Abstract. We prove that the Hilbert-Kunz function of the ideal \((I, It)\) of the Rees algebra \(R(I)\), where \(I\) is an \(m\)-primary ideal of a 1-dimensional local ring \((R, m)\), is a quasi-polynomial in \(e\), for large \(e\). For \(s \in \mathbb{N}\), we calculate the Hilbert-Samuel function of the \(R\)-module \(I^{[s]}\) and obtain an explicit description of the generalized Hilbert-Kunz function of the ideal \((I, It)\) when \(I\) is a parameter ideal in a Cohen-Macaulay local ring of dimension \(d \geq 2\), proving that the generalized Hilbert-Kunz function is a piecewise polynomial in this case.

1. Introduction

Let \((R, m)\) be a \(d\)-dimensional Noetherian local ring of positive prime characteristic \(p\) and let \(I\) be an \(m\)-primary ideal. The \(q^{th}\)-Frobenius power of \(I\) is the ideal \(I^{[q]} = (x^q \mid x \in I)\) where \(q = p^e\) for \(e \in \mathbb{N}\). The function \(e \mapsto \ell_R(R/I^{[q]})\) is called the Hilbert-Kunz function of the ideal \(I\) and was first considered by E. Kunz in [7]. In [9], P. Monsky showed that this function is of the form \(\ell_R(R/I^{[q]}) = \varepsilon_{HK}(I, R)q^d + O(q^{d-1})\) where \(\varepsilon_{HK}(I, R)\) is a positive real number called the Hilbert–Kunz multiplicity of \(I\) in \(R\). Besides the mysterious leading coefficient, the behavior of the Hilbert–Kunz function is also unpredictable.

Monsky [9] proved that in the case of 1-dimensional rings, the Hilbert–Kunz function \(\ell_R(R/I^{[q]}) = \varepsilon_{HK}(I, R)q + \delta_e\), where \(\delta_e\) is a periodic function of \(e\), for large \(e\). Precisely, take \(R = k[[X, Y]]/(X^5 - Y^5)\), where \(k\) is a field of prime characteristic \(p \equiv \pm 2\mod 5\). Let \(m\) be the maximal ideal of \(R\). Monsky [9] showed that for large \(e\), \(\ell_R(R/m^{[q]}) = 5q + \alpha_m(e)\), where \(\alpha_m(e) = -4\) when \(e\) is even and \(\alpha_m(e) = -6\) when \(e\) is odd. In [5], the authors determined the Hilbert-Kunz function of the maximal ideal of the ring \(R = \mathbb{Z}/p[[x_1, \ldots, x_s]]/(x_1^{d_1} + \cdots + x_s^{d_s})\), where \(d_i\) are positive integers and \(s = d + 1 \geq 3\). They proved that if \(\ell_R(R/m^{[q]}) = \varepsilon_{HK}(R)q^d + \delta_e\), then \(\varepsilon_{HK}(R)\) is rational and \(\delta_e\) is an eventually periodic function of \(e\) whenever \(s = 3\) or \(p = 2\). But when \(s = 4\) and \(p = 5\) with \(d_1 = \cdots = d_s = 4\), then \(\ell_R(R/m^{[5^e]}) = (168/61) \cdot 5^{3e} - (107/61) \cdot 3^e\). In [1], H. Brenner proved the following result

The first author is supported by a UGC fellowship, Govt. of India.

Key words and phrases: Hilbert-Kunz function, Hilbert–Kunz multiplicity, Rees algebra, generalized Hilbert-Kunz function.

2010 AMS Mathematics Subject Classification: 13A30, 13A35, 13D40.
Theorem 1.1. Let $k$ denote the algebraic closure of a finite field of characteristic $p$. Let $R$ denote a normal two-dimensional standard-graded $k$-domain and let $I$ denote a homogeneous $R_+$-primary ideal. Then the Hilbert–Kunz function of $I$ has the form $\ell_R(R/I^{[q]}) = e_{HK}(I,R)q^2 + \delta_e$, where $e_{HK}(I,R)$ is a rational number and $\delta_e$ is an eventually periodic function.

Several other authors including D. Brinkmann, V. Trivedi and P. Teixeira have worked on similar problems.

We recall a few definitions. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and $I$ be an $\mathfrak{m}$-primary ideal of $R$. The Hilbert-Samuel function $H_I(n)$ of $I$ is defined as $H_I(n) = \ell_R(R/I^n)$. It is known that $H_I(n)$ is a polynomial function of $n$ of degree $d$, for large $n$. In particular, there exists a polynomial $P_I(x) \in \mathbb{Q}[x]$ such that $H_I(n) = P_I(n)$ for all large $n$. Write

$$P_I(x) = e_0(I)\left(\frac{x + d - 1}{d}\right) - e_1(I)\left(\frac{x + d - 2}{d - 1}\right) + \cdots + (-1)^d e_d(I),$$

where $e_i(I)$ for $i = 0, 1, \ldots, d$ are integers, called the Hilbert coefficients of $I$. The leading coefficient $e_0(I) = e(I)$ is called the multiplicity of $I$ and $e_1(I)$ is called the Chern number of $I$. The postulation number of $I$ is defined as

$$n(I) = \max\{n \mid H_I(n) \neq P_I(n)\}.$$  

The notion of reduction of an ideal was introduced by D. G. Northcott and D. Rees. Let $J \subseteq I$ be ideals of $R$. If $JI^n = I^{n+1}$ for some $n$, then $J$ is called a reduction of $I$. The reduction number $r_J(I)$ of $I$ with respect to $J$ is the smallest $n$ such that $JI^n = I^{n+1}$. The reduction number of $I$ is defined as

$$r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$  

Moreover, if $R/\mathfrak{m}$ is infinite, then minimal reduction of $I$ exists.

In this paper, we calculate the Hilbert-Kunz function of certain ideals of the Rees algebra. Let $\mathcal{R}(I) = \oplus_{n \geq 0} I^n t^n$ denote the Rees algebra of $I$. The Hilbert-Kunz multiplicity of various blowup algebras was estimated by K. Eto and K.-i. Yoshida in their paper [3]. Put $c(d) = (d/2) + d/(d+1)!$. They proved the following.

Theorem 1.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$ with $d = \dim R \geq 1$. Then for any $\mathfrak{m}$-primary ideal $I$, we have

$$e_{HK}(\mathcal{R}(I)) \leq c(d) \cdot e(I).$$

Moreover, equality holds if and only if $e_{HK}(R) = e(I)$. When this is the case, $e_{HK}(R) = e(R)$ and $e_{HK}(I) = e(I)$.

Since a precise formula for the multiplicity of the maximal graded ideal of the Rees algebra is known (see [12]), it is natural to ask if there is a formula for the Hilbert-Kunz multiplicity of $(\mathfrak{m}, It)$, when $I$ is an $\mathfrak{m}$-primary ideal, in terms of invariants of the ideals $\mathfrak{m}$ and $I$. We aim to answer this question in some cases. In section 2, we begin by proving the following result.
Theorem 1.3. Let \((R, m)\) be a 1-dimensional Noetherian local ring with prime characteristic \(p > 0\). Let \(I, J\) be \(m\)-primary ideals. Then \(e_{HK}(((J, It)\mathcal{R}(I)) = e(J)\).

Next, we prove that the Hilbert-Kunz function of the ideal \((I, It)\) of the Rees algebra \(\mathcal{R}(I)\), where \(I\) is an \(m\)-primary ideal of a 1-dimensional ring, is a quasi-polynomial in \(e\), for large \(e\). Recall that a quasi-polynomial of degree \(d\) is a function \(f : \mathbb{Z} \to \mathbb{C}\) of the form

\[
\ell_R \left( \frac{R(I)}{I[a]} \right) = e(I)q^2 - e(I) \left( \frac{r}{2} \right) + r \cdot e(\alpha) + \sum_{n=0}^{r-1} \ell_R \left( \frac{R}{I^n} \right) + 2 \sum_{n=0}^{r-1} \alpha(I^n, e).
\]

where \(\alpha(I^n, e)\) is a constant and \(\alpha(I^n, e) = \ell_R(I^n/I[a]I^n) - e(I)q\) is a periodic function in \(e\), for large \(e\). In other words, \(\ell_R(\mathcal{R}(I)/I[a])\) is a quasi-polynomial for large \(e\).

