EIGENVALUES OF THE LAPLACE-BELTRAMI OPERATOR UNDER THE HOMOGENEOUS NEUMANN CONDITION ON A LARGE ZONAL DOMAIN IN THE UNIT SPHERE

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Dedicated to Professor Wei-Ming Ni on the occasion of his seventieth birthday

Abstract. We consider the eigenvalues of the Laplace-Beltrami operator under the homogeneous Neumann condition on a spherical domain. Especially, we investigate the case when the domain is a large zonal one and letting the zone larger so that the zone covers the whole sphere as a limit. We discuss the behavior of eigenvalues according to the rate of expansion of the zone.

1. Introduction. In this paper we study the eigenvalue problem

\[ \begin{cases}
\Lambda_n u + \lambda u = 0 \text{ in } \Omega_{\varepsilon, \eta} \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}, \\
\partial_n u = 0 \text{ on } \partial \Omega_{\varepsilon, \eta}
\end{cases} \tag{1.1} \]

where \( \Lambda_n \) is the Laplace-Beltrami operator on the unit sphere \( \mathbb{S}^n, n \geq 2 \), the domain

\[ \Omega_{\varepsilon, \eta} = \{(x_1, x_2, \ldots, x_{n+1}) | \sum_{i=1}^{n+1} x_i^2 = 1, \cos(\pi - \varepsilon) < x_{n+1} < \cos \eta \} \]

is the zonal domain with the geodesic distance from the North Pole \((0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}\) between \( \eta > 0 \) and \( \pi - \varepsilon \) (\( \eta > 0 \) and \( \varepsilon > 0 \) are small), and \( \partial_n \) denotes the derivative in the direction of the outer normal.

In the precedent work by Bandle, Kabeya and Ninomiya [1], we studied the case when \( \eta = 0 \) (large spherical cap, or one hole case) and showed the asymptotic behavior of the eigenvalues as \( \varepsilon \to 0 \). We also studied the nonlinear problem there from the bifurcation-theoretic point of view.

Here, we consider the large zonal domain on the unit sphere and enlarge the domain so that the domain covers the whole sphere.

Concerning the relation between the perturbation of domains and the behavior of eigenvalues, Ozawa [11, 12, 13] was one of the pioneers, who studied the dependence of the eigenvalues of the Laplacian with the homogeneous Dirichlet boundary condition, under small perturbations of the domain. Also, he assumed that the eigenvalue of the unperturbed problem is simple, which does not always hold, and which is not the case for our spherical domain.

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Extensions to manifolds are found in Courtois [3], see also the references cited therein. In contrast to the Dirichlet case very little is known for the behavior of eigenvalues with the Neumann boundary conditions under the domain perturbation. Some results for simple eigenvalues in planar domains with holes are found in Lanza de Cristoforis [6]. Also several interesting properties of eigenvalues of the Laplacian under the Neumann condition on a domain in the Euclidean space are discussed in Chapter 1 in Ni [9] based on Ni and Wang [10]. The authors stress that the domain monotonicity property of the eigenvalues does not necessarily hold.

The asymptotic behavior of eigenvalues of $-\Lambda_n$ under the Robin condition with $\eta = 0$, $\varepsilon > 0$ when $n = 2, 3$, was discussed in Kabeya, Kawakami, Kosaka and Ninomiya [5]. The difference of the asymptotic behavior of the eigenvalues subject to the boundary condition is observed. More precisely, the asymptotic behavior of those under the Dirichlet condition or under the Neumann condition is different from the Robin case. In fact, for example, the three dimensional case, the asymptotic behavior as $\varepsilon \to +0$ of the $k$-th eigenvalue whose eigenfunction depends only on $x_4 = \cos \theta$ is as follows:

\[ \lambda = k(k + 2) + C_i^* \varepsilon + o(\varepsilon) \]  for the Dirichlet boundary condition
\[ \lambda = k(k + 2) + C_i^* \varepsilon^2 + o(\varepsilon^2) \]  for the Robin boundary condition
\[ \lambda = k(k + 2) + C_i^* \varepsilon^3 + o(\varepsilon^3) \]  for the Neumann boundary condition

with $C_i^*$ ($i = 1, 2, 3$) being nonzero constant. The Dirichlet case is due to MacDonald [7] and the Neumann case is due to [1].

We shall treat the problem (1.1) as a perturbation in $\varepsilon$ and $\eta$ of the problem $\Lambda_n u + \lambda u = 0$ in $\mathbb{S}^n$. In the whole sphere case, all the eigenvalues are completely understood. That is, the eigenvalues of $\Lambda_n Y + \sigma Y = 0$ in $\mathbb{S}^n$

are

$\sigma_{k,n} := k(k + n - 1), \quad k = 0, 1, 2, \ldots$

and the corresponding eigenfunctions are the regular spherical harmonics $Y_{(k,n)}$ of degree $k$. The multiplicity of $\sigma_{k,n}$ is

$\quad d(k, n) := (2k + n - 1) \frac{(k + n - 2)!}{(n - 1)!k!},$

which can be found in Müller [8] or in Shimakura [14]. The corresponding eigenfunctions $Y_{(k,n)}$ are expressed in terms of the associated Legendre polynomials as it will be seen in Section 2 (see also [8]). The eigenfunction corresponding to $k = 0$ is the constant.

We may always assume that $\eta$ is at most of $\varepsilon$ order since the equation $\Lambda_n + \lambda u = 0$ is unchanged under the transformation $x_n \mapsto -x_n$.

The following theorem describes the behavior of the eigenvalues of $-\Lambda$ on the large zonal domain $\Omega_{\varepsilon,\eta}$ as $\varepsilon \to 0$ and as $\eta \to 0$.

**Theorem 1.1.** Suppose that $\varepsilon > 0$ and $\eta > 0$ are small. For each $n \geq 2$ and $k \in \mathbb{N}$, (1.1) has $(k+1)$ distinct eigenvalues $\lambda_{k,\varepsilon,\eta,m}$ (parametrized by $m = 0, 1, \ldots, k$) close to $k(k + n - 1)$. They are expanded as below:

(i): If $\eta/\varepsilon \to 0$ as $\varepsilon \to 0$, then there holds

$\lambda_{k,\varepsilon,\eta,m} - k(k + n - 1) = C_{k,m,n} \varepsilon^n + o(\varepsilon^n m) \quad \text{as} \quad \varepsilon \to 0.$
(ii): If \( \eta = \kappa \varepsilon + o(\varepsilon) \) as \( \varepsilon \to 0 \) with some \( \kappa > 0 \), then there holds \[
\lambda_{k,\varepsilon,\eta,m} - k(k + n - 1) = C_{k,m,n,\kappa} \varepsilon^{n_m} + o(\varepsilon^{n_m}) \quad \text{as} \quad \varepsilon \to 0
\]
with some constant \( C_{k,m,n,\kappa} \) and with \( C_{k,m,n,\kappa} \).

Here \( n_m = \max\{2m + n - 2, n\} \).

**Remark 1.** The constants \( C_{k,m,n} \) and \( C_{k,m,n,\kappa} \) can be computed explicitly and \( C_{k,0,n} \neq C_{k,1,n} \) (cf. Sections 4, 5 and 6). Moreover, we see the following:

(i): If \( \eta \) is of small order of \( \varepsilon \), then the first approximation does not depend on \( \eta \). In fact, \( C_{k,m,n} \) is the same as that obtained for the large spherical cap as in [1]. Only when \( \eta \) is of the same order of \( \varepsilon \), the dependence appears.

(ii): The multiplicity of \( \lambda_{k,\varepsilon,\eta,m} \) is \( d(m, n - 1) \), which is the same as that with \( \eta = 0 \) in [1].

(iii): If \( \eta \) is “exponentially small”, that is, \( \eta \) is of the same order as \( \varepsilon^r \) with some \( r(\geq 2) \in \mathbb{N} \), then in a higher order expansion, we will see that the difference in some order of \( \varepsilon \) would appear (see also Remark 2).

(iv): If \( \eta \) is of \( \exp(-1/\varepsilon) \) order or even smaller than this, then we will not see the difference of the expansion of the eigenvalues. Asymptotically, we cannot hear the difference of sound of the zonal drum if one of the holes radius is exponentially small to the other.

(v): Similarly, we can treat the asymptotic behavior of the eigenvalues under the Robin condition. However, more complicated calculations are necessary and we do not do that here and postpone to a future work. Similar difference as in [5] would be found.

This paper is organized as follows: In the next section, Section 2, we give the exact form of eigenfunctions. We give a strategy to a proof of Theorem 1.1 in Section 3. An actual proof is given in Sections 4, 5 and 6 depending on the dimension. Formulas on the associated Legendre functions, the Gauss hypergeometric functions, and the Gamma function and di-Gamma function are listed in Section 7 as appendix.

### 2. Eigenfunctions.

In this section as in [1], we introduce polar coordinates which are most suitable to compute the eigenfunctions of problem (1.1):

\[
\begin{align*}
x_1 &= \left( \prod_{j=1}^{n-1} \sin \theta_j \right) \sin \phi, \\
x_2 &= \left( \prod_{j=1}^{n-1} \sin \theta_j \right) \cos \phi, \\
x_{n-k+1} &= \left( \prod_{j=1}^{k} \sin \theta_j \right) \cos \theta_{k+1}, \quad k = 1, 2, \ldots, n - 2, \\
x_{n+1} &= \cos \theta_1,
\end{align*}
\]

where \( \eta < \theta_1 < \pi - \varepsilon \), \( 0 \leq \theta_j \leq \pi (j = 2, 3, \ldots, n - 1) \) and \( 0 \leq \phi < 2\pi \). The angle \( \theta_1 \) denotes the azimuthal variable. In these coordinates the Laplace-Beltrami operator \( \Lambda_n \) on \( \mathbb{S}^n \) becomes

\[
\Lambda_n = \frac{1}{\sin^{n-1} \theta_1} \frac{\partial}{\partial \theta_1} \left[ \sin^{n-1} \theta_1 \frac{\partial}{\partial \theta_1} \right]
\]
In the polar coordinate, the boundary condition yields
\[ \frac{\partial}{\partial \theta} \left[ \sin^{n-2} \theta \frac{\partial}{\partial \theta} \left[ \sin^{n-2} \theta \frac{\partial}{\partial \theta} \right] \right] + \cdots + \frac{1}{\sin^2 \theta \sin^2 \phi} \frac{\partial}{\partial \phi} \left[ \sin \theta \frac{\partial}{\partial \theta} \right] \]

Hence
\[ \Lambda_n \frac{1}{\sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left[ \sin^{n-1} \theta \frac{\partial}{\partial \theta} \right] + \frac{1}{\sin^2 \theta} \Lambda_n - 1, \quad \Lambda_1 = \frac{\partial^2}{\partial \phi^2}. \]

Here we note that \( \Lambda_{n-1} \) is a Laplace-Beltrami operator on \( S^{n-1} \) with a coordinate \( \{ \theta, \phi \} \). It is convenient to introduce the variables
\[ t := \cos \theta \quad \text{and} \quad \zeta \in S^{n-1}. \]

Then \( x = (\sqrt{1-t^2} \zeta, t) \) and
\[ \Lambda_n = \frac{1}{(1-t^2)^{n-1}} \frac{\partial}{\partial t} \left[ (1-t^2)^n \frac{\partial}{\partial t} \right] + \frac{1}{1-t^2} \Lambda_{n-1} =: L_{n-1} + \frac{1}{1-t^2} \Lambda_{n-1}. \]

