Algebras of conjugacy classes in symmetric groups and checker triangulated surfaces

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In 1999 V. Ivanov and S. Kerov observed that structure constants of algebras of conjugacy classes of symmetric groups $S_n$ admit a stabilization (in a non-obvious sense) as $n \to \infty$. We extend their construction to a class of pairs of groups $G \supset K$ and algebras of conjugacy classes of $G$ with respect to $K$. In our basic example, $G = S_n \times S_n$, $K$ is the diagonal subgroup $S_n$. In this case we get a geometric description of this algebra.

MSC. 20B30, 20C32, 20E45

Key words. Symmetric groups, group algebras, conjugacy classes, Ivanov–Kerov algebra, partial bijections, triangulated surfaces

1 Introduction

1.1. Notation. 1) Denote by $\#A$ the number of elements of a set $A$. By $\coprod A_j$ we denote the disjoint union of sets $A_j$.
2) Denote by $J_n$ the set $J_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$.

Let $Y$ be a finite or countable set. Denote by $S(Y)$ the group of all finitely supported permutations of $Y$. By $S_n$ we denote the group of permutations of $J_n$, by $S_\infty$ the group $S(\mathbb{N})$. We regard groups $S_n$ as subgroups in $S_\infty$.

3) Let $G$ be a group, $K \subset G$ a subgroup. Denote by $G\backslash\!\!\backslash K$ the set of double cosets of $G$ with respect to $K$, i.e., the quotient of $G$ with respect to the equivalence relation

$g \sim h_1gh_2$, where $g \in G$, $h_1, h_2 \in K$.

Denote by $G\!\!/K$ the set of conjugacy classes of $G$ with respect to $K$, i.e., the quotient of $G$ with respect to the equivalence relation

$g \sim hgh^{-1}$, where $g \in G$, $h \in H$.

Consider the group $G \times K$ and the subgroup $\bar{K}$ consisting of elements $(h, h) \in G \times K$, where $h$ ranges in $K$. We have a canonical identification

$\bar{K}\backslash(G \times K)/\bar{K} \simeq G\!\!/K$.

Indeed, let $(g, h) \in G \times K$. The corresponding double coset contains

$\quad (g, h)(h^{-1}, h^{-1}) = (gh^{-1}, 1)$

1Supported by the grants FWF, P28421, P31591.
and also all elements of the form
\[(r,r)(gh^{-1},1)(r^{-1},r^{-1}) = (r(gh^{-1})r^{-1},1).\]

For a group \(G\) consider its multiples \(G^{(m)} = G \times \cdots \times G\) \((m\ times)\), by \(\text{diag}(G) = \text{diag}_m(G) \subset G^{(m)}\) we denote the diagonal, i.e., the set of all tuples \((g,\ldots,g) \in G^{(m)}\). We have obvious identifications
\[
\begin{align*}
\text{diag}_2(G) \setminus (G \times G) / \text{diag}_2(G) & \simeq G / G; \\
\text{diag}_3(G) \setminus (G \times G \times G) / \text{diag}_3(G) & \simeq (G \times G) / \text{diag}_2(G),
\end{align*}
\]
etc.

4) For a finite group \(G\) denote by \(\mathbb{C}(G)\) the group algebra of \(G\), we denote the convolution by \(*\). For \(f \in \mathbb{C}(G)\) denote by \(f^*\) the function
\[f^*(g) := \overline{f(g^{-1})}.
\]
Clearly \(f \rightarrow f^*\) is an anti-involution of the group algebra
\[
(f_1 * f_2)^* = f_2^* * f_1^*, \quad (f^*)^* = f.
\]

Denote by \(\mathbb{C}(K \setminus G / K)\) \((\text{resp. } \mathbb{C}(G / K))\) the subalgebra of the group algebra consisting of functions that are constant on double cosets \((\text{resp. conjugacy classes})\), these subalgebras are closed with respect to the anti-involution.

Let \(\bar{K} \subset G \times K\) be as above. We have an obvious isomorphism
\[
\mathbb{C}(G / K) \simeq \mathbb{C}(\bar{K} \setminus (G \times K) / \bar{K}),
\]
i.e., convolution algebras of conjugacy classes are special cases of algebras of double cosets.

1.2. Bibliographical remarks on algebras of double cosets and their infinite-dimensional degenerations. Formally, this subsection is not necessary, however here we explain origins and purposes of this work.

Let \(\rho\) be a unitary representation of a finite group \(G\) in a space \(V\), denote by the same symbol \(\rho\) the corresponding representation of the group algebra. Denote by \(V^K\) the subspace of all \(K\)-fixed vectors, by \((V^K)^\perp\) its orthocomplement. The convolution algebra \(\mathbb{C}(K \setminus G / K)\) acts in \(V = V^K \oplus (V^K)^\perp\) by operators of block form
\[
\rho(f) = \begin{pmatrix}
\rho'(f) & 0 \\
0 & 0
\end{pmatrix}.
\]
Thus for any representation \(\rho\) of \(G\) we have a representation \(\rho'\) of \(\mathbb{C}(K \setminus G / K)\) in \(V^K\). It can be easily shown that if \(\rho\) is irreducible and \(V^K \neq 0\), then \(\rho'\) ’remembers’ \(\rho\). For this reason convolution algebras became tools of investigation of representations. We recall some well-known examples.

1) Hecke algebras (Iwahori [12]). Let \(G\) be the group \(GL(n, \mathbb{F}_q)\) of all invertible matrices over a finite field \(\mathbb{F}_q\) and \(K\) be the group of upper-triangular
matrices. These algebras admit explicit descriptions and an interpolation in $q$
(for $q = 1$ we get $\mathbb{C}(S_n)$). They can live their own lives independently of the
group $\text{GL}(n, \mathbb{F}_q)$, see, e.g., [3]. For the subgroup $T$ of strictly upper triangular
matrices the convolution algebra $\mathbb{C}(T \backslash \text{GL}(n, \mathbb{F}_q)/T)$ also admits a transparent
description, see Yokomuna [29].

