A Generalization of the One-Dimensional Boson-Fermion Duality Through the Path-Integral Formalism

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Abstract

We study boson-fermion dualities in one-dimensional many-body problems of identical particles interacting only through two-body contacts. By using the path-integral formalism as well as the configuration-space approach to indistinguishable particles, we find a generalization of the boson-fermion duality between the Lieb-Liniger model and the Cheon-Shigehara model. We present an explicit construction of \( n \)-boson and \( n \)-fermion models which are dual to each other and characterized by \( n - 1 \) distinct (coordinate-dependent) coupling constants. These models enjoy the spectral equivalence, the boson-fermion mapping, and the strong-weak duality. We also discuss a scale-invariant generalization of the boson-fermion duality.
1 Introduction

In his seminal paper [1] in 1960, Girardeau proved the one-to-one correspondence—the duality—between one-dimensional spinless bosons and fermions with hard-core interparticle interactions. By using this duality, he presented a celebrated example of the spectral equivalence between impenetrable bosons and free fermions. Since then, the one-dimensional boson-fermion duality has been a testing ground for studying strongly-interacting many-body problems, especially in the field of integrable models. So far there have been proposed several generalizations of the Girardeau’s finding, the most prominent of which was given by Cheon and Shigehara in 1998 [2]: they discovered the fermionic dual of the Lieb-Liniger model [3] by using the generalized pointlike interactions. The duality between the Lieb-Liniger model and the Cheon-Shigehara model is a natural generalization of [1] and consists of (i) the spectral equivalence between the bosonic and fermionic systems, (ii) the one-to-one mapping between bosonic and fermionic wavefunctions, and (iii) the one-to-one correspondence between a strong-coupling regime in one system and a weak-coupling regime in the other (i.e., the strong-weak duality). The purpose of the present paper is to derive and further generalize this duality by using the path-integral formalism. Before going into details, however, let us first briefly recall the basics of the boson-fermion duality by simplifying the argument of [2].

Let us consider $n$ identical particles which have no internal structures (i.e., spinless), move on the whole line $\mathbb{R}$, and interact through two-body contact interactions but otherwise freely propagate in the bulk. The Lieb-Liniger model and the Cheon-Shigehara model are particular examples of such $n$-body systems and described by the following Hamiltonians, respectively:

$$
H_B = -\frac{\hbar^2}{2m} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{m} \sum_{1<j<k<n} \delta(x_j - x_k; \frac{1}{a}),
$$

(1a)

$$
H_F = -\frac{\hbar^2}{2m} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{m} \sum_{1<j<k<n} \epsilon(x_j - x_k; a),
$$

(1b)

where $m$ is the mass of the particles and $x_j$ is the coordinate of the $j$th particle. Here $\delta(x; \frac{1}{a}) = \frac{1}{a} \delta(x)$ is the $\delta$-function potential with $a$ being a real constant that has the dimension of length and $\epsilon(x; a)$ is the so-called $\epsilon$-function potential defined by a limit of a particular linear combination of the $\delta$-functions [4]. To see the duality, however, there is no need to know the precise definition of $\epsilon(x; a)$ because both the $\delta$- and $\epsilon$-function potentials are prescribed by connection conditions for wavefunctions. In other words, the $n$-body systems described by the Hamiltonians (1a) and (1b) are equivalently described by the bulk free Hamiltonian $H_0 = -\frac{\hbar^2}{2m} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ with some connection conditions at the coincidence points $x_j = x_k$. For example, the $n$-body Hamiltonian (1a) is described by the bulk free Hamiltonian $H_0$ with the following connection conditions for the $\delta$-function potential $\delta(x_j - x_k; \frac{1}{a})$:

$$
\frac{\partial \psi_B}{\partial x_{jk}}|_{x_{jk}=0} - \frac{\partial \psi_B}{\partial x_{jk}}|_{x_{jk}=0} - \frac{1}{a} \left( \psi_B|_{x_{jk}=0} + \psi_B|_{x_{jk}=0} \right) = 0,
$$

(2a)

$$
\psi_B|_{x_{jk}=0} - \psi_B|_{x_{jk}=0} = 0,
$$

(2b)

where $x_{jk} = x_j - x_k$ and $\frac{\partial}{\partial x_{jk}} = \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}$. On the other hand, the $n$-body Hamiltonian (1b) is described by the bulk free Hamiltonian $H_0$ with the following connection conditions for the $\epsilon$-function potential $\epsilon(x_j - x_k; a)$ [2, 4]:

$$
\frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} - \frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} - a \left( \frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} + \frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} \right) = 0,
$$

(3a)

$$
\frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} - \frac{\partial \psi_F}{\partial x_{jk}}|_{x_{jk}=0} = 0.
$$

(3b)

Yet another pseudo-potential realization of the connection conditions (3a) and (3b) was discussed in [5]. Note that the contact interaction described by (3a) and (3b) is also widely called the "$\delta'$-interaction" in the literature; see, e.g., [6].
Notice that \( \psi_B \) and its derivatives become continuous in the limit \( 1/a \rightarrow 0 \), which indicates that \( 1/a \) describes the deviation from the smooth continuous theory (i.e., the free theory) and plays the role of a coupling constant in the Lieb-Liniger model (1a). In contrast, \( \psi_F \) and its derivatives become continuous in the limit \( a \rightarrow 0 \), which indicates that \( a \) describes the deviation from the free theory and plays the role of a coupling constant in the Cheon-Shigehara model (1b). This inverse relation of the coupling constants is already incorporated into the notations \( \delta(x; \frac{1}{a}) \) and \( \epsilon(x; a) \) and the heart of the strong-weak duality.

Now, all the above connection conditions are valid for generic wavefunctions without any symmetry. However, further simplifications occur if the wavefunctions are totally symmetric (antisymmetric) under the exchange of coordinates, which must hold for identical spinless bosons (fermions) due to the indistinguishability of identical particles in quantum mechanics. For example, if \( \psi_B \) is the totally symmetric function and satisfies the identity \( \psi_B(\ldots, x_j, \ldots, x_k, \ldots) = \psi_B(\ldots, x_k, \ldots, x_j, \ldots) \), there automatically hold the additional conditions \( \psi_B|_{x_k=0} = \psi_B|_{x_k=0} \) and \( \frac{\partial \psi_B}{\partial x_k}|_{x_k=0} = 0 \), which reduce (2a) to \( \frac{\partial \psi}{\partial x_k}|_{x_k=0} = 0 \). Similarly, if \( \psi_F \) is the totally antisymmetric function and satisfies the identity \( \psi_F(\ldots, x_j, \ldots, x_k, \ldots) = -\psi_F(\ldots, x_k, \ldots, x_j, \ldots) \), there automatically hold the additional conditions \( \psi_F|_{x_k=0} = -\psi_F|_{x_k=0} \) and \( \frac{\partial \psi_F}{\partial x_k}|_{x_k=0} = 0 \), which reduce (3a) to \( \psi_F|_{x_k=0} = 0 \). Putting all these things together, one immediately sees that the systems of \( n \) identical bosons and fermions described by (1a) and (1b) are both described by the bulk free Hamiltonian \( H_0 \) together with the following Robin boundary conditions at the coincidence points:

\[
\frac{\partial \psi_B}{\partial x_k}|_{x_k=0} - \frac{1}{a} \psi_B|_{x_k=0} = 0.
\]

Since both systems are described by the same bulk Hamiltonian and the same boundary conditions, they automatically become isospectral. This is the boson-fermion duality in one dimension, which consists of the spectral equivalence between \( H_B \) and \( H_F \), the equivalence between the strong coupling regime of \( H_B/F \) and the weak coupling regime of \( H_B/B \), and, as we will see in section 2.3, the one-to-one mapping between \( \psi_B \) and \( \psi_F \). Note that the extreme case \( a \rightarrow 0 \) corresponds to the simplest duality between the impenetrable bosons and the free fermions.

Now it is obvious from the above discussion that the one-dimensional boson-fermion duality just follows from simple connection condition arguments. However, it took more than thirty years to arrive at the above findings since the discovery of the simplest duality by Girardeau. One reason for this would be the lack of a systematic derivation of the dual contact interactions for identical bosons and fermions. The purpose of the present paper is to fill this gap and to present a systematic machinery for deriving (and generalizing) the boson-fermion duality. As we will see in the rest of the paper, this purpose is achieved by using the path-integral formalism as well as the configuration-space approach to identical particles [7–10].

