Fermion masses and mixing in $\Delta(27)$ flavour model

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An extension of the Standard Model (SM) based on the non-Abelian discrete group $\Delta(27)$ is considered. The $\Delta(27)$ flavour symmetry is spontaneously broken only by gauge singlet scalar fields, therefore our model is free from any flavour changing neural current. We show that the model accounts simultaneously for the observed quark and lepton masses and their mixing. In the quark sector, we find that the up quark mass matrix is flavour diagonal and the Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix arises from down quarks. In the lepton sector, we show that the charged lepton mass matrix is almost diagonal. We also adopt type-I seesaw mechanism to generate neutrino masses. A deviated mixing matrix from tri-bimaximal Maki-Nakagawa-Sakata (MNS), with $\sin^2\theta_{13} \sim 0.13$ and $\sin^2\theta_{23} \sim 0.41$, is naturally produced.

I. INTRODUCTION

The understanding of the origin of quark and lepton families and the observed pattern of their masses and mixing is still one of the major outstanding problems in particle physics. In the SM, these masses and mixing are derived from Yukawa couplings, which are not defined by the gauge symmetry. Therefore, they are arbitrary parameters and another type of symmetry, called flavour symmetry, is required to explain the observed fermion flavour structures. In particular, one aims to interpret the large mass ratios between generations: $m_u \ll m_c \ll m_t$; $m_d \ll m_s \ll m_b$; $m_e \ll m_{\mu} \ll m_{\tau}$, and the smallness of the off-diagonal elements of the quark weak coupling matrix, in addition to the tiny neutrino masses and their large mixings as recent data suggest [1].

Two standard approaches for dealing with flavour symmetries in particle physics. The first one is known as "top-down" approach, where one assumes that the SM Lagrangian is invariant under certain flavour group $G$ and a number of Higgs-like scalar bosons, called flavons, are coupled invariantly to SM fermions. The Vacuum Expectation Values (VEVs) of these flavons break the flavour symmetries and generate mass terms for SM fermions. The comparison of the resultant mixing matrices and the mass eigenvalues with the experimental data will confirm or refute if this group represents the correct flavour symmetry. In the second approach, which is known as "bottom-up", one studies the residual symmetry that manifests in the mass matrix and tries to relate it with the flavour symmetry group. For instance, by calculating the matrices $S_i$ that keep the neutrino mass matrix invariant and the matrices $T_i$ that keep the charged leptons mass matrix invariant [2]. The group $G$ generated by these matrices can be considered as the group of the flavour symmetry of lepton sector. In this regard, it was argued that for Majorana neutrinos, regardless the form of the mass matrix $M_\nu$, it has $\mathbb{Z}_2 \times \mathbb{Z}_2$ residual symmetry [3, 4], provided that
it has three distinct eigenvalues [5].

Many attempts were done for the interpretation of the flavour aspects by using discrete symmetry groups (see [6]). In particular, the non-abelian groups $A_4$ and $S_4$ have been significantly considered and shown to be useful for obtaining tri-bimaximal neutrino mixing matrix [7, 8]. Also $\Delta(27)$ has been considered in Ref.[9–11] as an example of discrete symmetries that may deviate MNS mixing matrix from tri-bimaximal. However in [9, 10] the attention has been devoted for lepton sector only. Also extra Higgs (flavons) doublets have been considered, which could make the model suffers from danger flavour changing neutral currents. It is worth noting that within flavour symmetry approaches, the Yukawa couplings are typically generated through non-renormalizable flavon interactions with the SM fermions, i.e., $Y \sim \langle \phi \rangle^n / \Lambda^n$, $n = 1, 2, \ldots$. In this respect, the hierarchy of fermion masses is related to the order of non-renormalizable interactions. For instance, third generation Yukawa coupling can be obtained from $\langle \phi \rangle / \Lambda$ while first and second generation Yukawa couplings should correspond to higher order terms.

In this paper we explore the possibility that the flavour symmetry based on the group $\Delta(27)$ leads to the correct quarks, charged leptons and neutrino masses, in addition to the quark and neutrino mixing matrices consistently with the latest experimental results. We present a new model based on the semi-direct product $\Delta(27) \rtimes S_2$, where $S_2$ is quite useful symmetry that grants the tri-bimaximal mixing as zero approximation in our model. We will show that deviation from tri-bimaximal neutrino mixing matrix is related to spontaneous breaking of this symmetry. The Higgs sector in our model consists of one Higgs doublet only to break the electroweak symmetry and SM singlet scalars to break the flavour symmetry. Therefore, our model is free from the famous Flavour Changing Neural Current (FCNC) constraints that most of constructed models suffer from, due to the existence of more than one $SU(2)$ doublet Higgs.

We will show that the observed hierarchical structure of quark and lepton masses can be accommodated. In addition, the small quark mixing in the $V_{CKM}$ and large neutrino mixing in $U_{MNS}$ can be simultaneously realised. If one assumes that left-handed quarks and right-handed up quarks transform as triplets under $\Delta(27)$, then one finds that the up quark mass matrix is flavour diagonal. With right-handed downs quarks transform as singlets under $\Delta(27)$, we will show that $V_{CKM}$ mixing matrix can be obtained from the down quark sector. In the lepton sector, the lepton doublet transform under $\Delta(27)$ as triplet, while the right-handed charged lepton as singlets. In this case, the charged lepton mass matrix is almost diagonal. Finally we assume right-handed neutrinos as singlets under $\Delta(27)$, thus with the appropriate singlet scalars (triplet and singlets under $\Delta(27)$) we will show that a generic MNS mixing matrix can be obtained and different interesting limits will be studied.

