Equilibrium solutions of axially moving Timoshenko beam with a supercritical speed

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Abstract. Analytical solution of the equilibrium configuration of an axially moving Timoshenko beam in supercritical regime is studied. Three kinds of classical boundary conditions are considered. An analytical solution of the equilibrium configuration in terms of the axial velocity in supercritical regime is determined. But most of all, for an axially moving Timoshenko beam with the fixed ends in supercritical regime, an anti-symmetric configuration is firstly detected. And, its analytical solution is solved. Besides, numerical example shows that the solution of the equilibrium configuration bifurcates with axial velocity. And the critical velocity derived from the Timoshenko beam theory is smaller than that derived from Euler-Bernoulli beam theory. The amplitude of the solution of equilibrium configuration derived from Timoshenko beam is larger.

1. Introduction

Axially moving beam has been widely used in many mechanical, aerospace, and automotive systems. Hence, mechanical behavior of axially moving beam has attracted extensive attention from scholars. The studies about axially moving beam can be divided into two aspects in terms of system modelling. The first aspect is studies in the subcritical regime, while the second is studies in the supercritical regime. The literature on the first aspects is large while for the second class is limited.

For the first group, there is many literatures. Chen \textit{et al.} \[1\] and Marynowski \textit{et al.} \[2\] gives a comprehensive summary of the axially moving beam in subcritical regime. For the second group, Wicker \textit{et al.} \[3\] is the first to examine the supercritical axially moving beams. Above the critical velocity, the straight equilibrium configuration loses stability and produces bifurcation to yield multiple equilibrium positions. Krzysztof \textit{et al.} \[4\] investigated the dynamical behavior and bifurcations of axially moving web and found that the moving web may encounter divergent or flutter instability in supercritical regime. Parker \textit{et al.} \[5\] studied instability of trivial equilibrium and the critical velocity of an axially moving string supported by elastic foundation. Based on the Galerkin method \[6\] and the discrete Fourier transform method \[7\], Ding \textit{et al.} investigated the natural frequencies of an axially supercritical moving beam. Making use of finite difference method, they also investigated the numerical solution of equilibrium configuration an axially supercritical moving beam with the classical boundary condition \[8\] and the hybrid boundary condition \[9\]. Applied the multiple scale method and the finite difference method, Zhang \textit{et al.} \[10\] studied the steady-state response and stability of an axially moving viscoelastic beam in supercritical regime. Ghayesh \textit{et al.} \[11,12\] numerically investigated the global dynamics behavior of an axially supercritical moving beam with
three-to-one internal resonance. Mao et al. [13] analyzed the state-state responses of an axially supercritical beam subjected to the combined excitation. Yang et al. [14] firstly detected anti-symmetric configuration of an axially moving Euler-Bernoulli beam with the fixed-fixed boundary condition.

All the studies mentioned above is derived from Euler-Bernoulli beam theory. There is a little research on axially moving Timoshenko beams in supercritical regime. Applied the direct time integration method, Ghayesh et al. [15] investigated the nonlinear forced vibration and stability of an axially moving Timoshenko beam with an intra-span spring-support. Ding et al. [16] utilizes trial and error to study the equilibrium configuration of an axially supercritical moving Timoshenko beam and determines an analytical expression. In their work, only symmetric configurations were found. The anti-symmetric configuration in the supercritical regime has not been reported. To fully understand the behavior of the supercritical Timoshenko beam, we use an analytical method to solve equilibrium configuration of an axially moving Timoshenko beam in the supercritical regime. Based on the solution, we derive an anti-symmetric equilibrium configuration for an axially moving Timoshenko beam with the fixed-fixed boundary condition.

