Factorisation of Lie Resolvents

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Abstract. Let $G$ be a group, $F$ a field of prime characteristic $p$ and $V$ a finite-dimensional $FG$-module. Let $L(V)$ denote the free Lie algebra on $V$, regarded as an $FG$-module, and, for each positive integer $r$, let $L^r(V)$ be the $r$th homogeneous component of $L(V)$, called the $r$th Lie power of $V$. In a previous paper we obtained a decomposition of $L^r(V)$ as a direct sum of modules of the form $L^s(W)$, where $s$ is a power of $p$. Here we derive some consequences. First we obtain a similar result for restricted Lie powers of $V$. Then we consider the ‘Lie resolvents’ $\Phi^r$: certain functions on the Green ring of $FG$ which determine Lie powers up to isomorphism. For $k$ not divisible by $p$, we obtain the factorisation $\Phi^{p^m k} = \Phi^{p^m} \circ \Phi^k$, separating out the key case of $p$-power degree. Finally we study certain functions on power series over the Green ring, denoted by $S^*$ and $L^*$, which encode symmetric powers and Lie powers, respectively. In characteristic 0, $L^*$ is the inverse of $S^*$. In characteristic $p$, the composite $L^* \circ S^*$ maps any $p$-typical power series to a $p$-typical power series.

Keywords: free Lie algebra, Lie power, Adams operation, Lie resolvent

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1. Introduction

Let $G$ be a group and $F$ a field. For any finite-dimensional $FG$-module $V$ let $L(V)$ be the free Lie algebra on $V$. Then $L(V)$ may be regarded as an $FG$-module on which each element of $G$ acts as a Lie algebra automorphism. Furthermore, each homogeneous component $L^r(V)$ is a finite-dimensional submodule, called the $r$th Lie power of $V$. We regard $L(V)$ as an $FG$-submodule of the free associative

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algebra $T(V)$. Thus $L^r(V)$ is a submodule of the $r$th homogeneous component $T^r(V)$.

When $F$ has prime characteristic $p$ we may also form the free restricted Lie algebra $R(V)$ which again may be regarded as an $FG$-submodule of $T(V)$. The homogeneous component $R^r(V)$ is called the $r$th restricted Lie power of $V$.

A general problem is to describe the modules $L^r(V)$ and $R^r(V)$ up to isomorphism. We refer to [4] and the papers cited there for a discussion of progress on $L^r(V)$. Although some use has been made of $R(V)$ in studying $L(V)$, results so far on the module structure of $R^r(V)$ are rather sparse: but see [9, 10, 17].

The main result of [4], called the ‘Decomposition Theorem’, reduces the study of arbitrary Lie powers in characteristic $p$ to the study of Lie powers of $p$-power degree. It states that, for each positive integer $r$, there is a submodule $B_r$ of $L^r(V)$ such that $B_r$ is a direct summand of $T^r(V)$ and, for $k$ not divisible by $p$ and $m \geq 0$,

$$L^{p^m k}(V) = L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^m k}).$$

In this paper we derive some consequences of the Decomposition Theorem.

Our first result, obtained in §2, is an analogous result for $R(V)$. With the modules $B_r$ and $k$ and $m$ as before, we find that

$$R^{p^m k}(V) = R^{p^m}(B_k) \oplus R^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus R^1(B_{p^m k}).$$

In the remainder of the paper we turn to applications in the Green ring $R_{FG}$. This is the ring spanned by the isomorphism classes of finite-dimensional $FG$-modules with addition and multiplication coming from direct sums and tensor products, respectively. In §3 we describe the general framework which may be used to study Lie powers and symmetric powers of finite-dimensional $FG$-modules. This is mainly a summary of material contained in [1]. In particular we describe the relevance of the Adams operations and the Lie resolvents. The latter are $\mathbb{Z}$-linear maps $\Phi^r : R_{FG} \to R_{FG}$. Knowledge of these maps is essentially equivalent to knowledge of the isomorphism classes of Lie powers of finite-dimensional $FG$-modules. We also introduce certain functions $L^*$ and $S^*$ defined on formal power series in an indeterminate $t$ with coefficients from $R_{FG}$. Here $L^*$ encodes Lie powers
and \( S^* \) encodes symmetric powers. Furthermore, \( L^* \) and \( S^* \) have properties like those of the logarithm function and exponential function, respectively.

In §4 we obtain the main result of the paper, called the ‘Factorisation Theorem’. It gives a ‘factorisation’ of Lie resolvents under composition: for every non-negative integer \( m \) and every positive integer \( k \) not divisible by \( p \), \( \Phi^{p^m k} = \Phi^{p^m} \circ \Phi^k \). The Lie resolvents \( \Phi^k \) for \( k \) not divisible by \( p \) are comparatively well understood, so the result can be interpreted as a reduction to the case of \( p \)-power degree. The Factorisation Theorem was conjectured in [3] and proved in [7] in the special case \( m = 1 \). It is interesting that the Witt polynomials (as used to define the operations on Witt vectors) arise here in connection with the Factorisation Theorem.

Finally, in §5, we describe relations between \( L^* \) and \( S^* \). In characteristic \( p \) we use the Factorisation Theorem to show that the composite function \( L^* \circ S^* \) takes any power series of the form \( Vt \) to a power series where the coefficient of \( t^r \) is 0 unless \( r \) is a power of \( p \). A power series of the latter form is called ‘\( p \)-typical’. It follows that \( L^* \circ S^* \) maps every \( p \)-typical power series to a \( p \)-typical power series. In characteristic 0 we find that \( L^* \) is the inverse of \( S^* \). This is closely related to results of Joyal [12] and Reutenauer [14].

2. Decomposition of Lie powers and restricted Lie powers

Let \( V \) be a finite-dimensional \( FG \)-module, where \( F \) is a field and \( G \) is a group. We write \( T(V) \) for the free associative algebra freely generated by any \( F \)-basis of \( V \). Thus \( T(V) \) has an \( F \)-space decomposition \( T(V) = \bigoplus_{r \geq 0} T^r(V) \), where, for each \( r \), \( T^r(V) \) is the \( r \)th homogeneous component of \( T(V) \) and \( T^1(V) \) is identified with \( V \). The action of \( G \) on \( V \) extends to \( T(V) \) so that \( G \) acts by algebra automorphisms. Thus \( T(V) \) becomes an \( FG \)-module, and each \( T^r(V) \) is a finite-dimensional submodule.

