Exponential stabilization on infinite dimensional system with impulse controls

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Abstract

This paper studies the exponential stabilization on infinite dimensional system with impulse controls, where impulse instants appear periodically. The first main result shows that exponential stabilizability of the control system with a periodic feedback law is equivalent to one kind of weak observability inequalities. The second main result presents that, in the setting of a discrete LQ problem, the exponential stabilizability of control system with a periodic feedback law is equivalent to the solvability of an algebraic Riccati-type equation which was built up in [Qin, Wang and Yu, SIAM J. Control Optim., 59 (2021), pp. 1136-1160] for finite dimensional system. As an application, some sufficient and necessary condition for the exponential stabilization of an impulse controlled system governed by coupled heat equations is given.

Keywords. Exponential stabilization, Infinite dimensional system, Impulse control, Periodic feedback

Mathematics Subject Classifications. 35K40, 93B05, 93C20

1 Introduction

We start this section with some notations. Let $\mathbb{R}^+ := (0, +\infty), \mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ := \{1, 2, \ldots\}$. Given a Hilbert space $V$, we denote its norm and inner product by $\| \cdot \|_V$ and $\langle \cdot, \cdot \rangle_V$ respectively. Let $\mathcal{L}(V; W)$ be the space of all linear bounded operators from a Hilbert space $V$ to another Hilbert space $W$ and we write $\mathcal{L}(V) := \mathcal{L}(V; V)$ for simplicity. We denote by $\text{Card}E$ the number of the set of $E$. Given $h \in \mathbb{N}^+$ arbitrarily. Let

$$J_h := \{\{t_j\}_{j \in \mathbb{N}} : t_0 = 0 < t_1 < t_2 < \cdots \text{ and } t_{j+h} - t_j = t_h \text{ for each } j \in \mathbb{N}\}.$$  

Then, for each $\{t_j\}_{j \in \mathbb{N}} \in J_h$, we have

$$t_{j+k} + k = t_j + kt_h \text{ for all } j, k \in \mathbb{N}. \quad (1.1)$$

Given an (bounded or unbounded) operator $O$, let $O^*$ be its adjoint operator. Given a Hilbert space $V$, we define the following sets:

$$\mathcal{S}_{\mathcal{L}}(V) := \{O \in \mathcal{L}(V) : O^* = O\}, \quad \mathcal{S}_{\mathcal{L}+}(V) := \{O \in \mathcal{S}_{\mathcal{L}}(V) : \langle v, Ov \rangle_V \geq 0 \text{ for any } v \in V\},$$

$$\mathcal{S}_{\mathcal{L}+\mathcal{L}}(V) := \{O \in \mathcal{S}_{\mathcal{L}+}(V) : \text{ there exists } \delta > 0 \text{ so that } O - \delta I \in \mathcal{S}_{\mathcal{L}+}(V)\},$$

$$\mathcal{M}_{h,+}(V) := \{(M_j)_{j \in \mathbb{N}^+} : M_j \in \mathcal{S}_{\mathcal{L}+}(V) \text{ and } M_{j+h} = M_j \text{ for each } j \in \mathbb{N}^+\}$$

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and 
\[ Mh := \{ (M_j)_{j \in \mathbb{N}^+} : M_j \in \mathcal{S}L(V) \text{ and } M_{j+h} = M_j \text{ for each } j \in \mathbb{N}^+ \} . \]

Throughout this paper, we use \( H \) and \( U \) to denote two reflexive Hilbert spaces. Assume that \( H \) and \( U \) are identified with their dual spaces respectively. Let the linear unbounded operator \( A \) with domain \( D(A) \subset H \) be a generator of a \( C_0 \)-semigroup \( \{ e^{At} \}_{t \geq 0} \) over \( H \). Let \( B_h := \{ B_k \}_{k=1}^h \subset \mathcal{L}(U; H) \) and \( B_h^* := \{ B_k^* \}_{k=1}^h \subset \mathcal{L}(H; U) \).

1.1 Control problem and aim

Given \( \Lambda^h = \{ t_j \}_{j \in \mathbb{N}} \in \mathcal{J}_h \), we consider the following impulse controlled system:
\[
\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \in \mathbb{R}^+ \setminus \Lambda^h, \\
x(t^+_j) &= x(t_j) + B_{\nu(j)}u_j, \quad j \in \mathbb{N}^+,
\end{aligned}
\tag{1.2}
\]
where \( \nu(j) := j - [j/h]h \) for each \( j \geq 1 \), \( x(t^+_j) \) denotes the right limit at \( t_j \) for the function \( x(\cdot) \), and \( (u_j)_{j \in \mathbb{N}^+} \in L^2([0, \infty); U) \). Here and in what follows, \( [s] := \max\{ k \in \mathbb{N} : k < s \} \) for each \( s > 0 \). We denote the system \((1.2)\) by \([A, B_h, \Lambda^h]\) and the solution of \((1.2)\) by \( x(\cdot; x_0, u) \) with control \( u \in L^2([0, \infty); U) \) and initial data \( x_0 \in H \).

Now, we introduce some definitions on exponential stabilization of the system \([A, B_h, \Lambda^h]\).

**Definition 1.1.** The system \([A, B_h, \Lambda^h]\) is called exponentially \( h \)-stabilizable if there is a sequence of feedback laws \( \{ F_k \}_{k=1}^h \subset \mathcal{L}(H; U) \) so that the following closed-loop system is exponentially stable:
\[
\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \in \mathbb{R}^+ \setminus \Lambda^h, \\
x(t^+_j) &= x(t_j) + B_{\nu(j)}F_{\nu(j)}x(t_j), \quad j \in \mathbb{N}^+,
\end{aligned}
\tag{1.3}
\]
i.e., there are constants \( C > 0 \) and \( \mu > 0 \) so that any solution of \((1.3)\), denoted by \( x_F(\cdot) \), satisfies that
\[ \| x_F(t) \|_H \leq Ce^{-\mu t}\| x_F(0) \|_H \text{ for any } t \in \mathbb{R}^+. \]

If the system \([A, B_h, \Lambda^h]\) is exponentially \( h \)-stabilizable with feedback law \( \{ F_k \}_{k=1}^h \), we simply write \( F := \{ F_k \}_{k=1}^h \) and denote the closed-loop system \((1.3)\) by \([A, \{ B_k F_k \}_{k=1}^h, \Lambda^h]\). Since \( \{ F_k \}_{k=1}^h \) appears at time instants \( \Lambda^h \) \( h \)-periodically, the feedback law \( F \) is indeed \( h \)-periodic time-varying.

**Aim.** In this paper, we shall mainly present some equivalence criteria for the exponential stabilization of the system \([A, B_h, \Lambda^h]\).

1.2 Related works and motivation

The stabilizability of infinite dimensional system with distributed controls have been studied widely. For instance, we can refer to \([1-4, 7-9, 11-14, 16, 19, 22, 23]\). Recently, the impulse control system has attracted a lot of attention. For example, we can refer to \([5, 6, 15, 17, 18, 21]\). In particular, in \([18]\), the authors studied the exponential stabilizability of finite dimensional system with impulse controls and obtained some equivalence results. Therefore, how to obtain the equivalence conditions for the exponential stabilizability of infinite dimensional system with impulse controls seems very interesting.

1.3 Main results

The first main result can be stated as follows.

**Theorem 1.2.** Given \([A, B_h, \Lambda^h]\). The following statements are equivalent:

(i) The system \([A, B_h, \Lambda^h]\) is exponentially \( h \)-stabilizable.
(ii) For each \( \sigma \in (0,1) \), there are \( k = k(\sigma) \in \mathbb{N}^+ \) and \( C(\sigma) > 0 \) so that
\[
\|e^{A^*t_k} \varphi\|_H \leq C(\sigma) \left( \sum_{j=1}^{k} \|B^*_{\nu(j)} e^{A^*(t_{k-j})} \varphi\|_U^2 \right)^{\frac{1}{2}} + \sigma \|\varphi\|_H \text{ for any } \varphi \in H. \tag{1.4}
\]

(iii) There are \( \sigma \in (0,1) \), \( k \in \mathbb{N}^+ \) and \( C > 0 \) so that
\[
\|e^{A^*t_k} \varphi\|_H \leq C \left( \sum_{j=1}^{k-1} \|B^*_{\nu(j)} e^{A^*(t_{k-j-1})} \varphi\|_U^2 \right)^{\frac{1}{2}} + \sigma \|\varphi\|_H \text{ for any } \varphi \in H. \tag{1.5}
\]

(iv) There are \( \sigma \in (0,1) \), \( k \in \mathbb{N}^+ \) and \( C > 0 \) so that, for any \( x_0 \in H \), there is a control \( u \in l^2(\mathbb{N}^+; U) \) so that
\[
\|x(t_k; x_0, u)\|_H \leq \sigma \|x_0\|_H \text{ and } \|u\|_{l^2(\mathbb{N}^+; U)} \leq C \|x_0\|_H. \tag{1.6}
\]

**Remark 1.3.** It is well known that, on \([0,t_k]\) \((k \in \mathbb{N}^+)\), the dual-observe system of \([A, B_h, \Lambda_h]\) is given by
\[
\begin{align*}
g'(t) + A^*g(t) &= 0, \quad t \in [0,t_k], \\
g(t_k) &= \varphi \in H, \\
z(t_j) &= B^*_{\nu(j)} y(t_j), \quad j \in \{1, 2, \ldots, k\}.
\end{align*}
\]

We denote this observe system by \([A^*, B^*_h, \Lambda^*_h]\). From the viewpoint of observability, the inequality (1.4) is called a weak observability of \([A^*, B^*_h, \Lambda^*_h]\) on \([0,t_k]\). It should be noted that, in infinite dimensional setting, the strong observability inequality (i.e., \( \sigma = 0 \) in (1.4)), which means the corresponding control system is null controllable, does not hold generally (see, for instance, [17]).

The proof of Theorem 1.2 is based on the study of a discrete LQ problem associated with the system \([A, B_h, \Lambda_h]\). We now introduce this problem and present another equivalence criterion for the exponential stabilization of the system \([A, B_h, \Lambda_h]\).

We arbitrarily fix \( \Lambda_h = \{t_j\}_{j \in \mathbb{N}} \in \mathcal{J}_h, (Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\geq}(H) \) and \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\geq}(U) \). For each \( x_0 \in H \), define the admissible control set:
\[
U_{ad}(x_0) := \{u = (u_j)_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; U) : (x(t_j; x_0, u))_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; H)\} \tag{1.7}
\]
and the cost functional:
\[
J(u; x_0) := \sum_{j=1}^{+\infty} \langle (Q_j x(t_j; x_0, u), x(t_j; x_0, u))_H + (R_j u_j, u_j)_U \rangle \text{ for any } u = (u_j)_{j \in \mathbb{N}^+} \in U_{ad}(x_0). \tag{1.8}
\]

Consider the following optimal control problem:

**I-I-LQ:** Given \( x_0 \in H \). Find a control \( u^* = (u_j^*)_{j \in \mathbb{N}^+} \in U_{ad}(x_0) \) so that
\[
J(u^*; x_0) = \inf_{u \in U_{ad}(x_0)} J(u; x_0).
\]

The problem **I-I-LQ** is a kind of discrete LQ problems associated with the system \([A, B_h, \Lambda_h]\). For solving it, we first introduce the following operator-valued algebraic equation:
\[
\begin{align*}
P_{k-1} &= e^{A^*(t_{k-1}) - t_{k-1}} P_k e^{A(t_{k-1})} - e^{A^*(t_{k-1})} Q_k e^{A(t_{k-1})} \\
&\quad - e^{A^*(t_{k-1})} P_k B_k (R_k + B_k^* P_k B_k)^{-1} B_k^* P_k e^{A(t_{k-1})}, \quad k \in \{1, 2, \ldots, h\}, \\
P_0 &= P_h.
\end{align*}
\tag{1.9}
\]

We call (1.9) a Riccati-type equation, in which unknowns \( P_k \) \((k = 1, \ldots, h)\) are in \( \mathcal{SL}_+(H) \). The solution to (1.9), if exists, is denoted by \( \{P_k\}_{k=0}^h \).

The second main result of this paper is given as follows.
Theorem 1.4. Given $[A, B_h, \Lambda_h]$. The following statements are equivalent:

(i) The system $[A, B_h, \Lambda_h]$ is exponentially $h$-stabilizable.

(ii) For each $x_0 \in H$, the set $\mathcal{U}_{ad}(x_0)$ is not empty.

(iii) For any $(Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(H)$ and $(R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(U)$, the Riccati-type equation (1.9) has a unique solution.

(iv) There are $(Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(H)$ and $(R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(H)$ so that the Riccati-type equation (1.9) has a unique solution.

Moreover, when the system (1.9) is exponentially $h$-stabilizable, the feedback law $F = \{F_k\}_{k=1}^h$ can be chosen as

$$F_k := -(R_k + B^*_kB_k)^{-1}B^*_kP$$

for each $k = 1, ..., h$,

where $\{P_k\}_{k=0}^h$ is the solution of (1.9) with arbitrarily fixed $(Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(H)$ and $(R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(U)$.

1.4 Novelties of this paper

- As an important control system, there is very little literature considering the stabilization problem with impulse controls in infinite dimensional setting. In essence, the controllers in the impulse control system are time-varying. Thus, the classic strategies (for instance, frequency domain method) do not work. Indeed, Theorem 1.2 can be regarded as a time domain method (for the time domain method in continuous control system, one can refer to [14, 22]).

- It seems for us that the equivalence characterization on the exponential stabilization for impulse control systems in infinite dimensional setting has not been touched upon (at least, we do not find any such literature). Thus, the equivalence results in Theorem 1.2 and Theorem 1.4 are new.

- In infinite dimensional setting, the discrete LQ problem used in our paper seems new for us. Although many literatures used LQ problem to consider the stabilizability of infinite dimensional systems (for instance, [7, 11]), but our LQ problem is discrete. It is completely different from continuous time horizon LQ problems.

