Dynamics of open quantum systems

J. Okołowicz\textsuperscript{1,2}, M. Płoszajczak\textsuperscript{1} and I. Rotter\textsuperscript{3}

\textsuperscript{1} Grand Accélérateur National d’Ions Lourds (GANIL), CEA/DSM – CNRS/IN2P3, BP 5027, F-14076 Caen Cedex 05, France

\textsuperscript{2} Institute of Nuclear Physics, Radzikowskiego 152, PL - 31342 Kraków, Poland

\textsuperscript{3} Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany

Abstract

The coupling between the states of a system and the continuum into which it is embedded, induces correlations that are especially large in the short time scale. These correlations cannot be calculated by using a statistical or perturbational approach. They are, however, involved in an approach describing structure and reaction aspects in a unified manner. Such a model is the SMEC (shell model embedded in the continuum). Some characteristic results obtained from SMEC as well as some aspects of the correlations induced by the coupling to the continuum are discussed.

I. INTRODUCTION

Most states of a nucleus are embedded in the continuum of decay channels due to which they get a finite life time. That means: the discrete states of a nucleus shade off into resonance states with complex energies $E_k = E_k - \frac{i}{2} \Gamma_k$. The $E_k$ give the positions in energy of the resonance states while the widths $\Gamma_k$ are characteristic of their life times. The $E_k$ may be different from the energies of the discrete states, and the widths $\Gamma_k$ may be large corresponding to a short life time. Nevertheless, there is a well defined relation between the discrete states characterizing the closed system, and the resonance states appearing in the open system. The main difference in the theoretical description of quantum systems without and with coupling to an environment is that the function space of the system is supposed to be complete in the first case while this is not so in the second case. Accordingly, the Hamilton operator is Hermitian in the first case, and the eigenvalues are discrete. The resonance states, however, characterize a subsystem described by a non-Hermitian Hamilton operator with complex eigenvalues. The function space containing everything consists, in the second case, of system plus environment.

The mathematical formulation of this problem goes back to Feshbach \cite{feshbach} who introduced the two subspaces $Q$ and $P$, with $Q + P = 1$, containing the discrete and scattering states, respectively. Feshbach was able to formulate a unified description of nuclear reactions with both direct processes in the short-time scale and compound nucleus processes in the long-time scale. Due to the high excitation energy and high level density in compound nuclei,
he introduced statistical approximations in order to describe the discrete states of the $Q$ subspace. A unified description of nuclear structure and nuclear reaction aspects is much more complicated and became possible only at the end of the last century (see [2] for a recent review). In this formulation, the states of both subspaces are described with the same accuracy. All the coupling matrix elements between different discrete states, different scattering states as well as between discrete and scattering states are calculated in order to get results that can be compared with experimental data. This method has been applied to the description of light nuclei by using the shell model approach for the discrete many-particle states of the $Q$ subspace [2].

In the unified description of structure and reaction aspects, the system is described by an effective Hamiltonian $\mathcal{H}$ that consists of two terms: the Hamiltonian matrix $H$ of the closed system with discrete eigenstates, and the coupling matrix between system and environment. The last term is responsible for the finite lifetime of the resonance states. The eigenvalues of $\mathcal{H}$ are complex and give the poles of the $S$ matrix.

The dynamics of quantum systems is determined by the $S$ matrix, more exactly by its poles and the postulation of unitarity. The unitarity is involved in the continuum shell model calculations [2], but is in conflict with the statistical assumptions when calculations in the overlapping regime are performed [4].

Characteristic of the motion of the poles of the $S$ matrix as a function of a certain parameter are the following generic results obtained for very different systems [5–8]: in the overlapping regime, the trajectories of the $S$ matrix poles avoid crossing with the only exception of exact crossing when the $S$ matrix has a double pole. At the avoided crossing, either level repulsion or level attraction occurs. The first case is caused by a predominantly real interaction between the crossing states and is accompanied by the tendency to form a uniform time scale of the system. Level attraction occurs, however, when the interaction is dominated by its imaginary part arising from the coupling via the continuum. It is accompanied by the formation of different time scales in the system: while some of the states decouple more or less completely from the continuum and become long-lived (trapped), a few of the states become short-lived and wrap the long-lived ones in the cross section. The dynamics of quantum systems at high level density is determined by the interplay of these two opposite tendencies. For a more detailed discussion see [2].

At large overall coupling strength, quick direct reaction processes may appear from slow resonance processes by means of the resonance trapping phenomenon. In recent calculations on microwave cavities with fixed large overall coupling strength, the details of resonance trapping are shown to depend on the position of the attached leads [8]. In microwave cavities of Bunimovich type, that are chaotic when closed, coherent whispering gallery and bouncing ball modes may be strongly enhanced by trapping other (incoherent) modes. Since the coherent modes determine the value of the conductivity, resonance trapping may cause observable effects that are not small. It is interesting to remark that the trapped long-lived states can be described by random matrix theory, as a shot-noise analysis of the numerical results has shown. The enhanced conductivity, however, is related to the short-lived whispering gallery modes that are regular and dominant when the leads are attached to the cavity in a suitable manner [9].

Meanwhile, the phenomenon of resonance trapping has been proven experimentally on a microwave cavity as a function of the degree of opening of the cavity to an attached
lead [10]. In this experiment, the parameter varied is the overall coupling strength between
discrete and scattering states. Resonance trapping may appear, however, as function of any
parameter [2].

