Weak cosmic censorship conjecture is not violated for a rotating linear dilaton black hole

Fei Qu\textsuperscript{a}, * Si-Jiang Yang\textsuperscript{a}, Zhi Wang\textsuperscript{a}, Ji-Rong Ren\textsuperscript{ab}\textsuperscript{†}

\textsuperscript{a}Institute of Theoretical Physics & Research Center of Gravitation, Lanzhou University, Lanzhou 730000, China
\textsuperscript{b}Key Laboratory for Magnetism and Magnetic of the Ministry of Education, Lanzhou University, Lanzhou 730000, China

In this paper, we investigate the validity of the weak cosmic censorship conjecture (WCCC) for a rotating linear dilaton black hole from two different methods. By using the classical ingoing test particle method, we obtain the same results as given in the new version of gedanken experiment recently proposed by Wald. We find that even for this rotating linear dilaton black hole the Iyer-Wald formalism is still functioning. By comparing these two methods, we find that the same result will be obtained in both cases up to first order. The nearly extremal black hole can be overspun, while the extremal one cannot be overspun. When we include the second order modification into consideration, the Iyer-Wald method show that even for the nearly extremal black hole, the WCCC is well protected. These results imply that weak cosmic censorship conjecture is still valid using the ingoing test particle method up to second order modification.

PACS numbers:

I. INTRODUCTION

Gravitational collapse inevitably leads to space-time singularity, which indicate the failure of the predictability of the theory. To protect the predictability of classical gravitational theory, Penrose proposed the weak cosmic censorship conjecture, which states that naked singularity cannot be formed by gravitational collapse[1]. Though more than fifty years has passed while a general proof of the conjecture is still beyond reach, and the conjecture has play an important role in black hole physics. Over the past fifty years, many ways have been proposed to test the conjecture[2–7]. One way of checking the conjecture is to consider whether we can destroy the event horizon to form a naked singularity.

Pioneer works to consider the destruction of a black hole event horizon was envisaged by Wald, in whose gedanken experiment a test particle with large charge or large angular momentum was dropped into an extremal Kerr-Newman black hole. The result suggests that particles causing the destruction of event horizon will not be captured by a black hole[8]. While, later work of Hubeny shows that near-extremal charged black holes can be overcharged by test particle[9], the results are supported by the following work of Jacobson and Sotiriou for Kerr black hole[10]. By carefully choosing the parameters of the particle, there are some possible counter-examples that the event horizon of near-extremal black holes can be destroyed[11–13]. But when self-force and radiation effects are taken into account, the above counter-example for weak cosmic censorship conjecture might be recovered[14–18].

Recently, Sorce and Wald proposed a new version of gedanken experiment by taking the second-order approximation of the perturbation that comes from the matter fields into account[19]. The new gedanken experiment is based on the Lagrangian method[20–23], and naturally incorporate the self-force and backreaction effects. They showed that the Kerr-Newmann black hole cannot be overcharged or overspun up to second order perturbation. After that, in several cases with an asymptotically flat metric the validity of the conjecture has been confirmed[24–35].

Using the Iyer-Wald formalism[36], the RN black hole in a non-asymptotically flat(AdS) background has been discussed. While, previous research has not discussed the linear dilaton case using this formulation. Inspired by this, we test the validity of the conjecture for a rotating linear dilaton black hole in this paper. We obtain the Komar-type integral for the Iyer-Wald formalism, and confirm it is consistent with the results given by[37]. After that, we test the weak cosmic censorship conjecture (WCCC) by perturbing it with an ingoing particle. We find that the WCCC is valid for the extremal black hole, while not for the near extremal case. Then we obtain the two variational inequality corrsponding to our case. After that, By using these inequality, we find that the same results as given using test particles method will be obtained up to first order approximation. And if we include the second order modification into the case, we find the WCCC is well preserved. The similarity implies that if we consider perturbation of classical ingoing particle up to second order, the WCCC might be well protected as well.

The rest of the paper is organised as follows. In Sec. II, we give an overview for the linear dilaton solution from the Einstein-Maxwell-Dilaton-Axion (EMDA) theory. In the Sec. III, we calculate the corresponding Komar-type integral for later convenience. In the Sec. IV, we discuss the classical version of the gedanken experiment. In the Sec. V, VI, we review the Iyer-Wald formulation, and obtain the variational inequality. In the section VII, we test the weak cosmic censorship conjecture with the
variational inequality. The last section is devoted to the conclusion.

II. A ROTATING LINEAR DILATON BLACK HOLE IN EMDA THEORY

In this section, we give a brief review concerned with a special black hole in the Einstein-Maxwell-Dilaton-Axion theory, then we will mention about its thermodynamics. The results is given in [38] in detail. The action for the Einstein-Maxwell-Dilaton-Axion is given by

$$ S = \frac{1}{16\pi} \int d^4x \left[ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa 
- e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right]. $$

(1)

Where $R$ is the usual Ricci scalar curvature, $\phi$ and $\kappa$ are the dilaton field and axion field respectively, $F_{\mu\nu}$ with its Hodge dual $\tilde{F}_{\mu\nu}$ corresponds to the Maxwell field.

