The nondynamical r-matrix structure for the elliptic $A_{n-1}$ Calogero-Moser model
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Abstract

In this paper, we construct a new Lax operator for the elliptic $A_{n-1}$ Calogero-Moser model with general $n$ ($2 \leq n$) from the classical dynamical twisting, in which the corresponding r-matrix is purely numeric (nondynamical one). The nondynamical r-matrix structure of this Lax operator is obtained, which is elliptic $\mathbb{Z}_n$-symmetric r-matrix.

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1 Introduction

A general description of classical completely integrable models of $n$ one-dimensional particles with two-body interactions $V(q_i - q_j)$ was given in Ref.[24]. To each simple Lie algebra and choice of one of these type interactions, one can associate a classically completely integrable systems[5,13,23,24]. The most general form of the potential in such models is so-called elliptic Calogero-Moser(CM) model with an elliptic interacting potential. The various degenerations of this general system yield rational CM model (type I in Ref.[5]), hyperbolic CM model (type II in Ref.[5]) and trigonometric CM model (type III in Ref.[5]). So, the study of the elliptic CM model is of great importance in the completely integrable particle systems.

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The Lax pair representation (Lax representation) of a completely integrable system, which means that the equations in the problem can be formulated in a Lax form, is the most effective way to show its integrability and construct the complete set of integrals of the motion. The Lax representation and its corresponding r-matrix structure for rational, hyperbolic and trigonometric $A_{n-1}$ CM models were constructed by Avan et al\[5\]; The Lax representation for the elliptic CM models was constructed by Krichever\[22\] and the corresponding r-matrix structure was given by Sklyanin\[30\] and Braden et al\[10\]. There exists a specific feature that the r-matrix of the Lax representation for these models turn out to be a dynamical one (i.e. it depends upon the dynamical variables) and satisfy a dynamical Yang-Baxter equations\[6,10,11,30\]. Such structures also appear in the study Ruijsenaars-Schneider models, which is known as the relativistic Calogero-Moser models and can be related to the soliton systems of the affine Toda field theories \[11,12,27,28\]. Moreover, such a dynamical r-matrix structure is connected with the Hamiltonian reduction of the cotangent bundle of Lie algebra for Calogero-Moser model and the contangent bundle of Lie group for Ruijsenaars-schneider model \[2,16,31\]. This greatly promotes the study of the classical (and quantum) dynamical r-matrices (and R-matrix). A partial classification scheme has very recently been proposed for the dynamical r-matrix obeying the particular version of the dynamical Yang-Baxter equation\[3,14,29\]. However, at this time one lacks a general classifying scheme such as exists in the case of nondynamical classical r-matrices thanks to Belavin and Drinfeld\[9\].

The other difficulties presented by the dynamical aspect of the r-matrix also occur: I. the fundamental Poisson algebra of Lax operator, which structural constants are given by a dynamical r-matrix, is generally speaking no longer closed (cf. the nondynamical one); II. To solve the quantization problem and its geometrical interpretation is still an open problem\[30\]. So far, only for one particular case—the spin generalization of the CM model—a proper algebraic setting (the Gervais-Neveu-Felder equation) was found\[4\] which allows one to quantize the model. On the other hand, it is well-known that the Lax representation for a completely integrable models is not unique. The different Lax representation of an integrable system is conjugated each other (only for finite particles system, but for the field system it should be transferred as gauge transformation from each other). But the corresponding r-matrix should be transferred as a “gauge” transformation (see Eq. (5)) which is the classical dynamically twisted relations\[4\] between r-matrix. So, to overcome the above difficulties which caused by the dynamical r-matrix may be that whether another Lax representation for the CM model which has a nondynamical r-matrix structure could be found. The plan of our work is to find such a “good” Lax representation for the elliptic $A_{n-1}$ CM model if possible. In our former work\[18\], we succeeded in constructing a new Lax operator (cf. Krichever’s\[22\]) for the elliptic $A_{n-1}$ CM model with $n = 2$ and showing that the corresponding r-matrix is a nondynamical one which is the classical eight-vertex r-matrix\[20\]. In present paper, extending our
former work in Ref.[18], we construct a new Lax operator (cf. Krichever’s) for the elliptic $A_{n-1}$ CM model with the general $n(2 \leq N)$ which be a “good” one in sense that the corresponding r-matrix is nondynamical one—the classical $Z_n$-symmetric r-matrix.

The paper is organized as follows. In section 2, from the classical dynamical twisting , the condition that the “good” Lax representation could exist is found.In section 3, after some review of quantum $Z_n$-symmetry Belavin model, we construct the classical $Z_n$-symmetry r-matrix. After some review of Sklyanin’s work on elliptic CM model in section 4, we construct the “good” Lax representation for elliptic $A_{n-1}$ CM model which enjoys in nondynamical r-matrix structure in section 5. Finally, we give summary and discussions in section 6. Appendix contains some detailed calculations.

