Robust Approximation Algorithms for the Detection of Attraction Basins in Dynamical Systems

Roberto Cavoretto1 · Alessandra De Rossi1 · Emma Perracchione1 · Ezio Venturino1

Received: 16 November 2014 / Revised: 17 November 2015 / Accepted: 19 November 2015 / Published online: 27 November 2015
© Springer Science+Business Media New York 2015

Abstract A particular solution of a dynamical system is completely determined by its initial condition. When the omega limit set reduces to a point, the solution settles at steady state. The possible steady states of the system are completely determined by its parameters. However, with the same parameter set, it is possible that several steady states can originate from different initial conditions (multi-stability). In that case the outcome depends on the chosen initial condition. Therefore, it is important to assess the domain of attraction for each possible attractor. The algorithms presented here are general and robust enough so as to solve the problem of reconstructing the basin of attraction of each stable equilibrium point. In order to have a graphical representation of the separatrix manifolds, we focus on systems of two and three ordinary differential equations exhibiting bi- or tri-stability. For this purpose we have implemented several MATLAB functions for the approximation of the points lying on the curves or on the surfaces determining the basins of attraction and for the reconstruction of such curves and surfaces. We approximate the latter with the implicit partition of unity method using radial basis functions as local approximants. Numerical results, obtained with a MATLAB package made available to the scientific community, support our findings.

Keywords Scattered data approximation · Partition of unity method · Radial basis functions · Dynamical systems · Competition population models · Basins of attraction

Mathematics Subject Classification 65D05 · 65D17 · 92D25 · 37M20

✉ Roberto Cavoretto
roberto.cavoretto@unito.it
Alessandra De Rossi
alessandra.derossi@unito.it
Emma Perracchione
emma.perracchione@unito.it
Ezio Venturino
ezio.venturino@unito.it

1 Department of Mathematics “G. Peano”, University of Turin, via Carlo Alberto 10, 10123 Turin, Italy
1 Introduction

The aim of this investigation is the formulation of a new reliable and accurate algorithm for the reconstruction of unknown surfaces partitioning the three-dimensional space in disjoint sets. We first illustrate the importance of having such a versatile tool available in applied sciences.

Mathematical modelling is nowadays commonly applied to major disciplines, such as biology, medicine and social sciences. By these models the prediction of the temporal evolution of the considered quantities, i.e. populations, cancer, divorces, is sought [1, 29]. This is obtained in general via dynamical systems.

In an initial value problem, involving a set of ordinary differential equations, a particular solution of the system is completely determined by the initial condition. The latter establishes the steady state toward which the solution will evolve. Depending on the initial state of the system and on conditions involving the model parameters, the trajectories may in fact tend toward different equilibria. Stated formally, in a dynamical system the trajectories from a given initial condition evolve possibly toward a certain equilibrium. However note that in what follows we will be more loose and write instead for short the initial condition evolves (or stabilizes) toward an equilibrium.

The phase state of the dynamical system is thus partitioned into different regions, called the basins of attraction of each equilibrium, depending on where the trajectories originating in them will ultimately stabilize. In such cases the final outcome of a mathematical model depends on the initial condition. If the latter lies in the basin of attraction of a certain equilibrium point, the system will finally settle to this specific steady state. To establish the ultimate system behavior it is therefore important to assess for each possible attractor its domain of attraction.

The separatrix manifolds generated by a saddle point are determined locally (by linearization) in well-known examples, see e.g. [20]. Specifically, even if some techniques to prove the existence of invariant sets have already been developed, none of them, except for particular and well-known cases, allows to have a graphical representation of the separatrix manifolds [14, 26]. Such techniques are based on results from algebraic topology, and thus such methods are not constructive in the sense that they do not give a precise structure and location of the invariant sets. Furthermore, numerical tools based on characterizing in (exponentially) asymptotically autonomous systems a Lyapunov function as a solution of a suitable linear first-order PDE have already been developed. Such equation is then approximated using meshless collocation methods [17, 18].

Our aim is instead more ambitious since, on the contrary, the software presented here allows to reconstruct the basin of attraction of each equilibrium in a three-dimensional dynamical system, providing a graphical representation of the separatrix manifolds. Moreover, we are not restricted to asymptotically autonomous systems and thus the transformations made in order to use powerful methods, which are well-suited only for autonomous models, are not here necessary.

To achieve our aim, a suitable scheme is at first constructed for the generation of these surfaces. It provides points that, within a certain tolerance, lie on these sought surfaces. This is obtained via a suitable bisection-like routine that employs pairs of points belonging to two different sets of the partition. These points are determined by an educated random procedure, as it will be apparent in what follows. They are not known in advance, as they represent the omega-limit sets of the trajectories of a dynamical system. As such, their location cannot be generally predicted a priori. Overall then, we propose a new robust and general
algorithm that allows to approximate surfaces partitioning the phase space into the regions of interest.

This research extends to three dimensional systems the work already undertaken in a particular squirrel competition model with safety niches which presents bistability \([7,10]\). It provides a more robust and general framework for the three stable equilibria case, allowing to approximate surfaces partitioning the phase space into the resulting three basins of attraction. As a bonus, the separatrix surfaces in case of bistability can be obtained as a particular case of this algorithm. Mostly by means of numerical experiments, we will see how such robust tool for three-dimensional dynamical systems can be easily adapted to the reconstruction of separatrix manifolds for two-dimensional models.

More specifically, the approximation of the attraction basins for two and three dimensional dynamical systems leads to a method consisting of two steps:

1. detection of the points lying on the separatrix manifolds,
2. interpolation of such points in a suitable way.

For this purpose we have implemented several routines for the approximation of the points lying on the curves or on the surfaces determining the basins of attraction. The \textsc{Matlab} software here discussed is available at:

\url{http://hdl.handle.net/2318/1520518}.

A basin of attraction can be described by an implicit equation. The reconstruction of curves and surfaces defined in terms of point cloud data is a common problem in computer aided design and computer graphics \([23,24,31]\). We approximate the curves and the surfaces with the implicit partition of unity method using radial basis functions (RBFs) as local approximants, and in particular the compactly supported Wendland’s functions \([4,5,11,12,15,32]\). This choice follows from the fact that meshfree methods are suited for changes in the problem geometry \([6,8,9,21]\). Especially in this context the importance of using such versatile methods will be pointed out. Moreover the partition of unity method turns out to be really meaningful if coupled with compactly supported RBFs since it leads to solve small sparse systems of equations. Therefore, acting in this way we can improve both computational cost and ill-conditioning arising from data which are not uniformly distributed, as in the case of separatrix points.

Meshless methods have already been used in a similar context to approximate the basins of attraction of periodic orbits \([16]\), or as collocation methods to approximate Lyapunov functions \([17,18]\). Even if here we are considering a different problem, the use of a tool independent from the problem geometry and mesh turns out to be essential also in this case.

The paper is organized as follows. Section 2 is devoted to the presentation of the designed algorithms for the detection of points lying on such curves and surfaces. In Section 3 we describe the method used for approximating the curves and the surfaces determining the basins of attraction. Section 4 contains our numerical results. A brief discussion concludes the paper.

