Magnetically Charged Black Holes and Their Stability

Peter C. Aichelburg, Piotr Bizon

Vienna, Preprint ESI 3 (1993)
Magnetically Charged Black Holes and Their Stability

P.C. Aichelburg
Institut für Theoretische Physik
Universität Wien

and

P. Bizon*
Internationales Erwin Schrödinger Institut für Mathematische Physik
Wien, Austria

Abstract

We study magnetically charged black holes in the Einstein-Yang-Mills-Higgs theory in the limit of infinitely strong coupling of the Higgs field. Using mixed analytical and numerical methods we give a complete description of static spherically symmetric black hole solutions, both abelian and nonabelian. In particular we find a new class of extremal nonabelian solutions. We show that all nonabelian solutions are stable against linear radial perturbations. The implications of our results for the semiclassical evolution of magnetically charged black holes are discussed.

*) On leave of absence from Institute of Physics, Jagellonian University, Cracow, Poland.
1 Introduction

It has long been thought that the only static black hole solution in spontaneously broken gauge theories coupled to gravity (Einstein-Yang-Mills-Higgs (EYMH) theories) is the Reissner-Nordström (RN) solution, with covariantly constant Higgs field and the electromagnetic field trivially embedded in a nonabelian gauge group [1]. This belief, based on no-hair properties of black holes, was put in doubt by the discovery of essentially nonabelian black holes in EYM theory (so called colored black holes) [2], because it demonstrated that the uniqueness properties of black holes in Einstein-Maxwell theory [3] are lost when the gauge field is nonabelian. This suggested that there might also exist nontrivial (i.e. different than RN) black holes when the Higgs field is included. Indeed, it was found recently that in the EYMH theory there are spherically symmetric magnetically charged black holes which are asymptotically indistinguishable from the RN solution but carry nontrivial YM and Higgs fields outside the horizon [4,5]. These essentially nonabelian solutions may be viewed as black holes inside 't Hooft-Polyakov magnetic monopoles. Because the EYMH field equations are rather complicated (even for spherical symmetry), one does not have yet a complete picture of these solutions. In particular, one would like to know how the solutions depend on external parameters (like coupling constants)\footnote{This question was extensively examined in the Prasad-Sommerfield limit in Ref. [5].}, whether there are solutions with degenerate horizon, and, most important, whether the solutions are stable.

The aim of this paper is to give answers to these questions within a model obtained from the EYMH system by taking the limit of infinitely strong coupling of the Higgs field. In this so called \( \sigma \)-model limit, the Higgs field is effectively “frozen” in its vacuum expectation value. Thus one degree of freedom is reduced which simplifies considerably the analysis of solutions. This model is interesting on its own for testing ideas about black holes and as we shall argue has the additional virtue that many of its features carry over to the full EYMH theory (albeit there are also some important differences).

Our model depends effectively on one dimensionless parameter \( \alpha = \sqrt{G} v \) (\( G \) – gravitational constant, \( v \) – vacuum expectation value of the Higgs field). We shall concentrate on static spherically symmetric black holes with magnetic charge (hereafter just called solutions). By analytical methods we derive necessary conditions for the existence of solutions. The solutions are then obtained by numerical integration.

Our results may be summarized as follows: For \( \alpha > 1 \) there is only one branch of solutions, namely the RN family (parametrized by the radius of the horizon \( r_H \geq \sqrt{G}/v \)). For \( \alpha < 1 \) there appears a second (fundamental) branch of essentially nonabelian solutions\footnote{For sufficiently small values of \( \alpha \) there appear other branches of nonabelian solutions, which may be viewed as excitations of the fundamental solution.} The radius of the horizon of these solutions is confined to the interval \( r_H^{\min}(\alpha) \leq r_H \leq r_H^{\max}(\alpha) \). The upper bound \( r_H^{\max}(\alpha) \) is a bifurcation point at which the two branches merge. The lower bound \( r_H^{\min}(\alpha) \) is nonzero only for \( 1/\sqrt{2} < \alpha < 1 \). The solutions with \( r_H = r_H^{\min}(\alpha) > 0 \) are extremal in the sense that they have a degenerate event horizon. The analysis of linear radial perturbations about the solutions shows that the nonabelian solutions are stable whereas the RN solution changes stability at the bifurcation point, i.e. it is stable for \( r_H > r_H^{\max}(\alpha) \) and unstable for \( r_H < r_H^{\max}(\alpha) \) [6].

The relation between the bifurcation structure of static solutions and their stability is the general phenomenon [7]. We shall use this relation to infer stability properties of solutions in...
The full EYMH theory.

The paper is organized as follows. In the next section we define the model and derive the spherically symmetric field equations. In Section 3 we analyze the a priori behaviour of solutions and obtain conditions on the parameters characterizing the solutions. In Section 4 we describe the numerical results and discuss the qualitative properties of solutions. Section 5 is devoted to stability analysis. Finally, in Section 6 we touch upon the question of evolution of black hole configurations due to Hawking radiation.

