On Marginally Correct Approximations of Dempster-Shafer Belief Functions from Data

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Abstract
Mathematical Theory of Evidence (MTE) is blamed to leave frequencies outside its framework. In this paper we consider this problem from the point of view of conditioning in the MTE. We describe the class of belief functions for which marginal consistency with observed frequencies may be achieved and conditional belief functions are proper belief functions, and deal with implications for approximation of general belief functions by this class and for inference models in MTE.

1 INTRODUCTION
The Dempster-Shafer Theory or the Mathematical Theory of Evidence (MTE) [21, 3] is intended to be a generalization of bayesian theory of subjective probability [24]. This theory offers capability of representing ignorance in a simple and direct way, compatibility with the classical probability theory, compatibility with boolean logic and feasible computational complexity [20]. MTE may be applied for (1) representation of incomplete knowledge, (2) belief updating, (3) and for combination of evidence [18]. MTE covers the statistics of random sets and may be applied for representation of incomplete statistical knowledge. Random set statistics is quite popular in analysis of opinion polls whenever partial indecisiveness of respondents is allowed [4]. Practical applications of MTE include: integration of knowledge from heterogeneous sources for object identification [2], technical diagnosis under unreliable measuring devices [5], medical applications: [7, 31].

In spite of indicated merits, MTE experienced sharp criticism from many sides. The basic line of criticism is connected with the relationship between the belief function (the basic concept of MTE) and frequencies [30, 8]. A number of attempts to interpret belief functions in terms of probabilities have failed so far to produce a fully compatible interpretation with MTE - see e.g. [14, 8, 6] etc. Shafer [24] and Smets [27], in defense of MTE, dismissed every attempt to interpret MTE frequentistically. Shafer stressed that even modern (that meant bayesian) statistics is not frequentistic at all (bayesian theory assigns subjective probabilities), hence frequencies be no matter at all. Wasserman [30] strongly opposed claims of Shafer [24] about frequencies and bayesian theory. Wasserman pointed out that the major success story of bayesian theory is the exchangeability theory of de Finetti, which treats frequency based probabilities as a special case of bayesian belief. Hence frequencies, as Wasserman claims, are inside the bayesian theory, but outside the Mathematical Theory of Evidence.

In this paper we consider this problem from the point of view of apriorical and aposteriorical conditioning in the MTE. We describe the class of belief functions for which: marginal consistency with observed frequencies may be achieved, apriorical conditional belief functions are proper belief functions.

We will assume that the Reader is familiar with basic concepts of MTE like: belief function (Bel), basic probability assignment function (bpa, or m), commonality function (Q), marginalization (projection) onto a subset s of the set of all variables (Bel↑s), vacuous extension onto the set of variables (Bel↑*), focal points (sets for which bpa is non-zero). These terms are explained in many standard papers and books on MTE [21, 25].

2 CASE-BASED UNDERSTANDING OF BELIEF FUNCTIONS

In an attempt to overcome the reason for those numerous failures to interpret consistently Dempster’s rule of combination, a new frequency interpretation of MTE has been proposed in [13]. It has been demon-
strated there that, in general, Dempster’s rule has a destructive impact on the data so that whatever we expect calculation of conditional probabilities (given an event) differs from whatever we obtain under conditioning (given an event) in MTE (see the definition of aposteriorical conditioning of Shafer in [22].

Let us have a look at SQL construct to calculate joint probability distribution from a database in two variables P(X,Y):

create view Total (Counted) as select count(*) from Cases;

create view Probability (X,Y,Prob) as select X,Y, count(*)/Counted from Cases,Total group by X,Y;

Let us look at calculation of conditional probability P(X,Y|X ∈ AX). We proceed as follows: we select first the proper subset of cases from the database and proceed to calculate the unconditional probability for the selected cases.

create view SelectedCases (X,Y) as select X,Y from Cases where X ∈ AX;

create view TotalCond (Counted) as select count(*) from SelectedCases;

create view CondProbability (X,Y,CondProb) as select Y,X,count(*)/Counted from SelectedCases,TotalCond where X ∈ AX group by X, Y;

