Adaptive fuzzy finite-time control with prescribed performance for waverider vehicles

Baoxu Jiang¹ and Xiangwei Bu²

Abstract
This paper proposes an adaptive fuzzy control method with prescribed performance for Waverider Vehicles (WVs), being able to guarantee finite-time convergence and small overshoot for tracking errors. Firstly, we design a new type of performance function that is independent of the initial error, and possess finite-time convergence and small overshoot. Then, we transform the inequality constraints on tracking errors into an unconstrained equation by introducing a transformed error. On this basis, we design a prescribed performance control (PPC) approach to limit the tracking errors within prescribed funnels utilizing the transformed error and fuzzy approximation, which ensures that satisfactory transient performance and steady-state accuracy can be guaranteed for tracking errors. Compared with the existing PPC, the improvement is to assure finite-time convergence of tracking errors with almost zero overshoot. Finally, compared simulations are given to verify the advantage.

Keywords
Waverider vehicles, prescribed performance control, finite-time, small overshoot, fuzzy approximation

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Introduction
Waverider Vehicles (WVs) have been the primary development direction for countries around the world because of their advantages of cost-effectiveness such as fast response time, high mobility, long range, and strong penetration ability.¹–⁶

The control system is the core of WVs, enabling them to execute and complete flight missions safely and efficiently. Many scholars and scientific research institutions have developed researches in this field. The motion model established for WVs has a lot of nonlinearity and uncertainty because the configuration of WVs is particularly complicated, and there are many unknown factors in the flight environment. Simultaneously, WVs' high velocity flight also puts forward extremely requirements on the transient performance of the control system. Therefore, the robustness and transient performance of the control system have also been the focus of attention. Sun et al.⁷ proposed a control method with prescribed performance for WVs by designing a new prescribed function, while the dependence of the control law on the initial error value was eliminated. Further, the new prescribed function proposed in Bu and Qi⁸ was extended to the control problem with unknown direction, and the Nussbaum function was used to estimate the control gain whose sign is unknown. This ensures the satisfactory transient performance and steady-state accuracy of the velocity tracking error and altitude tracking error of the WVs. In order to achieve faster error convergence, some scholars have carried out research on PPC with finite-time convergence. The main idea is to design a piecewise function that meets prescribed performance conditions to achieve finite-time convergence. However, there is still the problem of uncontrollable overshoot. For this reason, the small overshoot PPC was studied in Xiang and Liu,⁹ and the convergence of small overshoot or even zero overshoot of the WV tracking error was realized. Besides, to ensure the real-time performance of the control system, scholars mainly carry out work from two aspects: (1) reduce the structural complexity, (2) reduce the amount of online learning. Zheng et al.¹⁰ designed an inversion control law based on finite-time convergent differentiators, which

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estimated the derivatives of the virtual control inputs. Zheng and Xie\textsuperscript{11} exploited the Multi-Layer Perception method, and as a result, only one parameter was needed to be adaptively regulated, which reduces the learning amount of the neural network and ensures the good real-time performance of the controller.

Though the above methods are able to ensure the robustness and transient performance indicators for the control system, there still exist some challenging problems such as relying on the initial value of the tracking error, longer convergence time, and large overshoot. To overcome those shortcomings, this paper exploits a new PPC method for WVs to guarantee the tracking errors with satisfactory prescribed performance. The main contributions are summarized as:

(1) Different from the existing studies, the proposed new performance functions are able to guarantee the velocity tracking error and the altitude tracking error with finite-time prescribed performance with almost zero overshoot.

(2) A low-complexity control structure is obtained since the complex design procedure of back-stepping is avoided. Moreover, the computational burden is reduced by introducing an improved fuzzy approximation with less online learning parameters.

\textbf{WV model and preliminaries}

\textbf{WV model}

We consider the following integrated analytical two-dimensional model that describes the longitudinal motion of WVs.\textsuperscript{12}

\begin{equation}
\ddot{V} = \frac{T \cos(\theta - \gamma) - D}{m} - g \sin \gamma
\end{equation}

\begin{equation}
\dot{h} = v \sin \gamma
\end{equation}

\begin{equation}
\dot{\gamma} = \frac{L + T \sin(\theta - \gamma)}{m v} - \frac{g}{v} \cos \gamma
\end{equation}

\begin{equation}
\dot{\theta} = Q
\end{equation}

\begin{equation}
\dot{Q} = \frac{M + \psi_1 \eta_1 + \psi_2 \eta_2}{I_{yy}}
\end{equation}

\begin{equation}
k_1 \bar{\eta}_1 = -2 \zeta_1 \omega_1 \bar{\eta}_1 - \omega_1^2 \eta_1 + N_1
\end{equation}

\begin{equation}
-\psi_1 \frac{M}{I_{yy}} - \frac{\psi_2 \eta_2}{I_{yy}}
\end{equation}

\begin{equation}
k_2 \bar{\eta}_2 = -2 \zeta_2 \omega_2 \bar{\eta}_2 - \omega_2^2 \eta_2 + N_2
\end{equation}

\begin{equation}
-\psi_2 \frac{M}{I_{yy}} - \frac{\psi_1 \eta_1}{I_{yy}}
\end{equation}

where \(\psi_1, \psi_2, \phi_1(\xi),\) and \(\phi_2(\xi)\) are mode shape functions, \(I_{yy} = 5 \times 10^5\) slugs.ft\(^2\)/rad, \(\omega_1 = 16.021, \omega_2 = 19.582, \psi_1 = 4.2234 \times 10^3, \psi_2 = 4.2236 \times 10^3.\)

