A GENERAL RELATIVISTIC MODEL OF LIGHT PROPAGATION IN THE GRAVITATIONAL FIELD OF THE SOLAR SYSTEM: THE STATIC CASE

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ABSTRACT

We develop here a new approach for the relativistic modeling of the photons moving into a quasi-Minkowskian spacetime, where the metric is generated by an arbitrary N-body distribution within an isolated solar system. Our model is built on the prescriptions of the theory of general relativity and leaves the choices of the metric, as well as that of the motion of the observer, arbitrary. Adopting a quasi-Minkowskian expression of the metric accurate to order \((v/c)^2\), we conduct a thorough numerical test campaign to verify correctness and reliability of the model equations. The test results show that the model behaves according to predictions. Specifically, comparisons to true (simulated) data demonstrate that stellar distances are reconstructed up to the specified level of accuracy. Although the \((v/c)^2\) approximation is not always sufficient for its application to future astrometric experiments, which require modeling to \((v/c)^3\), this work serves also as a natural test ground for the higher order model, whose formulation is now close to completion and will be presented in a forthcoming paper.

Subject headings: astrometry — gravitation — relativity

On-line material: color figures

1. INTRODUCTION

It is known that the light signal carries most of the physical information on the celestial objects and the physical field it passed through (see Kopeikin & Schäfer 1999; Kopeikin & Mashhoon 2002 and references therein). Part of this information, like, e.g., the distance to the source and its velocity, can be extracted by means of (classical) astrometric techniques, but at a high level of accuracy, as demonstrated by ESA space mission Hipparcos, relativistic effects cannot be neglected, and this “relativistic astrometry” can be considered part of fundamental physics (for example, astrometric measurements are one of the natural choices to detect possible deviations from general relativity generated by a scalar field \(\phi\) that couples with the metric tensor \(g\) to generate gravity; Damour, Piazza, & Veneziano 2002a, 2002b), or they were used for a possible attempt at measuring the speed of gravity (Kopeikin 2001; Fomalont & Kopeikin 2003).

The purpose of this paper is to construct a model of the celestial sphere using the prescriptions of the general theory of relativity in order to take into account the relativistic effects suffered by light while propagating through the gravitational field of the solar system. The planned future astrometric experiments set the optimal target accuracy for such models to the microarcsecond level. To this level, it was shown that the light propagation will be affected not only by the mass of the Sun and of the other planets but also by their gravitational quadrupole and their translational and rotational motion (Epstein & Shapiro 1980; Klioner & Kopeikin 1992; Kopeikin & Mashhoon 2002). Assuming that the solar system is isolated and is the only source of gravity, the accuracy of 0.1 \(\mu\text{as}\) is achieved by considering terms of the background metric up to the order of \((v/c)^3\); here \(c\) is the velocity of light in vacuum and \(v\) is an average velocity for energy balance. In the solar system this is \(\sim 10 \text{ km s}^{-1}\).

In this sense, the next-generation astrometric missions like GAIA, of ESA (Perryman et al. 2001; Perryman 2003), and SIM, of NASA (Shao 1998; Marr 2003), offer a unique opportunity to test our model. On the other hand, the observational error of these satellites will be pushed to the microarcsecond level; therefore, the need is to implement a model of the celestial sphere and of the observables that is accurate to that order. As a reference, we recall that Hipparcos (of ESA), the only previous example of a global astrometric mission, had an accuracy of 1 mas only.

The construction of a general relativistic many-body astrometric model with the required accuracy is expected to be complicated not only by the mathematical structure of the relativistic equations but also by the numerical methods deployed to implement the model into software code. Therefore, it is crucial to have an efficient strategy for testing the model. With this goal in mind, we had already developed a relativistic astrometric model taking as background geometry the exact (unperturbed) Schwarzschild solution (de Felice et al. 1998, 2001). That model was used as a basic touchstone for comparison in the construction of our many-body model. Moreover, since a model limited to the \((v/c)^2\) order of accuracy is significantly simpler to handle and test, we decided for a full computer implementation of the \((v/c)^2\) model, with the intention to use it as a test ground for the higher order extension, which is our ultimate goal.

Finally, it is worth stressing that our aim was not to add to the theory of light propagation, which is well known, but to

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develop a relativistic astrometric model with a new theoretical approach that treats the light propagation in a curved spacetime in such a way that all the possible relativistic perturbations coming from the gravity sources are naturally included in the light-path reconstruction.

In §§2 and 3 we define the background geometry in a way compatible with our basic physical assumptions. In §4 we discuss how to handle light propagation through the solar system and provide the differential equations that enable us to reconstruct the light trajectory from a distant star to the observer. Then, in §§5 and 6 we define the “observables” as the measurements coming from a given satellite and link them to the mathematical boundary conditions needed to integrate the light trajectory. Finally, in §§7 and 8 we present the test campaign used to validate the model.

In what follows, Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

2. THE SPACETIME METRIC

The spacetime structure that underlies the development of relativistic astrometry must mirror the physical and operational reality experienced by the observer.

The basic step of this project is to identify the background geometry. Our first assumption is that the solar system is isolated; this means that there are no perturbing bodies intervening between the emitting stars and the solar system boundaries. It is clear that this assumption may not be fully justified since light rays from distant stars may suffer deflections due to microlensing induced by intervening bodies (e.g., as far as the project GALA is concerned, we expect nearly 1000 microlensing events along the Galactic disk during 5 yr; Dominik & Sahu 2000). Indeed, they could generate systematic errors in the data reduction of the celestial sphere; however, their number is very small as compared to the number of observations ($\geq10^4$). Furthermore, we know that most of the stars are in a binary system and have proper motion that generates an unsteady gravitational field causing a rotation of the celestial reference frame. However, the rotation coefficients have a temporal variation that amounts to 1 year every 20 yr, a period much longer than the lifetime of all the future space astrometric missions (Sazhin et al. 1998). Thus, we judge that the above cases do not affect the validity of the hypothesis that the solar system is isolated.

A second assumption is that the solar system generates a weak gravitational field. We can then adopt a quasi-Minkowskian coordinate system so that the spacetime metric can be written as a perturbation of the Minkowski metric $\eta_{\alpha\beta}$, namely,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2),$$

where the $h_{\alpha\beta}$ values describe effects generated by the bodies of the solar system and are small in the sense that $|h_{\alpha\beta}| \ll 1$, their spatial variations are of the order of $|h_{\alpha\beta}|$, while their time variations are at most of the order $(v/c)|h_{\alpha\beta}|$. Clearly the metric form given by equation (1) is preserved under gauge transformations of the order of $h$. The order of magnitude of the correction terms entering the $h_{\alpha\beta}$ values is expressed in terms of powers of $(v/c)$. The $h_{\alpha\beta}$ values are at least of the order of $(v/c)^2$ (Newtonian terms); hence, any higher level of accuracy within the model is fixed by the power of $(v/c)$ larger than 2 that one likes to include in the analysis.

Having the above considerations in mind, we choose a solution of Einstein’s equations of the type

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \sum_a h_{\alpha\beta}^{[a]} + O(h^2),$$

where now the sum is extended to the bodies of the solar system. In this approximation, the metric tensor given by equation (2) has in general a nonvanishing term $g_{ij} = O((v/c)^3)$ and the nonlinearity of the gravitational field is confined to terms $O((v/c)^4)$ in $h_{ab}$ and $h_{ij}$.

Moreover, in 2000 August the General Assembly of the IAU stated that a solution like equation (2) has to be adopted to define the reference frames and timescales in the solar system (Damour, Soffel, & Xu 1991; Brumberg & Kopeikin 1989; Brumberg 1991). At the first post-Newtonian level of approximation, the metric tensor nearby any planet of the solar system takes the form

$$g_{00} = -1 + \frac{2W}{c^2} - \frac{2W^2}{c^4} + O(c^{-5}),$$

$$g_{0i} = \frac{4W_i}{c^3} + O(c^{-5}),$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2W}{c^2}\right) + O(c^{-4}),$$

with $i, j = 1, 2, 3$.