The algebra \( T(V) \) may be made into a Lie algebra by defining \([a, b] = ab - ba\) for all \( a, b \in T(V) \). If \( F \) has prime characteristic \( p \) then \( T(V) \) may be made into a restricted Lie algebra by taking the additional powering operation \( a \mapsto a^p \). The Lie subalgebra of \( T(V) \) generated by \( V \) is denoted by \( L(V) \) and (when \( F \) has characteristic \( p \)) the restricted Lie subalgebra of \( T(V) \) generated by \( V \) is denoted...
by $R(V)$. As is well known, $L(V)$ is a free Lie algebra and $R(V)$ is a free restricted Lie algebra, both freely generated by any basis of $V$ (see, for example, [13, Sections 1.2, 1.6.3 and 2.5.2]). For $r \geq 1$ we write $L^r(V) = T^r(V) \cap L(V)$ and $R^r(V) = T^r(V) \cap R(V)$. Thus $L(V) = \bigoplus_{r \geq 1} L^r(V)$ and $R(V) = \bigoplus_{r \geq 1} R^r(V)$. Here $L^r(V)$ and $R^r(V)$ are $FG$-submodules of $T^r(V)$, called, respectively, the $r$th Lie power and $r$th restricted Lie power of $V$.

As observed in [4, §2], if $B$ is a submodule of $T^r(V)$, the associative subalgebra of $T(V)$ generated by $B$ may be identified with $T(B)$. The Lie subalgebra of $T(V)$ generated by $B$ is then the free Lie algebra $L(B)$ and the restricted Lie subalgebra of $T(V)$ generated by $B$ is the free restricted Lie algebra $R(B)$. We assume such identifications in the next two theorems. The first is [4, Theorem 4.4].

**Theorem 2.1** (Decomposition Theorem) Let $F$ be a field of prime characteristic $p$. Let $G$ be a group and $V$ a finite-dimensional $FG$-module. For each positive integer $r$ there is a submodule $B_r$ of $L^r(V)$ such that $B_r$ is a direct summand of $T^r(V)$ and, for $k$ not divisible by $p$ and $m \geq 0$,

$$L^{p^m k}(V) = L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^mk}). \quad (2.1)$$

We use this to prove a similar result for restricted Lie powers.

**Theorem 2.2** In the notation of Theorem 2.1,

$$R^{p^m k}(V) = R^{p^m}(B_k) \oplus R^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus R^1(B_{p^mk}).$$

**Proof:** For any subset $S$ of $L(V)$ and $i \geq 0$ we write $S^{[p^i]}$ for the set $\{s^{p^i} : s \in S\}$. We shall use the fact that if $\mathcal{X}$ is any basis of $L(V)$ then the elements $x^{p^i}$ with $x \in \mathcal{X}$ and $i \geq 0$ are distinct and form a basis of $R(V)$: they are linearly independent by the Poincaré–Birkhoff–Witt Theorem and it is easy to see that they span $R(V)$.

For each non-negative integer $i$ and each positive integer $k$ not divisible by $p$, let $\mathcal{X}(i, k)$ be a basis of $L^{p^i k}(V)$. Hence $\bigcup_{i,k} \mathcal{X}(i, k)$ is a basis of $L(V)$. Here, and
in the remainder of the proof, all unions are unions of disjoint sets. Furthermore, write
\[ \hat{X}(i, k) = X(0, k)^{[p]} \cup X(1, k)^{[p-1]} \cup \cdots \cup X(i, k)^{[1]} . \] (2.2)
Considerations of degree in the basis of \( R(V) \) obtained from \( \bigcup_{i, k} X(i, k) \) show that \( \hat{X}(i, k) \) is a basis of \( R^{p,i,k}(V) \).

For each triple \( (i, j, k) \) where \( i \) and \( j \) are non-negative integers and \( k \) is a positive integer not divisible by \( p \), let \( Y(i, j, k) \) be any basis of \( L^{p,i}(B^{p,j,k}) \) and write
\[ \hat{Y}(i, j, k) = Y(0, j, k)^{[p]} \cup Y(1, j, k)^{[p-1]} \cup \cdots \cup Y(i, j, k)^{[1]} . \] (2.3)
Thus \( \hat{Y}(i, j, k) \) is a basis of \( R^{p,i}(B^{p,j,k}) \).

By the Decomposition Theorem, we can choose the basis \( X(i, k) \) of \( L^{p,i,k}(V) \) to satisfy
\[ X(i, k) = Y(i, 0, k) \cup Y(i - 1, 1, k) \cup \cdots \cup Y(0, i, k) . \] (2.4)
Let \( m \) be a non-negative integer. It is easily verified from (2.2) and (2.4) that \( \hat{X}(m, k) \) is the union of the sets \( Y(i, j, k)^{[p]} \) with \( i + j + r = m \). Hence, by (2.3),
\[ \hat{X}(m, k) = \hat{Y}(m, 0, k) \cup \hat{Y}(m - 1, 1, k) \cup \cdots \cup \hat{Y}(0, m, k) . \]
Since \( \hat{X}(m, k) \) spans \( R^{p,m,k}(V) \) and \( \hat{Y}(i, m - i, k) \) spans \( R^{p,i}(B^{p,m-i,k}) \), we obtain the required result. \( \square \)

3. Symmetric powers and Lie powers

The underlying ideas that we use are described in [1]. However, we shall make the treatment here as self-contained as possible. We also formulate some of the ideas in a slightly new way, in order to emphasise the analogies between symmetric powers and Lie powers. Let \( G \) be a group and \( F \) a field. Let Mod\((FG)\) be the class of all finite-dimensional \( FG \)-modules and let \( R_{FG} \) be the associated Green ring (or representation ring), as defined in §1.

