One-dimensional lattice model with an exact matrix-product ground state describing the Laughlin wave function

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We introduce one-dimensional lattice models with exact matrix-product ground states describing the fractional quantum Hall (FQH) states in Laughlin series (given by filling factors $\nu = 1/q$) on torus geometry. Surprisingly, the exactly solvable Hamiltonian has the same mathematical structure as that of the pseudopotential for the Laughlin wave function, and naturally derives the general properties of the Laughlin wave function such as the $Z_2$ properties of the FQH states and the fermion-boson relation. The obtained exact ground states have high overlaps with the Laughlin states and well describe their properties. Using the matrix product method, density functions and correlation functions are calculated analytically. Especially, obtained entanglement spectra reflects gapless edge states as was discussed by Li and Haldane.

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I. INTRODUCTION

The fractional quantum Hall (FQH) effect has been one of the most intriguing research fields in condensed-matter physics during the past few decades, in both theory and experiment. Recently the FQH effect has attracted much attentions due to new developments including the observation in graphene flat band Chern insulators, rapidly rotating bosons, applications to topological quantum computing and so on.

A key property of FQH states is their topological order. One consequence thereof is that their physical properties are insensitive to smooth deformations of the manifold on which we choose to study them. Using toroidal boundary conditions, the two-dimensional (2D) continuum system in a magnetic field can be reduced to a one-dimensional (1D) lattice model. This fact has been exploited in a series of recent studies of the interacting many-body problem in the limit geometry of a thin torus, referred as the Tao-Thouless (TT) limit.

In this paper, based on the 1D approach, we introduce minimal models with exact ground states that describe the FQH states in the Laughlin series (filling factors $\nu = 1/q$). The physical quantities, such as density functions and correlation functions, are calculated analytically by the matrix product (MP) method. The simplest case of our model ($q = 3$) reproduces a model describing the $\nu = 1/3$ FQH state that has been discussed quite recently by the authors with Bergholtz and Jansen.

We also will show that the structure of the 1D models [Eq. (1) below] implies that of the pseudopotential for the Laughlin wave function [Eq. (10) below], so that properties of the Laughlin wave function such as the $Z_2$ classification (the FQH effect is limited only odd $q$ for fermions and even $q$ for bosons) and general relations between fermionic and bosonic systems are naturally derived from our model.

Among physical quantities obtained by the MP method, entanglement spectra (ES) have attracted the attention recently. ES are obtained as coefficients when the wave function is divided into two subsystems. According to Li and Haldane, ES describe the “edge states” of the subsystem. Based on this notion, we extract (chiral) Tomonaga-Luttinger liquid behavior of the edge state analytically.

This paper is organized as follows: In Sec. II we discuss how 1D models with exact ground states are related to the $\nu = 1/q$ Laughlin states. We also consider bosonic version of the model. In Sec. III we introduce MP representation of the exact ground state, and perform analytical calculation of the physical quantities such as density functions, correlation functions, and ES. In Appendices, we present the detailed calculation of the matrix elements of the pseudopotential, and those of the physical quantities based on the MP method.

II. MODELS WITH EXACT GROUND STATES

A. FQH system on a torus

As discussed in the preceeding works when we consider 2D fermion system in a magnetic field using the torus boundary condition with lengths $L_1$ in $x_1$ directions ($l = 1, 2$), and the single-particle wave function with the Landau gauge $A = Bx_2\hat{x}_1$ for the lowest Landau level, the system is described by the following model on a 1D discrete lattice on direction $x_2$:

$$\mathcal{H} = \sum_{i=1}^{N_s} \sum_{|m|<k\leq N_s/2} V_{km} c_{i+m}^\dagger c_i + k c_{i+m+k} c_i,$$

where $c_i^\dagger$ ($c_i$) creates (annihilates) a fermion at site $i$, $\hat{n}_i = c_i^\dagger c_i$, and $N_s = L_1/2\pi$ is the number of sites (the magnetic length $l_B \equiv \sqrt{\hbar/eB}$ is set to be unity). The matrix element $V_{km}$ specifies the amplitude for a process where particles with separation $k + m$ hop $m$ steps to a separation $k - m$. This model conserves the center-of-mass coordinate defined by $K = \sum_{j=1}^{N_s} j \hat{n}_j \mod N_s$,
which is related to the momentum of the $x_1$ direction.

B. $\nu = 1/3$

In the preceding paper a 1D model with an exact ground state describing $\nu = 1/3$ FQH state is constructed in the following way: First, as an approximation in the vicinity of the TT limit, we truncate the Hamiltonian at $\nu = 1/3$ up to the four leading terms ($k + |m| \leq 3$). In the condition

$$V_{21}^2 = V_{10}V_{30}, \quad V_{20} > 0,$$

(2)

which is satisfied for the $\nu = 1/3$ Laughlin state for any $L_1$ with $L_2 \to \infty$, this model can be rewritten as

$$\mathcal{H}_{1/3} = \sum_i [Q_i^1 Q_i + P_i^1 P_i],$$

(3)

where

$$Q_i = \alpha_1 c_i + c_i + \alpha_0 c_i c_i + c_i + \beta_0 c_i c_i + c_i + 2,$$

with $\alpha_0, \alpha_1, \beta_0 \in \mathbb{R}$. This Hamiltonian apparently has positive expectation values $\langle \mathcal{H}_{1/3} \rangle \geq 0$. Next, we introduce the wave function

$$|\Psi_{1/3}\rangle = \prod_i (\alpha_1 - \alpha_0 c_i^1 c_i + c_i + 3 c_i)|\Psi_0^0_{1/3}\rangle,$$

(5)

with $|\Psi_0^0_{1/3}\rangle \equiv |100110100 \cdots \rangle$. Then the relation $Q_i |\Psi_{1/3}\rangle = P_i |\Psi_{1/3}\rangle = 0, \forall i$ is satisfied, so that $|\Psi_{1/3}\rangle$ is shown to be an exact ground state of $\mathcal{H}_{1/3}$. This state is three-fold degenerate due to the center-of-mass conservation. This argument can also be applied to a $\nu = 1/2$ bosonic FQH state with simple replacements.