In (1)–(7), the parameter fitting forms of aerodynamic force and moment are \(T, D, L, M, N_1,\) and \(N_2.\)

\begin{equation}
\begin{aligned}
T &= C_T' \alpha^3 + C_T' \alpha^2 + C_T' \alpha + C_T' \\
D &= \dot{q} S(C_D' \alpha^2 + C_D' S^2 \delta_e + C_D' \delta_e + C_D) \\
L &= \dot{\alpha} S(C_L' \alpha + C_L' \delta_e + C_L') \\
M &= \zeta_T T + \dot{q} S[C_L' \alpha^2 + C_L' \alpha + C_L'] \\
N_1 &= N_1^2 \alpha^2 + N_1^2 \alpha + N_1^2 \\
N_2 &= N_2^2 \alpha^2 + N_2^2 \alpha + N_2^2 \\
C_T' &= \beta_1(h, \bar{h}) \Phi + \beta_2(h, \bar{h}) \\
C_D' &= \beta_1(h, \bar{h}) \Phi + \beta_2(h, \bar{h}) \\
C_L' &= \beta_2(h, \bar{h}) \Phi + \beta_3(h, \bar{h}) \\
\bar{\alpha} &= \frac{1}{2} \bar{V}^2 \\
\bar{\rho} &= \bar{\rho}_0 \exp \left(\frac{h_0 - h}{h_i}\right)
\end{aligned}
\end{equation}

The above motion model contains five rigid body states \((V, h, \gamma, \theta,\) and \(Q),\) two flexible states \((\eta_1\) and \(\eta_2)\) and two control inputs \((\Phi\) and \(\delta_e).\) \(\Phi\) and \(\delta_e\) are implicit in aerodynamic force and moment \((T, D, L, M, N_1,\) and \(N_2).\) Where the flexible states \(\eta_1\) and \(\eta_2\) are unpredictable, and there is no corresponding actuator to control the flexible states in actual engineering. Therefore, \(\eta_1\) and \(\eta_2\) are regarded as unknown disturbances in the control law design process. Moreover \(\eta_1\) and \(\eta_2\) are bounded under the condition that the rigid body states and the control inputs are bounded.\textsuperscript{13}

WVs require the controller to realize the robust tracking of the velocity \(V\) and the altitude \(h\) to the respective reference inputs \(V_{ref}\) and \(h_{ref}\) by adjusting the prescribed performance controllers \(\Phi\) and \(\delta_e\) based on fuzzy approximation, and moreover the tracking error of each subsystem has satisfactory transient performance and steady-state accuracy.

\textbf{Remark 1.} We consider that the flexible states \(\eta_1\) and \(\eta_2\) cannot be measured, and only the rigid body states \(V, h, \gamma, \theta,\) and \(Q\) are used for the control design.
Lemma 1. Define \( h(\theta) \) and \( Y_0(t) \geq 0 \) in \([0, t_f]\). If the following inequality holds
\[
Y_0(t) \leq \sum_{j=0}^{t} + \int_{0}^{t} \left( \lambda_0 h(\theta) + 1 \right) \delta(\tau) d\tau
\]
where \( \Sigma_0 \) is a suitable constant and \( \lambda_0 \neq 0 \), then \( Y_0(t) \), \( h(\theta) \) and \( \int_{0}^{t} \left( \lambda_0 h(\theta) + 1 \right) \delta(\tau) d\tau \) are bounded in \([0, t_f]\).

Lemma 2. (Implicit Function Theorem). Suppose \( H(\sigma, \alpha) : R^r \times R^r \rightarrow R^r \) is continuously differentiable at every \( Y_U \subset R^r \times R^r \) of open set \((\sigma, \alpha)\). Define \( (\sigma_0, \alpha_0) \) as a point in \( Y_U \), where \( H(\sigma_0, \alpha_0) = 0 \) and Jacobian matrix \( \frac{\partial H}{\partial \alpha}(\sigma_0, \alpha_0) \) are nonsingular. Then, there are neighborhoods \( U \subset R^r \) of \( \sigma_0 \) and \( G \subset R^r \) of \( \alpha_0 \) such that for each \( \alpha \in G \) the equation, \( H(\sigma, \alpha) = 0 \) has a unique solution \( \sigma = g_0(\alpha) \). Further, the solution can be expressed as \( \sigma = g_0(\alpha) \), where \( g_0(\cdot) \) is the continuous differentiable function at \( \sigma = \sigma_0 \).

Lemma 3. Assume that \( f(x_0, y_0) : R^r \times R \rightarrow R \) is differentiable at every point of open set \( R^r \times (a, b) \), and that it is continuous at the end points of \( y_0 = a \) and \( y_0 = b \). Then there must be a point \( y_0' \in (a, b) \) such that
\[
f(x_0, b) - f(x_0, a) = \frac{\partial f(x_0, y_0)}{\partial y_0} (b - a)
\]

Lemma 4. For \( \forall \mu_1, \mu_2 \in R \), we obtain
\[
\mu_1 \mu_2 \leq \frac{\mu_1}{p_0} \left| \mu_1 \right|^{p_0} + \frac{1}{q(\mu_2)^q} \left| \mu_2 \right|^{q}
\]
Define the prescribed performance as:

\[-f(t) < e(t) < f(t)\]  \hspace{1cm} (22)

where \(e(t)\) is a tracking error.

When \(k\) is small enough, \(f(t) \to +\infty\) and \(-f(t) \to -\infty\) can be known from the property (3) of \(f(t)\). Then for uncertain but bounded \(e(0)\), there is

\[-f(0) < e(0) < f(0)\]  \hspace{1cm} (23)

Therefore, \(e(0)\) must be within the prescribed range defined by equation (19), which can avoid the same problems as the control singularity caused by improper initial value setting of the traditional performance function.