If we calculate a conditional probability P(X,Y|X ∈ AX, X ∈ BX) on a series of conditions (X ∈ AX, X ∈ BX) we can proceed first by selecting cases for the first, then for the second condition etc., finding the intersection, and then calculating the probabilities (id - the identifier).

create view SelectedCasesA (id,X,Y) as select id,X,Y from Cases where X ∈ AX;

create view SelectedCasesB (id,X,Y) as select id,X,Y from Cases where X ∈ BX;

create view SelectedCases (id,X,Y) as select id,X,Y from SelectedCasesA intersection select id,X,Y from SelectedCasesB;

create view TotalCond (Counted) as select count(*) from SelectedCases;

create view CondProbability (X,Y,CondProb) as select Y,X,count(*)/Counted from SelectedCases,TotalCond where X ∈ AX group by X, Y;

Let us look at calculation of basic probability assignment (bpa) m(X,Y) from data:

create view Total (Counted) as select count(*) from Cases;

create view bpa (X,Y,m) as select X,Y, count(*)/TCounted from Cases,Total group by X,Y;

Let us look at calculation of aposterioric conditional bpa m(X,Y||X ∈ AX) We have to proceed as follows: we select first the proper subset of cases from the database and MODIFY the values for the variable X. Then we proceed to calculate the unconditional bpa for the selected and updated cases.

create view UpdatedCases (X,Y) as select X∩AX,Y from Cases where X∩AX ≠ ∅;

create view TotalCond (Counted) as select count(*) from UpdatedCases;

create view CondBpa (X,Y,Condbpa) as select Y,X,count(*)/Counted from UpdatedCases,TotalCond ;

If we calculate a conditional bpa m(||X ∈ AX, X ∈ BX) on a series of conditions (X ∈ AX, X ∈ BX) we CANNOT proceed by selecting cases for the first, then for the second condition etc., finding the intersection, and then calculating the bpa, because this would yield wrong results due to side effects stemming from case modifications.

This has a serious impact if we try to factorize a belief function into simpler components, e.g. for purposes of propagation of uncertainty (methods of propagation of uncertainty are presented e.g. in [1, 25]. It turns out that:

- It is, in general, impossible to factorize a joint belief distribution into components being conditional belief functions ¹ (see eg. [1] for a discussion why two different notions of conditioning

¹Notice that in the domain of probability distributions, EVERY probability distribution may be represented by a composition of conditional probability distributions as so-called bayesian network
are needed for MTE: the posteriori-conditionals as conditionals in the sense of Shafer, and a priori-conditionals invented by Cano et al.)

- A priori-conditional belief functions as proposed by Cano et al. in general do not exists (see a discussion on non-existence of a priori conditionals in MTE presented by [26])

- What is more, it is often impossible to factorize a belief function \( \text{Bel} \) in variables \( p, q, r \) into two factors one \( \text{Bel}_1 \) in variables \( p, r \) and the second \( \text{Bel}_2 \) in variables \( q, r \) even if in conditional distribution given any value of the variable \( r \), variables \( p, q \) are independent (see [29]), that is when for every subset \( r_i \) of the domain of the variable \( r \) the following holds:

\[
\text{Bel}(\{r_i\})^{p,q} = \text{Bel}(\{r_i\})^{p} \oplus \text{Bel}(\{r_i\})^{iq}
\]

Therefore in papers [9, 10, 13] we have presented another approach to factorization of belief functions in terms of anticonditional belief functions. It turns out, however, that anticonditional belief functions are in general not belief functions but only pseudo-belief functions (that is ones with non-negative commonality functions). Thus, anticonditional belief functions have no direct counterparts in the physical world, as the basic probability assignment may take negative values.

One can be tempted to suggest, that one shall then resign from modeling the joint belief distribution and instead try to find a marginally consistent decomposition of the joint belief distribution. But:

- What is the class of Dempster-Shafer (DS) belief functions for which apriori-conditional belief functions exist?

