Asymptotic Behavior of Solutions to a Model System of a Radiating Gas

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Abstract
In this paper we focus on the initial value problem for a hyperbolic-elliptic coupled system of a radiating gas in multi-dimensional space. By using a time-weighted energy method, we obtain the global existence and optimal decay estimates of solutions. Moreover, we show that the solution is asymptotic to the linear diffusion wave which is given in terms of the heat kernel.

keywords: radiating gas; initial value problem; asymptotic behavior.

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1 Introduction
In this paper we consider the initial value problem:

\[
\begin{align*}
\left\{
\begin{array}{l}
  u_t + \sum_{j=1}^n f_j(u)x_j + \text{div}q = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
  -\nabla\text{div}q + q + \nabla u = 0, \quad x \in \mathbb{R}^n, \quad t > 0,
\end{array}
\right.
\end{align*}
\]

(1.1)

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with the initial data
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n. \]  
(1.2)

Here \( f_j(u) \), \( j = 1, \cdots, n \), are smooth functions of \( u \) satisfying \( f_j(u) = O(u^2) \) for \( u \to 0 \), and \( u = u(x, t) \) and \( q = (q_1, \cdots, q_n)(x, t) \) are unknown functions of \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \) \( (n \geq 1) \) and \( t > 0 \). Typically, \( u \) and \( q \) represent the velocity and radiating heat flux of the gas, respectively.

The system (1.1) is a simplified version of a radiating gas model in \( n \)-dimensional space. More precisely, in a certain physical situation, the system (1.1) gives a good approximation to the following system of a radiating gas, that is a quite general model for compressible gas dynamics where the heat radiative transfer phenomena are taken into account:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u + pI) &= 0, \\
\{\rho(e + \frac{|u|^2}{2})\}_t + \text{div}\{\rho u(e + \frac{|u|^2}{2}) + pu + q\} &= 0, \\
-\nabla \text{div} q + a_1 q + a_2 \nabla \theta &= 0,
\end{align*}
\]
(1.3)

where \( \rho, u, p, e \) and \( \theta \) are respectively the mass density, velocity, pressure, internal energy and absolute temperature of the gas, while \( q \) is the radiative heat flux, and \( a_1 \) and \( a_2 \) are given positive constants depending on the gas itself. The first three equations form the usual Euler system, which describes the inviscid flow of a compressible fluid, and express the conservation of mass, momentum and energy, respectively. We refer to the book of Courant and Friedrichs \[1\] for a detailed derivation of several models in compressible gas dynamics. On the other hand, the physical motivation of the fourth equation, which takes into account the heat radiation phenomena, is given in \[32, 3\]. The simplified model (1.1) was first recovered by Hamer \[8\], and the reduction of the full system (1.3) to (1.1) was given in \[32, 8, 16, 7\].

There are many works on the study of the hyperbolic-elliptic coupled system for one-dimensional radiating gas. The earlier paper with application to this kind of systems is \[31\], where Schochet and Tadmor studied the regularized Chapman-Enskog expansion for scalar conservation laws. We refer to \[8, 12, 13, 14, 18, 23, 24, 25, 27, 26\] for shock waves, \[3, 10, 9, 11\] for nonlinear diffusion waves, \[16\] for rarefaction waves, \[15, 17, 21\] for a singular limit and relaxation limit, and \[4, 29, 30\] for \( L^1 \) stability results. In the multi-dimensional case, Di Francesco in \[2\] obtained the global-in-time existence and uniqueness of weak entropy solutions to the system (1.1) and analyzed the relaxation limits. In \[33\], recently Wang and Wang studied the initial value problem for the system (1.1) in multi-dimensions and obtained the pointwise estimates of classical solutions by using the method of Green
function combined with some energy estimates. Also, the stability of planar rarefaction waves was discussed in [7, 6]. Very recently, Ruan and Zhu [28] investigated the asymptotic decay rates toward the linear diffusion wave and also the rarefaction wave in $\mathbb{R}^n$, and proved the asymptotic relation of our Proposition 4.2 with $k = 0$ and the norm $L^2$ instead of $H^{s-1-k}$ in the case $2 \leq n \leq 7$ by using the energy method and the semigroup argument.

In this paper we investigate the decay rate not only to the same linear diffusion wave as in [28] which can be seen from Proposition 4.2, but also to another diffusion wave which is given in terms of the heat kernel as shown in Theorem 2.4 for the initial value problem (1.1), (1.2) in $\mathbb{R}^n$ for $n \geq 2$ by applying the time-weighted energy method together with the semigroup argument, which removes the restriction $n \leq 7$ assumed in [28] and also improves their results. To this end, we first transform the system (1.1) into the following equivalent decoupled system (1.4) which makes the derivation of our energy estimates easier, but is not essential for obtaining our main results,

$$\begin{cases}
  u_t - \Delta u - \Delta u + (1 - \Delta) \sum_{j=1}^n f_j(u) x_j = 0, & x \in \mathbb{R}^n, \ t > 0, \\
  q = -(1 - \Delta)^{-1} \nabla u, & x \in \mathbb{R}^n, \ t > 0.
\end{cases} \tag{1.4}$$

Then, by applying the time-weighted energy method to the decoupled system (1.4), we derive the optimal decay estimates of solutions for all $n \geq 1$. Finally, using the semigroup argument, we show that the solution is asymptotic to the linear diffusion wave as $t \to +\infty$, provided that $n \geq 2$. Our linear diffusion wave is given explicitly in terms of the heat kernel.

The contents of the paper are as follows. In Section 2 we give full statements of our main theorems. Section 3 gives the proof of the results on the global existence and decay estimates of solutions. The last section gives the proof of the theorem on the asymptotic convergence to the linear diffusion wave.

Before closing this section, we give some notations to be used below. Let $F[f]$ denote the Fourier transform of $f$ defined by

$$F[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and we denote its inverse transform by $F^{-1}$.

$L^p = L^p(\mathbb{R}^n) \ (1 \leq p \leq +\infty)$ is the usual Lebesgue space with the norm $\| \cdot \|_{L^p}$. For $\gamma \in \mathbb{R}$, let $L^1_\gamma = L^1_\gamma(\mathbb{R}^n)$ denote the weighted $L^1$ space with the norm

$$\| f \|_{L^1_\gamma} := \int_{\mathbb{R}^n} (1 + |x|)^\gamma |f(x)| \, dx.$$
Let $s$ be a nonnegative integer. Then $H^s = H^s(\mathbb{R}^n)$ denotes the Sobolev space of $L^2$ functions, equipped with the norm

$$\|f\|_{H^s} := \left( \sum_{k=0}^s \|\partial_x^k f\|_{L^2}^2 \right)^{\frac{1}{2}}.$$ 

Here, for a nonnegative integer $k$, $\partial_x^k$ denotes the totality of all the $k$-th order derivatives with respect to $x \in \mathbb{R}^n$. Also, $C^k(I; H^s(\mathbb{R}^n))$ denotes the space of $k$-times continuously differentiable functions on the interval $I$ with values in the Sobolev space $H^s = H^s(\mathbb{R}^n)$.