The \( \mathbb{Q} \)-algebra \( \mathbb{Q} \otimes_{\mathbb{Z}} R_{FG} \) will be denoted by \( \Gamma_{FG} \) and we regard \( R_{FG} \) as a subset of \( \Gamma_{FG} \). (In [1], \( \mathbb{C} \) was used instead of \( \mathbb{Q} \). But, for what is needed here, \( \mathbb{Q} \) works as well as \( \mathbb{C} \).)
Let $\Pi$ denote the power series ring over $\Gamma_{FG}$ in an indeterminate $t$; that is, $\Pi = \Gamma_{FG}[[t]]$. Thus $t\Pi$ and $1 + t\Pi$ are the subsets consisting of all power series with constant term 0 and 1, respectively. For $f \in t\Pi$ and $g \in 1 + t\Pi$ we may define $\exp(f) \in 1 + t\Pi$ and $\log(g) \in t\Pi$ by

$$
\exp(f) = 1 + f + f^2/2! + f^3/3! + \cdots,
$$

$$
\log(g) = (g - 1) - (g - 1)^2/2 + (g - 1)^3/3 - \cdots.
$$

Thus we have mutually inverse functions

$$\exp : t\Pi \to 1 + t\Pi, \quad \log : 1 + t\Pi \to t\Pi.$$  

If $V \in \text{Mod}(FG)$ we also write $V$ for the element of $R_{FG}$ or $\Gamma_{FG}$ determined by $V$. Thus, for example, $T^r(V)$ as an element of $R_{FG}$ denotes the isomorphism class of the module $T^r(V)$ and it is equal to $V^r$.

We start with a discussion of symmetric powers and Adams operations. Let $V \in \text{Mod}(FG)$. We write $S(V)$ for the polynomial algebra (free commutative associative algebra) over $F$ freely generated by any basis of $V$. The action of $G$ on $V$ extends to $S(V)$ so that $G$ acts by algebra automorphisms. For $r \geq 0$, the $r$th homogeneous component $S^r(V)$ is an $FG$-submodule of $S(V)$ called the $r$th symmetric power of $V$. Here $S^1(V)$ is identified with $V$.

For $V \in \text{Mod}(FG)$, let $S(V, t) \in 1 + t\Pi$ be defined by

$$S(V, t) = 1 + S^1(V)t + S^2(V)t^2 + \cdots.$$  

It is well known and easy to verify that for $U, V \in \text{Mod}(FG)$ we have

$$S(U \oplus V, t) = S(U, t)S(V, t).$$

It follows that there is a $\mathbb{Q}$-linear function $\psi : \Gamma_{FG} \to t\Pi$ satisfying

$$\psi(V) = \log(S(V, t)) \tag{3.1}$$

for all $V \in \text{Mod}(FG)$. Hence we may define a function

$$S^+ : t\Pi \to t\Pi.$$
by
\[
S^+(A_1 t + A_2 t^2 + A_3 t^3 + \cdots) = \psi(A_1) + \psi(A_2) t + \psi(A_3) t^2 + \cdots, \tag{3.2}
\]
for all \(A_1, A_2, \ldots \in \Gamma_{FG}\), where the subscript \(t \mapsto t^r\) denotes the operation of replacing a power series \(X_1 t + X_2 t^2 + \cdots\) by \(X_1 t^r + X_2 t^{2r} + \cdots\). The properties of \(S^+\) given in the following lemma are readily obtained from the definition.

**Lemma 3.1** The function \(S^+\) is \(\mathbb{Q}\)-linear and \(S^+(t^r \Pi) \subseteq t^r \Pi\), for all \(r \geq 1\). For all \(f \in t \Pi\) and all \(r \geq 1\), \(S^+(f_{t \mapsto t^r}) = S^+(f)_{t \mapsto t^r}\). If \(f_i \in t^i \Pi\) for \(i = 1, 2, \ldots\), then \(S^+(\sum f_i) = \sum S^+(f_i)\).

We now define a function
\[
S^*: t \Pi \longrightarrow 1 + t \Pi
\]
as the composite
\[
S^* = \exp \circ S^+. \tag{3.3}
\]
It is clear from (3.1) and (3.2) that \(S^*(V t) = S(V, t)\) for all \(V \in \text{Mod}(FG)\). Also, since \(S^+\) is additive,
\[
S^*(f + g) = S^*(f) S^*(g)
\]
for all \(f, g \in t \Pi\).

For \(A \in \Gamma_{FG}\) we may define \(\psi^1(A), \psi^2(A), \ldots \in \Gamma_{FG}\) by the equation
\[
S^+(At) = \psi(A) = \psi^1(A) t + \frac{1}{2} \psi^2(A) t^2 + \frac{1}{3} \psi^3(A) t^3 + \cdots. \tag{3.4}
\]
Thus we obtain \(\mathbb{Q}\)-linear functions \(\psi^r : \Gamma_{FG} \to \Gamma_{FG}\). These functions were denoted by \(\psi_S^r\) in [1, §5]: they are the *Adams operations* on \(\Gamma_{FG}\) determined by symmetric powers. (Adams operations, different in general, can also be defined using exterior powers.) By [1, §5], if \(V \in \text{Mod}(FG)\) then \(\psi^r(V) \in R_{FG}\). Thus \(\psi^r\) restricts to a \(\mathbb{Z}\)-linear function \(\psi^r : R_{FG} \to R_{FG}\).

If \(V \in \text{Mod}(FG)\) then, by (3.1) and (3.4),
\[
\log(S(V, t)) = \psi^1(V) t + \frac{1}{2} \psi^2(V) t^2 + \cdots.
\]
This equation shows that symmetric powers may be expressed in terms of Adams operations and vice versa.

**Lemma 3.2** If $r$ and $s$ are positive integers such that $r$ is not divisible by the characteristic of $F$ then $\psi^r$ is an algebra endomorphism of $\Gamma_{FG}$ and

$$\psi^{rs} = \psi^r \circ \psi^s.$$ 

In particular, if $F$ has prime characteristic $p$ then

$$\psi^{p^nk} = \psi^k \circ \psi^{p^m}$$

for all non-negative integers $m$ and all positive integers $k$ not divisible by $p$.

**Proof:** See [1, Theorem 5.4]. \hfill \Box

All the results stated above for symmetric powers have analogues for exterior powers. However, we shall not need exterior powers for the applications in this paper.

