The vector field of a rolling rigid body

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Abstract.
Nonholonomic systems are variational models commonly used for mechanical systems with ideal no-slip constraints. This note provides a differential-geometric derivation of the nonholonomic equations of motion for an arbitrary rigid body rolling on an arbitrary surface, via the semi-symplectic formalism, and in terms of shape operators (a.k.a. Weingarten maps). By a semi-symplectic reduction, the well-known differential equations in the case where the surface is a horizontal plane are shown to be semi-symplectic.

Given a configuration manifold \( \mathcal{Q} \), a smooth Lagrangian \( L: T\mathcal{Q} \to \mathbb{R} \), and a (generally non-involutive) distribution \( \mathcal{D} \) on \( \mathcal{Q} \), the evolutions \( q(t), t \in [a,b] \) in the Lagrange d’Alembert model are defined by the (fixed endpoint) variational problem

\[
\int_a^b L(q'(t)) \, dt = \text{constant},
\]

where the action \( S \) is defined by (\( q'(t) \) denotes the geometric derivative in \( T\mathcal{Q} \), i.e., including the base point)

\[
S(q(t)) = \int_a^b L(q'(t)) \, dt.
\]

The model is called holonomic if \( \mathcal{D} \) is integrable, and otherwise it is nonholonomic. The usual energy is conserved, although the usual symmetry-associated momentum may not be. For further information on such models, see Bates and Sniatycki 1993; Marle 1995; Marle 1998; Sniatycki 1998; Patrick 2007, and the references therein.

Given natural regularity conditions on \( L \), the critical curved of the Lagrange d’Alembert model correspond to (projections to \( \mathcal{Q} \)) of the integral curves of a vector field \( \mathcal{E}_L \) on the phase space \( \mathcal{D} \) itself. If \( \mathcal{D} = T\mathcal{Q} \) then the differential equations defined by \( \mathcal{E}_L \) are the Euler-Lagrange equations, and otherwise they are the Lagrange-d’Alembert equations. This note provides a derivation of \( \mathcal{E}_L \) for a single rigid body rolling on a surface in Euclidean space; see (12) and (14).

The derivation here uses the semi-symplectic formalism, where \( \mathcal{E}_L \) is determined by the system energy and a nondegenerate antisymmetric two from on a distribution of the relevant phase space. If the body rolls on a horizontal plane then the semi-symplectic system admits an \( SE(2) \) symmetry and a semi-symplectic reduction shows that the well-known differential equations of this planar system are semi-symplectic.

1 Lagrangian formalism

Assume a reference body with moments of inertia \( I \) and center of mass at the origin of the reference frame. Let the surface of the body in the reference frame be the 2-submanifold \( \mathcal{M} \) and let \( \mathcal{H} \subseteq \mathbb{R}^3 \) be the 2-submanifold on which the body rolls. Configuration of the body may be determined by elements \( (A, a, s) \in SE(3) \times \mathcal{M} \), with interpretation that a point \( X \) in the reference body is located at \( AX + a \), and that the body contacts the surface at \( As + a \). Left translating, \( \Omega = A^{-1} \dot{A} \) and \( v = A^{-1} \dot{a} \), and the Lagrangian on \( T(SE(3) \times \mathcal{M}) \) is the left invariant

\[
L = \frac{1}{2} \Omega^T I \Omega + \frac{1}{2} m|v|^2 - m a \cdot k.
\]

Establishing contact of the body with the fixed surface means imposing constraints. First, the (holonomic) constraint \( As + a \in \mathcal{H} \) imposes that the contact point lies on the surface. Second, assuming \( n_M \) and \( n_H \) are respectively smooth choices of unit normal for \( \mathcal{M} \) and \( \mathcal{H} \) (so both surfaces are assumed orientable), the (holonomic) constraint \( A n_M(s) = n_H(As + a) \) imposes that the surfaces do not infinitesimally interpenetrate at the contact point (global body-surface interpenetration issues are not considered here). Defining \( x = As + a \), and replacing \( a \), the
configuration space is

\[ Q = \{(A, s, x) \in SO(3) \times \mathcal{M} \times \mathcal{H} \mid A \eta_\mathcal{M}(s) = n_\mathcal{H}(x)\} \]  

\( \mathcal{Q} \) and \( T\mathcal{Q} \) are inserted into \( SE(3) \times \mathcal{M} \) and \( T(\mathcal{SE}(3) \times \mathcal{M}) \) by the equations

\[ a = x - As, \quad v = A^{-1} \dot{a} = A^{-1}(\dot{x} - \dot{A}s - A\dot{s}) = A^{-1}\dot{x} - \dot{s} - \Omega \times s, \]

and the Lagrangian is the pullback of \((1)\) by these, i.e., the result of substitution. It should be noted that the constraint \( A \eta_\mathcal{M}(s) = n_\mathcal{H}(x) \) implies physical meaning to the choice of the normals for \( \mathcal{M} \) and \( \mathcal{H} \). For example, if \( \mathcal{H} \) is the plane \( z = 0 \), and \( n_\mathcal{H} = -k \), then the choice of the outward normal for \( \mathcal{M} \) places the body above the plane, whereas the choice \( n_\mathcal{H} = k \) places it below.