The paper is organised as follows. In the next section we briefly introduce $\Delta(27)$ flavour symmetry. In section 3 we show that the charged lepton mass hierarchy can be naturally accounted for. Section 4 is devoted for neutrino masses and mixing, where the observed nearly tri-bimaximal mixing is realized. Quark sector is discussed in section 5. In our model the quark mixing matrix, $V_{CKM}$, is obtained from down quarks. Finally we give our conclusions in section 6.
TABLE I: The singlet multiplications of the group $\Delta(27)$

|  $1_2$ | $1_3$ | $1_4$ | $1_5$ | $1_6$ | $1_7$ | $1_8$ | $1_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $1_2$ | $1_3$ |       |       |       |       |       |       |
| $1_3$ | $1_1$ | $1_2$ |       |       |       |       |       |
| $1_4$ | $1_6$ | $1_5$ | $1_7$ |       |       |       |       |
| $1_5$ | $1_4$ | $1_6$ | $1_9$ | $1_8$ |       |       |       |
| $1_6$ | $1_5$ | $1_4$ | $1_8$ | $1_7$ | $1_9$ |       |       |
| $1_7$ | $1_8$ | $1_9$ | $1_1$ | $1_3$ | $1_2$ | $1_4$ |       |
| $1_8$ | $1_9$ | $1_7$ | $1_2$ | $1_1$ | $1_3$ | $1_6$ | $1_5$ |
| $1_9$ | $1_7$ | $1_8$ | $1_3$ | $1_2$ | $1_1$ | $1_5$ | $1_4$ | $1_6$ |

The discrete group $\Delta(27)$ is a subgroup of $SU(3)$ and an isomorphic to the semi-direct product group $(Z_3 \times Z'_3) \times Z''_3$. It is also one of the groups $\Delta(3n^2)$ with $n = 3$. It has 27 elements and 11 conjugacy classes, so it has 11 irreducible representations, two triplets, 3 and its conjugate 3, and 9 singlets $1_1 - 1_9$. The group multiplication rules for $\Delta(27)$ are

$$
\left(\begin{array}{c}
  x_1 \\
  x_2 \\
  x_3
\end{array}\right)_3 \times 
\left(\begin{array}{c}
  y_1 \\
  y_2 \\
  y_3
\end{array}\right)_3 =
\left(\begin{array}{c}
  x_1y_1 + x_2y_2 + x_3y_3 \\
  x_1y_1 + x_2y_2 + x_3y_3 \\
  x_1y_1 + x_2y_2 + x_3y_3
\end{array}\right)_3
$$

(1)

and $3 \times 3 = \sum_{i=1}^{9} 1_i$, where

$$
1_1 = x_1y_1 + x_2y_2 + x_3y_3, \quad 1_2 = x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3,
$$

$$
1_3 = x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3, \quad 1_4 = x_1y_2 + x_2y_3 + x_3y_1,
$$

$$
1_5 = x_1y_2 + \omega x_2y_3 + \omega^2 x_3y_1, \quad 1_6 = x_1y_2 + \omega^2 x_2y_3 + \omega x_3y_1,
$$

$$
1_7 = x_2y_1 + x_3y_2 + x_1y_3, \quad 1_8 = x_2y_1 + \omega x_3y_2 + \omega^2 x_1y_3,
$$

$$
1_9 = x_2y_1 + \omega x_3y_2 + \omega^2 x_1y_3,
$$

(2)

where $\omega = e^{2\pi i/3}$. The singlets multiplications are given in Table I

Non-vanishing neutrino masses imply the existence of three right handed neutrinos. Therefore, we consider the matter sector of SM besides three right handed neutrinos. We assign the lepton doublet to the triplet $3$ of $\Delta(27)$, while right handed components are ascribed to different singlet representations of $\Delta(27)$. As mentioned, we consider only one SM Higgs scalar ($H$) and the following singlets: $\chi, \xi, \eta, \sigma$ and $\phi$ that break the flavour symmetry.

In order to get the tri-bimaximal mixing as zero order approximation in our model, we find that an additional $S_2$ symmetry should be considered. The $S_2$, the group of permutation of two objects, has the following generators in the 3-dimensional representation:

$$
e = \left(\begin{array}{ccc}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{array}\right), \quad a = \left(\begin{array}{ccc}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
\end{array}\right).
$$

(3)
\[ \Delta(27) \]

| Fields | \( \ell \) | \( e_R \) | \( \mu_R \) | \( \tau_R \) | \( \nu_{R \alpha} \) | \( H \) | \( \chi \) | \( \eta_{\alpha} \) | \( \xi \) | \( \phi \) | \( \sigma \) |
|--------|---------|---------|---------|---------|-----------|------|-------|-----------|-------|-------|--------|
| \( Z_4 \) | 1       | 1       | -1      | 1       | i         | -i   | 1     | -1        | -i   | -1    | -1     |

TABLE II: Field transformations under \( \Delta(27) \), and \( Z_4 \). Here \( \alpha \) refers to 1, 2, 3.

The particle transformations under \( S_2 \) are given by:

\[ f_1 \leftrightarrow f_1, \quad f_2 \leftrightarrow f_3, \quad (4) \]

where \( f \) stands for \( L_i, \chi_i, \eta_i, \phi, \xi_i, \sigma \) and \( \nu_{Ri} \). The right handed charged leptons transform trivially under \( S_2 \) symmetry. Moreover, we consider extra \( Z_4 \) group to get the correct mass hierarchy of the charged leptons. In Table I, we present the field transformation under \( \Delta(27) \) and \( Z_4 \).

