CARTAN-TYPE CRITERIONS FOR CONSTANCY OF ALMOST HERMITIAN MANIFOLDS

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Abstract. We studied the axiom of anti-invariant 2-spheres and the axiom of co-holomorphic \((2n+1)\)-spheres. We proved that a nearly Kählerian manifold satisfying the axiom of anti-invariant 2-spheres is a space of constant holomorphic sectional curvature. We also showed that an almost Hermitian manifold of dimension \(2m \geq 6\) satisfying the axiom of co-holomorphic \((2n+1)\)-spheres for some \(n\), where \((1 \leq n \leq m-1)\), the manifold \(M\) has pointwise constant type \(\alpha\) if and only if \(M\) has pointwise constant anti-holomorphic sectional curvature \(\alpha\).

1. Introduction

E. Cartan [1] introduced the axiom of \(n\)-planes as: A Riemannian manifold \(M\) of dimension \(m \geq 3\) is said to satisfy the axiom of \(n\)-planes, where \(n\) is a fixed integer \(2 \leq n \leq m - 1\), if for each point \(p \in M\) and each \(n\)-dimensional subspace \(\sigma\) of the tangent space \(T_p(M)\), there exists an \(n\)-dimensional totally geodesic submanifold \(N\) such that \(p \in N\) and \(T_p(N) = \sigma\). He also gave a criterion for constancy of sectional curvature for any Riemannian manifold of dimension \(m \geq 3\) in the following theorem.

Theorem 1.1. A Riemannian manifold of dimension \(m \geq 3\) with the axiom of \(n\)-planes is a real space form.

D.S. Leung and K. Nomizu [16] introduced the axiom of \(n\)-spheres by using totally umbilical submanifold \(N\) with parallel mean curvature vector field instead of totally geodesic submanifold \(N\) in the axiom of \(n\)-planes. They proved a generalization of Theorem 1.1.

Later on, Cartan’s idea was applied to almost Hermitian manifolds in various studies. Kählerian manifolds were studied in [2, 5, 9, 14, 17, 22, 24]. The articles [20] and [21] discussed nearly Kählerian (almost Tachibana) manifolds. The results concerning larger classes of almost Hermitian manifolds can be found in [12, 13].
Here, we shall call the criterions used in all of the above papers as \textit{Cartan-type criterions}.

2. Preliminaries

2.1. Some classes of almost Hermitian manifolds. Let $M$ be an almost Hermitian manifold with an almost complex structure $J$ in its tangent bundle and a Riemannian metric $g$ such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of $C^\infty$ vector fields on $M$. Let $\nabla$ be the Riemannian connection on $M$. The Riemannian curvature tensor $R$ associated with $\nabla$ is defined by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [X, Y]$. We denote $g(R(X, Y)Z, U)$ by $R(X, Y, Z, U)$. Curvature identities are of fundamental importance for understanding the geometry of almost Hermitian manifolds. The following curvature identities are used in various studies e.g.\cite{7 8}:

1. $R(X, Y, Z, U) = R(X, Y, JZ, JU)$,
2. $R(X, Y, Z, U) = R(JX, JY, Z, U) + R(JX, Y, JZ, U)$
3. $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$.

Let $AH_i$ denote the subclass of the class $AH$ of almost Hermitian manifolds satisfying the curvature identity $(i)$, $i = 1, 2, 3$. We know that

$$AH_1 \subset AH_2 \subset AH_3 \subset AH,$$

from \cite{7}. Some authors call $AH_1$-manifold as a \textit{para-Kählerian manifold} and call $AH_3$-manifold as an \textit{RK-manifold} \cite{19}. An almost Hermitian manifold $M$ is called Kählerian if $\nabla_X J = 0$ for all $X \in \chi(M)$ and nearly Kählerian (almost Tachibana) if $(\nabla_X J)X = 0$ for all $X \in \chi(M)$. It is well-known that a Kählerian manifold is $AH_1$-manifold and a nearly Kählerian manifold (non para-Kählerian) manifold is $AH_2$-manifold, see \cite{8 19}.

A two-dimensional linear subspace of a tangent space $T_p(M)$ is called a \textit{plane section}. A plane section $\sigma$ is said to be \textit{holomorphic} (resp. \textit{anti-holomorphic} or \textit{totally real}) if $J\sigma = \sigma$ (resp. $J\sigma \perp \sigma$) \cite{2 22}. The sectional curvature $K$ of $M$ which is determined by orthonormal vector fields $X$ and $Y$ is given by $K(X, Y) = R(X, Y, X, Y)$. The sectional curvature of $M$ restricted to a holomorphic (resp. an anti-holomorphic) plane $\sigma$ is called \textit{holomorphic} (resp. \textit{anti-holomorphic} \textit{sectional curvature}). If the holomorphic (resp. anti-holomorphic) sectional curvature at each point $p \in M$ does not depend on $\sigma$, then $M$ is said to be \textit{pointwise constant holomorphic} (resp. \textit{pointwise constant anti-holomorphic}) \textit{sectional curvature}. A connected Riemannian (resp. Kählerian) manifold of (global) constant sectional curvature (resp. of constant holomorphic sectional curvature) is called a \textit{real space form} (resp. a \textit{complex space form}) \cite{2 20 12 23}. The following useful notion was defined by A. Gray in \cite{6}.