2. Problem formulation

The figure 1 shows the schematic representation of an axially moving Timoshenko beam, with length $L$, density $\rho$, cross-sectional area $A$, young’s modulus $E$, shearing modulus $G$, the moment of inertia of the cross section $I$, axial velocity $\gamma$ and axial tension $P$. $V(X,T)$ and $\Phi(X,T)$ denote the transverse displacement and slope of the deflection curve. The governing equation of axially moving Timoshenko beam is given by [16]

\[
\rho A v_{,TT} + 2\gamma v_{,xT} + \gamma^2 v_{,xx} - PV_{,xx} - \frac{EA}{2L} \int_0^L v_x^2 \, dx + \kappa GA (\Phi_x - V_{,xx}) = 0
\]  

\[\rho l (\Phi_{,TT} + 2\gamma \Phi_{,xT} + \gamma^2 \Phi_{,xx}) - E l \Phi_{,xx} + \kappa GA (\Phi - V_x) = 0\]  

In this paper, we consider three boundary conditions.

First: Fixed-Fixed boundary condition

\[V(0,T) = 0 \quad V(L,T) = 0 \quad \Phi(0,T) = 0 \quad \Phi(L,T) = 0\]  

Second: Fixed-Hinged boundary condition

\[V(0,T) = 0 \quad V(L,T) = 0 \quad \Phi(0,T) = 0 \quad E l \Phi_{,x}(L,T) = 0 \]  

Third: Hinged-Hinged boundary condition

\[V(0,T) = 0 \quad E l \Phi_{,x}(0,T) = 0 \quad \gamma \Phi_{,x}(0,T) + \gamma \Phi_{,x}(0,T) = 0 \quad E l \Phi_{,x}(L,T) = 0 \quad \gamma \Phi_{,x}(L,T) = 0\]  

For convenience, we introduce the following nondimensional variables.

\[\nu = \frac{V}{L}, \varphi = \frac{\Phi}{L}, \chi = \frac{X}{L}, t = \frac{T}{L}, \tau = \frac{\gamma}{L}, \rho A = \frac{E A}{L}, k_1 = \frac{\kappa GA}{P}, k_2 = \frac{1}{A L}, k_y = \frac{E A}{P}, k_x = \frac{E A}{P E^2}\]

As a result, we rewrite (1)-(4) as

\[\nu_{,x} + 2\nu_{,xT} + (\gamma^2 - 1) \nu_{,xx} + k_1 (\varphi_{,x} - \nu_{,x}) - \frac{1}{2} k_y \nu_{,xx} \int_0^L \nu_x^2 \, dx = 0\]

\[k_2 (\varphi_{,x} + 2\varphi_{,xT} + \gamma^2 \varphi_{,xx}) - k_x \varphi_{,xx} + k_1 (\varphi - \nu_{,x}) = 0\]
Fixed-Fixed boundary condition
\[ \nu(0,t) = 0 \quad \nu(1,t) = 0 \quad \varphi(0,t) = 0 \quad \varphi(1,t) = 0 \]  
(6)

Fixed-hinged boundary condition
\[ \nu(0,t) = 0 \quad \nu(1,t) = 0 \quad k_c \varphi_s(1,t) - k_2 \gamma \varphi_s(1,t) + \gamma \varphi_s(1,t) = 0 \]  
(7)

Hinged-hinged boundary condition
\[ \nu(0,t) = 0 \quad k_c \varphi_s(0,t) - k_2 \gamma \varphi_s(0,t) + \gamma \varphi_s(0,t) = 0 \]  
\[ \nu(1,t) = 0 \quad k_c \varphi_s(1,t) - k_2 \gamma \varphi_s(1,t) + \gamma \varphi_s(1,t) = 0 \]  
(8)

3. Bucking problem

3.1. Fixed-Fixed beam

In the case of the beam with two fixed ends, dropping the time-dependent terms of the governing equation and the boundary condition, the supercritical problem can be written as
\[(\gamma^2 - 1)\psi^\prime + k_i (\phi^\prime - \psi^\prime) - \frac{1}{2} k_n \psi^\prime \int_0^L \psi^\prime \, dx = 0 \]  
(9a)
\[(k_2 \gamma^2 - k_i) \phi^\prime + k_i (\phi - \psi^\prime) = 0 \]  
(9b)
\[\psi(0) = \phi(0) = 0 \quad \psi(1) = \phi(1) = 0 \]  
(9c)

