Online Weighted Matching: Breaking the $\frac{1}{2}$ Barrier

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Abstract

Online matching and its variants are some of the most fundamental problems in the online algorithms literature. In this paper, we study the online weighted bipartite matching problem. Karp et al. (STOC 1990) gave an elegant algorithm in the unweighted case that achieves a tight competitive ratio of $1 - \frac{1}{e}$. In the weighted case, however, we can easily show that no competitive ratio is obtainable without the commonly accepted free disposal assumption. Under this assumption, it is not hard to prove that the greedy algorithm is $\frac{1}{2}$ competitive, and that this is tight for deterministic algorithms. We present the first randomized algorithm that breaks this long-standing $\frac{1}{2}$ barrier and achieves a competitive ratio of at least 0.501. In light of the hardness result of Kapralov et al. (SODA 2013) that restricts beating a $\frac{1}{2}$ competitive ratio for the monotone submodular welfare maximization problem, our result can be seen as strong evidence that solving the weighted bipartite matching problem is strictly easier than submodular welfare maximization in the online setting. Our approach relies on a very controlled use of randomness, which allows our algorithm to safely make adaptive decisions based on its previous assignments.

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1 Introduction

Matchings are fundamental structures in graph theory that play a crucial role in combinatorial optimization. An enormous amount of effort has gone into designing efficient algorithms for finding maximum matchings in terms of cardinality or the total weight of its allocation. In particular, matchings in bipartite graphs have found countless applications in settings where it is desirable to assign entities from one set to those in another set (e.g., matching students to schools, physicians to hospitals, computing tasks to servers, and impressions in online media to advertisers). Due to the tremendous growth of matching markets in digital domains, efficient online matching algorithms have become increasingly important. In particular, search engine companies have created opportunities for online matching algorithms to have enormous impact in multi-billion dollar advertising markets. Motivated by these applications, we consider the problem of matching a set $I$ of impressions that arrive one by one to a set $A$ of advertisers that are given in advance. When an impression arrives, its edges to the advertisers are revealed and an irrevocable decision has to be made about which advertiser the impression should be assigned to. Karp et al. [KMT11] gave an elegant online algorithm called RANKING to find matchings in unweighted bipartite graphs with a competitive ratio of $1 - 1/e$, and they also proved that this is the best competitive ratio that can be achieved.

The situation in the weighted case is much less clear. This is partly due to the fact that no competitive algorithm can be designed without the free disposal assumption. To see this, consider two instances of the online bipartite weighted matching problem, each with one advertiser $a$ and two impressions. The weight of the first impression to $a$ is 1 in both instances, and the weight of the second impression to $a$ is zero in the first instance and $L$, for some arbitrarily large $L$, in the second instance. The online algorithm has no way to distinguish between these two instances, even after the first impression arrives. When the first impression arrives, the algorithm must decide whether or not to assign this impression to $a$. Not assigning it gives a competitive ratio of zero for the first instance, and assigning it gives a competitive ratio of $1/L$, which can be arbitrarily small, for the second. Note that assigning both impressions to $a$ is not an option in this setting. This example is the primary motivation for allowing assignments of multiple impressions to a single advertiser.

**Free Disposal Assumption.** In display advertising applications, assigning more impressions to an advertiser than they paid for only makes them happier. In other words, we can assign more than one impression to any given advertiser $a \in A$. However, instead of achieving the weights of all edges assigned to $a$, we only receive the maximum weight of the edges assigned to $a$. Concretely, the total weighted achieved by an allocation is equal to $\sum_{a \in A} \max_{i \in T_a} w_{i,a}$, where $T_a$ is the set of impressions assigned to $a$ and $w_{i,a}$ is the weight of the edge between $i$ and $a$. In the display advertising literature [FKM+09, KMZ13], the free disposal assumption is well received and widely applied because of its natural economic interpretation. With free disposal, it is not hard to reduce weighted bipartite matching to the monotone submodular welfare maximization problem, where we can apply known 1/2-competitive greedy algorithms [FNW78, LLN06].

1.1 Our Contributions

For almost thirty years since the seminal work of Karp et al. [KVV90], finding an algorithm for the online weighted bipartite matching problem that achieves a competitive ratio greater than 1/2 has been a tantalizing open problem. In this paper, we introduce the STOCHASTICGREEDY algorithm and answer this question in the affirmative, breaking the long-standing 1/2 competitive ratio barrier (under the free disposal assumption).

**Theorem 1.1.** There exists a 0.501-competitive algorithm for the online weighted bipartite matching problem.

Given the hardness result of Kapralov et al. [KPV13] that restricts beating a competitive ratio of 1/2 for
monotone submodular welfare maximization, our algorithm can be seen as strong evidence that solving weighted bipartite matching is strictly easier than submodular welfare maximization in an online setting.

One of our main technical contributions is the controlled use of randomness in **StochasticGreedy**, which allows the algorithm to safely make adaptive decisions based on past assignments and prevents the cascading of conditional probabilities in our analysis. A more subtle feature of our use of randomness is that in every step of the algorithm, expected values of random variables are computed over all possible branches of the randomized algorithm instead of being conditioned on the current state. This ensures that most sequences of variables in the algorithm are deterministic quantities governed solely by the input graph and arrival order of the impressions. Our method for making adaptive decisions combined with the limited randomness of the algorithm allows us to efficiently maintain these expected values (over all possible branches of the algorithm) using dynamic programming. Lastly, we introduce a mechanism called **DistributeExcess** in our analysis, which is not part of the algorithm but allows us to systematically redistribute the extra marginal gain that the algorithm produces and improve upon the greedy algorithm.

### 1.2 Related Works

We first draw attention to two very recent works of Huang and Tao [HT19, Hua19] that build on an earlier version of this paper that appeared on arXiv in 2017 [Zad17]. These works “distill a key ingredient from the algorithm of Zadimoghaddam” and enhance this idea by using the online primal-dual framework [DJK13] and an improved online correlated selection scheme to achieve an improved competitive ratio of 0.514 for the same online weighted bipartite matching problem. In this version of our paper, we have tried to improve and simplify the presentation of our algorithm and analysis so that our approach is easier to understand.

The literature online weighted bipartite matching algorithms is extensive, but most of these works are devoted to achieving competitive ratios greater than $1/2$ (usually $1 - 1/e$ or $1 - \varepsilon$) by assuming that advertisers have large capacities or that some stochastic information about the arrival order of the impressions is known in advance. There are many exciting papers in this area, but we only list a few of the leading works and refer interested readers to the excellent survey of Mehta [Meh13]. We note that there have recently been several significant advances in more general settings, including different arrival models and general (non-bipartite) graphs [HKT'18, GKS19, GKM'19, HPT'19].

**Large Capacities.** Exploiting the large capacities assumption to beat the $1/2$ competitive ratio barrier dates back two decades ago to [KP00]. Feldman et al. [FKM'09] gave a primal-dual algorithm that achieves a competitive ratio of $1 - 1/e$ assuming that each advertiser has a large capacity, where the capacity denotes the number of impressions that can be assigned to it (e.g., the Display Ads problem). Under similar assumptions, the same competitive ratio was obtained for the Budgeted Allocation problem [MSVV05, BJN07], in which advertisers have some budget constraint on the total weight that can be assigned to them rather than the number of impressions. From a theoretical point of view, one of the primary goals in the online matching literature is to provide algorithms with competitive ratio greater than $1/2$ without making any assumption on the capacities of advertisers. Without loss of generality, we can assume every advertiser has capacity one, since we can replace each advertiser $a$ with capacity $C_a$ by $C_a$ identical copies of $a$, each with unit capacity.

**Stochastic Arrivals.** If we have knowledge about the arrival patterns of impressions, then we can often leverage this information to design better algorithms. Typical stochastic assumptions include assuming the impressions are drawn from some known or unknown distribution [FMMM09, KMT11, DJSW11, HMZ11, MGS12, MP12, JL13] or that the impressions arrive in a random order [GM08, DH09, FHK'10, MY11, MGZ12, MWZ15, HTWZ19]. These works achieve a $1 - \varepsilon$ competitive ratio if the large capacity assumption holds in addition to the stochastic assumptions, or at least $1 - 1/e$ for arbitrary capacities. Korula et al. [KMZ18] show that the greedy algorithm is $0.505$-competitive for the more general problem of submodular welfare
maximization if the impressions arrive in random order, without making any assumptions on the capacities. The random order assumption is particularly justified because Kapralov et al. [KPV13] show that beating 1/2 for submodular welfare maximization in the oblivious adversary model is equivalent to proving \( \text{NP} = \text{RP} \).

## 2 Preliminaries

Let \( A \) be the set of advertisers, \( I \) be the set of \( n \) impressions, and \( w_{i,a} \) denote the nonnegative weight of the edge between impression \( i \) and advertiser \( a \). If there is not an edge between \( i \) and \( a \), we introduce an edge of weight zero to simplify the notation. The set of advertisers is given in advance, and the impressions arrive one by one at times \( t = 1, 2, \ldots, n \). We do not assume \( n \) is known to the algorithm. When an impression \( i \) arrives at time \( t_i \), all edge weights incident to \( i \) are revealed to the algorithm, that is, \( w_{i,a} \) for all \( a \in A \), and the algorithm has to assign \( i \) to one of the advertisers at this time. This is an irrevocable decision and cannot be changed later. At the end of the algorithm, if more than one impression is assigned to an advertiser, only the impression with the maximum weight is kept. The rest are discarded and do not counted towards the total weight of the allocation. This is known as the free disposal assumption. The objective is to maximize the total weight of maximum-valued edges assigned to the advertisers, that is, \( \sum_{a \in A} \max_{i \in T_a} w_{i,a} \), where \( T_a \) is the set of impressions assigned to \( a \).

In this paper, we assume that the impressions arrive in an adversarial order. Specifically, we deal with an oblivious adversary, that is, an adversary that does not have access to the outcomes of the random choices made by the algorithm, and instead has to fix the input graph and arrival order in advance. We use the standard notion of competitive ratio to measure the performance of our online algorithm. For a randomized bipartite matching algorithm in this adversarial model, the competitive ratio is defined to be the worst-case ratio \( \min_{G(A, I, w), \text{order of } i} E[\text{ALG}] / \text{OPT} \), where ALG is a random variable denoting the value of the objective function achieved by the algorithm and OPT is the maximum objective value attained offline.

We present our main randomized online algorithm \textsc{StochasticGreedy} in Section 3. This algorithm uses randomness in a very controlled manner, so we deliberately use a sans serif font and "upper camel case" to distinguish quantities that are random variables. To evaluate how much marginal value (i.e., increase in the objective function) can be achieved by assigning an impression \( i \) to an advertiser \( a \) at every point in the algorithm, we need to keep track of the maximum weight assigned to \( a \) by \textsc{StochasticGreedy}. Therefore, we let the random variable \( \text{MaxW}_a^t \) denote the maximum weight assigned to \( a \) by \textsc{StochasticGreedy} up to (and including) time \( t \) for every \( 0 \leq t \leq n \). Since assignments are made at times \( t = 1, 2, \ldots, n \), we define \( \text{MaxW}_a^0 \) to be zero for all \( a \in A \). Next, we define the random variable \( \text{Gain}_{i,a} \) to be the marginal gain of assigning impression \( i \) to advertiser \( a \). Formally, we have \( \text{Gain}_{i,a} = (w_{i,a} - \text{MaxW}_a^{t_i-1})^+ \), where \((x)^+\) is the function \( \max\{0, x\} \) and \( t_i \) is the arrival time of impression \( i \). We note that \( \text{Gain}_{i,a} \) depends on the random choices that \textsc{StochasticGreedy} makes before \( i \) arrives.

We let ALG be the total weight achieved by \textsc{StochasticGreedy}. Since only the maximum weight edge assigned to each advertiser contributes to the total weight of the final allocation, we have \( \text{ALG} = \sum_{a \in A} \text{MaxW}_a^n \). We can also interpret the total weight by letting \( \text{ALG} = \sum_{i \in I} \text{MarginalGain}_i \), where \( \text{MarginalGain}_i \) is the random variable that denotes how much the assignment of impression \( i \) increases the total weight of the allocation at the time of its assignment. We let OPT denote the maximum weight matching of the instance, and we let \( a'_i \) be the advertiser to which impression \( i \) is assigned in OPT. We overload the notation of OPT and also let it be the weight of the allocation, that is, \( \text{OPT} = \sum_{i \in I} w_{i,a'_i} \). For the sake of analysis, we can add a large enough number of dummy impressions (advertisers) with edges of weight zero to all advertisers (impressions) so that all vertices (impressions and advertisers) are matched in the optimal solution. For the completeness of our algorithm, we also introduce a null impression \( i = 0 \) with weight \( w_{0,a} = 0 \) for all \( a \in A \).
3  The StochasticGreedy Algorithm

In this section we introduce our randomized online algorithm StochasticGreedy (Algorithm 1). We start by describing the algorithm at a high level, and then we present it formally together with two important lemmas that highlight its deterministic features. In Section 3.1 we describe the variables in the algorithm. Then in Section 3.2 and Section 3.3 we discuss the two main cases the algorithm considers when assigning an impression. Lastly, in Section 3.4 we explain why this approach breaks the 1/2 competitive ratio barrier.

Our algorithm builds on the greedy approach. Upon the arrival of impression \( i \), StochasticGreedy first constructs a set \( B \subset A \) of candidate advertisers that can potentially be matched with \( i \). If there are multiple candidates, the algorithm considers the top two \( a_1 \) and \( a_2 \), and performs one of the three actions uniformly at random: (1) greedily assign \( i \) to \( a_1 \), (2) greedily assign \( i \) to \( a_2 \), or (3) adaptively choose between \( a_1 \) and \( a_2 \) by looking at a past decision of the algorithm. The top candidates are determined by their expected gain \( \mathbb{E}[\text{Gain}_{i,a}] \) (where the randomness is over all possible branches of the algorithm up to this point and not conditioned on the current state) and an adaptive gain value that originates in the proof of Lemma 4.6. In the event that there are not multiple candidates, the algorithm makes a nonadaptive assignment.

