Chow’s Theorem for Semi-abelian Varieties and Bounds for Splitting Fields of Algebraic Tori

Chia Fu YU
Institute of Mathematics, Academia Sinica & NCTS Astronomy Mathematics Building, No. 1, Roosevelt Rd. Sec. 4, Taipei 10617, Taiwan, China
E-mail: chiafu@math.sinica.edu.tw

Abstract  A theorem of Chow concerns homomorphisms of two abelian varieties under a primary field extension base change. In this paper, we generalize Chow’s theorem to semi-abelian varieties. This contributes to different proofs of a well-known result that every algebraic torus splits over a finite separable field extension. We also obtain the best bound for the degrees of splitting fields of tori.

Keywords  Algebraic tori, splitting fields, semi-abelian varieties, inverse Galois problem

MR(2010) Subject Classification  20G15, 20C10, 11G10, 12F12

1 Introduction

Let $k$ be a field, $\bar{k}$ an algebraic closure of $k$, and $k_s$ the separable closure of $k$ in $\bar{k}$. A connected algebraic $k$-group $T$ is an algebraic torus if there is a $\bar{k}$-isomorphism $T \otimes_k \bar{k} \simeq (\mathbb{G}_m)^d \otimes_k \bar{k}$ of algebraic groups for some integer $d \geq 0$. We say $T$ splits over a field extension $K$ of $k$ if there is a $K$-isomorphism $T \otimes_k K \simeq (\mathbb{G}_m)^d \otimes_k K$. This paper is motivated from the following fundamental result.

Theorem 1.1  Any algebraic $k$-torus $T$ splits over $k_s$. In other words, $T$ splits over a finite separable field extension of $k$.

This theorem is well known and it is stated and proved in the literature several times. Surprisingly, different authors choose their favorite proofs which are all quite different. As far as we know, the first proof is given by Takashi Ono [40, Proposition 1.2.1]. Borel gives a different proof in his book Linear Algebraic Groups; see [3, Proposition 8.11]. In the second edition of his book Linear Algebraic Groups [49], Springer includes a systematic treatment of the rationality problem of algebraic groups where he also gives another proof of Theorem 1.1; see [49, Proposition 13.1.1]. Another proof, due to John Tate, can be found in Borel and Tits [4, Proposition 1.5]. Tits himself also provides one proof in his Yale University Lectures Notes; see [53, Theorem 1.4.1].

A key point in Borel’s proof is that any $k_s$-valued point of $T$ is semi-simple. In fact, Borel proves Theorem 1.1 more generally for diagonalizable $k$-groups, because these groups also have this property. Springer’s proof has more flavor of differential geometry; a key ingredient uses derivations. In some sense Springer treats a purely inseparable descent using derivations and connections though this is not stated explicitly. It is known (Matsumura [36, Chap. 9]), that some hidden information of purely inseparable field extensions ignored by Galois theory can be revealed by derivations and connections. However, the way that Springer applies such an inseparable descent to Theorem 1.1 is rather interesting. Tits’ and Tate’s proofs use only...
algebraic properties of characters; the ideas of their proofs are similar. Both start with that a suitable \( p \)-power of a character \( \chi \) of \( T \) is defined over \( k_s \) and prove that \( \chi \) is defined over \( k_s \).

A main difference is that Tits works with the coordinate ring \( \overline{k}[T] \) of \( T \) while Tate works with its function field \( \overline{k}(T) \). That Tate proves the \( k_s \)-rationality of \( \chi \) uses the language in Weil’s foundation, while Tits’ argument is more elementary. We shall present only the proofs of Ono and Borel as they are mostly relevant to our results.

Chow’s theorem [9] states as follows: Let \( K/k \) be a primary field extension, that is, \( k \) is separably algebraically closed in \( K \). If \( X \) and \( Y \) are two abelian varieties over \( k \), then the map \( \text{Hom}_k(X,Y) \to \text{Hom}_K(X_K,Y_K) \) is bijective, where \( X_K := X \otimes_k K \) and \( Y_K := Y \otimes_k K \). Ono observes [40, Lemma 1.2.1] that Theorem 1.1 follows immediately from an analogue of Chow’s theorem for tori (i.e., the above bijection also holds if one replaces abelian varieties by tori), and he points out in the proof that Chow’s original proof (for abelian varieties) also works for tori.

As is well known, Chow’s theorem (also see [34, Chapter II, Theorem 5]) is proved under an old fashioned of Weil’s foundation. Thus, it is desirable to have a modern proof using the language of schemes for Chow’s theorem and its analogue for tori. Conrad [12, Theorem 3.19] gives a modern proof of the original Chow theorem. The central idea is Grothendieck’s faithfully flat descent.

Grothendieck’s descent theory has been a very powerful tool of algebraic geometry. The standard reference is SGA 1 [18]. The reader can also find the exposition in some books or articles working with moduli spaces or étale cohomology, for example, Milne [37, Chapter 1, Section 2], Freitag and Kiehl [16, Appendix A], Bosch, Lütkebohmert, and Raynaud [5, Chapter 6] and Conrad [12, Section 3]. The faithfully flat descent is a very clean formulation which reorganizes both the classical Galois descent and the purely inseparable descent through derivations over fields in one unified way (regardless the explicit structure of the flat base in question). More powerfully, this simple formulation works for arbitrary base schemes, so it is far beyond the combination of both separable and inseparable descent over fields.

The idea of Conrad’s proof of Chow’s theorem is pursued further in this article. Indeed, using the similar idea, we generalize Chow’s theorem to semi-abelian varieties. This includes the case of tori which is related to Theorem 1.1 by our discussion. We refer to Section 3.2 for the definition of semi-abelian varieties.

**Theorem 1.2** Let \( X \) and \( Y \) be two semi-abelian varieties over a field \( k \), and let \( K \) be a primary field extension of \( k \). Then the monomorphism of \( \mathbb{Z} \)-modules

\[
\text{Hom}_k(X,Y) \to \text{Hom}_K(X_K,Y_K)
\]

is bijective.

We actually give two proofs of this theorem. As mentioned, the first proof utilizes the flat descent. Our second proof is more elementary; it does not rely on the flat descent. Thus, we add another two (or one depending on how one counts) proofs of Theorem 1.1 to those given by Borel, Springer, Tate, Ono and Tits. We remark that Theorem 1.1 is equivalent to Theorem 1.2 for tori; see Proposition 2.5.

