A GENERALIZED EULER PROBABILITY DISTRIBUTION

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Abstract. From a new class of $q$-deformed coherent states we introduce a generalization of the Euler probability distribution for which the main statistical parameters are obtained explicitly. As application, we discuss the corresponding photon counting statistics with respect to all parameters labeling the coherent states under consideration.

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1. Introduction

The canonical coherent states (CS), denoted $|z\rangle$ and labeled by points $z \in \mathbb{C}$, which go back to the early years of quantum mechanics [1] may be defined in four ways: (i) as eigenstates of the annihilation operator $A$, (ii) by applying the displacement operator $\exp(zA^* - \bar{z}A)$ on the vacuum state $|0\rangle$ such that $A|0\rangle = 0$, where $A^*$ is the Hermitian conjugate of $A$, (iii) by finding states that minimize the Heisenberg uncertainty principle and (iv) as a specific superposition of eigenstates $\phi_j$ of the harmonic oscillator number operator $N = A^*A$ as

$$\Psi_z = (e^{\bar{z}z})^{-1/2} \sum_{j=0}^{+\infty} \frac{\bar{z}^j}{\sqrt{j!}} \phi_j.$$

(1.1)

Recently, much attention has been paid to a class of states named "non classical" which appeared in quantum optics and in other fields ranging from solid states physics to cosmology. These states exhibit some purely quantum-mechanical properties, such as squeezing and antibunching (sub-Poissonian statistics) [2]. Among them, there are the so-called generalized coherent states (GCS) arising by extending one of the four aforementioned ways defining the canonical CS. These extensions may lead to inequivalent definitions [3].

The GCS are usually associated with algebras other than the oscillator one [4]. An important example is provided by the $q$-deformed CS, related to deformations of the canonical commutation relation or, equivalently, to deformed boson operators [5, 6], where this type of CS have been constructed in a way that they reduce to their standard counterparts as $q \to 1$. Among the latter ones, those satisfying the relation

$$A_q A^*_q - q A^*_q A_q = 1$$

with $0 < q < 1$, see [6]. The operators $A_q$ are often termed maths-type $q$-bosons [7] because the 'basic' numbers and special functions attached to them have been extensively studied in the mathematical literature for a long time [8]. A $q$-analogue of the number states expansion (1.1) is

$$\Psi_q^z := (e_q(z\bar{z}))^{-1/2} \sum_{j=0}^{+\infty} \frac{\bar{z}^j}{\sqrt{j!}} \phi^q_j,$$

(1.2)

(1 - $q$)$\bar{z}z < 1$ and $e_q(z\bar{z})$ being a $q$-exponential function (see Eq. (2.14) below). Here, the $q$-number states $\phi^q_j$ are supposed to span a formal quantum Hilbert space $\mathcal{H}_q$. By averaging the density matrix $\hat{\rho}_j := |\phi^q_j\rangle \langle \phi^q_j|$ in the system of CS $\Psi_q^z$, we obtain the Husimi $Q$-function [9] $Q_{\hat{\rho}_j}(z) := \mathbb{E}_{\Psi_q^z}(\hat{\rho}_j) = \langle \hat{\rho}_j \Psi_q^z, \Psi_q^z \rangle_{\mathcal{H}_q}$. Fixing $z$ and varying $j$, the function

$$j \mapsto Q_{\hat{\rho}_j}(z) = \frac{\bar{z}^{2j}}{[j]_q!} (e_q(|z|^2))^{-1}, \quad j = 0, 1, 2, \ldots,$$

(1.3)
defines a $q$-deformed Poisson distribution, here denoted by $X \sim \mathcal{P}(\lambda; q)$, with $\lambda = z\bar{z}$ as a parameter. The latter one was first introduced in [10] under the name Euler distribution. It is unimodal, have increasing failure rates, overdispersed and is infinitely divisible [11]. Note that it reduces to the standard Poisson distribution $\mathcal{P}(\lambda)$ in the limit $q \to 1$.

In the present paper, we consider the following generalization of (1.2):

$$\Psi_{z,m}^q := (\mathcal{N}_{q,m}(z\bar{z}))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} C_{j}^{q,m}(z) \phi_j^q,$$

where

$$C_{j}^{q,m}(z) := \frac{(-1)^{\min(m,j)} (q; q)_{\max(m,j)} (m-j)_{\frac{1}{2}} \sqrt{1-q}^{m-j} |z|^{|m-j|} e^{i(m-j)\arg(z)} (q; q)_{|m-j|} q^{m-j} (q; q)_j}{(q; q)_{m-j} \sqrt{q^{m-j}} (q; q)_m (q; q)_j} P_{\min(m,j)} \left( (1-q)z\bar{z}; q^{|m-j|} |q \right),$$

are given in terms of Wall polynomials $P_n(\cdot, a|q)$ ([14], p.260). For fixed $m \in \mathbb{N}$, these coefficients generalize the above ones $z^j$ in the number states expansion (1.2). $\mathcal{N}_{q,m}(z\bar{z})$ is a factor ensuring the normalization condition $\langle \Psi_{z,m}^q, \Psi_{z,m}^q \rangle_q = 1$.

Here, also by the same procedure leading to (1.3), we fix $z$, and we associate to the generalized CS $\Psi_{z,m}^q$ a discrete probability distribution, denoted by $X \sim \mathcal{P}(\lambda; q, m)$, as

$$j \mapsto \Pr(X = j) = \langle \hat{\rho}_j \Psi_{z,m}^q, \Psi_{z,m}^q \rangle_q, \quad j = 0, 1, 2, \ldots, \lambda = z\bar{z},$$

which naturally, generalizes the Euler probability distribution in (1.3). Precisely, our goal is to introduce explicitly the discrete random variable $\mathcal{P}(\lambda; q, m)$. Next, we write down its generating function (p.g.f) from which we drive the main statistical parameters of the $q$-deformed number operator $N_q = A_q^* A_q$. For the latter one, we examine the photon counting statistics in the state $\Psi_{z,m}^q$ with respect to the location of the labeling point $z$ in the corresponding phase space.

