Formulas for partition $k$-tuples with $t$-cores

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Abstract. Let $A_{t,k}(n)$ denote the number of partition $k$-tuples of $n$ where each partition is $t$-core. In this paper, we establish formulas of $A_{t,k}(n)$ for some values of $t$ and $k$ by employing the method of modular forms, which extends Wang’s result for $t = 3$ and $k = 2, 3$.

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1. INTRODUCTION

Let $A_t(n)$ denote the number of $t$-core partitions of $n$, that is, the number of partitions of $n$ with no hook numbers being multiples of $t$. Garvan, Kim, and Stanton [7, Eq. (2.1)] showed that the generating function of $A_t(n)$ is given by

$$\sum_{n \geq 0} A_t(n)q^n = \frac{(q^t; q^t)_\infty}{(q; q)_\infty},$$

where, as usual, we denote $$(a; q)_\infty := \prod_{n \geq 0} (1 - aq^n),$$

and $$(a; q)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad (-\infty < n < \infty).$$

For convenience, we also write $$(a_1, a_2, \ldots, a_n; q)_\infty := (a_1; q)_\infty(a_2; q)_\infty \cdots (a_n; q)_\infty.$$ We say $(\lambda_1, \ldots, \lambda_k)$ is a partition $k$-tuple of $n$ if the sum of all the parts equals $n$. For example, $\{(1,1), \{1\}\}$ is a partition pair of 3. Furthermore, a partition $k$-tuple $(\lambda_1, \ldots, \lambda_k)$ of $n$ with $t$-cores means that each $\lambda_i$ is $t$-core. Let $A_{t,k}(n)$ denote the number of partition $k$-tuples of $n$ with $t$-cores. From (1.1), we readily obtain the generating function of $A_{t,k}(n)$, that is,

$$\sum_{n \geq 0} A_{t,k}(n)q^n = \left(\sum_{n \geq 0} A_t(n)q^n\right)^k = \frac{(q^t; q^t)^{kt}}{(q; q)^k_\infty}. \quad (1.2)$$

Here we write $A_{t,1}(n) = A_t(n)$.

Many authors have studied the number of partitions and partition pairs with $t$-cores and obtained sets of Ramanujan-like congruences (see, e.g., [1, 2, 3, 4, 5, 10, 12, 16]).
More recently, Wang [15] established formulas of $A_{3,2}(n)$ and $A_{3,3}(n)$ with the help of Ramanujan’s $1\psi_1$ formula and Bailey’s $6\psi_6$ formula. Let $\sigma_1(n) = \sum_{d|n} d$ and $\sigma_{2,\chi_3}^*(n) = \sum_{d|n} \chi_3(\frac{n}{d})d^2$ (here $\chi_3(n) = (n|3)$ denotes the Legendre symbol). He proved that for any integer $n \geq 0$,

$$A_{3,2}(n) = \frac{1}{3}\sigma_1(3n + 2),$$

and

$$A_{3,3}(n) = \sigma_{2,\chi_3}^*(n + 1).$$

However, we notice that Wang’s method expires for $k \geq 4$ due to a lack of corresponding $s\psi_s$ formulas. Recall that for the case $k = 1$, Granville and Ono [9] gave the formula

$$A_{3,1}(n) = \sigma_{0,\chi_3}(3n + 1),$$

where $\sigma_{0,\chi_3}(n) = \sum_{d|n} \chi_3(d)$, using the tools of modular forms. We also notice that in [13], Ono, Robins, and Wahl found formulas of the number of representations of $n$ as sums of $k$ triangular numbers (denoted by $\delta_k(n)$) for $k = 2, 4, 6, 8, 10, 12, $ and $24$ by applying the same method. It is known by Jacobi’s identity that

$$\sum_{n\geq 0} q^{T_n} = 1 + q + q^3 + q^6 + q^{10} + \cdots = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

We immediately see that their $\delta_k(n)$ is our $A_{2,k}(n)$. It is therefore natural to expect the method of modular forms will play a role in finding formulas of $A_{t,k}(n)$ for some other values of $t$ and $k$.

In Section 2, we will give a brief introduction to modular forms. In Section 3, we will discuss $A_{3,k}(n)$ and derive two new formulas for $k = 4$ and $6$. We should say that these formulas, which involve Fourier coefficients of some $\eta$-products, are not explicit. However, they can be regarded as analogues to $\delta_k(n)$ in [13]. In the following sections, we will present some formulas for $t \geq 4$. Here the formula of $A_{5,1}(n)$ is explicit.