### 2 Approximation of Manifolds Determining the Basins of Attraction

This section describes the algorithms implemented for the detection of points lying on the manifolds delimiting the basins of attraction, while in the next section we will consider the interpolation problem. We start by analyzing the problem in a dynamical system of dimension
three in Sect. 2.2, and then in Sect. 2.3 we adapt the designed algorithm for a dynamical system of dimension two. Specifically, in Sect. 2.2 we discuss the problem of partitioning the phase space into three subregions, when the system presents three stable equilibria.

Before analyzing such routines we consider an example in order to illustrate the goal of such numerical tool in Sect. 2.1.

2.1 Motivations and Targets

Let us consider the following model describing a population affected by a disease [22]:

\[
\begin{align*}
\frac{dP}{dt} &= r(1 - P)(P - u)P - \alpha I, \\
\frac{dI}{dt} &= [-\alpha - d - ru + (\sigma - 1)P - \sigma I]I,
\end{align*}
\]

where \( P \) is the dimensionless total population that is composed of infected individuals \( I \) and susceptibles \( P - I \). It is easy to verify that \( E_0 = (0, 0) \), \( E_1 = (1, 0) \) and \( E_2 = (u, 0) \) are equilibria of the system (2.1). For the study of the endemic steady states see [22].

As suggested by [22] we set \( r = 0.2, u = 0.1, d = 0.25 \) and \( \alpha = 0.1 \); furthermore we fix \( \sigma = 2.5 \). With this choice there exists exactly one endemic steady state \( E_4 \approx (0.9562, 0.0724) \) which is a stable equilibrium point. Moreover the origin \( E_0 \) is also stable.

This situation suggests the existence of a curve separating the paths tending to the disease-free equilibrium point from those tending to the endemic steady state. This curve is shown in Fig. 1. From this consideration the importance of having a graphical representation of the attraction basins follows. In fact, given an initial condition, we can suggest measures in order to move the initial condition in the region of interest.

A similar result has already been used to prevent the extinction of herbivores living in natural parks [30].

2.2 Detection of Points Determining the Basins of Attraction of Three Different Equilibria

In order to approximate the basins of attraction, when the system admits three stable equilibria, the general idea is to find the points lying on the surfaces determining the domains of attraction.

Fig. 1 Approximation of the separatrix curve. The blue circles represent the stable equilibria \( E_0 \) and \( E_4 \) (Color figure online)
and finally to interpolate them with a suitable method. The steps of the so-called detection-interpolation algorithm are summarized in the 3D-Detec-Interp Algorithm. At first, we need to consider a set of points as initial conditions, then we take points in pairs and we proceed with a bisection routine to determine a point lying on a surface dividing the domains of attraction [10]. The simplest idea, which turns out to be also reliable, consists in considering the set of initial conditions in a cube domain $[0, \gamma]^3$, where $\gamma \in \mathbb{R}^+$. We remark that it is always possible to find a cube containing the equilibria, since we are assuming to have finite trajectories, i.e. finite equilibrium points. Of course, other geometrical shapes can also be considered, but the cube-based scheme ensures to find well-distributed sets of separatrix points. Moreover, because of the symmetry of the cube the implementation turns out to be easy.

Essentially, the algorithm finds three sets of points, which, after being interpolated, graphically describe the basins of attraction.

Specifically, in the detection-interpolation algorithm, after considering $n$ equispaced initial conditions on each edge of the cube $[0, \gamma]^3$, we construct a grid on the faces of the cube, (see Step 2 in the 3D-Detec-Interp Algorithm):

\begin{align}
P_{i_1,i_2}^1 &= (x_{i_1}, y_{i_2}, 0) \quad \text{and} \quad P_{i_1,i_2}^2 = (x_{i_1}, y_{i_2}, \gamma), \quad i_1, i_2 = 1, \ldots, n, \\
P_{i_1,i_2}^3 &= (x_{i_1}, 0, z_{i_2}) \quad \text{and} \quad P_{i_1,i_2}^4 = (x_{i_1}, \gamma, z_{i_2}), \quad i_1, i_2 = 1, \ldots, n, \\
P_{i_1,i_2}^5 &= (0, y_{i_1}, z_{i_2}) \quad \text{and} \quad P_{i_1,i_2}^6 = (\gamma, y_{i_1}, z_{i_2}), \quad i_1, i_2 = 1, \ldots, n,
\end{align}

and a bisection routine is applied with initial conditions (2.2) [10].

At first, the bisection-like routine integrates the system with a pair of initial conditions in a time interval $t$. Then it checks toward which equilibrium the trajectories originating in such initial conditions ultimately stabilize. Finally, if the two initial conditions evolve toward two different stable equilibria, it provides a point named:

1. $q_3$, if the point lies on the surface delimiting the domain of attraction of both the first and the second equilibrium point, or
2. $q_2$, if lies on the surface determining the basin of attraction of both the first and the third equilibrium point, or
3. $q_1$, if the point lies on the surface delimiting the domain of attraction of both the second and the third equilibrium point.

Summarizing, once we apply the bisection algorithm, three different sets of points, which in pairs identify the basins of attraction, are detected, (see Steps 3–4 in the 3D-Detec-Interp Algorithm). Considering then the method described in Sect. 3,
the associated algorithm interpolates such points and returns values of the three interpolants. They approximate the basins of attraction of the first, second and third equilibrium point, respectively, (see Step 5 in the 3D-Detec-Interp Algorithm).

**Remark 1** If an initial condition coincides with an unstable steady state or if it lies on a stable submanifold of a saddle point we move such initial condition away from it in order to obtain, by the bisection routine, a point lying on a separatrix surface.

**Remark 2** When we perform the bisection between two initial conditions evolving toward two different equilibria, for instance the first and the second one, we suppose that none of the computed midpoints ultimately stabilizes toward the third stable equilibrium.

**Remark 3** The separatrix surfaces in case of bistability, investigated in [10], can be obtained as a particular case of the detection-interpolation algorithm analyzed in Sect. 2.2. Specifically, the case of two stable steady states can be seen as the case of three attractors, in which two equilibria coincide.

### 2.3 Detection of Points Determining the Basins of Attraction for a 2D Dynamical System

In Sect. 2.2 we have designed algorithms for the detection of points lying on and for the reconstruction of the surfaces bounding the basins of attraction.

It is easy to adapt the detection-interpolation algorithm to dynamical systems of dimension two which present three stable equilibria. In the 2D case, we start considering \( n \) equispaced initial conditions on each edge of the square \([0, \gamma]^2\); the bisection routine is applied with the following initial conditions [10]:

\[
P^1_i = (x_i, 0) \quad \text{and} \quad P^2_i = (x_i, \gamma), \quad i = 1, \ldots, n, \\
P^3_i = (0, y_i) \quad \text{and} \quad P^4_i = (\gamma, y_i), \quad i = 1, \ldots, n.
\]  

(2.3)

Once we apply the bisection algorithm with initial conditions (2.3), three different sets of points, as in the previous subsection, are detected. Finally, we interpolate the points lying on such curves with the method described in Sect. 3.

The considerations about bistability in Sect. 2.2 also hold for a 2D dynamical system. Thus, we can also approximate separatrix curves.

