Fermion Absorption Cross Section and Topology of Spherically Symmetric Black Holes

Yu-Xiao Liu, Li Zhao, Zhen-Bin Cao, and Yi-Shi Duan

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

Abstract

In 1997, Liberati and Pollifrone in Phys. Rev. D56 (1997) 6458 (hep-th/9708014) achieved a new formulation of the Bekenstein-Hawking formula, where the entropy and the Euler characteristic are related by $S = \chi \mathcal{A}/8$. In this work we present a relation between the low-energy absorption cross section for minimally coupled fermions and the Euler characteristic of (3+1)-dimensional spherically symmetric black holes, i.e. $\sigma = \chi \mathcal{g}_h^{-1} \mathcal{A}$. Based on the relation, using the Gauss–Bonnet–Chern theorem and the $\phi$-mapping method, an absorption cross section density is introduced to describe the topology of the absorption cross section. It is shown that the absorption cross section and its density are determined by the singularities of the timelike Killing vector field of the spacetime and these singularities carry the topological numbers, Hopf indices and Brouwer degrees, naturally.

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I. INTRODUCTION

Interest in the absorption of quantum waves by black holes was reignited in the 1970s, following Hawking’s discovery that black holes can emit, as well as scatter and absorb, radiation [1]. More recently, absorption cross sections have been of interest in the context of higher-dimensional string theories.

In a series of papers [2, 3, 4], Sanchez considered the scattering and absorption of massless scalar particles by an uncharged, spherically-symmetric black hole. Unruh [5] studied the absorption of massive spin-half particles by piecing together analytic solutions to the Dirac equation across three regions. He showed that, in the low-energy limit, the scattering cross section for the fermion is exactly 1/8 of that for the scalar particle. Doran et al. [6] studied the absorption of massive spin-half particles by a small Schwarzschild black hole by numerically solving the single-particle Dirac equation in Painlevé–Gullstrand coordinates.

In Ref. [7], Das et al. computed the low-energy absorption cross section for minimally coupled massless scalars and spin-half particles, into a general spherically symmetric black hole in arbitrary dimensions. It is interesting to note for extremal black holes, the absorption cross section for minimally coupled fermion vanishes in the limit of extremality. In 2005, Jung, Kim and Park [8] computed the ratio of the low-energy absorption cross section for Dirac fermion to that for minimally coupled scalar when the spacetimes are various types of the higher-dimensional spherically symmetric black holes. They found that the low-energy absorption cross section for the Dirac fermion always goes to zero in the extremal limit regardless of the detailed geometry of the spacetime. So, there should be some relation between the low-energy absorption cross section for minimally coupled fermions and the topology of spherically symmetric black holes. In this paper we will consider the relation between them.

This paper is organized as follows. In Section III using the Gauss–Bonnet–Chern (GBC) theorem, the Euler characteristic and the low-energy absorption cross section for Dirac fermions of both the extremal and nonextremal spherically symmetric black holes are briefly reviewed. Some useful notations are also prepared. In Section III from the opinion of the decomposition of spin connection, the density of the absorption cross section of spherically symmetric black holes is proposed in terms of the smooth unit tangent vector field. Then, in Section IV using the $\phi$-mapping method [9], the topology of the absorption cross section is
investigated at the singularities of the unit vector field. In Section V, we compute the value of the Hopf index for several example spacetimes. The conclusion of this paper is given in Section VI.

II. THE EULER CHARACTERISTIC AND THE ABSORPTION CROSS SECTION FOR DIRAC FERMION OF SPHERICALLY SYMMETRIC BLACK HOLES

Let us consider the general spherically symmetric metric in (3+1)-dimensional spacetime of the form

\[ ds^2 = f(\hat{r}) dt^2 - g(\hat{r}) \left( d\hat{r}^2 + \hat{r}^2 d\Omega^2 \right), \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the metric on the unit 2-sphere. We also assume

\[ \lim_{\hat{r} \to \infty} f(\hat{r}) = \lim_{\hat{r} \to \infty} g(\hat{r}) = 1 \]