In the above setup, if \(I\) is a parameter ideal and \(J\) is an \(m\)-primary ideal of a Cohen-Macaulay local ring \(R\), then we prove that for large \(e\),

\[
\ell_R \left( \frac{\mathcal{R}(I)}{(J, It)[a]} \right) = q^2 e(J) + q \cdot \alpha_J(e),
\]

where \(\alpha_J(e) = \ell_R(R/J[a]) - e(J)q\) is a periodic function in \(e\). As a consequence of the above results, we calculate the Hilbert-Kunz function of the ideals \((m, mt)\mathcal{R}(m)\) and \((m, It)\mathcal{R}(I)\), when \(R = k[[X, Y]]/(X^5 - Y^5)\), where \(k\) is a field of prime characteristic \(p \equiv \pm 2(\text{mod } 5)\), \(m\) is the maximal ideal of \(R\), \(I = (x)\) is a parameter ideal and \(x\) is the image of \(X\) in \(R\).

Let \(R\) be a \(d\)-dimensional Noetherian ring. Let \(I\) be an ideal of finite co-length. Aldo Conca introduced the concept of generalized Hilbert-Kunz function in [2]. For \(s \in \mathbb{N}\), let \(I^s = (a_1^s, \ldots, a_n^s)\) where \(\{a_1, a_2, \ldots, a_n\}\) is a fixed set of generators of \(I\). Then the generalized Hilbert-Kunz function is defined as

\[
HK_{R,I}(s) = \ell_R \left( \frac{R}{I^s} \right).
\]
The generalized Hilbert-Kunz multiplicity is defined as \( \lim_{s \to \infty} HK_{R,I}(s)/s^d \) whenever the limit exists. If \( \text{char}(R) = p > 0 \), then the generalized Hilbert-Kunz function (multiplicity) coincides with the Hilbert-Kunz function (multiplicity) and is independent of the choice of the generators of \( I \).

In section 3, we find the generalized Hilbert-Kunz function of the ideal \( I = (I, It) \) in \( \mathcal{R}(I) \), when \( I \) is generated by a regular sequence in a \( d \)-dimensional Cohen-Macaulay local ring. Our approach requires knowledge of the Hilbert-Samuel function of the \( R \)-module \( I^s \). We obtain an explicit description of the function \( F(n) = \ell_R(I^s/I^s I^n) \) for a fixed \( s \in \mathbb{N} \) and then use some properties of Stirling numbers of the second kind to prove the following result.

**Theorem 1.5.** Let \( R \) be a \( d \)-dimensional Cohen-Macaulay local ring and \( d \geq 2 \). Let \( I \) be a parameter ideal of \( R \) and \( \mathcal{I} = (I, It)\mathcal{R}(I) \). Let \( s \in \mathbb{N} \).

1. Let \( s < d \). Write \( d = k_1 s + k_2 \) where \( k_2 \in \{0, 1, \ldots, s - 1\} \). If \( k_2 = 0 \), then

\[
\ell_R \left( \frac{\mathcal{I}}{I^s} \right) = (d-k_1+1)s^{d+1}e(I) + d \cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left[ (-1)^i \binom{d}{i} e(I) \left( \frac{(d-i-k_1+1)s + d - 1}{d + 1} \right) \right].
\]

If \( k_2 \neq 0 \), then

\[
\ell_R \left( \frac{\mathcal{I}}{I^s} \right) = (d-k_1)s^{d+1}e(I) + d \cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left[ (-1)^i \binom{d}{i} e(I) \left( \frac{(d-i-k_1)s + d - 1}{d + 1} \right) \right].
\]

2. Let \( s \geq d \). Then

\[
\ell_R \left( \frac{\mathcal{I}}{I^s} \right) = d s^{d+1}e(I) + d \cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left[ (-1)^i \binom{d}{i} e(I) \left( \frac{(d-i)s + d - 1}{d + 1} \right) \right].
\]

In other words, for \( s \) large,

\[
\ell_R \left( \frac{\mathcal{I}}{I^s} \right) = c(d) e(I) s^{d+1} + e(I) \left( \frac{d-2}{2} \right) \left( \frac{1}{(d-1)!} - 1 \right) s^d + e(I) \frac{d(d-1)(3d-10)}{24(d-1)!} s^{d-1} + \ldots,
\]

implying that the generalized Hilbert-Kunz multiplicity \( e_{HK}((I, It)\mathcal{R}(I)) = c(d) \cdot e(I) \).

As a consequence, we obtain the following result.

**Corollary 1.6.** Let \( (R, \mathfrak{m}) \) be a \( d \)-dimensional regular local ring with \( d \geq 2 \). Then for \( s \geq d \),

\[
\ell_R \left( \frac{\mathcal{I}}{\mathfrak{m}^s} \right) = c(d) e(\mathfrak{m}) s^{d+1} + e(\mathfrak{m}) \left( \frac{d-2}{2} \right) \left( \frac{1}{(d-1)!} - 1 \right) s^d + e(\mathfrak{m}) \frac{d(d-1)(3d-10)}{24(d-1)!} s^{d-1} + \ldots.
\]

**Acknowledgements:** We thank K.-i. Watanabe and Anurag Singh for several discussions and their lectures at IIT Bombay on Hilbert-Kunz multiplicity and positive characteristic methods.
2. The Hilbert-Kunz function of the Rees Algebra

Let \((R, \mathfrak{m})\) be a 1-dimensional Noetherian local ring with prime characteristic \(p > 0\). In this section, we calculate the Hilbert-Kunz function and Hilbert-Kunz multiplicity of certain ideals of the Rees ring \(\mathcal{R}(I)\), where \(I\) is an \(\mathfrak{m}\)-primary ideal of \(R\).

Let \(I, J\) be \(\mathfrak{m}\)-primary ideals of \(R\). We begin by calculating \(e_{HK}((J, It)\mathcal{R}(I))\), the Hilbert-Kunz multiplicity of the ideal \((J, It)\) in \(\mathcal{R}(I)\). Recall that if a Noetherian ring \(R\) has prime characteristic \(p\), then \(x \in R\) is said to be in the **tight closure** \(I^*\) of an ideal \(I\) if there exists \(c \in R^\circ := \{a \in R \mid a \not\in \mathfrak{p}\} \) for any minimal prime \(\mathfrak{p} \subseteq R\) such that \(cx^q \in I^{[b]}\) for all large \(q = p^e\). In \([6, \text{Theorem } 8.17(a)]\), M. Hochster and C. Huneke proved that if \(J \subseteq I\) are \(\mathfrak{m}\)-primary ideals of \(R\) such that \(I^* = J^*\), then \(e_{HK}(I) = e_{HK}(J)\).

For an ideal \(I\) in a ring \(R\), the **integral closure** of \(I\), denoted by \(\overline{I}\), is the ideal which consists of elements \(x \in R\) such that \(x\) satisfies an equation of the form \(x^n + a_1x^{n-1} + \cdots + a_n = 0\) for some \(a_i \in \mathfrak{I}^i, 1 \leq i \leq n\). The following generalization of the Briançon-Skoda theorem was given by Hochster and Huneke.

**Theorem 2.1** ([6, Theorem 5.4]). Let \(R\) be a Noetherian ring of prime characteristic \(p\) and let \(I\) be an ideal of positive height generated by \(n\) elements. Then for every \(m \in \mathbb{N}\), \(\overline{I}^{1+m} \subseteq (I^{m+1})^*\).

It now follows that the tight closure and integral closure of principal ideals coincide.

**Theorem 2.2.** Let \((R, \mathfrak{m})\) be a 1-dimensional Noetherian local ring with prime characteristic \(p > 0\). Let \(I, J\) be \(\mathfrak{m}\)-primary ideals. Then \(e_{HK}((J, It)\mathcal{R}(I)) = e(J)\).