The use of an implicit interpolation method, described in the next section, is useful when the curve approaches the \( y \) axis; furthermore it allows to skip the refinement algorithm, improving the results obtained in [7,10].

### 3 Approximation of Manifolds Determining the Basins of Attraction: Interpolation Phase

In this section we describe the method used for the reconstruction of the basins of attraction. The latter are often defined by implicit equations, consequently to reconstruct the domains of attraction we use an implicit scheme, specifically the implicit partition of unity method.

We will describe such method for a 3D dataset, but it can easily be adapted to a 2D dataset so as to allow the reconstruction also of implicit curves [15]. However, an implicit approach is not always necessary for the approximation of some separatrix curves and surfaces. In fact we have already obtained good results with the explicit partition of unity method [7,10]. But the nature of a curve or of a surface is known only after detecting the points lying on them.
Thus, though such curves or surfaces might usually be expressed by an explicit equation, we use the more general implicit partition of unity technique. The use of such method is the key step which allows to reconstruct the basins of attraction of any stable equilibrium point.

3.1 Implicit Surface Reconstruction

Given a point cloud data set, i.e. data in the form $X_N = \{x_i \in \mathbb{R}^3, i = 1, \ldots, N\}$, belonging to an unknown two dimensional manifold $\mathcal{M}$, namely a surface in $\mathbb{R}^3$, we seek another surface $\mathcal{M}^*$ that is a reasonable approximation to $\mathcal{M}$. For the implicit approach, we think of $\mathcal{M}$ as the surface of all points $x \in \mathbb{R}^3$ satisfying the implicit equation:

$$f(x) = 0,$$  \hspace{1cm} (3.1)

for some function $f$, which implicitly defines the surface $\mathcal{M}$ [15,32]. This means that the equation (3.1) is the zero iso-surface of the trivariate function $f$, and therefore this iso-surface coincides with $\mathcal{M}$. The surface $\mathcal{M}^*$ is constructed via partition of unity interpolation [15,32], as shown in Sect. 3.2. Unfortunately, the solution of this problem, by imposing the interpolation conditions (3.1), leads to the trivial solution, given by the identically zero function [12]. The key to finding the interpolant of the trivariate function $f$, from the given data points $x_i$, $i = 1, \ldots, N$, is to use additional significant interpolation conditions, i.e. to add an extra set of off-surface points. Once we define the augmented data set, we can then compute a three dimensional interpolant $I$ to the total set of points [15]. A common practice is to assume that in addition to the point cloud data also the set of surface oriented normals $n_i \in \mathbb{R}^3$ to the surface $\mathcal{M}$ at the points $x_i$ is given. We construct the extra off-surface points by taking a small step away along the surface normals, i.e. we obtain for each data point $x_i$ two additional off-surface points. One point lies outside the manifold $\mathcal{M}$ and is given by

$$x_{N+i} = x_i + \delta n_i,$$

whereas the other point lies inside $\mathcal{M}$ and is given by

$$x_{2N+i} = x_i - \delta n_i,$$

$\delta$ being the stepsize. The union of the sets $X_\delta^+ = \{x_{N+1}, \ldots, x_{2N}\}$, $X_\delta^- = \{x_{2N+1}, \ldots, x_{3N}\}$ and $X_N$ gives the overall set of points on which the interpolation conditions are assigned. Note that if we have zero normals in the given normal data set, we must exclude such points [15].

Now, after creating the data set, we compute the interpolant $I$ whose zero contour (iso-surface $I = 0$) interpolates the given point cloud data, and whose iso-surfaces $I = 1$ and $I = -1$ interpolate $X_\delta^+$ and $X_\delta^-$, respectively, i.e.

$$I(x_i) = 0, \quad i = 1, \ldots, N,$$
$$I(x_i) = 1, \quad i = N + 1, \ldots, 2N,$$
$$I(x_i) = -1, \quad i = 2N + 1, \ldots, 3N.$$

The values $+1$ or $-1$ are arbitrary. Their precise value is not as critical as the choice of $\delta$. In fact the stepsize can be rather critical for a good surface fit [4,15]. A suitable value for such parameter will be discussed in Sect. 4. Finally, we just render the resulting approximating surface $\mathcal{M}^*$ as the zero contour of the 3D interpolant [15]. If the normals are not explicitly given, we illustrate some techniques to estimate them in Sect. 3.3 [23,24,32].
3.2 Partition of Unity Method and Radial Basis Function Interpolation

In Sect. 3.1 we have presented an approach, to obtain a surface that fits the given 3D scattered data set, based on the use of implicit surfaces defined in terms of some meshfree approximation methods such as the partition of unity interpolation [11,15,28,33,34].

Let \( \mathcal{X}_N = \{ x_i \in \mathbb{R}^3, i = 1, \ldots, N \} \) be a set of distinct data points or nodes, arbitrarily distributed in a domain \( \Omega \subseteq \mathbb{R}^3 \), with an associated set \( \mathcal{F}_N = \{ f_i, i = 1, \ldots, N \} \) of data values or function values, which are obtained by sampling some (unknown) function \( f : \Omega \rightarrow \mathbb{R} \) at the nodes, i.e., \( f_i = f(x_i), i = 1, \ldots, N \).

The basic idea of the partition of unity method is to start with a partition of the open and bounded domain \( \Omega \) into \( d \) subdomains \( \Omega_j \) such that \( \Omega \subseteq \bigcup_{j=1}^{d} \Omega_j \) with some mild overlap among the subdomains.

A 2D view of a partition of unity structure covering a set of scattered data in the unit square is shown in Fig. 2.

Associated with these subdomains we choose a partition of unity, i.e. a family of compactly supported, non-negative, continuous functions \( W_j \) with \( \text{supp}(W_j) \subseteq \Omega_j \) such that

\[
\sum_{j=1}^{d} W_j(x) = 1, \quad x \in \Omega. \tag{3.2}
\]

The global approximant thus assumes the following form

\[
\mathcal{I}(x) = \sum_{j=1}^{d} R_j(x) W_j(x), \quad x \in \Omega. \tag{3.3}
\]

For each subdomain \( \Omega_j \) we define a local radial basis function interpolant [25] \( R_j : \Omega \rightarrow \mathbb{R} \) of the form

\[
R_j(x) = \sum_{k=1}^{N_j} c_k \phi(||x - x_k||_2), \tag{3.4}
\]

Fig. 2 An illustrative example of partition of unity subdomains covering the domain \( \Omega = [0, 1]^2 \). The red stars represent a set of scattered data while the blue circles identify the partition of unity subdomains (Color figure online)
where $\phi : [0, \infty) \to \mathbb{R}$ is called radial basis function, $|| \cdot ||_2$ denotes the Euclidean norm, and $N_j$ indicates the number of data points in $\Omega_j$. Moreover, $R_j$ satisfies the interpolation conditions

$$R_j(x_i) = f_i, \quad i = 1, \ldots, N_j. \quad \text{(3.5)}$$

In particular, we observe that if the local approximants satisfy the interpolation conditions (3.5), then the global approximant also interpolates at $x_i$, i.e. $I(x_i) = f(x_i)$, for $i = 1, \ldots, N_j$.