The paper is organized as follows. In section 2 we first present the basics of the configuration-space approach to identical particles following the argument of Leinaas and Myrheim [10]. In this approach, the indistinguishability of identical particles is incorporated into the configuration space rather than the permutation symmetry of multiparticle wavefunctions. We first see that the configuration space of \( n \) identical particles on \( \mathbb{R} \) is given by the orbit space \( \mathcal{M}_n = (\mathbb{R}^n - \Lambda_n)/S_n \), where \( \Lambda_n \) is the set of coincidence points of two or more particles and \( S_n \) the symmetric group. We see that this space has a number of nontrivial boundaries, where \( k \)-body contact interactions take place at codimension-(\( k - 1 \)) boundaries. We then see that, irrespective of the particle statistics, two-body contact interactions are generally described by the Robin boundary conditions at the codimension-1 boundaries of \( \mathcal{M}_n \). We also discuss that the boson-fermion mapping holds irrespective of the boundary conditions. In section 3 we study the Feynman kernel for identical particles from the viewpoint of path integral. We will show that, by using the Feynman kernel on \( \mathcal{M}_n \), the boson-fermion duality between (1a) and (1b) is generalized to the duality between the models described by the Hamiltonians (49a) and (49b). In section 4 we summarize our results with possible future directions and discuss a scale-invariant generalization of the boson-fermion duality. Appendices A and B present proofs of some mathematical formulae.
2 Identical particles in one dimension

One of the principles of quantum mechanics is the indistinguishability of identical particles. There are two main approaches to implement this principle into a theory. The first is to consider the permutation symmetry of multiparticle wavefunctions. The second is to restrict the multiparticle configuration space by identifying all the permuted points. As we will see below, the one-dimensional boson-fermion duality is best described by the second approach. In order to fix the notations, however, let us first start with the first approach.

Let us consider $n$ identical particles moving on the whole line $\mathbb{R}$. Let us label each particle with a number $j \in \{1, \ldots, n\}$ and let $x_j \in \mathbb{R}$ be a spatial coordinate of the $j$th particle. Let $\sigma \in S_n$ be a permutation of $n$ indices that acts on a multiparticle wavefunction $\psi(x) = \psi(x_1, \ldots, x_n)$ as follows:

$$\sigma : \psi(x) \mapsto \psi(\sigma x),$$

where $\sigma x$ is defined as $\sigma x := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Note that this definition satisfies the identity $\sigma(\sigma') x = (\sigma \sigma') x = (x_{\sigma(\sigma'(1))}, \ldots, x_{\sigma(\sigma'(n))})$ for any $\sigma, \sigma' \in S_n$. The indistinguishability of identical particles then implies that the multiparticle wavefunctions $\psi(x)$ and $\psi(\sigma x)$ are physically equivalent; that is, the probability densities for these configurations must be the same. Thus we have

$$|\psi(\sigma x)|^2 = |\psi(x)|^2.$$

In other words, $\psi(\sigma x)$ and $\psi(x)$ must be identical up to a phase factor. Hence,

$$\psi(\sigma x) = \chi(\sigma) \psi(x),$$

where $\chi(\sigma) \in U(1)$ is a phase that may depend on $\sigma$. In addition, the identity $\psi(\sigma(\sigma') x) = \psi((\sigma \sigma') x)$ implies that $\chi$ must preserve the group multiplication law $\chi(\sigma) \chi(\sigma') = \chi(\sigma \sigma')$ for any $\sigma, \sigma' \in S_n$; that is, the map $\chi : S_n \mapsto U(1)$ must be a one-dimensional unitary representation of $S_n$. As is well-known, there are just two such representations of $S_n$, one is the totally symmetric representation (i.e., the trivial representation) $\chi^{[B]}$ and the other the totally antisymmetric representation (i.e., the sign representation) $\chi^{[F]}$, both of which are simply given by

$$\chi^{[B]}(\sigma) = 1,$$

$$\chi^{[F]}(\sigma) = \text{sgn}(\sigma).$$

Here $\text{sgn}(\sigma)$ stands for the signature of $\sigma$ and is given by $\text{sgn}(\sigma) = 1$ for even permutations and $\text{sgn}(\sigma) = -1$ for odd permutations. Of course, the totally symmetric representation $\chi^{[B]}$ corresponds to the Bose-Einstein statistics and the totally antisymmetric representation $\chi^{[F]}$ of the Fermi-Dirac statistics. In this way, the particle statistics is determined by the one-dimensional unitary representation of the symmetric group $S_n$.

There is another, more geometrical approach to identical particles, which was introduced independently by Souriau [7, 8] and by Laidlaw and DeWitt [9] and then thoroughly investigated by Leinaas and Myrheim [10] (see also the introduction of [14] for a nice pedagogical review). In this approach, the indistinguishability of identical particles is built into the configuration space by identifying all the permuted configurations of identical particles. More precisely, given a one-particle configuration space $X$, the configuration space of $n$ identical particles is generally given by first taking the Cartesian product of $n$ copies of $X$, and then subtracting the coincidence points of two or more particles, and then identifying points under the action of every permutation. The resulting space is the orbit space

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2 We here assume that $\chi(\sigma)$ is independent of the coordinates. This assumption excludes, for example, the anyon exchange relations in [11, 12]. For simplicity, we will not touch upon coordinate-dependent particle-exchange phases in the present paper. Note that a path-integral approach to one-dimensional anyons was discussed in [13].

3 This subtraction procedure is the heart of the configuration-space approach and is based on the assumption that two or more particles cannot occupy the same point simultaneously [9]. This assumption may be arguable, but it successfully leads to the braid-group statistics in two dimensions and the Bose-Fermi alternative in three and higher dimensions.
\( \mathcal{M}_n = (X^n - \Delta_n)/S_n \), where \( X^n \) is the Cartesian product of \( X \) and \( \Delta_n \) the set of coincidence points. In this setting, a particle exchange is no longer described by the permutation symmetry of multiparticle wavefunctions because all the permuted configurations are identified in \( \mathcal{M}_n \). Instead, it is described by dynamics: identical particles are said to be exchanged if they start from an initial point in \( \mathcal{M}_n \) and then return to the same point in \( \mathcal{M}_n \) in the course of the time-evolution. Clearly, such an exchange process corresponds to a closed loop in the configuration space. In addition, if \( \mathcal{M}_n \) is a multiply-connected space, multiparticle wavefunctions may not return to itself but rather acquire a phase after completing the loop. It is such a phase that determines the particle statistics. Moreover, it can be shown that such a phase must be a member of a one-dimensional unitary representation of the fundamental group \( \pi_1(\mathcal{M}_n) \). A typical example is the case \( X = \mathbb{R}^d \) with \( d \geq 3 \), in which the fundamental group is \( S_n \).

Therefore, in three and higher dimensions, particle-exchange phases must be \(+1\) or \(-1\), thus reproducing the Bose-Fermi alternative. Another typical example is the case \( X = \mathbb{R}^2 \), in which the fundamental group becomes the braid group \( B_n \). Since there is a one-parameter family of one-dimensional unitary representations of \( B_n \), the particle-exchange phase must be of the form \( e^{i\theta} \), thus predicting anyons that interpolate bosons and fermions. We note that, though the particle exchange is a dynamical process in the configuration-space approach, the particle statistics itself is still kinematical in the sense that it is determined by the representation theory of \( \pi_1(\mathcal{M}_n) \).

In one dimension, however, the situation is rather different: if \( X = \mathbb{R} \), the configuration space becomes a simply-connected convex set with boundary (see section 2.1) such that any closed loops become homotopically equivalent. One might therefore think that multiparticle wavefunctions would not acquire any nontrivial phase under the particle exchange and there would arise only the trivial statistics (i.e., the Bose-Einstein statistics). This is, however, not the case because—as we will see in the path-integral formalism—identical particles still acquire a nontrivial phase every time multiparticle trajectories hit the boundaries, just as in the case of a single particle on the half-line \([15, 16]\) or in a box \([17–19]\) (see also \([20]\) for a textbook exposition). In addition, such a phase turns out to be a member of a one-dimensional unitary representation of the symmetric group \( S_n \), thus reproducing the Bose-Fermi alternative again in one dimension. We will revisit these things in section 3.

The rest of this section is devoted to a detailed analysis of the configuration space for \( n \) identical particles on \( \mathbb{R} \). We see that a coincidence point of \( k + 1 \) particles corresponds to a codimension-\( k \) boundary of the configuration space. We then discuss that two-body contact interactions are generally described by the Robin boundary conditions at the codimension-1 boundaries. Since there are \( n - 1 \) such boundaries, it is possible to introduce \( n - 1 \) distinct coupling constants in the \( n \)-body problem of identical particles in one dimension. Furthermore, these coupling constants turn out to be able to depend on the coordinates. These are in stark contrast to the boundary conditions \((4)\) and lead us to generalize the boson-fermion duality between the Lieb-Liniger and Cheon-Shigehara models. Finally, we discuss the boson-fermion mapping in our setup.

### 2.1 Configuration space of identical particles

Let us begin with the definition of the \( n \)-body configuration space in one dimension. As noted before, the configuration space of \( n \) identical particles is given by first taking the Cartesian product of the one-particle configuration space, and then subtracting the coincidence points, and then identifying all the permuted points. The resulting space in one dimension is the following orbit space:

\[
\mathcal{M}_n = \hat{\mathbb{R}}^n / S_n.
\]

(9)

where \( \hat{\mathbb{R}}^n \) is the configuration space for \( n \) distinguishable particles on \( \mathbb{R} \) and given by

\[
\hat{\mathbb{R}}^n = \mathbb{R}^n - \Delta_n.
\]

(10)

Here \( \Delta_n \) is a set of coincidence points \( x_j = x_k \) where two or more particles occupy the same point. In one dimension, such a set can be defined as the following vanishing locus of the Vandermonde
polynomial:

$$\Delta_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \prod_{1 \leq j < k \leq n} (x_j - x_k) = 0 \right\}. \quad (11)$$

Note that \(\mathbb{R}^n\) consists of \(n!\) disconnected regions described by the ordering \(x_{\sigma(1)} > \cdots > x_{\sigma(n)}\), where \(\sigma\) runs through all possible permutations of \(n\) indices. Since all these regions are physically identified, it is sufficient to consider only a single region, say \(x_1 > \cdots > x_n\). Hence the orbit space (9) can be identified with the following bounded region in \(\mathbb{R}^n\):

$$\mathcal{M}_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > \cdots > x_n \}. \quad (12)$$

This is the configuration space of \(n\) identical particles in one dimension. Notice that this space is a convex set and hence simply connected\(^4\).

