Log-convex and Stieltjes moment sequences

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Abstract
We show that Stieltjes moment sequences are infinitely log-convex, which parallels a famous result that (finite) Pólya frequency sequences are infinitely log-concave. We introduce the concept of \(q\)-Stieltjes moment sequences of polynomials and show that many well-known polynomials in combinatorics are such sequences. We provide a criterion for linear transformations and convolutions preserving Stieltjes moment sequences. Many well-known combinatorial sequences are shown to be Stieltjes moment sequences in a unified approach and therefore infinitely log-convex, which in particular settles a conjecture of Chen and Xia about the infinite log-convexity of the Schröder numbers. We also list some interesting problems and conjectures about the log-convexity and the Stieltjes moment property of the (generalized) Apéry numbers.

Keywords: Log-convex sequence, Stieltjes moment sequence, Totally positive matrix

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1. Introduction

Let \(\alpha = (a_k)_{k \geq 0}\) be a sequence of nonnegative numbers. The sequence is called log-convex (log-concave, resp.) if \(a_k a_{k+2} \geq a_{k+1}^2\) (\(a_k a_{k+2} \leq a_{k+1}^2\), resp.) for all \(k \geq 0\). The log-convex and log-concave sequences arise often in combinatorics and have been extensively investigated. We refer the reader

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to [7, 23, 25] for the log-concavity and [17, 27] for the log-convexity. A basic approach to such problems comes from the theory of total positivity [6–11].

Let $A = [a_{n,k}]_{n,k \geq 0}$ be a finite or infinite matrix of real numbers. It is called totally positive (TP for short) if all its minors are nonnegative. It is called TP$_2$ if all minors of order $\leq 2$ are nonnegative. Given a sequence $\alpha = (a_k)_{k \geq 0}$, define its Toeplitz matrix $T(\alpha)$ and Hankel matrix $H(\alpha)$ by

$$T(\alpha) = [a_{i-j}]_{i,j \geq 0} = \begin{bmatrix} a_0 \\ a_1 & a_0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$H(\alpha) = [a_{i+j}]_{i,j \geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$}

Clearly, a sequence of positive numbers is log-concave (log-convex, resp.) if and only if the corresponding Toeplitz matrix (Hankel matrix, resp.) is TP$_2$.

We say that $\alpha$ is a Pólya frequency sequence (PF for short) if its Toeplitz matrix $T(\alpha)$ is TP. Such sequences have been deeply studied in the theory of total positivity [15] and in combinatorics [6]. For example, the fundamental representation theorem of Schoenberg and Edrei states that a sequence $a_0 = 1, a_1, a_2, \ldots$ of real numbers is PF if and only if its generating function has the form

$$\sum_{k \geq 0} a_k x^k = e^{\gamma z} \frac{\prod_{j \geq 1}(1 + \alpha_j z)}{\prod_{j \geq 1}(1 - \beta_j z)}$$

in some open disk centered at the origin, where $\alpha_j, \beta_j, \gamma \geq 0$ and $\sum_{j \geq 1}(\alpha_j + \beta_j) < +\infty$ (see [15, p. 412] for instance). In particular, a finite sequence of nonnegative numbers is PF if and only if its generating function has only real zeros [15, p. 399].

We say that $\alpha = (a_k)_{k \geq 0}$ is a Stieltjes moment (SM for short) sequence if its Hankel matrix $H(\alpha)$ is TP. It is well known that $\alpha$ is a Stieltjes moment...
sequence if and only if it has the form

\[ a_k = \int_0^{+\infty} x^k d\mu(x), \quad (1) \]

where \( \mu \) is a non-negative measure on \([0, +\infty)\) (see [19, Theorem 4.4] for instance). Stieltjes moment problem is one of classical moment problems and arises naturally in many branches of mathematics [21, 26]. It is well known that many counting coefficients form Stieltjes moment sequences, including the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers, the Schröder numbers. See [16] for details.

