Random Bulgarian solitaire

Serguei Popov*

March 29, 2022

Departamento de Estatística, Instituto de Matemática e Estatística, Universidade de São Paulo, rua do Matão 1010, CEP 05508–090, São Paulo SP, Brasil.
E-mail: popov@ime.usp.br

Abstract

We consider a stochastic variant of the game of Bulgarian solitaire [9]. For the stationary measure of the random Bulgarian solitaire, we prove that most of its mass is concentrated on (roughly) triangular configurations of certain type.

Keywords: shape theorem, triangular configuration, Markov chain, stationary measure

1 Introduction and results

Consider the following (random) game: a deck of $N$ cards is divided into several piles. Then, for each pile, we leave it intact with probability $1 - p$ and remove one card from there with probability $p$ ($p \in [0, 1]$ is a given parameter), independently of other piles. The cards that were removed are collected to form a new pile. The order of piles is not important and the piles of size zero are ignored. The case $p = 0$ is trivial (nothing moves) and will not be considered. When $p = 1$, this is the game of Bulgarian solitaire, made known by Martin Gardner in [9], and studied in [1, 3, 10, 11, 12] (cf. also [5, 15] for some variations of that game). The “truly random” model with parameter $0 < p < 1$ is a discrete-time irreducible and aperiodic Markov

*Partially supported by CNPq (302981/02–0)
chain on the space of all unordered partitions of \( N \); for obvious reasons, it will be referred to as the \textit{random Bulgarian solitaire}.

If the number of cards \( N \) is a triangular number, i.e., \( N = 1 + 2 + \cdots + k \) for some \( k \), a remarkable fact is that, starting from any initial configuration, after a finite number of moves the (deterministic) Bulgarian solitaire will reach the stable configuration formed by piles of sizes \( k, k-1, \ldots, 1 \). The above result was proved in [14] (see the solution to Problem 6.10) and in [4] independently, and later it was discovered that the maximal number of moves necessary to enter the stable configuration is \( k^2 - k \), and that that bound is sharp (see [11, 12]). If \( N \) is not a triangular number, then such a stable configuration does not exist. However, it is possible to prove that after at most \( O(k^2) = O(N) \) moves the game will enter into a cycle. Moreover, all the configurations of the cycle are “almost triangular” in the following sense. Let \( k = \max\{n : 1 + 2 + \cdots + n \leq N\} \); then all the configurations in that cycle can be constructed from the configuration \((k, k-1, \ldots, 1)\) by adding at most one card to each pile, and maybe adding one more pile of size 1, see [1, 3, 10, 14] for exact formulations and more details.

Thus, we see that Bulgarian solitaire “likes” triangular configurations, and so we may expect some kind of similar behaviour from the random Bulgarian solitaire. There is no possibility, however, to obtain exact results of the form of those of the previous paragraph, since random Bulgarian solitaire is a finite irreducible Markov chain, so it visits all its states infinitely many times a.s. Instead, we aim at the results of the following kind: the stationary measure of the set of configurations which are in some sense close to the (rescaled) triangular configuration is close to 1. This can be regarded as a “shape theorem” result even though it is substantially different from most of the shape results appearing in the literature. (In most cases some time-dependent random set is constructed, and then, when rescaled by time, it converges to some, usually nonrandom, shape. See e.g. [2, 8, 13, 16] for results of this kind.) The results we are aiming at resemble rather those of [6, 7].

Also, let us remark here that the question of how fast the deterministic Bulgarian solitaire \textit{approximates} the triangle has not been yet studied in the literature. To motivate this question, take \( N = 1 + 2 + \cdots + k \), and suppose that the initial configuration is \((k-1, k-1, k-2, k-3, \ldots, 3, 2, 1, 1)\), i.e., the exact triangular configuration is modified by removing one card from the biggest pile and forming one more pile of size 1 with that card. Then, macroscopically this configuration is already quite triangular; however, if we are aiming to reach \((k, k-1, \ldots, 1)\), this is the worst possible initial configu-
ration (the number of moves needed is exactly $k^2 - k$, cf. [11])! Here we prove that, whenever the initial configuration is "reasonable" (i.e., the number of piles is $O(N^{1/2})$ and the number of cards in the biggest pile is also $O(N^{1/2})$), we need only $O(N^{1/2})$ moves of deterministic Bulgarian solitaire to make the $(N^{1/2} \text{-rescaled})$ configuration close to the triangle. While such a result by itself may not be of great interest, the method of its proof will be an important tool in the course of the proof of the results about random Bulgarian solitaire.

Now, we introduce some notations and give the formal definition of the process. If $\ell(S)$ is the number of piles in the configuration $S$, we write $S = (k_1, \ldots, k_{\ell(S)})$, where $k_1 \geq \ldots \geq k_{\ell(S)}$. We denote also by $R(S) := k_1$ the size of the biggest pile and by $|S| := k_1 + \cdots + k_{\ell(S)}$ the number of cards in the configuration. Let $\text{ord}(n_1, \ldots, n_m)$ be the operation of putting $n_1, \ldots, n_m$ in the decreasing order and discarding zeros. Now, let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of i.i.d. random variables such that $P[\xi_1 = 1] = 1 - P[\xi_1 = 0] = p$. Then the operator $Q_p$ which transforms the configuration $S = (k_1, \ldots, k_{\ell(S)})$ in the game of random Bulgarian solitaire with parameter $p$ is defined by

$$Q_p S = \text{ord}(k_1 - \xi_1, \ldots, k_{\ell(S)} - \xi_{\ell(S)}; \xi_1 + \cdots + \xi_{\ell(S)}).$$

Denote also by $Q_p^{(n)} S$ the result of $n$ independent applications of $Q_p$ to $S$; clearly, the process is conservative in the sense that $|Q_p^{(n)} S| = |S|$ for all $n$. Suppose that $|S_0| = N$. As remarked above, for $0 < p < 1$ the stochastic process $S_0, Q_p S_0, Q_p^{(2)} S_0, \ldots$ is an irreducible aperiodic Markov chain with finite state space $\mathcal{X}_N := \{S : |S| = N\}$. We denote by $\pi_{p,N}(\cdot)$ its stationary measure.

To formulate our results, we need also to find a way to define sets of configurations that are close to a specific triangular configuration. To this end, for two configurations $S_1 = (n_1, \ldots, n_{\ell(S_1)})$, $S_2 = (m_1, \ldots, m_{\ell(S_2)})$ define the distance $\rho(S_1, S_2)$ by

$$\rho(S_1, S_2) = \max_{j \geq 1} |n_j - m_j|,$$

with the convention $n_j = 0$ for all $j > \ell(S_1)$ and $m_j = 0$ for all $j > \ell(S_2)$. Next, we define the triangular configuration $T(p, N) = (n_1, \ldots, n_m)$ by $n_j = \lceil (2Np)^{1/2} - pj \rceil$, $m_0 = \ell(T(p, N)) = \max\{j : \lceil (2Np)^{1/2} - pj \rceil \geq 1\}$. When $p$ is fixed and $N \to \infty$, we can write $R(T(p, N)) = \ell(T(p, N)) = (2Np)^{1/2} + O(1)$. Finally, for $\varepsilon > 0$ (which may depend on $N$) define the...
set $T(\varepsilon,p,N)$ of “roughly triangular” configurations by

$$T(\varepsilon,p,N) = \{ S : |S| = N, \rho(S,T(p,N)) \leq \varepsilon N^{1/2} \}.$$ 

Let $k = k(N) = \max\{n : 1+2+\cdots+n \leq N\}$, and define the configuration $T_0^N := (k + m_1, k - 1 + m_2, \ldots, 1 + m_k)$, where

$$m_i = \begin{cases} 1, & \text{if } i \leq N - \frac{k(k+1)}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $|T_0^N| = N$ (for example, for $N = 11$ we have $T_0^N = (5, 3, 2, 1)$). For the particular case $p = 1$ we say that $T(\varepsilon,1,N)$ is nondegenerate if it contains the configuration $T_0^N$, as well as all the configurations $S$ with $\rho(T_0^N,S) = 1$.

It is easy to see that for any fixed $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that $T(\varepsilon,1,N)$ is nondegenerate for all $N \geq N_0$, and the same is true when e.g. $\varepsilon \sim N^{-\alpha}$, $\alpha < 1/2$.

Now we are ready to formulate the main results of this paper. First, we state the result about the time to approximate the triangular configuration for the deterministic Bulgarian solitaire (i.e., with $p = 1$).

**Theorem 1.1** Take $\varepsilon > 0$ and suppose that $N$ is large enough to guarantee that $T(\varepsilon,1,N)$ is nondegenerate. Suppose that the initial configuration $S_0$ with $|S_0| = N$ has the following properties: $\ell(S_0) \leq \gamma_1 N^{1/2}$ and $R(S_0) \leq \gamma_2 N^{1/2}$ for some $\gamma_1, \gamma_2 > 0$. Then there exists $v_0 = v_0(\varepsilon,\gamma_1,\gamma_2)$ such that we have

$$Q_1^{(n)} S_0 \in T(\varepsilon,1,N)$$

(1.1)

for all $n \geq v_0 N^{1/2}$.

In words, this result means that if the initial configuration is “reasonable”, then the number of moves required to approximate the triangle is $O(N^{1/2})$.

