On the Epstein zeta function and the zeros of a class of Dirichlet series

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Abstract

By generalizing the classical Selberg-Chowla formula, we establish the analytic continuation and functional equation for a large class of Epstein zeta functions. This continuation is studied in order to provide new classes of theorems regarding the distribution of zeros of Dirichlet series in their critical lines and to produce a new method for the study of these problems. Due to the symmetries provided by the representation via the Selberg-Chowla formula, some generalizations of well-known formulas in analytic number theory are also deduced as examples.

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1 Introduction

Let \( Q(x, y) = ax^2 + bxy + cy^2 \) be a real and positive definite quadratic form. The classical Epstein zeta function is defined as the Dirichlet series

\[
Z_2(s, Q) = \sum_{m,n \neq 0} \frac{1}{Q(m,n)^s}, \quad \text{Re}(s) > 1,
\]

where the notation given in the subscript, \( m, n \neq 0 \), here means that only the term \( m = n = 0 \) is omitted from the infinite series.

In 1949, S. Chowla and A. Selberg \cite{64} announced the following formula for (1.1), valid in the entire complex plane,

\[
a^s \Gamma(s) Z_2(s, Q) = 2\Gamma(s) \zeta(2s) + 2k^{1-2s}\pi^{1/2}\Gamma\left(s - \frac{1}{2}\right) \zeta(2s-1) + 8k^{1-s}\pi^{s}\sum_{n=1}^{\infty} n^{v-\frac{1}{2}} \sigma_{1-2s}(n) \cos(n\pi b/a) K_{s-\frac{1}{2}}(2\pi kn),
\]

where \( d := b^2 - 4ac \) is the discriminant of the quadratic form, \( k^2 := |d|/4a^2 \) and \( \sigma_v(n) = \sum_{d|n} d^v \) is the generalized divisor function of index \( v \). Also, \( \zeta(s) \) denotes the classical Riemann zeta function and \( K_{\nu}(z) \) is the modified Bessel function.

According with Berndt’s revision \cite{10}, after its first announcement, (1.2) was proved by Rankin \cite{61}, Bateman and Grosswald \cite{4}, Chowla and Selberg \cite{65} (in a paper which appeared 18 years after the statement of (1.2)) and Motohashi \cite{54}, although the latter author had the main goal of proving directly the first Kronecker limit formula. As it should be expected, all these proofs used the theory of Fourier series and invoke directly the Poisson summation formula.

Berndt himself \cite{10} gave a very interesting proof of (1.2), which was based on a previous theorem due to him regarding the Bessel expansion of the so called generalized Dirichlet series \cite{6}. In fact, Berndt’s proof of (1.2) was considered for a more general Epstein zeta function of the form

\[
Z_2(s, Q; g, h) = \sum_{m,n \neq 0} \frac{e^{2\pi i (hm + h_2n)}}{Q(m+g_1, n+g_2)^s}, \quad \text{Re}(s) > 1,
\]

where \( g_1, g_2, h_1 \) and \( h_2 \) are real. Although only written for the particular case (1.2), a simplified version of Berndt’s argument was later given by Kuzumaki \cite{51}. It is remarkable that Kuzumaki’s proof, as well as Berndt’s, depends explicitly on the functional equation for the Hurwitz zeta function \( \zeta(s, a) \). Although this functional equation can be derived immediately from the reflection formula for the \( \theta \)–function \cite{36}

\[
\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\alpha n^2} \cos(\beta n) = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2}{\alpha}(n^2 + \frac{1}{4})^2},
\]

(1.3)
(which is obviously connected with the Poisson summation formula, thus with all original proofs of (1.2) it can nevertheless be established in strikingly different ways).

The generalization of (1.2) for Epstein zeta functions attached to quadratic forms with \( n \) variables was given by Terras [72], whose proof employed a multidimensional version of the Poisson summation formula. Suzuki [70] provided analogues of (1.2) for other Dirichlet series which apriori have nothing to do with (1.1). These Dirichlet series include the Riemann \( \zeta \)–function, the Dirichlet \( L \)–function and \( L \)–functions attached to holomorphic cusp forms.

The idea behind a Bessel expansion of a Dirichlet series of the form (1.1) seems to have been motivated by a formula discovered by Watson [78, p. 99, eq. (4)], which was later generalized by H. Kober [49] (see Example 5.1), and perhaps the first explicit form of (1.2) appearing in the literature is due to Taylor [71]. Although Taylor himself attributed (1.2) to Kober, it is in Taylor’s paper where the continuation of the Epstein zeta function via (1.2) is made explicit for the first time. Taylor’s proof employed an elegant Mellin-Barnes representation [71, p. 181] of the left-hand side of (1.2), involving the Gauss hypergeometric function. Although Taylor did not prove the functional equation for \( Z_2(s, Q) \) directly from (1.2), as Selberg and Chowla did, his derivation of both results used the same ideas and this seems to be the first indication of the connection between (1.2) and the symmetric properties of the continuation of \( Z_2(s, Q) \).

The Selberg-Chowla formula has found many applications in Number Theory. By using it, Selberg and Chowla [65] proved the existence of a real zero for \( Z_2(s, Q) \) in the open interval \( (\frac{1}{2}, 1) \), when the quadratic form is \( Q(x, y) = x^2 + cy^2 \) and \( c \) is large enough. This result was later improved by Bateman and Grosswald [4, p. 367, Thm 3.], who have shown that, for a general positive definite and real quadratic form \( Q \), \( Z_2(s, Q) \) has a real zero between \( \frac{1}{2} \) and 1 if \( k := \sqrt{|d|/2a} > 7.0556 \). A generalization of this argument to Epstein zeta functions attached to ternary quadratic forms was furnished by Terras [73], who employed prior results due to her concerning a multidimensional version of (1.2) (see [72, p. 480, eq. (2.3)] and 5.53 below).

With the aid of (1.2), Stark [69] and Fujii [37] gave very detailed descriptions about the distribution of the complex zeros of \( Z_2(s, Q) \) in bounded regions. Building on the work of many previous authors, H. Ki [47] proved a remarkable result regarding the zeros of each finite truncation of the right-hand side of (1.2), showing that all but finitely many of them lie on the critical line \( \text{Re}(s) = 1/2 \). The idea of studying the

\footnote{1}{see, for instance, a very recent paper by A. Dixit and R. Kumar [28], which established the functional equation for \( \zeta(s, \alpha) \) via its Hermite integral representation. See also the discussion in Remark 3.2.}

\footnote{2}{Although fairly unknown, Taylor’s approach has found interesting applications in some aspects of spectral theory. Since Maass cusp forms have Fourier expansions analogous to (1.2), a representation involving the hypergeometric function similar to Taylor’s holds for these forms, allowing to construct \( \psi_j(z) \) from the \( L \)–function attached to it, \( L_j(s) \). This representation is one of the main results in [30, p. 120, eq. (4.3)], where it is used to prove an analogue of Hardy-Littlewood’s Theorem to \( L \)–functions attached to Maass cusp forms.}
zeros of the Epstein zeta function via equivalent representations to (1.2) seems to have its origin in Deuring’s work [26].

In a very interesting paper by the same author [25], in which most of the approaches later used to prove (1.2) were already given3, Deuring connected the Fourier expansion for the simple Epstein \( \zeta \)–function \( Z_2(s, \mathbb{I}_2) \) (where \( b = 0, a = c = 1 \)) with the existence of infinitely many zeros of \( \zeta(s) \) on the critical line \( \text{Re}(s) = \frac{1}{2} \). Using this relation between two drastically different Dirichlet series, he was able to deduce Hardy’s Theorem for \( \zeta(s) \). Although Deuring’s proof of Hardy’s Theorem requires more motivation than the standard ones given by Hardy [40], Landau [53] or Fekete [35], it is nevertheless an interesting proof, since the way it seeks the contradiction is implied from a subconvex estimate for the Epstein zeta function on the critical line, due to E. C. Titchmarsh [74],

\[
Z_2 \left( \frac{1}{2} + it, \mathbb{I}_2 \right) = O \left( |t|^{\frac{1}{3} + \varepsilon} \right), \tag{1.4}
\]

which in its turn is analogous to the classical Van der Corput4 estimate for \( \zeta \left( \frac{1}{2} + it \right) \) [44].

In the most part of the classical proofs of Hardy’s Theorem, one arrives at a contradiction by using the reflection formula for the Jacobi \( \theta \)–function, which ensures a suitable decay for \( \theta \left( e^{\left( \frac{x}{2} - \varepsilon \right)} \right) \), as \( \varepsilon \to 0^+ \) [40, 53]. Standard variations of these arguments [21, 55, 57] employ the evaluation of certain exponential integrals and sums, by contrasting the behavior of the integrals \( \int_T^{2T} |\xi \left( \frac{1}{2} + it \right)| dt \) and \( \int_T^{2T} \xi \left( \frac{1}{2} + it \right) dt \) as \( T \to \infty \).

Although much more difficult to ensure than the asymptotic properties of Jacobi’s \( \theta \)–function, a condition like (1.4) is deep since it connects two fundamental properties of two distinct Dirichlet series.

From the correspondence between the zeros of \( \zeta(s) \) and the order of magnitude of \( Z_2(s, \mathbb{I}_2) \), motivated by the Selberg-Chowla formula, one gains an additional degree of freedom, which is the separate study of the properties of the latter Dirichlet series.

Our goal in this paper is to study this additional degree of freedom in a sufficiently general framework. One of the first steps towards a generalization of Deuring’s proof is to extend the scope of the Selberg-Chowla formula (1.2). We start by making a distinction between two cases:

1. **The diagonal Epstein zeta function**: for \( \text{Re}(s) \) sufficiently large, one of the goals of this paper is to study an infinite series of the form

\[
\sum_{m,n \neq 0}^{\infty} \frac{a_1(m)a_2(n)}{(\lambda_m + \lambda_n')^s},
\]

3For example, a closer look at Deuring’s argument shows also that the fundamental ideas employed by Motohashi in his proof of the Kronecker limit formula (see [54], p. 615) and the evaluation of (5) by using the Cahen-Mellin representation (1.22)) were already been used by Deuring [25], p. 587, eq. (5), (6)].

4As indicated in (4.16) in Theorem 4.1, we do not need the full strength of Titchmarsh’s estimate (1.4) when the Dirichlet series \( \phi(s) \) is the Riemann zeta function.
where $a_1(m), a_2(n), \lambda_m$ and $\lambda'_n$ will be specified later. This double series was also studied in Berndt’s paper [10] and a formula of Selberg-Chowla type for it has been indicated there. Under additional assumptions, in section 2 we give two formulas analogous to $\text{(1.2)}$ and use them to arrive at a functional equation for this double infinite series.

2. The non-diagonal Epstein zeta function: analogously to the previous item, for $\text{Re}(s)$ sufficiently large, we study a double infinite series of the form

$$
\sum_{m,n \neq 0} \frac{a_1(m)a_2(n)}{Q(\lambda_m, \lambda'_n)^s},
$$

where $Q(x,y)$ denotes, just as in the classical case $\text{(1.1)}$, a real and positive definite Quadratic Form.

We study some conditions ensuring the analytic continuation and functional equation of such double series.

The study of the double series outlined above is made in Sections 2 and 3 respectively. By proving analogues of the Selberg-Chowla formula $\text{(1.2)}$, we establish functional equations for these series. In section 4 we use the obtained Selberg-Chowla formulas to generalize Deuring’s argument [25] to a Class of Dirichlet series. In section 5 we furnish particular examples of diagonal and non-diagonal Epstein zeta functions and a description of their analytic continuation is given.

Most of our principal results are given in section 4 and generalize the Theorems presented in [12, 13, 42], where analogues of Hardy’s Theorem are proved for large classes of Dirichlet series. The corollaries derived from our main Theorem 4.1 contain also some of the key observations used in most of the well-known proofs of Hardy’s Theorem. In fact, by simplifying Deuring’s argument for $\zeta(s)$, we shall see how his proof of Hardy’s Theorem, as well as our general and "inductive" approach, extends to all the other proofs available in the literature, which usually involve the asymptotic behavior of the Jacobi $\theta$—function or classical estimates of some exponential integrals [21, 35, 40, 53, 55, 57, 76].

Although written in the general setting of generalized Dirichlet series, the method used by us to prove the existence of infinitely many zeros of $\phi(s)$ on the line $\text{Re}(s) = \frac{r}{2}$ seems to possess certain advantages. The first of these is its uniform applicability to a wide class of Dirichlet series, allowing to generalize certain results due to Berndt and Hecke [13, 42] (see Corollaries 4.2 and 4.3). A simple application of it also allowed us to reprove a known theorem due to Berndt [12] (see corollary 4.4) via “elementary” means, as well as to prove that a certain combination of bounded shifts of $\Gamma(s)\phi(s)$ has infinitely many zeros on the line $\text{Re}(s) = \frac{r}{2}$, extending a result by A. Dixit, N. Robles, A. Roy and A. Zaharescu [29].

\footnote{Throughout this paper we shall use the term "double series" or "multiple series" to denote Dirichlet series attached to more than one arithmetical function. This slightly contrasts with the standard terminology, in which these terms are applied to Dirichlet series with more than one complex variable.}
The second advantage is the fact that it provides a certain degree of freedom in applications, since in order to check that Hardy’s theorem is valid for a particular Dirichlet series, it suffices to verify (independently) that it holds for at least one (out of the infinitely many!) Epstein zeta functions attached to it. As proved by Suzuki [70], it is possible to obtain analogues of the Selberg-Chowla formula for \( \zeta(s) \) (or \( L(s, \chi) \) and \( L(s, a) \)), so that we can find Dirichlet series to which \( \zeta(s) \) (or \( L(s, \chi) \) and \( L(s, a) \)) plays the same role as the Epstein zeta function \((1.1)\) does with \( \zeta(s) \). Thus, the results described by Theorem 4.1 and Corollary 4.6 can be employed, for example, to study the zeros of the Dirichlet appearing in Suzuki’s paper.

As a rather simple example of our method, we will see that the final conclusion of Hardy’s Theorem for \( \zeta(s) \) is implied by the celebrated Jacobi’s 4-square Theorem, which constitutes a curious connection between a purely analytic theorem involving deep properties of \( \zeta(s) \) and a beautiful arithmetical property of \( r_4(n) \) (see Examples 5.3 and 5.6).

### 1.1 Notation and Definitions

To state the two generalizations of the Selberg-Chowla formula given in this paper, we need the following definition:

**Definition 1.1.** Let \((\lambda_n)_{n \in \mathbb{N}}\) and \((\mu_n)_{n \in \mathbb{N}}\) be two sequences of positive numbers strictly increasing to \( \infty \) and \((a(n))_{n \in \mathbb{N}}\) and \((b(n))_{n \in \mathbb{N}}\) two sequences of complex numbers not identically zero. Consider the functions \( \phi(s) \) and \( \psi(s) \) representable as Dirichlet series

\[
\phi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s} \tag{1.5}
\]

with finite abscissas of absolute convergence \( \sigma_\alpha \) and \( \sigma_\beta \) respectively. Let \( \Delta(s) \) denote one of the following three gamma factors: \( \Gamma(s) \), \( \Gamma\left(\frac{s}{2}\right) \) and \( \Gamma\left(\frac{s+1}{2}\right) \) and \( r \) be an arbitrary positive real number in the first case and 1 in the other two. We say that \( \phi \) and \( \psi \) satisfy the functional equation

\[
\Delta(s) \phi(s) = \Delta(r-s) \psi(r-s), \tag{1.6}
\]

if there exists a meromorphic function \( \chi(s) \) with the following properties:

1. \( \chi(s) = \Delta(s) \phi(s) \) for \( \text{Re}(s) > \sigma_\alpha \) and \( \chi(s) = \Delta(r-s) \psi(r-s) \) for \( \text{Re}(s) < r - \sigma_\beta \);

2. \( \lim_{\text{Im}(s) \to \infty} \chi(s) = 0 \) uniformly in every interval \( -\infty < \sigma_1 \leq \text{Re}(s) \leq \sigma_2 < \infty \).

3. The singularities \( \chi(s) \) are at most poles and are confined to some compact set.

Moreover, if \( \Delta(s) = \Gamma\left(\frac{s+\delta}{2}\right) \), \( \delta \in \{0, 1\} \), we take a convention which extends \( a(n) \) to the negative integers as follows: \( a(-n) = (-1)^{\delta} a(n) \).
We say that the pair of functions \((\phi, \psi)\) representable as Dirichlet series (1.5) satisfy **Hecke’s functional equation** if they satisfy the conditions of the previous definition for \(\Delta(s) = \Gamma(s)\). The particular case of (1.6) reads

\[
\Gamma(s) \phi(s) = \Gamma(r - s) \psi(r - s), \quad r > 0.
\]

Similarly, we say that the pair of functions \((\phi, \psi)\) representable as (1.5) with finite abscissas of absolute convergence \(\sigma_a\) and \(\sigma_b\) satisfy **Bochner’s functional equation** if they satisfy the functional equation

\[
\Gamma \left( s + \frac{\delta}{2} \right) \phi(s) = \Gamma \left( 1 + \frac{\delta - s}{2} \right) \psi(1 - s),
\]

in the sense of Definition 1.1. Moreover, if the pair \((\phi, \psi)\) satisfies (1.8) with \(\delta = 0\), then we will say that \(\phi(s)\) and \(\psi(s)\) are even Dirichlet series belonging to the Bochner class. Otherwise, if (1.8) is satisfied for \(\delta = 1\), we will say that \(\phi(s)\) and \(\psi(s)\) are odd Dirichlet series belonging to the Bochner class.

Although under slight variations, the class of Dirichlet series covered by Definition 1.1 was considered by a large number of authors. The Dirichlet series satisfying the functional equation (1.8) were studied by S. Bochner and K. Chandrasekharan in [19], where considerations of general analogues of Hamburger’s theorem were given. K. Chandrasekharan and R. Narasimhan [20] proved the equivalence between several identities in Number theory and Hecke’s functional equation (1.7). Included as a subclass in the context of generalized Dirichlet series are the classical Dirichlet series with signature \((\lambda, r, \gamma)\), considered in Hecke’s works [6].

Before proceeding further, we shall denote \(\sum_{m,n \neq 0}\) as the infinite sum over all integers \(m, n\) not simultaneously zero. If we assume that \(m\) and \(n\) are positive, we shall write \(\sum_{m,n \neq 0}\) instead and we will always use the convention \(\lambda_0 = \mu_0 = 0\) and \(\lambda_{-n} = -\lambda_n, \mu_{-n} = -\mu_n\). Also, we will always write \(\sigma_a := \max \{\sigma_{a_1}, \sigma_{a_2}\}\) and, analogously, \(\sigma_0 := \max \{\sigma_{b_1}, \sigma_{b_2}\}\).

We will now define a double Dirichlet series which exhibits a similar behavior as the classical Epstein zeta function (1.1). We will often call it by the name “diagonal Epstein zeta function”.

**Definition 1.2.** Let \((\lambda_n, \lambda'_n)_{n \in \mathbb{N}}\) be a pair of sequences of positive numbers strictly increasing to \(\infty\) and \((a_1(n), a_2(n))_{n \in \mathbb{N}}\) a pair of sequences of complex numbers not identically zero. Also, let \(\phi_1(s)\) and \(\phi_2(s)\) be two functions representable as Dirichlet series

\[
\phi_1(s) = \sum_{n=1}^{\infty} \frac{a_1(n)}{\lambda_n^s}, \quad \phi_2(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{\lambda'_n^s},
\]

with finite abscissas of absolute convergence \(\sigma_{a_1}\) and \(\sigma_{a_2}\) respectively. For the pair \((\phi_1, \phi_2)\), we define the generalized diagonal Epstein zeta function as the double Dirichlet series

\[
\mathcal{Z}_2(s, a_1, a_2, \lambda, \lambda') = \sum_{m,n \neq 0} \frac{a_1(m) a_2(n)}{(\lambda_m + \lambda'_n)^s}, \quad \text{Re}(s) > 2 \max \{\sigma_{a_1}, \sigma_{a_2}\}.
\]
It is easily seen that the double series \((1.10)\) is absolutely convergent if \(\Re(s) > 2 \max \{\sigma_{a_1}, \sigma_{a_2}\}\). Examples of \((1.10)\) include the classical Dirichlet associated with the sum of two squares

\[
\frac{1}{4} \zeta_2(s) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = \sum_{m,n\neq 0} \frac{1}{(m^2+n^2)^s}, \quad \Re(s) > 1 \tag{1.11}
\]

which can be constructed if we take \(\phi_1(s) = \phi_2(s) = \zeta(2s)\).

Just as \((1.11)\), we may represent a general diagonal Epstein \(\mathcal{E}_2^2(s, a_1, a_2, \lambda, \lambda')\) via the single Dirichlet series

\[
\mathcal{E}_2^2(s, a_1, a_2, \lambda, \lambda') = \sum_{m,n\neq 0} \frac{a_1(m)a_2(n)}{(\lambda_m + \lambda_n')}, \quad \Re(s) > 2 \sigma_a \tag{1.12}
\]

where \(\Lambda_2(n)\) generalizes the arithmetical function \(r_2(n)\) in the following way:

\[
\Lambda_2(n) = \sum_{j_1, j_2: \lambda_{j_1} + \lambda_{j_2}' = \lambda_n} a_1(j_1)a_2(j_2), \tag{1.13}
\]

and the sequence \(\Lambda_n\) is taken as \(\Lambda_n := (\lambda_{j_1} + \lambda_{j_2}')_{j_1, j_2 \in \mathbb{N}_0}\) and then rearranged in increasing order.

Since \((1.11)\) constitutes a particular case of the Epstein zeta function \(Z_2(s, Q)\) when \(Q\) is a diagonal quadratic form, we shall call to the double series \((1.10)\) by the name “diagonal Epstein zeta function”. In section 2 we prove that, if \(\phi_1\) and \(\phi_2\) satisfy definition \((1.1)\) with \(\Delta(s) = \Gamma(s)\), then it is possible to write two distinct Selberg-Chowla formulas for \((1.10)\). We will also introduce a subclass of Dirichlet, named class \(\mathcal{A}\), for which \((1.10)\) will also obey to Hecke’s functional equation (see definition \(2.2\)).

In section 3 we will study the analytic continuation of the following double Dirichlet series:

**Definition 1.3.** Let \(Q\) be a real, binary and positive definite quadratic form given by \(Q(x, y) = ax^2 + bxy + cy^2\) and \((\phi_1, \phi_2)\) a pair of Dirichlet series similar to \((1.9)\) with absciss of absolute convergence \(\sigma_{a_1}\) and \(\sigma_{a_2}\). Assume also that their arithmetical functions \(a_i(n)\) can be extended to the negative integers as \(a_i(n) = (-1)^{\delta}a_i(n), \delta \in \{0, 1\}\), and take the convention \(\lambda_0 = \lambda'_0 = 0\) and \(\lambda_{-n} = -\lambda_n, \lambda'_{-n} = -\lambda'_n\).

We define the non-diagonal Epstein zeta function associated with \(\phi_1\) and \(\phi_2\) as the double Dirichlet series

\[
\mathcal{E}_2^2(s; Q, a_1, a_2; \lambda, \lambda') = \sum_{m,n\neq 0} \frac{a_1(m)a_2(n)}{(a\lambda_m^2 + b\lambda_m\lambda_n' + c\lambda_n'^2)}, \quad \Re(s) > \sigma_a := \max \{\sigma_{a_1}, \sigma_{a_2}\} \tag{1.14}
\]

where the infinite series is taken over the pair of integers which are not both zero.

It is easily seen that the double Dirichlet series on the right-hand side converges absolutely if \(\Re(s) > \max \{\sigma_{a_1}, \sigma_{a_2}\}\). We will always adopt the convention that \(a_1(n)\) and \(a_2(n)\) have the same parity since, otherwise, the previous double series would be identically zero.

In section 3 we prove analogues of the Selberg-Chowla formula for \((1.14)\) when \(\phi_1\) and \(\phi_2\) satisfy definition \((1.1)\) with \(\Delta(s) = \Gamma \left( \frac{s+\delta}{2} \right)\).
Before giving the preliminary results, we remark that throughout this paper we shall use the following notation and conventions:

- Let $Q$ denote the matrix representation of the Quadratic form $Q(x, y)$ given in definition [1.2]. If $d$ is the discriminant of $Q$, we will define the inverse quadratic form as $Q^{-1}(x) = -\frac{1}{d}x^T Q^{-1} x$. Note that $Q^{-1}(x, y) = cx^2 - bxy + ay^2$.

- In order to make our general formulas resemble the original formulation [1.2], we will use the symbol $\sigma_\varepsilon(v_j; b_1, a_2; \mu, \lambda')$ to denote a generalized version of the divisor function (see Definition [2.1]). At some points (see Example [5.1]), we will simplify the notation and reduce it to $\sigma_\varepsilon(v_j; b_1, a_2)$.

- Whenever we use the term “critical line” for a given Dirichlet series $\phi(s)$ satisfying Definition [1.1], we will be referring to $\text{Re}(s) = \frac{r}{2}$ if $\phi(s)$ satisfies Hecke’s functional equation [1.7] and to $\text{Re}(s) = \frac{1}{2}$ if $\phi(s)$ satisfies Bochner’s functional equation [1.8]. The property “$\phi(s)$ has infinitely many zeros at its critical line” will sometimes be reformulated as “$\phi(s)$ satisfies Hardy’s Theorem”.

- We will write $H_1(s; b_1, a_2; \mu, \lambda')$ and $H_2(s; b_2, a_1; \mu', \lambda)$ to denote the entire complex functions [2.1] and [2.3] appearing in the Selberg-Chowla representations for the diagonal Epstein zeta function. For the entire complex functions appearing in [3.11] and [3.12], we shall use alternatively $H_1(s; Q; b_1, a_2; \mu, \lambda')$ and $H_2(s; Q; b_2, a_1, \mu', \lambda)$. In analogy with the generalized divisor function, we will sometimes simplify the notation by writing these entire functions as $H_1(s; Q; b_1, a_2)$ or $H_2(s; Q; b_2, a_1)$.

- If, in Definitions [1.2] and [1.3] the Dirichlet series are such that $\phi_1 = \phi_2 = \phi$, we will write $Z_2(s; a; \lambda)$ and $Z_2(s; Q; a; \lambda)$ to respectively denote [1.10] and [1.14].

- We shall use $R_\phi(s)$ and $R_\psi(s)$ to denote respectively $\Gamma(s)\phi(s)$ and $\Gamma(s)\psi(s)$. This notation will be employed mainly during the proof of Theorem [4.1]. Under some conditions given there, $R_\phi(s)$ and $R_\psi(s)$ will represent real functions on the critical line $\text{Re}(s) = r/2$.

- Throughout the proof of Theorem [4.1] $c$ will represent the least positive integer such that $a(c) \neq 0$, where $a(n)$ is the arithmetical function attached to a Dirichlet series satisfying definition [1.1]. This integer also appears in the statement of Corollary [4.3].

### 1.2 Preliminary results

In several occasions throughout this paper, we shall need to estimate the asymptotic order of certain integrals involving a combination of Gamma functions and Dirichlet series, $\phi(s)$, satisfying Definition [1.1]
To justify most of the steps, we will often invoke the following version of Stirling’s formula
\[
\Gamma(\sigma + it) = (2\pi)^{\frac{1}{2}} t^{\sigma + \frac{1}{2}} e^{-\frac{1}{2}t} \left(1 + \frac{1}{12\sigma + O(1/t^2)}\right),
\]
for \( t \to \infty \), uniformly for \( -\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty \). A similar formula can be written for \( t < 0 \) as \( t \) tends to \( -\infty \) by using the fact that \( \Gamma(\sigma) = \Gamma(\sigma) \).

To estimate the order of \( \phi(s) \) at the line \( \text{Re}(s) = \sigma \) we shall need a version of the classical Phragmén-Lindelöf theorem given in [75], p. 180, 5.65. Since, for \( \sigma > \sigma_b \), \( \psi(\sigma + it) = O(1) \) as \( |t| \to \infty \), it follows from the functional equation (1.6) and Stirling’s formula (1.15) that, for \( \sigma < r - \sigma_b \), \( \phi(s) \) satisfies
\[
\phi(\sigma + it) = O\left(\frac{\Delta(r-s)}{\Delta(s)} \psi(r-s)\right) = O\left(\frac{\Delta(r-s)}{\Delta(s)}\right),
\]
as \( |t| \) tends to \( \infty \). Since \( \phi(s) = O(1) \) for \( \sigma = \text{Re}(s) > \sigma_a \), it follows from property 2. in Definition 1.1 that:

1. If \( \phi(s) \) satisfies Hecke’s functional equation (1.7), then for any \( \delta > 0 \) and \( r - \sigma_a - \delta \leq \sigma := \text{Re}(s) \leq \sigma_a + \delta \),
\[
\phi(\sigma + it) = O\left(|t|^{\sigma_a + \delta - \sigma}\right), \quad |t| \to \infty.
\]
2. If \( \phi(s) \) satisfies Bochner’s functional equation (1.8), then for any \( \varepsilon > 0 \) and \( 1 - \sigma_a - \varepsilon \leq \sigma := \text{Re}(s) \leq \sigma_a + \varepsilon \),
\[
\phi(s) = O\left(|t|^{\frac{\sigma_a + \varepsilon - \sigma}{2}}\right), \quad |t| \to \infty.
\]

Since Hardy’s short note [40] and the subsequent quantitative proofs given by Landau, Fekete, Hardy and Littlewood [35,41,52], as well as later adaptations to other Dirichlet series by Kober [48] and Hecke [42], the study of the zeros of \( \zeta(s) \) and other general Dirichlet series is often dependent on the asymptotic behavior of the associated Jacobi \( \theta \)– function as a function in the upper half-plane. From now on, if \( \phi(s) \) satisfies Hecke’s functional equation (1.7) in the sense of Definition 1.1 and it is representable by the first Dirichlet series in (1.5), we shall use \( \Theta(z; a; \lambda) \) to denote the generalized \( \theta \)– function
\[
\Theta(z; a; \lambda) := \sum_{n=1}^{\infty} a(n) e^{-\lambda n z}, \quad \text{Re}(z) > 0.
\]

Following Bochner [18], for \( \text{Re}(z) > 0 \), let \( P(z) \) denote the residual function [8]
\[
P(z) = \frac{1}{2\pi i} \int_{C} \chi(s) z^{-s} ds,
\]
where \( C \) denotes a curve, or curves, encircling the singularities of \( \chi(s) \) given in Definition 1.1. It was proved by Bochner [18] that Hecke’s functional equation for \( \phi(s) \) and \( \psi(s) \) (1.7) and the modular relation
\[
\sum_{n=1}^{\infty} a(n) e^{-\lambda n z} = z^{-r} \sum_{n=1}^{\infty} b(n) e^{-\mu n / z} + P(z).
\]
are equivalent. It is interesting that the observation of this equivalence had its genesis in B. Riemann’s revolutionary memoir, where one of the implications was proved for the first time. There, a second proof of the functional equation for \( \pi^{-s} \zeta(2s) \) (which satisfies Hecke’s functional equation with \( r = \frac{1}{2} \)) was given by observing the reflection formula for the classical Jacobi \( \theta \)-function, which is a particular case of (1.21).

The converse is obtained upon the use of the Cahen-Mellin integral [41, p. 120, eq. (1.II)]

\[
e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)z^{-s}ds,
\]

valid for \( c > 0 \), \( \Re(z) > 0 \) and \( z^{-s} \) having its principal value.

We will also need a slightly more general version of (1.21) when \( \phi(s) \) satisfies one of Bochner’s functional equations (1.8). This will be done in Lemma 3.1 of the paper. In order to establish it, we will require the representation of the confluent hypergeometric function \( \mathbf{1}_F_1 \) as the following Mellin transform [39, p. 503, eq. 3.952.8], valid for \( \Re(\alpha) > 0 \), \( \Re(s) > 0 \) and \( \beta \in \mathbb{C} \),

\[
\int_0^\infty x^{s-1} e^{-\alpha x^2} \cos(\beta x)dx = \frac{\alpha^{-\frac{1}{2}} e^{-\frac{\beta^2}{4\alpha}}}{2} \Gamma \left(\frac{s}{2}\right) \mathbf{1}_F_1 \left(1 - \frac{s}{2}; \frac{1}{2}; \frac{\beta^2}{4\alpha}\right).
\]

Since the real function \( e^{-\alpha x^2} \cos(\beta x) \) is smooth, we are allowed to integrate the left-hand side of (1.23) by parts an arbitrary number of times. Doing so, we see that its asymptotic estimate must be of the form \( O(|s|^{-N}) \) as \( |s| \to \infty \) for any \( N \in \mathbb{N} \). A simple argument by continuation shows that an estimate like this holds for every \( s \in \mathbb{C} \). This gives a suitable decaying behavior for the right-hand side of (1.23) which will be useful in the proof of Lemma 3.1. Of course, stronger estimates for the right-hand side of (1.23) could be given by invoking asymptotic expansions for the Whittaker function [39], but since we will not need these expansions we shall not write them.