Boros and Moll [4, p. 157] introduced the concept of the infinite log-concavity. Given a sequence \( \alpha = (a_k)_{k \geq 0} \) of nonnegative numbers, define a new sequence \( \mathcal{L}(\alpha) = (b_k)_{k \geq 0} \) by \( b_0 = a_0^2 \) and \( b_{k+1} = a_{k+1}^2 - a_k a_{k+2} \). Then the sequence \( \alpha \) is log-concave if and only if the sequence \( \mathcal{L}(\alpha) \) is nonnegative, i.e., all \( b_k \) are nonnegative. Call \( \alpha \) infinitely log-concave if \( \mathcal{L}(\alpha) \) is nonnegative for all \( i \geq 1 \), where \( \mathcal{L}^i = \mathcal{L}(\mathcal{L}^{i-1}) \). The following result was conjectured independently by Fisk, Stanley, McNamara and Sagan [18] and shown by Brändén [5].

**Theorem 1.1.** The operator \( \mathcal{L} \) preserves the PF property of finite sequences. A finite Pólya frequency sequence is therefore infinitely log-concave.

Very recently, Chen and Xia [12] introduced the notion of the infinite log-convexity. Let \( \alpha = (a_k)_{k \geq 0} \) be an infinite sequence of positive numbers. Define a new sequence \( \mathcal{L}(\alpha) = (c_k)_{k \geq 0} \) by \( c_k = a_k a_{k+2} - a_{k+1}^2 \). Then \( \alpha \) is log-convex if and only if \( \mathcal{L}(\alpha) \) is nonnegative. Call \( \alpha \) \( m \)-log-convex if \( \mathcal{L}^i(\alpha) \) is nonnegative for all \( 1 \leq i \leq m \) and infinitely log-convex if \( \mathcal{L}^i(\alpha) \) is nonnegative for all \( i \geq 1 \). Chen and Xia [12] showed that some combinatorial sequences, including the Apéry numbers and the Schröder numbers, are \( 2 \)-log-convex via analytic methods. Based on numerical evidence they further suggested the infinite log-convexity of these sequences. However, no non-trivial infinitely log-convex sequences is presented.

In the next section we show that Stieltjes moment sequences are infinitely log-convex, a parallel result to Theorem 1.1. So many famous counting coefficients turn to be infinitely log-convex. For example, the sequence of the large Schröder numbers is a Stieltjes moment sequence, and is therefore infinitely log-convex. In §3, we introduce the concept of \( q \)-Stieltjes moment sequences
of polynomials and show that many well-known polynomials in combinatorics are such sequences, including the Bell polynomials, the Eulerian polynomials, the Narayana polynomials (of type B), the $q$-central Delannoy numbers and the $q$-Schröder numbers. In §4, we provide a criterion for the linear transformations and convolutions preserving Stieltjes moment sequences. The SM properties of many well-known combinatorial sequences are easily followed from this viewpoint. Finally in §5, we list some interesting problems and conjectures about the log-convexity and the SM property of the (generalized) Apéry numbers.

2. Infinitely log-convex sequences

In this section we show that Stieltjes moment sequences are infinitely log-convex. We need the following classical characterization of Stieltjes moment sequences.

Given a sequence $\alpha = (a_k)_{k \geq 0}$, let $\alpha_\uparrow = (a_{k+1})_{k \geq 0}$ denote the shifted sequence of $\alpha$.

Lemma 2.1 ([21, Theorem 1.3]). The sequence $\alpha$ is a Stieltjes moment sequence if and only if both $H(\alpha)$ and $H(\alpha_\uparrow)$ are positive definite matrices.

Corollary 2.2. Let $\alpha$ be a Stieltjes moment sequence. Then $\mathcal{L}(\alpha)$ is positive.

Proof. The positive definiteness of $H(\alpha)$ and $H(\alpha_\uparrow)$ imply that their leading principal minors of order 2 are positive:

$$a_{2k}a_{2k+2} - a_{2k+1}^2 = \det \begin{bmatrix} a_{2k} & a_{2k+1} \\ a_{2k+1} & a_{2k+2} \end{bmatrix} > 0$$

and

$$a_{2k-1}a_{2k+1} - a_{2k}^2 = \det \begin{bmatrix} a_{2k-1} & a_{2k} \\ a_{2k} & a_{2k+1} \end{bmatrix} > 0.$$ 

Thus the sequence $\alpha$ is strictly log-convex:

$$a_ka_{k+2} - a_{k+1}^2 > 0, \quad k = 0, 1, 2, \ldots.$$

In other words, $\mathcal{L}(\alpha)$ is positive. 

