Quantum interference of edge supercurrents in a two-dimensional topological insulator

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(Dated: October 3, 2014)

Josephson weak links made of two-dimensional topological insulators (TIs) exhibit magnetic oscillations of the supercurrent that are reminiscent of those in superconducting quantum interference devices (SQUIDs). We propose a microscopic theory of such a TI SQUID effect. The theory shows a temperature-driven crossover from the normal \( \Phi_0 \)-periodic SQUID pattern to a \( 2\Phi_0 \)-quasiperiodic interference pattern consisting of a series of alternating even and odd peaks (where \( \Phi_0 = \hbar c/2e \) is the magnetic flux quantum). The predicted even-odd effect is the signature of gapless (protected) Andreev bound states with a sawtooth dependence on the magnetic flux. Our findings may explain recently observed even-odd interference patterns in InAs/GaSb-based TI Josephson junctions, suggesting new operation regimes for nano-SQUIDs.

PACS numbers:

Introduction. In a two-dimensional topological insulator (2D TI) electric current flows near the edges of the system and is protected against elastic backscattering by time-reversal symmetry. This property has important implications for both normal and superconducting transport in 2D TIs. In very recent experiments, 2D TIs have been implemented as Josephson weak links between two superconductors. Remarkably, a 2D TI Josephson junction (JJ) acts as an intrinsic superconducting quantum interference device (SQUID) in which a magnetic flux, \( \Phi \), enclosed in the interior of the 2D TI controls the interference of the Josephson currents flowing at the opposite edges of the sample. The net supercurrent exhibits oscillations reminiscent of the SQUID pattern \( \propto |\cos(\pi \Phi/\Phi_0)| \) rather than the Fraunhofer pattern observed in nontopological weak links.

In view of the experimental progress in fabricating 2D TI JJs and their application potential as topologically protected nano-SQUIDs, there is an apparent need for a theoretical understanding of quantum interference phenomena in this type of JJs. One of the important questions is the following. The cosine-like SQUID pattern reflects the sinusoidal Josephson current-phase relation that can typically be attributed to the electronic states with energies close to or above the superconducting gap. However, the 2D TI JJs support also subgap Andreev bound states (ABSs) which are immune to non-magnetic disorder (see, e.g., Refs. \[4,5\]) and are highly anharmonic (e.g., sawtooth-like) with respect to the Josephson phase difference. One may therefore ask: How do such ABSs manifest themselves in a TI SQUID? We believe that the answer to that question may shed some light on the experimental findings of Ref. \[4\] in which an unusual even-odd interference pattern deviating from the cosine-like SQUID behavior has been observed in InAs/GaSb-based TI JJs. Also, understanding the role of the ABSs in TI SQUIDs may suggest an alternative scheme to detect Majorana fermion states in JJs.

Another unexplored issue concerns the mechanism by which an external magnetic field suppresses Andreev reflection in TI JJs. Such suppression has been observed previously in JJs (see, e.g., Refs. \[10,11\]) and is expected to have an effect on the performance of TI SQUIDs as well. This issue, however, cannot be resolved within a simple model in which the magnetic field enters only as a magnetic flux piercing the area of the weak link. In order to address all these points, in this paper we propose a microscopic theory of the Josephson effect in 2D TIs subject to an out-of-plane magnetic field. Interestingly, the influence of the magnetic field on the edge supercurrents in a 2D TI resembles the well-known Doppler effect, which allows for a compact analytic treatment of the problem.

Model. We consider a JJ created by depositing two s-wave superconducting (S) films on top of a 2D TI at a distance \( L \) from each other (see Fig. 1). The width of the 2D TI, \( w \), is assumed to be much larger than the total width of the two edge states. Then, each superconducting edge can be described by the quasi-one-dimensional Bogoliubov-de Gennes (BdG) Hamiltonian of the form

\[
H_{\text{BdG}} = \begin{bmatrix} h(x) & i\sigma_y \Delta(x) e^{i\varphi_0(x)} \\ -i\sigma_y \Delta(x) e^{-i\varphi_0(x)} & -h^*(x) \end{bmatrix},
\]

\[
h(x) = v\sigma x \left( -i\hbar \partial_x + \frac{p_x}{2} \right) + U(x) - \mu.
\]

Answering the question about the role of the ABSs in TI JJs may suggest an alternative scheme to detect Majorana fermion states in JJs.