In the following, we will discuss the interplay of the different time scales in nuclei. Most
interesting is the mechanism of formation of short-lived states in open quantum systems. In
Sect. 2, the effective Hamiltonian and the $S$ matrix are written down for a quantum system
embedded in a continuum while in Sect. 3, the basic relations for the spectroscopic information
are discussed. Characteristic features of the different approaches to the system and to the
environment are sketched in Sect. 4. In Sect. 5, some results obtained from calculations
with unified description of structure and reaction aspects are shown. The relation between
lifetimes and decay widths of resonance states in the overlapping region is discussed in Sect.
6 while in Sect. 7, properties of the system in the short time scale are illustrated. In any
case, the correlations induced by the coupling to the continuum are large. The last section
contains some concluding remarks.

II. EFFECTIVE HAMILTONIAN AND $S$ MATRIX FOR A QUANTUM SYSTEM
EMBEDDED IN A CONTINUUM

In the unified description of structure and reaction aspects of quantum systems, the
Schrödinger equation

$$(H - E)\Psi = 0 \quad (1)$$

is solved in a function space containing everything, i.e. discrete as well as continuous states.
The Hamilton operator $H$ is Hermitian, the wave functions $\Psi$ depend on energy as well as on
the decay channels and all the resonance states of the system. Knowing the wave functions
$\Psi$, an expression for the $S$ matrix can be derived that holds true also in the overlapping
regime, see the recent review [2]. In the continuum shell model, it reads

$$S_{cc'} = e^{i(\delta_c - \delta_{c'})} \left[ \delta_{cc'} - S^{(1)}_{cc'} - S^{(2)}_{cc'} \right], \quad (2)$$

where $S^{(1)}_{cc'}$ is the smooth direct reaction part related to the short-time scale, and

$$S^{(2)}_{cc'} = i \sum_{k=1}^{N} \frac{\tilde{\gamma}_c^c \tilde{\gamma}_c^{c'}}{E - \tilde{E}_k + \frac{i}{2} \tilde{\Gamma}_k} \quad (3)$$

is the resonance reaction part related to the long-time scale. Here, the $\tilde{\mathcal{E}}_k = \tilde{E}_k - \frac{i}{2} \tilde{\Gamma}_k$ are
the complex eigenvalues of the non-Hermitian Hamilton operator

$$\mathcal{H}_{QQ} = \mathcal{H}_{QP} G^{(+)} P \mathcal{H}_{PQ} \quad (4)$$

appearing effectively in the system ($Q$ subspace) after embedding it into the continuum ($P$
subspace). They are energy dependent functions and determine the positions $E_k = \tilde{E}_k (E=
E_k)$ and widths $\Gamma_k = \tilde{\Gamma}_k (E=E_k)$ of the resonance states $k$ [2]. The $G^{(+)}_{P}$ in (4) are the
Green functions in the $P$ subspace. The $\tilde{\gamma}_k$ are the coupling matrix elements between the
resonance states and the scattering states. They are also energy dependent functions. The
wave functions \( \tilde{\Omega}_k \) of the resonance states are related to the eigenfunctions \( \tilde{\Phi}_k \) of \( H_{QQ} \) by a Lippmann-Schwinger like relation [2],

\[
\tilde{\Omega}_k = (1 + G^{(+)}_P H_{PQ}) \tilde{\Phi}_k.
\] (5)

The eigenfunctions of \( H_{QQ} \) are bi-orthogonal,

\[
\langle \tilde{\Phi}^*_l | \tilde{\Phi}_k \rangle = \delta_{kl}
\] (6)

so that

\[
\langle \tilde{\Phi}_k | \tilde{\Phi}_k \rangle = \text{Re} (\langle \tilde{\Phi}_k | \tilde{\Phi}_k \rangle) ; \quad A_k \equiv \langle \tilde{\Phi}_k | \tilde{\Phi}_k \rangle \geq 1
\] (7)

\[
\langle \tilde{\Phi}_k | \tilde{\Phi}_{l \neq k} \rangle = i \text{ Im} (\langle \tilde{\Phi}_k | \tilde{\Phi}_{l \neq k} \rangle) = -\langle \tilde{\Phi}_{l \neq k} | \tilde{\Phi}_k \rangle ; \quad B^l_k \equiv |\langle \tilde{\Phi}_k | \tilde{\Phi}_{l \neq k} \rangle| \geq 0.
\] (8)

As a consequence of (7), it holds [2]

\[
\tilde{\Gamma}_k = \sum_c |\tilde{\gamma}^c_k|^2 / A_k \leq \sum_c |\tilde{\gamma}^c_k|^2.
\] (9)

The main difference to the standard theory is that the \( \tilde{\Gamma}_k, \tilde{\gamma}^c_k \) and \( \tilde{E}_k \) are not numbers but energy dependent functions [2]. The energy dependence of \( \text{Im} \{ \tilde{\mathcal{E}}_k \} = -\frac{1}{2} \tilde{\Gamma}_k \) is large near to the threshold for opening the first decay channel. This causes not only deviations from the Breit Wigner line shape of isolated resonances lying near to the threshold, but also an interference with the above-threshold "tail" of bound states, see Sect. 5.2 for an example. Also an inelastic threshold may have an influence on the line shape of a resonance when the resonance lies near to the threshold and is coupled strongly to the channel which opens [11]. Also in this case, \( \tilde{\Gamma}_k \) depends strongly on energy. In the cross section, a cusp may appear in the cross section instead of a resonance of Breit Wigner shape. Both types of threshold effects in the line shape of resonances can explain experimental data known in nuclear physics [2]. They can not be simulated by a parameter in the \( S \) matrix.