Recall that the Weingarten map of \( \mathcal{M} \) is the vector bundle map (over the identity) \( L_\mathcal{M}: TM \to TM \) defined by

\[ L_\mathcal{M} \frac{ds}{dt} = -\frac{d}{dt} \eta_\mathcal{M}(s(t)), \]

where \( s(t) \) is a smooth curve in \( \mathcal{M} \). Similarly, \( L_\mathcal{H} \) denotes the Weingarten map of \( \mathcal{H} \). If \( x \in \mathbb{R}^3 \) then \( x^\perp \) denotes the \( 3 \times 3 \) matrix such that \( x^\perp y = x \times y \) for all \( y \in \mathbb{R}^3 \).

**Lemma 3** *(The configuration space is a manifold).* \( \mathcal{Q} \) is a 5 dimensional submanifold of \( SO(3) \times \mathcal{M} \times \mathcal{H} \) with tangent bundle

\[ T\mathcal{Q} = \{ (A, s, x), (\Omega, \delta s, \delta x) \mid (A, s, x) \in \mathcal{Q}, \Omega \times \eta_\mathcal{M}(s) = L_\mathcal{M}(s) \delta s - A^{-1} L_\mathcal{H}(x) \delta x \}, \]

and the projection \((A, s, x) \mapsto (s, x)\) is a (trivial) principle \( SO(2) \)-bundle.

**Proof.** The map \( \mathcal{Q} \to S^2 \) defined by \((A, s, x) \mapsto A \eta_\mathcal{M}(s)\) has derivative (use left translation on the factor \( SO(3)\))

\[ (A, s, x), (\Omega, \delta s, \delta x) \mapsto \left. \frac{d}{dt}\right|_{t=0} \exp(t\Omega^\perp) \eta_\mathcal{M}(s(t)) = (\Omega \times \eta_\mathcal{M}(s) - L_\mathcal{M}(s) \delta s) \]

This is a submersion: take \( \delta s = 0 \) and then \( \Omega \mapsto \Omega \times \eta_\mathcal{M}(s) \) is clearly onto the orthogonal complement of \( \eta_\mathcal{M}(s) \), i.e., onto \( T_s S^2 \). So, \((A, s, x) \mapsto A \eta_\mathcal{M}(s) \) and \((A, s, x) \mapsto \eta_\mathcal{H}(x) \) are transversal, and similarly differentiating the second of these, \( \mathcal{Q} \) is smooth with tangent space at \((A, s, x)\) the solutions \((\Omega, \delta s, \delta x)\) to

\[ A(\Omega \times \eta_\mathcal{M}(s) - L_\mathcal{M}(s) \delta s) = -L_\mathcal{H}(x) \delta x, \]

and \( \dim \mathcal{Q} = \dim(SO(3) \times \mathcal{M} \times \mathcal{H}) - \dim S^2 = (3+2+2) - 2 = 5 \). If \( \theta \in \mathbb{R} \) then \( \theta(\cdot, A, s, x) = (A \exp(-\theta \eta_\mathcal{M}(s)^\perp), s, x) \) defines a right action of \( SO(2) \) which, for fixed \( s \) and \( x \), is free and transitive on the \( A \in SO(3) \) such that \( A \eta_\mathcal{M}(s) = n_\mathcal{H}(x) \); the assignment of the identity of \( SO(3) \) to each \((s, x)\) is a global section. \( \square \)

The holonomic Lagrangian \((1)\) is not regular because it does not involve \( s \)—there is no interaction of the body and the surface. To include that interaction, impose the rolling constraint that the point on the body at \( As + a \) is instantaneously at rest, i.e.,

\[ \frac{d}{dt}(A(t)s + a(t)) = \dot{A}s + \dot{a} = 0. \]

Ideal rolling without slipping means zero velocity of the physical location of the fixed point on the body (at \( s \)) in the (inertial) frame of the surface, so \( s \) is not differentiated here. Converting to the variable \( x, \dot{a} = \dot{x} - \dot{A}s - A\dot{s} = -\dot{A}s, \) and the rolling constraint becomes \( \dot{x} - A\dot{s} = 0. \)

Summarizing: the nonholonomic system for a body with surface \( \mathcal{M} \) rolling on a surface \( \mathcal{H} \) is the lagrangian system

\[
\begin{align*}
\mathcal{Q} &= \{(A, s, x) \in SO(3) \times \mathcal{M} \times \mathcal{H} \mid A \eta_\mathcal{M}(s) = n_\mathcal{H}(x)\}, \\
T\mathcal{Q} &= \{ (A, s, x), (\Omega, \delta s, \delta x) \mid (A, s, x) \in \mathcal{Q}, \Omega \times \eta_\mathcal{M}(s) = L_\mathcal{M}(s) \delta s - A^{-1} L_\mathcal{H}(x) \delta x \}, \\
\mathcal{D} &= \{ (A, s, x), (\Omega, \delta s, \delta x) \in T\mathcal{Q} \mid \delta x = A \delta s \}, \\
L &= \frac{1}{2} \Omega^T \Omega + \frac{1}{2} m|v|^2 - mga \cdot k, \quad a = x - As, \quad v = A^{-1} \dot{x} - \Omega \times s - \dot{s}.
\end{align*}
\]
2 Semi-symplectic derivation of the vector field