1.5 Plan of this paper

The rest of this paper is organized as follows: In section 2, we present some preliminaries. In section 3, we give the proof of our main results. In Section 4, we apply our main results to the system of heat equations coupled by constant matrices with impulse controls.

2 Preliminaries

2.1 Finite horizon LQ problem

Given $\Lambda_h = \{t_j\}_{j \in \mathbb{N}} \in \mathcal{J}_h$. We arbitrarily fix $(Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, +}(H)$ and $(R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \gg}(U)$ and $M \in \mathcal{SL}_+(H)$. We also fix a terminal instant $t_{\ell}$ with $\ell \in \mathbb{N}^+$. For each $\ell \in \mathbb{N}$ and any $x_0 \in H$, we denote by $x(\cdot; x_0, v, \ell)$ the solution of the following system:

$$\begin{align*}
{x'}(t) &= A{x}(t), & t &\in \mathbb{R}^+ \setminus \Lambda_h, \\
x(t_j^+) &= x(t_j) + B_{\nu(j)}v_j, & j &> \ell, \\
x(t_{\ell}^+) &= x_0,
\end{align*}$$

(2.1)
where \( v = (v_{j+\ell})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \) is the control. Here and in what follows, for a given Hilbert space \( V \), \((f_j)_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; V)\) means that, if we let \( g_j := f_{j+\ell} \) for each \( j \in \mathbb{N}^+ \), then \((g_j)_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; V)\).

Given \( x_0 \in H \). For each \( \ell \in \mathbb{N} \) with \( \ell < \hat{k} \), we define

\[
J(v; x_0, \ell, \hat{k}) := \sum_{j=\ell+1}^{\hat{k}} \left( (Q_j x(t_j; x_0, v, \ell), x(t_j; x_0, v, \ell))_H + (R_j v_j, v_j) \right) + \langle M x(t_{\hat{k}}^+; x_0, v, \ell), x(t_{\hat{k}}^+; x_0, v, \ell) \rangle_H \text{ for each } v = (v_{j+\ell})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U).
\]

(2.2)

It should be noted that, in the definition of \( J(v; x_0, \ell, \hat{k}) \), the effective controls are \( \{v_j\}_{j=\ell+1}^{\hat{k}} \).

Now, we consider the following finite horizon LQ problem associated with the system \([A, B_0, \Lambda_h]:\)

\[
(F-I-LQ)_{\ell, \hat{k}}: \text{Given } x_0 \in H, \text{ find a control } v(\ell, \hat{k}) = (v_{j+\ell}(\ell, \hat{k}))_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \text{ so that } J(v(\ell, \hat{k}); x_0, \ell, \hat{k}) = \inf_{v = (v_{j+\ell})_{j \in \mathbb{N}^+}} J(v; x_0, \ell, \hat{k}).
\]

Since \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \succ}(U)\), by some standard arguments, one can easily show that

(a1) \((F-I-LQ)_{\ell, \hat{k}}\) has an optimal control for any \( x_0 \in H \);

(a2) \((v_{\ell+1}(\ell, \hat{k}), \ldots, v_{\hat{k}}(\ell, \hat{k}))\) (called the restriction of \( v(\ell, \hat{k}) \) on \([\ell+1, \hat{k}]\)) is unique for any \( x_0 \in H \).

Fix \( \ell \in \mathbb{N} \) with \( \ell < \hat{k} \). Let

\[
V(x_0; \ell, \hat{k}) := \inf_{v = (v_{j+\ell})_{j \in \mathbb{N}^+}} J(u; x_0, \ell, \hat{k}) \text{ for any } x_0 \in H
\]

(2.3)

and \( \{P_j^{\ell, \hat{k}}\}_{j=1}^{\hat{k}} \) be the solution of the following Riccati-type equation:

\[
\begin{cases}
P_j^{\ell, \hat{k}} - e^{A^*(t_j-t_{j-1})} P_{j-1}^{\ell, \hat{k}} e^{A(t_{j-1}-t_j)} = e^{A^*(t_j-t_{j-1})} Q_j e^{A(t_{j-1}-t_j)} - e^{A^*(t_{j-1}-t_j)} P_j^{\ell, \hat{k}} B_{\nu(j)} (R_j + B_{\nu(j)}^* P_j^{\ell, \hat{k}} B_{\nu(j)})^{-1} B_{\nu(j)}^* e^{A(t_j-t_{j-1})}, & j \in \{1, 2, \ldots, \hat{k}\}, \\
P_{\hat{k}}^{\ell, \hat{k}} = M.
\end{cases}
\]

(2.4)

**Remark 2.1.** Since \((Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, +}(H)\) and \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \succ}(U)\) and \(M \in \mathcal{SL}_+(H)\), it is clear that, for any \( x_0 \in H \),

\[
J(v; x_0, \ell, \hat{k}) \geq 0 \text{ for all } v \in L^2(\mathbb{N}^+; U).
\]

This implies that \( V(x_0; \ell, \hat{k}) \geq 0 \) for all \( x_0 \in H \). Moreover, since \( M \in \mathcal{SL}_+(H) \), we can conclude that the equation (2.4) has a unique solution \( \{P_j^{\ell, \hat{k}}\}_{j=1}^{\hat{k}} \) satisfying \( P_j^{\ell, \hat{k}} = (P_j^{\ell, \hat{k}})^* \) for each \( j = 1, 2, \ldots, \hat{k} \).

Next, we state the relationship between the optimal control problem \((F-I-LQ)_{\ell, \hat{k}}\) and the solution of the Riccati-type equation (2.4).

**Proposition 2.2.** For each \( \ell \in \mathbb{N} \) with \( \ell < \hat{k} \), it stands that

\[
V(x_0; \ell, \hat{k}) = (P_{\ell, \hat{k}} x_0, x_0)_H \text{ for any } x_0 \in H.
\]

(2.5)

**Proof.** Let \( x_0 \in H \) be arbitrarily fixed. The proof is given by induction.

**Step 1.** We show that (2.5) stands for \( \ell = \hat{k} - 1 \).

In fact, by (2.2), one can easily check that, for any \( v = (v_{j+\ell-1})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \),

\[
J(v; x_0, \ell, \hat{k}) = J(v; x_0, \hat{k} - 1, \hat{k})
\]

\[
= (Q_{\hat{k}} x(t\hat{k}; x_0, v, \hat{k}-1), x(t\hat{k}; x_0, v, \hat{k}-1))_H + (R_{\hat{k}} v_{\hat{k}}, v_{\hat{k}}) + \langle M x(t_{\hat{k}}^+; x_0, v, \hat{k}-1), x(t_{\hat{k}}^+; x_0, v, \hat{k}-1) \rangle_H.
\]

(2.6)
By (2.1), it is obvious that
\[ x(t_{k}; x_0, v, \hat{k} - 1) = e^{A(t_{k} - t_{k-1})}x_0 \] and \( x(t_{k}^\pm; x_0, u, \hat{k} - 1) = e^{A(t_{k} - t_{k-1})}x_0 + B_{\nu}^{-1}u_{\nu}. \) (2.7)

Thus, by (2.6) and (2.7), we have
\[
J(v; x_0, \hat{k} - 1, \hat{k}) = \langle (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^U v_{\nu}, v_{\nu} \rangle + 2\langle v_{\nu}, B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \rangle_U \\
+ \langle (Q_{v} + M) e^{A(t_{k} - t_{k-1})}x_0, e^{A(t_{k} - t_{k-1})}x_0 \rangle_H
\]
\[
= \| (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^U v_{\nu} + (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \|_U^2 \\
- \langle (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0, B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \rangle_U \\
+ \langle (Q_{v} + M) e^{A(t_{k} - t_{k-1})}x_0, e^{A(t_{k} - t_{k-1})}x_0 \rangle_H. \) (2.8)

It follows that
\[
J(v; x_0, \hat{k} - 1, \hat{k}) \geq -\langle (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0, B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \rangle_U \\
+ \langle (Q_{v} + M) e^{A(t_{k} - t_{k-1})}x_0, e^{A(t_{k} - t_{k-1})}x_0 \rangle_H \text{ for any } v = (v_{j+\hat{k}-1})_{j \in \mathbb{N}^+} \in l^2 (\mathbb{N}^+; U). \) (2.9)

Taking \( v^* = (v_{j+\hat{k}-1})_{j \in \mathbb{N}^+} \in l^2 (\mathbb{N}^+; U) \) with
\[
v_{\nu} = -(R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \\
= -(R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M P_{\nu}^\infty x(t_{k}; x_0, v, \hat{k} - 1)
\]
into (2.8), by (2.9), we get
\[
J(v^*; x_0, \hat{k} - 1, \hat{k}) = -\langle (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0, B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \rangle_U \\
+ \langle (Q_{v} + M) e^{A(t_{k} - t_{k-1})}x_0, e^{A(t_{k} - t_{k-1})}x_0 \rangle_H. \) (2.10)

By (2.9) and (2.10), we can conclude that
\[
V(x_{0}; \hat{k} - 1, \hat{k}) = J(v^*; x_0, \hat{k} - 1, \hat{k}) = \langle (Q_{v} + M) e^{A(t_{k} - t_{k-1})}x_0, e^{A(t_{k} - t_{k-1})}x_0 \rangle_H \\
- \langle (R_{v} + B_{v}(\nu) M B_{v}(\nu) )^{-1} B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0, B_{v}(\nu) M e^{A(t_{k} - t_{k-1})}x_0 \rangle_U.
\]

This, together with (2.4), implies that (2.5) stands for \( \ell = k - 1. \)

Step 2. Let \( \ell' \in \mathbb{N} \) with \( \ell' < k - 1 \) be fixed arbitrarily. Suppose that for \( \ell = \ell' + 1, \ell' + 2, \ldots, k - 1, \) (2.5) stands. We prove that (2.5) holds for \( \ell = \ell'. \)

In fact, for each \( v = (v_{j+\ell'+1})_{j \in \mathbb{N}^+} \in l^2 (\mathbb{N}^+; U), \) we have
\[
J(v; x_0, \ell', \hat{k}) = \sum_{j=\ell'+1}^{\ell + \hat{k}} \langle (Q_{j} x(t_j; x_0, v, \ell'), x(t_j; x_0, v, \ell')) \rangle_H + \langle (R_{j} v_j, v_j) \rangle_U \\
+ \langle M x(t^+_k; x_0, v, \ell'), x(t^+_k; x_0, v, \ell')) \rangle_H \\
= \langle Q_{\ell'+1} x(t_{\ell'+1}; x_0, v, \ell'), x(t_{\ell'+1}; x_0, v, \ell') \rangle_H + \langle R_{\ell'+1} v_{\ell'+1}, v_{\ell'+1} \rangle_U \\
+ J \left( v'; x(t^+_{\ell'+1}; x_0, \ell'), \ell' + 1, \hat{k} \right),
\]
where \( v' := (v_{j+1}+1)_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \). This, along with the assumption that (2.5) stands for \( \ell = \ell' + 1 \), shows that, for any \( v = (v_{j+1}+1)_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \),

\[
J(v; x_0, \ell, \hat{k}) \geq \langle Qv_{\ell+1} + x(t_{\ell+1}; x_0, v, \ell' + 1) \rangle_H + \langle Rv_{\ell+1} + x(t_{\ell+1}; x_0, v, \ell') \rangle_H.
\]

Then by the same arguments as those in Step 1, we obtain that

\[
J(v; x_0, \ell', \hat{k}) \geq (P^k_{\ell'+1}x_0) \quad \text{for any} \quad v = (v_{j+1}+1)_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U),
\]

and when we take \( v = (v_{j+1}+1)_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \) with

\[
v_j = -(R_j + B_jP_j^B_j)^{-1}B_j^*P_j^k \quad \text{for each} \quad j = \ell' + 1, \ldots, \hat{k}
\]

into (2.11), the inequality (2.11) achieves an equality.

We complete the proof.

By Remark 2.1 and Proposition 2.2, we have the following corollary:

**Corollary 2.3.** The solution \( \{P^k_j\}_{j=1}^{\hat{k}} \) of the Riccati-type equation (2.4) belongs to \( SL_+(H) \).

### 2.2 Infinite horizon LQ problem

Arbitrarily fix \( \Lambda_h = \{(t_j)_{j\in\mathbb{N}} \in \mathcal{J}_h \mid (Q_j)_{j\in\mathbb{N}^+} \in \mathcal{M}_{h,\geq}(H) \} \) and \( (R_j)_{j\in\mathbb{N}^+} \in \mathcal{M}_{h,\geq}(U) \). Let \( U_{ad}() \) and \( J(\cdot; \cdot) \) be defined by (1.7) and (1.8), respectively. Throughout this subsection, we always suppose that

\[
U_{ad}(x_0) \neq \emptyset \quad \text{for any} \quad x_0 \in H.
\]

Now, we consider the problem (I-I-LQ). From (2.12), the following two facts should be noted:

(b1) By a standard argument, one can show that the problem (I-I-LQ) has a unique optimal control for any \( x_0 \in H \);

(b2) For any \( x_0 \in H \), \( \inf_{u \in U_{ad}(x_0)} J(u; x_0) = \inf_{u \in L^2(\mathbb{N}^+; U)} J(u; x_0) < +\infty \). Here and in what follows, we permit \( J(u; x_0) = +\infty \) if \( u \in L^2(\mathbb{N}^+; U) \setminus U_{ad}(x_0) \).