- How can general belief functions and uncertainty propagation for them be approximated by this class of belief functions and uncertainty propagation for them?

- How can the belief functions and the reasoning with them be related to frequencies (cases)?

\[\text{Bel}(\|r_i\|) = \text{Bel}(\|r_i\|)^{p} \oplus \text{Bel}(\|r_i\|)^{iq}\]

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- How can the belief functions and the reasoning with them be related to frequencies (cases)?

2In probability distributions, a-priori and a-posteriori conditionals from the point of view of frequencies coincide, and they always exist: Let us assume \( P(X,Y) \) is a frequency based (case-based) probability distribution of \( X \) and \( Y \). A posteriori conditional distribution of variable \( Y \) given \( Y = y \) is interpreted as uncertainty distribution of variable \( Y \) \( P(X|Y = y) \) if we restrict ourselves only to cases for which \( Y = y \). Apriori conditional distribution of \( X \) given \( Y = y \) is a function \( P(X|Y = y) \) in \( X \) defined as \( P(X|Y = y) = P(X,Y) / P(Y) \) for all values of \( y \) for which \( P(Y = y) > 0 \). Obviously: \( P(X = x|Y = y) = P(X = x,Y = y) \) if \( P(Y = y) > 0 \).

3 CORRECT APPROXIMATION OF CONDITIONALS

Shafer [22] suggested that a belief function may emerge if we observe a variable \( X \) with domain of values \( \Xi_X \) indirectly (via a mapping \( f \)) by observing actually another variable \( Y \) with a domain \( \Xi_Y \) such that the mapping \( f : \Xi_Y \rightarrow \Xi_X \) is not a function. Hence, in some cases, if we observe a single value \( y \) of the variable \( Y \), we can tell at most that the value of the variable \( X \) belongs to a non-empty set of elements \( \{\xi_{X_1}, \xi_{X_2}, \ldots, \xi_{X_n}\} \). So a probability distribution \( P(Y) \) in \( Y \) translates into a basic belief assignment function, from which one calculates easily belief distribution \( \text{Bel}^A \) in \( X \). In this way we could get a case-based distribution in variable \( X \).

| Variable | Corresponding set values of X (A) | Frequency | \( m^X(A) \) | \( \text{Bel}^X(A) \) |
|----------|----------------------------------|-----------|-------------|----------------|
| \( y_1 \) | \{\( x_1 \)\}                     | 10        | .10         | .10            |
| \( y_2 \) | \{\( x_1, x_2 \)\}                | 20        | .20         | .30            |
| \( y_3 \) | \{\( x_2, x_3 \)\}                | 30        | .30         | .70            |
| \( y_4 \) | \{\( x_3 \)\}                     | 40        | .40         | .40            |
|         | \{\( x_1, x_2, x_3 \)\}           | –         | .0          | 1.0            |

This could be extended simply to multivariate belief distributions. However, we encounter one interpretational problem:

Let us consider the following frequency table in variables \( X \) and \( Z \):

| \( X \) | \( Z \) | frequencies |
|---------|---------|-------------|
| \{\( x_1, x_2 \)\} | \{\( z_1, z_2 \)\} | 20 |

What shall be the focal points of the multivariate belief function \( \text{Bel}^{X,Z} \) in two variables \( X, Z \)? Marginal consistency will be achieved either if we assume:

\( m^{X, Z}_1(\{(x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2)\}) = 1 \)

and if we take:

\( m^{X, Z}_2(\{(x_1, z_1), (x_2, z_2)\}) = 1 \)

and if we suppose:

