Two-temperature accretion around rotating black holes: a description of the general advective flow paradigm in the presence of various cooling processes to explain low to high luminous sources

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ABSTRACT

We investigate viscous two-temperature accretion disc flows around rotating black holes. We describe the global solution of accretion flows with a sub-Keplerian angular momentum profile, by solving the underlying conservation equations including explicit cooling processes self-consistently. Bremsstrahlung, synchrotron and inverse Comptonization of soft photons are considered as possible cooling mechanisms. We focus on the set of solutions for sub-Eddington, Eddington and super-Eddington mass accretion rates around Schwarzschild and Kerr black holes with a Kerr parameter of 0.998. It is found that the flow, during its infall from the Keplerian to sub-Keplerian transition region to the black hole event horizon, passes through various phases of advection: the general advective paradigm to the radiatively inefficient phase, and vice versa. Hence, the flow governs a much lower electron temperature $\sim 10^8$–$10^9$ K, in the range of accretion rate in Eddington units $0.01 \lesssim \dot{M} \lesssim 100$, compared to the hot protons of temperature $\sim 10^{10.2}$–$10^{11.8}$ K. Therefore, the solution may potentially explain the hard X-rays and $\gamma$-rays emitted from active galactic nuclei (AGNs) and X-ray binaries. We then compare the solutions for two different regimes of viscosity. We conclude that a weakly viscous flow is expected to be cooling dominated, particularly at the inner region of the disc, compared to its highly viscous counterpart, which is radiatively inefficient. With all the solutions in hand, we finally reproduce the observed luminosities of the underfed AGNs and quasars (e.g. Sgr A*) to ultraluminous X-ray sources (e.g. SS433), at different combinations of input parameters, such as the mass accretion rate and the ratio of specific heats. The set of solutions also predicts appropriately the luminosity observed in highly luminous AGNs and ultraluminous quasars (e.g. PKS 0743-67).

Key words: accretion, accretion discs – black hole physics – hydrodynamics – radiative transfer.

1 INTRODUCTION

Pringle & Rees (1972), Shakura & Sunyaev (1973) and Novikov & Thorne (1973) found that it was inappropriate to use a cool Keplerian accretion disc to explain observed hard X-rays, for example, from Cyg X-1 (Lightman & Shapiro 1975). It was argued that the secular instability of the cool disc swells the optically thick, radiation-dominated region to a hot, optically thin, gas-dominated region, resulting in a hard component of spectrum $\sim 100$ keV (Thorne & Price 1975; Shapiro, Lightman & Eardley 1976). This region is strictly of two temperatures with electron and ion temperatures, respectively, of $\sim 10^9$ and $\sim 5 \times 10^{11}$ K. This confirms that the cool, one-temperature, pure Keplerian accretion solution is not unique. Indeed, Eardley & Lightman (1975) found that a Keplerian disc is unstable as a result of thermal and viscous effects when the viscosity parameter $\alpha$ (Shakura & Sunyaev 1973) is constant. Later, using numerical simulations, Eggum, Coroniti & Katz (1985) showed that a Keplerian disc with a constant $\alpha$ collapses.

In the 1980s, therefore, the idea of a two-component accretion disc was born. For example, Paczyński & Wiita (1980) described a geometrically thick regime of accretion disc in the optically thick limit, while Rees et al. (1982) introduced an accretion torus in the optically thin limit. Moreover, Muchotrzeb & Paczyński (1982) introduced the idea of sub-Keplerian, transonic accretion, which was later improved...
by other authors (Chakrabarti 1989, 1996; Mukhopadhyay 2003). Other models were proposed by, for example, Gierlisński et al. (1999), Coppi (1999) and Zdziarski et al. (2001), including a secondary component in the accretion disc. However, Narayan & Yi (1995) introduced a two-temperature disc model in the regime of inefficient cooling, resulting in a vertical thickening of the hot disc gas. Here, the pressure forces are expected to become important for modifying the disc dynamics, which is likely to be sub-Keplerian. Other models with similar properties were proposed by, for example, Begelman (1978), Liang & Thompson (1980), Rees et al. (1982) and Eggum, Coroniti & Katz (1988). Abramowicz et al. (1988) proposed a height-integrated disc model, the ‘slim disc’, which has a high optical depth of accreting gas at a super-Eddington accretion rate such that the diffusion time is longer than the viscous time. The model was further applied to study the thermal and viscous instabilities in optically thick accretion discs (Wallinder 1991; Chen & Taam 1993).

Shapiro et al. (1976) initiated a two-temperature Keplerian accretion disc at a low mass accretion rate, which is optically thin and significantly hotter than the single-temperature Keplerian disc of Shakura & Sunyaev (1973). The optically thin hot gas cools down through the bremsstrahlung and inverse-Compton processes, and could explain various states of Cyg X-1 (Melia & Misra 1993). Similarly, the ‘ion torus’ model of Rees et al. (1982) was applied to explain active galactic nuclei (AGNs) at a low mass accretion rate. However, the two-temperature model solutions by Shapiro et al. (1976) appear thermally unstable. Narayan & Popham (1993), and subsequently Narayan & Yi (1995), showed that the introduction of advection may stabilize the system. However, the solutions of Narayan & Yi (1995), while of two temperatures, could explain only a particular class of hot systems with inefficient cooling mechanisms. They also described the hot flow based on the assumption of ‘self-similarity’, which is just a ‘plausible choice’. They kept the electron heating decoupled from the disc hydrodynamical computations, which merely is an assumption. Later, Nakamura et al. (1997), Mammoto, Mineshige & Kasunose (1997) and Medvedev & Narayan (2001) attempted to generalize the solutions, by relaxing the efficiency of cooling into the systems, but concentrating only on specific classes of solutions. However, Chakrabarti & Titarchuk (1995), and later Mandal & Chakrabarti (2005), modelled two-temperature accretion flows around Schwarzschild black holes in the general ‘advective paradigm’, emphasizing the possible formation of shock and its consequences therein. However, they also did not include the effect of electron heating self-consistently into the hydrodynamical equation, and thus the hydrodynamical quantities are not coupled to the rate of electron heating (see also Rajesh & Mukhopadhyay 2009).

In this paper, we model a self-consistent accretion flow in the regime of a two-temperature transonic sub-Keplerian disc (see also Sinha, Rajesh & Mukhopadhyay 2009, which is a brief version of the present work, but around Schwarzschild black holes). We consider all the hydrodynamical equations of the disc along with the thermal components and we solve the coupled set of equations self-consistently. We neither restrict to the advection dominated regime nor the self-similar solutions. We allow the disc to cool self-consistently according to the thermo-hydrodynamical evolution and we compute the corresponding cooling efficiency factor as a function of the radial coordinate. We investigate when the disc switches from a radiatively inefficient nature to a general advective paradigm, and vice versa.

In order to implement our model to explain observed sources, we focus on underluminous AGNs and quasars (e.g. Sgr A‘), ultraluminous quasars and highly luminous AGNs (e.g. PKS 0743-67) and ultraluminous X-ray (ULX) sources (e.g. SS433); the latter are likely to be ‘radiation-trapped’ accretion discs. While the first two cases correspond, respectively, to sub-Eddington and super-Eddington accretion flows around supermassive black holes, the last case corresponds to super-Eddington accretors around stellar mass black holes.

In Section 2, we discuss the model equations describing the system and the procedure used to solve these. Subsequently, in Sections 3 and 4, we discuss two-temperature accretion disc flows around stellar mass and supermassive black holes, respectively, for sub-Eddington, Eddington and super-Eddington accretion rates. In Section 5 we compare the disc flow of low α (Shakura & Sunyaev 1973) with that of high α and then between the flows around corotating and counter-rotating black holes. In the summary in Section 6, we discuss the implications of the results.

2 MODEL EQUATIONS DESCRIBING THE SYSTEM AND SOLUTION PROCEDURE

For the present purpose, we set five coupled differential equations describing the law of conservation in the sub-Keplerian optically thin accretion regime. Necessarily, the set of equations describes the inner part of the accretion disc where the gravitational potential energy dominates over the centrifugal energy of the flow.