By the Weierstrass polynomial approximation theorem (see also Titchmarsh [15, 16]), any eigenfunction to (1.1) is obtained by separation of variables. We are therefore looking for solutions of (1.1) of the form \( v = U_1(t)Y(\zeta) \). \( Y \) is an eigenfunction of \( -\Lambda_{n-1} \) on \( S^{n-1} \) corresponding to the eigenvalue \( k(k+n-2) \) for any \( k \in \mathbb{N} \). Recalling the ingredients of [1], we see that the eigenfunctions \( Y \) of \( -\Lambda_{n-1} \) on \( S^{n-1} \) are expressed as
\[ Y = \hat{P}_k^{(n-2)/2} \left( \cos \theta_1 \right) \prod_{j=1}^{n-2} \frac{\hat{P}_k^{(n-j-2+j-1)/2} \left( \cos \theta_{n-j} \right)}{(\sin \theta_{n-j})^{(j-1)/2}} \times \left\{ \begin{array}{ll} \cos k_n \phi, \\ \sin k_n \phi \end{array} \right. \]

where \( \{k_j\} \) is a sequence of integers such that
\[ 0 \leq k_n \leq k_{n-1} \leq \cdots \leq k_{j+1} \leq k_j \leq \cdots \leq k_2 \leq k, \]

and
\[ \hat{P}_\mu(t) := \left\{ \begin{array}{ll} P_\mu(t), & \text{if } \mu \text{ is an integer}, \\ P_{\mu'}(t), & \text{if } \mu \text{ is a half integer}. \end{array} \right. \]

For the latter use, we set
\[ U_1(t) := (1-t^2)^{-n/2} \mathcal{P}(t) = \frac{\mathcal{P}(\cos \theta_1)}{\sin^{(n-2)/2} \theta_1}. \]

In the polar coordinate, the boundary condition yields
\[ \frac{d}{d\theta} \left( \frac{\mathcal{P}(\cos \theta_1)}{\sin^{(n-2)/2} \theta_1} \right) \bigg|_{\theta_1=\eta} = 0. \quad (2.1) \]

Then \( \mathcal{P}(t) \) is a solution to
\[ (1-t^2) \mathcal{P}''(t) - 2t \mathcal{P}'(t) + \left( \lambda + \frac{n(n-2)}{4} - \frac{(k+n-2)^2}{1-t^2} \right) \mathcal{P}(t) = 0 \text{ in } (t_\epsilon, \tau_0) \]
and the boundary condition (2.1) yields in terms of $t$ as
\[
\mathcal{P}'(\tau_\eta) = -\frac{(n-2)\tau_\eta}{2(1-\tau_\eta^2)} \mathcal{P}(\tau_\eta), \quad \mathcal{P}'(t_\epsilon) = -\frac{(n-2)t_\epsilon}{2(1-t_\epsilon^2)} \mathcal{P}(t_\epsilon). \tag{2.2}
\]
where $\tau_\eta = \cos \eta \sim 1 - \eta^2/2$, $t_\epsilon := \cos(\pi - \epsilon) \sim -1 + \epsilon^2/2$ and we have used the notation $' := d/dt$. Since the associated Legendre functions $P_\nu^\mu(t)$ and $Q_\nu^\mu(t)$ are fundamental solutions of the associated Legendre equation
\[
(1-t^2)P''(t) - 2tP'(t) + \left(\nu(\nu+1) - \frac{\mu^2}{1-t^2}\right)P(t) = 0
\]
(see, e.g., Section 9.5, p.318 of Beals and Wong [2]), we see that
\[
\mathcal{P}(t) = C_1 P_\nu^\mu(t) + C_2 Q_\nu^\mu(t),
\]
where $C_1, C_2$ are constants,
\[
\begin{align*}
\mu &= m + \frac{n-2}{2}, & \lambda &= \nu(\nu+1) - \frac{n(n-2)}{4},
\end{align*}
\]
with $m$ being a natural number. In the following, we fix $m$.

For a half-integer $\mu$, by (61) in p. 230 in Hobson [4], we see that
\[
Q_\nu^\mu(t) = c(\mu, \nu)P_{\nu}^{-\mu}(t)
\]
where $c(\mu, \nu)$ is a constant.

Thus the eigenfunction $\mathcal{P}$ has the form
\[
\begin{align*}
\mathcal{P}(\cos \theta) &= C_1 P_\nu^\mu(\cos \theta) + C_2 P_{\nu}^{-\mu}(\cos \theta) \\
\mathcal{P}(\cos \theta) &= C_1 P_\nu^\mu(\cos \theta) + C_2 Q_\nu^\mu(\cos \theta)
\end{align*}
\]
and we determine the relation between $C_1$ and $C_2$ then we do the behavior of $\nu$ from (2.2). In the case of a spherical cap, we use one of $P_\nu^\mu$ or $Q_\nu^\mu$ since one of them always has a singularity at $t = 1$ (at the North Pole). In this case, we need both of them to represent the eigenfunctions. According to the boundary conditions, $\nu$ is determined and hence the eigenvalues are done. In the following sections, we investigate the behavior of $\nu$ and give a proof of Theorem 1.1.

3. Basic strategy to a proof of Theorem 1.1. In this section we discuss the behavior of the eigenvalues as $\eta \to 0$ and $\epsilon \to 0$. We compute the eigenvalues by carefully analyzing the associated Legendre functions introduced in Section 2.

Before going into details, we use the lemma which assures that the eigenvalues depend continuously on $\epsilon$ and $\eta$. The argument here is identical to that in [5] so we omit the proof.

**Lemma 3.1.** For each $k$, $n \in \mathbb{N}$ with $n \geq 2$ and $m = 0, 1, \ldots, k$, there exist $\eta_0(n,k,m)$ and $\epsilon_0(n,k,m) > 0$ such that $(k+1)$ distinct eigenvalues
\[
\lambda_{k,m,\eta,\epsilon} = (k + \zeta_{k,m}(\eta, \epsilon))(k + n - 1 + \zeta_{k,m}(\eta, \epsilon))
\]
to (1.1) near $k(k+n-1)$ exist and the term $\zeta_{k,m}(\eta, \epsilon)$ is uniquely determined for any $\eta \in (0, \eta_0(n,k,m))$ and for any $\epsilon \in (0, \epsilon_0(n,k,m))$.

Since the existence of eigenvalues is ensured by Lemma 3.1, we concentrate on the asymptotic behavior of $\nu$ as $\eta \to 0$ and as $\epsilon \to 0$. 

Similarly, from (7.4) in Section 7, for even and
and

\begin{align*}
\sin^2 \nu \{ C_1(P^\mu_\nu)'(\cos \eta) + C_2(P^{-\mu}_\nu)'(\cos \eta) \}
&= -\frac{n-2}{2} \{ C_1P^\mu_\nu(\cos \eta) + C_2P^{-\mu}_\nu(\cos \eta) \} \cos \eta
\end{align*}

and

\begin{align*}
\sin^2 \varepsilon \{ C_1(P^\mu_\nu)'(\cos(\pi - \varepsilon)) + C_2(P^{-\mu}_\nu)'(\cos(\pi - \varepsilon)) \}
&= \frac{n-2}{2} \{ C_1P^\mu_\nu(\cos(\pi - \varepsilon)) + C_2P^{-\mu}_\nu(\cos(\pi - \varepsilon)) \} \cos \varepsilon.
\end{align*}

for odd \(n\). For \(n\) even, we have

\begin{align*}
\sin^2 \eta \{ C_1(P^\mu_\nu)'(\cos \eta) + C_2(Q^\mu_\nu)'(\cos \eta) \}
&= -\frac{n-2}{2} \{ C_1P^\mu_\nu(\cos \eta) + C_2Q^\mu_\nu(\cos \eta) \} \cos \eta
\end{align*}

and

\begin{align*}
\sin^2 \varepsilon \{ C_1(P^\mu_\nu)'(\cos(\pi - \varepsilon)) + C_2(Q^\mu_\nu)'(\cos(\pi - \varepsilon)) \}
&= \frac{n-2}{2} \{ C_1P^\mu_\nu(\cos(\pi - \varepsilon)) + C_2Q^\mu_\nu(\cos(\pi - \varepsilon)) \} \cos \varepsilon.
\end{align*}

Using the recursion formula (7.3) in Section 7 for odd \(n\), we have

\begin{align*}
\left\{ (\nu + \frac{n}{2}) \cos \eta P^\mu_\nu(\cos \eta) - (\nu - \mu + 1)P^\mu_{\nu+1}(\cos \eta) \right\} C_1
&= -\left\{ (\nu + \frac{n}{2}) \cos \eta P^{-\mu}_\nu(\cos \eta) - (\nu + \mu + 1)P^{-\mu}_{\nu+1}(\cos \eta) \right\} C_2 \quad (3.1)
\end{align*}

and

\begin{align*}
\left\{ -(\nu + \frac{n}{2}) \cos \varepsilon P^\mu_\nu(\cos(\pi - \varepsilon)) - (\nu - \mu + 1)P^\mu_{\nu+1}(\cos(\pi - \varepsilon)) \right\} C_1
&= \left\{ (\nu + \frac{n}{2}) \cos \varepsilon P^{-\mu}_\nu(\cos(\pi - \varepsilon)) + (\nu + \mu + 1)P^{-\mu}_{\nu+1}(\cos(\pi - \varepsilon)) \right\} C_2 \quad (3.2)
\end{align*}

Similarly, from (7.4) in Section 7, for even \(n\) we have

\begin{align*}
\left\{ (\nu + \frac{n}{2}) \cos \eta P^\mu_\nu(\cos \eta) - (\nu - \mu + 1)P^\mu_{\nu+1}(\cos \eta) \right\} C_1
&= -\left\{ (\nu + \frac{n}{2}) \cos \eta Q^\mu_\nu(\cos \eta) - (\nu - \mu + 1)Q^\mu_{\nu+1}(\cos \eta) \right\} C_2 \quad (3.3)
\end{align*}

and

\begin{align*}
\left\{ -(\nu + \frac{n}{2}) \cos \varepsilon P^\mu_\nu(\cos(\pi - \varepsilon)) - (\nu - \mu + 1)P^\mu_{\nu+1}(\cos(\pi - \varepsilon)) \right\} C_1
&= \left\{ (\nu + \frac{n}{2}) \cos \varepsilon Q^\mu_\nu(\cos(\pi - \varepsilon)) + (\nu - \mu + 1)Q^\mu_{\nu+1}(\cos(\pi - \varepsilon)) \right\} C_2 \quad (3.4)
\end{align*}

In the following three sections, we first deduce the relation between \(C_1\) and \(C_2\) and then determine the behavior of eigenvalues by investigating the behavior of \(\nu\) as \(\eta\) and \(\varepsilon\) go to zero.
4. Proof of Theorem 1.1: The odd dimensional case. In the proof, we use the symbol \( f(x) \approx g(x) \) in the sense that at least two leading terms are the same. That is, suppose that functions \( f(x) \) and \( g(x) \) defined for \( x > 0 \) are expanded near \( x = 0 \) as

\[
    f(x) = a_1 x^p + a_2 x^q + f_1(x), \quad g(x) = a_1 x^p + a_2 x^q + g_1(x)
\]

with

\[
    \lim_{x \to +0} x^{-\min(p,q)} f_1(x) = \lim_{x \to +0} x^{-\min(p,q)} g_1(x) = 0,
\]

where \( a_1, a_2, p, q \) are constants. In such a case, we write \( f(x) \approx g(x) \). If \( g_1 \equiv 0 \), \( g(x) \) is regarded as the second approximation of \( f(x) \) near \( x = 0 \). Hereafter, we do not use the symbol like \( o(x^q) \).