We illustrate how Theorem 1.1 is related to integral representations of finite groups, the inverse Galois problem and Noether’s problem. These are interesting research topics and remain very active even up to date. Let \( \Gamma_k := \text{Gal}(k_s/k) \) be the absolute Galois group of \( k \). Using Theorem 1.1, the functor

\[
a \text{-torus } T \mapsto X(T) := \text{Hom}_{k_s}(T_{k_s}, \mathbb{G}_{m,k_s})
\]

(1.2)
gives rise to an anti-equivalence of categories between that of algebraic \( k \)-tori and that of finitely generated free \( \mathbb{Z} \)-modules together with a continuous action of \( \Gamma_k \), or simply \( \mathbb{Z}\Gamma_k \)-lattices. The inverse functor is given by \( M \mapsto \text{Spec } k_s[M]^\Gamma_k \). If \( K/k \) is a finite Galois extension with Galois
group $G := \text{Gal}(K/k)$, then this functor induces a bijection

$$\left\{ \text{isomorphism classes of} \ k\text{-tori splitting over } K \right\} \sim \left\{ \text{isomorphism classes} \ \text{of ZG-lattices} \right\}. \quad (1.3)$$

Thus, classifying $k$-tori can be divided into two steps:

1. Classify all finite groups $G$ which appear as quotients of $\Gamma_k$.
2. Classify integral representations of $G$.

It is known that the symmetric group $S_n$ or the alternating group $A_n$ occurs as a Galois group over $\mathbb{Q}$ [46, Chapter 4]. A theorem of Shafarevich states that if $k$ is a global field and $G$ is a finite solvable group, then there exists a finite Galois extension $K/k$ with $\text{Gal}(K/k)$ isomorphic to $G$; see [39, Theorem 9.6.1] (or Theorem 9.5.1 in the 1st edition). Zywina [56] proves that the finite simple group $\text{PSL}_2(\mathbb{F}_p)$ for $p \geq 5$ occurs as a Galois group over $\mathbb{Q}$. We remind the reader the book by Jensen, Ledet and Yui [22] where one can find many explicit examples of generic polynomials for small or medium groups. This book also provides a convenient and detailed introduction to Galois theory and a nice exposition of Saltman’s paper [43]. We should also mention some published books related to the subject, for example, those of Serre [46], Völklein, and Malle and Matzat.

Noether’s problem is the most prominent and famous problem in the inverse Galois problem. For a finite group $G$, consider the field extension $k(G) := k(x_g|g \in G)^G$ of a field $k$, where the action of $G$ on the finite set $\{x_g|g \in G\}$ of indeterminates is defined by $g \cdot x_h = x_{gh}$ for all $g, h \in G$. Noether’s problem asks whether $k(G)$ is rational (= purely transcendental) over $k$. A standard survey of Noether’s problem is Swan’s paper [51]. For a more recent survey, see Kersten [28]. Lenstra [35] gives a complete solution for the case of finite abelian groups. Saltman [44] introduces the unramified Brauer group $\text{Br}_{ur}(k(G))$ of $k(G)$, which paves a way to construct counterexamples to Noether’s problem. If $k(G)$ is rational over $k$, then $\text{Br}_{ur}(k(G))$ is trivial. Saltman produces an example of a group $G$ of order $p^9$ where $p$ is a prime number different from $\text{char } k$ such that $\text{Br}_{ur}(k(G))$ is non-trivial. Bogomolov [6] gives an explicit formula for the unramified Brauer group and produces a similar example, with $G$ of order $p^6$. The notion of Saltman’s unramified Brauer group is extended to the higher degree unramified cohomology groups by Colliot-Thélène and Ojanguren [11]. For recent development of Noether’s problem, we refer to works of Hoshi, Kang, Kitayama, and Yamasaki and others [10, 19, 23–26, 29, 30, 41, 55]. Surely, the reader can also easily find more in the literature. We also refer to the excellent survey paper by Kunyavskii [32] for developments of more research topics related to algebraic tori.

We conclude this paper with the best bound for the degrees of splitting fields of tori.

**Proposition 1.3** (Corollary 4.4) For any $d \geq 1$ and any number field $k$, there exist infinitely many $d$-dimensional tori $T$ over $k$ such that $[k_T:k] = \text{Max}(d, \mathbb{Q})$, where $k_T$ is the (minimal) splitting field of $T$ and $\text{Max}(d, \mathbb{Q})$ is the maximal order of finite subgroups of $\text{GL}_d(\mathbb{Q})$.

The paper is organized as follows. Section 2 contains minimal preliminaries of diagonalizable groups and proofs of Theorem 1.1 due to Borel and Ono. Theorem 1.2 is proved in Section 3; we also give another proof of Theorem 1.2 and hence that of Theorem 1.1. In Section 4 we study bounds of splitting fields of algebraic tori.

## 2 Two Proofs of Theorem 1.1

### 2.1 Characters and Diagonalizable Groups

In this subsection we shall present a proof of Theorem 1.1 due to Borel [3]. A basic result is that any set of characters is linearly independent in the following sense.

**Lemma 2.1** Let $H$ be an abstract group, $k$ any field, and let $X$ be the set of all homomorphisms $H \to k^\times$. Then $X$ is $k$-linearly independent as a subset in the $k$-vector space $\mathcal{C}(H, k)$ of $k$-valued
functions on \( H \). That is, for any distinct characters \( \chi_1, \ldots, \chi_n \) and elements \( a_1, \ldots, a_n \in k \), then
\[
a_1\chi_1 + \cdots + a_n\chi_n = 0 \text{ in } \mathcal{C}(H, k) \implies a_1 = \cdots = a_n = 0. \tag{2.1}
\]

\textbf{Proof} \quad \text{See [3, Lemma 8.1].} \qed

Let \( G \) be a linear algebraic group over a field \( k \). Put \( K := \bar{k} \). Let \( X(G) := \text{Hom}_{K-\text{gp}}(G, \mathbb{G}_m) \) denote the group of all characters, which is a finitely generated abelian group. The subgroup of \( k \)-rational characters is denoted by \( X(G)_k \).

\textbf{Definition 2.2} \quad (1) \text{ We say that } G \text{ is diagonalizable if the coordinate ring } K[G] \text{ is spanned by } X(G) \text{ over } K.

(2) \text{ We say that a diagonalizable group } G \text{ splits over } k \text{ if the coordinate ring } k[G] \text{ is spanned by } X(G)_k \text{ over } k.