Now, we turn our attention to random Bulgarian solitaire:

**Theorem 1.2** Suppose that $0 < p < 1$. Then for any $a < 1/144$ there exist positive constants $v_1 = v_1(a,p)$ and $\delta = \delta(a,p)$ such that for all $N$

$$\pi_{p,N}(T(N^{-a},p,N)) \geq 1 - \exp(-v_1 N^{\delta}).$$

(1.2)

In Section 3 there are some more comments and open problems related to the Bulgarian solitaire (both deterministic and random). Also, the reader may find it interesting to look at JAVA simulation of the random Bulgarian solitaire (with $p = 1/2$) on the internet page of Kyle Petersen at

http://people.brandeis.edu/~tkpeters/reach/stuff/reach
2 Proofs

This section is organized in the following way. In Section 2.1 we introduce the notion of Etienne diagram, which is just another way to represent the configurations of the game. Then, we show how the moves of Bulgarian solitaire are performed on this diagram and discuss its other properties. In Section 2.2 we prove Theorem 1.1 and in Section 2.3 we prove Theorem 1.2 (in Section 2.3 some results and technique from Sections 2.1 and 2.2 are used, most notably the inequality (2.13)).

2.1 Representation via Etienne diagram and its properties

Before starting the proofs, we need to describe another representation of a particular state of (deterministic) Bulgarian solitaire, which we call an Etienne diagram (cf. [11]). In this approach the cards are identified with particles living in the cells of the set \( \mathcal{Z} = \{(i,j) \in \mathbb{Z}^2 : i \geq 1, 1 \leq j \leq i\} \), with at most one particle per cell. We write \( \mathcal{R}_{i,j} = 1 \) when the cell \((i,j)\) is occupied and \( \mathcal{R}_{i,j} = 0 \) when the cell \((i,j)\) is empty. Clearly, \( \mathcal{Z} \) is a half-quadrant of \( \mathbb{Z}^2 \), but we would like to visualize \( \mathcal{Z} \) in a little bit unconventional way (see Figure 1): the cell \((1,1)\) lies in the base and supports the column \( \{(i,1), i = 1, 2, 3, \ldots\} \), while the diagonal \( \{(i,i), i = 1, 2, 3, \ldots\} \) goes in the NW direction (so the rows of \( \mathcal{Z} \) are enumerated from right to left; notice that at this point we deviate from [11], where the rows were enumerated from left to right). Now, a configuration \( S = (k_1, \ldots, k_{\ell(S)}) \) is represented as follows (as on Figure 1): we put \( \mathcal{R}_{i,j} = \mathcal{R}_{i,j}(S) = 1 \) for

\[
(i,j) \in \bigcup_{n=1}^{\ell(S)} \bigcup_{m=1}^{k_n} \{(n+m-1,m)\},
\]

and \( \mathcal{R}_{i,j} = \mathcal{R}_{i,j}(S) = 0 \) for all other pairs \((i,j)\). From the fact that \( k_1 \geq \ldots \geq k_{\ell(S)} \) we immediately deduce that for any \( S \)

\[
\text{if } \mathcal{R}_{i,j} = 0 \text{ then } \mathcal{R}_{n,m} = 0 \text{ for all } n \geq i, j \leq m \leq j + n - i, \tag{2.1}
\]

and

\[
\text{if } \mathcal{R}_{i,j} = 1 \text{ then } \mathcal{R}_{n,m} = 1 \text{ for all } n \leq i, \max\{1,j-i+n\} \leq m \leq j. \tag{2.2}
\]
Figure 1: The Etienne diagram of $S = (7, 5, 3, 2, 1, 1)$. We have $\mathcal{R}_{i,j}(S) = 1$ for all $(i, j)$ with $i \leq 5$ and for $(i, j) = (6, 1), (6, 5), (6, 6), (7, 7)$.

One of the advantages of the representation via Etienne diagram is that it makes it more clear how the process approaches the triangular configuration. To see what we mean, first note that the move of the Bulgarian solitaire consists in applying the following two substeps to the corresponding Etienne diagram (see Figure 2):

- apply the cyclic shift (from left to right) to each row of the diagram;
- if after the shift there is a particle that is placed above an empty cell, then the particle falls there; this procedure is repeated until no further fall is possible.

Speaking formally, let $S' = Q_1 S$. Then the Etienne diagram of $S'$ is constructed using the following procedure:

1. **(I):** For all $i$ put $b^0_{i,j} = \mathcal{R}_{i,j+1}(S)$ for $j < i$ and $b^0_{i,i} = \mathcal{R}_{i,1}(S)$.

2. **(II):** Suppose that for the array $b^0$ we can find $(i_0, j_0)$ such that $b^0_{i_0,j_0} = 0$, $b^0_{i_0+1,j_0} = 1$. Then construct the new array $b^1$ by $b^1_{i_0,j_0} = 1$, $b^1_{i_0+1,j_0} = 0$, and $b^1_{i,j} = b^0_{i,j}$ for $(i, j) \neq (i_0, j_0), (i_0 + 1, j_0)$. 

6
Figure 2: Moves of Bulgarian solitaire on the Etienne diagram
(III): Repeat the previous procedure with $b^1$ instead of $b^0$, and so on. At some moment we will obtain an array $b^n$ for which we cannot find $(i_m, j_m)$ such that $b_{i_m,j_m}^n = 0, b_{i_m+1,j_m}^n = 1$. Then for all $(i, j) \in \mathbb{Z}$ put $\mathcal{R}_{i,j}(S') = b_{i,j}^n$.

Now, suppose that $|S| = 1 + 2 + \cdots + k$. On the Etienne diagram the triangular configuration corresponds to the configuration $(\mathcal{R}_{i,j} = 1\{i \leq k\})$. Note also that if the first $m$ rows of the diagram are occupied, then they will remain occupied during all the subsequent evolution. This shows that the falls of particles “help” to reach the stable configuration (more and more rows become all occupied). Moreover, in many concrete situations it is possible to know how many moves are needed to fill out some region which was originally empty. Arguments of this kind will be heavily used in the course of the proof of our results.

Consider the Etienne diagram of a configuration $S$. Since the system is conservative, there is a natural correspondence between particles (holes) in that diagram and particles (holes) in the diagram of the configuration $Q_1 S$. This shows that for each particle (hole) on the original diagram we can define its trajectory, i.e., we know its position after $n$ moves of the game. Let $J_{i,j}(n)$ be the second coordinate of the particle (hole) from $(i, j)$ after $n$ moves, and let $\mathcal{M}_{i,j}(n)$ be the number of falls (movements upwards) that the particle (hole) from $(i, j)$ was subjected to during $n$ moves. That means that, if $\mathcal{R}_{i,j}(S) = 1$, then $(i - \mathcal{M}_{i,j}(n), J_{i,j}(n))$ are the coordinates of the particle from $(i, j)$ after $n$ moves, while if $\mathcal{R}_{i,j}(S) = 0$, then $(i + \mathcal{M}_{i,j}(n), J_{i,j}(n))$ are the coordinates of the hole from $(i, j)$ after $n$ moves. It seems to be very difficult to calculate exactly $J_{i,j}(n)$ and $\mathcal{M}_{i,j}(n)$ (except in trivial situations, when, e.g., $\mathcal{R}_{i,j}(S) = 1$ and $\mathcal{R}_{i',j'}(S) = 1$ for all $i' < i$). However, we can establish some relation between these quantities by defining first

$$
\hat{J}_{i,j}(n) = \begin{cases} 
  j - n + i \left\lfloor \frac{n}{i} \right\rfloor, & \text{if } n - i \left\lfloor \frac{n}{i} \right\rfloor < j, \\
  j - n + i \left( \left\lfloor \frac{n}{i} \right\rfloor + 1 \right), & \text{if } n - i \left\lfloor \frac{n}{i} \right\rfloor \geq j.
\end{cases}
$$

In words, $(i, \hat{J}_{i,j}(n))$ would be the position of particle (hole) from $(i, j)$ at time $n$ if we know that $\mathcal{M}_{i,j}(n) = 0$ (the quantities $J, \mathcal{M}, \hat{J}$ depend also on $S$, but we do not indicate that in our notations).
Lemma 2.1 If \(R_{i,j}(S) = 0\) and \(n\) is such that \(i - \hat{J}_{i,j}(n) > \lfloor n/i \rfloor M_{i,j}(n)\), or \(R_{i,j}(S) = 1\) and \(n\) is such that \(\hat{J}_{i,j}(n) > \lfloor n/i \rfloor M_{i,j}(n)\), then

\[
|J_{i,j}(n) - \hat{J}_{i,j}(n)| \leq \left\lfloor \frac{n}{i} \right\rfloor M_{i,j}(n)
\]

Proof. Suppose for example that \(R_{i,j}(S) = 0\). Denote \(j' = j - M_{i,j}(n)\).

