ON DYSON’S CRANK CONJECTURE
AND THE UNIFORM ASYMPTOTIC BEHAVIOR
OF CERTAIN INVERSE Theta FUNCTIONS

KATHRIN BRINGMANN AND JEHANNE DOUSSE

Abstract. In this paper we prove a longstanding conjecture by Freeman
Dyson concerning the limiting shape of the crank generating function. We
fit this function in a more general family of inverse theta functions which play
a key role in physics.

1. Introduction and statement of results

Dyson’s crank was introduced to explain Ramanujan’s famous partition congru-
ences with modulus 5, 7, and 11. Denoting for \( n \in \mathbb{N} \) by \( p(n) \) the number of integer
partitions of \( n \), Ramanujan \cite{Ramanujan1920} proved that for \( n \geq 0 \)
\[
p(5n + 4) \equiv 0 \pmod{5},
p(7n + 5) \equiv 0 \pmod{7},
p(11n + 6) \equiv 0 \pmod{11}.
\]
A key ingredient of his proof is the modularity of the partition generating function
\[
P(q) := \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty} = \frac{q^{1/24}}{\eta(\tau)},
\]
where for \( j \in \mathbb{N}_0 \cup \{ \infty \} \) we set \((a)_j = (a;q)_j := \prod_{\ell=0}^{j-1} (1 - aq^\ell)\), \( q := e^{2\pi i \tau} \), and \( \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) is Dedekind’s \( \eta \)-function, a modular form of weight \( \frac{1}{2} \).

Ramanujan’s proof however gives little combinatorial insight into why the above
congruences hold. In order to provide such an explanation, Dyson \cite{Dyson1944} famously
introduced the rank of a partition, which is defined as its largest part minus the
number of its parts. He conjectured that the partitions of \( 5n + 4 \) (resp. \( 7n + 5 \))
form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp.
7). This conjecture was proven by Atkin and Swinnerton-Dyer \cite{Atkin1966}. Ono and the
first author \cite{Ono2003} showed that partitions with given rank also satisfy Ramanujan-type
congruences. Dyson further postulated the existence of another statistic which he
called the “crank” and which should explain all Ramanujan congruences. The crank
was later found by Andrews and Garvan \cite{Andrews1984, Andrews1985}. If for a partition \( \lambda \), \( o(\lambda) \) denotes
the number of ones in \( \lambda \), and \( \mu(\lambda) \) is the number of parts strictly larger than \( o(\lambda) \),
then the crank of \( \lambda \) is defined as
\[
\text{crank}(\lambda) := \begin{cases} 
\text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\
\mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0.
\end{cases}
\]
Denote by $M(m, n)$ the number of partitions of $n$ with crank $m$. Mahlburg \cite{Mah} then proved that partitions with fixed crank also satisfy Ramanujan-type congruences. In this paper, we solve a longstanding conjecture by Dyson \cite{Dy} concerning the limiting shape of the crank generating function.

**Conjecture 1.1** (Dyson). As $n \to \infty$ we have

$$M(m, n) \sim \frac{1}{4} \beta \text{sech}^2 \left( \frac{\beta m}{2} \right) p(n)$$

with $\beta := \frac{\pi}{\sqrt{6}}$.

Dyson then asked the question about the precise range of $m$ in which this asymptotic holds and about the error term. In this paper, we answer all of these questions.

**Theorem 1.2.** The Dyson-Conjecture is true. To be more precise, if $|m| \leq \beta \sqrt{\frac{1}{6}} \log n$, we have as $n \to \infty$

$$M(m, n) = \frac{\beta}{4} \text{sech}^2 \left( \frac{\beta m}{2} \right) p(n) \left( 1 + O \left( \beta^\frac{1}{2} |m|^{\frac{3}{2}} \right) \right).$$

**Remarks.**

1. For fixed $m$ one can directly obtain asymptotic formulas since the generating function is the convolution of a modular form and a partial theta function \cite{Bou}. However, Dyson’s conjecture is a bivariate asymptotic. Indeed, this fact is the source of the difficulty of this problem.

2. We note that Theorem 1.2 is of a very different nature than known asymptotics in the literature. For example, the partition function can be approximated as

$$p(n) = M(n) + O \left( n^{-\alpha} \right),$$

where $M(n)$ is the main term which is a sum of varying length of Kloosterman sums and Bessel functions and $\alpha > 0$. Rademacher \cite{Rad} obtained $\alpha = \frac{3}{8}$, Lehmer improved this to $\frac{1}{2} - \varepsilon$ and Folsom and Masri \cite{FoMa} in their recent work obtained an impressive error of $n^{-\delta}$ for some absolute $\delta > \frac{1}{2}$. Our result has a very different flavor due to the nonmodularity of the generating function and the bivariate asymptotics.

3. In fact we could replace the error by $O(\beta^\frac{1}{2} |m|^{\alpha}(m))$ for any $\alpha(m)$ such that $\frac{\log n}{n^\frac{1}{4}} = o(\alpha(m))$ for all $|m| \leq \frac{1}{\pi \sqrt{6}} \sqrt{n} \log n$ and $\beta m \alpha(m) \to 0$ as $n \to \infty$. Here we chose $\alpha(m) = |m|^{-\frac{1}{2}}$ to avoid complicated expressions in the proof.

