Klein–Gordon particles in Gödel-type Som-Raychaudhuri cosmic string spacetime and the phenomenon of spacetime associated degeneracies

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Abstract

We argue that only exact, comprehensive, and explicit solutions for the fundamental quantum mechanical models (i.e., the Klein–Gordon (KG) oscillators and the KG-Coulomb) would help to understand and describe the effects of gravitational fields on the dynamics of such systems. In the current methodical proposal, the effects of the gravitational fields generated by a Gödel-type Som-Raychaudhuri (SR) cosmic string spacetime on KG-oscillators (KG-particles in general) are studied and reported. In so doing, we revisit the KG-oscillators in a topologically trivial Gödel-type spacetime background and use textbook procedures to report its exact solution that covers even and odd parities. Next, we discuss the drawbacks associated with the power series expansion approach that implies the biconfluent Heun functions/polynomials solution. We, therefore, recollect the so called pseudo perturbative shifted $\ell$ expansion technique (PSLET) as an alternative and more sophisticated method/technique. Illustrative examples are used: (i) a KG-oscillator in a topologically trivial Gödel-type spacetime, (ii) a quasi-free KG-oscillator in Gödel SR-type cosmic string spacetime, (iii) a KG-Coulombic particle in Gödel SR-type cosmic string spacetime at zero vorticity, and (iv) a massless KG-particle in Gödel SR-type cosmic string spacetime in a Cornell-type Lorentz scalar potential. The corresponding exact energies are obtained from the zeroth (leading) order correction of PSLET, where all higher order correction identically vanish. The comprehensive exactness of the reported solutions manifestly suggest degeneracies associated with spacetime (STAD) phenomenon.

1. Introduction

Topological defects in spacetime have stimulated intensive research studies in quantum gravity. Amongst are, domain walls [1, 2], cosmic string [3, 4], global monopole [5] and textures [6]. The energy levels of relativistic/non-relativistic quantum particles are shown to be affected by the gravitational fields generated by different spacetime backgrounds with such topological defects, not only in general relativity but also in the geometrical theory of defects in condensed matter physics. In general relativity, for example, the Gödel spacetime metric [7], with an embedded cosmic string, introduces itself as the first cosmological solution to Einstein’s equation with rotating matter. Its compact form allowed analytical research studies of many physical and mathematical systems in gravitational backgrounds with rotation and causality violation. It has been shown that all spacetime homogeneous (ST-homogeneous) Gödel-type metrics characterized by the vorticity $\Omega$ of spacetime (a real number that is related to rotation of the material, i.e., $\Omega = \pm |\Omega|$ [8]) and a given value of the parameter $\tilde{\mu}; -\infty \leq \tilde{\mu}^2 \leq \infty$, can be transformed in cylindrical coordinates [8–18] to

$$ds^2 = -\left(dt + \frac{\Omega^2 \sinh^2(\tilde{\mu} r)}{\tilde{\mu}^2} d\varphi\right)^2 + \frac{\sinh^2(2 \tilde{\mu} r)}{4 \tilde{\mu}^2} d\varphi^2 + dr^2 + dz^2. \quad (1)$$

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Here, the disclination parameter $\alpha$ (curvature parameter) admits the values $0 < \alpha < 1$ in general relativity (for cosmic strings with positive curvature), $\alpha > 1$ in the geometric theory of defects in condensed matter (for a negative curvature), and $\Omega = 0$ and $\alpha = 1$ corresponds to Minkowski flat spacetime metric. Moreover, for $\tilde{\mu}^2 < 0$ there is an infinite number of successive causal and noncausal regions, and for $0 \leq \tilde{\mu}^2 < \Omega^2$ there is one noncausal region for a given $r > r_c$. Where, $r_c$ is the critical radius determined by $r_c = \frac{1}{\mu} \arctanh \left( \frac{\tilde{\mu}}{\sqrt{\Omega^2}} \right)$.

For $\tilde{\mu}^2 = \Omega^2$ the Gödel spacetime metric (the first exact Gödel solution of the Einstein equation describing a complete causal, ST-homogeneous rotating universe) is recovered, and for $\tilde{\mu} < \Omega$ there is a causal region (classified as chronologically safe) with radius $r_c$ surrounded by the outer noncausal space. Nevertheless, at the limit $\tilde{\mu} \rightarrow 0$ of the Gödel spacetime metric equation, we obtain the ST-homogeneous Som-Raychaudhuri (SR) solution

$$ds^2 = -(dt + \alpha \Omega r^2 d\varphi)^2 + \alpha^2 r^2 d\varphi^2 + dr^2 + dz^2$$

of the Einstein field equations, in the presence of cosmic string. Moreover, the covariant and contravariant metric tensors associated with the Som-Raychaudhuri spacetime are, respectively, given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & -\alpha \Omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha \Omega r^2 & 0 & \alpha^2 r^2 (1 - \Omega^2 r^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$g^{\mu\nu} = \begin{pmatrix} (\Omega^2 r^2 - 1) & 0 & -\frac{\Omega}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\Omega}{\alpha} & 0 & \frac{1}{\alpha^2 r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \det(g) = -\alpha^2 r^2.$$

The objective of the current methodical proposal is to explore in a comprehensive manner the effect of the gravitational field generated by the Gödel-type cosmic string SR-spacetime equation taking into account $E = \pm |E|$, $\Omega = \pm |\Omega|$, and including nodeless states. In so doing, nevertheless, it is unavoidably inviting to reconsider and report on the KG-oscillators in a topologically trivial Gödel del-type spacetime metric, used by Ahmed, where

$$ds^2 = -dt^2 + dx^2 + (1 - \alpha^2 x^2) dy^2 + 2\alpha x dt dy + dz^2,$$

where $H(x) = \alpha x$ and $D(x) = 1$ in the general Gödel metric.

The organization of our methodical proposal is in order. We start with the KG-oscillators in a topologically trivial Gödel-type spacetime, equation (4), considered by Ahmed, who has reported only the positive energy states (without the ground state) and missed all the negative energy states. In section 2, we give this problem a proper textbook quantum mechanical treatment and report the correct spectra that include all positive and negative energy states. In section 3, we discuss KG-particles in Gödel SR-type cosmic string spacetime in a Cornell-type Lorentz scalar potentials. Therein, we show the disadvantages of using a power series (Frobenius method) expansion method that leads to biconfluent Heun functions/polynomials. In section 4, we recollect (as a powerful alternative method) the so-called pseudo-perturbative shifted-\$C^\infty$ expansion technique (PSLET), to solve for Schrödinger equation (a common form that relativistic wave equations collapse into) in D-dimensions, to be used in the current methodical proposal. PSLET has shown, with a brut force evidence through comparison with numerical integration techniques, to provide very highly accurate results (even with its non-Hermitian $PT$-symmetric version by Mustafa and Znojil, using imaginary cubic oscillator with spikes as a model). It has been applied to quartic anharmonic oscillators, to harmonic oscillator plus Coulomb, to 2D-quantum dots, to D-dimensional spiked harmonic oscillator, etc. We use some illustrative examples, in section 5, that include a KG-oscillator (of section 2) in a topologically trivial Gödel-type...
spacetime equation (4), a quasi-free KG-oscillator in Gödel SR-type cosmic string spacetime equation (2), a KG-Coulombic particle in Gödel SR-type cosmic string spacetime equation (2) at zero vorticity (i.e., $\Omega = 0$), and a massless KG-particles in Gödel SR-type cosmic string spacetime in a Cornell-type Lorentz scalar potential [38] (used in quarkonium mass spectroscopy). We conclude in section 6.