\( m^{X, Z}_3(\{(x_1, z_2), (x_2, z_1)\}) = 1 \)

and for many other belief functions. But which one is the best? Let us try to calculate the conditional belief function \( \text{Bel}_i(\|X = x_1\|) \). (From Shafer’s formula: \( \text{Bel}_i(\|X = x_1\|) = (\text{Bel}_i \oplus \text{Bel}_{X=x_1}) \). With \( \text{Bel}_{X=x_1} \) being a belief function with the only focal point \( m_{X=x_1}(\{(x_1, z_1), (x_1, z_2)\}) = 1 \). We get three totally different results - three belief functions with differing focal points: \( m_1(\|X = x_1\|) = 1 \), \( m_2(\|X = x_1\|) = 1 \), \( m_3(\|X = x_1\|) = 1 \).
It is next to impossible to decide which of these conditionals is the correct one from the point of view of the observed data. So which belief function shall be treated as the most representative for the data? We suggest here the first one (Bel$^1$) for the following reasons:

1. if we observe $X, Z$ separately, we have no reason to assume that $z_1$ must co-occur with $z_2$ but never with $z_2$ etc - we assume we have no more information than actually visible from the data,

2. the methods of uncertainty propagation suggested both by Cano et al. [1] and Shenoy & Shafer [25] implicitly assume that the joint belief in values of observed variables $X_1, X_2, ..., X_n$ is the composition of the values of individual variables:

$$Bel_{obs} = Bel_{X_1=A_1} \oplus Bel_{X_2=A_2} \oplus ... \oplus Bel_{X_n=A_n}$$

with $A_i \subseteq \Xi_{X_i}$ being a subset of the domain of the $i^{th}$ variable. Violation of this assumption would invalidate the respective method of uncertainty propagation.

Let us consider now the following data in three (logical) variables $X, Y, Z$, giving the belief function Bel$^{and}$.

| $X$ | $Y$ | $Z$ | frequencies | $m_{and}$ $(A_X \times A_Y \times A_Z)$ |
|-----|-----|-----|-------------|----------------------------------|
| $\{x_1\}$ | $\{z_1\}$ | $\{z_2\}$ | 40 | .10 |
| $\{x_2\}$ | $\{z_2\}$ | $\{z_2\}$ | 60 | .20 |

Let us consider calculation of an apriori conditional belief function in the sense of Cano et al. [1] such that it would imply the value of $Z$ given $X, Y$. A look at the data would suggest that variables $X, Y$ and $Z$ are connected by the logical equation: $X \land Y = Z$ so that one might suggest a belief function with focal point $m_{\&}(\{(t_x, t_y, t_z), (t_x, f_y, f_z), (f_x, t_y, f_z), (f_x, f_y, f_z)\}) = 1$

is the apriori conditional connecting $X, Y$ and $Z$. But this is a wrong conclusion, because:

$$Bel_{and} \neq Bel_{and}^{\{X, Y\}} \oplus Bel_{\&}$$

What is more, the Cano et al a priori conditional of $Z$ given $X, Y$ does not exist at all!!! Therefore, decomposition of a joint belief distribution into conditionals cannot in general be achieved. However, we must acknowledge that marginal consistency is actually achieved, that is:

$$Bel_{and}^{\{X\}} = (Bel_{and}^{\{X, Y\}} \oplus Bel_{\&})^{\{X\}}$$
$$Bel_{and}^{\{Y\}} = (Bel_{and}^{\{X, Y\}} \oplus Bel_{\&})^{\{Y\}}$$
$$Bel_{and}^{\{Z\}} = (Bel_{and}^{\{X, Y\}} \oplus Bel_{\&})^{\{Z\}}$$

Can we always construct a marginally consistent apriori-conditional? We shall assume that we have found a marginally consistent apriori conditional Bel$^p$ of Bel given set of variables $p$ if Bel$^p$ is a belief function and for every variable $X$:

$$Bel^p = (Bel^p \oplus Bel^p)^{\{X\}}$$

let us consider the following belief distribution in variables $X, Z$.

| $X$ | $Z$ | frequencies |
|-----|-----|-------------|
| $\{x_1\}$ | $\{z_1\}$ | 40 |
| $\{x_2\}$ | $\{z_2\}$ | 60 |

It is easily to show that marginally consistent Bel$^X$ does not exist.

