The perfect $\mathcal{F}$-locality from the basic $\mathcal{F}$-locality over a Frobenius $P$-category $\mathcal{F}$

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Abstract: Let $p$ be a prime, $P$ a finite $p$-group, $\mathcal{F}$ a Frobenius $P$-category and $\mathcal{F}^{sc}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$. Recently, we have understood an easy way to obtain the perfect $\mathcal{F}^{sc}$-locality $\mathcal{P}^{sc}$ from the basic $\mathcal{F}^{sc}$-locality $\mathcal{L}^{b,sc}$: it depends on a suitable filtration of the basic $\mathcal{F}$-locality $\mathcal{L}^{b}$ and on a vanishing cohomology result, given with more generality in [11, Appendix].

1. Introduction

1.1. Let $p$ be a prime and $P$ a finite $p$-group. After our introduction of the Frobenius $P$-categories $\mathcal{F}$ [7] and the question of Dave Benson [1] on the existence of a suitable category $\mathcal{P}^{sc}$ — called linking system in [2] and perfect $\mathcal{F}^{sc}$-locality in [8, Chap. 17] — extending the full subcategory $\mathcal{F}^{sc}$ of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ [8, Chap. 3], the existence and the uniqueness of $\mathcal{P}^{sc}$ has concentrate some effort.

1.2. In [2] Carles Broto, Ran Levi and Bob Oliver formulate the existence and the uniqueness of the category $\mathcal{P}^{sc}$ in terms of the annulation of an obstruction 3-cohomology element and of the vanishing of a 2-cohomology group, respectively. They actually state a sufficient condition for the vanishing of the corresponding $n$-cohomology groups.

1.3. In [3] Andrew Chermak has proved the existence and the uniqueness of $\mathcal{P}^{sc}$ via his objective partial groups, but his proof depends on the so-called Classification of the finite simple groups and on some results by U. Meierfrankenfeld and B. Stellmacher. In [6] Bob Oliver, following some of Chermak’s methods, has also proved for $n \geq 2$ the vanishing of the $n$-cohomology groups mentioned above. In [5] George Glauberman and Justin Lynd remove the use of the Classification of the finite simple groups in [6].

1.4. Independently, with direct methods which already employ the basic $\mathcal{F}$-locality $\mathcal{L}^{b}$ [8, Chap. 22], in [9] and [10] we prove the existence and the uniqueness of an extension $\mathcal{P}$ of $\mathcal{F}$ — called perfect $\mathcal{F}$-locality in [8, Chap. 17] — which contains $\mathcal{P}^{sc}$ as the full subcategory over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ [8, Chap. 3].

1.5. But recently, we have understood an easier way to obtain $\mathcal{P}^{sc}$ from the full subcategory $\mathcal{L}^{b,sc}$ of $\mathcal{L}^{b}$ over the set of $\mathcal{F}$-selfcentralizing subgroups.

† Although they need a partial classification for $p=2$.

†† In [10] we give a full correction of the uniqueness argument for $\mathcal{P}^{sc}$ in [9].
of $P$ [8, Chap. 3]. Denoting by $Z^\geq : L^{h,sc} \to \mathfrak{Ab}$ the obvious contravariant functor from $L^{h,sc}$ to the category $\mathfrak{Ab}$ of finite Abelian groups, mapping any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ on its center $Z(Q)$, it is easy to see that we have a quotient category $\tilde{L}^{h,sc} = L^{h,sc} / Z^\geq$ and that the structural functor $\pi^\geq : L^{h,sc} \to \mathcal{F}^\geq$ factorizes through a functor $\tilde{\pi}^\geq : \tilde{L}^{h,sc} \to \mathcal{F}^\geq$.

1.6. The point is that $\tilde{\pi}^\geq$ admits an essentially unique section functor $\tilde{\sigma}^\geq : \mathcal{F}^\geq \to \tilde{L}^{h,sc}$, and then $\mathcal{P}^\geq$ is just the converse image in $\tilde{L}^{h,sc}$ of the image $\tilde{\pi}^\geq(F^\geq)$ of $F^\geq$ in $\tilde{L}^{h,sc}$; since in [9, Theorem 7.2] we prove that any perfect $\mathcal{F}^\geq$-locality $P^\geq$ can be extended to a unique perfect $\mathcal{F}$-locality $P$, this proves the existence of $\mathcal{P}$. Moreover, in [9, 8.5 and Theorem 8.10] we prove that there is an $\mathcal{F}$-locality functor $\sigma$ from any perfect $\mathcal{F}$-locality $P$ to $\tilde{L}^h$; then, it is easy to check that $\sigma$ induces a functor $\tilde{\sigma}^\geq : F^\geq \to \tilde{L}^{h,sc}$ which is a section of $\tilde{\pi}^\geq$, proving the uniqueness of $\mathcal{P}^\geq$ and therefore, by [9, Theorem 7.2], the uniqueness of $\mathcal{P}$.

1.7. In Section 2 we recall all the definitions and state properly all the quoted results. The existence and the essential uniqueness of the section functor $\tilde{\sigma}^\geq$ mentioned above depend on a suitable filtration of $\tilde{L}^h$ and on a vanishing cohomology result; this filtration has been already introduced in [9, 8.3 and Corollary 8.4], but it seems necessary to give here a complete account in Section 3. The vanishing cohomology result we need here is given in [11, Appendix] in a more general framework. In Section 4 we give explicit proofs of all the results announced in 1.6 above and, in particular, an independent proof of the existence of the functor $\sigma : \mathcal{P} \to \tilde{L}^h$ mentioned above.

2. Definitions and quoted results

2.1. Denote by $\mathfrak{Ab}$ the category of Abelian groups and by $\mathfrak{iGr}$ the category formed by the finite groups and by the injective group homomorphisms. Recall that, for any category $\mathcal{C}$, we denote by $\mathcal{C}^\circ$ the opposite category and, for any $\mathcal{C}$-object $A$, by $\mathcal{C}_A$ (or $(\mathcal{C})_A$ to avoid confusion) the category of “$\mathcal{C}$-morphisms to $A$” [8, 1.7]; moreover, for any pair of objects $A$ and $B$, $\mathcal{C}(B, A)$ denote the set of $\mathcal{C}$-morphisms from $A$ to $B$ and we set $\mathcal{C}(A) = \mathcal{C}(A, A)$ for short.

2.2. For any finite subgroup $G$ and any $p$-subgroup $P$ of $G$, denote by $\mathcal{F}_{G,P}$ and $\mathcal{T}_{G,P}$ the respective categories where the objects are all the subgroups of $P$ and, for two of them $Q$ and $R$, the respective sets of morphisms $\mathcal{F}_{G,P}(Q, R)$ and $\mathcal{T}_{G,P}(Q, R)$ are formed by the group homomorphisms from $R$ to $Q$ respectively induced by the conjugation by elements of $G$, and by the set $\mathcal{T}_{G}(R, Q)$ of such elements, the compositions being the obvious ones.

2.3. For a finite $p$-group $P$, a Frobenius $P$-category $\mathcal{F}$ is a subcategory of $\mathfrak{iGr}$ containing $\mathcal{F}_{P} = \mathcal{F}_{P, P}$ where the objects are all the subgroups of $P$.
and the morphisms fulfill the following three conditions [8, 2.8 and Proposition 2.11]

2.3.1 For any subgroup $Q$ of $P$, the inclusion functor $(F)_Q \rightarrow (i\mathcal{E}t)_Q$ is full.

2.3.2 $F_P(P)$ is a Sylow $p$-subgroup of $F(P)$.

We say that a subgroup $Q$ of $P$ is fully centralized in $F$ if for any $F$-morphism $\xi: Q \rightarrow C_P(Q)$ we have $\xi(C_P(Q)) = C_P(\xi(Q))$; similarly, replacing in this condition the centralizer by the normalizer, we say that $Q$ is fully normalized.

2.3.3 For any subgroup $Q$ of $P$ fully centralized in $F$, any $F$-morphism $\varphi: Q \rightarrow P$ and any subgroup $R$ of $N_P(\varphi(Q))$ such that $\varphi(Q) \subseteq R$ and that $F_P(Q)$ contains the action of $F_R(\varphi(Q))$ over $Q$ via $\varphi$, there exists an $F$-morphism $\zeta: R \rightarrow P$ fulfilling $\zeta(\varphi(u)) = u$ for any $u \in Q$.

We denote by $\tilde{F}$ — called the exterior quotient of $F$ — the quotient of $F$ by the inner automorphisms of the $F$-objects [8, 1.3] and by $\epsilon_F: F \rightarrow \tilde{F}$ the canonical functor. Note that, with the notation above, if $P$ is Sylow $p$-subgroup of $G$ then $F_G,P$ is a Frobenius $P$-category; often, we write $F_G$ instead of $F_G,P$.

2.4. Then, a (divisible) $F$-locality† is a triple $(\tau, \mathcal{L}, \pi)$ formed by a finite category $\mathcal{L}$, a surjective functor $\pi: \mathcal{L} \rightarrow F$ and a functor $\tau: \mathcal{T}_P \rightarrow \mathcal{L}$ from the transporter category $\mathcal{T}_P = \mathcal{T}_{P,P}$ of $P$, fulfilling the following two conditions [8, 17.3 and 17.8]

2.4.1 The composition $\pi \circ \tau$ coincides with the composition of the canonical functor defined by the conjugation $\kappa_P: \mathcal{T}_P \rightarrow F_P$ with the inclusion $F_P \subseteq F$.

We denote by $\tilde{\kappa}_P: \mathcal{T}_P \rightarrow \tilde{F}_P$ the composition of $\kappa_P$ with $\epsilon_F$, above.

2.4.2 For any pair of subgroups $Q$ and $R$ of $P$, $\text{Ker}(\pi_R)$ acts regularly on the fibers of the following maps determined by $\pi$

$$\pi_{Q,R}: \mathcal{L}(Q,R) \rightarrow F(Q,R)$$

Analogously, for any pair of subgroups $Q$ and $R$ of $P$, we denote by

$$\tau_{Q,R}: \mathcal{T}_P(Q,R) \rightarrow \mathcal{L}(Q,R)$$

the map determined by $\tau$, and whenever $R \subseteq Q$ we set $i_R^Q = \tau_{Q,R}(1)$; if $R = Q$ then we write $Q$ once for short.

2.5. We say that an $F$-locality $(\tau, \mathcal{L}, \pi)$, or $\mathcal{L}$ for short, is coherent if it fulfills the following condition [8, 17.9]

(Q) For any pair of subgroups $Q$ and $R$ of $P$, any $x \in \mathcal{L}(Q,R)$ and any $v \in R$, we have $x \cdot \tau_R(v) = \tau_Q(\pi_{Q,R}(x)(v)) \cdot x$.

† Here we only consider divisible $F$-localities in the sense of [8, Chap. 17].
In this case,, if \(Q'\) and \(R'\) are subgroups of \(P\), and we have the inclusions \(R \subset Q\) and \(R' \subset Q'\), denoting by \(L(Q', Q)_{R', R}\) the set of \(y \in L(Q', Q)\) such that \((\pi_{Q', Q}(y))(R) \subset R'\), we get a restriction map (possibly empty!)

\[
\tau_{Q', Q}^{Q, R} : L(Q', Q)_{R', R} \longrightarrow L(R', R) \quad 2.5.1
\]

fulfilling \(y^{\tau_{Q', Q}^{Q, R}} = \iota_{Q', Q}^{Q, R} \iota_{R', R}^{Q', Q}(y)\) for any \(y \in L(Q', Q)_{R', R}\). Note that, with the notation above, if \(P\) is Sylow \(p\)-subgroup of \(G\) then \(T_G = T_{G, P}\) endowed with the obvious functors

\[
\tau_G : T_P \longrightarrow T_G \quad \text{and} \quad \pi_G : T_G \longrightarrow F_G = F_{G, P} \quad 2.5.2
\]

becomes a coherent \(F\)-locality. Moreover, we say that a coherent \(F\)-locality \((\tau, L, \pi)\) is \(p\)-coherent (resp. \(ab\)-coherent) when \(\text{Ker}(\pi_Q)\) is a finite \(p\)-group (resp. a finite abelian group) for any subgroup \(Q\) of \(P\).

2.6. Recall that the \(F\)-hyperfocal subgroup is the subgroup \(H_F\) of \(P\) generated by the union of the sets \(\{u^{-1}\sigma(u)\}_{u \in Q}\) where \(Q\) runs over the set of subgroups of \(P\) and \(\sigma\) over the set of \(p'\)-elements of \(F(Q)\). We say that an \(F\)-locality \((\tau, L, \pi)\) is perfect if \(P\) is coherent and, for any subgroup \(Q\) of \(P\) fully centralized in \(F\), the \(C_{F}^{P}(Q)\)-hyperfocal subgroup \(H_{C_{F}^{P}(Q)}\) coincides with \(\text{Ker}(\hat{\pi}_Q)\) \([8, 17.13]\); actually, this is equivalent to say that \(P(Q)\) endowed with

\[
\tau_Q : T_{N_F(Q)} \longrightarrow \mathcal{P}(Q) \quad \text{and} \quad \pi_Q : \mathcal{P}(Q) \longrightarrow F(Q) \quad 2.6.1
\]

is an \(F\)-localizer of \(Q\) \([8, 18.5\) and Theorem 18.6], for any subgroup \(Q\) of \(P\) fully centralized in \(F\).

2.7. Further, for any \(F\)-locality \((\tau, L, \pi)\) we get a contravariant functor from \(L\) to the category \(\mathcal{Gr}\) of finite groups \([8, 17.8.2]\)

\[
\text{Ker}(\pi) : L \longrightarrow \mathcal{Gr} \quad 2.7.1
\]

sending any subgroup \(Q\) of \(P\) to \(\text{Ker}(\pi_Q)\) and any \(L\)-morphism \(x : R \rightarrow Q\) to the group homomorphism

\[
\text{Ker}(\pi)_x : \text{Ker}(\pi_Q) \longrightarrow \text{Ker}(\pi_R) \quad 2.7.2
\]

fulfilling \(u \cdot x = x \cdot (\text{Ker}(\pi)_x(u))\) for any \(u \in \text{Ker}(\pi_Q)\). If \(L\) is \(ab\)-coherent then the functor \(\text{Ker}(\pi)\) factorizes through the exterior quotient \(\hat{F}\), inducing a functor

\[
\hat{\text{Ker}}(\pi) = \hat{\tau}_L : \hat{F} \longrightarrow \mathcal{Gr} \quad 2.7.3
\]

indeed, in this case it follows from \([8,\) Proposition 17.10\] that, for any subgroup \(Q\) of \(P\), \(\tau_Q(Q)\) centralizes \(\text{Ker}(\pi_Q)\) and therefore, for any \(u \in \text{Ker}(\pi_Q)\) and any \(v \in Q\), we have

\[
\tau_Q(v) \cdot u = u \cdot \tau_Q(v) = \tau_Q(v) \cdot (\text{Ker}(\pi)_{\tau_Q(v)}(u)) \quad 2.7.4
\]
so that \( \ker(\pi)_{\tau_Q(v)} = \id_{\ker(\pi_Q)} \); the same argument holds for \( w \in \ker(\pi_Q) \).

2.8. If \( \mathcal{L}' \) is a second \( \mathcal{F} \)-locality with structural functors \( \tau' \) and \( \pi' \), we call \( \mathcal{F} \)-locality functor from \( \mathcal{L} \) to \( \mathcal{L}' \) any functor \( l: \mathcal{L} \to \mathcal{L}' \) fulfilling

\[
\tau' = \id \circ \tau \quad \text{and} \quad \pi' \circ \id = \pi
\]

2.8.1; the composition of two \( \mathcal{F} \)-locality functors is obviously an \( \mathcal{F} \)-locality functor. It is easily checked that any \( \mathcal{F} \)-locality functor \( l: \mathcal{L} \to \mathcal{L}' \) determines a natural map

\[
\chi_l: \ker(\pi) \to \ker(\pi')
\]

2.8.2; conversely, it is quite clear that any subfunctor \( h \) of \( \ker(\pi) \) determines a quotient \( \mathcal{F} \)-locality \( \mathcal{L}/h \) defined by the quotient sets

\[
(\mathcal{L}/h)(Q, R) = \mathcal{L}(Q, R)/h(R)
\]

2.8.3, for any pair of subgroups \( Q \) and \( R \) of \( P \), and by the corresponding induced composition; moreover, \( \mathcal{L}/h \) is coherent whenever \( \mathcal{L} \) is it.

2.9. We say that two \( \mathcal{F} \)-locality functors \( l \) and \( \bar{l} \) from \( \mathcal{L} \) to \( \mathcal{L}' \) are naturally \( \mathcal{F} \)-isomorphic if we have a natural isomorphism \( \lambda: l \cong \bar{l} \) fulfilling \( \pi' \circ \lambda = \id_{\pi} \) and \( \lambda \circ \tau = \id_{\tau} \); in this case, \( \lambda_Q \) belongs to \( \ker(\pi_Q') \) for any subgroup \( Q \) of \( P \) and, since \( \lambda(\iota_Q^\rho) = \iota_Q^\rho = \bar{l}(\iota_Q^\rho) \), \( \lambda \) is uniquely determined by \( \lambda_P \); indeed, we have

\[
\lambda_P \cdot i_Q^\rho = i_Q^\rho \cdot \lambda_Q
\]

2.9.1. Once again, the composition of a natural \( \mathcal{F} \)-isomorphism with an \( \mathcal{F} \)-locality functor or with another such a natural \( \mathcal{F} \)-isomorphism is a natural \( \mathcal{F} \)-isomorphism.

2.10. Moreover, from two \( \mathcal{F} \)-localities \( (\tau, \mathcal{L}, \pi) \) and \( (\tau', \mathcal{L}', \pi') \), we can construct a third \( \mathcal{F} \)-locality \( \mathcal{L}'' = \mathcal{L} \times_{\mathcal{F}} \mathcal{L}' \) from the corresponding category defined by the pull-back of sets

\[
\mathcal{L}''(Q, R) = \mathcal{L}(Q, R) \times_{\mathcal{F}(Q, R)} \mathcal{L}'(Q, R)
\]

2.10.1 with the obvious composition and with the structural maps

\[
\tau_{Q,R}'' \quad \pi_{Q,R}''
\]

2.10.2 respectively induced by \( \tau \) and \( \tau' \), and by \( \pi \) and \( \pi' \). Note that we have obvious \( \mathcal{F} \)-locality functors

\[
\mathcal{L} \leftarrow \mathcal{L} \times_{\mathcal{F}} \mathcal{L}' \to \mathcal{L}'
\]

2.10.3 and that \( \mathcal{L} \times_{\mathcal{F}} \mathcal{L}' \) is coherent if \( \mathcal{L} \) and \( \mathcal{L}' \) are so.
2.11. In order to define the basic $\mathcal{F}$-locality, we have to consider the $\mathcal{F}$-basic $P \times P$-sets; recall that an $\mathcal{F}$-basic $P \times P$-set $\Omega$ is a finite nonempty $P \times P$-set fulfilling the following three conditions \cite{8, 21.4 and 21.5}, where $\Omega^\circ$ denotes the $P \times P$-set obtained from $\Omega$ by exchanging both factors, for any subgroup $Q$ of $P$ we denote by $i_Q^P$ the corresponding inclusion map, and for any $\varphi \in \mathcal{F}(P,Q)$ we set
\[ \Delta_\varphi(Q) = \{ (\varphi(u), u) \}_{u \in Q} \] 2.11.1.

2.11.2 We have $\Omega^\circ \cong \Omega$, $\{1\} \times P$ acts freely on $\Omega$ and $|\Omega|/|P| \equiv 0 \mod p$.

2.11.3 For any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$ we have a $Q \times P$-set isomorphism
\[ \text{Res}_{\varphi \times \text{id}_P}(\Omega) \cong \text{Res}_{i_Q^P \times \text{id}_P}(\Omega) \]

2.11.4 Any $P \times P$-orbit in $\Omega$ is isomorphic to $(P \times P)/\Delta_\varphi(Q)$ for a suitable subgroup $Q$ of $P$ and some $\varphi \in \mathcal{F}(P,Q)$.

Moreover, we say that an $\mathcal{F}$-basic $P \times P$-set $\Omega$ is thick if the multiplicity of the indecomposable $P \times P$-set $(P \times P)/\Delta_\varphi(Q)$ is at least two for any subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$ \cite{8, 21.4}.

2.12. The existence of a thick $\mathcal{F}$-basic $P \times P$-set is guaranteed by \cite{8, Proposition 21.12}; we choose one of them $\Omega$ and denote by $G$ the group of $\{1\} \times P$-set automorphisms of $\text{Res}_{\{1\} \times P}(\Omega)$; it is clear that we have an injective map from $P \times \{1\}$ into $G$ and we identify this image with the $p$-group $P$ itself so that, from now on, $P$ is contained in $G$ and acts freely on $\Omega$. Then, it follows from the conditions above that we have
\[ \mathcal{F}_{G,P} = \mathcal{F} \] 2.12.1

and it is quite clear that, as in 2.5.2, $\mathcal{T}_{G,P} = \mathcal{T}_{G,P}$ becomes a coherent $\mathcal{F}$-locality.

2.13. For any subgroup $Q$ of $P$, it is clear that the centralizer $C_G(Q)$ coincides with the group of $Q \times P$-set automorphisms of $\text{Res}_{Q \times P}(\Omega)$; moreover, since any $Q \times P$-orbit in $\Omega$ is isomorphic to the $Q \times P$-set $(Q \times P)/\Delta_\eta(T)$, for a suitable subgroup $T$ of $P$ such that $\mathcal{F}(Q,T) \neq \emptyset$ and some $\eta \in \mathcal{F}(Q,T)$ (cf. condition 2.11.3), and since we have \cite{8, 22.3}
\[ \text{Aut}_{Q \times P}((Q \times P)/\Delta_\eta(T)) \cong \tilde{N}_{Q \times P}(\Delta_\eta(T)) \] 2.13.1

denoting by $k_\eta$ the multiplicity of $(Q \times P)/\Delta_\eta(T)$ in $\Omega$ and by $\mathcal{G}_{k_\eta}$ the corresponding $k_\eta$-symmetric group, we actually get obvious group isomorphisms
\[ C_G(Q) \cong \prod_{T \in \mathcal{C}_P} \prod_{\eta \in \mathcal{D}_Q^T} \tilde{N}_{Q \times P}(\Delta_\eta(T)) \wr \mathcal{G}_{k_\eta} \] 2.13.2

where $\wr$ denotes the wreath product, $\mathcal{C}_P$ is a set of representatives for the set of $P$-conjugacy classes of subgroups $T$ of $P$ and, for any $T \in \mathcal{C}_P$, ...
$$\mathfrak{O}_Q \subset \mathcal{F}(Q, T)$$ is a (possibly empty) set of representatives for the quotient set \(Q \setminus \mathcal{F}(Q, T)/N_P(T)\). For short, let us set

$$\mathfrak{O}_Q = \bigsqcup_{T \in C_P} \mathfrak{O}_Q^T$$

2.13.3;

this set actually indexes the set of isomorphic classes of transitive \(Q \times P\)-sets; to avoid confusion, we note by \((T, \eta)\) the element \(\eta\) in \(\mathfrak{O}_Q^T\).