There are some widely explored examples of convolution algebras related to
locally compact groups (in this case $K$ must be compact):

2) Let $G$ be a reductive Lie group and $K$ its maximal compact subgroup.
Algebras $\mathbb{C}(K \backslash G/K)$ were widely explored in classical representation theory
of Lie groups at least after Gelfand [7]. If $G$ is a rank one classical group,
i.e., pseudounitary group $\text{SU}(1, n)$, or quaternions (in usual notation, $G = \text{SO}(1, n)$, $\text{SU}(1, n)$, $\text{Sp}(1, n)$), then multiplication in this algebra is
determined by an explicit hypergeometric kernel, these algebras have an explicit
two-parametric interpolation with respect to $n$ and a dimension $d = 1, 2, 4$ of a
field $\mathbb{K}$, this interpolation also includes one real form of the exceptional group
$F_4$, see, Flensted-Jensen, Koornwinder [5], see also [15].

3) Affine Hecke algebras (Iwahori, Matsumoto [13]). Let $\mathbb{Q}_p$ be a $p$-adic
field, $\mathbb{O}_p$ be the ring of $p$-adic integers. Let $G$ be the group $\text{GL}(n, \mathbb{Q}_p)$, and
$K$ be the Iwahori subgroup. Recall that the Iwahori subgroup is a subgroup
on $\text{GL}(n, \mathbb{O}_p)$ consisting of matrices whose elements under the diagonal are
contained in $p\mathbb{O}_p$. Such algebras $\mathbb{C}(K \backslash G/K)$ also admit an explicit description
and an interpolation in $p$ and can live their own lives.

These examples have further extensions, however in all these cases subgroups
$K$ are quite large in $G$, for smaller subgroups algebras of double cosets usually
seem to be too complicated.

It appears that for infinite-dimensional groups double coset spaces $K \backslash G/K$
often admit a natural structure of a semigroup, and for each unitary representa-
tion of $G$ this semigroup acts in the subspace of $K$-fixed vectors. First example
of such semigroup was discovered by Ismagilov in [10]. Many cases were exam-
ined by Olshansky [23], [24], he showed that semigroups $K \backslash G/K$ admit explicit
descriptions in some cases when their finite-dimensional counterparts seem non-
handable. In [18] it was observed that existence of such semigroup structures is
a relatively usual phenomenon. In [19]–[22], [4] descriptions of such semigroups
were proposed on a quite general setting.

A basic example in [21] was semigroups $K \backslash G/K$, where

$$G = S_\infty \times S_\infty \times S_\infty$$

and $K$ is a subgroup in the diagonal $S_\infty$ fixing points $1, 2, \ldots, \alpha \in \mathbb{N}$.

The present paper is an attempt to look from infinity to finite objects. We present
a kind of description (or quasi-description) of a family of convolution semigroups
related to symmetric groups, our basic example is

$$\mathbb{C}\left(\text{diag}_3(S_n) \backslash (S_n \times S_n \times S_n) / \text{diag}_3(S_n)\right) \simeq \mathbb{C}\left((S_n \times S_n) / \text{diag}_2(S_n)\right).$$
We use arguments of Ivanov and Kerov [11] who observed that the algebras $\mathbb{C}(S_n//S_n)$ admit a stabilization (in a non-obvious sense) as $n \to \infty$.

1.3. Ivanov-Kerov algebra of conjugacy classes of groups $S_j \times S_j$ with respect to the diagonal subgroup. Denote

$$G_j := S_j \times S_j, \quad K_j := \text{diag}_2(S_j).$$

For
g, h, r, \ldots \in \bigoplus_{j=0}^{\infty} G_j

denote by

$$\overline{g}, \overline{h}, \overline{r}, \ldots \in \bigoplus_{j=0}^{\infty} G_j//K_j$$

the corresponding conjugacy classes.

Let $j \leq N$ and $g \in G_j$. Denote by $\overline{g}$ the corresponding element of $G_N \supset G_j$.

We define an element $A_N[\overline{g}]$ of $\mathbb{C}(G_N//K_N)$ by the formula

$$A_N[\overline{g}] = A_N^{(j)}[\overline{g}] := \frac{1}{(N-j)!} \sum_{\tau \in K_N} \tau^{-1}\overline{g}.$$  

Sometimes we write superscript $(j)$ to emphasize that $\overline{g}$ is considered as an element of $G_j//K_j$. If $j > N$ and $g \in G_j$, we set

$$A_N[\overline{g}] := 0.$$

Thus for each $N$ we get a family of elements of $\mathbb{C}(G_N//K_N)$.

Remark. Elements $A_N[\overline{g}] = A_N^{(N)}[\overline{g}]$, where $\overline{g}$ ranges in $G_N//K_N$, form a basis in $\mathbb{C}(G_N//G_N)$. However, if $g \in G_N$ actually is an element of a subgroup $G_k$, our family contains also elements

$$A_N^{(N-1)}[\overline{g}] = \frac{1}{N} A_N[\overline{g}], \quad A_N^{(N-2)}[\overline{g}] = \frac{1}{2!} A_N[\overline{g}], \quad \ldots, \quad A_N^{(N-k)}[\overline{g}] = \frac{1}{(N-k)!} A_N[\overline{g}].$$

Theorem 1.1 Let $\overline{g}, \overline{h}, \overline{r}$ range in the disjoint union $\bigoplus_{j=0}^{\infty} G_j//K_j$. Then there are non-negative integers $a_{\overline{g}, \overline{h}, \overline{r}}$ which do not depend on $N$, satisfying the following properties:

- For each $N$

  $$A_N[\overline{g}] * A_N[\overline{h}] = \sum_{\overline{r}} a_{\overline{g}, \overline{h}, \overline{r}} A_N[\overline{r}].$$  

(1.1)
Consider a linear space $\mathcal{B}$ with a basis consisting of symbols $A[\overline{g}]$, where $\overline{g}$ ranges in $\prod_{j=0}^{\infty} G_j/K_j$. Then the formula
\[
A[\overline{g}] \ast A[\overline{h}] = \sum_{\overline{r}} a_{\overline{g}, \overline{h}}^{\overline{r}} A[\overline{r}],
\]  
(1.2)
determines a structure of an associative algebra on $\mathcal{B}$.

