Asymptotic behaviors of governing equation of Gauged Sigma model for Heisenberg ferromagnet

HUYUAN CHEN
Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China
Email: chenhuyuan@yeah.net

FENG ZHOU
Center for PDEs, School of Mathematical Sciences, East China Normal University, Shanghai 200241, PR China
Email: fzhou@math.ecnu.edu.cn

Abstract

In this note, we study weak solutions of equation
\[ \Delta u = \frac{4e^u}{1 + e^u} - 4\pi \sum_{i=1}^{N} \delta_{p_i} + 4\pi \sum_{j=1}^{M} \delta_{q_j} \quad \text{in } \mathbb{R}^2, \]
(0.1)
where \(\{\delta_{p_i}\}_{i=1}^{N}\) (resp. \(\{\delta_{q_j}\}_{j=1}^{M}\)) are Dirac masses concentrated at the points \(p_i, i = 1, \ldots, N\), (resp. \(q_j, j = 1, \ldots, M\)) and \(N - M > 1\). The equation (0.1) represents a governing equation of gauged sigma model for Heisenberg ferromagnet. We show that it has a sequence of solutions \(u_\beta\) having behaviors as \(-\beta \ln |x| + O(1)\) at infinity with a free parameter \(\beta \in (2, 2(N - M))\), and our concern in this paper is to study the asymptotic behavior of \(b_\beta\) as \(\beta\) approaching the extremal values 2 and \(2(N - M)\).

Key words: Gauged sigma model; Dirac mass; Asymptotic behavior.

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1 Introduction

Vortices appear in various planar condensed-matter systems and have important applications in many fundamental areas of physics including superconductivity [1, 10, 14], particle physics [13], optics [4] and cosmology [23]. The study of multiple charges vortex construction in gauged field theory was studied by Taubes [14, 20, 21], initiated the existence and asymptotic behaviors of static solutions of the sigma model. Later on, Schroers [19] extended the classical \(O(3)\) sigma model solved by Belavin-Polyakov [3] to incorporate an Abelian gauged field and allow the existence of vortices of opposite local charges so that the vortices of negative local charges viewed as poles of a complex scalar field \(u\) makes contribute to, but those positive local charges viewed as zero of \(u\) do not affect, the total energy, although they give some magnetic manifestation for their existence [24]. In fact, these peculiar properties are all due to the absence of symmetry breaking and in order to obtain vortices of opposite magnetic alignments with an energy that takes account of both type of vortices, it suffices to impose a broken symmetry. After that, Yang in [26] established an Abelian field theory model that allows the coexistence of vortices and anti-vortices, showed how vortices and anti-vortices with the coupling of gravity, namely, cosmic strings and anti-strings, can be constructed in the Abelian gauged field model.

After involving the magnetic field, the sigma model for Heisenberg ferromagnet, would be transformed into the local \(U(1)\)-invariant action density,
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi - \frac{1}{2} (1 - \vec{n} \cdot \phi)^2, \]
where \(\vec{n} = (0, 0, 1)\), \(\phi : S^2 \to \mathbb{R}^3\) with \(|\phi| = 1\), \(D_\mu\) is gauge-covariant derivatives on \(\phi\), defined by
\[ D_\mu \phi = \partial_\mu \phi + A_\mu (\vec{n} \times \phi), \quad \mu = 0, 1, 2 \]
and
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]
Assuming the temporal gauge \( A_0 = 0 \), the total energy is derived as
\[
E(\phi, A) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ (D_1 \phi)^2 + (D_2 \phi)^2 + (1 - \vec{n} \cdot \phi)^2 + F_{12}^2 \right\} \]
\[
= 4\pi |\text{deg}(\phi)| + \frac{1}{2} \int_{\mathbb{R}^2} \left\{ (D_1 \phi \pm \phi \times D_2 \phi)^2 + (F_{12} \mp (1 - \vec{n} \cdot \phi))^2 \right\},
\]
where \( \text{deg}(\phi) \) represents the Brouwer’s degree of \( \phi \).

The related Bogomol’nyi equations could be stated as
\[
\begin{cases}
D_1 v + iD_2 v = 0, \\
F_{12} = \frac{2|v|^2}{1 + |v|^2},
\end{cases}
\]
then, setting \( u = \ln |v|^2 \), it reduces into the following governing equation of the gauged sigma model
\[
-\Delta u + \frac{4e^u}{1 + e^u} = 4\pi \sum_{i=1}^N \delta_{p_i} - 4\pi \sum_{j=1}^M \delta_{q_j} \quad \text{in } \mathbb{R}^2,
\]
(1.1)
where \( \delta_p \) is the Dirac mass concentrated at \( p \in \mathbb{R}^2 \). This subject has been expanded extensively in recent years, see the works of Chern-Yang [9], Lin-Yang [16], Yang [25] and the references therein. In particular, Yang [24, 25] obtained a sequence of solutions \( u_\beta \) with the asymptotic behavior
\[
u_\beta(x) = -\beta \ln |x| + b_\beta + o(1) \quad \text{at infinity for } \beta \in (2, 2(N - M)),
\]
(1.2)
for some constant \( b_\beta \) under the restriction that \( N - M > 1 \). When \( N > 1, M = 0 \) and there is only one magnetic monopole, it was proved in [12] by using ODE analysis that a radial extremal non-topological solution of (1.1) has the asymptotic behavior
\[
u_2(r) = -2\ln r - 2\ln \ln r + O(1) \quad \text{as } \quad r \to +\infty.
\]

Our aim in this paper is to consider the behavior of \( b_\beta \) in (1.2) as \( \beta \) approaches the extremal points 2 and \( 2(N - M) \), and more asymptotic behavior estimates for the solutions of (1.1). For convenience of readers, we use some notations and follow some presentations of known results mainly from the book of Yang [25] (see e.g. [24]).

We first introduce some auxiliary functions. Let \( \rho \) be a smooth monotone increasing function over \((0, +\infty)\) such that
\[
\rho(t) = \begin{cases}
\ln t, & 0 < t \leq 1/2 \\
0, & t \geq 1.
\end{cases}
\]
Let \( \{p_i\}_{i=1}^N \) and \( \{q_j\}_{j=1}^M \) are different points in \( \mathbb{R}^2 \). Set \( v_1(x) = 2\sum_{i=1}^N \rho\left(\frac{|x - p_i|}{\varrho}\right) \) and \( v_2(x) = 2\sum_{j=1}^M \rho\left(\frac{|x - q_j|}{\varrho}\right) \), where \( \varrho \in (0, 1) \) such that any two balls of
\[
\{B_\varrho(p_i) : i = 1, \cdots, N\} \cup \{B_\varrho(q_j) : j = 1, \cdots, M\}
\]
do not intersect. We fix a positive number \( r_0 \geq 4\varrho \) large enough such that \( B_\varrho(p_i), B_\varrho(q_j) \subset B_{r_0}(0) \) for \( i = 1, \cdots, N \) and \( j = 1, \cdots, M \). Let \( \eta_0 : [0, +\infty) \to [0, 2] \) be a smooth, non-increasing function with compact support in \([0, 1]\) such that \( \int_0^1 \eta_0(r)rdr = 1 \), and we take also the notation \( \eta_0(x) = \eta_0(|x|) \) for \( x \in \mathbb{R}^2 \). Denote \( v_3 = \Gamma \ast \eta_0 - c_0 \), where \( c_0 = \int_0^1 (-\ln r)\eta_0(r)rdr > 0 \), \( \ast \) means the standard convolution operator and \( \Gamma(x) = -\frac{1}{2\pi} \ln |x| \) is the fundamental solution of Laplacian in \( \mathbb{R}^2 \), i.e.
\[
-\Delta \Gamma = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).
\]
Notice that $v_3 \leq 0$ is a smooth function in $\mathbb{R}^2$ satisfying $-\Delta v_3 = \eta_0 \geq 0$ in $B_1(0)$ and

$$|v_3(x) + \ln |x| + c_0| \leq 3\tau_0 |x|^{-1} \quad \text{for } |x| \geq 2\tau_0.$$  \hspace{1cm} (1.3)

See Section 2 for the proof.