We now turn to Lie powers and Lie resolvents. We shall summarise the general theory described in [1] and formulate some of it in a slightly different way. The theory is based on a function

$$\mathcal{L}_{FG} : t\Pi \to t\Pi$$

called the **Lie module function**. When $F$ and $G$ are understood we write $\mathcal{L}$ instead of $\mathcal{L}_{FG}$. The properties of this function are described in [1, §2 and §3]. In particular, by [1, Lemma 3.1],

$$\mathcal{L}(V t^r) = L^1(V) t^r + L^2(V) t^{2r} + \cdots ,$$

(3.5)

for all $V \in \text{Mod}(FG)$ and $r \geq 1$, and

$$\mathcal{L}(f) + \mathcal{L}(g) = \mathcal{L}(f + g - fg),$$

(3.6)

for all $f, g \in t\Pi$. 

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There are advantages in considering a modification of $L$, namely the function

$$L^* : 1 + t \Pi \rightarrow t \Pi$$

defined by

$$L^*(f) = -L(1-f)$$  \hspace{1cm} (3.7)

for all $f \in 1 + t \Pi$. It follows from (3.5), (3.6) and (3.7) that

$$-L^*(1 - Vt) = L^1(V)t + L^2(V)t^2 + \cdots,$$

for all $V \in \text{Mod}(FG)$, and

$$L^*(fg) = L^*(f) + L^*(g),$$

for all $f, g \in 1 + t \Pi$.

We may define a function

$$L^+ : t \Pi \rightarrow t \Pi$$

by $L^+ = L^* \circ \exp$. Thus $L^+$ is additive and

$$L^* = L^+ \circ \log.$$  \hspace{1cm} (3.8)

Let $\text{Exp} : t \Pi \rightarrow t \Pi$ be defined by $\text{Exp}(f) = 1 - \exp(-f)$ for all $f \in t \Pi$. Since

$$L^+(f) = -L^+(-f) = -L^*(\exp(-f)) = L(1 - \exp(-f)),$$

we obtain

$$L^+ = L \circ \text{Exp}.$$  \hspace{1cm} (3.9)

Additional properties of $L^+$ are given in [1, §3], where $L^+$ is denoted by $F_{FG}$. These properties yield the following result.

**Lemma 3.3** The function $L^+$ has the properties of $S^+$ stated in Lemma 3.1.

In particular $L^+$ is $\mathbb{Q}$-linear.

For $A \in \Gamma_{FG}$ we may define $\Phi^1(A), \Phi^2(A), \ldots \in \Gamma_{FG}$ by the equation

$$L^+(At) = \Phi^1(A)t + \frac{1}{2}\Phi^2(A)t^2 + \frac{1}{3}\Phi^3(A)t^3 + \cdots.$$  \hspace{1cm} (3.10)
Thus we obtain $\mathbb{Q}$-linear functions $\Phi^r : \Gamma_{FG} \to \Gamma_{FG}$. These functions were considered in [1], where they were written $\Phi^r_{FG}$ and called the Lie resolvents for $G$ over $F$. They restrict to $\mathbb{Z}$-linear functions $\Phi^r : R_{FG} \to R_{FG}$. Lie powers may be expressed in terms of Lie resolvents and vice versa by means of the following expressions given in [1, Corollary 3.3]: for all $V \in \text{Mod}(FG)$ and every positive integer $r$,

$$L^r(V) = \frac{1}{r} \sum_{d|r} \Phi^d(V^{r/d}) \quad (3.11)$$

and

$$\Phi^r(V) = \sum_{d|r} \mu(r/d) d L^d(V^{r/d}), \quad (3.12)$$

where $\mu$ denotes the Möbius function.

We conclude this section with two further lemmas.

**Lemma 3.4** If $r$ is not divisible by the characteristic of $F$, then

$$\Phi^r = \mu(r) \psi^r.$$ 

**Proof:** See [1, Corollary 6.2].

**Lemma 3.5** Let $f \in t\Pi$, where $f = A_1 t + A_2 t^2 + \cdots$. Then

$$L^+(f) = \sum_{r \geq 1} \left( \sum_{d|r} \frac{1}{d} \Phi^d(A_{r/d}) \right) t^r$$

and

$$S^+(f) = \sum_{r \geq 1} \left( \sum_{d|r} \frac{1}{d} \psi^d(A_{r/d}) \right) t^r.$$ 

**Proof:** By (3.10) and (3.4), and in the terminology of [1, §2], the functions $\frac{1}{r} \Phi^r$ and $\frac{1}{r} \psi^r$ are the ‘components’ of $L^+$ and $S^+$, respectively. Hence the result follows from [1, (2.5)].
4. The Factorisation Theorem

Let $F$ be an infinite field and $n$ a positive integer. We write $G(n) = \text{GL}(n, F)$. We require some facts about polynomial modules and their characters. Most of these have already been collected in [1, §5], so we use this as a convenient reference: but see also [8]. As in [1, §5], let $R_{FG(n)}^{\text{poly}}$ denote the subring of the Green ring $R_{FG(n)}$ spanned by all the isomorphism classes of finite-dimensional polynomial modules.

It is clear from (3.12) that, for each $r$, the Lie resolvent $\Phi^r$ restricts to a map $\Phi^r : R_{FG(n)}^{\text{poly}} \to R_{FG(n)}^{\text{poly}}$.

Let $t_1, \ldots, t_n$ be indeterminates and let $\Delta$ be the subring of $\mathbb{Z}[t_1, \ldots, t_n]$ consisting of all symmetric polynomials. For each positive integer $r$, the endomorphism of $\mathbb{Z}[t_1, \ldots, t_n]$ satisfying $t_i \mapsto t_i^r$ for all $i$ restricts to an endomorphism of $\Delta$, which we denote by $\chi^r$. As explained in [1, §5], there is a ring homomorphism $\text{ch} : R_{FG(n)}^{\text{poly}} \to \Delta$ such that, for every finite-dimensional polynomial $FG(n)$-module $U$, $\text{ch}(U)$ is the formal character of $U$.