What is then the class of belief functions possessing marginally consistent conditionals? What is the class of marginally consistent conditional belief functions?

It is easy to show that if there exists a marginally consistent conditional belief function Bel$^p$ of the belief function Bel given set of variables $p$ then there exists another marginally consistent conditional belief function Bel$^p_\Xi$ of the belief function Bel given set of variables $p$ such that (Bel$^p_\Xi$) has only one focal point $(m_\Xi^p)^{\{p\}}(\Xi_p) = 1$ with $\Xi_p$ being the joint domain of variables from set $p$. In other words, Cano et al. apriori-conditionals represent completely the class of marginally consistent conditional belief functions.

Hence it is nearly obvious that in general case-based (separately measured) belief functions do not possess marginally consistent conditional belief functions.

Therefore it is in general of primary interest to find an appropriate approximation of general belief functions by means of decompositions into Cano’s et al apriori-conditionals. Let us say that an approximation $Bel'$ of belief function Bel is correct iff for every set $A$ Bel$'$(A) < Bel(A). An approximation $Bel'$ of belief function Bel is marginally correct iff for every variable $X$ and every set $A$ Bel$'$(X,A) < Bel(X,A).
Let us consider the following algorithm for calculation of a correct approximation of the function \( Bel \) in variables \( X,Y \):

0. We initialize the basic probability assignment function \( m_{\text{cond}} \) defined over variables \( X,Y \) with 0 for every subset of \( \Xi_X \times \Xi_Y \).

1. For each set \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \) we calculate the quantity \( g(A_X, A_Y) := m(A_X \times A_Y) / m^{i,X}(A_X) \).

2. \( i:=1, q:=0 \)

3. For each set \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \) we select a set \( r_i(A_X) \subseteq \Xi_Y \) with \( g(A_X, r_i(A_X)) > 0 \). \( g_{i,r,min} \) be the minimum of \( g(A_X, r_i(A_X)) \) over all \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \).

4. For the relation \( r_i \) we select a function \( a_i : \Xi_X \to 2^{\Xi_Y} \) such that for each set \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \) have \( r_i(A_X) \subseteq \bigcup_{\xi_X \in A_X} a_i(\xi_X) \).

5. We update the function \( m_{\text{cond}} \) as follows: We calculate the set \( A = \bigcup_{\xi_X \in \Xi_X} \{\xi_X\} \times a_i(\xi_X) \). Then \( m_{\text{cond}}(A) := m_{\text{cond}}(A) + g_{i,r,min} \).

6. We update the \( g \) function as follows: For each set \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \) \( g(A_X, r_i(A_X)) := g(A_X, r_i(A_X)) - g_{i,r,min} \).

7. For each set \( A_X \subseteq \Xi_X \) with \( m^{i,X}(A_X) > 0 \)

\[
q := \frac{g + g_{i,r,min} \cdot m^{i,X}(A_X) \cdot \text{card}(A_X)}{\text{card}(\bigcup_{\xi_X \in A_X} a_i(\xi_X))}
\]

8. \( i:=i+1 \). If \( q \) is equal zero everywhere then terminate, otherwise continue with step 3.

The marginal quality of an approximation constructed by the above algorithm be the quantity \( q \). The quality \( q \) can range from zero to one. If the quality is equal one then we have constructed a marginally consistent conditional of Bel given X. If Bel is a probabilistic belief function (with focal points being singleton sets), then a marginally consistent Cano’s conditional of Bel given X always exists. If Bel is a general belief function possessing a marginally consistent Cano’s conditional of Bel given X, then the construction by the above algorithm of the conditional is characterized by the fact that in step 4 we have \( r_i(A_X) = \bigcup_{\xi_X \in A_X} a_i(\xi_X) \). However, the construction task as such is hard. In particular, we cannot assume that if there exists a Cano’s conditional of Bel given X, and if and if for i=1,...,k we managed to obtain \( r_i(A_X) = \bigcup_{\xi_X \in A_X} a_i(\xi_X) \), then we will get a \( r_i(A_X) = \bigcup_{\xi_X \in A_X} a_i(\xi_X) \) for i=k+1. Consider the following (counter)example: Let Bel be a belief function separately measurable in X,Y, marginally consistent with \( Bel_1 \oplus Bel_2 \) where \( Bel_1 \) being a belief function in X, \( Bel_2 \) in X,Y with focal points:

