Stylized facts in minority games with memory: a new challenge

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Abstract

A finite memory is introduced in the score dynamics of Minority Games. As expected, this removes the dependence of the stationary state on the initial conditions. However, it also causes an unexpected increase of fluctuations in grand-canonical models for very large times. Current analytical methods are inadequate to solve this simple and natural extension.

Key words: Minority Game, financial markets, finite memory, volatility clustering, fat tails, price return
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1 Introduction

Remarkable new phenomenology arises when agents are granted the possibility of not participating in a game [1,2]. In the context of the Minority Game (MG) [3], this extension – which takes the name of Grand Canonical MG (GCMG) – has been introduced independently by several authors [4,5,6]. The MG captures some aspects of financial market dynamics, namely the interplay between exploitable predictability, price fluctuation and trader behaviour (we refer the interested reader to [2,7,8,9] for more details). Furthermore, GCMGs
reproduce some of the main stylized facts of financial market phenomenology, such as large price changes and volatility clustering [2,5,10,11]. Remarkably, these features emerge close to a phase transition, thus suggesting that financial markets operate in the vicinity of a critical point [2,9].

For physicists, this is almost intuitive. Nevertheless, it is hard to convince economists that it is actually true. The MG is able to propose a coherent account of why financial markets are at criticality, based on market (in)efficiency (or predictability) [2,9]. The idea of explaining anomalous fluctuations as a critical phenomenon is not new. Other models, notably those based on two-dimensional percolation [12], also relate stylized facts with criticality. There is however little reason to consider the market as bi-dimensional (although the trading floors are), or to believe it should spontaneously reach the critical percolation probability. The fact that all kinds of financial markets generate power-law-distributed price changes implies that a reasonable model of a financial market should not need the fine-tuning of a parameter.

An annoying feature of the GCMG is that, close to the phase transition, its dynamics displays a peculiar dependence on initial conditions: anomalous fluctuations only materialize in some realizations of the dynamics, whereas others exhibit plain Gaussian price fluctuations [2]. The introduction of a finite memory in the learning dynamics of agents, such that the impact that events occurred far in the past have on the agents’ choices in the present vanishes with the time lag, is the natural remedy to this inconvenient. This modification was introduced in [13] for a model in which agents play the MG strategically. Here we focus on the standard MG, where agents adopt a naïve price-taking behavior [9].

We shall first show that indeed a finite memory eliminates the dependence on initial conditions. A direct extrapolation of theoretical results [13] suggests that a finite memory leads to an increase of fluctuations. This is confirmed by numerical simulations. Then we shall see that, however, memory brings about new non-trivial effects in the GCMG. In particular, it causes a build-up of fluctuations in the stationary state that sets in after very large times. Our discussion will be mainly informal and we shall give only the minimal technical details. We will refer to original papers for the detailed definition of the models.

2 Finite memory score in the standard MG

In few words, the MG models a system of $N$ interacting agents who should repeatedly take a binary decision, such as buy/sell. Each agent aims at taking the minority decision. This captures the fact that it is generally convenient to
buy when the majority sells and vice-versa. Agents resort to trading strategies to process market information. The latter takes one of $P = \alpha N$ possible values. The performance of each strategy is monitored by a score function. The dynamics of the score of strategy $s$ has the form

$$U_{i,s}(t + 1) = U_{i,s}(t) - a_{i,s}^{\mu(t)} A(t)$$

(1)

where $a_{i,s}^{\mu(t)} = \pm 1$ is the action prescribed by strategy $s$ when the information takes the value $\mu(t)$ and $A(t)$ is the sum of all actions ($\pm 1$) of agents. At each time step, agents adopt the strategy with the highest score. The last term in (1) is the payoff of agent $i$ for using strategy $s$ at time $t$. As the particular form of the dynamics is not at stake in what follows, we refer to [9] for more details. We just remind that the key observables are the magnitude of fluctuations $\sigma^2 = \langle A^2 \rangle$ and the information content $H = (1/P) \sum_{\mu} \langle A | \mu \rangle^2$, which details how the outcome $A(t)$ is correlated with the information $\mu(t)$ [9,14,15]. We also remind that the value $N = P/\alpha_c$, with $\alpha_c = 0.3374 \ldots$, separates an asymmetric, information rich phase with $H > 0$ (for $N < P/\alpha_c$) from a symmetric phase with $H = 0$ ($N > P/\alpha_c$). In the latter, the stationary state depends on the asymmetry

