General Navier-Stokes-like Momentum and Mass-Energy Equations

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A new system of general Navier-Stokes-like equations is proposed to model electromagnetic flow analogous to hydrodynamic flow. While most attempts to derive analogues of hydrodynamic to electromagnetic flow, and vice versa, start with Navier-Stokes or a Euler approximation, we propose general conservation equations as a starting point. Such equations provide a structured framework from which additional insights into the problem at hand could be obtained. To that end, we propose a system of momentum and mass-energy conservation equations coupled through both momentum density and velocity vectors.

I. BACKGROUND

A. System of Navier-Stokes Equations

Several groups have applied the Navier-Stokes (NS) equations to Electromagnetic (EM) fields through analogies of EM field flows to hydrodynamic fluid flow. Most recently, Boriskina and Reinhard made a hydrodynamic analogy utilizing Euler’s approximation to the Navier-Stokes equation in order to describe their concept of Vortex Nanogear Transmissions (VNT), which arise from complex electromagnetic interactions in plasmonic nanostructures [1]. In 1998, H. Marmanis published a paper that described hydrodynamic turbulence and made direct analogies between components of the NS equation and Maxwell’s equations of electromagnetism[2]. Kambe formulated equations of compressible fluids using analogous Maxwell’s relation and the Euler approximation to the NS equation[3]. Lastly, in a recently published paper John B. Pendry, et. al. developed a general hydrodynamic model approach to plasmonics [4].

In the cases of Kambe and Boriskina, et.al., the groups built their models through analogous Euler-like equations along with relevant mass continuity analogues, respectively shown below.

\[
\frac{Dv}{Dt} = -\frac{\nabla p}{\rho}, \quad (1)
\]

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot v = 0 \quad (2)
\]

where \(v\) is the velocity vector, \(\nabla = \frac{\partial}{\partial x}\) is the del operator, \(p\) is pressure, \(\rho\) is fluid density, and \(D/Dt = \partial/\partial t + v \cdot \nabla\) is a material derivative operator. Kambe assumed the advective term in the momentum equation was zero. He then compared Maxwell’s equations in vector potential form under a Lorenz gage condition through analogy with a fluid under an isentropic flow. Under those conditions, Kambe used the EM vector potential as analogy to the velocity vector in the hydrodynamic model. In a similar fashion, the EM scalar potential was analogous to enthalpy.

In contrast, Boriskina, et. al. assumed steady state in both photon “fluid” velocity and density. Thus, the second term of the momentum equation (1), was equated with the gradient of an internal and external potential obtained through application of a Madelung transformation and, thus, replaced the pressure potential term in the Euler equation. Similarly, the second and third term of the mass continuity equation above was equated with a material loss or gain term times the photon fluid density, in which the EM wave intensity played such role. The EM wave phase gradient became the velocity vector in this case.

Marminis was primarily concerned with describing turbulent hydrodynamics. The application of EM analogy to NS equation of the form below led to a new turbulence description by the introduction of two new hydrodynamic concepts: turbulent charge and turbulent current. Making the assumptions of an incompressible, inviscid fluid led him to offer a theory of metafluid dynamics.

As such, Marminis and others [5, 6] utilized the Navier-Stokes equation (3) to build their EM analogues:

\[
\rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla p + \mu \nabla^2 v + f. \quad (3)
\]

The terms on the left side of the equation represent the fluid’s inertia per volume. The \(\frac{\partial v}{\partial t}\) term represents an unsteady state acceleration, while \(v \cdot \nabla v\) is a non-linear advection term. On the right hand side, the sum of the pressure gradient, \(\nabla p\), and the viscosity, \(\mu \nabla^2 v\), represent the divergence of a stress tensor. Finally, \(f\) represents the sum of all other body forces acting on the system. Equation (3) is the momentum equation that describes fluid flow, while equation (1) is its approximation under zero body forces and inviscid flow, neglecting heat conduction, also termed the Euler approximation.