**Proof.** We may assume that \(R\) has an infinite residue field. Let \((x), (y)\) be minimal reductions of \(J\) and \(I\) respectively in \(R\). We claim that \(e_{HK}((J, It)\mathcal{R}(I)) = e_{HK}((x, yt)\mathcal{R}(I))\). Note that it is sufficient to show that \((J, It)\mathcal{R}(I))^* = ((x, yt)\mathcal{R}(I))^*\). Using generalized Briançon-Skoda theorem, it follows that \((x)\mathcal{R}(I))^* = (x)\mathcal{R}(I)\) and \((yt)\mathcal{R}(I))^* = (yt)\mathcal{R}(I)\). As \((x)\mathcal{R}(I)\) and \((yt)\mathcal{R}(I)\) are reductions of \(J\mathcal{R}(I)\) and \((It)\mathcal{R}(I)\) respectively, we have

\[
(x)\mathcal{R}(I))^* = (x)\mathcal{R}(I) = J\mathcal{R}(I) \quad \text{and} \quad (yt)\mathcal{R}(I))^* = (yt)\mathcal{R}(I) = (It)\mathcal{R}(I).
\]

Since \((J\mathcal{R}(I))^* = (x)\mathcal{R}(I))^* \subseteq (J\mathcal{R}(I))^*\), we get \((x)\mathcal{R}(I))^* = (J\mathcal{R}(I))^*\). Similarly, \((yt)\mathcal{R}(I))^* = ((It)\mathcal{R}(I))^*\). Consider

\[
(x, yt)\mathcal{R}(I))^* = (((x)\mathcal{R}(I))^* + ((yt)\mathcal{R}(I))^*)^* = ((J\mathcal{R}(I))^* + ((It)\mathcal{R}(I))^*)^* = (J, It)\mathcal{R}(I))^*.
\]

This proves the claim. Therefore, \(e_{HK}((J, It)\mathcal{R}(I)) = e_{HK}((x, yt)\mathcal{R}(I))\). As \(x, yt\) form a system of parameters in \(\mathcal{R}(I)_{(m, It)}\), it follows that

\[
e_{HK}((J, It)\mathcal{R}(I)) = e_{HK}((x, yt)\mathcal{R}(I)) = e((x, yt)\mathcal{R}(I)) = e((J, It)\mathcal{R}(I)) = e(J),
\]

where the last equality follows from \([12, \text{Theorem } 3.1]\). \(\square\)
Let $I$ be an $m$-primary ideal of $R$. We prove that $\ell_R(\mathcal{R}(I)/(I, It)^{[p^e]})$ is a quasi-polynomial in $e$, for large $e$.

**Theorem 2.3.** Let $(R, m)$ be a 1-dimensional Noetherian local ring with prime characteristic $p > 0$. Let $I$ be an $m$-primary ideal. Let $r$ be the reduction number of $I$ and $\rho$ be the postulation number of $I$. Put $\mathcal{I} = (I, It)\mathcal{R}(I)$. Let $q = p^e$, where $e \in \mathbb{N}$ is large.

1. If $\rho + 1 \leq r$, then
   \[
   \ell_R\left(\frac{\mathcal{R}(I)}{\mathcal{I}^{[q]}}\right) = e(I)q^2 - e(I)\left(\frac{r}{2}\right) + r \cdot e_1(I) + \sum_{n=0}^{r-1} \ell_R\left(\frac{R}{I^n}\right) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e).
   \]

2. If $r < \rho + 1$, then
   \[
   \ell_R\left(\frac{\mathcal{R}(I)}{\mathcal{I}^{[q]}}\right) = e(I)q^2 - e(I)\left(r(r - 1) - \frac{\rho(\rho + 1)}{2}\right) + (2r - \rho - 1)e_1(I) + \beta + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e),
   \]
   where $\beta = \sum_{n=0}^{r-1} \ell_R\left(\frac{R}{I^n}\right) - \sum_{n=r}^{\rho} \ell_R\left(\frac{R}{I^n}\right)$ is a constant and $\alpha_I(I^n, e) = \ell_R(I^n/I^{[q]}I^n) - e(I)q$ is a periodic function in $e$, for large $e$. In other words, $\ell_R(\mathcal{R}(I)/\mathcal{I}^{[q]})$ is a quasi-polynomial for large $e$.

**Proof.** Fix $q = p^e$ large. Observe that
\[
\mathcal{I}^{[q]} = (I^{[q]}, I^{[q]}I^n) = \left(\bigoplus_{n=0}^{q-1} I^{[q]}I^n \right) + \left(\bigoplus_{n=q}^{q+r-1} I^{[q]}I^{n-q}I^n \right) + \left(\bigoplus_{n=q+r} I^nI^n \right).
\]
We may assume that $R$ has an infinite residue field. Let $(x)$ be a minimal reduction of $I$ and let $r$ be the reduction number of $I$. Then $x^kI^l = I^{k+l}$ for all $k \geq 1$ and $l \geq r$. Write $I = x + J$, for some ideal $J \subseteq I$. Then $I^{[q]}I^{n-q} = (x^q + J^{[q]})I^{n-q} = I^n$, for all $n \geq q + r$. Therefore,
\[
\mathcal{I}^{[q]} = \left(\bigoplus_{n=0}^{q-1} I^{[q]}I^n \right) + \left(\bigoplus_{n=q}^{q+r-1} I^{[q]}I^{n-q}I^n \right) + \left(\bigoplus_{n=q+r} I^nI^n \right)
\]
and hence
\[
\ell_R\left(\frac{\mathcal{R}(I)}{\mathcal{I}^{[q]}}\right) = \sum_{n=0}^{r-1} \ell_R\left(\frac{I^n}{I^{[q]}I^n}\right) + \sum_{n=q}^{q-1} \ell_R\left(\frac{I^n}{I^{[q]}I^n}\right) + \sum_{n=q}^{q+r-1} \ell_R\left(\frac{I^n}{I^{[q]}I^{n-q}}\right) + \sum_{n=q+r} \ell_R\left(\frac{I^n}{I^n}\right).
\]
For $n \geq r$, $I^{[q]}I^n = (x^q + J^{[q]})I^n = I^{n+q}$. Using [9, Theorem 3.11], we can write $\ell_R(I^n/I^{[q]}I^n) = e(I, I^n)q + \alpha_I(I^n, e)$, where $\alpha_I(I^n, e)$ is a periodic function of $e$. Using the associativity formula for $e(I)$, one may check that $e(I, I^n) = e(I, R)$. Thus,
\[
\ell_R\left(\frac{\mathcal{R}(I)}{\mathcal{I}^{[q]}}\right) = 2 \sum_{n=0}^{r-1} (e(I)q + \alpha_I(I^n, e)) + \sum_{n=q}^{q-1} \ell_R\left(\frac{R}{I^{n+q}}\right) + \sum_{n=0}^{r-1} \ell_R\left(\frac{R}{I^n}\right) - \sum_{n=r} \ell_R\left(\frac{R}{I^n}\right).
\]
Case 1: Let $\rho + 1 \leq r$. Then
\[
\ell_R \left( \frac{R(I)}{I[\ell]} \right) = 2rq \cdot e(I) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e) + 2q - 1 \sum_{n=r+q} e(I)n - e_1(I) + \sum_{n=0}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)} - \sum_{n=r}^{q+r-1} (e(I)n - e_1(I))
\]
\[
= 2rq \cdot e(I) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e) + e(I) \left[ \frac{2q}{2} - 2 \left( \frac{r + q}{2} \right) + \left( \frac{r}{2} \right) \right] + r \cdot e_1(I) + \sum_{n=0}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)}
\]
\[
= e(I)q^2 - e(I) \left( \frac{r}{2} \right) + r \cdot e_1(I) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e).
\]

Case 2: Let $r < \rho + 1$. Then
\[
\ell_R \left( \frac{R(I)}{I[\ell]} \right) = 2rq \cdot e(I) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e) + 2q - 1 \sum_{n=r+q} e(I)n - e_1(I) + \sum_{n=0}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)} - \sum_{n=\rho+1}^{q+r-1} (e(I)n - e_1(I))
\]
\[
= 2rq \cdot e(I) + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e) + e(I) \left[ \frac{2q}{2} - 2 \left( \frac{r + q}{2} \right) + \left( \frac{\rho + 1}{2} \right) \right] + (2r - \rho - 1)e_1(I) + \beta
\]
\[
= e(I)q^2 - e(I) \left( r(r - 1) - \frac{\rho(\rho + 1)}{2} \right) + (2r - \rho - 1)e_1(I) + \beta + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e),
\]
where $\beta = \sum_{n=0}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)} - \sum_{n=\rho}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)}$ is a constant. Let $N$ be a common period of $\alpha_I(I^n, e)$, for $0 \leq n \leq r - 1$. Then it follows that $\ell_R(R(I)/I[\ell]) - e(I)q^2$ is a periodic function of $e$. In other words, $\ell_R(R(I)/I[\ell])$ is a quasi-polynomial in $e$, for large $e$. \[\Box\]

In particular, if $R$ is a 1-dimensional Cohen-Macaulay local ring we get the following result.