For any \( x_0 \in H \), we define the value function of the problem (I-I-LQ) associated with \( x_0 \in H \) as follows:

\[
V(x_0) := \inf_{u \in U_{ad}(x_0)} J(u; x_0).
\]

Since (b2) stands, it is clear that \( V(x_0) < +\infty \) for any \( x_0 \in H \). For each \( \ell \in \mathbb{N} \) and any \( x_0 \in H \), we define

\[
U_{ad}(x_0; \ell) := \{ u = (u_{j+\ell})_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U) : (x(t_{j+\ell}; u, x_0, \ell))_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; H) \},
\]

and

\[
J(u; x_0, \ell) := \sum_{j=\ell+1}^{\infty} \langle (Q_jx(t_j; x_0, u, \ell)), x(t_j; x_0, u, \ell) \rangle_H + \langle R_ju_j, u_j \rangle_H
\]

for any \( u = (u_{j+\ell})_{j\in\mathbb{N}^+} \in U_{ad}(x_0; \ell) \), where \( x(\cdot; x_0, u, \ell) \) is the solution of the equation (2.1) with initial data \( x_0 \) and control \( u = (u_{j+\ell})_{j\in\mathbb{N}^+} \in L^2(\mathbb{N}^+; U) \). One can easily check that, for any \( x_0 \in H \),

\[
U_{ad}(x_0; 0) = U_{ad}(x_0), \quad J(u; x_0, 0) = J(u; x_0) \quad \text{for any} \quad u \in U_{ad}(x_0).
\]
Lemma 2.4. For each \( \ell \in \mathbb{N} \) and \( x_0 \in H \), \( U_{ad}(x_0; \ell) \neq \emptyset \).

Proof. Arbitrarily fix \( \ell \in \mathbb{N} \) and \( x_0 \in H \). First of all, by (2.12) and (2.14), we have

\[ U_{ad}(x_0; 0) \neq \emptyset, \]

i.e., there exists a control \( \tilde{u} = (\tilde{u}_j)_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; U) \) so that

\[ (x(t_j; x_0, \tilde{u}))_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; H). \tag{2.15} \]

We now claim that

\[ U_{ad}(x_0; N\ell) \neq \emptyset \] for any \( N \in \mathbb{N}^+. \) (2.16)

Actually, let \( \tilde{u}^N = (\tilde{u}_j^N)_{j \in \mathbb{N}^+} \) be defined as

\[ \tilde{u}_j^N := \tilde{u}_{j-N\ell} \text{ for each } j > N\ell. \]

It is clear that \( (\tilde{u}_j^N)_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; U) \). Moreover, we can directly check that

\[ x(t_j+N\ell; x_0, \tilde{u}_j^N, N\ell) = x(t_j; x_0, \tilde{u}, 0) \text{ for all } j \in \mathbb{N}^+. \tag{2.17} \]

Thus, by (2.15) and (2.17), we have \( \tilde{u}_j^N \in U_{ad}(x_0; N\ell) \). It follows that (2.16) is true.

Let \( N := \lceil \ell / h \rceil \). It is obvious that \( N\ell < \ell \leq (N+1)\ell \). By the arbitrariness of \( x_0 \) and (2.16), there exists a control

\[ \hat{u} = (\hat{u}_j)_{j \in \mathbb{N}^+} \in U_{ad}(e^{A\ell t_1}x_0; (N+1)\ell), \]

and

\[ (x(t_j+(N+1)\ell; \hat{u}_j, (N+1)\ell))_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; H). \tag{2.19} \]

Define \( \hat{u} = (\hat{u}_j+\ell)_{j \in \mathbb{N}^+} \) in the manner:

\[ \hat{u}_j := \begin{cases} 0 & \text{when } j < (N+1)\ell, \\ \hat{u}_j & \text{when } j > (N+1)\ell. \end{cases} \]

By (2.18), it is clear that \( \hat{u} \in l^2(\mathbb{N}^+; U) \). Moreover, one can easily check that

\[ x(t_j; \hat{u}_j, x_0, \ell) = \begin{cases} e^{A(t_j-t_1)}x_0, & \text{if } \ell \leq j \leq (N+1)\ell, \\ x(t_j; \hat{u}_j, e^{A(t_j+(N+1)\ell-t_1)}x_0, (N+1)\ell), & \text{if } j > (N+1)\ell. \end{cases} \]

This, along with (2.19), yields that \( (x(t_j+\ell; \hat{u}_j, x_0, \ell))_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; H) \). Hence, \( \hat{u} \in U_{ad}(x_0; \ell) \), i.e., \( U_{ad}(x_0; \ell) \neq \emptyset \). This ends the proof.

Now, for each \( \ell \in \mathbb{N} \), we study the following LQ problem:

\textbf{(I-I-LQ)\ell}: Given \( x_0 \in H \). Find a control \( u^\ell \in U_{ad}(x_0; \ell) \) so that

\[ V(x_0; \ell) := \inf_{u \in U_{ad}(x_0; \ell)} J(u; x_0, \ell) = J(u^\ell; x_0, \ell). \tag{2.20} \]

We call \( V(\cdot; \ell) \) the value function of \((\textbf{I-I-LQ})_\ell\). It is clear that \((\textbf{I-I-LQ})_0\) coincides with \((\textbf{I-I-LQ})\) and \( V(\cdot) = V(\cdot; 0) \).

Lemma 2.5. The following two statements are true:

(i) For each \( \ell \in \mathbb{N} \), there is a unique \( P_\ell \in \mathcal{S}\mathcal{L}_+(H) \), so that

\[ V(x_0; \ell) = \langle P_\ell x_0, x_0 \rangle_H \text{ for any } x_0 \in H. \tag{2.21} \]
(ii) \( P_{t+h} = P_t \) for all \( \ell \in \mathbb{N} \).

Proof. Let \( \ell \in \mathbb{N} \) be arbitrarily fixed. We first present the proof of (i). The proof is divided into 3 steps.

Step 1. We prove that there exists a constant \( K > 0 \) so that
\[
V(x_0; \ell) \leq K\|x_0\|_H^2 \quad \text{for any} \quad x_0 \in H. \tag{2.22}
\]

Let \( x_0 \in H \) be arbitrarily fixed. By Lemma 2.4, we know that \( \mathcal{U}_{a,b}(x_0; \ell) \neq \emptyset \) which implies that \( 0 \leq V(x_0; \ell) < +\infty \) obviously. Thus the problem (I-I-LQ)\( \ell \) has a unique optimal control \( u^* = (u^*_{j+\ell})_{j} \in \mathbb{N}^+ \) (see (b1) above for the case \( \ell = 0 \)).

For each \( k > \ell \), we consider the finite horizon LQ problem (F-I-LQ)\( \ell,k \) with \( M = 0 \) and the same \( (Q_j)_{j} \in \mathbb{N}^+ \), \( (R_j)_{j} \in \mathbb{N}^+ \) as in the problem (I-I-LQ)\( \ell \) with respect to the initial data \( x_0 \in H \) (for the definition of (F-I-LQ)\( \ell,k \), one can see Section 2.1 with \( k = \hat{k} \)). Then, it is obvious that
\[
V(x_0; \ell,k) \leq J(u^*; x_0, \ell) \leq V(x_0; \ell) \quad \text{for each} \quad k \in \mathbb{N}^+. \tag{2.23}
\]

Therefore, there exist a subsequence of \( (Q_j)_{j} \in \mathbb{N}^+ \), \( (R_j)_{j} \in \mathbb{N}^+ \) so that
\[
\sum_{j=\ell+1}^k (\langle Q_j x^\ell,k,x_0(t_j), x^\ell,k,x_0(t_j) \rangle_H + \langle R_j u^\ell,k,x_0, u^\ell,k,x_0 \rangle_U) \leq V(x_0; \ell) \quad \text{for each} \quad k \in \mathbb{N}^+. \tag{2.24}
\]

Since the effective interval of (F-I-LQ)\( \ell,k \) is \([\ell, k]\), we permit that
\[
u_j^{\ell,k,x_0} = 0 \quad \text{for each} \quad j > k. \tag{2.25}
\]

Define
\[
\tilde{x}^\ell,k,x_0 = \begin{cases} x^{\ell,k,x_0}(t_j), & \ell \leq j \leq k, \\ 0, & j > k, \end{cases} \tag{2.26}
\]
and \( \tilde{x}^{\ell,k,x_0} = (\tilde{x}^\ell,k,x_0)_{j} \in \mathbb{N}^+ \). Because \( (Q_j)_{j} \in \mathbb{N}^+ \in \mathcal{M}(H) \) and \( (R_j)_{j} \in \mathbb{N}^+ \in \mathcal{M}(U) \), there exists a constant \( \delta > 0 \) so that \( Q_j - \delta I \in \mathcal{S}\mathcal{L}_+ (H) \) and \( R_j - \delta I \in \mathcal{S}\mathcal{L}_+ (U) \) for each \( j \in \mathbb{N}^+ \). By (2.25), (2.26) and (2.24), we have that
\[
V(x_0; \ell) \geq \sum_{j=\ell+1}^{+\infty} (\|Q_j^{\ell} x^\ell,k,x_0\|_H^2 + \|R_j^{\ell} u_j^{\ell,k,x_0}\|_U^2) \geq \delta \sum_{j=\ell+1}^{+\infty} \left( \|Q_j^{\ell} x^\ell,k,x_0\|_H^2 + \|u_j^{\ell,k,x_0}\|_U^2 \right) \quad \text{for each} \quad k > \ell. \tag{2.27}
\]

Thus, there exist a subsequence of \( (x^{\ell,k,x_0}, u^{\ell,k,x_0})_{k} \) and \( x^{\ell,x_0} := (x^{\ell,x_0})_{j} \in \mathbb{N}^+ \), \( u^{\ell,x_0} := (u^{\ell,x_0})_{j} \in \mathbb{N}^+ \) so that
\[
\begin{cases} x^{\ell,k,x_0} \rightarrow x^{\ell,x_0} \text{ weakly in } l^2(\mathbb{N}^+; H) \\ u^{\ell,k,x_0} \rightarrow u^{\ell,x_0} \text{ weakly in } l^2(\mathbb{N}^+; U) \end{cases} \quad \text{as} \quad k \rightarrow \infty. \tag{2.28}
\]

By (2.28) and (2.26), we have that, for each \( j > \ell \),
\[
\langle x^\ell,x_0, \varphi \rangle_H = \lim_{k \rightarrow +\infty} \langle \tilde{x}^\ell,k,x_0, \varphi \rangle_H = \lim_{k \rightarrow +\infty} \langle \tilde{x}^{\ell,k,x_0}(t_j), \varphi \rangle_H = \lim_{k \rightarrow +\infty} \left( e^{At_{j-\ell}} x_0 + \sum_{i=\ell+1}^{j-1} e^{At_{j-\ell}} B_{ij} u^\ell_{t_{j}}, \varphi \right) \quad \text{for any} \quad \varphi \in H. \tag{2.29}
\]
This implies that
\[ x^{\ell,x_0} = (x(t_j+\ell; x_0, u^{\ell,x_0}, \ell))_{j \in \mathbb{N}^+}. \]  
(2.29)

It follows that
\[ (x(t_j+\ell; x_0, u^{\ell,x_0}, \ell))_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; H) \]
and
\[ u^{\ell,x_0} \in U_{ad}(x_0; \ell). \]
Moreover, since \( Q \in M_{h,\geq}(H) \) and \( R \in M_{h,\geq}(U), \) by (2.28) and (2.29), we have
\[
\left\{ \begin{array}{l}
\| (Q^+_{j+\ell} x(t_j+\ell; x_0, u^{\ell,x_0}, \ell))_{j \in \mathbb{N}^+} \|_{l^2(\mathbb{N}^+; H)} \\
\| (R^+_{j+\ell} u^{\ell,x_0})_{j \in \mathbb{N}^+} \|_{l^2(\mathbb{N}^+; U)}
\end{array} \right. 
\leq \liminf_{k \to \infty} \| (Q^+_{j+\ell} x^{\ell,k,x_0})_{j \in \mathbb{N}^+} \|_{l^2(\mathbb{N}^+; H)},
\leq \liminf_{k \to \infty} \| (R^+_{j+\ell} u^{\ell,k,x_0})_{j \in \mathbb{N}^+} \|_{l^2(\mathbb{N}^+; U)}.
\]

This, together with the first inequality in (2.27), gives that
\[ J(u^{\ell,x_0}; x_0, \ell) \leq \liminf_{k \to +\infty} V(x_0; \ell, k). \]  
(2.30)

This, along with (2.23), yields that
\[ J(u^{\ell,x_0}; x_0, \ell) \leq V(x_0; \ell). \]

By this, the arbitrariness of \( x_0, \) and the optimality of \( V(x_0; \ell), \) we can conclude that \( u^{\ell,x_0} \) is the optimal control to the problem (I-I-LQ)\( \ell \) (i.e., \( u^{\ell,x_0} = u^\ell \)) and
\[ J(u^{\ell,x_0}; x_0, \ell) = V(x_0; \ell) \text{ for any } x_0 \in H. \]

This, along with (2.30) and (2.23), gives that
\[ V(x_0; \ell) = \lim_{k \to +\infty} V(x_0; \ell, k). \]  
(2.31)

For each \( k > \ell, \) let \( \{P^k_{\ell}\}_{k=1}^\infty \) be the solution of the equation (2.4) with \( \widehat{k} = k. \) Thus, by Proposition 2.2 and Corollary 2.3, we obtain
\[ V(x_0; \ell, k) = \langle P^k_{\ell} x_0, x_0 \rangle_H = \| (P^k_{\ell})^\frac{1}{2} x_0 \|_H^2 \text{ for each } x_0 \in H. \]  
(2.32)

This, along with (2.23), implies that
\[ \sup_{k > \ell} \| (P^k_{\ell})^\frac{1}{2} x_0 \|_H \leq \sqrt{V(x_0; \ell)} < +\infty \text{ for each } x_0 \in H. \]

Thus, by the uniform boundedness theorem, there is a constant \( K > 0, \) so that
\[ \| (P^k_{\ell})^\frac{1}{2} \|_{L(H)} \leq \sqrt{K} \text{ for each } k > \ell. \]

This, together with (2.32) again, implies that
\[ V(x_0; \ell, k) \leq K \| x_0 \|_H^2 \text{ for each } x_0 \in H. \]  
(2.33)

Thus, by (2.31) and (2.33), we can conclude that (2.22) holds.

