The lattice point discrepancy of a body of revolution: Improving the lower bound by Soundararajan’s method

Manfred Kühleitner and Werner Georg Nowak

Abstract. For a convex body $B$ in $\mathbb{R}^3$ which is invariant under rotations around one coordinate axis and has a smooth boundary of bounded nonzero curvature, the lattice point discrepancy $P_B(t)$ (number of integer points minus volume) of a linearly dilated copy $\sqrt{t}B$ is estimated from below. On the basis of a recent method of K. Soundararajan [16] an $\Omega$-bound is obtained that improves upon all earlier results of this kind.

1. Introduction. We consider a compact convex body $B$ in $\mathbb{R}^3$ which contains the origin as an inner point and assume that its boundary $\partial B$ is a $C^\infty$ surface$^{(1)}$ with bounded nonzero Gaussian curvature throughout. For a large real parameter $t$, we consider a linearly dilated copy $\sqrt{t}B$ of $B$, and in particular its lattice point discrepancy

$$P_B(t) := \# (\sqrt{t}B \cap \mathbb{Z}^3) - \operatorname{vol}(B)t^{3/2}. \quad (1.1)$$

There is a rich and very classic theory dealing with the estimation of such quantities $P_B(t)$, both in arbitrary dimensions and for very special cases. An enlightening survey can be found in E. Krätzel’s monographs [8] and [9] which have to be supplemented by M. Huxley’s book [7] where he exposed his breakthrough in planar lattice point theory (Discrete Hardy-Littlewood method).

For our specific setting stated above, the sharpest results read

$$P_B(t) = O(t^{63/86+\varepsilon}) \quad (1.2)$$

and$^{(2)}$

$$P_B(t) = \Omega_-(t^{1/2}(\log t)^{1/3}). \quad (1.3)$$

These are due to W. Müller [14] (who improved earlier results by E. Hlawka [5] and Krätzel and Nowak [10], [11]), and the second named author [15], respectively.

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$^{(1)}$ This assumption will be made a bit more precise at the end of section 2.

$^{(2)}$ For the definitions of the different $\Omega$-symbols, cf. Krätzel [8], p. 14.
In recent years, it has been noted that sharper estimates are true for a body $B$ which is invariant under rotations around one of the coordinate axes. In this case,

$$P_B(t) = O\left(t^{11/16}\right),$$

according to F. Chamizo [1], and

$$P_B(t) = \Omega_\pm \left(t^{1/2}(\log t)^{1/3}(\log_2 t)^{1/2}\log^2\exp(-c\sqrt{\log_3 t})\right), \quad c > 0,$$

as was shown by the first named author [12], on the basis of a deep and fairly general method of J.L. Hafner [3].

Quite recently, K. Soundararajan [16] exploited a brilliant new idea to obtain sharper $\Omega$-estimates in the classic circle and divisor problems. In the present note we will apply this ingenious new approach to improve the lower bound of (1.5).

**Theorem.** Let $B$ be a compact convex body in $\mathbb{R}^3$ which is invariant under rotations around one of the coordinate axes and contains $(0,0,0)$ as an inner point. Assume that its boundary $\partial B$ is of class $C^\infty$ and has bounded nonzero Gaussian curvature throughout. Then

$$P_B(t) = \Omega_\pm \left(t^{1/2}(\log t)^{1/3}(\log_2 t)^{1/2}\left(\sqrt{2} - 1\right)(\log_3 t)^{-2/3}\right).$$

We remark parenthetically that still much sharper estimates are known for the special case that $B$ is the unit ball $B_0$ in $\mathbb{R}^3$ (sphere problem). In fact, Heath-Brown [4] obtained

$$P_{B_0}(t) = O\left(t^{21/32+\varepsilon}\right),$$

thereby improving a result of Chamizo and Iwaniec [2] and earlier classic work of I.M. Vinogradov [20]. In the other direction, K.-M. Tsang [19] showed that

$$P_{B_0}(t) = \Omega_\pm \left(t^{1/2}(\log t)^{1/2}\right),$$

the $\Omega_\pm$-part of this result being much older and actually due to G. Szegö [17].