In Theorem 1.2 we present a geometric description (or quasi-description) of this algebra.

1.4. Checker triangulated surfaces and symmetric groups. We say that a checker triangulated surface is a two-dimensional oriented closed (generally, disconnected) surface (see Fig. 1) equipped with the following data:

- a graph separating surface into triangles (faces);
Figure 3: Surfaces $\mathcal{R}$, $\mathcal{Q}$ and a partial bijection $\mathcal{R}_-$ to $\mathcal{Q}_+$. 
• for each triangle we assign a sign (+) or (−), pluses and minuses are arranged in a checker order;
• edges are colored by red, yellow, blue; colors of edges of each face are different; in plus-triangles these colors are located clockwise, on minus-triangles anti-clockwise.

We admit non-connected surfaces. Also, we admit pairs of triangles, which have two or three common edges as on Fig. 2.

Surfaces are defined up to the combinatorial equivalence. Denote by $\Xi_N$ the set of all checker triangulated surfaces with $2N$ faces.

We say that a labeling of a surface $\mathcal{R} \in \Xi_N$ is a bijective map from the set $J_N = \{1, 2, \ldots, N\}$ to the set of all plus-triangles. We automatically assign labels to minus-triangles assuming that triangles separated by blue edges have same labels. Denote by $\tilde{\Xi}_N$ the set of all labeled surfaces with $2N$ triangles defined up to a combinatorial equivalence.

There is a natural one-to-one correspondence between the set $\tilde{\Xi}_N$ and the group $G_N = S_N \times S_N$.

Indeed, fix a labeled checker surface $\tilde{\mathcal{R}}$. For each red edge $v$ we consider labels $i_+(v)$ and $i_-(v)$ on its plus and minus sides. Assuming $\sigma_{\text{red}} : i_+(v) \mapsto i_-(v)$ we get an element $\sigma_{\text{red}}$ of $S_N$. Considering yellow edges we obtain another permutation $\sigma_{\text{yellow}} \in S_N$.

A permutation $\tau \in S_N$ of labels corresponds to a simultaneous conjugation

$$(\sigma_{\text{red}}, \sigma_{\text{yellow}}) \mapsto (\tau \sigma_{\text{red}} \tau^{-1}, \tau \sigma_{\text{yellow}} \tau^{-1}).$$

Therefore we get a canonical bijection between sets $\Xi_N$ and $(S_N \times S_N) / / S_N$.

Next, let us describe a product in $S_N \times S_N$ in terms of labeled surfaces. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{R}}$ be labeled surfaces. For each $j = 1, \ldots, N$, we identify the $j$-th minus-triangle of $\tilde{\mathcal{R}}$ with the $j$-th plus-triangle of $\tilde{\mathcal{P}}$ according colors of their sides. Thus we get a two-dimensional simplicial cell complex consisting of $N$ labeled plus-triangles inherited from $\mathcal{R}$, labeled $N$ minus-triangles inherited from $\mathcal{P}$ and $N$ plus-minus-triangles obtained as result of gluing. Each edge is contained in 3 triangles. Removing interiors of all plus-minus-triangles we come to a simplicial cell complex such that each edge is contained in two triangles. In fact, this is a surface, but some vertices of the surface can be glued one with an other (see Fig. 4). Cutting all such gluings we get a new surface whose triangles are equipped with labels. Some of blue edges were contained in removed triangles. Labels on both sides of such edge coincide with a label on the removed triangle and therefore coincide. So our new surface is correctly labeled.

1.5. Remark. Belyi data. Checker triangulated surfaces arise in a natural way in algebraic geometry under names Belyi data or dessins d’enfant.

Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers. Consider an algebraic curve $C$ and a meromorphic function (Belyi function) $f$ on $C$ whose critical values

\footnote{If a surface $\mathcal{R}$ admits combinatorial automorphisms, then different labelings of $\mathcal{R}$ can give equivalent labeled surfaces.}
are contained in the set 0, 1, ∞. According the famous Belyi theorem \[2\] such a function on a given curve \(C\) exists if and only if \(C\) can be determined by a system of algebraic equations with coefficients in \(\overline{\mathbb{Q}}\).

Consider the Riemann sphere \(\overline{\mathbb{C}} = \mathbb{C} \cup \infty\) and the real projective line \(\overline{\mathbb{R}} = \mathbb{R} \cup \infty\) in \(\overline{\mathbb{C}}\). Let us say that the upper half-plane is a plus-triangle, lower half-plane is a minus-triangle, the segment \([1, \infty]\) is red, the segment \([0, 1]\) is yellow, and the segment \([\infty, 0]\) is blue. Thus the Riemann sphere \(\overline{\mathbb{C}}\) becomes a checker triangulated surface. The preimage of \(\overline{\mathbb{R}}\) on \(\overline{\mathbb{C}}\) is colored graph splitting \(\overline{\mathbb{C}}\) into triangles. Clearly, we come to a checker triangulated surface.

