Generalization of Schl"afli formula to the volume of a spherically faced simplex

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Abstract

The purpose of this note is to present two identities (contiguity relation and variation formula) concerning the volume of a spherically faced simplex in the Euclidean space. These identities are described in terms of Cayley-Menger determinants and their differentials involved with hypersphere arrangements. They are derived as a limit of fundamental identities for hypergeometric integrals associated with hypersphere arrangements obtained by the authors in the preceding article. The corrected version of the variational formula of Schl"afli type for the volume of a spherically faced simplex is presented.

1 Introduction and preliminaries

In this article, we give a new variation formula for the volume of a pseudo simplex with spherical faces in the Euclidean space (see Theorems I and II). To derive it, we apply the variation formula obtained in [8], which is involved in hypergeometric integrals associated with hypersphere arrangements. This procedure can be done by regularization of integrals (the method of generalized functions) i.e., by taking the zero limit of exponents for hypergeometric integrals (see [11]). A hypersphere arrangement in the $n$ dimensional Euclidean space can be realized by the stereographic projection as the restriction to the fundamental unit hypersphere of a hyperplane arrangement in the $(n + 1)$ dimensional Euclidean space. The theory of hypergeometric integrals associated with hypersphere arrangements has been developed in

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this framework in terms of twisted rational de Rham cohomology (see [6], [8]). It is described in terms of Cayley-Menger determinants. In Theorem III, we give a correct version of some errors in the variation volume formula in [4], which is an extension of L.Schl"afli formula of a geodesic simplex in the unit hypersphere (see [1], [2], [3], [9], [12], [15], [16], [20], [22], [23], and see also [9], [20], [21] related to the bellows conjecture).

1. Let \( n + 1 \) real quadratic polynomials \( f_j \) in \( n \) variables \( x = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \)

\[
f_j(x) = Q(x) + \sum_{\nu=1}^{n} 2\alpha_{j\nu} x_{\nu} + \alpha_{j0} \quad (1 \leq j \leq n + 1)
\]

be given, where \( Q(x) \) denotes the quadratic form

\[
Q(x) = \sum_{j=1}^{n} x_j^2,
\]

and \( \alpha_{j\nu}, \alpha_{j0} \in \mathbb{R} \).

Let \( S_j \) be the \( n - 1 \) dimensional hypersphere defined by \( f_j(x) = 0 \). Denote by \( O_j \) the center of \( S_j \), by \( r_j \) the radius of \( S_j \) and by \( \rho_{jk} \) the distance of \( O_j \) and \( O_k \).

Then

\[
r_j^2 = -\alpha_{j0} + \sum_{\nu=1}^{n} \alpha_{j\nu}^2,
\]

\[
\rho_{jk}^2 = \sum_{\nu=1}^{n} (\alpha_{j\nu} - \alpha_{k\nu})^2.
\]

Let \( \mathcal{A} = \bigcup_{j=1}^{n+1} S_j \) be the arrangement of hyperspheres consisting of \( n + 1 \) hyperspheres \( S_j \).

2. Denote by \( N \) the set of indices \( \{1, 2, \ldots, n + 1\} \).

**Definition 1** Denote by \( J, K \) the two non-empty subsets of indices \( J = \{j_1, \ldots, j_p\}, K = \{k_1, \ldots, k_p\} \subset N \) of size \( p \). Cayley-Menger determinants associated with \( \mathcal{A} \) are given by the following system of determinants (see [9],
For example, $B(0) = -1$, $B(0j) = 2\rho_{jk}$, $B(0* j) = 2r_j^2$ and

$$B(0* jk) = r_j^4 + r_k^4 + \rho_{jk}^2 - 2r_j^2 r_k^2 - 2r_j r_k \rho_{jk},$$

$$B(0 jk l) = \rho_{jk}^4 + \rho_{jl}^4 + \rho_{kl}^4 - 2\rho_{jk}^2 \rho_{jl}^2 - 2\rho_{jk} \rho_{jl} \rho_{kl}^2 - 2\rho_{jk}^2 \rho_{kl}^2.$$  

From now on, we shall assume the following condition:

$$(H1) \quad (-1)^p B(0 J) > 0, \quad (-1)^{p+1} B(0* J) > 0 \quad (1 \leq p \leq n + 1),$$
for $|J| = p$ (the size of $J$ being denoted by $|J|$).

3. Denote the inside and the outside of $S_j$ in $\mathbb{R}^n$ by

$$D_j^- := \{ f_j \leq 0 \}, \quad D_j^+ := \{ f_j \geq 0 \}$$

respectively. For $K \subset N$, denote the real domains $D_K^- = \cap_{j \in K} D_j^-$ and $D_K^+ = \cap_{j \in K} D_j^+$ respectively.

In particular, we can take as the (non-empty) domain $D = D_{12 \ldots n+1}$ defined by

$$D_{12 \ldots n+1} = \cap_{j=1}^{n+1} D_j^-,$$

$$D_j^- : f_j \leq 0 \subset \mathbb{R}^n \quad (1 \leq j \leq n+1).$$

It is a non-empty spherically faced $n$-simplex (which will be called pseudo $n$-simplex in the sequel). The boundary of $D$ consists of the faces $D \cap S_k$ where, for any $K \subset N$ such that $|K| = p$, $1 \leq p \leq n$, the intersection $S_k = \cap_{j \in K} S_j$ defines an $n-p$ dimensional sphere. In particular, $\cap_{k \in \partial_j N} S_k$ consists of two points. Here the symbol $\partial_j K$ denotes the set of indices obtained by the deletion of the element $j$ from the set $K$. $K^c$ denotes the complement of $K$ in $N$. We can also see that $D_K^- \cap D_K^+$ are a pseudo $n$-simplex.

The orientation of $\mathbb{R}^n$ and $D$ is determined such that the standard $n$ form $\varpi$ is positive:

$$\varpi = dx_1 \wedge \ldots \wedge dx_n > 0.$$ 

Let $P_j$ be the point in $\mathbb{R}^n$ such that

$$\{ P_j \} = \cap_{k \in \partial_j N} S_k \cap \partial D \quad (\partial D \text{ denotes the boundary of } D).$$

4. We can take the Euclidean coordinates $x_1, \ldots, x_n$ such that the polynomials $f_j$ have the following expressions:

$$f_j(x) = Q(x) + \sum_{\nu=1}^{n+1-j} 2\alpha_{j\nu} x_\nu + \alpha_j 0, \quad (1 \leq j \leq n) \quad (1)$$

$$f_{n+1}(x) = Q(x) + \alpha_{n+1} 0. \quad (2)$$

We assume for simplicity that $\alpha_{j,n+1-j} > 0$ ($1 \leq j \leq n+1$) and that $P_j$ satisfies the following condition:

(\(\mathcal{H}2\)) The $x_{n+1-j}$-coordinate of $P_j$ is negative for every $j$.
We have the equalities
\[ \prod_{j=p}^{n} \alpha_{j} n+1-j = \sqrt{(-1)^{n-p} B(0 \ldots n n + 1)} \quad (1 \leq p \leq n). \tag{3} \]

Denote by \( \Delta[P_1, P_2, \ldots, P_{n+1}] \) and \( \widetilde{\Delta}[P_1, P_2, \ldots, P_{n+1}] \) be the linear \( n \)-simplex and the pseudo \( n \)-simplex respectively with hyperspherical faces both with vertices \( P_j \) such that their sign of orientation is \((-1)^{\frac{n(n+1)}{2}}\).

By definition, the following properties are valid.

**Lemma 2**

(i) \((-1)^{\frac{n(n+1)}{2}+\nu-1} df_1 \wedge \cdots \wedge df_{\nu} \cdots \wedge df_{n+1} > 0 \quad (1 \leq \nu \leq n + 1)\) on \( D \).

(ii) The simplex \([P_1, P_2, \ldots, P_{n+1}]\) has the sign of orientation \((-1)^{\frac{n(n-1)}{2}}\) such that
\[ [P_1, P_2, \ldots, P_{n+1}] = (-1)^{\frac{n(n-1)}{2}} D. \]

**Proof.** Indeed, we can show that
\[ df_2 \wedge \cdots \wedge df_{n+1} = 2^n (-1)^{\frac{(n-1)(n-2)}{2}} \prod_{j=2}^{n} \alpha_{j} n+1-j x_n \varpi. \tag{4} \]

Seeing that \( x_n < 0 \), (i) is proved. (ii) follows from \((H2)\). \( \square \)

**Lemma 3** For \( J \subset N \ (|J| = p, 1 \leq p \leq n - 1) \), \( S_J \) is a sphere of dimension \( n - p \). Its radius \( r_J \) equals
\[ r_J = \sqrt{\frac{1}{2} B(0 \star J)/B(0 J)}. \]

By the use of coordinates \( x_j \), \( S_{n-p+2,\ldots,n+1} \) represent the spheres of dimension \( n - p \) satisfying the following equalities:
\[ S_{n+1} : Q(x) = r_{n+1}^2, \]
\[ S_{n-p+2,\ldots,n+1} : \sum_{j=p}^{n} x_j^2 = r_{n-p+2,\ldots,n+1}^2, \]
\[ S_{2,\ldots,n+1} : \{ \text{two points} \}. \]

5. Denote by \( f_J \) the product \( \prod_{j \in J} f_j \). The residues of the forms \( \varpi \) along \( S_J \) can be computed explicitly as follows.
**Proposition 4** For $J = \{j_1, \ldots, j_p\} \subset N$, we have

\[
\text{Res}_{J} \left[ \frac{\omega}{f_J} \right] = \frac{\omega}{df_{j_1} \wedge \cdots \wedge df_{j_p}}_{S_J} = \frac{(-1)^{p(p-2)/2}}{\sqrt{(-1)^{p-1}2^p B(0 \star J)}} \omega_J, \quad (1 \leq p \leq n)
\]

where $\omega_J$ denote the standard sphere volume elements on $S_J$ respectively such that

\[
\omega_{n+1} = \sum_{\nu=1}^{n} (-1)^{\nu-1} x_{\nu} dx_1 \wedge \cdots \wedge dx_n, \\
\omega_{n-p+2\ldots n+1} = \sum_{\nu=1}^{n-p+1} (-1)^{\nu-1} x_{p+\nu-1} dx_p \wedge \cdots \wedge dx_{p+\nu-1} \wedge dx_n, \quad (1 \leq p \leq n-1)
\]

\[
\omega_{2\ldots n+1} = -1,
\]

and

\[
r_{n-p+2\ldots n+1} = \sqrt{-\frac{1}{2} B(0 \star n - p + 2 \ldots n + 1)}.
\]

$\omega_J$ are obtained respectively from $\omega_{n-p+2\ldots n+1}$ by permutations of elements in the set of indices $N$.