\hfill \Box
If the Hankel matrix of a sequence is positive definite, then we say that the sequence is a positive definite sequence, or a Hamburger moment sequence. Such a sequence has the form

$$a_k = \int_{-\infty}^{+\infty} x^k d\mu(x), \quad k = 0, 1, 2, \ldots,$$

where $\mu$ is a positive Borel measure on $(-\infty, +\infty)$ (see [21] for instance). By Lemma 2.1, a sequence $\alpha$ is a Stieltjes moment sequence if and only if both $\alpha$ and $\overline{\alpha}$ are positive definite sequences.

**Theorem 2.3.** The operator $\mathcal{L}$ preserves the SM property. A Stieltjes moment sequence is therefore infinitely log-convex.

**Proof.** Let $A = [a_{ij}]_{0 \leq i, j \leq n}$ be an $n \times n$ matrix. The compound matrix $C(A)$ of $A$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix, whose elements are all minors of order 2 of $A$, arranged lexicographically according to the row and column indices of the minors. The compound operation has the following properties:

(i) $C(A^T) = C^T(A)$;

(ii) $C(AB) = C(A)C(B)$; and

(iii) if $A$ is invertible, then so is $C(A)$.

See [13, p. 1] or [14, p. 21] for details. Clearly, if $A$ is a positive definite matrix, then so is $C(A)$. Indeed, if the matrix $A = P^TP$ is congruent to the identity matrix, then so is its compound matrix $C(A) = C^T(P)C(P)$.

We first show that the operator $\mathcal{L}$ preserves the positive definiteness of sequences. Let $\alpha = (a_k)_{k \geq 0}$ be a positive definite sequence. Then all $H_n(\alpha) = [a_{i+j}]_{0 \leq i, j \leq n}$ are positive definite by the definition. Thus the compound matrix $C(H_n(\alpha))$ is also positive definite. We need to prove that all $H_n(\mathcal{L}(\alpha))$ are positive definite. The key observation behind our proof is that $H_{n-1}(\mathcal{L}(\alpha))$ is a principal submatrix of $C(H_n(\alpha))$ by symmetry. For example, consider the case $n = 3$. Then $H_3(\alpha), C(H_3(\alpha))$ and $H_2(\mathcal{L}(\alpha))$ are

$$
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 \\
a_1 & a_2 & a_3 & a_4 \\
a_2 & a_3 & a_4 & a_5 \\
a_3 & a_4 & a_5 & a_6
\end{pmatrix},
$$
On the other hand, the Delannoy numbers, the factorial numbers and the large Schröder numbers are a Stieltjes moment sequence. For example, Liang et al. [16] showed that

\[
\begin{bmatrix}
a_0 & a_1 & a_0 & a_2 & a_0 & a_3 & a_1 & a_2 & a_1 & a_3 & a_2 & a_3 \\
a_1 & a_2 & a_1 & a_3 & a_1 & a_4 & a_2 & a_3 & a_2 & a_4 & a_3 & a_4 \\
a_0 & a_1 & a_0 & a_2 & a_0 & a_3 & a_1 & a_2 & a_1 & a_3 & a_2 & a_3 \\
a_2 & a_3 & a_2 & a_4 & a_2 & a_5 & a_3 & a_4 & a_3 & a_5 & a_4 & a_5 \\
a_0 & a_1 & a_0 & a_2 & a_0 & a_3 & a_1 & a_2 & a_1 & a_3 & a_2 & a_3 \\
a_3 & a_4 & a_3 & a_5 & a_3 & a_6 & a_4 & a_5 & a_4 & a_6 & a_5 & a_6 \\
a_1 & a_2 & a_1 & a_3 & a_1 & a_4 & a_2 & a_3 & a_2 & a_4 & a_3 & a_4 \\
a_2 & a_3 & a_2 & a_4 & a_2 & a_5 & a_3 & a_4 & a_3 & a_5 & a_4 & a_5 \\
a_1 & a_2 & a_1 & a_3 & a_1 & a_4 & a_2 & a_3 & a_2 & a_4 & a_3 & a_4 \\
a_3 & a_4 & a_3 & a_5 & a_3 & a_6 & a_4 & a_5 & a_4 & a_6 & a_5 & a_6 \\
a_2 & a_3 & a_2 & a_4 & a_2 & a_5 & a_3 & a_4 & a_3 & a_5 & a_4 & a_5 \\
a_3 & a_4 & a_3 & a_5 & a_3 & a_6 & a_4 & a_5 & a_4 & a_6 & a_5 & a_6 
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
a_0 & a_1 & a_1 & a_2 & a_2 & a_3 \\
a_1 & a_2 & a_2 & a_3 & a_3 & a_4 \\
a_2 & a_3 & a_3 & a_4 & a_4 & a_5 \\
a_3 & a_4 & a_4 & a_5 & a_5 & a_6 \\
\end{bmatrix}
\]

