Involute-Evolute Curves in Galilean Space $G_3$

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March 17, 2010

Abstract

In this paper, definition of involute-evolute curve couple in Galilean space is given and some well-known theorems for the involute-evolute curves are obtained in 3-dimensional Galilean space.

Mathematics Subject Classification (2010). 53A35, 51M30.

Keywords: Galilean space, Involute-evolute curve.

1 Introduction

C. Huggens discovered involutes while trying to build a more accurate clock, [1]. Later, the relations Frenet apparatus of involute-evolute curve couple in the space $\mathbb{E}^3$ were given in [2]. A. Turgut examined involute-evolute curve couple in $\mathbb{R}^n$, [3]. At the beginning of the twentieth century, Cayley-Klein discussed Galilean geometry which is one of the nine geometries of projective space. After that, the studies with related to the curvature theory were maintained [4, 5, 6] and the properties of the curves in the Galilean space were studied in [7]. In this paper, we define involute-evolute curves couple and give some theorems and conclusions, which are known from the classical differential geometry, in the three dimensional Galilean space $G_3$. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2 Preliminaries

The Galilean space $G_3$ is a Cayley-Klein space equipped with the projective metric of signature $(0,0,+,+)$, as in [8]. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w,f,I\}$, where $w$ is the ideal (absolute) plane, $f$ is the line (absolute line) in $w$ and $I$ is the fixed elliptic involution of points of $f$, [9]. In the non-homogeneous coordinates the similarity group $H_8$
has the form
\[
\begin{align*}
\mathbf{x} &= a_{11} + a_{12}x \\
\mathbf{y} &= a_{21} + a_{22}x + a_{23}y \cos \varphi + a_{23}z \sin \varphi \\
\mathbf{z} &= a_{31} + a_{32}x - a_{23}y \sin \varphi + a_{23}z \cos \varphi
\end{align*}
\]
(2.1)

where $a_{ij}$ and $\varphi$ are real numbers, \[5\].

In what follows the coefficients $a_{12}$ and $a_{23}$ will play the special role. In particular, for $a_{12} = a_{23} = 1$, (2.1) defines the group $B_6 \subset H_8$ of isometries of Galilean space $G_3$.

In $G_3$ there are four classes of lines:

i) (proper) non-isotropic lines- they don’t meet the absolute line $f$.

ii) (proper) isotropic lines- lines that don’t belong to the plane $w$ but meet the absolute line $f$.

iii) unproper non-isotropic lines-all lines of $w$ but $f$.

iv) the absolute line $f$.

Planes $x =$constant are Euclidean and so is the plane $w$. Other planes are isotropic.

If a curve $C$ of the class $C^r$ ($r \geq 3$) is given by the parametrization
\[
r = r(x, y(x), z(x))
\]
then $x$ is a Galilean invariant the arc length on $C$.

The curvature is
\[
\kappa = \sqrt{y''(x)^2 + z''(x)^2}
\]
and torsion
\[
\tau = \frac{1}{\kappa^2} \det(r'(x), r''(x), r'''(x)).
\]

The orthonormal trihedron is defined
\[
\begin{align*}
T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\
N(s) &= \frac{1}{\kappa(s)} (0, y''(s), z''(s)) \\
B(s) &= \frac{1}{\kappa(s)} (0, -z''(s), y''(s)).
\end{align*}
\]

The vectors $T, N, B$ are called the vectors of tangent, principal normal and bi-normal line of $\alpha$, respectively. For their derivatives the following Frenet formulas hold
\[
\begin{align*}
T'(s) &= \kappa(s) N(s) \\
N'(s) &= \tau(s) B(s) \\
B'(s) &= -\tau(s) N(s),
\end{align*}
\]
\[9\].

Galilean scalar product can be written as
\[
\langle u_1, u_2 \rangle = \begin{cases} 
  x_1 x_2 , & \text{if } x_1 \neq 0 \lor x_2 \neq 0 \\
  y_1 y_2 + z_1 z_2 , & \text{if } x_1 = 0 \land x_2 = 0
\end{cases}
\]
where $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector $u = (x, y, z)$ defined by
\[
\|u\| = \begin{cases} 
  \frac{x}{\sqrt{y^2 + z^2}} , & x \neq 0 \\
  0 , & \text{if } x = 0
\end{cases}
\]
Let $\alpha$ be a curve given first by

$$\alpha : I \rightarrow G_3, \quad I \subset \mathbb{R}$$

$$t \rightarrow \alpha(t) = (x(t), y(t), z(t))$$

(2.2)

where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and $t$ run through a real interval.

3 Involute- Evolute Curves in Galilean Space

In this section, we give a definition of involute-evolute curve and obtain some theorems about these curves in $G_3$.

**Definition 3.1** Let $\alpha$ and $\alpha^*$ be two curves in the Galilean space $G_3$. The curve $\alpha^*$ is called involute of the curve $\alpha$ if the tangent vector of the curve $\alpha$ at the point $\alpha(s)$ passes through the tangent vector of the curve $\alpha^*$ at the point $\alpha^*(s)$ and

$$\langle T, T^* \rangle = 0,$$

where $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are Frenet frames of $\alpha$ and $\alpha^*$, respectively. Also, the curve $\alpha$ is called the evolute of the curve $\alpha^*$.

This definition suffices to define this curve mate as

$$\alpha^* = \alpha + \lambda T$$

(see Figure 1).

![Figure 1. Involute-evolute curves](image)

**Theorem 3.1** Let $\alpha$ and $\alpha^*$ be two curves in the Galilean space $G_3$. If the curve $\alpha^*$ is an involute of the curve $\alpha$, then the distance between the curves $\alpha$ and $\alpha^*$ is constant.