2. KG-oscillators in a topologically trivial Gödel-type spacetime background

2.1. KG-oscillator I as a topologically trivial Gödel-type spacetime byproduct

In this section we recollect the topologically trivial Gödel-type spacetime metric equation (4), along with the metric tensor elements in equation (5) and $\det (g) = -1$, used by Ahmed [15]. Then, a KG-particle of rest mass energy $m$ (denoting $mc^2$ in $c = h = 1$ units) in such a Gödel-type spacetime is described by the KG-equation

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \psi) = m^2 \psi.$$  \hspace{1cm} (6)

Consequently, with

$$\psi(t, x, y, z) = \exp (i [k_{xy} y + k_{xz} z - E t]) \phi(x),$$  \hspace{1cm} (7)

it yields

$$\phi''(x) - (\gamma^2 x^2 + 2 \gamma k_{xy} x) \phi(x) = \tilde{\beta} \phi(x); \quad \tilde{\beta} = m^2 + k_{xy}^2 + k_{xz}^2 - E^2, \quad \gamma^2 = \alpha^2 E^2.$$ \hspace{1cm} (8)

At this point, one should notice that $\gamma = \pm \alpha |E|$ as mandated by the very nature of the Gödel-type spacetime in equation (4). Obviously, this equation resembles the well known one-dimensional Schrödinger equation for the so called shifted-oscillator (but does not inherit its parametric characterizations) that can be treated, in a straightforward textbook manner, by rewriting equation (8) as

$$\phi''(u) - \tilde{\gamma}^2 u^2 \phi(u) + \lambda \phi(u) = 0; \quad \lambda = - \frac{\tilde{\beta}}{\tilde{\gamma}^2}, \quad \tilde{\gamma}^2 = \frac{1}{\gamma^2}.$$ \hspace{1cm} (9)

One would then use the change of variable $u = \gamma x + k_{xy}$ to obtain

$$\phi''(u) - \tilde{\gamma}^2 u^2 \phi(u) + \lambda \phi(u) = 0; \quad \lambda = - \frac{\tilde{\beta}}{\tilde{\gamma}^2}, \quad \tilde{\gamma}^2 = \frac{1}{\gamma^2}.$$ \hspace{1cm} (10)

This is the one-dimensional Schrödinger-like harmonic oscillator form that admits exact textbook solution with the eigenvalues and eigenfunctions, respectively, given by

$$\lambda_n = |\gamma| (2n + 1), \quad \phi(u) = N_n e^{-|\gamma|^2 u^2} H_n(|\gamma| u); \quad n = 0, 1, 2, \ldots,$$ \hspace{1cm} (11)

where $H_n(|\gamma| u)$ are the Hermite polynomials and $N_n$ are the normalization constants. The finiteness and square integrability of the wave function enforces the quantum mechanical condition that $|\gamma| = \pm \tilde{\gamma}_n \geq 0$ (i.e., $\tilde{\gamma} = \tilde{\gamma}_n; \quad \tilde{\gamma}_n = 1/\alpha E, \quad \tilde{\gamma}_n = 1/\alpha E \implies |\gamma| = \pm \tilde{\gamma}_n$, where $E_{\pm} = \pm |E|$). Consequently, with the parametric substitutions above equation (8), one obtains

$$\frac{m^2 + k_{xy}^2 - E_{\pm}^2}{\alpha^2 E_{\pm}^2} = - |\gamma| (2n + 1) \Rightarrow E_{\pm}^2 = \alpha E_{\pm} (2n + 1) - (m^2 + k_{xy}^2) = 0; \quad n = 0, 1, 2, \ldots.$$ \hspace{1cm} (12)

Hence,

$$E_n = E_{\pm} = \pm \alpha \left( n + \frac{1}{2} \right) \pm \sqrt{\frac{\alpha^2 \left( n + \frac{1}{2} \right)^2}{\alpha^2 E_{\pm}^2} + m^2 + k_{xy}^2}.$$ \hspace{1cm} (13)

Which in turn implies

$$E_+ = + \alpha \left( n + \frac{1}{2} \right) + \sqrt{\frac{\alpha^2 \left( n + \frac{1}{2} \right)^2}{\alpha^2 E_{\pm}^2} + m^2 + k_{xy}^2}$$ \hspace{1cm} (14)

as the positive energy solution, and

$$E_- = - \alpha \left( n + \frac{1}{2} \right) - \sqrt{\frac{\alpha^2 \left( n + \frac{1}{2} \right)^2}{\alpha^2 E_{\pm}^2} + m^2 + k_{xy}^2}$$ \hspace{1cm} (15)

for the negative energy solution. Notably, unlike the solution reported by Ahmed [15] (see equation (58) in [15]), the textbook solution above includes both even and odd parities, positive and negative energies, as well as the ground state [25].
2.2. KG-oscillator II as a topologically trivial Gödel-type spacetime and momentum operator deformation byproduct

Using similar recipe as that of Moshinsky and Szczepaniak [39] and Mirza and Mohadesi [40] for the Dirac and KG oscillators, we redefine the momentum operator as

\[ p_\mu \longrightarrow p_\mu + \eta \chi_\mu, \quad \chi_\mu = (0, x, 0, 0), \]

where the dimensions of the second term suits the dimensions of the momentum operator \( p_\mu \) and here we consider \( \eta \) to have positive and/or negative values. Then the KG-equation reads

\[ \frac{1}{\sqrt{g}} \left( \partial_\mu + \eta \chi_\mu \right) \sqrt{-g} g^\mu\nu \partial_\nu \Psi = m^2 \Psi. \]

At this point, one may use \( \eta = m \omega \geq 0 \) (with \( m \) denoting rest mass of the KG-particle) to recover the traditionally used values as in \( [15, 22–24, 39, 40] \) and other related references cited therein. However, we shall use a general parameter \( \eta \) and allow it to take positive and/or negative values. Yet, in this case we avoid eminent confusion and inconsistency between \( m^2 c^2 = m m c^2 \), rest mass multiplied by rest mass energy) on the R.H.S. and the rest mass of the particle on the L.H.S. of the KG-equation equation (17) for \( \eta = m \omega \). This point is made clear by Moshinsky and Szczepaniak [39] and Mirza and Mohadesi [40] while dealing with the Dirac and KG oscillators. We therefore stick with our assumption and use the spacetime metric tensor elements in equation (5), to recast equation (17) as

\[ \phi(x) - (\omega^2 x^2 + 2 \gamma k_j x) \phi(x) = \zeta \phi(x); \quad \zeta = m^2 + k_j^2 + k_j^2 - E + \eta, \quad \omega^2 = \gamma^2 + n^2, \quad \gamma = \alpha E. \]

Hereby, one should be aware that \( n^2 x^2 \) would take the form of \( \eta^2 x^2 = m (m \alpha^2 x^2) \) as in [39, 40] when the speed of light is kept intact in Dirac and/or KG equations (i.e., consistent with \( m^2 c^2 = m m c^2 \)) on the R.H.S. of equation (18). The source of confusion is therefore clear and should be avoided. Moreover, equation equation (18) admits solution in the form of hypergeometric functions [25] given by

\[ \phi(x) = e^{-\frac{x^2}{4|\omega|^2}} \left[ A F \left( \frac{3}{4}, 1, \frac{(\omega^2 x + \delta)^2}{|\omega|^2} \right) + B (\omega^2 x + \delta) \right] F \left( \frac{3}{4}, 1 - \frac{\delta^2}{4|\omega|^2} \right), \]

where \( \delta = \gamma k_j = \alpha E k_j, |\omega| \geq 0 \) to secure finiteness and square integrability of the wave function, and

\[ \tilde{b} = \frac{3}{4} + \frac{\delta^2}{4|\omega|^2} + \frac{\zeta}{4|\omega|^2} = \tilde{a} + \frac{1}{2}. \]

Again, the condition for the hypergeometric functions to become polynomials of degree \( n \geq 0 \) suggests that \( a = -n \) and \( b = a + 1/2 = -n \). This would imply that

\[ \tilde{a} = \frac{1}{4} - \frac{\delta^2}{4|\omega|^2} + \frac{\zeta}{4|\omega|^2} = -n \implies \zeta_{2n} = -|\omega|(4n + 1) + \frac{\delta^2}{\omega^2}, \]

for even parity, and

\[ \tilde{b} = -n \implies \zeta_{2n+1} = -|\omega|(4n + 3) + \frac{\delta^2}{\omega^2}. \]

for odd parity solutions. The two results equation (21) and equation (22) would reduce to a common form

\[ \zeta_n = -|\omega|(2n + 1) + \frac{\delta^2}{\omega^2}, \quad |\omega| = \pm \alpha E \pm \sqrt{1 + \frac{\eta^2}{\alpha^2 E^2}} \geq 0, \]

to imply that

\[ E_\pm^2 - (m^2 + k_j^2 + k_j^2 + \eta) = \pm \alpha E \pm \sqrt{1 + \frac{\eta^2}{\alpha^2 E^2}} (2n + 1) - \frac{\alpha^2 E^2 k_j^2}{\alpha^2 E^2 + \eta^2}. \]

Obviously, this result collapses into that in equation (13) for \( \eta = 0 \), where the effect of even and odd parities is reflected on the related spectrum.