In the numerical calculations in the framework of the continuum shell model, the coupling matrix elements \( \tilde{\gamma}^c_k \) between resonance states and continuum are obtained by representing the eigenfunctions \( \tilde{\Phi}_k \) of the effective non-Hermitian Hamilton operator \( \tilde{H}_{QQ} \) in the set of eigenfunctions \( \{ \Phi_k \} \) of the Hermitian Hamilton operator \( H_{QQ} \),

\[
\tilde{\Phi}_k = \sum_l b_{kl} \Phi_l.
\] (10)

The \( \Phi_k \) are real, while the \( \tilde{\Phi}_k \) are complex and energy dependent. The coefficients \( b_{kl} \) and the \( (\tilde{\gamma}^c_k)^2 \) are complex and energy dependent, too. The \( (\tilde{\gamma}^c_k)^2 \) characterizing the coupling of the resonance state \( k \) to the continuum, are related to the width of this state. In the overlapping regime, their sum over all channels is, however, not equal to the width even in the one-channel case, eq. (9). Both functions, \( (\tilde{\gamma}^c_k)^2 \) and \( \tilde{\Gamma}_k \), may show a different energy dependence. An example is shown in Sect. 5.3.
III. SPECTROSCOPY OF RESONANCE STATES

A. Isolated resonance states

The energies and widths of the resonance states follow from the solutions of the fixed-point equations:

$$E_k = \tilde{E}_k(E=E_k)$$

and

$$\Gamma_k = \tilde{\Gamma}_k(E=E_k)$$

on condition that the two subspaces are defined adequately [2]. The values $E_k$ and $\Gamma_k$ correspond to the standard spectroscopic observables. The functions $\tilde{E}_k(E)$ and $\tilde{\Gamma}_k(E)$ follow from the eigenvalues $\tilde{E}_k$ of $\mathcal{H}_{QQ}$. The wave functions of the resonance states are defined by the functions $\tilde{\Omega}_k$, Eq. (5), at the energy $E = E_k$. The partial widths are related to the coupling matrix elements $(\tilde{\gamma}_k^c)^2$ that are calculated independently by means of the eigenfunctions $\tilde{\Phi}_k$ of $\mathcal{H}_{QQ}$. For isolated resonances, $A_k = 1$ according to (7) and $(\tilde{\gamma}_k^c)^2 = |\gamma_k^c|^2$. In this case the standard relation $\Gamma_k = \sum_c |\gamma_k^c|^2$ follows from (9).

It should be underlined that different $\tilde{\Phi}_k(E=E_k)$ are neither strictly orthogonal nor bi-orthogonal since the bi-orthogonality relation (6) holds only when the energies of both states $k$ and $l$ are equal. The spectroscopic studies on resonance states are performed therefore with the wave functions being only approximately bi-orthogonal. The deviations from the bi-orthogonality relation (6) are small, however, since the $\tilde{\Phi}_k$ depend only weakly on the energy.

This drawback of the spectroscopic studies of resonance states has to be contrasted with the advantage it has for the study of observable values: the $S$ matrix and therefore the cross section is calculated with the resonance wave functions being strictly bi-orthogonal at every energy $E$ of the system. Furthermore, the full energy dependence of $\tilde{E}_k, \tilde{\Gamma}_k$ and, above all, of the coupling matrix elements $\tilde{\gamma}_k^c$ is taken into account in the $S$ matrix and therefore in all calculations for observable values.

As a result of the formalism sketched in Sect. 2 for describing the nucleus as an open quantum system, the influence of the continuum of scattering states on the spectroscopic values consist mainly in the following: there is (i) an additional shift in energy of the states and (ii) an additional mixing of the states through the continuum of decay channels.

For isolated resonances, the additional shift is usually taken into account by simulating $\text{Re}(\mathcal{H}_{QQ}) = \mathcal{H}_{QQ} + \text{Re}(W)$ (see Eq. (4)) by $H_0 + V'$, where $V'$ contains the two-body effective residual forces and $W \equiv H_{PQ}G^{(+)}_{PQ}H_{PQ}$. Furthermore, the widths of isolated states are not calculated from $\text{Im}(W)$, but from the sum of the partial widths. The amplitudes of the partial widths are the coupling matrix elements between the discrete states of the $Q$ subspace and the scattering wave functions of the $P$ subspace. The additional mixing of the states via the continuum is neglected in the standard calculations.

It should be mentioned, however, that $\text{Re}(W)$ can not completely be simulated by an additional contribution to the residual two-body interaction since it contains many-body effects, as follows from the analytical structure of $W$. $\text{Re}(W)$ is an integral over energy and
depends explicitly on the energies $\epsilon_c$ at which the channels $c$ open. As a matter of fact, the thresholds for neutron and proton channels in nuclei open at different energies. Therefore, $\text{Re}(W)$ causes some charge dependence of the effective nuclear forces in spite of the charge symmetry of the Hamiltonian $H_{QQ}$. It arises as a many-body effect depending on shell closures, and is directly related to the different binding energies of neutrons and protons in nuclei [2,11].

Since only a few data on isolated resonances are sensitive to the many-body effects involved in $\text{Re}(W)$, the standard calculations performed by using a Hermitian operator are mostly justified. However, the standard calculations can not be justified for closely-lying levels which are coupled via the continuum of decay channels, as well as for well isolated levels in the neighbourhood of thresholds where new decay channels open.

**B. Correlations induced by the coupling via the continuum**

The coupling of the resonance states via the continuum induces correlations between the states that are described by the term $H_{QP} G^{(+)}_P H_{PQ} \equiv W$ of the effective Hamiltonian $H_{QQ}$, Eq. (4). $W$ is complex and energy dependent [2]. The real part $\text{Re}(W)$ causes level repulsion in energy and is accompanied by the tendency to form a uniform time scale in the system. In contrast to this behaviour, the imaginary part $\text{Im}(W)$ causes different time scales in the system and is accompanied by level attraction in energy. That means, the formation of correlations at short-time scales is essentially influenced by $\text{Im}(W)$.