Denote

$$u_0(x) = -v_1(x) + v_2(x) + \beta v_3(x),$$

where $\beta$ is a positive number to be chosen later, then $u_0$ contains all singularities at the points $p_i, q_j$ and the infinity. We observe that a solution $u$ of (1.1) could be written as $u = u_0 + v$, where the remainder term $v$ is a bounded solution of

$$\Delta v = \frac{4K_{\beta}e^{\beta v_2 + v}}{e^{v_1} + K_{\beta}e^{\beta v_2 + v}} - g_\beta \quad \text{in } \mathbb{R}^2,$$  \hspace{1cm} (1.4)

with $K_{\beta} = e^{\beta v_3}$, $g_1 = \sum_{i=1}^{N} 4\pi \delta_{p_i} - \Delta v_1$, $g_2 = \sum_{j=1}^{M} 4\pi \delta_{q_j} - \Delta v_2$ and

$$g_\beta = g_1 - g_2 + \beta \Delta v_3,$$  \hspace{1cm} (1.5)

which is a smooth function with compact support in $B_{\tau_0}(0)$ and verifies that

$$\int_{\mathbb{R}^2} g_\beta \, dx = 2\pi [2(N - M) - \beta].$$

Our main results on asymptotic behavior of solutions states as follows.

**Theorem 1.1** Let $N - M > 1$, then for any $\beta \in (2, 2(N - M))$, problem (1.4) has a unique solution $v_\beta$ such that

$$v_\beta(x) = b_\beta + O(|x|^{-\frac{\beta}{2}}) \quad \text{as } |x| \to +\infty,$$  \hspace{1cm} (1.6)

where the constant $b_\beta \in \mathbb{R}$ depends on $\beta$ satisfying

$$\lim_{\beta \to 2(N-M)^-} \frac{b_\beta}{\ln(2(N - M) - \beta)} = 1,$$  \hspace{1cm} (1.7)

$$1 \leq \liminf_{\beta \to 2^+} \frac{b_\beta}{\ln(\beta - 2)} \leq \limsup_{\beta \to 2^+} \frac{b_\beta}{\ln(\beta - 2)} = 2.$$  \hspace{1cm} (1.8)

An interesting phenomena in Theorem 1.1 is that, by (1.7) and (1.8), the asymptotic behavior of $b_\beta$ as $\beta \to (2(N - M))^-$ (resp. $\beta \to 2^+$) is of order $\ln(2(N - M) - \beta)$ (resp. $\ln(\beta - 2)$). Back to problem (1.1), a sequence of solutions are constructed with the behaviors as $-\beta \ln |x| + O(1)$ at infinity with a free parameter $\beta \in (2, 2(N - M))$. Normally, this type of solutions are called as non-topological solutions, for instance [2, 6, 8, 13] on non-topological solutions of Chern-Simon equation or systems. In particular, for $\beta \in (2, 4)$, the author in [25, Chapter 2], see also [24], gave an existence result of (1.4) and asserted that the solution converges to a constant at infinity as mentioned by (1.2). Therefore our result extend the existence of solutions for (1.4) with the free parameter $\beta$ in the range $(2, 2(N - M))$ and the uniqueness follows by comparison principle.

Our idea for the estimates (1.7) and (1.8) is to construct suitable super and sub solutions of (1.4), by the uniqueness to see the asymptotic behavior from the super and sub solutions. These super and sub solutions are constructed by adding some suitable constants depending on $\beta$ to the solution of (1.4) with $\beta = N - M + 1$.

The rest of this paper is organized as follows. In Section 2, we show some estimates at infinity of the convolution function $\Gamma * F$ for function $F$ satisfying $\int_{\mathbb{R}^2} F \, dx = 0$ and sketch the proof of the existence in Theorem 1.1. Section 3 is devoted to the estimates for (1.7) and (1.8) by constructing super and subsolutions.
2 Existence and Uniqueness

In this section, we show the results on existence and uniqueness. To this end, we first claim that (1.3) holds for $|x| > 2r_0$. In fact, by direct computation, we observe that

\[
v_3(x) + \ln |x| + c_0 = \frac{1}{2\pi} \int_{B_1(0)} (-\ln |x - y|)\eta_0(y)dy + \frac{1}{2\pi} \int_{B_1(0)} \ln |x|\eta_0(y)dy
\]

\[
= -\frac{1}{2\pi} \int_{B_1(0)} \ln(|x - y|/|x|)\eta_0(y)dy.
\]

For $|x| > 2r_0$ and $|y| \leq r_0$ (since $|y| \leq 1$), we have that

\[
1 - \frac{r_0}{|x|} \leq |x - y|/|x| \leq 1 + \frac{r_0}{|x|},
\]

then

\[
|\ln(|x - y|/|x|)| \leq \max\{\ln(1 + r_0/|x|), -\ln(1 - r_0/|x|)\} \leq \frac{2r_0}{|x|},
\]

and thus,

\[
|v_3(x) + \ln |x| + c_0| \leq \frac{2r_0}{|x|},
\]

the claim is true. \(\square\)

Observe that $K_\beta = e^{\beta v_3}$ is a positive smooth function verifying that

\[
e^{-2(N-M)\epsilon_1|x|^{-\beta}} \leq K_\beta(x) \leq e^{2(N-M)\epsilon_1|x|^{-\beta}} \text{ for } |x| \geq 2r_0,
\]

(2.1)

when $\beta$ varies from 2 to $2(N-M)$ and $-\epsilon_1 + \ln |x| \leq v_3 \leq \epsilon_1 + \ln |x|$ for $|x| \geq 2r_0 > 2\epsilon$ by (1.3).

Let

\[
X_\beta = \left\{ w : \mathbb{R}^2 \to \mathbb{R} \mid \|\nabla w\|_{L^2(\mathbb{R}^2)} + \|w\|_{L^2(\mathbb{R}^2, K_\beta dx)} < \infty \right\}.
\]

Now we show the following result.

**Proposition 2.1** Let $N - M > 1$ and $2 < \beta < 2(N - M)$. Then problem (1.4) has a unique solution $v_\beta \in X_\beta$ and there exists $b_\beta \in \mathbb{R}$ such that

\[
v_\beta(x) = b_\beta + O(|x|^{-\frac{2}{\beta - 1}}) \text{ as } |x| \to \infty.
\]

(2.2)

To prove this result, we start the analysis by doing the decay estimates at infinity.