**Corollary 2.4.** Let $(R, m)$ be a 1-dimensional Cohen-Macaulay local ring with prime characteristic $p > 0$. Let $I$ be an $m$-primary ideal. Let $r$ be the reduction number of $I$. Put $I = (I, It)R(I)$. Then for $q = p^e$, where $e \in \mathbb{N}$ is large,
\[
\ell_R \left( \frac{R(I)}{I[\ell]} \right) = e(I)q^2 - e(I) \left( \frac{r}{2} \right) + r \cdot e_1(I) + \sum_{n=0}^{r-1} \frac{\ell_R \left( \frac{R}{I^n} \right)}{\ell_R \left( \frac{R}{I^n} \right)} + 2 \sum_{n=0}^{r-1} \alpha_I(I^n, e),
\]
where $\alpha_I(I^n, e) = \ell_R(I^n/It_I^n) - e(I)q$ is a periodic function in $e$, for large $e$. In other words, $\ell_R(R(I)/I[\ell])$ is a quasi-polynomial for large $e$.

**Proof.** Since $R$ is Cohen-Macaulay, using [8, Theorem 2.15] it follows that the postulation number of $I$, $\rho = r - 1$. Substitute the same in Theorem 2.3 to conclude. \[\Box\]

Next, we calculate the Hilbert-Kunz function of the ideal $J = (J, It)R(I)$ when $I$ is a parameter ideal and $J$ is an $m$-primary ideal of a Cohen-Macaulay ring $R$. 

Theorem 2.5. Let \((R, m)\) be a 1-dimensional Cohen-Macaulay local ring with prime characteristic \(p > 0\). Let \(I = (a)\) be a parameter ideal and \(J\) be an \(m\)-primary ideal. Put \(\mathcal{J} = (J, It)\mathcal{R}(I)\). Then for \(q = p^e\), where \(e \in \mathbb{N}\) is large,

\[
\ell_R \left( \frac{\mathcal{R}(I)}{\mathcal{J}^{[q]}} \right) = q^2 e(J) + q \cdot \alpha_J(e),
\]

where \(\alpha_J(e) = \ell_R(R/J^{[q]}) - e(J)q\) is a periodic function in \(e\).

Proof. Observe that

\[
\mathcal{J}^{[q]} = (J^{[q]}, I^{[q]}t^q) = \left( \bigoplus_{n=0}^{q-1} J^{[q]}(a^n)t^n \right) + \left( \bigoplus_{n \geq q} (a^n)t^n \right)
\]

which implies that

\[
\ell_R \left( \frac{\mathcal{R}(I)}{\mathcal{J}^{[q]}} \right) = \sum_{n=0}^{q-1} \ell_R \left( \frac{(a^n)}{J^{[q]}(a^n)} \right) = q \cdot \ell_R \left( \frac{R}{J^{[q]}} \right) = e(J)q^2 + q \cdot \alpha_J(e).
\]

We illustrate the above results in the following example.

Example 2.6. Let \(R = k[[X, Y]]/(X^5 - Y^5)\), where \(k\) is a field of prime characteristic \(p \equiv \pm 2 \pmod{5}\). Let \(m\) be the maximal ideal of \(R\). Put \(q = p^e\) for some \(e \in \mathbb{N}\). Monsky [9] showed that for large \(e\), \(\ell_R(R/m^{[q]}) = 5q + \alpha_m(e)\), where \(\alpha_m(e) = -4\) when \(e\) is even and \(\alpha_m(e) = -6\) when \(e\) is odd.

(1) We first calculate \(\ell_R(\mathcal{R}(m)/(m, mt)^{[q]})\). It is easy to check that \(e(m) = 5\) and \(e_1(m) = 10\). Let \(I = (x)\), where \(x\) denotes the image of \(X\) in \(R\). Then \(I\) is a minimal reduction of \(m\) and \(r(m) = 4\).

Using Corollary 2.4, it follows that

\[
\ell_R \left( \frac{\mathcal{R}(m)}{(m, mt)^{[q]}} \right) = 5q^2 + 10 + \sum_{n=0}^{3} \ell_R \left( \frac{R}{m^n} \right) + 2 \sum_{n=0}^{3} \alpha_m(m^n, e) = 5q^2 + 20 + 2 \sum_{n=0}^{3} \alpha_m(m^n, e). \tag{2.1}
\]

For \(n = 1, 2, 3\), writing

\[
\ell_R \left( \frac{m^n}{m^{[q]}m^n} \right) = \ell_R \left( \frac{m^{[q]}m^n}{m^{[q]}m^n} \right) - \ell_R \left( \frac{R}{m^n} \right) + \ell_R \left( \frac{R}{m^{[q]}} \right) = \sum_{i=0}^{n-1} \mu(m^{[q]}m^i) - \ell_R \left( \frac{R}{m^{[q]}} \right) + \ell_R \left( \frac{R}{m^{[q]}} \right),
\]

we obtain

\[
\alpha_m(m, e) = \begin{cases} -3 & \text{if } e \text{ is even} \\ -5 & \text{if } e \text{ is odd} \end{cases}, \quad \alpha_m(m^2, e) = \begin{cases} -2 & \text{if } e \text{ is even} \\ -3 & \text{if } e \text{ is odd} \end{cases}, \quad \alpha_m(m^3, e) = -1.
\]

Substituting in (2.1), we get \(\ell_R(R(m)/(m, mt)^{[q]}) = 5q^2 + \beta_e\), where \(\beta_e = 0\) when \(e\) is even and \(\beta_e = -10\) when \(e\) is odd.
(2) We now calculate $\ell_R(\mathcal{R}(I)/(m, It)[q])$. Using Theorem 2.5, we get

$$
\ell_R\left(\frac{\mathcal{R}(I)}{(m, It)[q]}\right) = e(m)q^2 + q \cdot \alpha_m(e) = 5q^2 + \begin{cases} 
-4q & \text{if } e \text{ is even} \\
-6q & \text{if } e \text{ is odd}.
\end{cases}
$$

One can also verify this using the following arguments. Observe that $\mathcal{R}(I) \simeq k[[X, Y]][Z]/(X^5 - Y^5)$. In order to find $\ell(k[X, Y, Z]/(X^5 - Y^5, X^q, Y^q, Z^q))$, we find the Gröbner basis of the ideal $M_q = (X^5 - Y^5, X^q, Y^q, Z^q)$ in $k[X, Y, Z]$. Let ‘$>$’ be any monomial ordering on $k[X, Y, Z]$ with $X > Y > Z$. Since $q$ is large, the $S$-polynomials are:

$$
S(X^5 - Y^5, X^q) = X^{q-5}Y^5, \quad S(X^5 - Y^5, X^{q-5}Y^5) = X^{q-10}Y^{10}, \ldots
$$

Therefore, Gröbner basis of $M_q$ is $G = \{X^5 - Y^5, X^q, Y^q, Z^q, X^{q-5}Y^{5i}|q - 5i > 0\}$. Observe that

$$
\ell_R\left(\frac{\mathcal{R}(I)}{(m, It)[q]}\right) = \ell\left(\frac{k[X, Y, Z]}{\text{in}_>(M_q)}\right) = \ell\left(\frac{k[X, Y, Z]}{(X^5, Y^q, Z^q, X^{q-5}Y^{5i}|q - 5i < 5)}\right).
$$

We now explore the condition $q - 5i$ such that $q - 5i < 5$. Since $q$ is of the form $(5k + 2)^e$ or $(5k + 3)^e$ for some $k, e \in \mathbb{N}$, it is sufficient to find the values $(5k + 2)^e$ and $(5k + 3)^e$ modulo 5. First, consider the case $(5k + 2)^e$ modulo 5. Using the binomial theorem, we only need to find $2^e (\mod 5)$.

If $e$ is even, then $2^e \equiv 1, 4 (\mod 5)$. This implies that $X^{q-5i}Y^{5i} = XY^{q-1}$ or $X^{q-5i}Y^{5i} = X^4Y^{q-4}$. In either of the case, we get $\ell_R(\mathcal{R}(I)/(m, It)[q]) = 5q^2 - 4q$.

If $e$ is odd, then $2^e \equiv 2, 3 (\mod 5)$. This implies that $X^{q-5i}Y^{5i} = X^2Y^{q-2}$ or $X^{q-5i}Y^{5i} = X^3Y^{q-3}$. In either of the case, $\ell_R(\mathcal{R}(I)/(m, It)[q]) = 5q^2 - 6q$. We get the same conclusion in the case $(5k + 3)^e$ modulo 5.