Solving the $j$-th interpolation problem (3.5) leads to a system of linear equations of the form

$$
\begin{bmatrix}
\phi(||x_1 - x_1||_2) & \phi(||x_1 - x_2||_2) & \cdots & \phi(||x_1 - x_{N_j}||_2) \\
\phi(||x_2 - x_1||_2) & \phi(||x_2 - x_2||_2) & \cdots & \phi(||x_2 - x_{N_j}||_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi(||x_{N_j} - x_1||_2) & \phi(||x_{N_j} - x_2||_2) & \cdots & \phi(||x_{N_j} - x_{N_j}||_2)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{N_j}
\end{bmatrix}
= 
\begin{bmatrix}
f_1 \\
f_2 \\
\vdots \\
f_{N_j}
\end{bmatrix},
$$

or simply

$$\Phi \mathbf{c} = \mathbf{f}. $$

Now, we give the following definition (see [34]).

**Definition 1** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded set. Let $\{\Omega\}_{j=1}^{d}$ be an open and bounded covering of $\Omega$. This means that all $\Omega_j$ are open and bounded and that $\Omega$ is contained in their union. A family of nonnegative functions $\{W_j\}_{j=1}^{d}$ with $W_j \in C^k(\mathbb{R}^3)$ is called a $k$-stable partition of unity with respect to the covering $\{\Omega_j\}_{j=1}^{d}$ if

1. $\text{supp}(W_j) \subseteq \Omega_j$;
2. $\sum_{j=1}^{d} W_j(x) \equiv 1$ on $\Omega$;
3. for every $\beta \in \mathbb{N}_0^3$ with $|\beta| \leq k$ there exists a constant $C_{\beta} > 0$ such that

$$||D^\beta W_j||_{L^\infty(\Omega_j)} \leq C_{\beta} \delta_j^{|eta|}, \quad j = 1, \ldots, d,$$

where $\delta_j = \text{diam}(\Omega_j) = \sup_{x,y \in \Omega_j} ||x - y||_2$.

In agreement with the statements in [33] we require some additional regularity assumptions on the covering $\{\Omega_j\}_{j=1}^{d}$.

**Definition 2** Suppose that $\Omega \subseteq \mathbb{R}^3$ is bounded and $\chi_N = \{x_i, i = 1, \ldots, N\} \subseteq \Omega$ are given. An open and bounded covering $\{\Omega_j\}_{j=1}^{d}$ is called regular for $(\Omega, \chi_N)$ if the following properties are satisfied:

(a) for each $x \in \Omega$, the number of subdomains $\Omega_j$ with $x \in \Omega_j$ is bounded by a global constant $K$;
(b) each subdomain $\Omega_j$ satisfies an interior cone condition [34];
(c) the local fill distances $h_{\chi_N, \Omega_j}$, where $\chi_{N_j} = \chi_N \cap \Omega_j$, are uniformly bounded by the global fill distance $h_{\chi_N, \Omega}$, i.e.

$$h_{\chi_N, \Omega} = \sup_{x \in \Omega} \min_{x_k \in \chi_N} ||x - x_k||_2.$$

Let $C^k(\mathbb{R}^3)$ be the space of all functions $f \in C^k$ whose derivatives of order $|\beta| = k$ satisfy $D^\beta f(x) = O(||x||_2^2)$ for $||x||_2 \to 0$. The following convergence result is well known (see, e.g. [15,34]).
Theorem 1  Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded and assume that $X_N = \{x_i, i = 1, \ldots, N\} \subseteq \Omega$. Let $\phi \in C^k(\mathbb{R}^3)$ be a strictly positive definite function. Let $\{\Omega_j\}_{j=1}^d$ be a regular covering for $(\Omega, X_N)$ and let $\{W_j\}_{j=1}^d$ be $k$-stable for $\{\Omega_j\}_{j=1}^d$. Then the error between $f \in \mathcal{N}_\phi(\Omega)$, where $\mathcal{N}_\phi$ is the native space of $\phi$, and its partition of unity interpolant (3.3) is bounded by

$$\left|D^\beta f(x) - D^\beta I(x)\right| \leq C h^{(k+\nu)/2 - |\beta|} |f|_{\mathcal{N}_\phi(\Omega)},$$

for all $x \in \Omega$ and all $|\beta| \leq k/2$.

If we compare this result with the global error estimates (see e.g. [34]), we can see that the partition of unity preserves the local approximation order for the global fit. This means that we can efficiently compute large RBF interpolants by solving small RBF interpolation problems and then glue them together with the global partition of unity $\{W_j\}_{j=1}^d$. In other words, the partition of unity approach is a simple and effective technique to decompose a large problem into many small problems while at the same time ensuring that the accuracy obtained for the local fits is carried over to the global one. In particular, the partition of unity method can be thought as a Shepard’s method with higher-order data, since local approximations $R_j$ are used instead of data values $f_j$.

We end this section with an example (in 2D) illustrating how the implicit partition of unity works. Let us consider the following data set:

$$x_i = ([2 + \sin(t_i)] \cos(t_i), [2 + \cos(t_i)] \sin(t_i)), \quad i = 1, \ldots, N,$$

where $t_i$ is a Halton sequence [15]. The normals are given by:

$$n_i = ([2 + \cos(t_i)] \cos(t_i) - \sin(t_i)^2, [2 + \sin(t_i)] \sin(t_i) - \cos(t_i)^2), \quad i = 1, \ldots, N,$$

Let us fix $N = 75$. In Fig. 3 (left) the data set is shown.

To obtain the set of off-surface points, we add the function values. Specifically, we assign the value 0 to each original data point and the value 1 or $-1$ respectively to the outside and the inside points (obtained by marching a small distance $\delta$ along the normals), as described in Fig. 3 (right).

Now the problem is turned into a full 2D interpolation problem. Thus we use the partition of unity method, in order to reconstruct the surface that interpolates the augmented data set.

![Fig. 3](image-url)  

Fig. 3  The original and augmented point cloud data set (left and right, respectively)
The result is shown in Fig. 4 (left). The interpolant curve, shown in Fig. 4 (right), is the zero contour of the interpolant surface.

### 3.3 Normals Estimation

To implement the method described in Sect. 3.1, the question on how the normal vectors are computed and consistently oriented needs to be answered. To be more precise, for every point, we find a normal to the surface and next we orient all normals consistently. Of course, we have to assume that the surface is indeed orientable [32].

Given data in the form $\mathcal{X}_N = \{x_i \in \mathbb{R}^3, i = 1, \ldots, N\}$, we fix a number $K < N$, and we find, for every point $x_i$, the $K$ nearest neighbors. The set of the neighbors of $x_i$ is denoted by $\mathcal{N}(x_i)$. The first step is to compute an oriented tangent plane for each data point [24]. The elements that describe the tangent plane $T_p(x_i)$ are a point $o_i$, called the center, and a unit normal vector $n_i$. The latter is computed so that the plane is the least squares best fitting plane to $\mathcal{N}(x_i)$ [23,24,32]. So, the center $o_i$ is taken to be the centroid of $\mathcal{N}(x_i)$, and the normal $n_i$ is determined using principal component analysis [2,24,32].