As stated in [6, §3.2], if $V$ is the natural $FG(n)$-module then

$$\text{ch}(L^r(V)) = \frac{1}{r} \sum_{d|r} \mu(d) (t_1^d + \cdots + t_n^d)^{r/d}. \tag{4.1}$$

(The right-hand side is formally an element of $\mathbb{Q} \otimes_{\mathbb{Z}} \Delta$, but is found to belong to $\Delta$.) Suppose that $U$ is a finite-dimensional polynomial $FG(n)$-module. Then we may write $\text{ch}(U) = w_1 + \cdots + w_m$ where $m = \dim U$ and $w_1, \ldots, w_m$ are monomials in $t_1, \ldots, t_n$. We may choose a basis of $U$ consisting of elements from weight spaces. Then every diagonal element of $G(n)$ is represented on $U$ by a diagonal element of $\text{GL}(m, F)$. By (4.1) with $m$ instead of $n$, we obtain

$$\text{ch}(L^r(U)) = \frac{1}{r} \sum_{d|r} \mu(d) (w_1^d + \cdots + w_m^d)^{r/d}.$$  

Hence

$$\text{ch}(L^r(U)) = \frac{1}{r} \sum_{d|r} \mu(d) \chi^d(\text{ch}(U)^{r/d}). \tag{4.2}$$

Lemma 4.1 Let $U$ be a finite-dimensional polynomial $FG(n)$-module.

(i) For all $r \geq 1$, $\text{ch}(\Phi^r(U)) = \mu(r) \chi^r(\text{ch}(U))$. 

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(ii) If \((r, s) = 1\) then
\[
\text{ch}(\Phi^{rs}(U)) = \text{ch}(\Phi^r(\Phi^s(U))).
\]

Proof: By (3.11),
\[
\text{ch}(L^r(U)) = \frac{1}{r} \sum_{d|r} \text{ch}(\Phi^d(U^{r/d})).
\]
Comparing this with (4.2) and using induction on \(r\) gives (i). Hence, for all \(W \in R_{FG(n)}^{\text{poly}}\),
\[
\text{ch}(\Phi^r(W)) = \mu(r)\chi^r(\text{ch}(W)). \quad (4.3)
\]
Therefore
\[
\text{ch}(\Phi^r(\Phi^s(U))) = \mu(r)\chi^r(\text{ch}(\Phi^s(U)))
\]
\[
= \mu(r)\chi^r(\mu(s)\chi^s(\text{ch}(U))).
\]
However, \(\chi^r \circ \chi^s = \chi^{rs}\) and, if \(r\) and \(s\) are coprime, \(\mu(r)\mu(s) = \mu(rs)\). Therefore
\[
\text{ch}(\Phi^r(\Phi^s(U))) = \mu(rs)\chi^{rs}(\text{ch}(U)).
\]
Hence, by (4.3), \(\text{ch}(\Phi^r(\Phi^s(U))) = \text{ch}(\Phi^{rs}(U))\), as required for (ii). \(\square\)

We recall the Decomposition Theorem (Theorem 2.1). The submodules \(B_r\) are not uniquely determined by \(V\), as easy examples show. However, it follows from equations (2.1), by induction, that the \(B_r\) are uniquely determined up to isomorphism. Thus they give uniquely determined elements of \(R_{FG}\). We shall deduce the following result from the Decomposition Theorem.

**Theorem 4.2 (Factorisation Theorem)** Let \(F\) be a field of prime characteristic \(p\). Let \(G\) be a group. For every non-negative integer \(m\) and every positive integer \(k\) not divisible by \(p\),
\[
\Phi^{pmk} = \Phi^{pm} \circ \Phi^k \quad (4.4)
\]
and, for every finite-dimensional \(FG\)-module \(V\), the elements \(B_r\) of \(R_{FG}\) obtained from the Decomposition Theorem satisfy
\[
p^m B_{pmk} + p^{m-1} B_{pm-1k} + \cdots + p B_{pk} + B_{pk}^{r-1} + B_{pk}^r = L^r(V^{pm}). \quad (4.5)
\]
**Proof:** When we wish to show the dependence on \( V \) we write \( B_r(\cdot) \), and when we wish to show the role of \( F \) and \( G \) in \( \Phi^r \) we write it as \( \Phi^r_{FG} \). We first reduce to the case where \( F \) is infinite. Let \( E \) be an infinite extension field of \( F \). Assume that the theorem holds with \( E \) in place of \( F \). Each \( FG \)-module \( V \) determines an \( EG \)-module \( E \otimes_F V \) by extension of scalars. Thus we obtain a ring homomorphism \( \iota : R_{FG} \to R_{EG} \). It follows from the Noether–Deuring Theorem (see [5, (29.7)]) that \( \iota \) is an embedding. By [3, Lemma 2.4], for all \( r \geq 1 \) and all \( V \in \text{Mod}(FG) \), we have \( L_r(\iota(V)) = \iota(L_r(V)) \) and \( \Phi^r_{EG} \circ \iota = \iota \circ \Phi^r_{FG} \).

By (4.4) over \( E \), \( \Phi^{p^m k}_{EG} \circ \iota = \Phi^{p^m k}_{EG} \circ \Phi^{k}_{FG} \circ \iota \) and hence \( \iota \circ \Phi^{p^m k}_{FG} = \iota \circ \Phi^{p^m k}_{FG} \circ \Phi^{k}_{FG} \). Since \( \iota \) is an embedding we obtain (4.4) over \( F \). Let \( V \in \text{Mod}(FG) \). Applying \( \iota \) to equations (2.1) over \( F \) we find that \( \iota(B_r(V)) = B_r(\iota(V)) \) for all \( r \). By (4.5) over \( E \) for the module \( \iota(V) \),

\[
p^m B^{p^m k}(\iota(V)) + p^{m-1}(B^{p^m-1 k}(\iota(V))) + \cdots + (B^k(\iota(V)))^p = L^k(\iota(V)^{p^m}).
\]

Thus

\[
\iota(p^m B^{p^m k}(V) + \cdots + (B^k(V))^{p^m}) = \iota(L^k(V^{p^m})).
\]

Since \( \iota \) is an embedding we obtain equation (4.5) over \( F \).

Hence it is enough to prove the theorem over \( E \). In other words we may assume that \( F \) is infinite.