In the overlapping regime, many calculations have shown the phenomenon of resonance trapping caused by $\text{Im}(W)$,

$$\sum_{k=1}^{N} \tilde{\Gamma}_k \approx \sum_{K=1}^{K} \tilde{\Gamma}_k \quad ; \quad \sum_{k=K+1}^{N} \tilde{\Gamma}_k \approx 0. \quad (13)$$

It means almost complete decoupling of $N - K$ resonance states from the continuum while $K$ of them become short-lived. Usually, $K \ll N - K$. The long-lived resonance states in the overlapping regime appear often to be well isolated from one another. The few short-lived resonance states determine the evolution of the system (short time scale).

The formation of different time scales in an open quantum system that is accompanied by level attraction, is accompanied also by the appearance of a non-trivial energy dependence of the $W$ [3]. This energy dependence can directly be expressed by non-linear terms appearing in the overlapping regime. As a consequence, the use of an effective Hamiltonian in describing scattering processes is meaningful only when, at the same time, the energy dependence of the $W$ is considered.

In the framework of statistical approaches, the coupling matrix elements between resonance states and continuum are assumed to be parameters being energy independent. Also in the different versions of $R$ matrix approaches, the correlations induced by $W$ cannot be studied. The interplay between the different time scales of open quantum systems at high level density can be studied only microscopically, without any statistical assumptions on the level distribution or perturbation theory approaches.
C. Overlapping resonance states

The solutions $E_k$ and $\Gamma_k$ of the fixed point equations (11) and (12) are basic for spectroscopic studies not only of isolated but also of overlapping resonances since the energy dependence of the eigenvalues $\tilde{E}_k = \tilde{E}_k - i/2 \tilde{\Gamma}_k$ of the effective Hamiltonian $H_{QQ}$ is smooth everywhere. The $E_k$ and $\Gamma_k$ are therefore well defined and it makes sense to use them for spectroscopic studies. The coupling coefficients $\tilde{\gamma}_c^k$ are however worse defined since the wave functions $\tilde{\Phi}_k(E=E_k)$ are bi-orthogonal. The bi-orthogonality relations (7) and (8) become important at the avoided level crossings where $A_k > 1$. In approaching a double pole of the $S$ matrix, $A_k \to \infty$. The same holds for the modulus square of the coupling coefficients: $|\tilde{\gamma}_c^k|^2 \to \infty$, in accordance with the relation (9).

The numerator of the resonance part of the $S$ matrix (3) is

$$\langle \tilde{\Phi}_k^* | \tilde{W}_{cc} | \tilde{\Phi}_k \rangle = 2\pi \langle \tilde{\Phi}_k^* | V^\dagger | \xi^c_E \rangle \langle \xi^c_E | V | \tilde{\Phi}_k \rangle = \tilde{\gamma}_c^k \tilde{\gamma}_c^{c'}_k .$$

(14)

For $c = c'$, this is $(\tilde{\gamma}_c^k)^2$ and not $|\tilde{\gamma}_c^k|^2$ as often assumed [12]. Expression (14) remains meaningful also in approaching the double pole of the $S$ matrix [2], and the $S$ matrix (2) with (3) is unitary also in the overlapping regime. When the energy difference $\Delta E = |E_k - E_l|$ between two neighbouring resonance states is smaller than their widths, higher-order terms in the $S$ matrix that are related to the bi-orthogonality of the eigenfunctions of the non-Hermitian Hamilton operator $H_{QQ}$, can not be neglected. At a double pole of the $S$ matrix, $(\tilde{\gamma}_c^k)^2 \to -(\tilde{\gamma}_l^l)^2$ corresponding to $\tilde{\Phi}_k \to \pm i \tilde{\Phi}_l$ [2]. Here, the two resonance terms cancel, and the system decouples from the continuum at the energy of the double pole. The same relations hold when the two states avoid crossing in the complex plane by varying a certain parameter [6]. The point is, however, that in such a case the transition $\tilde{\Phi}_k \to \pm i \tilde{\Phi}_l$ influences the wave functions not only at the critical point but in a certain region around the critical value of the parameter [6]. At high level density, this fact will cause deviations from the relation $\tilde{\Gamma}_k = \sum_c (\tilde{\gamma}_c^k)^2$. For numerical results on the relation between $\tilde{\Gamma}_k$ and $(\tilde{\gamma}_c^k)^2$, see Sect. 5.3.

Furthermore, the energies and widths of overlapping resonance states are given by the values $E_k$ and $\Gamma_k$ (Eqs. (11) and (12)), at which the $S$ matrix has poles. However, the positions of the maxima in the cross section do, generally, not appear at the energies $E_k$ when the resonance states overlap [2].

The relation between $\tilde{\Gamma}_k = -2 \text{Im} \{ \langle \tilde{\Phi}_k^* | H_{QQ} | \tilde{\Phi}_k \rangle \}$ and the sum of the coupling coefficients $\sum_c (\tilde{\gamma}_c^k)^2$ is, in general, more complicated than for isolated resonances due to the avoidance of level crossings in the complex plane [2]. The $S$ matrix behaves smoothly in the neighbourhood of a double pole. The same is true for measurable values due to their relation to the $S$ matrix. The value $|\tilde{\gamma}_c^k|^2$ loses its physical meaning in the overlapping regime.

IV. DIFFERENT APPROACHES

A. Statistical approach to the system

More than 40 years ago, the unified theory of nuclear reactions has been formulated by Feshbach [1]. Feshbach introduced the projection operator technique in order to make possible the concurrent numerical solution of equations with discrete and scattering states in spite
of their very different mathematical properties. By means of the projection operator technique, the whole function space is divided into the subspace of discrete states ($Q$ subspace) and the subspace of scattering states ($P$ subspace). Then, the problem in the $P$ subspace is solved numerically by coupled-channel methods while the problem in the $Q$ subspace is not solved directly. Here, statistical assumptions are introduced by which the mean properties of the discrete states are described. Also the coupling matrix elements between discrete and scattering states are determined statistically and characterized by their mean values.