In the works mentioned above, the approach was to start with a hydrodynamic model, find appropriate analogies to EM field flow, then proceed to work out details applicable to the problem. Our aim is to provide a general starting point from which one can proceed to a solution through the use of appropriate assumptions.*



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B. General Momentum, Mass, Energy Conservation Hydrodynamic Equations

Equation (3) is not in its most general form to describe fluid momentum. A more general equation is the Cauchy Momentum equation into which one substitutes in an appropriate stress tensor and constitutive relations relative to the problem at hand. Such substitution then leads to the NS equation. Making use of the material derivative operator, the Cauchy Momentum Equation is:

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \mathbf{\tau} + \mathbf{f}.$$  \hfill (4)

where $\nabla \cdot \mathbf{\sigma}$ is the divergence of a stress tensor, which can be further broken down into the sum of a pressure tensor, $-\nabla p$, and a deviatoric tensor, $\nabla \cdot \mathbf{\tau}$. Here we have opted to represent tensors as boldface lower-case Greek letters. \cite{7}

Given the above, the question then becomes: What is necessary to generally define a hydrodynamic model obeying Navier-Stokes-type equations. The answer comes in the form of conservation of momentum, mass and energy. In terms of the material derivative operator these three are:

- **Momentum**: $\rho \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \mathbf{\tau} + \mathbf{f}$ \hfill (5)
- **Mass**: $\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0$ \hfill (6)
- **Energy**: $\frac{\partial S}{\partial t} - \frac{Q}{T} = 0$ \hfill (7)

where $Q$ and $T$ are the heat transfer rate and temperature, respectively. The above equations (5-7) plus relative constitutive equations lead to hydrodynamic models for non-relativistic flows within continuum space dynamics.

II. ELECTROMAGNETIC “FLOW” DIFFERENTIAL EQUATIONS

A. General Momentum and Mass-Energy Relations

The general idea behind this paper is to find out if we could apply similar general models to Electromagnetic field ‘fluids’. If we are to apply a hydrodynamic analogy to electromagnetic flow, we must first start with both analogous conservation equations and constitutive relations. Then one can utilize a relevant tensor, velocity vector and fluid density which could be placed into such analogous conservation equations. A first step is to look for a term analogous to the hydrodynamic stress tensor, $\sigma_{ij}$, that can be placed into the Cauchy momentum equation. For EM flow, a logical choice is the Maxwell stress tensor. In this paper, we will choose to use two choices, we, thus, have the Maxwell stress tensor of the form \cite{8}

$$\mathbf{\tau} = \tau_{ij} = \varepsilon_0 [E_i E_j + c^2 B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + c^2 B^2)]$$ \hfill (8)

where $E$ and $B$ are the electric and magnetic fields, respectively, $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum and $\delta_{ij}$ is the Kronecker delta. We have also used $\varepsilon_0$ as permittivity and $\mu_0$ as permeability both of free space. In addition, the third term of equation (8), without $\delta_{ij}$, is the energy density defined as, $u = \frac{1}{2} [\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}]$ where the constitutive relations for $\mathbf{D}$ and $\mathbf{H}$ were used to obtain $u = \varepsilon_0 \frac{1}{2} (E^2 + c^2 B^2)$.

With the Maxwell stress tensor, EM conservation of linear momentum, derived from forces on a charged particle of arbitrary volume traveling through an EM field, is \cite{8,9}:

$$\nabla \cdot \mathbf{\tau} = \frac{\partial \mathbf{g}}{\partial t} + \mathbf{f}$$ \hfill (9)

Here $\mathbf{g}$ is defined as EM field momentum density ($\mathbf{g} \equiv S/c^2$) and $S = \mathbf{E} \times \mathbf{H}$ is the Poynting vector. The $\mathbf{f}$ in this case is the Lorentz force, which under our vacuum assumption will be zero. On the left-hand side of (9), $\nabla \cdot \mathbf{\tau}$ represents the total momentum flowing through the surface of an arbitrary volume per unit time, while on the right-hand side $\partial \mathbf{g}/\partial t$ represents the rate of change of field momentum density within such volume. If we apply the above to equation (4) with $\mathbf{f} = 0$ we get

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \nabla \cdot \mathbf{\tau} = 0$$ \hfill (10)

or, if we substitute in the right hand side of equation (9), again with $\mathbf{f} = 0$,

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{g}}{\partial t} = 0.$$ \hfill (11)

Thus, we have a EM field linear momentum conservation equation in two forms: (10) and (11).