Throughout, we express all the variables in dimensionless units, unless stated otherwise. The radial velocity \( \dot{\vartheta} \) and speed of sound \( c_s \) are expressed in units of light speed \( c \). The specific angular momentum \( \lambda \) is expressed in units of \( GM/c \), where \( G \) is the Newton gravitational constant and \( M \) is the mass of the compact object, for the present purpose a black hole, expressed in units of solar mass \( M_\odot \), the radial coordinate \( x \) is expressed in units of \( GM/c^2 \), the density \( \rho \) and the total pressure \( P \), accordingly. The disc fluid under consideration consists of ions and electrons, and is thus a two-fluid/temperature system, apart from radiation. Furthermore, at high temperatures, the disc flow with ions/electrons behaves (almost) as a non-interacting gas.

2.1 Conservation laws

(i) Mass transfer.

\[
\frac{1}{x} \frac{\partial}{\partial x} (x \rho \dot{\vartheta}) = 0, \tag{1}
\]

whose integrated form gives the mass accretion rate

\[
\dot{M} = -4\pi x \Sigma \dot{\vartheta}, \tag{2}
\]
where the surface density
\[ \Sigma = I_n \rho h(x), \]  
(3)

\[ I_n = \frac{(2^n n!)^2}{(2n + 1)!}, \]  
(4)

(Matsumoto et al. 1984). Here, \( n \) is the polytropic index, which is equal to \( 1/(\gamma - 1) \) when \( \gamma \) is the ratio of specific heats, and half-thickness, based on the vertical equilibrium assumption, of the disc
\[ h(x) = c_s x^{1/2} F^{-1/2}. \]  
(5)

(ii) Radial momentum balance.
\[ \frac{d}{dx} \left( \frac{1}{\rho} \frac{dP}{dx} \right) - \frac{\lambda^2}{x^3} + F = 0, \]  
(6)

when following the pseudo-Newtonian approach of Mukhopadhyay (2002)
\[ F = \frac{(x^2 - 2a \sqrt{x} + a^2)^2}{x^3 \sqrt{x(x - 2) + a^2}}, \]  
(7)

where \( a \) is the specific angular momentum (Kerr parameter) of the black hole. We also define a parameter
\[ \beta = \frac{\text{gas pressure } P_{\text{gas}}}{\text{total pressure } P} = \frac{6\gamma - 8}{3(\gamma - 1)}. \]  
(8)

(e.g. Ghosh & Mukhopadhyay 2009), where \( \gamma \) may range from 4/3 to 5/3, \( P_{\text{gas}} = P_i \) (ion pressure) + \( P_e \) (electron pressure), such that
\[ P = \frac{\rho}{\beta c^2} \left( \frac{k T_i}{\mu_i m_i} + \frac{k T_e}{\mu_e m_e} \right) = \rho c_s^2. \]  
(9)

Here, \( T_i \) and \( T_e \) are, respectively, the ion and electron temperatures in K, \( m_i \) is the proton mass in gm, \( \mu_i \) and \( \mu_e \) are the corresponding effective molecular weights, respectively, and \( k \) is the Boltzmann constant. We assume \( \beta \) (and then \( \gamma \)) to be constant throughout the flow.

(iii) Azimuthal momentum balance.
\[ \frac{d}{dx} \left( \frac{1}{\Sigma} \frac{d}{dx} \left( x^2 |W_{\phi}| \right) \right), \]  
(10)

where, following Mukhopadhyay & Ghosh (2003, hereafter MG03), the shearing stress can be expressed in terms of the pressure and density as
\[ W_{\phi} = -\alpha (I_{n+1} P_{eq} + I_n \rho h(x)) \frac{d}{dx}, \]  
(11)

Here, \( \alpha \) is the dimensionless viscosity parameter and \( P_{eq} \) and \( \rho_{eq} \) are the pressure and density, respectively, at the equatorial plane. Note that we assume \( P_{eq} \sim P \) and \( \rho_{eq} \sim \rho \) when obtaining the solutions.

(iv) Energy production rate.
\[ \frac{d}{dx} \left( \frac{dP}{dx} - \Gamma_1 \frac{P}{\rho} \frac{d\rho}{dx} \right) = Q^+ - Q_w, \]  
(12)

where, following MG03,
\[ Q^+ = \alpha (I_{n+1} P + I_n \beta^2 \rho) h(x) \frac{d\rho}{dx}, \]  
(13)

which is the heat generated by viscous dissipation. \( Q_w \) is the Coulomb coupling (Bisnovatyi-Kogan & Lovelace 2000) given in dimensionful form as
\[ q_w = \frac{8(2\pi)^{1/2} e^2 n_i n_e}{m_i m_e} \left( \frac{T_i}{m_e} + \frac{T_e}{m_i} \right)^{-3/2} \ln(A) (T_i - T_e) \text{ erg cm}^{-3} \text{ s}^{-1} \]  
(14)

when \( q_w = Q_w c^{-1} / (hG^4 M^3) \). Here, \( n_i \) and \( n_e \) denote the number densities of ions and electrons, respectively, \( e \) is the charge of an electron and \( \ln(A) \) is the Coulomb logarithm. We also define (MG03)
\[ \Gamma_1 = 1 + \frac{\Gamma_1 - \beta}{4 - 3\beta}, \]  
(15)

\[ \Gamma_1 = \beta + \frac{(4 - 3\beta)^2(\gamma - 1)}{\beta + 12(\gamma - 1)(1 - \beta)}. \]  
(16)

(v) Energy radiation rate.
\[ \frac{d}{dx} \left( \frac{dP_e}{dx} - \Gamma_1 \frac{P_e}{\rho_e} \frac{d\rho_e}{dx} \right) = Q_e - Q_{\text{comp}}, \]  
(17)

where \( Q_{\text{comp}} \) is the heat radiated away by the bremsstrahlung (\( q_{\text{br}} \)) and synchrotron (\( q_{\text{syn}} \)) processes and inverse Comptonization (\( q_{\text{comp}} \)) due to soft synchrotron photons, given in dimensionful form as
\[ q_{\text{comp}} = q_{\text{br}} + q_{\text{syn}} + q_{\text{comp}}. \]  
(18)
when $q^- = Q^- c^{11}/(hG^2 M^3)$. Various components of the cooling processes may be read as (see Narayan & Yi 1995 and Mandal & Chakrabarti 2005 for a detailed description, which we do not repeat here):

$$q_{br} = 1.4 \times 10^{-27} n_i n_f T_{in}^{1/2} (1 + 4.4 \times 10^{-10} T_e) \text{erg cm}^{-3} \text{s}^{-1},$$

$$q_{\text{syn}} = \frac{2\pi}{3c^2} k T_e \frac{v_0^3}{R} \text{erg cm}^{-3} \text{s}^{-1}, \quad R = x GM/c^2,$$

$$q_{\text{comp}} = \mathcal{F} q_{\text{syn}}, \quad \mathcal{F} = \eta_1 \left[ 1 - \left( \frac{x_a}{3e} \right)^{n_2} \right], \quad \eta_1 = \frac{p(A - 1)}{(1 - p A)}, \quad p = 1 - \exp(-\tau_{es}),$$

$$A = 1 + 4\theta_e + 16\theta_e^2, \quad \theta_e = k T_e/m_e c^2, \quad \eta_2 = 1 + \left( \frac{\ln(p) + 1}{\ln(A)} \right), \quad x_a = h \nu_a/m_e c^2.$$  \hspace{1cm} (19)

Here, $\tau_{es}$ is the scattering optical depth given by

$$\tau_{es} = \kappa_{es} \rho h,$$  \hspace{1cm} (20)

where $\kappa_{es} = 0.38 \text{ cm}^2 \text{ gm}^{-1}$ and $v_a$ is the synchrotron self-absorption cut-off frequency determined by following Narayan & Yi (1995).

Note that without satisfactory knowledge of the magnetic field in accretion discs, following Mandal & Chakrabarti (2005), we assume the maximum possible magnetic energy density to be the gravitational energy density of the flow. As the total optical depth should include the effects of absorption resulting from non-thermal processes, the effective optical depth is computed as

$$\tau_{\text{eff}} \simeq \sqrt{\tau_{es} \tau_{abs}},$$  \hspace{1cm} (21)

where

$$\tau_{abs} = \frac{h}{4\pi T_e^2} \left( q_{br} + q_{\text{syn}} + q_{\text{comp}} \right) \frac{GM}{c^2},$$  \hspace{1cm} (22)

where $\sigma$ is the Stefan–Boltzmann constant.