Figure 1 shows the prescribed performance defined by equation (17). \(L_A(T)\) and \(L_r(T)\) represent the range of the steady-state value of \(e(t)\), that is \(L_A(T) < e(\infty) < L_r(T)\), which can ensure that \(e(t)\) has an ideal steady-state accuracy. By selecting appropriate \(L_A(T)\) and \(L_r(T)\), it is also ensured that \(e_{ss}\) has an ideal range, that is \(L_A(T) < e_{ss} < L_r(T)\). The maximum overshoot allowed by \(e(t)\) is limited by \(L_A(0)\) and \(L_r(0)\). The convergence speed of \(L_A(T)\) and \(L_r(T)\) is directly affected by \(r\). The smaller \(r\) is, the faster the falling speed of \(L_A(t)\) and \(L_r(t)\) will be.

**Remark 3.** The control law will be designed based on the equation (20). As long as \(\eta(t)\) is bounded, \(e(t)\) can be limited to the prescribed performance defined by equation (10). By selecting appropriate design parameters for \(L_A(t)\) and \(L_r(t)\), it can be ensured that \(e(t)\) has satisfactory transient performance and steady-state accuracy.

**Controller design and stability analysis**

**Velocity control law design**

Define velocity tracking error as

\[\dot{V} = V - V_{ref}\]  \hspace{1cm} (24)
Using (8), we get \( \dot{V} \)
\[
\dot{V} = F_V + \Phi - \dot{V}_{\text{ref}} 
\]
(25)

A transformed error \( e_1(t) \) is defined as
\[
e_1(t) = \ln \left( \frac{\partial_1(t)}{1 - \partial_1(t)} \right) 
\]
(26)

where
\[
\partial_1(t) = \frac{\dot{V} - P_{n}(t)}{P_{r}(t) - P_{n}(t)},
\]

\[
P_{n}(t) = \left[ \text{sign} \left( \dot{V}(0) \right) - \delta_{11} \right] \rho_{1}(t) - \rho_{1-} \text{sign}(\dot{V}(0)),
\]

\[
P_{r}(t) = \left[ \text{sign} \left( \dot{V}(0) \right) + \delta_{12} \right] \rho_{1}(t) - \rho_{1-} \text{sign}(\dot{V}(0)),
\]

\[
\rho_{1}(t) = (\rho_{10} - \rho_{1-}) e^{-\delta_{11}t} + \rho_{1+},
\]

with \( 0 \leq \delta_{11} \leq 1, \ 0 \leq \delta_{12} \leq 1, \ \rho_{10} > \rho_{1-} > 0, \ \delta_{1} > 0 \).

Using (26), we get \( \dot{e}_1(t) \)
\[
\dot{e}_1(t) = r_1 \left[ F_V + \Phi - \dot{V}_{\text{ref}} - \frac{\dot{V}[P_{r}(t) - P_{n}(t)]}{P_{r}(t) - P_{n}(t)} \right] 
\]
\[ + \frac{P_{n}(t)\dot{P}_{r}(t) - \dot{P}_{n}(t)P_{r}(t)}{P_{r}(t) - P_{n}(t)} \]
(27)

with
\[
r_1 = \frac{1}{1 - \partial_1(t)[P_{r}(t) - P_{n}(t)]}, \quad \dot{P}_{n}(t) = \left[ \text{sign} \left( \dot{V}(0) \right) - \delta_{11} \right] \dot{\rho}_{1}(t),
\]

\[
\dot{P}_{r}(t) = \left[ \text{sign} \left( \dot{V}(0) \right) + \delta_{12} \right] \dot{\rho}_{1}(t),
\]

\[
\dot{\rho}_{1} = - l_1 (\rho_{10} - \rho_{1-}) e^{-l_1 t} + \rho_{1+}.
\]

During the flight of WV, by considering the problems of parameter perturbation, input limitation and external disturbance, an adaptive control law is introduced to ensure the robustness of the system.

The velocity controller \( \Phi \) is selected as
\[
\Phi = - k_{V1} e_1(t) - k_{V2} \int_{0}^{t} e_1(\tau) d\tau
\]
\[- \frac{1}{2} e_1(t) \dot{\phi}_1 h^T_1(X_1) h(X_1)
\]
\[- \frac{P_{n}(t)\dot{P}_{r}(t) - \dot{P}_{n}(t)P_{r}(t)}{P_{r}(t) - P_{n}(t)} \]
\[ + \frac{[\dot{P}_{r}(t) - \dot{P}_{n}(t)]}{P_{r}(t) - P_{n}(t)} + \dot{V}_{\text{ref}} \]
(28)

where \( k_{V1} > 0, \ k_{V2} > 0 \) are design parameters; \( \dot{\phi}_1 \)

denotes the estimation of \( \phi_1 \) with
\[
\dot{\phi}_1 = \frac{\lambda_1}{2} e_{1}^2(t) h_1^T(X_1) h_1(X_1) - 2 \dot{\phi}_1 
\]
(29)

with \( \lambda_1 > 0 \).

**Theorem 2.** Consider the closed-loop system consisting of plant (8) with controller (28) and adaptive law (29). Then all the signals involved are semi-globally uniformly bounded.