2. A BRIEF INTRODUCTION TO MODULAR FORMS

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We first define the following congruence subgroups of level $N$:

(1) $\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$.

(2) $\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$.

(3) $\Gamma(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$. 


Here “∗” means “unspecified.”

Let $A \in SL_2(\mathbb{Z})$ act on the complex upper half plane $\mathcal{H}$ by the linear fractional transformation

$$A \tau = \frac{a \tau + b}{c \tau + d}.$$  

Let $\chi$ be a Dirichlet character mod $N$ and $k \in \mathbb{Z}^+$ satisfying $\chi(-1) = (-1)^k$. Let $f(\tau)$ be a holomorphic function on $\mathcal{H}$ such that

$$f(A \tau) = \chi(d)(c \tau + d)^k f(\tau)$$

for all $A \in \Gamma_0(n)$ and all $\tau \in \mathcal{H}$. We call $f(\tau)$ a modular form of weight $k$ and nebentypus $\chi$ on $\Gamma_0(N)$. Furthermore, we say $f(\tau)$ is a holomorphic modular form (resp. cusp form) if $f(\tau)$ is holomorphic (resp. vanishes) at the cusps of $\Gamma_0(N)$. It is known that the holomorphic modular forms (resp. cusp forms) of weight $k$ and nebentypus $\chi$ form finite dimensional $\mathbb{C}$-vector spaces, denoted by $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$). Moreover, $M_k(\Gamma_0(N), \chi)$ is the direct sum of $S_k(\Gamma_0(N), \chi)$ and Eisenstein series.

Every modular form $f(\tau) \in M_k(\Gamma_0(N), \chi)$ admits a Fourier expansion at infinity of the form

$$f(\tau) = \sum_{n \geq 0} a(n) q^n$$

where $q := e^{2\pi i \tau}$. Since spaces of modular forms are finite dimensional, given two modular forms with the same level $N$ and weight $k$, it is known that they are equal if their Fourier expansions agree for the first $k[SL_2(\mathbb{Z}) : \Gamma_0(N)]/12$ terms (see [11, 14]).

For more details on modular forms, the reader may refer to [6].

3. Formulas of $A_{3,k}(n)$

Taking $t = 3$ in (1.1), we have

$$\Phi(q) := \sum_{n \geq 0} A_3(n) q^n = \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty},$$

which is essentially a quotient of Dedekind $\eta$-functions, where $\eta$ is defined by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) = q^{1/24}(q; q)_\infty$$

with $q := e^{2\pi i \tau}$. In fact, we have

$$\Phi(q) = q^{-1/3} \frac{\eta^3(3\tau)}{\eta(\tau)}.$$  \hspace{1cm} (3.2)

Now we focus on $\Phi^k(q)$. To make our argument more complete, we include the proofs of (1.3), (1.4), and (1.5) as part of this section. We should mention at first that for odd $k$ our modular forms have nebentypus $\chi = (-3|n)$, whereas the nebentypus is $\chi = (9|n)$ when $k$ is even (here $(\cdot|n)$ denotes the Jacobi symbol).
3.1. The case \( k = 1 \). We consider the weight 1 modular form \( q\Phi(q^3) \) on \( \Gamma_0(9) \) defined by

\[
q\Phi(q^3) = \frac{\eta^3(9\tau)}{\eta(3\tau)} = \sum_{n \geq 0} A_{3,1}(n)q^{3n+1}.
\]

The first few terms of its Fourier expansion are

\[
q\Phi(q^3) = q + q^4 + 2q^7 + 2q^{13} + q^{16} + \cdots.
\]

Let \( \chi_3(n) = (n|3) \) denote the Legendre symbol, and write

\[
\sigma_{0,\chi_3}(n) = \sum_{d|n} \chi_3(d).
\]

We define the following weight 2 Eisenstein series on \( \Gamma_0(9) \)

\[
\sum_{n \geq 0} \sigma_{0,\chi_3}(3n+1)q^{3n+1}.
\]