Lagrange-d’Alembert models have an equivalent semi-symplectic formalism (Bates and Sniatycki 1993; Sniatycki 1998; Patrick 2007): Given $Q$ and $D$, a lagrangian $L: Q \rightarrow \mathbb{R}$ is called $D$-regular if its second fiber derivative is nonsingular when restricted to $D$. The distribution $K_D = TD \cap (T\tau_Q)^{-1}D$, where $\tau_Q: TQ \rightarrow Q$ is the projection, has fiber dimension twice that of $D$, and $\omega_L$ is nonsingular on $K_D$ if and only if $L$ is $D$-regular, in which case

$$(i_{Y_E} \omega_L - dE)|_{K_D} = 0, \text{ where } Y_E(Q) \subseteq K_D,$$

defines a vector field $Y_E$ with integral curves exactly the solutions of the Lagrange-d’Alembert variational principle. In general one is led to a category defined by a nondegenerate antisymmetric two form with domain a (generally nonintegrable) distribution. The semi-symplectic formulation is advantageous because it has this formula for the evolution vector field—the Lagrange-d’Alembert equations have already been geometrically determined as (5).

Since $L$ is fiberwise bilinear, regularity is equivalent to $\Omega = 0$, $\dot{s} = 0$, and $\dot{x} = 0$ whenever $\Omega^t I \Omega + m|v|^2 = 0$. Assuming $I$ is positive definite and $m > 0$, the latter is equivalent to $\Omega = 0$ and $v = 0$, i.e., $\dot{x} = A\dot{s}$ (within $TQ$). Restricting to $TQ$ leads to $L_M(s)\dot{s} - A^{-1}L_H(x) A\dot{s} = 0$, so $L$ is $D$-regular if and only

$$L_{A,s,x}: T_{\pi}M \rightarrow T_{\pi}M, \quad L_{A,s,x} = L_M(s) - A^{-1}L_H(x) A$$

is fiberwise nonsingular for all $(A,s,x) \in Q$.

Another advantage of the semi-symplectic formalism is an early clear emphasis and identification of the relevant phase space $D$, which by Lemma 3 is the subset of $TQ$ satisfying $\dot{x} = A\dot{s}$ and $L_{A,s,x}\dot{s} = \Omega \times n_M(s)$, and which, if $L$ is regular, may be identified with $P = Q \times \mathbb{R}^3$ by

$$(A,s,x,\Omega) \leftrightarrow \left( (A,s,x), (\Omega, L_{A,s,x}^{-1}(\Omega \times n_M(s)), A L_{A,s,x}^{-1}(\Omega \times n_M(s))) \right).$$

So from the outset one seeks differential equations for $dA/dt, ds/dt, dx/dt,$ and $d\Omega/dt$, which is not entirely obvious apriori because from the variational principle one might have anticipated second order differential equations for $s$ or $x$. Since every evolution has derivative in $D$ and is second order, three of the required differential equations are known:

$$\frac{dA}{dt} = A^{-1}\Omega, \quad \Lambda_{A,s,x} \frac{ds}{dt} = \Omega \times n_M, \quad \frac{dx}{dt} = A \frac{ds}{dt}.$$  

Only the differential equation for $d\Omega/dt$ need be determined.

To identify the rolling body system as semi-symplectic, assuming regularity, it is required to find on $P$ the distribution $K_P$, the Lagrange two-form $\omega_L$, and the pullback of the energy $E$, all of which are defined by pullback to $P$:

Using left translation with the first factor $SO(3)$ of $P$, the pullback of $K_D$ to the distribution on $K_P$ on $P$ is the pullback of $D$ by the projection $(A,s,x,\Omega) \rightarrow (A,s,x)$, i.e.

$$K_P = \{( (A,s,x,\Omega), (\delta A, \delta s, \delta x, \Omega) ) | (A,s,x) \in Q, \delta A \times n_M(s) = L_{\delta(s,x)} \delta s, \delta s = A \delta x \}. \quad (7)$$

This may be viewed as determining $\delta s$ and $\delta x$ with free and uncoupled $\delta A$ and $\delta \Omega$, and hence has fiber dimension 6. Analogously, in (4), $\delta s$ and $\delta x$ are determined from a free $\Omega$, so the fiber dimension of $D$ is 3.

It is an error to substitute the constraint distribution into the Lagrangian before calculating the Lagrange one form. This is the point in the semi-symplectic formalism which avoids obtaining incorrect evolution equations by substituting the constraint into the Lagrangian before varying the action.

The Lagrange forms are natural with respect to lifts of diffeomorphisms, so it suffices to pull back $\omega_L$ defined by (1) as a function on $T(SE(3)) \times \mathbb{R}^3$, and (1) is independent of $s$, so a formula for $\omega_L$ with $L$ regarded as a left invariant Lagrangian of $T(SE(3))$ will do. The general formula for the Lagrange two-form of a left invariant Lagrangian $L(\xi)$ on a Lie group $G = \{ g \}$, where $g = \{ \xi \}$ is the Lie algebra and $\xi \in g$, is

$$\omega_L(g,\xi)\left( (g,\xi,\delta g_1,\delta \xi_1), (g,\xi,\delta g_2,\delta \xi_2) \right) = D^2L(\xi)(\delta \xi_2,\delta g_1) - D^2L(\xi)(\delta \xi_1,\delta g_2) + DL(\xi)[\delta g_1,\delta g_2],$$

and since Lie bracket of se(3) is $\{ (\xi, \eta) \}$ is

$$[(\xi, \eta), (\eta, \eta)] = (\xi \times \eta, \xi \times \eta - \eta \times \eta),$$