In the equations above, $W$ represents a generalization of the Newtonian potential and $W_i$ is a vector potential describing the dynamical contribution to the background geometry by the relative motion of the gravitating sources, as well as by the peculiarities of their extended structures. The same form (eqs. [3]–[5]) is adopted to describe the metric tensor generated by the whole solar system referenced hereafter to the Barycentric Celestial Reference System. Assuming the spacetime to be asymptotically flat, the potentials $W$ and $W_i$ are a sum of integrals containing terms of gravitational mass and mass current density. Such integrals are taken over the support of each body of the solar system. It should be stressed here that the linearized form of the metric, in the weak field and slow motion ($v \ll c$) approximation, is the current support for calculating the ephemerides at the Jet Propulsion Laboratory (Soffel 2000), California Institute of Technology.

A light ray, on its way to the observer from a distant star, first feels the gravitational field of the solar system as a pointlike mass centered in its barycenter; then, as it gets closer, it feels the gravitational perturbations of the individual bodies of the system. To the order of $(v/c)^2$, the light ray will feel the contributions from the individual mass structures, while the effects arising from their relative motion and spin would enter terms of higher order. Since a time derivative adds a factor of the order of $(v/c)$, time derivatives of the metric coefficients generate terms at least of the order of $(v/c)^3$. Our third assumption then is that light rays will not feel perturbations of the order $O((v/c)^3)$; so, within each integration of the light trajectories, the solar system is approximated in our model to a static, nonrotating, and nonexpanding gravitating system. A natural consequence of this assumption is that, along with time variations of the metric, we shall also neglect mixed metric coefficients of type $h_{0i}$. Furthermore, in calculating the background metric coefficients that affect the light signal at each spatial point of its trajectory, we need not consider the spatial location of the individual gravitational source at the
corresponding retarded time. This does not mean that the positions of the perturbing bodies are considered fixed during the whole integration, but simply that the ephemerides used do not include the retarded time and that we neglect, to \((v/c)^2\), the effects due to their velocities.

3. SETTING THE GEOMETRY OF THE CELESTIAL SPHERE

Let \((\xi^i, \xi^0 = \tau)\) be a quasi-Minkowskian coordinate system with respect to which the spacetime metric takes the form given by equation (1); as stated, the choice to consider only terms up to \((v/c)^2\) amounts to assuming that spacetime is static. Hence, there exists a timelike Killing vector field \(\eta\), say, along which the physical properties of the spacetime do not change and therefore uniquely identify a time direction. Let the coordinate time \(\tau\) be a parameter along \(\eta\) so that \(\eta^\mu = \delta_\tau^\mu\).

From the Killing equation \(\eta_{\alpha; \beta} = 0\), where a semicolon means covariant derivative with respect to the given metric and parentheses mean symmetrization, one easily deduces that the congruence \(C_\eta\) of Killing lines is, to the required order, vorticity-free. In fact, denoting as \(P(\eta)^\alpha_\beta = \delta_\tau^\beta + \eta^\nu_\alpha \eta^\mu_\nu\) the tensor operator that projects orthogonally to \(\eta\), one finds

\[
\omega_{\alpha\beta} = P(\eta)^\alpha_\alpha P(\eta)^\sigma_\beta \eta_{\eta\sigma} = O\left[(v/c)^3\right],
\]

where \(\omega_{\alpha\beta}\) is the vorticity tensor and square brackets mean antisymmetrization. A spacetime that admits a vorticity-free congruence of lines can be foliated, namely, it admits a family of three-dimensional spacelike hypersurfaces \(S(\xi^i, \tau)\) that are everywhere orthogonal to \(C_\eta\). It is always possible to choose a coordinate system such that the surfaces \(S(\xi^i, \tau)\) have equation \(\tau = \text{const.}\); in this case the spatial coordinates can be fixed within each slice up to spatial transformations only. We shall term these surfaces \(S(\tau)\).

Let us now consider the unit vector field \(u\), which is everywhere proportional to \(\eta\), namely, \(u^\alpha = e^{\nu} \delta^\alpha_{\nu}\), where \(e^{\nu} = (-\gamma_{00})^{-1/2}\) is the normalization factor that assures that \(u_\mu u^\mu = -1\); the associated one-form has components \(u_\alpha = -e^{\nu}(\partial\tau/\partial e^{\alpha})\) so it is proportional to the gradient of \(S(\tau)\), as expected. Through each point of a slice \(S(\tau)\), for any \(\tau\), there is a timelike curve, orthogonal to \(S(\tau)\) and having as a tangent a vector of the vector field \(u\); the totality of these curves through all the points of \(S(\tau)\) form a nonintersecting family of curves, or a congruence \(C_u\), which identifies a physical observer. The property of this observer is to be static with respect to the selected coordinate representation; that is, the spatial coordinates do not vary along its world lines. The parameter \(\sigma\) on the congruence, such that \(u^\alpha = d\xi^\alpha/d\sigma\), is the proper time of the observer \(u\). We then require that the geometry that each photon feels before reaching the target is described by the metric given by equation (1); moreover, we also require that within each photon travel time the world lines of the bodies of the solar system belong to the congruence \(C_u\) and in particular the barycenter of the solar system is fixed at the origin of the spatial coordinates on each slice (Fig. 1). With this choice, the observer \(u\) will be termed “locally barycentric.” This observer is an essential prerequisitely of our relativistic astrometric model because at any spacetime point and apart from a position-dependent rescaling of its time rate, it plays the role of the barycentric observer that is located at the origin of the spatial coordinates fixed, as said, at the barycenter of the solar system. Before concluding this section, let us recall here that the constraints on the metric, namely, \(h_{0i} = 0\) and \(C_\eta\) being Killing and vorticity-free, are gauge invariant only up to the order of \((v/c)^2\).

4. THE LIGHT TRAJECTORIES

A photon traveling from a distant star to the observer within the solar system would see the spacetime as a time development of slices of constant \(\tau\). As stated, we shall treat each light trajectory assuming that the bodies of the solar system were fixed at the spatial position they had, say, at the time of observation. Evidently, any subsequent light ray will be considered updating the positions of the bodies of the solar system according to their actual motion. Let us then prepare the way to a suitable treatment of a light ray. Let

\[
P(u)^\alpha_\beta = \delta^\beta_\alpha + u_\alpha u^\beta
\]

be the operator that projects orthogonally to \(u\). Because of the unitary condition, the parameter \(\sigma\) on the trajectories of \(u\) is not constant on the slices \(S(\tau)\) but varies differentially with the position as \(\sigma = \sigma(\xi^i, \tau)\). Since the spatial coordinates \(\xi^i\) are constant along the unique normal going through the point with those coordinates, the parameter \(\sigma\) along it will be a function of \(\tau\) only, i.e., \(\sigma = \sigma_{\xi}(\tau)\). Let us now consider a null geodesic \(\gamma\) with tangent vector field \(k^\alpha = d\xi^\alpha/d\lambda\), which satisfies the following equations:

\[
k^\alpha k_\alpha = 0, \\
\frac{dk^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} k^\mu k^\nu = 0.
\]

The latter express, respectively, the null and the geodesic conditions; here \(\lambda\) is a real parameter on \(\gamma\) and \(\Gamma^\alpha_{\mu\nu}\) are the
connection coefficients of the given metric (Fig. 2). Assume that the trajectory starts at a point \( P_* \) on a slice \( S(\tau_0) \) (say) and with spatial coordinates \( \xi^i_0 \). The light trajectory will end at the observation place on a slice \( S(\tau) \) and at a point with spatial coordinates \( \xi^i(\tau) \). We remember that the origin of the coordinate system is meant to be the barycenter of the solar system. The purpose of this work is to determine \( \xi^i_0 \), namely, the coordinates of the star, from a prescribed set of observables.