| \( A_X \) | \( m_1(A_X) \) |
|---|---|
| \{x_1\} | \( p_1 \) |
| \{x_2\} | \( p_2 \) |
| \{x_3\} | \( p_2 \) |
| \{x_1, x_2\} | \( p_{12} \) |
| \{x_1, x_3\} | \( p_{13} \) |
| \{x_2, x_3\} | \( p_{23} \) |

| \( A_{X,Y} \) | \( m_2(A_{X,Y}) \) |
|---|---|
| \{(x_1, y_1), (x_2, y_2) \} | \( 1/3 \) |
| \{(x_1, y_1), (x_2, y_2), (x_2, y_3) \} | \( 1/3 \) |
| \{(x_1, y_2), (x_2, y_1) \} | \( 1/3 \) |

Obviously then \( Bel_2 \) is the marginally consistent Cano conditional of Bel. Let us select \( r_1 \) as follows:

| \( A_X \) | \( r_1(A_X) \) |
|---|---|
| \{x_1\} | \{y_1\} |
| \{x_2\} | \{y_1\} |
| \{x_3\} | \{y_1\} |
| \{x_1, x_2\} | \{y_1\} |
| \{x_1, x_3\} | \{y_1\} |
| \{x_2, x_3\} | \{y_1\} |

We can then easily construct function \( a_1 \) as \( a_1(x_1) = y_1, a_1(x_2) = y_1, a_1(x_3) = y_1 \) so that obviously \( r_1(A_X) = \bigcup_{\xi_X \in A_X} a_1(\xi_X) \). However, if we increase in step 8 i to i=2 and reenter step 3, then it will not be possible any more to construct such an \( a_2 \) that \( r_2(A_X) = \bigcup_{\xi_X \in A_X} a_2(\xi_X) \) because necessarily a fragment of \( r_2 \) will be:

| \( A_X \) | \( r_2(A_X) \) |
|---|---|
| \{x_1, x_2\} | \{y_1, y_2\} |
| \{x_1, x_3\} | \{y_1, y_2\} |
| \{x_2, x_3\} | \{y_1, y_2\} |

Hence in general finding a marginally consistent Cano’s conditional, even if it exists, is hard and requires backtracking. So probably one will be satisfied already if one finds a high quality marginally correct approximate conditional belief distribution (Application of genetic algorithms is advised here.).

## 4 IMPACT ON REASONING

Let Bel be a belief function in X,Y and \( Bel^{i,X} \) its marginally correctly approximate conditional belief distribution. Then \( Bel^{i,X} \oplus Bel^{i,X} \) is a marginally correct approximation of Bel. Then Shafer’s conditioning Bel(\{A_X \times \Xi_Y\}) with \( A_X \) being any non-empty subset of the domain \( \Xi_X \) of X is marginally correctly approximated by (Bel^{i,X} \oplus Bel^{i,X})(\{A_X \times \Xi_Y\}), that is
given the "decomposition" of Bel into its marginal on X and the approximately correct conditional on X we can reason approximately correctly about the a posteriori distribution of Y given any event of observation of the intrinsic value of X. However, we cannot do it in the reverse direction: we cannot derive the a posteriori distribution of X by observation of Y because in general Bel(\|X \times A_Y\|) with A_Y being any non-empty subset of the domain \Xi_Y of Y is NOT marginally correctly approximated by (Bel_{\|X\|} \oplus Bel_{\|X\|})(\|X \times A_Y\|). What is worse, even if the conditionals are marginally consistent, we will not achieve marginal correctnesses, not to say marginal consistency.

