Research Article

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Quasilinear Riccati-Type Equations with Oscillatory and Singular Data

https://doi.org/10.1515/ans-2020-2079
Received December 30, 2019; accepted February 17, 2020

Abstract: We characterize the existence of solutions to the quasilinear Riccati-type equation

\[
\begin{aligned}
- \text{div} A(x, \nabla u) &= |\nabla u|^q + \sigma \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with a distributional or measure datum \( \sigma \). Here \( \text{div} A(x, \nabla u) \) is a quasilinear elliptic operator modeled after the \( p \)-Laplacian \((p > 1)\), and \( \Omega \) is a bounded domain whose boundary is sufficiently flat (in the sense of Reifenberg). For distributional data, we assume that \( p > 1 \) and \( q > p \). For measure data, we assume that they are compactly supported in \( \Omega \), \( p > \frac{2n}{n+1} \), and \( q \) is in the sub-linear range \( p - 1 < q < 1 \). We also assume more regularity conditions on \( A \) and on \( \partial \Omega \) in this case.

Keywords: Quasilinear Equations, Wolff and Riesz Potentials, Hardy–Littlewood Maximal Function, Renormalized Solutions, Bessel and Riesz Capacities

MSC 2010: 31C15, 35J62, 35J92, 35R06, 45G15

Communicated by: Julián López-Gómez and Patrizia Pucci

1 Introduction and Main Results

We address in this note the question of existence for the quasilinear Riccati-type equation

\[
\begin{aligned}
- \text{div} A(x, \nabla u) &= |\nabla u|^q + \sigma \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  \hspace{1cm} (1.1)

where the datum \( \sigma \) is generally a signed distribution given on a bounded domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \).

In (1.1) the nonlinearity \( A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is a Carathéodory vector-valued function, i.e., \( A(x, \xi) \) is measurable in \( x \) for every \( \xi \) and continuous in \( \xi \) for a.e. \( x \). Moreover, for a.e. \( x \), \( A(x, \xi) \) is differentiable in \( \xi \) away from the origin. Our standing assumption is that \( A \) satisfies the following growth and monotonicity conditions: for some \( 1 < p < \infty \) and \( \Lambda \geq 1 \) there hold

\[
|A(x, \xi)| \leq \Lambda|\xi|^{p-1}, \quad |\nabla_\xi A(x, \xi)| \leq \Lambda|\xi|^{p-2}
\]

and

\[
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \Lambda^{-1}(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2
\]

for any \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus (0,0)\) and a.e. \( x \in \mathbb{R}^n \). The special case \( A(x, \xi) = |\xi|^{p-2}\xi \) gives rise to the standard

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$p$-Laplacian $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$. Note that these conditions imply that $\mathcal{A}(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$, and

$$\langle \nabla \mathcal{A}(x, \xi), \lambda, \lambda \rangle \geq 2 \frac{\varepsilon^2}{n} \Lambda^{-1} |\xi|^{p-2} |\lambda|^2$$

for every $(\lambda, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ and a.e. $x \in \mathbb{R}^n$.

More regularity conditions will be imposed later on the nonlinearity $\mathcal{A}(x, \xi)$ in the $x$-variable and on the boundary $\partial \Omega$ of $\Omega$.

One can view (1.1) as a quasilinear stationary viscous Hamilton–Jacobi equation or Kardar–Parisi–Zhang equation, which appears in the physical theory of surface growth [18, 19].

**Necessary Conditions.** For $q > p - 1$, it is known (see [15, 26]) that in order for (1.1) to have a $u$ with $|\nabla u| \in L^q_{\text{loc}}(\Omega)$ it is necessary that $\sigma$ be regular and small enough. In particular, if $\sigma$ is a signed measure, these necessary conditions can be quantified as

$$\int \frac{|\varphi|^q}{|\nabla \varphi|^p} \, d\sigma \leq \Lambda \frac{q}{q - p + 1} \left( \int \frac{|\nabla u|^{p-1}}{|\nabla \varphi|^q} |\nabla \varphi| \, dx \right) \int \frac{|\nabla u|^q}{|\nabla \varphi|^q} \, dx$$

for all $\varphi \in C^0_0(\Omega)$. This can be seen by using $|\varphi|^q / |\nabla \varphi|^p$ as a test function in (1.1) and applying the first inequality in (1.2) to get

$$\int \frac{|\varphi|^q}{|\nabla \varphi|^p} \, d\sigma \leq \Lambda q \frac{q - p + 1}{q - p} \int \frac{|\nabla u|^{p-1}}{|\nabla \varphi|^q} |\nabla \varphi| \, dx - \int |\nabla u|^q \frac{|\varphi|^q}{|\nabla \varphi|^q} \, dx.$$

Then by an appropriate Young’s inequality one arrives at (1.4) (see also [26] and [17]). Note that estimate (1.4) also holds when $\sigma$ is a distribution in $W^{-1, \frac{q}{p-1}}_{\text{loc}}(\Omega)$ in which case the left-hand side should be understood as $\langle \sigma, |\varphi|^q / |\nabla \varphi|^p \rangle$.

Thus if $\sigma$ is a *nonnegative measure* (or equivalently a nonnegative distribution) compactly supported in $\Omega$, then condition (1.4) implies the capacitary condition

$$\sigma(K) \leq C \text{Cap}_{1, \frac{q}{p-1}}(K)$$

for every compact set $K \subset \Omega$ and a constant $C$ independent of $K$. Here $\text{Cap}_{1, s}$, $s > 1$, is the capacity associated to the Sobolev space $W^{1, s}(\mathbb{R}^n)$ defined for each compact set $K \subset \mathbb{R}^n$ by

$$\text{Cap}_{1, s}(K) = \inf \left\{ \int_{\mathbb{R}^n} (|\nabla \varphi|^s + \varphi^s) \, dx : \varphi \in C^0_0(\mathbb{R}^n), \varphi \geq \chi_K \right\},$$

where $\chi_K$ is the characteristic function of $K$.