3. KG-particles in a 4-vector and scalar Lorentz potentials in Gödel SR-type cosmic string spacetime background

The KG-equation for a spin-0 particle in a 4-vector \( A_\mu \) and a radial scalar \( S(r) \) potentials in the Gödel SR-type spacetime background equation (2) is given by
\[
\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu \nu} D_\nu \Psi) = (m + S(r))^2 \Psi, \tag{25}
\]

where the gauge-covariant derivative is given by \( D_\mu = \partial_\mu - ieA_\mu \). This would in turn allow us to cast equation equation (25) as

\[
\left\{ -D_t^2 + \frac{1}{r} D_t (r D_t) + \left( \Omega t D_t - \frac{1}{\alpha r} D_r \right)^2 + D_r^2 - (m + S(r))^2 \right\} \Psi = 0. \tag{26}
\]

At this point, we may now define the corresponding gauge-covariant derivatives so that

\[
D_t = \partial_t - ieA_t = \partial_t - iV(r), \quad D_r = \partial_r, \quad D_\theta = \partial_\theta - ieA_\theta, \quad D_\phi = \partial_\phi,
\]

where, \( V(r) = eA_t \) is the Lorentz 4-vector potential (i.e., transforms within the 4-vector potential \( A_\mu \)), \( S(r) \) is the Lorentz scalar potential (i.e., transforms like the rest mass energy \( m \rightarrow m + S(r) \)), and \( eA_\phi \) may include both magnetic and Aharonov–Bohm flux fields effects \([41, 42]\). We now use the assumption that

\[
\Psi(t, r, \varphi, z) = \exp(i[\varphi \varphi + k_z z - Et]) \psi(r) = \exp(i[\varphi \varphi + k_z z - Et]) \frac{R(r)}{\sqrt{r}}, \tag{28}
\]

in equation (26) to obtain

\[
\left\{ \partial_t^2 + \frac{1}{4r^2} + (E + V(r))^2 - \left[ \Omega t (E + V(r)) + \left( \frac{\varphi}{\alpha r} - eA_\phi \right) \right]^2 - k_z^2 - (m + S(r))^2 \right\} R(r) = 0, \tag{29}
\]

This result would describe KG-particles in Gödel SR-type cosmic string spacetime background.

In what follows, we shall be interested in a set of KG-particles in Gödel SR-type cosmic string spacetime under the influence of \( A_\phi = 0 \) (i.e., no magnetic or Aharonov–Bohm flux fields effects), and no Lorentz 4-vector potential (i.e., \( V(r) = 0 \)). Under such settings, the KG-equation equation (29) reduces to

\[
\left\{ \partial_t^2 - \left( \frac{\varphi^2 - 1/4}{r^2} \right) - \Omega^2 E^2 r^2 - 2ms(r) - S(r)^2 + \Lambda \right\} R(r) = 0; \quad \varphi = \frac{\varphi}{\alpha}, \quad \Lambda = E^2 - 2 \varphi \Omega E - k_z^2 - m^2. \tag{30}
\]

Obviously, this equation resembles the one-dimensional form of the two-dimensional radial Schrödinger equation. Which, with a Cornell-type Lorentz scalar potential \( S(r) = ar + b/r \) would result

\[
\left\{ \partial_t^2 - \left( \frac{\varphi^2 - 1/4}{r^2} \right) - \bar{\Omega}^2 r^2 - 2m\varphi r + \Lambda \right\} R(r) = 0, \tag{31}
\]

where

\[
\bar{\Omega}^2 = \Omega^2 E^2 + a^2, \quad \varphi = \hat{\varphi}^2 + b^2, \quad \Lambda = E^2 - 2 \varphi \Omega E - (k_z^2 + m^2 + 2ab). \tag{32}
\]

equation equation (31) is known to admit, using power series method, a solution in the form of the biconfluent Heun functions

\[
R(r) = C r^{i|\varphi|+1/2} \exp\left( -\frac{|\varphi|}{2} r^2 - \frac{am}{|\Omega|} r \right) H_0(\alpha, \beta, \gamma, \delta, \frac{\varphi}{\Omega}, r), \tag{33}
\]

where \( \alpha = 2|\varphi|, \beta = \frac{2am}{|\Omega|^2}, \gamma = \frac{am + M\varphi^2}{|\Omega|^2}, \) and \( \delta = \frac{4mb}{|\Omega|^2} \). For the case \( a = 0 = b \), this solution implies

\[
R(r) = C r^{i|\varphi|+1/2} \exp\left( -\frac{|\Omega|}{2} r^2 \right) H_0\left( 2|\varphi|, 0, \frac{\Lambda}{|\Omega|}, 0, \sqrt{|\Omega|} r \right)
= C r^{i|\varphi|+1/2} \exp\left( -\frac{|\Omega|}{2} r^2 \right) F_2\left( \left[ 1 + \frac{\alpha}{4}, \frac{\gamma}{4} \right], \left[ 1 + \frac{\alpha}{2}, \frac{\gamma}{2} \right], |\Omega| r^2 \right), \tag{34}
\]

where the condition that

\[
1 + \frac{\alpha}{4} - \frac{\gamma}{4} = 1 + \frac{|\varphi|}{2} - \frac{\Lambda}{4|\Omega|}, \quad n_\varphi \tag{35}
\]

is required for the confluent hypergeometric function to become a polynomial of degree \( n_\varphi \geq 0 \) and to secure the finiteness and square integrability of the wave function. This would immediately result, in terms of the associated Laguerre polynomials \( L_n^{(\alpha)}(z^2) \), that

\[
\Lambda = 2|\Omega|E(2n_\varphi + |\varphi| + 1), \quad \text{and} \quad R(r) = C r^{i|\varphi|+1/2} \exp\left( -\frac{|\Omega|}{2} r^2 \right) L_n^{(\alpha)}\left( |\Omega| r^2 \right) \tag{36}
\]
as the exact eigen energies and eigen functions for the two dimensional Schrödinger harmonic oscillator. Moreover, the same condition equation (35) should also be imposed on the biconfluent Heun function equation (33) so that a biconfluent Heun polynomial [43] of degree \( n = 2n_r \geq 0 \) is obtained (c.f., e.g., [23, 41, 44, 45]) to imply

\[
\Lambda = 2|\Omega E| \sqrt{1 + \frac{a^2}{\Omega^2 E^2}} (2n_r + |\mathcal{L}| + 1),
\]

and

\[
E^2 - 2 \ell \Omega E - (k_0^2 + m^2 + 2ab) = 2|\Omega E| \sqrt{1 + \frac{a^2}{\Omega^2 E^2}} (2n_r + |\mathcal{L}| + 1).
\]

(38)

Where the detailed analysis of such a quadratic equation are given in section 5-B, taking into account that

\[
|\Omega E| = \left\{ \frac{\Omega E_+}{-\Omega E_+} \right\} \geq 0; \quad E_\pm = \pm|E|, \quad \Omega_\pm = \pm|\Omega|,
\]

(39)

At this point, the truncation order \( n \) of a power series (biconfluent Heun function here) does not make it a valid quantum number, but rather it should be correlated to the well known quantum numbers as done above (i.e., \( n = 2n_r \geq 0 \)). Hereby, one should give credentials for Neto and co-workers [23] as they resemble a group of the very few who have correctly used condition equation (35) on the biconfluent Heun polynomials. Only under such condition that the result of equation (38) would recover those of equation (36) for \( a = 0 \neq b \). Obviously, however, for the case \( a = 0 = \Omega \) the solution in equation (38) collapses into a free relativistic particle eigenvalues

\[
E^2 - (k_0^2 + m^2) = 0,
\]

(40)

although we have an effective two-dimensional Coulomb problem (i.e., a KG-Coulombic particle). This is a clear drawback of the Heun power series (Frobenius method) expansion approach. Moreover, if condition equation (35) is not implemented then the biconfluent Heun polynomial solution would fail to address the exact solution for the KG-oscillator for \( a = 0 \neq b \) (the reader is advised to see the detailed discussion on this issue in appendix B below). One should, therefore, search for an alternative method to deal with such a case in equation (31).

As a more reliable alternative method, to the power series (Frobenius method) expansion, we recollect the so called pseudo-perturbative shifted-\( \ell \) expansion technique (PSLET) for the \( D \)-dimensional radial Schrödinger-type equation and solve for different KG-particles’ settings in the Gödel SR-type cosmic string spacetime backgrounds of equation (31). In order to make our proposal self contained, we, in short, discuss PSLET in the following section. We shall also use Ahmed’s model equation (10) [15] as one of the illustrative examples to be used in the current methodical proposal.