**Proof.** Because of symmetry, it sufficient to prove (4) in the case where $J = \{n - p + 2, \ldots, n + 1\}$.

First, we prove (4) in the case of (5) and (6). Since

\[
df_{n-p+2} \wedge \cdots \wedge df_{n+1} = 2^p (-1)^{(p-1)(p-2)/2} \prod_{j=n-p+2}^{n} \alpha_{j, n-j+1} \sum_{j=p}^{n} x_j dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dx_j,
\]

(3) (7) and (9) imply

\[
df_{n-p+2} \wedge \cdots \wedge df_{n+1} \wedge \omega_{n-p+2\ldots n+1}
\]

\[
= 2^p (-1)^{(p-1)(p-2)/2} \prod_{j=n-p+2}^{n} \alpha_{j, n-j+1} \frac{(\sum_{j=p}^{n} x_j^2)}{r_{n-p+2\ldots n+1}} \omega
\]

\[
= 2^p (-1)^{(p-1)(p-2)/2} \sqrt{(-1)^{p-1} B(0 \star n - p + 2 \ldots n + 1) \over 2^p} \omega \quad (1 \leq p \leq n - 1).
\]

(11)
Hence, along $S_J$ it follows that

$$\left[ \frac{\omega}{df_{n-p+2} \wedge \cdots \wedge df_{n+1}} \right]_{S_J} = \frac{(-1)^{\frac{n-1}{2}(p-2)}}{\sqrt{(-1)^{p+1} 2^p B(0 \ast n - p + 2 \ldots n + 1)}} \omega_{n-p+2 \ldots n+1}. $$

When $p = n$, in view of (3), (4) and $x_n < 0$, we have the identity

$$x_n = -r_{2 \ldots n+1} < 0.$$ 

Hence, at $P_1$ it follows that

$$\left[ \frac{\omega}{df_2 \wedge \cdots \wedge df_{n+1}} \right]_{P_1} = -\frac{(-1)^{\frac{n-1}{2}(n-2)}}{\sqrt{(-1)^{n+1} 2^n B(0 \ast 2 \ldots n + 1)}}.$$ 

(12)

\[ \square \]

**Notation 1** For $J \subset N$, denote by $F_J$ the rational $n$-form and by $W_0(J) \omega$ a linear combination of $F_K$ ($K \subset J$) as follows:

$$F_J = \frac{\omega}{f_{j}},$$

$$W_0(J) \omega = -\sum_{\nu \in J} B\left( \begin{array}{c} 0 \ast \partial_{\nu} J \\ 0 \nu \partial_{\nu} J \end{array} \right) F_{\partial_{\nu} J} + B(0 \ast J) F_J.$$ 

Remark that $F_J$ is also a linear combination of $W_0(K) \omega$ ($K \subset J$, $|K| \geq 1$).

The following Lemma can be proved by a direct calculation (see [8] Lemma 12).

**Lemma 5**

$$\sum_{\nu=1}^{n+1} (-1)^{\nu-1} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_{\nu}}{f_{\nu}} \wedge \frac{df_{n+1}}{f_{n+1}} = \frac{2^n (-1) \frac{n(n-1)}{2} + 1}{\sqrt{(-1)^{n+1} 2^n B(0N)}} W_0(N) \omega. $$

(13)

$s_{\partial_j N}$ consists of two points (denoted by $P_j, P_j'$) satisfying the equations

$$f_k = 0 \quad (k \in \partial_j N).$$

The following Proposition gives the values of $f_j$ at the points $P_j, P_j'$.  

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Proposition 6 The values of $\frac{1}{f_j}$ at $P_j, P'_j$ are negative and positive respectively. They are evaluated as

\[ \left[ \frac{1}{f_j} \right]_{P_j} = \frac{(-1)^{n+1} \sqrt{B(0 \star \partial_j N) B(0 N)} + B\left( \begin{array}{cc} 0 & \partial_j N \\ 0 & j \partial_j N \end{array} \right)}{B(0 \star N)} < 0, \]

(14)

\[ \left[ \frac{1}{f_j} \right]_{P'_j} = \frac{(-1)^n \sqrt{B(0 \star \partial_j N) B(0 N)} + B\left( \begin{array}{cc} 0 & \partial_j N \\ 0 & j \partial_j N \end{array} \right)}{B(0 \star N)} > 0. \]

(15)

Due to the product formula for resultant,

\[ \left[ \frac{1}{f_j} \right]_{P_j} \left[ \frac{1}{f_j} \right]_{P'_j} = -\frac{B(0 \partial_j N)}{B(0 \star N)} < 0. \]

Proof. For simplicity, we may assume that $j = 1$. First notice that $f_1 \neq 0$ at $P_1$. By taking the residues of both sides of (12) at $P_1$ (see [24]), we have from (11)

\[ 1 = \text{Res}_{P_1} \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \]

\[ = \frac{2^n(-1)^{n(n-1)+1}}{\sqrt{(-1)^{n+1} 2^n B(0 N)}} \left\{ -B\left( \begin{array}{cc} 0 & \partial_1 N \\ 0 & 1 \partial_1 N \end{array} \right) + B(0 \star N) \left[ \frac{1}{f_1} \right]_{P_1} \text{Res}_{P_1} \left[ \frac{\varpi}{f_2 \cdots f_{n+1}} \right] \right\} \]

\[ = \frac{(-1)^{n+1}}{\sqrt{B(0 \star \partial_1 N) B(0 N)}} \left\{ -B\left( \begin{array}{cc} 0 & \partial_1 N \\ 0 & 1 \partial_1 N \end{array} \right) + B(0 \star N) \left[ \frac{1}{f_1} \right]_{P_1} \right\}. \]

(16)

We can solve the equation (16) with respect to $\left[ \frac{1}{f_1} \right]_{P_1}$ and gets the formula (14). (15) can be deduced in a similar way.

\[ \square \]

2 Main Theorems

Main Theorems are given by “regularization procedure of integrals” and are a consequence from some identities of hypergeometric integrals defined on the $n$ dimensional complex affine space $\mathbb{C}^n$. 

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Suppose that the system of exponents $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ are given such that all $\lambda_j > 0$.

Let $\Phi(x)$ be the multiplicative meromorphic function
\[
\Phi(x) = \prod_{j \in N} f_j^{\lambda_j}.
\]

For each $J \subset N \ (1 \leq |J|)$, consider the integral of $|\Phi(x)|$ over the domain $D = D_J^- \cap D_J^+$:
\[
\mathcal{J}_\lambda(\varphi) = \int_D |\Phi(x)| \varphi \omega,
\]
where we take the branch of $\Phi(x)$ such that $\Phi(x) > 0$ for $x \in D$. There exists a twisted $n$-cycle $\mathfrak{z}$ such that
\[
\mathcal{J}_\lambda(\varphi) = \int_\mathfrak{z} \Phi(x) \varphi \omega.
\]

Then the following Proposition holds true (refer to [8]):

**Proposition 7** For each $D = D_J^- \cap D_J^+$, the following identity holds true:
\[
(2\lambda_\infty + n) \mathcal{J}_\lambda(1) = \sum_{p=1}^{n+1} \sum_{J \subset N, |J| = p} (-1)^p \frac{\prod_{j \in J} \lambda_j}{\prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)| W_0(J) \omega,
\]
where the sum ranges over the family of all unordered sets $J$ such that $J \subset N \ (1 \leq p \leq n + 1)$ and $\lambda_\infty = \sum_{j=1}^{n+1} \lambda_j$.

On the other hand, the variation of $\mathcal{J}_\lambda(1)$ is defined by
\[
d_B \mathcal{J}_\lambda(1) = \sum_{j=1}^{n+1} \sum_{\nu=0}^{n} d\alpha_{j\nu} \frac{\partial}{\partial \alpha_{j\nu}} \mathcal{J}_\lambda(1).
\]

We want to give an explicit variation formula for $\mathcal{J}_\lambda(1)$ with respect to the parameters $r_j^2, \rho_{kl}^2$. To do that, it is necessary to introduce the system of special 1-forms $\theta_j$:
Definition 8

\[ \theta_j = -\frac{1}{2} d \log r_j^2; \]
\[ \theta_{jk} = \frac{1}{2} d \log \rho_{jk}^2; \]
\[ \theta_J = (-1)^p \sum_{j,k \in J, j < k} \frac{1}{2} d \log B(0jk) \cdot \sum_{\mu_1, \ldots, \mu_{p-2}} \prod_{\nu=1}^{p-2} B\left( \begin{array}{c} 0 \ast \mu_{\nu-1} \ldots \mu_1 \ j \ k \\ 0 \mu_\nu \mu_{\nu-1} \ldots \mu_1 \ j \ k \end{array} \right) \]
\[ \cdot \prod_{j \in J} \lambda_j \prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu) \theta_J \int_D |\Phi(x)| W_0(J) \varpi. \]

where \( \mu_1, \ldots, \mu_{p-2} \) ranges over the family of all ordered sequences consisting of \( p - 2 \) different elements of \( \partial_j \partial_k J \).

Then we have the following (refer to [8]).

Proposition 9
For each \( D = D_+^J \cap D_0^J \), we have

\[ d_B J_\lambda(1) = \sum_{p=1}^{n+1} \sum_{J \subset N, |J| = p} \frac{\prod_{j \in J} \lambda_j}{\prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu)} \theta_J \int_D |\Phi(x)| W_0(J) \varpi. \]

\( J \) ranges over the family of unordered sets such that \( J \subset N \) and \( 1 \leq p \leq n+1 \).

We now take \( D = D_N^- \). The following Theorem is an immediate consequence of Propositions 8 and 9 tending \( \lambda_j \to 0 \) for all positive \( \lambda_j \).