A straightforward calculation shows

**Corollary 1.3.** Almost all partitions satisfy Dyson’s conjecture. To be more precise

$$\sharp \left\{ \lambda \vdash n | \text{crank}(\lambda) | \leq \frac{\sqrt{n}}{\pi \sqrt{6}} \log n \right\} \sim p(n).$$

**Remarks.**

1. We thank Karl Mahlburg for pointing out Corollary 1.3 to us.

2. We can improve (1.2) and give the size of the error term.
Dyson’s conjecture follows from a more general result concerning the coefficients $M_k(m, n)$ defined for $k \in \mathbb{N}$ by

$$C_k(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m, n) \zeta^m q^n := \frac{(q)_{\infty}^{2-k}}{(\zeta q)_{\infty} ((\zeta^{-1} q)_{\infty}^k)}.$$

Note that $M(m, n) = M_1(m, n)$. Denoting by $p_k(n)$ the number of partitions of $n$ allowing $k$ colors, we have

**Theorem 1.4.** For $k$ fixed and $|m| \leq \frac{1}{6\beta_k} \log n$, we have as $n \to \infty$

$$M_k(m, n) = \beta_k \frac{4}{2} \text{sech}^2 \left( \frac{\beta_km}{2} \right) p_k(n) \left( 1 + O \left( \beta_k^2 |m|^\frac{1}{2} \right) \right),$$

with $\beta_k := \pi \sqrt{\frac{k}{6n}}$.

**Remarks.**

1. We note that for $k \geq 3$, the functions $C_k(\zeta; q)$ are well known to be generating functions of Betti numbers of moduli spaces of Hilbert schemes on $(k-3)$-point blow-ups of the projective plane [13] (see also [6] and the references therein). The results of this paper immediately give the limiting profile of the Betti numbers for large second Chern class of the sheaves. Recently, Hausel and Rodriguez-Villegas [16] also determined profiles of Betti numbers for other moduli spaces.

2. Note that our method of proof would allow determining further terms in the asymptotic expansion of $M_k(m, n)$.

3. Again we could replace the error by $O(\beta_k^2 |m|^\alpha)$ for any $\alpha_k(m)$ such that $\log n = o(\alpha_k(m))$ for all $|m| \leq \frac{1}{6\beta_k} \log n$ and $\beta_k |m| \to 0$ as $n \to \infty$.

4. The function $C_k$ can also be represented as a so-called Lerch sum. To be more precise, we have [1]

$$\sum_{m \in \mathbb{Z}} M(m, n) \zeta^m q^n = \frac{1 - \zeta}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(n+1)}{2}} \frac{1 - \zeta q^n}{1 - \zeta q^n}.$$

This representation, which was a key representation in [6], is not used in this paper.

5. The special case $k = 2$ yields the birank of partitions [14].

6. We expect that our methods also apply to show an analogue of (1.1) for the rank. The case of fixed $m$ is considered in upcoming work by Byungchan Kim, Eunmi Kim, and Jeehyeon Seo [17].

This paper is organized as follows. In Section 2, we recall basic facts on modular and Jacobi forms which are the base components of $C_k$ and collect properties on Euler polynomials. In Section 3, we determine the asymptotic behavior of $C_k$. In Section 4, we use Wright’s version of the Circle Method to finish the proof of Theorem 1.4. In Section 5, we illustrate Theorem 1.2 numerically.
2. Preliminaries

2.1. Modularity of the generating functions. A key ingredient of our asymptotic results is to employ the modularity of the functions $C_k$. To be more precise, we write (throughout $q := e^{2\pi i \tau}$, $\zeta := e^{2\pi i w}$ with $\tau \in \mathbb{H}, w \in \mathbb{C}$)

$$C_k(\zeta; q) = \frac{i \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) q^{\frac{k}{2}} \eta^3 - k(\tau)}{\vartheta(w; \tau)},$$

where

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

$$\vartheta(w; \tau) := i \zeta^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n) \left( 1 - \zeta q^n \right) \left( 1 - \zeta^{-1} q^{-n-1} \right).$$

The function $\eta$ is a modular form, whereas $\vartheta$ is a Jacobi form. To be more precise, we have the following transformation laws (see e.g. [20]).

Lemma 2.1. We have

$$\eta \left( \frac{-1}{\tau} \right) = \sqrt{-i\pi} \eta(\tau),$$

$$\vartheta \left( \frac{w}{\tau}; \frac{-1}{\tau} \right) = -i \sqrt{-i\pi} e^{\frac{\pi i w^2}{\tau}} \vartheta(w; \tau).$$

2.2. Euler polynomials. Recall that the Euler polynomials may be defined by their generating function

$$(2.2) \quad \frac{2e^{xt}}{e^t + 1} =: \sum_{r=0}^{\infty} E_r(x) \frac{t^r}{r!}.$$ 

The following lemma may easily be concluded by differentiating the generating function (2.2). For the reader’s convenience we give a proof.

Lemma 2.2. We have

$$-\frac{1}{2} \sech^2 \left( \frac{t}{2} \right) = \sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r}}{(2r)!}.$$ 

Proof. We have

$$\sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r}}{(2r)!} = \frac{d}{dt} \sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r+1}}{(2r+1)!}.$$ 

Now

$$\frac{2}{e^t + 1} = \sum_{r=0}^{\infty} E_r(0) \frac{t^r}{r!},$$

$$\frac{2}{e^{-t} + 1} = \sum_{r=0}^{\infty} E_r(0) \frac{(-t)^r}{r!}.$$ 

Taking the difference gives the claim of the lemma since

$$\frac{d}{dt} \left( \frac{1}{e^t + 1} - \frac{1}{e^{-t} + 1} \right) = -\frac{1}{2} \sech^2 \left( \frac{t}{2} \right).$$
We also require an integral representation of Euler polynomials. To be more precise, setting for \( j \in \mathbb{N}_0 \)

\[
E_j := \int_0^\infty \frac{w^{2j+1}}{\sinh(\pi w)} dw,
\]

we obtain

**Lemma 2.3.** We have

\[
E_j = \frac{(-1)^{j+1} E_{2j+1}(0)}{2}.
\]

**Proof.** We make the change of variables \( w \to w + \frac{i}{2} \) and then use the Residue Theorem to shift the path of integration back to the real line. Using the Binomial Theorem, we may thus write

\[
E_j = -\frac{i}{2} \int_\mathbb{R} \frac{(w + \frac{i}{2})^{2j+1}}{\cosh(\pi w)} dw = -\frac{i}{2} \sum_{\ell=0}^{2j+1} \binom{2j+1}{\ell} \left( \frac{i}{2} \right)^{2j+1-\ell} \int_\mathbb{R} \frac{w^\ell}{\cosh(\pi w)} dw.
\]

