Fermionic Character Sums and the Corner Transfer Matrix

Ezer Melzer

Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794-3840

Abstract

We present a “natural finitization” of the fermionic $q$-series (certain generalizations of the Rogers-Ramanujan sums) which were recently conjectured to be equal to Virasoro characters of the unitary minimal conformal field theory (CFT) $M(p, p + 1)$. Within the quasi-particle interpretation of the fermionic $q$-series this finitization amounts to introducing an ultraviolet cutoff, which – contrary to a lattice spacing – does not modify the linear dispersion relation. The resulting polynomials are conjectured (proven, for $p=3,4$) to be equal to corner transfer matrix (CTM) sums which arise in the computation of order parameters in regime III of the $r=p+1$ RSOS model of Andrews, Baxter, and Forrester. Following Schur’s proof of the Rogers-Ramanujan identities, these authors have shown that the infinite-lattice limit of the CTM sums gives what later became known as the Rocha-Caridi formula for the Virasoro characters. Thus we provide a proof of the fermionic $q$-series representation for the Virasoro characters for $p=4$ (the case $p=3$ is “trivial”), in addition to extending the remarkable connection between CFT and off-critical RSOS models. We also discuss finitizations of the CFT modular-invariant partition functions.

1 Address after Sept. 1, 1993: School of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel.
1. Prelude

This paper is concerned with generalizations of the following identities:

\[
\sum_{m=0}^{\infty} q^{m(m+a)} \frac{(q)_m}{(q)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4+a})(1-q^{5n-1-a})}
= \lim_{L \to \infty} \sum_{\sigma_i \in \{0,1\}, \sigma_i \sigma_{i+1} = 0} q^{\sum_{j=1}^{L-1} j \sigma_{j+1}}
= (q)_{\infty} \sum_{k \in \mathbb{Z}} \left( q^{k(10k+1+2a)} - q^{(2k+1)(5k+2-a)} \right)
\]

for \( a = 0, 1 \), where\(^2\)

\[
(q)_0 = 1, \quad (q)_m = \prod_{n=1}^{m} (1-q^n) \quad \text{for} \quad m = 1, 2, 3, \ldots
\]

Though equal, the four sides of (1.1) represent for us different objects. In order to distinguish between them in the discussion below, we will refer to (generalizations of) the infinite sums on the first, second, and third lines of (1.1) as fermionic, corner-transfer-matrix (CTM), and bosonic, respectively. (Since (1.1) contains only a single infinite product expression it is not necessary to give it a name; if pressed, we will call it a bosonic product.) In order to avoid the introduction of unnecessary notation which might obscure the general picture at an early stage, we defer the presentation of the specific generalizations of (1.1) we are interested in to the main body of the paper; cf. eqs. (3.20) and (3.22) for the fermionic sums, (3.12) for the CTM sums, and (2.2) for the bosonic ones.

The first line of (1.1), stating the equality (fermionic sum = product), is due to Rogers [1][2], Schur [3], and Ramanujan [2], and is usually referred to as the Rogers-Ramanujan(-Schur) identities. The equality (product = CTM sum) has a combinatorial interpretation in terms of certain restricted partitions of integers [3]-[5] (see Corollaries 7.6–7.7 in [6]). A direct proof of (CTM sum = fermionic sum) is given in [7], and finally the bosonic side is found in [1]-[3].

\(^2\) Throughout the paper, \(\mathbb{Z}, \mathbb{R}, \text{and} \mathbb{C}\) denote the sets of all integers, real numbers, and complex numbers, respectively. Also \(\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})\) is the cyclic (additive) group of \(N\) elements.
2. Introduction

The relevance of the celebrated identities (1.1) and their generalizations to two-dimensional physics, as well as string theory, has been recognized in the last two decades; see e.g. [8] and the reviews [9][10]. The “physics connection” goes through (at least) two independent routes, which we will call the critical and the off-critical. Describing these connections will motivate the names – completely arbitrary, up to this point – which were associated above with the sums in (1.1).

2.1. The critical connection

It turns out that many \( q \)-series of the type (1.1) are characters of irreducible highest-weight representations of infinite-dimensional algebras, called chiral or vertex operator algebras, which serve as symmetry algebras of two-dimensional conformal field theories (CFTs). In particular, the two \( q \)-series in (1.1) (with \( a=0 \) and 1) are the Virasoro characters corresponding to the two primary fields (of conformal dimensions \( \Delta = -\frac{1}{5} \) and 0, respectively) of the minimal model \( \mathcal{M}(2,5) \). This model belongs to the minimal series \( \mathcal{M}(p,p') \), labeled by two coprime positive integers \( p' > p \geq 2 \). The central charge and highest-weights of the corresponding irreducible representations \( V_{r,s}^{(p,p')} \) of the Virasoro algebra are\(^3\)

\[
\Delta_{r,s}^{(p,p')} = \frac{(rp' - sp)^2 - (p' - p)^2}{4pp'} \quad (r = 1, \ldots, p - 1, \quad s = 1, \ldots, p' - 1). \quad (2.1)
\]

The characters of these representations are [14]

\[
\chi_{r,s}^{(p,p')}(q) \equiv q^{-\Delta_{r,s}^{(p,p')}} \text{Tr}_{V_{r,s}^{(p,p')}} q^{L_0} = \frac{1}{(q)_\infty} \sum_{k \in \mathbb{Z}} \left( q^{k(kpp' + rp' - sp)} - q^{(kp+r)(kp'+s)} \right), \quad (2.2)
\]

where the overall normalization is chosen such that \( \chi_{r,s}^{(p,p')}(q) = 1 + \sum_{n=1}^\infty a_n q^n \), with the \( a_n \) non-negative integers which depend on \( p, p', r, s \). Note the symmetry of the “conformal grid” \( \{(r, s) \mid r \in \{1, \ldots, p - 1\}, \quad s \in \{1, \ldots, p' - 1\}\} \),

\[
(r, s) \leftrightarrow (p - r, p' - s) : \quad \Delta_{r,s}^{(p,p')} = \Delta_{p-r,p'-s}^{(p,p')} \quad \Rightarrow \quad \chi_{r,s}^{(p,p')}(q) = \chi_{p-r,p'-s}^{(p,p')}(q). \quad (2.3)
\]

The bosonic sum in (1.1) is precisely \( \chi_{1,2-a}^{(2,5)}(q) = \chi_{1,3+a}^{(2,5)}(q) \), as given by the rhs of (2.2).

\(^3\) Whenever the superscript \( p' \) is suppressed in formulas below, it will be understood as being equal to \( p + 1 \), corresponding to the unitary series \( \mathcal{M}(p, p+1) \).
We remark that “purely mathematical” derivations (see e.g. [13]-[15]) of characters of highest-weight representations of chiral algebras, as well as their computation using free-field resolutions ([16] and references therein) based on the work of [17], usually lead to bosonic expressions for the characters. The signature of these expressions are factors of $(q)^{-1}$ which is the character of a (freely generated) bosonic Fock space. [Note, however, that when the free-field realization of the chiral algebra involves Fermi fields, as is the case for superconformal algebras, the resulting character formulas also include factors of $\prod_{n=1}^{\infty} (1 + q^n - \epsilon)$. This “fermionic product”, to be contrasted with $(q)^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$ or the product in (1.4), is the character of a fermionic Fock space, with $\epsilon = 0, \frac{1}{2}$ corresponding to periodic or anti-periodic boundary conditions, respectively, on the free massless Fermi field. For this reason it is perhaps more appropriate to call the sum on the third line of (1.1) a free-field sum.]

However, direct Lie-theoretic derivations of fermionic sum representations for characters which are branching functions of affine Lie algebras are known in some cases [18], based on the construction of $Z$-algebras [19]. Also, cf. [20] for an analysis which leads to product expressions for the characters of $\mathcal{M}(2, p')$, the products being the ones in the Gordon-Andrews identities [5] (which include the Rogers-Ramanujan-Schur identities as a special case).

The Hilbert space $\mathcal{H}$ of any conformal field theory is a direct sum $\bigoplus_{\imath, \bar{\imath}} (\mathcal{V}_{\imath} \otimes \mathcal{V}_{\bar{\imath}})$ of products of irreducible highest-weight representations of two commuting copies of the Virasoro algebra, or some extended chiral algebra, governing the “right-moving” (holomorphic) and “left-moving” (antiholomorphic) sectors. Once the multiplicities $N_{i \bar{i}}$ in the direct sum are specified, all the remaining information about the spectrum of the conformal field theory (i.e. the conformal dimensions of all the fields, which are the eigenvalues of $L_0$ and $\bar{L}_0$, cf. (2.2)) is encoded in the characters. The partition function of the conformal field theory on a torus of modulus $\tau$ is then written as

$$Z_{\text{CFT}}(q, \bar{q}) = |q|^{-c/12} \text{Tr}_\mathcal{H} q^{L_0} \bar{q}^{\bar{L}_0} = |q|^{-c/12} \sum_{i, \bar{i}} N_{i \bar{i}} q^{\Delta_i} \chi_i(q) \bar{q}^{\Delta_{\bar{i}}} \chi_{\bar{i}}(\bar{q}) \quad (2.4)$$

where $q=e^{2\pi i \tau}$ and $\bar{q}=q^* = e^{-2\pi i \tau^*}$. The prefactor $|q|^{-c/12}$ accounts for the Casimir effect [21] and ensures modular invariance of $Z_{\text{CFT}}$ for appropriately chosen multiplicities $N_{i \bar{i}}$ (see [22] [23] for the case of the minimal models $\mathcal{M}(p, p')$).