2 Field Equations

The Einstein-Yang-Mills-Higgs (EYMH) theory is defined by the action

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R + \mathcal{L}_{\text{matter}} \right)$$

with

$$\mathcal{L}_{\text{matter}} = -\frac{1}{16\pi} F^2 - \frac{1}{2} (D\phi)^2 - \lambda (\phi^2 - v^2)^2$$

$$F = dA + e[A, A]$$

$$D\phi = d\phi + e[A, \phi]$$

where the YM connection $A$ and the Higgs field $\phi$ take values in the Lie algebra of the gauge group $G$. Here we consider $G = SU(2)$.

There are four dimensional parameters in the theory: Newton’s constant $G$, the gauge coupling constant $e$, the Higgs coupling constant $\lambda$ and the vacuum expectation value of the Higgs field $v$. Out of these parameters one can form two scales of length:

$$\left[ \frac{\sqrt{G}}{e} \right] = \left[ \frac{1}{ev} \right] = \text{length}.$$

The ratio of these two scales

$$\alpha = \sqrt{G} v$$

plays a rôle of the effective coupling constant in the model. The second dimensionless parameter

$$\beta = \frac{\sqrt{\lambda}}{2}$$

measures the strength of the Higgs field coupling. In this paper we consider in detail the case $\beta = \infty$ (σ-model limit). In this limit the Higgs field is “frozen” in its vacuum expectation value, which simplifies considerably the analysis.

We are interested in spherically symmetric configurations. It is convenient to parametrize the metric in the following way

$$ds^2 = -B^2 N dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

where $B$ and $N$ are functions of $(t, r)$. 

We assume that the electric part of the YM field vanishes. Then the purely magnetic spherically symmetric $SU(2)$ YM connection can be written, in the abelian gauge, as

$$eA = w\tau_1 d\theta + (\cot \tau_3 + w\tau_2) \sin \theta d\varphi$$  \hspace{1cm} (8)

where $\tau_i$ ($i = 1, 2, 3$) are Pauli matrices and $w$ is a function of $(t, r)$.

The spherically symmetric Ansatz for the Higgs field in the abelian gauge is

$$\phi = \nu \tau_3 h(t, r).$$  \hspace{1cm} (9)

As we said we concentrate on the limit $\beta = \infty$, which implies $h(r, t) \equiv 1$. Note that although the Higgs field is constant, its covariant derivative is nonzero and generates a mass term for the YM field:

$$(D\phi)^2 = v^2 w^2.$$  \hspace{1cm} (10)

The YM curvature is given by

$$eF = w dt \wedge \Omega + w' dr \wedge \Omega - (1 - w^2)\tau_3 d\theta \wedge \sin \theta d\varphi$$  \hspace{1cm} (11)

where $\Omega = \tau_1 d\theta + \tau_2 \sin \theta d\varphi$.

From the EYMH Lagrangian (1) one derives the following components (in orthonormal frame) of the stress energy tensor $T_{ab}$

$$\rho \equiv \frac{i}{4\pi} T_{\hat{\theta}\hat{\theta}} = \frac{1}{e^2 r^2} (N w' / 2 + B^{-2} N^{-1} w^2) + \frac{(1 - w^2)^2}{2e^2 r^4} + v^2 \frac{w^2}{r^2}$$  \hspace{1cm} (12)

$$\tau \equiv -\frac{1}{4\pi} T_{\hat{\varphi}\hat{\varphi}} = \frac{1}{e^2 r^2} (N w' / 2 + B^{-2} N^{-1} w^2) + \frac{(1 - w^2)^2}{2e^2 r^4} + v^2 \frac{w^2}{r^2}$$  \hspace{1cm} (13)

$$(\text{In what follows we do not make use of the other nonzero components } T_{\hat{\theta}\hat{\varphi}} = T_{\hat{\varphi}\hat{\theta}}).$$

The Einstein equations reduce to the system

$$N' = \frac{1}{r} (1 - N) - 2Gr \rho$$  \hspace{1cm} (14)

$$B' = Gr BN^{-1}(\rho - \tau)$$  \hspace{1cm} (15)

$$\ddot{N} = -2Gr BN \sigma.$$  \hspace{1cm} (16)

The remaining $\hat{\theta}\hat{\theta}$-component of the Einstein equations is equivalent to the YM equation

$$-(B^{-1} N^{-1} \dot{w}) + (BNw')' + \frac{1}{r^2} Bw(1 - w^2) - Be^2 v^2 w = 0.$$  \hspace{1cm} (17)