C. $\nu = 1/q$

Now we discuss how an exactly solvable model is constructed for general Laughlin series $\nu = 1/q$ with odd $q$. For $\nu = 1/q$ with $q > 3$, the Hamiltonian should include longer range interactions than $\mathcal{H}_{1/3}$. The dimension of the subspace increases as $(q + 1)/2 = [q/2] + 1$ where $[\cdots]$ is the integer part of $\cdots$ (e.g. $\{11001\}, \{01110\}$ for $q = 3$ and $\{100001\}, \{010110\}, \{001110\}$ for $q = 5$). We find that the extended model should be given in the following form:

$$\mathcal{H}_{1/q} = \sum_{i=1}^{[q/2]} \sum_{\mu=0}^{|q/2|-1} [Q_i^{(j)} Q_i + P_i^{(j)} P_i],$$

(6a)

$$Q_i^{(j)} = \sum_{\mu=0}^{|q/2|-1} \alpha^{(j)}_{\mu} c_i + \mu c_i + q - \mu,$$

(6b)

$$P_i^{(j)} = \sum_{\mu=0}^{|q/2|-1} \beta^{(j)}_{\mu} c_i + \mu c_i + q - \mu - 1.$$

(6c)

This is also positive semidefinite $\langle \mathcal{H}_{1/q} \rangle \geq 0$. We can construct a similar Hamiltonian with zero energy eigenstates if we take only single-$l$ term in (6a) into account as was done in Refs. [13] and [14]. However, if there are no degrees of freedoms of $l$, we cannot fix the parameters $\alpha^{(j)}_{\mu}$ uniquely, as will be discussed below.

The corresponding wave function which gives the zero energy eigenstate is

$$|\Psi_{1/q}\rangle = \prod_i \sum_{\mu=0}^{|q/2|} s_j \hat{U}_{q-j,i}(i) |\Psi^0_{1/q}\rangle,$$

(7)

where the squeezing operators are defined by $\hat{U}_{k,0} \equiv 1$ and $\hat{U}_{k,m}(i) \equiv c_i + m c_i + k c_i + m c_i$ for $m \neq 0$. This operator is identical with a part of the pair hopping terms in Eq. (1). The root wave function $|\Psi^0_{1/q}\rangle$ is given by the charge-density-wave state in the TT limit,

$$|\Psi^0_{1/q}\rangle = |\cdots 10 \cdots 0 \cdots 0 \cdots \rangle,$$

(8)

where every qth site is occupied. The parameters in Eqs. (6b), (6c) and (7) are determined so that they satisfy orthogonal conditions, $\alpha^{(j)}_s = 0, \forall l$ where $\alpha^{(j)}_s = (\alpha^{(1)}_0, \alpha^{(1)}_1, \cdots, \alpha^{(j)}_{[q/2]})$ and $s = (1, 1, \cdots, s_{[q/2]})$. Since the number of the subspace (components of $s$) is larger than the number that $l$ may take by 1, these equations fix the parameters $s$ uniquely. On the other hand, there is no constraint for the parameter $\beta^{(l)}_{\mu}$, since the operator $P^{(l)}_i$ gives pair hopping processes between electrons separated even number of lattice sites which do not exist in Eq. (7).

Thus $Q^{(l)}_i |\Psi_{1/q}\rangle = P^{(l)}_i |\Psi_{1/q}\rangle = 0, \forall i$ is satisfied which means that (7) is the exact ground state of Eq. (6a) with zero energy. Due to the center-of-mass conservation, the ground state is $q$-fold degenerate, even if parameters $\alpha, \beta$, and $s$ have site dependence. Uniqueness of the ground state in each center-of-mass sector can be shown considering that other states have finite positive energies.

D. Bosonic systems

The similar argument can also be applied to bosonic FQH systems with even $q$. The bosonic version of (11) is given by

$$\mathcal{H} = \sum_{i=1}^{N_x} \sum_{|m| \leq k \leq N_x/2} V_{km} b^{\dagger}_{i+m} b^{\dagger}_{i+k} b_{i+m+k} b_{i},$$

(9)

where $b^{\dagger}_i$ ($b_i$) creates (annihilates) a boson at site $i$. In this case the dimension of the subspace is $(q + 2)/2$ for even $q$ (e.g. $\{110001\}, \{010101\}, \{002000\}$ for $q = 4$). Our exact argument for fermions can be applied straightforwardly to the bosonic cases only replacing the fermionic operators by the bosonic ones $c^{(j)}_i \to b^{(j)}_i$ in Eqs. (6b), (6c) and (7).
E. Relationship with Laughlin states

We discuss that our exactly solvable model well describes the $\nu = 1/q$ Laughlin states. In order to discuss the Laughlin states in the present 1D framework, we introduce a pseudopotential \cite{16,17} which is known to have the $\nu = 1/q$ Laughlin state as the exact ground state,

$$V(x) = \sum_{\lambda=0}^{q-1} c_{\lambda} b^{2\lambda} \nabla^{2\lambda} \delta^2(x),$$

(10)

where $x \equiv (x_1, x_2)$, $c_{\lambda} = (-1)^\lambda |c_{\lambda}|$ are constants to keep the energy positive, and $b > 0$ stands for the range of the interactions. Note that number of the terms increases as $q$ is increased. This reminds us that our exactly solvable model \cite{6a} has a similar structure. For $N_s \gg L_1$, the single-particle wave function for the lowest Landau level in Landau gauge $A = Bx_2 \hat{x}_1$ is given by

$$\psi_k(x) \propto e^{i(2\pi k/L_1)x_1 - \frac{1}{2}(x_2 + 2\pi k/L_1)^2},$$

(11)

where $2\pi k/L_1$ is the momentum along the $x_1$-direction. In this basis, we derive the general form of the matrix elements of the Hamiltonian \cite{1} as the second quantization of (10),

$$V'_{km} = \sum_{\lambda=0}^{q-1} V^{(\lambda)}_{km};$$

$$V''_{km}^{(\lambda)} = \sum_{\nu=0}^{\lambda} \sum_{\mu=0}^{\lambda-\nu} C_{\mu\nu}(k^2 - m^2)^\nu \times \left[(-1)^\nu k^{2\nu + m 2\nu} e^{-\frac{2\pi^2}{L_1^2}(k^2 + m^2)} \right],$$

(12)

where $C_{\mu\nu}$ are given by $c_{\lambda}$, $b$, and $L_1$ (see Appendix A). Note that the function is antisymmetric (symmetric) under the exchange of $k^2$ and $m^2$ as a consequence of the Fermi (Bose) statistics.