If \( H \) is an abstract group, then the group algebra \( K[H] \) of \( H \) over \( K \) admits a natural structure of Hopf algebra, with the co-multiplication \( \Delta : K[H] \to K[H] \otimes_K K[H] \) defined by \( \Delta(h) = h \otimes h \). By Lemma 2.1, the abelian group \( X(G) \) is a linearly independent subset in \( K[G] \). So if \( G \) is diagonalizable, then \( K[G] \) is equal to the group algebra \( K[X(G)] \) of the abelian group \( X(G) \). Moreover, this equality respects the Hopf algebra structures of \( K[G] \) and of \( K[X(G)] \). Particularly, \( G \) is commutative if \( G \) is diagonalizable.

It is clear from the definition that an algebraic torus is precisely a connected diagonalizable algebraic group. We recall basic properties of diagonalizable groups. Let \( D_n \subset \text{GL}_n \), for \( n \geq 1 \), denote the diagonal split torus of dimension \( n \).

\textbf{Proposition 2.3} \quad Let \( G \) be a linear algebraic group over \( K \). The following statements are equivalent:

(1) \( G \) is diagonalizable.

(2) \( G \) is isomorphic to a subgroup of \( D_n \) for some \( n \geq 1 \).

(3) For each rational representation \( \pi : G \to \text{GL}_n \), the subgroup \( \pi(G) \) is conjugate to a subgroup of \( D_n \).

(4) \( G \) contains a dense commutative subgroup consisting of semi-simple elements.

\textbf{Proof} \quad \text{See [3, Proposition 8.4].} \qed

\textbf{Theorem 2.4} \quad Every diagonalizable \( k \)-group \( G \) splits over \( k_s \).

\textbf{Proof} \quad \text{Choose a } k \text{-embedding } G \subset \text{GL}_n \text{. By Proposition 2.3, } G(k_s) \text{ contains a dense commutative subgroup } S \text{ consisting of semi-simple elements. As } S \text{ is commutative and every element } s \text{ in } S \text{ is diagonalizable in } \text{GL}_n(k_s), \text{ we can diagonalize simultaneously the matrices } s \text{ for all } s \in S \text{. That is, there is an element } g \in \text{GL}_n(k_s) \text{ such that } gSg^{-1} \subset D_n(k_s). \text{ Since } S \text{ is dense, the inner automorphism } \text{Int}(g) \text{ sends } G \text{ into a subgroup of } D_n. \text{ Therefore, } G \text{ is } k_s \text{-isomorphic to a subgroup of } D_n. \text{ Note that every subgroup of } D_n \text{ splits over } k \text{ and hence } G \otimes_k k_s \text{ is a split torus.} \qed

Theorem 1.1 follows from Theorem 2.4, because every algebraic torus is a diagonalizable group. We remark that Proposition 2.3 (4) is also known due to Rosenlicht who uses it to show that every algebraic \( k \)-torus is unirational; see [42, Proposition 10].

2.2 Ono’s Proof

We now present Ono’s proof of Theorem 1.1 based on Theorem 1.2 for tori. Let \( T \) be an algebraic torus over \( k \). It suffices to show that any character \( \chi \in X(T)_{\bar{k}} = \text{Hom}_{\bar{k}}(T, \mathbb{G}_m) \) is defined over \( k_s \).

Since \( \bar{k}/k_s \) is primary, applying Theorem 1.2 to \( A = T \) and \( B = \mathbb{G}_m \), we have an isomorphism
\[
\text{Hom}_{k_s}(T, \mathbb{G}_m) \cong \text{Hom}_{\bar{k}}(T, \mathbb{G}_m).
\]

Thus, every character is defined over \( k_s \).
We observe that Theorems 1.1 and 1.2 for tori are equivalent.

**Proposition 2.5** The following two statements are equivalent.

(a) Every $k$-torus splits over $k_s$.

(b) For any two $k$-tori $X$ and $Y$ and any primary field extension $K/k$, we have the bijection $\text{Hom}_K(X,Y) \simeq \text{Hom}_K(X_K,Y_K)$.

**Proof** We have proven (b) ⇒ (a) and now prove the other direction. Choose a separable closure $K_s$ of $K$ containing $k_s$. By (a), we have $\text{Hom}_{k_s}(X_{k_s}, Y_{k_s}) \simeq \text{Hom}_K(X_{k_s}, Y_{k_s}) = \text{Hom}_{k_s}(X_{k_s}, Y_{k_s})$. As $K/k$ is primary, $K \cap k_s = k$ and $\text{Gal}(k_s/K) \simeq \Gamma_k = \text{Gal}(k_s/k)$. Thus,

$$\text{Hom}_K(X_K, Y_K) = \text{Hom}_{k_s}(X_{k_s}, Y_{k_s})^{\Gamma_k}$$

$$= \text{Hom}_{k_s}(X_{k_s}, Y_{k_s})^{\text{Gal}(k_s/K)}$$

$$\simeq \text{Hom}_{k_s}(X_{k_s}, Y_{k_s})^{\Gamma_k}$$

$$= \text{Hom}_{k_s}(X, Y).$$

\[\square\]

3 Chow’s Theorem for Semi-abelian Varieties

In this section we shall give a proof of Theorem 1.2. As mentioned in Section 1, the main ingredient is Grothendieck’s faithfully flat descent.

3.1 Faithfully Flat Descent

We recall some basic terminology needed to describe the flat descent.

**Definition 3.1** (1) A ring homomorphism $A \to B$ of commutative rings is said to be flat if the functor $\otimes_A B : A \text{-Mod} \to B \text{-Mod}$ is exact.

(2) A morphism $f : X \to Y$ of schemes is said to be flat if for any point $x \in X$ with $y = f(x)$, the ring homomorphism $O_{Y,y} \to O_{X,x}$ of local rings is flat. We say $f$ is faithfully flat if it is flat and surjective.

(3) We say $f$ is quasi-compact if the pre-image $f^{-1}(U)$ of every open affine subscheme $U$ of $Y$ is quasi-compact, that is, it is a finite union of open affine subschemes.

(4) A scheme $X$ is said to be quasi-affine if it is quasi-compact and it is contained in an affine scheme. A morphism $f$ of schemes is said to be quasi-affine if it is quasi-compact and the pre-image $f^{-1}(U)$ of every open affine subscheme $U$ of $Y$ is quasi-affine.

(5) Let $Y$ be a Noetherian scheme. A morphism $f : X \to Y$ of schemes of finite type is said to be projective (resp. quasi-projective) if $X$ is isomorphic to a closed (resp. locally closed) subscheme of the projective scheme $\mathbb{P}_Y^N$ for some positive integer $N$.