for ensuring that the metric is asymptotically flat. Ref. [7] has shown that the low-energy absorption cross section for minimally coupled fermions is given by the area measured in the flat spatial metric conformally related to the true metric in the form

\[ \sigma = 2 g_h^{-1} A, \]

where \( A \equiv 4\pi R_h^2 \) is the horizon area, \( R_h = \hat{r}_h \sqrt{g_h} \), \( g_h = g(\hat{r}_h) \), and \( \hat{r}_h \) is a horizon radius usually determined by the largest solution of \( f(\hat{r}) = 0 \), the factor 2 comes from the number of spinors.

As a fundamental but very important case of (1), the Reissner–Nordström (RN) black hole has the metric form

\[ ds^2_{RN} = F(r) dt^2 - F(r)^{-1} dr^2 - r^2 d\Omega^2. \]

with

\[ F(r) = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right). \]

There are two horizons, the event horizon \( r_+ \) and Cauchy horizon \( r_- \),

\[ r_\pm = M \pm \sqrt{M^2 - Q^2}. \]

The extremal RN black hole corresponds to the case \( r_+ = r_- \) or, equivalently, \( M = Q \). The metric (4) can also be written in the form

\[ ds^2_{RN} = f_{RN}(\hat{r}) dt^2 - g_{RN}(\hat{r}) \left( d\hat{r}^2 + \hat{r}^2 d\Omega^2 \right) \]
with
\[ f_{RN}(\hat{r}) = \left[ \frac{\hat{r}^2 - C^2}{(\hat{r} + C)^2 + \hat{r}r_-} \right]^2, \tag{8} \]
\[ g_{RN}(\hat{r}) = \left[ \frac{(\hat{r} + C)^2}{\hat{r}^2} + \frac{r_-}{\hat{r}} \right]^2, \]
where \( C = (r_+ - r_-)/4 \), and the relation between \( r \) and \( \hat{r} \) is
\[ r = \frac{(\hat{r} + C)^2}{\hat{r}} + r_. \tag{9} \]

Using the GBC theorem and the boundary conditions, it is shown that the Euler characteristic
\[ \chi = 2 \tag{10} \]
for the nonextremal RN black hole \( [10, 11, 12] \) and
\[ \chi = 0 \tag{11} \]
for the extremal RN black hole \( [13, 14] \). In the later sections, we will prove that the above two formulas are held for general spherically symmetric black holes as well as for the Kerr and Kerr–Newman black holes.

It is interesting to note that the low-energy absorption cross section \( \sigma \) for the Dirac fermion always vanishes in the extremal limit regardless of the detailed geometry of the spacetime \( [7, 8] \). So we introduce the relation between the absorption cross section and the topology of spherically symmetric black holes
\[ \sigma = g_h^{-1} A \chi. \tag{12} \]
Thus, the topologies \( \text{[10]} \) and \( \text{[11]} \) of spherically symmetric black holes lead directly to the nonzero and zero results of the absorption cross section by taking account of the relation \( \text{[12]} \).

III. THE ABSORPTION CROSS SECTION DENSITY IN SPHERICALLY SYMMETRIC BLACK HOLES

The topologies \( \text{[10]} \) and \( \text{[11]} \) give the global structure of spherically symmetric black holes. In this section, in order to show the relation between the absorption cross section and
the topology of spherically symmetric black holes, let us consider the Euler characteristic in detail and introduce the absorption cross section density of spherically symmetric black holes.