[Theorem 1]
Assume the conditions (H1) and (H2). Let \( v(D) \) be the volume of the pseudo \( n \)-simplex \( D = D_N^- \):

\[ v(D) = \int_D \varpi. \]

Then
(i) we have the identity:

\[ n \ v(D) = - \sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{J \subset N, |J| = p} (-1)^p \sqrt{\frac{(-1)^{p+1} B(0 \ast J)}{2^p}} v_J 
+ (-1)^n \frac{1}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(0N)}{2^n}}. \]
(ii) the following variational formula holds true:

\[
\begin{align*}
    d_B v(D) &= -\sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{J \subseteq N, |J| = p} \sqrt{\frac{(-1)^p B(0 \star J)}{2^p}} \theta_J v_J \\
    &\quad - \frac{1}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(0 \star N)}{2^n}} \theta_N.
\end{align*}
\]

(18)

Here \(v_j, v_{jk}, v_J\) denote the lower dimensional volumes corresponding to the boundaries \(S_j \cap \partial D, S_j \cap S_k \cap \partial D, S_J \cap \partial D (|J| \geq 3)\) respectively such that

\[
v_J = \int_{S_J \cap \partial D} |\varpi_J| \quad (1 \leq |J| \leq n).
\]

(19)

**Remark** The formula (18) is just an analog of the classical variational formula due to L.Schl"afi concerning the volume of a \(n\) dimensional geodesic simplex in the unit hypersphere.

**Proof.** Since both identities can be proved in the same way, we only give a proof for the latter identity (18).

Proposition 9 shows

\[
    d_B J_{\lambda}(1) = \sum_{j=1}^{n+1} \lambda_j \int_D |\Phi(x)| W_0(j) \varpi \theta_j + \sum_{j<k} \frac{\lambda_j \lambda_k}{\lambda_\infty + n - 1} \int_D |\Phi(x)| W_0(jk) \varpi \theta_{jk} \\
    + \sum_{p=3}^{n} \frac{\prod_{j \in J} \lambda_j}{\prod_{\nu=1}^{p-1} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)| W_0(J) \varpi \theta_J \\
    + \frac{\prod_{j \in N} \lambda_j}{\prod_{\nu=1}^{n} (\lambda_\infty + n - \nu)} \int_D |\Phi(x)| W_0(N) \varpi \theta_N.
\]

(20)

Let us take the limit for \(\lambda_j \to 0\) \((1 \leq j \leq n+1)\) on both sides of (19) in the LHS such that

\[
    \lambda_j = \varepsilon (\varepsilon \downarrow 0).
\]

In the LHS,

\[
    \lim_{\lambda \to 0 \,(1 \leq j \leq n+1)} d_B J_\lambda(1) = d_B v(D).
\]

In the RHS, first remark that

\[
    \lim_{\varepsilon \downarrow 0} \prod_{j \in J} \lambda_j \int_D |\Phi(x)| \varphi(x) F_K = 0,
\]

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provided $K \not\subset J$.

In the RHS, seeing that $f_j < 0$ in $D$, due to Proposition 4, the following equalities hold by the method of generalized functions (see [11] Chap III, 2):

$$\lim_{\lambda \to 0} \lambda_j \int_D |\Phi(x)| \varphi(x) W_0(j) \varpi = \lim_{\lambda \to 0} \lambda_j \int_D |\Phi(x)| \varphi(x) B(0 * j) \frac{\varpi}{f_j}$$

$$= -B(0 * j) \lim_{\lambda \to 0} \lambda_j \int_D |\Phi(x)| \varphi(x) \frac{\varpi}{|f_j|}$$

$$= -\sqrt{\frac{B(0 * j)}{2}} \int_{S_j \cap \partial D} [\varphi]_{S_j} |\varpi_j|,$$  \hspace{1cm} (21)

$$\lim_{\lambda \to 0} \lambda_j \lambda_k \int_D |\Phi(x)| \varphi(x) W_0(jk) \varpi = B(0 * jk) \lim_{\lambda \to 0} \lambda_j \lambda_k \int_D |\Phi(x)| \varphi(x) \frac{\varpi}{f_j f_k}$$

$$= B(0 * jk) \lim_{\lambda \to 0} \lambda_j \lambda_k \int_D |\Phi(x)| \varphi(x) \frac{\varpi}{|f_j f_k|}$$

$$= B(0 * jk) \int_D |\Phi(x)| \varphi(x) \frac{\varpi}{|df_j \wedge df_k|}$$

$$= -\sqrt{-\frac{-B(0 * jk)}{4}} \int_{S_j \cap \partial D} [\varphi]_{S_j} |\varpi_{jk}|.$$  \hspace{1cm} (22)

In general, we have, for $|J| \leq n$,

$$\lim_{\lambda \to 0} \prod_{j \in J} \lambda_j \int_D |\Phi(x)| \varphi(x) W_0(J) \varpi = B(0 * J) \lim_{\lambda \to 0} \prod_{j \in J} \lambda_j \int_D |\Phi(x)| \varphi(x) F_J$$

$$= (-1)^p B(0 * J) \lim_{\lambda \to 0} \prod_{j \in J} \lambda_j \int_D |\Phi(x)| \varphi(x) |F_J|$$

$$= (-1)^p B(0 * J) \int_{S_j \cap \partial D} [\varphi]_{S_j} \frac{\varpi}{|df_{j_1} \wedge \ldots \wedge df_p|} |s_j|$$

$$= -\sqrt{\frac{(-1)^{p+1} B(0 * J)}{2^p}} \int_{S_j \cap \partial D} [\varphi]_{S_j} [\varphi]_{S_j} |\varpi_j|,$$  \hspace{1cm} (23)

where $J = \{j_1, \ldots, j_p\}, p \leq n$.

As for the last term of the RHS of (20), when $\lambda \to 0$, the limit value is divided into the ones at $P_j$ ($1 \leq j \leq n + 1$).

We assume that $D$ is divided into $n + 1$ domains $D_j^*$ ($j \in N$) such that

$$P_j \text{ lies only in the inside of } \partial D_j^*$$

$$D = \bigcup_{j \in N} D_j^*.$$
Then
\[ \int_D |\Phi(x)| \varpi = \sum_{j \in N} \int_{D_j^*} |\Phi(x)| \varpi. \]

We restrict ourselves to the integral over \(D_1^*\). Since \(f_k < 0\) (1 \(\leq k \leq n+1\)) in the inside of \(D_1^*\) and \([f_k]_{P_i} = 0\) (2 \(\leq k \leq n+1\) and \([f_1]_{P_i} < 0\),
\[
\lim_{\lambda \to 0} \frac{\prod_{j=1}^{n+1} \lambda_j}{\prod_{\nu=0}^{n-1} (\lambda_\infty + \nu)} \int_{D_1^*} |\Phi(x)| \ W_0(N) \varpi
\]
\[
= \lim_{\epsilon \downarrow 0} \frac{(-1)^n}{(n + 1) \prod_{\nu=1}^{n-1} ((n + 1)\epsilon + \nu)} \cdot \left( -B \left( \begin{array}{cc} 0 & \partial_1 N \\ 0 & \partial_1 N \end{array} \right) + B(0 \ast N)[\frac{1}{f_1}]_{P_i} \right) \left[ \frac{\varpi}{df_2 \wedge \cdots \wedge df_{n+1}} \right]_{P_i} \right]
\]
\[
= - \frac{1}{(n + 1) (n - 1)!} \sqrt{\frac{(-1)^{n+1} B(0N)}{2^n}},
\]
in view of (12) and Proposition 6.

Similarly, the integral over each \(D_j^*\) has the same value as above:
\[
\lim_{\epsilon \downarrow 0} \frac{\prod_{j=1}^{n+1} \lambda_j}{\prod_{\nu=0}^{n-1} (\lambda + \nu)} \int_{D_j^*} |\Phi(x)| \ W_0(N) \varpi
\]
\[
= - \frac{1}{(n + 1) (n - 1)!} \sqrt{\frac{(-1)^{n+1} B(0N)}{2^n}}.
\] (24)

(20) and (24) imply Main Theorem I (ii). (i) can be proved in the same way from Proposition 7.

For \(D = D_j \cap D_{j+c}^*\), the similar formulae to (17) and (18) for \(v(D)\) can be described and be proved in the same way:
\[ nv(D) = - \sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{J \subset N, |J|=p} (-1)^p \sqrt{\frac{(-1)^{p+1} B(0 \ast J)}{2^p}} v_J \]
\[ + (-1)^n \frac{1}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(0N)}{2^n}}. \]
\[ d_B v(D) = - \sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{J \subset N, |J|=p} (-1)^p \sqrt{\frac{(-1)^{p+1} B(0 \ast J)}{2^p}} \theta_J v_J \]
\[ - \frac{1}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(0N)}{2^n}} \theta_N. \]
In place of \((\mathcal{H}1)\), we can also consider the following condition:

\[(\mathcal{H}1') \quad (-1)^p B(0 J) > 0, \quad (-1)^{p+1} B(0 \star J) > 0 \quad (1 \leq p \leq n \text{ for } |J| = p), \]
\[-(-1)^{n+1} B(0 N) > 0, \quad (-1)^n B(0 \star N) < 0,\]

instead of \((-1)^n B(0 \star N) > 0\) (For \(n = 2\), refer to Figure 2).

There are \(2^{n+1} - 1\) non-empty bounded chambers

\[D^-_j : f_j \leq 0 \quad (j \in J), \quad f_j \geq 0 \quad (j \in J^c),\]

where \(J\) ranges over the family of subsets in \(N\) such that \(|J| = p, 0 \leq p \leq n\).

When \(p = 0\), i.e., when \(J\) is empty, we may denote \(D^-_j\) by \(D^+_N\):

\[D^+_N : f_j \geq 0 \quad (1 \leq j \leq n + 1).\]

We assume that the inside of \(D^+_N\) is a non-empty domain. \(D^+_N\) is the support of the \(n\)-simplex \(\tilde{\Delta}[P_1 \ldots P_{n+1}]\) such that \(P_j \in S_{\partial_j N}\). \(S_{\partial_j N}\) consists of two points \(\{P_j, P'_j\}\) (see Figure 2 for 2 dimensional case). Then

\[
\left[ \frac{1}{f_j} \right]_{P_j} > 0, \quad \left[ \frac{1}{f_j} \right]_{P'_j} > 0,
\]

and

\[
\left[ \frac{1}{f_j} \right]_{P_j} \left[ \frac{1}{f_j} \right]_{P'_j} = \frac{B(0 \partial_j N)}{B(0 \star N)} > 0.
\]

For the volume of \(D^+_N\), the following theorem holds true.