We first describe the flat descent for morphisms. Let $p : S' \to S$ be a morphism of base schemes, and let $X \to S$ be a morphism of schemes. For any integer $n > 1$, write $S^{(n)} := S' \times_S \cdots \times_S S'$ ($n$ times), and $X^{(n)} := X \times_S S^{(n)}$. Let $p_1, p_2 : S'' := S^{(2)} \to S'$ be two projection maps.

**Proposition 3.2** Let $p : S' \to S$ be a faithfully flat and quasi-compact morphism of base schemes. Let $X$ and $Y$ be two schemes over $S$ and let $f' : X' \to Y'$ a morphism of schemes over $S'$. If $p_1(f') = p_2(f')$, then there is a unique morphism $f : X \to Y$ over $S$ such that $f' = p^*(f)$.

**Proof** See [16, A.III.1 Lemma]. \[\square\]

We now describe the flat descent for objects. For any two integers $1 \leq i < j \leq 3$, denote by $p_{ij} : S^{(3)} \to S^{(2)}$ the projection map onto the $i$-th and $j$-th components. Let $p_i^{(n)} : S^{(n)} \to S'$ denote the $i$-th projection map. Clearly, one has $p_1 p_{ij} = p_1^{ij}$ and $p_2 p_{ij} = p_2^{ij}$ in $\text{Hom}(S^{(3)}, S')$. If $X'/S'$ is a scheme over $S'$ and $\alpha : p_1^{ij}(X') \to p_2^{ij}(X')$ is a morphism over $S^{(2)}$, then the pull-back
morphism \( p_{ij}^*(\alpha) \) is a morphism
\[
p_{ij}^*(\alpha) : (p_i^3)^*(X') \to (p_j^3)^*(X').
\]

**Definition 3.3** (1) Let \( p : S' \to S \) be a faithfully flat and quasi-compact morphism of schemes. A descent datum for \( p \) consists of a pair \((X'/S',\alpha)\), where \( X'/S' \) is a scheme over \( S' \) and \( \alpha : p_1^*(X') \to p_2^*(X') \) is an isomorphism of schemes over \( S'' \) satisfying the condition
\[
p^*_\alpha \circ p^*_1(\alpha) = p^*_1(\alpha).
\]

(2) A descent datum \((X'/S',\alpha)\) is said to be effective if there exist a scheme \( X/S \) over \( S \) and an isomorphism \( p^*(X) \cong X' \) over \( S' \).

**Theorem 3.4** Let \((X'/S',\alpha)\) be a descent datum for a faithfully flat and quasi-compact morphism \( p : S' \to S \) of base schemes. If \( X'/S' \) is quasi-affine, then \((X'/S',\alpha)\) is effective.

**Proof** See [16, A.III.6 Proposition].

**Remark 3.5** A classical Weil descent states that if \( p : S' \to S \) is Spec \( K \to Spec \) \( k \) for an algebraic separable field extension \( K/k \) and \( X' \) is a quasi-projective algebraic variety over \( K \), then any descent datum \((X'/K',\alpha)\) is effective. Comparing Weil’s descent and Theorem 3.4, one may ask whether the assumption of \( X' \) in Grothendieck’s flat descent can be weakened by assuming only that \( X' \) is quasi-projective. However, this is not the case. Indeed, there exists an étale covering \( S' \to S \) of schemes and a descent datum \((X'/S',\alpha)\) relative to \( S' \to S \) such that \( X' \to S' \) is projective, but the descent datum is not effective in the category of schemes. See [50, Tag 08KF] for a counterexample.

### 3.2 Proof of Theorem 1.2
Recall that a semi-abelian variety is a connected commutative smooth algebraic group \( G \) which is an extension of an abelian variety by an algebraic torus, that is, the maximal affine subgroup of \( G \) is an algebraic torus.

Recall the statement of Theorem 1.2 that \( X \) and \( Y \) are two semi-abelian varieties over a field \( k \), and \( K/k \) is a primary field extension. We must show that any morphism \( f : X_K \to Y_K \) over \( K \) is defined over \( k \).

Let \( p : S := Spec \ K \to Spec \ k \) and \( p_1, p_2 : S'' := S \times_{Spec \ k} S \to S \) be the projection maps. Put \( K' := K \otimes_k S \). Since \( K/k \) is a primary extension, the scheme \( S'' \) is irreducible and hence connected. Now let \( f \in \text{Hom}_K(X_K, Y_K) \). By Proposition 3.2, it suffices to show that \( p_1^*(f) = p_2^*(f) \).

Let \( x = \Delta : Spec \ K = S \to S'' = S \times_{Spec \ k} S \) be the \( K \)-valued point of \( S'' \) defined by the diagonal morphism. As \( p_i \circ \Delta = \text{id} \), one has \( x^*p_1^*(f) = x^*p_2^*(f) \), i.e., the morphisms \( p_1^*(f) \) and \( p_2^*(f) \) agree on the fiber over the point \( x \). Let \( \ell \) be any prime different from \( char \ k \). The morphism \( p_i^*(f) : X_{K'} \to Y_{K'} \) induces a morphism \( X_{K'}[\ell^n] \to Y_{K'}[\ell^n] \), where \( X_{K'}[\ell^n] \) denotes the \( \ell^n \)-torsion finite subgroup scheme of \( X_{K'} \). Since \( X_{K'}[\ell^n] \) has order prime to \( char \ k \), it is a finite étale group scheme. Denote by \( p_i^*(f)[\ell^n] \) the restriction of \( p_i^*(f) \) to the finite group scheme \( X_{K'}[\ell^n] \). As \( p_1^*(f) = p_2^*(f) \) on \( X_{K'}[\ell^n] \), it follows that \( p_1^*(f) = p_2^*(f) \). This proves Theorem 1.2.

### 3.3 A Descent Lemma
The purpose of this subsection is to prove another descent result. This yields a second and simpler proof of Theorem 1.2 and hence yields another proof of Theorem 1.1.

**Lemma 3.6** Let \( X \) and \( Y \) be \( k \)-schemes locally of finite type. Let \( \{X_n\}_{n \geq 1} \) be a sequence of closed \( k \)-subschemas of \( X \). Suppose that the scheme-theoretic closure of the image \( \bigsqcup X_n \to X \)
is equal to $X$. Let $K/k$ be a field extension, and $f : X \otimes_k K \to Y \otimes_k K$ a $K$-morphism. If the morphisms $f_n := f|_{X_n} : X_n \otimes K \to Y \otimes K$ are defined over $k$ for all $n$, then $f$ is defined over $k$.