respectively. Thus the positive definiteness of \( C(H_n(\alpha)) \) implies that of \( H_{n-1}(L'(\alpha)) \). In other words, the positive definiteness of \( \alpha \) implies that of \( L(\alpha) \), as desired.

Now let \( \alpha \) be a Stieltjes moment sequence. Then \( \alpha \) and \( \overline{\alpha} \) are positive definite sequences by Lemma 2.1, so are \( L(\alpha) \) and \( L(\overline{\alpha}) \). Note that \( L(\alpha) = L(\overline{\alpha}) \) is also a Stieltjes moment sequence again by Lemma 2.1. On the other hand, \( L(\alpha) > 0 \) by Corollary 2.2, the sequence \( \alpha \) is therefore infinitely log-convex. This completes the proof of the theorem.

Stieltjes moment sequences are much better behaved than infinitely log-convex sequences and there are various approaches to show that a sequence is a Stieltjes moment sequence. For example, Liang et al. [16] showed that many Catalan-like numbers form Stieltjes moment sequences via the total positivity of the corresponding Aigner’s recursive matrices [1], including the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the factorial numbers and the large Schröder numbers.
These Catalan-like numbers are therefore infinitely log-convex, which, in particular, settles a conjecture of Chen and Xia about the infinite log-convexity of the Schröder numbers \[12, Conjecture 5.4\].

3. Stieltjes moment sequences of polynomials

Let \( f(q) \) and \( g(q) \) be two real polynomials in \( q \). We say that \( f(q) \) is \( q \)-nonnegative if \( f(q) \) has nonnegative coefficients. Denote \( f(q) \geq_q g(q) \) if \( f(q) - g(q) \) is \( q \)-nonnegative. Let \( A(q) = [a_{n,k}(q)]_{n,k \geq 0} \) be a matrix whose entries are all real polynomials in \( q \). We say that \( A(q) \) is \( q \)-TP if all minors are \( q \)-nonnegative. Let \( \alpha(q) = (a_n(q))_{n \geq 0} \) be a sequence of real polynomials in \( q \). We say that the sequence is strongly \( q \)-log-convex (\( q \)-SLCX for short) if

\[
a_{n+1}(q)a_{m-1}(q) \geq_q a_n(q)a_m(q)
\]

for \( n \geq m \geq 0 \). If the Hankel matrix \( H(\alpha(q)) = [a_{i+j}(q)]_{i,j \geq 0} \) is \( q \)-TP, then we say that \( \alpha(q) \) is a \( q \)-Stieltjes moment (\( q \)-SM for short) sequence. If \( \alpha(q) \) is a Stieltjes moment sequence for any fixed \( q \geq 0 \), then we say that \( \alpha(q) \) is a pointwise Stieltjes moment (PSM for short) sequence of polynomials. Clearly, a \( q \)-SM sequence is both \( q \)-SLCX and PSM.

The simplest non-trivial \( q \)-SM sequence in combinatorics should be \(((q + 1)^n)_{n \geq 0}\). Zhu \[27\] showed that the Bell polynomials, the Eulerian polynomials, the Narayana polynomials (of type B), the \( q \)-central Delannoy numbers, the \( q \)-Schröder numbers are \( q \)-SLCX. In this section we show that these polynomials actually form \( q \)-SM sequences.