Unfortunately, there is no conservation of mass equation for an EM field. But, the conservation of energy equation with electron current $j = \rho \mathbf{v} = 0$ is given by the following relation \cite{9}:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0$$ \hfill (12)

where $u$ and $S$ are as before.

Now, equations (10-11) are a combination of a Navier-Stokes conservation equation with an EM field conservation equation. Utilizing Einstein’s non-relativistic mass-energy relationship, $E = mc^2$, we can establish another similar combination equation by taking equation (6), after expanding the material derivative operator, multiplying it by $c^2$ and setting it equal to equation (12) to get the following relation

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right\} \times c^2$$ \hfill (13)
dividing by $c^2$ and re-arranging terms we get
\[ \frac{\partial \rho_{\text{em}}}{\partial t} + \nabla \cdot \mathbf{g} - \nabla \cdot (\rho_{\text{ns}} \mathbf{v}) = 0 \] (14)
where $\rho_{\text{em}} = \rho_{\text{ed}} - \rho_{\text{ns}}$ and $\rho_{\text{ed}} = u/c^2$, while $\rho_{\text{ns}}$ is a material density of the medium.

In summary, we have the following two EM field conservation relations here derived that lead to a system of Navier-Stokes-like equations useful in representing an analogy between a hydrodynamic model and EM field equations, whereby additional insight could be gained into optical systems.

**Momentum:** $\rho_{\text{ns}} \frac{D \mathbf{v}}{Dt} = \nabla \cdot \mathbf{g} = \nabla \cdot (\rho_{\text{ns}} \mathbf{v})$ (15)

**Mass-Energy:** $\frac{\partial \rho_{\text{em}}}{\partial t} = -\nabla \cdot \mathbf{g} + \nabla \cdot (\rho_{\text{ns}} \mathbf{v})$ (16)

The momentum equation is a vector equation that describes the time rate of change of the momentum density vector, $\mathbf{g}$, and the velocity vector, $\mathbf{v}$.

### B. Euler-like equation

Interestingly, we obtain a Euler-like approximation for equation (15) when utilizing $\nabla \cdot \mathbf{v}$ on the right-hand side of the equation. First, let us represent the divergence of Maxwell stress tensor in component notation. Since it is a second rank tensor we will have the following,
\[ \nabla \cdot \mathbf{v} = \frac{\partial \tau_{ij}}{\partial x_j} \mathbf{e}_i \]
\[ = \varepsilon_o \left\{ (E_i \frac{\partial E_j}{\partial x_j} + c^2 B_j \frac{\partial B_i}{\partial x_j} - \delta_{ij} \frac{\partial (u/\varepsilon_o)}{\partial x_j}) \right\} \mathbf{e}_i \] (17)
where the $\mathbf{e}_i$ are basis vectors. Therefore, equation (15) in component notation becomes
\[ \rho_{\text{ns}} \left( \frac{\partial \mathbf{v}_i}{\partial t} + v_j \frac{\partial \mathbf{v}_i}{\partial x_j} \right) \]
\[ = \varepsilon_o \left\{ (E_j \frac{\partial E_i}{\partial x_j} + c^2 B_i \frac{\partial B_j}{\partial x_j} \mathbf{e}_i - \frac{\partial (u/\varepsilon_o)}{\partial x_j}) \right\} \mathbf{e}_i \] (18)

or in vector notation
\[ \rho_{\text{ns}} \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) \]
\[ = \varepsilon_o \left\{ (\mathbf{E} \cdot \nabla)\mathbf{E} + c^2 (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla (u/\varepsilon_o) \right\} \] (19)