And it is well known that the lagrangian of this action is equivalent to following formulation

$$ S = \int \sqrt{-g} d^4x \left[ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-4\phi} H_{\mu\nu\tau} H^{\mu\nu\tau} 
- \frac{1}{8} e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right]. $$

(2)

By equivalence, here we mean that if we substitute

$$ H_{\mu\nu\rho} = -\frac{1}{\sqrt{-g}} e^{4\phi} \epsilon_{\mu\nu\rho\sigma} \partial_\sigma \kappa. $$

(3)

into the lagrangian and make a rescale, we can obtain the same equation of motion. One may seek more information about the equivalence of these two theory in [39, 40]. Where $\epsilon$ is the total antisymmetric symbol. For which, asymptotically flat black hole solution has been discussed in [30, 41].

The stationary black hole solution without nut charge can be given as in spherical coordinates

$$ ds^2 = -\frac{\Gamma}{r_0^2} d\tau^2 + r_0 r \left[ \frac{dr^2}{\Gamma} + d\theta^2 + \sin^2 \theta \left( d\varphi - \frac{a}{r_0 r} dt \right)^2 \right], $$

(4)

with

$$ \Gamma = r^2 - 2Mr + a^2. $$

(5)

The background field is given as

$$ F = \sqrt{e^{-2\phi}} \left[ \frac{r^2 - a^2 \cos^2 \theta}{r_0^2 r^2} dr \wedge dt + a \sin 2\theta d\theta \wedge \left( d\varphi - \frac{a}{r_0 r} dt \right) \right], $$

$$ e^{-2\phi} = \frac{\Gamma}{r_0^2 \sin^2 \theta}, \quad \kappa = \frac{\Gamma}{r_0^2 \cos^2 \theta}. $$

(6)

Which gives the explicit expression of the electromagnetic field, dilaton field, and axion field respectively.

The $(M, a)$ pair is given as the parameter to determine the black hole properties. It has been discussed in detail[38] that $r_0$ is a fixed background constant representing the charge of the metric, which actually means that we set the background with fixed charge. This is why we call the metric is a rotating linear dilaton black hole.

The event horizon is given by $\Gamma = 0$. In our case, this corresponds to two different event horizon, given as

$$ r_+ = M + \sqrt{M^2 - a^2}, $$

$$ r_- = M - \sqrt{M^2 - a^2}. $$

(7)

(8)

For simplicity, we rename $\sqrt{M^2 - a^2}$ as $\Delta$. The black hole is extremal when $\Delta = 0$ is satisfied, and the non-extremal case corresponds to $\Delta > 0$. For imaginary value of $\Delta$ corresponds to the naked singularity.

The Hawking temperature $T_H$, the Bekenstein-Hawking entropy $S_{BH}$ and the angular velocity $\Omega_H$ of the black hole are

$$ T_H = \frac{r_+ - r_-}{4\pi r_0 r_+ r_-}, $$

$$ \Omega_H = \frac{a}{r_0 r_+}, $$

$$ S_{BH} = \pi r_0 r^2. $$

(9)

Since that this metric is asymptotically dilaton, so that its thermodynamics needs to be carefully defined because the action is linearly divergent. Then one should set the charge as the invariant background property to refine its conserved quantity. In [38] they use the Hamiltonian method developed by Brown and York in [37] with a renormalized action,

$$ \tilde{S} = S_{(g)} + S_{(m)} - S_{(0)}. $$

(10)

Where the first two part is the total action given by our classical solution eq.(4) and eq.(6), while the third term is given by the corresponding divergent background. They argue that the Brown-York(BY) mass for Black Hole first law need to be defined as [38]

$$ \mathcal{M} = - \int_B d^2x \sqrt{\sigma} \tilde{u}_0 \varepsilon, $$

$$ = - \int_B d^2x \sqrt{\sigma} \tilde{u}_0 \left( \varepsilon_{(g)} + \varepsilon_{(m)} - \varepsilon_{(0)} \right). $$

(11)

Here the second term is linear divergent while can be exactly cancelled by electromagnetic contribution of the linear dilaton background evaluated with the same boundary data from fixed background field, which accounts for why in our case $r_0$ is constant and charge is fixed, such that the total BY mass is conserved, and in this case it is

$$ \mathcal{M} = \frac{M}{2}. $$

(12)

And the angular momentum is defined similarly as

$$ J = \int_B d^2x \sqrt{\sigma} j_\varphi = \int_B d^2x \sqrt{\sigma} \left( j_{(g)} \varphi + j_{(m)} \varphi \right). $$

(13)
It is calculated as
\[ J = \frac{ar_0}{2}. \] (14)
And it is the fixed charge that make the loss of Q term in its first law of black hole. Eventually, we have
\[ d\mathcal{M} = T_H dS_{BH} + \Omega_H dJ. \] (15)
as its first law of black hole thermodynamics.