Since we will apply Mellin’s inversion formula to the right-hand side of (1.23) (see eq. (3.3) below), the confluent hypergeometric function will have to satisfy a reflection formula compatible with Bochner’s functional equation (1.8). As we shall see, the right symmetries are expected due to Kummer’s transformation formula for \( \mathbf{1}_F_1 \) [1, p. 191, eq. (4.1.11)]

\[
\mathbf{1}_F_1(a; c; x) = e^x \mathbf{1}_F_1(c-a; c; -x).
\]

In the sequel, we will employ several times well-known integral representations for the modified Bessel function \( K_\nu(z) \). For a matter of clarity in our exposition, we will invoke them solely when they are needed.

2 The Selberg-Chowla formula and the analytic continuation of the Diagonal Epstein zeta function

Before stating the properties of the special class \( \mathcal{A} \), for which functional equations will be given, we
will consider the following general Theorem, which establishes the analytic continuation of the Diagonal Epstein zeta function (1.10). As usual, the following result provides a formula valid in a region contained in the plane of absolute convergence of the Double series (1.10). However, as one should expect, they may be taken to other regions of \( \mathbb{C} \) by analytic continuation.

**Theorem 2.1** (A general Selberg-Chowla formula [ [10], p. 166 ]). Let \( \phi_1 \) and \( \phi_2 \) be the pair of Dirichlet series (1.9) satisfying the above definition (1.1) and obeying to Hecke’s functional equation (1.7). Also, let \( s \) be a complex number such that \( \Re(s) > \mu \geq \max \{ 0, 2\sigma_a, 2\sigma_b, \frac{\nu}{2} \} \), where \( \sigma_a = \max \{ \sigma_{a_1}, \sigma_{a_2} \} \), \( \sigma_b = \max \{ \sigma_{b_1}, \sigma_{b_2} \} \) and \( \nu' = \max \{ r_1, r_2 \} \). Let \( H_{r_1} (s; b_1, a_2; \mu, \lambda') \) denote the double series

\[
H_{r_1} (s; b_1, a_2; \mu, \lambda') = \sum_{m, n=1}^{\infty} b_1(m) a_2(n) \left( \frac{\mu}{\lambda'_n} \right)^{s-1} K_{r_1-s} \left( 2\sqrt{\mu_1 \lambda'_n} \right).
\]  

Then, for \( \Re(s) > \mu \), the following identity for the generalized Epstein zeta function holds

\[
\Gamma(s) \mathcal{Z}_2 (s; a_1, a_2; \lambda, \lambda') = a_1(0) \Gamma(s) \phi_2(s) + a_2(0) \Gamma(s) \phi_1(s) + \sum_{n=1}^{\infty} a_2(n) \frac{\Gamma(s)}{\lambda'_n} R_1 \left( s, \frac{1}{\lambda'_n} \right) + 2 H_{r_1} (s; b_1, a_2; \mu, \lambda'),
\]

which is the identity for the generalized Epstein zeta function holds

\[
\Gamma(s) \mathcal{Z}_2 (s; a_1, a_2; \lambda, \lambda') = a_2(0) \Gamma(s) \phi_2(s) + a_2(0) \Gamma(s) \phi_1(s) + \sum_{n=1}^{\infty} a_2(n) \frac{\Gamma(s)}{\lambda'_n} R_1 \left( s, \frac{1}{\lambda'_n} \right) + 2 H_{r_1} (s; b_1, a_2; \mu, \lambda'),
\]

where \( b_1(m) \) is the arithmetical function associated to the Dirichlet series \( \psi_1(s) \) and \( R_1(s, x) \) denotes the sum of the residues of the meromorphic function \( R_1(z) = \Gamma(z) \phi_1(z) \Gamma(s - z) x^{-z} \) at the poles of \( \Gamma(z) \phi_1(z) \).

Equivalently, if \( H_{r_2} (s; b_2, a_1; \mu', \lambda) \) denotes the double infinite series,

\[
H_{r_2} (s; b_2, a_1; \mu', \lambda) = \sum_{m, n=1}^{\infty} b_2(m) a_1(n) \left( \frac{\mu'}{\lambda_m} \right)^{s-1} K_{r_2-s} \left( 2\sqrt{\mu' \lambda_m} \right),
\]

then \( \mathcal{Z}_2 (s; a_1, a_2; \lambda, \lambda') \) can be also described by the formula

\[
\Gamma(s) \mathcal{Z}_2 (s; a_1, a_2; \lambda, \lambda') = a_1(0) \Gamma(s) \phi_2(s) + a_2(0) \Gamma(s) \phi_1(s) + \sum_{m=1}^{\infty} a_1(m) \frac{\Gamma(s)}{\lambda_m} R_2 \left( s, \frac{1}{\lambda_m} \right) + 2 H_{r_2} (s; b_2, a_1; \mu', \lambda),
\]

whenever \( \Re(s) > \mu \). Here, in analogy with \( R_1(s, x) \), \( R_2(s, x) \) denotes the sum of the residues of the meromorphic function \( R_2(z) = \Gamma(z) \phi_2(z) \Gamma(s - z) x^{-z} \) at the poles of \( \Gamma(z) \phi_2(z) \).

**Proof:** The idea of the proof follows [ [66], p. 311] and its sketched version given in [10]. Let \( \mu > \max \{ 0, 2\sigma_a, 2\sigma_b, \frac{\nu}{2} \} \) as in the statement and consider a fixed \( s \in \mathbb{C} \) such that \( \Re(s) > \mu \). Under this condition, the Beta transform integral [ [33], p. 349, 7.3.15]

\[
\frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(z) \Gamma(s-z) x^{-z} dz = \frac{\Gamma(s)}{(1+x)^s}, \quad x > 0
\]  

(2.5)
holds. Since our Epstein series (1.10) is defined as a double infinite series which converges absolutely for \( \text{Re}(s) > \mu \geq 2\sigma_a \), we may write it as

\[
\mathcal{D}_2(s, a_1, a_2; \lambda, \lambda') = a_1(0)\phi_2(s) + a_2(0)\phi_1(s) + \sum_{n=1}^{\infty} a_2(n) \ell_n(s, a_1),
\]

(2.6)

where \( \ell_n(s, a_1) \) denotes the generalized Dirichlet series \[6\] given by

\[
\ell_n(s, a_1) = \sum_{m=1}^{\infty} \frac{a_1(m)}{(\lambda_m + \lambda'_n)^s}, \quad \text{Re}(s) > \sigma_a.
\]

By the Mellin representation (2.5), we see that we can write \( \ell_n(s, a_1) \) as the contour integral

\[
\lambda_n^{s} \Gamma(s) \ell_n(s, a_1) = \sum_{m=1}^{\infty} \frac{a_1(m) \Gamma(s)}{(1 + \lambda_m)^s} = \sum_{m=1}^{\infty} \frac{a_1(m)}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(z) \Gamma(s-z) \left( \frac{\lambda_m}{\lambda'_n} \right)^{-z} dz,
\]

(2.7)

valid whenever \( \text{Re}(s) > \sigma_a \), which is the case as \( \text{Re}(s) > \mu > \max\{0, 2\sigma_a\} \geq \max\{0, 2\sigma_a\} \). From the definition of \( \mu \), we know that \( \sum a_1(m) \lambda_m^{-\mu} \) converges absolutely. Moreover, from a corollary of Stirling’s formula (1.15),

\[
|\Gamma(\mu + it) \Gamma(s - \mu - it)| = O\left( |t|^{|\text{Re}(s)| - 1} e^{-\pi|t|} \right), \quad |t| \to \infty,
\]

(2.8)

we are allowed to interchange the orders of integration and summation in (2.7). This gives the integral representation for the generalized Dirichlet series \( \ell_n(s, a_1) \),

\[
\lambda_n^{s} \Gamma(s) \ell_n(s, a_1) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \phi_1(z) \Gamma(z) \Gamma(s-z) \left( \frac{1}{\lambda'_n} \right)^{-z} dz, \quad \mu > \max\{0, \sigma_a, \sigma_b, r_1/2\}.
\]

Let us now move the line of integration to \( z = r_1 - \mu + it \) (which is on the left of \( \text{Re}(z) = \mu \) because \( \mu > 2 \sigma_a \geq r_1 \) by hypothesis) and integrate along a positively oriented rectangular contour \( \mathcal{R}_1 \) containing the vertices \( \mu \pm iT \) and \( r_1 - \mu \pm iT \), where \( T > \sup_{\rho \in \mathcal{S}} |\text{Im}(\rho)| \), with \( \mathcal{S} \) denoting the set of singularities of \( \chi_1(z) \Gamma(s-z) \). By the Residue Theorem, we have the equality

\[
\frac{1}{2\pi i} \int_{\mu - iT}^{\mu + iT} \phi_1(z) \Gamma(z) \Gamma(s-z) \lambda'_n^{-z} dz = \frac{1}{2\pi i} \int_{r_1 - \mu + iT}^{r_1 - \mu - iT} \phi_1(z) \Gamma(z) \Gamma(s-z) \lambda'_n^{-z} dz
\]

\[
+ \frac{1}{2\pi i} \int_{\mu \pm iT} \phi_1(z) \Gamma(z) \Gamma(s-z) \lambda'_n^{-z} dz + \sum_{\rho \in \mathcal{S}_1} \text{Res}_{z=\rho} \left( \phi_1(z) \Gamma(z) \Gamma(s-z) \left( 1/\lambda'_n \right)^{-z} \right),
\]

(2.9)

where the second integral represents the integrals over the horizontal lines. We now see that

\[
\int_{\mu \pm iT} \phi_1(z) \Gamma(z) \Gamma(s-z) \lambda'_n^{-z} dz \to 0 \quad \text{as} \quad T \to \infty,
\]

(2.10)

Note that this supremum taken over \( \mathcal{S} \) is finite from the assumption that the singularities of \( \chi_1(z) \) are contained in a compact set and the observation that all the singularities of \( \Gamma(s-z) \) have their imaginary part equal to \( \text{Im}(s) \).
by condition 2 on Definition \([1, 1]\). Letting \(T \to \infty\) in \([2.9]\), we obtain

\[
\lambda_n^{(s)} \Gamma(s) \ell_n(s, a_1) = \frac{1}{2\pi i} \int_{r_1 - \mu - i\infty}^{r_1 - \mu + i\infty} \phi_1(z) \Gamma(z) \left( \frac{1}{\lambda_n^s} \right)^{-z} dz + \sum_{\rho \in \mathfrak{R}_i} \text{Res}_{z=\rho} \left( \phi_1(z) \Gamma(z) \left( \frac{1}{\lambda_n^s} \right)^{-z} \right).
\]

We now claim that the last term of the previous equality is precisely \(R_1(s, 1/\lambda_n')\), whose definition is given at the statement of this result. It is easy to check by the functional equation for \(\phi_1(z)\), \([1, 7]\), and the definition of \(\mu\) that all poles of \(\chi(z) = \Gamma(z) \phi_1(z)\) are on the interior of \(\mathcal{R}_1\). Moreover, since \(\text{Re}(s) > \mu\) by hypothesis, we clearly have that \(\text{Re}(s) > 0\) for any \(z \in \mathcal{R}_1\), so that \(\Gamma(s - z)\) has no poles inside \(\mathcal{R}_1\) and the only poles taken into account in the sum in \([2.11]\) are precisely the ones coming from the function \(\Gamma(z) \phi_1(z)\). Thus, the previous equality reduces to

\[
\lambda_n^{(s)} \Gamma(s) \ell_n(s, a_1) = \frac{1}{2\pi i} \int_{r_1 - \mu - i\infty}^{r_1 - \mu + i\infty} \phi_1(z) \Gamma(z) \left( \frac{1}{\lambda_n^s} \right)^{-z} dz + R_1 \left( s, \frac{1}{\lambda_n'} \right).
\]

Let us now invoke the functional equation for the Dirichlet series \(\phi_1(z)\) \([1, 7]\) and use it in the contour integral given above. Doing this and taking the change of variables \(z \leftrightarrow r_1 - z\), we arrive at

\[
\lambda_n^{(s)} \Gamma(s) \ell_n(s, a_1) = \lambda_n^{(m)} \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \psi_1(z) \Gamma(z) \left( s - r_1 + z \right) \left( \lambda_n \right)^{-z} \left( \mu \right)^{1-z} dz + R_1 \left( s, \frac{1}{\lambda_n'} \right)
\]

\[
= \lambda_n^{(m)} \sum_{m=1}^{\infty} \frac{b_1(m)}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(z) \left( s - r_1 + z \right) \left( \mu \lambda_n' \right)^{-z} \left( \mu \right)^{1-z} dz + R_1 \left( s, \frac{1}{\lambda_n'} \right),
\]

where in the last step we have also used the absolute convergence of the Dirichlet series \(\psi_1(z)\) for \(\text{Re}(z) = \mu > \sigma_b \geq \sigma_{b_1}\) and Stirling’s formula for the product of \(\Gamma\)-functions.

Recalling the Mellin representation for the Modified Bessel function \([33, \text{p. 349, 7.3 (17)}]\)

\[
K_v(x) = \left( \frac{x}{2} \right)^v 1 \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(z - v) \Gamma(z) \left( \frac{x}{4} \right)^{-z} dz, \quad \lambda > \max\{0, \text{Re}(v)\},
\]

we see that, since \(\mu > \max\{0, r_1 - \text{Re}(s)\}\), we may represent the contour integral on the right-hand side of \([2.12]\) as

\[
\frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(z) \left( s - r_1 + z \right) \left( \mu \lambda_n' \right)^{-z} dz = 2 \left( \mu \lambda_n' \right)^{-s} K_{r_1 - s} \left( 2 \sqrt{\mu \lambda_n'} \right).
\]

Combining \([2.12]\) and \([2.14]\), we arrive at the following representation of the generalized Dirichlet series \(\ell_n(s, a_1)\) (c.f. \([6]\))

\[
\lambda_n^{(s)} \Gamma(s) \ell_n(s, a_1) = 2 \lambda_n^{(m)} \sum_{m=1}^{\infty} b_1(m) \left( \mu \lambda_n' \right)^{-\frac{r_1 - s}{2}} K_{r_1 - s} \left( 2 \sqrt{\mu \lambda_n'} \right) + R_1 \left( s, \frac{1}{\lambda_n'} \right).
\]
We can easily see that the double series on the right side of (2.6) converges absolutely if $\text{Re}(s) > \sigma_1 + \sigma_2$ and so it must converge absolutely for $\text{Re}(s) > \mu$. Since $\phi_1(s)$ is analytic in the region $\text{Re}(s) > \sigma_1$, it is clear from the expression for $R_1(s, x)$ that the series $\sum a_2(n) \lambda_n^{-s} R_1(s, \lambda_n^{-1})$ converges absolutely whenever $\text{Re}(s) > \sigma_2 + \max \{0, \sigma_1\}$. Hence, assuming that $\text{Re}(s) > \mu \geq \sigma_2 + \max \{0, \sigma_1\}$ and summing $a_2(n) \ell_n(s, a_1)$ with respect to $n$ as in (2.6), we arrive at the representation (2.2). To finish, we only need to argue that, under the assumption $\text{Re}(s) > \mu$, the double series which we have obtained by summing with respect to $n$, (2.1), converges absolutely. This is immediate due to the asymptotic estimate for the modified Bessel function

$$K_\nu(x) = O\left(x^{-\frac{\nu}{2}} e^{-x}\right), \quad x \to \infty$$

which implies that, for some positive constant $C$,

$$|H_{r_1}(s; b_1, a_2; \mu, \lambda')| \leq C \sum_{m,n=1}^\infty b_1(m) a_2(n) \left(\frac{\mu_m}{\lambda_n}\right)^{s-r_1} \left(\frac{\lambda_m}{\mu_n}\right)^{\frac{1}{\nu}} e^{-2\sqrt{\nu \mu_n \lambda_m}}$$

For any $\alpha > 0$, we know that $e^{-2\sqrt{\nu \mu_n \lambda_m}} \leq (\mu_m \lambda_n)^{-\alpha}$ for $m, n > N$, where $N$ is sufficiently large.$^8$ Choose $\alpha > \max \{\sigma_a - 1 - 2\alpha, \sigma_b - 1 + 2\alpha\}$. From this choice and the absolute convergence of $\phi_2$ and $\psi_1$ for $\text{Re}(s) > \max \{\sigma_a, \sigma_b\}$, we immediately see that the series defining $H_{r_1}$ converges absolutely for $\text{Re}(s) > \mu$.

In a completely analogous way, the second formula (2.4) may be obtained, with the same choice of $\mu$. In order to ensure that the identities (2.2) and (2.4) represent the same diagonal Epstein zeta function, we need to impose the possibility of reversing the order of summation in the double series (1.10). This is the case when we take $\text{Re}(s) > \max \{0, 2\sigma_a\}$, so our choice of $\mu$ allows to reverse the order. This concludes the proof.

Since the number of poles of $\phi_1$ and $\phi_2$ is finite (as they are contained in a compact set, by condition 3. in definition [11]), we can easily study the continuation of the residual series

$$\varphi_1(s) := \sum_{m=1}^\infty a_1(m) \frac{R_2(s, 1/\lambda_m)}{\lambda_m^s}, \quad \varphi_2(s) := \sum_{n=1}^\infty a_2(n) \frac{R_1(s, 1/\lambda_n)}{\lambda_n^s}.$$  (2.16)

Thus, if we warrant that any one of the Dirichlet series $(\phi_1, \phi_2, \psi_1, \psi_2)$ has analytic continuation to the entire complex plane, the only remaining step towards the extension of the Selberg-Chowla formula to any $s \in \mathbb{C}$ is the analytic continuation of the functions $H_{r_j}(s, a)$. As in the case of the classical Selberg-Chowla formula [4], we will show that $H_{r_1}$ and $H_{r_2}$ represent entire functions of $s$. Although the proof by Bateman and Grosswald [4] works mutatis mutandis in our case, we add, for completeness, an alternative proof of

$^8$note that $(\nu_j)_{j \in \mathbb{N}} := (\mu_m \lambda_n)_{m,n \in \mathbb{N}}$ may be arranged into an increasing sequence and we can impose the inequality $e^{-2\sqrt{\nu}} \leq \nu^{-\alpha}$ for $j \geq N_0$. 

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this fact, which employs the Mellin representation (2.13) of the modified Bessel function. A similar proof, adapted in a different context, is given in [10], p. 181.

**Proposition 2.1** (The analytic continuation of \( H \)). Let \( H_1 \) and \( H_2 \) be the double series given in (2.17) and (2.12). Then they represent entire functions in the variable \( s \in \mathbb{C} \) and obey to the functional equation

\[
H_1 \left( s; a_1, b_2; \lambda, \mu' \right) = H_2 \left( r_1 + s; b_2, a_1; \mu', \lambda \right).
\]

(2.17)

**Proof.** As in the previous Theorem, let \( \mu > \max \left\{ 0, 2\sigma_a, 2\sigma_b, \frac{r'}{2} \right\} \) and take \( \eta > \mu \). For any \( \varepsilon > 0 \), consider a rectangle \( \mathcal{R}_{\varepsilon, \eta}(T) \), \( T > 0 \), whose vertices are \( r' - \mu + \sigma_a + \varepsilon i T \) and \( \eta \pm i T \). Assuming that \( s \in \mathcal{R}_{\varepsilon, \eta}(T) \) and returning to the representation (2.12), we can write the function \( H_1 \) as

\[
H_1 \left( s; a_1, b_2; \lambda, \mu' \right) := \sum_{m,n=1}^{\infty} b_1(m) a_2(n) \lambda_{mn}^{r_1-s} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(z) \Gamma(s-r_1+z) (\mu_{mn} \lambda_{n}^{r})^{-z} dz.
\]

(2.18)

It is readily seen from (2.18) that we can argue the uniform convergence of the previous double series in \( \mathcal{R}_{\varepsilon, \eta}(T) \) by estimating the contour integral appearing on the right-hand side of (2.18). To do this, we will set \( s := \sigma + it \in \mathcal{R}_{\varepsilon, \eta}(T) \) and split the integral concerning the vertical line \( \mu - i\infty, \mu + i\infty \) into three parts: \( (\mu - iT, \mu - iT + 1) \), \( (\mu - iT + 1, \mu + iT + 1) \) and \( (\mu + iT + 1, \mu + i\infty) \). The bound for the first of these integrals is easy to obtain and is simply given by

\[
\left| \int_{\mu-i(T+1)}^{\mu+i(T+1)} \Gamma(z) \Gamma(s-r_1+z) (\mu_{mn} \lambda_{n}^{r})^{-z} dz \right| \leq C (\mu_{mn} \lambda_{n}^{r})^{-\mu} \Gamma(\mu) \leq C' (\mu_{mn} \lambda_{n}^{r})^{-\mu} \Gamma(\sigma-r_1+\mu) \Gamma(\mu) \leq C' (\mu_{mn} \lambda_{n}^{r})^{-\mu} \Gamma(\sigma-r_1+\mu-\varepsilon)
\]

(2.19)

where \( C' \) only depends on the maximum value that \( \Gamma(x) \) attains in the interval \( \sigma_a + r' - r_1 + \varepsilon < x < \eta + \mu - r_1 \) and thus (2.19) does not depend of \( s \in \mathcal{R}_{\varepsilon, \eta}(T) \). From Stirling’s formula, it is also immediate to arrive at the second bound

\[
\left| \int_{\mu+i(T+1)}^{\mu+i\infty} \Gamma(z) \Gamma(s-r_1+z) (\mu_{mn} \lambda_{n}^{r})^{-z} dz \right| \leq C (\mu_{mn} \lambda_{n}^{r})^{-\mu} \int_{T+1}^{\infty} y^{\mu-\frac{1}{2}} e^{-\pi y} (y+1)^{\sigma-r_1+\mu-\frac{1}{2}} dy
\]

\[
\leq C (\mu_{mn} \lambda_{n}^{r})^{-\mu} \int_{T+1}^{\infty} y^{\mu-\frac{1}{2}} e^{-\pi y} (y+1)^{\sigma-r_1+\mu-\frac{1}{2}} dy \leq C' (\mu_{mn} \lambda_{n}^{r})^{-\mu}
\]

(2.20)

where, once more, \( C' \) does not depend also on \( s \in \mathcal{R}_{\varepsilon, \eta}(T) \). Due to the symmetry \( \Gamma(\zeta) = \overline{\Gamma(\zeta)} \), it is simple to see that a similar estimate to (2.20) holds when we integrate from \( \mu - i\infty \) to \( \mu - iT + 1 \). By (2.18), we see that

\[
|H_1 (s; a_1, b_2; \lambda, \mu')| \leq C \sum_{m=1}^{\infty} \frac{|b_1(m)|}{\mu_m} \sum_{n=1}^{\infty} \frac{|a_2(n)|}{\lambda_n^{\mu+\sigma-r_1}}
\]

where \( \sigma > r' - \mu + \sigma_a + \varepsilon \) (since \( s \in \mathcal{R}_{\varepsilon, \eta}(T) \)) and \( C \) does not depend on \( s \). Since the right-hand side of the previous inequality is the product of two absolutely convergent series, an application of the Weierstrass M-test leads to the conclusion that the infinite series appearing in (2.18) converges absolutely and uniformly on \( \mathcal{R}_{\varepsilon, \eta}(T) \). Since \( \mu \) can be taken sufficiently large, we know that every bounded subset of \( \mathbb{C} \) is contained
in some rectangle of the form \( \mathcal{R}_{E, \eta}(T) \) and this shows that the double series (2.47) converges absolutely and uniformly in every bounded subset of the complex plane. From the fact that the double sequence of functions \( f_{m,n}(s) = 2 (\mu_m / \lambda_n^2) \sum_{r=1}^{s} \left( \frac{1}{2} \sqrt{\mu_m \lambda_n^2} \right) \) represents, for each pair of integers \((m, n) \in \mathbb{N}^2\), an analytic function of \( s \), we thus have that \( H_{r_1}(s; b_1, a_2; \mu, \lambda') \) is an entire function.

Analogously, one may see that \( H_{r_2} \) converges uniformly on any rectangle \( \mathcal{R}_{E, \eta}(T) \). The same can be concluded after an application of the reflection formula (2.17). By analytic continuation, this shows that both representations (2.2) and (2.4) hold for the Epstein zeta function (1.10). Finally, it is clear that (2.17) comes immediately from the property for the Modified Bessel function \( K_{\nu}(z) = K_{-\nu}(z) \).

\[ \text{Remark 2.1.} \] It should be also pointed out that, even if one of the Dirichlet series \( \phi_1 \) or \( \phi_2 \) does not possess a functional equation of Hecke type (1.7), one of the formulas (2.2) or (2.4) is still valid, since the only tool required to prove these is the Functional equation of only one of the Dirichlet series. Although the appropriate Selberg-Chowla formula provides the analytic continuation of \( \mathcal{Z}_2 \) in this case, as it will be clear, we cannot obtain a suitable functional equation for it.

**Definition 2.1 (A Generalized Divisor function).** Let \( \{v_j\}_{j \in \mathbb{N}} \) represent the "product sequence" \( v_j = \{\mu_m \cdot \lambda_n'\}_{m,n \in \mathbb{N}} \) and then arrange this product in increasing order. We define the generalized divisor function as

\[
\sigma_{\nu}(v_j; b_1, a_2; \mu, \lambda') = \sum_{(m,n) \in \mathbb{N}^2: \mu_m \lambda_n' = v_j} b_1(m) a_2(n) \mu_m^2. \tag{2.21}
\]

With the notation introduced in Definition 2.1, we have that \( H_{r_1} \) can be described by the series

\[
H_{r_1}(s; b_1, a_2; \mu, \lambda') = \sum_{j=1}^{\infty} \sigma_{s-r_1}(v_j; b_1, a_2; \mu, \lambda') v_j^{s-r_1} K_{r_1-s}(2 \sqrt{v_j}); \tag{2.22}
\]

which strongly resembles the entire part appearing in the classical Selberg-Chowla formula (1.2). Analogously, if we write \( \{v'_j\}_{j \in \mathbb{N}} = \{\mu_m' \lambda_n\}_{m,n \in \mathbb{N}} \), it is also evident that we can write \( H_{r_2} \) as

\[
H_{r_2}(s; b_2, a_1; \mu', \lambda) = \sum_{j=1}^{\infty} \sigma_{s-r_2}(v'_j; b_2, a_1; \mu', \lambda) v'_j^{s-r_2} K_{r_2-s}(2 \sqrt{v'_j}).
\]

Taking \( z = 0 \) in (2.21), we also get a generalization of the standard divisor function \( d(n) = \sum_{d|n} 1 \),

\[
d(v_j; b_1, a_2; \mu, \lambda') = \sum_{(m,n) \in \mathbb{N}^2: \mu_m \lambda_n' = v_j} b_1(m) a_2(n). \tag{2.23}
\]

Having proved that \( H_{r_i} \), for each \( i = 1, 2 \), is entire, we are ready to establish the analytic continuation of the Epstein series \( \mathcal{Z}_2(s; a_1, a_2; \lambda, \lambda') \). To do this, we need to understand the continuation of the residual functions given in the proof of Theorem 2.1 \( \phi_1(s) \) and \( \phi_2(s) \), appearing in (2.16). Although we can proceed the study of the analytic continuation of \( \mathcal{Z}_2(s; \cdot) \) in a general form (see Remark 2.3), we state all the next corollaries in the following class of Dirichlet series:
**Definition 2.2.** Let \( \phi(s) \) be a Dirichlet series satisfying Definition [1.1] with \( \Delta(s) = \Gamma(s) \). We say that \( \phi \) belongs to the class \( \mathcal{A} \) if additionally:

1. \( \phi \) and \( \psi \) have analytic continuations to the entire complex plane and are analytic on \( \mathbb{C} \) except for possible simple poles located at \( s = r \) with residues \( \rho \) and \( \rho^* \) respectively.

2. \( a(n) \) can be defined at 0 and has the value \(-\phi(0)\). \( b(n) \) can be analogously extended to \( n = 0 \) as \(-\psi(0)\).

**Remark 2.2.** Note that, by the functional equation (2.27) and condition 1 in the previous definition, \( \phi_i(0) = -\rho_i^* \Gamma(r_i) \), while \( \psi_i(0) = -\rho_i \Gamma(r_i) \). Under the symmetry imposed by the functional equation and the first condition, it is clear that \( \phi \in \mathcal{A} \) if and only if \( \psi \in \mathcal{A} \). The purpose of introducing this class is to mimic as much as possible the class of Dirichlet series with a given signature, which is a clear subclass of \( \mathcal{A} \). Besides, this class reproduces in a general form what happens in the classical cases of diagonal Epstein zeta functions. As a corollary of the Selberg-Chowla formulas given in Theorem 2.1 in the next result we prove that, if \( \phi_1, \phi_2 \in \mathcal{A} \), then \( \mathcal{A}_2 \in \mathcal{A} \). Hence, this class is closed under composition of Dirichlet series attached to diagonal Epstein zeta functions. A similar class was also considered in [5, p. 221].

From the functional equation for \( \phi \) and under the hypotheses given for the class \( \mathcal{A} \), it is simple to see that the residual function \( R_1(s, x) \) appearing in (2.2) is given by

\[
R_1(s, x) = \phi_1(0) \Gamma(s) + \phi_1 \Gamma(r_1) \Gamma(s-r_1) x^{-r_1},
\]

so that, for \( \text{Re}(s) > \mu > \max \left\{ 0, 2\sigma_0, 2\sigma_b, \frac{\pi}{2} \right\} \), we can easily see that

\[
\phi_2(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{\lambda_2^n} R_1(s, 1/\lambda_2) = \phi_1(0) \Gamma(s) \phi_2(s) + \phi_1 \Gamma(r_1) \Gamma(s-r_1) \phi_2(s-r_1). \tag{2.25}
\]

Likewise, one may find \( R_2(s, x) \) and \( \phi_1(s) \). Invoking the fact that \( a_i(0) = -\phi_i(0) \), we see that formulas (2.2) and (2.4), when considered over the class \( \mathcal{A} \), are

\[
\Gamma(s) \mathcal{Z}_2 \left( s; a_1, a_2; \lambda, \lambda' \right) = -\phi_2(0) \Gamma(s) \phi_1(s) + \phi_1 \Gamma(r_1) \Gamma(s-r_1) \phi_2(s-r_1)
+ 2 \mathcal{H}_1 \left( s; b_1, a_2; \mu, \lambda' \right) \tag{2.26}
\]

and

\[
\Gamma(s) \mathcal{Z}_2 \left( s; a_1, a_2; \lambda, \lambda' \right) = -\phi_1(0) \Gamma(s) \phi_2(s) + \phi_2 \Gamma(r_2) \Gamma(s-r_2) \phi_1(s-r_2)
+ 2 \mathcal{H}_2 \left( s; b_2, a_1; \mu', \lambda \right). \tag{2.27}
\]

Using both representations (2.26) and (2.27) as well as the type of continuation which they yield, we now establish the following Corollary.
Corollary 2.1. Let \( \phi_1 \) and \( \phi_2 \) satisfy the conditions of the class \( \mathcal{A} \) and having residues \( \rho_1 \) and \( \rho_2 \) at \( s = r_1 \) and \( s = r_2 \) respectively. Then the Selberg-Chowla formula (2.2) provides the analytic continuation of the diagonal Epstein \( \zeta \)-function (1.10) as a meromorphic function having a simple pole at \( s = r_1 + r_2 \) with residue \( \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)} \rho_1\rho_2 \). Moreover, the continuation of \( \mathcal{Z}_2 \) obeys to the functional equation

\[
\Gamma(s) \mathcal{Z}_2 \left(s; a_1, a_2; \lambda, \lambda'\right) = \Gamma(r_1 + r_2 - s) \mathcal{Z}_2 \left(r_1 + r_2 - s; b_1, b_2; \mu, \mu'\right).
\] (2.28)

Proof. First, we shall use the first representation for \( \mathcal{Z}_2 \) (2.2), which, under restriction to the class \( \mathcal{A} \), reduces to (2.26). We have seen in the previous proposition that \( H_{r_1} \) is an entire function, so the meromorphic part of the Epstein \( \zeta \)-function \( \mathcal{Z}_2 \left(s; a_1, a_2; \lambda, \lambda'\right) \) comes from the first two terms in (2.26), i.e., from the meromorphic function

\[
G_{r_1} \left(s; a_1, a_2; \lambda, \lambda'\right) = -\phi_2(0) \phi_1(s) + \rho_1 \Gamma(r_1) \frac{\Gamma(s-r_1)}{\Gamma(s)} \phi_2(s-r_1).
\] (2.29)

Clearly, from a standard verification, \( G_{r_1} \) has removable singularities at \( s = r_1 - k \), \( k \in \mathbb{N}_0 \). Since, by the definition of the class \( \mathcal{A} \), \( \phi_1(s) \) and \( \Gamma(s-r_1) \) are analytic in a neighbourhood of \( s = r_1 + r_2 \), \( G_{r_1} \) has at most one pole at \( s = r_1 + r_2 \) coming from the factor \( \phi_2(s-r_1) \). The residue is easily seen to be \( \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)} \rho_1\rho_2 \).