To adaptively decide between the top two candidates, the algorithm looks at the result of the last coin flip associated with \( a_k \), where \( a_k \) is the advertiser in \( \{ a_1, a_2 \} \) with the greater adaptive gain. If the assignment based on this coin flip was adaptive, then the algorithm chooses between \( a_1 \) and \( a_2 \) uniformly at random. Otherwise, the assignment of the past impression \( i' \) associated with this coin flip was nonadaptive, and the algorithm looks at whether or not \( i' \) was matched to \( a_k \). If \( i' \) was matched to \( a_k \) then the algorithm assigns \( i \) to the other top candidate in \( \{ a_1, a_2 \} \), and if \( i' \) was not matched to \( a_k \) then \( i \) is assigned to \( a_k \). In general, adaptive decisions can cause cascading effects of conditional probabilities that can severely alter the distribution of many MaxW variables. However, by ensuring that the adaptive decisions are based on an earlier nonadaptive (i.e., random) assignment, we can prevent this effect and analyze the algorithm. We continue this discussion about the benefits of this kind of adaptive decision in more detail in Section 3.4.

Now we formally present StochasticGreedy in Algorithm 1. This algorithm takes as input two nonnegative parameters \( \epsilon \) and \( \delta \) that control the thresholds for how greedy and adaptive the algorithm is, respectively. We optimize these constants later in our analysis in Section 4. While Algorithm 1 is initially cumbersome to parse, we point out that it is comprised of two separate cases that can be analyzed individually (see Section 3.2 and Section 3.3). We also acknowledge that the definitions of the adaptive_gain_{i,a} variables and the set \( B \) initially appear to be unnatural, but these conditions arise in our analysis and allow us to overcome the shortcomings of the greedy algorithm. Before stepping through the details of Algorithm 1, we first make two critical observations about how the algorithm uses randomness.

**Lemma 3.1.** The only random variables in StochasticGreedy are the assignments of the impressions and the values of Priority_{a} and R_i. All other variables (e.g., the maximum gains M_{i}, all values of adaptive_gain_{i,a}, the sequence of sets B and choices a_1 and a_2, all updates to active(a), index(a), partner(a), and the sums S_{a}) are deterministic quantities that depend solely on the instance and the arrival order of the impressions.

**Lemma 3.2.** We can maintain the probability mass function for all random variables Gain_{i,a} over the course of algorithm. In particular, we can efficiently compute the value \( \mathbb{E}[\text{Gain}_{i,a}] \) at any point in StochasticGreedy.

In particular, Lemma 3.1 guarantees that the top candidates in each step are predetermined by the input instance, even though the assignments of past impressions to these advertisers could have been random. This property is simple to show by induction but easy to miss because of the complexity of Algorithm 1. Lemma 3.2 states that we can efficiently compute \( \mathbb{E}[\text{Gain}_{i,a}] \) for all \( a \in A \) in line 5 of Algorithm 1. This is a
consequence of the limited randomness in the algorithm and dynamic programming. We defer the proofs of both of these lemmas to Appendix A.

We begin the description of StochasticGreedy by explaining the preprocessing stage in lines 5–11 of Algorithm 1. When impression $i$ arrives, the algorithm first computes the maximum expected marginal gain $M_i = \max_{a \in A} \mathbb{E}[\text{Gain}_{i,a}]$ as a benchmark. We remark that it is not hard to show that the standard $1/2$ competitive ratio proof of the greedy algorithm goes through if we use $\mathbb{E}[\text{Gain}_{i,a}]$ instead of their realized values. Using expected values, however, has the advantage that if there are two choices with high expected gains, then the algorithm can occasionally realize them in a controlled way and exploit the gap between them to achieve a better result. Next, for every $a \in A$ the algorithm computes their adaptive gain value, and then it constructs the set of candidates $B$. There is some slack in how greedy Algorithm 1 is, but an advertiser must be able to contribute a value of at least $(1 - \epsilon)M_i$ to be considered. We explain the meaning of the variables used in lines 7–11 in the next subsection, but for now we note that all of the quantities in these formulas are deterministic and well-defined. If there are at least two candidates in $B$, the algorithm assigns $i$ in Case I (lines 13–30). Otherwise, the algorithm jumps to line 31 and assigns $i$ in Case II (lines 31–42). We explain these cases in Section 3.2 and Section 3.3, respectively.

### 3.1 Variable Descriptions

All of the following variables with the exception of adaptive\textunderscore gain\textunderscore $i,a$ help maintain the state of Algorithm 1. We reiterate that $\text{Priority}_a$ and $R_i$ are the only variables that are randomized. All other variables (at every time step of the algorithm) are predetermined by the input graph and the arrival order of the impressions.

- active($a$) is a boolean value that indicates whether or not advertiser $a$ potentially has an adaptive gain. At its core, this variable serves as a mechanism for checking if $a$ is a partner in a valid pairing.
- adaptive\textunderscore gain\textunderscore $i,a$ represents how much extra marginal gain that algorithm can achieve by adaptively assigning $i$ to $a$. If an adaptive assignment is made, the algorithm achieves an additional marginal gain that is at most proportional to $\mathbb{E}[\text{Gain}_{\text{index}(a),a}]$ since it knows the earlier impression $\text{index}(a)$ was nonadaptively assigned to partner($a$) and not $a$. We give further intuition for this mechanic in Section 3.4 and the derivation of this formula in the proof of Lemma 4.6.
- $\text{index}(a)$ records the last impression for which $a$ was considered in a potentially adaptive assignment. If $i$ is to be matched to $a_1$ or $a_2$ in lines 13–30, the algorithm sets $\text{index}(a_1) \leftarrow i$ and $\text{index}(a_2) \leftarrow i$.
- partner($a$) records the last advertiser with which $a$ was paired in a potentially adaptive assignment. If $i$ is to be matched to $a_1$ or $a_2$ in lines 13–30, then we set partner($a_1$) $\leftarrow a_2$ and partner($a_2$) $\leftarrow a_1$.
- $\text{Priority}_a$ is a random variable for the outcome of the last potentially adaptive assignment involving $a$. If $i$ is to be matched to $a_1$ or $a_2$ in lines 13–30, one of the following actions is performed uniformly at random: (1) greedily assign $i$ to $a_1$ and set $\text{Priority}_{a_1} \leftarrow 1$, $\text{Priority}_{a_2} \leftarrow 2$; (2) greedily assign $i$ to $a_2$ and set $\text{Priority}_{a_1} \leftarrow 2$, $\text{Priority}_{a_2} \leftarrow 1$; or (3) adaptively assign $i$ to $a_1$ or $a_2$ by looking at $\text{Priority}_{a_k}$, where $k \in \{1, 2\}$ is defined in line 21, and reset $\text{Priority}_{a_1} \leftarrow 0$, $\text{Priority}_{a_2} \leftarrow 0$. The intuition here is that in the first two actions, the advertiser that is not matched with $i$ receives a higher priority value and is therefore more likely to be adaptively assigned a future impression. In the third action, an adaptive decision is potentially made based on these priority values, and then the priorities are reset.
- $S_a$ is an upper bound for the sum of expected gains assigned to $a$ in lines 31–42 since the last time $a$ was chosen as a candidate in lines 13–30. Whenever $a$ is a choice in lines 13–30, $S_a$ is reset to zero.
Algorithm 1: Online weighted bipartite matching algorithm.

```plaintext
function StochasticGreedy(ε, δ)
    Set active(a) ← false, index(a) ← 0, partner(a) ← 0, Priority_a ← 0, S_a ← 0 for all a ∈ A
    for t = 1, 2, ..., |I| do
        Let i be the impression that arrives at time t, i.e., t_i = t
        M_i ← max_{a∈A} E[Gain_{i,a}] // Not conditioned on the algorithm’s state (see Lemma 3.1)
        for a ∈ A do // Compute adaptive gain values
            if active(a) = true and w_{i,a} ≥ w_{index(a),a} - δM_i then
                adaptive_gain{i,a} ← (E[Gain_{index(a),a}]/3 - (w_{index(a),a} - w_{i,a})^2/3 - S_a)^+ / 12
            else
                adaptive_gain_{i,a} ← 0
            end
            B ← {a ∈ A : (w_{i,a} ≥ w_{index(a),a} - δM_i) and (E[Gain_{i,a}] + 2/3 · adaptive_gain_{i,a} ≥ (1 - ε)M_i)}
            Let R_i be a uniformly random real number in the interval [0, 1)
            if |B| ≥ 2 then // Case I: There are enough candidates to exploit adaptivity
                a_1 ← arg max_{a∈B} E[Gain_{i,a}] + 2/3 · adaptive_gain_{i,a}
                a_2 ← arg max_{a∈B \setminus \{a_1\}} E[Gain_{i,a}] + 2/3 · adaptive_gain_{i,a}
                for a ∈ {a_1, a_2} do // Couple the top pair of advertisers a_1 and a_2
                    Set active(a) ← true, S_a ← 0, index(a) ← i
                    if partner(a) /∈ {a_1, a_2} then
                        active(partner(a)) ← false
                    end
                end
                Set partner(a_1) ← a_2 and partner(a_2) ← a_1
                k ← arg max_{j∈\{1,2\}} adaptive_gain_{i,a_j}
                if R_i ∈ [1/3, 2/3) or adaptive_gain_{i,a_k} = 0 then // Make an adaptive decision if possible
                    if Priority_{a_k} = 2 and adaptive_gain_{i,a_k} > 0 then Assign i to a_k
                    if Priority_{a_1} = 1 and adaptive_gain_{i,a_1} > 0 then Assign i to partner(a_1)
                    if Priority_{a_1} = 0 or adaptive_gain_{i,a_1} = 0 then
                        Assign i to a_1 or a_2 each with probability 1/2
                    end
                    Set Priority_{a_1} ← 0 and Priority_{a_2} ← 0
                else // Make a random assignment and prepare for future adaptive decisions
                    if R_i ∈ [1/3, 2/3) then Assign i to a_1 and set Priority_{a_1} ← 1, Priority_{a_2} ← 2
                    if R_i ∈ [2/3, 1) then Assign i to a_2 and set Priority_{a_1} ← 2, Priority_{a_2} ← 1
                end
            else // Case II: There is no adaptivity
                B' ← {a ∈ A : (w_{i,a} ≥ w_{index(a),a} - δM_i) and (E[Gain_{i,a}] ≥ (1 - ε)M_i)} // Note B ⊆ B'
                C ← {a ∈ A : (w_{i,a} < w_{index(a),a} - δM_i) and (E[Gain_{i,a}] ≥ (1 - ε)M_i)}
                if |B' ∪ C| = 1 then // Advertiser a_2 does not exist
                    Assign i to a_1 ← arg max_{a∈A} E[Gain_{i,a}] 
                    Set S_{a_1} ← S_{a_1} + M_i
                else // Make a random assignment to the top two choices
                    if B' ≠ ∅ then a_1 ← the only advertiser in B'
                    else a_1 ← arg max_{a∈C} E[Gain_{i,a}]
                    a_2 ← arg max_{a∈C \setminus \{a_1\}} E[Gain_{i,a}]
                    Assign i to a_1 or a_2 each with probability 1/2
                    Set S_{a_1} ← S_{a_1} + M_i/2 and S_{a_2} ← S_{a_2} + M_i/2
                end
            end
        end
    end
end
```

3.2 Case I: Lines 13–30 and Adaptive Assignments

If B contains at least two advertisers, then \( a_1 \) and \( a_2 \) are chosen as the top two candidates in lines 14–15 based on their value of \( \mathbb{E}[\text{Gain}_{i,a}] + 2/3 \cdot \text{adaptive_gain}_{i,a} \). Before assigning \( i \) in lines 21–30, the algorithm performs an update procedure in lines 16–20 to couple the pair of advertisers \( a_1 \) and \( a_2 \). For each of the top candidates \( a \in \{ a_1, a_2 \} \), this step activates \( a \) for future adaptive decision, deactivates its previous partner (unless this partner is the other top candidate), and updates \( \text{partner}(a) \) to be the other top candidate. This routine ensures that the active variables are set to true if and only if their advertiser is in a proper pairing. This is important because the algorithm uses the active state of an advertiser to ensure that \( \text{adaptive_gain}_{i,a} \) is set to zero for all unpaired advertisers \( a \) in lines 7–10. Finally, for each \( a \in \{ a_1, a_2 \} \), line 17 also updates \( \text{index}(a) \) to the current impression and resets the value of \( S_a \).

Now we focus on the assignment of \( i \) in lines 21–30. The algorithm starts by drawing a uniformly random real number \( R_i \) from the interval \([0,1)\). We start by describing the assignment rules in lines 29–30, as they are simpler to explain and give insight into how the adaptive decision works. With probability 1/3 (i.e., the event \( R_i \in [\frac{1}{3}, \frac{2}{3}] \) in line 29), the algorithm matches \( i \) to \( a_1 \). We note that this assignment does not depend on any previous coin flips of the algorithm because the sequence of top candidates over the course of the algorithm is determined by the input instance (Lemma 3.1). It will be useful for future adaptive assignments to record that the algorithm nonadaptively assigned \( i \) to \( a_1 \), so we update \( \text{Priority}_{a_1} \leftarrow 1 \) and \( \text{Priority}_{a_2} \leftarrow 2 \). Formally, \( \text{Priority}_{a_1} = 1 \) means that the last time \( a \) was chosen as a candidate in lines 13–30, the impression that arrived at this time was nonadaptively assigned to \( a \). The event \( \text{Priority}_{a_2} = 2 \) is in some sense the complement and means that the last time \( a \) was chosen as a candidate in lines 13–30, the impression that arrived was nonadaptively assigned to the other top candidate (i.e., not \( a \)). We refer to this variable for the adaptive state of an advertiser as a priority because if in lines 13–30 a nonadaptive decision is made and the top candidate does not receive the impression, then it is given a higher priority and is more likely to be assigned an impression in the future. With another probability of 1/3 (i.e., when \( R_i \in [\frac{2}{3}, 1) \) in line 30), the algorithm assigns \( i \) to \( a_2 \) and updates the \( \text{Priority} \) variables accordingly.