It turns out that \( q\Phi(q^3) \) is the Eisenstein series on \( \Gamma_0(9) \) given by

\[
q\Phi(q^3) = \sum_{n \geq 0} \sigma_{0,\chi_3}(3n+1)q^{3n+1}.
\]

We therefore obtain

**Theorem 3.1** (Granville and Ono). For any integer \( n \geq 0 \),

\[
A_{3,1}(n) = \sigma_{0,\chi_3}(3n+1). \tag{3.3}
\]

3.2. The case \( k = 2 \). We consider the modular form \( q^2\Phi^2(q^3) \in \mathcal{M}_2(\Gamma_0(9)) \).

\[
q^2\Phi^2(q^3) = \frac{\eta^6(9\tau)}{\eta^2(3\tau)} = \sum_{n \geq 0} A_{3,2}(n)q^{3n+2}.
\]

The first few terms of its Fourier expansion are

\[
q^2\Phi^2(q^3) = q^2 + 2q^5 + 5q^8 + 4q^{11} + 8q^{14} + \cdots.
\]

Let

\[
\sigma_1(n) = \sum_{d|n} d.
\]

We define the following weight 2 Eisenstein series on \( \Gamma_0(9) \)

\[
\sum_{n \geq 0} \sigma_1(3n+2)q^{3n+2}.
\]

One readily verifies our generating function \( q^2\Phi^2(q^3) \) satisfies

\[
q^2\Phi^2(q^3) = \frac{1}{3} \sum_{n \geq 0} \sigma_1(3n+2)q^{3n+2}.
\]

We therefore prove

**Theorem 3.2** (Wang). For any integer \( n \geq 0 \),

\[
A_{3,2}(n) = \frac{1}{3} \sigma_1(3n+2). \tag{3.4}
\]
3.3. The case $k = 3$. Here we consider the weight 3 modular form $q\Phi^3(q)$ on $\Gamma_0(3)$:

$$q\Phi^3(q) = \frac{\eta^3(3\tau)}{\eta^3(\tau)} = \sum_{n \geq 0} A_{3,3}(n)q^{n+1}.$$ 

We give the first few terms of its Fourier expansion as follows

$$q\Phi^3(q) = q + 3q^2 + 9q^3 + 13q^4 + 24q^5 + \cdots.$$ 

Let 

$$\sigma_{2,\chi_3}^*(n) = \sum_{d | n} \chi_3(n/d)d^2.$$ 

Now we consider the weight 3 Eisenstein series on $\Gamma_0(3)$

$$\sum_{n \geq 1} \sigma_{2,\chi_3}^*(n)q^n.$$ 

By equating Fourier coefficients we find that

$$q\Phi^3(q) = \sum_{n \geq 1} \sigma_{2,\chi_3}^*(n)q^n.$$ 

It follows

**Theorem 3.3** (Wang). For any integer $n \geq 0$,

$$A_{3,3}(n) = \sigma_{2,\chi_3}^*(n + 1). \quad (3.5)$$

3.4. The case $k = 4$. We consider the weight 4 modular form $q^4\Phi^4(q^3)$ on $\Gamma_0(9)$ given by

$$q^4\Phi^4(q^3) = \frac{\eta^{12}(9\tau)}{\eta^4(3\tau)} = \sum_{n \geq 0} A_{3,4}(n)q^{3n+4}.$$ 

Here are the first few terms of the Fourier expansion of $q^4\Phi^4(q^3)$:

$$q^4\Phi^4(q^3) = q^4 + 4q^7 + 14q^{10} + 28q^{13} + 57q^{16} + \cdots.$$ 

Let 

$$\sigma_3(n) = \sum_{d | n} d^3.$$ 

We consider the following weight 4 Eisenstein series on $\Gamma_0(9)$:

$$E(\tau) = \sum_{n \geq 0} \sigma_3(3n + 1)q^{3n+1}.$$ 

Note also that the space of cusp forms $S_4(\Gamma_0(9))$ is 1 dimensional and is spanned by the $\eta$-product

$$\eta^8(3\tau) = q - 8q^4 + 20q^7 - 70q^{10} + 64q^{13} + \cdots.$$ 

By equating Fourier coefficients we obtain the following identity:

$$q^4\Phi^4(q^3) = \frac{1}{81}(E(\tau) - \eta^8(3\tau)).$$ 

It implies
**Theorem 3.4.** Let $\eta^8(3\tau) = \sum_{n \geq 1} a(n)q^n$. For any integer $n \geq 0$,

$$A_{3,4}(n) = \frac{1}{81}(\sigma_3(3n+4) - a(3n+4)). \quad (3.6)$$

**Remark 3.1.** It is known that $\eta^8(3\tau)$ is a cusp with complex multiplication. Since all forms with complex multiplication are lacunary, that is, the arithmetic density of their non-zero Fourier coefficients is 0, we immediately see $A_{3,4}(n) = \sigma_3(3n+4)/81$ almost always.