Though it is not necessary for deriving the boson-fermion duality, it may be instructive to point out here that \(\mathcal{M}_n\) can be factorized into the direct product of three distinct pieces—the space of the center-of-mass motion, the space of the hyperradial motion, and the space of the hyperangular motion. To see this, it is convenient to introduce the following normalized Jacobi coordinates:

$$\xi_j = \frac{x_1 + \cdots + x_j - jx_{j+1}}{\sqrt{j(j+1)}}, \quad j \in \{1, \ldots, n-1\}, \quad (13a)$$

$$\xi_n = \frac{x_1 + \cdots + x_n}{\sqrt{n}}. \quad (13b)$$

We note that these are normalized in the sense that the coordinate transformation \((x_1, \ldots, x_n) \mapsto (\xi_1, \ldots, \xi_n)\) is an \(SO(n)\) transformation and hence preserves the dot product. Note also that \(\xi_n\) corresponds to the center-of-mass coordinate and is invariant under \(S_n\). It then follows from the definitions (13a) and (13b) that there hold the identities \(\frac{x_1-x_n}{\sqrt{2}} = \xi_1\) and \(\frac{x_i-x_j}{\sqrt{2}} = -\sqrt{\frac{j}{j+1}}\xi_{j+1} + \sqrt{\frac{j+1}{j}}\xi_j\) for \(j = \{2, \ldots, n-1\}\), from which one can show that the condition \(x_1 > x_2 > \cdots > x_n\) is translated into the condition \(0 < \xi_1 < \cdots < \sqrt{\frac{n(n-1)}{2}}\xi_{n-1}\). Note that there is no constraint on \(\xi_n\), meaning that the one-dimensional space \(\mathbb{R}(\exists \xi_n)\) is factored out from \(\mathcal{M}_n\). In order to see further factorizations, let us next introduce the hyperradius \(r\) as follows:

$$r = \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2}$$

$$= \sqrt{\xi \cdot \xi - \xi_n^2}$$

$$= \sqrt{x \cdot x - \frac{1}{n}(x_1 + \cdots + x_n)^2}$$

$$= \sqrt{n \sum_{1 \leq j < k \leq n} (x_j - x_k)^2}, \quad (14)$$

where \(\xi = (\xi_1, \ldots, \xi_n)\), \(x = (x_1, \ldots, x_n)\), and the dot stands for the dot product. Now we write \(\xi_j = r \tilde{\xi}_j\) for \(j \in \{1, \ldots, n-1\}\). Then it follows from the above discussion that \(\tilde{\xi}_j\) should satisfy \(\tilde{\xi}_1^2 + \cdots + \tilde{\xi}_{n-1}^2 = 1\) and \(0 < \tilde{\xi}_1 < \cdots < \sqrt{\frac{n(n-1)}{2}}\tilde{\xi}_{n-1}\). Putting all the above things together, we arrive at the following factorization of the configuration space:

$$\mathcal{M}_n = \mathbb{R} \times \mathbb{R}_+ \times \Omega_{n-2}, \quad (15)$$

\(^4\)It is easy to see that, if \(x, y \in \mathcal{M}_s\), then \((1-s)x + sy \in \mathcal{M}_s\) for any \(s \in [0,1]\). Thus \(\mathcal{M}_s\) is a convex set. Note that any convex set is simply connected.
Figure 1: Configuration spaces for the relative motion of two, three, and four identical particles in one dimension.
(a) $\mathcal{R}_1 = \{\xi_1 \in \mathbb{R} : 0 < \xi_1\}$ is just the half-line. The blue dot represents the codimension-1 boundary at which a two-body contact interaction takes place. (b) $\mathcal{R}_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : 0 < \xi_1 < \sqrt{3}\xi_2\}$ is the infinite sector with the angle $\pi/3$. The gray shaded region represents the impenetrable domain for the identical particles. The blue lines and the red dot represent the codimension-1 and -2 boundaries at which two- and three-body contact interactions take place. (c) $\mathcal{R}_3 = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : 0 < \xi_1 < \sqrt{3}\xi_2 < \sqrt{6}\xi_3\}$ is the infinite triangular pyramid. (To visualize $\mathcal{R}_3$, consider, e.g., the tetrakis hexahedron.) The gray shaded region represents the impenetrable domain. The blank white surfaces (including the blue curves), the red lines, and the green dot represent the codimension-1, -2, and -3 boundaries at which two-, three-, and four-body contact interactions take place.

where $\mathbb{R} = \{\xi_n : -\infty < \xi_n < \infty\}$ is the space of the center-of-mass motion, $\mathbb{R}_+ = \{r : r > 0\}$ the space of the hyperradial motion, and $\Omega_{n-2}$ the space of the hyperangular motion given by

$$\Omega_{n-2} = \left\{(\hat{\xi}_1, \ldots, \hat{\xi}_{n-1}) \in \mathbb{R}^{n-1} : \hat{\xi}_1^2 + \cdots + \hat{\xi}_{n-1}^2 = 1 \& 0 < \hat{\xi}_1 < \cdots < \sqrt{\frac{n(n-1)}{2}} \hat{\xi}_{n-1} \right\}. \quad (16)$$

We note that the last factor $\Omega_{n-2}$ in (15) must be discarded if $n = 2$. Note also that the subspace $\mathcal{R}_{n-1} = \mathbb{R}_+ \times \Omega_{n-2}$ is nothing but the relative space in [10] that describes the relative motion of identical particles. Typical examples of the relative space are depicted in figure 1.

Now, as can be observed from figure 1, there are a number of nontrivial boundaries in the relative space $\mathcal{R}_{n-1}$. For example, for $n = 4$ (see figure 1c), there are (i) three codimension-1 boundaries, (ii) three codimension-2 boundaries, and (iii) a single codimension-3 boundary, which, in the original Cartesian coordinates, correspond to (i) \{ $x_1 \geq x_2 \geq x_3 \geq x_4$, $x_1 > x_2 = x_3 > x_4$, $x_1 > x_2 > x_3 = x_4$, \}, (ii) \{ $x_1 = x_2 = x_3 > x_4$, $x_1 > x_2 = x_3 = x_4$, $x_1 > x_2 > x_3 = x_4$, \}, and (iii) \{ $x_1 = x_2 = x_3 = x_4$, \}, respectively. In general, a coincidence point of $k$ particles corresponds to one of the codimension-$(k-1)$ boundaries of $\mathcal{R}_{n-1}$ (or $\mathcal{M}_n$).

To summarize, we have seen that $k$-body contact interactions take place at a codimension-$(k - 1)$ boundary of the configuration space $\mathcal{M}_n$. Since the purpose of the present paper is to derive and generalize the dual two-body contact interactions for bosons and fermions, in what follows we will concentrate on only the codimension-1 boundaries.

### 2.2 Two-body boundary conditions

In order to construct a quantum theory on $\mathcal{M}_n$, we have to specify boundary conditions for wavefunctions. Below we will do this by imposing the probability conservation at the codimension-1 boundaries.

Let us first note that a two-body contact interaction between the $j$th and $(j + 1)$th particles takes place at the following codimension-1 boundary:

$$\partial \mathcal{M}_{n,j}^{2\text{-body}} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > \cdots > x_j = x_{j+1} > \cdots > x_n\}, \quad (17)$$

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5The relative space can also be written as $\mathcal{R}_{n-1} = \{(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} : 0 < \xi_1 < \cdots < \sqrt{\frac{n(n-1)}{2}} \xi_{n-1}\}$. 

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where \( j \in \{ 1, \ldots, n - 1 \} \). As originally discussed by Leinaas and Myrheim [10], in order to ensure the probability conservation, the normal component of the probability current density must vanish at the boundary. Thus we impose the following condition:

\[
n_j \cdot j = 0 \quad \text{on} \quad \partial M^{2\text{-body}}_{n,j},
\]

(18)

where \( n_j \) stands for a normal vector to the boundary \( \partial M^{2\text{-body}}_{n,j} \) and \( j \) is the \( n \)-body probability current density defined by

\[
j = \frac{\hbar}{2im} \left( \overline{\psi} \nabla \psi - (\nabla \overline{\psi}) \psi \right).
\]

(19)

Here \( \psi = \psi(x_1, \ldots, x_n) \) is a multiparticle wavefunction on \( M_n \), \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \) is the \( n \)-body differential operator, and the overline stands for the complex conjugate. Substituting (19) into (18) we get

\[
\overline{\psi} (n_j \cdot \nabla \psi) - (n_j \cdot \overline{\nabla \psi}) \psi = 0 \quad \text{on} \quad \partial M^{2\text{-body}}_{n,j}.
\]

(20)

Note that this is a quadratic equation of \( \psi \). However, it can in fact be linearized and enjoys a one-parameter family of solutions. As is well-known, the solution to the equation (20) is given by the following Robin boundary condition:

\[
n_j \cdot \nabla \psi - \frac{1}{a_j} \psi = 0 \quad \text{on} \quad \partial M^{2\text{-body}}_{n,j},
\]

(21)

where \( a_j \) is a real parameter with the dimension of length. We emphasize that \( a_j \) may depend on the coordinates parallel to the boundary. Naively, such coordinate dependence would break the translation invariance. As originally noted in [10], this is true for \( n = 2 \). However, for \( n \geq 3 \), \( a_j \) can depend on the coordinates without spoiling the translation invariance. We will revisit this possibility and resulting scale- and translation-invariant two-body contact interactions in section 4.