4. PSLET for the \( D \)-dimensional Schrödinger-type equation

Let us recollect that the \( D \)-dimensional radial Schrödinger equation (\( \hbar = 2m = 1 \) units are used in this section, e.g., see the sample of references [27, 28, 29, 30, 31] and related references cited therein) reads

\[
\left\{ -\partial^2_r + \frac{\ell_D (\ell_D + 1)}{r^2} + V(r) \right\} \Phi_{k,\ell}(r) = E_{k,\ell} \Phi_{k,\ell}(r),
\]

(41)

where \( \ell_D = \ell + (D - 3)/2 \) to incorporate interdimensional degeneracies associated with the isomorphism between angular momentum and the dimensionality \( D \). Hereby, one should notice that for \( D = 1, \ell_1 = \ell - 1 \) and hence \( \ell_D (\ell_D + 1) = \ell (\ell - 1) = 0 \) for \( \ell = 0 \) and \( \ell = 1 \) to denote even and odd parity, respectively, for the one-dimensional case with \( r = x \in (-\infty, \infty) \). In the two dimensional case with radially cylindrical symmetry, \( r = \sqrt{x^2 + y^2} \in (0, \infty) \) and \( \ell_2 = |\ell| - 1/2 \) with \( \ell \) denoting the magnetic quantum number. Moreover, in the three-dimensional radially spherical symmetry, \( r = \sqrt{x^2 + y^2 + z^2} \in (0, \infty) \) and \( \ell_3 = \ell \) where \( \ell \) denotes angular momentum quantum number. Since we are going to use \( 1/\ell_D \) as a small perturbation expansion parameter at large-\( \ell_D \) limit (i.e., pseudo-classical limit, e.g., [27, 28, 29, 30, 31–33]) we avoid the trivial case for \( \ell_D = 0 \) (for \( D = 3 \) and \( \ell = 0 \)) and use a shift \( \beta \) so that \( \ell^\prime = \ell_D - \beta \). Under such settings, our Schrödinger equation in equation (41) reads

\[
\left\{ -\partial^2_r + \frac{\ell^\prime_2 + (2\beta + 1) + \beta (\beta + 1)}{r^2} + V(r) \right\} \Phi_{k,\ell}(r) = E_{k,\ell} \Phi_{k,\ell}(r).
\]

(42)
For convenience, we now shift the origin of the coordinate system and use
\[ x = \frac{\hat{x}^{1/2}}{\kappa_0} (r - \eta), \]
and expand about \( x = 0 \) to obtain
\[ \left( \frac{x}{\hat{x}^{1/2}} + 1 \right)^{\frac{3}{2}} = 1 - \frac{2x}{\hat{x}^{1/2}} + \frac{3x^2}{\hat{x}^{3/2}} + \frac{4x^3}{\hat{x}^{5/2}} + \cdots, \]
\[ V(x(r)) = \frac{\hat{x}^{1/2}}{Q} \left[ V(\eta) + V'(\eta) \frac{r_0 x}{\hat{x}^{1/2}} + V''(\eta) \frac{r_0^2 x^2}{2\hat{x}} + V'''(\eta) \frac{r_0^3 x^3}{6\hat{x}^{3/2}} + \cdots \right]. \]

Moreover, let us expand the energy so that
\[ E_k = \frac{\hat{x}^{1/2}}{Q} \left[ E_0 + E_1 + E_2 + E_3 + \cdots \right], \]
where \( \hat{x}^{1/2} \) is a constant parameter that scales the potential at large-\( \ell_D \) limit and is set equal to \( \frac{\hat{x}^{1/2}}{Q} \) at the end of the calculation. Consequently, we may now re-write equation (42) as
\[ \left\{- \partial_x^2 + \left( \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} + 2(\beta + 1) + \frac{\beta(\beta + 1)}{\hat{x}^{1/2}} \right) \left( 1 - \frac{2x}{\hat{x}^{1/2}} + \frac{3x^2}{\hat{x}^{3/2}} + \frac{4x^3}{\hat{x}^{5/2}} + \cdots \right) \right\} \Phi_k(x(r)) = \xi_k \Phi_k(x(r)) \]
with
\[ \xi_k = \frac{\hat{x}^{1/2}}{Q} \left[ E_0 + E_1 + E_2 + E_3 + \cdots \right]. \]
This equation is to be compared with that of the one-dimensional anharmonic oscillator
\[ \left\{- \partial_x^2 + \frac{1}{4} \omega^2 x^2 + \varepsilon_0 + P(x) \right\} \chi_k(x) = \lambda_k \chi_k(x) \]
with \( P(x) \) is a perturbation given by
\[ P(x) = \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} \left[ \varepsilon_1 x + \varepsilon_3 x^3 \right] + \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} \left[ \varepsilon_2 x^2 + \varepsilon_4 x^4 \right] \\
+ \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} \left[ \varepsilon_5 x^5 + \varepsilon_6 x^6 \right] + \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} \left[ \varepsilon_7 x^7 + \varepsilon_8 x^8 \right]. \]

It is obvious that equation equation (49) represents the one-dimensional Schrödinger anharmonic oscillator that has been readily discussed by Imbo et al\[27, 28\] and by Mustafa and Barakat \[29\] and reported to admit the eigenvalues
\[ \lambda_k = \varepsilon_0 + \left( k + \frac{1}{2} \right) \omega + \frac{\alpha_1}{\hat{x}} + \frac{\alpha_2}{\hat{x}^{3/2}} \]
where \( \alpha_1 \) and \( \alpha_2 \) are given in the Appendix, and \( \left( k + \frac{1}{2} \right) \omega \) are the energies of the unperturbed one-dimensional Schrödinger oscillator with \( k = 0, 1, 2, \cdots \) that denotes the number of nodes in the wave function (for \( D = 2, 3 \) it denotes the radial quantum number or if you wish, the number of nodes in the wave function). A comparison between equation (47) and equation (49) would imply that
\[ \lambda_k = \frac{\dot{\hat{x}}}{\hat{x}^{1/2}} \left[ 1 + \frac{2}{Q} V(\eta) \right] + 2(\beta + 1) + \left( \frac{1}{2} + \frac{1}{2} \right) \omega + \frac{1}{\hat{x}} \left[ \beta(\beta + 1) + \alpha_1 \right] + \frac{\alpha_2}{\hat{x}^{3/2}} + \cdots. \]
We may now simply compare equation (48) with equation (52) to obtain
\[ E_0 = Q \frac{V(\eta)}{r_0}, \quad E_1 = Q \frac{V(\eta)}{r_0} \left( 2(\beta + 1) + \left( \frac{1}{2} + \frac{1}{2} \right) \omega + \frac{1}{\hat{x}} \left[ \beta(\beta + 1) + \alpha_1 \right] \right), \quad E_2 = \frac{Q}{r_0^2} \alpha_2, \]
where \( r_0 \) is chosen to minimize the zeroth order term so that it satisfies the relations
\[ \frac{dE_0}{dr_0} = 0, \quad \frac{d^2E_0}{dr_0^2} > 0 \implies (\ell_D - \beta) = \sqrt{\frac{r_0^3 V(\eta)}{2}}. \]
and the shifting parameter \( \beta \) is determined through the requirement that the first order term vanishes. This would result in
One would therefore recast our anharmonic oscillator energies of equation (46) as

\[ E_k = E_0 + \frac{1}{r_0^2} \left[ \beta (\beta + 1) + \alpha_1 \right] + \frac{\alpha_2}{r_0^2}. \]  

Now we recall the corresponding Schrödinger potential for our KG-particles in the Gödel SR-type cosmic string spacetime with a Cornell-type Lorentz scalar interaction of equation (31). This would result, using equations (56), (55) and (54), that

\[ \omega = 2 \sqrt{\frac{4 \beta^3 r_0^3 + 3 am \beta r_0 - bm}{\Omega^3 r_0^3 + am \beta r_0 - bm}}, \beta = \frac{1}{2} (k + \frac{1}{2}) \omega - \frac{1}{2}, \Omega = \Omega E \]  

and

\[ \ell = (\ell_0 - \beta) = \sqrt{\Omega^3 r_0^3 + am \beta r_0 - bm}; \ell_0 = \ell + (D - 3)/2. \]  

Various especial cases of such settings are given in the following illustrative examples.

5. Illustrative examples

In this section, we use PSLET and consider the following illustrative examples.

