Addendum: Level spacings for integrable quantum maps in genus zero

Steve Zelditch
Johns Hopkins University, Baltimore, Maryland 21218
April 29, 1998

In this note we continue our study [Z] of the pair correlation functions (PCF’s)

\[ \rho_n^2(f) = \frac{1}{n^2} \sum_{\ell} |\tilde{f}(\frac{\ell}{n})| \left| \text{Tr} U_n^\ell \right|^2 = \frac{1}{n^2} \sum_{\ell} \tilde{f}(\frac{\ell}{n}) \sum_{k=1}^{n} e^{i n \ell (\alpha \phi(\frac{k}{n}) + \beta \frac{k}{n})} \]

of completely integrable quantum maps over \( \mathbb{CP}^1 \). To be specific, the quantum maps are assumed to have the form \( U_{n,\alpha,\beta} = e^{i n (\alpha \tilde{I} + \beta \tilde{I})} \) where \( \tilde{I} \) is an action operator (i.e. an angular momentum operator) with eigenvalues \( \frac{k}{n} \) (\( k = -n, \ldots, n \)), acting on the quantum Hilbert space \( \mathcal{H}_n \) of nth degree spherical harmonics at Planck constant \( \frac{1}{n} \). Also \( \phi \) is a smooth function satisfying \( \phi'' \neq 0 \) on \([-1, 1]\). Our main result was

**Theorem 0.0.1 (Z)** Let \( n_m = [m (\log m)^4] \). Then for almost all \((\alpha, \beta)\) (in the Lebesgue sense), \( \rho_{n_m}^{2,\alpha,\beta} \to \rho_2^{\text{POISSON}} \) as \( m \to \infty \).

Our aim in this addendum is to strengthen this result to almost everywhere convergence to Poisson along the entire sequence of Planck constants. The price we pay is that the results apply not to the individual \( \rho_n^2 \)'s but to the average

\[ \bar{\rho}_{2,\alpha,\beta}^N := \frac{1}{N} \sum_{n=1}^{N} \rho_{2,\alpha,\beta}^n. \]  

(1)

Here we change the notation from \( \rho_N^2 \) in [Z] to \( \rho_2^2 \) so that \( N \) is reserved for the cumulative PCF \( \bar{\rho}_2^N \) up to level \( N \).

**Theorem 0.0.2** Suppose that \( \phi(x) \) is a polynomial satisfying \( \phi'' \neq 0 \) on \([-1, 1]\). Then, for almost all \((\alpha, \beta)\) we have:

\[ \bar{\rho}_{2,\alpha,\beta}^N \to \rho_2^{\text{POISSON}}. \]

This addendum was motivated by a comparison of the results of [Z] with those of Rudnick-Sarnak [R.S] on the PCF of fractional parts of polynomials. Independently, both [R.S] and [Z] established mean square convergence to Poisson of their respective PCF’s. However, [R.S] went on to prove a.e. convergence. Their technique was first to prove that the local PCF’s \( \rho_{2,\alpha,\beta}^{n_m} \) tend to Poisson almost everywhere along a sparse subsequence \( \{n_m\} \) of Planck constants, and then to show that for \( n \in [n_m, n_{m+1}] \) the oscillation \( \rho_2^n - \rho_2^{n_m} \) was relatively small and hence the full sequence converged to Poisson. This latter step seemed (and still seems) intractable in the quantum maps situation [Z]. The main difference is that the local spectra in [R.S] increase with \( n \) whereas for quantum maps [Z] they change in rather uncontrollable ways. However we can re-establish a parallel to their situation by focussing on the mean PCF’s \( \bar{\rho}_{2,\alpha,\beta}^N \) rather than the individual \( \rho_n^2 \)'s. Our spectra then increase with \( N \) and there is much less oscillation between Planck constants.

As in [R.S], the proof of this last step is based on the use of Weyl estimates of exponential sums and seems limited to polynomial phases. In addition to the Weyl method, it also uses some considerations from the measure theory of continued fractions.