**Lemma 2.1** Suppose that $\Gamma$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^2$, $F \in L^\infty(\mathbb{R}^2)$ has compact support, i.e. $\text{supp } F \subset B_R(0)$ for some $R > 0$, and satisfies that

\[
\int_{\mathbb{R}^2} F(x)dx = 0.
\]

(2.3)

Then we have that

\[
\|\Gamma * F\|_{L^\infty(\mathbb{R}^2)} \leq \|F\|_{L^1(\mathbb{R}^2)} + R^2 \ln R \|F\|_{L^\infty(\mathbb{R}^2)}
\]

(2.4)

and

\[
|\Gamma * F(x)| \leq \frac{R}{\pi |x|} \|F\|_{L^1(\mathbb{R}^2)} \text{ for } |x| > 4R.
\]

(2.5)
Proof. Since \( \text{supp} F \subset B_{R}(0) \) and \( F \in L^{\infty}(\mathbb{R}^{2}) \), then \( F \in L^{1}(\mathbb{R}^{2}) \) and for \( |x| > 4R \),

\[
|\Gamma \ast F(x)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^{2}} \ln |x - y| F(y) dy - \int_{\mathbb{R}^{2}} \ln |x| F(y) dy \right|
\]

\[
= \frac{|x|^{2}}{2\pi} \left| \int_{B_{R}(0)} \ln |e_{x} - z| F(|x| z) dz \right|
\]

\[
\leq \frac{|x|^{2}}{\pi} \int_{\mathbb{R}^{2}} \ln |z| |F(|x| z)| dz
\]

\[
\leq \frac{R}{\pi |x|} \| F \|_{L^{1}(\mathbb{R}^{2})},
\]

where \( e_{x} = \frac{x}{|x|} \) and we have used (2.3) and the fact that

\[
| \ln |e_{x} - z|| \leq 2|z| \leq \frac{2R}{|x|} \quad \text{for} \quad z \in B_{R/|x|}(0) \subset B_{1/4}(0).
\]

Therefore (2.5) holds. Moreover, for \( |x| \leq 4R \), we have that

\[
|\Gamma \ast F(x)| = \frac{1}{2\pi} \left| \int_{B_{R}(0)} F(y) \ln |x - y| dy \right| \leq \| F \|_{L^{\infty}(\mathbb{R}^{2})} R^{2} \ln R,
\]

which completes the proof of (2.4). \( \square \)

Replacing the compact support assumption for \( F \) by some decay at infinity, we have the following estimate.

Lemma 2.2 Let \( \beta \in (2, 2(N - M)) \), \( F \in L^{1}(\mathbb{R}^{2}) \) verifies (2.3) and

\[
|F(x)| \leq c_{2}|x|^{-\beta}, \quad \forall |x| \geq 1
\]

for some \( c_{2} \geq 1 \). Then we have that

\[
|\Gamma \ast F(x)| \leq \frac{c_{2}c_{3}}{\beta - 2} |x|^{-\frac{\beta - 2}{\beta - 1}} \quad \text{for large} \quad |x| > 4e,
\]

where \( c_{3} > 0 \) depends on \( \| F \|_{L^{1}(\mathbb{R}^{2})} \), but it is independent of \( c_{2} \) and \( \beta \).

Proof. Since \( F \) satisfies (2.3), then for all \( |x| > 4e \), we have that

\[
2\pi \Gamma \ast F(x) = |x|^{2} \int_{\mathbb{R}^{2}} \ln |e_{x} - z| F(|x| z) dz + |x|^{2} \ln |x| \int_{\mathbb{R}^{2}} F(|x| z) dz
\]

\[
= |x|^{2} \int_{B_{R/|x|}(0)} \ln |e_{x} - z| F(|x| z) dz + |x|^{2} \int_{B_{1/2}(e_{x})} \ln |e_{x} - z| F(|x| z) dz
\]

\[
+ |x|^{2} \left( \int_{\mathbb{R}^{2}\setminus(B_{R/|x|}(0) \setminus B_{1/2}(e_{x}))} \ln |e_{x} - z| F(|x| z) dz \right)
\]

\[
= : I_{1}(x) + I_{2}(x) + I_{3}(x),
\]

where \( R \in (e, \frac{|x|}{4}) \) will be chosen latter. Here we have that \( B_{R/|x|}(0) \cap B_{1/2}(e_{x}) = \emptyset \) for \( R \leq \frac{|x|}{4} \).

By directly computation, we have that

\[
|I_{1}(x)| \leq |x|^{2} \int_{B_{R/|x|}(0)} |z| |F(|x| z)| dz
\]

\[
= \frac{R}{|x|} \int_{B_{R}(0)} |F(y)| dy \leq \frac{R}{|x|} \| F \|_{L^{1}(\mathbb{R}^{2})}.
\]

5
For } z \in B_{1/2}(e_x), \text{ we have that } |x| |z| \geq \frac{1}{2} |x| > \epsilon, \text{ then } |F(|x| z)| \leq c_2 |x|^{-\beta}|z|^{-\beta} \text{ and }

\begin{align*}
|I_2(x)| & \leq c_2 |x|^{2-\beta} \int_{B_{1/2}(e_x)} \left( -\ln |e_x - z| \right) |z|^{-\beta} dz \\
& \leq c_2 2^\beta \left( \int_{B_{1/2}(e_x)} \left( -\ln |e_x - z| \right) dz \right) |x|^{2-\beta} = c_2 2^\beta \int_{B_{1/2}(0)} \left( -\ln |z| \right) dz |x|^{2-\beta} \leq c_4 c_2 R^{2-\beta},
\end{align*}

where } c_4 = 2^{N-M} \left( \int_{B_{1/2}(0)} (-\ln |z|) dz \right).

For } z \in \mathbb{R}^2 \setminus (B_{R/|z|}(0) \cup B_{1/2}(e_x)), \text{ we have that } |\ln |e_x - z|| \leq \ln (2 + |z|). \text{ In fact, if } |e_x - z| \geq 1, \text{ then it follows by the fact that } |e_x - z| \leq |e_x| + |z| = 1 + |z|; \text{ if } |e_x - z| \leq 1, \text{ we have that } |e_x - z| \geq \frac{1}{2} \text{ for } z \in \mathbb{R}^2 \setminus (B_{R/|z|}(0) \cup B_{1/2}(e_x)), \text{ then } |\ln |e_x - z|| \leq \ln 2 \leq \ln (2 + |z|). \text{ Together with the fact that } |F(|x| z)| \leq c_2 |x|^{-\beta}|z|^{-\beta}, \text{ since } |z| \geq \frac{R}{|x|} > \frac{\epsilon}{|x|}, \text{ and thus the integration by parts gives }

\begin{align*}
|I_3(x)| & \leq c_2 |x|^{2-\beta} \int_{\mathbb{R}^2 \setminus B_{R/|z|}(0)} \ln (2 + |z|) |z|^{-\beta} dz \\
& \leq \frac{2\pi c_2}{\beta - 2} R^{2-\beta} \ln (2 + \frac{R}{|x|}) + \frac{2\pi c_2}{(\beta - 2)^2} R^{2-\beta} \\
& \leq \frac{2\pi c_2}{(\beta - 2)^2} \left( 2(N - M - 1) \ln 3 + 1 \right) R^{2-\beta}.
\end{align*}

Thus taking } R = \frac{|x|}{\pi^{1/2}} \text{ and } |x| \text{ sufficient large (certainly } R \in (e, \frac{2|z|}{\pi} \text{ is satisfied), we have that }

\begin{align*}
|\Gamma * F(x)| & \leq \frac{R}{2 \pi |x|} \|F\|_{L^1(\mathbb{R}^2)} + \frac{c_2 c_4}{\pi} R^{2-\beta} + \frac{c_2}{(\beta - 2)^2} \left( 2(N - M - 1) \ln 3 + 1 \right) R^{2-\beta} \\
& \leq \frac{c_5}{(\beta - 2)^2} |x|^{-\frac{\beta - 2}{\nu - 1}},
\end{align*}

where } c_5 > 0. \text{ This ends the proof. } \square

Now for } \sigma \in \mathbb{R} \text{ and } s \in \mathbb{N}, \text{ we define } W_{s,\sigma}^2 \text{ as the closure of the set of } C^\infty \text{ functions over } \mathbb{R}^2 \text{ with compact supports under the norm }

\begin{align*}
\|\xi\|_{W_{s,\sigma}^2}^2 = \sum_{|\alpha| \leq s} \| (1 + |x|)^{\sigma + |\alpha|} D^\alpha \xi \|_{L^2(\mathbb{R}^2)}^2.
\end{align*}

For more details of properties of these weighted Sobolev spaces, see e.g. [3], [17]. Let } C_0(\mathbb{R}^2) \text{ be the set of continuous functions on } \mathbb{R}^2 \text{ vanishing at infinity.