For a closed $N$(even)-dimensional Riemannian manifold $M^N$, the Euler characteristic $\chi(M^N)$ can be expressed as the volume integral of the GBC differential $N$-form $\Lambda$:

$$\chi(M^N) = \int_{M^N} \Lambda, \quad \ldots \quad (13)$$

$$\Lambda = \left(-1\right)^{N/2-1} \frac{\pi^N}{2^N N!} \epsilon_{A_1A_2 \ldots A_{N-1}A_N} F^{A_1A_2} \wedge \ldots \wedge F^{A_{N-1}A_N}, \quad \ldots \quad (14)$$

in which $F^{AB}$ is the curvature tensor of $SO(N)$ principal bundle of the Riemannian manifold $M$, i.e., the $SO(N)$ gauge field 2-form

$$F^{AB} = d\omega^{AB} - \omega^{AC} \wedge \omega^{CB}, \quad \ldots \quad (15)$$

where $\omega^{AB}$ is the spin connection 1-form. Chern has shown that the GBC $N$-form $\Lambda$ on $M^N$ can be pulled back to $S(M^N)$ as the exterior derivative of a differential $(N-1)$-form $\Omega$:

$$\pi^* \Lambda = d\Omega, \quad \ldots \quad (16)$$

where $\pi^*$ denotes the pullback of the projection $\pi : S(M^N) \to M^N$. Using a recursion method, Chern has proved that the $(N-1)$-form on $S(M^N)$ can be written as

$$\Omega = \frac{1}{(2\pi)^{N/2}} \sum_{k=0}^{N/2-1} \left(-1\right)^k \frac{2^{-k}}{(N-2k-1)!! k!} \Theta_k, \quad \ldots \quad (17)$$

which is called the Chern form with

$$\Theta_k = \epsilon_{A_1A_2 \ldots A_{N-2k} A_{N-2k+1} A_{N-2k+2} \ldots A_{N-1} A_N} n^{A_1} \theta^{A_2} \wedge \ldots \wedge \theta^{A_{N-2k}} \wedge F^{A_{N-2k+1} A_{N-2k+2} \ldots A_{N-1} A_N}, \quad \ldots \quad (18)$$

and

$$\theta^A \equiv Dn^A = dn^A - \omega^{AB} n^B, \quad \ldots \quad (19)$$

in which $n^A$ is the section of the sphere bundle $S^{N-1}(M)$

$$n : \partial M \to S^{N-1}(M). \quad \ldots \quad (20)$$
It is noted that $\pi^*$ maps the cohomology of $M^N$ into that of $S(M^N)$, where $n^*$ performs the inverse operation. Thus $n^*\pi^*$ amounts to the identity and the Euler characteristic $\chi(M^N)$ in Eq. (13) can be written as

$$\chi(M^N) = \int_{M^N} \Lambda = \int_{M^N} n^*\pi^* \Lambda = \int_{M^N} n^*d\Omega. \quad (21)$$

In the opinion of the decomposition of gauge potential, Duan et al. ([9, 15, 16]) showed that the $(N - 1)$-form can be formulated in terms of the unit tangent vector field $n^a(x)$ cleanly as

$$\Omega = \frac{\Gamma^{N/2}}{(N-1)!(2\pi)^{N/2}} \epsilon_{a_1a_2...a_N} n^{a_1}dn^{a_2} \wedge ... \wedge dn^{a_N}. \quad (22)$$

Therefore the Euler characteristic $\chi(M^N)$ is

$$\chi(M^N) = \frac{1}{(N-1)!A(S^{N-1})} \int_{M^N} \epsilon^{\mu_1\mu_2...\mu_N} \epsilon_{a_1a_2...a_N} \times \partial_{\mu_1} n^{a_1} \partial_{\mu_2} n^{a_2} ... \partial_{\mu_N} n^{a_N} d^N x, \quad (23)$$

where $A(S^{N-1})$ is the surface area of $(N - 1)$-dimensional unit sphere $S^{N-1}$,

$$A(S^{N-1}) = \frac{(2\pi)^{N/2}}{\Gamma^{N/2}}. \quad (24)$$

For (3+1)-dimensional spherically symmetric black holes, Eq. (15) becomes

$$\chi = \frac{1}{12\pi^2} \int_M \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d d^4 x, \quad (25)$$

and $n^a(x)$ coincides with the timelike Killing vector field of the spherically symmetric spacetime. According to the relation (12), we introduce the absorption cross section density $\rho$ of spherically symmetric black holes

$$\rho = \frac{A}{gh} \cdot \frac{1}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{abcd} \partial_\mu n^a \partial_\nu n^b \partial_\lambda n^c \partial_\rho n^d. \quad (26)$$