III. KOMAR INTEGRAL APPROACH TO THE CONSERVED QUANTITY

It is well known that the first law of black hole can be deduced from different methods. Wald have discussed the first law of asymptotically flat black hole in this section, we want to deduce the conserved integral in our case using the Komar-type formulae for later simplicity. To do this, we need to rewrite the metric in a more applicable form.

\[ ds^2 = -\frac{1}{a^2} \sin^2 \theta \frac{dt^2}{r_0} + \frac{dr^2}{r} + \frac{r^2 d\Omega}{\sin^2 \theta} - 2asin^2 \theta \theta r_0 dtd\phi. \] (16)
From which, we can easily read its coefficients. The Komar-type conserved integral is given by (in our notation)

\[ \mathcal{M} = -\frac{1}{16\pi} \int_{S_\infty} \nabla^\alpha \xi^\beta dS_{\alpha\beta}, \]
\[ J = \frac{1}{16\pi} \int_{S_\infty} \nabla^\alpha \xi^\beta \sin \theta dS_{\alpha\beta}. \] (17)
The two-surface \( S_\infty \) should be our spatial infinity. And the \( \xi + \Omega_H \xi_\phi \) is a time-like killing vector of our metric, the surface element is given as

\[ dS_{\alpha\beta} = -2n_{[\alpha} r_{\beta]} \sqrt{\sigma} d^2 \theta. \] (18)
The \( \sqrt{\sigma} \) is the invariant surface element on \( S_\infty \). The \( n \) is a normal vector to our chosen three dimensional spatial hypersurface \( t = const \). For which \( S_\infty \) is a boundary. The \( r \) is a normal vector live in our hypersurface, while is orthogonal to the \( S_\infty \) := \( r = const \).

\[ n_\alpha = -r \frac{1}{r_0} \frac{M}{r} \partial_\alpha t, \]
\[ r^\alpha = \sqrt{\frac{r}{r_0}} \frac{M}{r} \partial_\alpha. \] (19)
Then if one just calculate the integral given in eq.(17), one may find it is linearly divergent with the growing of the radius, while fortunately this term will not be parameter dependent if we set \( r_0 \) as constant. We can calculate the same integral for the chosen background. One might find that if we subtract the background contribution, the results will be \( \frac{2\pi}{r_0} \).

One can substitute this equation into the first line of eq.(17), then the finite part will be
\[ \mathcal{M} = \frac{M}{2}. \] (21)
Which gives exactly the same answer as eq.(12). Similarly, eq.(14) will be recovered from the second line of eq.(17).

From above discussion, One may see that in this linear rotating black hole, the conserved quantity will consist of two parts: one part is parameter dependent, and is convergent; while another part is parameter independent as well as infinite. And the finite part is related to our black hole thermodynamic[38, 42, 43].

IV. CLASSICAL GEDANKEN EXPERIMENT

There are many ways to conduct the gedanken experiment to test the validity of the WCCC. In this section, we briefly discuss whether we can violate the WCCC with a test particle with a large enough angular momentum.

The lagrange for a test particle is given by
\[ L = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \] (22)
From which, the equation of motion can be derived as
\[ \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0. \] (23)
The energy and angular momentum is given as
\[ \delta E = -P_t = -\frac{\partial L}{\partial \dot{t}} = -mg_{00} \frac{dx^0}{d\tau}, \]
\[ \delta J = P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mg_{03} \frac{dx^3}{d\tau}. \] (24)
According to our signature, the 4-velocity of a massive particle satisfies
\[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{1}{m^2} g^{\mu\nu} P_\mu P_\nu = -1. \] (25)
Combining above expression, we have
\[ \delta E = \frac{g^{03}}{g^{00}} \delta J - \frac{1}{g^{00}} \left[ (g^{03})^2 \delta J^2 - g^{00} g^{33} \delta J^2 \right. \]
\[ -g^{00} \left( g^{11} P_r^2 + g^{22} P_\theta^2 + m^2 \right) \right]^{\frac{1}{2}}. \] (26)
Where we have used the future directed condition \( dt/d\tau > 0 \), which implies
\[ \delta E > \frac{g^{03}}{g^{33}} \delta J. \] (27)
Hence, for particles crossing the event horizon, the energy \( \delta E \) and angular momentum \( \delta J \) must satisfy
\[ \delta E > \Omega H \delta J. \] (28)

On the other hand, if we want to overspin the black hole, we need condition
\[ M + \delta E < a + \delta a. \] (29)
For the extremal black hole \( M = a \), eq.(28) and eq.(29) can be rewritten as
\[
\begin{align*}
\delta J &< r_0\delta E, \quad (30) \\
\delta J &> r_0\delta E, \quad (31)
\end{align*}
\]
which cannot be satisfied simultaneously as Wald’s foundation work suggested.

While if we consider the near-extremal black hole, that might not be the same case as extremal. To do this, we rewrite eq.(28) and eq.(29) as
\[
\begin{align*}
\delta J &< \frac{r_0^2(E + \Delta)}{J}\delta E, \quad (32) \\
\delta J &> r_0E - J + r_0\delta E. \quad (33)
\end{align*}
\]
And we know that for the near-extremal case, it is true \( \mathcal{M} > \frac{r_0^2}{J} \), which result in that the \( \frac{r_0^2\mathcal{M}}{J} > r_0 \). Certainly there must be values existing within these two inequality. Thus, the event horizon of the near-extremal black hole can be destroyed and the weak cosmic censorship conjecture can be violated.