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the Lagrange two-form of $L$ on $T(SE(3)) \times M$ is
\[
\omega_L(A, s, \Omega, v)((\delta A_1, \delta a_1, \delta \Omega_1, \delta x_1), (\delta A_2, \delta a_2, \delta \Omega_2, \delta x_2))
\]
\[
= ((I \delta \Omega_2) \cdot \delta A_1 + m \delta x_2 \cdot \delta a_1) - ((I \delta \Omega_1) \cdot \delta A_2 + m \delta x_1 \cdot \delta a_2)
\]
\[
+ (I \Omega) \cdot (\delta A_1 \times \delta A_2) + mv \cdot (\delta A_1 \times \delta a_2 - \delta A_2 \times \delta a_1).
\]  
\tag{8}

Obtaining the pullback of (8) to $\mathcal{D}$ means substituting the derivatives of (2), i.e.,
\[
a = x - As, \quad \delta a = \delta x - A(\delta A \times s) - A \delta s,
\]
\[
v = A^{-1} \dot{x} - \dot{s} - \Omega \times s, \quad \delta v = -\delta A \times (A^{-1} \dot{x}) + A^{-1} \delta \dot{x} - \delta \dot{s} - \delta \Omega \times s - \Omega \times \delta s.
\]  
\tag{9}

Here it is useful to realize that the semi-symplectic form is only required on $K_\mathcal{D}$ and may be replaced by any two-form with equal values on that. Since $K_\mathcal{D}$ is defined by
\[
\dot{x} = A \dot{s}, \quad \delta x = A \delta s, \quad \delta \dot{x} = A(\delta A \times \dot{s}) + A \delta \dot{s},
\]
these may be substituted into (9) to obtain the simpler
\[
\delta a = -\delta A \times s, \quad v = -\Omega \times s, \quad \delta x = -\delta \Omega \times s - \Omega \times \delta s.
\]

with the result (the symbol $\cong$ means equal on $K_\mathcal{D}$)
\[
\omega_L \cong (I \delta \Omega_2) \cdot \delta A_1 + m(-\delta \Omega_2 \times s - \Omega \times \delta s_2) \cdot (-\delta A_1 \times s)
\]
\[
- (I \delta \Omega_1) \cdot \delta A_2 - m(-\delta \Omega_1 \times s - \Omega \times \delta s_1) \cdot (-\delta A_2 \times s)
\]
\[
+ (I \Omega) \cdot (\delta A_1 \times \delta A_2) - m(\Omega \times \delta s_1) \cdot (s \times \delta A_1)
\]
\[
= (I \delta \Omega_1 - ms \times (s \times \delta A_1)) \cdot \delta \Omega_2 - m(\Omega \times \delta s_1) \cdot (s \times \delta A_1)
\]
\[
\tag{10}
\]

To view this as a two-form restricted to $K_\mathcal{P}$, regard $\delta A$ and $\delta \Omega$ as free and restrict $\delta s$ and $\delta x$ as in (7), assuming of course that $A n_M(s) = n_H(x)$. In the same way, one requires the energy $E$ only restricted to $\mathcal{D}$, so
\[
E \cong \frac{1}{2} \Omega^T I \Omega + \frac{1}{2} m |\Omega \times s|^2 + mg(x - As) \cdot k = \frac{1}{2} \Omega^T \tilde{I} \Omega + mg(x - As) \cdot k,
\]
and,
\[
dE \cong (I \Omega) \cdot \delta \Omega_2 + mv \cdot \delta x - mgA(\delta A_2 \times s) \cdot k
\]
\[
= (I \Omega) \cdot \delta \Omega_2 + m(\Omega \times s) \cdot (\delta \Omega_2 \times s + \Omega \times \delta s_2) - mg(s \times A^{-1} k) \cdot \delta A_2
\]
\[
= (\tilde{I} \Omega) \cdot \delta \Omega_2 - m(s \times \Omega) \cdot (\Omega \times \delta s_2) - mg(s \times A^{-1} k) \cdot \delta A_2.
\]  
\tag{11}

To find the vector field $Y_E$ replace 1-subscripted quantities such as $\delta \Omega_1$ with their corresponding derivatives $d\Omega/dt$, and set (10) to the derivative (11) of $E$, for all $(\delta A_2, \delta s_2, \delta x_2, \delta \Omega_2) \in K_\mathcal{P}$. As already noted, in this context $\delta \Omega_2$ and $\delta A_2$ are free and uncoupled, and $\delta s_2 = 0$ if $\delta A_2 = 0$. Set $\delta A_2 = 0$ and $\delta s_2 = 0$ to obtain $\tilde{I} \delta A_1 = \tilde{I} \Omega$ i.e. $A^{-1} dA/dt = \Omega$, which is already known (6). Substituting back, all the $\delta s_2$ cancel,
\[
- ((\tilde{I} \delta A_2) \cdot \delta \Omega_1 + (\tilde{I} \Omega) \cdot (\Omega \times \delta A_2) + m(\Omega \times \delta s_1) \cdot (s \times \delta A_2)
\]
\[
= -(\tilde{I} \delta \Omega_1 + (\tilde{I} \Omega) \times \Omega + m(\Omega \times \delta s_1) \times s) \cdot \delta A_2
\]
\[
= -mg(s \times A^{-1} k) \cdot \delta A_2,
\]
and finally
\[
\int \frac{d\Omega}{dt} = (\tilde{I} \Omega) \times \Omega + ms \times \left( \frac{ds}{dt} \times \Omega \right) + mg s \times (A^{-1} k).
\]
Summarizing: the dynamical system corresponding to the Lagrange-d’Alembert variational principle (4) are

\[ P = \{(A, s, x, \Omega) \in SO(3) \times M \times H \times \mathbb{R}^3 \mid A_n M(s) = n M(x)\}, \quad E = \frac{1}{2} \Omega_\Omega + mg(x - As) \cdot k, \]

\[ \Lambda_{A, s, x} \frac{ds}{dt} = \Omega \times n_M(s), \quad \tilde{I} \frac{d\Omega}{dt} = (\tilde{I} \Omega) \times \Omega + ms \times \left( \frac{ds}{dt} \times \Omega + g(A^{-1} k) \right), \quad A^{-1} \frac{dA}{dt} = \Omega^\wedge, \]