For $G = 0$ the Einstein equations (14-16) are solved either by the Minkowski spacetime $N = B = 1$, or by the Schwarzschild spacetime $B = 1$, $N = 1 - 2m/r$, and then the Eq. (17) reduces to the YMH equation on fixed background. For $v = 0$, the system (14 – 17) reduces to the EYM equations.
3 Static Solutions

In this section we wish to consider static black hole solutions of the system (14 – 17). In terms of the dimensionless variable \( x = \frac{r}{\sqrt{G}} \) the static equations are

\[
N' = \frac{1}{x}(1 - N) - \frac{2}{x}(NW'^{2} + \frac{(1 - w^{2})^{2}}{2x^{2}} + \alpha^{2}w^{2})
\]

\[
B' = \frac{2}{x}BW^{2}
\]

\[
(NBW')' + B\frac{1}{x^{2}}w(1 - w^{2} - \alpha^{2}x^{2}) = 0.
\]

(18)  

(19)  

(20)

Note that the function \( B \) can be eliminated from Eq. (20) by using Eq. (19). This system of equations has been studied previously by Breitenlohner et al. [4] who found asymptotically flat solutions with naked singularity at \( x = 0 \). This singularity is an artifact of the limit \( \bar{\beta} = 1 \), since then there is no symmetry restoration at the origin and the last term in the expression (11) for the energy density diverges at \( r = 0 \). In the black hole case the singularity is hidden inside the horizon.

We shall consider the solutions of Eqs. (18 – 20) in the region \( x \in [x_{H}, \infty) \), where \( x_{H} \) is the radius of the (outermost) horizon. The boundary conditions at \( x = x_{H} \) are

\[
N(x_{H}) = 0, \quad N'(x_{H}) \geq 0,
\]

(21)

and the functions \( w, w', B \) are assumed to be finite.

At infinity we impose asymptotic flatness conditions which are ensured by

\[
N(\infty) = 1, \quad w(\infty) = w'(\infty) = 0, \quad B(\infty) \text{ finite.}
\]

(22)

One explicit solution satisfying these boundary conditions is well known. This is the Reissner-Nordström (RN) solution

\[
w = 0, \quad B = 1, \quad N = 1 - \frac{2m}{x} + \frac{1}{x^{2}}
\]

(23)

where \( m \geq 1 \) (mass is measured in units \( 1/\sqrt{G} \)). The corresponding YM curvature is

\[
F = -\tau_{3}d\theta \wedge \sin \theta d\varphi
\]

showing that it is an abelian solution describing a black hole with mass \( m \) and unit magnetic charge (the unit of charge is \( 1/e \)).

It turns out that the system (14-16) admits also nonabelian solutions which we have found numerically. Before discussing the numerical results we want to make several elementary observations about the global behaviour of solutions satisfying the boundary conditions (21) and (22). In what follows we assume that \( \alpha \) is nonzero.

First, consider the behaviour of solutions at infinity. The asymptotic solution of Eqs. (18 – 20) is

\[
w \simeq e^{-\alpha x}
\]

\[
N \simeq 1 - \frac{2m}{x} + \frac{1}{x^{2}} + O(e^{-2\alpha x})
\]

\[
B \simeq 1 + O(e^{-2\alpha x})
\]
hence for large \( x \) the solution is very well approximated by the RN solution (23).

Second, notice that the function \( w \) has no maxima for \( w > 1 \) and no minima for \( w < -1 \), which follows immediately from Eq. (20). Thus a function \( w \) which once leaves the region \( w \in [-1, 1] \), cannot reenter it. Actually, the solution \( w \) stays within this region for all \( x \geq x_H \), because \( w(x_H) < 1 \) (without loss of generality we may assume that \( w(x_H) > 0 \), because there is a reflection invariance \( w \rightarrow -w \)). To see this, consider Eq. (20) at \( x = x_H \). Assuming that \( w''(x_H) \) is finite one has

\[
N'w' + \frac{1}{x^2}w(1 - w^2 - \alpha^2x^2) \bigg|_{x=x_H} = 0. \tag{24}
\]

If \( w(x_H) \geq 1 \), then \( w'(x_H) > 0 \), but since there are no maxima for \( w > 1 \) the condition \( w(\infty) = 0 \) cannot be fulfilled. Thus \( w(x_H) < 1 \). Next one can show that the solution \( w \) has no positive minima and negative maxima. Suppose that there is a positive minimum at some \( x_0 > x_H \). Then Eq. (20) implies that the function \( f(x) = 1 - w^2 - \alpha^2x^2 \) is negative at \( x_0 \). For \( x \geq x_0 \), \( f(x) \) decreases (because \( w' \geq 0 \)), hence there cannot exist a maximum of \( w \) for \( x > x_0 \), so again the condition \( w(\infty) = 0 \) cannot be met. We can repeat this argument to show that \( w'(x_H) < 0 \), because if \( w'(x_H) > 0 \), then \( f(x_H) < 0 \), as follows from Eq. (20).