To prove the functional equation (2.28), let us use both representations (2.26) (2.27) for \( \mathcal{Z}_2 \) and replace \( s \) by \( r_1 + r_2 - s \) on (2.26). From the reflection formula for \( H_{r_1} \) (2.17), we see that

\[
H_{r_1} \left(r_1 + r_2 - s; b_1, a_2; \mu, \lambda'\right) = H_{r_2} \left(s; a_2, b_1; \lambda', \mu\right).
\] (2.30)

Thus, to describe \( \mathcal{Z}_2 \left(r_1 + r_2 - s; \cdot\right) \), we need to see how \( G_{r_1} \) behaves under the reflection \( s \leftrightarrow r_1 + r_2 - s \). Since \( \phi_1(0) = -\rho_1^* \Gamma(r_1) \) and \( \psi_i(0) = -\rho_i \Gamma(r_1) \) (by condition 2. on the class \( \mathcal{A} \) and the functional equation for \( \phi_i(s) \)), we have that

\[
\Gamma(r_1 + r_2 - s) G_{r_1} \left(r_1 + r_2 - s; a_1, a_2; \lambda, \lambda'\right) = -\phi_2(0) \Gamma(s-r_2) \psi_1(s-r_2) + \rho_1 \Gamma(r_1) \Gamma(s) \psi_2(s)
\]

\[
= \rho_2^* \Gamma(r_2) \Gamma(s-r_2) \psi_1(s-r_2) - \psi_2(0) \Gamma(s) \psi_2(s),
\] (2.31)

where in the penultimate equality we have used the functional equation for \( \phi_i \) directly. Now, by equations (2.26), (2.30) and (2.31), we see that

\[
\Gamma(r_1 + r_2 - s) \mathcal{Z}_2 \left(r_1 + r_2 - s; a_1, a_2; \lambda, \lambda'\right) = -\psi_1(0) \Gamma(s) \psi_2(s) + \rho_2^* \Gamma(r_2) \Gamma(s-r_2) \psi_1(s-r_2) +
\]

\[
+ 2H_{r_2} \left(s; a_2, b_1; \lambda', \mu\right).
\] (2.32)

Using the second representation of the Selberg-Chowla formula (2.27), and replacing there the arithmetical functions \( a_1(n) \) by \( b_1(n) \) and \( a_2(n) \) by \( b_2(n) \) or, by other words, using the fact that \( \psi_1, \psi_2 \in \mathcal{A} \) and applying the work done in Theorem 2.1 for the diagonal Epstein \( \zeta \)-function,

\[
\mathcal{Z}_2 \left(s; b_1, b_2; \mu, \mu'\right) = \sum_{m,n \neq 0} b_1(m) b_2(n) (\mu_m + \mu'_n)^s, \quad \text{Re}(s) > 2\sigma_b,
\] (2.33)
we find immediately that the right-hand side of (2.32) is precisely the same as the one given by (2.27) under the above mentioned substitutions. This establishes the functional equation (2.28) and completes the proof.

**Remark 2.3.** Although the totality of the examples given in the fifth section of this paper concerns the class \( \mathcal{A} \), we should remark that the previous corollary, as well as the results given in section 4, hold for a (much more) general class of Dirichlet series satisfying Definition 1.1 and admitting analytic continuations into the entire complex plane. However, in order to have a complete description of the analytic continuations of the Epstein zeta functions (1.10) and (1.14), we have decided to omit the details of this general case. In this remark we briefly see that the properties of a function satisfying definition 1.1 are preserved by the Epstein zeta function. Note that, if \( C_1 \) denotes a curve, or curves, chosen so that it encircles all the singularities of \( \Gamma(z) \phi_1(z) \), it is then clear that \( R_1(s, x) \) (given in the statement of Theorem 2.1) can be written as

\[
R_1(s, x) = \frac{1}{2\pi i} \int_{C_1} \Gamma(s - w) \Gamma(w) \phi_1(w) x^{-w} dw, \quad \text{Re}(s) > \mu,
\]

and so the residual series \( \varphi_2(s) \) appearing in (2.16) may be expressed as

\[
\varphi_2(s) = \frac{1}{2\pi i} \int_{C_1} \Gamma(s - w) \varphi_2(s - w) \Gamma(w) \phi_1(w) dw, \quad \text{Re}(s) > 2\mu,
\]

which can be extended to other values of \( s \) by analytic continuation (with an analogous expression holding for \( \varphi_1(s) \)). Note that the right-hand side of (2.34) is analytic when \( \text{Re}(s) \) is taken sufficiently large. Supposing that each \( \phi_i(s) \) has an analytic continuation to the entire complex plane, (2.34) (resp. \( \varphi_1(s) \)) represents a complex meromorphic function, having its singularities also confined to a compact set. Thus, the first assertion proved in Corollary 2.1 holds outside the class \( \mathcal{A} \).

To deduce that we still have the functional equation (2.28) in this general case, note that if \( C_1 \) encircles the singularities of \( \Gamma(w) \phi_1(w) \), then \( C_1' := r_1 - C_1 = \{r_1 - z, z \in C_1\} \) encircles all the singularities of \( \Gamma(w) \psi_1(w) \), by virtue of definition 1.1. Note now that \( \varphi_2(s) \) satisfies a functional equation of the form

\[
\varphi_2(r_1 + r_2 - s) = \frac{1}{2\pi i} \int_{C_1} \Gamma(s - w) \varphi_2(s - w) \Gamma(w) \phi_1(w) dw = \frac{1}{2\pi i} \int_{C_1'} \Gamma(s - w) \psi_2(s - w) \Gamma(w) \psi_1(w) dw,
\]

where the right-hand side of the previous equality represents the residual series appearing in the first Selberg-Chowla formula for \( \mathcal{Z}_2(s; b_1, b_2; \mu, \mu') \).

**Remark 2.4.** Alternative proofs of Theorem 2.1 and Corollary 2.1 can be given by using Bochner’s modular relation (1.21) to one of the Dirichlet series \( \phi_1(s) \) or \( \phi_2(s) \). Note that the left-hand side of (2.2) can be
described, for \( \text{Re}(s) > \mu > \max \left\{ 0, 2\sigma_d, 2\sigma_0, \frac{\rho}{2} \right\} \), as the Mellin transform

\[
\Gamma(s) \mathcal{L}_2 (s; a_1, a_2; \lambda, \lambda') = \int_0^\infty x^{s-1} \sum_{m,n \neq 0} a_1(m) a_2(n) e^{-\left(\lambda_m + \lambda'_n\right)x} \, dx
= a_1(0) \int_0^\infty x^{s-1} \Theta(x; a_2; \lambda') \, dx + a_2(0) \int_0^\infty x^{s-1} \Theta(x; a_1; \lambda) \, dx + \int_0^\infty x^{s-1} \Theta(x; a_1; \lambda) \Theta(x; a_2; \lambda') \, dx.
\]

(2.36)

It is then evident that, if we invoke the reflection formula for the Theta function \( \Theta(x; a_1; \lambda) \) \eqref{reflection_formula} on the third integral above and then use the definition of the residual function \( P_1(x) \) \eqref{residual_function_definition}, as well as arguing by absolute convergence, the first two summands and the residual terms coming on the third integral give the first three terms appearing on the right-hand side of \eqref{integral_representation}. Lastly, the remaining integral can be evaluated as follows

\[
\int_0^\infty x^{r_1-1} \Theta\left(\frac{1}{x}; b_1; \mu\right) \Theta(x; a_2; \lambda') \, dx = \sum_{m,n=1}^\infty b_1(m) a_2(n) \int_0^\infty x^{r_1-1} \exp \left(-\lambda_n x - \mu_m / x\right) \, dx
= 2 \sum_{m,n=1}^\infty b_1(m) a_2(n) \left(\frac{\mu_m}{\lambda_n}\right)^{\frac{r_1}{2}} K_{r_1-\delta} \left(2 \sqrt{\mu_m \lambda_n}\right) := 2 \kappa_1\left(s; b_1, a_2, \mu, \lambda'\right),
\]

where we have used the representation of the Modified Bessel function \( K_\nu(x) \) as the Mellin convolution of exponentials [39, eq. 3.471.9, p. 368]

\[
\int_0^\infty x^{s-1} e^{-\beta x} e^{-\gamma / x} \, dx = 2 \left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}} K_s \left(2 \sqrt{\beta \gamma}\right), \quad \text{Re}(\beta), \text{Re}(\gamma) > 0.
\]

(2.37)

The proof of the functional equation \eqref{functional_equation} for \( \phi_1, \phi_2 \in \mathcal{A} \) can be given by invoking the reflection formula for both theta functions \( \Theta(x; a_1; \lambda) \) and \( \Theta(x; a_2; \lambda') \). If we denote the theta function of the Dirichlet series \( \mathcal{L}_2(s; \cdot) \) by \( \Theta_2(x; a_1, a_2; \lambda, \lambda') \) we can derive a particular case of the modular relation \eqref{reflection_formula},

\[
\Theta_2(x; a_1, a_2; \lambda, \lambda') = x^{-r_1-r_2} \Theta_2\left(x^{-1}; b_1, b_2; \mu, \mu'\right) + \Gamma(r_1) \Gamma(r_2) \rho_1 \rho_2 x^{-r_1-r_2}
- \Gamma(r_1) \Gamma(r_2) \rho_1^* \rho_2^*, \quad x > 0,
\]

(2.38)

which is equivalent to \eqref{functional_equation} (see also [8, p. 342, Thm. 8.1] for an analogous argument with a single series). In Theorem \ref{main_theorem} we shall prove the Selberg-Chowla for the non-diagonal Epstein zeta function \eqref{selberg_chowla} using this observation. Note also that Suzuki’s proof [70] of the analogues of the Selberg-Chowla formula for \( \zeta(s) \) and other Dirichlet series such as \( L(s, \chi) \) and \( L(s, a) \) (see example 3.5) follows the lines given in this remark.

We now introduce a class of multidimensional Epstein zeta functions and prove functional equations for them by mimicking the argument given in Corollary \ref{corollary}. If \( \{\phi_i\}_{i=1, \ldots, k} \) denotes a k-tuple of Dirichlet series
Assume where, in analogy with representable by the series
\[ V = \sum_{n=1}^{\infty} \frac{a_i(n)}{\lambda_{n,i}}, \text{ Re}(s) > \sigma_{a_i}, \]
then we can define the multidimensional Epstein zeta function as
\[ \mathcal{Z}_k(s; a_1, \ldots, a_k; \lambda_1, \ldots, \lambda_k) := \sum_{n_1, \ldots, n_k \neq 0} \frac{a_1(n_1) \cdots a_k(n_k)}{(\lambda_{n_1,1} + \cdots + \lambda_{n_k,k})^s}, \text{ Re}(s) > k\sigma_{a_k}, \]
where \( \sigma_{a_k} = \max_{1 \leq i \leq k} \sigma_{a_i} \). As in representation (1.12), we can write \( \mathcal{Z}_k \) as the Dirichlet series
\[ \mathcal{Z}_k(s; a_1, \ldots, a_k; \lambda_1, \ldots, \lambda_k) = \sum_{n=1}^{\infty} \frac{\Omega_k(n)}{\Lambda_{a_k}^s}, \text{ Re}(s) > k\sigma_{a_k} \]
where, in analogy with \( \Omega_2(n) \) given in (1.13),
\[ \Omega_k(n) = \sum_{\lambda_{j_1} + \cdots + \lambda_{j_k} = \Lambda_a} a_1(j_1) \cdots a_k(j_k). \]

Suppose also that \( \{ \psi_i \}_{i=1}^{k} \) denotes the k-tuple of Dirichlet series conjugate to each \( \phi_i \) via Hecke’s functional equation (1.7) and being representable by the series
\[ \psi_i(s) = \sum_{n=1}^{\infty} \frac{b_i(n)}{\mu_{n,i}^s}, \text{ Re}(s) > \sigma_{b_i}, \]
then we can write \( \mathcal{Z}_k(s; b_1, \ldots, b_k; \mu_1, \ldots, \mu_k) \) also as a single Dirichlet series of the form
\[ \mathcal{Z}_k(s; b_1, \ldots, b_k; \mu_1, \ldots, \mu_k) = \sum_{n=1}^{\infty} \frac{\Omega_k(n)}{\Omega_{\mu_k}^s}, \text{ Re}(s) > k\sigma_{\mu_k}, \]
where \( \Omega_k(n) \) plays the same role as \( \Omega_k(n) \) and \( \sigma_{b_k} = \max_{1 \leq i \leq k} \sigma_{b_i} \). In analogy with (2.28) one should expect a functional equation connecting (2.40) and (2.41). In fact, this is the case and the next corollary establishes it.

**Corollary 2.2 (The Analytic Continuation of the Multidimensional diagonal Epstein zeta function).** Assume that \( \{ \phi_i \}_{i=1}^{k} \) are Dirichlet series satisfying Hecke’s functional equation with parameter \( r_i > 0 \) and belonging to the class \( \mathcal{A} \). Then (2.39) has an analytic continuation as a Dirichlet series belonging to the class \( \mathcal{A} \). This continuation has at most a simple pole located at \( s = \sum_{j=1}^{k} r_j \) and with residue
\[ \text{Res}_{s=\sum_{j=1}^{k} r_j} \mathcal{Z}_k(s; a_1, \ldots, a_k; \lambda_1, \ldots, \lambda_k) = \frac{\prod_{j=1}^{k} \Gamma(r_j) \rho_j}{\Gamma(\sum_{j=1}^{k} r_j)}. \]
Moreover, \( \mathcal{Z}_k \) obeys to the functional equation
\[ \Gamma(s) \mathcal{Z}_k(s; a_1, \ldots, a_k; \lambda_1, \ldots, \lambda_k) = \Gamma(r_1 + \cdots + r_k - s) \mathcal{Z}_k(r_1 + \cdots + r_k - s; b_1, \ldots, b_k; \mu_1, \ldots, \mu_k). \]
Proof. We have seen in Corollary 2.1 that (2.42) and (2.43) hold for \( k = 2 \). By induction, assume now that \( \mathscr{Z}_{k-1} (s; a_1, \ldots, a_{k-1}; \lambda_1, \ldots, \lambda_{k-1}) \) belongs to the class \( \mathscr{A} \) and satisfies Hecke’s functional equation with parameter \( r = \sum_{j=1}^{k-1} r_j \) and has a residue at \( s = \sum_{j=1}^{k-1} r_j \) given by

\[
\text{Res}_{s=\sum_{j=1}^{k-1} r_j} \mathscr{Z}_{k-1} = \frac{\prod_{j=1}^{k-1} \Gamma(r_j) \rho_j}{\Gamma(\sum_{j=1}^{k-1} r_j)},
\]

By the first Selberg-Chowla formula (2.26) applied to \( \mathscr{Z}_{k-1} \) and \( \phi_k(s) \) (both satisfying Hecke’s functional equation (1.7) by the induction hypothesis) and using the representation (2.41), we see that \( \mathscr{Z}_k \) can be written as

\[
-\phi_k(0) \mathscr{Z}_{k-1} (s; a_1, \ldots, a_{k-1}; \lambda_1, \ldots, \lambda_{k-1}) + \prod_{j=1}^{k-1} \Gamma(r_j) \rho_j \frac{\Gamma(s-r_1-\ldots-r_{k-1})}{\Gamma(s)} \phi_k (s-r_1-\ldots-r_{k-1})
+ \frac{2}{\Gamma(s)} \sum_{m,n=1}^{\infty} \Omega_k(m) \Omega_{k-1}(n) \left( \frac{\mu_m}{\Lambda_{m,k-1}} \right)^{s+r_1+\ldots+r_{k-1}} K_{r_1+\ldots+r_{k-1}-s} \left( 2 \sqrt{\Omega_{m,k-1} \Lambda_{m,k-1}} \right). \tag{2.44}
\]

It is clear from a simple adaptation of Proposition 2.1 that the last term in (2.44) represents an entire function. Hence, the possible poles of (2.39) come from the sum of the first two terms. It is clear once more that \( \mathscr{Z}_k \) has removable singularities at \( s = r_1 + \ldots + r_{k-1} - n, n \in \mathbb{N}_0 \). This comes from the functional equation for \( \phi_k \) and the induction hypothesis on \( \mathscr{Z}_{k-1} \). Thus, the only possible pole of \( \mathscr{Z}_k \) comes from the factor \( \phi_k (s-r_1-\ldots-r_{k-1}) \) on the second term of the previous expression and it is located at \( s = r_1 + \ldots + r_k \).

The computation of its residue gives (2.42).

For proving the functional equation, it suffices to compare the representation (2.44) with the second representation coming from (2.27).

\[
-\mathscr{Z}_{k-1} (0; a_1, \ldots, a_{k-1}; \lambda_1, \ldots, \lambda_{k-1}) \phi_k(s) + \rho_k \Gamma(r_k) \frac{\Gamma(s-r_k)}{\Gamma(s)} \times \mathscr{Z}_{k-1} (s-r_k; a_1, \ldots, a_{k-1}; \lambda_1, \ldots, \lambda_{k-1})
+ \frac{2}{\Gamma(s)} \sum_{m,n=1}^{\infty} b_k(m) \Omega_k(n) \left( \frac{\mu_m}{\Lambda_{m,k-1}} \right)^{s+r_k+\ldots+r_{k-1}} K_{r_k+\ldots+r_{k-1}-s} \left( 2 \sqrt{\mu_m \Lambda_{m,k-1}} \right),
\]
and use the induction hypothesis. By adapting the argument given in the proof of Corollary 2.1 and replacing the roles of \( \phi_1 \) and \( \phi_2 \) there by \( \phi_k \) and \( \mathscr{Z}_{k-1} \), we arrive immediately at (2.43). \( \square \)

Remark 2.5 (The dyadic Epstein zeta function). Let \( k \geq 0 \) and consider the case where \( \phi_1 = \phi_2 = \ldots = \phi_{2^k} : = \phi \in \mathscr{A} \), with \( \phi(s) \) satisfying Hecke’s functional equation with parameter \( r \) and abcissa of absolute convergence \( \sigma_r \). One of the \( 2^{k+1} - 1 \) ways of constructing the Epstein zeta function \( \mathscr{Z}_{2^k+1} \) is by taking the symmetric construction

\[
\mathscr{Z}_{2^k} (s; a; \lambda) = \sum_{m=1}^{\infty} \frac{\Omega_{2^k}(m)}{\Lambda_m}, \quad \mathscr{Z}_{2^{k+1}} (s; a; \lambda) = \sum_{m, n \neq 0} \frac{\Omega_{2^k}(m) \Omega_{2^k}(n)}{(\Lambda_m + \Lambda_n)^3}, \tag{2.45}
\]
with each series converging absolutely for \( \text{Re}(s) > 2^k\sigma_a, \ 2^{k+1}\sigma_a \) respectively (note the convention \( Z_{2x}(s; a; \lambda) := \phi(s) \)). Each \( Z_{2x}(s; a; \lambda) \) plays the role of \( \phi_1 \) and \( \phi_2 \) in Theorem 2.1 and Corollary 2.2. By an application of Corollary 2.2, it is clear that we can write a Selberg-Chowla formula in the form

\[
\Gamma(s) Z_{2s+1}(s; a; \lambda) = - Z_{2s}(0; a; \lambda) \Gamma(s) Z_{2s}(s; a; \lambda) + \Gamma(s-2^k r) \rho^{2^k \Gamma(s-2^k r)} Z_{2s} (s-2^k r; a; \lambda) + 2H_{2s}(s; a; \lambda),
\]

(2.46)

where \( H_{2s}(s; a; \lambda) \) denotes the entire function

\[
H_{2s}(s; a; \lambda) = \sum_{m,n=1}^{\infty} \mathbb{Q}_{2s}(m) \mathbb{Q}_{2s}(n) \left( \frac{\Omega_m}{\Lambda_n} \right)^{\frac{s}{2^k-1}} K_{2s-r-s} \left( 2\sqrt{\frac{\Omega_m \Lambda_n}{4}} \right),
\]

(2.47)

with the arithmetical functions \( \mathbb{Q}_{2s}(n) \) and \( \mathbb{Q}_{2s}(n) \) appearing respectively in the Dirichlet series representations (2.40) and (2.41). From Corollary 2.2, we know that the multidimensional Epstein zeta function \( Z_{2s}(s; a; \lambda) \) satisfies Hecke’s functional equation (2.43) with parameter \( 2^k r \) and so its critical line is \( \text{Re}(s) = 2^{k-1} r \). From the Phragmén-Lindelöf principle (1.17), we have that, for any \( \delta > 0 \) and \( 2^k r - 2^k \sigma_a - \delta \leq \sigma \leq 2^k \sigma_a + \delta \),

\[
Z_{2s}(\sigma+it; a; \lambda) = O\left( |t|^{2^k \sigma_a - \sigma + \delta} \right), \quad |t| \to \infty.
\]

(2.48)

Equations (2.46) and (2.48) will be used in the penultimate section of the paper, where we shall prove a class of Hardy Theorems for Dirichlet series with narrow critical strips (see Lemma 4.1 and Theorem 4.1).

3 The analytic continuation of a Non-Diagonal Epstein Zeta function

Although the previous section furnished an analogue of the Epstein \( \zeta \)-function, the double Dirichlet series there presented is only useful when our study is reduced to diagonal quadratic forms.

In order to have a proper environment to develop Selberg-Chowla formulas for Epstein zeta functions attached to non-diagonal quadratic forms, we need to replicate in a certain way the conditions obeyed by the Riemann \( \zeta \)-function, which is behind the analytic continuation of the classical Epstein zeta function \( Z_2(s, Q) \). Doing so requires to introduce the following subclass \( \mathcal{B} \) of Bochner Dirichlet series, which has a similar role as the class \( \mathcal{A} \) had in the previous section.

Definition 3.1 (A Subclass of Bochner Dirichlet series). Let \( \phi(s) \) be a Dirichlet series satisfying Definition 1.1 with \( \Delta(s) = \Gamma \left( \frac{s+\delta}{2} \right) \), \( \delta \in \{0, 1\}, r = 1 \) (Bochner class). We say that \( \phi \) belongs to the class \( \mathcal{B} \) if

- For \( \delta = 0 \), \( \phi \) and \( \psi \) have analytic continuations to the entire complex plane and are analytic everywhere in \( \mathbb{C} \) except for possible simple poles located at \( s = 1 \) with residues \( \rho \) and \( \rho^* \) respectively. For \( \delta = 1 \), \( \phi \) and \( \psi \) can be analytically continued as entire Dirichlet series.
• $a(n)$ can be defined at 0 as $a(0) := -2\phi(-\delta)$. Moreover, $a(n)$ can be defined for negative values of $n$ as $a(-n) = (-1)^\delta a(n)$ and the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ can be extended to $n \in \mathbb{Z}$ with the values $\lambda_0 = \mu_0 = 0$ and $\lambda_{-n} = -\lambda_n$, $\mu_{-n} = -\mu_n$. A similar extension holds for $b(n)$ with $b(0) := -2\psi(-\delta)$ and $b(-n)$ analogously taken.

**Remark 3.1.** Note that definition 3.1 mimics in some way the properties that the Dirichlet $L$–functions and the Riemann $\zeta$–function have. By the functional equation for Bochner Dirichlet series (1.8), it is effortless to see that $\phi(0) = -\frac{\rho}{2} \sqrt{\pi}$, while $\psi(0) = -\frac{\rho}{2} \sqrt{\pi}$. Moreover, if $\phi(s)$ is entire, then it comes from the functional equation and the second item that $a(0) = 0$. We remark that, in analogy with Theorem 2.1, it is possible to prove a general Selberg-Chowla formula for $\mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda')$ with the general setting of Definition 1.1. However, in order to simplify the foregoing argument, we shall prove a Selberg-Chowla formula for (1.14) assuming apriori the conditions of the previous definition.

To prove a Selberg-Chowla formula for the non-diagonal Epstein zeta function $\mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda')$, we use a generalization of Bochner’s modular relation (1.21) for the class $\mathcal{B}$. In the well-known case $\phi_1(s) = \phi_2(s) = \pi^{-\frac{s}{2}} \zeta(s)$ the formula given below reduces to Jacobi’s reflection formula. We remark that, when the non-diagonal Epstein zeta function reduces to (1.1), our method of proof of the generalized Selberg-Chowla formula for $\mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda')$ reduces to the one presented on the second section of Selberg and Chowla’s paper [32].

**Lemma 3.1 (A generalization of the Theta reflection formula for the Class $\mathcal{B}$).** Let $\phi(s)$ be a Dirichlet series belonging to the class $\mathcal{B}$. If $a(n)$ is an even arithmetical function (i.e., $\delta = 0$), then, for any $x > 0$, $\text{Re}(\alpha) > 0$ and $\beta \in \mathbb{C}$, the following identity holds

$$
\sum_{n=1}^{\infty} a(n) e^{-\alpha \lambda_n^2 x} \cos(\beta \lambda_n x) = \phi(0) + \sqrt{\frac{\pi \rho}{2 \alpha}} e^{-\frac{\rho^2}{4 \alpha x}} + \frac{1}{2 \sqrt{\alpha x}} \sum_{n \neq 0} b(n) e^{-\frac{1}{2 \alpha x} \left(\frac{\rho}{\alpha} + \frac{x}{\alpha} \right)^2}.
$$

(3.1)

If, by other hand, $a(n)$ is an odd arithmetical function, we have the identity

$$
\sum_{n=1}^{\infty} a(n) e^{-\alpha \lambda_n^2 x} \sin(\beta \lambda_n x) = \frac{1}{2 \sqrt{\alpha x}} \sum_{n \neq 0} b(n) e^{-\frac{1}{2 \alpha x} \left(\frac{\rho}{\alpha} + \frac{x}{\alpha} \right)^2}.
$$

(3.2)

**Proof.** As in the proof of Theorem 2.1 we start by choosing a suitable parameter $\mu$ which allows absolute convergence in every side of the equality to be proven. In our case, assume $\mu > \max \{\sigma_a, \sigma_b, 1\}$ and use

---

\[9\] It is also possible to generalize to this scope the proof given by Taylor [71], mentioned at the Introduction. We remark that Taylor’s proof is incomplete, since it only provides the Selberg-Chowla formula for the subset of positive definite binary quadratic forms satisfying $b^2 < 2ac$ (see page 182 on Taylor’s paper and the assumption on the variable $Y$). However, as remarked in the historical introduction, the purpose of Taylor’s analysis was only to obtain Kober’s result [49], which was itself a particular case of the now called Selberg-Chowla formula. Nevertheless, a simple argument of continuation provided by the contiguous relations for the hypergeometric function $\mathcal{F}_1$ [39, p. 1009, eq. (9.132.2)] allows to extend Taylor’s proof to all positive definite binary quadratic forms.

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the observation that, for Re(α) > 0, σ = Re(z) > 0 and β ∈ C, the following Mellin inverse representation holds [39, p. 503, eq. 3.952.8]

\[ e^{-\alpha x^2} \cos(\beta x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \left(\frac{1-z}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha}\right) x^{-z} dz, \]  

which can be proved by invoking (1.23) together with Kummer’s transformation (1.24). Although the key of this comparison is Mellin’s inversion formula, (3.3) could be alternatively proved by invoking the absolutely convergent power series for 1\(F_1\) and then interchanging the orders of summation and integration.

From the fact that the integrand in (3.3) is the Mellin transform of a smooth function and decays faster than any polynomial as |s| → ∞, we can write the left-hand side of (3.1) as the contour integral

\[ \sum_{n=1}^{\infty} a(n) e^{-\alpha \lambda_n^2 x^2} \cos(\alpha \lambda_n x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma\left(\frac{z}{2}\right) \phi(z) \left(\frac{1-z}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha}\right) (\sqrt{\alpha x})^{-z} dz. \]  

Proceeding once more as in Theorem 2.1 let us now move the line of integration on (3.4) to Re(z) = 1 − μ < 0 and integrate along a positively oriented rectangular contour containing the vertices μ ± iT and 1 − μ ± iT, T > 0. Once again, we are allowed to invoke the Phragmén-Lindelöf principle for φ(z) (1.18), together with the decay properties of the integrand on the right-hand side of (3.3) in order to conclude that the integrals along the horizontal segments [1 − μ − iT, μ − iT] and [μ + iT, 1 − μ + iT] must vanish when T → ∞.

By the residue Theorem, we obtain the equality

\[ \sum_{n=1}^{\infty} a(n) e^{-\alpha \lambda_n^2 x^2} \cos(\alpha \lambda_n x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{1-\mu-i\infty}^{1-\mu+i\infty} \Gamma\left(\frac{z}{2}\right) \phi(z) \left(\frac{1-z}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha}\right) (\sqrt{\alpha x})^{-z} dz \]

\[ + \frac{e^{-\frac{\beta^2}{4\alpha}}}{2} \sum_{\rho \in \delta} \text{Res}_{z=\rho} \left\{ \Gamma\left(\frac{z}{2}\right) \phi(z) \left(\frac{1-z}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha}\right) (\sqrt{\alpha x})^{-z} dz \right\}. \]  

By the conditions of the class \( \delta \), it is evident that \( \phi(z) \) has simple zeros located at the negative even integers if \( a(n) = a(-n) \). Also, from the fact that \( 1F_1 \left( \frac{1}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha} \right) \) is an entire function of \( z \), we observe that all the poles that we need to take into consideration on the shift of the line are located at \( z = 0 \) and \( z = 1 \). It is immediate to see that their residues are

\[ \text{Res}_{z=0} = 2\phi(0) e^{-\frac{\beta^2}{4\alpha}}, \quad \text{Res}_{z=1} = \sqrt{\frac{\pi \rho}{\alpha \alpha x}}. \]

Now, invoking the functional equation for \( \phi(z) \) (1.18) with \( \delta = 0 \) and performing the change of variables \( z \leftrightarrow 1 - z \), we see that the first term on the right-hand side of (3.5) is equal to

\[ \frac{e^{-\frac{\beta^2}{4\alpha}}}{2\sqrt{\alpha x}} \sum_{n=1}^{\infty} \frac{b(n)}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma\left(\frac{z}{2}\right) 1F_1 \left( \frac{z}{2} ; \frac{1}{2} ; \frac{\beta^2}{4\alpha} \right) \left( \frac{\mu_n}{\sqrt{\alpha x}} \right)^{-z} dz + \phi(0) + \sqrt{\frac{\pi \rho}{\alpha \alpha x}} e^{-\frac{\beta^2}{4\alpha}}. \]  

(3.6)
The contour integral inside the infinite series can be evaluated via (3.3) upon a substitution of $\beta$ by $i\beta$ and the use of Kummer’s reflection formula (1.24). Nevertheless, for completeness, we may present a closed evaluation. Note that the power series defining the confluent Hypergeometric function is absolutely convergent for arbitrary $\beta \in \mathbb{C}$ and $\alpha$ such that $\text{Re}(\alpha) > 0$. Moreover, a simple application of Stirling’s formula (1.15) allows to reverse the orders of summation and integration once we employ this series representation and we get

\[
\frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma \left( \frac{z}{2} \right) {}_1F_1 \left( \frac{1}{2}; \frac{3}{2}; \frac{\beta^2}{4\alpha} \right) y^{-z} dz = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{k!} \left( \frac{\beta^2}{4\alpha} \right)^k \int_{\mu - i\infty}^{\mu + i\infty} \Gamma \left( \frac{z}{2} + k \right) y^{-z} dz
\]

\[
= \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{k!} \left( \frac{\beta^2 y^2}{4\alpha} \right)^k \int_{\mu + 2k - i\infty}^{\mu + 2k + i\infty} \Gamma \left( \frac{z}{2} \right) y^{-z} dz = 2e^{-y^2} \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \frac{\beta y}{\sqrt{\alpha}} \right)^{2k} = 2e^{-y^2} \cosh \left( \frac{\beta y}{\sqrt{\alpha}} \right). \quad (3.7)
\]

Combining (3.7), (3.6) and (3.5) we obtain the generalization of the reflection formula for Jacobi’s $\theta$-function,

\[
\sum_{n=1}^{\infty} a(n) e^{-\alpha \lambda_n^2 x^2} \cos (\beta \lambda_n x) = \phi(0) + \frac{\sqrt{\pi}}{\sqrt{2\alpha}} e^{-\frac{\beta^2}{2\alpha}} + e^{-\frac{\beta^2}{2\alpha}} \sum_{n=1}^{\infty} b(n) e^{-\frac{\beta^2}{2\alpha x}} \cosh \left( \frac{\beta \mu_n}{\alpha x} \right). \quad (3.8)
\]

Since $b(n)$ is even by hypothesis and we have assumed that $\mu_{-n} = -\mu_n$, we can rewrite the last series in a symmetric form by summing over the non-zero integers. This gives the desired reflection formula (3.1).