Since \(n\) is such that \(i - \hat{J}_{i,j}(n) > \lfloor n/i \rfloor M_{i,j}(n)\), we have that \(\hat{J}_{i,j}(n) \leq \hat{J}_{i,j}(n)\). The lemma then follows from the fact that \(J_{i,j}(n)\) should be somewhere in between \(\hat{J}_{i,j}(n)\) and \(\hat{J}_{i,j}(n)\). The other case is treated analogously. \qed

Next, we define some quantities which concern the geometric structure of the representation via Etienne diagram, and prove some relations between them. For \(N \geq 1\) define

\[
\theta_N = \max \left\{ k : \frac{k(k+1)}{2} \leq N \right\};
\]

when \(N \to \infty\), we have \(\theta_N = (2N)^{1/2} + O(1)\). Using the Etienne representation of a configuration \(S\), define

\[
E_-(S) = \sum_{i \leq \theta_N} \sum_{j : R_{i,j}(S) = 0} \left( \theta_{|S|} - i + \frac{1}{2} \right), \quad (2.3)
\]

\[
E_+(S) = \sum_{i > \theta_N} \sum_{j : R_{i,j}(S) = 1} \left( i - \theta_{|S|} - \frac{1}{2} \right), \quad (2.4)
\]

and put \(E(S) = E_-(S) + E_+(S)\). The quantity \(E(S)\) can be thought of as the “energy” of the configuration: the bigger is \(E(S)\), the “more distant” (not necessarily in the sense of the distance \(\rho\)) is \(S\) from \(T_0^N\). Denote also \(G_{\alpha,\beta}^N = \{ S : |S| = N, \ell(S) \leq \alpha N^{1/2}, R(S) \leq \beta N^{1/2} \}\). The next lemma establishes some elementary properties of the energy \(E(S)\).

Lemma 2.2 (i) There exists a constant \(\gamma = \gamma(\alpha, \beta)\) such that for all \(N\) and all \(S \in G_{\alpha,\beta}^N\) we have \(E(S) \leq \gamma N^{3/2}\).

(ii) For all \(S\) it holds that \(E(Q_1S) \leq E(S)\). Moreover, \(E(S) - E(Q_1S)\) is equal to the number of falls of particles during the second substep of the move of the Bulgarian solitaire represented by the Etienne diagram (i.e., it is equal to \(\bar{m}\) in (III)).
Proof. Define \( \mathcal{H}(S) = \max\{i : \text{there exists } j \text{ such that } R_{i,j}(S) = 1\} \). From (2.2) one easily gets that there exists \( \gamma' = \gamma'(\alpha, \beta) \) such that for all \( S \in G^N_{\alpha, \beta} \) we have \( \mathcal{H}(S) \leq \gamma' N^{1/2} \). The proof of (i) then reduces to an elementary computation (roughly speaking, to compute \( E(S) \) we have at worst \( O(N) \) terms, each of order \( O(N^{1/2}) \)).

As for the proof of (ii), note first that the operation of cyclic shift does not change the quantities defined in (2.3)–(2.4). Then, it is straightforward to see that each particle's fall decreases \( E \) by one unit, which concludes the proof of the lemma.

Define
\[
\begin{align*}
  h_+(S) &= \max\{i : \text{there exists } j \in [i - \theta|S|, \theta|S|] \text{ such that } R_{i,j}(S) = 1\} - \theta|S|, \\
  h_-(S) &= \theta|S| - \min\{i : \text{there exists } j \text{ such that } R_{i,j}(S) = 0\},
\end{align*}
\]

and
\[
\begin{align*}
  V_+(S) &= |\{(i, j) \in Z : i > \theta|S|, R_{i,j}(S) = 1\}|, \\
  V_-(S) &= |\{(i, j) \in Z : i \leq \theta|S|, R_{i,j}(S) = 0\}|.
\end{align*}
\]

(cf. Figure 3). In words,

- \( h_-(S) \) is the maximal vertical distance between \( \theta_N \) and the holes below \( \theta_N \);
- \( h_+(S) \) is the maximal vertical distance between \( \theta_N \) and the particles above \( \theta_N \) which also lie inside the area indicated by the dashed lines on Figure 3
- \( V_- \) is the total area covered by the holes below \( \theta_N \);
- \( V_+ \) is the total area covered by the particles above \( \theta_N \).

Similarly to the energy \( E(S) \), all those quantities could be used to measure the deviation of \( S \) from the "almost triangular" configuration \( T_0^N \). Consider also the normalized quantities \( \tilde{h}_\pm(S) = |S|^{-1/2} h_\pm(S) \), \( \tilde{V}_\pm(S) = |S|^{-1} V_\pm(S) \), and \( \tilde{E}_\pm(S) = |S|^{-3/2} \tilde{E}_\pm(S) \), \( \tilde{E}(S) = |S|^{-3/2} E(S) \).
Lemma 2.3 For all $S \in G_{\gamma_1, \gamma_2}^{\lfloor |S| \rfloor}$ there exist constants $\alpha_i, i = 1, \ldots, 8$ (depending on $\gamma_1, \gamma_2$) such that

\begin{align*}
\alpha_1 \tilde{h}^3_-(S) &\leq \tilde{E}_-(S) \leq \alpha_2 \tilde{h}^2_-(S), & (2.5) \\
\alpha_3 \tilde{h}^3_+(S) &\leq \tilde{E}_+(S) \leq \alpha_4 \tilde{h}_+(S), & (2.6) \\
\alpha_5 \tilde{h}^2_-(S) &\leq \tilde{V}_-(S) \leq \alpha_6 \tilde{h}_-(S), & (2.7) \\
\alpha_7 \tilde{h}^2_+(S) &\leq \tilde{V}_+(S) \leq \alpha_8 \tilde{h}_+(S). & (2.8)
\end{align*}

Proof. It is elementary to obtain the inequalities (2.5) and (2.7) from (2.1). Analogously, to obtain (2.6) and (2.8), one can use (2.2) and the fact that $S \in G_{\gamma_1, \gamma_2}^{\lfloor |S| \rfloor}$ together with the following observation. If $R_{i,j}(S) = 1$ for some $i > \theta_{|S|}$, then either $i - \theta_{|S|} \leq h_+(S)$ or $\min\{j, i-j\} \leq h_+(S)$.

Consider a configuration $S$ such that $|S| = N$. By definition of $\theta_N$, there exists a constant $\hat{\beta} > 0$ such that

$$0 \leq V_+(S) - V_-(S) \leq \hat{\beta} N^{1/2}.$$  

(2.9)
Also, we will always tacitly assume that $V_-(S) \geq \beta N^{1/2}$, i.e., we will not consider configurations that are “too close” to the triangle. In this case there are constants $\beta_1, \beta_2 > 0$ such that

$$\beta_1 < \frac{\tilde{V}_+(S)}{V_-(S)} < \beta_2 \quad (2.10)$$

(note also that if $N$ is a triangular number, then $\tilde{V}_+(S)/V_-(S) = 1$ for any $S \in \mathcal{X}_N$). Using (2.6), (2.7), and (2.10), we obtain

$$\tilde{E}_-(S) \leq \alpha_2 h_-(S) \leq \frac{\alpha_2}{\alpha_5} \tilde{V}_-(S) \leq \frac{\alpha_2\beta_1}{\alpha_5} h_+(S),$$

and, by (2.6), $\tilde{E}_+(S) \leq \alpha_4 h_+(S)$. This shows that there is a constant $C_1$ such that $\tilde{h}_+(S) \geq C_1 \tilde{E}(S)$ for all $S \in G_{\gamma_1,\gamma_2}$. Analogously, we obtain that for some $C_2, C_3$ it holds that $\tilde{h}_-(S) \geq C_2 \tilde{E}_+^2(S)$ and $\tilde{h}_-(S) \geq C_3 \tilde{E}_-^2(S)$. By Lemma 2.2 (i) the quantity $\tilde{E}$ is bounded on $G_{\gamma_1,\gamma_2}$, so there is $C_4$ such that $\tilde{E}_-^{1/2}(S) \geq C_4 \tilde{E}_+^2(S)$, which implies that $\tilde{h}_-(S) \geq C_5 \tilde{E}_+^2(S)$ for some $C_5$. Finally, we use Lemma 2.2 (i) once again to obtain that there exists $\beta = \beta(\gamma_1, \gamma_2)$ such that

$$\min\{\tilde{h}_+(S), \tilde{h}_-(S)\} \geq \beta \tilde{E}_+^2(S) \quad (2.11)$$

when $S \in G_{\gamma_1, \gamma_2}$.

### 2.2 Proof of Theorem 1.1

First, the idea is to prove that after $O(N^{1/2})$ moves, the “normalized energy” $\tilde{E}$ will decrease by a considerable amount. Consider a configuration $S$ with $|S| = N$. Abbreviate $\hat{h} = \lfloor \beta \tilde{E}_+(S) N^{1/2} \rfloor$; by (2.11), we can find $m_1, m_2$ such that $R_{\theta_N - \hat{h}, m_1}(S) = 0$ and $R_{\theta_N + \hat{h}, m_2}(S) = 1$. Moreover, without loss of generality one can suppose that $\hat{h}$ is divisible by 5. Define also $j_1 = m_1 + 4\hat{h}/5$, $j_2 = m_2 - 4\hat{h}/5$, and $\tilde{a} = \sqrt{2h^2}/25N^{1/2}$. Define two sets $U_1, U_2 \subset \mathcal{Z}$ by

$$U_1 = \{(i, j) : \theta_N - \tilde{a} - \frac{\hat{h}}{5} \leq i \leq \theta_N - \frac{\hat{h}}{5}, m_1 \leq j \leq j_1 + i - \theta_N + \frac{\hat{h}}{5}\},$$

$$U_2 = \{(i, j) : \theta_N + \frac{\hat{h}}{5} \leq i \leq \theta_N + \tilde{a} + \frac{\hat{h}}{5}, j_2 + i - \theta_N - \frac{\hat{h}}{5} \leq j \leq m_2\},$$
(see Figure 4). Note that from (2.1) and (2.2) it follows that $R_{i,j}(S) = 0$ for all $(i,j) \in U_1$ and $R_{i,j}(S) = 1$ for all $(i,j) \in U_2$.