We use induction on \( m \) and \( k \). If \( m = 0 \) the result is clear because \( \Phi^1 \) is the identity map and, for all \( V \), \( B^k(V) = L^k(V) \), by (2.1). Hence we may assume that \( m \geq 1 \) and that the result holds for \( p^i d \) with \( d \) not divisible by \( p \) if \( i < m \) or \( i = m \) and \( d < k \).

Since the functions \( \Phi^r \) are linear, in order to prove (4.4) it suffices to prove

\[
\Phi^{p^m k}(V) = \Phi^{p^m \circ k}(V) \tag{4.6}
\]

for all \( V \in \text{Mod}(FG) \).

Let \( V \in \text{Mod}(FG) \) and \( n = \dim V \). By choice of a basis of \( V \), the representation of \( G \) on \( V \) gives a homomorphism \( \theta : G \to G(n) \). Furthermore, \( \theta \) induces a ring homomorphism \( \theta^* : R_{FG(n)} \to R_{FG} \). We can regard \( V \) as the natural module for
$G(n)$, in which case we write it as $V(n)$. Thus $\theta^*(V(n)) = V$. We now show that it is enough to prove the required results for $G(n)$ and $V(n)$. Suppose that (4.6) and (4.5) hold for $G(n)$ and $V(n)$. By [3, Lemmas 2.2 and 2.3], for all $r \geq 1$ and all $U \in \text{Mod}(FG(n))$, we have $L^r(\theta^*(U)) = \theta^*(L^r(U))$ and $\theta^* \circ \Phi_{FG(n)}^r = \Phi_{FG} \circ \theta^*$.

Thus, by applying $\theta^*$ to (4.6) for $G(n)$ and $V(n)$, we get (4.6) for $G$ and $V$. By applying $\theta^*$ to equations (2.1) for $G(n)$ and $V(n)$ we get $\theta^*(B^r(V(n))) = B^r(V)$ for all $r$. Hence, by applying $\theta^*$ to (4.5) for $G(n)$ and $V(n)$, we get (4.5) for $G$ and $V$.

Thus it remains to prove (4.6) and (4.5) for $G(n)$ and $V(n)$. To simplify the notation, write $V$ for $V(n)$ and $B_r$ for $B^r(V(n))$.

As in §3, let $\Pi = \Gamma_{FG(n)}[[t]]$. Let $\text{Log}: t\Pi \to t\Pi$ be defined by

$$\text{Log}(f) = -\log(1 - f) = f + f^2/2 + f^3/3 + \cdots$$

for all $f \in t\Pi$. Thus $\text{Log}$ is the inverse of the function $\text{Exp}$ defined in §3.

For $f \in t\Pi$ and $r \geq 1$, we write $f^{(r)}$ for the coefficient of $t^r$ in $f$; thus $f^{(r)} \in \Gamma_{FG(n)}$ and $f = \sum_{r \geq 1} f^{(r)} t^r$.

Let $Q' = \sum_{r \geq 0} \text{Log}(B_{p^r t^{p^r}})$, $Q'' = \sum_{i \geq 0} p^{-i} L^k(V^{p^i}) t^{p^i}$ and $Q = Q' - Q''$. Then it is easily calculated that

$$Q_{(p^i)} = \frac{1}{p^{i+1}} B_{p^i k}^{p^i} + \frac{1}{p^{i-1}} B_{p^i pk}^{p^{i-1}} + \cdots + B_{p^{i} k} - \frac{1}{p^i} L^k(V^{p^i}).$$

By the inductive hypothesis, $Q_{(p^i)} = 0$ for $i < m$. Hence, by Lemma 3.5, we obtain

$$\mathbf{L}^+(Q)_{(p^m)} = \Phi^1(Q_{(p^m)}) = Q_{(p^m)} = \frac{1}{p^m} B_{p^m k}^{p^m} + \frac{1}{p^{m-1}} B_{p^m pk}^{p^{m-1}} + \cdots + B_{p^{m} k} - \frac{1}{p^m} L^k(V^{p^m}).$$

Since $\mathbf{L}^+$ is additive and $\mathbf{L}^+ \circ \text{Log} = \mathcal{L}$, by (3.9), we have

$$\mathbf{L}^+(Q') = \sum_{r \geq 0} \mathcal{L}(B_{p^r t^{p^r}}).$$

Thus, by (3.5),

$$\mathbf{L}^+(Q')_{(p^m)} = L_{p^m}(B_k) + L_{p^{m-1}}(B_{pk}) + \cdots + L^1(B_{p^m k}).$$
Therefore, by the Decomposition Theorem,

\[ L^+(Q'_{(p^m)}) = L^{p^m k}(V). \]

By Lemma 3.5,

\[ L^+(Q''_{(p^m)}) = \frac{1}{p^m} \sum_{d | p^m} \Phi^d(L^{k}(V^{p^m/d})). \]

Hence, by (3.11),

\[
L^+(Q''_{(p^m)}) = \frac{1}{p^m k} \sum_{d | p^m, e | k} (\Phi^d \circ \Phi^e)(V^{p^m k/de}).
\]

We write

\[ \Phi^{p^m} \circ \Phi^k = \Phi^{p^m k} + (\Phi^{p^m} \circ \Phi^k - \Phi^{p^m k}) \]

and use the inductive hypothesis to write \( \Phi^d \circ \Phi^e = \Phi^{de} \) for \( de < p^m k \). Thus

\[
L^+(Q''_{(p^m)}) = \frac{1}{p^m k} \sum_{s | p^m k} \Phi^s(V^{p^m k/s}) + \frac{1}{p^m k} (\Phi^{p^m}(\Phi^k(V)) - \Phi^{p^m k}(V))
\]

\[
= L^{p^m k}(V) + \frac{1}{p^m k} (\Phi^{p^m}(\Phi^k(V)) - \Phi^{p^m k}(V)).
\]

Since \( L^+(Q)_{(p^m)} = L^+(Q')_{(p^m)} - L^+(Q'')_{(p^m)} \), we obtain

\[
\frac{1}{p^m} B^p_{p^m k} + \frac{1}{p^{m-1}} B^p_{p^m k} - \frac{1}{p^m} L^k(V^{p^m}) = \frac{1}{p^m k} (\Phi^{p^m k}(V) - \Phi^{p^m}(\Phi^k(V))).
\]