[Theorem II]

Let \(v = v(D^+_N)\) be the volume of \(D^+_N\) and \(v_J\) be similarly defined as in (19) by

\[
\begin{align*}
v &= v(D^+_N) = \int_{D^+_N} \varpi, \\
v_J &= \int_{s_J \cap D^+_N} |\varpi_J| \quad (1 \leq p \leq n, \ p = |J|).
\end{align*}
\]

Then the following identities hold true:
\[ n \, v(D^+_N) = - \sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{|J|=p} \sqrt{(-1)^{p+1} B(0 \star J)} \frac{2^p}{v_J} \]
\[ + \frac{1}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(N)}{2^n}}. \]

(ii)
\[ d_B v(D^+_N) = - \sum_{p=1}^{n} \frac{(n-p)!}{(n-1)!} \sum_{|J|=p} (-1)^p \theta_J \sqrt{\frac{(-1)^{p+1} B(0 \star J)}{2^p}} v_J \]
\[ + \frac{(-1)^{n+1}}{(n-1)!} \sqrt{\frac{(-1)^{n+1} B(N)}{2^n}} \theta_N. \]

**Proof.** Theorem II can also be proved in the same way as Theorem I. This theorem also follows from the identities in Theorem 1 by an analytic continuation (i.e., Picard-Lefschetz transformation of twisted cycles) of \( v(D^-_N) \) with respect to the parameters \( r^2_j, \rho^2_{jk} \) such that
\[ B(0 \star N) \rightarrow -B(0 \star N), \]
\[ v(D^-_N) \rightarrow (-1)^n v(D^+_N), \]
\[ v(D^-_J) \rightarrow (-1)^{p-1} v(D^+_J). \]

\[ \square \]

**Remark** An elementary proof of Theorem II (i) (and therefore of Theorem I (i)) will be given in the Appendix.

In the following, we give a simple application of Main Theorems.

### 3 Examples

[example1]

In the \( n \) dimensional Euclidean space, consider two hyperspheres \( S_1, S_2 \) with the centers \( O_1, O_2 \) and with radii \( r_1, r_2 \) such that the distance between \( O_1 \) and \( O_2 \) is equal to \( \rho_{12} \).
Assume \( S_1 \cap S_2 \) is a non-empty \( n-2 \) dimensional sphere. \( S_1 \cap S_2 \) is contained in the hyperplane \( L \) which intersects the segment \( O_1O_2 \) at a point \( M \).

The radius \( h \) of \( S_1 \cap S_2 \), the distance \( O_1M \) and \( O_2M \) are expressed as

\[
h = \frac{\sqrt{-B(0 \star 12)}}{2\rho_{12}} = r_1 \sin \frac{1}{2} \psi_{12} = r_2 \sin \frac{1}{2} \psi_{21},
\]

\[
O_1M = r_1 \cos \frac{1}{2} \psi_{12} = \frac{B(0 2 1)}{2\rho_{12}},
\]

\[
O_2M = r_2 \cos \frac{1}{2} \psi_{21} = \frac{B(0 1 2)}{2\rho_{12}}
\]

such that

\[
\rho_{12} = r_1 \cos \frac{1}{2} \psi_{12} + r_2 \cos \frac{1}{2} \psi_{21},
\]

where \( \psi_{12}, \psi_{21} \) satisfy \( 0 < \psi_{12} < \pi, 0 < \psi_{21} < \pi \).

Denote by \( D_{12} \) the common domain (lens domain) surrounded by \( S_1, S_2 \). \( S_1 \cap S_2 \) is an \( (n-2) \) dimensional sphere.

The volume \( v(D_{12}) \) of \( D_{12} \) can be evaluated by an elementary calculus as follows:

\[
v(D_{12}) = v_1 + v_2,
\]

(25)

where

\[
v_1 = \frac{1}{n-1} C_{n-2} r_1^n \int_{\cos \frac{1}{2} \psi_{12}}^{1} (1 - \tau^2)^{\frac{n-1}{2}} d\tau,
\]

\[
v_2 = \frac{1}{n-1} C_{n-2} r_2^n \int_{\cos \frac{1}{2} \psi_{21}}^{1} (1 - \tau^2)^{\frac{n-1}{2}} d\tau,
\]

\( v_1, v_2 \) denote the volumes of the domains surrounded by \( S_1, L \) and \( S_2, L \) respectively, and \( C_{n-2} \) denotes the volume of the \( n-2 \) dimensional unit hypersphere:

\[
C_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}.
\]
The lower dimensional volumes of $S_1 \cap S_2$, $S_1 \cap \partial D_{12}$, $S_2 \cap \partial D_{12}$ equal respectively

$$v(S_1 \cap S_2) = C_{n-2} h^{n-2},$$

$$v(S_1 \cap \partial D_{12}) = \frac{\partial v}{\partial r_1} = C_{n-2} r_1^{n-1} \int_0^{\psi_{12}} \sin^{n-2} t \, dt,$$  \hspace{1em} (26)

$$v(S_2 \cap \partial D_{12}) = \frac{\partial v}{\partial r_2} = C_{n-2} r_2^{n-1} \int_0^{\psi_{21}} \sin^{n-2} t \, dt.$$  \hspace{1em} (27)

The integral in the RHS can be expressed in the following expansion:

$$\int_0^{\psi_{jk}} \sin^{n-2} t \, dt = - \sum_{0 \leq 2\nu \leq n-3} \cos \frac{\psi_{jk}}{2} \frac{(n-3) \cdots (n-2\nu+1)}{(n-2) \cdots (n-2\nu)} \left( \sin \frac{\psi_{jk}}{2} \right)^{n-3-2\nu}$$

$$+ \left\{ \begin{array}{cc}
C''_{n-2} \left( 1 - \cos \frac{\psi_{jk}}{2} \right) \\
C''_{n-2} \frac{\psi_{jk}}{2} \end{array} \right\} (\{j,k\} = \{1,2\}),$$  \hspace{1em} (28)

where $2C''_{n-2}$ or $2\pi C''_{n-2}$ equals $\frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$ according as $n$ is odd or even.

**Remark** In the case where $n$ is odd, $\dim S_j \cap \partial D_{12}$ ($j = 1, 2$) is even, (25) - (27) are related with the generalized Gauss-Bonnet formula. Indeed, the second formula due to Allendoerfer-Weil (see [2]) can be applied to $v(S_1 \cap \partial D_{12})$ or $v(S_2 \cap \partial D_{12})$. The above formulae coincide with it.

The derivation of $v(D_{12})$ with respect to $r_1, r_2, \rho_{12}$ in (25) leads to the following formula

$$dv(D_{12}) = v(S_1 \cap \partial D_{12}) \, dr_1 + v(S_2 \cap \partial D_{12}) \, dr_2$$

$$- \frac{1}{n-1} v(S_1 \cap S_2) \sqrt{-B(0 \ast 12)} \, \theta_{12}$$  \hspace{1em} (29)

with $\theta_{12} = \frac{1}{2} \log \rho_{12}^2$.

(29) is nothing else than a special case in the $n$ dimensional space derived from (18) for $\epsilon \to 0$, after putting to be $\lambda_1 = \lambda_2 = \epsilon$ and $\lambda_j = 0$ ($3 \leq j \leq n + 1$).

In particular, in the case where $n = 2, 3$, the volumes $v(D_{12})$ are simply
written as

\[ v(D_{12}) = \frac{1}{2} r_1^2 (\psi_{12} - \sin \psi_{12}) + \frac{1}{2} r_2^2 (\psi_{21} - \sin \psi_{21}) \quad (n = 2), \quad (30) \]

\[ v(D_{12}^-) = \pi r_1^3 \left( \frac{2}{3} - \cos \frac{1}{2} \psi_{12} + \frac{1}{3} \cos^3 \frac{1}{2} \psi_{12} \right) \\
+ \pi r_2^3 \left( \frac{2}{3} - \cos \frac{1}{2} \psi_{21} + \frac{1}{3} \cos^3 \frac{1}{2} \psi_{21} \right) \quad (n = 3), \quad (31) \]

where \( \psi_{12}, \psi_{21} \) denote the angles at \( O_1, O_2 \) respectively subtended by the diameter of \( S_1 \cap S_2 \). Remark that

\[ e^{i \frac{\psi_{12}}{2}} = \frac{B \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} + i \sqrt{-B(0 \ast 12)}}{2 \rho_{12} r_1}, \]

\[ e^{i \frac{\psi_{21}}{2}} = \frac{B \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} + i \sqrt{-B(0 \ast 12)}}{2 \rho_{12} r_2}. \]

In the case where \( n = 2, 3 \), the formula (28) becomes

\[ \bullet \ dv(D_{12}^-) = r_1 \psi_{12} \, dr_1 + r_2 \psi_{21} \, dr_2 - \sqrt{-B(0 \ast 12)} \frac{d\rho_{12}}{\rho_{12}}, \quad (32) \]

\[ \bullet \ dv(D_{12}) = \frac{\pi r_1}{\rho_{12}} \{ r_2^2 - (r_1 - \rho_{12})^2 \} \, dr_1 + \frac{\pi r_2}{\rho_{12}} \{ r_1^2 - (r_2 - \rho_{12})^2 \} \, dr_2 \\
- \frac{\pi}{4 \rho_{12}^2} B(0 \ast 12) \, d\rho_{12}, \quad (33) \]

in view of the identity

\[ \frac{1}{2} d\psi_{jk} = \frac{1}{\sqrt{-B(0 \ast jk)}} \left\{ -B \begin{pmatrix} 0 & j \\ 0 & k \end{pmatrix} \frac{dr_j}{r_j} + 2 r_k \, dr_k - B \begin{pmatrix} 0 & j \\ 0 & k \end{pmatrix} \frac{d\rho_{jk}}{\rho_{jk}} \right\} \quad (34) \]

for \( j, k = 1, 2 \) or \( 2, 1 \) respectively.