Then the absorption cross section $\sigma$ of spherically symmetric black holes reads as

$$\sigma = \int_M \rho \ d^4 x. \quad (27)$$

In the following, we will consider the topology of the absorption cross section of spherically symmetric black holes through the absorption cross section density $\rho$ and the so-called $\phi$-mapping method.
IV. THE TOPOLOGY OF THE ABSORPTION CROSS SECTION IN SPHERICALLY SYMMETRIC BLACK HOLES

The unit tangent vector \( n^a \) over \( M \) satisfies

\[
n^a n^a = 1, \quad a = 1, 2 \cdots 4, \tag{28}
\]

and can, in general, be further expressed as

\[
n^a = \frac{\phi^a}{\| \phi \|}, \quad \| \phi \| = \sqrt{\phi^a \phi^a}, \tag{29}
\]

where \( \phi^a = e^a_{\mu} \phi^\mu \), \( e^a_{\mu} \) and \( \phi^\mu \) are the vierbein and the timelike Killing vector field of spherically symmetric black holes, respectively. Substituting (29) into (26) and considering that

\[
\partial_\mu n^a = \frac{\partial_\mu \phi^a}{\| \phi \|} + \phi^a \partial_\mu \frac{1}{\| \phi \|}, \tag{30}
\]

we have

\[
\rho = -\frac{A}{24 \pi^2 g_h} \epsilon^{\mu \nu \lambda \rho} e_{abcd} \partial_\mu \phi^k \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \frac{\partial}{\partial \phi^a} \frac{1}{\| \phi \|^2}. \tag{31}
\]

If we define the Jacobian \( J(\phi/x) \) as

\[
\epsilon^{kbed} J(\phi/x) = \epsilon^{\mu \nu \lambda \rho} e_{abcd} \partial_\mu \phi^k \partial_\nu \phi^b \partial_\lambda \phi^c \partial_\rho \phi^d \tag{32}
\]

and notice \( \epsilon_{abcd} \epsilon^{kbed} = 3! \delta^k_a \), we obtain

\[
\rho = -\frac{A}{4 \pi^2 g_h} \left( \frac{\partial^2}{\partial \phi^a \partial \phi^a} \frac{1}{\| \phi \|^2} \right) J(\frac{\phi}{x}), \tag{33}
\]

Via the general Green function relation in \( \phi \)-space

\[
\frac{\partial^2}{\partial \phi^a \partial \phi^a} \frac{1}{\| \phi \|^2} = -4 \pi^2 \delta(\phi), \tag{34}
\]

we do obtain the \( \delta \)-function like absorption cross section density \( \rho \) of spherically symmetric black holes

\[
\rho = \frac{A}{g_h} \delta(\phi) J(\frac{\phi}{x}). \tag{35}
\]