The last integral is known to equal \(-2i\)^{\ell} E_\ell, where \( E_\ell := 2\ell E_\ell(\frac{1}{2}) \) denotes the \( \ell \)th Euler number (see page 41 of [10]). The claim now follows using the well-known identity (see page 41 of [10])

\[
E_j(x) = \sum_{\ell=0}^{j} \binom{j}{\ell} \left( x - \frac{1}{2} \right)^{j-\ell} E_\ell \frac{2^\ell}{2^\ell}.
\]

\[\Box\]

### 3. Asymptotic behavior of the function \( C_k \).

Since \( M_k(-m,n) = M_k(m,n) \) we from now on assume that \( m \geq 0 \). The goal of this section is to study the asymptotic behavior of the generating function of \( M_k(m,n) \). We define

\[
C_{m,k}(q) := \sum_{n=0}^\infty M_k(m,n) q^n = \int_{-\frac{i}{2}}^{\frac{i}{2}} C_k(e^{2\pi i w};q) e^{-2\pi i mw} dw
\]

\[
= 2 \frac{\eta^{\pi i \tau}}{\eta^k(\tau)} \int_0^{\frac{1}{2}} g(w;\tau) \cos(2\pi mw) dw,
\]

where

\[
g(w;\tau) := \frac{i \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \eta^3(\tau)}{\vartheta(w;\tau)}
\]

Here we used that \( g(-w;\tau) = g(w;\tau) \). In this section we determine the asymptotic behavior of \( C_{m,k}(q) \), when \( q \) is near an essential singularity on the unit circle. It turns out that the dominant pole lies at \( q = 1 \). Throughout the rest of the paper let \( \tau = \frac{iz}{2\pi}, z = \beta_k(1 + ixm^{-\frac{1}{3}}) \) with \( x \in \mathbb{R} \) satisfying \( |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta_k} \).

#### 3.1. Bounds near the dominant pole.

In this section we consider the range \( |x| \leq 1 \). We start by determining the asymptotic main term of \( g \). Lemma 2.1 and the definition of \( \vartheta \) and \( \eta \) immediately imply

**Lemma 3.1.** For \( 0 \leq w \leq 1 \) we have for \( |x| \leq 1 \) as \( n \to \infty \)

\[
g \left( w; \frac{iz}{2\pi} \right) = \frac{2\pi \sin(\pi w)}{z \sinh(\frac{\pi z}{2})} e^{\frac{2\pi^2 w^2}{z}} \left( 1 + O \left( e^{-4\pi^2(1-w)Re(\frac{1}{2})} \right) \right).
\]
In view of Lemma 3.1 it is therefore natural to define
\[ G_{m,1}(z) := \frac{4\pi}{z} \int_0^{\frac{1}{2}} \frac{\sin(\pi w)}{\sinh\left(\frac{2\pi^2 w}{z}\right)} e^{\frac{2\pi^2 w^2}{z}} \cos(2\pi mw) \, dw, \]
\[ G_{m,2}(z) := 2 \int_0^{\frac{1}{2}} \left( g\left( \frac{i z}{2\pi} \right) - \frac{2\pi \sin(\pi w)}{z \sinh\left(\frac{2\pi^2 w}{z}\right)} e^{\frac{2\pi^2 w^2}{z}} \right) \cos(2\pi mw) \, dw. \]
Thus
\[ (3.1) \quad C_{m,k}(q) = \frac{q \pi}{\eta^k(\tau)} (G_{m,1}(z) + G_{m,2}(z)). \]
The dominant contribution comes from \( G_{m,1}. \)

**Lemma 3.2.** Assume that \(|x| \leq 1\) and \(m \leq \frac{1}{6\beta_k} \log n\). Then we have as \(n \to \infty\)
\[ G_{m,1}(z) = \frac{z}{4} \operatorname{sech}\left(\frac{\beta_k m}{2}\right) + O\left(\beta^2 m^2 \operatorname{sech}\left(\frac{\beta_k m}{2}\right)\right). \]