Now, since the boundary \( \partial M^{2\text{-body}}_{n,j} \) is the codimension-1 surface \( x_j - x_{j+1} = 0 \) in \( \mathbb{R}^n \), the normal vector can be simply written as follows:

\[
n_j = \nabla(x_j - x_{j+1}) = (0, \ldots, 0, 1, -1, 0, \ldots, 0),
\]

(22)

from which we find that the Robin boundary condition (21) is cast into the following form:

\[
\left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) \psi - \frac{1}{a_j} \psi = 0 \quad \text{on} \quad \partial M^{2\text{-body}}_{n,j}.
\]

(23)

This is the boundary condition that describes the two-body contact interaction for identical particles.

It should be noted that Leinaas and Myrheim interpreted the parameter \( a_j \) as a statistics parameter that interpolates the Bose-Einstein and Fermi-Dirac statistics [10], because \( a_j \) continuously interpolates the Neumann boundary condition (i.e., the boundary condition for free bosons) and the Dirichlet boundary condition (i.e., the boundary condition for free fermions). They then advocated the existence of intermediate statistics in one dimension. In the present paper, however, our take is different: the parameter \( a_j \) just describes the two-body contact interaction rather than the statistics.

As we will see in section 3, for bosons, the Robin boundary condition (23) is translated into the \( \delta \)-function potential whose coupling constant is \( 1/a_j \) and whose support is the codimension-1 singularity \( \{ x_1 > \cdots > x_j = x_{j+1} > \cdots > x_n \} \) in \( \mathbb{R}^n \). For fermions, on the other hand, (23) turns out to become the \( \epsilon \)-function potential whose coupling constant is \( a_j \) and whose support is \( \{ x_1 > \cdots > x_j = x_{j+1} > \cdots > x_n \} \).

\(^{6}\text{Note that normal vectors become ill-defined at the codimension-}k(\geq 2)\text{ boundaries, because these boundaries are corner singularities in general; see figures 1b and 1c. In this work we will not touch upon boundary conditions at these singularities.}\)
2.3 Boson-fermion mapping

Before closing this section, let us discuss here the boson-fermion mapping in terms of multiparticle wavefunctions on \( \mathbb{R}^n \). To this end, let us first suppose that we find a normalized wavefunction \( \psi(x) \) on the region \( x_1 > \cdots > x_n \) (i.e., the configuration space \( \mathcal{M}_n \)). Let us then extend this wavefunction by introducing the following two distinct wavefunctions on the region \( x_{\sigma(1)} > \cdots > x_{\sigma(n)} \):

\[
\psi_B(x) := \frac{1}{\sqrt{n!}} \psi(\sigma x), \tag{24a}
\]

\[
\psi_F(x) := \frac{1}{\sqrt{n!}} \text{sgn}(\sigma) \psi(\sigma x). \tag{24b}
\]

As \( \sigma \) runs through all possible permutations, eqs. (24a) and (24b) define normalized wavefunctions on \( \mathbb{R}^n \). By construction, it is obvious that \( \psi_B \) and \( \psi_F \) are totally symmetric and antisymmetric under the permutation of coordinates, thus providing the wavefunctions of identical spinless bosons and fermions on \( \mathbb{R}^n \). It is also obvious by construction that there holds the identity \( \psi_F(x) = \text{sgn}(\sigma) \psi_B(x) \) on the region \( x_{\sigma(1)} > \cdots > x_{\sigma(n)} \). An alternative equivalent expression for this is the following identity on \( \mathbb{R}^n \):

\[
\psi_F(x) = \left( \prod_{1 \leq j < k \leq n} \text{sgn}(x_j - x_k) \right) \psi_B(x), \quad \forall x \in \mathbb{R}^n, \tag{25}
\]

where \( \text{sgn}(x) \) here stands for the sign function, \( \text{sgn}(x) = x/|x| \). This is the celebrated boson-fermion mapping in one dimension [1]. It is now clear that the fundamental ingredient of the boson-fermion duality is the wavefunction \( \psi \) on the configuration space \( \mathcal{M}_n \); if \( \psi_B \) and \( \psi_F \) are constructed from the same \( \psi \), the bosonic and fermionic systems are automatically isospectral. It should be emphasized that the identity (25) holds irrespective of boundary conditions.

3 Dual description of the Feynman kernel on \( \mathcal{M}_n = \mathbb{R}^n/S_n \)

In the previous section, we have presented a detailed analysis of the configuration space of \( n \) identical particles in one dimension. Note, however, that the particle statistics is still unclear at this stage: it is not clear whether and how multiparticle wavefunctions on \( \mathcal{M}_n \) acquire a phase under the process of particle exchange. As noted at the beginning of section 2, the particle exchange is a dynamical process in the configuration-space approach. Hence it is natural to expect that particle-exchange phases may show up by studying dynamics. In general, dynamics in quantum mechanics is described by the Feynman kernel—the integral kernel of the time-evolution operator—which, as is well-known, can be studied by the path-integral formalism. In fact, the configuration-space approach and the path-integral formalism were known to be intimately connected. The key was the covering-space approach to the path integral on multiply-connected spaces, which was initiated by Schulman [21] and later generalized by Laidlaw and DeWitt [9] and by Dowker [22] (see also [23–25]). In this approach, one first constructs a multiply-connected space \( Q \) as \( Q = \widetilde{Q}/\Gamma \), where \( \widetilde{Q} \) is a (simply-connected) universal covering space of \( Q \) and \( \Gamma \) is a discrete subgroup of the isometry of \( \widetilde{Q} \) that acts freely on \( \widetilde{Q} \) (without fixed points). Then, the path integral on \( Q \) is generally given by a weighted sum of the path integrals on \( \widetilde{Q} \) with weight factors given by a one-dimensional unitary representation of \( \Gamma \). All the weight factors are linked with homotopically distinct paths and—since \( \Gamma \) is isomorphic to the fundamental group \( \pi_1(Q) \)—coincide with the particle-exchange phases in the configuration-space approach. In this way, the path-integral approach to the particle statistics was successful in deriving the Bose-Fermi alternative for \( d \geq 3 \) [9] and the braid-group statistics for \( d = 2 \) [26].

In one dimension, however, the situation is rather different again. This is because the configuration space \( \mathcal{M}_n = \mathbb{R}^n/S_n \) is not a multiply-connected space and does not fit into the form \( Q = \widetilde{Q}/\Gamma \). In fact, to the best of our knowledge, the path-integral derivation of the particle statistics in one dimension is still missing.
In this section we study the Feynman kernel for \(n\) identical particles in one dimension by using the path-integral formalism. We will see that—though \(\mathbb{R}^n\) is not a universal covering space of \(\mathcal{M}_a\) and \(S_n\) is not the fundamental group of \(\mathcal{M}_a\)—the covering-space-like approach still works in one dimension: the Feynman kernel on \(\mathcal{M}_n = \mathbb{R}^n/S_n\) turns out to be given by a weighted sum of the Feynman kernels on \(\mathbb{R}^n\) with weight factors given by a one-dimensional unitary representation of \(S_n\).\(^7\) As we will see shortly, this leads to the Bose-Fermi alternative as well as a generalized form of the boson-fermion duality.

To begin with, let us consider the simplest situation where the particles freely propagate in the bulk yet interact only through the two-body contact interactions described by (23). The dynamics of such a system is described by the following time-dependent Schrödinger equation on \(\mathcal{M}_n\):

\[
\left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) \psi(x, t) = 0, \quad \forall x \in \mathcal{M}_n,
\]

with the Robin boundary conditions (23). A standard approach to solve this problem is to find out the Feynman kernel \(K_{\mathcal{M}_a}(x, y; t) = \langle x|\exp(-\frac{i}{\hbar}Ht)|y \rangle\) on \(\mathcal{M}_n\), which is the coordinate representation of the time-evolution operator and must satisfy the following properties:

- **Property 1. (Composition law)**
  \[
  \int_{\mathcal{M}_n} dz K_{\mathcal{M}_a}(x, z; t_1)K_{\mathcal{M}_a}(z, y; t_2) = K_{\mathcal{M}_a}(x, y; t_1 + t_2), \quad \forall x, y \in \mathcal{M}_n.
  \]

- **Property 2. (Initial condition)**
  \[
  K_{\mathcal{M}_a}(x, y; 0) = \delta(x - y), \quad \forall x, y \in \mathcal{M}_n.
  \]

- **Property 3. (Unitarity)**
  \[
  K_{\mathcal{M}_a}(x, y; -t) = K_{\mathcal{M}_a}(y, x; t), \quad \forall x, y \in \mathcal{M}_n.
  \]