**Proof.** Define
\[
\tilde{\phi}_1 = \phi_1 - \phi_1 
\]
(30)

Substituting (16) and (28) into (27), we have
\[
\dot{e}_1(t) = r_1 \left[ - k_{V1} e_1(t) - k_{V2} \int_{0}^{t} e_1(\tau) d\tau + W_1^T h_1(X_1) \right]
\[- \frac{1}{2} e_1(t) \dot{\phi}_1 h_1^T(X_1) h_1(X_1) + e_1 \]
(31)

Define
\[
W_V = \frac{1}{2r_1} e_{1}^2(t) + \frac{1}{2} k_{V2} \left( \int_{0}^{t} e_1(\tau) d\tau \right)^2 + \frac{\tilde{\phi}_1}{2\lambda_1} 
\]
(32)

Utilizing (29)–(31), we get \( \dot{W}_V \)
\[
\dot{W}_V = \frac{1}{r_1} e_{1}^2(t) \dot{\phi}_1(t) + k_{V2} \int_{0}^{t} e_1(\tau) d\tau + \frac{\tilde{\phi}_1}{\lambda_1} \dot{\phi}_1
\]
\[- k_{V1} e_1(t) \dot{\phi}_1 h_1^T(X_1) h_1(X_1) - \frac{2 \dot{\phi}_1}{\lambda_1} \dot{\phi}_1
\]
\[- \frac{1}{2} e_1^2(t) \dot{\phi}_1 h_1^T(X_1) h_1(X_1) + e_1 \dot{e}_1(t) \]
(33)

Since \( 2 \dot{\phi}_1 \dot{\phi}_1 \geq \dot{\phi}_1^2 - \dot{\phi}_1^2 \), (33) becomes
\[
\dot{W}_V \leq - k_{V1} e_1^2(t) + e_1(t) W_1^T h_1(X_1)
\[- \frac{1}{2} e_1(t) \dot{\phi}_1 h_1^T(X_1) h_1(X_1) + e_1 \dot{e}_1(t) \]
(34)

Note that
where

\[ h(t) = \frac{\dot{h} - L_2(t)}{L_2(t) - L_2(t)} \]

Thus, (34) becomes

\[ \dot{V} \leq \left(k_f - \frac{1}{2} \right) \leq \frac{1}{2} e_1(t) - \frac{1}{2} L_2(t) \leq \frac{1}{2} \]

Let \( k_f > \frac{1}{2} \) and choose the following compact sets

\[ \Omega_{e_1(t)} = \left\{ e_1(t) \left| e_1(t) \leq \frac{1}{2} \frac{1}{2} \right. \right\} \]

\[ \Omega_{\dot{h}} = \left\{ \dot{h} \left| \dot{h} \leq \frac{1}{2} \frac{1}{2} \right. \right\} \]

It is obvious that \( \dot{V} \) will be negative if \( e_1(t) \notin \Omega_{e_1(t)} \) or \( \dot{h} \notin \Omega_{\dot{h}} \). Therefore, \( e_1(t) \) and \( \dot{h} \) are semi-globally uniformly ultimately bounded. This is the end of the proof.

**Remark 4.** Theorem 2 proves the boundedness of \( e_1(t) \).

Based on Theorem 1, it is further concluded that \( P_1(t) < \dot{V} < P_1(t) \).

**Altitude control law design and stability analysis**

The control goal for the altitude subsystem (equations (2)–(5)) is to design a prescribed performance control law \( \delta_x \) based on neural approximation such that the altitude \( h \) tracks its reference input \( h_{\text{ref}} \). Moreover, the tracking error is limited to a prescribed area to ensure satisfactory transient performance and steady-state accuracy.

Define the altitude error as

\[ \tilde{h} = h - h_{\text{ref}} \]

Select the track angle reference input as

\[ \gamma_d = \arcsin \left( -k_x \hat{h} + \frac{h_{\text{ref}}}{V} \right) \]

with \( k_x \in \mathbb{R}^+ \).

Define the transformed error \( \eta(t) \) as

\[ \eta(t) = \ln \left( \frac{\dot{h} - \dot{h}}{1 - \dot{h}} \right) \]

where

\[ \dot{h} = \frac{\dot{h} - L_2(t)}{L_2(t) - L_2(t)} \]

\[ L_2(t) = [\text{sign}(\dot{h}(0)) - \delta_2] f_2(t) - f_2, \text{sign}(\dot{h}(0)) \]

\[ f_2(t) = \left\{ \begin{array}{ll} \frac{[r_0 - T_3]}{f_{2T_3}} \left( f_{20} - f_{2T_3} \right) + f_{2T_3}, & 0 \leq t \leq T_2 \\ 0, & t \geq T_2 \end{array} \right. \]

with \( 0 \leq \delta_2 \leq 1, 0 \leq \delta_2 \leq 1, 0 \leq \delta_2 \leq 1, f_{20} > f_{2T_3} > 0, r_0 \in (-1, 1) \).

Take the track angle reference input as

\[ \gamma_d = \arcsin \left( -k_x \hat{h} + \frac{h_{\text{ref}}}{V} \right) - \frac{L_2(t) \dot{L}_2(t)}{V(L_2(t) - L_2(t))} \]

\[ \dot{L}_2(t) = [\text{sign}(\dot{h}(0)) - \delta_2] \dot{f}_2(t), \]

\[ \dot{f}_2(t) = \left\{ \begin{array}{ll} \frac{[r_0 - T_3]}{f_{2T_3}} \left( f_{20} - f_{2T_3} \right) + f_{2T_3}, & 0 \leq t \leq T_2 \\ 0, & t \geq T_2 \end{array} \right. \]

If \( \gamma \rightarrow \gamma_d \), then the response of \( \eta(t) \) is \( \eta_0(t) + k_x \eta_0(t) = 0 \). Thus \( \eta(t) \) must be bounded. Furthermore, the control task becomes \( \gamma \rightarrow \gamma_d \).