in which \(\psi(x)\) and \(\phi(x)\) denote supercritical configuration. Note that the integral in equation (9a) is a constant value for a given \(\psi(x)\). Hence, let
\[H = \frac{1}{2} k_n \int_0^L \psi^\prime \, dx \]  
(10)

where \(H\) is a constant. Convert equation (9a) and (9b) to the following equation:
\[(c + 1)\psi^\prime + \phi^\prime = 0 \quad \psi^\prime + d \phi^\prime - \phi^\prime = 0 \]  
(11)

where \(c = (H + 1 - \gamma^2)/k_i\) and \(d = (k_t - k_2 \gamma^2)/k_i\). Equation (11) is an ordinary-differential equation with the constant coefficients. Its general solution takes the following forms
\[\psi(x) = C_1 x + C_2 \lambda \sin \lambda x - C_3 \lambda \cos \lambda x + C_4 \]  
(12a)
\[\phi(x) = C_1 + C_2 \lambda \cos \lambda x + C_3 \lambda \sin \lambda x \]  
(12b)

where \(C_i\) are the undetermined constants and \(\lambda\) is the constant and
\[\lambda^2 = -\frac{c}{d(c + 1)} \quad K = 1 + \frac{\lambda^2 d}{\lambda} \]  
(13)

Substituting equation (12) into the boundary conditions (9c) obtain an eigenvalue problem for \(\lambda\) as follows
\[C_1 + C_2 = 0 \]  
\[C_1 + C_2 \cos \lambda - C_4 = 0 \]  
\[C_1 \lambda \cos \lambda - C_4 = 0 \]  
\[C_1 + C_2 \lambda \sin \lambda - C_4 \lambda + C_4 = 0 \]  
(14)

To ensure equation set (14) has a non-zero solution, demanding that the determinant of the coefficient matrix equals zero. The following characteristic equation for \(\lambda\) can be obtain:
\[2K(1 - \cos \lambda) - \sin \lambda = 0 \]  
(15)

The corresponding mode shapes \(\psi(x)\) and \(\phi(x)\) can be obtained as
\[\psi(x) = C K \left(1 - \cos \lambda \right) \sin \lambda x - \sin \lambda x - \cos \lambda x \]  
(16a)
\[\phi(x) = C \left(- \frac{K(1 - \cos \lambda)}{1 - K \sin \lambda} + 1 \right) \cos \lambda x + \sin \lambda x \]  
(16b)
where \( C \) is a constant to be determined. \( \psi(x) \) has two different modes: symmetric mode and antisymmetric mode. Next, the expressions for symmetric and antisymmetric modes are gained by using trigonometric identities. Equation (15) is manipulated as follows:

\[
2K(1 - \cos \lambda) - \sin \lambda = 4\sin \frac{\lambda}{2} \left( K \sin \frac{\lambda}{2} - \cos \frac{\lambda}{2} \right) = 0 \quad (17)
\]

It follows from equation (17) that there are two possible solutions. Frist,

\[
\sin \frac{\lambda}{2} = 0 \quad \text{or} \quad \lambda = 2m\pi \quad m = 1, 2, \ldots \quad (18)
\]

and equation (16) yields the symmetric configuration

\[
\psi(x) = CK \left[ 1 - \cos(2m\pi x) \right] \quad \phi(x) = C \sin(2m\pi x) \quad (19a, b)
\]

Bucking configuration \( \psi(x) \) also must satisfied equation (13). Substituting equations (10) and (19) into equation (13), using equation (18) and trigonometric identities, we obtain

\[
C = \pm \frac{2}{k_x \lambda K} \sqrt{\gamma^2 - 1 - \frac{k_x \lambda d}{K}} \quad (20)
\]

Second,

\[
\tan \frac{\lambda}{2} = \frac{1}{2K} \quad (21)
\]

and equation (16) yields the antisymmetric mode

\[
\psi(x) = CK \left( 1 - 2x - \cos \lambda x + 2K \sin \lambda x \right) \quad (22a)
\]

\[
\phi(x) = C \left[ 2K (\cos \lambda x - 1) + \sin \lambda x \right] \quad (22b)
\]

Substituting equations (10) and (22) into equation (13), using equation (18), we obtain

\[
C = \pm \frac{2}{k_x \lambda K \sqrt{1 + 12d + 4\lambda^2 d^2}} \sqrt{\gamma^2 - 1 - \frac{k_x \lambda d}{K}} \quad (23)
\]

Thus, for a given axial velocity, \( C \) corresponding to any eigenvalue \( \lambda \) can be determined.