First, we calculate the ratio between \( C_1 \) and \( C_2 \) from (3.1). We use the expression of \( P_0^\mu \) in terms of the Gauss hypergeometric function, which can be found in p. 319 of [2]

\[
P_0^\mu(t) = \frac{1}{\Gamma(1 - \mu)} \left( \frac{1 + t}{1 - t} \right)^{\mu/2} F(-\nu, \nu + 1, 1 - \mu; \frac{1 - t}{2}), \quad t \in (-1, 1).
\]

(4.1)

Since \( \eta \) is close to 0, we may regard \( \cos \eta = 1 - \eta^2/2 \) and put \( \mu = m + (n - 2)/2 \). Then we have

\[
P_0^{m + \frac{n - 2}{2}} (\cos \eta) \approx \frac{2^{m + \frac{n - 2}{2}}}{\Gamma(2 - m - \frac{n}{2}) \eta^{m + \frac{n - 2}{2}}} \left( 1 - \frac{\nu(\nu + 1)}{8 - 4m - 2n \eta^2} \right).
\]

and similarly, we get

\[
P_0^{m + \frac{n - 2}{2} + \frac{2}{n}} (\cos \eta) \approx \frac{2^{m + \frac{n - 2}{2}}}{\Gamma(2 - m - \frac{n}{2}) \eta^{m + \frac{n - 2}{2} + \frac{2}{n}}} \left( 1 - \frac{\nu(\nu + 1)}{8 - 4m - 2n \eta^2} \right).
\]

As for the negative super suffix, we have

\[
P_0^{-m - \frac{n - 2}{2}} (\cos \eta) \approx \frac{\eta^{m + \frac{n - 2}{2}}}{\Gamma(m + \frac{n}{2}) 2^{m + \frac{n - 2}{2}}} F(-\nu, \nu + 1, m + \frac{n}{2}; \frac{1}{4} \eta^2)
\]

\[
\approx \frac{\eta^{m + \frac{n - 2}{2}}}{\Gamma(m + \frac{n}{2}) 2^{m + \frac{n - 2}{2}}} \left( 1 - \frac{\nu(\nu + 1)}{m + \frac{n}{2} \eta^2} \right)
\]

and

\[
P_0^{-m - \frac{n - 2}{2} + \frac{2}{n}} (\cos \eta) \approx \frac{\eta^{m + \frac{n - 2}{2} + \frac{2}{n}}}{\Gamma(m + \frac{n}{2}) 2^{m + \frac{n - 2}{2} + \frac{2}{n}}} \left( 1 - \frac{\nu(\nu + 1)}{m + \frac{n}{2} \eta^2} \right).
\]

Hence we obtain

\[
\frac{2^{m + \frac{n - 2}{2}}}{\Gamma(2 - m - \frac{n}{2}) \eta^{m + \frac{n - 2}{2}}} \left( \nu + \frac{n}{2} \right)(\nu^2)(1 - \frac{\eta^2}{2}) \left( 1 - \frac{\nu(\nu + 1)}{8 - 4m - 2n \eta^2} \right)
\]

\[
\frac{2^{m + \frac{n - 2}{2}}}{\Gamma(2 - m - \frac{n}{2}) \eta^{m + \frac{n - 2}{2} + \frac{2}{n}}} \left( \nu + \frac{n}{2} \right)(\nu^2) \left( 1 - \frac{\nu(\nu + 1)}{8 - 4m - 2n \eta^2} \right).
\]
Substituting (4.3) for (3.2), we see that

\[-(\nu - m - \frac{n}{2} + 2) \left( 1 - \frac{(\nu + 1)(\nu + 2)}{8 - 4m - 2n} \right) \right\} C_1

\[
\times \frac{\eta^{m+n+\frac{n}{2}}}{{\Gamma(m + \frac{n}{2})2^{m+n+\frac{n}{2}}} \left\{ - (\nu + \frac{n}{2})(1 - \frac{\eta^2}{2}) \left( 1 - \frac{\nu(\nu + 1)}{4m + 2n} \eta^2 \right) 
\right. 
\left. + (\nu + m + \frac{n}{2}) \left( 1 - \frac{(\nu + 1)(\nu + 2)}{4m + 2n} \eta^2 \right) \right\} C_2.
\]

Simplifying the relation above, we have

\[
\left\{ m + n - 2 + \frac{2(m + n - 4)\nu(\nu + 1) - (3n^2 - 6m + 8)\eta^2}{4(n + 2m - 4)} \right\} C_1
\]

\[
\times \left\{ \frac{\Gamma(2 - m - \frac{n}{2})\eta^{2m+n-2}}{\Gamma(m + \frac{n}{2})2^{2m+n-2}} \left\{ m - \frac{(m + 2)\nu(\nu + 1) + (n + 2m)\eta^2}{2(n + 2m)} \right\} \right\} C_2.
\]

Using the relation (7.6) in Section 7, we have

\[
\frac{\Gamma(2 - m - \frac{n}{2})}{\Gamma(m + \frac{n}{2})} = (-1)^{m+n+\frac{n}{2}+1}2^{m+2n-3}\left( \frac{(2m + n - 2)!}{(m + n - 2)!} \right)^2 \]

\[
= C_0(m).
\]

Hence we see that

\[
C_1 \asymp (-1)^{m+n+\frac{n}{2}+1}2^{2m+n-1}m(2m + n - 2)\left( \frac{(m + n - 1)!}{(2m + n - 1)!} \right)^2 \eta^{2m+n-2}C_2
\]

if \( m \geq 1 \).

If \( m = 0 \), we have

\[
\nu = k + \frac{n - 2}{2} + \nu_{q,\epsilon}, \quad \nu_{q,\epsilon} \to 0 \text{ as } \eta, \epsilon \to 0
\]

and we see that

\[
C_1 \asymp (-1)^{n-1}2^{n-3}\left( \frac{(n-1)!}{(n-1)!} \right)^2 \eta^nC_2.
\]

For the moment, we do not need the exact value, we write

\[
C_1 \asymp K\eta^{\max\{2m+n-2,n\}}C_2
\]  

(4.3)

with \( K \) as

\[
K = \begin{cases} 
(1)^{m+n+\frac{n}{2}}2^{2m+n-1}m(2m + n - 2)\left( \frac{(m + n - 1)!}{(2m + n - 1)!} \right)^2 \text{ if } m \geq 1, \\
(-1)^{n+1}2^{n-3}\left( \frac{(n-1)!}{(n-1)!} \right)^2 \eta^n \text{ if } m = 0.
\end{cases}
\]

Substituting (4.3) for (3.2), we see that

\[
K\eta^{\max\{2m+n-2,n\}} \times
\left\{ -(\nu + \frac{n}{2}) \cos P_{\nu}^\epsilon (\cos(\pi - \epsilon)) - (\nu - \mu + 1)P_{\nu}^\epsilon (\cos(\pi - \epsilon)) \right\}
\]

\[
\asymp (\nu + \frac{n}{2}) \cos P_{\nu}^-\epsilon (\cos(\pi - \epsilon)) + (\nu + \mu + 1)P_{\nu+1}^-\epsilon (\cos(\pi - \epsilon)) =: A.
\]
We recall that we have defined \( \mu = m + (n - 2)/2 \) and denote \( n_m := \max\{2m + n - 2, n\} \).

In the odd dimensional case, however, the Legendre function \( P_{\mu}^\nu(t) \) has a singularity at \( t = -1 \) and we need to use the conversion formula (7.2) in Section 7 and the relation between the associated Legendre function and the Gauss hypergeometric functions (4.1) to analyze the behavior of \( P_{\mu}^\nu \) near the singularity.

In terms of the Gauss hypergeometric functions, the right-hand side of (4.4), which is denoted by \( A \), is expressed as

\[
A = \left( \frac{1 + \cos(\pi - \varepsilon)}{1 + \cos(\pi - \varepsilon)} \right)^{-\frac{n}{2}} \left\{ \left( \nu + \frac{n}{2} \right) \cos \varepsilon \times 
F(-\nu, \nu + 1, 1 + \mu; \frac{1 - \cos(\pi - \varepsilon)}{2}) 
+ (\nu + \mu + 1) F(-\nu - 1, \nu + 2, 1 + \mu; \frac{1 - \cos(\pi - \varepsilon)}{2}) \right\} \frac{1}{\Gamma(1 + \mu)}
\]

(4.5)

According to the conversion formulas (7.2) in Section 7, we see from the right-hand side of (4.4) that

\[
A = \left( \frac{1 + \cos(\pi - \varepsilon)}{1 + \cos(\pi - \varepsilon)} \right)^{-\frac{n}{2}} \left\{ \left( \nu + \frac{n}{2} \right) \cos \varepsilon \times 
F(-\nu, \nu + 1, 1 - \mu; \frac{1 + \cos(\pi - \varepsilon)}{2}) 
+ \frac{\Gamma(-\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left( \frac{1 + \cos(\pi - \varepsilon)}{2} \right)^{\mu} \times 
F(\mu + \nu + 1, \mu - \nu + 1, 1 + \mu; \frac{1 + \cos(\pi - \varepsilon)}{2}) \right\} \frac{1}{\Gamma(1 + \mu)}
\]

Also, from the left-hand side of (4.4), we have

\[
A \approx K \eta^{n_m} \left( \frac{1 + \cos(\pi - \varepsilon)}{1 - \cos(\pi - \varepsilon)} \right)^{n/2} \left[ - (\nu + \frac{n}{2}) \cos \varepsilon \times 
\left\{ \frac{\Gamma(-\mu)}{\Gamma(1 - \mu + \nu)\Gamma(-\mu - \nu)} F(-\nu, \nu + 1, 1 + \mu; \frac{1 + \cos(\pi - \varepsilon)}{2}) 
+ \frac{\Gamma(\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left( \frac{1 + \cos(\pi - \varepsilon)}{2} \right)^{-\mu} \times 
\right\} \frac{1}{\Gamma(1 + \mu)} \right]
\]
\begin{equation}
\sum_{k=0}^{\infty} \int_0^1 \left[ 1 - \sum_{k=0}^{\infty} \frac{(\tau \lambda)^k}{k!} \right] \frac{e^{-\lambda t}}{\lambda t} dt = \sum_{k=0}^{\infty} \frac{(\tau \lambda)^k}{k!} \int_0^1 e^{-\lambda t} \frac{dt}{\lambda t}.
\end{equation}
\[
\begin{aligned}
+ \frac{\Gamma(-\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 1)\Gamma(\mu - \nu - 1)} F(\mu + \nu + 1, \mu - \nu + 1, 1 + \mu; \frac{\varepsilon^2}{4}) \right. \\
+ (\nu + \mu + 1) \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 2)\Gamma(\mu - \nu - 1)} F(\mu + \nu + 2, 1 - \mu; \frac{\varepsilon^2}{4}) \right. \\
+ \frac{\Gamma(-\mu)}{\Gamma(-\nu - 1)\Gamma(\nu + 2)} \left( \frac{\varepsilon^2}{4} \right)^\mu F(\mu + \nu + 2, \mu - \nu - 1, 1 + \mu; \frac{\varepsilon^2}{4}) \left\} \times \frac{1}{\Gamma(1 + \mu)}.
\end{aligned}
\]

Hence we obtain
\[
K\eta^{\mu \nu m} \left[ -(\nu + \frac{n}{2}) \left( \frac{\varepsilon^2}{4} \right)^{\mu/2} \left\{ \frac{\Gamma(-\mu)}{\Gamma(1 - \mu + \nu)\Gamma(-\mu - \nu)} \right. \\
+ \frac{\Gamma(\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} \left\} \right. \\
- (\nu - \mu + 1) \left( \frac{\varepsilon^2}{4} \right)^{\mu/2} \left\{ \frac{\Gamma(-\mu)}{\Gamma(2 - \mu + \nu)\Gamma(-\mu - \nu - 1)} \right. \\
+ \frac{\Gamma(\mu)}{\Gamma(-\nu - 1)\Gamma(\nu + 2)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} \left\} \right. \\
\times \left( \frac{\varepsilon^2}{4} \right)^{-\mu/2} \left[ (\nu + \frac{n}{2}) \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 1)\Gamma(\mu - \nu)} \right. \\
+ \frac{\Gamma(-\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left( \frac{\varepsilon^2}{4} \right)^{\mu} \left\} \right. \\
+ (\nu + \mu + 1) \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 2)\Gamma(\mu - \nu - 1)} \right. \\
+ \frac{\Gamma(-\mu)}{\Gamma(-\nu - 1)\Gamma(\nu + 2)} \left( \frac{\varepsilon^2}{4} \right)^{\mu} \left\} \right. \left] \frac{1}{\Gamma(1 + \mu)}. \right)
\]

We multiply both sides of (4.7) by \((\varepsilon^2/4)^{-\mu/2}\), we have
\[
K\eta^{\mu \nu m} \left[ -(\nu + \frac{n}{2}) \left\{ \frac{\Gamma(-\mu)}{\Gamma(1 - \mu + \nu)\Gamma(-\mu - \nu)} \right. \\
+ \frac{\Gamma(\mu)}{\Gamma(-\nu)\Gamma(\nu + 1)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} \left\} \right. \\
- (\nu - \mu + 1) \left\{ \frac{\Gamma(-\mu)}{\Gamma(2 - \mu + \nu)\Gamma(-\mu - \nu - 1)} \right. \\
+ \frac{\Gamma(\mu)}{\Gamma(-\nu - 1)\Gamma(\nu + 2)} \left( \frac{\varepsilon^2}{4} \right)^{\mu} \left\} \right. \left] \frac{1}{\Gamma(1 + \mu)}. \right)
\]
prove (iii) and (iv) in Remark 1.