Next, we will design a low-computation fuzzy control law, so that \( \gamma \rightarrow \gamma_d \).

Define \( x_1 = \gamma, x_2 = \theta, x_3 = Q \). Then the rest of the WV altitude subsystem (equations (3)–(5)) can be expressed as the following non-affine form

\[ \begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= f_3(x, \delta_x)
\end{aligned} \]

where \( x = [x_1, x_2, x_3]^T \), and \( f_1(x_1, x_2) \) and \( f_3(x, \delta_x) \) are continuously differentiable functions.

We give the following reasonable assumption.

**Assumption 1.** For any \((x, \delta_x) \in \Gamma \times \mathbb{R} \), the following inequalities hold\(^{21}\)
where $\Omega_c$ is a controllable domain.

**Remark 5.** According to the literature\textsuperscript{21} and the value range of the rigid body state of the WV flight envelope (see Table 1), it can be seen that Assumption 1 holds.

In order to avoid the complicated design process of backstepping control, the model (40) is equivalently transformed as follows.

**Step 1:** Define $z_1 = x_1 = \gamma$, $z_2 = \dot{z}_1 = f_1(x_1, x_2)$. Perform the following equivalent transformation to model (40).

$$
\dot{z}_2 = \frac{\partial f_1(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial f_1(x_1, x_2)}{\partial x_2} \dot{x}_2 = \frac{\partial f_1(x_1, x_2)}{\partial x_1} f_1(x_1, x_2) + \frac{\partial f_1(x_1, x_2)}{\partial x_2} x_3 \tag{42}
$$

**Step 2:** Define $z_3 = \dot{z}_2 = f_{\delta_1}(x)$. According to equation (40), the first derivative of $z_3$ with respect to time is

$$
\dot{z}_3 = \frac{\partial f_{\delta_1}(x)}{\partial x_1} \dot{x}_1 + \frac{\partial f_{\delta_1}(x)}{\partial x_2} \dot{x}_2 + \frac{\partial f_{\delta_1}(x)}{\partial x_3} \dot{x}_3 = \frac{\partial f_{\delta_1}(x)}{\partial x_1} f_1(x_1, x_2) + \frac{\partial f_{\delta_1}(x)}{\partial x_2} x_3 + \frac{\partial f_{\delta_1}(x)}{\partial x_3} f_{\delta_1}(x, \delta_e) \tag{43}
$$

After the above model transformation, equation (40) becomes the following non-affine pure feedback model

$$
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= f_{\delta_2}(x, \delta_e)
\end{align*} \tag{44}
$$

where $f_{\delta_2}(x, \delta_e)$ is a continuously differentiable unknown function.

**Remark 6.** Assumption 1 imposes the global controllability condition on (44), which is also the condition of satisfying Lemma 2. Different from other work\textsuperscript{22}, the control gain $\frac{\partial f_{\delta_2}(x, \delta_e)}{\partial \delta_e}$ must be positive. Through the method based on Nussbaum-type function, the tracking controller does not need such strict assumptions.

**Remark 7.** From equations (41) to (43), we can get

$$
\frac{\partial f_{\delta_2}(x, \delta_e)}{\partial \delta_e} = \frac{\partial f_1(x_1, x_2)}{\partial x_1} \frac{\partial f_2(x_1, x_2)}{\partial x_2} + \frac{\partial f_1(x_1, x_2)}{\partial x_2} x_3 + \frac{\partial f_1(x_1, x_2)}{\partial x_3} f_{\delta_1}(x, \delta_e) > 0 \tag{45}
$$

**Remark 8.** Compared with equation (40), equation (44) is not only simple in form, but also contains only an unknown function. Based on equation (44) to design the control law, the cumbersome backstepping design process is no longer needed.

Define track angle tracking error $e_0$ and error function $E$

$$
\begin{align*}
e_0 &= \gamma - \gamma_d = z_1 - \gamma_d \\
E &= \left( \frac{d}{dt} + \mu \right)^3 \int_{0}^{t} e_0 d\tau \tag{46}
\end{align*}
$$

where $\mu \in \mathbb{R}^+$. Since $(s + \mu)^3$ is a Hurwitz polynomial, when $E$ is bounded, $e_0$ must be bounded.

The first third derivative of $e_0$ with respect to time is

$$
\begin{align*}
\dot{e}_0 &= \frac{\partial}{\partial x_1} f_1(x_1, x_2) + \frac{\partial}{\partial x_2} f_1(x_1, x_2) x_3 + \frac{\partial}{\partial x_3} f_{\delta_1}(x, \delta_e) \\
\dot{e}_0 &= \frac{\partial}{\partial x_1} f_1(x_1, x_2) + \frac{\partial}{\partial x_2} f_1(x_1, x_2) x_3 + \frac{\partial}{\partial x_3} f_{\delta_1}(x, \delta_e) \tag{47}
\end{align*}
$$

where $\mu \in \mathbb{R}^+$ is the parameter to be designed, $F_{\delta_1}(x, \delta_e) = f_{\delta_1}(x, \delta_e)$ is a continuously differentiable unknown function.

The first derivative of $E$ with respect to time is

$$
\dot{E} = e_0^{(3)} + 3\eta e_0 + 3\eta^2 \dot{e}_0 + \eta^3 \ddot{e}_0 \tag{48}
$$

where $e_0^{(3)} = \frac{\partial f_{\delta_1}(x, \delta_e)}{\partial \delta_e}$ is a continuously differentiable unknown function.