Since the electromagnetic wave is propagating in a vacuum, from Maxwell’s equations $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$. After rearranging, equation (19) becomes
\[ \rho_{\text{ns}} \frac{D \mathbf{v}}{Dt} = -\nabla u / \rho_{\text{ns}} \] (20)
Now, the energy density term, $u$, on the right hand side of the equation is measured in energy per unit volume, which is also a measure of pressure as a force per unit area. Through simple dimensional analysis one can ascertain that
\[ \text{Energy} \quad \frac{F \cdot d}{A \cdot d} = \frac{F}{A} = P \]
where $F$ is a force, $A$ is unit area and $d$ is distance. As a consequence, the energy density term can be thought of as a pressure component so that we can let $p_{\text{em}} = u$, as a representation of pressure. With this substitution, the analogy with Euler approximation to Navier-Stokes equation (1) is evident as we can write equation (20) as
\[ \frac{D \mathbf{v}}{Dt} = -\nabla p_{\text{em}} / \rho_{\text{ns}} \] (21)
Comparison of equation (21) with Euler equation (1) yields a readily apparent similarity. We have distinguished electromagnetic pressure from NS pressure through the subscript, em.

### C. Lamb vector form

We can make some modifications to the left-hand side of equation (15) that will further allow us to understand its dynamics. First, using the vector identity below we can describe the advection operator in Lamb vector form used quite often in hydrodynamics\[2,10].
\[ (\mathbf{v} \cdot \nabla) \mathbf{v} = (-\mathbf{v} \times \nabla) \times \mathbf{v} + \frac{1}{2} \nabla v^2 \]
\[ = \Lambda + \frac{1}{2} \nabla v^2 \] (22)
where $\Lambda = \Omega \times \mathbf{v}$ is the Lamb vector and $\Omega = \nabla \times \mathbf{v}$ is vorticity. The Lamb vector describes a local Coriolis acceleration of a velocity field due to its vorticity\[10]. Thus we describe equation (15) in Lamb vector form as below.
\[ \rho_{\text{ns}} \left( \frac{\partial \mathbf{v}}{\partial t} + \Lambda + \frac{1}{2} \nabla v^2 \right) = \nabla \cdot \mathbf{v} = \frac{\partial \mathbf{g}}{\partial t} \] (23)
This form gives additional insight into the type of flow the EM field experiences. While obvious, we mention here that the following three expressions are equivalent:
\[ \frac{D \mathbf{v}}{Dt} = \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \left( \frac{\partial \mathbf{v}}{\partial t} + \Lambda + \frac{1}{2} \nabla v^2 \right) \]
\[ = \left( \frac{\partial \mathbf{v}}{\partial t} \right) + (\mathbf{v} \cdot \nabla) \mathbf{v} \]
Yet, another form of the advection operator, $(\mathbf{v} \cdot \nabla) \mathbf{v}$, is the skew symmetric advection operator shown below\[11,12].
\[ \frac{1}{2} (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) \]
It is skew symmetric for fixed non-divergent $\mathbf{v}$ such that $\nabla \cdot \mathbf{v} = 0$. We will make use of this form in our examples to follow in the next section.
III. APPLICATION EXAMPLES

To find the usefulness of these new equations we apply them to a non-dispersive, homogeneous medium and choose steady-state electromagnetic energy density such that $\partial \rho_{em}/\partial t = 0$. Specifically, we have a linearly polarized Gaussian, laser beam traveling in the $z$-direction which encounters a small spherical dielectric particle as in Fig. 1. We begin this section by first describing the EM field then applying the mass-energy relation to find the field’s momentum flux and intensity.

A. EM field momentum flux and intensity

We define an electromagnetic wave traveling in the $z$-direction, with the usual convention that the electric field $E(\mathbf{r})$ is polarized parallel to the $x$ axis, the magnetic field $B(\mathbf{r})$ is perpendicular to $E(\mathbf{r})$ and both are perpendicular to the propagation direction.

$E \perp B$ and $E, B \perp k$

For simplicity, let us take a plane wave described by the following electric and magnetic fields:

$$E = E_0 e^{i(\Phi(\mathbf{r}) - \omega t)}, \quad \text{and} \quad B = B_0 e^{i(\Phi(\mathbf{r}) - \omega t)}$$

(24)

where $\Phi(\mathbf{r}) = k \cdot \mathbf{r}$. Now using the constitutive relation $\mathbf{H} = B/\mu_0$ as before, we can establish the following relationships as in [13] for the time-independent electric and magnetic field vectors at $\mathbf{r}$ given as follows:

$$E(\mathbf{r}) = E(\mathbf{r}) \hat{x}$$

(25)

for which the associated magnetic field vector would be

$$\mathbf{H}(\mathbf{r}) = \hat{z} \times \mu_0 \varepsilon_0 E(\mathbf{r}) = n\varepsilon_0 c E(\mathbf{r}) \hat{y} = H(\mathbf{r}) \hat{y}.$$  

(26)

where $n$ is the refractive index of the medium and other parameters are defined as before.