Finally, notice that for \( x > 1/\alpha \), the solution \( w \) cannot have any extrema because then \( f(x) < 0 \), hence there are no positive maxima and negative minima (whereas other extrema we have already excluded). To summarize, we have shown that if a global solution of Eqs. (18 – 20) exists, then the function \( w \) stays in the region \( w \in [-1, 1] \) and either monotonically tends to zero or oscillates around \( w = 0 \) for \( x < 1/\alpha \) and then monotonically goes to zero.

Now, we will show that global solutions may exist only if the parameters \( \alpha \) and \( x_H \) satisfy certain inequalities. Let \( b \equiv w(x_H) \). We have shown above that \( w'(x_H) < 0 \) for \( b > 0 \), hence from (24) we obtain

\[
\frac{1 - b^2}{x_H^2} \geq \alpha^2 \tag{25}
\]

which implies the necessary condition for the existence of a black hole solution

\[
\alpha x_H \leq 1. \tag{26}
\]

As we will see in the next section this is not a sufficient condition.

For \( x_H < 1 \) we can improve the inequality (26). From Eq. (18) we have

\[
N'(x_H) = \frac{1}{x_H} - \frac{(1 - b^2)}{x_H^3} - \frac{2\alpha^2b^2}{x_H} \tag{27}
\]

hence

\[
(1 - b^2)^2 - x_H^2 + 2\alpha^2x_H^2b^2 \leq 0. \tag{28}
\]

This inequality has a real solution for \( b \) only if the discriminant

\[
\frac{1}{4}\Delta = (\alpha^4x_H^2 - 2\alpha^2 + 1)x_H^2 \tag{29}
\]

is nonnegative, which is always satisfied for \( x_H \geq 1 \) but for \( x_H < 1 \) this implies the additional condition

\[
\alpha^2 \leq \alpha_{\max}^2 = \frac{1}{x_H^2}(1 - \sqrt{1 - x_H^2}). \tag{30}
\]
It turns out from numerical results that the inequality (30) is also a sufficient condition for the existence of black holes. Note that when \( x_H \to 0 \) then \( \alpha_{\text{max}} \to 1/\sqrt{2} \), which is the upper bound for “regular” solutions found by Breitenlohner et al. [5].

Finally, let us see whether there may exist extremal black holes in the model. By extremal we mean a solution with degenerate horizon, i.e. \( N'(x_H) = 0 \). For such solutions the inequalities (25) and (28) are saturated. Eliminating \( b^2 \) we obtain the condition

\[
\alpha^4 x_H^2 - 2\alpha^2 + 1 = 0
\]

(31)

which is equivalent to \( \Delta = 0 \) and is solved by \( \alpha = \alpha_{\text{max}} \) given by (30) provided that \( x_H \leq 1 \). Let us point out that the necessary conditions for the existence of extremal black hole solution in EYMH theories were derived some time ago by Hajicek [8]. In the terminology of Hajicek, Eq. (31) is the special case of the zero-order condition. In the next section we shall find numerically the extremal solutions satisfying (31).

4 Numerical Results

The formal power-series expansion of a solution near the horizon is

\[
\begin{align*}
  w(x) &= b + \sum_{k=1}^{\infty} \frac{1}{k!} w^{(k)}(x_H)(x - x_H)^k, \\
  N(x) &= \sum_{k=1}^{\infty} \frac{1}{k!} N^{(k)}(x_H)(x - x_H)^k.
\end{align*}
\]

(32)

All coefficients in the above series are determined, through recurrence relations, by \( b \), in particular the expressions for \( w'(x_H) \) and \( N'(x_H) \) are given by Eqs. (24) and (27). Thus this expansion (assuming that its radius of convergence is nonzero) defines a one-parameter family of local solutions labelled by initial value \( b \). We use a standard numerical procedure, called the shooting method, to find such values of \( b \) for which the local solution extends to a global solution satisfying the asymptotic boundary conditions (22). Note that \( b \) is not arbitrary but, as follows from Eq. (28), must lie in the interval

\[
1 - \alpha^2 x_H^2 - x_H \sqrt{\alpha^4 x_H^2 - 2\alpha^2 + 1} \leq b^2 \leq 1 - \alpha^2 x_H^2.
\]

(33)

We find that for every \( x_H \), there is a maximal value of \( \alpha \), call it \( \alpha_{\text{max}}(x_H) \), such that for \( \alpha > \alpha_{\text{max}}(x_H) \) there are no solutions, while for \( \alpha \leq \alpha_{\text{max}} \) there is exactly one solution for which \( w \) is monotonically decreasing. This solution we call a fundamental solution, in contrast to solutions with oscillating \( w \), which we call excitations. In what follows we restrict our attention to fundamental solutions. At the end we will briefly describe excitations.

