$K$-theory of Hermitian Mackey functors and a reformulation of
the Novikov Conjecture

Emanuele Dotto
Dept. of Mathematics
University of Bonn

Crichton Ogle
Dept. of Mathematics
The Ohio State University

Abstract

From a genuine $\mathbb{Z}/2$-equivariant spectrum $A$ equipped with a compatible multiplicative structure we produce a genuine $\mathbb{Z}/2$-equivariant spectrum $KR(A)$. This construction extends the real $K$-theory framework of Hesselholt-Madsen for discrete rings and the Hermitian $K$-theory framework of Burghelea-Fiedorowicz for simplicial rings. We construct a natural trace map of $\mathbb{Z}/2$-spectra $\text{tr} : KR(A) \to \text{THR}(A)$ to the real topological Hochschild homology spectrum, which extends the $K$-theoretic trace of Bökstedt-Hsiang-Madsen.

We use the trace to show that a certain lift of the rational assembly map in $L$-theory to the rational Hermitian $K$-theory of the Burnside Mackey-functor is split injective. This allows us to reformulate the Novikov conjecture in terms of the vanishing of the trace on a summand of the Hermitian $K$-theory of the Mackey functor of components of the spherical group-ring.

Contents

1 Hermitian Mackey functors and their $K$-theory
1.1 Hermitian Mackey functors ........................................ 4
1.2 The Hermitian $K$-theory of a Hermitian Mackey functor .......... 10
1.3 The assembly map for the Burnside group-ring ................... 12
1.4 The Hermitian cyclic $K$-theory of a Hermitian Mackey-functor .... 13

2 Application to the Novikov conjecture .............................. 16
2.1 Splitting the restricted assembly map ............................ 16
2.2 Reformulation of the Novikov conjecture .......................... 17

3 Real $K$-theory .......................................................... 20
3.1 The real and dihedral Bar constructions .......................... 20
3.2 Ring-spectra with anti-involution .................................. 23
3.3 The real $K$-theory $\mathbb{Z}/2$-space of a ring spectrum with anti-involution .... 25
3.4 Connective equivariant deloopings of real $K$-theory .............. 27
3.5 Hermitian $K$-theory of ring-spectra with anti-involution ....... 30

4 The real trace map ....................................................... 32
4.1 Real topological Hochschild homology ............................ 32
4.2 The definition of the real trace map ................................ 34
4.3 The splitting of the restricted assembly map of the Burnside group-ring ... 36
Introduction

In [HM17] Hesselholt and Madsen construct a \( \mathbb{Z}/2 \)-equivariant spectrum \( \text{KR}(C) \) from an exact category with duality \( C \), whose underlying spectrum is the \( K \)-theory spectrum \( K(C) \) of [Qui73] and whose fixed-points spectrum is the connective Hermitian \( K \)-theory spectrum \( \text{KH}(C) \) of [Sch10]. Specified to the category of free modules over a discrete ring with anti-involution \( A \) this construction provides a \( \mathbb{Z}/2 \)-equivariant spectrum \( \text{KR}(A) \) whose fixed-points are the connective Hermitian \( K \)-theory \( \text{KH}(A) \) of [Kar73]. The construction of \( \text{KH}(A) \) is extended in [BF84] from discrete rings to simplicial rings, and the homotopy type of \( \text{KH}(A) \) depends both on the homotopy type of \( A \) and of the fixed-points space \( A/\mathbb{Z}/2 \).

In this paper we propose a further extension of the real \( K \)-theory functor \( \text{KR} \) to the category of ring-spectra with anti-involution. These are genuine \( \mathbb{Z}/2 \)-equivariant spectra with a suitably compatible multiplication (see [32], and the output \( \text{KR}(A) \) is a genuine \( \mathbb{Z}/2 \)-equivariant spectrum. The fixed-points spectrum \( \text{KH}(A) := \text{KR}(A)^{\mathbb{Z}/2} \) depends on the equivariant homotopy type of \( A \) and differs form other constructions in the literature, for example from the one of [Spi16].

There is an algebraic case of particular interest that lies between discrete rings and ring spectra, namely those ring spectra with anti-involution whose underlying \( \mathbb{Z}/2 \)-spectrum is the Eilenberg-MacLane spectrum \( HM \) of a Mackey functor \( M \). In this case the multiplicative structure on \( HM \) specifies to a multiplication on the underlying Abelian group \( \pi_0HM \) together with a multiplicative action of \( \pi_0HM \) on \( \pi_0(HM)^{\mathbb{Z}/2} \) suitably compatible with the restriction and the transfer maps. We call such an object a Hermitian Mackey functor (see definition [11]). A class of examples comes from Tambara functors, where the underlying ring acts on the fixed-points datum via the multiplicative transfer (see example [12]). We start our paper by constructing the Hermitian \( K \)-theory of a Hermitian Mackey functor in [11] as the group completion of a certain symmetric monoidal category of Hermitian forms \( \text{Herm}_M \) over \( M \)

\[
\text{KH}(M) := \Omega B \text{Bi Herm}_M, \oplus.
\]

The key idea for the definition of \( \text{Herm}_M \) is that the fixed-points datum of the Mackey functor specifies a refinement of the notion of “symmetry” used in the classical definition of Hermitian forms over a ring. In [3] we extend these ideas to ring spectra and we give the full construction of the KR functor.

The main feature of our real \( K \)-theory construction is that it comes equipped with a natural trace map to the real topological Hochschild homology spectrum \( \text{THH}(A) \) defined in [HM17] (see also [Dot12], [Hg16] and [DMPPR17]). The following is in [12]

**Theorem.** Let \( A \) be a ring spectrum with anti-involution (see [32]). There is a natural transformation of \( \mathbb{Z}/2 \)-spectra

\[
\text{tr} : \text{KR}(A) \to \text{THH}(A)
\]

whose underlying map of spectra is Bökstedt-Hsiang-Madsen’s trace map \( \text{K}(A) \to \text{THH}(A) \) from [BHM93].

In the case of discrete rings this map provides a map of spectra \( \text{KH}(A) \to \text{THH}(A)^{\mathbb{Z}/2} \) which is a refinement of earlier constructions of the Chern Character from Hermitian \( K \)-theory to dihedral homology appearing in [Cn93].

As an application of our constructions we reformulate the Novikov conjecture in terms of the vanishing of a map closely related to the trace on a summand of the Hermitian \( K \)-theory of the “\( \mathbb{Z}/2 \)-Burnside group-ring”. The Novikov conjecture of [Nov68] is equivalent to injectivity of \( L \)-theoretic assembly map

\[
W(\mathbb{Z}) \wedge B\pi_+ \to W(\mathbb{Z}[\pi])
\]

on rational homotopy groups (see e.g. [KL05]). Burghelea an Fiedorowicz show in [BFS4] that there is a rational decomposition

\[
\text{KH}_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong (W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \oplus (\text{K}_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{\mathbb{Z}/2}
\]
where $K_*(\mathbb{Z}[\pi])$ are the algebraic $K$-theory groups, and the fixed-points are taken with respect to the action induced by the anti-involution of $\mathbb{Z}[\pi]$ defined by inversion in $\pi$. Thus the trace map induces a map on the rational homotopy groups of the fixed-points spectra

$$tr: (W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \oplus (K_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{Z/2} \longrightarrow \text{THR}^{Z/2}(\mathbb{Z}[\pi]) \otimes \mathbb{Q},$$

that we can try to exploit to detect the assembly map. Schlichting’s theorem [Sch17 7.6] combined with the main result of [BF84] identifies, away from 2, the connective that we can try to exploit to detect the assembly map. Schlichting’s theorem [Sch17 7.6] combined with the main result of [BF84] identifies, away from 2, the connective $L$-theory spectrum with the geometric fixed-points spectrum $\Phi^{Z/2}KR(\mathbb{Z}[\pi])$, and the splitting above occurs as the splitting

$$KR_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong (\Phi^{Z/2}KR_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \oplus (K_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{Z/2}$$

of the rational isotropy separation sequence. Thus the restriction of the trace to the $L$-theory summand maps to the geometric fixed-points of THR

$$tr: W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \longrightarrow \Phi^{Z/2}(\text{THR}(\mathbb{Z}[\pi])) \otimes \mathbb{Q}.$$

The rational geometric fixed points $\Phi^{Z/2} \text{THR}(A) \otimes \mathbb{Q}$ have however been computed to be contractible in [DMPPR17], for every discrete ring $A$. This is in line with the construction of [Ch33], where the Chern Character to dihedral homology factors through algebraic $K$-theory via the forgetful map, and cannot be used to detect the image of the rationalized assembly map for the Witt theory summand of rational Hermitian $K$-theory.

The rational geometric fixed-points $\Phi^{Z/2}(\text{THR}(HM)) \otimes \mathbb{Q}$ are however generally non-trivial when the input is a Hermitian Mackey functor that does not come from a ring with anti-involution. The starting point of our analysis is to replace the integers with the Burnside Mackey functor, much in the same way one replaces the integers with the sphere spectrum in the proof of the $K$-theoretical Novikov conjecture of [BHM89]. However, since the integers are not rationally equivalent to the Burnside Mackey functor our result will depend on the difference between the corresponding Hermitian $K$-theories.

We define a Hermitian Mackey functor $\mathcal{B}[\pi] := \mathcal{B}(\mathbb{Z} \wedge \pi_+)$, the “Burnside group-ring” of a discrete group $\pi$ (see 3.7). There is a restriction map $d: \mathcal{B}[\pi] \to \mathbb{Z}[\pi]$ coming from the Hurewicz map, where $\mathbb{Z}[\pi]$ is the Mackey functor associated to the integral group-ring $\mathbb{Z}[\pi]$. The trace map is constructed as a composition

$$tr: KH(A) \xrightarrow{s_1} K^{cy}H(A) \xrightarrow{tr^{cy}} \text{THR}(A)^{Z/2}$$

where $K^{cy}H(A)$ is a Hermitian version of cyclic $K$-theory, and $s_1$ is a split inclusion (see 3.7). We let $T^{cy}$ be the map

$$T^{cy}: K^{cy}H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} \xrightarrow{tr^{cy}} \text{THR}_*(\mathcal{B}[\pi])^{Z/2} \otimes \mathbb{Q} \cong H_*(B^{cy} \pi \Pi (B^{di} \pi)^{Z/2}; \mathbb{Q}) \xrightarrow{p^{Z/2}} H_*(B\pi; \mathbb{Q}),$$

where $B^{cy} \pi$ and $B^{di} \pi$ are respectively the cyclic and dihedral nerve of $\pi$ (see 3.7), and $p: B^{di} \pi \to B\pi$ is the canonical projection map. The calculation of the rational homotopy groups of $\text{THR}(\mathcal{B}[\pi])^{Z/2}$ follows from a theorem of [Hg16]. The following is proved in 2.4.

**Theorem.** Let $\pi$ be a discrete group. The Novikov conjecture holds for $\pi$ if and only if the map $T^{cy}$ is zero on the kernel of $d^{cy}: K^{cy}H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} \to K^{cy}H_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.

The proof of the theorem is based on the construction of a lift of the restricted assembly map

$$A^0: H_*(B\pi; \mathbb{Q}) \cong W_0(\mathbb{Z}) \otimes H_*(B\pi; \mathbb{Q}) \to W_*(\mathbb{Z}) \otimes H_*(B\pi; \mathbb{Q}) \to W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$
to $K^{cy}H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q}$, which is split by $T^{cy}$. That is, in 3.4 we construct a commutative diagram
where \( i \) is the summand inclusion. We then use Bott periodicity and the naturality of \( T^{cyc} \) to access the full assembly map in \([2,3]\) and \([4,4]\).

We are of course left with the problem of calculating the kernel of \( d^{cyc} \). In \([2,3]\) we prove in fact a stronger statement. There are two maps of interest

\[
\sigma_1, \sigma_\sigma : \text{KH}(B[\pi]) \to K^{cyc}H(B[\pi]).
\]

The map \( \sigma_1 \) is constructed as the canonical splitting of a projection \( B^{di}G \to BG \), and \( \sigma_\sigma \) is a certain twisting of \( \sigma_1 \) by the element \( \sigma \) of the Burnside ring represented by the virtual \( \mathbb{Z}/2 \)-set \( \mathbb{Z}/2 - 1 \) (see \([4,4]\)). We show that our lift \( \overline{\sigma} \) lifts further to \( (\text{KH}(B[\pi]) \oplus \text{KH}(B[\pi])) \otimes \mathbb{Q} \) along \( \sigma_1 + \sigma_\sigma \). The advantage of working with two summands \( \text{KH}(B[\pi]) \otimes \mathbb{Q} \) instead of \( K^{cyc}H(B[\pi]) \otimes \mathbb{Q} \) is that the trace map

\[
\text{tr} : \text{KH}(B[\pi]) \xrightarrow{\sigma_1} K^{cyc}H(B[\pi]) \xrightarrow{\text{tr}^{cyc}} \text{THR}(B[\pi])^{\mathbb{Z}/2}
\]

lifts to the fixed-points of the real topological cyclic homology spectrum \( TCR(B[\pi]) \) of \([Hg16]\) along the canonical map \( TCR(B[\pi])^{\mathbb{Z}/2} \to \text{THR}(B[\pi])^{\mathbb{Z}/2} \). If one can lift also the other summand

\[
\text{tr}_\sigma : \text{KH}(B[\pi]) \xrightarrow{\sigma_\sigma} K^{cyc}H(B[\pi]) \xrightarrow{\text{tr}^{cyc}} \text{THR}(B[\pi])^{\mathbb{Z}/2}
\]

to the fixed-points of \( TCR(B[\pi])^{\mathbb{Z}/2} \), the Novikov conjecture would hold if the composite

\[
TCR_{cyc}^{\mathbb{Z}/2}(B[\pi]) \otimes \mathbb{Q} \to \text{THR}_{cyc}^{\mathbb{Z}/2}(B[\pi])^{\mathbb{Z}/2} \otimes \mathbb{Q} \xrightarrow{p \cdot \mathbb{Z}/2} H_*(B\pi; \mathbb{Q})
\]

is zero on the kernel of \( d : TCR^{\mathbb{Z}/2}(B[\pi]) \otimes \mathbb{Q} \to TCR^{\mathbb{Z}/2}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \). We believe that this kernel is accessible by computations.

One can in fact reduce even further. The homotopy groups constructed above are the homotopy groups of fixed-points spectra. Thus rationally they split off a copy of the rational homotopy groups of the geometric fixed-points spectra. All the maps constructed are induced by maps of \( \mathbb{Z}/2 \)-spectra and therefore they preserve these splittings. Thus one would only need to calculate the kernel of the map

\[
d : \pi_* \Phi^{\mathbb{Z}/2} TCR(B[\pi]) \otimes \mathbb{Q} \to \pi_* \Phi^{\mathbb{Z}/2} TCR(\mathbb{Z}[\pi]) \otimes \mathbb{Q}.
\]

on geometric fixed-points. This will be addressed in future work.

A brief outline of the paper follows. In \([1]\) we define Hermitian Mackey functors, we construct some examples, and we define their Hermitian \( K \)-theory and Hermitian cyclic \( K \)-theory. In \([2]\) we prove the reformulation of the Novikov conjecture, except for the definition of the map \( T^{cyc} \) which needs the trace and is postponed to \([1]\). In \([3]\) we construct the real \( K \)-theory of ring spectra with anti-involution, and prove that its fixed-points recovers the connective Hermitian \( K \)-theory of simplicial rings of \([BFS4]\), and the Hermitian \( K \)-theory of Hermitian Mackey functors of \([2]\). Finally \([4]\) contains the construction of the real trace map, and the construction of the splitting of the lifted assembly map from \([2]\).

### Acknowledgments

We would like to sincerely thank Ib Madsen and Lars Hesselholt for sharing so much of their current work on real \( K \)-theory with the first author, and for the guidance offered over several years. We thank Irakli Patchkoria for pointing out a missing condition in definition \([1]\). We also thank Markus Land, Wolfgang Lück, Kristian Moi, Thomas Nikolaus, Irakli Patchkoria, Marco Schlichting and Stefan Schwede for many valuable conversations.

### 1 Hermitian Mackey functors and their \( K \)-theory

#### 1.1 Hermitian Mackey functors

The standard input of Hermitian \( K \)-theory is a ring \( A \) with an anti-involution \( w : A^{\text{op}} \to A \), or in other words an Abelian group \( A \) with a \( \mathbb{Z}/2 \)-action and a ring structure

\[
A \otimes_{\mathbb{Z}/2} A \to A
\]
which is equivariant with respect to the \( \mathbb{Z}/2 \)-action on the tensor product that swaps the two factors. In equivariant homotopy theory Abelian groups with \( \mathbb{Z}/2 \)-actions are replaced by the more general notion of \( \mathbb{Z}/2 \)-Mackey functors. In what follows, we will define a suitable multiplicative structure on a Mackey functor which extends the notion of ring with anti-involution.

We recall that a \( \mathbb{Z}/2 \)-Mackey functor \( M \) consists of two Abelian groups \( M(\mathbb{Z}/2) \) and \( M(*) \), a \( \mathbb{Z}/2 \)-action \( w \) on \( M(\mathbb{Z}/2) \), and \( \mathbb{Z}/2 \)-equivariant maps (with respect to the trivial action on \( M(*) \))

\[
R: M(*) \to M(\mathbb{Z}/2) \quad T: M(\mathbb{Z}/2) \to M(*)
\]
called respectively the restriction and the transfer, subject to the relation

\[
RT(a) = a + w(a)
\]
for every \( a \in M(\mathbb{Z}/2) \).

**Definition 1.1.** A Hermitian Mackey functor is a \( \mathbb{Z}/2 \)-Mackey functor \( M \), together with a multiplication on \( M(\mathbb{Z}/2) \) and a multiplicative left action of \( M(\mathbb{Z}/2) \) on \( M(*) \) which satisfy the following conditions:

i) \( w(aa') = w(a')w(a) \) for all \( a, a' \in M(\mathbb{Z}/2) \),

ii) \( R(a \cdot b) = aR(b)w(a) \) for all \( a \in M(\mathbb{Z}/2) \) and \( b \in M(*) \),

iii) \( a \cdot T(c) = T(acw(a)) \) for all \( a, c \in M(\mathbb{Z}/2) \),

iv) \( (a + a') \cdot b = a \cdot b + a' \cdot b + T(aR(b)w(a')) \) for all \( a, a' \in M(\mathbb{Z}/2) \) and \( b \in M(*) \).

**Example 1.2.** Let \( A \) be a ring with anti-involution \( w: A^{op} \to A \). The Mackey functor \( M_A \) associated to \( A \) has values \( M_A(\mathbb{Z}/2) = A \) and \( M_A(*) = A^{\mathbb{Z}/2} \), the Abelian subgroup of fixed-points. The restriction is the inclusion of fixed-points \( R: A^{\mathbb{Z}/2} \to A \), and the transfer is \( T(a) = a + w(a) \). The multiplication on \( A \) defines an action of \( A \) on \( A^{\mathbb{Z}/2} \) by

\[
a \cdot b = abw(a)
\]
for \( a \in A \) and \( b \in A^{\mathbb{Z}/2} \). This gives \( M_A \) the structure of a Hermitian Mackey functor.

**Example 1.3.** Let \( B \) be the \( \mathbb{Z}/2 \)-Burnside Mackey functor. The Abelian group \( B(\mathbb{Z}/2) \) is the group completion of the monoid of isomorphism classes of finite sets, and it has the trivial involution. The Abelian group \( B(*) \) is the group completion of the monoid of isomorphism classes of finite \( \mathbb{Z}/2 \)-sets. The restriction forgets the \( \mathbb{Z}/2 \)-action, and the transfer sends a set \( A \) to the free \( \mathbb{Z}/2 \)-set \( A \times \mathbb{Z}/2 \). The underlying Abelian group \( B(\mathbb{Z}/2) \) has a multiplication induced by the cartesian product, and it acts on \( B(*) \) by

\[
A \cdot B = \left( \coprod_{\mathbb{Z}/2} A \right) \times B.
\]

Explicitly, \( B(\mathbb{Z}/2) \) is isomorphic to \( \mathbb{Z} \) as a ring, \( B(*) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) with generators the trivial \( \mathbb{Z}/2 \)-set with one element and the free \( \mathbb{Z}/2 \)-set \( \mathbb{Z}/2 \). The restriction is the identity on the first summand and multiplication by 2 on the second summand, and the transfer sends the generator of \( \mathbb{Z} \) to the generator of the second \( \mathbb{Z} \)-summand. The underlying ring \( \mathbb{Z} \) then acts on \( \mathbb{Z} \oplus \mathbb{Z} \) by

\[
a \cdot (b, c) = (ab, ba(a - 1)/2 + a^2c).
\]

The Hermitian structure on the \( \mathbb{Z}/2 \)-Burnside Mackey functor is a special case of the following construction. If the multiplication of a ring \( A \) is commutative, then an anti-involution on \( A \) is simply an action of \( \mathbb{Z}/2 \) by ring maps. The Mackey-version of a commutative ring is a Tambara functor, and we show that there is indeed a forgetful functor from \( \mathbb{Z}/2 \)-Tambara functor to Hermitian Mackey functors. We recall that a \( \mathbb{Z}/2 \)-Tambara functor is a Mackey functor where both \( M(\mathbb{Z}/2) \) and \( M(*) \) are commutative rings, and with an extra equivariant multiplicative transfer \( N: M(\mathbb{Z}/2) \to M(*) \), called the norm, which satisfies the properties
Lemma 1.6. The anti-involution on \( M \) restricted to \( M^* \) follows by matrix transposition. The restriction of an element \( a \) of \( M \) because the multiplication is commutative and equivariant. The second axiom is followed by matrix transposition. The restriction of an element \( a \) of \( M \) because the multiplication is commutative and equivariant. The second axiom is

\[ R(a \cdot b) = R(N(a)b) = R(N(a))R(b) = aR(b)w(a) \]

and the third is

\[ a \cdot T(c) = N(a)T(c) = T(R(N(a)))c = T(aw(a)c) = T(acw(a)). \]

Let us verify the axioms of a Hermitian Mackey functor. The first axiom is satisfied because the multiplication is commutative and equivariant. The second axiom is

\[ R(a \cdot b) = R(N(a)b) = R(N(a))R(b) = aR(b)w(a) \]

and the third is

\[ a \cdot T(c) = N(a)T(c) = T(R(N(a)))c = T(aw(a)c) = T(acw(a)). \]

The last axiom is clear from the third condition of a Tambara functor.

Definition 1.5. Let \( M \) be a Hermitian Mackey functor. The Hermitian Mackey functor \( M_n(M) \) of \( n \times n \)-matrices in \( M \) is defined by the Abelian groups

\[ M_n(M)(\mathbb{Z}/2) = M_n(M(\mathbb{Z}/2)) \quad M_n(M)(*) = ( \bigoplus_{1 \leq i < j \leq n} M(\mathbb{Z}/2)) \oplus ( \bigoplus_{1 \leq i = j < n} M(*)). \]

The anti-involution on \( M_n(M)(\mathbb{Z}/2) \) is the anti-involution of \( M(\mathbb{Z}/2) \) applied entrywise followed by matrix transposition. The restriction of an element \( B \) of \( M(*) \) has entries

\[ R(B)_{ij} = \begin{cases} B_{ij} & \text{if } i < j \\ w(B_{ji}) & \text{if } i > j \\ R(B_{ii}) & \text{if } i = j \end{cases} \]

The transfer of an \( n \times n \)-matrix \( A \) with coefficients in \( M(\mathbb{Z}/2) \) has components

\[ T(A)_{ij} = \begin{cases} A_{ij} + w(A_{ji}) & \text{if } i < j \\ T(A_{ii}) & \text{if } i = j \end{cases} \]

The multiplication on \( M_n(M)(\mathbb{Z}/2) \) is the standard matrix multiplication. The action of \( M_n(M)(\mathbb{Z}/2) \) on \( M_n(M)(*) \) is defined by

\[ (A \cdot B)_{ij} = \begin{cases} (AR(B)w(A))_{ij} & \text{if } i < j \\ T( \sum_{1 \leq k < i \leq n} A_{ik}B_{ki}w(A_{kl})) + \sum_{1 \leq k \leq n} A_{ik} \cdot B_{kk} & \text{if } i = j \end{cases} \]

that is by the conjugation action on the off-diagonal entries, and through the Hermitian structure on the diagonal entries.