More precisely, we compute the center of gravity of $\{x_k, k \in \mathcal{N}(x_i)\}$, i.e.:

$$o_i = \frac{1}{K} \sum_{k \in \mathcal{N}(x_i)} x_k, \quad (3.6)$$

and the associated covariance matrix:

$$Cov(x_i) = \sum_{k \in \mathcal{N}(x_i)} (x_k - o_i)(x_k - o_i)^T, \quad (3.7)$$

which is a symmetric $3 \times 3$ positive semi-definite matrix. The eigenvalues $\lambda_1^i \geq \lambda_2^i \geq \lambda_3^i$ and the corresponding unit eigenvectors $v_1^i, v_2^i, v_3^i$ of this positive semi-definite matrix represent the plane and the normal to this plane [24]. Specifically, let us assume that two eigenvalues $v_1^i$ and $v_2^i$ are close together and the third one is significantly smaller. The eigenvectors for the first two eigenvalues determine the plane, while the eigenvector $v_3^i$ determines the normal to this plane. The second step is to orient the normal consistently, in fact $n_i$ is chosen to be either $v_3^i$ or $-v_3^i$. Note that if two data points $x_i$ and $x_k$ are close, their associated normals $n_i$ and
$n_k$ are nearly parallel, i.e. $n_in_k^T \approx \pm 1$. Consequently, if $n_in_k^T \approx -1$ either $n_i$ or $n_k$ should be flipped. The difficulty in finding a consistent global orientation is that this condition should hold between all pairs of sufficiently close data points. A common practice is to model this problem as graph optimization [24,32]. At first, we build the Riemann graph $G = \{V,E\}$, with each node in $V$ corresponding to one of the 3D data points. We remark that the Riemann graph is defined as the undirect graph among which there exists an edge $e_{ik}$ in $E$ if $v_k$ is one of the $K$ nearest neighbours of $v_i$, and vice versa. More precisely, we build a weighted graph, for example we could choose the weights $w(e_{ik}) = n_in_k^T$, and so the cost of the edge connecting the vertices $n_i$ and $n_k$ measures the deviation of the normals [32]. Hence, the normals are consistently oriented if we find directions $b_i = \{-1,1\}$, so that $\sum_{e_{ik}} b_ib_k w(e_{ik})$ is maximized. Unfortunately, this problem is NP-hard, i.e. no solution method can guarantee to find its exact solution in a reasonable time, as shown in [23]. We use the approximate solution described in [23]. To assign orientation to an initial plane, the unit normal of the tangent plane whose center has the largest $z$ coordinate is made to point toward the positive $z$ axis. We assign to each edge $e_{ik}$ the cost $w(e_{ik}) = 1 - |n_in_k^T|$, as suggested in [32]. Note that $w(e_{ik})$ is small if the unoriented tangent planes are nearly parallel. A favourable propagation order can therefore be achieved by traversing the minimal spanning tree of the Riemann graph. The advantage of this order consists in tending to propagate orientation along directions of low curvature in the data. To compute the minimal spanning tree of the resulting graph, we use Kruskal’s algorithm, as suggested in [32].

4 Numerical Experiments

In this section we summarize the extensive experiments performed to test our detection and approximation techniques. Specifically, in Sects. 4.1 and 4.2 respectively, we test the routines for 3D and 2D dynamical systems, considering the cases in which such models admit both two and three stable equilibria.

For the dynamical systems in consideration we establish conditions to be imposed on the parameters so that the separatrix manifolds exist. Here, after detecting the points lying on the latter with the algorithm described in Sect. 2, at first we compute the normal vectors and consistently orient them to the surfaces by choosing, for the three different manifolds, the nearest neighbours $K_i$, $i = 1,2,3$. Typically we set $K_i = 1,2,3$ between 5 and 10. Then we build the extra off-surface points by marching a small distance $\delta$ along the surface normals, as shown in Sect. 3.1; following [32], we take $\delta$ to be 1% of the maximum dimension of the data. Finally we interpolate the points lying on the separatrix surfaces with the implicit partition of unity method, described in Sect. 3.2, using in (3.4) the compactly supported Wendland’s $C^2$ function [34], as local approximants

$$\phi(r) = (1 - \epsilon r)^4 (4\epsilon r + 1).$$

Here $r = \|\cdot\|_2$ is the Euclidean norm, $(\cdot)_+$ denotes the truncated power function and $\epsilon > 0$ is the shape parameter. Such parameter determines the size of the support of the basis function. Its choice can significantly affect the final result. Specifically, for the three manifolds, we choose the shape parameters so that $0.01 \leq \epsilon_i \leq 0.1$, $i = 1,2,3$. Assuming to have a nearly uniform node distribution such as the Halton points, according to [6], a possible choice for the number of subdomains centers consists in constructing a uniform grid of $d_i = (d_{PU})^s$ centers, where $s$ is the dimension of the dynamical system and $d_{PU} = \lceil 1/2(N_i/2)^{1/s} \rceil$,
\[ i = 1, 2, 3. \] However, we point out that in our tests we find good results even with different \( d_i \), \( i = 1, 2, 3. \) This is due to the fact that we deal with concrete and unstructured data.

Such choices, described above, are suitable assuming to start with \( 8 \leq n \leq 15 \) equispaced initial conditions on each edge of the cube \([0, 1]^3\). For the tolerance used in the bisection routine, a recommended value is \( 10^{-3} \leq \text{tol} \leq 10^{-5} \), since it allows to achieve a good trade-off between accuracy and computational cost.

### 4.1 3D Detection-Interpolation Tests

A model chosen to test the detection-interpolation algorithm is the classical three-populations competition model. Letting \( x, y \) and \( z \) denote the populations, we consider the following system

\[
\begin{align*}
\frac{dx}{dt} &= p(1 - \frac{x}{u})x - axy - bxz, \\
\frac{dy}{dt} &= q(1 - \frac{y}{v})y - cxy - eyz, \\
\frac{dz}{dt} &= r(1 - \frac{z}{w})z - fzx - gyz,
\end{align*}
\]

where \( p, q \) and \( r \) are the growth rates of \( x, y \) and \( z \), respectively, \( a, b, c, e, f \) and \( g \) are the competition rates, \( u, v \) and \( w \) are the carrying capacities of the three populations. The model (4.1) describes the interaction of three competing populations within the same environment (see e.g. [19]).