5.1. KG-oscillator I as a topologically trivial Gödel-type spacetime byproduct of equation (10)

We hereby recall the KG-oscillator I as a trivial Gödel-type spacetime of section 2 described by equation (10) as

\[ \phi''(u) - \gamma^2 \phi(u) = -\lambda \phi(u); \lambda = -\frac{\beta}{\gamma^2}, \gamma^2 = \frac{1}{\gamma^2}, \mu = \gamma x + k y. \]  

Obviously, the central repulsive/attractive core \( \ell_0 (\ell_0 - 1)/r^2 \) in the D-dimensional Schrödinger equation equation (41) must vanish to resemble the one dimensional form of Schrödinger equation equation (60). In this case, \( D = 1 \Rightarrow \ell_0 (\ell_0 - 1) = \ell (\ell - 1) = 0 \) for \( \ell = 0 \) and \( \ell = 1 \) to denote even and odd parities \((-1)\ell\), respectively. Moreover, \( \mu \equiv r \in (-\infty, \infty) \) and \( u_0 \) is determined by equation (54) to read

\[ u_0 = \sqrt{\left[ \ell^2 / \gamma^2 \right]}; \ell = \ell_0 - \beta = \ell - 1 - \beta. \]  

This would in turn imply that \( \omega = 4 \) by equation (56) and \( \beta = -2k^2 - 3/2 \) by equation (55). Consequently, equations equation (53) would yield the zeroth-order correction

\[ E_0 = 2|\gamma| \left( 2k + \ell + 1/2 \right); \text{ for even parity } \ell = 0 \] 

\[ E_0 = 2|\gamma| \left( 2k + \ell + 1/2 \right); \text{ for odd parity } \ell = 1 \]  

which results in a common form

\[ E_0 = 2|\gamma| \left( n + 1/2 \right); \text{ for } n = 0, 1, 2, \ldots \]  

with \( E_1, E_2, \) and \( E_3 \) identically vanish. Therefore, the energies associated with the shifted Schrödinger oscillator are

\[ \lambda_n = 2|\gamma| \left( n + 1/2 \right); \text{ for } n = 0, 1, 2, \ldots \]  

Consequently, the KG-oscillator I above admits the exact energies reported in equation (13) and goes through the same procedure discussed in section 2-A.

5.2. Quasi-free KG-oscillator in Gödel SR-type cosmic string spacetime equation (2)

Next, we consider a KG-particle in Gödel SR-type cosmic string spacetime background described by equation (29) with the assumption that \( V(r) = S(r) = 0 \) (no interaction potential is involved, hence the notion "quasi-free KG-particle" is introduced). In this case one would rewrite equation (29) as
\[
\left\{ \partial_r^2 - \frac{\left( \frac{\partial^2}{r^2} - 1/4 \right)}{r^2} - \Omega^2 r^2 + \lambda \right\} R(r) = 0,
\]
(64)
where
\[
\lambda = E^2 - 2 \tilde{\ell} \Omega E - (k^2 + m^2); \quad \tilde{\ell} = \frac{\ell}{\alpha}, \quad \tilde{\Omega}^2 = \Omega^2 E^2.
\]
(65)

Which resembles the two-dimensional radial Schrödinger-oscillator with an irrational angular frequency \( \Omega = \pm |\Omega| E \) (now angular velocity). Under such settings, our PSLET implies that for \( D = 2 \implies \ell_D = |\tilde{\ell}| - 1/2 \) (with \( \tilde{\ell} \) denoting irrational magnetic quantum number and \( \ell \) replaces our \( \ell \) of PSLET in equation (41)), \( \mathbb{R} \ni r \in (0, \infty) \) and \( r_0 \) is determined by equation (54) to read \( r_0 = \sqrt{\tilde{\ell} / |\tilde{\Omega}|} \in \mathbb{R} \) with \( \omega = 4, \beta = -2k - 3/2 \) by equation (58), and \( \tilde{\ell} = \ell_D - \beta = 2k + |\tilde{\ell}| + 1 \) by equation (59). Consequently, equations equation (53) would yield the zeroth-order correction
\[
E_0 = 2|\Omega|E(2k + |\tilde{\ell}| + 1).
\]
(66)

One should be aware that \( r_0 = \sqrt{\tilde{\ell} / |\tilde{\Omega}|} \in \mathbb{R} \) is manifestly a condition that secures the finiteness and square integrability wave function, and \( E_1, E_2, \) and \( E_3 \) identically vanish.

Therefore, the energies associated with quasi-free KG-oscillator in Gödel SR-type cosmic string spacetime are given by equation (65) as
\[
E^2 - 2 \tilde{\ell} \Omega E - (k^2 + m^2) = 2|\Omega|E(2k + |\tilde{\ell}| + 1); \quad |\Omega|E = \left\{ \frac{+\Omega \pm E_{\pm}}{-\Omega \mp E_{\pm}} \right\} \geq 0.
\]
(67)

This would allow us to obtain
\[
E_{\pm,1} = \pm |\Omega| \hat{n}_+ \pm \sqrt{\Omega^2 \hat{n}_+^2 + k^2 + m^2}; \quad \hat{n}_+ = 2k + |\tilde{\ell}| + 1, \quad \text{for } |\Omega|E = +\Omega \pm E_{\pm}
\]
\[
E_{\pm,2} = \pm |\Omega| \hat{n}_- \pm \sqrt{\Omega^2 \hat{n}_-^2 + k^2 + m^2}; \quad \hat{n}_- = 2k + |\tilde{\ell}| - 1, \quad \text{for } |\Omega|E = -\Omega \pm E_{\pm}.
\]
(68)

However, we may rearrange such energies so that
\[
E^{(+)}_{k,\ell,\pm} = \pm |\Omega| \hat{n}_+ \pm \sqrt{\Omega^2 \hat{n}_+^2 + k^2 + m^2} \quad \Rightarrow \quad \begin{cases} E^{(+)}_{k,\ell,\pm} = +|\Omega| \hat{n}_+ + \sqrt{\Omega^2 \hat{n}_+^2 + k^2 + m^2} & \text{for } |\Omega| = +|\Omega| \\ E^{(-)}_{k,\ell,\pm} = -|\Omega| \hat{n}_- - \sqrt{\Omega^2 \hat{n}_-^2 + k^2 + m^2} & \text{for } |\Omega| = -|\Omega| 
\end{cases}
\]
\[
E^{(-)}_{k,\ell,\pm} = \pm |\Omega| \hat{n}_- \pm \sqrt{\Omega^2 \hat{n}_-^2 + k^2 + m^2} \quad \Rightarrow \quad \begin{cases} E^{(+)}_{k,\ell,\pm} = +|\Omega| \hat{n}_- + \sqrt{\Omega^2 \hat{n}_-^2 + k^2 + m^2} & \text{for } |\Omega| = +|\Omega| \\ E^{(-)}_{k,\ell,\pm} = -|\Omega| \hat{n}_- - \sqrt{\Omega^2 \hat{n}_-^2 + k^2 + m^2} & \text{for } |\Omega| = +|\Omega| 
\end{cases}
\]
(69)

where in \( E^{(+)}_{k,\ell,\pm} \) and \( E^{(-)}_{k,\ell,\pm} \) the superscripts (+) and (−) identify states with positive and negative vorticities, \( \Omega = \Omega_+ \) and \( \Omega = \Omega_- = -|\Omega| \), respectively. Moreover, one should notice that \( \hat{n}_\pm(\tilde{\ell} = -|\tilde{\ell}|) = \hat{n}_\pm(\tilde{\ell} = |\tilde{\ell}|) \) and consequently \( E^{(+)}_{k,\ell,\pm}(\tilde{\ell} = -|\tilde{\ell}|) = E^{(-)}_{k,\ell,\pm}(\tilde{\ell} = |\tilde{\ell}|) \) indicating space-time-type associated degeneracies (STAD). Moreover, when this result is compared with that of Carvalho et al [12] (i.e., equation (14) in [12]) we observe that not only the negative energies are missed but also the STADs discussed above. Their results should, therefore, be generalized into those reported in equation (69). So should be the case with the results reported on the linear confinement of a scalar particle in Gödel-type space-time by Vitória [22] (their equation (23)) and in the related comment by Neto et al [23] (their equation (18)) for zero linear confinement. That would also include the results reported by Ahmed [46], where he also missed all nodeless (i.e., states with \( n_i = 0 \) states).