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\]

\[
h(x) = v\sigma x \left( -i\hbar \partial_x + \frac{p_x}{2} \right) + U(x) - \mu.
\]
In the equations above, $h(x)$ is the Hamiltonian for a given edge in the normal state, $\sigma_x$ and $\sigma_y$ denote spin Pauli matrices, $v$ and $\mu$ are the edge velocity and Fermi energy, respectively, whereas the potential $U(x)$ accounts for quasiparticle scattering in the JJ. The off-diagonal entries in $H_{BDG}$ incorporate a spin-singlet $s$-wave pair potential induced in the 2DIT underneath the S contacts. We assume that the superconducting gap $\Delta(x)$ and order-parameter phase $\varphi(x)$ both vary across the JJ as

$$
\Delta(x) = \begin{cases} 
0, & |x| < \frac{L}{2}, \\
\Delta, & |x| \geq \frac{L}{2}, 
\end{cases} \\
\varphi(x) = \begin{cases} 
\pm \frac{\pi}{4}, & x \leq -\frac{L}{2}, \\
\pm \frac{\pi}{4}, & x \geq \frac{L}{2}, 
\end{cases}
$$

(3)

where $\phi_0$ denotes the Josephson phase difference between the $S$ regions in the absence of the external magnetic field. The magnetic field induces a local gradient of the superconducting phase, resulting in a finite Cooper-pair field. The magnetic field induces a local gradient of the magnetic field 

$$
p_s(B) = -\frac{2e}{c} A_x \left( \pm \frac{w}{2} \right) = \pm \frac{2e}{c} \frac{Bw}{\Phi_0},
$$

(4)

where $\pm$ correspond to the upper ($u$) and lower ($l$) edges in Fig. 1 respectively.

The Josephson currents for both upper and lower edges, $J_{u,l}$, can be obtained from the well-known scattering theory formula in the Matsubara representation (see, e.g., Refs. [13,14])

$$
J_{u,l}(\phi_0, B) = -\frac{2e}{\hbar} k_B T \frac{\partial}{\partial \phi_0} \sum_{\omega_n=0}^{\infty} \ln D(\phi_0, B, \epsilon)|_{\epsilon=\omega_n},
$$

(5)

where $\omega_n = (2n+1)\pi k_B T$ are the fermionic Matsubara frequencies for temperature $T$ ($k_B$ is the Boltzmann constant), and $D(\phi_0, B, \epsilon)$ is the characteristic function of energy $\epsilon$ whose zeros yield the energy spectrum of the JJ. Eq. (5) implicitly assumes that quasiparticle poisoning lifts constraints associated with the fermion parity of the ground state. To find $D(\phi_0, B, \epsilon)$ explicitly, one should calculate the determinant of the system of eigenvalue equations obtained by matching the scattering states of $H_{BDG}$ at the boundaries $x = \pm L/2$ and at the scattering region [13]. We have done this straightforward calculation, taking into account the finite condensate momentum $p_s$ and assuming large Fermi energy $\mu \gg |p_s|$. The result is

$$
D(\phi_0, B, \epsilon) = \left( 1 - \alpha_\epsilon \alpha_{\epsilon}^* e^{i \beta_\epsilon - \beta_{\epsilon}} \right)^2 - T \left( \alpha_\epsilon e^{i \phi_0 + \beta_{\epsilon}} - \alpha_{\epsilon}^* e^{-i \phi_0 - \beta_{\epsilon}} \right)^2.
$$

(7)

