Strong Banach Property (T) for Simple Algebraic Groups of Higher Rank

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Abstract

In [Laf08, Laf09], Vincent Lafforgue proved strong Banach property (T) for $SL_3$ over a non archimedean local field $F$. In this paper, we extend his results to $Sp_4$ and therefore to any connected almost $F$-simple algebraic group with $F$-split rank $\geq 2$. As applications, the family of expanders constructed from any lattice of such groups does not admit a uniform embedding in any Banach space of type $> 1$, and any affine isometric action of such groups in a Banach space of type $> 1$ has a fixed point.

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1 Introduction

In [Laf08, Laf09], Vincent Lafforgue proved strong Banach property (T) for $SL_3$ over a non archimedean local field $F$. In this paper, we

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extend his results to $Sp_4$ and therefore to any connected almost $F$-simple algebraic group with $F$-split rank $\geq 2$. As applications, the family of expanders constructed from any lattice of such groups does not admit a uniform embedding in any Banach space of type $> 1$, and any affine isometric action of such groups in a Banach space of type $> 1$ has a fixed point.

To announce the precise statements, we begin by recalling some definitions and notations from [Laf09].

**Definition 1.1** A class of Banach spaces $\mathcal{E}$ is of type $> 1$ if one of the following two equivalent conditions holds.

- i) There exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for any Banach space $E \in \mathcal{E}$, $E$ does not contain $\ell_1^n (1 + \varepsilon)$-isometrically;
- ii) There exist $p > 1$ (called the type) and $T \in \mathbb{R}_+$ such that for any $E \in \mathcal{E}$, $n \in \mathbb{N}^*$ and $x_1, ..., x_n \in E$, we have
  \[
  \left( \sum_{\varepsilon_i = \pm 1} \frac{n}{n} \sum_{i=1}^n \varepsilon_i x_i \right)^{\frac{1}{p}} \leq T \left( \sum_{i=1}^n \|x_i\|_E^p \right)^{\frac{1}{p}}.
  \]

**Remark 1.** We say that a class of Banach spaces $\mathcal{E}$ is given by a super-property, if any Banach space $F$ finitely representable in $\mathcal{E}$ (i.e. for any finite dimensional subspace $V \subset F$ and $\varepsilon > 0$ there exists $E \in \mathcal{E}$ which contains $V (1 + \varepsilon)$-isometrically) is an element of $\mathcal{E}$. It is clear that a class of type $> 1$ is given by a super-property.

**Remark 2.** If $\mathcal{E}$ is a class of Banach spaces given by a super-property and not a class of type $> 1$, then $\mathcal{E}$ contains $L_1(\mu)$, where $\mu$ is any $\sigma$-finite measure. In fact, by the classification of $\sigma$-finite measures it suffices to show that $\ell_1$ and $L_1(\{0, 1\}^\infty)$ are elements of $\mathcal{E}$. $L_1(\{0, 1\}^\infty)$ is finitely representable in $\ell_1$. By condition i) in the definition, $\ell_1$ is finitely representable in the class $\mathcal{E}$. Since $\mathcal{E}$ is given by a super-property, we conclude that $L_1(\{0, 1\}^\infty)$ and $\ell_1$ belong to $\mathcal{E}$.

Let $\mathcal{E}$ be a class of Banach spaces stable under complex conjugation and duality. Let $\ell$ be a length function of $G$. Denote $\mathcal{E}_{G, \ell}$ the set of isomorphic classes of representations $(E, \pi)$ of $G$ such that $E \in \mathcal{E}$ and

\[
\|\pi(g)\|_{\mathcal{L}(E)} \leq e^{\ell(g)}
\]

for any $g \in G$. Denote $\mathcal{C}_{\ell}^G(G)$ the completion of compactly supported functions $C_c(G)$ on $G$ by the norm

\[
\|f\|_{\mathcal{C}_{\ell}^G(G)} = \sup_{(E, \pi) \in \mathcal{E}_{G, \ell}} \|f(g)\pi(g)dg\|_{\mathcal{L}(E)}.
\]
Definition 1.2 We say that a locally compact group $G$ has strong Banach property (T) if for any class of Banach spaces $E$ of type $> 1$ stable complex conjugation and duality, and any length function $\ell$ over $G$, there exists $s_0 > 0$ such that the following holds. For any $C > 0$ and $s_0 > s > 0$, there exists a real self-adjoint idempotent element $p$ in $C^E_{C+d}(G)$, such that for any representation $(E, \pi) \in E_{G,C+d}$, the image of $\pi(p)$ consists of all $G$-invariant vectors in $E$.

Remark. In this definition, the condition of type $> 1$ cannot be replaced by a weaker condition given by a super-property because otherwise it would be satisfied only for compact groups. Indeed when $G$ is non compact, suppose that $E$ is a class of Banach spaces (stable under complex conjugation and duality) given by a super-property, and that there exists a real self-adjoint idempotent $p \in C_0^E(G)$ such that for any $(E, \pi) \in E_{G,0}$ we have $\pi(p) E = E^G$ (the space of $G$-invariant vectors), we show that $E$ is a class of Banach spaces of type $> 1$. If not, by remark 2 below definition 1.1 $E$ must contain $L^1(G)$. Note that for any $(E_1, \pi_1), (E_2, \pi_2) \in E_{G,0}$, any surjective morphism $E_1 \to E_2$ in the category $E_{G,0}$ induces a surjective morphism from $E_1^G = \pi_1(p)E_1$ to $E_2^G = \pi_2(p)E_2$. Now consider the morphism from $L^1(G)$ (with the left regular representation of $G$) to $C$ (with the trivial action of $G$) by integration on $G$. Since $G$ is non compact, there is no non zero $G$-invariant integrable function on $G$, therefore $L^1(G)^G = \{0\}$. However, $C^G = C$, and this is a contradiction to that $L^1(G)^G \to C^G$ must be a surjective morphism. Therefore, $E$ must be a class of type $> 1$ (see the second remark below definition 0.2 in [La09]).

Let $F$ be a non archimedean local field. The purpose of this paper is to prove the following theorem.

Theorem 1.3 Any connected almost $F$-simple algebraic group with $F$-split rank $\geq 2$ has strong Banach property (T).

Remark. This result cannot be extended to any almost $F$-simple algebraic group with $F$-split rank $= 1$ because they do not even have Kazhdan’s property (T).

The following definition corresponds to the special case of isometric actions.

Definition 1.4 We say that a locally compact group $G$ has Banach property (T) if for any class of Banach spaces $E$ of type $> 1$ stable under complex conjugation and duality, there exists a real self-adjoint idempotent element $p$ in $C_0^E(G)$, such that for any representation $(E, \pi) \in E_{G,0}$, the image of $\pi(p)$ consists of all $G$-invariant vectors in $E$. 
Remark. If a locally compact group $G$ has (strong) Banach property $(T)$ with $p \in C^\infty_c(G)$ being the corresponding idempotent, there always exist $p_n \in C_c(G)$ of integral 1, such that $p_n$ converges to $p$ in $C^\infty_c(G)$. In fact, let $\tilde{p}_n \in C_c(G)$ be any sequence such that $\tilde{p}_n \to p$. Let $s_n = \int_G \tilde{p}_n(g) dg$. Then

$$\|p - s_n p\|_{C^\infty_c(G)} = \|p^2 - \tilde{p}_n p\|_{C^\infty_c(G)} \leq \|p - \tilde{p}_n\|_{C^\infty_c(G)} \|p\|_{C^\infty_c(G)},$$

and hence $|1 - s_n| \leq \|p - \tilde{p}_n\|_{C^\infty_c(G)} \to 1$ when $n \to \infty$. Therefore, $s_n \neq 0$ for big enough $n$ and $p_n = \tilde{p}_n / s_n$ has integral 1 and tends to $p$.

With the remark above and the same argument as in theorem 5.4 in [Laf09], we obtain the following theorem on application to expanders. Let $\Gamma$ be a discrete group with Banach property $(T)$. Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a family of subgroups of $\Gamma$ such that $|\Gamma/\Gamma_i|$ tends to infinity. Let $S$ a finite symmetric system of generators of $\Gamma$ which contains 1. For any $i \geq 0$, $X_i = \Gamma/\Gamma_i$ is endowed with a graph structure associated to $S$ and we denote by $d_i$ the associated metric. As $\Gamma$ has the usual property $(T)$, $X_i$ forms a family of expanders. We say that the family $X_i$ is embedded uniformly in a Banach space $E$, if there exists a function $\rho : \mathbb{N} \to \mathbb{R}_+$ tends to infinity at infinity and 1-Lipschitz maps $f_i : X_i \to E$ such that

$$\|f_i(x) - f_i(y)\|_E \geq \rho(d_i(x, y))$$

for any $i \in \mathbb{N}$ and $x, y \in X_i$.

**Theorem 1.5** Let $\Gamma$ be any discrete group with Banach property $(T)$. Then the family of expanders $(X_i, d_i)$ constructed above does not admit a uniform embedding in any Banach space of type $> 1$.

Since strong Banach property clearly implies Banach property $(T)$, and Banach property $(T)$ is inherited by lattices (proposition 5.3 in [Laf09]), when $\Gamma$ is a lattice of a connected almost $F$-simple algebraic groups of $F$-split rank $\geq 2$, we see that the family of expanders constructed above does not admit a uniform embedding in any Banach space of type $> 1$.

We recall that it is still unknown whether or not such a family of expanders (or in fact any family of expanders) admits a uniform embedding in a Banach of finite cotype (see [Laf09], [Pis10] and [MN]).

We turn to application to fixed-point property. As a consequence of proposition 5.6 in [Laf09], we immediately obtain:
Proposition 1.6 Let $G$ be a connected almost $F$-simple algebraic group with $F$-split rank $\geq 2$. Then any affine isometric action of $G(F)$ on a Banach space of type $>1$ has a fixed point.

Remark. This result cannot be strengthened to affine isometric action on any Banach space in a class containing $L^1$-spaces. In fact, denote $d\mu$ the Haar measure on $G$, and $L^1(G)$ the space of functions $f \in L^1(G)$ such that $\int_G f(g) d\mu(g) = i, i = 0, 1$. Then $L^1(G)$ is an affine Banach space with $L^0_0(G)$ as the underlying Banach space. Let $G$ act on $L^1_0(G)$ by left translation. It is an affine isometric action of $G$ without fixed point, since $G$ is not compact.

This paper will be part of my PhD thesis in Université Paris Diderot-Paris 7. I would like to thank my thesis adviser Vincent Lafforgue for his encouragement and guidance, and very helpful discussions about this paper. I also thank Yanqi Qiu for the discussion of type of a Banach space.

Here is how the paper is organized. In section 2, we review the theorem of strong Banach property (T) for $SL_3$ in [Laf09] and announce the corresponding theorem 2.3 for $Sp_4$. In section 3, we prove theorem 2.3 when $\text{char}(F) \neq 2$ by constructing matrices for $Sp_4$ and adapting the arguments in [Laf09]. In section 4, we prove theorem 2.3 when $\text{char}(F) = 2$ by constructing new matrices for the local estimate of the move $(0,2)$ and establishing the existence of two limits in the spherical proposition. In section 5, we adapt a well known argument [DK, Vas, Wang] and extend the results of $SL_3$ and $Sp_4$ to any almost $F$-simple algebraic groups with $F$-split rank $\geq 2$.