On the other hand, the matrix elements of our exactly solvable model $V_{km}$ ($k + |m| \leq q$) are identified in the following way. First, we divide the matrix elements as $V_{km} = \sum_{\lambda} V^{(\lambda)}_{km}$ and relate $\lambda$ to the variable in Eq. (6a) as $\lambda = 2l - 1$ for fermions $\lambda = 2l$ for bosons. The matrix elements for each $\lambda$ which stem from $[\cdots]$ in Eq. (6a) satisfy the relation

$$\left[V^{(\lambda)}_{km}\right]^2 = V^{(\lambda)}_{k-m,0} V^{(\lambda)}_{k+m,0}.$$

(13)

This relation is a counterpart of Eq. (2) for the $\nu = 1/q$ case, and is also satisfied if we choose the matrix elements in the following form:

$$V_{km} = \sum_{\lambda=1, \text{odd}, \lambda=0, \text{even}}^{q-1} V^{(\lambda)}_{km},$$

$$V^{(\lambda)}_{km} = C_{\lambda}(k^2 - m^2)^\lambda e^{-\frac{2\pi^2}{L_1^2}(k^2 + m^2)},$$

(14)

where $C_{\lambda}$ is a constant and $\lambda$ is restricted to be odd (even) for fermions (bosons) due to the statistics, as already mentioned. Although Eqs. (12) and (14) coincide at $q = 3$ ($q = 2$) for fermions (bosons), they are different in general. However, we may interpret the truncated Hamiltonian of (11) with (14) as an approximation of the full Hamiltonian with (12) for $L_1 \to 0$ limit (see Appendix A).

In order to show that our model well describes the Laughlin state, we numerically calculate overlap of the ground-state wave function of the Laughlin state and that of our exactly solvable model (7) for $q = 3, 5$ and $N_s \equiv N_s/q$ as functions of $L_1$ in finite-size systems, calculated by exact diagonalization. The data are plotted up to the isotropic points of the torus $(L_1 = L_2)$. The approximation becomes better for larger-$q$ cases, since the number of long-range terms are added as $q$ is increased.

![FIG. 1: (Color online) Overlap of the ground-state wave function of the fermionic Laughlin state and that of our exactly solvable model (7) for $q = 3, 5$ and $N_s \equiv N_s/q$ as functions of $L_1$ in finite-size systems, calculated by exact diagonalization. The data are plotted up to the isotropic points of the torus $(L_1 = L_2)$. The approximation becomes better for larger-$q$ cases, since the number of long-range terms are added as $q$ is increased.](image)

F. $Z_2$ properties

The present exact result seems to be applicable to fillings $\nu = 1/q$ with even (odd) denominator $q$ for fermions (bosons) as well. However, the structure of our model naturally excludes such a situation, in accordance with the symmetry of the Laughlin wave function. For fermions (bosons) with even (odd) $q$, the expres-
sions of $Q_i^{(0)}$, $P_i^{(0)}$, and Eq. (14) remain unchanged, while the summations in Eqs. (16) and (17) are replaced by $\sum_{j=1}^{[(q+1)/2]}$ and $\sum_{j=0}^{[(q-1)/2]}$, respectively. This is because the number of subspace for fermions (bosons) with even (odd) $q$ system becomes $[(q + 1)/2]$ (e.g., $(10001), (01010)$) for $q = 4$ fermions. Thus the number that $l$ may take becomes the same as the number of the subspace. Therefore, the parameter $s$ cannot be determined uniquely for these cases by the equation $\alpha_i^{(s)} = 0, \forall l$, which means that models with the exact-ground-states of $\Psi_0^{(s)}$ cannot be constructed.

**G. Compressibility**

We can show the vanishing of the compressibility in this system as is expected for FQH states. As discussed in Ref. 11, a zero energy eigenstate can be created by inserting 0 everywhere in $|\Psi_0^{(s)}\rangle$. This process makes a particle pair separated $q + 1$ site while the range of interactions is $q$ sites, so that it is equivalent to making an open boundary. Therefore the ground-state energy of such cases gives $E_q(N_s + 1) = E_q(N_s) = 0$. On the other hand, if we shrink the system size by removing 0, a defect corresponding to the fractional charge $e^* = e/q$ appears. Since this defect localizes, the excitation energy $E_q(N_s - 1)$ has very small size dependence as was explicitly shown in the $\nu = 1/3$ case. Therefore, the inverse compressibility

$$\frac{1}{\kappa} = \lim_{N_s \rightarrow \infty} \frac{N_s}{4\pi l_s^2} [E(N_s - 1) + E(N_s + 1) - 2E_q(N_s)]$$

clearly diverges.

**III. MATRIX PRODUCT METHOD**

From the exact solution of the ground-state expectation values. For this purpose, it is convenient to introduce a matrix product (MP) representation of the ground-state density matrix (see Appendix B),

$$|\Psi_{\nu/n}^q\rangle = \text{tr} \left[ g_1 g_2 \cdots g_N \right],$$

where $g_i$ is identified as the following $(m + 1) \times (m + 1)$ matrix (see Appendix B),

$$g_i = \begin{bmatrix} |o_i\rangle & |+1_i\rangle & |+2_i\rangle & \cdots & |+m_i\rangle \\ s_1 |1_i\rangle & 0 & 0 & \cdots & 0 \\ s_2 |2_i\rangle & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_m |m_i\rangle & 0 & 0 & \cdots & 0 \end{bmatrix}.$$  

The matrix (17) has the same size as that of the valence-bond-solid states of an $S = m$ quantum spin chain, however, there is a $m \times m$ zero entry part and the transfer matrix $G(g_1, g_2)$ has only two eigenvalues $\lambda_{\pm} = (1 \pm \sqrt{4s^2 - 3})/2$. Due to this property, the matrix $G$ can be reduced to $4 \times 4$ form for all $s$, so that the calculation of expectation values is extremely simplified.