Step 2. We prove that \( V(\cdot; \ell) \) satisfies the parallelogram law, i.e.,
\[ V(x_0 + y_0; \ell) + V(x_0 - y_0; \ell) = 2 (V(x_0; \ell) + V(y_0; \ell)) \text{ for all } x_0, y_0 \in H. \]  
(2.34)

In fact, by Proposition 2.2, for each \( k > \ell, \) it is obvious that
\[ V(x_0 + y_0; k, \ell) + V(x_0 - y_0; k, \ell) = 2(V(x_0; k, \ell) + V(y_0; k, \ell)) \text{ for any } x_0, y_0 \in H. \]
This, together with (2.31), yields that (2.34) holds.

**Step 3. We complete the proof of (i).**

Indeed, by (2.22), we can conclude that $V(\cdot; \ell)$ is continuous in 0. Thus, by (2.34) and [10, Theorem 2], $V(\cdot; \ell)$ is continuous at any point. Thus, by (2.34) again and [10, Theorem 3], there is a unique symmetric and bounded linear operator $P_\ell$ so that

$$V(x_0; \ell) = \langle P_\ell x_0, x_0 \rangle_H \text{ for each } x_0 \in H.$$  

By the fact that $V(x_0; \ell) \geq 0$ for each $x_0 \in H$, it is clear that $P_\ell \in \mathcal{SL}_+(H)$. Hence, we complete the proof of (i).

Next, we give the proof of (ii). To this end, by (2.21), it suffices to show that

$$V(x_0; \ell) = V(x_0; \ell + h) \text{ for each } x_0 \in H. \quad (2.35)$$

In fact, let $x_0 \in H$ be arbitrarily fixed. Let $u^\ell = (u^\ell_{j+1})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U)$ be the optimal control of $\text{(I-I-LQ)}_\ell$ with respect to the initial state $x_0$, i.e.,

$$V(x_0; \ell) = J(u^\ell; x_0, \ell). \quad (2.36)$$

Let $u^{\ell+h} = (u^{\ell+h}_{j+1})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U)$ be defined as

$$u^{\ell+h}_j := u^{\ell}_{j-h}, \forall j > \ell + h. \quad (2.37)$$

By this and (1.1), it is easy to check that

$$x(t_j; x_0, u^{\ell+h}, \ell + h) = x(t_{j-h}; x_0, u^\ell, \ell) \text{ for each } j \geq \ell + h. \quad (2.38)$$

Thus, by (2.13), (2.37) and (2.38), we have

$$V(x_0; \ell + h) \leq J(u^{\ell+h}; x_0, \ell + h) = J(u^\ell; x_0, \ell).$$

This, along with (2.36), implies that $V(x_0; \ell + h) \leq V(x_0; \ell)$. Similarly, it holds that $V(x_0; \ell) \leq V(x_0; \ell + h)$. Therefore, (2.35) holds. Thus, by the arbitrariness of $x_0$ and (2.35), the claim (ii) is true.

This completes the proof. \hfill \Box

The following result is the dynamic programming principle to the problem $\text{(I-I-LQ)}_\ell$, which has been proved in [18] for finite dimensional case. One can easily check that, by the same arguments used in [18], the dynamic programming principle also holds for infinite dimensional setting. Thus, we omit its proof.

**Lemma 2.6.** For each $k > \ell$ ($\ell, k \in \mathbb{N}$), it holds that

$$V(x_0; \ell) = \inf_{u=(u_{j+1})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U)} \left\{ \sum_{j=\ell+1}^{k+1} \left( \langle Q_j x(t_j; x_0, u, \ell), x(t_j; x_0, u, \ell) \rangle_H + \langle R_j u_j, u_j \rangle_U \right) + V(x(t_{k+1}; x_0, u, \ell); k) \right\} \text{ for any } x_0 \in H.$$  

The results in Proposition 2.2, Lemma 2.5 and Lemma 2.6 lead to the following proposition.

**Proposition 2.7.** Let the operators $P_\ell \in \mathcal{SL}_+(H)$ be so that (2.21) in Lemma 2.5 holds for each $\ell = 0, 1, \ldots, h$. Then $\{P_\ell\}_{\ell=0}^h$ satisfy the Riccati-type equation (1.9).

**Proof.** For each $0 \leq \ell < h$, by Lemma 2.6 and the claim (i) in Lemma 2.5, we have that

$$\langle P_\ell x_0, x_0 \rangle_H = \inf_{u=(u_{j+1})_{j \in \mathbb{N}^+} \in L^2(\mathbb{N}^+; U)} \left\{ \sum_{j=\ell+1}^{h} \left( \langle Q_j x(t_j; x_0, u, \ell), x(t_j; x_0, u, \ell) \rangle_H + \langle R_j u_j, u_j \rangle_U \right) + \langle P_h x_0, x_0 \rangle_H \right\} \text{ for any } x_0 \in H. \quad (2.39)$$
It is obvious that the right side of (2.39) is a finite horizon LQ problem \((F-I-LQ)_{k,h}\) with \(M = P_h\) (for the definition of \((F-I-LQ)_{k,h}\), one can see Section 2.1 with \(k = h\)). Then, from Proposition 2.2, we can claim that \(\{P_{\ell}\}_{\ell=1}^{h}\) verifies the following Riccati-type equation:

\[
P_{\ell-1} - e^{A^* (t_{\ell-1} - t_{\ell-1})} P_{\ell} e^{A (t_{\ell-1} - t_{\ell-1})} = e^{A^* (t_{\ell-1} - t_{\ell-1})} Q_{\ell} e^{A (t_{\ell-1} - t_{\ell-1})} P_{\ell} B_{\ell} (R_{\ell} + B_{\ell}^* P_{\ell} B_{\ell})^{-1} B_{\ell}^* P_{\ell} e^{A (t_{\ell-1} - t_{\ell-1})}
\]

for each \(\ell \in \{1, 2, \ldots, h\}\). This, together with the claim (ii) in Lemma 2.5, yields that \(\{P_{\ell}\}_{\ell=0}^{h}\) satisfies (1.9).

3 Proof of the main theorems

In this section, we shall prove Theorem 1.2 and Theorem 1.4. As stated in Section 1, the proof of Theorem 1.2 is based on Theorem 1.4. Thus, we first prove Theorem 1.4 and then continue to prove Theorem 1.2.

3.1 The proof of Theorem 1.4

We divide our proof into 4 steps.

**Step 1.** The proof of (i) \(\Rightarrow\) (ii).

Suppose the system \([A, \{B_k\}_{k=1}^{h}, \Lambda_h]\) is exponentially \(h\)-stabilizable, i.e., there exists a feedback law

\[\mathcal{F} = \{F_k\}_{k=1}^{h} \subset \mathcal{L}(H; U),\]

so that the solution \(x_{\mathcal{F}}(t)\) of the closed-loop system (1.3) satisfies that

\[\|x_{\mathcal{F}}(t)\|_H \leq Ce^{-\mu t}\|x_{\mathcal{F}}(0)\|_H, \quad \forall \, t > 0\]  

for some \(C > 0\) and \(\mu > 0\). We denote the solution of (1.3) with initial data \(x_0\) by \(x_{\mathcal{F}}(\cdot; x_0)\). For any \(x_0 \in H\), we take \(u = (u_j)_{j \in \mathbb{N}^+}\) in (1.2) (with \(x(0) = x_0\)) with

\[u_j := F_{\nu(j)} x(j; x_0, u), \quad \forall \, j \in \mathbb{N}^+.\]

One can easily check that \(x_{\mathcal{F}}(\cdot; x_0) = x(\cdot; x_0, u)\). Then, by (3.1), we have

\[
\sum_{j=1}^{+\infty} \|x(t_j; x_0, u)\|_H^2 \leq C^2 \|x_0\|_H^2 \left( \sum_{k=0}^{+\infty} e^{-2k\mu t_k} \right) \left( \sum_{j=1}^{h} e^{-2\mu t_j} \right) \\
\leq C^2 \|x_0\|_H^2 \left( \sum_{j=1}^{h} e^{-2\mu t_j} \right) \frac{1}{1 - e^{-2\mu t_k}} < +\infty. \tag{3.3}
\]

This, together with (3.2), gives that

\[
\sum_{j=1}^{+\infty} \|u_j\|_U^2 \leq \max_{1 \leq j \leq h} \|F_j\|_{\mathcal{L}(U; H)}^2 \sum_{j=1}^{+\infty} \|x(t_j; x_0, u)\|_H^2 < +\infty. \tag{3.4}
\]

Thus, by (3.3) and (3.4), we have \(u \in \mathcal{U}_{ad}(x_0)\), which indicates that \(\mathcal{U}_{ad}(x_0) \neq \emptyset\). Hence, the claim (ii) is true.

**Step 2.** The proof of (ii) \(\Rightarrow\) (iii).

Suppose the claim (ii) is true. Let \((Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \geq} (H)\) and \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h, \geq} (U)\) be arbitrarily fixed. From Proposition 2.7, we know that the claim (ii) implies that the Riccati-type equation (1.9) has
a solution \( \{P_\ell\}_{\ell=0}^h \). We only need to prove this solution is unique. Suppose \( \{\tilde{P}_\ell\}_{\ell=0}^h \) is a solution of the Riccati-type equation (1.9). By Lemma 2.5, it suffices to prove that

\[
V(x_0; \ell) = \langle \tilde{P}_\ell x_0, x_0 \rangle_H, \quad \forall x_0 \in H, \quad \forall 0 \leq \ell \leq h,
\]

where \( V(\cdot; \ell) \) is defined by (2.20).

Let \( x_0 \in H \) and \( 0 \leq \ell \leq h \) be arbitrarily fixed. Then, for any \( u = (u_j + \ell)_{j \in \mathbb{N}^+} \in \mathcal{U}_{ad}(x_0; \ell) \) and \( k > \ell \), it stands that

\[
x(t_k; x_0, u, \ell) = e^{A(t_k-t_\ell-1)}x(t_{\ell+1}; x_0, u, \ell) \tag{3.6}
\]

and

\[
x(t_{k+1}^+; x_0, u, \ell) = e^{A(t_k-t_{\ell}+1)}x(t_{k+1}^+; x_0, u, \ell) + B_k u_k \tag{3.7}
\]

where \( x(\cdot; x_0, u, \ell) \) is the solution of the equation (2.1) with \( v = u \). Denote \( x(\cdot) := x(\cdot; x_0, u, \ell) \) for simplicity. By (3.6), (1.1) and the fact that \( \{\tilde{P}_\ell\}_{\ell=0}^h \) satisfies the equation (1.9), we obtain that, for each \( k > \ell \),

\[
\langle Q_k x(t_k), x(t_k) \rangle_H + \langle R_k u_k, u_k \rangle_U
\]

\[
= \langle \tilde{P}_{k+1} x(t_{k+1}^-), x(t_{k+1}^-) \rangle_H - \langle \tilde{P}_{k+1} x(t_{k+1}^-), x(t_{\ell+1}^-) \rangle_H + \langle (R_k + B_k u_k) \tilde{P}_{k+1} x(t_{\ell+1}^-), u_k \rangle_U
\]

\[
+ \langle (R_k + B_k u_k) \tilde{P}_{k+1} x(t_k), x(t_k) \rangle_U + 2 \langle \tilde{P}_{k+1} x(t_k), x(t_k) \rangle_H
\]

\[
= (\tilde{P}_{k+1} x(t_{k+1}^-), x(t_{k+1}^-)) - (\tilde{P}_{k+1} x(t_{k+1}^-), x(t_{\ell+1}^-)) + (R_k + B_k u_k) \tilde{P}_{k+1} x(t_{\ell+1}^-) u_k + (R_k + B_k u_k) \tilde{P}_{k+1} x(t_k) x(t_k) + 2 \langle \tilde{P}_{k+1} x(t_k), x(t_k) \rangle_H
\]

\[
+ \| (R_k + B_k u_k) \tilde{P}_{k+1} x(t_k) \|_U^2 \tag{3.8}
\]

Since \( u \in \mathcal{U}_{ad}(x_0) \), one can easily check that \( \lim_{j \to +\infty} \langle \tilde{P}_{\ell} x(t_j^-), x(t_j^-) \rangle_H = 0 \). Therefore, by (2.13) and (3.8), we can conclude that, for any \( u = (u_j + \ell)_{j \in \mathbb{N}^+} \in \mathcal{U}_{ad}(x_0; \ell) \),

\[
J(u; x_0, \ell) = \sum_{k=\ell+1}^{+\infty} \langle Q_k x(t_k), x(t_k) \rangle_H + \langle R_k u_k, u_k \rangle_U
\]

\[
= (\tilde{P}_\ell x_0, x_0)_H + \sum_{k=\ell+1}^{+\infty} \| (R_k + B_k u_k) \tilde{P}_{k+1} x(t_k) \|_U^2 \tag{3.9}
\]

This, together with the definition of \( V(\cdot; \ell) \), implies that

\[
V(x_0; \ell) \geq \langle \tilde{P}_\ell x_0, x_0 \rangle_H. 
\tag{3.10}
\]

Further, we define \( u^* = (u_j^* + \ell)_{j \in \mathbb{N}^+} \) as

\[
u_k^* := -(R_k + B_k u_k) \tilde{P}_{k+1} x(t_k) u_k \quad \text{for any} \quad k \in \mathbb{N}^+.
\]

By similar proofs of (3.9), one can show that

\[
J(u^*; x_0, \ell) = \langle \tilde{P}_\ell x_0, x_0 \rangle_H. 
\tag{3.11}
\]
This, along with the facts \((Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\gg}(H)\) and \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\gg}(U)\), yields that
\[
(x(t_{j+1};x_0,u^*,\ell))_{j \in \mathbb{N}^+} \in \ell^2(N^+;H) \quad \text{and} \quad u^* \in \ell^2(N^+;U).
\]
These imply that \(u^* \in \mathcal{U}_{ad}(x_0;\ell)\) and by (3.11),
\[
V(x_0;\ell) \leq \langle \tilde{P}_H x_0, x_0 \rangle_H.
\] (3.12)
Therefore, by (3.10), (3.12) and the arbitrariness of \(x_0\) and \(\ell\), we have (3.5), i.e., the claim (iii) holds.

