Two-neutrino Double-Beta Decay within the Realistic Shell Model

L Coraggio¹, L De Angelis¹, T Fukui¹, A Gargano¹, and N Itaco¹,²
¹Istituto Nazionale di Fisica Nucleare, Complesso Universitario di Monte S. Angelo, Via Cintia - I-80126 Napoli, Italy
²Dipartimento di Matematica e Fisica, Università degli Studi della Campania “Luigi Vanvitelli”, viale Abramo Lincoln 5 - I-81100 Caserta, Italy

Abstract. We report on the calculation of double-β decay properties of $^{130}$Te and $^{136}$Xe within the framework of the realistic shell model. The effective shell-model Hamiltonian and Gamow-Teller transition operator are derived by way of many-body perturbation theory, in order to not to resort to an empirical quenching of the axial coupling constant $g_A$. The results compare well with experimental data, paving the way to the calculation of the neutrinoless double-β decay nuclear matrix element for the nuclei that are currently the experimental target for the detection of this process. In this work we investigate about the perturbative properties of our approach, and the theoretical reliability of the realistic shell model.

1. Introduction
One of the crossovers of current interest between nuclear structure and the physics “beyond the Standard Model” is the calculation of the nuclear matrix element (NME) of the neutrinoless double-β (0νββ) decay, that is directly related to the half life of such a process in the following form:

$$\left[ T_{1/2}^{0\nu} \right]^{-1} = G_{0\nu} \left| M_{0\nu} \right|^2 \langle m_{\nu} \rangle^2,$$

where $G_{0\nu}$ is the so-called phase-space factor (or kinematic factor), $\langle m_{\nu} \rangle$ is the effective neutrino mass that takes into account the neutrino parameters associated with the mechanisms of light- and heavy-neutrino exchange, and $M_{0\nu}$ is the NME of the decay connecting the wave functions of the parent and grand-daughter nuclei.

The detection of 0νββ decay is nowadays the target of experiments in many laboratories, since its observation would correspond to a lepton number violation and provide direct information on the nature and properties of the neutrino [1, 2].

From Eq. 1, it is very clear that a crucial role is played by a reliable estimate of the NME in order to assess the necessary sensitivity of a detecting device and link the experimental results to a measurement of $\langle |m_{\nu}| \rangle$.

Currently, calculations of the NME are carried out by resorting to different nuclear structure models, such as the Interacting Boson Model (IBM) [3–5], the Quasiparticle Random-Phase Approximation (QRPA) [6–9], Energy Density Functional methods [10], and the Shell Model (SM). [11–17]. In the present work we report about calculations performed within the realistic
shell model, where all the parameters appearing in the SM Hamiltonian, namely the single-particle (SP) energies and the two-body matrix elements of the residual interaction (TBME), and in the transition operators are derived from a realistic free nucleon-nucleon (NN) potential $V_{NN}$.

As a matter of fact, effective operators derived by way of perturbation many-body theory take into account the truncation of the full Hilbert space into the reduced SM model space, and calculations may be performed without resorting to any empirical parameter. In other words, in our approach we do not employ, for example, effective charges to calculate electromagnetic transition strengths, and we do not quench empirically the axial and vector current coupling constants. As a first step, we check this approach to the calculation of the $0\nu\beta\beta$ NME by calculating the two-neutrino double-beta ($2\nu\beta\beta$) decay NME and Gamow-Teller (GT) strengths of nuclei involved in possible $0\nu\beta\beta$ decays, and comparing the obtained results with the experimental counterparts. In Ref. [18] we have reported the results of our realistic SM calculation for the GT strengths and $2\nu\beta\beta$-decay NME of $^{130}$Te and $^{136}$Xe, that compare well with the available data.

In this paper, we will present a study of the convergence properties of the perturbative expansion of the effective SM Hamiltonian and operators we have employed in Ref. [18], derived within a model space spanned by the five proton and neutron orbitals $0g_{7/2}, 1d_{5/2}, 1d_{3/2}, 2s_{1/2}, 0h_{11/2}$ outside the doubly-closed $^{100}$Sn, starting from the high-precision NN potential CD-Bonn [19].