\textbf{Definition 2.1.} Let $M$ be an almost Hermitian manifold. Then $M$ is said to be of \textit{constant type} at $p \in M$ provided that for all $X \in T_p(M)$, we have $\lambda(X, Y) = \lambda(X, Z)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic and $g(Y, Y) = g(Z, Z)$, where the function $\lambda$ is defined by $\lambda(X, Y) = R(X, Y, X, Y) - R(X, Y, JX, JY)$. If this holds for all $p \in M$, then we say that $M$ has \textit{(pointwise) constant type}. Finally, if for $X, Y \in \chi(M)$ with $g(X, Y) = g(JX, Y) = 0$, the value
\( \lambda(X,Y) \) is constant whenever \( g(X,X) = g(Y,Y) = 1 \), then we say that \( M \) has global constant type.

It follows that any \( AH_1 \)-manifold has global vanishing constant type from Definition 2.1.

Let \( M \) be a \( 2m \)-dimensional Kählerian manifold, for all \( X,Y,Z,U \in T_p(M) \) and \( p \in M \), the Bochner curvature tensor \( B [15] \) is defined by

\[
B(X,Y,Z,U) = R(X,Y,Z,U) - L(Y,Z)g(X,U) + L(Y,U)g(X,Z) - L(X,U)g(Y,Z) + L(X,Z)g(Y,U)
+ 2L(Z,JU)g(X,JY) + 2L(X,JY)g(Z,JU),
\]

where \( L = \frac{\theta}{2^{m+2}} - \frac{\tau}{8(m+1)(m+2)}g \), \( \theta \) is the Ricci tensor and \( \tau \) is the scalar curvature of \( M \). It is well-known that the Bochner curvature tensor is a complex analogue of the Weyl conformal curvature tensor \([23]\) on a Riemannian manifold.

2.2. Submanifolds of a Riemannian manifold. Let \( N \) be a submanifold of a Riemannian manifold \( M \) with a Riemannian metric \( g \). Then Gauss and Weingarten formulas are respectively given by

\[
\nabla_X Y = \tilde{\nabla}_X Y + h(X,Y) \quad \text{and} \quad \nabla_X \xi = -A_\xi X + \nabla_\xi X \quad \text{for all} \quad X,Y \in \chi(N) \quad \text{and} \quad \xi \in \chi^\perp(N).
\]

Here \( \nabla, \tilde{\nabla}, \) and \( \nabla^\perp \) are respectively the Riemannian, induced Riemannian, and induced normal connection in \( M,N \), and the normal bundle \( \chi^\perp(N) \) of \( N \), and \( h \) is the second fundamental form related to shape operator \( A \) corresponding to the normal vector field \( \xi \) by \( g(h(X,Y),\xi) = g(A_\xi X, Y) \). A submanifold \( N \) is said to be totally geodesic if its second fundamental form identically vanishes: \( h = 0 \), or equivalently \( A_\xi = 0 \). We say that \( N \) is totally umbilical submanifold in \( M \) if for all \( X,Y \in \chi(N) \), we have

\[
h(X,Y) = g(X,Y)\eta,
\]

where \( \eta \in \chi^\perp(N) \) is the mean curvature vector field of \( N \) in \( M \). A vector field \( \xi \in \chi^\perp(N) \) is said to be parallel if \( \nabla_X \xi = 0 \) for each \( X \in \chi(N) \). The Codazzi equation is given by

\[
(R(X,Y)Z)^\perp = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),
\]

for all \( X,Y,Z \in \chi(N) \), where \(^\perp \) denotes the normal component and the covariant derivative of \( h \) denoted by \( \nabla_X h \) is defined by

\[
(\nabla_X h)(Y,Z) = \nabla_X^\perp (h(Y,Z)) - h(\tilde{\nabla}_X Y,Z) - h(Y,\tilde{\nabla}_X Z),
\]

for all \( X,Y,Z \in \chi(N) \) \((2, 5, 9, 22)\).