**Step 3. The proof of (iii) \(\Rightarrow\) (iv).** It is trivial.

**Step 4. The proof of (iv) \(\Rightarrow\) (i).**

Given \((Q_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\gg}(H)\) and \((R_j)_{j \in \mathbb{N}^+} \in \mathcal{M}_{h,\gg}(U)\) so that the claim (iv) is true. Then, by the definitions of \(\mathcal{M}_{h,\gg}(H)\) and \(\mathcal{M}_{h,\gg}(U)\), there exist constants \(q > 0\) and \(\delta > 0\) so that
\[
Q_j - qI > 0 \quad \text{and} \quad R_j - \delta I > 0 \quad \text{for each} \quad j \in \mathbb{N}^+.
\] (3.13)
Let \(\{P_l\}_{l=1}^\infty\) be the unique solution of the equation (1.9). Consider the following equation
\[
\begin{cases}
x'(t) = Ax(t), & t \in \mathbb{R}^+, \lambda_h,
\end{cases}
\begin{cases}
x(t^+_j) = x(t_j) + B_{\nu(j)} F_{\nu(j)} x(t_j), & j \in \mathbb{N}^+,
\end{cases}
\] (3.14)
where
\[
F_j := -(R_j + B_j^* P_j B_j)^{-1} B_j^* P_j, \quad \forall 1 \leq j \leq h.
\] (3.15)
Denote the solution of (3.14) by \(x_F(\cdot; x_0)\) with initial data \(x_0 \in H\). Let \(\Phi_h : H \rightarrow H\) be defined as
\[
\Phi_h := (I + B_h F_h)e^{A(t_h-t_{h-1})} \cdots (I + B_2 F_2)e^{A(t_2-t_1)} (I + B_1 F_1)e^{A t_1}.
\] (3.16)
It is clear that \(\Phi_h \in L(H)\). By (3.16) and (1.1), one can easily check that, for any \(x_0 \in H\),
\[
x_F(t^+_{kH}; x_0) = (\Phi_h)^k x_0 \quad \text{for each} \quad k \in \mathbb{N}^+.
\] (3.17)
By (3.13) and the same arguments as those lead to (3.12), we know that
\[
\sum_{k=1}^{+\infty} \|x_F(t^+_{kH}; x_0)\|_H^2 \leq \|I + B_h F_h\|_{L(H)}^2 \sum_{k=1}^{+\infty} \|x_F(t_{kH}; x_0)\|_H^2 \leq \frac{1}{q} \|(I + B_h F_h)\|_{L(H)}^2 \langle P_h x_0, x_0 \rangle_H\] for any \(x_0 \in H\).

This, together with (3.17), yields that
\[
\sum_{k=1}^{+\infty} \|\Phi_h^k x_0\|_H^2 \leq C_{h,q} \|x_0\|_H^2 \quad \text{for any} \quad x_0 \in H,
\] (3.18)
where \(C_{h,q} := \frac{1}{q}\|I + B_h F_h\|_{L(H)}^2 \|P_h\|_{L(H)}\). It follows that
\[
\sup_{k \in \mathbb{N}^+} \|\Phi_h^k\|_{L(H)} \leq \sup_{\|x_0\|_H = 1} \left( \sum_{k=1}^{+\infty} \|\Phi_h^k x_0\|_H^2 \right)^{\frac{1}{2}} \leq \sqrt{C_{h,q}}.
\] (3.19)
Thus, by (3.18) and (3.19), for each \(k \in \mathbb{N}^+\) and any \(x_0 \in H\),
\[
k \|\Phi_h^k x_0\|_H^2 = \sum_{j=1}^{k} \|\Phi_h^j x_0\|_H^2 \leq \sum_{j=1}^{k} \|\Phi_h^{k-j}\|_{L(H)}^2 \|\Phi_h^j x_0\|_H^2 \leq C_{h,q} \sum_{j=1}^{k} \|\Phi_h^j x_0\|_H^2 \leq C_{h,q}^2 \|x_0\|^2.
\] (3.20)
Let \( k_{h,q} := [4C_{h,q}^2] + 1 \). By (3.20), we have
\[
\| (\Phi_h)^k \|_{\mathcal{L}(H)} \leq \frac{1}{2} \quad \text{for each } k \geq k_{h,q}.
\] (3.21)

Fix \( \hat{k} \geq k_{h,q} \). Then, by (3.21), we have \( \| (\Phi_h)^{\hat{k}} \|_{\mathcal{L}(H)} \leq \frac{1}{2} \). It follows that
\[
\| (\Phi_h)^{n\hat{k}} \|_{\mathcal{L}(H)} \leq \left( \frac{1}{2} \right)^n \quad \text{for each } n \in \mathbb{N}^+.
\]
This gives that
\[
\limsup_{n \to +\infty} \| (\Phi_h)^{n\hat{k}} \|_{\mathcal{L}(H)}^{\frac{1}{n}} \leq \left( \frac{1}{2} \right)^{\frac{1}{\hat{k}}} < 1.
\]
This implies that, there exists a \( n^* \in \mathbb{N}^+ \) so that
\[
\| (\Phi_h)^{n^*\hat{k}} \|_{\mathcal{L}(H)} \leq e^{-\alpha t_{n^*\hat{k}}},
\]
where \( \alpha = \alpha(n^*, \hat{k}) := t^{-1}_{n^*\hat{k}} \ln \left( 2 \left( 1 + \left( \frac{1}{2} \right)^{\hat{k}} \right) \right) > 0 \). This, together with (3.17) and (1.1), yields that, for any \( x_0 \in H \),
\[
\| x_F(t^+_{j^*n^*\hat{k}}; x_0) \|_H \leq e^{-j\alpha t_{n^*\hat{k}}} \| x_0 \|_H = e^{-\alpha t_{n^*\hat{k}}} \| x_0 \|_H \quad \text{for each } j \in \mathbb{N}^+.
\] (3.22)

One can easily check that, for any \( x_0 \in H \),
\[
\| x_F(t; x_0) \|_H \leq C_{n^*, \hat{k}} \| x_0 \|_H \quad \text{for all } t \in [0, t_{j^*n^*\hat{k}}],
\] (3.23)
where \( C_{n^*, \hat{k}} > 0 \) is a constant. Let \( t \in \mathbb{R}^+ \) be arbitrarily fixed. It is clear that there is \( j^* \in \mathbb{N} \) so that \( t_{j^*n^*\hat{k}} < t \leq t_{(j^* + 1)\alpha n^*\hat{k}} \). Thus, by (3.22) and (3.23), we have that, for any \( x_0 \in H \),
\[
\| x_F(t; x_0) \|_H = \| x_F(t - t_{j^*n^*\hat{k}}; x_F(t^+_{j^*n^*\hat{k}}; x_0)) \|_H \leq C_{n^*, \hat{k}} \| x_F(t^+_{j^*n^*\hat{k}}; x_0) \|_H \\
\leq C_{n^*, \hat{k}} e^{\alpha (t - t_{j^*n^*\hat{k}})} e^{-\alpha t} \| x_0 \|_H \leq C_{n^*, \hat{k}} e^{\alpha (t - t_{j^*n^*\hat{k}})} e^{-\alpha t} \| x_0 \|_H.
\]
This, together with the arbitrariness of \( t \), implies the system (1.2) is exponentially stable with the feedback law \( F \) defined by (3.15). Thus, the claim (i) holds and the corresponding feedback law can be given as (1.4) (see (3.17)).

In summary, we complete the proof.

### 3.2 The proof of Theorem 1.2

The proof is divided into 4 steps.

#### Step 1. The proof of (i) \( \Rightarrow \) (ii).

Suppose that the system \( [A, \{B_k\}_{k=1}^\infty, \Lambda_h] \) is exponentially \( h \)-stabilizable. Then, there is a feedback law \( F = \{F_j\}_{j=1}^\infty \subset \mathcal{L}(H; U) \) so that the solution \( x_F(\cdot) \) of the closed-loop system (1.3) verifies
\[
\| x_F(t) \|_H \leq C_1 e^{-\mu t} \| x_F(0) \|_H \quad \text{for any } t \in \mathbb{R}^+,
\] (3.24)
where \( C_1 \) and \( \mu \) are two positive constants. We fix the feedback law \( F \). Since \( x_F(t_j^+) = (I + B_{\nu(j)} F_{\nu(j)}) x_F(t_j) \) for each \( j \in \mathbb{N}^+ \), by (3.24), we have
\[
\| x_F(t_j^+) \|_H \leq C_2 e^{-\mu t_j} \| x_F(0) \|_H \quad \text{for any } j \in \mathbb{N}^+,
\] (3.25)
where \( C_2 := C_1 \sup_{1 \leq k \leq h} \| I + B_k F_k \|_{\mathcal{L}(H)} \).
Let $\sigma \in (0, 1)$ and $x_0 \in H$ be arbitrarily fixed. It is obvious that there is a $\hat{k} = \hat{k}(\sigma) \in \mathbb{N}^+$ so that
\[
C_2 e^{-\mu \hat{k}} \leq \sigma. \tag{3.26}
\]
We denote the closed-loop system (1.3) with the initial data $x_0$ by $x_F(:x_0)$. Let $u_j(x_0) := F_{\nu(j)x_F(t_j,x_0)}$ for each $j \in \mathbb{N}$ and $u(x_0) := (u_j(x_0))_{j \in \mathbb{N}^+}$ in the control system (1.2). By (3.24), one can easily check that
\[
\|u_j(x_0)\|_{U} \leq C_3 e^{-\mu \hat{k}} \|x_0\|_H \quad \text{for each} \quad j \in \mathbb{N}^+, \tag{3.27}
\]
where $C_3 := \max_{1 \leq k \leq h} \|F_k\|_{L(H,U)}$. It should be noted that, by (1.1),
\[
\sum_{j=1}^{+\infty} e^{-2\mu \hat{k}} \leq \max_{1 \leq k \leq h} \frac{1}{|t_k - t_{k-1}|} \int_0^{+\infty} e^{-2\mu s} ds = \frac{1}{2\mu} \max_{1 \leq k \leq h} \frac{1}{|t_k - t_{k-1}|}.
\]
This, along with (3.27), yields that
\[
\|u(x_0)\|_{U(N^+,U)} = \left( \sum_{j=1}^{+\infty} \|u_j(x_0)\|_{E}^2 \right)^{\frac{1}{2}} \leq C_4 \|x_0\|_H \quad \text{for any} \quad x_0 \in H, \tag{3.28}
\]
where $C_4 := C_3 \left( \frac{1}{2\mu} \max_{1 \leq k \leq h} \frac{1}{|t_k - t_{k-1}|} \right)^{\frac{1}{2}}$. Moreover, one can easily check that
\[
x(t_k^+; x_0, u(x_0)) = x_F(t_k^+;x_0) \quad \text{for each} \quad k \in \mathbb{N}^+.
\]
This, together with (3.25) and (3.26), implies that
\[
\|x(t_k^+; x_0, u(x_0))\|_H \leq \sigma \|x_0\|_H. \tag{3.29}
\]
Furthermore, it is obvious that
\[
x(t_k^+; x_0, u(x_0)) = e^{A t_k} x_0 + \sum_{j=1}^{\hat{k}} e^{A(t_k-t_j)} B_{\nu(j)} u_j(x_0).
\]
Hence,
\[
\langle x(t_k^+; x_0, u(x_0)), \varphi \rangle_H = \langle x_0, e^{A t_k} \varphi \rangle_H + \sum_{j=1}^{\hat{k}} \langle u_j(x_0), B_{\nu(j)} e^{A(t_k-t_j)} \varphi \rangle_U \quad \text{for any} \quad \varphi \in H.
\]
This, along with (3.28) and (3.29), yields that
\[
\langle x_0, e^{A t_k} \varphi \rangle_H \leq \sigma \|x_0\|_H \|\varphi\|_H + C_4 \|x_0\|_H \left( \sum_{j=1}^{\hat{k}} \|B_{\nu(j)} e^{A(t_k-t_j)} \varphi \|_H \right)^{\frac{1}{2}} \quad \text{for any} \quad \varphi \in H.
\]
This, together with the arbitrariness of $x_0 \in H$, implies (1.4) with $C(\sigma) := C_4$. Therefore, the claim (ii) is true.

Step 2. The proof of (ii) $\Rightarrow$ (iii).
Suppose that the claim (ii) is true. We take $\tilde{\sigma} \in (0, \min\{1, (\max_{0 \leq \ell \leq h-1} \|e^{A(t_k-t_0)}\|_{L(H)})^{-1}\})$ arbitrarily. By the claim (ii), there are $\tilde{k} = \tilde{k}(\tilde{\sigma}) \in \mathbb{N}^+$ and $C(\tilde{\sigma}) > 0$ so that (1.4) holds with $\sigma = \tilde{\sigma}$ and
$C(\sigma) = C(\hat{\sigma})$. We take $k^* \in \mathbb{N}^+$ so that $(k^* - 1)h \leq \hat{k} < k^*h$. Thus, by (1.4), we have

$$
\|e^{A^*t_{k^*h}}\varphi\|_H \leq C(\hat{\sigma}) \left( \sum_{j=1}^{\hat{k}} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \varphi\|_{L^2_U}^2 \right)^{\frac{1}{2}} + \hat{\sigma} \|e^{A^*(t_{k^*h-\ell})} \varphi\|_H \\
\leq C(\hat{\sigma}) \left( \sum_{j=1}^{k^*-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \varphi\|_{L^2_U}^2 \right)^{\frac{1}{2}} + \max_{0 \leq \ell \leq h-1} \|e^{A^*(t_{\ell})}\|_{L^2(U)} \|\varphi\|_H \leq C(\hat{\sigma}) \left( \sum_{j=1}^{k^*-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \varphi\|_{L^2_U}^2 \right)^{\frac{1}{2}} + \sigma \|\varphi\|_H,
$$

where $\sigma := \hat{\sigma} \max_{0 \leq \ell \leq h-1} \|e^{A^*(t_{k^*h-\ell})}\|_{L^2(U)}$. It is clear that $\sigma \in (0, 1)$. Thus, the claim (iii) is true with $k = k^*$ and $C = C(\hat{\sigma})$.