Now, let us apply equation (16), with our assumptions stated above, where $\partial \rho_{em}/\partial t = 0$ leads to

$$\nabla \cdot \mathbf{g} = \rho_{ns} \nabla \cdot \mathbf{v}.$$  

(27)

Since we have a divergence on both sides, we can rearrange the equation as follows

$$\nabla \cdot (\mathbf{g} - \rho_{ns} \mathbf{v}) = 0$$

from which we get

$$\mathbf{g} = \rho_{ns} \mathbf{v} + \mathbf{b},$$  

(28)

where $\mathbf{b}$ is a constant vector which we can define as $\mathbf{b} = \nabla s$, $s$ being a constant curve. Now, we obtain $\mathbf{g}$, as previously defined, by first evaluating the time-averaged Poynting vector. From its usual definition, the time-averaged Poynting vector is

$$\langle \mathbf{S}(\mathbf{r}) \rangle = \frac{1}{2} \Re[\mathbf{E}^*(\mathbf{r}) \times \mathbf{H}(\mathbf{r})]$$

where we have applied the EM field above to obtain,

$$\langle \mathbf{S} \rangle = \frac{1}{2} n\varepsilon_0 c |E(\mathbf{r})|^2 \hat{z}.$$  

(29)

Upon application of the relationship to momentum flux we get,

$$\mathbf{g} = \frac{n\varepsilon_0}{2c} |E(\mathbf{r})|^2 \hat{z}.$$  

(30)

Equation (30) resembles the Minkowski formulation for momentum flux. We now proceed to define the electromagnetic wave intensity (light intensity) as a function of position $\mathbf{r}$ as below [13]

$$I(\mathbf{r}) = \frac{1}{2} n\varepsilon_0 c |E(\mathbf{r})|^2$$

(31)

so we have $\mathbf{g}$ as

$$\mathbf{g} = \frac{I(\mathbf{r})}{c^2} \hat{z}$$  

(32)

and from (28) we find that

$$\mathbf{v} = \frac{I(\mathbf{r})}{\rho_{ns} c^2} \hat{z} + \frac{\mathbf{b}}{\rho_{ns}}.$$  

(33)

We call Eq. (33) the momentum flux velocity, where the choice of constant vector is arbitrary and we will henceforth ignore it.

B. Bernoulli-like equation

Now we proceed with equation (21) in Lamb vector form so that we have the following:

$$-\frac{\nabla \rho_{em}}{\rho_{ns}} = \frac{\partial \mathbf{v}}{\partial t} + \Lambda + \frac{1}{2} \nabla v^2.$$  

(34)

We choose a constant momentum flux velocity term so that $\partial v/\partial t = 0$, meaning no acceleration of momentum flux is present. Further, we wish to study a “non-turbulent”, irrotational system where $\Omega = 0$. Thus, for an irrotational, steady-state flow, equation (34) becomes

$$\frac{\nabla \rho_{em}}{\rho_{ns}} = -\frac{1}{2} \nabla v^2.$$  

(35)
We can re-write as
\[ \nabla \left( \frac{p_{em}}{\rho_{ns}} + \frac{1}{2} v^2 \right) = 0 \]  
(36)

so that
\[ \frac{p_{em}}{\rho_{ns}} + \frac{1}{2} v^2 = C(\gamma) = (p_{em})_o \]  
(37)

where \( C(\gamma) \) is some constant along a streamline, \( \gamma \), which in the case of irrotational, steady-state flow is \((p_{em})_o\) everywhere. Equation (37) is the electromagnetic equivalent of Bernoulli’s equation.

C. Dielectric particle velocity

We are now interested in calculating the particle velocity, \( v_p \), along the optical axis. Due to non-slip inviscid flow, such velocity will be in some way proportional to the velocity of momentum flux in the system represented in Fig. 1.