2 Strong Banach property (T) for $Sp_4(F)$

Let $\mathcal{E}$ be any class of Banach spaces of type $>1$, stable under complex conjugation and duality. Let $F$ be a non archimedean local field, $\mathcal{O}$ the ring of integers of $F$, $\pi$ one of its uniformizer, $F$ the residue field, and $q$ the cardinality of $F$. The following proposition from [Laf09] (corollary 2.3) introduces parameters $\alpha > 0$ and $h \in \mathbb{N}^*$ for the class $\mathcal{E}$.

Proposition 2.1 There exist $\alpha > 0$ and $h \in \mathbb{N}^*$ such that for any $E \in \mathcal{E}$ we have

$$\|T_{\mathcal{O}/\pi^h \mathcal{O}} \otimes 1_E\| \leq e^{-\alpha},$$

where $T_{\mathcal{O}/\pi^h \mathcal{O}} \otimes 1_E \in \mathcal{L}(\ell^2(\mathcal{O}/\pi^h \mathcal{O}, E), \ell^2(\mathcal{O}/\pi^h \mathcal{O}, E))$ is defined by

$$(T_{\mathcal{O}/\pi^h \mathcal{O}} \otimes 1_E)(f)(\chi) = \sum_{a \in \mathcal{O}/\pi^h \mathcal{O}} \chi(a)f(a),$$
for any \( \chi \in \hat{O}/\pi^h O \) and \( f \in \ell^2(O/\pi^h O, E) \).

It is proved in \cite{Laf09} that \( SL_3(F) \) has strong Banach property (T).

\textbf{Theorem 2.2} \textbf{(Theorem 4.1 of \cite{Laf09})} Let \( G = SL_3(F) \), and \( \ell \) be the length function on \( G \) defined by

\[
\ell\left( k (\pi^{-i-j} \begin{pmatrix} \pi^{-i-j} & \pi^{-j} & \pi^j & \pi^i \\ \pi^j & \pi^i & \pi^{-j} & \pi^{-i-j} \\ \pi^i & \pi^{-i-j} & \pi^{-j} & \pi^{-i-j} \\ \pi^{-i-j} & \pi^{-j} & \pi^j & \pi^i \end{pmatrix} k') \right) = i + j,
\]

for any \( k, k' \in SL_3(O) \) and \( i, j \geq 0 \) with \( i - j \in 3\mathbb{Z} \). Let \( \beta \in [0, \frac{\alpha}{3\pi h}) \).

There exist \( t, C' > 0 \) such that for any \( C \in \mathbb{R}_+ \), there exists a real and self-adjoint idempotent element \( p \in C^\ell_{E+\beta\ell}(G) \) such that

- (i) for any representation \( (E, \pi) \in E_{G, C+\beta\ell} \), the image of \( \pi(p) \) is the subspace of \( E \) consisting of all \( G \)-invariant vectors,
- (ii) there exists a sequence \( p_n \in C_c\ell(G) \), such that \( \int_G |p_n(g)| dg \leq 1 \), \( p_n \) has support in \( \{ g \in G, \ell(g) \leq n \} \), and

\[
\|p - p_n\|_{C^\ell_{E+\beta\ell}(G)} \leq C'e^{2C - tn}.
\]

Now we turn to \( Sp_4 \). Let \( G = Sp_4(F) \), which is the group of \( 4 \times 4 \) matrices \( g \) over \( F \) such that \( ^tg J g = J \) where \( J \) is the skew-symmetric matrix,

\[
J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \( K = Sp_4(O) \) (i.e. the subgroup in \( Sp_4(F) \) whose matrix elements are in \( O \)). For any \( (i, j) \in \mathbb{Z}^2 \) let

\[
D(i, j) = \begin{pmatrix}
\pi^{-i} & \pi^{-j} & \pi^j & \pi^i \\
\pi^{-j} & \pi^i & \pi^{-j} & \pi^{-i}
\end{pmatrix}.
\]

By \( \|g\| \) we denote the norm of the operator \( g \in \text{End}(F^4) \) w.r.t. the standard norm on \( F^4 \), i.e. \( \|g\| = \max_{1 \leq \alpha, \beta \leq 4} |g_{\alpha\beta}| \). Similarly, denote \( \|\Lambda^2 g\| \) the biggest norm of all \( 2 \times 2 \) minors of \( g \in G \), which is the norm of \( \Lambda^2 \in \text{End}(\Lambda^2 F^4) \) w.r.t. the standard norm on \( \Lambda^2 F^4 \). Let \( \Lambda = \{(i, j) \in \mathbb{N}^2, i \geq j \} \). Any element in \( G \) has the form \( kD(i, j)k' \) for some \( (i, j) \in \Lambda \) and \( k, k' \in K \). For such a \( g = kD(i, j)k' \in G \), we have \( \|g\| = q^i \) and \( \|\Lambda^2 g\| = q^{i+j} \), and this gives a bijection from \( K \backslash G / K \)
to $\Lambda$ by $g \mapsto (i,j)$, which is the inverse of $(i,j) \mapsto KD(i,j)K$. Let $\ell$ be the length function of $G$ defined by $\ell(kD(i,j)k') = i+j$, for any $k,k' \in K$ and $(i,j) \in \Lambda$.

We will prove the following theorem with the argument used in [Laf09] for the proof of theorem 2.2 (note that the statement is the same except for the range of $\beta$).

**Theorem 2.3** Let $\alpha$ and $h$ be as in proposition 2.1, and $\beta \in [0, \frac{\alpha}{4h})$. There exist $t, C' > 0$ such that for any $C \in \mathbb{R}_+$, there exists a real and self-adjoint idempotent element $p \in C^E + \beta \ell(G)$ such that

1. for any representation $(E, \pi) \in \mathcal{E}_{G, C+\beta\ell}$, the image of $\pi(p)$ is the subspace of $E$ consisting of all $G$-invariant vectors,

2. there exists a sequence $p_n \in C_c(G)$, such that $\int_G |p_n(g)| dg \leq 1$, $p_n$ has support in $\{g \in G, \ell(g) \leq n\}$, and

$$
\|p - p_n\|_{C^E + \beta \ell(G)} \leq C' e^{2C-tn}.
$$

3 Proof of theorem [2.3] when char$(F) \neq 2$

This section is dedicated to the proof of theorem 2.3 when the characteristic of $F$ is different from 2. We will first reduce the theorem to two propositions on matrix coefficients, and then prove them by a zig-zag argument in the Weyl chamber with two local estimates of the matrix coefficients.

Most of the claims in this section are only true when char$(F) \neq 2$, but some are still valid in characteristic 2 and will be used in the next section for the proof in characteristic 2.

When char$F \neq 2$, we denote $v_0$ the valuation of 2 in $\mathcal{O}$. For any $a \in \mathbb{R}$, denote $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the biggest (resp. smallest) integer $\leq a$ (resp. $\geq a$).

Let $(E, \pi)$ be any continuous representation of $G$ of a Banach space $E$, $(V, \tau)$ any irreducible unitary representation of $K$. For fixed $\xi \in E$ and $\eta \in V \otimes E^*$, we denote $c(g) = \langle \eta, \pi(g)\xi \rangle \in V$ for any $g \in G$. By abuse of notation we write

$$
c(i,j) = \langle \eta, \pi(D(i,j))\xi \rangle.
$$

The following is the proposition on spherical matrix coefficients, which will be used to construct the idempotent element $p$ in theorem 2.3.
Proposition 3.1 Suppose that \( \text{char}(F) \neq 2 \). Let \( \alpha \) be as in proposition 2.1. \( \beta \in [0, \frac{n}{2}] \). There exists \( C' > 0 \), such that the following holds. Let \( C \in \mathbb{R}^+_* \), \((E, \pi)\) any element in \( \mathcal{E}_{G,C+\beta E} \), and \( \xi \in \mathbb{E}, \eta \in \mathbb{E}^* \) any \( K \)-invariant vectors of norm 1. There exists \( c_\infty \in \mathbb{C} \), such that for any \( i \geq j \geq 0 \),

\[
|c(i, j) - c_\infty| \leq C'e^{2C' - \left(\frac{\alpha}{2} - 2\beta\right)i}.
\]

Next we turn to the proposition on non spherical matrix coefficients.

Proposition 3.2 Suppose that \( \text{char}(F) \neq 2 \). Let \( \alpha \) be as in proposition 2.1. \( \beta \in [0, \frac{n}{2}] \), and \((V, \tau)\) a non trivial irreducible unitary representation of \( K \). There exists \( C' > 0 \), such that the following holds. Let \( C \in \mathbb{R}^+_* \), \((E, \pi)\) any element in \( \mathcal{E}_{G,C+\beta E} \), and \( \xi \in \mathbb{E}, \eta \in \mathbb{E} \) any \( K \)-invariant vectors of norm 1. We have for any \( i \geq j \geq 0 \),

\[
\|\xi(\eta, j)\|_V \leq C'e^{2C' - \left(\frac{\alpha}{2} - 2\beta\right)i}.
\]

Proof of theorem 2.3 when \( \text{char}(F) \neq 2 \) assuming proposition 3.1 and 3.2 Let \( P_g = e_K e_{g\pi} \), where \( e_K = \int_K e_k dk \) and \( dk \) is the Haar measure on \( K \) such that \( K \) has volume 1. As a consequence of proposition 3.1 we see that the limit \( p = \lim_{\ell(g) \to \infty} P_g \) exists in \( C^\infty_{G+\beta E}(\mathbb{G}) \). It is a real and self-adjoint element because \( P_g = P_g \), and \( P_g' = P_{g^{-1}} \). Moreover for any \( k \in K \) and \( g, g' \in G \) we have \( \ell(\eta gk) \geq \ell(g') - \ell(g^{-1}) \), which gives

\[
e_K e_{g\pi} = \lim_{\ell(g') \to \infty} e_K \int_K P_{gkg} dke_K = p,
\]

and therefore \( p^2 = p \).

On the other hand, for any non trivial irreducible representation \((V, \pi)\) of \( K \), denote \( e_K^V = n \int_K \text{Tr}(\pi(k)) e_k dk \in C^\infty_{G+\beta E}(\mathbb{G}) \), where \( n = \text{dim} V \). For any \((E, \pi) \in \mathcal{E}_{G,C+\beta E} \), denote \( \pi^* : G \to \mathcal{L}(E^*) \) the contragredient representation of \( \pi \), i.e. \( \pi^*(g) = \overline{\pi(g^{-1})} \), then \( \pi^*(e_K^V)E^* \) is the subspace of vectors in \( E^* \) whose \( K \)-type is \( V \). For any \( \xi \in \pi^*(e_K^V)E^* \) there exist \( K \)-invariant vectors \( \eta_i \in \mathbb{E}^* \) and vectors \( v_i \in \mathbb{E}, 1 \leq i \leq n \), such that \( \xi = \sum_{i=1}^n \langle \eta_i, v_i \rangle \). By applying proposition 3.2 to \( V^* \) and \( E \) we have \( e_K^V e_{g\pi} \to 0 \) in \( C^\infty_{G+\beta E}(\mathbb{G}) \) when \( \ell(g) \to \infty \), and therefore \( e_K^V e_{g\pi} = 0 \).

For any \((E, \pi) \in \mathcal{E}_{G,C+\beta E}, \) \( x \in \mathbb{E} \) and any \( K \)-finite vector \( y \in E^* \), we have

\[
\langle y, \pi(e_{g\pi})x \rangle = \sum_{\text{finitely many } V} \langle \pi^*(e_K^V)y, \pi(e_{g\pi})x \rangle
\]

\[
= \sum_{\text{finitely many } V} \langle y, \pi(e_K^V e_{g\pi})x \rangle = \langle y, \pi(e_{K e_{g\pi}})x \rangle = \langle y, \pi(p)x \rangle.
\]
Since the linear space of $K$-finite vectors is dense in $E^*$, we have

$$\pi(e_0 p) = \pi(p),$$

and therefore $\pi(p)E$ is the subspace of $G$-invariant vectors in $E$.