**Step 3. The proof of (iii) $\Rightarrow$ (iv).**

Assume that (iii) holds. We take $\sigma \in (0, 1)$, $k \in \mathbb{N}^+$ and $C > 0$ so that (1.5) holds. Now, we prove (iv) is true. We take $x_0 \in H$ arbitrarily and, for each $\varepsilon > 0$, introduce a functional $\mathcal{J}_\varepsilon : H \to \mathbb{R}$ as follows:

$$
\mathcal{J}_\varepsilon(\varphi) := \frac{1}{2} \sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \varphi\|_{L^2_U}^2 + \langle \varphi, e^{A^*(t_{k^*h})}x_0 \rangle_H + (\sigma \|x_0\|_H + \varepsilon) \|\varphi\|_H \text{ for each } \varphi \in H.
$$

First, we show that $\mathcal{J}_\varepsilon$ is coercive. For this purpose, given any sequence $\{\varphi_n\}_{n \in \mathbb{N}^+} \subset H \setminus \{0\}$ with $\|\varphi_n\|_H \to +\infty$ as $n \to +\infty$, we only need to prove that

$$
\liminf_{n \to +\infty} \frac{\mathcal{J}_\varepsilon(\varphi_n)}{\|\varphi_n\|_H} \geq \varepsilon.
$$

(3.30)

Indeed, let $\bar{\varphi}_n := \varphi_n/\|\varphi_n\|_H$. Then we have

$$
\mathcal{J}_\varepsilon(\varphi_n) = \frac{1}{2} \|\varphi_n\|_H \sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \bar{\varphi}_n\|_{L^2_U}^2 + \langle \bar{\varphi}_n, e^{A^*(t_{k^*h})}x_0 \rangle_H + \sigma \|x_0\|_H + \varepsilon
$$

(3.31)

for each $n \in \mathbb{N}^+$. There are only two cases may happen.

**Case 1.** $\liminf_{n \to +\infty} \sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \bar{\varphi}_n\|_{L^2_U}^2 > 0$. In this case, by (3.31), it is clear that

$$
\liminf_{n \to +\infty} \frac{\mathcal{J}_\varepsilon(\varphi_n)}{\|\varphi_n\|_H} = +\infty.
$$

Thus, (3.30) is true.

**Case 2.** $\liminf_{n \to +\infty} \sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \bar{\varphi}_n\|_{L^2_U}^2 = 0$. Since $\|\varphi_n\|_H = 1$ for each $n \in \mathbb{N}^+$, there is a subsequence of $\{\bar{\varphi}_n\}_{n \in \mathbb{N}^+}$, still denoted by the same way, and $\bar{\varphi}$ so that

$$
\bar{\varphi}_n \rightharpoonup \bar{\varphi} \text{ weakly in } H \text{ as } n \to +\infty.
$$

(3.32)

It is obvious that $\|\bar{\varphi}\|_H \leq 1$. This, together with the assumption that $\liminf_{n \to +\infty} \sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \bar{\varphi}_n\|_{L^2_U}^2 = 0$, yields that

$$
\sum_{j=1}^{kh-1} \|B_{\nu(j)}^* e^{A^*(t_{k^*h-\ell})} \bar{\varphi}\|_{L^2_U}^2 = 0.
$$
Hence, by (1.5), we have
\[ 0 \leq (-\|e^{A^t x_0}\| H + \sigma \|\bar{\varphi}\| H)\|x_0\| H \leq \langle \varphi, e^{A^t x_0}x_0 \rangle H + \sigma \|\varphi\| H \|x_0\| H. \]

This, along with (3.31) and (3.32), gives that (3.30) holds in this case.

In summary, we have proved that \( J_\varepsilon \) is coercive for any \( \varepsilon > 0 \). It is obvious that \( J_\varepsilon \) is continuous and convex. Thus, for each \( \varepsilon > 0 \), \( J_\varepsilon \) has a minimizer \( \varphi_\varepsilon \in \mathcal{H} \). Therefore, after some simple calculations, we have that for any \( \xi \in \mathcal{H} \),
\[
0 \leq \lim_{\lambda \to 0^+} \frac{J_\varepsilon(\varphi_\varepsilon^* + \lambda \xi) - J_\varepsilon(\varphi_\varepsilon^*)}{\lambda} = \langle e^{A^t x_0}, \xi \rangle_H + \sum_{j=1}^{kh-1} \left\{ \begin{array}{ll}
\|B_{i(j)}e^{A^t (t_j - t_j)}\|_{\mathcal{H}} \varphi_\varepsilon^* B_{i(j)}e^{A^t (t_j - t_j)} \xi \rangle_U & \text{if } \varphi_\varepsilon^* \neq 0, \\
(\|\varphi_\varepsilon^*\| H + \varepsilon) \|\xi\| H & \text{if } \varphi_\varepsilon^* = 0.
\end{array} \right.
\]

(3.33)

Let \( u_\varepsilon^* := (B_1 e^{A^t (t_k - t_j)} \varphi_\varepsilon^*, B_2 e^{A^t (t_k - t_j)} \varphi_\varepsilon^*, \ldots, B_k e^{A^t (t_k - t_j)} \varphi_\varepsilon^*, 0, \ldots) \). One can easily check that
\[
\langle x(t_k; x_0, u_\varepsilon^*), \xi \rangle_H = \langle e^{A^t x_0}, \xi \rangle_H + \sum_{j=1}^{kh-1} \left\{ \begin{array}{ll}
\|B_{i(j)}e^{A^t (t_j - t_j)}\|_{\mathcal{H}} \varphi_\varepsilon^* B_{i(j)}e^{A^t (t_j - t_j)} \xi \rangle_U & \text{if } \varphi_\varepsilon^* \neq 0, \\
(\|\varphi_\varepsilon^*\| H + \varepsilon) \|\xi\| H & \text{if } \varphi_\varepsilon^* = 0.
\end{array} \right.
\]

(3.34)

for any \( \xi \in \mathcal{H} \). This, together with (3.33), yields that
\[
\|\varphi_\varepsilon^*\|_{\mathcal{H}} \leq (\sigma \|x_0\| H + \varepsilon) \|\xi\| H \text{ for any } \xi \in \mathcal{H},
\]

which implies that
\[
\|x(t_k; x_0, u_\varepsilon^*)\|_H \leq \sigma \|x_0\| H + \varepsilon.
\]

(3.35)

Moreover, by the definition of \( u_\varepsilon^* \), it is obvious that
\[
\|u_\varepsilon^*\|^2_{\mathcal{H}(N+U)} = \sum_{j=1}^{kh-1} \|B_{i(j)}e^{A^t (t_j - t_j)}\|_{\mathcal{H}} \varphi_\varepsilon^* \|_{\mathcal{H}}^2.
\]

(3.36)

Thus, by (1.5), we have
\[
-C\|u_\varepsilon^*\|^2_{\mathcal{H}(N+U)} \leq \langle \varphi_\varepsilon^* \rangle_H - \|e^{A^t x_0} \varphi_\varepsilon^*\| H)\|x_0\| H \leq \sigma \|\varphi_\varepsilon^*\| H \|x_0\| H + \langle \varphi_\varepsilon^*, e^{A^t x_0} \rangle H.
\]

This, together with (3.36) and the facts that \( \varepsilon > 0 \) and \( J_\varepsilon(\varphi_\varepsilon^*) \leq J_\varepsilon(0) = 0 \), implies that
\[
\|u_\varepsilon^*\|^2_{\mathcal{H}(N+U)} - 2C\|u_\varepsilon^*\|^2_{\mathcal{H}(N+U)} \|x_0\| H \leq 0.
\]

Hence, by the arbitrariness of \( \varepsilon > 0 \), we get
\[
\|u_\varepsilon^*\|^2_{\mathcal{H}(N+U)} \leq 2C\|x_0\| H \text{ for any } \varepsilon > 0.
\]

(3.37)

It follows that, there is a sequence \( \{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) with \( \varepsilon_n \to 0 \) as \( n \to +\infty \), and \( u^* \in \mathcal{L}^2(N^+_U) \) so that
\[
u^*_{\varepsilon_n} \to u^* \text{ weakly in } \mathcal{L}^2(N^+_U) \text{ as } n \to +\infty.
\]

(3.38)

By (3.37), it is clear that
\[
\|u^*\|^2_{\mathcal{H}(N+U)} \leq 2C\|x_0\| H.
\]

(3.39)

Moreover, by (3.38), one can easily check that
\[
x(t_k; x_0, u^*_{\varepsilon_n}) \to x(t_k; x_0, u^*) \text{ weakly in } \mathcal{H} \text{ as } n \to +\infty.
\]
This, along with (3.35), yields that
\[ \|x(t_{kh}; x_0, u^*)\|_H \leq \sigma \|x_0\|_H. \]  
(3.40)

Therefore, by (3.39) and (3.40), we conclude that (1.6) is true with \(u = u^*\). Thus (iv) holds.

**Step 4. The proof of (iv) ⇒ (i).**

To this end, by Theorem 1.4, we only need to show
\[ \mathcal{U}_{ad}(x_0) \neq \emptyset \] for any \(x_0 \in H\),
(3.41)
where \(\mathcal{U}_{ad}(\cdot)\) is defined by (1.7). Let \(\sigma \in (0, 1), k \in \mathbb{N}^+\) and \(C > 0\) be such that the claim (iv) holds.

Let \(x_0 \in H\) be arbitrarily fixed. Write \(U^{kh} := U \times U \times \cdots \times U\). It is clear that, for any \(w = (w_1, \ldots, w_{kh}) \in U^{kh}\),
\[ x(t; x_0, w) = e^{A(t-t_1)}x(t_1; x_0, w) + e^{A(t-t_{j-1})}B_{j-1}w_{j-1} \text{ for each } j \in \{1, 2, \ldots, kh\}. \]
Here and in what follows we agree that \(B_0 = 0\) and \(w_0 = 0\). It follows that
\[ \|x(t; x_0, w)\|_H^2 \leq C_A\|x(t_1; x_0, w)\|_H^2 + C_{A, B_h}\|w_{j-1}\|_U^2 \text{ for each } j \in \{1, 2, \ldots, kh\}, \]
where
\[ C_A := \max \left\{ 1, 2 \max_{1 \leq l \leq h} \|e^{A(t_l-t_{l-1})}\|_{\mathcal{L}(H)}^2 \right\} \text{ and } C_{A, B_h} := 2 \max_{1 \leq l \leq h} \left\| e^{A(t_l-t_{l-1})}\|_{\mathcal{L}(H)}^2 \right\|_{\mathcal{L}(U^l; H)}^2 ) \]

Therefore, for each \(j \in \{1, 2, \ldots, kh\}\),
\[ \|x(t; x_0, w)\|_H^2 \leq C_A^h \|x_0\|_H^2 + C_{A, B_h} \left( \sum_{l=1}^{j-1} C_A^{j-1-l} \|w_l\|_U^2 \right) \]
\[ \leq C_A^{kh} \|x_0\|_H^2 + C_{A, B_h} C_A^{kh-1} \left( \sum_{l=1}^{kh-1} \|w_l\|_U^2 \right) \]  
(3.42)

(Here, we used the fact \(C_A \geq 1\).)

By the claim (iv), there is a control \(u = (u_j)_{j \in \mathbb{N}^+} \in l^2(\mathbb{N}^+; U)\) so that (1.6) is true. Define \(v^1 := (v_{11}, v_{21}, \ldots, v_{kh}) \in U^{kh}\) by
\[ v_j^1 = u_j \text{ for each } j \in \{1, 2, \ldots, kh - 1\} \text{ and } v_{kh}^1 = 0. \]

This, together with (1.6), yields that
\[ \|x(t_{kh}; x_0, v^1)\|_H = \|x(t_{kh}; x_0, v^1)\|_H = \|x(t_{kh}; x_0, u)\|_H \leq \sigma \|x_0\|_H \]  
(3.43)

and
\[ \|v^1\|_{U^{kh}} \leq \|u\|_{l^2(\mathbb{N}^+; U)} \leq C \|x_0\|_H. \]  
(3.44)

By (3.44) and (3.42), it is obvious that
\[ \sum_{j=1}^{kh} \|x(t; x_0, v^1)\|_H^2 \leq C_{A, B_h} \|x_0\|_H^2, \]
where \(C_{A, B_h, k} := kh \left( C_A^{kh} + C^2_{A, B_h} C_A^{kh-1} \right) \).