The constant \( \tilde{c} \) when the \( \Gamma(\mu) \) converges to 0 as \( \epsilon \to 0 \) can be calculated as in Theorem 1.1 in [1] by letting the left-hand side of (4.8) be zero and we have

\[
\frac{\Gamma(\mu)}{\Gamma(-\nu-1)\Gamma(\nu+2)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} \right] \frac{1}{\Gamma(1+\mu)}
\]

\[
\times \left[ \left( \nu + \frac{n}{2} \right) \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 1)\Gamma(\mu - \nu)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} + \frac{\Gamma(-\mu)}{\Gamma(-\nu)\Gamma(\nu+1)} \right\} + (\nu + \mu + 1) \left\{ \frac{\Gamma(\mu)}{\Gamma(\mu + \nu + 2)\Gamma(\mu - \nu - 1)} \left( \frac{\varepsilon^2}{4} \right)^{-\mu} \right. \right.
\]

\[
\left. \left. + \frac{\Gamma(-\mu)}{\Gamma(-\nu-1)\Gamma(\nu+2)} \right\} \right] \frac{1}{\Gamma(1+\mu)}.
\]

As before we recall \( \nu = k + (n - 2)/2 + \nu_{\eta,\epsilon} \) with \( \nu_{\eta,\epsilon} \to 0 \) as \( \eta, \epsilon \to 0 \) and \( \mu = m + (n - 2)/2 \). Then we see that

\[
\mu - \nu = m - k - \nu_{\eta,\epsilon}, \quad \mu + \nu = m + k + n - 2 + \nu_{\eta,\epsilon} > 0.
\]

We note the fact that the Gamma function has a pole (see Lemma 7.1) of order one for \( m \leq k \) and that the following holds

\[
\lim_{\delta \to +0} \delta \Gamma(m - k - \delta) = \frac{(-1)^{k-m+1}}{(k-m)!},
\]

Thus, the terms having \( \Gamma(m - \nu) \), \( \Gamma(-\mu - \nu - 1) \) in the left-hand side of (4.8) converges to 0 as \( \varepsilon \to 0 \). On the other hand, as we have seen in [1], only the case when the \( \Gamma(m - \nu) \) and \( \varepsilon^{-2\mu} \) are of the same order is admissible. If \( \eta \to 0 \) in such a way as \( \eta/\varepsilon \to 0 \) as \( \varepsilon \to 0 \), then the left-hand side of (4.8) converges to 0 when \( m \geq 1 \), while the right-hand side of (4.8) is of order \( \varepsilon^2 \) when \( m = 0 \) as in Subsection 3.1 in [1]. Even in this case, if \( \eta/\varepsilon \to 0 \) as \( \varepsilon \to 0 \), then the left-hand side is of smaller order of \( \varepsilon^2 \). Thus, we conclude that \( \nu_\varepsilon \) has the same order as shown in Subsection 3.1 of [1] and the leading term depends only on \( \varepsilon \) and is determined as

\[
\nu_{\eta,\epsilon} = c_{k,m,n}\varepsilon^{\max\{2m+n-2,n\}}.
\]

The constant \( c_{k,m,n} \) is calculated as in Theorem 1.1 in [1] by letting the left-hand side of (4.8) be zero and we have

\[
c_{k,m,n} = \begin{cases} 
-\frac{2^n m(n - m)(2m - 1)!(2m - n)!}{n(2m - n - 1)!} & \text{if } m \geq 1, \\
\frac{2^{n-2}(n-1)!n!}{n(2m - n - 1)!} & \text{if } m = 0.
\end{cases}
\]

\[
(4.9)
\]

Remark 2. Since the Gauss hypergeometric functions are expanded in power series, we can expand (4.5) and (4.6) more. If \( \eta = o(\varepsilon^2) \), then we will see that

\[
\nu_{\eta,\epsilon} = c_{k,m,n}\varepsilon^{\max\{2m+n-2,n\}} + \tilde{c}_{k,m,n}\varepsilon^{\max\{2m+n-2,n\}+2} + o(\varepsilon^{\max\{2m+n-2,n\}+2})
\]

The constant \( \tilde{c}_{k,m,n} \) is determined in the similar way. Thus in principle, we can prove (iii) and (iv) in Remark 1.
What is left is to study the case when $\eta = \kappa \varepsilon + o(\varepsilon)$.

Substituting $\nu_{\eta, \varepsilon} = c_{k,m,n} \varepsilon^{\max\{(2m+n-2,n)\}}(1 + o(1))$ for (4.8) if $m \geq 1$, we have

$$K_{\kappa}^{n} 2^{2m+n-2} \Gamma(m + \frac{n-2}{2}) \times$$

$$\left\{ -\frac{k+n-1}{\Gamma(-k - \frac{n}{2} + 1)\Gamma(k + \frac{n}{2})} + \frac{m-k-1}{\Gamma(-k - \frac{n}{2})\Gamma(k + \frac{n}{2} + 1)} \right\} \Gamma(2 - m - \frac{n}{2})$$

$$= (-1)^{k-m+1} \Gamma(m + \frac{n}{2} - 1) \left\{ \frac{(k+n-1)(k-m)!}{(m+k+n-2)!} \right\} - \frac{(k+1-m)!}{(m+k+n-2)!} c_{k,m,n}$$

$$+ 2^{-(2m+n-2)} \Gamma(-m - \frac{n}{2} + 1) \left\{ \frac{k+n-1}{\Gamma(-k - \frac{n}{2} + 1)\Gamma(k + \frac{n}{2})} \Gamma(m + \frac{n}{2} - 1) \right\}$$

$$\left\{ \frac{k+m+n-1}{\Gamma(-k - \frac{n}{2})\Gamma(k + \frac{n}{2} + 1)} \right\}.$$

Hence, the equality above is written as

$$K_{\kappa}^{n} 2^{2m+n-2} \frac{\Gamma(m + \frac{n}{2} - 1)(m+n-2)}{\Gamma(k + \frac{n}{2})\Gamma(-k - \frac{n}{2})(k + \frac{n}{2})} \frac{\Gamma(2 - m - \frac{n}{2})}{\Gamma(m + \frac{n}{2})}$$

$$- 2^{-(2m+n-2)} \frac{\Gamma(-m - \frac{n}{2} + 1)m}{\Gamma(-k - \frac{n}{2})\Gamma(k + \frac{n}{2})(k + \frac{n}{2})}$$

$$= (-1)^{k-m+1} \Gamma(m + \frac{n}{2} - 1) \frac{(k-m)!(n+m-2)}{(m+k+n-2)!} c_{k,m,n,\kappa}.$$

In view of the famous formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad (4.10)$$

we see that

$$\Gamma(k + \frac{n}{2})\Gamma(-k - \frac{n}{2}) = \frac{(-1)^{k+(n+1)/2}\pi}{k + \frac{n}{2}}. \quad (4.11)$$

Thus, we get

$$\frac{(-1)^{k+(n+1)/2}}{\pi} \left\{ KC_{0}(m)k^{n} 2^{2m+n-2} \Gamma(m + \frac{n}{2} - 1)(m+n-2) \right.$$

$$- 2^{-(2m+n-2)} \Gamma(-m - \frac{n}{2} + 1)m \right\}$$

$$= (-1)^{k-m+1} \Gamma(m + \frac{n}{2} - 1) \frac{(k-m)!(n+m-2)}{(m+k+n-2)!} c_{k,m,n,\kappa}.$$

Substituting (4.2) for $C_{0}(m)$, we obtain

$$c_{k,m,n,\kappa} = \frac{(-1)^{\frac{n+1}{2}+m}2^{2m+n-4}(m+k+n-2)!}{(k-m)!\pi} \times$$
Remark 3. We see that the odd dimensional case is complete.

When \( m = 0 \). We may assume that \( \nu_{\eta, \varepsilon} = c_{k, 0, n, \kappa} \varepsilon^n (1 + o(1)) \). Dividing both side of (4.8) by \( \varepsilon^n \) and taking the limit, we have

\[
\frac{(-1)^{k+(n-1)/2}2^{n-2}}{\pi} KC_0(0) \kappa^n (n-2) ! \Gamma \left( \frac{n}{2} - 1 \right)
\]

\[\] \]

In view of (7.6) in Section 7, we see that

\[
\Gamma \left( \frac{n}{2} \right) = \frac{(n-1)!}{2^{n-1}(\frac{n-1}{2})!} \sqrt{\pi}, \quad \Gamma \left( \frac{n}{2} - 1 \right) = \frac{(n-1)!}{2^{n-2}(n-2)(\frac{n-1}{2})!} \sqrt{\pi}.
\]

Again from (4.2), we have

\[
C_0(0) = (-1)^{\frac{n-1}{2}} 2^{2n-3} (n-2) ! \left\{ \frac{(n+1)!}{(n-1)!} \right\}^2.
\]

Using (4.10) and (4.11), we obtain

\[
c_{k, 0, n, \kappa} = \left\{ (-1)^{\frac{n-1}{2}} 2^{2n+3} \frac{n(n-2)}{\sqrt{\pi}} (\frac{n+1}{2})! K \kappa^n + (-1)^{\frac{k}{2}} 2^{2n-3} (\frac{n-1}{2})! \right\} \times \frac{2^{2n-3} (n-2) (n+1)!}{(n-1)!}.
\]

The odd dimensional case is complete.

Remark 3. We see that \( c_{k, 1, n, \kappa} \neq c_{k, 0, n, \kappa} \) and that \( \lim_{\kappa \to +0} c_{k, m, n, \kappa} = c_{k, m, n} \).

5. Proof of Theorem 1.1: In the case of the even dimensional case (except for \( n = 2 \) and \( m = 0 \)). In this section, we put

\[
M = m + \frac{n-2}{2}, \quad \nu = k + \frac{n-2}{2} + \nu_{\eta, \varepsilon}
\]

with \( \nu_{\eta, \varepsilon} \to 0 \) as \( \nu \to 0 \), \( \varepsilon \to 0 \). If \( n \) is even, \( M \) is an integer and \( \nu \) converges to an integer. This causes difficulty. In this case, \( Q^M_{\nu}(t) \) is expressed as

\[
Q^M_{\nu}(t) = \frac{\pi}{2 \sin(M + \nu) \pi} \left\{ \cos(M + \nu) \pi P^M_{\nu}(t) - P^M_{\nu}(-t) \right\}.
\]

We recall that

\[
P^M_{\nu}(t) = \frac{\Gamma(1+\nu+M)}{\Gamma(1+\nu-M)}(1-\nu^2)^{M/2} F(M-\nu, M+\nu+1, 1 + M; \frac{1-t}{2}).
\]

Since \( 1 + M - (M-\nu) = M + \nu + 1 < 0 \), this solution is singular at \( t = -1 \). Thus \( Q^M_{\nu}(t) \) has singularities at \( t = \pm 1 \). As in [1], we effectively use the function \( U \) defined as below.
Let $\alpha, \beta$ be non-integer values and $\ell$ be a positive integer. We define the new function $U(\alpha, \beta, \ell, x)$ as follows

\[
U(\alpha, \beta, \ell; x) = (-1)^\ell \frac{\Gamma(\alpha + 1 - \ell)\Gamma(\beta + 1 - \ell)(\ell - 1)!}{\Gamma(\alpha)\Gamma(\beta)} \times
\]

\[
F(\alpha, \beta, \ell; x) \log x
\]

\[
+ \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta)_i}{(\ell)_i i!} \{\psi(\alpha + i) + \psi(\beta + i) - \psi(i + 1) - \psi(\ell + i)\} x^i
\]

\[
+ \frac{(\ell - 2)!}{\Gamma(\alpha)\Gamma(\beta)} x^{1-i} \sum_{i=0}^{\ell-2} \frac{(\alpha + 1 - \ell)_i(\beta + 1 - \ell)_i}{(2 - \ell)_i i!} x^i,
\]

where $\psi(z)$ is the psi (or di-Gamma) function, whose properties are listed in Lemmas 7.1 and 7.2 in Section 7, defined as

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},
\]

$(z)_0 = 1$ and $(z)_i = z(z + 1) \cdots (z + i - 1)$ with $i \in \mathbb{N}$. The last term in (5.3) does not exist if $\ell = 0, 1$. Then the conclusion in [2] in p. 276 is

\[
F(\alpha, \beta, \ell; x) = \Gamma(\ell)U(\alpha, \beta; \alpha + \beta + 1 - \ell; 1 - x)
\]

(5.4)

provided $\alpha + \beta + 1 - \ell$ is a non-positive integer.