Lemma 2.3 [24 Lemma 2.4.5] The following statements hold:
(i) If } s > 1 \text{ and } \sigma > -1, \text{ then } W_{s,\sigma}^2 \subset C_0(\mathbb{R}^2).
(ii) For } -1 < \sigma < 0, \text{ the Laplace operator } \Delta : W_{s,\sigma}^2 \rightarrow W_{0,\sigma+2}^2 \text{ is one to one and the range of } \Delta \text{ has the characterization }

\begin{align*}
\Delta(W_{s,\sigma}^2) &= \left\{ F \in W_{0,\sigma+2}^2 \left| \int_{\mathbb{R}^2} F dx = 0 \right. \right\}.
\end{align*}

(iii) If } \xi \in X_\beta \text{ and } \Delta \xi = 0, \text{ then } \xi \text{ is a constant.

Now we are ready to prove Proposition 2.7.

Proof of Proposition 2.7. The key point to study problem (1.4) is the following equation

\begin{align*}
\Delta w = \frac{4K_\beta e^w}{e^{v_1} + K_\beta e^w} - h \quad \text{in } \mathbb{R}^2, \quad (2.8)
\end{align*}
where $K_\beta$ is a positive smooth function, $h \geq 0$ is a function in $C_c^\infty(\mathbb{R}^2)$, i.e. with compact support such that

$$
\int_{\mathbb{R}^2} h(x) \, dx = 2\pi [2(N - M) - \beta]. \tag{2.9}
$$

Existence: As it is proved in [25, Chapter 2, Section 2.4.2], problem (2.8) has a solution $w_1$, which is derived by considering the critical point of the energy functional

$$
I(w) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\nabla w|^2 + 4 \ln(e^{v_1} + K_\beta e^w) - hw \right\} \, dx
$$

in the admissible space

$$
A = \{ w \in X_\beta \mid \int_{\mathbb{R}^2} \frac{4K_\beta e^w}{e^{v_1} + K_\beta e^w} \, dx = \int_{\mathbb{R}^2} h \, dx \}.
$$

Moreover, $w_1$ is a classical solution of (2.8).

The subsolution of (1.4) could be constructed as $w_- = w_1 - \Gamma \ast (h - g) - c_6$, where $c_6 > 0$ is a constant such that $w_1 - c_6 \leq 0$. The supersolution is given by $w^+ = w_1 - \tilde{w}_2 - v_2$, where $\tilde{w}_1$ is the solution of

$$
\Delta w = \frac{4K_\beta e^w}{1 + K_\beta e^w} - \tilde{h},
$$

$\tilde{w}_2 = \Gamma \ast (\tilde{h} - \tilde{g})$ with $\tilde{h} \geq 0$ being a function in $C_c^\infty(\mathbb{R}^2)$ such that

$$
\int_{\mathbb{R}^2} \tilde{h} \, dx = \int_{\mathbb{R}^2} \tilde{g} \, dx = 2\pi (2N - \beta),
$$

and $\tilde{g} = (\sum_{i=1}^N 4\pi \delta_{p_i} - \Delta v_1) + \beta \Delta v_3$. Then a solution $v$ of (1.4) is derived by the method of super and subsolutions. Furthermore,

$$
\int_{\mathbb{R}^2} \frac{4K_\beta e^v}{e^{v_1} + K_\beta e^v} \, dx = \int_{\mathbb{R}^2} h \, dx,
$$

by Lemma 2.3, it is known that there is a constant $b$ such that

$$
v(x) \to b \quad \text{as} \quad |x| \to +\infty.
$$

Uniqueness. Assume that $w_i$ with $i = 1, 2$ are two solutions of problem (1.4), by Lemma 2.4.5 in [25], verifying that

$$
w_i(x) \to b_i \quad \text{as} \quad |x| \to +\infty,
$$

where we may assume that $b_1 \geq b_2$. We claim that

$$
w_1 \geq w_2 \quad \text{in} \quad \mathbb{R}^2. \tag{2.10}
$$

Otherwise, it follows by $b_1 \geq b_2$, there exists $x_0 \in \mathbb{R}^2$ such that

$$
w_1(x_0) - w_2(x_0) = \min_{x \in \mathbb{R}^2} (w_1 - w_2)(x) < 0,
$$

then

$$
\Delta(w_1 - w_2)(x_0) \geq 0,
$$

which contradicts the fact that

$$
\Delta(w_1 - w_2)(x_0) = \frac{4K_\beta e^{w_1(x_0)}}{e^{v_1(x_0) - v_2(x_0)} + K_\beta e^{w_1(x_0)}} - \frac{4K_\beta e^{w_2(x_0)}}{e^{v_3(x_0) - v_2(x_0)} + K_\beta e^{w_2(x_0)}} < 0.
$$
Thus, \( w_1 \geq w_2 \) in \( \mathbb{R}^2 \). If \( w_1 \neq w_2 \), then it implies from (2.10) that
\[
\int_{\mathbb{R}^2} g_\beta dx = \int_{\mathbb{R}^2} \frac{4K_\beta e^{v_1}}{e^{v_1-v_2} + K_\beta e^{v_1}} dx > \int_{\mathbb{R}^2} \frac{4K_\beta e^{w_2}}{e^{v_1-v_2} + K_\beta e^{w_2}} dx = \int_{\mathbb{R}^2} g_\beta dx,
\]
which is impossible. Therefore, we have that \( w_1 \equiv w_2 \) in \( \mathbb{R}^2 \).

We conclude that for \( \beta \in (2, 2(N-M)) \), problem (1.4) has a unique solution \( v_\beta \) and \( v_\beta(x) \to b_\beta \) as \( |x| \to +\infty \). Then we may rewrite that
\[
v_\beta = b_\beta + \Gamma \left( \frac{4K_\beta e^{v_\beta}}{e^{v_1} + K_\beta e^{v_\beta}} - g_\beta \right),
\]
where \( \int_{\mathbb{R}^2} \left( \frac{4K_\beta e^{v_\beta}}{e^{v_1} + K_\beta e^{v_\beta}} - g_\beta \right) dx = 0 \). Then by applying Lemma 2.2 that
\[
\left| \Gamma \left( \frac{4K_\beta e^{v_\beta}}{e^{v_1-v_2} + K_\beta e^{v_\beta}} - g_\beta \right) \right| \leq c_0 |x|^{-\frac{\beta-2}{\beta-1}}, \quad \forall \ x > 1
\]
for some constant \( c_6 \) depending on \( \beta \). This completes the proof. \( \square \)

Remark 2.1 (i) Since the mapping \( t \mapsto \frac{4K_\beta e^t}{e^{v_1-v_2} + K_\beta e^t} \) is increasing, the method of super and sub solutions is valid to find out the solution. By the uniqueness and constructing a super solution \( w_1 \) and a sub solution \( w_2 \) such that \( w_1 \geq w_2 \), then the unique solution of (1.4) stays between \( w_1 \) and \( w_2 \). Furthermore, we have that
\[
\int_{\mathbb{R}^2} \frac{4K_\beta e^{v_1}}{e^{v_1-v_2} + K_\beta e^{v_1}} dx \geq \int_{\mathbb{R}^2} g_\beta dx = 2\pi [2(N-M) - \beta]
\]
and
\[
\int_{\mathbb{R}^2} \frac{4K_\beta e^{w_2}}{e^{v_1-v_2} + K_\beta e^{w_2}} dx \leq 2\pi [2(N-M) - \beta].
\]