**3.5. The case $k = 6$.** Here we consider the modular form $q^2\Phi_6(q) \in \mathcal{M}_6(\Gamma_0(3)).$

$$q^2\Phi_6(q) = \frac{\eta^{18}(3\tau)}{\eta^6(\tau)} = \sum_{n \geq 0} A_{3,6}(n)q^{n+2}. \quad (3.6)$$

We give the first few terms of its Fourier expansion

$$q^2\Phi_6(q) = q^2 + 6q^3 + 27q^4 + 80q^5 + 207q^6 + \cdots .$$

Given a prime $p$, let $\nu_p(n)$ denote the largest integer $e$ such that $p^e \mid n$. Write

$$\sigma_{5,3}^\#(n) = 3^{5\nu_3(n)} \sum_{d \mid n, \, d \not= 0 \mod 3} d^5.$$

Now we define the following weight 6 Eisenstein series on $\Gamma_0(3)$

$$E(\tau) = \sum_{n \geq 1} \sigma_{5,3}^\#(n)q^n.$$

It is easy to see $S_6(\Gamma_0(3))$ is 1 dimensional and is spanned by

$$\eta^6(\tau)\eta^6(3\tau) = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + \cdots .$$

Our form $q^2\Phi_6(q)$ satisfies

$$q^2\Phi_6(q) = \frac{1}{39}(E(\tau) - \eta^6(\tau)\eta^6(3\tau)).$$

We thus conclude

**Theorem 3.5.** Let $\eta^6(\tau)\eta^6(3\tau) = \sum_{n \geq 1} a(n)q^n$. For any integer $n \geq 0$,

$$A_{3,6}(n) = \frac{1}{39}(\sigma_{5,3}^\#(n+2) - a(n+2)). \quad (3.7)$$

**4. Formula of $A_{4,2}(n)$**

Taking $t = 4$ in (1.1), we have

$$\Phi(q) := \sum_{n \geq 0} A_4(n)q^n = \frac{(q^4; q^4)_\infty}{(q; q)_\infty}. \quad (4.1)$$

One readily sees it equals the following $\eta$-quotient

$$\Phi(q) = q^{-5/8}\frac{\eta^4(4\tau)}{\eta(\tau)}. \quad (4.2)$$
Now we consider the modular form $q^5 \Phi^2(q^4) \in \mathcal{M}_3(\Gamma_0(16), (-4|n))$.

$$q^5 \Phi^2(q^4) = \frac{\eta^8(16\tau)}{\eta^2(4\tau)} = \sum_{n \geq 0} A_{4,2}(n)q^{4n+5}.$$ 

The first few terms of its Fourier expansion are

$$q^5 \Phi^2(q^4) = q^5 + 2q^9 + 5q^{13} + 10q^{17} + 12q^{21} + \cdots.$$ 

Let $\chi_{4,2}$ be the Dirichlet character mod 4 given by

$$\chi_{4,2}(n) = \begin{cases} 1 & n \equiv 1 \mod 4, \\ -1 & n \equiv 3 \mod 4, \\ 0 & (n,4) > 1, \end{cases}$$

and write

$$\sigma_{2,\chi_{4,2}}(n) = \sum_{d|n} \chi_{4,2}(d)d^2.$$ 

We consider the following weight 3 Eisenstein series on $\Gamma_0(16)$:

$$E(\tau) = \sum_{n \geq 0} \sigma_{2,\chi_{4,2}}(4n + 1)q^{4n+1}.$$ 

It is also known that the space of cusp forms $\mathcal{S}_3(\Gamma_0(16), (-4|n))$ is 1 dimensional and is spanned by

$$\eta^6(4\tau) = q - 6q^5 + 9q^9 + 10q^{13} - 30q^{17} + \cdots.$$ 

One readily verifies

$$q^5 \Phi^2(q^4) = \frac{1}{32}(E(\tau) - \eta^6(4\tau)).$$ 

We therefore prove

**Theorem 4.1.** Let $\eta^6(4\tau) = \sum_{n \geq 1} a(n)q^n$. For any integer $n \geq 0$,

$$A_{4,2}(n) = \frac{1}{32}(\sigma_{2,\chi_{4,2}}(4n + 5) - a(4n + 5)). \quad (4.3)$$