From the above formula one see that the absorption cross section density does not vanish only at the zero points of the vector field \( \phi(x) \). This result shows that the topology of the absorption cross section is determined by the zeros of \( \phi(x) \), i.e. the singularities of \( n(x) \) which coincides with the timelike Killing vector field of the spherically symmetric spacetime. Therefore, it is essential to investigate the solutions of \( \phi(x) = 0 \).
Suppose that the vector field $\phi(x)$ has $l$ zero points which are denoted as $\vec{z}_i$ ($i = 1, \cdots, l$). According to the implicit function theorem [18], when the zero points $\vec{z}_i$ are the regular points of $\phi(x)$, i.e. when the Jacobian determinant
\[ J \left( \frac{\phi}{x} \right) \Bigg|_{\vec{z}_i} = \frac{\partial (\phi^1, \phi^2, \phi^3, \phi^4)}{\partial (x^1, x^2, x^3, x^4)} \Bigg|_{\vec{z}_i} \neq 0, \] where $J (\phi/x) = J^0 (\phi/x)$ is the usual Jacobian determinant, there exists one and only one continuous solution of $\phi^a(x) = 0$. The solution can be expressed as $\vec{z}_i = \vec{z}_i(t)$, which is the trajectory of the $i$th zero point of $\phi(x)$. According to the $\delta$-function theory [19] and the $\phi$-mapping theory, we know that $\delta(\phi)$ can be expanded by these zeros as
\[ \delta(\phi) = \sum_{i=1}^{l} \frac{\beta_i}{|J(\phi/x)|_{\vec{z}_i}} \delta(\vec{x} - \vec{z}_i), \] where the positive integer $\beta_i$ is called the Hopf index of the $\phi$-mapping at $\vec{z}_i$ and it means that, when the point $\vec{x}$ covers the neighborhood of $\vec{z}_i$ once, the function $\phi(x)$ covers the corresponding region $\beta_i$ times, which is a topological number of first Chern class and relates to the generalized winding number of the $\phi$-mapping. Substituting (37) into (35), the absorption cross section density $\rho$ is formulated by
\[ \rho = \frac{A}{g_h} \sum_{i=1}^{l} \beta_i \eta_i \delta(\vec{x} - \vec{z}_i), \] where $\eta_i$ is called the Brouwer degree of the $\phi$-mapping at $\vec{z}_i$ [9, 20] and
\[ \eta_i = \text{sign} J(\phi/x)|_{\vec{z}_i} = \pm 1 \] according to the clockwise or anti-clockwise rotation of $\phi(x)$ when $\vec{x}$ circles $\vec{z}_i$ clockwise. So, from (27), the absorption cross section $\sigma$ of spherically symmetric black holes is given by the sum of these Hopf indices and Brouwer degrees of the Killing vector field at its singularities, i.e.
\[ \sigma = \frac{A}{g_h} \sum_{i=1}^{l} \beta_i \eta_i, \] which is the direct result of the combination of the relation (12) and the Hopf index theorem.

From (38) and (40) we see that the absorption cross section density is determined by the singularities of the timelike Killing vector field and the absorption cross section is characterized by the topological numbers, Hopf indices and Brouwer degrees of $\phi$-mapping, at these singularities.
V. THE VALUE OF THE HOPF INDEX FOR SEVERAL EXAMPLE SPACE-TIMES

In this section, we compute the value of the Hopf index for several example black holes and for a general static spherically symmetric metric.

A. The Schwarzschild black hole

The Schwarzschild black hole has the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d^2\Omega.$$  \hfill (41)

with the horizon located at $r = 2M$. Using the GBC theorem, it has been shown (Liberati and Pollifrone, 1997) that the Euler characteristic of the Schwarzschild black hole is $\chi = 2$, \hfill (42)

which leads directly to the Bekenstein–Hawking entropy $S = A/4$ by taking account of the relation $S = \chi A/8$. The topology (42) gives the global property of the Schwarzschild black hole. In the following, let us compute the value of the Hopf index for the Schwarzschild black hole.

By using the Killing equation

$$(\partial_\lambda g_{\mu\nu})\phi^\lambda + g_{\lambda\mu}\partial_\nu \phi^\lambda + g_{\lambda\nu}\partial_\mu \phi^\lambda = 0,$$ \hfill (43)

and the static and spherically symmetric properties of the Schwarzschild black hole that imply

$$\partial_0 \phi^\mu = \partial_3 \phi^\mu, \quad \mu = 0, 1, 2, 3,$$ \hfill (44)

one can solve the Killing vector field $\phi^\mu(x)$ and, by transforming into the local orthonormal vierbein index “a”: $\phi^a = e^a_\mu \phi^\mu(x)$, one obtains

$$\phi^0 = \sqrt{1 - \frac{2M}{r}}, \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = r \sin \theta.$$ \hfill (45)