Now combining all the above equations we obtain

$$\frac{d\phi}{dx} = \frac{N(x, \vartheta, c_e, \lambda, T_e)}{D(\vartheta, c_e)} .$$  \hspace{1cm} (23)

where

$$N = \frac{\Gamma_1 + 1}{\Gamma_3 - 1} \vartheta^2 c_s J - \frac{\alpha e^2}{m_e} \left( \frac{\alpha_{n+1}}{\alpha_n} \right) H \left( \frac{c_n^2}{\vartheta} - \vartheta \right) - \alpha^2 \left( \frac{\alpha_{n+1}}{\alpha_n} \right) 2H J + \frac{\Gamma_1 - 1}{\Gamma_3 - 1} \vartheta^2 c_s^3 L + \alpha H \left( \frac{2\vartheta c_s}{x^2} \right)$$

$$+ \frac{4\pi Q_{\text{es}}}{M} \vartheta^2 c_s x^{3/2} F^{-1/2},$$  \hspace{1cm} (24)

$$D = \frac{1 - \Gamma_1}{\Gamma_3 - 1} \vartheta^3 + 2\alpha c_e \frac{\alpha_{n+1}}{\alpha_n} H \left( \frac{c_n^2}{\vartheta} - \vartheta \right) + \frac{\Gamma_1 + 1}{\Gamma_3 - 1} \vartheta^2 c_s \left( \vartheta - \frac{c_s^2}{\vartheta} \right) + \alpha^2 \vartheta H \left( \frac{H}{\vartheta} \right)$$  \hspace{1cm} (25)

and

$$L = \left( \frac{3}{2x} - \frac{1}{2F} \frac{dF}{dx} \right), \quad H = (\alpha_{n+1} c_n^2 + I_n \vartheta^2), \quad J = \left( c_s^2 L + \frac{\lambda^2}{x^3} - F \right) .$$  \hspace{1cm} (26)

We know that around the sonic radius $N = D = 0$ (Mukhopadhyay 2003) and hence we obtain the Mach number at this critical radius from $D = 0$:

$$M_s = \frac{\vartheta_c}{c_s} \sqrt{-B + (B^2 - 4AC)^{1/2}} \frac{1}{2A} .$$  \hspace{1cm} (27)

Here

$$A = \frac{\Gamma_1 + 1}{1 - \Gamma_3} - 2\alpha^2 (\alpha_{n+1} - I_n), \quad B = \frac{2(\Gamma_1 + 1)}{1 - \Gamma_3} - 2\alpha^2 \frac{\alpha_{n+1}}{\alpha_n} (I_{n+1} - I_n),$$

$$C = \alpha^2 \frac{\alpha_{n+1}}{\alpha_n} (1 - 2I_{n+1}) .$$  \hspace{1cm} (28)

Also, from $N = 0$, we can compute explicitly $c_s$ as a function of sonic/critical radius $x_c$. For a physical $x_c$, what one has to adjust in order to obtain a physical solution connecting outer boundary to black hole horizon through $x_c$ and $c_s$ and then $\vartheta_c$ can be assigned, discussed in Appendix A in detail. Note that an improper $x_c$ may lead to an unphysical/imaginary $c_s$ and $\vartheta_c$.

Finally, combining equations (6), (10) and (17), we obtain

$$\frac{dc_s}{dx} = \left( \frac{c_s - \vartheta}{c_s} \right) \frac{d\vartheta}{dx} + \frac{J}{c_s},$$  \hspace{1cm} (29)

$$\frac{d\vartheta}{dx} = \left[ 2\alpha x \frac{\alpha_{n+1}}{\alpha_n} \left( \frac{c_s^2}{\vartheta} - \vartheta c_s \right) + \alpha x \right] \frac{d\vartheta}{dx} + \left( \frac{c_s^2}{c_s} + \frac{\lambda^2}{x^3} + \vartheta \right) .$$  \hspace{1cm} (30)
\[
\frac{dT_e}{dx} = (1 - \Gamma_1)T_e \frac{\partial}{c_s^2 \partial x} + (1 - \Gamma_1)T_e \left( \frac{J}{c_s^2} + L \right) + \frac{(\Gamma_1 - 1)4\pi c_s^3/2}{M^{4/3}} (Q^e - Q^-).
\]

(31)

Thus, knowing \( \vartheta \) we can obtain the other variables \( c_s, \lambda \) and \( T_e \). Note that \( d\vartheta/dx \) is indeterminate (of 0/0 form) at \( x_c \). In Appendix B we discuss the procedure to obtain \( d\vartheta/dx \) at \( x_c \).

As the entropy increases inwards in advective flows (see, for example, Narayan & Yi 1994; Chakrabarti 1996; MG03), there is the possibility of convective instability and then corresponding transport, as proposed by Narayan & Yi (1994). Dynamical convective instability arises when the square of effective frequency

\[v_{\text{eff}}^2 = v_{BV}^2 + v_r^2 < 0,\]

(32)

where \( v_{BV} \) is the Brunt–Väisälä frequency given by

\[v_{BV}^2 = -\frac{1}{\rho} \frac{dP}{dx} \frac{d}{dx} \ln \left( \frac{P^{1/\gamma}}{\rho} \right)\]

(33)

and \( v_r \) is the radial epicyclic frequency.

### 2.2 Solution procedure

In order to obtain the steady-state solution, as in previous work (Mukhopadhyay 2003; MG03), primarily we need to fix the appropriate critical radius \( x_c \) (in fact, the energy at the critical radius, which is not conserved in the present case) and the corresponding specific angular momentum \( \lambda_c \) of the flow. A detailed description of the procedure to obtain physically meaningful values of \( x_c \) and \( \lambda_c \), to be determined iteratively, is given in Appendix A. As the flow is considered to be of two temperatures, at \( x_c \) an appropriate electron temperature \( T_{ec} \) also needs to be determined (this is also discussed in Appendix A). Note that we have to adjust the set of values \( x_c, \lambda_c, T_{ec} \) appropriately/iteratively to obtain a self-consistent solution connecting the outer boundary and the black hole event horizon through \( x_c \).

Depending on the type of accreting system to model, we then have to specify the related inputs: \( \dot{M}, M, \gamma \) and \( a \). An important point to note is that, unlike in previous work (e.g. Chakrabarti & Titarchuk 1995; Chakrabarti 1996; MG03), here \( x_c \) changes when \( \dot{M} \) changes, because the various cooling processes considered here explicitly depend on \( \dot{M} \). Finally, we have to solve equation (23) from \( x_c \) inwards (up to the black hole event horizon) and outwards (up to the transition radius \( x_o \) where the disc deviates from the Keplerian to the sub-Keplerian regime such that \( \lambda/\lambda_K = 1 \), with \( \lambda_K \) being the specific angular momentum of the Keplerian part of the disc). Fig. 1 shows how the ratio \( \lambda/\lambda_K \) varies as a function of radial coordinate for different \( a \). Note that a higher value of \( a \), which corresponds to a lower disc angular momentum (Mukhopadhyay 2003), reassembles the Keplerian part advancing with a smaller size of the sub-Keplerian disc. However, for a lower value of \( a \), the inner edge of the
Keplerian component recedes. The fact of moving in and out of the inner edge of the disc reassembles, respectively, the soft and hard states of the black hole (e.g. Gilfanov, Churazov & Sunyaev 1997). Hence, it is naturally expected to link with the spin of the black hole.

However, an important point to note is that there is no self-consistent model to describe the transition region where $\lambda / \lambda_K = 1$. Therefore, the transition of the flow from the Keplerian to sub-Keplerian regime does not appear smooth. This is mainly because the set of equations used to model the sub-Keplerian flow is not valid to explain the cold Keplerian flow, unless an extra boundary condition is imposed at the outer edge of the sub-Keplerian disc. However, in this paper we do not intend to address the transition zone; rather, we prefer to concentrate on the sub-Keplerian flow. Narayan, Kato & Honma (1997) imposed boundary conditions at both the ends of accretion flows along with at the critical radius to fix the problem, at the cost of more input parameters than the parameters chosen in our work. However, the transition of the flow from the Keplerian to sub-Keplerian regime still remains undefined. Yuan (1999) later discussed how the solutions vary with the change of outer boundary conditions influencing the structure of an optically thin accretion flow.

Below, we discuss solutions in various parameter regimes to understand the properties of the accretion disc, first around the stellar mass ($M = 10$) and then the supermassive ($M = 10^7$) black holes.