(12)

\[ \frac{dx}{dt} = A \frac{ds}{dt}, \quad \Lambda_{A, s, x} = L_M(s) - A^{-1} L_H(x) A, \quad \tilde{I} = I - m(s^\wedge)^2. \]

3 Rolling on a horizontal plane; semi-symplectic reduction

This is the special case where \( H \) is the \( x, y \) plane, \( L_H = 0 \) and \( n_H = -k \) (so that the body is above the plane when \( n_M \) is the outward normal). The group

\[ \mathcal{G} \equiv \{(B, b) \in SE(3) \mid Bk = k, b \cdot k = 0\} \cong SE(2) \]

acts on the semisymplectic phase space \( P \) by

\[ (B, b)(A, s, x, \Omega) = (BA, s, Bx + b, \Omega) \]

(13)

and the projection to \( M \times \mathbb{R}^3 = \{(s, \Omega)\} \) is a quotient map. The vector field (12) is equivariant and the differential equations for \( s \) and \( \Omega \) close: on \( P \), \( A_n M(s) = n_H(x) = -k \) so \( n_M(s) = -A^{-1} k \) can be substituted. Also, the energy drops to the quotient by using \( k \cdot (x - As) = -k \cdot As = n_M \cdot s \), leading to the dynamical system

\[ \mathcal{P} = \{(s, \Omega) \in M \times \mathbb{R}^3 \}, \quad E = \frac{1}{2} \Omega_\Omega + mg n_M \cdot s. \]

\[ L_M \frac{ds}{dt} = \Omega \times n_M(s), \quad \tilde{I} \frac{d\Omega}{dt} = (\tilde{I} \Omega) \times \Omega + ms \times \left( \frac{ds}{dt} \times \Omega - g n_M \right), \quad \tilde{I} = I - m(s^\wedge)^2. \]

(14)

Equations (14) are the same as equations (8a–c) in Garcia and Hubbard 1988, equations (1.1) and (1.2) of Borisov and Mamaev 2002, and equations (4) of Borisov and Mamaev 2003 (after replacing \( dn_M/dt = -L_M ds/dt \) and accounting for the choice of unit normal).

In passing, if \( y^a, a = 1, 2 \), are coordinates on \( M \), so that \( M \) is the image of an immersion \( s(y) \), arranged so that the outward normal is

\[ n_M = \frac{\partial s}{\partial y^1} \times \frac{\partial s}{\partial y^2}. \]

Let \( g_{ab} \) and \( L_{ab} \) be the first and second fundamental forms of \( M \), and let \( L_b^a \) be the Weingarten map, so

\[ g_{ab} = \frac{\partial s}{\partial y^a} \cdot \frac{\partial s}{\partial y^b}, \quad L_{ab} = n_M \cdot \frac{\partial^2 s}{\partial y^a \partial y^b}, \quad L_b^a = g^{ac} L_{bc}, \]

giving a 2 × 2 matrix \( L = [L_{ab}] \) that depends on \( y \). From the left side of the differential equation for \( ds/dt \) in (14),

\[ \frac{\partial s}{\partial y^a} \cdot L_M \frac{ds}{dt} = \frac{\partial s}{\partial y^a} \cdot \left( L_M \frac{\partial s}{\partial y^b} \right) \frac{dy^b}{dt} = \frac{\partial s}{\partial y^a} \cdot \left( L_b^c \frac{\partial s}{\partial y^c} \right) \frac{dy^b}{dt} = g_{ac} L_b^c \frac{dy^b}{dt} = L_{ab} \frac{dy^b}{dt} = \left[ L \frac{dy^b}{dt} \right]_a, \]

while on the right side,

\[ \frac{\partial s}{\partial y^a} \cdot (\Omega \times n_M) = \left( n_M \times \frac{\partial s}{\partial y^a} \right) \cdot \Omega = [B^a \Omega]_a, \quad B = n_M \wedge \frac{\partial s}{\partial y}, \]

giving the equations of motion

\[ L \frac{dy}{dt} = B^a \Omega, \quad \tilde{I} \frac{d\Omega}{dt} + \left( ms^\wedge \Omega \wedge \frac{\partial s}{\partial y} \right) \frac{dy}{dt} = (\tilde{I} \Omega) \times \Omega + mg n_M \times s, \]

where \( \tilde{I} = I - m(s^\wedge)^2 \), the 2 × 2 matrix \( L \), the 3 × 2 matrix \( B \), and the three-vectors \( s \) and \( n_M \), are all given functions of \( y \).
The formula \((B, b)(A, s, x) = (BA, s, Bx + b), (B, b) \in \mathcal{G}\), is an action on \(Q\), because \((A, s, x) \in Q\) implies \((BA, s, Bx + b) \in \mathcal{Q}\), since

\[
(BA)n_M(s) = BAn_M(s) = Bn_M(x) = Bk = n_M(Bx + b),
\]