Abbreviate also $i_0 = \theta_N - \hat{h}/5$, $i_0' = \theta_N + \hat{h}/5$. Now, the idea is to consider the evolution of sets $U_1, U_2$ at times $i_0 k$, $k = 0, 1, 2, \ldots$. First, note that $\hat{J}_{i_0,j}(k|i_0) = j$, for any $j$ and $k$. Then, each time we make a complete turn (i.e., $i_0$ moves) a particle which was on the level $\theta_N + \hat{h}/5$ will be $2\hat{h}/5$ units to the left of its initial position (provided it did not fall). This shows that there exists $k_0 \leq \frac{5\sqrt{2}}{2\hat{h}} N^{1/2}$ such that

$$\left| [\hat{J}_{i_0,j_2}(k_0 i_0), \hat{J}_{i_0',m_2}(k_0 i_0)] \cap [m_1, j_1] \right| \geq \frac{2\hat{h}}{5}$$

(when $\hat{J}_{i_0,j_2}(k_0 i_0) > \hat{J}_{i_0',m_2}(k_0 i_0)$, by $[\hat{J}_{i_0,j_2}(k_0 i_0), \hat{J}_{i_0',m_2}(k_0 i_0)]$ we mean in fact $[0, \hat{J}_{i_0',m_2}(k_0 i_0)] \cup [\hat{J}_{i_0,j_2}(k_0 i_0), i_0]$). Take $j_3, j_4$ such that

$$[j_3, j_4] \subset ([\hat{J}_{i_0,j_2}(k_0 i_0), \hat{J}_{i_0',m_2}(k_0 i_0)] \cap [m_1, j_1])$$

and $j_4 - j_3 + 1 = \frac{2\hat{h}}{5}$. We consider two cases:

Case 1: at time $k_0 i_0$ in the set

$$U'_1 = \{(i,j) : i \in [i_0 - \hat{a}, i_0], j \in [j_3, j_3 - 1 + \hat{h}/5]\}$$

there is at least one hole, i.e., $R_{i,j}(Q_{k_0 i_0}^1 S) = 0$ for at least one $(i,j) \in U'_1$. In this case, by (2.1) no particle can be in the set

$$U'_2 = \{(i,j) : i \in [i_0', i_0' + \hat{a}], j \in [j_4 - \hat{h}/5, j_4]\},$$
i.e., for all \((i, j) \in U_2'\) we have that \(\mathcal{R}_{i,j}(Q_1^{(k_0i_0)}) S = 0\). Note that \(\hat{a} k_0 \leq \hat{h}/5\), so the “image” of \(U_2\) after \(k_0\) turns completely covers \(U'_2\). On the other hand, \(U'_2\) must be completely empty, so there should have been a lot of particle falls in order to avoid \(U'_2\). In what follows we estimate the minimal number of falls necessary (and, consequently, we find the minimal amount by which the energy \(E\) should decrease). Define the set

\[
U''_2 = \{(i, j) : i \in [i_0' + \hat{a}/2, i_0' + \hat{a}], j \in [j_4 - \hat{h}/10, j_4]\} \subset U'_2.
\]

For any \((i, j) \in U''_2\) there is a unique \(j'\) such that \(\hat{J}_{i,j'}(k_0i_0) = j\), and, by the above observation, \((i, j') \in U_2\), so the cell \((i, j')\) originally contained a particle. To guarantee that that particle is not in \(U''_2\) at time \(k_0i_0\), at least one of the following two possibilities must occur:

- either \(M_{i,j'}(k_0i_0) \geq \hat{a}/2\),
- or \(\hat{J}_{i,j'}(k_0i_0) - \hat{J}_{i,j}(k_0i_0) > \hat{h}/10\), but in this case, by Lemma 2.1, \(M_{i,j'}(k_0i_0) \geq \frac{\hat{h}}{10 \hat{k}_0} > \frac{\hat{h}^2}{25 \sqrt{2} N^{3/2}} = \hat{a}/2\).

Denote \(h_0 = \hat{h}N^{-1/2}\); for the both of the above possibilities, we obtained in fact that \(M_{i,j'}(k_0i_0) \geq C_1 h_0^2 N^{1/2}\). Since the number of cells in the set \(U''_2\) is at least \(C_2 h_0^3 N\), the number of particle falls until time \(k_0i_0\) should be at least \(C_1 h_0^2 N^{1/2} \times C_2 h_0^3 N = C_1 C_2 h_0^5 N^{3/2}\). By Lemma 2.2 (ii), it means that, for the Case 1,

\[
E(Q_1^{(k_0i_0)}) S - \bar{E}(S) \leq -C_1 C_2 h_0^5.
\]  

Case 2: there are no holes at time \(k_0i_0\) in the set \(U'_1\), i.e., \(\mathcal{R}_{i,j}(Q_1^{(k_0i_0)}) S = 1\) for all \((i, j) \in U'_1\). Using the duality between holes and particles, this case can be treated quite analogously to the Case 1. Namely, we note first that the “image” of \(U'_1\) after \(k_0\) turns completely covers \(U'_1\). So, in order to escape \(U'_1\), the holes that are “candidates” to be there must make a sufficient number of movements in the upwards direction. In the same way as in the Case 1, one can work out all the details to obtain that (2.12) is valid for the Case 2 as well.

We continue proving Theorem 1.1. By (2.11) and (2.12), there exist \(\lambda_1, \lambda_2\) such that

\[
\bar{E}(Q_1^{(ns)}) S - \bar{E}(S) \leq -\lambda_1 \bar{E}^{10}(S),
\]

where \(n_s = \lambda_2 \bar{E}^{-2}(S) N^{1/2}\) (the formula (2.13) will play an important role in the proof of Theorem 1.2 as well). Consider now the initial configuration \(S_0 \in \)
by Lemma 2.2 (i), \(a_0 := \tilde{E}(S_0) \leq \Gamma\) for some \(\Gamma\). Fix an arbitrary \(\varepsilon > 0\) and define \(\varphi(x) = x - \lambda_1 x^{10}\); then there exists \(k_1\) (depending only on \(\Gamma, \varepsilon, \lambda_1\)) such that \(\varphi(k_1)(a_0) < \varepsilon\). By (2.13) this means that

\[
\tilde{E}(Q_1^{(n_S')} S) \leq \varepsilon,
\]

where \(n_S' = k_1 \lambda_2 \varepsilon^{-2} N^{1/2}\), i.e., after \(O(N^{1/2})\) moves we will arrive to a configuration with small normalized energy \(\tilde{E}\).

Now we are almost done with the proof of Theorem 1.1, and it remains only to make one small effort: we have to prove that if the energy \(\tilde{E}(S)\) is small, then either \(S\) is already close to the triangular configuration \(T(1, |S|)\) (in the sense of the distance \(\rho\)), or it will come close to \(T(1, |S|)\) after \(O(N^{1/2})\) moves.

Define the sets

\[
\mathcal{V}(\varepsilon, N) = \{S : |S| = N, \max\{h_+(S), h_-(S)\} \leq \varepsilon N^{1/2}\},
\]

\[
\hat{\mathcal{V}}(\varepsilon, N) = \{S : |S| = N, \max\{i : \text{there exists } j \text{ such that } R_{i,j}(S) = 1\} \leq \theta_N + \varepsilon N^{1/2}\} \cap \mathcal{V}(\varepsilon, N).
\]

It is elementary to see that, for fixed \(\varepsilon\) and for all \(N\) large enough

\[
\hat{\mathcal{V}}(\varepsilon, N) \subset T(2\varepsilon, 1, N). \tag{2.15}
\]

We need the following

**Lemma 2.4** Suppose that \(S \in \mathcal{V}(\varepsilon, N) \cap G_{\gamma_1, \gamma_2}^N\), and put \(n_0 = [\max\{\gamma_1, \gamma_2\}] + 2\). Then \(Q_1^{(n_0 \theta_N)} S \in \hat{\mathcal{V}}(2n_0 \varepsilon, N)\).

**Proof.** Define

\[
\mathcal{V}'(\varepsilon, N) = \{S : |S| = N, \max\{i : \text{there exists } j \in [1, \theta_N] \text{ such that } R_{i,j}(S) = 1\} \leq \theta_N + \varepsilon N^{1/2}\} \cap \mathcal{V}(\varepsilon, N).
\]

First, if \(\varepsilon < 1/\sqrt{2}\) and \(S \in \mathcal{V}(\varepsilon, N) \cap G_{\gamma_1, \gamma_2}^N\), then the set \(\{(i, j) : i \geq \theta_N + \varepsilon N^{1/2}, j \in [1, \varepsilon N^{1/2}]\}\) will be empty of particles after \(\varepsilon N^{1/2}\) moves. By examining where those particles could go, we see that \(Q_1^{(\varepsilon N^{1/2})} S \in \mathcal{V}'(2\varepsilon, N)\) and that

\[
\max\{i : \text{there exists } j \text{ such that } R_{i,j}(S') = 1\} - \theta_N \leq (\varepsilon + \max\{\gamma_1, \gamma_2\}) N^{1/2},
\]

where \(S' = Q_1^{(\varepsilon N^{1/2})} S\). To conclude the proof of the Lemma 2.4 note the following two facts:
• Suppose that at some moment the configuration belongs to the set \( \mathcal{V}'(\epsilon', N) \). Then if \( i' > \theta N + \epsilon' N^{1/2} \) and there are some particles in the set \( \{(i, j) : i \geq i', j = i'\} \), then at the next moment all those particles will fall one unit.