Here we use the subscripts $\geq$ to indicate the quasiparticle momentum direction – downstream ($>$) or upstream ($<$) – with respect to the condensate flow. In particular, $\alpha_\epsilon$ are the amplitudes of Andreev reflection for the downstream and upstream moving quasiparticles:

$$
\alpha_\epsilon = \frac{\Delta}{\epsilon + i \sqrt{\Delta^2 - \epsilon^2}}, \quad \epsilon = \epsilon \mp \frac{v p_s}{2},
$$

(8)

while $\beta_\epsilon$ are the phase differences between particle and hole acquired in the normal region for the down- and upstream momentum directions:

$$
\beta_\epsilon = \frac{2e}{\hbar v} L = \frac{2e}{\hbar v} k_s L, \quad k_s = p_s/\hbar.
$$

(9)

Importantly, $\alpha_\epsilon \neq \alpha_\epsilon^*$ and $\beta_\epsilon \neq \beta_\epsilon^*$ because the energies of the down- and upstream moving quasiparticles, $\epsilon_\epsilon$, acquire opposite-sign shifts $\pm v p_s/2$, a phenomenon analogous to the Doppler effect. This is a qualitatively new ingredient of the model. For $p_s = 0$, Eqs. (8) – (9) reproduce the results for a TR-symmetric S/2DTI/S junction (see, e.g., Ref. [8]). We note that the topological protection of the edge states guarantees the perfect transmission $|T| = 1$ in Eq. (7) despite the presence of the potential $U(x)$.17

Inserting Eqs. (8) – (9) into Eq. (5), we arrive at the following expressions for the Josephson currents at the upper and lower edges:

$$
J_u(\phi_0, B) = -\frac{2e}{\hbar} k_B T \sum_{n=0}^{\infty} \frac{A_n^2(-B)e^{i\phi_0} - A_n^2(B)e^{-i\phi}}{1 + A_n(-B)A_n(B)} + \frac{|A_n(-B)e^{i\phi/2} - A_n(B)e^{-i\phi/2}|^2}{2}.
$$

(10)

where the Josephson phase difference, $\phi$, includes the contribution of the external magnetic field,

$$
\phi = \phi_0 + k_s L = \phi_0 + \frac{\Phi}{\Phi_0}, \quad \Phi = BLw,
$$

(11)

proportional to the flux, $\Phi$, piercing the area of the weak link, $Lw$, while the coefficients $A_n(B)$ absorb theAndreev reflection amplitude together with the dynamical phase factor $e^{i\phi L/\hbar v}$, both taken at imaginary energy
$\epsilon = i\omega_n$:

$$A_n(B) = \frac{\Delta_0 - \omega_n L/\hbar}{\omega_n + \frac{i}{2} vp_s(B) + \sqrt{[\omega_n + \frac{i}{2} vp_s(B)]^2 + \Delta^2}}$$  \hspace{1cm} (12)

The two current edges differ only by the sign of the momentum $p_s$ [see Eq. (4)], which yields the second relation in Eq. (10).

Eqs. (10) – (12) are our main results and merit a few comments here. As we can see, the external magnetic field has a two-fold effect on the supercurrent. On the one hand, it leads to oscillations of the current with the magnetic field, the junction current-phase relation where

$$\phi \approx \Phi_0 \cos(\pi k_B T),$$

or, explicitly,

$$B \geq B_{AR} = \frac{2\Phi_0}{\pi w \xi_*}, \quad \xi_* = \min \left(\frac{\hbar}{\Delta}, \frac{\hbar}{\pi k_B T}\right),$$  \hspace{1cm} (14)

where $B_{AR}$ is the characteristic field for the suppression of Andreev reflection. This field scale competes with $B_{osc} = \Phi_0/\pi w L$ on which the oscillations occur, violating the periodicity of the currents $J_{u,l}$ with respect to the magnetic flux (field).