V. REVIEW OF WALD’S GEOMETRICAL FORMULATION

In this section, we will discuss the Wald formulation for a linear dilaton background. We will review the steps taken in [20–23, 44]. Hereafter we will use bold character to denote differential form.

For simplicity, we note that
\[
\delta \phi = \frac{d\phi}{d\lambda}|_{\lambda=0}. \quad (34)
\]
\[
\delta^2 \phi = \frac{d^2\phi}{d\lambda^2}|_{\lambda=0}. \quad (35)
\]
The off-shell variation is given as
\[
\delta E = E_\phi \delta \phi + d\Theta. \quad (36)
\]
Where \( E_\phi \) corresponds to the equation of motion, while \( \Theta = \partial (\phi, \delta \phi) \) is the boundary term from the variation. If we substitute \( \delta \phi = \mathcal{L}\phi \) into the variation(where \( \xi \) is a killing vector associated with the metric), and use the well-known identity \( \mathcal{L}\phi = dl + e_i \xi d \), where \( e_i \xi \) is the interior product of the differential form, and \( l \) is the differential operator. One may rewrite eq.(36) as
\[
d (i_\xi l) = E_\phi \delta \phi + d\Theta. \quad (37)
\]
If we take \( E_\phi = 0 \), Then we have
\[
d (i_\xi l - \Theta) = 0. \quad (38)
\]
We denote
\[
i_\xi l - \Theta = -J_\xi. \quad (39)
\]
And it is easy to see that the \( J_\xi \) is a closed 3-form defined by \( \xi \) iff the equation of motion is satisfied. According to the Poincaré lemma, it is locally exact. Wald further showed that[22]
\[
J_\xi = C_\xi + d\mathcal{Q}_\xi. \quad (40)
\]
Where \( C_\xi \) and \( Q_\xi \) denote the constraints and noether charge 2-form associated with \( \xi \) separately. If we further assume that \( \xi \) is invariant w.r.t the variation, and take another variation of \( J_\xi \). Then combine eq.(38), eq.(39) and eq.(40)
\[
d [\delta \mathcal{Q}_\xi - \xi \cdot \Theta(\phi, \delta \phi)] = \omega (\phi, \delta \phi, \mathcal{L}\xi \phi) - \xi \cdot E_\phi \delta \phi - \delta C_\xi. \quad (41)
\]
Where the symplectic 2-form \( \omega \) defines as
\[
\omega (\phi, \delta \phi) = \delta_1 \Theta (\phi, \delta_2 \phi) - \delta_2 \Theta (\phi, \delta_1 \phi). \quad (42)
\]
Then one may further obtain from eq.(41) higher-rank variation equality
\[
d [\delta^2 \mathcal{Q}_\xi - \xi \cdot \delta \Theta(\phi, \delta \phi)] = \omega (\phi, \delta \phi, \mathcal{L}\xi \delta \phi) - \xi \cdot E_\phi \delta \phi - \delta^2 C_\xi. \quad (43)
\]
Even further like in [33], but in this article, what will be useful is only the first two order variational identity.

In Wald’s paper [23], the corresponding conserved quantity for the black hole in the asymptotically-flat case is given as
\[
\mathcal{M} = \int_\infty \mathcal{Q}[\xi] - t \cdot \mathcal{B}, \quad \mathcal{J} = -\int_\infty \mathcal{Q}[\delta \phi]. \quad (44)
\]
Where \( \xi^a = t^a + \Omega \phi^a \) is a tame-like killing vector of the black hole.

While this is not the case in our discussion. to use the right hand side of the above equation, for non-asymptotically-flat black hole, we need to subtract the corresponding divergence from the conserved quantity. And following with the discussion in section II, III and IV, we may define
\[
\mathcal{M} + \mathcal{M}_0 = \int_\infty \mathcal{Q}[\xi] - t \cdot \mathcal{B}, \quad \mathcal{J} = -\int_\infty \mathcal{Q}[\phi]. \quad (45)
\]
The \( \mathcal{E}_0 \) is the divergent part from linear dilaton background, and \( \mathcal{E} \) is the finite part that contributes to the first law of thermodynamics hence we may rewrite this as
\[
\mathcal{M} = \int_\infty (\mathcal{Q}[\xi] - t \cdot \mathcal{B}) - \mathcal{M}_0, \quad \mathcal{J} = -\int_\infty \mathcal{Q}[\phi]. \quad (46)
\]
And for more general case, We may generalise the eq.(40) and eq.(20) in [23] with
\[
\mathcal{M} + \mathcal{M}_0 = \int_\infty (\mathcal{Q}[\xi] - t \cdot \mathcal{B}), \quad \mathcal{J} = -\int_\infty \mathcal{Q}[\phi]. \quad (47)
\]
Using results from above discussion, we can rewrite eq.(41) and eq.(43) in its integration form. (While we
choose the domain as $\Sigma = \mathcal{H} \cup \Sigma_0$, and these 3-surface is
bounded by the bifurcate surface and spatial infinity of
the $\Sigma_0$. )

$$\int_{\partial \Sigma} [\delta Q_\xi - \eta \Theta(\phi, \delta \phi)] = \int_{\Sigma} \phi(\delta \phi, L_\xi \phi) - \int \delta C_\xi$$

$$- \int l_\xi (E(\phi) \cdot \delta \phi).$$

(48)

$$\mathcal{E}_\Sigma(\phi; \delta \phi) = \int_{\partial \Sigma} [\delta^2 Q_\xi - \eta \delta \Theta(\phi, \delta \phi)] + \int \delta^2 C_\xi$$

$$+ \int l_\xi (\delta E \cdot \delta \phi),$$

(49)

where

$$\mathcal{E}_\Sigma(\phi; \delta \phi) = \int_{\Sigma} \omega(\phi; \delta \phi, L_\xi \delta \phi).$$

(50)

And Stokes theorem has been used in between.