2 The dynamical twisting of classical r-matrix

In this paper, we only deal with the completely integrable finite particles systems. In this section we will review some general theories of the completely integrable finite particles systems

A Lax pair $(L,M)$ consists of two functions on the phase space of the system with values in some Lie algebra $g$, such that the evolution equations may be written in the following form

$$\frac{dL}{dt} = [L,M]$$

where $[,]$ denotes the bracket in the Lie algebra $g$. The interest in the existence of such a pair lies in the fact that it allows for an easy construction of conserved quantities (integrals of motion)— it follows that the adjoint-invariant quantities $trL^n$ are the integrals of the motion. In order to implement Liouville theorem onto this set of possible action variables we need them to be Poisson-commuting. As shown in Ref.[7], for the commutativity of the integrals $trL^n$ of the Lax operator it is necessary and sufficient that the fundamental Poisson bracket $\{L_1(u),L_2(v)\}$ could be represented in the commutator form

$$\{L_1(u),L_2(v)\} = [r_{12}(u,v),L_1(u)] - [r_{21}(v,u),L_2(v)]$$

where we use the notation

$$L_1 \equiv L \otimes 1 \quad , \quad L_2 \equiv 1 \otimes L \quad , \quad r_{21} = Pr_{12}P$$

and $P$ is the permutation operator such that $Px \otimes y = y \otimes x$.

Generally speaking, r-matrix $r_{12}(u,v)$ does depend on dynamical variables. For some special case of $r_{12}(u,v)$ independent on dynamical variables, the r-matrix is called as the nondynamical r-matrix which has been well studied[15]. In contrast with the well-studied case of the nondynamical r-matrix, no general theory of the dynamical r-matrix exists at the moment, apart few concrete examples and observations.
Still, the collection of examples is rather sparse, and any new example of dynamical r-matrix could possibly contribute to better understanding of their algebraic and geometric nature.

The Poisson bracket structure Eq.(2) obeys a Jacobi identity which implies an algebraic constraint for the r-matrix. Since r-matrix may depend on dynamical variables this constraint takes a complicated form

\[
[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cycl.perm} = 0
\]

Relevant particular cases of this general identity, one can obtain the classical Yang-Baxter equation for the nondynamical r-matrix and the classical dynamical Yang-Baxter equation [3,7] for the dynamical one.

It should be remarked that such a classification is by no means unique, which drastically depend on the Lax representation that one choose for a system. Namely, there is no one-to-one correspondence between a given dynamical system and a defined r-matrix, a same dynamical system may have several Lax representations and several r-matrix. The different Lax representation of a system is conjugated each other: if \((\widetilde{L}, \widetilde{M})\) is one of other Lax pair of the same dynamical system conjugated with the old one \((L, M)\), it means that

\[
\frac{d\widetilde{L}}{dt} = [\widetilde{L}, \widetilde{M}]
\]

\[
\widetilde{L}(u) = g(u)L(u)g^{-1}(u) , \quad \widetilde{M}(u) = g(u)M(u)g^{-1}(u) - \left(\frac{d}{dt}g(u)\right)g^{-1}(u)
\]

where \(g(u) \in G\) whose Lie algebra is \(g\). Then, we have

**Proposition 1.** The Lax pair \((\widetilde{L}, \widetilde{M})\) has the following r-matrix structure

\[
[\widetilde{L}_1 (u), \widetilde{L}_2 (v)] = [\tilde{r}_{12} (u,v), \widetilde{L}_1 (u)] - [\tilde{r}_{21} (v,u), \widetilde{L}_2 (v)]
\]

where

\[
\tilde{r}_{12} (u,v) = g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v)
\]

\[
+ \frac{1}{2}\{(g_1(u), g_2(v))g_1^{-1}(u)g_2^{-1}(v) , \quad g_2(v)L_2(v)g_2^{-1}(v)\}
\]

**Proof:** The proof is direct substituting Eq.(4) and Eq.(4a) into the fundamental Poisson bracket and use the following identity

\[
[[s_{12}, L_1], L_2] = [[s_{12}, L_2], L_1]
\]
where \( s_{12} \) is any matrix on \( g \otimes g \).

It can be seen that: I. The Lax operator \( L \) is transferred as a similarity transformation from the different Lax representation (only for finite particles system); II. The corresponding \( M \) is undergone the usual gauge transformation; III. The r-matrix is transferred as some generalized gauge transformation, which can be considered as the classical version of the dynamically twisted relation between the quantum R-matrix[4]. Therefore, it is of great value to find a “good” Lax representation for a system if it exists, in which the corresponding r-matrix is nondynamical one and the well-studied theories[4,15] can be directly applied in the system—such as the dressing transformation, quantization......

**Corollary to proposition 1.** For given Lax pair \((L, M)\) and the corresponding r-matrix, if there exist such a \( g \) that

\[
h_{12} = \{g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u),L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) + \frac{1}{2}\{[g_1(u),g_2(v)]g_1^{-1}(u)g_2^{-1}(v),g_2(v)L_2(v)g_2^{-1}(v)\}\}
\]

and \( \partial_\theta h_{12} = \partial_p h_{12} = 0 \)

the nondynamical Lax representation of the system exist.

