2m(p) \leq \max_{\Sigma}(2m), \tag{2.14}
\]

where the first inequality is by Eq. \(2.4\). \(\Box\)

Next, we have the following global upper bound for \(r\).

**Lemma 3.** Fix any spherically symmetric globally hyperbolic spacetime \((M, g_{ab})\) that satisfies the non-negative-pressures and dominant-energy conditions. For any spherically symmetric Cauchy surface \(\Sigma\) of \((M, g_{ab})\) we have

\[
r \leq \max(\sup_{\Sigma}(r), \sup_{\Sigma}(2m)). \tag{2.15}
\]

**Proof.** It suffices, since we can always reverse the roles of past and future, to establish this bound for any \(p \in D^+ (\Sigma)\).

Consider any point \(q\), where \(r\) reaches its maximum value on the compact set \(C = J^- (p) \cap D^+ (\Sigma)\). If \(q \in C \cap \Sigma\), then \(r(p) \leq r(q) \leq \sup_{\Sigma}(r)\). If \(q \notin C \cap \Sigma\), then \(D^a r\) must be either past-directed timelike, past-directed null, or zero, at \(q\), for otherwise there would
exist a past-directed timelike direction along which \( r \) would increase. We now show, in all three cases, that \( r(q) \leq \sup_{\Sigma}(2m) \).

If \( D^a r \) is past-directed timelike or past-directed null at \( q \), then, by Eq. (2.4), \( r(q) \leq 2m(q) \). Consider the maximal integral curve \( \sigma \) of \( D^a r \) with future end point \( q \) on which \( D^a r \) is past-directed timelike or past-directed null. The curve \( \sigma \) does not have a past endpoint. [Proof: Using the non-negative-pressures condition and Eq. (2.9) we have \((D^a r)D_a(D_m r D^m r) \leq 0\), on \( \sigma \), showing that for \( \sigma \) to have a past endpoint \( q' \), then \( D^a r \) must be past-directed null all along \( \sigma \) and zero at \( q' \). In this case, since \( \sigma \) is radial and null, it must be a geodesic curve. Affinely parameterize \( \sigma \) by \( \lambda \) and denote its associated tangent vector by \( k^a \). Then, using the null-convergence condition \((G_{ab} k^a k^b \geq 0 \) for all null \( k^a \), which follows from such energy conditions as the dominant-energy condition) and Eq. (2.9), we find that \( d^2 r/d\lambda^2 \leq 0 \). But, since \( dr/d\lambda < 0 \) at \( q \), it is impossible for \( dr/d\lambda \) and hence \( D^a r \) to be zero at \( q' \). Thus, a past endpoint \( q' \) cannot exist.] So, since \( \sigma \) is a past-directed causal curve with future endpoint \( q \in D^+(\Sigma) \) and without past endpoint, it must, by global hyperbolicity, intersect \( \Sigma \). Again using the non-negative-pressures condition and Eq. (2.4), we find that \((D^a r)D_a(2m) \geq 0\), on \( \sigma \), so that \( r(q) \leq 2m(q) \leq 2m(\sigma \cap \Sigma) \leq \sup_{\Sigma}(2m) \).

If \( D^a r \) vanishes at \( q \), then so does \( D_a(-D_m r D^m r) \). Using Eq. (2.9) we find that, at \( q \), for radial unit past-directed timelike \( t^a \),

\[
t^a t^b D_a D_b(-D_m r D^m r) = \frac{1}{2r^2} + \tau^{mn} \epsilon_{ma} \epsilon_{nb} t^a t^b \\
+ \frac{r^2}{2} (\tau^{mn} \epsilon_{ma} t^a)(\tau^{pq} \epsilon_{pb} t^b) h_{mq}.
\]

(2.16)