Since our approximation permits a global foliation of the spacetime, we find it more convenient for the determination and physical interpretation of the gravitational effects on light propagation to consider the spatial projection of the light ray \( \Upsilon \) on the slice \( S(\tau_0) \); this projection is a curve made of the points \( \xi^i(\tau) \) that one gets to by moving along the unique normal through the point of intersection of the light ray with the slice \( S(\tau) \), for any \( \tau \). This curve, which we denote as \( \Upsilon \), is smooth and has a tangent vector field \( \vec{k} \) whose coordinate components are equal to those of the projection of the tangent \( \vec{k} \) to the null ray into the rest frame of the local barycentric observer at each of its points, namely,

\[
I^\alpha = P(u)^\alpha \vec{k}^\beta.
\]  (9)

(see Appendix A for a mathematical description of this mapping procedure). The vector \( I \) physically identifies the local line of sight of the local barycentric observer at each space-time point of the light-ray trajectory. Indeed, the knowledge of \( I \) is essential to reconstruct the whole story of the light ray. The curve \( \Upsilon \) will be natural parameterized by \( \lambda \).

From \( u^\alpha u_\alpha = -1 \), it follows that

\[
I^\alpha = k^\alpha + u^\alpha (u_\beta k^\beta). \tag{10}
\]

Clearly it is \( I^\alpha = 0 \) showing that \( I^\alpha \) lies everywhere on the slice \( S(\tau_0) \) as expected. Since each point of \( \Upsilon \) is the image under the above projection operation of a point of \( \Upsilon \) at time \( \tau \), it is more convenient to label the points of \( \Upsilon \) with the value of the parameter \( \sigma_{\xi^i}(\tau) \), which, as we have already said, uniquely identifies that point on the normal to the slice \( S(\tau) \) that contains it. Hence, being

\[
d\sigma = -(u_\alpha k^\alpha) d\lambda,
\]  (11)

we define the new tangent vector field:

\[
\vec{\tilde{I}}^\alpha = -\frac{d\sigma}{d\sigma} = -\frac{I^\alpha}{(u_\beta k^\beta)}.
\]  (12)

In the same way we denote

\[
\vec{\tilde{k}}^\alpha = \vec{\tilde{I}}^\alpha + u^\alpha,
\]  (13)

which implies

\[
\vec{\tilde{I}} \cdot \vec{\tilde{k}} = 1. \tag{15}
\]

In what follows we shall denote \( \sigma_{\xi^i}(\tau) \) as \( \sigma \). We can now write the differential equation that is satisfied by the vector field \( \vec{\tilde{I}} \). From equations (11) and (13), the second equation in equation (8) writes

\[
\frac{d\vec{\tilde{I}}^\alpha}{d\sigma} + \frac{d u^\alpha}{d\sigma}(\vec{\tilde{I}}^\alpha + u^\alpha)(\vec{\tilde{I}}^\beta u_\beta + \vec{\tilde{I}}^\gamma \nabla_{\gamma} u_\beta) + \Gamma^\alpha_{\beta\gamma} (\vec{\tilde{I}}^\beta + u^\beta)(\vec{\tilde{I}}^\gamma + u^\gamma) = 0. \tag{16}
\]

In this equation the quantity \( \vec{\tilde{I}}^\beta \nabla_{\gamma} u_\beta \) can be written explicitly in terms of the expansion \( \Theta_{\rho\sigma} \) of \( C_\rho \) (de Felice & Clarke 1990), as

\[
\vec{\tilde{I}}^\beta \nabla_{\gamma} u_\beta = \Theta_{\rho\sigma} \vec{\tilde{I}}^\rho, \tag{17}
\]

where \( \Theta_{\rho\sigma} = P(\rho)^\rho \delta_{\rho}(\gamma)^\alpha \nabla_{\alpha} u_\beta \). Since the only nonvanishing components of the expansion are \( \Theta_{\rho\sigma} = \frac{1}{2} \partial_\rho h_{\sigma\beta} \), the expansion vanishes identically as a consequence of the assumption to neglect time variations of the metric. From this condition and imposing \( h_{0\gamma} = 0 \), equation (16) becomes to the required order and after some algebra

\[
\frac{d\vec{\tilde{I}}^\alpha}{d\sigma} + \frac{1}{2} (\vec{\tilde{I}}^\beta \partial_\beta h_{0\alpha}) \vec{\tilde{I}}^\alpha + \frac{1}{2} (\vec{\tilde{I}}^\beta \partial_\beta h_{00}) (\vec{\tilde{I}}^\alpha + \delta^\alpha_0) \\
+ \eta^\alpha \delta_{\rho\sigma} \left( \partial_\rho h_{\sigma\beta} - \frac{1}{2} \partial_\sigma h_\beta \right) \vec{\tilde{I}}^\rho + \eta^\alpha \partial_\rho h_{0\alpha} \vec{\tilde{I}}^\rho \\
- \frac{1}{2} \eta^\alpha \partial_\rho h_{00} = 0. \tag{18}
\]

If \( \alpha = 0 \), equation (18) leads to \( d\tilde{I}^0/d\sigma = 0 \), assuring that condition \( \tilde{I}^0 = 0 \) holds true all along the curve \( \Upsilon \); if \( \alpha = \lambda \),
equation (18) gives the set of differential equations that we need to integrate to identify the coordinate position of the star:

\[ \frac{d \vec{l}^k}{d \sigma} + \vec{r}^k \left( \frac{1}{2} \vec{\gamma} \partial_i h_{00} + \delta^{k}\left( \partial_j h_{ij} - \frac{1}{2} \partial_i h_{ij} \right) \vec{\gamma} \vec{\gamma} \right) - \frac{1}{2} \delta^{k}(\partial_i h_{00} = 0, \right) \]

\[ \vec{l}^k = \frac{d\epsilon^k}{d\sigma}. \tag{19} \]

Here we remind that Latin indices take values 1, 2, 3.

5. OBSERVABLES AND BOUNDARY CONDITIONS

Our aim is to determine the coordinate positions of a star, from a prescribed set of observables. As a first step we express the observables in a way that reflects a convenient setup for the observer. The latter carries a frame, namely, a clock that measures its proper time and a rest space, which are different from the barycentric proper time and rest space, respectively, given by the parameter \( \sigma(r) \) along the congruence \( C_u \) and the spacelike slices \( S(\tau) \). To determine the boundary conditions needed to integrate equation (19), we need both frames; the observer frame allows us to define the measurements and the barycentric one to express all the coordinate tensorial components. Each measurement is made at a coordinate time \( \tau_0 \) when the observer was at a spatial position with respect to the barycenter given by the coordinates \( \xi(\eta) \). Hence, we consider as observables the angles between the direction of the incoming photon and the three spatial directions of a frame adapted to the observer. These three angles provide the required boundary values for \( \vec{l} \). Let \( \vec{u}' \) be the vector field tangent to the observer’s world line and let \( \{ \lambda_{\alpha} \} \) (where \( \alpha = 1, 2, 3 \)) be a spacelike triad carried by the observer. The angles \( \psi(\lambda_{\alpha}, \tau) \) that the incoming light ray forms with each of the triad directions are given by (de Felice & Clarke 1990)

\[ \cos \psi(\lambda_{\alpha}, \tau) = \epsilon_{\alpha} = \frac{\mathcal{P}(u')_{\alpha} \lambda^{\alpha}_{\beta}}{\mathcal{P}(u')_{\alpha} \lambda^{\alpha}_{\beta} \lambda^{\beta}_{\alpha}} = \frac{u_{\alpha}u_{\alpha}'}{u_{\alpha}u_{\alpha}'}, \tag{20} \]

where no sum is meant over \( \alpha \) and \( \mathcal{P}(u')_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} + u_{\beta}u_{\beta}' \) is the operator that projects into the observer’s rest frame (Fig. 3).