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1 Partially supported by NSF grant #DMS-9703775.
1 Preliminary results on $\tilde{\rho}_2^{\alpha, \beta}$

Up until the last step, the analysis of $\tilde{\rho}_2^{N, \alpha, \beta}$ is analogous to the analysis of $\tilde{\rho}_2^{N, \alpha, \beta}$ in [Z]. As in [Z, Theorem (5.1.1)] we have:

**Theorem 1.0.3** Let $\hat{H}_{\alpha, \beta} = \alpha \phi(\hat{I}) + \beta \hat{I}$ where $|\phi''| \geq C_{\alpha} > 0$ on $[-1, 1]$. Let $\tilde{\rho}_2^{N, \alpha, \beta}$ be as above. Then for any $f$ with $\text{supp} \hat{f}$ compact:

$$\int_{-T}^{T} \int_{-T}^{T} \left| \tilde{\rho}_2^{N, \alpha, \beta}(f) - \rho_2^{\text{POISSON}}(f) \right|^2 d\alpha d\beta = O\left(\frac{(\log N)^2}{N}\right).$$

**Corollary 1.0.4** Let $N_m = \lfloor m(\log m)^4 \rfloor$. Then for almost all $(\alpha, \beta)$ in the Lebesgue sense,

$$\lim_{m \to \infty} \rho_2^{N_m}(f) = \rho_2^{\text{POISSON}}(f).$$

To fill in the gaps in the sparse sequence $\{N_m\}$, consider $\tilde{\rho}_2^{M, \alpha, \beta}$ for $N_m < M < N_{m+1}$. Obviously,

$$\tilde{\rho}_2^{M, \alpha, \beta}(f) - \tilde{\rho}_2^{N_m}(f) = \frac{N_m - M}{M} \rho_2^{N_m}(f) + \frac{1}{M} \sum_{n=N_m}^{M} \rho_2^{n, \alpha, \beta}(f). \quad (2)$$

We have $M - N_m << (N_{m+1} - N_m) \sim (m+1)(\log(m+1)^4) - m(\log m)^4 << (\log m)^4$. So in the first sum $\frac{N_m - M}{M} << m^{-1+\epsilon}$. In the second we have $O((\log m)^4)$ terms. Under the assumption supp $\hat{f} \subset [-1, 1]$ the trivial bound $\tilde{\rho}_2^{n, \alpha, \beta}(f) << n$ already gives

$$\frac{N_m - M}{M} \rho_2^{N_m}(f) + \frac{1}{M} \sum_{n=N_m}^{M} \rho_2^{n, \alpha, \beta}(f) << (M - N_m) << (\log m)^4. \quad (3)$$

So we just need a tiny improvement on the trivial bound to prove that these terms tend to zero. In the following section we will prove that for almost all $(\alpha, \beta)$, $\tilde{\rho}_2^{N, \alpha, \beta}(f) \leq C(\alpha, \beta) n^{-1-\frac{2k-1}{K}}$ where $K = 2^{k-1}$ with $k$ the degree of $\phi$. From this it also follows by standard density arguments that $\tilde{\rho}_2^{N, \alpha, \beta}|[a, b] \to \rho_2^{\text{POISSON}}[a, b]$ for all intervals $[a, b]$. We refer to [R.S] for the details of the density argument.

2 The Main Lemma

The purpose of this section is to prove:

**Lemma 2.0.5** Suppose that $\phi$ is a polynomial of degree $k$ satisfying the hypotheses: (i) $|\phi''| > 0$ and (ii) $|\alpha \phi' + \beta| > 0$ on $[-1, 1]$. Then for any $f \in C_0(\mathbb{R})$ and almost all $(\alpha, \beta)$, we have: $n^2 \tilde{\rho}_2^{n, \alpha, \beta}(f) \leq C(\alpha, \beta)n^{1-\frac{2k-1}{K}}$, where $K = 2^{k-1}$.

Recall that the local PCF’s have the form

$$\tilde{\rho}_2^{n, \alpha, \beta} = \sum_{\ell \in 2\mathbb{Z}} \hat{f}(\ell n) \sum_{k=1}^{n} e(\alpha\ell[\phi]\ell_n + \beta k_n)|^2.$$

Since $\hat{f}$ is compactly supported, the $\ell$-sum runs over an interval of integers of the form $[-Cn, Cn]$ for some $C > 0$. For simplicity of notation, and with no loss of generality, we will assume the sum over $\ell$ runs over the interval $[-n, n]$. Throughout we use the notation $e(x) = e^{2\pi ix}$.
2.1 The quadratic case