Lemma 1.6. The object \( M_n(M) \) defined above is a Hermitian Mackey functor, and if \( A \) is a ring with anti-involution \( M_n(A) \cong M_n(M_A) \).
Proof. It is clearly a well-defined Mackey functor, since

\[
RT(A)_{ij} = \begin{cases} 
T(A)_{ij} & \text{if } i < j \\
w(T(A)_{ji}) & \text{if } i > j \\
R(T(A)_{ii}) & \text{if } i = j
\end{cases} = \begin{cases} 
A_{ij} + w(A_{ji}) & \text{if } i < j \\
w(A_{ji} + w(A_{ij})) = w(A_{ji}) + A_{ij} & \text{if } i > j = (A + w(A))_{ij}.
\end{cases}
\]

Let us verify that the formula above indeed defines a monoid action. This is immediate for the components \(i < j\). For the diagonal components let us first verify that the identity matrix \(I\) acts trivially. This is because

\[
(I \cdot B)_{ii} = T(\sum_{1 \leq k < l \leq n} I_{ik} B_{kl} w(I_{il})) + \sum_{1 \leq k \leq n} I_{ik} \cdot B_{kk} = 0 + B_{kk}.
\]

In order to show associativity we start by calculating the diagonal components of \((AC) \cdot B\) for matrices \(A, C \in M_n(M(Z/2))\) and \(B \in M_n(M)(\ast)\). These are

\[
((AC) \cdot B)_{ii} = T(\sum_{p < q} (AC)_{ip} B_{pq} w((AC)_{iq})) + \sum_{t} (AC)_{it} \cdot B_{tt}
\]

An easy induction argument on the fourth axiom of a Hermitian functor shows that

\[
(\sum_{1 \leq k \leq n} a_{ik}) \cdot b = \sum_{1 \leq k \leq n} (a_{ik} \cdot b) + \sum_{1 \leq k < l \leq n} T(a_{ik} R(b) w(a_{kl})).
\]

Thus the expression above becomes

\[
((AC) \cdot B)_{ii} = T(\sum_{p < q} \sum_{k,l} A_{ik} C_{kp} B_{pq} w(A_{il} C_{lq})) + \sum_{t} (\sum_{u} A_{iu} C_{ut} \cdot B_{tt}) +
\]

\[
+ \sum_{t} \sum_{k < l} T(A_{ik} C_{kt} B_{tt} w(A_{il} C_{lt})).
\]

On the other hand the diagonal components of \((A \cdot (C \cdot B))_{ii}\) are

\[
(A \cdot (C \cdot B))_{ii} = T(\sum_{k < l} A_{ik} (C \cdot B)_{kl} w(A_{il})) + \sum_{u} A_{iu} \cdot (C \cdot B)_{uu}
\]

\[
= T(\sum_{k < l} \sum_{p,q} A_{ik} C_{kp} R(B)_{pq} w(A_{il} C_{lq})) + \sum_{u} A_{iu} \cdot (T(\sum_{p < q} C_{up} B_{pq} w(C_{uq})) + \sum_{t} C_{ut} \cdot B_{tt})
\]

\[
= T(\sum_{k < l} \sum_{p,q} A_{ik} C_{kp} R(B)_{pq} w(A_{il} C_{lq})) +
\]

\[
+ T(\sum_{u} \sum_{p < q} A_{iu} C_{up} B_{pq} w(A_{iu} C_{uq})) + \sum_{u} \sum_{t} A_{iu} \cdot C_{ut} \cdot B_{tt}
\]

\[
= T(\sum_{k < l} \sum_{p < q} A_{ik} C_{kp} (B_{pq} + w(B_{pq})) w(A_{il} C_{lq})) + T(\sum_{k < l} \sum_{t} A_{ik} C_{kt} R(B_{tt}) w(A_{il} C_{lt}))
\]

\[
+ T(\sum_{u} \sum_{p < q} A_{iu} C_{up} B_{pq} w(A_{iu} C_{uq})) + \sum_{u} \sum_{t} A_{iu} \cdot C_{ut} \cdot B_{tt}.
\]

We see that the second and the fourth term of this expression cancel respectively with the third and second term of the expression of \((AC) \cdot B\). Finally, by using that the transfer is equivariant we rewrite the sum of the first and third term as

\[
T(\sum_{k < l} \sum_{p < q} A_{ik} C_{kp} (B_{pq} + w(B_{pq})) w(A_{il} C_{lq})) + T(\sum_{u} \sum_{p < q} A_{iu} C_{up} B_{pq} w(A_{iu} C_{uq}))
\]

\[
= T(\sum_{k < l} \sum_{p < q} A_{ik} C_{kp} B_{pq} w(A_{il} C_{lq})) + T(\sum_{u} \sum_{p < q} A_{iu} C_{up} B_{pq} w(A_{iu} C_{uq}))
\]

\[
= T(\sum_{k < l} \sum_{p < q} A_{ik} C_{kp} B_{pq} w(A_{il} C_{lq})) + T(\sum_{k > l} \sum_{p < q} A_{ik} C_{kp} B_{pq} w(A_{il} C_{lp})) + T(\sum_{u} \sum_{p < q} A_{iu} C_{up} B_{pq} w(A_{iu} C_{uq}))
\]

\[
= T(\sum_{k,l} \sum_{p < q} A_{ik} C_{kp} B_{pq} w(A_{il} C_{lq})).
\]
Let us now verify the last three axioms of a Hermitian Mackey functor. The compatibility between the action and the restriction is

\[
R(A \cdot B)_{ij} = \begin{cases} 
(A \cdot B)_{ij} & \text{if } i < j \\
R((A \cdot B)_{ii}) = w((AR(B)w(A))_{ii}) + \sum_{1 \leq k \leq n} R(A_{ik} \cdot B_{kk}) & \text{if } i = j
\end{cases}
\]

= \begin{cases} 
(AR(B)w(A))_{ij} & \text{if } i \neq j \\
R((AR(B)w(A))_{ii}) + \sum_{1 \leq k \leq n} A_{ik} R(B_{kk})w(A_{ik}) & \text{if } i = j
\end{cases}

\]

The compatibility between the action and the transfer is

\[
(A \cdot T(C))_{ij} = \begin{cases} 
(AR(T(C)))_{ij} = (ACw(A) + w(ACw(A)))_{ij} & \text{if } i < j \\
T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} T(C_{kl})w(A_{il}) \right) + \sum_{1 \leq k \leq n} A_{ik} \cdot T(C_{kk}) & \text{if } i = j
\end{cases}
\]

\]

The distributivity of the action over the sum in \(M_n(M)(*)\) is easy to verify for the components \(i < j\). In the diagonal components we have that

\[
((A + A') \cdot B)_{ii} = T \left( \sum_{1 \leq k \leq l \leq n} (A + A')_{ik} B_{kl}w((A + A')_{il}) \right) + \sum_{1 \leq k \leq n} (A + A')_{ik} \cdot B_{kk} =
\]

\[
= T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} B_{kl}w(A_{il}) \right) + T \left( \sum_{1 \leq k \leq l \leq n} A'_{ik} B_{kl}w(A'_{il}) \right) +
\]

\[
+ T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} B_{kl}w(A_{il}) \right) + T \left( \sum_{1 \leq k \leq l \leq n} A'_{ik} B_{kl}w(A'_{il}) \right) +
\]

\[
+ \sum_{1 \leq k \leq n} A_{ik} \cdot B_{kk} + \sum_{1 \leq k \leq n} A'_{ik} \cdot B_{kk} + \sum_{1 \leq k \leq n} T(A_{ik} R(B_{kk})w(A_{ik})) =
\]

\[
= (A \cdot B)_{ii} + (A' \cdot B)_{ii} + T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} B_{kl}w(A_{il}) \right) + T \left( \sum_{1 \leq k \leq l \leq n} A'_{ik} B_{kl}w(A_{il}) \right) +
\]

\[
+ \sum_{1 \leq k \leq n} T(A_{ik} R(B_{kk})w(A_{ik})),
\]

By using that the transfer is equivariant and by reindexing the sum we rewrite the fourth summand as

\[
T \left( \sum_{1 \leq k \leq l \leq n} A'_{ik} B_{kl}w(A_{il}) \right) = T \left( \sum_{1 \leq k \leq l \leq n} A'_{ik} B_{kl}w(A_{ik}) \right) = T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} w(B_{ik})w(A'_{il}) \right).
\]

Thus the expression above is equal to

\[
(A \cdot B)_{ii} + (A' \cdot B)_{ii} + T \left( \sum_{1 \leq k \leq l \leq n} A_{ik} R(B_{kl})w(A'_{il}) \right) = (A \cdot B)_{ii} + (A' \cdot B)_{ii} + T(AR(B)(A'))_{ii}.
\]

Finally, by inspection, we see that \(M_{M_n}(A) \cong M_n(M_A)\). \(\Box\)

We end the section with an extension to Hermitian Mackey functors of the group-ring construction. Let \(\pi\) be a discrete group with an anti-involution \(\tau: \pi^{op} \to \pi\) (for example inversion). If \(A\) is a ring with anti-involution, the group-ring \(A[\pi] = \oplus_{\pi} A\) inherits an anti-involution

\[
w(\sum_{g \in \pi} a_g g) = \sum_{g \in \pi} w(a_{\tau g}) g.
\]
A choice of section $s$ of the quotient map $\pi \to \pi/(\mathbb{Z}/2)$ determines an isomorphism
\[
(A[\pi][\mathbb{Z}/2] / A^\mathbb{Z}/2) \cong A^{\mathbb{Z}/2}[\pi^{\mathbb{Z}/2}] \oplus A[\pi^{\text{free}}/(\mathbb{Z}/2)]
\]
where $\pi^{\text{free}} = \pi \setminus \mathbb{Z}/2$ is the subset of $\pi$ on which $\mathbb{Z}/2$ acts freely. It is defined on the $A^{\mathbb{Z}/2}[\pi^{\mathbb{Z}/2}]$ summand by the inclusion, and on the second summand by sending $cx$ to $cs(x) + w(c)\tau(s(x))$.

**Definition 1.7 (Group-Mackey functor).** Let $M$ be a Hermitian Mackey functor and $\pi$ a discrete group with anti-involution $\tau: \pi^{op} \to \pi$. The associated group-Mackey functor is the Hermitian Mackey functor $M[\pi]$ defined by the Abelian groups
\[
M[\pi](\mathbb{Z}/2) = M(\mathbb{Z}/2)[\pi] \quad M[\pi](*) = M(*)[\pi^{\mathbb{Z}/2}] \oplus M(\mathbb{Z}/2)[\pi^{\text{free}}]/\mathbb{Z}/2.
\]
The anti-involution on $M(\mathbb{Z}/2)[\pi]$ is the standard anti-involution on the group-ring. The restriction is induced by the restriction map $R: M(*) \to M(\mathbb{Z}/2)$ and by the inclusion of the fixed-points of $\pi$ on the first summand, and by the map
\[
R(cx) = cs(x) + w(c)\tau(s(x))
\]
on the second summand. The transfer is defined by
\[
T(ag) = \begin{cases} T(a)g & \text{if } g \in \pi^{\mathbb{Z}/2} \\ a[g] & \text{if } g \in \pi^{\text{free}} \text{ and } g = s[g] \\ w(a)[g] & \text{if } g \in \pi^{\text{free}} \text{ and } g = \tau s[g] \end{cases}
\]
The multiplication on $M[\pi](\mathbb{Z}/2)$ is that of the group-ring $M(\mathbb{Z}/2)[\pi]$. The action of a generator $ag \in M[\pi](\mathbb{Z}/2)$ on $M[\pi](*)$ is extended linearly from
\[
ag \cdot bh = (a \cdot b)(gh\tau(g))
\]
for $bh \in M(*)[\pi^{\mathbb{Z}/2}]$, and
\[
ag \cdot cx = \begin{cases} acw(a)[gs(x)\tau(g)] & \text{if } gs(x)\tau(g) = s[gs(x)\tau(g)] \\ aw(c)w(a)[gs(x)\tau(g)] & \text{if } gs(x)\tau(g) = \tau s[gs(x)\tau(g)] \end{cases}
\]
for $cx \in M(\mathbb{Z}/2)[\pi^{\text{free}}]/(\mathbb{Z}/2)$. It is then extended to the whole group-ring $M[\pi](\mathbb{Z}/2)$ by enforcing condition iv), namely by defining
\[
(\sum_{g \in \pi} a_g g) \cdot \xi = \sum_{g \in \pi} (a_g g \cdot \xi) + \sum_{g < g'} T(a_g g R(\xi)w(a_{g'})\tau(g'))
\]
for some choice of total order on the finite subset of $\pi$ on which $a_g \neq 0$.

When $M = \mathcal{B}$ is the Burnside Mackey functor, we will call $\mathcal{B}[\pi]$ the Burnside group-ring.

**Remark 1.8.** The definition of $M[\pi]$ depends on the choice of section up to isomorphism, and it is therefore not strictly functorial in $\pi$. However it is independent of such choice for the Mackey functors that have trivial action $w$. This will be the case for example for the Burnside Mackey functor $\mathcal{B}$. This construction is always functorial in $M$.

**Lemma 1.9.** The functor $M[\pi]$ is a well-defined Hermitian Mackey functor, and if $A$ is a ring with anti-involution $M_A[\pi] \cong M_A[\pi]$.

**Proof.** We see that $M[\pi]$ is a Mackey functor, since
\[
RT(ag) = \begin{cases} R(T(a)g) = (RT(a))g = (a + w(a))g & \text{if } g \in \pi^{\mathbb{Z}/2} \\ R(a[g]) = ag + w(a)\tau(g) & \text{if } g \in \pi^{\text{free}} \text{ and } g = s[g] \\ R(w(a))[g] = w(a)\tau(g) + ag & \text{if } g \in \pi^{\text{free}} \text{ and } g = \tau s[g] \end{cases}
\]
A calculation completely analogous to the one of lemma[4] shows that the action of $M[\pi](\mathbb{Z}/2)$ on $M[\pi](*)$ is indeed associative. The compatibility between the action and the restriction is
\[
R(ag \cdot bh) = R((a \cdot b)(gh\tau(g))) = R(a \cdot b)(gh\tau(g)) = aR(b)w(a)(gh\tau(g)) = (ag)R(bh)w(a)\tau(g)
\]
for the action on the first summand, and
\[
R(ag \cdot cx) = \begin{cases} 
R(acw(a)[gs(x)\tau(g)]) & \text{if } gs(x)\tau(g) = s[gs(x)\tau(g)] \\
R(aw(c)w(a)[gs(x)\tau(g)]) & \text{if } gs(x)\tau(g) = \tau s[gs(x)\tau(g)]
\end{cases}
\]

\[
\begin{cases}
acw(a)gs(x)\tau(g) + w(acw(a))\tau(g) & \text{if } gs(x)\tau(g) = s[gs(x)\tau(g)] \\
aw(c)w(a)\tau(g) + w(aw(c)w(a)gs(x)\tau(g)) & \text{if } gs(x)\tau(g) = \tau s[gs(x)\tau(g)]
\end{cases} = (ag)R(cx)(w(a)\tau(g))
\]

for the action on the second summand. Let us verify the compatibility between the action and the transfer. We have that \(ag \cdot T(bh)\) is equal to
\[
\begin{cases}
(ag) \cdot (T(b)h) = (a \cdot T(b))ghτ(g) = T(abw(a))ghτ(g) & \text{if } h \in \pi^{Z/2} \\
(ag) \cdot (b[h]) = abw(a)[ghτ(g)] & \text{if } h \in \pi^{free}, h = \pi^{free}, h = \tau s[ghτ(g)] \\
(ag) \cdot (w(b)[h]) = aw(b)w(a)[ghτ(g)] & \text{if } h \in \pi^{free}, h = \tau s[ghτ(g)] \\
(ag) \cdot (w(b)[h]) = aw^2(b)w(a)[ghτ(g)] & \text{if } h \in \pi^{free}, h = \tau s[ghτ(g)] \\
T(abw(a))ghτ(g) & \text{if } ghτ(g) \in \pi^{Z/2} \\
abw(a)[ghτ(g)] & \text{if } h \in \pi^{free} \text{ and } ghτ(g) = s[ghτ(g)] = T(abw(a)ghτ(g)) \\
aw(b)w(a)[ghτ(g)] & \text{if } h \in \pi^{free} \text{ and } ghτ(g) = \tau s[ghτ(g)]
\end{cases}
\]

The last axiom is satisfied by construction. By inspection we see that \(MA[π] \cong MA[π].\)

1.2 The Hermitian K-theory of a Hermitian Mackey functor

Let \(M\) be a Hermitian Mackey functor. We use the Hermitian Mackey functors of matrices constructed in definition 1.3 to define a symmetric monoidal category of Hermitian forms, whose group completion will be the Hermitian \(K\)-theory of \(M\).

Definition 1.10. Let \(M\) be a Hermitian Mackey functor. An \(n\)-dimensional Hermitian form on \(M\) is an element of \(M_n(M)(\ast)\) which restricts to an element of \(GL_n(M(\mathbb{Z}/2))\) under the restriction map
\[
R: M_n(M)(\ast) \rightarrow M_n(M(\mathbb{Z}/2)).
\]

We write \(GL_n(M)(\ast)\) for the set of \(n\)-dimensional Hermitian forms. An morphism \(A \rightarrow B\) of Hermitian forms is a matrix \(λ\) in \(M_n(M(\mathbb{Z}/2))\) which satisfies
\[
A = w(λ) \cdot B,
\]
where the operation is the action of \(M_n(M(\mathbb{Z}/2))\) on \(M_n(M)(\ast)\). Multiplication of matrices defines a category of Hermitian forms, which we denote by \(\text{Herm}_M\).

Remark 1.11. Let \(A\) be a ring with anti-involution. An \(n\)-dimensional Hermitian form on the associated Hermitian Mackey functor \(M_A\) is an invertible anti-symmetric matrix with entries in \(A\). This is the data of an anti-symmetric non-degenerate bilinear pairing \(A^\oplus \otimes A^\oplus \rightarrow A\), that is a Hermitian form on \(A^\oplus\). Since the action of \(M_n(M_A(\mathbb{Z}/2))\) on \(M_n(M_A)(\ast)\) is by conjugation, a morphism of Hermitian forms in the sense of definition 1.10 corresponds to the classical notion of isometry.

Block sum of matrices on objects and morphisms defines the structure of a permutative category on \(\text{Herm}_M\), and thus a monoid structure on the classifying space \(Bi\text{Herm}_M\) of the category of invertible morphisms.

Definition 1.12. Let \(M\) be a Hermitian Mackey functor. The Hermitian \(K\)-theory of \(M\) is the group completion of the topological monoid \(Bi\text{Herm}_M\), that is
\[
KH(M) = ΩB(Bi\text{Herm}_M, \oplus).
\]

Segal’s \(Γ\)-space construction for the symmetric monoidal category \((i\text{Herm}_M, \oplus)\) defines a spectrum \(KH(M)\) whose infinite loop space is equivalent to \(KH(M)\).
Remark 1.13. If $\lambda: A \to B$ is a morphism of Hermitian forms, the form $A$ is determined by $B$ and the matrix $\lambda$. Thus a string of composable morphisms

$$A_0 \xrightarrow{\lambda_0} A_1 \xrightarrow{\lambda_1} \ldots \xrightarrow{\lambda_n} A_n$$

is determined by the sequence of matrices $\lambda_1, \ldots, \lambda_n$, and by the form $A_n$. It follows that there is an isomorphism

$$\text{KH}(M) \cong \Omega B \prod_{n \geq 0} B(*, \text{Gl}_n(M(Z/2)), \text{Gl}_n(M)(*))$$

where $B(*, \text{Gl}_n(M(Z/2)), \text{Gl}_n(M)(*))$ is the Bar construction of the right action of $\text{Gl}_n(M(Z/2))$ on the set of $n$-dimensional Hermitian forms $w(\lambda) \cdot A$, given by the Hermitian structure of the Mackey functor $M_n(M)$. The action indeed restricts to an action on $\text{Gl}_n(M)(*)$ because if $\lambda$ is in $\text{Gl}_n(M(Z/2))$ and the restriction of $A \in M_n(M)(*)$ is invertible, then

$$R(w(\lambda) \cdot A) = w(\lambda)R(A)\lambda$$

is also invertible.

Remark 1.14. Since the notion of Hermitian forms on Hermitian Mackey functors extends that of Hermitian forms on rings with anti-involution, it follows that our definition of Hermitian $K$-theory extends the Hermitian $K$-theory construction of [BF84], of the category of free modules over a discrete ring with anti-involution.

Now we make our Hermitian $K$-theory construction functorial.

Definition 1.15. A morphism of Hermitian Mackey functors is a map of Mackey functors $f: M \to N$ such that $f_{Z/2}: M(Z/2) \to N(Z/2)$ is a ring map, and such that $f_*: M(*) \to N(*)$ is a map of $M(Z/2)$-modules, where $N(*)$ is a $M(Z/2)$ via $f_{Z/2}$.

Clearly a map of Hermitian Mackey functors $f: M \to N$ induces a symmetric monoidal functor $f_*: \text{Herm}_M \to \text{Herm}_N$, by applying $f_{Z/2}$ and $f_*$ entrywise. Thus it induces a continuous map $f_*: \text{KH}(M) \to \text{KH}(N)$, and a map of spectra $f_*: \text{KH}(M) \to \text{KH}(N)$. We will be mostly interested with the following example.

Example 1.16. Let $Z$ be the ring of integers with the trivial anti-involution, and $M_Z$ the corresponding Hermitian Mackey functor. There is a morphism of Hermitian Mackey functors

$$d: \mathcal{B} \to M_Z$$

from the Burnside Mackey functor. The map $d_{Z/2}$ is the identity of $Z$, and the map

$$d_*: Z \oplus Z \to Z$$

is the identity on the first summand and multiplication by 2 on the second. In terms of finite $Z/2$-sets it sends a set to its cardinality. This is in fact a morphism of Tambara functors for the standard multiplicative structures on $\mathcal{B}$ and $M_Z$, and since the Hermitian structures are defined via the multiplicative norms it follows that $d$ is a map of Hermitian Mackey functors.