• If \( S \in \mathcal{V}'(\epsilon', N) \) and
\[
\max_{i > \theta N + \epsilon' N^{1/2}} (i - \min \{ j : \mathcal{R}_{i,j}(S) = 1 \}) \leq \epsilon'' N^{1/2},
\]
then \( Q_1^{(\theta N + \epsilon' N^{1/2})} S \in \mathcal{V}'(\epsilon' + \epsilon'', N) \) (to see this, it is enough to figure out what happens with the configuration \( S'' \) after \( \theta N + \epsilon' N^{1/2} \) moves, where \( S'' \) is defined by \( \mathcal{R}_{i,j}(S'') = 1 \) whenever either \( i \leq \theta N + \epsilon' N^{1/2} \) or \( i - j < \epsilon'' N^{1/2} \)).

2.3 Proof of Theorem 1.2

Consider a finite irreducible discrete-time Markov chain with state space \( X \), transition matrix \( P \), and stationary measure \( \pi \). The following elementary fact will be useful in the course of the proof of Theorem 1.2: for any \( A \subset X \) and \( n \geq 1 \)
\[
\sum_{x \in A, y \in A^c} \pi(x) P_{xy}^{(n)} = \sum_{x \in A, y \in A^c} \pi(y) P_{yx}^{(n)}. \tag{2.16}
\]

Let us describe the main steps of the proof of Theorem 1.2:

• first, in Lemma 2.5 we prove (using (2.16)) that a typical configuration of the random game should be reasonable, i.e., the number of piles and the biggest pile should be \( O(N^{1/2}) \);

• then, the idea is the following: starting from a reasonable configuration, the macroscopic evolution of the profiles will be very similar in the
random game and in the deterministic game where the initial sizes of the piles are $p^{-1}$ times bigger (indeed, if the initial size of the pile in the random game is $k = O(N^{1/2})$, then it will be emptied typically in time $k/p \pm O(N^{1/4})$);

- unfortunately, it seems to be difficult to dominate the stochastic game by the deterministic one directly. So, we introduce another deterministic process by allowing the immigration of particles to the system on each step. In Lemma 2.6 we prove that the random Bulgarian solitaire is in some sense dominated by this new deterministic process;

- it is then possible to see that the process with immigration of particles does not differ much (when the time interval in question is not too long) from the deterministic Bulgarian solitaire, because the total number of added particles is relatively small, and they cannot be very concentrated (Lemma 2.7 takes care of the latter statement). Using this observation and applying inequality (2.13) from the previous section, we obtain that the (suitably defined) energy will decrease with large probability after a sufficiently large number of steps (this is Lemma 2.8);

- the rest of the proof is a straightforward (although somewhat lengthy) application of (2.16) and Lemma 2.8.

So, the first step is to prove that a typical configuration $S \in \mathcal{X}_N$ should be “reasonable”, i.e., it should belong to $G_{\gamma_1', \gamma_2'}^N$ for some $\gamma_1', \gamma_2'$.

**Lemma 2.5**

(i) For any $p \in (0, 1)$ there exist positive constants $\sigma_0, \gamma_1', \gamma_2'$ (depending on $p$) such that

$$\pi_{p,N}(G_{\gamma_1', \gamma_2'}^N) \geq 1 - e^{-\sigma_0 N^{1/2}}. \quad (2.17)$$

for all $N$.

(ii) Also, suppose that $S \in G_{\gamma_1', \gamma_2'}^N$, where $\gamma_1', \gamma_2'$ are the quantities from item (i). Then there exist $\gamma_1'', \gamma_2''$ and $\sigma_1$ such that for any $M > 1$

$$\mathbf{P}[Q_{p}^{(n)} S \in G_{\gamma_1'', \gamma_2''}^N] \geq 1 - N^M e^{-\sigma_1 N^{1/2}}.$$

**Proof.** We begin by proving (i). Consider the sets

$$A_N = \{S : |S| = N, \ell(S) > 3N^{1/2}/p\}$$
\[ A'_N = \{ S : |S| = N, \ell(S) > 5N^{1/2}/p \}. \]

Note that if \( S = (k_1, \ldots, k_{\ell(S)}) \in A_N \), then \(|\{ i : k_i > pN^{1/2} \}| < N^{1/2}/p\), i.e., in \( S \) there are at most \( N^{1/2}/p \) piles with at least \( pN^{1/2} \) cards. Clearly, if a pile had no more than \( pN^{1/2} \) cards, then there is \( C_1 > 0 \) such that by the time \( 3N^{1/2}/2 \) that pile will be empty with probability at least \( 1 - e^{-C_1N^{1/2}} \). During the time \( 3N^{1/2}/2 \) only \( 3N^{1/2}/2 \) new piles can appear, so, since \( 1/p + 3/2 < 3/p \), for any \( S \in A_N \) we have that,

\[
P[Q_p^{(3N^{1/2})} S \in \mathcal{X}_N \setminus A_N] \geq 1 - Ne^{-C_1N^{1/2}}. \tag{2.18}
\]

Since \( A'_N \subset A_N \), \( \text{(2.18)} \) also shows that for any \( S \in A_N \setminus A'_N \) we have \( \mathbb{P}[Q_p^{(3N^{1/2})} S \in A'_N] \leq Ne^{-C_1N^{1/2}} \). On the other hand, if \( S \in \mathcal{X}_N \setminus A_N \), then clearly \( \mathbb{P}[Q_p^{(3N^{1/2})} S \in A'_N] = 0 \), so (since \( 3/2 < 2/p \)) for any \( S \in \mathcal{X}_N \setminus A'_N \) we have

\[
\mathbb{P}[Q_p^{(3N^{1/2})} S \in A'_N] \leq Ne^{-C_1N^{1/2}}. \tag{2.19}
\]

Now we use \( \text{(2.16)} \) with \( A = A'_N \) and \( n = 3N^{1/2}/2 \) to obtain from \( \text{(2.18)} \) and \( \text{(2.19)} \) that for some \( C_2 > 0 \)

\[
\pi_{p,N}(\mathcal{X}_N \setminus A'_N) = \pi_{p,N}(S : \ell(S) \leq 5N^{1/2}/p) \geq 1 - e^{-C_2N^{1/2}}. \tag{2.20}
\]

For \( k = 1, \ldots, \lfloor N/2 \rfloor \) define

\[
B_N^{(k)} = \{ S : |S| = N, (k-1)pN^{1/2} < R(S) \leq kpN^{1/2} \} \cap (\mathcal{X}_N \setminus A'_N).
\]

Suppose that \( S \in B_N^{(k)} \) for some \( k > 1 + \frac{5}{p^2} + \frac{3}{2p} \), and let us try to figure out what the configuration \( Q_p^{(3N^{1/2})} S \) should look like. Note that

- \( \ell(S) \leq \frac{5}{p}N^{1/2} \), and moreover \( \ell(Q_p^{(m)} S) \leq (\frac{5}{p} + \frac{3}{2})N^{1/2} \) for all \( m \leq \frac{3}{2}N^{1/2} \), so, since \( (k-1)p > \frac{5}{p} + \frac{3}{2} \), no new pile of size greater than \((k-1)pN^{1/2}\) can appear until the moment \( \frac{3}{2}N^{1/2} \);

- the evolution of a single pile can be modeled by a random walk on \( \mathbb{Z}_+ \) which jumps one unit to the left with probability \( p \) and holds its position with probability \( 1 - p \). This shows that if the size of the pile was less than \( kpN^{1/2} \), then after \( \frac{3}{2}N^{1/2} \) moves it will be less than \((k-1)pN^{1/2}\) with probability at least \( 1 - e^{-C_3N^{1/2}} \) for some \( C_3 > 0 \).
From the above facts we deduce that for any $S \in B_N^{(k)}$
\[ \mathbf{P}[Q_p^{(3N^{1/2})}S \notin B_N^{(k)}] \leq Ne^{-C_N^{1/2}} , \]
as long as $k > 1 + \frac{5}{p^2} + \frac{3}{2p}$. Now using (2.16) with $A = B_N^{(k)}$ and $n = 3N^{1/2}/2$, we obtain that for some $C_4 > 0$
\[ \pi_{p,N}(B_N^{(k)}) \leq e^{-C_4N^{1/2}} + \sum_{m \geq k} \pi_{p,N}(B_m^{(m)}) , \]
so by induction one can show that $\pi_{p,N}(B_N^{(k)}) \leq C_N^{5N^2}e^{-C_N^{4N^{1/2}}}$. Summing over $k > 1 + \frac{5}{p^2} + \frac{3}{2p}$ and recalling (2.20), we conclude the proof of the part (i) of Lemma 2.5 (with $\gamma_1 = \frac{5}{p}$, $\gamma_2 = \frac{5}{p} + \frac{3}{2}$).