**Proof.** Inserting the Taylor expansion of \(\sin, \exp, \) and \(\cos,\)
\[ \sin(\pi w) e^{\frac{2\pi^2 w^2}{z}} \cos(2\pi mw) = \sum_{j,\nu,r \geq 0} \frac{(-1)^j + \nu}{(2j + 1)! (2\nu)! r!} \pi^{2j+1} (2\pi m)^{2\nu} \left(\frac{2\pi^2}{z}\right)^r w^{2j+2\nu+2r+1}. \]
This yields that
\[ G_{m,1}(z) = \frac{4\pi}{z} \sum_{j,\nu,r \geq 0} \frac{(-1)^j + \nu}{(2j + 1)! (2\nu)! r!} \pi^{2j+1} (2\pi m)^{2\nu} \left(\frac{2\pi^2}{z}\right)^r I_{j+\nu+r} \]
where for \(\ell \in \mathbb{N}_0\) we define
\[ I_\ell := \int_0^{\frac{1}{2}} \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)} \, dw. \]
We next relate \(I_\ell\) to \(E_\ell\) defined in (2.3). For this, we note that
\[ (3.2) \quad I_\ell = \int_0^{\infty} \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)} \, dw - I'_\ell \]
with
\[ I'_\ell := \int_{\frac{1}{2}}^{\infty} \frac{w^{2\ell+1}}{\sinh\left(\frac{2\pi^2 w}{z}\right)} \, dw \ll \int_{\frac{1}{2}}^{\infty} w^{2\ell+1} e^{-2\pi^2 w \operatorname{Re}(\frac{1}{z})} \, dw \]
\[ \ll \left( \operatorname{Re}\left(\frac{1}{z}\right) \right)^{-2\ell-2} \Gamma \left(2\ell + 2; \pi^2 \operatorname{Re}\left(\frac{1}{z}\right) \right). \]
Here \(\Gamma(\alpha; x) := \int_x^{\infty} e^{-w} w^{\alpha-1} \, dw\) denotes the incomplete gamma function and throughout \(g(x) \ll f(x)\) means that \(g(x) = O\left(f(x)\right).\) Using that as \(x \to \infty\)
\[ (3.3) \quad \Gamma(\ell; x) \sim x^{\ell-1} e^{-x} \]
thus yields that
\[ I'_\ell \ll \left( \operatorname{Re}\left(\frac{1}{z}\right) \right)^{-1} e^{-\pi^2 \operatorname{Re}(\frac{1}{z})} \leq e^{-\pi^2 \operatorname{Re}(\frac{1}{z})}. \]
In the first summand in (3.2) we make the change of variables $w \rightarrow zw^2\pi$ and then shift the path of integration back to the real line by the Residue Theorem. Thus we obtain that
\[
\int_0^\infty \frac{w^{2\ell+1}}{\sinh \left(\frac{2\pi w}{z}\right)} dw = \left(\frac{z}{2\pi}\right)^{2\ell+2} \mathcal{E}_\ell = \left(\frac{z}{2\pi}\right)^{2\ell+2} \frac{(-1)^{\ell+1}E_{2\ell+1}(0)}{2},
\]
where for the last equality we used Lemma 2.3. Thus
\[
G_{m,1}(z) = \sum_{j,\nu,r \geq 0} \frac{(-1)^{r+1}}{2^{2j+r+1}(2j+1)!(2\nu)!r!} m^{2\nu} z^{2j+2\nu+r+1}
\times \left(E_{2j+2\nu+2r+1}(0) + O\left(|z|^{2j-2\nu-2r-2}e^{-\pi^2\text{Re}\left(\frac{1}{z}\right)}\right)\right)
= \sum_{\nu=0}^{\infty} \frac{(mz)^{2\nu}}{(2\nu)!} \left(- \frac{z}{2} E_{2\nu+1}(0) + O\left(|z|^2\right)\right) = \frac{z}{4} \text{sech}^2 \left(\frac{mz}{2}\right) + O\left(|z|^2 \cosh(mz)\right),
\]
where for the last equality we used Lemma 2.2. To finish the proof we have to approximate $\text{sech}^2 \left(\frac{mz}{2}\right)$ and $\cosh(mz)$. We have
\[
\cosh(mz) = \cosh \left(\beta_k m + i\beta_k m^\frac{2}{3} x\right)
= \cosh(\beta_k m) \cos \left(\beta_k m^\frac{2}{3} x\right) + i \sinh(\beta_k m) \sin \left(\beta_k m^\frac{2}{3} x\right)
= \cosh(\beta_k m) \left(1 + O \left(\beta_k m^\frac{2}{3}\right)\right).
\]
This implies that
\[
\text{sech} \left(\frac{mz}{2}\right) = \frac{1}{\cosh \left(\frac{mz}{2}\right)} = \frac{1}{\cosh \left(\frac{\beta_k m}{2}\right) \left(1 + O \left(\beta_k m^\frac{2}{3}\right)\right)},
\]
yielding
\[
\text{sech}^2 \left(\frac{mz}{2}\right) = \text{sech}^2 \left(\frac{\beta_k m}{2}\right) \left(1 + O \left(\beta_k m^\frac{2}{3}\right)\right).
\]
Thus we obtain
\[
G_{m,1}(z) = \frac{z}{4} \text{sech}^2 \left(\frac{\beta_k m}{2}\right) \left(1 + O \left(\beta_k m^\frac{2}{3}\right)\right) + O \left(\beta_k^2 \left(1 + \frac{1}{m^\frac{2}{3}}\right) \cosh(\beta_k m)\right)
= \frac{z}{4} \text{sech}^2 \left(\frac{\beta_k m}{2}\right) + O \left(\beta_k^2 m^\frac{2}{3} \text{sech}^2 \left(\frac{\beta_k m}{2}\right) + O \left(\beta_k^2 \cosh(\beta_k m)\right)\right).
\]

We may now easily finish the proof distinguishing the cases on whether $\beta_k m$ is bounded or goes to $\infty$. □

We next turn to bounding $G_{m,2}$.

**Lemma 3.3.** Assume that $|x| \leq 1$. Then we have as $n \to \infty$
\[
G_{m,2}(q) \ll \frac{1}{\beta_k} e^{-\frac{5n^2}{4\beta_k}}.
\]

**Proof.** By Lemma 3.1 we obtain that
\[
G_{m,2}(z) \ll \frac{1}{|z|} \int_0^\frac{\pi}{2} \left|\frac{\sin(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}}\right| e^{2\pi^2 \text{Re}\left(\frac{1}{z}\right)(w^2+w-2)} dw.
\]
It is not hard to see that \[
\left| \frac{\sin(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}} \right| \ll 1.
\]

Moreover,
\[
|z| = \beta_k \sqrt{1 + m - \frac{2}{3}x^2} \gg \beta_k,
\]
\[
\Re \left( \frac{1}{z} \right) \geq \frac{1}{2\beta_k}.
\]

The claim now follows, using that the maximum of \(w^2 + w - 2\) on \([0, \frac{1}{2}]\) is obtained for \(w = \frac{1}{2}\).