Finally, in this paper, we denote every positive constant by the same symbol $C$ or $c$ without confusion. $[\cdot]$ is the Gauss’ symbol.

2 Main theorems

Our first theorem is on the global existence and uniform energy estimate of solutions to the problem (1.1), (1.2).

**Theorem 2.1.** Let $n \geq 1$ and let $s \geq \left[ \frac{n}{2} \right] + 2$ be an integer. Assume that $u_0 \in H^s(\mathbb{R}^n)$ and put $E_0 := \|u_0\|_{H^s}$. Then there exists a small positive constant $\delta_0$ such that if $E_0 \leq \delta_0$, then the problem (1.1), (1.2) has a unique global solution $(u, q)(x, t)$ with

$$u \in C([0, +\infty); H^s(\mathbb{R}^n)), \quad \partial_x u \in L^2([0, +\infty); H^{s-1}(\mathbb{R}^n)),
q \in C([0, +\infty); H^{s+1}(\mathbb{R}^n)) \cap L^2([0, +\infty); H^{s+1}(\mathbb{R}^n)).$$

The solution verifies the uniform energy estimate

$$\|u(t)\|_{H^s}^2 + \|q(t)\|_{H^{s+1}}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 + \|q(\tau)\|_{H^{s+1}}^2 d\tau \leq CE_0^2. \quad (2.1)$$

When $n \geq 2$, the solution obtained in Theorem 2.1 verifies the following decay estimates.

**Corollary 2.2.** Let $n \geq 2$ and let $s \geq \left[ \frac{n}{2} \right] + 2$ be an integer. If $E_0 = \|u_0\|_{H^s}$ is suitably small, then the global solution obtained in Theorem 2.1 verifies the decay estimates

$$\|\partial_x^k u(t)\|_{H^{s-k}} \leq CE_0(1 + t)^{-\frac{k}{2}} \quad (2.2)$$

for $k$ with $0 \leq k \leq s$ and

$$\|\partial_x^k q(t)\|_{H^{s+1-k}} \leq CE_0(1 + t)^{-\frac{k+1}{2}} \quad (2.3)$$

for $k$ with $0 \leq k \leq s - 1$. 

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The next theorem is on the optimal decay estimates of solutions for initial data in $H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

**Theorem 2.3.** Let $n \geq 1$, and let $s \geq 3$ for $n = 1$ and $s \geq \left[\frac{n}{2}\right] + 2$ for $n \geq 2$. Assume that $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and put $E_1 := \|u_0\|_{H^s} + \|u_0\|_{L^1}$. Then there exists a small positive constant $\delta_1$ such that if $E_1 \leq \delta_1$, then the global solution obtained in Theorem 2.1 satisfies the decay estimates

\[
\|\partial_x^k u(t)\|_{H^{s-k}} \leq C E_1 (1 + t)^{-\frac{n}{4} - \frac{k}{2}} \quad (2.4)
\]

for $k$ with $0 \leq k \leq s - 1$ and

\[
\|\partial_x^k q(t)\|_{H^{s+1-k}} \leq C E_1 (1 + t)^{-\frac{n}{4} - \frac{k}{2} + \frac{1}{2}} \quad (2.5)
\]

for $k$ with $0 \leq k \leq s - 2$.

**Remark 1.** To obtain the optimal decay estimates of solutions stated in Corollary 2.2 and Theorem 2.3, we will use the following decay estimate for $L^\infty$ norm of the derivative $\partial_x u$ as shown by $N(T)$ in (3.12):

\[
\|\partial_x u(t)\|_{L^\infty} \leq C (1 + t)^{-1}. \tag{2.8}
\]

Our final result is concerning the asymptotic profile of the global solution obtained in Theorem 2.3 for $n \geq 2$. First we show that for $n \geq 2$, the solution to the problem (1.1), (1.2) can be approximated by the solution to the corresponding linear problem,

\[
\begin{cases}
\bar{u}_t - \Delta \bar{u}_t - \Delta \bar{u} = 0, & \bar{u}(x, 0) = u_0(x), \\
\bar{q} = -(1 - \Delta)^{-1} \nabla \bar{u}.
\end{cases} \tag{2.6}
\]

Then we prove that the solution to this linear problem can be further approximated by the following simpler problem (2.7) based on the linear heat equation $u_t - \Delta u = 0$,

\[
\begin{cases}
\tilde{u}_t - \Delta \tilde{u} = 0, & \tilde{u}(x, 0) = u_0(x), \\
\tilde{q} = -(1 - \Delta)^{-1} \nabla \tilde{u}.
\end{cases} \tag{2.7}
\]

Since the solution to the linear heat equation is asymptotic to the heat kernel

\[
G_0(x, t) = \mathcal{F}^{-1}[e^{-|\xi|^2}(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}], \tag{2.8}
\]

we thus conclude that the asymptotic profile of our global solution is given by the following linear diffusion wave $(u^*, q^*)(x, t)$:

\[
u^*(x, t) = MG_0(x, t + 1), \quad q^*(x, t) = -\nabla u^*(x, t), \tag{2.9}
\]

where $M = \int_{\mathbb{R}^n} u_0(x) dx$ denotes the “mass”. The result is precisely stated as follows.
Theorem 2.4. Let \( n \geq 2 \) and \( s \geq \left[ \frac{n}{2} \right] + 2 \). Assume that \( u_0 \in H^s(\mathbb{R}^n) \cap L^1_1(\mathbb{R}^n) \) and put \( E_2 := \| u_0 \|_{H^s} + \| u_0 \|_{L^1_1} \). Then the global solution \((u, q)\) to the problem (1.1), (1.2), which is constructed in Theorem 2.3, is asymptotic to the linear diffusion wave \((u^*, q^*)\) in (2.9) as \( t \to +\infty \):

\[
\| \partial_x^k (u - u^*)(t) \|_{H^{s-1-k}} \leq CE_2 \rho(t)(1 + t)^{-\frac{n}{4} - \frac{k+1}{2}}
\]

for \( 0 \leq k \leq s - 1 \) and

\[
\| \partial_x^k (q - q^*)(t) \|_{H^{s-k}} \leq CE_2 \rho(t)(1 + t)^{-\frac{n}{4} - \frac{k+2}{2}}
\]

for \( 0 \leq k \leq s - 2 \), where \( \rho(t) = \ln(1 + t) \) for \( n = 2 \) and \( \rho(t) = 1 \) for \( n \geq 3 \).