5.3. KG-Coulombic particle in Gödel SR-type cosmic string spacetime equation (2) at zero vorticity, \( \Omega = 0 \).

Now, we consider a KG-particle in a Gödel SR-type cosmic string spacetime equation (2) at zero vorticity, \( \Omega = 0 \), and subjected to a Lorentz scalar potential \( S(r) = \beta_s/r \) in equation (29) to obtain
\[
\left\{ \partial_r^2 - \frac{\left( \frac{\partial^2}{r^2} - 1/4 \right)}{r^2} - \frac{\beta_s}{r} + \lambda \right\} R(r) = 0,
\]
(70)
where
\[
\lambda = E^2 - (k^2 + m^2); \quad \tilde{\ell} = \frac{\ell}{\alpha}, \quad \tilde{\Omega}^2 = \tilde{\Omega}^2 + \beta_s^2; \quad \beta_s = 2m\beta_s.
\]
(71)

In this case, our PSLET recipe above implies that \( D = 2 \implies \ell_D = |\tilde{\ell}| - 1/2 \) (with \( \tilde{\ell} \) now denotes irrational magnetic quantum number and \( \tilde{\ell} \) replaces our \( \ell \) of PSLET in equation (41)), \( r \in (0, \infty) \) and \( r_0 \) is determined.
by equation (59) to read \( r_\circ = 2/\sqrt{3} \) with \( \omega = 2, \beta = -k - 1 \) by equation (58), and 
\( \tilde{\epsilon} = \tilde{\epsilon}_D - \beta = k + |L| + 1/2 \). Consequently, equations equation (53) would yield

\[
E_0 = -\frac{\beta_2^2}{4(k + |L| + 1/2)^2}
\]  

(72)

with \( E_1, E_2, \) and \( E_3 \) identically vanish. Therefore, the energies associated with KG-Coulombic particle in Gödel SR-type cosmic string spacetime are given by

\[
E_{k,\ell} = \pm \sqrt{k^2 + m^2 - \frac{m^2\beta_2^2}{(k + |L| + 1/2)^2}}.
\]  

(73)

One should notice that the result of the Heun polynomials solution for the KG-Coulomb in equation (40) for the case \( a = 0 = \Omega \) and \( b = \beta_2 \) does not match the exact solution equation (73), where the effect of the Coulombic parameter \( b = \beta_2 \) have no contribution in equation (40).

### 5.4. Massless KG-particles in Gödel SR-type cosmic string spacetime in a Cornell-type Lorentz scalar potential

We hereby use the massless KG equation (30) with a Cornell-type Lorentz scalar potential \( S(r) = \beta_1 r + \beta_2/r \) to obtain

\[
\left\{ \frac{\partial^2}{r^2} - \frac{(\tilde{\epsilon}^2 - 1)}{r^2} - \frac{\Omega^2}{r^2} + \tilde{\lambda} \right\} R(r) = 0,
\]  

(74)

\( \tilde{\Omega}^2 = \Omega^2 E^2 + \beta_1^2, \) \( \tilde{\epsilon}^2 = \tilde{\beta}^2 + \beta_2^2, \) \( \tilde{\beta} = \frac{\tilde{\epsilon}}{\alpha}, \) \( \tilde{\lambda} = E^2 - 2 \tilde{\epsilon} \Omega E - k_1^2 - 2\beta_1\beta_2. \)

(75)

Then, our PSLET recipe implies that \( D = 2 \implies \epsilon_D = |L| - 1/2 \) (with \( L \) denoting irrational magnetic quantum number and \( L \) replaces our \( \ell \) of PSLET in equation (41)), where \( r_\circ \) is determined by equation (59) to read \( r_\circ = \sqrt{\tilde{\epsilon} / |\Omega|} \in \mathbb{R}, \omega = 4 \) and \( \beta = -(2k + 3/2) \) by equation (58) to imply 
\( \tilde{\epsilon} = \epsilon_D - \beta = 2k + |L| + 1 \). Under such settings, one obtains

\[
E_0 = 2|\Omega| \sqrt{1 + \frac{\beta_1^2}{\Omega^2 E^2}}(2k + |L| + 1) \implies \tilde{\lambda} = 2|\Omega| \sqrt{1 + \frac{\beta_1^2}{\Omega^2 E^2}}(2k + |L| + 1),
\]  

(76)

where \( E_2 = 0, E_3 = 0, \) and \( |\tilde{\Omega}| = |\Omega| \sqrt{1 + \beta_1^2/\Omega^2 E^2} \geq 0 \) is used. Consequently, we get

\[
\tilde{\lambda} = E^2 - 2\tilde{\epsilon} \Omega E - (k_2^2 + 2\beta_1\beta_2) = 2|\Omega| \sqrt{1 + \frac{\beta_1^2}{\Omega^2 E^2}}(2k + |L| + 1)
\]  

(77)

to yield

\[
E_{k,\ell}^\pm = 2\tilde{\epsilon} \Omega_{k,\ell} E_{k,\ell} \pm (k_2^2 + 2\beta_1\beta_2) = 2\Omega_{k,\ell} \sqrt{1 + \frac{\beta_1^2}{\Omega^2 E^2}}(2k + |L| + 1)
\]  

(78)

for \( |\Omega| = \Omega_{k,\ell} E_{k,\ell} \), and

\[
E_{k,\ell}^\pm = -2\tilde{\epsilon} \Omega_{k,\ell} E_{k,\ell} \pm (k_2^2 + 2\beta_1\beta_2) = -2\Omega_{k,\ell} E_{k,\ell} \sqrt{1 + \frac{\beta_1^2}{\Omega^2 E^2}}(2k + |L| + 1). \tag{79}
\]

for \( |\Omega| = -\Omega_{k,\ell} E_{k,\ell} \). However, the two results above would eventually imply that

\[
E_{k,\ell}^\pm = 2|\Omega| \pm (k_2^2 + 2\beta_1\beta_2) = 0 \implies E_{k,\ell,\pm} = \pm |\Omega| N_{k,\ell} \pm \sqrt{\Omega^2 N_{k,\ell}^2 + k_2^2 + 2\beta_1\beta_2},
\]  

(80)

to read

\[
E_{k,\ell,+}^{(+)\pm} = + |\Omega| N_+ + \sqrt{\Omega^2 N_+^2 + k_2^2 + 2\beta_1\beta_2} \quad \text{and} \quad E_{k,\ell,-}^{(+)\pm} = - |\Omega| N_- - \sqrt{\Omega^2 N_-^2 + k_2^2 + 2\beta_1\beta_2},
\]  

(81)

for \( \Omega = \Omega_+ = |\Omega| \), and

\[
E_{k,\ell}^\pm = 2|\Omega| \pm (k_2^2 + 2\beta_1\beta_2) = 0 \implies E_{k,\ell,\pm} = \pm |\Omega| N_{k,\ell} \pm \sqrt{\Omega^2 N_{k,\ell}^2 + k_2^2 + 2\beta_1\beta_2},
\]  

(82)

to read

\[
E_{k,\ell,+,+}^{(-)\pm} = + |\Omega| N_- + \sqrt{\Omega^2 N_-^2 + k_2^2 + 2\beta_1\beta_2} \quad \text{and} \quad E_{k,\ell,+,\pm}^{(-)} = - |\Omega| N_- - \sqrt{\Omega^2 N_-^2 + k_2^2 + 2\beta_1\beta_2},
\]  

(83)
for \( \Omega = \Omega_+ = -|\Omega| \), where

\[
N_\pm = \sqrt{1 + \frac{\beta^2}{\Omega^2} (2k + |\mathcal{L}| + 1) \pm \hat{\ell}}.
\]  

(84)

Again, one observes that \( N_{\Omega} (\hat{\ell} = -|\hat{\ell}|) = N_{\Omega} (\hat{\ell} = |\hat{\ell}|) \) and consequently

\[
E^{\pm 1}_{k, l, \pm} (\hat{\ell} = -|\hat{\ell}|) = E^{\pm 1}_{k, l, \pm} (\hat{\ell} = |\hat{\ell}|)
\]

to represent spacetime-type associated degeneracies (STAD).

### 6. Concluding remarks

In this paper, we started with KG-oscillators in a topologically trivial Gödel-type spacetime background. A KG-oscillator I as a topologically trivial Gödel-type spacetime byproduct, and a KG-oscillator II as a topologically trivial Gödel-type spacetime plus momentum operator deformation byproduct. We have used simple and straightforward textbook procedures and reported, for the sake of scientific correctness, their exact relativistic energies in equations (14), (15), and (24). This would make the results reported by [15] on KG-oscillators in topologically trivial Gödel-type spacetime equation (4) only partially correct and consequently should be generalized into those reported in equation (24). Next, we have discussed the drawbacks associated with the power series (Frobenius method) expansion approach that implied the biconfluent Heun functions/polynomial solution equation (33). We have shown that for the effective interaction potential \( V_{gf}(r) = \hat{\Omega}^2 r^2 + 2mar + 2mb/\hat{r} \) in the Schrödinger-like quantum model equation (31), the biconfluent Heun polynomial solution equation (33) cease to be able to address the especial case of KG-Coulombic particle when \( a = 0 = \Omega \). Consequently, the biconfluent Heun polynomial solution equation (33) can only be classified as a conditionally exact solution that may address the KG-oscillator equation (36), when \( a = 0 = b \), but tragically fails to address the KG-Coulomb case when \( a = 0 = \Omega \). Detailed discussion on this model is also given in appendix B.