The paper is organized as follows. In Section 2, we recall some notations of $q$-calculus. In Section 3, we define a new class of generalized coherent states after a brief review of the coherent state formalism and its $q$-analog. In Section 4, we introduce a generalized Euler probability distribution and we give a formula for its generating function (p.g.f). In Section 5, the main statistical parameters of the $q$-deformed number operator are derived. Section 6 is devoted to discuss the domain of classicality/non-classicality of the generalized $q$-deformed CS we have introduced. In section 7, we conclude with some remarks.

2. Notations

This section collects the basic notations of $q$-calculus and definitions used in the rest of the paper. For more details we refer to [8, 14, 15]. We assume that $0 < q < 1$.

1. For $a \in \mathbb{C}$, the number

$$[a]_q = \frac{1 - q^a}{1 - q}$$

is called a $q$-number which satisfies $[a]_q \to a, q \to 1$. In particular, $[n]_q$ is called a $q$-integer.

2. For $a \in \mathbb{C}$, the $q$-shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - a q^k), \quad n \in \mathbb{N}, \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k),$$

and, for any $\alpha \in \mathbb{C}$, we shall also use

$$(a; q)_n = \frac{(a; q)_\infty}{(q^\alpha; q)_\infty}, \quad \alpha q^n \neq q^{-n}, \quad n \in \mathbb{N}. \quad (2.3)$$
7. The hypergeometric series is defined by
\[
{}_{r}F_{s}\left(\begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s
\end{array} \mid \frac{\xi}{q}\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k (\xi q^{k+1})}{(b_1)_k \cdots (b_s)_k k!}. 
\]
where \((a)_0 := 1\) and \((a)_k := (a(a+1)(a+2) \cdots (a+k-1)), k = 1, 2, 3, \cdots\), is the shifted factorial. A basic hypergeometric series is defined by
\[
{}_{r}\phi_{s}\left(\begin{array}{c}
a_1, \ldots, a_r \\
b_1, \ldots, b_s
\end{array} \mid \frac{\xi}{q}\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k (\xi q^{k+1})}{(b_1, \ldots, b_s; q)_k} (-1)^{(1+r-s)k} q^{(1+s-r)k}(\xi q^{k}) (1+s-r)(2)_k (q; q)_k. 
\]
We note that both of the series $rF_s$ and $r\phi_s$ converges absolutely for all $\xi$ if $r \leq s$ and for $|\xi| < 1$ if $r = s + 1$. In this special case, Eq. \eqref{2.18} reduces to

\[ s+1\phi_s \left( a_1, \ldots, a_{s+1} | q; \xi \right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{\xi^k}{(q; q)_k}. \]  

\[ 2.19 \]

8. The Laguerre polynomial is defined by

\[ L_n^{(\alpha)}(x) := \frac{(\alpha + 1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(\alpha + 1)_j} \frac{x^j}{j!}, \quad \alpha > -1, \]  

\[ 2.20 \]

and can be expressed in terms of the confluent hypergeometric function $1F_1$ (which is defined by $r = s = 1$ in \eqref{2.17}) as

\[ L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} 1F_1 \left( -n \left| \frac{\alpha + 1}{x} \right. \right). \]  

\[ 2.21 \]

9. The little $q$-Laguerre or Wall polynomial is defined by means of the basic hypergeometric series $2\phi_1$ or $2\phi_0$ as

\[ P_n(x; a|q) = 2\phi_1 \left( q^{-n}; 0 | q; qx \right) = \frac{1}{(a^{-1} q^{-n}; q)_n} 2\phi_0 \left( q^{-n}, x^{-1} - | q; \frac{x}{a} \right) \]  

\[ 2.22 \]

and satisfy

\[ \lim_{q \to 1} P_n(x(1-q); q^n|q) = \frac{n!}{(\alpha + 1)_n} L_n^{(\alpha)}(x). \]  

\[ 2.23 \]

3. A CLASS OF GENERALIZED $q$-DEFORMED COHERENT STATES

The original idea of coherent states was introduced by E. Schrödinger in 1926 [1] in order to obtain quantum states in $L^2(\mathbb{R})$ that follow the classical flow associated to the harmonic oscillator Hamiltonian

\[ \hat{H} = -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2}. \]

Namely, we have a set $\{\Psi_z \in L^2(\mathbb{R}), \ z \in \mathbb{C}\}$ labeled by elements of $\mathbb{C} \simeq T^* \mathbb{R}$ (the phase space of a particle moving on $\mathbb{R}$) given by

\[ \Psi_z(x) = e^{-\frac{1}{2} \hbar z^2} \frac{1}{(\pi \hbar)^{1/4}} \exp \left( -\frac{1}{2\hbar} (\bar{z}^2 + x^2 - 2\sqrt{\hbar} z x) \right) \]  

\[ 3.1 \]

where $\hbar$ denotes the Planck parameter. If we denote $\{\phi_j\}$ an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions of $\hat{H}$, i.e. $\hat{H} \phi_j = j \phi_j$ (called number states) and we set $\hbar = 1$ for the sake of simplicity. Then, the function in \eqref{3.1} also admit an expansion over the basis vectors $\{\phi_j\}$ as

\[ \Psi_z(x) = (e^{z\bar{z}})^{-1/2} \sum_{j=0}^{+\infty} \frac{z^j}{\sqrt{j!}} \phi_j(x). \]  

\[ 3.2 \]