Therefore

\[
\frac{1}{k} (\Phi^{p^m k}(V) - \Phi^{p^m}(\Phi^k(V))) = W,
\]

where

\[
W = p^m B^p_{p^m k} + p^{m-1} B^p_{p^m k} + \cdots + p B^p_{p^m} - L^k(V^{p^m}).
\]

By Lemma 4.1, ch(\( \Phi^{p^m k}(V) - \Phi^{p^m}(\Phi^k(V)) \)) = 0. Thus ch(W) = 0. By the Decomposition Theorem, the modules \( B^p_{p^m k} \), \( B^p_{p^m k} \), \( \ldots \), \( B^p_{p^m} \) are isomorphic to direct summands of the tensor power \( V^{p^m k} \). Hence they are tilting modules: see [1, §5]. Since \( p \nmid k \), \( L^k(V^{p^m}) \) is also a direct summand of \( V^{p^m k} \) (see, for example, [6,
Thus it too is a tilting module. However, tilting modules are determined up to isomorphism by their formal characters (see, for example, [6]). Since \( \text{ch}(W) = 0 \) it follows that \( W = 0 \). Thus \( \Phi^{p^m k}(V) - \Phi^{p^m}(\Phi^k(V)) = 0 \). This gives (4.5) and (4.6). □

Note that equations (4.5) give a recursive description, within \( \Gamma_{FG} \), of \( B_{p^m k} \) as a polynomial in \( L^k(V), L^k(V^p), \ldots, L^k(V^{p^m}) \). The polynomials in \( B_k, B_{p k}, B_{p^2 k}, \ldots \) occurring on the left-hand side of (4.5) may be recognised as the Witt polynomials associated to the prime \( p \): see [16, Chapter II, §6] or [11, Chapter 3]. Thus, in the terminology often used in relation to Witt vectors, \( L^k(V), L^k(V^p), L^k(V^{p^2}), \ldots \) are the ‘ghost’ components of \( (B_k, B_{p k}, B_{p^2 k}, \ldots) \). However, we have not yet been able to deduce from this more explicit information on the modules \( B_{p^m k} \).

Of particular interest is the case where \( V \) is the natural module for \( \text{GL}(n,F) \). Then, since the modules \( B_k, B_{p k}, \ldots \) and \( L^k(V), L^k(V^p), \ldots \) are direct summands of tensor powers of \( V \), they are determined, up to isomorphism, by their formal characters. We may therefore translate equations (4.5) into equations in symmetric functions. Closely related equations have been considered by Reutenauer [14] and Scharf and Thibon [15].

Statement (4.4) in the Factorisation Theorem is analogous to the property of Adams operations given by the second part of Lemma 3.2. The Factorisation Theorem and Lemma 3.4 reduce the study of Lie resolvents to the study of Adams operations and \( p \)-power Lie resolvents.

5. Symmetric powers and Lie powers revisited

Let \( F \) be a field and \( G \) a group. For each positive integer \( r \) we may define a function \( \rho^r : \Gamma_{FG} \to \Gamma_{FG} \) by

\[
\rho^r = \frac{1}{r} \sum_{d|r} \Phi^d \circ \psi^{r/d}.
\]  

(5.1)

Thus, since \( \psi^1 \) is the identity map, the Lie resolvents \( \Phi^r \) may be expressed recursively in terms of the functions \( \rho^r \) and the Adams operations. Hence, if we assume
the Adams operations, knowledge of the functions $\rho^r$ is equivalent to knowledge of
the Lie resolvents.

**Theorem 5.1**  Let $G$ be a group and $F$ a field of prime characteristic $p$. Then
$\rho^r = 0$ unless $r$ is a power of $p$.

**Proof:** Write $r = p^m k$ where $m \geq 0$ and $k$ is not divisible by $p$. By (5.1) and
the Factorisation Theorem,

$$\rho^r = \frac{1}{r} \sum_{d|r} \Phi^d \circ \psi^{r/d}$$

$$= \frac{1}{r} \sum_{0 \leq i \leq m} \sum_{e|k} \Phi^{p^i e} \circ \psi^{p^m - i k/e}$$

$$= \frac{1}{r} \sum_{0 \leq i \leq m} \sum_{e|k} \Phi^{p^i} \circ \Phi^e \circ \psi^{p^m - i k/e}.$$  

Hence, by Lemmas 3.4 and 3.2,

$$\rho^r = \frac{1}{r} \sum_{0 \leq i \leq m} \sum_{e|k} \mu(e) (\Phi^{p^i} \circ \psi^{p^m - i k/e})$$

$$= \frac{1}{r} \sum_{0 \leq i \leq m} \sum_{e|k} \mu(e) (\Phi^{p^i} \circ \psi^{p^m - i k}).$$

If $r$ is not a power of $p$ then $k > 1$ and so $\sum_{e|k} \mu(e) = 0$, which gives $\rho^r = 0$.  □

**Lemma 5.2**  We have $L^* \circ S^* = L^+ \circ S^+$ and, for all $A \in \Gamma_{FG},$

$$(L^* \circ S^*)(At) = \rho^1(A)t + \rho^2(A)t^2 + \cdots.$$  

**Proof:** This result is essentially the same as [2, Lemma 4.1], but, for complete-
ness, we give a proof.

By (3.3) and (3.8), $L^* \circ S^* = L^+ \circ S^+$. Also, by (3.4) and Lemma 3.5,

$$L^+(S^+(At)) = L^+(\psi^1(A)t + \frac{1}{2}\psi^2(A)t^2 + \cdots)$$

$$= \sum_{r \geq 1} \left( \sum_{d|r} \frac{1}{d} \Phi^d((d/r)\psi^{r/d}(A)) \right) t^r.$$
Hence, by the linearity of $\Phi^d$ and (5.1),

$$(L^* \circ S^*)(At) = (L^+ \circ S^+)(At) = \sum_{r \geq 1} \rho^r(A)t^r.$$  

□

Recall that $\Pi = \Gamma_{FG}[[t]]$. An element of $t\Pi$ will be called $p$-typical if, for every positive integer $r$, the coefficient of $t^r$ is zero unless $r$ is a power of $p$.