The case of quadratic polynomials is more elementary than that of polynomials of general degree and we can prove our main result without analysing continued fraction convergents to $\alpha$. Hence we begin by discussing this case. The relevant exponential sum is

$$\left| \sum_{k=1}^{n} e(\alpha n \ell (\frac{k}{n} + \beta \frac{k}{n})) \right|^2 = \sum_{h=-n}^{n} \sum_{x=1}^{2n} e(\ell h (\alpha \frac{x}{n} + \beta)).$$

For $f$ with $\text{supp} \hat{f}$ in $[-1,1]$ we have

$$n^2 \rho_{2,\alpha,\beta}^n(f) << \left| \sum_{|\ell| \leq n} \sum_{h=-n}^{n} \sum_{x=1}^{2n} e(\ell h (\alpha \frac{x}{n} + \beta)) \right|.$$

The following estimate is weaker than that claimed in the Main Lemma but is sufficient for the proof of the theorem.

**Lemma 2.1.1** Let $\alpha$ be a diophantine number satisfying $|\alpha - \frac{a}{q}| \geq \frac{K(\alpha)}{q^2}$ for any rational number $\frac{a}{q}$. Then for all $\beta$, $\rho_{2,\alpha,\beta}^n(f) << n^{\frac{1}{2} + \epsilon}$.

**Proof:**

We begin with the standard estimate (e.g. [K, Lemma 1])

$$\left| \sum_{x=1}^{2n} e(\ell h (\alpha \frac{x}{n} + \beta)) \right| = \left| \sum_{x=1}^{2n} e(\ell h (\alpha \frac{x}{n})) \right| \leq \min(2n, \frac{1}{2 ||\ell h \frac{2}{a}||}).$$

where $|| \cdot ||$ denotes the distance to the nearest integer. This gives

$$n^2 \rho_{2,\alpha,\beta}^n(f) << \sum_{h=-n}^{n} \sum_{x=-n}^{n} \min(2n, \frac{1}{2 ||\ell h \frac{2}{a}||}).$$

The variable $x = h\ell$ runs over $[-n^2, n^2]$; when $x \neq 0$, the multiplicity $c_x = \# \{(h, \ell) : h\ell = x\}$ is well-known to have order $n^\epsilon$ (e.g [V, Lemma 2.5]). Then there are $2n$ terms where $h\ell = 0$, each contributing $n$ to the sum. Hence,

$$n^2 \rho_{2,\alpha,\beta}^n(f) << n^{2} + n^\epsilon \sum_{x=-n^2}^{n^2} \min(2n, \frac{1}{2 ||\ell h \frac{2}{a}||}). \quad (4)$$

At this point we are close to the well-known estimate (e.g. Korobov [K, Lemma 14])

$$\sum_{x=1}^{Q} \min(P, \frac{1}{||\alpha x + \beta||}) << (1 + \frac{Q}{q})(P + q \log P)$$

where $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$ with $|\theta| < 1$ and with $(a, q) = 1$. In our situation $Q = n^2, P = n$, giving $(1 + \frac{n^2}{q})(n + q \log n)$, but the estimate does not apply because our ‘$\alpha$’ is $\frac{a}{q}$; the rational approximation $\frac{a}{q}n$ to $\frac{a}{q}$ has a remainder of only $\frac{1}{q^2}$ rather than $\frac{1}{(qn)^2}$. This complicates the argument and worsens the resulting estimate.

Since we do not know the continued fraction expansion of $\frac{a}{q}$, we use the rational approximation $\frac{a}{qn} + \frac{\theta}{qn^2}$. It is not necessary that $(a, n) = 1$ so we rewrite $\frac{a}{qn} = \frac{a'}{q'n'}$ with $(a', n') = 1$ (hence $(a', n'q) = 1$). Then

$$\frac{a}{n} = \frac{a'}{n'q} + \frac{\theta}{n'q^2}, \quad (a', n') = 1, \quad |\theta| < 1.$$
Now break up \([-n^2, n^2]\) into blocks of length \(n'q\). There are at most \(\frac{n^2}{n'q} + 1\) such blocks. Hence

\[
n' \sum_{x=-n^2}^{n^2} \min(2n, \frac{1}{2||x/n'||}) \ll n' \sum_{y=0}^{\frac{n^2}{n'q}} \min(2n, \frac{1}{2||x + yq'\theta/n'||}). \tag{5}
\]