This rather abstract description of the CFT spectrum can be made more concrete. One way, still within the framework of continuum quantum field theory, is to construct $Z_{\text{CFT}}$
as an (appropriately regularized) euclidean path integral on the torus, see e.g. [24]; this of course requires knowledge of the measure, namely an action for the CFT. Alternatively, one can consider realizations of the given CFT as appropriate scaling limits of certain critical two-dimensional models of (classical) statistical mechanics, or gapless one-dimensional (quantum) spin chains. In the first case, the CFT partition function is obtained from the classical partition sum on a doubly-periodic \( L \times L' \) lattice in the limit \( L, L' \to \infty \) with \( L'/L \) fixed. Then the variable \( q \) in (2.4) is \( e^{\alpha L'/L} \), where \( \alpha \in \mathbb{C} \) depends on the anisotropy. In the second case \( Z_{\text{CFT}} \) is obtained by considering the partition function of the \( L \)-site spin chain at temperature \( T \) in the limit \( L, T^{-1} \to \infty \) with \( LT \) fixed. Now \( q = e^{-2\pi v/LT} \), where \( v \in \mathbb{R} \) is the fermi velocity characterizing the massless dispersion relation of (all) the excitations. (In both cases one has to factor out a certain “bulk contribution” in order to obtain \( Z_{\text{CFT}} \); see e.g. sect. 3 of [28].) Clearly it is very difficult to fully derive \( Z_{\text{CFT}} \) from the definition of particular lattice/chain models. With the exception of the Ising model [25], results in this direction are rather limited and often incomplete.

In [26][27] the spectrum of the gapless three-state Potts spin chain was analyzed in the conformal scaling limit. This analysis, in which the eigenvalues of the hamiltonian are computed from solutions of Bethe equations, leads to a description of the spectrum in terms of excitations which were called \emph{quasi-particles}. These quasi-particles obey a fermionic exclusion rule in momentum space, and moreover the momentum ranges are subject to particular restrictions which depend on the number of quasi-particles in a given state. From this description of the spectrum in the scaling limit, fermionic expressions for the characters of the CFT – of \((\mathbb{Z}_4)^3\) \( \mathbb{Z}_3 \) parafermions in the case of the (anti-) ferromagnetic three-state Potts chain – were obtained [26][27]. (In the case of \( \mathbb{Z}_4 \) parafermions the fermionic expressions are the ones found earlier by Lepowsky and Primc [18] using Lie-theoretic methods.) These results suggested generalizations\(^4\) which led to the discovery/conjecture of new fermionic sum representations for many classes of CFT characters [11][30]-[32], all of them allowing an interpretation of the CFT spectrum in terms of fermionic quasi-particles.

\(^4\) Important clues for obtaining these generalizations come from the connection between the analysis of the leading \( q \to 1 \) behavior of the fermionic sums [29]-[32] and thermodynamic Bethe Ansatz computations. This connection, which involves sum rules for the Rogers dilogarithm function, is beyond the scope of the present article.
2.2. The off-critical connection

CTM sums have been encountered in computations of order parameters in off-critical exactly solvable models using Baxter’s corner transfer matrix technique \[33\]. In particular, the sum on the second line of (1.1) appears in the analysis of regime I of the generalized hard-hexagon model \[7\] (in addition, as noted in \[7\], the ratio of the products with \(a = 0, 1\) emerged already in the elliptic parametrization of the Boltzmann weights which define the model!). The fact that CTM sums are equal to characters of chiral algebras seems \textit{a priori} very mysterious. Note in particular that the variable \(q\) in the CTM context is a temperature-like parameter, which measures the deviation from criticality \((q_c=1)\). Hence it appears to have nothing to do with any of the three “critical meanings” of \(q\) mentioned in subsect. 2.1. Nevertheless, the connection has been recently elucidated in \[34\] based on the work of \[35\]. The key idea (see \[34\] for the details) is to relate the order parameters, which are one-point functions on the plane with fixed boundary conditions at “infinity”, to the partition function at criticality on a cylinder with fixed boundary conditions on its rims. CFT predicts \[36\] that the partition function in such a geometry is a linear combination of characters, rather than the toroidal bilinear form (2.4).

An interesting feature of the CTM sums is that they provide a natural (in the CTM context) \textit{finitization} of the \(q\)-series into an infinite set of polynomials, which we call \textit{CTM polynomials}. For the particular case of (1.1), they are defined simply by removing the \textit{lim} sign from the second line of that equation:

\[
C_a^{(L)}(q) = \sum_{\sigma_1=a, \sigma_i\sigma_{i+1}=0, \sigma_L=0} q^{\sum_{j=1}^{L-1} j\sigma_{j+1}} \quad (a = 0, 1) \quad (2.5)
\]

for \(L = 1, 2, \ldots\), with \(C_a^{(0)}(q) = \delta_{a,0}\). (Here \(C_a^{(L)}(q)\) is \(F(a)\) of eq. (44) of \[7\], with \(m=L\), so that \(L\) is the size of the edge of the corner on which the CTM acts.)

It is natural to ask whether there exist “natural” \textit{finitizations} of the fermionic and bosonic sums in (1.1), which coincide with the \(C_a^{(L)}\). The answer is positive: For \(a = 0, 1\) let \[37\]

\[
F_a^{(L)}(q) = \sum_{m\in\mathbb{Z}} q^{m(m+a)} \left[ \frac{L-m-a}{m} \right]_q, \quad (2.6)
\]

\[5\] We use this awkward word in contradistinction with ‘truncation’, to make sure that the procedure we talk about is not confused with that of eliminating from the \(q\)-series powers which are bigger than some threshold.
\[ B_{n}^{(L)}(q) = \sum_{L \in \mathbb{Z}} (-1)^{L} q^{(5L-1-2a)/2} \left[ \frac{L}{2} (L + 5L - a) \right]_{q} \]

\[ = \sum_{k \in \mathbb{Z}} \left( q^{k(10k+1+2a)} \left[ \frac{L}{2} - 5k \right]_{q} - q^{(2k+1)(5k+2-a)} \left[ \frac{L+a}{2} - 5k - 2 \right]_{q} \right), \]

where \([x]\) denotes the integer part of \(x\). Both expressions here involve \(q\)-binomial coefficients, which are defined (for \(m, n \in \mathbb{Z}\)) by

\[ \left\lfloor \frac{n}{m} \right\rfloor_{q} = \left\lfloor \frac{n}{n-m} \right\rfloor_{q} = \left\{ \begin{array}{ll} \frac{(q)^{n}}{q^{n-m}} & \text{if } 0 \leq m \leq n \\ 0 & \text{otherwise} \end{array} \right. \]  \hspace{1cm} (2.8)

The fact that \(F_{a}^{(\infty)}(q)\) and \(B_{n}^{(\infty)}(q)\) are the fermionic and bosonic sums in (1.1), respectively, follows from the properties

\[ \lim_{n \to \infty} \left\lfloor \frac{n}{m} \right\rfloor_{q} = 1, \quad \lim_{n,m \to \infty} \left\lfloor \frac{n+m}{m} \right\rfloor_{q} = 1 \]  \hspace{1cm} (2.9)

The finitized version of (1.1) (without a finitized product) is

\[ F_{a}^{(L)}(q) = C_{a}^{(L)}(q) = B_{a}^{(L)}(q) \quad \text{for } a = 0, 1, \quad L = 0, 1, 2, \ldots \]  \hspace{1cm} (2.10)

The proof of (2.10), following [3], will serve as a warm-up for the more involved discussion in the next section. The idea is to compare the recursion relations (in \(L\)) and initial conditions which uniquely specify the \(C_{a}^{(L)}, F_{a}^{(L)},\) and \(B_{a}^{(L)}\).

Consider first the \(C_{a}^{(L)}\). From the definition (2.5) we have for \(L \geq 3\)

\[ C_{a}^{(L)} = \sum_{\sigma_{2}, \ldots, \sigma_{L-1} \in \{0, 1\}} \sum_{\sigma_{1} = a, \sigma_{i+1} = 0} q^{\sum_{j=1}^{L-2} j \sigma_{j+1}} \sum_{\sigma_{L} \in \{0, 1\} \atop \sigma_{L-1} = \sigma_{L}} q^{(L-1)\sigma_{L}} \]  \hspace{1cm} (2.11)

\[ = (1 + q^{L-1}) C_{a}^{(L-2)} + q^{L-2} C_{a}^{(L-3)} \]

where the first (second) term on the last line arises from the summation on the first line when restricted to \(\sigma_{L-1} = 0\) (1). The recursion relation (2.11), the same for both \(a = 0, 1,\) together with the initial conditions

\[ C_{0}^{(0)} = 1, \quad C_{0}^{(1)} = 1, \quad C_{0}^{(2)} = 1 + q \]
\[ C_{1}^{(0)} = 0, \quad C_{1}^{(1)} = 1, \quad C_{1}^{(2)} = 1 \]  \hspace{1cm} (2.12)
are sufficient to uniquely specify all the \( C_a^{(L)} \).