According to Lemma 2 and Assumption 1, we know that $\delta_e$ satisfies $F_{\delta_1}(x, \delta_e) = \gamma_\delta^{(3)} + 3\eta \dot{e}_0 + 3\eta^2 \ddot{e}_0 + \eta^3 e_0$. Then (48) becomes

$$
\dot{E} = F_{\delta_1}(x, \delta_e) - F_{\delta_1}(x, \delta_e) \tag{49}
$$
According to Lemma 3, we get
\[ F_h(x, \delta_e) - F_h(x, \delta^*_e) = G_h(x, \delta^*_e)(\delta_e - \delta^*_e) \]  
(50)

with \( G_h(\cdot) = G_h(x, \delta^*_e) = \frac{\partial F_h}{\partial \delta_e} \neq 0, \ \ell = \theta_h \delta_e + (1 - \theta_h) \delta^*_e, \theta_h \in [0, 1]. \)

According to (49) and (50), we have
\[ \hat{E} = G_h(x, \delta)_e(\delta_e - \delta^*_e) = G_h(x, \delta^*_e)\delta_e - G_h(x, \delta^*_e)\delta^*_e \]
(51)

For the unknown term \( \delta^*_e, \) the fuzzy system is applied to approximate it.
\[ \delta^*_e = \phi_2^T P_2(x) + \epsilon_{2M} \]
(52)

where \( X_2 = [y, \theta, \mathbf{Q}]^T \in \mathbb{R}^3 \) is the input vector, and \( \epsilon_{2M} > 0 \) is the approximate error.

Define \( \phi_2 = ||\phi_2||_2 \) and select the following control law
\[
\begin{align*}
\delta_e &= k_h \mathbb{E}l(\theta) + \frac{\partial F_h}{\partial \delta_e} P_1(x) P_2(x) \\
\hat{\theta} &= k_h E + E \phi_2 P_1(x) P_2(x) \\
\end{align*}
\]
(53)

where \( k_h \) is the design parameter, \( \phi_2 \) is the estimated value of \( \phi_2. \)

The adaptive law of \( \phi_2 \) is selected as
\[ \dot{\phi}_2 = \frac{k_0}{k_1} E^2 + \frac{E}{2} \phi_2 P_1(x) P_2(x) \]
(54)

with \( k_1 > 0. \)

**Theorem 3.** Consider a closed-loop system composed of non-affine formula (37), controller (46) and adaptive law (47) under assumption 1. Then, all parameters involved are bounded.

**Proof.** Consider the following Lyapunov function
\[ W = \frac{E^2}{2|G_h(x, \delta^*_e)|} + \frac{\phi^2_2}{2k_2} \]
(55)

with \( \phi_2 = \phi_2 - \phi_2. \)

Take the time derivative of equation (55) to get
\[ \dot{W} = \frac{E \hat{E}}{|G_h(x, \delta^*_e)|} - \frac{\dot{G}_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|^2} E^2 + \frac{\phi_2 \dot{\phi}_2}{k_2} \]
(56)

Substitute equations (51), (52) and (54) into equation (56) to get
\[ \dot{W} = \frac{E G_h(x, \delta^*_e)(\delta_e - \delta^*_e)}{|G_h(x, \delta^*_e)|} - \frac{\dot{G}_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|^2} E^2 + \frac{\phi_2 \dot{\phi}_2}{k_2} \]

with \( \chi = \frac{G_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|} \in \{-1, 1\}. \)

Considering equation (53) and \( \phi_2 = \phi_2 - \phi_2, \) then equation (57) becomes
\[ \dot{W} = \chi \mathbb{E}l(\theta) \left[ k_h E + \frac{1}{2} E \phi_2 P_1(x) P_2(x) \right] E \\
- \chi E \phi_2^T P_2(x) + \chi E \epsilon_{2M} - \frac{\dot{G}_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|^2} E^2 \\
- \frac{\dot{\phi}_2}{k_2} = \frac{1}{2} (\phi_2 - \phi_2) E^2 + \frac{2 \phi_2 \phi_2}{k_2} \\
- \chi E \phi_2^T P_1(x) P_2(x) \]
(58)

Add and subtract \( k_h E^2 \) on the right side of equation (58) to get
\[ \dot{W} = -k_h E^2 + \frac{1}{2} \phi_2 E^2 P_1(x) P_2(x) \]

with
\[ \chi = \frac{G_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|} \in \{-1, 1\}. \]

Considering equation (53) and \( \phi_2 = \phi_2 - \phi_2, \) then equation (57) becomes
\[ \dot{W} = \chi \mathbb{E}l(\theta) \left[ k_h E + \frac{1}{2} E \phi_2 P_1(x) P_2(x) \right] E \\
- \chi E \phi_2^T P_2(x) + \chi E \epsilon_{2M} - \frac{\dot{G}_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|^2} E^2 \\
- \frac{\dot{\phi}_2}{k_2} = \frac{1}{2} (\phi_2 - \phi_2) E^2 + \frac{2 \phi_2 \phi_2}{k_2} \\
- \chi E \phi_2^T P_1(x) P_2(x) \]
(58)