With the remaining 1/3 probability (i.e., when \( R_i \in [0, \frac{1}{3}) \) in line 22), impression \( i \) is assigned adaptively. We first note that after this assignment, \( \text{Priority}_{a_1} \) and \( \text{Priority}_{a_2} \) are reset to 0. To simplify the description of this part of the algorithm, assume that \( \text{adaptive_gain}_{i,a_1} \geq \text{adaptive_gain}_{i,a_2} \). This means \( k \) is set to 1 in line 21. The assignment of \( i \) is conditioned on the assignment of \( i' \), where \( i' \) is the last impression that chose \( a_1 \) as a candidate in lines 13–30. Note that \( i' \) was the value of \( \text{index}(a_1) \) immediately before its update in line 17. The algorithm adaptively makes its assignment conditioned on past events by looking at the value of \( \text{Priority}_{a_1} \). If \( i' \) was matched nonadaptively to \( a_1 \), then the algorithm makes an adaptive choice and assigns \( i \) to \( a_2 \) (i.e., the case where \( \text{Priority}_{a_1} = 1 \) and \( \text{Priority}_{a_2} = 2 \)). Intuitively, this is beneficial because conditioning on the event where \( i' \) was assigned to \( a_1 \) decreases the expected gain of assigning \( i \) to \( a_1 \). Thus, we should consider \( a_2 \) as the better option. Similarly, if \( \text{Priority}_{a_1} = 2 \) then the algorithm adaptively assigns \( i \) to \( a_1 \). The last case is when \( \text{Priority}_{a_1} = 0 \). Since we want to prevent the chaining of conditional probabilities, the algorithm does not make an adaptive choice here and instead assigns \( i \) to \( a_1 \) or \( a_2 \) uniformly at random using a new coin toss that is independent of \( R_i \). In the event that \( \text{adaptive_gain}_{i,a_1} = 0 \) (which means that \( \text{adaptive_gain}_{i,a_2} = 0 \) because we assumed \( \text{adaptive_gain}_{i,a_1} \geq \text{adaptive_gain}_{i,a_2} \)), it suffices for the algorithm to make a random assignment in line 26.

3.3 Case II: Lines 31–42

If the set \( B \) contains zero or one advertiser, then the algorithm is forced to make a nonadaptive assignment. The high level idea in this case is that we want to choose at most two advertisers to be matched with \( i \).
while ensuring that both of the following conditions are met:

- The advertiser with maximum expected gain, \( \arg\max_{a \in A} E[\text{Gain}_{i,a}] \), is chosen as one of the candidates.
- If \( B \) is not empty and the only advertiser in \( B \) has expected gain at least \((1 - \epsilon)M_i \), it should be chosen.

First observe that the definitions of the sets \( B' \) and \( C \) in lines 32–33 imply that their union \( B' \cup C \) consists of all advertisers with expected gain at least \((1 - \epsilon)M_i \). Therefore, if the conditional statement on line 34 holds, the advertiser \( \arg\max_{a \in A} E[\text{Gain}_{i,a}] \) is the only advertiser that meets the \((1 - \epsilon)M_i \) threshold. The algorithm assigns \( i \) to this advertiser and does not consider a second option. Otherwise, the algorithm selects the only advertiser in \( B' \subset B \) (if it exists) as a candidate and chooses one or two additional advertisers in \( C \) with the highest expected gains to be the candidates \( a_1 \) and \( a_2 \). The impression \( i \) is then assigned to \( a_1 \) or \( a_2 \) with equal probability. The only remaining detail is the variable \( S_a \), which maintains an upper bound for the sum of expected gains assigned to \( a \) since the last time \( a \) was chosen as a candidate in lines 31–42. We note that \( S_a \) is reset to zero in line 17, and is otherwise incremented in line 36 or line 42 in a way that is consistent with the probability of impression \( i \) being assigned to \( a \).

### 3.4 Intuition for Breaking the \( \frac{1}{2} \) Barrier

The key result that gives us a chance to break the \( 1/2 \) barrier is Lemma 4.6, which states that all assignments in lines 13–30 satisfy \( E[\text{MarginalGain}_i] \geq (E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}]) / 2 + \text{adaptive_gain}_{i,a_1} + \text{adaptive_gain}_{i,a_2} \). Recall from line 7 of Algorithm 1 that \( \text{adaptive_gain}_{i,a} = (E[\text{Gain}_{\text{index}(a),a}] / 3 - (w_{\text{index}(a),a} - w_{i,a})^+ / 3 - S_a)^+ / 12 \). We are able to prove Lemma 4.6 because we ensure that adaptive decisions are based on a past nonadaptive assignment, which stops the chaining of conditional dependencies. We first argue that \( \text{adaptive_gain}_{i,a} \) can be thought of as some constant fraction of \( E[\text{Gain}_{\text{index}(a),a}] \). If this is not the case, then \((w_{\text{index}(a),a} - w_{i,a})^+ \) or \( S_a \) is large. The condition \( w_{i,a} \geq w_{\text{index}(a),a} - \delta M_i \) on line 7 implies that \((w_{\text{index}(a),a} - w_{i,a})^+ \leq \delta M_i \) is not too large, so we can bound the drop in \( E[\text{Gain}_{\text{index}(a),a}] \). If \( S_a \) is large, then the algorithm made potential assignments to \( a \) in lines 31–42. Assignments to \( a \) in lines 31–42 are favorable because they either agree with \( \text{OPT} \) or yield substantially more gain than \( E[\text{Gain}_{i,a}] \). Ultimately, this allows us to charge the additional gain from these assignments to \( \text{adaptive_gain}_{i,a} \).

We proceed by assuming that every \( \text{adaptive_gain}_{i,a} \) variable is proportional to \( E[\text{Gain}_{\text{index}(a),a}] \). Note that we are giving the intuition behind our approach and that many details are omitted. For any \( a \in A \), let \( L = \{i_1, i_2, \ldots, i_L\} \) be the set of impressions potentially matched with \( a \) in lines 13–30. It follows that \( \text{index}(a) \) takes values in \( L \) over the course of the algorithm. Each of the impressions in \( L \setminus \{i_1\} \) generates enough extra value in the form of adaptive gain to increase the marginal gain of the previous impression. The only expected marginal gain value that is lacking is the one associated with the \( i_L \). For this last impression, we consider a few different cases and forward reference to the additional sources of marginal gain \( \text{Y}_i \) and \( \text{Z}_i \) that arise in our lower bound for \( E[\text{ALG}] \) in Lemma 4.1. The following argument is formalized by the \text{DistributeExcess} \ mechanism in Section 4.2 for reallocating extra marginal gain. Let \( i' \) be the impression matched with \( a \) in \( \text{OPT} \) (i.e., \( a = a_i' \)). If \( i' \) arrives before \( i_L \), then \( \text{Y}_i' \) is large and the mechanism borrows from it to achieve enough value. Now assume that \( i' \) arrives after \( i_L \). If \( i' \) is assigned in lines 31–42, there is enough extra value to allocate to \( E[\text{Gain}_{i,a}] \) since this is the favorable case. Otherwise, \( i' \) is assigned in lines 13–30 and \( a \) is not one of the top candidates in \{\( a_1, a_2 \)\}. If \( a \) is not one of the top candidates because \( w_{i',a} < w_{\text{index}(a),a} - \delta M_i \) (which makes the condition on line 7 false), then the value of \( Z_i \) is large enough to make up for this difference. Otherwise, the sum \( \text{adaptive_gain}_{i',a} + \text{adaptive_gain}_{i,a} \) is large enough to cover the marginal gain of all three impressions \( i, \text{index}(a), \) and \( \text{index}(a) \) since the Algorithm 1 accounts for this \( 2/3 \) split in lines 14–15 when choosing \( a_1 \) and \( a_2 \). Putting everything together, we show that all assignments in lines 13–30 satisfy \( E[\text{MarginalGain}_i] \geq (1/2 + \epsilon') \cdot (E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}]) \), for some \( \epsilon' > 0 \).
4 Analysis of the Competitive Ratio

In this section we analyze the competitive ratio of STOCHASTICGREEDY and show that it breaks the 1/2 barrier. We start by presenting the high-level structure of our argument in Section 4.1, deferring the proofs of our two core lemmas (Lemma 4.2 and Lemma 4.3) to the following subsections. In Section 4.2 we introduce a mechanism called DISTRIBUTEEXCESS, which we use in our analysis to systematically redistribute excess marginal gain. We stress that DISTRIBUTEEXCESS exists solely for the sake of analysis and is not a component of STOCHASTICGREEDY. Then in Section 4.3 we prove Lemma 4.3 by cases and show that DISTRIBUTEEXCESS allocates enough excess marginal gain to every impression for STOCHASTICGREEDY to be 0.501-competitive.

4.1 Outline of the Main Proof

For every impression \( i \in I \), we compare \( E[\text{MarginalGain}_i] \) to the expected value that STOCHASTICGREEDY could have achieved by assigning \( i \) to \( a'_i \), namely \( E[\text{Gain}_{i,a'_i}] = E[(w_{i,a'_i} - \text{MaxW}^{i-1}_{a'_i})^+] \). To prove that greedy algorithms achieve a 1/2 competitive ratio, it usually suffices to show that \( E[\text{MarginalGain}_i] \geq E[\text{Gain}_{i,a'_i}] \). Our algorithm STOCHASTICGREEDY, however, is designed in such a way that this condition is not necessarily satisfied for every impression. Instead, we show that the sum of expected marginal gains over all impressions is significantly greater than this lower bound in aggregate. Intuitively, impressions that beat this benchmark share their excess marginal gain with other impressions so that at the end of the algorithm, every impression contributes enough to exceed the standard 1/2 competitive ratio. One of the major technical contributions of this paper is our carefully designed mechanism DISTRIBUTEEXCESS (Mechanism 2), which reallocates the potential excess marginal gain of an assignment to guarantee a uniform lower bound for every impression.

We begin by lower bounding the expected weight of the assignment that STOCHASTICGREEDY makes in such a way that reveals three additional sources of potential excess gain that can be exploited.

Lemma 4.1. The expected weight of the assignment of STOCHASTICGREEDY, namely \( E[\text{ALG}] \), is at least

\[
\frac{1}{2} \text{OPT} + \frac{1}{2} \sum_{i \in I} E[\text{MarginalGain}_i - \text{Gain}_{i,a'_i}] + E\left[\text{MaxW}^i_{a'_i} - \text{MaxW}^{i-1}_{a'_i}\right] + E\left[\left(\text{MaxW}^{i-1}_{a'_i} - w_{i,a'_i}\right)^+\right].
\]

Proof. We know that \( E[\text{ALG}] = \sum_{i \in I} E[\text{MarginalGain}_i] = \sum_{i \in I} X_i + E[\text{Gain}_{i,a'_i}] \). By the definition of the \((x)^+\) operator, we have \( \text{Gain}_{i,a'_i} = w_{i,a'_i} - \text{MaxW}^{i-1}_{a'_i} + (\text{MaxW}^{i-1}_{a'_i} - w_{i,a'_i})^+ \), which gives us the \( Z_i \) term. So far we have shown that

\[
E[\text{ALG}] = \sum_{i \in I} X_i + E\left[w_{i,a'_i} - \text{MaxW}^{i-1}_{a'_i}\right] + Z_i.
\]

On the other hand, we know \( E[\text{ALG}] = \sum_{a \in A} E[\text{MaxW}^i_{a_i}] \) and \( \text{OPT} = \sum_{i \in I} w_{i,a'_i} \). Writing \( w_{i,a'_i} - \text{MaxW}^{i-1}_{a'_i} \) as

\[
\left(w_{i,a'_i} - \text{MaxW}^{i}_{a'_i}\right) + \left(\text{MaxW}^{i}_{a'_i} - \text{MaxW}^{i-1}_{a'_i}\right)
\]

yields with the \( Y_i \) term. Since \( \sum_{i \in I} E[\text{MaxW}^i_{a'_i}] \leq \sum_{a \in A} E[\text{MaxW}^i_{a}] = E[\text{ALG}] \), it follows that

\[
E[\text{ALG}] = \sum_{i \in I} X_i + E\left[w_{i,a'_i} - \text{MaxW}^{i}_{a'_i}\right] + Y_i + Z_i \geq \text{OPT} - E[\text{ALG}] + \sum_{i \in I} X_i + Y_i + Z_i,
\]

which completes the proof. \( \square \)
The expectations $Y_i$ and $Z_i$ are nonnegative for all $i \in I$ since the random variables $\text{MaxW}_{\lambda i}$ are nondecreasing in $t$ and by the definition of the $(\chi)^+$ operator. The expectations $X_i$, however, can sometimes be negative.

For the sake of analysis, we define an auxiliary variable $\text{excess}_i$ for each impression $i \in I$ and show how to assign its value at any given step of the algorithm. We reiterate that the excess variables are not actually used in \textsc{StochasticGreedy} and are only defined to help us prove the competitive ratio. As noted above, the sum $X_i + Y_i + Z_i$ is not necessarily nonnegative. Therefore, we introduce the mechanism \textsc{DistributeExcess} to systematically redistribute excess marginal gain that is incurred over the course of the algorithm. This allows us to assign values to all the excess variables so that $\sum_{i \in I} \text{excess}_i \leq \sum_{i \in I} X_i + Y_i + Z_i$, and more importantly, for every impression $i \in I$, we have $\text{excess}_i \geq \lambda M_i$ for some universal constant $\lambda > 0$. In Theorem 4.4, we exploit these two properties of the excess variables to prove that \textsc{StochasticGreedy} is at least $\frac{1+\lambda}{2+\lambda} > 1/2$ competitive. Note that we call the routine of assigning values to the excess variables a mechanism (and not an algorithm) because it is only used in our analysis as a means to argue about the aggregate excess marginal gain that \textsc{StochasticGreedy} produces.