Let \( \sigma = (s_k(q))_{k \geq 0} \) and \( \tau = (t_{k+1}(q))_{k \geq 0} \) be two sequences of polynomials. Define an infinite lower triangular matrix \( R(q) := R^{\sigma,\tau}(q) = [r_{n,k}(q)]_{n,k \geq 0} \) by the recurrence

\[
r_{0,0}(q) = 1,
\]

\[
r_{n+1,k}(q) = r_{n,k-1}(q) + s_k(q)r_{n,k}(q) + t_{k+1}(q)r_{n,k+1}(q),
\]

(2)

where \( r_{n,k}(q) = 0 \) unless \( n \geq k \geq 0 \). Similar to Aigner \[2\], we say that \( R(q) \) is the \( q \)-recursive matrix and \( r_{n,0}(q) \) are the \( q \)-Catalan-like numbers corresponding to \( (\sigma, \tau) \). Call the tridiagonal matrix (Jacobi matrix)

\[
J(q) = \begin{bmatrix}
s_0(q) & 1 & & & \\
t_1(q) & s_1(q) & 1 & & \\
t_2(q) & s_2(q) & 1 & & \\
t_3(q) & s_3(q) & & & \\
& & & & \ddots
\end{bmatrix}
\]
the coefficient matrix of the recurrence \( (2) \).

**Example 3.1.** Many well-known polynomials are the \( q \)-Catalan-like numbers.

(i) The Bell polynomials \( B_n(q) = \sum_{k=0}^{n} S(n, k)q^k \) when \( s_k = k + q \) and \( t_k = kq; \)

(ii) The Eulerian polynomials \( A_n(q) = \sum_{k=0}^{n} A(n, k)q^k \) when \( s_k(q) = (k + 1)q + k \) and \( t_k = k^2 q; \)

(iii) The \( q \)-Schröder numbers \( r_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} q^k \) when \( s_0 = q + 1, s_k = 2q + 1 \) and \( t_k = q(q + 1); \)

(iv) The \( q \)-central Delannoy numbers \( D_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} q^k \) when \( s_k = 1 + 2q, t_1 = 2q(1 + q) \) and \( t_k = q(1 + q); \)

(v) The Narayana polynomials \( N_n(q) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} q^k \) when \( s_0 = q, s_k = 1 + q \) and \( t_k = q; \)

(vi) The Narayana polynomials \( W_n(q) = \sum_{k=0}^{n} \binom{n}{k} q^k \) of type B when \( s_k = 1 + q, t_1 = 2q \) and \( t_k = q \) for \( k > 1. \)

Liang et al. \[16, Theorem 2.1\] showed that if the coefficient matrix of a recursive matrix is TP, then the corresponding Catalan-like numbers form a SM sequence. The method of the proof used in \[16, Theorem 2.1\] can be carried over verbatim to its \( q \)-analogue. Here we omit the details for brevity.

**Theorem 3.2.** If the coefficient matrix \( J(q) \) is \( q \)-TP, then the corresponding \( q \)-Catalan-like numbers \( r_{n,0}(q) \) form a \( q \)-SM sequence.

The following criterion for the total positivity of a tridiagonal matrix will be very useful.

**Lemma 3.3.** Let \( b_n(q) \) and \( c_n(q) \) be all \( q \)-nonnegative. Then the tridiagonal matrix

\[
\begin{bmatrix}
1 & b_1(q) + c_1(q) \\
& 1 & b_2(q) + c_2(q) \\
& & & 1 & b_3(q) + c_3(q) \\
& & & & &\ddots
\end{bmatrix}
\]

is \( q \)-TP.
Proof. We have the decomposition

\[
\begin{bmatrix}
  b_1 + c_1 & 1 & & \\
  b_2 c_1 & b_2 + c_2 & 1 & \\
  & b_3 c_2 & b_3 + c_3 & \\
  & & & \ddots
\end{bmatrix}
= \begin{bmatrix}
  b_1 & 1 & & \\
  & b_2 & 1 & \\
  & & b_3 & \ddots \\
  & & & \ddots
\end{bmatrix}
\begin{bmatrix}
  1 & & & \\
  c_1 & 1 & & \\
  & c_2 & 1 & \\
  & & & \ddots
\end{bmatrix}
\]

Clearly, bidiagonal matrices whose entries are \(q\)-nonnegative are \(q\)-TP, and the product of \(q\)-TP matrices is still \(q\)-TP. So the statement follows.

\[\square\]

**Corollary 3.4.** The six sequences of polynomials in Example 3.1 are all \(q\)-SM.

Proof. It suffices to show that the corresponding coefficient matrices are \(q\)-TP. The \(q\)-total positivity of the first five coefficient matrices may be obtained directly by Lemma 3.3.