A derivation of a recursion relation for the \( F_a^{(L)} \) requires the use of (one of) the following recursions for the \( q \)-binomial coefficients:

\[
\binom{n}{m}_q = \binom{n-1}{m-1}_q + q^{n-m} \binom{n-1}{m-1}_q = \binom{n-1}{m-1}_q + q^m \binom{n-1}{m}_q .
\] (2.13)

We see that for \( L \geq 2 \)

\[
F_a^{(L)} = \sum_{m \in \mathbb{Z}} q^{m(m+a)} \left( \binom{L-1-m-a}{m}_q + q^{L-2m-a} \binom{L-1-m-a}{m-1}_q \right) = F_a^{(L-1)} + q^{L-1} F_a^{(L-2)} ,
\] (2.14)

where the second line is obtained after changing the summation variable in the second sum on the first line from \( m \) to \( m' = m - 1 \). It follows that (2.14) together with the initial conditions, which are readily obtained from the definition (2.6),

\[
F_0^{(0)} = 1 , \quad F_0^{(1)} = 1 ; \quad F_1^{(0)} = 0 , \quad F_1^{(1)} = 1 ,
\] (2.15)

fully determine all the \( F_a^{(L)} \). Now note that by iterating (2.14) once (replacing the term \( F_a^{(L-1)} \) on the rhs by the rhs of (2.14) with \( L \rightarrow L - 1 \)), one finds that the \( F_a^{(L)} \) actually satisfy the recursion relation (2.11) for the \( C_a^{(L)} \), and since the initial conditions (2.15) and (2.12) coincide we conclude that the first equality in (2.10) holds. A corollary of this result is the equality of the fermionic and CTM sums in (1.1) (note that Baxter's proof [4] of this equality is obtained along different lines).

Regarding the \( B_a^{(L)} \), we state (omitting the details, which are slightly more involved, cf. [3][37]) that the recursion relations

\[
B_a^{(L)} = B_a^{(L-1)} + q^{L-1} B_a^{(L-2)} \quad \text{for} \quad a = 0, 1, \quad L = 2, 3, \ldots
\] (2.16)

are obtained from the definition (2.7), using (2.13). These recursions are the same as (2.14), and also the initial conditions for the \( B_a^{(L)} \) can be seen to coincide with (2.15). Hence the equality of the \( B_a^{(L)} \) and the \( F_a^{(L)} \) in (2.10) follows.

3. (Finitized) characters of unitary minimal models

Up to this point we reviewed known material. We now generalize the discussion above to the case of the Virasoro characters (2.2) with \( p' = p + 1 \) (which will henceforth be suppressed in formulas below). We start by introducing some extensive
3.1. Notation

Denote by $C_n$ and $I_n = 2 - C_n$ the Cartan and incidence matrices, respectively, of the simple Lie algebra $A_n$, i.e.

$$(I_n)_{ab} = \delta_{a,b+1} + \delta_{a,b-1} \quad \text{for } a, b = 1, \ldots, n. \quad (3.1)$$

Also let $e_a$ denotes the $n$-dimensional unit vector in the $a$-direction (i.e. $(e_a)_b = \delta_{ab}$), and set $e_a = 0$ for $a \not\in \{1, \ldots, n\}$. For $L$ a non-negative integer, $A, u \in \mathbb{Z}^n$, and $Q \in (\mathbb{Z}_2)^n$ such that $QI_n + u + Le_1 \in 2\mathbb{Z}^n$, we define

$$F^{(L)}_{n}(A)(u|q) = \sum_{m \in 2\mathbb{Z}^n + Q} q^{\frac{1}{2}mC_n + \frac{1}{2}A \cdot m} \prod_{a=1}^{n} \left[ \frac{1}{2}(mI_n + u + Le_1)_a \right]_q, \quad (3.2)$$

where $A \cdot m = \sum_{a=1}^{n} A_am_a$.

Now let us fix $n = p - 2 \geq 1$, and set

$$Q_{r,s} = (s-1)\rho + (e_{r-1} + e_{r-3} + \ldots) + (e_{p+1-s} + e_{p+3-s} + \ldots)$$

$$\tilde{Q}_{r,s} = (e_{p-1-r} + e_{p-3-r} + \ldots) + (e_{p-1-s} + e_{p-3-s} + \ldots) \quad (3.3)$$

$$u_{r,s} = e_r + e_{p-s}, \quad \tilde{u}_{r,s} = e_{p-r} + e_{p-s}, \quad A_s = e_{p-s},$$

where $\rho = e_1 + \ldots + e_n$. The following relations are valid for all $r = 1, 2, \ldots, p - 1$ and $s = 1, 2, \ldots, p$ (recall that $Q \in (\mathbb{Z}_2)^n$, and note that $A_1 = A_p = 0$):

$$Q_{r,s}I_n + u_{r,s} + (s-r-1)e_1 \in 2\mathbb{Z}^n, \quad \tilde{Q}_{r,s}I_n + \tilde{u}_{r,s} + (s-r)e_1 \in 2\mathbb{Z}^n \quad (3.4)$$

$$(\tilde{Q}_{r,1}, A_1, \tilde{u}_{r,1}) = (Q_{p-r, p}, A_p, u_{p-r, p}) \quad (3.5)$$

$$(\tilde{Q}_{p-r, p}, A_p, \tilde{u}_{p-r, p}) = (Q_{p-r, 1}, A_1, u_{p-r, 1})$$

$$Q_{1,s} = \tilde{Q}_{p-1,s} = (Q_{1,s}, A_s, u_{1,s} - e_1) \quad (3.6)$$

Next we define the objects of most interest to us:

$$F^{(L)}_{p; r, s}(q) = q^{-(s-r)(s-r-1)/4} \times \begin{cases} 
  F^{(L)}_{p-2}(Q_{r,s}, A_s)(u_{r,s}|q) & \text{if } L \not\equiv s - r \pmod{2} \\
  F^{(L)}_{p-2}(Q_{p-r, p+1-s}, A_{p+1-s})(u_{p-r, p+1-s}|q) & \text{if } L \equiv s - r \pmod{2}, 
\end{cases} \quad (3.7)$$

$$\tilde{F}^{(L)}_{p; r, s}(q) = q^{-(s-r)(s-r-1)/4} \times \begin{cases} 
  \tilde{F}^{(L)}_{p-2}(Q_{r,s}, A_s)(\tilde{u}_{r,s}|q) & \text{if } L \equiv s - r \pmod{2} \\
  \tilde{F}^{(L)}_{p-2}(Q_{p-r, p+1-s}, A_{p+1-s})(\tilde{u}_{r,s}|q) & \text{if } L \not\equiv s - r \pmod{2}, 
\end{cases}$$
The relations (3.4) ensure that these definitions are consistent, namely that the upper entries of all the \( q \)-binomials on the rhs (cf. (3.2)) are integers. Furthermore, from the definition it is manifest that

\[
F_{p;r,s}^{(L)}(q) = F_{p;p-r,p+1-s}^{(L)}(q), \quad \hat{F}_{p;r,s}^{(L)}(q) = \hat{F}_{p;p-r,p+1-s}^{(L)}(q),
\]  

(3.8)

and the relations (3.5)–(3.6) imply

\[
\hat{F}_{p;r,1}^{(L)}(q) = F_{p;r,1}^{(L)}(q), \quad \hat{F}_{p;1,s}^{(L)}(q) = \begin{cases} 
F_{p;1,s}^{(L+1)}(q) & \text{if } L \equiv s \pmod{2} \\
F_{p;1,s}^{(L-1)}(q) & \text{if } L \not\equiv s \pmod{2}.
\end{cases}
\]  

(3.9)

We move on to a different set of definitions, essentially borrowed from the work of Andrews, Baxter, and Forrester [38] on the RSOS model in regime III. For \( a, b, c \in \{1, 2, \ldots, p\} \) with \( |b - c| = 1 \), and \( L \) a non-negative integer such that \( L \equiv a - b \pmod{2} \), let

\[
C_{p}^{(L)}(a, b, c; q) = \sum_{h_1, h_{L+1}, h_{L+2} = (a, b, c), |h_i - h_{i+1}| = 1} q^{\sum_{j=1}^{L} j|h_{j+2} - h_j|/4}.
\]

(3.10)

(This sum is precisely \( X_L(a, b, c; q) \), eq. (1.5.11) of [38] with \( r=p+1 \).) The following symmetry property is obvious (just change summation variables from \( h_i \) to \( p+1-h_i \)):

\[
C_{p}^{(L)}(a, b, c; q) = C_{p}^{(L)}(p+1-a, p+1-b, p+1-c; q).
\]

(3.11)

Next define for \( r = 1, 2, \ldots, p-1 \) and \( s = 1, 2, \ldots, p \)

\[
C_{p;r,s}^{(L)}(q) = q^{-(s-r)(s-r-1)/4} \begin{cases} 
C_{p}^{(L)}(s, r, r+1; q) & \text{if } L \equiv s-r \pmod{2} \\
C_{p}^{(L)}(s, r+1, r; q) & \text{if } L \not\equiv s-r \pmod{2}.
\end{cases}
\]