**A. Correlation functions**

Using the MP formalism we obtain the density function in the infinite systems $N \rightarrow \infty$, which has $q$-site periodicity as (see Appendix C2),

$$\langle n_{qi} \rangle = \frac{s^2}{2(s^2 - 1)} \left( 1 - \frac{1}{\sqrt{4s^2 - 3}} \right),$$

$$\langle n_{qi} \pm 1 \rangle = \frac{1}{\sqrt{4s^2 - 3}}.$$  

Figure 2(a) shows the $L_1$ dependence of the density functions for $q = 3, 5, 7$. Since our wave function is not trans-
The correlation length becomes longer in large \( L \) state, however, these results indicate that the system’s approach to homogeneous liquid states as \( L_1 \) is increased. This is consistent with the result obtained by Rezayi and Haldane which showed that the charge-ordered state in the TT limit is adiabatically connected to the FQH liquid-state through the smooth deformation of the torus geometries at \( L_1 = 9 \) as functions of the momentum for \( x_1 \) direction \( \Delta K \). They form the conformal tower structure of (chiral) Tomonaga-Luttinger liquids describing the edge states.

Actually, our exact result describes many important aspects of the FQH states, as we have already seen. Similarly, the density-density correlation functions are calculated, which have exponentially decaying forms as (see Appendix C2)

\[
\langle \hat{n}_i \hat{n}_j \rangle - \langle \hat{n}_i \rangle \langle \hat{n}_j \rangle \sim \left( \frac{1 - \sqrt{4s^2 - 3}}{1 + \sqrt{s^2 - 3}} \right)^{|i-j|} \equiv e^{-q|i-j|/\xi},
\]

where \( \xi \) is the correlation length. As shown in Fig. 2(b), the correlation length becomes longer in large \( L_1 \) regions as \( q \) is increased. Since the correlation length is proportional to the inverse of the energy gap, this result is consistent with the fact that the \( \nu = 1/q \) FQH state becomes unstable for larger \( q \). Similar behavior has also been confirmed in the \( \nu = p/(2p + 1) \) Jain series.

**B. Entanglement spectra**

The MP formalism enables us to derive entanglement spectra (ES) of the FQH system \( \{\xi_j\} \) analytically (see Appendix D). First we consider the cylinder geometry (open boundary conditions), and divide the system into two blocks \( A \) and \( B \) with length \( L = N/2 \). Then the wave function \( \Psi \) is written as the Schmidt decomposition

\[
|\Psi_{1/q}\rangle = \sum_j e^{-\xi_j/2} |\psi_j^A \rangle \otimes |\psi_j^B \rangle,
\]

where \( |\psi_j^A \rangle = [\mathcal{N}_{j_{1j_2}^A}]^{-1/2} \prod_{i=1}^L g_{i(\pm L)} \) are orthogonal states in the subsystem \( A(B) \), and \( \mathcal{N}_{j_{1j_2}^A} \) are their norms. The ES are given by \( \xi_j = -\ln(\mathcal{N}_{j_{1j}^A}/\mathcal{N}) \).

As was pointed out by Li and Haldane, the ES characterize the “edge states” of the subsystem. Although the number of the ES is limited as \( m+1 \), the number of entanglement levels for bulk FQH states grows with the size of the system. However, when the ES are plotted as functions of the center-of-mass \( \Delta K \) (which corresponds to momentum of \( x_1 \) direction) of the subsystem as shown in Fig. 3, they give low-lying liner spectra of the gapless edge state described as a chiral Tomonaga-Luttinger liquid.

Next, we evaluate ES in the torus geometry where the Schmidt decomposition is given by

\[
|\Psi_{1/q}\rangle = \sum_{ij} e^{-\xi_{ij}/2} |\psi_{ij}^A \rangle \otimes |\psi_{ij}^B \rangle,
\]

The ES \( \xi_{ij} = -\ln(\mathcal{N}_{ij}^A \mathcal{N}_{ij}^B / \mathcal{N}) \) are characterized by two separated “edge states” of the subsystems. The number of the ES is \( m + 1)^2 \), and \( m(m+1)/2 \) of them are two-fold degenerate. As shown in Fig. 4 the ES show a “diamond shape” conformal tower structure of a nonchiral Tomonaga-Luttinger liquid which has both left and right movers. This result can be explained by the relation obtained by the MP method

\[
\xi(\Delta K_1 - \Delta K_2) = \xi(\Delta K_1) + \xi(\Delta K_2) - \xi_0
\]

for \( N \to \infty \), where \( \xi_0 \) is the lowest ES. This means that the ES are given by the spectra of an effective noninteracting 1D “entanglement Hamiltonian” with right and
left modes with momenta $\Delta K_1 = -\Delta K_2$. We can also evaluate the difference between ES on a torus and those of cylinder as $\ln[\lambda_+/(\lambda_+ - \lambda_-)]$.

In Fig. 3 $L_1$ dependence of the ES and entanglement entropy (EE) for $q = 5$ with torus boundary conditions are shown. The diamond structure of ES is broken for the large $L_1$ regions and the EE $S$ approach to $\ln(m+1)^2$.

IV. CONCLUSION

We have introduced a 1D minimal reference model with exact ground states which describes $\nu = 1/q$ FQH states with odd (even) $q$ for fermions (bosons). This model is constructed as the second quantization of the pseudopotential for the Laughlin wave function, around the TT limit of the torus boundary system. The model has the same structures of the pseudopotential, and naturally derives general properties of the Laughlin wave function. The ground states are described by the MP wave functions that enable us to calculate physical quantities analytically. Especially, we have analytically obtained the conformal tower structure of the ES which reflects chiral Tomonaga-Luttinger liquid behavior of the edge state.