- **Property 4. (Schrödinger equation)**
  \[
  \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) K_{\mathcal{M}_a}(x, y; t) = 0, \quad \forall x, y \in \mathcal{M}_n.
  \]

- **Property 5. (Two-body boundary conditions)**
  \[
  \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) K_{\mathcal{M}_a}(x, y; t) - \frac{1}{\delta_j} K_{\mathcal{M}_a}(x, y; t) = 0, \quad \forall x \in \partial \mathcal{M}_n^{2\text{-body}}, \quad \forall y \in \mathcal{M}_n.
  \]

Notice that the first four properties are just the coordinate representations of the composition law \(U_{t_1}U_{t_2} = U_{t_1+t_2}\), the initial condition \(U_0 = 1\), the unitarity \(U_{-t}^* = U_{-t}^{-1} = U_{-t}\), and the Schrödinger equation \((i\hbar \frac{\partial}{\partial t} - H)U_t = 0\) for the time-evolution operator \(U_t = \exp(-\frac{i}{\hbar}Ht)\), where \(H\) stands for the Hamiltonian of the system. Once we find such a Feynman kernel, the solution to the time-dependent Schrödinger equation (26) can be written as the following integral transform of the initial wavefunction at \(t = 0\):

\[
\psi(x, t) = \int_{\mathcal{M}_n} dy K_{\mathcal{M}_a}(x, y; t)\psi(y, 0), \quad \forall x \in \mathcal{M}_n.
\]

\(^7\)If one wants to use covering-space language, one should discard the subtraction procedure and consider the orbifold \(\mathcal{M}_a = \mathbb{R}^n/S_n\) [27]. In this case, one can say that the path integral on \(\mathcal{M}_n\) is given by a weighted sum of the path integrals on the orbifold universal cover \(\mathbb{R}^n\) and that weight factors are given by a one-dimensional unitary representation of the orbifold fundamental group \(\pi_1^{orb}(\mathcal{M}_a) \cong S_n\). (For orbifolds, see, e.g., the Thurston’s lecture notes [28].) Notice that, in one dimension, the difference between \(\mathcal{M}_a = \mathbb{R}^n/S_n\) = \{\(x_1, \ldots, x_n\) : \(x_1 \geq \cdots \geq x_n\}\) and \(\mathcal{M}_a = \mathbb{R}^n/S_n\) = \{\(x_1, \ldots, x_n\) : \(x_1 > \cdots > x_n\}\) is merely the boundary: the former includes the boundary but the latter does not. Hence it is almost a matter of preference which one we should use. In this paper we will use \(\mathcal{M}_a = \mathbb{R}^n/S_n\) in order to make the conceptual transition from \(d = 1\) to \(d \geq 2\) smooth.
It should be emphasized that, if the Feynman kernel satisfies (31), the wavefunction given by the integral transform (32) automatically satisfies the Robin boundary conditions (23).

Now, as we will prove in appendix A, the Feynman kernel on $\mathcal{M}_n = \mathbb{R}^n/S_n$ can be constructed in almost the same way as the Dowker’s covering-space method [22] and written as follows:

$$K_{\mathcal{M}_n}(x, y; t) = \sum_{\sigma \in S_n} \chi(\sigma)K_{\hat{R}^n}(x, \sigma y; t), \quad \forall x, y \in \mathcal{M}_n, \quad (33)$$

where $\chi : S_n \to U(1)$ is a one-dimensional unitary representation of $S_n$. Here $K_{\hat{R}^n}$ is the Feynman kernel on $\hat{R}^n$ and assumed to satisfy the following conditions:

- **Assumption 1. (Composition law)**
  $$\int_{\hat{R}^n} dz K_{\hat{R}^n}(x, z; t_1)K_{\hat{R}^n}(z, y; t_2) = K_{\hat{R}^n}(x, y; t_1 + t_2), \quad \forall x, y \in \hat{R}^n, \quad (34)$$

- **Assumption 2. (Initial condition)**
  $$K_{\hat{R}^n}(x, y; 0) = \delta(x - y), \quad \forall x, y \in \hat{R}^n, \quad (35)$$

- **Assumption 3. (Unitarity)**
  $$K_{\hat{R}^n}(x, y; t) = K_{\hat{R}^n}(y, x; -t), \quad \forall x, y \in \hat{R}^n, \quad (36)$$

- **Assumption 4. (Schrödinger equation)**
  $$\left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right) K_{\hat{R}^n}(x, y; t) = 0, \quad \forall x, y \in \hat{R}^n, \quad (37)$$

- **Assumption 5. (Permutation invariance)**
  $$K_{\hat{R}^n}(x, y; t) = K_{\hat{R}^n}(\sigma x, \sigma y; t), \quad \forall x, y \in \hat{R}^n, \quad \forall \sigma \in S_n, \quad (38)$$

with some connection conditions at the vanishing locus $\Delta_n$ in order to glue the $n!$ disconnected regions together. It is these connection conditions that we wish to uncover below. Before doing this, however, let us first present a brief derivation of the formula (33) by following the Dowker’s argument [22].

Let $\psi(x, t)$ be an equivariant function on $\hat{R}^n$ that satisfies $\psi(\sigma x, t) = \chi(\sigma)\psi(x, t)$ for any $\sigma \in S_n$. Notice that, if $\chi = \chi^{[1]} (\chi = \chi^{[b]})$, such an equivariant function is nothing but the wavefunction of identical bosons (fermions) on $\hat{R}^n$. Then, the solution to the time-dependent Schrödinger equation on $\hat{R}^n$ is given by the following integral transform of the initial wavefunction:

$$\psi(x, t) = \int_{\hat{R}^n} dy K_{\hat{R}^n}(x, y; t)\psi(y, 0)$$

$$= \int_{\mathcal{M}_n} dy \left( \sum_{\sigma \in S_n} K_{\hat{R}^n}(x, \sigma y; t)\psi(\sigma y, 0) \right)$$

$$= \int_{\mathcal{M}_n} dy \left( \sum_{\sigma \in S_n} K_{\hat{R}^n}(x, \sigma y; t)\chi(\sigma)\psi(y, 0) \right)$$

$$= \int_{\mathcal{M}_n} dy \left( \sum_{\sigma \in S_n} \chi(\sigma)K_{\hat{R}^n}(x, \sigma y; t) \right) \psi(y, 0), \quad \forall x \in \hat{R}^n. \quad (39)$$

Here the second equality follows from the following integral formula (for the proof, see appendix B):

$$\int_{\hat{R}^n} dy f(y) = \int_{\mathcal{M}_n} dy \left( \sum_{\sigma \in S_n} f(\sigma y) \right). \quad (40)$$
For the case of $\delta_n$ with $n=3$, one arrives at the formula (33). A proof that eq. (33) indeed satisfies the conditions (27)–(30) is presented in appendix A. It is now obvious from the above derivation that the weight factor $\chi(\sigma)$ in (33) describes a particle-exchange phase. More precisely, $\chi(\sigma)$ is an accumulation of particle-exchange phases for two adjacent particles in the course of the time-evolution; see figure 2 for the case of $n = 3$.

Now, as noted repeatedly, there are just two distinct one-dimensional unitary representations of $S_n$: the totally symmetric representation $\chi^{[B]}$ and the totally antisymmetric representation $\chi^{[F]}$. This simple mathematical fact has two distinct physical meanings here. To see this, let us first suppose that $K_{R^n}$ is given. Then the formula (33) implies that there exist two distinct Feynman kernels on $M_n$:

$$K_{M_n}^{[B]}(x, y; t) = \sum_{\sigma \in S_n} \chi^{[B]}(\sigma) K_{R^n}(x, \sigma y; t),$$

$$K_{M_n}^{[F]}(x, y; t) = \sum_{\sigma \in S_n} \chi^{[F]}(\sigma) K_{R^n}(x, \sigma y; t).$$

Since $K_{R^n}$ is normally constructed from classical theory via the Feynman’s path-integral quantization, eqs. (41a) and (41b) indicate that there exist two inequivalent quantizations depending on the particle–exchange phases $\chi^{[B,F]}(\sigma)$. Hence the first meaning is about the particle statistics: in one dimension, there exists just the Bose–Fermi alternative rather than the intermediate statistics of Leinaas and Myrheim [10]. To see another meaning, let us next suppose that $K_{M_n}$ is given. Then the formula (33) implies that there exist two distinct Feynman kernels on $R^n$ satisfying the following equalities:

$$K_{M_n}(x, y; t) = \sum_{\sigma \in S_n} \chi^{[B]}(\sigma) K_{R^n}^{[B]}(x, \sigma y; t) = \sum_{\sigma \in S_n} \chi^{[F]}(\sigma) K_{R^n}^{[F]}(x, \sigma y; t).$$

Since $K_{M_n}$ determines the dynamics as well as energy spectrum of identical-particle systems, eq. (42) indicates that the two distinct systems on $R^n$ described by $K_{R^n}^{[B]}$ and $K_{R^n}^{[F]}$ are, if they exist, completely isospectral. Hence the second meaning is about the boson–fermion duality: there may exist bosonic and fermionic systems whose energy spectra are completely equivalent.