2.14. Then, it follows from [8, Proposition 22.11] that the correspondence sending \(Q\) to the minimal normal subgroup \(G_{\mathfrak{O}_Q}(Q)\) of \(C_G(Q)\) containing the image of \(\prod_{\langle T, \eta \rangle \in \mathfrak{O}_Q} \mathfrak{S}_{\mathfrak{O}_Q, \eta}\) for any isomorphism 2.13.2 induces a functor

$$G_G : T_G \rightarrow i\mathfrak{S}_{\mathfrak{O}}$$

2.14.1;

it is actually a subfunctor of \(\mathfrak{S}_{\mathfrak{O}}(\pi^o)\) (cf. 2.5.2) and therefore determines a coherent \(\mathcal{F}\)-locality \(L^b = T_G/G_G\) (cf. 2.8) — called the basic \(\mathcal{F}\)-locality [8, Chap. 22] — which, according to [9, Corollary 4.11], does not depend on the choice of the thick \(\mathcal{F}\)-basic \(P \times P\)-set \(\Omega\). Moreover, denoting by

\(\tau^b : T_P \rightarrow L^b \quad \text{and} \quad \pi^b : L^b \rightarrow \mathcal{F}\)

2.14.2

the structural functors, it follows from [8, Proposition 22.7] that, for any subgroup \(Q\) of \(P\), isomorphisms in 2.13.2 induce a canonical isomorphism

$$(\mathfrak{S}_{\mathfrak{O}}(\pi^b))(Q) \cong \prod_{\langle T, \eta \rangle \in \mathfrak{O}_Q} \mathfrak{S}_{\mathfrak{O}_Q, \eta}(\Delta_{\eta}(T))$$

2.14.3

where \(\mathfrak{S}_{\mathfrak{O}} \rightarrow \mathfrak{A}\mathfrak{B}\) denotes the obvious functor mapping any finite group \(H\) on its maximal Abelian quotient \(H/[H, H]\); in particular, note that \(L^b\) is \(p\)-coherent (cf. 2.5).

2.15. Moreover, any \(L^b\)-morphism \(x : R \rightarrow Q\) can be lifted to an element \(\hat{x} \in G\) fulfilling \(\hat{x} \circ R \circ \hat{x}^{-1} \subset Q\) in the group of bijections of \(\Omega\); in particular, we also have

$$\hat{x}^{-1} \circ C_G(Q) \circ \hat{x} \subset C_G(R)$$

2.15.1

and, considering isomorphisms 2.13.2 for both \(C_G(Q)\) and \(C_G(R)\), it is clear that the conjugation by \(\hat{x}^{-1}\) sends the factor determined by \(T \in C_P\) and by \(\eta \in \mathfrak{O}_Q^T\) in some factors determined by \(U \in C_P\) and by \(\theta \in \mathfrak{O}_Q^T\) in such a way that, setting \(\varphi = \pi^b(x)\), there exists an injective \(R \times P\)-set homomorphism

$$f : (R \times P)/\Delta_{\theta}(U) \rightarrow \text{Res}_{\varphi \times \text{id}_P}((Q \times P)/\Delta_{\eta}(T))$$

2.15.2

or, equivalently, we have

$$\Delta_{\varphi \circ \theta}(U) = (\varphi(R) \times P) \cap (u, n)\Delta_{\eta}(T)$$

2.15.3

for suitable \(u \in Q\) and \(n \in P\).
2.16. More precisely, the $L^b$-morphism $x: R \to Q$ determines the group homomorphism
\[
(\text{Re}r(\pi^b))(x) : (\text{Re}r(\pi^b))(Q) \to (\text{Re}r(\pi^b))(R)
\]
considering isomorphisms 2.14.3 for both $(\text{Re}r(\pi^b))(Q)$ and $(\text{Re}r(\pi^b))(R)$, it makes sense to introduce the projection in $ab\left(\bar{N}_{R \times P}(\Delta_\varphi(U))\right)$ of the restriction of $(\text{Re}r(\pi^b))(x)$ to 2.17 $ab\left(\bar{N}_{Q \times P}(\Delta_\eta(T))\right)$ — noted $(\text{Re}r(\pi^b))(x)_{(U, \theta)}$ for any $(T, \eta) \in \mathcal{D}_Q$ and any $(U, \theta) \in \mathcal{D}_R$; according to 2.15 above, $(\text{Re}r(\pi^b))(x)_{(U, \theta)} \neq 0$ forces
\[
\Delta_{\varphi_00}(U) = \left(\varphi(R) \times P\right) \cap \{(u, n)\Delta_\eta(T)\}
\]
for suitable $u \in Q$ and $n \in P$.

2.17. In this case, in [8, Proposition 22.17] we describe $(\text{Re}r(\pi^b))(x)_{(U, \theta)}$ as follows. Consider the set of injective $R \times P$-set homomorphisms as in 2.15.2 above; it is clear that $\bar{N}_{R \times P}(\Delta_\varphi(U)) \times \bar{N}_{Q \times P}(\Delta_\eta(T))$ acts on this set by left- and right-hand composition and, denoting by $\mathcal{I}_{(T, \eta)}(\varphi)$ a set of representatives for the set of $\bar{N}_{R \times P}(\Delta_\varphi(U)) \times \bar{N}_{Q \times P}(\Delta_\eta(T))$-orbits, for any $f \in \mathcal{I}_{(T, \eta)}(\varphi)$ consider the stabilizer $\bar{N}_{Q \times P}(\Delta_\eta(T))_{\text{Im}(f)}$ of $\text{Im}(f)$ in $\bar{N}_{Q \times P}(\Delta_\eta(T))$, so that we get an inclusion and an obvious group homomorphism
\[
\varepsilon_f : \bar{N}_{Q \times P}(\Delta_\eta(T))_{\text{Im}(f)} \to \bar{N}_{Q \times P}(\Delta_\eta(T))
\]
\[
\delta_f : \bar{N}_{Q \times P}(\Delta_\eta(T))_{\text{Im}(f)} \to \bar{N}_{R \times P}(\Delta_\varphi(U))
\]
fulfilling $\bar{a} \cdot f = f \delta_f(\bar{a})$ for any $\bar{a} \in \bar{N}_{Q \times P}(\Delta_\eta(T))_{\text{Im}(f)}$. Then, denoting by $ab^\epsilon : \text{Re}r \to \text{Ab}$ the contravariant functor mapping any finite group $H$ on its maximal Abelian quotient $H/[H, H]$ and any injective group homomorphism on the group homomorphism induced by the transfert, it follows from [8, Proposition 22.17] that for any $(T, \eta) \in \mathcal{D}_Q$ and any $(U, \theta) \in \mathcal{D}_R$ fulfilling condition 2.16.2 for suitable $u \in Q$ and $n \in P$ we have
\[
(\text{Re}r(\pi^b))(x)_{(U, \theta)} = \sum_{f \in \mathcal{I}_{(T, \eta)}(\varphi)} ab(\delta_f) \circ ab^\epsilon(\varepsilon_f)
\]

3. A filtration for the basic $\mathcal{F}$-locality

3.1. Let $P$ be a finite $p$-group, $\mathcal{F}$ a Frobenius $P$-category and $(\pi^b, L^b, \pi^b)$ the corresponding basic $\mathcal{F}$-locality; we already know that the contravariant functor
\[
\text{Re}r(\pi^b) : L^b \to \text{Ab}
\]
factorizes throughout the extrem quotient \( \tilde{F} \) of \( F \) (cf. 2.7), so that it defines a contravariant functor

\[
\tilde{\xi}_{\mathcal{L}} := \tilde{\xi}_{\mathcal{F}}^b : \tilde{F} \rightarrow \mathbb{Ab}
\]

which, up to suitable identifications, maps any \( \tilde{F} \)-morphism \( \tilde{\phi} : R \rightarrow Q \) on the group homomorphism (cf. 2.17.2)

\[
\tilde{\xi}_{\mathcal{F}}^b(\tilde{\phi}) = \sum_{(T,n) \in \mathcal{D}_Q} \sum_{(U,\theta) \in \mathcal{D}_R} \sum_{f \in \mathcal{F}_{(T,n)}(\varphi)} ab(\delta_f) \circ ab(\varepsilon_f)
\]

from \( \bigoplus_{(T,n) \in \mathcal{D}_Q} ab\left(\tilde{N}_{Q \times P}(\Delta_\eta(T))\right) \) to \( \bigoplus_{(U,\theta) \in \mathcal{D}_R} ab\left(\tilde{N}_{R \times P}(\Delta_\theta(U))\right) \), where we set \( \mathcal{F}_{(T,n)}(\varphi) = \emptyset \) whenever condition 2.16.2 is not fulfilled for any \( u \in Q \) and any \( n \in P \).

3.2. In particular, note that the homomorphism \( \tilde{\xi}_{\mathcal{F}}^b(\tilde{\phi}) \) sends an element of \( ab\left(\tilde{N}_{Q \times P}(\Delta_\eta(T))\right) \) to a sum of terms indexed by elements \((U,\theta)\) in \( \mathcal{D}_R \) such that \( U \) is contained in a \( P \)-conjugated of \( T \); hence, for any subset \( N \) of \( C_P \) which fulfills

3.2.1 any \( U \in C_P \) which is contained in a \( P \)-conjugated of \( T \in N \) belongs to \( N \),

setting \( \mathcal{D}^N_Q = \bigsqcup_{T \in N} \mathcal{D}^T_Q \) for any subgroup \( Q \) of \( P \), it is quite clear that we get a contravariant subfunctor \( \tilde{\xi}^N_{\mathcal{F}} : F \rightarrow \mathbb{Ab} \) of \( \tilde{\xi}^b_{\mathcal{F}} \) sending \( Q \) to

\[
\bigoplus_{(T,n) \in \mathcal{D}^N_Q} ab\left(\tilde{N}_{Q \times P}(\Delta_\eta(T))\right)
\]

and we consider the corresponding quotient \( F \)-locality \( \mathcal{L}^b/\left(\tilde{\xi}^N_{\mathcal{F}} \circ \psi^b\right) \) (cf. 2.8) — denoted by \( (\tilde{\tau}^{b,N}, \mathcal{L}^{b,N}, \psi^{b,N}) \) — of the basic \( F \)-locality \( \mathcal{L}^b \) above.

3.3. It is quite clear that if \( M \) is another subset of \( C_P \) fulfilling condition 3.2.1 and containing \( N \), we have a canonical functor \( \tilde{\tau}^{M,N}_{\mathcal{F}} : \mathcal{L}^{b,N} \rightarrow \mathcal{L}^{b,M} \).

From now on, we fix a proper subset \( N \) of \( C_P \) fulfilling condition 3.2.1 and, in order to argue by induction on \( |C_P - N| \), we also fix a minimal element \( U \) in \( C_P - N \), setting \( M = N \cup \{U\} \). Hence, it makes sense to consider the quotient contravariant functor

\[
\tilde{\tau}^U_{\mathcal{F}} = \ker(\tilde{\tau}^{M,N}_{\mathcal{F}}) = \tilde{\tau}^M_{\mathcal{F}}/\tilde{\tau}^N_{\mathcal{F}} : \tilde{F} \rightarrow \mathbb{Ab}
\]

which only depends on \( U \) as we show in 3.4 and 3.5 below. More precisely, for any \( m \in \mathbb{N} \) let us consider the subfunctor \( p^m \cdot \mathcal{I} : \mathbb{Ab} \rightarrow \mathbb{Ab} \) of the identity functor \( \mathcal{I} : \mathbb{Ab} \rightarrow \mathbb{Ab} \) sending any Abelian group \( A \) to \( p^m \cdot A \), setting \( s_m = p^m \cdot \mathcal{I} / p^{m+1} \cdot \mathcal{I} \).
Then, the key point to prove the main results announced in 1.6 above is that, for any \( m \geq 0 \) and any \( n \geq 1 \), the \( n \)-th stable cohomology group — noted \( H^*_v(F, s_m \circ \tilde{\Delta}_F) \) (see [8, A3.17]) — of \( \tilde{\Delta}_F \) over \( s_m \circ \tilde{\Delta}_F \) vanish; that is to say, that the differential subcomplex in [11, A2.2], where \( B = \tilde{\Delta}_F \) and \( a = s_m \circ \tilde{\Delta}_F \), and where we only consider the elements which are stable by \( \tilde{\Delta}_F \)-isomorphisms, is exact.

3.4. This vanishing result will follow from Theorem 3.11 below and from [11, Theorem A5.5]; that is to say, with the terminology introduced in [11, 45.1], in Theorem 3.11 below we prove that, for any \( m \in \mathbb{N} \), the contravariant functor \( s_m \circ \tilde{\Delta}_F \) above admits indeed a compatible complement. From definition 3.3.1 above it is clear that, for any subgroup \( Q \) of \( P \)

\[
\tilde{\Delta}_F^U(Q) = \bigoplus_{(T, \eta) \in \Omega_Q^M - \Omega_Q^N} ab\left( \tilde{N}_{Q \times P}(\Delta_\eta(U)) \right)
\]

and then, for any \( (T, \eta) \in \Omega_Q^M - \Omega_Q^N \), we necessarily have \( T = U \); hence, we get

\[
\tilde{\Delta}_F^U(Q) = \bigoplus_{(u, n) \in \Omega_Q^U} ab\left( \tilde{N}_{Q \times P}(\Delta_\eta(U)) \right)
\]

where \( \Omega_Q^U \subset F(Q, U) \) is a set of representatives for \( Q \setminus F(Q, U)/N_P(U) \); more precisely, the group \( Q \times N_P(U) \) acts on \( F(Q, U) \) and if \( \eta, \eta' \in \mathcal{F}(Q, U) \) are in the same \( Q \times N_P(U) \)-orbit then the conjugation by a suitable element \( (u, n) \) in \( Q \times N_P(U) \) induces a group isomorphism

\[
ab\left( \tilde{N}_{Q \times P}(\Delta_\eta(U)) \right) \cong ab\left( \tilde{N}_{Q \times P}(\Delta_{\eta'}(U)) \right)
\]

which is clearly independent of the choice of \( (u, n) \) fulfilling \( \eta' = (u, n) \eta \). Consequently, from 3.4.2 we get a canonical isomorphism

\[
\tilde{\Delta}_F^U(Q) \cong \bigg( \bigoplus_{\eta \in \mathcal{F}(Q, U)} ab\left( \tilde{N}_{Q \times P}(\Delta_\eta(U)) \right) \bigg)^{Q \times N_P(U)}
\]

3.5. Moreover, for any \( \tilde{\Delta} \)-morphism \( \tilde{\varphi} : R \rightarrow Q \), from 3.1.3 above we still get

\[
\tilde{\Delta}_F^U(\tilde{\varphi}) = \sum_{(u, n) \in \Omega_Q^U} \sum_{(U, \theta) \in \Omega_R^U} \sum_{f \in \mathcal{F}(U, \theta)} ab(\delta_f) \circ ab(\varepsilon_f)
\]

in this case, it follows from 2.16 that \( \gamma(U, \theta)(\tilde{\varphi}) \) is empty unless for suitable \( u \in Q \) and \( n \in P \) we have

\[
\Delta_{\tilde{\varphi} \circ \theta}(U) = (u, n) \Delta_\eta(U)
\]
or, equivalently, $n$ belongs to $N_P(U)$ and $\varphi \circ \theta$ belongs to the classe of $\eta$ in $Q\backslash \mathcal{F}(Q,U)/N_P(U)$; in this case we have an injective $R \times P$-set homomorphism

$$f : (R \times P)/\Delta_\theta(U) \longrightarrow \text{Res}_{\varphi \times \text{id}_P} \left( (Q \times P)/\Delta_\eta(U) \right)$$

3.5.3

sending the class of $(1,1)$ to the class of $(u,n)$ . Then, denoting by

$$\varphi_\theta : \tilde{N}_{R \times P}(\Delta_\theta(U)) \longrightarrow \tilde{N}_{Q \times P}(\Delta_{\varphi \circ \theta}(U))$$

3.5.4

the group homomorphism induced by $\varphi \times \text{id}_P$, and by

$$\kappa^\eta_{(u,n)} : \tilde{N}_{Q \times P}(\Delta_\eta(U)) \cong \tilde{N}_{Q \times P}(\Delta_{\varphi \circ \theta}(U))$$

3.5.5

the conjugation by $(u,n)$ , it is quite clear that the image of $\varphi_\theta$ stabilizes the image of $f$ and therefore that $f$ is the unique element of $\mathcal{F}^?_{(U,\eta)}(\varphi)$ , that $\delta_f$ is an isomorphism in 2.17.1 and that we get [9, 8.8]

$$\text{ab}(\delta_f) \circ \text{ab}^f(\varepsilon_f) = \text{ab}^f((\kappa^\eta_{(u,n)})^{-1} \circ \varphi_\theta) = \text{ab}^f(\varphi_\theta) \circ \text{ab}(\kappa^\eta_{(u,n)})$$

3.5.6

Consequently, equality 3.5.1 becomes

$$\text{Gr}^U_{\mathcal{F}}(\hat{\varphi}) = \sum_{(U,\theta) \in \mathcal{D}_R^U} \text{ab}^f(\varphi_\theta) \circ \text{ab}(\kappa^\eta_{(u,n)})$$

3.5.7

where, for any $(U,\theta) \in \mathcal{D}_R^U$, $(U,\eta)$ belongs to $\mathcal{D}_Q^U$ and $(u,n)$ fulfills 3.5.2.

3.6. But, for our purposes, we need a better description as follows for the functor $\tilde{\text{Gr}}^U_{\mathcal{F}}$. It is quite clear that we have a functor $\mathcal{n}^U_{\mathcal{F}} : \mathcal{F} \rightarrow \Theta$ mapping any subgroup $Q$ of $P$ on the direct product of groups

$$\mathcal{n}^U_{\mathcal{F}}(Q) = \prod_{\eta \in \mathcal{F}(Q,U)} \tilde{N}_{Q \times P}(\Delta_\eta(U))$$

3.6.1

and any $\mathcal{F}$-morphism $\varphi : R \rightarrow Q$ on the direct product of group homomorphisms (cf. 3.5.4)

$$\mathcal{n}^U_{\mathcal{F}}(\varphi) = \prod_{\theta \in \mathcal{F}(R,U)} \varphi_\theta : \prod_{\theta \in \mathcal{F}(R,U)} \tilde{N}_{R \times P}(\Delta_\theta(U)) \longrightarrow \prod_{\eta \in \mathcal{F}(Q,U)} \tilde{N}_{Q \times P}(\Delta_\eta(U))$$

3.6.2;

note that, for any $u \in Q$ denoting by $\kappa_{Q,u} : Q \cong Q$ the conjugation by $u$ , the action of $(u,1) \in Q \times N_P(U)$ on $\mathcal{n}^U_{\mathcal{F}}(\kappa_{Q,u})$ coincides with $\mathcal{n}^U_{\mathcal{F}}(\kappa_{Q,u})$ . Similarly, as in 3.4.3 above, for any $n \in N_P(U)$ the action of $(1,n) \in Q \times N_P(U)$ on $\mathcal{n}^U_{\mathcal{F}}(Q)$ induces obvious isomorphisms

$$\left( (1,n) \right)^\eta : \tilde{N}_{Q \times P}(\Delta_\eta(U)) \cong \tilde{N}_{Q \times P}(\Delta_{\eta \circ \kappa_{U,n,-1}}(U))$$

3.6.3

for any $\eta \in \mathcal{F}(Q,U)$; moreover, for any $\theta \in \mathcal{F}(R,U)$ , we obviously get

$$\left( (1,n) \right)^\circ \varphi_\theta = \varphi_\theta \circ \kappa_{U,n,-1}$$

3.6.4.
3.7. Consequently, we also get the functors (cf. 2.14 and 2.17)
\[ ab^c \circ n_{\mathcal{F}}^U : \mathcal{F} \to \mathcal{A}b^o \quad \text{and} \quad ab \circ n_{\mathcal{F}}^U : \mathcal{F} \to \mathcal{A}b \]
which send any subgroup \( Q \) of \( P \) to
\[ (ab^c \circ n_{\mathcal{F}}^U)(Q) = \bigoplus_{\eta \in \mathcal{F}(Q,U)} ab\left(\hat{N}_{Q \times P}(\Delta_{\eta}(U))\right) = (ab \circ n_{\mathcal{F}}^U)(Q) \]
and we know that \( Q \times N_P(U) \) acts on this Abelian \( p \)-group; then, it is quite easy to check that we have a subfunctor of \( ab^c \circ n_{\mathcal{F}}^U \) and a quotient functor of \( ab \circ n_{\mathcal{F}}^U \), respectively denoted by
\[ h^o(ab^c \circ n_{\mathcal{F}}^U) : \mathcal{F} \to \mathcal{A}b^o \quad \text{and} \quad h_o(ab \circ n_{\mathcal{F}}^U) : \mathcal{F} \to \mathcal{A}b \]
3.7.3. sending any subgroup \( Q \) of \( P \) to the subgroup \( (ab^c \circ n_{\mathcal{F}}^U)(Q)Q \times N_P(U) \) of fixed elements and to the quotient \( (ab \circ n_{\mathcal{F}}^U)(Q)Q \times N_P(U) \) of co-fixed elements of \( \bigoplus_{\eta \in \mathcal{F}(Q,U)} ab\left(\hat{N}_{Q \times P}(\Delta_{\eta}(U))\right) \); actually, it is easily checked that both factorize through the exterior quotient \( \hat{\mathcal{F}} \) (cf. 3.1) yielding respective functors
\[ \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U) : \hat{\mathcal{F}} \to \mathcal{A}b^o \quad \text{and} \quad \tilde{h}_o(ab \circ n_{\mathcal{F}}^U) : \hat{\mathcal{F}} \to \mathcal{A}b \]
3.7.4. In particular, it follows from 3.4.4 that for any subgroup \( Q \) of \( P \) we have
\[ \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(Q) \cong \tilde{\mathcal{F}}_x^U(Q) \]
3.7.5.
3.8. Actually, we claim that for any \( \hat{\mathcal{F}} \)-morphism \( \hat{\varphi} : R \to Q \) we also have the commutative diagram
\[ \begin{array}{ccc}
\tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(Q) & \cong & \tilde{\mathcal{F}}_x^U(Q) \\
\tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(\hat{\varphi}) & \downarrow & \tilde{\mathcal{F}}_x^U(\hat{\varphi}) \\
\tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(R) & \cong & \tilde{\mathcal{F}}_x^U(R)
\end{array} \]
so that the functors \( \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U) \) and \( \tilde{\mathcal{F}}_x^U \) from \( \hat{\mathcal{F}} \) to \( \mathcal{A}b^o \) are isomorphic. Indeed, \( \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(\hat{\varphi}) \) sends any element
\[ a = \sum_{\eta \in \mathcal{F}(Q,U)} a_\eta \in \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(Q) \]
3.8.2, where \( a_\eta \) belongs to \( ab\left(\hat{N}_{Q \times P}(\Delta_{\eta}(U))\right) \) for any \( \eta \in \mathcal{F}(Q,U) \), to the element
\[ \sum_{\theta \in \mathcal{F}(R,U)} ab^c(\varphi_\theta)(a_{\varphi_\theta}) \in \tilde{h}^o(ab^c \circ n_{\mathcal{F}}^U)(R) \]
3.8.3, then, the commutativity of diagram 3.8.1 follows from equality 3.5.7.
3.9. Moreover, for any subgroup $Q$ of $P$ we clearly have a canonical group isomorphism

$$\tau_Q : (ab \circ n^U_P)(Q \times N_P(U)) \cong (ab^c \circ n^U_P)(Q \times N_P(U))$$

which, for any $\eta \in F(Q, U)$, maps the class in $(ab \circ n^U_P)(Q \times N_P(U))$ of an element $a_\eta$ of $\text{ab} \left( \bar{N}_Q \times P(\Delta_\eta(U)) \right)$ on the element (cf. 3.6)

$$\tau_{Q \times N_P(U)}(\eta)(a_\eta) = \sum_{(u, n)} (u, n)^\eta a_\eta$$

in $(ab^c \circ n^U_P)(Q \times N_P(U))$ (cf. 3.7.2), where $(u, n)$ runs over a set of representatives for the quotient set $(Q \times N_P(U))/N_Q \times P(\Delta_\eta(U))$.