**VI. VARIATIONAL INEQUALITY**

In this section, we will obtain the inequality in EMDA
theory required to discuss the equality in the preceding
section with the null energy condition. While as
shown before, the variational inequality is easily to be
obtained from the lagrangian 4-from with the assistance
of Lie derivative, so we start from the lagrangian descrip-
tion of the theory discussed in part II.

$$L = \frac{\epsilon}{16 \pi} \left[ R - 2 \partial_\mu \phi \partial^\mu \phi - \frac{e^2 \kappa}{2} \partial_\mu \phi \partial^\mu \kappa - e^{2 \phi} F_{\mu \nu} F^{\mu \nu} - \kappa F_{\mu \nu} F^{\mu \nu} \right].$$

(51)

Actually, combining the analysis in [30], as well as that
we assume the charge is fixed background property. We
may consider it into two parts, the gravitational part and
the matter parts

$$L = \frac{\epsilon}{16 \pi} R + L_{\text{other}}.$$  

(52)

As we introduce the extra matter source(perturbation)
term $T_{ab}(\lambda)$ with $T_{ab}(0) = 0$ into the the system, It turns
out that the equation of motion and $\Theta$ can be given like

$$R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi (T_{ab}^{\text{D}L} + T_{ab}^{\text{E}M} + T_{ab}^{\text{axion}} + T_{ab}),$$

(53)

$$\nabla_\mu \left( e^{-2 \Phi} F^{\mu \nu} + \kappa^* F^{\mu \nu} \right) = 4 \pi j^\nu,$$  

(54)

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{2} e^{-2 \phi} F^2 + \frac{1}{2} e^{4 \Phi} (\partial a)^2,$$  

(55)

$$\nabla_\mu (e^{4 \Phi} g^{\mu \nu} \partial_\nu a) + F_{\mu \nu} F^{\mu \nu} = 0,$$  

(56)

$$\Theta(\phi, \delta \phi) = \Theta^{\text{GR}}(\phi, \delta \phi) + \Theta^{\text{Matter}}(\phi, \delta \phi).$$  

(57)

For latter convenience, we may divide the
$\Theta^{\text{Matter}}(\phi, \delta \phi)$ as

$$\Theta^{\text{Matter}}(\phi, \delta \phi) = \Theta^1(\phi, \delta \phi) + \Theta^{\text{CS}}(\phi, \delta \phi),$$  

(58)

where $\Theta^{\text{CS}}(\phi, \delta \phi)$ comes from Chern-Simons(CS) part,
and the remaining stuff is $\Theta^1(\phi, \delta \phi)$.

$$\Theta_{abc}^{\text{GR}}(\phi, \delta \phi) = \frac{1}{16 \pi} \epsilon_{dabc} g^{dr} g^t g \left( \nabla g_{ef}^t - \nabla e g_{df} \right).$$  

(59)

$$\Theta_{abc}^1(\phi, \delta \phi) = - \frac{1}{4 \pi} \epsilon_{dabc} e^{-2 \phi} F_{de}^t \delta A_e - \frac{1}{4 \pi} \epsilon_{dabc} \left( \nabla^d \phi \right) \delta \phi$$

$$- \frac{1}{16 \pi} \epsilon_{dabc} e^{4 \phi} \left( \nabla^d \kappa \right) \delta \kappa.$$  

(60)

$$\Theta_{abc}^{\text{CS}}(\phi, \delta \phi) = - \frac{3}{4 \pi} \kappa F_{[abc]} \delta A_c.$$  

(61)

Hence the $\omega$ associated with $\Theta$ are

$$\omega_{abc}^{\text{GR}} = \frac{1}{16 \pi} \epsilon_{dabc} w^d.$$  

(62)

$$\omega_{abc}^{\text{CS}} = - \delta_1 \left( \frac{3}{4 \pi} \kappa F_{[abc]} \delta A_c \right) + \delta_2 \left( \frac{3}{4 \pi} \kappa F_{[abc]} \delta A_c \right).$$  

(63)

$$\omega_{abc}^1 = \delta_1 \Theta_{abc}^1(\phi, \delta_2 \phi) - \delta_2 \Theta_{abc}^1(\phi, \delta_1 \phi).$$  

(64)

$$w^a = P_{abcdef}(\delta_2 g_{bc} \nabla_d \delta_1 g_{ef} - \delta_1 g_{bc} \nabla_d \delta_2 g_{ef}).$$  