**Example 2**

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Assume that $n = 2$. Then $D_{123}$ is the pseudo-triangle $\tilde{\Delta}[P_1 P_2 P_3]$ with vertices $P_1 = (\xi_1, \xi_2)$, $P_2(\eta_1, \eta_2)$, $P_3(\zeta_1, \zeta_2)$ (see Figure 1), where

\[
\xi_1 = \frac{B \begin{pmatrix} 0 & 2 & 3 \\ 0 & * & 3 \end{pmatrix}}{\sqrt{2B(023)}}, \quad \xi_2 = -\sqrt{\frac{-B(0*23)}{2B(023)}}, \\
\eta_1 = \frac{1}{B(013)\sqrt{2B(023)}} \{-B \begin{pmatrix} 0 & 1 & 3 \\ 0 & * & 3 \end{pmatrix} B \begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix} - \sqrt{B(0*13)B(0123)} \}, \\
\eta_2 = \frac{1}{B(013)\sqrt{2B(023)}} \{-B \begin{pmatrix} 0 & 1 & 3 \\ 0 & * & 3 \end{pmatrix} \sqrt{-B(0123)} - B \begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix} \sqrt{-B(0*13)} \}, \\
\zeta_1 = \frac{1}{B(012)\sqrt{2B(023)}} \{B \begin{pmatrix} 0 & 1 & 2 \\ 0 & * & 2 \end{pmatrix} B \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} - B(012)B(023) \\
+ \sqrt{B(0123)B(0*12)} \}, \\
\zeta_2 = \frac{1}{B(012)\sqrt{2B(023)}} \{-B \begin{pmatrix} 0 & 1 & 2 \\ 0 & * & 2 \end{pmatrix} \sqrt{-B(0123)} + B \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} \sqrt{-B(0*12)} \}.
\]

Note that $\xi_2 < 0$, $\eta_1 < 0$.

The area of $\Delta(O_1 O_3 O_2)$ is expressed by

\[
|\Delta(O_1 O_3 O_2)| = \frac{1}{2} |\delta|,
\]

where $\delta$ denotes

\[
\delta = \begin{vmatrix} 1 & \xi_1 & \xi_2 \\ 1 & \eta_1 & \eta_2 \\ 1 & \zeta_1 & \zeta_2 \end{vmatrix} = -\frac{1}{2} \sqrt{-B(0123)} < 0.
\]

Denote by $\varphi_j$ the angle of the triangle $\Delta(O_1 O_3 O_2)$ at the vertex $O_j$. Then

\[
e^{i\varphi_j} = \frac{B \begin{pmatrix} 0 & k & j \\ 0 & l & j \end{pmatrix} + i\sqrt{-B(0123)}}{2p_{jk}p_{jl}} (j, k, l \text{ different indices}).
\]

Denote by $P'_1$, $P'_2$, $P'_3$ the intersection points of $S_2 \cap S_3$, $S_3 \cap S_1$, $S_1 \cap S_2$ which are different from $P_1$, $P_2$, $P_3$ respectively. Also denote by $\psi_{jk}$ the angle at $O_j$ subtended by the arc $\overline{P_k P'_k} \cap S_j$.

Then Theorem I (i) shows
Lemma 10

\[ v(D) = \tilde{\Delta}(P_1P_3P_2) \]
\[ = |\Delta(O_1O_3O_2)| - \sum_{j=1}^{3} |\Delta(O_1O_3O_2) \cap D_j| + \sum_{1 \leq j < k \leq 3} |\Delta(O_1O_3O_2) \cap D_j^- \cap D_k^-|, \]

(37)

where owing to (30)

\[ |\Delta(O_1O_3O_2) \cap D_j^-| = \frac{1}{2} r_j^2 \varphi_j, \]
\[ |\Delta(O_1O_3O_2) \cap D_j^- \cap D_k^-| = \frac{1}{2} |D_j^- \cap D_k^-| \]
\[ = \frac{1}{4} r_j^2 (\psi_{jk} - \sin \psi_{jk}) + \frac{1}{4} r_k^2 (\psi_{kj} - \sin \psi_{kj}). \]

Denote by \( \psi_1, \psi_2, \psi_3 \) the angles at \( O_j \) subtended by the sides \( \hat{P}_2P_3, \hat{P}_3P_1, \hat{P}_1P_2 \) of the pseudo triangle \( \tilde{\Delta}(P_1, P_3, P_2) \) respectively such that the arc length \( s_j \) of \( \hat{P}_kP_i \) is equal to

\[ s_j = r_j \psi_j. \]

\( \psi_j \) are also related with \( \psi_{jk}, \varphi_j \) as follows:

\[ \psi_j = \frac{1}{2} \psi_{jk} + \frac{1}{2} \psi_{jl} - \varphi_j. \]

The formula (37) can be identified with the special case \( n = 2 \) of (17).

On the other, since \( \varphi_1 + \varphi_2 + \varphi_3 = 2\pi \),

\[ \psi_1 + \psi_2 + \psi_3 = \frac{1}{2} \sum_{j \neq k} \psi_{jk} \]
\[ = 2\pi - \angle P_1P_3P_2 - \angle P_2P_1P_3 - \angle P_3P_2P_3. \]

This identity is a special case of the second Allendoerfer-Weil formula in the Euclidean plane (see [2] Theorem II). Furthermore,

\[ 2v(D) = r_1 v_1 + r_2 v_2 + r_3 v_3 - \frac{1}{2} \sum_{1 \leq j < k \leq 3} \sqrt{-B(0 \star j k)} v_{jk} + \frac{1}{2} \sqrt{-B(0 1 2 3)}, \]

with \( v_j = r_j \psi_j \). This identity coincides with (17) in the two dimensional case.
Taking into consideration the identities (30), (31), (33) and the following equalities \((j, k, l\) are different indices of 1, 2, 3)

\[
\begin{align*}
\frac{dB(0123)}{dB} &= \frac{1}{2} \rho_{jk} B \begin{pmatrix} 0 & j & l \\ 0 & k & l \end{pmatrix}, \\
d\varphi_j &= \frac{1}{\sqrt{-B(0123)}} \left\{ -B \begin{pmatrix} 0 & j & k \\ 0 & l & k \end{pmatrix} \frac{d\rho_{jk}}{\rho_{jk}} - B \begin{pmatrix} 0 & j & l \\ 0 & k & l \end{pmatrix} \frac{d\rho_{jl}}{\rho_{jl}} + 2 \rho_{kl} d\rho_{kl} \right\},
\end{align*}
\]

we get the formula

\[
Dv(D) = \sum_{j=1}^{3} r_j \psi_j dr_j - \frac{1}{2} \sum_{j<k} \sqrt{-B(0*jk)} d\rho_{jk} \frac{d\rho_{jk}}{\rho_{jk}}
\]

\[
- \frac{1}{2\sqrt{-B(0123)}} \left\{ \sum_{j<k} B \begin{pmatrix} 0 & * & j & k \\ 0 & 0 & l & j & k \end{pmatrix} \frac{d\rho_{jk}}{\rho_{jk}} \right\},
\]

which is nothing else than (18) for \(n = 2\) in view of the identity

\[
\theta_{123} = -\frac{1}{B(0123)} \left\{ B \begin{pmatrix} 0 & * & 1 & 2 \\ 0 & 3 & 1 & 2 \end{pmatrix} \frac{d\rho_{12}}{\rho_{12}} + B \begin{pmatrix} 0 & * & 1 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix} \frac{d\rho_{13}}{\rho_{13}} \right\}
\]

\[
+ B \begin{pmatrix} 0 & * & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \frac{d\rho_{23}}{\rho_{23}},
\]

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Figure 1: $B(0 \ast 1 \ast 2 \ast 3) > 0$

Figure 2: $B(0 \ast 1 \ast 2 \ast 3) < 0$
4 Restriction to the unit hypersphere

We assume further
\[ f_{n+1}(x) = Q(x) - 1, \]
i.e., \( S_{n+1} \) is the unit hypersphere with center \( O_{n+1} \) at the origin.
We may assume the linear functions
\[ f'_j(x) := f_j(x) - Q(x) + 1 = \sum_{\nu=1}^{n} u_{j\nu} x_\nu + u_{j0} \quad (1 \leq j \leq n) \]
are normalized such that the configuration matrix \( A' = (a'_{jk}) \) (0 \leq j, k \leq n) of order \( n+1 \) consisting of
\[ a'_{j0} = a'_{0j} = u_{j0}, \]
\[ a'_{jk} = \sum_{\nu=1}^{n} u_{j\nu} u_{k\nu} - u_{j0} u_{k0}, \]
satisfies \( a'_{00} = -1, \) \( a'_{jj} = 1 \) (1 \leq j \leq n). We put further
\[ f'_{n+1} = 1 - Q(x). \]

For the set of indices \( J = \{j_1, \ldots, j_p\}, \) \( K = \{k_1, \ldots, k_p\} \subset \{0, 1, \ldots, n, n+1\} \), we denote by \( A'\left(\begin{array}{c} J \\ K \end{array}\right) \) the subdeterminant with the \( j_1, \ldots, j_p \)th rows and the \( k_1, \ldots, k_p \)th columns. In particular, we abbreviate \( A'\left(\begin{array}{c} J \\ J \end{array}\right) \) by \( A'(J) \).

The family of the hyperplanes \( H_j : f'_j(x) = 0 \) define the arrangement of hyperplanes \( \mathcal{A}' = \bigcup_{j=1}^{n} H_j \) which correspond to \( \mathcal{A} = \bigcup_{j=1}^{n} S_j, \) \( S_j : f_j(x) = 0, \) one-to-one.

The components of the matrix \( A' \) are described by the Cayley-Menger determinants as follows:
\[ a'_{j0} = \frac{B\left(\begin{array}{cc} 0 & j \\ 0 & \ast \end{array}\right) n + 1}{\sqrt{-B(0 \ast j n + 1)}}, \quad (38) \]
\[ a'_{jk} = \frac{-B\left(\begin{array}{cc} 0 & j \\ 0 & \ast \end{array}\right) n + 1)}{\sqrt{B(0 \ast j n + 1) B(0 \ast k n + 1)}}. \quad (39) \]
$H_j$ has the same intersection with $S_{n+1}$ as the intersection $S_j \cap S_{n+1}$.

From now on, we shall assume the condition ($\mathcal{H}1$).

($\mathcal{H}1$) can be rephrased in terms of the minors of $A'$ as follows:

($\mathcal{H}1$) $A'(0 \ J) < 0 \ (J \subset \partial_{n+1} N), \ A'(J) > 0 \ (1 \leq |J|, J \subset \partial_{n+1} N)$.

Remark that it always holds: $-A'(0 \ J) > A'(J) > 0$.

Since $S_{n+1}$ is the unit hypersphere, we have the identity

$$B(0 \star n + 1) = 2, \ B(0 \star j n + 1) = -1,$$

so that

$$d'_{jk} = -B(0 \star j k n + 1) = -\cos\langle j, k\rangle,$$

where $\langle j, k\rangle$ denotes the angle subtended by $S_j, S_k$ in $S_{n+1}$.