The Galois group of \(\mathbb{Q}\) over \(\mathbb{Q}\) acts on the set of all Belyi functions. A. Grothendieck proposed a program of investigation of the Galois groups using Belyi functions and graphs on surfaces, see, e.g., \[26\], \[27\], \[28\], \[8\], \[25\]. Relations of this topic and infinite symmetric group remain to be non-clear.

1.6. **Partial bijections.** Recall that a partial bijection \(\lambda\) of a set \(Y\) to a set \(Z\) is a bijection of a subset \(A \subset Y\) to a subset \(B \subset Y\) (we admit the case \(A = B = \emptyset\)). We define rank, domain, and image of a partial bijection by

\[
\operatorname{rk} \lambda := \# A = \# B, \quad \operatorname{dom} \lambda := A, \quad \operatorname{im} \lambda := B.
\]

Denote by \(\text{PB}(Y, Z)\) the set of all partial bijections \(Y \to Z\).

For partial bijections \(\mu : W \to Y, \lambda : Y \to Z\) we define their product \(\lambda \mu : W \to Z\). We say that \(w \in \operatorname{dom} \lambda \mu\) if \(w \in \operatorname{dom} \mu\) and \(\mu(w) \in \operatorname{dom} \lambda\). In this case we set \(\lambda \mu(w) = \lambda(\mu(w))\)
1.7. Construction of the algebra $B$. The basis of the algebra is numerated by the set
\[ \prod_{n=0}^{\infty} \Xi_n. \]
Let $R \in \Xi_n$, $Q \in \Xi_k$. Let $\lambda$ be a partial bijection from the set of minus-triangles of $R$ to the set of plus-triangles of $Q$ (see Fig. 3). Consider the disjoint union $R \coprod Q$ and let us perform the following transformations. For each face $A \in \text{dom} \lambda$, we take the face $\lambda(A) \in \text{im} \lambda$, remove both faces and identify their boundaries according colors of edges. After this, we get a compact two-dimensional simplicial complex, and each edge of the complex is contained in precisely two faces. As above some vertices of the surface can be glued one with another (see Fig. 4). Cutting all such gluings we get a new surface
\[ R \circledast \lambda Q. \]
In this notation, the product is defined by
\[ R \ast Q = \sum_{\lambda \in \text{PB}(R_-, Q_+)} R \circledast \lambda Q. \]

**Theorem 1.2** This algebra coincides with the algebra $B$ defined in Theorem 1.1.

2 A more general construction

2.1. Groups $G_n$. Fix a finite set $I$. Consider a finite or countable set $X$, the product $I \times \mathbb{N}$ and the unions
\[ V = X \cup (I \times \mathbb{N}), \quad V_n = X \cup (I \times J_n) \subset V, \]
see Fig. 5. The group $S_\infty$ acts on $\mathbb{N}$, therefore it acts on the product $I \times \mathbb{N}$. We can imagine $I \times \mathbb{N}$ as a table with $\# I$ infinite rows. The group $S_\infty$ acts by permutations of columns.

Next, we consider the trivial action of $S_\infty$ on $X$, this determines an action of $S_\infty$ on the whole $V$.

Consider a subgroup $G_\infty \subset S(V)$ containing $S_\infty$. Set
\[ G_n := G_\infty \cap S(V_n). \]
In other words, we consider elements of $G$ supported by $V_n$. As above, we denote
\[ K_n := S_n, \quad K_\infty = S_\infty. \]

2.2. Some natural examples. a) The case discussed above corresponds to $X = \emptyset$ and a 2-element set $I$. The group $G_\infty = S_\infty \times S_\infty$ acts by permutations preserving rows.

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*3The case $X = \emptyset$ is sufficiently interesting.*
Figure 5: Sets $V \supset V_n$.

b) $X = \emptyset$, $G = S(I \times \mathbb{N})$.

c) Let $X = \emptyset$, $G = S_\infty \times \cdots \times S_\infty$ consists of permutations of $I \times V$ preserving each row.

d) $X$ is a finite set, $\#I = 1$, $G = S(X \cup \mathbb{N})$.

e) $X = \emptyset$, $\#I = k$, $G$ is a semidirect product of $S_\infty$ acting by permutations of columns and the group $(S_k)^{\infty}$ acting by permutations of elements of each column.

In all these cases the sets of conjugacy classes admit geometric descriptions by tricks of [21].

2.3. Algebras $\mathbb{C}(G_n/\!\!\!\!\!\!\!\!\!\!\!/K_n)$.

Let $g, h, r, \ldots$ range in $\bigsqcup_{j=0}^{\infty} G_j/\!\!\!\!\!\!\!\!\!\!\!\!/K_j$. Denote by $g, h, r$ their representatives.

Fix $n = 0, 1, 2, \ldots$, let $\tilde{g} \in G_n/\!\!\!\!\!\!\!\!\!\!\!\!/K_n$, $g \in \tilde{g}$, and $N \geq n$. For any $n$-element set $\Omega \subset \mathbb{N}$ we define the sum

$$R(g, \Omega) := \sum_{\sigma : J_n \to \Omega} \sigma g \sigma^{-1} \in \mathbb{C}(G_\infty),$$

where $\sigma$ ranges in the set of bijective maps $J_n \to \Omega$. We explain this more carefully. Any $\sigma$ determines a bijection $I \times J_n \to I \times \Omega$, we denote it by the same symbol $\sigma$. Thus, $\sigma g \sigma^{-1}$ is a map

$$\sigma g \sigma^{-1} : X \cup (I \times \Omega) \to X \cup (I \times \Omega).$$

We extend it to a map $V_N \to V_N$ in a trivial way,

$$\sigma g \sigma^{-1}(w) := w, \quad \text{if } w \in I \times (J_N \setminus \Omega).$$

Next, we consider the sum

$$A_N[\tilde{g}] = \sum_{\Omega \subset J_N: \#\Omega = n} R(g, \Omega).$$

For $N < n$, we set $A_N[\tilde{g}] = 0$.