For now, we abstract away the details of \textsc{DistributeExcess} so that we can understand its role in our analysis of the competitive ratio. We present the mechanics of \textsc{DistributeExcess} and the proof of the following lemma in Section 4.2.

**Lemma 4.2.** The mechanism \textsc{DistributeExcess}(ζ, γ, σ) computes a value excess$_i$ for each impression $i \in I$ such that $\sum_{i \in I} \text{excess}_i \leq \sum_{i \in I} X_i + Y_i + Z_i$, where the terms $X_i$, $Y_i$, and $Z_i$ are defined in Lemma 4.1. Furthermore, for every $i \in I$, at least a $\zeta$ fraction of $Z_i$ is distributed to excess$_i$.

The next lemma covers a variety of different cases that \textsc{StochasticGreedy} can encounter and is the crux of our analysis. In particular, Lemma 4.3 shows that we can use \textsc{DistributeExcess} to uniformly lower bound each excess$_i$ variable in terms of the maximum expected gain $M_i$ when $i$ arrives. We note that the inputs $\epsilon, \delta$ to \textsc{StochasticGreedy} and $\zeta, \gamma, \sigma$ to \textsc{DistributeExcess} are intentionally left as variables so that we can optimize them retroactively and so that the case analysis in the proof of Lemma 4.3 is simpler. We discuss our approach and the supporting lemmas for proving Lemma 4.3 in more detail in Section 4.3.

**Lemma 4.3.** For any $\epsilon, \delta \geq 0$, the mechanism \textsc{DistributeExcess}(ζ, γ, σ) finds a value excess$_i$ for each impression $i \in I$, where $\lambda$ is defined to be $\lambda = \lambda(\epsilon, \delta) = \max_{0 \leq \epsilon, \delta, \gamma, \sigma \leq 1} \min \{\frac{\epsilon - 2y - 2t}{2}, \frac{1 - 3\epsilon - 4y - 4t}{4}, \frac{2(\delta - 3\epsilon - 6y - 6t)}{6}, \frac{3 - 2\epsilon}{19}, \frac{6\epsilon - 1 - 18\epsilon}{18}, \frac{32(1 - \epsilon)^2 - 36\epsilon}{18468(1 - \epsilon)}, \frac{32(1 - \epsilon)^2 - 36\epsilon}{18468(1 - \epsilon)} \times 18\gamma, \frac{2(1 - \epsilon)}{19}, (1 - \epsilon) \times \frac{5}{118}, \frac{6\epsilon - 1 - 18\epsilon}{18}, \frac{32(1 - \epsilon)^2 - 36\epsilon}{18468(1 - \epsilon)} \times \gamma, \frac{2\gamma}{118} \times \frac{18\gamma(1 - \epsilon)}{19}\}$.

In particular, by setting $\epsilon = 0.082, \delta = 0.445, \zeta = 0.955, \gamma = 0.00337198$, and $\sigma = 0.03362$, we have $\lambda \geq 0.00400802$.

Now that we have presented our key prerequisite lemmas, we show how to assemble them to prove our main result about \textsc{StochasticGreedy}.

**Theorem 4.4.** For any $\epsilon, \delta \geq 0$, the algorithm \textsc{StochasticGreedy}(ε, δ) is $\frac{1+\lambda}{2+\lambda}$-competitive, where $\lambda = \lambda(\epsilon, \delta)$ is defined in Lemma 4.3. In particular, if $\epsilon = 0.082$ and $\delta = 0.445$, then $\lambda \geq 0.00400802$ and \textsc{StochasticGreedy} is 0.501-competitive.

**Proof.** By combining Lemma 4.1, Lemma 4.2, and Lemma 4.3, we know that

$$E[\text{ALG}] \geq \frac{1}{2} \text{OPT} + \frac{1}{2} \sum_{i \in I} X_i + Y_i + Z_i \geq \frac{1}{2} \text{OPT} + \frac{1}{2} \sum_{i \in I} \text{excess}_i \geq \frac{1}{2} \text{OPT} + \frac{1}{2} \sum_{i \in I} \lambda M_i.$$ 

For each impression we also have $M_i \geq E[\text{Gain}_{L,i}]$, which is at least $E[w_{l,i} - \text{MaxW}_{\lambda l}] = E[w_{l,i} - \text{MaxW}_{\lambda l}]$. Summing this lower bound over all impressions gives us $\sum_{i \in I} M_i \geq \text{OPT} - E[\text{ALG}]$. Therefore, it follows that $E[\text{ALG}] \geq \frac{1}{2} \text{OPT} + \frac{1}{2}(\text{OPT} - E[\text{ALG}])$, or equivalently $E[\text{ALG}] \geq \frac{1+\lambda}{2+\lambda} \text{OPT}$. 

\qed
4.2 Mechanism for Distributing Excess Marginal Gain

In this subsection we introduce the \textsc{DistributeExcess} mechanism, which reallocates the excess marginal gain \(\sum_{i} X_i + Y_i + Z_i\) defined in Lemma 4.1 to a set of auxiliary variables called \(\text{excess}_i\). This mechanism and the \(\text{excess}_i\) variables are not part of the \textsc{StochasticGreedy} algorithm, and are used only to analyze the competitive ratio. At any time during the online algorithm, this mechanism assumes oracle access to the optimal assignment and assigns a value to each of the \(\text{excess}_i\) variables in a way that allows us to show that \(E[\text{ALG}] \geq 0.501 \cdot \text{OPT}\) for the current sequence of impressions.

At a high level, \textsc{DistributeExcess} mirrors the execution of \textsc{StochasticGreedy} and systematically redistributes the sum \(\sum_{i} X_i + Y_i + Z_i\) across all of the \(\text{excess}_i\) variables. Its execution path relies solely on deterministic quantities in \textsc{StochasticGreedy} (see Lemma 3.1) and is thus independent of its randomness. The mechanism takes as input three parameters \(\zeta, \gamma, \sigma\) representing allocation proportions that we optimize later. The allocations of the \(X_i\) and \(Z_i\) variables are direct, but the \(Y_i\) variables are partitioned and distributed over time. Before presenting \textsc{DistributeExcess}, we define a refinement of \(Y_i\) to captures how it evolves.

\textbf{Definition 4.5}. For every \(i \in I\), we define the time sequence of \(Y_i\) (introduced in Lemma 4.1) to be

\[
Y_{i,t} = \begin{cases} 
0 & \text{if } 0 \leq t < t_i, \\
E[\max W_{a_i}^t - \max W_{a_i}^{t-1}] & \text{if } t_i \leq t \leq n,
\end{cases}
\]

We let \(\Delta'\) denote the backwards difference for time sequence values, implicitly defined as \(\Delta'(Y_i) = Y_{i,t} - Y_{i,t-1}\).

\begin{algorithm}
\caption{Mechanism to populate the \(\text{excess}_i\) variables.}
\begin{algorithmic}[1]
\Function{DistributeExcess}{$\zeta, \gamma, \sigma$}
\State Initialize \(\text{excess}_i \leftarrow 0\) for all \(i \in I \cup \{0\}\) \hfill \Comment{Recall that 0 is the initial value of index(a)}
\For{$t = 1, 2, ..., |I|$}
\State Let \(i\) be the impression that arrives at time \(t\), i.e., \(t_i = t\)
\State Increase \(\text{excess}_i\) by \(\zeta Z_i\) \hfill \Comment{Distribute \(\zeta Z_i\)}
\For{$a \in \{a_1, a_2\}$ (if \(a_2\) does not exist, only consider \(a_1\))}
\State Let \(i'\) be the impression for which \(a = a_i'\) \hfill \Comment{Distribute \(\Delta'(Y_{i'})\)}
\If{$i'$ arrived at or before time \(t\) then}
\State \(K \leftarrow \{i, i', \text{index}(a)\}\)
\For{$j \in K$}
\State Increase \(\text{excess}_j\) by \(\Delta'(Y_{i'})/|K|\)
\EndFor
\EndIf
\If{$|B| \geq 2$ then}
\For{$a \in \{a_1, a_2, a_i'\}$}
\State Increase \(\text{excess}_{\text{index}(a)}\) by \(2/3 \cdot \text{adaptive\_gain}_{i,a}\)
\State Set \(\text{excess}_i = \text{excess}_i + X_i - \sum_{a \in \{a_1, a_2, a_i'\}} 2/3 \cdot \text{adaptive\_gain}_{i,a}\)
\EndFor
\For{$a \in \{a_1, a_2\}$}
\State Increase \(\text{excess}_{\text{index}(a)}\) by \(2/3 \cdot \text{adaptive\_gain}_{i,a}\) \hfill \Comment{Use the value of index(a) at the beginning of time \(t\)}
\If{index(a) remains unchanged after time \(t\) then}
\State Let \(i'\) be the impression for which \(a = a_i'\) \hfill \Comment{Distribute \((1-\zeta)Z_i\)}
\State Increase \(\text{excess}_i\) by \((1-\zeta)Z_i\)
\EndIf
\Else
\If{\(a_2\) exists then}
\State Increase \(\text{excess}_{\text{index}(a_2)}\) and \(\text{excess}_{\text{index}(a_1)}\) each by \(|\sigma M_i/2|
\Else
\State Increase \(\text{excess}_{\text{index}(a_1)}\) by \(|\sigma M_i|
\EndIf
\State Increase \(\text{excess}_{\text{index}(a_2)}\) by \(|\gamma M_i|
\EndIf
\EndIf
\EndFor
\EndFor
\EndFor
\EndFunction
\end{algorithmic}
\end{algorithm}

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We conclude this subsection by proving Lemma 4.2, which guarantees that the mechanism performs a valid reallocation. In particular, we show that the final assignments satisfy $\sum_{j \in I_i} \text{excess}_j \leq \sum_{j \in I_i} X_j + Y_j + Z_i$.

Proof of Lemma 4.2. The second claim is immediate from line 5 of Mechanism 2, so we focus on the first part of the statement. At each time step $t = t_i$, lines 13–15 and lines 21–24 change the sum $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$ by exactly $X_j$. Now we analyze lines 6–11 of Mechanism 2. Assuming that $a_2$ exists, let $i'_2$ be the impression for which $a_2 = a'_2$. Since $a_1 \neq a_2$ we know $i'_2 \neq i'_1$. Therefore, the change in $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$ at time $t = t_i$ is at most $\Delta'(Y_{i'_2}) + \Delta'(Y_{i'_1}) = \sum_{j \in I_i} \Delta'(Y_j)$. If $a_2$ does not exist, we can also upper bound the change in $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$ at time $t = t_i$ by $\Delta'(Y_{i'_1}) = \sum_{j \in I_i} \Delta'(Y_j)$. Recall that we have $\sum_{i = 1}^n \Delta'(Y_j) = Y_j$ by Definition 4.5. Summing over all time steps $t$, the total contribution of lines 6–11 at the end of Mechanism 2 to $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$, is at most $\sum_{i \in I} Y_i$. Now we analyze the contribution of the $Z_i$ variables. The sum $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$ is clearly increased at time $t = t_i$ by $\zeta Z_i$ on line 5. Next, observe that the total contribution from lines 16–19 to the sum $\sum_{j \in I_i \cup \{0\}} \text{excess}_j$ over the course of the mechanism is at most $(1 - \zeta) \sum_{j \in I_i} Z_i$. This follows from the facts that the conditional statement on line 17 evaluates to true at most once for each $a \in A$, and that each impression is matched to at most one advertiser in the optimal assignment. Since all of these contributions are disjoint, we have $\sum_{j \in I_i \cup \{0\}} \text{excess}_j \leq \sum_{j \in I_i} X_j + Y_j + Z_i$ at the end of Mechanism 2. The claim follows because $\text{excess}_0$ is always increased by nonnegative amounts.

4.3 Lower Bounding the Excess Allocated to Each Impression

The rest of our analysis is devoted to proving Lemma 4.3. We will need the following three inequalities in addition to Lemma 4.2, so we state them together for ease of reference and defer the longer proofs to Appendix B. First, we show in Lemma 4.6 how the adaptive decisions in lines 23–24 of Algorithm 1 in addition to Lemma 4.2, so we state them together for ease of reference and defer the longer proofs to Appendix B. First, we show in Lemma 4.6 how the adaptive decisions in lines 23–24 of Algorithm 1 (e.g., lines 41–42), we use adaptiveness in a very controlled way and limit its use to the most beneficial parts of the algorithm.

Lemma 4.6. If impression $i$ is assigned in lines 13–30 of StochasticGreedy, i.e., case $|B| \geq 2$, we have

$$E[\text{MarginalGain}_{i,a}] \geq \frac{E[\text{Gain}_{i,a}]}{2} + \text{adaptive_gain}_{i,a} + \text{adaptive_gain}_{i,a_2}.$$ 

The following two inequalities are derivatives of Lemma 4.6 that show how adaptive_gain_{i,a}, E[Gain_{i,a}], and $M_i$ relate to one another. We note that although there are other potential adaptive opportunities to exploit in Algorithm 1 (e.g., lines 41–42), we use adaptiveness in a very controlled way and limit its use to the most beneficial parts of the algorithm.

Lemma 4.7. For any advertiser $a \in A$ and impression $i \in I$, we have $\text{adaptive_gain}_{i,a} \leq \frac{1}{15} E[\text{Gain}_{i,a}]$.

Lemma 4.8. For any advertiser $a \in B$ at time $t = t_i$ for impression $i$, we have $E[\text{Gain}_{i,a}] \geq \frac{19(1 - \zeta)}{19} M_i$.