Before concluding this section, we comment on possible vacuum alignments for the VEVs of the singlet scalars. The \( \Delta(27) \times S_2 \) invariant scalar potential at the renormalizable level is given by:

\[ V = m_1^2 \phi^\dagger \phi + m_2^2 \chi^\dagger \chi + m_3^2 \eta^\dagger \eta + m_4^2 \xi^\dagger \xi + m_5^2 \sigma^\dagger \sigma + m_6^2 \phi^\dagger \phi + m_7^2 \eta^\dagger \eta + m_8^2 \xi^\dagger \xi + m_9^2 \sigma^\dagger \sigma + \text{h.c.} \]

\[ \text{where} \quad m_1 = g_1 \quad \text{and} \quad g_3 = g_4. \]

From this equation, one can notice that the potential contains 6 free mass parameters and 23 free self-interacting couplings. This large number of free parameters is one of the features of any flavour scalar (non-supersymmetric) potential. Therefore, the minimization conditions of this potential can imply the following extremum solutions:

\[ \langle \xi \rangle = (0, w, 0), \quad \langle \phi \rangle = (0, 0, w'), \quad \langle \chi \rangle = (v, v, v) \]
\[ \langle \sigma \rangle = (v', v', v'), \quad \langle \eta_1 \rangle = u_1, \quad \langle \eta_2 \rangle = u_2, \quad \langle \eta_1 \rangle = u_3 \]

(6)

with the conditions:

\[ u_1^2 \sim u_2^2 \sim u_3^2 \sim w^2 \sim w'^2 \sim v'^2 \sim v^2 \]
\[ \sim -\frac{m_1^2}{k_1} \sim -\frac{m_2^2}{k_2} \sim -\frac{m_3^2}{k_3} \sim -\frac{m_4^2}{k_4} \sim -\frac{m_5^2}{k_5} \sim -\frac{m_6^2}{k_6} \sim -\frac{m_7^2}{k_7} \]

(7)

where

\[ k_1 = 2g_1 + g_2 + 2g_3 + h_5 + 3h_6 + h_7 + 3h_8, \]
\[ k_2 = g_2 + g_3 + g_5 + h_5 + 3h_{10} + h_{11} + 3h_{12}, \]
\[ k_3 = g_2 + g_3 + g_5 + h_0 + 3h_{10} + h_{11} + 3h_{12}, \]
\[ k_4 = 2h_1 + h_{13} + 3h_{14} + h_5 + h_9 + 3h_{16}, \]
\[ k_5 = 2h_3 + h_{13} + 3h_{15} + h_7 + h_{11} + 3h_{18}, \]
\[ k_6 = 8h_4 + h_{14} + h_{15} + h_8 + h_{12} + 3h_{17}, \]
\[ k_7 = 8h_2 + h_{18} + h_{16} + h_6 + h_{10} + 3h_{17}. \]

(8)
III. CHARGED LEPTON MASSES AND $\mathbb{Z}_4$ SYMMETRY

As shown in Table II, the lepton doublet is assigned to the triplet $3$ of $\Delta(27)$ while right-handed lepton $l_c$ are ascribed as singlet representations of $\Delta(27)$. We find that the hierarchy between the charged lepton masses may be achieved by imposing an extra discrete symmetry $\mathbb{Z}_4$. A possible set of charge assignments of $\mathbb{Z}_4$ that lead to Yukawa interactions compatible with the experimental data is also given in Table II. Therefore, the charged lepton Yukawa Lagrangian, invariant under $\Delta(27) \ltimes S_2 \times \mathbb{Z}_4$ is given by

$$L_l = \frac{\lambda_e}{\Lambda^4} \bar{\ell} H e_R (\chi^4 + \phi^4) + \frac{\lambda_\mu}{\Lambda^2} \bar{\ell} H \mu_R (\xi \xi) + \frac{\lambda_\tau}{\Lambda^2} \bar{\ell} H \tau_R (\phi) + h.c.,$$

(9)

where $\Lambda$ is non-renormalization scale, which is much larger than TeV. The Yukawa couplings $\lambda_e, \lambda_\mu, \lambda_\tau$ are of order one. As mentioned in the previous section, the scalar potential $V(\phi, \chi, \xi)$ contains several free parameters that can be adjusted to generate the VEVs for the flavons as given in Eq.(6). To get the hierarchal mass spectra of charged leptons, one assumes that the VEVs $w, w', v'$ and $v$ are of the same order and satisfy the following relation:

$$\frac{w}{\Lambda} \sim \frac{w'}{\Lambda} \sim \frac{v}{\Lambda} \sim \frac{v'}{\Lambda} \sim \mathcal{O}(\lambda_C^2),$$

(10)

where $\lambda_C$ is the Cabibbo angle, i.e, $\lambda_C \sim 0.22$. In this case, one finds that the charged lepton mass matrix $m_\ell$ is given by

$$m_\ell = \begin{pmatrix} \lambda_C^6 & 0 & 0 \\ \lambda_C^6 & \lambda_C^2 & 0 \\ \lambda_C^6 & 0 & 1 \end{pmatrix} \lambda_\tau \lambda_C^2 \langle H \rangle.$$  

(11)

The matrix $m_\ell$ is not symmetric or Hermition, so it can be diagonalized by two unitary matrices.