(65)

where $P_{abcdef}$ is given like

$$P_{abcdef} = g^{ae} g^{fb} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{cf} - \frac{1}{2} g^{ab} g^{cd} g^{ef}$$

$$- \frac{1}{2} g^{bc} g^{ae} g^{df} + \frac{1}{2} g^{bc} g^{ad} g^{ef}.$$

(66)

After a simple calculation the constraint will be ob-
tained as

$$C_{abcd} = \epsilon_{abcd} (T_a^e + A_{ad}^f).$$  

(67)

And noether charge is

$$Q_\xi = Q_\xi^{\text{GR}} + Q_\xi^{\text{EM}} + Q_\xi^{\text{CS}},$$  

(68)

where

$$\left(Q_\xi^{\text{GR}}\right)_{ab} = - \frac{1}{16 \pi} \epsilon_{abcd} \nabla^c \xi^d,$$  

(69)
\[ (Q_{\xi})^{\text{CS}}_{ab} = -\frac{1}{4\pi}\kappa F_{ab}\xi e^e A_e, \]  
\[ (Q^1_{\xi})_{ab} = -\frac{1}{8\pi}F_{abcd}e^{2\phi}F^{cd}A_e\xi e^e. \]

We note that due to the specialty of this black hole, we set charge as background without perturbing it. And following the set-ups in [30, 31], we choose \(\Sigma = \Sigma_0 \cup \mathcal{H}\). The 3-hypersurface start from the bifurcate surface where no collision occurs, to its future horizon after which the collision has occurred, then extend spatially to spatial infinity. From this we know that \(\Sigma\) is bounded by a bifurcate surface noted as \(B\), and spatial infinity \(S_\infty\).

And we assume the stability of the non-extremal black hole, which means that it will evolve into the same black hole with different parameters, such that after the first order it decays to another stationary final state. With all these set-ups, One may rewrite eq. (48) as

\[-\int_B \left[ \delta Q_\xi - i_\xi \Theta(\phi, \delta \phi) \right] + \int_\infty \left[ \delta Q_\xi - i_\xi \Theta(\phi, \delta \phi) \right] = -\int_\Sigma \mathcal{C}_\xi. \]  
\[ (72) \]

It is worth noting that, in the above equation, the variation is defined on a set of \(S_\infty\), which is the boundary of the \( t = \text{const} \) surface of the metric with different parameters 16. We note that \( r_0 \) is fixed during the variation, the electric charge is not free parameter, but a background constant. Certainly the above set-up will be adopted automatically.

Where \( \omega = 0 \) because that \( \xi \) is a symmetry of \( \phi \), and the equation of motion is satisfied so that the last term vanishes. And the first term vanishes due to there is no perturbation near the bifurcate surface \( B \) till the very late time. Then as obtained in section III, this equation can be calculated such that

\[ \delta M - \Omega_H \delta J \geq 0. \]  
\[ (73) \]

Where the Null Energy Condition has been used[31]. It’s worth noting that this result exactly corresponds to what we obtain in section IV, which confirms [19].

With \( \mathcal{E}_\Sigma = \mathcal{E}_\mathcal{H} + \mathcal{E}_{\Sigma_0} \), eq.(49) can be represented as

\[ \mathcal{E}_\mathcal{H} + \mathcal{E}_{\Sigma_0} = \int_{\partial \Sigma_0} [\delta^2 Q_\xi - i_\xi \delta \Theta(\phi, \delta \phi)] + \int_\Sigma \delta^2 C_\xi 
+ \int_\Sigma l_\xi (\delta E \cdot \delta \phi). \]  
\[ (74) \]

The first term satisfies [30]

\[ \mathcal{E}_\mathcal{H} \geq \mathcal{E}_\mathcal{H}^{\text{CS}}. \]  
\[ (75) \]

According to eq.(61), we have

\[ \frac{4\pi}{3}C^{\text{CS}}_\mathcal{H} = -\int_\mathcal{H} \delta_1 (\kappa F_{(ab}L_{\xi} \delta A_{c)} + \int_\mathcal{H} \mathcal{L}_\xi \delta (\kappa F_{(ab} \delta_1 A_{c)} , \]

\[ = \int_\mathcal{H} \mathcal{L}_\xi \delta F_{(ab} \delta A_{c)} + \int_\mathcal{H} \kappa \mathcal{L}_\xi \delta F_{(ab} \delta A_{c]} 
- \int_\mathcal{H} \delta F_{(a} \delta \mathcal{L}_{b} \delta A_{c)} + \int_\mathcal{H} \kappa \delta F_{(ab} \mathcal{L}_{\xi} \delta A_{c)}. \]  
\[ (76) \]

With the consideration that \( \xi^a = 0 \) at the bifurcate surface, and the gauge condition \( \delta A_\xi = 0 \), the right hand side can be further massaged as

\[ \frac{4\pi}{3}C^{\text{CS}}_\mathcal{H} = \int_\mathcal{H} d(\xi \cdot (\kappa F_{(ab} \delta A_{c)})) = 0. \]  
\[ (77) \]