Let us finally derive a generalization of the boson–fermion duality by using (42), which can be achieved by studying the connection conditions for $K_{R^n}^{[B]}$ and $K_{R^n}^{[F]}$ at the codimension-1 singularities. To this end, let us first note that any element of $S_n$ can be classified into either even or odd permutations. In other words, the symmetric group $S_n$ can be decomposed into the following coset decomposition:

$$S_n = A_n \cup A_n \tau,$$

where $A_n$ is the alternating group that consists of only even permutations. $\tau \in S_n$ is an arbitrary transposition and $A_n \tau = \{ \sigma \tau : \sigma \in A_n \}$ is the right coset that consists of only odd permutations.
Then, corresponding to (43), the formula (33) can be decomposed into the following form:

\[
K_{M_n}(x, y; t) = \sum_{\sigma \in A_n} \left( \chi(\sigma)K_{\hat{R}^n}(x, \sigma y; t) + \chi(\tau)K_{\hat{R}^n}(x, \sigma \tau y; t) \right)
\]

\[
= \sum_{\sigma \in A_n} \chi(\sigma) \left( K_{\hat{R}^n}(\sigma^{-1} x, y; t) + \chi(\tau)K_{\hat{R}^n}(\tau^{-1} \sigma^{-1} x, y; t) \right),
\]

(44)

where we have used \( \chi(\tau) = \chi(\sigma)\chi(\tau) \) and \( K_{\hat{R}^n}(x, \sigma y; t) = K_{\hat{R}^n}(\sigma x, y; t) = K_{\hat{R}^n}(x, \sigma y; t) \) (\( \forall \sigma \in S_n \)) which follows from the permutation invariance (38). Now we choose \( \tau \) as the adjacent transposition \( \tau = \tau_j = (i, j + 1) \) which just swaps \( x_j \) and \( x_{j+1} \). Then, by applying the differential operator \( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} - \frac{1}{a_j} \) to (44), we get

\[
\left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) K_{M_n}(x, y; t) - \frac{1}{a_j} K_{M_n}(x, y; t)
\]

\[
= \sum_{\sigma \in A_n} \chi(\sigma) \left[ \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) K_{\hat{R}^n}(\sigma^{-1} x, y; t) \right.
\]

\[
- \chi(\tau_j) \left( \frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} \right) K_{\hat{R}^n}(\tau_j^{-1} \sigma^{-1} x, y; t)
\]

\[
- \frac{1}{a_j} \left( K_{\hat{R}^n}(\sigma^{-1} x, y; t) + \chi(\tau_j)K_{\hat{R}^n}(\tau_j^{-1} \sigma^{-1} x, y; t) \right)
\]

\[
= \sum_{\sigma \in A_n} \chi(\sigma) \left[ \left( \frac{\partial}{\partial z_{\sigma(j)}} - \frac{\partial}{\partial z_{\sigma(j+1)}} \right) K_{\hat{R}^n}(z, y; t) \right|_{z = \sigma^{-1} x}
\]

\[
- \chi(\tau_j) \left( \frac{\partial}{\partial z_{\sigma(j)}} - \frac{\partial}{\partial z_{\sigma(j+1)}} \right) K_{\hat{R}^n}(z, y; t) \right|_{z = \tau_j^{-1} \sigma^{-1} x}
\]

\[
- \frac{1}{a_j} \left( K_{\hat{R}^n}(z, y; t) \right|_{z = \sigma^{-1} x} + \chi(\tau_j) K_{\hat{R}^n}(z, y; t) \right|_{z = \tau_j^{-1} \sigma^{-1} x}
\]

(45)

where in the last equality we have introduced a new variable \( z = \sigma^{-1} x \) and used the relation \( x = \sigma z \) (i.e., \( x_k = z_{\sigma(k)} \)) in any \( k \) in the first and third terms. Similarly, in the second and fourth terms of the last line we have introduced \( z = \tau_j^{-1} \sigma^{-1} x \) and used the relation \( x = \sigma \tau_j z \), which gives \( x_{j+1} = z_{\sigma(\tau_j+1)} = z_{\sigma(j+1)} \), and \( x_k = z_{\sigma(\tau_j(k))} = z_{\sigma(k)} \) for \( k \notin \{j, j+1\} \). It should be noted that \( x \) in (45) is in the region \( x_1 > \cdots > x_n \), which means that \( z \) in the first and third terms is in the region \( z_{\sigma(1)} > \cdots > z_{\sigma(j)} > z_{\sigma(j+1)} > \cdots > z_{\sigma(n)} \) whereas \( z \) in the second and fourth terms is in the region \( z_{\sigma(1)} > \cdots > z_{\sigma(j+1)} > z_{\sigma(j)} > \cdots > z_{\sigma(n)} \). Hence, in order for \( K_{M_n} \) to satisfy the Robin boundary condition (31) as \( x_j - x_{j+1} \to 0_+ \), the Feynman kernel on \( \hat{R}^n \) should satisfy the following connection condition at the codimension-1 singularity \( \{z_{\sigma(1)} > \cdots > z_{\sigma(j)} = z_{\sigma(j+1)} > \cdots > z_{\sigma(n)} \} \):

\[
0 = \left( \frac{\partial}{\partial z_{\sigma(j)}} - \frac{\partial}{\partial z_{\sigma(j+1)}} \right) K_{\hat{R}^n}(z, y; t) \bigg|_{z_{\sigma(j)} - z_{\sigma(j+1)} = 0_+}
\]

\[
- \chi(\tau_j) \left( \frac{\partial}{\partial z_{\sigma(j)}} - \frac{\partial}{\partial z_{\sigma(j+1)}} \right) K_{\hat{R}^n}(z, y; t) \bigg|_{z_{\sigma(j+1)} - z_{\sigma(j)} = 0_+}
\]

\[
- \frac{1}{a_j} \left( K_{\hat{R}^n}(z, y; t) \bigg|_{z_{\sigma(j)} - z_{\sigma(j+1)} = 0_+} + \chi(\tau_j) K_{\hat{R}^n}(z, y; t) \bigg|_{z_{\sigma(j+1)} - z_{\sigma(j)} = 0_+} \right).
\]

(46)

where \( j \in \{1, \ldots, n-1\} \) and \( \sigma \in A_n \). Now it is obvious that, in the totally symmetric representation
\[ \chi = \chi^{(B)}, \text{ where } \chi^{(B)}(\tau) = 1, \text{ we have the following connection condition for } K_{R^a} = K^{(B)}: \]

\[
0 = \left( \frac{\partial}{\partial \chi_{\sigma(j)}} - \frac{\partial}{\partial \chi_{\sigma(j+1)}} \right) K^{(B)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0}, \]

\[
- \left( \frac{\partial}{\partial \chi_{\sigma(j)}} - \frac{\partial}{\partial \chi_{\sigma(j+1)}} \right) K^{(B)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0}, \]

\[
- \frac{1}{a_j} \left( K^{(B)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0} + K^{(B)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0} \right), \quad (47) \]

where we have renamed \( z \) to \( x \). In contrast, in the totally antisymmetric representation \( \chi = \chi^{(F)} \), where \( \chi^{(F)}(\tau) = \text{sgn}(\tau) = -1, \) we have the following connection condition for \( K_{R^a} = K^{(F)}: \)

\[
0 = \left( \frac{\partial}{\partial \chi_{\sigma(j)}} - \frac{\partial}{\partial \chi_{\sigma(j+1)}} \right) K^{(F)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0}, \]

\[
+ \left( \frac{\partial}{\partial \chi_{\sigma(j)}} - \frac{\partial}{\partial \chi_{\sigma(j+1)}} \right) K^{(F)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0}, \]

\[
- \frac{1}{a_j} \left( K^{(F)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0} - K^{(F)}_{R^a}(x, y; t) \bigg|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0} \right), \quad (48) \]

Now, these connection conditions are nothing but (a part of) the connection conditions for the \( \delta \)- and \( \epsilon \)-function potentials. In fact, it follows from these connection conditions that the wavefunctions constructed from the integral transform (39) satisfy the conditions (2a) or (3a) with \( a \) being replaced by \( a_j \). Note that the continuity conditions for the wavefunction (2b) and its derivative (3b) are not necessary here because the totally symmetric function and the derivative of totally antisymmetric function automatically becomes continuous at the coincidence points. Thus, reversing the argument used to arrive at (2a)–(3b) from (1a) and (1b), we obtain the following dual Hamiltonians for \( n \) identical bosons and fermions:

\[ H_B = \frac{\hbar^2}{2m} \nabla^2 + V_B(x), \quad (49a) \]

\[ H_F = -\frac{\hbar^2}{2m} \nabla^2 + V_F(x), \quad (49b) \]

where

\[ V_B(x) = \frac{\hbar^2}{m} \sum_{j=1}^{n-1} \sum_{\sigma \in A_n} \left( \prod_{k \in 1, \ldots, \sigma(j)} \theta(x_{\sigma(k)} - x_{\sigma(k+1)}) \right) \delta(x_{\sigma(j)} - x_{\sigma(j+1)}; \frac{1}{a_j}), \quad (50a) \]

\[ V_F(x) = \frac{\hbar^2}{m} \sum_{j=1}^{n-1} \sum_{\sigma \in A_n} \left( \prod_{k \in 1, \ldots, \sigma(j)} \theta(x_{\sigma(k)} - x_{\sigma(k+1)}) \right) \epsilon(x_{\sigma(j)} - x_{\sigma(j+1)}; a_j). \quad (50b) \]

Note that the factor \( \prod_{k \in 1, \ldots, \sigma(j-1) \setminus \{j\}} \theta(x_{\sigma(k)} - x_{\sigma(k+1)}) \), where \( \theta \) is the step function, is introduced in order to guarantee the ordering \( x_{\sigma(1)} > \cdots > x_{\sigma(j)} > x_{\sigma(j+1)} > \cdots > x_{\sigma(n)} \). Note also that the total number of the codimension-1 singularities in \( \mathbb{R}^n \) is \( \frac{2(n-1)}{n} \times (n-1) = (n-1) \times \frac{n}{2} = (n-1) \times |A_n| \), where \( |A_n| \) stands for the order of the alternating group \( A_n \). Hence the summation \( \sum_{j=1}^{n-1} \sum_{\sigma \in A_n} \) is indeed summing over all the codimension-1 singularities. Finally, it is easy to see that the potential energies (50a) and (50b) are invariant under \( S_n \) and satisfy the identities \( V_{B,F}(\sigma x) = V_{B,F}(x) \) for any \( \sigma \in S_n \).