Proof. This is a direct consequence of Lemma 4.7 and the definition of the set $B$ in line 11 of Algorithm 1.

Now we present the proof of Lemma 4.3, which completes the analysis of Mechanism 2 and consequently the competitive ratio of Algorithm 1. To show that $\text{excess}_i \geq \lambda M_i$ for every impression $i \in I$, we first consider three different scenarios that can occur when $i$ arrives. For each of the three top-level cases, we analyze a series of subcases, all of which result in lower bounds for $\text{excess}_j$, that are a multiple of $M_i$. The branching structure of these cases is initially difficult to discern, but it is somewhat unavoidable given the adaptivity of Algorithm 1 and the design of Mechanism 2. We note, however, that the subcases themselves are relatively easy to verify. We present a distilled version of the subcases and their implications in Figure 1.
why we say the only advertiser in 

Therefore, 

It suffices to show that for any 

Case 1 (i.e., Case 2). 

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main cases and prove each separately: (1) impression 

because the excess 

during Mechanism 2 for a variety of reasons. Since the sequence 

• index(i) or index(a2) changes after 

- index(i) and index(a2) remain unchanged after time t 

Impression t1 arrived at or before time t where 

Impression t2 arrived at or before time t where 

Impressions t1 and t2 arrive after time t 

(1 − ζ) \frac{\delta}{1+\delta} \times \frac{6(1-\epsilon)}{19}, \frac{18(1-\epsilon)}{19} \times \sigma \right) M_i 

M_i.

We start by observing that excessi might change during Mechanism 2 for a variety of reasons. Since the sequence \(Y_{i,t}\) is nonnegative and nondecreasing in t, the updates to the excess variables in line 11 are nonnegative. Similarly, \(Z_i\) is nonnegative so the changes in line 5 and line 19 are nonnegative. The excess variables also do not decrease in line 14 or lines 21–23 because the adaptive_gain and \(M_i\) variables are nonnegative. The only place where excessi might be reduced is in line 15 or line 24 at \(t = t_i\). In particular, this happens for small or negative values of \(X_i\). In this proof, instead of tracking all changes to excessi, we bound this one-time reduction to excessi, and show that excessi is increased enough elsewhere to compensate for this potential decrease. We consider three main cases and prove each separately: (1) impression i is assigned in lines 31–42 of Algorithm 1, (2) i is assigned in lines 13–30 and \(a'_i \in \{a_1, a_2\}\), and (3) impression i is assigned in lines 13–30 and \(a'_i \notin \{a_1, a_2\}\).

Case 1 (i is assigned in lines 31–42). We start by proving the claim in the simplest case when i is assigned in lines 31–42 of Algorithm 1. Since adaptive_gain is always nonnegative, \(B'\) is a subset of \(B\). According to the else condition on line 31 of Algorithm 1, the set \(B\), and thus \(B'\), contains at most one advertiser. This is why we say the only advertiser in \(B'\) on line 38. Recall that \(X_i = E[\text{MarginalGain}_{i} ] - E[\text{Gain}_{i,a'_i}]\). If \(a'_i \notin B' \cup C\), then 

\(E[\text{Gain}_{i,a'_i}] < (1 - \epsilon)M_i\) by the definition of sets \(B'\) and \(C\). On the other hand, 

arg \max_{a \in A} E[\text{Gain}_{i,a}] will be selected as one of the advertisers to which i is assigned, and it achieves a gain of \(M_i\) by definition. If there is a second choice (i.e., \(a_2\) exists), its gain is at least \((1 - \epsilon)M_i\) by the definition of the sets \(B'\) and \(C\). Therefore, \(E[\text{MarginalGain}_i]\) is at least \((1 - \frac{\epsilon}{2})M_i\), which implies that \(X_i \geq \frac{\epsilon}{2}M_i\). It follows from line 24 of
Mechanism 2 that \( excess_i \) is increased by at least \( \frac{\varepsilon^2 - 2\varepsilon - 3\varepsilon}{2} M_i \), which proves the claim.

Next, we consider the scenario \( a'_i \in B' \). In this subcase, Algorithm 1 selects \( a'_i \) as one of at most two candidates for \( i \). This potential assignment increases \( Y_i \) at time \( t \) by at least \( E[\text{Gain}_{i,a'_i}] / 2 \), which implies that \( \Lambda'(Y_i) \geq E[\text{Gain}_{i,a'_i}] / 2 \geq (1 - \varepsilon) M_i / 2 \). Observe that this situation causes Mechanism 2 to increase \( excess_i \) in line 11. Since \( i' = i \) in this subcase, we have \( |K| \leq 2 \), which means \( excess_i \) is increased by at least \( (1 - \varepsilon) M_i / 4 \). On the other hand, similar to the argument above, we can show that the change in \( excess_i \) in line 24 of Mechanism 2 is at least \( (1 - \varepsilon) M_i - M_i - (\gamma + \sigma) M_i = -\frac{\varepsilon^2 + 2\varepsilon}{2} M_i \). Therefore, at the end of Mechanism 2, the value of \( excess_i \) is at least \( \frac{1 - \varepsilon}{4} M_i - \frac{\varepsilon^2 + 2\varepsilon}{2} M_i = \frac{1 - 3\varepsilon - 4\sigma}{6} M_i \), which again proves the claim.

To conclude Case 1, we assume that \( a'_i \in C \). Similar to the argument in the previous paragraph, we can show that the change in \( excess_i \) is at least \( -\frac{\varepsilon^2 + 2\varepsilon}{2} M_i \) in line 24 of Mechanism 2. Since \( a'_i \in C \), we know that \( w_i,a'_i < w_i,\text{index}(a'_i),a'_i - \delta M_i \). Observe that \( \text{index}(a'_i) \neq 0 \), for otherwise we would have \( w_i,a'_i < 0 \). Therefore, impression \( \text{index}(a'_i) \) arrived before \( i \) and was assigned to \( a'_i \) with probability at least \( 1/3 \). This potential assignment of \( \text{index}(a'_i) \) to \( a'_i \) at time \( t_{\text{index}(a'_i)} \) implies that \( Z_i \geq (w_i,\text{index}(a'_i),a'_i - w_i,a'_i) / 3 \geq \delta M_i / 3 \) by the definitions of \( Z_i \) and \( C \). Lemma 4.2 implies that a \( \varepsilon \) fraction of \( Z_i \) is distributed to \( excess_i \), so when Mechanism 2 ends, \( excess_i \) is at least \( \frac{\varepsilon}{3} Z_i - \frac{\varepsilon^2 + 2\varepsilon}{2} M_i \geq \frac{1}{3} M_i - \frac{\varepsilon^2 + 2\varepsilon}{2} M_i = \frac{2\varepsilon^2 - 3\varepsilon + 6\varepsilon - 6\varepsilon^2}{6} M_i \), as desired.

Case 2 (\( i \) is assigned in lines 13–30 and \( a'_i \in \{a_1, a_2\} \)). Now we consider the case where \( i \) is assigned in lines 13–30 of Algorithm 1 and \( a'_i \) is equal to either \( a_1 \) or \( a_2 \). We show that \( excess_i \) does not decrease too much in line 15 of Mechanism 2, and then we lower bound its increments on other occasions. Using Lemma 4.6, we know \( E[\text{MarginalGain}_{i,a'_i}] \) is at least \( E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}] / 2 + \text{adaptive_gain}_{i,a_1} + \text{adaptive_gain}_{i,a_2} \). Since \( a'_i \in \{a_1, a_2\}, \) the change in \( excess_i \) in line 15 at time \( t = t_i \) is at least

\[
X_i - \sum_{a \in \{i, a_1 \}} \frac{2}{3} \cdot \text{adaptive_gain}_{i,a} = E[\text{MarginalGain}_{i,a'_i}] - E[\text{Gain}_{i,a'_i}] - \sum_{a \in \{i, a_1 \}} \frac{2}{3} \cdot \text{adaptive_gain}_{i,a} \geq \left( \sum_{a \in \{i, a_1 \}} 1/2 \cdot E[\text{Gain}_{i,a}] + \text{adaptive_gain}_{i,a} \right) - E[\text{Gain}_{i,a'_i}] - \sum_{a \in \{i, a_1 \}} \frac{2}{3} \cdot \text{adaptive_gain}_{i,a} = \frac{1}{2} \left( E[\text{Gain}_{i,a_1}] + 2/3 \cdot \text{adaptive_gain}_{i,a_1} + E[\text{Gain}_{i,a_2}] + 2/3 \cdot \text{adaptive_gain}_{i,a_2} \right) - E[\text{Gain}_{i,a'_i}] \geq (1 - \varepsilon) M_i - M_i = -\varepsilon M_i,
\]

where the last inequality holds because \( a_1 \) and \( a_2 \) are both in the set \( B \) and because \( E[\text{Gain}_{i,a'_i}] \leq M_i \).

Now, since \( a'_i \) is the same as \( a_1 \) or \( a_2 \), variable \( Y_i \) increases by at least \( E[\text{Gain}_{i,a'_i}] / 3 \) at time \( t \) because \( 1/3 \) is a lower bound on the probability of assigning \( i \) to \( a'_i \). Furthermore, Lemma 4.8 implies \( E[\text{Gain}_{i,a'_i}] \geq \frac{18(1 - 2\varepsilon)}{19} M_i \) because \( a'_i \in B \). This means that at time \( t = t_i \) on line 11 of Mechanism 2, \( excess_i \) is increased by at least \( \Lambda'(Y_i) / |K| \geq E[\text{Gain}_{i,a'_i}] / 6 \geq \frac{3(1 - \varepsilon)}{19} M_i \) since we have \( i = i' \) and \( |K| \leq 2 \). Therefore, we conclude that at the end of Mechanism 2, the value \( excess_i \) is at least \( \frac{3(1 - \varepsilon)}{19} M_i - \varepsilon M_i = \frac{1 - 2\varepsilon}{19} M_i \), which proves the claim for Case 2.

Case 3 (\( i \) is assigned in lines 13–30 and \( a'_i \notin \{a_1, a_2\} \)). The final case is when \( i \) is assigned in lines 13–30 and \( a'_i \notin \{a_1, a_2\} \). We first bound the reduction of \( excess_i \) in line 14 of Mechanism 2 at time \( t = t_i \), and then we prove it is increased enough in other occasions. Like in Case 2, we start by applying Lemma 4.6. The new idea we can use here is that since \( a_1 \) and \( a_2 \) have been selected as the top two choices in the set \( B \) (lines 14–15 of Algorithm 1) and \( a'_i \) has not been chosen, at least one of the following inequalities holds: \( w_{i,a'_i} < w_{i,\text{index}(a'_i),a'_i} - 8\sigma M_i \) or \( E[\text{Gain}_{i,a'_i}] + 2/3 \cdot \text{adaptive_gain}_{i,a'_i} \leq E[\text{Gain}_{i,a_1}] + 2/3 \cdot \text{adaptive_gain}_{i,a_2} \), for both \( j \in \{1, 2\} \). We start by proving the claim in the first scenario. To do this, we need to introduce the new
2/3 \cdot \text{adaptive}_i \text{ gain}_{i,a} \text{ term into (1) from Case 2 to address the fact that } a'_i \notin \{a_1, a_2\}. Working from (1), we can say that the change in excess, in line 15 of Mechanism 2 is at least

\[-\varepsilon M_i - \frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a} \geq -\varepsilon M_i - \frac{2}{3} \cdot \frac{\mathbb{E}[\text{Gain}_{i,a}]}{12} \geq -\left(\frac{1}{18} + \varepsilon\right) M_i,\]

where the first inequality holds by Lemma 4.7. Since we first assume \( w_{i,a'_i} < w_{i,\text{index}(a'_i),a'_i} - \delta M_i \), impression index \( (a'_i) \) exists (i.e., it is not zero). We know that impression index \( (a'_i) \) arrived before \( i \) and is assigned to \( a'_i \) with probability at least 1/3 in Algorithm 1. Therefore, it follows that \( Z_i \geq (w_{i,\text{index}(a'_i),a'_i} - w_{i,a_i})/3 > \delta M_i/3 \).

Applying Lemma 4.2, we know Mechanism 2 increases excess, on line 5 by at least \( \zeta \delta M_i/3 \) upon termination. Therefore, the final value of excess, is at least \( \left(\frac{\zeta^6}{3} - \frac{1}{18} - \varepsilon\right) M_i = \frac{6\zeta^6 - 1 - 18\varepsilon}{18} M_i \), which proves the claim for the first scenario.

To complete the analysis for Case 3, we focus on the second subcase where we assume for \( j \in \{1, 2\} \) that

\[\mathbb{E}[\text{Gain}_{i,a_j}] + \frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a_j} \geq \mathbb{E}[\text{Gain}_{i,a_i}] + \frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a_i}.\]

To address the assumption that \( a'_i \notin \{a_1, a_2\} \), we adapt (1) as follows and then combine it with the inequalities above to get the following lower bound on the change of excess, in line 15 of Mechanism 2:

\[\frac{1}{2} \left( \sum_{j \in \{1,2\}} \mathbb{E}[\text{Gain}_{i,a_j}] + \frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a_j} \right) - \mathbb{E}[\text{Gain}_{i,a_i}] - \frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a_i} \geq 0.\]

Therefore, excess, is not reduced in line 15 of Mechanism 2. To complete Case 3, it suffices to show that excess, is increased enough in other places.