Remark. Clearly, for $N \geq n$, we have

$$A_N[\tilde{g}] = \frac{1}{(N-n)!} \sum_{\tau \in K_N} \tau g \tau^{-1}. \quad (2.1)$$
However, our long definition will be used below.

**Theorem 2.1** For the groups $G_n$ defined in this subsection Theorem 1.1 remains true.

Thus, for any group $G_\infty$, we get an associative algebra $\mathcal{B} = \mathcal{B}[G_\infty/K_\infty]$ and canonical epimorphisms $\mathcal{B}[G_\infty/K_\infty] \to \mathbb{C}(G_N/K_N)$.

**2.4. Local bijections.** Following Ivanov and Kerov [11] we define a semigroup of local bijections.

Let $Y$ be a finite or countable set. A local bijection $\omega$ is a bijection from a finite subset $\Omega \subset Y$ to $\Omega$. We denote such local bijection by $((\omega, \Omega))$. Any local bijection admits a canonical extension to an element $\bar{\omega} \in S(Y)$, we set $\bar{\omega}y = y$ if $y \notin \Omega$. We define the product of local bijections by

$$((\omega, \Omega)) \circ ((\mu, M)) = \left(\overline{\omega \cdot \mu}_{\Omega \cup M}, \Omega \cup M\right).$$

Denote by $\mathcal{L}(Y)$ the semigroup of all local bijections of $Y$.

**Remark.** A local bijection determines a partial bijection. But the $\circ$-product differs from the product of partial bijections.

The group $S(Y)$ acts on $\mathcal{L}(Y)$ by conjugations in the obvious way,

$$\sigma((w, \Omega))\sigma^{-1} = ((\sigma\bar{w}\sigma^{-1}_{|_{\sigma\Omega}}), \sigma\Omega).$$

On the other hand, we have a natural forgetting homomorphism

$$\iota: \mathcal{L}(Y) \to S(Y)$$

defined by

$$\iota((\omega, \Omega)) = \bar{\omega}.$$

**2.5. The semigroup algebra for local bijections.** Denote by $\mathbb{C}[\mathcal{L}(Y)]$ the space of all formal series of the form

$$\sum_{((\omega, \Omega)) \in \mathcal{L}(Y)} a_{((\omega, \Omega))} ((\omega, \Omega)), \quad a_{((\omega, \Omega))} \in \mathbb{C}.$$

This space is equipped with a convolution

$$\sum_{((\omega, \Omega))} a_{((\omega, \Omega))} ((\omega, \Omega)) \ast \sum_{((\mu, M))} b_{((\mu, M))} ((\mu, M)) = \sum_{((\nu, N))} c_{((\nu, N))} ((\nu, N)),$$

where

$$c_{((\nu, N))} = \sum_{((\omega, \Omega)), ((\mu, M))} a_{((\omega, \Omega))} b_{((\mu, M))}$$

(this sum is finite).

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4See [3], [17], [16], [13] on continuations of the work [11].
For any subset $Z$ in $Y$ we have a homomorphism

$$\pi^Y_Z : \mathbb{C}[\mathcal{L}(Y)] \to \mathbb{C}[\mathcal{L}(Z)]$$

defined on generators by

$$\pi((\omega, \Omega)) = \begin{cases} (\omega, \Omega), & \text{if } \Omega \subset Z; \\ 0, & \text{otherwise.} \end{cases}$$

For a finite set $Y$ we have a homomorphism of algebras

$$\iota : \mathbb{C}[\mathcal{L}(Y)] \to \mathbb{C}[\mathcal{S}(Y)]$$
defined by

$$\iota\left(\sum_{(\sigma, \omega)} \sum_{\Omega} a_{(\sigma, \omega)} \omega \right) = \sum_{g \in \mathcal{S}(Y)} \left[ \sum_{\Omega \supset \{\text{support of } g\}} \left( \sum_{u \in \mathcal{S}(\Omega)} a_{(g | \omega)} \right) \right] g$$

(for an infinite set $Y$ the sum in square brackets is infinite).

2.6. Elements $B[g]$. Let $g \in G_\infty // K_\infty, g \in \Xi$. For any $n$-element subset $\Omega \subset \mathbb{N}$ we define an element of $\mathbb{C}[\mathcal{L}(V)]$ by

$$R(\overline{g}, \Omega) = \sum_{\sigma : J_n \to \Omega} ((\omega g \sigma^{-1}, X \cup (I \times \Omega)),$$

where the summation is taken over all bijections $\sigma : J_n \to \Omega$. Equivalently, we can choose one bijection $\sigma_0 : J_n \to \Omega$ and write the formula in the form

$$R(\overline{g}, \Omega) = \sum_{u \in \mathcal{S}_n} ((\sigma_0 u g^{-1} \sigma_0^{-1}, X \cup (I \times \Omega))).$$

Next, we define

$$B[\overline{g}] := \sum_{\Omega : \# \Omega = n} R(\overline{g}, \Omega).$$

By construction these elements are invariant with respect to conjugations by elements of $K_\infty$.