3.10. Explicitly, an element $(u, n)$ of $Q \times P$ belongs to $N_Q \times P(\Delta_\eta(U))$ if and only if we have $^uU = U$ and $\eta(\nu v) = \eta(v)$ for any $v \in U$; in particular, $u$ normalizes $\eta(U)$ and, denoting by $\eta_* : U \cong \eta(U)$ the isomorphism induced by $\eta$, this element belongs to the converse image $Q_\eta$ of

$$F_Q(\eta(U)) \cap (\eta_* \circ F_P(U) \circ \eta_*^{-1})$$

in $N_Q(\eta(U))$; then, the conjugation by $\eta_*^{-1}$ induces a group homomorphism $\nu_\eta : Q_\eta \to F_P(U)$; thus, setting

$$\Delta^\nu_\eta(\eta) = \{(u, \nu_\eta(u)) \mid u \in Q_\eta \subset Q \times F_P(U) \}$$

we get the exact sequence

$$1 \to \{1\} \times C_P(U) \to N_Q \times P(\Delta_\eta(U)) \to \Delta^\nu_\eta(\eta) \to 1$$

and, in particular, denoting by $a_\eta$ the classe of $a_\eta \in \text{ab} \left( \bar{N}_Q \times P(\Delta_\eta(U)) \right)$ in $(ab \circ n^U_P)(Q \times N_P(U))$ and by $Q^\eta \subset Q$ a set of representatives for $Q/Q_\eta$, since $\{1\} \times C_P(U)$ acts trivially on $\text{ab} \left( \bar{N}_Q \times P(\Delta_\eta(U)) \right)$, we still get

$$\tau_Q(a_\eta) = \sum_{\nu \in F_P(U)} \sum_{u \in Q^\eta} (u, \nu) \cdot a_\eta$$

Finally, for any $m \in \mathbb{N}$, for short we set

$$\gamma_{F,m}^U = s_m \circ h_\circ(ab \circ n^U_P)$$

and

$$\gamma_{F,0}^U := s_m \circ h_\circ(ab \circ n^U_P)$$

moreover, for any subgroup $Q$ of $P$, it is clear that $\tau_Q$ induces a group isomorphism

$$\tau_Q : \gamma_{F,0}^U(Q) \cong \gamma_{F,m}^U(Q)$$
Theorem 3.11.† With the notation above, the functor \( \mathcal{F}_{\mathcal{F},m}^{U}: \mathcal{F} \to \mathbb{A}^\circ \) admits a compatible complement \((\mathcal{F}_{\mathcal{F},m}^{U,\circ})': \mathcal{F} \to \mathbb{A}^\circ \) sending any \( \mathcal{F} \)-morphism \( \varphi: R \to Q \) to the group homomorphism

\[
(\mathcal{F}_{\mathcal{F},m}^{U,\circ})'(\varphi) = \text{tr}^m_Q \circ \mathcal{F}_{\mathcal{F},m}^{U,\circ} (\varphi) \circ (\text{tr}^m_R)^{-1} \tag{3.11.1}
\]

**Proof:** It is clear that equalities 3.11.1 define a functor \( \mathcal{F} \to \mathbb{A}^\circ \) sending any subgroup \( Q \) of \( P \) to \( s_m \left( (ab^c \circ n^U_{x})(Q)^{Q \times N_P(U)} \right) \); thus, it remains to prove that \((\mathcal{F}_{\mathcal{F},m}^{U,\circ})'\) fulfills the conditions A5.1.2 and A5.1.3 in [11]. With the notation in 3.7 above, assuming that \( a = \sum_{\eta \in \mathcal{F}(Q,U)} a_\eta \) belongs to \( p^m \cdot h^\circ(ab^c \circ n^U_x)(Q) \) and denoting by \( \bar{a}^m \) its image in \( s_m \left( (ab^c \circ n^U_x)(Q)^{Q \times N_P(U)} \right) \), for condition A5.1.3 we have to compute \((\mathcal{F}_{\mathcal{F},m}^{U,\circ})'(\varphi) \circ \mathcal{F}_{\mathcal{F},m}^{U,\circ}(\varphi)(\bar{a}^m)\) in \( \mathcal{F}_{\mathcal{F},m}^{U,\circ}(Q) \); clearly, this element is the image in \( s_m \left( (ab^c \circ n^U_x)(Q)^{Q \times N_P(U)} \right) \) of

\[
\left( \text{tr}^m_Q \circ \hat{h}_x(ab^c \circ n^U_x)(\varphi) \circ (\text{tr}^m_R)^{-1} \right) h^\circ(ab^c \circ n^U_x)(\varphi)(a) = \left( \text{tr}^m_Q \circ \hat{h}_x(ab^c \circ n^U_x)(\varphi) \circ (\text{tr}^m_R)^{-1} \right) \left( \sum_{\theta \in \mathcal{F}(R,U)} ab^\circ(\varphi_\theta)(a_{\varphi_\theta}) \right) \tag{3.11.2}
\]

which is equal to zero whenever \( \mathcal{F}(R,U) \) is empty.

Otherwise, for any element \((y, n)\) in \( R \times N_P(U)\), \((\varphi(y), n)\) belongs to \( Q \times N_P(U) \) and therefore, with the obvious action of \( R \times N_P(U) \) on \( \mathcal{F}(R,U) \), we have \( a_{\varphi_\theta(y, \cdot, n)}(y, a_{\varphi_\theta}) \); consequently, this element coincides with (cf. 2.13 and 3.9)

\[
\sum_{\theta \in \mathcal{F}(R)} \text{tr}^Q_{N_{Q \times R}(\Delta \varphi_\theta(U))} \left( (ab(\varphi_\theta) \circ ab^\circ(\varphi_\theta))(a_{\varphi_\theta(U)}) \right) \tag{3.11.3}
\]

and we know that for any \( \theta \in \mathcal{D}_{R}^{U} \) we have

\[
(ab(\varphi_\theta) \circ ab^\circ(\varphi_\theta))(a_{\varphi_\theta}) = \left| \frac{\hat{N}_{Q \times P}(\Delta \varphi_\theta(U))}{N_{R \times P}(\Delta \varphi_\theta(U))} \right| a_{\varphi_\theta} \tag{3.11.4}
\]

thus, either \( |\hat{N}_{Q \times P}(\Delta \varphi_\theta(U))| \neq |N_{R \times P}(\Delta \varphi_\theta(U))| \) and the term

\[
\text{tr}^Q_{N_{Q \times R}(\Delta \varphi_\theta(U))} \left( (ab(\varphi_\theta) \circ ab^\circ(\varphi_\theta))(a_{\varphi_\theta}) \right) \tag{3.11.5}
\]

belongs to \( p^{m+1} \cdot h^\circ(ab^c \circ n^U_x)(Q) \), or we have \( \varphi(R_\theta) = Q_{\varphi_\theta} \) (cf. 3.10.2).

† In [9, Proposition 8.9] the statement and the proof are far from correction.
But, for any \( \eta \in \mathcal{F}(Q, U) \) and any element \((u, n)\) in \( Q \times N_P(U) \) we still have \( a_{u, \eta^{-1}} = (u, n) \cdot a_{\eta} \); consequently, for any \( \eta \) in the set \( \varphi \circ \mathcal{F}(R, U) \), setting
\[
\mathcal{T}_{Q \times N_P(U)}(\eta, \varphi \circ \theta) = \{(u, n) \in Q \times N_P(U) \mid \eta = (u, n) \cdot (\varphi \circ \theta)\}
\]
3.11.6,
in the second case we have
\[
\text{tr}_{Q \times N_P(U)}^{N_{Q \times F}(\Delta_{\varphi \circ \theta}(U))} \left( (a \cdot b)(\varphi \circ \theta) \right) = \sum_{\eta \in \varphi \circ \mathcal{F}(R, U)} \frac{|\mathcal{T}_{Q \times N_P(U)}(\eta, \varphi \circ \theta)|}{|N_{Q \times P}(\Delta_{\eta}(U))|} \cdot a_{\eta}
\]
3.11.7.
Moreover, for any element \( u \) in the transporter \( \mathcal{T}_Q(\varphi(R), \eta(U)) \) (cf. 2.2), the following diagram
\[
\begin{array}{ccc}
R & \cong & u^{-1}(R) \\
\varphi' & \uparrow & \eta(U) \\
U & \cong & \varphi(R)
\end{array}
\]
3.11.8
determines a pair formed by \( \varphi' \in \tilde{\varphi} \) and by \( \theta' \) in the \( \{1\} \times N_P(U) \)-orbit of \( \theta \) such that \( \eta = \varphi' \circ \theta' \) and therefore it is quite clear that
\[
\mathcal{T}_{Q \times N_P(U)}(\eta, \varphi \circ \theta) = \mathcal{T}_Q(\varphi(R), \eta(U)) \times N_P(U)
\]
3.11.9.
Finally, note that the map
\[
\varphi(R) \setminus \mathcal{T}_Q(\varphi(R), \eta(U)) \longrightarrow \varphi(R) \setminus \eta(U)
\]
3.11.10
sending the class of \( u \in \mathcal{T}_Q(\varphi(R), \eta(U)) \) to \( \varphi(R)u\eta(U) \) is injective and its image is the set of double classes of cardinal equal to \( |\varphi(R)| \), so that \( p \) divides \( |\varphi(R) \setminus \mathcal{T}_{Q}(\varphi(R), \eta(U))| \); in conclusion, \( p \) also divides the quotient \( |\mathcal{T}_{Q \times N_P(U)}(\eta, \varphi \circ \theta)|/|N_{Q \times P}(\Delta_{\eta}(U))| \). Consequently, in both cases we obtain
\[
\left( (\mathbb{1}^U_{\mathcal{F}, m}(\tilde{\varphi}')) \circ \mathbb{1}^U_{\mathcal{F}, m}(\varphi) \right)(\mathbb{1}^U_{m}) = 0
\]
3.11.11.

In order to show condition A5.1.3 in [11], for any pair of \( \tilde{\mathcal{F}} \)-morphisms \( \tilde{\varphi} : R \rightarrow Q \) and \( \tilde{\psi} : T \rightarrow Q \) we have to prove that
\[
\mathbb{1}^U_{\mathcal{F}, m}(\tilde{\psi}) \circ (\mathbb{1}^U_{\mathcal{F}, m}(\tilde{\varphi})) = \sum_{w \in W} (\mathbb{1}^U_{\mathcal{F}, m}(\tilde{\varphi}))(\mathbb{1}^U_{m}(\varphi_w))
\]
3.11.12
where, choosing a pair of representatives \( \varphi \) of \( \tilde{\varphi} \) and \( \psi \) of \( \tilde{\psi} \), and a set of representatives \( W \subset Q \) for the set of double classes \( \varphi(R) \setminus Q \setminus \psi(T) \), for any \( w \in W \) we set \( S_w = \varphi(R)^w \cap \psi(T) \) and denote by
\[
\varphi_w : S_w \longrightarrow R \quad \text{and} \quad \psi_w : S_w \longrightarrow T
\]
3.11.13
the \( \mathcal{F} \)-morphisms fulfilling \( \varphi(\varphi_w(u)) = wuw^{-1} \) and \( \psi(\psi_w(u)) = u \) for any element \( u \in S_w \).
For any \( \theta \in \mathcal{F}(R, U) \), let \( b_\theta \) be an element of \( p^m \cdot ab \left( \bar{N}_{R \times P}(\Delta_\theta(U)) \right) \) and denote by \( \bar{b}_\theta \) the image of \( b_\theta \) in \( \left( ab \circ n_{F}^U \right)(R)_{R \times N_F(U)} \) (cf. 3.7.2); thus, \( \bar{r}_{m}^{m}(\bar{b}_\theta) \) is an element of \( \bar{v}_{m}^{m}(R) \) (cf. 3.10.6) and we have to compute (cf. 3.11.1)

\[
(\bar{r}_{X,m}^{U,\circ}(\bar{\psi}) \circ (\bar{r}_{X,m}^{U,\circ})^t(\bar{\varphi})) \left( \bar{r}_{m}^{m}(\bar{b}_\theta) \right) 
= (\bar{r}_{m}^{m}(\bar{\psi}) \circ \bar{r}_{m}^{m}(\bar{\varphi})(\bar{b}_\theta)) 
\]

3.11.14;

this element is clearly the image in \( \bar{r}_{X,m}^{U,\circ}(T) \) of the element (cf. 3.9.2)

\[
a = \tilde{b}_\theta^t(ab^t \circ n_{F}^U)(\bar{\psi}) \left( \operatorname{tr} \left( \bar{h}_\theta(ab \circ n_{F}^U)(\bar{\varphi})(b_\theta) \right) \right) 
= (ab^t \circ n_{F}^U)(\bar{\psi}) \left( \operatorname{tr} \left( N_{Q \times P}(\Delta_{\eta}(U)) (ab(\varphi_\theta)(b_\theta)) \right) \right) 
\]

3.11.15

where \( \bar{h}_\theta(ab \circ n_{F}^U)(\bar{\varphi})(b_\theta) \) denotes the image of \( \bar{h}_\theta(ab \circ n_{F}^U)(\bar{\varphi})(b_\theta) \) in the quotient \( (ab \circ n_{F}^U)(Q)(Q_{Q \times N_F(U)}) \) and we set \( \eta = \varphi \circ \theta \).

Then, as in 3.10 above, denoting by \( Q_\eta \) the converse image of the intersection \( \mathcal{F}_Q(\eta(U)) \cap (\eta_* \circ \mathcal{F}_P(U) \circ \eta_*^{-1}) \) in \( N_Q(\eta(U)) \) and by \( Q^\eta \subset Q \) a set of representatives for \( Q/Q_\eta \), we have (cf. 3.10.4)

\[
\operatorname{tr} \left( N_{Q \times P}(\Delta_{\eta}(U)) (ab(\varphi_\theta)(b_\theta)) \right) = \sum_{u \in Q^\eta} \sum_{\nu \in F_P(U)} (u, \nu) \cdot ab(\varphi_\theta)(b_\theta) 
\]

3.11.16;

but, for any \( u \in Q^\eta \) and any \( \nu \in F_P(U) \), the element \( (u, \nu) \cdot ab(\varphi_\theta)(b_\theta) \) belongs to \( p^m \cdot ab \left( \bar{N}_{Q \times P}(\Delta_{u \cdot n^\nu_{Q \times P}}(U)) \right) \) and therefore it follows from definition 3.6.2 that the element (cf. 3.6)

\[
a_{u, \nu} = (ab^t \circ n_{F}^U)(\bar{\psi})(u, \nu) \cdot ab(\varphi_\theta)(b_\theta) 
= (1, \nu) \cdot ab^t(\circ n_{F}^U)(\kappa_{Q \cdot u^{-1}} \circ \psi)(ab(\varphi_\theta)(b_\theta)) 
\]

3.11.17

is equal to zero unless \( (\kappa_{Q \cdot u^{-1}} \circ \psi)(T) \) contains \( \eta(U) \), so that there is a unique \( \zeta_u \in \mathcal{F}(T, U) \) fulfilling \( \kappa_{Q, \eta \circ \psi} \circ \eta = \psi \circ \zeta_u \); in this case, setting \( \psi^u = \kappa_{Q, u^{-1}} \circ \psi \) we get

\[
a_{u, \nu} = (1, \nu) \cdot ab^t \left( (\psi^u)_{\zeta_u} \right) \left( ab(\varphi_\theta)(b_\theta) \right) 
\]

3.11.18;

let us denote by \( \hat{Q}^\eta \subset Q^\eta \) the subset of \( u \in Q^\eta \) fulfilling this condition.

In this situation, note that we have the two injective group homomorphisms

\[
\varphi_\theta \cdot \bar{N}_{Q \times P}(\Delta_{\eta}(U)) 
\]

3.11.19;
thus, since $ab^r$ (cf. 2.17) is a Mackey complement of $ab$, for any $u \in Q^u$ we need to consider the set of double classes

$$X_u = N_{\varphi(R) \times P}(\Delta_\eta(U)) \setminus N_{Q^u \times P}(\Delta_\eta(U)) / N_{\psi^u(T) \times P}(\Delta_\eta(U))$$  

which, according to the exact sequence 3.10.3, admits an obvious canonical bijection with the set of double classes $\{\varphi(R) \cap Q_\eta \} \setminus \{\psi^u(T) \cap Q_\eta\}$; hence, choosing a set $X_u \subset Q_\eta$ of representatives for this last set of double classes, we get

$$ab^r((\psi_u)_u) \circ ab(\varphi_\eta) = \sum_{x \in X_u} ab(\psi_{\eta,u,x}) \circ ab^r(\varphi_{\theta,u,x})$$  

where for any $x \in X_u$ we set $S_{u,x} = \varphi(R)^x \cap \psi(T)^u$ and denote by

$$\varphi_{\theta,u,x} : \tilde{N}_{S_{u,x} \times P}(\Delta_\eta(U)) \rightarrow \tilde{N}_{R \times P}(\Delta_\eta(U))$$

$$\psi_{\eta,u,x} : \tilde{N}_{S_{u,x} \times P}(\Delta_\eta(U)) \rightarrow \tilde{N}_{T \times P}(\Delta_\eta(U))$$

the $F$-morphisms fulfilling (cf. 3.10.2)

$$\varphi_\eta(\varphi_{\theta,u,x}(s,n)) = (s,\tilde{x},n) = (x,\tilde{x}) \in S_{u,x}$$

$$\psi_{\eta,u,x}(\psi_{\eta,u,x}(s,n)) = (s,n)$$

for any element $(s,n) \in N_{S_{u,x} \times P}(\Delta_\eta(U))$ and for a choice of $\tilde{x} \in Q_\eta$ lifting $\nu_\eta(x)$ (cf. 3.10.2); note that the element $(x,\tilde{x}) \in Q_\eta \times P$ normalizes $\Delta_\eta(U)$. Hence, from 3.11.15, 3.11.18, 3.11.19 and 3.11.21 we obtain

$$a = \sum_{\nu \in \mathcal{F}_R(U)} (1,\nu) \sum_{u \in Q^u} \sum_{x \in X_u} (ab(\psi_{\eta,u,x}) \circ ab^r(\varphi_{\theta,u,x}))(b_\theta)$$

On the other hand, we have to prove that the element (cf. 3.11.12)

$$\bar{c} = \left( \sum_{w \in W} (\tilde{r}_{U,m}^U)_{\varphi}(\tilde{\psi}_w) \circ \tilde{r}_{F,m}^U(\tilde{\psi}_w) \right)(tr_{R}^m(b_\theta))$$

is also the image of $a$. But, according to 3.9, $tr_{R}^m(b_\theta)$ is the image in $\tilde{r}_{F,m}^U(R)$ of

$$tr_{N_R \times R}(U)^{(\Delta_\eta(U))}(b_\theta) = \sum_{y \in R^u} \sum_{\nu \in \mathcal{F}_R(U)} (y,\nu) \cdot b_\theta$$

in $p^m(\varphi \circ n_{U}^r)(R)^{R \times N_R(U)}$ (cf. 3.10.4) where, denoting by $\theta_\eta : U \cong \theta(U)$ the isomorphism induced by $\theta$ and by $R_\theta$ the converse image of the intersection $\mathcal{F}_R(\theta(U)) \cap (\theta_\eta \circ \mathcal{F}_R(U) \cap \theta_\eta^{-1})$ in $N_R(\theta(U))$, $R_\theta \subset R$ is a set of representatives for $R/R_\theta$. Note that, according to 3.10 we have $R_\theta = \varphi^{-1}(Q_\eta)$.
In particular, for any \( w \in W \), the element \( t_{F,m}^{U,R}(\tilde{\varphi}_w)(\text{tr}_R^m(\tilde{b}_\theta)) \) is the image in \( t_{F,m}^{U,R}(S_w) \) of the element (cf. 3.10.5)

\[
d_w = \sum_{y \in R^\theta} \sum_{\nu \in \mathcal{F}_P(U)} (ab^\nu \circ n^U_y)(\varphi_w)((y, \nu) \cdot b_\theta)
\]

3.11.27;

but, as above, for any \( y \in R^\theta \) and any \( \nu \in \mathcal{F}_P(U) \), the element \((y, \nu) \cdot b_\theta\) belongs to \( p^m \cdot ab\left( \Delta_{y,0} \cdot (U) \right) \) and therefore it follows from definition 3.6.2 that the element

\[
d_{w, y, \nu} = (ab^\nu \circ n^U_y)(\varphi_w)((y, \nu) \cdot b_\theta)
\]

3.11.28

is equal to zero unless \( \varphi_w(S_w) \) contains \( y \theta(U) \), so that there is a unique \( \xi_{w,y} \in \mathcal{F}(S_w, U) \) fulfilling \( \kappa_{R,y} \circ \theta = \varphi_w \circ \xi_{w,y} \), which forces the equality \( \kappa_{Q,w^{-1}}(\varphi)(y) \circ \eta = \underleftarrow{Q}_{w,y} \circ \xi_{w,y} \); in this case, we have

\[
d_{w, y, \nu} = (1, \nu) \cdot ab^\nu((\kappa_{R,y}^{-1} \circ \varphi_w)_{\xi_{w,y}})(b_\theta)
\]

3.11.29;

let us denote by \( \tilde{R}_w^\theta \subset R^\theta \) the subset of \( y \in R^\theta \) fulfilling this condition; thus, we get

\[
d_w = \sum_{\nu \in \mathcal{F}_P(U)} (1, \nu) \cdot \sum_{y \in \tilde{R}_w^\theta} ab^\nu((\kappa_{R,y}^{-1} \circ \varphi_w)_{\xi_{w,y}})(b_\theta)
\]

3.11.30.