The above rational approximation brings

\[
\frac{\alpha x}{n} + yqn'\alpha n = \frac{a' x}{n'q} + \frac{x\theta}{nq} + y\alpha' + \frac{yn'\theta}{nq}.
\]

Hence

\[
||\frac{\alpha x}{n} + yqn'\alpha n|| = ||\frac{a' x}{n'q} + \frac{x\theta}{nq} + \beta||
\]

where \(\beta = \left\{\frac{ya'n}{nq}\right\}\). Write \(\beta = \frac{b(y)}{nq} + \frac{\theta_1}{nq}\) with \(b(y) \in \mathbb{Z}\) and with \(|\theta_1| < 1\). Since \(|x| \leq n'q\) we have

\[
||\frac{a' x + b(y)}{n'q}|| = ||\frac{x}{n} + yqn'\alpha n - \frac{x\theta}{nq} - \frac{\theta_1}{n'q}|| \leq ||\frac{x}{n} + yqn'\alpha n|| + \frac{1}{n'q} + \frac{1}{nq}
\]

The remainder \(\frac{n'}{nq}\) is much larger than occurs in the standard argument and since it is possible that \(n' = n\) we can only be sure that the remainder is \(O\left(\frac{1}{q}\right)\).

Therefore we are only sure that our sum is

\[
\ll \sum_{y=0}^{\frac{n^2}{n'q}} \sum_{x=1}^{n'q} \min(2n, \frac{1}{2||\frac{x}{n'q} + \frac{b(y)}{nq} + O(\frac{1}{q})||})
\]

Since \((a', n'q) = 1\), the numbers \(a'x + b(y)\) run thru a complete residue system modulo \(n'q\) as \(x\) runs thru \(1, \ldots, n'q\). Hence, the \(x\)-sum is independent of \(a', b(y)\) and we may rewrite it as

\[
\ll \left(\frac{n^2 + 1}{n'q}\right) \sum_{2 \leq x \leq n'q - 1} \min(2n, \frac{2}{||\frac{x}{n'q} + O(\frac{1}{q})||})
\]

The distance \(||\frac{x}{n'q} + O(\frac{1}{q})||\) can be less than \(\frac{1}{q}\) over the range of terms \(x \in [0, Cn']\) and \(x \in [n'q - Cn', n'q]\) where \(C\) is the implicit constant in \(O(\frac{1}{q})\). For these we must take \(n\) in the minimum. Since there are \(O(n)\) such terms in the \(x\)-sum, their contribution to the entire sum is \(\ll n^2 + \frac{n^2}{n'q}\).

For the remaining terms we use that \(\min(2n, \frac{2}{||\frac{x}{n'q} + O(\frac{1}{q})||})\) is an even function of \(x\) to put the \(x\)-sum in the form

\[
\sum_{Cn' \leq x \leq \frac{n'q}{2}} \min(2n, \frac{2}{||\frac{x}{n'q} + O(\frac{1}{q})||})
\]

The minimum is now surely attained by \(\frac{2}{||\frac{x}{n'q} + O(\frac{1}{q})||}\) and since it stays in the left half of the interval we have

\[
\frac{1}{||\frac{x}{n'q} + O(\frac{1}{q})||} = \frac{1}{\frac{x}{n'q} + O(\frac{1}{q})}
\]

Therefore

\[
\sum_{Cn' \leq x \leq \frac{n'q}{2}} \min(2n, \frac{2}{||\frac{x}{n'q} + O(\frac{1}{q})||}) \ll n'q \sum_{Cn' \leq x \leq \frac{n'q}{2}} \frac{1}{x - O(n')} \ll n'q \log(n'q).
\]
The first parenthetical term is of size \( n^{1+r}/q \) when \( n' = n \) while the trivial bound was \( n \). It is at this point that we must restrict to diophantine numbers satisfying \(|\alpha - \frac{a}{q}| \geq \frac{k(a)}{q} \) for all rational \( \frac{a}{q} \). By Dirichlet’s box principle there exists \( q \leq n' \) and a rational \( \frac{a}{q} \) with \( (a, q) = 1 \) such that \(|\alpha - \frac{a}{q}| \leq \frac{1}{q n' \epsilon} \). It follows that \( q > n'^{-\epsilon} \). Substituting into our estimate, we get

\[
\rho_{2, \alpha, \beta} \ll 1 + \left( \frac{n^{1+r}}{n'} + n^{-2} \right)[n^2 + n' n \log(n)] \ll n^r ((a,n)n^{1-r} + \frac{1}{(a,n)} n^{-1+r}).
\]