Using formulas above, we calculate

\[
(\nu + n/2)(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta).
\]

Since $\eta > 0$ is close to 0, we have

\[
(\nu + \frac{n}{2})(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta)
\]

\[
\approx (\nu + \frac{n}{2})(1 - \frac{\eta^2}{2})\frac{\pi}{2\sin(M + \nu)\pi} \left\{ (\cos(M + \nu)\pi)P_{\nu}^M(\cos \eta) - P_{\nu+1}^M(-\cos \eta) \right\}
\]

\[
- (\nu - M + 1)\frac{\pi}{2\sin(M + \nu + 1)\pi} \times
\]

\[
\left\{ (\cos(M + \nu + 1)\pi)P_{\nu+1}^M(\cos \eta) - P_{\nu+1}^M(-\cos \eta) \right\}.
\]

Now we write $(\nu + n/2)(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta)$ by using (5.2). We see

\[
(\nu + \frac{n}{2})(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta)
\]

\[
\approx \frac{\pi}{2\sin(M + \nu)\pi} \left[ (\nu + \frac{n}{2})(1 - \frac{\eta^2}{2})\frac{\Gamma(M + \nu + 1)}{\Gamma(-M + \nu + 1)}\sin \eta \right.\]

\[
\times \left\{ (\cos(M + \nu)\pi)P_{\nu}^M(\cos \eta) - P_{\nu+1}^M(-\cos \eta) \right\}.
\]
\[
F(M - \nu, M + \nu + 1, M + 1; \frac{1 - \cos \eta}{2})
\]

\[
- \frac{\Gamma(M + \nu + 1)}{\Gamma(-M + \nu + 1)} (\sin^M \eta) F(M - \nu, M + \nu + 1, M + 1; \frac{1 + \cos \eta}{2})
\]

\[
+(\nu - M + 1) \left\{ - (\cos(M + \nu)\pi) \frac{\Gamma(M + \nu + 2)}{\Gamma(-M + \nu + 2)} (\sin^M \eta) \times
\right.
\]

\[
F(M - \nu - 1, M + \nu + 2, M + 1; \frac{1 - \cos \eta}{2})
\]

\[
- \frac{\Gamma(M + \nu + 2)}{\Gamma(-M + \nu + 2)} (\sin^M \eta) F(M - \nu - 1, M + \nu + 2, M + 1; \frac{1 + \cos \eta}{2})
\}\].

\(\vdash B\)

(5.5)

As before, we have

\[
F(M - \nu, M + \nu + 1, M + 1; \frac{1 - \cos \eta}{2}) \approx
\]

\[
1 + \frac{(M - \nu)(M + \nu + 1) \eta^2}{M + 1}
\]

\[
F(M - \nu - 1, M + \nu + 2, M + 1; \frac{1 - \cos \eta}{2}) \approx
\]

\[
1 + \frac{(M - \nu - 1)(M + \nu + 2) \eta^2}{M + 1}
\]

Also, by (5.4), we see

\[
F(M - \nu, M + \nu + 1, M + 1; \frac{1 + \cos \eta}{2}) \approx
\]

\[
\Gamma(M + 1) U(M - \nu, M + \nu + 1, M + 1; \frac{\eta^2}{4})
\]

and

\[
F(M - \nu - 1, M + \nu + 2, M + 1; \frac{1 + \cos \eta}{2}) \approx
\]

\[
\Gamma(M + 1) U(M - \nu - 1, M + \nu + 2, M + 1; \frac{\eta^2}{4})
\]

Moreover, using (5.3), we obtain

\[
F(M - \nu, M + \nu + 1, M + 1; \frac{1 + \cos \eta}{2})
\]

\[
\approx \Gamma(M + 1) \left\{ \frac{(-1)^M}{\Gamma(-\nu)\Gamma(\nu + 1)M!} \psi(M - \nu)
\right.
\]

\[
+ \frac{(M - 1)!}{\Gamma(M - \nu)\Gamma(M + \nu + 1)} \left( \frac{\eta^2}{4} \right)^{-M} \}
\]
As we treat the even dimensional case, we substitute $M$ and we see that $B$ right-hand side of (5.5) always substitute those expressions for $M = \nu$ with $k \geq m$ and with $\tilde{\nu} \to 0$. In the following sections, we always substitute those expressions for $M$ and $\nu$, respectively. Now we evaluate the right-hand side of (5.5) $B$:

$$B \geq \frac{(-1)^{k+m+n-2}(k+m+n-2)!}{2(k-m)!\tilde{\nu}} \times$$

$$\left\{ \begin{align*}
(k+n-1)(1-\frac{\eta^2}{2})\left( (-1)^{k+m+n-2}\eta^{m+n-2}\left( 1 + \frac{(m-k)(k+m+n-1)}{2(2m+n)} \right) \eta^2 \\
-\frac{(m+n-2)!}{2}\eta^{m+n-2} \times \right. \\
\left. \frac{(-1)^{k+m+n+1}(m+n-2)!}{(k+m+n-1)!} \left( 1 + \frac{(k-m-1)(k+m+n)}{2(2m+n)} \eta^2 \right) \right. \\
\left. -\frac{k+m+n-1}{k-m+1}\left( m+n-2 \right)!\eta^{m+n-2}/2 \times \right. \\
\left. \frac{2^{2m+n-2}(-1)^{k-m+1}(m-n-2)!\tilde{\nu}}{(k+m-n-1)!} \eta^{2m+n-2} \right\} 
\right\}$$

and we see that

$$B \geq \frac{(-1)^{k+m+n-2}(k+m+n-2)!\eta^{m+n-2}}{2(k-m)!\tilde{\nu}} \left\{ (-1)^{k+m+n-1}\mu \\
-(-1)^{k+m+n-2}\left\{ \frac{k+n-1}{2}\frac{(m-k)(k+m+n-1)(k+n-1)}{2(2m+n)} \\
-\frac{(k-m+1)(k-m-1)(k+m+n)}{2(2m+n)} \right\} \eta^2 \\
-\frac{2^{2m+n-2}(-1)^{k-m+1}(k-m)!n+k+n-2}{2} \frac{(m+n-2)!\tilde{\nu}}{(k+m+n-2)!} \eta^{2m+n-2} \right\}$$
\[-(k + m + n - 1)(m + \frac{n}{2} - 1)! \frac{(-1)^{k-m+2}2^{m+n-2}(m + \frac{n}{2} - 2)! (k + 1 - m)!}{(k + m + n - 1)!} \times \frac{\tilde{\nu}}{\eta^{2m+n-2}}.\]

Finally, we obtain

\[B \approx \frac{(-1)^{k+m+n-2}(k + m + n - 2)! \pi}{2(k - m)! \tilde{\nu}} \eta^{m + \frac{n-2}{2}} \times \]
\[
\left[ (-1)^{k+m+n-1}m + (-1)^{k+m+n-2} \left\{ -\frac{k + n - 1}{2} \right. \right.
\]
\[
+ \frac{(m - k)(k + m + n - 1)(k + n - 1) - (k - m + 1)(k - m - 1)(k + m + n)}{2(2m + n)} \right. \eta^2
\]
\[
- \frac{(-1)^{k-m+1}2^{m+n-2}(m + \frac{n}{2} - 2)! (m + \frac{n}{2} - 1)! (k - m)! (n + m - 2)! \tilde{\nu}}{(k + m + n - 2)! \eta^{2m+n-2}} \]
\[
= (K_1 m + K_2 \eta^2) \frac{\eta^{m + \frac{n-2}{2}}}{\tilde{\nu}} + K_3 \eta^{-m - \frac{n-2}{2}},
\]

where

\[K_1 = -\frac{(k + m + n - 2)! \pi}{2(k - m)!},\]

\[K_2 = -K_1 \left\{ -\frac{k + n - 1}{2} \right. \]
\[
- \frac{(m - k)(k + m + n - 1)(k + n - 1) - (k - m + 1)(k - m - 1)(k + m + n)}{2(2m + n)} \}
\[
\text{and}
\]
\[K_3 = (-1)^{n-2} \frac{2^{2m+n-3}(m + \frac{n}{2} - 2)! (m + \frac{n}{2} - 1)! (k - m)! (n + m - 2)! \pi}{(k - m)!}.
\]

Next, we calculate

\[(\nu + \frac{n}{2})(\cos \eta) P_M^\nu (\cos \eta) - (\nu - M + 1) P_M^{\nu+1} (\cos \eta).\]

We use (5.2) to express the terms by using the Gauss hypergeometric functions. We have

\[(\nu + \frac{n}{2})(\cos \eta) P_M^\nu (\cos \eta) - (\nu - M + 1) P_M^{\nu+1} (\cos \eta) \]
\[
\approx (\nu + \frac{n}{2})(1 - \frac{\eta^2}{2}) \frac{\Gamma(M + \nu + 1)}{\Gamma(-M + \nu + 1)} \eta^M F(M - \nu, M + \nu + 1, M + 1; \frac{\eta^2}{4}). \quad (5.6)
\]
\[-(\nu - M + 1) \frac{\Gamma(M + \nu + 2)}{\Gamma(-M + \nu + 2)} \eta^M F(M - \nu - 1, M + \nu + 2, M + 1; \frac{\eta^2}{4}) \]
\[
\times \frac{(m + k + n - 2)!}{(k - m)!} \eta^{m + \frac{n-2}{2}} \times
\left[ - m - \left( \frac{k + n - 1}{2} + \frac{(k + n - 1)(k - m)(m + k + n - 1)}{4m + 2n} \right) \right.
\left. + \frac{(m + k + n - 1)(k - m + 1)(m + k + n)}{4m + 2n} \right] \eta^2
\]
\[
= (K_4m + K_5\eta^2)\eta^{m + \frac{n-2}{2}}.
\]

Here we set
\[K_4 = -\frac{(m + k + n - 2)!}{(k - m)!},\]
and
\[K_5 = \left\{ \frac{k + n - 1}{2} + \frac{(k + n - 1)(k - m)(m + k + n - 1)}{4m + 2n} \right\} K_4.\]

Hence, (3.3) yields
\[(K_4m + K_5\eta^2)\eta^{m + \frac{n-2}{2}} C_1 = \left\{ (K_1m + K_2\eta^2) \eta^{m + \frac{n-2}{2}} + K_3\eta^{m - \frac{n-2}{2}} \right\} C_2.\]

Hence we obtain
\[C_1 \approx \frac{(K_1m + K_2\eta^2)\eta^{m - \frac{n-2}{2}} + K_3\eta^{m - \frac{n-2}{2}}}{K_4m + K_5\eta^2} C_2.\] (5.8)

Now, we consider (3.4). The left-hand side of (3.4) has been calculated in Subsection 3.2 in [1] and from that, we see
\[-(\nu - M + 1) (\nu \pm \frac{1}{2})(\cos \epsilon) P^M_\nu (\cos(\pi - \epsilon)) - (\nu - M + 1) P^M_\nu (\cos(\pi - \epsilon)) \]
\[
\approx (-1)^{m-k} \left\{ \frac{m}{(m + \frac{3}{2} - 1)!} \right. 
\left. + \frac{2^{m+n-2}(m + \frac{3}{2} - 2)!(k - m)!n(n - 2)!}{(m + k + n - 2)!} \frac{\tilde{\nu}}{\varepsilon^{2m+n-2}} \right\} e^{m + \frac{n-2}{2}}. \] (5.9)