It is easy to see that for the Schwarzschild black hole the Killing vector field $\phi^a$ has two zeros located at

$$(r = 2M, \theta = 0), \quad (r = 2M, \theta = \pi).$$ \hfill (46)
One can see the distribution of Killing vector field in the figures in Ref. [22]. For the zero $(2M, 0)$, the Killing vector field $\phi^a$ rotates from 0 to $\pi$ anti-clockwise when the spacetime point $(r, \theta)$ circles the zero in the same way. So we have the Hopf index and the Brouwer degree

$$\beta_1 = 1, \quad \eta_1 = +1,$$

at the zero $(2M, 0)$. Similarly, for the zero $(2M, \pi)$, we get

$$\beta_2 = 1, \quad \eta_2 = +1.$$  

Then, from the general results in Eqs. (38) and (40), we obtain the Euler characteristic and the absorption cross section of the Schwarzschild black hole

$$\chi = \sum_{i=1}^{2} \beta_i \eta_i = 2,$$

$$\sigma = \frac{A}{gh} \sum_{i=1}^{2} \beta_i \eta_i = \frac{2A}{gh},$$

and particularly the absorption cross section density

$$\rho = \frac{A}{gh} \delta(r - 2M) \left[ \delta(\theta) + \delta(\theta - \pi) \right].$$

B. The RN black hole

For the RN black hole the metric is given in Eq. (4). By using the Killing equation (43) and the condition for the Killing vector field of the RN black hole (44), we can also solve the Killing vector field $\phi^\mu(x)$. The result for $\phi^a = e^\mu_a \phi^\mu(x)$ is of the form

$$\phi^0 = \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}, \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = r \sin \theta.$$  

For the nonextremal RN black hole, it turns out that the Killing vector field $\phi^a$ has four zeros located at

$$(r = r_+, \theta = 0), \quad (r = r_-, \theta = 0), \quad (r = r_+, \theta = \pi), \quad (r = r_-, \theta = \pi),$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. For the zero $(r_+, 0)$, the Killing vector field $\phi^a$ rotates from 0 to $\pi$ anti-clockwise when the spacetime point $(r, \theta)$ circles the zero in the same way. So we have the Hopf index and the Brouwer degree

$$\beta_1 = 1, \quad \eta_1 = +1.$$
at the zero \((r_+, 0)\). While for the zero \((r_-, 0)\), when \((r, \theta)\) circles the zero from \(\pi\) to 0 anti-clockwise, \(\phi^a\) rotates from \(\pi\) to 0 clockwise, which leads to the Hopf index and the Brouwer degree

\[
\beta_2 = 1, \quad \eta_2 = -1, \quad (55)
\]

Similarly, one can get

\[
\beta_3 = 1, \quad \eta_3 = -1, \quad (56)
\]

for the zero \((r_-, \pi)\) and

\[
\beta_4 = 1, \quad \eta_4 = +1, \quad (57)
\]

for the zero \((r_+, \pi)\). Since the event horizon is the boundary of the outer spacetime of the RN black hole, it is only the zeros \((r_+, 0)\) and \((r_+, \pi)\) that have contribution to the Euler characteristic and the absorption cross section. Then, the Euler characteristic, the absorption cross section and its density of the RN black hole are

\[
\chi = \sum_{i=1,4} \beta_i \eta_i = 2, \quad (58)
\]

\[
\sigma = \frac{A}{g_h} \sum_{i=1,4} \beta_i \eta_i = 2 \frac{A}{g_h}, \quad (59)
\]

\[
\rho = \frac{A}{g_h} \delta(r - r_+) \left[ \delta(\theta) + \delta(\theta - \pi) \right]. \quad (60)
\]

respectively.

For the extremal RN black hole, the Killing vector field \(\phi^a\) has two zeros located at

\[
(r = M, \theta = 0), \quad (r = M, \theta = \pi). \quad (61)
\]

In this case, when the spacetime point \((r, \theta)\) circles these zeros anti-clockwise, \(\phi^a\) rotates anti-clockwise at the outer spacetime of horizon but clockwise at the inner of horizon. So we have the Hopf indices

\[
\beta_1 = \beta_2 = 0, \quad (62)
\]

which gives the Euler characteristic, the absorption cross section, and its density of the extremal RN black hole

\[
\chi = 0, \quad \sigma = 0, \quad \rho = 0. \quad (63)
\]

This result is in agreement with the statements that the extremal black hole has a unique internal state [13] and its temperature is zero due to the vanishing of surface gravity on horizon.
C. General spherically symmetric black holes

In above two subsection, we have consider two example of spherically symmetric black holes. Now let us compute the value of the Hopf index for general spherically symmetric black holes.