The first term is manifestly positive; the second term is non-negative by the non-negative-pressures condition; and by the dominant-energy condition, there exist \( t^a \) for which the last term is non-negative. [Sketch of proof: Use the fact that \( h^{ab} = -2k^a(k^b) \) where \( k^a \) and \( l^b \) are two linearly-independent radial past-directed null vectors. Set \( u^a = -\tau^a_k k^b \) and \( v^a = -\tau^a_l l^b \). If either \( u^a \) or \( v^a \) is timelike (necessarily being past-directed) then taking \( t^a \) to be colinear with either (timelike) vector guarantees that the last term is zero. Otherwise,
if $u^a$ and $v^a$ are null or zero, then any $t^a$ will do.] Consider the past-directed timelike geodesic starting at $q$, with initial tangent vector $t^a$ such that the last term, in Eq. (2.16), is non-negative. Then, at $q$, $t^a D_a (t^b D_b (\cdot D_m r D^m r)) > 0$, and, by the non-negative-pressures condition, $t^a D_a (t^b D_b r) < 0$. Hence, $D^a r$ immediately becomes past-directed timelike along the curve. From our analysis of such points, we conclude that $r(q) = 2 m(q) \leq \sup_{\Sigma} (2 m)$. $\square$

Combining Lemmas 2 and 3, the global upper bound $r \leq \max_{\Sigma} (2 m)$ is established.

### III. TOLMAN RESULT

The Tolman spacetimes are the spherically symmetric spacetimes whose matter content is dust (a perfect fluid with zero pressure) [9]. The Einstein equation is then

$$G_{ab} = 8 \pi \rho u_a u_b$$  \hspace{1cm} (3.1)

where $\rho$ is the energy-density of the dust and $u^a$ is a (spherically symmetric) unit future-directed timelike vector field. Decomposing $G_{ab}$, as described in Sec. II, we have $\tau_{ab} = 8 \pi \rho u_a u_b$ and $P = 0$. Thus, by Eq. (2.8), we find that $u^a$ is geodetic and that $D_a (\rho r^2 u^a) = 0$. For a Tolman spacetime to satisfy the dominant-energy and non-negative-pressures conditions it is necessary and sufficient that $\rho$ be non-negative.

Strictly speaking, $u^a$ need only be defined where $\rho$ is positive. It is for this reason that we have restricted ourselves to the nowhere empty Tolman spacetimes. However, the more general case where $\rho$ can vanish in a region (and so $u^a$ is undefined there) can be analyzed if $u^a$ can be extended to these regions in such a way that it is spherically symmetric and geodetic. However, whether this extension can be accomplished in general is unclear, though it seems likely. If it can be accomplished, then all the results that follow also apply for these spacetimes.

One problem with dust as a matter source is that dust lines may cross thus destroying the dust assumption embodied in Eq. (3.1). In this work we shall adopt a strict definition of a Tolman spacetime by demanding that there are no such crossings. So, for example, if one
where to evolve one of these spacetimes from a given initial data surface and such a crossing were to develop, then the evolution is to be stopped where the crossing occurs and continued elsewhere only to the extent that the constructed spacetime is globally hyperbolic.

One feature of the Tolman spacetimes that makes their study tractable is that $m$ is constant along the integral curves of $u^a$. For proof, using Eq. (2.11) we have $u^aD_a(2m) = (8\pi \rho u^m u^a)(\epsilon_{ma} u^a)(\epsilon_{nb} D^b r) = 0$. Further, by Theorem 2, $r$ is bounded from above by

$$r_M = \max_{\Sigma}(2m),$$

(3.2)

where $\Sigma$ is any spherically symmetric Cauchy surface for this spacetime. Using these facts we have

Lemma 4. Any integral curve $\gamma$ of $u^a$ for which $m(\gamma) > 0$ is bounded in length by

$$T[\gamma] = \pi \sqrt{\frac{r_M^3}{2m(\gamma)}},$$

(3.3)

Proof. Since $m(\gamma) > 0$, $r$ must be strictly positive on $\gamma$ and so $0 < r \leq r_M$ on $\gamma$. Further, since $\gamma$ is geodetic it follows that

$$\frac{d^2 r}{dt^2} = u^a u^b D_a D_b r = -\frac{m(\gamma)}{r^2},$$

(3.4)

where the second equality follows from Eq. (2.9) and the vanishing of the pressure $\tau^{mn}(\epsilon_{ma} u^a)(\epsilon_{nb} u^b)$. However, it is a straightforward exercise to show that it is impossible to satisfy Eq. (3.4) and the inequalities $0 < r \leq r_M$ for a time $T[\gamma]$ or greater. □