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(3) By $\log_j$, $j = 2, 3, \ldots$, we denote throughout the $j$-fold iterated logarithm.

(4) Note that $\frac{1}{3}\log 2 = 0.2310\ldots$ while $\frac{1}{6}\left(\sqrt{2} - 1\right) = 0.2761\ldots$.

(5) It is instructive to compare the numerical values of the exponents in (1.2), (1.4), and (1.6): $\frac{63}{86} = 0.7325\ldots$, $\frac{11}{16} = 0.6875$, $\frac{21}{32} = 0.65625$. 
2. Preliminaries.

**Soundararajan’s Lemma** [16]. Let \((f(n))_{n=1}^{\infty}\) and \((\lambda_n)_{n=1}^{\infty}\) be sequences of non-negative real numbers, \((\lambda_n)_{n=1}^{\infty}\) non-decreasing, and \(\sum_{n=1}^{\infty} f(n) < \infty\). Let \(L \geq 2\) be an integer and \(\Lambda\) a positive real parameter. Suppose further that \(\mathcal{M}\) is a finite set of positive integers, such that \(\{\lambda_m : m \in \mathcal{M}\} \subset \left[\frac{1}{2} \Lambda, \frac{3}{2} \Lambda\right]\). Then, for any real \(T \geq 2\), there exists some \(t \in \left[\frac{1}{2} T, (6L)^{|\mathcal{M}|+1} T\right]\) with

\[
\sum_{n=1}^{\infty} f(n) \cos(2\pi \lambda_n t) \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L-1} \sum_{n: \lambda_n \leq 2\Lambda} f(n) - \frac{2}{\pi^2 T \Lambda} \sum_{n=1}^{\infty} f(n).
\]

We further notice some important properties of the tac function \(H\) of a convex body \(\mathcal{B}\) with the properties stated above. This is defined by

\[
H(w) = \max_{x \in \mathcal{B}} (x \cdot w) \quad (w \in \mathbb{R}^3)
\]

where \(\cdot\) denotes the standard inner product. From this the following facts are evident:

(i) \(H\) is positive and homogeneous of degree 1.

(ii) There exist constants \(c_2 > c_1 > 0\), depending on \(\mathcal{B}\), such that for all \(w \in \mathbb{R}^3\)

\[
c_1 \|w\| \leq H(w) \leq c_2 \|w\|,
\]

where \(\|\|\) stands for the Euclidean norm throughout.

(iii) If \(\mathcal{B}\) is invariant with respect to rotations around the third coordinate axis (say), then so is \(H\), i.e., for all \((w_1, w_2, w_3) \in \mathbb{R}^3\),

\[
H(w_1, w_2, w_3) = H(\sqrt{w_1^2 + w_2^2}, 0, w_3).
\]

It seems appropriate to say a bit more about the smoothness condition that \(\partial \mathcal{B}\) be of class \(C^\infty\). Properly speaking, this is supposed to mean that for every point of \(\partial \mathcal{B}\) there exists a neighbourhood in which the corresponding portion of \(\partial \mathcal{B}\) has a regular (6) parametrization \(x = x(u_1, u_2)\) whose components are all of class \(C^\infty\). However, as has been neatly worked out in W. Müller [13], Lemmas 1 and 2, this local property implies that the spherical map, which sends every point of the unit sphere into that point of \(\partial \mathcal{B}\) where the outward normal has the same direction, is globally one-one and \(C^\infty\). Under these latter conditions, Hlawka’s asymptotic formulas for the Fourier transform of the indicator function of \(\mathcal{B}\) had been established [5], [6]. These in turn have been used in [15], upon which our present analysis will be based.