3 TWO-TEMPERATURE ACCRETION DISC AROUND STELLAR MASS BLACK HOLES

Primarily, we concentrate on two extreme regimes: (i) the sub-Eddington and Eddington limits of accretion; (ii) the super-Eddington accretion. Furthermore, for each accretion rate, we focus on solutions around non-rotating (Schwarzschild) and rotating (Kerr with $a = 0.998$) black holes.

One of our aims is to understand how the explicit cooling processes affect the disc dynamics and then how the cooling efficiency varies over the disc radii. The cooling efficiency $f$ is defined as the ratio of the energy advected by the flow to the energy dissipated, which is 1 for the advection-dominated accretion flow (ADAF; Narayan & Yi 1994, 1995) and less than 1 for the general advective accretion flow (GAAF; Chakrabarti 1996; Mukhopadhyay 2003; MG03). Therefore, $f$ directly controls the ion and electron temperatures of the disc. Far from the black hole where the gravitational power is weaker, the angular momentum profile becomes Keplerian and thus the disc becomes (or tends to become) of one temperature in the presence of efficient cooling.

3.1 Sub-Eddington and Eddington accretors

3.1.1 Schwarzschild black holes

We first consider flows around static black holes where the Kerr parameter $a = 0$. Fig. 2 shows the behaviour of flow variables as functions of the radial coordinate for $\dot{M} = 0.01, 0.1$ and 1; throughout the paper, we express $\dot{M}$ in units of the Eddington limit. The sets of input parameters are $a = 0$, $\alpha = 0.01$ and $M = 10$ (see Table 1 for details).

![Figure 2](https://academic.oup.com/mnras/article-abstract/402/2/961/1103036/fig2.png)

**Figure 2.** Variation of dimensionless (a) radial velocity, (b) density, (c) cooling factor and (d) square of convective frequency, as functions of the radial coordinate for sub-Eddington and Eddington accretion flows. The solid, dashed and dotted curves represent $\dot{M} = 0.01, 0.1$ and 1, respectively. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10$ (see Table 1 for details).
parameters for the model cases described here are given in Table 1. Fig. 2(a) verifies that a higher radial velocity corresponds to a lower mass accretion rate of the flow (~0.01). This results in less possible accumulation of matter in a particular radius, attributing to a lower disc density (Fig. 2b). This hinders the bremsstrahlung process from cooling the flow. At around $x = 30$ the centrifugal barrier dominates and decreases the velocity $\vartheta$, particularly for $M = 0.01$, which finally merges with that of higher $M$. However, for a lower $M$ corresponds to a gas-dominated hot flow, which is radiatively less efficient and quasi-spherical in nature. As a result, $\vartheta$ is high, as seen in Fig. 2(a). The efficiency of cooling is shown in Fig. 2(c). Naturally, a low $M$ corresponds to a radiatively inefficient flow, rendering $f \gtrsim 0.9$ up to $x = 30$. At $x < 30$, the dominance of the centrifugal barrier slows down the infall, which increases the residence time of matter in the disc before plunging into the black hole. This allows matter to have enough time to radiate by the synchrotron process and inverse Comptonization as a result of synchrotron soft photons, rendering $f \rightarrow 0$ close to the black hole. In other words, for $M = 0.01$, the disc is essentially radiatively inefficient, up to $x \sim 30$, and therefore the electron temperature never decreases. However, the density sharply increases in the vicinity of the black hole (Fig. 2b), which favours efficient cooling at a high temperature.

Therefore, although far from the black hole, a sub-Eddington flow appears to be radiatively inefficient; from $x = 30$ onwards, it turns out to be a radiatively efficient advective flow with $f$ much less than unity. However, for $M = 0.1$ and 1, the density is higher than that for $M = 0.01$ and hence the bremsstrahlung effect starts to play a role in radiation mechanisms at radii further out. This decreases the ion–electron temperature difference at the transition radius $x_c$. However, as the flow advances, the synchrotron and corresponding inverse Compton effects dominate, attributing to strong radiative loss. This renders $f \lesssim 0.5$ up to $x = 10$. Further in, a strong radial infall, in the absence of any centrifugal barrier compared to a low $M$ case, does not permit matter to radiate enough, rendering $f$ up to 1. This is particularly because the strong advection decreases the residence time of the flow before plunging into the black hole, and thus renders a weaker ion–electron coupling. This, in turn, hinders the transfer of energy from the ions to electrons, meaning that the ions remain hot, while the electrons continue to be cooled further by radiative processes.

However, Fig. 2(d) shows that neither of the cases exhibit convective instability (see, however, Narayan, Igumenshchev & Abramowicz 2000; Quataert & Gruzinov 2000) up to the very inner edge, even if the radiatively inefficient flow deviates to a radiatively efficient GAFF. At the very inner edge, discs with $M = 0.1$ and 1 particularly appear to be marginally unstable, which, however does not seem to play any role in angular momentum transfer.

Fig. 3 shows how the various cooling processes and corresponding temperature profiles vary as functions of the radial coordinate. At a low $M (\approx 0.01)$ the system is radiatively inefficient, relative to that at a higher $M (= 0.1, 1)$, which brings out a hot two-temperature Keplerian–sub-Keplerian transition region. As the flow advances through the sub-Keplerian part, the strong two-temperature nature remains intact. For a higher $M (= 1)$, however, the transition region is of marginally two temperatures. This is due to the efficient bremsstrahlung radiation at high density. In the vicinity of the black hole $T_e \sim 10^9$ K in the flow with $M = 0.01$ when $f \rightarrow 0$, as explained above, in contrast to the cases with $M = 0.1, 1$ when $f \rightarrow 1$ and $T_e$ decreases sharply. Note that the accretion disc around a stellar mass black hole is arrested by significant magnetic field. This results in the dominance of the synchrotron effect over bremsstrahlung as the flow advances.

3.1.2 Kerr black holes

We consider rotating black holes with $a = 0.998$. As discussed earlier (Mukhopadhyay 2003), the angular momentum of the flow should be smaller around a rotating black hole compared to that around a static black hole. This reassembles advancing the Keplerian component, which decreases the size of the sub-Keplerian part. As the disc remains Keplerian (which is radiatively efficient) up to, for example, $x \sim 100$ (see the outer radius in Fig. 4), the flow cools significantly before deviating to the sub-Keplerian regime. Fig. 4(a) shows that the velocity profiles for all $M$ are similar to each other, in the absence of a strong centrifugal barrier. However, $f$, while very small at $x \sim 100$, increases as the sub-Keplerian flow advances. This is because the residence time of the flow decreases in an element of the sub-Keplerian disc, hindering the cooling processes from completing. Fig. 4(c) shows that even $f \rightarrow 0.9$ at $x \sim 30$ for $M = 0.01$, when the density is lowest (see Fig. 4b).
Figure 3. Variation of (a) the dimensionless energy of Coulomb coupling (thick line), the bremsstrahlung process (dotted line), synchrotron process (solid), inverse Comptonization process as a result of synchrotron photons (dashed line) in logarithmic scale, and (b) the corresponding ion (solid) and electron (dotted) temperatures in units of $10^8$ K, as functions of the radial coordinate for $\dot{M} = 0.01$. (c), (e) Same as (a) except $\dot{M} = 0.1$ and 1, respectively. (d), (f) Same as (b) except $\dot{M} = 0.1$ and 1, respectively. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10$ (see Table 1 for details).

Figure 4. Same as Fig. 2, except $a = 0.998$. 

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However, as the flow approaches the black hole, the synchrotron emission increases, and hence the system acquires enough soft photons to help the inverse Compton process to occur. As a result, the flow cools further. When $\dot{M} = 0.01$, the cooling process at the very inner edge of the accretion disc is dominant because of the relatively high residence time of the flow, compared to that of a higher $\dot{M}$, rendering high $T_e$ and then $f \to 0$ very close to the black hole. Fig. 4(d) proves that the flow is convectionally stable all the way up to the black hole event horizon. For $\dot{M} \gtrsim 0.1$, however, the flow is strongly advective and then unable to cool before plunging into the black hole. Fig. 5 shows the profiles of the cooling processes and ion–electron temperatures. The basic nature of the profiles is similar to that of the Schwarzschild cases, except that all of them advance in.