Moreover, in the case of nonnegative measure datum $\sigma$, all solutions of (1.1) must obey the regularity condition

$$\int_K |\nabla u|^q \, dx \leq C \text{Cap}_{1, \frac{q}{p-1}}(K)$$

for every compact set $K \subset \Omega$. However, unlike (1.5), the constant $C$ in (1.6) might depend on the distance from $K$ to the boundary of $\Omega$ (see [15, 26]).

Motivated from (1.5), we now introduce the following definition.

**Definition 1.1.** Given $s > 1$ and a domain $\Omega \subset \mathbb{R}^n$, we define the space $M^{1, s}(\Omega)$ to be the set of all signed measures $\mu$ with bounded total variation in $\Omega$ such that the quantity $|\mu|_{M^{1, s}(\Omega)} < +\infty$, where

$$[\mu]_{M^{1, s}(\Omega)} := \sup \left\{ \frac{|\mu(K)|}{\text{Cap}_{1, s}(K)} : \text{Cap}_{1, s}(K) > 0 \right\},$$

with the supremum being taken over all compact sets $K \subset \Omega$.

It is well known that a measure $\mu \in M^{1, s}(\Omega)$ if and only if the trace inequality

$$\int_{\mathbb{R}^n} |\varphi|^s \, d|\mu| \leq C \int_{\mathbb{R}^n} (|\nabla \varphi|^s + |\varphi|^s) \, dx$$

holds for all $\varphi \in C^0_0(\mathbb{R}^n)$, with a constant $C$ independent of $\varphi$. Here $\mu$ is extended by zero outside $\Omega$. For this characterization see, e.g., [1]. Other characterizations are also available (see [20]).
In practice, it is useful to realize that the condition $\mu \in M^{1,2}(\Omega)$ is satisfied if $\mu$ is a function verifying the Fefferman–Phong condition $\mu \in L^{1+c\ell(1+\varepsilon)}(\Omega)$ for some $c > 0$ (see [10]). Here $L^{1+c\ell(1+\varepsilon)}(\Omega)$ is a Morrey space (see, e.g., [21]). In particular, it is satisfied provided $\mu$ is a function in the weak Lebesgue space $L^{2,\infty}(\Omega)$, $s < n$. Another sufficient condition is given by $(G_1 + |\mu|)^{s/n} \in L^{1+c\ell(1+\varepsilon)}(\Omega)$ for some $c > 0$ (see [20]), where $G_1$ is the Bessel kernel of order 1 defined via its Fourier transform by $G_1(\xi) = (1 + |\xi|^2)^{s/2}$.

Now in view of (1.6), it is natural to look for a solution $u$ of (1.1) such that $|\nabla u|^q$ belongs to $M^{2/q}(\Omega)$. In this paper, we will be interested in only such a space of solutions.

**Sufficient Conditions in Capacitary Terms.** There are many papers that obtain existence results for equation (1.1) under certain integrability conditions on the datum $\sigma$ which are generally not sharp. The pioneering work [15] originally used capacities to treat (1.1) in the ‘linear’ case $p = 2$ in $\mathbb{R}^n$ ($q > 1$), or in a bounded domain $\Omega$ ($q > 2$). For $p > 2 - \frac{1}{q}$ and $q \geq 1$, it was shown in [28, 30] (see also [13, 27] for the sub-critical case $p - 1 < q < \frac{2n-2}{n-2}$) that, under certain regularity conditions on $A$ and $\partial \Omega$, if $\sigma$ is a finite signed measure in $M^{2/q}(\Omega)$, with $[\sigma]_{M^{2/q}(\Omega)}$ being sufficiently small, then equation (1.1) admits a solution $u \in W^{1,q}_0(\Omega)$ such that $|\nabla u|^q \in M^{2/q}(\Omega)$. Similar existence results have recently been extended to the case $\frac{3n-2}{2n-2} < p \leq 2 - \frac{1}{q}$, $q \geq 1$, in [23] and to the case $1 < p \leq \frac{3n-2}{2n-4}$, $q \geq 1$, in [25]. We also mention that the earlier work [26, 29] covers all $p > 1$ but only for $q > p$.

We observe that whereas the existence results of [15, 23, 25, 26, 28] are sharp when $\sigma$ is a nonnegative measure, they could not be applied to a large class distributional data $\sigma$ with strong oscillation. Take for example the function

$$f(x) = |x|^{-\varepsilon-d} \sin(|x|^s),$$

where $s = \frac{q}{q-p+1}$ and $\varepsilon > 0$ such that $\varepsilon + s < n$. Then $\sigma = |f(x)| \, dx$ fails to satisfy the capacitary inequality (1.5), but it is possible to show that the equation

$$-\Delta_p u = |\nabla u|^q + \lambda |f|, \quad q \geq p,$$

admits a solution $u \in W^{1,q}_0(B_1(0))$ provided $|\lambda|$ is sufficiently small. For this see [21] which addresses oscillatory data in the Morrey space framework. See also [2, 5, 11, 12] in which the case $q = p$ is considered.

Note that in this special case, the Riccati-type equation $-\text{div} \, A(x, \nabla u) = |\nabla u|^p + \sigma$ is strongly related to the Schrödinger-type equation $-\text{div} \, A(x, \nabla u) = |\nabla u|^{p-2} u$ (see [14]). This relation has been employed in an essential way in [16, 17] to study the existence of local solutions in this case. Here by a local solution we mean one that belongs to $W^{1,p}(\Omega)$ and has no pre-specified boundary condition.