For the odd case the proof is even simpler: just proceed as before and use an integral representation similar to (3.3), i.e., [39], p. 503, eq. 3.952.7

\[
e^{-\alpha x^2} \sin(\beta x) = \frac{\beta e^{-\frac{\beta^2}{4\alpha}}}{4\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-s} \Gamma \left( \frac{1+s}{2} \right) {}_1F_1 \left( 1 - \frac{s}{2}; \frac{3}{2}; \frac{\beta^2}{4\alpha} \right) x^{-s} ds, \quad (3.9)
\]

valid for $\text{Re}(\alpha) > 0$ and $\sigma > -1$.

\[\square\]

Remark 3.2. The previous lemma is connected with N. J. Fine’s proof of the functional equation for the classical Hurwitz zeta function $\zeta(s, \alpha)$ [36], p. 361, eq. (3)]. As done by Berndt for the case of Dirichlet series satisfying Hecke’s functional equation [6], we could define the generalized Hurwitz zeta function in the following way

\[
\phi(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + \alpha)^s}, \quad \text{Re}(s) > \sigma_\alpha, \; \alpha > 0 \quad (3.10)
\]

and derive the analytic continuation and functional equation for it by using the previous lemma. This observation connects our forthcoming proof of the Selberg-Chowla formula with the proofs by Berndt and Kuzumaki, which naturally employ the functional equation for $\zeta(s, \alpha)$ (see [10], p. 160) and [51], eq. (16), (24) and (25)].

\[\text{The Dirichlet series appearing in (3.10) is not a generalized/perturbed Dirichlet series in the sense of [6], p.309 since it is not constructed from a sequence $(\lambda_n)_{n \in \mathbb{N}}$ attached to a Dirichlet series satisfying Hecke’s functional equation.}\]

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**Theorem 3.1** (Selberg-Chowla for positive definite binary quadratic forms). Let \( \phi_1 \) and \( \phi_2 \) be two Dirichlet series belonging to the class \( \mathcal{B} \) and \( s \) be a complex number such that \( \text{Re}(s) > \mu > \max \{ \sigma_a, \sigma_b \} \). If \( a_1(n), a_2(n) \) are even arithmetical functions, then we have the following Selberg-Chowla formula

\[
a^s \Gamma(s) \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = -4 \Phi_2(0) \Gamma(s) \Phi_2(2s) + 2 \sqrt{\pi} \rho_1 \Gamma \left( s - \frac{1}{2} \right) k^{1-2s} \Phi_2(2s-1) + 4k^{1-s} \sum_{j=1}^{\infty} \sigma_{2j-1} (v_j; b_1, a_2; \mu, \lambda') \cos \left( \frac{b_v}{a} \right) v_j^{1-s} K_{s-\frac{1}{2}} (2kv_j),
\]

(3.11)

where \( (v_j)_{j \in \mathbb{N}} = \{ \mu_m \lambda_m' \}_{m=1}^\infty \) is the product sequence arranged in increasing order and \( \sigma_{2s-1} (v_j) \) denotes the generalized weighted divisor function of power \( 2s - 1 \) (see Definition 2.1). Moreover, \( d := b^2 - 4ac \) is the discriminant of the quadratic form \( Q \) and \( k^2 := |d|/4a^2 \).

Analogously, if \( a_1(n) \) and \( a_2(n) \) are odd arithmetical functions, we have the Selberg-Chowla formula

\[
a^s \Gamma(s) \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = -4k^{1-s} \sum_{j=1}^{\infty} \sigma_{2j-1} (v_j; b_1, a_2; \mu, \lambda') \sin \left( \frac{b_v}{a} \right) v_j^{1-s} K_{s-\frac{1}{2}} (2kv_j).
\]

(3.12)

Under the same assumptions, \( \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') \) can be described by the second Selberg-Chowla formulas,

\[
c^s \Gamma(s) \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = -4 \Phi_1(0) \Gamma(s) \Phi_2(2s) + 2 \sqrt{\pi} \rho_2 \Gamma \left( s - \frac{1}{2} \right) k^{1-2s} \Phi_2(2s-1) + 4k^{1-s} \sum_{j=1}^{\infty} \sigma_{2j-1} (v_j'; b_2, a_1; \mu', \lambda) \cos \left( \frac{b_v'}{c} \right) v_j'^{1-s} K_{s-\frac{1}{2}} (2kv_j'),
\]

(3.13)

for \( a_1(n), a_2(n) \) being even arithmetical functions and

\[
c^s \Gamma(s) \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = -4k^{1-s} \sum_{j=1}^{\infty} \sigma_{2j-1} (v_j'; b_2, a_1; \mu', \lambda) \sin \left( \frac{b_v'}{c} \right) v_j'^{1-s} K_{s-\frac{1}{2}} (2kv_j'),
\]

(3.14)

for \( a_1(n), a_2(n) \) representing odd arithmetical functions. Here, \( k^2 := |d|/4c^2 \) and the sequence \( v_j' \) is analogously defined by \( v_j' = \mu_m \lambda_n \).

**Proof.** Throughout this proof we shall assume that \( a_i(n), i = 1, 2 \), are even arithmetical functions, as analogous computations can be given for the odd case. As in the proof of Theorem 2.1, we start the argument by writing the double series (1.14) as

\[
\mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = 2a_2(0)a^{-s} \Phi_2(2s) + 2a_1(0)c^{-s} \Phi_2(2s) + \sum_{m \neq 0, n \neq 0} \frac{a_1(m)a_2(n)}{(a \lambda_m^2 + b \lambda_m \lambda_n' + c \lambda_n'^2)^s}.
\]

Also, we choose a parameter \( \mu \) satisfying \( \mu > \max \{ \sigma_a, \sigma_b \} \) and a fixed \( s \in \mathbb{C} \) such that \( \text{Re}(s) > \mu \). Of course, in order to compute \( \mathcal{Z}_2 \), we need to evaluate the double series

\[
\sum_{n \neq 0} a_2(n) \sum_{m \neq 0} \frac{a_1(m)}{(a \lambda_m^2 + b \lambda_m \lambda_n' + c \lambda_n'^2)^s}.
\]

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In analogy to the generalized Dirichlet series appearing in the proof of Theorem 2.1 for a fixed $n$ let us denote by $\ell_n(s, Q, a_1)$ the inner infinite series with respect to $m$ appearing above. For $\Re(s) > \mu$, $\ell_n(s, Q, a_1)$ is explicitly given by

$$
\ell_n(s, Q, a_1) = \sum_{m \neq 0} \frac{a_1(m)}{(a\lambda_m^2 + b\lambda_m'\lambda'_m + c\lambda_m'^2)} = \frac{1}{\Gamma(s)} \sum_{m \neq 0} a_1(m) \int_0^\infty y^{s-1} e^{-(a\lambda_m^2 + b\lambda_m'\lambda'_m + c\lambda_m'^2)y} dy
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} e^{-\frac{1}{\alpha}\lambda'_m y} \sum_{m \neq 0} a_1(m) e^{-\alpha y(\lambda_m + \frac{b}{a}\lambda'_m)^2} dy
$$

where the last step can be justified by the fact that $\Re(s) > \mu > \sigma_o$ and by absolute convergence. Invoking the reflection formula (3.1) with the roles of $\phi$ and $\psi$ reversed and taking the substitutions $\alpha = \frac{1}{ay} > 0$, $\beta = \frac{b}{a}\lambda'_m$ and $x = 1$, we obtain that

$$
\sum_{m \neq 0} a_1(m) e^{-\alpha y(\lambda_m + \frac{b}{a}\lambda'_m)^2} = \frac{2}{\sqrt{\alpha}} \left\{ \sum_{m=1}^\infty b_1(m) e^{-\frac{b}{a\lambda'_m}} \cos \left( \frac{b}{a\mu_m\lambda'_m} \right) - \psi_1(0) - \frac{\beta \sqrt{\pi ay}}{2} e^{-\frac{b^2}{4\alpha\lambda'_m y}} \right\}
$$

and so $\ell_n(s, Q, a_1)$ can be given by

$$
\frac{2a^{-\frac{s}{2}} \Gamma(s)}{\lambda'_m^{2s}} e^{-\frac{s}{2} \frac{|d|}{4a}} \left\{ \sum_{m=1}^\infty b_1(m) e^{-\frac{b}{a\lambda'_m}} \cos \left( \frac{b}{a\mu_m\lambda'_m} \right) - \psi_1(0) - \frac{\beta \sqrt{\pi ay}}{2} e^{-\frac{b^2}{4\alpha\lambda'_m y}} \right\} dy
$$

$$
= \frac{2\phi_1(0)}{\lambda'_m^{2s}} e^{-\frac{s}{2} \frac{|d|}{4a}} + \frac{2a^{-\frac{s}{2}} \Gamma(s) \psi_1(0) (|d|/4a)^{\frac{s}{2}}}{\Gamma(s) \lambda'_m^{2s-1}} + 2a^{-\frac{s}{2}} \Gamma(s) \sum_{m=1}^\infty b_1(m) \cos \left( \frac{b}{a\mu_m\lambda'_m} \right) \int_0^\infty y^{s-\frac{1}{2}} e^{-\frac{b^2}{4\alpha\lambda'_m y}} dy
$$

$$
= \frac{2\phi_1(0)}{\lambda'_m^{2s}} e^{-\frac{s}{2} \frac{|d|}{4a}} + \frac{2}{\Gamma(s) \sqrt{\pi}} a^{-\frac{s}{2}} \Gamma(s) \psi_1(0) (|d|/4a)^{\frac{s}{2}} + \frac{4}{\Gamma(s) \sqrt{\pi}} (ka)^{\frac{s}{2}} \sum_{m=1}^\infty b_1(m) \cos \left( \frac{b}{a\mu_m\lambda'_m} \right) \left( \frac{\mu_m}{\lambda'_m} \right)^{\frac{s}{2}} K_{s-\frac{1}{2}}(2k\mu_m\lambda'_m)
$$

with the second equality being justified again by absolute convergence. We have also used the fact that $\psi_1(0) = -\frac{\phi_1}{2} \sqrt{\pi}$ and the integral representation of the modified Bessel function given in (2.37).

It suffices now to multiply the previous equality by $a_2(n)$ and to sum over the index $n \in \mathbb{Z} \setminus \{0\}$. Since we are assuming that $\Re(s) > \mu$, all the Dirichlet series involved in this process converge absolutely. Similarly to what we have done in the final part of the proof of Theorem 2.1 we can argue that the resulting double series of the form

$$
\sum_{m,n=1}^\infty b_1(m) a_2(n) \cos \left( \frac{b}{a\mu_m\lambda'_m} \right) \left( \frac{\mu_m}{\lambda'_m} \right)^{\frac{s}{2}} K_{s-\frac{1}{2}}(2k\mu_m\lambda'_m)
$$

will converge absolutely due to the classical estimate for the Modified Bessel function (2.15). From the condition $\Re(s) > \mu > \max \{\sigma_o, \sigma_0\}$, we see that the series defining the non-diagonal Epstein zeta function is absolutely convergent as a double series and the order of the summation of both series can be interchanged. This gives the second Selberg-Chowla formula for the even case (3.13), which is obtained after replacing the role of $\phi_1$ by the one of $\phi_2$ and then replace $a$ by $c$ and $k$ by $k'$. Invoking the definition of the generalized divisor function (2.21), we obtain all the Selberg-Chowla formulas in the form stated. □
By analytic continuation, it is possible to establish the validity of the formulas \([3.11, 3.12, 3.13, 3.14]\) for all complex values of \(s\). Mimicking the steps leading to the proof of Proposition \([2.11]\), it is simple to see that the infinite series involving the generalized divisor function are entire functions of \(s\) and obey to a reflection formula which is analogous to \([2.17]\).

Similarly to the entire functions \(H_r(s; \cdot)\) and \(H_s(s; \cdot)\) \([2.1, 2.3]\), we may see that the series on the first Selberg-Chowla formulas \([3.11]\) and \([3.12]\) can be written in a unified way as

\[
H_1^\delta(s; Q; b_1, a_2) = \sum_{m,n=1}^\infty b_1(m) a_2(n) \left\{ (1 - \delta) \cos \left( \frac{b}{a} \mu_m \lambda_n \right) + \delta \sin \left( \frac{b}{a} \mu_m \lambda_n \right) \right\} \left( \frac{\mu_m}{\lambda_n} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2k \mu_m \lambda_n) .
\]

Analogously, the divisor series on the second Selberg-Chowla formulas \([3.13]\) and \([3.14]\) can be also written as

\[
H_2^\delta(s; Q; b_2, a_1) = \sum_{m,n=1}^\infty b_2(m) a_1(n) \left\{ (1 - \delta) \cos \left( \frac{b}{c} \mu_m \lambda_n \right) + \delta \sin \left( \frac{b}{c} \mu_m \lambda_n \right) \right\} \left( \frac{\mu_m}{\lambda_n} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2k' \mu_m \lambda_n) .
\]

To prove a reflection formula connecting both entire functions, substitute on the argument of \(H_1^\delta(s; Q; \cdot)\) \(s\) by \(1 - s\) and \(Q\) by \(Q^{-1}\). Since \(Q^{-1}(x,y) = cx^2 - bxy + ay^2\) and \(K_v(z) = K_{-v}(z)\), one obtains without effort

\[
H_1^\delta(1 - s; Q^{-1}; a_1, b_2; \lambda, \mu') = (-1)^\delta H_2^\delta(s; Q; b_2, a_1, \mu', \lambda) ,
\]

which is employed to prove our next result.

**Corollary 3.1 (The Analytic Continuation of the non-diagonal Epstein zeta function).** Under the hypotheses given on the statement of Theorem \([3.1]\) the non-diagonal Epstein zeta function \([1.1, 4]\) can be continued to the entire complex plane as:

1. A meromorphic function with a simple pole located at \(s = 1\) with residue \(\frac{2\pi i p_1 p_2}{\sqrt{|d|}}\) if \(a_1(n)\) and \(a_2(n)\) are even arithmetical functions and \(p_1, p_2\) respectively denote the residues of \(\phi_1, \phi_2\) at \(s = 1\). In particular, if at least one of the Dirichlet series \(\phi_i\) is entire, then \(\mathcal{Z}_2\) is entire.

2. An entire function if \(a_1(n), a_2(n)\) are odd arithmetical functions.

Thus, \(\mathcal{Z}_2\) is a Dirichlet series belonging to the class \(\mathcal{A}\) and satisfying Hecke’s functional equation

\[
\left( \frac{2}{\sqrt{|d|}} \right)^{-s} \Gamma(s) \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') = (-1)^\delta \left( \frac{2}{\sqrt{|d|}} \right)^{s-1} \Gamma(1-s) \mathcal{Z}_2(1-s; Q^{-1}; b_1, b_2; \mu, \mu') .
\]

**Proof.** The proof of the continuation is similar to the proof of Corollary \([2.1]\) since \(H_1^\delta(s, Q; \cdot)\) and \(H_2^\delta(s, Q; \cdot)\) are entire functions of the complex variable \(s\), then it follows from the Selberg-Chowla formula \([3.12]\) that \(\mathcal{Z}_2\) is entire if \(a_1(n)\) and \(a_2(n)\) are odd arithmetical functions. Furthermore, the functional equation \([3.16]\) with \(\delta = 1\) is an immediate consequence of \([3.15]\).
It remains to show the corollary for the case where \( a_1(n) \) and \( a_2(n) \) are even arithmetical functions: of course, the first terms on (3.11) are the only ones contributing to the singularities of \( \mathcal{Z}_2 \) so that we just need to study the meromorphic part

\[
G(s, \mathcal{Q}) = -4^{a}s \phi_2(0) \phi_1(2s) + 2\sqrt{\pi} \rho_1 \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} a^{-s} k^{1-2s} \phi_2(2s-1).
\]

It is clear that \( G(s, \mathcal{Q}) \) has removable singularities at the points \( s = \frac{1}{2} - k, k \in \mathbb{N} \). This easily comes from the functional equation and the fact that \( \phi_1 \in \mathcal{B} \). Now, since \( \phi_1(2s) \) and \( \Gamma\left(s - \frac{1}{2}\right) \) are analytic in a neighbourhood of \( s = 1 \), we see that \( \mathcal{Z}_2 \) must have a pole at \( s = 1 \) coming from the contribution of \( \phi_2(2s-1) \). The residue is very easy to compute and it is given by \( \frac{2\pi \rho_1 \rho_2}{\sqrt{|d|}} \).

As in Corollary 2.4, the proof of the functional equation is made by comparing both representations of \( \mathcal{Z}_2 \), (3.11) and (3.13): start by multiplying (3.11) by \( a^{-s} \) and then substitute \( s \) by \( 1-s, \mathcal{Q} \) by \( \mathcal{Q}^{-1} \) and \( a_1(n), a_2(n) \) by \( b_1(n), b_2(n) \). We obtain

\[
\Gamma(1-s) \mathcal{Z}_2(1-s; \mathcal{Q}^{-1}; b_1, b_2; \mu, \mu') = -4 c^{s-1} \psi_2(0) \Gamma(1-s) \psi_1(2-2s) + 2\sqrt{\pi} \rho_1 \Gamma\left(s - \frac{1}{2}\right) c^{s-1} k^{2s-1} \psi_2(1-2s) + 4 \left(k'c\right)^{s-\frac{1}{2}} \sqrt{\frac{1}{c} H^0_1(1-s; \mathcal{Q}^{-1}; a_1, b_2; \lambda, \mu')},
\]

which, when combined with the reflection formula (3.15) and the functional equation for \( \psi_1 \) and \( \psi_2 \), yields

\[
\Gamma(1-s) \mathcal{Z}_2(1-s; \mathcal{Q}^{-1}; b_1, b_2; \mu, \mu') = -4 c^{s-1} \psi_2(0) \Gamma\left(s - \frac{1}{2}\right) \phi_1(2s-1) + 2\sqrt{\pi} \rho_1 c^{s-1} k^{2s-1} \Gamma(s) \psi_2(2s) + 4 \left(k'c\right)^{s-\frac{1}{2}} \sqrt{\frac{1}{c} H^0_2(s; \mathcal{Q}; b_2, a_1, \mu', \lambda)}.
\]

Invoking now the simple properties of the class \( \mathcal{B} \), \( \phi(0) = -\frac{\rho_1}{2} \sqrt{\pi} \) and \( \psi_0(0) = -\frac{\rho_2}{2} \sqrt{\pi}, \) and making these substitutions on the first two terms of (3.18), we arrive precisely at the right-hand side of the second Selberg-Chowla formula (3.13) multiplied by an extra factor of \( (2/\sqrt{|d|})^{1-2s} c^{-s} \). This completes the proof.

\[ \square \]

**Remark 3.3.** If \( a(n) \) is an even arithmetical function and \( \phi \in \mathcal{B} \), then \( \phi(2s) \in \mathcal{A} \) with \( r = 1 \). Analogously, if \( a(n) \) is an odd arithmetical function and \( \phi \in \mathcal{B} \) then \( \phi(2s-1) \in \mathcal{A} \) with \( r = \frac{3}{2} \). In fact, by letting \( Q(x, y) = x^2 + y^2 \), we have that (3.11) is a reformulation of the first Selberg-Chowla formula restricted to the class \( \mathcal{A} \), (2.26), for \( r_1 = r_2 = \frac{1}{2} \). This is why we have named the Epstein zeta function (1.10) as “diagonal”. Despite this, it is clear that the Selberg-Chowla formulas for odd arithmetical functions and some particular quadratic form \( Q(x, y) \) cannot be seen as reformulations or as particular cases of (2.26) and (2.27). This happens because the arithmetical function associated to the Dirichlet series \( \varphi(s) := \phi(2s-1) \) is \( c(n) = a(n) \lambda_n \) and not \( a(n) \). Note that, by item 2. in Definition 3.1, \( c(n) \) is an even arithmetical function and it is possible to establish a Selberg-Chowla formula for its non-diagonal Epstein zeta function. Although we
cannot apply Theorem 3.1 in a direct way (because $a(n)\lambda_n$ is not attached to a Bochner Dirichlet series), it is not difficult to derive an analogue of Lemma 3.1. Let $\mathcal{Z}_2(s;\,Q;\,c_1,\,c_2;\,\lambda,\,\lambda')$ denote the following version of the non-diagonal Epstein zeta function

$$\mathcal{Z}_2(s;\,Q;\,c_1,\,c_2;\,\lambda,\,\lambda') = \sum_{m,n \neq 0} \frac{a_1(m)\lambda_m a_2(n)\lambda'_n}{Q(\lambda_m,\lambda'_n)s}, \quad \text{Re}(s) > \sigma_0 + 1.$$  \hspace{1cm} (3.19)

Then, for any $\text{Re}(s) > \mu > \max\{\sigma_0 + 1, \sigma_b + 1\}$, the following Selberg-Chowla formula holds

$$a^r \Gamma(s) \mathcal{Z}_2(s;\,Q;\,c_1,\,c_2;\,\lambda,\,\lambda') = 4k^{\frac{3}{2}} - s \sum_{j=1}^{\infty} \sigma_{2s-3} (v_j;\,b_1,\,a_2;\,\mu,\,\lambda') v_j^{\frac{3}{2} - s}$$

$$\times \left\{ \cos \left( \frac{b}{a} v_j \right) K_{\frac{3}{2} - s}(2k v_j) + \frac{2b}{\sqrt{|d|}} \sin \left( \frac{b}{a} v_j \right) K_{\frac{3}{2} - s}(2k v_j) \right\}, \hspace{1cm} (3.20)$$

which gives the analytic continuation of the Dirichlet series (3.19) as an entire complex function. In particular, if $b = 0$ and $a = c = 1$, (3.20) reduces to (2.26) with $r_1 = r_2 = \frac{3}{2}$.

4 The zeros of a class of Dirichlet series

It is immediate to check from the Selberg-Chowla (or the functional equation) that $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\lambda')$ has trivial zeros located at the negative integers, this is, $s = -n, n \in \mathbb{N}$. A similar comment can be made for $\mathcal{Z}_2(s;\,Q;\,a_1,\,a_2;\,\lambda,\,\lambda')$. Moreover, in a completely analogous way to the results given in [4, p. 367, Theorem 3] and [73], one can exhibit subclasses of the Epstein zeta functions studied in the previous sections which do not obey to a generalized Riemann hypothesis, i.e., which fail the condition of having all their nontrivial zeros located in the critical line $\text{Re}(s) = r$. The next corollary of the Selberg-Chowla formula gives this result. In order to state it, we introduce the following notation: for a given $\xi > 0$, we write $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\xi\lambda')$ as the analogue of the diagonal Epstein zeta function given by the double Dirichlet series

$$\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\xi\lambda') = \sum_{m,n \neq 0} \frac{a_1(m)a_2(n)}{(\lambda_m + \lambda'_n\xi)^s}, \quad \text{Re}(s) > 2\sigma_0. \hspace{1cm} (4.1)$$

**Corollary 4.1 (Failure of the Riemann Hypothesis for $\mathcal{Z}_2$).** Suppose that $\phi_1$ and $\phi_2$ are Dirichlet series with real arithmetical functions $a_1(n)$ and $a_2(n)$, satisfying Hecke’s functional equation and belonging to the class $\mathcal{A}$. Assume also that $\phi_i(0) < 0$ for any $i = 1, 2$. Then, for a sufficiently large $\xi > 0$, $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\xi\lambda')$ has a real zero on the interval $(r_1, \,r_1 + r_2)$, for $r_2 \leq r_1$.

Moreover, if $\phi_i, \,i = 1, 2,$ are also real Bochner Dirichlet series belonging to the class $\mathcal{B}$ and such that $\phi_i(0) < 0$ then, for a quadratic form $Q$ having a sufficiently large $k := \sqrt{|d|}/2a$, $\mathcal{Z}_2(s;\,Q;\,a_1,\,a_2;\,\lambda,\,\lambda')$ has a real zero on the interval $(\frac{1}{2}, \,1)$. 

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Proof. We shall prove the result only for the diagonal Epstein zeta function \( \mathcal{Z}_2(s; a_1, a_2; \lambda, \xi \lambda') \), since similar considerations hold for \( \mathcal{Z}_2(s; Q; a_1, a_2; \lambda, \lambda') \). The idea of the proof is to write a Selberg-Chowla formula for (4.1). To do this, let us replace on (2.26) the sequence \( \lambda_n' \) by \( \xi \lambda_n' \), where \( \xi > 0 \). By virtue of Hecke’s functional equation, we also need to replace \( \mu_n' \) by \( \mu_n'/\xi \), \( b_2(n) \) by \( \xi^{-r_2}b_2(n) \) and \( \rho_2 \) by \( \xi^{-r_2}\rho_2 \). From (2.26), the following Selberg-Chowla formula holds

\[
\Gamma(s) \sum_{m,n \neq 0} \frac{a_1(m) a_2(n)}{(\lambda_m + \lambda_n' \xi)} = -\phi_2(0) \Gamma(s) \phi_1(s) + \rho_1 \Gamma(r_1) \Gamma(s - r_1) \xi^{r_1-s} \phi_2(s - r_1) + 2 \xi^{\frac{1}{2}} \sum_{m,n=1}^{\infty} \sigma_{s-r_1}(v_j; a_1, a_2) v_j^{r_1-s} K_{r_1-s} \left( 2 \sqrt{\mu_m \xi} \right),
\]

which provides the analytic continuation of (4.1). Without loss of generality, assume that \( r_2 \leq r_1 \): from Corollary 2.1 we know that the only pole that \( \mathcal{Z}_2 \) possesses is located at \( s = r_1 + r_2 \) with residue given by

\[
\text{Res}_{s=r_1+r_2} \mathcal{Z}_2(s; a_1, a_2; \lambda, \xi \lambda') = \frac{\Gamma(r_1) \Gamma(r_2)}{\Gamma(r_1 + r_2)} \xi^{-r_2} \rho_1 \rho_2.
\]

Thus, since \( \rho_1 > 0 \) (because \( \phi(0) < 0 \) by hypothesis) and \( \xi > 0 \), the latter is a positive real number, so that \( \lim_{s \to r_1 + r_2} \mathcal{Z}_2(s; a; \lambda; \xi) = -\infty \). We may now compute \( \mathcal{Z}_2 \) at \( s = r_1 \): to do this, let us write the Laurent series for \( \phi_1(s) \) around \( s = r_1 \) as

\[
\phi_1(s) = \frac{\rho_1}{s-r_1} + \rho_{0,1} + O(s - r_1),
\]

and use the meromorphic expansions for \( \Gamma(s) \) and \( \phi_1(s) \) around \( s = 0 \),

\[
\Gamma(s) = \frac{1}{s} - \gamma + O(s),
\]

\[
\phi_1(s) = \phi_1(0) + \phi_1'(0) s + O(s^2),
\]

where \( \gamma \) is Euler’s constant. Due to the uniform convergence of the infinite series in (4.2), (by Proposition 2.1), taking the limit \( s \to r_1 \) on the right-hand side of (4.2) gives

\[
-\phi_2(0) \left\{ \rho_{0,1} + \rho_1 \log(\xi) + \rho_1 \gamma + \rho_1 \Gamma'(r_1) \right\} + \rho_1 \phi_2'(0) + \frac{2}{\Gamma(r_1)} \sum_{j=1}^{\infty} d(v_j; b_1, a_1; \mu, \lambda') K_0(2\xi \sqrt{v_j}),
\]

where the arithmetical function \( d \) is the generalization of the usual divisor function \( d(n) \) provided by formula (2.23).

Now, we study the infinite series at the right-hand side of (4.6): from the inequality [4, p. 368, Lemma 3]

\[
K_0(x) \leq \left( \frac{\pi}{2x} \right)^{1/2} e^{-x} \left[ 1 - \frac{1}{8x} + \frac{9}{128x^2} \right],
\]

we can deduce that the infinite series in (4.6) is dominated by

\[
\left( \frac{\pi}{4 \xi} \right)^{1/2} \sum_{m,n=1}^{\infty} b(m) a(n) (\mu_m \lambda_n)^{-1/4} e^{-2\xi \sqrt{\mu_m \lambda_n}} \left[ 1 - \frac{1}{16 \xi} \sqrt{\mu_m \lambda_n} + \frac{9}{512 \xi^2 \mu_m \lambda_n} \right],
\]

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which clearly goes to zero when $\xi \to \infty$. Hence, it suffices to choose $\xi$ for which the residual term in (4.6) is positive. But this condition is satisfied whenever
\[
\xi > \exp \left\{ \frac{\phi'_1(0)}{\phi_1(0)} - \frac{P_{0,1}}{P_1} - \gamma - \frac{\Gamma'(r_1)}{\Gamma(r_1)} \right\}.
\] (4.9)

Since there exists some $\xi_0$ such that, for any $\xi > \xi_0$, (4.8) is less than the right-hand side of (4.9), if we take $\xi > \xi_0$ we have that $Z_2(r_1; a_1, a_2; \lambda, \xi \lambda') > 0$ and so there must be some real zero of the Epstein zeta function in the interval $(r_1, r_1 + r_2)$.

Despite the fact that, in general, most particular cases of the Epstein zeta functions defined in the previous sections may have infinitely many zeros off their critical lines, as proved by Davenport, Heilbronn and Voronin for the classical Epstein zeta function \[2.3\] \[2.77\], a large class of them will certainly admit infinitely many zeros on the critical line, even under our general setting. For the classical Epstein zeta function $Z_2(s, Q)$, this fact was proved for the first time by Potter and Titchmarsh \[57\]. A simpler proof, similar to Hardy’s proof of the same theorem for $\zeta(s)$, was given by Kober \[48\].

Since we have generalized the Selberg-Chowla formula and the continuation of Epstein zeta functions in the previous sections, we now use this advantage of representing the double Dirichlet series $\mathcal{Z}_2(s; a_1, a_2; \lambda, \lambda')$ in terms of the single series $\phi_1(s)$ and $\phi_2(s)$ in order to relate the distribution of zeros of $\phi_1(s)$ and $\phi_2(s)$ with the asymptotic order of $\mathcal{Z}_2$ in its critical line $\text{Re}(s) = r$.

We also prove that if $\mathcal{Z}_2$ satisfies Hardy’s Theorem, then $\phi(s)$ must satisfy Hardy’s Theorem as well.

The bridge connecting the truth of Hardy’s theorem for $\phi(s)$ and for $\mathcal{Z}_2$ lies precisely in Selberg-Chowla’s formula (2.46). However, to derive in a precise form this connection, we still need to write it in a suitable form, i.e., involving an integral representation of $\phi$ on the critical line. In the next lemma we rewrite (2.46) in an integral form. There, we shall consider the subsequence of multidimensional diagonal Epstein zeta functions, $\mathcal{Z}_{2k}^k(s; a; \lambda)$, defined in Remark 2.5.