To summarize, by imposing the Robin boundary conditions to the Feynman kernel (42), we have found that \( K_{R^a}^{(B)} \) and \( K_{R^a}^{(F)} \), respectively, must satisfy the connection conditions for the \( \delta \)- and \( \epsilon \)-function potentials at the codimension-1 singularities in \( \mathbb{R}^n \). The \( n \)-body Hamiltonians that realize these connection conditions for identical bosons and fermions are given by (49a) and (49b), both of which possess \( n-1 \) distinct (coordinate-dependent) coupling constants. By construction, these models enjoy (i) the spectral equivalence, (ii) the boson-fermion mapping, and (iii) the strong-weak duality.
4 Summary and discussion

In the present paper, we have revisited the boson-fermion duality in one dimension by using the configuration-space approach and the path-integral formalism. In section 1, we have first presented the detailed analysis of the configuration space for \( n \) identical particles on \( \mathbb{R} \). We have shown that the two-body contact interactions for \( n \) identical particles are generally described by the \( n-1 \) distinct Robin boundary conditions (23), where the boundary-condition parameters \( a_j (j = 1, \ldots, n-1) \) may depend on the coordinates parallel to the codimension-1 boundaries of \( \mathcal{M}_n = \mathbb{R}^n / S_n \). In section 2, we have then studied the dynamics of \( n \) identical particles on \( \mathbb{R} \) by using the Feynman kernel. We have first shown that the Feynman kernel on \( \mathcal{M}_n \) is generally given by (33)—the weighted sum of the Feynman kernels on \( \mathbb{R}^n \), where the weight factor \( \chi(\sigma) \) is a member of either the totally symmetric or totally antisymmetric representations of \( S_n \). This result has two distinct physical meanings: the first is the existence of the Bose-Fermi alternative in one dimension, which is expressed by the equations (41a) and (41b), and the second is the existence of the boson-fermion duality in one dimension, which is expressed by the identity (42). Then, by using (42), we have shown that—in order for the Feynman kernel on \( \mathcal{M}_n \) to satisfy the Robin boundary conditions—the Feynman kernel on \( \mathbb{R}^n \) must satisfy the connection conditions for the \( \delta \)-function (\( \epsilon \)-function) potential if \( \chi \) is the totally (anti)symmetric representation. This proves the boson-fermion duality between the systems described by the \( n \)-boson Hamiltonian (49a) and the \( n \)-fermion Hamiltonian (49b), which are natural generalizations of the Lieb-Liniger model (1a) and the Cheon-Shigehara model (1b), respectively.

There are a number of directions for future work. One interesting direction is to generalize the one-particle configuration space \( X \) to other geometries, such as the half-line \( (X = \mathbb{R} / \mathbb{Z}) \), a finite interval \( (X = \mathbb{R} / \mathbb{Z}) \), and graphs. Because the type and the number of particle statistics depend on \( X \), one may go beyond the boson-fermion duality by suitably choosing \( X \). For example, if there exist several particle statistics, one may have triality, quadrality, pentality and so on rather than duality. Another interesting direction is to generalize to the case in which wavefunctions become multi-component. In this case, the weight factor \( \chi(\sigma) \) can be a higher-dimensional unitary representation of the symmetric group \( S_n \); that is, there can arise the parastatistics rather than the Bose-Fermi alternative. In addition, multi-component wavefunctions offer much more variety of two-body contact interactions. For example, if wavefunctions have \( N \) components, two-body contact interactions are generally described by \( U(N) \)-family of boundary conditions. Note that such a generalization can also be applied to many-body problems of non-identical particles: if wavefunctions have \( N \) components, two-body contact interactions for distinguishable particles are generally described by \( U(2N) \)-family of connection conditions at the codimension-1 singularities. We hope to return to all these things in the future.

Finally, let us comment on an implication for the possible coordinate dependence on \( a_j \). As noted in section 2.2, \( a_j \) may depend on the coordinates parallel to the codimension-1 boundaries. This opens up a possibility to realize scale-invariant two-body contact interactions which do not contain any dimensionful parameters. A typical example for such coordinate dependence is given by

\[
a_j = g_j r, \quad j \in \{1, \ldots, n-1\},
\]

where \( g_j \) are dimensionless reals and \( r \) is the hyperradius defined in (14). Note that the hyperradius (14) does not contain the center-of-mass coordinate \( \xi_0 \) and is invariant under the spatial translation \( x_j \mapsto x_j + c \) for any \( c \in \mathbb{R} \). Hence in this case the Hamiltonians (49a) and (49b) are also invariant under the spatial translation such that the center-of-mass momenta are conserved. Note also that, for \( n \geq 3 \), the hyperradius \( r \) is nonvanishing at the codimension-1 boundary \( \partial \mathcal{M}_{n/2}^{2\text{-body}} \). In fact, \( r \) vanishes only at the codimension-(\( n-1 \)) boundary \( \{x_1 = x_2 = \cdots = x_n\} \). Therefore, in the three- or more-body problems of identical particles, we can construct scale-invariant models without spoiling the translation invariance. Note, however, that this continuous scale invariance could be broken down

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\footnote{For particle statistics on graphs, see [14, 29–32].}

\footnote{The configuration space of non-identical particles in one dimension was discussed, e.g., in [33–35].}
where in the second equality we have used the assumptions that \( \tilde{\chi} \) is a representation that satisfies \( \tilde{\chi}(\sigma)\tilde{\chi}(\sigma') = \tilde{\chi}(\sigma\sigma') \) and \( \tilde{K}_{\tilde{R}^n} \) satisfies the permutation invariance (38). The third equality follows from the change of the integration variable from \( z \) to \( w = \sigma z \), the fourth equality the change of the summation variables from \( \sigma \) and \( \sigma' \) to \( \sigma \) and \( \sigma'' := \sigma\sigma' \), the fifth equality the fact that the region \( w_{\sigma^{-1}(1)} > \cdots > w_{\sigma^{-1}(n)} \) covers the whole \( \tilde{R}^n \) as \( \sigma \) runs through all possible permutations, and the sixth equality the assumption (34). Hence we have shown that (33) satisfies the composition law (27) if \( \tilde{\chi} \) is a representation of \( S_n \) and if \( \tilde{K}_{\tilde{R}^n} \) satisfies the permutation invariance (38) and the composition law (34).

**Property 2. (Initial condition)** Let us next prove the initial condition (28). By substituting (33) into the left-hand side of (28) we have

\[
K_{\mathcal{M}_n}(x, y; 0) = \sum_{\sigma \in S_n} \chi(\sigma)K_{\tilde{R}^n}(x, \sigma y; 0)
= \sum_{\sigma \in S_n} \chi(\sigma)\delta(x - \sigma y)
= \chi(e)\delta(x - ey)
= \delta(x - y),
\]

(53)

where the second equality follows from the assumption (35). In the third equality we have used the fact that, if \( \sigma \) is not the identity element \( e \), \( x - \sigma y \) cannot be zero for any \( x, y \in \mathcal{M}_n \), which leads to \( \delta(x - \sigma y) = 0 \) for \( \sigma \neq e \). The last equality follows from \( \chi(e) = 1 \) and \( ey = y \). Hence we have shown that (33) satisfies the initial condition (28) if \( \tilde{\chi} \) is a representation of \( S_n \) and if \( \tilde{K}_{\tilde{R}^n} \) satisfies the initial condition (35).

A Proof of the path-integral formula (33)

Following the ideas presented in [37, 38], in this section we show that the formula (33) satisfies the properties (27)–(30) if \( \chi \) is a one-dimensional unitary representation of \( S_n \) and if \( K_{\tilde{R}^n} \) fulfills the conditions (34)–(38). Below we prove these four properties separately.