Since \( 1 \leq (a,n) \leq n \) the final estimate is

\[
\ll n^r (n^{2-r} + n^{-1+r}).
\]

The terms balance when \( r = \frac{3}{2} \) to give

\[
\rho_{2, \alpha, \beta}(f) \ll n^r. 
\]

**Remark** In the next section we will see that there are rational numbers \( \frac{a}{q} \) satisfying the above requirements and also satisfying \( (a,n) \leq C(\alpha)n^r \). This changes the final estimate to \( \ll n^r (n^{1-r} + n^{-1+r}) \) and gives \( \rho_{2, \alpha, \beta}(f) \ll n^r \).

### 2.2 The general polynomial case

Now let \( \phi(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \ldots + \alpha_k \) be a general polynomial. We would like to estimate

\[
\rho_{2, \alpha, \beta}(f) = \frac{1}{n^2} \sum_{x \in \mathbb{Z}} \left| f(x) \right|^2 \leq \sum_{x=1}^n \sum_{k=1}^n \sum_{|h_j| < n} e(n \ell \phi(x/n)).
\]

As in the classical Weyl inequality (cf. [V, Lemma 2.4]) we will estimate \( \left| \sum_{x=1}^n e(n \ell \phi(x/n)) \right|^2 \) by squaring and differencing repeatedly until we reach the linear case. Let \( \Delta_j \) be the jth iterate of the forward difference operator, so that

\[
\Delta_j \phi(x; h) = \phi(x + h) - \phi(x)
\]

\[
\Delta_{j+1} \phi(x; h_1, \ldots, h_{j+1}) = \Delta_1 (\Delta_j \phi(x; h_1, \ldots, h_{j+1})).
\]

We recall (cf. [V, Lemma 2.3]):

**Lemma 2.2.1** We have

\[
\left| \sum_{x=1}^n e(f(x)) \right|^2 \leq (2n)^{2^j - 1} \sum_{|h_1| < n} \cdots \sum_{|h_j| < n} \left| \sum_{x \in I_j} e(\Delta_j f(x; h_1, \ldots, h_j)) \right|
\]

where the intervals \( I_j = I_j(h_1, \ldots, h_j) \) satisfy \( I_1 \subset [1, n], I_j \subset I_{j-1} \).
Now let

\[ T(\phi; n, \ell) = \sum_{x=1}^{n} e(n\phi(\frac{x}{n})) \]

with \( \phi(x) = \alpha_o x^k + \ldots + \alpha_o \) and put \( K = 2^{k-1} \). Apply the previous lemma with \( j = k - 1 \) to get:

\[ |T(\phi; n, \ell)|^K < n^{K-k} \]

\[ \sum_{h_1} \cdots \sum_{h_{k-1}} \sum_{x \in I_{k-1}} e(h_1 \ldots h_{k-1} \ell p_{k-1}(x; h_1, \ldots, h_{k-1}; n, \ell)). \]

Here, the sum runs over \( h \) with \(|h| \leq n \) and

\[ p_{k-1}(x; h_1, \ldots, h_{k-1}; n) = k! n^{-k+1} \alpha_o (x + \frac{1}{2} h_1 + \ldots + \frac{1}{2} h_{k-1}) + (k - 1)! n^{-k+2} \alpha_1. \]

This is just as in the standard Weyl estimate ([V][D, §3]) except for the powers of \( n \) in the coefficients of \( p_{k-1} \).

Then write

\[ \rho_2^p(f) = \frac{1}{n} \sum_{\ell} \sqrt{\frac{n}{\ell}} \left| T(\phi; n, \ell) \right|^2 \ll \frac{1}{n} \sum_{\ell \leq n} \left( \frac{1}{n} \left| T(\phi; n, \ell) \right|^2 \right) \]

(6)

Since the \( \ell \)-sum is an average, we may apply Holder's inequality with exponent \( \frac{2}{2} \) to get

\[ \rho_2^p(f) \ll \frac{1}{n} \sum_{\ell \leq n} \left| T(\phi; n, \ell) \right|^2 \]

(7)