(i) For the Bell polynomials, \(b_k = k - 1\) and \(c_k = q\) for \(k \geq 1\).

(ii) For the Eulerian polynomials, \(b_k = (k - 1)q\) and \(c_k = k\) for \(k \geq 1\).

(iii) For the \(q\)-Schröder numbers, \(b_1 = 0, b_{k+1} = q\) and \(c_k = q + 1\) for \(k \geq 1\).

(iv) For the \(q\)-central Delannoy numbers, \(b_1 = 1, b_{k+1} = q + 1\) and \(c_1 = 2q, c_{k+1} = q\).

(v) For the Narayana polynomials, \(b_k = 1\) and \(c_k = q\) for \(k \geq 1\).

(vi) For the Narayana polynomials of type B, the corresponding coefficient matrix is

\[
J(q) = \begin{bmatrix}
  q + 1 & 1 & & \\
  2q & q + 1 & 1 & \\
  & q & q + 1 & 1 \\
  & & & \ddots
\end{bmatrix}
\]

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We show that $J(q)$ is $q$-TP by showing that all minors of $J(P)$ are $q$-nonnegative. Clearly, it suffices to consider the following minors of two kinds:

$$
\begin{vmatrix}
q + 1 & 1 \\
q & q + 1 & 1 \\
\vdots & \ddots & \ddots & 1 \\
q & q + 1
\end{vmatrix}
$$

and

$$
\begin{vmatrix}
q + 1 & 1 \\
2q & q + 1 & 1 \\
\vdots & \ddots & \ddots & 1 \\
q & q + 1
\end{vmatrix}
$$

By an inductive argument, we obtain $d^{(1)}_n(q) = q^n + \cdots + q + 1$ and $d^{(2)}_n(q) = q^n + 1$. It is clear that both of them are $q$-nonnegative, as required. Thus $J(q)$ is $q$-TP.

**Remark 3.5.** Given a sequence of polynomials, sometimes we may construct a $q$-recursive matrix such that these polynomials are precisely the $q$-Catalan-like numbers of this matrix. As an example, consider the Morgan-Voyce polynomials $M_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} q^k$. Define a $q$-recursive matrix $M(q)$ by setting $s_0 = q + 1, s_k = 1, t_1 = q, t_{k+1} = 0$ for $k \geq 1$. Then it is not difficult to verify that $M_n(q)$ is precisely the corresponding $q$-Catalan-like numbers. By Lemma 3.3 the coefficient matrix of $M(q)$ can be decomposed into the product of two bidiagonal matrices with $b_k = 1, c_1 = q$ and $c_{k+1} = 0$ for $k \geq 1$, and is therefore $q$-TP. Thus $M_n(q)$ form a $q$-SM sequence.

**Corollary 3.6.** The following numbers form Stieltjes moment sequences respectively.

1. The Bell numbers $B_n = \sum_{k=0}^{n} S(n, k) = B_n(1)$;
2. The factorial numbers $n! = \sum_{k=0}^{n} A(n, k) = A_n(1)$;
3. The Schröder numbers $r_n = r_n(1)$;
4. The central Delannoy numbers $D_n = D_n(1)$;
(v) The Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n} = N_n(1)$;

(vi) The central binomial coefficients $\binom{2n}{n} = W_n(1)$.

4. Linear transformations and convolutions

In this section we present some sufficient conditions for linear transformations and convolutions that preserve Stieltjes moment sequences. Similar studies have been carried out for Pólya frequency sequences [6], log-concave sequences [25], log-convex sequences [13, 17], as well as Stieltjes moment sequences [3].

Let $A = [a_{n,k}]_{n,k \geq 0}$ be an infinite nonnegative lower triangular matrix. Define the $A$-linear transformation

$$z_n = \sum_{k=0}^{n} a_{n,k} x_k, \quad n = 0, 1, 2, \ldots \quad (3)$$

and the $A$-convolution

$$z_n = \sum_{k=0}^{n} a_{n,k} y_{n-k}, \quad n = 0, 1, 2, \ldots \quad (4)$$

We say that (3) preserves the SM property: if $(x_n)_{n \geq 0}$ is a Stieltjes moment sequence, then so is $(z_n)_{n \geq 0}$. Similarly, we say that (4) preserves the SM property: if both $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are Stieltjes moment sequences, then so is $(z_n)_{n \geq 0}$.