Let $D = D_{12...n+1}$ be the (non-empty) real $n$ dimensional domain defined by

$$D_{12...n+1} = \bigcap_{j=1}^{n+1} D_j^-, \ D_j^- : f_j' \leq 0 \subset \mathbb{R}^n \ (1 \leq j \leq n + 1).$$

Then, for any $J \subset \partial_{n+1} N$ such that $|J| = p, 1 \leq p \leq n - 1$, the intersection $S_{J_{n+1}} = S_{n+1} \cap \bigcap_{j \in J} S_j$ defines an $n - p - 1$ dimensional sphere. In particular, $\bigcap_{k \in \partial_{n+1} N} S_k$ consists of two points.

The orientation of $\mathbb{R}^n$ and $D$ is determined such that the standard $n$-form $\varpi$ is positive:

$$\varpi = dx_1 \wedge \cdots \wedge dx_n > 0.$$

We can define the standard volume form on $S_{n+1}$ as

$$\varpi_{n+1} := \sum_{\nu=1}^{n} (-1)^\nu x_\nu dx_1 \wedge \cdots \wedge \widehat{dx_\nu} \cdots \wedge dx_n = 2 \left[ \frac{\varpi}{d f_n'} \right] s_{n+1}.$$

Let $\Phi'(x)$ be the multiplicative function

$$\Phi'(x) = \prod_{j \in \partial_{n+1} N} f_j'(x)^{\lambda_j} \ (\lambda_j \in \mathbb{R}_{\geq 0}).$$

We take the value of the many valued function $\Phi'(x)$ such that $\Phi'(x) > 0$ at the infinity in $\mathbb{R}^n$. 

24
Denote the twisted rational de Rham \((n-1)\)-cohomology by \(H^{n-1}_\nabla(X, \Omega (\ast S))\) and its dual by \(H_{n-1}(X, \mathcal{L}^\ast)\), where \(\mathcal{L}^\ast\) denotes the dual local system on the space \(X = S_{n+1} - \bigcup_{j \in A} S_j\) associated with \(\Phi'\). The covariant differentiation \(\nabla\) is given by

\[
\nabla \psi = d \psi + d \log \Phi' \wedge \psi.
\]

The corresponding integral can be expressed as the pairing

\[
H^{n-1}_\nabla(X, \Omega (\ast S)) \times H_{n-1}(X, \mathcal{L}^\ast) \ni (\varphi, \mathfrak{z}) \mapsto J'(\varphi) = \int_{\mathfrak{z}} \Phi'(x) \varphi(x) \varpi_{n+1}.
\]

for \(\varphi \varpi \in \Omega^{n-1}(\ast S)\) and a twisted \((n-1)\)-cycle \(\mathfrak{z}\).

The following has been proved in [4].

**Proposition 11** \(H^{n-1}_\nabla(X, \Omega (\ast S))\) is of dimension \(2^n\) and has a basis

\[
F'_\lambda = \frac{\varpi_{n+1}}{f'_J},
\]

where \(f'_J\) means the product \(\prod_{j \in J} f'_j\) and \(J\) ranges over the family of all unordered subsets of indices such that \(J \subset \partial_{n+1} N\) including the empty set \(\emptyset\).

From now on, we choose a twisted cycle \((n-1)\)-cycle \(\mathfrak{z}\) such that

\[
\int_{\mathfrak{z}} \Phi'(x) \varphi \varpi_{n+1} = \int_{\bar{D}_{-1} \ldots n+1} |\Phi'(x)| \varphi \varpi_{n+1} \quad (\varphi \varpi_{n+1} \in \Omega^{n-1}(\ast S)).
\]

\(F'_\emptyset\) means \(\varpi_{n+1}\), and we define

\[
J'_\lambda(\varphi) = \int_{\mathfrak{z}} \Phi'(x) \varphi \varpi_{n+1}.
\]

The derivation of the integral \(J'_\lambda(\varphi)\) with respect to the parameters \(a'_{jk}, a'_{j0}\) can be expressed as

\[
d'_{A'} J'_\lambda(\varphi) = \sum_{j=1}^{n} da'_{j0} \frac{\partial}{\partial a'_{j0}} J'_\lambda(\varphi) + \sum_{1 \leq j, k \leq n} da'_{jk} \frac{\partial}{\partial a'_{jk}} J'_\lambda(\varphi)
\]

\[
= \int_{\mathfrak{z}} \Phi'(x) \nabla'_{A'}(\varphi \varpi_{n+1}),
\]

where

\[
\nabla'_{A'}(\varphi \varpi_{n+1}) = d_{A'}(\varphi \varpi_{n+1}) + d_{A'} \log \Phi'(x) \wedge \varphi \varpi_{n+1}.
\]

In addition to the above basis, it is convenient to introduce the following basis which we call “of second kind”:

25
Definition 12
\[ F_{*J} := F_J + \sum_{\nu \in J} A\left( \frac{0}{\nu} \frac{\partial_{\nu} J}{A(J)} \right) F_{\partial_{\nu} J}. \]
In particular, \( F_{*\emptyset} = F_{\emptyset} = \infty_{n+1} \).

The differential 1-forms defined below will play an essential role in the sequel.

Definition 13
\[ \theta'_{j} := da'_{j0}, \]
\[ \theta'_{jk} := da'_{jk} - \frac{A\left( \frac{0}{j} \frac{k}{k} \right)}{A'(0 k)} da'_{k0} - \frac{A\left( \frac{0}{k} \frac{j}{j} \right)}{A'(0 j)} da'_{j0}. \]

General \( \theta'_J \) for \( |J| \geq 3 \) are defined by induction:
\[ \theta'_J := -\sum_{\nu \in J} A\left( \frac{0}{\nu} \frac{\partial_{\nu} J}{A'(0 \partial_{\nu} J)} \right) \theta'_{\partial_{\nu} J} \quad (3 \leq |J| \leq n). \]

Denote \( \lambda'_\infty = \sum_{j=1}^{n} \lambda_j \) and \( J = \{j_1, \ldots, j_p\}, |J| = p \).

The following fact has been proved in [4].

Proposition 14 The following variation formula holds:
\[ \nabla A'(F'_{\emptyset}) \sim \sum_{p=1}^{n} \sum_{1 \leq j_1 < \cdots < j_p \leq n} \frac{\lambda_{j_1} \cdots \lambda_{j_p}}{\prod_{q=1}^{p-1}(-\lambda_{\infty} - n + q + 1)} (-1)^p \theta'_{J} A'(0 J) F'_{*J}. \tag{41} \]
(The formula (4.12) in [4] has an error. In the RHS, the sign \((-1)^p\) should be added as above to the original formula).

For example,
\[ \nabla A'(F'_{\emptyset}) \sim -\lambda_1 \frac{1}{A'(0 1)} \theta'_1 (F'_1 + a'_{10} F'_0) - \lambda_2 \frac{1}{A'(0 2)} \theta'_2 (F'_2 + a'_{20} F'_0) \]
\[ - \frac{\lambda_1 \lambda_2}{\lambda_{\infty}} \frac{A'(12)}{A'(0 1 2)} \theta'_{123} F'_{*123} \quad (n = 2), \]
\[ \nabla A'(F'_{\emptyset}) \sim -\sum_{j=1}^{3} \lambda_j da'_{j0} F'_{*J} - \sum_{1 \leq j < k \leq 3} \frac{\lambda_j \lambda_k}{\lambda_{\infty} + 1} \frac{A'(jk)}{A'(0 jk)} \theta'_{jk} F'_{*jk} \]
\[ - \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_{\infty} (\lambda_{\infty} + 1)} \frac{A'(123)}{A'(0123)} \theta'_{123} F'_{*123} \quad (n = 3). \]
5 Analog of Schl"afli formula (errata and some comments)

The variational formula for the volume of a spherically faced simplex in the unit hypersphere was presented in [4]. In addition to the formulae stated in Theorem I and II here, they make a completely integrable system.

However, some formulae stated there have a few errors. In this section, we present a correct version as in Theorem III.

Let \( P_j (1 \leq j \leq n) \) be the points in \( \mathbb{R}^n \) such that

\[
\{ P_j \} = \bigcap_{k \in \partial_j N} H_k \cap S_{n+1}.
\]

We can take the Euclidean coordinates \( x_1, \ldots, x_n \) such that the polynomials \( f_j \) have the following expressions:

\[
f_j' (x) = \sum_{\nu=1}^{n+1-j} u_{j\nu} x_\nu + u_{j0} \quad (1 \leq j \leq n).
\]

We assume for simplicity that \( u_{j, n+1-j} = 2 \alpha_{j, n+1-j} > 0 \) (1 \( \leq j \leq n \)) and that \( P_j \) satisfies the condition (\( \mathcal{H}2 \)).

We have the equalities

\[
\prod_{j=p+1}^{n} u_{j, n-j+1} = \sqrt{-A'(0 \ldots p+1 \ldots n)} \quad (1 \leq p \leq n).
\]

The affine subspace \( \bigcap_{j=n-p+1}^{n} H_j \) contains the \( n-p-1 \) dimensional sphere \( S_{n-p+1 \ldots n+1} = \bigcap_{j=n-p+1}^{n} S_j \cap S_{n+1} \) with radius

\[
r_{n-p+1 \ldots n+1} = \sqrt{-A'(n-p+1 \ldots n) / A'(0 \ldots n-p+1 \ldots n)}.
\]

Denote by \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \) be the pseudo \( n-1 \) simplex in \( S_{n+1} \) with hyperspherical faces with vertices \( P_j \) such that their sign of orientation is \( (-1)^{\frac{a(n-1)}{2}} \). The support of \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \) coincides with \( D = D_{12 \ldots n+1} \).

By definition, the following properties are valid.

**Lemma 15 (i)**

\[
df_n' \wedge \cdots \wedge df_1' > 0
\]
on $D$.