(ii) From the proof of uniqueness in Proposition 2.7, we conclude a type of Comparison Principle: Let \( w_1, w_2 \) be super and subsolutions of (1.4) respectively, verifying that \( b_1 \geq b_2 \), where
\[
w_i(x) \to b_i \quad \text{as} \quad |x| \to +\infty, \quad i = 1, 2.
\]

Then
\[
w_1 \geq w_2 \quad \text{in} \quad \mathbb{R}^2.
\]

3 Analysis of \( b_\beta \) and \( \Gamma \ast v_\beta \)

In this section, we refine the estimates of \( b_\beta \) and \( \Gamma \ast v_\beta \) by constructing suitable super and sub solutions of problem (1.4) when \( \beta \) approaches to the end points of interval \((2, 2(N-M))\). To this end, let us first set
\[
\beta_0 := N - M + 1
\]
and then (1.4) has a unique solution \( v_{\beta_0} \) and \( \int_{\mathbb{R}^2} g_{\beta_0} dx = 2\pi [2(N-M) - \beta_0] \).

For \( \beta \in (2, 2(N-M)) \), let \( w_{\beta, \tau} = v_{\beta_0} + \tau \) with \( \tau \in \mathbb{R} \) being a given free parameter, and our super and subsolutions of (1.4) will be constructed by varying the parameter \( \tau \).

Proposition 3.1 Let \( \beta \in (\beta_0, 2(N-M)) \) and \( b_\beta \) be derived by Proposition 2.7, then there exists a positive constant \( c_7 > 0 \) independent of \( \beta \) such that
\[
|b_\beta - \ln(2(N-M) - \beta)| \leq c_7.
\]
Proof. Part 1: subsolution. For $\beta \in (\beta_0, 2(N - M))$, denote

$$\tau_{\beta,1} := \left( \ln \left( \frac{(2(N - M) - \beta)2\pi}{d_1} \right) \right)_-,$$

where $a_\pm = \min\{0, a\}$ and $d_1 = \int_{\mathbb{R}^2} 4K_{\beta_0} e^{v_{\beta_0} - v_1} dx$. Let

$$w_{\beta,1} = v_{\beta_0} + \tau_{\beta,1} \quad \text{and} \quad F_{\beta,1} = \frac{4e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}} - g_{\beta},$$

(3.2)

where $g_{\beta}$ is given by (1.5). In particular, we take $\tau_{\beta_0,1} = 0$. Note that

$$-\Delta w_{\beta,1} + \frac{4K_{\beta} e^{v_{\beta,1}}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}} - g_{\beta}
= \left( \frac{4e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}} - g_{\beta} \right) - \left( \frac{4K_{\beta_0} e^{v_{\beta_0}}}{e^{v_1} - v_1 + K_{\beta_0} e^{v_{\beta_0}}} - g_{\beta} \right)
=: F_{\beta,1} - F_{\beta_0,1}.$$ (3.3)

Since the mapping $\beta \mapsto K_{\beta}$ is decreasing, $v_2 \leq 0$, then

$$0 < \int_{\mathbb{R}^2} \frac{4e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}} dx
\leq e^{\tau_{\beta,1}} \int_{\mathbb{R}^2} 4K_{\beta} e^{v_{\beta_0}} e^{-v_1} dx
\leq e^{\tau_{\beta,1}} \int_{\mathbb{R}^2} 4K_{\beta_0} e^{v_{\beta_0} - v_1} dx
= 2\pi \left( 2(N - M) - \beta \right).$$

From the fact that $\int_{\mathbb{R}^2} g_{\beta} dx = 2\pi[2(N - M) - \beta]$, we have that

$$\int_{\mathbb{R}^2} F_{\beta,1} dx \leq 0.$$ (3.4)

Let

$$T_{\beta,1} := \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{\beta,1} dx \quad \text{and} \quad \tilde{F}_{\beta,1} := -F_{\beta,1} + T_{\beta,1}(-\Delta v_3),$$

then we have that $\int_{\mathbb{R}^2} \tilde{F}_{\beta,1} dx = 0$. Obviously, $\int_{\mathbb{R}^2} F_{\beta_0} dx = \int_{\mathbb{R}^2} \tilde{F}_{\beta_0} dx = 0.$

Claim 1: There exist $\nu, c_8 > 0$ such that for any $\beta \in [\beta_0, 2(N - M))$,

$$\|\Gamma * \tilde{F}_{\beta,1}\|_{L^\infty(\mathbb{R}^2)} \leq \nu$$ (3.5)

and

$$|\Gamma * \tilde{F}_{\beta,1}(x)| \leq c_8 |x|^{\frac{\beta + 2}{\beta - 1}} \quad \text{for} \quad |x| > 4\varepsilon.$$ (3.6)

Proof of Claim 1: Since the function $t \mapsto \frac{4e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0} + v_3}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}}$ is increasing and $\tau_{\beta,1} \leq 0$, then we have that

$$\frac{4e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}}{e^{v_1} - v_1 + e^{\tau_{\beta,1}}K_{\beta} e^{v_{\beta_0}}} \leq \frac{4K_{\beta_0} e^{v_{\beta_0}}}{e^{v_1} - v_1 + K_{\beta_0} e^{v_{\beta_0}}},$$

thus,

$$-e^{2(N - M)c_1} |x|^{-\beta} \leq \tilde{F}_{\beta,1}(x) \leq -e^{2(N - M)c_1} |x|^{-\beta}, \quad \forall \ |x| > 2\varepsilon,$$

and

$$\|\tilde{F}_{\beta,1}\|_{L^\infty(\mathbb{R}^2)} \leq 4 + \|g_1\|_{L^\infty(\mathbb{R}^2)} + \|g_2\|_{L^\infty(\mathbb{R}^2)} + (4\pi + 2)(N - M)\|\Delta v_3\|_{L^\infty(\mathbb{R}^2)} := a_0.$$
From Lemma 3.2, we have that (3.6) holds true for \( x \in \mathbb{R}^N \setminus B_{4e}(0) \) and then \( \Gamma * \tilde{F}_{\beta,1} \) is bounded in \( \mathbb{R}^N \setminus B_{4e}(0) \). So we only have to prove that
\[
2\pi|\Gamma * \tilde{F}_{\beta,1}(x)| = \left| \int_{\mathbb{R}^2} \ln |x-y| \tilde{F}_{\beta,1}(y) dy \right| \leq \nu_0, \quad \forall |x| \leq 4e. \tag{3.7}
\]
In fact, for \( |x| \leq 4e \), we observe that
\[
\left| \int_{B_{r_0}(0)} \ln |x-y| \tilde{F}_{\beta,1}(y) dy \right| \leq \|\tilde{F}_{\beta,1}\|_{L^\infty(\mathbb{R}^2)} \int_{B_{r_0}(0)} |\ln |x-y|| dy \leq \pi a_0 \left( r_0^2 (|\ln r_0| + 1) + 4e \right)
\]
and
\[
\left| \int_{B_{r_0}(0)} \ln |x-y| \tilde{F}_{\beta,1}(y) dy \right| \leq e^{2(N-M)c_1} \int_{B_{r_0}(0)} \ln(4e + |y|)|y|^{-\beta} dy \leq e^{2(N-M)c_1} \int_{B_{r_0}(0)} \ln(4e + |y|)|y|^{-\beta_0} dy,
\]
which imply (3.5), where \( r_0 > 1 \). Thus, Claim 1 holds true.