3.2 Super-Eddington accretors

The ‘radiation-trapped’ accretion disc can be attributed to the radiatively driven outflow or jet. This is likely to occur when the accretion rate is super-Eddington (Lovelace, Romanova & Newman 1994; Fabbiano 2004; Begelman, King & Pringle 2006; Ghosh & Mukhopadhyay 2009), as seen in ULX sources, such as SS433 (with luminosity $\sim 10^{40}$ erg s$^{-1}$ or so; Fabrika 2004). In order to describe such sources, the models described below are the meaningful candidates. We consider $\dot{M} = 10$ and 100.

3.2.1 Schwarzschild black holes

A high mass accretion rate significantly enhances density, up to two orders of magnitude compared to that of a low $\dot{M}$, which severely affects $f$ and finally temperature profiles. The profiles of velocity shown in Fig. 6 are similar to those of the sub-Eddington and Eddington cases. For similar reasons as explained in Section 3.1.1, the profile exhibits a stronger centrifugal barrier for a lower $\dot{M}(=10)$. A lower $\dot{M}$ flow will have relatively more gas and then a quasi-spherical structure, compared to that of a higher $\dot{M}(=100)$. This results in a lower velocity in the latter case. However, a lower velocity corresponds to a higher density, which results in strong bremsstrahlung radiation rendering $f \to 0$. For a lower $\dot{M}$, at $x \sim 50$, the energy radiated because of the bremsstrahlung process becomes weaker than the energy transferred from protons to electrons through the Coulomb coupling (see Fig. 7), which increases $f$ (see Fig. 6c). Subsequently, the synchrotron process becomes dominant (see Fig. 7), reassembling $f \to 0$. However, very close to the black hole, strong advection does not allow the flow, independent of $\dot{M}$, to radiate efficiently, rendering $f \to 1$ again. This also results in marginal convective instability at $x < 10$, as shown in Fig. 6(d).
Figure 6. Variation of dimensionless (a) radial velocity, (b) density, (c) cooling factor and (d) square of convective frequency, as functions of the radial coordinate for super-Eddington accretion flows. The solid and dashed curves are for $\dot{M} = 10$ and $100$, respectively. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10$ (see Table 1 for details).

Fig. 7 shows that discs remain of one temperature at the transition radius. As the flows advance with a sub-Keplerian angular momentum, the $T_p$ profile deviates from that of $T_e$. For $M = 100$, the high-density flow is dominated by very efficient bremsstrahlung radiation all the way. In the vicinity of the black hole, an efficient cooling reassembles a sharp downfall of $T_e$.

3.2.2 Kerr black holes

As $\lambda$ decreases in the case of higher $a$, as in the low $\dot{M}$ cases (see Mukhopadhyay 2003), the transition region advances an order of magnitude compared to that of Schwarzschild black holes. Similar to the cases of low $\dot{M}$, as shown in Fig. 8(a), any centrifugal barrier is smeared. However, unlike the flow around a static black hole, here the disc with $\dot{M} = 10$ remains stable until very close to the black hole. This is because high $a$ corresponds to a larger inner edge of the disc and thus the residence time of matter in the disc is higher. As a result, the radiative processes continue to cool and then stabilize the flow up to the very inner edge.

Fig. 9 shows that although a high $\dot{M}$ exhibits a one-temperature transition zone as a result of extremely efficient cooling processes, particularly because of bremsstrahlung radiation, as $\dot{M}$ decreases the Keplerian disc itself becomes of two temperatures before deviating to the sub-Keplerian zone, unlike that of the Schwarzschild case. This is mainly because a flow with high $a$ brings the Keplerian disc further in, where the transport of angular momentum increases, leading to the decrease of the residence time of matter, which does not allow efficient cooling. However, the basic behaviours of various cooling processes are similar to the behaviour around a static black hole.

4 TWO-TEMPERATURE ACCRETION DISC AROUND SUPERMASSIVE BLACK HOLES

As for stellar mass black holes, here we also concentrate on two extreme regimes: (i) the sub-Eddington and Eddington limits of accretion; (ii) the super-Eddington accretion. We focus on both non-rotating (Schwarzschild) and rotating (Kerr with $a = 0.998$) black holes.

4.1 Sub-Eddington and Eddington accretors

Underluminous AGNs and quasars (e.g. Sgr $A^*$) have already been described by an advection-dominated model, where the flow is expected to be substantially subcritical/sub-Eddington with a very low luminosity ($\lesssim 10^{35}$ erg s$^{-1}$). Therefore, the present cases, particularly of $M \lesssim 0.01$, could be potential models to describe underluminous sources.
Two-temperature accretion around black holes

Figure 7. Variation of (a) the dimensionless energy of Coulomb coupling (thicker line), the bremsstrahlung process (dotted line), the synchrotron process (solid) and the inverse Comptonization process as a result of synchrotron photons (dashed line) in logarithmic scale, and (b) the corresponding ion (solid) and electron (dotted) temperatures in units of $10^8$ K, as functions of the radial coordinate for $\dot{M} = 10$. (c) Same as (a) except $\dot{M} = 100$. (d) Same as (b) except $\dot{M} = 100$. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10$ (see Table 1 for details).

Figure 8. Same as Fig. 6, except $a = 0.998$. 

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4.1.1 Schwarzschild black holes

Table 2 lists the sets of input parameters for the model cases described here. Naturally, a disc around a supermassive black hole will have a much lower density compared to that around a stellar mass black hole. Therefore, the cooling processes, particularly the bremsstrahlung radiation, which is density-dependent, are expected to be inefficient, leading to high $f$. However, the velocity profiles shown in Fig. 10(a) are very similar to those around a stellar mass black hole. Fig. 10(c) shows that $f \to 1$ in most of the sub-Keplerian regime for $M = 0.01$. As $\dot{M}$ increases, the density increases and thus the bremsstrahlung radiation increases, as shown in Fig. 11. This leads to the transition of radiatively inefficient flow to GAAF. When $\dot{M} = 1$, the bremsstrahlung effect is very high, resulting in a GAAF with $f$ much smaller than unity until very close to the black hole. Fig. 10(d) shows that close to the black hole there is a possible convective instability for all $\dot{M}$. This is because a strong advection of matter close to the black hole hinders the cooling processes, which results in $f \to 1$. This reassembles a possible convective instability at the inner edge. For higher $\dot{M}$, the density is high, which favours convection and thus brings the convective instability earlier in, at a relatively outer radius.

Table 2. Parameters for accretion with $\alpha = 0.01$ around black holes of $M = 10^7$, where the subscript ‘c’ indicates the quantity at the critical radius and $T_{ec}$ is expressed in units of $m_1 c^2/k$.

| $\dot{M}$ | $\alpha$ | $\gamma$ | $x_c$ | $\lambda_c$ | $T_{ec}$ |
|-----------|----------|----------|-------|-------------|----------|
| Sub-Eddington and Eddington accretors | | | | | |
| 0.01 | 0 | 1.5 | 5.5 | 3.2 | 0.0001 |
| 0.01 | 0.998 | 1.5 | 3.5 | 1.7 | 0.0001 |
| 0.1 | 0 | 1.4 | 5.5 | 3.2 | 0.000178 |
| 0.1 | 0.998 | 1.4 | 3.5 | 1.7 | 0.00023 |
| 1 | 0 | 1.35 | 5.5 | 3.2 | 0.0002493 |
| 1 | 0.998 | 1.35 | 3.5 | 1.7 | 0.000295 |
| Super-Eddington accretors | | | | | |
| 10 | 0 | 1.345 | 5.5 | 3.2 | 0.000427 |
| 10 | 0.998 | 1.345 | 3.5 | 1.7 | 0.0006 |
| 100 | 0 | 1.34 | 5.5 | 3.2 | 0.0003874 |
| 100 | 0.998 | 1.34 | 3.5 | 1.7 | 0.00059 |
Two-temperature accretion around black holes

The temperature profiles shown in Fig. 11 are similar to what we obtain for stellar mass black holes, so we do not repeat the explanation here. However, note that, unlike stellar mass black holes, only the bremsstrahlung radiation is effective in cooling the flow around a supermassive black hole, particularly for $\dot{M} = 1$.