Finally we complete the proof by taking $p_n = P_{D(n,0)}$ and $t = \frac{\alpha}{h} - 2\beta$. □

Now we turn to the proof of proposition 3.1 on spherical matrix coefficients, which is based on two local estimates on spherical matrix coefficients corresponding to the move $(0,1)$ and $(1,-1)$ in the Weyl chamber.

**Lemma 3.3** Suppose $\text{char}(F) \neq 2$. Let $\alpha$ be as in proposition 2.1. Let $\beta \in [0, \frac{\alpha^2}{2h})$. Then there exists $C' > 0$, such that for any $C \in \mathbb{R}^*_+$, any $(E, \pi) \in \mathcal{E}_{G, C + \beta l}$, and any $K$-invariant vectors $\xi \in E, \eta \in E^*$ of norm 1, and any $(i, j) \in \Lambda$ with $i - j \geq v_0 + 1$, we have

$$|c(i, j) - c(i, j + 1)| \leq C' e^{2C(\frac{\alpha}{2h} - 2\beta) i} e^{\frac{2\alpha}{h} j},$$

where $C'$ is a constant depending on $q, h, v_0, \alpha, \beta$.

**Lemma 3.4** Let $F$ be of any characteristic. Let $\alpha$ be as in proposition 2.1 and $\beta \in [0, \frac{\alpha^2}{2h})$. Then there exists $C' > 0$, such that for any $C \in \mathbb{R}^*_+$, any $(E, \pi) \in \mathcal{E}_{G, C + \beta l}$, and any $K$-invariant vectors $\xi \in E, \eta \in E^*$ of norm 1, and $(i, j) \in \Lambda$ with $j \geq 2$, we have

$$|c(i, j) - c(i + 1, j - 1)| \leq C' e^{2C + \beta i (\frac{2\alpha}{h} - \beta) j}.$$

**Proof of proposition 3.1 assuming lemma 3.3 and 3.4:** We adopt the zig-zag argument from [La08] to $Sp_4$. We put

$$S_\alpha = \{(i, j) \in \Lambda | 0 \leq i - 2j \leq \alpha\}.$$

First we move any $(i, j) \in \Lambda$ to the strip $S_3$. Then we show that we can move any $(i, j) \in S_3$ to the line $i = 2j$ using the moves inside $S_4$, and then we move $(i, j)$ to infinity along this line as illustrated below.
Precisely, when \( i \geq 2j \geq 0 \), we have \((i, \lfloor i/2 \rfloor) \in S_2 \subset S_3\) and

\[
|c(i, j) - c(i, \lfloor i/2 \rfloor)| \\
\leq C'e^{2C-2(\frac{n}{3}-\beta)i+\frac{\alpha h}{2}j + \ldots + C'e^{2C-2(\frac{n}{3}-\beta)i+\frac{\alpha h}{2}((1/2)-1)}} \\
\leq C'e^{2C-(\frac{n}{3}-2\beta)i}.
\] (1)

When \( 2j \geq i \geq 0 \), we have \((i + \lfloor \frac{2j-i}{3} \rfloor, j - \lfloor \frac{2j-i}{3} \rfloor) \in S_3\), and

\[
|c(i, j) - c(i + \lfloor \frac{2j-i}{3} \rfloor, j - \lfloor \frac{2j-i}{3} \rfloor)| \\
\leq C'e^{2C-(\frac{n}{3}-\beta)i+\beta j + \ldots + C'e^{2C-(\frac{n}{3}-\beta)i+\lfloor \frac{2j-i}{3} \rfloor-1)+\beta(j-\lfloor \frac{2j-i}{3} \rfloor+1)} \\
\leq C'e^{2C-(\frac{n}{3}-\beta)(i+j)}.
\] (2)

For any \((i, j) \in S_3\), if \( i \in 2N + k, k \in \{0, 1\} \) then

\[
|c(i, j) - c(i + k, (i + k)/2)| \leq C'e^{2C-(\frac{n}{3}-2\beta)i}.
\] (3)

In fact, by lemmas 3.3 and 3.4 when \((i, j) \in S_4\) we have

\[
\max\{|c(i, j) - c(i, j + 1)|, |c(i, j) - c(i + 1, j - 1)|\} \leq C'e^{2C-(\frac{n}{3}-2\beta)i}.
\]

When \( i \in 2N \) and \((i, j) \in S_3\), we get inequality (3) by considering the move \((i, j) \mapsto (i, i/2)\). When \( i \in 2N + 1 \) and \((i, j) \in S_3\), there exists \( k \in \{0, 1\} \), such that \((i + 1, j + k - 1) \in S_4\). Therefore, we obtain inequality (3) by considering the following moves inside \(S_4\) : \((i, j) \mapsto (i, j + k) \mapsto (i + 1, j + k - 1) \mapsto (i + 1, (i + 1)/2)\).

Combining inequalities (1), (2) and (3) we obtain: when \( i \geq 2j \geq 0 \), and \( i \in 2N + k, k \in \{0, 1\} \),

\[
|c(i, j) - c(i + k, (i + k)/2)| \leq C'e^{2C-(\frac{n}{3}-2\beta)i};
\] (4)
when $2j \geq i \geq j \geq 0$, there exists $k \in \{0, 1, 2\}$ such that

$$|c(i,j) - c\left(\frac{2}{3}(i + j)\right) + k, \frac{1}{2}\left(\frac{2}{3}(i + j)\right) + k)| \leq C\varepsilon^{2C - \left(\frac{2}{3} - 2\beta\right)j}. \quad (5)$$

Finally for any $j \geq 0$, we have

$$|c(2j,j) - c(2j + 2, j + 1)| \leq C\varepsilon^{2C - 2\left(\frac{2}{3} - 2\beta\right)j}.$$ 

Proposition 3.1 is then proved. \qed

It remains to prove lemmas 3.3 and 3.4. To prove these two lemmas, we use the following lemma in [La09] which is a variant of fast Fourier transform.

**Lemma 3.5** (Lemma 4.4 in [La09]) Let $\chi : F \to \mathbb{C}$ be a non trivial character. Let $h \in \mathbb{N}^\ast, \alpha \in \mathbb{R}^+ \times \mathbb{N}^\ast$. Let $E$ be a Banach space such that $\|T_{O/\pi^nO} \otimes 1_E\| \leq e^{-\alpha}$, and let $(\xi_{x,y})_{x,y \in O/\pi^nO}$ be a family of vectors of $E$. Then

$$\mathbb{E}_{a,b \in O/\pi^nO} \mathbb{E}_{x \in O/\pi^nO, \varepsilon \in F} \chi(\varepsilon) \xi_{x,ax+b+\pi^n-1}^{\varepsilon} \leq q^{2h-2}e^{-2\left(\frac{3}{2} - 1\right)\alpha} \mathbb{E}_{x,y \in O/\pi^nO} \|\xi_{x,y}\|^2.$$

**Proof of lemma 3.3**: Denote $m = \lfloor \frac{i + j}{2} \rfloor$, and $n_1 = 2m - 2j - \nu_0$. Let $x, y, a, b \in O/\pi^{n_1}O$, and let $\sigma : O/\pi^{n_1}O \to O$ be a section. Let $\beta(a, b)^{-1}, \alpha(x, y)$ be the elements in $G$ defined as follows,

$$\beta(a, b)^{-1} = \begin{pmatrix} \pi^m & 0 \\ \pi^{i-m+j} & 1 \\ \pi^{-i+m-j} & 1 \\ \pi^{-m} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 \\ \sigma(a)^2 - 2\sigma(b) & 0 & 1 & 0 \\ \sigma(a) & 0 & 1 & 0 \\ \sigma(a) \end{pmatrix},$$

$$\alpha(x, y) = \begin{pmatrix} \pi^{-m+j} & 0 \\ \pi^{-m+j} & 1 \\ \pi^{-m+j} & 0 \\ \pi^{-m+j} & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi^{m-j} & 0 \\ \pi^{m-j} & 1 \\ \pi^{m-j} & 0 \\ \pi^{m-j} & 1 \end{pmatrix}.$$

Then

$$\beta(a, b)^{-1}\alpha(x, y) = \begin{pmatrix} \pi^m & 0 \\ \pi^{i-m+j} & 1 \\ \pi^{-i+m-j} & 1 \\ \pi^{-m} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 & 1 \\ \sigma(a) + \sigma(x) & 1 & 1 & 0 \\ \sigma(a) + \sigma(x) & 1 & 1 & 0 \\ \sigma(a) + \sigma(x) & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \pi^{-m+j} & 0 \\ \pi^{-m+j} & 1 \\ \pi^{-m+j} & 0 \\ \pi^{-m+j} & 1 \end{pmatrix}.$$
Recall from the second section that for any $g \in KD(k,l)K$, $q^k$ is the biggest norm of all matrix elements in $g$, and $q^{k+l}$ is the biggest norm of all $2 \times 2$ minors of $g$. It is easy to see that

$$\| \Lambda^2(\beta(a,b)) \| = q^{i+j}, \| \Lambda^2(\alpha(x,y)) \| = q^{2m-2j},$$

and

$$\| \beta(a,b)^{-1}\alpha(x,y) \| = q^i.$$

On the other hand, we calculate the minor of rows 3, 4 and columns 1, 2,

$$\det \left( \begin{pmatrix} \pi^{-i+m-j} & \pi^{-m} \\ \pi^{-m+j} & \pi^{-m+j} \end{pmatrix} \begin{pmatrix} \sigma(a) + \sigma(x) \\ \sigma(a)^2 - 2\sigma(b) + \sigma(x)^2 + 2\sigma(y) \\ \sigma(a) + \sigma(x) \end{pmatrix} \right) \times -2\pi^{-i-2m+j}(\sigma(y) - \sigma(a)\sigma(x) - \sigma(b)) \times \frac{1}{q^m}.$$

Since the norm of the minor of rows 3, 4 and columns 2, 4 is $q^{i+j}$, we have

$$\| \Lambda^2(\beta(a,b)^{-1}\alpha(x,y)) \| = \max(q^{i+2m-j-v}, q^{i+j}),$$

where $v \in \{0, 1, \ldots, 2m - 2j\}$ is the valuation of $2(y - ax - b) \in O/\pi^{2m-2j}O$. Let $y = ax + b + \pi^{n_1-1}x$, where $\varepsilon \in \mathbb{F}$. When $\varepsilon = 0$, we have $v = 2m - 2j$ and

$$\beta(a,b)^{-1}\alpha(x,y) \in KD(i,j)K.$$

When $\varepsilon \in \mathbb{F}^*$ we have $v = 2(m - j) - 1$, and then

$$\beta(a,b)^{-1}\alpha(x,y) \in KD(i,j+1)K.$$

Let $\chi : \mathbb{F} \to \mathbb{C}^*$ be a non trivial character. By Cauchy-Schwarz inequality and lemma 3.3 we have

$$|c(i,j) - c(i,j+1)|$$

$$= q \left| \sum_{a,b \in O/\pi^{n_1}O, c \in \mathbb{F}} \chi(c) \pi^{(\beta(a,b))} \eta(x, ax + b + \pi^{n_1-1}x) |\xi| \right|$$

$$\leq q \left\| \sum_{a,b \in O/\pi^{n_1}O} \left\| \pi^{(\beta(a,b))} \eta \right\| \pi^{(\alpha(x, ax + b + \pi^{n_1-1}x))} |\xi| \right\|$$

$$\leq q^{C+\beta(i+j)} \cdot q^{2h-2} \cdot e^{-2(\frac{m}{2} - 1)\alpha} \cdot e^{C + 2\beta(m-j)} \cdot e^{2C - (2\beta - 2\beta)i + \frac{m}{2}},$$

and the lemma follows immediately. \qed
Proof of lemma 3.4 Let \( x, y, a, b \in O/\pi^{j-1}O \), and let \( \sigma : O/\pi^{j-1}O \to O \) be a section. Define

\[
\beta(a, b)^{-1} = \begin{pmatrix} \pi^i & 1 \\ 1 & 1 \\ \pi^{-i} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 + \pi \sigma(a) \\ 0 \\ 0 \\ 1 \\ -\pi \sigma(b) \\ 0 \\ -1 - \pi \sigma(a) \end{pmatrix} \in G,
\]

\[
\alpha(x, y) = \begin{pmatrix} 1 \\ \sigma(x) \\ \pi \sigma(y) + \sigma(x) \\ 1 \\ \sigma(x) \end{pmatrix} \cdot \begin{pmatrix} \pi^{-j} \\ 0 \\ 1 \\ \sigma(x) \\ -1 - \pi \sigma(a) \end{pmatrix} \in G.
\]