Moreover, for any \( y \in R^\theta \) fulfilling the above condition and any \( s \in S_w \), the product \( \varphi_w(s) \cdot y \) still fulfills this condition and we have

\[
\varphi_w \circ \xi_{w, \varphi_w(s) \cdot y} = \kappa_{R, \varphi_w(s) \cdot y} \circ \theta = \kappa_{R, \varphi_w(s)} \circ \kappa_{R,y} \circ \theta
\]

\[
= \kappa_{R, \varphi_w(s)} \circ \varphi_w \circ \xi_{w,y} = \varphi_w \circ \kappa_{S_w, s} \circ \xi_{w,y}
\]

3.11.31,

so that we get \( \xi_{w, \varphi_w(s) \cdot y} = \kappa_{S_{w,s}} \circ \xi_{w,y} \); in particular, \( \varphi_w(S_w) \) has an action on \( \tilde{R}_w^\theta \) and, choosing a set of representatives \( \tilde{\gamma}_w^\theta \subset \tilde{R}_w^\theta \) for the set of \( \varphi_w(S_w) \)-orbits, the element \( d_w \) above is also equal to

\[
\sum_{\nu \in \mathcal{F}_P(U)} (1, \nu) \cdot \sum_{y \in \tilde{\gamma}_w^\theta} \sum_{s \in S_w^{\theta,y}} ab^\nu((\kappa_{R,v}^{-1} \circ \varphi_w)_{\xi_{w,v, \varphi_w(s) \cdot y}})(b_\theta)
\]

3.11.32

where for any \( y \in \tilde{\gamma}_w^\theta \), setting \( S_{w, \theta,y} = \varphi_w^{-1}(R_\theta)^y \), \( S_{w,y}^\theta \subset S_w \) is a set of representatives for \( S_w/S_{w, \theta,y} \); but, it is quite clear that

\[
ab^\nu((\kappa_{R,v}^{-1} \circ \varphi_w)_{\xi_{w,v, \varphi_w(s) \cdot y}})
\]

\[
= ab^\nu((\kappa_{R,v}^{-1} \circ \varphi_w \circ \kappa_{S_{w,s}^{-1}})_{\xi_{w,v, \varphi_w(s) \cdot y}})
\]

\[
= ab((\kappa_{S_{w,s}})_{\xi_{w,y}}) \circ ab^\nu((\kappa_{R,v}^{-1} \circ \varphi_w)_{\xi_{w,y}})
\]

3.11.33;
hence, setting \( \varphi_w^y = \kappa_{R,y}^{-1} \circ \varphi_w \) and denoting by \( \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}})(b_0) \) the image of \( \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}})(b_0) \) in the quotient \( \mathbf{(ab \circ n_y^U)}(S_w)_{S_w \times N_T(U)} \); according to 3.10.4 we easily obtain

\[
\sum_{\nu \in \mathcal{F}_P(U)} (1, \nu) \sum_{n \in S_w^{\theta,w}} \left( \mathbf{ab}^\ell((\kappa_{R,(\varphi_{w}^y(z,y))}\circ \varphi_w)_{\xi_{w,y},\varphi_{w}(z,y)})(b_0) \right) = \text{tr}^m_{S_w} \left( \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}})(b_0) \right)
\]

Consequently, it follows from definition 3.11.1 that we have (cf. 3.11.25)

\[
\tilde{c} = \sum_{w \in W} \left( \text{tr}^m_{\mathcal{F}_P(U)}(\tilde{\psi}_w) \right) \left( \sum_{y \in \tilde{Y}^{\theta,w}} \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}})(b_0) \right)
\]

this element is clearly the image in \( \tilde{\mathcal{F}}^m_{\mathcal{C}}(T) \) of the element

\[
\sum_{w \in W} \sum_{y \in \tilde{Y}^{\theta,w}} \text{tr}^m_{N_T \times P\left(\Delta_{\xi_{w,y}}(U)\right)} \left( \mathbf{ab}\left((\tilde{\psi}_w)_{\xi_{w,y}}\right) \circ \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}}) \right)(b_0)
\]

\[
= \sum_{w \in W} \sum_{y \in \tilde{Y}^{\theta,w}} \sum_{z \in \tilde{Z}^{w,y}} \left( \mathbf{ab}\left((\tilde{\psi}_w)_{\xi_{w,y}}\right) \circ \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}}) \right)(b_0)
\]

in \( p^m(\mathbf{ab} \circ n_U^U(T))^{T \times N_T(U)} \) where, for any \( w \in W \) and any \( y \in \tilde{Y}^{\theta,w} \), setting \( \xi_{w,y} = \psi_w \circ \xi_{w,y} \) and denoting by \( (\xi_{w,y}):U \cong \xi_{w,y}(U) \) the isomorphism induced by \( \xi_{w,y} \) and by \( T_{w,y} \) the converse image of the intersection \( \mathcal{F}_T(\xi_{w,y}(U)) \cap (\xi_{w,y})_{*} \circ \mathcal{F}_P(U) \circ (\xi_{w,y})_{*}^{-1} \) in \( N_T(\xi_{w,y}(U)) \), we choose as above a set of representatives \( Z^{w,y} \) for \( T/T_{w,y} \) and, for any \( z \in Z^{w,y} \), we set \( \tilde{\psi}_w = \kappa_{T,z} \circ \psi_w \).

Finally, we claim that this element \( \tilde{c} \) coincides with \( a \) in 3.11.24 above; that is to say, considering the sets

\[
X = \bigcup_{u \in \tilde{Q}^\theta} \{u\} \times X_u \quad \text{and} \quad Z = \bigcup_{w \in W} \{w\} \times \bigcup_{y \in \tilde{Y}^{\theta,w}} \{y\} \times \tilde{Z}^{w,y}
\]

in \( p^m(\mathbf{ab} \circ n_U^U(T))^{T \times N_T(U)} \) we have to prove the equality

\[
\sum_{(u,z) \in X} \sum_{\nu \in \mathcal{F}_P(U)} (1, \nu) \left( \mathbf{ab}(\psi_{\theta,u,z}) \circ \mathbf{ab}^\ell((\varphi_{\theta,u,z})_{\xi_{\theta,u,z}}) \right)(b_0)
\]

\[
= \sum_{(w,y,z) \in Z} \sum_{\nu \in \mathcal{F}_P(U)} (1, \nu) \left( \mathbf{ab}(\tilde{\psi}_{w,y}) \circ \mathbf{ab}^\ell((\varphi_{w}^y)_{\xi_{w,y}}) \right)(b_0)
\]
actually, we will define a bijection between $X$ and $Z$ such that the corresponding terms in both sums coincide with each other.

Indeed, for any $w \in W$, any $y \in Y^\theta, w$ and any $z \in Z^w, y$ let us consider the element $\varphi(y)^{-1}w\psi(z)^{-1}$ of $Q$; this element certainly belongs to $Q_{\eta}u^{-1}$ for some $u$ in $Q^\eta$ so that we have $\varphi(y)^{-1}w\psi(z)^{-1} = vu^{-1}$ for some $v$ in $Q_{\eta}$; but, since $y$ belongs to $Y^w$, $\varphi_w(S_w)$ contains $\psi(U)$ and therefore $wS_w$ contains $\varphi(y)\eta(U)$ (cf. 3.11.13); thus, $\psi(w^{-1}\varphi(y)\eta(U)$ is contained in $\psi(T)$ and, since $Q_{\eta} \subset N_Q(\eta(U))$ (cf. 3.9), $\psi(U) = \psi(z)^{-1}\varphi(y)\psi(v)\eta(U)$ is also contained in $\psi(T)$, so that $u$ belongs to $\bar{Q}^\eta$ (cf. 3.11.18). Moreover, the double class of $v$ in $(\varphi(R) \cap Q_{\eta}) \setminus Q_{\eta}/(\psi^U(T) \cap Q_{\eta})$ determines an element $x$ in $X_u$ such that we have $v = \varphi(r)xu^{-1}\psi(t)u$ for some $r \in R_{\theta}$ and some $t \in T$ fulfilling $\psi(t)^{\eta} \in Q_{\eta}$, so that we get

$$\varphi(y)^{-1}w\psi(z)^{-1} = \varphi(r)xu^{-1}\psi(t)$$ 3.11.39.

Thus, we obtain a map from $Z$ to $X$ sending $(w, y, z)$ to $(u, x)$.

Moreover, with the same notation, setting

$$q = \psi(tz)^{-1}u = w^{-1}\varphi(y)x$$ 3.11.40,

it is clear that the automorphism $\kappa_{Q_{\eta}}$ of $Q$ (cf. 3.6) maps $S_{u, x}$ onto $S_w$ inducing a group isomorphism $\chi : S_{u, x} \cong S_w$; hence, since we have (cf. 3.10.3)

$$\kappa_{Q, q} \circ \eta = \kappa_{Q, w^{-1}\varphi(y)} \circ \eta \circ \nu_{\theta}(\varphi(r)x) = i_{S_w}^Q \circ \xi_{w, y} \circ \nu_{\theta}(\varphi(r)x)$$ 3.11.41,

we get the group isomorphism (cf. 3.5.4)

$$\chi_{\eta} : \bar{N}_{S_{u, x}} \times p(\Delta_U(\eta)) \cong \bar{N}_{S_w} \times p(\Delta_{\xi_{w, y}, \nu_{\theta}(\varphi(r)x)}(U))$$ 3.11.42.

Then, we claim that

$$\begin{align*}
(1, \nu_{\theta}(\varphi(r)x)^{-1})ab^\xi(x_{\theta, u, x}) &= ab^\xi(\chi_{\eta}) \circ ab^\xi((\varphi(y)_{w, y})^\xi_{w, y}) \\
(1, \nu_{\theta}(\varphi(r)x)^{\psi^U(t)})ab^\xi(x_{\theta, u, x}) &= ab^\xi(\psi_w)^\xi_{w, y} \circ ab^\xi(\chi_{\eta})
\end{align*}$$ 3.11.43.

Indeed, for any $(s, n) \in N_{S_{u, x}} \times p(\Delta_U(\eta))$ it is easily checked that we have (cf. 3.11.23)

$$\begin{align*}
(\varphi_{\theta, u, x}(\varphi(r)x) \circ (\varphi(y)_{w, y} \circ \nu_{\theta}(\varphi(r)x)) \circ \chi_{\eta})(s, n) &= (\varphi_{\theta, u, x}(\varphi(r)x) \circ (\varphi(y)_{w, y} \circ \nu_{\theta}(\varphi(r)x))(s, n) \\
&= (\varphi(r)x, \psi_w)(s, n) = (\varphi(r)x, \psi_w)(s, n) \\
&= (\varphi(r), \nu_{\theta}(x)^{-1}) \cdot (\varphi(y) \circ \varphi_{\theta, u, x})(s, n) \\
&= (\varphi_{\theta, u, x}(\varphi(r)x) \circ \varphi_{\theta, u, x})(s, n)
\end{align*}$$ 3.11.44

since it follows from 3.10.3 that $\kappa_{R, r} \circ \theta = \theta \circ \nu_{\theta}(r) = \theta \circ \nu_{\theta}(\varphi(r))$; thus, since the homomorphism $\varphi_{\theta, u, x}(\varphi(r)x)$ is injective, we get

$$(\varphi(y)_{w, y} \circ \nu_{\theta}(\varphi(r)x)) \circ \chi_{\eta} = \varphi_{\theta, u, x}$$ 3.11.45
and, according to 3.6.4 above, we still get

$$(\varphi_w^u)_{\xi_{w,y}} \circ \chi_\eta = (1, \nu_\eta(\varphi(r)x)) \cdot \varphi_{\theta, u,x}$$ \hspace{1cm} \text{3.11.46.}$$

Similarly, since $q = (\psi(z)^{-1}u)\psi^u(t)^{-1}$, we have (cf. 3.11.23)

\[
\begin{align*}
((\psi^u)_{\zeta_{w,y}} \circ \nu_\eta(\varphi(r)x) \circ \chi_\eta)(s,n) & = ((\psi^u)_{\zeta_{w,y}} \circ \nu_\eta(\varphi(r)x) \circ \chi_\eta)(q,s,n) \\
& = (\psi^u(t^{-1}), id_P) \cdot (\psi^u)_{\xi_{w,y}} \circ \psi_{\eta,u,x}(s,n) \\
& = ((\psi^u)_{\zeta_{w,y}} \circ (t^{-1}, id_P) \cdot \psi_{\eta,u,x})(s,n)
\end{align*}
\]

\hspace{1cm} \text{3.11.47.}$$

where, as in 3.11.23 above, $\zeta_w \in \mathcal{F}(T, U)$ is the unique element fulfilling $\eta = \psi^u \circ \zeta_w$; but, it is easily checked that $\zeta_{w,y} = \zeta_{w,y} \circ \nu_\eta(\varphi(r)x)$; thus, since the homomorphism $(\psi^u)_{\zeta_{w,y}}$ is injective, we get

$$((\psi^u)_{\zeta_{w,y}} \circ \nu_\eta(\varphi(r)x) \circ \chi_\eta = (t^{-1}, id_P) \cdot \psi_{\eta,u,x}$$ \hspace{1cm} \text{3.11.48.}$$

and, according to 3.6.4 above, we still get

$$((\psi^u)_{\zeta_{w,y}} \circ \chi_\eta = (t^{-1}, \nu_\eta(\varphi(r)x)) \cdot \psi_{\eta,u,x}$$ \hspace{1cm} \text{3.11.49.}$$

Moreover, since $\psi^u(t) \in Q_\eta$, in 3.10.3 above $(t, \nu_\eta(\psi^u(t)))$ is the image of an element of $N_{T \times P}(\Delta_{\zeta_w}(U))$; hence, it acts trivially over $ab\left(\tilde{N}_{T \times P}(\Delta_{\zeta_w}(U))\right)$ and therefore we obtain

$$ab((\psi^u)_{\xi_{w,y}}) \circ ab(\chi_\eta) = (1, \nu_\eta(\varphi(r)x) \psi^u(t)) \cdot ab(\psi_{\eta,u,x})$$ \hspace{1cm} \text{3.11.50.}$$

Finally, since $ab(\chi_\eta) = ab(\chi_\eta)^{-1}$, the composition of both equalities in 3.11.43 yields

$$\begin{align*}
(1, \nu_\eta(\psi^u(t))) \cdot ab(\psi_{\eta,u,x}) \circ ab(\varphi_{\theta, u,x}) \\
= ab((\psi^u)_{\xi_{w,y}}) \circ ab((\varphi^\theta)^{\xi_{w,y}})
\end{align*}$$ \hspace{1cm} \text{3.11.51.}$$

and therefore in 3.11.38 we get

$$\sum_{\nu \in \mathcal{F}(U)} \left(ab(\psi_{\eta,u,x}) \circ ab((\varphi^\theta)^{\nu, u,x})\right)(b_\eta)$$ \hspace{1cm} \text{3.11.52.}$$
Conversely, for any \( u \in \hat{Q}^o \) and any \( x \in X_u \) let us consider the element \( w \in W \) determined by the double class of \( xu^{-1} \) in \( \varphi(R) \setminus Q/\psi(T) \), so that we have
\[
xu^{-1} = \varphi(y)^{-1}w\psi(z)
\]
for suitable \( y \in R \) and \( z \in T \); then, with the notation above, we claim that \( \varphi_w(S_w) \) contains \( \theta(U) \) or, equivalently, that \( ^wS_w = \varphi(R) \cap ^w\psi(T) \) contains \( \varphi(y)\eta(U) \). Indeed, since \( \theta(U) \) is contained in \( R \), it is clear that \( \varphi(y)\eta(U) \) is contained in \( \varphi(R) \); it remains to prove that \( \eta(U) \) is contained in \( \varphi(R) \) or, equivalently, in \( ^xu^{-1}\psi(T) \); but, \( x \) normalizes \( \eta(U) \) and \( \eta = \psi^u \circ \zeta_u \), so that \( \eta(U) \) is contained in \( \psi^n(T) \); this proves the claim.

Consequently, from the very definitions of \( \hat{R}^o,w \), of \( \hat{Y}^o,w \) and of \( \hat{S}_w^o \) above, we actually have \( y = \varphi_w(s)\hat{y}r \) for a unique \( \hat{y} \in \hat{Y}^o,w \), a unique \( s \in \hat{S}_w^o \) and a unique \( r \in R_o \); now, the equality 3.11.40 becomes
\[
\varphi(r)xu^{-1} = \varphi(\hat{y})^{-1}(\varphi \circ \varphi_w)(s^{-1})w\psi(z)
\]
and, since \( s \in S_w \subset \psi(T) \), there exist a unique \( \hat{z} \in Z^{w,\hat{y}} \) and a unique \( t \in T_{w,\hat{y}} \) fulfilling \( s^{-1}\psi(z) = \psi(\hat{z}t^{-1}) \), so that equality 3.11.55 becomes
\[
\varphi(r)xu^{-1}\psi(t) = \varphi(\hat{y})^{-1}w\psi(\hat{z})
\]
Thus, we obtain a map from \( X \) to \( Z \) sending \( (u,x) \) to \( (w,\hat{y},\hat{z}) \) which is clearly the inverse of the map from \( Z \) to \( X \) defined above. We are done.

3.12. For the next result, we borrow the notation from A5 in [11]. Recall that in 3.10.5 above, for any \( m \in \mathbb{N} \) we actually define the functors
\[
\hat{\tau}_{F,m}^U : \hat{F} \to \hat{\mathcal{O}-mod}^o \quad \text{and} \quad \hat{\tau}_{F,o}^U : \hat{F} \to \mathcal{O}-\mathfrak{mod}
\]

**Corollary 3.13.** Let \( \mathcal{G} \) be a subcategory of \( \hat{F} \) having the same objects, only having \( \mathcal{G} \)-isomorphisms and containing all the \( \hat{F}_p \)-isomorphisms. Then, with the notation above, for any \( m \in \mathbb{N} \) and any \( n \geq 1 \) we have
\[
\mathcal{H}^o_{\mathcal{G}}(\hat{F}, \hat{\tau}_{F,m}^U) = \{0\}.
\]

**Proof:** It is an immediate consequence of Theorems 3.11 above and Theorem A5.5 in [11].

4. **Existence and uniqueness of the perfect \( F \)-locality**

4.1. As in 3.1 above, let \( P \) be a finite \( p \)-group, \( F \) a Frobenius \( P \)-category and \((\tau^h, \mathcal{L}^o, \pi^h)\) the corresponding basic \( F \)-locality. Recall that we have a contravariant functor [8, Proposition 13.14]
\[
\epsilon^p_F : F \to \mathfrak{Ab}
\]
mapping any subgroup $Q$ of $P$ fully centralized in $\mathcal{F}$ on $C_P(Q)/F_{C_P(Q)}$, where $F_{C_P(Q)}$ denotes the $C_P(Q)$-focal subgroup of $C_P(Q)$ [8, 13.1], and any $\mathcal{F}$-morphism $\varphi : R \to Q$ between subgroups of $P$ fully centralized in $\mathcal{F}$, on the group homomorphism

\[ C_P(Q)/F_{C_P(Q)} \to C_P(R)/F_{C_P(R)} \]

induced by an $\mathcal{F}$-morphism [8, 2.8.2]

\[ \zeta : \varphi(R)C_P(Q) \to R\cdot C_P(R) \]

fulfilling $\zeta(\varphi(v)) = v$ for any $v \in R$. Actually, it is easily checked that this contravariant functor factorizes through the exterior quotient $\hat{\mathcal{F}}$ inducing a new contravariant functor

\[ \hat{\mathcal{F}}^1 : \hat{\mathcal{F}} \to \mathcal{Ab} \]

**Proposition 4.2.** The structural functor $\tau^b : T_P \to \mathcal{L}^b$ induces a natural map $\hat{\tau}^b$ from $\hat{\mathcal{F}}^1$ to $\mathcal{F}^1$.

**Proof:** For any subgroup $Q$ of $P$, the functor $\tau^b$ induces a group homomorphism $\tau_Q^b$ from $N_P(Q)$ to $\mathcal{L}^b(Q)$ which clearly maps $C_P(Q)$ in $(\mathcal{F} \ker(\tau^b))(Q)$; we claim that this correspondence defines a natural map (cf. 3.1.2)

\[ \hat{\tau}^b : \hat{\mathcal{F}}^1 \to \mathcal{F}^1 \]

First of all, we claim that $\tau_Q^b$ maps the $C_P(Q)$-focal subgroup above on the trivial subgroup of $\mathcal{L}^b(Q)$; we may assume that $Q$ is fully centralized in $\mathcal{F}$ and then we know that $F_{C_P(Q)}$ is generated by the elements $u^{-1}\theta(u)$ where $u$ runs over any subgroup $T$ of $C_P(Q)$ and $\theta$ runs over $\mathcal{F}(T,Q)$ stabilizing $T$ and acting trivially on $Q$ [8, 13.1]; but, according to 2.12 above, $\theta$ can be lifted to $\hat{\theta} \in N_G(T,Q)$ normalizing $T$ and centralizing $Q$; hence, the element $u^{-1}\theta(u) = [u, \hat{\theta}^{-1}]$ belongs to $[C_G(Q), C_G(Q)]$ and therefore it has indeed a trivial image in $\mathcal{L}^b(Q)$; consequently, the canonical homomorphism

\[ C_P(Q) \subset C_G(Q) \to \ker(\pi_Q) \]

factorizes through a group homomorphism $\hat{\tau}^b_Q : \hat{\mathcal{F}}^1(Q) \to \ker(\pi_Q)$.

In order to prove the naturality of this correspondence, let $x : R \to Q$ be an $\mathcal{L}^b$-morphism between subgroups of $P$ fully centralized in $\mathcal{F}$ and set $\varphi = \pi_{Q,R}^b(x)$; it follows from [8, 2.8.2] that there exists an $\mathcal{F}$-morphism $\zeta$ from $\varphi(R)C_P(Q)$ to $R\cdot C_P(R)$ fulfilling $\zeta(\varphi(v)) = v$ for any $v \in R$; then, $\zeta$ can be lifted to an $\mathcal{L}^b$-morphism

\[ y : \varphi(R)C_P(Q) \to R\cdot C_P(R) \]
fulfilling \((\pi^b_{\mathcal{L}^b_{\mathcal{F}}(Q)(R)})(\varphi(v)) = v\) for any \(v \in R\); in particular, by

the divisibility of \(\mathcal{L}^b\), \(y\) induces an \(\mathcal{L}^b\)-isomorphism \(y_R : \varphi(R) \cong R\) and then, setting \(xy_R = z\), the \(\mathcal{L}^b\)-morphism \(z : \varphi(R) \to Q\) fulfills \(\pi^b_{\mathcal{L}^b_{\mathcal{F}}(Q)(R)}(z) = \iota^b_{2}(\varphi)\) (cf. 2.4); consequently, we easily get the following commutative diagram

\[
\begin{CD}
\mathcal{L}^b_{\mathcal{F}}(Q) @> \iota^b_{Q} >> \text{Ker}(\pi_Q) \\
\iota^b_{\mathcal{F}}(\varphi) \downarrow @. \downarrow \iota^b_{\varphi}(\varphi) \\
\mathcal{L}^b_{\mathcal{F}}(R) @> \iota^b_{R} >> \text{Ker}(\pi_R)
\end{CD}
\]

We are done.

4.3. The image \(\hat{\tau}^b(\hat{\mathcal{L}}^b_{\mathcal{F}})\) of \(\hat{\tau}^b\) is a subfunctor of \(\hat{\mathcal{L}}^b\) and therefore, by 2.8 above, it determines a quotient \(\mathcal{F}\)-locality \(\hat{\mathcal{L}}^b = \mathcal{L}^b / (\hat{\tau}^b(\hat{\mathcal{L}}^b_{\mathcal{F}}) \circ \hat{\tau}^b)\) of \(\mathcal{L}^b\) (cf. 2.3); we denote by

\[
\tau^b : \mathcal{T}P \to \hat{\mathcal{L}}^b \text{ and } \tau^b : \hat{\mathcal{L}}^b \to \mathcal{F}
\]

the corresponding structural functors; the point is that \(\tau^b\) admits an essentially unique section as proves the theorem below. First of all, we need the following lemma.

**Lemma 4.4.** For any subgroup \(Q\) of \(P\) there is a group homomorphism \(\mu_Q : \mathcal{F}(Q) \to \hat{\mathcal{L}}^b(Q)\) fulfilling \(\mu_Q \circ \kappa_Q = \tau^b_Q\).