4.1.2 Kerr black holes

The specific angular momentum of the black hole is chosen to be $a = 0.998$. The basic hydrodynamical properties are similar to those in the Schwarzschild cases, except, like the flows around stellar mass black holes, the transition region advances because of a smaller angular momentum of the flow (as shown in Fig. 12). As discussed in Section 3.2.2, higher $a$ corresponds to smaller $\lambda$, which in turn decreases $\vartheta$ at a particular radius of the inner edge of the disc, when the inner edge is stretched in, compared to that around a Schwarzschild black hole. This results in the increase of the residence time of the flow in the sub-Keplerian disc before plunging into the black hole. Therefore, the bremsstrahlung process continues cooling and then stabilizing the flow up to the very inner edge. An important point to note, as a consequence, is that the disc flow around a rotating black hole is convectively more stable compared to that around a non-rotating black hole. This is particularly because the density gradient of the inner flow (e.g. the vicinity of $x = 2$, which is the event horizon for a non-rotating black hole) around a rotating black hole is less steep compared to that around a non-rotating black hole. Hence, the flow is convectively more stable in the case of rotating black holes.

Fig. 13 shows that basic features of the temperature profiles are similar to the cases of static black holes. However, the transition region reveals that for lower $\dot{M}$, the Keplerian flow exhibits inverse Comptonization via synchrotron photons. As the flow advances with a sub-Keplerian angular momentum, the residence time decreases and thus inverse Comptonization decreases, resulting in a hotter flow.

4.2 Super-Eddington accretors

Ultraluminous accretors with a high kinetic luminosity ($\sim 10^{46}$–$10^{49}$ erg s$^{-1}$) radio jet have been observed in highly luminous AGNs and ultraluminous quasars (e.g. PKS 0743-67; Punsly & Tingay 2005), possibly in ultra luminous infrared galaxies (Genzel et al. 1998) and narrow-line Seyfert 1 galaxies (e.g. Mineshige et al. 2000). Therefore, the following cases could potentially be models to explain such sources.
Figure 11. Variation of (a) the dimensionless energy of Coulomb coupling (thicker line), the bremsstrahlung process (dotted line), the synchrotron process (solid line) and the inverse Comptonization process as a result of synchrotron photons (dashed line) in logarithmic scale, and (b) the corresponding ion (solid) and electron (dotted) temperatures in units of $10^8$ K, as functions of the radial coordinate for $\dot{M} = 0.01$. (c), (e) Same as (a) except $\dot{M} = 0.1$ and 1, respectively. (d), (f) Same as (b) except $\dot{M} = 0.1$ and 1, respectively. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10^7$ (see Table 2 for details).

4.2.1 Schwarzschild black holes

Figs 14 and 15 show that the basic flow properties are similar to those around stellar mass black holes, except in the present cases the centrifugal barrier smears out. This is because a high black hole mass corresponds to a low density of the flow and thus a fast infall. Unlike for stellar mass black holes, this also results in inefficient synchrotron radiation even at the inner edge of the disc.

4.2.2 Kerr black holes

Figs 16 and 17 show a similar situation as that around stellar mass black holes, but with a smeared centrifugal barrier, as described above for static black holes. However, because of decreasing density, the overall cooling effects decrease, keeping the disc hotter, particularly for $\dot{M} = 10$.

5 COMPARISON BETWEEN FLOWS WITH DIFFERENT $\alpha$ AND AROUND COROTATING AND COUNTER-ROTATING BLACK HOLES

So far, we have restricted ourselves to a typical Shakura–Sunyaev viscosity parameter $\alpha = 0.01$, for corotating black holes. Now, we plan to explore lower values of $\alpha$, as well as a counter-rotating black hole, in order to understand any significant change in the flow behaviour.

5.1 Comparison between flows with $\alpha = 0.01$ and $\alpha = 0.0001$

A decreasing value of $\alpha$ naturally decreases the rate of energy-momentum transfer between any two successive layers of the fluid element, and increases the residence time of the flow in the sub-Keplerian disc. This also means that the Keplerian–sub-Keplerian transition region recedes further. This is mainly because a low value of $\alpha$ cannot continue the outward angular momentum transport efficiently in the Keplerian flow below a certain radial coordinate. Therefore, the disc flow cannot remain Keplerian and becomes sub-Keplerian at a larger radius, compared to a flow with high $\alpha$.

We know, however, that an increasing residence time increases the possibility of completing various radiative processes in the disc flow, before the infalling matter plunges into the black hole. Therefore, the flow is expected to appear cooler with smaller $f$. Hence, for the purposes
of comparison, we consider a flow with $\dot{M} = 0.01$ around a supermassive black hole of $M = 10^7$ (e.g. Sgr A*, which is radiatively inefficient and hot for $\alpha = 0.01$).

Fig. 18 shows that although the velocity profiles are similar for both values of $\alpha$, the size of the sub-Keplerian disc is about five times greater for $\alpha = 0.0001$ than for $\alpha = 0.01$. Inside $x = 17$, the low $\alpha$ disc flow becomes cooler very fast, rendering $f \to 0$ at $x > 10$ (see Fig. 18c). Therefore, the flow sharply transits from being radiatively inefficient in nature to a GAAF. As a consequence, the low $\alpha$ flow remains stable, as shown in Fig. 18(b), all the way to the event horizon. As the sub-Keplerian flow of smaller $\alpha$ extends further, where the influence of black hole is very weak, $T_e$ and $T_i$ merge (see Fig. 18d) before the flow crosses the transition radius, unlike the flow with $\alpha = 0.01$ when $T_i > T_e$.

### 5.2 Comparison between flows around corotating and counter-rotating black holes

We have already discussed that the model with a low mass accretion rate around a supermassive black hole can potentially explain the observed dim source, Sgr A*. However, the ULX sources presumably correspond to models with a high mass accretion flow around a stellar mass black hole. Therefore, in order to compare the flow properties between corotating and counter-rotating black holes, we choose these two extreme cases.

Qualitatively, flows with similar initial conditions around corotating and counter-rotating black holes of the same mass are similar, as shown in Figs 19 and 20 for $a = \pm 0.5$. However, the sub-Keplerian disc size around the black hole with $a = -0.5$ is smaller because of the smaller value of the effective angular momentum (Mukhopadhyay 2003) of the system. Hence, the radial velocity is almost an order of magnitude higher, particularly at the inner edge, for $a = -0.5$.

### 6 DISCUSSION AND SUMMARY

We have modelled the two-temperature accretion flow, particularly around black holes, combining the equations of conservation and comprehensive cooling processes. We have considered self-consistently the important cooling mechanisms: bremsstrahlung, synchrotron and inverse Comptonization as a result of synchrotron photons, where ions and electrons are allowed to have different temperatures. As matter falls in, hot electrons cool through the various cooling mechanisms, particularly by synchrotron emission when the magnetic field is high. This is particularly the case for the flow around stellar mass black holes where the magnetic field may also act as a boost in transporting the angular momentum. However, in this paper, we have not considered such processes in detail, but we have stuck with the standard $\alpha$-prescription.

By solving a complete set of disc equations, we show that in general the disc system exhibits a GAAF. However, in certain circumstances the GAAF becomes radiatively inefficient, depending on the flow parameters and hence the efficiency of the cooling mechanisms. Transitions
from GAAF to radiatively inefficient flow, and vice versa, are clearly explained by the cooling efficiency factor $f$, as shown for each model. While previous authors who proposed ADAF (Narayan & Yi 1994, 1995) were especially restricted with flows having $f = 1$ (inefficient cooling), here we do not impose any restriction on the flow parameters at the start and we allow the parameter $f$ to be determined self-consistently as the system evolves. Therefore, our model is very general whose special case may be understood as a radiatively inefficient advection-dominated flow.

In particular, we have explored the optically thin flows incorporating the bremsstrahlung, synchrotron and inverse Comptonization processes. Fig. 21 shows the variation of the effective optical depth as a function of disc radii for two limiting cases. While flows around rotating black holes appear thinner compared to the corresponding cases of static black holes, in general $\tau_{\text{eff}} \lesssim 5 \times 10^{-4}$. This verifies our choice of optically thin flows. However, for the present purposes, when the main aim is to understand disc dynamics in the global, viscous, two-temperature regime, we have ignored inverse Comptonization resulting from bremsstrahlung photons, if any. This may be important in extreme super-Eddington accretion flows, which we plan to explore in future, in particular analysing the underlying spectra.