Then we have

\[
\beta(a, b)^{-1} \alpha(x, y) = \begin{pmatrix} \pi^i \\ 1 \\ \pi^{-i} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 + \pi \sigma(a) \\ \sigma(x) \\ \pi (\sigma(y) - \sigma(a) \sigma(x) - \sigma(b)) \\ \sigma(x) \sigma(x) - 1 - \pi \sigma(a) \end{pmatrix} \times \begin{pmatrix} \pi^{-j} \\ 1 \\ \pi^j \end{pmatrix}.
\]

Firstly, we see that

\[
\| \Lambda^2(\beta(a, b)) \| = q^i, \| \Lambda^2(\alpha(x, y)) \| = q^j,
\]

and

\[
\| \Lambda^2(\beta(a, b)^{-1} \alpha(x, y)) \| = q^{i+j},
\]

which is the norm of the determinant of the submatrix of rows 2, 4 and columns 1, 3. Denote the valuation of \( y - ax - b \in O/\pi^{j-1}O \) by \( v \in \{0, 1, \ldots, j-1\} \), and we have

\[
\| \beta(a, b)^{-1} \alpha(x, y) \| = \max(q^i, q^{i+j-v-1}).
\]

Let

\[
y = ax + b + \pi^{j-2} \varepsilon, \varepsilon \in F.
\]

When \( \varepsilon = 0 \), we see that \( v = j - 1 \) and

\[
\beta(a, b)^{-1} \alpha(x, ax + b) \in KD(i, j)K.
\]

When \( \varepsilon \in F^* \) we have \( v = j - 2 \), and therefore

\[
\beta(a, b)^{-1} \alpha(x, ax + b + \pi^{j-2} \varepsilon) \in KD(i + 1, j - 1)K.
\]
Let \( \chi : F \rightarrow \mathbb{C}^* \) be a non trivial character. By the same estimates as in the end of the proof of lemma 3.3 (\( n_1 \) replaced by \( j - 1 \)), we have

\[
|c(i, j) - c(i + 1, j - 1)| \\
\leq \varepsilon^{C + \beta j} \cdot q^{2h - 1} \cdot e^{-2(\frac{i}{n} - 1)\alpha} \cdot e^{C + \beta i} \\
= q^{2h - 2} \cdot e^{2(\frac{j}{n} + 1)\alpha} \cdot e^{2C + \beta i - (\frac{2n}{\pi} - \beta)j}.
\]

\( \square \)

As for proposition 3.2, we need two similar lemmas as follows for its proof.

Lemma 3.6 Suppose \( \text{char}(F) \neq 2 \). Let \( \alpha \) be as in proposition 2.1 \( \beta \in [0, \frac{\pi}{2}) \), and \((V, \tau)\) a non trivial irreducible unitary representation of \( K \) which factorizes through \( \text{Sp}_4(O/\pi^kO) \) for \( k \geq 1 \). There exists \( C' > 0 \), such that the following holds. Let \( C \in \mathbb{R}_+^* \), \( \lambda \) any element in \( \mathcal{E}_{G, C + \beta \tau} \), and \( \xi \in E, \eta \in V \otimes E^* \) any \( K \)-invariant vectors of norm 1. Then for any \((i, j) \in \Lambda \) with \( i - j \geq 2k + v_0 \), we have

\[
\|c(i, j) - c(i + 1, j - 1)\|_V \leq C' e^{2C - (\frac{2n}{\pi} - \beta)j}.
\]

Lemma 3.7 Let \( F \) be of any characteristic. Let \( \alpha \) be as in proposition 2.1 \( \beta \in [0, \frac{\pi}{2}) \), and \((V, \tau)\) a non trivial irreducible unitary representation of \( K \) which factorizes through \( \text{Sp}_4(O/\pi^kO) \) for \( k \geq 1 \). There exists \( C' > 0 \), such that the following holds. Let \( C \in \mathbb{R}_+^* \), \( \lambda \) any element in \( \mathcal{E}_{G, C + \beta \tau} \), and \( \xi \in E, \eta \in V \otimes E^* \) any \( K \)-invariant vectors of norm 1. Then for any \((i, j) \in \mathbb{Z}^2 \) with \( i + 1 \geq j \geq 2k + 2 \), we have

\[
\|c(i, j) - c(i + 1, j - 1)\|_V \leq C' e^{2C + \beta i - (\frac{2n}{\pi} - \beta)j}.
\]

In particular,

\[
\|c(i - 1, j) - c(i, j - 1)\|_V \leq C' e^{2C - (\frac{2n}{\pi} - \beta)j}.
\]

Lemma 3.8 Let \( h, \alpha, n, E \) as in lemma 3.7. Let \( k \in \{0, \ldots, \lfloor n/2 \rfloor \} \), \( \varepsilon_0 \in \mathbb{F}^* \), and let \( (\xi_{x, y})_{x \in \pi^kO/\pi^nO, y \in \pi^{2k}O/\pi^nO} \) be a family of vectors of \( E \). Then there exists a constant \( C_2 \) depending only on \( q \), such that

\[
\mathbb{E}_{\alpha \in \pi^kO/\pi^nO, \beta \in \pi^{2k}O/\pi^nO} \left\| \mathbb{E}_{x \in \pi^kO/\pi^nO} \xi_{x, ax + b + \pi^{n-1}x_0} - \mathbb{E}_{x \in \pi^kO/\pi^nO} \xi_{x, ax + b} \right\|^2 \\
\leq C_2 q^{2h - 2} e^{-2(\frac{n - 2k}{n} - 1)\alpha} \mathbb{E}_{x \in \pi^kO/\pi^nO, \gamma \in \pi^{2k}O/\pi^nO} \|\xi_{x, y}\|^2.
\]

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Proof: When $k = 0$, let $f$ be the function on $\mathbb{F}$ defined by $f(\varepsilon_0) = q$, $f(0) = -q$, and zero elsewhere. The left hand side of the inequality is equal to

$$\mathbb{E}_{a, b \in \mathcal{O}/\pi^n \mathcal{O}} \mathbb{E}_{\mathcal{F} \in \mathcal{O}} f(\varepsilon) \xi_{ax+b+\pi^{n-1}\varepsilon}^2.$$

Write $f = \sum_{\chi \in \mathcal{F}, \chi \neq 1} f_\chi \chi$ with $f_\chi \in \mathbb{C}$, then by the triangular inequality and and lemma 3.5, the left hand side is equal to

$$\sum_{\chi \in \mathcal{F}, \chi \neq 1} f_\chi \chi \mathbb{E}_{a, b \in \mathcal{O}/\pi^n \mathcal{O}} \mathbb{E}_{\mathcal{F} \in \mathcal{O}} \chi(\varepsilon) \xi_{ax+b+\pi^{n-1}\varepsilon}^2 \leq C_2 \max_{\chi \in \mathcal{F}, \chi \neq 1} \mathbb{E}_{a, b \in \mathcal{O}/\pi^n \mathcal{O}} \mathbb{E}_{\mathcal{F} \in \mathcal{O}} \chi(\varepsilon) \xi_{ax+b+\pi^{n-1}\varepsilon}^2 \leq C_2 q^{2h-2} e^{-2(\frac{\pi}{\alpha} - 1)} \mathbb{E}_{a, b \in \mathcal{O}/\pi^n \mathcal{O}} \Vert \xi_{xy} \Vert^2,$$

where $C_2 = (\sum_{\chi \in \mathcal{F}, \chi \neq 1} |f_\chi|)^2$.

In general, let $s : \mathcal{O}/\pi^n - k \mathcal{O} \to \mathcal{O}/\pi^n - k \mathcal{O}$ be a section, and for any $x_1, y_1 \in \mathcal{O}/\pi^n - k \mathcal{O}$ let

$$\xi'_{x_1, y_1} = z \mathbb{E}_{\mathcal{F} \in \mathcal{O}/\pi^n - k \mathcal{O}} \xi_{s(x_1)+z, \pi^k y_1}.$$

For any $a, x \in \pi^k \mathcal{O}/\pi^n \mathcal{O}$, the product $ax \in \mathcal{O}/\pi^n \mathcal{O}$ only depends on the images of $a, x$ in $\pi^k \mathcal{O}/\pi^n - k \mathcal{O}$. So the left hand side of the inequality is equal to

$$\mathbb{E}_{a_1, b_1 \in \mathcal{O}/\pi^n - k \mathcal{O}} \mathbb{E}_{x_1 \in \mathcal{O}/\pi^n - k \mathcal{O}} \left( \xi'_{x_1, a_1 x_1 + b_1 + \pi^{n-2k-1} \varepsilon_0} - \xi'_{x_1, a_1 x_1 + b_1} \right)^2.$$

By applying the lemma when $k = 0$ to $(\xi'_{x_1, y_1})_{x_1, y_1 \in \mathcal{O}/\pi^n - 2k \mathcal{O}}$ we get the inequality in the lemma with the same $C_2$. □

Proof of lemma 3.6

Let $m, x, y, a, b, \varepsilon, \sigma, \alpha(x, y), \beta(a, b)$ be as in the proof of lemma 3.3

We recall also from the proof that

$$\|\Lambda^2(\alpha(x, y))\| = q^{i+j}, \|\Lambda^2(\beta(a, b))\| = q^{2m-2j}.$$

Let $\varepsilon_0$ be image of $\pi^{m_0}/2$ in $\mathbb{F}$, and let

$$\varepsilon_1 = 2\pi^{-2m+2j+1}(\sigma(y) - \sigma(a)\sigma(x) - \sigma(b)) \in \mathcal{O}.$$

Recall that $y = ax + b + \pi^{2m-2j-m_0-1}\varepsilon$, we have $\varepsilon_1 \bmod \pi \mathcal{O} = \varepsilon_0^{-1}\varepsilon$.