Combining the above yields

**Proposition 3.4.** Assume that \(|x| \leq 1\). Then we have as \(n \to \infty\)
\[
C_{m,k}(q) = \frac{z^{\frac{k}{2}+1}}{4(2\pi)^{\frac{k}{2}}} sech^2 \left( \frac{\beta_km}{2} \right) e^{\frac{\kappa n^2}{2\pi}} + O \left( \beta_k^2 + \frac{m^3}{2} \right) e^{\frac{\pi}{\sqrt{6n}}}. \]

**Proof.** Recall from (3.1) that
\[
C_{m,k}(q) = \frac{q^{\frac{k}{2}}}{\eta_k(\tau)} (G_{m,1}(z) + G_{m,2}(z)).
\]

Lemma 2.1 easily gives that
\[
\frac{q^{\frac{k}{2}}}{\eta_k(\tau)} = \left( \frac{z}{2\pi} \right)^{\frac{k}{2}} e^{\frac{\kappa n^2}{2\pi}} (1 + O(\beta_k)).
\]

The functions \(G_{m,1}\) and \(G_{m,2}\) are now approximated using Lemma 3.2 and Lemma 3.3, respectively. It is not hard to see that the main error term arises from approximation \(G_{m,1}\). We thus obtain
\[
C_{m,k}(q) = \frac{z^{\frac{k}{2}+1}}{4(2\pi)^{\frac{k}{2}}} sech^2 \left( \frac{\beta_km}{2} \right) + O \left( |z| \beta_k^2 m^3 sech^2 \left( \frac{\beta_km}{2} \right) e^{\frac{\pi}{\sqrt{6n}} \Re(\frac{1}{z})} \right).
\]

The claim follows now using that
\[
|z| \ll \beta_k,
\]
\[
\Re \left( \frac{1}{z} \right) \leq \frac{1}{\beta_k} = \frac{\sqrt{6n}}{\pi \sqrt{k}}.
\]

**3.2. Bounds away from the dominant pole.** We next investigate the behavior of \(C_{m,k}\) away from the dominant cusp \(q = 1\). To be more precise, we consider the range \(1 \leq x \leq \frac{\pi m \sqrt{3}}{\beta_k}\). Let us start with the following lemma, which proof uses the same idea as in [23].

**Lemma 3.5.** Assume that \(\tau = u + iv \in \mathbb{H}\) with \(Mv \leq |u| \leq \frac{1}{2}\) for \(u > 0\) and \(v \to 0\). We have that
\[
|P(q)| \ll \sqrt{v} \exp \left[ \frac{1}{v} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + M^2}} \right) \right) \right].
\]
Proof. We rewrite
\[
\log(P(q)) = -\sum_{n=1}^{\infty} \log(1 - q^n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{nm}}{m} = \sum_{m=1}^{\infty} \frac{q^m}{m(1 - q^m)}.
\]
Therefore we may estimate
\[
|\log(P(q))| \leq \sum_{m=1}^{\infty} \frac{|q|^m}{m|1 - q^m|} \leq \frac{|q|}{|1 - q|} - \frac{|q|}{1 - |q|} + \sum_{m=1}^{\infty} \frac{|q|^m}{m(1 - |q|^m)}.
\]
We now split \( u \) into 2 ranges. If \( Mv \leq |u| \leq \frac{1}{2} \), then we have \( \cos(2\pi u) \leq \cos(2\pi Mv) \). Therefore
\[
|1 - q|^2 = 1 - 2e^{-2\pi v} \cos(2\pi u) + e^{-4\pi v} \geq 1 - 2e^{-2\pi v} \cos(2\pi Mv) + e^{-4\pi v}.
\]
Taylor expanding around \( v = 0 \) we find that
\[
|1 - q| \geq 2\pi v \sqrt{1 + M^2} + O(v^2).
\]
For \( \frac{1}{4} \leq |u| \leq \frac{1}{2} \) we have \( \cos(2\pi u) \leq 0 \). Therefore
\[
|1 - q| \geq 1 > 2\pi v \sqrt{1 + M^2}.
\]
Hence, for all \( Mv \leq |u| \leq \frac{1}{2} \),
\[
|1 - q| \geq 2\pi v \sqrt{1 + M^2} + O(v^2).
\]
Furthermore we have
\[
1 - |q| = 1 - e^{-2\pi v} = 2\pi v + O(v^2).
\]
By Lemma 2.1 we have
\[
P(|q|) = e^{-\frac{2\pi v}{\eta(iv)}} = \sqrt{\pi} e^{\frac{\pi}{12v}} (1 + O(v)).
\]
Thus
\[
\log(P(|q|)) = \frac{\pi}{12v} + \frac{1}{2} \log(v) + O(v).
\]
Combining (3.5), (3.6), and (3.7), we obtain
\[
|\log(P(q))| \leq \frac{\pi}{12v} + \frac{1}{2} \log(v) + O(v) - \frac{1}{2\pi v} \left( 1 - \frac{1}{\sqrt{1 + M^2}} \right) + O(1)
\]
\[
= \frac{1}{v} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + M^2}} \right) \right) + \frac{1}{2} \log(v) + O(1).
\]
Exponentiating yields the desired result. \( \square \)

We are now able to bound \( |C_{m,k}(q)| \) away from \( q = 1 \).

Proposition 3.6. Assume that \( 1 \leq |x| \leq \frac{\pi m^\frac{3}{2}}{\beta} \). Then we have, as \( n \to \infty \),
\[
|C_{m,k}(q)| \ll n^{\frac{2-k}{4}} \exp \left( \frac{\pi}{6} \sqrt{\frac{kn}{6}} \frac{\sqrt{6kn}}{8\pi} m^{-\frac{3}{4}} \right).
\]
Proof. We have by definition

\[ \mathcal{C}_{m,k}(q) = 2P^k(q) \int_0^{\frac{1}{2}} g(w; \tau) \cos(2\pi mw) \, dw. \]

Note that by (1.3)

\[ g(w; \tau) = 1 + (1 - \zeta) \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n^2+n}{2}}}{1 - \zeta q^n} + (1 - \zeta^{-1}) \sum_{n \geq 1} \frac{(-1)^n q^{\frac{n^2+n}{2}}}{1 - \zeta^{-1} q^n}. \]

We thus may bound

\[ g(w; \tau) \ll \sum_{n \geq 1} \frac{|q|^{\frac{n^2+n}{2}}}{1 - |q|^n} \ll \frac{1}{1 - |q|} \sum_{n \geq 1} e^{-\frac{\beta_k n^2}{2}} \ll \beta_k^{-\frac{3}{2}} \ll n^{\frac{3}{2}}. \]