3. The Hilbert-Kunz function in dimension $\geq 2$.

Let $R$ be a Cohen-Macaulay Stanley-Reisner ring of a simplicial complex over an infinite field with prime characteristic $p > 0$. Let $m$ be the maximal homogeneous ideal of $R$ and $I$ be an ideal generated by a linear system of parameters. It is proved in [4, Theorem 6.1] that $m = I^*$. Therefore, using [3, Corollary 4.5] we get $e_{HK}((m, mt)\mathcal{R}(m)) = e_{HK}((I, It)\mathcal{R}(I))$. Thus in order to calculate $e_{HK}(\mathcal{R}(m))$, it is sufficient to calculate $e_{HK}((I, It)\mathcal{R}(I))$. We start with a more general setup.

In this section, we find the generalized Hilbert-Kunz function of the ideal $(I, It)\mathcal{R}(I)$, where $I$ is a parameter ideal in a Cohen-Macaulay local ring. It turns out that in this case, the generalized Hilbert-Kunz function is eventually a polynomial. Let $G(I) = \oplus_{n\geq 0}I^n/I^{n+1}$ be the associated graded ring of $I$.

We shall use the following result to find the reduction number of powers of an $m$-primary ideal.

**Theorem 3.1** ([8, Corollary 2.21]). Let $(R, m)$ be a $d$-dimensional Cohen-Macaulay local ring with infinite residue field and $I$ be an $m$-primary ideal such that $\text{grade}(G(I)_+) \geq d - 1$. Then for $k \geq 1$,

$$
r(I^k) = \left\lfloor \frac{n(I)}{k} \right\rfloor + d.
$$
Let $R$ be a Cohen-Macaulay local ring and $I$ be a parameter ideal. Fix $s \in \mathbb{N}$. For a fixed set of generators of $I$, define functions

$$F(n) := H_I(I^s, n) = \ell_R \left( \frac{I^s}{I^s I^n} \right) \quad \text{and} \quad H(n) := H_I(R, n) = \ell_R \left( \frac{R}{I^n} \right) = e(I) \left( \frac{n + d - 1}{d} \right)$$

for all $n$. Note that if $R$ is 1-dimensional, then $F(n) = H(n)$ for all $n$.

**Theorem 3.2.** Let $R$ be a $d$-dimensional Cohen-Macaulay local ring and let $I$ be a parameter ideal. Let $d \geq 2$. For a fixed $s \in \mathbb{N}$,

$$F(n) = \begin{cases} d \cdot H(n) & \text{if } 1 \leq n \leq s, \\ \sum_{i=1}^{d-1} (-1)^{i+1} \binom{d}{i} H(n - (i - 1)s) & \text{if } s + 1 \leq n \leq (d - 1)s - 1, \\ H(n + s) - s^d e(I) & \text{if } n \geq (d - 1)s. \end{cases}$$

**Proof.**

**Case 1:** Let $1 \leq n \leq s$. Let $I = (x_1, \ldots, x_d)$. Consider the following epimorphism

$$\left( \frac{R}{I^n} \right)^{\oplus d} \xrightarrow{\phi} \frac{I^s}{I^s I^n} \rightarrow 0$$

$$(a_1 + I^n, \ldots, a_d + I^n) \mapsto (a_1 x_1^s + \cdots + a_d x_d^s) + I^s I^n.$$ Let $(a_1 + I^n, \ldots, a_d + I^n) \in \ker(\phi)$. Then $a_1 x_1^s + \cdots + a_d x_d^s \in I^s I^n$ implies that

$$a_1 x_1^s + \cdots + a_d x_d^s = b_1 x_1^s + \cdots + b_d x_d^s,$$

where $b_1, \ldots, b_d \in I^n$. Since $x_1^s, \ldots, x_d^s$ is an $R$-regular sequence, it follows that $(a_i - b_i) \in I^s \subseteq I^n$ for all $i = 1, \ldots, d$. Thus $a_1, \ldots, a_d \in I^n$. Therefore, $\ker(\phi) = 0$ and $I^s/I^s I^n \simeq (R/I^n)^{\oplus d}$. Hence $F(n) = d \cdot H(n)$, for all $1 \leq n \leq s$.

**Case 2:** Let $s + 1 \leq n \leq (d - 1)s - 1$. Let $x_i^s$ denote the image of $x_i$ in $I/I^2$, for all $i = 1, \ldots, d$. As $G(I)$ is Cohen-Macaulay, $(x^s)^{[s]} = (x_1^s)^s, \ldots, (x_d^s)^s \in I^s/I^{s+1}$ is a $G(I)$-regular sequence. Hence the following exact sequence is obtained from the Koszul complex of $G(I)$ with respect to $(x^s)^{[s]}$.

$$0 \rightarrow G(I)(-(d - 1)s) \rightarrow G(I)(-(d - 2)s)^{[t]} \rightarrow \cdots \rightarrow G(I(s)) \rightarrow G(I(s)) \rightarrow H_0((x^s)^{[s]}; G(I(s))) \rightarrow 0.$$

As we have an exact sequence of graded $G(I)$-modules, taking the $n^{th}$-graded component of each of these modules gives us the following exact sequence

$$0 \rightarrow \frac{I^{n-(d-1)s}}{I^{n-(d-1)s+1}} \rightarrow \left( \frac{I^{n-(d-2)s}}{I^{n-(d-2)s+1}} \right)^{[t]} \rightarrow \cdots \rightarrow \left( \frac{I^n}{I^{n+1}} \right)^{[t]} \rightarrow \frac{I^{n+s}}{I^{n+s+1}} \rightarrow \frac{I^{n+s}}{I^n I^n + I^{n+s+1}} \rightarrow 0.$$ For $s \leq n \leq (d - 1)s - 1$, it follows that

$$\ell_R \left( \frac{I^{n+s}}{I^n I^n + I^{n+s+1}} \right) = \ell_R \left( \frac{I^{n+s}}{I^{n+s+1}} \right) + \sum_{i=1}^{d-1} (-1)^i \binom{d}{i} \ell_R \left( \frac{I^{n-(i-1)s}}{I^{n-(i-1)s+1}} \right).$$
which implies

\[
\ell_R \left( \frac{R}{I^{[s]} I^n + I^{n+s+1}} \right) = \ell_R \left( \frac{R}{I^{n+s+1}} \right) + \sum_{i=1}^{d-1} (-1)^i \binom{d}{i} \ell_R \left( \frac{I^{n-(i-1)s}}{I^{n-(i-1)s+1}} \right). \tag{3.1}
\]

We can also write

\[
\ell_R \left( \frac{R}{I^{[s]} I^n + I^{n+s+1}} \right) = \ell_R \left( \frac{R}{I^{[s]}} \right) + \ell_R \left( \frac{I^{[s]} I^n}{I^{[s]}} \right) - \ell_R \left( \frac{I^{[s]} I^{n+s+1}}{I^{[s]} I^n} \right)
= \ell_R \left( \frac{R}{I^{[s]}} \right) + \ell_R \left( \frac{I^{[s]} I^n}{I^{[s]}} \right) - \ell_R \left( \frac{I^{n+s+1}}{I^{[s]} I^n} \right). \tag{3.2}
\]

since using Valabrega-Valla criterion ([11, Corollary 2.7]),

\[
\frac{I^{[s]} I^n + I^{n+s+1}}{I^{[s]} I^n} \simeq \frac{I^{n+s+1}}{I^{[s]} \cap I^{n+s+1}} \simeq \frac{I^{n+s+1}}{I^{[s]} \cap I^n} \simeq \frac{I^{n+s+1}}{I^{[s]} I^n}.
\]

Combining equations (3.1) and (3.2), we get

\[
\ell_R \left( \frac{I^{[s]} I^n}{I^{[s]}} \right) - \ell_R \left( \frac{I^{[s]} I^{n+s+1}}{I^{[s]} I^n} \right) = \sum_{i=1}^{d-1} (-1)^i \binom{d}{i} \ell_R \left( \frac{I^{n-(i-1)s}}{I^{n-(i-1)s+1}} \right)
\]

and hence for all \( s \leq n \leq (d-1)s - 1, \)

\[
F(n+1) - F(n) = \sum_{i=1}^{d-1} (-1)^{i+1} \binom{d}{i} \left[ H(n-(i-1)s+1) - H(n-(i-1)s) \right].
\]