(3.12)

The power of \( q \) here is such that the \( C_{p;r,s}^{(L)}(q) \) are polynomials in \( q \), which satisfy, as follows from (3.11),

\[
C_{p;r,s}^{(L)}(q) = C_{p;p-r,p+1-s}^{(L)}(q).
\]

(3.13)

From the analysis in subsect. 2.3 of [38] one can infer that the \( C_{p;r,s}^{(L)}(q) \) are uniquely determined from the recursion relation

\[
C_{p;r,s}^{(L)}(q) = C_{p;r,s}^{(L-1)}(q) + \begin{cases} 
q^{(L+s-r)/2}C_{p;r-1,s}^{(L-1)}(q) & \text{if } L \equiv s - r \pmod{2} \\
q^{(L-s+r+1)/2}C_{p;r+1,s}^{(L-1)}(q) & \text{if } L \not\equiv s - r \pmod{2}
\end{cases}
\]

(3.14)
(where it is understood that \( C_{p;r,s}(q)=0 \) for \( r \not\in \{1, \ldots, p-1\} \)), and the initial conditions
\[
C_{p;r,s}^{(0)}(q) = \delta_{r,s} + \delta_{r+1,s} .
\] (3.15)

Using the above definitions, Theorem 2.3.1 of [38] can be restated as
\[
C_{p;r,s}(L)(q) = B_{p;r,s}(q) ,
\] (3.16)
where the \( B_{p;r,s}(q) \) are defined by
\[
B_{p;r,s}(q) = \sum_{k \in \mathbb{Z}} \left( q^{k(kp+1)+r(p+1)-sp} \left[ \frac{L+s-r}{2} - k(p+1) \right]_q - q^{(kp+r)(k(p+1)+s)} \left[ \frac{L-s-r}{2} - k(p+1) \right]_q \right).
\] (3.17)

The proof of Theorem 2.3.1 of [38], as given there, demonstrates that the \( B_{p;r,s}(q) \) satisfy (3.14)–(3.15), which implies the identity (3.16).

3.2. Main claim

We are now in a position to state our

\[ \text{Conjecture : } \tilde{F}_{p;r,s}(q) = F_{p;r,s}(q) = B_{p;r,s}(q) = C_{p;r,s}(q) \]

for all \( L = 0, 1, 2, \ldots, \ p = 3, 4, 5, \ldots, \ r = 1, 2, \ldots, \ p-1, \ s = 1, 2, \ldots, p. \) (3.18)

(Of course the third equality is just (3.16), which was proved in [38].) Before describing the evidence we have in support of (3.18), cf. subsect. 3.4, let us discuss its implications for

3.3. Infinite \( L \)

When \( L \) becomes infinite, we see from (2.9) that \( F_{n}(L) [Q]_{A}^{n}(u|q) \) reduces to
\[
S_{n} [Q]_{A}^{n}(u|q) = \sum_{m \in \mathbb{Z}^{n+1} \geq 0} q^{\frac{1}{4} m_{C} m' - \frac{1}{2} A \cdot m} \frac{1}{(q)_{m_{1}} \prod_{a=2}^{n} \left[ \frac{1}{2} (m_{I_{a}} + u)_{a} \right]_q} ,
\] (3.19)

whereas \( B_{p;r,s}(q) \) of (3.17) becomes the Virasoro character \( \chi_{r,s}^{(p)}(q) \), as given by the rhs of (2.2) with \( p'=p+1 \). Therefore, using (3.7), the second equality in (3.18) is equivalent when \( L \to \infty \) to
\[
\chi_{r,s}^{(p)}(q) = q^{-(s-r)(s-r-1)/4} S_{p-2} [Q]_{A}^{r,s} (u_{r,s}|q)
\]
\[
= q^{-(s-r)(s-r-1)/4} S_{p-2} [Q]_{A}^{r,p+1-s} (u_{p-r,p+1-s}|q) .
\] (3.20)
The two expressions on the right here correspond to the limits taken with \( L \) even or odd. Their equality is immediate for the “corners” of the conformal grid, \((r, s)=(1,1)\) or \((1,p)\) and their mirror partners \((p − 1,p)\) and \((p − 1,1)\); the reason is that the difference between \((Q_{1,1}, A_{1}, u_{1,1})=(0,0,e_1)\) and \((Q_{p−1,p}, A_{p}, u_{p−1,p})=(0,0,0)\), and between \((Q_{1,p}, A_{p}, u_{1,p})=(e_{p−2}+ e_{p−4} + \ldots ,0,e_1)\) and \((Q_{p−1,1}, A_{1}, u_{p−1,1})=(e_{p−2}+ e_{p−4} + \ldots ,0,0)\), is just in the first component \(u_1\) of \(u\) on which the \(S_{\;\;p−2}[Q_A](u|q)\) do not depend at all. However, for other \((r, s)\) the second equality in (3.20) appears to be quite nontrivial.

Eq. (3.20) was conjectured in [32], based on the analysis of the spectrum of the ferromagnetic three-state Potts chain [27] which corresponds to the case \(p=5\). It was verified to hold as an equality of power series to high order in \(q\) in many cases. As another consistency check, pertaining to the infinitely high powers in the \(q\)-series, the leading \(q \rightarrow 1\) behavior of the \(S_{\;\;p−2}[Q_A](u|q)\) was shown in [32] to agree with the one obtained from the modular properties [22] of the characters \(\chi^{(p)}(q)\) as determined from (2.2), namely

\[
S_{\;\;p−2}[Q_A](u|q) \sim \tilde{q}^{-c^{(p)}/24} \quad \text{as} \quad q \rightarrow 1^{-}, \tag{3.21}
\]

where \(\tilde{q}=e^{-2\pi i/\tau}\) for \(q=e^{2\pi i\tau}\).

Similarly to (3.20), equality of the \(\tilde{F}_{p;r,s}^{(L)}\) and the \(B_{p;r,s}^{(L)}\) in (3.18) leads, when \(L \rightarrow \infty\), to

\[
\chi^{(p)}_{r,s}(q) = q^{−(s−r)(s−r−1)/4}S_{\;\;p−2}[Q^{r,s}_A]_{s}(\tilde{u}_{r,s}|q) = q^{−(s−r)(s−r−1)/4}S_{\;\;p−2}[Q^{p−r,p+1−s}_{A_{p+1−s}}](\tilde{u}_{p−r,p+1−s}|q). \tag{3.22}
\]

For all \((r, s)\) along the “boundary” of the conformal grid, \(i.e.\) for \(r \in \{1, p−1\}\) and/or \(s \in \{1, p\}\), this equation simply reduces to (3.20), due to the relations (3.5)–(3.6). For all other pairs \((r, s)\), constituting the “interior” of the conformal grid, eq. (3.22) gives new fermionic sum representations for the characters \(\chi^{(p)}_{r,s}\) (for \(p=5, s=3\) they were found in [27]).

To summarize, eqs. (3.20) and (3.22) are conjectured to provide fermionic sums of the form (3.19) for the Virasoro characters \(\chi^{(p)}_{r,s}\), one for each of the characters at the “corners” of the conformal grid, two different (looking) sums for each one of all the other characters along the “boundary”, and four different fermionic sums for characters in the “interior” of the conformal grid. (Of course one should in fact restrict attention to half of
the conformal grid, due to the symmetry (2.3).) Bosonic expressions for the characters are given in (2.2), and CTM forms are obtained by taking the $L \to \infty$ limit of (3.12). Infinite product expressions for some of the characters of each model $\mathcal{M}(p,p+1)$ can be found in [39].

The main motivation for the present work is to try and prove (3.20) and (3.22) by considering the (finitized) fermionic sums $F_{p,r,s}^{(L)}$, $\tilde{F}_{p,r,s}^{(L)}$ of eq. (3.7) at

3.4. Finite $L$

Using Mathematica, we have checked that for many choices of the parameters $p, r, s, L$ the $F_{p,r,s}^{(L)}(q)$ and $\tilde{F}_{p,r,s}^{(L)}(q)$ are equal and satisfy the same recursion relations and initial conditions as the ones (3.14)–(3.15) satisfied by the $C_{p,r,s}^{(L)}(q)$. These successful “experiments” encourage an attempt to prove these recursion relations analytically. At present, we have accomplished this task only for $p = 3, 4$, as we now describe.

- $p = 3$ (Ising): This case is rather simple, and in fact an analysis of the finitized sums is not really necessary for proving (3.18) at $L = \infty$. The $F_{3,r,s}^{(\infty)}(q)$ and $\tilde{F}_{3,r,s}^{(\infty)}(q)$ can be expressed as (linear combinations of) certain infinite products – see finitized versions below – through simple combinatorial considerations. The corresponding identities (see e.g. item (2) on Slater’s list [10], known as Euler’s identity), and the equality of the infinite products and the bosonic sums $B_{3, r, s}^{(\infty)}(q)$ (cf. [14]), are well known. Nevertheless, we think that the forthcoming analysis is illuminating.