In the next section we sketch out the derivation of our SM Hamiltonian and of the effective transition and decay operators. In Section 3, a detailed discussion about the perturbative expansion of the products of the effective SM Hamiltonian $H_{\text{eff}}$, namely SP energies and TBME, and of the effective Gamow-Teller operator $GT_{\text{eff}}$. In the last section some considerations and conclusions about the outcome of our study are are drawn.

2. Outline of calculations

We derive the SM effective Hamiltonian within the framework of the time-dependent degenerate linked-diagram expansion [20].

To this end, an auxiliary one-body potential $U$ is introduced to write down the many-body Hamiltonian as the sum of an unperturbed term $H_0$, which describes the independent motion of the nucleons, and a residual interaction $H_1$:

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2m} + \sum_{i<j} V^{ij}_{NN} = T + V_{NN} = (T + U) + (V_{NN} - U) = H_0 + H_1,$$

where $V_{NN}$ represents the input NN potential.

The effective Hamiltonian $H_{\text{eff}}$ is obtained by way of the Kuo-Lee-Ratcliff folded-diagram expansion in terms of the vertex function $\hat{Q}$-box, which in the perturbative approach is built by a collection of irreducible valence-linked diagrams [21]. We calculate the $\hat{Q}$-box including one- and two-body Goldstone diagrams through third order in $H_1$ [22]. Calculations beyond the third order in perturbation theory are computationally prohibitive, so we have calculated also the Padé approximant [2/1] [23] of the $\hat{Q}$-box in order to obtain the convergence value of the perturbation series, as suggested in [24]. The folded-diagram series is then summed up to all orders using the Lee-Suzuki iteration method [25].

The $\hat{Q}$-box contains one-body contributions, the collection of which is the so-called $\hat{S}$-box [26]. The folded-diagram expansion of the $S$-box represents the theoretical SP energies, that will be employed in the SM calculations. The TBME will be then obtained by way of a subtraction procedure of these SP energies from $H_{\text{eff}}$. 
We employ as $V_{NN}$ a low-momentum potential $V_{\text{low} - k}$ defined within a cutoff momentum $\Lambda$ by way of a similarity transformation [27]. This is a smooth potential which preserves exactly the on-shell properties of the original $V_{NN}$ and is suitable for being used directly in nuclear structure calculations [28]. This has been done starting from the high-precision CD-Bonn $NN$ potential [19], and then deriving a $V_{\text{low} - k}$ corresponding to $\Lambda = 2.6 \text{ fm}^{-1}$.

Employing this $V_{\text{low} - k}$, we have calculated effective decay operators, namely the matrix elements of the effective GT operator, consistently with SM $H_{\text{eff}}$ up to third-order in perturbation theory, within an approach that is strictly based on the one presented by Suzuki and Okamoto in Ref. [29].

In the following Section, we will present the results of our study about the convergence properties of theoretical SP energy spectra, TBME, and $2\nu\beta\beta$ NMEs as a function both of the dimension of the intermediate-state space and of the order of the perturbative expansion in order to assess the reliability of our realistic SM calculation.

3. Results

In Fig. 1, we report the single-neutron energy spectrum relative to the $0g_{7/2}$ orbit, as a function of the maximum allowed excitation energy of the intermediate states expressed in terms of the oscillator quanta $N_{\text{max}}$. From the inspection of Fig. 1, it is clear that our results have practically achieved convergence at $N_{\text{max}} = 14$.

![Figure 1. Neutron SP energies as a function of $N_{\text{max}}$.](image1)

![Figure 2. Same as in Fig. 1 as a function of the perturbative order.](image2)

In Fig. 2, the calculated neutron SP energies, using a number of intermediate states corresponding to $N_{\text{max}} = 16$, are reported as a function of the order of the perturbative expansion up to the perturbative order $n = 3$, and compared with the Padé approximant [21] of the $\hat{Q}$-box. It can be seen that the results at third order are very close to those obtained with the Padé approximant, indicating that the truncation at third order may provide a good estimate of the sum of the series.