As a consequence of constant fluid flow along a streamline, we expect that \( v_1 A_1 = v_2 A_2 \), where \( A_i \) are cross-sectional areas traced out by the laser beam’s gaussian profile and \( v_i \) are particle velocities along the beam’s optical axis as a function of distance from beam waist. As the medium density doesn’t change along the streamline, we have the following relationship, which would be the case along any streamline due to irrotational flow.

\[ v_2 = v_1 \frac{A_1}{A_2} \]  
(38)

In addition, the electromagnetic field influences particle motility to a large extent so that its initial state of motion is small compared to a final state of motion relative to EM phase velocity. As such, particle velocity in a laser beam could seemingly be modeled under a system analogous to Rankine-Hugoniot hydrodynamic shock wave generation in 1-D. Following [14] as analogy, we start with a conservation of mass relation given by equation (6) such that,
\[ \nabla \cdot (\rho_{ns} \mathbf{v}) = 0 \]

This leads to the relation
\[ \rho_2 (v_f - v_p) = \rho_1 v_f, \]
(39)

where \( v_f = c/n \) is phase velocity, \( v_p \) is particle velocity, and essentially \( \rho_{ns} = \rho_1 = \rho_2 \). The left hand side of (39) explains the mass flux after the EM field contacts the particle. Utilizing conservation of momentum through equation (37) together with relation (39), we arrive at the conclusion that
\[ v_{p1} = \frac{2 \Delta p_{em}}{\rho_{ns} v_f} \]  
(40)

A derivation is included in footnote [15]. From (39) and (40), we find that
\[ v_{p2} = v_{p1} \frac{A_1}{A_2} \]  
(41)

Now, \( A_1 \) is the cross-sectional area of the laser beam at the beam waist radius, \( w_o \), which is \( A_1 = \pi w_o^2 \), while \( A_2 \) is at some other position, \( z \), along the optical axis of the beam so that \( A_2 = \pi w(z)^2 \), where \( w(z) \) is the spot size radius at \( z \). Therefore, we have that \( A_1/A_2 = (w_o/w(z))^2 \). Using the well known relation for beam waist as a function of \( z \) for a Gaussian laser beam, \( w(z) = w_o \sqrt{1 + \left( \frac{z}{z_r} \right)^2} \), we find that
\[ \frac{A_1}{A_2} = \left( \frac{w_o}{w(z)} \right)^2 \frac{1}{1 + \left( \frac{z}{z_r} \right)^2}. \]  
(42)

Inserting Eq. (42) into Eq. (41), we find that
\[ v_{p2} = \left( \frac{2 \Delta p}{\rho_{ns} (c/n)} \right) \left\{ \frac{1}{1 + \left( \frac{z}{z_r} \right)^2} \right\}. \]  
(43)

This makes sense as higher velocities occur for \( 0 \leq |z| \leq |z_r| \) while maximum velocity is at \( z=0 \), where the laser beam has its highest irradiance. We also see that as the particle moves farther away from the Rayleigh length (\( |z| > z_r \)), its velocity decreases.

Now, recalling that \( p_{em} = \varepsilon_0 u = 1/2 \varepsilon_0 [E^2 + c^2 B^2] \). We further find that for a plane wave \( B = E/c \) so that \( E^2 = c^2 B^2 \). Therefore, we have that \( p_{em} = 1/2 \varepsilon_0 [2E^2] = \varepsilon_0 E^2 \), but per our definition of intensity, we also have that \( E^2 = 2I/(c \varepsilon_0 n) \). These relations give us that \( p_{em} = 2I/cn \), or, likewise, \( \Delta p_{em} = 2\Delta I/cn \). Using this in equation (43) we get that
\[ v_{p2} = \left( \frac{4 \Delta I(r)}{c^2 \rho_{ns}} \right) \left\{ \frac{1}{1 + \left( \frac{z}{z_r} \right)^2} \right\}. \]  
(44)

Thus, a particle’s velocity along the optical axis is inversely proportional to distance away from the center of the beam and varies with change in intensity. Let \( \Delta q(r) = \Delta I(r)/c^2 \). From equation (44) we see that, for a given \( \rho_{ns} \), higher particle velocities occur in the range \( 2\Delta q(r) \leq v \leq 4\Delta q(r) \), \( \forall |z| \in [0, |z_r]| \), whereas \( v_p \to 0 \) for \( |z| \to \infty \). This makes physical sense as within the Rayleigh range the largest amount of momentum flux is imparted onto the dielectric particle by the electromagnetic field of the laser.