The advantage of using different approximations in the two subspaces consists, above all, in the possibility to solve the coupled-channel problem with high accuracy. Since the $P$ subspace is constructed from all open decay channels, it changes with energy since new channels open in passing the corresponding thresholds. Furthermore, the inclusion of, e.g., $\alpha$ decay channels into the $P$ subspace is not a problem. The method is applied successfully to the description of nuclear reactions in energy regions with high level density of the excited nucleus which makes it possible for a statistical treatment of the discrete states of the $Q$ subspace. It represents the standard method in analyzing nuclear reaction data on medium and heavy nuclei at low energy.

The shell model approach to nuclear reactions \cite{12} is formulated by Mahaux and Weidenmüller. Also in this approach, the whole function space is divided into the two subspaces. However, the $P$ subspace contains open as well as closed decay channels and, therefore, does not change with energy. The inclusion of more than one particle in the continuum becomes a principal problem. The bi-orthogonality of the eigenfunctions of the effective Hamiltonian is not considered what causes problems with the unitarity of the $S$ matrix in the overlapping regime due to $\Gamma_k < \sum_c |\gamma_k^c|^2$ \cite{12}. Eventually, the states of the $Q$ subspace are treated by means of statistical methods in the same manner as in the Feshbach formulation \cite{1}. The restrictions in the applicability of both treatments are therefore the same: as long as the (long-lived) resonance states are isolated from each other and their individual properties can be neglected to a good approximation, the method gives reliable results.

The formation of different time scales in a realistic system cannot be studied by using a statistical description of the states, since the interplay between the real and imaginary parts of the interaction in the effective Hamiltonian $\mathcal{H}_{QQ}$ is not taken into account.

**B. $R$ matrix approach**

In contrast to the Feshbach unified theory of nuclear reactions, different approaches for the description of decaying states are worked out by starting from well established nuclear structure models. These approaches are based on the $R$ matrix theory of nuclear reactions that is justified at low level density \cite{13}. Here, the resonance levels are assumed to be isolated, \emph{i.e.} the influence of resonance overlapping on the nuclear structure is not considered.

The advantage of these studies consists, above all, in the integration of proven nuclear structure models into the calculations. That means, the wave functions of the $Q$ subspace are realistic. The coupling to the supplementary $P$ subspace (continuum of decay channels) is described in a straightforward manner. The feedback from the continuum of decay channels on the nuclear structure is however hidden, if at all taken into account, in the results of the numerical studies. When the resonances overlap, some averaging over many levels is performed in the $R$ matrix theory of nuclear reactions \cite{13}.
The formation of different time scales in the system cannot be studied since it arises from the feedback from the continuum to the states of the system that is not taken into account in the $R$ matrix approach.

C. Shell model approach to the system

In reactions on light nuclei and in studying nuclei near to the drip line, the level density is low and the individual properties of the nuclear states cannot be neglected. In these nuclei, the restriction to a description of the mean properties of the states is not justified. The problem in the $Q$ subspace has to be solved with a higher accuracy.

The spectroscopic properties of light nuclei are described successfully in the framework of the shell model. It is therefore reasonable to identify the $Q$ subspace with the function space of the shell model used in performing numerical calculations for these nuclei. Two different approaches have been developed: (i) the CSM-FDP approach (continuum shell model with finite depth potential), that generates the single particle basis states in a Woods-Saxon potential [11,14], and has been used mainly for a description of giant resonances in light nuclei, and (ii) the SMEC (shell model embedded in the continuum) which uses the shell model effective interaction in the $Q$ subspace and provides, in particular, a realistic description of resonance phenomena near particle decay thresholds [15]. Common to both approaches is that Eq. (1) is solved numerically by using similar approximations in the two subspaces. The bi-orthogonality of the eigenfunctions of the effective Hamiltonian (Eqs. (6) to (8)) is taken into account in both approaches. As a consequence, the unitarity of the $S$ matrix is ensured also in the overlapping regime. These calculations provide a unified description of nuclear structure and nuclear reaction aspects.

In the SMEC, the nuclear shell model is involved what makes it possible for a realistic description of the nuclear structure, as in the models based on the $R$ matrix approach. However, in contrast to these models, the feedback from the continuum of decay channels on the nuclear structure is explicitly taken into account. Therefore, the formation of different time scales in the system can be studied by means of SMEC.

V. SOME RESULTS OBTAINED FOR $^{24}$Mg IN SMEC

A. The $^{24}$Mg nucleus

Let us consider $^{24}$Mg with the inner core $^{16}$O and the phenomenological $sd$-shell interaction among the valence nucleons. Within this configuration space, the $^{24}$Mg nucleus has 325 states with $J^\pi = 0^+ , T = 0$. These states can couple to a number of open channels which correspond to excited states in the neighboring $(A - 1)$ nucleus. For details see [2,16].

For illustration, we show in Fig. 1 the dependence of energies $\tilde{E}_k$ and widths $\tilde{\Gamma}_k$ of the ten lowest $0^+$ states of $^{24}$Mg on the energy $\tilde{E}$ of the particle in the continuum, as well as the eigenvalue picture with the energy $E$ parametrically varied. The number of channels is one (the l.h.s plot) and two (the r.h.s. plot). We can see the non-random features occurring at this edge of the spectrum. The coupling between the channels reduces the differences between the widths of the different states. It has almost no influence onto their positions.
The positions \( \tilde{E}_k \) of the resonance states are almost independent of a variation of the energy of the system (Fig. 1). The widths \( \tilde{\Gamma}_k \) however depend on energy: they rise at low energies above the particle decay threshold and decrease again at energies beyond the positions \( E_k \) of the states. Most of the resonances have therefore a tail at the high energy side. This feature is well pronounced especially for the lowest-lying state which is bound. Due to its large width, it can contribute to the cross section in the threshold region.