There are three characters in this case, which can be taken to be $\chi_{1,1}^{(3)}$, $\chi_{1,2}^{(3)}$, and $\chi_{2,1}^{(3)}$. According to (3.7)–(3.9) we have to consider only four sums. Explicitly, they are

$$F_{3,1,1}^{(L)}(q) = \sum_{m \in 2\mathbb{Z}} q^{m^2} \left[ \frac{L+1}{2} \right]_q^m, \quad F_{3,1,2}^{(L)}(q) = \sum_{m \in 2\mathbb{Z}+L+1} q^{m(m-1)} \left[ \frac{L+2}{2} \right]_q^m$$

$$F_{3,2,1}^{(L)}(q) = \sum_{m \in 2\mathbb{Z}+1} q^{m^2-1} \left[ \frac{L+1}{2} \right]_q^m, \quad F_{3,1,2}^{(L)}(q) = \sum_{m \in 2\mathbb{Z}+L} q^{m(m-1)} \left[ \frac{L+2}{2} \right]_q^m. \quad (3.23)$$

Using (2.13), followed by a change of summation variable $(m - 1) \to m$ in one of the two resulting sums, one obtains the recursion relations

$$F_{3,1,1}^{(L)} = F_{3,1,1}^{(L-1)} + \begin{cases} 0, & q^{\frac{L+1}{2}} F_{3,2,1}^{(L-1)} \\ q^{\frac{L+1}{2}} F_{3,1,1}^{(L-1)} & \end{cases}, \quad F_{3,1,2}^{(L)} = F_{3,1,2}^{(L-1)} + \begin{cases} q^{\frac{L}{2}} F_{3,1,2}^{(L-1)} \\ 0 & \end{cases}$$

$$F_{3,2,1}^{(L)} = F_{3,2,1}^{(L-1)} + \begin{cases} 0, & q^{\frac{L-1}{2}} F_{3,1,1}^{(L-1)} \\ q^{\frac{L-1}{2}} F_{3,1,1}^{(L-1)} & \end{cases}, \quad \tilde{F}_{3,1,2}^{(L)} = \tilde{F}_{3,1,2}^{(L-1)} + \begin{cases} q^{\frac{L}{2}} \tilde{F}_{3,1,2}^{(L-1)} \\ 0 & \end{cases} \quad (3.24)$$
where the upper (lower) cases apply when $L$ is even (odd). Hence it is sufficient to supplement them by the initial conditions

$$F_{3,1,1}^{(0)} = F_{3,1,2}^{(0)} = \tilde{F}_{3,1,2}^{(0)} = 1, \quad F_{3,2,1}^{(0)} = 0,$$

which are obtained from (3.23). Comparing with (3.14)–(3.15) for $p=3$, we conclude that the unique solution to (3.23)–(3.25) is

$$F_{3;1,1}^{(L)} = C_{3;1,1}^{(L)}, \quad F_{3;1,2}^{(L)} = \tilde{F}_{3;1,2}^{(L)} = C_{3;1,2}, \quad F_{3;2,1}^{(L)} = C_{3;2,1}^{(L)},$$

which completes the proof of (3.18) for this case.

For completeness, let us write down finite product formulas for (linear combinations of) the $F_{3;r,s}^{(L)}$, which can be deduced from (3.24)–(3.25):

$$F_{3;1,1}^{(L)}(q) + q^{1/2} F_{3;2,1}^{(L)}(q) = \prod_{n=1}^{[\frac{1}{2}(L+1)]} (1 + q^{n-1/2})$$

$$F_{3;1,1}^{(L)}(q) - q^{1/2} F_{3;2,1}^{(L)}(q) = \prod_{n=1}^{[\frac{1}{2}(L+1)]} (1 - q^{n-1/2})$$

$$F_{3;1,2}^{(L)}(q) = \tilde{F}_{3;1,2}^{(L)}(q) = \prod_{n=1}^{[\frac{4}{2}L]} (1 + q^n).$$

These expressions – in the $L \to \infty$ limit – correspond to the familiar construction of the Ising CFT characters in terms of a single free massless chiral (right-moving, say) Majorana fermion, with certain projections on sectors of even or odd number of particles and periodic or anti-periodic boundary conditions. The finitization of the products here simply corresponds in this language to putting a cutoff on the allowed single-particle momenta (or energies). In fact, all the fermionic sums – both infinite and finitized – have fermionic quasi-particle interpretations which generalize the above picture, cf. [24] [27] [31] [32], but generically no “fermionic product” expressions of the type (3.27) seem to be available. It is apparently the free nature of the Majorana fermion underlying the Ising model [41] which is responsible for this further simplification.

As a final comment on the $p=3$ case, note that (3.27) immediately gives

$$F_{3;1,1}^{(L)}(1) = F_{3;2,1}^{(L)}(1) = 2^{[\frac{1}{2}(L-1)]}, \quad F_{3;1,2}^{(L)}(1) = \tilde{F}_{3;1,2}^{(L)}(1) = 2^{[\frac{1}{2}L]}.$$
This can of course be seen also directly from (3.23), as the \( q \)-binomial coefficient reduces to the ordinary binomial coefficient when \( q = 1 \), as well as from the definition of the CTM polynomials \( C_{3;r,s}(q) \) (see sect. 4).

\( \bullet \) \( p = 4 \): Using the symmetries (3.8) and the first relation in (3.9), the number of sums which must be considered in this case is reduced to nine. We take them to be \( F_{4;r,s}^{(L)} \) with \( r = 1,2,3 \) and \( s = 1,2 \), and \( \tilde{F}_{4;r,2}^{(L)} \) with \( r = 1,2,3 \), as defined by (3.7). After some tedious elementary algebra we arrive at the following recursion relations (we use (2.13), twice for cases with \( r = 2 \), followed by changes of summation variables \((m_1 - 1) \rightarrow m_1 \) or \((m_2 - 1) \rightarrow m_2 \) whenever necessary):

\[
\begin{align*}
F_{4;1,1}^{(L)} &= \begin{cases} 
F_{4;1,1}^{(L-1)} \\
F_{4;1,1}^{(L-2)} + q^{\frac{L+1}{2}} F_{4;2,1}^{(L-1)}
\end{cases}, \\
F_{4;1,2}^{(L)} &= \begin{cases} 
F_{4;1,2}^{(L-2)} + q^\frac{1}{2} F_{4;2,2}^{(L-1)} \\
\tilde{F}_{4;1,2}^{(L-1)}
\end{cases}, \\
\tilde{F}_{4;1,2}^{(L)} &= \begin{cases} 
\tilde{F}_{4;1,2}^{(L-2)} + q^\frac{1}{2} \tilde{F}_{4;2,2}^{(L-1)} \\
\tilde{F}_{4;1,2}^{(L-1)}
\end{cases}, \\
F_{4;3,1}^{(L)} &= \begin{cases} 
F_{4;3,1}^{(L-2)} + q^{\frac{L+1}{2}} F_{4;2,1}^{(L-1)} \\
F_{4;3,1}^{(L-1)}
\end{cases}, \\
F_{4;3,2}^{(L)} &= \begin{cases} 
F_{4;3,2}^{(L-2)} + q^{\frac{L+1}{2}} F_{4;2,2}^{(L-1)} \\
\tilde{F}_{4;3,2}^{(L-1)}
\end{cases}, \\
\tilde{F}_{4;3,2}^{(L)} &= \begin{cases} 
\tilde{F}_{4;3,2}^{(L-2)} + q^{\frac{L+1}{2}} \tilde{F}_{4;2,2}^{(L-1)} \\
\tilde{F}_{4;3,2}^{(L-1)}
\end{cases}
\end{align*}
\]

where the upper (lower) cases apply when \( L \) is even (odd). In fact, invoking the second relation in (3.9) one can reexpress all the \( F_{4;r,s}^{(L-2)} \) and \( \tilde{F}_{4;r,2}^{(L-2)} \) in (3.29) in terms of \( F_{4;r,s}^{(L-1)} \) and \( \tilde{F}_{4;r,2}^{(L-1)} \).

To fully (and uniquely) determine all the \( F_{4;r,s}^{(L)} \) and \( \tilde{F}_{4;r,2}^{(L)} \), it suffices to specify the following initial conditions, obtained directly from (3.7):

\[
\begin{align*}
F_{4;1,1}^{(0)} &= F_{4;1,2}^{(0)} = \tilde{F}_{4;1,2}^{(0)} = F_{4;2,2}^{(0)} = \tilde{F}_{4;2,2}^{(0)} = 1 \\
F_{4;3,1}^{(0)} &= F_{4;3,2}^{(0)} = F_{4;3,2}^{(0)} = \tilde{F}_{4;3,2}^{(0)} = 0 \\
F_{4;1,1}^{(1)} &= F_{4;1,2}^{(1)} = \tilde{F}_{4;1,2}^{(1)} = F_{4;2,2}^{(1)} = \tilde{F}_{4;2,2}^{(1)} = 1 \\
F_{4;3,1}^{(1)} &= F_{4;3,2}^{(1)} = \tilde{F}_{4;3,2}^{(1)} = 1, \quad F_{4;3,3}^{(1)} = 0 .
\end{align*}
\]
Now using (3.14) at $p=4$ (iterated, when necessary) and (3.15), we verify that

$$
\begin{align*}
F_{4;1,1}^{(L)} &= C_{4;1,1}^{(L)} , \quad F_{4;1,2}^{(L)} = \tilde{F}_{4;1,2}^{(L)} = C_{4;1,2}^{(L)} , \quad F_{4;2,1}^{(L)} = C_{4;2,1}^{(L)} , \\
F_{4;2,2}^{(L)} &= \tilde{F}_{4;2,2}^{(L)} = C_{4;2,2}^{(L)} , \quad F_{4;3,1}^{(L)} = C_{4;3,1}^{(L)} , \quad F_{4;3,2}^{(L)} = \tilde{F}_{4;3,2}^{(L)} = C_{4;3,2}^{(L)}
\end{align*}
$$

(3.32)

is a – and therefore the – solution of (3.29)–(3.31), which proves (3.18) for $p=4$.