We note that in line 17 of Algorithm 1, \( \text{index}(a_1) \) and \( \text{index}(a_2) \) are set to \( i \) and \( \text{active}_{a_1} \) and \( \text{active}_{a_2} \) are set to true. For now, we assume that at least one of these two indices changes after time \( t \). (We prove the claim later if this assumption does not hold.) Let \( t' \) be the first time that this happens and let \( t' \) be the impression that arrives at time \( t' \). Without loss of generality, we assume that \( \text{index}(a_1) \) is the one among these two that changes from \( i \) to \( i' \) at time \( t' \). (The following argument also holds even if both of them change at \( t' \).) At time \( t' \), line 14 of Mechanism 2 increases excess, by \( 2/3 \cdot \text{adaptive}_i \text{ gain}_{i,a_1} \) because \( \text{index}(a_1) \) is equal to \( i \) before it is set to \( i' \). In the time period \( \{t + 1, t' - 1\} \), the indices of \( a_1 \) and \( a_2 \) remain unchanged, and they are always active. Note that if \( t' = t + 1 \), the time period is empty and the claim about their active state still holds. Since \( a_1 \) is one of the two advertisers that Algorithm 1 selects for \( i' \), we know \( a_1 \) is in the set \( B \) at time \( t' \), which further implies \( w_{i',a_1} \geq w_{i,a_1} - \delta M_i \). Therefore, the if condition on line 7 of Algorithm 1 is true for \( a_1 \) at time \( t' \), and \( \text{adaptive}_i \text{ gain}_{i,a_1} \) is set to \( (\mathbb{E}[\text{Gain}_{i,a_1}]/3 - (w_{i,a_1} - w_{i,a_1})^+/3 - S_{a_i})^+/12 \). Recall that excess, is increased at time \( t' \) by

\[\frac{2}{3} \cdot \text{adaptive}_i \text{ gain}_{i,a_1} = \left( \frac{\mathbb{E}[\text{Gain}_{i,a_1}]}{54} - \frac{(w_{i,a_1} - w_{i,a_1})^+}{54} - \frac{S_{a_i}}{18} \right)^+.\]

We bound each of the three terms in (2) separately. Since \( a_1 \) is in set \( B \) when both \( i \) and \( i' \) arrive, we have \( \mathbb{E}[\text{Gain}_{i,a_1}] \geq \frac{18(1 - \varepsilon)}{19} M_i \) and \( \mathbb{E}[\text{Gain}_{i,a_1}] \geq \frac{18(1 - \varepsilon)}{19} M_i \) by Lemma 4.8. The term \( (w_{i,a_1} - w_{i,a_1})^+ \) is zero if the weights satisfy \( w_{i,a_1} \leq w_{i,a_1} \). If we have \( w_{i,a_1} > w_{i,a_1} \), then \( \mathbb{E}[\text{Gain}_{i,a_1}] \geq \mathbb{E}[\text{Gain}_{i,a_1}] \) because \( i \) arrives before \( i' \) and also has a larger weight to advertiser \( a_1 \). Since \( M_i \geq \mathbb{E}[\text{Gain}_{i,a_1}] \), we also have \( M_i \geq \frac{18(1 - \varepsilon)}{19} M_i \). Applying the inequality \( w_{i,a_1} - w_{i,a_1} \leq \delta M_i \) from the definition of set \( B \) shows that excess, is increased by at least

\[\left( \frac{18(1 - \varepsilon) M_i}{54 \cdot 19} - \frac{19 \delta M_i}{54 \cdot 18} \cdot \frac{S_{a_i}}{18} \right)^+ = \left( \frac{324(1 - \varepsilon)^2 - 361 \delta}{18468(1 - \varepsilon)} M_i - S_{a_i} \right)^+.\]
Now we focus on lower bounding $S_{a_1}$. At time $t'$, $S_{a_1}$ is the expected sum of $M_{i'}$ for every impression $i'$ that has been assigned to $a_1$ in lines 31–42 of Algorithm 1 during the time period $[t + 1, t' - 1]$. For each of these impressions, Mechanism 2 increases excess $s_i$ in lines 21–22 by $\sigma M_{i''}/2$ or $\sigma M_{i''}$ depending on the existence of $a_2$ for $i''$. Note that this is consistent with how Algorithm 1 increases $S_{a_1}$. Therefore, excess $s_i$ is increased in total by at least $\left(\frac{32(1-\epsilon)^2 - 361}{186468(1-\epsilon)} M_i - \frac{S_{a_1}}{18}\right)^{+} + \sigma S_{a_1}$. This lower bound proves the claim because its minimum occurs when either $S_{a_1}$ or the expression in the $(x)^+$ operator is zero. In both cases, the lower bound is at least $\lambda M_i$.

We have reached the final step of Case 3 where we consider the scenario when $\text{index}(a_1)$ and $\text{index}(a_2)$ both remain unchanged after time $t$, and thus are active until the end of the algorithm. If for some $a \in \{a_1, a_2\}$, impression $i'$ (the impression with $a = a'_i$) arrived at or before time $t = t_i$, we can lower bound the increase in excess $s_i$ similar to in Case 2. Since $i$ is assigned to $a$ with probability at least 1/3, $\Lambda(Y_r) \geq \text{E}[\text{Gain}_{i,a}] / 3$. Mechanism 2 increases excess $s_i$ by at least one third of this amount in line 11. Since $a \in B$, we also know that $\text{E}[\text{Gain}_{i,a}] \geq \frac{18(1-\epsilon)M_i}{19}$ by Lemma 4.8. Therefore, excess $s_i$ is increased by at least $\frac{2}{19}(1-\epsilon)M_i$, which proves the claim.

Now we consider the case where the impressions $i_1$ and $i_2$ arrive after time $t$, where $i_1$ and $i_2$ are defined such that $a_1 = a'_{i_1}$ and $a_2 = a'_{i_2}$. If $w_{i_1,a_1} < w_{i_2,a_2} - \delta M_{i_1}$, then we prove the claim as follows. First, let $\Delta_w$ be the left-hand side of the inequality $w_{i_1,a_1} - w_{i_2,a_2} > \delta M_{i_1}$. We note that $\text{E}[\text{Gain}_{i_1,a_1}]$ is at least $\text{E}[\text{Gain}_{i_2,a_2}] - \Delta_w - S$, where $S$ is the value of $S_{a_1}$ at the time $t'$ when $i_1$ arrives. This lower bound holds because $\Delta_w$ compensates for how much smaller $w_{i_1,a_1}$ is compared to $w_{i_2,a_2}$ and $S$ is an upper bound on the total marginal gains of the edges assigned to $a_1$ between the times between the arrivals of $i_1$ and $i_1$. By definition, $M_{i_1} \geq \text{E}[\text{Gain}_{i_1,a_1}]$. Using the assumption $\Delta_w > 3M_{i_1}$, we have $\Delta_w > \delta \text{E}[\text{Gain}_{i_1,a_1}] \geq \delta \text{E}[\text{Gain}_{i_2,a_2}] - \Delta_w - S$. Applying Lemma 4.8, we know that $\text{E}[\text{Gain}_{i_2,a_2}] \geq \frac{18(1-\epsilon)M_i}{19}$, which implies $\Delta_w > \delta \left(\frac{18(1-\epsilon)M_i - S}{19}\right)^{+} - \sigma S \geq \lambda M_i$ at the end of the mechanism, which proves the claim. The subcase when $w_{i_1,a_1} < w_{i_2,a_2} - \delta M_{i_1}$ follows similarly.

To complete the proof, we now assume $w_{i_1,a_1} \geq w_{i_2,a_2} - \delta M_{i_1}$ and $w_{i_2,a_2} \geq w_{i_2,a_2} - \delta M_{i_2}$. If one of $i_1$ or $i_2$ is assigned in lines 13–30, the proof of the claim is identical to the part above starting near (2) where we showed the claim for the scenario that at least one of $\text{index}(a_1)$ or $\text{index}(a_2)$ changes after time $t$. To see this, observe that if $i_1$ is assigned in lines 13–30, then Mechanism 2 increases excess $\text{index}(a''_1) = \text{index}(s)$ by $2/3 \cdot \text{adaptive\_gain}_{i_1,a_1}$ on line 14 at time $t_k$. Otherwise, excess $i_2$ is increased in line 23 by $\gamma(M_{i_1} + M_{i_2})$ when $i_1$ and $i_2$ arrive since $\text{index}(a_1')$ and $\text{index}(a_2')$ are still equal to $i$. Like in the previous paragraph, we can show that $M_{i_1} \geq \text{E}[\text{Gain}_{i_1,a_1}] \geq \text{E}[\text{Gain}_{i_2,a_2}] - \Delta_w^{+} - S'$, where $\Delta_w^{+} = (w_{i_1,a_1} - w_{i_1,a_1})^{+}$ and $S'$ is the value of $S_{a_1}$ when $i_1$ arrives. By noting that $\Delta_w^{+} \leq \Delta_w$ and applying Lemma 4.8, we have the inequality $M_{i_1} \geq \frac{1}{19}(18(1-\epsilon)M_i - S')^{+}$. Observe that we can apply the $(x)^+$ operator because $M_k$ is nonnegative. We can derive an analogous lower bound for $M_{i_1}$ by replacing $\Delta_w^{+}$ with $\Delta_w^{+}$ and $S'$ with $S''$. Therefore, excess $i_2$ is increased by at least $\gamma \frac{1}{19}(18(1-\epsilon)M_i - S')^{+} + \sigma S' + \gamma \frac{1}{19}(18(1-\epsilon)M_i - S'')^{+} + \sigma S''$. This lower bound proves the claim because its minimum occurs when either one of the $(x)^+$ terms are zero, or both $S'$ and $S''$ are zero. This completes Case 3 and therefore concludes the proof of Lemma 4.3.

5 Conclusion

We give the first algorithm for online weighted bipartite matching with competitive ratio greater than $1/2$ (under the free disposal assumption), resolving a central open problem in the literature of online algorithms.
since the seminal work of Karp et al. [KVV90] thirty years ago. Given the hardness result of Kapralov et al. [KPV13], our algorithm can be seen as strong evidence that solving the weighted bipartite matching problem is strictly easier than submodular welfare maximization in online settings. Our main technical contributions in this work include a novel method for making adaptive decisions that is amenable to analysis, using the expectation of random variables over all possible branches of the randomized algorithm to force key variables of the algorithm to be deterministic, and a mechanism that we design solely for the sake of analysis to systematically reallocate extra marginal gain that the algorithm produces.

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A Missing Analysis from Section 3

A.1 Proof of Lemma 3.1

Lemma 3.1. The only random variables in STOCHASTICGREEDY are the assignments of the impressions and the values of Priority_a and Ri. All other variables (e.g., the maximum gains Mi, all values of adaptive_gaini,a, the sequence of sets B and choices a1 and a2, all updates to active(a), index(a), partner(a), and the sums Sa) are deterministic quantities that depend solely on the instance and the arrival order of the impressions.

Proof. We proceed by induction on t. At the beginning of Algorithm 1 when t = 0, all variables are initialized deterministically. Assuming the claim as the induction hypothesis, we proceed by analyzing the state of all the variables at time t ≥ 1. Recall that impression i arrives at time t = ti and is predetermined by the arrival order. We begin by considering lines 4–12. The maximum expected gain Mi is an expected value over all branches of the randomized algorithm up to time t and is deterministic by definition. For each a ∈ A, the current values of active(a), index(a), and Sa are deterministic by the induction hypothesis. Therefore, all values of adaptive_gaini,a and the set of candidates B at time t are also deterministic.

If we have |B| ≥ 2, then the algorithm executes lines 13–30. In this case, the advertisers a1 and a2 are the maximizers of deterministic quantities and therefore deterministic themselves, assuming ties are broken lexicographically. The updates that occur in lines 16–21 do not rely on any randomness since the value of partner(a) in line 18 is fixed by the induction hypothesis. The branching in lines 22–30 possibly depend on the random values of R and Priorityai, but in all of these conditional statements, only the assignment of impression i and updates to the variables Priorityai and Priorityai are made. In the second case, if |B| ≤ 1, then the algorithm executes lines 31–42. The only randomness here is the assignment of i to either a1 or a2. Therefore, the claim holds for all time steps t by induction.

A.2 Proof of Lemma 3.2

Lemma 3.2. We can maintain the probability mass function for all random variables Gaini,a over the course of algorithm. In particular, we can efficiently compute the value E[Gaini,a] at any point in STOCHASTICGREEDY.

Proof. First, recall that Algorithm 1 is essentially deterministic except for the impression assignments and the values of Priority_a. To be specific, Lemma 3.1 shows that the values of adaptive_gaini,a and the top candidates a1 and a2 are deterministic and depend solely on the underlying instance and the arrival order of the impressions. Furthermore, recall that Gaini,a = (wia − MaxWi_a−1). Therefore, it suffices to maintain the probability mass function for the random variables MaxWi_a at each time step of the algorithm.

We proceed by induction on t. Let W_t^i_a = {w_0,a, w_1,i,a, w_2,a, ..., w_i−1,a} denote the set of possible weights assigned to advertiser a at the beginning of time step t ≥ 1, and recall that w_0,a = 0. Note that we use i to denote the impression that arrives at time t. Next, let D_t^i : W_t^i_a → R≥0 be the probability mass function for the random variable MaxWi_a at the beginning of iteration t. Since the state of advertiser a, namely Priority_a, is randomized, we refine the distribution D_t^i into three conditional distributions. Let p_t^i,a be the probability that Priority_a = 2 at the beginning of time t, and let D_t^i be the distribution for MaxWi_a given that Priority_a = 2 at the beginning of time t. The probabilities p_t^i,a, p_t^i,a, and conditional distributions D_t^i, D_t^i, are defined similarly. We note that if a distribution is not well-defined because p_t^i,a = 0 for some j ∈ {0, 1, 2}, our analysis still holds since because these update rules simulate all branches of the randomized algorithm and always pull information from reachable states. Note also that we have D_t^i = ∑_j=0^2 p_t^i,a D_t^i by the law of total probability if we treat the addition and scalar multiplication of probability distributions like vectors.