When \( x_H \leq 1 \), then \( \alpha_{\text{max}}(x_H) \) is given by Eq. (30). For \( x_H > 1 \), \( \alpha_{\text{max}}(x_H) \) is displayed in Table 2 (in this case the analytical bound given by Eq. (26) is not sharp). The numerical results for several values of \( x_H \) are summarized in Table 1.
Table 1: Shooting parameter $b$ and mass $m$ (in units $1/\sqrt{G}c$) as functions of $\alpha$ for $x_H = 0.5$, $x_H = 1$, and $x_H = 2$.

|        | $x_H = 0.5$ |       | $x_H = 1$ |       | $x_H = 2$ |       |
|--------|-------------|-------|-----------|-------|-----------|-------|
| $\alpha$ | $b$       | $m$   | $\alpha$ | $b$   | $m$       | $\alpha$ | $b$   | $m$   |
| 0.01   | 0.99993    | 0.268 | 0.01      | 0.99975 | 0.517     | 0.01    | 0.99907 | 1.018 |
| 0.2    | 0.98296    | 0.572 | 0.1       | 0.98191 | 0.663     | 0.1     | 0.9317  | 1.147 |
| 0.4    | 0.94782    | 0.809 | 0.5       | 0.63099 | 0.9805    | 0.2     | 0.7142  | 1.2286 |
| 0.6    | 0.923028   | 0.954 | 0.74      | 0.1852  | 0.99937   | 0.26    | 0.43074 | 1.248 |
| 0.7    | 0.928359   | 0.990 | 0.9       | 0.00868 | 0.99999   | 0.28    | 0.2414  | 1.2498 |
| 0.73   | 0.930534   | 0.996 |          |        |           | 0.28867 | 0.012  | 1.25  |

On Fig. 1 we plot solutions $w(x)$ for $x_H = 2$ and several values of $\alpha$.

Table 2: Maximal value of $\alpha$ as a function of $x_H$ for $x_H > 1$.

| $x_H$ | $\alpha_{\text{max}}$ |
|-------|-----------------------|
| 1     | 1                     |
| 1.000001 | 0.822            |
| 1.01  | 0.744                 |
| 1.1   | 0.605                 |
| 1.5   | 0.398                 |
| 2     | 0.288                 |
| 10    | 0.055                 |
| 20    | 0.027                 |

On the basis of analytical and numerical results we have a pretty clear picture of the qualitative behaviour of solutions. Let us discuss in more detail how solutions depend on the parameters $\alpha$ and $x_H$. First consider the limit $\alpha \to 0$. In our units this corresponds to $v \to 0$. In this limit a solution on any finite region outside the horizon is well approximated by the Schwarzschild solution

$$w = 1, \quad N = 1 - \frac{x_H}{x},$$

with mass $m = x_H/2$. However, for large $x$ and small but finite $\alpha$ the term $\alpha^2 x^2$ in Eqs. (18) and (20), becomes dominant and asymptotically the solution tends to the RN solution.

Next, consider the limit $\alpha \to \alpha_{\text{max}}(x_H)$. In this case the behaviour of solutions depends on whether $x_H$ is less or greater than one. For $x_H \leq 1$, the interval of allowed values of $b$, given by Eq. (33), shrinks to zero as $\alpha$ goes to $\alpha_{\text{max}}$, and for $\alpha = \alpha_{\text{max}}$, $b$ is uniquely determined by $x_H$. The corresponding limiting solution describes an extremal black hole which is essentially nonabelian. For $x_H \geq 1$, when $\alpha$ goes to $\alpha_{\text{max}}$ the solution tends to the RN solution and for $\alpha = \alpha_{\text{max}}$ coalesce with it. Thus for given $x_H \geq 1$, the point $\alpha = \alpha_{\text{max}}(x_H)$ is a bifurcation point: for $\alpha > \alpha_{\text{max}}$ there is only one solution (RN), while for $\alpha \leq \alpha_{\text{max}}$ the second (nonabelian) solution appears. The bifurcation diagram in the plane $(\alpha, x_H)$ is graphed in Fig. 2. The nonabelian solutions exist only in the region below the curve ABC. Along the curve AB, the nonabelian solutions are extremal. Along the curve BC the nonabelian and RN solutions coalesce. The RN solutions exist for $x_H \geq 1$ and do not depend on $\alpha$ (for $x_H = 1$ the RN solution is extremal).
The bifurcation of solutions is also seen in Fig. 3, where the mass $m$ as a function of $\alpha$ is graphed for given $x_H \geq 1$. Notice that the mass of the RN solution $m_{RN} = \frac{1}{2}(x_H + \frac{1}{x_H})$ is larger than the mass of the nonabelian solution with the same $x_H$. This suggests that for $\alpha < \alpha_{max}$, where there are two distinct solutions with the same radius of the horizon $x_H$, the RN solution is unstable. We will show in the next section that this is indeed the case.