For the case that \(\mathcal{B}\) is a body of revolution (with respect to the \(x_3\)-axis, say), the conditions of our Theorem can be stated in a more concise form. It suffices to assume that

\[
\partial \mathcal{B} = \{x = (x_1, x_2, x_3) = (\rho(\theta) \sin(\theta) \cos(\phi), \rho(\theta) \sin(\theta) \sin(\phi), \rho(\theta) \cos(\theta)) : 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\},
\]

(6) I.e., \(\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}\) are linearly independent.
where \( \rho : \mathbb{R} \to \mathbb{R}_{>0} \) is an even function, periodic with period \( 2\pi \) and everywhere of class \( C^\infty \), which satisfies throughout
\[
\rho \rho'' - 2\rho'^2 - \rho^2 \neq 0. \tag{2.3}
\]
In fact, the Gaussian curvature \( \kappa_3 \) of this surface \( \partial B \) is readily computed as
\[
\kappa_3(\theta) = \frac{\frac{dx_3}{d\theta}}{\rho(\theta) \sin(\theta)} \frac{\rho(\theta) \rho''(\theta) - 2\rho'^2(\theta) - \rho^2(\theta)}{(\rho^2(\theta) + \rho'^2(\theta))^{3/2}}.
\]
We may imagine \( \partial B \) to be generated by rotation of the meridian
\[
\{(x_1, x_3) = (\rho(\theta) \sin(\theta), \rho(\theta) \cos(\theta)) : 0 \leq \theta \leq \pi \}
\]
around the \( x_3 \)-axis. The curvature \( \kappa_2 \) of the latter satisfies
\[
|\kappa_2(\theta)| = \left| \frac{\rho(\theta) \rho''(\theta) - 2\rho'^2(\theta) - \rho^2(\theta)}{(\rho^2(\theta) + \rho'^2(\theta))^{3/2}} \right|.
\]
Therefore, (2.3) guarantees the nonvanishing of \( \kappa_2 \), and also that of \( \kappa_3 \), since by geometric evidence \( \frac{dx_3}{d\theta} > 0 \) for \( 0 < \theta < \pi \).

3. Proof of the Theorem. For real \( t > 0 \), we put
\[
X = X(t) = (\log t)^{-1}, \quad k = k(t) = t^2 \log t, \tag{3.1}
\]
then the Borel mean-value of the lattice rest \( P_B \) is defined as
\[
B(t) := \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u^k} P_B(Xu) \, du. \tag{3.2}
\]
We start from formula (13) in [15]: For large \( t \), and arbitrary \( \epsilon > 0 \),
\[
B(t) = -\frac{1}{2\pi} t S(t) + O \left( t^{3/8+\epsilon} \right), \tag{3.3}
\]
where
\[
S(t) := \sum_{0 < \|m\| \leq t^{\epsilon_0} X^{-1/2}} \frac{\alpha(m)}{\|m\|^2} \exp \left( -\frac{1}{2} \pi^2 X H(m)^2 \right) \cos(2\pi H(m)t). \tag{3.4}
\]
Here \( \epsilon_0 > 0 \) is a sufficiently small constant, \( m = (m_1, m_2, m_3) \) denotes elements of \( \mathbb{Z}^3 \) throughout, and the coefficients \( \alpha(m) \) are positive reals bounded both from above and away from 0. By (2.2), we can rewrite this last formula as
\[
S(t) = \sum_{0 < \ell + m_3^2 \leq t^{2\epsilon_0} \log t} \frac{\ell g(\ell, m_3)}{\ell + m_3^2} \exp \left( -\frac{1}{2} \pi^2 X H(\sqrt{\ell}, 0, m_3)^2 \right) \cos(2\pi H(\sqrt{\ell}, 0, m_3)t),
\]
with
\[ g(\ell, m_3) := \sum_{(m_1, m_2) \in \mathbb{Z}^2; \quad m_1^2 + m_2^2 = \ell} \alpha(m_1, m_2, m_3) \sim r(\ell), \quad (3.5) \]

\( r(\ell) \) the number of ways to write \( \ell \in \mathbb{N} \) as a sum of two squares of integers.