To prove the part (ii), first observe that in the proof of (i) we have constructed $\gamma_1', \gamma_2'$ in such a way that for any $S \in G_N^{\gamma_1', \gamma_2'}$
\[ \mathbf{P}[Q_p^{(3N^{1/2})}S \in \mathcal{X}_N \setminus G_N^{\gamma_1', \gamma_2'}] \leq e^{-C_NN^{1/2}} . \]
To complete the proof of (ii), it is enough to take $\gamma_1'' = \gamma_1' + \frac{3}{2}$, $\gamma_2'' = \max\{\gamma_1', \gamma_2'\} + \frac{3}{2}$ (note that for any $S$ we have $\ell(Q_pS) - \ell(S) \leq 1$, $R(Q_pS) \leq \max\{R(S), \ell(S)\}$).

We continue proving Theorem 1.2. Now, the main idea is the following: first, to dominate the random Bulgarian solitaire by a certain deterministic process (that we will call Bulgarian solitaire with immigration of particles), and then apply to that process some methods from the proof of Theorem 1.1.

Fix $\delta_0 \in (0, \frac{1}{4} - 36a)$, and abbreviate $\kappa_N = \lfloor N^{\delta_0 + \frac{1}{4}} \rfloor$. Denote also $N_p := \lfloor N/p \rfloor$. For any $S$, let us define configurations $\mathfrak{D}(S)$, $\mathfrak{D}(S)$ in the following way: if $S = (n_1, \ldots, n_{\ell(S)})$, let
\[ \mathfrak{D}(S) = \left( \left\lfloor \frac{n_1}{p} \right\rfloor - z_1, \ldots, \left\lfloor \frac{n_{\ell(S)}}{p} \right\rfloor - z_{\ell(S)} \right) , \]
\[ \mathfrak{D}(S) = \left( \left\lfloor \frac{n_1}{p} \right\rfloor - z_1 + \kappa_N, \ldots, \left\lfloor \frac{n_{\ell(S)}}{p} \right\rfloor - z_{\ell(S)} + \kappa_N \right) , \]
where $z_i = z_i(S) \in \{0, 1\}$ are chosen in such a way that $z_1 \geq z_2 \geq \ldots \geq z_{\ell(S)}$ and for any $S \in \mathcal{X}_N$ we have $|\mathfrak{D}(S)| = N_p$. Define the operator $\tilde{Q}$ by
\[ \tilde{Q}S = \text{ord}(n_1 - 1, \ldots, n_{\ell(S)} - 1, \ell(S) + \kappa_N) , \]
i.e., making the $\tilde{Q}$-move consists of making a move of deterministic Bulgarian solitaire, and then adding $\kappa_N$ cards to the new pile (so that $|\tilde{Q}S| - |S| = \kappa_N$). For the simplicity of notations, we do not indicate in $\tilde{Q}$ the dependence on $N$ and $\delta_0$; note also that in the above display we do not assume that $|S| = N$, so $\tilde{Q}$ need not apply to only $S \in \mathcal{X}_N$.

For two configurations $S_1 = (n_1, \ldots, n_{\ell(S_1)})$, $S_2 = (m_1, \ldots, m_{\ell(S_2)})$ we say that $S_1 \leq S_2$ if $\ell(S_1) \leq \ell(S_2)$ and $n_j \leq m_j$ for all $j = 1, \ldots, \ell(S_1)$.

Lemma 2.6 Suppose that $|S| = N$ and $S \in G_{\gamma_1', \gamma_2'}^N$ (where $\gamma_1', \gamma_2'$ are the quantities from Lemma 2.5). Then for any $M > 0$ there exists $\sigma_2$ such that

$$P[\tilde{Q}(n)D(S) \geq D(Q_p(n)S) \text{ for all } n \leq N^M] \geq 1 - N^M e^{-\sigma_2 N^\delta_0}. \quad (2.21)$$

Proof. Let us refer to the $\ell(S)$ piles of $S$ and $\tilde{D}(S)$ as $P_1, \ldots, P_{\ell(S)}$ and $\tilde{P}_1, \ldots, \tilde{P}_{\ell(S)}$ respectively. Then, the piles born at the moment $i$ are referred to as $P_{\ell(S)+i}$ and $\tilde{P}_{\ell(S)+i}$. Using the notation $(x)^+ := \max\{x, 0\}$, for $n \geq (i - \ell(S))^+$, let $P_i(n)$ and $\tilde{P}_i(n)$ stand for the sizes of the piles $P_i$ and $\tilde{P}_i$ at the moment $n$, respectively (if a pile is emptied at some moment $n^* < n$, then we mean that the size remains 0 for all $m \geq n^*$).

Clearly, the event

$$\{\tilde{Q}(n)\tilde{D}(S) \geq D(Q_p(n)S) \text{ for all } n \leq N^M\} \subset \bigcap_{i=1}^{N^M+\ell(S)} \Lambda_i, \quad (2.22)$$

where we define the event $\Lambda_i$ by

$$\Lambda_i = \{P_i(n) \leq n\tilde{P}_i(n) \text{ for all } n \geq (i - \ell(S))^+\}.$$

Define also the event $D = \{Q_p(n)S \in G_{\gamma_1', \gamma_2'}^N\}$; by Lemma 2.5 (ii) we know that $P[D] \geq 1 - N^M e^{-\sigma_1 N^{1/2}}$. On the other hand,

$$P[\Lambda_i \mid \Lambda_1, \ldots, \Lambda_{i-1}, D] \geq P[H_i]P[\Lambda_i \mid H_i, D], \quad (2.23)$$

where $H_i = \{n\tilde{P}_i((i - \ell(S))^+) \geq P_i((i - \ell(S))^+) + \kappa_N/2\}$. Now, on $D$ we have that $\ell(Q_p(n)S) = O(N^{1/2})$ for all $n \leq N^M$, and on $\Lambda_1 \cap \ldots \cap \Lambda_{i-1}$ it holds that $\ell(\tilde{Q}(n)\tilde{D}(S)) \geq \ell(Q_p(n)S)$. Using the Large Deviation bound for the Binomial distribution, we see that the first term in the right-hand side of (2.23) is at least $1 - e^{-C_1 N^{\delta_0}}$. As for the second term, note that the
difference between $p\tilde{P}_t(\cdot)$ and $P_t(\cdot)$ is a random walk with drift 0. Since the time that the pile $P_t$ needs to be emptied is $O(N^{1/2})$, the second term in (2.23) is in fact the probability that such a random walk does not deviate from its initial position by more than $\kappa_N/2$ by time $O(N^{1/2})$; clearly, that probability is bounded from below by $1 - e^{-C_2N^{\delta_0}}$. Then, it is immediate to deduce Lemma 2.6 from (2.22) and (2.23).

Recall that (cf. the proof of Lemma 2.7) for any configuration $S$ we use the notation

$$\mathcal{H}(S) = \max\{i: \text{ there exists } j \text{ such that } R_{i,j}(S) = 1\}.$$

**Lemma 2.7** Suppose that $S \in G^N_{\delta_1,\gamma_2}$ and let $\tilde{\beta}$ be such that $\tilde{\beta} \leq \frac{1}{4} - \delta_0$. Then there exists $L_0$ such that $\mathcal{H}(\mathcal{Q}^{(n)} \mathcal{D}(S)) \leq L_0 N^{1/2}$ for all $n \leq N^{1/2 + \tilde{\beta}}$.

**Proof.** Let $b_0 = \mathcal{H}(\mathcal{D}(S))$ and denote $\tilde{b}_0 = b_0(b_0 + 1)/2$. Define the triangular configuration $\tilde{T}_0$ by $R_{i,j}(\tilde{T}_0) = 1\{i \leq \tilde{b}_0\}$; then, clearly, $\mathcal{D}(S) \leq \tilde{T}_0$. Denote $b_1 = b_0 + [\kappa_N/\sqrt{2}]$, $\tilde{b}_1 = b_1(b_1 + 1)/2$ and define the configuration $\tilde{T}_1$ by $R_{i,j}(\tilde{T}_1) = 1\{i \leq \tilde{b}_1\}$. By examining the $\tilde{Q}$-evolution of $\tilde{T}_0$ on the Etienne diagram, it is clear that $\tilde{Q}^{(n)} \tilde{T}_0 \leq \tilde{T}_1$ for all $n \leq b_0$. We then repeat this construction by defining $b_{m+1} = b_m + [\kappa_N/\sqrt{2}]$, $\tilde{b}_{m+1} = b_{m+1}(b_{m+1} + 1)/2$ and the configuration $\tilde{T}_{m+1}$ by $R_{i,j}(\tilde{T}_{m+1}) = 1\{i \leq \tilde{b}_{m+1}\}$. Analogously, we obtain that $\tilde{Q}^{(n)} \tilde{T}_m \leq \tilde{T}_{m+1}$ for all $n \leq b_n$. A simple monotonicity argument then shows that $\tilde{Q}^{(n)} \mathcal{D}(S) \leq \tilde{T}_{m+1}$ for all $n \leq b_0 + \cdots + b_m$. We have $b_0 + \cdots + b_m \geq (m+1)b_0$ and $b_0 \leq C_1 N^{1/2}$ for some $C_1$, so $\tilde{Q}^{(n)} \mathcal{D}(S) \leq \tilde{T}_{C_1^{-1}N^{\tilde{\beta}}}$ for all $n \leq N^{1/2 + \tilde{\beta}}$. So, since $\frac{1}{4} + \delta_0 + \tilde{\beta} \leq \frac{1}{2}$, for some $L_0$ we have

$$\mathcal{H}(\tilde{Q}^{(n)} \mathcal{D}(S)) \leq C_1 N^{1/2} + \left[\frac{\kappa_N}{\sqrt{2}}\right] C_1^{-1} N^{\tilde{\beta}} \leq L_0 N^{1/2}$$

for all $N$, thus concluding the proof of Lemma 2.7.