No impressions have been assigned at the beginning of time t = 1, so for every advertiser a ∈ A the probabilities are set to p_t^i,a = 1, p_t^i,a = 0, p_t^i,a = 0 and the distributions satisfy D_t^i(0) = 1 for all j ∈ {0, 1, 2}.
This completes the base case when \( t = 1 \), so now assume \( t \geq 1 \) and that we have computed \( p_{a,j}^t \) and \( D_{a,j}^t \) for all \( a \in A \) and \( j \in \{0, 1, 2\} \). We show how to compute all of these quantities at time \( t + 1 \). Let \( a_1 \) and \( a_2 \) be the top candidates at time \( t \), and let \( i = i_t \). (If \( a_2 \) does not exist, we can ignore this term.) For every advertiser \( a \in A \setminus \{a_1, a_2\} \), the state of \( \text{Priority}_a \) does not change and \( a \) does not receive impression \( i \). Therefore, we have \( p_{a,j}^{t+1} \leftarrow p_{a,j}^t \) and \( D_{a,j}^{t+1} \leftarrow D_{a,j}^t \) for \( j \in \{0, 1, 2\} \). Now we focus on updating these values for \( a_1 \) and \( a_2 \).

**Case 1** (\( |B| \leq 1 \) and \( |B \cup C| = 1 \)). First consider the case where \( a_2 \) does not exist. The state \( \text{Priority}_{a_1} \) does not change, so set \( p_{a_1,j}^{t+1} \leftarrow p_{a_1,j}^t \) and \( D_{a_1,j}^{t+1} \leftarrow D_{a_1,j}^t \) for \( j \in \{0, 1, 2\} \). Since \( i \) is assigned to \( a_1 \), we update \( D_{a_1,j}^{t+1} \leftarrow \max(w_{i,a_1}, D_{a_1,j}^t) \) for each \( j \in \{0, 1, 2\} \). Here, the max operator transfers all probability mass from entries less than \( w_{i,a_1} \) to the value \( w_{i,a_1} \).

**Case 2** (\( |B| \leq 1 \) and \( |B \cup C| \geq 2 \)). Now suppose \( |B| \leq 1 \) and that there are two candidates \( a_1 \) and \( a_2 \). The states \( \text{Priority}_{a_1} \) and \( \text{Priority}_{a_2} \) remain unchanged, so set \( p_{a_1,j}^{t+1} \leftarrow p_{a_1,j}^t \) and \( p_{a_2,j}^{t+1} \leftarrow p_{a_2,j}^t \) for \( j \in \{0, 1, 2\} \). Impression \( i \) is assigned to \( a_1 \) or \( a_2 \) with equal probability \( 1/2 \), independent of their current states. Therefore, we update the conditional distributions to be \( D_{a_1,j}^{t+1} \leftarrow \frac{1}{2} \max(w_{i,a_1}, D_{a_1,j}^t) + \frac{1}{2} D_{a_1,j}^t \) and \( D_{a_2,j}^{t+1} \leftarrow \frac{1}{2} D_{a_2,j}^t + \frac{1}{2} \max(w_{i,a_2}, D_{a_2,j}^t) \), for each \( j \in \{0, 1, 2\} \).

**Case 3** (\( |B| \geq 2 \) and \( \text{adaptive\_gain}_{i,a_k} = 0 \)). Now suppose that \( |B| \geq 2 \) and that \( \text{adaptive\_gain}_{i,a_k} = 0 \). Recall that \( \text{adaptive\_gain}_{i,a_k} \) is a deterministic quantity governed solely by the instance and arrival order. The algorithm is guaranteed to ensure the conditional statement on line 25, so we update the priority probabilities to be \( p_{a_1,0}^{t+1} \leftarrow 1, p_{a_1,1}^{t+1} \leftarrow 0, p_{a_1,2}^{t+1} \leftarrow 0 \) and \( p_{a_2,0}^{t+1} \leftarrow 1, p_{a_2,1}^{t+1} \leftarrow 0, p_{a_2,2}^{t+1} \leftarrow 0 \) since the algorithm sets \( \text{Priority}_{a_1} \leftarrow 0 \) and \( \text{Priority}_{a_2} \leftarrow 0 \) on line 27. The impression \( i \) is randomly assigned to \( a_1 \) or \( a_2 \) with equal probability on line 26, so we update the two well-defined conditional distributions to be

\[
D_{a_1,0}^{t+1} \leftarrow \frac{1}{2} \sum_{j=0}^{2} p_{a_1,j}^t \max(w_{i,a_1}, D_{a_1,j}^t) + \frac{1}{2} \sum_{j=0}^{2} p_{a_1,j}^t D_{a_1,j}^t,
\]

\[
D_{a_2,0}^{t+1} \leftarrow \frac{1}{2} \sum_{j=0}^{2} p_{a_2,j}^t D_{a_2,j}^t + \frac{1}{2} \sum_{j=0}^{2} p_{a_2,j}^t \max(w_{i,a_2}, D_{a_2,j}^t).
\]

The values of the other distributions \( D_{a_1,1}^{t+1}, D_{a_1,2}^{t+1}, D_{a_2,1}^{t+1}, D_{a_2,2}^{t+1} \) do not matter, so we leave them unchanged. We note that the equations above can be simplified using the distributive property of the max operator, but the recurrences are easier to verify in their current form.

**Case 4** (\( |B| \geq 2 \) and \( \text{adaptive\_gain}_{i,a_k} > 0 \)). Now assume \( |B| \geq 2 \) and \( \text{adaptive\_gain}_{i,a_k} > 0 \), where \( a_k \) is defined in line 21 of Algorithm 1. In this case, the randomness of \( R_i \) and the current states of \( a_1 \) and \( a_2 \) determine how \( i \) is assigned. First, observe that we should set \( p_{a_1,j}^{t+1} \leftarrow \frac{1}{3} \) and \( p_{a_2,j}^{t+1} \leftarrow \frac{1}{3} \) for \( j \in \{0, 1, 2\} \) since each of the three main branches is equally likely. Now let us focus on computing the conditional distributions \( D_{a_1,j}^{t+1} \) for \( j \in \{0, 1, 2\} \). If we condition on the value of \( \text{Priority}_{a_k} \) at the beginning of time \( t + 1 \), then we can determine how the algorithm branched at time \( t \) based on the value of \( R_i \). This provides us with recurrence relations for the distributions \( D_{a_k,j}^{t+1} \), where we consider all possible previous states of \( \text{Priority}_{a_k} \) in each equation:

\[
D_{a_1,0}^{t+1} \leftarrow p_{a_1,2}^t \max(w_{i,a_1}, D_{a_1,2}^t) + p_{a_1,1}^t D_{a_1,1}^t + p_{a_1,0}^t \left( \frac{1}{2} \max(w_{i,a_1}, D_{a_1,0}^t) + \frac{1}{2} D_{a_1,2}^t \right),
\]

\[
D_{a_1,1}^{t+1} \leftarrow p_{a_1,2}^t \max(w_{i,a_1}, D_{a_1,2}^t) + p_{a_1,1}^t \max(w_{i,a_1}, D_{a_1,1}^t) + p_{a_1,0}^t \max(w_{i,a_1}, D_{a_1,0}^t),
\]

\[
D_{a_1,2}^{t+1} \leftarrow p_{a_1,2}^t D_{a_1,2}^t + p_{a_1,1}^t D_{a_1,1}^t + p_{a_1,0}^t D_{a_1,0}^t.
\]
We can compute the distributions $D_{a_{i-1},j}^{t+1}$ for $j \in \{0, 1, 2\}$ similarly, though in the adaptive decisions we need to account for the probabilities $p_{a_{i,j}}^t$. We use the equality $D_a^{t} = \sum_{j=0}^{2} p_{a_{i,j}}^t D_{a_{i,j}}^{t}$ in the following recurrences:

$$
D_{a_{i-1},0}^{t+1} \leftarrow p_{a_{i,2}}^t D_{a_{i-1},2}^{t} + p_{a_{i,1}}^t \sum_{j=0}^{2} p_{a_{i-1,j}}^t \max \left( w_{i,a_{i-1},j}, D_{a_{i-1,j}}^{t} \right) + p_{a_{i,0}}^t \left( \frac{1}{2} D_{a_{i-1},2}^{t} + \frac{1}{2} \sum_{j=0}^{2} p_{a_{i-1,j}}^t \max \left( w_{i,a_{i-1},j}, D_{a_{i-1,j}}^{t} \right) \right),
$$

$$
D_{a_{i-1},1}^{t+1} \leftarrow p_{a_{i,2}}^t \max \left( w_{i,a_{i-1},1}, D_{a_{i-1,2}}^{t} \right) + p_{a_{i,1}}^t \max \left( w_{i,a_{i-1},1}, D_{a_{i-1,1}}^{t} \right) + p_{a_{i,0}}^t \max \left( w_{i,a_{i-1},1}, D_{a_{i-1,0}}^{t} \right),
$$

$$
D_{a_{i-1},2}^{t+1} \leftarrow p_{a_{i,2}}^t D_{a_{i-1,2}}^{t} + p_{a_{i,1}}^t D_{a_{i-1,1}}^{t} + p_{a_{i,0}}^t D_{a_{i-1,0}}^{t}.
$$

In all four cases, we have shown how to compute the probabilities $p_{a_{i,j}}^{t+1}$ and conditional distributions $D_{a_{i,j}}^{t+1}$ for each $a \in A$ and $j \in \{0, 1, 2\}$. Therefore, by induction, we can maintain the distribution $D_a^t$ for $\text{MaxW}_a^t$ at each time step $t$, and hence compute the exact values of $E[\text{Gain}_{i,a}]$. □

## B Missing Analysis from Section 4

Before we give the proofs of Lemma 4.6 and Lemma 4.7, we present two self-contained, prerequisite lemmas. For any random variable $X$ and event $C$, define $E[X : C]$ to be $E[X | C] \Pr(C)$ where $E[X | C]$ is the expected value of $X$ conditioned on $C$. If this conditional probability is not well-defined, we assume that $E[X : C] = 0$. We use the following properties of the operators $(x)^+$ and $E[X : C]$ throughout this section.

**Lemma B.1.** For any three real numbers $u, v, w \in \mathbb{R}$, we have:

1. $(w - \max\{u, v\})^+ \geq (w - u)^+ - (\max\{u, v\} - u),$
2. $(w - u)^+ \geq (w - \max\{v, u\})^+ + (v - u)^+ - (v - w)^+.$

**Proof.** There are $3! = 6$ possible orderings for $u, v,$ and $w$. We consider each case separately:

- **Case 1** ($w \geq u \geq v$). Property 1 is equivalent to $w - u \geq (w - u) - 0$, which is true. Property 2 is equivalent to $w - u \geq (w - u) + 0 - 0$, which is also true.

- **Case 2** ($w \geq v \geq u$). Property 1 is equivalent to $w - v \geq (w - u) - (v - u)$, which is true. Property 2 is equivalent to $w - u \geq (w - v) + (v - u) - 0$, which is also true.

- **Case 3** ($u \geq w \geq v$). Property 1 is equivalent to $0 \geq 0 - (u - u)$, which is true. Property 2 is equivalent to $0 \geq 0 + 0 - 0$, which is also true.

- **Case 4** ($u \geq v \geq w$). Property 1 is equivalent to $0 \geq 0 - (u - u)$, which is true. Property 2 is equivalent to $0 \geq 0 + 0 - (v - w)$, which is also true.

- **Case 5** ($v \geq w \geq u$). Property 1 is equivalent to $0 \geq (w - u) - (v - u)$, which is true. Property 2 is equivalent to $w - u \geq 0 + (v - u) - (v - w)$, which is also true.

- **Case 6** ($v \geq u \geq w$). Property 1 is equivalent to $0 \geq 0 - (v - u)$, which is true. Property 2 is equivalent to $0 \geq 0 + (v - u) - (v - w)$, which is also true.

This completes the proof. □
**Lemma B.2.** For any \( k \) disjoint events \( C_1, C_2, \ldots, C_k \) and nonnegative random variable \( X \), we have

\[
E[X] \geq \sum_{i=1}^{k} E[X : C_i].
\]

If these disjoint events span the probability space, that is, \( \sum_{i=1}^{k} \Pr(C_i) = 1 \), the inequality can be replaced by equality, even if \( X \) is not nonnegative.

**Proof.** Let \( \overline{C} \) be the event that none of \( C_1, C_2, \ldots, C_k \) occur. The \( k + 1 \) events \( \overline{C}, C_1, C_2, \ldots, C_k \) are all disjoint and span the entire probability space. Therefore, for any random variable \( X \) (not necessarily nonnegative), the law of total expectation gives us

\[
E[X] = E[X \mid \overline{C}] \Pr(\overline{C}) + \sum_{i=1}^{k} E[X \mid C_i] \Pr(C_i) = E[X \mid \overline{C}] + \sum_{i=1}^{k} E[X \mid C_i],
\]

which proves the second part of the claim since \( \Pr(\overline{C}) = 0 \) if \( C_1, C_2, \ldots, C_k \) span the probability space. For the first part, if \( X \) is nonnegative then \( E[X \mid \overline{C}] \) is also nonnegative, and therefore \( E[X] = \sum_{i=1}^{k} E[X \mid C_i] \), which concludes the proof. \( \square \)

**B.1 Proof of Lemma 4.6**

**Lemma 4.6.** If impression \( i \) is assigned in lines 13–30 of STOCHASTICGREEDY, i.e., case \( |B| = 2 \), we have

\[
E[\text{MarginalGain}_i] \geq \frac{E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}]}{2} + \text{adaptive_gain}_{i,a_1} + \text{adaptive_gain}_{i,a_2}.
\]

**Proof.** Recall that the variables \( \text{adaptive_gain}_{i,a} \) and \( S_a \) are not random variables as show in Lemma 3.1—they are completely determined by the arrival order of the impressions. Recall also that \( k \in \{1, 2\} \) is set in line 21 of Algorithm 1 such that \( \text{adaptive_gain}_{i,a_k} \geq \text{adaptive_gain}_{i,a_{k-1}} \). If \( \text{adaptive_gain}_{i,a_k} = 0 \), then we also have \( \text{adaptive_gain}_{i,a_{k-1}} = 0 \), so we need to show that

\[
E[\text{MarginalGain}_i] \geq \frac{E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}]}{2}.
\]

This is evident because \( i \) is assigned to \( a_1 \) or \( a_2 \) with equal probability on line 26. Therefore, we focus on the case \( \text{adaptive_gain}_{i,a_k} > 0 \) for the rest of the proof.