Similarly to the standard harmonic oscillator, the $q$-analog of the number states are supposed to span a formal quantum Hilbert space $\mathcal{H}_q$ and are given by

\[ \phi_j^q = \frac{(A_q^n)^n}{\sqrt{n}_q} |0\rangle_q, \]  

\[ 3.3 \]
$A_q$ and $A_q$ are $q$-creation and $q$-annihilation operators satisfying the commutation relation $A_qA_q^* - qA_q^*A_q = 1$. Here, the oscillator-like Hamiltonian may be defined by $\hat{H}_q := \frac{1}{2}(A_qA_q^* + A_q^*A_q)$. Therefore, a generalization of coherent states (3.2) is provided by states with the form

$$\Psi_z^q := (e_q(z\bar{z}))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \frac{z^j}{\sqrt{|j|}} \phi_j^q$$

(3.4)

where $z\bar{z} < (1 - q)^{-1}$ which determines the domain of convergence of $e_q(z\bar{z})$. We also observe that coefficients $C_j^q(z) := \frac{z^j}{\sqrt{|j|}}$, $j = 0, 1, 2, \ldots$, (3.5) appearing in (3.4) form an orthonormal system in the Hilbert space $L^2(\mathbb{C}, d\mu_q(z))$

$$d\mu_q(z) = \frac{d\theta}{2\pi} \sum_{l=0}^{q^l(q; q)_\infty} q^l(q; q)_l d\mu_l(z),$$

(3.6)

and $d\mu_l(z)$ is the Lebesgue measure on the circle of radius $r_l = q^{\frac{l}{2}}/\sqrt{1 - q}$. Actually, these coefficients $C_j^q(z)$ constitute a particular case of a larger class of $2D$ complex $q$-orthogonal polynomials introduced in [12] as

$$H_{m,j}(z, \zeta|q) := \sum_{k=0}^{m\wedge j} \left[ m \atop k \right] q^{\left[ j \atop k \right]} (1 - q)^{m-j} q^{m(j-k)} q^m e^{i(m-j)\arg(z)} P_{m\wedge j}((1 - q)z\bar{z}; q^{m-j}|q),$$

(3.7)

where $m, j \in \mathbb{N}$ and $m \wedge j = \min(m, j)$. Indeed, by taking $\zeta = \bar{z}$ in (3.7) and making a slight modification, we will consider, as in [13], the functions

$$C_j^m(z) := (-1)^{m\wedge j} q^{m\wedge j} q^{(m\wedge j)} \sqrt{1 - q^{m-j}} |z|^{m-j} e^{i(m-j)\arg(z)} P_{m\wedge j}((1 - q)z\bar{z}; q^{m-j}|q),$$

(3.8)

$P_n(\cdot, a|q)$ denote a Wall polynomial ([14], p.109) and $m \vee j = \max(m, j)$. These coefficients generalize the above ones in (3.5) in the sense that for $m = 0$ we have that $C_j^0(z) = C_j^q(z)$. Therefore, by fixing $m \in \mathbb{N}$ in the expression given by (3.8), we can extend the expression (3.4) of the $q$-deformed CS by setting

$$\Psi_{z,m}^q := (N_{q,m}(z\bar{z}))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} C_j^m(z) \phi_j^q,$$

(3.9)

where

$$N_{q,m}(z\bar{z}) := \frac{(q^{1-m}(1 - q)z\bar{z}; q)_m}{q^m(q^{-m}(1 - q)z\bar{z}; q)_\infty}, \quad z\bar{z} < q^m(1 - q)^{-1},$$

(3.10)

is a factor ensuring the normalization condition $\langle \Psi_{z,m}^q, \Psi_{z,m}^q \rangle_{q_i} = 1$. These $q$-deformed CS will, naturally, allows us to generalize the Euler distribution [10] with respect to the parameter $m = 0, 1, 2, \ldots$.

**Remark 3.1.** In [12], Ismail and Zhang have also introduced a second class of 2D complex $q$-orthogonal polynomials by setting

$$h_{m,j}(z, \zeta|q) = q^{m-j} z^m \zeta^j \sum_{k=0}^{+\infty} \frac{(q^{-m}, q^{-j}; q)_k}{(q; q)_k} \left( \frac{-q}{z\zeta} \right)^k$$

(3.11)

which are connected to the previous polynomials (3.7) by

$$h_{m,j}(z, \zeta|q^{-1}) = q^{-m-j} i^{-m-j} H_{m,j}(iz, i\zeta|q).$$

(3.12)

This last relation may also suggest the construction of another extension of $q$-deformed CS via the same procedure. In a such case, this would lead to a new generalization (with respect to $m$) of the Heine probability distribution [10].
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4. A generalized Euler distribution

Averaging the density matrix \( \hat{\rho}_j \) = \( |q_j\rangle \langle q_j| \) in the system of generalized CS \( \Psi_{z;m}^q \) enables us to define, in the usual way, a photon counting probability distribution by setting \( p_j(\lambda; q, m) := \langle \hat{\rho}_j \Psi_{z;m}^q, \Psi_{z;m}^q \rangle_\lambda, \lambda = z \bar{z} \). Explicit calculations lead to the following definition.

**Definition 4.1.** The discrete random variable having the probability distribution

\[
p_j(\lambda; q, m) := \frac{q^{2(m_j^\lambda)}(1 - q)^{|m - j|\lambda}}{N_{m,q}(\lambda)} (q; q)^{m_j^\lambda} (q; q_j)^{m_j^\lambda} (1 - q)^{|m - j|\lambda} |q|^{2|m - j|}, \quad j = 0, 1, \ldots ,
\]

is denoted by \( X \sim \mathcal{P}(\lambda; q, m) \) and will be called a generalized Euler probability distribution of index \( m \).