**Main Results.** The first main result of this paper is to treat (1.1) with oscillatory data in the framework of the natural space $M^{1-p/q}(\Omega)$. This provides non-trivial improvements of the results of [15, 23, 25, 26, 28] and [21] at least in the case $q > p$. We first observe the following necessary condition on $\sigma$ so that (1.1) has a solution $u$ such that $|\nabla u|^q \in M^{1-p/q}(\Omega)$.

**Theorem 1.2.** Let $p > 1$, $q \geq 1$, and let $A$ satisfy the first inequality in (1.2). Suppose that $\sigma$ is a distribution in a bounded domain $\Omega$ such that the Riccati-type equation

$$-\text{div} \, A(x, \nabla u) = |\nabla u|^q + \sigma \quad \text{in} \, \mathcal{D}'(\Omega)$$

admits a solution $u \in W^{1,q}_0(\Omega)$ with $|\nabla u|^q \in M^{1-p/q}(\Omega)$. Then there exists a vector field $f$ on $\Omega$ such that $\sigma = \text{div} \, f$ and $|f|^p \in M^{1-p/q}(\Omega)$. In particular, we have $\sigma \in W^{-1,\frac{p}{p-1}}(\Omega)$, and moreover

$$|\langle \sigma, |\varphi|^{\frac{p}{p-1}} \rangle| \leq C \int_{\Omega} |\nabla \varphi|^{\frac{p}{p-1}} \, dx$$

for all $\varphi \in C^\infty_0(\Omega)$, with a constant $C$ independent of $\varphi$.

Conversely, when $q > p$, we obtain the following existence result.

**Theorem 1.3.** Let $1 < p < q < \infty$, $R_0 > 0$, and assume that $A$ satisfies (1.2)–(1.3). Then there exists a constant $\delta = \delta(n, p, A, q) \in (0, 1)$ such that the following holds. Let $\omega \in M^{1-p/q}(\Omega)$ and let $f$ be a vector field on $\Omega$
such that \(|f|^{\frac{p}{q}} \in M^{1,\frac{p}{q}}(\Omega)\). Assume that \(\Omega\) is \((\delta, R_0)\)-Reifenberg flat and that \(A\) satisfies the \((\delta, R_0)\)-BMO condition. Then there exists a positive constant \(c_0 = c_0(n, p, \Lambda, q, \text{diam}(\Omega), \frac{\text{diam}(\Omega)}{R_0})\) such that whenever
\[
|\omega|^{\frac{p}{q}}_{M^{1,\frac{p}{q}}(\Omega)} + |[f]^{\frac{p}{q}}|_{M^{1,\frac{p}{q}}(\Omega)} \leq c_0,
\]
there exists a solution \(u \in W^{1,\frac{p}{q}}(\Omega)\) to the Riccati-type equation
\[
\begin{cases}
- \text{div} \, A(x, \nabla u) = |\nabla u|^q + \omega + \text{div} \, f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \(|\nabla u|^q \in M^{1,\frac{p}{q}}(\Omega)\).

**Remark 1.4.** Under a slightly different condition on \(A(x, \xi)\), it is possible to use the results of [3, 4] and the method of this paper to extend Theorem 1.3 to the end-point case \(q = p\). However, this case has been treated in [2] by using a different method (see also [5]).

The notion of \((\delta, R_0)\)-Reifenberg flat domains mentioned in Theorem 1.3 is made precise by the following definition.

**Definition 1.5.** Given \(\delta \in (0, 1)\) and \(R_0 > 0\), we say that \(\Omega\) is a \((\delta, R_0)\)-Reifenberg flat domain if for every \(x_0 \in \partial \Omega\) and every \(r \in (0, R_0)\), there exists a system of coordinates \([y_1, y_2, \ldots, y_n]\), which may depend on \(r\) and \(x_0\), so that in this coordinate system \(x_0 = 0\) and that
\[
B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n < -\delta r\}.
\]

Examples of such domains include those with \(C^1\) boundaries or Lipschitz domains with sufficiently small Lipschitz constants. They also include certain domains with fractal boundaries.

On the other hand, the \((\delta, R_0)\)-BMO condition imposed on \(A(x, \xi)\) allows it to have small jump discontinuities in the \(x\)-variable. More precisely, given two positive numbers \(\delta\) and \(R_0\), we say that \(A(x, \xi)\) satisfies the \((\delta, R_0)\)-BMO condition if
\[
[A]_{R_0} := \sup_{y \in \mathbb{R}^n, \delta < r < R_0} \int_{B_r(y)} Y(A, B_r(y))(x) \, dx \leq \delta,
\]
where for each ball \(B = B_r(y)\) we let
\[
Y(A, B)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x, \xi) - \overline{A}_B(\xi)|}{|\xi|^{p-1}},
\]
with
\[
\overline{A}_B(\xi) = \int_B A(x, \xi) \, dx.
\]
Thus one can think of the \((\delta, R_0)\)-BMO condition as an appropriate substitute for the Sarason VMO condition.

The second main result of the paper is to treat (1.1) for the case \(p > \frac{2n-2}{2n-1}, \ p - 1 < q < 1, \) and \(\sigma\) is a signed measure compactly supported in \(\Omega\). This extends the results of [23] to the sublinear range \(p - 1 < q < 1\), which cannot be dealt with by the method of [23] due to the lack of convexity. However, here we cannot assume that \(A(x, \xi)\) is Hölder continuous in the \(x\)-variable, i.e.,
\[
|A(x, \xi) - A(x_0, \xi)| \leq A|x - x_0|^{\theta} |\xi|^{p-1}
\]
for some \(\theta \in (0, 1)\) and all \(x, x_0, \xi \in \mathbb{R}^n\). We note that this regularity assumption can be relaxed by using a weaker Dini’s condition as in [24]. Moreover, for \(\Omega\) we further assume the following integrability condition (besides the \((\delta, R_0)\)-Reifenberg flatness condition):
\[
\int_{\Omega} d(x)^{-\epsilon_0} \, dx < +\infty
\]
Thus by (1.8) we have that
\[
|x \in \Omega : \tau < d(x) \leq 2\tau| \leq C\epsilon
\]
holds for all small \( \tau > 0 \).