In order to apply Soundararajan’s Lemma, we consider a one-to-one map \( q \) of \( \mathbb{N}_* \) onto \( \mathbb{N} \times \mathbb{Z} \setminus \{ (0, 0) \} \), \( n \mapsto q(n) = (\ell, m_3) \) such that the sequence \( (\lambda_n)_{n=1}^{\infty} \) defined by

\[ \lambda_n := H(\sqrt{\ell}, 0, m_3)(\ell, m_3) = q(n) \quad (3.6) \]

is non-decreasing\(^{(7)} \). Putting further

\[ f(n) := \frac{g(\ell, m_3)}{\ell + m_3^2} \exp(-\frac{1}{2}\pi^2 X H(\sqrt{\ell}, 0, m_3)^2)(\ell, m_3) = q(n) \quad (3.7) \]

if \( \ell + m_3^2 \leq t^{2\varepsilon_0} \log t \), and \( f(n) = 0 \) else, we obtain in fact

\[ S(t) = \sum_{n=1}^{\infty} f(n) \cos(2\pi \lambda_n t), \]

and are thus prepared to apply Soundararajan’s Lemma. For \( T \geq 40 \) a large real parameter, we put \( L = [(\log_2 T)^{20}] \) and assume that the set \( M \) will be chosen such that

\[ (6L)^{|M|+1} \leq T. \quad (*) \]

Then, by Soundararajan’s Lemma, there exists a value \( t \in [\frac{1}{2} T, T^2] \) for which

\[ S(t) \geq \frac{1}{8} \sum_{m \in M} f(m) - \frac{1}{L-1} \sum_{n: \lambda_n \leq 2\Lambda} f(n) - \frac{2}{\pi^2 T \Lambda} \sum_{n=1}^{\infty} f(n), \quad (3.8) \]

where \( \Lambda > 0 \) is a parameter remaining to be determined.

By homogeneity of the tac-function \( H \), there exist positive constants \( a_2 > a_1 > 0 \) and \( a_3 > a_4 > 0 \) depending on \( B \) such that the two-dimensional interval \( [a_1, a_2] \times [a_3, a_4] \) in the \( (w_1, w_3) \)-plane, say, lies between the two curves \( H(w_1, 0, w_3) = \frac{1}{2} \) and \( H(w_1, 0, w_3) = \frac{3}{2} \). Consequently, for integers \( \ell > 0 \) and \( m_3 \), the condition \( (\sqrt{\ell}, m_3) \in [a_1\Lambda, a_2\Lambda] \times [a_3\Lambda, a_4\Lambda] \) always implies that \( H(\sqrt{\ell}, 0, m_3) \in [\frac{1}{2}\Lambda, \frac{3}{2}\Lambda] \).

Let us denote by \( A_1 \) the set of positive integers whose prime divisors are all congruent to 1 mod 4, and by \( \omega(\ell) \) the number of prime divisors of \( \ell \in \mathbb{N}_* \).

\(^{(7)} \) In other words: We arrange the elements \( (\ell, m_3) \) of \( \mathbb{N} \times \mathbb{Z} \setminus \{ (0, 0) \} \) according to the size of the values \( H(\sqrt{\ell}, 0, m_3) \).
Then we define
\[ \hat{\mathcal{M}} = \{ (\ell, m_3) \in \mathbb{N}_*^2 : a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ a_3 \Lambda \leq m_3 \leq a_4 \Lambda, \ \ell \in \mathbb{A}_1, \ \omega(\ell) = [\beta \log_2 \Lambda] \}, \]
where \( \beta > 0 \) is a coefficient whose optimal choice ultimately will be \( \beta = \sqrt{2} \).

Let \( \mathcal{M} \) be the preimage of \( \hat{\mathcal{M}} \) under the map \( q \). By construction, \( \{ \lambda_m : m \in \mathcal{M} \} \subset [\frac{1}{2} \Lambda, \frac{3}{2} \Lambda] \), as required in Soundararajan’s Lemma.