In this general case, the metric can be taken the form given in Eq. (1). By using the Killing equation (43) and the condition for the Killing vector field (44), we can also get the general form of the Killing vector field $\phi^a$

$$\phi^0 = \sqrt{F(r)}, \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = r \sin \theta. \quad (64)$$

We consider here the case that there are two horizons, the event horizon $r_+$ and Cauchy horizon $r_-$ (which are the solutions of $F(r) = 0$) for general spherically symmetric black holes. For the nonextremal black hole, the Killing vector field $\phi^a$ (64) has four zeros $(r_i, \theta_i)$ located at

$$(r_+, 0), \quad (r_+, \pi), \quad (r_-, 0), \quad (r_-, \pi). \quad (65)$$

For these zeros, the corresponding Hopf indexes and the Brouwer degrees $(\beta_i, \eta_i)$ are

$$(1, +1), \quad (1, +1), \quad (1, -1), \quad (1, -1). \quad (66)$$

Again, it is only the zeros $(r_+, 0)$ and $(r_+, \pi)$ that have contribution to $\chi$, $\sigma$ and $\rho$. And the Euler characteristic, the absorption cross section and its density are the same as the case of the nonextremal RN black hole.

$$\chi = 2, \quad \sigma = 2 \frac{A}{gh}, \quad \rho = \frac{A}{gh} \delta(r - r_+) \left[ \delta(\theta) + \delta(\theta - \pi) \right]. \quad (67)$$

For extremal black holes $(r_+ = r_-)$, the Killing vector field $\phi^a$ has two zeros located at $(r_+, 0)$ and $(r_+, \pi)$, and the corresponding Hopf indices are $\beta_1 = \beta_2 = 0$, which gives the following result

$$\chi = 0, \quad \sigma = 0, \quad \rho = 0. \quad (68)$$

This result is in agreement with the statements that the extremal black hole has a unique internal state [13] and its temperature is zero due to the vanishing of surface gravity on horizon.
D. The Kerr black hole

In this subsection, we extend this work to the Kerr black hole. The Kerr solution describes both the stationary axisymmetric asymptotically flat gravitational field outside a massive rotating body and a rotating black hole with mass $M$ and angular momentum $J$. The Kerr black hole can also be viewed as the final state of a collapsing star, uniquely determined by its mass and rate of rotation. Moreover, its thermodynamical behavior is very different from the Schwarzschild black hole or the RN black hole, because of its much more complicated causal structure. Hence its study is of great interest in understanding physical properties of astrophysical objects, as well as in checking any conjecture about thermodynamical properties of black holes.

In terms of Boyer–Lindquist coordinates, the Euclidean Kerr metric reads

$$ds^2 = \frac{\Delta}{\varrho^2} (dt - a \sin^2 \theta d\varphi)^2 + \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\theta^2 + \frac{\sin^2 \theta}{\varrho^2} \left[ (r^2 + a^2) d\varphi - adt \right]^2,$$

where $\Delta$ is the Kerr horizon function

$$\Delta = r^2 - 2Mr + a^2,$$

and

$$\varrho = r^2 + a^2 \cos^2 \theta.$$ 

Here $a$ is the angular momentum for unit mass as measured from the infinity; it vanishes in the Schwarzschild limit. The nonextremal Kerr black hole has the event horizon $r_+$ and the Cauchy horizon $r_-$ at

$$r_{\pm} = M \pm \sqrt{M^2 - a^2},$$

The extreme case corresponds to $r_+ = r_-$ or $M = a$. Using the GBC theorem, it has been proved that $\chi = 2$ for the nonextremal Kerr black hole and $\chi = 0$ for the extremal one.