3 Global existence and decay estimates

3.1 Global existence of solutions

This subsection is devoted to the proof of the global existence result stated in Theorem 2.1. Since a local existence result can be obtained by the standard method based on the successive approximation sequence, we omit its details and only derive the desired a priori estimates of solutions.

First we give a lemma which will be used in the derivation of our energy estimates.

Lemma 3.1. Let \( 1 \leq p, q, r \leq +\infty \) and \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). Then the following estimates hold:

\[
\| \partial_x^k (uv) \|_{L^p} \leq C(\| u \|_{L^q} \| \partial_x^k v \|_{L^r} + \| v \|_{L^q} \| \partial_x^k u \|_{L^r}), \quad k \geq 0.
\]

\[
\| [\partial_x^k, u] \partial_x v \|_{L^p} \leq C(\| \partial_x u \|_{L^q} \| \partial_x^k v \|_{L^r} + \| \partial_x v \|_{L^q} \| \partial_x^k u \|_{L^r}), \quad k \geq 1.
\]

The proof of this lemma can be found in [20]. The next lemma shows the equivalence of the system (1.1) and (1.4).

Lemma 3.2. The system (1.1) is equivalent to the decoupled system (1.4).

Proof. First we show that (1.1) implies (1.4). It follows from (1.1) that \( q = \nabla \phi \) with \( \phi = \text{div}q - u \). Therefore we see that \( \nabla \text{div}q = \nabla \Delta \phi = \Delta \nabla \phi = \Delta q \).

Consequently, (1.1) becomes \((1 - \Delta)q + \nabla u = 0\), which gives (1.4). Also, applying \( \text{div} \) to (1.4)\(_2\), we have

\[
\text{div}q = -(1 - \Delta)^{-1} \Delta u.
\]
We substitute (3.1) into (1.1) and then apply $1 - \Delta$ to the resulting equation. This yields (1.4)_1.

Next we derive (1.1) from (1.4). We have (3.1) from (1.4)_2. Apply $(1 - \Delta)^{-1}$ to (1.4)_1 and substitute (3.1) to the result. This yields (1.1)_1. Also, we rewrite (3.1) as

$$\text{div } q = u - (1 - \Delta)^{-1}u$$

and apply $\nabla$, obtaining

$$\nabla \text{div } q = \nabla u - (1 - \Delta)^{-1}\nabla u = \nabla u + q,$$

where we have used (1.4)_2. This gives (1.1)_2. Thus the proof of Lemma 3.2 is complete.

In view of Lemma 3.2, we only need to consider the decoupled system (1.4), which makes the derivation of our energy estimates easier. As a priori estimates of solutions, we show the uniform energy estimate for $u$ by using the equation (1.4)_1 and then a similar estimate for $q$ by making use of the relation (1.4)_2. To this end, we consider solutions $u$ to the problem (1.4)_1, (1.2), which are defined on the time interval $[0, T]$ for $T > 0$ and satisfy

$$\|u(t)\|_{L^\infty} \leq C, \quad \|\partial_x u(t)\|_{L^\infty} \leq C$$

for $0 \leq t \leq T$. Notice that these estimates hold true if $\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq C$ for $s \geq \left[\frac{n}{2}\right] + 2$.

Now we multiply (1.4)_1 by $u$. A direct computation gives

$$\begin{align*}
(u^2 + |\nabla u|^2)_t + 2|\nabla u|^2 - 2\nabla \cdot \{u \nabla (u_t + u + \sum_j f_j(u)x_j)\} \\
+ \sum_j \{2g_j(u) + f'_j(u)|\nabla u|^2\}x_j = - \sum_j f''_j(u)u x_j |\nabla u|^2 \!
\end{align*}$$

where $g_j(u) = \int_0^u f'_j(\eta) \eta \, d\eta$. Integrating this equality with respect to $x$, we have

$$\frac{d}{dt} \|u(t)\|_{H^1}^2 + 2\|\partial_x u(t)\|_{L^2}^2 \leq C\|\partial_x u(t)\|_{L^\infty}^2 \|\partial_x u(t)\|_{L^2}^2.$$

To get similar estimates for the derivatives, we apply $\partial_x^l$ to (3.1)_1, obtaining

$$\begin{align*}
\partial_x^l u_t - \Delta \partial_x^l u_t - \Delta \partial_x^l u + \sum_j f'_j(u) \partial_x^l u x_j - \nabla \cdot \sum_j f''_j(u) \nabla \partial_x^l u x_j \\
= -h^l + \nabla \cdot H^l,
\end{align*}$$

where $h^l = \sum_j [\partial_x^l, f'_j(u)] u x_j$ and $H^l = \sum_j [\partial_x^l \partial_x, f'_j(u)] \nabla u$. We multiply this
equation by \( \partial_t u \) and compute directly to get
\[
(|\partial_t^2 u|^2 + |\nabla \partial_t^2 u|^2)_t + 2|\nabla \partial_t^2 u|^2 - 2 \nabla \cdot \{ \partial_t^2 u \nabla \partial_t^2 (u_t + u) \\
+ \partial_t u \sum_j f_j(u) \nabla \partial_t^2 u_{x_j} \} + \sum_j \{ f_j(u)(|\partial_t^2 u|^2 + |\nabla \partial_t^2 u|^2) \}_{x_j}
\]
\[
= \sum_j f''_j(u) u_{x_j} (|\partial_t^2 u|^2 + |\nabla \partial_t^2 u|^2) - 2(h^1 \partial_t^2 u + H^1 \cdot \nabla \partial_t^2 u) + 2 \nabla \cdot (H^1 \partial_t^2 u).
\]

Integrating this equality with respect to \( x \) and estimating the right hand side by applying Lemma 3.1, we obtain
\[
\frac{d}{dt} \| \partial_t^2 u(t) \|_{L^2}^2 \leq C \| \partial_t u(t) \| \| \partial_t^2 u(t) \|_{L^2},
\]
where \( 1 \leq l \leq s - 1 \). Here we have used the fact that \( f_j(u) = O(u^2) \), \( j = 1, \ldots, n \) for \( u \to 0 \). We add (3.3) and (3.4) for \( 1 \leq l \leq s - 1 \) and integrate over \((0, t)\). This yields
\[
\int_0^t \| \partial_t u(\tau) \|_{L^2}^2 d\tau \leq C \int_0^t \| \partial_t u(\tau) \|_{L^2} \| \partial_t^2 u(\tau) \|_{L^2} d\tau,
\]
where \( E_0 = \| u_0 \|_{H^s} \). Let \( \delta \) be a positive number (independent of \( T \)) and assume that \( \sup_{0 \leq t \leq T} \| u(t) \|_{H^s} \leq \delta \), where \( s \geq \left\lceil \frac{n}{2} \right\rceil + 2 \). Then the second term on the right hand side of (3.5) is estimated by \( C\delta \int_0^t \| \partial_t u(\tau) \|_{L^2}^2 d\tau \). Therefore, choosing \( \delta \) so small that \( C\delta \leq \frac{1}{2} \), we arrive at the uniform energy estimate
\[
\int_0^t \| \partial_t u(\tau) \|_{H^{s-1}}^2 d\tau \leq C E_0^2.
\]
On the other hand, it follows from (1.4) that \( \| q \|_{H^{s+1}} \leq \| \partial_t u \|_{H^{s-1}} \), which combined with (3.6) yields
\[
\int_0^t \| q(\tau) \|_{H^{s+1}}^2 d\tau \leq C E_0^2.
\]
These observations are summarized as follows.