By (3.5) and (3.7),
\[ \sum_{m \in \mathcal{M}} f(m) \gg \frac{1}{\Lambda^2} \sum_{a_3 \Lambda \leq m_3 \leq a_4 \Lambda} \sum_{\substack{a_2^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ \ell \in \mathbb{A}_1, \ \omega(\ell) = [\beta \log_2 \Lambda]}} r(\ell) \]
\[ \gg \frac{1}{\Lambda} \sum_{a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ \ell \in \mathbb{A}_1, \ \omega(\ell) = [\beta \log_2 \Lambda]} r(\ell), \tag{3.9} \]
where we have been assuming for the moment that
\[ XH(\sqrt{\ell}, 0, m_3)^2 \ll 1 \] (**)
for the values of \( \ell \) and \( m_3 \) involved.

Furthermore, \( r(\ell) \geq 2^{\omega(\ell)} \) for \( \ell \in \mathbb{A}_1 \), and the cardinality of
\[ S_{\Lambda,K} := \{ \ell \in \mathbb{N}_* : a_1^2 \Lambda^2 \leq \ell \leq a_2^2 \Lambda^2, \ \ell \in \mathbb{A}_1, \ \omega(\ell) = K \} \]
is readily estimated after the example of Tenenbaum [18], section II.6. One may start from the observation that, for \( \Re(s) > 1 \), \( z \in \mathbb{C} \) arbitrary,
\[ \sum_{n \in \mathbb{A}_1} z^{\omega(n)} n^{-s} = \prod_{p \equiv 1 \mod 4} \left( 1 + \frac{z}{p^s - 1} \right) = (\zeta_{Q(i)}(s))^{z/2} G(s; z), \]
where \( \zeta_{Q(i)} \) is the Dedekind zeta-function of the Gaussian field, and \( G(s; z) \) is holomorphic and bounded in every half-plane \( \Re(s) \geq \sigma_0 > \frac{1}{2} \). It follows (8) that, as long as \( K \ll \log_2 \Lambda \),
\[ |S_{\Lambda,K}| \asymp \frac{\Lambda^2}{\log \Lambda} \frac{(\frac{1}{2} \log_2 \Lambda)^{K-1}}{(K-1)!}. \]
With Stirling’s formula in the shape \( (K-1)! \asymp K^{K-1/2} e^{-K} \) and the choice \( K = [\beta \log_2 \Lambda] \), this gives
\[ |S_{\Lambda,K}| \asymp \frac{\Lambda^2}{\sqrt{\log_2 \Lambda}} \frac{(\log \Lambda)^{\beta-1-\beta \log(2\beta)}}, \]

(8) This has been noticed already by Soundararajan [16], f. (3.7). The authors intend to carry out the details for the case of a general number field \( K \) in a forthcoming article.
and thus
\[ |M| = |\hat{M}| \simeq \frac{\Lambda^3}{\sqrt{\log_2 \Lambda}} \left( \log \Lambda \right)^{\beta - 1 - \beta \log(2\beta)}, \tag{3.10} \]

Therefore, recalling (3.9) and the fact that \( r(\ell) \geq 2^{\epsilon(\ell)} \) for \( \ell \in A_1 \), we obtain
\[ \sum_{m \in M} f(m) \gg \frac{\Lambda}{\sqrt{\log_2 \Lambda}} \left( \log \Lambda \right)^{\beta - 1 - \beta \log \beta}. \tag{3.11} \]

We now have to choose \( \Lambda \) such that \((\ast)\) is satisfied. This is done optimally as
\[ \Lambda = c_0 (\log T)^{1/3} (\log_2 T)^{\frac{1}{2} \left( 1 - \beta + \beta \log(2\beta) \right)} (\log_3 T)^{-1/6}, \tag{3.12} \]

where \( c_0 \) is an appropriate small constant. As a consequence, \((\ast\ast)\) is verified, since \( X \ll (\log T)^{-1} \) and \( H(\sqrt{T}, 0, m_3) \ll \Lambda \) for the values of \( \ell \) and \( m_3 \) involved. Furthermore, \( \log \Lambda \asymp \log_2 T \) and \( \log_2 \Lambda \asymp \log_3 T \), thus ultimately
\[ \sum_{m \in M} f(m) \gg (\log T)^{1/3} (\log_2 T)^{\frac{1}{2} \left( \beta - 1 - \beta \log \beta \right) + \frac{1}{2} \beta \log 2 (\log_3 T)^{-2/3}. \tag{3.13} \]