**Theorem 5.3**  
Let $G$ be a group and $F$ a field of prime characteristic $p$. Then $(L^* \circ S^*)(f)$ is $p$-typical for every $p$-typical element $f$ of $t\Pi$.

**Proof:**  
By Lemma 5.2, $L^* \circ S^* = L^+ \circ S^+$. By Theorem 5.1 and Lemma 5.2, $(L^+ \circ S^+)(At)$ is $p$-typical for all $A \in \Gamma_{FG}$. By Lemmas 3.1 and 3.3, $L^+ \circ S^+$ has the properties of $S^+$ stated in Lemma 3.1. It follows that $(L^+ \circ S^+)(f)$ is $p$-typical for every $p$-typical element $f$ of $t\Pi$. □

The functions $\rho^r$ seem to have some importance in the study of Lie powers. As we have already seen, they may be used instead of the Lie resolvents, but, on the available evidence, their properties seem to be smoother. Theorem 5.1 shows that $\rho^r = 0$ unless $r$ is a power of $p$. It is remarkable also that, in the cases which have so far been calculated, even the functions $\rho^1, \rho^p, \rho^{p^2}, \ldots$ are well behaved. Of course, $\rho^1$ is the identity function. It follows from [2, Corollary 4.5 and Lemma 4.6] that if $G$ is cyclic of order $p$ (and $F$ has characteristic $p$) then $\rho^{p^m} = 0$ for all $m > 1$. In fact, from the results in [2] and [3] it is not difficult to deduce the same fact for every finite group $G$ such that $p^2 \nmid |G|$. It is interesting to speculate on how far this generalises to other groups: perhaps, if $G$ is finite, we have $\rho^{p^m} = 0$ for all sufficiently large $m$.

We have used the Factorisation Theorem in the proof of Theorem 5.1 to obtain the fact that $\rho^r = 0$ unless $r$ is a power of $p$. However, by [2, Lemma 5.1 (ii)], this fact itself yields the Factorisation Theorem: hence it is ‘equivalent’ to the Factorisation Theorem.
We conclude with a few remarks about the easier case where \( F \) has characteristic 0. In this case, by Lemma 3.4, \( \Phi^r = \mu(r)\psi^r \) for all \( r \). By a simplified version of the calculation of Theorem 5.1 we obtain \( \rho^r = 0 \) for all \( r > 1 \). Thus, by Lemma 5.2, \((L^+ \circ S^+)(At) = At\) for all \( A \in \Gamma_{FG}\). By the properties of \( S^+ \) and \( L^+ \) it follows that \( L^+ \circ S^+ \) is the identity function on \( t\Pi \). A similar calculation shows that \( S^+ \circ L^+ \) is the identity function. However, \( L^* \circ S^* = L^+ \circ S^+ \) and

\[
S^* \circ L^* = \exp \circ S^+ \circ L^+ \circ \log.
\]

Thus we have the following result, closely related to results of Joyal [12] and Reutenauer [14].

**Theorem 5.4** Let \( G \) be a group and \( F \) a field of characteristic 0. Then \( L^* \circ S^* \) is the identity function on \( t\Pi \) and \( S^* \circ L^* \) is the identity function on \( 1 + t\Pi \).

Recall that, by (3.7), \( L^*(f) = -L(1 - f) \) for all \( f \in 1 + t\Pi \). Since \( S^* \circ L^* \) is the identity function in characteristic 0, it follows easily that

\[
S^*(L(g)) = (1 - g)^{-1} = 1 + g + g^2 + \cdots
\]

for all \( g \in t\Pi \). This may be regarded as a version of the Poincaré–Birkhoff–Witt Theorem (by taking \( g = Vt \)) and it is similar to a result of Joyal [12, Chapter 4, Proposition 1] (see also [14, Lemma 3.2]). Similarly, the fact that \( L^* \circ S^* \) is the identity function is a version of a result of Reutenauer [14, Theorem 3.1].

**References**

1. R. M. Bryant, “Free Lie algebras and Adams operations”, *J. London Math. Soc. (2)* 68 (2003), 355–370.
2. R. M. Bryant, “Modular Lie representations of groups of prime order”, *Math. Z.* 246 (2004), 603–617.
3. R. M. Bryant, “Modular Lie representations of finite groups”, *J. Austral. Math. Soc.* 77 (2004), 401–423.
4. R. M. Bryant and M. Schocker, “The decomposition of Lie powers”, preprint arXiv:math.RT/0505325.

5. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley–Interscience, New York, 1962.

6. S. Donkin and K. Erdmann, “Tilting modules, symmetric functions, and the module structure of the free Lie algebra”, J. Algebra 203 (1998), 69–90.

7. K. Erdmann and M. Schocker, “Modular Lie powers and the Solomon descent algebra”, preprint arXiv:math.RT/0408211.

8. J. A. Green, Polynomial Representations of $GL_n$, Lecture Notes in Mathematics 830, Springer, Berlin, 1980.

9. S. Guilfoyle, On Lie Powers of Modules for Cyclic Groups Ph. D. thesis, Manchester, 2000.

10. S. Guilfoyle and R. Stöhr, “Invariant bases for free Lie algebras”, J. Algebra 204 (1998), 337–346.

11. M. Hazewinkel, Formal Groups and Applications, Academic Press, New York, 1978.

12. A. Joyal, “Foncteurs analytiques et espèces de structures”, in Combinatoire Énumérative, edited by G. Labelle and P. Leroux, Lecture Notes in Mathematics 1234, Springer, Berlin, 1986, pp. 126–159.

13. C. Reutenauer, Free Lie Algebras, Clarendon Press, Oxford, 1993.

14. C. Reutenauer, “On symmetric functions related to Witt vectors and the free Lie algebra”, Adv. Math. 110 (1995), 234–246.

15. T. Scharf and J.-Y. Thibon, “On Witt vectors and symmetric functions”, Algebra Colloq. 3 (1996), 231–238.

16. J.-P. Serre, Local Fields, Springer, New York, 1979.

17. R. Stöhr, “Restricted Lazard elimination and modular Lie powers”, J. Austral. Math. Soc. 71 (2001), 259–277.