**Proof** We first show that we can reduce the statement to the case where both $X$ and $Y$ are affine. Let $U_i$ and $V_i$ be affine coverings of $X$ and $Y$, respectively, such that $f(U_i \otimes K) \subseteq V_i \otimes K$. Clearly, $\{X_n \cap U_i\}_{n \geq 1}$ is a sequence of closed subschemes of $U_i$ satisfying the same condition of the lemma. If each morphism $f_i := f|_{U_i \otimes K}$ is defined over $k$, then we can glue $f_i$ to be a map $g$ which is defined over $k$ and one has $g \otimes K = f$.

Write $X = \text{Spec} A$ and $Y = \text{Spec} B$. Let $I_n$ be the ideal of $A$ defining the closed subscheme $X_n$. The map $f$ is given by a map also denoted by $f : B \otimes K \to A \otimes K$. The assumptions say that the induced map $f_n : B \otimes_k K \to (A/I_n) \otimes_k K$ is defined over $k$, that is, $f_n(B) \subseteq A/I_n$, and that the natural map $A \to \prod_n A/I_n$ is injective. Since the image $f(B)$ is contained in $A \otimes K$ and $\prod_n A/I_n$, it is contained in $A$. This proves the lemma. \[\square\]

### 3.4 Second Proof of Theorem 1.2

Let $f \in \text{Hom}_K(X_K, Y_K)$. Let $\ell$ be a prime different from char $k$. Since $X[\ell^n]$ and $Y[\ell^n]$ are finite étale group schemes, the functor $\mathcal{H}(S) := \text{Hom}_S(X[\ell^n] \times S, Y[\ell^n] \times S)$ for any $k$-scheme $S$ is representable by a finite $\mathbb{Z}/\ell^n$-module scheme over $k$. In particular, one has $\mathcal{H}(K) = \mathcal{H}(k)$ for any primary field extension $K/k$. Thus, the restriction of $f$ to $X[\ell^n] \otimes_k K$ is defined over $k$ for any $n$. Since the collection $\{X[\ell^n]\}_{n \geq 1}$ of finite étale group subschemes forms a Zariski dense subset of $X$ and $X$ is reduced, it follows from Lemma 3.6 that $f$ is defined over $k$.

## 4 Bounds for Splitting Fields of Tori

### 4.1 Splitting Fields

Let $T$ be an algebraic torus of dimension $d$ over a field $k$. The group of cocharacters of $T$, denoted $X_s(T)$, is a free $\mathbb{Z}$-module of rank $d$ equipped with a continuous action of the Galois group $\Gamma_k := \text{Gal}(k_s/k)$. Thus, one has a group homomorphism

$$\rho_T : \Gamma_k \to \text{Aut}(X_s(T)) \simeq \text{GL}_d(\mathbb{Z})$$

(4.1)

for a choice of a basis of $X_s(T)$. The splitting field of $T$ by definition is the smallest field extension $k_T$ of $k$ such that $T$ splits over $k_T$. The field $k_T$ is characterized by the property $\ker \rho_T = \text{Gal}(k_s/k_T) = : \Gamma_{k_T}$. Thus, the map $\rho_T$ induces a faithful representation of $\text{Gal}(k_T/k)$ on $X_s(T)$. In particular, $k_T$ is a finite Galois extension of $k$. For studying algebraic tori, it is useful to bound the degree of the splitting field of an algebraic torus.

For any positive integer $d \geq 1$, let $\text{Max}(d, \mathbb{Q})$ denote the maximal order of finite subgroups in $\text{GL}_d(\mathbb{Q})$. Clearly, one has $[k_T : k] \leq \text{Max}(d, \mathbb{Q})$ for any $d$-dimensional algebraic torus $T/k$. The following lemma provides explicit bounds for $[k_T : k]$.

For any integer $N \geq 1$, let $T[N]$ denote the $N$-torsion finite group subscheme of $T$. When $N$ is prime-to-char $k$, let $k(T[N])$ be the field extension of $k$ in $k_s$ joining all the coordinates of points in $T[N](k_s)$, and let

$$\rho_{T,N} : \Gamma_k \to \text{Aut}(T[N]_{k_s}) \simeq \text{GL}_d(\mathbb{Z}/N\mathbb{Z}).$$

(4.2)

Clearly, $k(T[N])$ is the Galois separable extension with $\Gamma_{k(T[N])} = \ker \rho_{T,N}$.

**Lemma 4.1** Let $T$ be a $d$-dimensional algebraic torus over $k$.

1. For any prime-to-char $k$ positive integer $N$ with $N \geq 3$, one has $k_T \subset k(T[N])$ and $[k_T : k] \leq \text{Max}(d, \mathbb{Q})$.
2. If char $k \neq 2$, then $[k_T : k] \leq (2 \cdot \# \text{GL}_d(\mathbb{F}_2))$.

**Proof** This follows from the fact that the reduction map $\text{GL}_d(\mathbb{Z}) \to \text{GL}_d(\mathbb{Z}/N\mathbb{Z})$ induces an injective map on any finite subgroup $G$ if $N \geq 3$, and a map $\rho : G \to \text{GL}_d(\mathbb{Z}/N\mathbb{Z})$ with...
ker $\rho \cap G \subset \{\pm 1\}$ if $N = 2$. This fact follows immediately from a lemma of Serre [38, Lemma, p. 207].

Lemma 4.1 (1) states that if $N \geq 3$ with $(\text{char } k, N) = 1$ and all $N$-torsion points of $T$ are defined over $k$, then $T$ splits over $k$. This reminds a theorem of Raynaud, stating that if an abelian variety and all its $N$-torsion points are defined over $k$, and $N \geq 3$, then the abelian variety has semistable reduction away from $N$ ([48, p. 403]).

**Definition 4.2** We say that a field $k$ is Hilbertian if it satisfies one of the following variants of the Hilbert irreducibility property.

(a) For any irreducible and separable polynomial $f(x, t) = a_d(t)x^d + \cdots + a_0(t) \in k(t)[x]$ over $k(t)$ of degree $d \geq 1$, where $x$ and $t$ are indeterminates, there exist infinitely many specializations $t = t_0 \in k$ such that $f(x, t_0)$ is an irreducible and separable polynomial over $k$ of degree $d$.

(b) For any $n \geq 1$ and any finite separable extension $K_t/k(t, t_1, \ldots, t_n)$ of $K_t$ of transcendental degree $n$, there exist infinitely many specializations $t \sim t_0 \in k$ such that $K_{t_0}$ is a finite separable extension of $k$ of the same degree $[K_t : k(t_1, \ldots, t_n)]$.