Let $k_1$ be the element in $K$ defined by

$$k_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -\pi^{-2m+j}(\sigma(a) + \sigma(x)) & 1 \\
-1 & 0 & -\pi^{-2m+3j+1}(\sigma(a) + \sigma(x)) & \pi^{2j+1} \\
-\pi^{-2m+j}(\sigma(a) + \sigma(x)) & -1 & \pi^{2m+2j} & 0
\end{pmatrix}.$$
and let
\[ g_1 = k_1 \beta(a, b)^{-1} \alpha(x, y) \]
\[
= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -\pi^{i-2m+j} (\sigma(a) + \sigma(x)) & 1 \\
-1 & 0 & -\pi^{i-2m+3j+1} (\sigma(a) + \sigma(x)) & \pi^{2j+1} \\
-\pi^{i-2m+j} (\sigma(a) + \sigma(x)) & -1 & 0 & 0
\end{pmatrix} \times
\begin{pmatrix}
\pi^j & 0 & \pi^{i-2m+2j} & 0 \\
0 & \pi^j (\sigma(a) + \sigma(x)) & \pi^{-i} & \pi^{-i-2m-2j} \\
\pi^{-2m+j} (\sigma(a) + \sigma(x))^2 & \pi^{-j-1} \varepsilon_1 & \pi^{-2m+j} (\sigma(a) + \sigma(x)) & 0 \\
\pi^{-j-1} \varepsilon_1 & 0 & -\pi^{-j} (\sigma(a) + \sigma(x)) & \pi^{-j}
\end{pmatrix}
\]

When \( \varepsilon = 0 \), we have
\[
\begin{pmatrix}
\pi^i \\
\pi^j \\
\pi^{-j} \\
\pi^{-i}
\end{pmatrix} g_1 =
\begin{pmatrix}
(\sigma(a) + \sigma(x)) & 1 & \pi^{2m-2j} & 0 \\
\pi^{-1} \varepsilon_1 & 0 & -1 & \pi \\
\varepsilon_1 - 1 & 0 & -\pi (\sigma(a) + \sigma(x)) & \pi \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
which is an element in \( K \). When \( \varepsilon = \varepsilon_0 \), we have
\[
\begin{pmatrix}
\pi^i \\
\pi^{j+1} \\
\pi^{-j-1} \\
\pi^{-i}
\end{pmatrix} g_1 =
\begin{pmatrix}
(\sigma(a) + \sigma(x)) & 1 & \pi^{2m-2j} & 0 \\
\pi^{-1} (\varepsilon_1 - 1) & 0 & -1 & \pi \\
\varepsilon_1 - 1 & 0 & -\pi (\sigma(a) + \sigma(x)) & \pi \\
0 & 0 & 1 & 0
\end{pmatrix},
\]
which is also in \( K \). Denote \( \xi_{x,y} = \pi(\alpha(x, y)) \xi \), \( \eta_{a,b} = (\text{Id}_V \otimes \pi(\beta(a, b))) \eta \) and \( n_1 = 2(m - j) - v_0 \). Note that \( c(k'gk'') = \tau(k')c(g) \) for any \( k', k'' \in K, g \in G \), we then have
\[
\|c(i, j) - c(i, j + 1)\|_V = q\|_{\alpha, x \in \pi^k \mathcal{O} / \pi^{n_1} \mathcal{O}, \beta \in \pi^{2k} \mathcal{O} / \pi^{n_1} \mathcal{O}} \tau(k_1)\left(\langle \eta_{a,b}, \xi_{x,ax+b+\pi^{n_1-1} \varepsilon_0} \rangle - \langle \eta_{a,b}, \xi_{x,ax+b} \rangle\right)\|_V.
\]

\[ (6) \]

When \( i - j \geq k + v_0 \) and \( a, x \in \pi^k \mathcal{O} / \pi^{n_1} \mathcal{O} \), we have
\[
k_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \mod \pi^k \mathcal{O},
\]
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so (6) becomes
\[
q \left\| \sum_{a,b \in \pi^k \mathcal{O}/\pi^{n_1} \mathcal{O}, b \in \pi^{2k} \mathcal{O}/\pi^{n_1} \mathcal{O}} \left( \langle \eta_{a,b}, \xi_{a,ax+b} \rangle - \langle \eta_{a,b}, \xi_{ax+b} \rangle \right) \right\|_1.
\]
By Cauchy-Schwarz inequality and lemma 3.8 (when \( i - j \geq 2k + n_0 \)), it is less than
\[
q \sqrt{\sum_{a,b \in \pi^k \mathcal{O}/\pi^{n_1} \mathcal{O}, b \in \pi^{2k} \mathcal{O}/\pi^{n_1} \mathcal{O}} \mathbb{E} \left[ \left( \xi_{a,ax+b} \right)^2 \right]} \cdot \mathbb{E} \left[ \left( \mathbb{E} \left[ \xi_{a,ax+b}^2 \right] \right) \right]^{1/2} \leq q e^{C + \beta(i+j)} \cdot C_2 q^{2h_2} \cdot a^{-2(\frac{n_2-2k}{n}+1) \alpha} \cdot e^{C+2\beta(m-j)} \leq C_2 q^{2h_1} \cdot a^{2(\frac{n_2+2k}{n}+1) \alpha} \cdot e^{2C-(\frac{2n}{n}+2)i+\frac{2n_{ij}}{n}}.
\]
\[
\square
\]

**Proof of lemma 3.7.**

Let \( x, a, b, \varepsilon, \sigma, \alpha(x, y), \beta(a, b) \) be as in the proof of lemma 3.4. From the proof we have
\[
\| \Lambda^2 (\beta(a, b)) \| = q^i, \| \Lambda^2 (\alpha(x, y)) \| = q^j.
\]
Denote \( \varepsilon_1 = \pi^{-j+2}(\sigma(y) - \sigma(a)\sigma(x) - \sigma(b)) \in \mathcal{O}, \) and we have \( \varepsilon_1 \mod \pi \mathcal{O} = \varepsilon. \) Denote \( a_1 = 1 + \pi \sigma(a) \in \mathcal{O}. \) For any \( i + 1 \geq j \geq 1, \) let \( k_1 \) be the element in \( K \) defined by
\[
k_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\pi^{i-j+1}a_1 \\
0 & -\pi^{i-j-1}a_1^{-2}\varepsilon - a_1^{-1}\sigma(x) & 1 & \pi^{-1}a_1^{-1} \\
-1 & \pi^{i-j+1}a_1(1 - \varepsilon_1) & \pi^{-j+1}a_1 & \pi^{2i-j+1}
\end{pmatrix}.
\]
Denote
\[
g_1 = k_1 \beta(a, b)^{-1}\alpha(x, y)
\]
\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\pi^{i-j+1}a_1 \\
0 & -\pi^{i-j-1}a_1^{-2}\varepsilon - a_1^{-1}\sigma(x) & 1 & \pi^{-1}a_1^{-1} \\
-1 & \pi^{i-j+1}a_1(1 - \varepsilon_1) & \pi^{-j+1}a_1 & \pi^{2i-j+1}
\end{pmatrix} \times 
\begin{pmatrix}
\varepsilon^{-j}a_1 & 1 \\
\varepsilon^{-j}\sigma(x) & 0 & 1 \\
\varepsilon^{-1} & \varepsilon^{-1}\sigma(x) & -\varepsilon^{-1}a_1 & \varepsilon^{-i+j}
\end{pmatrix} = \begin{pmatrix}
\varepsilon^{-i-1}\varepsilon_1 & \varepsilon^{-i-1}a_1 & \varepsilon^{-i-j+1}a_1 & \varepsilon^{-i+j} \\
\varepsilon^{-i-1}a_1(1 - \varepsilon_1) & 1 - \varepsilon^{-i-j+1}a_1 & \varepsilon^{-j+1}a_1 & \varepsilon^{-i+j} \\
0 & -\varepsilon^{-i-j-1}a_1^{-2}\varepsilon_1 & 0 & \varepsilon^{-i} \\
0 & \varepsilon^{-i-1}a_1^{-1} & \varepsilon^{-i-1} & \varepsilon^{-i+1}
\end{pmatrix}
\]
and we have
When \( \varepsilon = 0 \), we have

\[
\left( \begin{array}{cccc}
\pi^i & \pi^j & \pi^{-j} & \pi^{-1} \\
\pi^{-1} \varepsilon_1 & \sigma(x) & -a_1 & \pi^j \\
a_1(1 - \varepsilon_1) & \pi^j - \pi a_1 \sigma(x) & -a_1^2 & -\pi^{-1} a_1 \sigma(x) \\
0 & -\pi^{-1} a_1^2 \varepsilon_1 & 0 & a_1^{-1} \\
0 & \pi^{-1} a_1^{-2} (1 - \varepsilon_1) & 0 & \pi \\
\end{array} \right) \in K.
\]

When \( \varepsilon = 1 \), we have

\[
\left( \begin{array}{cccc}
\pi^{i+1} & \pi^{j-1} & \pi^{-j+1} & \pi^{-i-1} \\
\pi^{-1} a_1 (1 - \varepsilon_1) & \sigma(x) & -a_1 & \pi^{j+1} \\
\pi^{-1} a_1^{-1} (1 - \varepsilon_1) & \pi^{j-1} - a_1 \sigma(x) & a_1^2 & -\pi^{-1} a_1 \\
0 & -\pi^{-1} a_1^2 \varepsilon_1 & 0 & \pi a_1^{-1} \\
0 & \pi^{-1} a_1^{-2} (1 - \varepsilon_1) & 0 & 1 \\
\end{array} \right) \in K
\]

When \( j \geq 2k + 2 \) and \( a, x \in \pi^k \mathcal{O}/\pi^{j-1} \mathcal{O} \), we have

\[
k_1 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -\pi^{i-j+1} \\
0 & 0 & 1 & 0 \\
-1 & 0 & \pi^{i-j+1} & 0 \\
\end{pmatrix} \mod \pi^k \mathcal{O}.
\]

By the same estimates (with \( n_1 \) replaced by \( j - 1 \) and \( \varepsilon_0 \) by 1 \( \in \mathbb{F}^* \)) as in the end of the proof of lemma 3.6 we have

\[
\|c(i, j) - c(i + 1, j - 1)\|_V \leq e^{C + \beta j} \cdot C_2 q^{2h-1} \cdot e^{-2(\frac{j-1+2k}{2})\alpha} \cdot e^{C + \beta i} = C_2 q^{2h-1} \cdot e^{2(\frac{j+2k}{2}+1)\alpha} \cdot e^{2C + \beta i - \frac{\alpha}{2} - \beta j}.
\]

Let \( K_1 \) be the subgroup of \( K \) consisting of elements of the form

\[
\begin{pmatrix}
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\end{pmatrix}
\]

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and $K_2$ consisting of elements of the form
\[
\begin{pmatrix}
1 & * & * \\
* & * & * \\
* & * & 1
\end{pmatrix},
\]
i.e.
\[
K_1 = \{ \begin{pmatrix} A & Q^t A^{-1} Q \\ Q^t A^{-1} Q & 1 \end{pmatrix} | A \in GL_2(\mathcal{O}) \},
\]
where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and
\[
K_2 = \{ \begin{pmatrix} 1 & B \\ B & 1 \end{pmatrix} | B \in SL_2(\mathcal{O}) \}.
\]

**Lemma 3.9** Let $F$ be of any characteristic. Then $K = (K_1K_2)^3$.

**Proof:** Denote $B$ the lower triangular matrices in $K$, and $W$ the Weyl group associated to $G = Sp_4(F)$. Denote
\[
w_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},
w_{32} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.
\]
The dihedral group $W$ (of order 8) is generated by $w_{21}$ and $w_{32}$, which are reflections w.r.t. the axes $x=y$ and $x=0$, respectively. Since $w_{21} \in K_1$ and $w_{32} \in K_2$ we obtain $W \subset (K_1K_2)^4$.