The \( \text{adaptive_gain}_{i,a} \) variables are computed in terms of the expected values \( E[\text{Gain}_{i,a}] \) and therefore do not depend on the state of the algorithm (i.e., independent of the previous coin tosses \( R_i \)) by Lemma 3.2. Therefore, conditioning on the event \( \text{adaptive_gain}_{i,a_k} > 0 \) does not affect the distribution of \( R_i \) variables.

Impression \( i \) is assigned to \( a_1 \) or \( a_2 \) with equal probability (and also independently to prior decisions of the algorithm) unless \( R_i \in \langle 0, \frac{1}{3} \rangle \) and \( \text{Priority}_{a_k} \in \{1, 2\} \). This is the only cases where \( i \) is not assigned to \( a_1 \) or \( a_2 \) symmetrically. We also note that \( \text{Priority}_{a_k} \in \{1, 2\} \) is associated with the events \( R_{i'} \in \langle \frac{1}{3}, \frac{2}{3} \rangle \) or \( R_{i'} \in \langle \frac{2}{3}, 1 \rangle \), where \( i' \) is equal to \( \text{index}(a_k) \) at the beginning of time \( t = t_i \) when \( i \) arrives. Let \( t' = t_{i'} < t \) be the time that \( i' \) arrives. Observe that the random variables \( R_i \) and \( R_{i'} \) are independent of each other. Therefore, by the law of total expectation, the expected marginal gain \( E[\text{MarginalGain}_i] \) is equal to

\[
\frac{E[\text{Gain}_{i,a_1}] + E[\text{Gain}_{i,a_2}]}{3} + \frac{E[\text{Gain}_{i,a_1} : C_2] + E[\text{Gain}_{i,a_{k-1}} : C_1]}{3} + E[\text{Gain}_{i,a_k} : C_0] + E[\text{Gain}_{i,a_{k-1}} : C_0],
\]

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where \( C_2, C_1, \) and \( C_0 \) are the events that \( \text{Priority}_{a_k} \) is equal to 2, 1, and 0, respectively. These events partition the space, so Lemma B.2 implies \( \mathbb{E}[\text{Gain}_{i,a_k} : C_2] + \mathbb{E}[\text{Gain}_{i,a_k} : C_0]/6 = (\mathbb{E}[\text{Gain}_{i,a_k}] - \mathbb{E}[\text{Gain}_{i,a_k} : C_1])/6. \) An analogous decomposition also holds for \( \mathbb{E}[\text{Gain}_{i,a_{1:k}}]/6. \) Therefore, \( \mathbb{E}[\text{MarginalGain}_{i} \text{]} \) is equal to
\[
\mathbb{E}[\text{Gain}_{i,a_k} : C_2] + \mathbb{E}[\text{Gain}_{i,a_k} : C_0]/6 = (\mathbb{E}[\text{Gain}_{i,a_k}] - \mathbb{E}[\text{Gain}_{i,a_k} : C_1])/6.
\]

Since \( \text{adaptive\_gain}_{i,a_k} \geq \text{adaptive\_gain}_{i,a_{1:k}} \), it suffices to prove the inequalities
\[
\mathbb{E}[[\text{Gain}_{i,a_k} : C_2] - \mathbb{E}[\text{Gain}_{i,a_k} : C_0]/6 \geq 2 \cdot \text{adaptive\_gain}_{i,a_k}
\]
and
\[
\mathbb{E}[\text{Gain}_{i,a_{1:k}} : C_1] - \mathbb{E}[\text{Gain}_{i,a_{1:k}} : C_2] \geq 0.
\]
to complete the proof.

**Proof of Inequality (3).** Conditioning on the event \( C_2 \), we can apply Property 1 in Lemma B.1 to get
\[
\text{Gain}_{i,a_k} = (w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ \geq (w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ - (\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r).
\]
Recall that \( t' = t_{\text{index}(a_k)} \) and note that no impression is assigned to \( a_k \) at time \( t' \) according to the event \( C_2 \). Therefore, we have \( \text{MaxW}_{a_k}^{r-1} \geq \text{MaxW}_{a_k}^r \). Taking the conditional expectation implies that
\[
\mathbb{E}[\text{Gain}_{i,a_k} : C_2] \geq \mathbb{E}[(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ : C_2] - \mathbb{E}[\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r : C_2] 
\]
\[
\geq \mathbb{E}(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ - \mathbb{E}[\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r],
\]
where the second inequality holds for the following reasons. The random variable \( \text{MaxW}_{a_k}^{r-1} \) is independent of event \( C_2 \), so \( \mathbb{E}[(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ : C_2] = \mathbb{E}[(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+]/3. \) We can also apply Lemma B.2 for the nonnegative term \( \text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r \), and to get \( \mathbb{E}[\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r : C_2] \leq \mathbb{E}[\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r]. \)

The term \( \text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r \) represents all assignments to \( a_k \) in the time range \([t' + 1, t - 1]\), inclusive. In this interval, \( \text{index}(a_k) \) has remained the same since it was last set to \( t' \) at time \( t' \). Therefore, any impression \( i' \) assigned to \( a_k \) in this time period has been allocated in lines 31–42 of the algorithm (i.e., the case \( |B| \leq 1 \)). Since we increment \( S_{a_k} \) for each of these assignments accordingly (by either \( M_{a_k} \) or \( M_{a_k}/2 \) depending on whether \( a_k \) is the only candidate or not), the expected increase of \( \text{MaxW}_{a_k} \) in this time period is upper bounded by \( S_{a_k} \). Formally, we have \( \mathbb{E}[\text{MaxW}_{a_k}^{r-1} - \text{MaxW}_{a_k}^r] \leq S_{a_k} \). Therefore, it follows from (5) that
\[
\mathbb{E}[\text{Gain}_{i,a_k} : C_2] \geq \frac{\mathbb{E}(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+}{3} - S_{a_k}.
\]

By Property 2 of Lemma B.1, we always have the inequality
\[
(w_{i,a_k} - \text{MaxW}_{a_k}^{r-1})^+ \geq (w_{i,a_k} - \max\{w_{i',a_k}, \text{MaxW}_{a_k}^{r-1}\})^+ + (w_{i',a_k} - \text{MaxW}_{a_k}^{r-1})^+ - (w_{i',a_k} - w_{i,a_k})^+.
\]
By combining inequalities (6) and (7), it follows that
\[
\mathbb{E}[\text{Gain}_{i,a_k} : C_2] \geq \frac{\mathbb{E}(w_{i,a_k} - \max\{w_{i',a_k}, \text{MaxW}_{a_k}^{r-1}\})^+}{3} + \frac{\mathbb{E}(w_{i',a_k} - \text{MaxW}_{a_k}^{r-1})^+}{3} - \frac{\mathbb{E}(w_{i',a_k} - w_{i,a_k})^+}{3} - S_{a_k}
\]
\[
\geq \mathbb{E}[\text{Gain}_{i,a_k} : C_1] + \frac{\mathbb{E}[\text{Gain}_{i,a_k}]}{3} - \frac{(w_{i',a_k} - w_{i,a_k})^+}{3} - S_{a_k},
\]
where (8) is proved as follows. Conditioning on event \( C_1 \), we know that impression \( i' \) was assigned to \( a_k \). Thus, \( \text{Gain}_{i,a_k} = (w_{i,a_k} - \text{Max}W^{t-1}_{a_k})^+ \) is at most \((w_{i,a_k} - \max\{w_{r,a_k}, \text{Max}W^{t-1}_{a_k}\})^+ \). Observing that \( \text{Max}W^{t-1}_{a_k} \) is independent of \( C_1 \), we have
\[
\mathbb{E}\left[\left(w_{i,a_k} - \max\{w_{r,a_k}, \text{Max}W^{t-1}_{a_k}\}\right)^+\right] \geq \mathbb{E}\left[\text{Gain}_{i,a_k} : C_1\right],
\]
which proves (8). Note that the only assumption we used to prove (8) is that \( \text{adaptive\_gain}_{i,a_k} \) is positive. Therefore, the definition of \( \text{adaptive\_gain}_{i,a_k} \) in line 8 of Algorithm 1 and inequality (8) imply that
\[
\mathbb{E}[\text{Gain}_{i,a_k} : C_2] - \mathbb{E}[\text{Gain}_{i,a_k} : C_1] \geq 12 \cdot \text{adaptive\_gain}_{i,a_k},
\]
which completes the proof of inequality (3).

**Proof of Inequality (4).** Let \( a' \) be partner\((a_k)\) at time \( t-1 \) (right before \( a_{3-k} \) becomes the partner of \( a_k \)). Impression \( i' \) is assigned to either \( a_k \) or \( a' \). The index values of these two advertisers are unchanged in the interval \( [t' + 1, t - 1] \), otherwise active\((a_k)\) would be false at the beginning of time \( t \), which contradicts the positivity of \( \text{adaptive\_gain}_{i,a_k} \). Moreover, the advertiser \( a' \neq 0 \) is well-defined by the positivity of \( \text{adaptive\_gain}_{i,a_k} \) since active\((a_k)\) = true. Therefore, all impressions assigned to \( a_k \) or \( a' \) in this interval must be non-adaptive decisions (i.e., are assigned in lines 31–42). These impressions are assigned independent to the value that \( R_{i'} \) takes for impression \( i' \). Each impression that arrives before time \( t \) and has neither \( a_k \) nor \( a' \) as its candidates is also assigned independent to the value of \( R_{i'} \). In particular, conditioning on the events \( \text{Priority}_{a_k} = 1 \) or \( \text{Priority}_{a_k} = 2 \) does not change the distribution of MaxW \(_a^{t-1}\) for any \( a \in A \setminus \{a_k, a'\} \). Therefore, if \( a_{3-k} \) is not the same as \( a' \), then \( \mathbb{E}[\text{Gain}_{i,a_{3-k}} : C_1] = \mathbb{E}[\text{Gain}_{i,a_{3-k}, a_k} : C_2] \), which proves the inequality. If \( a_{3-k} = a' \), then all assignments to \( a_{3-k} \) at times \( T = \{1, 2, ..., t - 1\} \setminus \{t'\} \) are independent of both events \( \text{Priority}_{a_k} = 1 \) and \( \text{Priority}_{a_k} = 2 \). If we fix the set of impressions assigned to \( a_{3-k} \) at times in \( T \), conditioning on \( \text{Priority}_{a_k} = 1 \) compared to conditioning on \( \text{Priority}_{a_k} = 2 \) can only decrease \( \text{Max}W^{t-1}_{a_{3-k}} \), which further implies \( \mathbb{E}[\text{Gain}_{i,a_{3-k}} : C_1] \geq \mathbb{E}[\text{Gain}_{i,a_{3-k}} : C_2] \). This completes the proof of inequality (4), and therefore the proof of the lemma. \( \square \)

**B.2 Proof of Lemma 4.7**

**Lemma 4.7.** For any advertiser \( a \in A \) and impression \( i \in I \), we have \( \text{adaptive\_gain}_{i,a} \leq \frac{1}{12} \mathbb{E}[\text{Gain}_{i,a}] \).

**Proof.** If \( \text{adaptive\_gain}_{i,a} = 0 \), then the claim is trivial. Therefore, assume that
\[
\text{adaptive\_gain}_{i,a} = \mathbb{E}\left[\frac{\text{Gain}_{i',a}}{12} - \left(\frac{w_{i',a} - w_{i,a}}{3} - S_a\right)^+\right] > 0,
\]
where \( i' \) is equal to index\((a)\) at the beginning of time \( t = t_i \). Indices are set on line 17 of Algorithm 1, so \( \text{adaptive\_gain}_{i,a} > 0 \) implies that \( i' \) is assigned to an advertiser in lines 13–30. Without loss of generality, assume \( a \) is the first choice of \( i' \) (i.e., \( a = a_1 \) and \( a \neq a_2 \)). The proof for the other case is identical by swapping the \( \text{Priority}_{a} = 1 \) and \( \text{Priority}_{a} = 2 \) terms.

We focus on the events where \( i' \) is assigned in lines 29–30 of Algorithm 1, which correspond to setting \( \text{Priority}_a = 1 \) or \( \text{Priority}_a = 2 \). Let \( C_1 \) be the event that \( i' \) is assigned to \( a \) in line 29, or, equivalently, the event \( \text{Priority}_a = 1 \). We note that \( i' \) might be assigned to \( a \) in lines 13–30 but this is not included in the
event $C_1$. Similarly, let $C_2$ be the event that $i'$ is assigned to the second candidate in line 30. Let $t' = t_r$ be the time $i'$ arrives. Conditioning on $C_2$, we can apply (8) from the proof of Lemma 4.6, which gives us

$$
E \left[ \text{Gain}_{i,a} : C_2 \right] \geq E \left[ \text{Gain}_{i,a} : C_1 \right] + \frac{E \left[ \text{Gain}_{r,a} \right]}{3} - \frac{(w_{r,a} - w_{l,a})^+}{3} - S_a
$$

$$
\geq \frac{E \left[ \text{Gain}_{r,a} \right]}{3} - \frac{(w_{r,a} - w_{l,a})^+}{3} - S_a.
$$

(9)

Since $\text{adaptive\_gain}_{i,a} > 0$, inequality (9) implies $E[\text{Gain}_{i,a} : C_2] \geq 12 \cdot \text{adaptive\_gain}_{i,a}$. Therefore, we have $E[\text{Gain}_{i,a}] \geq E[\text{Gain}_{i,a} : C_2] \geq 12 \cdot \text{adaptive\_gain}_{i,a}$ by Lemma B.2 since Gain random variables are always nonnegative. 

\qed