The action lifts to \(TQ\) as

\[
(B, b)((A, s, x), (\Omega, \dot{s}, \dot{x})) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (B, b)(A + \epsilon A\omega^\wedge, \dot{s}, x + \epsilon \dot{x})
\]

\[
= ((BA, s, Bx + b), (A^{-1}B^{-1}BA\omega^\wedge, s, B\dot{x}))
\]

\[
= ((BA, s, Bx + b), (\Omega, \dot{s}, B\dot{x})).
\]

(15)

If \(\dot{x} = A\dot{s}\) then \(B\dot{x} = B(A\dot{s}) = (BA)\dot{s}\) so the (15) preserves the rolling constraint, restricts to an action on \(D\), and induces on \(P\) the action (13). The Lagrangian is invariant:

\[
L((B, b)((A, s, x), (\Omega, \dot{s}, \dot{x})))
\]

\[
= L((BA, s, Bx + b), (\Omega, \dot{s}, B\dot{x}))
\]

\[
= \frac{1}{2}\Omega^I\Omega^j + \frac{1}{2}|(BA)^{-1}B\dot{x} - \Omega \times s - \dot{s}|^2 - mg(Bx + b - BAS) \cdot k
\]

\[
= \frac{1}{2}\Omega^I\Omega^j + \frac{1}{2}|A^{-1}\dot{x} - \Omega \times s - \dot{s}|^2 - mg(x - As) \cdot B^I k - mgb \cdot k
\]

\[
= \frac{1}{2}\Omega^I\Omega^j + \frac{1}{2}|A^{-1}\dot{x} - \Omega \times s - \dot{s}|^2 - mg(x - As) \cdot k
\]

\[
= L((A, s, x), (\Omega, \dot{s}, \dot{x})).
\]

Consequently, \(\mathcal{G}\) acts symplectically with respect to \(\omega_L\), and hence acts by semi-symplectomorphisms.

The Lie algebra of \(\mathcal{G}\) is \(\mathbb{R} \times \mathbb{R}^2 = \{(\xi^r, \xi^a)\}\) and the infinitesimal generator of the action is

\[
\frac{d}{d\epsilon}\bigg|_{\epsilon=0} (A^{-1}(1 + \epsilon \xi^r)k^\wedge A, s, (1 + \epsilon \xi^r k^\wedge) x + \epsilon \xi^a) = ((A, s, x), (\xi^r A^{-1} k, 0, \xi^r k \times x + \xi^a)).
\]

The momentum associated to \(\xi = (\xi^r, \xi^a)\) at the state \(((A, s, x), (\Omega, \dot{s}, \dot{x}))\) is

\[
J_\xi = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L((A, s, x), (\Omega + \epsilon \xi^r A^{-1} k, \dot{s}, \dot{x} + \epsilon \xi^r k \times x + \epsilon \xi^a))
\]

\[
= \Omega^I I (\xi^r A^{-1} k) + mv^I \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(A^{-1}(\dot{x} + \epsilon \xi^r k \times x + \epsilon \xi^a) - (\Omega + \epsilon \xi^r A^{-1} k) \times s - \dot{s}\right)
\]

\[
= \Omega^I I (\xi^r A^{-1} k) + mv^I \left(A^{-1}(\xi^r k \times x + \xi^a) - (\xi^r A^{-1} k) \times s\right)
\]

\[
= \xi^r \left(AI\Omega + m(x - As) \times Av\right) \cdot k + mA \cdot \xi^a.
\]

Pulling this back to \(P\) means \(\dot{x} = A\dot{s}\) and \(A_{A,s,x}\dot{s} = \Omega \times n_M(s)\), resulting in \(v = -\Omega \times s\) and

\[
J_\xi = \xi^r \left(AI\Omega - m(x - As) \times (\Omega \times s)\right) \cdot k - mA((\Omega \times s) \cdot \xi^a)
\]

\[
= \xi^r \left(AI\Omega - mx \times (\Omega \times s)\right) \cdot k - mA((\Omega \times s) \cdot \xi^a)
\]

The phase space is \(P\), which assuming regularity is diffeomorphic to the original rolling distribution \(D\). The distribution \(K_P\) within \(P\) corresponds to the second order part of \(TD\). From the above, the relevant symmetric
semi-symplectic formalism is
\[ \mathcal{P} = \{ (A, s, x, \Omega) \in SO(3) \times M \times H \times \mathbb{R}^3 \mid A_{n_M} = -k \}, \]
\[ T\mathcal{P} = \{ ((A, s, x, \Omega), (\delta A, \delta s, \delta x, \delta \Omega)) \mid (A, s, x) \in Q, \delta A \times n_M = L_M \delta s \}, \]
\[ K_\mathcal{P} = \{ ((A, s, x, \Omega), (\delta A, \delta s, \delta x, \delta \Omega)) \in T\mathcal{P} \mid \delta x = A \delta s \}, \]
\[ \omega_L \equiv (\tilde{I} \delta A_1) \cdot \delta \Omega_2 - (\tilde{I} \delta A_2) \cdot \delta \Omega_1 + (\tilde{I} \delta A_1) \cdot (\delta A_1 \times \delta A_2) + m(\Omega \times \delta s_1) \cdot (s \times \delta A_2) - m(\Omega \times \delta s_2) \cdot (s \times \delta A_1), \]
\[ E = \frac{1}{2} \Omega^T \tilde{I} \Omega + mg(x - A s) \cdot k, \quad (16) \]
\[ dE \equiv ((\tilde{I} \delta \Omega) \cdot \delta \Omega - m(s \times \Omega) \cdot (\Omega \times \delta s) + mg(s \times n_M) \cdot \delta A, \]
\[ G = \{ (B, b) \in SE(3) \mid Bk = k, b \cdot k = 0 \} \equiv SE(2), \]
\[ (B, b)(A, s, x, \Omega) = (BA, s, Bx + b, \Omega), \]
\[ \xi(A, s, x, \Omega) = (\xi^\prime A^{-1}k, 0, \xi^0k \times x + \xi^a, 0), \]
\[ J = \left(A \tilde{H} \Omega - mx \times (\Omega \times s)\right) \cdot (\xi, k) - mA(\Omega \times s) \cdot \xi^a. \]