Now we continue to construct a subsolution. Let
\[
v_\beta = w_{\beta,1} + \Gamma * (\tilde{F}_{\beta,1} + F_{\beta_0,1}) - 2\nu_1,
\]
where \( \Gamma * (\tilde{F}_{\beta,1} + F_{\beta_0,1}) - 2\nu_1 \leq 0 \) for some \( \nu_1 \) independent of \( \beta \) by Claim 1. Then we have that
\[
-\Delta v_\beta + \frac{4K_\beta e^{w_{\beta,1}}}{e^{v_1-v_2} + K_\beta e^{v_0}} - g_\beta \leq -\Delta w_{\beta,1} + \tilde{F}_{\beta,1} + F_{\beta_0,1} + \frac{4K_\beta e^{w_{\beta,1}}}{e^{v_1-v_2} + K_\beta e^{w_{\beta,1}}} - g_\beta = T_{\beta,1}(-\Delta v_\beta) \leq 0,
\]
where we used (3.4). Then \( v_\beta \) is a subsolution of (1.3) for \( \beta \in [\beta_0, 2(N-M)] \).

**Part 2: supersolution.** For \( \beta \in (\beta_0, 2(N-M)) \), we denote
\[
\tau_{\beta,2} = \left( \ln(2(N-M) - \beta) + \ln \frac{2\pi}{d_2} \right) - , w_{\beta,2} = v_{\beta_0} + \tau_{\beta,2} \quad \text{and} \quad F_{\beta,2} = \frac{4e^{\tau_{\beta,2}} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_{\beta,2}} K_\beta e^{v_{\beta_0}}} - g_\beta,
\]
where \( d_2 = \int_{\mathbb{R}^2} \frac{4K_2(2(N-M)) e^{v_{\beta_0}}}{e^{v_1-v_2} + K_2(2(N-M)) e^{v_{\beta_0}}} dx \). Then we derive that
\[
-\Delta w_{\beta,2} + \frac{4K_\beta e^{w_{\beta,2}}}{e^{v_1-v_2} + K_\beta e^{w_{\beta,2}}} - g_\beta = F_{\beta,2} - F_{\beta_0,2},
\]
where \( F_{\beta_0,2} = F_{\beta_0,1} \) with \( \tau_{\beta_0,2} = 0 \). By the decreasing monotonicity of the function \( \beta \mapsto K_\beta \), we have that
\[
\int_{\mathbb{R}^2} \frac{4e^{\tau_{\beta,2}} K_\beta e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_{\beta,2}} K_\beta e^{v_{\beta_0}}} dx \geq \int_{\mathbb{R}^2} \frac{4e^{\tau_{\beta,2}} K_2(2(N-M)) e^{v_{\beta_0}}}{e^{v_1-v_2} + e^{\tau_{\beta,2}} K_2(2(N-M)) e^{v_{\beta_0}}} dx \geq e^{\tau_{\beta,2}} \int_{\mathbb{R}^2} \frac{4K_2(2(N-M)) e^{v_{\beta_0}}}{e^{v_1-v_2} + K_2(2(N-M)) e^{v_{\beta_0}}} dx = 2\pi [2(N-M) - \beta],
\]
which, together with \(\int_{\mathbb{R}^2} g_\beta dx = 2\pi[2(N - M) - \beta]\), implies that \(\int_{\mathbb{R}^2} F_{\beta, 2} dx \geq 0\). Thus we have \(T_{\beta, 2} := \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{\beta, 2} dx \geq 0\). Let \(\tilde{F}_{\beta, 2} := -F_{\beta, 2} + T_{\beta, 2}(-\Delta v_3)\). For the choice of \(\tau_{\beta, 2}\), Claim 1 holds true also, that is, there exists \(\nu_2 > 0\) such that for \(\beta \in [\beta_0, 2(N - M)]\),

\[
\|\tilde{F}_{\beta, 2}\|_{L^\infty(\mathbb{R}^2)} \leq \nu_2.
\]

Let

\[
\tilde{v}_\beta = w_{\beta, 2} + \nu_1 + \nu_2 + \Gamma * (\tilde{F}_{\beta, 2} + F_{\beta_0, 2}) + \nu_3,
\]

where \(\nu_3 = \max\{0, \ln \frac{d_1}{d_3}\}\). Therefore \(\nu_2 \leq \tilde{v}_\beta\) in \(\mathbb{R}^2\). Since \(\nu_1 + \nu_2 + \Gamma * (\tilde{F}_{\beta, 2} + F_{\beta_0, 2}) \geq 0\), then we have that

\[
-\Delta \tilde{v}_\beta + \frac{4K_\beta e^{v_\beta}}{e^{u_1 - v_2} + K_\beta e^{v_\beta}} - g_\beta \\
\geq -\Delta w_{\beta, 2} + \tilde{F}_{\beta, 2} + F_{\beta_0, 2} + \frac{4K_\beta e^{u_\beta, 2}}{e^{u_1 - v_2} + K_\beta e^{u_\beta, 2}} - g_\beta \\
= T_{\beta, 2}(-\Delta v_3) \\
\geq 0,
\]

where \(T_{\beta, 2} \geq 0\). Then \(\tilde{v}_\beta\) is a supersolution of (1.4) for \(\beta \in (\beta_0, 2(N - M))\).

From Remark 2.1 (i), we have that for any \(\beta \in (\beta_0, 2(N - M))\),

\[
\tau_\beta \leq \nu_2 \leq \tilde{v}_\beta \quad \text{in} \quad \mathbb{R}^2,
\]

which implies that (3.1) holds. We complete the proof. \(\square\)

**Proposition 3.2** Let \(\beta \in (2, \beta_0)\) and \(b_\beta\) be derived by Proposition 2.1, then there exists a positive constant \(c_9 > 0\) independent of \(\beta\) such that

\[
2\ln(\beta - 2) - c_9 \leq b_\beta \leq \ln(\beta - 2) + c_9.
\]

**Proof.** Part I: subsolution. For \(\beta \in (2, \beta_0)\), we denote

\[
\tau_{\beta, 3} = \left(2\ln(\beta - 2) + \ln\left(\frac{2\pi(\beta_0 - 2)}{d_3}\right)\right)^-,
\]

\[w_{\beta, 3} = \nu_{\beta_0} + \tau_{\beta, 3}\quad\text{and}\quad F_{\beta, 3} = \frac{4e^{v_{\beta_0}}K_\beta e^{v_{\beta_0}}}{e^{u_1 - v_2} + e^{v_{\beta_0}}K_\beta e^{v_{\beta_0}}} - g_\beta,
\]

where \(d_3 = 4e^{v_{\beta_0}}L_\infty(\mathbb{R}^2)\left((\beta_0 - 2)\int_{B_{\beta_0}(0)} e^{-v_1} dx + 2\pi\right)\). In particular, we set \(F_{\beta_0, 3} = F_{\beta_0, 1}\). Direct computation shows that

\[
-\Delta w_{\beta, 3} + \frac{4K_\beta e^{u_{\beta, 3}}}{e^{u_1 - v_2} + K_\beta e^{u_{\beta, 3}}} - g_\beta = F_{\beta, 3} - F_{\beta_0, 1}.
\]