Note that for \( m = 0 \), Eq. (1.1) reduces to

\[
p_j(\lambda; q, 0) = \frac{\lambda^j}{|j|!} E_q(-\lambda), \quad j = 0, 1, 2, \ldots ,
\]

which is the standard Euler distribution \( \mathcal{P}(\lambda; q) \) with parameter \( \lambda \) (see [10]). The \( q \)-exponential function \( E_q(\cdot) \) is defined by (2.15).

**Remark 4.1.** In ([17], p. 495) the author has merged into one formula:

\[
P(j; \alpha, q) = \frac{1}{|j|!} \left( \frac{\alpha}{1 - q} \right)^j \left( e_q(\frac{\alpha}{1 - q}) \right)^{-1}
\]

the probability mass functions (PMF) of two distributions. That is, for \( 0 < q < 1 \) and \( 0 < \alpha < 1 \) with \( \alpha = (1 - q)\lambda \), Eq. (4.3) reproduces the Euler distribution (1.3) while for \( q > 1 \) and \( \alpha < 0 \) it defines the Heine distribution [10].

For \( m \neq 0 \), one can check that, as \( q \rightarrow 1 \), the following limit

\[
p_j(\lambda; q, m) \rightarrow \frac{\lambda^{|m - j|} e^{-\lambda}}{|j|!} \frac{L_{m\wedge j}^{(|m - j|)}}{\sqrt{m!}}
\]

holds true, \( L_{m\wedge j}^{\alpha}(\cdot) \) being the Laguerre polynomial in (2.20). So that we recover the generalized Poisson distribution \( \mathcal{P}(\lambda; q, m) \) having the quantity in the R.H.S of (4.4) as its mass probability function [16].

A convenient way to summarize all properties of the distribution \( X \sim \mathcal{P}(\lambda; q, m), m \geq 0 \), is with the probability generating function (p.g.f) which is defined by the expectation

\[
G_X(t) := \mathbb{E} (t^X),
\]

where \( t \) is a real number. We precisely establish the following result (see Appendix A for the proof).

**Proposition 4.1.** For \( \lambda \in [0, q^m(1 - q)^{-1}] \), the p.g.f of \( X \sim \mathcal{P}(\lambda; q, m) \) is given by

\[
G_X(t) = \frac{t^m(q^m(1 - q)\lambda)_{\infty}}{(q^m t - q(1 - q)\lambda)_{\infty}} 3\phi_2 \left( \begin{array}{c} q^{-m}, q/t, t \\ q^{-1-m}(1 - q)\lambda, q \end{array} ; q; q\lambda(1 - q) \right), \quad |t| \leq 1,
\]

in terms of the basic terminating hypergeometric series \( 3\phi_2 \).

**Corollary 4.1.** For \( m = 0 \), the expression of \( G_X(t) \) reduces to

\[
G_X(t) \rightarrow t^m \exp(\lambda(t - 1)) \frac{(1 - q)z\bar{z}; q}_{\infty} \frac{(1 - q)z\bar{t}; q}_{\infty},
\]

which is the well known p.g.f of the Euler distribution. For \( m \neq 0 \), we have the following limit

\[
G_X(t) \rightarrow t^m \exp(\lambda(t - 1)) L_{m}^{(0)}(\lambda(2 - \frac{1}{t})), \quad |t| \leq 1;
\]

as \( q \rightarrow 1 \).
Proof. Assuming that $|t| \leq 1$ and taking into account the relation (2.10), we get that
\[
\lim_{q \to 1} \frac{(q^{-m}(1-q)\lambda; q)_\infty}{(q^{-mt}\lambda(1-q); q)_\infty} = \lim_{q \to 1} e_q(q^{-m}t\lambda)E_q(-q^{-m}\lambda) = \exp(\lambda(t-1)).
\] (4.9)

By another side, using Eq.(2.19) together with the fact that (identity (2.3), Eq.(5.5) reduces to
\[
\text{Thus, from identities (2.13)-(2.12) we, successively, have}
\]
\[
\sum_{k=0}^{m} \frac{(q^{-m}, q/t, t; q)_k}{(q^{1-m}(1-q)\lambda, q; q)_k}\frac{(q^{-1}\lambda(1-q))}{(q; q)_k}.
\] (4.10)

Thus, from identities (2.13)-(2.12) we, successively, have
\[
\lim_{q \to 1} \sum_{k=0}^{m} \frac{(q^{-m}, q/t, t; q)_k}{(q^{1-m}(1-q)\lambda, q; q)_k}\frac{(q^{-1}\lambda(1-q))}{(q; q)_k} = \sum_{k=0}^{m} \lim_{q \to 1} \frac{(q^{-m}; q)_k}{(q; q)_k} \frac{(q/t, t; q)_k}{(q^{1-m}(1-q)\lambda; q)_k}(1-q^k)q^{-k}q^{-k}\lambda^k
\]
\[
= \sum_{k=0}^{m} \lim_{q \to 1} \frac{(-1)^kq^{k-1}}{q}[q]_q^{-mk}\frac{(q/t, t; q)_k}{(q^{1-m}(1-q)\lambda; q)_k}q^{-k}\lambda^k
\]
\[
= \sum_{k=0}^{m} \frac{(-1)^k(\lambda(1-t)(1-t^{-1}))}{k!}q^{-k}\lambda^k.
\] (4.11)

Finally, from (4.20) we can see that the last sum in (4.11) is the evaluation of the Laguerre polynomial $L_m^{(0)}$ at $\lambda(2-t(1-t^{-1}))$. This ends the proof. □

Remark 4.2. By setting $t = q^{\mu}$ in the R.H.S of (4.2), we recover the characteristic function $\Phi^m_X(u)$ of the generalized Poisson distribution $\mathcal{P}(\lambda; m)$, which was obtained in [16, Prop 4.1, p.264].