From the straightforward calculation, we also have

**Proposition 2.** The twisted Lax pair \((\tilde{L}, \tilde{M})\) and the corresponding r-matrix \( \tilde{r}_{12} \) satisfies

\[
[\tilde{L}_1, [\tilde{r}_{12}, \tilde{r}_{13}]] + [\tilde{r}_{12}, \tilde{r}_{23}] + [\tilde{r}_{32}, \tilde{r}_{13}] + \{\tilde{L}_2, \tilde{r}_{13}\} - \{\tilde{L}_3, \tilde{r}_{12}\} + cycl.\ perm = 0
\]

The main purpose of this paper is to find a “good” Lax representation for the elliptic \( A_{n-1} \) CM model.

### 3 The elliptic function and elliptic \( Z_n \)-symmetric R-matrix and r-matrix

We first briefly review the elliptic \( Z_n \)-symmetric quantum R-matrix which is related to \( Z_n \)-symmetric Belavin model[8,19,21,25]. For \( n \in Z_+, 2 \leq n \leq \), we define \( n \times n \) matrices \( h, g, I \), as

\[
h_{ij} = \delta_{i+1,j}\mod n , \quad g_{ij} = \omega^{i} \delta_{i,j} , \quad I_{\alpha_1,\alpha_2} \equiv I_{\alpha} = g^{\alpha_2}h^{\alpha_1}
\]

where \( \alpha_1,\alpha_2 \in Z_n \) and \( \omega = exp(2\pi \sqrt{-1}/n) \). We also define some elliptic functions

\[
\theta^{(j)}(u) = \theta \left[ \frac{\frac{1}{2} - \frac{j}{n}}{\frac{1}{2}} \right] (u, n\tau) , \quad \sigma(u) = \theta \left[ \frac{\frac{1}{2}}{\frac{1}{2}} \right] (u, \tau)
\]
here

A

τ

where

s

is to make Eq.(15) satisfied. The R-matrix satisfy quantum Yang-Baxter equation (QYBE)

Moreover, the R-matrix enjoys in following

Z

as

related to the coupling constant of elliptic

w

equation (also called QYBE)

Introduce an

n

Set an

n

(1)

−

u, q

1

face model [21] in Ref. [19]. Construct the operator

A

im

⊗

correspond to the interwiner function

j

i

matrix

(12)

if

i + j = l + k \mod n

otherwise

where w is a complex number which is called as the crossing parameter of the R-matrix. We should remark that our R-matrix coincide with the usual one [19, 21] up to a scalar factor

\frac{\theta''(u)v\sigma(u)\theta''(v)\sigma(v)}{\sigma(u)\sigma(v)\theta''(u)v\sigma(v)\theta''(v)\sigma(u)} \Pi_{j=1}^{n-1} \frac{\theta''(u)\sigma(u)}{\sigma(u)\theta''(v)\sigma(v)},

which is to make Eq.(15) satisfied. The R-matrix satisfy quantum Yang-Baxter equation (QYBE)

\begin{align}
R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2)
\end{align}

Moreover, the R-matrix enjoys in following

Z_n \otimes Z_n symmetric properties

\begin{align}
R_{12}(v) = (a \otimes a)R_{12}(v)(a \otimes a)^{-1} \quad \text{for} \quad a = g, h
\end{align}

Introduce an \( n \otimes n \) matrix \( \hat{T}(v) \), with the matrix element \( \hat{T}(v)_{ij} \) being operators, which satisfy the equation (also called QYBE)

\begin{align}
R_{12}(v_1 - v_2)\hat{T}_1(v_1)\hat{T}_2(v_2) = \hat{T}_2(v_2)\hat{T}_1(v_1)R_{12}(v_1 - v_2)
\end{align}

We next turn to the factorized difference representation for the operator \( \hat{T}(v) \)[17, 19, 25]

Set an \( n \otimes n \) matrix \( A(u; q) \)

\begin{align}
A(u; q)_j^i \equiv A(u; q_1, q_2, \ldots, q_n)_j^i = \theta^{(i)}(u + nq_j - \sum_{k=1}^{n} q_k + \frac{n-1}{2})
\end{align}

here \( A(u, q)_j^i \) correspond to the interwiner function \( \varphi_j^{(i)} \) between the \( Z_n \)-symmetric Belavin R-matrix and the \( A^{(i)}_{n-1} \) face model [21] in Ref. [19]. Construct the operator \( \hat{T}(u) \)

\begin{align}
\hat{T}(u)_j^i = A(u + sw; q)_k^i A^{-1}(u; q)_j^k D_k
\end{align}

where \( s \) is a complex number which associate with the representation of Sklyanin algebra [19] and will be related to the coupling constant of elliptic \( A_{n-1} \) CM model Eq.(36), and \( D_k \) is a difference operator such as

\begin{align}
D_k f(q) \equiv D_k f(q_1, q_2, \ldots, q_n) = f(q_1, \ldots, q_{k-1}, q_k - w, q_{k+1}, \ldots, q_n)
\end{align}
Then following the results in Ref.[16], we have