Observe that

\[
0 < \int_{\mathbb{R}^2} \frac{4e^{v_{\beta, 3}}K_\beta e^{v_{\beta_0}}}{e^{u_1 - v_2} + e^{v_{\beta, 3}}K_\beta e^{v_{\beta_0}}} dx
\]

\[
\leq 4e^{\beta, 3}e^{v_{\beta_0}}L_\infty(\mathbb{R}^2) \int_{\mathbb{R}^2} K_\beta e^{-v_1} dx
\]

\[
\leq 4e^{\beta, 3}e^{v_{\beta_0}}L_\infty(\mathbb{R}^2) \left(\int_{B_{\beta_0}(0)} e^{-v_1} dx + \int_{\mathbb{R}^2 \setminus B_{\beta_0}(0)} |x|^{-\beta} dx\right)
\]

\[
\leq 4e^{\beta, 3}e^{v_{\beta_0}}L_\infty(\mathbb{R}^2) \left(\int_{B_{\beta_0}(0)} e^{-v_1} dx + \frac{2\pi}{\beta - 2}\right)
\]

\[
\leq \frac{4e^{\beta, 3}e^{v_{\beta_0}}L_\infty(\mathbb{R}^2)}{\beta - 2} \left((\beta_0 - 2)\int_{B_{\beta_0}(0)} e^{-v_1} dx + 2\pi\right)
\]

\[
\leq 2\pi(\beta_0 - 2),
\]

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where $e^{\gamma_{\beta,3}} \leq \frac{2\beta(\beta_0 - 2)}{d_3}(\beta - 2)^2$. Then we have that for $\beta \in (2, \beta_0)$,

$$\beta - 2(N - M) < \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{\beta,3} \, dx \leq (\beta_0 - 2) - [2(N - M) - \beta] < 0.$$  

Thus,

$$T_{\beta,3} := \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{\beta,3} \, dx \in \left(2 - 2(N - M), 0\right).$$  

Claim 2: Let $\tilde{F}_{\beta,3} := -F_{\beta,3} + T_{\beta,3}(-\Delta v_3)$, then there exists $\nu_3 > 0$ such that for any $\beta \in (2, \beta_0)$,

$$|\Gamma \ast \tilde{F}_{\beta,3}(x)| \leq \nu_3, \quad \forall \ x \in \mathbb{R}^2. \quad (3.12)$$

The proof of Claim 2 is postponed to the end of this proof and we continue to prove Proposition 3.2 Reset

$$\nu_\beta = w_{\beta,3} - \nu_3 - \nu_1 + \Gamma \ast (\tilde{F}_{\beta,3} + F_{\beta_0,1}).$$

Note that $-\nu_3 - \nu_1 + \Gamma \ast (\tilde{F}_{\beta,3} + F_{\beta_0}) \leq 0$ and then by (3.11),

$$-\Delta \nu_\beta + \frac{4K_\beta e^{\nu_{\beta,3}}}{e^{u_{\beta,3}} + K_\beta e^{w_{\beta,3}}} - g_\beta \leq -\Delta w_{\beta,3} + \tilde{F}_{\beta,3} + F_{\beta_0,1} + \frac{4K_\beta e^{w_{\beta,3}}}{e^{u_{\beta,3}} + K_\beta e^{w_{\beta,3}}} - g_\beta \leq T_{\beta,3}(-\Delta v_3) \leq 0.$$

Then $\nu_\beta$ is a subsolution of (3.14) for $\beta \in (2, \beta_0)$. From Lemma 2.2, we have that

$$\lim_{|x| \to +\infty} \Gamma \ast (\tilde{F}_{\beta,3} + F_{\beta_0,3})(x) = 0,$$

and then

$$\lim_{|x| \to +\infty} \nu_\beta(x) = b_{\beta_0} + 2\ln(\beta - 2) + \ln(2\pi(\beta_0 - 1)/d_3) - \nu_3 - \nu_1. \quad (3.13)$$

Part II: supersolution. For $\beta \in (2, \beta_0)$, let

$$\tau_{\beta,4} = (\ln(\beta - 2) - \ln d_4) \quad (3.14)$$

with $d_4 = \frac{e^{-2(N - M)c_1} e^{b_{\beta_0}(2\beta_0 - 2)^2} + e^{b_{\beta_0}2\beta_0}}{e^{b_{\beta_0}(2\beta_0 - 2)^2} + e^{b_{\beta_0}2\beta_0}}$.

Since $\lim_{|x| \to +\infty} v_{\beta_0}(x) = b_{\beta_0}$, there exists $R_0 > 2r_0$ such that

$$\frac{1}{2} e^{b_{\beta_0}} \leq e^{\nu_{\beta_0}} \leq 2e^{b_{\beta_0}}, \quad \forall \ |x| \geq R_0$$

and then

$$e^{\gamma_{\beta,4}} K_{\beta} e^{\nu_{\beta_0}} \leq 2(\beta_0 - 2)e^{b_{\beta_0}}, \quad \forall \ |x| \geq R_0,$$

for $R \geq 2R_0$,

$$\int_{B_R(0)} \frac{4e^{\gamma_{\beta,4}} K_{\beta} e^{\nu_{\beta_0}}}{e^{u_{\beta,3}} + e^{\gamma_{\beta,4}} K_{\beta} e^{\nu_{\beta_0}}} \, dx \geq \int_{B_{R_0}(0) \setminus B_{2r_0}(0)} \frac{4e^{\gamma_{\beta,4}} K_{\beta} e^{\nu_{\beta_0}}}{1 + e^{\gamma_{\beta,4}} K_{\beta} e^{\nu_{\beta_0}}} \, dx$$

$$\geq \frac{2e^{\gamma_{\beta,4}} e^{b_{\beta_0}}}{1 + 2(\beta_0 - 2)e^{b_{\beta_0}}} \int_{B_R(0) \setminus B_{2r_0}(0)} e^{-2(N - M)c_1} |x|^{-\beta} \, dx$$

$$\geq \frac{e^{-2(N - M)c_1}}{d_4} \frac{4\pi e^{b_{\beta_0}}}{1 + 2(\beta_0 - 2)e^{b_{\beta_0}}}(2r_0)^{2-\beta_0} \left(1 - \left(\frac{R}{2r_0}\right)^{2-\beta}\right)$$

$$\geq 4\pi (N - M - 1) \left(1 - \left(\frac{R}{2r_0}\right)^{2-\beta}\right),$$

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where we have used the estimate (2.1). Thus, passing to the limit as \( R \to +\infty \), there holds
\[
\int_{\mathbb{R}^2} \frac{4e^{\tau_3.4}K(e^{\nu_0})}{e^{v_1-v_2}+e^{\tau_3.4}K} \, dx \geq 4\pi(N-M-1).
\]

Let \( F_{\beta,4} = \frac{4e^{\tau_3.4}K(e^{\nu_0})}{e^{v_1-v_2}+e^{\tau_3.4}K} - g_\beta \), then from \( \int_{\mathbb{R}^2} g_\beta \, dx = 2\pi[N(M-M) - \beta] \), we have that
\[
T_{\beta,4} = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{\beta,4} \, dx \geq \beta - 2 > 0. \tag{3.15}
\]

Let
\[
E_\beta = -F_{\beta,4}(\chi_{B_{2r_0}}(0)) + T_{\beta,4}(-\Delta v_3)
\]
then \( \int_{\mathbb{R}^2} E_\beta \, dx = 0 \). From Claim 1 and Lemma 2.1, we obtain that
\[
\| \Gamma * F_{\beta,0,1} \|_{L^\infty(\mathbb{R}^2)} \leq \nu_1 \quad \text{and} \quad \| \Gamma * E_\beta \|_{L^\infty(\mathbb{R}^2)} \leq \nu_4, \tag{3.16}
\]
where \( \nu_4 > 0 \) is independent of \( \beta \).