$$m_\ell = U_L^T m_\ell^{\text{diag}} U_R = \lambda_\tau \lambda_C^2 \langle H \rangle \begin{pmatrix} 1 & -\lambda_C^8 & 0 \\ \lambda_C^8 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_C^2 & \lambda_C^6 \\ \lambda_C^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_C^4 & \lambda_C^8 \\ \lambda_C^4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(12)

Therefore the charged lepton masses are given by:

$$m_e \sim \lambda_\tau \lambda_C^8 \langle H \rangle,$$

$$m_\mu \sim \lambda_\tau \lambda_C^2 \langle H \rangle,$$

$$m_\tau \sim \lambda_\tau \lambda_C^2 \langle H \rangle.$$  

(13)

Hence, the following mass relations are satisfied

$$m_\tau : m_\mu : m_e \approx 1 : \lambda_C^2 : \lambda_C^6,$$

(14)

which are consistent with the hierarchy between the charged lepton masses:

$$m_e = 0.511 \text{ MeV}, \quad m_\mu = 105.658 \text{ MeV}, \quad m_\tau = 1.776 \text{ GeV}.$$  

It is worth noting that the left handed mixing matrix $U_L$ is close to the identity matrix, so the lepton mixing should arise mainly from the neutrino sector.
IV. NEUTRINO MASSES AND MIXING

From solar and atmospheric neutrino oscillation data [12], the neutrino mass squared differences are given by:

\[ \Delta m^2_{21} = 7.54^{+0.26}_{-0.22} \times 10^{-5} \text{eV}^2, \quad |\Delta m^2_{31}| = 2.47^{+0.06}_{-0.22} \times 10^{-3} \text{eV}^2, \quad |\Delta m^2_{32}| = 2.46^{+0.07}_{-0.11} \times 10^{-3} \text{eV}^2. \]  

(15)

In addition, the latest best-fit results for the mixing pattern in the lepton sector is given by [1]

\[ \sin^2 \theta_{12} = 0.308^{+0.017}_{-0.017}, \quad \sin^2 \theta_{23} = 0.437^{+0.033}_{-0.023}, \quad \sin^2 \theta_{13} = 0.0234^{+0.0020}_{-0.0019}. \]  

(16)

Having the lepton doublet, \( \ell_i \), as \( \Delta(27) \) triplet and right-handed neutrinos, \( \nu_{Rj} \), as singlets, then one can construct the following invariant interaction terms:

\[ \mathcal{L}_D = \frac{v}{\Lambda} \lambda_{ijk} \bar{\ell}_i \nu_{Rj} H \chi_k, \]  

(17)

where \( \chi \) is \( \Delta(27) \) triplet scalar. Therefore, one gets the following terms:

\[ \frac{1}{\Lambda} \lambda_1 (\bar{\ell}_1 \chi_1 + \bar{\ell}_2 \chi_2 + \bar{\ell}_3 \chi_3) \nu_{R_1} H, \]

\[ \frac{1}{\Lambda} \lambda_2 (\bar{\ell}_1 \chi_1 + \omega^2 \bar{\ell}_2 \chi_2 + \omega \bar{\ell}_3 \chi_3) \nu_{R_2} H, \]

\[ \frac{1}{\Lambda} \lambda_3 (\bar{\ell}_1 \chi_1 + \bar{\ell}_2 \chi_2 + \omega^2 \bar{\ell}_3 \chi_3) \nu_{R_3} H. \]

The \( S_2 \) flavour symmetry imposes the equality of the second and third couplings: \( \lambda_2 = \lambda_3 \). After the flavour symmetry breaking through the aligned vacuum: \( \langle \chi \rangle = (v, v, v) \), the following Dirac neutrino mass matrix is obtained

\[ m_D = \frac{v}{\Lambda} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_1 & \omega^2 \lambda_2 & \omega \lambda_2 \\ \lambda_1 & \omega \lambda_2 & \omega^2 \lambda_2 \end{pmatrix} \langle H \rangle, \]  

(18)

which can be expressed as

\[ m_D = \frac{v}{\Lambda} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \langle H \rangle. \]  

(19)

Note that here all Dirac neutrino masses are generated from the same non-renormalizable interactions of order \( v/\Lambda \). Therefore, one would not expect any hierarchy between them.

Furthermore from the invariant interactions of right-handed neutrinos with \( \Delta(27) \) singlets \( \eta_i \), Majorana mass terms for \( \nu_{Rj} \) can be obtained from the following renormalizable interactions:

\[ \mathcal{L}_M = f_{ij} \bar{\nu}_{R_i} \nu_{R_j} \eta_k. \]  

(20)

According to the \( \Delta(27) \) multiplication rules of singlet representations, the invariants that give right-handed neutrino masses are:

\[ f_1 \bar{\nu}_{R_1} \nu_{R_1} \eta_1, \quad f_2 \bar{\nu}_{R_1} \nu_{R_2} \eta_3, \quad f_3 \bar{\nu}_{R_1} \nu_{R_3} \eta_2, \]

\[ f_4 \bar{\nu}_{R_2} \nu_{R_2} \eta_2, \quad f_5 \bar{\nu}_{R_2} \nu_{R_3} \eta_1, \quad f_6 \bar{\nu}_{R_3} \nu_{R_3} \eta_3. \]  

(21)
The symmetry $S_2$ imposed the following constraints:

\[ f_2 = f_3 \quad f_4 = f_6. \]

Therefore, after $\Delta(27)$ symmetry breaking through the VEVs of $\eta_k$, one obtains the following right-handed Majorana mass matrix:

\[
M_R = \begin{pmatrix}
    f_1 u_1 & f_3 u_3 & f_3 u_2 \\
    f_3 u_3 & f_4 u_2 & f_5 u_1 \\
    f_3 u_2 & f_5 u_1 & f_4 u_3
\end{pmatrix}.
\]  