If we assume a particle starts with zero momentum flux, prior to laser light being shone on it, then we can get rid of the \( \Delta \) and re-write (45), with \( \rho_{ns} = 1 \ kg \ m^3 \), as
\[ v_{p2} = g(r) \left\{ \frac{1}{\frac{1}{4} + \left( \frac{z}{b} \right)^2} \right\}, \]  
(45)

where \( b = 2z_r = \frac{2w_o}{N_A} = \frac{2w_o^2}{\pi} \) is the confocal parameter of the beam and N.A. the numerical aperture.

Equation (45) shows a direct relationship to wavelength, N.A. and momentum flux. Indeed, Svoboda, et.
al. found that forces on a particle in the Rayleigh regime were directly determined by wavelength and N.A. InCREASE in the later directly increasing the light gradient to ensure stable trapping [16]. Also, Roichman, et. al., imposed a transverse phase profile, \( \varphi(r) \) on plane waves utilized for holographic optical traps such that the transverse momentum flux was proportional to intensity times a phase gradient, \( g_x(r) \propto I(r) \nabla \varphi. \) Remarkably, it was found that for a linear phase gradient of the form, \( \nabla \varphi = q\hat{x}, \) where \( q \) is a wave vector component in the \( \hat{x} \) direction, a particle’s velocity, \( v_p, \) varied in linear proportion to \( q \) for a range of \( q = \pm 12 \) rad/\( \mu m \) [17]. Assuming such particle stayed at the beam waist, \( z=0, \) equation (45) would show this direct relationship as \( v_p = 4g(r) \) along the desired wave vector direction.

### D. Optical forces

A measure of the particle size relative to wavelength is typically the ratio \( x = 2\pi a/\lambda, \) where \( a \) is the particle radius and \( \lambda \) is the incident laser beam wavelength. The Rayleigh approximation, which describes a particle as a point dipole, can be used when \( x << 1. \)

Per Fig. 1, an incident electromagnetic wave exerts a scattering and gradient force on the particle due to an interaction of the electromagnetic wave with the particle’s dipole oscillation which radiates scattered waves isotropically. If the particle has a time-averaged dipole moment, \( m_d(r) = \alpha E(r), \) where \( \alpha \) is the polarizability, and a scattering cross-section, \( \sigma_s, \) we can calculate both the scattering and gradient forces using our hydrodynamic analogy.

1. Scattering force:
   This is electromagnetic momentum flux exerted on a dielectric particle’s scattering cross-section. If we think of the scattering cross-section as the cross-sectional area, then this is the area over which electromagnetic radiation momentum flux (i.e. ‘fluid’ momentum) can be exerted. The scattering force is given by \( F_{sc}(r) = \frac{n}{c}\sigma_s I(r)\hat{z} = v_f \sigma_s g \hat{z}, \) (46) where \( v_f = c/n \) is the phase velocity.

2. Gradient Force:
   For this treatment, the gradient force can be found by letting

   \[
   v(r) = |v| = \sqrt{\nabla \cdot \nabla} = \sqrt{\left( \frac{I(r)}{\rho_{ns}c^2} \right)^2} = \frac{I(r)}{\rho_{ns}c^2},
   \]
   (47)

   where we used equation (33), without the constant vector, as the velocity. Following the same rationale as in part B. Section III with equation (34) we find that energy density for this system is conserved. As such, the gradient force on the particle can be expressed in terms of the negative gradient of a potential energy according to, \( F = -\nabla \Psi, \) where \( \Psi \) is a potential energy. In general, a pressure gradient is potential flow that causes fluid acceleration such that, \( a = -\nabla P/\rho [19], \) where \( P \) is pressure and \( \rho \) is density. From Newton’s second law, \( F = ma, \) taking \( \rho \) as \( m \) upon substitution we find:

   \[
   F = -\nabla P,
   \]

   where we can substitute in \( p_{em} \) for \( P \) in our system.