**Theorem 1.6.** Let \( p > \frac{3n-2}{2n-1} \), \( p - 1 < q < 1 \), \( R_0 > 0 \), and assume that \( A \) satisfies (1.2), (1.3), and (1.11). Suppose that (1.12) holds for an \( \epsilon_0 > 0 \) and that \( \omega \in M^{1, \frac{q}{p-1}}(\Omega) \) with \( \text{supp}(\omega) \in \Omega \). Then there exists a constant \( \delta = \delta(n, p, \Lambda, q, \epsilon_0) \in (0, 1) \) such that the following holds. If \( \Omega \) is the \((\delta, R_0)\)-Reifenberg flat, then there exists a positive constant
\[
c_0 = c_0(n, p, \Lambda, q, \theta, \epsilon_0, \text{diam}(\Omega), \frac{\text{diam}(\Omega)}{R_0}, \text{dist}(\text{supp}(\omega), \partial \Omega))
\]
such that whenever
\[
[\omega]_{M^{1, \frac{q}{p-1}}(\Omega)} \leq c_0,
\]
there exists a renormalized solution \( u \), with \( |\nabla u|^q \in M^{1, \frac{q}{p-1}}(\Omega) \), to the Riccati-type equation
\[
\begin{cases}
-\text{div} A(x, \nabla u) = |\nabla u|^q + \omega & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We refer to [9] for the notion of renormalized solutions. Note that in the case \( p \leq 2 - \frac{1}{n} \) the gradients of such solutions should be interpreted appropriately.

**Remark 1.7.** It is worth mentioning that the case \( p > 2 - \frac{1}{n} \) and \( p - 1 < q < 1 \), which is a sub-critical case, has been addressed in [13, 27] by different methods that require no compact support condition on \( \omega \). However, our proof of Theorem 1.6 produces a solution to (1.14) whose gradient is well controlled pointwise. Moreover, our proof also works in the super-linear case \( q \geq 1 \) that was considered earlier in [23].

## 2 Proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3. We begin with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Here we employ an idea of [16, 17] that treated the case \( q = p \). Let \( B \) be a ball of radius \( \text{diam}(\Omega) \) containing \( \Omega \) and let \( G(x, y) \) be the Green function with zero boundary condition associated to \( -\Delta \) on \( B \). Then it follows that
\[
|\nabla u(x)|^q = -\text{div} \int_B \nabla_x G(x, y)|\nabla u(y)|^q \chi_{\Omega}(y) \, dy \quad \text{in } \mathcal{D}'(\Omega).
\]

Thus by (1.8) we have that \( \sigma = \text{div} f \) in \( \mathcal{D}'(\Omega) \) with
\[
f = -A(x, \nabla u) + \int_B \nabla_x G(x, y)|\nabla u(y)|^q \chi_{\Omega}(y) \, dy.
\]

Note that by the first inequality in (1.2) we find
\[
\left[ |A(x, \nabla u)| \right]^{\frac{p-1}{q}}_{M^{1, \frac{q}{p-1}}(\Omega)} \leq \Lambda \left[ |\nabla u|^q \right]^{\frac{p-1}{q}}_{M^{1, \frac{q}{p-1}}(\Omega)}.
\]

On the other hand, using the pointwise estimate
\[
|\nabla_x G(x, y)| \leq C(n, \text{diam}(\Omega))|x - y|^{1-n} \quad \text{for all } x, y \in B, x \neq y,
\]

for some \( \epsilon_0 > 0 \). Here \( d(x) \) is the distance from \( x \) to \( \partial \Omega \), i.e., \( d(x) = \inf \{|x - y| : y \in \partial \Omega\} \). It is not clear to us if the \((\delta, R_0)\)-Reifenberg flatness condition for a sufficiently small \( \delta \) will imply (1.12). Note that (1.12) holds (even with any \( 0 < \epsilon_0 < 1 \)) for any bounded Lipschitz domain. More generally, (1.12) holds for some \( \epsilon_0 > 0 \) provided we can find an \( \epsilon > 0 \) such that
\[
|\nabla \sigma(x)| \leq C\epsilon \quad \text{in } B.
\]
and [26, Corollary 2.5] we obtain
\[
\left[ \int_B \left( \nabla G(\cdot, y) \nabla |u(y)|^q \chi(y) \right) \, dy \right]^{\frac{q}{q-1}}_{M^{1, \frac{q}{p-1}}(\Omega)} \leq C \left[ \int \left| \nabla u \right|^q \right]_{M^{1, \frac{q}{p-1}}(\Omega)}.
\]
These show that \( |f|^{\frac{q}{p-1}} \in M^{1, \frac{q}{p-1}}(\Omega) \) with the estimate
\[
\left[ \int |f|^{\frac{q}{p-1}} \right]_{M^{1, \frac{q}{p-1}}(\Omega)} \leq C \left( \int |\nabla u|^q \right)_{M^{1, \frac{q}{p-1}}(\Omega)} + \left[ \int |\nabla u|^q \right]_{M^{1, \frac{q}{p-1}}(\Omega)}.
\]
Finally, given any \( \varphi \in C^0_0(\Omega) \) we have
\[
\left| \langle \sigma, |\varphi|^{\frac{q}{p-1}} \right| = \int_{\Omega} \varphi \cdot \nabla (|\varphi|^{\frac{q}{p-1}}) \, dx \leq \frac{q}{q-p+1} \int_{\Omega} |\varphi|^q |\nabla \varphi| \, dx
\]
\[
\leq \frac{q}{q-p+1} \left( \int_{\Omega} |f|^{\frac{q}{p-1}} |\varphi|^{\frac{q}{p-1}} \, dx \right)^{\frac{q-p+1}{q}} \left( \int_{\Omega} |\nabla \varphi|^{\frac{p}{p-1}} \, dx \right)^{\frac{q-p+1}{q}}
\]
\[
\leq C \left[ |\nabla \varphi|^{\frac{p}{p-1}} \right]_{\Omega}.
\]
Here the last inequality follows since by (1.7) and Poincaré’s inequality we have
\[
\int_{\Omega} |f|^{\frac{q}{p-1}} |\varphi|^{\frac{q}{p-1}} \, dx \leq C(\text{diam}(\Omega)) \int_{\Omega} |\nabla \varphi|^{\frac{p}{p-1}} \, dx.
\]
Thus (1.9) is verified, which completes the proof of the theorem.