(22)

As usual, the light neutrino mass matrix is obtained in terms of Dirac neutrino mass matrix and right-handed neutrino through type I seesaw mechanism as

\[
M_\nu = -m_D M_R^{-1} m_D^T.
\]  

(23)

It is noticeable that the mass matrix $M_R$ in Eq.(22) is generic matrix that can lead to different type of neutrino mixing matrix (tri-bimaximal or nearly tri-bimaximal mixing matrix), depending on the coupling $f_3$ and the difference between the VEVs $u_2$ and $u_3$. In general the tri-bimaximal mixing matrix, $U_{TBM}$, can be written as [13, 14]:

\[
U_{TBM} = \Gamma_{mag} U',
\]  

(24)

where $\Gamma_{mag}$ is the magic matrix proposed by Cabibbo [15] and Wolfenstein [16] and has the form:

\[
\Gamma_{mag} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},
\]  

(25)

and

\[
U' = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
    0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & i
\end{pmatrix} = \begin{pmatrix}
    0 & 1 & 0 \\
    \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
    \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}}
\end{pmatrix}.
\]  

(26)

From Eqs. (19) and (25):

\[
M_\nu = -\frac{3 v^2}{\Lambda^2} \langle H \rangle^2 \Gamma_{mag} D_\lambda M_R^{-1} D_\lambda \Gamma_{mag},
\]  

(27)

where $D_\lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_2)$. If $M_\nu$ is diagonalized by tri-bimaximal mixing matrix, then we can determine the corresponding form of the right-handed neutrino mass matrix, which typically takes the form:

\[
(M_R)_{TBM} = \begin{pmatrix}
    x & 0 & 0 \\
    0 & z & y \\
    0 & y & z
\end{pmatrix}.
\]  

(28)
Therefore, the exact tri-bimaximal can be naturally obtained within $\Delta(27)$ flavour symmetry if the coupling $f_3 = 0$ and the VEVs $u_2 = u_3 = u$, which ensures the $S_2$ invariance. In this case, one obtains

$$M_\nu^{\text{diag}} = -3 v^2 \frac{\lambda^2}{\Lambda^2} (H)^2 \begin{pmatrix} \frac{\lambda^2}{f_5 u_1 + f_4 u} & 0 & 0 \\ 0 & \frac{\lambda^2}{f_5 u_1} & 0 \\ 0 & 0 & \frac{\lambda^2}{f_5 u_1 - f_4 u} \end{pmatrix}. \quad (29)$$

As expected, unlike the charged lepton masses, here there is no clear argument for neutrino mass hierarchy. Instead one should assume a hierarchy among the involved couplings of flavon VEVs to achieve the type of desired neutrino mass spectrum. For instance if one considers $f_4 \sim f_5 \gg f_1$, $u_1 \sim u$ and the couplings $\lambda$, are of the same order, the normal neutrino mass hierarchy is realized. While inverted neutrino mass hierarchy is obtained if $f_4 \sim f_5 \gg f_1$ and $u_1 \sim -u$. Finally, degenerate scenario is obtained if $f_1 \sim f_5 \gg f_4$ and $u_1 \gg u$.

Now we consider the case of $f_3 \neq 0$ and $u_2 = u_3 = u$ (i.e., $M_\nu$ is still invariant under $S_2$ symmetry). In this case, the neutrino mass matrix is given by

$$M_\nu = \frac{v^2}{\Lambda^2} (H)^2 \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix}, \quad (30)$$

where,

$$A = \frac{f_5 u_1 \lambda^2_1 + (f_4 \lambda_1 - 4 f_3 \lambda_2) u \lambda_1 + 2 f_1 \lambda^2_2 u_1}{f_1 u_1 (f_5 u_1 + f_4 u) - 2 f_3^2 u^2},$$

$$B = \frac{f_5 u_1 \lambda^2_1 + (f_4 \lambda_1 - f_3 \lambda_2) u \lambda_1 - f_1 \lambda^2_2 u_1}{f_1 u_1 (f_5 u_1 + f_4 u) - 2 f_3^2 u^2},$$

$$C = \frac{f_5^2 \lambda^2_1 u_1^2 + 2 f_5 \lambda_2 u_1 (f_1 \lambda_2 u_1 + f_3 \lambda_1 u) - u (-f_1 f_4 \lambda^2_2 u_1 + (f_2^2 \lambda^2_2 + 2 f_3 f_4 \lambda_1 \lambda_2 + 3 f_3^2 \lambda^2_2) u)}{(f_5 u_1 - f_4 u) (f_1 u_1 (f_5 u_1 + f_4 u) - 2 f_3^2 u^2)},$$

$$D = \frac{f_5^2 \lambda^2_1 u_1^2 + f_5 \lambda_2 u_1 (-f_1 \lambda_2 u_1 + 2 f_3 \lambda_1 u) - u (2 f_1 f_4 \lambda^2_2 u_1 + (f_2^2 \lambda^2_2 + 2 f_3 f_4 \lambda_1 \lambda_2 - 3 f_3^2 \lambda^2_2) u)}{(f_5 u_1 - f_4 u) (f_1 u_1 (f_5 u_1 + f_4 u) - 2 f_3^2 u^2)}.$$