Thus, for the Tolman spacetimes, we quite easily arrive at the result that all of integral curves of the fluid flow (with $m > 0$) must be incomplete. However, this is quite far from showing that there is an upper bound to the lengths of all the timelike curves in such a spacetime. For, consider the upper bound given by Eq. (3.3). As we approach either $\gamma_n$ or $\gamma_s$, $m(\gamma)$ approaches zero so that, at this stage in the argument, it is still conceivable that while curves “between the poles” will be finite in length, $\gamma_n$, $\gamma_s$, and “nearby” curves could be infinite in length! This is by no means a minor technicality. For instance, it is conceivable that integral curves of $u^a$ initially near $\gamma_n$ “peel away” in such a way as to hide their demise from $\gamma_n$.  

16
While this “peeling” of neighboring curves from $\gamma_n (\gamma_s)$ may occur, they can’t do it in such a way that their demise can’t be seen by $\gamma_n (\gamma_s)$. Or, said better, we will see that an integral curve of $u^a$ sufficiently near $\gamma_n (\gamma_s)$ having a finite length requires that $\gamma_n (\gamma_s)$ have a finite length (Lemma 7 (8)).

A. The “time function” $t$

It follows directly from the facts that the vector field $u^a$ is spherically symmetric, unit, and geodetic that $D_a u_b = (D_m u^m) (h_{ab} + u_a u_b)$ from which it is apparent that $u_a$ is closed: $(du)_{ab} = 0$. Thus, since $M$ is simply connected, there exists a scalar field $t$ (unique up to the addition of a constant) such that

$$u_a = -(dt)_a. \tag{3.5}$$

The “time function” $t$ is a great aid in establishing an upper bound to the lengths of timelike curves in the Tolman spacetimes. Noting that $u^a (dt)_a = 1$ we see, by Lemma 4, that $t$ is bounded from above and below along any integral curve of $u^a$ for which $m$ is positive.

Those familiar with the Tolman spacetimes may recognize $t$ as being one of the coordinates in which the Tolman spacetimes are usually presented. In fact, if a hypersurface everywhere orthogonal to $u_a$ exists, then $t$ is one of the synchronous (Gaussian normal) coordinates associated with this hypersurface and the geodetic vector field $u^a$ [16]. However, since there is no guarantee that such a surface will exist, the above construction of $t$ is preferred. (We further caution the reader that some or all surfaces of constant $t$, though spacelike, may not be Cauchy surfaces.)

Recall that the distance function $d(p^-, p^+)$ is defined for $p^+ \in I^+(p^-)$ to be the least upper bound of the lengths of timelike curves from $p^-$ to $p^+$ (and to is defined to be zero otherwise) [13]. Thus, the least upper bound to the lengths of timelike curves in a spacetime $(M, g_{ab})$ will be finite iff $d(p^-, p^+)$ is bounded for all $p^\pm \in M$.

For any two points $p^\pm \in M$ with $p^+ \in I^+(p^-)$, let $\mu$ be a future directed timelike curve connecting $p^-$ to $p^+$ and having length $d(p^-, p^+)$. (In other words, $\mu$ is a maximal geodesic
connecting the two points.) Parameterize $\mu$ by $\tau$ so that its tangent vector $v^a$ has unit norm. We then have

$$d(p^-, p^+) = \int_\mu d\tau = \int_\mu \frac{dt}{(v^a(dt)_a)} \leq \int_\mu dt = t(p^+) - t(p^-). \quad (3.6)$$

The second equality follows from the fact that $dt/d\tau = v^a(dt)_a$. The inequality follows from the fact that $v^a(dt)_a = -v^a u_a$ is no less than unity as both $v^a$ and $u^a$ are unit future-directed timelike vectors. So, to bound the length of a timelike curves, we need only find a global bound on the difference $t(p^+) - t(p^-)$.