Thus

\[ |\mathcal{C}_{m,k}(q)| \ll |P^k(q)| n^{\frac{3}{2}}. \]

Using Lemma 3.5 with \( v = \frac{\beta_k}{2\pi}, u = \frac{\beta_k m^{-\frac{3}{2}}}{2\pi}, \) and \( M = m^{-\frac{3}{2}}, \) yields for \( 1 \leq |x| \leq \pi m^{\frac{1}{3}} \beta_k, \)

\[ |P(q)| \ll n^{-\frac{3}{2}} \exp \left[ \frac{2\pi}{\beta_k} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{3}{2}}}} \right) \right) \right]. \]

Therefore

\[ |\mathcal{C}_{m,k}(q)| \ll n^{\frac{3-k}{4}} \exp \left[ \frac{2\pi k}{\beta_k} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{3}{2}}}} \right) \right) \right] \]

\[ \ll n^{\frac{3-k}{4}} \exp \left[ \frac{\pi \sqrt{kn}}{6} - \frac{\sqrt{6kn}}{\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{3}{2}}}} \right) \right] \]

\[ \ll n^{\frac{3-k}{4}} \exp \left( \frac{\pi \sqrt{kn}}{6} - \frac{\sqrt{6kn}}{8\pi} m^{-\frac{3}{2}} \right). \] \( \square \)

4. The circle method

In this section we use Wright’s variant of the Circle Method and complete the proof of Theorem 1.4 and thus the proof of Dyson’s conjecture. We start by using Cauchy’s Theorem to express \( M_k \) as an integral of its generating function \( \mathcal{C}_{m,k} \):

\[ (4.1) \quad M_k(m,n) = \frac{1}{2\pi i} \int_C \frac{\mathcal{C}_{m,k}(q)}{q^{n+1}} \, dq, \]

where the contour is the counterclockwise transversal of the circle \( C := \{ q \in \mathbb{C}; |q| = e^{-\beta_k} \} \). Recall that \( z = \beta_k(1 + ixm^{-\frac{3}{2}}) \). Changing variables we may write

\[ M_k(m,n) = \frac{\beta_k}{2\pi m^{\frac{3}{2}}} \int_{|x| \leq \frac{\alpha m^{\frac{1}{3}}}{\beta_k}} \mathcal{C}_{m,k}(e^{-z}) e^{nx} \, dx. \]

We split this integral into two pieces

\[ M_k(m,n) = M + E \]
with
\[ M := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq 1} C_{m,k} \left( e^{-z} \right) e^{nz} dx, \]
\[ E := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{1 \leq |x| \leq \frac{m^{\frac{1}{3}}}{\pi k}} C_{m,k} \left( e^{-z} \right) e^{nz} dx. \]

In the following we show that \( M \) contributes to the asymptotic main term whereas \( E \) is part of the error term.

4.1. Approximating the main term. The goal of this section is to determine the asymptotic behavior of \( M \). We show

Proposition 4.1. We have
\[ M = \frac{\beta_k}{4} \text{sech}^{2} \left( \frac{\beta km}{2} \right) p_k(n) \left( 1 + O \left( \frac{m^{\frac{1}{3}}}{n^{\frac{1}{4}}} \right) \right). \]

A key step for proving this proposition is the investigation of
\[ P_{s,k} := \frac{1}{2\pi i} \int_{1-im^{\frac{1}{3}}}^{1+im^{\frac{1}{3}}} v^s e^{\pi \sqrt{\frac{kn}{6}} (v+\frac{1}{v})} dv \]
for \( s > 0 \). These integrals may be related to Bessel functions. Denoting by \( I_s \) the usual \( I \)-Bessel function of order \( s \), we have.

Lemma 4.2. As \( n \to \infty \)
\[ P_{s,k} = I_{-s-1} \left( \pi \sqrt{\frac{2kn}{3}} \right) + O \left( \exp \left( \pi \sqrt{\frac{k}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right). \]

Proof. We use the following loop integral representation for the \( I \)-Bessel function \([4]\) \( (x > 0) \)
\[ I_\ell(2x) = \frac{1}{2\pi i} \int_{\Gamma} t^{-\ell-1} e^{x(t + \frac{1}{t})} dt, \]
where the contour \( \Gamma \) starts in the lower half plane at \( -\infty \), surrounds the origin counterclockwise and then returns to \( -\infty \) in the upper half plane. We choose for \( \Gamma \) the piecewise linear path that consists of the line segments
\[ \gamma_4 : \left( -\infty - \frac{i}{2m^{\frac{1}{3}}}, -1 - \frac{i}{2m^{\frac{1}{3}}} \right), \quad \gamma_3 : \left( -1 - \frac{i}{2m^{\frac{1}{3}}}, -1 - \frac{i}{m^{\frac{1}{3}}} \right), \]
\[ \gamma_2 : \left( -1 - \frac{i}{m^{\frac{1}{3}}}, 1 - \frac{i}{m^{\frac{1}{3}}} \right), \quad \gamma_1 : \left( 1 - \frac{i}{m^{\frac{1}{3}}}, 1 + \frac{i}{m^{\frac{1}{3}}} \right), \]
which are then followed by the corresponding mirror images \( \gamma_4', \gamma_3', \) and \( \gamma_2' \). Note that \( P_{s,k} = \int_{\gamma_1}. \) Thus, to finish the proof, we have to bound the integrals along \( \gamma_4, \gamma_3, \) and \( \gamma_2 - \) the corresponding mirror images follow in the same way.
First
\[
\int \ll \int_{-\infty}^{1} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( t - \frac{im - \frac{1}{2}t}{2} \right) \right) \left| t - \frac{im - \frac{1}{2}t}{2} \right|^s dt
\ll \int_{1}^{\infty} e^{-\pi \sqrt{\frac{kn}{6}} \left| t + \frac{im - \frac{1}{2}t}{2} \right|^s} dt
\ll \int_{1}^{\infty} t^s e^{-\pi \sqrt{\frac{kn}{6}} t} dt \ll n^{-\frac{s+1}{2}} \Gamma \left( s + 1; \pi \sqrt{\frac{kn}{6}} \right) \ll n^{-\frac{1}{2}} e^{-\pi \sqrt{\frac{kn}{6}}},
\]
using (3.3).