The temperature of the flow depends on the accretion rate. If the accretion rate is low and thus the flow is radiatively inefficient, then the disc is hot. Attempts to model such a hot flow have been made since 1976 (Shapiro et al. 1976) when it was assumed that locally $Q^+ \sim Q^-$ and thus $f \rightarrow 0$. While the model was successful in explaining observed hard X-rays from Cyg X-1, it turned out to be thermally unstable. Rees et al. (1982) proposed a hot ion torus model avoiding $f$ to unity. In a similar way, Narayan & Yi (1995) proposed a hot two-temperature solution, assuming $f \rightarrow 1$ and including strong advection in the flow. Abramowicz et al. (1995), based on the single temperature model, showed that the optically thin disc flow of accretion rate more than one Eddington does not have an equilibrium solution. However, they did not attempt to solve the complete set of differential equations. Based on some simplistic assumptions, they showed the importance of advective cooling. Moreover, a single-temperature description does not allow the inclusion of all the underlying physics necessary to describe the cooling processes. In due course, Mandal & Chakrabarti (2005) proposed a two-temperature disc solution where the ion temperature could be as high as $\sim 10^{12}$ K. However, they placed particular emphasis on how the shock in the disc flow enables cooling through the synchrotron mechanism, without carrying out a complete analysis of the dynamics. In our present paper, to the best of our knowledge, we provide the first comprehensive work to model the two-temperature accretion flow self-consistently by solving the complete set of underlying equations without any presumptive choice of the flow variables at the start.

The generality lies not only in its construction but also in its ability to explain the underluminous to ultraluminous sources, stellar mass to supermassive black holes. Table 3 lists the luminosities for a wide range of parameter sets, obtained by our model. It reveals that for a very low mass accretion rate $M = 0.0001$ around a supermassive black hole, the luminosity is $L \sim 10^{34}$ erg s$^{-1}$, which is indeed similar...
Figure 14. Variation of dimensionless (a) radial velocity, (b) density, (c) cooling factor and (d) square of convective frequency, as functions of the radial coordinate for super-Eddington accretion flows. The solid and dashed curves are for $\dot{M} = 10$ and 100, respectively. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10^7$ (see Table 2 for details).

In general, an increase of accretion rate increases the density of the flow, which may lead to a high rate of cooling and thus a decrease of the cooling factor $f$. Hence, $f$ is higher, close to unity reassembling radiatively inefficient flows, for sub-Eddington accretors, and is lower, sometimes close to zero, for super-Eddington flows. The actual value of $f$ in a flow also depends on the behaviour of the hydrodynamic variables that determine the rate of the cooling processes. Naturally, as the flow advances from the transition region to the event horizon, $f$ varies between 0 and 1. However, if the black hole is considered to be of stellar mass, then at a high $\dot{M} = 100$, the model reveals $L \sim 10^{40}$ erg s$^{-1}$, which is similar to the observed luminosity from ULX sources (e.g. SS433).

In all cases, the ion and electron temperatures merge or tend to merge at around the transition radius. This is because the electrons are in thermal equilibrium with the ions, and thus virial around the transition radius, particularly when $\dot{M} \gtrsim 1$. As the sub-Keplerian flow advances, the ions become hotter and the corresponding temperature increases, rendering the ion–electron Coulomb collisions weaker. The electrons, however, cool down via processes such as bremsstrahlung, synchrotron emissions, etc., keeping the electron temperature roughly constant up to the very inner disc. This reveals the two-temperature flow strictly.

An important point to note is that we have assumed throughout that the coupling between the ions and electrons is a result of Coulomb scattering. However, the inclusion of possible non-thermal processes of transferring energy from the ions and electrons (Phinney 1981; Begelman & Chieu 1988) might modify the results. However, as argued by Narayan & Yi (1995), the collective mechanism discussed by Begelman & Chieu (1988) may dominate over the Coulomb coupling at either a very low $\alpha$ or a very low $\dot{M}$. Instead, the viscous heating rate of ions is much larger than the collective rate of non-thermal heating of electrons, unless $\alpha$ is too small, which we have not considered here. Therefore, the assumption to neglect the non-thermal heating of electrons is justified.

In future, we need to understand the radiation emitted by the flows discussed here and to model the corresponding spectra. This will be the ultimate test of the model to explain the observed data.
Figure 15. Variation of (a) the dimensionless energy of Coulomb coupling (thick line), the bremsstrahlung process (dotted line), the synchrotron process (solid line) and the inverse Comptonization process as a result of synchrotron photons (dashed line) in logarithmic scale, and (b) the corresponding ion (solid) and electron (dotted) temperatures in units of $10^8$ K, as functions of the radial coordinate for $M = 10$. (c) Same as (a) except $M = 100$. (d) Same as (b) except $M = 100$. The other parameters are $a = 0$, $\alpha = 0.01$ and $M = 10^7$ (see Table 2 for details).

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Figure 16. Same as Fig. 14, except $a = 0.998$. 

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Figure 17. Same as Fig. 15, except \( \alpha = 0.998 \).

Figure 18. Comparison of solutions for high and low \( \alpha \). Variation of (a) velocity, (b) square of convective frequency, (c) cooling factor, and (d) ion (upper set of lines) and electron (lower set of lines) temperatures, as functions of the radial coordinate. The solid lines correspond to \( \alpha = 0.01 \) and the dashed lines correspond to \( \alpha = 0.0001 \). The other parameters are \( \dot{M} = 0.01 \), \( M = 10^7 \) and \( a = 0 \).
\[ M(\lambda - \lambda_{\text{in}}) = -4\pi x^2 |W_{\phi}|, \]

where \( \lambda_{\text{in}} \) is the specific angular momentum at the inner edge of the disc, to be fixed by no torque inner boundary condition. Note that \( \lambda_{\text{in}} \leq \lambda_c \) (see, for example, Chakrabarti 1996).

We therefore need the initial values of \( \vartheta, c_c, \lambda \) and \( T_s \) to solve the set of equations. When we impose the condition that the flow must pass through a critical radius \( x_c \) (around a sonic radius) where \( D = 0, \vartheta \) and \( c_c \) at \( x_c \) are related by a quadratic equation of Mach number given by equation (24).

For the continuity of \( d\vartheta/dx, N = 0 \) at \( x_c \). Therefore, from \( N = 0 \), which an algebraic non-linear equation, \( c_c \) at \( x_c \) can be computed iteratively (using the bisection method), which in turn also fixes \( \vartheta \) at \( x_c \), provided \( \lambda \) is known at that radius.

Now we need to set appropriate values of \( x_c \) and the corresponding specific angular momentum \( \lambda_c \). This is fixed iteratively by invoking the condition that the critical point is saddle-type. This can be seen as follows. First, we impose that \( \lambda_{\text{in}} = \lambda_c \). Then, by fixing the value of \( \lambda_c \) we find that if \( x_c \) is greater than a certain critical value \( x_{cc} \), then the type of \( x_c \) changes from saddle-type to \( O \)-type, through which matter never can pass. Fig. A1(a) shows how the type of critical point and then the solution topology change with a slight increase of \( x_c \). However, as \( x_c \) decreases from \( x_{cc} \), which corresponds to the energy at \( x_c (E_c) \) increases, the sub-Keplerian disc decreases in size. This advances the Keplerian disc. The reason is that increasing \( E_c \) corresponds to decreasing \( x_c \) and then increasing centrifugal energy (~\( x^2/c^2 \)), which keeps the flow Keplerian until the inner region. However, in principle, it is possible to obtain the solution of the model equations for any value of \( x_c \) from \( x_{cc} \) to the marginally bound orbit \( x_b \).

Once \( x_c \) is fixed at \( x_{cc} \), we have to obtain the best value of \( \lambda_c \). By increasing the value of \( \lambda_c \) beyond a certain critical value \( \lambda_{cc} \) at a particular \( x_c \), we again find a transition from a saddle-type to an \( O \)-type critical point. Fig. A1(b) shows how the type of critical point and then the solution topology change with a slight increase of \( \lambda_c \). However, decreasing \( \lambda_c \) from \( \lambda_{cc} \) will tend to make the disc a more Bondi-type. Now, for \( \lambda_{cc} \) we again have to obtain a new value of \( x_{cc} \) following the procedure outlined above and thereafter corresponding \( \lambda_{cc} \). This needs to be continued iteratively until a specific combination of critical radius \( x_{cc} \) and corresponding specific angular momentum \( \lambda_{cc} \), lying in a narrow

**APPENDIX A: DISCUSSION OF BOUNDARY VALUES**

We have four coupled non-linear differential equations (6), (10), (12) and (17) to be solved for \( \vartheta, c_c, \lambda \) and \( T_s \); the equations also involve \( \vartheta \) and \( P \). To eliminate \( \varrho \) and \( P \), we use the mass transfer equation (1) and the equation of state (9). Therefore, in total we have five differential equations supplemented by an equation of state. Hence, we need five boundary conditions to start integration. Equation (1) can be integrated to obtain \( M \) already given in equation (2), which is supplied as an input parameter. Similarly, by integrating equation (10) we can obtain the angular momentum flux

\[ M(\lambda - \lambda_{\text{in}}) = -4\pi x^2 |W_{\phi}|, \]

for

\[ \vartheta, c_c, \lambda, T_s, \]

We therefore need the initial values of \( \vartheta, c_c, \lambda \) and \( T_s \) to solve the set of equations. When we impose the condition that the flow must pass through a critical radius \( x_c \) (around a sonic radius) where \( D = 0, \vartheta \) and \( c_c \) at \( x_c \) are related by a quadratic equation of Mach number given by equation (24).