From equation (14), the above equation can be written more conveniently as

\[ \epsilon_{\alpha} = \frac{\vec{l}_{\alpha} \lambda_{\beta}}{u_{\alpha}u_{\alpha}', u_{\beta}u_{\beta}'}, \tag{21} \]

where \( \vec{l}_{\alpha} \equiv \vec{l}(\tau_0) \) and recalling that \( \gamma_{\alpha\beta}^{\alpha\beta} = 1 \); this is a matrix equation where the unknowns are the photon’s spatial directions \( \vec{l}_{\alpha} \) at the time of observation; they can be singled out as

\[ \vec{l}_{\alpha}^{i}(u'_{\alpha} - \lambda_{\alpha}) = u_{\alpha}^{i} - \epsilon_{\alpha}^{i}(u_{\alpha}u_{\alpha}'). \tag{22} \]

The direction of the light ray, as it is seen from the point of observation, depends on the motion of the latter relative to the center of mass. This dependence gives rise to the stellar aberration. If the observer does not move with respect to the spatial coordinate grid, namely, if \( u_{\alpha}^{\alpha} = u^{\alpha} \), then equation (22) becomes \( \vec{l}_{\alpha}^{i} \lambda_{\alpha} = \epsilon_{\alpha}^{i} \) as expected. Equation (19) and the three conditions given by equation (22), together with equation (12) and the coordinate positions \( \xi_{\alpha}(\tau_0) \) of the observer at the time of observation, form a closed system of equations whose solutions are the coordinate positions of the star provided that one is able to identify the value of the parameter \( \sigma \) at the emission. The solution of equation (19), in fact, is of the type \( \xi(\sigma, l_{\alpha}, \xi_{\alpha}); \) hence, \( \sigma_{\tau_0} = \sigma(\tau) \), which marks the photon emission, is an implicit unknown. The way to determine \( \sigma_{\tau_0} \) is discussed in the following section.

In order to find the boundary conditions \( l_{\alpha} \) for equation (19), we have first to construct a tetrad frame adapted to the observer. The tetrad frame \( \{ \lambda_{\alpha} \} \) forms a system of local Cartesian axes, which equips the observer with an instantaneous inertial frame; in fact, it is

\[ \left( \lambda_{\alpha} \mid \lambda_{\beta} \right) = \eta_{\alpha\beta}. \tag{23} \]

Let us relabel for convenience the coordinates \( \lambda_{\alpha} \) as \( (\tau, x, y, z) \); each tetrad vector \( \lambda_{\alpha} \) can then be expressed in terms of coordinate components with respect to the coordinate basis, as

\[ u'_{\alpha} = \lambda_{\alpha} = e^{\alpha'}(T_{\alpha}x_{\alpha} + X_{\alpha}x_{\alpha} + Y_{\alpha}x_{\alpha} + Z_{\alpha}x_{\alpha}), \tag{24} \]

\[ \lambda_{\alpha} = T_{\alpha}x_{\alpha} + X_{\alpha}x_{\alpha} + Y_{\alpha}x_{\alpha} + Z_{\alpha}x_{\alpha}, \tag{25} \]

where \( e^{\alpha'} \) is the normalization function that makes \( u' \) unitary and \( \alpha = 1, 2, 3 \). Conditions given by equation (23) must

3 The letter \( s \) is for “satellite observer.”
be simultaneously satisfied; hence, after some algebra, we obtain

$$e^{\omega'} = \left[-\left(T^2 g_{00} + X^2 g_{xx} + Y^2 g_{yy} + Z^2 g_{zz}\right)\right]^{-1/2}$$  \hspace{1cm} (26)

and

$$T_a T_b g_{00} + X_a X_b g_{xx} + Y_a Y_b g_{yy} + Z_a Z_b g_{zz} = 0,$$  \hspace{1cm} (27)

$$T_a T_b g_{00} + X_a X_b g_{xx} + Y_a Y_b g_{yy} + Z_a Z_b g_{zz} = 0,$$  \hspace{1cm} (28)

$$T^2_a g_{00} + X^2_a g_{xx} + Y^2_a g_{yy} + Z^2_a g_{zz} = 1,$$  \hspace{1cm} (29)

where $a \neq b$. A general solution of equations (27)–(29) is given by

$$\lambda_0 = e^{\omega'} \left(\partial_0 - \omega(y_0 - y_\odot) \partial_x + \omega(x_0 - x_\odot) \partial_y\right),$$  \hspace{1cm} (30)

$$\lambda_1 = X_1 \partial_x + Y_1 \partial_y,$$  \hspace{1cm} (31)

$$\lambda_2 = T_2 \partial_0 + X_2 \partial_x + Y_2 \partial_y,$$  \hspace{1cm} (32)

$$\lambda_3 = T_3 \partial_0 + Z_3 \partial_z,$$  \hspace{1cm} (33)

where the components are explicitly given in Appendix B. This solution describes the instantaneous inertial frame of an observer endowed with a general motion in a gravitational field described by the metric given by equation (2).

It is crucial that our model is consistent, under the same conditions, with the one discussed in de Felice et al. (1998, 2001) where the Sun was the only source of gravity. In order to make comparison easier, let us assume that the observer moves on a circular orbit around the barycenter of the Sun-Earth system and has, at the time of observation, spatial coordinates $x_0, y_0,$ and $z_0 = 0$. The 4-velocity of the observer now reads

$$u^\alpha = e^{\omega'} \left[\partial_0 - \omega(y_0 - y_\odot) \partial_x + \omega(x_0 - x_\odot) \partial_y\right],$$  \hspace{1cm} (34)

where $\omega$ is the coordinate Keplerian angular velocity of revolution and $(x_\odot, y_\odot)$ are the coordinates of the barycenter of the Sun-Earth with respect to the barycenter of the solar system. In this case, the tetrad frame becomes

$$\lambda_0 = e^{\omega'} \left(\partial_0 - \omega(y_0 - y_\odot) \partial_x + \omega(x_0 - x_\odot) \partial_y\right),$$  \hspace{1cm} (35)

$$\lambda_1 = X_1 \partial_x + Y_1 \partial_y,$$  \hspace{1cm} (36)

$$\lambda_2 = T_2 \partial_0 + X_2 \partial_x + Y_2 \partial_y,$$  \hspace{1cm} (37)

$$\lambda_3 = Z_3 \partial_z,$$  \hspace{1cm} (38)

where

$$e^{\omega'} = \left[\left[g_{00} + \omega^2 (y_0 - y_\odot)^2 g_{xx} + \omega^2 (x_0 - x_\odot)^2 g_{yy}\right]\right]^{-1/2},$$  \hspace{1cm} (39)

$$X_1 = -\frac{\sqrt{g_{yy}(x_0 - x_\odot)}}{\sqrt{g_{xx} \Sigma}},$$  \hspace{1cm} (40)

$$Y_1 = -\frac{\sqrt{g_{xx}(y_0 - y_\odot)}}{\sqrt{g_{yy} \Sigma}},$$  \hspace{1cm} (41)

$$T_2 = \frac{\omega \sqrt{\Sigma}}{\sqrt{-g_{00} \Pi}},$$  \hspace{1cm} (42)

$$X_2 = -\frac{(y_0 - y_\odot) \sqrt{-g_{00}}}{\sqrt{\Sigma} \sqrt{-\Pi}},$$  \hspace{1cm} (43)

and where we have named

$$\Sigma = g_{xx} (y_0 - y_\odot)^2 + g_{yy} (x_0 - x_\odot)^2,$$

$$\Pi = g_{00} + \left[g_{xx} (y_0 - y_\odot)^2 + g_{yy} (x_0 - x_\odot)^2\right] \omega^2.$$  \hspace{1cm} (44)

We can now solve the system given by equation (22) explicitly, relating the observed quantities $e_i'$ to the unknowns $\tilde{t}_0'$ by means of the observer’s comoving frame $\{\lambda_i\}$, i.e.,

$$\tilde{t}_0' = \frac{1}{\sqrt{g_{xx} \Sigma}} \left\{e_i' (y_0 - y_\odot) \sqrt{g_{yy} - \Pi} - (x_0 - x_\odot) \sqrt{g_{xx} \Sigma}\right\}$$

$$\times \left(\omega \sqrt{\Sigma} + e_2' \sqrt{g_{00}}\right)$$

$$\left(e_2' \omega \sqrt{\Sigma} - \sqrt{-g_{00}}\right)^{-1},$$  \hspace{1cm} (46)

$$\tilde{t}_0' = \frac{1}{\sqrt{g_{yy} \Sigma}} \left\{e_i' (y_0 - y_\odot) \sqrt{g_{xx} \Sigma} - (x_0 - x_\odot) \sqrt{g_{yy} \Sigma}\right\}$$

$$\times \left(\omega \sqrt{\Sigma} + e_2' \sqrt{g_{00}}\right)$$

$$\left(e_2' \omega \sqrt{\Sigma} - \sqrt{-g_{00}}\right)^{-1},$$  \hspace{1cm} (47)

$$\tilde{t}_0' = -\frac{e_i' \sqrt{-\Pi}}{\sqrt{g_{zz} \left(e_2' \omega \sqrt{\Sigma} - \sqrt{-g_{00}}\right)}},$$  \hspace{1cm} (48)