**Theorem 1.** ([16],[29],[30]) The L-operator $\hat{T}(u)$ defined in Eq.(14a) satisfies the QYBE Eq.(13).

We can define a corresponding $Z_n$-symmetric (classical) r-matrix which has the following relation with the R-matrix

$$R_{12}(v)|_{w=0} = 1 \otimes 1$$
$$R_{12}(v) = 1 \otimes 1 + wr_{12}(v) + O(w^2) \quad \text{when the crossing parameter } w \to 0 \quad (15)$$

Then we have

**Proposition 3.** The corresponding elliptic $Z_n$-symmetric r-matrix is

$$r_{ij}^{kl}(v) = \begin{cases} (1 - \delta_i^j) \left\{ \theta'(0)(v) \theta(i-j)(v) + \delta_i^j \left( \theta'(i-j)(v) - \frac{\sigma'(v)}{\sigma(v)} \right) \right\} & \text{if } i + j = l + k \mod n \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

and satisfy the nondynamical (classical) Yang-Baxter equation and antisymmetric properties

$$[r_{12}(v_1 - v_2), r_{13}(v_1 - v_3)] + [r_{12}(v_1 - v_2), r_{23}(v_2 - v_3)] + [r_{13}(v_1 - v_3), r_{23}(v_2 - v_3)] = 0 \quad (17)$$
$$-r_{21}(-v) = r_{12}(v) \quad (18)$$

**Proof:** When $w \to 0$, we have following asymptotic properties

$$\sigma(w) = w\sigma'(0) + 0(w^3) \quad , \quad \theta'(0)(w) = w\theta'(0)(0) + 0(w^3)$$
$$\theta(i)(w) = \theta(i)(0) + w\theta(i)(0) + 0(w^2) \quad , \quad i \neq 0 \mod n$$

Then one have ,when $w \to 0$

$$\frac{\theta'(0)(0)\sigma(v)}{\theta'(0)(v)} \prod_{m=1}^{n-1} \frac{\theta(m)(v)}{\theta(m)(0)} P_{ij}^{kl} = w(1 - \delta_i^j) \frac{\sigma'(0)}{\sigma(v)} \prod_{m=1}^{n-1} \frac{\theta(m)(v)}{\theta(m)(0)} \frac{\theta(i-j)(v)}{\theta(i-j)(0)} + 0(w^2)$$
$$+ \delta_i^j \frac{\sigma'(0) + 0(w^2)}{\sigma(v) + w\sigma'(0) + 0(w^2)} \prod_{m=1}^{n-1} \frac{\theta(m)(v)}{\theta(m)(0)} \frac{\theta(i-j)(v)}{\theta(i-j)(0)} + 0(w^2)$$
$$= \prod_{m=1}^{n-1} \frac{\theta(m)(v)}{\theta(m)(0)} \frac{\sigma'(0)\theta'(0)(v)}{\theta'(0)(0)\sigma(v)} \delta_i^j \delta_k^l$$
$$+ w \prod_{m=1}^{n-1} \frac{\theta(m)(v)}{\theta(m)(0)} \left\{ (1 - \delta_i^j) \frac{\sigma'(0)\theta'(0)(v)\theta(i-j)(v)}{\sigma(v)\theta'(0)(v)\theta(i-j)(0)} + \delta_i^j \delta_k^l \frac{\sigma'(0)\theta'(0)(v)}{\sigma(v)\theta'(0)(0)} \right\} + 0(w^2)$$
By the definition of classical r-matrix from the quantum one Eq.(15), we have Eq.(16). The classical Yang-Baxter equation Eq.(17) is the direct results of the QYBE and the asymptotic properties Eq.(18). The antisymmetric properties of the r-matrix can be derived from the following relations between the \( \theta \)-functions

\[
\theta^{(\alpha)}(v) = -e^{2\sqrt{-1}\pi \alpha \theta^{(-\alpha)}(-v)} , \quad \frac{\theta^{(\alpha)}(v)}{\theta^{(\alpha)}(v)} = -\frac{\theta^{(-\alpha)}(-v)}{\theta^{(-\alpha)}(-v)} \quad \square
\]

One can also check that the classical r-matrix \( r_{12}(u) \) enjoys in the \( Z_n \otimes Z_n \)-symmetric

\[
r_{12}(v) = (a \otimes a) r_{12}(v)(a \otimes a)^{-1} , \quad \text{for} \quad a = g, h
\]

4 Review of the elliptic \( A_{n-1} \) CM model

The elliptic \( A_{n-1} \) CM model is a system of \( n \) one-dimensional particles interaction by the two-body potential

\[
V(q_{ij}) = \gamma Q(q_{ij}) , \quad q_{ij} = q_i - q_j , \quad i, j = 1, ..., n
\]

\[
Q(v) - Q(u) = E(u, v)E(u, -v)
\]

where \( \gamma \) is the coupling constant, \( Q(u) \) is a Weierstrass function and the elliptic function \( E(u, v) \) is defined in Eq.(9). In terms of the canonical variables \( \{ p_i, q_j \} \) \( (i, j = 1, ..., n) \) with the canonical Poisson bracket

\[
\{ p_i, p_j \} = \{ q_i, q_j \} = 0 , \quad \{ p_i, q_j \} = \delta_{ij} , \quad i, j, k, l = 1, ..., n
\]

the Hamiltonian of the system is expressed as

\[
H = \sum_{i=1}^{n} p_i^2 + \sum_{i \neq j} V(q_{ij})
\]