The following is a classic characterization of Stieltjes moment sequences (see, e.g., [21, Theorem 1.3] or [26, p. 132–135]).

**Lemma 4.1.** The sequence $\alpha = (a_n)_{n \geq 0}$ is a Stieltjes moment sequence if and only if

$$\sum_{n=0}^{N} c_n a_n \geq 0$$

for every polynomial

$$\sum_{n=0}^{N} c_n q^n \geq 0$$

on $[0, \infty)$. 

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Our main result in this section is the following.

**Theorem 4.2.** Let \( A_n(q) = \sum_{k=0}^{n} a_{n,k}q^k \) be the \( n \)th row generating function of the triangle \( A \). Assume that \( (A_n(q))_{n \geq 0} \) is a Stieltjes moment sequence for any fixed \( q \geq 0 \). Then both the \( A \)-linear transformation (3) and the \( A \)-convolution (4) preserve the SM property.

**Proof.** It is known \([3, \text{Theorem 6}]\) that if the \( A \)-linear transformation preserves the SM property, then the same goes for the \( A \)-convolution. So it suffices to show that the \( A \)-linear transformation preserves the SM property.

Let the polynomial
\[
\sum_{n=0}^{N} c_n q^n
\]
be nonnegative on \([0, +\infty)\). By the assumption, the sequence
\[
A_n(q) = \sum_{k=0}^{n} a_{n,k}q^k, \quad n = 0, 1, 2, \ldots
\]
is a Stieltjes moment sequence for any fixed \( q \geq 0 \). Hence by Lemma 4.1, we have
\[
\sum_{n=0}^{N} c_n \sum_{k=0}^{n} a_{n,k}q^k \geq 0
\]
for \( q \geq 0 \). Now let \((x_n)_{n \geq 0}\) be a Stieltjes moment sequence. Then
\[
\sum_{n=0}^{N} c_n \sum_{k=0}^{n} a_{n,k}x_k \geq 0
\]
by Lemma 4.1. Thus the sequence
\[
z_n = \sum_{k=0}^{n} a_{n,k}x_k, \quad n = 0, 1, 2, \ldots
\]
is a Stieltjes moment sequence again by Lemma 4.1. In other words, the linear transformation (3) preserves the SM property, as desired.

The \( n \)th row generating function of the Pascal triangle is \( \sum_{k=0}^{n} \binom{n}{k}q^k = (q + 1)^n \). So the following corollary is an immediate consequence of Theorem 4.2 which is due to Pólya and Szegö \([20, \text{Part VII, Theorem 42}]\).
Corollary 4.3. If both \((x_n)_{n \geq 0}\) and \((y_n)_{n \geq 0}\) are Stieltjes moment sequences, then so is their binomial convolution

\[
z_n = \sum_{k=0}^{n} \binom{n}{k} x_k y_{n-k}, \quad n = 0, 1, 2, \ldots
\]

Since a \(q\)-Stieltjes moment sequence of polynomials is a pointwise Stieltjes moment sequence, we have the following result.

Corollary 4.4. The following convolutions preserve the SM properties:

1. \(z_n = \sum_{k=0}^{n} S(n, k) x_k y_{n-k};\)
2. \(z_n = \sum_{k=0}^{n} A(n, k) x_k y_{n-k};\)
3. \(z_n = \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x_k y_{n-k};\)
4. \(z_n = \sum_{k=0}^{n} \left( \frac{n}{k} \right)^2 x_k y_{n-k};\)
5. \(z_n = \sum_{k=0}^{n} \binom{n+k}{n-k} x_k y_{n-k}.\)

Remark 4.5. Taking \(x_k \equiv 1\) and \(y_k \equiv 1\) in Corollary 4.4 (i)-(iv), we obtain the SM property of the Bell numbers \(B_n\), the factorial numbers \(n!\), the Catalan numbers \(C_n\), and the central binomial coefficients \(b(n) = \binom{2n}{n}\) again.

Taking \(y_k \equiv 1, x_k = b(k), C_k\) and 1 in (v), we obtain the SM property of the central Delannoy numbers \(D_n\), the Schröder numbers \(r_n\), and the Fibonacci numbers \(F_{2n+1}\) with odd indices respectively.