**Lemma 2.8** Fix some $\bar{a} \in (0, 1/16)$ and suppose that a configuration $S \in G^N_{\delta_1,\gamma_2}$ is such that $\tilde{E}(\mathcal{D}(S)) \geq \lambda_3 N^{-\bar{a}}$ for some $\lambda_3$. Then, with $\lambda_1, \lambda_2$ from (2.13), we have for some $\sigma_3, \delta_1 > 0$

$$P[\tilde{E}(\mathcal{D}(Q_p^{(n_S)} S)) - \tilde{E}(\mathcal{D}(S)) \leq -\lambda_1 \tilde{E}^{10}(\mathcal{D}(S))/2] \geq 1 - e^{-\sigma_3 N^{\delta_1}},$$

(2.24)

where $n_S = \lambda_2 \tilde{E}^{-2}(\mathcal{D}(S)) N^{1/2}/p$. Moreover, (2.24) remains true when $Q_p^{(n_S)}$ is substituted by $Q_p^{(n)}$, for any $n \in [n_S, 2n_S]$.
Proof. First, each particle added to \( \mathfrak{D}(S) \) changes the energy \( E \) by at most \( O(N^{1/2}) \), so we have for some constants \( C_1, C_2 \)

\[
|E(\mathfrak{D}(S)) - E(\mathfrak{D}(S))| \leq \ell(\mathfrak{D}(S)) \times C_1 N^{1/2} \kappa_N \leq C_2 N^{5/4 + \delta_0}. \tag{2.25}
\]

Using the same sort of argument and the fact that \( \tilde{Q}^{(m)} S' \geq Q_1^{(m)} S'' \) for any \( m, S' \geq S'' \), with the help of Lemma 2.2 and (2.25) we obtain

\[
|E(Q_1^{(n_S)} \mathfrak{D}(S)) - E(\mathfrak{D}(S))| \leq C_2 N^{5/4 + \delta_0} + C_3 n'_S N^{3/4 + \delta_0}. \tag{2.26}
\]

Introduce the event \( D_1 = \{ \tilde{Q}^{(n_S)} \mathfrak{D}(S) \geq \mathfrak{D}(Q_p^{(n_S)} S) \} \). By Lemma 2.6, we have

\[
P[D_1] \geq 1 - n'_S e^{-\sigma_2 N^{\delta_0}}, \tag{2.27}
\]

and, since

\[
|\tilde{Q}^{(n_S)} \mathfrak{D}(S)| - |\mathfrak{D}(Q_p^{(n_S)} S)| \leq (O(N^{1/2}) + n'_S) \kappa_N,
\]

analogously to \( (2.25) - (2.26) \) we obtain that on \( D_1 \)

\[
|E(Q_1^{(n_S)} \mathfrak{D}(S)) - E(\mathfrak{D}(Q_p^{(n_S)} S))| \leq C_4 n'_S N^{\frac{3}{4} + \delta_0}. \tag{2.28}
\]

Now, we have that \( \bar{E}_{10}(\mathfrak{D}(S)) \geq \lambda_3^{10} N^{-10\alpha} \), and \( n'_S N^{\frac{3}{4} + \delta_0} \leq C_5 N^{\frac{3}{4} + 2\alpha + \delta_0} \). Since \( \frac{3}{4} - 10\alpha > \frac{3}{2} + 2\alpha + \delta_0 \), we obtain the proof of \( (2.24) \) from \( (2.13), (2.26), (2.27), \) and \( (2.28) \).

As for the second claim of Lemma 2.8, we note that for \( n \geq n'_S \), by Lemma 2.2 (ii) it holds that \( \tilde{E}(Q_1^{(n)} \mathfrak{D}(S)) \leq \bar{E}(Q_1^{(n_S)} \mathfrak{D}(S)) \), and then use the same kind of estimates as used above. \( \Box \)

Now we are ready to finish the proof of Theorem 1.2. By Lemma 2.5 (i) there are \( \sigma_0, \gamma_1', \gamma_2' \) such that \( (2.17) \) holds. Note that there exists \( \Gamma' = \Gamma'(\gamma_1', \gamma_2') \) such that if \( S \in G_N^{\gamma_1', \gamma_2'} \), then \( \tilde{E}(\mathfrak{D}(S)) \leq \Gamma' \). Define \( \psi(x) = x - \frac{1}{2} \lambda_1 x^{-10} \). Let \( y_0 = \Gamma' \) and \( y_{i+1} = \psi(y_i) \) for \( i \geq 0 \). Take \( \varepsilon = N^{-a}, \ a < 1/144, \) and define \( \hat{\varepsilon} = \min\{\alpha_1, \alpha_3\} \varepsilon^3 \) (cf. \( (2.6) \) and \( (2.4) \)). Let \( \hat{n} = \min\{n : y_n < \hat{\varepsilon}\} \); since \( \hat{\varepsilon} = O(N^{1/48}) \), by examining the iteration scheme \( x \mapsto \psi(x) \) we obtain that there exists \( C_1 \) such that \( \hat{n} \leq C_1 N^{5/24} \). Let

\[
L_n = \{ S \in G_N^{\sigma_0, \gamma_1', \gamma_2'} : \tilde{E}(\mathfrak{D}(S)) \in (y_{n+1}, y_n) \},
\]

22
and define also \( \mathcal{L}_{>n} = \bigcup_{k>n} \mathcal{L}_k \), \( \mathcal{L}_{<n} = \bigcup_{k<n} \mathcal{L}_k \). Take any \( n \leq \hat{n} \) and denote 
\[
m_n = \lambda_2 y_n^{-2} N^{1/2}/p.
\]
By (2.16) and Lemma 2.5 (i) we can write
\[
\sum_{S_1 \in \mathcal{L}_n} \sum_{S_2 \in \mathcal{L}_{<n}} \pi_{p,N}(S_2) P_{S_2 S_1}^{(m_n)} \geq \pi_{p,N}(\mathcal{L}_n)(1 - e^{-\sigma_0 N^{1/2}}),
\]
(2.31)

Again using Lemma 2.8 we write
\[
T_1 \leq \sum_{S_2 \in \mathcal{L}_{>n}} \pi_{p,N}(S_2) e^{-\sigma_3 N^{3/4}} \leq e^{-\sigma_3 N^{3/4}}.
\]
(2.32)

Using now (2.31) and (2.32) together with the trivial bound \( T_2 \leq \pi_{p,N}(\mathcal{L}_{<n}) \), we obtain from (2.29)–(2.30) that for some \( C_2 > 0 \)
\[
\pi_{p,N}(\mathcal{L}_n) \leq C_2 \left( e^{-\sigma_0 N^{1/2}} + e^{-\sigma_3 N^{3/4}} + \sum_{k<n} \pi_{p,N}(\mathcal{L}_k) \right).
\]
(2.33)

By induction, we then obtain that there is \( C_3 > 0 \) such that for any \( n < \hat{n} \)
\[
\pi_{p,N}(\mathcal{L}_n) \leq C_3 n^2 e^{-\sigma_3 N^{3/4}},
\]
(2.34)
so, since \( \hat{n} \leq C_1 N^{5/24} \), taking summation in (2.34) we obtain for some \( C_4 > 0 \) that
\[
\pi_{p,N}(S \in \mathcal{X}_N : \hat{E}(\mathcal{D}(S)) \geq \hat{\epsilon}) \leq C_4 N^{15/24} e^{-\sigma_3 N^{4/3}}. \tag{2.35}
\]

Now, the last step of the proof of Theorem 1.2 is analogous to what was done in Lemma 2.4. Note that if \( \hat{E}(\mathcal{D}(S)) < \hat{\epsilon} \), then
\[
\max\{\tilde{h}_-(\mathcal{D}(S)), \tilde{h}_+(\mathcal{D}(S))\} \leq \left( \frac{\hat{\epsilon}}{\min\{\alpha_1, \alpha_3\}} \right)^{1/3} = \epsilon,
\]
so if \( \hat{E}(\mathcal{D}(S)) < \hat{\epsilon} \), then \( \mathcal{D}(S) \in \mathcal{V}(\epsilon, |\mathcal{D}(S)|) \), thus showing that
\[
\pi_{p,N}(S \in \mathcal{X}_N : \mathcal{D}(S) \in \mathcal{V}(\epsilon, N_p)) \geq 1 - C_4 N^{15/24} e^{-\sigma_3 N^{4/3}}. \tag{2.36}
\]

Define
\[
H_0 = \{ S : \mathcal{D}(S) \in \mathcal{V}(\epsilon, |\mathcal{D}(S)|), \max\{i : \text{there exists } j \leq \epsilon |\mathcal{D}(S)|^{1/2} \text{ such that } \mathcal{R}_{i,j}(\mathcal{D}(S)) = 1 \} \geq \theta |\mathcal{D}(S)| + 2 \epsilon |\mathcal{D}(S)|^{1/2} \}.
\]