**Example.** Let $G_\infty$ be $S_\infty \times S_\infty$ acting on $\{1, 2\} \times \mathbb{N}$ as above. For $\overline{g} \in (S_N \times S_N) // S_N$ we take the corresponding $\mathcal{R} \in \Xi_N$. Then

$$B[\overline{g}] = \sum_{\Omega \subset \mathbb{N}, \# \Omega = N} \sum_{\pi} \mathcal{R}_\pi,$$

where the summation in the interior sum is taken over all bijective maps $\pi$ from $\Omega$ to the set of plus-triangles of $\mathcal{R}$. A checker triangulated surface equipped with such a map determines an element of the group $S(\Omega) \times S(\Omega)$. We consider this element as a local bijection of $\{1, 2\} \times \mathbb{N}$ with domain $\{1, 2\} \times \Omega$. 

\[ \square \]
For any $\overline{g} \in G_{n}/K_n$, $\overline{h} \in G_{m}/K_m$ the convolution $B[\overline{g}] * B[\overline{h}]$ is $K_{\infty}$-invariant, therefore it has the form

$$B[\overline{g}] * B[\overline{h}] := \sum_{\overline{r} \in \prod_{G_{j} \not\subset K_{j}} a_{\overline{g}, \overline{h}} B[\overline{r}].$$

### 2.7. Proof of Theorem 2.1

First, we apply the map $\pi_{V_{N}}$. For any $\overline{g}$ and $n \in \mathbb{N}$ we get an element $B[\overline{g}] := \pi_{V_{N}} B[\overline{g}]$. Since $\pi_{V_{N}}$ is a homomorphism of algebras, we get

$$B[\overline{g}] * B[\overline{h}] := \sum_{r \in \prod_{G_{j} \not\subset K_{j}} a_{\overline{g}, \overline{h}} B[\overline{r}],}$$

where $g$, $h$ range in $\prod_{k=0}^{N} G_{K_{k}}/S_{k}$.

Next, we apply the forgetting map $\iota : C[L(V_{n})] \to C[S(V_{n})]$ to elements $B[\overline{g}]$. Evidently, we get $\iota(B[\overline{g}]) = A[\overline{g}]$.

This implies our statement.

### 2.8. An expression for products

Let $g \in G_{n}$, $h \in G_{k}$. Let $\lambda$ ranges in the set of partial bijections $J_{k} \to J_{n}$. Fix $\lambda$, denote $d = \text{rk} \lambda$. Fix any pair of injective maps $\sigma_{0} : J_{n} \to J_{n+k-d}$, $\tau_{0} : J_{k} \to J_{n+k-d}$ such that the product $\sigma_{0}^{-1} \tau_{0}$ as a product of partial bijections is $\lambda$. Define the conjugacy class $g \circlearrowright_{\lambda} h := \sigma_{0}(\sigma_{V_{n}}) \sigma_{0}^{-1} \circ \tau_{0}(\tau_{V_{k}}) \tau_{0}^{-1} \in G_{n+k-d}/S_{n+k-d}$.

It does not depend on the choice of $\sigma_{0}$ and $\tau_{0}$.

**Theorem 2.2**

$$B[\overline{g}] * B[\overline{h}] = \sum_{\lambda \in \text{PB}(J_{k}, J_{n})} B[\overline{g} \circlearrowright_{\lambda} \overline{h}].$$ (2.2)

**Proof.** We expand the product $B[\overline{g}] * B[\overline{h}]$ by definition as a sum

$$\sum_{\sigma, \tau} \sigma((g, V_{n})) \sigma^{-1} \circ \tau((h, V_{k})) \tau^{-1} = \sum_{\sigma, \tau} ((\sigma g \sigma^{-1}, \sigma V_{n}) \circ ((\tau h \tau^{-1}, \tau V_{k})),$$ (2.3)

where $\sigma : J_{n} \to \mathbb{N}$, $\tau : J_{k} \to \mathbb{N}$ are injective maps. We wish to show that this expansion coincides with (2.2). It is sufficient to identify the following sub-sum of (2.3)

$$\sum_{\sigma, \tau : \sigma J_{n} \circlearrowright \tau J_{k} = J_{n+k-d}} ((\sigma g \sigma^{-1}, \sigma V_{n}) \circ ((\tau h \tau^{-1}, \tau V_{k}))$$ (2.4)
Figure 6: The left part. A product $\sigma((g, V_n)) \sigma^{-1} \circ \tau((h, V_k)) \tau^{-1}$. Here $k = 5$, $n = 7$, $d = 3$, $n + k - d = 9$. We draw 7 copies of the set $J_{n-k+1}$. Maps $\tau^{-1}$, $\tau$, $\sigma^{-1}$, $\sigma$ are shown by arcs. We mark $\sigma \dom \lambda = \tau^{-1} \im \lambda$ as fat points. Arcs containing fat points are thick. Boxes correspond to elements $((h, V_k)) \in G_k$ and $((g, V_n)) \in G_n$.

The right part. A canonical form of a pair $(\tau, \sigma)$. 
and the sub-sum of (2.2) consisting of all summands of the form

$$
\sum_{\lambda \in PB(J_k, J_n)} R(g \otimes h, J_{n+k-d}) =
\sum_{\lambda \in PB(J_k, J_n)} \sum_{u \in S_{n+k-d}} \langle u^{-1}(g \otimes h)u, V_{n+k-d} \rangle.
$$

(2.5)

Notice that the group $S_{n+k-d}$ acts on the set of summands of (2.5) permuting
summands of the interior sum. The corresponding action on (2.4) has the form

$u : (\sigma, \tau) \mapsto (u^{-1}\sigma, u^{-1}\tau)$.

We refer to Fig. 6. The second action corresponds to simultaneous application
of a substitution $u$ to rows number 1, 4, 7. Rows 2, 3, 5, 6 remain to be fixed,
and arcs are moved by corresponding permutations of their ends. Applying an
appropriate $u$ we can put fat points to positions 1, 2, ..., $d$. Moreover, we can
make $\tau^{-1}u$ monotone on $J_d$:

$$i < j \leq d \quad \Rightarrow \quad \tau^{-1}u(i) < \tau^{-1}u(j).$$

(on Fig. 6 this means that the corresponding arcs have no intersections).