### B. Bounding $t$

Denote the one-parameter local pseudo group of diffeomorphisms associated with the vector field $u^a$ by $\exp: M \to M$ \cite{[L]}. For any set $C$ define $\exp_\pm(C)$ be the set of those $p \in M$ such that $p = \exp_s(c)$ for some $c \in C$ and some $\pm s \geq 0$. Set $\exp(C)$ to be the union of these two sets. So, in particular for a point $p \in M$, $\exp(p)$ is the integral curve of $u^a$ that passes through the point $p$ while $\exp(p) \cap \Sigma$ is the point where this integral curve intersects $\Sigma$.

To establish bounds on $t$, we divide the spacetime $(M, g_{ab})$ into three regions. We do this by first dividing up the Cauchy surface $\Sigma$. Let $C_n$ and $C_s$ be any two spherically symmetric compact connected subsets of $\Sigma$ such that

(i) $C_n$, $C_s$ intersects $\gamma_n$, $\gamma_s$ respectively,

(ii) $D^a r$ is spacelike on $C_n$ and $C_s$,

(iii) $m[\partial C_n], m[\partial C_s] > 0$.

Note that such sets always exist and that they are always disjoint. Define $C_m$ to be the closure of $\Sigma - (C_n \cup C_s)$ in $\Sigma$. (Note that $\partial C_m = \partial C_n \cup \partial C_s$.) Using the three sets $C_m$, $C_n$, and $C_s$, we divide the full spacetime $M$ into three regions: a “middle” region, $\exp(C_m)$;
a “northern” region, exp($C_n$); and a “southern” region, exp($C_s$). Denote the boundaries of these regions (being timelike three-surfaces with $u^a$ tangent thereto) by $T_n = \partial \exp(C_n) = \exp(\partial C_n)$ and $T_s = \partial \exp(C_s) = \exp(\partial C_s)$.

1. Bounds on the middle region

Setting

$$T_m = \pi \sqrt{\frac{r^3}{\min C_m(2m)}}.$$  \hspace{1cm} (3.7)

we have

Lemma 5. For any $p^\pm \in \exp_\pm(C_m)$ there are $a^\pm \in C_m$ such that

$$\pm t(p^\pm) < \pm t(a^\pm) + T_m. \hspace{1cm} (3.8)$$

Proof. Setting $a^\pm = \exp(p^\pm) \cap \Sigma \in C_m$ this is a simple application of Lemma 4 as $p^\pm$ and $a^\pm$ lie on the same integral curve of $u^a$ with $2m \geq \min C_m(2m)$. (Lemma 1 gives a positive lower bound for this last quantity.) \square

2. Bounds on the northern region

To obtain bounds on $t$ in the region exp($C_n$), we first define $\pm k^a$ to be the spherically symmetric null vector fields such that $\pm k^a(dt)_a = \pm 1$ and $\pm k^a D_a r = +1$ on $\gamma_n$. (Actually, $\pm k^a$ are not well-defined on $\gamma_n$. However, this is a mere nuisance and not a fundamental problem.) Set

$$\kappa_n^\pm = \min_{C_n} (\pm k^a D_a r) > 0. \hspace{1cm} (3.9)$$

Lemma 6. $\pm k^a D_a r \geq \kappa_n^\pm$ on exp$_\pm(C_n)$.

Proof. Consider

$$u^a D_a (\pm k^b D_b r) = u^a \pm k^b D_a D_b r = \mp \frac{m}{r^2}, \hspace{1cm} (3.10)$$
where the first equality follows from the fact that $u^a D_a \pm k^b = 0$ and the second from the fact that $\pm k^a u_a = \mp 1$ and the vanishing of the pressure for these spacetimes ($\tau^{mn}(\epsilon_{ma} u^a)\epsilon_{nb} = 0$).

Set $Q^\pm = \pm k^a D_a r$. For any point $p \in \text{exp}_\pm(C_n)$, there is a unique integral curve $\gamma$ of $u^a$ from $C_n$ to $p$. But, by Eq. (3.10), the quantity $Q^-$ is non-decreasing along $\gamma$. Thus, $Q^-(p) \geq Q^-(\gamma \cap C_n) \geq \kappa_n^-$. Likewise, for any point $p \in \text{exp}_-(C_n)$, there is a unique integral curve $\gamma$ of $u^a$ from $p$ to $C_n$. But, by Eq. (3.10), the quantity $Q^+$ is non-increasing along $\gamma$. Thus, $Q^+(p) \geq Q^+(\gamma \cap C_n) \geq \kappa_n^+$. □