And the second term can be obtained by reusing eq.(49) on \( \Sigma_0 \), which is

\[ \mathcal{E}_{\Sigma_0}(\phi, \delta \phi^{BH}) = \int_{\partial \Sigma_0} [\delta^2 Q_\xi - i_\xi \delta \Theta(\phi, \delta \phi^{BH})] + \int_{\Sigma_0} \delta^2 C_\xi 
+ \int_{\Sigma_0} l_\xi (\delta E \cdot \delta \phi^{BH}). \]  
\[ (78) \]

Which can be reduced to [19]

\[ \mathcal{E}_{\Sigma_0}(\phi, \delta \phi^{BH}) = -T_H \delta^2 S_{BH}. \]  
\[ (79) \]

Considering the null energy condition as well as \( \delta \phi \) satisfies the linearized equation of motion. After computation, our second order inequality is[19]

\[ \delta^2 M + T_H \delta^2 S_{BH} - \Omega_H \delta^2 J \geq 0. \]  
\[ (80) \]

One may refer to [19, 30] for more details.

**VII. GEDENKEN EXPERIMENT**

From eq.(5), We may define a function as

\[ j(\lambda) = M^2(\lambda) - a^2(\lambda). \]  
\[ (81) \]

With that

\[ M(0) = M, \]  
\[ a(0) = a. \]  
\[ (82) \]

It is worth noting that the \( M \) appearing here does not correspond to \( M \) because of the eq.(12). Then we may expand it with respect to \( \lambda \) perturbatively.

\[ j(\lambda) = M^2 - a^2 + (2M \delta M - 2a \delta a) \lambda + (\delta M^2 - \delta a^2 - M \delta^2 M - a \delta^2 a) \lambda^2. \]  
\[ (83) \]

Actually, one may consider more complicated case involving higher rank variation here, but one still need higher rank inequality to evaluate our \( j(\lambda) \). Here what we need
is just to consider whether if this $j(\lambda) \geq 0$ is held strictly in our case. And we refer to inequalities obtained for help.

$$\frac{1}{2} \delta M - \Omega_H \delta J \geq 0. \quad (84)$$

Following the same settings as in[19], Here we rewrite

$$\delta^2 r_{BH} = -\frac{1}{\Delta}(M \delta M - a \delta a)^2 + \frac{1}{\Delta}(M^2 - \delta a^2). \quad (86)$$

From here, we may reexpress eq.(84) and eq.(85) as

$$M \delta M - a \delta a \geq -\Delta \delta M. \quad (87)$$

$$\delta^2 M - \frac{a \delta^2 a}{r_+} \geq \frac{(M \delta M - a \delta a)^2}{\Delta^2 r_+} - \frac{(\delta M)^2 - (\delta a)^2}{r_+}. \quad (88)$$

And if we substitute the first inequality into the second one, then

$$\delta^2 M - \frac{a \delta^2 a}{r_+} \geq \frac{(\delta M)^2}{r_+} - \frac{(\delta M)^2 - (\delta a)^2}{r_+}. \quad (89)$$

And ignoring the $\Delta$-infinitesimal term, we have

$$\delta M^2 - \delta a^2 + M \delta^2 M - a \delta^2 a \geq (\delta M)^2. \quad (90)$$

With these conditions for help, we may firstly assess $j(\lambda)$ to its first order. Which means that, we only consider the $\lambda$-term in eq.(83) and eq.(84). Then we will automatically receive the same results as in section IV. For which the near extremal case may not respect the WCCC in our case to first order. But for the extremal case($\Delta = 0$), no violation occurs as discussed in section III.

If we consider eq.(85) and $j(\lambda)$ to its second order, then we will obtain

$$j(\lambda) \geq (\Delta - \delta M \lambda)^2 \geq 0. \quad (91)$$

As expected, the WCCC is restored for the near-extremal black hole if we consider the modification up to second order.

**VIII. CONCLUSION**

In this paper, we use modified Iyer-Wald formalism to confirm the validity of the black hole first thermodynamics [38], then we use two different methods to discuss the WCCC for a rotating linear dilaton black hole in the Einstein-Maxwell-Dilaton-Axion theory. To first order, we can come to the same conclusion for both extremal and near-extremal black hole via these two methods as Wald said in[19]. That is, WCCC is well preserved for extremal black hole, but not so for near-extremal black hole. To second order, The WCCC is preserved in our case as expected for both extremal and nearly extremal black hole, which implies that if we test the nearly extremal black hole with the test particle to second order precision, WCCC is well protected.

**Acknowledgements**

We thank Jie Jiang for enlightening discussions and many useful suggestions. This work was supported by the basic scientific research business expenses of the central university and the Open Project of Key Laboratory for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University(LZUMMM2020010). Fei Qu acknowledges support from the Fundamental Research Funds for the Central Universities (Grants No. lzujbky-2020-it04). Si-Jiang Yang acknowledges support from National Natural Science Foundation of China (Grants No. 11875151, No. 11522541, and No. 11675064), and the Fundamental Research Funds for the Central Universities (Grants No. lzujbky-2019-it21).