**Proposition 3.3.** Let \( n \geq 1 \) and let \( s \geq \left\lceil \frac{n}{2} \right\rceil + 2 \) be an integer. Assume that \( u_0 \in H^s(\mathbb{R}^n) \) and put \( E_0 := \| u_0 \|_{H^s} \). Let \((u, q)\) be a solution to the problem (1.1), (1.2) on the time interval \([0, T]\). Then there is a small positive constant
\(\bar{\delta}\) independent of \(T\) such that if \(\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq \bar{\delta}\), then the solution verifies the following uniform energy estimate for \(t \in [0, T]\):

\[
\|u(t)\|_{H^s}^2 + \|q(t)\|_{H^{s+1}}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 + \|q(\tau)\|_{H^{s+1}}^2 d\tau \leq C E_0^2. \tag{3.8}
\]

By virtue of the a priori estimate (3.8) for small solutions stated in Proposition 3.3, we can apply the continuity argument and obtain a unique global solution to the problem (1.1), (1.2), provided that \(E_0\) is suitably small, say, \(E_0 \leq \delta_0\). The solution obtained verifies (3.8) for \(t \geq 0\). This proves Theorem 2.1.

### 3.2 Optimal decay estimates

In this subsection, we obtain the optimal decay estimates of the solution constructed in Theorem 2.1 by using the time-weighted energy method. To this end, we define two time-weighted energy norms \(E(T)\) and \(M(T)\). Also, we introduce \(D(T)\) as the dissipation norm corresponding to \(E(T)\).

\[
E(T)^2 := \sum_{j=0}^{s} \sup_{0 \leq t \leq T} \sum_{j=0}^{n} (1 + t)^j \|\partial_x^j u(t)\|_{H^{s-j}}^2, \tag{3.9}
\]

\[
D(T)^2 := \sum_{j=0}^{s} \int_0^T (1 + \tau)^j \|\partial_x^{j+1} u(\tau)\|_{H^{s-j-1}}^2 d\tau, \tag{3.10}
\]

\[
M(T)^2 := \sum_{j=0}^{s-1} \sup_{0 \leq t \leq T} (1 + t)^{\frac{n}{2} + j} \|\partial_x^j u(t)\|_{H^{s-j}}^2, \tag{3.11}
\]

where \(s \geq 1\). To derive estimates for \(E(T)\), \(D(T)\) and \(M(T)\), we make use of the following time-weighted norm \(N(T)\):

\[
N(T) := \sup_{0 \leq t \leq T} (1 + t) \|\partial_x u(t)\|_{L^\infty}. \tag{3.12}
\]

As for the energy \(E(T)\) and \(D(T)\), we have the following estimate.

**Proposition 3.4.** Let \(n \geq 1\) and \(s \geq \left\lceil \frac{n}{2} \right\rceil + 2\). Assume that \(u_0 \in H^s(\mathbb{R}^n)\) and put \(E_0 := \|u_0\|_{H^s}\). Then the solution to the problem (1.1), (1.2) constructed in Theorem 2.1 satisfies following energy estimate:

\[
E(T)^2 + D(T)^2 \leq CE_0^2 + CN(T) D(T)^2.
\]
Proof. In order to prove this proposition, it is enough to show the following estimates for any \( t \in [0, T] \) and \( 0 \leq j \leq s \):

\[
(1 + t)^{j} \| \partial_{x}^{j} u(t) \|_{H^{s-j}}^{2} + \int_{0}^{t} (1 + \tau)^{j} \| \partial_{x}^{j+1} u(\tau) \|_{H^{s-j-1}}^{2} d\tau \leq CE_{0}^{2} + CN(T)D(T)^{2}. \tag{3.13}
\]

We know from (2.10) that (3.13) holds true for \( j = 0 \). Now, let \( 0 \leq k \leq s - 1 \) and suppose that (3.13) holds true for \( j = k \). Then we show (3.13) for \( j = k + 1 \). Multiplying (3.4) by \( (1 + t)^{k+1} \), integrating with respect to \( t \) over \((0, t)\) and adding for \( l \) with \( k + 1 \leq l \leq s - 1 \), we have

\[
(1 + t)^{k+1} \| \partial_{x}^{k+1} u(t) \|_{H^{s-k-1}}^{2} + \int_{0}^{t} (1 + \tau)^{k+1} \| \partial_{x}^{k+2} u(\tau) \|_{H^{s-k-2}}^{2} d\tau \]
\[
\leq CE_{0}^{2} + C \int_{0}^{t} (1 + \tau)^{k} \| \partial_{x}^{k+1} u(\tau) \|_{H^{s-k-1}}^{2} d\tau
\]
\[
+ CE_{0}^{2} + CN(T)D(T)^{2}.
\]

The second term on the right hand side is estimated by the induction hypothesis (3.13) with \( j = k \), while the last term can be estimated by \( CN(T)D(T)^{2} \). Consequently, we have

\[
(1 + t)^{k+1} \| \partial_{x}^{k+1} u(t) \|_{H^{s-k-1}}^{2} + \int_{0}^{t} (1 + \tau)^{k+1} \| \partial_{x}^{k+2} u(\tau) \|_{H^{s-k-2}}^{2} d\tau \]
\[
\leq CE_{0}^{2} + CN(T)D(T)^{2},
\]

which shows that (3.13) holds true for \( j = k + 1 \). Thus, by induction, we have proved Proposition 3.3. \( \square \)

By employing the optimal decay results expressed in \( E(T) \) and \( M(T) \), we obtain the following estimates for \( N(T) \).

Lemma 3.5. (i) If \( n \geq 2 \) and \( s \geq \left[ \frac{n}{2} \right] + 2 \), then \( N(T) \leq CE(T) \).

(ii) If \( n = 1 \) and \( s \geq 3 \), then \( N(T) \leq CM(T) \).