5. Conjectures and open problems

The Apéry numbers

\[
A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \ldots
\]

and

\[
B_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \quad n = 0, 1, 2, \ldots
\]

were introduced by Apéry in his famous proof to the irrationality of \(\zeta(3) = \sum_{n \geq 1} n^{-3}\). Chen and Xia [12] proposed the following conjecture.
Conjecture 5.1. Both $(A_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ are infinitely log-convex sequences.

Independently, Sokal [22] suggested the following conjecture.

Conjecture 5.2. Both $(A_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ are Stieltjes moment sequences.

Let $r$ and $s$ be two positive integers. Define the generalized Apéry polynomials

$$A_n(r, s; q) = \sum_{k=0}^{n} \left( \binom{n}{k}^r \binom{n+k}{k}^s q^k \right), \quad n = 0, 1, 2, \ldots$$

A general problem to ask is whether $A_n(r, s; 1)$ form an infinitely log-convex sequence or even a Stieltjes moment sequence. We have known that the $q$-central Delannoy numbers $D_n(q) = A_n(1, 1; q)$ are $q$-SM. Sun [24] conjectured that $A_n(2, 1; q)$ are $q$-log-convex. In which case, $A_n(r, s; q)$ are PSM, $q$-SLCX, or even $q$-SM?

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References

References

[1] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33–51.

[2] M. Aigner, Catalan and other numbers: a recurrent theme, in: H. Crapo, D. Senato (Eds.), Algebraic Combinatorics and Computer Science, Springer, Berlin, 2001, 347–390.

[3] G. Bennett, Hausdorff means and moment sequences, Positivity 15 (2011) 17–48.

[4] G. Boros, V. Moll, Irresistible Integrals, Cambridge University Press, Cambridge, 2004.
[5] P. Brändén, Iterated sequences and the geometry of zeros, J. Reine Angew. Math 658 (2011) 115–131.

[6] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. 413 (1989).

[7] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Contemp. Math. 178 (1994) 71–89.

[8] F. Brenti, Combinatorics and total positivity, J. Combin. Theory Ser. A 71 (1995) 175–218.

[9] F. Brenti, The applications of total positivity to combinatorics, and conversely, Total positivity and its applications (Jaca, 1994), 451–473, Math. Appl., 359, Kluwer Acad. Publ., Dordrecht, 1996.

[10] X. Chen, H.Y.L. Liang, Y. Wang, Total positivity of Riordan arrays, European J. Combin. 46 (2015) 68–74.

[11] X. Chen, H.Y.L. Liang, Y. Wang, Total positivity of recursive matrices, Linear Algebra Appl. 471 (2015) 383–393.

[12] W.Y.C. Chen, E.X.W. Xia, The 2-log-convexity of the Apéry numbers, Proc. Amer. Math. Soc. 139 (2011) 391–400.

[13] H. Davenport, G. Pólya, On the product of two power series, Canad. J. Math. 1 (1949) 1–5.

[14] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[15] S. Karlin, Total Positivity, Vol. I, Stanford University Press, Stanford, 1968.

[16] H.Y.L. Liang, L.-L. Mu, Y. Wang, Catalan-like numbers and Stieltjes moment sequences, Discrete Math. 339 (2016) 484–488.

[17] L.L. Liu, Y. Wang, On the log-convexity of combinatorial sequences, Adv. in Appl. Math. 39 (2007) 453–476.

[18] P.R.W. McNamara, B.E. Sagan, Infinite log-concavity: developments and conjectures, Adv. in Appl. Math. 44 (2010) 1–15.
[19] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, 2010.

[20] G. Pólya, G. Szegö, Problems and Theorems in Analysis, Vol. II, Springer-Verlag, Berlin, 1976.

[21] J.A. Shohat, J.D. Tamarkin, The Problem of Moments, Amer. Math. Soc., New York, 1943.

[22] A. Sokal, Personal communication, 2014.

[23] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. 576 (1989) 500–534.

[24] Z.-W. Sun, List of conjectural series for powers of $\pi$ and other constants, arXiv: 1102.5649.

[25] Y. Wang, Y.-N. Yeh, Log-concavity and LC-positivity, J. Combin. Theory Ser. A 114 (2007) 195–210.

[26] D.V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1946.

[27] B.-X. Zhu, Log-convexity and strong $q$-log-convexity for some triangular arrays, Adv. in Appl. Math. 50 (2013) 595–606.