Denote for any $a \in \mathcal{O}$,
\[
\mu_{21}(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \\ 1 & 0 \\ -a & 1 \end{pmatrix},
\mu_{32}(a) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ a & 0 & 1 \end{pmatrix},
\mu_{31}(a) = \begin{pmatrix} 1 \\ a \\ 0 \\ 1 \\ a \\ 0 \\ 1 \\ 0 \\ a \\ 0 \\ 1 \\ 0 \end{pmatrix},
\mu_{41}(a) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ a & 0 & 1 \end{pmatrix}.
\]
By calculations we have
\[
\mu_{41}(a) = w_{21} \mu_{32}(a) w_{21} \in (K_1K_2)^3
\]
Lemma 3.10 Let \( \mu_{31}(a) = \mu_{21}(-a)\mu_{32}(1)\mu_{21}(a)\mu_{32}(-1)\mu_{41}(-a^2) \in (K_1K_2)^7 \). Any element in \( B \) has the form
\[
\begin{pmatrix}
1 & a & c \\
0 & b & d \\
0 & c - ab & -a
\end{pmatrix}
\begin{pmatrix}
e & f & e^{-1} \\
f & f^{-1} & e^{-1}
\end{pmatrix}
\]
where \( a, b, c, d \in O \) and \( e, f \in O^\times \), which is equal to
\[
\mu_{21}(a)\mu_{32}(b)\mu_{31}(c)\mu_{41}(ac + d) \cdot \begin{pmatrix}
e & f & e^{-1} \\
f & f^{-1} & e^{-1}
\end{pmatrix}.
\]
So we have \( B \subset (K_1K_2)^{13} \).

By the Bruhat decomposition, we have \( K = BWB = (K_1K_2)^2_7 \).

**Lemma 3.10** Let \( K \) be any compact group, \( \{K_i\}_{1 \leq i \leq n} \) a family of subgroups such that \( K = (K_1K_2\ldots K_n)^N \) for some \( N \in \mathbb{N}^* \). Then for any finite dimensional unitary representation \( (V, \tau) \) of \( K \) without invariant vector, and any \( x \in V \), and \( y_i \in V \) invariant by \( K_i \) for each \( 1 \leq i \leq n \), we have
\[
\|x\|_V \leq 2nN \max_{1 \leq i \leq n} \{\|x - y_i\|_V\}.
\]

**Proof:** Since \( \int_K \|\tau(k)x - x\|_V^2 \, dk = 2\|x\|_V^2 \geq \|x\|_V^2 \) we see that there exists a \( k \in K \) such that \( \|\tau(k)x - x\|_V \geq \|x\|_V \). Suppose that \( k = (k_{11}\ldots k_{n1})\ldots(k_{1N}\ldots k_{nN}) \) with \( k_{ij} \in K_i(1 \leq i \leq n, 1 \leq j \leq N) \). We then have
\[
\|x\|_V \leq \|\tau(k)x - x\|_V \leq \sum_{1 \leq i \leq n, 1 \leq j \leq N} \|\tau(k_{ij})x - x\|_V
\]
\[
\leq 2 \sum_{1 \leq i \leq n, 1 \leq j \leq N} \|y_i - x\|_V \leq 2nN \max_{1 \leq i \leq n} \{\|x - y_i\|_V\} \]
\[
\]
**Proof of proposition 3.2:** By lemmas 3.6 and 3.7 we obtain two similar inequalities as (4) and (5) in the proof of proposition 3.1 (using the same argument): when \( i \geq 2j \geq 0 \), and \( i \in 2\mathbb{N} + k, k \in \{0, 1\} \),
\[
\|c(i, j) - c(i + k, (i + k)/2)\|_V \leq C'e^{2C-(\frac{2i}{3} - 2\beta)i};
\]

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when $2j \geq i \geq j \geq 0$, there exists $k \in \{0, 1, 2\}$ such that

$$\|c(i, j) - c\left(\frac{2}{3}(i + j)\right) + k, \frac{1}{2}\left(\frac{2}{3}(i + j) + k\right)\|_{V} \leq C'e^{2C' - \frac{\alpha}{6} - 2\beta}i.$$ 

So it remains to prove

$$\|c(2j, j)\|_{V} \leq C'e^{2C' - \frac{\alpha}{6} - 2\beta}2^j.$$ 

First we see that

$$\max\left(\|c(2j, j) - c(2j, 0)\|_{V}, \|c(2j, j) - c(\lfloor 3j/2 \rfloor, \lfloor 3j/2 \rfloor)\|_{V}\right) \leq C'e^{2C' - \frac{\alpha}{6} - 2\beta}2^j.$$ 

Moreover, $c(k'gk'') = \tau(k')c(g)$, $\forall k', k'' \in K, g \in G$, and it follows that $c(\lfloor 3j/2 \rfloor, \lfloor 3j/2 \rfloor)$ is invariant by $K_1$, and that $c(2j, 0)$ invariant by $K_2$. By applying lemma 3.10 to $K = (K_1K_2)^{30}$, we complete the proof of the proposition. 

4 Proof of theorem 2.3 when char$(F') = 2$

In this section we prove theorem 2.3 when char$(F) = 2$. The proof for char$(F) = 2$ is technically more difficult because it is only possible to prove a local estimate for the move $(0, 2)$, and therefore we have two limits in the spherical propositions (proposition 4.2).

Throughout this section we assume $F$ is of characteristic 2.

**Lemma 4.1** Let $\alpha > 0$ as in proposition 2.1, $\beta \in [0, \frac{\alpha}{6}]$. Let $(V, \tau)$ be an irreducible unitary representation of $K$ which factorizes through $Sp_4(\mathcal{O}/\pi^k\mathcal{O})$ for $k \geq 0$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, $(E, \pi)$ any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any $K$-invariant vectors of norm 1. Then for any $(i, j) \in \Lambda$ with $i - j \geq 4k + 2$, we have

$$\|c(i, j) - c(i, j + 2)\|_{V} \leq C'e^{2C' - \frac{\alpha}{6} - 2\beta}i + \frac{\alpha}{6}j.$$ 

In particular when $(V, \tau)$ is the trivial representation of $K$ (and $V = \mathbb{C}$), we have

$$|c(i, j) - c(i, j + 2)| \leq C'e^{2C' - \frac{\alpha}{6} - 2\beta}i + \frac{\alpha}{6}j,$$

for any $(i, j) \in \Lambda$ with $i - j \geq 2$. 

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\textbf{Proof:} Since $\text{char}(F) = 2$, we have $-1 = 1$ in $F$. Let $m = \lfloor \frac{i+j}{2} \rfloor$, $x, y, a, b \in \mathcal{O}/\pi^{m-j-1}\mathcal{O}$, $\sigma : \mathcal{O}/\pi^{m-j-1}\mathcal{O} \to \mathcal{O}$ a section. Let
\[
\beta(a, b)^{-1} = \begin{pmatrix}
\pi^m & \pi^{i-m+j} \\
0 & \pi^{-i+m-j+1}\sigma(b) \\
\pi^{-i+m-j+1}\sigma(b) & \pi^{i-m+j} \\
0 & \pi^{-i+m-j}
\end{pmatrix},
\]
\[
\alpha(x, y) = \begin{pmatrix}
\pi^{-m+j} & \pi^{-m+j} \\
\pi^{-m+j}(\sigma(x) + \pi\sigma(y)) & \pi^{-m+j}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) \\
\pi^{-m+j}\sigma(x)^2 & \pi^{-m+j}(\sigma(x) + \pi\sigma(y)) \\
0 & \pi^{-m-j}
\end{pmatrix}.
\]
Then
\[
\beta(a, b)^{-1}\alpha(x, y) =
\begin{pmatrix}
\pi^j & \pi^{-i+2m+2j} \\
0 & \pi^{-i}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) \\
\pi^{-i}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) & \pi^{-i+2m+2j} \\
\pi^{-2m+j}\sigma(x)^2 & \pi^{-2m+j}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y))
\end{pmatrix}.
\]
We see that
\[
\|\Lambda^2(\beta(a, b))\| = q^{j}, \|\Lambda^2(\alpha(x, y))\| = q^{2m-2j}.
\]
Denote
\[
a_1 = (\pi\sigma(b) + \sigma(x) + \pi\sigma(y))(1 + \pi\sigma(a))^{-2},
\]
and
\[
\varepsilon_1 = \pi^{-m+j+2}(\sigma(y) + \sigma(a)\sigma(x) + \sigma(b)) \in \mathcal{O}.
\]
Let $k_1$ be the element in $K$ defined by
\[
k_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & \pi^{-i}a_1 & \pi^{i-2m+j}a_1 \\
1 & 0 & \pi^{i-2m+3j+2}a_1 & \pi^{2j+2} \\
\pi^{-i}a_1 & 1 & \pi^{2i-2m+2j}(1 + \pi\sigma(a))^{-2} & 0
\end{pmatrix},
\]
and let
\[
g_1 = k_1\beta(a, b)^{-1}\alpha(x, y)
\]
\[
= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & \pi^{-i}a_1 & \pi^{i-2m+j}a_1 \\
1 & 0 & \pi^{i-2m+3j+2}a_1 & \pi^{2j+2} \\
\pi^{-i}a_1 & 1 & \pi^{2i-2m+2j}(1 + \pi\sigma(a))^{-2} & 0
\end{pmatrix} \times
\begin{pmatrix}
\pi^j \\
\pi^{-i}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) \\
\pi^{-2m+j}\sigma(x)^2 \\
\pi^{-i}(1 + \pi\sigma(a))^2
\end{pmatrix} \times
\begin{pmatrix}
\pi^{-i+2m+2j} \\
\pi^{-i}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) \\
\pi^{-2m+j}(\pi\sigma(b) + \sigma(x) + \pi\sigma(y)) \\
\pi^{-2m+j}(1 + \pi\sigma(a))^{-2}
\end{pmatrix}
\]
...
\[
= \begin{pmatrix}
\pi^{-i}a_1(1 + \pi\sigma(a))^2 & \pi^{-i}(1 + \pi\sigma(a))^2 & \pi^{-i+2m-2j} & 0 \\
\pi^{-i-2\varepsilon_1^2}(1 + \pi\sigma(a))^{-2} & 0 & \pi^{-j}a_1 & \pi^{-j} \\
\pi^j\varepsilon_1^2(1 + \pi\sigma(a))^{-2} + \pi^j & 0 & \pi^{j+2}a_1 & \pi^{j+2} \\
0 & 0 & \pi^i(1 + \pi\sigma(a))^{-2} & 0
\end{pmatrix}.
\]

When \(\varepsilon = 0\), we have \(|\varepsilon_1^2| \leq q^{-2}\) and

\[
= \begin{pmatrix}
\pi^i & \pi^j & \pi^{-j} & \pi^{-i} \\
\pi^i & \pi^j & \pi^{-j} & \pi^{-i} \\
\pi^{-2}\varepsilon_1^2(1 + \pi\sigma(a))^{-2} & 0 & \pi^{-j}a_1 & \pi^{-j} \\
\varepsilon_1^2(1 + \pi\sigma(a))^{-2} + 1 & 0 & \pi^2a_1 & \pi^2 \\
0 & 0 & (1 + \pi\sigma(a))^{-2} & 0
\end{pmatrix} \in K.
\]

When \(\varepsilon = 1\), we have

\[|\varepsilon_1^2(1 + \pi\sigma(a))^{-2} + 1| = |(\varepsilon_1(1 + \pi\sigma(a))^{-1} + 1)^2| \leq q^{-2},\]

and then

\[
= \begin{pmatrix}
\pi^i & \pi^j+2 & \pi^{-j-2} & \pi^{-i} \\
\pi^i & \pi^j+2 & \pi^{-j-2} & \pi^{-i} \\
\pi^{-2}\varepsilon_1^2(1 + \pi\sigma(a))^{-2} & 0 & \pi^{-j}a_1 & \pi^{-j} \\
\varepsilon_1^2(1 + \pi\sigma(a))^{-2} + 1 & 0 & \pi^2a_1 & \pi^2 \\
0 & 0 & (1 + \pi\sigma(a))^{-2} & 0
\end{pmatrix} \in K.
\]