5. Expectation and variance of the operator $[N]_q$

Here, we first start by observing that the expectation and the variance of the q-deformed number operator $[N]_q$ in the state $\Psi_{z,m}^q$ in the state $\Psi_{z,m}^q$ coincide with those of the $q$-deformed random variable whose values are the $q$-numbers $[j]_q$, $j = 0, 1, 2, \cdots$, as defined by (2.1). Namely, we have
\[
\langle [N]_q \rangle = \sum_{j \geq 0} [j]_q p_j(\lambda; q, m).
\] (5.1)

Proposition 5.1. The mean value and the mean square deviation of the number operator $[N]_q = A_q^* A_q$ in the state $\Psi_{z,m}^q$ are respectively given by
\[
\langle [N]_q \rangle = \lambda + [m]_q,
\] (5.2)
\[
\langle [N]_q - \langle [N]_q \rangle \rangle = \lambda^2 q^m (1 + q - 2q^{-m}) + \lambda q^m (2[m]_q + q^m).
\] (5.3)

Proof. From (5.1) and (2.1), one can write
\[
\langle [N]_q \rangle = \frac{1}{1-q}(1 - G_X(q))
\] (5.4)
in terms of the p.g.f (1.6). In view of Proposition 4.1, we get that
\[
\langle [N]_q \rangle = \frac{1}{1-q}\left(1 - \frac{q^m(q^{-m}\lambda(1-q); q)_\infty}{(q^{-mt}\lambda(1-q); q)_\infty}\right)\Phi_2(\frac{q^{-m}, 1, q}{q^{1-m}(1-q)\lambda, q}; q; q\lambda(1-q))
\] (5.5)
Now, by using (4.10) for $t = q$, the series $\Phi_2$ in (5.5) reduces to 1. On the other hand, by applying the identity (2.3), Eq. (5.5) reduces to
\[
\langle [N]_q \rangle = \frac{1}{1-q}\left(1 - q^m(q^{-m}\lambda(1-q); q)_1\right) = [m]_q + \lambda.
\] (5.6)
For the mean square deviation, Eq. (6.1) leads to
\[
\langle ([N]_q - \langle [N]_q \rangle)^2 \rangle = \sum_{j \geq 0} ([j]_q - \langle [N]_q \rangle)^2 p_j(\lambda; q, m) = \sum_{j \geq 0} [j]^2 p_j(\lambda; q, m) - \langle [N]_q \rangle^2, \tag{5.7}
\]
and by using the identity (2.1), we, successively, obtain
\[
\sum_{j \geq 0} [j]^2 p_j(\lambda; q, m) = \frac{1}{(1 - q)^2} \left( \sum_{j \geq 0} (1 - 2q^j + q^{2j}) p_j(\lambda; q, m) \right) = \frac{1}{(1 - q)^2} \left( \sum_{j \geq 0} p_j(\lambda; q, m) - 2q \sum_{j \geq 0} q^j p_j(\lambda; q, m) + \sum_{j \geq 0} q^{2j} p_j(\lambda; q, m) \right) = \frac{1}{(1 - q)^2} \left( 1 - 2G_X(q) + G_X(q^2) \right). \tag{5.8}
\]
Next, we replace the quantities occurring in the R.H.S of (5.7) by their respective expressions in Eq. (5.8) and Eq. (5.4). This leads, after simplifications, to
\[
\langle ([N]_q - \langle [N]_q \rangle)^2 \rangle = \frac{1}{(1 - q)^2} \left( 1 - 2G_X(q) + G_X(q^2) \right) - \frac{1}{(1 - q)^2} \left( 1 - 2G_X(q) + (G_X(q))^2 \right) = \frac{1}{(1 - q)^2} \left( G_X(q^2) - (G_X(q))^2 \right). \tag{5.9}
\]
Finally, a direct evaluation of $G_X$ at $q^2$ using (4.6) gives us
\[
G_X(q^2) = q^{2m} (1 - q^{-m} \lambda(1 - q)) (1 - q^{1-m} \lambda(1 - q) + q^{-m} \lambda[q]_q(1 + q)(1 - q^2)). \tag{5.10}
\]
Summarizing the above calculations, we arrive at the expression of $\langle ([N]_q - \langle [N]_q \rangle)^2 \rangle$ as announced in (5.3). \( \square \)

6. Photon counting statistics for $[N]_q$

To define a measure of non-classicality of a quantum state, one can follow several different approaches. An early attempt was initiated by Mandel [20] who investigated radiation fields and introduced the parameter
\[
Q = \frac{\text{Var}(Y)}{\mathbb{E}(Y)} - 1, \tag{6.1}
\]
for $q = 0$. If $Q < 0$, then the underlying statistics are said to be sub-Poissonian and describes the anti-bunching of the light. Such anti-bunching is an explicit feature of a quantum field and its observation would provide rather direct evidence of existence of optical photons. Super-Poisson statistics corresponds rather to $Q > 0$ and the bunching phenomenon occurs. It is possible to look on this phenomena as a characteristic quantum feature of thermal photons.

In our context, the Mandel parameter reads
\[
Q_{m,q}(\lambda) := \frac{\langle ([N]_q - \langle [N]_q \rangle)^2 \rangle - \langle [N]_q \rangle^2}{\langle [N]_q \rangle}, \tag{6.2}
\]
For $m = 0$, it reduces to $Q_{0,q} = (q - 1)\lambda$ where $\lambda = z\bar{z}$ which shows the sub-Poissonian nature of the photon statistics of $[N]_q$ inside the domain $0 < z\bar{z} < (1 - q)^{-1}$. For $m \neq 0$, we may use the results of Proposition 5.1 to find out that sign of $Q_{m,q}$ is the sign of the quantity
\[
P(\lambda) := \lambda^2 q^m (1 + q - 2q^{-m}) + \lambda (q^m (2[m]_q + q^m) - 1) - [m]_q \tag{6.3}
\]
which is polynomial in the variable $\lambda = z\bar{z}$ and whose discriminant is
\[
\Delta := q^{2m} (2[m]_q + q^m)^2 + 4[m]_q(q^{m+1} - 2q^m) + 1 = \frac{q^{m} - 1}{(1 - q)^2} \delta \tag{6.4}
\]
where
\[ \delta := (1 + q)^2q^{3m} + (q - 3)(q + 1)q^{2m} + (q - 1)(3q + 1)q^m + 7 - q(6 + q). \] (6.5)

The dependence of the sign of \( \delta \) on the values of \( m \) is discussed in Appendix B.