5. **Formulas of $A_{5,k}(n)$**

Taking $t = 5$ in (1.1), we have

$$\Phi(q) := \sum_{n \geq 0} A_5(n)q^n = \frac{(q^5; q^5)_\infty}{(q; q)_\infty}. \quad (5.1)$$

It is easy to obtain the following identity:

$$\Phi(q) = q^{-1}\frac{\eta^5(5\tau)}{\eta(\tau)}. \quad (5.2)$$
5.1. The case \( k = 1 \). Here we consider the weight 2 modular form \( q\Phi(q) \) on \( \Gamma_0(5) \) with nebentypus \( (5|n) \).

\[
q\Phi(q) = \frac{\eta^5(5\tau)}{\eta(\tau)} = \sum_{n \geq 0} A_{5,1}(n)q^{n+1}.
\]

The first few terms of its Fourier expansion are

\[
q\Phi(q) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots.
\]

Let \( \chi_{5,3} \) be the Dirichlet character mod 5 given by

\[
\chi_{5,3}(n) = \begin{cases} 
1 & n \equiv 1, 4 \mod 5, \\
-1 & n \equiv 2, 3 \mod 5, \\
0 & (n, 5) > 1.
\end{cases}
\]

Now write

\[
\sigma^*_{1,\chi_{5,3}}(n) = \sum_{d|n} \chi_{5,3}(n/d)d.
\]

Define the following weight 2 Eisenstein series on \( \Gamma_0(5) \)

\[
E(\tau) = \sum_{n \geq 1} \sigma^*_{1,\chi_{5,3}}(n)q^n.
\]

It follows by equating Fourier coefficients that

\[
q\Phi(q) = E(\tau).
\]

We conclude

**Theorem 5.1.** For any integer \( n \geq 0 \),

\[
A_{5,1}(n) = \sigma^*_{1,\chi_{5,3}}(n + 1).
\]

**Remark 5.1.** It is of interest to mention that we can prove Theorem 5.1 through Bailey’s \( 6\psi_6 \) formula. Recall

**Lemma 5.2 (Bailey’s \( 6\psi_6 \) formula).** For \( |aq^2/(bcde)| < 1 \),

\[
6\psi_6 \left[ \begin{array}{cccccc}
q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\
\sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e \\
\end{array} \right] \\
= \frac{(aq, aq/(bc), aq/(bd), aq/(be), aq/(cd), aq/(ce), aq/(de), q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/(bcde); q)_\infty},
\]

where the \( s\psi_s \) function is defined as

\[
s\psi_s \left[ \begin{array}{ccc}
\frac{a_1, \ldots, a_s}{b_1, \ldots, b_s}; q, z \\
\end{array} \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_s; q)_n z^n}{(b_1, \ldots, b_s; q)_n}.
\]

For its proof, the reader may refer to [8, Sec. 5.3]. Now taking

\[
(a, b, c, d, e, q) \to (q^4, q, q^3, q^3, q^5)
\]

in (5.4), we deduce that

\[
\sum_{n \geq 0} A_{5,1}(n)q^{n+1} = q(q^5; q^5)_\infty/(q; q)_\infty
\]
Note that for $|q| < 1$ we have

$$\frac{q}{(1 - q)^2} = \sum_{d \geq 1} dq^d.$$ 

It therefore follows

$$\sum_{n \geq 0} A_{5,1}(n)q^{n+1} = \sum_{n \geq 0} \sum_{d \geq 1} \left\{ dq^{(5n+1)d} - dq^{(5n+2)d} - dq^{(5n+3)d} + dq^{(5n+4)d} \right\}.$$ 

By equating coefficients we have

$$A_{5,1}(n) = \sum_{d \mid n+1} \chi_{5,3}(d) \frac{n+1}{d} = \sum_{d \mid n+1} \chi_{5,3} \left( \frac{n+1}{d} \right) d = \sigma^*_1 \chi_{5,3}(n+1).$$