Replacing \(x_0\) by \(x(t^+_1; x_0, v^1)\) in the above arguments, we can show that there exists \(v^2 := (v_{12}, v_{22}, \ldots, v_{kh}) \in U^{kh}\) so that
\[ \|v^2\|_{U^{kh}} \leq C \|x(t_{kh}; x_0, v^1)\|_H \]
and
\[ \sum_{j=1}^{kh} \|x(t_j; x(t_{kh}^+; x_0, v^1), v^2)\|_H^2 \leq C_{A,B_h,k} \|x(t_{kh}^+; x_0, v^1)\|_H^2. \]
These, along with (3.43), imply that
\[ \|v^2\|_{U^{kh}} \leq C\sigma \|x_0\|_H \]
and
\[ \sum_{j=1}^{kh} \|x(t_j; x(t_{kh}^+; x_0, v^1), v^2)\|_H \leq C_{A,B_h,k} \sigma^2 \|x_0\|_H^2. \]
By the same way, we can deduce that, for each \( l \in \mathbb{N}^+ \) with \( l \geq 2 \), there is \( v^l := (v^l_1, v^l_2, \ldots, v^l_{kh}) \in U^{kh} \) so that
\[ \|v^l\|_{U^{kh}} \leq C\sigma^{l-1} \|x_0\|_H \]
and
\[ \sum_{j=1}^{kh} \|x(t_j; x(t_{kh}^+; x_0, v^1), v^l)\|_H \leq C_{A,B_h,k} \sigma^{2(l-1)} \|x_0\|^2, \] (3.46)
where
\[ \begin{align*}
&x_1(t_{kh}^+) := x(t_{kh}^+; x_0, v^1), \\
x_j(t_{kh}^+) := x(t_{kh}^+; x_{j-1}(t_{kh}^+), v^j) \quad \text{for each } l \geq 2.
\end{align*} \] (3.47)
Define
\[ v := (v^1_1, v^1_2, \ldots, v^1_{kh}, v^2_1, v^2_2, \ldots, v^2_{kh}, \ldots). \]
Since \( \sigma \in (0, 1) \), by (3.44) and (3.45), we obtain
\[ \|v\|^2_{L^2(N^+; U)} \leq \sum_{l=1}^{\infty} \|v^l\|^2_{U^{kh}} \leq C^2 \left( \sum_{l=1}^{+\infty} \sigma^{2(l-1)} \right) \|x_0\|^2_H < +\infty. \] (3.48)
Moreover, by (3.47), for any \( l \in \mathbb{N}^+ \), one can easily check that
\[ x(t_j; x_0, v) = x(t_j; x_{l-1}(t_{kh}^+), v^j) \quad \text{for each } j \in \{(l-1)h + 1, \ldots, lh\}. \]
This, together with (3.46), yields that
\[ \sum_{j=1}^{+\infty} \|x(t_j; x_0, v)\|_H^2 \leq \sum_{l=1}^{+\infty} \sum_{j=1}^{kh} \|x(t_j; x_{l-1}(t_{kh}^+), v^j)\|_H \leq C_{A,B_h,k} \left( \sum_{l=1}^{+\infty} \sigma^{2(l-1)} \right) \|x_0\|_H^2 < +\infty. \] (3.49)
Thus, by (3.48) and (3.49), we conclude that \( v \in L^2(N^+; U) \) and \( (x(t_j; x_0, v))_{j \in \mathbb{N}^+} \in L^2(N^+; H) \), i.e., \( v \in \mathcal{U}_ad(x_0) \). This, along with the arbitrariness of \( x_0 \), implies that (3.41) is true. Hence, the claim (i) is true.

In summary, we complete our proof.

4 Application to the coupled heat system with impulse controls

In this section, we consider the system of heat equations coupled by constant matrices, which is a special case of (1.2). Let \( \Omega \subset \mathbb{R}^N(N \subset \mathbb{N}^+) \) be a bounded domain with a smooth boundary \( \partial \Omega \). Let \( \omega_k \subset \Omega \) \( (1 \leq k \leq h) \) be an open and nonempty subset of \( \Omega \) with \( \omega := \cap_{k=1}^{h} \omega_k \neq \emptyset \). Denote by \( \chi_E \) the characteristic function of the set \( E \subset \mathbb{R}^N \). Write \( \triangle_n := I_n \triangle = \text{diag}(\triangle, \triangle, \ldots, \triangle) \), where \( I_n \) is the identity matrix in
$\mathbb{R}^{n \times n}$ and $\Delta$ is the Laplace operator with domain $D(\Delta) := H_0^1(\Omega) \cap H^2(\Omega)$. In this section, we consider the system (1.2) with

$$A := \Delta_n + S, \quad D(A) := (H^1_0(\Omega) \cap H^2(\Omega))^n \text{ and } B_k := \chi_{\omega_k} D_k \text{ for each } 1 \leq k \leq h,$$

where $S \in \mathbb{R}^{n \times n}$ and $\{D_k\}_{k=1}^h \subset \mathbb{R}^{n \times m}$ ($n, m \in \mathbb{N}^+$). We let $\mathcal{D} := (D_1, D_2, \cdots, D_h)$. It is obvious that if we let $H := (L^2(\Omega))^n$ and $U := (L^2(\Omega))^m$, then $H = H^*$ and $U = U^*$. Moreover, $(A, D(A))$ generates a $C_0$-semigroup $\{e^{tA}\}_{t \in \mathbb{R}^+}$ over $(L^2(\Omega))^n$. For simplicity, we denote the system

$$\begin{align*}
&x_t - \Delta_n x - Sx = 0, \quad \text{in } \Omega \times (\mathbb{R}^+ \setminus \Lambda_h), \\
x &= 0, \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
x(t^+) = x(t_j) + \chi_{\omega_j} D_{\nu(j)} u_j, \quad \text{in } \Omega, \quad j \in \mathbb{N}^+
\end{align*}$$

by $[S, \{\chi_{\omega_k} D_k\}_{k=1}^h, \Lambda_h]$ instead of $[A, B_0, \Lambda_h]$ in this section, where $\Lambda_h := \{t_j\}_{j \in \mathbb{N}} \subset \mathcal{J}_h$. We will prove the following stabilization results for the system (4.1), which is an application of Theorem 1.2.

**Theorem 4.1.** Given $S \in \mathbb{R}^{n \times n}$ and $\{D_k\}_{k=1}^h \subset \mathbb{R}^{n \times m}$. The following statements are equivalent.

(i) There exists $\Lambda_h \in \mathcal{J}_h$ so that the system $[S, \{\chi_{\omega_k} D_k\}_{k=1}^h, \Lambda_h]$ is exponentially $h$-stabilizable.

(ii) $\text{rank}(\lambda I_n - S, \mathcal{D}) = n$ for any $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq \lambda_1$.

(iii) $\text{rank}(\lambda I_n - S, \mathcal{D}) = n$ for any $\lambda \in \sigma(S)$ with $\text{Re}(\lambda) \geq \lambda_1$.

Here and throughout this section, $\lambda_1$ denotes the first eigenvalue of $-\Delta$ and $\sigma(S)$ the spectrum of the matrix $S$.

In order to prove Theorem 4.1, we first introduce some definitions and two lemmas. For any $E \in \mathbb{R}^{k \times \ell}$, $F \in \mathbb{R}^{k \times \ell}$ ($k, \ell \in \mathbb{N}^+$), we define (see [17] and [18])

$$d_E := \min \left\{ \frac{\pi}{\text{Im}(\lambda)} : \lambda \in \sigma(E) \right\} \quad \text{and} \quad q(E, F) := \max \{\dim V_E(f) : f \text{ is a column of } F\},$$

where $\text{Im}(\lambda) = \lambda_2$ if $\lambda = \lambda_1 + i\lambda_2$, $V_E(f) := \text{span}\{f, Ef, \cdots, E^{k-1}f\}$ and we agree that $\frac{1}{0} = +\infty$.

Moreover, we define

$$L_{E,F,h} := \left\{ \{\tau_j\}_{j \in \mathbb{N}} : 0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots \right\}, \quad \text{Card}((s, s + d_E) \cap \{\tau_j\}_{j \in \mathbb{N}}) \geq hq(E,F) + 2 \text{ for any } s \in \mathbb{R}^+.$$}

The following result can be found in the proof of Theorem 2.2 in [17].

**Lemma 4.2.** Let $E \in \mathbb{R}^{k \times k}$, $F \in \mathbb{R}^{k \times \ell}$ ($k, \ell \in \mathbb{N}^+$). For each strictly increasing sequence $\{\tau_j\}_{j=1}^{q(E,F)}$ with $\tau_{q(E,F)} - \tau_1 < d_F$, we have that

$$\text{span}\left\{e^{-\tau_1 E} F, e^{-\tau_2 E} F, \cdots, e^{-\tau_{q(E,F)} E} F\right\} = \text{span}\{F, EF, \cdots, E^{k-1}F\}.$$

In the following of this section, we discuss for fixed $h \in \mathbb{N}^+$, $S \in \mathbb{R}^{n \times n}$ and $\{D_k\}_{k=1}^h \subset \mathbb{R}^{n \times m}$. Write $\mathcal{D} := (D_1, \cdots, D_h) \in \mathbb{R}^{n \times mh}$. Lemma 4.3 is quoted from [24].

**Lemma 4.3.** Let $\Lambda_h = \{t_j\}_{j \in \mathbb{N}} \subset \mathcal{J}_h$ and $\hat{k} \in \mathbb{N}^+$ be fixed. The following two claims are equivalent:

(i) $\text{rank}\left(\{e^{-St_1} D_{\nu(1)}, e^{-St_2} D_{\nu(2)}, \cdots, e^{-St_{\hat{k}}} D_{\nu(\hat{k})}\}\right) = n$.

(ii) There are two constants $\theta \in (0, 1)$ (independent of $\hat{k}$) and $C(\hat{k}) > 0$ so that

$$\|e^{\lambda I_n + \Phi} \|_{(L^2(\Omega))^m} \leq C(\hat{k}) \left(\sum_{j=1}^{\hat{k}} \|B_{\nu(j)} e^{\lambda I_n + \Phi} \|_{(L^2(\Omega))^m}^\theta\right)^\theta \|\Phi\|_{(L^2(\Omega))^m}^{1-\theta} \text{ for all } \Phi \in (L^2(\Omega))^m.$$

(4.3)
Based on Lemma 4.2 and Lemma 4.3, we obtain the following Lemma.

**Lemma 4.4.** Let \( \Lambda_h = \{ t_j \}_{j \in \mathbb{N}} \subseteq J_h \cap L_{S,D,h} \). Suppose that \( \text{rank} (D, SD, \cdots, S^{n-1}D) = n \). Then the system \([S, \{ \chi_{\omega_k} D_k \}_{k=1}^{h}, \Lambda_h]\) is exponentially \( h \)-stabilizable.

**Proof.** Since \( \Lambda_h \subseteq L_{S,D,h} \), by the definition of \( q(S,D) \) (see (4.2)) and the fact that \( D = (D_1, \cdots, D_h) \), it is easy to see
\[
t_{j+q(S,D)h} - t_j \leq t_{j+q(S,D)h} - t_j < ds \quad \text{for each } j \in \{1, \ldots, h\}.
\]
This, along with Lemma 4.2, shows that
\[
\text{span} \{ e^{-St_j D_j}, \cdots, e^{-St_j q(S,D)D_j} \} = \text{span} \{ D_j, SD_j, \cdots, S^{n-1}D_j \} \quad \text{for each } j \in \{1, \ldots, h\}.
\]
From this, we obtain that
\[
\sum_{j=1}^{h} \text{span} \{ e^{-St_j D_j}, \cdots, e^{-St_j q(S,D)D_j} \} = \sum_{j=1}^{h} \text{span} \{ D_j, SD_j, \cdots, S^{n-1}D_j \} = \text{span} \{ D, SD, \cdots, S^{n-1}D \}.
\]
(4.4)

Here, for two linear spaces \( V_1 \) and \( V_2 \), we let \( V_1 + V_2 = \text{span} \{ V_1, V_2 \} \). Noting that for each \( j \in \{1, \ldots, h\} \), \( \nu(j) = \nu(j + \ell h) \) for any \( \ell \in \mathbb{N} \), (4.4) implies that
\[
\text{span} \{ e^{-St_1 D_{\nu(1)}}, \cdots, e^{-St_\ell q(S,D)D_{\nu(\ell q(S,D)h)}} \} = \text{span} \{ D, SD, \cdots, S^{n-1}D \}.
\]
(4.5)

Denote \( \tilde{n} = h + q(S,D)h \). By (4.5) and our assumption, we have that \( \text{rank} (e^{-St_1 D_{\nu(1)}}, \cdots, e^{-St_\ell D_{\nu(\ell q(S,D)h)}}) = n \). This, along with Lemma 4.3, shows that there exists \( \theta \in (0, 1) \) and \( C(\tilde{n}) > 0 \) so that (4.3) stands with \( \tilde{k} \) replaced by \( \tilde{n} \). Then for each \( \alpha > 0 \), there exists a constant \( C(\tilde{n}, \alpha) > 0 \) so that
\[
\| e^{A t_{\tilde{n}+1} \varphi} \|_{L^2(\Omega)^n} \leq C(\tilde{n}, \alpha) \left( \sum_{j=1}^{\tilde{n}} \| B_{\nu(j)} e^{A(t_{\tilde{n}+1}-t_j) \varphi} \|_{L^2(\Omega)^n}^2 \right)^{\frac{1}{2}} + \alpha \| \varphi \|_{L^2(\Omega)^n} \quad \text{for all } \varphi \in (L^2(\Omega))^n.
\]
(4.6)

This, along with Theorem 1.2, completes the proof.

**Lemma 4.5.** (Kalman controllability decomposition (see [20])) The following statements are equivalent:

(i) \( \text{rank} (D, SD, \cdots, S^{n-1}D) = n < n \);

(ii) There exists an invertible matrix \( J \in \mathbb{R}^{n \times n} \) so that
\[
J^{-1} SJ = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{and} \quad J^{-1} D = \begin{pmatrix} \tilde{D} \\ 0 \end{pmatrix},
\]
(4.6)

where \( S_1 \in \mathbb{R}^{n_1 \times n_1}, \tilde{D} \in \mathbb{R}^{n_1 \times m_h} \) and \( \text{rank}(\tilde{D}, S_1 \tilde{D}, \cdots, S_1^{n_1-1} \tilde{D}) = n_1 \).

**Proposition 4.6.** Let \( \Lambda_h = \{ t_j \}_{j \in \mathbb{N}} \subseteq J_h \cap L_{S,D,h} \). Suppose that
\[
\text{rank}(\lambda I - S, D) = n \quad \text{for any } \lambda \in \mathbb{C} \text{ with } \text{Re}(\lambda) \geq \lambda_1.
\]
(4.7)

Then the system \([S, \{ \chi_{\omega_k} D_k \}_{k=1}^{h}, \Lambda_h]\) is exponentially \( h \)-stabilizable.