Thus, if we have a marginally consistent factorization of a belief function then we can in general use neither Shafer & Shenoy nor Cano's at al. framework for propagation of uncertainty because the results will be inconsistent with the data. A way out of this problem, for the framework of Cano, is to consider separate marginally correct approximations in each direction of reasoning. Given the polytree representing correctly dependencies among variables, we need to transform this polytree, for every variable the value of which we want to infer, into a such a (belief) network graph that every arrow points towards that variable. In case of plain directed trees the result of this transformation is a polytree (with undirected backbone identical with that of the original tree) such that edges connecting the target variable with its neighbours are reoriented to point at the target variable and every other variable is connected via a directed path with the target variable. In case of general polytree the result of the transformation is a network with more edges than the original polytree. Edges are added when an edge is reversed which originally pointed at a node with many ingoing edges. E.g. if we had the situation that X → Z < −Y and we want to invert the edge Z < −Y, then we need to add the edge X → Y. In case of general polytree we obtain in this way networks with considerably different undirected backbones.

If we assume that we adopt a structure (factorization) of Cano type, that is in form of a generalized polytree (singly connected "bayesian" network), then products of above transformations are directly convertible into a Shenoy’s and Shafer’s Markov tree. Let us now shift to Shenoy’s and Shafer’s belief propagation in Markov tree: The general principle there is "message passing" - if a node of Markov-tree gets information from its all but one neighbors, then it sends, to the remaining node, a "message", that is \oplus combination of those messages plus its own factor of the belief function factorization. In the original Shenoy/Shafer algorithm, this node’s own factor of the belief function factorization is exactly the same independently to which neighbor the message is sent. We propose to have separate hypertrees for each target variable and to reason within each of them in one direction only (resp. modifications of propagation algorithm are known). Then it is guaranteed that the results of reasoning (a posteriori marginal distributions) will be marginally correct approximations with respect to the intrinsic distribution.

Cano et al. conditionals are in fact sets of mappings between sets of variables selected with some probability. This gives a new meaning to the belief function. Instead of thinking in the way probability functions do that is that given some value of one variable, the conditional probability distribution assigns a value to another variable, we can think of objects that are assigned with some probability a belief.

5 DISCUSSION

Smets [27] stated that domains of MTE applications are those where "we are ignorant of the existence of probabilities", and warns that MTE is "not a model for poorly known probabilities" ([27], p.324). Smets states further "Far too often, authors concentrate on the static component (how beliefs are allocated?) and discover many relations between TBM (transferable belief model of Smets) and ULP (upper lower probability) models, inner and outer measures (Fagin and Halpern [6]), random sets (Nguyen [15]), probabilities of provability (Pearl [16]), probabilities of necessity (Ruspini [19]) etc. But these authors usually do not explain or justify the dynamic component (how are beliefs updated?), that is, how updating (conditioning) is to be handled (except in some cases by defining conditioning as a special case of combination). So I (that is Smets) feel that these partial comparisons are incomplete, especially as all these interpretations lead to different updating rules." ([27], pp. 324-325). Ironically, Smets gives later in the same paper an example of belief function ("hostile-Mother-Nature-Example") which may be clearly considered as lower probability interpretation of belief function, just, at further consideration, leading to very same pitfalls as approaches criticized himself.

In order to explicate the reasons of failures of various attempts to establish a case-based interpretation of MTE, in this paper we drew our attention to the questions related to updating (conditioning) in this theory. We have investigated some fundamental problems of case-based interpretation of the Dempster-Shafer Mathematical Theory of Evidence.
6 CONCLUSION

1. The case-based derivation of aposteriori MTE belief function requires combination of operations of case selection with operation of case updating.

2. Therefore, marginally correct inference rules for MTE derived from data can be applied only uni-directionally - reversal of direction of reasoning leads to contradictions with the data. For this reason, original Cano et al. and Shenoy/Shafer uncertainty propagation methods are not suitable for case-based MTE belief functions.

3. Hence in case of direct dependence of a set of n variables, n inferences networks - one for each variable as dependent on the remaining ones - have to be established and used depending on target variable.

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