V. ACKNOWLEDGMENTS

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Appendix A: Matrix elements of the pseudo potential [Eqs. (12) and (14)]

We consider a model of $N$ interacting electrons on a torus with lengths $L_i$ in $x_i$ directions ($i = 1, 2$). When the torus is pierced by $N_s$ magnetic flux quanta, boundary conditions require $L_1L_2 = 2\pi N_s$. In Landau gauge, $A = B x_2 \hat{e}_1$, a complete basis of $N_s$ degenerate single-particle states in the lowest Landau level, labeled by $k = 0, \ldots, N_s - 1$, can be chosen as

$$\psi_k(x) = \frac{1}{\sqrt{\pi^{1/2} L_1}} \sum_{m \in \mathbb{Z}} \exp \left[ i \left( \frac{2\pi}{L_1} k + L_2 m \right) x_1 \right] \exp \left[ -\frac{1}{2} \left( x_2 + \frac{2\pi}{L_1} k + L_2 m \right)^2 \right],$$

where unit of $x_i$ and $L_i$ is the magnetic length $l_B$. In $L_2 \to \infty$ limit, it becomes Eq. (11). In this basis, any translation-invariant 2D Hamiltonian with two-body interactions assumes the following 1D lattice model:

$$\hat{H} = \sum_{k_1k_2k_3k_4} \hat{V}_{k_1k_2k_3k_4} = \sum_{|m| \leq k \leq N_s/2} \hat{V}_k,$$

where the upper (lower) signs correspond to fermions (bosons) and the operators $c_k(c_k^\dagger)$ annihilate (create) a particle at site (orbital) $k$. The matrix elements $\hat{V}_{q_1q_2q_3q_4}$ can be calculated according to the standard second quantization

$$\hat{V}_{k_1k_2k_3k_4} = V_{k_1k_2k_3k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3} c_{k_4} = \frac{1}{2} \int \psi_{k_1}(x_1) \psi_{k_2}(x_2) V(x_1 - x_2) \psi_{k_3}(x_2) \psi_{k_4}(x_1) d^2x_1 d^2x_2,$$

where $\psi_k(x) \equiv \psi_k(x)c_k$. One can find that for fermionic (bosonic) systems the matrix elements are real and antisymmetric (symmetric) functions for $k$ and $m$ reflecting Fermi (Bose) statistics. Using the Fourier transform $V(p)$ as the interaction potential $V(x_1 - x_2)$, the matrix elements $V_{k_1k_2k_3k_4}$ are given by

$$V_{k_1k_2k_3k_4} = \frac{\delta_{k_1+k_2,k_3+k_4}}{2L_1L_2} \sum_{p} \delta_{k_1-k_4,p_1L_1/2\pi} V(p) e^{-p^2/2-i(k_1-k_3)p_2L_2/N_s},$$

where $p = (p_1, p_2)$, $p_1 = 2\pi m_1/L_1$, $p_2 = 2\pi m_2/L_2$, $m_1, m_2 \in \mathbb{Z}$, and $\delta'$ is the Kronecker delta function with a period $N_s$. The pseudo potential proposed by Trugman and Kivelson is known that it has the $\nu = 1/q$ Laughlin wave function as the exact ground state,\(^\dagger\)

$$V(x) = \sum_{\lambda=0}^{q-1} V^{(\lambda)} = \sum_{\lambda=0}^{q-1} c_\lambda b^{2\lambda} \nabla^{2\lambda} \delta_2(x), \quad x = x_1 - x_2,$$
where \( b \) is the range of the interaction. For \( N_s \gg L_1 \), the matrix elements \( V_{km} \) are calculated in the \( L_2 \to \infty \) limit using the binomial theorem and Gauss integrals \( S_n = \int_{-\infty}^{\infty} dx x^{2n} e^{-x^2/2} \) as

\[
V_{km}^{(\lambda)} = \frac{(-b^2)^{\lambda} c_{\lambda}}{4\pi L_1} e^{-(K^2 + M^2)/2} \times \int dx \left[ (M^2 + x^2)^{\lambda} \left( e^{-x-iK^2/2} + e^{-x+iK^2/2} \right) \mp (K^2 + x^2)^{\lambda} \left( e^{-x-iM^2/2} + e^{-x+iM^2/2} \right) \right] \\
= \frac{(-b^2)^{\lambda} c_{\lambda}}{4\pi L_1} e^{-(K^2 + M^2)/2} \sum_{\nu=0}^{\lambda} \lambda C_{\nu} (K^2 - M^2)^{\nu} \times \int dx \left[ (-1)^{\nu} (K^2 + x^2)^{\lambda-\nu} \left( e^{-x-iK^2/2} + e^{-x+iK^2/2} \right) \mp (M^2 + x^2)^{\lambda-\nu} \left( e^{-x-iM^2/2} + e^{-x+iM^2/2} \right) \right] \\
= \frac{(-b^2)^{\lambda} c_{\lambda}}{2\sqrt{2\pi L_1}} e^{-(K^2 + M^2)/2} \sum_{\nu=0}^{\lambda} \lambda C_{\nu} (K^2 - M^2)^{\nu} \left[ (-1)^{\mu} 2^{\nu+1} \sum_{\nu=0}^{\nu} \frac{C_{\nu} S_{\lambda-\nu-\mu} \left[ (-1)^{\nu} K^2 \mp M^2 \right]}{\nu! (\lambda - \nu - 2\mu)!} \right]
\]

where \( K = \frac{2\pi k}{L_1} \) and \( M = \frac{2\pi m}{L_1} \). One can find that in fermion (boson) systems the powers of \( (K^2 - M^2) \) are always odd (even) for each term, and the leading term of \( K, M \) appears as \( (K^2 - M^2)^{\lambda} \) when \( \lambda \) is odd (even), because \( \left[ (-1)^{\nu} K^{2\mu} \mp M^{2\mu} \right] \) vanishes for \( (\nu, \mu) = (\lambda, 0) \) when \( \lambda \) is even (odd). Therefore Eq. (12) can be approximated as Eq. (13) in the small \( L_1 \) regions.

**Appendix B: Matrix Product Representation [Eq. (9)]**

Let us consider the case of \( \nu = 1/3 \) as the simplest example. The ground-state manifold of this model is threefold degenerate, spanned by charge ordered states with one electron per a three-site unit cell: \( \cdots |010 010 010 \cdots \). The \( \hat{V}_{21} \) term induces fluctuations upon these ground states through the process

\[
|010 010\rangle \leftrightarrow |001 100\rangle. \quad (B1)
\]