Denote
\[
v_\beta = w_\beta + \nu_1 + \nu_4 + \nu_5 + \Gamma * (E_\beta + F_{\beta,0,1}),
\]
where \( \nu_5 = \ln(2(\beta_0 - 2)/d_3) + \ln d_4 \).

Note that
\[
\nu_1 + \nu_4 + \Gamma * (E_\beta + F_{\beta,0,1}) \geq 0,
\]
by (3.15), we get
\[
-\Delta \bar{v}_\beta + \frac{4K_\beta e^{\bar{v}_\beta}}{e^{v_1-v_2}+K_\beta e^{\nu_0}} - g_\beta
\]
\[
\geq -\Delta w_{\beta,4} + E_\beta + F_{\beta,0,1} + \frac{4K_\beta e^{\nu_3}}{e^{v_1-v_2}+K_\beta e^{\nu_3}} - g_\beta
\]
\[
= T_{\beta,4}(-\Delta v_3) + F_{\beta,4}(\chi_{\mathbb{R}^2\setminus B_{2r_0}}(0))
\]
\[
\geq 0.
\]

Then \( \bar{v}_\beta \) is a supersolution of (1.4) for \( \beta \in (2, \beta_0) \). By the definition of \( \nu_5 \), we have that
\[
\lim_{|x| \to +\infty} v_\beta(x) \leq \lim_{|x| \to +\infty} \bar{v}_\beta(x),
\]
then it infers by Remark 2.1 (ii) that \( v_\beta \leq \bar{v}_\beta \) in \( \mathbb{R}^2 \). From Remark 2.1 (i), for any \( \beta \in (2, \beta_1) \), we get that \( \bar{v}_\beta \leq v_\beta \leq \bar{v}_\beta \) in \( \mathbb{R}^2 \), which implies (3.9).

**Finally we prove Claim 2.** Note that there exists \( z_0 > 0 \) such that for \( \beta \in (2, \beta_0) \),
\[
-\frac{z_0}{\Gamma} \leq F_{\beta,3}(x) \leq c_9(1 + |x|)^{-\frac{\nu_3}{\nu_3 + 1}} + z_0, \quad \forall x \in \mathbb{R}^2. \tag{3.17}
\]

In fact, since \( g_\beta \) has compact support, then
\[
\| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)} \leq 4 + \| g_\beta \|_{L^\infty(\mathbb{R}^2)}
\]
\[
= 4 + \| g_1 \|_{L^\infty(\mathbb{R}^2)} + \| g_2 \|_{L^\infty(\mathbb{R}^2)} + 2(N-M)\| \eta_0 \|_{L^\infty(\mathbb{R}^2)}
\]
and for \( x \in \mathbb{R}^2 \setminus B_{r_0}(0) \),
\[
\frac{1}{c_{10}} e^{\tau_{\beta,3}} |x|^{-\beta} \leq \tilde{F}_{\beta,3}(x) \leq c_{10} e^{\tau_{\beta,3}} |x|^{-\beta},
\]
where \( c_{10} > 1 \) is independent of \( \beta \) and we recall that \( e^{\tau_{\beta,3}} \leq \frac{2\pi(\beta_0 - 2)}{d_3} (\beta - 2)^2 \).
Estimates for $|x| \leq 4e$. Observe that

$$
\int_{\mathbb{R}^2} \ln |x-y| \tilde{F}_{\beta,3}(y) \, dy = \left( \int_{B_{r_0}(0)} + \int_{B_2(x) \setminus B_{r_0}(0)} + \int_{\mathbb{R}^2 \setminus (B_{r_0}(0) \cup B_2(x))} \right) \ln |x-y| \tilde{F}_{\beta,3}(y) \, dy,
$$

where

$$
\left| \int_{B_{r_0}(0)} \ln |x-y| \tilde{F}_{\beta,3}(y) \, dy \right| \leq \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)} \int_{B_{r_0}(0)} \ln |x-y| \, dy \leq \pi r_0^2 (|\ln r_0| + 1) \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)},
$$

$$
\left| \int_{B_2(x) \setminus B_{r_0}(0)} \ln |x-y| \tilde{F}_{\beta,3}(y) \, dy \right| \leq \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)} \int_{B_2(x)} \ln |x-y| \, dy \leq 4(1 + \ln 2) \pi \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)}
$$

and

$$
0 \leq \int_{\mathbb{R}^2 \setminus (B_{r_0}(0) \cup B_2(x))} \ln |x-y| \tilde{F}_{\beta,3}(y) \, dy \leq c_{11} e^{r_{\beta,3}},
$$

where $c_{11} > 1$ is independent of $\beta$. Thus, (3.12) holds true for $|x| \leq 4e$.

Estimates for $|x| > 4e$. This is very similar to the proof of Lemma 2.2. We rewrite

$$
2\pi \Gamma * \tilde{F}_{\beta,3}(x) =: I_1(x) + I_2(x) + I_3(x), \quad \text{for } |x| > 4e.
$$

Then

$$
|I_1(x)| \leq 2 \frac{R}{|x|} \int_{B_R(0)} |\tilde{F}_{\beta,3}(y)| \, dy \leq 2\pi \frac{R^3}{|x|} \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)}.
$$

For $z \in B_{1/2}(e_x)$, we have that $0 < \tilde{F}_{\beta,3}(|x-z|) \leq c_{10} e^{r_{\beta,3}} |x-z|^{-\beta}$ and then

$$
|I_2(x)| \leq c_{10} e^{r_{\beta,3}} |x|^{2-\beta} \int_{B_{1/2}(e_x)} \ln |e_x - z| \, dz \leq c_{12} |x|^{2-\beta},
$$

where $c_{12} > 0$ is independent of $\beta$.

For $z \in \mathbb{R}^2 \setminus (B_R/|x|(0) \cup B_{1/2}(e_x))$, we have that $0 < \tilde{F}_{\beta,3}(|x-z|) \leq c_{10} e^{r_{\beta,3}} |x-z|^{-\beta}$, then

$$
0 \leq I_3(x) \leq 2\pi c_{13} e^{r_{\beta,3}} \left( \frac{R^{2-\beta}}{\beta - 2} \ln(e + \frac{R}{|x|}) + \frac{2\pi c_{13}}{(\beta - 2)^2} R^{2-\beta} \right) \leq c_{14} R^{2-\beta},
$$

where $c_{13}, c_{14} > 0$ are independent of $\beta$.

Thus, taking $R = |x|^{\frac{\beta}{\beta - 2}}$ and $|x| > 4e$, we have that

$$
2\pi \Gamma * \tilde{F}_{\beta,3}(x) \geq -2 \frac{R^3}{|x|} \| \tilde{F}_{\beta,3} \|_{L^\infty(\mathbb{R}^2)} - c_{12} |x|^{2-\beta} \geq -c_{15} |x|^{-\frac{\beta-2}{\beta+1}}
$$

and

$$
2\pi \Gamma * \tilde{F}_{\beta,3}(x) \leq c_{15} |x|^{-\frac{\beta-2}{\beta+1}},
$$

where $c_{15} > 0$ is independent of $\beta$. Therefore, (3.17) holds true. \qed
Proof of Theorem 1.1. From Proposition 2.1, problem (1.4) has a unique solution $w_\beta$, which verifies (2.2). Proposition 3.1 and Proposition 3.2 show that

$$|b_\beta - \ln(2(N - M) - \beta)| \leq c_7 \quad \text{for} \quad \beta \in (\beta_0, 2(N - M)) \quad (3.18)$$

and

$$2\ln(\beta - 2) - c_9 \leq b_\beta \leq \ln(\beta - 2) + c_9 \quad \text{for} \quad \beta \in (2, \beta_0), \quad (3.19)$$

which imply (1.7) and (1.8) respectively. □

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