The conditions (a) and (b) are equivalent [47, §9.5, Remark (1)]. It is well known that any global field is Hilbertian (see [17, Theorem 13.4.2]). If $k$ is any field, then $k(t)$ and any finitely generated extension of it are Hilbertian (see [17, Theorem 13.4.2], also see p. 155 and Theorem 12.10 in the first edition).

We shall prove

**Theorem 4.3** For any $d \geq 1$ and any Hilbertian field $k$ of characteristic zero, there exist infinitely many Galois extensions $K/k$ with group isomorphic to a finite subgroup $G \subset \text{GL}_d(\mathbb{Q})$ of order $\text{Max}(d, \mathbb{Q})$.

As an immediate consequence of Theorem 4.3, we attain the best bound for $[k_T : k]$.

**Corollary 4.4** For any $d \geq 1$ and any number field $k$, there exist infinitely many $d$-dimensional algebraic tori $T$ over $k$ such that $[k_T : k] = \text{Max}(d, \mathbb{Q})$.

4.2 Proof of Theorem 4.3

According to [15], the signed permutation group $\{\pm 1^d \times S_d \subset \text{GL}_d(\mathbb{Q})$ attains the maximal order of finite subgroups of $\text{GL}_d(\mathbb{Q})$ except for $d \in \{2, 4, 6, 7, 8, 9, 10\}$. The exceptional cases are listed in Table 1 (see [2, Table 1]) with maximal-order finite subgroups. Here $W(D)$ denotes the Weyl group of the root system with Dynkin diagram $D$.

| $d$ | Maximal-order subgroup $G$ | $\text{Max}(d, \mathbb{Q}) = \#G$ |
|-----|---------------------------|---------------------|
| 2   | $W(G_2)$                  | 12                  |
| 4   | $W(F_4)$                  | 1152                |
| 6   | $(W(E_6), -I)$            | 103680              |
| 7   | $W(E_7)$                  | 2903040             |
| 8   | $W(E_8)$                  | 696729600           |
| 9   | $W(E_8) \times W(A_1)$   | 1393459200          |
| 10  | $W(E_8) \times W(G_2)$   | 8360755200          |
| all other $d$ | $W(B_d) = W(C_d) = \{\pm 1^d \times S_d$ | $2^d d!$ |

Table 1  Maximal-order finite subgroups of $\text{GL}_d(\mathbb{Q})$

Theorem 4.3 follows from the following proposition.

**Proposition 4.5** Let $G$ be the finite subgroup as in Table 1. For any Hilbertian field $k$ of characteristic zero, there exist infinitely many finite Galois extensions $K/k$ with group isomorphic to $G$. 
Proof. Note that $G$ is a finite reflection group $W$ except $d = 6$. Regarding $GL_n(Q) = GL(V)$ and putting $V_k = V \otimes Q k$, where $V = Q^n$ and $V_k = k^n$, one has $GL_n(k) = GL(V_k)$. In this case, $W \subset GL_n(V_k)$ is also a finite reflection group acting on $V_k$. By [2, Proposition 6] (mainly following from Chevalley’s theorem [8, Theorem (A)]), the fixed subfield $k(x_1, \ldots, x_d)^G = k(I_1, \ldots, I_d)$ is a purely transcendental extension of $k$. By the Hilbert irreducibility property, there exist infinitely many finite Galois extensions $K$ of $k$ with Galois group isomorphic to $G$. □

4.3 Remarks on Classification of Tori

Consider pairs $(\Gamma, \rho)$ which consist of a finite group $\Gamma$ together with a group monomorphism $\rho : \Gamma \rightarrow GL_d(Z)$ for some positive integer $d$. We call $d$ the degree of $(\Gamma, \rho)$ (or of $\rho$). Two such pairs $(\Gamma, i, \rho_i : \Gamma_i \rightarrow GL_d(Z))$ $(i = 1, 2)$ are said to have the same type if $d_1 = d_2$ and there exist an isomorphism $\alpha : \Gamma_1 \rightarrow \Gamma_2$ and an element $g \in GL_d(Z)$ such that $\rho_1(\gamma)g^{-1} = \rho_2(\alpha(\gamma))$ for all $\gamma \in \Gamma_1$. We say that two integral representations $(M_1, \rho_1)$ and $(M_2, \rho_2)$ of $\Gamma$ of finite rank have the same type if there exists an automorphism $\alpha$ of $\Gamma$ such that $(M_1, \rho_1 \circ \alpha) \simeq (M_2, \rho_2)$ as $Z\Gamma$-lattices. Let $(Z^d, \rho)$ be the integral representation of $\Gamma$ associated to $(\Gamma, \rho)$. Then two integral representations $(Z^{d_1}, \rho_1)$ and $(Z^{d_2}, \rho_2)$ of $\Gamma$ associated to $(\Gamma, \rho)$ have the same type if and only if so do $(\Gamma, \rho_1)$ and $(\Gamma, \rho_2)$. In general, there may exist non-isomorphic faithful integral representations of a finite group $\Gamma$ which have the same type. Let $T_p$ (resp. $T_{d,p}$) be the set of types of such pairs $(\Gamma, \rho)$ (resp. those of degree $d$). It is easy to see that the map $(\Gamma, \rho) \mapsto \rho(\Gamma)$ induces a bijection between $T_p$ and the set of $GL_d(Z)$-conjugacy classes of finite subgroups of $GL_d(Z)$.

Similarly, we consider pairs $(\Gamma, \rho_Q : \Gamma \rightarrow GL_d(Q))$ and define two pairs to have the same type in the same way. Let $T_{p,q}$ (resp. $T_{d,p,q}$) be the set of types of all such pairs $(\Gamma, \rho_Q)$ (resp. those of degree $d$). Similarly, we can identify the set $T_{d,p,q}$ with that of $GL_d(Q)$-conjugacy classes of finite subgroups of $GL_d(Q)$. For each pair $(\Gamma, \rho : \Gamma \rightarrow GL_d(Z))$ we denote by $\rho_Q : \Gamma \rightarrow GL_d(Q)$ the map induced from the inclusion $GL_d(Z) \subset GL_d(Q)$. This gives rise to a natural map $T_p \rightarrow T_{p,q}$. This map is surjective, because any finite subgroup of $GL_d(Q)$ preserves a lattice of $Q^d$ and then it is conjugate to a subgroup of $GL_d(Z)$.

To any algebraic torus $T$ over a field $k$ we associate a triple $(k_T, \text{Gal}(k_T/k), \rho_T)$, where

- $k_T$ is the splitting field of $T$,
- $\text{Gal}(k_T/k)$ is the Galois group of $k_T/k$, and
- $\rho_T : \text{Gal}(k_T/k) \rightarrow GL_d(Z)$ $(d = \text{dim } T)$ is the faithful representation induced by (4.1).