The truncated model can be mapped to an \( S = 1 \) quantum spin chain by identifying the states of the unit cell as \( |010\rangle \to |\sigma\rangle \), \( |001\rangle \to |+\rangle \), and \( |100\rangle \to |-\rangle \). We consider the matrix product representation of the ground-state wave function in periodic boundary conditions is given by

\[
|\Psi_{1/3}\rangle_p = \prod_{i=1}^{N} (1 + s_1 \hat{U}_{3i-1}) |01001001000\ldots\rangle, \quad (B3)
\]

\[
= \prod_{i=1}^{N} (1 + s_1 \hat{U}_{i}) |000\ldots\rangle \\
= \text{tr}[g_1 g_2 \cdots g_N], \quad (B4)
\]

where \( s_1 \equiv \alpha_0 / \alpha_1 \), and

\[
\hat{U}_k = c_{k+1}^\dagger c_k + c_k^3 c_{k+3}^\dagger, \quad \hat{U}_k = -\frac{1}{2} \hat{S}_x^k \hat{S}_y^k \hat{S}_z^k \hat{S}_y^{k+1} \hat{S}_z^{k+1}, \quad (B6)
\]

with \( \hat{S}_k^\sigma \) being an \( S = 1 \) spin operator. One can find that the states of the nearest two spins can be written as

\[
\psi_{-1} \otimes \psi_1 = |00\rangle + s_1 |+\rangle + |0+\rangle + |10\rangle + |1-\rangle. \quad (B7)
\]

Therefore, we can set

\[
f_{-} = f_{0-} = f_{i0} = f_{i+} = 0 \quad (B8)
\]

\[
f_{00} = f_{0+} = f_{-0} = f_{-+} = 1 \quad (B9)
\]

\[
f_{+-} = s_1. \quad (B10)
\]

Thus, the matrix Eq. (12) then becomes

\[
g_i = \begin{bmatrix}
0 & |0\rangle_i & |+\rangle_i \\
0 & |0\rangle_i & |+\rangle_i \\
s_1 |\rangle_i & 0 & 0
\end{bmatrix}. \quad (B11)
\]

If we change the matrix (12) in different basis \( s_1 |\rangle_i + |0\rangle_i, |+\rangle_i \), the above matrix can be reduced to \( 2 \times 2 \) form
without losing generality.

\[ g_i = \begin{bmatrix} 0 \cdots 0 |o \rangle \mid +1 \rangle \cdots | +m \rangle \\ \vdots & \ddots & \vdots & \vdots \\ 0 \cdots 0 |o \rangle \mid +1 \rangle \cdots | +m \rangle \\ \vdots & \ddots & | -1 \rangle & 0 \cdots 0 \\ | -m \rangle & \cdots & 0 \cdots 0 \end{bmatrix} \]  

(B12)

Although Eqs. (B11) and (B12) give the same result, we use the latter representation for simplicity.

In the diagonal component of the matrix product, the upper one is both for open and periodic boundary conditions, while the lower one is only for periodic boundary systems. Therefore, for open boundary systems, the ground-state wave function is given as

\[ |\Psi_{1/3} \rangle_o = \text{tr}'[g_1 g_2 \cdots g_N], \]  

(B13)

where tr' means partial trace for the upper component.

For general \( \nu = 1/q \) with \( q = 2m + 1 \), we consider in a similar way as the above using the spin-\( m \) representation \[15\], and obtain a \( q \times q \) matrix as

\[ g_i = \begin{bmatrix} 0 \cdots 0 |o \rangle \mid +1 \rangle \cdots | +m \rangle \\ \vdots & \ddots & \vdots & \vdots \\ 0 \cdots 0 |o \rangle \mid +1 \rangle \cdots | +m \rangle \\ \vdots & \ddots & | -1 \rangle & 0 \cdots 0 \\ | -m \rangle & \cdots & 0 \cdots 0 \end{bmatrix} \]  

(B14)

The matrix with the reduced dimension \( m + 1 \) can be obtained as Eq. (17) in the text.

Appendix C: Correlation functions

1. Transfer matrix

We consider the matrix product representation of the \( \nu = 1/(2m + 1) \) ground-state wave function in \( (2m + 1)N \) sites periodic system. The normalized ground-state wave function can be written as

\[ |\Psi_m \rangle = N^{-1/2} \text{Tr}[g_1 g_2 \cdots g_N], \]  

(C1)

\[ N = \text{Tr}[G^N], \quad G \equiv g_i \otimes g_i, \]  

(C2)

where \( g_i \) is identified as Eq. (17). The \( (m + 1)^2 \times (m + 1)^2 \) transfer matrix \( G(r_1, r_2; t_1, t_2) \equiv g_i^\dagger(r_1, t_1) g_j(r_2, t_2) \) has only two eigenvalues,

\[ \lambda_0 = 0, \quad \lambda_{\pm} = \frac{1 \pm \sqrt{4s^2 - 3}}{2}, \]  

(C3)

that are obtained from the following secular equation:

\[ 0 = \det[G - \lambda] = (1 - \lambda)(-\lambda)^m - (-\lambda)^{m-1} \sum_{j=1}^{m} s_j^2. \]  

(C4)

We define \( (m + 2) \times (m + 2) \) local matrix \( \tilde{G} \) to reduce the dimension of \( G \) to \( (m + 1) \times (m + 1) \):

\[ G = \begin{bmatrix} \tilde{G} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} 1 & 0 \cdots & 0 \\ 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \]  

(C5)

Thus the transfer matrix is diagonalized by finding a unitary matrix \( U \) as

\[ \begin{bmatrix} \lambda_- \tilde{G} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} 1 & 0 \cdots & 0 \\ 0 & \lambda_+ \tilde{G} & 0 \cdots 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]  

(C6)

Thus we can calculate physical quantities in just the same way as in the case of \( \nu = 1/3 \).