Adding the above equality from \( s \) to \( t-1, \) for any \( t \) such that \( s+1 \leq t \leq (d-1)s - 1, \) we get

\[
F(t) - F(s) = \sum_{i=1}^{d-1} (-1)^{i+1} \binom{d}{i} \left[ H(t-(i-1)s) - H((2-i)s) \right]
= d[H(t) - H(s)] + \sum_{i=2}^{d-1} (-1)^{i+1} \binom{d}{i} H(t-(i-1)s).
\]

Since \( F(s) = d \cdot H(s), \) we get the result for \( F(n), \) \( s+1 \leq n \leq (d-1)s - 1. \)

**Case 3:** Let \( n \geq s(d-1). \) As \( R \) is a Cohen-Macaulay ring and \( I \) is a parameter ideal, it follows that the associated graded ring \( G(I) \) is a polynomial ring and its \( a \)-invariant \( a_d(G(I)) = d < 0. \)

Using Theorem 3.1, it follows that \( r(I^s) = d - 1 \) if \( s \geq d. \) Let \( s < d. \) Write \( d = k_1 s + k_2, \) where \( k_2 \in \{0, 1, \ldots, s-1\}. \) Then using Theorem 3.1 it follows that

\[
r(I^s) = [-k_1 - \frac{k_2}{s}] + d = \begin{cases} 
    d - k_1 & \text{if } k_2 = 0, \\
    d - k_1 - 1 & \text{if } k_2 \neq 0.
\end{cases}
\]

In all these cases, it follows that \( I^{[s]} I^n = I^{n+s} \) for all \( n \geq (d-1)s. \) Thus for all \( n \geq (d-1)s, \)

\[
F(n) = \ell_R \left( \frac{I^{[s]} I^n}{I^{n+s}} \right) = \ell_R \left( \frac{R}{I^{n+s}} \right) - \ell_R \left( \frac{R}{I^{[s]}} \right) = H(n+s) - s^d e(I).
\]

\[\square\]
Observe that $F(n)$ is the Hilbert-Samuel function of $I$ for the $R$-module $I^{[a]}$. Let $n(I)$ denote the postulation number of $I$ in $R$ and $P_I(n)$ denote the Hilbert-Samuel polynomial of $I$ in $R$. Using Theorem 3.2, it follows that $F(n)$ is a polynomial, given by $P(n + s) - s^d e(I)$, for $n > \max\{n(I) + s + 1, (d - 1)s\}$.

We recall some properties of Stirling numbers of the second kind.

**Remark 3.3** ([10, Chapter 1, Sec. 1.4]). The Stirling number of the second kind, denoted by $S(n, k)$, admits the following characterizations:

1. $S(n, k)$ is equal to the number of partitions of the set $[n] = \{1, \ldots, n\}$ into $k$ blocks.

2. $S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$.

Using (1), it is easy to check that

- (a) $\binom{d + 1}{2} = S(d + 1, d) = \frac{1}{d!} \sum_{i=0}^{d} (-1)^{d-i} \binom{d}{i} i^{d+1},$
- (b) $S(d, d) = 1$, and
- (c) $S(d - 1, d) = 0$.

**Lemma 3.4.** For $d \geq 2$, put $\beta_1 = (d - 2)/2$ and $\beta_2 = (d - 1)(3d - 10)/24$. Then for any $\alpha \in \mathbb{N}$,

$$
\binom{\alpha q + d - 1}{d + 1} = \frac{\alpha^{d+1}}{(d+1)!} q^{d+1} + \frac{\alpha^d \beta_1}{d!} q^d + \frac{\alpha^{d-1} \beta_2}{(d-1)!} q^{d-1} + \cdots,
$$

**Proof.**

$$
\binom{\alpha q + d - 1}{d + 1} = \frac{(\alpha q + d - 1)(\alpha q + d - 2) \cdots (\alpha q - 1)}{(d+1)!} = \frac{\alpha^{d+1}}{(d+1)!} \left[ \frac{q - (1 - d)}{\alpha} \right] \left[ \frac{q - (2 - d)}{\alpha} \right] \cdots \left[ \frac{q - 1}{\alpha} \right] = \frac{\alpha^{d+1}}{(d+1)!} \cdot f(q).
$$

If we write $f(q) = q^{d+1} - \sigma_1 q^d + \sigma_2 q^{d-1} + \cdots + (-1)^{d+1} \sigma_{d+1}$, then

$$
\sigma_1 = \sum_{n=1}^{d+1} \frac{(n - d)}{\alpha} = \frac{1}{\alpha} \left[ \frac{(d+2)(d+1)}{2} - d(d+1) \right] = -\frac{(d-2)(d+1)}{2\alpha} = -\frac{(d+1)\beta_1}{\alpha}.
$$

Let $w_2 = \sum_{n=1}^{d+1} \left( \frac{n - d}{\alpha} \right)^2$. Then using Newton’s identities, $w_2 = \sigma_1^2 - 2\sigma_2$. Consider

$$
w_2 = \sum_{n=1}^{d+1} \left( \frac{n - d}{\alpha} \right)^2 = \frac{1}{\alpha^2} \sum_{n=1}^{d+1} \left( n^2 - 2nd + d^2 \right) = \frac{1}{\alpha^2} \left( \frac{(d+1)(d+2)(2d+3)}{6} - d(d+2)(d+1) + d^2(d+1) \right).
$$

$$
= \frac{1}{\alpha^2} \left( \frac{(d^2 + 3d + 2)(2d+3)}{6} - d^2 + 3d + 2 + d^2 + d^2 \right) = \frac{1}{\alpha^2} \left( \frac{2d^3 + 9d^2 + 13d + 6}{6} - 2d^2 - 2d \right) = \frac{2d^3 - 3d^2 + d + 6}{6\alpha^2}.
$$
Then
\[
\sigma_2 = \frac{\sigma_1^2 - w_2}{2} = \frac{1}{2} \left( \frac{(d - 2)^2(d + 1)^2}{4\alpha^2} - \frac{2d^3 - 3d^2 + d + 6}{6\alpha^2} \right)
= \frac{1}{2} \left( \frac{d^4 - 2d^3 - 3d^2 + 4d + 4}{4\alpha^2} - \frac{2d^3 - 3d^2 + d + 6}{6\alpha^2} \right)
= \frac{3d^4 - 10d^3 - 3d^2 + 10d}{24\alpha^2} = \frac{d(d + 1)(d - 1)(3d - 10)}{24\alpha^2} = \frac{d(d + 1)\beta_2}{\alpha^2}.
\]

Therefore,
\[
\left( \frac{\alpha q + d - 1}{d + 1} \right) = \frac{\alpha^{d+1}}{(d+1)!} q^{d+1} + \frac{\alpha^d \beta_1}{d!} q^d + \frac{\alpha^{d-1} \beta_2}{(d-1)!} q^{d-1} + \ldots,
\]
where \( \beta_1 = (d - 2)/2 \) and \( \beta_2 = (d - 1)(3d - 10)/24 \). \( \square \)

**Theorem 3.5.** Let \( R \) be a \( d \)-dimensional Cohen-Macaulay local ring and \( d \geq 2 \). Let \( I \) be a parameter ideal of \( R \) and \( \mathcal{I} = (I, It) \mathcal{R}(I) \). Let \( s \in \mathbb{N} \).

1. Let \( s < d \). Write \( d = k_1 s + k_2 \) where \( k_2 \in \{0, 1, \ldots, s - 1\} \). If \( k_2 = 0 \), then
\[
\ell_R \left( \frac{\mathcal{R}(I)}{I^{[s]}} \right) = (d-k_1+1)s^{d+1}e(I)+d\cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left(-1\right)^i \left( \begin{array}{c} d+1 \\ i \end{array} \right) e(I) \left( \frac{(d-i-k_1+1)s + d - 1}{d + 1} \right).
\]

If \( k_2 \neq 0 \), then
\[
\ell_R \left( \frac{\mathcal{R}(I)}{I^{[s]}} \right) = (d-k_1)s^{d+1}e(I)+d\cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left(-1\right)^i \left( \begin{array}{c} d+1 \\ i \end{array} \right) e(I) \left( \frac{(d-i-k_1)s + d - 1}{d + 1} \right).
\]

2. Let \( s \geq d \). Then
\[
\ell_R \left( \frac{\mathcal{R}(I)}{I^{[s]}} \right) = ds^{d+1}e(I) + d \cdot e(I) \left( \frac{s + d - 1}{d + 1} \right) - \sum_{i=0}^{d-1} \left(-1\right)^i \left( \begin{array}{c} d+1 \\ i \end{array} \right) e(I) \left( \frac{(d-i)s + d - 1}{d + 1} \right).
\]

In other words, for \( s \) large,
\[
\ell_R \left( \frac{\mathcal{R}(I)}{I^{[s]}} \right) = c(d)e(I)s^{d+1} + e(I) \left( \frac{d - 2}{2} \right) \left( \frac{1}{(d - 1)!} - 1 \right) s^d + e(I) \frac{d(d - 1)(3d - 10)}{24(d - 1)!} s^{d-1} + \ldots,
\]
implicity that the generalized Hilbert-Kunz multiplicity \( e_{HK}((I, It) \mathcal{R}(I)) = c(d) \cdot e(I) \).