**Proof:** Since we can choose an \(\mathcal{F}\)-isomorphism \(\theta : Q \cong Q'\) such that \(Q'\) is fully normalized in \(\mathcal{F}\) and \(\theta\) can be lifted to \(\hat{\mathcal{L}}^b(Q',Q)\), we may assume that \(Q\) is fully normalized in \(\mathcal{F}\).

We apply [8, Lemma 18.8] to the groups \(\mathcal{F}(Q)\) and \(\hat{\mathcal{L}}^b(Q)\), to the normal \(p\)-subgroup \(\text{Ker}(\pi^b_Q)\) of \(\hat{\mathcal{L}}^b(Q)\) and to the group homomorphism \(\text{id}_{\mathcal{F}(Q)}\). We consider the group homomorphism \(\tau^b_Q : N_{\mathcal{F}(Q)} \to \hat{\mathcal{L}}^b(Q)\) and, for any subgroup \(R\) of \(N_{\mathcal{F}(Q)}\) and any \(\alpha \in \mathcal{F}(Q)\) such that \(\alpha \circ \mathcal{F}(Q) \circ \alpha^{-1} \subset \mathcal{F}(Q)\), it follows from [8, Proposition 2.11] that there exists \(\zeta \in \mathcal{F}(N_{\mathcal{F}(Q)};Q\cdot R)\) extending \(\alpha\); then, it follows from [8, 17.11.2] that there exists \(x \in \hat{\mathcal{L}}^b(Q)\) fulfilling

\[
\tau^b_Q(\zeta(v)) = \pi^b_Q(v)
\]

for any \(v \in R\). That is to say, condition 18.8.1 in [8, Lemma 18.8] is fulfilled and therefore this lemma proves the existence of \(\mu_Q\) as announced.

**Theorem 4.5.** With the notation above, the structural functor \(\tau^b\) admits a unique natural \(\mathcal{F}\)-isomorphism class of \(\mathcal{F}\)-locality functorial sections.
Proof: We consider the filtration of \( \tilde{\mathcal{L}}^N \) induced by the filtration of the basic \( \mathcal{F} \)-locality introduced in section 3 and then argue by induction. That is to say, recall that we denote by \( \mathcal{C}_P \) a set of representatives for the set of \( P \)-conjugacy classes of subgroups \( U \) of \( P \) (cf. 2.13); now, for any subset \( \mathcal{N} \) of \( \mathcal{C}_P \) fulfilling condition 3.2.1, we consider the obvious functor (cf. 3.2)

\[
\hat{\tau}^b(\tilde{\mathcal{L}}^N) : \tilde{\mathcal{F}} \to \mathfrak{Af}
\]

sending any subgroup \( Q \) of \( P \) to \( \hat{\tau}^b(\tilde{\mathcal{L}}^N(Q)) \), and the quotient \( \mathcal{F} \)-locality \( \tilde{\mathcal{L}}^{\mathcal{N}} = \mathcal{L}^b / (\hat{\tau}^b(\tilde{\mathcal{L}}^N) \circ \tilde{\mathcal{F}}) \) with the structural functors

\[
\tilde{\mathcal{L}}^{\mathcal{N}} : \mathcal{T}_P \to \tilde{\mathcal{L}}^{\mathcal{N}} \quad \text{and} \quad \tilde{\mathcal{L}}^{\mathcal{N}} : \tilde{\mathcal{L}}^{\mathcal{N}} \to \mathcal{F}
\]

Note that if \( \mathcal{N} = \emptyset \) then \( \tilde{\mathcal{L}}^{\mathcal{N}} = \tilde{\mathcal{L}}^b \); hence, arguing by induction on \( |\mathcal{C}_P - \mathcal{N}| \), it suffices to prove that \( \tilde{\mathcal{L}}^{\mathcal{N}} \) admits a unique natural \( \mathcal{F} \)-isomorphism class of \( \mathcal{F} \)-locality functorial sections.

Moreover, if \( \mathcal{N} = \mathcal{C}_P \) then \( \tilde{\mathcal{L}}^{\mathcal{N}} = \tilde{\mathcal{L}}^b \); therefore \( \tilde{\mathcal{L}}^{\mathcal{N}} = \mathcal{F} \) and \( \tilde{\mathcal{L}}^{\mathcal{N}} = \mathfrak{d}_F \), so that we may assume that \( \mathcal{N} \neq \mathcal{C}_P \); then, we fix a minimal element \( U \) in \( \mathcal{C}_P - \mathcal{N} \), setting \( \mathcal{M} = \mathcal{N} \cup \{U\} \) and \( \tilde{\mathcal{L}}^{\mathcal{M}} = \tilde{\mathcal{L}}^{\mathcal{N}} / \tilde{\mathcal{F}} \). If \( U \neq P \) then \( \mathcal{M} \neq \mathcal{C}_P \) and, as a matter of fact, we have \( \hat{\tau}^b(\tilde{\mathcal{F}}) \cap \tilde{\mathcal{F}} = \{0\} \), so that

\[
(\hat{\tau}^b(\tilde{\mathcal{L}}^N) \circ \tilde{\mathcal{F}}) / (\hat{\tau}^b(\tilde{\mathcal{L}}^N) \circ \tilde{\mathcal{F}}) \cong \tilde{\mathcal{F}}
\]

in this case, for any \( m \in \mathbb{N} \) we simply denote by \( \mathfrak{l}^N \circ \mathfrak{d}_F \) the converse image of \( p^m \tilde{\mathcal{L}}^N \) in \( \mathfrak{l}^N \circ \tilde{\mathcal{F}} \); set \( \mathfrak{l}^N \circ \mathfrak{d}_F = \mathfrak{l}^N \circ \tilde{\mathcal{F}} \), and, coherently, denote by \( \mathfrak{l}^N \circ \mathfrak{d}_F \) and \( \mathfrak{l}^N \circ \tilde{\mathcal{F}} \) the corresponding structural functors. Note that, by 3.8 and 3.10.5 above we get

\[
\mathfrak{l}^N \circ \mathfrak{d}_F / \mathfrak{l}^N \circ \tilde{\mathcal{F}} \cong \mathfrak{d}_F / \tilde{\mathcal{F}}
\]

and in particular, by Corollary 3.13, for any \( n \in \mathbb{N} \) we still get

\[
\mathfrak{l}^N \circ \mathfrak{d}_F / \mathfrak{l}^N \circ \tilde{\mathcal{F}} = \{0\}
\]

If \( U = P \) then \( \mathcal{M} = \mathcal{C}_P \), so that in this case \( \mathfrak{l}^N \circ \mathfrak{d}_F = \mathfrak{l}^N \circ \tilde{\mathcal{F}} \) and, denoting by \( \mathfrak{d}_P : \tilde{\mathcal{F}} \to \mathfrak{Af} \) the functor mapping \( P \) on \( Z(P) \) and any other subgroup of \( P \) on \( \{0\} \), from 3.7 and 3.8 it is easily checked that

\[
\tilde{\mathcal{L}}^N \circ \tilde{\mathcal{F}} / (\hat{\tau}^b(\tilde{\mathcal{L}}^N) \circ \tilde{\mathcal{F}}) \cong \prod_{\sigma \in \tilde{\mathcal{F}}(P)} \mathfrak{d}_P / \Delta(\mathfrak{d}_P)
\]

where \( \Delta \) denotes the usual diagonal map; but, similarly we have

\[
\mathfrak{l}^N \circ \mathfrak{d}_P \cong \prod_{\sigma \in \tilde{\mathcal{F}}(P)} \delta_m \circ \mathfrak{d}_P
\]
and, according to Corollary 3.13, we get \( H^n_{\mathcal{F}, s_m \circ \partial_P} = \{0\} \); moreover, since \( \rho \) does not divide \( |\mathcal{F}(P)| \), we still have

\[
\prod_{\sigma \in \mathcal{F}(P)} \partial_P / \Delta(\partial_P) \cong \prod_{\sigma \in \mathcal{F}(P) - \{\text{id}_P\}} \partial_P
\]

hence, still setting \( \mathring{I}_F^{p,m} = p^m \mathring{F}_F^p \) and \( \mathcal{L}^{b,F,m} = \mathcal{L}^{b,F/m} \), we still get

\[
H^n_{\mathcal{F}, \mathcal{I}_F^{p,m} / \mathcal{I}_F^{p,m+1}} = \{0\}
\]

Further, we denote by \( \mathcal{C}_F \) a set of representatives, fully normalized in \( \mathcal{F} \), for the \( \mathcal{F} \)-isomorphism classes of subgroups of \( P \) and, for any subgroup \( Q \) in \( \mathcal{C}_F \), we choose a group homomorphism \( \mu_Q : \mathcal{F}(Q) \to \mathcal{L}(Q) \) as in Lemma 4.4 above and, for any \( m \in \mathbb{N} \), simply denote by \( \mu_Q^m \) the corresponding group homomorphism from \( \mathcal{F}(Q) \) to \( \mathcal{L}^{b,F,m}(Q) \). For any \( \mathcal{F} \)-morphism \( \phi : R \to Q \) denote by \( \mathcal{F}(Q)_\phi \) and by \( \mathcal{L}^{b,F,m}(Q)_\phi \) the respective stabilizers of \( \phi(R) \) in \( \mathcal{F}(Q) \) and in \( \mathcal{L}^{b,F,m}(Q) \); it is clear that we have a group homomorphism \( a_\phi : \mathcal{F}(Q)_\phi \to \mathcal{F}(R) \) fulfilling \( \eta \circ \phi = \phi \circ a_\phi(\eta) \) for any \( \eta \in \mathcal{F}(Q)_\phi \); similarly, for any \( x^m \in \mathcal{L}^{b,F,m}(Q,R) \) we have a group homomorphism

\[
a_{x^m} : \mathcal{L}^{b,F,m}(Q)_\phi \to \mathcal{L}^{b,F,m}(R)
\]

fulfilling \( y^m \cdot x^m = x^m \cdot a_{x^m}(y^m) \) for any \( y^m \in \mathcal{L}^{b,F,m}(Q)_\phi \).

For any subgroups \( Q \) and \( R \) in \( \mathcal{C}_F \) and any \( \mathcal{F} \)-morphism \( \phi : R \to Q \), \( \mathcal{F}_P(Q) \) and \( \mathcal{F}_P(R) \) are respective Sylow \( p \)-subgroups of \( \mathcal{F}(Q) \) and \( \mathcal{F}(R) \) [8, Proposition 2.11]; therefore, there are \( \alpha \in \mathcal{F}(Q) \) such that \( \mathcal{F}_P(Q)_\alpha \) contains a Sylow \( \mathcal{F}_P(Q)_{\phi,\alpha} \) \( p \)-subgroup of \( \mathcal{F}(Q)_\phi \) and \( \beta \in \mathcal{F}(R) \) such that \( a_\phi(\mathcal{F}_P(Q)_{\phi,\alpha}) \) is contained in \( \mathcal{F}_P(R)_\beta \). Thus, we choose a set of representatives \( \mathcal{F}_{Q,R} \) for the set of double classes \( \mathcal{F}(Q) \setminus \mathcal{F}(Q,R) / \mathcal{F}(R) \) such that, for any \( \phi \) in \( \mathcal{F}_{Q,R} \), \( \mathcal{F}_P(Q) \) contains a Sylow \( p \)-subgroup of \( \mathcal{F}(Q)_\phi \) and \( a_\phi(\mathcal{F}_P(Q)_\phi) \) is contained in \( \mathcal{F}_P(R) \); of course, we choose \( \mathcal{F}_{Q,R} = \{\text{id}_Q\} \).

With all this notation and arguing by induction on \( |\mathcal{C}_P - \mathcal{N}| \) and on \( m \), we will prove that there is a functorial section

\[
\sigma^m : \mathcal{F} \to \mathcal{L}^{b,F,m}
\]

such that, for any \( Q \in \mathcal{C}_F \) and any \( u \in Q \), we have \( \sigma^m(\kappa_Q(u)) = \mathcal{L}^{b,F,m}(u) \), and that, for any groups \( Q \) and \( R \) in \( \mathcal{C}_F \), and any \( \mathcal{F} \)-morphism \( \phi : Q \to R \)
in $\mathcal{F}_{Q,R}$, we have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{F}(Q) & \xrightarrow{\mu^m_Q} & \mathcal{L}^{b,U,m} (Q) \\
\downarrow a_\varphi & & \downarrow a_{\varphi m} \\
\mathcal{F}(R) & \xrightarrow{\mu^m_R} & \mathcal{L}^{b,U,m} (R)
\end{array}
\]
4.5.12.

Since we have $\pi^{b,U,0} = \pi^{b,U}$ and $|M| = |N| + 1$, by the induction hypothesis we actually may assume that $m \neq 0$, that $\pi^{b,U,m-1}$ admits a functorial section $\sigma^{m-1}$ which fulfills the conditions above.

Then, for any $\varphi \in \mathcal{F}_{Q,R}$ it follows from [8, Proposition 2.11], applied to the inverse $\varphi^*$ of the isomorphism $\varphi : R \cong \varphi(R)$ induced by $\varphi$, that there exists an $\mathcal{F}$-morphism $\zeta : N_\varphi(Q) \rightarrow N_\varphi(R)$ fulfilling $\zeta(\varphi(v)) = v$ for any $v \in R$, so that we easily get the following commutative diagram:
\[
\begin{array}{ccc}
N_\varphi(Q) & \xrightarrow{\kappa_Q} & \mathcal{F}(Q) \\
\downarrow \zeta & & \downarrow a_\varphi \\
N_\varphi(R) & \xrightarrow{\kappa_R} & \mathcal{F}(R)
\end{array}
\]
4.5.14;

note that, if $Q = R$ and $\varphi = \kappa_Q(u)$ for some $u \in Q$, we may assume that $\zeta = \kappa_{N_\varphi(Q)}(u)$. In particular, since $\sigma^{m-1}$ fulfills the corresponding commutative diagram 4.5.12, we still get the following commutative diagram
\[
\begin{array}{ccc}
N_\varphi(Q) & \xrightarrow{\tau_{N_\varphi(Q),\varphi}^{b,U,m-1}} & \mathcal{L}^{b,U,m-1} (Q) \\
\downarrow \zeta & & \downarrow a_{\varphi m-1} \\
N_\varphi(R) & \xrightarrow{\tau_{N_\varphi(R),\varphi}^{b,U,m-1}} & \mathcal{L}^{b,U,m-1} (R)
\end{array}
\]
4.5.15.

The first step is, for any $\mathcal{F}$-morphism $\varphi$ in $\mathcal{F}_{Q,R}$, to choose a suitable lifting $\sigma^{m-1}(\varphi)$ of $\sigma^{m-1}(\varphi)$ in $\mathcal{L}^{b,U,m-1} (Q,R)$. We start by choosing a lifting $\sigma^{m-1}(\zeta)$ of $\sigma^{m-1}(\zeta)$ in the obvious stabilizer $\mathcal{L}^{b,U,m} (N_\varphi(R), N_\varphi(Q))_{R,\varphi(R)}$; thus, by the coherence of $\mathcal{L}^{b,U,m}$ (cf. (Q)), for any $u \in N_\varphi(Q)$ we have
\[
\sigma^{m-1}(\zeta) \cdot \tau_{N_\varphi(Q),\varphi}^{b,U,m}(u) = \tau_{N_\varphi(R),\varphi}^{b,U,m}(\zeta(u)) \cdot \sigma^{m-1}(\zeta)
\]
4.5.16;

moreover, by the divisibility of $\mathcal{L}^{b,U,m}$ (cf. 2.4), we find $z_\varphi \in \mathcal{L}^{b,U,m} (R, \varphi(R))$ fulfilling
\[
\sigma^{m-1}(\zeta) \cdot \tau_{N_\varphi(Q),\varphi}^{b,U,m}(1) = \tau_{N_\varphi(R),\varphi}^{b,U,m}(1) \cdot z_\varphi
\]
4.5.17;

similarly, $\sigma^{m-1}(\zeta)$ restricts to $\sigma^{m-1}(\varphi^*) \in \mathcal{L}^{b,U,m-1} (R, \varphi(R))$, so that it is easily checked that $z_\varphi$ lifts $\sigma^{m-1}(\varphi^*)$ to $\mathcal{L}^{b,U,m} (R, \varphi(R))$ and therefore $\sigma^{m-1}(\varphi) = \tau_{Q,\varphi(R)}^{b,U,m}(1) z_\varphi^{-1}$ lifts $\sigma^{m-1}(\varphi)$ to $\mathcal{L}^{b,U,m}(Q,R)$. 

Then, from 4.5.16 and 4.5.17 above, for any $u \in N_P(Q)_{\varphi}$ we get
\[\sigma_{m-1}(\zeta) \cdot \tau_{N_P(Q)_{\varphi}}(u) \cdot \tau_{N_P(Q)_{\varphi}}(1) = \tau_{N_P(R)_{\varphi}}(1) \cdot z_{\varphi} \cdot \tau_{N_P(R)_{\varphi}}(u)\]
\[= \tau_{N_P(R)_{\varphi}}(1) \cdot \tau_{N_P(R)_{\varphi}}(\zeta(u)) \cdot z_{\varphi}\]
\[= \tau_{Q}(u) \cdot \sigma_{m-1}(\varphi) = \tau_{Q}(u) \cdot \sigma_{m-1}(\varphi)\]
and therefore we still get $z_{\varphi} \cdot \tau_{Q}(u) = \tau_{Q}(u) \cdot z_{\varphi}$, so that
\[\tau_{Q}(u) = \tau_{Q}(u) \cdot \sigma_{m-1}(\varphi)\]
or, equivalently, we have $a_{\sigma_{m-1}(\varphi)}(\tau_{Q}(u)) = \tau_{Q}(u) \cdot \sigma_{m-1}(\varphi)$.

At this point, we will apply the uniqueness part of [8, Lemma 18.8] to the groups $F(Q)_{\varphi}$ and $L^{h, U, m}(R)$ and to the composition of group homomorphisms
\[a_{\sigma_{m-1}(\varphi)} \circ \mu_{Q}^{-1} : F(Q)_{\varphi} \longrightarrow L^{h, U, m-1}(Q)_{\varphi} \longrightarrow L^{h, U, m-1}(R)\]
\[= \tau_{R} \circ \zeta : N_P(Q)_{\varphi} \longrightarrow N_P(R) \longrightarrow L^{h, U, m}(R)\]
\[= \tau_{R} \circ \zeta : N_P(Q)_{\varphi} \longrightarrow N_P(R) \longrightarrow L^{h, U, m}(R)\]
Now, according to the commutative diagrams 4.5.12 for $m - 1$ and 4.5.14, and to equality 4.5.18 above, the two group homomorphisms
\[a_{\sigma_{m-1}(\varphi)} \circ \mu_{Q}^{-1} : F(Q)_{\varphi} \longrightarrow L^{h, U, m}(Q)_{\varphi} \longrightarrow L^{h, U, m}(R)\]
\[= \mu_{R} \circ a_{\varphi} : F(R) \longrightarrow L^{h, U, m}(R)\]
both fulfill the conclusion of [8, Lemma 18.8]; consequently, according to this lemma, there is $k_{\varphi}$ in the kernel of the canonical homomorphism from $L^{h, U, m}(R)$ to $L^{h, U, m-1}(R)$ such that, denoting by $\text{int}_{L^{h, U, m}(R)}(k_{\varphi})$ the conjugation by $k_{\varphi}$ in $L^{h, U, m}(R)$, we have
\[\text{int}_{L^{h, U, m}(R)}(k_{\varphi}) \circ a_{\sigma_{m-1}(\varphi)} \circ \mu_{Q}^{-1} = \mu_{R} \circ a_{\varphi}\]
\[\text{int}_{L^{h, U, m}(R)}(k_{\varphi}) \circ a_{\sigma_{m-1}(\varphi)} = a_{\sigma_{m-1}(\varphi)} \cdot k_{\varphi}^{-1}\]
Finally, we choose \( \sigma^{m-1}(\varphi) = \sigma^{m-1}(\varphi) \cdot \kappa^{-1}_\varphi \), lifting indeed \( \sigma^{m-1}(\varphi) \) to \( \mathcal{L}^{h, U, m}(Q, R) \) and, according to equalities 4.5.23 and 4.5.24, fulfilling the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}(Q)_{\varphi} & \xrightarrow{\mu_Q^m} & \mathcal{L}^{h, U, m}(Q)_{\varphi} \\
\downarrow a_\varphi & & \downarrow a_{\sigma^{m-1}(\varphi)} \\
\mathcal{F}(R) & \xrightarrow{\mu_R^m} & \mathcal{L}^{h, U, m}(R) \\
\end{array}
\]

4.5.25;

note that, if \( Q = R \) and \( \varphi = \kappa_Q(u) \) for some \( u \in Q \), this choice is compatible with \( \sigma^{m-1}(\kappa_Q(u)) = \tau_Q^{h, U, m}(u) \). In particular, considering the action of \( \mathcal{F}(Q) \times \mathcal{F}(R) \), by composition on the left- and on the right-hand, on \( \mathcal{F}(Q, R) \) and on \( \mathcal{L}^{h, U, m}(Q, R) \) via \( \mu_Q^m \) and \( \mu_R^m \), we have the inclusion of stabilizers

\[
(\mathcal{F}(Q) \times \mathcal{F}(R))_{\varphi} \subset (\mathcal{F}(Q) \times \mathcal{F}(R))_{\sigma^{m-1}(\varphi)}
\]

4.5.26;

indeed, it is quite clear that \( (\alpha, \beta) \in (\mathcal{F}(Q) \times \mathcal{F}(R))_{\varphi} \) forces \( \alpha \in \mathcal{F}(Q)_{\varphi} \); then, since \( \alpha \circ \varphi = \varphi \circ a_\varphi(\alpha) \), we get \( \beta = \alpha_\varphi(\alpha) \) and the inclusion above follows from the commutativity of diagram 4.5.26.