When \(i - j \geq 4k+2\), \(a, x \in \pi^k\mathcal{O}/\pi^{m-j-1}\mathcal{O}\), and \(b, y \in \pi^{2k}\mathcal{O}/\pi^{m-j-1}\mathcal{O}\), we have

\[
k_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & \pi^{2j+2} \\
0 & 1 & 0 & 0
\end{pmatrix} \mod \pi^k\mathcal{O}.
\]

By replacing \(n_1\) by \(m - j - 1\) at the end of the proof of lemma 3.6, we get

\[
\|e(i, j) - c(i, j + 2)\|_V \\
\leq qe^{C + \beta(i+j)} \cdot C_2q^{2h-2} \cdot e^{-2(\frac{m-j+1}{h}-\frac{2k}{h})} \cdot e^{C+2\beta(m-j)} \\
\leq C' e^{2C - (\frac{k}{h} - 2\beta) + \frac{\alpha}{h}}.
\]

\[\square\]
Proposition 4.2 Let $\alpha > 0$, $\beta \in [0, \frac{\alpha}{2\pi})$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}^*_+, (E, \pi)$ any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in E^*$ any $K$-invariant vectors of norm 1. There exist $c_0, c_1 \in \mathbb{C}$, such that

$$|c(i, j) - c_1| \leq C'e^{2C-(\frac{\alpha}{2\pi}-2\beta)i},$$

for any $(i, j) \in \Lambda$ with $i + j \in 2\mathbb{N} + l, l = 0, 1$.

Proof: We apply the same argument as in the proof of proposition 3.1 (which is still true in characteristic 2) and lemma 4.1 (in the particular case when $(V, \tau)$ is the trivial representation of $K$). We will get two limits because the moves $(i, j) \mapsto (i+1, j-1)$ and $(i, j) \mapsto (i, j+2)$ generate a sublattice of $\mathbb{Z}^2$ of index 2.

First, we put $S_{\alpha} = \{(i, j) \in \Lambda | 0 \leq i - 2j \leq \alpha\}$. When $0 \leq 2j \leq i$, we have $(i, j + 2[\frac{2j-i}{3}]) \in S_4$, and by the particular case of lemma 4.1 when $(V, \tau)$ is the trivial representation of $K$, we get

$$|c(i, j) - c(i, j + 2[\frac{i-2j}{4}])| \leq C'e^{2C-(\frac{\alpha}{2\pi}-2\beta)i}.$$ 

When $0 \leq i \leq 2j$, we have $(i + [\frac{2j-i}{3}], j - [\frac{2j-i}{3}]) \in S_3 \subset S_4$. By lemmas 3.4 we have

$$|c(i, j) - c(i + \frac{2j-i}{3}, j - \frac{2j-i}{3})| \leq C'e^{2C-(\frac{\alpha}{2\pi}-\beta)(i+j)}.$$

Moreover, when $(i, j) \in S_4$, there exists $k \in \{0, 1, 2\}$ such that

$$|c(i, j) - c(i + k, \frac{1}{2}(i+k))| \leq C'e^{2C-(\frac{\alpha}{2\pi}-2\beta)2j}.$$

In fact, when $(i, j) \in S_8$, we first have

$$\max\{|c(i, j) - c(i, j + 2)|, |c(i, j) - c(i + 1, j - 1)|\} \leq C'e^{2C-(\frac{\alpha}{2\pi}-2\beta)i}.$$

It suffices to show the inequality when $i - 2j \in \{1, 2, 3, 4\}$, by considering the following moves inside $S_8$. When $i - 2j = 1$, we obtain the inequality by considering $(2j+1, j) \mapsto (2j+2, j-1) \mapsto (2j+2, j+1)$. When $i - 2j = 2$, we consider $(2j+2, j) \mapsto (2j+4, j-2) \mapsto (2j+4, j+2)$. When $i - 2j = 3$ or 4, use the moves $(2j+3, j) \mapsto (2j+2, j+1)$ and $(2j+4, j) \mapsto (2j+4, j+2)$ respectively.

In sum, when $i \geq 2j \geq 0$, there exists $k \in \{0, 1, 2\}$, such that

$$|c(i, j) - c(i + k, \frac{1}{2}(i+k))| \leq C'e^{2C-(\frac{\alpha}{2\pi}-2\beta)i}; \quad (7)$$
when $2j \geq i \geq j \geq 0$, there exists $k \in \{0, 1, 2, 3\}$ such that

$$|c(i, j) - c \left( \left\lfloor \frac{2}{3} (i + j) \right\rfloor + k, \frac{1}{2} \left( \left\lfloor \frac{2}{3} (i + j) \right\rfloor + k \right) \right)| \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)i} \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)i}. \tag{8}$$

Finally the proposition follows from the inequality

$$|c(2j, j) - c(2j + 4, j + 2)| \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)2j}. \tag{9}$$

\[\square\]

**Proposition 4.3** Let $\alpha > 0$, $\beta \in [0, \frac{\alpha}{2\alpha})$, and $(V, \tau)$ a non trivial irreducible unitary representation of $K$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}^+_+$, $(E, \pi)$ any element in $\mathcal{E}_{G, C + \beta E}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any $K$-invariant vectors of norm 1. We have

$$\|c(i, j)\|_V \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)i}.$$

**Proof:** As (7) and (8) in the proof of the above proposition 4.2 by lemmas 3.7 and 4.1, we have the following inequalities. When $i \geq j \geq 0$, there exists $k \in \{0, 1, 2\}$, such that

$$\|c(i, j) - c(i + k, \frac{1}{2}(i + k))\|_V \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)i}.$$

When $2j \geq i \geq j \geq 0$, there exists $k \in \{0, 1, 2, 3\}$ such that

$$\|c(i, j) - c \left( \left\lfloor \frac{2}{3} (i + j) \right\rfloor + k, \frac{1}{2} \left( \left\lfloor \frac{2}{3} (i + j) \right\rfloor + k \right) \right)\|_V \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)i}.$$

So it remains to prove that for any $j \in \mathbb{N}$ we have

$$\|c(2j, j)\|_V \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)2j}. \tag{9}$$

First when $j \in 2\mathbb{N}$, we know inequality (9) holds. In fact, by lemmas 3.7 and 4.1, when $j \in 2\mathbb{N}$ we have

$$\max \left( \|c(2j, j) - c(2j, 0)\|_V, \|c(2j, j) - c(3j/2, 3j/2)\|_V \right) \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)2j}.$$

Let $K_1, K_2$ be the subgroups of the group $K$ as lemma 3.9. By lemmas 3.9 and 3.10 we get inequality (9).

It remains to show inequality (9) when $j \in 2\mathbb{N} + 1$. We first have

$$\max \left( \|c(2j, j) - c(2j + 1, 0)\|_V, \|c(2j, j) - c(2j - \lfloor \frac{j}{2} \rfloor, \lfloor \frac{j}{2} \rfloor)\|_V \right) \leq C'e^{2C - (\frac{|h|}{\alpha} - 2\beta)2j}.$$
Note that lemma 3.7 is still valid for \( i = j - 1 \), i.e.
\[
\|c(2j - \lfloor \frac{j}{2} \rfloor, j + \lfloor \frac{j}{2} \rfloor) - c(2j - \lfloor \frac{j}{2} \rfloor - 1, j + \lfloor \frac{j}{2} \rfloor + 1)\|_V \leq C'e^{2C'-(\frac{3\alpha}{\pi}-3\beta)j}.
\]

Then we have
\[
\|c(2j, j) - c(2j - \lfloor \frac{j}{2} \rfloor - 1, j + \lfloor \frac{j}{2} \rfloor + 1)\|_V \leq C'e^{2C'-(\frac{\alpha}{\pi}-2\beta)2j}.
\]

Denote \( B_1, B_2 \) the image in \( K_1 \) of \( \left( \begin{array}{cc} O^X & \pi O \\ O & O^X \end{array} \right) \) and \( \left( \begin{array}{cc} O^X & O \\ \pi O & O^X \end{array} \right) \) respectively, under the group isomorphism \( GL_2(O) \to K_1 \). We see that \( K_1 = (B_1B_2)^2 \). Moreover \( c(k'gk'') = \tau(k')c(g) \) for any \( k', k'' \in K, g \in G \), it follows that \( c(2j + 1, 0), c(2j - \lfloor \frac{j}{2} \rfloor, j + \lfloor \frac{j}{2} \rfloor), c(2j - \lfloor \frac{j}{2} \rfloor - 1, j + \lfloor \frac{j}{2} \rfloor + 1) \) are invariant by \( K_2, B_1, B_2 \) repectively. By applying lemma 3.10 to \( K = (B_1B_2K_2)^6 \), we obtain inequality 9 for \( j \in 2\mathbb{N} + 1 \). □

**Proof of theorem 2.3 when \( \text{char}(F) = 2 \):** For simplicity we say that an element \( g \in G \) is even (resp. odd) when \( g \in KD(i, j)K, i \geq j \geq 0 \) and \( i + j \) is even (resp. odd). By proposition 1.2 we see that when \( g \) is even (resp. odd) and tends to infinity, the limit of \( e_K e_g e_K \) exists in \( C^E_{\alpha+\beta}(G) \), which we denote by \( T_0 \) (resp. \( T_1 \)).

First for any \( g \in G \) we have
\[
e_K e_g T_0 = \alpha(g)T_0 + \beta(g)T_1, \tag{10}
\]
and
\[
e_K e_g T_1 = \beta(g)T_0 + \alpha(g)T_1, \tag{11}
\]
where \( \alpha(g) \) (resp. \( \beta(g) \)) denotes the volume of the set of elements \( (k_1, k_2, k_3, k_4) \in K \) such that
\[
\| gk_1 \land gk_2 \| \in q^{2j} (\text{resp. } q^{2j+1}).
\]

In fact, when \( i + j \in 2\mathbb{N} \) (resp. \( 2\mathbb{N} + 1 \)) with \( (i, j) \in \Lambda \) and when \( \| gk_1 \land gk_2 \| \geq q^{-2j} \), \( gKD(i, j) \) is even exactly when \( \| gk_1 \land gk_2 \| \in q^{2j} \) (resp. \( q^{2j+1} \)). Hence we have
\[
\lim_{i+j \in 2\mathbb{N}, \text{ resp. } 2\mathbb{N}+1, j \to \infty} \text{vol}\{ k \in K, gKD(i, j) \text{ is even} \} = \alpha(g) (\text{resp. } \beta(g)),
\]
and also
\[
\lim_{i+j \in 2\mathbb{N}, \text{ resp. } 2\mathbb{N}+1, j \to \infty} \text{vol}\{ k \in K, gKD(i, j) \text{ is odd} \} = \beta(g) (\text{resp. } \alpha(g)).
\]

Next let \( p = \frac{1}{2}(T_0 + T_1) \). We have for any \( g \in G, e_K e_g p = p \), and thus \( p^2 = p \). In fact, denote \( \alpha \) (resp. \( \beta \)) the volume of the set of elements \( (k_{ij})_{1 \leq i,j \leq 4} \in K \) such that
\[
|k_{11}k_{22} - k_{21}k_{12}| \in q^{2j},
\]