• $p > 4$: For the general case we are unable to report complete analytic results. Deriving recursion relations for the $F_{p;r,s}^{(L)}$ and $\tilde{F}_{p;r,s}^{(L)}$, which “close” on these polynomials with fixed $s$ and lower $L$, are not so easily obtained. As examples of what we were able to show for arbitrary $p$, we list

$$
\begin{align*}
F_{p;1,1}^{(L)} &= \tilde{F}_{p;1,1}^{(L-1)} + \begin{cases} q^{(L-s+2)/2}F_{p;2,1}^{(L-1)} & \text{if } L \equiv s \pmod{2} \\
0 & \text{if } L \not\equiv s \pmod{2} \end{cases} \\
\tilde{F}_{p;1,1}^{(L)} &= F_{p;1,1}^{(L-1)} + \begin{cases} q^{(L-s+2)/2}F_{p;2,1}^{(L-1)} & \text{if } L \equiv s \pmod{2} \\
0 & \text{if } L \not\equiv s \pmod{2} \end{cases} \\
F_{p;2,2}^{(L)} &= F_{p;2,2}^{(L-2)} + q^{(L-s-3)/2}F_{p;1,1}^{(L-2)} + q^{(L-s+3)/2}F_{p;1,1}^{(L-1)} & \text{if } L \not\equiv s \pmod{2} \\
\tilde{F}_{p;2,2}^{(L)} &= \tilde{F}_{p;2,2}^{(L-2)} + q^{(L-s-3)/2}F_{p;1,1}^{(L-2)} + q^{(L-s+3)/2}F_{p;1,1}^{(L-1)} & \text{if } L \not\equiv s \pmod{2},
\end{align*}
$$

(3.33)

which can be seen to be consistent with (3.14) if (3.18) holds, but is certainly not sufficient to prove the latter.

4. Discussion

The coincidence (3.18) of finitized fermionic sum representations for Virasoro characters and finite corner-transfer-matrix sums is quite remarkable. The former have a “natural interpretation” in terms of the spectrum of gapless spin chains, while the latter arise in computations of order parameters in off-critical two-dimensional systems. Let us make some further comments on the two different types of objects which have been claimed (in some cases proven) to be equal.

As was demonstrated in [26,27] (cf. also subsec. 2.1), fermionic character sums arise in the analysis of those energy levels of a gapless spin chain hamiltonian which scale as the inverse size of the system when it becomes infinite. The analysis, which is based on the study of Bethe equations obtained from functional equations for the (diagonal-to-diagonal) transfer matrix of a corresponding critical two-dimensional lattice model, provides an interpretation of the fermionic sums in terms of massless quasi-particles whose
momenta obey certain restrictions in addition to a fermionic exclusion rule. In particular, as discussed in detail in [32], the fermionic sums of the form (3.19) for characters of the unitary Virasoro minimal model $\mathcal{M}(p, p + 1)$, are partition functions of such a “gas” of massless fermionic quasi-particles. The single-particle momenta of these quasi-particles are quantized in spacing of $2\pi/L$, where $L$ is the size of the system which serves as an infrared cutoff. These momenta are also restricted in a way (different for different characters) which depends on the number $m_a$ of quasi-particles of type $a=1, 2, \ldots, n$ in the state. The feature common to all fermionic sums (3.19) for a given model $\mathcal{M}(p, p + 1)$ (where $n=p-2$), is that for finite $m_a$ the single-particle momenta of the $a=1$ quasi-particles can take values in a semi-infinite range (which becomes the positive or negative half of the real axis as $L\to\infty$, corresponding to either right- or left-moving quasi-particles), whereas the momenta of all other quasi-particles are restricted to a finite range (which shrinks to 0 when the size of the system becomes infinite).

Finitizing the fermionic sums, namely considering the $L$-dependent polynomials $F^{(L)}_{p;r,s}(q)$ of (3.7) instead of the $q$-series (3.20), has the effect of further restricting the momenta of the $a=1$ quasi-particles to lie in a finite range, whose upper limit depends on $L$. This procedure can be thought of as introducing an ultraviolet cutoff, which is of order $L^0=1$ (i.e. of order $L$ in the quantization unit $2\pi/L$). This is not to be confused with the “original” ultraviolet cutoff – the distance between neighboring sites along the chain – which is present in the problem of studying the spectrum of the spin chain. It is important to note that the effect of the built-in chain cutoff is to restrict the momentum to some (periodic) Brillouin zone, in which the dependence of the single-particle energies on the momentum is nontrivial. On the other hand, the ultraviolet cutoff implied by the finitization of the fermionic character sums has the appealing property of not changing the linear dispersion relation of the quasi-particle excitations which survive the (conformal) continuum limit.

Now the CTM polynomials $C^{(L)}_{p;r,s}(q)$, eq. (3.12), are obtained from the sums (3.10) which arise in the analysis [38] of the $r=p+1$ RSOS model (in regime III). In this context $L$ is the length of the corner edge on which the corner transfer matrix acts, namely $L$ is half the diagonal of the big square on which the model is defined, this big square being tilted by $45^0$ with respect to the orientation of the square plaquettes of the lattice. (Hence, if we envisage the spectrum of the corner transfer matrix as built out of some (fictitious) “CTM excitations”, then $L$ serves as an infrared cutoff for them.) It is interesting to count
the number of states which are summed over to give the CTM polynomials, namely to
evaluate these polynomials at $q=1$. They are simply obtained from (3.10) as

$$C^{(L)}_{p,r,s}(1) = \begin{cases} ((I_p)^L)_{s,r} & \text{if } L \equiv s - r \pmod{2} \\ ((I_p)^L)_{s,r+1} & \text{if } L \not\equiv s - r \pmod{2} \end{cases} = ((I_p)^L)_{s,r} + ((I_p)^L)_{s,r+1}, \quad (4.1)$$

where $I_p$ is the incidence matrix of $A_p$ (cf. (3.1)), which enforces the RSOS restrictions
$h_i \in \{1, \ldots, p\}$, $|h_i - h_{i+1}| = 1$ in (3.10). (The second equality in (4.1) is valid since
$((I_p)^L)_{r,s} = 0$ if $(L + s + r)$ is odd.) Our claim (3.18) then implies, using the definitions
(3.7), (3.2), and (3.17), the following infinite set of identities for the (ordinary) binomial
coefficients $\binom{n}{m}$:

$$\sum Q_{r,s} p - 2 \prod_{a=1}^{p-2} \left( \frac{1}{2} (mI_{p-2} + u_{r,s} + Le_1) \right)_{m_a} = \sum \tilde{Q}_{p-r,p+1-s} p - 2 \prod_{a=1}^{p-2} \left( \frac{1}{2} (mI_{p-2} + \tilde{u}_{p-r,p+1-s} + Le_1) \right)_{m_a}$$

$$= \sum k \in \mathbb{Z} \left( \frac{L+s-r-1}{2} - k(p+1) \right) - \left( \frac{L-s-r-1}{2} - k(p+1) \right) = ((I_p)^L)_{s,r+1} \quad \text{for } (L + s + r) \text{ odd} \quad (4.2)$$

$$\sum Q_{r,s} p - 2 \prod_{a=1}^{p-2} \left( \frac{1}{2} (mI_{p-2} + \tilde{u}_{r,s} + Le_1) \right)_{m_a} = \sum Q_{p-r,p+1-s} p - 2 \prod_{a=1}^{p-2} \left( \frac{1}{2} (mI_{p-2} + u_{p-r,p+1-s} + Le_1) \right)_{m_a}$$

$$= \sum k \in \mathbb{Z} \left( \frac{L+s-r-1}{2} - k(p+1) \right) - \left( \frac{L-s-r-1}{2} - k(p+1) \right) = ((I_p)^L)_{s,r} \quad \text{for } (L + s + r) \text{ even},$$

where $\sum Q$ indicates summation over $m \in 2\mathbb{Z}^{p-2} + Q$, $L$ is a non-negative integer, $p \geq 3$
an integer, $r=1, \ldots, p-1$, $s=1, \ldots, p$, and the various vectors are defined in (3.3). We
note that when $p=3$ the elements of the matrix $(I_3)^L$ can be simplified, as in (3.28), and
that our analysis in subsect. 3.4 also provides a proof of (4.2) for $p=4$.

Identities similar to (4.2) for the case of $p=5$ were encountered in [42] in the analysis
of completeness of the solutions to the Bethe equations for the three-state Potts spin chain.