For the continuity of \( d\vartheta/dx, N = 0 \) at \( x_c \). Therefore, from \( N = 0 \), which an algebraic non-linear equation, \( c_c \) at \( x_c \) can be computed iteratively (using the bisection method), which in turn also fixes \( \vartheta \) at \( x_c \), provided \( \lambda \) is known at that radius.

Now we need to set appropriate values of \( x_c \) and the corresponding specific angular momentum \( \lambda_c \). This is fixed iteratively by invoking the condition that the critical point is saddle-type. This can be seen as follows. First, we impose that \( \lambda_{\text{in}} = \lambda_c \). Then, by fixing the value of \( \lambda_c \) we find that if \( x_c \) is greater than a certain critical value \( x_{cc} \), then the type of \( x_c \) changes from saddle-type to \( O \)-type, through which matter never can pass. Fig. A1(a) shows how the type of critical point and then the solution topology change with a slight increase of \( x_c \). However, as \( x_c \) decreases from \( x_{cc} \), which corresponds to the energy at \( x_c (E_c) \) increases, the sub-Keplerian disc decreases in size. This advances the Keplerian disc. The reason is that increasing \( E_c \) corresponds to decreasing \( x_c \) and then increasing centrifugal energy (~\( x^2/c^2 \)), which keeps the flow Keplerian until the inner region. However, in principle, it is possible to obtain the solution of the model equations for any value of \( x_c \) from \( x_{cc} \) to the marginally bound orbit \( x_b \).

Once \( x_c \) is fixed at \( x_{cc} \), we have to obtain the best value of \( \lambda_c \). By increasing the value of \( \lambda_c \) beyond a certain critical value \( \lambda_{cc} \) at a particular \( x_c \), we again find a transition from a saddle-type to an \( O \)-type critical point. Fig. A1(b) shows how the type of critical point and then the solution topology change with a slight increase of \( \lambda_c \). However, decreasing \( \lambda_c \) from \( \lambda_{cc} \) will tend to make the disc a more Bondi-type. Now, for \( \lambda_{cc} \) we again have to obtain a new value of \( x_{cc} \) following the procedure outlined above and thereafter corresponding \( \lambda_{cc} \). This needs to be continued iteratively until a specific combination of critical radius \( x_{cc} \) and corresponding specific angular momentum \( \lambda_{cc} \), lying in a narrow

**Figure 19.** Comparison of solutions for corotating and counter-rotating stellar mass black holes. Variation of (a) velocity, (b) square of convective frequency, (c) cooling factor and (d) ion (upper set of lines) and electron (lower set of lines) temperatures, as functions of the radial coordinate. The solid lines correspond to \( b = 0.5 \) and the dashed lines correspond to \( b = -0.5 \). The other parameters are \( M = 100, M = 10 \) and \( \alpha = 0.01 \).
range, is obtained. This leads to a physically interesting large sub-Keplerian accretion disc, with matter infalling from the largest possible transition radius to a black hole event horizon through a saddle-type critical point. However, in principle, it is possible to obtain the solution of the model equations for a range of \( \lambda \) such that \( \lambda_{cc} \geq \lambda > 0 \).

There is, however, another initial value (i.e. \( T_e \) at \( x_c \) (\( T_{ec} \)) to be assigned. The choice of \( T_{ec} \) depends on the observed non-thermal radiation, which restricts the value of \( T_e \) in general. However, this restriction can only provide an order of magnitude of \( T_e \). An exact value of

Figure 20. Same as Fig. 19, except \( M = 0.01 \) and \( M = 10^7 \).

Figure 21. Variation of the effective optical depth as a function of the radial coordinate. The solid (\( a = 0 \)) and dotted (\( a = 0.998 \)) curves correspond to \( M = 10 \) and \( M = 100 \) and the dashed (\( a = 0 \)) and dot-dashed (\( a = 0.998 \)) curves correspond to \( M = 10^7 \) and \( M = 0.01 \) (see Tables 1 and 2 for details).
Table 3. Luminosity in erg s$^{-1}$.

| $M$   | $M$   | $\gamma$ | $L$   |
|-------|-------|-----------|-------|
| 0.0001| $10^7$| 1.6       | $10^{34}$|
| 0.01  | $10^7$| 1.5       | $10^{38}$|
| 1     | $10^7$| 1.35      | $5 \times 10^{42}$|
| 100   | $10^7$| 1.34      | $10^{47}$|
| 0.01  | 1     | 1.5       | $10^{33}$|
| 1     | 1     | 1.35      | $7 \times 10^{36}$|
| 100   | 1     | 1.34      | $10^{40}$|

Figure A1. Comparison of the variation of radial velocity as a function of the radial coordinate (a) between solutions with $x_c = 5.5$ (dotted curve) and $x_c = 5.7$ (solid curve), when $\lambda_c = 3.2$, and (b) between solutions with $\lambda_c = 3.2$ (dotted curve) and $\lambda_c = 3.3$ (solid curve), when $x_c = 5.5$. The other parameters are the same as in Fig. 2 for $M = 0.01$.

$T_{ec}$ should be obtained iteratively from a plausible range of $T_e$ at $x_c$ so that the values of $x_{cc}$ and $\lambda_{cc}$ (obtained following the above-mentioned procedure) converge.

**APPENDIX B: COMPUTATION OF THE DERIVATIVE OF VELOCITY AT THE CRITICAL RADIUS**

First, we recall the derivative of velocity from equation (20)

$$\frac{d\vartheta}{dx} = \frac{N(x, \vartheta, c_s, \lambda, T_e)}{D(\vartheta, c_s)}.$$  \hspace{1cm} (B1)

At the critical radius

$$\frac{d\vartheta}{dx} = 0.$$ \hspace{1cm} (B2)

Therefore, we apply the l’Hospital rule and obtain

$$\frac{d\vartheta}{dx} = \left(\frac{D/Dx}[N(x, \vartheta, c_s, \lambda, T_e)]\right)^{-1}.$$ \hspace{1cm} (B3)

Now, combining with equations (26)–(28), we obtain

$$\frac{d\vartheta}{dx} = \frac{N_1 + N_2(d\vartheta/dx)}{D_1 + D_2(d\vartheta/dx)},$$ \hspace{1cm} (B4)

where

$$N_1 = \frac{dN}{dx} + \frac{dN}{dx} \frac{J}{d\vartheta} c_s + \frac{dN}{dx} \frac{J}{d\vartheta} c_s + \frac{dN}{dx} \frac{J}{d\vartheta} c_s$$

and

$$N_2 = \frac{dN}{dx} \left(\frac{c_s^2 - 2\pi a J}{c_s} + \vartheta\right) + \frac{dN}{dx} \frac{J}{d\vartheta} \left(\frac{(\Gamma_1 - 1)4\pi c_s x^{3/2} (Q^{ec} - Q^\nu)}{M} F^{1/2} (1 - \Gamma_1) T_e \left(J \frac{c_s^2}{r_n^2} + G\right)\right).$$ \hspace{1cm} (B5)
Finally, cross-multiplying in equation (B4) we obtain a quadratic equation

\[ D_2 \left( \frac{d\vartheta}{dx} \right)^2 + (D_1 - N_2) \left( \frac{d\vartheta}{dx} \right) - N_1 = 0, \]  

such that

\[ \frac{d\vartheta}{dx} \bigg|_{c_s} = \frac{N_2 - D_1 \pm \sqrt{(D_1 - N_2)^2 + 4D_2N_1}}{2D_2}, \]  

where the upper and lower signs before square root correspond to wind and accretion, respectively.