As emphasised in Ref.[17], the tri-bimaximal mixing matrix corresponds to neutrino mass matrix that satisfies the following three conditions:

$$
M_\nu \begin{pmatrix} 12 \\ 22 \\ 23 \end{pmatrix} = M_\nu \begin{pmatrix} 13 \\ 33 \\ 23 \end{pmatrix}, \\
(M_\nu)_{12} = (M_\nu)_{33}, \\
(M_\nu)_{11} + (M_\nu)_{12} = (M_\nu)_{22} + (M_\nu)_{23}. \quad (31)
$$

It is clear that the neutrino mass matrix in our case satisfies the first two conditions only while the third condition is violated. Therefore, this neutrino mass matrix can be diagonalized by a matrix which is very close to tri-bimaximal. However, we find that the resulting mixing matrix still has zero $\theta_{13}$ and maximal $\theta_{23}$. It essentially deviates from tri-bimaximal in the first and the second
columns only. Also the corresponding eigenvalues of neutrino masses are given by

\[
\begin{align*}
    m_1 &= -3 u^2 \langle H \rangle^2 \frac{2 \lambda_1^2 \lambda_2^2}{(x + \sqrt{x^2 - y^2})}, \\
    m_2 &= -3 u^2 \langle H \rangle^2 \frac{2 \lambda_1^2 \lambda_3^2}{(x - \sqrt{x^2 - y^2})}, \\
    m_3 &= -3 u^2 \langle H \rangle^2 \frac{\lambda_2^2}{(f_5 u_1 - f_4 u)}.
\end{align*}
\]

where \( x = (f_1 \lambda_1^2 + f_5 \lambda_1^2)u_1 + f_4 u \lambda_1^2 \) and \( y^2 = 4 \lambda_1^2 \lambda_2^2 (-2 f_3^2 u^2 + f_1 u_1 (f_5 u_1 + f_4 u)) \). Here the normal hierarchy is achieved if \( f_4 \sim f_5 \gg f_1 \gg f_3 \), \( u_1 \sim u \) and the couplings \( \lambda_1 \sim \lambda_2 \). The degenerate scenario is obtained if \( f_1 \sim f_5 \gg f_1 \gg f_3 \) and \( u_1 \gg u \). Finally, the inverted hierarchy is obtained if \( f_4 \sim f_5 \gg f_1 \gg f_3 \) and \( u_1 \sim -u \).

Now we turn to the case of spontaneous \( S_2 \) symmetry breaking, i.e., \( u_2 \neq u_3 \) with \( f_3 \sim 0 \). In this case, all the three relations, in Eq. (31), are violated. The consequences of the deviation from tri-bimaximal mixing on the symmetry manifesting in neutrino mass matrix were studied in [17]. Following the notations used in this reference, we define the parameters which characterize the deviation of mixing angles from the tri-bimaximal values as

\[
D_{12} \equiv \frac{1}{3} - s_{12}^2, \quad D_{23} \equiv \frac{1}{2} - s_{23}^2, \quad D_{13} \equiv s_{13},
\]

where \( s_{ij} \equiv \sin \theta_{ij} \). The violation of tri-bimaximal symmetry of neutrino mass matrix in Eq. (31) can be written in terms of deviation parameters \( D_{23} \) and \( s_{13} \) as follows:

\[
\Delta_1 = (M_\nu)_{12} - (M_\nu)_{13} = \frac{\sqrt{2}}{3}((2m_1 + m_2)e^{2i\delta} - 3m_3)s_{13}e^{-i\delta} + \frac{2}{3}(m_2 - m_1)D_{23},
\]

\[
\Delta_2 = (M_\nu)_{22} - (M_\nu)_{33} = \frac{2\sqrt{2}}{3}(m_2 - m_1)s_{13}e^{i\delta} + \frac{1}{3}(m_1 + 2m_2 - 3m_3)D_{23},
\]

\[
\Delta_3 = (M_\nu)_{11} + (M_\nu)_{12} - ((M_\nu)_{22} + (M_\nu)_{23})
\]

\[
= \left( \frac{1}{3\sqrt{2}}(3m_3 - (2m_1 + m_2)e^{2i\delta})e^{-i\delta} - \frac{\sqrt{2}}{3}(m_2 - m_1)e^{i\delta} \right)s_{13}
\]

\[
+ \left( \frac{2}{3}(3m_3 - (2m_1 + m_2)e^{2i\delta})e^{-2i\delta} - \frac{1}{3}(m_2 - m_1) \right) \frac{s_{13}}{2}
\]

\[
- \frac{1}{3}(2m_1 + m_2 - 3m_3)D_{23} - \frac{9}{4}(m_2 - m_1)D_{12},
\]

where \( m_i \) are the masses of effective neutrino and \( \delta \) is the leptonic Dirac phase. In our model the deviations from tri-bimaximal conditions in Eq. (31) can give constrains on our parameters (couplings and VEVs) in order to get the correct mixing angles and desired scenario of mass spectra,
From Eqs. (34) and (35) we can calculate the deviation parameters from tri-bimaximal mixing (33) as follows

\begin{align*}
    s_{13} &= \frac{v^2}{\lambda_2^2} \langle H \rangle^2 \frac{i\sqrt{3}f_4\lambda_2^2(u_2 - u_3)}{(m_1 - m_3)(f_5^2u_1^2 - f_4^2u_2u_3)}, \\
    D_{23} &= -\frac{v^2}{\lambda_2^2} \langle H \rangle^2 \frac{i\sqrt{3}f_4\lambda_3^2(u_2 - u_3)}{2(m_1 - m_3)(f_5^2u_1^2 - f_4^2u_2u_3)}, \\
    D_{12} &= -\frac{v^2}{\lambda_2^2} \langle H \rangle^2 \frac{(u_2 - u_3)^2 f_4^2\lambda_3^2(m_1 + m_2 - 2m_3)}{3(m_1 - m_2)(m_1 - m_3)^2 (f_5^2u_1^2 - f_4^2u_2u_3)}.
\end{align*}