\[ \xi(A, s, x, \Omega) \in K_\mathcal{P} \] for all \((A, s, x, \Omega)\) implies \(\xi = 0\) (the \(\delta s = 0\), so there is no semi-hamiltonian part of the symmetry and there is no conserved momentum (Patrick 2007). Consequently, the nonholonomic reduced phase space is \(\pi: \mathcal{P} \rightarrow \mathcal{P}/G\) with nonholonomic distribution \(\tilde{K}_\mathcal{P} = T\pi((\ker T\pi \cap \mathcal{K})\omega_L)\). \(\omega_L\) drops to a nondegenerate two form on \(\tilde{K}_\mathcal{P}\), \(E\) also drops. The resulting semi-symplectic equations must be the same as \((14)\) — this is verified below.

The map \(SO(3) \times M \times H \times \mathbb{R}^3 \rightarrow S^2 \times M \times \mathbb{R}^3\) by \((A, s, x, \Omega) \mapsto (A^{-1}k, s, \Omega)\) is a quotient for the action of \(G\), because
\[ (B, b)(A, s, x, \Omega) = (BA, s, Bx + b, \Omega) \mapsto ((BA)^{-1}k, s, \Omega) = \left(A^{-1}B^{-1}k, s, \Omega\right) = \left(A^{-1}k, s, \Omega\right), \]
while \(\pi(A, s, x, \Omega) = \pi(A, s, x, \tilde{\Omega})\) implies \(Ak = \tilde{A}k, s = \tilde{s}, \) and \(x = \tilde{x}\), from which \(B = AA^{-1}B = \tilde{x}\) provides \((B, b) \in G\) such that \(B, b)(A, s, x, \Omega) = (A, s, \tilde{x}, \Omega)\). The restriction to \(\mathcal{P}\) has \(n_M = -A^{-1}k\) so
\[ \tilde{P} = \mathcal{P}/G = M \times \mathbb{R}^3 = \{(s, \Omega)\}, \quad \pi(A, s, x, \Omega) = (s, \Omega) \quad \text{(restricted to } A_{n_M} = -k). \]
This makes
\[ \ker(T\pi \cap \mathcal{K}) = \{ ((A, s, x, \Omega), (\delta A, \delta s, \delta x, \delta \Omega)) \in T\mathcal{P} \mid \delta s = 0, \delta \Omega = 0, \delta x = A \delta s \}, \]
\[ = \{ ((A, s, x, \Omega), (\delta A, 0, 0, 0)) \mid \delta A \in \mathbb{R} n_M \}, \]
and required is the symplectic complement of this in \(\mathcal{K}\). For that, put \(\delta A_1 = n_M, \delta s_1 = 0, \delta \Omega_1 = 0, \) and \(L_M \delta s_2 = \delta A_2 \times n_M\) into \(\omega_L = 0\), obtaining
\[ (\tilde{I} n_M) \cdot \delta \Omega_2 - (\tilde{I} \delta \Omega) \cdot L_M \delta s_2 - m(\Omega \times \delta s_2) \cdot (s \times n_M) = 0, \quad (\delta A_2, \delta s_2, \delta x_2, \delta \Omega_2) \in K_\mathcal{P}. \]
which refers only to \(\delta s_2\) and \(\delta \Omega_2\). Any such can be arranged into \(K_\mathcal{P}\), so \(\tilde{K}_\mathcal{P} = T\pi((\ker T\pi \cap \mathcal{K})\omega_L)\) is defined by
\[ (\tilde{I} n_M) \cdot \delta \Omega - (\tilde{I} \delta \Omega)^T L_M \delta s + m(n_M \cdot \Omega)(s \cdot \delta s) = 0, \]
or equivalently,
\[ n_M^T \tilde{I} \delta \Omega - \Omega^T \tilde{I} L_M \delta s + m(n_M \cdot \Omega)s \cdot \delta s = 0. \quad (17) \]
To calculate \(\tilde{\omega}_L(s, \Omega)((\delta s_1, \delta \Omega_1), (\delta s_2, \delta \Omega_2))\) use those same \(\delta s_i\) and \(\delta \Omega_i\) and substitute into the expression for \(\omega_L\) in \((16)\) any \(\delta A_i\), such that \(\delta A_i \times n_M = L_M \delta s_i\), e.g., \(\delta A_i = n_M \times L_M \delta s_i\) (there is no \(\delta x\) in the formula for \(\omega_L\) anyway).
Since \(x \cdot k = 0\), the reduced energy is
\[ \tilde{E} = \frac{1}{2} \Omega^T \tilde{I} \Omega + mgs \cdot n_M. \]
To verify the reduced vector field, using a multiplier $\lambda$ for the constraint (17) to $\bar{K}$ (and remembering that $\delta A_i = n_M \times L_M \delta s_i$), the equations for the reduced vector field are