17
In particular, one of the results proven there (eq. (5.12) of [42]) reads in our notation

\[ F_{5;1}^{(L)}(1) + F_{5;1,5}^{(L)}(1) = \sum_{m_1 + m_3 \in \mathbb{Z}} \left( \frac{1}{2} \left( L + m_2 \right) \right) \left( \frac{1}{2} \left( m_1 + m_3 \right) \right) \left( \frac{2 m_2}{m_3} \right) = 3^{\frac{L}{2} - 1}, \] (4.3)

for \( L \) even. Our conjectured identities (4.2) can be seen to be consistent with this, as

\((I_5)^L_{1,1} + (I_5)^L_{5,1} = \frac{1}{2} (3^{\frac{L}{2} - 1} + 1) + \frac{1}{2} (3^{\frac{L}{2} - 1} - 1) = 3^{\frac{L}{2} - 1} \) for \( L \) even.

It should be noted, though, that when analyzing completeness of the Bethe-equation solutions, all excitations (involving both left- and right-moving quasi-particles) are included. Each finitized character sum, on the other hand, describes just a certain sector of one chiral half of the theory. Recalling the factorization (2.4) of the CFT partition function into characters, we are faced with an intriguing question of whether there is some “natural” finitization of the partition function as a whole, where the building blocks are the finitized characters. In the case \( p=3 \) we can construct the following two finitized partition functions, corresponding to the Ising model and the theory of a free Majorana fermion with antiperiodic boundary conditions (we take \( q \in \mathbb{R} \) here):

\[ Z_{\text{Ising}}^{(L)}(q) = q^{-1/24} \sum_{s=1}^{3} q^{2 \Delta_{3;1,s}^{(3)}} F_{3;1,s}^{(L-1)}(q) F_{3;1,s}^{(L)}(q), \] (4.4)

\[ Z_{\text{Maj}}^{(L)}(q) = q^{-1/24} (F_{3;1,1}^{(L-1)}(q) + q^{1/2} F_{3;2,1}^{(L-1)}(q)) (F_{3;1,1}^{(L)}(q) + q^{1/2} F_{3;2,1}^{(L)}(q)), \]

where \( \Delta_{3;1,s}^{(3)} = 0, \frac{1}{16}, \) and \( \frac{1}{2} \) for \( s=1,2, \) and 3, respectively. One reason why we regard these specific finitizations to be natural is the existence – for \( L \) even – of the following neat product forms, which can be obtained from (3.27):

\[ Z_{\text{Ising}}^{(L)}(q) = q^{-\frac{L+1}{2}} \left\{ \prod_{n=-\frac{L-1}{2}}^{\frac{L-1}{2}} \left( 1 + q^{|n|} \right) + \prod_{n=-\frac{L-1}{2}}^{\frac{L-1}{2}} \left( 1 - q^{|n|} \right) + q^{\frac{L}{8}} \prod_{n=-\frac{L-1}{2}}^{\frac{L}{4}} \left( 1 - q^{|n|} \right) \right\}, \] (4.5)

\[ Z_{\text{Maj}}^{(L)}(q) = q^{-\frac{L}{2}} \prod_{n=-\frac{L-1}{2}}^{\frac{L-1}{2}} (1 + q^{|n|}), \]

where the product variable \( n \) is incremented in steps of one. Another nice property, valid for either parity of \( L \), is

\[ Z_{\text{Ising}}^{(L)}(1) = Z_{\text{Maj}}^{(L)}(1) = 2^L, \] (4.6)

which is the total number of states of an Ising/Majorana spin chain of \( L \) sites.
We emphasize that from the point of view of the RSOS model per se, the “completeness rule” ([1,6]) is somewhat surprising: recall that the $F_{3; r,s}^{(L)}$ were in fact constructed to be equal to the CTM polynomials ([3,12]) which arise in the RSOS model of [38], at $r=4$ in this case. The origin of this rule lies in the fact that for $r=4$ the system is equivalent [43] to (two decoupled copies of the) Ising model, and so the heights $h_i \in \{1, 2, 3\}$, which are restricted by $|h_i - h_{i+1}|=1$, cf. (3.10), can be traded in for the unrestricted two-valued Ising spins.

There is another case where a similar phenomenon occurs, namely at $r=6$. Here an orbifold construction [44] leads to the $D_4$ RSOS model of Pasquier [45], which is equivalent at criticality to two decoupled copies of the three-state Potts model describing unrestricted three-valued “spins”. We are therefore tempted to write down the obvious generalization of $Z_{\text{Ising}}^{(L)}$ to the $\mathbb{Z}_3$ case, obtained by finitizing the partition function of the three-state Potts CFT [22]:

$$Z_{3sP}^{(L)}(q) = q^{-1/15} \left\{ (F_{5; 1,1}^{(L-1)}(q) + q^3 F_{5; 4,1}^{(L-1)}(q))(F_{5; 1,1}^{(L)}(q) + q^3 F_{5; 4,1}^{(L)}(q)) \\
+ q^{4/5} (F_{5; 2,1}^{(L-1)}(q) + q F_{5; 3,1}^{(L-1)}(q))(F_{5; 2,1}^{(L)}(q) + q F_{5; 3,1}^{(L)}(q)) \\
+ 2(q^{2/15} F_{5; 2,3}^{(L-1)}(q) F_{5; 2,3}^{(L)}(q) + q^{4/3} F_{5; 1,3}^{(L-1)}(q) F_{5; 1,3}^{(L)}(q)) \right\}. \tag{4.7}$$

Amusingly, using (4.2), we find for all $L$

$$Z_{3sP}^{(L)}(1) = \{3^{L-1} + 2 \cdot 3^{L-1}\} = 3^L, \tag{4.8}$$

with each sector of definite $\mathbb{Z}_3$ charge contributing equally. On the other hand, the finitized version of the partition function of the tetracritical Ising CFT [22] is

$$Z_{\text{tetra}}^{(L)}(q) = q^{-1/15} \sum_{r=1}^{2} \sum_{s=1}^{5} q^{2\Delta_{r,s}^{(5)}} F_{5; r,s}^{(L-1)}(q) F_{5; r,s}^{(L)}(q), \tag{4.9}$$

where the dimensions $\Delta_{r,s}^{(5)}$ are given by (2.7), and we find that

$$Z_{\text{tetra}}^{(L)}(1) = 3^L + 1. \tag{4.10}$$

The above observations can be generalized further to the whole $ADE$-series [23] of modular-invariant partition functions of the unitary minimal models $\mathcal{M}(p, p+1)$. Namely, consider the corresponding finitized partition functions

$$Z_{p(X)}^{(L)}(q) = q^{-c(p)/12} \sum_{r, \bar{r}=1}^{p-1} \sum_{s, \bar{s}=1}^{p} \sum_{r,s; \bar{r}, \bar{s}} N_{r,s; \bar{r}, \bar{s}}^{p(X)} q^{\Delta_{r,s}^{(p)} + \Delta_{\bar{r},\bar{s}}^{(p)}} F_{p; r,s}^{(L-1)}(q) F_{p; \bar{r},\bar{s}}^{(L)}(q). \tag{4.11}$$
Here the multiplicities $N_{r,s;\bar{r},\bar{s}}^p(X)$, corresponding to the partition function in the $X=A,D,E$ series (so that $X=A$ for $p=3,4$, $X=A$ or $D$ for $p \geq 5$, and for $p=11,12,17,18,29,30$ also $X=E$ is possible), are given in [23]. Remarkably, we find the following “completeness rules” for arbitrary $L$:

\[
Z_{p(A)}^{(L)}(1) = \frac{1}{2} \text{Tr} \left( I_{p(A)} \right)^{2L} \quad \text{for } p = 3, 4, 5, \ldots ,
\]

\[
Z_{p(D)}^{(L)}(1) = \frac{1}{2} \text{Tr} \left( I_{D(p+3)/2} \right)^{2L} \quad \text{for } p = 5, 7, 9, \ldots ,
\]

\[
Z_{p(E)}^{(L)}(1) = \frac{1}{2} \text{Tr} \left( I_{E_n} \right)^{2L} \quad \text{with } n = 6, 7, 8 \text{ for } p = 11, 17, 29, \text{ resp.},
\]

where $I_X$ is the incidence matrix of the Dynkin diagram of the Lie algebra $X$, whose rank is such that its Coxeter number is $p+1$. (The traces can be evaluated explicitly using the known eigenvalues of the $I_X$, which are given by $2 \cos \frac{\pi m}{g}$ where the $m$ and $g$ are the Coxeter exponents and Coxeter number of $X$, respectively; using this fact, the particularly simple rhs’s of (4.6), (4.8), and (4.10) can easily be seen to follow from (4.12).) The general $A_p$ case of these identities can be proved using (4.1), assuming (3.18) holds. For the $D$- and $E$-cases we conjecture them based on their validity for many small values of $p$ and $L$, which we have verified. [One can also finitize the fermionic semi-modular-invariant partition functions of [46], using a generalization of (4.11). We denote the resulting functions by $Z_{p(f)}^{(L)}(q)$, where the $N_{r,s;\bar{r},\bar{s}}^{p(f)}$ are read off from the ‘fermionic $D$-series’ expressions in eq. (5.2) of [46], so that $Z_{3(f)}^{(L)}(q) = Z_{Maj}^{(L)}(q)$ of (4.4) and (4.5). Results for small $p$ and $L$ lead us to conjecture that $Z_{p(f)}^{(L)}(1) = Z_{p(D)}^{(L)}(1)$ for all odd $p \geq 5$, generalizing (1.6).]