Here we set Dirac phase \( \delta = 0 \). Thus, one can write the following relations

\begin{align*}
    |s_{13}| &= \sqrt{2}D_{23}, \\
    D_{12} &= \frac{4(m_1 + m_2 - 2m_3)}{9(m_1 - m_2)} D_{23}^2.
\end{align*}

From the first relation, one finds that \( s_{13} \sim 0.13 \) (lower 3\( \sigma \) experimental limit) if \( D_{23} \sim 0.09 \), which corresponds to 1\( \sigma \) limit of atmospheric neutrino mixing angle [1]. In addition, if \( D_{23} \sim 0.11 \) (2\( \sigma \) limit), one gets \( s_{13} \sim 0.155 \) (best fit value). In Fig. 1 we plot the relation between the lightest neutrino mass \( m_1 \) and \( D_{12} \) for \( D_{23} \) is given by its best value 0.066, and 1\( \sigma \) limits 0.09 and 0.11. As can be seen from this figure, for \( D_{23} = 0.09 - 0.11 \), which lead to a consistent \( s_{13} \), the mass spectrum of neutrino should be strongly hierarchical, i.e. \( m_1 < 0.01 \) eV, in order to get \( D_{12} \) in the allowed range. We also plot \( D_{12} \) versus \( D_{23} \) for different values of \( m_1 \), namely \( m_1 = 0, 0.01, 0.015 \) eV. It confirms the same conclusion that the allowed range for \( D_{12} \) can be achieved if \( m_1 \leq 0.01 \) eV for \( 0.09 \leq D_{23} \leq 0.11 \).
assume that the left handed quarks and up right quarks transform under the same order. The degenerate scenario is obtained if $f_\theta = 0$. The approximated neutrino mass eigenvalues are given by

$$m_1 \simeq -3 \frac{v^2}{\Lambda^2} \langle H \rangle^2 \left( \frac{2 \lambda_2^2}{\sqrt{4 f_5^2 u_1^2 + f_4^2 (u_2 - u_3)^2 + f_4 (u_2 + u_3)}} \right)$$

$$m_2 \simeq -3 \frac{v^2}{\Lambda^2} \langle H \rangle^2 \left( \frac{\lambda_2^2}{f_1 u_1} \right)$$

$$m_3 \simeq -3 \frac{v^2}{\Lambda^2} \langle H \rangle^2 \left( \frac{2 \lambda_2^2}{\sqrt{4 f_5^2 u_1^2 + f_4^2 (u_2 - u_3)^2 - f_4 (u_2 + u_3)}} \right)$$

We can tune the parameters to obtain the various mass hierarchy spectra as the following: The normal hierarchy is achieved if $f_4 \sim f_5 \gg f_1$, $u_1 \sim (u_2 + u_3)$ and the couplings $\lambda_4$ are of the same order. The degenerate scenario is obtained if $f_1 \sim f_5 \gg f_4$ and $u_1 \gg u_2, u_3$. The inverted hierarchy is obtained if $f_4 \sim f_5 \gg f_1$ and $u_1 \sim -(u_2 + u_3)$.

To ensure that there are various values of parameter space (consists of flavon VEVs and couplings) that can account for the recent value of mixing angle $\theta_{13}$ and neutrino masses simultaneously, we show in Fig.2 a correlation between $s_{13}$ and $m_3$, where all free parameters vary in their allowed ranges. In Fig.2 we display the deviation of solar neutrino parameters $D_{12}$ versus the light neutrino mass $m_1$ and the deviation of atmospheric neutrino parameter $D_{23}$.

It is important to notice the essential role of $S_2$ symmetry, which permutes the second flavour to the third one, that leads to equal couplings in the Dirac mass matrix and right handed mass matrix. In this case, the neutrino mass matrix has the form of Eq.(28) and hence tri-bimaximal mixing is realized. When we break the $S_2$ spontaneously by imposing different VEVs to the second and the third flavour of the flavon $\eta$, the deviation from tri-bimaximal is achieved.

V. QUARK MASSES AND CKM MIXING

In this section we analyse the quark masses and mixing in the framework of the symmetry group $\Delta(27) \times S_2 \times Z_4$. The quark transformations under $\Delta(27)$ are shown in Table III. We also assume that the left handed quarks and up right quarks transform under $S_2$ such that $Q_{L2} \leftrightarrow Q_{L3}$.
Table III, one can write the following invariants:  

For instance, if three masses are different and can account for the hierarchial mass spectrum of the up quark sector. The following mass matrix of down quarks is obtained:  

\[ \begin{pmatrix}  \delta^2 & \delta & \delta \\ \delta & \delta^2 & \delta \\ \delta & \delta & \delta^2 \end{pmatrix} \]

where \( \delta = h_1 - h_2 \). Thus, the following invariants terms are explicitly found:

\[ \frac{1}{\Lambda} h_1^u H(\bar{Q}_1 u_R + \bar{Q}_2 c_R + \bar{Q}_3 t_R) \eta_1, \]

\[ \frac{1}{\Lambda} h_2^u H(\bar{Q}_1 u_R + \omega^2 \bar{Q}_2 c_R + \omega \bar{Q}_3 t_R) \eta_2, \]

\[ \frac{1}{\Lambda} h_3^u H(\bar{Q}_1 u_R + \omega \bar{Q}_2 c_R + \omega^2 \bar{Q}_3 t_R) \eta_3. \]