6. IDENTIFYING THE EMISSION TIME

Let us assume that the star has no proper motion with respect to the given spatial coordinate grid whose origin is at the barycenter of the solar system. In this case, the spatial coordinates $\xi_i = \xi_i(\tau_*)$ of the star remain fixed with time $\tau$ and the star’s world line is one of the curves of the congruence $C_u$. If a photon is emitted at $\tau_*$, say, then its trajectory all the way to the observer at $(\tilde{t}_0, \xi_0)$ is mapped into a spatial path on the slice $S(\tau_0)$ having the boundary conditions fixed as explained in §4. Integration along this path leads to a solution $\xi = \xi(\sigma(\tau))$. Let the same star be observed at a subsequent coordinate time $\tau'_0 = \tau_0 + \Delta \tau$; the received photon is emitted at a coordinate time $\tau'_* = \tau_* + \Delta \tau$ and its trajectory is now mapped into a slice $S(\tau'_0)$. Evidently, this second observation implies a different set of boundary conditions for the integration along the second path; hence, the solution will be a new function $\xi = \xi(\sigma(\tau))$. Since the spatial coordinates of the star are preserved under the mapping, they will be identified by the value of the parameter $\sigma(\tau_*)$ such that $\xi(\sigma(\tau_* + \Delta \tau)) = \xi(\sigma(\tau_*))$ (see Fig. 4).

A first check of the procedure is shown in Figure 5; here we assume that the gravitational field of the Sun acts only on the light path. We considered two stars at coordinate distances, respectively, of 1 and 2 pc. Fixing the boundary conditions corresponding to observations of the stars in two symmetrically opposite directions with respect to the Sun and assuming $\Delta \tau$ equal to 6 months, we find, in logarithmic units, the points of intersections of the two integrated spatial paths. The actual distances are slightly less than 1 and 2 pc,
respectively, as expected since we calculate proper rather coordinate distances.

7. TESTING THE MODEL

The perturbative model presented in the last chapter is expected to include all terms to the order of \((v/c)^2\). This means that, for a typical velocity in the solar system of \(\approx 10 \text{ km s}^{-1}\), one can expect deviations from the predictions of exact models of \(10\) arcsec. Equation (19) cannot be integrated analytically, so numerical techniques are needed to reconstruct the light path. We then needed a test campaign to check the correctness and accuracy of the model.

We used the implementation of the RADAU integrator given in Everhart (1985) to integrate the system of differential equations for the null geodesic because it has been thoroughly tested and can be easily implemented to reach very high orders of accuracy. Since the geodesic equations include both \(\dot{\xi}_i = \xi''_i\) and \(\xi'_i\), the system of differential equations is of class II according to the definition used in Everhart (1985), that is,

\[ y'' = F(y', y, t). \]

The tests we devised verify the following:

1. The perturbative model is self-consistent.
2. The amount of light deflection caused by each individual body of the solar system, as evaluated in our perturbative model, coincides with that expected for an analytical Schwarzschild solution at the same order of accuracy and under the same observational conditions.
3. The model is able to reconstruct stellar distances.

It is clear from these items that our tests do not consider comparisons with other models besides the Schwarzschild solution, like those in Kopeikin & Mashhoon (2002) and Klioner (2003); however, we believe that it is still too early for such comparisons. First of all, our model is at the \((v/c)^2\) order, so it is not strictly needed to confront it with higher order models yet. Moreover, although with the same assumptions some results can be recovered, the background mathematical framework is quite different, and this is another reason for not attempting a detailed comparison. Our model is evolving to the \((v/c)^3\) order, and this future version will be presented in a forthcoming paper and, indeed, compared to the two works cited above.

Before illustrating each specific test, let us make some general remarks on the numerical solution of the system of differential equations given by equation (19). This system requires a set of six boundary conditions, and, as said in § 5, the natural choice is the three spatial coordinates of the observer at the observation time and the three components of the vector tangent to the spatial line of sight.

We are dealing with a numerical problem; therefore, we need to define methods to stop the integration procedure.

One possibility is to fix a finite range of integration, the starting point being the position of the observer and the ending point that of the star. This approach will be adopted for the test on stellar distances.

The other choice is to stop the integration when the tangent vector to the light trajectory becomes constant at the precision level of the computer. This method was used in the light deflection test. In particular, when we consider only the gravitational field generated by the Sun, the total amount of deflection produced beyond 100 AU is under the 0.1 \(\mu\)as level; therefore, this distance turns out to be a pretty good choice for stopping the integration.

As a final consideration, we mention that the geodesic equations can be adapted to various physical situations; in
fact, as explained in § 2, the metric has not been made explicit, the only requirement being that it take the form

\[ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2). \]  

(49)

In our approximation it is

\[ (\eta_{\alpha\beta} + h_{\alpha\beta}) \vec{i} \cdot \vec{j} = 1 + O\left(\frac{v}{c}\right)^3, \]

(50)

so the first step was to verify whether the spatial vectors \( \vec{i} \) obtained at each step of integration satisfied this unitarity condition up to the correct order of accuracy of \( \sim 10^{-12} \). The results show that this always happens along the integration path.

### 7.1. Self-Consistency Test

The basic test is whether our model satisfies spherical symmetry in the case that the Sun is the only source of gravity. Therefore, we took the \((v/c)^2\) approximation of the Schwarzschild metric; the fundamental property of this space-time solution is its spherical symmetry. This means that the amount of light deflection measured by an observer in a given position must depend only on the angular distance of the celestial object from the gravitational source.

We considered an observer at \( r_0 = 1 \) AU and a set of stars placed at different angular distances \( \psi \) from the Sun. For each \( \psi \), we have taken four stars symmetrically positioned with respect to the Sun as in Figure 6.

The results are reported in Table 1 where \( \delta \psi_i \) are the deflections calculated in the four cases. They were calculated to the 0.1 \( \mu \text{as} \) level and, as expected, are the same for a given angular distance: this is a verification of the physical consistency of the model.

The way we calculated the light deflections is discussed below.

#### 7.1.1. Numerical Calculation of the Light Deflection

The integrator takes as boundary conditions the Cartesian coordinates of the observer at the time of observation and the component of the unit vector representing the direction of the line of sight (that is, the tangent unit vector to the light path at the moment of observation). The quantities \( \xi \) and \( \eta \) are recalculated at each step of integration, so the total light deflection angle is the angle between the viewing direction \( \vec{l} \) at the observation time and the direction \( \vec{l}' \) of the unit vector tangent to the light path at the end of integration (Fig. 7).

The most straightforward formula for the computation of an angle between two directions would be just

\[ \Delta = \arccos(v \cdot w), \]  

(51)

where \( v \) and \( w \) are any two given unit tangent vectors. Unfortunately, from a numerical point of view, this formula is of little utility because of the smallness of the deflection angle. In fact, the numerical accuracy that we need for \( \Delta \) is \( \delta \Delta \sim 1 \mu \text{as} \), but for angles as small as those expected for the relativistic deflections, namely, \( \Delta \lesssim 2'' \), the accuracy of the built-in arccos function drops at the \( 10^{-4} \) arcsec level. Let us show this with a simple test. Consider two points \( R_1 \) and \( R_2 \) on the unit celestial sphere with polar coordinates \((\alpha_1, \delta_1)\) and \((\alpha_2, \delta_2)\), and let \( \alpha_1 = \delta_1 = \delta_2 = 0 \) and \( \alpha_2 \) be variable. Then the angle \( \Delta \) between the two directions \( v \) and \( w \) pointing to them is simply given by \( \alpha_2 \). However, application of equation (51) leads to the results with poor accuracy shown in Table 2.