If moreover $\pi$ is a discrete group with anti-involution, the map $d$ induces a morphism on the associated group-Mackey functors $d: \mathcal{B}[\pi] \to M_Z[\pi]$. The underlying map $d_{Z/2}$ is again the identity on $Z[\pi]$, and the map

$$d_*: \mathcal{B}[\pi](*) = (Z \oplus Z)[\pi] \oplus Z[\pi^{free}](Z/2) \to (Z[\pi])^{\mathcal{B}/2} = Z[\pi] \oplus Z[\pi^{free}]/(Z/2)$$

is $d[\pi]$ on the first summand and the identity on the second summand. Thus this map induces a map on Hermitian $K$-theory spectra

$$d_*: \text{KH}(\mathcal{B}[\pi]) \to \text{KH}(M_Z[\pi]) = \text{KH}(Z[\pi]).$$
1.3 The assembly map for the Burnside group-ring

Let $M$ be a Hermitian Mackey functor. We define a pairing of categories

$$\text{Herm}_\mathbb{Z} \times \text{Herm}_M \rightarrow \text{Herm}_M$$

by means of an extension of the standard tensor product of matrices. On objects, we send a pair $(P, (A, B))$ of a non-singular antisymmetric $m \times m$-matrix $P$ with integral coefficients and an $n$-dimensional form $(A, B)$ on $M$ to the $mn$-dimensional form $P \otimes (A, B)$ with diagonal components

$$(P \otimes (A, B))_{ij} = P_{kk} \cdot B_{uu} \quad \text{where} \quad k = \left\lfloor \frac{i - 1}{n} \right\rfloor + 1, \quad u = i - n \left\lfloor \frac{i - 1}{n} \right\rfloor$$

where the multiplication is the action of $Z$ on the Abelian group $M(*)$. The off-diagonal term $1 \leq i < j \leq mn$ of $P \otimes (A, B)$ is defined by

$$(P \otimes (A, B))_{ij} = \begin{cases} P_{kl} \cdot A_{uv}, & u < v \\ P_{kl} \cdot R( B_{uu} ), & u = v \\ P_{kl} \cdot w(A_{vu}), & u > v \end{cases} \quad \text{where} \quad k = \left\lfloor \frac{l - 1}{n} \right\rfloor + 1, \quad l = \left\lfloor \frac{j - 1}{n} \right\rfloor + 1, \quad u = i - n \left\lfloor \frac{i - 1}{n} \right\rfloor, \quad v = j - n \left\lfloor \frac{j - 1}{n} \right\rfloor$$

This is the standard formula of the Kronecker product of matrices (see [BF84, §6]), adjusted with the relevant restrictions when the index $(u, v)$ is diagonal.

**Example 1.17.** In the case $m = n = 2$ the product above is given by the blocks

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & A_{12} \\ w(A_{12}) & B_{22} \end{pmatrix} = \begin{pmatrix} P_{11} \left( \begin{array}{cc} B_{11} & A_{12} \\ w(A_{12}) & B_{22} \end{array} \right) & P_{12} \left( \begin{array}{cc} R(B_{11}) & A_{12} \\ w(A_{12}) & R(B_{22}) \end{array} \right) \\ P_{12} \left( \begin{array}{cc} R(B_{11}) & A_{12} \\ w(A_{12}) & R(B_{22}) \end{array} \right) & P_{22} \left( \begin{array}{cc} B_{11} & A_{12} \\ w(A_{12}) & B_{22} \end{array} \right) \end{pmatrix}$$

This operation lifts the standard Kronecker product of matrices, in the sense that

$$R(P \otimes (A, B)) = P \otimes R(A, B)$$

as $mn \times mn$-matrices in $M(\mathbb{Z}/2)$. We define the pairing $\text{Herm}_\mathbb{Z} \times \text{Herm}_M \rightarrow \text{Herm}_M$ on morphisms by the standard Kronecker product of matrices. Since the restriction map of $M$ is additive it is immediate to verify the standard identities for the product of forms:

i) $(P \oplus P') \otimes (A, B) = (P \otimes (A, B)) \oplus (P' \otimes (A, B))$,
ii) $P \otimes ((A, B) \oplus (A', B')) = \sigma(P \otimes (A, B)) \oplus (P \otimes (A', B'))\sigma^{-1}$, where $\sigma$ is a permutation matrix,
iii) $0 \otimes (A, B) = (0, 0)$ and $P \otimes (0, 0) = (0, 0)$.

By property ii) the permutation $\sigma$ defines an isomorphism of forms

$$P \otimes ((A, B) \oplus (A', B')) \cong (P \otimes (A, B)) \oplus (P \otimes (A', B'))$$

and one can easily verify that this satisfies the higher coherences required to give the following.

**Lemma 1.18.** The pairing $\text{Herm}_\mathbb{Z} \times \text{Herm}_M \rightarrow \text{Herm}_M$ is a pairing of permutative categories. Hence there is an induced pairing of spectra

$$\otimes: \text{KH}(\mathbb{Z}) \otimes \text{KH}(M) \rightarrow \text{KH}(M)$$

exhibiting $\text{KH}(M)$ as a $\text{KH}(\mathbb{Z})$-module.

The assembly map for the Hermitian $K$-theory of the Burnside group-ring is defined as the composite

$$\mathcal{A}_{\mathcal{B}[\pi]}: \text{KH}(\mathbb{Z}) \wedge B\pi_+ \xrightarrow{1 \wedge \gamma} \text{KH}(\mathbb{Z}) \wedge \text{KH}(\mathcal{B}[\pi]) \xrightarrow{\otimes} \text{KH}(\mathcal{B}[\pi])$$
where \( \gamma: S \wedge B\pi_+ \to KH(B[\pi]) \) is the map of spectra induced by a certain map of \( \Gamma \)-spaces

\[
\gamma: n+ \wedge B\pi_+ = B(\Pi_n \pi)_+ \to B(\text{Herm}_{\mathbb{G}[\pi]\, [n]}),
\]

where \( \text{Herm}_{\mathbb{G}[\pi]\, [n]} \) denotes the \( n \)-simplices of the Segal \( \Gamma \)-category associated to the symmetric monoidal category \( (\text{Herm}_{\mathbb{G}[\pi]}, \oplus) \). The map \( \gamma \) is induced by the functor

\[
\Pi_n \pi \to \text{Herm}_{\mathbb{M}[\pi]}^n \overset{\sim}{\to} \text{Herm}_{\mathbb{M}[\pi][n]}
\]

where the second map is the standard equivalence of categories. The first map sends the unique object in the \( i \)-component to the object \((0, \ldots, 0, (1,0)e), 0, \ldots, 0)\), where the non-zero entry is in the \( i \)-th slot. Here \((1,0)e\) denotes the unit of the group-ring \((\mathbb{Z} \oplus \mathbb{Z}[\pi] \subset B[\pi](*)\) considered as a \( 1 \times 1 \)-matrix. A morphism \((i, g)\) is sent to the morphism with non-zero \( i \)-th component \( g: (1,0)e \to (1,0)e \). It is immediate to see that this defines a map of \( \Gamma \)-categories.

**Lemma 1.19.** The assembly map \( A_{\mathbb{G}[\pi]} \) is natural in \( \pi \), and it lifts the connective assembly map of the integral group ring of \( [BF\overline{S}] \). That is the diagram

\[
\begin{array}{ccc}
A_{\mathbb{G}[\pi]} & \to & KH(B[\pi]) \\
\downarrow & & \downarrow d \\
KH(\mathbb{Z}) \wedge B\pi_+ & \xrightarrow{\delta} & \text{KH}(\mathbb{Z}[\pi])
\end{array}
\]

where \( B[\pi] \to \mathbb{Z}[\pi] \) is the dimension map of \([1.10]\).

**Proof.** The assembly \( A_{\mathbb{Z}[\pi]} \) is constructed as a similar composite

\[
A_{\mathbb{Z}[\pi]}: KH(\mathbb{Z}) \wedge B\pi_+ \overset{1 \wedge \delta}{\to} KH(\mathbb{Z}) \wedge KH(\mathbb{Z}[\pi]) \overset{\oplus}{\to} \text{KH}(\mathbb{Z}[\pi])
\]

where \( \delta: S \wedge B\pi_+ \to \text{KH}(\mathbb{Z}[\pi]) \) is constructed just like \( \gamma \), but by replacing the unit \((1,0)e\) of \( B(*) \) with the unit of \( \mathbb{Z} \). Since \( d \) preserves the unit we obtain a commutative diagram

\[
\begin{array}{ccc}
& KH(\mathbb{Z}) \wedge KH(B[\pi]) & \overset{\oplus}{\to} \text{KH}(B[\pi]) \\
\downarrow & & \downarrow d \\
KH(\mathbb{Z}) \wedge B\pi_+ & \xrightarrow{\text{KH}(\mathbb{Z}) \wedge 1 \wedge \delta} & \text{KH}(\mathbb{Z}) \wedge KH(\mathbb{Z}[\pi]) \overset{\oplus}{\to} \text{KH}(\mathbb{Z}[\pi])
\end{array}
\]

where the right-hand square commutes by naturality of the pairing in the Hermitian Mackey-functor.

\( \square \)

1.4 **The Hermitian cyclic K-theory of a Hermitian Mackey-functor**

Let \( M \) be a Hermitian Mackey functor. We remind the reader that the Hermitian \( K \)-theory space of \( M \) is isomorphic to the group completion

\[
KH(M) \cong \Omega B \prod_{n \geq 0} B(*, GL_n(M(\mathbb{Z}/2)), GL_n(M)(*))
\]

of the Bar construction of the right action of \( GL_n(M(\mathbb{Z}/2)) \) on the set of \( n \)-dimensional Hermitian forms \( GL_n(M)(*) \) given by the Hermitian structure of the Mackey functor \( M_n(M) \), that is by \( A \cdot \lambda := w(\lambda) \cdot A \) (see \([1.13]\)). We extend this construction to the two-sided Bar construction, for the actions \( A \cdot \lambda \) and \( \lambda \cdot A \).

**Definition 1.20.** The Hermitian cyclic \( K \)-theory space of a Hermitian Mackey functor \( M \) is the group completion of the two-sided Bar construction of the actions of \( GL_n(M(\mathbb{Z}/2)) \) on the set of \( n \)-dimensional Hermitian forms, that is

\[
K^\text{cy}H(M) = \Omega B \prod_n B(GL_n(M)(*), GL_n(M(\mathbb{Z}/2)), GL_n(M)(*))
\]

where the monoid structure on the disjoint union is defined by componentwise block-sum of matrices.
In order to deloop $K\text{cy}H(M)$ we express this space as the group completion of the classifying space of a symmetric monoidal category. A two-sided Bar construction is always the nerve of a category, and in our case it the category $i\text{Herm}^M_{cy}$ defined as follows. Its objects are pairs $(A, B)$ of $n$-dimensional Hermitian forms over $M$. A morphism $(A, B) \to (A', B')$ is a matrix $\lambda \in \operatorname{Gl}_n(M(\mathbb{Z}/2))$ such that $A' = \lambda \cdot A$ and $B = B' \cdot \lambda$. This category has a symmetric monoidal structure defined by block-sum of matrices.

**Definition 1.21.** The Hermitian cyclic $K$-theory spectrum $K\text{cy}H(M)$ of a Hermitian Mackey functor is the spectrum associated to Segal’s $\Gamma$-space construction of the symmetric monoidal category $(i\text{Herm}^M_{cy}, \oplus)$.

**Remark 1.22.** The notation $K\text{cy}H$ has the following origins. Suppose that $M$ is the Hermitian Mackey functor associated to a ring with anti-involution $A$. The one-sided Bar construction $B(*, \text{Gl}_n(A), \text{Gl}_n(A)^{\mathbb{Z}/2})$ is the fixed-points space of an involution on the Bar construction $B\text{Gl}_n(A)$ (see [3.1]). Thus $KH(A)$ is the fixed-points space of an involution on the $K$-theory space $K(A)$. Similarly, the two-sided Bar construction $B(\text{Gl}_n(A)^{\mathbb{Z}/2}, \text{Gl}_n(A), \text{Gl}_n(A)^{\mathbb{Z}/2})$ is the fixed-points of an involution on the cyclic Bar construction $B^\text{cy}\text{Gl}_n(A)$ (see [3.1]), and consequently $K\text{cy}H(A)$ is the fixed-points of an involution on the cyclic $K$-theory space

$$K\text{cy}H(A) := \Omega B \bigoplus_n B^\text{cy}\text{Gl}_n(A),$$

hence the terminology. We will promote these constructions to the realm of $\mathbb{Z}/2$-equivariant spectra and ring spectra with anti-involution in [3.3].

There is a symmetric monoidal functor $p: (i\text{Herm}^M_{cy}, \oplus) \to (i\text{Herm}_M, \oplus)$ that sends an object $(A, B)$ to $B$, and a morphism $\lambda$ to $\lambda$. We conclude this section by examining the sections of this functor. Clearly $p$ has a section that sends an object $B$ to $(0, B)$. We will need a larger set of sections, associated to certain “central” elements in $M(*)$.

Let us suppose that $2$ is invertible in $M$, that is that $2$ is invertible in the ring $M(\mathbb{Z}/2)$ and the map $2(-): M(*) \to M(*)$ is bijective. Then $2$ is invertible also in the Hermitian Mackey-functor of matrices $M_n(M)$. Given an $n$-dimensional Hermitian form $A \in \text{Gl}_n(M)(*)$, we define

$$A^{-1} := \frac{1}{2} T(R(A)^{-1}) \in M_n(M)(*).$$

**Lemma 1.23.** The matrix $A^{-1}$ is an $n$-dimensional Hermitian form over $M$, such that $R(A^{-1}) = R(A)^{-1}$. Moreover

$$(\lambda \cdot A)^{-1} = w(\lambda)^{-1} \cdot A^{-1}$$

for every $n$-dimensional form $A \in \text{Gl}_n(M)(*)$ and matrix $\lambda \in \text{Gl}_n(M(\mathbb{Z}/2))$.

**Proof.** Clearly $A^{-1}$ belongs to $\text{Gl}_n(M)(*)$ since

$$R(A^{-1}) = \frac{1}{2} RT(R(A)^{-1}) = \frac{R(A)^{-1} + w(R(A))^{-1}}{2} = R(A)^{-1}$$

is invertible. Moreover by the compatibility of the action of $M_n(M(\mathbb{Z}/2))$ on $M_n(M)(*)$ with the restriction and the transfer we see that

$$(\lambda \cdot A)^{-1} = \frac{1}{2} T(R(\lambda \cdot A)^{-1}) = \frac{1}{2} T((\lambda R(\lambda)w(\lambda))^{-1}) = \frac{1}{2} T(w(\lambda)^{-1} R(A)^{-1} \lambda^{-1})$$

$$= \frac{1}{2} w(\lambda)^{-1} T(R(A)^{-1}) = w(\lambda)^{-1} \cdot A^{-1}.$$

\[ \square \]

When $2$ is invertible in $M$ we can use this inversion function to define a section

$$(i\text{Herm}^M_{cy}, \oplus) \xrightarrow{s_i} (i\text{Herm}_M, \oplus).$$
for the projection functor $p$. We let $s_1$ be the symmetric monoidal functor that sends an object $A$ to the pair $(A^{-1}, A)$, and a morphism $\lambda: w(\lambda) \cdot B \to B$ to

$$
\lambda: ((w(\lambda) \cdot B)^{-1}, w(\lambda) \cdot B) \to (B^{-1}, B).
$$

This is a well-defined morphism because by the previous lemma $\lambda \cdot (w(\lambda) \cdot B)^{-1} = \lambda \cdot \lambda^{-1} \cdot B^{-1} = B^{-1}$. This induces a splitting of spectra

$$
K^O(M) \sim KH(M) \vee K^O(M).
$$

**Remark 1.24.** Suppose that $M$ is associated to a ring with anti-involution $A$, where $2 \in A$ is invertible. Under the isomorphism

$$
NiHerm_A^{cy} \cong \prod_n N\left(\text{Gl}_n(A)^{2/2}, \text{Gl}_n(A), \text{Gl}_n(A)^{2/2}\right)
$$

the section $s_1$ corresponds to the section

$$
N(\ast, \text{Gl}_n(A), \text{Gl}_n(A)^{2/2}) \to N(\text{Gl}_n(A)^{2/2}, \text{Gl}_n(A), \text{Gl}_n(A)^{2/2})
$$

that sends $(\lambda_1, \ldots, \lambda_k, A)$ to $((A \cdot (\lambda_k \ldots \lambda_1))^{-1}, \lambda_1, \ldots, \lambda_k, A)$. If one identifies the two-sided Bar construction with the fixed-points of the involution on the cyclic nerve $N^{cy}\text{Gl}_n(A)$ (see [3.1]), this is the restriction of the standard section

$$
s_1: \text{NGl}_n(A) \to N^{cy}\text{Gl}_n(A)
$$

that sends $(\lambda_1, \ldots, \lambda_k)$ to $((\lambda_k \ldots \lambda_1)^{-1}, \lambda_1, \ldots, \lambda_k)$. We observe that this section can be twisted by any element $c$ in the center of $\text{Gl}_n(A)$, simply by multiplying the first coordinate of $s_1$ by $c$. We proceed by defining an analogue of this twisting for the Burnside group-ring.

Let $\mathcal{B}_Q$ be the rationalized Burnside Mackey functor, $\pi$ a discrete group, and $\mathcal{B}_Q[\pi]$ the associated Burnside group-ring. Let $\sigma \in \mathcal{B}_Q(\ast)$ the element of the Burnside ring defined by the virtual $\mathbb{Z}/2$-set $\sigma := \mathbb{Z}/2 - 1$. We observe that this element is invertible and central with respect to the multiplication of the Burnside ring. Given an $n$-dimensional Hermitian form $A \in \text{Gl}_n(M)(\ast)$ we let $\sigma A$ denote the $n$-dimensional Hermitian form obtained by multiplying the diagonal entries of $A$ with $\sigma$, and leaving the off-diagonal entries untouched. Since $R(\sigma) = 1$ we see that $R(\sigma A) = R(A)$ is invertible. We let

$$
(i \text{Herm}_M^{cy}, \oplus) \xrightarrow{s_\pi} (i \text{Herm}_M, \oplus)
$$

be the section defined on objects by $s_\pi(B) = (\sigma B^{-1}, B)$, and by sending a morphism $\lambda: B \to w(\lambda) \cdot B$ to $s_\pi(\lambda) = \lambda$. Again, this is well-defined because

$$
\lambda \cdot \sigma (w(\lambda) \cdot B)^{-1} = \sigma \lambda \cdot \lambda^{-1} \cdot B^{-1} = \sigma B^{-1}.
$$

**Remark 1.25.** We observe that when projected to the rational group-ring $\mathbb{Q}[\pi]$ the two sections coincide with the section $s_1$, that is both squares

$$
\begin{array}{ccc}
\mathbb{K}H(\mathcal{B}_Q[\pi]) & \xrightarrow{s_1} & \mathbb{K}^O(\mathcal{B}_Q[\pi]) \\
\downarrow d & & \downarrow d \\
\mathbb{K}H(\mathbb{Q}[\pi]) & \xrightarrow{s_1} & \mathbb{K}^O(\mathbb{Q}[\pi])
\end{array}
\quad
\begin{array}{ccc}
\mathbb{K}H(\mathcal{B}_Q[\pi]) & \xrightarrow{s_\pi} & \mathbb{K}^O(\mathcal{B}_Q[\pi]) \\
\downarrow d & & \downarrow d \\
\mathbb{K}H(\mathbb{Q}[\pi]) & \xrightarrow{s_\pi} & \mathbb{K}^O(\mathbb{Q}[\pi])
\end{array}
$$

commute.

We will make use of the sections $s_1$ and $s_\pi$ in the construction of the splitting of the assembly map in theorem 4.8.
2 Application to the Novikov conjecture

In this section we state our main application, that reformulates the Novikov conjecture in terms of a splitting of a lift to the Burnside group-ring \( B[\pi] \) of the restricted assembly map of the Hermitian \( K \)-theory of the integral group-ring. We give a proof of this theorem except for the existence of this splitting, which is technical and that we postpone to the remainder of the paper.

2.1 Splitting the restricted assembly map

We recall that the restricted assembly map for the Hermitian \( K \)-theory of the integral group-ring is the map \( A_{B[\pi]}^{0} \) defined as the composite

\[
A_{B[\pi]}^{0} : KH_{0}(\mathbb{Z}) \otimes \pi_{*}(S \wedge B\pi_{+}) \longrightarrow KH_{*}(\mathbb{Z}) \otimes \pi_{*}(S \wedge B\pi_{+}) \longrightarrow \pi_{*}(KH(\mathbb{Z}) \wedge B\pi_{+}) \xrightarrow{A_{B[\pi]}} KH_{*}(\mathbb{Z}[\pi])
\]

where the first map is the inclusion in degree zero and the second map is the canonical map. The group \( \pi_{0}KH(\mathbb{Z}) \) is canonically isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \), where the summands are respectively generated by the 1-dimensional forms \((1)\) and \((-1)\). Thus rationally, the Hurewicz map identifies the source of \( A_{B[\pi]}^{0} \) with two copies of the rational homology of \( B\pi \). The restricted assembly map is then the sum of two maps

\[
\pi_{*}(S \wedge B\pi_{+}) \longrightarrow KH_{*}(\mathbb{Z}[\pi]).
\]

The first map is the map induced on homotopy groups by the map \( \delta \) defined in 1.3. The second is the map \( \overline{\delta} \), defined just as \( \delta \) but by replacing the 1-dimensional form \( \langle e \rangle \) given by the unit \( e \) of the group-ring \( \mathbb{Z}[\pi] \) with the form \( \langle -e \rangle \).

**Theorem 2.1.** Let \( \pi \) be a discrete group. The restricted rational assembly map of the Burnside group-ring \( A_{B[\pi]}^{0} \) admits a lift to \( (KH_{*}(B[\pi]) \oplus KH_{*}(B[\pi])) \otimes \mathbb{Q} \) which is rationally split-injective. That is, there is a commutative diagram, natural in \( \pi \),

\[
\begin{array}{ccc}
H_{*}(B\pi; \mathbb{Q} \oplus \mathbb{Q}) & \xrightarrow{A_{B[\pi]}^{0}} &KH_{*}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \\
& & \downarrow d+d \\
& & (KH_{*}(B[\pi]) \oplus KH_{*}(B[\pi])) \otimes \mathbb{Q} \\
& & \xrightarrow{T}
\end{array}
\]

where \( d \) is induced by the morphism of Hermitian Mackey functors \( B[\pi] \to \mathbb{Z}[\pi] \) from 1.10.

Thus the restricted assembly \( A_{B[\pi]}^{0} \) admits a splitting to \( KH_{*}(B[\pi]) \) if and only if the restriction of the retraction \( T \) to the kernel of \( d + d \) is zero.

**Remark 2.2.** The restriction of the assembly \( A_{B[\pi]} \) constructed in 1.3 provides a lift of \( A_{B[\pi]}^{0} \) to \( KH_{*}(B[\pi]) \). However, the trace methods we employ do not allow us to show that this map is rationally split-injective, and we need to lift to two copies of \( KH_{*}(B[\pi]) \). See also 4.10.

**Proof of 2.1.** We define two separate lifts

\[
\gamma, \overline{\gamma} : S \wedge B\pi_{+} \longrightarrow KH(B[\pi])
\]

of the respective maps \( \delta \) and \( \overline{\delta} \). The map \( \gamma \) is the one defined in 1.3 induced by sending the unique object of the category \( \pi \) to the 1-dimensional form \( \langle 1e \rangle \), where 1 is the unit of the Burnside ring. The map \( \overline{\gamma} \) is defined by a similar construction, but by replacing \( \langle 1e \rangle \) by the form \( \langle -1e \rangle \). As \( d(1e) = e \) and \( d(-1e) = -e \), it follows that the sum of these maps \( A_{B[\pi]}^{0} := \gamma + \overline{\gamma} \) lifts the restricted assembly map \( A_{B[\pi]}^{0} \).