Such a metric corresponds to the following vierbein 1-forms:

$$e^0 = \frac{\sqrt{\Delta}}{\varrho} (dt - a \sin^2 \theta d\varphi), \quad e^1 = \frac{\varrho}{\sqrt{\Delta}} dr,$$

$$e^2 = \varrho d\theta, \quad e^3 = \frac{\sin \theta}{\varrho} \left[ (r^2 + a^2) d\varphi - adt \right].$$
where $\varrho$ is the positive square root of $\varrho^2$. The Killing equation and the condition for the Killing vector field $\phi^\mu$ are same as Eqs. (43) and (44). By the relation $\phi^a = e^a_\mu \phi^\mu$ and the vierbein given in (73) and (74), the solution for $\phi^a$ is read as

$$\phi^0 = \sqrt{\Delta} \varrho, \quad \phi^1 = \phi^2 = 0, \quad \phi^3 = \frac{a \sin \theta}{\sqrt{\varrho}}.$$  \hspace{1cm} (75)

For the nonextremal Kerr black hole, the Killing vector field $\phi^a$ (75) has four zeros $(r_i, \theta_i)$ located at

$$(r_+, 0), \quad (r_+, \pi), \quad (r_-, 0), \quad (r_-, \pi).$$  \hspace{1cm} (76)

The Hopf indexes and the Brouwer degrees $(\beta_i, \eta_i)$ correspond to these zeros are

$$(1, +1), \quad (1, +1), \quad (1, -1), \quad (1, -1).$$  \hspace{1cm} (77)

Again, it is only the zeros $(r_+, 0)$ and $(r_+, \pi)$ that have contribution to the Euler characteristic $\chi$ which is calculated to be

$$\chi = 2.$$  \hspace{1cm} (78)

For the extremal Kerr black hole, the Killing vector field $\phi^a$ has two zeros located at $(r_+, 0)$ and $(r_+, \pi)$, and the corresponding Hopf indices are $\beta_1 = \beta_2 = 0$, which gives the Euler characteristic

$$\chi = 0.$$  \hspace{1cm} (79)

Lastly, let us give a comment on a more general spacetime—the Kerr–Newman black hole, which metric has Petrov type $D$ and has also the form of the Kerr black hole (69) but with $\Delta = r^2 - 2M r + a^2 + Q^2$. From the formalized solution of the Killing vector field for the Kerr black hole (75), one can conclude that the results (78) and (79) are also held for the Kerr–Newman black hole. We will assume that the result (3) generalizes to arbitrary charged rotating black holes and so the relations (67) and (68) continues to hold for the Kerr black hole and the Kerr–Newman black hole.

VI. CONCLUSIONS

In summary, we first present a relation between the low-energy absorption cross section for minimally coupled fermions and the Euler characteristic of (3+1)-dimensional spherically symmetric black holes. From the relation, one can see clearly that the topologies (10) and
of spherically symmetric black holes, which correspond to the nonextremal and extremal black holes, respectively, lead directly to the nonzero and zero results of the absorption cross section. Then using the relation and the GBC theorem, we introduce the absorption cross section density to describe the relation between them. It is shown that the absorption cross section and its density are determined by the singularities of the timelike Killing vector field of spacetime and these singularities are labeled by the topological numbers, Hopf indices and Brouwer degrees. For nonextremal black holes, there are two singularities with the Hopf indices $\beta = 1$ and Brouwer degrees $\eta = +1$ on the north pole and south pole of event horizon and two singularities with $\beta = 1$ and $\eta = -1$ on the north pole and south pole of Cauchy horizon. In this case, only the singularities on event horizon have contribution to the Euler characteristic and the absorption cross section and one obtains $\chi = 2$ and $\sigma = 2g_h^{-1}A$. For extremal ones, there are only two singularities with $\beta = 0$ on the north pole and south pole of horizon. In this case, the singularities have no contribution to the Euler characteristic and the absorption cross section and then $\chi = 0$ and $\sigma = 0$. The results (38) and (40) give information of quantization through the singularities and the topological numbers, which can be looked upon as the topological quantization of the absorption cross section.

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