Finally, we need to calculate the right-hand side of (3.4). Using (5.1) and (5.2), we first write the right-hand side of (3.4) in terms of the Gauss hypergeometric functions. We have
\[(\nu + \frac{n}{2})(\cos \epsilon) Q^M_\nu (\cos(\pi - \epsilon)) + (\nu - M + 1) Q^M_\nu (\cos(\pi - \epsilon)) \] (5.10)
Now we express \( F \) in terms of \( U \). Recalling (5.4) and (5.3), we have

\[
\begin{align*}
F &\propto \frac{(k + n - 1)(1 - \frac{\varepsilon^2}{2})}{2\sin(k + m + n - 2 + \tilde{\nu})} \left\{ (\cos(m + k + n - 2)\pi)P^{m+\frac{n-2}{2}}_\nu (-1 + \frac{\varepsilon^2}{2}) \\ &\quad - P^{m+\frac{n-2}{2}}_{\nu+1} (1 - \frac{\varepsilon^2}{2}) \right\} \\
&\quad + \frac{(k - m + 1)\pi}{2\sin(k + m + n - 1 + \tilde{\nu})} \left\{ (\cos(m + k + n - 1)\pi)P^{m+\frac{n-2}{2}}_{\nu+1} (-1 + \frac{\varepsilon^2}{2}) \\ &\quad - P^{m+\frac{n-2}{2}}_\nu (1 - \frac{\varepsilon^2}{2}) \right\} \\
&\propto \frac{(-1)^{k+m+n-2}(k + n - 1)(1 - \frac{\varepsilon^2}{2})}{2\tilde{\nu}} \left\{ (-1)^{k+m+n-2}(k + m + n - 2)! \frac{\Gamma(m - k + n + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(k - m)!} \\ &\quad - \frac{(k + m + n - 1)!}{(k - m)!} \varepsilon^{m+\frac{n-2}{2}} F(m - k, m + k + n - 1, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4}) \right\} \\
&\quad + \frac{(-1)^{k+m+n-1}(k - m + 1)\pi}{2\tilde{\nu}} \left\{ (-1)^{k+m+n-1}(k + m + n - 1)! \frac{\Gamma(m - k + n + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(k - m + 1)!} \\ &\quad - \frac{(k + m + n - 2)!}{(k - m + 1)!} \varepsilon^{m+\frac{n-2}{2}} F(m - k - 1, m + k + n, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4}) \right\} \\
&\quad \times \frac{\varepsilon^{m+n-2} \Gamma(m - k - \tilde{\nu}, m + k + n - 1 + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(m + \frac{n}{2}) \frac{\varepsilon^{2m+n-2}(m + \frac{n}{2} - 2)!}{\Gamma(m - k - \tilde{\nu}) \Gamma(m + k + n - 1)} \varepsilon^{-2m-(n-2)} \} \\
&\quad \times \frac{\varepsilon^{m+n-2} \Gamma(m - k - \tilde{\nu}, m + k + n - 1 + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(m + \frac{n}{2}) \frac{\varepsilon^{2m+n-2}(m + \frac{n}{2} - 2)!}{\Gamma(m - k - \tilde{\nu}) \Gamma(m + k + n - 1)} \varepsilon^{-2m-(n-2)} \} \\
&\quad \times \frac{\varepsilon^{m+n-2} \Gamma(m - k - \tilde{\nu}, m + k + n - 1 + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(m + \frac{n}{2}) \frac{\varepsilon^{2m+n-2}(m + \frac{n}{2} - 2)!}{\Gamma(m - k - \tilde{\nu}) \Gamma(m + k + n - 1)} \varepsilon^{-2m-(n-2)} \} \\
&\quad \times \frac{\varepsilon^{m+n-2} \Gamma(m - k - \tilde{\nu}, m + k + n - 1 + \tilde{\nu}, m + \frac{n}{2}; 1 - \frac{\varepsilon^2}{4})}{(m + \frac{n}{2}) \frac{\varepsilon^{2m+n-2}(m + \frac{n}{2} - 2)!}{\Gamma(m - k - \tilde{\nu}) \Gamma(m + k + n - 1)} \varepsilon^{-2m-(n-2)} \} \\
\end{align*}
\]

(5.11)
Let us denote
\[ \Gamma(m + \frac{n}{2}) \left\{ \frac{(-1)^{m + \frac{n}{2}}}{\Gamma(-k - \frac{n}{2} + 1 - \tilde{\nu}) \Gamma(k + \frac{n}{2})(m + \frac{n}{2} - 1)!} \psi(m - k - 1 - \tilde{\nu}) + \frac{2^{2m+n-2}(m + \frac{n}{2} - 2)!}{\Gamma(m - k - 1 - \tilde{\nu}) \Gamma(m + k + n - 1)} \varepsilon^{-2m-(n-2)} \right\}. \]

Also, by (7.1) in Section 7, we get
\[ F(m - k - \tilde{\nu}, m + k + n - 1, m + \frac{n}{2}; \varepsilon^2) \approx 1 + \frac{(m - k)(m + k + n - 1)}{2(2m + n)} \varepsilon^2, \]
\[ F(m - k - 1 - \tilde{\nu}, m + k + n, m + \frac{n}{2}; \varepsilon^2) \approx 1 + \frac{(m - k - 1)(m + k + n)}{2(2m + n)} \varepsilon^2. \]

Substituting these for the right-hand side of (5.11), we obtain
\[ (\nu + \frac{n}{2})(\cos \varepsilon)Q^M_\nu(\cos(\pi - \varepsilon)) + (\nu - M + 1)Q^M_{\nu+1}(\cos(\pi - \varepsilon)) \]
\[ \approx \frac{(-1)^{k+m+n-1}\pi}{2\tilde{\nu}} \left\{ \frac{(k + m + n - 2)!}{(k - m)!} \varepsilon^{m + \frac{n+2}{2}} - \frac{(k + n - 1)(k + m + n - 2)!}{2(2m + n)(k - m)!} \right\} \varepsilon^{m + \frac{n+2}{2}} \]
\[ + \frac{\varepsilon^{m + \frac{n+2}{2}}}{2\tilde{\nu}} \left\{ \frac{(k + n - 1)(m - k - \tilde{\nu})}{(k - m)!} \left\{ \frac{(k + m + n - 1)!}{\Gamma(-k - \frac{n}{2} + 1 - \tilde{\nu})(k + \frac{n}{2} - 1)!} + \frac{(k + m + n - 1)(m + \frac{n}{2} - 1)!}{\Gamma(-k - \frac{n}{2} - \tilde{\nu})(k + \frac{n}{2} - 1)!} \right\} \varepsilon^{-2m-(n-2)} \right\}. \]

Let us denote
\[ \frac{(-1)^{k+m+n-1}\pi}{2\tilde{\nu}} \left\{ \frac{(k + m + n - 2)!}{(k - m)!} \varepsilon^{m + \frac{n+2}{2}} - \frac{(k + n - 1)(k + m + n - 2)!}{2(k - m)!} \right\} \varepsilon^{-2m-(n-2)}. \]
where
\[
K_6 = \frac{(-1)^{k+m+n-1}(k+m+n-2)!\pi}{2(k-m)!}
\]
and
\[
K_7 = \frac{(-1)^{k+m+n}(k+m+n-2)!\pi}{4(k-m)!} \left\{ k + n - 1 \right. \\
+ \left. \frac{(k+n-1)(m-k)(m+k+n-1) + (m-k-1)(m+n)(m+n-2)}{2m+n} \right\}.
\]

Using the formulas in Lemma 7.1, we see that
\[
(\nu + \frac{n}{2})(\cos{\varepsilon}Q_{\nu}^{M}(\cos{\pi - \varepsilon})) + (\nu - M + 1)Q_{\nu+1}^{M}(\cos{\pi - \varepsilon})
\]
\[
\approx (K_{6}\bar{m} + K_{7}\varepsilon^{2})\frac{\varepsilon^{m+\frac{n-2}{2}}}{\bar{\nu}} + (-1)^{k+m+n-1}(k+m+n-2)!\pi m\pi \frac{\varepsilon^{m+\frac{n-2}{2}}}{(k-m)!} \\
+ \frac{(-1)^{k+m+1}2^{m+n-2}(k+m+n-2)!}{(k-m)!(k+m+n-2)!} \\
\times (m+n-2)\varepsilon^{-m-\frac{n-2}{2}}
\]
\[
= \left\{ (K_{6} + K_{8}\bar{m} + K_{7}\varepsilon^{2})\frac{\varepsilon^{m+\frac{n-2}{2}}}{\bar{\nu}} + K_{9}(m+n-2)\varepsilon^{-m-\frac{n-2}{2}} \right\},
\]
where
\[
K_{8} = \frac{(-1)^{k+m+n-1}(k+m+n-2)!\pi}{(k-m)!},
\]
\[
K_{9} = \frac{(-1)^{k+m+1}2^{m+n-2}(k+m+n-2)!(m+\frac{n}{2}-2)!(m+\frac{n}{2}-1)!(k-m)!}{(k-m)!(k+m+n-2)!},
\]
Hence, (3.4) in view of (5.9) yields
\[
\left\{ K_{10}\bar{m} + K_{11}(m+n-2)\frac{\bar{\nu}}{2^{m+n-2}} \right\} C_{1}\varepsilon^{m+\frac{n-2}{2}}
\]
\[
= \left\{ \left( K_{6} + K_{8}\bar{m} + K_{7}\varepsilon^{2} \right)\frac{\varepsilon^{m+\frac{n-2}{2}}}{\bar{\nu}} + K_{9}(m+n-2)\varepsilon^{-m-\frac{n-2}{2}} \right\} C_{2},
\]
where
\[
K_{10} = \frac{(-1)^{m-k}}{(m+\frac{n}{2}-1)!}, \quad K_{11} = \frac{(-1)^{m-k}2^{m+n-2}(m+\frac{n}{2}-2)!(k-m)!}{(m+k+n-2)!}.
\]
We substitute (5.8) for (5.12) to have
\[
\left\{ K_{10}\bar{m} + K_{11}(m+n-2)\frac{\bar{\nu}}{2^{m+n-2}} \right\} \left\{ \frac{K_{1}m + K_{2}\eta^{2}}{\bar{\nu}} + K_{3}\eta^{-2m-(n-2)} \right\}
\]
\[
\approx \left( K_{4}\bar{m} + K_{5}\eta^{2} \right) \left\{ \frac{(K_{6} + K_{8})m - K_{7}\varepsilon^{2}}{\bar{\nu}} + K_{9}(m+n-2)e^{-2m-(n-2)} \right\}.
\]
\[
\left\{ K_{10}\bar{m} + K_{11}(m+n-2)\frac{\bar{\nu}}{2^{m+n-2}} \right\} \left\{ \frac{K_{1}m + K_{2}\eta^{2}}{\bar{\nu}} + K_{3}\eta^{-2m-(n-2)} \right\}
\]
\[
\approx \left( K_{4}\bar{m} + K_{5}\eta^{2} \right) \left\{ \frac{(K_{6} + K_{8})m - K_{7}\varepsilon^{2}}{\bar{\nu}} + K_{9}(m+n-2)e^{-2m-(n-2)} \right\}.
\]
If $\eta = o(\varepsilon)$, in order to hold (5.13), we see that

$$K_{10}m + K_{11}(m + n - 2)\varepsilon^{-2m-(n-2)}\bar{\nu} \to 0$$

as $\varepsilon \to 0$. If otherwise, there is no term comparable to $K_{3}\eta^{-2m+n-2}$ as in the deduction of (4.9). This argument is valid even for the case $n \geq 4$ and $m = 0$, or the case $n = 2$ and $m \geq 1$. In these cases, as in Subsection 3.2 of [1], we see that the leading order of $\bar{\nu}$ is $\varepsilon^{2m+n-2}$ and if we write $\bar{\nu} = c_{k,m,n}\varepsilon^{\max(2m+n-2,n)} + o(\varepsilon^{\max(2m+n-2,n)})$ is obtained in the same way as below similarly to the way in [1]:

$$c_{k,m,n} = -\frac{m(m+k+n-2)!}{2^{2m+n-2}(m+n-2)(m+n-2)!}(m+n-2)! (k-m)!$$

when $m \geq 1$ and

$$c_{k,0,n} = -\frac{k(k+n)!}{2^{n-1}(n+2k)!((n-2)!((k-1)!)}$$

when $m = 0$.