**Lemma 4.1.** Let $\mathcal{Z}_{2k}^k(s; a; \lambda)$ be the dyadic Epstein zeta function defined by (2.45). Then the following representation of $\mathcal{Z}_{2k}^{k+1}(s; a; \lambda)$ at the critical line, $\text{Re}(s) = 2^k r$, holds

\[
\Gamma(2^k r + 2it) \mathcal{Z}_{2k+1}^k \left( 2^k r + 2it; a; \lambda \right) = -2 \mathcal{Z}_{2k}^k \left( 0; a; \lambda \right) \Gamma(2^k r + 2it) \mathcal{Z}_{2k}^k \left( 2^k r + 2it; a; \lambda \right) +
\]

\[
+2 \Gamma^{2k} (r) \rho^{2k} \Gamma(2it) \mathcal{Z}_{2k}^k (2it; a; \lambda) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma \left( 2^{k-1} r + i(y-t) \right) \mathcal{Z}_{2k}^k \left( 2^{k-1} r + i(y-t); b; \lambda \right) \times
\]

\[
\times \Gamma \left( 2^{k-1} r + i(y+t) \right) \mathcal{Z}_{2k}^k \left( 2^{k-1} r + i(y+t); a; \lambda \right) dy.
\] (4.10)

**Proof.** Start with the Selberg-Chowla formula (2.46) and write $s = 2^k r + 2it, t \in \mathbb{R}$. We obtain the represent-
Remark where the definitions of the arithmetical functions $U_n$ as formula is located on the left of the line Ration oriented rectangular contour $R$. To proceed further, we need to give an integral representation for the double series appearing on the right-hand side of (4.11). Our idea is then to return to the integral representation for the modified Bessel function (2.13) and rewrite it in a more symmetric form

$$K_r(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} 2^{w-2} \Gamma \left( \frac{w+1}{2} \right) \Gamma \left( \frac{w-1}{2} \right) x^{-w} \, dw, \quad \mu > \text{Re}(v). \quad (4.12)$$

Replacing $v$ by $2it$ in (4.12) and letting $\mu > \max \left\{ 0, 2^k \sigma_2, 2^k \sigma_b, 2^{k+1}r \right\}$, we can write $2H_{2r}(2^k r + 2it; a; \lambda)$ as

$$\sum_{m=1}^{\infty} \Omega_{2r}(m) \Omega_{2r}(n) \left( \frac{\Omega_m}{\Lambda_n} \right) \frac{i}{\pi} \int_{\mu - i\infty}^{\mu + i\infty} 2^{w-2} \Gamma \left( \frac{w+2it}{2} \right) \Gamma \left( \frac{w-2it}{2} \right) \left( 2\sqrt{\Omega_m \Lambda_n} \right)^{-w} \, dw \quad \text{(4.13)}$$

since this choice of $\mu$ allowed to interchange the double series and the Mellin integral (see the proof of Theorem 2.1). Let us now shift the line of integration in (4.13) to Re$(w) = 2^k r$ which, by the choice of $\mu$, is located on the left of the line Re$(w) = \mu$. Doing this requires once more to integrate along a positively oriented rectangular contour $\mathcal{R}$ with vertices $\mu \pm iT$ and $2^k r \pm iT$, $T > 0$. The application of the Phragmén-Lindelöf principle for $\mathcal{Z}_{2r}$ (2.48) and Stirling’s formula allows to conclude that the integrals on the horizontal segments vanish when $T \to \infty$. By shifting the line of integration, we note that the integrand has two simple poles on the interior of $\mathcal{R}$ in the form $w = 2^{k+1} r \pm 2it$ and their residues are, respectively,

$$R_{2^{k+1} r + 2it} = 2\Gamma(2^k r + 2it) \Gamma(2^{k+1} r + 2it) \mathcal{Z}_{2r} \left( 2^k r + 2it; a; \lambda \right)$$

$$+ 2\Gamma(2^k r + 2it) \Gamma(2^{k+1} r + 2it) \mathcal{Z}_{2r} \left( 2^{k+1} r + 2it; a; \lambda \right), \quad (4.14)$$

where we have used the fact that $\mathcal{Z}_{2r}(s; a; \lambda) \in \mathcal{A}$ (see Corollary 2.2), and

$$R_{2^{k+1} r - 2it} = 2\Gamma(2^k r + 2it) \mathcal{Z}_{2r} \left( 2^k r + 2it; a; \lambda \right). \quad (4.15)$$

Using the residue residue theorem and adding the residues (4.14) and (4.15) to (4.11), we obtain the formula

$$\Gamma(2^k r + 2it) \mathcal{Z}_{2k+1} \left( 2^k r + 2it; a; \lambda \right) = -2 \mathcal{Z}_{2r} \left( 0; a; \lambda \right) \Gamma(2^k r + 2it) \mathcal{Z}_{2r} \left( 2^k r + 2it; a; \lambda \right)$$

$$+ 2\Gamma(2^k r + 2it) \mathcal{Z}_{2r} \left( 2^k r + 2it; a; \lambda \right) + \frac{1}{4\pi i} \int_{2^k r - i\infty}^{2^k r + i\infty} \Gamma \left( \frac{w-2it}{2} \right) \mathcal{Z}_{2r} \left( \frac{w+2it}{2}; b; \lambda \right) \times \Gamma \left( \frac{w+2it}{2} \right) \mathcal{Z}_{2r} \left( \frac{w+2it}{2}; a; \lambda \right) \, dw, \quad (4.16)$$

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which yields (4.10) after we write \( w = 2^k r + 2i y \) on the contour integral.

\[ \square \]

We now establish our Main Theorem:

**Theorem 4.1.** Let \( \phi(s) \) be a Dirichlet series satisfying Hecke’s functional equation in the sense of Definition 1.1 and belonging to the class \( \mathbb{A} \). Suppose that the parameters of \( \phi \) and \( \psi \) satisfy \( \bar{\alpha}(n) = b(n) \) and \( \lambda_n = \mu_n \). Also, let \( \sigma_a \) denote the abcissa of absolute convergence of \( \phi \). If one of the following conditions holds:

1. The abcissa of absolute convergence satisfies \( \sigma_a < \min\{1,r\} + \frac{1}{2} \max\{1,r\} \) and, for some \( k \geq 0 \), the diagonal Epstein zeta function \( \mathcal{Z}_{2k+1} \), (2.45), associated with \( \phi \) has infinitely many zeros in its critical line \( \text{Re}(s) = 2^k r \).

2. The abcissa of absolute convergence satisfies \( \sigma_a < \min\{1,r\} + \frac{1}{2} \max\{1,r\} \) and, for some \( k \geq 0 \), the diagonal Epstein zeta function \( \mathcal{Z}_{2k+1} \) obeys to a subconvex estimate on the critical line of the form

\[ \mathcal{Z}_{2k+1}(2^k r + it; a; \lambda) = O \left( |t|^{\sigma_a - \frac{k}{2}} \right), \quad |t| \to \infty. \] (4.16)

Then \( \phi(s) \) has infinitely many zeros on the critical line \( \text{Re}(s) = \frac{1}{2} \).

**Proof.** Since \( \sigma_a = \sigma_b \) (because \( \bar{\alpha}(n) = b(n) \) and \( \lambda_n = \mu_n \)) and \( \phi_1 = \phi_2 = \phi \in \mathbb{A} \), the Selberg-Chowla representation for the diagonal Epstein zeta function \( \mathcal{Z}_2(s; \lambda_1, \lambda_2; \lambda, \lambda') \) reduces to (4.11) with \( k = 0 \). For the case \( k = 0 \), (4.10) gives

\[ \Gamma(r + 2it) \mathcal{Z}_2(r + 2it; a; \lambda) = -2\phi(0) \Gamma(r + 2it) \phi(r + 2it) + 2 \Gamma(r) \rho \Gamma(2it) \phi(2it) \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma \left( \frac{r}{2} + i(y-t) \right) \psi \left( \frac{r}{2} + i(y-t) \right) \Gamma \left( \frac{r}{2} + i(y+t) \right) \phi \left( \frac{r}{2} + i(y+t) \right) dy. \] (4.17)

The idea of this proof is to estimate both sides of the previous equality when \( |t| \to \infty \). We shall give an estimate for the right-hand side of (4.17) under the assumption that \( \phi(s) \) does not possess infinitely many zeros at the line \( \text{Re}(s) = \frac{1}{2} \) which will contradict the assumptions of our Theorem.

By the hypothesis \( b(n) = \bar{\alpha}(n) \), we know that the integrand in the previous equation must be a real function of \( y \in \mathbb{R} \). For convenience, we shall write \( \Gamma(s) \phi(s) \) as \( R_0(s) \) and \( \Gamma(s) \psi(s) \) as \( R_{\psi}(s) \) (see subsection 1.1). Supposing that \( \phi(s) \) has finitely many zeros on the line \( \text{Re}(s) = \frac{1}{2} \), there exists some \( T_0 > 0 \) such that, if \( |y-t| > T_0 \) and \( |y+t| > T_0 \), the integrand has a constant sign. Hence, the following equality holds

\[ \left| \int_{-t-T_0}^{t-T_0} R_{\psi} \left( \frac{r}{2} + i(y-t) \right) R_0 \left( \frac{r}{2} + i(y+t) \right) dy \right| = \int_{-t-T_0}^{t-T_0} \left| R_{\psi} \left( \frac{r}{2} + i(y-t) \right) R_0 \left( \frac{r}{2} + i(y+t) \right) \right| dy. \] (4.18)

The idea now is to see that the right-hand side of (4.18) will provide a lower bound for the integral on the right-hand side of (4.17) which, in its turn, will contradict hypothesis 1. and 2. of our statement. If we
take a partition of the integral in (4.17) as
\[
\left\{ \int_{t-T_0}^{t+t_0} + \int_{t-t_0}^{t-t_0} \right\} R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy,
\]
we see that, in the third of these, we can use the contradiction hypothesis (4.18).

Let us denote each integral in the previous partition by \( \mathcal{A}_i(t) \). We first check that, for any \( \delta \geq 0 \), the first
integrals satisfy the asymptotic order
\[
\mathcal{A}_1(t), \mathcal{A}_2(t) = O \left( \left| t \right|^{\sigma_a - \frac{1}{2} + \delta} e^{-\pi t} \right).
\]

The estimate (4.20) comes directly from Stirling’s formula and a simple application of the Phragmén-Lindelöf principle (1.17) for \( \phi \) and \( \psi \) on the critical line \( \text{Re}(s) = \frac{1}{2} \). It suffices to verify (4.20) only for \( \mathcal{A}_1(t) \), which in its turn concludes the same verification for \( \mathcal{A}_2(t) \) since, from the functional equation (1.7) for \( \phi(s) \), \( \mathcal{A}_1(t) = \mathcal{A}_2(t) \). From the fact that \( \phi \) and \( \psi \) have the same abcissa of absolute convergence, \( \sigma_a \), \( R_\phi \) and \( R_\psi \) obey to the estimates
\[
R_\phi \left( \frac{r}{2} + it \right), R_\psi \left( \frac{r}{2} + it \right) = O \left( \left| t \right|^{\sigma_a - \frac{1}{2} + \delta} e^{-\frac{\pi}{2} t} \right), \quad \left| t \right| \to \infty,
\]
which can be used inside the integral for \( \mathcal{A}_1(t) \), after we split \( \mathcal{A}_1 \) as follows
\[
\int_{-T_0+t}^{\infty} R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy = \left\{ \int_{-T_0}^{T_0} + \int_{T_0}^{\infty} \right\} R_\psi \left( \frac{r}{2} + it \right) R_\phi \left( \frac{r}{2} + i(u+2t) \right) du.
\]

Thus, by a simple application of Stirling’s formula one sees immediately that
\[
\int_{-T_0}^{T_0} R_\psi \left( \frac{r}{2} + uit \right) R_\phi \left( \frac{r}{2} + i(u+2t) \right) du = O \left( \left| t \right|^{\sigma_a - \frac{1}{2} + \delta} e^{-\pi |t|} \right)
\]
while, analogously,
\[
\int_{T_0}^{\infty} R_\psi \left( \frac{r}{2} + uit \right) R_\phi \left( \frac{r}{2} + i(u+2t) \right) du = O \left( \left| t \right|^{\sigma_a - \frac{1}{2} + \delta} e^{-\pi |t|} \right)
\]
providing (4.20).

In the interval of integration considered on the third integral \( \mathcal{A}_3(t) \), we may invoke Stirling’s formula
once more, since we can take \( T_0 \) sufficiently large and, in this interval, we have \( |y-t| > T_0 \) and \( |y+t| > T_0 \).
From (1.15), we have that
\[
\left| \Gamma \left( \frac{r}{2} + i(y-t) \right) \Gamma \left( \frac{r}{2} + i(y+t) \right) \right| = 2\pi \left( t^2 - y^2 \right) \frac{|t|}{2\pi} e^{-\pi |t|} \left( 1 + O(|t|^{-1}) \right).
\]

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Let \( c \) be the least positive integer such that \( a(c) \neq 0 \): invoking (4.24) on the right-hand side of (4.18), we obtain the inequality
\[
\int_{t-T_0}^{t} R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy = \lambda_c^{-r} \int_{t-T_0}^{t} \left| \frac{dr}{2} + 2i \right| R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy
\]
\[
\geq 2\pi \lambda_c^{-r} e^{-\pi|t|} \left| \int_{\frac{T}{2}-i(t-T_0)}^{\frac{T}{2}+i(t-T_0)} \lambda_c^{2c} \left( t^2 + (z - \frac{r}{2})^2 \right)^{\frac{1}{4}} \psi(z-it) \phi(z+it) dz \right| \left\{ 1 + O \left( |t|^{-1} \right) \right\}.
\]
(4.25)

By Cauchy’s Theorem applied to the positively oriented rectangular contour with vertices \( \frac{T}{2} \pm i(t-T_0) \) and \( \sigma_0 + 1 \pm i(t-T_0) \), we can write the contour integral on the right-hand side of (4.25) as
\[
\left\{ \int_{\frac{T}{2}-i(t-T_0)}^{\sigma_0+1-i(t-T_0)} + \int_{\sigma_0+1+i(t-T_0)}^{\frac{T}{2}+i(t-T_0)} + \int_{\frac{T}{2}+i(t-T_0)}^{\sigma_0+1+i(t-T_0)} \right\} \lambda_c^{2c} \left( t^2 + (z - \frac{r}{2})^2 \right)^{\frac{1}{4}} \psi(z-it) \phi(z+it) dz.
\]
(4.26)

The first and third integrals in the above partition can be easily evaluated with the aid of Phragmén-Lindelöf estimates. Note that in the first of these, the factor \( \phi(z+it) \) does not depend on \( t \) and since \( T_0 \) is a fixed number then we just need to apply (1.17) for \( \psi(z-it) \). By symmetry, the same happens on the third integral with the roles of \( \phi \) and \( \psi \) being reversed. Denoting each integral on (4.26) by \( \mathcal{A}_{3j} \), we find that, for every \( \delta > 0 \),
\[
\mathcal{A}_{31}(t), \mathcal{A}_{33}(t) = O \left( |t|^{\sigma_0+\frac{r}{2}+1+\delta} \right).
\]
(4.27)

In order to estimate the middle integral, \( \mathcal{A}_{32} \), we can use the Dirichlet series representation of the product \( \psi(z-it) \phi(z+it) \). We arrive at (\( k := \sigma_0 + 1 - \frac{r}{2} \))
\[
\int_{\sigma_0+1-i(t-T_0)}^{\sigma_0+1+i(t-T_0)} \lambda_c^{2c} \left( t^2 + (z - \frac{r}{2})^2 \right)^{\frac{1}{4}} \psi(z-it) \phi(z+it) dz = |a(c)|^2 \int_{t-i(t-T_0)}^{k+i(t-T_0)} (t^2 + z^2)^{\frac{1}{4}} dz
\]
\[
+ \sum_{n=c+1}^{\infty} \tau(m) a(n) \left( \frac{\lambda_m}{\lambda_n} \right)^{i(\frac{2}{r} + \frac{1}{2})} \int_{k-i(t-T_0)}^{t+i(t-T_0)} (t^2 + z^2)^{\frac{1}{4}} \left( \frac{\lambda_n}{\lambda_m} \right)^{\frac{2}{r}} dz = 2i |a(c)|^2 |t|^r + O \left( |t|^{r-1} \right)
\]
upon an integration by parts for the integrals in the infinite series. Since \( \sigma_0 < \min\{1,r\} + \frac{1}{2} \max\{1,r\} \) by hypothesis, (4.25) gives
\[
\int_{t-T_0}^{t} R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy \geq 2\pi \lambda_c^{-r} e^{-\pi|t|} \left| 2i |a(c)|^2 |t|^r + O \left( |t|^{r-1} \right) + O \left( |t|^{|a(c)|+\frac{r}{2}+1+\delta} \right) \right|
\]
\[
= 4\pi |a(c)|^2 \lambda_c^{-r} |t|^r e^{-\pi|t|} + O \left( |t|^{r-1} e^{-\pi|t|} \right) + O \left( |t|^{|a(c)|+\frac{r}{2}+1} e^{-\pi|t|} \right).
\]
(4.29)

Now, let us note that the second \( O \) term in (4.29) always bounds the first one. As usual, define
\[
\mu(\sigma) = \inf \left\{ \xi : \phi(\sigma+it) = O \left( |t|^\xi \right) \right\}.
\]
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From (1.16), the general theory of $\mu(\sigma)$ [75, p. 299] and the hypothesis of Theorem 4.1, we know that $\mu(\sigma) \leq r - 2\sigma$ for $\sigma \leq r - \sigma_a$. Since $\mu(\sigma)$ is decreasing, we must have that $0 \leq \mu(r - \sigma_a) \leq 2\sigma_a - r$ and this implies $\sigma_a \geq \frac{r}{2}$. From the contradiction hypothesis (4.18), (4.29) gives the lower bound for $\mathcal{A}_3(t)$

$$|\mathcal{A}_3(t)| \geq 4\pi |a(c)|^2 \lambda e^{-r |t|} e^{-\pi |t|} + O \left(|t|^{|\sigma_a + \frac{r}{2} - 1} e^{-\pi |t|}\right), \quad |t| > T_0.$$ (4.30)

Returning to the Selberg-Chowla formula (4.17), together with (4.30), the hypothesis (4.18) and the definition of the integrals $\mathcal{A}_i(t)$, we deduce the following inequality

$$2 |a(c)|^2 \lambda e^{-r |t|} e^{-\pi |t|} + O \left(|t|^{|\sigma_a + \frac{r}{2} - 1} e^{-\pi |t|}\right) \leq \frac{1}{2\pi} |\mathcal{A}_3(t)| = |\Gamma(r + it) \mathcal{Z}_2(r + it; a; \lambda) - 2\phi(0) \Gamma(r + 2it) \phi(r + 2it) + 2\Gamma(r) \rho \Gamma(2it) \phi(2it) - \frac{1}{2\pi} \mathcal{A}_1(t) - \frac{1}{2\pi} \mathcal{A}_2(t)|.$$ (4.31)

To contradict hypothesis 1. and 2. for $k = 1$, we just need to estimate the remaining terms appearing on the right-hand side of (4.31). We have seen already that $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ satisfy (4.20), so we just need to estimate the remaining two. From (1.17) and (1.15), it is immediate to see that, for every positive $\delta$,

$$2\Gamma(r) \rho \Gamma(2it) \phi(2it) - 2\phi(0) \Gamma(r + 2it) \phi(r + 2it) = O \left(|t|^{|\sigma_a - \frac{r}{2} + \delta} e^{-\pi |t|}\right), \quad |t| \to \infty.$$ (4.32)

Using (4.31) and the fact that $\sigma_a < r + \frac{1}{2}$, we obtain the lower bound for the Epstein zeta function

$$|\Gamma(r + 2it) \mathcal{Z}_2(r + 2it; a; \lambda)| \geq 2 |a(c)|^2 \lambda e^{-r |t|} e^{-\pi |t|} + O \left(|t|^{|\sigma_a + \frac{r}{2} - 1} e^{-\pi |t|}\right) + O \left(|t|^{|\sigma_a - \frac{r}{2} + \delta} e^{-\pi |t|}\right)$$ (4.33)

valid for any $|t| > T_0$. Using Stirling’s formula (1.15) and enlarging $T_0$ if needed, (4.33) immediately implies that, for some positive constant $B$, the order of $\mathcal{Z}_2(s; a; \lambda)$ at the critical line satisfies the lower bound

$$|\mathcal{Z}_2(r + 2it; a; \lambda)| > B |t|^{|\frac{r}{2}}, \quad |t| > T_0,$$ (4.34)

and so $\mathcal{Z}_2(s; a; \lambda)$ cannot have infinitely many zeros on the critical line $\text{Re}(s) = r$, since it is always greater than a positive function for every $|\text{Im}(s)| > 2T_0$. This also contradicts (4.16) for $k = 1$. To prove the Theorem for a general $k \in \mathbb{N}$, we shall use Lemma 4.1 and prove by induction that (4.34) implies (4.16).

The following claim contains the necessary assertion to finish the proof:

**Claim 4.1.** Suppose that, for some $T_k > 0$, the $2^k$-th Epstein zeta function has the lower bound on its critical line

$$|\mathcal{Z}_{2^k}(2^{k-1}r + it; a; \lambda)| > B |t|^{|\frac{r}{2^k}}, \quad |t| > T_k,$$ (4.35)

for some $B > 0$. Then we have that $\mathcal{Z}_{2^{k+1}}$ has a similar bound of the form

$$|\mathcal{Z}_{2^{k+1}}(2^{k}r + it; a; \lambda)| > B' |t|^{|\frac{r}{2^{k+1}}}, \quad |t| > 2T_k, \quad B' > 0.$$ (4.36)
The idea is to mimic the argument just given. Use (4.10) and estimate the residual terms. It is clear from (2.48) and Stirling’s formula that these are bounded as \( O \left( |t|^{2k-a_r - \frac{1}{2} + \delta} \right) \). To estimate the integral, we take once more a partition similar to (4.19). As before, if we denote each one of these integrals by \( \mathcal{A}_i^{(k)}(t) \), \( i = 1, 2, 3 \), we can mimic the proof of (4.20) to obtain
\[
\mathcal{A}_1^{(k)}(t), \mathcal{A}_2^{(k)}(t) = O \left( |t|^{2k-a_r - \frac{1}{2} + \delta} e^{-\pi|t|} \right), \ |t| \to \infty.
\]

On the third integral, \( \mathcal{A}_3^{(k)}(t) \), we proceed as in the equality (4.18); by the inductive hypothesis (4.35), we know that \( \mathcal{Z}_2(s; a; \lambda) \) has constant sign on the line \( \text{Re}(s) = 2^{k-1}r \) for \( |\text{Im}(s)| > T_k \). Employing directly (4.35) on the third integral \( \mathcal{A}_3^{(k)}(t) \) we obtain, for any \( |t| > T_k \),
\[
\left| \mathcal{A}_3^{(k)}(t) \right| = \left| \int_{t-T_k}^{t+T_k} \Gamma \left( 2k-1-r+i(y-t) \right) \mathcal{Z}_2(k-1-r+i(y-t); b; \lambda) \times \times \Gamma \left( 2k-1-r+i(y+t) \right) \mathcal{Z}_2(k-1-r+i(y+t); a; \lambda) \right| \, dy >
\]
\[
> B e^{-\pi|t|} \left| \int_{t-T_k}^{t+T_k} \left( t^2 - y^2 \right)^{2k-1(r+1)-1} \, dy \right| = D |t|^{2k(r+1)-1} e^{-\pi|t|} + O \left( |t|^{2k(r+1)-2} e^{-\pi|t|} \right),
\]
for some \( D > 0 \). From the Selberg-Chowla formula (4.10) we have that, for \( |t| > T_k \) and some positive constant \( D' \),
\[
|\Gamma(2k-1+r) \mathcal{Z}_2(k+1)(2k+r+2it; a; \lambda)| > D' |t|^{2k(r+1)-1} e^{-\pi|t|},
\]
which implies (4.36).

Returning to the proof of Theorem 4.1 the assumption that \( \phi(s) \) does not have infinitely many zeros on the critical line implies (4.35) for \( k = 1 \) and \( T_1 := 2T_0 \). But then the previous claim shows that this contradicts conditions 1. and 2. on the statement and the proof follows.

**Remark 4.1.** It is clear from the previous proof that the argument can be extended to all Dirichlet series satisfying definition [1.1] such that \( \Gamma(s) \phi(s) \) is analytic on the critical line \( \text{Re}(s) = r/2 \). Hence, the imposition \( \phi \in \mathcal{A} \) in the statement of Theorem 4.1 can be removed. Note that the only time we have used the fact that \( \phi \in \mathcal{A} \) (besides in assuming that \( R_\phi(s) \) is holomorphic on the critical line) was in the estimate (4.32) of the residual terms appearing in (2.26), (2.27). Writing \( s = \sigma + it \) in (2.34), we know that, as \( |t| \) tends to infinity,
\[
\Gamma(\sigma + it - w) \phi(\sigma + it - w) = O \left( |t|^{a_r - \frac{1}{2} + \delta} e^{-\frac{\pi}{2}|t|} \right)
\]
uniformly in the curve \( C_1 \), so that
\[
\phi_2(\sigma + it; a; \lambda) = O \left( |t|^{a_r - \frac{1}{2} + \delta} e^{-\frac{\pi}{2}|t|} \int_{C_1} |\Gamma(w) \phi(w)| \, dw \right) = O \left( |t|^{a_r - \frac{1}{2} + \delta} e^{-\frac{\pi}{2}|t|} \right), \ |t| \to \infty,
\]
which is precisely (4.32).

**Remark 4.2.** Note that the condition \( \sigma_a < 1 + \frac{r}{2} \) was essential to move the line of integration on the third integral \( \mathcal{A}_3(t) \) from the critical line \( \text{Re}(s) = \frac{r}{2} \) to \( \text{Re}(s) = \sigma_a + 1 \). In fact, this condition is an essential tool.
to prove that the estimate (4.27) of the horizontal integrals is actually bounded by $|t|^r$, hence allowing to establish the lower bound (4.30). If, however, we disregard such generality and assume that the generalized Dirichlet series $\phi(s)$ is a Titchmarsh series [55, Chapter 2, p. 39], then a Theorem of Ramachandra [3, p. 570, Thm. 3] would still allow to conclude a lower bound of the form (4.34). Examples of “Titchmarsh series” include the class of Hecke Dirichlet series with signature $(\lambda, r, \gamma)$ and so the condition $\sigma_a < 1 + \frac{r}{2}$ in the statement of the previous theorem can be suspended for this class. In any case, the imposition $\sigma_a < r + \frac{1}{2}$ needs to be kept since it assures that $|t|^r$ exceeds the estimates of the integrals $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ (4.20), as well as the estimates for the residual terms (4.32) appearing in the Selberg-Chowla formula. For an application of Ramachandra’s result, see [55], where an adaptation of the Potter-Titchmarsh method [57], together with Ramachandra’s Theorem, are essential to give a proof of Hardy’s Theorem for certain Dirichlet series belonging to the Selberg class with degree two.

**Remark 4.3.** Condition 2. (4.16) on the previous theorem was invoked by Deuring to prove the infinitude of zeros of $\zeta(s)$ on $\text{Re}(s) = \frac{1}{2}$. This argument reduces the proof of Hardy’s Theorem to the establishment of the order of magnitude of a Dirichlet series (in this case $\zeta_2(s)$) on its critical line $\text{Re}(s) = \frac{1}{2}$. Although this may be extraordinarily difficult to establish in general, note that we do not need explicitly (4.16) to apply our method. What is actually required to complete the proof of Theorem 4.1 is to verify that one lower bound of the form (4.36) cannot hold, i.e., to find at least one dimension in which the absolute value of the Epstein zeta function, $|\mathcal{Z}_{2k}(s; \cdot)|$, "oscillates" along a curve of the form $C|t|^\alpha$, $0 \leq \alpha \leq 2^{k-1} - \frac{1}{2}$. This is why the first condition may add simplicity to the theorem, as the infinitude of zeros of $\mathcal{Z}_2(s; a; \lambda)$ can be established independently. This is certainly the case for the Dirichlet series $\phi(s) = \zeta_2(s) = \sum r_2(n)/n^s$, whose infinitude of zeros on the critical line $\text{Re}(s) = \frac{1}{2}$ can be easily proved by invoking the fact that $\zeta_4(s)$ has infinitely many zeros on the line $\text{Re}(s) = 1$. In its turn, the infinitude of zeros of $\zeta_4(s)$ in the line $\text{Re}(s) = 1$ may be obtained in an independent way, by invoking, for instance, Jacobi’s 4-square Theorem, which is a quantitative way of telling what is the projection of the modular form $\theta^4(z)$ onto the subspace of Eisenstein series. To prove Hardy’s Theorem to multidimensional Epstein zeta functions, a similar reasoning is made by using a version of a formula of Siegel (see Examples 5.3 and 5.6).

Based upon the previous Theorem, we can generalize several parallel results which are scattered in the literature, including proofs involving the behavior of the $\theta$–function. The following Corollary appears to be new.

**Corollary 4.2.** Assume that $\phi(s)$ is a Dirichlet series satisfying the conditions of Theorem 4.1 and $\Theta(z; a; \lambda)$ denote its generalized $\theta$–function (1.19). If

$$
\Theta \left( e^{\left( \frac{z}{2} - \varepsilon \right)}; a; \lambda \right) = o \left( e^{-\frac{r+1}{2}} \right), \quad \varepsilon \to 0^+;
$$

(4.38)

then $\phi(s)$ has infinitely many zeros on the critical line $\text{Re}(s) = \frac{\xi}{2}$. 

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Assume that $\phi(s)$ has finitely many zeros on the critical line $\text{Re}(s) = \frac{1}{2}$. It follows from the proof of the previous Theorem that, at its critical line $\text{Re}(s) = r$, the Epstein zeta function $\mathcal{Z}(s; a; \lambda)$ satisfies the inequality

$$|\Gamma(r + 2it) \mathcal{Z}(r + 2it; a; \lambda)| \geq 2 |a(c)|^2 \lambda_c^{-r} |t|^{\varepsilon} + O \left( |t|^\sigma_1 \e^{-|t|} \right) + O \left( |t|^\sigma_2 \e^{-|t|} \right), \quad |t| > T_0.$$  

(4.39)

For $\text{Re}(z) > 0$ and $\mu > 2\sigma_n$, we have from (1,22) that $\Theta_2(z; a; \lambda)$ admits the representation as the Mellin integral

$$\Theta_2(z; a; \lambda) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(s) \mathcal{Z}(s; a; \lambda) z^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{C_1} \Gamma(s) \mathcal{Z}(s; a; \lambda) z^{-s} ds + \frac{1}{2\pi i} \int_{C_2} \Gamma(s) \mathcal{Z}(s; a; \lambda) z^{-s} ds,$$  

(4.40)

where $C_1$ and $C_2$ are the paths given by

$$C_1 = (r - i\infty, r - 2iT_0) \cup (r + 2iT_0, r + i\infty)$$

and

$$C_2 = (r - 2iT_0, \mu - 2iT_0) \cup (\mu - 2iT_0, \mu + 2iT_0) \cup (\mu + 2iT_0, r + 2iT_0).$$

If we now take $z = \exp \left\{ i \left( \frac{\pi}{4} - \varepsilon \right) \right\}$ in (4.40), we have that

$$\Theta_2 \left( e^{i \left( \frac{\pi}{4} - \varepsilon \right)}; a; \lambda \right) = \frac{1}{2\pi i} \int_{C_1} \Gamma(s) \mathcal{Z}(s; a; \lambda) e^{-i \left( \frac{\pi}{4} - \varepsilon \right)s} ds + O(1).$$  

(4.41)

From the functional equation for $\mathcal{Z}(s; a; \lambda)$ (2.28) and the assumption that $b(n) = \overline{a(n)}$, $\Gamma(s) \mathcal{Z}(s; a; \lambda)$ is a real valued function on the line $\text{Re}(s) = r$. From this observation and combining (4.41) with (4.39), we obtain that the inequality

$$|\Theta_2 \left( e^{i \left( \frac{\pi}{4} - \varepsilon \right)}; a; \lambda \right)| = \frac{1}{2\pi} \int_{|y| > 2T_0} |\Gamma(r + iy) \mathcal{Z}(r + iy; a; \lambda)| e^{i \left( \frac{\pi}{4} - \varepsilon \right)y} dy + O(1)$$

$$\geq \frac{|a(c)|^2 (2\lambda_c)^{-r}}{\pi} \int_{2T_0}^\infty y^{r} e^{-\varepsilon y} dy + O \left( \int_{2T_0}^\infty y^{\sigma_1 + \max \{1, r\}} e^{-\varepsilon y} dy \right)$$

$$= \frac{|a(c)|^2 (2\lambda_c)^{-r} \Gamma(r + 1)}{\pi} e^{-(r + 1)} + O \left( e^{-\sigma_2 - \max \{1, r\}} \right),$$  

(4.42)

is valid for every positive $\varepsilon$. However, since

$$\Theta_2(z; a; \lambda) = \sum_{m, n \neq 0} a(m) a(n) e^{-\left( \lambda_n + \lambda_m \right) z} = 2a(0) \Theta(z; a; \lambda) + \Theta^2(z; a; \lambda),$$  

(4.43)

it follows from (4.38) that $\Theta_2 \left( e^{i \left( \frac{\pi}{4} - \varepsilon \right)}; a; \lambda \right) = o \left( e^{-r - 1} \right)$ as $\varepsilon \to 0^+$, which contradicts (4.42). 

\[ \square \]
Remark 4.4. Note that, when restricted to the class \( \mathcal{A} \), as well as to the Dirichlet series satisfying definition 4.1, the previous corollary extends Berndt’s result [13, p. 679, Theorem 1]. When restricted to the class of Dirichlet series satisfying Theorem 4.4 (4.38), improves Hecke’s condition (see [42, p. 74, Satz 1]),

\[
\Theta \left( e^{(\frac{\xi}{2} - \epsilon)}; a; \lambda \right) = O(e^{-\beta}), \quad 0 \leq \beta < \frac{r + 1}{2}.
\] (4.44)

Corollary 4.3. Let \( \phi(s) \) be a Dirichlet series with \( r < 1 \) and satisfying the conditions of Theorem 4.1. Assume also that \( a(n) \) satisfies \( \arg \{a(n)\}_{n \in \mathbb{N}} = \text{constant} \). Then \( \phi(s) \) has infinitely many zeros on the critical line \( \text{Re}(s) = \frac{r}{2} \). Moreover, if \( r = 1 \) and if \( a(n) \) satisfies the condition

\[
|a(c)| > \sqrt{2\pi \lambda_c} |\rho|,
\] (4.45)

where \( \rho \) denotes the residue of \( \phi \) at \( s = r = 1 \) and \( c \) denotes the least positive integer for which \( a(c) \neq 0 \), then \( \phi(s) \) has infinitely many zeros at the critical line \( \text{Re}(s) = \frac{1}{2} \).