There are eight equilibrium points. The origin \( E_0 = (0, 0, 0) \) and the points associated with the survival of only one population \( E_1 = (u, 0, 0) \), \( E_2 = (0, v, 0) \) and \( E_3 = (0, 0, w) \). Then we have the equilibria with two coexisting populations:

\[
E_4 = \left( \frac{uq(aw - p)(uv - pq)}{cuwa - pq}, \frac{pv(cu - q)}{cuva - pq}, 0 \right); \quad E_5 = \left( \frac{ur(bw - p)}{f_{uw} - rp}, 0, \frac{wp(fu - r)}{f_{uw} - rp} \right).
\]

\[
E_6 = \left( 0, \frac{uv(we - q)}{gwve - qr}, \frac{uw(vg - r)}{gwwe - qr} \right).
\]

Finally we have the coexistence equilibrium,

\[
E_7 = \left( \frac{u[p(aw - p)]}{p(aw - p)} - aq(we - q) - bwp(vg - r)}, \frac{vq(aw - p)}{vq(aw - p)} + upc(wb - p) - pew(fu - r)}{uq(aw - p) + upc(wb - p) - pew(fu - r)}, \frac{r[(cuva - pq) - gwp(cu - q) - upq(va - p)]}{r[(cuva - pq) + bpu(fu - vg) + evw(gp - fau)]}).
\]

Letting \( p = 1, q = 2, r = 2, a = 5, b = 4, c = 3, e = 7, f = 7, g = 10, u = 3, v = 2, w = 1 \), the points associated with the survival of only one population, i.e. \( E_1 = (3, 0, 0) \), \( E_2 = (0, 2, 0) \) and \( E_3 = (0, 0, 1) \), are stable, the origin \( E_0 = (0, 0, 0) \) is an unstable equilibrium and the coexistence equilibrium \( E_7 \approx (0.1899, 0.0270, 0.2005) \) is a saddle point. The remaining equilibria \( E_4 \approx (0.6163, 0.1591, 0) \), \( E_5 \approx (0.2195, 0, 0.5317) \) and \( E_6 \approx (0, 0.1714, 0.2647) \) are other saddle points. The manifolds joining these saddles partition the phase space into the different basins of attraction, but intersect only at the coexistence saddle point, labeled \( E_7 \).\(^1\) In this situation we can use the detection-interpolation

\(^1\) In case of bistability the manifold through the origin and a saddle point partitions the phase space into two regions. In case of a system with three equilibria instead, more saddles are involved in the dynamics. But the three separating manifolds all intersect only at one saddle with all nonnegative populations.
routine to approximate the basins of attraction. More precisely, we choose \( n = 15, \gamma = 6, \)
\( toll = 10^{-3}, t = 90, \epsilon = (0.1, 0.09, 0.08), d^{PU} = (3, 4, 4), K = (7, 8, 6). \) Figure 5 shows
the separatrix points and the basins of attraction of \( E_1, E_2 \) and \( E_3, \) (left to right, top to
bottom). Finally, in Fig. 6 we plot together the three basins of attraction.

To test our detection-interpolation routine when bistability occurs, we consider the follow-
ing model, describing a three level food web, with a top predator indicated by \( W, \) the
intermediate population \( V \) and the bottom prey \( N \) that is affected by an epidemic. It is
subdivided into the two subpopulations of susceptibles \( S \) and infected \( I \) [13],

\[
\begin{align*}
\frac{dW}{dt} &= -mW + pVW, \\
\frac{dV}{dt} &= -lV + eSV - hWW + qIV, \\
\frac{dI}{dt} &= \beta IS - nIV - \gamma I - \nu I,
\end{align*}
\]
intermediate population is corresponding loss rate of infected individuals in the lowest trophic level due to capture by the infected prey. In this lowest trophic level, only the healthy prey reproduce at net rate $K$, the prey environment carrying capacity is $K$ which must clearly be smaller than the damage inflicted to the susceptibles $e$, gain obtained by the intermediate population from hunting of susceptibles is denoted by $E$. Then we have the point at which just the bottom prey thrive, with endemic disease, $E_1 = (0, 0, 0, K)$, the disease-free equilibrium with all the trophic levels $E_2$ and the steady state in which only the intermediate population and the bottom healthy prey thrive $E_3$:

$$E_2 = \left( \frac{apKe - mecK - apl}{ahp}, \frac{m}{p}, 0, \frac{K ap - cm}{ap} \right), \quad E_3 = \left( 0, \frac{a(K e - l)}{ecK}, 0, \frac{l}{e} \right).$$

Then we have the point at which just the bottom prey thrives, with endemic disease, $E_4$ and two equilibria in which the top predators disappear, $E_5$ and $E_6$:

$$E_4 = \left( 0, 0, \frac{a(K \beta \gamma + k \beta \nu - \gamma^2 - 2 \gamma \nu - \nu^2)}{\beta(a \gamma \alpha v + K \beta \nu)}, \frac{\gamma + \nu}{\beta} \right),$$

$$E_{5,6} = \left( 0, \frac{\beta \hat{S} - \gamma - \nu}{\hat{S}}, 1 - e \hat{S}, \frac{\hat{S}}{q} \right),$$

where $\hat{S}$ are the roots of $\hat{S}^2 + \hat{B}S + C = 0$.

With the parameters values $l = 10, e = 2, q = 1, \beta = 1.6, n = 5, \gamma = 1, \nu = 3, a = 8, K = 6, c = 0.5$, the equilibria $E_3 \approx (0, 2.6666, 0, 5)$ and $E_4 \approx (0, 0, 1.8421, 2.5)$ are both stable and $E_5 \approx (0, 0.7244, 0.4721, 4.7639)$ is the saddle point that partitions the domain in the $W = 0$ three-dimensional phase subspace. Thus system (4.2) is reduced to a system of three equations and therefore we can reconstruct the separatrix surface in such subspace with the routine described in Sect. 2.2. The separatrix points and the separatrix surface, shown in Fig. 7 (left) and (right) respectively, are the result of the detection-interpolation algorithm with $n = 11, \gamma = 10, tol = 10^{-4}, t = 30, \varepsilon = 0.6, d^{PU} = 4, K = 7$. 

$$\frac{dS}{dt} = aS \left(1 - \frac{S + I}{K}\right) - cVS - \beta SI + \gamma I,$$  (4.2)
4.2 2D Detection-Interpolation Tests

To give an example for a dynamical system of dimension two, we can consider the competition model analyzed in [27]. Letting $P$ and $Q$ denote two populations gathering in herds, we consider the following system describing the competition of two different populations within the same environment:

\[
\begin{align*}
\frac{dQ}{d\tau} &= r \left(1 - \frac{Q}{K_Q}\right)Q - q\sqrt{Q}\sqrt{P}, \\
\frac{dP}{d\tau} &= m \left(1 - \frac{P}{K_P}\right)P - p\sqrt{Q}\sqrt{P},
\end{align*}
\]  

(4.3)

where $r$ and $m$ are the growth rates of $Q$ and $P$, respectively, $q$ and $p$ are the competition rates, $K_Q$, and $K_P$ are the carrying capacities of the two populations.