A solution to this problem is to consider an approximate formula for \( \Delta \) that is better suitable for our case. From spherical trigonometry the exact formulae for the angular
distance $\Delta$ and the position $p$ of two points on the unit celestial sphere (Fig. 8) $\mathbf{R}_1 = (\alpha_1, \delta_1)$ and $\mathbf{R}_2 = (\alpha_2, \delta_2)$ are

$$\sin \Delta \sin p = \cos \delta_2 \sin(\alpha_2 - \alpha_1),$$  \hfill (52)  
$$\sin \Delta \cos p = \sin \delta_2 \cos \delta_1 - \cos \delta_2 \sin \delta_1 \cos(\alpha_2 - \alpha_1).$$ \hfill (53)

The Taylor series of the sine and cosine functions about $x = 0$ are

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),$$ \hfill (54)  
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6).$$ \hfill (55)

We want to stress here that these formulae require the polar coordinates of a point given by the transformations $x = \arctan(x/y)$, $\delta = \arcsin(z/r)$, and $r = (x^2 + y^2 + z^2)^{1/2}$, where $x, y, z$ are the respective Cartesian components.

**TABLE 2**

| $\Delta = \alpha^3$ (arcsec) | arccos (arcsec) | $\Delta - \text{arccos}$ (mas) |
|------------------------------|-----------------|-------------------------------|
| 3240.003043                  | 3240.000043     | 0.000                         |
| 3240.003040                  | 324.0000030     | -0.001                        |
| 3240.000030                  | 324.000031      | -0.069                        |
| 3.2400030                    | 3.2400003       | -0.485                        |
| 0.324000                    | 0.323998        | 2.721                         |
| 0.032400                    | 0.032382        | 17.771                        |
| 0.0032400                   | 0.003074        | 166.415                       |
| 0.00032400                  | 0.000000        | 324.000                       |

Notes.—The angular coordinates of two points are given as follows: the first one has $\alpha_1 = \delta_1 = 0$, the second one has $\alpha_2 = 0$ while $\delta_2$ decreases. The angle $\Delta$ between the two directions $v$ and $w$ pointing to $\mathbf{R}_1$ and $\mathbf{R}_2$ is exactly given by $\alpha_2$, but is also calculated taking the arccos of the scalar product of the two unit vectors. The results show that when $\Delta \approx 10^{-4}$ arcsec the accuracy of the built-in arccos is less than 1 mas.
This is the formula that we shall use to compute the angle between any two vectors.\(^5\)

### 7.2. The Light Deflection Test

A formula for the light deflection is given in Misner, Thorne, & Wheeler (1973), for the case of the Schwarzschild metric, as

\[
\Delta m = \frac{2 M_s}{r_o} \sqrt{\frac{1 + \cos \psi}{1 - \cos \psi}},
\]

where \(M_s\) is the solar mass in geometrized units, \(r_o\) is the distance of the observer from the Sun, and \(\psi\) is the angular displacement of a star from the Sun.

Since equation (62) is an analytical formula to the same order of \((v/c)^2\), we expect that its predictions coincide with those of our model, namely, to \(\leq 10^{-9}\) rad.

Taking the same stars and computing the light deflection using the methods seen previously, the tests show that the difference between the two predictions is always much less than the limit of the approximation. The maximum value (\(\leq 15 \mu\)as) is reached for limb-grazing rays and the difference becomes rapidly less than 0.1 \(\mu\)as for \(\psi > \frac{\pi}{2}\) (see Table 3).

### 7.3. Stellar Distances Test: Comparison with a Schwarzschild Model

Recently we developed a Schwarzschild model for astrometric observations (Vecchiato 1996; de Felice et al. 1998). In that model we express the components of the vector \(k^i\) tangent to a null geodesic relative to the spatial axes \(\hat{r}, \hat{\theta}, \hat{\phi}\) of a phase-locked tetrad adapted to an observer on a circular orbit around the Sun (Fig. 9), namely (de Felice & Usseglio-Tomasset 1992).

\[
e_r \equiv \cos \Theta (\hat{r}, k) = \pm \sqrt{\frac{(1 - 2M_s/r_o)^{-1} - \Lambda^2/r_o^2}{(1 - \omega_k \lambda)^2/1 - 3M_s/r_o}},
\]

\[
e_{\theta} \equiv \cos \Theta (\hat{\theta}, k) = \pm \sqrt{\frac{(1/r_o)^2 \Lambda^2 - \lambda^2}{(1 - \omega_k \lambda)^2/1 - 3M_s/r_o}},
\]

\[
e_{\phi} \equiv \cos \Theta (\hat{\phi}, k) = \sqrt{1 - 2M_s/r_o \lambda - \omega_k \lambda r_o (1 - 2M/r_o)}.
\]

Here \(r_o\) is the distance of the observer from the Sun, \(\omega\) is the coordinate angular velocity of the observer, while \(\Lambda\) and \(\lambda\) are two constants of motion of the null geodesic.\(^6\) The constants of motion \(\lambda\) and \(\Lambda\) can be expressed as functions of the impact parameter of the light geodesic with respect to the Sun \((r_o)\) and of the angular Schwarzschild coordinates of the star \((\theta, \phi)\).

\(\footnotesize{\text{TABLE 4}}\)

**Difference between the Parallax \(p_s^*\) Calculated from the Exact Schwarzschild Model and That Reconstructed Using Our \((v/c)^2\) Model (\(p_s^\prime\))**

| \(r_o^* \) (pc) | \( \varepsilon_{c_s} = 4.8 \) | \( \varepsilon_{c_s} = 9.6 \times 10^{-2} \) | \( \varepsilon_{c_s} = 1.9 \times 10^{-3} \) | \( \varepsilon_{c_s} = 4 \times 10^{-5} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1 ................ | -262.2          | -12.5           | -17.8           | -17.8           |
| 10 ............ | 124.6           | -19.1           | -17.8           | -17.8           |
| 100 ........... | 150.9           | -23.1           | -17.8           | -17.8           |
| 1000 .......... | 229.1           | -20.6           | -17.7           | -17.8           |
| 10000 ......   | 85.6            | -12.8           | -17.8           | -17.8           |

\(\footnotesize{\text{Note}}:\) The first column reports the exact distance, and the remaining four give \((p_s^* - p_s^\prime)\) as obtained for each of the corresponding \(\varepsilon_{c_s}\), the level of accuracy for the impact parameter \(r_o^*\) expressed in meters.

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\(\footnotesize{\text{\(^5\) To the \(\zeta^2\) order, we can neglect the difference between \(\cos \delta_1\) and \(\cos \delta_2\), in fact \(\cos \delta_2 = \cos (\delta_1 + \epsilon) \approx \cos \delta_1(1 - \frac{1}{2} \epsilon^2) - \epsilon \sin \delta_1\); substituting this expression into eq. (61) and remembering that \((\alpha_2 - \alpha_1) = \zeta \sim \epsilon\), it is easy to show that \((\alpha_2 - \alpha_1)^2 \cos^2 \delta_2 = (\alpha_2 - \alpha_1)^2 \cos^2 \delta_1 + O(\epsilon^4))\).}

\(\footnotesize{\text{\(^6\) We have denoted the constants of motion consistently with the notation of the cited work; hence, \(\lambda\) should not be confused with the affine parameter of the null geodesic mentioned in \(\S\) 4.}}\)
Finally, $r_c$ can be implicitly expressed as a function of the complete set of Schwarzschild coordinates of the star $(r, \theta, \phi)$. This means that, given the position of a star and of an observer (in Schwarzschild coordinates), we are able to derive the Cartesian components of the tangent to the null geodesic at the position of the observer by means of an almost completely analytical procedure. These components allow us to define the observables according to equation (21).

In the present model we can reproduce the same physical situation and the same observables by taking $h_{\alpha \beta}$ as the approximation to the Schwarzschild metric and by adopting the tetrad, defined by equations (35)–(38), of an observer on a circular orbit. Hence, we can use those observables and the position of the observer as the boundary conditions needed to integrate backward the set of differential equations given by equation (19). It was this complementarity of the two algorithms that was exploited for our test.