[1] R. Penrose, Riv. Nuovo Cim. 1 (1969), 252-276.
[2] W. E. East, Phys. Rev. Lett. 122, 231103 (2019), [arXiv:1901.04498 [gr-qc]].
[3] P. Figueras, M. Kunesch and S. Tunyasuvunakool, Phys. Rev. Lett. 116, 071102 (2016), [arXiv:1512.04532 [hep-th]].
[4] P. Figueras, M. Kunesch, L. Lehner and S. Tunyasuvunakool, Phys. Rev. Lett. 118, 151103 (2017), [arXiv:1702.01755 [hep-th]].
[5] T. Crisford and J. E. Santos, Phys. Rev. Lett. 118,
181101 (2017), [arXiv:1702.05490 [hep-th]].
[6] T. T. Hu, Y. Song, S. Sun, H. B. Li and Y. Q. Wang, Eur. Phys. J. C 80, 147 (2020), [arXiv:1906.00235 [hep-th]].
[7] Y. Song, T. T. Hu and Y. Q. Wang, [arXiv:2008.02513 [hep-th]].
[8] R. M. Wald, Ann. Phys. 83, 548 (1974).
[9] V. E. Hubeny, Phys. Rev. D 59 (1999), 064013.
[10] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. 103 (2009), 141101.
[11] S. J. Yang, J. Chen, J. J. Wan, S. W. Wei and Y. X. Liu, Phys. Rev. D 101, 064048 (2020).
[12] S. J. Yang, J. J. Wan, J. Chen, J. Yang and Y. Q. Wang, arXiv:2004.07934 [gr-qc].
[13] B. Liang, S. W. Wei and Y. X. Liu, Mod. Phys. Lett. A 34, 1950037 (2019).
[14] E. Barausse, V. Cardoso and G. Khanna, Phys. Rev. Lett. 105, 261102 (2010).
[15] E. Barausse, V. Cardoso and G. Khanna, Phys. Rev. D 84, 104006 (2011).
[16] P. Zimmerman, I. Vega, E. Poisson and R. Haas, Phys. Rev. D 87, 041501 (2013).
[17] M. Colleoni and L. Barack, Phys. Rev. D 91, 104024 (2015).
[18] M. Colleoni, L. Barack, A. G. Shah and M. van de Meent, Phys. Rev. D 92, 084044 (2015).
[19] J. Sorce and R. M. Wald, Phys. Rev. D 96, no.10, 104014 (2017).
[20] J. Lee and R. M. Wald, J. Math. Phys. 31, 725-743 (1990).
[21] V. Iyer and R. M. Wald, Phys. Rev. D 50, 846-864 (1994).
[22] V. Iyer and R. M. Wald, Phys. Rev. D 52, 4430-4439 (1995).
[23] R. M. Wald and A. Zoupas, Phys. Rev. D 61, 084027 (2000).
[24] J. Jiang and Y. Gao, Phys. Rev. D 101 (2020) no.8, 084005.
[25] J. Jiang, Phys. Lett. B 804 (2020), 135365.
[26] Y. L. He and J. Jiang, Phys. Rev. D 100 (2019) no.12, 124060.
[27] S. Shaymatov, N. Dadhich and B. Ahmedov, Phys. Rev. D 101 (2020) no.4, 044028.
[28] S. Shaymatov, N. Dadhich and B. Ahmedov, Eur. Phys. J. C 79, no.7, 585 (2019).
[29] B. Ge, Y. Mo, S. Zhao and J. Zheng, Phys. Lett. B 783, 440-445 (2018).
[30] J. Jiang, X. Liu and M. Zhang, Phys. Rev. D 100, no.8, 084059 (2019).
[31] S. Gao and R. M. Wald, Phys. Rev. D 64, 084020 (2001).
[32] J. Jiang, B. Deng and Z. Chen, Phys. Rev. D 100, no.6, 066024 (2019).
[33] X. Y. Wang and J. Jiang, JHEP 05, 161 (2020).
[34] J. An, J. Shan, H. Zhang and S. Zhao, Phys. Rev. D 97 (2018) no.10, 104007.
[35] B. Ning, B. Chen and F. L. Lin, Phys. Rev. D 100 (2019) no.4, 044043.
[36] X. Y. Wang and J. Jiang, JCAP 07, 052 (2020).
[37] J. D. Brown and J. W. York, Jr., Phys. Rev. D 47, 1407-1419 (1993).
[38] Gérard Clément, Dmitri Gal’tsov, and Cédric Leygnac Phys. Rev. D 67, 024012(2003).
[39] P. E. D. Goulart Santos, “Einstein-Maxwell-dilaton theory: Black holes, wormholes, and applications to AdS/CMT.”
[40] T. Matos, G. Miranda, R. Sanchez-Sanchez and P. Wiederhold, Phys. Rev. D 79, 124016 (2009).
[41] R. Li and J. R. Ren, JHEP 09 (2010), 039.
[42] S. W. Hawking and S. F. Ross, Phys. Rev. D 52 (1995), 5865-5876.
[43] R. M. Wald, Phys. Rev. D 48 (1993) no.8, 3427-3431.
[44] S. Hollands and R. M. Wald, Commun. Math. Phys. 321, 629-680 (2013).
[45] S. Hod, Phys. Rev. D 66 (2002), 024016.