**Proof.** From definition of involute-evolute curve couple, we know

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s)$$

(3.1)

Differentiating both sides of the equation (3.1) with respect to $s$ and use the Frenet formulas, we obtain

$$T^*(s) = T(s) + \frac{d\lambda}{ds}T(s) + \lambda(s)\kappa(s)N(s).$$
Since the curve $\alpha^*$ is involute of $\alpha$, $\langle T, T^* \rangle = 0$.

Then we have

$$\frac{d\lambda}{ds} + 1 = 0. \quad (3.2)$$

From the last equation, we easily get

$$\lambda(s) = c - s \quad (3.3)$$

where $c$ is constant. Thus, the equation (3.1) can be written as

$$\alpha^*(s) - \alpha(s) = (c - s)T(s). \quad (3.4)$$

Taking the norm of the equation (3.4), we reach

$$\|\alpha^*(s) - \alpha(s)\| = |c - s|. \quad (3.5)$$

This completes the proof. $\blacksquare$

**Theorem 3.2** Let $\alpha$ and $\alpha^*$ be two curves in Galilean space $G_3$. $\kappa$, $\tau$ and $\kappa^*$, $\tau^*$ be the curvature functions of $\alpha$ and $\alpha^*$, respectively. If $\alpha$ is evolute of $\alpha^*(s)$ then there is a relationship

$$\kappa^* = \frac{\tau}{(c - s)\kappa},$$

where $c$ is constant and $s$ is arc length parameter of $\alpha$.

**Proof.** Let Frenet frames be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ at the points $\alpha(s)$ and $\alpha^*(s)$, respectively. Differentiating both sides of equation (3.4) with respect to $s$ and using Frenet formulas, we have following equation

$$T^*(s)\frac{ds^*}{ds} = (c - s)\kappa(s)N(s) \quad (3.6)$$

where $s$ and $s^*$ are the arc length parameter of the curves $\alpha$ and $\alpha^*$, respectively. Taking the norm of the equation (3.6), we reach

$$T^* = N \quad (3.7)$$

and

$$\frac{ds^*}{ds} = (c - s)\kappa(s). \quad (3.8)$$

By taking the derivative of equation (3.7) and using the Frenet formulas and equation (3.8), we obtain

$$\kappa^*N^* = \frac{\tau}{(c - s)\kappa}B. \quad (3.9)$$

From the last equation, we get

$$\kappa^* = \frac{\tau}{(c - s)\kappa}. \quad \blacksquare$$
Theorem 3.3 Let $\alpha$ be the non-planar evolute of curve $\alpha^*$, then $\alpha$ is a helix.

Proof. Under assumption $s$ and $s^*$ are arc length parameter of the curves $\alpha$ and $\alpha^*$, respectively. We take the derivative of the following equation with respect to $s$

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s)$$

we obtain that

$$T^* \frac{ds^*}{ds} = \lambda \kappa N.$$ 

$\{T^*, N\}$ are linearly dependent. We may define function as

$$f = \langle T, T^* \wedge N^* \rangle$$

and take the derivative of the function $f$ with respect to $s$, we obtain

$$f' = -\tau \langle T, N^* \rangle.$$ \hspace{1cm} (3.10)

From the equation (3.10) and the scalar product in Galilean space, we have

$$f' = 0.$$

That is,

$$f = \text{const.}$$

The velocity vector of the curve $\alpha$ always composes a constant angle with the normal of the plane which consist of $\alpha^*$. Then the non-planar evolute of the curve $\alpha^*$ is a helix. $\blacksquare$

Theorem 3.4 Let the curves $\beta$ and $\gamma$ be two evolutes of $\alpha$ in the Galilean space $G_3$. If the points $P_1$ and $P_2$ correspond to the point of $\alpha$, then the angle $\hat{\angle} P_1 \hat{P} P_2$ is constant.

Proof. Let’s assume that the curves $\beta$ and $\gamma$ be two evolutes of $\alpha$ (see Figure 2). And let the Frenet vectors of the curves $\alpha, \beta$ and $\gamma$ be $\{T, N, B\}, \{T^*, N^*, B^*\}$ and $\{\hat{T}, \hat{N}, \hat{B}\}$, respectively.

![Figure 2. Evolutes of $\alpha$ curve](5)
Following the same way in the proof of the Theorem 3.3, it is easily seen that \( \{T, N^*\} \) and \( \{T, \tilde{N}\} \) are linearly dependent. Thus,

\[
\langle T, T^* \rangle = 0 \quad (3.11)
\]

and

\[
\langle T, \tilde{T} \rangle = 0. \quad (3.12)
\]

We define a function \( f \) as an angle between tangent vector \( T^* \) and \( \tilde{T} \), that is,

\[
f(s) = \langle T^*, \tilde{T} \rangle \quad (3.13)
\]

Then, differentiating equation (3.13) with respect to \( s \), we have

\[
f'(s) = \kappa^* \frac{ds^*}{ds} \langle N^*, \tilde{T} \rangle + \tilde{\kappa} \frac{d\tilde{s}}{ds} \langle T^*, \tilde{N} \rangle + \kappa \frac{d\kappa}{ds} \langle T^*, \tilde{T} \rangle + \tilde{\kappa} \frac{d\tilde{\kappa}}{ds} \langle T^*, \tilde{T} \rangle \quad (3.14)
\]

where \( s, s^* \) and \( \tilde{s} \) are arc length parameter of the curves \( \alpha, \beta, \gamma \), respectively. Also \( \kappa, \kappa^* \) and \( \tilde{\kappa} \) are the curvatures of the curves \( \alpha, \beta, \gamma \), respectively.

Considering the equations (3.11), (3.12) and (3.14), we get

\[
f'(s) = 0.
\]

This means that

\[
f = \text{const.}
\]

So, the proof is completed. ■

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