We start by giving the "true" position of a star $(r^*, \theta^*, \phi^*)$ and two symmetric observers (with respect to the Sun) $(r_1, \theta_1, \phi_1)$ and $(r_2, \theta_2, \phi_2)$ in the "Schwarzschild model," from which we calculate the tetrad components of the local light directions for both observers using equations (63)–(65). Let us call those components $(e_{r_1}, e_{\theta_1}, e_{\phi_1})$ and $(e_{r_2}, e_{\theta_2}, e_{\phi_2})$. Note that $(r_1, \theta_1, \phi_1)$ and $(e_{r_1}, e_{\theta_1}, e_{\phi_1})$ represent the boundary conditions needed in our model to integrate the light trajectory for the first observer, and analogously $(r_2, \theta_2, \phi_2)$ and $(e_{r_2}, e_{\theta_2}, e_{\phi_2})$ are the boundary conditions for the second observer.

The point of intersection of the two paths obtained in this way should be the position of the star at $(r^*, \theta^*, \phi^*)$; however, we expect to find slightly different coordinates $(r^*, \theta^*, \phi^*)$. Indeed, there are two reasons for this difference: the approximation of our perturbative model, which is up to $(u/c)^2$, and possible numerical errors. The latter should always be maintained below the level of the intrinsic approximation.

8. N-BODY TESTS

The tests described in § 7 consider the Sun as the unique source of gravity. This is because in that section we wanted...
to compare our \((v/c)^2\) model with an exact one (or at least with a model that was already tested at the level of accuracy we require) and the obvious choice was the Schwarzschild model.

However, our model has been built to include an arbitrary number of gravity sources. We shall then subject our model with more than one gravitating body to similar tests to see whether in this case the results are consistent with expectations. This will also serve as the proper testing ground for our forthcoming \((v/c)^3\) model and whatever others would be available. Indeed, to the best of our knowledge, we do not know of any extensive numerical testing for multibody relativistic models.

### 8.1. Deflection due to a Variable Number of Aligned Bodies

In this test we have calculated the total deflection due to the Sun plus a variable number of planets. The geometry is depicted in Figure 10; here the planets are aligned behind the Sun with respect to the observer and the light ray was assumed to be grazing the solar limb. The results are reported in Table 5 and Figure 11.

The first row of the table provides the total deflection in four different cases: the first one gives the result for the Sun plus Jupiter, the second one for the Sun, Jupiter, and Saturn, and so on.

The second row shows the differences between the deflection due to the Sun (i.e., the value \(\Delta \psi_0\)) and the amount of total deflection immediately above. This difference gives the deflection due to the planets alone.

The same effect can be approximately estimated calculating the light deflection due to each planet, with the analytical formula for the Schwarzschild solution, and summing them up. This is what is represented in the third row. Finally, the fourth row simply reports the differences among the values in the second row and those in the third.

We stress here that the predictions contained in the third row are not well justified from a theoretical point of view. However, they can be considered reasonably close to reality; hence, this and the following tests just tell us that our model behaves "well" with respect to a reasonable (but not exact) model under the same physical situation.

### 8.2. Deflection of Two Bodies with Variable Relative Position

In this test we considered only two bodies, the Sun and Jupiter, and calculated the total light deflection experienced by a light ray grazing the solar limb and positioning Jupiter in several different places along its orbit. In particular, the planet’s position varied in a range of 90°, from conjunction to quadrature (Fig. 12).

The results (Table 6 and Fig. 13) show that the contribution of Jupiter, as it should be, adds to that of the Sun when the two masses are both "on the same side" with respect to the light path; their influences subtract when the light passes between the bodies. Moreover, when the light ray grazes the Jovian...

#### Table 6

| Angle (deg) | \(\Delta \psi_{\text{tot}}\) (arcsec) | \(\Delta \psi\) (arcsec) |
|------------|---------------------------------|-----------------|
| 0.0000000 | 1.7509921 | 0.0002699 |
| 0.0355333 | 1.7510261 | 0.0003039 |
| 0.0794550 | 1.7510823 | 0.0003601 |
| 0.1123663 | 1.7511401 | 0.0004179 |
| 0.1376200 | 1.7511988 | 0.0004766 |
| 0.1589100 | 1.7512629 | 0.0005407 |
| 0.1776667 | 1.7513355 | 0.0006133 |
| 0.1946242 | 1.7514202 | 0.0006980 |
| 0.2247322 | 1.7516472 | 0.0008900 |
| 0.2512588 | 1.7520185 | 0.0012963 |
| 0.2752403 | 1.7525750 | 0.0020348 |
| 0.2972935 | 1.7549956 | 0.0042734 |
| 0.3077281 | 1.7563532 | 0.0089130 |
| 0.3097728 | 1.7620437 | 0.0113125 |
| 0.3118041 | 1.7661978 | 0.0154756 |
| 0.3120065 | 1.7667852 | 0.0160630 |
| 0.3120672 | 1.7699702 | 0.0162480 |
| 0.3120773 | 1.7670014 | 0.0162792 |
| 0.3226012 | 1.7344451 | -0.0162771 |
| 0.3226110 | 1.7344754 | -0.0162468 |
| 0.3227480 | 1.7348868 | -0.0158354 |
| 0.3294355 | 1.7354393 | -0.0152829 |
| 0.3327245 | 1.7373081 | -0.0134141 |
| 0.3356888 | 1.7404386 | -0.0102836 |
| 0.3376016 | 1.7423752 | -0.0083470 |
| 0.3370993 | 1.7463871 | -0.0043351 |
| 0.3553359 | 1.7484676 | -0.0025314 |
| 0.3726774 | 1.7491742 | -0.0015480 |
| 0.3892469 | 1.7495309 | -0.0011913 |
| 0.4051432 | 1.7497466 | -0.0009756 |
| 0.4204370 | 1.7498130 | -0.0009092 |
| 0.4351937 | 1.7499953 | -0.0007269 |
| 0.5025188 | 1.7502596 | -0.0004626 |
| 0.7945559 | 1.7505427 | -0.0001795 |
| 0.9401342 | 1.7505846 | -0.0001376 |
| 1.1236807 | 1.7506159 | -0.0001063 |
| 1.3762332 | 1.7506413 | -0.0000809 |
| 1.5891409 | 1.7506548 | -0.0000674 |
| 1.9463344 | 1.7506096 | -0.0000526 |
| 2.2474695 | 1.7506778 | -0.0000444 |
| 2.5127875 | 1.7506832 | -0.0000390 |
| 3.5590511 | 1.7506957 | -0.0000265 |
| 7.9518754 | 1.7507110 | -0.0000112 |
| 11.2547126 | 1.7507143 | -0.0000079 |
| 15.9423686 | 1.7507167 | -0.0000055 |
| 19.5572137 | 1.7507177 | -0.0000045 |
| 30.0733476 | 1.7507193 | -0.0000029 |
| 39.7151372 | 1.7507209 | -0.0000016 |
| 54.7655820 | 1.7507206 | -0.0000013 |
| 67.3801351 | 1.7507209 | -0.0000011 |
| 78.9125108 | 1.7507211 | -0.0000009 |
| 90.0000000 | 1.7507213 | -0.0000009 |

**Fig. 12.**—Total deflection due to the Sun and Jupiter, calculated varying the position of Jupiter along its orbit. [See the electronic edition of the Journal for a color version of this figure.]
limb (rows 18 and 19 of Table 6), the contribution to the total deflection due to the planet approaches the predicted theoretical value of 16.28 mas (de Felice et al. 2000; Klioner 2000) as shown in Table 7.

Finally, it is also worth noting that the amount of Jupiter’s contribution to the total deflection is of the order of 1 μas when the planet is 90° far from the light path, again in very good agreement with the theoretical predictions (Klioner 2000).