Proof. Assume first that \( 0 < r < 1 \) and that \( \phi(s) \) does not have infinitely many zeros on the critical line \( \text{Re}(s) = \frac{r}{2} \). Then an application of (4.42) gives

\[
\epsilon^{r+1} \left| \Theta_2 \left( e^{(\frac{\xi}{2} - \epsilon)}; a; \lambda \right) \right| \geq C,
\] (4.46)

for some positive constant \( C \). On the other hand, using Bochner’s modular relation for \( \Theta_2(z; a; \lambda) \) (2.38) and the hypothesis of Theorem 4.1, \( b(n) = \overline{a}(n) \), we know that

\[
\Theta_2(z; a; \lambda) = z^{-2r} \sum_{m,n=1}^{\infty} \overline{a}(m) \overline{a}(n) e^{-(\lambda_m + \lambda_n)/z} + \Gamma^2(r) \rho^2 z^{-2r} + \mathcal{O}_2(0, a, \lambda).
\] (4.47)

Using the fact that \( \arg \{a(n)\}_{n \in \mathbb{N}} = \text{constant} \), we have from the previous relation,

\[
\left| \Theta_2 \left( e^{(\frac{\xi}{2} - \epsilon)}; a; \lambda \right) \right| \leq \sum_{m,n \neq 0} a(m) a(n) e^{-(\lambda_m + \lambda_n) \sin(\epsilon)} = \sum_{m,n=1}^{\infty} a(m) a(n) e^{-(\lambda_m + \lambda_n) \sin(\epsilon)}
\]

\[
= \left| \sin^{-2r}(\epsilon) \sum_{m,n \neq 0} \overline{a}(m) \overline{a}(n) e^{-(\lambda_m + \lambda_n) \sin(\epsilon)} + \Gamma^2(r) \rho^2 \sin^{-2r}(\epsilon) + \mathcal{O}_2(0, a, \lambda) \right|
\]

\[
= \Gamma^2(r) |\rho|^2 \sin^{-2r}(\epsilon) + O(1).
\] (4.48)

Since, by hypothesis, (4.46) holds, then we must have

\[
0 < C \leq \epsilon^{r+1} \left| \Theta_2 \left( e^{(\frac{\xi}{2} - \epsilon)}; a; \lambda \right) \right| = \epsilon^{r+1} \Gamma^2(r) |\rho|^2 \sin^{-2r}(\epsilon) + O(\epsilon^{r+1})
\] (4.49)

for every \( \epsilon > 0 \). However, since \( r < 1 \), we see that the right-hand side of (4.49) tends to zero as \( \epsilon \to 0^+ \), contradicting (4.46). This concludes the proof for the first case.

Assume now that \( r = 1 \): from (4.42) with \( r = 1 \), we know that the constant \( C \) given above can be explicitly written as \( C = |a(c)|^2 / 2\pi \lambda_c \), and so (4.49) contradicts (4.45). \( \square \)
Remark 4.5. By a Theorem of Fekete \[34\], which in its turn extends a classical Theorem of Landau on Dirichlet series, the condition \(\arg_{n \in \mathbb{N}} \{a(n)\} = \text{const.}\) implies that \(\phi(s)\) cannot be entire. Moreover, if we assume that \(\phi \in \mathcal{A}\), then the previous Corollary is only valid for \(\sigma_a = r\). In any case it is not strictly necessary that \(\arg_{n \in \mathbb{N}} \{a(n)\} = \text{const.}\) to arrive at the conclusion of Corollary 4.2. In fact, it suffices to suppose that \(a(n)\) satisfies \(|a(n)| \leq c(n)\) for every \(n \in \mathbb{N}\), with \(c(n)\) being such that the Dirichlet series \(\phi(s) = \sum c(n) \lambda^{-s}_n\) satisfies Hecke’s functional equation with \(r \leq 1\), as well as the conditions of Theorem 4.1. For example, Dirichlet \(L\)-functions attached to even primitive Dirichlet characters satisfy this condition.

It is also of interest to note that (4.34) contradicts any bound of the type \(Z_2(r + it; a; \lambda) = o\left(|t|^{\frac{1}{2}}\right)\), which may be obtained independently via more sophisticated methods employing exponential sums, as those in [46]. Under our general setting, however, it is not simple to assure these subconvex estimates for the diagonal Epstein zeta functions \(Z_{2k}\) on their critical lines. In general, all the information about the behavior of \(Z_{2k}\) on the line \(\text{Re}(s) = 2^{k-1}r\) comes exclusively from the Phragmén-Lindelöf principle.

Based solely on these classical estimates, in the next result we impose a condition which implies that, for a sufficiently large \(k\), the order of the dyadic Epstein zeta function \(Z_{2k+1}\) at the critical line \(\text{Re}(s) = 2^k r\) will eventually contradict (4.35).

We remark that the next result was already proved by Berndt [12] under a different setting. Since our proof is drastically different from his and it has the advantage of avoiding the exponential integrals typical in most part of the proofs of Hardy’s Theorem [11], we present our alternative short proof. Being an immediate application of the argument developed in the proof of Theorem 4.1, we remark once more that the foregoing corollary can be extended to the Dirichlet series satisfying definition 1.1, not necessarily in \(\mathcal{A}\).

**Corollary 4.4.** Let \(\phi(s)\) be a Dirichlet series satisfying the conditions of Theorem 4.1. Moreover, assume that its abscissa of absolute convergence satisfies

\[
\sigma_a < \frac{r + 1}{2}. \tag{4.50}
\]

Then \(\phi(s)\) has infinitely many zeros on the critical line \(\text{Re}(s) = \frac{r^2}{2}\). 

---

\[\text{[11]}\text{see, for instance, [21], [52], [55], [57], where exponential integrals are used to deduce analogues of Hardy’s Theorem. The idea behind all these proofs, originally due to Landau [55], is to contrast the behaviors of }|\int_T^{2T} R_{\phi}(\frac{s}{2} + it) dt| \text{ and } |\int_T^{2T} |R_{\phi}(\frac{s}{2} + it)| dt| \text{ as } T \to \infty. \text{ Due to the oscillations resulting from Stirling’s formula, the integral } |\int_T^{2T} R_{\phi}(\frac{s}{2} + it) dt| \text{ has substantial cancellation and the bound for it is usually found by appealing to exponential integrals, which cannot be avoided even if we use, for the case of Riemann’s } \zeta- \text{function, weak versions of the approximate functional equation and the Riemann-Siegel formula (see [62], p. 43) and [45], Chpt. 2). By appealing to the freedom of selecting a suitable dimension for the Epstein zeta function, our proof of the corollary 4.4 avoids non-elementary estimates and it seeks the contradiction by finding an Epstein zeta function not satisfying the lower bound (4.35) in its critical line.} \]
By the Phragmén-Lindelöf principle applied to the Dirichlet series $\mathcal{L}_{2^k+1}(s; a; \lambda)$, we have that, for every positive $\delta$, \begin{equation}
abla \mathcal{L}_{2^k+1}(2^k r + 2i t; a; \lambda) = O \left( |t|^{2^k+1} \sigma_0 - 2^k r + \delta \right), \quad |t| \to \infty. \tag{4.51} \end{equation}

However, if $\phi(s)$ has finitely many zeros on the line $\text{Re}(s) = \frac{r}{2}$, then there is some $T_0 > 0$ such that, for every $k \in \mathbb{N}_0$ (see claim 4.1), \begin{equation} |\mathcal{L}_{2^k+1}(2^k r + 2i t; a; \lambda)| > B |t|^{\frac{2^k+1}{2}}, \quad |t| > T_k := 2^k T_0, \tag{4.52} \end{equation}
for some $B > 0$ depending on $k$. After a direct comparison between (4.51) and (4.52), and letting $|t|$ tend to infinity, we deduce that the inequality $\sigma_0 \geq \frac{r+1}{2} - \frac{1}{2^k} - \delta'$ must hold for every $k \in \mathbb{N}_0$ and every $\delta' > 0$. This contradicts (4.50) and the corollary follows. \hfill $\square$

Another application of the method developed in Theorem 4.1 is the proof that, for the class of Dirichlet series considered in this section, every combination of bounded vertical shifts of $R_\phi(s)$ has infinitely many zeros on the critical line $\text{Re}(s) = \frac{r}{2}$.

Let $(c_j)_{j \in \mathbb{N}}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty} |c_j| < \infty$ and $(\tau_j)_{j \in \mathbb{N}}$ be a bounded sequence of real numbers. For all $s \in \mathbb{C}$, define \begin{equation} F_\phi(s) := \sum_{j=1}^{\infty} c_j \Gamma(s + i \tau_j) \phi(s + i \tau_j) = \sum_{j=1}^{\infty} c_j R_\phi(s + i \tau_j). \end{equation}

Note that, if $\phi \in \mathcal{A}$, $F_\phi(s)$ has poles located at $s = -i \tau_j$ and $s = r - i \tau_j$ for all $j \in \mathbb{N}$. From the fact that the poles of $\phi$ are, in general, confined to a compact set (see definition 1.1) and the boundedness of the sequence $(\tau_j)_{j \in \mathbb{N}}$, one can show that there exists a bounded set $D \subset \mathbb{C}$ such that $F_\phi(s)$ is analytic on $\mathbb{C} \setminus D$. For the case where $\phi \in \mathcal{A}$, $D$ can be taken as the union of two bounded vertical intervals containing $s = 0$ and $s = r$. If $(\tau_j)_{j \in \mathbb{N}}$ is an infinite sequence, then $F_\phi(s)$ has essential singularities inside $D$, but despite this it can be seen that the function $F_\phi \left( \frac{r}{2} + iy \right)$ is well defined for $y \in \mathbb{R}$ if $\Gamma(s) \phi(s)$ is analytic on the line $\text{Re}(s) = \frac{r}{2}$. Moreover, if the parameters of $\phi$ and $\psi$ satisfy $\overline{\psi}(n) = b(n)$ and $\lambda_n = \mu_n$, then $F_\phi \left( \frac{r}{2} + iy \right)$ is real valued for $y \in \mathbb{R}$.

In the next corollary we give sufficient conditions for $F_\phi(s)$ to have infinitely many zeros on the critical line $\text{Re}(s) = \frac{r}{2}$. For the case where $\phi(s) = \pi^{-s} \zeta(2s)$, this result has been proved by A. Dixit, N. Robles, A. Roy and A. Zaharescu [29]. Their proof employed a variation of Hardy’s method and involved the computation of the shifted moments of the Riemann $\zeta$–function. Due to the limitations of generalizing the method of the moments to other classes of Dirichlet series, their proof cannot be easily extended to the class of Dirichlet series satisfying definition 1.1. However, the method employed by us in the proof of Theorem 4.1 and Corollary 4.2 can. Although the following result is stated for the class $\mathcal{A}$ only, by Remarks 2.3 and 4.1 it can be also expected to hold for a more general class of Dirichlet series.
Corollary 4.5. Let \((c_j)_{j \in \mathbb{N}}\) be a sequence of non-zero real numbers such that \(\sum |c_j| < \infty\) and \(\tau_j\) be a bounded sequence of distinct real numbers. Assume also that \(\phi\) satisfies the conditions given in the statement of Theorem 4.4. If one of the following condition holds:

1. The generalized \(\Theta\)–function, \(\Theta(z; a; \lambda)\), satisfies \(\Theta \left( e^{i \left( \frac{z}{2} - \epsilon \right)}; a; \lambda \right) = o \left( \epsilon^{-\frac{1}{2}} \right)\) as \(\epsilon \to 0^+\).

2. The abscissa of absolute convergence of \(\phi, \sigma_a\), satisfies the condition \(\sigma_a < \frac{1}{2}\).

Then the function \(F_\phi(s) := \sum_{j=1}^{\infty} c_j \Gamma(s+i\tau_j) \phi(s+i\tau_j)\) has infinitely many zeros on the critical line \(\text{Re}(s) = \frac{1}{2}\).

Proof: Throughout this proof we shall denote \(F_\psi(s) := \sum_{j=1}^{\infty} c_j \Gamma(s+i\tau_j) \psi(s+i\tau_j)\). We just show this corollary for the first item, as for the second a simple adaptation of the proofs of Claim 4.1 and Corollary 4.1 Recalling (4.17), it is simple to see that, for any elements \(\tau_j\) and \(\tau_k\) belonging to the bounded sequence \((\tau_j)_{j \in \mathbb{N}}\), the following representation holds

\[
\Gamma(r + 2it + i(\tau_k - \tau_j)) \mathcal{D}_2(r + 2it + i(\tau_k - \tau_j); a; \lambda) = -2\phi(0) \Gamma(r + 2it + i(\tau_k - \tau_j)) \phi(r + 2it + i(\tau_k - \tau_j)) +\]

\[+ 2\Gamma(r) \rho \Gamma(2it + i(\tau_k - \tau_j)) \phi(2it + i(\tau_k - \tau_j)) + \frac{1}{2\pi} \int_{-\infty}^{\infty} R_\psi \left( \frac{r}{2} + i(y + \tau_j - t) \right) R_{\phi} \left( \frac{r}{2} + i(y + \tau_k + t) \right) dy.
\]

Multiplying the previous equality by \(c_j c_k\) and summing over \(j\) and \(k\) and using the hypothesis \((c_j)_{j \in \mathbb{N}} \in \ell^1\), we arrive at the equality

\[
\sum_{j,k=1}^{\infty} c_j c_k \Gamma(r + 2it + i(\tau_k - \tau_j)) \mathcal{D}_2(r + 2it + i(\tau_k - \tau_j); a; \lambda) = -2\phi(0) \sum_{j,k=1}^{\infty} c_j c_k R_\phi(r + 2it + i(\tau_k - \tau_j)) +\]

\[+ 2\Gamma(r) \rho \sum_{j,k=1}^{\infty} c_j c_k R_\phi(2it + i(\tau_k - \tau_j)) + \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\psi \left( \frac{r}{2} + i(y - t) \right) F_\phi \left( \frac{r}{2} + i(y + t) \right) dy,
\]

(4.53)

which is now useful to employ the contradiction hypothesis. Assume that \(F_\phi(s)\) does not have infinitely many zeros on the line \(\text{Re}(s) = \frac{1}{2}\). Following the proof of Theorem 4.1 and using the fact that \((\tau_j)_{j \in \mathbb{N}}\) is bounded and that all infinite series converge absolutely (so that we can apply all the estimates given in (4.27) (4.29) and (4.32) uniformly), we arrive at a lower bound similar to (4.29) of the form

\[
\left| \sum_{j,k=1}^{\infty} c_j c_k \Gamma(r + 2it + i(\tau_k - \tau_j)) \mathcal{D}_2(r + 2it + i(\tau_k - \tau_j); a; \lambda) \right| > C |t|^\varepsilon e^{-\pi|t|}, \quad |t| > T_0,
\]

(4.54)

where \(T_0\) is such that \(F_\phi \left( \frac{1}{2} + iy \right)\) has no zeros for \(|y| > T_0\) and \(C\) is some positive constant. Note also that, by the functional equation for \(\mathcal{D}_2(s; a; \lambda)\), the double series on the left-hand side of (4.54) is a real function of \(t\).
The inequality \( (4.54) \) clearly contradicts the first hypothesis of our corollary: indeed, from \( (4.41) \),
\[
e^{-\zeta_{\tau}(\tau_{\tau}-\xi)} \Theta_2 \left( e^{i(\xi-e)} ; a; \lambda \right) = \frac{1}{2\pi i} \int_{C_1} \Gamma(s+i(\tau_k-\tau_j)) \mathcal{Z}_2(s+i(\tau_k-\tau_j)); a; \lambda) e^{-i(\xi-e)} ds + O(1),
\]
where \( C_1 \) denotes the path \( (r-i\infty, r-2iT_0) \cup (r+iT_0, r+i\infty) \). Following the proof of Corollary \( 4.2 \) \( (4.54) \) yields
\[
\sum_{j,k=1}^{\infty} |c_j c_k| e^{-\zeta_{\tau}(\tau_{\tau}-\xi)} \left| \Theta_2 \left( e^{i(\xi-e)} ; a; \lambda \right) \right| > D e^{-(r+1)} + O(1), \tag{4.55}
\]
for every positive \( \varepsilon \). Since the double sequence \( A_{j,k} := (\tau_k-\tau_j)_{j,k \in \mathbb{N}} \) is bounded and \( \sum_{j,k} |c_j c_k| < \infty \) by hypothesis, it is clear that the double series in the left-hand side of the previous inequality is convergent. Henceforth, \( (4.55) \) immediately contradicts the first assumption, by virtue of \( (4.43) \).

As an application of the method described in Theorem \( 4.1 \) and Corollary \( 4.4 \), we now give a quantitative result regarding the distribution of the zeros of \( \phi(s) \) on the critical line, i.e., for a sufficiently large \( T \) we find \( H = H(T) \) such that the interval \( [T, T+H] \) contains the ordinate of a zero of \( \phi(s) \) located at the critical line \( \text{Re}(s) = \frac{1}{2} \). This method may be useful in future approaches to the problem of finding \( H(T) \) for which a given Dirichlet series possesses a zero on its critical line with ordinate between \( T \) and \( T+H \).

**Corollary 4.6.** Let \( \phi(s) \) be a Dirichlet series satisfying the conditions of Corollary \( 4.2 \). Then, for any fixed \( \varepsilon > 0 \), there exists a positive number \( T_0(\varepsilon) \) such that, for all \( T \geq T_0(\varepsilon) \), there is a zero \( s = \frac{1}{2} + iT \) of \( \phi(s) \) with
\[
|\tau - T| \leq T^{\sigma_0 + \frac{1}{2} - \varepsilon}, \tag{4.56}
\]

**Proof.** We just indicate the main steps: instead of taking a fixed parameter \( T_0 \) as in the proof of Theorem \( 4.1 \), we will take a sufficiently large \( T \). If one assumes that \( \phi(\frac{1}{2} + iy) \) does not possess a zero whenever \( y \in [T, T+H] \), then the integrand on \( (4.17) \) does not possess a zero when \( y \in [t - T - H, t - T] \). If we let \( t \) vary on the interval \( (T + \frac{H}{2}, T + \frac{H}{2}) \) and take a partition of the integral exactly as \( (4.19) \), we will now obtain a partition of the form
\[
\left\{ \int_{aH}^{\infty} + \int_{-\infty}^{-aH} + \int_{aH}^{\alpha H} \right\} R_\psi \left( \frac{r}{2} + i(y-t) \right) R_\phi \left( \frac{r}{2} + i(y+t) \right) dy,
\]
where \( \frac{1}{4} < \alpha \leq \frac{1}{2} \). Estimates of the first two integrals will be similar to the already given \( (4.20) \) and by the contradiction hypothesis, we are allowed to take the modulus inside the third integral as in \( (4.18) \) and then we can invoke Cauchy’s Theorem to arrive at a lower bound. At this point, since the amplitude of the interval considered for \( \mathcal{Z}_2(t) \) is \( 2\alpha H \), the quantity \( H(T) \) needs to be chosen so that it must be greater than the estimates for the integrals concerning the horizontal segments in \( (4.26) \). An immediate adaptation of the steps leading to \( (4.28) \) will give a lower bound for \( \mathcal{Z}_2(r + 2it; a, \lambda) \) of the form
\[
|\mathcal{Z}_2(r + 2it; a, \lambda)| > B'T^{-\frac{1}{2}}H, \quad T + \frac{H}{4} < t \leq T + \frac{H}{2},
\]
for some positive $B'$. An inductive reasoning similar to the one given in Claim 4.1 will also yield

$$|Z_{2k+1}^2(2^k r + 2it; a; \lambda)| > B' T - \frac{2^{k+1} - 1}{2} H^{2^k - 1} - 1, \quad T + \frac{H}{2^k+1} < t \leq T + \frac{H}{2^k},$$

valid for all $k \in \mathbb{N}_0$. If our Dirichlet series is such that (4.50) holds, the choice $H(T) = T^{\sigma_a + \frac{1}{2} + \varepsilon}$ satisfies all of our requirements. In fact, under the hypothesis (4.50), (4.57) will contradict the Phragmén-Lindelöf bound (4.51) for $Z_{2k+1}^2$.

**Remark 4.6.** Although (4.56) is valid in the general conditions of Corollary 4.4, in most of the particular cases direct adaptations of the method described in Theorem 4.1 and Corollary 4.6 yield far better estimates than (4.56) (see Examples 5.3 and 5.6 below). This happens when we have enough information regarding the Epstein zeta functions associated with $\phi(s)$. Once we have such information, the condition (4.50) used in Corollary 4.4 may be discarded, since we can find a lower bound for $\mathcal{A}_3(T)$ by invoking Ramachandra’s Theorem [3].

## 5 A class of examples and identities arising from the Selberg-Chowla formula

**Example 5.1.** (A Generalized Watson formula and its corollaries) On Taylor’s paper regarding the functional equation for the Epstein zeta function and its Selberg-Chowla representation, formula (1.2) was attributed to Kober [71, p. 182]. Although Kober [49] hadn’t the explicit purpose of studying the analytic continuation of the Epstein zeta function, he did study a generalization of a previous formula due to Watson [78, eq. (4)]. By using the Poisson summation formula, G. N. Watson proved that, for $\text{Re}(s) > 0$ and $x > 0$, the following identity holds

$$\sum_{n=1}^{\infty} n^r K_s(2\pi nx) + \frac{1}{4} \left( \frac{\pi x}{\pi} \right)^{-s} \Gamma(s) - \frac{\sqrt{\pi}}{4} \left( \frac{x}{\pi} \right)^{-s-1} \Gamma \left( s + \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2x} \left( \frac{x}{\pi} \right)^{s+1} \Gamma \left( s + \frac{1}{2} \right) \sum_{n=1}^{\infty} \frac{1}{(n^2 + x^2)^{s+1}}. \quad (5.1)$$

In this first example, we look at generalizations of (5.1). These generalizations also include Kober’s formulas given in [49]. Looking at our derivation of formulas (3.11) and (3.12) on Theorem 3.1, it is not hard to write generalizations of Watson’s formula for Bochner Dirichlet series. Let $\phi(s) = \sum a(n) \lambda_n^{-s}$, $\text{Re}(s) > \sigma_a$, be a Bochner Dirichlet series belonging to the class $\mathcal{B}$. If $v$ is a complex number satisfying $\text{Re}(v) > \frac{\sigma_a}{2}$, consider the infinite series

$$\mathcal{S}(v) = \sum_{n \in \mathbb{Z}} \frac{a(n)}{(a \lambda_n^2 + b \lambda_n + c)^v},$$
Theorem 3.1 if we perform a summation with respect to only one variable, we obtain the following formulas

\[-2\phi(0) c^{-\nu} + \sum_{n \neq 0} \frac{a(n)}{(a\lambda_n^2 + b\lambda_n + c)^\nu} = \rho \sqrt{\pi} \frac{\Gamma \left( v - \frac{1}{2} \right)}{\Gamma(v)} a^{-\nu} k^{1-2\nu} \]

\[+ \frac{4k^{1-2\nu} c^{-\nu}}{\Gamma(v)} \sum_{n=1}^{\infty} b(n) \cos \left( \frac{b}{a} \mu_n \right) \mu_n^{-\frac{1}{2}} K_{v-\frac{1}{2}}(2k \mu_n), \ a(n) \text{ even}, \tag{5.2} \]

\[\sum_{n \neq 0} \frac{a(n)}{(a\lambda_n^2 + b\lambda_n + c)^\nu} = -\frac{4k^{1-2\nu} c^{-\nu}}{\Gamma(v)} \sum_{n=1}^{\infty} b(n) \sin \left( \frac{b}{a} \mu_n \right) \mu_n^{-\frac{1}{2}} K_{v-\frac{1}{2}}(2k \mu_n), \ a(n) \text{ odd}. \tag{5.3} \]

The first of these identities generalize Watson’s formula and its extension by Kober [49, p. 614, eq. (2b)]. Furthermore, the Selberg-Chowla formula (3.20) given in Remark 3.3 gives another formula of Watson-type

\[\sum_{n \neq 0} \frac{a(n) \lambda_n}{(a\lambda_n^2 + b\lambda_n + c)^\nu} = \frac{4k^{1-2\nu} c^{-\nu}}{\Gamma(v)} \sum_{n=1}^{\infty} b(n) \mu_n^{-\frac{1}{2}} \times \left\{ \cos \left( \frac{b}{a} \mu_n \right) K_{\frac{1}{2}-(2k \mu_n)} + \frac{2b}{\sqrt{d}} \sin \left( \frac{b}{a} \mu_n \right) K_{\frac{1}{2}-v}(2k \mu_n) \right\}, \ a(n) \text{ odd}. \tag{5.4} \]

Let \(\chi\) be a non-principal and primitive Dirichlet character modulo \(\ell\): since \(\zeta(s)\) and \(L(s, \chi)\) both satisfy functional equations of Bochner-type (1.8), we can replace the sequences appearing in (5.2) and (5.4) by those of these Dirichlet series. For example, if \(\phi(s) = \zeta(s)\) and \(b = 0\), (5.2) reduces to an equivalent form of Watson’s formula (5.1). Under the same hypothesis of diagonal \(Q(x, y)\), if we replace, respectively, \(a(n)\) by \(\chi(n)\) in (5.2) and (5.4) we obtain, respectively, formulas (2.9) and (2.10) given in [15, p. 3, Thm 2.1].

There are several known applications of the classical Watson formula. In his Lost Notebook [17], Ramanujan recorded the following formula. Let \(s \in \mathbb{C}\) and assume that \(\alpha, \beta > 0\) are such that \(\alpha \beta = \pi^2\). Then the following identity holds

\[\sqrt{\alpha} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_{\frac{s}{2}}(2n\alpha) - \sqrt{\beta} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_{\frac{s}{2}}(2n\beta) = \]

\[\frac{1}{4} \Gamma \left( -\frac{s}{2} \right) \zeta(-s) \left\{ \beta^{1+s/2} - \alpha^{1+s/2} \right\} + \frac{1}{4} \Gamma \left( \frac{s}{2} \right) \zeta(s) \left\{ \beta^{1-s/2} - \alpha^{1-s/2} \right\}. \tag{5.5} \]

This formula was rediscovered by Guinand in 1955 [17, 38], who employed Watson’s formula (5.1) in order to derive it.

By using a similar idea, we can also provide a generalization of (5.5) with the generalized divisor function introduced in this paper. To do so, for \(\xi > 0\), recall the analogue of the Epstein zeta function \(\mathcal{L}_2(s; a_1, a_2; \lambda, \xi \lambda')\) given in (4.1). By using the first Selberg-Chowla representation (2.26), we have deduced (4.2). By using the same substitutions there mentioned and using the second Selberg-Chowla formula
we can analogously derive
\[
\Gamma(s) \sum_{m,n \neq 0} a_1(m) a_2(n) \left( \lambda_m + \lambda_n \right) = -\xi^{-s} \phi_1(0) \Gamma(s) \phi_2(0) + \xi^{-s} \rho_2 \Gamma(r_2) \Gamma(s-r_2) \phi_1(s-r_2) \\
+ 2 \xi^{-s} \sum_{m,n=1} b_2(m) a_1(n) \left( \frac{\mu_m}{\lambda_n} \right) K_{r_2-s} \left( 2 \sqrt{\frac{\mu_m \lambda_n}{\xi}} \right) .
\]

(5.6)

Viewed in this way, the Selberg-Chowla formula not only provides the analytic continuation of a Dirichlet series \( \mathcal{D}_2(s; a_1, a_2; \lambda, \xi \lambda') \) depending on \( \xi > 0 \) but it also gives a reformulation of this continuation in terms of a formula of modular type. A comparison between (4.2) and (5.6) yields a generalization of the Ramanujan-Guinand formula (5.3), valid for all \( s \in \mathbb{C} \) by Proposition 2.1 and the assumptions on the class \( \mathcal{A} \).

Stated in a clear way, let \( \phi_i(s), i = 1, 2 \) represent the pair of Dirichlet series given in (1.9) which also satisfy Hecke’s functional equation (1.7) and belong to the class \( \mathcal{A} \). Denote also by \( \rho_i \) the residue that \( \phi_i \) has at \( s = r_i \). Then, for all \( s \in \mathbb{C} \) and \( x > 0 \), the following generalization of Guinand’s formula holds
\[
2x^{r_1-s} \sum_{j=1}^{x} \sigma_{r_1-r} \left( \nu_j; b_1, a_2 \right) r_1-s \left( 2x \sqrt{v_j} \right) - \frac{2}{x^{2r_1+s}} \sum_{j=1}^{x} \sigma_{r_1-r} \left( \nu_j; b_2, a_1 \right) r_1-s \left( \frac{2 \sqrt{v_j}}{x} \right) \\
= \Gamma(s) \left\{ \phi_2(0) \phi_1(s) - \phi_1(0) \phi_2(s) x^{-2r} \right\} + \rho_2 x^{-2r_2} \Gamma(r_2) \Gamma(s-r_2) \phi_1(s-r_2) - \rho_1 x^{-2r_1-2r} \Gamma(r_1) \Gamma(s-r_1) \phi_2(s-r_1),
\]

(5.7)

where \( \sigma_2 \) denotes the generalized weighted divisor function described by (2.21) \([8]\).

From (5.7) it is possible to derive generalizations of the classical Koshliakov formula (50), by mimicking the steps given in (17). Writing the Laurent series for \( \phi_i(s) \) around \( s = r_i = r \) as
\[
\phi_i(s) = \frac{\rho_i}{s-r} + \rho_{0,i} + O(s-r), \quad i = 1, 2,
\]

(5.8)

assuming the same condition as the ones given in Ramanujan-Guinand’s formula (5.7) and taking \( r_1 = r_2 = r \) we have, for any \( x > 0 \), that the following formula of Koshliakov type holds
\[
2 \sum_{j=1}^{x} d \left( \nu_j; b_1, a_2 \right) K_0 \left( 2x \sqrt{v_j} \right) - 2x^{-2r} \sum_{j=1}^{x} d \left( \nu_j; b_2, a_1 \right) K_0 \left( \frac{2 \sqrt{v_j}}{x} \right) \\
= \Gamma(r) \left\{ \phi_2(0) \rho_{0,1} + \phi_2(0) \rho_1 \frac{\Gamma(r)}{\Gamma(r)} - \rho_1 \left\{ \rho_2 \Gamma'(r) + \rho_0 \Gamma(r) \right\} + \frac{2 \log(x) \phi_2(0) \rho_1}{2} \right\} +
\]
\[
+ x^{-2r} \Gamma(r) \left\{ \rho_1 \Gamma'(r) + \rho_1 \Gamma(r) \right\} \rho_2 - \phi_1(0) \rho_{0,2} - \phi_1(0) \rho_2 \frac{\Gamma(r)}{\Gamma(r)} + \frac{2 \log(x) \phi_1(0) \rho_2}{2},
\]

(5.9)

\[\text{Berndt [8], p. 343, eq. (9.1)] established other analogues of Guinand’s formula. But these formulas of Berndt are only valid for arithmetical functions whose Dirichlet series satisfy Hecke’s functional equation, while in general the arithmetical function of divisor type } c(z) := \sigma_i(\nu_j; b_1, a_2) \text{ does not give rise to a Dirichlet series satisfying (17). In fact, the Dirichlet series attached to this general divisor function is actually } \phi_z := \psi z \phi_2(0) \text{ and the assumption that } \psi z \text{ and } \phi_2 \text{ satisfy Hecke’s functional equation (1.7) allows to write a functional equation (involving a product of two } \Gamma-\text{functions) for } \phi_z(0). \text{ Thus, (5.7) is actually equivalent to this functional equation that } \phi_z(0) \text{ possesses (see also [8], p. 324 and [17], pp. 39-40).} \]
where \( \rho^*_j \) and \( \rho^*_0 \) denote the coefficients of the meromorphic expansion (5.8) for \( \psi_1(s) \) and \( d(v_j; b_1, a_2) \) is defined by (2.23). We obtain (5.9) by letting \( s \to r \) in (5.7) and using the meromorphic expansions for \( \Gamma(s) \) and \( \phi_i(s) \) around \( s = 0 \), (4.4) and (4.5), as well as (5.8).