Proof. (i) Let \( s_{0} = \left[ \frac{n}{2} \right] + 1 \) and \( \theta = \frac{n}{2s_{0}} \). By applying the Gagliardo-Nirenberg inequality, we see that

\[
\| \partial_{x} u(t) \|_{L^{\infty}} \leq C \| \partial_{x} u(t) \|_{L^{2}}^{\frac{1}{2}} \| \partial_{x}^{s_{0}+1} u(t) \|_{L^{2}}^{\theta}
\]
\[
\leq CE(T)(1 + t)^{-\frac{1}{2}(1-\theta)-\frac{n-1}{2s_{0}}} \theta
\]
\[
= CE(T)(1 + t)^{-\frac{4}{2}+\frac{1}{2}}.
\]

Since \( n \geq 2 \), it yields that \( N(T) \leq CE(T) \).

(ii) By using the one-dimensional Gagliardo-Nirenberg inequality, we have

\[
\| \partial_{x} u(t) \|_{L^{\infty}} \leq C \| \partial_{x} u(t) \|_{L^{2}}^{\frac{1}{2}} \| \partial_{x}^{2} u(t) \|_{L^{2}}^{\frac{1}{2}} \leq CM(T)(1 + t)^{-1},
\]

which gives \( N(T) \leq CM(T) \). This completes the proof. \( \square \)
Proof of Corollary 2.2. By virtue of Lemma 3.5 (i), we have $N(T) \leq CE(T)$ for $n \geq 2$, which together with Proposition 3.4 gives

$$E(T)^2 + D(T)^2 \leq CE_0^2 + CE(T)D(T)^2.$$ 

Put $X(T) := E(T) + D(T)$. Then we have $X(T)^2 \leq CE_0^2 + CX(T)^3$. This inequality is solved as $X(T) \leq CE_0$, provided that $E_0$ is suitably small. Moreover, by virtue of (1.4), we have

$$\|\partial_x^k q(t)\|_{H^{s+1-k}} \leq\|\partial_x^{k+1} u(t)\|_{H^{s-1-k}} \leq CE_0(1 + t)^{-\frac{k+1}{2}},$$

for $0 \leq k \leq s - 1$, where we have used (2.2) with $k$ replaced by $k + 1$. This completes the proof of Corollary 2.2.

To estimate the energy $M(T)$, we need the following $L^1$ estimate of the solution.

Lemma 3.6. Under the same assumptions as in Theorem 2.3, the solution to the problem (1.1), (1.2) satisfies the following $L^1$ estimate for $u$:

$$\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}. \quad (3.14)$$

Proof. Applying $(1 - \Delta)^{-1}$ to (1.4), we have

$$u_t + \sum_{j=1}^n f^j(u)x_j + u - (I - \Delta)^{-1}u = 0. \quad (3.15)$$

We denote by $K(x)$ the fundamental solution to the operator $I - \Delta$, that is, $(I - \Delta)^{-1}u = K * u$. We know that $K(x) \geq 0$, $K \in L^1$ and $\int_{\mathbb{R}^n} K(x)dx = 1$. See [2] for the details. Let $j_\delta$ be the Friedrichs mollifier and put

$$\phi_\delta(u) = j_\delta * \text{sign}(u), \quad \Phi_\delta(u) = \int_0^u \phi_\delta(\xi)d\xi.$$ 

We multiply (3.15) by $\phi_\delta(u)$ and integrate the resulting equation over $\mathbb{R}^n \times (0,t)$. Then, letting $\delta \rightarrow 0$, we obtain the desired $L^1$ estimate (3.14) just in the same way as in [2, 16, 28]. The details are omitted.

By employing the time-weighted energy method together with the $L^1$ estimate (3.14), we get the following estimate for $M(T)$. 

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**Proposition 3.7.** Let \( n \geq 1 \) and \( s \geq \left[ \frac{n}{2} \right] + 2 \). Assume that \( u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \), and put \( E_0 := \| u_0 \|_{H^s} \) and \( E_1 := \| u_0 \|_{H^s} + \| u_0 \|_{L^1} \). Then, if \( E_0 \) is suitably small, then the solution to the problem (1.1), (1.2) constructed in Theorem 2.1 satisfies following estimate:

\[
M(T)^2 \leq CE_1^2(1 + N(T))^{s-1}.
\]

**Proof.** In order to prove this proposition, it is enough to show the following estimate for any \( t \in [0, T] \) and \( 0 \leq j \leq s - 1 \):

\[
\| \partial_x u(t) \|_{H^{s-j}}^2 \leq CE_1^2(1 + N(T))^{s-1}(1 + t)^{-\frac{2}{n}}.
\]

For this purpose, it is sufficient to prove

\[
(1 + t)^{j+\alpha} \| \partial_x^j u(t) \|_{H^{s-j}}^2 + \int_0^t (1 + \tau)^{j+\alpha} \| \partial_x^{j+1} u(\tau) \|_{H^{s-j-1}}^2 d\tau \leq CE_1^2(1 + N(T))^{s-1}(1 + t)^{\alpha - \frac{2}{n}}
\]

for \( t \in [0, T] \) and \( 0 \leq j \leq s - 1 \), where \( \alpha > \frac{n}{2} \).

First we show (3.17) for \( j = 0 \). We add (3.3) and (3.4) for \( 1 \leq l \leq s - 1 \), multiply the resulting inequality by \((1 + t)^{\alpha}\), and then integrate over \((0, t)\). This yields

\[
(1 + t)^{\alpha} \| u(t) \|_{H^s}^2 + \int_0^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau \leq CE_0^2 + C \int_0^t (1 + \tau)^{\alpha-1} \| u(\tau) \|_{H^s}^2 d\tau + C \int_0^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{L^\infty} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau =: CE_0^2 + I_1 + I_2.
\]

Since \( \| u(t) \|_{H^s} \leq CE_0 \) by (2.1), we can estimate the term \( I_2 \) as

\[
I_2 \leq CE_0 \int_0^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau \leq \frac{1}{4} \int_0^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau,
\]

where \( E_0 \) is assumed to be small as \( CE_0 \leq \frac{1}{4} \). On the other hand, we divide \( I_1 \) into two parts:

\[
I_1 = C \int_0^t (1 + \tau)^{\alpha-1} \| u(\tau) \|_{L^2}^2 d\tau + C \int_0^t (1 + \tau)^{\alpha-1} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau =: I_{11} + I_{12}.
\]