Now we consider the case $\eta = \kappa\varepsilon + o(\varepsilon)$. First we consider the case when $m \geq 1$. In this case, in (5.13), we have

$$\left\{K_{10}m + K_{11}(m + n - 2)\frac{\bar{\nu}}{\varepsilon^{2m+n-2}\varepsilon^{-2m-(n-2)}}\right\} \times$$

$$\left\{\frac{K_{1}m + K_{2}\kappa^{2}\varepsilon^{2}}{\bar{\nu}} + K_{3}\kappa^{-2m-(n-2)}\varepsilon^{-2m-(n-2)}\right\}$$

$$\asymp \left(K_{4}m + K_{5}\kappa^{2}\varepsilon^{2}\right)\left\{(K_{6} + K_{9})m + K_{7}\varepsilon^{2} + K_{8} + K_{10}(m + n - 2)\varepsilon^{-2m-(n-2)}\right\}.$$ (5.14)

In this case, we see that $\bar{\nu} = c_{k,m,n,\kappa}\varepsilon^{2m+n-2} + o(\varepsilon^{2m+n-2})$. Also, in (5.14) we see that the terms having $\varepsilon^{-2m-(n-2)}$ are the leading ones. Thus we have

$$\left\{K_{10}m + K_{11}(m + n - 2)c_{k,m,n,\kappa}\left(\frac{K_{1}m}{c_{k,m,n,\kappa}} + K_{3}\kappa^{-2m-(n-2)}\right)\right\}$$

$$= K_{4}m\left\{(K_{6} + K_{9})m + K_{10}(m + n - 2)\right\}.$$ (5.15)

c_{k,m,n,\kappa} is a root of the quadratic equation

$$K_{3}K_{11}(m + n - 2)\kappa^{-2m-(n-2)}(c_{k,m,n,\kappa})^{2}$$

$$+ \left\{(K_{1}K_{11}m(m + n - 2) + K_{3}K_{10}m\kappa^{-2m-(n-2)} - K_{4}K_{10}m(m + n - 2)\right\}c_{k,m,n,\kappa}$$

$$+ (K_{1}K_{10} - K_{4}K_{6} - K_{4}K_{9})m^{2} = 0.$$ (5.16)

Next we consider the case when $m = 0$ and $n \geq 4$. Then (5.14) yields

$$K_{11}(n - 2)\varepsilon^{-(n-2)}\bar{\nu}\left\{\frac{K_{2}\kappa^{2}\varepsilon^{2}}{\bar{\nu}} + K_{3}\kappa^{-(n-2)}\varepsilon^{-(n-2)}\right\}$$

$$\asymp K_{5}\kappa^{2}\varepsilon^{2}\left\{\frac{K_{2}\varepsilon^{2}}{\bar{\nu}} + K_{8} + K_{10}(n - 2)\varepsilon^{-(n-2)}\right\}.$$ (5.17)
Hence we see that $\tilde{\nu} = c_{k,0,n,\kappa} \varepsilon^n + o(\varepsilon^n)$ in view of (5.15). The leading terms in (5.15) is the terms of $\varepsilon^{-n}$ order and we have

$$K_{11}(n-2)c_{k,0,n,\kappa}(\frac{K_2K^2}{c_{k,0,n,\kappa}} + K_3 \kappa^{-(n-2)}) = \frac{K_3K_7K^2}{c_{k,0,n,\kappa}} + K_5K_{10}(n-2)\kappa^2.$$ 

Thus, the coefficient $c_{k,0,n,\kappa}$ is a root of the quadratic equation $K_3K_{11}(n-2)\kappa^{-(n-2)}(c_{k,0,n,\kappa})^2 + (n-2)\kappa^2(K_2K_{11} - K_5K_{10})c_{k,0,n,\kappa} - K_5K_7\kappa^2 = 0$.

In either case, due to Lemma 3.1, $c_{k,0,n,\kappa}$ is uniquely determined (we do not write down its exact form as its is too complicated).

The proof of Theorem 1.1 is complete except for $n = 2$ and $m = 0$.

6. Proof of Theorem 1.1: Two dimensional and azimuthal case. In this section, we treat the case $n = 2$ and $m = 0$. We substitute $M = 0$, $n = 2$ and $\nu = k + \tilde{\nu}$ for (5.5) to have

$$(\nu + \frac{n}{2})(\cos \eta)(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta)$$

$$\approx \frac{(-1)^k}{2\tilde{\nu}} \left[ (k + 1 + \tilde{\nu})(1 - \frac{\eta^2}{2}) \left\{ (-1)^k F(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) ight. ight.$$

$$\left. \left. - F(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) \right\} \right]$$

By using (6.1), (6.1) yields

$$(\nu + \frac{n}{2})(\cos \eta)Q_{\nu}^M(\cos \eta) - (\nu - M + 1)Q_{\nu+1}^M(\cos \eta)$$

$$\approx \frac{(-1)^k}{2\tilde{\nu}} \left\{ (-1)^k \left( \frac{(k + 1)^2}{2} \eta^2 - (k + 1 + \tilde{\nu})(1 - \frac{\eta^2}{2}) F(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) \right) \right.$$

$$\left. + (k + 1 + \tilde{\nu}) F(-k - \tilde{\nu}, k + 2 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) \right\}$$

We expand $F$ (in fact $U$) a little more than we treated for $n \geq 4$ case. We first note that

$$F(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) = \Gamma(1)U(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; \frac{\eta^2}{4})$$

$$= U(-k - \tilde{\nu}, k + 1 + \tilde{\nu}, 1; \frac{\eta^2}{4})$$

$$F(-k - 1 - \tilde{\nu}, k + 2 + \tilde{\nu}, 1; 1 - \frac{\eta^2}{4}) = \Gamma(1)U(-k - 1 - \tilde{\nu}, k + 3 + \tilde{\nu}, 1; \frac{\eta^2}{4})$$

$$= U(-k - 1 - \tilde{\nu}, k + 2 + \tilde{\nu}, 1; \frac{\eta^2}{4})$$
in view of $\Gamma(1) = 1$. According to (5.3), we expand $U$ finer than in previous case to have

$$U(-k - \tilde{\nu}, k + 1 + \tilde{\nu}; \frac{\eta^2}{4}) \simeq \frac{-1}{\Gamma(-k - \tilde{\nu})\Gamma(k + 1 + \tilde{\nu})} \times$$

$$\left[ \frac{\eta^2}{4} + \psi(-k - \tilde{\nu}) + \psi(k + 1 + \tilde{\nu}) - 2\psi(1)
\right.$$

$$- (k + \tilde{\nu})(k + 1 + \tilde{\nu})\psi(1 - k - \tilde{\nu}) \frac{\eta^2}{4} \biggr]\right]$$

$$\simeq (-1)^k \left\{ 1 - \frac{k(k + 1)}{4} \eta^2 \right\} \log \frac{\eta^2}{4} + \psi(-k - \tilde{\nu}) + \psi(k + 1) - 2\psi(1)$$

$$- \frac{k(k + 1)}{4} \psi(1 - k - \tilde{\nu}) \eta^2 \biggr].$$

and

$$U(-k - 1 - \tilde{\nu}, k + 2 + \tilde{\nu}; \frac{\eta^2}{4}) \simeq \frac{-1}{\Gamma(-k - 1 - \tilde{\nu})\Gamma(k + 2 + \tilde{\nu})} \times$$

$$\left[ \frac{\eta^2}{4} + \psi(-k - 1 - \tilde{\nu}) + \psi(k + 2 + \tilde{\nu}) - 2\psi(1)$$

$$- (k + 1 + \tilde{\nu})(k + 2 + \tilde{\nu})\psi(1 - k - \tilde{\nu}) \frac{\eta^2}{4} \biggr]\right]$$

$$\simeq (-1)^{k+1} \tilde{\nu} \left\{ 1 - \frac{(k + 1)(k + 2)}{4} \eta^2 \right\} \log \frac{\eta^2}{4} + \psi(-k - 1 - \tilde{\nu}) + \psi(k + 2) - 2\psi(1)$$

$$- \frac{(k + 1)(k + 2)}{4} \psi(1 - k - \tilde{\nu}) \eta^2 \biggr].$$

Hence we see that (6.2) yields

$$(k + 1)(\cos \eta)Q_{k+\tilde{\nu}}^0(\cos \eta) - (k + 1)Q_{k+1+\tilde{\nu}}^0(\cos \eta)$$

$$\simeq \frac{(-1)^k}{2\tilde{\nu}} \left\{ (-1)^k \frac{(k + 1)^2}{2} \eta^2 - (-1)^k(k + 1)(1 - \frac{\eta^2}{2})\tilde{\nu} \times$$

$$\left\{ (1 - \frac{k(k + 1)}{4}) \eta^2 \log \frac{\eta^2}{4} + \psi(-k - \tilde{\nu}) + \psi(k + 1) - 2\psi(1)$$

$$- \frac{k(k + 1)}{4} \psi(1 - k - \tilde{\nu}) \eta^2 \right\} \right.$$
\[
\left\{ 1 - \frac{(k+1)(k+2)}{4} \eta^2 \log \frac{\eta^2}{4} + \psi(-k-1-\tilde{\nu}) + \psi(k+2) - 2\psi(1) \right. \\
- \frac{(k+1)(k+2)}{4} \psi(1-k-\tilde{\nu}) \eta^2 \right\} \\
\times \frac{1}{2\tilde{\nu}} \left[ \frac{(k+1)^2}{2} \eta^2 + (k+1)\tilde{\nu} \left\{ -\frac{k+1}{2} \eta^2 \log \frac{\eta^2}{4} + \frac{\eta^2}{2} \log \frac{\eta^2}{4} + \psi(-k-1-\tilde{\nu}) \\
- \psi(k+1) + \psi(k+2) - \psi(k+1) + \frac{k(k+1)}{4} \psi(1-k-\tilde{\nu}) \eta^2 \\
- \frac{(k+1)(k+2)}{4} \psi(-k-\tilde{\nu}) \eta^2 \right\} \right].
\]

We use the relation (7.5) in Section 7 to have
\[
(k+1)(\cos \eta)Q^0_{k+\tilde{\nu}}(\cos \eta) - (k+1)Q^0_{k+1+\tilde{\nu}}(\cos \eta) \\
\asymp -\left\{ \frac{k(k+1)}{4} \eta^2 \log \frac{\eta^2}{4} + \frac{\tilde{\nu}}{k+1} + \frac{\eta^2}{\tilde{\nu}} \right\}. \\
\]
The relation (5.7) is valid for \(n=2\) and \(m=0\), we see that
\[
K_5 \eta^2 C_1 \asymp -\left\{ \frac{k(k+1)}{4} \eta^2 \log \frac{\eta^2}{4} + \frac{\tilde{\nu}}{k+1} + \frac{\eta^2}{\tilde{\nu}} \right\} C_2. \\
\]
that is, we obtain
\[
C_1 \asymp -\frac{k(k+1)}{4K_5} \log \frac{\eta^2}{4} + \frac{1}{(k+1)K_5} + \frac{\tilde{\nu}}{(k+1)\eta^2} + \frac{1}{4K_5\tilde{\nu}} \} C_2. \\
\]
The relation (3.3) has been established.