Since singularities could arise in the Jacobian when one or both populations vanish, we define the following new variables, as suggested in [27]:

\[
\begin{align*}
X(t) &= \sqrt{\frac{Q(t)}{K_Q}}, & Y(t) &= \sqrt{\frac{P(t)}{K_P}}, & t &= \tau \frac{q\sqrt{K_P}}{2\sqrt{K_Q}}, \\
a &= \frac{pK_Q}{qK_P}, & b &= \frac{r\sqrt{K_Q}}{q\sqrt{K_P}}, & c &= \frac{m\sqrt{K_Q}}{q\sqrt{K_P}}.
\end{align*}
\]  

(4.4)

Thus the adimensionalized, singularity-free system for (4.3) is

\[
\begin{align*}
\frac{dX}{dt} &= b \left(1 - X^2\right)X - Y, \\
\frac{dY}{dt} &= c \left(1 - Y^2\right)Y - aX.
\end{align*}
\]  

(4.5)

We can easily verify that the origin $E_0 = (0, 0)$, and the points associated with the survival of only one population $E_1 = (K_Q, 0)$, $E_2 = (0, K_P)$ are equilibria of (4.3). To study the remaining equilibria we consider the adimensionalized system, in fact the coexistence
equilibria are the roots of the eighth degree equation
\[
cb^3 X^8 - 3cb^3 X^6 + 3cb^3 X^4 - cb(b^2 + 1)X^2a + cb = 0.
\]

Observe that in our test we have to take into account that \( E_1' = (1, 0) \) and \( E_2' = (0, 1) \), corresponding to \( E_1 = (K_Q, 0) \) and \( E_2 = (0, K_P) \) of system (4.3), are not critical points of the system (4.5).

With the parameters \( r = 0.7895, m = 0.7885, p = 0.225, q = 0.2085, K_P = 12 \) and \( K_Q = 10 \), the points \( E_1 = (10, 0), E_2 = (0, 12) \) and \( E_3 \approx (7.0127, 8.9727) \) are stable equilibria of the system (4.3). Instead of integrating the latter we consider the model (4.5), whose three stable equilibria are \( E_1^* \approx (-1.1342, 1.1237), E_2^* \approx (1.1342, -1.1237), E_3^* \approx (0.8374, 0.8647) \), whereas the origin is the saddle point through which all the three curves go. Note that when three stable attractors are present there are also other saddles involved in the dynamics, namely \( E_4 \approx (-0.9585, -0.2692), E_5 \approx (0.9585, 0.2692) \) and \( E_6 \approx (0.3055, 0.9575) \). Observe that, applying the transformations (4.4), obviously \( E_3^* \) corresponds to \( E_3 \), while \( E_1^* \) and \( E_2^* \) are not feasible, but roughly speaking, they represent \( E_1 \) and \( E_2 \). In fact the trajectories converging to \( E_1^* \) and \( E_2^* \), under the biological constraint \( X \geq 0, Y \geq 0 \), stop on the axes evolving toward the biological equilibria \( E_1' \) and \( E_2' \). Therefore we consider \( E_1^*, E_2^* \) and \( E_3^* \). To apply the algorithm with initial conditions (2.3) we need a further consideration. Specifically, we have to translate the problem in the positive plane with the substitutions

\[
X' = X + \frac{Y}{2} \quad \text{and} \quad Y' = Y + \frac{Y}{2}, \tag{4.6}
\]

where \( \gamma \) is the length of the square. At this point we can apply the detection-interpolation algorithm, described in Sect. 2.3. More precisely, we choose: \( n = 13, \gamma = 3, \text{tol} = 10^{-4}, t = 40, \varepsilon = (0.1, 0.06, 0.08), d^PU = (4, 3, 3), K = (4, 6, 6) \). Figure 8 shows how the algorithm works. It generates first the points lying on the curves determining the domains of attraction (top left), then subsequently the basins of attraction of \( E_1^* \) (top right), \( E_2^* \) (bottom left) and \( E_3^* \) (bottom right), in the original system \( X \) and \( Y \). Finally, in Fig. 9 we plot together the three basins of attraction, always in the original system. Using again the transformation (4.4) we obtain the curves separating the basins of attraction of \( E_1, E_2 \) and \( E_3 \), shown in Fig. 11 (left).

To test our detection-interpolation algorithm when bistability occurs we choose the parameters as follows: \( r = 0.7895, m = 0.7885, p = 0.225, q = 0.2085, K_P = 12 \) and \( K_Q = 10 \). With this choice the equilibria \( E_1 = (10, 0) \) and \( E_2 = (0, 16.5) \) are stable, the origin \( E_0 \) is unstable and \( E_3 \approx (3.8757, 3.1919) \) is the saddle coexistence equilibrium point partitioning the phase space domain of the system (4.3). The stable equilibria of (4.5) are \( E_1^* \approx (1.3436, -1.2482), E_2^* \approx (-1.3436, 1.2482) \) and the coexistence saddle point is \( E_3^* \approx (0.6717, 0.4252) \). In view of the above considerations we can identify \( E_1^* \) and \( E_2^* \) with \( E_1' = (1, 0) \) and \( E_2' = (0, 1) \). After translating the problem in the positive plane with the substitutions (4.6), we can apply the detection-interpolation routine. In this case we choose: \( n = 15, \gamma = 4, \text{tol} = 10^{-4}, t = 40, \varepsilon = 0.1, d^PU = 3, K = 4 \). Figure 10 shows the separatrix points (left) and the separatrix curve (right) in the phase plane of the system (4.5). Using again the transformation (4.4) we obtain the curve separating the basins of attraction of \( E_1, E_2 \), shown in Fig. 11 (right).
Fig. 8  Set of points lying on the curves determining the domains of attraction (top left) and the reconstruction of the basin of attraction of $E_1$ (top right), $E_2$ and $E_3$ (bottom, left to right). The four figures (left to right, top to bottom) show the progress of the algorithm: first it generates the points on the separatrices, then in turn each individual basin of attraction. The black and blue circles represent the origin and the stable equilibria, respectively. Moreover the other saddles ($E_4, E_5$ and $E_6$) that lie on the separatrix manifolds of the attraction basins are identified by green circles (Color figure online).

Fig. 9  Reconstruction of the basins of attraction with parameters $r = 0.7895$, $m = 0.7885$, $p = 0.225$, $q = 0.2085$, $Kp = 12$ and $Kq = 10$.
Fig. 10 Set of points lying on the curve separating the domains of attraction of $E_1$ and $E_2$ (left) and the reconstruction of the separatrix curve (right). The black and blue circles represent the unstable origin, the coexistence saddle point and the stable equilibria, respectively (Color figure online).

Fig. 11 The basin of attraction of $E_1$, $E_2$ and $E_3$ with parameters $r = 0.7895$, $m = 0.7885$, $p = 0.225$, $q = 0.2085$, $K_p = 12$ and $K_q = 10$ (left), and the curve separating the basin of attraction of $E_1$, $E_2$ with parameters $r = 0.8888$, $m = 0.602$, $p = 0.401$, $q = 0.5998$, $K_p = 16.5$, $K_q = 10$.

5 Conclusion and Future Work

In this paper we present a novel algorithm for the detection of the attraction basins of equilibria of dynamical systems. It is robust enough to work for dynamical systems presenting two or three stable equilibrium points. Such routine allows to have a graphical representation of the domains of attraction. In many applications, an accurate representation turns out to be very useful. In fact, the knowledge of the state of the system (together with the computation of the attraction basins) allows to eventually suggest measures and strategies to move the initial condition far away from an unwanted attraction basin.

Work in progress consists in extending our simple but powerful routine in case of dynamical systems presenting periodic orbits. In that case, during the bisection routine, a different stopping criterion, which enables us to test if a trajectory follows a cyclic orbit around an equilibrium (not necessarily an orbit of a simple shape as a circle), should be adopted.
Acknowledgments  The authors sincerely thank the two anonymous referees for helping to significantly improve our paper. This work was supported by the University of Turin via grant “Metodi numerici nelle scienze applicate”.