In order to Theorem 1.3, we need the following equi-integrability result.

**Lemma 2.1.** For each \( j = 1, 2, 3, \ldots \), let \( f_j \in L^{p_j}(\Omega, \mathbb{R}^n) \), \( q > p \), and \( u_j \in W^{1,q}_0(\Omega) \) be the solution of
\[
\text{div} \mathcal{A}(x, \nabla u) = \text{div} f_j \quad \text{in} \ \Omega.
\]
Assume that \( \{|f_j|^{\frac{p_j}{p-1}}\}_j \) is a bounded and equi-integrable subset of \( L^1(\Omega) \). Then there exists \( \delta = \delta(n, p, \Lambda, q) \in (0, 1) \) such that if \( \Omega \) is (\( \delta, R_0 \))-Reifenberg flat and \( |\mathcal{A}|_{L_p} \leq \delta \) for some \( R_0 > 0 \), then the set \( \{|\nabla u_j|^q\}_j \) is also a bounded and equi-integrable subset of \( L^1(\Omega) \).

**Proof.** By de la Vallée–Poussin Lemma on equi-integrability we can find an increasing and convex function \( G : [0, \infty) \to [0, \infty) \) with \( G(0) = 0 \) and \( \lim_{t \to \infty} \frac{G(t)}{t} = \infty \) such that
\[
\sup_j \int_{\Omega} G(|f_j|^{\frac{p_j}{p-1}}) \, dx \leq C.
\]
Moreover, we may assume that \( G \) satisfies a \( \Delta_2 \) (moderate growth) condition (see, e.g., [22]): there exists \( c_1 > 1 \) such that
\[
G(2t) \leq c_1 G(t) \quad \text{for all} \ t \geq 0.
\]
It follows that the function \( \Phi(t) := G\left( \frac{t}{p-1} \right) \) also satisfies a \( \Delta_2 \) condition since
\[
\Phi(2t) = G\left( \frac{2t}{p-1} \right) \leq G\left( \frac{t}{p-1} + \frac{t}{p-1} \right) \leq c_1 \left( \frac{t}{p-1} + \frac{t}{p-1} \right) \Phi(t),
\]
where \( \left[ \frac{q}{p} \right] \) is the integral part of \( \frac{q}{p} \).

On the other hand, as \( G \) is convex and \( G(0) = 0 \), for \( c_2 = \left( \frac{p}{q} \right)^{\frac{q}{p-1}} > 1 \) we have
\[
\Phi(t) = G\left( c_2 \frac{t}{p-1} \right) \leq c_2 \frac{t}{p-1} G\left( c_2 \frac{t}{p-1} \right) = \frac{1}{2c_2} \Phi(c_2 t).
\]
In other words, \( \Phi \) satisfies a \( \nabla_2 \) condition.
Also, by the above properties of $G$ we have that $\Phi$ is an increasing and convex Young function, i.e.,

$$
\Phi(0) = 0, \quad \lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0, \quad \text{and} \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.
$$

With these properties of $\Phi$, by the main result of [7] (see also [8]), we have that

$$
\sup_{\Omega} \int \Phi(|\nabla u|^p) \, dx = \sup_{\Omega} \int G(|\nabla u|^q) \, dx \leq C.
$$

Here the constant $C$ depends only on $n$, $p$, $q$, $G$, $\Lambda$, $\Omega$, and $\delta$. Hence by de la Vallée–Poussin Lemma, it follows that the sequence $\{ |\nabla u|^q \}$ is equi-integrable in $\Omega$.

We now recall that $G_1$ is the Bessel kernel of order 1. For any nonnegative measure $\nu$, we define a Bessel potential of $\nu$ by

$$
G_1(\nu)(x) := G_1 * \nu(x) = \int_{\mathbb{R}^n} G_1(x-y) \, d\nu(y), \quad x \in \mathbb{R}^n.
$$

**Lemma 2.2.** Let $q > p > 1$ and suppose that $\mu \in M^{1, \frac{q}{p}}(\Omega)$ and that $g$ is a vector field on $\Omega$ such that $|g|^{\frac{q}{p-1}} \in M^{1, \frac{q}{p-1}}(\Omega)$. There exists a constant $\delta = \delta(n, p, \Lambda, q) \in (0, 1)$ such that if $\Omega$ is $(\delta, R_0)$-Reifenberg flat and $|A|_{R_0} \leq \delta$ for some $R_0 > 0$, then the equation

$$
\begin{cases}
\text{div} A(x, \nabla U) = \mu + \text{div} g & \text{in} \Omega, \\
u = 0 & \text{on} \partial \Omega,
\end{cases}
$$

admits a unique solution $U \in W^{1,q}_0(\Omega)$ with

$$
G_1(|\nabla U|^q) \leq C \left[ G_1(|g|^{\frac{q}{p-1}}) + |\mu|^{\frac{q}{p-1}} \frac{q}{p-1} G_1(|\mu|) \right] \quad \text{a.e. in} \, \mathbb{R}^n.
$$

Here $U$, $g$, and $\mu$ are extended by zero outside $\Omega$. The constant $C$ in (2.3) depends only on $n$, $p$, $\Lambda$, $q$, $\text{diam}(\Omega)$, and $\frac{\text{diam}(\Omega)}{R_0}$.