Next, we consider (3.4). According to [1], concerning the left-hand side of (3.4), we have
\[
-(k+1+\tilde{\nu})(\cos \varepsilon)P^0_{k+\tilde{\nu}}(\cos(\pi - \varepsilon)) - (k+1+\tilde{\nu})P^0_{k+1+\tilde{\nu}}(\cos(\pi - \varepsilon)) \\
\asymp (k+1) \left[ -\left( 1 - \frac{\varepsilon^2}{2} \right) \left\{ 1 - \frac{k(k+1)}{4} \varepsilon^2 \right\} \log \frac{\varepsilon^2}{4} + \psi(-k-\tilde{\nu}) + \psi(k+1) \\
-2\psi(1) - \frac{k(k+1)}{4} \psi(-k+1-\tilde{\nu}) \varepsilon^2 \right\} \\
+ \left\{ 1 - \frac{(k+1)(k+2)}{4} \varepsilon^2 \right\} \log \frac{\varepsilon^2}{4} + \psi(-k-1-\tilde{\nu}) + \psi(k+2) \\
-2\psi(1) - \frac{(k+1)(k+2)}{4} \psi(-k-\tilde{\nu}) \varepsilon^2 \right\} \\
\asymp (k+1) \left( \frac{2}{k+1} - \frac{k}{2} \varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{k}{2} \frac{\varepsilon^2}{\tilde{\nu}} \right). \\
\]
Finally, we calculate the right hand-side of (3.4). We substitute $M = m = 0$, $n = 2$ for (5.11). Then the right hand-side of (3.4) yields

\[(k + 1 + \hat{\nu}) \left\{ (\cos \varepsilon)Q^0_{k+\hat{\nu}}(\cos(\pi - \varepsilon)) + Q^0_{k+1+\hat{\nu}}(\cos(\pi - \varepsilon)) \right\} \]

\[\simeq \frac{k + 1}{2\hat{\nu}} \left[ P^0_{k+\hat{\nu}}(-1 + \frac{\varepsilon^2}{2}) + P^0_{k+1+\hat{\nu}}(-1 + \frac{\varepsilon^2}{2}) \right. \]

\[\left. + (-1)^{k+1} \left\{ P^0_{k+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) - P^0_{k+1+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) \right\} \right] \]

\[= \frac{\varepsilon^2}{2} \left\{ P^0_{k+\hat{\nu}}(-1 + \frac{\varepsilon^2}{2}) - P^0_{k+1+\hat{\nu}}(-1 + \frac{\varepsilon^2}{2}) \right\} \]

In view of (7.1) in Section 7, we see that

\[P^0_{k+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) = F(-k - \hat{\nu}, k + 1 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \simeq 1 - \frac{k(k+1)}{4} \varepsilon^2 \]

and

\[P^0_{k+1+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) = F(-k - 1 - \hat{\nu}, k + 2 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \simeq 1 - \frac{(k+1)(k+2)}{4} \varepsilon^2. \]

Moreover, there hold

\[P^0_{k+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) = F(-k - \hat{\nu}, k + 1 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) = U(-k - \hat{\nu}, k + 1 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \]

\[\simeq - \frac{1}{\Gamma(-k - \hat{\nu})\Gamma(k + 1 + \hat{\nu})} \left[ F(-k - \hat{\nu}, k + 1 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \log \frac{\varepsilon^2}{4} + \psi(-k - \hat{\nu}) \right. \]

\[\left. + \psi(k + 1 + \hat{\nu}) - 2\psi(1) - \frac{k(k+1)}{4} \psi(-k + 1 - \hat{\nu}) \varepsilon^2 \right] \]

\[\simeq (-1)^{k+2}\hat{\nu} \left\{ 1 - \frac{k(k+1)}{4} \varepsilon^2 \right\} \log \frac{\varepsilon^2}{4} + \psi(-k - \hat{\nu}) + \psi(k + 1 + \hat{\nu}) - 2\psi(1) \]

\[- \frac{k(k+1)}{4} \psi(-k + 1 - \hat{\nu}) \varepsilon^2 \]

and

\[P^0_{k+1+\hat{\nu}}(1 - \frac{\varepsilon^2}{2}) = F(-k - 1 - \hat{\nu}, k + 2 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \]

\[= U(-k - 1 - \hat{\nu}, k + 2 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \]

\[\simeq - \frac{1}{\Gamma(-k - 1 - \hat{\nu})\Gamma(k + 2 + \hat{\nu})} \left[ F(-k - 1 - \hat{\nu}, k + 2 + \hat{\nu}, 1; \frac{\varepsilon^2}{4}) \log \frac{\varepsilon^2}{4} \right. \]

\[\left. + \psi(-k - 1 - \hat{\nu}) + \psi(k + 2 + \hat{\nu}) - 2\psi(1) - \frac{(k+1)(k+2)}{4} \psi(-k - \hat{\nu}) \varepsilon^2 \right] \]
\[
\approx (-1)^{k+3}\tilde{\nu} \left\{ 1 - \frac{(k+1)(k+2)}{4}\varepsilon^2 \right\} \log \frac{\varepsilon^2}{4} + \psi(-k - 1 - \tilde{\nu}) + \psi(k + 2 + \tilde{\nu}) - 2\psi(1) - \frac{k(k+1)}{4}\psi(-k - \tilde{\nu})\varepsilon^2 \right].
\]

Hence, we see that

\[
P^0_{k+\tilde{\nu}}(-1 + \frac{\varepsilon^2}{2}) + P^0_{k+1+\tilde{\nu}}(-1 + \frac{\varepsilon^2}{2}) \approx (-1)^{k+1}\tilde{\nu} \left\{ \frac{k+1}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} + \psi(-k - \tilde{\nu}) - \psi(-k - 1 - \tilde{\nu}) + \psi(k + 1 + \tilde{\nu}) - \psi(k + 2 + \tilde{\nu}) - \frac{(k+1)}{2}\varepsilon^2 \right\}
\]

\[
\approx (-1)^{k+1}\tilde{\nu} \left\{ \frac{(k+1)}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{2}{k+1} + \frac{k+1}{2}\tilde{\nu} \right\}
\]

in view of (ii) of Lemma 7.1. Hence we obtain

\[
(k + 1 + \tilde{\nu})\left\{ (\cos \varepsilon)Q^0_{k+\tilde{\nu}}(\cos(\pi - \varepsilon)) + Q^0_{k+1+\tilde{\nu}}(\cos(\pi - \varepsilon)) \right\} \approx \frac{k + 1}{\tilde{\nu}} \left\{ (-1)^{k+1}\frac{k}{k+1}\varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{1}{k+1}\tilde{\nu} + \frac{1}{2} - \frac{(1)^k}{8}\varepsilon^4 \right\}
\]

Finally, in view of (6.3), we see that (3.4) yields

\[
- (k + 1) \left( \frac{2}{k+1} - \frac{k}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{k\varepsilon^2}{2}\tilde{\nu} \right) \times \frac{\{ k(k+1) \log \frac{\eta^2}{4} + \frac{1}{(k+1)K_5} + \frac{\tilde{\nu}}{\eta^2} + \frac{1}{4K_5\tilde{\nu}} \}}{(k+1) - \frac{1}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{(1)^k}{2}\varepsilon^2 \left( \frac{1}{\tilde{\nu}} \right) + \frac{(1)^k k(k+1)}{8} \varepsilon^4} \]  

\[
\approx \left[ (-1)^{k+1}\frac{k(k+1)}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} - 2(-1)^k \frac{(1)(k+1)}{2}\varepsilon^2 \left( \frac{1}{\tilde{\nu}} \right) + \frac{(1)^k k(k+1)}{8} \varepsilon^4 \right] \tag{6.4}
\]

If \( \eta = o(\varepsilon) \), then in order to hold (6.4) for any small \( \varepsilon \),

\[
\frac{2}{k+1} - \frac{k}{2}\varepsilon^2 \log \frac{\varepsilon^2}{4} - \frac{k\varepsilon^2}{2}\tilde{\nu} \to 0
\]

as \( \varepsilon \to 0 \). This implies that

\[
\tilde{\nu} = \frac{k(k+1)}{4}\varepsilon^2 + o(\varepsilon^2),
\]

which is the same as in Subsection 3.2 of [1] for the case of \( n = 2 \) and \( m = 0 \).
If \( \eta = \kappa \varepsilon + o(\varepsilon) \), then we also see that \( \tilde{\nu} = c_{k,0,2,\kappa} \varepsilon^2 + o(\varepsilon^2) \) and the terms having \( \log \varepsilon \) are the leading ones. That is, there holds
\[
\left\{ \frac{k(k + 1)}{2c_{k,0,2,\kappa}} - 2 \right\} \frac{k(k + 1)}{2K_5} \log \varepsilon = (-1)^{k+1} \frac{k(k + 1)}{c_{k,0,2,\kappa}} \log \varepsilon.
\]
In this case \( K_5 = (k + 1)(k^2 + 2k - 1)/4 \) and we obtain
\[
c_{k,0,2,\kappa} = \frac{(-1)^k(k + 1)(k^2 + 2k - 1)}{8} + \frac{k(k + 1)}{4}.
\]
All the cases have been examined and the proof of Theorem 1.1 is complete.

**Remark 4.** \( c_{k,0,2,\kappa} \) does not depend on \( \kappa \), however, \( c_{k,0,2,\kappa} \neq c_{k,0,2} \). Also, Remark 2 holds true for even dimensional cases.

7. **Appendix.** We here collect fundamental properties of the Gauss hypergeometric functions \( F(a, b, c; z) \), a recurrence formula for the associated Legendre functions, \( \Gamma(z) \) and \( \psi(z) \).

7.1. **Definition of the Gauss hypergeometric functions.** The definition of the Gauss hypergeometric function \( F(a, b, c; z) \) is as follows:
\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!},
\]
with \( a, b, c \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\} \) and \( z \) is a complex variable. If \( |z| \) is very small, we have
\[
F(a, b, c, z) = 1 + \frac{ab}{c} z + O(|z|^2).
\]  
(7.1)

7.2. **Conversion formula.** For the Gauss hypergeometric function, there is a conversion formula, which is useful for the analysis on an odd dimensional case.

Let \( a, b, c \) be real numbers such that none of \( c, a + b + 1 - c \) and \( c + 1 - a - b \) is a non-positive integer. Then for \( x \in (-1, 1) \), there holds
\[
F(a, b, c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b + 1 - c; 1 - x)
\]
\[
+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - x)^{c-a-b} F(c - a, c - b, 1 + c - a - b; 1 - x).
\]  
(7.2)

7.3. **Recurrence formula.** The general recurrence formulas for \( P_\nu^\mu \) and \( Q_\nu^\mu \) with its derivative is as follows. Here, \( \mu \) and \( \nu \) are real numbers and \( t \) varies within \((-1, 1)\). There holds (see, e.g., (162) in p. 289 in [4])
\[
(1 - t^2) \frac{d}{dt} P_\nu^\mu(t) = (\nu + 1)t P_\nu^\mu(t) - (\nu - \mu + 1)P_\nu^{\mu+1}(t).
\]  
(7.3)
\[
(1 - t^2) \frac{d}{dt} Q_\nu^\mu(t) = (\nu + 1)t Q_\nu^\mu(t) - (\nu - \mu + 1)Q_\nu^{\mu+1}(t).
\]  
(7.4)
7.4. Properties of the Gamma function and the psi function. In this subsection, we enumerate the definitions and several properties of the Gamma function and the psi (di-Gamma) function.

One of the definitions of the Gamma function $\Gamma(z)$ is

$$\Gamma(z) = \lim_{n \to \infty} \frac{(n-1)!n^z}{z(z+1)\cdots(z+n-1)}.$$ 

By this definition, we see that $\Gamma(z)$ has a pole of order 1 at $z = 0, -1, -2, \ldots$ and the local behavior around one of these poles.

The definition of the psi function is

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz} \log \Gamma(z).$$

The psi function has the following property.

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \quad (7.5)$$

The limiting behavior of the Gamma function and the di-psi function as follows.

**Lemma 7.1.** Let $n$ and $k$ be positive integers. Then there hold

(i) $\lim_{\zeta \to 0} \zeta \Gamma(-n - \zeta) = \frac{(-1)^{n+1}}{n!}$.

(ii) $\lim_{\zeta \to 0} \zeta \psi(-n - \zeta) = 1$.

(iii) $\lim_{\zeta \to 0} \frac{\psi(-n - \zeta)}{\Gamma(-k - \zeta)} = (-1)^{k+1}k!$.

Also, the power series representation of $\psi$ is useful.

**Lemma 7.2.** The following expansions hold

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty} \left( \frac{1}{x+n} - \frac{1}{n+1} \right),$$

where $\gamma$ is the Euler number defined as

$$\gamma = \lim_{n \to \infty} \left( \sum_{m=1}^{n} \frac{1}{m} - \log n \right).$$

Also, the following relations are often used in the determination of the coefficient of eigenvalues.

Applying the recursion formula $\Gamma(z+1) = z\Gamma(z)$ several times and $\Gamma(1/2) = \sqrt{\pi}$, we have

$$\Gamma(-m - \frac{1}{2}) = (-1)^{m+1} \frac{2^{2(m+1)}(m+1)!}{\sqrt{\pi}}, \quad \Gamma(m + \frac{1}{2}) = \frac{(2m)!}{2^{2m}m!} \sqrt{\pi} \quad (7.6)$$

for any nonnegative integer $m$.

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