References

1. Arrowsmith, D.K., Place, C.K.: An Introduction to Dynamical Systems. Cambridge University Press, Cambridge (1990)
2. Belton, D.: Improving and extending the information on principal component analysis for local neighborhoods in 3D point clouds. The International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences 37, (2008) B5: 477 ff
3. Buhmann, M.D.: Radial Basis Functions: Theory and Implementation. Cambridge Monogr. Appl. Comput. Math., vol. 12, Cambridge University Press, Cambridge (2003)
4. Carr, J.C., Beatson, R.K., Cherrie, J.B., Mitchell, T.J., Fright, W.R., Mccallum, B.C., Evans, T.R.: Reconstruction and representation of 3D objects with radial basis functions. In: Proceedings of the 28th Annual Conference on Computer Graphics and Interactive Techniques, Los Angeles, CA, USA, pp. 67–76 (2001)
5. Carr, J.C., Fright, W.R., Beatson, R.K.: Surface interpolation with radial basis functions for medical imaging. IEEE Trans. Med. Imaging 16, 96–107 (1997)
6. Cavoretto, R.: A numerical algorithm for multidimensional modeling of scattered data points. Comput. Appl. Math. 34, 65–80 (2015)
7. Cavoretto, R., Chaudhuri, S., De Rossi, A., Menduni, E., Moretti, F., Rodi, M., Venturino, E.: Approximation of dynamical system’s separatrix curves. In: Proceedings of the ICNAAM 2011. Simos T.E., et al. (eds.) AIP Conference Proceedings, vol. 1389, Melville, NY, pp. 1220–1223 (2011)
8. Cavoretto, R., De Rossi, A.: A meshless interpolation algorithm using a cell-based searching procedure. Comput. Math. Appl. 67, 1024–1038 (2014)
9. Cavoretto, R., De Rossi, A.: A trivariate interpolation algorithm using a cube-partition searching procedure. SIAM J. Sci. Comput. 37, A1891–A1908 (2015)
10. Cavoretto, R., De Rossi, A., Perracchione, E., Venturino, E.: Reliable approximation of separatrix manifolds in competition models with safety niches. Int. J. Comput. Math. 92, 1826–1837 (2015)
11. Chen, Y.L., Lai, S.H.: A partition of unity based algorithm for implicit surface reconstruction using belief propagation. In: Proceedings of the 2007 International Conference on Shape Modeling and Applications, Lyon, France, pp. 147–155 (2007)
12. Cuomo, S., Galletti, A., Giunta, G., Starace, A.: Surface reconstruction from scattered point via RBF interpolation on GPU. In: Ganza M. et al. (eds.) Proceedings of the 2013 Federated Conference on Computer Science and Information Systems, IEEE, pp. 433–440 (2013)
13. De Rossi, A., Lisa, F., Rubini, L., Zappavigna, A., Venturino, E.: A food chain ecoepidemic model: infection at the bottom trophic level. Ecol. Complex. 21, 233–245 (2015)
14. Dellnitz, M., Junge, O., Rumpf, M., Strzodka, R.: The computation of an unstable invariant set inside a cylinder containing a knotted flow. In: Fiedler B. et al. (eds.) Proceedings of EQUADIFF 99, World Scientific, pp. 1015–1020 (2000)
15. Fasshauer, G.E.: Meshfree Approximation Methods with Matlab. World Scientific Publishers Co. Inc, River Edge, NJ (2007)
16. Giesl, P., Wendland, H.: Approximating the basin of attraction of time-periodic odes by meshless collocation. Discrete Contin. Dyn. Syst. 25, 1249–1274 (2009)
17. Giesl, P., Wendland, H.: Numerical determination of the basin of attraction for exponentially asymptotically autonomous dynamical systems. Nonlinear Anal. Theor. 74, 3191–3203 (2011)
18. Giesl, P., Wendland, H.: Numerical determination of the basin of attraction for asymptotically autonomous dynamical systems. Nonlinear Anal. Theor. 75, 2823–2840 (2012)
19. Gosso, A., La Morgia, V., Marchisio, P., Telve, O., Venturino, E.: Does a larger carrying capacity for an exotic species allow environment invasion?—Some considerations on the competition of red and grey squirrels. J. Biol. Syst. 20, 221–234 (2012)
20. Hale, J.K., Kocak, H.: Dyn. Bifurc. Springer, New York (1991)
21. Heryudono, A.R.H., Driscoll, T.A.: Radial basis function interpolation on irregular domain through conformal transplantation. J. Sci. Comput. 44, 286–300 (2010)
22. Hilker, F.M., Langlais, M., Malchow, H.: The Allee effect and infectious diseases: extinction, multistability, and the (dis-)appearance of oscillations. Am. Nat. 173, 72–88 (2009)
23. Hoppe, H.: Surface Reconstruction from Unorganized Points. Ph.D. Thesis, University of Washington (1994)
24. Hoppe, H., Derose, T., Duchamp, T., Mcdonald, J., Stuetzle, W.: Surface reconstruction from unorganized points. In: Brown, M. et al. (eds.) Proceedings of 19th Annual Conference and Exhibition on Computer Graphics and Interactive Techniques. ACM SIGGRAPH Computer Graphics, vol. 26, New York, USA, pp. 71–78 (1992)
25. Iske, A.: Scattered data approximation by positive definite kernel functions. Rend. Sem. Mat. Univ. Pol. Torino 69, 217–246 (2011)
26. Johnson, T., Tucker, W.: Automated computation of robust normal forms of planar analytic vector fields. Discrete Contin. Dyn. Syst. Ser. B 12, 769–782 (2009)
27. Melchionda, D., Pastacaldi, E., Perri, C., Venturino, E.: Interacting population models with pack behavior. Submitted for publication (2014), arXiv:1403.4419v1
28. Melenk, J.M., Babuška, I.: The partition of unity finite element method: basic theory and applications. Comput. Methods Appl. Mech. Eng. 139, 289–314 (1996)
29. Murray, J.D.: Mathematical Biology. Springer, Berlin (1993)
30. Sabetta, G., Perracchione, E., Venturino E.: Wild herbivores in forests: four case studies. In: Mondaini RP (ed.) Proceedings of Biomat 2014. World Scientific, Singapore, pp. 56–77 (2015)
31. Turk, G., O’Brien, J.F.: Modelling with implicit surfaces that interpolate. ACM Trans. Graph. 21, 855–873 (2002)
32. Wendland, H.: Surface reconstruction from unorganized points, http://people.maths.ox.ac.uk/wendland/research/old/reconhtml/reco-nhtml.html (2002)
33. Wendland, H.: Fast evaluation of radial basis functions: methods based on partition of unity. In: Chui, C.K., Schumaker, L.L., Stockler, J. (eds.) Approximation Theory X: Wavelets, Splines, and Applications, pp. 473–483. Vanderbilt University Press, Nashville (2002)
34. Wendland, H.: Scattered Data Approximation. Camb. Monogr. Appl. Comput. Math., vol. 17, Cambridge University Press, Cambridge (2005)