Next
\[
\int \ll m^{-\frac{1}{3}} \int_{\frac{1}{3}}^{1} \exp \left( -\pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}t^2}} \right) \right) \left| 1 + im^{-\frac{1}{3}}t \right|^s dt \ll e^{-\pi \sqrt{\frac{kn}{6}}}.\]

Finally
\[
\int \ll \int_{-1}^{1} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( t + \frac{t}{t^2 + m^{-\frac{2}{3}}} \right) \right) \left| t - im^{-\frac{1}{3}} \right|^s dt
\ll \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right),\]
where we used that \( t + \frac{t}{t^2 + m^{-\frac{2}{3}}} \) obtains its maximum at \( t = 1 \). This finishes the proof. \( \square \)

We now turn to the proof of Proposition 4.1.

**Proof of Proposition 4.1.** Using Proposition 3.4 and making a change of variables, we obtain by Lemma 4.2
\[
M = \frac{\beta_k^{\frac{1}{2} + 2}}{4(2^\pi)^{\frac{1}{2}}} \text{sech}^2 \left( \frac{\beta_k m}{2} \right) P_{\frac{1}{2} + 1, k} + O \left( \beta_k^{\frac{1}{2} + 3} m^\frac{1}{4} \text{sech}^2 \left( \frac{\beta_k m}{2} \right) e^x \sqrt{\frac{kn}{6}} \right)
\ll \frac{\beta_k^{\frac{1}{2} + 2}}{4(2^\pi)^{\frac{1}{2}}} \text{sech}^2 \left( \frac{\beta_k m}{2} \right) \left( I_{-\frac{1}{2} - 2} \left( \pi \sqrt{\frac{2kn}{3}} \right) \right)
+ O \left( \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right) + O \left( \beta_k^{\frac{1}{2} + 3} m^\frac{1}{4} \text{sech}^2 \left( \frac{\beta_k m}{2} \right) e^x \sqrt{\frac{2\pi}{3}} \right).
\]

Using the Bessel function asymptotic (see (4.12.7) in [3])
\[
I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O \left( \frac{e^x}{x^{\frac{3}{2}}} \right)
\]
yields
\[ M = \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}} \text{sech}^2\left(\frac{\beta_km}{2}\right) \left(\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{\pi\sqrt{2\left(\frac{2kn}{3}\right)^{\frac{3}{2}}}} + O\left(\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{n^{\frac{3}{2}}}\right)\right) + O\left(\exp\left(\pi\sqrt{\frac{kn}{6}}\left(1 + \frac{1}{1+m^{-\frac{3}{2}}}\right)\right)\right) + O\left(\beta_k^{\frac{k}{2}+3}m^{\frac{3}{4}}\text{sech}^2\left(\frac{\beta_km}{2}\right)e^{\pi\sqrt{\frac{2kn}{3}}}\right).\]

It is not hard to see that the last error term is the dominant one. Thus
\[ M = \frac{\beta_k^{\frac{k}{2}+2}}{4(2\pi)^{\frac{k}{2}}} \text{sech}^2\left(\frac{\beta_km}{2}\right) \left(\frac{e^{\pi\sqrt{\frac{2kn}{3}}}}{\pi\sqrt{2\left(\frac{2kn}{3}\right)^{\frac{3}{2}}}} \left(1 + O\left(m^{\frac{3}{4}}n^{-\frac{1}{4}}\right)\right)\right).\]

Using that \[15, 21\]
\[ p_k(n) = 2 \left(\frac{k}{3}\right)^{1+k} (8n)^{-\frac{3+k}{4}} e^{\pi\sqrt{\frac{2kn}{3}}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\]
now easily gives the claim. \(\square\)

4.2. The error arc. We finally bound \(E\) and show that it is exponentially smaller than \(M\). The following proposition then immediately implies Theorem 1.4.

**Proposition 4.3.** As \(n \to \infty\)
\[ E \ll n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{2kn}{3}} - \sqrt{\frac{6kn}{8\pi}} m^{-\frac{3}{2}}\right).\]

**Proof.** Using Proposition 3.6 we may bound
\[ E \ll \frac{\beta_k}{m^{\frac{3}{2}}} \int_{1 \leq x \leq \frac{n^{\frac{3+k}{4}}}{k}} n^{\frac{3-k}{2}} \exp\left(\pi\sqrt{\frac{kn}{6}} - \sqrt{\frac{6kn}{8\pi}} m^{-\frac{3}{2}}\right) e^{\beta_k n} dx \ll n^{\frac{3-k}{4}} \exp\left(\pi\sqrt{\frac{2kn}{3}} - \sqrt{\frac{6kn}{8\pi}} m^{-\frac{3}{2}}\right). \] \(\square\)

5. Numerical data

We illustrate our results in two tables:

| \(n\) | \(M(0, n)\) | \(\tilde{M}(0, n)\) | \(\frac{M(0, n)}{\tilde{M}(0, n)}\) |
|------|----------|----------------|------------------|
| 20   | 41       | ~ 45           | ~ 0.912          |
| 50   | 8626     | ~ 9261         | ~ 0.931          |
| 500  | 3.228743492·10^{19} | ~ 3.298285542·10^{19} | ~ 0.979          |
| 1000 | 2.403603986·10^{29} | ~ 2.439699707·10^{29} | ~ 0.985          |

| \(n\) | \(M(1, n)\) | \(\tilde{M}(1, n)\) | \(\frac{M(1, n)}{\tilde{M}(1, n)}\) |
|------|----------|----------------|------------------|
| 20   | 38       | ~ 44           | ~ 0.863          |
| 50   | 8541     | ~ 9185         | ~ 0.930          |
| 500  | 3.226300403·10^{19} | ~ 3.295574297·10^{19} | ~ 0.979          |
| 1000 | 2.402671309·10^{29} | ~ 2.438696696·10^{29} | ~ 0.985          |

where we set \(\tilde{M}(m, n) := \frac{\beta}{4} \text{sech}^2\left(\frac{\beta_m}{2}\right)p(n).\)
ACKNOWLEDGEMENTS