Let us comment on the status of the no-hair conjecture in our model. The no-hair conjecture (in its strong version) states that black hole solutions, within a given model, are uniquely determined by global charges defined as surface integral at infinity. In our case, all solutions have unit magnetic charge, so the only global parameter which may differentiate solutions at infinity is their mass. For $\alpha \geq 1$ there is only one solution for given mass $m$, namely the RN solution, hence the strong no-hair conjecture is valid. However, for $\alpha < 1$ the situation is different. This is illustrated in Table 3 where we show how the masses of nonabelian and RN solutions depend on $x_H$ for given $\alpha$.

**Table 3:** Masses versus $x_H$ for $\alpha = 0.1$

| $x_H$ | $m_{NA}$ | $m_{RN}$ |
|-------|----------|----------|
| 0.1   | 0.226    |          |
| 0.2   | 0.274    |          |
| 1     | 0.663    | 0.75     |
| 2     | 1.147    | 1.25     |
| 5     | 2.599    | 2.6      |
| 5.55  | 2.86508  | 2.86509  |
| 5.595 | 2.886865 | 2.886865 |
| 6     | 3.08333  |          |
| 8     | 4.0625   |          |

For the RN solution $x_H \geq 1$, so its mass $m_{RN} = \frac{1}{2}(x_H + \frac{1}{x_H})$ is bounded from below by 0.75. At $x_H^{max}(\alpha)$ the nonabelian and RN solutions coalesce, so the maximal mass of the nonabelian solution is equal to $m_{RN}(x_H^{max})$. For given $\alpha < 1$ and $0.75 \leq m < m_{RN}(x_H^{max})$ there are two distinct (i.e. RN and nonabelian) solutions with the same mass. This violates the strong no-hair conjecture.

We will show in the next section that in the region where two distinct solutions coexist, only one of them is stable. Thus, in our model, a weak no-hair conjecture holds, i.e. global charges determine uniquely the stable black hole solution. However, as we shall argue, in the full EYMH theory, even the weak no-hair conjecture is violated.

Finally, let us briefly consider solutions for which $w$ is not monotonic. Such solutions may be viewed as excitations of the fundamental solutions described above. The existence of excitations in the EYMH model was first noticed in Ref. 5. To understand the existence of excitations it is useful to consider the limit $\alpha \to 0$. Then, the Eqs. (18 – 20) become the EYM equations, which have a countable family of so-called colored black holes [2]. These solutions are labelled by an integer $n = \text{number of nodes of } w$, and in contrast to the $\alpha \neq 0$ case, they have zero magnetic charge (because $w(\infty) = \pm 1$). For sufficiently small $\alpha$, the excitations may be viewed as singular perturbations (in $\alpha$) of colored black holes. As long as $\alpha x \ll 1$, the excitations are well approximated by colored black hole solutions (with the same number of nodes for $w$), but for large $x$ the term $\alpha^2 x^2$ in Eqs. (18) and (19) becomes dominant and forces $w$ to decay.
exponentially. This is shown in Fig. 5, where \( n = 1 \) excitations are graphed for two values of \( \alpha \) and compared to the \( n = 1 \) colored black hole solution.

It is rather difficult to find numerically the excitations with large \( n \), however we expect that for given \( x_H \) and \( \alpha > 0 \), the number of excitations is finite (in contrast to the \( \alpha = 0 \) case, where there are infinitely many solutions). Our expectation is based on monotonicity of solutions for \( x < 1/\alpha \), proven in Section 3. The existence of excitations with arbitrarily large \( n \) would be inconsistent with this property, because, in analogy to the colored black holes, the location of nodes is expected to extend to infinity as \( n \) increases.

The bifurcation structure of excitations is similar to that for the fundamental solution. For given \( x_H \geq 1 \) there is a decreasing sequence \( \{\alpha_{\text{max}}^0, \alpha_{\text{max}}^1, \ldots\} \) of bifurcation points such that at \( \alpha_{\text{max}}^n \) the \( n \)-th excitation bifurcates from the RN branch (by zeroth excitation we mean the fundamental solution). As we shall discuss in the next section this picture is closely related to the stability properties of the RN solution.

5 Stability Analysis

In this section we address the issue of linear stability of the black hole solutions described above. To that purpose we have to study the evolution of linear perturbations about the equilibrium configurations. Since we do not expect nonspherical instability (and since, admittedly, the analysis of nonspherical perturbations would be extremely complicated), we restrict our study to radial perturbations. For radial perturbation the stability analysis is relatively simple, because the spherically symmetric gravitational field has no dynamical degrees of freedom and therefore the perturbations of metric coefficients are determined by the perturbations of matter fields. This was explicitly demonstrated by Straumann and Zhou [9] for EYM system who derived the pulsation equations governing the evolution of radial normal modes of the YM field. Below we repeat their derivation with a slight modification due to the presence of the mass term in Eqs. (14 – 17).