**Proof.** Again, let $B$ be a ball of radius $\text{diam}(\Omega)$ containing $\Omega$ and let $G(x, y)$ be the Green function with zero boundary condition associated to $-\Delta$ on $B$. Then we can write $\mu = -\text{div} h_\mu$ in $D'(\Omega)$, where $h_\mu$ is a gradient vector field on $B$ given by

$$
h_\mu(x) = \int_B \nabla_x G(x, y) \, d\mu(y).
$$

In what follows, we say that a function $w \in A_1$ if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$, $w \geq 0$, and

$$
\sup_{r > 0} \int_{B_r(x)} w(y) \, dy \leq Aw(x) \quad \text{for a.e.} \, x \in \mathbb{R}^n.
$$

The least possible constant $A$ in the above inequality is called the $A_1$ constant of $w$ and is denoted by $[w]_{A_1}$.

By [21, Theorem 1.10], for any weights $w \in A_1$, there exists a constant $\delta = \delta(n, p, \Lambda, q, [w]_{A_1}) \in (0, 1)$ such that if $\Omega$ is $(\delta, R_0)$-Reifenberg flat and $|A|_{R_0} \leq \delta$, then (2.2) admits a unique solution $U \in W^{1,q}_0(\Omega)$ such that

$$
\int_{\Omega} |\nabla U|^q w \, dx \leq C \int_{\Omega} |g - h_\mu|^{\frac{q}{p-1}} \, w \, dx.
$$

Moreover, the constant $C$ in (2.5) depends on $w$ only through $[w]_{A_1}$.

We now observe from the asymptotic behavior of $G_1$ (see [1, Section 1.2.4]) that the function

$$
w(x) = G_1(g)(x),
$$

where $g$ is any nonnegative and bounded function with compact support, satisfies the following local $A_1$ condition:

$$
\sup_{0 < r \leq 1} \int_{B_r(x)} w(y) \, dy \leq Aw(x) \quad \text{for a.e.} \, x \in \mathbb{R}^n.
$$
The constant $A$ is independent of $g$. Thus by [31, Lemma 1.1] there exists a weight $\overline{w} \in A_1$ such that $w = \overline{w}$ in $B$ and $[\overline{w}]_{A_1} \leq C = C(n, \text{diam}(\Omega), A)$. Then using $\overline{w}$ in (2.5) and applying Fubini’s Theorem, we find

$$
\int_{\mathbb{R}^n} G_1(|\nabla U|^q(x)) dx \leq C \int_{\mathbb{R}^n} G_1(|g - h_\mu|^q(x)) dx.
$$

Due to the arbitrariness of $g$, this yields

$$
G_1(|\nabla U|^q(x)) \leq C G_1(|g - h_\mu|^q(x)) \quad \text{a.e. in } \mathbb{R}^n
$$

(2.6)

for a constant $C$ that depends only on $n, p, \Lambda, q, \text{diam}(\Omega)$, and $\text{diam}(\Omega)/R_0$.

Note that by (2.4) and the pointwise estimate (2.1) it follows that

$$
|h_\mu(x)| \leq C G_1(|\mu|)(x) \quad \text{a.e. in } \mathbb{R}^n.
$$

(2.7)

On the other hand, by [20, Theorem 1.2] we find

$$
G_1[|\mu|^q(x)] \leq C[\mu] \frac{q+1}{M^q + q + q^2 + 1} G_1(|\mu|)(x) \quad \text{a.e. in } \mathbb{R}^n.
$$

(2.8)

Thus in view of (2.7) we see that

$$
G_1[|h_\mu|^q] \leq C[\mu] \frac{q+1}{M^q + q + q^2 + 1} G_1(|\mu|)(x) \quad \text{a.e. in } \mathbb{R}^n.
$$

(2.9)

Combining (2.6) and (2.9) we arrive at the pointwise estimate (2.3) as desired.

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let $\omega$ and $f$ be as in the theorem. Our strategy is to apply Schauder Fixed Point Theorem to the following closed and convex subset of $W^{1,q}_0(\Omega)$:

$$
E := \{ v \in W^{1,q}_0(\Omega) : G_1(|\nabla v|^q) \leq T G_1(|\mu|^q) + G_1(|\omega|^q) \} \quad \text{a.e.},
$$

where $T > 0$ is to be chosen. Note that by (2.8) we have

$$
G_1[|\omega|^q] \leq C[\omega] \frac{q+1}{M^q + q + q^2 + 1} G_1(|\omega|).
$$

Thus by [20, Theorems 1.1 and 1.2] (see also [26, Theorem 2.3]), from the definition of $E$ we obtain for any $v \in E$,

$$
[|\nabla v|^q]_{M^q + q + q^2 + 1} \leq C_0 T \left[ [\omega] \frac{q}{M^q} + [f] \frac{q}{M^q + q + q^2 + 1} \right]
$$

for a constant $C_0$ depends only on $n, p, \Lambda, q, \text{diam}(\Omega)$, and $\frac{\text{diam}(\Omega)}{R_0}$. Therefore, if we assume that

$$
[\omega] \frac{q}{M^q + q + q^2 + 1} + [f] \frac{q}{M^q + q + q^2 + 1} \leq c_0,
$$

where $c_0$ is to be determined, then we have for any $v \in E$,

$$
[|\nabla v|^q]_{M^q + q + q^2 + 1} \leq c_0 C_0 T.
$$

(2.10)

Let $S : E \to W^{1,q}_0(\Omega)$ be defined by $S(v) = u$, where $u \in W^{1,q}_0(\Omega)$ is the unique solution of

\[
\begin{cases}
- \text{div} A(x, \nabla u) = |\nabla v|^q + \omega + \text{div} f & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We claim that there are $T > 0$ and $c_0 > 0$ such that $S : E \to E$.