Therefore

\[ \left| \rho_2^p(f) \right|^\frac{2}{2} \ll n^{K-k} n^{-\frac{d}{2}} \sum_{\ell \leq n} \cdots \sum_{h_{k-1}} \sum_{x \in I_{k-1}} e(h_1 \ldots h_{k-1} \ell p_{k-1}(x; h_1, \ldots, h_{k-1}; n)). \]

There are \( n^{k-1} \) terms with \( h_1 \ldots h_{k-1} \ell = 0 \), each contributing \( n \) to the \( x \)-sum. So the contributions of such terms to the total sum is \( O(n^k) \), and we get

\[ \left| \rho_2^p(f) \right|^\frac{2}{2} \ll n^{k-1} \sum_{\ell \leq n} \cdots \sum_{h_{k-1}} \sum_{x \in I_{k-1}} e(h_1 \ldots h_{k-1} \ell p_{k-1}(x; h_1, \ldots, h_{k-1}; n)) \]

(8)

where the primed sum runs only over non-zero values of \( h_1 \ldots h_{k-1} \ell \).

As in the case with \( k = 2 \) above we sum over \( x \) to get

\[ \left| \rho_2^p(f) \right|^\frac{2}{2} \ll n^{k-1} \sum_{\ell \leq n} \cdots \sum_{h} \sum_{x} \min(n, \frac{1}{\|k! h_1 \ldots h_{k-1} \ell n^{-k+1} \alpha\|}) \]

(9)

and then rewrite the variable \( k! h_1 \ldots h_{k-1} \ell \) as a new variable \( x \) ranging over \([0, k! n^k]\). As before, the number \( c_x \) of ways of representing \( x \neq 0 \) as a product \( k! h_1 \ldots h_{k-1} \ell \) is \( O(n^r) \) so

\[ \left| \rho_2^p(f) \right|^\frac{2}{2} \ll n^{k-1} [k^{n^k} \sum_{x \leq k! n^k} \min(n, \frac{1}{\|x n^{-k+1} \alpha\|})]. \]

(10)
We now repeat the steps of the quadratic case but with $\frac{\alpha}{n}$ replacing $\frac{\alpha}{n^2}$. Thus, the rational approximation $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$ gives the approximation $\frac{\alpha}{n^2} = \frac{a}{n} + \frac{\theta}{n^2}$ and hence requires us to break up the sum over $[0, k!n^k]$ into blocks of size $n^{k-1}q/(a, n^{k-1})$. Precisely the same argument (with $n'_k = \frac{n^{k-1}}{(a, n^{k-1})}$) then gives

$$\sum_{x \leq kn^k} \min(n, \frac{1}{|nx^{-k+1}a|}) \ll (n^k \frac{q}{qq'_k} + 1)(n^k + qn'_k \log(qn'_k)).$$

Hence we get

$$[p_2^n(f)]^{\frac{1}{k}} \ll n^{\frac{k}{2} - k - 1 + \varepsilon} n^k + (n^k \frac{q}{qq'_k} + 1)(n^k + qn'_k \log(qn'_k)) \ll n^{\frac{k}{2} - k - 1 + \varepsilon} n^k + n^{2k} qn'_k + qn'_k]. \quad (11)$$

Recalling that $n'_k = n^{k-1}/(a, n^{k-1})$, the last expression is

$$\ll n^{\frac{k}{2} - 1 + \varepsilon} [1 + n(a, n^{k-1}) n^k qn'_k + qn'_k n(a, n^{k-1})].$$

Thus,

$$[p_2^n(f)] \ll n^{1 - \varepsilon} [1 + n(a, n^{k-1}) n^k qn'_k + \frac{q}{n(a, n^{k-1})} n^k]. \quad (12)$$