Add and subtract \( k_h E^2 \) on the right side of equation (58) to get
\[ \dot{W} = -k_h E^2 + \frac{1}{2} \phi_2 E^2 P_1(x) P_2(x) \]

with
\[ \chi = \frac{G_h(x, \delta^*_e)}{|G_h(x, \delta^*_e)|} \in \{-1, 1\}. \]
From the fact that $\tilde{\phi}_2^2 + 2\tilde{\phi}_2 (\tilde{\phi}_2 - \phi_2) + \phi_2^2 = \tilde{\phi}_2^2 + 2 \tilde{\phi}_2 \phi_2 + \phi_2^2 = (\tilde{\phi}_2 + \phi_2)^2 \geq 0$, we have $2\tilde{\phi}_2 \phi_2 \geq \phi_2^2 - \tilde{\phi}_2^2$. By further attention

$$-\chi E\tilde{\phi}_2^2 P_2(X_2) = E\tilde{\phi}_2^2 P_2(X_2)$$

$$= E\tilde{\phi}_2^2 \|P_2(X_2)\|^2 + \frac{1}{2}$$

$$= E\tilde{\phi}_2^2 \|P_2(X_2)\|^2 + \frac{1}{2}$$

$$= \frac{1}{2} \tilde{\phi}_2^2 P_2^2(X_2) + \frac{1}{2}$$

$$= \frac{1}{2} \tilde{\phi}_2^2 P_2^2(X_2) + \frac{1}{2}$$

According to Lemma 4, we get

$$\chi E\tilde{\phi}_2^2 \leq \frac{E^2}{2} + \tilde{\phi}_2^2$$

According to Lemma 1, we know that $W(t)$ is bounded. At the same time, $E$ and $\tilde{\phi}_2$ are bounded. Since the polynomial $(s + r)^3$ is a Hurwitz polynomial, the tracking error $e$ is also bounded. The certificate is complete.

**Remark 9.** Theorem 3 shows that the proposed altitude controller allows $\gamma \rightarrow \gamma_t$ to ensure the boundedness of $e_0(t)$. According to Theorem 1, we further obtain that the expected performance of $h$ can be guaranteed.

**Remark 10.** Note that there is only one fuzzy system and one learning parameter $\phi_2$. Therefore, the computational load of the proposed controller is lower than the existing study.  

### Simulation analysis

In this section, a numerical simulation is given to test the effectiveness of the control. The membership function of the fuzzy system is selected as

$$\sigma_{\mu_1}(\gamma) = \exp \left[-\left(\frac{\gamma - 0.0044}{0.009}\right)^2\right],$$

$$\sigma_{\mu_2}(\gamma) = \exp \left[-\left(\frac{\gamma - 0.0044}{0.009}\right)^2\right],$$

$$\sigma_{\mu}(\gamma) = \exp \left[-\left(\frac{\gamma - 0.0044}{0.009}\right)^2\right],$$

$$\sigma_{\mu_2}(\theta) = \exp \left[-\left(\frac{\theta - 0.087}{0.022}\right)^2\right],$$

$$\sigma_{\mu_2}(\theta) = \exp \left[-\left(\frac{\theta - 0.044}{0.022}\right)^2\right],$$

$$\sigma_{\mu_2}(\theta) = \exp \left[-\left(\frac{\theta - 0.044}{0.022}\right)^2\right],$$

$$\sigma_{\mu_2}(\theta) = \exp \left[-\left(\frac{\theta - 0.044}{0.022}\right)^2\right],$$

where $\mu = \min \left\{ \left[2G_2(x, \delta) \left[k_1 + \frac{\delta_1 G_2(x, \delta)}{2G_2(x, \delta)} - \frac{1}{4} \right], 2 \right\}$. Multiply $e^\mu$ by equation (61) to get

$$\frac{d}{dt}(W e^\mu) \leq -tW + [1 + \chi h(\theta)] \hat{\theta} e^\mu$$

$$+ \left(\frac{1}{2} + \tilde{\phi}_2^2 + \frac{\phi_2^2}{\kappa_2}\right) e^\mu \quad (62)$$

Integrating equation (62) on $[0, t]$, we get

$$W(t) \leq e^{-t} \left[1 + \chi h(\theta)] \hat{\theta} e^\mu dt + \left(\frac{1}{2} + \tilde{\phi}_2^2 + \frac{\phi_2^2}{\kappa_2}\right) \right]$$

$$+ \left[ W(0) - \left(\frac{1}{2} + \tilde{\phi}_2^2 + \frac{\phi_2^2}{\kappa_2}\right) \right] e^{-t} \quad (63)$$
The design parameters are taken as: $k_V = 0.9$, $k_{f1} = 0.3$, $k_{f2} = 0.8$, $k_\gamma = 2$, $k_h = 0.9$, $k_h1 = 50$, $\mu = 7$, $r = 7$, $\kappa_1 = 0.01$. The velocity and altitude reference inputs are both given by the second-order reference model shown in Figure 2. In order to test robustness, we assume that all aerodynamic coefficients are uncertain. Define

$$C = \begin{cases} C_0, & 0s \leq t \leq 40 s \\ C_0[1 + 0.4 \sin(0.1\pi t)], & else \end{cases}$$

where $C$ is the value of the uncertainty coefficient, and $C_0$ is the nominal value of $C$. In this way, the parameter uncertainty below 40% of the nominal value is considered. In order to prove the superiority, the proposed prescribed performance controller (PPC) is compared with a neural inversion controller (NBC) addressed.23

**Scenario 1:** Take $\bar{v}(0) = 0.76m/s$, $\bar{h}(0) = 0.15m$.

**Scenario 2:** Take $\bar{v}(0) = -0.76m/s$, $\bar{h}(0) = -0.15m$.

The simulation results of scenario 1 are presented in Figures 3 to 17. It can be seen from Figures 3 to 6 that when the model parameters are perturbed, the PPC method in this paper can ensure that the velocity tracking error and altitude tracking error have better transient performance and better steady-state accuracy, and also achieve the velocity tracking error and the altitude tracking error with small overshoot finite-time convergence. Although the velocity tracking error fluctuates greatly in the first 5s, it does not exceed the designed envelope, so the phenomenon of control failure will not occur. In addition, the prescribed performance function designed in this paper can adjust the convergence time. If the convergence time is increased, the fluctuation will not occur. The situation will improve. Figures 7 to 13 show that the attitude angle, flexible states and control inputs of the two control methods are relatively smooth, and there is no high-frequency chattering phenomenon. Figures 14 and 15 show the learning effects of $\phi_1$ and $\phi_2$. Figures 16 and 17 show that $\eta_1(t)$ and $\eta_2(t)$ are bounded. Besides, the simulation results of scenario 2, depicted in Figures 8 to 32 also proves the advantage of the proposed PPC in comparison with
Figure 4. Velocity tracking error of Scenario 1.