By Lemma 2.2 we may assume that

$$
G_1(|\nabla S(v)|^q) \leq C_1 \left[ G_1(|g|^q) \right]_{M^q + q + q^2 + 1} G_1(|\nabla v|^q) \quad \text{a.e. in } \mathbb{R}^n.
$$

(2.11)
where \( g = f - h_\omega \) and \( h_\omega \) is the gradient vector field associated to \( \omega \) as in the proof of Lemma 2.2. We next note from (2.7) that
\[
|g|^{\frac{q}{p-1}} \leq C_2(|f|^{\frac{q}{p-1}} + G_1(|\omega|)^{\frac{q}{p-1}}) \quad \text{a.e.} \tag{2.12}
\]
Moreover, in view of (2.10) we have
\[
||\nabla v||^{\frac{q}{p-1}} \leq (c_0 C_0 T)^{\frac{q}{p-1}} T G_1(|f|^{\frac{q}{p-1}} + G_1(|\omega|)^{\frac{q}{p-1}}). \tag{2.13}
\]
Combining (2.11), (2.12), and (2.13) yields
\[
G_1(|\nabla (\nabla v)|^q) \leq (\max\{C_1, C_2\} + 1)^2 (c_0 C_0 T)^{\frac{q}{p-1}} T (G_1(|f|^{\frac{q}{p-1}} + G_1(|\omega|)^{\frac{q}{p-1}}).
\]
We now choose \( T = 2(\max\{C_1, C_2\} + 1)^2 \) and then choose \( c_0 > 0 \) so that \( (c_0 C_0 T)^{\frac{q}{p-1}} T \leq 1 \). Then it follows that
\[
G_1(|\nabla (\nabla v)|^q) \leq T \left[ G_1(|f|^{\frac{q}{p-1}}) + G_1(|\omega|) \right],
\]
and thus \( S(v) \in E \) as desired.

We next show that the set \( S(E) \) is precompact in the strong topology of \( W^{1,q}_0(\Omega) \). Let \( u_k = S(v_k) \), where \( \{v_k\} \) is a sequence in \( E \). We have
\[
\left\{- \text{div} A(x, \nabla u_k) = |\nabla v_k|^q + \omega + \text{div} f \quad \text{in} \ \Omega, \right.
\]
\[
\left. u_k = 0 \quad \text{on} \ \partial \Omega. \right.
\]
As \( |\nabla v_k|^q + \omega + \text{div} f = \text{div}(f - h_\omega - h_{\nabla v_k}) \) in \( \mathcal{D}'(\Omega) \), where
\[
|h_\omega| + |h_{\nabla v_k}| \leq C G_1(|\omega| + |\nabla v_k|^q)
\]
\[
\leq C (G_1(|\omega|) + T G_1(|f|^{\frac{q}{p-1}} + G_1(|\omega|)^{\frac{q}{p-1}}) \]
\[
\leq C (G_1(|\omega|) + G_1(|\omega|)^{\frac{q}{p-1}}),
\]
we may apply Lemma 2.1 to see that \( |\nabla u_k|^q \) is a bounded and equi-integrable subset of \( L^1(\Omega) \).

On the other hand, by [6, Theorem 2.1] there exists a subsequence \( \{u_{k'}\} \) and a function \( u \in W^{1,q}_0(\Omega) \) such that
\[
\nabla u_{k'} \to \nabla u \quad \text{a.e. in} \ \Omega.
\]
Thus the Vitali Convergence Theorem yields that \( u_{k'} \to u \) in \( W^{1,q}_0(\Omega) \) as desired.

Similarly, by uniqueness we see that the map \( S \) is continuous on \( E \) (in the strong topology of \( W^{1,q}_0(\Omega) \)). Then by Schauder Fixed Point Theorem, \( S \) has a fixed point in \( E \), which gives a solution \( u \) to problem (1.10). This completes the proof of the theorem.

\[\Box\]

### 3 Proof of Theorem 1.6

For any nonnegative measure \( \nu \) we define
\[
P^\beta [\nu](x) = \left( \int_0^R \left( \frac{\nu(B(x))}{r^{n-1}} \right)^{\beta} \frac{dr}{T} \right)^{\frac{1}{\beta}}, \quad R = 2 \text{diam}(\Omega),
\]
where \( \beta = 1 \) if \( p > 2 - \frac{1}{n} \) and \( \beta \) is any number in \( (0, (p-1)n) \) if \( \frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n} \). For \( x > 0 \), we also let
\[
T^\nu [\nu](x) = d(x)^{-x} P^\beta [\nu](x) \chi_{\Omega}(x),
\]
where recall that \( d(x) \) is the distance from \( x \) to \( \partial \Omega \).

It is clear that if \( c_0 \) is a positive number for which (1.12) holds, then for any \( 0 < k \leq \frac{c_0}{nT} \)
\[
\|d^{-x}\|_{L^\infty(\Omega)} \leq C. \tag{3.1}
\]
On the other hand, note that for any $f \in L^{2n}(\Omega)$,
\[
\|P^R[|f|]\|_{L^{\infty}(\mathbb{R}^n)} \leq C(R, \beta)\|f\|^{\frac{1}{p}}_{L^{2n}(\Omega)}.
\]
Thus we have that, for any $0 < \kappa \leq \frac{c_0}{4^{n+1}}$,
\[
\|P^R[d(\cdot)^{-k}\chi(\cdot)]\|_{L^{\infty}(\mathbb{R}^n)} \leq C.
\] (3.2)

We now record the following result that was obtained in [24].