The splitting \( T \) is constructed as a composite of the form

\[
(KH_{*}(B[\pi]) \oplus KH_{*}(B[\pi])) \otimes \mathbb{Q} \xrightarrow{s+1} K^{cy}H_{*}(B[\pi]) \otimes \mathbb{Q} \xrightarrow{tr^{\mathbb{Q}}} THR_{*}^{\mathbb{Q}/2}(B[\pi]) \otimes \mathbb{Q} \xrightarrow{T} H_{*}(B\pi; \mathbb{Q} \oplus \mathbb{Q}) \xrightarrow{p_{0}} H_{*}(B^{cy}\pi; \mathbb{Q}) \oplus H_{*}((B^{cy}\pi)^{2}; \mathbb{Q})
\]
where $s_1$ and $s_2$ are the sections of the projection off the cyclic nerve defined at the end of §1.3. The map $\text{tr}^{cy}$ is a Hermitian analogue of the trace map of [BHM93]. The last map is induced by the standard projection $p_0 : B^{cy} \pi \to B\pi$ from the cyclic Bar construction to the Bar construction, and by the inclusion of fixed-points spaces $(B^{cy} \pi)^{Z/2} \to B^{cy} \pi$ followed by the projection $p_0$ to $B\pi$. The construction of the trace map $\text{tr}^{cy}$ is technical and it is postponed to §1.2. The proof that the map $T$ indeed splits $A^0_{\mathbb{B}[\pi]}$ is Theorem 1.8.

Now if we assume that $A^0_{\mathbb{B}[\pi]}$ is injective, we see that $A^0_{\mathbb{B}[\pi]}$ is injective if and only if $d + d$ is injective on the image of $A^0_{\mathbb{B}[\pi]}$. Since $T$ splits $A^0_{\mathbb{B}[\pi]}$ the source of $d$ decomposes as the direct sum of the image of $A^0_{\mathbb{B}[\pi]}$ and the kernel of $T$. Thus $d + d$ is injective on the image of $A^0_{\mathbb{B}[\pi]}$ if and only if its kernel is included in the kernel of $T$, that is if $T$ is zero on the kernel of $d + d$. □

2.2 Reformulation of the Novikov conjecture

Let $\pi$ be a discrete group and $W(\mathbb{Z}[\pi])$ the Witt spectrum of the corresponding integral group-ring. We let $W_*(\mathbb{Z}[\pi])$ denote its homotopy groups. The assembly map for Witt theory is a map of spectra

$$W(\mathbb{Z}) \wedge B\pi_+ \rightarrow W(\mathbb{Z}[\pi]).$$

Rationally the map $KH_*(\mathbb{Z}[\pi]) \to W_*(\mathbb{Z}[\pi])$ splits canonically on positive degrees, determining a natural decomposition

$$KH_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong (W_{* \geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \oplus (K_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{Z/2}$$

where $(K_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{Z/2}$ denotes the subgroup of fixed-points of the involution on the $K$-groups $K_*(\mathbb{Z}[\pi])$ induced by the anti-involution of $\mathbb{Z}[\pi]$. Then the connective part of the rational assembly for Witt theory is the restriction of the rational assembly for Hermitian $K$-theory to this summand. This splitting can be extended to negative degrees using non-connective deloopings of $KH$ (see [BF84]). The Novikov conjecture for a group $\pi$ states that the non-connective rational assembly in Witt theory for the group ring $\mathbb{Z}[\pi]$ is split-injective.

**Theorem 2.3.** Let $\pi$ be a discrete group. Then the Novikov conjecture holds for $\pi$ if and only if the map

$$(KH_*(\mathbb{B}[\pi]) \oplus KH_*(\mathbb{B}[\pi])) \otimes \mathbb{Q} \xrightarrow{T} H_*(B\pi; \mathbb{Q} \oplus \mathbb{Q}) \xrightarrow{(1,-1)} H_*(B\pi; \mathbb{Q})$$

is zero on the kernel of

$$d + d: (KH_*(\mathbb{B}[\pi]) \oplus KH_*(\mathbb{B}[\pi])) \otimes \mathbb{Q} \to KH_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q},$$

where $T$ is the splitting of $\mathbb{Z}[\pi]$ and $d: \mathbb{B}[\pi] \to \mathbb{Z}[\pi]$ is the dimension map of $\mathbb{Z}[\pi]$.

**Proof.** Let us start by proving that under the hypothesis above the Novikov conjecture holds for $\pi$. Rationally $W_*(\mathbb{Z})$ is a polynomial algebra on one generator in degree 4. Thus the rational assembly map in Witt theory takes the form

$$A^W_{\mathbb{Z}[\pi]}: \bigoplus_{k \in \mathbb{Z}} H_*(B\pi; \mathbb{Q}) \cdot \beta^k \cong W_*(\mathbb{Z}[\pi]) \otimes H_*(B\pi; \mathbb{Q}) \rightarrow W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

where $\beta \in W_4(\mathbb{Z})$ is the Bott element. The restricted assembly map is the composite

$$A^W_{\mathbb{Z}[\pi]}: H_*(B\pi; \mathbb{Q}) \rightarrow \bigoplus_{k \in \mathbb{Z}} H_*(B\pi; \mathbb{Q}) \cdot \beta^k A^W_{\mathbb{Z}[\pi]} W_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

where the first map is the inclusion of the summand $k = 0$. An element $x$ of $W_*(\mathbb{Z}[\pi]) \otimes H_*(B\pi; \mathbb{Q})$ of degree $n$ can be written as a polynomial

$$x \beta^{-k} + x_{n+4(k+1)} \beta^{-k-1} + \cdots + x_{n+4} \beta^{-1} + x_n + x_{n-4} \beta + \cdots + x_{n-4} \beta^j$$
In order to emphasize the naturality of our transformations in the group \( \tilde{B}_\pi \homotopy \), by multiplying by the appropriate power of \( \beta \) it suffices to show that the assembly is injective on elements of the form
\[
\underline{x} = x_n + x_{n-4} \beta + \cdots + x_{n-4j} \beta^j
\]
where \( x_n \neq 0 \).

Now we consider the commutative diagram

\[
\begin{array}{ccc}
\text{Im}(d + d) & \rightarrow & H_*(B\pi; \mathbb{Q} \oplus \mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \mathbb{Z}
\end{array}
\]

where the vertical maps \( p \) and \( i \) are respectively the projection and the inclusion of the Witt summand of the decomposition \( KH_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong (W_{\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \oplus (K_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q})^{\mathbb{Z}/2} \). The vertical maps on the left are the inclusion and the projection of the subgroup \( W(\mathbb{Z})_0 \cong \mathbb{Z} \) into \( KH(\mathbb{Z})_0 \cong \mathbb{Z}(1) \oplus \mathbb{Z}(-1) \). Our assumption guarantees that the map \((1, -1)T\) descends to a map
\[
\overline{T}: \text{Im}(d + d) \cong KH(B[\pi]) \otimes \mathbb{Q}/ \text{ker}(d + d) \rightarrow H_*(B\pi; \mathbb{Q}).
\]

As the connective assembly map for the Hermitian \( K \)-theory of the integral group-ring admits a lift to \( KH_*(B[\pi]) \) (see [1, 19]), if \( \underline{x} \) has the form above then \( iA^W_{\mathbb{Z}[\pi]}(\underline{x}) \) lies in the image of \( d + d \). Thus in order to show that \( A^W_{\mathbb{Z}[\pi]}(\underline{x}) \) is non-zero it is sufficient to show that \( \overline{T}iA^W_{\mathbb{Z}[\pi]}(\underline{x}) \) is non-zero in \( H_*(B\pi; \mathbb{Q}) \). Clearly we can write
\[
\overline{T}iA^W_{\mathbb{Z}[\pi]}(\underline{x}) = \overline{T}iA^W_{\mathbb{Z}[\pi]}(x_n) + \overline{T}iA^W_{\mathbb{Z}[\pi]}(x_{n-4} \beta + \cdots + x_{n-4j} \beta^j).
\]

Since \( T \) splits \( A^0_{\mathbb{Z}[\pi]} \) by [2, 1] it follows that \( \overline{T} \) splits \( iA^W_{\mathbb{Z}[\pi]} \). As \( x_n \neq 0 \), we know that \( \overline{T}iA^W_{\mathbb{Z}[\pi]}(x_n) \neq 0 \). Thus it remains to show that
\[
\overline{T}iA^W_{\mathbb{Z}[\pi]}(x_{n-4} \beta^{-1} + \cdots + x_{n-4j} \beta^j) = 0.
\]

Let \( H_<(B\pi)(\beta) \) be the subgroup of \( H_*(B\pi; \mathbb{Q}) \otimes W_*(\mathbb{Z}) \) consisting of elements of the form
\[
x_{n-4} \beta + \cdots + x_{n-4j} \beta^j.
\]

By the Kan-Thurston theorem there exists a group \( \pi_{(n-1)} \) and a map \( \lambda_{n-1}: B\pi_{(n-1)} \rightarrow (B\pi)^{(n-1)} \) to the \((n-1)\)-skeleton of \( B\pi \) which is a homology isomorphism. Then the first homotopy group of the composite \( B\pi_{(n-1)} \rightarrow B\pi^{(n-1)} \rightarrow B\pi \) gives a group homomorphism \( \lambda_{n-1}: \pi_{(n-1)} \rightarrow \pi \) which induces an isomorphism
\[
\lambda_{n-1}: H_<(B\pi_{(n-1)})(\beta) \xrightarrow{\cong} H_<(B\pi^{(n-1)})(\beta) \cong H_<(B\pi)(\beta).
\]

In order to emphasize the naturality of our transformations in the group \( \pi \) we add a super-
script to our notations. By naturality of $T^*$ and $d^*$ the diagram

\[
\begin{array}{cccccc}
\lambda_{n-1} & \lambda_{n-1} & & \lambda_{n-1} \\
\ker(d^* + d^*) & \subset & (KH_*(\mathcal{B}[\pi_{(n-1)}]) + KH_*(\mathcal{B}[\pi_{(n-1)}])) \otimes \mathbb{Q} & \rightarrow & H_*(B\pi_{(n-1)}; \mathbb{Q}) & (1, -1) \circ T^*(n-1) \\
\end{array}
\]

commutes. Because $\lambda_{n-1} : H_*(B\pi_{(n-1)}; \mathbb{Q}) \rightarrow H_*(B\pi; \mathbb{Q})$ is injective we must have that $(1, -1) \circ T^*(n-1)$ is also zero on $\ker(d^* + d^*)$, and it descends to a map

\[
\tilde{T}^*(n-1) : \text{im} \, d^*(n-1) \rightarrow H_*(B\pi_{(n-1)}; \mathbb{Q}).
\]

This results in a commutative diagram

\[
\begin{array}{cccccc}
\lambda_{n-1} & \lambda_{n-1} & & \lambda_{n-1} \\
H_{<n}(B\pi_{(n-1)})(\beta) & A^W_{\pi_{(n-1)}} & W_\pi(\mathbb{Z}[\pi_{(n-1)}]) & \tilde{T}^*(n-1) & H_{<n}(B\pi_{(n-1)}; \mathbb{Q}) = 0 \\
& & & H_\pi(B\pi; \mathbb{Q}) & &
\end{array}
\]

and it follows that $\tilde{T}^* i_\pi A^W_{\pi_{(n-1)}}$ is zero on $H_{<n}(B\pi)(\beta)$.

Now suppose that the Novikov conjecture holds for $\pi$. In particular the restricted assembly map $A^W_{\pi_{(n-1)}}$ is injective, and so is $i_\pi A^W_{\pi_{(n-1)}} = A^W_{\mathbb{Z}[\pi]} (1/2)$. The assembly for the Burnside ring $A^W_{\mathbb{Z}[\pi]} (1/2)$ is also injective, and this forces $d^* + d$ to be injective on the image of $A^W_{\mathbb{Z}[\pi]} (1/2)$. Since $(-1, 1)T$ splits $A^W_{\mathbb{Z}[\pi]} (1/2)$, this is the same as requiring that the kernel of $d^* + d$ is included in the kernel of $(-1, 1)T$, that is $(-1, 1)T$ is zero on the kernel of $d^* + d$. \[
\square
\]

We recall that the map $T$ is constructed as a rational composite

\[
(KH_*(\mathcal{B}[\pi]) \oplus KH_*(\mathcal{B}[\pi])) \otimes \mathbb{Q} \xrightarrow{s_1 +_s s_\pi} K^\text{cy} H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} \xrightarrow{\text{tr}^{\text{cy}}} \text{THR}^2(Q^{\mathbb{Z}/2}(\mathcal{B}[\pi]) \otimes \mathbb{Q}) \xrightarrow{1-1} H_*(B\pi; \mathbb{Q}).
\]

**Corollary 2.4.** The Novikov conjecture holds for $\pi$ if and only if the map

\[
K^\text{cy} H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} \xrightarrow{\text{tr}^{\text{cy}}} \text{THR}^2(Q^{\mathbb{Z}/2}(\mathcal{B}[\pi])) \otimes \mathbb{Q} \xrightarrow{(1-1)p_0 \otimes \text{tr}^{\text{cy}}} H_*(B\pi; \mathbb{Q})
\]

is zero on the kernel of $d^{\text{cy}}$ : $K^\text{cy} H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} \rightarrow K^\text{cy} H_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.

**Proof.** The commutative diagram

\[
\begin{array}{cccccc}
& & s_1 & & \\
& & KH_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} & \xrightarrow{d^{\text{cy}}} & K^\text{cy} H_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} & &
\end{array}
\]

shows that the restriction of $T$ to the kernel of $d^* + d$ factors through the restriction of $(1, -1)p_0 \otimes \text{tr}^{\text{cy}}$ to the kernel of $d^{\text{cy}}$. Thus if the latter is zero, so is the former and the Novikov conjecture for $\pi$ holds by \cite{23}. Conversely, if the Novikov conjecture holds for $\pi$ the bottom row of the commutative diagram

\[
\begin{array}{cccccc}
& & (1-1)p_0 \otimes \text{tr}^{\text{cy}} & & \\
& & K^\text{cy} H_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} & \xrightarrow{(s_1 + s_\pi)A^W_{\mathbb{Z}[\pi]} (1/2)} & K^\text{cy} H_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} & &
\end{array}
\]

\[
\begin{array}{cccccc}
& & (1, -1)p_0 \otimes \text{tr}^{\text{cy}} & & \\
& & KH_*(\mathcal{B}[\pi]) \otimes \mathbb{Q} & \xrightarrow{s_1} & K^\text{cy} H_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} & &
\end{array}
\]
is injective. Thus \((1, -1)p_0 \text{tr}^g\) is zero on the kernel of \(d^g\).

3 Real \(K\)-theory

We give a construction of the free real \(K\)-theory of a ring spectrum with anti-involution which supports a trace map to the real topological Hochschild homology of \([HM17]\) and \([Dot12]\). We prove in 3.22 that for discrete and simplicial rings with anti-involution its fixed-points are equivalent to the connective Hermitian \(K\)-theory of \([BFS84]\) and \([Kar73]\). For Eilenberg MacLane spectra with anti-involution the fixed-points recover the Hermitian \(K\)-theory of Hermitian Mackey functors defined in §1.2.

For general ring spectra our definition of real \(K\)-theory differs from other known constructions, for example the one of \([Spi16]\). The input of our construction is a structured genuine \(\mathbb{Z}/2\)-spectrum, and the output depends on its genuine equivariant homotopy type. The intuition is that fixed-points spectrum of the input ring spectrum \(A\) determine a notion of “symmetry” for the Hermitian forms over \(A\).

3.1 The real and dihedral Bar constructions

In this section we investigate the Bar construction associated to a monoid with an anti-involution. This is essentially a recollection of materials from \([Lod87]\), but we need a context without degeneracies which requires a bit of care.

By a non-unital topological monoid we mean a possibly non-unital monoid in the monoidal category of spaces with respect to the cartesian product. Let \(M\) be a non-unital topological monoid which is equipped with an anti-involution, that is a continuous monoids map \(w: M^{op} \rightarrow M\) that satisfies \(w^2 = \text{id}\). The Bar construction \(BM\) is the geometric realization of the semi-simplicial space \(NM\), the nerve of \(M\), with \(n\)-simplices \(N_n M = M^*\)

and with the standard face maps induced by the projections and by the multiplication of \(M\). The anti-involution induces a semi-simplicial map \((NM)^{op} = N(M^{op}) \rightarrow NM\), defined degreewise by

\[(m_1, \ldots, m_n) \mapsto (w(m_n), \ldots, w(m_1)).\]

The resulting semi-simplicial space, together with its involution, is denoted \(N^{1,1}M\). In order to obtain a semi-simplicial space with a semi-simplicial \(\mathbb{Z}/2\)-action we need to apply Segal’s edgewise subdivision, obtaining a simplicial space \(sdNM\) with \(n\)-simplices \((sdNM)_n = N_{2n+1}M\) and \(i\)-th face maps \(d_i d_{2n+1-i}: (sdNM)_n \rightarrow (sdNM)_{n-1}\). The levelwise involution on \(sdNM\) now commutes with the face maps, and \(sdN^{1,1}M\) is a semi-simplicial \(\mathbb{Z}/2\)-space.

Definition 3.1. The classifying space of a non-unital topological monoid with anti-involution \(M\) is the \(\mathbb{Z}/2\)-space defined as the geometric realization

\[B^{1,1}M = |sdN^{1,1}M|\]

with the involution induced by the semi-simplicial involution of \(sdN^{1,1}M\).

Remark 3.2. i) In contrast with the simplicial case, the geometric realization of a semi-simplicial space is in general not equivalent to the geometric realization of its subdivision. However, if our monoid \(M\) is non-equivariantly equivalent to a unital monoid \(M'\), and if \(M'\) is well pointed at the unit, then the canonical map \(|sdNM| \rightarrow |NM|\) is a weak homotopy equivalence. This is because of the following commutative diagram

\[
\begin{array}{ccc}
|sdNM| & \xrightarrow{\sim} & |sdNM'| \\
\downarrow & & \downarrow \\
|NM| & \xrightarrow{\sim} & |NM'|
\end{array}
\]

where \(|-|_s\) denotes the geometric realization of a simplicial space. In the examples of interest in this paper we will always be in this situation.
ii) There is another possible definition for $B^{1,1}M$, which is the space $|NM|$ with the involution
\[ [x \in N_n M, (t_0, \ldots, t_n) \in \Delta^n] \mapsto [w(x) \in N_n M, (t_n, \ldots, t_0) \in \Delta^n]. \]
If $M$ is unital and well-pointed, this $\mathbb{Z}/2$-space is equivariantly equivalent to $B^{1,1}M$, in view of the commutative square
\[
\begin{array}{ccc}
|sdNM| & \sim & |sdNM|_s \\
\downarrow & & \downarrow \\
|NM| & \sim & |NM|_s
\end{array}
\]
where the canonical homeomorphism is easily seen to be equivariant. These two constructions do not agree in general. We choose to work with definition 3.1 because it gives us better control over the fixed-points:

iii) The geometric realization of semi-simplicial sets commutes with fixed-points of finite groups. This can be easily proved by induction on the skeleton filtration, since fixed-points commute with pushouts along cofibrations and with filtered colimits. Thus the fixed-points space $(B^{1,1}M)^{\mathbb{Z}/2}$ is homeomorphic to the geometric realization of the semi-simplicial space $(sdNM)^{\mathbb{Z}/2}$.

The fixed-points of $B^{1,1}M$ are modeled not by a monoid, but by a category. Let us define a topological category $Sym M$ (without identities) as follows. Its space of objects is the fixed-points space $M^{\mathbb{Z}/2}$, and the morphisms $m \to n$ consist of the subspace of elements $l \in M$ with $n = l \cdot m \cdot w(l)$. Composition is defined by $l \circ k = k \cdot l$.

Lemma 3.3. Let $M$ be a non-unital topological monoid with anti-involution. The $\mathbb{Z}/2$-fixed points of $B^{1,1}M$ are naturally homeomorphic to the classifying space of $Sym M$.

Proof. By the previous remark, it is sufficient to understand the fixed-points of $sdNM$ levelwise. There is an equivariant isomorphism of semi-simplicial $\mathbb{Z}/2$-spaces between $sdNM$ and $NsdM$, where $sdM$ is the edgewise subdivision of the category $M$ (aka twisted arrow category). This is the topological category with $\mathbb{Z}/2$-action whose space of objects is $M$, and where the space of morphisms $m \to n$ is the subspace of $M \times M$ of pairs $(l, k)$ such that $n = lmk$. Composition is defined by $(l, k) \circ (l', k') = (ll', kk')$.

The involution on $sdM$ sends an object $m$ to $w(m)$, and a morphism $(l, k) \mapsto (w(k), w(l))$. Since the nerve functor commute with fixed-points, the fixed-points of $sdNM$ are isomorphic to the nerve of the fixed-points category of $sdM$. It’s objects are the fixed-objects $M^{\mathbb{Z}/2}$, and its morphisms the pairs $(l, k)$ where $k = w(l)$. This is isomorphic to the category $Sym M$. □

Lemma 3.4. Let $f : M \to M'$ be a map of non-unital topological monoids with anti-involutions, and suppose that $f$ is a weak equivalence of $\mathbb{Z}/2$-spaces. Then
\[ B^{1,1}f : B^{1,1}M \longrightarrow B^{1,1}M' \]
is a $\mathbb{Z}/2$-equivalence.

Proof. Non-equivariantly $Bf$ is an equivalence, since realizations of semi-simplicial spaces preserve levelwise equivalences. Since realizations commute with fixed-points it remains to show that $(sdNM)^{\mathbb{Z}/2} \to (sdNM')^{\mathbb{Z}/2}$ is a levelwise equivalence, that is that for every $n$
\[ (M^{\times 2n+1})^{\mathbb{Z}/2} \longrightarrow ((M')^{\times 2n+1})^{\mathbb{Z}/2} \]
is a weak equivalence. By inspection, this is the map
\[ f^{\times n} \times f^{\mathbb{Z}/2} : M^{\times n} \times M^{\mathbb{Z}/2} \longrightarrow (M')^{\times n} \times (M')^{\mathbb{Z}/2} \]
which is an equivalence by assumption. □
Example 3.5. Let \( \pi \) be a discrete group with the anti-involution defined by inversion \( w = (-1)^{-1} \pi^{op} \rightarrow \pi \). The \( \mathbb{Z}/2 \)-space \( B^{1,1}_\pi \) is a classifying space for principal \( \pi \)-bundles of \( \mathbb{Z}/2 \)-spaces. A model for such a universal bundle is constructed in [May90] as the map

\[
\text{Map}(E\mathbb{Z}/2, E\pi) \longrightarrow \text{Map}(E\mathbb{Z}/2, B\pi),
\]

where \( E\pi \) denotes the free and contractible \( \pi \)-space. The base-space is equivalent to the nerve of the functor category \( \text{Cat}(E\mathbb{Z}/2, \pi) \) where \( E\mathbb{Z}/2 \) is the translation category of the left \( \mathbb{Z}/2 \)-set \( EZ/2 \) (whose nerve is the classical model for \( E\mathbb{Z}/2 \)), see [GH94]. It is easy to see that the nerve of \( \text{Cat}(E\mathbb{Z}/2, \pi) \) and the edgewise subdivision of \( N^{1,1}_\pi \) are equivariantly isomorphic.

The following property will be used in the definition of the splitting of \( A_{B[n]} \) in [ES].