**Lemma 6.1.** (i) If \( 0 < q \leq q_0 \), \( q_0 = \frac{\sqrt{5} - 2}{11} \) we have that \( \Delta < 0 \) for all \( m \neq 0 \). (ii) If \( q_0 < q < 1 \) then :
(a) \( \Delta < 0 \) if \( m > m_q := \left\lfloor \frac{\log q}{\log q} \right\rfloor \),
(b) \( \Delta > 0 \) if \( m \leq m_q \). Here, \( \zeta_q \) is the real solution of the equation
\[ (1 + q)^2x^3 + (q - 3)(1 + q)x^2 + (q(3q - 2) - 1)x + 7 - q(6 + q) = 0 \] (6.6)
and \( \lfloor s \rfloor \) denotes the greatest integer not exceeding \( s \). In this last case, we denote the two roots of the polynomial \( P(\lambda) \) in (6.3) by
\[ \lambda_{\pm}^{m,q} = \left( \frac{1 - q^m(2[m]q + q^m) \pm \sqrt{\Delta}}{2q^m(1 + q - 2q^{-m})} \right)^{1/2}. \] (6.7)

We summarize the discussion on the classicality/nonclassicality of \( \Psi_q^{q,z,m} \) with respect to the location of the labeling point \( z \) in the complex plane.

**Proposition 6.1.** The photon number statistics for \( [N]_q \) in the state \( \Psi_q^{q,z,m} \) are
(i) sub-Poissonian in the following cases :
(a) \( 0 < q \leq q_0 \) and \( m \neq 0 \) for \( zz(1 - q) < q^m \).
(b) \( q_0 < q < 1 \) and \( m > m_q \) for \( zz(1 - q) < q^m \).
(c) \( q_0 < q < 1 \) and \( m \leq m_q \) for \( zz < \lambda_{+}^{m,q} \) or \( \lambda_{-}^{m,q} < zz < q^m \).
(ii) super-Poissonian if \( q_0 < q < 1 \) and \( m \leq m_q \) for \( \lambda_{+}^{m,q} < zz < \lambda_{-}^{m,q} \).
(iii) Poissonian if \( q_0 < q < 1 \) and \( m \leq m_q \) for \( zz = \lambda_{+}^{m,q} \) or \( zz = \lambda_{-}^{m,q} \).

As we have already mentioned, for \( m = 0 \) the photon statistics of \( [N]_q \) is sub-Poissonian inside the whole domain \( zz < (1 - q)^{-1} \). This fact may be known in the literature since it is associated with the Euler distribution. However, when \( m \neq 0 \) we can conclude from Proposition 6.1 that for specific ranges of parameters \( q \) and \( m \) namely \( q \in [q_0, 1] \) and \( m \in [0, m_q] \), the \( m \)-deformation of the Euler distribution as defined by (4.1) gives rise, inside the previous domain \( zz < (1 - q)^{-1} \), to two subdomains where the photon counting of \( [N]_q \) exhibits Poissonian (coherent) and super-Poissonian (thermal) statistics. Since \( m_q \) is a threshold value depending on the deformation parameter \( q \), we describe below the behavior of \( m_q \) with respect to \( q \).
7. Concluding remarks

While dealing with a new class of $q$-deformed CS, denoted $\Psi^{q,m}_z$, that we have defined "à la Iwata" by their number states expansion, $0 < q < 1$ and $z$ being a labeling complex number, we have introduced a deformation with respect to the parameter $m = 0, 1, 2, \cdots$. For this distribution, we have obtained the generating function from which the main statistical parameters of the $q$-deformed number operator $[N]_q$ have been derived. As application, we have examined the photon counting statistics of $\{N\}_q$ in the state $\Psi^{q,m}_z$ with respect to the ranges of parameters $q$, $z$ and $m$. For $m = 0$, these statistics are sub-Poissonian (antibunching) inside the domain $z \bar{z} < (1 - q)^{-1}$ while for $m \neq 0$ specific ranges of $q \in ]q_0, 1[$ and $m \in ]0, m_q]$ reveal that the $m$-deformation of the Euler distribution gives rise, inside the previous domain of $z$, to the subdomains: circles and annulus where the statistics under consideration are Poissonian (coherent) and super-Poissonian (bunching) respectively. Finally, this analysis may be better understood if we were able to recover our generalized $q$-deformed CS $\Psi^{q,m}_z$ as a kind of displaced Fock states as $D_q(z)|\phi_q^m\rangle$ where the operator $D_q(z)$ and the ket vector $|\phi_q^m\rangle$ should be fixed. This would require further investigation which will be deferred to a later paper.