The above Hamiltonian with the potential Eq.(19) is known to be completely integrable[13,22,23,24]. The most effective way to show its integrability is to construct the Lax representation for the system. One Lax pair \((L, M)\) was first found by Krichever[22]. The Lax operator (or L-operator) of Krichever is

\[
L_j^i(u) = p_i \delta_j^i + (1 - \delta_j^i) \sqrt{\gamma} E(u, q_{ji})
\]

where \( u \) is spectra parameter and the motion equation can be rewritten in the Lax form

\[
\frac{d}{dt} L(u) = \{ L(u), H \} = [L(u), M(u)]
\]
The Hamiltonian defined in Eq. (22) can be rewritten in terms of the Poisson-commuting family \( \{ \text{tr} L^l(u) \} \) \( (l = 1, \ldots, n) \), which forms of enough independent integrals

\[
H = \text{tr}(L^2(u)) + V(u)
\]  

(24)

\( V(u) \) does not depend upon the dynamical variables and the identity Eq. (20) is used. The r-matrix structure of this Lax operator was given by Sklyanin[30] and Braden et al[10]. The fundamental Poisson bracket of the Lax operator can be described in terms of r-matrix form[30]

\[
\{ L_1(u), L_2(v) \} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)]
\]  

(25)

and the dynamical r-matrix \( r_{12}(u, v) \) is

\[
r_{12}(u, v) = a \sum_{i=1}^n E^i_{ii} + \sum_{i \neq j} c_{ij} E^i_{jj} + \sum_{i \neq j} d_{ij} (E^i_{ij} + E^j_{ji})
\]  

(26)

where

\[
E^i_{jk} = E^i_j \otimes E^k_j, \quad a = r^i_{ii} = -\xi(u - v) - \xi(v)
\]  

(27)

\[
c_{ij} = r^i_{ji} = \sqrt{\gamma} E(u - v, q_{ij}), \quad d_{ij} = r^i_{ij} = r^j_{ij} = 1/2 \sqrt{\gamma} E(v, q_{ij})
\]  

(28)

where the elliptic function \( \xi(u) \) is defined in Eq. (8). Sklyanin also shown that the dynamical r-matrix \( r_{12}(u, v) \) defined in Eq. (26) satisfies the dynamical Yang-Baxter equation (or generalized Yang-Baxter equation)[30]

\[
[R^{(123)}, L_1] + [R^{(231)}, L_2] + [R^{(321)}, L_3] = 0
\]  

(29)

where

\[
R^{(123)} = r_{(123)} - \{ r_{13}, L_2 \} + \{ r_{12}, L_3 \}
\]

and

\[
r_{(123)} = [r_{12}, r_{13}] + [r_{12}, r_{23}] - [r_{13}, r_{23}]
\]

The Jacobi identity of the fundamental Poisson bracket is the results of Eq. (29). Due to the dynamical properties of the r-matrix \( r_{12}(u, v) \), the Poisson bracket of L-operator is no longer closed. The quantum version of Eq. (29) and the generalized (dynamical) Yang-Baxter equation is still not found except the spin generalization of CM model in which the Gervais-Neveu-Felder equation was found[4,30].
5 The “good” Lax representation of elliptic $A_{n-1}$ CM model and its r-matrix

The L-operator of the elliptic $A_{n-1}$ CM model given by Krichever in Eq.(23) and corresponding r-matrix $r_{12}(u,v)$ given by Sklyanin in Eq.(26) leads to some difficulties[30] in the investigation of CM model. This motivate us to find a “good” Lax representation of the CM model. As see from the proposition 1 and its corollary in section 3, this means to find such a $g(u)$ in Eq.(4) which satisfies Eq.(6). Fortunately, we could find such a $g(u)$, from which we construct a new L-operator $\tilde{L}(u)$ of the elliptic $A_{n-1}$ CM model. (This kind L-operator does not always exist for general completely integrable system). The corresponding r-matrix of $\tilde{L}(u)$ is purely numeric one, and is equal to classical $Z_n$-symmetric r-matrix In order to compare with the L-operator given by Krichever, we call this L-operator found by us as the new Lax operator.

Define

$$g(u) = A(u; q)\Lambda(q)\ , \quad \Lambda(q)^i_j = h_i(q)\delta^i_j$$

$$h_j(q) \equiv h_j(q_1, ..., q_n) = \frac{1}{\prod_{i \neq j} \sigma(q_{il})}$$

where $A(u; q)^i_j$ is defined in Eq.(14). Let us construct the new L-operator $\tilde{L}(u)$

$$\tilde{L}(u) = g(u)L(u)g^{-1}(u)$$

$$\tilde{M}(u) = g(u)M(u)g^{-1}(u) - \left(\frac{d}{dt}g(u)\right)g^{-1}(u)$$

Then we

**Proposition 4.** The fundamental Poisson bracket of L-operator $\tilde{L}(u)$ can be written in the usual Poisson-Lie form with a purely numerical r-matrix

$$\{\tilde{L}_1(u), \tilde{L}_2(v)\} = [\tilde{r}_{12}(u-v), \tilde{L}_1(u) + \tilde{L}_2(v)]$$

and the corresponding r-matrix $\tilde{r}_{12}(u)$ is a nondynamical one—$Z_n$-symmetric r-matrix defined in Eq.(16).