**Property 1. (Composition law)** Let us first prove the composition law (27). By substituting (33) into the left-hand side of (27) we have

\[
\int_{\mathcal{M}_n} dz K_{\mathcal{M}_n}(x, z; t_1)K_{\mathcal{M}_n}(z, y; t_2)
= \sum_{\sigma \in S_n} \sum_{\sigma' \in S_n} \chi(\sigma)\chi(\sigma') \int_{z_1 \cdots \cdots z_n} dz K_{\tilde{R}^n}(x, \sigma z; t_1)K_{\tilde{R}^n}(z, \sigma' y; t_2)
= \sum_{\sigma \in S_n} \sum_{\sigma' \in S_n} \chi(\sigma)\chi(\sigma') \int_{z_1 \cdots \cdots z_n} dz K_{\tilde{R}^n}(x, \sigma z; t_1)K_{\tilde{R}^n}(z, \sigma' y; t_2)
= \sum_{\sigma'' \in S_n} \chi(\sigma'') \int_{\tilde{R}^n} dw K_{\tilde{R}^n}(x, w; t_1)K_{\tilde{R}^n}(w, \sigma'' y; t_2)
= \sum_{\sigma'' \in S_n} \chi(\sigma'')K_{\tilde{R}^n}(x, \sigma'' y; t_1 + t_2)
= K_{\tilde{M}_n}(x, y; t_1 + t_2),
\]

(52)
Property 3. (Unitarity) Let us next prove the unitarity (29). By substituting (33) into the left-hand side of (29) we have

\[ K_{M_{n}}(x, y; t) = \sum_{\sigma \in S_{n}} \chi(\sigma) K_{\hat{R}^{n}}(x, \sigma y; t) \]

\[ = \sum_{\sigma \in S_{n}} \chi(\sigma)^{-1} K_{\hat{R}^{n}}(\sigma y, x; -t) \]

\[ = \sum_{\sigma \in S_{n}} \chi(\sigma^{-1}) K_{\hat{R}^{n}}(y, \sigma^{-1} x; -t) \]

\[ = K_{M_{n}}(y, x; -t), \quad (54) \]

where the second equality follows from \( \chi(\sigma) \in U(1) \) and the assumption (36), the third equality \( \chi(\sigma)^{-1} = \chi(\sigma^{-1}) \) and the assumption (38). Hence we have shown that (33) satisfies the unitarity (29) if \( \chi \) is a one-dimensional unitary representation of \( S_{n} \) and if \( K_{\hat{R}^{n}} \) satisfies the unitarity (36) as well as the permutation invariance (38).

Property 4. (Schrödinger equation) Let us finally prove that (33) satisfies the Schrödinger equation (30). By substituting (33) into the left-hand side of (30) we have

\[ \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_{x}^2 \right) K_{M_{n}}(x, y; t) = \sum_{\sigma \in S_{n}} \chi(\sigma) \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_{x}^2 \right) K_{\hat{R}^{n}}(x, \sigma y; t) \]

\[ = 0, \quad (55) \]

where we have used the assumption (37). Hence we have shown that (33) satisfies the Schrödinger equation (30) if \( K_{\hat{R}^{n}} \) satisfies the Schrödinger equation (37).

Now, as discussed in section 3, eq. (33) also satisfies the Robin boundary conditions (31) if \( K_{\hat{R}^{n}} \) satisfies the connection conditions (46). Thus the formula (33) fulfills all the required properties (27)–(31) if \( \chi \) is a one-dimensional unitary representation of \( S_{n} \) and if \( K_{\hat{R}^{n}} \) satisfies the assumptions (34)–(38) as well as the connection conditions (46).

B Proof of the integral formula (40)

In this section we prove the integral formula (40). To this end, let \( f \) be an arbitrary test function on \( \hat{R}^{n} \). Then we have

\[ \int_{\hat{R}^{n}} dy f(y) = \sum_{\sigma \in S_{n}} \int_{y_{\sigma(1)} \cdots y_{\sigma(n)}} dy f(y) \]

\[ = \sum_{\sigma \in S_{n}} \int_{z_{1} \cdots z_{n}} dz f(\sigma^{-1} z) \]

\[ = \int_{z_{1} \cdots z_{n}} dz \left( \sum_{\sigma \in S_{n}} f(\sigma^{-1} z) \right), \quad (56) \]

where in the second line we have changed the integration variable as \( y = \sigma^{-1} z \). By changing the notations as \( \sigma^{-1} \rightarrow \sigma \) and \( z \rightarrow y \) in the last line, we arrive at the formula (40).

References

[1] M. Girardeau,
“Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension”,
J. Math. Phys. 1, 516–523 (1960).
[2] T. Cheon and T. Shigehara, “Fermion-Boson Duality of One-Dimensional Quantum Particles with Generalized Contact Interactions”, Phys. Rev. Lett. 82, 2536–2539 (1999), arXiv:quant-ph/9806041 [quant-ph].
[3] E. H. Lieb and W. Liniger, “Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State”, Phys. Rev. 130, 1605–1616 (1963).
[4] T. Cheon and T. Shigehara, “Realizing discontinuous wave functions with renormalized short-range potentials”, Phys. Lett. A 243, 111–116 (1998), arXiv:quant-ph/9709035 [quant-ph].
[5] M. D. Girardeau and M. Olshanii, “Theory of spinor Fermi and Bose gases in tight atom waveguides”, Phys. Rev. A 70, 023608 (2004), arXiv:cond-mat/0401402 [cond-math.soft].
[6] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, Solvable Models in Quantum Mechanics (Springer, Berlin, 1988) Chap. I.4.
[7] J.-M. Souriau, “Quantification géométrique. Applications”, Ann. Inst. Henri Poincaré A 6, 311–341 (1967).
[8] J.-M. Souriau, Structure des Systèmes Dynamiques (Dunod, Paris, 1969) Chap. V.
[9] M. G. G. Laidlaw and C. M. DeWitt, “Feynman Functional Integrals for Systems of Indistinguishable Particles”, Phys. Rev. D 3, 1375–1378 (1971).
[10] J. M. Leinaas and J. Myrheim, “On the theory of identical particles”, Nuovo Cim. B 37, 1–23 (1977).
[11] A. Kundu, “Exact Solution of Double δ Function Bose Gas through an Interacting Anyon Gas”, Phys. Rev. Lett. 83, 1275–1278 (1999), arXiv:hep-th/9811247 [hep-th].
[12] M. D. Girardeau, “Anyon-Fermion Mapping and Applications to Ultracold Gases in Tight Waveguides”, Phys. Rev. Lett. 97, 100402 (2006), arXiv:cond-mat/0604357 [cond-mat.soft].
[13] R.-G. Zhu and A.-M. Wang, “Theoretical construction of 1D anyon models”, arXiv:0712.1264 [cond-mat.stat-mech].
[14] J. M. Harrison, J. P. Keating, and J. M. Robbins, “Quantum statistics on graphs”, Proc. Roy. Soc. A 467, 212–233 (2010), arXiv:1101.1535 [math-ph].
[15] T. E. Clark, R. Menikoff, and D. H. Sharp, “Quantum mechanics on the half-line using path integrals”, Phys. Rev. D 22, 3012 (1980).
[16] E. Farhi and S. Gutmann, “The Functional Integral on the Half-Line”, Int. J. Mod. Phys. A 5, 3029–3052 (1990).
[17] W. Janke and H. Kleinert, “Summing Paths for a Particle in a Box”, Lett. Nuovo Cim. 25, 297–300 (1979).
[18] A. Inomata and V. A. Singh, “Path integrals and constraints: Particle in a box”, Phys. Lett. A 80, 105–108 (1980).
[19] M. Goodman, “Path integral solution to the infinite square well”, Am. J. Phys. 49, 843–847 (1981).
[20] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, 5th (World Scientific, Singapore, 2009) Chap. 6.
[21] L. Schulman, “A Path Integral for Spin”, Phys. Rev. 176, 1558–1569 (1968).
[22] J. S. Dowker, “Quantum mechanics and field theory on multiply connected and on homogeneous spaces”, J. Phys. A 5, 936–943 (1972).

[23] P. A. Horváthy, “Quantisation in multiply connected spaces”, Phys. Lett. A 76, 11–14 (1980).

[24] H. P. Berg, “Feynman path integrals on manifolds and geometric methods”, Nuovo Cim. A 66, 441–449 (1981).

[25] P. A. Horvathy, G. Morandi, and E. C. G. Sudarshan, “Inequivalent quantizations in multiply connected spaces”, Nuovo Cim. D 11, 201–228 (1989).

[26] Y.-S. Wu, “General Theory for Quantum Statistics in Two-Dimensions”, Phys. Rev. Lett. 52, 2103–2106 (1984).

[27] N. P. Landsman, “Quantity and superselection sectors III: Multiply connected spaces and indistinguishable particles”, Rev. Math. Phys. 28, 1650019 (2016), arXiv:1302.3637 [math-ph].

[28] W. P. Thurston, The Geometry and Topology of Three-Manifolds, Princeton lecture notes (2002) Chap. 13, http://library.msri.org/books/gt3m/.

[29] T. Maciążek and A. Sawicki, “Non-abelian Quantum Statistics on Graphs”, Commun. Math. Phys. 330, 1293–1326 (2014), arXiv:1304.5781 [math-ph].

[30] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[31] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[32] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[33] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[34] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[35] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[36] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[37] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[38] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[39] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[40] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[41] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[42] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[43] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[44] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[45] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[46] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[47] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[48] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[49] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[50] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[51] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[52] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[53] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[54] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[55] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[56] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[57] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[58] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].

[59] T. Maciążek and B. H. An, “Universal properties of anyon braiding on one-dimensional wire networks”, Phys. Rev. B 102, 201407 (2020), arXiv:2007.01207 [quant-ph].