**Proof.** We first find \( \ell_R(\mathcal{R}(I)/I^{[s]}) \). Consider
\[
\mathcal{I}^{[s]} = (I^{[s]}, I^{[s]}I^s) = \left( \bigoplus_{n=0}^{s-1} I^{[s]}I^n \right) + \left( \bigoplus_{n \geq s} I^{[s]}I^{n-s}t^n \right).
\]
Since \( G(I) \) is Cohen-Macaulay and the \( a \)-invariant \( a_d(G(I)) = -d < 0 \), using Theorem 3.1 it follows that \( r(I^s) = d - 1 \) if \( s \geq d \). Let \( s < d \). Write \( d = k_1 s + k_2 \), where \( k_2 \in \{0, 1, \ldots, s - 1\} \). Then using Theorem 3.1 it follows that
\[
r(I^s) = \left[ -k_1 - \frac{k_2}{s} \right] + d = \begin{cases} d - k_1 & \text{if } k_2 = 0, \\ d - k_1 - 1 & \text{if } k_2 \neq 0. \end{cases}
\]
Case 1: Let \( s < d \). Write \( d = k_1 s + k_2 \), where \( k_2 \in \{0,1, \ldots, s-1\} \). Then as observed above, \( r(I^s) = d - k_1 - j \), \( j \in \{0,1\} \). As \( I^{[s]} \) is a minimal reduction of \( I^s \), we get, \( I^{[s]} I^{(d-k_1-j)s} = I^{(d-k_1-j+1)s} \). In other words, \( I^{[s]} I^{n-s} = I^n \), for all \( n \geq (d - k_1 - j + 1)s \). Therefore,

\[
T^{[s]} = \left( \bigoplus_{n=0}^{s-1} I^{[s]} I^n t^n \right) + \left( \bigoplus_{n=s}^{(d-k_1-j+1)s-1} I^{[s]} I^{n-s} t^n \right) + \left( \bigoplus_{n \geq (d-k_1-j+1)s} I^n t^n \right).
\]

Consider

\[
\ell_R \left( \frac{R(I)}{T^{[s]}} \right) = \sum_{n=0}^{s-1} \ell_R \left( \frac{I^n}{I^{[s]} I^n} \right) + \sum_{n=s}^{(d-k_1-j+1)s-1} \ell_R \left( \frac{I^n}{I^{[s]} I^{n-s}} \right) + \sum_{n=(d-k_1-j+1)s} \ell_R \left( \frac{I^n}{I^{[s]} I^n} \right).
\]

As \( s < d \), we get \( k_1 \geq 1 \). Thus, \( d - k_1 - j \leq d - 1 \). Using Theorem 3.2, we get

\[
\ell_R \left( \frac{R(I)}{T^{[s]}} \right) = (d - k_1 - j + 1)s^{d+1} e(I) + 2d \cdot e(I) \sum_{n=1}^{s-1} \binom{n + d - 1}{d} + d \cdot e(I) \sum_{n=s}^{(d-k_1-j)s-1} \binom{n + d - 1}{d} \]

\[
+ \sum_{n=s}^{(d-k_1-j)s-1} \left[ \sum_{i=2}^{d-1} (-1)^{i+1} \binom{d}{i} e(I) \binom{n - (i-1)s + d - 1}{d} \right] - \sum_{n=1}^{(d-k_1-j+1)s-1} e(I) \binom{n + d - 1}{d}.
\]

Case 2: Let \( s \geq d \). Then \( r(I^s) = d - 1 \). As \( I^{[s]} \) is a minimal reduction of \( I^s \), we get, \( I^{[s]} I^{(d-1)s} = I^{ds} \). In other words, \( I^{[s]} I^{n-s} = I^n \), for all \( n \geq ds \). Therefore,

\[
T^{[s]} = \left( \bigoplus_{n=0}^{s-1} I^{[s]} I^n t^n \right) + \left( \bigoplus_{n=s}^{ds-1} I^{[s]} I^{n-s} t^n \right) + \left( \bigoplus_{n \geq ds} I^n t^n \right).
\]
Consider
\[
\ell_R \left( \frac{\mathcal{R}(I)}{T^{[s]}} \right) = \sum_{n=0}^{s-1} \ell_R \left( \frac{I^n}{T^{[s]}I^n} \right) + \sum_{n=s}^{ds-1} \ell_R \left( \frac{I^n}{T^{[s]}I^{n-s}} \right)
\]
\[
= \sum_{n=0}^{s-1} \ell_R \left( \frac{R}{T^{[s]}I^n} \right) + \sum_{n=s}^{ds-1} \ell_R \left( \frac{R}{T^{[s]}I^{n-s}} \right) - \sum_{n=0}^{ds-1} \ell_R \left( \frac{R}{I^n} \right)
\]
\[
= ds \cdot \ell_R \left( \frac{R}{T^{[s]}} \right) + 2 \sum_{n=1}^{s-1} \ell_R \left( \frac{I^{[s]}}{T^{[s]}I^n} \right) + \sum_{n=s}^{(d-1)s-1} \ell_R \left( \frac{I^{[s]}}{T^{[s]}I^n} \right) - \sum_{n=1}^{(d-1)s-1} \ell_R \left( \frac{R}{I^n} \right)
\]
\[
= ds^{d+1}e(I) + 2 \sum_{n=1}^{s-1} \ell_R \left( \frac{I^{[s]}}{T^{[s]}I^n} \right) + \sum_{n=s}^{(d-1)s-1} \ell_R \left( \frac{I^{[s]}}{T^{[s]}I^n} \right) - \sum_{n=1}^{(d-1)s-1} e(I) \left( n + d - 1 \right) \frac{n + d - 1}{d}.
\]

Using Theorem 3.2, we get
\[
\ell_R \left( \frac{\mathcal{R}(I)}{T^{[s]}} \right) = ds^{d+1}e(I) + 2d \cdot e(I) \sum_{n=1}^{s-1} \binom{n + d - 1}{d} + d \cdot e(I) \sum_{n=s}^{(d-1)s-1} \binom{n + d - 1}{d}
\]
\[
+ \sum_{n=s}^{(d-1)s-1} \sum_{i=2}^{d-1} \left( -1 \right)^{i+1} \binom{d}{i} e(I) \binom{n - (i - 1)s + d - 1}{d} - \sum_{n=1}^{(d-1)s-1} e(I) \binom{n + d - 1}{d}
\]
\[
= ds^{d+1}e(I) + 2d \cdot e(I) \binom{s + d - 1}{d + 1} + d \cdot e(I) \left[ \binom{(d-1)s + d - 1}{d + 1} - \binom{s + d - 1}{d + 1} \right]
\]
\[
- \sum_{i=2}^{d-1} \left( -1 \right)^{i} \binom{d}{i} e(I) \binom{(d - i)s + d - 1}{d + 1} - e(I) \binom{ds + d - 1}{d + 1}
\]
\[
= ds^{d+1}e(I) + d \cdot e(I) \binom{s + d - 1}{d + 1} - \sum_{i=0}^{d-1} \left( -1 \right)^{i} \binom{d}{i} e(I) \binom{(d - i)s + d - 1}{d + 1}.
\]  \hspace{1cm} (3.3)

This implies that the Hilbert-Kunz function is a polynomial for \( s \geq d \). The coefficient of \( s^{d+1} \) in the above expression is
\[
e(I) \left[ d + \frac{d}{(d + 1)!} - \sum_{i=0}^{d-1} (-1)^{i} \binom{d}{i} \frac{(d - i)^{d+1}}{(d + 1)!} \right] = d \cdot e(I) \left[ 1 + \frac{1}{(d + 1)!} - \sum_{i=0}^{d-1} (-1)^{i} \binom{d}{i} \frac{(d - i)^{d+1}}{(d + 1)!} \right]
\]
\[
= d \cdot e(I) \left[ 1 + \frac{1}{(d + 1)!} - \frac{1}{d(d + 1)} S(d + d, d) \right].
\]