**Proof:** The proof is shifted to appendix A. The most important properties of this new Lax operator is that the corresponding r-matrix does not depend upon the dynamical variables. Consequently, the well-studied theory for the nondynamical system[15] can be used to study the elliptic $A_{n-1}$ CM model.
The $Z_n$-symmetric r-matrix $\tilde{r}_{12}(u)$ can also be obtained from the classical dynamical twisting as follows

$$
\tilde{r}_{12}(u,v) = g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u),L_2(v)\}g_1^{-1}(u)g_2^{-1}(v)
$$  \(33\)

up to some matrix which commute with $L_1(u) + L_2(v)$.

The standard Poisson-Lie bracket Eq.(32) of L-operator $\tilde{L}(u)$ and the numerical r-matrix $\tilde{r}_{12}(u)$ enjoying in the classical Yang-Baxter equation Eq.(17) and antisymmetry Eq.(18), make it possible to construct the quantum theory of the elliptic $A_{n-1}$ CM model. Moreover, the numerical r-matrix $\tilde{r}_{12}(u)$ could provide a mean to construct a seperation of variables for the elliptic $A_{n-1}$ CM model in the same manner as that in the case of the integrable magnetic chain[30]. It also make it possible that one can construct the dressing transformation for the model. The dressing group of this system would be an analogue to the semi-classical limit of $Z_n$ Sklyanin algebra.

6 Discussions

In this paper, we only construct the nondynamical r-matrix structure for the elliptic $A_{n-1}$ Calogero-Moser model. Such a “good” Lax representation for the degenerated case—rational, trigonometric and hyperbolic CM model also could be constructed. It is also very interesting to construct such a classical dynamical twisting for the Ruijsenaars-Schneider model.

Appendix

Appendix A. The proof of Proposition 4

In this appendix we give the proof the proposition 4 which is main results of our paper.

Set the classical L-operator $T(u)$ as follow

$$
T(u)_j^i = \sum_k A(u;q)_j^k A^{-1}(u;q)_i^k p_k - s(\partial_u A(u;q))_j^k A^{-1}(u;q)_i^k
$$

where $\{p_k\}$ is the classical moment which is conjugated with $\{q_k\}$ and $\{p_i,q_j\}$ satisfies the canonical Poisson bracket Eq.(21).

Lemma 1. The classical operator $T(u)$ has the standard Poisson-Lie bracket

$$
\{T_1(u),T_2(v)\} = [\tilde{r}_{12}(u-v),T_1(u)+T_2(v)]
$$  \(34\)

where r-matrix $\tilde{r}_{12}(u)$ is the $Z_n$-symmetric r-matrix defined in Eq.(16).
Proof: When \( w \to 0 \), the difference quantum L-operator \( \hat{T}(u) \) has the following asymptotic properties

\[
\hat{T}(u)_j^i = \sum_k A(u; q)_k^i A^{-1}(u; q)_j^k - w \sum_k A(u; q)_k^i A^{-1}(u; q)_j^k \frac{\partial}{\partial q_k} + w \sum_k (\partial_u A(u; q))_j^k A^{-1}(u; q)_j^k + 0(w^2)
\]

\[
\equiv \delta_j^i - w\hat{T}^{(1)}(u)_j^i + 0(w^2)
\]

where

\[
\hat{T}(u)_j^i = \sum_k A(u; q)_k^i A^{-1}(u; q)_j^k \frac{\partial}{\partial q_k} - s \sum_k (\partial_u A(u; q))_k^i A^{-1}(u; q)_j^k
\]

From the QYBE Eq.(13), we have

\[
[\hat{T}_1^{(1)}(u), \hat{T}_2^{(1)}(v)] = \hat{r}_{12} (u - v), \hat{T}_1^{(1)}(u) + \hat{T}_2^{(1)}(v)
\]

If we use \( p_k \) instead of the differential \( \frac{\partial}{\partial q_k} \) and the classical L-operator \( T(u) \) instead of \( \hat{T}^{(1)}(u) \), we have the Eq.(34).

\[ \square \]