To estimate \( I_{12} \), we choose \( T_1 \) so large that \( C(1 + T_1)^{-1} \leq \frac{1}{4} \). Then we divide the time interval \([0, t]\) into two parts \([0, T_1]\) and \([T_1, t]\); here we treat the case \( t \geq T_1 \) because the case \( t \leq T_1 \) is easier. Thus we have

\[
I_{12} \leq C \max_{0 \leq s \leq T_1} \{ (1 + t)^{\alpha-1} \} \int_0^{T_1} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau + \frac{1}{4} \int_{T_1}^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau \leq CE_0^2 + \frac{1}{4} \int_0^t (1 + \tau)^{\alpha} \| \partial_x u(\tau) \|_{H^{s-1}}^2 d\tau.
\]
where we have used (2.1). Finally, we estimate the term $I_{11}$. By using the Gagliardo-Nirenberg inequality $\|u\|_{L^2} \leq C \|\partial_x u\|_{L^2}^{\theta} \|u\|_{L^1}^{1-\theta}$ with $\theta = \frac{2}{n+2}$ and applying the Young inequality, we can estimate $I_{11}$ as

$$I_{11} \leq C \int_0^t (1 + \tau)^{\alpha - 1} \|\partial_x u(\tau)\|_{L^2}^{2(1-\theta)} \|u(\tau)\|_{L^1}^{2(1-\theta)} d\tau$$

$$\leq \frac{1}{4} \int_0^t (1 + \tau)^{\alpha} \|\partial_x u(\tau)\|_{L^2}^2 d\tau + C \int_0^t (1 + \tau)^{\alpha-\frac{2}{3}-1} \|u(\tau)\|_{L^1}^2 d\tau$$

$$\leq \frac{1}{4} \int_0^t (1 + \tau)^{\alpha} \|\partial_x u(\tau)\|_{L^2}^2 d\tau + C \|u_0\|_{L^1}^2 (1 + t)^{\alpha - \frac{2}{3}},$$

where we have used the $L^1$ estimate (3.14) and the condition $\alpha > \frac{2}{3}$. Consequently, under the smallness assumption on $E_0$, we arrive at the estimate

$$(1 + t)^{\alpha} \|u(t)\|_{H^s}^2 + \int_0^t (1 + \tau)^{\alpha} \|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau \leq CE^2_1 (1 + t)^{\alpha - \frac{2}{3}},$$

which proves (3.17) for $j = 0$.

Now, let $0 \leq k \leq s - 2$ and suppose that (3.17) holds true for $j = k$. Then we show (3.17) for $j = k + 1$. Multiplying (3.4) by $(1 + t)^{k+1+\alpha}$, integrating with respect to $t$ over $(0, t)$ and adding up for $t$ with $k + 1 \leq l \leq s - 1$, we have

$$(1 + t)^{k+1+\alpha} \|\tilde{\partial}_x^{k+1} u(t)\|_{H^{s-l-2}}^2 + \int_0^t (1 + \tau)^{k+1+\alpha} \|\tilde{\partial}_x^{k+2} u(\tau)\|_{H^{s-l-3}}^2 d\tau$$

$$\leq CE^2_0 + C \int_0^t (1 + \tau)^{k+\alpha} \|\tilde{\partial}_x^{k+1} u(\tau)\|_{H^{s-l-1}}^2 d\tau$$

$$+ C \int_0^t (1 + \tau)^{k+1+\alpha} \|\tilde{\partial}_x u(\tau)\|_{L^\infty} \|\tilde{\partial}_x^{k+1} u(\tau)\|_{H^{s-l-1}}^2 d\tau.$$

Here, using the induction hypothesis (3.17) with $j = k$, we can estimate the second term on the right hand side by $CE^2_1 (1 + N(T))(1 + t)^{\alpha - \frac{2}{3}}$. Similarly, we estimate the last term as

$$C \int_0^t (1 + \tau)^{k+1+\alpha} \|\tilde{\partial}_x u(\tau)\|_{L^\infty} \|\tilde{\partial}_x^{k+1} u(\tau)\|_{H^{s-l-1}}^2 d\tau$$

$$\leq CN(T) \int_0^t (1 + \tau)^{k+\alpha} \|\tilde{\partial}_x^{k+1} u(\tau)\|_{H^{s-l-1}}^2 d\tau$$

$$\leq CE^2_1 N(T)(1 + N(T))(1 + t)^{\alpha - \frac{2}{3}}.$$

Thus we obtain

$$(1 + t)^{k+1+\alpha} \|\tilde{\partial}_x^{k+1} u(t)\|_{H^{s-l-2}}^2 + \int_0^t (1 + \tau)^{k+1+\alpha} \|\tilde{\partial}_x^{k+2} u(\tau)\|_{H^{s-l-3}}^2 d\tau$$

$$\leq CE^2_1 (1 + N(T))(1 + t)^{\alpha - \frac{2}{3}}.$$

This shows that (3.17) holds true for $j = k + 1$. Thus, by induction, we have proved Proposition 3.7. \qed
Proof of Theorem 2.3. By virtue of Lemma 3.5, we have $N(T) \leq C(E(T) + M(T))$. Therefore we have from Propositions 3.4 and 3.7 that

\[ E(T)^2 + D(T)^2 \leq CE_1^2 + C(E(T) + M(T))D(T)^2, \]
\[ M(T)^2 \leq CE_1^2(1 + E(T) + M(T))^{s-1}. \]

Put $Y(T) := E(T) + D(T) + M(T)$. Then we have $Y(T)^2 \leq CE_1^2(1 + Y(T))^{s-1} + CY(T)^3$. This inequality is solved as $Y(T) \leq CE_1$, provided that $E_1$ is suitably small, say, $E_1 \leq \delta_1$. In particular, we have $M(T) \leq CE_1$, which proves the decay estimate (2.4) for $0 \leq k \leq s - 1$. Moreover, using (1.3) and (2.3) with $k$ replaced by $k + 1$, we obtain

\[ \|\partial_x^k q(t)\|_{H^{s+1-k}} \leq \|\partial_x^{k+1} u(t)\|_{H^{s-1-k}} \leq CE_1(1 + t)^{-\delta + \frac{1}{q+1}} \]

for $0 \leq k \leq s - 2$. This completes the proof of Theorem 2.3.

\[ \square \]

4 Asymptotic profile

The aim of this section is to prove Theorem 2.4 on the asymptotic profile. To this end, we first consider the corresponding linear problem (2.6). The fundamental solution to (2.6), is given by $G(x, t) = \mathcal{F}^{-1}[e^{-\frac{|\xi|^2}{1 + |\xi|^4}}](x)$, and the solution to (2.6) can be expressed in terms of the fundamental solution as

\[ \tilde{u}(x, t) = (G(t) * u_0)(x). \] (4.1)

The solution operator $G(t)*$ verifies the following decay property:

**Lemma 4.1.** Let $n \geq 1$ and $s \geq 0$. If $\phi \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then we have the following decay estimate:

\[ \|\partial_x^k G(t) * \phi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2} - \frac{k}{q}} \|\phi\|_{L^1} + Ce^{-\frac{t}{2}} \|\partial_x^k \phi\|_{L^2}, \quad 0 \leq k \leq s. \]