**Lemma 3.6.** Let \( \pi \) be a discrete group with anti-involution \( w \). There is a map \( B\pi \rightarrow (B^{1,1}_\pi)^{2/2} \), and its composition with the fixed-points inclusion \( B\pi \rightarrow (B^{1,1}_\pi)^{2/2} \rightarrow B\pi \) is homotopic to the identity. This exhibits \( B\pi \) as a retract of \( (B^{1,1}_\pi)^{2/2} \).

**Proof.** We define a simplicial map \( \phi : N_p \pi \rightarrow (sdN^{1,1}_\pi)_p = N^{1,1}_{2p+1}_\pi \) by

\[
(g_0, \ldots, g_p) \mapsto (g_0, \ldots, g_p, 1, w(g_p), \ldots, w(g_1)).
\]

This map clearly lands in the fixed-points of \( sdN^{1,1}_\pi \). Non-equivariantly, the composition of \( \phi \) followed by the last-vertex map \( sdN\pi \rightarrow N\pi \) is the identity. After identifying \( B\pi \) and \( |sdN\pi| \) the last-vertex map is homotopic to the identity. Thus so is \( \phi \).

We conclude the section with the construction of a real analogue of the cyclic nerve. Let \( M \) be a non-unital topological monoid with anti-involution. The cyclic nerve of \( M \) is the semi-simplicial space \( N^{cy}_\pi M \) with \( n \)-simplices

\[
N^{cy}_n M = M^{\times n+1},
\]

and with face maps

\[
d_i(m_0, \ldots, m_n) = (m_0, \ldots, m_{i-1}, m_{i+1}m_i, m_{i+2}, \ldots, m_n), \quad 0 \leq i \leq n-1
\]

\[
d_n(m_0, \ldots, m_n) = (m_0m_n, m_1, \ldots, m_{n-2}, m_{n-1}).
\]

It has a levelwise involution, defined by

\[
(m_0, \ldots, m_n) \mapsto (w(m_0), w(m_n), \ldots, w(m_1)).
\]

**Definition 3.7.** The semi-simplicial space \( N^{cy}_\pi M \) equipped with the levelwise involution above is called the dihedral nerve of \( M \), and it is denoted by \( N^{di}_\pi M \). Its edgewise subdivision \( sdN^{di}_\pi M \) is a semi-simplicial \( \mathbb{Z}/2 \)-space, and its geometric realization is denoted

\[
B^{di}_\pi M := |sdN^{di}_\pi M|
\]

and called the dihedral Bar construction.

The fixed-points of the subdivided dihedral nerve of \( M \) are isomorphic to the two-sided Bar construction

\[
(N^{2n+1}_{2n+1} M)^{2/2} \cong N_0(M^{2/2}, M, M^{2/2})
\]

of the left action of \( M \) on \( M^{2/2} \) defined by \( m \cdot n := mnnv(m) \) and the right action \( n \cdot m := w(m)n \). Thus the simplicial set \( (sdN^{di}_\pi M)^{2/2} \) is isomorphic to the nerve of a category \( \text{Sym}^{cy}_\pi M \). Its objects are the pairs \( (n_0, n_1) \) of fixed-points of \( M^{2/2} \). A morphism \( m : (n_0, n_1) \rightarrow (n'_0, n'_1) \) is an element \( m \in M \) such that \( n'_0 = m \cdot n_0 \) and \( n_1 = n'_1 \cdot m \). Thus we obtain the following.

**Proposition 3.8.** There is a natural isomorphism of simplicial sets

\[
(sdN^{di}_\pi M)^{2/2} \cong N \text{Sym}^{cy}_\pi M.
\]
The constructions \( N^{1,1} \) and \( N^{di} \) extend to categories with duality. We will use this generalization occasionally, mostly in \( \S 3.4 \).

**Remark 3.9.** We recall that a category with strict duality is a category (possibly without identities) \( C \) equipped with a functor \( D: \text{C}^{\text{op}} \to \text{C} \) such that \( D^2 = \text{id} \). If \( C \) has one object this is the same as a monoid with anti-involution. There is a levelwise involution on the nerve \( NC \) which is defined by

\[
( c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n ) \mapsto ( Dc_n \xrightarrow{Df_n} Dc_{n-1} \to \cdots \xrightarrow{Df_1} Dc_0 ).
\]

We define \( B^{1,1}C := |sdN^{1,1}C| \). There is a category \( \text{Sym}^{C} \) whose objects are the morphisms \( f: c \to Dc \) such that \( Df = f \), and the morphisms \( f \to f' \) are the maps \( \gamma: c \to c' \) such that \( f = D(\gamma)f'\gamma \). The previous considerations extend to give an isomorphism

\[
|sdN^{1,1}C| \cong N\text{Sym}^{C}.
\]

**Remark 3.10.** Similarly, there is a construction of the dihedral nerve of a category with strict duality. An \( n \)-simplex of the cyclic nerve \( N^{\text{cyc}}C \) is a string of composable morphisms

\[
c_n \xrightarrow{f_n} c_{n-1} \to \cdots \xrightarrow{f_2} c_1 \to \xrightarrow{f_1} c_0,
\]

and the levelwise involution of the dihedral nerve sends this string to

\[
Dc_0 \xrightarrow{Df_0} Dc_n \xrightarrow{Df_n} Dc_{n-1} \to \cdots \xrightarrow{Df_2} Dc_1 \xrightarrow{Df_1} Dc_0.
\]

We define \( B^{1,1}C := |sdN^{1,1}C| \).

### 3.2 Ring-spectra with anti-involution

Let \( A \) be an orthogonal ring spectrum. An anti-involution on \( A \) is a map of ring spectra \( w: A^{\text{op}} \to A \) such that \( w \circ w = \text{id} \). We observe that the map \( w \) gives the underlying spectrum of \( A \) the structure of a \( \mathbb{Z}/2 \)-spectrum, but that the (genuine) fixed-points spectrum \( A^{\mathbb{Z}/2} \) is no longer a ring spectrum. In this section we will explain how such an object generalizes the Hermitian Mackey functors of \( \text{HL} \) and we will define a genuine spectral versions of Hermitian forms over \( A \).

We let \( I \) be Bökstedt’s category of finite sets and injective maps. Its objects are the natural numbers (zero included), and a morphism \( i \to j \) is an injective map \( \{1, \ldots, i\} \to \{1, \ldots, j\} \). We recall that the spectrum \( A \) induces a diagram \( \Omega^{\bullet}A: I \to \text{Top}^* \) (see e.g. \( \text{Sch04} \)) by sending an integer \( i \) to the \( i \)-fold loop space \( \Omega^iA \). We denote its homotopy colimit by

\[
\Omega^\infty_I A := \text{hocolim}_I \Omega^{\bullet}A.
\]

On the one hand the multiplication of \( A \) endows \( \Omega^\infty_I A \) with the structure of a topological monoid (see \( \text{Sch04} \)). On the other hand, the category \( I \) has an involution which is trivial on objects and that sends a morphism \( \alpha: i \to j \) to

\[
\alpha(s) = j + 1 - \alpha(i + 1 - s).
\]

The diagram \( \Omega^{\bullet}A \) has a \( \mathbb{Z}/2 \)-structure in the sense of \( \text{DM16} \), defined by the maps

\[
\Omega^iA \xrightarrow{\Omega^\omega} \Omega^iA \xrightarrow{\Omega^\chi_i} \Omega^iA \xrightarrow{(-)\circ \chi_i} \Omega^iA
\]

where \( \chi_i: i \to i \) is the permutation the reverses the order on \( i \), and \( \Omega^\infty_I A \) has an induced \( \mathbb{Z}/2 \)-action. These two structures make \( \Omega^\infty_I A \) into a topological monoid with anti-involution. In case \( A \) is non-unital, \( \Omega^\infty_I A \) is a non-unital topological monoid with anti-involution.

**Remark 3.11.** Throughout the paper, we will make extensive use of the fact that, as a \( \mathbb{Z}/2 \)-space, \( \Omega^\infty_I A \) is equivalent to the genuine equivariant infinite loop space of \( A \). There is a comparison map

\[
\text{hocolim}_{n \in \mathbb{N}} \Omega^{n\rho+1}A_{n\rho+1} \to \Omega^\infty_I A,
\]
where $\rho$ is the regular representation of $\mathbb{Z}/2$, coming from the inclusion $\mathbb{N} \to I$ that sends $n$ to $2n + 1$ and the unique morphism $n \to m$ to the map

$$\iota(s) = \begin{cases} 
s & \text{if } 1 \leq s \leq n \\
m + 1 & \text{if } s = n + 1 \\
s + 2(m - n) & \text{if } n + 2 \leq s \leq 2n + 1 \end{cases}$$

The failure of this map from being a non-equivariant equivalence is measured by the action of the monoid of self-injections of $\mathbb{N}$ on the poset $\mathbb{N}$, and this action is homotopically trivial since $A$ is an orthogonal spectrum (see [SS13, §2.5]). A similar comparison exists equivariantly, and the comparison map is an equivariant equivalence since $A$ is an orthogonal $\mathbb{Z}/2$-spectrum. The details can be found in [DMPPR17]).

Since $\Omega^n A$ is a topological monoid with anti-involution, there is an action

$$\Omega^n A \times (\Omega^n A)^{\mathbb{Z}/2} \to (\Omega^n A)^{\mathbb{Z}/2}$$

defined by $a \cdot b := abw(a)$ where we denoted by $w$ the anti-involution of $\Omega^n A$. We use this action to define a category of Hermitian forms over $A$ in a way somewhat analogous to the Mackey functor case of [1.2].

**Definition 3.12.** We let $M_n^\vee(A)$ be the (non-unital) ring spectrum

$$M_n^\vee(A) = \bigvee_{n \times n} A,$$

where the multiplication is defined by the maps $M_n^\vee(A_i) \wedge M_n^\vee(A_j) \to M_n^\vee(A_{i+j})$ that send $((k, l), a) \wedge ((k', l'), a')$, where $(k, l), (k', l') \in n \times n$ indicate the wedge component, to $((k, l'), a \cdot a')$ if $k' = l$, and to the basepoint otherwise.

The anti-involution $w: A^\op \to A$ induces an anti-involution on $M_n^\vee(A)$, defined as the composite

$$M_n^\vee(A)^{\op} = (\bigvee_{n \times n} A)^{\op} \xrightarrow{\tau} \bigvee_{n \times n} A^\op \xrightarrow{\wedge w} \bigvee_{n \times n} A = M_n^\vee(A)$$

where $\tau$ is the automorphism of $n \times n$ which swaps the product factors. We now let $\widetilde{M}_n^\vee(A)$ be the non-unital topological monoid with anti-involution

$$\widetilde{M}_n^\vee(A) := \Omega^n A_n^\vee M_n^\vee(A).$$

We let $\widetilde{GL}_n^\vee$ be the subspace of invertible components, defined as the pullback of non-unital topological monoids with anti-involution

$$\begin{array}{ccc}
\widetilde{GL}_n^\vee(A) & \longrightarrow & \widetilde{M}_n^\vee(A) \\
\downarrow & & \downarrow \\
GL_n(\pi_0 A) & \longrightarrow & M_n(\pi_0 A)
\end{array}$$

where $\pi_0 A$ is the ring of components with the induced anti-involution, and $GL_n(\pi_0 A)$ its ring of invertible $(n \times n)$-matrices with involution given by entrywise involution and transposition. The right vertical map is the composite

$$\Omega^n A_n^\vee M_n^\vee(A) \xrightarrow{\sim} \Omega^n \prod_{n \times n} A \longrightarrow M_n(\pi_0 A)$$

which is both equivariant and multiplicative.

**Definition 3.13.** An $n$-dimensional Hermitian form on $A$ is an element of the fixed-points space $\widetilde{GL}_n^\vee(A)^{\mathbb{Z}/2}$. These form a category $\text{Sym} \widetilde{GL}_n^\vee(A)$ as in [3.3] and we define

$$\text{Herm}_A := \prod_{n \geq 0} \text{Sym} \widetilde{GL}_n^\vee(A).$$
Remark 3.14. The anti-involution of \( A \) induces a functor \( D: \mathcal{M}_A^{\text{op}} \to \mathcal{M}_A \) on the category \( \mathcal{M}_A \) of right module spectra. It is defined by the spectrum of module maps

\[
D(P) = \text{Hom}_A(P, A_w),
\]

where \( A_w \) is the spectrum \( A \) equipped with the right \( A \)-module structure

\[
A \wedge A \xrightarrow{\tau} A \wedge A \xrightarrow{w \wedge \text{id}} A \wedge A \xrightarrow{\mu} A,
\]

where \( \tau \) is the symmetry isomorphisms and \( \mu \) is the multiplication of \( A \). The ring spectrum of \((n \times n)\)-matrices on \( A \) is usually defined as the endomorphism spectrum \( \text{End}(\bigvee_n A) \) of the sum of \( n \)-copies of \( A \). Since \( \text{Hom}_A(A, P) \) is canonically isomorphic to \( P \), the endomorphism ring \( \text{End}(\bigvee_n A) \) is canonically isomorphic to \( \prod_n \bigvee_n A \). The module \( \bigvee_n A \) is homotopically self-dual, as the inclusion of wedges into products

\[
\bigvee_n A \xrightarrow{\sim} \prod_n A \cong D(\bigvee_n A)
\]

is a natural equivalence. This defines a homotopy coherent action on \( \text{End}(\bigvee_n A) \), and one could define Hermitian forms as the homotopy fixed-points of this action (this is essentially the approach of [Spi16]). The inclusion \( M'_n(\bigvee_n A) \to \text{End}(\bigvee_n A) \) is a weak equivalence, and it is coherently equivariant. The point of our construction is to exploit the fact that the action on \( M'_n(\bigvee_n A) \) is strict, and that it therefore gives rise to a genuine equivariant homotopy type. The resulting fixed-points spectrum \( M'_n(\bigvee_n A)^{\mathbb{Z}/2} \) is an equivariant refinement of the homotopy coherent action on \( \text{End}(\bigvee_n A) \), and it generally differs from the homotopy fixed-points. Morally speaking, this equivariant homotopy type determines the notion of "symmetry" for the associated Hermitian forms.

3.3 The real \( K \)-theory \( \mathbb{Z}/2 \)-space of a ring spectrum with anti-involution

The goal of this section is to define a \( \mathbb{Z}/2 \)-action on the \( K \)-theory space of a ring spectrum \( A \) with anti-involution \( w: A^{\text{op}} \to A \). We define this action by adapting the group completion construction of the free \( K \)-theory space

\[
K(A) = \Omega B \prod_n B\text{GL}_n(A),
\]

where \( \prod_n B\text{GL}_n(A) \) is group-completed with respect to block-sum, to the model for the equivariant matrix ring constructed in the previous section.

We recall from [LS] that the classifying space of a non-unital monoid with anti-involution \( M \) inherits a natural \( \mathbb{Z}/2 \)-action, which we denote by \( B^{1,1} M \). Thus the anti-involution on \( \tilde{\text{GL}}_n(A) \) gives rise to a \( \mathbb{Z}/2 \)-space \( B^{1,1} \tilde{\text{GL}}_n(A) \). However, the space \( \Pi_n B^{1,1} \tilde{\text{GL}}_n(A) \) does not have a strict monoid structure, since the standard block-sum of matrices does not restrict to the matrix rings \( \tilde{M}'_n(A) \). We can however define a Bar construction using a technique similar to Segal's group completion of partial monoids. The block sum operation on the ring spectra \( \tilde{M}'_n(A) \) is a collection of maps

\[
\oplus: \tilde{M}'_n(A) \vee \tilde{M}'_k(A) \to \tilde{M}'_{n+k}(A)
\]

induced by the inclusions \( n \to n + k \) and \( k \to n + k \). We observe that this map commutes with the anti-involutions. There is a simplicial \( \mathbb{Z}/2 \)-space with \( p \)-simplices

\[
\prod_{m_1, \ldots, m_p} B^{1,1} \Omega_\infty^\infty(\tilde{M}'_{m_1}(A) \vee \cdots \vee \tilde{M}'_{m_p}(A)).
\]

The face maps are induced by the block-sum maps, and the degeneracies are the summand inclusions. This results into a well-defined simplicial object since \( B^{1,1} \Omega_\infty^\infty \) is functorial with
Respect to maps of ring spectra with anti-involution. This simplicial structure restricts to the \( \mathbb{Z}/2 \)-spaces

\[
\prod_{n_1, \ldots, n_p} B^{1,1} GL_{n_1, \ldots, n_p}^\vee(A)
\]

where \( GL_{n_1, \ldots, n_p}^\vee(A) \) is the pull-back of non-unital monoids with anti-involution

\[
\begin{array}{ccc}
\widehat{GL}_{n_1, \ldots, n_p}^\vee(A) & \longrightarrow & \Omega^\infty_1(M_{n_1}^\vee(A) \vee \cdots \vee M_{n_p}^\vee(A)) \\
\downarrow & & \downarrow \\
(GL_{n_1}^\vee(\pi_0 A) \times \cdots \times GL_{n_p}^\vee(\pi_0 A)) & \longrightarrow & (M_{n_1}(\pi_0 A) \times \cdots \times M_{n_p}(\pi_0 A))
\end{array}
\]

**Definition 3.15.** The free real \( K \)-theory space of a ring spectrum with anti-involution \( A \) is the \( \mathbb{Z}/2 \)-space defined as the loop space of the thick geometric realization

\[
KR(A) := \Omega | \prod_{n_1, \ldots, n_\bullet} B^{1,1} GL_{n_1, \ldots, n_\bullet}^\vee(A) |.
\]

We observe that all of our constructions are homotopy invariant, and therefore the functor \( KR \) sends maps of ring spectra with anti-involution which are stable equivalences of underlying \( \mathbb{Z}/2 \)-spectra to equivalences of \( \mathbb{Z}/2 \)-spaces. We conclude this section by verifying that the underlying space of \( KR(A) \) has the right homotopy type.

**Proposition 3.16.** Let \( A \) be a ring spectrum with anti-involution, and let us denote \( A|_1 \) the underlying ring spectrum of \( A \) and \( KR(A)|_1 \) the underlying space of \( KR(A) \). There is a weak equivalence

\[
KR(A)|_1 \sim K(A|_1)
\]

**Proof.** For convenience, let us drop the restriction notation. The inclusion of wedges into products defines an equivalence of ring spectra \( M_n^\vee(A) \rightarrow M_n(A) := \prod_n V_n A \), and therefore an equivalence of monoids \( \hat{M}_n^\vee(A) \rightarrow \hat{M}_n(A) \) on infinite loop spaces. This induces an equivalence of spaces

\[
\prod_n B\hat{M}_n^\vee(A) \longrightarrow \prod_n B\hat{M}_n(A)
\]

after taking the thick realization. The block-sum maps of \( M_n^\vee(A) \) and \( M_n(A) \) are compatible, in the sense that the diagram

\[
\begin{array}{ccc}
M_n^\vee(A) \vee M_k^\vee(A) & \longrightarrow & M_{n+k}^\vee(A) \\
\sim & & \sim \\
M_n(A) \times M_k(A) & \longrightarrow & M_{n+k}(A)
\end{array}
\]

commutes. It follows that the level-wise equivalences on the Bar constructions

\[
\prod_{n_1, \ldots, n_p} B\Omega^\infty_1(M_{n_1}^\vee(A) \vee \cdots \vee M_{n_p}^\vee(A)) \sim \prod_{n_1, \ldots, n_p} (B\Omega^\infty_1 M_{n_1}(A) \times \cdots \times (B\Omega^\infty_1 M_{n_p}(A))
\]

commute with the face maps. After restricting to invertible components and taking thick geometric realizations this gives the equivalence \( KR(A)|_1 \sim K(A|_1) \). \( \square \)

We conclude this section with a cyclic version of this construction. There is a similar simplicial structure on the spaces

\[
\prod_{n_1, \ldots, n_p} B^{di} GL_{n_1, \ldots, n_p}^\vee(A),
\]

induced by the same block-sum maps using the functoriality of the dihedral Bar construction \( B^{di} \) from \([3,7]\).
**Definition 3.17.** The free cyclic real $K$-theory space of a ring spectrum with anti-involution $A$ is the $\mathbb{Z}/2$-space
\[
\text{KR}^c(A) := \Omega | \prod_{k_1, \ldots, k_n} B^{\text{d}i \hat{G}_{k_1, \ldots, k_n}}(A)|.
\]

### 3.4 Connective equivariant deloopings of real $K$-theory

We show that the real $K$-theory space of a ring spectrum with anti-involution defined in 3.3 is the equivariant infinite loop space of a (special) $\mathbb{Z}/2$-equivariant $\Gamma$-space. Our construction of these deloopings is an adaptation of Segal’s construction (Seg74 and SS79) for spectrally enriched symmetric monoidal categories, to a set-up where the symmetric monoidal structure is partially defined.

We start with an explicit definition of the $\mathbb{Z}/2$-$\Gamma$-space, and relate it to Segal’s construction in the proof of 3.1. We recall from [Shi91] that a $G$-space, where $G$ is a finite group, is simply a functor $X: \Gamma^G \to \text{Top}^G$, from the category $\Gamma^G$ which is a skeleton for the category of pointed finite sets and pointed maps, to the category of pointed $G$-spaces. This induces a symmetric $G$-spectrum whose $n$-th space is the value at the $n$-sphere of the left Kan-extension of $X$ to the category of finite pointed simplicial sets.

For every natural number $n$ and sequence of non-negative integers $\mathbf{a} = (a_1, \ldots, a_n)$ we consider the collections of permutations $\alpha = \{\alpha_{S,T} \in \Sigma_{S \subseteq \Pi T} a_i\}$ where the indices $S, T$ run through the pairs of disjoint subsets $\Pi T \subset \{1, \ldots, n\}$. We require that these permutations satisfy the standard conditions of Segal’s construction, see e.g. [DGM13 2.3.1.1]. We denote by $\langle \mathbf{a} \rangle$ the set of such collections $\alpha$ for the $n$-tuple $\mathbf{a}$.

Given a ring spectrum with anti-involution $A$ we let $\text{KR}(A): \Gamma^G \to \text{Top}^{\mathbb{Z}/2}$ be the functor that sends the pointed set $n_+ = \{+, 1, \ldots, n\}$ to
\[
\text{KR}(A)_n := \prod_{\mathbf{a}} B^{1,1}(\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A))
\]
where $\hat{G}_{a_1, \ldots, a_n}(A)$ is defined in 3.3 and $\langle \mathbf{a} \rangle$ is the category with objects set $\langle \mathbf{a} \rangle$, and with a unique morphism for any pair of objects. It has a duality that is the identity on objects, and that sends the unique morphism $\alpha \to \beta$ to the unique morphism $\beta \to \alpha$. Thus $\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A)$ is a non-unital topological category with duality, and $B^{1,1}$ is the functor of 3.3.

**Remark 3.18.** Since every object in $\langle \mathbf{a} \rangle$ is both initial and final, the projection map
\[
\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A) \to \hat{G}_{a_1, \ldots, a_n}(A)
\]
is an equivalence of topological categories. Moreover by the uniqueness of the morphisms of $\langle \mathbf{a} \rangle$ we see that $\text{Sym}(\langle \mathbf{a} \rangle) = \langle \mathbf{a} \rangle$. Thus the projection map
\[
\text{Sym}(\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A)) \cong \text{Sym}(\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A)) \to \text{Sym} \hat{G}_{a_1, \ldots, a_n}(A)
\]
is also an equivalence of categories. It follows that $B^{1,1}(\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A)) \cong B^{1,1} \hat{G}_{a_1, \ldots, a_n}(A)$.