B. Near-threshold behaviour of the cross section

In Fig. 2, we summarize the generic features of cross sections near thresholds. As an example we show the cross section for the reaction \( n + {}^{23}\text{Mg} \rightarrow {}^{24}\text{Mg}(0^+) \) (one open channel, the \( s \)-wave scattering). The upper part shows the cross section with only one excited (resonance) state \( 0^+_2 \) of \( {}^{24}\text{Mg} \). The minimum in the cross section is an effect of destructive interference of the resonance \( (E^* = 2.23 \text{ MeV}, \Gamma = 1.76 \text{ MeV}) \) with the background of the potential scattering (the direct part of the reaction cross section), denoted by the dashed line. In the middle part, the cross section with only the ground state (bound state) \( 0^+_1 \) in \( {}^{24}\text{Mg} \) is shown. The cross section exhibits a strong increase for \( E \rightarrow 0 \). This is caused by the bound state for which \( \tilde{\Gamma}_k \neq 0 \) at \( E > 0 \) in spite of \( \tilde{\Gamma}_k = 0 \) at \( E = \tilde{E}_k < 0 \). Finally, the cross section with both ground state \( 0^+_1 \) and resonance state \( 0^+_2 \) is shown in the bottom part of the figure. The interference picture of these two states shows level repulsion accompanied by a decrease of the width of the higher-lying state \( (E^* = 2.40 \text{ MeV}, \Gamma = 0.47 \text{ MeV}) \). The line shape of the resonance resembles a typical interference picture for overlapping resonances in spite of the fact that the calculation is performed with only one resonance state while the other state is bound.

C. Relation between total and partial widths

The resonance states shown in Fig. 1 do not overlap strongly. Nevertheless, the relation between their widths \( \tilde{\Gamma}_k \) and the coupling matrix elements \( (\tilde{\gamma}_c^k)^2 \) is far from being both well defined and energy independent, even in the one-channel case. In Figs. 3 and 4, we show the total widths \( \tilde{\Gamma}_k \) and the real and imaginary parts of the coupling matrix elements \( (\tilde{\gamma}_c^k)^2 \) for six of these states and, additionally, the partial widths \( |\gamma_c^k|^2 \) (the unambiguous identification of the label for different states can be obtained from Fig. 5). All results are for the one-channel case and therefore \( \Gamma_k = |\gamma_c^k|^2 \) is assumed in the \( R \) matrix theory.

The states ‘1’ and ‘8’ are well isolated from the other ones due to the large distance in energy (state ‘1’) and small width (state ‘8’), respectively. For these two states the relation \( \tilde{\Gamma}_k \approx \text{Re}(\tilde{\gamma}_c^k)^2 \) holds in the whole energy region considered. The values \( \tilde{\Gamma}_k \) and \( |\gamma_c^k|^2 \) differ from each other, but show a similar energy dependence (Fig. 4).

States ‘5’ and ‘6’ are coming near to one another at an energy higher than their position. As a consequence, the total widths \( \tilde{\Gamma}_k \) are different from the \( \text{Re}(\tilde{\gamma}_c^k)^2 \) that are the real parts of the coupling matrix elements of the resonance states to the continuum. They differ also from the \( |\gamma_c^k|^2 \) that are the coupling matrix elements of the discrete states to the continuum. The differences are noticeable in the whole energy region considered, and not only at their nearest distance in energy (Fig. 3). This happens for the states ‘7’ and ‘10’ in a similar
manner (Fig. 4) as for the states ‘5’ and ‘6’ although the distance to the neighbouring states is, in these cases, much larger than in the case of two states ‘5’ and ‘6’.

Figs. 3 and 4 illustrate that the standard one-level formula for the cross section (the Breit-Wigner representation) can be applied only for well isolated resonances. Only in such a case, unitarity of the S matrix provides a clear relation between the total widths $\tilde{\Gamma}_k$ and the coupling coefficients between system and environment. Well isolated resonances appear, however, seldom in realistic situations. Therefore, a generic relation of the type $\tilde{\Gamma}_k = \sum |\gamma_{ck}|^2$ (or $\tilde{\Gamma}_k = \sum (\tilde{\gamma}_{ck})^2$) does not exist, even in the one-channel case. The partial widths $|\gamma_{ck}|^2$ of the state $k$ relative to the channels $c$ lose their physical meaning when the resonance states are not well isolated. Furthermore, the $\tilde{\Gamma}_k$ and even the $|\gamma_{ck}|^2$ are energy dependent.

These numerical results show that, in general, the influence of the different states onto the properties of the system can not be restricted to the small energy region that is determined by their energies and widths. This restriction being one of the basic approximations of $R$ matrix approaches, is justified neither for bound states lying just below the first particle decay threshold nor for resonance states at high level density.

Thus, even though the SMEC and the different $R$ matrix approaches start both from a reliable nuclear structure model, the coupling of the resonance states via the continuum of decay channels is taken into account correctly only in the SMEC.

D. Statistical versus dynamical aspects of resonance states at high level density

The statistical properties of $^{24}$Mg are studied in [16]. As a result, the dynamics of the $^{24}$Mg nucleus in the short-time scale is determined by the states at the edges of the spectra of the parent and daughter nuclei. These states are strongly related to each other with the result that the corresponding resonance states have short lifetimes. Randomness in an open quantum system can be found only in the long-time scale and, even here, only in the one-channel case. Since the short-lived and long-lived states are created together at avoided level crossings, both time scales exist simultaneously in the nucleus. This statement is in agreement with experimental results on different nuclei of the $sd$-shell, including $^{24}$Mg. The experimental data show the interplay of various reaction times, ranging from the lifetime of the compound nucleus to the time associated with shape resonances in the ion-ion potentials [17]. For a more detailed discussion see [2].