**Proof.** By direct calculation, we have

\[ \|\partial_x^k G(t) * \phi\|_{L^2}^2 \leq C \left( \int_{|\xi| \leq 1} + \int_{|\xi| > 1} \right) \|\xi|^{2k} e^{-\frac{|\xi|^2}{1 + |\xi|^4}} \hat{\phi}(\xi)\|^2 d\xi \]
\[ \leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-|\xi|^2} |\hat{\phi}(\xi)|^2 d\xi + C \int_{|\xi| > 1} |\xi|^{2k} e^{-t} |\hat{\phi}(\xi)|^2 d\xi \]
\[ \leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-|\xi|^2} |\hat{\phi}(\xi)|^2 d\xi + C \int_{|\xi| > 1} |\xi|^{2k} e^{-t} |\hat{\phi}(\xi)|^2 d\xi \]
\[ \leq C(1 + t)^{-\frac{q}{2} - \frac{k}{q}} \|\phi\|_{L^1}^2 + Ce^{-\frac{t}{2}} \|\partial_x^k \phi\|_{L^2}^2. \]

This completes the proof. \[ \square \]
By Duhamel principle, we can express the solution of (1.4) or (3.15), (1.2) as follows:
\[
    u(x, t) = G(t) * u_0 - \int_0^t G(t - \tau) * \text{div} f(u)(\tau) d\tau.
\]
(4.2)

Here and in the following, we use the abbreviation \( f(u) = (f_1(u), \cdots, f_n(u)) \).
We decompose the solution formula (4.2) in the form \( u(t) = \bar{u}(t) - F(u)(t) \),
where \( \bar{u} \) is the linear solution given in (4.1) and
\[
    F(u)(x, t) = \int_0^t G(t - \tau) * \text{div} f(u)(\tau) d\tau.
\]
(4.3)

First we prove that the solution to the problem (1.4), (1.2) can be approximated by the solution to the corresponding linear problem (2.6).

**Proposition 4.2.** Let \( n \geq 2 \) and \( s \geq \left\lceil \frac{n}{2} \right\rceil + 2 \). Assume that \( u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) and put \( E_1 := \|u_0\|_{H^s} + \|u_0\|_{L^1} \). Let \((u, q)\) be the global solution to the problem (1.1), (1.2) which is obtained in Theorem 2.1, and let \( \bar{u} \) be the solution to the corresponding linear problem (2.6), which is given by the formula (4.1). Then we have
\[
    \|\partial_x^k(u - \bar{u})(t)\|_{H^{s-1-k}} \leq CE_1^2 \rho(t)(1 + t)^{-\frac{n}{4} - \frac{k+1}{2}}
\]
for \( k \) with \( 0 \leq k \leq s - 1 \), where \( \rho(t) = \ln(1 + t) \) for \( n = 2 \) and \( \rho(t) = 1 \) for \( n \geq 3 \).

**Proof.** Let \( k \) and \( m \) be nonnegative integers. We apply \( \partial_x^{k+m} \) to \( F(u) \) in (4.3) and take the \( L^2 \) norm, obtaining
\[
    \|\partial_x^{k+m} F(u)(t)\|_{L^2} \leq \left( \int_0^t + \int_t^\infty \right) \|\partial_x^{k+m+1} G(t - \tau) * f(u)(\tau)\|_{L^2} d\tau =: I_1 + I_2.
\]
(4.4)

By applying Lemma 4.1 using Lemma 3.1 and noticing that \( f_j(u) = O(u^2), j = 1, \cdots, n \) for \( u \to 0 \), we have
\[
    I_1 \leq C \int_0^t (1 + t - \tau)^{-\frac{n}{4} - \frac{k+m+1}{2}} \|f(u)\|_{L^1} d\tau
\]
\[
    + C \int_0^t e^{-\frac{t-\tau}{2}} \|\partial_x^{k+m+1} f(u)\|_{L^2} d\tau
\]
\[
    \leq C \int_0^t (1 + t - \tau)^{-\frac{n}{4} - \frac{k+m+1}{2}} \|u\|_{L^\infty}^2 d\tau
\]
\[
    + C \int_0^t e^{-\frac{t-\tau}{2}} \|u\|_{L^\infty} \|\partial_x^{k+m+1} u\|_{L^2} d\tau.
\]
Similarly, we have
\[
I_2 \leq CE^2 \int_0^t \left(1 + t - \tau\right)^{-\frac{\theta}{2} - \frac{k + 1}{2}} \left|\partial_{x}^{k+1} f(u)\right| L^1 d\tau \\
+ CE^2 \int_0^t e^{-\frac{t - \tau}{2}} \left|\partial_{x}^{k+1} f(u)\right| L^2 d\tau \\
\leq CE^2 \int_0^t \left(1 + t - \tau\right)^{-\frac{\theta}{2} - \frac{k + 1}{2}} \left|u\right| L^2 \left|\partial_{x}^{k+1} u\right| L^2 d\tau \\
+ CE^2 \int_0^t e^{-\frac{t - \tau}{2}} \left|u\right| L^\infty \left|\partial_{x}^{k+1} u\right| L^2 d\tau.
\]

We estimate the terms \(I_1\) and \(I_2\). Let \(s_0 = \left[\frac{n}{2}\right] + 1\) and \(\theta = \frac{n}{2s_0}\). By using the Gagliardo-Nirenberg inequality and (2.2), we see that
\[
\left\|u(t)\right\|_{L^\infty} \leq C\left\|u(t)\right\|_{L^2}^{1 - \theta/2} \left\|\partial_{x}^{s_0} u(t)\right\|_{L^2}^\theta \\
\leq CE_1 (1 + t)^{-\frac{\theta}{2} = CE_1 (1 + t)^{-\frac{n}{2s_0}}.}
\]

By using (2.2), (2.4) and (4.5), we estimate \(I_1\) as
\[
I_1 \leq CE^2 \int_0^t \left(1 + t - \tau\right)^{-\frac{\theta}{2} - \frac{k + 1}{2}} \left(1 + \tau\right)^{-\frac{n}{2s_0}} d\tau \\
+ CE^2 \int_0^t e^{-\frac{t - \tau}{2}} \left(1 + \tau\right)^{-\frac{n}{2s_0}} d\tau \\
\leq CE^2 \rho(t)(1 + t)^{-\frac{n}{2s_0} - \frac{k + 1}{2}}
\]
for \(k\) satisfying \(k + m + 1 \leq s\), where \(\rho(t)\) is given above. Similarly, we can estimate \(I_2\) as
\[
I_2 \leq CE^2 \int_0^t \left(1 + t - \tau\right)^{-\frac{\theta}{2} - \frac{k + 1}{2}} \left(1 + \tau\right)^{-\frac{n}{2s_0}} d\tau \\
+ CE^2 \int_0^t e^{-\frac{t - \tau}{2}} \left(1 + \tau\right)^{-\frac{n}{2s_0}} d\tau \\
\leq CE^2 \rho(t)(1 + t)^{-\frac{n}{2s_0} - \frac{k + 1}{2}}
\]
for \(k\) satisfying \(k + m + 1 \leq s\).