Appendix A

According to (4.5) the p.g.f of $X \sim \mathcal{D}(\lambda; m, q)$ is given by

$$E(t^X) = \sum_{j=0}^{+\infty} t^j p_j(\lambda, q, m) = \frac{1}{N_{m,q}(\lambda)(q; q)_m} \mathcal{G}^{q,m}(\lambda, t)$$

(7.1)

where

$$\mathcal{G}^{q,m}(\lambda, t) = \sum_{j=0}^{+\infty} \frac{t^j q^{2jm}(1 - q)^{m-j} \lambda^{m-j}}{(q; q)_j} \left( \frac{(q; q)_m}{(q; q)_m} P_j((1 - q)\lambda; q^{m-j}|q)_j \right)^2.$$

(7.2)

We decompose this last sum as

$$\mathcal{G}^{q,m}(\lambda, t) = \mathcal{S}^{q,m}_{(\infty)}(\lambda, t) + \mathcal{S}^{q,m}_{(\infty > \infty)}(\lambda, t)$$

(7.3)

where

$$\mathcal{S}^{q,m}_{(\infty)}(\lambda, t) = \sum_{j=0}^{m-1} \frac{t^j q^{2jm}(1 - q)^{m-j} \lambda^{m-j}}{(q; q)_j} \left( \frac{(q; q)_m}{(q; q)_m} P_j((1 - q)\lambda; q^{m-j}|q)_j \right)^2.$$

(7.4)

and

$$\mathcal{S}^{q,m}_{(\infty > \infty)}(\lambda, t) = \sum_{j=0}^{+\infty} \frac{t^j q^{2jm}(1 - q)^{j-m} \lambda^{j-m}}{(q; q)_j} \left( \frac{(q; q)_m}{(q; q)_m} P_m((1 - q)\lambda; q^{j-m}|q)_j \right)^2.$$

By making use of the identity (188, p.3) :

$$P_n(x; q^{-N}|q) = x^{N-1}q^{-N}q^{(N+1-2n)}(q^{N+1}; q)_n x^{-N}q^{N}P_n(x; q^{-N}|q)$$

(7.5)

for parameters $N = j - m, n = j$ and $x = (1 - q)\lambda$, we obtain that $\mathcal{S}^{q,m}_{(\infty)}(\lambda, t) = 0$. For the infinite sum in (7.4), let us rewrite the Wall polynomial as (114, p.260) :

$$P_n(x; a|q) = \frac{(x^{-1}; q)_n (x)^n q^{-\frac{a}{2}}}{(aq; q)_n} \frac{(q^{-n}; 0)}{(x^{-1-n}; q)_n} \frac{(aq^{n+1})}{(a q^{n+1})}$$

(7.6)
where \( n = m, x = (1 - q)\lambda \) and \( a = q^{j-m} \). Next, by using (2.8) and (2.7) respectively, Eq.(7.3) becomes

\[
\mathcal{S}_m^q(t, \lambda) = \frac{q^{2m}(q_1^m - q_m^m)}{\xi_m} \eta_m(t, \lambda)
\]

(7.7)

where

\[
\eta_m^q(t, \lambda) = \sum_{j > 0} \frac{Y_j(q; q)_j}{(q; q)_j} \left( \xi q^{-1} \right)^2 \phi_1 \left( q^{-m}; 0 | q; q^2 \right)^2
\]

(7.8)

with \( \lambda = (1 - q)\lambda \) and \( Y = q^{-m}t\lambda(1 - q) \). By identity (2.7), it follows that \( (q; q)_j^2(q^{j-m}; q)_m^2 = (q; q)_j^2 \), and with (2.19), we get

\[
\eta_m^q(t, \lambda) = \sum_{j > 0} \frac{Y_j(q; q)_j}{(q; q)_j} \sum_{j = 0}^m \frac{(q^{-m}; q)_k}{(q; q)_k} \frac{(q^{j+1}; q)_l}{(q; q)_l} \sum_{l = 0}^m \frac{(q^{j+1}; q)_l}{(q; q)_l} \frac{(q^{-m}; q)_l}{(q; q)_l} \frac{(q^{j+1}; q)_l}{(q; q)_l} \frac{(q; q)_k}{(q; q)_k} \frac{(q; q)_l}{(q; q)_l}
\]

(7.9)

Now, by applying the \( q \)-binomial theorem ([14], p.17):

\[
\sum_{n=0}^a \frac{a^n}{(q; q)_n} = \frac{1}{(a; q)_\infty}, \quad |a| < 1,
\]

(7.10)

for \( a = q^{k+l}Y \), which requires the condition \(|t| \leq 1\), the R.H.S of (7.9) takes the form

\[
\eta_m^q(t, \lambda) = \sum_{k,l=0}^m \frac{(q^{-m}; q)_k}{(q; q)_k} \frac{(q^{k}; q)_l}{(q; q)_l} \frac{(q^{m}; q)_l}{(q; q)_l} \frac{1}{(Y q^{k+l}; q)_\infty}
\]

(7.11)

By applying the identity (2.3) to the factor \( \frac{1}{(Y q^{k+l}; q)_\infty} \), Eq.(7.11) can be rewritten as

\[
\eta_m^q(t, \lambda) = \frac{1}{(Y; q)_\infty} \sum_{k,l=0}^m \frac{(q^{-m}; q)_k}{(q; q)_k} \frac{(q^{k}; q)_l}{(q; q)_l} \frac{(q^{m}; q)_l}{(q; q)_l} \frac{1}{(Y q^{k+l}; q)_t}
\]

(7.12)

Next, by writing \( (Y; q)_t = (Y; q)_k(q^{k}Y; q)_t \), it follows that

\[
\eta_m^q(t, \lambda) = \frac{1}{(Y; q)_\infty} \sum_{k=0}^m \frac{(q^{-m}, Y; q)_k}{(q; q)_k} \frac{(q^{k}; q)_k}{(q; q)_k} \frac{(q^{m}; q)_l}{(q; q)_l} \frac{1}{(Y q^{k+l}; q)_t}
\]

(7.13)

which can also be expressed as

\[
\eta_m^q(t, \lambda) = \frac{1}{(Y; q)_\infty} \sum_{k=0}^m \frac{(q^{-m}, Y; q)_k}{(q; q)_k} \frac{(q^{k}; q)_k}{(q; q)_k} \frac{2\phi_1 \left( q^{-m}, q^{k}Y \right)}{(q^{1-m}; q)_k (q; q)_k} \frac{1}{(Y q^{k+l}; q)_t}
\]