   Now, in terms of the system’s physical arrangement, particle velocity is due to a non-slip, albeit with phase delay due to polarizability, inviscid flow due to the beam’s electric field. Any velocity gradient of the particle will primarily be due to its polarizability, induced by the electric field, within the medium. We can, thus, think of the electromagnetic field as the advection term operating on the particle. Therefore, the gradient force per volume can be found through:

   \[
   \frac{F(r)}{V} = -\nabla p_{em} = \rho_{ns} (\mathbf{v}_{field} \cdot \nabla) \mathbf{v}_{particle},
   \]
   (48)

   where we have made use of the advective term with \( \frac{\partial \rho_{particle}}{\partial t} = 0 \) from Euler Eq. (21). The volume over which this force is exerted is the three dimensional space taken up by the particle acting as a point dipole. Now, making use of the skew-symmetric advection operator for equation (48), we have

   \[
   \frac{F(r)}{V} = \rho_{ns} \left( \frac{1}{2} (\mathbf{v}_f \cdot \nabla) \mathbf{v}_p + \frac{1}{2} \nabla (\mathbf{v}_f \mathbf{v}_p) \right).
   \]
   (49)

   For irrotational flow where there is no fluid divergence, \( \nabla \cdot \mathbf{v} = 0, \) we have, \( \mathbf{v} \cdot \hat{n} = 0, \) where \( \hat{n} \) is a direction normal to fluid flow. Thus, the first term in (49) vanishes since \( \mathbf{v}_f \cdot \mathbf{v}_p = 0. \) Further, we have \( v_{fx} = v_{fy} = 0, \) where \( v_{fx} = v_f = c/n, \) the phase velocity. We also assume, \( v_p = v_{px} = v_{py} = v_{pz} \) since momentum flux velocity drives particle velocity, so (49) can be expressed as

   \[
   \frac{F(r)}{V} = \frac{1}{2} \rho_{ns} \nabla v_{fx} v_{pz} \hat{k} \hat{k} = \frac{1}{2} \rho_{ns} v_f \nabla v_p.
   \]
   (50)

Now, the particle behaves as a point dipole due to its polarization, \( \alpha, \) therefore, it must take up a volume in space as it oscillates perpendicular to the incident EM wave and parallel to the polarization axis. That volume is given by \( \alpha / \varepsilon_0. \) Multiplication of equation (50) by polarization volume and using (47) for momentum flux velocity imparted onto the particle we arrive at the gradient force:

   \[
   F_{gr}(r) = \frac{1}{2} \left( \frac{\alpha}{\varepsilon_0} \left( \frac{1}{nc} \right) \nabla I(r). \right)
   \]
   (51)

For a particle acting as a point dipole as in Fig. 1, the polarizability is given by

   \[
   \alpha = 4\pi \varepsilon_0 \left( \frac{n^2 \left( m^2 - 1 \right)}{m^2 + 2} \right) a^3
   \]
   (52)
where $m = n_p/n$ and $n_p$ is particle refractive index while $n$ is that of the medium. Substituting in the polarizability $\alpha$ above we obtain the well-known gradient force obtained in [13, 17, 18]:

$$F_{gr}(r) = 2\pi a^3 \left( \frac{n_p}{c} \right) \left( \frac{m^2 - 1}{m^2 + 2} \right) \nabla I(r). \quad (53)$$

IV. CONCLUSION

We have shown that by starting with general forms of momentum, mass and energy hydrodynamic conservation equations we can arrive at analogous general forms of momentum and mass-energy equations applicable to electromagnetic “fluid” flow. While, by no means, can these generalized EM Navier-Stokes-like equations be applied to any electromagnetic flow at hand, they could be useful if applied carefully to a system. One must first decide on appropriate flow assumptions regarding steady-state, vorticity, and constitutive relations appropriate to the medium within which EM flow occurs. While here a plane wave was utilized for the chosen application examples, it is expected that most arbitrary electromagnetic waves can be fit into the model.

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