COMPLEX POLYNOMIAL REPRESENTATION OF $\pi_{n+1}(S^n)$ AND $\pi_{n+2}(S^n)$.

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Abstract. The complex affine quadric $Q^{m} = \{ z \in \mathbb{C}^{m+1} | z_1^2 + ... + z_{m+1}^2 = 1 \}$ deforms by retraction onto $S^m$; this allows us to identify $[Q^k, Q^n]$ and $[S^k, S^n] = \pi_k(S^n)$. Thus one will say that an element of $\pi_k(S^n)$ is complex representable if there exists a complex polynomial map from $Q^k$ to $Q^n$ corresponding to this class. In this Note we show that $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are complex representable.

A map from $X \subset \mathbb{K}^m$ to $Y \subset \mathbb{K}^r$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is called polynomial if it is the restriction to $X$ of a polynomial (or algebraic) map $f : \mathbb{K}^m \to \mathbb{K}^r$ such that $f(X) \subset Y$. Representing elements of each homotopy group $\pi_k(S^n)$, where $S^n = \{ x \in \mathbb{R}^{n+1} | x_1^2 + ... + x_{n+1}^2 = 1 \}$ is the $n$-dimensional sphere, by real polynomial maps is a classic question now. A first result by P.F. Baum (see[1]) suggested a wide affirmative answer but, later on, R. Wood showed that if $k$ is a power of 2 then all polynomial maps from $S^k$ to $S^{k-1}$ are constant. Consequently the real polynomial representation is impossible in many cases among them, for example, the non-trivial element of $\pi_4(S^3)$. Therefore it seems reasonable to consider the complex representation, where this obstacle disappears and nothing changes up to homotopy.

Let $Q^{m-1}$ be the affine quadric in $\mathbb{C}^m$ (often called the complex sphere) defined by the equation $z_1^2 + ... + z_m^2 = 1$. If one set $z_j = x_j + iy_j$, that is to say, if one identifies $\mathbb{C}^m$ to $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ endowed with coordinates $(x, y) = (x_1, ..., x_m, y_1, ..., y_m)$ then $Q^{m-1}$ is given by the equations $x_1^2 + ... + x_m^2 = y_1^2 + ... + y_m^2 + 1$, $x_1 y_1 + ... + x_m y_m = 0$. The map $g(x, y) = ((y_1^2 + ... + y_m^2 + 1)^{-1/2}x, y)$ gives rise to a diffeomorphism between $Q^{m-1}$ and the tangent bundle of the sphere $T S^{m-1} \subset \mathbb{R}^m \times \mathbb{R}^m$. On the other hand $S^{m-1} = Q^{m-1} \cap (\mathbb{R}^m \times \{0\})$, and the homotopy $H(x, y, t) = (((1-t)^2 y_1^2 + ... + (1-t)^2 y_m^2 + 1)^{1/2}(y_1^2 + ... + y_m^2 + 1)^{-1/2}x, (1-t)y)$ shows that
$Q^{m-1}$ deforms by retraction onto $S^{m-1}$, which translates the usual retraction of a vector bundle onto the zero section. Therefore $[Q^k, Q^n]$ is naturally isomorphic to $[S^k, S^n] = \pi_k(S^n)$ under the inclusion of $S^k$ in $Q^k$ and the retraction $H_1 : Q^n \to S^n$ (see [5] for details). From now on the homotopy class of a map from $Q^k$ to $Q^n$ will be regarded as an element of $\pi_k(S^n)$. We will say that an element of $\pi_k(S^n)$ is complex representable if there exists a complex polynomial map from $Q^k$ to $Q^n$ corresponding to this class; $\pi_k(S^n)$ complex representable will mean that all its elements are complex representable. Note that if $f : \mathbb{R}^m \to \mathbb{R}^r$ is a polynomial map and $f(S^{m-1}) \subset S^{r-1}$ then $f(Q^{m-1}) \subset Q^{r-1}$ when $f$ is extended to a polynomial map from $\mathbb{C}^m$ to $\mathbb{C}^r$. In short, real representable implies complex representable.

Few things are known about the complex representation. In [5] it is proved that $\pi_n(S^n)$ and $\pi_{2k+1}(S^{2k})$ are complex representable. In this Note one shows that $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are complex representable for any dimension $n$.