We define the functions \( a \) and \( b \) by \( e^a = BN \) and \( e^b = N \) and write the perturbed fields as

\[
\begin{align*}
    w(r,t) &= w_0(r) + \delta w(r,t) \\
    a(r,t) &= a_0(r) + \delta a(r,t) \\
    b(r,t) &= b_0(r) + \delta b(r,t)
\end{align*}
\]  

(34)

where \((w_0, a_0, b_0)\) is a static solution. We insert the expressions (34) into Eqs. (14 – 17) and keep only terms of the first order in the perturbations. Hereafter we use dimensionless variables

\[
\tau = \frac{e}{\sqrt{G}} t \quad \text{and} \quad x = \frac{e}{\sqrt{G}} r
\]

and we omit a subscript 0 for static solutions. From (16) we obtain

\[
\delta b = -\frac{4}{x} w' \delta w.
\]  

(35)

The asymptotically flat solution of this equation is

\[
\delta b = -\frac{4}{x} w' \delta w.
\]  

(36)
Linearization of Eq. (15) yields
\[ \delta a' - \delta b' = \frac{4}{x} w' \delta w'. \]  
(37)

Thus, using (36), we obtain
\[ \delta a' = \frac{4}{x^2} w' \delta w - \frac{4}{x} w'' \delta w. \]  
(38)

Multiplying Eq. (17) by \( e^{-a} \) and linearizing we get
\[ -e^{-2a} \delta \dot{w} + e^{-a} (e^a \delta w')' + \delta a' w' - \delta b e^{-b} w \left[ \frac{1}{x^2} (1 - w^2) - \alpha^2 \right] + e^{-b} \left[ \frac{1}{x^2} (1 - 3w^2) - \alpha^2 \right] \delta w = 0. \]  
(39)

Now, we insert (36) and (38) into (39) and make an Ansatz
\[ \delta w = e^{i\omega x} \xi(x) \]  
(40)
to get the eigenmode equation
\[ -e^a (e^a \xi')' + U \xi = \omega^2 \xi \]  
(41)
where
\[ e^{-a} U = -\frac{4}{x^2} e^a (1 + a' x) w^2 - \frac{8}{x} e^{-b} w w' \left[ \frac{1 - w^2}{x^2} - \alpha^2 \right] - e^{-b} \left[ \frac{1}{x^2} (1 - 3w^2) - \alpha^2 \right]. \]  
(42)

The potential \( U \) is a smooth bounded function which vanishes at \( x_H \) and tends to \( \alpha^2 \) for \( x \to \infty \).

One can introduce the tortoise radial coordinate to transform Eq. (41) into the one-dimensional Schrödinger equation. However we shall not do so, because the numerical analysis is easier when one uses the coordinate \( x \).

A static solution \((w, a, b)\) is stable if there are no integrable eigenmodes \( \xi \) with negative \( \omega^2 \). To check this we have applied the rule of nodes for Sturm-Liouville systems [10], which states that the number of negative eigenmodes is equal to the number of nodes of the zero eigenvalue solution. Namely we have considered the Eq. (41) with negative \( \omega^2 \) and looked how the solution satisfying \( \xi(x_H) = 0 \) and \( \xi'(x_H) > 0 \) behaves as \( \omega^2 \to 0^- \).

We have found that the function \( \xi \) has no nodes, when \( U \) is determined by a nonabelian solution (for all allowed values of the parameters). Thus we conclude that the nonabelian solutions are stable. When \( x_H \to x_H^{\text{max}}(\alpha) \) the (everywhere positive) function \( \xi \) tends to zero at infinity which signals the existence of a static perturbation (bifurcation point).

One can apply the same method to examine the stability of the RN solution. This was done by Lee et al. [6] in the full EYMH theory. They showed that the pulsation equation for the Higgs field (which in the case of RN solution decouples from the pulsation equation for the YM field) has no unstable modes. Thus the question of stability of the RN solution does not depend on the parameter \( \beta \) (strength of the Higgs coupling) and reduces for every \( \beta \) to the Eq. (41), where now
\[ U = e^{-b} \frac{\alpha^2 x^2 - 1}{x^2}. \]  
(43)