From the symmetry, \( h_2^u = h_3^u \). Therefore, the masses of the up quarks are given by:

\[ m_u = \frac{1}{\Lambda} \langle H \rangle (h_1^u u_1 + h_2^u (2u_2 + \delta)), \]

\[ m_c = \frac{1}{\Lambda} \langle H \rangle (h_1^u u_1 + h_2^u (\omega u_2 + \omega^2 (u_2 + \delta))), \]

\[ m_t = \frac{1}{\Lambda} \langle H \rangle (h_1^u u_1 + h_2^u (\omega^2 u_2 + \omega (u_2 + \delta))), \]

where \( \delta = u_3 - u_2 \). In general the coupling constants \( h_i^u \) and VEVs \( u_i \) are complex, so the previous three masses are different and can account for the hierarchical mass spectrum of the up quark sector. For instance, if \( h_1^u \sim \mathcal{O}(\lambda_0^2) \) and \( h_2^u \simeq 6.85, h_3^u \simeq -6.85 \ e^{i\pi/3}, \) \( \lambda_0 \simeq -0.083 \ e^{i\pi/6}, \) we can get the up quark masses consistent with the following experimental results:

\[ m_u(1\text{GeV}) = 4.5 \pm 1 \text{ MeV}, \quad m_c(m_c) = 1.25 \pm 0.15 \text{ GeV}, \quad m_t(m_t) = 166 \pm 5 \text{ GeV}. \]

Finally, we consider the down quark mass and mixing. From the charge assignments given in Table III, one can write the following invariants:

\[ L_d = \frac{1}{\Lambda^3} h_d Q H d_R \phi^2 \sigma + \bar{Q} H s_R(\frac{1}{\Lambda^3} h_{s1} \xi^2 + \frac{1}{\Lambda^3} h_{s2} \sigma^2 \eta_1) + Q H h_R(\frac{1}{\Lambda} h_1 \phi + \frac{1}{\Lambda^3} h_2 \chi^2 \xi). \]

If \( h_d \sim h_{s1} \sim h_{b1} \sim \mathcal{O}(1) \) while \( h_{s2} \sim \mathcal{O}(0.1) \), then after spontaneous symmetry breaking the following mass matrix of down quarks is obtained:

\[ m_d \simeq \begin{pmatrix} \lambda_0^4 & \lambda_0^3 & \lambda_0^4 \\ \lambda_0^2 & \lambda_0^2 & \lambda_0^3 \\ \lambda_0 & \lambda_0 & 1 \end{pmatrix} h_2 \lambda_0^2 \langle H \rangle, \]
which can be diagonalized by two matrices,

\[ m_d = V_L^T m_u \text{diag} V_R \]

\[ = h_b \lambda_C^2 \langle H \rangle \begin{pmatrix} 1 - \lambda_C^2 & \lambda_C & \lambda_C^4 \\ -\lambda_C & 1 - \lambda_C^2 & \lambda_C^4 \\ -\lambda_C^4/2 & -\lambda_C^4 & 1 \end{pmatrix} \begin{pmatrix} \lambda_C^2 & 0 & 0 \\ 0 & \lambda_C^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_C^2 & \lambda_C^4 \\ -\lambda_C^4 & 1 & \lambda_C^2 \\ -2\lambda_C^4 & -\lambda_C^2 & 1 \end{pmatrix}. \quad (45) \]

It is clear that the left handed rotation matrix \( V_L \) is close to the quark mixing matrix, \( V_{CKM} \), and the hierarchical spectrum of down quark masses is obtained with the following mass ratios:

\[ m_b : m_s : m_d \approx 1 : \lambda_C^2 : \lambda_C^4, \quad (46) \]

which are compatible with measured down quark masses:

\[ m_d(1\text{GeV}) = 8.0 \pm 2 \text{ MeV}, \quad m_s(1\text{GeV}) = 150 \pm 50 \text{ MeV}, \quad m_b(m_b) = 4.25 \pm 0.15 \text{ GeV} \]

VI. CONCLUSIONS

In this paper we have constructed a model of fermion masses and mixing based on an extension of the SM with a discrete flavour symmetry \( \Delta(27) \). Our study is different from the previous \( \Delta(27) \) analyses in two main points: i) Our model is FCNC free, since one Higgs doublet is used to break the electroweak symmetry and SM singles only are involved in spontaneous breaking of \( \Delta(27) \) ii) Both quark and lepton masses and their mixing are simultaneously analysed under the same flavour symmetry. In fact most of the work in the literature focuses on the lepton sector only.

By assigning lepton doublets to \( \Delta(27) \) triplet and right-handed leptons to singlets, we have shown that the charged lepton mass matrix is almost diagonal with the desired hierarchy. Therefore, the MNS lepton mixing matrix is generated from the neutrino sector. We also argued that deviation from tri-bimaximal is due to spontaneous violation of the imposed \( S_2 \) symmetry. Similarly by assigning quark doublets and right-handed up quarks to \( \Delta(27) \) triplets and right-handed down quarks to singlets, we obtained diagonal up-quark mass matrix and CKM quark mixing matrix arises from down sector only.

Finally, our model predicts that for \( \sin \theta_{13} \simeq 0.13 \), the mass of lightest neutrino is \( \lesssim O(0.1) \) eV and \( \sin^2 \theta_{23} \simeq 0.41 \), which is a remarkable deviation from maximal mixing.

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