The map $\rho_T$ is only well-defined up to $GL_d(Z)$-conjugacy. It is known that the triple $(k_T, \text{Gal}(k_T/k), \rho_T)$ determines $T$ up to $k$-isomorphism. Clearly the type $[(\text{Gal}(k_T/k), \rho_T)]$ of $(\text{Gal}(k_T/k), \rho_T)$ is well-defined, which we call the type of $T$. Thus, the association $[(\text{Gal}(k_T/k), \rho_T)]$ to $T$ induces a well-defined map $\text{Tori}_k \rightarrow T_p$, where $\text{Tori}_k$ is the set of isomorphism classes of $k$-tori. The pre-image of each type $[(\Gamma, \rho)]$ consists of all pairs $(k', \rho')$, where $k'$ runs through finite Galois extensions $k'$ of $k$ such that $\text{Gal}(k'/k) \simeq \Gamma$, and $\rho'$ runs through non-isomorphic integral representations of $\Gamma$ of the same type as $\rho$. Let $\text{Tori}_{k}^{\text{isog}}$ be the set of isogeny classes of $k$-tori. Similarly the association $[(\text{Gal}(k_T/k), \rho_T, \rho_Q)]$ to $T$ induces a well-defined map $\text{Tori}_{k}^{\text{isog}} \rightarrow T_{p,q}$. The pre-image of $[(\Gamma, \rho_Q)]$ consists of pairs $(k', \rho'_Q)$ with $k'$ as above and $\rho'_Q$ running over non-isomorphic $Q$-representations of $\Gamma$ that have the same type as $\rho_Q$. Thus, classifying $\text{Tori}_k$ and $\text{Tori}_{k}^{\text{isog}}$ can be reduced essentially to classifying $T_p$ and $T_{p,q}$, respectively, and solving the inverse Galois problem.

The sets $T_{d,p}$ and $T_{d,p,q}$ have been classified for lower degrees $d$. It is obvious that $|T_{p,1}| = 2$ and $|T_{p,1,q}| = 2$. Two-dimensional tori have been classified by Voskresenskii [54] (Seligman [45]); particularly they prove $|T_{p,2}| = 13$. See [33, Lemma 4.7] for a list of 9 indecomposable subgroups in $GL_2(Z)$ (labeled $G_1, \ldots, G_9$ there). The remaining 4 decomposable subgroups are $C_1 = \langle \text{diag}(1,1) \rangle$, $C_2 = \langle \text{diag}(-1, -1) \rangle$, $C'_2 = \langle \text{diag}(1, -1) \rangle$ and $C_2 \times C_2 = \{ \text{diag}(\pm 1, \pm 1) \}$.
One can check using characters that $C_2'$ and $G_1$, $C_2 \times C_2$ and $G_2$, $G_5$ and $G_6$ are conjugate in $\text{GL}(2, \mathbb{Q})$. Thus, $|\text{Tp}_{2,\mathbb{Q}}| = 10$. Tahara [52] classifies finite subgroups of $\text{GL}(3, \mathbb{Z})$ up to conjugacy. However, there are overlapping 2 same classes in Tahara’s list (corrected in Ascher and Grimmer [1]) and one has $|\text{Tp}_3| = 73$. According to [13], we know $|\text{Tp}_4| = 710$. Conjugacy classes of finite subgroups of $\text{GL}_d(\mathbb{Q})$ for $d \leq 4$ are classified in [7]. From this we have $|\text{Tp}_{3,\mathbb{Q}}| = 32$ and $|\text{Tp}_{4,\mathbb{Q}}| = 227$; also see [27, pp. 54, 69]. We make a short list:

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $\text{Tp}_d$ | 2 | 10 | 32 | 227 | 955 | 7103 |
| $\text{Tp}_d,\mathbb{Q}$ | 2 | 13 | 73 | 710 | 6079 | 85308 |

(our reference for $d = 5, 6$ is [21, Section 3]). These explicit classifications have been used to make further development of the Noether problem for finite subgroups of $\text{GL}_d(\mathbb{Q})$ (for $d = 3, 4$); see [27] and the references therein for details. These also play an important role in the fundamental work of Hoshi and Yamasaki [21] on the rationality problem of tori of dimension up to 5.

Acknowledgements This paper grew out from lectures the author gave in the 2017 Fall NCTS course. He thanks the audience, Nai-Heng Sheu and Jiangwei Xue for their input and discussions. He is grateful to Ming-Chang Kang and Boris Kunyavskii for pointing out mistakes and helpful comments on an earlier version of this paper. Hethanks the referee for a careful reading and helpful comments.