### 3.2. Fixed-hinged beam

In the case of fixed-hinged supports, the boundary condition (9c) is replaced with

\[
\psi(0) = \phi(0) = 0 \quad \psi(1) = \phi'(1) = 0 \quad (24)
\]

The characteristic equation and the solution of equilibrium configuration are obtained

\[
K \sin \lambda - \cos \lambda = 0 \quad (25)
\]

\[
\psi(x) = CK \left( K \sin \lambda x - \cos \lambda x - x + 1 \right) \quad (26a)
\]

\[
\phi(x) = C (\sin \lambda x + K \cos \lambda x - K) \quad (26b)
\]

Substituting equations (10) and (26) into (15), using equation (25), we obtain

\[
C = \pm \frac{2}{k_x \lambda K \sqrt{1 + 3d + \lambda^2 d^2}} \sqrt{\gamma^2 - 1 - \frac{k_x \lambda d}{K}} \quad (27)
\]

### 3.3. Hinged-hinged beams

In the case of hinged-hinged supports, the boundary condition (9c) is replaced with

\[
\psi(0) = \phi'(0) = 0 \quad \psi(1) = \phi'(1) = 0 \quad (28)
\]

The characteristic equation and the solution of equilibrium configuration are obtained

\[
\lambda = m\pi \quad m = 1, 2, \ldots \quad (29)
\]

\[
\psi(x) = CK \sin \lambda x \quad \phi(x) = C \cos \lambda x \quad (30a, b)
\]

Substituting equations (10) and (30) into (13), using equation (29), we obtain
\[ C = \pm \frac{2}{k_n \lambda K} \sqrt{\gamma^2 - 1 - \frac{k_n \lambda d}{K}} \]  

(31)

4. Numerical examples
In this section, we consider a V-belt with \( E = 200 \text{Mpa}, \rho = 1200 \text{kg/m}^3, L = 0.15 \text{m}, G = 68 \text{Mpa}, P = 80 \text{N}, \) and \( I = 7.4250 \times 10^{-9} \text{m}^4. \) For the fixed-fixed boundary condition, let the axial velocity \( \gamma = 109 \text{m/s} \) that is beyond the second critical velocity, the first and the second supercritical configuration is showed in figure 2 and figure 3, respectively. Figure 2 represents the symmetric supercritical configuration for the axially moving Timoshenko beam, the first supercritical configuration. Figure 3 shows the anti-symmetric configuration of the Timoshenko beam, the second supercritical configuration. More remarkable, the anti-symmetric configuration is first discovered, which is different from previous research.

Figure 2. Symmetric supercritical configuration of the beam with fixed-fixed boundary condition.

Figure 3. Anti-symmetric supercritical configuration of the beam with fixed-fixed boundary condition.

For the fixed-hinged boundary condition, let the axial velocity \( \gamma = 64 \text{m/s} \). Figure 4 depicts the supercritical configuration the axially moving Timoshenko beam. Figure 5 shows the supercritical configuration the beam for the fixed-hinged boundary condition when the axial velocity \( \gamma = 46 \text{m/s} \).

Figure 4. Supercritical configuration of the beam with fixed-hinged boundary condition.
Figure 5. Supercritical configuration of the beam with hinged-hinged boundary condition.

To compare the differences between Euler-Bernoulli beams theory [14] and Timoshenko beam theory beam, the bifurcation diagram of the beam with the three different boundary conditions is showed in figure 6, figure 7 and figure 8, respectively. It is obvious that when the axial velocity is less than the critical velocity the straight configuration remains stable. When the axial velocity is beyond a critical value, the equilibrium produces a pitchfork bifurcation. By comparing the critical velocity of the beam with different boundary condition, it is shown that the critical velocity of the beam with fixed ends is the largest, and that with hinged ends is the minimum. And the critical velocity derived from the Timoshenko beam theory is smaller than that derived from Euler-Bernoulli beam theory. The amplitude of the solution of equilibrium configuration derived from Timoshenko beam theory is larger. The differences between the outcomes derived from the two beam theories are more sensitive to higher order critical velocity. Moreover, under the fixed boundary condition, the differences between two beam theories are the most obvious.