$$U_0 = U_{i,1}(0) - U_{i,2}(0)$$

of the initial conditions (we focus on the MG with two strategies per agent). This dependence is due to the fact that, under (1), all market fluctuations since time $t = 0$ are remembered and contribute with the same weight to $U_{i,s}(t)$, irrespective of how far in the past they took place.

This suggests to generalize (1) to

$$U_{i,s}(t + 1) = (1 - \lambda/P)U_{i,s}(t) - a_{i,s}^{\mu(t)} A(t).$$

(2)

with $\lambda > 0$ a constraint. Fig. 1 shows that a time dependent $\sigma^2 = \langle A^2 \rangle_t$ converges, for long times, to a value which is independent of initial conditions $U_0$. This implies that the initial asymmetry $U_0$ has no influence on the stationary state. This is confirmed by Fig. 2, where we plot $\sigma^2/N$ as a function of $\alpha = 1/n_s$ for different $U_0$ and $\lambda$. For a fixed $\alpha$ and for a given realization of the game, $\sigma^2$ and $H$ are increasing functions of $\lambda$, although in the symmetric phase ($\alpha < \alpha_c = 0.3374 \ldots$) $\sigma^2$ varies very slowly with $\lambda$ (see Fig. 3). While only small values of $\lambda$ make physical sense, for completeness the same figure reports also the queer effects of very large $\lambda$. From (2), $\lambda = P$ cancels the contribution of $U_i(t)$ in $U_i(t + 1)$; larger $\lambda$ implies that this contribution is of opposite sign to the usual MG.

\[\text{Here } \langle \ldots \rangle_t \text{ stands for an average over a long but finite time interval around } t.\]
Fig. 1. The same realization of the standard minority game with unbiased initial strategy scores and $\lambda = 0$ (circles), biased initial scores and $\lambda = 0$ (squares), biased initial scores and $\lambda = 0.1$ (triangles).

Fig. 2. Average volatility $\sigma^2/N = \langle A^2 \rangle / N$ versus $\alpha = P/N$ of the standard MG with unbiased initial scores and $\lambda = 0$ (circles), $\lambda = 0.1$ (squares), $\lambda = 0.2$ (diamonds), with biased initial scores and $\lambda = 0.2$ (triangles).

3 Finite memory in the Grand Canonical MG

Let us focus now on the simplest GCMG [2], where the agents (speculators) possess a single strategy each and may abstain from playing if the game is not profitable enough. A measure of profitability is given by a parameter $\epsilon$, which measures the gains from investing outside of the market. Again, we refer to the original paper [2] for a detailed account. The important new feature
of this model is that the number of participants is not fixed a priori, but is
dynamically determined by market gains. The model, which includes a number
$N_p$ of producers who trade no matter what, according to a given strategy, can
be solved exactly in the limit of infinitely many speculators $N_s \to \infty$ (with
$\alpha = P/N_s = 1/n_s$ and $n_p = N_p/P$ finite) via a static replica calculation
(this solution can be also recovered with the dynamical generating functional
technique [16]). This allows for a systematic understanding of the model’s
properties. It turns out that a phase transition occurs for $\epsilon = 0$ at $n_s = 1/\alpha > n_s^{c}$. The transition is discontinuous across the $\epsilon = 0$ line, where the number of
active agents jumps abruptly. Stylized facts are observed, for $\epsilon > 0$, close to
the point $\epsilon = 0, n_s \approx n_s^{*}(P) > n_s^{c}$ in finite systems, and the size of the critical
region can also be characterized. More precisely, we observe the occurrence of
distribution of returns $A(t)$ and volatility clustering [7], or long-time correlations
$\langle |A(t)||A(t+\tau)| \rangle \propto \tau^{-\gamma}$. These findings are summarized
in Fig. 4. First we see that fat tails or volatility clustering both emerge close
to the phase transition $n_s \approx n_s^{*}(P)$ and $\epsilon \approx 0$. Then, these two features do not
appear in all runs. Some runs, even in the critical region, show plain Gaussian
returns, sometimes with long range correlations and sometimes without. In the
real world, this would imply that some markets show fat-tailed price changes,
and some others not, which is at odds with the observed universality.