We wish to understand how the aforementioned magnetic-field effects manifest themselves in the dc supercurrent. A suitable experimental observable can be defined as follows

$$J_m(B) = |J(\phi_{\text{max}}, B)|,$$  \hspace{1cm} (15)

where $J(\phi_{\text{max}}, B) = J_u(\phi_{\text{max}}, B) + J_l(\phi_{\text{max}}, B)$ is the net Josephson current, and the phase $\phi_{\text{max}}$ is locked to the maximum possible current $J_m = |J(\phi_{\text{max}}, 0)|$ which is an intrinsic characteristic of the device at a given temperature. In the following we discuss $J_m(B)$ in detail for the two most representative cases - the long and short Josephson junctions.

**Long junction.** It is a junction with the separation between the S terminals, $L$, much larger than the superconducting coherence length, $\xi = \hbar/\Delta$. In the absence of the magnetic field, the junction current-phase relation has the well-known sawtooth shape at low temperatures $\pi k_B T \ll \hbar/\ell$ [15], turning sinusoidal in the opposite temperature limit $\pi k_B T \gg \hbar/\ell$ (see Fig. 2). For each temperature we determine the phase difference $\phi_{\text{max}}$ yielding the maximum current $J_m$ and use that as an input for calculating $J_m(B)$ in Eq. (15).

Figure 3 shows the dependence $J_m(B)$ expressed in terms of the flux, $\Phi$, enclosed between the S terminals [20]. At elevated temperatures (see Fig. 3a), we find SQUID-like oscillations due to the interference of two harmonic edge currents (cf. Fig. 2). The peak spacing is approximately $\Phi_0$ for the first two interference lobes, becoming progressively shorter at higher fields. As we mentioned above, the current periodicity is broken by the superimposed reduction of the Andreev reflection amplitude due to the Doppler effect in the magnetic field. For the same reason, the interference lobes decrease in magnitude when the magnetic field approaches the characteristic field $B_{AR}$ which is considerably higher than $B_{osc} = \Phi_0/\pi w L$ for the chosen parameters. From Eqs. (10) and (14) with $\pi k_B T \gg \hbar/\ell$, we derive an analytical formula for the SQUID-like current shown in Fig. 3:

$$J_m(B) \approx \frac{8 e k_B T}{\hbar} \Re \left( A_0^2(B) e^{-i\Phi_0/\Phi_0} \right).$$  \hspace{1cm} (16)

Here $\Re$ means the real part and $A_0(B)$ is the zeroth-order coefficient from Eq. (12). Eq. (16) accounts for the magnetic-field dependence of $A_0(B)$ and is more general than the standard SQUID result $J_m(B) \propto \cos(\pi \Phi/\Phi_0)$. The latter follows from Eq. (16) for $B \ll B_{AR}$.

With decreasing temperature, a $2\Phi_0$-quasiperiodic interference pattern emerge (see Figs. 3b and c). This effect is most pronounced at low temperatures $\pi k_B T \ll \hbar/\ell$ when the current displays a series of alternating peaks with different heights and widths, as shown in Fig. 3b. Without the field dependence of Andreev reflection (i.e. for $B_{AR} \rightarrow \infty$), the alternating peaks would be centered exactly at the even and odd numbers of the flux
quantum. Such an even-odd interference pattern reflects the contribution of ABSs to the Josephson current at these temperatures. In order to see this, let us examine the ABS spectrum which can be obtained from the zeros of the denominator in Eq. (10) as

$$\epsilon^\pm_k(\phi_0, B) = \frac{\hbar v}{2L} [2k] \pm (\phi_0 + \frac{\Phi}{\Phi_0})$$, \hspace{1cm} (17)$$