This allows us to choose a family of liftings \( \{\sigma^{m-1}(\varphi)\}_\varphi \), where \( \varphi \) runs over the set of \( \mathcal{F} \)-morphisms, which is compatible with \( \mathcal{F} \)-isomorphisms; precisely, for any pair of subgroups \( Q \) and \( R \) in \( \mathcal{C}_\mathcal{F} \), and any \( \varphi \in \mathcal{F}_{Q,R} \), we choose a lifting \( \sigma^{m-1}(\varphi) \) of \( \sigma^{m-1}(\varphi) \) in \( \mathcal{L}^{h, U, m}(Q, R) \) as above. Then, any subgroup \( Q \) of \( P \) determines a unique \( \hat{Q} \) in \( \mathcal{C}_\mathcal{F} \) which is \( \mathcal{F} \)-isomorphic to \( Q \) and we choose an \( \mathcal{F} \)-isomorphism \( \omega_Q : \hat{Q} \cong Q \) and a lifting \( x_Q \in \mathcal{L}^{h, U, m}(Q, \hat{Q}) \) of \( \omega_Q \); in particular, we choose \( \omega_Q = \text{id}_{\hat{Q}} \) and \( x_Q = \tau_Q^{h, U, m}(1) \). Thus, any \( \mathcal{F} \)-morphism \( \varphi : R \to Q \) determines subgroups \( \hat{Q} \) and \( \hat{R} \) in \( \mathcal{C}_\mathcal{F} \) and an element \( \hat{\varphi} \) in \( \mathcal{F}_{Q, R} \) in such a way that there are \( \alpha_\varphi \in \mathcal{F}(\hat{Q}) \) and \( \beta_\varphi \in \mathcal{F}(\hat{R}) \) fulfilling

\[
\varphi = \omega_Q \circ \alpha_\varphi \circ \hat{\varphi} \circ \beta_\varphi^{-1} \circ \omega_R^{-1}
\]

4.5.27

and we define

\[
\sigma^{m-1}(\varphi) = x_Q \cdot \mu_Q^m(\alpha_\varphi) \cdot \sigma^{m-1}(\hat{\varphi}) \cdot \mu_R^m(\beta_\varphi)^{-1} \cdot x_R^{-1}
\]

4.5.28;

once again, if \( Q = R \) and \( \varphi = \kappa_Q(u) \) for some \( u \in Q \), we actually get \( \sigma^{m-1}(\kappa_Q(u)) = \tau_Q^{h, U, m}(u) \). This definition does not depend on the choice of \( (\alpha_\varphi, \beta_\varphi) \) since for another choice \( (\alpha', \beta') \) we clearly have \( \alpha' = \alpha_\varphi \circ \alpha'' \) and \( \beta' = \beta_\varphi \circ \beta'' \) for a suitable \( (\alpha'', \beta'') \) in \( (\mathcal{F}(\hat{Q}) \times \mathcal{F}(\hat{R}))_{\hat{\varphi}} \) and it suffices to apply inclusion 4.5.26.
Moreover, for any pair of $\mathcal{F}$-isomorphisms $\zeta : Q \cong Q'$ and $\xi : R \cong R'$, considering $\varphi' = \zeta \circ \varphi \circ \xi^{-1}$ we claim that

$$\sigma^{-1}(\varphi') = \sigma^{-1}(\zeta) \cdot \sigma^{-1}(\varphi) \cdot \sigma^{-1}(\xi)^{-1}$$  \hspace{0.05\textwidth} 4.5.29;

indeed, it is clear that $Q'$ also determines $\hat{Q}$ in $\mathcal{C}_F$ and therefore, if we have $\zeta = \omega_Q \circ \gamma \circ \omega_{Q'}$ then we obtain $\sigma^{-1}(\zeta) = x_{Q'} \cdot \mu^\alpha_Q(\alpha \cdot x_Q^{-1})$; similarly, if we have $\xi = \omega_R \circ \beta \circ \omega_{R'}$ we also obtain $\sigma^{-1}(\xi)^{-1} = x_R \cdot \mu_R(\beta \cdot x_R^{-1})$; further, $\varphi'$ also determines $\hat{\varphi}$ in $\mathcal{F}_{\hat{Q}, \hat{R}}$; consequently, we get

$$\sigma^{-1}(\zeta) \cdot \sigma^{-1}(\varphi) \cdot \sigma^{-1}(\xi)^{-1} = (x_{Q'} \cdot \mu^\alpha_Q(\alpha \cdot x_Q^{-1}) \cdot (x_R \cdot \mu_R(\beta \cdot x_R^{-1}))^{-1} = \sigma^{-1}(\varphi')$$  \hspace{0.05\textwidth} 4.5.30.

Recall that we have the exact sequence of contravariant functors from $\mathcal{F}$ to $\mathcal{F}^b$ (cf. 2.7 and 2.8)

$$0 \to \mathcal{I}_m \to \mathcal{R}(\mathcal{F}) \to \mathcal{R}(\mathcal{F}) \to 0$$  \hspace{0.05\textwidth} 4.5.31;

hence, for another $\mathcal{F}$-isomorphism $\psi : T \to R$ we clearly have

$$\sigma^{-1}(\varphi) \cdot \sigma^{-1}(\psi) = \sigma^{-1}(\varphi \circ \psi) \cdot \gamma^m_{\psi, \varphi}$$  \hspace{0.05\textwidth} 4.5.32

for some $\gamma^m_{\psi, \varphi}$ in $\mathcal{I}^m(\mathcal{F})/\mathcal{I}_m^m(\mathcal{F})$. That is to say, borrowing notation and terminology from $\mathcal{C}_F$ [8, A2.8], we get a correspondence sending any $\mathcal{F}$-chain $q : \Delta_2 \to \mathcal{F}$ to the element $\gamma^m_{q(0), q(1)}$ in $\mathcal{I}^m(\mathcal{F})/\mathcal{I}_m^m(\mathcal{F})$ and, setting

$$\mathcal{C}^n(\mathcal{F}, \mathcal{I}^m_\mathcal{F}) = \prod_{\tilde{z} \in \mathcal{Z}(\Delta_n, \mathcal{F})} (\mathcal{I}^m_\mathcal{F} / \mathcal{I}_m^m(\mathcal{F}))_{\tilde{z}}(0)$$  \hspace{0.05\textwidth} 4.5.33

for any $n \in \mathbb{N}$, we claim that this correspondence determines an $\textit{stable}$ element $\gamma^m$ of $\mathcal{C}^2(\mathcal{F}, \mathcal{I}^m_\mathcal{F})$ [8, A3.17].

Indeed, for another $\mathcal{F}$-isomorphic $\mathcal{F}$-chain $q' : \Delta_2 \to \mathcal{F}$ and a natural $\mathcal{F}$-isomorphism $\nu : q \cong q'$, setting

$$T = q(0), \ T' = q'(0), \ R = q(1), \ R' = q'(1), \ Q = q(2), \ Q' = q'(2)$$

$$\psi = q(0 \bullet 1), \ \varphi = q(1 \bullet 2), \ \psi' = q'(0 \bullet 1), \ \varphi' = q'(1 \bullet 2)$$  \hspace{0.05\textwidth} 4.5.34,

$$\nu_0 = \eta, \ \nu_1 = \xi \ \text{and} \ \nu_2 = \zeta$$

from 4.5.30 we have

$$\sigma^{-1}(\varphi') = \sigma^{-1}(\zeta) \cdot \sigma^{-1}(\varphi) \cdot \sigma^{-1}(\eta)^{-1}$$

$$\sigma^{-1}(\psi') = \sigma^{-1}(\xi) \cdot \sigma^{-1}(\psi) \cdot \sigma^{-1}(\eta)^{-1}$$  \hspace{0.05\textwidth} 4.5.35

$$\sigma^{-1}(\varphi' \circ \psi') = \sigma^{-1}(\zeta) \cdot \sigma^{-1}(\varphi \circ \psi) \cdot \sigma^{-1}(\eta)^{-1}$$
and therefore we get
\[
\sigma^{m-1}(\varphi \circ \psi) \cdot \gamma_{\varphi,\psi}^{m} = \sigma^{m-1}(\varphi) \cdot \sigma^{m-1}(\psi) = \sigma^{m-1}(\zeta) \cdot (\sigma^{m-1}(\varphi) \cdot \sigma^{m-1}(\xi) - 1) \cdot \sigma^{m-1}(\eta) - 1
\]
4.5.36,
so that, by the divisibility of \(\mathcal{L}^{n,U,m}\), we have
\[
\gamma_{\varphi,\psi}^{m} = ((I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}}) \cdot (\sigma^{m-1}(\eta) - 1)) \cdot (\gamma_{\varphi,\psi}^{m})
\]
4.5.37;
this proves that the correspondence \(\gamma^{m}\) sending \((\varphi, \psi)\) to \(\gamma_{\varphi,\psi}^{m}\) is stable and, in particular, that \(\gamma_{\varphi,\psi}^{m}\) only depends on the corresponding \(\mathcal{F}\)-morphisms \(\varphi\) and \(\psi\); thus we set \(\gamma_{\varphi,\psi}^{m} = \gamma_{\varphi,\psi}^{m}\).

On the other hand, considering the usual differential map
\[
d_{2} : C^{2}(\mathcal{F}, I^{U,m-1}_{\overline{U}^{m}}) \longrightarrow C^{3}(\mathcal{F}, I^{U,m-1}_{\overline{U}^{m}})
\]
4.5.38,
we claim that \(d_{2}(\gamma^{m}) = 0\); indeed, for a third \(\mathcal{F}\)-morphism \(\varepsilon : W \rightarrow T\) we get
\[
(\sigma^{m-1}(\varphi) \cdot \sigma^{m-1}(\psi)) \cdot \sigma^{m-1}(\varepsilon) = (\sigma^{m-1}(\varphi \circ \psi) \cdot \gamma_{\varphi,\psi}^{m}) \cdot \sigma^{m-1}(\varepsilon)
\]
\[
= (\sigma^{m-1}(\varphi \circ \psi) \cdot \sigma^{m-1}(\varepsilon)) \cdot (I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}})(\varepsilon)(\gamma_{\varphi,\psi}^{m})
\]
\[
= \sigma^{m-1}(\varphi \circ \psi \circ \varepsilon) \cdot \gamma_{\varphi,\psi,\varepsilon}^{m} \cdot (I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}})(\varepsilon)(\gamma_{\varphi,\psi}^{m})
\]
4.5.39
\[
\sigma^{m-1}(\varphi) \cdot (\sigma^{m-1}(\psi)) \cdot \sigma^{m-1}(\varepsilon) = \sigma^{m-1}(\varphi) \cdot (\sigma^{m-1}(\psi \circ \varepsilon)) \cdot \gamma_{\varphi,\psi,\varepsilon}^{m}
\]
\[
= \sigma^{m-1}(\varphi \circ \psi \circ \varepsilon) \cdot \gamma_{\varphi,\psi,\varepsilon}^{m}
\]
and the divisibility of \(\mathcal{L}^{n,U,m}\) forces
\[
\gamma_{\varphi,\psi,\varepsilon}^{m} \cdot (I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}})(\varepsilon)(\gamma_{\varphi,\psi}^{m}) = \gamma_{\varphi,\psi,\varepsilon}^{m} \cdot \gamma_{\varphi,\psi,\varepsilon}^{m}
\]
4.5.40;
since \(\text{Ker}(\pi_{W}^{n,U,m})\) is abelian, with the additive notation we obtain
\[
0 = (I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}})(\varepsilon)(\gamma_{\varphi,\psi}^{m}) - \gamma_{\varphi,\psi,\varepsilon}^{m} + \gamma_{\varphi,\psi,\varepsilon}^{m} - \gamma_{\varphi,\psi,\varepsilon}^{m}
\]
4.5.41,
proving our claim.

At this point, it follows from equalities 4.5.5 and 4.5.9 that \(\gamma^{m} = d_{1}(\beta^{m})\) for some stable element \(\beta^{m} = (\beta_{m}^{m})_{m} \in \mathcal{F}\)-morphisms \((\Delta_{1}, \mathcal{F})\) in \(C^{1}(\mathcal{F}, I^{U,m-1}_{\overline{U}^{m}})\); that is to say, with the notation above we get
\[
\gamma_{\varphi,\psi}^{m} = ((I^{U,m-1}_{\overline{U}^{m}} - I^{U,m}_{\overline{U}^{m}})(\psi)) \cdot (\beta_{m}^{m}) \cdot (\beta_{\varphi,\psi}^{m})^{-1} \cdot \beta_{\psi}^{m}
\]
4.5.42.
where we identify any \( \tilde{F} \)-morphism with the obvious \( \tilde{F} \)-chain \( \Delta_1 \to \tilde{F} \); hence, from equality 4.5.32 we obtain
\[
(\sigma^{-1}(\varphi)(\beta^m_{\tilde{\varphi}})^{-1})(\sigma^{-1}(\psi)(\beta^m_{\tilde{\psi}})^{-1}) = ((\sigma^{-1}(\varphi)(\sigma^{-1}(\psi)) \cdot (\beta^m_{\tilde{\varphi}} \cdot ((\hat{\varphi}) \cdot (\hat{\psi})) \cdot (\beta^m_{\tilde{\psi}})))^{-1}
\]
4.5.43,
\[
= \sigma^{-1}(\varphi \circ \psi)(\beta^m_{\tilde{\varphi} \circ \tilde{\psi}})^{-1}
\]
which amounts to saying that the correspondence \( \sigma^m \) sending \( \varphi \in \mathcal{F}(Q, R) \) to \( \sigma^{-1}(\varphi)(\beta^m_{\tilde{\varphi}})^{-1} \) \( \in \mathcal{L}^{h, U, m}(Q, R) \) defines a functorial section of \( \pi^{h, U, m}_R \); note that, if \( Q = R \) and \( \varphi = \kappa_Q(u) \) for some \( u \in Q \), we have \( \tilde{\varphi} = \tilde{\kappa}_Q \) and \( \beta^m_{\tilde{\varphi}} = 1 \), so that \( \sigma^m(\kappa_Q(u)) = \tau^{h, U, m}_Q(u) \). It remains to prove that this functorial section fulfills the commutativity of the corresponding diagram 4.5.12; since we already have the commutativity of diagram 4.5.25, it suffices to get the commutativity of the following diagram
\[
\begin{array}{ccc}
\mathcal{F}(R) & \xrightarrow{\mu^m_R} & \mathcal{L}^{h, U, m}(R) \\
\text{id}_{\mathcal{F}(R)} & \downarrow & \downarrow \delta (\beta^m)^{-1} \\
\mathcal{F}(R) & \xrightarrow{\mu^m_R} & \mathcal{L}^{h, U, m}(R)
\end{array}
\]
4.5.44
which follows from the fact that \( \beta^m \) is stable and therefore \( (\beta^m_{\tilde{\varphi}})^{-1} \) fixes the image of \( \mu^m_R \).

We can modify this correspondence in order to get an \( \mathcal{F} \)-locality functorial section; indeed, for any \( \mathcal{F}_P \)-morphism \( \kappa_{Q, R}(u): R \to Q \) where \( u \) belongs to \( \mathcal{T}_P(Q, R) \), the \( \mathcal{L}^{h, U, m}(Q, R) \)-morphisms \( \sigma^m(\kappa_{Q, R}(u)) \) and \( \tau^{h, U, m}_{Q, R}(u) \) both lift \( \kappa_{Q, R}(u) \) \( \in \mathcal{F}(Q, R) \); thus, the divisibility of \( \mathcal{L}^{h, U, m}_{Q, R} \) guarantees the existence and the uniqueness of \( \delta_{\kappa_{Q, R}(u)} \) \( \in \text{Ker}(\pi^{h, U, m}_R) \) fulfilling
\[
\tau^{h, U, m}_{Q, R}(u) = \sigma^m(\kappa_{Q, R}(u)) \cdot \delta_{\kappa_{Q, R}(u)}
\]
4.5.45
and, since we have \( \sigma^m(\kappa_Q(w)) = \tau^{h, U, m}_Q(w) \) for any \( w \in Q \), it is quite clear that \( \delta_{\kappa_{Q, R}(u)} \) only depends on the class of \( \kappa_{Q, R}(u) \) in \( \tilde{F}(Q, R) \).

For a second \( \mathcal{F}_P \)-morphism \( \kappa_{R, T}(v): T \to R \), setting \( \xi = \kappa_{R, T}(u) \) and \( \eta = \kappa_{R, T}(v) \) we get
\[
\sigma^m(\xi \circ \eta) \cdot \delta_{\xi \circ \eta} = \tau^{h, U, m}_{Q, T}(uv) = \tau^{h, U, m}_{Q, R}(u) \cdot \tau^{h, U, m}_{R, T}(v) = \sigma^m(\xi) \cdot \delta_{\xi} \cdot \sigma^m(\eta) \cdot \delta_{\eta}
\]
4.5.46;
\[
= \sigma^m(\xi \circ \eta) \cdot (\text{Ker}(\pi^{h, U, m}_R)(\tilde{\xi})) \cdot (\tilde{\delta}_{\xi}) \cdot (\tilde{\delta}_{\eta})
\]
then, once again the *divisibility* of $L^{h,u,m}$ forces
\[ \delta_{\xi_0\eta} = (\widehat{\text{Reff}(\pi^{h,u,m})}(\eta))(\delta_{\xi}) \cdot \delta_{\eta} \] 4.5.47
and, since $\text{Ker}(\pi_T^{h,u,m})$ is abelian, with the additive notation we obtain
\[ 0 = (\widehat{\text{Reff}(\pi^{h,u,m})}(\eta))(\delta_{\xi}) - \delta_{\xi_0\eta} + \delta_{\eta} \] 4.5.48.

That is to say, denoting by $i: \tilde{F}_P \subset \tilde{F}$ the obvious *inclusion functor*, the correspondence $\delta$ sending any $\tilde{F}_P$-morphism $\xi: R \to Q$ to $\delta_{\xi}$ defines a 1-cocycle in $C^i(\tilde{F}_P, \tilde{\text{Reff}(\pi^{h,u,m})})$; but, since the category $\tilde{F}_P$ has a final object, we actually have [8, Corollary A4.8]
\[ \mathbb{H}^1(\tilde{F}_P, \tilde{\text{Reff}(\pi^{h,u,m})}) \circ i) = \{0\} \] 4.5.49;
consequently, we obtain $\delta = d_0(w)$ for some element $w = (w_Q)_{Q \in P}$ in
\[ C^0(\tilde{F}_P, \tilde{\text{Reff}(\pi^{h,u,m})}) = C^0(\tilde{\tilde{F}}, \tilde{\text{Reff}(\pi^{h,u,m})})) \] 4.5.50.
In conclusion, equality 4.5.45 becomes
\[ \tilde{\tau}_{Q,R}^{h,u,m}(u) = \sigma^m(\kappa_{Q,R}(u)) \cdot (\tilde{\text{Reff}(\pi^{h,u,m})}(\kappa_{Q,R}(u))(w_Q) \cdot w_R^{-1} \] 4.5.51
and therefore the new correspondence sending $\varphi \in \tilde{F}(Q, R)$ to $w_Q \cdot \sigma^m(\varphi) \cdot w_R^{-1}$ defines a $\tilde{\tilde{F}}$-locality functorial section of $\pi^{h,u,m}$. From now on, we still denote by $\sigma^m$ this $\tilde{\tilde{F}}$-locality functorial section of $\pi^{h,u,m}$.

Let $\sigma'^m: \tilde{F} \to \tilde{\tilde{L}}^{h,u,m}$ be another $\tilde{\tilde{F}}$-locality functorial section of $\pi^{h,u,m}$; arguing by induction on $|\mathcal{C}_P - \mathcal{N}|$ and on $m$, and up to natural $\tilde{\tilde{F}}$-isomorphisms, we clearly may assume that $\sigma'^m$ also lifts $\sigma'^{m-1}$; in this case, for any $\tilde{\tilde{F}}$-morphism $\varphi: R \to Q$, we have $\sigma'^m(\varphi) = \sigma^m(\varphi) \cdot \varepsilon^m_{\varphi}$ for some $\varepsilon^m$ in $(\tilde{l}^{u,m-1}_{F}/\tilde{l}^{u,m}_{F})(R)$; that is to say, as above we get a correspondence sending any $\tilde{\tilde{F}}$-chain $q: \Delta_1 \to \tilde{\tilde{F}}$ to $\varepsilon^m_{q(\bullet_1)}$, in $(\tilde{l}^{u,m-1}_{F}/\tilde{l}^{u,m}_{F})(q(0))$ and we claim that this correspondence determines an $\tilde{F}_P$-*stable* element $\varepsilon^m$ of $C^1(\tilde{\tilde{F}}, l_{F}^{u,m-1}/l_{F}^{u,m})$ [8, A3.17].

Indeed, for another $\tilde{F}_P$-isomorphic $\tilde{\tilde{F}}$-*chain* $q': \Delta_1 \to \tilde{\tilde{F}}$ and a *natural* $\tilde{F}_P$-*isomorphism* $\nu: q \cong q'$, as in 4.5.34 above setting
\[ R = q(0), R' = q'(0), Q = q(1), Q' = q'(1) \]
\[ \varphi = q(0 \bullet 1), \quad \varphi' = q'(0 \bullet 1) \] 4.5.52,
\[ \nu_0 = \kappa_{R',H}(u) \quad \text{and} \quad \nu_1 = \kappa_{Q',Q}(u) \]
from 4.5.29 we get
\[
\sigma^m(\varphi') = \kappa_{Q',Q}(u) \cdot \sigma^m(\varphi) \cdot \varepsilon^m_\varphi \cdot \kappa_{R',R}(v)^{-1}
\]
\[
= \sigma^m(\varphi') \cdot \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\widetilde{k}_{R',R}(v)^{-1}) \right) (\varepsilon^m_\varphi)
\]
\[\sigma^m(\varphi') = \sigma^m(\varphi') \cdot \varepsilon^m_\varphi \]
and the divisibility of \( L_{b,U} \) forces
\[
\varepsilon^m_\varphi = \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\widetilde{k}_{R',R}(v)^{-1}) \right) (\varepsilon^m_\varphi)
\]
this proves that the correspondence \( \varepsilon^m \) sending \( \varphi \) to \( \varepsilon^m_\varphi \) is \( \mathcal{F}_P \)-stable and, in particular, that \( \varepsilon^m_\varphi \) only depends on the corresponding \( \mathcal{F} \)-morphism \( \tilde{\varphi} \), thus we set \( \varepsilon^m_\tilde{\varphi} = \varepsilon^m_\varphi \).

On the other hand, considering the usual differential map
\[
d_1 : C^1(\tilde{\mathcal{F}}, \frac{U^{U,m-1}}{U^U}) \longrightarrow C^2(\tilde{\mathcal{F}}, \frac{U^{U,m-1}}{U^U})
\]
we claim that \( d_1(\varepsilon^m) = 0 \); indeed, for a second \( \mathcal{F} \)-morphism \( \psi : T \rightarrow R \) we get
\[
\sigma^m(\varphi) \cdot \sigma^m(\psi) = \sigma^m(\varphi) \cdot \varepsilon^m_\varphi \cdot \sigma^m(\psi) \cdot \varepsilon^m_\psi
\]
\[
= \sigma^m(\varphi \circ \psi) \cdot \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\tilde{\psi}) \right) (\varepsilon^m_\varphi) \cdot \varepsilon^m_\psi
\]
\[\sigma^m(\varphi) \cdot \sigma^m(\psi) = \sigma^m(\varphi \circ \psi) \cdot \varepsilon^m_{\tilde{\varphi} \circ \tilde{\psi}} \]
and the divisibility of \( L_{b,U} \) forces
\[
\left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\tilde{\psi}) \right) (\varepsilon^m_{\tilde{\varphi}}) \cdot \varepsilon^m_{\tilde{\psi}} = \varepsilon^m_{\tilde{\varphi} \circ \tilde{\psi}}
\]
since \( \ker(\pi^m_{b,U}) \) is Abelian, with the additive notation we obtain
\[
0 = \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\tilde{\psi}) \right) (\varepsilon^m_{\tilde{\psi}}) - \varepsilon^m_{\tilde{\varphi} \circ \tilde{\psi}} + \varepsilon^m_{\tilde{\psi}}
\]
proving our claim.

At this point, it follows from equalities 4.5.5 and 4.5.9 that \( \varepsilon^m = d_0(y) \) for some stable element \( y = (y_Q)_{Q \in \mathcal{P}} \) in \( C^0(\tilde{\mathcal{F}}, \frac{U^{U,m-1}}{U^U}) \); that is to say, with the notation above we get
\[
\varepsilon^m_{\tilde{\varphi}} = \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\tilde{\varphi}) \right) (y_Q) \cdot y_R^{-1}
\]
hence, we obtain
\[
\sigma^m(\varphi) = \sigma^m(\varphi) \cdot \left( (\widetilde{l}_X^{U,m-1}/\widetilde{l}_X^U)(\tilde{\varphi}) \right) (y_Q) \cdot y_R^{-1} = y_Q \cdot \sigma^m(\varphi) \cdot y_R^{-1}
\]
which amounts to saying that \( \sigma^m \) is naturally \( \mathcal{F} \)-isomorphic to \( \sigma^m \). We are done.
Corollary 4.6. There exists a perfect $\mathcal{F}$-locality $\mathcal{P}$.