As regards the TBME, we report in Figs. 3 and 4 the neutron-neutron diagonal $J^\pi = 0^+$ TBME as a function both of $N_{\text{max}}$ and the perturbative order $n$. We have chosen these TBMEs - that contain the pairing properties of our effective Hamiltonian - to be shown, since they are
the largest in size of our calculated matrix elements, and the most sensitive to the behavior of the perturbative expansion.

From the inspection of Fig. 3 we see that the convergence with respect to $N_{\text{max}}$ is very fast for diagonal matrix elements $(1d_{5/2})^2$, $(1d_{3/2})^2$, and $(2s_{1/2})^2$, while those corresponding to orbitals lacking of their own spin-orbit partner, $(0g_{7/2})^2$ and $(0h_{11/2})^2$, show a slower convergence.

The order-by-order convergence presented in Fig. 4 is satisfactory and as for the SP energies the results at third order are very close to those obtained with the Padé approximant, we may therefore conclude that our $H_{\text{eff}}$, obtained from a $V_{\text{low}}−k$ with a cutoff equal to 2.6 fm$^{-1}$, is a good estimate of the sum of its perturbative expansion.

Table 1. Order-by-order $2\nu\beta\beta$ NME (in MeV$^{-1}$) for $^{130}\text{Te}$ and $^{136}\text{Xe}$.

| Decay       | 1st ord $M_{2\nu}^{GT}$ | 2nd ord $M_{2\nu}^{GT}$ | 3rd ord $M_{2\nu}^{GT}$ | Expt.       |
|-------------|-------------------------|-------------------------|-------------------------|-------------|
| $^{130}\text{Te} \rightarrow ^{130}\text{Xe}$ | 0.085                   | 0.034                   | 0.044                   | 0.034 ± 0.003 |
| $^{136}\text{Xe} \rightarrow ^{136}\text{Ba}$ | 0.0564                  | 0.0239                  | 0.0285                  | 0.0218 ± 0.0003 |

In order to focus on the convergence of the GT$_{\text{eff}}$ matrix elements, in Table 2 we have reported also the results obtained with first-, second-, and third-order matrix elements, but employing always the same $H_{\text{eff}}$, namely the one calculated at third order in the perturbative expansion. As can be seen, the results change for both transitions about 260% from the first- to second-order
calculations, while the change is 9% and 5% from the second- to third-order results for $^{130}$Te and $^{136}$Xe decays, respectively.

**Table 2.** Same as in Table 1, but all calculated with the same third-order $H_{\text{eff}}$.

| Decay     | 1st ord $M_{2\nu}^{\text{GT}}$ | 2nd ord $M_{2\nu}^{\text{GT}}$ | 3rd ord $M_{2\nu}^{\text{GT}}$ | Expt.          |
|-----------|---------------------------------|---------------------------------|---------------------------------|----------------|
| $^{130}$Te $\rightarrow^{130}$Xe | 0.142                           | 0.040                           | 0.044                           | 0.034 $\pm$ 0.003 |
| $^{136}$Xe $\rightarrow^{136}$Ba | 0.0975                          | 0.0272                          | 0.0285                          | 0.0218 $\pm$ 0.0003 |

4. Conclusions
In this paper we have presented the outcome of a study of the convergence properties of the effective Hamiltonian and GT operator, derived starting from a $V_\text{low-k}$, that aims to assess the reliability of the calculation of the 2$\nu\beta\beta$ NME for $^{130}$Te and $^{136}$Xe decays within the framework of the realistic shell model. We have calculated the dependence of SP energies, TBME, and 2$\nu\beta\beta$ NME with respect to both the size of the intermediate-state space and the order by order convergence, up to the third order of the perturbative expansion. Our results show a rather good behavior of our perturbative expansion, besides a good agreement with the available data, providing support for our nearby goal, that will be the calculation of 0$\nu\beta\beta$ NME for $^{130}$Te and $^{136}$Xe by way of the realistic shell model.

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