As an alternative method/technique, we have recollected, in short, the so called pseudo perturbative shifted \( \ell \) expansion technique PSLET (e.g., [27, 28, 31–33, 29, 30, 34, 35]) to solve for the \( D \)-dimensional Schrödinger-type equations (which is a common form the relativistic wave equations collapse into, like equation (60), (31), etc). Some illustrative examples on the straightforward/simplistic applicability of PSLET procedure are used. Amongst, a KG-oscillator as a topologically trivial Gödel-type spacetime byproduct of equation (10) that collapses into the one dimensional form of Schrödinger equation, and three examples that resemble the radial two-dimensional form of Schrödinger equation: a Quasi-free KG-oscillator in Gödel SR-type cosmic string spacetime equation (2), a KG-Coulombic particle in Gödel SR-type cosmic string spacetime equation (2) at zero vorticity \( \Omega = 0 \), and a massless KG-particle in Gödel SR-type cosmic string spacetime in a Cornell-type Lorentz scalar potential. In all of our illustrative examples above, the exact energies for the harmonic oscillator (i.e., Equations (60), (64), and (74)) like and Coulomb like equation (70) interactions are obtained from the zeroth (leading) order correction, where all higher order correction identically vanish. In connection with the effective potential \( V_{gf}(r) = \hat{\Omega}^2 r^2 + 2mar + 2mb/\hat{r} \) in the Schrödinger-like quantum model equation (31), nevertheless, one would need to numerically solve for \( \Sigma \) and \( r_{\Sigma} \) in equations (58) and (59) for each parametric value involved in the Gödel SR-type cosmic string spacetime metric equation (2). In this case, although tedious, one would work out the corresponding eigenvalues for such a quantum mechanical problem. It can be applied, moreover, to quantum Newtonian cosmology [47, 48]. The accuracy of PSLET is shown to be very satisfactory and reliable for similar Schrödinger-like models [29–37]. However, this issue already lies far beyond the scope of the current proposal.

Finally, we believe that the effects of gravitational fields introduced by different spacetime structures can be clarified if only if the fundamental models like the KG-oscillators and KG-Coulomb are comprehensively and explicitly reported. Moreover, the comprehensive exactness of the reported solutions manifestly suggest degeneracies associated with spacetime (STAD) phenomenon. To the best of our knowledge, the results of the current methodical proposal have never been published elsewhere.

### Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
Appendix A. Explicit forms of $\alpha_1$ and $\alpha_2$

In this section we give the explicit forms of $\alpha_1$ and $\alpha_2$ as given by [28, 29],

$$\alpha_1 = [(1 + 2k) e_2 + 3(1 + 2k + 2k^2)e_1] - \frac{1}{\omega} [e_1^2 + 6(1 + 2k)e_1e_3 + (11 + 30k + 30k^2)e_3^2], \quad (85)$$

$$\alpha_2 = [(1 + 2k)d_2 + 3(1 + 2k + 2k^2)d_1 + 5(3 + 8k + 6k^2 + 4k^3)d_3] - \frac{1}{\omega} [(1 + 2k)e_2^2 + 12(1 + 2k + 2k^2)e_2e_4 + 2e_1d_1 + 2(21 + 59k + 5k^2 + 34k^3)e_1^2 + 6(1 + 2k)e_1d_3 + 6(1 + 2k)e_3d_1 + 2(11 + 30k + 30k^2)e_3d_3 + 10(13 + 40k + 42k^2 + 28k^3)e_3d_3] + \frac{1}{\omega^2} [4e_1^2 + e_2^2 + 2e_3^2 + 3e_1e_2e_3 + 30(1 + 2k)e_1e_3d_3 + 30(1 + 2k + 2k^2)e_1d_3 + 6(1 + 2k)e_1d_1 + 2(11 + 30k + 30k^2)e_3d_3 + 10(13 + 40k + 42k^2 + 28k^3)e_3d_3] + \frac{1}{\omega^3} [8e_1^3 + 3e_2^3 + 48(1 + 2k)e_1^2e_2 + 8(11 + 30k + 30k^2)e_1e_3 + 30(1 + 109k + 141k^2 + 94k^3)e_3^2], \quad (86)$$

with

$$e_i = \varepsilon_i/\omega^{i/2}; \quad d_i = \delta_i/\omega^{i/2}; \quad j = 1, 2, 3, 4; \quad i = 1, 2, 3, 4, 5, 6, \quad (87)$$

$$e_1 = -4\beta, \quad e_2 = 6\beta, \quad e_3 = -4 + r^6 V''(r_0)/6Q, \quad e_4 = 5 + r^6 V'''(r_0)/24Q, \quad (88)$$

$$\delta_1 = -2(\beta^2 - 1/4), \quad \delta_2 = 3(\beta^2 - 1/4), \quad \delta_3 = -8\beta, \quad \delta_4 = 10\beta, \quad \delta_5 = -6 + r^6 V''''(r_0)/120Q, \quad \delta_6 = 7 + r^6 V'''''(r_0)/120Q. \quad (89)$$

Appendix B. On Schrödinger oscillators in Cornell-type potentials

Let us rewrite (31) in terms of $\psi(r)$ so that

$$\psi''(r) + \frac{1}{r} \psi'(r) + \left[ \Lambda - \frac{L^2}{r^2} - \tilde{\Omega}^2 r^2 - A \frac{B}{r} \right] \psi(r) = 0; \quad A = 2ma, \quad B = 2mb, \quad (90)$$

with

$$\tilde{\Omega}^2 = \Omega^2 E^2 + a^2, \quad \mathcal{L}^2 = \tilde{\Omega}^2 + b^2, \quad \Lambda = E^2 - 2 \tilde{\Omega} E - (k_z^2 + m^2 + 2ab),$$

and define

$$\psi(r) = r^{\mathcal{L}^2} \exp \left[- \frac{\Omega r^2}{2} - \frac{A r}{2\tilde{\Omega}} \right] H(r), \quad (91)$$

to imply

$$r H''(r) + \left[ 1 + 2\mathcal{L}, -2 \tilde{\Omega} r^2 - \frac{A}{\Omega} r \right] H'(r) + (X + \omega') H(r) = 0. \quad (92)$$

Where

$$X = -\frac{A}{2\tilde{\Omega}} \gamma' - B, \quad \gamma' = 1 + 2\mathcal{L}, \quad \omega' = \frac{A^2}{4 \Omega^2} + \Lambda - \Omega \gamma' - \tilde{\Omega}. \quad (93)$$

With

$$H(r) = \sum_{j=0}^{\infty} C_j \ r^j, \quad (94)$$

in equation (92) would result

$$\sum_{j=0}^{\infty} \left\{ C_{j+2} [(j + 2)(j + \gamma' + 1)] + C_{j+1} [X - \frac{A}{\Omega} (j + 1)] + C_j [\omega' - 2 \tilde{\Omega} j] \right\} r^{j+1} + \gamma' C_i + X C_0 = 0. \quad (95)$$

Which in effect implies that

$$\gamma' C_i + X C_0 = 0 \Rightarrow C_i = -\frac{X}{\gamma'}; \quad C_0 = 1. \quad (96)$$
\[ C_{j+2}(j + 2)(j + \gamma' + 1) - C_{j+1} \left[ \frac{A}{\Omega} (j + 1) - \lambda' \right] + C_j [\omega' - 2 \Omega j] = 0; \quad j = -1, 0, 1, 2, \ldots, C_{-1} = 0. \]  

(97)

In order for the biconfluent Heun series to become a polynomial of degree \( n \geq 0 \), we truncate the power series by requiring that for \( j = n \), \( C_{n+1} = 0 \) and \( C_{n+2} = 0 \). Consequently, equation (97) would allow one to write

\[ C_n(\omega' - 2 \Omega n) = 0 \Rightarrow \omega' = 2 \Omega n \Rightarrow \Lambda = 2 \Omega(n + |\mathcal{L}| + 1) - \frac{A^2}{4 \Omega^2}; \quad n \geq 0. \]

(98)

In this case, following equation (97) we retrieve equation (96) for \( j = -1 \) and get, respectively, for \( j = 0, 1, \ldots \)

\[ C_2 = \frac{1}{2(1 + \gamma')} \left( \frac{A}{\Omega} - \lambda' \right) C_1 - \omega' \]

(99)

\[ C_3 = \frac{1}{3(2|\mathcal{L}| + 3)} \left( \frac{2 A}{\Omega} + \lambda' \right) C_2 - (\omega' - 2\Omega) C_1 \]

(100)

and

\[ C_{j+1} = \bar{A}_{j-1} C_j + \bar{B}_{j-1} C_{j-1}; \quad j = 0, 1, \ldots, n, \quad C_{-1} = 0, \]

(101)

where

\[ \bar{A}_j = \frac{A}{\Omega} \left( j + \frac{\gamma'}{2} + 1 \right) + B, \quad \bar{B}_j = \frac{2\Omega(j - n)}{(j + 2)(j + \gamma' + 1)}. \]

(102)

Hereby, the recursion relation equation (101) along with equation (102) would identify the relations between \( C_j's \) for \( 0 \leq j \leq n \).