where \(k = 0, \pm 1, \pm 2... \) is an integer \[21\]. Equation (17) holds for the upper edge; for the lower one we should change \(B \rightarrow -B\). The ABS spectrum is \(2\Phi_0\)-periodic in the magnetic flux, which, along with the superimposed field dependence of Andreev reflection, accounts for the \(2\Phi_0\)-quasiperiodicity of the interference pattern. To elaborate on that, in Fig. 3i we plot the current-flux relations for the upper and lower edges separately [see Eq. (10)]. The two currents have non-sinusoidal sawtooth profiles reproducing the flux dependence of the ABSs. Such sawtooth currents tend to cancel each other except for the narrow regions near the current jumps (indicated by the shaded areas) where both currents flow in the same direction. In those regions \(J_m(B)\) reaches the highest peaks. Remarkably, the peaks become narrower with decreasing temperature due to the increasing sharpness of the sawtooth pattern. This is the hallmark of the gapless ABSs in Eq. (17).

**Short junction.** To complete the physical picture, in the following we discuss the case of a short junction with \(L \ll \xi\). In this case, the current oscillations are strongly suppressed, showing no even-odd effect (see Fig. 1). This happens because the behavior of the current is dominated largely by the magnetic-field dependence of Andreev reflection. Indeed, with decreasing junction length the field \(B_{OS} = \Phi_0/\pi wL\) becomes comparable or even larger than \(B_{AR}\). For the interference pattern shown in Fig. 4b, the fields \(B_{OS} \) and \(B_{AR} \) are of the same order, while Fig. 4b corresponds to \(B_{AR} < B_{OS}\). In the latter case, Andreev reflection is suppressed before the first side lobes develop. The corresponding current-flux relations for the upper and lower edges are shown in Fig. 4c.

**Conclusions.** Our findings may have implications for interpreting recent experiments on 2D TI JJs [3, 4]. In particular, in Ref. [4], a gated InAs/GaSb quantum well was used as a Josephson weak link between superconducting Al leads. When subject to a strong gate potential, InAs/GaSb quantum wells exhibit a 2D TI state with helical edge modes [22]. Reference [4] reported a clear even-odd interference pattern of the supercurrent in JJs with an effective length of \(\sim 600\) nm at temperatures of 16, 40, 70, and 100 mK. With increasing temperature, a crossover to a more conventional SQUID-like pattern was observed. This behavior is consistent with our predictions for a 600 nm-long junction with \(\Delta = 0.125\) meV and \(\hbar v = 10\) meV-nm \[23\] for temperatures \(T = 15, 40, 500\) mK (see Fig. 4). The fact that the even-odd pattern emerges only at sufficiently low temperatures indicates that it could be due to the edge ABSs with a sawtooth \(2\Phi_0\)-periodic dependence on the magnetic flux [see Eq. (17)]. It is also worth noting that no even-odd interference patterns have been observed in nontopological superconducting devices, e.g., in carbon nanotube SQUIDS \[24\]. This may be due to potential disorder.
that generates an energy gap in unprotected ABSs, altering their flux dependence and reducing their contribution to the supercurrent. On the contrary, our results for the Josephson current-flux relation are insensitive to potential scattering. Another interesting feature of the topological Josephson currents (10) is that they appear to be independent of the position of the Fermi level $\mu$.

This work was supported by the German Research Foundation (DFG Grant No TK60/1-1, FOR 1162, SPP1666, DFG-JST research unit "Topotronics") and the ENB graduate school "Topological insulators". We also thank Y. Ando, C. W. J. Beenakker, L. I. Glazman, D. Goldhaber-Gordon, T. M. Klapwijk, L. P. Kouwenhoven, A. Levy Yeyati, and L. W. Molenkamp for valuable discussions.

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Equation (11) implies no screening of the external magnetic field by the S films, which is the case if the Pearl penetration length into the films, $\lambda_P = 2\lambda_L/d_S$, is much larger than their half-width $w/2$ (where $\lambda_L$ is the London penetration length of the bulk S material, and $d_S$ is the out-of-plane film dimension). Also, since $w$ is much larger than the transverse dimension of the edge state, the variation of the vector potential on the edge-state width is negligible. For more details, see Refs. [11, 13, 14].

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