Note that the rhs’s in (4.12) are precisely half the number of allowed height configurations along a row of $2L$ sites (with periodic boundary conditions $h_{2L+1} = h_1$) of the RSOS models [45] based on the Dynkin diagrams of $A_p$, $D_{(p+3)/2}$ and $E_n$. Since the lhs’s are defined in terms of conformal field theory characters, which have been “finitized” via a procedure inspired by a quasi-particle description of the spectrum of spin chains, the identities (4.12) reflect an intriguing connection between all these frameworks.

Finally, let us return to the initial motivation for the present work, namely proving the conjectured [27] [32] fermionic representations (3.20) (and their extensions (3.22)) for the Virasoro characters of $\mathcal{M}(p,p+1)$. As demonstrated, we have achieved this goal for $p=4$ using the finitization procedure. However, for bigger $p$ this procedure seems to be increasingly cumbersome and we have not been able yet to fully carry it out and obtain a proof.
As a possible hint for an alternative way of attacking the problem, which does not require finitization of the \( q \)-series involved, we would like to draw the reader’s attention to an interesting feature of the fermionic \( q \)-series on the rhs’s of (3.20) and (3.22): in all cases the vector \( \mathbf{A} \), specifying the linear shift of the quadratic form in the exponent of \( q \), is either zero or a unit vector. (In fact, the extension (3.22) of the conjecture (3.20) of [32] was arrived at by searching for possible right choices of \( \mathbf{Q}, \mathbf{u}, \) and \( \mathbf{A} \), allowing for other types of vectors; our – clearly not exhaustive – search did not yield any viable fermionic sum representations where \( \mathbf{A} \) is not a unit vector.) This observation suggests that we consider the following formal series in \( n \) variables \( z_a \) in addition to \( q \),

\[
S_n^\mathbf{Q}(\mathbf{u}|q,z) = \sum_{\mathbf{m} \in \mathbb{Z}^n + \mathbf{Q}} q^{\frac{1}{2}m \mathbf{C}_n \mathbf{m} \mathbf{t}} \left( \prod_{a=1}^{n} z_a^{\frac{1}{2} m_a} \right) \frac{1}{(q)_{m_1}} \prod_{a=2}^{n} \left[ \frac{1}{2} (m I_n + \mathbf{u})_a \right]_q ,
\]

which reduces to \( S_n^\mathbf{A}^Q(\mathbf{u}|q) \) of (3.19) when specializing to \( z_a = q^{A_a} \) for \( a=1, \ldots, n \). When inserted into (3.20) and (3.22), this means that all the \( z_a \) are in fact evaluated at 1, except possibly for one which is evaluated at \( q \), since \( \mathbf{A}_s = \mathbf{e}_{n+2-s} \). The series \( S_n^\mathbf{Q}(\mathbf{u}|q,z) \) are then generalizations of

\[
S(q,z) = \sum_{m=0}^{\infty} \frac{q^{m^2 z_m}}{(q)_m} ,
\]

giving the two fermionic sums with \( a=0 \) and 1 in (1.1) when evaluated at \( z=1 \) and \( z=q \), respectively. This function, which satisfies a linear second order \( q \)-difference equation, plays an important role in the proofs [2][3] of the Rogers-Ramanujan-Schur identities (1.1).

**Acknowledgements**

I would like to thank G. Albertini, R. Kedem, T.R. Klassen, and B.M. McCoy for useful discussions. This work was supported by the NSF, grant 91-08054.
References

[1] L.J. Rogers, Proc. London Math. Soc. (ser. 1) 25 (1894) 318; (ser. 2) 16 (1917) 315.
[2] L.J. Rogers and S. Ramanujan, Proc. Camb. Phil. Soc. 19 (1919) 211.
[3] I. Schur, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1917) 302.
[4] I. Schur, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. (1926) 488.
[5] B. Gordon, Amer. J. Math. 83 (1961) 363; G.E. Andrews, Proc. Nat. Sci. USA 71 (1974) 4082.
[6] G.E. Andrews, The Theory of Partitions (Addison-Wesley, London, 1976).
[7] R.J. Baxter, J. Stat. Phys. 26 (1981) 427.
[8] Vertex Operators in Mathematics and Physics, ed. J. Lepowsky, S. Mandelstam and I.M. Singer (Springer, Berlin, 1985).
[9] M. Okado, M. Jimbo and T. Miwa, Sugaku Expositions 2 (1989) 29.
[10] R. Kedem, B.M. McCoy and E. Melzer, Stony Brook preprint, [hep-th/9304056], to appear in C.N. Yang’s 70th birthday Festschrift, ed. S.T. Yau.
[11] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
[12] D. Friedan, Z. Qiu and S.H. Shenker, Phys. Rev. Lett. 52 (1984) 1575.
[13] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105.
[14] A. Rocha-Caridi, in [8].
[15] V.G. Kac, Infinite dimensional Lie algebras, third edition (Cambridge University Press, 1990);
A.J. Feingold and J. Lepowsky, Adv. in Math. 29 (1978) 271;
B.L Feigin and D.B. Fuchs, Funct. Anal. Appl. 17 (1983) 241;
V.G. Kac and D.H. Peterson, Adv. in Math. 53 (1984) 125;
M. Jimbo and T. Miwa, Adv. Stud. in Pure Math. 4 (1984) 97;
V.G. Kac and M. Wakimoto, Adv. in Math. 70 (1988) 156.
[16] P. Bouwknegt, J. McCarthy and K. Pilch, Nucl. Phys. B352 (1991) 139.
[17] G. Felder, Nucl. Phys. B317 (1989) 215.
[18] J. Lepowsky and M. Primc, Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$, Contemporary Mathematics, Vol. 46 (AMS, Providence, 1985).
[19] J. Lepowsky and R.L. Wilson, Proc. Nat. Acad. Sci. USA 78 (1981) 7254; Adv. in Math. 45 (1982) 21; Invent. Math. 77 (1984) 199.
[20] B.L. Feigin, T. Nakanishi and H. Ooguri, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 217
[21] H.W.J. Blôte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; I. Affleck, Phys. Rev. Lett. 56 (1986) 746.
[22] J.L. Cardy, Nucl. Phys. B270 (1986) 186.
[23] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987) 445;
D. Gepner, Nucl. Phys. B287 (1987) 111.
[24] C. Itzykson and J.-B. Zuber, Nucl. Phys. B275 (1986) 580; D.Z. Freedman and K. Pilch, Ann. Phys. 192 (1989) 331.
[25] A.E. Ferdinand and M.E. Fisher, Phys. Rev. 185 (1969) 832.
[26] R. Kedem and B.M. McCoy, Stony Brook preprint, hep-th/9210129. J. Stat. Phys. (in press).
[27] S. Dasmahapatra, R. Kedem, B.M. McCoy and E. Melzer, Stony Brook preprint, hep-th/9304150.
[28] S. Dasmahapatra, R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Stony Brook/Rutgers preprint, hep-th/9303013, to appear in the proceedings of “Yang-Baxter Equations in Paris”, ed. J.-M. Maillard (World Scientific).
[29] B. Richmond and G. Szekeres, J. Austral. Math. Soc. (ser. A) 31 (1981) 362; W. Nahm, A. Recknagel and M. Terhoeven, Bonn preprint, hep-th/9211034.
[30] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Phys. Lett. B304 (1993) 263 (hep-th/9211102).
[31] M. Terhoeven, Bonn preprint, hep-th/9111120; A. Kuniba, T. Nakanishi, J. Suzuki, Harvard preprint, hep-th/9301018.
[32] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, Stony Brook/Rutgers preprint, hep-th/9301046, Phys. Lett. B (in press).
[33] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[34] O. Foda, Nijmegen preprint (1991); K.A. Seaton and B. Nienhuis, Nucl. Phys. B384 (1992) 507.
[35] H. Saleur and M. Bauer, Nucl. Phys. B320 (1989) 591.
[36] J.L. Cardy, Nucl. Phys. B275 (1986) 200.
[37] G.E. Andrews, Scripta Math. 28 (1970) 297.
[38] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193.
[39] P. Christe, Int. J. Mod. Phys. A6 (1991) 5271; J. Kellendonk, M. Rössgen and R. Varnhagen, Bonn preprint, hep-th/9301086.
[40] L.J. Slater, Proc. London Math. Soc. (ser. 2) 54 (1951-52) 147.
[41] B. Kaufman, Phys. Rev. 76 (1949) 1232.
[42] G. Albertini, S. Dasmahapatra and B.M. McCoy, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 1.
[43] D.A. Huse, Phys. Rev. B30 (1984) 3908.
[44] P. Fendley and P. Ginsparg, Nucl. Phys. B324 (1989) 549.
[45] V. Pasquier, Nucl. Phys. B285 (1987) 162; J. Phys. A20 (1987) L217.
[46] T.R. Klassen and E. Melzer, Cornell/Stony Brook preprint, hep-th/9206114, Int. J. Mod. Phys. A (in press).