(ii) The pseudo $(n-1)$-simplex \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \) has the sign \((-1)^{\frac{n(n-1)}{2}}\) of orientation such that

\[ \tilde{\Delta}[P_1, P_2, \ldots, P_n] = (-1)^{\frac{n(n-1)}{2}} S_{n+1} \cap D. \]

**Proof.** Indeed, we can show that

\[ df'_n \wedge \cdots \wedge df'_1 = \prod_{j=1}^{n} u_{j, n-j+1} \varpi > 0. \quad (44) \]

(ii) follows from \((H2)\). \(\square\)

Let \( v_\emptyset \) be the volume of the pseudo $(n-1)$-simplex \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \) defined by

\[ v_\emptyset = \int_{\tilde{\Delta}[P_1, P_2, \ldots, P_n]} \varpi_{n+1} > 0, \]

where the orientation of \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \) is chosen such that \( \varpi_{n+1} \) should be positive.

We are interested in the variation formula for \( v_\emptyset \), which can be expressed in terms of the lower dimensional volumes of the faces of \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \).

Every face of the pseudo simplex is included in some \( S_{J, n+1} \). The \( S_{J, n+1} \) is defined as an \( n-p-1 \) dimensional sphere with radius

\[ r_{J, n+1} = \sqrt{-\frac{A'(J)}{A'(0, J)}}. \]

We can consider the \( n-p-1 \) dimensional volume \( v_J (|J| = p) \) relative to the corresponding standard volume form \( \varpi'_{J, n+1} \) on the \( n-p-1 \) dimensional sphere:

\[ v_J = \int_{\tilde{\Delta}[P_1, P_2, \ldots, P_n] \cap S_J} |\varpi'_{J, n+1}|, \]

where

\[ |\varpi'_{J, n+1}| = r_{J, n+1}^{n-p-1} |\varpi_{J, n+1}| > 0. \]

The orientation of \( \tilde{\Delta}[P_1, P_2, \ldots, P_n] \cap S_J \) is chosen such that \( \varpi'_{J, n+1} \) should be positive: \( \varpi'_J \cap S_J \) is chosen such that \( \varpi'_J \cap S_J \) is the absolute value of \( \varpi'_J \).
When $J = \{n - p + 1 \ldots n\}$, we can give an explicit expression for $\omega_{n-p+1 \ldots n+1}$ as follows:

$$f'_j(x) = 0 \quad (n - p + 1 \leq j \leq n + 1),$$

$$\sum_{j=p+1}^{n} x_j^2 = r_{n-p+1 \ldots n+1}^2,$$

where

$$r_{n-p+1 \ldots n+1} = \sqrt{-\frac{A'(n - p + 1 \ldots n)}{A'(0 n - p + 1 \ldots n)}}.$$

The standard volume form on $S_{n-p+1 \ldots n+1}$ is given by

$$\omega'_{n-p+1 \ldots n+1} = \sum_{\nu=p+1}^{n} (-1)^\nu \frac{x_{\nu} dx_{p+1} \wedge \cdots \wedge dx_{\nu} \wedge \cdots \wedge dx_n}{r_{n-p+1 \ldots n+1}}$$

$$= r_{n-p+1 \ldots n+1}^{n-p} \omega_{n-p+1 \ldots n+1},$$

(45)

where

$$\omega_{n-p+1 \ldots n+1} = \sum_{\nu=p+1}^{n} (-1)^\nu \xi_{\nu} d\xi_{p+1} \wedge \cdots \wedge d\xi_\nu \wedge \cdots \wedge d\xi_n.$$

through the transformation

$$x_\nu = r_{n-p+1 \ldots n+1} \xi_\nu \quad (p + 1 \leq \nu \leq n),$$

such that $\sum_{\nu=p+1}^{n} \xi_\nu^2 = 1$.

The following Lemma follows by definition of residue formula.

**Lemma 16** For $J = \{j_1, \ldots, j_p\}$ (1 \leq j_1 < \cdots < j_p \leq n),

$$\left[ \frac{\omega}{df_{j_p} \wedge \cdots \wedge df_{j_1}} \right]_{S_{j_1 \ldots j_p}} = \frac{1}{\sqrt{A'(J)}} \omega'_{Jn+1}.$$  

In particular,

$$\left[ \frac{\omega}{df_{n} \wedge \cdots \wedge df_{j} \wedge \cdots \wedge df_{1}} \right]_{P_j} = \frac{(-1)^{n-j}}{\sqrt{A'(\partial_j \partial_{n+1} N)}} \quad (1 \leq j \leq n),$$

since $[f'_j]_{P_j}$ at the point $P_j$ of $S_{\partial_j \partial_{n+1} N} \cap D$ is negative.
Proof. To prove Lemma 16, we may assume that \( j_1 = n - p + 1, \ldots, j_p = n \) and \( f'_j \) are represented by the reduced form (42). A direct calculation and (43) show the following identity

\[
d(1 - Q(x)) \wedge df'_n \wedge \cdots \wedge df'_{n-p+1} \wedge \sum_{\nu=p+1}^{n} (-1)^\nu x_\nu dx_{p+1} \wedge \cdots \wedge dx_\nu \wedge \cdots \wedge dx_n
\]

\[
= 2 \prod_{q=1}^{p} u_{n-q+1} q \left( \sum_{\nu=p+1}^{n} x_\nu^2 \right) \varpi
\]

\[
= 2 \sqrt{-A'(0 \ n - p + 1 \cdots n)} \ r_{n-p+1\cdots n+1}^2 \varpi.
\]

Hence,

\[
[ df'_n \wedge \cdots \wedge df'_{n-p+1} \wedge \sum_{\nu=p+1}^{n} (-1)^\nu x_\nu dx_{p+1} \wedge \cdots \wedge dx_\nu \wedge \cdots \wedge dx_n ]_{s_{n+1}}
\]

\[
= \sqrt{-A'(0 \ n - p + 1 \cdots n)} \ r_{n-p+1\cdots n+1}^2 \varpi_{n+1}.
\]

Namely,

\[
\varpi'_{n-p+1\cdots n+1} = \frac{\sum_{\nu=p+1}^{n} (-1)^\nu x_\nu dx_{p+1} \wedge \cdots \wedge dx_\nu \wedge \cdots \wedge dx_n}{r_{n-p+1\cdots n+1} \varpi_{n+1}}
\]

\[
= \sqrt{A'(n - p + 1 \cdots n)} \ [ df'_n \wedge \cdots \wedge df'_{n-p+1} ]_{s_{n-p+1\cdots n+1}}.
\]

General volume forms \( \varpi'_j \mid_{n+1} \) can be explicitly written by the use of suitable coordinates transformed by isometry. \( \square \)

The next Theorem has been essentially stated in [4] (Theorem 8), but has some errors in the formulae (5.6) therein. Here we state a correct version, which follows from Proposition 14.

[Theorem III]

For \( v_\emptyset = v(D_{12\cdots n}) \), we have

\[
d_{\mathcal{A}} v_\emptyset = - \sum_{p=1}^{n-1} \sum_{|J|=p} (-1)^p \frac{(n-p-1)!}{(n-2)!} \theta'_J \frac{\sqrt{A'(J)}}{A'(0 J)} v_J
\]

\[
+ (-1)^n \frac{1}{(n-2)!} \frac{1}{\sqrt{-A'(01, \ldots, n)}} \theta'_{12\cdots n},
\]

(46)
where $J$ ranges over the collection of unordered subsets of $\{1, 2, \ldots, n\}$ and $|J| = p$.

To prove this Theorem, we need the following Lemma equivalent to Proposition 6.

**Lemma 17** We have the identity

$$\left[ \frac{1}{f_j} \right] p_j = \left[ \frac{1}{f_j} \right] p_j$$

$$= \sqrt{-A'(\partial_j \partial_{n+1}N)} A'(0 \partial_{n+1}N) + A'\left( \begin{array}{c} 0 \\ j \end{array} \right) \left( \begin{array}{c} \partial_j \partial_{n+1}N \\ 0 \partial_{n+1}N \end{array} \right) < 0,$$

so that

$$\left[ \frac{1}{f_j} \right] p_j + \frac{A'\left( \begin{array}{c} 0 \\ j \end{array} \right) \left( \begin{array}{c} \partial_j \partial_{n+1}N \\ 0 \partial_{n+1}N \end{array} \right)}{A'(\partial_{n+1}N)} = -\sqrt{-A'(\partial_j \partial_{n+1}N)} A'(0 \partial_{n+1}N)\frac{A'(0 \partial_{n+1}N)}{A'(\partial_{n+1}N)}.$$

**Proof of Theorem III.**

Take $\lambda_j$ such that all $\lambda_j = \varepsilon > 0$ in the formula (41). Then (40) shows that

$$d_{A'} V_0 = \lim_{\varepsilon \downarrow 0} d_{A'} \int_{\Delta[p_1, \ldots, p_n]} |\Phi'(x)| \varpi_{n+1}$$

$$= \lim_{\varepsilon \downarrow 0} \int_\Delta \Phi'(x) \nabla A'(\varpi_{n+1})$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\Delta[p_1, \ldots, p_n]} |\Phi'(x)| \nabla A'(\varpi_{n+1}).$$

In view of the formula (4.11) and the proof of Theorem 7 in [3], we have only to check the following fact:

$$\lim_{\varepsilon \downarrow 0} \frac{\prod_{j=1}^n \lambda_j}{\prod_{q=1}^{n-1} (-\lambda_\infty - n + q + 1)} \frac{A'(0 \partial_{n+1}N)}{A'(\partial_{n+1}N)} \left\{ J_n(\frac{1}{f_{\partial_{n+1}N}}) \right\}$$

$$+ \sum_{j \in \partial_{n+1}N} \frac{A'\left( \begin{array}{c} 0 \\ j \end{array} \right) \left( \begin{array}{c} \partial_j \partial_{n+1}N \\ 0 \partial_{n+1}N \end{array} \right)}{A'(\partial_j \partial_{n+1}N)} J_n(\frac{1}{f_{\partial_j \partial_{n+1}N}}) = \frac{(-1)^n}{(n-2)! \sqrt{-A'(01 \ldots n)}}.$$  