The exponent of the right side will be less than one if and only if the exponent of $[1 + n(a, n^{k-1}) n^k qn'_k + \frac{q}{n(a, n^{k-1})} n^k]$ is less than one. Thus we are in very much the same situation as in the quadratic case (although the resulting exponent will be increasingly bad as $K \to \infty$). However, the estimate $(a, n) \leq n$ used in the quadratic case does not generalize well to higher degree: In higher degree, the estimate $(a, n^k) \leq n^{k-1}$ leads to $r = \frac{k+1}{2}$ and an exponent larger than one. Therefore we need to choose a rational approximation satisfying $(a, q) = 1$ and $|\alpha - \frac{a}{q}| < \frac{1}{q^2}$ and with low value of $(a, n^{k-1})$. The natural candidates for such numbers are the continued fraction convergents $\frac{\alpha}{m} = [a_0, a_1, \ldots, a_m]$ to $\alpha = [a_0, a_1, \ldots]$. Therefore we need to study the behaviour of

$$f_n(\alpha) := \min\left\{ \frac{n(p_m(\alpha), n^{k-1})}{m(\alpha)} + \frac{q_m(\alpha)}{n(p_m(\alpha), n^{k-1})} \right\}. \quad (13)$$

Since $\frac{\alpha}{q_m} = \alpha + O\left(\frac{1}{q_m}\right)$ we can (and will) replace the $q_m$ in this definition by $p_m$. Since it is presumably hard to arrange for $(p_m(\alpha), n^{k-1})$ to be large, we will require that $p_m(\alpha) \in [n^{r-\varepsilon}, n^r]$ for some exponent $r$ to be determined later. Before proceeding let us recall how the index $m$ is related to $n, r$.

**Proposition 2.2.2** For any $r, \varepsilon > 0$, any $M \in \mathbb{N}$ and almost any $\alpha \in \mathbb{R}$, there exists $n_o \in \mathbb{N}$ with the following property: for $n \geq n_o$ there exist at least $M$ consecutive convergents $p_m-1(\alpha), p_m-2(\alpha), \ldots, p_m \in [n^{r-\varepsilon}, n^r]$ with $m \leq C(\alpha) \log n$.

**Proof:** By a theorem of Khinchin and Levy [Kh], one knows that for almost all $\alpha$ the convergents satisfy

$$\lim_{m \to \infty} q_m^{-\frac{1}{r}} = \gamma; \quad \gamma := \frac{\pi^2}{12 \log 12}. \quad (14)$$

The first claim is equivalent to the statement that there exists $m$ such that, for $0 \leq j \leq M$,

$$(r - \varepsilon) \log n < \log p_m-j = m \log \gamma + o(m) < r \log n.$$ 

Evidently there exists $C(\alpha) > 0$ such that $m \leq rC(\alpha) \log n$, proving the second claim. The first claim is states that for sufficiently large $n$, there are at least $k$ consecutive solutions $m$ of

$$\left[\frac{(r-\varepsilon)}{\gamma} + o(1)\right] \log n \leq m \leq \left[\frac{r}{\gamma} + o(1)\right] \log n.$$
This is obvious since the width of the interval equals \([\frac{1}{2} + o(1)] \log n\), which is positive and unbounded. ■

We then have:

**Proposition 2.2.3** Fix \(k, r, \varepsilon > 0\). Then for almost all \(\alpha \in \mathbb{R}\) there exists a convergent \(\frac{p_m(\alpha)}{q_m(\alpha)}\) with \(p_m(\alpha) \in [n^{r-\varepsilon}, n^r]\) and with \((p_m(\alpha), n^{k-1}) \leq n^\varepsilon\).

**Proof** By the previous proposition, for any \(M > 0\), there are at least \(M\) consecutive \(p_m\)'s in \([n^{r-\varepsilon}, n^r]\) for sufficiently large \(n\). Our goal is to find one satisfying \((p_m(\alpha), n^{k-1}) \leq n^{1+\varepsilon}\).

To this end we recall [Kh] that

\[
\begin{cases}
p_m = a_mp_{m-1} + p_{m-2} \\
q_m = a_qq_{m-1} + q_{m-2}
\end{cases}
\]

and hence that \(p_mq_{m-1} - p_{m-1}q_m = \pm 1\). It follows that \((p_m(\alpha), p_{m-1}(\alpha))\) are relatively prime. This pattern continues in a sufficiently useful way. By a simple induction we find that for \(k < m\),

\[
p_mq_{m-k} - p_{m-k}q_k = \pm E_{k-1}(a_m, a_{m-1}, \ldots, a_{m-k+1})
\]

where \(E_0 = 1, E_1(a_m) = a_m, E_2(a_m, a_{m-1}) = a_ma_m + 1\) and where

\[
E_k(a_m, a_{m-1}, \ldots, a_{m-k}) = a_{m-k}E_{k-1}(a_m, a_{m-1}, \ldots, a_{m-k+1}) + E_{k-2}(a_m, a_{m-1}, \ldots, a_{m-k+2}).
\]

Hence any common divisor of \(p_m, p_{m-1}, p_{m-2}\) is a divisor of \(a_m\), and so on.