On each $\mathbb{C}^m$ we consider the symmetric bilinear form $b$ defined by $b(z, u) = z_1 u_1 + ... + z_m u_m$ and its associated quadratic form $q(z) = b(z, z)$; then $Q^{m-1} = q^{-1}(1)$. A map $f : \mathbb{C}^m \to \mathbb{C}^r$ will called pseudo-homogeneous of order a natural number $k$ when $q(f(z)) = q(z)^k$ whatever $z \in \mathbb{C}^m$; in this case $f(Q^{m-1}) \subset Q^{r-1}$. Note that order and degree may be quite different. A polynomial map from $\mathbb{C}^m$ to $\mathbb{C}^r$, homogeneous of degree $k$ and sending $f(Q^{m-1})$ onto $Q^{r-1}$, is pseudo-homogeneous of order $k$ as well, but the converse does not hold unless the polynomial map is real.

**Lemma 1.** For every natural number $k \geq 1$ there exist three polynomials in two variables $\rho$, $\beta_1$ and $\beta_2$, the first and the second ones with real coefficients and the third one with imaginary coefficients, such that $(t - s)\rho^2(s, t) + s^{2k-1} = t^k(\beta_1^2(s, t) + \beta_2^2(s, t))$ for any $(s, t) \in \mathbb{C}^2$, and $\rho(1, t) > 0$ when $t \in \mathbb{R}$ and $t \geq 0$.

**Proof.** by Lemma 2 of [3], for each natural number $\ell \geq 0$, there exist two real polynomials in one variable $\varphi_\ell$ and $\lambda_\ell$ such that $(t - 1)\varphi_\ell^2(t) + 1 = t^{\ell+1}\lambda_\ell(t)$, the degree of $\lambda_\ell$ equals $\ell$ and $\varphi_\ell(t) > 0$ whatsoever $t \geq 0$. So the degree of $\varphi_\ell$ equals $\ell$ too. Now set $k = \ell + 1$, $\rho(s, t) = s^\ell \varphi_\ell(t/s)$, $\beta(s, t) = s^\ell \lambda_\ell(t/s)$, $\beta_1 = \beta + (1/4)$ and $\beta_2 = i\beta - (i/4)$. $\square$

Let $f, g : \mathbb{C}^m \to \mathbb{C}^r$ be two polynomial maps pseudo-homogeneous of order $k \geq 1$ and b-orthogonal (that is to say $b(f(z), g(z)) = 0$ for each $z \in \mathbb{C}^m$). Then $f, g : Q^{m-1} \to Q^{r-1}$ are
homotopic by means of the homotopy $\tilde{H}(z,t) = \cos(t\pi/2)f(z) + \sin(t\pi/2)g(z)$ among others. Moreover, for any natural number $\ell$, we may define a polynomial map $F : \mathbb{C}^{m+\ell} = \mathbb{C}^{m} \times \mathbb{C}^{\ell} \to \mathbb{C}^{r+\ell} = \mathbb{C}^{r} \times \mathbb{C}^{\ell}$ by setting:

$$F(z, u) = (\beta_1(z_1^2 + \cdots + z_m^2 + u_1^2 + \cdots + u_{\ell}^2, z_1^2 + \cdots + z_m^2 + u_1^2 + \cdots + u_{\ell}^2, z_1^2 + \cdots + z_m^2)g(z), \rho(z_1^2 + \cdots + z_m^2 + u_1^2 + \cdots + u_{\ell}^2, z_1^2 + \cdots + z_m^2)u)$$

where $\rho$, $\beta_1$ and $\beta_2$ are like in Lemma 1 and $(z, u) = (z_1, \ldots, z_m, u_1, \ldots, u_{\ell})$.

A straightforward calculation shows that $F$ is pseudo-homogeneous of order $2k - 1$.

**Theorem 1.** The element of $\pi_{m+\ell-1}(S^{r+\ell-1})$ represented by $F : Q^{m+\ell-1} \to Q^{r+\ell-1}$ is the (iterated) suspension of the element of $\pi_{m-1}(S^{r-1})$ represented by $f : Q^m \to Q^r$.

**Proof.** Recall that the retraction of $Q^{n-1}$ onto $S^{n-1}$ was given by $H_1(x, y) = ((y_1^2 + \cdots + y_n^2 + 1)^{-1/2} x, 0)$. To calculate the class of $F : Q^{m+\ell-1} \to Q^{r+\ell-1}$ it is enough to consider $H_1 \circ F : S^{m+\ell-1} \to S^{r+\ell-1}$ when $n = m + \ell$. As usual $\mathbb{K}^{n_1}$ is identified to the vector subspace $\mathbb{K}^{n_1} \times \{0\}$ of $\mathbb{K}^{n_1+n_2}$, therefore $S^{n-1}$ and $Q^{n-1}$ are subspaces of $S^{n_1+n_2-1}$ and $Q^{n_1+n_2-1}$ respectively.