2. Density and Charge correlation

In the spin-\( m \) representation for the \( \nu = 1/(2m + 1) \) state [15], the density operator and its transfer matrix can be written as

\[ \hat{n}_{q_1} = \delta \hat{s}_{q_1} \hat{s}_{q_1}, \quad G[\hat{n}](j) \equiv \tilde{g}_i \otimes \delta \hat{s}_{q_i} \hat{s}_{q_i} g_i. \]  

(C7)

Using the unitary transformation \( U \) defined in Eq. (C6), we obtain the density function with \( q \) sites periodicity as

\[ D(j; N) \equiv \langle \hat{n}_{q_1} \rangle = \langle \hat{n}_{q_1} \rangle = \frac{\text{Tr}[U^{-1}G[\hat{n}](j)U G_0^{N-1}]]}{\text{Tr}[G_0^N]}. \]  

\[ \lim_{N \to \infty} D(j; N) = \begin{cases} \frac{s^2}{\sqrt{s^2 - 3}} \left( 1 - \frac{1}{\sqrt{4s^2 - 3}} \right) & (j \neq 0) \\ \frac{1}{\sqrt{4s^2 - 3}} & (j = 0) \end{cases}. \]  

(C8)

The charge-charge correlation function is calculated as

\[ C(i, j; i', j'; N) \equiv \langle \hat{n}_{q_{i+1}} \hat{n}_{q_{i'+1}} \rangle = \frac{\text{Tr}[U^{-1}G[\hat{n}](j)U G_0^{i'-1}U^{-1}G[\hat{n}](j')U G_0^{N-1-i'+i}]]}{\text{Tr}[G_0^N]}. \]  

(C9)

\[ \lim_{N \to \infty, |i-i'| \to \infty} C(i, j; i', j'; N) \sim \left( \frac{\lambda_-}{\lambda_+} \right)^{|i-i'|} \equiv e^{-q|i-i'|/\xi}, \]  

(C10)

where \( \xi \) is the correlation length for the infinite system.
Appendix D: Entanglement Spectra and Entropy

In this section, we compute the entanglement spectrum (ES) and the entanglement entropy (EE) both in cylinder and torus geometries. We obtain the conformal tower structure of the ES in Fig. [B] analytically.

1. Cylinder

For a open boundary system, the normalized ground-state wave function can be written as:

$$|\Psi_m\rangle = N^{-1/2} \text{Tr}[g_1^i \cdots g_N^i], \quad N \equiv \text{Tr} \left[ G^i G^{N-2} G^{m} \right],$$  \hspace{1cm} (D1)

where the terminators $g_i^i, g'^i_i$ are identified as the following matrices

$$g_i^i \equiv \begin{bmatrix} [0]_i & [+1]_i & \cdots & [+m]_i \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$  \hspace{1cm} (D2)

$$g_i'^i \equiv \begin{bmatrix} [0]_i & 0 & \cdots & 0 \\ s_1 [-1]_i & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_m [-m]_i & 0 & \cdots & 0 \end{bmatrix},$$  \hspace{1cm} (D3)

and the transfer matrices $G^i, G'^i$ are given by

$$G^i \equiv g_i^i \otimes g_i^i = \begin{bmatrix} \hat{1} & \hat{1} & \cdots & \hat{1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$  \hspace{1cm} (D4)

$$G'^i \equiv g_i'^i \otimes g_i'^i = \begin{bmatrix} \hat{1} & 0 & \cdots & 0 \\ (s_1^2 \hat{1}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (s_m^2 \hat{1}) & 0 & \cdots & 0 \end{bmatrix}. $$  \hspace{1cm} (D5)

We cut this system into blocks $A$ and $B$ with length $L$ and $N - L$, respectively. The wave function of the whole system is divided as

$$|\Psi_m\rangle = N^{-1/2} \text{Tr} [g_1^i g_2^i \cdots g_N^i] = N^{-1/2} \sum_{j=1}^{m+1} \left\{ [g_1^i g_2^i \cdots g_L^i]_{1, j} [g_{L+1}^i g_{L+2}^i \cdots g_N^i]_{j, 1} \right\}. $$  \hspace{1cm} (D7)

We can then define the block states as

$$|\psi^A_j\rangle \equiv \left(N_j^A\right)^{-1/2} [g_1^i g_2^i \cdots g_L^i]_{1,j},$$  \hspace{1cm} (D8)

$$|\psi^B_j\rangle \equiv \left(N_j^B\right)^{-1/2} [g_{L+1}^i g_{L+2}^i \cdots g_N^i]_{j,1}. $$  \hspace{1cm} (D9)

where $N_j^A(B)$ is the normalization factors that is given by

$$N_j^A \equiv \left[G^i G^{L-1} \right]_{j,(m+2)-(m+1), j},$$  \hspace{1cm} (D10)

$$N_j^B \equiv \left[G^{N-L-2} G^{m} \right]_{1,j,(m+2)-(m+1)}. $$  \hspace{1cm} (D11)

The reduced density matrix is given by

$$\hat{\rho}^A = \sum_j \frac{N_j^A N_j^B}{N} |\psi^A_j\rangle \langle \psi^A_j|. $$  \hspace{1cm} (D12)

The ES $\xi_j^A$ and EE $S_j^A \equiv -\text{Tr}_A [\hat{\rho}_A \log \hat{\rho}_A]$ can be written as

$$\xi_j^A = -\log \left( \frac{N_j^A N_j^B}{N} \right), $$  \hspace{1cm} (D13)

$$S_j^A = -\sum_j \frac{N_j^A N_j^B}{N} \log \left( \frac{N_j^A N_j^B}{N} \right). $$  \hspace{1cm} (D14)

The normalization factor for the wave function is calculated by using Eq. (D1) as

$$N = \lambda_{N+1}^{N+1} - \lambda_{N+1} \lambda_{N}. $$  \hspace{1cm} (D15)

For $L = N/2 \Rightarrow G^L = G^{N-L} = G^{N/2}$, the normalization factors $N_j^A, N_j^B$ are written as

$$N_j^A = \begin{cases} \frac{\lambda_{j+1}^{L+1} - \lambda_{j}^{L}}{\lambda_{j+1} - \lambda_{j}} & (j = 1) \\ \frac{\lambda_{j+1}^{L} - \lambda_{j}^{L}}{\lambda_{j+1} - \lambda_{j}} & (j > 1) \end{cases}, $$  \hspace{1cm} (D16)

$$N_j^B = \begin{cases} \frac{\lambda_{j-1}^{L+1} - \lambda_{j}^{L}}{\lambda_{j-1}^{L} - \lambda_{j}} & (j = 1) \\ \frac{\lambda_{j}^{L+1} - \lambda_{j}^{L}}{\lambda_{j-1}^{L} - \lambda_{j}} & (j > 1) \end{cases}. $$  \hspace{1cm} (D17)

We now can obtain the ES and EE by substituting the $N_j^A, N_j^B$ into Eq. (D13). For $L = N/2 \rightarrow \infty$, the

$$N_j^A N_j^B \frac{1}{N} \rightarrow \begin{cases} \frac{-\lambda_{j+1} - \lambda_{j}}{-\lambda_{j+1} + \lambda_{j}} \frac{s_{j+1}^2}{s_{j+1}^2} & (j = 1) \\ \frac{\lambda_{j}^{L+1} - \lambda_{j}^{L}}{\lambda_{j-1}^{L} - \lambda_{j}} \frac{s_{j+2}^2}{s_{j+2}^2} & (j > 1) \end{cases}. $$  \hspace{1cm} (D18)

When the ES are plotted as functions of the shift of the center of mass $\Delta K$ of the subsystem, it shows approximately a linear relation which corresponds to a conformal tower of the chiral Tomonaga-Luttinger liquid.