(7.14)

in terms of the series \( 2\phi_1 \). The latter one satisfies the identity ([8], p.10):

\[
2\phi_1 \left( q^{-n}; \frac{b}{c}; q \right) = \frac{(b^{-1}; q)_n}{(c; q)_n} \beta_n, \quad n = 0, 1, 2, \ldots
\]

(7.15)

which, with the parameters \( n = m, b = q^kY \) and \( c = q^{1-m}\xi \), allows us to rewrite (7.14) as

\[
\eta_m^q(t, \lambda) = \frac{1}{(Y; q)_\infty} \sum_{k=0}^m \frac{(q^{-m}, Y; q)_k}{(q; q)_k} \frac{(q^{k}; q)_k}{(q; q)_k} \frac{(q^{1-m-k}\xi / Y; q)_m}{(q^{1-m}; q)_m} (q^{k}Y)^m \frac{(q^{1-k}/t; q)_m}{(q^{1-m}\xi; q)_m}
\]

(7.16)
Furthermore, applying the identity \((6.9)\) to \((q^{1-k}/t;q)_m\), gives that
\[
\eta^m_q(t, \lambda) = \frac{Y^m(q/t; q)_m}{(Y; q)\infty(q^{1-m}\xi; q)_m} 3\phi_2 \left( \begin{array}{c} q^{-m}, Y, t \\ q^{1-m}\xi, q^{-m}t \end{array} \big| q; q \right).
\] (7.17)

By making appeal to the finite Heine transformation \((19), p.2\):
\[
3\phi_2 \left( \begin{array}{c} q^{-m}, \alpha, \beta \\ \gamma, q^{1-n}/\tau \end{array} \big| q; q \right) = \frac{(\alpha \tau; q)_n}{(\tau; q)_n} 3\phi_2 \left( \begin{array}{c} q^{-m}, \gamma/\beta, \alpha \\ \gamma, \alpha \tau \end{array} \big| q; \beta \tau q^n \right)
\] (7.18)
for the parameters \(\alpha = t, \beta = q^{-m}t\xi, \gamma = q^{1-m}\xi, \tau = q/t\), Eq.(7.17) reads
\[
\eta^m_q(t, \lambda) = \frac{Y^m(q; q)_m}{(Y; q)\infty(q^{1-m}\xi; q)_m} 3\phi_2 \left( \begin{array}{c} q^{-m}, q/t, t \\ q^{1-m}(1-q)\lambda, q \end{array} \big| q; q\lambda(1-q) \right).
\] (7.19)

Taking into account the prefactor in (7.7), we arrive, after some simplifications, at the expression
\[
G_X(t) = \frac{t^m(q^{-m}(1-q)\lambda; q)_\infty}{(q^{-m}t\lambda(1-q); q)_\infty} 3\phi_2 \left( \begin{array}{c} q^{-m}, q/t, t \\ q^{1-m}(1-q)\lambda, q \end{array} \big| q; q\lambda(1-q) \right).
\] (7.20)

This completes the proof. \(\square\)

**Appendix B**

In order to determine the sign of \(\delta\) in (6.4) with respect to \(m\), we set \(\zeta = q^m\) and we look for the solutions \(\zeta\) of the equation
\[
(1 + q)^2\zeta^3 + (q - 3)(q + 1)\zeta^2 + (q - 1)(3q + 1)\zeta - q(6 + q) + 7 = 0.
\] (7.21)

This is equivalent to solve the cubic equation
\[
\delta(\zeta) = a\zeta^3 + b\zeta^2 + c\zeta + d = 0
\] (7.22)
where \(a = (1+q)^2, b = (q-3)(1+q), c = (q-1)(3q+1)\) and \(d = 7-q(6+q)\). By using the Cardan’s method \((21), pp. 4-8\), we obtain the following discriminant
\[
\tilde{\delta}_q := \alpha^2 + \frac{4\beta^3}{27}
\] (7.23)
where \(\alpha = \frac{2\beta^3 - 9abc + 27a^2d}{27a^3}\) and \(\beta = \frac{3ac-b^2}{3a^2}\). One can check that \(\tilde{\delta}_q \geq 0\) for \(0 < q < q_0, q_0 = \frac{5\sqrt{11} - 2}{11}\), in this case all real solutions of Eq.(7.22) do not belong to the interval \([0,q]\). Therefore, \(\delta > 0\) for all \(m \neq 0\). From the relation (6.4) and the fact that \(\zeta - 1 < 0\), we conclude that \(\Delta < 0\). On the other hand, for \(q_0 < q < 1\), we have \(\tilde{\delta}_q < 0\). In this irreducible case, the roots cannot be extracted directly by Cardan’s algebraic formulas. Hence, we arrive at the so called trigonometrical solution of the cubic equation for the three distinct real roots \((21), pp. 18-19\). Here, only one of them belonging to the interval \([0,q]\) and it is given by
\[
\zeta = \frac{-b}{3\alpha} + 2\sqrt{-\frac{\beta}{3}} \cos \left( \theta + \frac{4\pi}{3} \right), \quad \theta = \arccos \left( \frac{3\sqrt{3}\alpha}{2\beta\sqrt{-\beta}} \right).
\] (7.24)

Since \(\delta < 0\) for \(\zeta\) such that \(\zeta_q < \zeta < q\), or equivalently, \(m \leq m_q := \left\lfloor \frac{\log \zeta_q}{\log q} \right\rfloor\), we deduce that \(\Delta > 0\). Here, \(\lfloor s \rfloor\) denotes the greatest integer not exceeding \(s\). \(\square\)

**Acknowledgments.** The authors would like to thank the Moroccan Association of Harmonic Analysis & Spectral Geometry.
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