The extra $\langle \mathbf{a} \rangle$-coordinate is used for defining $\text{KR}(A)$ on morphisms. Given a pointed map $f: n_+ \to k_+$ and $\alpha \in \langle \mathbf{a} \rangle$ we let $f_* \mathbf{a} \in N^{\mathbf{a}}$ and $f_* \alpha \in \langle f_* \mathbf{a} \rangle$ denote respectively
\[
(f_* \mathbf{a})_i := \sum_{j \in f^{-1}(i)} a_j, \quad (f_* \alpha)_{S,T} := \alpha_{f^{-1} S, f^{-1} T}
\]
for every $1 \leq i \leq k$ and $S \Pi T \subset \{1, \ldots, k\}$. We define $f_*: \text{KR}(A) \to \text{KR}(A)$ by mapping the $\mathbf{a}$-summand to the $f_* \mathbf{a}$-summand by a map whose first component is $B^{1,1}$ of the functor
\[
\langle \mathbf{a} \rangle \times \hat{G}_{a_1, \ldots, a_n}(A) \to \langle \mathbf{a} \rangle \xrightarrow{f_*} \langle f_* \mathbf{a} \rangle.
\]
where the first map is the projection. The second component of this map is defined as follows. A pair of permutations \( \alpha, \beta \in \langle \varnothing \rangle \) gives rise to a morphism of real monoids

\[
(\alpha, \beta)_*: GL_{a_1, \ldots, a_n}(A) \to GL^\vee_{f,\varnothing}(A)
\]

which is induced by the map of ring spectra with anti-involution obtained by wedging over \( i \in \{1, \ldots, k\} \) the maps

\[
(\alpha, \beta)_j: \bigvee_{j \in f^{-1}(i)} M^\vee_{a_j}(A) \to M^\vee_{(f,\varnothing),i}(A)
\]

defined by sending \( x \in M^\vee_{a_j}(A) \) to \( (\alpha, \beta)_j(x) := \beta_{(f^{-1}(i))\cup j}(0 \oplus x)\alpha^{-1}_{(f^{-1}(i))\cup j} \), where \( 0 \oplus x \) is the value at \( x \) of the block-sum map \( \oplus: M^\vee_{(f,\varnothing),i}(A) \vee M^\vee_{a_j}(A) \to M^\vee_{(f,\varnothing),i}(A) \). \( \alpha \) and \( \beta \) are considered as permutation matrices. More explicitly, an element of \( M^\vee_{a_j}(A) \) consists of a pair \( (c, d) \in a_j \times a_j \) and a point \( a \in A \) in some spectrum level. This is sent to

\[
(\alpha, \beta)_j(c, d, a) = (\beta_{(f^{-1}(i))\cup j}(a c), \alpha_{(f^{-1}(i))\cup j}(a d), a)
\]

where \( \iota: a_j \to (f_* a)_j \) is the inclusion. The second component of the map \( KR(f) \) is then induced by the functor

\[
(\varnothing) \times GL^\vee_{a_1, \ldots, a_n}(A) \to GL^\vee_{f,\varnothing}(A)
\]

which is the projection on objects and that sends a morphism \( (\alpha, \beta, x) \) in \( (\varnothing) \times GL_{a_1, \ldots, a_n}(A) \) to \( (\alpha, \beta), x \).

**Proposition 3.19.** Let \( A \) be a ring spectrum with anti-involution. The functor \( KR(A) \) is a special \( \mathbb{Z}/2\)–\( \Gamma \)-space in the sense of [Shi87](#). The first \( \mathbb{Z}/2 \)-space of the associated positively fibrant \( \mathbb{Z}/2 \)-spectrum \( KR(A)_{\mathbb{Z}/2} \) is equivalent to the real \( K \)-theory \( \mathbb{Z}/2 \)-space \( KR(A) \) of [DGM13](#). The underlying \( \Gamma \)-space of \( KR(A) \) is equivalent the \( K \)-theory \( \Gamma \)-space of \( A \).

**Proof.** Let \( F_A \) be the spectrally enriched category whose objects are the non-negative integers and where the endomorphisms of \( k \) consist of the matrix ring \( \prod_k \mathbb{V}_k \). We recall that the Segal construction on \( F_A \) is the \( \Gamma \)-category enriched in symmetric spectra defined by sending \( n_+ \) to the category \( F_A[n] \). Its objects are the pairs \( (\varnothing, \alpha) \) where \( \varnothing = (a_1, \ldots, a_n) \) is a collection of non-negative integers, and \( \alpha \) is a collection of isomorphisms \( \alpha = \{\alpha_{S, T}: \sum_{a \in S} a \alpha_{S, T} + \sum_{a \in T} a \alpha_{S, T} \to \sum_{a \in S \cup T} a \alpha_{S, T}\} \) in the underlying category of \( F_A \) satisfying the conditions of [DGM13](#). The spectrum of morphisms \( (\varnothing, \alpha) \to (\varnothing, \beta) \) is non-trivial only if \( \varnothing = \varnothing \), and it is defined by the collection of elements \( \{f_S \in M_{\sum_{a \in S} a \alpha_{S, T}}\}_{S \subseteq \varnothing} \) which satisfy \( \beta_{S, T}(f_S \oplus f_T) = f_{S \cup T} \alpha_{S, T} \).

There is an equivalence of spectral categories \( F_A \to F_A[n] \) that sends \( \varnothing = (a_1, \ldots, a_n) \) to \( (\varnothing, \alpha) \) where \( \alpha_{S, T} \) is the permutation matrix of the permutation of \( S \sqcup T \) that sends the order on \( S \sqcup T \) induced by the disjoint union of the orders of \( S \) and \( T \) to the order of \( S \sqcup T \) as a subset of \( n \) (the point is that \( F_A[n] \) is functorial in \( n \) with respect to all maps of pointed sets, whereas \( F_A^n \) only for order-preserving maps).

Now let \( F_A^n \) be the equivalent subcategory (without identities) of \( F_A \) with all the objects, but where the endomorphisms of \( a \) are the ring spectra \( M^\vee_{a\alpha}(A) = \bigvee_{n \times n} A \). The space \( KR(A)_n \) is roughly the invertible components of the classifying space of the image of \( (F_A^n) \) inside \( F_A[n] \). More precisely, there is a commutative square of spectrally enriched categories

\[
\begin{array}{ccc}
(F_A^n)^{\times n} & \to & F_A^n[n] \\
\downarrow & & \downarrow \\
F_A^n \times \sim & F_A^n[n]
\end{array}
\]

where \( F_A^n[n] \) is defined as the subcategory of \( F_A[n] \) on the objects \( (\varnothing, \alpha) \) where \( \alpha_{S, T} \) is a permutation representation, and where the morphisms \( \{f_S\}_{S \subseteq \varnothing} \) are such that there is a \( j \in \varnothing \) such that \( f_S = 0 \) if \( j \not\in S \). The top horizontal arrow is simply the restriction of the bottom horizontal one.
The spectral category $\mathcal{F}_A^\prime[n]$ has a strict duality, which is the identity on objects and the anti-involution on the matrix ring $M_\ast^\prime(A)$ on morphism. We observe that a morphism $\{f_s\}_{S \subset \underline{a}}$ in $\mathcal{F}_A^\prime[n]$ is determined by the value $f_j$, since for every $S \subset n$ containing $j$ we have that

$$f_S = f_{(S \setminus j) \cup j} = \beta_{S \setminus j, j}(0_{S \setminus j} \oplus f_j)\alpha_{S \setminus j, j}^{-1}.$$ 

Moreover this equation determines the relation $\beta_{S, T}(f_S \oplus f_T) = f_{S \cup T}\alpha_{S, T}$, and it follows that the value at $n_+$ of the corresponding $\mathbb{Z}/2$-$\Gamma$-space is

$$\prod_{\underline{a}} B^{1,1}(\underline{a}) \times \Omega^\infty_n(M_{a_1}^\prime(A) \vee \cdots \vee M_{a_n}^\prime(A)).$$

Its invertible components are then defined $\text{KR}_{\ast}(A)$ and the functoriality in $\Gamma^{op}$ induced by the ambient category $\mathcal{F}_A^\prime[n]$ is the one described above. The fact that $f_S := \beta_{S \setminus j, j}(0_{S \setminus j} \oplus f_j)\alpha_{S \setminus j, j}^{-1}$ determines a well-defined morphism $\langle \underline{a}, \alpha \rangle \to \langle \underline{a}, \beta \rangle$ follows from the following calculation:

$$\beta_{S, T}(f_S \oplus 0_T)\alpha_{S, T}^{-1} = \beta_{S, T}(\beta_{S \setminus j, j}(0_{S \setminus j} \oplus f_j)\alpha_{S \setminus j, j}^{-1} \oplus 0_T)\alpha_{S, T}^{-1}$$

$$= \beta_{S, T}(\beta_{S \setminus j, j} \Pi \text{id}_T)(0_{S \setminus j} \oplus f_j \oplus 0_T)(\alpha_{S \setminus j, j}^{-1} \Pi \text{id}_T)\alpha_{S, T}^{-1}$$

$$= \beta_{j, \text{SIT}_{\setminus j}}(\text{id}_j \Pi \beta_{S \setminus j, T})(\tau_{S \setminus j, j}) \Pi \text{id}_T)(0_{S \setminus j} \oplus f_j \oplus 0_T)(\alpha_{S \setminus j, j}^{-1} \Pi \text{id}_T)\alpha_{S, T}^{-1}$$

$$= \beta_{j, \text{SIT}_{\setminus j}}(\text{id}_j \Pi \beta_{S \setminus j, T})(f_j \oplus \text{SIT}_{\setminus j})(\text{id}_j \Pi \alpha_{S \setminus j, T}^{-1})\alpha_{S \setminus j, j}^{-1}$$

$$= \beta_{j, \text{SIT}_{\setminus j}}(f_j \oplus \text{SIT}_{\setminus j})\alpha_{S \setminus j, j}^{-1}$$

where $\tau_{S, T} : \sum_{s \in S} a_s + \sum_{t \in T} a_t \to \sum_{t \in T} a_t + \sum_{s \in S} a_s$ is the symmetry isomorphism of the symmetric monoidal structure. From this description of the morphisms of $\mathcal{F}_A^\prime[n]$ one can easily see that the top horizontal map of the square above, and hence all its maps, is an equivalence of categories. Thus the $\Gamma$-space underlying $\text{KR}_{\ast}(A)$ is equivalent to the $K$-theory of $A$.

We show that $\text{KR}$ is a special $\mathbb{Z}/2$-$\Gamma$-space, that is for every group homomorphism $\rho : \mathbb{Z}/2 \to \Sigma_n$ the map

$$\mathcal{F}_A^\prime[n] \longrightarrow (\mathcal{F}_A^\prime[1])^{\times n},$$

whose $j$-component is induced by the map $n_+ \to 1_+$ that sends $j$ to 1 and the rest to the basepoint, is a $\mathbb{Z}/2$-equivariant equivalence. Here the involution is induced by $\rho : \mathbb{Z}/2 \to \Sigma_n$ through the functoriality in $n$. The square above provides an equivalence of spectrally enriched categories $\mathcal{F}_A^\prime \to \mathcal{F}_A^\prime[1]$. This functor is in fact an isomorphism on mapping spectra, and it is therefore an equivariant equivalence. We show that the left vertical arrow of

$$\mathcal{F}_A^\prime[n] \longrightarrow (\mathcal{F}_A^\prime[1])^{\times n}$$

$$\mathcal{F}_A^\prime[n] \longrightarrow \mathcal{F}_A^\prime[n]$$

defined as the restriction of $\mathcal{F}_A^{\times n} \to \mathcal{F}_A[n]$ is an equivariant equivalence, which will finish the proof. It has an equivariant inverse $\mathcal{F}_A^\prime[n] \to (\mathcal{F}_A^\prime)^{\times n}$ that sends an object $\langle \underline{a}, \alpha \rangle$ to $\underline{a}$ and a morphism $\{f_s\}$ to $f_{(j)}$. 

Similarly, there is a special $\mathbb{Z}/2$-$\Gamma$-space whose value at the pointed set $n_+$ is the $\mathbb{Z}/2$-space

$$\text{KR}_{eq}(A)_n := \prod_{\underline{a}} B^{di}(\underline{a}) \times \overline{GL}_{a_1, \ldots, a_n}(A)$$

where $B^{di}$ is the dihedral nerve functor of $\underline{a}$.10
3.5 Hermitian K-theory of ring-spectra with anti-involution

Let $A$ be a ring spectrum with anti-involution.

**Definition 3.20.** The free Hermitian $K$-theory space of $A$ is the fixed-points space

$$KH(A) = KR(A)^{Z/2} \cong \Omega B\left( \prod_n \left( B^{1,1}\hat{GL}_n(A) \right)^{Z/2} \right).$$

The free Hermitian $K$-theory spectrum of $A$ is the spectrum $\hat{KH}(A)$ associated to the fixed-points $\Gamma$-space

$$\hat{KH}(A)_n := KR(A)^{Z/2} \cong \prod_n \left( B^{1,1}(\hat{a}) \times \hat{GL}^\vee_{a_1,\ldots,a_n}(A) \right)^{Z/2}.$$

**Remark 3.21.** The spectrum associated to the $\Gamma$-space $\hat{KH}(A)$ is the naïve fixed-points spectrum of the $\mathbb{Z}/2$-spectrum associated to $KR(A)$. However since $KR(A)$ is special as an equivariant $\Gamma$-space (see [3.19]), the canonical map of spectra $KH(A) \to KR(A)^{Z/2}$ is a stable equivalence, where $KR(A)^{Z/2}$ is the genuine fixed-points spectrum of $KR(A)$.

We analyze the $\Gamma$-space $\hat{KH}(A)$ and we interpret it as the Segal construction of a symmetric monoidal category of Hermitian forms on $A$. For the Eilenberg-MacLane spectra of simplicial rings with anti-involution we show that our construction agrees with the connective cover of Burghelea and Fiedorowicz’s Hermitian $K$-theory of simplicial rings, from [BFS4]. We also show that for Eilenberg MacLane spectra of Hermitian Mackey functors this construction is equivalent to the one of [1.2].

We recall from lemma 3.3 that the the fixed-points space of $B^{1,1}\hat{GL}^\vee_{a_1,\ldots,a_n}(A)$ is homeomorphic to the classifying space of a topological category $\mathcal{M}$. The space of objects of this category is the space of invertible components of the fixed-points space

$$\hat{M}^\vee_{a_1,\ldots,a_n}(A)^{Z/2} := (\Omega^\infty (M^\vee_{a_1}(A) \vee \cdots \vee M^\vee_{a_n}(A)))^{Z/2},$$

which is equivalent to the infinite loop space of the fixed-points spectrum $M^\vee_{a_1}(A)^{Z/2} \times \cdots \times M^\vee_{a_n}(A)^{Z/2}$ (see [3.11]). A morphism $l: m \to n$ of $\text{Sym} \hat{GL}^\vee_{a_1,\ldots,a_n}(A)$ is a homotopy invertible element of $\hat{M}^\vee_{a_1,\ldots,a_n}(A)^{Z/2}$ which satisfies $n = l \cdot m \cdot w(l)$, where $w$ denotes the involution on $\hat{GL}^\vee_{a_1,\ldots,a_n}(A)$. Thus we think of $\hat{KH}(A)$ as the Segal construction of a symmetric monoidal category

$$\text{Herm}_A = \prod_n \text{Sym} \hat{GL}_n^\vee(A)$$

of spectral Hermitian forms on $A$.

**Proposition 3.22.** Suppose that $A$ is a simplicial ring with an anti-involution $w: A^{op} \to A$. There is a weak equivalence between $\hat{KH}(HA)$ and the connective cover of the Hermitian $K$-theory spectrum $\hat{L}(A)$ of [BFS4]. In particular if $A$ is discrete this is equivalent to the connective Hermitian $K$-theory of free $A$-modules of [Kar73].

**Proof.** The inclusion of wedges into products defines a map of ring spectra with anti-involution

$$M^\vee_n(HA) = \bigvee_{\times n} HA \longrightarrow \prod_{\times n} HA \cong HM_n(A),$$

where $M_n(A) = \bigoplus_{n \times n} A$ is the ring of $n \times n$-matrices with entries in $A$. On the underlying $\mathbb{Z}/2$-spectra, this is an inclusion of indexed wedges into indexed products and it is therefore a stable equivalence. On the level of $\Gamma$-categories this shows that the composite

$$\mathcal{F}_{HA}[n] \longrightarrow \mathcal{F}_{HA}[n] \longrightarrow H\mathcal{F}_A[n]$$

is an equivalence, where $\mathcal{F}_A[n]$ is Segal’s construction of the symmetric monoidal category of free $A$-modules $(\mathcal{F}_A, \oplus)$, with the duality induced by conjugate transposition of matrices.
we observe that the middle term $\mathcal{F}_{HA}[n]$ does not have a duality). At the level of $\Gamma$-spaces this induces an equivalence

\[(B^{1,1}\Omega^\infty H\mathcal{F}_A[n])^{\mathbb{Z}/2} \overset{\sim}{\longrightarrow} (B^{1,1}\Omega^\infty H\mathcal{F}_A[n])^{\mathbb{Z}/2} \overset{\sim}{\leftarrow} (B^{1,1}\mathcal{F}_{A}[n])^{\mathbb{Z}/2} \cong B\text{Sym}(\mathcal{F}_{A}[n]).\]

which restricted to invertible components gives an equivalence

\[KH_n(A) \simeq B\text{Sym}(i\mathcal{F}_{A}[n]).\]

Moreover there is a functor of $\Gamma$-categories $(\text{Sym } i\mathcal{F}_{A}[n]) \rightarrow \text{Sym}(i\mathcal{F}_{A}[n])$, and since both categories are equivalent to $\text{Sym } i\mathcal{F}_{A}[n]$ it is an equivalence. Finally, $\text{Sym } i\mathcal{F}_{A}[n]$ is the category of Hermitian forms over the simplicial ring $A$ of [BF84].

We relate the Hermitian $K$-theory of ring spectra with that of Hermitian Mackey functors defined in §1.2. Let $HM$ be a ring spectrum with anti-involution such that the underlying $\mathbb{Z}/2$-spectrum is the Eilenberg-MacLane spectrum of a Mackey functor $M$.

**Proposition 3.23.** The multiplication on $HM$ induces a Hermitian structure on the Mackey functor $M$, and there is a stable equivalence of $\Gamma$-spaces

\[KH(HM) \overset{\sim}{\longrightarrow} KH(M)\]

induced by the projection maps $\Omega^\infty HM \rightarrow M(\mathbb{Z}/2)$ and $(\Omega^\infty HM)^{\mathbb{Z}/2} \rightarrow M(*)$ onto $\pi_0$.

**Proof.** We recall that since $\Omega^\infty HM$ is a topological monoid with anti-involution, there is an action

\[\Omega^\infty HM \times (\Omega^\infty HM)^{\mathbb{Z}/2} \rightarrow (\Omega^\infty HM)^{\mathbb{Z}/2}\]

Defined by sending $(m, n)$ to $mnw(m)$ where $w$ is the involution on $\Omega^\infty HM$. By taking $\pi_0$ this gives a Hermitian structure on the Mackey functor $M$.

The inclusion of wedges into products gives an equivalence

\[M_n^\vee(HM) = \bigvee_{n \times n} HM \rightarrow \prod_{n \times n} HM.\]

The fixed-points spectrum of the target is equivalent to

\[(\prod_{n \times n} HM)^{\mathbb{Z}/2} \cong \prod_{1 \leq i < j \leq n} HM(\mathbb{Z}/2) \times \prod_n HM(*).\]

Thus $M_n^\vee(HM)$ is a model for the Eilenberg-MacLane spectrum of the Hermitian Mackey functor of matrices $M_n(M)$ of [1.5] Thus the projections onto $\pi_0$ define an equivalence of topological categories

\[\prod_n \text{Sym } \widetilde{GL}_n^\vee(HM) \overset{\sim}{\rightarrow} i\text{Herm}_M\]

onto the category of Hermitian forms on $M$ and isomorphisms. At the level of $\Gamma$-spaces this gives an equivalence

\[KH(HM)_n \cong \prod_a B(a) \times \text{Sym } \widetilde{GL}_{a_1, \ldots, a_n}^\vee(HM) \overset{\sim}{\rightarrow} i\text{Herm}_M[n]\]

onto the Segal $\Gamma$-category associated to $(i\text{Herm}_M, \oplus)$, by the same argument of [3.22].

**Definition 3.24.** The free Hermitian cyclic $K$-theory of $A$ is the spectrum $K^{cy}H(A)$ associated to the fixed-points $\Gamma$-space

\[K^{cy}H(A)_n := KH^{cy}(A)^{\mathbb{Z}/2} \cong \prod_a (B_{a_1, \ldots, a_n}(A))^{\mathbb{Z}/2}.\]
Proposition 3.25. Suppose that $HM$ is a ring spectrum with anti-involution whose underlying spectrum is the Eilenberg MacLane spectrum of a Mackey functor $M$. The projection onto $\pi_0$ induces a stable equivalence of $\Gamma$-spaces

$$K^{cy}_\Gamma(HM) \simrightarrow K^{cy}_\Gamma(M)$$

where $K^{cy}_\Gamma(M)$ is the Hermitian cyclic $K$-theory of $\Gamma$.

Proof. We recall that by 3.8 there are canonical isomorphisms

$$(B^d_\Gamma GL_k(A))^\mathbb{Z}/2 \cong B(GL_k(A))^\mathbb{Z}/2, GL_k(A), GL_k(A)^\mathbb{Z}/2 \cong B\text{Sym}^q GL_k(A),$$

which combined with an argument analogous to 3.23 gives the desired equivalence. \qed

4 The real trace map

4.1 Real topological Hochschild homology

We recollect some of the constructions of [HM17] and [Dot12]. Let $A$ be a ring-spectrum with anti-involution, possibly non-unital. The real topological Hochschild homology of $A$ is a genuine $\mathbb{Z}/2$-spectrum $\text{THR}(A)$. It is determined by a strict $\mathbb{Z}/2$-action on the Bokstedt model for topological Hochschild homology $\text{THH}(A)$ of the underlying ring spectrum. We recall its construction.

Let $I$ be the category of finite sets and injective maps. For any non-negative integer $k$ there is a functor $\Omega^k A: I^{k+1} \to Sp$ that sends $\xi = (i_0, i_1, \ldots, i_k)$ to $\Omega^{i_0 + i_1 + \cdots + i_k}(S \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k})$.

Its homotopy colimit constitutes the $k$-simplices of a semi-simplicial spectrum

$$\text{THH}_k(A) := \text{hocolim}_{\xi \in I^{k+1}} \Omega^{i_0 + i_1 + \cdots + i_k}(S \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}),$$

see e.g. [DGM13]. The involution on $I$ described in 3.2 induces an involution on $I^{k+1}$, by sending $(i_0, i_1, \ldots, i_k)$ to $(i_0, i_k, \ldots, i_1)$ (it is the $k$-simplices of the dihedral Bar construction on $I$ with respect to disjoint union). The diagram $\Omega^k A$ admits a $\mathbb{Z}/2$-structure in the sense of [DMI16], defined by conjugating a loop with the maps

$$S^{i_0 + i_1 + \cdots + i_k} \xrightarrow{\chi_0 \wedge \chi_1 \wedge \cdots \wedge \chi_k} S^{i_0 + i_1 + \cdots + i_k} \xrightarrow{id_{i_0} \wedge \chi_k} S^{i_0 + i_1 + \cdots + i_k}$$

$$A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \xrightarrow{\chi_0 \wedge \chi_1 \wedge \cdots \wedge \chi_k} A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \xrightarrow{id_{i_0} \wedge \chi_k} A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \xrightarrow{id_{i_0} \wedge \chi_k} A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}$$

where $\chi_j \in \Sigma_j$ is the permutation that reverses the order of $\{1, \ldots, j\}$. Thus the homotopy colimit $\text{THH}_k(A)$ inherits a $\mathbb{Z}/2$-action, which induces a semi-simplicial map $\text{THH}_k(A)^{op} \to \text{THH}_k(A)$.