VI. RELATION BETWEEN LIFETIMES AND DECAY WIDTHS

The phenomenon of resonance trapping has been discussed also in quantum chemistry for unimolecular reactions. For illustration, let us consider here the unimolecular decay processes in the regime of overlapping resonances with the goal to elucidate how unimolecular reaction rates depend on resonance widths [18]. Using the definition

$$k^{\text{eff}} = -\frac{d}{dt} \ln\langle \phi(t)|\phi(t) \rangle \quad (15)$$

for the decay rate, and
\[ \langle \Gamma \rangle = \frac{1}{N} \sum_{k=1}^{N} \Gamma_k , \tag{16} \]

where the sum runs over all \( N \) resonance states in the energy region considered, the result is as follows [18]: in all studied cases, the dependence of the average decay rate on \( \langle \Gamma \rangle \) for a given energy interval is characterized by a saturation curve. In other words: in the regime of nonoverlapping resonances (the weak coupling regime), the standard relation between decay rate and \( \langle \Gamma \rangle \) holds, i.e. the unimolecular decay rate is equal to the resonance width divided by \( \hbar \). When, however, the resonance overlap increases (the strong coupling regime), the decay rate saturates as a function of increasing \( \langle \Gamma \rangle \). Identifying the average resonance width \( \Gamma_{av} \) with \( \langle \Gamma \rangle \), it was claimed [18] that the fundamental quantum mechanical relation between the average decay rate and the average resonance width does not hold in the strong overlapping limit.

This conclusion is, however, justified only under the assumption of uniform level broadening that makes it possible to identify \( \langle \Gamma \rangle \) with \( \Gamma_{av} \) [19]. According to the phenomenon of resonance trapping, Eq. (13), the levels are, however, broadened non-uniformly in the overlapping regime due to the reordering processes taking place under the influence of the environment into which the system is embedded. As a consequence

\[ \sum_{k=1}^{M} \Gamma_k \gg \sum_{k=M+1}^{N} \Gamma_k , \tag{17} \]

and \( \Gamma_{av} \) is different from \( \langle \Gamma \rangle \) in the overlapping region.

A meaningful definition of the average width of the long-lived states is [20]

\[ \Gamma_{av} = \frac{1}{N-M} \sum_{k=M+1}^{N} \Gamma_k . \tag{18} \]

The sum in (18) runs over the \( N - M \) long-lived (trapped) states only. These states do not overlap and the value \( \Gamma_{av} \) saturates in the long-time scale.

The saturation of the decay rate in the overlapping regime may be related also to the broadening of the widths distribution occurring in this regime [18]. This result is not in contradiction with the conclusion that the saturation is related to resonance trapping. The point is that resonance trapping creates differences in the transmission coefficients for the different states that cause a broadening of the widths distribution [20]. Consequently both, the broadening of the widths distribution and the saturation of the decay widths in the overlapping regime, can be traced back to the same origin, i.e. to resonance trapping or, more generally, to avoided level crossings in the complex plane.

It is possible to invert the discussion: it is not the standard relation between decay rate and average decay width which ceases to hold in the overlapping regime. The saturation of the decay rate is rather a proof of the formation of different time scales. A uniform level broadening does not take place in the system at high level density, and the unimolecular decay rate in the long-time scale is equal to the average resonance width \( \Gamma_{av} \) divided by \( \hbar \). It should be mentioned here, that also in atomic nuclei a similar saturation effect is known: the spreading width obtained from an analysis of experimental data on isobaric analogue resonances in different nuclei saturates [21].
We conclude that the standard relation between decay rate and average decay width $\Gamma^{av}$ holds in the regime of overlapping resonances in the long-time scale. The point is that different time scales exist in this regime that are caused by resonance trapping, i.e. by the bifurcation of the widths at the avoided level crossings.

VII. PROPERTIES OF THE BROADENED STATES

A. Dynamical localization

An answer to the question whether resonance trapping is accompanied by dynamical changes of the shape of the system, can not be found directly from studies on microwave cavities, since their shape is fixed from outside. Nevertheless, a dynamical localization of the wave function density inside the cavity may occur. A study of different wave functions in open microwave cavities showed, indeed, that the localization of the probability density for short-lived and long-lived states inside the cavity is different. While the short-lived states are localized along particular short paths related to the position(s) of the attached wave guide(s), all the long-lived trapped states have pronounced nodal structure that is distributed over the whole cavity [7,8]. The long-lived states can be described well by random matrix theory [9].

A similar result has been obtained in calculations for nuclei [22]. The numerical results obtained for the radial profile of partial widths of $1^{-}$ resonance states with 2p-2h structure in $^{16}$O show that, also in this case, resonance trapping is accompanied by a dynamical localization of the short-lived states. In other words: structures in space and time are created that are characterized by a small radial extension, a short lifetime and a small information entropy [22,23].

B. Classical description

As has been discussed in Sect. 3.2, resonance trapping is accompanied by the broadening of some states when the system is opened to the continuum. An example are the whispering gallery modes that appear in microwave cavities and may give an important contribution to the conductance when the cavity is opened by attaching wave guides to it [8].