Special cases of (5.7) and (5.9) are also possible to obtain for Dirichlet series in the Bochner class. Another important corollary of Koshliakov’s formula (or, in fact, a reformulation of it for \( r = \frac{1}{2} \) [60,68]) is due to K. Soni. According to an account in [79], Soni’s formula already appeared in Ramanujan’s lost notebook, written more than fifty years earlier (see also a multidimensional analogue of Soni’s formula in [80], p. 813, eq. (3.12)).

Indeed, taking \( r = \frac{1}{2}, \frac{3}{4} \) in (5.9) and mimicking the steps in [17] [68], it is possible to generalize Soni’s formula [68, p. 543, eq. (4)] and some character analogues appearing in [15]. For the case where \( r = \frac{1}{2} \) and assuming convergence of all the infinite series involved, we have the following formula

\[
2 \sum_{j=1}^{\infty} d(v_j; b_1, a_2) \frac{\log(\alpha/2\sqrt{v_j})}{\alpha^2 - 4v_j} - 2\pi \sum_{j=1}^{\infty} d(v_j; b_2, a_1) K_0\left(2\sqrt{v_j} \alpha^{\frac{1}{2}}\right) \\
= \sqrt{\frac{\pi}{\alpha^2}} \left\{ \phi_2(0)\rho_{0,1} - \sqrt{\pi}\rho_1\rho^*_0 - 2\rho_1\phi_2(0)(2\gamma + \log(2\alpha)) \right\} \\
+ \frac{\pi^{\frac{3}{2}}}{2\alpha} \left\{ -\phi_1(0)\rho_{0,2} + \sqrt{\pi}\rho_2\rho^*_1 + 2\rho_2\phi_1(0) \log\left(\frac{2}{\alpha}\right) \right\},
\]

where \( \alpha \in \mathbb{R}_+ \setminus \{2\sqrt{v_j}| j \in \mathbb{N}\} \).

**Example 5.2** (A character analogue of the Epstein zeta function). In this example we study two particular cases of the non-diagonal Epstein zeta function considered in section 3. Let \( Q \) denote an integral, positive definite and binary Quadratic form and let \( \chi_1 \) and \( \chi_2 \) be two non-principal, primitive Dirichlet characters (with the same parity) having moduli \( \ell_1 \) and \( \ell_2 \). We introduce the character analogue of the Epstein zeta function in the following form

\[
Z_2(s, Q, \chi_1, \chi_2) = \sum_{(m,n) \neq (0,0)} \frac{\chi_1(m)\chi_2(n)}{Q(m,n)^s}, \quad \text{Re}(s) > 1.
\]

Let us assume first that \( \chi_1 \) and \( \chi_2 \) are even Dirichlet characters and let us take in Theorem 3.1 \( \phi_1(s) = \pi^{-\frac{s}{2}}L(s, \chi_1) \) and \( \phi_2(s) = \pi^{-\frac{s}{2}}L(s, \chi_2) \). The functional equation for \( L(s, \chi) \) is given by [22]

\[
\left(\frac{\pi}{\ell}\right)^{-\frac{s}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} L(s, \chi) = \frac{G(\chi)}{\sqrt{\ell}} \left(\frac{\pi}{\ell}\right)^{-\frac{s}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} L(1-s, \chi),
\]

where \( G(\chi) \) denotes the Gauss sum associated to the character \( \chi \),

\[
G(\chi) = \sum_{r=1}^{\ell-1} \chi(r)e^{2\pi ir/\ell}.
\]

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Thus $\phi_1(s), \phi_2(s) \in \mathcal{B}$ with $\delta = 0$ and in order to apply our Theorem [3.1] we need to perform the following substitutions in (3.11):

$$a_j(n) = \chi_j(n), \quad b_j(n) = \frac{G(\chi_j)}{\ell_j} \overline{\chi_j}(n), \quad \lambda_{n,j} = \sqrt{\pi n}, \quad \mu_{n,j} = \frac{\sqrt{\pi n}}{\ell_j}. \quad (5.14)$$

Analogously, if $\chi_1$ and $\chi_2$ are odd and primitive Dirichlet characters having moduli $\ell_1$ and $\ell_2$, let us take $\phi_1(s) = \pi^{-\frac{s}{2}}L(s, \chi_1)$ and $\phi_2(s) = \pi^{-\frac{s}{2}}L(s, \chi_2)$. The functional equation for a Dirichlet $L$–function attached to an odd and primitive character $\chi$ modulo $\ell$ reads [22]

$$\left(\frac{\pi}{\ell}\right)^{-\frac{s}{2}} \Gamma\left(s + \frac{1}{2}\right) L(s, \chi) = -iG(\chi) \left(\frac{\pi}{\ell}\right)^{\frac{s}{2}} \Gamma\left(\frac{2 - s}{2}\right) L(1 - s, \overline{\chi}), \quad (5.15)$$

which implies that $\phi_1, \phi_2 \in \mathcal{B}$ with $\delta = 1$. In order to use Theorem [3.1] we also take the substitutions

$$a_j(n) = \chi_j(n), \quad b_j(n) = -\frac{iG(\chi_j)}{\ell_j} \overline{\chi_j}(n), \quad \lambda_{n,j} = \sqrt{\pi n}, \quad \mu_{n,j} = \frac{\sqrt{\pi n}}{\ell_j}. \quad (5.16)$$

After a simple application of Theorem [3.1] and Corollary [3.1] and using the substitutions outlined in (5.14) and (5.16), we see that $Z_2(s, Q, \chi_1, \chi_2)$ has a continuation to the complex plane as an entire function. Indeed, it possesses the following Selberg-Chowla formulas:

1. If $\chi_1$ and $\chi_2$ are even, then an application of the Selberg-Chowla formulas (3.11) and (3.13) yields

$$Z_2(s, Q, \chi_1, \chi_2) = \frac{8\pi^s a^{-s}}{\Gamma(s)} G(\chi_1) k^{\frac{s}{2} - s} \xi_1^{-\frac{s}{2} + \frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{1-2s}(n, \chi_1, \chi_2) n^{s} \cos \left(\frac{b}{a} \frac{n}{\ell_1}\right) K_{\frac{1}{2} - s} \left(\frac{2\pi kn}{\ell_1}\right), \quad (5.17)$$

as well as

$$Z_2(s, Q, \chi_1, \chi_2) = \frac{8\pi^s e^{-s}}{\Gamma(s)} G(\chi_2) k^{\frac{s}{2} - s} \xi_2^{-\frac{s}{2} + \frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{1-2s}(n, \chi_2, \chi_1) n^{s} \cos \left(\frac{b}{c} \frac{n}{\ell_2}\right) K_{\frac{1}{2} - s} \left(\frac{2\pi k'n}{\ell_2}\right), \quad (5.18)$$

2. If $\chi_1$ and $\chi_2$ are odd and primitive, then we equivalently have the Selberg-Chowla formula (from (3.12))

$$Z_2(s, Q, \chi_1, \chi_2) = -\frac{8i\pi^s a^{-s}}{\Gamma(s)} G(\chi_1) k^{\frac{s}{2} - s} \xi_1^{-\frac{s}{2} + \frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{1-2s}(n, \chi_1, \chi_2) n^{s} \sin \left(\frac{b}{a} \frac{n}{\ell_1}\right) K_{\frac{1}{2} - s} \left(\frac{2\pi kn}{\ell_1}\right), \quad (5.19)$$

with an analogous second Selberg-Chowla formula taking place, being of the form

$$Z_2(s, Q, \chi_1, \chi_2) = -\frac{8i\pi^s e^{-s}}{\Gamma(s)} G(\chi_2) k^{\frac{s}{2} - s} \xi_2^{-\frac{s}{2} + \frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{1-2s}(n, \chi_2, \chi_1) n^{s} \sin \left(\frac{b}{c} \frac{n}{\ell_2}\right) K_{\frac{1}{2} - s} \left(\frac{2\pi k'n}{\ell_2}\right). \quad (5.20)$$
In all of the previous representations, \( \sigma_z(n, \chi_a, \chi_b) \) is a character analogue of the divisor function, defined by
\[
\sigma_z(n, \chi_a, \chi_b) = \sum_{d|n} \chi_a(d) \chi_b \left( \frac{n}{d} \right) d^z. \tag{5.21}
\]

Moreover, it follows from Corollary 3.1 that \( Z_2(s; Q, \chi_1, \chi_2) \) satisfies the functional equation
\[
G(\chi_1) G(\chi_2) \left( \frac{2\pi}{\sqrt{|d|}} \right)^{-s} \Gamma(s) Z_2(s, Q, \chi_1, \chi_2) = \left( \frac{2\pi}{\sqrt{|d|}} \right)^{s-1} \Gamma(1-s) Z_2 \left( 1-s, Q_{\ell_1, \ell_2}^{-1}, \chi_1, \chi_2 \right), \tag{5.22}
\]
where \( Q_{\ell_1, \ell_2}^{-1} \) denotes the quadratic form \( Q^{-1} \left( \frac{x}{\ell_1}, \frac{y}{\ell_2} \right) \).

By using the previous example, we can establish analogues of the Ramanujan-Guinand formula for the character analogue of the divisor function (5.21). We only do this for the first case, i.e., assuming that the characters \( \chi_1 \) and \( \chi_2 \) are even. It is effortless to see that the comparison of (5.17) and (5.18) and the substitution \( a = c = 1, b = 0 \), yield the double-weighted Ramanujan-Guinand’s formula
\[
\sqrt{\alpha} G(\chi_1) \ell_1^{\frac{s}{2}-1} \sum_{n=1}^\infty \sigma_{-s}(n, \chi_1, \chi_2) n^{s/2} K_\chi \left( \frac{2\alpha n}{\ell_1} \right) = \sqrt{\beta} G(\chi_2) \ell_2^{\frac{s}{2}-1} \sum_{n=1}^\infty \sigma_{-s}(n, \chi_2, \chi_1) n^{s/2} K_\chi \left( \frac{2\beta n}{\ell_2} \right), \tag{5.23}
\]
where \( \alpha, \beta > 0 \) are such that \( \alpha \beta = \pi^2 \). Note that (5.23) can also be obtained from (5.7) and by using the fact that \( \pi^{-s} L(2s, \chi) \in \mathcal{S} \) and satisfies Hecke’s functional equation with \( r = \frac{1}{2} \).

Taking the limit \( s \to 0 \) in (5.23) yields Koshliakov’s formula for double characters. To write it in the form (5.9), take the substitution \( \alpha = \pi \sqrt{\ell_2} \) and replace \( \chi_1 \) by \( \chi_1 \). Use also the relation for Gauss sums (22),
\[
G(\chi) G(\overline{\chi}) = \chi(-1) \ell.
\]

Then a particular case of (5.9) is formulated as
\[
\sum_{n=1}^\infty d_{\chi_1, \chi_2}(n) K_0 \left( \frac{2\pi n \chi}{\sqrt{\ell_1 \ell_2}} \right) = \frac{G(\chi_1) G(\chi_2)}{\chi \sqrt{\ell_1 \ell_2}} \sum_{n=1}^\infty d_{\chi_1, \chi_2}(n) K_0 \left( \frac{2\pi n \chi}{\sqrt{\ell_1 \ell_2}} \right), \chi_1, \chi_2 \text{ even}, \tag{5.25}
\]
where \( d_{\chi_1, \chi_2}(n) \) represents (5.21) at \( z = 0 \), this is, the double-weighted divisor function, \( \sum_{d|n} \chi_1(d) \chi_2(n/d) \).

Identity (5.25) is given in [16, p. 45, eq. (3.7.)]. From (5.21), one can also derive character analogues of Soni’s formula: applying (5.10) gives (after replacing also \( \alpha \) by \( 2\pi \sqrt{\ell_1 \ell_2} \alpha \))
\[
\sum_{n=1}^\infty d_{\chi_1, \chi_2}(n) \log \left( \frac{\ell_1 \ell_2 \alpha}{n} \right) \frac{(\ell_1 \ell_2 \alpha)^2 - n^2}{(\ell_1 \ell_2 \alpha)^2} = 2\pi^2 G(\chi_1) G(\chi_2) \frac{\alpha \ell_1^2 \ell_2^2}{(\ell_1 \ell_2 \alpha)^2} \sum_{n=1}^\infty d_{\chi_1, \chi_2}(n) K_0 \left( 4\pi \sqrt{\alpha n} \right), \tag{5.26}
\]
which is valid for every \( \alpha \neq n/\ell_1 \ell_2, n \in \mathbb{N} \). A particular case of (5.26) where \( \chi_1(n) = \chi_2(n) = \chi(n), \ell_1 = \ell_2 = \ell \), has been observed in [15, p. 5, eq. (3.8)] and is equivalent to
\[
\sum_{n=1}^\infty \chi(n) d(n) \log \left( \frac{\alpha \ell^2}{n} \right) \frac{\alpha^2 \ell^4 - n^2}{\alpha^2 \ell^4} = 2\pi^2 \frac{G(\chi)^2}{\alpha \ell^4} \sum_{n=1}^\infty \overline{\chi}(n) d(n) K_0 \left( 4\pi \sqrt{\alpha n} \right).
\]
Extensions of other formulas given in [15] are also possible. For instance, it is not hard to derive generalizations of [15, eq. (3.15), (3.16) and (3.19)] for the character analogue of the divisor function $d_{\chi_1, \chi_2}(n)$.

Some particular cases of (5.23) include identities derived by Dixit [27] by an entirely different method [27, p. 322, Theorem 1.5., eq. (1-15)], employing the symmetric properties of an integral involving Riemann’s $\Xi-$function and a character analogue of it. Our formula (5.23) applied to the case $\chi_1 = \chi_2 = \chi$ yields $F(z, \alpha, \chi) = F(-z, \beta, \chi)$ in formula (1-15) of [27].

Example 5.3 (Hardy’s Theorem and the 4-square Theorem). Based upon Theorem 4.1 this example gives a new proof of Hardy’s Theorem for $\zeta(s)$, which is curious as its conclusion is derived from presumably independent properties of the arithmetical function $r_4(n)$. As it is well known, $r_4(n)$ counts the number of representations of $n$ as a sum of 4 squares with different signs and different orders of the summands giving distinct representations. Let us invoke Theorem 4.1 to $\phi(s) = \pi^{-s} \zeta(2s)$, which satisfies Hecke’s functional equation with $r = \frac{1}{2}$. If $\phi(s)$ does not have infinitely many zeros on the critical line $\text{Re}(s) = \frac{1}{2}$, then all of its dyadic Epstein zeta functions (2.45) will only have finitely many zeros at their critical lines. Consider $\mathcal{Z}_4(s)$: it comes immediately from the definition of multidimensional Epstein zeta function that, for $\text{Re}(s) > 2$,

$$
\mathcal{Z}_4(s) = \pi^{-s} \sum_{m_1, \ldots, m_4 \neq 0} \frac{1}{(m_1^2 + m_2^2 + m_3^2 + m_4^2)^s} = \frac{\pi^{-s}}{4} \sum_{n=1}^{\infty} \frac{r_4(n)}{n^s} = \frac{\pi^{-s}}{4} \zeta_4(s).
$$

Hence, according with our assumption, $\zeta_4(s)$ cannot have infinitely many zeros at its critical line $\text{Re}(s) = 1$. However, by the celebrated Jacobi’s four square theorem, $\zeta_4(s)$ is described by

$$
\zeta_4(s) = 8 \left(1 - 2^{2s-2}\right) \zeta(s) \zeta(s-1).
$$

Note that the right-hand side of (5.27) has infinitely many zeros in the line $\text{Re}(s) = 1$, all of them being of the form $1 \pm \frac{\pi j}{\log 2} k$, with $k \in \mathbb{N}$. This means that $\mathcal{Z}_4(s)$ has infinitely many zeros on the line $\text{Re}(s) = 1$ and a contradiction is derived, proving Hardy’s Theorem in its classical form. By using a formula due to Siegel [67], the argument presented here can be generalized to Epstein zeta functions attached to quadratic forms whose matrices have determinant 1.

---

13Although Dixit’s results may seem to be of a different nature, the integral representation involving the Riemann $\Xi-$function appearing in his paper [22, eq. (1-14)] can be generalized to our class of Dirichlet series. In fact, it is a generalization of Dixit’s integral representation that we employ to study the zeros of the Dirichlet series in Theorem 4.1 since we can represent the entire function $H_s(z; b; a)$ in a similar form (c.f. eq. 4.10 and 4.13 in the previous section).

14The novelty in this proof is the observation that a purely arithmetical property such as (5.27) implies a deep analytic theorem regarding the distribution of the zeros of the Riemann zeta function. One may argue that since (5.27) admits proofs using modular forms and a particular modular form (the $\theta-$function) is used in Hardy’s proof of Hardy’s Theorem [40], the reason should be this, as a look at a generalization of this argument in Example 5.3 shows. Despite this, the fact that there are elementary proofs of (5.27) available, such as in [43], adds a certain curiosity to this connection.
This observation can also be used to improve the quantitative result given in Corollary 4.6. For the case where \( \phi(s) = \pi^{-s} \zeta(2s) \) or \( \pi^{-\frac{1}{2}+\delta} L(2s-\delta, \chi) \), for \( \chi \) being a primitive Dirichlet character modulo \( \ell \), the result on Corollary 4.6 gives \( H(T) = T^{\frac{1}{2}+\epsilon} \), which is a result attributed to Landau. The argument invoking the 4-square theorem can actually improve this result for \( \zeta(s) \) to \( H(T) = T^{\frac{1}{2}+\epsilon} \), a result which was recorded for the first time in a paper by de La Vallée Poussin [33]. As it is clear from the brief sketch of the proof, the condition \( H(T) = T^{\sigma_0 + \frac{1}{2}+\epsilon} \) is only necessary in order to (4.57) contradict the Phragmén-Lindelöf estimates (4.51) combined with the assumption (4.50). In fact, for the case where \( \phi(s) = \pi^{-s} \zeta(2s) \) it suffices to assume that \( H(T) = T^{\frac{1}{2}+\epsilon} \) until the point we arrive at the condition (4.57). In this particular case, (4.57) gives the inequality

\[
|\zeta(s)(1+2it)| > B' \left( \frac{H}{T^{\frac{1}{2}}} \right)^3, \quad B' > 0
\]  

valid for every \( T + \frac{H}{\delta} < t \leq T + \frac{H}{1} \). This obviously contradicts (5.27), which implies that there is always a zero of \( \zeta(s)(1+it) \) in any interval of length \( \frac{\pi}{\log(\xi)} \).

Other proofs of Hardy’s Theorem for \( \zeta(s) \) can be derived from corollaries 4.3 and 4.4 yielding even easier arguments. As it was proved by Dixit et al. [29], \( \phi(s) = \pi^{-2s} \zeta(2s) \) satisfies Corollary 4.5.

If \( \chi \) is a primitive Dirichlet character modulo \( \ell \), then an application of Corollary 4.5 (or Corollary 4.3 together with remark 4.3 for the case where \( \chi \) is even), shows that the function

\[
F(s, \chi) := \sum_{j=1}^{\infty} c_j \left( \frac{\pi}{\ell} \right)^{(s+i\tau_j)} \Gamma \left( \frac{s+i\tau_j+\delta}{2} \right) L(s+i\tau_j; \chi), \quad \delta \in \{0, 1\},
\]

where \( c_j \in \ell^1(\mathbb{R}) \) and \( \tau_j \in \ell^{\infty}(\mathbb{R}) \), has infinitely many zeros on the critical line \( \text{Re}(s) = \frac{1}{2} \).

**Example 5.4** (Character analogue for the divisor function and Dedekind zeta function). Let \( \chi \) be an odd and primitive Dirichlet character modulo \( \ell \). By the functional equations for \( \zeta(s) \) and \( L(s, \chi) \) (5.15), one can check that \( \zeta(s) L(s, \chi) \) satisfies the functional equation

\[
\left( \frac{2\pi}{\sqrt{\ell}} \right)^{-s} \Gamma(s) \zeta(s) L(s, \chi) = -\frac{iG(\chi)}{\sqrt{\ell}} \left( \frac{2\pi}{\sqrt{\ell}} \right)^{-(1-s)} \Gamma(1-s) \zeta(1-s) L(1-s, \overline{\chi}).
\]  

Let us write the associated Dirichlet series satisfying (5.29) as

\[
\phi(s) = \zeta(s) L(s, \chi) = \sum_{n=1}^{\infty} \frac{d_{\chi}(n)}{n^s}, \quad \text{Re}(s) > 1,
\]

where \( d_{\chi}(n) \) denotes the character analogue of the divisor function [16, p. 42, eq. (3.1)],

\[
d_{\chi}(n) = \sum_{d|n} \chi(d).
\]

If \( \chi_1 \) and \( \chi_2 \) denote two odd and primitive Dirichlet characters modulo \( \ell_1 \) and \( \ell_2 \), we know by (5.29) that \( \phi_j(s) = \zeta(s) L(s, \chi_j) \) satisfies Hecke’s functional equation with \( r_j = 1, \ j = 1, 2 \). Moreover, each \( \phi_j(s) \)
Analogously, for $Z$ associated Dirichlet series is the classical Dedekind zeta function $L(1, \chi)$. We define the diagonal Epstein zeta function for $d_{\chi_1}(n)$ and $d_{\chi_2}(n)$ as the double Dirichlet series

$$\mathcal{Z}_2(s, d_{\chi_1}, d_{\chi_2}) := \sum_{m,n \neq 0} \frac{d_{\chi_1}(m) d_{\chi_2}(n)}{(m+n)^s}, \quad \text{Re}(s) > 2. \quad (5.30)$$

As in Example 5.2, in order to match the functional equation (5.29) with the formulations of Theorem 2.1 and Corollary 2.1, we need to perform the following substitutions for $\phi_j(s)$,

$$a_j(n) = d_{\chi_j}(n), \quad b_j(n) = -\frac{2\pi i G(\chi_j)}{\ell_j} d_{\chi_j}(n), \quad \lambda_{n,j} = n, \quad \mu_{n,j} = \frac{4\pi^2}{\ell_j} n.$$

Using the first Selberg-Chowla formula (2.26), we obtain the following continuation for (5.30)

$$\Gamma(s) \mathcal{Z}_2(s, d_{\chi_1}, d_{\chi_2}) = \frac{1}{2} L(0, \chi_2) \Gamma(s) \zeta(s) L(s, \chi_1) + L(1, \chi_1) \Gamma(s) L(s-1, \chi_2) - \frac{4\pi i G(\chi_2)}{\ell_2^{s-1}} (2\pi)^{s-1} \sum_{m,n=1}^{\infty} d_{\chi_1}(m) d_{\chi_2}(n) \left(\frac{m}{n}\right)^{1/2} K_{s-1} \left(\frac{4\pi}{\ell_1} \sqrt{mn}\right).$$

Analogously, (2.27) gives a second Selberg-Chowla formula of the form

$$\Gamma(s) \mathcal{Z}_2(s, d_{\chi_1}, d_{\chi_2}) = \frac{1}{2} L(0, \chi_2) \Gamma(s) \zeta(s) L(s, \chi_2) + L(1, \chi_2) \Gamma(s) L(s-1, \chi_1) - \frac{4\pi i G(\chi_1)}{\ell_1^{s-1}} (2\pi)^{s-1} \sum_{m,n=1}^{\infty} d_{\chi_1}(m) d_{\chi_2}(n) \left(\frac{m}{n}\right)^{1/2} K_{s-1} \left(\frac{4\pi}{\ell_2} \sqrt{mn}\right).$$

Both formulas given above provide the analytic continuation of (5.30) as a meromorphic function with a simple pole located at $s = 2$ with residue $L(1, \chi_1) \cdot L(1, \chi_2)$. The functional equation for (5.30) can be written as

$$\left(\frac{2\pi}{\sqrt{\ell_1 \ell_2}}\right)^{-s} \Gamma(s) \mathcal{Z}_2(s, d_{\chi_1}, d_{\chi_2}) = -G(\chi_1) G(\chi_2) \left(\frac{2\pi}{\sqrt{\ell_1 \ell_2}}\right)^{(2-s)} \Gamma(2-s) \mathcal{Z}_2(2-s, \mathbb{1}_{\ell_1, \ell_2}; d_{\chi_1}, d_{\chi_2}),$$

where $\mathcal{Z}_2(s, \mathbb{1}_{\ell_1, \ell_2}; d_{\chi_1}, d_{\chi_2})$ is described as the double Dirichlet series

$$\mathcal{Z}_2(s, \mathbb{1}_{\ell_1, \ell_2}; d_{\chi_1}, d_{\chi_2}) := \sum_{m,n \neq 0} \frac{d_{\chi_1}(m) d_{\chi_2}(n)}{(\ell_2 m + \ell_1 n)^s}, \quad \text{Re}(s) > 2. \quad (5.31)$$

It is also possible to construct analogues of Guinand’s formula via the above Selberg-Chowla formulas for $\mathcal{Z}_2(s, d_{\chi_1}, d_{\chi_2})$. Analogues of Koshliakov’s formula can be also obtained, although we need some additional computations of the values of $L'(1, \chi)$ [24].

In the same spirit, for $i = 1, 2$, let $F_i(n)$ denote, as usual, the number of integral ideals of norm $n$ in an imaginary quadratic number field with discriminant $D_i, K_i = \mathbb{Q} \left(\sqrt{-D_i}\right)$. Then it is well-known that the associated Dirichlet series is the classical Dedekind zeta function

$$\xi_{K_i}(s) = \sum_{n=1}^{\infty} \frac{F_i(n)}{n^s}, \quad \text{Re}(s) > 1, \quad (5.32)$$

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which satisfies Hecke’s functional equation

\[
\left( \frac{2\pi}{\sqrt{D_i}} \right)^{-s} \Gamma(s) \zeta_{K_i}(s) = \left( \frac{2\pi}{\sqrt{D_i}} \right)^{s-1} \Gamma(1-s) \zeta_{K_i}(1-s).
\]  

(5.33)

Since \( \zeta_{K_i}(s) \) has a continuation to the entire complex plane as an analytic function except at a simple pole located at \( s = 1 \), we see that, for \( i = 1, 2 \), \( \zeta_{K_i}(s) \in \mathcal{A} \) with \( n_i = 1 \). Moreover, the residue of \( \zeta_{K_i}(s) \) at \( s = 1 \) can be explicitly computed as

\[
\rho_i = \frac{2\pi h(K_i)R(K_i)}{\sqrt{D_i}w(K_i)},
\]

where \( h(K_i) \), \( R(K_i) \) and \( w(K_i) \) denote, respectively, the class number of \( K_i \), the regulator of \( K_i \) and the number of roots of unity in \( K_i \). We now take \( \phi_i(s) := \zeta_{K_i}(s) \): in order to apply Corollary 2.11 we need to take the following substitutions

\[
a_i(n) = F_i(n), \quad b_i(n) = \frac{2\pi}{\sqrt{D_i}}F_i(n), \quad \lambda_{n,i} = n, \quad \mu_n = \frac{4\pi^2}{D_i}n.
\]

By the conditions on the class \( \mathcal{A} \), we have \( \phi_i(0) = -h(K_i)R(K_i)/w(K_i) \), so that we take \( F_i(0) := h(K_i)R(K_i)/w(K_i) \). We now construct the diagonal Epstein zeta function attached to \( K_1 \) and \( K_2 \) as follows

\[
\mathcal{Z}_2(s; K_1, K_2) := \sum_{m,n \neq 0}^\infty \frac{F_1(m)F_2(n)}{(m+n)^s}, \quad \text{Re}(s) > 2.
\]  

(5.34)

Under the substitutions above mentioned, we have from the Selberg-Chowla formula (2.26) that the following identity holds for (5.34)

\[
\Gamma(s) \mathcal{Z}_2(s; K_1, K_2) = \frac{h(K_2)R(K_2)}{w(K_2)} \Gamma(s) \zeta_{K_1}(s) + \frac{2\pi h(K_1)R(K_1)}{\sqrt{D_1}w(K_1)} \Gamma(s-1) \zeta_{K_2}(s-1)
\]

\[+ 2 \left( \frac{2\pi}{\sqrt{D_1}} \right)^s \sum_{m,n=1}^\infty F_1(m)F_2(n) \left( \frac{m}{n} \right) \frac{i^n}{\mu_n} K_{s-1} \left( \frac{4\pi\sqrt{mn}}{D_1} \right),
\]

giving the analytic continuation of \( \mathcal{Z}_2(s; K_1, K_2) \) as a meromorphic function with one simple pole located at \( s = 2 \) whose residue is precisely

\[
\text{Res}_{s=2} \mathcal{Z}_2(s; K_1, K_2) = \frac{4\pi^2}{\sqrt{D_1D_2}} \frac{h(K_1)h(K_2)R(K_1)R(K_2)}{w(K_1)w(K_2)}.
\]

Moreover, if we write a second Selberg-Chowla for (5.34), we can deduce that it satisfies the following functional equation:

\[
\left( \frac{2\pi}{\sqrt{D_1D_2}} \right)^{-s} \Gamma(s) \mathcal{Z}_2(s; K_1, K_2) = \sqrt{D_1D_2} \left( \frac{2\pi}{\sqrt{D_1D_2}} \right)^{-2-s} \Gamma(2-s) \mathcal{Z}_2(s; \mathbb{I}_{D_1D_2}, K_1, K_2),
\]

where, just as in (5.31), \( \mathcal{Z}_2(s; \mathbb{I}_{D_1D_2}, K_1, K_2) \) denotes the Dirichlet series

\[
\mathcal{Z}_2^\varepsilon(s; \mathbb{I}_{D_1D_2}, K_1, K_2) := \sum_{m,n \neq 0}^\infty \frac{F_1(m)F_2(n)}{(D_2m+D_1n)^s}, \quad \text{Re}(s) > 2.
\]  

(5.35)

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If, for the quadratic fields \( K_1 \) and \( K_2 \), we define an analogue of the divisor function, \( \sigma_v(n; K_1, K_2) \), in the following way

\[
\sigma_v(n; K_1, K_2) = \sum_{d|n} F_1(d) F_2 \left( \frac{n}{d} \right) d^v,
\]

we have that, for any \( x > 0 \), the identity of Ramanujan-Guinand type \((5.7)\) holds

\[
\left( \frac{2\pi}{\sqrt{D_1}} \right)^s x^{1-s} \sum_{n=1}^{\infty} \sigma_{s-1}(n; K_1, K_2) n^{\frac{i\pi}{2} K_{s-1}} \left( \frac{4\pi \sqrt{n}}{\sqrt{D_1}} x \right)^s
- \left( \frac{2\pi}{\sqrt{D_2}} \right)^s x^{-1-s} \sum_{m,n=1}^{\infty} \sigma_{s-1}(n; K_2, K_1) n^{\frac{i\pi}{2} K_{s-1}} \left( \frac{4\pi \sqrt{n}}{\sqrt{D_2}} x \right)^s
= \frac{h(K_1) R(K_1)}{2w(K_1)} x^{-2s} \left\{ \Gamma(s) \zeta_{K_2}(s) - \frac{2\pi x^2}{\sqrt{D_1}} \Gamma(s-1) \zeta_{K_2}(s-1) \right\}
+ \frac{h(K_2) R(K_2)}{2w(K_2)} \left\{ \frac{2\pi}{\sqrt{D_2}} x^{1-s} \Gamma(s-1) \zeta_{K_1}(s-1) - \Gamma(s) \zeta_{K_1}(s) \right\}.
\]

From the Kronecker limit formula for imaginary quadratic fields, it is also possible to take the limit \( s \to 1 \) on \((5.37)\) in order to derive analogues of Koshliakov’s formula for the divisor function \((5.36)\). One dimensional analogues of \((5.37)\) are also given in \([14]\).

**Example 5.5** (Dirichlet series attached to Eisenstein Series and Cusp Forms). Let \( k \geq 3 \) be an odd positive integer and consider the divisor function \( \sigma_k(n) \). We know that the associated Dirichlet series is given by the product of zeta functions

\[
\zeta(s) \zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}, \quad \text{Re}(s) > k+1,
\]

and that \( \phi(s) = (2\pi)^{-s} \zeta(s) \zeta(s-k) \) obeys to the Hecke’s functional equation \([20\text{, p. 17, eq. (5.7)}]\)

\[
(2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s-k) = (-1)^{\frac{k+1}{2}} (2\pi)^{-s} \Gamma(k+1-s) \zeta(k+1-s) \zeta(1-s).
\]

Moreover, since \( \zeta(1-k) = 0 \), we know that \( \phi \in \mathcal{A}_0 \), once we take the convention \( \sigma_k(0) := -B_{k+1}/2k+2 \).