The research of the first author was supported by the Alfried Krupp Prize for Young University Teachers of the Krupp foundation and the research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP/2007–2013) / ERC Grant agreement n. 335220 - AQSER. Most of this research was conducted while the second author was visiting the University of Cologne funded by the Krupp foundation. The first author thanks Freeman Dyson and Jan Manschot for enlightening conversations. Moreover we thank Michael Mertens, Karl Mahlburg, and Larry Rolen for their comments on an earlier version of this paper. Moreover we thank the referee for valuable comments that improved the exposition of this paper.

REFERENCES

[1] George E. Andrews and F. G. Garvan, Dyson’s crank of a partition, Bull. Amer. Math. Soc. (N.S.) 18 (1988), no. 2, 167–171, DOI 10.1090/S0273-0979-1988-15637-6. MR1029094 (89b:11079)

[2] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, Proc. London Math. Soc. (3) 4 (1954), 84–106. MR0060535 (15,685d)

[3] George E. Andrews, Richard Askey, and Ranjan Roy, Special functions, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999. MR1688958 (2000g:33001)

[4] G. Arken, Modified Bessel functions, Mathematical Methods for Physicists, 3rd ed., Orlando, FL: Academic Press (1985), 610–616.

[5] Kathrin Bringmann, Karl Mahlburg, and Robert C. Rhoades, Asymptotics for rank and crank moments, Bull. Lond. Math. Soc. 43 (2011), no. 4, 661–672, DOI 10.1112/blms/bdq129. MR2820152 (2012k:11163)

[6] K. Bringmann and J. Manschot, Asymptotic formulas for coefficients of inverse theta functions, submitted for publication.

[7] Kathrin Bringmann and Ken Ono, Dyson’s ranks and Maass forms, Ann. of Math. (2) 171 (2010), no. 1, 419–449, DOI 10.4007/annals.2010.171.171.419. MR2630043 (2011e:11095)

[8] F. J. Dyson, Some guesses in the theory of partitions, Eureka 8 (1944), 10–15. MR3077150

[9] Freeman J. Dyson, Mappings and symmetries of partitions, J. Combin. Theory Ser. A 51 (1989), no. 2, 169–180, DOI 10.1016/0097-3165(89)90043-5. MR1001259 (90f:05009)

[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, Higher transcendental functions, Vol. I, Robert E. Krieger Publishing Co., Inc., Melbourne, Fla. (1981)

[11] Amanda Folsom and Riad Masri, Equidistribution of Heegner points and the partition function, Math. Ann. 348 (2010), no. 2, 289–317, DOI 10.1007/s00208-010-0478-6. MR2672303 (2011m:11028)

[12] F. G. Garvan, New combinatorial interpretations of Ramanujan’s partition congruences mod 5, 7 and 11, Trans. Amer. Math. Soc. 305 (1988), no. 1, 47–77, DOI 10.2307/2001040. MR920146 (89b:11081)

[13] Lothar Göttsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, Math. Ann. 286 (1990), no. 1-3, 193–207, DOI 10.1007/BF01453572. MR1032930 (91h:14007)

[14] Paul Hammond and Richard Lewis, Congruences in ordered pairs of partitions, Int. J. Math. Math. Sci. 45-48 (2004), 2509–2512, DOI 10.1155/S0161171204311439. MR2102870 (2005f:11234)

[15] G. Hardy and S. Ramanujan, Asymptotic formulae for the distribution of integers of various types, Proc. Lond. Math. Soc. 16 (1918), 112–132.

[16] T. Hausel and Rodriguez-Villegas, Cohomology of large semiprojective hyperkahler varieties, preprint.

[17] B. Kim, E. Kim, and J. Seo, Asymptotics for $q$-expansions involving partial theta functions, in preparation.

[18] D. H. Lehmer, On the remainders and convergence of the series for the partition function, Trans. Amer. Math. Soc. 46 (1939), 362–373. MR000410 (1,69c)
[19] Karl Mahlburg, *Partition congruences and the Andrews-Garvan-Dyson crank*, Proc. Natl. Acad. Sci. USA 102 (2005), no. 43, 15373–15376 (electronic), DOI 10.1073/pnas.0507051102. MR2188922 (2006k:11200)

[20] Hans Rademacher, *Topics in analytic number theory*, Springer-Verlag, New York-Heidelberg, 1973. Edited by E. Grosswald, J. Lehner and M. Newman; Die Grundlehren der mathematischen Wissenschaften, Band 169. MR0364103 (51 #358)

[21] Hans Rademacher and Herbert S. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. (2) 39 (1938), no. 2, 433–462, DOI 10.2307/1968796. MR1503417

[22] S. Ramanujan, *Congruence properties of partitions*, Math. Z. 9 (1921), no. 1-2, 147–153, DOI 10.1007/BF01378341. MR1544457

[23] E. Maitland Wright, *Asymptotic Partition Formulae: (II) Weighted Partitions*, Proc. London Math. Soc. S2-36, no. 1, 117, DOI 10.1112/plms/s2-36.1.117. MR1575956

[24] E. M. Wright, *Stacks. II*, Quart. J. Math. Oxford Ser. (2) 22 (1971), 107–116. MR0282940 (44 #174)

Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany

E-mail address: kbringma@math.uni-koeln.de

LIAFA, Universite Denis Diderot - Paris 7, 75205 Paris Cedex 13, France

E-mail address: jehanne.dousse@liafa.univ-paris-diderot.fr