**Proof.** Without loss of generality, we assume that \( \text{rank} (D, SD, \cdots, S^{n-1}D) = n_1 \) with \( 1 \leq n_1 < n \), since when \( n_1 = n \) the conclusion follows from Lemma 4.4 at once. By Lemma 4.5, there exists an invertible matrix \( J \in \mathbb{R}^{n \times n} \) so that (4.6) holds with \( S_1 \in \mathbb{R}^{n_1 \times n_1}, \tilde{D} \in \mathbb{R}^{n_1 \times m_h} \) and
\[
\text{rank}(\tilde{D}, S_1 \tilde{D}, \cdots, S_1^{n_1-1} \tilde{D}) = n_1.
\]
(4.8)
Denote $J^{-1}D_1 = \begin{pmatrix} \tilde{D}_1 \\ 0 \end{pmatrix}, \ldots, J^{-1}D_h = \begin{pmatrix} \tilde{D}_h \\ 0 \end{pmatrix}$, where $\tilde{D}_j \in \mathbb{R}^{n_1 \times m}$. Then
\begin{equation}
\tilde{D} = (\tilde{D}_1, \ldots, \tilde{D}_h).
\end{equation}

First, we show that
\begin{equation}
\text{Re}(\lambda) < \lambda_1 \text{ for any } \lambda \in \sigma(S_3).
\end{equation}
For otherwise, there would exist $\lambda_0 \in \sigma(S_3)$ so that $\text{Re}(\lambda_0) \geq \lambda_1$. It follows that
\begin{equation}
\text{rank}(\lambda_0 I - S, D) = \text{rank}(\lambda_0 I - J^{-1}JS, J^{-1}D) = \text{rank} \begin{pmatrix} \lambda_0 I - S_1 & S_2 \\ 0 & \lambda_0 I - S_3 \end{pmatrix} < n,
\end{equation}
which contradicts (4.7). Thus (4.10) stands. Therefore, there exist constants $M_0 > 0$ and $\varepsilon_0 > 0$ so that the solution $\tilde{y}(\cdot)$ of system
\begin{equation}
\begin{aligned}
\tilde{y}_t - \Delta_n \tilde{y} - S_0^T \tilde{y} &= 0, & \text{in } \Omega \times \mathbb{R}^+,
\tilde{y} &= 0, & \text{on } \partial\Omega \times \mathbb{R}^+
\end{aligned}
\end{equation}
satisfies that
\begin{equation}
\|\tilde{y}(t)\|_{(L^2(\Omega))^{n_1}} \leq M_0 e^{-\varepsilon_0 t} \|\tilde{y}(0)\|_{(L^2(\Omega))^{n_1}} \text{ for any } t > 0.
\end{equation}

Second, by the definitions of $d_S$ and $q(S, D)$ (see (4.2)), we obtain that $d_S \leq d_{S_1}$ and $q(S_1, \tilde{D}) \leq q(S, D)$. This, along with the fact that $\Lambda_h \in \mathcal{J}_h \cap \mathcal{L}_S, D, h$, shows that $\Lambda_h \in \mathcal{J}_h \cap \mathcal{L}_{S_1}, \tilde{D}, h$. Then, by (4.8), (4.9) and Lemma 4.4, we have that the system $\{S_1, \{\chi_{\omega_j} \tilde{D}_k\}_{k=1}^h, \Lambda_h\}$ is exponentially $h$-stabilizable. Thus, there exists a feedback control law $\tilde{F} = \{\hat{F}_j\}_{j=1}^h \subset \mathcal{L}(\mathcal{L}^2(\Omega)^{n_1}; (\mathcal{L}^2(\Omega)^m))$ so that the system
\begin{equation}
\begin{aligned}
z_t - \Delta_{n_1} z - S_1 z &= 0, & \text{in } \Omega \times \mathbb{R}^+\setminus \Lambda_h, \\
z &= 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\
z(t_j^+) = z(t_j) + \chi_{\omega_j} \tilde{D}_k \hat{F}_k \hat{z}_1(t_j), & \text{in } \Omega, j \in \mathbb{N}^+.
\end{aligned}
\end{equation}
is stable, i.e., if we denote the solution of the system (4.13) by $z(t) = S_{\tilde{F}}(t)z(0)$, then there exist constants $M_1 > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$, so that
\begin{equation}
\|z(t)\|_{(L^2(\Omega))^{n_1}} = \|S_{\tilde{F}}(t)z(0)\|_{(L^2(\Omega))^{n_1}} \leq M_1 e^{-\varepsilon_1 t} \|z(0)\|_{(L^2(\Omega))^{n_1}} \text{ for any } t \geq 0.
\end{equation}
This indicates that
\begin{equation}
\|S_{\tilde{F}}(t)\|_{\mathcal{L}((L^2(\Omega))^{n_1})} \leq M_1 e^{-\varepsilon_1 t} \text{ for any } t \geq 0.
\end{equation}

Third, we consider the system
\begin{equation}
\begin{aligned}
\tilde{y}_t - \Delta_n \tilde{y} - S_1 \tilde{y} &= S_2 \tilde{y}, & \text{in } \Omega \times \mathbb{R}^+\setminus \Lambda_h, \\
\tilde{y} &= 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\
\tilde{y}(t_j^+) &= \tilde{y}(t_j) + \chi_{\omega_j} \tilde{D}_k \hat{F}_k \hat{y}(t_j), & \text{in } \Omega, j \in \mathbb{N}^+.
\end{aligned}
\end{equation}
where $\tilde{y}(\cdot)$ is a solution of (4.11). One can easily verify that the solution to the system (4.15) satisfies that
\begin{equation}
\tilde{y}(t) = S_{\tilde{F}}(t)\tilde{y}(0) + \int_0^t S_{\tilde{F}}(t - s) S_2 \tilde{y}(s) ds \text{ for any } t \geq 0.
\end{equation}
This, combined with (4.14) and (4.12), shows that
\begin{equation}
\|\tilde{y}(t)\|_{(L^2(\Omega))^{n_1}} \leq M_2 e^{-\varepsilon_1 t} \left( \|\tilde{y}(0)\|_{(L^2(\Omega))^{n_1}} + \|\tilde{y}(0)\|_{(L^2(\Omega))^{n_1}}} \right) \text{ for any } t \geq 0,
\end{equation}
where $M_2 := \max\{M_1, M_1 M_0 \|S_2\|_{\mathcal{L}(\mathbb{R}^{n_1}; \mathbb{R}^{n_1})}\}$. 

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Finally, let \( x(t) = J \left( \frac{\hat{y}}{\bar{y}} \right) (t) \) for any \( t \geq 0 \) and \( F_j = \left( \tilde{F}_j \ 0 \right) \) \( J^{-1} \) for each \( j \in \{1, \ldots, h\} \). Then by (4.6), (4.11) and (4.15), we have \( x(\cdot) \) satisfies

\[
\begin{align*}
&x_t - \Delta_n x - S x = 0, &\text{in } \Omega \times (\mathbb{R}^+ \setminus \Lambda_h), \\
&x = 0, &\text{on } \partial \Omega \times \mathbb{R}^+ \\
&x(t_j^+) = x(t_j) + \chi_{\omega_{\nu(j)}} D_{\nu(j)} F_{\nu(j)} x(t_j), &\text{in } \Omega, j \geq 1.
\end{align*}
\]

Moreover, from (4.12) and (4.16) and the fact that \( 0 < \varepsilon_1 < \varepsilon_0 \), we obtain that

\[
\|x(t)\|_{(L^2(\Omega))^n} \leq \|J\|_{L(\mathbb{R}^n;\mathbb{R}^n)} \cdot \|y(t)\|_{(L^2(\Omega))^n} \leq M_3 e^{-\varepsilon_1 t} \|x(0)\|_{(L^2(\Omega))^n} \text{ for any } t \geq 0,
\]

where \( M_3 := 2(M_0 + M_2) \|J\|_{L(\mathbb{R}^n;\mathbb{R}^n)} \|J^{-1}\|_{L(\mathbb{R}^n;\mathbb{R}^n)} \). This completes the proof. \( \square \)

**Proof of Theorem 4.1.** It is easy to see that (ii) and (iii) are equivalent. It follows from Proposition 4.6 that (ii) \( \Rightarrow \) (i).

Next, we prove (i) \( \Rightarrow \) (iii) by contradiction. Suppose there exists a complex number \( \lambda_0 \in \sigma(S) \) with \( \text{Re}(\lambda_0) \geq \lambda_1 \) so that

\[
\text{rank}(\lambda_0 I_n - S, D) := n_1 < n.
\]

Thus by Lemma 4.5, there exists an invertible matrix \( J \in \mathbb{R}^{n \times n} \) so that

\[
J^{-1} S J = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{and} \quad J^{-1} D_1 = \begin{pmatrix} \tilde{D}_1 \\ 0 \end{pmatrix}, \ldots, J^{-1} D_h = \begin{pmatrix} \tilde{D}_h \\ 0 \end{pmatrix},
\]

where \( S_1 \in \mathbb{R}^{n_1 \times n_1}, \tilde{D}_k \in \mathbb{R}^{n_1 \times n_m} \) for each \( k = 1, 2, \ldots, h \), and by writing \( \tilde{D} = (\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_h) \),

\[
\text{rank}(\tilde{D}, S_1 \tilde{D}, \ldots, S_1^{n_1-1} \tilde{D}) = n_1.
\]

By (4.17) and (4.18), we have that

\[
\text{rank}(\lambda_0 I_n - S, D) = \text{rank}(\lambda_0 I_n - J^{-1} S J, J^{-1} D) = \text{rank} \left( \begin{pmatrix} \lambda_0 I_{n_1} - S_1 & S_2 \\ 0 & \lambda_0 I_{n-n_1} - S_3 \end{pmatrix} \tilde{D} \right) = n_1.
\]

This, along with (4.19), shows that \( \lambda_0 \) is an eigenvalue of \( S_3 \).

Since (i) is true, let \( \Lambda_h = \{t_j\}_{j \in \mathbb{N}} \subset \mathcal{J}_h \) be such that the system \( [S, \{\chi_{\omega_{\nu(j)}} D_k\}_{k=1}^{h}] \) is exponentially \( h \)-stabilizable. Then there exists a feedback control law \( F = \{F_k\}_{k=1}^{h} \), two positive constants \( C \) and \( \alpha \) so that the solution \( x_F(\cdot) \) to the closed-loop system

\[
\begin{align*}
&x_t - \Delta_n x - S x = 0, &\text{in } \Omega \times (\mathbb{R}^+ \setminus \Lambda_h), \\
&x = 0, &\text{on } \partial \Omega \times \mathbb{R}^+ \\
&x(t_j^+) = x(t_j) + \chi_{\omega_{\nu(j)}} D_{\nu(j)} F_{\nu(j)} x(t_j), &\text{in } \Omega, j \in \mathbb{N}^+,
\end{align*}
\]

satisfies

\[
\|x_F(t)\|_{(L^2(\Omega))^n} \leq C e^{-\alpha t} \|x_F(0)\|_{(L^2(\Omega))^n} \text{ for any } t \geq 0.
\]

Let \( y(\cdot) = J^{-1} x_F(\cdot) = \left( \frac{\hat{y}(\cdot)}{\bar{y}(\cdot)} \right) \), where \( \hat{y}(t) \in (L^2(\Omega))^{n_1} \) and \( \bar{y}(t) \in (L^2(\Omega))^{n-n_1} \) for each \( t \geq 0 \). Then by (4.21), there exists \( C' > 0 \) so that

\[
\|\hat{y}(t)\|_{(L^2(\Omega))^{n-n_1}} \leq C' e^{-\alpha t} \|y(0)\|_{(L^2(\Omega))^n} \text{ for any } t \geq 0,
\]

and by (4.18) and (4.20), \( \hat{y}(\cdot) \) verifies that

\[
\begin{align*}
&\hat{y}_t - \Delta_{n-n_1} \hat{y} - S_3 \hat{y} = 0, &\text{in } \Omega \times \mathbb{R}^+, \\
&\hat{y} = 0, &\text{on } \partial \Omega \times \mathbb{R}^+, \\
&\hat{y}(0) = L J^{-1} x_F(0), &\text{in } \Omega,
\end{align*}
\]

and by (4.18) and (4.20), \( \bar{y}(\cdot) \) verifies that

\[
\begin{align*}
&\bar{y}_t = \Delta_{n-n_1} \bar{y} - S_3 \bar{y} = 0, &\text{in } \Omega \times \mathbb{R}^+, \\
&\bar{y} = 0, &\text{on } \partial \Omega \times \mathbb{R}^+, \\
&\bar{y}(0) = L J^{-1} x_F(0), &\text{in } \Omega,
\end{align*}
\]
where $L := \text{diag}(0, I_{n_1}, I_{n-n_1})$. Let $\xi \in \mathbb{C}^{n-n_1}$ be an eigenvector of $S_3$ with respect to $\lambda_0$ and $e_1$ be the unitary eigenvector of $-\Delta$ with respect to $\lambda_1$. Let $x_F(0) = J \begin{pmatrix} 0 \\ \xi e_1 \end{pmatrix}$. One can easily check that $\tilde{y}(0) = \xi e_1$ and $\tilde{y}(t) = e^{(\lambda_0 - \lambda_1)t} \xi e_1$ for any $t \geq 0$. It contradicts (4.22) since $\text{Re}(\lambda_0) \geq \lambda_1$. Thus, the claim (iii) holds. This completes the proof of (i) $\Rightarrow$ (iii).

In summary, we complete the proof of Theorem 4.1.

Remark 4.7. In this section, we give a necessary and sufficient condition for the exponentially $h$-stabilization of the coupled heat equations (see Theorem 4.1). Moreover, Proposition 4.6 tells how to choose the impulse instants so that the system can be exponentially $h$-stabilized.

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