Here the second exponent is maximized for \( \beta = \sqrt{2} \), and we finally obtain
\[ \sum_{m \in M} f(m) \gg (\log T)^{1/3} (\log_2 T)^{\frac{1}{2} \left( \sqrt{2} - 1 \right) (\log_3 T)^{-2/3}. \tag{3.13} \]

It remains to show that the two other terms on the right hand side of (3.8) are small. In fact,
\[ \sum_{n: \lambda_n \leq 2\Lambda} f(n) \ll \sum_{0 < H(\sqrt{T}, 0, m_3) \leq 2\Lambda} \frac{r(\ell)}{\ell + m_3^2} = \sum_{0 < H(m) \leq 2\Lambda} \|m\|^{-2} \leq \sum_{0 < c_1 \|m\| \leq 2\Lambda} \|m\|^{-2} = \sum_{1 \leq n \leq (4/c_1^2)\Lambda^2} \frac{r_3(n)}{n} = \int_1^u \frac{1}{u} \mathrm{d} \left( \sum_{1 \leq n \leq u} r_3(n) \right) \ll \Lambda, \]

using integration by parts of Stieltjes integrals and the well-known bound \( \sum_{1 \leq n \leq u} r_3(n) \ll u^{3/2}. \) After division by \( L - 1 \), which by construction is \( \asymp (\log_2 T)^{20} \), this is small compared to the right-hand side of (3.13).

Similarly (for the value of \( t \in [\frac{1}{2} T, T^2] \) specified by Soundararajan’s Lemma),
\[ \frac{2}{\pi^2 TA} \sum_{n=1}^{\infty} f(n) \ll \frac{1}{TA} \sum_{0 < \|m\| \leq t^{\epsilon_0}/\sqrt{T}} \|m\|^{-2} = \]
\[ = \frac{1}{TA} \int_1^{t^{\epsilon_0} \log t} \frac{1}{u} \mathrm{d} \left( \sum_{1 \leq n \leq u} r_3(n) \right) \ll T^{3\epsilon_0 - 1}. \]
Combining the last two bounds with (3.8) and (3.3), we conclude that for arbitrary $T \geq 40$, there exists a value $t \in \left[\frac{1}{2}T, T^2\right]$ with

$$-B(t) \gg t (\log t)^{1/3} (\log_2 t)^{\frac{2}{3}(\sqrt{2}-1)} (\log_3 t)^{-2/3}.$$  

(3.14)

Let us assume that, with some constants $C$ and $\varepsilon_1 > 0$, and for all $u > 0$,

$$-P_B(u) \leq C + \varepsilon_1 u^{1/2} \mathcal{L}(u),$$

where

$$\mathcal{L}(u) := (\log u)^{1/3} (\log_2 u)^{\frac{2}{3}(\sqrt{2}-1)} (\log_3 u)^{-2/3}$$

for $u \geq 20$, and $\mathcal{L}(u) = \mathcal{L}(20)$ else. By the definition (3.2) of $B(t)$, this implies that

$$-B(t) \leq C + \frac{\varepsilon_1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k (X u)^{1/2} \mathcal{L}(X u) \, du,$$

for all $t > 0$. Estimating this integral by Hafner’s Lemma 2.3.6 in [3], we obtain

$$-B(t) \leq C + C_1 \varepsilon_1 (kX)^{1/2} \mathcal{L}(kX) = C + C_1 \varepsilon_1 t \mathcal{L}(t^2),$$

recalling (3.1). Together with (3.14), this yields a positive lower bound for $\varepsilon_1$ and thus completes the proof of our Theorem.

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Manfred Kühleitner & Werner Georg Nowak

Institut für Mathematik
Department für Integrative Biologie
Universität für Bodenkultur Wien
Peter Jordan-Straße 82
A-1190 Wien, Österreich

E-mail: kleitner@edv1.boku.ac.at, nowak@mail.boku.ac.at

Web: http://www.boku.ac.at/math/nth.html