Using Remark 3.3, we get \( e_{HK}((I, It), \mathcal{R}(I)) = e(I) \cdot c(d) \). We now use Lemma 3.4 to find coefficients of \( s^d \) and \( s^{d-1} \) in the expression (3.3). Coefficient of \( s^d \) in expression (3.3) is
\[
e(I) \left[ \frac{d \beta_1}{d!} - \sum_{i=0}^{d} (-1)^{i} \binom{d}{i} \frac{(d - i)^{d+1}}{d!} \beta_1 \right] = e(I) \left[ \frac{d - 2}{2} \right] \left[ \frac{1}{(d - 1)!} - S(d, d) \right]
\]
\[
= e(I) \left( \frac{d - 2}{2} \right) \left[ \frac{1}{(d - 1)!} - 1 \right]
\]
and coefficient of $s^{d-1}$ is
\[
e(I) \left[ \frac{d \beta_2}{(d-1)!} - \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{(d-i)^{d-1} \beta_2}{(d-1)!} \right] = e(I) \left( \frac{(d-1)(3d-10)}{24} \right) \frac{d}{(d-1)!} - dS(d-1,d)
\]
\[
e(I) \frac{d(d-1)(3d-10)}{24(d-1)!}.
\]

$\square$

**Corollary 3.6.** Let $(R, \mathfrak{m})$ be a $d$-dimensional regular local ring with $d \geq 2$. Then for $s \geq d$,
\[
\ell_R \left( \frac{\mathcal{R}(\mathfrak{m})}{(\mathfrak{m}, \mathfrak{m}t)^s} \right) = c(d)e(\mathfrak{m})s^{d+1} + c(\mathfrak{m}) \left( \frac{d-2}{2} \right) \left( \frac{1}{(d-1)!} - 1 \right) s^{d} + c(\mathfrak{m}) \frac{d(d-1)(3d-10)s^{d-1} + \cdots}{24(d-1)!}.
\]

**Proof.** Use $I = \mathfrak{m}$ in Theorem 3.5. $\square$

**Corollary 3.7.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$ with positive prime characteristic $p > 0$. If there exists a parameter ideal $I$ of $R$ such that $I^* = \mathfrak{m}$, then
\[
e_{HK}(\mathcal{R}(\mathfrak{m})) = c(d)e(\mathfrak{m}).
\]

**Proof.** If there exists a parameter ideal $I$ of $R$ such that $I^* = \mathfrak{m}$, then using the same arguments as in the proof of [3, Corollary 4.5], it follows that $((I,l)\mathcal{R}(\mathfrak{m}))^* = ((\mathfrak{m}, \mathfrak{m}t)\mathcal{R}(\mathfrak{m}))^*$ and hence $e_{HK}((\mathfrak{m}, \mathfrak{m}t)\mathcal{R}(\mathfrak{m})) = e_{HK}((I,l)\mathcal{R}(I))$. Therefore,
\[
e_{HK}(\mathcal{R}(\mathfrak{m})) = e_{HK}((\mathfrak{m}, \mathfrak{m}t)\mathcal{R}(\mathfrak{m})) = e_{HK}((I,l)\mathcal{R}(I)) = c(d)e(I) = c(d)e(\mathfrak{m}).
\]
The latter equality holds as $I^* = \mathfrak{m}$ implies that $I = \mathfrak{m}$. $\square$

**Corollary 3.8.** Let $R$ be a $d$-dimensional Cohen-Macaulay Stanley-Reisner ring of a simplicial complex over an infinite field with prime characteristic $p > 0$. Let $\mathfrak{m}$ be a maximal homogeneous ideal of $R$. Then $e_{HK}(\mathcal{R}(\mathfrak{m})) = c(d) \cdot f_{d-1}$, where $f_{d-1}$ is the number of facets in the simplicial complex.

**Proof.** Let $I$ be an ideal generated by a linear system of parameters. It is proved in [4, Theorem 6.1] that $\mathfrak{m} = I^*$. Therefore, using Corollary 3.7 it follows that $e_{HK}(\mathcal{R}(\mathfrak{m})) = c(d) \cdot e(\mathfrak{m})$. Since $e(\mathfrak{m}) = f_{d-1}$, we are done. $\square$

**Example 3.9.** Let $k[[X_1, X_2, X_3]]$ be a power series ring in 3 variables over a field $k$. Let $I = (X_1^{n_1}, X_2^{n_2}, X_3^{n_3})$, where $n_1, n_2, n_3 \in \mathbb{N}$. Then
\[
\ell_R \left( \frac{\mathcal{R}(I)}{(I,H)^{[2]}} \right) = e(I) \left[ 32 + 3 \sum_{i=0}^{2} (-1)^i \binom{3}{i} \binom{6-2i}{4} \right] = 23 \cdot e(I) = 23 \cdot n_1n_2n_3
\]
and for $s \geq 3$,
\[
\ell_R \left( \frac{\mathcal{R}(I)}{I^{[s]}} \right) = n_1n_2n_3 \cdot \left[ 3s^4 + 3 \left( s + \frac{2}{4} \right) - \sum_{i=0}^{2} (-1)^i \binom{3}{i} \binom{(3-i)s + 2}{4} \right].
\]
In particular,

\[
\ell_R \left( \frac{R(m)}{(m, mt)^s} \right) = \left[ 3s^4 + 3 \binom{s+2}{4} - \sum_{i=0}^{2} \left( -1 \right)^i \binom{3}{i} \left( \binom{3-i}{s+2} \right) \right] \\
= \frac{13}{8} s^4 - \frac{1}{4} s^3 - \frac{1}{8} s^2 - \frac{1}{4} s,
\]

for \( s \geq 3 \).

**Example 3.10.** Let \( R = k[[X,Y,Z]]/(XY - Z^n) \) for some positive integer \( n \geq 2 \) and let \( I \) be a parameter ideal of \( R \). Then for \( s \geq 2 \),

\[
\ell_R \left( \frac{R(I)}{I^s} \right) = e(I) \left[ 2s^3 + 2 \binom{s+1}{3} - \sum_{i=0}^{1} \left( -1 \right)^i \binom{2}{i} \left( \binom{2-i}{s+1} \right) \right] \\
= e(I) \left[ \frac{4}{3} s^3 - \frac{1}{3} s \right].
\]

**References**

[1] Holger Brenner. The Hilbert-Kunz function in graded dimension two. *Comm. Algebra*, 35(10):3199–3213, 2007.

[2] Aldo Conca. Hilbert-Kunz function of monomial ideals and binomial hypersurfaces. *Manuscripta Math.*, 90(3):287–300, 1996.

[3] Kazufumi Eto and Ken-ichi Yoshida. Notes on Hilbert-Kunz multiplicity of Rees algebras. *Comm. Algebra*, 31(12):5943–5976, 2003.

[4] Kriti Goel, Vivek Mukundan, and J. K. Verma. Tight closure of powers of ideals and tight Hilbert polynomials. *Math. Proc. Cambridge Philos. Soc.*, 2019.

[5] C. Han and P. Monsky. Some surprising Hilbert-Kunz functions. *Math. Z.*, 214(1):119–135, 1993.

[6] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990.

[7] Ernst Kunz. Characterizations of regular local rings of characteristic \( p \). *Amer. J. Math.*, 91:772–784, 1969.

[8] Thomas John Marley. *Hilbert functions of ideals in Cohen-Macaulay rings*. ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–Purdue University.

[9] P. Monsky. The Hilbert-Kunz function. *Math. Ann.*, 263(1):43–49, 1983.

[10] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[11] Paolo Valabrega and Giuseppe Valla. Form rings and regular sequences. *Nagoya Math. J.*, 72:93–101, 1978.
[12] J. K. Verma. Rees algebras and mixed multiplicities. *Proc. Amer. Math. Soc.*, 104(4):1036–1044, 1988. 2, 5

**Indian Institute of Technology Bombay, Mumbai, India 400076**

*E-mail address: kritigoel.maths@gmail.com*

**School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai, India 400005**

*E-mail address: mitrak@math.tifr.res.in*

**Indian Institute of Technology Bombay, Mumbai, India 400076**

*E-mail address: jkv@math.iitb.ac.in*