2. Torus

We cut a periodic system into blocks $A$ and $B$ with lengths $L$ and $N - L$, respectively. The wave function of the whole system (C1) can be divided as

$$|\Psi_m\rangle = N^{-1/2} \text{Tr} [g_1^i g_2^i \cdots g_N^i] = N^{-1/2} \sum_{j_1, j_2=1}^{m+1} \left\{ [g_1^i g_2^i \cdots g_L^i]_{j_1 j_2} [g_{L+1}^i g_{L+2}^i \cdots g_N^i]_{j_2 j_1} \right\}. $$  \hspace{1cm} (D19)

$$= N^{-1/2} \sum_{j_1, j_2=1}^{m+1} \left\{ [g_1^i g_2^i \cdots g_L^i]_{j_1 j_2} [g_{L+1}^i g_{L+2}^i \cdots g_N^i]_{j_2 j_1} \right\}. $$  \hspace{1cm} (D20)
We can then define the block states as
\[ |\psi_A^{(j_1, j_2)} \rangle \equiv \left( N_{j_1 j_2}^A \right)^{-1/2} g_1 g_2 \cdots g_L |j_1 j_2 \rangle, \quad \text{(D21)} \]
\[ |\psi_B^{(j_1, j_2)} \rangle \equiv \left( N_{j_1 j_2}^B \right)^{-1/2} g_{L+1} g_{L+2} \cdots g_N |j_1 j_2 \rangle, \quad \text{(D22)} \]
where \( N_{j_1 j_2}^{A(B)} \) is the normalization factors for A(B) block,
\[ N_{j_1 j_2}^A \equiv \left[ G^L \right]_{j_1 (m+2) - (m+1), j_2 (m+2) - (m+1)}, \quad \text{(D23)} \]
\[ N_{j_1 j_2}^B \equiv \left[ G^{N-L} \right]_{j_2 (m+2) - (m+1), j_1 (m+2) - (m+1)}. \quad \text{(D24)} \]

One can now rewrite the ground-state wave function as Schmidt decomposition,
\[ |\Psi_m \rangle = \sum_{j_1, j_2} \left( N_{j_1 j_2}^A N_{j_1 j_2}^B \right)^{1/2} |\psi_A^{(j_1, j_2)} \rangle \otimes |\psi_B^{(j_1, j_2)} \rangle. \quad \text{(D25)} \]

The reduced density matrix is given by
\[ \hat{\rho} = \sum_{j_1, j_2} \frac{N_{j_1 j_2}^A N_{j_1 j_2}^B}{N} |\psi_A^{(j_1, j_2)} \rangle \langle \psi_A^{(j_1, j_2)} |. \quad \text{(D26)} \]

The ES \( \xi_{j_1 j_2}^A \) and EE \( S^A \) can be written as
\[ \xi_{j_1 j_2}^A = -\ln \left( \frac{N_{j_1 j_2}^A N_{j_1 j_2}^B}{N} \right), \quad \text{(D27)} \]
\[ S^A = \sum_{j_1, j_2} \frac{N_{j_1 j_2}^A N_{j_1 j_2}^B}{N} \ln \left( \frac{N_{j_1 j_2}^A N_{j_1 j_2}^B}{N} \right). \quad \text{(D28)} \]

For \( L = N/2 \), the normalization factors \( N_{j_1 j_2}^A, N_{j_1 j_2}^B \) are calculated as
\[ N_{j_1 j_2}^A = \begin{cases} \frac{\lambda_{j_1+1} - \lambda_{j_2+1}}{\lambda_+ - \lambda_-} & (j_1 = 1, j_2 = 1) \\ \frac{\lambda_{j_1} - \lambda_{j_2}}{\lambda_+ - \lambda_-} s_{j_1-1} & (j_1 > 1, j_2 = 1) \\ \frac{\lambda_{j_1} - \lambda_{j_2}}{\lambda_+ - \lambda_-} s_{j_2-1} & (j_1 = 1, j_2 > 1) \end{cases}, \quad \text{(D29)} \]
\[ N_{j_1 j_2}^B = \begin{cases} N_{j_1 j_2}^A & (j_1 > 1, j_2 > 1) \end{cases}. \quad \text{(D30)} \]

Now we can use Eq. (D27) to calculate the ES and the EE. For \( L = N/2 \to \infty \), we get
\[ N_{j_1 j_2}^A \to \begin{cases} \frac{\lambda_{j_1+1} - \lambda_{j_2+1}}{\lambda_+ - \lambda_-} & (j_1 = 1, j_2 = 1) \\ \frac{\lambda_{j_1} - \lambda_{j_2}}{\lambda_+ - \lambda_-} s_{j_1-1} & (j_1 > 1, j_2 = 1) \\ \frac{\lambda_{j_1} - \lambda_{j_2}}{\lambda_+ - \lambda_-} s_{j_2-1} & (j_1 = 1, j_2 > 1) \end{cases}, \quad \text{(D31)} \]
\[ N_{j_1 j_2}^B = N_{j_1 j_2}^A. \quad \text{(D32)} \]

When the ES are plotted as functions of the shift of the center-of-mass \( \Delta K \) of the subsystem, for \( N \to \infty \), we can show the relation \[ \lambda_{j_1+1} - \lambda_{j_2+1} \right) \text{ which means that the ES of the torus are given by the combination of the right and the left movers of the chiral Tomonaga-Luttinger liquids. Therefore, the ES shows the conformal tower structure of the nonchiral Tomonaga-Luttinger liquid. We can estimate the difference of the ES on a cylinder and that on a torus as } \ln(\lambda_+/(\lambda_+ - \lambda_-)).} \]
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