**Proof:** Denote by $\bar{\mathcal{P}}$ the converse image in $\mathcal{L}^b$ of the image of $\mathcal{F}$ in $\bar{\mathcal{L}}^b$ by a section of $\pi^b$; since $\hat{\tau}(\mathcal{F})$ is contained in the image of $\tau^b$, we still have a functor $\tau^b : \mathcal{P} \to \mathcal{P}$; thus, together with the restriction of $\pi^b$ to $\bar{\mathcal{P}}$, $\bar{\mathcal{P}}$ becomes an $\mathcal{F}$-locality and, since $\bar{\mathcal{L}}^b$ is coherent, $\bar{\mathcal{P}}$ is coherent too.

We claim that $\bar{\mathcal{P}}^e$ is a perfect $\mathcal{F}^e$-locality; indeed, for any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$, since $C_P(Q)/F_{C_F(Q)} = Z(Q)$ we have a group extension (cf. 4.3)

$$1 \to Z(Q) \to \bar{\mathcal{P}}(Q) \to \mathcal{F}(Q) \to 1$$

4.6.1

together with an injective group homomorphism $\tau_Q^b : N_P(Q) \to \bar{\mathcal{P}}(Q)$; consequently, it follows from [8, 18.5] that $\bar{\mathcal{P}}(Q)$ is the $\mathcal{F}$-localizer of $Q$; thus, by the very definition in [8, 17.4 and 17.13], $\bar{\mathcal{P}}^e$ is a perfect $\mathcal{F}^e$-locality.

But, in [8, Ch. 20] we prove that any perfect $\mathcal{F}^e$-locality can be extended to a unique perfect $\mathcal{F}$-locality $\mathcal{P}$. We are done.

4.7. The uniqueness of the perfect $\mathcal{F}$-locality is an easy consequence of the following theorem; the proof of this result follows the same pattern than the proof of Theorem 4.5, but we firstly need the following lemmas.

Lemma 4.8. Let $(\tau, \mathcal{P}, \pi)$ be a perfect $\mathcal{F}$-locality and $\hat{\varphi} : Q \to P$ be a $\mathcal{P}$-morphism such that $\pi^u(Q)$ is fully normalized in $\mathcal{F}$. Then there is a $\mathcal{P}$-morphism $\hat{\zeta} : N_P(Q) \to P$ such that $\hat{\varphi} = \hat{\zeta} \cdot \tau_{N_P(Q), Q}(1)$.

**Proof:** Denoting by $\varphi$ the image of $\hat{\varphi}$ in $\mathcal{F}(P, Q)$, it follows from [8, 2.8.2] that there is an $\mathcal{F}$-morphism $\zeta : N_P(Q) \to P$ extending $\varphi$; then, lifting $\zeta$ to $\hat{\zeta}$ in $\mathcal{P}(P, N_P(R))$, it is clear that the $\mathcal{P}$-morphisms $\hat{\zeta} \cdot \tau_{N_P(Q), Q}(1)$ and $\hat{\varphi}$ have the same image $\varphi$ in $\mathcal{F}(P, Q)$ and therefore, by the very definition of $\mathcal{P}$ in [8, 17.13], there is $z \in C_P(Q)$ such that $\hat{\zeta} \cdot \tau_{N_P(Q), Q}(1) \cdot \tau_Q(z) = \hat{\varphi}$; but, it is clear that

$$\tau_{N_P(Q), Q}(1) \cdot \tau_Q(z) = \tau_{N_P(Q), Q}(z) = \tau_{N_P(Q), Q}(z) \cdot \tau_{N_P(Q), Q}(1)$$

4.8.1

consequently, $\hat{\zeta} \cdot \tau_{N_P(Q), Q}(z)$ extends $\hat{\varphi}$ in $\mathcal{P}$. We are done.

Lemma 4.9. Let $(\tau, \mathcal{P}, \pi)$ be a perfect $\mathcal{F}$-locality. For any subgroup $Q$ of $P$ there is a group homomorphism $\hat{\mu}_Q : \mathcal{P}(Q) \to \mathcal{L}^b(Q)$ fulfilling $\hat{\mu}_Q \circ \tau_Q = \tau_Q^b$.

**Proof:** Since we can choose an $\mathcal{F}$-isomorphism $\theta : Q \cong Q'$ such that $Q'$ is fully normalized in $\mathcal{F}$ and $\theta$ can be lifted to $\mathcal{P}(Q', Q)$ and to $\mathcal{L}^b(Q', Q)$, we may assume that $Q$ is fully normalized in $\mathcal{F}$.
We apply [8, Lemma 18.8] to the groups $\mathcal{P}(Q)$ and $\mathcal{L}^b(Q)$, to the normal $p$-subgroup $\text{Ker}(\pi^Q)$ of $\mathcal{L}^b(Q)$ and to the group homomorphism $\tau_Q$ from $\mathcal{P}(Q)$ to $\mathcal{F}(Q) \cong \mathcal{L}^b(Q)/\text{Ker}(\pi^Q)$. We consider the group homomorphism $\tau^Q_\alpha : N_P(Q) \to \mathcal{L}^b(Q)$ and, for any subgroup $R$ of $N_P(Q)$ and any $\hat{\alpha} \in \mathcal{P}(Q)$ such that $\hat{\alpha} \cdot \tau_Q(R) \cdot \hat{\alpha}^{-1} \subset \tau_Q(N_P(Q))$, it follows from [8, 2.10.1] that there exists $\zeta \in \mathcal{F}(N_P(Q), Q \cdot R)$ extending the image of $\hat{\alpha}$ in $\mathcal{F}(Q)$; then, it follows from [8, 17.11.2] that there exists $x \in \mathcal{L}^b(Q)$ fulfilling

$$\tau^Q_\alpha(\zeta(v)) = \pi^Q_\alpha(v)$$

for any $v \in Q \cdot R$. That is to say, condition 18.8.1 in [8, Lemma 18.8] is fulfilled and therefore this lemma proves the existence of $\hat{\mu}_Q$ as announced.

**Theorem 4.10.** For any perfect $\mathcal{F}$-locality $\mathcal{P}$ there exists a unique natural $\mathcal{F}$-isomorphism class of $\mathcal{F}$-locality functors to $\mathcal{L}^b$.

**Proof:** Let $\mathcal{P}$ be a perfect $\mathcal{F}$-locality with the structural functors

$$\tau : \mathcal{P} \to \mathcal{F} \quad \text{and} \quad \pi : \mathcal{P} \to \mathcal{F} \quad \text{4.10.1}$$

and for any subgroups $Q$ of $P$ and $R$ of $Q$ we set $i^Q_R = \tau_Q(1)$. We consider the filtration of the basic $\mathcal{F}$-locality introduced in section 3 and then argue by induction. That is to say, recall that we denote by $C_P$ a set of representatives for the set of $P$-conjugacy classes of subgroups $U$ of $P$ (cf. 2.13); now, for any subset $\mathcal{N}$ of $C_P$ fulfilling condition 3.2.1, we have the functor $\tilde{\xi}_\mathcal{N} : \mathcal{F} \to \mathfrak{Ab}$ (cf. 3.2) and we consider the quotient $\mathcal{F}$-locality $\mathcal{L}^{b,\mathcal{N}} = \mathcal{L}^b / (\tilde{\xi}_\mathcal{N} \circ \tilde{\pi}^b)$ with the structural functors

$$\tau^{b,\mathcal{N}} : \mathcal{P} \to \mathcal{L}^{b,\mathcal{N}} \quad \text{and} \quad \pi^{b,\mathcal{N}} : \mathcal{L}^{b,\mathcal{N}} \to \mathcal{F} \quad \text{4.10.2}.$$ 

Note that if $\mathcal{N} = \emptyset$ then $\mathcal{L}^{b,\emptyset} = \mathcal{L}^b$; hence, arguing by induction on $|\mathcal{C}_P - \mathcal{N}|$, it suffices to prove the existence of a unique natural $\mathcal{F}$-isomorphism class of $\mathcal{F}$-locality functors from $\mathcal{P}$ to $\mathcal{L}^{b,\mathcal{N}}$.

Moreover, if $\mathcal{N} = C_P$ then $\tilde{\xi}_\mathcal{N} = \tilde{\xi}^b$ and therefore $\mathcal{L}^{b,\mathcal{N}} = \mathcal{F}$, so that we may assume that $\mathcal{N} \neq C_P$; in this situation, we fix a minimal element $U$ in $\mathcal{C}_P - \mathcal{N}$, setting $\mathcal{M} = \mathcal{N} \cup \{U\}$ and $\tilde{\xi}^U = \tilde{\xi}_\mathcal{M} / \tilde{\xi}_U^b$; for any $m \in \mathbb{N}$ we simply denote by $\tilde{\xi}^{U,m}$ the converse image of $\tilde{\pi}^U$ in $\tilde{\xi}_U^b$; set $\mathcal{L}^{b,\mathcal{U},m} = \mathcal{L}^b / \tilde{\xi}^{U,m}$ and, coherently, denote by $\pi^{b,\mathcal{U},m}$ and $\tau^{b,\mathcal{U},m}$ the corresponding structural functors. Note that, by 3.8 and 3.10.5 above we get

$$\tilde{\xi}^{U,m}/\tilde{\xi}^{U,m+1}_U \cong \tilde{\xi}_{\mathcal{F},m}^{U,0} \quad \text{4.10.3}$$

and in particular, by Corollary 3.13, for any $n \in \mathbb{N}$ we still get

$$\mathfrak{N}_n((\mathcal{F}, \tilde{\xi}^{U,m}/\tilde{\xi}^{U,m+1}_U) = \{0\} \quad \text{4.10.4}.$$
As above, we denote by $C_F$ a set of representatives, fully normalized in $F$, for the $F$-isomorphism classes of subgroups of $P$ and, for any subgroup $Q$ in $C_F$, we choose a group homomorphism $\hat{\mu}_Q : P(Q) \rightarrow L^{h,m}(Q)$ as in Lemma 4.9 above and, for any $m \in \mathbb{N}$, simply denote by $\hat{\mu}_Q^m$ the corresponding group homomorphism from $P(Q)$ to $L^{h,m}(Q)$. For any $F$-morphism $\varphi : R \rightarrow Q$ denote by $P(Q)_\varphi$ and by $L^{h,m}(Q)_\varphi$ the respective stabilizers of $\varphi(R)$ in $P(Q)$ and in $L^{h,m}(Q)$. As above, for any $\varphi \in P(Q, R)$ and any $x^m \in L^{h,m}(Q, R)$ we have group homomorphisms

$$a_\varphi : P(Q)_\varphi \longrightarrow P(R) \text{ and } a^{x^m}_\varphi : L^{h,m}(Q)_\varphi \longrightarrow L^{h,m}(R) \quad 4.10.5$$

For any subgroups $Q$ and $R$ in $C_F$, we choose as above a set of representatives $P_{Q,R}$ for the set of double classes $P(Q) \backslash P(Q, R) / P(R)$ such that, for any $\hat{\varphi}$ in $P_{Q,R}$, denoting by $\varphi$ its image in $F(Q, R)$, $F_P(Q)$ contains a Sylow $p$-subgroup of $F(Q)_\varphi$ and $a_\varphi(F_P(Q)_\varphi)$ is contains in $F_P(R)$; of course, we choose $P_{Q,Q} = \{q_Q(1)\}$.

With all this notation and arguing by induction on $|C_P - N|$ and on $m$, we will prove that there is a functor

$$\lambda^m : P \longrightarrow L^{h,m} \quad 4.10.6$$

such that, for any $Q \in C_F$ and any $u \in Q$, we have $\lambda^m(q_Q(u)) = q^{h,m}(u)$, and that, for any groups $Q$ and $R$ in $C_F$, and any $\hat{\varphi}$ in $P_{Q,R}$, denoting by $\varphi$ its image in $F(Q, R)$, we have the commutative diagram

$$\begin{array}{ccc}
P(Q)_\varphi & \xrightarrow{a_\varphi} & L^{h,m}(Q)_\varphi \\
\downarrow \hat{\mu}_Q^m & & \downarrow a^{\lambda^m(\hat{\varphi})} \\
P(R) & \xrightarrow{\hat{\mu}_R^m} & L^{h,m}(R) \\
\end{array} \quad 4.10.7$$

Since we have $\pi^{h,0} = \pi^{h,M}$ and $|M| = |N| + 1$, by the induction hypothesis we actually may assume that $m \neq 0$ and that we have a functor

$$\lambda^{m-1} : P \longrightarrow L^{h,m-1} \quad 4.10.8$$

which fulfills the conditions above.

As above, for any $\hat{\varphi} \in P_{Q,R}$, denoting by $\varphi$ its image in $F(Q, R)$, it follows from [8, Proposition 2.11], applied to the inverse $\varphi^*$ of the isomorphism $\varphi_* : R \cong \varphi(R)$ induced by $\varphi$, that there exists an $F$-morphism $\zeta : N_P(Q)_\varphi \rightarrow N_P(R)$ fulfilling $\zeta(\varphi(v)) = v$ for any $v \in R$, so that we easily get the following commutative diagram

$$\begin{array}{ccc}
N_P(Q)_\varphi & \xrightarrow{\tau_Q} & P(Q)_\varphi \\
\zeta \downarrow & & \downarrow a_\varphi \\
N_P(R) & \xrightarrow{\tau_R} & P(R) \\
\end{array} \quad 4.10.9$$
note that, if $Q = R$ and $\hat{\phi} = \tau_Q(u)$ for some $u \in Q$, we may assume that $\hat{\zeta} = \kappa_{N_P(Q)}(u)$. In particular, since $\lambda^{m-1}$ fulfills the corresponding commutative diagram 4.10.7, we still get the following commutative diagram

$$
\begin{array}{ccc}
N_P(Q) & \xrightarrow{\tau_Q^{b,U,m-1}} & L^{b,U,m-1}(Q) \\
\downarrow & & \downarrow_{a_{\lambda^{m-1}(\hat{\phi})}} \\
N_P(R) & \xrightarrow{\tau_R^{b,U,m-1}} & L^{b,U,m-1}(R)
\end{array}
$$

4.10.10

With the notation above, the first step is to choose a suitable lifting $\lambda^{m-1}(\hat{\phi})$ of $\lambda^{m-1}(\hat{\phi})$ in $L^{b,U,m}(Q,R)$. Choosing a lifting $\hat{\zeta}$ of $\zeta$ in the obvious stabilizer $P\langle N_P(R), N_P(Q) \rangle_{R,\varphi(R)}$, we start by choosing a lifting $\lambda^{m-1}(\hat{\zeta})$ of $\lambda^{m-1}(\hat{\zeta})$ in $L^{b,U,m}(N_P(R), N_P(Q) \rangle_{R,\varphi(R)}$; thus, by the coherence of $L^{b,U,m}$ (cf. (Q)), for any $u \in N_P(Q)$ we have

$$
\lambda^{m-1}(\hat{\zeta}) \cdot \tau_{N_P(Q)\varphi}(u) = \tau_{N_P(R)}^{b,U,m} \langle \zeta(u) \rangle \cdot \lambda^{m-1}(\hat{\zeta})
$$

4.10.11;

moreover, by the divisibility of $L^{b,U,m}$ (cf. 2.4), we find $z_{\hat{\phi}} \in L^{b,U,m}(R, \varphi(R))$ fulfilling

$$
\lambda^{m-1}(\hat{\zeta}) \cdot \tau_{N_P(Q)\varphi}(R,\varphi(R))(1) = \tau_{N_P(R)}^{b,U,m} \langle z_{\hat{\phi}} \rangle
$$

4.10.12;

similarly, denoting by $\hat{\varphi}^*:\varphi(R) \cong R$ the $P$-isomorphism determined by $\hat{\varphi}$, $\lambda^{m-1}(\hat{\zeta})$ restricts to $\lambda^{m-1}(\hat{\phi}^*)$ and it is easily checked that $z_{\hat{\phi}}$ lifts $\lambda^{m-1}(\hat{\phi}^*)$ to $L^{b,U,m}(R, \varphi(R))$; consequently, $\lambda^{m-1}(\hat{\phi}) = \tau_{Q,\varphi(R)}^{b,U,m} \langle z_{\hat{\phi}} \rangle$ lifts $\lambda^{m-1}(\hat{\phi})$ to $L^{b,U,m}(Q,R)$.

Then, from 4.10.11 and 4.10.12 above, for any $u \in N_P(Q)$ we get

$$
\begin{aligned}
\lambda^{m-1}(\hat{\zeta}) \cdot \tau_{N_P(Q)\varphi}(u) \cdot \tau_{N_P(Q)\varphi}(R,\varphi(R))(1) &= \tau_{N_P(R)}^{b,U,m} \langle z_{\hat{\phi}} \rangle \cdot \tau_{R}^{b,U,m} \langle \zeta(u) \rangle \\
&= \tau_{N_P(R)}^{b,U,m} \langle \zeta(u) \rangle \cdot \tau_R^{b,U,m} \langle z_{\hat{\phi}} \rangle
\end{aligned}
$$

4.10.13

and therefore we still get $z_{\hat{\phi}} \cdot \tau_{R}^{b,U,m} \langle \zeta(u) \rangle = \tau_R^{b,U,m} \langle \zeta(u) \rangle \cdot z_{\hat{\phi}}$, so that

$$
\tau_{Q}^{b,U,m} \langle u \rangle \cdot \lambda^{m-1}(\hat{\phi}) = \lambda^{m-1}(\hat{\phi}) \cdot \tau_{R}^{b,U,m} \langle \zeta(u) \rangle
$$

4.10.14

or, equivalently, we have a $\lambda^{m-1}(\hat{\phi}) \cdot \tau_{Q}^{b,U,m} \langle u \rangle = \tau_{R}^{b,U,m} \langle \zeta(u) \rangle$. 

At this point, we will apply the uniqueness part of [8, Lemma 18.8] to the groups $\mathcal{P}(Q)_Q$ and $L^{b,u,m}(R)$, to the kernel of the canonical homomorphism from $L^{b,u,m}(R)$ to $L^{b,u,m-1}(R)$, and to the composition of group homomorphisms

$$a_{\lambda^{-1}(\bar{\varphi})} \circ \hat{\mu}_Q^m : \mathcal{P}(Q)_Q \longrightarrow L^{b,u,m-1}(Q)_Q \longrightarrow L^{b,u,m-1}(R)$$ 4.10.15,

together with the composition of group homomorphisms

$$\tau_R \circ \zeta : N^P(Q)_Q \longrightarrow N^P(R) \longrightarrow L^{b,u,m}(R)$$ 4.10.16.

Now, according to the commutative diagrams 4.10.7 for $m-1$ and 4.10.9, and to equality 4.10.14 above, the two group homomorphisms

$$a_{\lambda^{-1}(\bar{\varphi})} \circ \hat{\mu}_Q^m : \mathcal{P}(Q)_Q \longrightarrow L^{b,u,m}(Q)_Q \longrightarrow L^{b,u,m}(R)$$ 4.10.17,

$$\hat{\mu}_R^m \circ a_{\bar{\varphi}} : \mathcal{P}(Q)_Q \longrightarrow \mathcal{P}(R) \longrightarrow L^{b,u,m}(R)$$

both fulfill the conclusion of [8, Lemma 18.8]; consequently, according to this lemma, there is $k_{\bar{\varphi}}$ in the kernel of the canonical homomorphism from $L^{b,u,m}(R)$ to $L^{b,u,m-1}(R)$ such that, denoting by $\text{int}_{L^{b,u,m}(R)}(k_{\bar{\varphi}})$ the conjugation by $k_{\bar{\varphi}}$ in $L^{b,u,m}(R)$, we have

$$\text{int}_{L^{b,u,m}(R)}(k_{\bar{\varphi}}) \circ a_{\lambda^{-1}(\bar{\varphi})} \circ \hat{\mu}_Q^m = \hat{\mu}_R^m \circ a_{\bar{\varphi}}$$ 4.10.18;

but, it is easily checked that

$$\text{int}_{L^{b,u,m}(R)}(k_{\bar{\varphi}}) \circ a_{\lambda^{-1}(\bar{\varphi})} = a_{\lambda^{-1}(\bar{\varphi})} \cdot k_{\bar{\varphi}}^{-1}$$ 4.10.19.

Finally, we choose $\lambda^{-1}(\bar{\varphi}) = \lambda^{-1}(\bar{\varphi}) \cdot k_{\bar{\varphi}}^{-1}$, lifting indeed $\sigma^{-1}(\varphi)$ to $L^{b,u,m}(Q,R)$ and, according to equalities 4.10.18 and 4.10.19, fulfilling the following commutative diagram

$$\begin{array}{ccc}
\mathcal{P}(Q)_Q & \xrightarrow{\hat{\mu}_Q^m} & \mathcal{L}^{b,u,m}(Q)_Q \\
\downarrow a_{\bar{\varphi}} & & \downarrow a_{\lambda^{-1}(\bar{\varphi})} \\
\mathcal{P}(R) & \xrightarrow{\hat{\mu}_R^m} & \mathcal{L}^{b,u,m}(R)
\end{array}$$ 4.10.20;

note that, if $Q = R$ and $\bar{\varphi} = \tau_Q(u)$ for some $u \in Q$, this choice is compatible with $\lambda^{-1}(\tau_Q(u)) = \tau_Q(u)$. In particular, considering the action of $\mathcal{P}(Q) \times \mathcal{P}(R)$, by composition on the left- and on the right-hand, on $\mathcal{P}(Q,R)$ and on $L^{b,u,m}(Q,R)$ via $\hat{\mu}_Q^m$ and $\hat{\mu}_R^m$, we have the inclusion of stabilizers

$$(\mathcal{P}(Q) \times \mathcal{P}(R)) \cdot \bar{\varphi} \subseteq (\mathcal{P}(Q) \times \mathcal{P}(R)) \lambda^{-1}(\bar{\varphi})$$ 4.10.21;
indeed, it is quite clear that $(\hat{\alpha}, \hat{\beta}) \in (\mathcal{P}(Q) \times \mathcal{P}(R))_{\hat{}\phi}$ forces $\hat{\alpha} \in \mathcal{P}(Q)_{\hat{}\phi}$; then, since $\hat{\alpha} \cdot \hat{\phi} = \hat{\phi} \cdot a_{\hat{\phi}}(\hat{\alpha})$, we get $\hat{\beta} = a_{\hat{\phi}}(\hat{\alpha})$ by the divisibility of $\mathcal{P}$, and the inclusion above follows from the commutativity of diagram 4.10.20.