Figure 5. Altitude tracking of Scenario 1.

Figure 6. Altitude tracking error of Scenario 1.

Figure 7. Flight-path angle of Scenario 1.

Figure 8. Pitch angle of Scenario 1.

Figure 9. Pitch rate of Scenario 1.
Figure 10. The flexible state $\eta_1$ of Scenario 1.

Figure 11. The flexible state $\eta_2$ of Scenario 1.

Figure 12. Fuel equivalence ratio of Scenario 1.

Figure 13. Elevator angular deflection of Scenario 1.

Figure 14. The change curve of $\varphi_1$ of Scenario 1.

Figure 15. The change curve of $\varphi_2$ of Scenario 1.
Figure 16. The change curve of $e_1(t)$ of Scenario 1.

Figure 17. The change curve of $e_2(t)$ of Scenario 1.

Figure 18. Velocity tracking of Scenario 2.

Figure 19. Velocity tracking error of Scenario 2.

Figure 20. Altitude tracking of Scenario 2.

Figure 21. Altitude tracking error of Scenario 2.
Figure 22. Flight-path angle of Scenario 2.

Figure 23. Pitch angle of Scenario 2.

Figure 24. Pitch rate of Scenario 2.

Figure 25. The flexible state $\eta_1$ of Scenario 2.

Figure 26. The flexible state $\eta_2$ of Scenario 2.

Figure 27. Fuel equivalence ratio of Scenario 2.
From the simulation results, the finite-time prescribed performance control method proposed in this paper has obvious improvements in terms of overshoot and convergence time.\textsuperscript{21,22}

**Conclusions**

This paper studies the new non-affine PPC method for WVVs. By designing a new type of performance function, the control law gets rid of the dependence on the accurate initial value of the tracking error, and it can ensure that all tracking errors have good transient performance and reach a steady state within a limited time. Fuzzy systems are used to approximate the unknown parameters in the altitude control subsystem. The stability of closed-loop control system is proved via Lyapunov method. Finally, the given simulation

![Figure 28. Elevator angular deflection of Scenario 2.](image1)

![Figure 30. The change curve of $\varphi_2$ of Scenario 2.](image2)

![Figure 29. The change curve of $\varphi_1$ of Scenario 2.](image3)

![Figure 31. The change curve of $e_1(t)$ of Scenario 2.](image4)

![Figure 32. The change curve of $e_2(t)$ of Scenario 2.](image5)
results show that the proposed method can ensure the velocity and altitude tracking errors with small overshoot and finite-time convergence.

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Data availability
The experimental data used to support the findings of this study are available from the corresponding author upon request.

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Appendix

| Notation | Description |
| --- | --- |
| $m$ | vehicle mass |
| $\rho$ | density of air |
| $h$ | altitude |
| $S$ | reference area |
| $V$ | velocity |
| $\gamma$ | flight-path angle |
| $\theta$ | pitch angle |
| $\alpha$ | angle of attack |
$Q$  pitch rate  \hfill $C_i^q$  \text{i}th order coefficient of $\delta_q$ in $D$
$T$  thrust  \hfill $C_i^D$  constant coefficient in $D$
$D$  drag  \hfill $C_i^D$  \text{i}th order coefficient of $\alpha$ in $D$
$L$  lift  \hfill $C_{\delta^e}$  coefficient of $\delta_e$ contribution in $L$
$M$  pitching moment  \hfill $C_{\alpha^e}$  constant coefficient in $L$
$I_{yy}$  moment of inertia  \hfill $C_{\beta^e}$  \text{i}th order coefficient of $\alpha$ in $M$
$e_a$  aerodynamic chord  \hfill $C_{\beta^e}$  constant coefficient in $M$
$z_T$  thrust moment arm  \hfill $C_{\beta^e}$  \text{i}th order coefficient of $\alpha$ in $T$
$\phi$  fuel equivalence ratio  \hfill $C_{\beta^e}$  constant coefficient in $T$
$\delta_e$  elevator angular deflection  \hfill $h_0$  nominal altitude for air density approximation
$N_i$  \text{i}th generalized force  \hfill $\rho_0$  \text{air density at the altitude } $h_0$
$N_{i\alpha}$  \text{j}th order contribution of $\alpha$ to \hfill $\psi_i$  \text{constrained beam coupling constant for } $\eta_i$
$N_i^0$  constant term in $N_i$  \hfill $c_e$  coefficient of $\delta_e$ in $M$
$N_{i\delta_e}$  contribution of $\delta_e$ to $N$  \hfill $1/h_s$  \text{air density decay rate}$
$\beta_i(h, \dot{\overline{q}})$ \text{\text{i}th trust fit parameter}  \hfill $\dot{\overline{q}}$  dynamic pressure
$\eta_i$  \text{i}th generalized elastic coordinate  
$\zeta_i$  damping ratio for elastic mode  
$\omega_i$  natural frequency for elastic mode  
$C_{\alpha}^D$  \text{i}th order coefficient of $\alpha$ in $D$