For given \( x_H \geq 1 \), the existence of negative eigenmodes in this potential depends on \( \alpha \). For \( \alpha > \alpha_{\text{max}}(x_H) \) there are no negative eigenvalues, hence the RN solution is stable. For \( \alpha < \alpha_{\text{max}}(x_H) \), the RN is unstable. Equivalently, we can say that for given \( \alpha \), the RN is stable if \( x_H > x_H^{\text{max}}(\alpha) \), otherwise it is unstable.
We have found numerically that when $\alpha$ decreases the RN solution picks up additional unstable modes. That is, there is a decreasing sequence $\{\alpha_n\}$, where $\alpha_0 = \alpha_{\text{max}}$, such that in the interval $\alpha_{n+1} < \alpha < \alpha_n$ there are exactly $n$ unstable modes (we have checked this up to $n = 4$). This is consistent with the fact that the RN solution has infinitely many unstable modes for $\alpha = 0$, as follows easily from (43). We are convinced that the sequence $\{\alpha_n\}$ coincides with the sequence $\{\alpha^n_{\text{max}}\}$ of the bifurcation points at which the $n$-th excitation appears. Due to highly unstable behaviour of solutions near $\alpha_{\text{max}}$ we were able to verify this assertion with sufficient numerical accuracy only up to $n = 1$. However, as long as there are no other bifurcation points (as we believe), our conclusion is based on general results of the bifurcation theory [7]. Namely, if the operator governing small fluctuations if self-adjoint (hence its eigenvalues are real), there is the theorem which states that at the bifurcation point one eigenvalue passes through zero, whereas elsewhere the eigenvalues cannot change a sign.

One could have anticipated the stability properties of the solution from a mere comparison of masses. In the region of the plane $\alpha, x_H$ where two distinct solutions coexist (with the same $x_H$), the solution with lower mass (i.e. nonabelian) is stable, while the solution with higher mass (i.e. RN) is unstable. However, it should be emphasized that in more complicated situations the naive comparison of masses is not conclusive for stability. Actually, this happens in the full EYMH theory. To see this, let us consider the bifurcation diagram $(m, \alpha)$ for $x_H > 1$ in the full EYMH theory. As follows from the results of Ref. 5 in the Prasad-Sommerfield limit and our preliminary results for finite $\beta$, this diagram is more complicated than that shown in Fig. 3. We sketch it in Fig. 5.

The horizontal line in Fig. 5 corresponds to the mass of the RN solution. There are two branches of (fundamental) nonabelian solutions: the upper branch AB and the lower branch CB. These two branches merge at the bifurcation point B, which corresponds to $\alpha_{\text{max}}(x_H)$. There is a second bifurcation point A, corresponding to some $\alpha_0(x_H)$, where the upper branch merges with the RN solution. We will use these facts to infer the stability properties of solutions.

As we have pointed out above the existence of a bifurcation point is a necessary condition for the transition between stability and instability. However, it is not a sufficient condition, since it is only when the lowest eigenvalue passes through zero that the stability changes. Therefore, if there are two branches which merge at the bifurcation point and one of them is stable then the other one will have exactly one unstable mode.

Applying this reasoning in the present context, we see that if the lower branch is stable for small $\alpha$ (what we expect), than the whole lower branch must be stable, whereas the upper one is unstable with exactly one negative mode. By a similar argument the RN solution becomes unstable at $\alpha_0$. Therefore, in the region between $\alpha_0$ and $\alpha_{\text{max}}$ there are two stable solutions and one unstable. Note that in this region there is no relation between stability and mass. In particular the point in Fig. 5 at which the lower branch crosses the horizontal line is not related to the change in stability (contrary to the suggestion in Ref. 5).

The analogous bistability region exists in the plane $(m, x_H)$ for fixed $\alpha$. This means that for given mass $m$ there are two distinct stable black hole solutions which violates the weak no-hair conjecture [11].
6 Evaporation

Let us briefly consider the Hawking evaporation process in our model. As the initial configuration we take the RN black hole with unit magnetic charge and large mass. There are three different scenarios of its evolution depending on the value of \( \alpha \). For \( \alpha > 1 \) the situation is the same as in the Einstein-Maxwell theory. The RN black hole emits thermal radiation and loses mass. When the horizon shrinks to \( r_H = \sqrt{G/e} \) (extremal limit), the temperature drops to zero and the evaporation stops.

For \( \alpha < 1 \), the RN black hole contracts to \( r_H = r_H^{\text{max}}(\alpha) \) where it becomes classically unstable. The further evolution proceeds along the nonabelian branch and depends on whether \( \alpha \) is smaller or greater than \( 1/\sqrt{2} \). For \( \alpha > 1/\sqrt{2} \), the temperature drops to zero when the horizon shrinks to \( r_H^{\text{min}}(\alpha) \) (extremal limit) and the black hole settles down as an extremal nonabelian solution.

For \( \alpha < 1/\sqrt{2} \), the temperature grows as the black hole contracts, so the black hole evaporates completely leaving behind a magnetic monopole remnant.

This last scenario was first suggested by Lee et al., as a possible consequence of the instability of RN solution [6].

Of course, the above scenario might change if one includes back reaction effects.

Acknowledgement

P.B. would like to acknowledge the hospitality of the Aspen Center of Physics where part of this work was carried out. Partial support by the Fundacion Federico.