We substitute (4.6) and (4.7) into (4.4) and take the sum for \(m\) with \(0 \leq m \leq s - k - 1\). This yields
\[
\left\|\partial_{x}^{k} F(u(t))\right\|_{H^{s - k - 1}} \leq CE^2 \rho(t)(1 + t)^{-\frac{n}{2s_0} - \frac{k + 1}{2}}
\]
for \(k\) with \(0 \leq k \leq s - 1\). This completes the proof of Proposition 1.2. \(\Box\)

Next we consider the simpler problem for the linear heat equation (2.7). Notice that the solution to (2.7) is expressed as \(\bar{u}(x, t) = (G_0(t) * u_0)(x)\), where \(G_0(x, t)\) is the heat kernel given in (2.8). By direct computation we see that the solution \(\bar{u}(x, t)\) of (2.6) is well approximated by the solution \(\tilde{u}(x, t)\) of (2.7) as \(t \to +\infty\):
Lemma 4.3. Let \( n \geq 1 \) and \( s \geq 0 \). Assume that \( u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) and put \( E_1 = \| u_0 \|_{H^s} + \| u_0 \|_{L^1} \). Let \( \tilde{u}(x, t) \) and \( \check{u}(x, t) \) be solutions to (2.6) and (2.7), respectively. Then we have the following asymptotic relation:

\[
\| \partial_t^k (\tilde{u} - \check{u})(t) \|_{H^{s-k}} \leq C E_1 (1 + t)^{-\frac{n}{2} - \frac{k}{2} - 1}, \quad 0 \leq k \leq s.
\]

Proof. Since \( |(\hat{G}(\xi, t) - \hat{G}_0(\xi, t))| \leq C s e^{-c|\xi|^2 t} \) for \( |\xi| \leq 1 \), and \( |(\hat{G}(\xi, t) - \hat{G}_0(\xi, t)| \leq C e^{-ct} \) for \( |\xi| \geq 1 \), by direct calculation we have the following decay property for \( G - G_0)(t) \): \( \forall \phi \in H^s \cap L^1 \),

\[
\| \partial_t^k (G - G_0)(t) \ast \phi \|_{L^2} \leq C (1 + t)^{-\frac{n}{2} - \frac{k}{2} - 1} \| \phi \|_{L^1} + C e^{-ct} \| \partial_t^k \phi \|_{L^2},
\]

here \( k \leq s \). Replace \( k \) with \( k + m \), \( \phi \) with \( u_0 \), then we have

\[
\| \partial_t^{k+m} (G - G_0)(t) \ast u_0 \|_{L^2} \leq C (1 + t)^{-\frac{n}{2} - \frac{k+m}{2} - 1} \| u_0 \|_{L^1} + C e^{-ct} \| \partial_t^{k+m} u_0 \|_{L^2} \leq C (1 + t)^{-\frac{n}{2} - \frac{k}{2} - 1} \| u_0 \|_{H^{s-k}}.
\]

Take sum with \( 0 \leq m \leq s - k \), then we complete the proof of Lemma 4.3. \( \square \)

Finally, we show in the solution to (2.7) is well approximated by the linear diffusion wave defined in (2.9). For this purpose, we recall the following well known result for the heat kernel.

Lemma 4.4. Let \( n \geq 1 \) and \( 1 \leq q \leq 2 \). If \( \phi \in L^q(\mathbb{R}^n) \), then we have

\[
\| \partial_t^k G(t) \ast \phi \|_{L^2} \leq C t^{-\frac{n}{2} + \frac{k}{2}} \| \phi \|_{L^q}, \quad k \geq 0.
\]

Also, if \( \phi \in L^1(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} \phi(x) dx = 0 \), then we have

\[
\| \partial_t^k G(t) \ast \phi \|_{L^2} \leq C t^{-\frac{n}{2} + \frac{k+1}{2}} \| \phi \|_{L^1}, \quad k \geq 0.
\]

Proof of Theorem 2.4. We recall the linear diffusion wave \( (u^*, q^*)(x, t) \) in (2.9). Since \( G_0(x, t+1) \) is a solution of the linear heat equation \( u_t - \Delta u = 0 \) with the initial data \( G_0(x, 1) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}} \) : \( \phi_0(x) \), we see that \( G_0(x, t+1) = (G_0(t) \ast \phi_0)(x) \). Consequently, we can write

\[
(\tilde{u} - u^*)(t) = G_0(t) \ast (u_0 - M \phi_0),
\]

where \( M = \int_{\mathbb{R}^n} u_0(x) dx \). Here \( \phi_0(x) \) is a rapidly decreasing function satisfying \( \int_{\mathbb{R}^n} \phi_0(x) dx = 1 \), so that we have \( \int_{\mathbb{R}^n} (u_0 - M \phi_0)(x) dx = 0 \). Therefore, applying Lemma 4.4 we deduce that

\[
\| \partial_t^k (\tilde{u} - u^*)(t) \|_{H^{s-k}} = \| \partial_t^k G_0(t) \ast (u_0 - M \phi_0) \|_{H^{s-k}} \leq C E_2 (1 + t)^{-\frac{n}{2} + \frac{k+1}{2}}.
\]
for $0 \leq k \leq s$. Now we write $u - u^* = (u - \bar{u}) + (\bar{u} - \tilde{u}) + (\tilde{u} - u^*)$ and apply Proposition 4.2, Lemma 4.3 and (4.10). This yields the desired estimate (2.10) for $0 \leq k \leq s - 1$.

On the other hand, we have $q^* = -\nabla u^*$ by (2.9), which can be rewritten as $q^* = -(1 - \Delta)^{-1}\nabla u^* + (1 - \Delta)^{-1}\Delta \nabla u^*$. Therefore, noting (1.4), we find that

$$q - q^* = -(1 - \Delta)^{-1}\nabla (u - u^*) - (1 - \Delta)^{-1}\Delta \nabla u^*.$$ 

Consequently, we have

$$\|\partial_x^k (q - q^*)(t)\|_{H^{s-k}} \leq \|\partial_x^{k+1}(u - u^*)(t)\|_{H^{s-2-k}} + \|\partial_x^{k+3} u^*(t)\|_{H^{s-2-k}}.$$ 

Here, applying (2.10) with $k$ replaced by $k + 1$, we see that the first term on the right hand side is estimated by $C E_2 \rho(t) (1 + t)^{-\frac{n}{4} - \frac{k-1}{2}}$ for $0 \leq k \leq s - 2$, where $\rho(t)$ is given in Theorem 2.4. Also, using the expression $u^* = MG_0(t) \ast \phi_0$ and applying (4.8), we know that the second term can be estimated by $C \|u_0\|_{L^1}(1 + t)^{-\frac{n}{4} - \frac{k+3}{2}}$. These observations prove the estimate (2.11). Thus the proof of Theorem 2.4 is complete.

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