In the GCMG, this issue signals a non-ergodic behavior and is related to the
dependence on initial conditions. Both effects arise because of the presence of
infinite memory in the strategy scores dynamics. This suggests to introduce

\footnote{2} $n_s^{*}(P)$ increases with $P$ [2].

\footnote{3} Note that other functions for describing the long-range correlation have been proposed [7,17]
a finite memory $\lambda > 0$ in the dynamics\textsuperscript{4}, as done in the previous section for the standard MG (see (1) and (2)). We remark that, technically, finite score memory can also be implemented by fixing a time window $T$ such that only the $T$ most recent scores are kept by the agents, as in [5,18]\textsuperscript{5}. The two procedures are equivalent for $T \sim P/\lambda$.

The dynamics of the GCMG with finite memory is surprising. At first, fluctuations are reduced and are Gaussian. Many iterations are needed before stylized facts emerge again. Remarkably, the typical time for the emergence of anomalous fluctuations is much larger than characteristic times, which are of order $P/\lambda$. When one moves deeper into the crowded market region $N_s/P \gg n_c^s$, one sees even stronger effects, as shown in Fig. 6. Still, one has to wait a quite long time before wild fluctuations set in. But now large price fluctuations are proportional to the number of speculators ($A(t) \propto N_s$) and they appear associated to a particular information pattern $\mu$. The sample of Fig. 6 shows that the value of $\mu$ for which $A(t) \propto N_s$ may even change in the course of time. What is striking is that a modification like memory, which naively should make the dynamics smoother by introducing a finite cutoff in time dependencies, actually provokes the boost of wild fluctuations. The fact that this effects sets in for times much longer than the memory decay strongly

\textsuperscript{4} A second solution may be to allow for the evolution of agent’s strategies. In that case, as observed in [13], an infinite memory would make no sense.

\textsuperscript{5} Such games with binary payoffs [3] can be written as a matrix; predicting large price changes amounts to studying its eigenvalues [19].
Fig. 5. The same game with infinite memory score keeping (red) and finite (black). $PN_s = 16000, n_s = 20, n_p = 1, \epsilon = 0.01$. 

Fig. 6. The time series of $A(t)$ at fixed $\mu(t) = 0$ (red), 11 (green) and 26 (blue) in a run of the GCMG with $P = 32, N_s = 1600, \epsilon = 0.01$ and $\lambda = 0.1$. 

suggests that its origin lies in collective fluctuation phenomena. Finally, as for the plain MG, a finite memory $\lambda > 0$ eliminates all dependences on initial conditions and non-ergodicity effects.

4 Conclusions

In short, we have shown that finite score memory suppresses non-ergodic behavior and dependence on initial conditions in MGs. At the same time, it brings about new spectacular fluctuation phenomena in the GCMG. Interestingly, all the analytical tools used so far for the study of MG fail when $\lambda > 0$. On one side, as soon as $\lambda > 0$ the stationary state can no longer be related to the minimum of a global function, in contrast with the $\lambda = 0$ case.
This seems to rule out replica approaches. On the other, a finite memory destroys the frozen component of the dynamical variables which allows one to extract information from the generating functional approach [21,22]. This makes finite memory MG a quite challenging problem.

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