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(resp. $q^{2z+1}$). We see that
\[
\alpha = \lim_{i+j \in 2N, j \to \infty} \alpha(D(i,j)) = \lim_{i+j \in 2N+1, j \to \infty} \beta(D(i,j)),
\]and that
\[
\beta = \lim_{i+j \in 2N, j \to \infty} \beta(D(i,j)) = \lim_{i+j \in 2N+1, j \to \infty} \alpha(D(i,j)).
\]
In (10), let $g$ be even or odd and tends to infinity, we get the following inequalities respectively:
\[
T_0^2 = \alpha T_0 + \beta T_1,
\]
and
\[
T_0 T_1 = \beta T_0 + \alpha T_1.
\]
Similarly in (11), let $g$ be even or odd and tends to infinity, we have
\[
T_0 T_1 = \beta T_0 + \alpha T_1,
\]
and
\[
T_1^2 = \alpha T_0 + \beta T_1,
\]
respectively. Therefore we have $T_0 p = T_1 p = p$ (since $\alpha + \beta = 1$) and
\[
e_K e_g p = e_K e_g T_0 p = (\alpha(g) T_0 + \beta(g) T_1)p = p.
\]
By proposition 4.3 for any non trivial irreducible representation $V$ of $K$ we have $e_K e_g T_0 = e_K e_g T_1 = 0$. By the same argument as in the proof of theorem when $\text{char}(F) \neq 2$ in section 2, we have
\[
e_g p = e_K e_g p = p.
\]
We complete the proof by taking
\[
p_n = \frac{1}{2} \left( e_K e_{D(2 \lfloor \frac{n}{2} \rfloor, 0)} e_K + e_K e_{D(2 \lfloor \frac{n}{2} \rfloor - 1, 0)} e_K \right),
\]
and $t = \frac{\alpha}{2n} - 2\beta$. \hfill \Box

5 \quad \textbf{Extension to simple algebraic groups of higher split rank}

Let $F$ be a non archimedean local field. This section is dedicated to the proof the the following theorem, which is theorem 1.3 in the introduction.

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**Theorem 5.1** Let $G$ be a connected almost $F$-simple algebraic group. Then $G(F)$ has strong Banach property (T).

We begin the proof with some lemmas. The following lemma is proposition 8.2 in [Bor].

**Lemma 5.2** Let $k$ be a field and $H$ an abelian $k$-group. Let $\pi : H \to GL_n$ be a $k$-rational representation. Then $\pi(H)$ is conjugate over $k$ to some subgroup of the group of diagonal elements in $GL_n$.

The following lemma is a consequence of theorem 7.2 in [BT], which is also proposition I.1.6.2 in [Mar].

**Lemma 5.3** Let $k$ be any field and $G$ a connected almost $k$-simple group with $k$-split rank $\geq 2$. Then there exists a $k$-rational group homomorphism with finite kernel from $SL_3$ or $Sp_4$ to $G$.

The following lemma is a direct consequence of propositions I.1.3.3 (ii) and I.1.5.4 (iii), and theorem I.2.3.1 (a) in [Mar].

**Lemma 5.4** Let $G$ be a simply connected and almost $F$-simple group. Let $S$ be a maximal $F$-split torus of $G$, $\Phi(G, S)$ the root system with some ordering and $\vartheta$ a proper subset of simple roots. Then there exist two unipotent $F$-subgroups $V_\vartheta, V^-_\vartheta$ of $G$, and two $S$-equivariant $F$-isomorphisms $\text{Lie}V_\vartheta \to V_\vartheta, \text{Lie}V^-_\vartheta \to V^-_\vartheta$, such that

- (i) $\text{Lie}V_\vartheta$ (resp. $\text{Lie}V^-_\vartheta$) is the direct sum of eigenspaces of positive (resp. negative) roots which are not integral linear combinations of $\vartheta$, and
- (ii) $V_\vartheta(F) \cup V^-_\vartheta(F)$ generates $G(F)$.

The next two lemmas reduce the proof to the simply connected covering of our algebraic group.

**Lemma 5.5** (proposition I.1.4.11 in [Mar]) Let $k$ be a field, and let $G$ be connected semisimple $k$-group. Then there exists a simply connected $k$-group $\tilde{G}$ and a $k$-isogeny (i.e. surjective $k$-group homomorphism with finite kernel) from $\tilde{G}$ to $G$.

**Lemma 5.6** Let $G_1$ be a locally compact group and $G_2$ its quotient by a finite normal subgroup. Then $G_1$ has strong Banach property (T) if and only if $G_2$ has strong Banach property (T).

**Proof:** Let $H$ be the kernel of $G_1 \to G_2$. Suppose $G_1$ has strong Banach property (T), and let $p_n \in C_c(G_1)$ be real and self-adjoint elements (otherwise take $p_n + p_n^* + p_n^*$) that tends to the idempotent element in $C^*_\text{C+}(G_1)$. Then $\left( \mathbb{E}_{h \in H} p_n \right)$ tends to a real and self-adjoint
(since $H$ is normal) idempotent element $p'$ in $\mathcal{C}_{C+s\ell}^g(G_2)$ such that $e_gp' = p'$ for any $g \in G_2$. On the other direction, if $G_2$ has strong Banach property (T), let $p_n \in C_c(G_2)$ tend to the idempotent element in $\mathcal{C}_{C+s\ell}^g(G_2)$, and denote its lifting to $C_c(G_1)$ by $\tilde{p}_n$ (i.e. $\tilde{p}_n(gh) = p_n(g)$ for any $g \in G_1, h \in H$). Since for any $(E, \pi) \in \mathcal{E}_{G_1,C+s\ell}$ we have

$$\|\pi(\tilde{p}_n) - \pi(\tilde{p}_m)\|_{L(E)} = \|\pi(\tilde{p}_n) - \pi(\tilde{p}_m)\|_{L(E^H)},$$

where $E^H$ denotes the space of $H$-invariant vectors, we conclude that $\tilde{p}_n$ tends to a real and self-adjoint idempotent element $p$ in $\mathcal{C}_{C+s\ell}^g(G_1)$ such that $e_gp = p$ for any $g \in G_1$.

**Proof of theorem 5.1.** In view of lemmas 5.5 and 5.6, we can assume $G$ is simply connected. By lemma 5.7 there exist a subgroup $R$ of $G(F)$ and a surjective group homomorphism $I$ from $SL_3(F)$ or $Sp_4(F)$ to $R$ with finite kernel. Let $\rho : F^* \to SL_3(F)$ (resp. $Sp_4(F)$) be the group homomorphism defined by

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{pmatrix} \quad \text{(resp.} \quad \begin{pmatrix} x & 0 \\ -1 & 1 \\ x^{-1} \end{pmatrix} \text{)}$$

for any $x \in F$, and let $a = I \circ \rho(\pi)$, where $\pi$ is a uniformizer of $F$. By lemma 5.2 the set of eigenvalues of $Ad(a)$ is a subset of $\pi \mathbb{Z}$ which contains $\{1\}$ as a proper subset. Let $S$ be a maximal $F$-split torus of $G$ whose $F$ points contains $a$. We can choose an ordering of $\Phi(S, G)$ such that $|\chi(a)| \leq 1$ for any simple root $\chi$. Let $\vartheta$ be the proper subset of simple roots $\chi$ such that $|\chi(a)| = 1$, and let $V_{\vartheta}, V_{\vartheta}^-$ be as in lemma 5.4.

For simplicity denote $G(F)$ and $V_{\vartheta}(F), V_{\vartheta}^-(F)$ by $G$ and $V_{\vartheta}, V_{\vartheta}^-$ from now on. Let $\| \cdot \|$ be the norm on $\text{Lie} G$ defined w.r.t. some $F$-basis. Let $\ell'$ be the length function on $G$ defined by

$$\ell'(g) = \log \|Ad(g)\|_{\text{End}(\text{Lie} G)}.$$ 

Let $\mathcal{E}$ be a class of Banach spaces of type $> 1$ stable under duality and complex conjugation. Let $s, t, C, C' \in \mathbb{R}_+, p \in \mathcal{C}_{s\ell+C}(R), p_m \in C_c(R)$ verify the conditions (i) and (ii) of theorem 2.2 if $R$ is isogenous to $SL_3(F)$, or of theorem 2.3 if $R$ is isogenous to $Sp_4(F)$, where $\kappa \in \mathbb{R}_+$ such that $\ell'|_{R} \leq \kappa \ell$. Let $U$ be an open compact subgroup of $G$ and $f = \frac{e_f}{\varphi(\sigma_U)}$. Then to establish that $G$ has strong Banach property (T) it suffices to show that if $s$ is small enough the series $p_m f \in C_c(G)$ converges in $\mathcal{C}_{s\ell+C}(G)$ to a self adjoint idempotent $p'$ such that for any $(E, \pi) \in \mathcal{E}_{G,s\ell+C}$ the image of $\pi(p')$ consists of all $G$-invariant vectors of $E$. First it is clear that the series $p_m f$ is a Cauchy series in
When $\ell$ tends to 0 when $n$ tends to 0 when $\alpha$ is big enough such that $p > m > 0$, we have

$$
\pi(a^{-n}) (\pi(E(Ad(a^n)) - 1) \pi(pm) f x
$$

When $\ell(g) \leq m$, we have

$$
\|Ad(g^{-1}a^n)Y\| \leq e^{\ell(g)} \max_{\lambda \in \Lambda} |\lambda|^n \|Y_{\lambda}\|_{\text{Lie} V_{\bar{\vartheta}}} \leq r,
$$

and hence

$$
(\pi(E(Ad(g^{-1}a^n)) - 1) \pi(f) x = 0.
$$

Therefore we have

$$
\pi(a^{-n}) (\pi(E(Ad(a^n)) - 1) \pi(pm) f x = 0
$$

when $n$ is big enough.

On the other hand for any $n \in \mathbb{N}$, we always have

$$
Ad(a^n)Y = \sum_{\lambda \in \Lambda} \lambda^n Y_{\lambda} \in \bigoplus_{\lambda \in \Lambda} OY_{\lambda}.
$$
Hence
\[ \| \pi(a^{-n})(\pi(E(Ad(a^n)Y)) - 1)\pi(p' - p_m f)x \|_E \leq e^{C' + s\ell'(a)n}(1 + C'')\|\pi(p' - p_m f)x\|_E, \]
where
\[ C'' = \sup_{t_\lambda \in \mathcal{O}} \|\pi(E(\max_{\lambda \in \Lambda} t_\lambda Y_\lambda))\|_{L(E)} < \infty \]
depends only on \( Y \). But
\[ \|\pi(p' - p_m f)x\|_E \leq C'e^{2C-tm}\|\pi(f)x\|_E \]
by statement (ii) of theorem 2.2 if \( R \) is isogenous to \( SL_3(F) \), or of theorem 2.3 if \( R \) is isogenous to \( Sp_4(F) \) (we recall that \( C' \) and \( t \) are the constants of theorem 2.2 and theorem 2.3). In total, when \( n \) is big enough
\[ \|\pi(a^{-n})(\pi(E(Ad(a^n)Y)) - 1)\pi((p' - p_m f))x \|_E \leq e^{C' + s\ell'(a)n}(1 + C'')e^{2C-tm}\|\pi(f)x\|_E, \]
and if
\[ s < \frac{t}{\kappa \ell'(a)} \log \min_{\lambda \in \Lambda} |\lambda|^{-1}, \]
it tends to 0 when \( n \) tends to infinity.

We prove \( \pi(p')x \) is \( V^{-\vartheta}_0 \)-invariant by exactly the same argument (with \( a \) replaced by \( a^{-1} \) and the ordering of \( \Phi(S,G) \) by its inverse). □

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