Take any \( S \in G^N_{\gamma_1', \gamma_2'} \), and denote \( W = \{ S \in \mathcal{X}_N : \mathcal{D}(S) \in \mathcal{V}(\epsilon, N_p) \} \) (recall that \( |\mathcal{D}(S)| = N_p \)). Using (2.16), we write
\[
\sum_{S_1 \in H_0} \sum_{S_2 \in H_0} \pi_{p,N}(S_1) P_{S_1 S_2}^{(2\epsilon N_p^{1/2})} = \sum_{S_1 \in H_0} \sum_{S_2 \in H_0} \pi_{p,N}(S_2) P_{S_2 S_1}^{(2\epsilon N_p^{1/2})} = T_1' + T_2', \tag{2.37}
\]
where
\[
T_1' = \sum_{S_1 \in H_0} \pi_{p,N}(S_2) P_{S_2 S_1}^{(2\epsilon N_p^{1/2})},
\]
\[
T_2' = \sum_{S_1 \in H_0} \pi_{p,N}(S_2) P_{S_2 S_1}^{(2\epsilon N_p^{1/2})}.
\]

Observe that if \( \mathcal{D}(S) \in \mathcal{V}(\epsilon, N_p) \) and \( \epsilon \) is small enough, then after \( 2\epsilon N_p^{1/2} \) moves there will be no particles in the set \( \{(i, j) : i \geq \theta N_p + 2 \epsilon N_p^{1/2}, j \leq \}

24
$\varepsilon N_p^{1/2}$, with probability at least $1 - e^{-C_5 N_p^{1/2}}$ for some $C_5$. So, for the left-hand side of (2.37) we can write
\[ \sum_{S_1 \in H_0, S_2 \in H_0} \pi_{p,N}(S_1) P^{(2\varepsilon N_p^{1/2})}_{S_1 S_2} \geq \pi_{p,N}(H_0)(1 - e^{-C_5 N_p^{1/2}}). \quad (2.39) \]

On the other hand, the same argument implies that $T'_1 \leq e^{-C_5 N_p^{1/2}}$ and the bound $T'_2 \leq \pi_{p,N_p}(W^c)$ is trivial. So, using (2.35) and (2.39), we obtain from (2.37) that
\[ \pi_{p,N}(H_0) \leq C_6 N_p^{15/24} e^{-\sigma_3 N_p^{1/2}}. \quad (2.40) \]

Abbreviate $\hat{H} = H_0^c \cap W$ and define
\[
F_k = \{ S : D(S) \in \hat{H}, \max\{i : \text{there exists } j \geq \theta_{N_p} \text{ such that } R_{i,j}(D(S)) = 1\} - \theta_{N_p} - \varepsilon N_p^{1/2} \in (2\varepsilon k N_p^{1/2}, 2\varepsilon (k+1) N_p^{1/2})\},
\]

$F_{<k} = \bigcup_{m<k} F_k$, $F_{>k} = \bigcup_{m>k} F_k$. Analogously to (2.29)–(2.30) and (2.37)–(2.38), we write
\[
\sum_{S_1 \in F_k, S_2 \in F_k} \pi_{p,N}(S_1) P^{(4\varepsilon N_p^{1/2})}_{S_1 S_2} = \sum_{S_1 \in F_k, S_2 \in F_k} \pi_{p,N}(S_2) P^{(4\varepsilon N_p^{1/2})}_{S_2 S_1} = T_1'' + T_2'' + T_3'' \quad (2.41)
\]
\[
T_1'' = \sum_{S_1 \in F_k, S_2 \in F_{>k}} \pi_{p,N}(S_2) P^{(4\varepsilon N_p^{1/2})}_{S_2 S_1},
\]
\[
T_2'' = \sum_{S_1 \in F_k, S_2 \in F_{<k}} \pi_{p,N}(S_2) P^{(4\varepsilon N_p^{1/2})}_{S_2 S_1},
\]
\[
T_3'' = \sum_{S_1 \in F_k, S_2 \in \hat{H}^c} \pi_{p,N}(S_2) P^{(4\varepsilon N_p^{1/2})}_{S_2 S_1}.
\]

The following fact can be deduced from (2.2): if $D(S) \in \hat{H}$ and $R_{i,j}(D(S)) = 1$ for some $i > \theta_{N_p} + \frac{5}{2} \varepsilon N_p^{1/2}$, then $i - j \leq \varepsilon N_p^{1/2}$. Then, by examining
the evolution of $\mathcal{D}(S)$ on the Etienne diagram and using Lemma 2.6, it is elementary to obtain that for any $S \in F_k$, $k \geq 1$

\[
P[\mathcal{D}(Q_p^{4\varepsilon N^{1/2}})S \notin F_k \cup F_{\geq k}] \geq 1 - e^{-C_7 N^{1/2}}.
\]

Using that fact, one can bound the left-hand side of (2.41) from below by $\pi_{p,N}(F_k)(1 - e^{-C_7 N^{1/2}})$ and the term $T''_3$ can be bounded from above by $e^{-C_7 N^{1/2}}$. Then, it is straightforward to write $T'''_3 \leq m > k \pi_{p,N}(H^c)$. Denoting now $\tilde{m} = \frac{\gamma_1 N^{1/2}}{2 \varepsilon N^{1/2}} = \frac{\gamma_1}{2 \varepsilon}$, analogously to (2.33)–(2.34) we obtain

\[
\pi_{p,N}(F_k) \leq C_8 (\tilde{m} - k)^2 N^{15/24} e^{-\sigma_3 N^\delta}. \tag{2.43}
\]

Summing over $k = 1, \ldots, \tilde{m}$ and taking (2.36) and (2.40) into account, we finally obtain that for some $C_9, \delta > 0$ (depending on $a$)

\[
\pi_{p,N}(\hat{V}(3\varepsilon, N_p)) \geq 1 - e^{-C_9 N^\delta}.
\]

Since $\varepsilon = N^{-a}$ and $a < 1/144$ is arbitrary, we complete the proof of Theorem 1.2 (note that $\mathcal{D}^{-1}(\hat{V}(\varepsilon, N_p)) \subset T(2\varepsilon, p, N)$ for $\varepsilon \gg N^{-1/2}$).

3 Final remarks

A natural question that one may ask is: starting from an initial configuration $S$ with $\ell(S) = O(N^{1/2}), R(S) = O(N^{1/2})$, how many steps (of the deterministic game) are necessary to reach $T(\varepsilon(N), 1, N)$ where $\varepsilon(N) \to 0$ as $N \to \infty$. From the proof of Theorem 1.1 it can be deduced that if $\varepsilon(N) \sim N^{-\alpha}$, $0 < \alpha < 1/2$, then $O(N^{1/2 + 36\alpha})$ moves suffice. However, this result is only nontrivial when $\alpha < 1/72$ (since $O(N)$ moves are always enough to reach the “exact” triangle), and even then it is almost certainly far from being precise.

Also, loosely speaking, Theorem 1.2 shows that the typical deviation from the triangle is of order at most $O(N^{1/2})$. Again, we do not believe that that result is the best possible one. In fact, the author has strong reasons to conjecture that the typical deviation should be of order $N^{1/2}$; however, the proof of that is still beyond our reach.
Acknowledgements

The author is thankful to Pablo Ferrari for many useful discussions about the random Bulgarian solitaire, and to Ira Gessel, who posed the problem of finding the limiting shape for that model during the Open Problems session at the conference *Discrete Random Walks 2003* (IHP, Paris). Also, the author thanks the anonymous referees for careful reading of the manuscript and useful comments and suggestions.

References

[1] E. Akin, M. Davis (1985) Bulgarian solitaire. *Amer. Math. Monthly* 4, 237–250.

[2] O.S.M. Alves, F.P. Machado, S.Yu. Popov (2002) The shape theorem for the frog model. *Ann. Appl. Probab.* 12 (2), 533–546.

[3] H.-J. Bentz (1987) Proof of the Bulgarian solitaire conjectures. *Ars Combin.* 23, 151–170.

[4] J. Brandt (1982) Cycles of partitions. *Proc. Amer. Math. Soc.* 85, 483–486.

[5] C. Cannings, J. Haigh (1992) Montreal solitaire. *J. Combin. Theory Ser. A* 60 (1), 50–66.

[6] H. Cohn, N. Elkies, J. Propp (1996) Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.* 85 (1), 117–166.

[7] H. Cohn, M. Larsen, J. Propp (1998) The shape of a typical boxed plane partition. *New York J. Math.* 4, 137–165.

[8] R. Durrett, T.M. Liggett (1981) The shape of the limit set in Richardson’s growth model. *Ann. Probab.* 9 (2), 186–193.

[9] M. Gardner (1983) Mathematical games. *Scientific American* 249 (2), 8–13.

[10] J.R. Griggs, C.-C. Ho (1998) The cycling of partitions and compositions under repeated shifts. *Adv. Appl. Math.* 21 (2), 205–227.
[11] G. Etienne (1991) Tableaux de Young et solitaire bulgare. J. Combin. Theory Ser. A 58 (2), 181–197.

[12] K. Igusa (1985) Solution of the Bulgarian solitaire conjecture. Math. Mag. 58 (5), 259–271.

[13] D. Richardson (1973) Random growth in a tessellation. Proc. Cambridge Philos. Soc. 74, 515–528.

[14] N.B. Vassil’ev, V.L. Gutenmaher, J.M. Rabbot, A.L. Toom (1981) Mathematical Olympiads by Mail. (In Russian). Nauka, Moscow.

[15] Y.N. Yeh (1995) A remarkable endofunction involving compositions. Stud. Appl. Math. 95 (4), 419–432.

[16] Y. Zhang (1993) A shape theorem for epidemics and forest fires with finite range interactions. Ann. Probab. 21 (4), 1755–1781.