From a physical point of view, most interesting is the following fact. Special states of a certain type are characterized by their structure that is similar for all the states of this type. For example, all whispering gallery modes are spatially localized in different groups parallel to each other and near to the convex boundary of the cavity. The states of each group differ by the number of nodes, but not by the localization region. They couple therefore coherently to the continuum of decay channels when the leads are attached in a suitable manner. As a consequence, special states existing in closed systems among other states, may become dominant by opening the system to a small number of decay channels [8]. The widths of these states may increase strongly by trapping other incoherent or less coherent resonance states lying in the same energy region. The special states align with the channels while the trapped states decouple more or less completely from the environment. Eventually, the properties of the system as a whole are determined in the short-time scale mainly by the
special states. Since these states are localized, they lose partly their wave character, and it is even possible to describe some properties of the system by using the methods of classical physics.

The conductance of the cavity is determined by the non-diagonal terms of the $S$ matrix. The transmission coefficients between channel $n$ and $m$ may be represented by a Fourier transform in order to get the length spectra of the quantum mechanical calculations [8]. They are compared to the histograms of trajectories calculated classically as a function of the length $L$ of the path for the same cavity. In the classical calculations, the trajectories are obtained from paths of different lengths corresponding to a different number of bouncings of the particle at the convex boundary. The results display a remarkable and surprisingly good agreement between the quantum mechanical results of the Fourier analysis and the classical results in spite of the small value of the wave vector of the propagating waves [8].

The correspondence between the quantum mechanical and purely classical results holds not only in the length scale but also in the time scale: varying parametrically the length $L$ of the paths causes corresponding changes in the widths of the special states that agree with the changes of the times for transmission of a classical particle through the cavity [8]. Obviously, the correspondence is related to the spatial localization of the whispering gallery modes due to which the wave properties of the states are somewhat suppressed.

These results illustrate the strong correlations which may be induced in the system due to its coupling to the continuum. The correlations appear in the long time scale and, above all, in the short time scale. While the long-lived trapped states can be described well by random matrix theory, the short-lived coherent modes are regular [9].

**VIII. CONCLUDING REMARKS**

All the studies of open quantum systems have shown that the coupling of the system to the environment may change the properties of the system. The changes are small as long as the coupling strength between system and environment is smaller than the distance between the individual states of the unperturbed system, i.e. smaller than the distance between the eigenstates of the Hamiltonian $H$. The changes can, however, not be neglected when the coupling to the continuum is of the same order of magnitude as the level distance or larger. In such a case, the changes can be described neither by perturbation theory nor by introducing statistical assumptions for the level distribution. The point is that non-linear effects become important which cause a redistribution of the spectroscopic properties of the system and, consequently, changes of its properties. Under the influence of the coupling to the continuum, level repulsion as well as level attraction may appear that are accompanied by the tendency to form a uniform time scale for the system in the first case, but different time scales in the second case.

The resonance phenomena are described well by two ingredients also at high level density. The first ingredient is the effective Hamiltonian $\mathcal{H}$ that contains all the basic structure information involved in the Hamiltonian $H$, i.e. in the Hamiltonian of the corresponding closed system with discrete eigenstates. Moreover, $\mathcal{H}$ contains the coupling matrix elements between discrete and continuous states that account for the changes of the system under the influence of its coupling to the continuum. These matrix elements are responsible for the
non-Hermiticity of $\mathcal{H}$ and its complex eigenvalues which determine not only the positions of the resonance states but also their (finite) lifetimes.

The second ingredient is the unitarity of the $S$ matrix that has to be fulfilled in all calculations of resonance phenomena. It is taken into account in the unified description of structure and reaction aspects since any statistical or perturbative assumptions are avoided in solving the basic equation (1).

The studies within the formalism of a unified description of structure and reaction phenomena show that the coupling of the states via the continuum induces correlations that are not small. The correlations are important especially in the short time scale, but appear also in the long time scale. The short-lived states, involving information on the environment, characterize the system far from equilibrium. The long-lived states, however, are described well by random matrix theory. They are more or less decoupled from the environment.
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FIG. 1. Energy dependence of the positions $\tilde{E}_i$ (upper row) and widths $\tilde{\Gamma}_i$ (lower row) of the ten lowest $0^+$ states of $^{24}\text{Mg}$ as a function of the energy $E$ of the particle in the continuum. In the first column, the calculations include coupling to only one channel in $^{23}\text{Mg}$. In the other column, two channels are taken into account. The stars at the trajectories mark the fixed-point solutions.
FIG. 2. Cross section for the reaction $n + ^{23}\text{Mg} \rightarrow ^{24}\text{Mg}$ calculated for one open neutron channel and (a) the resonance state $0^+_2$ of $^{24}\text{Mg}$, (b) the ground state $0^+_1$ of $^{24}\text{Mg}$, (c) the bound $0^+_1$ and the resonance state $0^+_2$ of $^{24}\text{Mg}$. The dashed lines show the direct reaction part of the cross section. The arrows denote the position of the resonances.
FIG. 3. Energy dependence of the coupling coefficients $(\tilde{\gamma}_i)^2$ (solid lines) for crossing resonances in $^{24}$Mg (resonances 5 and 6 in Fig. 1, one-channel case). The real parts are shown on the left-hand side and the imaginary parts on the right-hand side. In addition, the figure on the left-hand side shows the dependence of the widths $\tilde{\Gamma}_i$ (dashed lines) and $|\gamma_i|^2$ (dotted lines) on the energy of the particle in the continuum.
FIG. 4. Energy dependence of various ‘widths’ for resonances ‘1’, ‘7’, ‘8’, ‘10’ (see Fig. 1) in $^{24}\text{Mg}$. The different lines denote: $\tilde{\Gamma}_i$ (solid line), $\text{Re}(\tilde{\gamma}_i)^2$ (dashed line), $\text{Im}(\tilde{\gamma}_i)^2$ (dashed-dotted line) and $|\gamma_i|^2$ (dotted line).
FIG. 5. The eigenvalue picture with the energy $E$ of the particle in the continuum parametrically varied for ten lowest $0^+$ states of $^{24}\text{Mg}$. 