Next, we can put points of $\tau J_k \setminus \tau J_d$ to points of $J_k \setminus J_d$. Moreover, we can
make $\tau^{-1}u$ monotone on $J_k \setminus J_d$. Finally, we can put points of $\sigma J_n \setminus \sigma J_d$
to points of $J_{n+k-d} \setminus J_k$. Moreover, we can make $\sigma^{-1}u$ monotone on $J_{n+k-d} \setminus J_k$.

This determines $u$ in a unique way. On the other hand, the partial bijection $\sigma^{-1}\tau$ does not changed under this transformation, and the new pair
$(u^{-1}\sigma, u^{-1}\tau)$ is uniquely determined by dom $\sigma^{-1}\tau$, im $\sigma^{-1}\tau$ and the map $\sigma^{-1}u : J_d \rightarrow$ im $\sigma^{-1}\tau$.

Thus, we see that orbits of the group $S_{n+k-d}$ on the set of summands of
(2.4) are enumerated by partial bijections $\lambda$ and stabilizers are trivial. □

3 Final remarks

3.1. The involution. The map $g \mapsto g^{-1}$ determines an anti-involution in the
algebra $L[V]$ and anti-involution in $B[G_{\infty} \varprojlim K_{\infty}]$,

$$B[\bar{g}]^* := B[\bar{g^{-1}}].$$

Evidently,

$$(B[\bar{g}] * B[\bar{h}])^* = B[\bar{h}]^* * B[\bar{g}]^*.$$  

3.2. The filtration. Fix $G_{\infty}$ and set

$$B := B[G_{\infty} \varprojlim K_{\infty}].$$

Let $B_n$ be the subspace in $B$ generated by all

$$B[\bar{g}], \quad where \ g \ ranges \ in \ \prod_{j=0}^{n} G_j \varprojlim S_j.$$

15
We get an increasing filtration,

\[ \cdots \subset B_n \subset B_{n+1} \subset \cdots \]

Evidently,

\[ V \in B_k, \quad W \in B_l \Rightarrow V \ast W \in B_{k+l}. \]

3.3. The associated graded algebra. We construct the graded algebra \( \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] \) in the usual way. Namely, the product

\[ B_k \times B_l \to B_{k+l} \]
determines a map

\[ B_k / B_{k-1} \times B_l / B_{l-1} \to B_{k+l} / B_{k+l-1}. \]

In this way, we get a structure of an associative algebra on

\[ \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] = \bigoplus_{k=0}^{\infty} B_k / B_{k-1}. \]

Each subspace \( B_k / B_{k-1} \) has a natural basis enumerated by elements of \( G_k / S_k \).

It is easy to describe the multiplication in \( \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] \): in the sum in the right-hand side of (2.2) we leave only the first summand (corresponding to the partial bijection of rank 0).

Formulate this more precisely. For \( n, k \in \mathbb{Z}_+ \) denote by \( \theta_{n,k} \) the partial bijection \( J_n \to J_{n+k} \setminus J_k \) defined by \( j \mapsto j + k \).

\textbf{Proposition 3.1} a) For \( \overline{g} \in G_n / S_n, \overline{h} \in G_k / S_k \), their product in graded algebra \( \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] \) is

\[ B[\overline{g}] \circ B[\overline{h}] = B[\theta_{n,k} \langle g, V_n \rangle \theta_{n,k}^{-1} \circ \langle h, V_k \rangle]. \]

b) If \( X = \emptyset \), then the algebra \( \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] \) is commutative.

In fact, we get a semigroup structure on \( \prod_{j=0}^{\infty} G_j / S_j \) given by

\[ \overline{g} \bullet \overline{h} := \theta_{n,k} \langle g, V_n \rangle \theta_{n,k}^{-1} \circ \langle h, V_k \rangle. \]

The algebra \( \text{gr} \mathcal{B}[G_{\infty} / K_{\infty}] \) is the semigroup algebra of this semigroup.

\textbf{Example}. For \( G_{\infty} = S_{\infty} \times S_{\infty} \) the \( \bullet \)-product corresponds to the disjoint union of checker triangulated surfaces.

\textbf{Remark}. These semigroups are similar to semigroups of double cosets, which were considered in [21], [20]. However, degree of generality in [21] is wider (as we noticed in Introduction, conjugacy classes are special cases of double cosets and not vice versa), even in the same situations we get slightly different structures. For instance, for \( G_{\infty} = S_{\infty} \times S_{\infty} \) adding to a checker triangulated
surface $R \in \Xi_j$ a collection of $k$ double triangles (drawn on Fig. 2) we get different objects (this corresponds to embeddings $S_j \times S_j \to S_{k+j} \times S_{k+j}$). However, such elements in [20] are identified. Notice that we have a natural identification of sets

$$
\prod_{j=0}^{\infty} G_j \lmod K_j \simeq (G_{\infty} \lmod K_{\infty}) \times \mathbb{Z}_+,
$$

where $\mathbb{Z}_+$ denotes the set of non-negative integers. The left-hand side enumerates elements of the basis in $B[G_{\infty} \lmod K_{\infty}]$. The semigroup

$$G_{\infty} \lmod K_{\infty} \simeq K_{\infty} \setminus G_{\infty} / K_{\infty}$$

is one of objects of [20].

3.4. The Poisson bracket. Let $X = \emptyset$, in particular, in this case $\gr B[G_{\infty} \lmod K_{\infty}]$ is commutative. Consider the map

$$B_k \times B_l \to B_{k+l-1}$$

given by

$$(V, W) \mapsto [V, W] = V \ast W - W \ast V.$$  

As usual, we get a map

$$B_k / B_{k-1} \times B_l / B_{l-1} \to B_{k+l-1} / B_{k+l-2}$$

and a structure of a Lie algebra on the space $\gr B[G_{\infty} \lmod K_{\infty}]$. It is easy to write a formula for the bracket

$$[B[\overline{g}], B[\overline{h}]]_{\gr} = \sum_{\lambda \in \PB(J_k, J_n), \rk \lambda = 1} \left( B[\overline{g} \oplus_{\lambda} \overline{h}] - B[\overline{h} \oplus_{\lambda^{-1}} \overline{g}] \right).$$

Of course, a partial bijection $J_k \to J_n$ of rank 1 is determined by a pair $\alpha \in J_k$, $\beta \in J_n$.