5.2. **The case $k = 2$.** We consider the modular form $q^2 \Phi^2(q) \in M_4(\Gamma_0(5))$.

$$q^2 \Phi^2(q) = \frac{\eta^{10}(5\tau)}{\eta^2(\tau)} = \sum_{n \geq 0} A_{5,2}(n)q^{n+2}.$$ 

The first few terms of its Fourier expansion are

$$q^2 \Phi^2(q) = q^2 + 2q^3 + 5q^4 + 10q^5 + 20q^6 + \cdots.$$ 

Let

$$\sigma^{\#}_{3,5}(n) = 5^{\nu_5(n)} \sum_{d \mid n, d \equiv 0 \mod 5} d^3,$$

where $\nu_5(n)$ denotes the largest integer $e$ such that $5^e \mid n$. Now we define the following weight 4 Eisenstein series on $\Gamma_0(5)$

$$E(\tau) = \sum_{n \geq 1} \sigma^{\#}_{3,5}(n)q^n.$$ 

We also notice that the space of cusp forms $S_4(\Gamma_0(5))$ is 1 dimensional and is spanned by

$$\eta^4(\tau) \eta^4(5\tau) = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 + \cdots.$$ 

Our form $q^2 \Phi^2(q)$ satisfies

$$q^2 \Phi^2(q) = \frac{1}{13} (E(\tau) - \eta^4(\tau) \eta^4(5\tau)).$$ 

We thus conclude

**Theorem 5.3.** Let $\eta^4(\tau) \eta^4(5\tau) = \sum_{n \geq 1} a(n)q^n$. For any integer $n \geq 0$,

$$A_{5,2}(n) = \frac{1}{13} (\sigma^{\#}_{3,5}(n + 2) - a(n + 2)).$$

(5.5)
6. Formula of $A_{7,1}(n)$

Taking $t = 7$ in (1.1), we have

$$\Phi(q) := \sum_{n \geq 0} A_7(n)q^n = \frac{(q^7: q^7)_{\infty}}{(q; q)_{\infty}}.$$  \hspace{1cm} (6.1)

One easily finds

$$\Phi(q) = q^{-2} \frac{\eta^7(7\tau)}{\eta(\tau)}. \hspace{1cm} (6.2)$$

Now we consider the modular form $q^2 \Phi(q) \in M_3(\Gamma_0(7), (-7|n))$.

$$q^2 \Phi(q) = \frac{\eta^7(7\tau)}{\eta(\tau)} = \sum_{n \geq 0} A_{7,1}(n)q^{n+2}. \hspace{1cm} (6.3)$$

The first few terms of its Fourier expansion are

$$q^2 \Phi(q) = q^2 + q^3 + 2q^4 + 3q^5 + 5q^6 + \cdots.$$ 

Let $\chi_{7,4}$ be the Dirichlet character mod 7 given by

$$\chi_{7,4}(n) = \begin{cases} 
1 & n \equiv 1, 2, 4 \pmod{7}, \\
-1 & n \equiv 3, 5, 6 \pmod{7}, \\
0 & (n, 7) > 1.
\end{cases}$$

We write

$$\sigma^*_{2,\chi_{7,4}}(n) = \sum_{d|n} \chi_{7,4}(n/d)d^2.$$ 

Consider the following weight 3 Eisenstein series on $\Gamma_0(7)$:

$$E(\tau) = \sum_{n \geq 1} \sigma^*_{2,\chi_{7,4}}(n)q^n.$$ 

Note also that $S_3(\Gamma_0(7), (-7|n))$ is 1 dimensional and is spanned by

$$\eta^3(\tau)\eta^3(7\tau) = q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + \cdots.$$

It is easy to verify that

$$q^2 \Phi(q) = \frac{1}{8}(E(\tau) - \eta^3(\tau)\eta^3(7\tau)).$$

We therefore prove

**Theorem 6.1.** Let $\eta^3(\tau)\eta^3(7\tau) = \sum_{n \geq 1} a(n)q^n$. For any integer $n \geq 0$,

$$A_{7,1}(n) = \frac{1}{8}(\sigma^*_{2,\chi_{7,4}}(n + 2) - a(n + 2)). \hspace{1cm} (6.3)$$

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