Let \( k_1, k_2 \geq 3 \) be two odd positive integers and consider two Dirichlet series \( \phi_i(s) \), \( i = 1, 2 \), given by \( \phi_i(s) = (2\pi)^{-s} \zeta(s) \zeta(s-k_i) \). We construct the diagonal Epstein zeta function \((1.10)\) as

\[
\mathcal{Z}_2(s; k_1, k_2) = (2\pi)^{-s} \sum_{m,n \neq 0}^{\infty} \frac{\sigma_{k_1}(m) \sigma_{k_2}(n)}{(m+n)^s} := (2\pi)^{-s} Z_2(s, k_1, k_2), \quad \text{Re}(s) > 2 \max\{k_1, k_2\} + 2.
\]

In order to apply directly Theorem \((2.2)\) and its corollary \((2.1)\) we take the simple substitutions

\[
a_i(n) = \sigma_{k_i}(n), \quad b_i(n) = (-1)^{\frac{k_i+1}{2}} \sigma_{k_i}(n), \quad \lambda_{n,i} = \mu_{n,i} = 2\pi n.
\]

Using Euler’s formula for \( \zeta(2n) \), it is very simple to find that

\[
\rho_i = \frac{(-1)^{\frac{k_i+1}{2}}}{2(k_i+1)!} B_{k_i+1}, \quad \phi_i(0) := \zeta(0) \zeta(-k_i) = \frac{B_{k_i+1}}{2k_i+2}.
\]
where \((B_n)_{n \in \mathbb{N}}\) denotes the sequence of Bernoulli numbers. Applying these computations to the first Selberg-Chowla formula on the class \(\mathcal{A}\) (2.26), we arrive at the representation

\[
(2\pi)^{-s} \Gamma(s) Z_2(s; k_1, k_2) = -\frac{B_{k_2+1}(2\pi)^{-s}}{2k_2 + 2} \Gamma(s) \zeta(s) \zeta(s-k_1) + \frac{(-1)^{k_2+1} B_{k_1+1}(2\pi)^{-s+k_1+1}}{2k_1 + 2} \Gamma(s-k_1-1) \times \zeta(s-k_1-1) \zeta(s-k_1-k_2-1) + 2 \sum_{m,n=1}^{\infty} (-1)^{k_1+1} \sigma_k(m) \sigma_{k_2}(n) \left( \frac{m}{n} \right)^{s-k_1-1} K_{s-k_1-1} \left( 4\pi \sqrt{mn} \right),
\]

which gives the analytic continuation of \(Z_2(s; k_1, k_2)\) as a meromorphic function with a simple pole located at \(s = k_1 + k_2 + 2\) whose residue is given by

\[
\text{Res}_{s=k_1+k_2+2} Z_2(s; k_1, k_2) = \frac{(-1)^{k_1+k_2+1} (2\pi)^{k_1+k_2+2} B_{k_1+1} B_{k_2+1}}{4(k_1+1)(k_2+1)(k_1+k_2+1)}.
\]

(5.43)

The second Selberg-Chowla formula can be analogously written and an application of Corollary 2.1 shows that \(Z_2(s; k_1, k_2)\) must satisfy the functional equation

\[
(2\pi)^{-s} \Gamma(s) Z_2(s; k_1, k_2) = (-1)^{k_1+k_2+2} (2\pi)^{-s} \Gamma(k_1+k_2+2-s) \Gamma(k_1+k_2+2-s) Z_2(k_1+k_2+2-s; k_1, k_2).
\]

(5.44)

An application of Theorem 4.1 together with the absence of zeros (at the critical line) of the Dirichlet series given in (5.38), allows to prove some lower bounds (at the critical line) for some Epstein zeta functions. In [42, Beispiel 11, p. 95] it is constructed an octonary quadratic form whose Epstein zeta function does not possess a single zero on the critical line \(\text{Re}(s) = 2\). Indeed, if \(I_n\) denotes the \(n\)-dimensional identity matrix and \(\mathcal{Q}(x_1, \ldots, x_8)\) is the octonary quadratic form represented by the following matrix

\[
\mathcal{Q} = \begin{pmatrix} 2 \cdot I_4 & \mathcal{J} \\ \mathcal{J} & 2 \cdot I_4 \end{pmatrix}
\]

with \(\mathcal{J} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix},
\]

then its associated Epstein zeta function, \(Z_8(s, \mathcal{Q})\), can be written explicitly as

\[
Z_8(s, \mathcal{Q}) = 240 \cdot 2^{-s} \zeta(s) \zeta(s-3),
\]

(5.45)

which obviously does not possess any zero at the critical line \(\text{Re}(s) = 2\). \(\text{Note that this violation of the Riemann hypothesis is not implied by the Selberg-Chowla formula in the sense of Corollary 4.1 and that } Z_8(s, \mathcal{Q}) \text{ possesses an Euler product, not falling under the class of Epstein zeta functions studied by Davenport and Heilbronn [23].}\)
we see by a simple application of item 2. of Theorem[4.1] that the order of the Epstein zeta function satisfies a lower bound of the form $|Z_{16}(4+it; \mathcal{P})| > C|t|^2$, $C > 0$, whenever $|t|$ is taken sufficiently large $^{16}$. This proves that, at most, $Z_{16}(s, \mathcal{P})$ has a finite number of zeros in the line $\text{Re}(s) = 4$. Hence, besides violating the Riemann hypothesis quite badly, as the construction (5.45) shows, Epstein zeta functions with sufficiently high dimension do not satisfy the Lindelöf hypothesis.

As a complementary case of the previous Dirichlet series, let $f$ be a cusp form with weight $k \geq 12$ (with $k$ always being an even integer) for the full modular group and $L(s, a)$ the associated $L$–function,

$$L(s, a) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad (5.46)$$

where the arithmetical functions $a(n)$ represent the Fourier coefficients of $f$. Then (5.46) is known to be absolutely convergent in the half-plane $\text{Re}(s) > \frac{k+1}{2}$ and can be analytically continued to an entire function $^{[2], \text{Theorem } 6.20}$. Furthermore, it obeys to Hecke’s functional equation

$$(2\pi)^{-s} \Gamma(s) L(s, a) = (-1)^{k/2} (2\pi)^{-(k-s)} \Gamma(k-s) L(k-s, a). \quad (5.47)$$

Consider two holomorphic cusp forms $f_1$ and $f_2$ having weights $k_1$ and $k_2$ respectively and the pairs of Dirichlet series $\phi_i(s) = (2\pi)^{-s} L(s, a_i)$ and $\psi_i(s) = (-1)^{k_i/2} (2\pi)^{-s} L(s, a_i)$, $i = 1, 2$, associated with them. Here, the substitutions analogous to (5.14) are obvious to consider. Since each $\phi_i(s)$ is entire, it follows from Corollary[2.1] that its (diagonal) Epstein zeta function given by

$$Z_2(s; f_1, f_2) = (2\pi)^{-s} \sum_{m,n \neq 0} \frac{a_1(m)a_2(n)}{(m+n)^s} = (2\pi)^{-s} Z_2(s; f_1, f_2), \quad \text{Re}(s) > \max \{k_1, k_2\} + 1, \quad (5.48)$$

is also entire and its continuation satisfies the Selberg-Chowla formulas

$$(2\pi)^{-s} \Gamma(s) Z_2(s; f_1, f_2) = 2(-1)^{k_1+k_2} \sum_{m,n=1}^{\infty} a_1(m)a_2(n) \left(\frac{m}{n}\right)^{\frac{k_1+k_2}{2}} K_{k_1-s}(4\pi \sqrt{mn})$$

and

$$(2\pi)^{-s} \Gamma(s) Z_2(s; f_1, f_2) = 2(-1)^{k_1+k_2} \sum_{m,n=1}^{\infty} a_1(m)a_2(n) \left(\frac{m}{n}\right)^{\frac{k_1+k_2}{2}} K_{k_2-s}(4\pi \sqrt{mn}).$$

It follows also from Corollary[2.1] that $Z_2(s; f_1, f_2)$ satisfies the functional equation

$$(2\pi)^{-s} \Gamma(s) Z_2(s; f_1, f_2) = \frac{\gamma^k}{(2\pi)^{-(k_1+k_2+s)}} \Gamma(k_1+k_2-s) Z_2(k_1+k_2-s; f_1, f_2). \quad (5.49)$$

$^{16}$By the functional equation for the Riemann zeta-function and Stirling’s formula, we have that $|\zeta(2+it, \mathcal{O})| \gg |t|^\frac{1}{2}, |t| \gg 1$. Applying this to the Selberg-Chowla formula (4.10) and proceeding just as in the proof of Claim[4.1] we can derive the lower bound $|Z_{16}(4+it, \mathcal{P})| \gg |t|^\frac{1}{2}, |t| \gg 1$. 

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Due to the fact that $\phi_1$ and $\phi_2$ are entire, it is now very easy to establish analogues of formulas of Ramanujan-Guinand and Koshliakov type. In particular, for the case $k_1 = k_2 = 12$, $\phi_1(s) = \phi_2(s) = \phi(s)$, we could consider the Dirichlet series associated to the Ramanujan $\tau-$function,

$$\phi(s) := (2\pi)^{-s} L(s, \tau) = (2\pi)^{-s} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}, \quad \text{Re}(s) > \frac{13}{2}$$

and the resulting diagonal Epstein zeta function (5.48) would satisfy Hecke’s functional equation with $r := k_1 + k_2 = 24$. For this case, a curious reflection formula\(^{17}\) of Koshliakov type takes place in the form

$$\sum_{j=1}^{\infty} d_2(n) K_0(4\pi \sqrt{n} x) = x^{-24} \sum_{j=1}^{\infty} d_2(n) K_0\left(\frac{4\pi \sqrt{n}}{x}\right), \quad x > 0, \quad (5.50)$$

where $d_2(n)$ denotes the divisor function associated with $\tau(n)$,

$$d_2(n) = \sum_{d|n} \tau(d) \tau\left(\frac{n}{d}\right).$$

Now assume that $a_i(n)$, $i = 1, 2$, is real. By (5.49), we know that $Z_2(s; f_1, f_2)$ is a real and entire Hecke Dirichlet series with signature $\left(1, k_1 + k_2, (-1)^{\frac{k_1 + k_2}{2}}\right)$. From a classical application of Hecke’s theory [42, p. 79, Satz 3], the fact that the signature parameter $\lambda = 1$ of $Z_2(s; f_1, f_2)$ satisfies $0 < \lambda < 2$, then one has that $\Theta_2(e^{\frac{\pi i \tau}{\lambda}}; f_1, f_2) = O(e^{-\beta})$, with $\beta < \frac{k_1 + k_2 + 1}{2}$. Thus, according with Remark 4.4, $Z_2(s; f_1, f_2)$ must have infinitely many zeros on the critical line $\text{Re}(s) = \frac{k_1 + k_2}{2}$. In particular, this shows that $Z_2(s; f, f)$ has infinitely many zeros at the critical line $\text{Re}(s) = k$ and so, by Theorem 4.1, $L(s, a)$ has infinitely many zeros at the critical line $\text{Re}(s) = \frac{k}{2}$. Since the $L$-functions (5.46) admit Selberg-Chowla expansions (see [70, p. 485, eq. (1.24)]), it would be also interesting to study more analogues of Hardy’s Theorem for the Dirichlet series composing these.

**Example 5.6 (The Epstein zeta function).** Kober [48, p. 5] established the condition given in Corollary 4.2 for a class of Epstein zeta functions attached to binary and positive definite quadratic forms whenever $a/\sqrt{|d|}$ is irrational. For the case where $a/\sqrt{|d|}$ is rational, then by using a set of conditions similar to that given in Corollary 4.3, we are only left with a finite set of Epstein zeta functions for which we need to verify Hardy’s Theorem (see [48, p. 8]). In all of them, a similar argument mimicking the one using Jacobi’s 4-square Theorem would suffice, but we would have to know apriori a formula for the representation of a given integer as the particular quaternary forms involved.

Since the theta function attached to $Z_2(s, Q)$ satisfies (4.38) whenever $a/\sqrt{|d|} \notin \mathbb{Q}$, it is now possible to derive from Corollary 4.5 that, for any $c_j \in \ell^1(\mathbb{R})$ and $\tau_j \in \ell^\infty(\mathbb{R})$, the infinite combination

$$F_Q(s) = \sum_{j=1}^{\infty} c_j \left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s-i\tau_j} \Gamma(s+i\tau_j) Z_2(s+i\tau_j, Q)$$

A different proof of (5.50) is given in [7, p. 357, Example 2]. This result is obtained from a generalization of Bochner’s modular relation (1.21) for Dirichlet series satisfying the functional equation (1.6) with $\Delta(s) = \Gamma^N(s)$.

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has infinitely many zeros on the critical line \( \text{Re}(s) = \frac{1}{2} \).

A still larger subclass of Epstein zeta functions satisfy the conditions of Corollary [4.5][48], p. 5], as well as Dedekind zeta functions attached to imaginary quadratic fields [13], p. 689].

By virtue of their functional equations (see [72], p. 481, Thm 2] and eq. (5.51) below) and remark[4.2] it is also possible to derive Hardy’s Theorem for certain multidimensional Epstein zeta functions satisfying the symmetric conditions given in Theorem[4.1] Assume that \( Q \) is a \( n \times n \) matrix attached to a real positive-definite Quadratic form and let \( s \) be a complex number such that \( \text{Re}(s) > \frac{1}{2} \). Then the \( n \)-dimensional Epstein’s \( \zeta \)–function is defined as the series \( Z_n(s, Q) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} Q(m)^{-s} \), where \( Q(x) = x^T Q x \) and \( m \in \mathbb{Z}^n \setminus \{0\} \).[18] It is well-known that \( Z_n(s, Q) \) can be extended to the complex plane as a meromorphic function with a simple pole at \( s = \frac{n}{2} \) and with residue given by \( |Q|^{-\frac{s}{2}} \pi^{n/2} / \Gamma(n/2) \), where \( |Q| = \det(Q) \). Moreover, it satisfies Hecke’s functional equation

\[
\pi^{-s} \Gamma(s) Z_n(s, Q) = |Q|^{-\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma \left( \frac{n}{2} - s \right) Z_n \left( \frac{n}{2} - s, Q^{-1} \right), \tag{5.51}
\]

where \( |Q| \) is the determinant of \( Q \). Note that, from (5.51), \( \pi^{-s} \Gamma(s) Z_n(s, Q) \) is real valued on the critical line \( r = \frac{1}{2} \) whenever \( Q \) is unimodular.

Let \( Q_1 \) and \( Q_2 \) denote, respectively, two matrices representing two positive-definite quadratic forms in \( \mathbb{R}^{m_1} \) and \( \mathbb{R}^{m_2} \) with \( m_1 \leq m_2 \). We shall apply Theorem [2.1] with \( \phi_i(s) = \pi^{-s} Z_{m_i}(s, Q_i) \) and \( \psi_i(s) = |Q|^{-\frac{s}{2}} \pi^{-s} Z_n(s, Q^{-1}) \). By definition of (1.10), the diagonal Epstein zeta function attached to \( \phi_1(s) \) and \( \phi_2(s) \) is

\[
\zeta^e(s; Q_1, Q_2) = \pi^{-s} \sum_{m \in \mathbb{Z}^{m_1+m_2} \setminus \{0\}} \frac{1}{Q_1^2(m)^{s}}, \quad \text{Re}(s) > \frac{m_1+m_2}{2},
\]

where \( Q^{(2)}_{1,2} \) is the \( m_1 + m_2 \)-dimensional quadratic form represented by the matrix

\[
Q^{(2)}_{1,2} = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}. \tag{5.52}
\]

A straightforward application of Theorem [2.1] gives the following Selberg-Chowla formula (c.f. [72], p. 480, eq. (2.3))

\[
\pi^{-s} \Gamma(s) Z_{m_1+m_2}(s, Q_{1,2}) = \pi^{-s} \Gamma(s) Z_{m_1}(s, Q_1) + |Q_1|^{-\frac{s}{2}} \pi^{-s} Z_{m_2}(s - \frac{m_1}{2}, Q_2)
\]

\[
+ 2 |Q_1|^{-\frac{s}{2}} \sum_{m_1, m_2} \left( \frac{Q_1^{-1}(m_1)}{Q_2(m_2)} \right)^{\frac{s}{2} + \frac{m_2}{2}} K_{m_2-s} \left( 2 \pi \sqrt{Q_1^{-1}(m_1) Q_2(m_2)} \right), \tag{5.53}
\]

\[18\text{Note now that this notation slightly differs from the conventions given in the section 3: here, the Dirichlet series } Z_n(s, Q) \text{ is attached to the matrix representation of the quadratic form } Q \text{ so that } Z_n(s, Q^{-1}) \text{ represents the Epstein zeta function attached to Quadratic form having as matrix } Q^{-1} \text{.} \]
where $m_1 \in \mathbb{Z}^{m_1} \setminus \{0\}$ and $m_2 \in \mathbb{Z}^{m_2} \setminus \{0\}$.

Using (5.33) and the formalism of Theorem 4.1, we are ready to prove Hardy’s Theorem for the multi-dimensional Epstein zeta functions $Z_k(s, Q)$. Assume that $k \geq 3$ (the case $k = 2$ is already mentioned above) and let $Q$ be a quadratic form attached to a matrix with $|Q| = \det(Q) = 1$ and having the genus of the identity matrix $I_k, \gamma_k$. Let $\phi(s) = \pi^{-s}Z_k(s, Q)$: then the four-dimensional Epstein zeta function (2.45) constructed from $\phi, \mathcal{Z}_4(s; \cdot)$, is actually $\pi^{-s}Z_{4k}(s; Q^{(4)})$, where $Q^{(4)}$ denotes the $4k$-dimensional matrix

$$Q^{(4)} = \begin{pmatrix} Q^{(2)} & 0 \\ \ast & Q^{(2)} \end{pmatrix},$$

with $Q^{(2)}$ being essentially (5.52) with $Q_1 = Q_2 = Q$.

It is clear that $Q^{(4)}$ has the same genus, $\gamma_{4k}$, as $\mathcal{I}_{4k}$. If $Q_1, \ldots, Q_h$ denote the representatives of all different classes of $\gamma_{4k}$ and $\text{Aut}(Q)$ the automorphism group of $Q$, we can consider the Eisenstein series

$$E(z; \gamma_{4k}) = \frac{\sum_{t=1}^h \theta_{Q_t}(z)/|\text{Aut}(Q_t)|}{\sum_{t=1}^h 1/|\text{Aut}(Q_t)|}$$

and attach to it a Dirichlet series $Z(s; \gamma_{4k})$. It is well-known [66,67] that this Dirichlet series is of the form

$$Z(s; \gamma_{4k}) = c_k \left\{ 1 - 2^{-s} + (-1)^k \left( 2^{2k-2s} - 2^{-s} \right) \right\} \zeta(s) \zeta(s+1-2k),$$

where $c_k > 0$. It is clear from the expression on the braces that $Z(s; \gamma_{4k})$ has infinitely many zeros on the line $\text{Re}(s) = k$, all of them of the form

$$\begin{align*}
&\left\{ \begin{array}{l}
\{ s = k \pm \frac{im}{\text{mg}(z)} \}, \ n \in \mathbb{N} \\
\{ s = k + \frac{im}{\text{mg}(z)} \left( 2n\pi \pm \arctan \left( \sqrt{2^{2k} - 1} \right) \right) \}, \ n \in \mathbb{Z} \\
k \text{ even.}
\end{array} \right.
\end{align*}$$

Since $\theta_{Q^{(4)}}(z) - E(z; \gamma_{4k})$ is a cusp form [66,67], the Dirichlet series attached to it, $L(s; S_{4k})$, satisfies the convex estimate $L(k+i t; S_{4k}) = O(| t |^{\frac{1}{2} + \varepsilon})$, $|t| \to \infty$. If we assume that $Z_k(s, Q)$ has only a finite number of zeros on the line $\text{Re}(s) = \frac{1}{2}$, then from (4.37) we must have that $|Z_{4k} \left( k + i t, Q^{(4)} \right) | > C | t |^{\frac{3}{2}}$ for $|t|$ sufficiently large. Since

$$Z_{4k} \left( k + i t; Q^{(4)} \right) = Z(k + i t; \gamma_{4k}) + O \left( |t|^{\frac{1}{2} + \varepsilon} \right),$$

we have that $| Z(k + i t; \gamma_{4k}) | > C' | t |^{\frac{3}{2}}$ for some positive $C'$ and $|t|$ large enough. This is absurd as $Z(s; \gamma_{4k})$ possesses infinitely many zeros on the line $\text{Re}(s) = k$.

Adapting the argument outlined in Example 5.3 we can also show the existence of a zero of $Z_k(s, Q)$ in any interval of the form $[T, T + T^{\frac{1}{2} + \varepsilon}]$ for a sufficiently large $T$. Although far from being as sharp as Siegel’s results in [66] (valid for $k \geq 12$), it seems that this method gives a quantitative estimate not covered by the methods employed in [66]. Siegel writes [66, p. 363] “... it is possible to discuss the remaining cases for $3 < k < 12$, but this requires some numerical computations and we omit it. For the function $\zeta_{3}(s)$, however,
our method does not lead to any result”. Since our method, when applied to the situation described above, gives $N_0(T, \zeta_3(s)) \gg T^{\frac{1}{2} - \varepsilon}$, $\varepsilon > 0$, this result seems, as far as we know, to be the first quantitative estimate regarding the number of zeros of $\zeta_3(s)$ on the line $\text{Re}(s) = \frac{3}{4}$.

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References

[1] G. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press (1999) [paperback edition: 2000].
[2] T. M. Apostol, Modular Functions and Dirichlet Series in Number Theory, 2nd edition, Springer, New York (1990).
[3] R. Balasubramanian, An improvement of a theorem of Titchmarsh on the mean square of $|\zeta(1/2 + it)|$, Proc. London Math. Soc. 36 (1978), 540–576.
[4] P. T. Bateman and E. Grosswald, On Epstein’s zeta function, Acta Arith. 9 (1964), 365–373.
[5] B. C. Berndt, Arithmetical identities and Hecke’s functional equation, Proc. Edinburgh Math. Soc. 16 (1969), 221–226.
[6] ———, Generalized Dirichlet series and Hecke’s functional equation, Proc. Edinburgh Math. Soc., 15 (1967), 309–313.
[7] ———, Identities involving the coefficients of a class of Dirichlet series. I, Trans. Amer. Math. Soc., 137 (1969), 345–359.
[8] ———, Identities involving the coefficients of a class of Dirichlet series. III., Trans. Amer. Math. Soc., 146 (1969), 323–348.
[9] ———, Identities involving the coefficients of a class of Dirichlet series. IV, Trans. Amer. Math. Soc., 149 (1970), 179–185.
[10] ———, Identities involving the coefficients of a class of Dirichlet series. VI, Trans. Amer. Math. Soc. 160 (1971), 157-167.
[11] ———, The functional equation of some Dirichlet series, II, Proc. Amer. Math. Soc., 31 (1972), 24–26.
[12] ———, On the Zeros of a Class of Dirichlet Series I, Illinois J. Mathematics, 14 (1970), 244–258.
[13] ———, On the Zeros of a Class of Dirichlet Series II, Illinois J. Mathematics, 14 (1970), 678–691.
[14] B. C. Berndt, A. Dixit, R. Gupta, A. Zaharescu, A Class of Identities Associated with Dirichlet Series Satisfying Hecke’s Functional Equation, preprint available on arxiv [arXiv:2108.13991v1].
[15] B. C. Berndt, A. Dixit, J. Sohn, Character analogues of theorems of Ramanujan, Koshliakov, and Guinand, Adv. Appl. Math., 46 (2011), 54–70.
[16] B. C. Berndt, S. Kim, A. Zaharescu, Analogues of Koshliakov’s formula, Ramanujan 125, Contemp. Math. 627, K. Alladi, F. Garvan, and A. J. Yee, editors, American Mathematical Society, Providence, RI, 2014, 41–48.
[17] B. C. Berndt, Y. Lee, and J. Sohn, Koshliakov’s formula and Guinand’s formula in Ramanujan’s lost notebook, Surveys in number theory, Dev. Math., 17 (2012), 21-42.
[18] S. Bochner, Some properties of modular relations, Ann. Math., 53 (1951), 332-360.
[19] S. Bochner and K. Chandrasekharan, On Riemann’s Functional Equation, Ann. of Math., 63 (1956), 336–360.
[20] K. Chandrasekharan and R. Narasimhan, Hecke’s functional equation and arithmetical identities, Ann. of Math., 74 (1961), 1-23.
[21] ———, Zeta-functions of ideal classes in quadratic fields and their zeros on the critical line, Comm. Math. Helv, 43 (1968), 18–30.
[22] H. Davenport, Multiplicative Number Theory, Springer-Verlag, Berlin Heidelberg, 2nd Edition, 1980.
[23] H. Davenport and H. Heilbronn, On the zeros of certain Dirichlet series, J. Lond. Math. Soc., 11 (1936), 181–185, 307–312.
[24] C. Deninger, On the analogue of the formula of Chowla and Selberg for real quadratic fields, J. Reine Angew. Math., 351 (1984), 171–191.
[25] Max F. Deuring, On Epstein’s zeta function, Annals of Mathematics, Second Series, 38 (1937), 585-593.
[26] ———, Zeta Funktionen Quadratischer Formen, J. Reine Angew. Math. (1935), 226–252.
[27] A. Dixit, Character analogues of Ramanujan type integrals involving the Riemann Ξ-function, Pacific J. Math., 255 (2012), 317–348.

[28] A. Dixit, R. Kumar, On Hurwitz zeta function and Lommel functions, International Journal of Number Theory, 17 (2021) 393–404.

[29] A. Dixit, N. Robles, A. Roy, and A. Zaharescu, Zeros of combinations of the Riemann ξ-function on bounded vertical shifts, Journal of Number Theory, 149 (2015), 404–434.

[30] C. Epstein, J. Hafner, P. Sarnak, Zeros of L-functions attached to Maass forms, Math. Zeit., 190 (1985) 113–128.

[31] P. Epstein, Zur Theorie allgemeiner Zetafunctionen, Math. Ann. 56 (1903), 615–644

[32] ———, Zur Theorie allgemeiner Zetafunktionen. II, Math. Ann. 63 (1907), 205–216.

[33] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Tables of integral transforms, Vol. 1 and 2, McGraw-Hill, New York, 1954.

[34] M. Fekete, Sur les séries de Dirichlet, Comptes Rendus hebdomadaires des séances de l’Académie des sciences, 150 (1910) 1033–1036.

[35] ———, The zeros of Riemann’s Zeta-Function on the Critical Line, Journal of the London Math. Society, 1 (1926), 15–19.

[36] N. J. Fine, Note on the Hurwitz Zeta-Function, Proceedings of the American Mathematical Society, 2 (1951), 361–364.

[37] A. Fujii, On the zeros of the Epstein zeta functions, J. Math. Kyoto Univ., 36 (1996), 697–770.

[38] A. P. Guinand, Some rapidly convergent series for the Riemann ζ-function, Quart J. Math. (Oxford) 6 (1955), 156—160.

[39] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th ed., Academic Press, San Diego, 2007.

[40] G. H. Hardy, Sur les zeros de la fonction ζ(s) de Riemann, Comptes rendus, 158 (1914), 1012–1014.

[41] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the distribution of primes, Acta Math., 41 (1918), 119–196.

[42] E. Hecke, Über Dirichlet-Reihen mit Funktionalgleichung und ihre Nullstellen auf der Mittelgeraden, Bayer. Akad. Wiss. Math. – Natur., Vol.2 (1937), 73–95.

[43] M. D. Hirschorn, A Simple Proof of Jacobi’s 4-square Theorem, Proc. Amer. Math. Soc., 101 (1987), 436–438.

[44] A. Ivić, The Riemann Zeta-function, John Wiley and Sons, Inc., New York, 1985.

[45] A. Ivić, The theory of Hardy’s Z-function, Cambridge University Press, UK, 2013.

[46] M. Jutila, Lectures on a method in the theory of exponential sums, Tata Inst. Fund. Res. Lect. Math. 80 (Springer, Berlin/Heidelberg/New York/Tokyo, 1987).

[47] H. Ki, All but finitely many zeros of the approximations of the Epstein zeta function are simple and lie on the critical line, Proc. London Math. Soc., 90 (2005), 321–344.

[48] H. Kober, Nullstellen Epsteinscher Zetafunktionen, Proc. London Math. Soc., 42 (1936), 1–8.

[49] ———, Transformationsformel gewisser Besselscher Reihen, Beziehungen zu Zeta-Funktionen, Math Zeitschrift, 39 (1934), 609–624.

[50] N. S. Koshliakov, On Voronoï’s sum-formula, Mess. Math. 58 (1929), 30—32 (in Russian).

[51] T. Kuzumaki, Asymptotic expansions for a class of zeta-functions, Ramanujan Journal, 24 (2011), 331-343.

[52] E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Volume II, Druck und Verlag von B.G. Teubner, Leipzig and Berlin, 1909

67
[53] ———, Über die Hardysche Entdeckung unendlich vieler Nullstellen der Zetafunktion mit reelem Teil \( \frac{1}{2} \), Math. Annalen, 76 (1915), 212–243.
[54] Y. Motohashi, A new proof of the limit formula of Kronecker, Proc. Jap. Acad, 44 (1968), 614–616.
[55] A. Mukhopadhyay, K. Srinivas, K. Rajkumar, On the zeros of functions in the Selberg class, Functiones et Approximatio Commentarii Mathematici 38 (2008), 121–130.
[56] F. Oberhettinger and K. L. Soni, On some relations which are equivalent to functional equations involving the Riemann zeta-function, Math. Z., 127 (1972), 17–34.
[57] H. S. A. Potter, E. C. Titchmarsh, The zeros of Epstein’s zeta-functions, Proc. London Math. Soc. (2), 39 (1935), 372-384.
[58] K. Ramachandra, On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function, Tata Inst. Fund. Res. Lect. Math. 85 (Springer, Berlin/Heidelberg/New York/Tokyo, 1995).
[59] ———, On the Zeros of a Class of Generalized Dirichlet series, Journal für die reine und angewandte Mathematik, 273 (1975), 31–40.
[60] ———, On the Zeros of a Class of Generalized Dirichlet series. II, Journal für die reine und angewandte Mathematik, 289 (1977), 174–180.
[61] R. A. Rankin, A minimum problem for the Epstein zeta-function, Proc. Glasgow Math. Association, 1 (1953), 149–158.
[62] Usha K. Sangale, A note on Hardy’s Theorem, Hardy-Ramanujan Journal, Hardy-Ramanujan Society, 39 (2017), 38-45.
[63] A. Sankaranarayanan, Zeros of quadratic zeta-functions on the critical line, Acta Arith. 69 (1995), 21–37.
[64] S. Chowla and A. Selberg, On Epstein’s zeta-function, Proc. Nat. Acad. Sci., 35 (1949), 371–374.
[65] A. Selberg and S. Chowla, On Epstein’s zeta function, J. Reine Angew. Math., 227 (1967), 86-110.
[66] C. L. Siegel, Contributions to the Theory of Dirichlet L-series and the Epstein zeta-functions, Ann. of Math., 44 (1943), 143–172.
[67] ———, Lectures on the analytical theory of quadratic forms. Notes by Morgan Ward. Third revised edition. Buchhandlung Robert Peppmüller, Göttingen, 1963.
[68] K. Soni, Some relations associated with an extension of Koshliakov’s formula, Proc. Amer. Math. Soc., 17 (1966), 543–551.
[69] H. M. Stark, On the zeros of Epstein’s zeta functions, Mathematika, 14 (1967), 47–55.
[70] M. Suzuki, An analogue of the Chowla–Selberg formula for several automorphic L-functions, Advanced Studies in Pure Mathematics, 49 (2007), 479–506.
[71] P. R. Taylor, The functional equation for Epstein’s zeta-function, The Quarterly Journal of Mathematics, 11 (1940), 177–182.
[72] A. Terras, Bessel series expansion of the Epstein zeta function and the functional equation, Trans. Amer. Math. Soc., 183 (1973), 477–486.
[73] ———, Real zeroes of Epstein’s zeta function for ternary positive definite quadratic forms, Illinois J. Math., 23 (1979), 1–14.
[74] E. C. Titchmarsh, On Epstein’s zeta-function, Proc. London Math. Soc. 36 (1934), 485—500.
[75] ———, The Theory of Functions, 2nd ed., Oxford University Press, London, 1939.
[76] ———, The Theory of The Riemann Zeta-Function, 2nd ed., The Clarendon Press, Oxford, 1986.
[77] S. M. Voronin, The zeros of zeta-functions of quadratic forms (in Russian), Tr. Mat. Inst. Steklova, 142 (1976), 135–147.
[78] G. N. Watson, Some self-reciprocal functions (1), Quart. J. of Math. (Oxford), 2 (1931), 298–309.
[79] ———, A Treatise on the Theory of Bessel Functions, second ed., Cambridge University Press, London, 1966.

[80] S. B. Yakubovich, A general class of Voronoi’s and Koshliakov-Ramanujan’s summation formulas involving $d_k(n)$, Integral Transforms Spec. Funct., 22 (2011), 801–821.