The last part of $H_1 \circ F_{|S^{m+\ell-1}}$ equals $\rho(1, z_1^2 + \cdots + z_m^2)u$ multiplied by a real positive function, where $(z, u)$ is real as well. But $\rho(1, z_1^2 + \cdots + z_m^2) > 0$ since $z_1^2 + \cdots + z_m^2$ is real non-negative, therefore $H_1 \circ F : S^{m+\ell-1} \to S^{r+\ell-1}$ sends equator onto equator, north hemisphere onto north hemisphere and south one onto south one. This implies that its homotopy class is the suspension of the class of $H_1 \circ F : S^{m+\ell-2} \to S^{r+\ell-2}$. But, if $\ell \geq 2$, this last map sends equator to equator, north to north and south to south and we can start the process again. In short the class of $H_1 \circ F : S^{m+\ell-1} \to S^{r+\ell-1}$ is the suspension of the one of $H_1 \circ \varphi : S^{m-1} \to S^{r-1}$, where $\varphi : Q^{m-1} \to Q^{r-1}$ is defined by $\varphi(z) = \beta_1(1, 1)f(z) + \beta_2(1, 1)g(z)$; note that $\beta_1^2(1, 1) + \beta_2^2(1, 1) = 1$ as it follows from Lemma 1. For finishing it is enough to show that $f, \varphi : Q^{m-1} \to Q^{r-1}$ are homotopic, which is obvious because $(\beta_1(1, 1), \beta_2(1, 1))$ and $(1, 0)$ belong to $Q^1$ and this last space is path-connected. \(\square\)

Let us show that $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are complex representable. As the unit element of a homotopy group is representable by a constant map and $\pi_2(S^1) = \pi_3(S^1) = 0$ we will assume $n \geq 2$. The Hopf-Whitehead method, applied to complex multiplication, gives us the polynomial map $f(x) = (x_1^2 + x_2^2 - x_3^2, 2x_1x_3 - 2x_2x_4, 2x_1x_4 + 2x_2x_3)$ from $\mathbb{R}^4$ to $\mathbb{R}^3$, sending $S^3$ to $S^2$. 

3
and whose homotopy class spans $\pi_3(S^2) = \mathbb{Z}$ (Hopf map). Composing on the right with the real polynomial maps from $S^3$ to itself, constructed by R. Wood for any topological degree (see theorem 1 of [4]), yields a real representation of $\pi_3(S^2)$; so this group is complex representable too.

On the other hand the extension of $f$ to $\mathbb{C}^4$, which is written $f(z) = (z_1^2 + z_2^2 - z_3^2 - z_4^2; 2z_1z_3 - 2z_2z_4, 2z_1z_4 + 2z_2z_3)$, sends $Q^3$ onto $Q^2$ and is quadratic homogeneous, therefore it is pseudo-homogeneous of order 2. But $g(z) = (2z_1z_4 - 2z_2z_3, 2z_1z_2 + 2z_3z_4, z_2^2 + z_3^2 - z_1^2 - z_4^2)$ is pseudo-homogeneous of order 2 as well and $b$-orthogonal to $f$. Now from Theorem 1 follows that $\pi_{n+1}(S^n) = \mathbb{Z}_2$, $n \geq 3$, is complex representable since the suspension of the class of $f : S^3 \to S^2$ is the non-trivial element of this group.

One has just shown that the non-trivial element of $\pi_4(S^3)$ may be represented by a complex polynomial map $\varphi : \mathbb{C}^5 \to \mathbb{C}^4$ pseudo-homogeneous of order 3. So $f_1 = f \circ \varphi : \mathbb{C}^5 \to \mathbb{C}^3$ and $g_1 = g \circ \varphi : \mathbb{C}^5 \to \mathbb{C}^3$ are pseudo-homogeneous of order 6; moreover they are $b$-orthogonal since $f$ and $g$ were $b$-orthogonal. As $f_* : \pi_4(S^3) \to \pi_4(S^2)$ is an isomorphism, $f_1$ represents the non-trivial element of $\pi_4(S^2)$. But the suspension of this element is the non-trivial element of $\pi_{n+2}(S^n)$, $n \geq 3$ (recall that the image of $S : \pi_4(S^2) \to \pi_5(S^3)$ is the kernel of the Hopf morphism $H : \pi_5(S^3) = \mathbb{Z}_2 \to \pi_3(S^3) = \mathbb{Z}$, so $S$ is an isomorphism). Therefore from Theorem 1 follows that each $\pi_{n+2}(S^n)$ is complex representable. In short:

**Theorem 2.** The homotopy groups $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are complex representable.

**Remark.** a) The non-trivial element of $\pi_5(S^3)$ can be represented by a complex polynomial map $\Phi : \mathbb{C}^6 \to \mathbb{C}^4$ pseudo-homogeneous of order some $k$. So $f_2 = f \circ \Phi : \mathbb{C}^6 \to \mathbb{C}^3$ is pseudo-homogeneous of order $2k$ and represents the non-trivial element of $\pi_5(S^2) = \mathbb{Z}_2$ because $f_* : \pi_5(S^3) \to \pi_5(S^2)$ is an isomorphism. On the other hand $g_2 = g \circ \Phi$ is also pseudo-homogeneous of order $2k$ and $b$-orthogonal to $f_2$ so, by Theorem 1, the suspension of this element is always complex representable.

A routine calculation shows that this suspension never vanishes. In other words the element $\alpha \in \pi_{n+3}(S^n) - \{0\}$, $n \geq 2$, such that $2\alpha = 0$ is complex representable because $\pi_6(S^3) = \mathbb{Z}_{12}$, $\pi_7(S^3) = \mathbb{Z} \oplus \mathbb{Z}_{12}$ and $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$, $n \geq 5$.

b) Since $\pi_1(S^3)$ may be represented by real homogeneous polynomial maps and each of them
has an orthogonal map of the same kind (take \( g = (-f_2, f_1) \) when \( f = (f_1, f_2) \)), by complexifying we can apply Theorem 1; therefore \( \pi_n(S^n) \) is complex representable by means of pseudo-homogeneous maps of order odd. More exactly, if the absolute value of the topological degree of an element of \( \pi_n(S^n) - \{0\} \) equals \( k \), then one can take \( 2k - 1 \) as order (when \( n = 2 \) any topological degree \( \pm k \) can be represented by a complex polynomial map of degree \( 2k - 1 \), which is not pseudo-homogeneous; see[2]).

Note that the odd order is essential for even dimension. Indeed, if \( \varphi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1} \) is pseudo-homogeneous of order \( 2r \) then \( H(z, t) = \gamma(t)^{-2r}\varphi(\gamma(t)z) \), where \( \gamma(t) \) is a path from 1 to \(-1\) in \( \mathbb{C} - \{0\} \), is a homotopy between \( \varphi : Q^n \rightarrow Q^n \) and \( \varphi \circ (-Id) : Q^n \rightarrow Q^n \); consequently the class of \( \varphi \) vanishes for \( n \) even.

c) The Hopf maps from \( S^7 \) and \( S^{15} \) to \( S^4 \) and \( S^8 \) respectively have no \( b \)-orthogonal map of the same kind. Indeed, if not the element \( \alpha \in \pi_8(S^5) \) corresponding to the suspension of the Hopf map from \( S^7 \) to \( S^4 \) (the other case is analogous) can be represented by a pseudo-homogeneous map \( \varphi : Q^8 \rightarrow Q^5 \) of order 3, which is homotopic to \(-\varphi \circ (-Id) : Q^8 \rightarrow Q^8 \) by considering \( H(z, t) = \gamma(t)^{-3}\varphi(\gamma(t)z) \) and the path \( \gamma(t) \) as previously. Therefore \( 2\alpha = 0 \), contradiction because \( \alpha \) spans \( \pi_8(S^5) \).

References

1. P. F. Baum, Quadratic maps and stable homotopy groups of spheres, Illinois J.Math. 11 (1967), 586–595.
2. G.M. Golasiński and F. Gómez Ruiz, Polynomial and regular maps into Grassmannians, K-Theory 26 (2002), 51-58.
3. F.J. Turiel, Polynomial maps and even dimensional spheres, Proc. Amer. Math.Soc. (2007) to appear.
4. R. Wood, Polynomial maps from spheres to spheres, Invent. Math. 5 (1968), 163–168.
5. R. Wood, Polynomial maps of affine quadrics, Bull. London Math. Soc. 25 (1993), 491–497.