With this technical lemma we have

**Lemma 7.** For any $p^\pm \in \text{exp}_\pm(C_n)$ there are $a^\pm \in C_n$ such that

$$\pm t(p^\pm) < \pm t(a^\pm) + T_m + \frac{r_M}{\kappa_n^\pm}. \tag{3.11}$$

**Proof.** Set $q^\pm = \lambda^\mp \cap \partial(\text{exp}_\pm(C_n))$ where $\lambda^\mp$ is the integral curve of $\mp k^a$ starting from $p^\pm$. Consider

$$r(q^\pm) - r(p^\pm) = \int_{\lambda^\pm} dr = \int_{\lambda^\pm} (\mp k^a(dr)_a)(\mp dt)$$

$$\geq \kappa_n^\mp \int_{\lambda^\pm} (\mp dt)$$

$$= \mp \kappa_n^\mp (t(q^\pm) - t(p^\pm)). \tag{3.12}$$

The second equality follows from the facts that $\mp k^a$ is tangent to the curve $\lambda^\mp$ and that $\mp k^a(dt)_a = \mp 1$. This together with the inequality $r_M \geq r(q^\pm) - r(p^\pm)$ gives us

$$\pm t(p^\pm) \leq \pm t(q^\pm) + \frac{r_M}{\kappa_n^\mp}. \tag{3.13}$$

Now, if $q^\pm \in C_n$ then taking $a^\pm = q^\pm$ Eq. (3.11) follows. Otherwise, if $q^\pm \in \mathcal{T}_n$ then take $a^\pm = \exp(q^\pm) \cap C_n$. Then Eq. (3.11) follows from the fact that

$$\pm t(q^\pm) = \pm t(a^\pm) \pm (t(q^\pm) - t(a^\pm))$$

$$< \pm t(a^\pm) + T_m. \tag{3.14}$$

This last inequality follows from again applying Lemma 4 and the fact that $2m(\mathcal{T}_n) = 2m(\partial C_n) \geq \min_{C_n}(2m).$ □
3. Bounds on the southern region

Now define $\pm k^a$ to be the null vector fields such that $\pm k^a(dt)_a = \pm 1$ and $\pm k^a D_a r = \pm 1$ on $\gamma_s$. Set

$$\kappa^\pm_s = \min_{C_s}(\pm k^a D_a r) > 0. \quad (3.15)$$

Then, by exactly the same methods as above we have

Lemma 8. For any $p^\pm \in \exp_\pm(C_s)$ there are $a^\pm \in C_s$ such that

$$\pm t(p^\pm) < \pm (t(a^\pm) + T_m + \frac{r_M}{\kappa^\pm_s}). \quad (3.16)$$

C. Bound on the Lengths of Timelike Curves

With the bounds established in Secs. III B 1, III B 2, we now prove Theorem 1 by proving

Theorem 5. In any Tolman spacetime that possesses $S^3$ Cauchy surfaces and whose energy density is positive

$$d(p^-, p^+) < 2T_m + \tau^+ + \tau^- + \Delta t, \quad (3.17)$$

where $T_m$ is given by Eq. (3.7), $\tau^\pm$ and $\Delta t$ are given by

$$\tau^\pm = \max_{\kappa_n^\pm, \kappa_s^\pm} \left(\frac{r_M}{\kappa_n^\pm}, \frac{r_M}{\kappa_s^\pm}\right), \quad (3.18)$$

$$\Delta t = \max_{a,b \in \Sigma} (t(b) - t(a)), \quad (3.19)$$

$\kappa_n^\pm$ and $\kappa_s^\pm$ are given by Eqs. (3.9) and (3.15) respectively, and $r_M$ is given by Eq. (3.2).