Definition 4.1 ([HM17], [Dot12]). The real topological Hochschild homology of $A$ is the $\mathbb{Z}/2$-spectrum $\text{THR}(A)$ defined as the thick geometric realization of semi-simplicial $\mathbb{Z}/2$-spectrum $sd\text{THH}_k(A)$, where $sd$ is Segal’s edgewise subdivision functor.

Lemma 4.2 ([Dot12]). Suppose that there is a constant $c$ such that for any finite dimensional $\mathbb{Z}/2$-representation $V$ the space $A_V$ is $(\dim V - c)$-connected, and $A_V^{\mathbb{Z}/2}$ is $(\dim V^{\mathbb{Z}/2} - c)$-connected. Then the $\mathbb{Z}/2$-spectrum $\text{THR}(A)$ is fibrant, thus the map

$$\text{THR}(A) := \left| sd \text{hocolim}_{\xi \in I^{k+1}} \Omega^{i_0 + i_1 + \cdots + i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}) \right| \to \Omega^{\infty\mathbb{Z}/2} \text{THR}(A)$$

is an equivalence, where $\Omega^{\infty\mathbb{Z}/2}$ denotes the genuine equivariant infinite loop space functor.
Example 4.3.  

i) Any suspension spectrum satisfies the hypotheses of the lemma. Indeed \((S\wedge X)_V = S^V \wedge X\) is \((\dim V + \conn X)\)-connected non-equivariantly, and its fixed-points 
\((S \wedge X)^{\mathbb{Z}/2}_V = S^{V/2} \wedge X^{\mathbb{Z}/2}\) are \((\dim V/2 + \conn X^{\mathbb{Z}/2})\)-connected.

ii) Eilenberg-MacLane spectra of Abelian groups with \(\mathbb{Z}/2\)-action satisfy this condition as well, see e.g. [Dot16 A.1.1].

Proof of 4.2. We show that the map 
\[ hocolim_{I \in I^{\mathbb{Z}/2}+1} \Omega^i \Omega(u_0 \wedge u_1 \wedge \cdots \wedge u_{i+1}) \to \Omega(n+1) \rho \hocolim_{I \in I^{\mathbb{Z}/2}+1} \Omega(S^{n+1}) \rho \wedge A^{\mathbb{Z}/2}_{m+1} \]

is an equivalence for every non-negative integer \(n\), where \(\rho\) is the regular representation of \(\mathbb{Z}/2\). By our connectivity assumption the subdivision of \(\text{TH}(\mathcal{A})_{S_\rho}\) is levelwise connected, both non-equivariantly and on fixed-points. It follows from [HM97 2.4] that the loops commute with the realization, proving that the adjoint structure map for the representation \((n+1)\rho\) is an equivalence. Since the direct sums of copies of \((n+1)\rho\) form a cofinal system for the representations of \(\mathbb{Z}/2\) all the structure maps are equivalences.

We recall from 3.11 that the homotopy colimit over \(I\) is naturally equivalent to the equivariant infinite loop space. Thus the map above is equivalent to the top row of the commutative diagram

\[
\begin{array}{ccc}
\hocolim_{m \in \mathbb{N}} \Omega^{(m+1)(2k+2)} A^{\mathbb{Z}/2}_{m+1} & \to & \Omega^{(n+1)\rho} \hocolim_{m \in \mathbb{N}} \Omega^{(m+1)(2k+2)} (S^{n+1}) \rho \wedge A^{\mathbb{Z}/2}_{m+1} \\
\hocolim_{m \in \mathbb{N}} \Omega^{(n+1)\rho} \Omega^{(m+1)(2k+2)} (S^{n+1}) \rho \wedge A^{\mathbb{Z}/2}_{m+1} & \sim & \hocolim_{m \in \mathbb{N}} \Omega^{(n+1)\rho} (S^{n+1}) \rho \wedge A^{\mathbb{Z}/2}_{m+1}
\end{array}
\]

where \(2k + 2 = \{0, \ldots, 2k + 1\}\) has the involution that fixes 0 and reverses the order of the other elements. We show that the diagonal map is an equivalence. By the Freudenthal Suspension Theorem and our connectivity hypotheses the map

\[ A^{\mathbb{Z}/2}_{m+1} \to \Omega^{(n+1)\rho} (S^{n+1}) \rho \wedge A^{\mathbb{Z}/2}_{m+1} \]

is roughly twice as connected as \(A^{\mathbb{Z}/2}_{m+1}\) non-equivariantly, and roughly

\[ \min\{2 \conn(A^{\mathbb{Z}/2}_{m+1}, A^{\mathbb{Z}/2}_{m+1})\} \]

on fixed-points. Since there is an isomorphism \((A^{\mathbb{Z}/2}_{m+1})^{\mathbb{Z}/2} \cong A^{\mathbb{Z}/2}_{m+1} \wedge A^{\mathbb{Z}/2}_{m+1}\), by our connectivity assumption this fixed-points space is roughly

\[ 2(m \dim \rho^{\mathbb{Z}/2} + 1 - c) + k(m \dim \rho + 1 - c) + k + 1 \approx 2m(k + 1) \]

connected. Thus the minimum above is approximately

\[ \min\{4m(k + 1), (2m + 1)(2k + 2) + 2k + 1\} = 4m(k + 1) \]

Thus for any given \(m\) the corresponding map in the homotopy colimit system is non-equivariantly approximately

\[ c_1(m) = 2((2m + 1)(2k + 2) + 2k + 1) - (2m + 1)(2k + 2) \approx (2m + 1)(2k + 2) \]

connected, and approximately

\[ c_2(m) = \min\{c_1(m), 4m(k + 1) - (m \dim \rho^{\mathbb{Z}/2} + 1)(k + 2)\} \approx 3mk \]

connected on fixed-points. Since both \(c_1(m)\) and \(c_2(m)\) diverge with \(m\) the map is an equivalence on homotopy colimits. \(\square\)
Remark 4.4. Under the connectivity assumptions of lemma 4.2 the $\mathbb{Z}/2$-spectrum $\text{THR}(A)$ arises as the $\mathbb{Z}/2$-spectrum of a $\mathbb{Z}/2$-$\Gamma$-space with value at the pointed set $n_+ = \{+ , 1 , \ldots , n \}$ the $\mathbb{Z}/2$-space

$$\text{THR}(A)_n := [sd \text{hocolim} \Omega^{|i_0 + i_1 + \cdots + i_k|} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge n_+)].$$

Indeed the value of the associated spectrum at a sphere $S^n$ is the geometric realization

$$\text{THR}(A)_S^n := |[p] \rightarrow \text{THR}(A)_S p| \cong [sd \text{hocolim} \Omega^{|i_0 + i_1 + \cdots + i_k|} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge S^n_p)],$$

and under our connectivity assumptions the canonical map

$$\Omega^{|i_0 + i_1 + \cdots + i_k|} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge S^n_p) | \rightarrow \Omega^{|i_0 + i_1 + \cdots + i_k|} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge |S^n_p|)$$

is an equivariant equivalence for the action of the stabilizer group $(i_0, \ldots , i_k) \in I^{xk+1}$ (see [HM97, 2.4]). It follows from [DMPPR17] that the map on homotopy colimits is an equivariant equivalence.

Lemma 4.5. The real topological Hochschild homology functor $\text{THR}$ commutes with rationalizations on ring spectra with anti-involution with cofibrant underlying $\mathbb{Z}/2$-spectrum.

Proof. Under this cofibrancy condition the spectrum $\text{THR}(A)$ is naturally equivalent to the dihedral Bar construction of $A$ with respect to the smash product. This result is a generalization of a theorem of [Shi00] and [PS16], and a proof can be found in [DMPPR17].

Let $S_Q$ be a cofibrant model for the rational $\mathbb{Z}/2$-equivariant sphere spectrum. We notice that if $K_+$ is any finite pointed $\mathbb{Z}/2$-set, the map

$$S_Q \cong S_Q \wedge \bigwedge_{k} S \xrightarrow{\sim} \bigwedge_{K_+} S_Q$$

given by the $K$-fold smash product of the unit maps of $S_Q$ smashed with $S_Q$ is an equivalence. Non-equivariantly this is clear since $S_Q \sim H\mathbb{Q}$ is idempotent. On geometric fixed-points this is the map

$$\Phi_{\mathbb{Z}/2}^S \wedge \bigwedge_{[k] \in K/(\mathbb{Z}/2)} \Phi_{\mathbb{Z}/2}^{S_Q} \cong \Phi_{\mathbb{Z}/2}^{S_Q} \wedge \bigwedge_{[k] \in K/(\mathbb{Z}/2)} S \xrightarrow{\sim} \Phi_{\mathbb{Z}/2}^{S_Q} \wedge \bigwedge_{K/(\mathbb{Z}/2)} \Phi_{\mathbb{Z}/2}^{S_Q}$$

which is the smash of the identity with the $K/(\mathbb{Z}/2)$-fold smash of the unit maps of $\Phi_{\mathbb{Z}/2}^{S_Q}$, where $(\mathbb{Z}/2)_k$ is the stabilizer group of $k \in K$. Since the geometric fixed-points $\Phi_{\mathbb{Z}/2}^{S_Q}$ are also equivalent to $H\mathbb{Q}$ they are idempotent, and the map is an equivalence.

Thus we have natural equivalences

$$\text{THR}(A) \wedge S_Q \cong |A^{*+1} \wedge S_Q| \xrightarrow{\sim} |A^{*+1} \wedge S_Q^{*+1}| \cong |(A \wedge S_Q)^{*+1}| \cong \text{THR}(A \wedge S_Q).$$

4.2 The definition of the real trace map

We adapt the construction of the trace of [BHM93] to define a map of $\mathbb{Z}/2$-$\Gamma$-spaces

$$\text{tr} : \text{KR}(A) \rightarrow \text{THR}(A)$$

for every ring spectrum with anti-involution $A$. Under the connectivity hypothesis of [LZ] the $\mathbb{Z}/2$-$\Gamma$-space $\text{THR}(A)$ models the real topological Hochschild spectrum of $A$ (see [LZ]). At an
object \(n_+ \in \Gamma^{\text{op}}\) the trace is defined as the composite

\[
\text{KR}(A)_n = \coprod_{\underline{a}} B^{1,1}(\w{a} \times \w{GL}_a(A)) \xrightarrow{s_1} \coprod_{\underline{a}} B^{d_i}(\w{a} \times \w{GL}_a(A))
\]

\[
\coprod_{\underline{a}} B^{d_i}(\w{a} \times \Omega_1^\infty(M_{a_1}(A) \vee \cdots \vee M_{a_n}(A))) \xleftarrow{} \coprod_{\underline{a}} B^{d_i}(\w{a} \times \Omega_1^\infty(M_{a_1}(A) \vee \cdots \vee M_{a_n}(A)));
\]

\[
\coprod_{\underline{a}} B^{d_i}(\w{a}) \times \text{THR}(M_{a_1}(A) \vee \cdots \vee M_{a_n}(A)) \xrightarrow{} \text{THR}(A)_n
\]

All the maps leave the \(\underline{a}\)-coordinate untouched. The first map is the section \(s_1 : B^{1,1}G \to B_{K^1}^G\) that sends \((g_1, \ldots, g_k)\) to \(((g_k \cdots g_1)^{-1}, g_1, \ldots, g_k)\). For this map to be well defined one needs \(G\) to have strict inverses. Thus we should functorially replace \(\w{GL}_a\) with an equivalent group with anti-involution. This can be achieved for example by group completing \(\w{GL}_a\) using the Kan loop-group, since for every group-like topological monoid \(M\) with anti-involution the canonical map

\[
M \to \Omega_1^1 B^{1,1}M
\]

is a \(\mathbb{Z}/2\)-equivalence, where \(\Omega_1^1\) denotes the Kan loop-group with the anti-involution on the free group that reverses the order of the letters within a word. The second map is the inclusion of the invertible components. The third map is induced by the canonical map from products to smash products, where \(B_{K^1}^{d_i}\) denotes the dihedral Bar construction with respect to the smash product. The fourth map commutes the smash products and the loops. The fifth map projects off the \(\underline{a}\)-component, and on the THR factor it is induced by the maps

\[
(M_{a_1}(A_{n_0}) \vee \cdots \vee M_{a_n}(A_{n_0})) \wedge \cdots \wedge (M_{a_1}(A_{i_k}) \vee \cdots \vee M_{a_n}(A_{i_k})) \to A_{i_0} \wedge \cdots \wedge A_{i_k} \wedge n_+
\]

defined as follows. An element of \(M_{a_j}(A_1) \vee \cdots \vee M_{a_n}(A_1)\) is the data of an integer \(1 \leq j \leq n\), a pair \((c, d) \in a_j \times a_j\) and an element \(x \in A_i\). The map above is then defined by

\[
(j_0, c_0, d_0, x_0) \wedge \cdots \wedge (j_k, c_k, d_k, x_k) \mapsto \begin{cases} 
  x_0 \wedge \cdots \wedge x_k \wedge j_0 & \text{if } j_0 = \cdots = j_k \\
  & \text{if } d_0 = c_1, d_1 = c_2, \ldots, d_{k-1} = c_k, d_k = c_0 \\
  * & \text{else}
\end{cases}
\]

This map remembers the entries of a string of matrices precisely when they are all composable, and sends the rest to the basepoint. The underlying map is the trace map of [DM96 §1.6.17] and it is a homotopy inverse for the map induced by the inclusion \(A \to M_{a_i}(A)\) into the \(1 \times 1\)-component. Although we won’t use this here, it is also an equivariant equivalence (see [Dot12 4.8.5]).

**Definition 4.6.** The trace will play a central role in [LS] for the construction of the splitting of the lifted assembly map of [2.1]. We will in fact only use the trace map from real cyclic homology, which we denote by

\[
\text{tr}^c : \text{KR}^c(A) = \coprod_{\underline{a}} B^{d_i}(\w{a} \times \w{GL}_a(A)) \to \text{THR}(A)
\]

All the spaces above extend to \(\mathbb{Z}/2\)-\(\Gamma\)-spaces by a construction similar to the definition of the \(\Gamma\)-structure on \(\text{KR}^c\) and all the maps in the diagram are maps of \(\mathbb{Z}/2\)-\(\Gamma\)-spaces. We will however only verify that \(\text{tr}^c\) is compatible with the \(\Gamma\)-structure.

**Proposition 4.7.** The trace map defined levelwise above gives rise to a map of \(\mathbb{Z}/2\)-\(\Gamma\)-spaces

\[
\text{tr}^c : \text{KR}^c(A) \to \text{THR}(A).
\]
Proof. Let \( f : n_+ \to k_+ \) be a pointed map. We need to verify that for every collection of non-negative integers \( \underline{a} = (a_1, \ldots, a_n) \) the square

\[
\begin{array}{ccc}
B^d(\underline{a}) \times B^dGL_{a_1,\ldots,a_n}(A) & \xrightarrow{1_{\underline{a}} \circ \Omega} & |hocolim_{j \in \mathbb{Z}^{p+1}} \Omega^{i_0+i_1+\cdots+i_p} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_p} \wedge n_+)| \\
\downarrow f & & \downarrow f \\
B^d(\theta \underline{a}) \times B^dGL_{f \underline{a}}(A) & \xrightarrow{1_{\underline{a}} \circ \Omega} & |hocolim_{j \in \mathbb{Z}^{p+1}} \Omega^{i_0+i_1+\cdots+i_p} (A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_p} \wedge k_+)|
\end{array}
\]

commutes. We prove that this diagram commutes in simplicial degree \( p = 1 \), and since the upper left corner is the nerve of a category this will suffice. A 1-simplex of the upper left corner consists of two pairs of families of permutations \((\beta, \alpha)\) and \((\alpha, \beta)\), where \( \alpha, \beta \in (\underline{a}) \), and a pair of elements \( x, y \in \Omega^\infty (M_{a_1}^\vee (A) \vee \cdots \vee M_{a_n}^\vee (A)) \) belonging to an invertible component.

For a fixed pair \((\alpha, \beta)\), we need to show that the square

\[
\begin{array}{c}
(M_{a_1}^\vee (A_{i_0}) \vee \cdots \vee M_{a_n}^\vee (A_{i_0})) \wedge (M_{a_1}^\vee (A_{i_1}) \vee \cdots \vee M_{a_n}^\vee (A_{i_1})) \rightarrow A_{i_0} \wedge A_{i_1} \wedge n_+ \\
\end{array}
\]

and the horizontal maps are defined at the beginning of the section. The upper composite takes \((j_0, (c_0, d_0), x_0) \wedge (j_1, (c_1, d_1), x_1)\), where \( 1 \leq j \leq n \), \((c, d) \in a_j \times a_j \) and \( x \in A_j \), to

\[
x_0 \wedge x_1 \wedge f(j_0) \quad \text{if} \quad j_0 = j_1 \quad \text{and} \quad d_0 = c_1, \quad d_1 = c_0,
\]

and to the basepoint otherwise. The lower composite takes it to

\[
x_0 \wedge x_1 \wedge f(j_0) \quad \text{if} \quad f(j_0) = f(j_1) \quad \text{and} \quad \beta f^{-1}(j_0) \wedge j_0, j_0(t_0 d_0) = \beta f^{-1}(j_1) \wedge j_1, j_1(t_1 c_1),
\]

\[
\alpha f^{-1}(j_0) \wedge j_1, j_1(t_1 d_1) = \alpha f^{-1}(j_0) \wedge j_0, j_0(t_0 c_0),
\]

where \( t_0 : a_j \rightarrow \Pi_{i \in f^{-1}(j_0)} a_i \) is the inclusion, and similarly for \( t_1 \). We need to show that these conditions are equivalent. Clearly the first condition implies the second one.

Suppose that the second condition holds, and set \( i := f(j_0) = f(j_1) \). By construction, the family of permutations \( \alpha \) satisfies the condition

\[
\alpha(f^{-1}i) \wedge j_1, j_1 \circ (\alpha_{j_0, (f^{-1}i)} \wedge j_0, j_1) \Pi id_{a_{j_1}} = \alpha(f^{-1}i) \wedge j_0, j_0 \circ (id_{a_{j_0}} \Pi \alpha(f^{-1}i) \wedge j_0, j_1, j_1).
\]

By evaluating this expression at \( t_0 c_0 \) we obtain that

\[
\alpha(f^{-1}i) \wedge j_1, j_1 \circ (\alpha_{j_0, (f^{-1}i)} \wedge j_0, j_1) \Pi id_{a_{j_1}}(t_0 c_0) = \alpha(f^{-1}i) \wedge j_0, j_0(t_0 c_0) = \alpha(f^{-1}i) \wedge j_1, j_1(t_1 d_1).
\]

Since \( \alpha(f^{-1}i) \wedge j_1, j_1 \) is invertible we must have that

\[
(\alpha_{j_0, (f^{-1}i)} \wedge j_0, j_1) \Pi id_{a_{j_1}}(t_0 c_0) = t_1 d_1,
\]

but since the left-hand map is the identity on \( a_{j_1} \) and \( t_1 \) includes in \( a_{j_1} \) we must have that \( j_0 = j_1 \) and \( c_0 = d_1 \). A similar argument shows that \( d_0 = c_1 \). \( \square \)

4.3 The splitting of the restricted assembly map of the Burnside group-ring

Let \( \pi \) be a discrete group, and \( HB \) a strictly commutative cofibrant \( \mathbb{Z}/2 \)-ring spectrum model for the Eilenberg MacLane spectrum of the Burnside Mackey functor \( B \) (for the existence see \([UIB3]\)).

Let us observe that \( HB[\pi] := HB \wedge \pi_+ \) is a model for the Eilenberg-MacLane spectrum of the Burnside group-ring \( B[\pi] \), since its fixed-points spectrum is equivalent to

\[
(HB \wedge \pi_+)^{\mathbb{Z}/2} \xrightarrow{\sim} \left( \prod_\pi HB \right)^{\mathbb{Z}/2} \xrightarrow{\sim} \prod_{\pi \text{free} / \mathbb{Z}/2} H \mathbb{Z} \times \prod_{\mathbb{Z}/2} H(\mathbb{Z} \oplus \mathbb{Z}).
\]
Thus by the projection onto $\pi_0$ gives an equivalence $\text{KH}(B\pi[\pi]) \cong \text{KH}(B[\pi])$. The aim of this section is to prove the following, which is the missing step in the proof of Theorem 4.1.

**Theorem 4.8.** The lifted restricted assembly map $A_{B[\pi]}^0: S \wedge B\pi^2 \to \text{KH}(B[\pi])^2$ defined in [2,1] is naturally rationally split-injective.

**Proof.** The splitting $T$ is induced on rational homotopy groups by the outer clockwise zig-zag of maps in the diagram

\[
\begin{array}{cccccc}
\text{KH}(B[\pi])^2 & \overset{s_1 \vee \sigma}{\to} & \text{KV}(B_0[\pi]) & \sim & \text{KV}(B_0[\pi]) & \overset{\text{KR}^{s_1}(B_0[\pi])}{\to} & \text{THR}(B_0[\pi])^2 \\
\sim & & \sim & & \sim & & \sim \QL  \\
(\text{S} \wedge B\pi^2) & \overset{\rho \nu \rho}{\sim} & (\text{S} \wedge B\pi^2) & \sim & (\text{S} \wedge B\pi^2) & \sim & (\text{S} \wedge B\pi^2) \\
\end{array}
\]

We start by defining the solid arrows, clockwise from the top left to the bottom left vertex. The first map is the composition

\[
\text{KH}(B[\pi]) \vee \text{KH}(B[\pi]) \to \text{KH}(B_0[\pi]) \vee \text{KH}(B_0[\pi]) \overset{s_1 \vee \sigma}{\to} \text{KV}(B_0[\pi])
\]

where the first map is induced by the rationalization map $B \to B_0$, and $s_1$ and $s_\sigma$ are the sections defined in [1,1] where $\sigma := \mathbb{Z}/2 - 1$ in the Burnside ring. By abuse of notation we denote the composite also by $s_1 \vee s_\sigma$. The second map is the equivalence of [3,25] induced by the projection onto $\pi_0$. The third map is the equivalence of [3,21] that includes the naive fixed-points of $\text{KR}^{s_\sigma}$ into its genuine fixed-points. The map $\text{tr}^{s_\sigma}$ is the map on genuine fixed-points spectra induced by the trace map of [4,2]. The next map is induced by the unit map $S \to HB$, and it is a rational equivalence by [4,5]. Below it is the fixed-points of the composite

\[
S \wedge B^{d_i} \pi \to \text{THR}(S) \wedge B^{d_i} \pi \to \text{THR}(S)[\pi]
\]

of the unit map of $\text{THR}(S)$, which includes $S$ as the 0-th spectrum of the colimit system $\text{hocolim}_f \Omega^i(S \wedge S)$ of the 0-simplices of $\text{THR}$, and of the assembly map that commutes with the smash by $B^{d_i} \pi$, with realizations, homotopy colimits, and loops. The composite is an equivariant equivalence by a result of [Hg16], a generalization of [BHM93]. We recall that the dihedral Bar construction $B^{d_i} \pi$ is the realization of the subdivision of the simplicial set $N^{d_i} \pi$ with $n$-simplices $\pi^{n+1}$, and with $\mathbb{Z}/2$-action induced by the map $(N^{d_i} \pi)^{op} \to N^{d_i} \pi$ that sends

\[
(g_0, g_1, \ldots, g_n) \mapsto (g_0^{-1}, g_1^{-1}, \ldots, g_n^{-1}).
\]

The map $p_0: B^{d_i} \pi \to B^{1,1} \pi$ is induced by projecting off the first factor. The equivalence labelled $TD$ is the Tom-Dieck splitting. The next map is induced by the equivariant map $\nu: B \pi \to B^{1,1} \pi$ of [3,6]. It is a non-equivariant equivalence, and therefore it induces an equivalence on homotopy orbits. Moreover this map is a split monomorphism on fixed-points, and the last map is the retraction $\rho: (B^{1,1} \pi)^{\mathbb{Z}/2} \to B \pi$ wedged with the projection $p: B \pi \times \mathbb{Z}/2 \to B \pi$.