$$(\bar{I} \delta A_1) \cdot \delta \Omega_1 - (\bar{I} \delta A_2) \cdot \delta \Omega_2 + (\bar{I} \Omega) \cdot (\delta A_1 \times \delta A_2) + m(\Omega \times \delta s_1) \cdot (s \times \delta A_2) - m(\Omega \times \delta s_2) \cdot (s \times \delta A_1)
= (\bar{I} \Omega) \cdot \delta \Omega_2 - m(s \times \Omega) \cdot (\Omega \times \delta s_2) - mg(s \times A^{-1} k) \cdot \delta A_2
+ \lambda (n_M \times I \delta \Omega_2 - \Omega^t \bar{I} L_M \delta s_2 + m(n_M \cdot \Omega)s^t \delta s_2),$$

$$n_M \times I \delta \Omega_1 - \Omega^t \bar{I} L_M \delta s_2 + m(n_M \cdot \Omega)s^t \delta s_1 = 0,$$

where $\delta s_2 \in TM$ and $\delta \Omega_2 \in \mathbb{R}^3$ are arbitrary. Setting $\delta s_2 = 0$ (so then $\delta A_2 = 0$), (18) become

$$(\bar{I} \delta A_1) \cdot \delta \Omega_2 = (\bar{I} \Omega) \cdot \delta \Omega_2 + \lambda n_M \cdot I \delta \Omega_2,
= n_M \cdot I \delta \Omega_1 - \Omega^t \bar{I} L_M \delta s_1 + m(n_M \cdot \Omega)s^t \delta s_1 = 0.$$

From the first $n_M \times L_M \delta s_1 - \Omega = \lambda n_M$, because $I \delta \Omega_2$ is arbitrary. Then the cross-product with $n_M$ provides $-L_M \delta s_1 - n_M \times \Omega = 0$, i.e., $L_M \delta s_1 = \Omega \times n_M$, which is the $ds/dt$ equation of (14), while the dot-product with $n_M$ obtains $\lambda = -n_M \cdot \Omega$. Noting that

$$n_M \cdot (s \times (\delta s_1 \times \Omega)) = n_M \cdot ((s \cdot \Omega)\delta s_1 - (s \cdot \delta s_1)\Omega) = -(n_M \cdot \Omega)(s \cdot \delta s_1),$$

the second equation of (19) is

$$n_M \cdot (\bar{I} \delta \Omega_1 - (\bar{I} \Omega) \times \Omega - ms \times (\delta s_1 \times \Omega)) = 0$$

corresponding to the $n_M$ component of the $d\Omega/dt$ equation of (14). For the component orthogonal to $n_M$, assuming $\delta s_2$ is arbitrary and setting $\delta \Omega_2 = 0$, (18) becomes

$$-(\bar{I} \delta A_2) \cdot \delta \Omega_1 + (\bar{I} \Omega) \cdot (\delta A_1 \times \delta A_2) + m(\Omega \times \delta s_1) \cdot (s \times \delta A_2) - m(\Omega \times \delta s_2) \cdot (s \times \delta A_1)
= -m(s \times \Omega) \cdot (\Omega \times \delta s_2) + mg(s \times n_M) \cdot \delta A_2 - (n_M \cdot \Omega)(\Omega^t \bar{I} L_M \delta s_2 + m(n_M \cdot \Omega)s^t \delta s_2).$$

But $\delta A_1 = n_M \times L_M \delta s_1 = n_M \times (\Omega \times n_M) = \Omega - (n_M \cdot \Omega)n_M$, so the second term on the left of (20) is

$$(\bar{I} \Omega) \cdot (\delta A_1 \times \delta A_2) = ((\bar{I} \Omega) \times (\Omega - (n_M \cdot \Omega)n_M)) \cdot \delta A_2
= (\bar{I} \Omega) \times \Omega - (n_M \cdot \Omega)((\bar{I} \Omega) \times n_M) \cdot \delta A_2,$$

while the fourth term on the left of (20) is

$$-m(\Omega \times \delta s_2) \cdot (s \times \delta A_1)
= -m(\Omega \times \delta s_2) \cdot (s \times (\Omega - (n_M \cdot \Omega)n_M))
= -m(s \times \Omega) \cdot (\Omega \times \delta s_2) + m(n_M \cdot \Omega)((s \cdot \Omega) \cdot (n_M \cdot \delta s_2) - (n_M \cdot \Omega)(\delta s_2 \cdot s))
= -m(s \times \Omega) \cdot (\Omega \times \delta s_2) - m(n_M \cdot \Omega)^2(\delta s_2 \cdot s).$$

So (20) is $-(\bar{I} \delta \Omega_1) + (\bar{I} \Omega) \times \Omega + (m(\Omega \times \delta s_1) \times s)) \cdot \delta A_2 = mg(s \times n_M) \cdot \delta A_2$, i.e., the component of (14) orthogonal to $n_M$.

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