We now claim that for the \(M\) consecutive \(p_m\)'s in \([n^{r-\varepsilon}, n^r]\) we have:

\[
(p_m-M, n^{k-1})(p_m-M+1, n^{k-1}) \cdots (p_m, n^{k-1}) \leq n^{k-1} \prod_{j=0}^{M-1} E_{\ell}(a_m, a_{m-j-1}, \ldots, a_{m-j+1})
\]

(16)

The idea of the argument is that, were all the \(p_{m-j}\)'s relatively prime, then each \((p_{m-j}, n^{k-1})\) would contribute a distinct factor of \(n^{k-1}\) and hence the product would be \(\leq n^{k-1}\). The \(p_{m-j}\)'s are of course not relatively prime but (15) gives an upper bound on the greatest common divisors of each pair.

Thus, let us start with \(p_m\) and consider the degree to which factors in \((p_m, n^{k-1})\) are replicated by the lower \((p_{m-j}, n^{k-1})\)'s. Since \((p_m, p_{m-1}) = 1\) there is no duplication of factors due to the nearest neighbor. Since \((p_m, p_{m-2})|a_m\) the greatest common factor of \((p_{m-2}, n^{k-1}), (p_m, n^{k-1})\) is less than \((a_m, n^{k-1})\) and hence less than \(a_m\). Similarly the greatest common factor of \((p_{m-3}, n^{k-1}), (p_m, n^{k-1})\) is less than \(E_2(a_m, a_{m-1})\). In all, the product \((p_m-M, n^{k-1})(p_m-M+1, n^{k-1}) \cdots (p_m, n^{k-1})\) replicates factors of \((p_m, n^{k-1})\) by at most \(E_1(a_m) \cdots E_M(a_m, a_{m-1}, \ldots, a_{m-M+1})\).

Next, move on to \((p_m-1, n^{k-1})\). These factors of \(n^{k-1}\) can get duplicated in \((p_m-3, n^{k-1})\) and so on down to \((p_m-k, n^{k-1})\). One gets a similar estimate as in the first case but with the indices lowered by one. Proceeding down to \((p_m-M, n^{k-1})\) proves the claim.

To complete the proof of the proposition, we use another fact from the metric theory of continued fractions [Kh, Theorem 30]: For almost any \(\alpha \in \mathbb{R}\), there exists \(C(\alpha) > 0\) such that \(a_m(\alpha) \leq C(\alpha)m^{1+\varepsilon}\). By Proposition (2.2.2), the relevant values of \(m\) are of order \(\log(n)\). Therefore, for the \(p_m, p_{m-1}, \ldots, p_{m-M}\) under consideration we have \(a_{m-j} << \log n\). Since \(E_\ell\) is a polynomial in the \(a_{m-j}\)'s of degree \(\ell\), we have

\[E_\ell(a_m, a_{m-j-1}, \ldots, a_{m-j+1}) << (\log n)^\ell.\]

Therefore

\[
\prod_{j=0}^{M-1} E_{\ell}(a_m, a_{m-j-1}, \ldots, a_{m-j+1}) << (\log n)^M.
\]

It follows that

\[
\prod_{j=0}^{M} (p_m-j, n^{k-1}) \leq C(\alpha)n^{k-1}(\log n)^M.
\]
Hence at least one factor must be $\leq C(\alpha)^{1/M} n^{-M} (\log n)^M$. The proposition follows from the fact that $M$ can be arbitrarily large. ■

We now complete the proof of the lemma and of our main result. We have proved the existence of $(p_m, q_m)$ with all the necessary properties and such that $q_m \in [n^{r-\epsilon}, n^r], (p_m, n^{k-1}) << n^r$. It follows that

$$\frac{n(p_m, n^{k-1})}{q_m} + \frac{q_m}{n(p_m, n^{k-1})} << n^{1+\epsilon-r} + n^{-1}.$$  \hspace{1cm} (19)

The terms balance when $r = 1/2$ and give the power $n^{1r}$. It follows from (12) that $\rho_n^p(f) << n^{1-2K+\epsilon}$. ■

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