8.3. Stellar Distances Revisited: Effect of Many Bodies

As a final test, we have repeated the experiment of § 7.3, but with an increasing number of aligned planets as in § 8.1 (the sequence of planets was kept the same). However, unlike the previous case, in this test we cannot compare our model with any other, so this is in fact another test of self-consistency for the \( (v/c)^2 \) model. The sense of this statement is clarified below.

We consider the same boundary conditions as those used for the test with the Schwarzschild model. These were unchanged as we added Jupiter, Saturn, and the other planets.

Therefore, the expectation was that the distance to a star decreased as we added more planets (and the corresponding parallax increased) because of the increased total deflection (Fig. 14).

The results are reported in Table 8. Stellar distances, as before, ranged from 1 pc to 10 kpc. The first row in each set of three contains the difference between \( p_{\odot} \) and the parallax of the emitting star to an observer (satellite) orbiting around the Sun. The self-consistency of the model assures that these last two values coincide up to numerical errors.

The results show that, as we add more planets, the distances decrease, confirming our qualitative expectations. Moreover, the numerical residuals are \( \Delta p_i \approx 10^{-1} \) μas (\( i = \text{Jup, Sat, Ura, Nep} \)), and this again is compatible with our accuracy for double precision numbers.

9. CONCLUSIONS

We have developed a general relativistic astrometric \( N \)-body model capable of following a light ray all the way from the emitting star to an observer (satellite) orbiting around the Sun.

This model has an intrinsic accuracy of \( \sim 0.1 \) mas \( ((v/c)^2) \), and its computer implementation was validated via a thorough

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**TABLE 7**

**MAXIMUM APPROACH TO THE JUPITER LIMP**

| Row | \( r_{\text{min}} \) (km) | \( \Delta \psi \) (arcsec) |
|-----|-----------------|-----------------|
| 18 | 71461 | 0.0162792 |
| 19 | 71474 | -0.0162771 |

Notes.—The distance of maximum approach of the light ray to Jupiter \( (r_{\text{max}}) \) is calculated in the two extremal cases in which the photons pass nearby the limb of the planet. It is always slightly more than the planet’s actual radius \( R_{\text{Jup}} = 71,398 \) km (Murray & Dermott 1999).

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8 Since, as in § 8.1, each column gives the results for a given number of planets (i.e., the single Sun in first column, the Sun plus Jupiter in the second, and so on), we refer to the quantities in the first row as \( p_{\odot}, p_{\text{Jup}} \), etc., and those in the second row as \( p_{\odot}, p_{\text{Jup}} \), etc.
test campaign that proved (1) the self-consistency of the model; (2) that the amount of light deflection, in the case of a spherical, nonrotating, single-body metric, coincides with that produced by an analytical solution; (3) that, under the same assumptions, our model is able to reconstruct stellar distances; and (4) that in the case of a multibody configuration, for which we do not have any exact or even well-founded analytical approximation, the outcomes for the light deflection and the reconstruction of stellar distances are consistent with the results from semiquantitative derivations.

Finally, this model will be the natural test bed for the more advanced astrometric model, accurate to \((r/c)^3\), which will be presented in a forthcoming article.

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**APPENDIX A**

**MATHEMATICAL DESCRIPTION OF THE MAPPING PROCEDURE**

Here we describe rigorously the mathematical procedure used to obtain the spatial projection of the light ray on the slice \(S(\tau_0)\).

The null geodesic crosses each slice \(S(\tau)\) at a point with coordinates \(\xi^I = \xi^I(\lambda(\tau))\), but this point also belongs to the unique normal to the slice \(S(\tau)\), crossing it with a value of the parameter \(\sigma = \sigma(\xi^I(\lambda), \tau) \equiv \sigma_{\xi(\lambda)}(\tau)\) (Fig. 2).

Let us now define the one-parameter local diffeomorphism:

\[
\phi_{\Delta \sigma} = \phi_{[\sigma(\xi(\lambda)) \rightarrow \sigma(\xi(\lambda))]} : \mathrm{Y} \cap S(\tau) \rightarrow S(\tau_0),
\]  

(A1)

which maps each point of the null geodesic \(\mathrm{Y}\) to the point on the slice \(S(\tau_0)\) that one gets to by moving along the unique normal through the point \(T(\lambda) \cap S(\tau)\), by a parameter distance \(\Delta \sigma = \sigma_{\xi(\lambda)}(\tau_0) - \sigma_{\xi(\lambda)}(\tau)\) (Fig. 15). Since the spatial coordinates \(\xi^I\) are Lie
transported along the normals to the slices, the points in \( S(\gamma) \), which are images of those on the null geodesic under \( \phi_{\Delta\gamma} \), have coordinates \( \gamma^i = \gamma^i(\sigma(\gamma)) \). The curve in \( S(\gamma_0) \), which is the image of \( \gamma \) under \( \phi_{\Delta\gamma} \), is

\[
\hat{\gamma} = \phi_{\Delta\gamma} \circ \gamma,
\]

where the circle means composition of maps and the curve \( \hat{\gamma} \) has tangent vector

\[
\dot{\hat{\gamma}}^\alpha = \left( \phi_{\Delta\gamma} \right)^\alpha \frac{\partial \gamma^\alpha}{\partial \gamma^\beta} \dot{\gamma}^\beta \equiv F^\alpha.
\]

The map given by equation (A1) acts on a four-dimensional manifold with images in a three-dimensional one; hence, the coordinates of the target points \( \gamma \cap S(\gamma) \) and those of the image points on \( S(\gamma_0) \) are, respectively,

\[
\xi^\alpha(\sigma(\tau_0)) = \tau \delta^\alpha_0 + \xi^\alpha(\sigma(\tau)) \delta^\alpha_0,
\]

\[
\xi^\alpha(\sigma(\tau)) = \xi^\alpha(\sigma(\tau)) - \tau \delta^\alpha_0.
\]

From this and equation (A2) it follows that (Fig. 15)

\[
F^\alpha = \gamma^\alpha_{\beta} \dot{\gamma}^\beta;
\]

hence, the curve \( \hat{\gamma} \) is the spatial projection of the null geodesic on the slice \( S(\gamma_0) \) at the time of observation.

APPENDIX B

EXPLICIT EXPRESSION FOR THE TETRAD COMPONENTS

The following expressions represent the coordinate components of each tetrad vector \( \lambda_\alpha \) (\( \alpha = 1, 2, 3 \)) with respect to the coordinate basis \( \partial_\alpha \) (\( \alpha = 0, x, y, z \)) when the observer moves on the orbit that has general barycentric components \( X_s, Y_s, Z_s \) (to be specified):

\[
T_0 = 1,
\]

\[
T_1 = 0,
\]
\[ X_1 = \pm \frac{Y_s \sqrt{g_{yy}} \sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}}{\sqrt{g_{xx} \sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}}}, \]
\[ Y_1 = \pm \frac{X_s \sqrt{g_{xx}} \sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}}{\sqrt{g_{yy} \sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}}}, \]
\[ Z_1 = 0, \]
\[ T_2 = \pm \sqrt{\frac{(g_{xx} X_x^2 + g_{yy} Y_y^2)}{(g_{xx} X_x^2 + g_{yy} Y_y^2)}} \]
\[ X_2 = \pm \frac{X_s \sqrt{(g_{xx} X_x^2 + g_{yy} Y_y^2)}}{\sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}} \]
\[ Y_2 = \pm \frac{Y_s \sqrt{(g_{xx} X_x^2 + g_{yy} Y_y^2)}}{\sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}} \]
\[ Z_2 = \pm \frac{Z_s \sqrt{(g_{xx} X_x^2 + g_{yy} Y_y^2)}}{\sqrt{g_{xx} X_x^2 + g_{yy} Y_y^2}} \]
\[ T_3 = \pm \sqrt{\frac{g_{zz} g_{xx} X_x^2 + g_{yy} Y_y^2}}{g_{xx} X_x^2 + g_{yy} Y_y^2}} \]
\[ X_3 = 0, \]
\[ Y_3 = 0, \]
\[ Z_3 = \pm \sqrt{\frac{g_{zz} g_{xx} X_x^2 + g_{yy} Y_y^2}}{g_{xx} X_x^2 + g_{yy} Y_y^2}} \]

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