References

[1] Ascher, E., Grimmer, H.: Comment on a paper by Tahara on the finite subgroups of $\text{GL}(3, \mathbb{Z})$. Nagoya Math. J., 48, 203 (1972)
[2] Berry, N., Dubickas, A., Elkies, N., et al.: The conjugate dimension of algebraic numbers. Q. J. Math., 55(3), 237–252 (2004)
[3] Borel, A.: Linear Algebraic Groups. Second Edition. Graduate Texts in Mathematics, 126, Springer-Verlag, New York, 1991
[4] Borel, A., Tits, J.: Groupes réductifs. Inst. Hautes Études Sci. Publ. Math., 27, 55–150 (1965)
[5] Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990
[6] Bogomolov, F. A.: The Brauer group of quotient spaces of linear representations. Math. USSR-Izv., 30(3), 455–485 (1988)
[7] Brown, H., Bulow, R., Neubuser, J., et al.: Crystallographic groups of four-dimensional space. Wiley Monographs in Crystallography. Wiley-Interscience, New York-Chichester-Brisbane, 1978
[8] Chevalley, C.: Invariants of finite groups generated by reflections. Amer. J. Math., 77, 778–782 (1955)
[9] Chow, W. L.: Abelian varieties over function fields. Trans. Amer. Math. Sci., 78, 253–275 (1955)
[10] Chu, H., Hoshi, A., Hu, S. J., et al.: Noether’s problem for groups of order 243. J. Algebra, 442, 233–259 (2015)
[11] Colliot-Thélène, J. L., Ojanguren, M.: Variétés unirationnelles non rationnelles: au-delà de l’exemple d’Artin et Mumford. Invent. Math., 97(1), 141–158 (1989)
[12] Conrad, B.: Chow’s $K/k$-image and $K/k$-trace, and the Lang-Néron theorem. Enseign. Math., 52, 37–108 (2006)
[13] Crystallographic group. Encyclopedia of Mathematics. URL: http://www.encyclopediaofmath.org/index.php?title=Crystallographic_group&oldid=39414
[14] ENDO, S., KANG, M. C.: Function fields of algebraic tori revisited. Asian J. Math., 21(2), 197–224 (2017)
[15] Feit, W.: Orders of finite linear groups. Proceedings of the First Jamaican Conference on Group Theory and its Applications, 9–11, Univ. West Indies, Kingston, 1996
[16] Freitag, E., Kiehl, R.: Étale cohomology and the Weil conjecture. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 13, 317 pp, Springer-Verlag, Berlin, 1988
[17] Fried, M. D., Jarden, M.: Field arithmetic. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 11. Springer-Verlag, Berlin, 2008
[18] Grothendieck, A.: Revêtements étalés et groupe fondamental (SGA 1), Lecture Notes in Math., vol. 224, Springer-Verlag, Berlin-Heidelberg-New-York, 1971
[19] Hoshi, A.: On Noether's problem for cyclic groups of prime order. Proc. Japan Acad. Ser. A Math. Sci., 91(3), 39–44 (2015)
[20] Hoshi, A., Kang, M. C., Yamasaki, A.: Degree three unramified cohomology groups. J. Algebra, 458, 120–133 (2016)
[21] Hoshi, A., Yamasaki, A.: Rationality problem for algebraic tori. Mem. Amer. Math. Soc., 248(1176), v+215 pp. (2017)
[22] Jensen, C. U., Ledet, A., Yui, N.: Generic polynomials. Constructive aspects of the inverse Galois problem. Mathematical Sciences Research Institute Publications, 45, 258 pp. Cambridge University Press, 2002
[23] Kang, M. C.: Noether's problem for metacyclic $p$-groups. Adv. Math., 203(2), 554–567 (2006)
[24] Kang, M. C.: Retract rationality and Noether’s problem. Int. Math. Res. Not., IMRN, no. 15, 2760–2788 (2009)
[25] Kang, M. C.: Noether's problem for $p$-groups with a cyclic subgroup of index $p^2$. Adv. Math., 226(1), 218–234 (2011)
[26] Kang, M. C.: Noether’s problem for cyclic groups of prime order. Arch. Math. (Basel), 110(1), 1–8 (2018)
[27] Kang, M. C., Zhou, J.: The rationality problem for four and five dimensional linear actions. J. Algebra, 324(4), 591–597 (2010)
[28] Kersten, I.: Noether's problem and normalization. Jahresber. Deutsch. Math. Verein., 100(1), 3–22 (1998)
[29] Kitayama, H.: Noether’s problem for four and five dimensional linear actions. J. Algebra, 457–483 (2001)
[30] Kitayama, H., Yamasaki, A.: The rationality problem for four-dimensional linear actions. J. Math. Kyoto Univ., 49(2), 359–380 (2009)
[31] Kunyavskii, B. E.: Three-dimensional algebraic tori. Selecta Math. Soviet., 9, 1–21 (1990)
[32] Kunyavskii, B.: Algebraic tori — thirty years after. Vestnik Samara State Univ., 7, 198–214 (2007)
[33] Kunyavskii, B., Sansuc, J. J.: Réduction des groupes algébriques commutatifs. J. Math. Soc. Japan, 53(2), 457–483 (2001)
[34] Lang, S.: Abelian Varieties. Interscience, New York, 1959
[35] Lenstra Jr., H. W.: Rational functions invariant under a finite abelian group. Invent. Math., 25, 299–325 (1974)
[36] Matsumura, H.: Commutative ring theory. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1986
[37] Milne, J. S.: Etale cohomology. Princeton Mathematical Series, 33, Princeton University Press, 1980
[38] Mumford, D.: Abelian Varieties, Oxford University Press, 1974
[39] Neukirch, J., Schmidt, A., Wingberg, K.: Cohomology of number fields. Second edition. Grundlehren der Mathematischen Wissenschaften 323, Springer-Verlag, Berlin, 825 pp, 2008
[40] Ono, T.: Arithmetic of algebraic tori. Ann. Math., 74, 101–139 (1961)
[41] Plans, B.: On Noether’s rationality problem for cyclic groups over Q. Proc. Amer. Math. Soc., 145(6), 2407–2409 (2017)
[42] Rosenlicht, M.: Some rationality questions on algebraic groups. Ann. Mat. Pura Appl., 43(4), 25–50 (1957)
[43] Saltman, D. J.: Generic Galois extensions and problems in field theory. Adv. in Math., 43(3), 250–283 (1982)
[44] Saltman, D. J.: Noether’s problem over an algebraically closed field. Invent. Math., 77(1), 71–84 (1984)
[45] Seligman, G. B.: On two-dimensional algebraic groups. Scripta Math., 29(3–4), 453–465
[46] Serre, J. P.: Topics in Galois theory. Lecture notes prepared by Henri Darmon. Research Notes in Mathematics, 1. Jones and Bartlett Publishers, Boston, MA, 1992
[47] Serre, J. P.: Lectures on the Mordell-Weil theorem. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt. With a foreword by Brown and Serre. Third edition. Aspects of Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1997
[48] Silverberg, A., Zarhin, Y. G.: Semistable reduction and torsion subgroups of abelian varieties. Ann. Inst. Fourier (Grenoble), 45(2), 403–420 (1995)
[49] Springer, T. A.: Linear algebraic groups. Second edition. Progress in Mathematics, 9, Birkhauser Boston, 1998
[50] The Stack Project Authors, Stacks Project, http://stacks.math.columbia.edu, (2017)
[51] Swan, R. G.: Noether’s problem in Galois theory, Emmy Noether in Bryn Mawr (Bryn Mawr, Pa., 1982), 21–40, Springer, New York-Berlin, 1983
[52] Tahara, K.: On the finite subgroups of GL(3, Z). Nagoya Math. J., 41, 169–209 (1971)
[53] Tits, J.: Lectures on Algebraic Groups. Department of Mathematics, Yale University, 1970
[54] Voskresenskii, V. E.: On two-dimensional algebraic tori. Izv. Acad. Nauk SSSR Ser. Mat., 29, 239–244 (1965)
[55] Yamasaki, A.: Some cases of four dimensional linear Noether’s problem. J. Math. Soc. Japan, 62(4), 1201–1211 (2010)
[56] Zywina, D.: The inverse Galois problem for $\mathrm{PSL}_2(F_p)$. Duke Math. J., 164(12), 2253–2292 (2015)