Nevertheless, it has been a tradition/belief that one should again use the condition \( C_{n+1} = 0 \) for \( \forall j > n \). In so doing, one would, using equations (101) and (102), obtain

\[ C_{n+1} = 0 = \bar{A}_{n-1} C_n + \bar{B}_{n-1} C_{n-1} \implies C_n \left[ \frac{A}{2\Omega} (2n + \gamma') + B \right] = 2\Omega C_{n-1}. \]

(103)

It is obvious that this equation would correlate \( A \) and \( B \) for each value of the truncation order \( n \). For example, for

\[ n = 0 \implies A = -\frac{2\Omega B}{\gamma'}, \]

(104)

\[ n = 1 \implies \frac{A^2}{2\Omega} \gamma'(2 + \gamma') + 2AB(1 + \gamma') + 2\Omega B^2 = 4\Omega^2 \gamma', \]

(105)

and so on. This means that for every \( n \) value we have a different correlation between \( A = A(n, B) \) and \( B = B(n, A) \) (which are in exact accord with those reported in (9) of Fernández [49]). Under such severe restrictions on the parameters of the Cornell-type potential, one would recast the eigenvalues of equation (98) as

\[ \Lambda = 2 \sqrt{\Omega E^2 + a^2 (n + |\mathcal{L}| + 1)} \left( \frac{A^2}{4(\Omega E^2 + a)} \right); \quad n \geq 0, \quad A = A(n, B). \]

(106)

However, in order to retrieve the results of equation (36) we set \( A = 0 = B \implies \forall A(n, B)'s = 0 \) in equation (106) and compare the two equations to come out with the correlation between the truncation order \( n \) and the radial quantum number \( n_r \), so that \( n = 2n_r \geq 0 \). Therefore,

\[ \omega_{n_r}' = 4\Omega n_r \Rightarrow \Lambda = 2 \sqrt{\Omega E^2 + a^2 (2n_r + |\mathcal{L}| + 1)} \left( \frac{A^2}{4(\Omega E^2 + a)} \right), \]

(107)

to represent the eigenvalues of equation (90). At this point, one should notice that the condition of truncation of the biconfluent Heun series into a biconfluent Heun polynomial of degree \( n = 2n_r \geq 0 \) is not violated. This is not a new practice. It has been discussed in [23, 41, 44, 45, 50–52, 49, 53]. Obviously, moreover, this result shows that the biconfluent Heun polynomial solution discussed above is one of the so called conditionally exact solutions and not ‘the exact solution’ for equation (90). In this case, we rewrite our biconfluent Heun polynomial of degree \( n = 2n_r \) as

\[ H(r) = \sum_{j=0}^{n_r-1} C_j r^j \Rightarrow H_{n_r}(r) = 1 + \sum_{j=1}^{n_r} C_j r^j \]

(108)
This polynomial is, in fact, responsible for the nodes in the corresponding radial wave function \( \psi(r) \). One should also notice that the coefficient \( C_{n+1} \) is not involved in the solution. Intuitively, therefore, one should not worry about its structure and/or whatever it yields to.

In this appendix section, we have followed, more or less, the usual procedure followed by many authors (e.g., [22, 23, 50, 54, 51, 55, 52, 49, 53] and references cited therein). In fact, we have very closely followed Fernández [49, 53] to work out the above results. Fernández [49, 53] has very carefully and righteously detailed the most misunderstood conditionally-exact solvability of the model above, related to the three terms recursion relation equation (101). Yet, the result reported in equation (107) have derived some authors to conclude/claim that there exist some quantization recipe for the parameter \( \lambda = A(n, B) \) (hence, \( B = B(n, B) \)) mandated by the condition \( C_{n+1} = C_{2n+1} = 0 \) in equation (103). However, if we put this above procedure to the test, we may then pin point the problem associated with such assumptions.

Let us consider that \( B = 0 \) and \( |L| = |\tilde{L}| = 1/2 \) in equation (90) (i.e., we are hereby within the geometric theory of topological defects in Condense matter territories) and consequently equation (90), with \( \psi(r) = R(r)/\sqrt{r} \), now reads

\[
R''(r) + \left[ \lambda - \tilde{\Omega}^2 r^2 - A \, r \right] R(r) = 0; \quad \lambda = E^2 - 2 \tilde{\Omega} \Omega - (k_f^2 + m^2),
\]

where the effect of the central repulsive/attractive core \((L^2 - 1/4)/r^2\) is removed. Clearly, this equation resembles a shifted-harmonic oscillator and reduces to

\[
R''(r) + \left[ \tilde{\lambda} - \tilde{\Omega}^2 (r + \zeta)^2 \right] R(r) = 0; \quad \tilde{\lambda} = \lambda + \frac{A^2}{4\Omega^2}, \quad \zeta = \frac{A}{2\Omega},
\]

that can be rewritten with \( \tilde{r} = r + \zeta \) as

\[
R''(\tilde{r}) + \left[ \tilde{\lambda} - \tilde{\Omega}^2 \tilde{r}^2 \right] R(\tilde{r}) = 0.
\]

This equation denotes a radial harmonic Schrödinger oscillator, without the central repulsive/attractive core \((L^2 - 1/4)/r^2\), that admits the exact textbook eigenvalues

\[
\tilde{\lambda} = 2\tilde{\Omega} \left( 2n_r + \frac{3}{2} \right) \implies \lambda = 2\Omega \left( 2n_r + \frac{3}{2} \right) - \frac{A^2}{4\Omega^2} = 2\sqrt{\Omega^2E^2 + a^2} \left( 2n_r + \frac{3}{2} \right) - \frac{A^2}{4(\Omega^2E^2 + a^2)},
\]

and eigen functions

\[
\psi(\tilde{r}) \sim \tilde{r}^{1/2} \exp \left( -\frac{\tilde{\Omega} \tilde{r}^2}{2} \right) L_{n_r}^{1/2}(\tilde{\Omega} \tilde{r}^2) \iff \psi(r) \sim r^{1/2} \exp \left( -\frac{\tilde{\Omega} \tilde{r}^2}{2} \right) L_{n_r}^{1/2}(\tilde{\Omega} \tilde{r}^2).
\]

At this point one should notice that equation (107) with \( |L| = |\tilde{L}| = 1/2 \) is in exact accord with equation (112). Moreover, the result in equation (112) suggests that there is no quantization characterization associated with the parameter \( A \) or \( \tilde{\Omega} \), they are both \( (n = 2n_r) \)-independent parameters. This is a brute-force evidence that should be taken into account while dealing with this problem.

A final note on the procedures discussed above is unavoidably necessary and vital. For \( A = 0 = B \), the recursion relation equation (104) is safely satisfied but not that of equation (105). The relation of equation (105) implies that \( \gamma = 0 \implies |L| = 1/2 \) which is neither physically nor mathematically acceptable (similar consequence appear in equation (9) of Fernández [49] where the angular momentum quantum number \( \ell \) takes the value \( \ell = -1 \) as \( A = B = 0 \) in his relation equation (4))). Yet, for example, in the results reported by Medeirosa and de Mello [51] (section 4.3), we notice that things are more tragic in the sense that if one sets \( \eta = 0 \) in their equation (40), then their \( \tilde{\omega}_{n,m} \) of their equation (52) takes the value \( \tilde{\omega}_{n,m} = 0 \). As a result, their reported energy spectrum collapses into that of free particle energy \( E_{n,m,n} = \pm \sqrt{k^2 + M^2} \) (although they still have the harmonic oscillator term but their solution tragically fails for such parametric settings). This is also reflected on their general solution (their section 4.4). The same happens with the results reported by Verçin [56] in his equation (22). This should lead us to one conclusion. As long as quantum mechanics is in point, using the condition \( C_{n+1} = 0 \) more than one time yields catastrophic results that are neither physically nor mathematically acceptable. Therefore, the repeated usage of this condition should be labeled, hereinafter, as a quantum mechanically redundant condition. It could be of interest for pure mathematical curiosity and endeavour.

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