(47)
By the residue theorem, the LHS reduces to $n$ pieces of point measures at $P_j$ and equals
\[
\lim_{\varepsilon \to 0} \varepsilon^n \prod_{q=1}^{n-1} (-n\varepsilon - n + q + 1) \frac{\mathcal{J}(F',\partial_{n+1}N)}{A'(0 \partial_{n+1}N)} \frac{A'((\partial_{n+1}N))}{A'(0 \partial_{n+1}N)} = -\lim_{\varepsilon \to 0} \frac{\varepsilon^{n-1}}{n} \prod_{q=1}^{n-2} (-n\varepsilon - n + q + 1) \frac{\mathcal{J}(F',\partial_{n+1}N)}{A'(0 \partial_{n+1}N)} \frac{A'((\partial_{n+1}N))}{A'(0 \partial_{n+1}N)} = \sum_{j=1}^{n} \frac{(-1)^{n-1}}{n(n-2)!} \left\{ \left[ \frac{1}{f_j^1} \right]_{P_j} + \frac{A' \begin{pmatrix} 0 & \partial_j \partial_{n+1}N \\ j & \partial_j \partial_{n+1}N \end{pmatrix}}{A'(\partial_{n+1}N)} \right\} \frac{A'((\partial_{n+1}N))}{A'(0 \partial_{n+1}N)} \left[ \frac{\bar{\omega}_{n+1}}{df_n^1 \ldots df_j^1 \ldots df_1^1} \right]_{P_j}.
\]

On the other hand, we have
\[
\left[ \frac{\bar{\omega}_{n+1}}{df_n^1 \ldots df_j^1 \ldots df_1^1} \right]_{P_j} = \frac{(-1)^{n+1-j}}{\sqrt{A'(\partial_j \partial_{n+1}N)}}.
\]

Each term in the summand of the RHS does not depend on $j$ and is equal to
\[
\frac{(-1)^{n-1}}{n(n-2)!} \left\{ \sqrt{-A'(\partial_j \partial_{n+1}N)} A'(0 \partial_{n+1}N) \right\} \frac{1}{\sqrt{A'(\partial_j \partial_{n+1}N)}} = \frac{(-1)^{n-1}}{n(n-2)!} \frac{\sqrt{-A'(0 \partial_{n+1}N)}}{A'(\partial_{n+1}N)}.
\]

Hence, the LHS of (47) becomes
\[
\lim_{\varepsilon \to 0} \varepsilon^n \prod_{q=1}^{n-1} (-n\varepsilon - n + q + 1) \frac{A'((\partial_{n+1}N))}{A'(0 \partial_{n+1}N)} \mathcal{J}(F',\partial_{n+1}N) = \frac{(-1)^n}{(n-2)!} \frac{1}{\sqrt{-A'(0 \partial_{n+1}N)}} \theta'_{\partial_{n+1}N}.
\]

In this way, we have proved Theorem III. \qed

**Remark** In three dimensional case, i.e., for $n = 3$, $D_{123}$ is a pseudo triangle $\tilde{\Delta}P_1P_2P_3$ with circular arc sides. Theorem III shows the identity
\[
d_{A'} v_0 = \sum_{j=1}^{3} \frac{1}{A'(0 j)} \theta_j' v_j - \sum_{j<k} \frac{\sqrt{A'(jk)}}{A'(0 j k)} \theta_{jk} - \frac{1}{\sqrt{-A'(0 1 2 3)}} \theta_{123}.
\]
On the other hand, Gauss-Bonnet theorem shows the identity

\[ v_0 = 2\pi - \sum_{j=1}^{3} a'_{j0} v_j - \sum_{j < k} (\pi - \langle jk \rangle), \quad (49) \]

where \( \langle jk \rangle \) denotes the angle of the triangle at \( P_i (\{j, k, l\} : \text{a permutation of } \{1, 2, 3\}) \) such that

\[ a'_{jk} = - \cos \langle jk \rangle, \]

and \( a'_{j0} \) is the geodesic curvature of the arc \( \partial D_{123} \cap S_j \).

We can see by a direct calculation that the differential of (49) coincides with (48). Gauss-Bonnet theorem was extended into a higher dimensional polyhedral domain by Allendoerfer-Weil (see the second formula in [2]). However, in the case of a spherically faced simplex, the formula (46) does not seem to generally coincide with the differential of the latter.

**Appendix  Elementary proof of Theorem II (i)**

Denote by \( P_j (1 \leq j \leq n+1) \) the vertex points of the \( n \)-simplex \( D_N^+ \) such that \( P_j \in \partial D_N^+ \cap \bigcap_{k \neq j} S_k \). For the ordered set \( J = \{j_1, \ldots, j_p\} \subset N \) such that \( j_1 > j_2 > \cdots > j_p \), \( |J| = p \) and \( J^c = \{j_1^* > \cdots > j_p^*\} \), \( \tilde{\Delta} O_j P_{J^c} \) means the \( n \)-cell

\[ \tilde{\Delta} O_j P_{J^c} = \tilde{\Delta} O_{j_1} \cdots O_{j_p} P_{j_1^*} \cdots P_{j_p^*}, \]

with the vertices \( O_{j_1}, \ldots, O_{j_p} \) and \( P_{j_1^*}, \ldots, P_{j_p^*} \). Notice that \( \tilde{\Delta} P_{j_1^*} \cdots P_{j_p^*} = S_j \cap D_N^+ \) are \((p-1)\)-pseudo simplex with the faces \( S_k \cap S_j \cap D_N^+ (k \in J^c) \) in the \( n - p \) dimensional sphere \( S_j = \bigcap_{1 \leq k \leq n} S_j \).

We have the cell decomposition of \( \Delta O_{n+1} \cdots O_2 O_1 \):

\[ \Delta O_{n+1} \cdots O_2 O_1 = - \sum_{p=0}^{n} \sum_{|J|=p} \tilde{\Delta} O_j P_{J^c} \varepsilon_j, \]

where \( \varepsilon_j \) denotes \((-1)^{\sum_{j \in J^c} j} (-1)^{(n-p)(n-p+1)/2} \).

Hence we have the identity for their volumes

\[ v(\Delta O_{n+1} \cdots O_1) = \sum_{J \subseteq N, |J| \leq n} v(\Delta O_j P_{J^c}), \]
or equivalently,
\[ v(\tilde{\Delta}P_N) = v(\Delta O_{n+1} \ldots O_1) - \sum_{J \subseteq N, 1 \leq |J| \leq n} v(\Delta O_J P_J^c). \]

The identity stated in Theorem II (i) is a direct consequence of the following Lemma.

Lemma 18
\[ v(\tilde{\Delta}O_J P_J^c) = \frac{(n - p)!}{(n - 1)!} \sqrt{\frac{(-1)^{p+1} B(0 \ast J)}{2p}} v_J. \]

Proof. Without losing generality, we may assume that \( f_j \) have the standard form (1), (2) and \( J = \{n + 1, n, \ldots, n - p + 2\} \).

\( O_j \) \((n - p + 2 \leq j \leq n + 1)\) can be expressed as
\[ O_j = (-\alpha_{j1}, \ldots, -\alpha_{j,n-j+1}, 0, \ldots, 0) \quad (\alpha_{j,n-j+1} > 0). \]

The spherical \((n - p)\)-simplex \( \tilde{\Delta}P_{n-p+1} \ldots P_1 \) with support \( D^N_X \cap S_J \) is defined by the equations for \( \xi = (\xi_1, \ldots, \xi_n) \):
\[ f_j(\xi) = 0 \quad (n - p + 2 \leq j \leq n + 1), \quad f_k(\xi) \geq 0 \quad (1 \leq k \leq n - p + 1). \] (50)
The coordinates \( \xi_j \) \((1 \leq j \leq p - 1)\) are uniquely determined by (50) and denoted by \( \gamma_j \).

\( \xi \) ranges over the \( n - p \) dimensional sphere
\[ \xi = (\gamma_1, \ldots, \gamma_{p-1}, \xi_p, \ldots, \xi_n) \]
under the condition
\[ f_k(\xi) \geq 0 \quad (1 \leq k \leq n - p + 1), \]
\[ \sum_{j=p}^{n+1} \xi_j^2 = r_{n-p+2 \ldots n+1}^2, \]
where \( r_{n+1\ldots n-p+2} \) denotes the radius of the hypersphere \( S_J \):
\[ r_{n-p+2 \ldots n+1} = \sqrt{\frac{(-1)^p B(0 n - p + 2 \ldots n + 1)}{2p-1}}. \]

The \( n \)-pseudo simplex \( \tilde{\Delta}O_{n+1} \ldots O_{n-p+2} P_{n-p+1} \ldots P_1 \) consists of the union of the \( p \)-simplex \( \Delta O_{n+1} \ldots O_{n-p+2} \xi \) with \( \xi \in \Delta(P_{n-p+1} \ldots P_1) \):
\[ \tilde{\Delta}O_{n+1} \ldots O_{n-p+2} P_{n-p+1} \ldots P_1 = \bigcup_{\xi \in \Delta(P_{n-p+1} \ldots P_1)} \Delta O_{n+1} \ldots O_{n-p+2} \xi. \]
Namely, every point of $\hat{\Delta}O_{n+1}\ldots O_{n-p+2}P_{n-p+1}\ldots P_1$ is parametrized by the expressions:

$$x_j = -\sum_{k=1}^{j} y_k \alpha_{n-k+1,j} + y_0 \gamma_j \quad (1 \leq j \leq p - 1),$$

$$x_j = y_0 \xi_j \quad (p \leq j \leq n)$$

such that $y = (y_0, \ldots, y_{p-1})$ ranges over the $p$-convex set

$$\delta_p : y_j \geq 0 \quad (0 \leq j \leq p - 1), \quad \sum_{j=0}^{p-1} y_j \leq 1.$$

As a result in view of (3), (9) and (45),

$$v(\hat{\Delta}O_{n+1}\ldots O_{n-p+2}P_{n-p+1}\ldots P_1) = \int_{\hat{\Delta}O_{n+1}\ldots O_{n-p+2}P_{n-p+1}\ldots P_1} |dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dx_p \wedge \cdots \wedge dx_n|$$

$$= \prod_{j=1}^{p-1} \alpha_{n-j+1,j} \int_{\delta} y_0^{n-p} dy_1 \wedge \cdots \wedge dy_{p-1} \wedge dy_0,$$

$$\int_{\hat{\Delta}P_{n-p+1}\ldots P_1} \left| \sum_{\nu=p}^{n} (-1)^{\nu-p} \xi_\nu d\xi_\nu \wedge \cdots \wedge d\xi_n \right|$$

$$= \frac{(n-p)!}{(n-1)!} \sqrt{\frac{(-1)^{p+1} B(0 \ast n-p+2\ldots n+1)}{2^p}} v_{n-p+2\ldots n+1}.$$

In this way, Lemma 18 has been proved in case where $J = \{n+1 \ldots n-p+2\}$.

Therefore it also holds true for general $J$ because of symmetry. \hfill $\square$

Theorem I (i) can also be proved in a similar way.

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