Lemma 2. The \( T(u) \) can be explicitly written as follows

\[
T(u)_j^i = \sum_{k,k'} \tilde{g}(u; q)_k^i T(u)_k^{i'} \tilde{g}^{-1}(u)_j^{i'} \equiv \{(A(u; q)\Lambda(q))T(u)(A(u; q)\Lambda(q))^{-1}\}_j^i \quad (35)
\]

where \( T(u) \) and \( \Lambda(q) \) are

\[
T(u)_j^i = (p_i - \frac{\partial}{\partial q_k} \ln \Delta^{\hat{z}}(q)) \delta_j^i + \sqrt{\gamma(1 - \delta_j^i)} E(u; q_{ji})
\]

\[
\Delta(q) = \prod_{i<j} \sigma(q_{ij}) \quad \text{coupling constant} \quad \gamma = (\frac{sn(0)}{n})^2 \quad \Lambda(q)_j^i = \frac{1}{\prod_{i\neq i} \sigma(q_{ii})} \delta_j^i \quad (36)
\]

Proof: In order to calculate the matrix element \( (\partial_u (A(u; q))) A^{-1}(u; q) \), we first consider

\[
A(u + w; q)A^{-1}(u; q) = A(u; q)[A^{-1}(u; q)A(u + w; q)]A^{-1}(u; q)
\]

From the definition of \( A(u; q)^i_j \) Eq.(14) and the determinat formula of Vandermonde type

\[
det[\theta^{(j)}(u_k)] = \text{const.} \times \sigma(\frac{1}{n} \sum_k u_k - \frac{n-1}{2}) \prod_{1 \leq j < k \leq n} \sigma(u_k - u_j) \sigma(u_k - u_j)
\]

where the \text{const.} does not depend upon \( \{u_k\} \), we have

\[
[A^{-1}(u; q)A(u + w; q)]_j^i = \sum_k A^{-1}(u; q)_k A(u + w; q)_j^k = \frac{\sigma(\frac{w}{n} + u + q_{ji})}{\sigma(u)} \prod_{k \neq i} \frac{\sigma(\frac{w}{n} + q_{jk})}{\sigma(q_{ik})}
\]
Lemma 3. The map defined in Eq.(37) is a Poisson map [1] (or a canonical transformation).

Then, we have

\[
(A^{-1}(u; q) \partial_u A(u; q))_j^i = \frac{\partial}{\partial w} \left\{ \frac{\sigma(u + q_j)}{\sigma(u)} \prod_{k \neq i} \frac{\sigma(u + q_k)}{\sigma(q_{ik})} \right\} \bigg|_{w=0}
\]

\[
= \frac{1}{n} \left( \frac{\sigma'(u)}{\sigma(u)} \delta_j^i + \frac{\sigma(u + q_j)}{\sigma(u)} \left( \delta_j^i \sum_{k \neq i} \frac{\sigma'(q_{ik})}{\sigma(q_{ik})} + (1 - \delta_j^i)\sigma'(0) \prod_{k \neq j, k \neq i} \frac{\sigma(q_{jk})}{\sigma(q_{ik})} \right) \right)
\]

\[
= \frac{1}{n} \left( \frac{\sigma'(u)}{\sigma(u)} + \sum_{k \neq i} \frac{\sigma'(q_{ik})}{\sigma(q_{ik})} \right) \delta_j^i \delta_{jk} + (1 - \delta_j^i)\sigma'(0) \prod_{k \neq j, k \neq i} \frac{\sigma(q_{jk})}{\sigma(q_{ik})} \right)
\]

\[
= \frac{1}{n} \left( \frac{\sigma'(u)}{\sigma(u)} + \frac{\partial}{\partial q_j} (ln \Delta(q)) \right) \delta_j^i - (1 - \delta_j^i)(-\sigma'(0)) E(u; q_j) \prod_{k \neq j, k \neq i} \frac{\sigma(q_{jk})}{\sigma(q_{ik})} \right)
\]

\[
= \prod_{k \neq i} \frac{1}{\sigma(q_{ik})} \left( \frac{\sigma'(u)}{n\sigma(u)} + \frac{\partial}{\partial q_j} (ln \Delta^\pm(q)) \right) \delta_j^i - (1 - \delta_j^i)(-\sigma'(0)) E(u; q_j) \prod_{k \neq j, k \neq i} \frac{\sigma(q_{jk})}{\sigma(q_{ik})} \right)
\]

Substituting \(A^{-1}(u; q) \partial_u A(u; q)\) into the definition of \(T(u)\), we have

\[
T(u)_j^i = \sum_k A(u; q)_k^i p_k A^{-1}(u; q)_k^j - s \sum_{k, k'} A(u; q)_k^i \left( A^{-1}(u; q) \partial_u A(u; q) \right)_k^{k'} A^{-1}(u; q)_j^{k'}
\]

\[
= (A(u; q) \Lambda(q))_j^i \left\{ (p_k - \frac{s\sigma'(u)}{n\sigma(u)} + \frac{\partial}{\partial q_j} (ln \Delta^\pm(q)) \right) \delta_k^j + (1 - \delta_k^j)(\sqrt{\sigma E(u; q_j)}) (\Lambda(q)^{-1}) A(u; q)^{-1})_j^{k'}
\]

We consider a map:

\[
\begin{cases}
  p_i & \rightarrow p_i - \frac{\partial}{\partial q_i} (ln \Delta^\pm(q)) \\
  q_i & \rightarrow q_i
\end{cases}
\]

(37)

Lemma 3. The map defined in Eq.(37) is a Poisson map [1] (or a canonical transformation).