Proof. Without loss in generality, taking $p^\pm \in D^\pm(\Sigma)$ it follows directly from Eq. (3.6) and Lemmas 5, 7, and 8 that

$$d(p^-, p^+) \leq t(p^+ - t(p^-)$$

$$< 2T_m + \tau^+ + \tau^- + (t(a^+) - t(a^-)) \quad (3.20)$$

for some $a^\pm \in \Sigma$. Eq. (3.17) now follows from the fact that $(t(a^+) - t(a^-)) \leq \Delta t. \Box$
All the quantities that appear in the upper bound given by Eq. (3.17) are calculable from the initial data on an initial data surface. Once \( C_n \) and \( C_s \) are chosen, that \( T_m \) and \( \tau^\pm \) can be calculated is clear. Further, the quantity \( \Delta t \) can be calculated since \( t(b) - t(a) = -\int_a^b u_m \), where the integral is along any path from \( a \) to \( b \) in the Cauchy surface. (Note that if \( \Sigma \) is everywhere normal to \( u_m \), then \( \Delta t = 0 \). However, as mentioned earlier, such a surface may not exist!)

As a check on the bound given by Theorem 5, we use the \( k = +1 \) Friedmann-Robertson-Walker spacetimes with dust as a source. For these spacetimes it is a straightforward exercise to show that the least upper bound to \( d(p^-, p^+) \) is \( \pi \max_\Sigma (2m) \) where \( \Sigma \) is any Cauchy surface. Since \( T_m > \pi \max_\Sigma (2m) \), the inequality given by Eq. (3.17) does hold for these spacetimes.

Lastly, we note that the bound given by Theorem 5 may be a bit weaker than can be argued. As evidence for this, it is not too difficult to show that for \( p^\pm \in \exp(C_m) \) that \( d(p^-, p^+) < T_m. \) (The idea is to show that for such points, the geodesic \( \mu \) attaining the length \( d(p^+, p^-) \) will remain in \( \exp(C_m) \) and thus on \( \mu \), we have \( 2m \geq \min_{C_m} (2m) \). Using these facts it follows by the argument used in Ref. [5] that \( d(p^-, p^+) < T_m. \))

**IV. DISCUSSION**

Now that we have a proof of the closed-universe recollapse conjecture for the Tolman spacetimes, we ask whether its proof can be generalized. After all, such a result would be of far greater interest than the more limited Tolman result. Unfortunately, too many properties of the Tolman spacetimes are used in the proof to make any generalizations apparent. First, how does one generalize the “time-function” \( t? \) Even for a perfect fluid, where there is a preferred vector field \( u^a \), a similar construction of a “time-function” fails as \( u_a \) fails to be closed. One possible generalization might involve the construction of a (locally defined) geodetic vector field. Then, as for Tolman, a (locally defined) “time-function” would exist. Yet, how such a construction might arise is not clear. Second, even if such a geodetic
congruence were to be constructed, the proof of Lemma 6 would not go through. With the presence of a pressure, the right-hand side of Eq. (3.10) does not have any definite sign. In light of these difficulties, for the time being, the Tolman result presented here will remain the only proof of the closed-universe recollapse conjecture for a class of spacetimes that are spherically symmetric, have $S^3$ Cauchy surfaces, and are spatially inhomogeneous.

Lastly, we offer the following piece of evidence, taking the form of a gedanken experiment, which offers hope that the closed-universe recollapse conjecture is true for the spherically symmetric spacetimes with $S^3$ Cauchy surfaces. Imagine two observers, one at $\gamma_n$, the other at $\gamma_s$, each surrounded by a spherically symmetric space station of mass $\mu$ designed to protect them from any outside dangers—e.g., infalling matter. (The argument, as currently formulated, requires both observers, but it seems plausible that one such observer is sufficient for the argument to go through.) Can such observers be protected forever, and therefore live forever, using such an arrangement? For simplicity, suppose we demand that their protection is so good that the spacetime inside the space stations is static. In that case consider the longest timelike curve connecting the outer surface of one of the stations to itself at two different times. Such a curve can be shown to lie completely in the region between the two stations. Further, it can be shown that in this region $m$ is bounded from below by a positive constant (no greater than $\mu$). It follows from the same argument used to place an upper bound on the lengths of timelike curves in the $S^1 \times S^2$ spherically symmetric spacetimes that there is an upper bound to the length of such a curve connecting the outer surface of the station at two different times. Thus, the length of time such an arrangement can be maintained (being less than this upper bound) is finite. Unfortunately, how this scenario can be parlayed into a more general theorem is not clear.
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