In order to verify that the composition of these maps on rational homotopy groups splits the restricted assembly $A_{B[\pi]}^0$ we need to fill in the dashed arrows in the diagram above.
Given elements $g \in \pi$ and $x \in \Omega_1^\infty(\mathbb{H}BQ)$ we let $xg$ denote the image of the pair $x \land g$ under the $\mathbb{Z}/2$-equivariant assembly map

$$
\Omega_1^\infty(\mathbb{H}BQ) \land \pi_+ \longrightarrow \Omega_1^\infty(\mathbb{H}BQ \land \pi_+) = \Omega_1^\infty(\mathbb{H}BQ[\pi]).
$$

The map $A^1$ is defined by a map of $\Gamma$-spaces

$$
n_+ \land B\pi_+ \lor B\pi_+ \cong B(\Pi_+ \pi \Pi_+ \pi)_+ \longrightarrow \prod a B(\widehat{\mathcal{G}} \times \text{Sym}^{cy} \mathcal{G}L_{a_1, \ldots, a_n}(\mathbb{H}BQ[\pi])) \cong \text{KH}(\mathbb{H}BQ[\pi])_n,
$$

where $\text{Sym}^{cy} G$ is the category such that $\text{N Sym}^{cy} G = N(\text{Sym} G, G, \text{Sym} G) = (\text{N cy} G)^{Z/2}$, from [5,8]. We observe that for $\mathcal{G} = (0, \ldots, a_1 = 1, \ldots, 0)$ the set $\langle a \rangle$ has only one element.

The map $A^1$ is induced by the functor that sends an object $1 \leq i \leq n$ in the first copy of $\Pi_+ \pi$ to the summand $\mathcal{G} = (0, \ldots, a_1 = 1, \ldots, 0)$, to the object $(1e, 1e)$ of

$$
\text{Ob Sym}^{cy} \mathcal{GL}_\mathcal{G}(\mathbb{H}BQ[\pi]) \cong (\Omega_1^\infty(\mathbb{H}BQ[\pi]))^{Z/2} \times (\Omega_1^\infty(\mathbb{H}BQ[\pi]))^{Z/2}
$$

where $e \in \pi$ is the unit and 1 is the unit of $\Omega_1^\infty \mathbb{H}BQ$. An object $i$ in the second copy of $\Pi_+ \pi$ is sent to the same summand, to the object $(-\sigma e, -1e)$ where $-\sigma$ is a choice of element that represents $(1, -1)$ in

$$
\pi_0(\Omega_1^\infty \mathbb{H}BQ)^{Z/2} \cong \mathbb{Q} \oplus \mathbb{Q}.
$$

A morphism $(i, g)$ in either copy of $\Pi_+ \pi$ is sent to the morphism $1g \in \Omega_1^\infty(\mathbb{H}BQ[\pi])$. Since $1g$ commutes strictly with both $1e$ and $-\sigma e$ these are well-defined morphisms in $\text{Sym}^{cy} \mathcal{GL}_\mathcal{G}(\mathbb{H}BQ[\pi])$. The composite $(s_1 \lor s_\sigma) A^0_{B[\pi]}$ takes an object $i$ respectively in the first and second component to

$$
s_1(1e) = (1e, 1e) \quad \text{and} \quad s_\sigma(-1e) = (-\sigma e, -1e)
$$

(see [4,3]). Since the equivalence $\text{KH}^\mathcal{G}(\mathbb{B}Q[\pi]) \cong \text{KH}^\mathcal{G}(\mathbb{B}Q[\pi])$ is induced by the projection onto $\pi_0$ the upper left diagram commutes by construction.

The map $A^2$ is defined as follows. On the first summand of $B\pi$ it is the composite

$$
A^2_1 : (S \land B\pi) \longrightarrow (S \land B^{1,1_\pi})^{Z/2} \stackrel{\iota}{\longrightarrow} (S \land B^{1,1_\pi})^{Z/2} \rightarrow (S \land B^{1,\pi})^{Z/2}
$$

where the first map is the inclusion of naïve into genuine fixed-points spectra (it is well defined since $Z/2$ acts trivially on $B\pi$), and $s_e$ is the canonical section of $p_0$. The composition $s_\epsilon : B\pi \rightarrow B^{1,\pi}$ is the geometric realization of the simplicial map $N \pi \rightarrow s_\epsilon N^{cy} \pi$ that sends $(g_1, \ldots, g_k)$ to $(e, g_1, \ldots, g_k, e, g_1^{-1}, \ldots, g_k^{-1})$. On the second summand it is defined on infinite loop spaces, as the composite

$$
A^2_2 : \Omega_1^\infty (S \land B\pi) \longrightarrow \Omega_1^\infty (S \land B^{1,\pi})^{Z/2} \rightarrow \Omega_1^\infty (S \land B^{1,\pi})^{Z/2}
$$

of the same map $A^2_1$ followed by multiplication by an element $\sigma \in (\Omega_1^\infty)_{Z/2}$ representing $(-1, 1) = Z/2 - 1$ in the Burnside ring. We verify that the middle square commutes in lemma [4,9] below.

Finally, the map $A^3$ is defined as the wedge of the inclusion $\iota : B\pi \rightarrow (B^{1,1_\pi})^{Z/2}$ and the inclusion $j : B\pi \rightarrow B\pi_{hZ/2} \cong B\pi \times BZ/2$ at the basepoint of $BZ/2$. Clearly $A^3$ is a section for $\rho \lor p$. Finally, we show that the square involving the Tom Dieck splitting commutes. There is a diagram

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
where \( q : S \wedge B_{\pi^+} \to (S \wedge B_{\pi^+})^{\mathbb{Z}/2} \) is the inclusion from naive into genuine fixed-points. The bottom right square commutes by construction, the square above it commutes by naturality of the Tom Dieck splitting, and \( A^2 = s_t(q, \sigma q) \) by definition. The top part of the diagram commutes because \( p_0s_e = \text{id} \). For the remaining part of the diagram, we remark that

\[
TD(1 \lor j) = (q, \mathbb{Z}/2q)
\]

where \( \mathbb{Z}/2 \) is multiplication by a representative of \( (0, 1) \in \pi_0 \Omega^\gamma S \cong \mathbb{Z} \oplus \mathbb{Z} \). Thus

\[
TD(1 \lor j) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (q, \mathbb{Z}/2q) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (q, q(\mathbb{Z}/2 - 1)) = (q, \sigma q).
\]

\[\Box\]

**Lemma 4.9.** The middle square in the diagram at the beginning of the proof of 4.8 commutes.

**Proof.** The upper composite factors as

\[
S \wedge B_{\pi^+} \to A^1 \to K^{\nu}\mathbb{H}(HBQ[\pi]) = \text{KR}^{\nu}(HBQ[\pi])^{N\mathbb{Z}/2} \to \text{THR}(HBQ[\pi])^{N\mathbb{Z}/2} \to \text{THR}(HBQ[\pi])^{\mathbb{Z}/2}
\]

where the superscript \( N\mathbb{Z}/2 \) denotes naive fixed-points spectra, and the last map is the inclusion into genuine fixed-points. It is an equivalence by [L2]. Let us start by computing the composite \( \text{tr}^{\nu} A^1 \). This is a map of \( \Gamma \)-spaces given at the pointed set \( n^+ \) by the composite

\[
\begin{array}{ccc}
n^+ \wedge (B_{\pi^+} \vee B_{\pi^+}) & \xrightarrow{A^1} & \prod_{\mathbb{Z}} B^d_i((\widetilde{\alpha}) \times \tilde{G}_{a_1, \ldots, a_n}(HBQ[\pi]))^{\mathbb{Z}/2} \\
|sd_e \text{ hocollim}_{i \in k+1} \Omega^{\nu}(HBQ[\pi])_{i_0} \wedge \cdots \wedge HBQ[\pi]_{i_{k+1}} \wedge n^+|^\mathbb{Z}/2 & \xrightarrow{\text{tr}^{\nu}} & \\
\end{array}
\]

Let \( g = (g_1, \ldots, g_k) \) be an element in the nerve \( N_k \pi \). This map takes an element \((i, g)\) in the first summand to

\[
\text{tr}^{\nu} ((0, \ldots, a_i = 1, \ldots, 0), 1g_1, \ldots, 1g_k, 1e, w(1g_k), \ldots, w(1g_1)).
\]

Since the assembly \( \Omega^{\nu}(HBQ) \wedge \pi^+ \to \Omega^{\nu}(HBQ \vee \pi^+) \) is equivariant we have that \( w(1g) = w(1g)^{-1} = 1g^{-1} \). Chasing through the definition of the trace map on the summand \((0, \ldots, a_i = 1, \ldots, 0)\) one can easily see that the value of the trace at \((1g_1, \ldots, 1g_k, 1e, 1g_k^{-1}, \ldots, 1g_1^{-1})\) is the element of

\[
\text{hocollim}_{i \in k+2} \Omega^{\nu}(HBQ[\pi])_{i_0} \wedge \cdots \wedge HBQ[\pi]_{i_{k+1}} \wedge n^+
\]

represented by the element \((1e \wedge 1g_1 \wedge \cdots \wedge 1g_k \wedge 1e \wedge 1g_k^{-1} \wedge \cdots \wedge 1g_1^{-1} \wedge i)\) in the \( i = 0 \) space of the homotopy colimit. On the second summand \( \text{tr}^{\nu} A^1 \) sends \((i, g)\) to the element of the homotopy colimit represented by the loop

\[
(-\sigma e \wedge 1g_1 \wedge \cdots \wedge 1g_k \wedge -1e \wedge 1g_k^{-1} \wedge \cdots \wedge 1g_1^{-1} \wedge i).
\]

The upper composite of the square on the infinite loop space of the first summand then sends a loop \( f = x \wedge g \in \Omega^{\nu}(S^n \wedge N_k \pi) \) to the loop determined by the \( k \)-simplex in

\[
\Omega^{\nu S^{k+2}/2} \text{ hocollim}_{i \in k+2} \Omega^{\nu}(HBQ[\pi])_{i_0} \wedge \cdots \wedge HBQ[\pi]_{i_{k+1}} \wedge S^n\rho
\]

that sends \( t \vee s \in S^n \wedge S^{n\sigma} \cong S^{n\rho} \) to the element of the homotopy colimit represented by the loop

\[
(1e \wedge 1g_1(t) \wedge \cdots \wedge 1g_k(t) \wedge 1e \wedge 1g_k^{-1}(t) \wedge \cdots \wedge 1g_1^{-1}(t) \wedge (x(t) \wedge s)).
\]

Similarly, a loop \( f = x \wedge g \in \Omega^{\nu}(S^n \wedge N_k \pi) \) in the infinite loop space of the second \( B_{\pi} \) summand is sent to the element represented by the loop

\[
(-\sigma e \wedge 1g_1(t) \wedge \cdots \wedge 1g_k(t) \wedge -1e \wedge 1g_k^{-1}(t) \wedge \cdots \wedge 1g_1^{-1}(t) \wedge (x(t) \wedge s)).
\]
where $-1$ is a loop representing $-1$ in the Burnside ring. We observe that $\text{THR}(B_{Q}[\pi])$ is the realization of a bisimplicial object with $(k,l)$-simplices

$$\text{THR}_{2k+1,2l+1}(B_{Q}; \pi) := \text{hocolim}_{I^{+2}} \Omega^{2}((HB_{Q})_{i_{0}} \wedge \ldots \wedge (HB_{Q})_{i_{2k+1}} \wedge (N^{2k+1}_{2\pi} \wedge S)$$

where the simplicial structure in the $l$-direction is defined as for $\text{THR}(HB_{Q})$, and in the $k$-direction it is induced by the simplicial structure of the subdivided dihedral nerve. Since the only non-constant loops in the expression above are $-\sigma e$ and $-1e$, the upper composite of the square lands in the fixed-points of

$$\Omega^{\infty}(\text{THR}_{1}(B_{Q}; \pi))^{\mathbb{Z}/2},$$

the equivariant infinite loop space of the realization in the $l$-simplicial direction of the $k = 0$-simplices of the bisimplicial spectrum $\text{THR}(B_{Q}; \pi)$.

Now let us analyze the lower composite. Similarly, we write $\text{THR}_{*}(S; \pi)$ for the bisimplicial spectrum that realizes to $\text{THR}(S[\pi])$. Since the map of spectra $S \wedge B^{d_{i} \pi} \to \text{THR}(S[\pi])$ is induced by the inclusion of zero-simplices $S \to \text{THR}(S)$, it factors as

$$S \wedge B^{d_{i} \pi} = (\text{THR}_{0}(S; \pi)) \xrightarrow{s_{0}} (\text{THR}_{1}(S; \pi)) \xrightarrow{\gamma} (\text{THR}_{*}(S; \pi)) \cong \text{THR}(S[\pi])$$

where the first map is induced by the 0-th degeneracy of $\text{THR}(S)$, the second map the inclusion of the 0-simplices in the $k$-simplicial direction, and the last map is the canonical isomorphism between the geometric realization taken one simplicial direction at the time and the geometric realization of the diagonal. Thus the lower composite of the square on infinite loop spaces is the map

$${\Omega^{\infty}(S \wedge (B_{\pi} \vee B_{\pi}))} \xrightarrow{A^{2}} {\Omega^{\infty}((S \wedge B^{d_{i} \pi})^{\mathbb{Z}/2})} \xrightarrow{s_{0}} {\Omega^{\infty}((\text{THR}_{1}(S; \pi))^{\mathbb{Z}/2})} \xrightarrow{\eta} {\Omega^{\infty}(\text{THR}_{*}(B_{Q}; \pi))^{\mathbb{Z}/2}}$$

where $\eta$ is induced by the unit map $S \to HB_{Q}$. We show that $\eta s_{0}A^{2}$ and the upper composite $tr^{\gamma} A^{1}$ are homotopic inside $\Omega^{\infty}(\text{THR}_{1}(B_{Q}; \pi))^{\mathbb{Z}/2}$, which will conclude the proof. On the first $B_{\pi}$-summand $\eta s_{0}A^{2}$ takes a loop $f = x \wedge g \in \Omega^{0}(S^{n} \wedge N_{k}\pi)$ to

$$(1 \wedge 1 \wedge g_{1}(t) \wedge \ldots \wedge g_{k}(t) \wedge e \wedge g_{k}^{-1}(t) \wedge \ldots \wedge g_{1}^{-1}(t) \wedge (x(t) \wedge s),$$

in the $i_{0} = i_{1} = 0$ stage of the homotopy colimit defining $\text{THR}_{1}(B_{Q}; \pi)$. Therefore it agrees with $tr^{\gamma} A^{1}$ on the first summand. On the second summand $\eta s_{0}A^{2}$ takes $f = x \wedge g \in \Omega^{0}(S^{n} \wedge N_{k}\pi)$ to the element in

$$\Omega^{(n+m)p} \lim_{x \times I} ((HB_{Q})_{i_{0}} \wedge (HB_{Q})_{i_{1}} \wedge (N^{2k+1}_{2\pi} \wedge S^{(n+m)p})$$

that sends $t \wedge s \wedge r \in S^{n} \wedge S^{n} \wedge S^{np} \cong S^{(n+m)p}$ to the element of the homotopy colimit represented by the loop

$$(1 \wedge 1 \wedge g_{1}(t) \wedge \ldots \wedge g_{k}(t) \wedge e \wedge g_{k}^{-1}(t) \wedge \ldots \wedge g_{1}^{-1}(t) \wedge (x(t) \wedge s \wedge r))$$

in the $i_{0} = i_{1} = 0$ stage of the homotopy colimit, where $\sigma : S^{np} \to S^{mp}$ is an equivariant map in the component $(-1, 1)$ of the Burnside ring. The smash factors belonging to the dihedral nerve agree with the ones of $tr^{\gamma} A^{1}$. Thus in order to compare the two composites we need to show that the loops

$$(\sigma \wedge 1 \wedge (x \wedge id_{S^{mp}})) \in \Omega^{0}((HB_{Q})_{m_{p}} \wedge (HB_{Q})_{0} \wedge S^{np})$$

$$(1 \wedge (x \wedge id_{S^{np}} \wedge \sigma)) \in \Omega^{(n+m)p}((HB_{Q})_{0} \wedge (HB_{Q})_{0} \wedge S^{(n+m)p})$$

define the same element of $\pi_{0}(\lim_{x \times I} \Omega^{(n+m)p}((HB_{Q})_{i_{0}} \wedge (HB_{Q})_{i_{1}} \wedge S^{np}))^{\mathbb{Z}/2}$. This follows from the fact that this homotopy group is isomorphic to

$$\pi_{0}(\lim_{n,i,j \in \mathbb{N}} \Omega^{(n+i+j)p}S^{(n+i+j)p})^{\mathbb{Z}/2}.$$
Remark 4.10. It is imperative that in the construction of our splitting we include the two copies of $\text{KH}(B[\pi])$ inside $\text{K}^{\text{cy}}H(B[\pi])$ using the two distinct sections $s_1$ and $s_\sigma$. The construction of theorem 2.1 allows one to lift the restricted assembly $S \wedge B\pi^+[2] \to \text{KH}(\mathbb{Z}[\pi])$ to $\text{KH}(B[\pi])$ along $d$. In fact, any choice of lifts of 1 and $-1$ to units in the Burnside ring determine such a lift. However, one can see from the argument of 4.9 that the trace map

$$\text{tr}: \text{KH}(B[\pi]) \to \text{K}^{\text{cy}}H(B[\pi]) \to \text{THR}(B[\pi])$$

equalizes the two summands of the lifted assembly, no matter the choice of lifts. This is essentially because if $\tau$ is a unit in the Burnside ring, the composite of the associated lift and the trace sends $(g_1, \ldots, g_k)$ to the point in THR represented by

$$(\tau^{-1}, g_1, \ldots, g_k, \tau, g_k^{-1}, \ldots, g_1^{-1}),$$

which by the argument of 4.9 represent the same point than $(1, g_1, \ldots, g_k, 1, g_k^{-1}, \ldots, g_1^{-1})$.

References

[BF84] D. Burghelea and Z. Fiedorowicz, Hermitian algebraic $K$-theory of topological spaces, Algebraic $K$-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math., vol. 1046, Springer, Berlin, 1984, pp. 32–46. MR 750675

[BHM89] M. Bökstedt, W. C. Hsiang, and I. Madsen, The cyclotomic trace and the $K$-theoretic analogue of Novikov’s conjecture, Proc. Nat. Acad. Sci. U.S.A. 86 (1989), no. 22, 8607–8609. MR 1023810 (91e:57051)

[BHM93] ———, The cyclotomic trace and algebraic $K$-theory of spaces, Invent. Math. 111 (1993), no. 3, 465–539. MR 1202133 (94g:55011)

[Cn93] Guillermo Cortiñas, $L$-theory and dihedral homology. II, Topology Appl. 51 (1993), no. 1, 53–69. MR 1229500

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy, The local structure of algebraic $K$-theory, Algebra and Applications, vol. 18, Springer-Verlag London Ltd., London, 2013. MR 3013261

[DM96] Bjørn Ian Dundas and Randy McCarthy, Topological Hochschild homology of ring functors and exact categories, J. Pure Appl. Algebra 109 (1996), no. 3, 231–294. MR 1388700 (97i:19001)

[DM16] Emanuele Dotto and Kristian Moi, Homotopy theory of $G$-diagrams and equivariant excision, Algebr. Geom. Topol. 16 (2016), no. 1, 325–395. MR 3470703

[DMPPR17] E. Dotto, K. Moi, I. Patchkoria, and S. Precht-Reeh, Real topological Hochschild homology, To Appear, 2017.

[Dot12] Emanuele Dotto, Stable Real $K$-theory and Real topological Hochschild homology, Ph.D. thesis, University of Copenhagen, 2012, arXiv:1212.4310.

[Dot16] Emanuele Dotto, Equivariant calculus of functors and $\mathbb{Z}/2$-analyticity of Real algebraic $K$-theory, Journal of the Institute of Mathematics of Jussieu 15(4) (2016), pp. 829–883.

[GM14] B. Guillou and J.P. May, Enriched model categories and diagram categories, arXiv: 1201.5178, 2014.

[Hg16] Amalie Høgenhaven, Real topological cyclic homology of spherical group rings, arXiv: 1611.01204, 2016.
Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology 36 (1997), no. 1, 29–101. MR 1410465 (97i:19002)

**HM17**

Real algebraic K-theory, http://www.math.ku.dk/~larsh/papers/s05/, 2017.

Max Karoubi, *Périodicité de la K-théorie hermitienne*, Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 301–411. Lecture Notes in Math., Vol. 343. MR 0382400 (52 #3284)

Matthias Kreck and Wolfgang Lück, *The Novikov conjecture*, Oberwolfach Seminars, vol. 33, Birkhäuser Verlag, Basel, 2005, Geometry and algebra. MR 2117411 (2005i:19003)

Jean-Louis Loday, *Homologies diédrale et quaternionique*, Adv. in Math. 66 (1987), no. 2, 119–148. MR 917736

J. P. May, *Some remarks on equivariant bundles and classifying spaces*, Astérisque (1990), no. 191, 7, 239–253, International Conference on Homotopy Theory (Marseille-Luminy, 1988). MR 1098973

S. P. Novikov, *Pontrjagin classes, the fundamental group and some problems of stable algebra*, Amer. Math. Soc. Translations Ser. 2, Vol. 70: 31 Invited Addresses (8 in Abstract) at the Internat. Congr. Math. (Moscow, 1966), Amer. Math. Soc., Providence, R.I., 1968, pp. 172–179. MR 0231401

Irakli Patchkoria and Steffen Sagave, *Topological Hochschild homology and the cyclic bar construction in symmetric spectra*, Proc. Amer. Math. Soc. 144 (2016), no. 9, 4099–4106. MR 3513565

Daniel Quillen, *Higher algebraic K-theory, I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR 0338129 (49 #2895)

Christian Schlichtkrull, *Units of ring spectra and their traces in algebraic K-theory*, Geom. Topol. 8 (2004), 645–673. MR 2057776

Marco Schlichting, *Hermitian K-theory of exact categories*, J. K-Theory 5 (2010), no. 1, 105–165. MR 2600285 (2011b:19007)

Hiroshi Tambara, *On multiplicative transfer*, Comm. Algebra 21 (1993), no. 4, 1393–1420. MR 1209937
[Ull13] John Ullman, *Tambara functors and commutative ring spectra*, arXiv:1304.4912, 2013.