**Proof:** The Lemma 3 can be proven from considering the symplectic two-form

\[
\sum_i d(p_i - \frac{\partial}{\partial q_i} (ln \Delta^\pm(q))) \wedge dq_i = \sum_i dp_i \wedge dq_i + \sum_{ij} \left( \frac{\partial^2}{\partial q_i \partial q_j} (ln \Delta^\pm(q)) \right) dq_i \wedge dq_j
\]

\[
= \sum_i dp_i \wedge dq_i \quad \square
\]

Since the Poisson bracket is invariant under the Poisson map [1], we could have proposition 4 from Lemma 1 and Lemma 3.
References

1. Arnold, V.I. : Mathematical methods of classical mechanics, Springer Verlag (1978).

2. Arutyunov, G.E., Frolov, S.A., Medredev, P.B. : Elliptic Ruijsenaars-Schneider model via the Poisson reducution of the affine Heisenberg Double, hep-th/9607177 (1996); Elliptic Ruijsenaars-Schneider model form the cotangent bundle over the two-dimensional current group, hep-th/9608013 (1996).

3. Avan, J. : Classical dynamical r-matrices for Calogero-Moser systems and their generalizations,

4. Avan, J., Babelon, O., Billey, E.: The Gervais-Neveu-Felder equation and the quantum Calogero-Moser systems, hep-th/9505091 (1995) Comm. Math. Phys. 178, 281 (1996).

5. Avan, J., Talon, T.: Phys. Lett. B 303, 33 (1993).

6. Babelon, O., Bernard, D.: Phys. Lett. B 317, 363 (1993).

7. Babelon, O., Viallet, C.M.: Phys. Lett. B237, 411 (1989).

8. Belavin, A.A.: Nucl. Phys. B180, 189 (1981).

9. Belavin, A.A., Drinfeld, V.G.: Triangle equation and simple Lie algebras, Soviet Sci. reviews, Sect. C 4, 93 (1984).

10. Braden, H.W., Suzaki, T.: Lett. Math. Phys. Vol. 30, 147 (1994).

11. Braden, H.W., Andrew, N.W.Hone: Affine Toda solitons and systems of Calogero-Moser type, hep-th/9603178 (1996).

12. Braden, H.W., Corrigan, E., Dorey, P.E., Sasaki, R.: Nucl. Phys. B338, 689 (1990); Nucl. Phys. B356, 469 (1991).

13. Calogero, F.: Lett. Nuovo. Cim. 13, 411 (1975); Lett. Nuovo. Cim. 16, 77 (1976).

14. Etingof, P., Varchenko, A.: Geometry and classification of solutions of the dynamical Yang-Baxter equation, q-alg/9703040.

15. Faddeev, L.D., Takhtajan, L.: Hamiltonian methods in the theory of solitons, Springer Verlag (1987).

16. Gorsky, A., Nekrasov, N.: Nucl. Phys. B414, 213 (1994); Nucl. Phys. B436, 582 (1995).

17. Hasegawa, K.: Ruijsenaars’ commuting difference operators as commuting transfer matrices, q-alg/9512029. Jour. Math. Phys. 35, 6158 (1994).
18. Hou, B.Y., Yang, W.L.: The nondynamical r-matrix structure of the elliptic Calogero-Moser model (n=2), Preprint: IMPNWU-960810.

19. Hou, B.Y., Shi, K.J., Yang, Z.X.: J. Phys. A 26, 4951 (1993).

20. Hou, B.Y., Wei, H.: J. Math. Phys. 30, 2750 (1989).

21. Jimbo, M., Miwa, T., Okado, M.: Nucl. Phys. B300, 74 (1988).

22. Krichever, I.M.: Func. Annal. Appl. 14, 282 (1980).

23. Moser, J.: Adv. Math. 16, 1 (1975).

24. Olshanetsky, M.A., Perelomov, A.M.: Phys. Rep. 71, 313 (1981).

25. Quano, Y.H., Fujii, A.: Mod. Phys. Lett. A6, 3635 (1991).

26. Richey, M.P., Tracy, C.A.: J. Stat. Phys. bf 42, 311 (1986); Tracy, C.A.: Physica D16, 203 (1985).

27. Ruijsenaars, S.N.M., Schneider, H.: Ann. Phys. Vol.170, 370 (1986).

28. Ruijsenaars, S.N.M.: Comm. Math. Phys. Vol.115, 127 (1988).

29. Schiffman, O.: On classification of dynamical r-matrices, q-alg/9706014 (1997).

30. Sklyanin, E.K.: Funct. Anal. Appl. 16, 263 (1982); Funct. Anal. Appl. 17, 320 (1983); Comm. Math. Phys. Vol.150, 181 (1992); Dynamical r-matrices for the elliptic Calogero-Moser model, hep-th/9308060 (1993).

31. Suris, Yuri B.: Why are the rational and hyperbolic Ruijsenaars-Schneider hierarchies governed by the the same R-operators as the Calogero-Moser ones, hep-th/9602164 (1996).