**Lemma 3.1.** Let $p > \frac{3n-2}{2n-1}$ and suppose that $\mu$ is finite signed measure in $\Omega$. If $u$ is a renormalized solution to
\[
\begin{cases}
-\text{div} A(x, \nabla u) = \mu & \text{in } \Omega,
\end{cases}
\]
and $\partial \Omega$ is sufficiently flat, then
\[
|\nabla u(x)| \leq C T[|\mu|](x)
\]
for a.e. $x \in \mathbb{R}^n$, where $|\nabla u(x)|$ is set to be zero outside $\Omega$.

We can now prove Theorem 1.6.

**Proof of Theorem 1.6.** Let $c_0$ be as in the theorem and suppose that $\text{supp}(\omega) \subset \Omega_{\delta_0}$. In this proof, we shall fix $\kappa \in (0, \frac{c_0}{4^{n+1}})$.

By [26, inequality (2.10)] and condition (1.13) we have
\[
P^R[|P^R[|\omega|]|^q](x) \leq C[\omega]^\frac{1}{p} \|P^R[|\omega|]\|_{L^{\infty}(\mathbb{R}^n)} \leq C(c_0) \frac{1}{p} \|P^R[|\omega|]\|_{L^{\infty}(\mathbb{R}^n)} (3.3)
\]
for a.e. $x \in \Omega$. Moreover, since $\text{supp}(\omega) \subset \Omega_{\frac{\delta_0}{p}}$, we also have
\[
\|P^R[|\omega|]\|_{L^{\infty}(\mathbb{R}^n) \setminus \Omega_{\frac{\delta_0}{p}}} \leq C(\delta_0, \beta, p, n, R)|\omega(\Omega)\|_{p}^{\frac{1}{p}},
\]
and thus by (3.2),
\[
P^R[|d(\cdot)^{-k}P^R[|\omega|\chi(\cdot)]|^q](x) \leq C(\delta_0)P^R[|P^R[|\omega|]|^q](x) + C(|\omega|(|\Omega|)^{\frac{1}{p}}).
\]
Combining this with (3.3) and condition (1.13), we find
\[
P^R[|d(\cdot)^{-k}P^R[|\omega|\chi(\cdot)]|^q](x) \leq C(\delta_0) \frac{1}{p} P^R[|\omega|](x) + C(|\omega|(|\Omega|)^{\frac{1}{p}})
\]
\[
\leq C(\delta_0) \frac{1}{p} P^R[|\omega|](x) + C(\delta_0) \frac{1}{p} |\omega(\Omega)|^{\frac{1}{p}}
\]
\[
\leq C(\delta_0) \frac{1}{p} + (c_0) \frac{1}{p} |\omega(\Omega)|^{\frac{1}{p}}
\]
for a.e. $x \in \Omega$. This gives
\[
T|T[|\omega|]|^q](x) \leq C(c_0) \frac{1}{p} + (c_0) \frac{1}{p} |\omega(\Omega)|^{\frac{1}{p}} (3.4)
\]
for a.e. $x \in \mathbb{R}^n$.

**Step 1.** In this step, we assume that $\omega \in C^0_c$ with $\text{supp}(\omega) \subset \Omega_{\delta_0}$. Let us set
\[
V = \{v \in W^{1,1}_0 : |\nabla v| \leq N T[|\omega|] \text{ a.e.}\},
\]
where $N$ is to be determined. Since $\omega \in C^0_c(\Omega)$, in view of (3.1) we have that
\[
|\nabla v(x)|^q \leq C(\omega) d(x)^{-qk} \in L^{2n}(\Omega),
\]
and in particular, $|\nabla v|^q \in W^{-1,\gamma\kappa}(\Omega)$ for any $v \in V$. 
We next define a map \( S : V \to W^{1,1}_0 \) by letting \( S(v) = u \), where \( v \in V \) and \( u \) is the unique renormalized solution to
\[
\begin{align*}
- \operatorname{div} A(x, \nabla u) &= |\nabla v|^q + \omega \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

By Lemma 3.1 and (3.4) we have
\[
|\nabla u|^p \leq T|\nabla v|^q + \omega
\]
\[
\leq C N^q T|\nabla \chi^q | + CT|\omega|
\]
\[
\leq C N^q T|\chi^q | + CT|\omega|.
\]
Thus if we choose \( N = 2C \) and \( c_0 \) sufficiently small, we obtain that \( S(V) \subset V \). Moreover, using the results of [9], it can be shown that \( S \) is continuous and compact (see also [23]). Thus by Schauder Fixed Point Theorem, there exists a solution \( u \in V \) to the equation (1.14).

Step 2. Let \( \omega_k = \rho_k \ast \omega \), where \( \{\rho_k\}_{k \in \mathbb{N}} \) is a standard sequence of mollifiers. Choose \( k \) sufficiently large so that \( \omega_k \in C^{0}\_c(\Omega_{\delta/2}) \) for all such \( k \). It is easy to see from condition (1.13) that
\[
\frac{\omega_k}{\nabla v} \leq AC_0,
\]
where \( A \) is independent of \( k \). Thus we may apply Step 1 with \( \omega = \omega_k \) to obtain a sequence of solutions \( \{u_k\} \subset V \) to the equation
\[
\begin{align*}
- \operatorname{div} A(x, \nabla u_k) &= |\nabla u_k|^q + \omega_k \quad \text{in } \Omega, \\
\quad u_k &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]
Then we apply the results of [9] to get a subsequence \( \{u_{k'}\} \) and function \( u \) such that \( \nabla u_{k'} \to \nabla u \in L^q(\Omega) \) and \( u \) is a renormalized solution of (1.14) (see also [23]).

\[\square\]

**Funding:** Quoc-Hung Nguyen is supported by the ShanghaiTech University startup fund. Nguyen Cong Phuc is supported in part by Simons Foundation, award number 426071.

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