Figure 6. Bifurcation diagram for the supercritical beam with a fixed-fixed boundary condition.

Figure 7. Bifurcation diagram for the supercritical beam with a fixed-hinged boundary condition.
Figure 8. Bifurcation diagram for the supercritical beam with a hinged-hinged boundary condition.

Figures 9-11 show the effect of the length, the modulus and the tension on the equilibrium configuration, respectively. It is observed from figures 9-11 that the critical velocity increases with the tension and modulus of elasticity and decreases with the length. Moreover, the amplitude of the solution for equilibrium configuration is increasing with the growing the tension and the length of the Timoshenko beam and decreasing with growth the modulus of elasticity. Therefore, the critical velocity is sensitive to the length of beam, especially the beam with the fixed boundary condition.

Figure 9. Effects of the length of the beam on the configuration (a) fixed-fixed (b) hinged-hinged.

Figure 10. Effects of the modulus of the beam on the configuration (a) fixed-fixed (b) hinged-hinged.
Figure 11. Effects of the tension of the beam on the configuration (a) fixed-fixed (b) hinged-hinged.

5. Conclusions
The present paper studies the analytical solution of the equilibrium configuration for an axially moving Timoshenko beam in supercritical regime. Three different boundary conditions are considered. The expression of equilibrium configuration in supercritical regime is deduced analytically. For axially moving Timoshenko beam with the fixed ends, we firstly detect an anti-symmetric configuration and obtain analytical expression. Numerical example shows that when the axial velocity beyond the critical, the solution of the equilibrium configuration generates bifurcation. And the critical velocity derived from the Timoshenko beam theory is smaller than that derived from Euler-Bernoulli beam theory. The amplitude of the solution derived from Timoshenko beam is larger. Therefore, Euler-Bernoulli beam theory underestimates the amplitude of equilibrium configuration of the axially moving beam and overestimates the critical velocity.

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References
[1] L. Q. Chen. Appl.Mech. Rev. 58,7 (2005)
[2] K. Marynowski, T. Kapitaniak, Int. J. Mech. Sci. 81, 26-41(2014)
[3] J. A. Wickert, Int. J. Non-Lin Mech. 27, 503-517(1992)
[4] K. Marynowski, J. Theor. App. Mech-Pol. 37, 335-347(1999)
[5] R. G.Parker, J. Sound Vib. 221, 205-219 (1999)
[6] H. Ding, L. Q. Chen, J. Sound Vib. 329, 3484-3494(2010)
[7] H. Ding, G. C. Zhang, L. Q. Chen, Int. J. Non-Lin. Mech. 47, 1095-1104(2010)
[8] H. Ding, L. Q. Chen, Arch. Appl. Mech. 81, 51-64(2011)
[9] H. Ding, G. C. Zhang, L. Q. Chen, Mech. Res. Commun. 38, 52-56(2011)
[10] G. C. Zhang, H. Ding, L. Q. Chen, J. Sound Vib. 331,1612-1623(2012)
[11] M. H. Ghayesh, H. A. Kafiabad, Int. J. Solids Struct. 49, 227-243(2012)
[12] M. H. Ghayesh, M. Amabili, Int. J. Mech. Sci. 68, 76-91(2013)
[13] X. Y. Mao, H. Ding, L. Q. Chen, Nonlinear Dynam. 89,1-13 (2013)
[14] T. Z. Yang, X. D. Yang, Arch. Appl. Mech. 83,899-906 (2013)
[15] M. H. Ghayesh, M. Amabili, Mech. Mach. Theory. 67,1-16 (2013)
[16] H. Ding, X. Tan, G. C. Zhang, Acta. Mech. 227, 1-14(2016)