Remark. Recall that there is a well-known Poisson structure on spaces $(K \times \cdots \times K) \lmod \diag K$, where $K$ is a compact Lie group, see [9], [10].

References

[1] Alexeevski A. V., Natanzon S. M. Algebras of conjugacy classes of partial elements. In Topology, geometry, integrable systems and mathematical physics. Novikov’s seminar: 2012–2014. Providence, RI: American Mathematical Society, 1-11 (2014).

[2] Belyi G. V. Galois extensions of a maximal cyclotomic field. Math. USSR-Izvestiya, 1980, 14:2, 247–256.

[3] Bump D. Lie groups. Second edition. Springer, New York, 2013.

17
[4] Gaifullin A. A., Neretin Yu. A., Infinite symmetric group, pseudomanifolds, and combinatorial cobordism-like structures. J. Topol. Anal., J. Topol. Anal. 10 (2018), no. 3, 605-625.

[5] Flensted-Jensen M., Koornwinder T. H. Jacobi functions: the addition formula and the positivity of the dual convolution structure. Ark. Mat. 17 (1979), no. 1, 139-151.

[6] Fock, V. V., Rosly, A. A. Flat connections and polyubles. Theoret. and Math. Phys. 95 (1993), no. 2, 526-534.

[7] Gelfand I. M., Spherical functions on Riemannian symmetric spaces. (Russian) Doklady, 70 (1950), 5-8.

[8] Gironio, E., González-Diez, G. Introduction to compact Riemann surfaces and dessins d’enfants. Cambridge University Press, Cambridge, 2012.

[9] Goldman, W. M. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math. 85 (1986), no. 2, 263-302.

[10] Ismagilov R. S., Elementary spherical functions on the group SL(2, P) over a field P, which is not locally compact, with respect to the subgroup of matrices with integral elements, Math. USSR-Izv., 1:2 (1967), 349-380.

[11] Ivanov V., Kerov S., The Algebra of Conjugacy Classes in Symmetric Groups and Partial Permutations, Zapiski nauchn.semin. POMI RAN, 256 (1999), 95–120; J. of Math. Sci. (New York), 2001, 107:5, 4212-4230.

[12] Iwahori N., On the structure of a Hecke ring of a Chevalley group over a finite field., J. Fac. Sci. Univ. Tokyo Sect. I, 10 (1964), 215-236.

[13] Iwahori N., Matsumoto H., On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Inst. Hautes Études Sci. Publ. Math., 25:1 (1965), 5-48.

[14] Kannan A. S., Ryba Ch. Stable Centres II: Finite Classical Groups. Preprint, https://arxiv.org/abs/2112.01467

[15] Koornwinder T. H., Jacobi functions and analysis on noncompact semisimple Lie groups, in Special Functions: Group Theoretical Aspects and Applications, Reidel, Dordrecht, 1984, 1-85.

[16] Méliot P.-L. Partial isomorphisms over finite fields. J. Algebraic Combin. 40 (2014), no. 1, 83-136.

[17] Mironov A., Morozov A., Natanzon, S. Infinite-dimensional topological field theories from Hurwitz numbers. J. Knot Theory Ramifications 23, No. 6, Article ID 1450033, 16 p. (2014).

[18] Neretin Yu.A. Categories of symmetries and infinite-dimensional groups. Oxford University Press, 1996.
[19] Neretin Yu. A., *Multi-operator colligations and multivariate characteristic functions*, Anal. Math. Phys., 1:2-3 (2011), 121-138.

[20] Neretin Yu. A., *Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories*. Int. Math. Res. Not., 2012:3 (2012), 501-523.

[21] Neretin Yu.A., *Infinite symmetric groups and combinatorial constructions of topological field theory type*. Russ. Math. Surv., 2015, 70:4, 715-773.

[22] Neretin Yu. A., *Multiplication of conjugacy classes, colligations, and characteristic functions of matrix argument*. Funct. Anal. Appl., 51:2 (2017), 98-111.

[23] Olshanski G. I., *Unitary representations of infinite dimensional pairs (G, K) and the formalism of R. Howe*, in *Representation of Lie groups and related topics*, Adv. Stud. Contemp. Math., 7, Gordon and Breach, New York, 1990, 269-463.

[24] Olshanski G. I., *Unitary representations of (G, K)-pairs that are connected with the infinite symmetric group S(∞)*. Leningrad Math. J., 1:4 (1990), 983-1014.

[25] Shabat G. *Calculating and drawing Belyi pairs*. Zapiski nauchn. semin. POMI RAN 446, 2016, 182-220; also, J. Math. Sci. (N.Y.) 226 (2017), no. 5, 667-693.

[26] Shabat, G. B., Voevodsky, V. A. *Drawing curves over number fields*. in *The Grothendieck Festschrift*, Vol. III, 199-227, Birkhäuser, Boston, Boston, MA, 1990.

[27] Schneps L. (ed.), *The Grothendieck theory of dessins d’enfants*. Cambridge University Press, Cambridge, 1994.

[28] Schneps L., Lochak L. (eds.) *Geometric Galois actions*. V. 1, 2. Cambridge University Press, Cambridge, 1997.

[29] Yokonuma T. *Sur la structure des anneaux de Hecke d’un groupe de Chevalley fini*. C. R. Acad. Sci. Paris Sér. A-B 264 (1967), A344-A347

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