This allows us to choose a family of liftings $\left\{\lambda : Q \rightarrow Q, \hat{}\phi \right\}$, where $\hat{}\phi$ runs over the set of $\mathcal{P}$-morphisms, which is compatible with $\mathcal{P}$-isomorphisms; precisely, for any pair of subgroups $Q$ and $R$ in $\mathcal{C}_X$, and any $\hat{}\phi \in \mathcal{P}(Q,R)$, we choose as above a lifting $\lambda = \lambda(\hat{}\phi)$ of $\mathcal{P}$ in $L^{b_{\lambda},m}(Q,R)$. Then, any subgroup $Q$ of $\mathcal{P}$ determines a unique $Q$ in $\mathcal{C}_X$ which is $\mathcal{F}$-isomorphic to $Q$ and we choose a $\mathcal{P}$-isomorphism $\hat{}\omega : \hat{}Q \cong Q$ and a lifting $x_Q \in L^{b_{\lambda},m}(Q,\hat{}Q)$ of the image $\hat{}\omega_Q \in \mathcal{F}(Q,\hat{}Q)$; in particular, we choose $\hat{}\omega_Q = \hat{}\tau_Q(1)$ and $x_Q = \hat{}\tau_Q^{-1}(1)$. Thus, any $\mathcal{P}$-morphism $\hat{}\phi : R \rightarrow Q$ determines subgroups $\hat{}Q$ and $\hat{}R$ in $\mathcal{C}_X$ and an element $\hat{}\phi$ in $\mathcal{P}(\hat{}Q,\hat{}R)$ in such a way that there are $\alpha_{\hat{\phi}} \in \mathcal{P}(\hat{}Q)$ and $\beta_{\hat{\phi}} \in \mathcal{P}(\hat{}R)$ fulfilling

$$\hat{}\phi = \hat{}\omega_Q \cdot \alpha_{\hat{\phi}} \cdot \hat{}\beta_{\hat{\phi}} \cdot \hat{}\omega_R^{-1}$$

4.10.22

and we define

$$\lambda^{-1}(\hat{}\phi) = x_Q \cdot \hat{}\mu_{\hat{}Q}(\alpha_{\hat{\phi}}) \cdot \lambda^{-1}(\hat{}\phi) \cdot \hat{}\mu_{\hat{}R}(\beta_{\hat{\phi}})^{-1} \cdot x_R^{-1}$$

4.10.23; once again, if $Q = R$ and $\hat{}\phi = \hat{}\tau_Q(u)$ for some $u \in Q$, we actually get $\lambda^{-1}(\hat{}\tau_Q(u)) = \mathcal{P}_{\lambda,m}(Q)$. This definition does not depend on the choice of $(\alpha_{\hat{\phi}}, \beta_{\hat{\phi}})$ since for another choice $(\hat{\alpha}', \beta')$ we clearly have $\beta' = \alpha_{\hat{\phi}} \cdot \hat{\alpha}'$ and $\beta' = \beta_{\hat{\phi}} \cdot \beta''$ for a suitable $(\alpha', \beta'')$ in $(\mathcal{P}(\hat{}Q) \times \mathcal{P}(\hat{}R))_{\hat{}\phi}$ and it suffices to apply inclusion 4.10.21.

Moreover, for any pair of $\mathcal{P}$-isomorphisms $\hat{}\xi : Q \cong Q'$ and $\hat{}\xi : R \cong R'$, considering $\phi' = \hat{}\xi \cdot \hat{}\phi \cdot \xi^{-1}$ we claim that

$$\lambda^{-1}(\hat{}\phi') = \lambda^{-1}(\hat{}\xi) \cdot \lambda^{-1}(\hat{}\phi) \cdot \lambda^{-1}(\hat{}\xi)^{-1}$$

4.10.24; indeed, it is clear that $Q'$ also determines $\hat{}Q$ in $\mathcal{C}_X$ and therefore, if we have $\hat{}\xi = \hat{}\omega_Q \cdot \hat{}\hat{\alpha}_Q \cdot \hat{}\omega_Q^{-1}$ then we obtain $\lambda^{-1}(\hat{}\xi) = x_Q \cdot \hat{}\mu_{\hat{}Q}(\hat{}\alpha_{\hat{}Q}) \cdot x_Q^{-1}$; similarly, if we have $\hat{}\xi = \hat{}\omega_{\hat{}R} \cdot \hat{}\hat{\alpha}_{\hat{}R} \cdot \hat{}\omega_{\hat{}R}^{-1}$ we also obtain $\lambda^{-1}(\hat{}\xi) = x_{\hat{}R} \cdot \hat{}\mu_{\hat{}R}(\hat{}\alpha_{\hat{}R})^{-1} \cdot x_{\hat{}R}^{-1}$; further, $\phi'$ also determines $\hat{}\phi$ in $\mathcal{P}(\hat{}Q,\hat{}R)$; consequently, we get

$$\lambda^{-1}(\hat{}\xi) \cdot \lambda^{-1}(\hat{}\phi) \cdot \lambda^{-1}(\hat{}\xi)^{-1} = (x_Q \cdot \hat{}\mu_{\hat{}Q}(\hat{}\alpha_{\hat{}Q}) \cdot x_Q^{-1}) \cdot \lambda^{-1}(\hat{}\phi) \cdot (x_{\hat{}R} \cdot \hat{}\mu_{\hat{}R}(\hat{}\alpha_{\hat{}R})^{-1} \cdot x_{\hat{}R}^{-1})$$

4.10.25.
Recall that we have the exact sequence of contravariant functors from \( F \) to \( \mathfrak{U} \) (cf. 2.7 and 2.8)

\[
0 \rightarrow \mathfrak{U}_{m-1}/\mathfrak{U}_m \rightarrow \mathfrak{Re}t(\pi^U_{m,0}) \rightarrow \mathfrak{Re}t(\pi^U_{m-1,0}) \rightarrow 0
\]

hence, for another \( \mathcal{P} \)-morphism \( \hat{\psi} : T \rightarrow R \) we clearly have

\[
\lambda^{m-1}(\hat{\varphi}) \cdot \gamma^{m-1}(\hat{\psi}) = \chi^{m-1}(\lambda \cdot \hat{\varphi} \cdot \hat{\psi})
\]

for some \( \gamma^m_{\overline{\varphi}, \overline{\psi}} \) in \( (\mathfrak{U}_{m-1}/\mathfrak{U}_m)(T) \). That is to say, borrowing notation and terminology from [8, A2.8], we get a correspondence sending any \( P \) and therefore we get

\[
\tilde{L} = \lambda^m(P) \rightarrow \tilde{q}
\]

so that, by the divisibility of \( \psi^m \), we have

\[
\gamma^m_{\overline{\varphi}, \overline{\psi}} = (\lambda^{m-1}(\tilde{q})^{-1})(\gamma^m_{\overline{\varphi}, \overline{\psi}})
\]
this proves that the correspondence \( \gamma^m \) sending \((\hat{\varphi}, \hat{\psi})\) to \( \gamma^m_{\varphi, \tilde{\psi}} \) is stable and, in particular, that \( \gamma^m_{\varphi, \tilde{\psi}} \) only depends on the corresponding \( \tilde{\Phi} \)-morphisms \( \tilde{\varphi} \) and \( \tilde{\psi} \); thus we set \( \gamma^m_{\varphi, \tilde{\psi}} = \gamma^m_{\tilde{\varphi}, \psi} \), where \( \varphi \) and \( \psi \) are the corresponding \( \Phi \)-morphisms.

On the other hand, considering the usual differential map

\[
d_2 : \mathbb{C}^2(\tilde{\Phi}, \tilde{I}_{\tilde{\Phi}}) \to \mathbb{C}^3(\tilde{\Phi}, \tilde{I}_{\tilde{\Phi}})
\]

we claim that \( d_2(\gamma^m) = 0 \); indeed, for a third \( \Phi \)-morphism \( \epsilon : W \to T \) we get

\[
(\lambda^{m-1}(\hat{\varphi}) \cdot \lambda^{m-1}(\hat{\psi})) \cdot \lambda^{m-1}(\hat{\epsilon}) = (\lambda^{m-1}(\hat{\varphi} \cdot \hat{\psi}) \cdot \lambda^{m-1}(\hat{\epsilon})) \cdot \lambda^{m-1}(\hat{\varphi}, \hat{\psi})
\]

\[
= (\lambda^{m-1}(\hat{\varphi} \cdot \hat{\psi}) \cdot \lambda^{m-1}(\hat{\epsilon})) \cdot \lambda^{m-1}(\hat{\varphi}, \hat{\psi})
\]

\[
\lambda^{m-1}(\hat{\varphi}) \cdot \lambda^{m-1}(\hat{\psi}) \cdot \lambda^{m-1}(\hat{\epsilon}) = \lambda^{m-1}(\hat{\varphi}) \cdot (\lambda^{m-1}(\hat{\psi}) \cdot \lambda^{m-1}(\hat{\epsilon}))
\]

and the divisibility of \( L_{h, \mathcal{U}, m} \) forces

\[
\gamma^m_{\varphi, \psi, \tilde{\epsilon}} \cdot ((\tilde{I}_{\tilde{\Phi}}^{m-1} / \tilde{I}_{\tilde{\Phi}}^{m})((\gamma^m_{\varphi, \psi})) = \gamma^m_{\varphi, \tilde{\psi}, \sigma} \cdot \gamma^m_{\varphi, \tilde{\epsilon}}
\]

4.10.35;

since \( \text{Ker}(\pi^m_{\tilde{\Phi}}) \) is abelian, with the additive notation we obtain

\[
0 = ((\tilde{I}_{\tilde{\Phi}}^{m-1} / \tilde{I}_{\tilde{\Phi}}^{m})((\gamma^m_{\varphi, \psi})) - \gamma^m_{\varphi, \tilde{\psi}, \sigma} + \gamma^m_{\varphi, \tilde{\psi}, \sigma} - \gamma^m_{\varphi, \tilde{\sigma}}
\]

4.10.36;

proving our claim.

At this point, it follows from equality 4.10.4 that \( \gamma^m = d_1(\beta^m) \) for some stable element \( \beta^m = (\beta^m_{\varphi})_{\varphi \in \Phi} \) in \( \mathbb{C}^1(\tilde{\Phi}, \tilde{I}_{\tilde{\Phi}}^{m-1} / \tilde{I}_{\tilde{\Phi}}^{m}) \); that is to say, with the notation above we get

\[
\gamma^m_{\varphi, \tilde{\psi}} = ((\tilde{I}_{\tilde{\Phi}}^{m-1} / \tilde{I}_{\tilde{\Phi}}^{m})((\psi)) \cdot (\beta^m_{\varphi} \cdot (\beta^m_{\varphi})^{-1})
\]

4.10.37;

hence, from equality 4.10.27 we obtain

\[
(\lambda^{m-1}(\hat{\varphi}) \cdot (\beta^m_{\varphi})^{-1}) \cdot (\lambda^{m-1}(\hat{\psi}) \cdot (\beta^m_{\varphi})^{-1})
\]

\[
= (\lambda^{m-1}(\hat{\varphi}) \cdot (\beta^m_{\varphi})^{-1}) \cdot ((\tilde{I}_{\tilde{\Phi}}^{m-1} / \tilde{I}_{\tilde{\Phi}}^{m})((\psi)) \cdot (\beta^m_{\varphi})^{-1})
\]

4.10.38,

\[
= \lambda^{m-1}(\hat{\varphi} \cdot \hat{\psi}) \cdot (\beta^m_{\varphi \cdot \psi})^{-1}
\]
which amounts to saying that the correspondence \( \lambda^m \) sending \( \hat{\varphi} \in \mathcal{P}(Q, R) \) to \( \lambda^{m-1}(\hat{\varphi}) \cdot (\beta^m_{\hat{\varphi}})^{-1} \in \mathcal{L}^{b, U, m}(Q, R) \) defines the announced functor; note that, if \( Q = R \) and \( \hat{\varphi} = \tau_Q(u) \) for some \( u \in Q \), we have \( \hat{\varphi} = \text{id}_Q \) and \( \beta^m_{\hat{\varphi}} = 1 \), so that \( \lambda^m(\tau_Q(u)) = \tau_Q^{b, U, m}(u) \). It remains to prove that this functorial section fulfills the commutativity of the corresponding diagram 4.10.7; since we already have the commutativity of diagram 4.10.20, it suffices to get the \[ \mathcal{P}(R) \xrightarrow{\mu^m} \mathcal{L}^{b, U, m}(R) \]

\[
\text{id}_{\mathcal{P}(R)} \downarrow \quad \downarrow \delta(\beta^m_{\hat{\varphi}})^{-1}
\]

\[
\mathcal{P}(R) \xrightarrow{\mu^m} \mathcal{L}^{b, U, m}(R)
\]

which follows from the fact that \( \beta^m \) is stable and therefore \( (\beta^m_{\hat{\varphi}})^{-1} \) fixes the image of \( \mu^m \).

We can modify this correspondence in order to get an \( \mathcal{F} \)-locality functor; indeed, for any \( \mathcal{P} \)-morphism \( \tau_{Q,R}(u): R \to Q \) where \( u \) belongs to \( \mathcal{T}(Q, R) \), the \( \mathcal{L}^{b, U, m}(Q, R) \)-morphisms \( \lambda^m(\tau_{Q,R}(u)) \) and \( \tau_{Q,R}^{b, U, m}(u) \), both lift \( \kappa_{Q,R}(u) \) in \( \mathcal{F}(Q, R) \); thus, the divisibility of \( \mathcal{L}^{b, U, m} \) guarantees the existence and the uniqueness of \( \delta_{\kappa_{Q,R}(u)} \in \text{Ker}(\pi^{b, U, m}_R) \) fulfilling

\[
\tau_{Q,R}^{b, U, m}(u) = \lambda^m(\tau_{Q,R}(u)) \cdot \delta_{\kappa_{Q,R}(u)}
\]

and, since we have \( \lambda^m(\tau_Q(w)) = \tau_Q^{b, U, m}(w) \) for any \( w \in Q \), it is quite clear that \( \delta_{\kappa_{Q,R}(u)} \) only depends on the class of \( \kappa_{Q,R}(u) \) in \( \tilde{\mathcal{F}}(Q, R) \).

For a second \( \mathcal{P} \)-morphism \( \tau_{R,T}(v): T \to R \), setting \( \hat{\xi} = \tau_{R,T}(u) \) and \( \hat{\eta} = \tau_{R,T}(v) \) we get

\[
\lambda^m(\hat{\xi} \cdot \hat{\eta}) \cdot \delta_{\hat{\xi} \cdot \hat{\eta}} = \tau_{Q,T}^{b, U, m}(uv) = \tau_{Q,R}^{b, U, m}(u) \cdot \tau_{R,T}^{b, U, m}(v) = \lambda^m(\hat{\xi}) \cdot \delta_{\hat{\xi}} \cdot \lambda^m(\hat{\eta}) \cdot \delta_{\hat{\eta}} = \lambda^m(\hat{\xi} \cdot \hat{\eta}) \cdot \delta_{\hat{\xi} \cdot \hat{\eta}}
\]

then, once again the divisibility of \( \mathcal{L}^{b, U, m} \) forces

\[
\delta_{\hat{\xi} \cdot \hat{\eta}} = \left( \widetilde{\text{Ker}}(\pi^{b, U, m})(\hat{\eta}) \right) (\delta_{\hat{\xi}}) \cdot \delta_{\hat{\eta}}
\]

and, since \( \text{Ker}(\pi^{b, U, m}_T) \) is abelian, with the additive notation we obtain

\[
0 = \left( \widetilde{\text{Ker}}(\pi^{b, U, m})(\hat{\eta}) \right) (\delta_{\hat{\xi}}) - \delta_{\hat{\xi} \cdot \hat{\eta}} + \delta_{\hat{\eta}}
\]

4.10.43.
That is to say, denoting by \( i: \tilde{\mathcal{F}}_\Pi \subset \tilde{\mathcal{F}} \) the obvious \textit{inclusion functor}, the correspondence \( \delta \) sends any \( \tilde{\mathcal{F}}_\Pi \)-morphism \( \xi: R \to Q \) to \( \delta \xi \) defines a \( 1 \)-cocycle in \( \mathcal{C}^1(\tilde{\mathcal{F}}_\Pi, \tilde{\text{Ret}}(\pi^{b,U,m}) \circ i) \); but, since the category \( \tilde{\mathcal{F}}_\Pi \) has a final object, we actually have [8, Corollary A4.8]

\[
\mathbb{H}^1(\tilde{\mathcal{F}}_\Pi, \tilde{\text{Ret}}(\pi^{b,U,m}) \circ i) = \{0\}
\]

4.10.44;

consequently, we obtain \( \delta = d_0(w) \) for some element \( w = (w_Q)_{Q \in \mathcal{P}} \) in

\[
\mathcal{C}^0(\tilde{\mathcal{F}}_\Pi, \tilde{\text{Ret}}(\pi^{b,U,m}) \circ i) = \mathcal{C}^0(\tilde{\mathcal{F}}, \tilde{\text{Ret}}(\pi^{b,U,m}))
\]

4.10.45.

In conclusion, equality 4.10.40 becomes

\[
\tau_{\mathcal{P}^m}^{b,U,m}(u) = \lambda^m(\tau_{\mathcal{P}^m}(u)) \cdot \left( \tilde{\text{Ret}}(\pi^{b,U,m})(\tau_{\mathcal{P}^m}(u)) \right) (w_Q) w_R^{-1}
\]

4.10.46

and therefore the new correspondence sending \( \hat{\phi} \in \mathcal{P}(Q,R) \) to \( w_Q \lambda^m(\hat{\phi}) w_R^{-1} \) defines a \( \mathcal{F} \)-\textit{locality functor}. From now on, we still denote by \( \lambda^m \) this \( \mathcal{F} \)-\textit{locality functor}.

Let \( \lambda^m: \mathcal{P} \to \mathcal{L}^{b,U,m} \) be another \( \mathcal{F} \)-locality functor; arguing by induction on \(|\mathcal{P} - \mathcal{N}| \) and on \( m \), and up to natural \( \mathcal{F} \)-isomorphisms, we clearly may assume that \( \lambda^m \) also lifts \( \lambda^{m-1} \); in this case, for any \( \mathcal{P} \)-morphism \( \hat{\phi}: R \to Q \), we have \( \lambda^m(\phi) = \lambda^m(\hat{\phi}) \varepsilon^m_\phi \) for some \( \varepsilon^m_\phi \) in \( (I_{\mathcal{P}}^{U,m-1}/I_{\mathcal{P}}^{U,m})(R) \); that is to say, as above we get a correspondence sending any \( \mathcal{P} \)-\textit{chain} \( q: \Delta_1 \to \mathcal{P} \) to \( \varepsilon^m_q(0,1) \), in \( (I_{\mathcal{P}}^{U,m-1}/I_{\mathcal{P}}^{U,m})(q(0)) \) and we claim that this correspondence determines a \( \mathcal{P} \)-\textit{stable} element \( \varepsilon^m \) of \( \mathcal{C}^1(\mathcal{P}, I_{\mathcal{P}}^{U,m-1}/I_{\mathcal{P}}^{U,m}) \) [8, A3.17].

Indeed, for another \( \mathcal{P} \)-\textit{isomorphic} \( \mathcal{P} \)-\textit{chain} \( q': \Delta_1 \to \mathcal{P} \) and a \textit{natural} \( \mathcal{P} \)-\textit{isomorphism} \( \nu: q \equiv q' \), as in 4.10.29 above setting

\[
R = q(0), \quad R' = q'(0), \quad Q = q(1), \quad Q' = q'(1)
\]

\[
\hat{\phi} = q(0 \ast 1), \quad \hat{\phi}' = q'(0 \ast 1)
\]

\[
\nu_0 = \hat{\xi} \quad \text{and} \quad \nu_1 = \hat{\xi}
\]

from 4.10.24 we get

\[
\lambda^m(\phi') = \hat{\xi} \lambda^m(\phi) \varepsilon^m_\phi \hat{\xi}^{-1}
\]

4.10.48

\[
\lambda^m(\phi') = \lambda^m(\phi') \varepsilon^m_\phi
\]

and the divisibility of \( \mathcal{L}^{b,U,m} \) forces

\[
\varepsilon^m_{\phi'} = ((I_{\mathcal{P}}^{U,m-1}/I_{\mathcal{P}}^{U,m})(\hat{\xi}^{-1}))(\varepsilon^m_\phi)
\]

4.10.49.
this proves that the correspondence \( \varepsilon^m \) sending \( \hat{\varphi} \) to \( \varepsilon^m_{\hat{\varphi}} \) is \( \mathcal{P} \)-stable and, in particular, that \( \varepsilon^m_{\hat{\varphi}} \) only depends on the corresponding \( \mathcal{F} \)-morphism \( \hat{\varphi} \); thus we set \( \varepsilon^m_{\hat{\varphi}} = \varepsilon^m_{\tilde{\varphi}} \).

On the other hand, considering the usual differential map

\[
d_1 : \mathbb{C}^1(\mathcal{P}, \mathcal{C}^{U,m-1}_{\mathcal{F}}) \rightarrow \mathbb{C}^2(\mathcal{P}, \mathcal{C}^{U,m-1}_{\mathcal{F}})
\]

we claim that \( d_1(\varepsilon^m) = 0 \); indeed, for a second \( \mathcal{P} \)-morphism \( \hat{\psi} : T \rightarrow R \) we get

\[
\lambda^m(\hat{\varphi}) \cdot \lambda^m(\hat{\psi}) = \lambda^m(\hat{\varphi}) \cdot \varepsilon^m_{\hat{\psi}} \cdot \lambda^m(\hat{\psi}) \cdot \varepsilon^m_{\hat{\varphi}} = \lambda^m(\hat{\varphi} \cdot \hat{\psi}) \cdot \varepsilon^m_{\hat{\varphi}} \cdot \varepsilon^m_{\hat{\psi}}
\]

and the divisibility of \( \mathcal{L}^{b,U,m} \) forces

\[
((\mathcal{C}^{U,m-1}_{\mathcal{F}}(\hat{\psi})))(\varepsilon^m_{\hat{\varphi}}) \cdot \varepsilon^m_{\hat{\psi}} = \varepsilon^m_{\hat{\varphi} \cdot \hat{\psi}}
\]

since \( \text{Ker}(\pi_{b,U,m}) \) is Abelian, with the additive notation we obtain

\[
0 = ((\mathcal{C}^{U,m-1}_{\mathcal{F}}(\hat{\psi})))(\varepsilon^m_{\hat{\varphi}}) - \varepsilon^m_{\hat{\varphi} \cdot \hat{\psi}} + \varepsilon^m_{\hat{\psi}}
\]

proving our claim.

At this point, it follows from equality 4.10.4 that \( \varepsilon^m = d_0(\hat{\nu}) \) for some stable element \( \hat{\nu} = (\hat{\nu}_Q)_{Q \in \mathcal{P}} \) in \( \mathbb{C}^0(\mathcal{P}, \mathcal{C}^{U,m-1}_{\mathcal{F}}) \); that is to say, with the notation above we get

\[
\varepsilon^m_{\hat{\varphi}} = ((\mathcal{C}^{U,m-1}_{\mathcal{F}}(\hat{\varphi}))(\hat{\nu}_Q) \cdot \hat{\nu}_R^{-1}
\]

hence, we obtain

\[
\lambda^m(\hat{\varphi}) = \lambda^m(\hat{\psi}) \cdot ((\mathcal{C}^{U,m-1}_{\mathcal{F}}(\hat{\varphi}))(\hat{\nu}_Q) \cdot \hat{\nu}_R^{-1} = \hat{\nu}_Q \cdot \lambda^m(\hat{\varphi}) \cdot \hat{\nu}_R^{-1}
\]

which amounts to saying that \( \lambda^m \) is naturally \( \mathcal{F} \)-isomorphic to \( \lambda^m \). We are done.

**Corollary 4.11.** There exists a unique perfect \( \mathcal{F} \)-locality \( \mathcal{P} \) up to natural \( \mathcal{F} \)-isomorphisms.

**Proof:** The existence has been proved in Corollary 4.6 above and the uniqueness is an easy consequence of Theorem 4.10.
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