2.3. Anti-invariant submanifolds of an almost Hermitian manifold. Let \( M \) be a \( 2m \)-dimensional almost Hermitian manifold endowed with an almost complex structure \( J \) and a Hermitian metric \( g \). An \( n \)-dimensional Riemannian manifold \( N \) isometrically immersed in \( M \) is called an anti-invariant submanifold of \( M \) (or totally real submanifold of \( M \)) if \( JT_p(N) \subset T_p(N)^\perp \) for each point \( p \) of \( N \). Then we have \( m \geq n \) \((23)\).
3. The axiom of anti-invariant 2-spheres

S. Yamaguchi and M. Kon [22] introduced the axiom of anti-invariant 2-spheres as: An almost Hermitian manifold $M$ is said to satisfy the axiom of anti-invariant 2-spheres, if for each point $p \in M$ and each anti-holomorphic 2-plane $\sigma$ of the tangent space $T_p(M)$, there exists a 2-dimensional totally umbilical anti-invariant submanifold $N$ such that $p \in N$ and $T_p(N) = \sigma$. They proved in Theorem 1([22]) that a Kählerian manifold with the axiom of anti-invariant 2-spheres is a complex space form. Here, we give a generalization of Theorem 1([22]). From now on, we shall assume that all manifolds are connected throughout this study.

We shall need the following Lemma for the proof of the main Theorem 3.1 below.

**Lemma 3.1.** ([11]) Let $N$ be an anti-invariant submanifold of a nearly Kählerian manifold $M$. Then

$$A_{JZ}X = A_{JX}Z$$

holds for any two vectors $X$ and $Z$ tangent to $N$, where $A$ is the shape operator of $N$.

Let $N$ be given as in Lemma 3.1, then

$$g(A_{JZ}X,Y) = g(A_{JX}Z,Y) = g(A_{JY}X,Z)$$

for any vectors $X, Y$ and $Z$ tangent to $N$, from (3.1). This equation is equivalent to

$$g(h(X,Y), JZ) = g(h(Z,Y), JX) = g(h(X,Z), JY)$$

where $h$ is the second fundamental form of $N$.

**Theorem 3.1.** Let $M$ be a nearly Kählerian manifold of dimension $2m \geq 6$. If $M$ satisfies the axiom of anti-invariant 2-spheres, then $M$ is a space of constant holomorphic sectional curvature.

**Proof.** Let $p$ be any point of $M$. Let $X$ and $Y$ be any orthonormal vectors of $T_p(M)$ spanning an anti-holomorphic 2-plane $\sigma$. By the axiom of anti-invariant 2-spheres, there exists a two-dimensional totally umbilical anti-invariant submanifold $N$ such that $p \in N$ and $T_p(N) = \sigma$. Then, we have

$$R(X,Y)Y^\perp = \nabla^\perp_X \eta$$

with the help of (2.1) and (2.3) from (2.2), where $\eta$ is the mean curvature vector field of $N$ in $M$. We find

$$R(X,Y,Y,JX) = g(\nabla^\perp_X \eta, JX)$$

from (3.4), since $JX$ is normal to $N$. If we put $Y = Z$ into (3.3), we obtain

$$g(\eta, JX) = 0$$

by using (2.1). Similarly, we also have $g(\eta, JY) = g(\eta, JZ) = 0$, for any vectors $X, Y, Z$ tangent to $N$. If we differentiate the equation (3.6) with respect to $X$, then we have

$$0 = X[g(\eta, JX)] = g(\nabla^\perp_X \eta, JX) + g(\eta, \nabla^\perp_X JX)$$

Upon straightforward calculation, we see that $g(\eta, \nabla^\perp_X JX) = 0$. Thus, we obtain

$$g(\nabla^\perp_X \eta, JX) = 0$$
from (3.7). We get
\begin{equation}
R(X, Y, Y, JX) = 0
\end{equation}
by combining (3.8) with (3.5). For all orthonormal vectors \(X, Y \in T_p(M)\) with \(g(X, JY) = 0\), we derive
\begin{equation}
R(JY, Y, Y, JX) = 0,
\end{equation}
by replacing \(X\) by \(\frac{\sqrt{2}}{2}(X + JY)\) in (3.9). In this case, it follows that \(M\) has pointwise constant holomorphic sectional curvature, using (3.10) and Lemma 1(12). We conclude that \(M\) is a space of constant holomorphic sectional curvature by Theorem 6(10).

4. The axiom of co-holomorphic \((2n + 1)\)-spheres

Let \(M\) be a 2\(m\)-dimensional almost Hermitian manifold. L. Vanhecke [20] defined a co-holomorphic \((2n + 1)\)-plane as a \((2n + 1)\)-plane containing a holomorphic \(2n\)-plane for the manifold \(M\). It is not difficult to see that a co-holomorphic \((2n + 1)\)-plane contains an anti-holomorphic \((n + 1)\)-plane and that \(1 \leq n \leq m - 1\). He also gave the axiom of co-holomorphic \((2n + 1)\)-spheres as: A 2\(m\)-dimensional almost Hermitian manifold \(M\) which is said to satisfy the axiom of co-holomorphic \((2n + 1)\)-spheres, if for each point \(p \in M\) and each co-holomorphic \((2n + 1)\)-plane \(\sigma\) of the tangent space \(T_p(M)\), there exists a \((2n + 1)\)-dimensional totally umbilical submanifold \(N\) such that \(p \in N\) and \(T_p(N) = \sigma\). He studied this axiom for \(AH_3\)-manifolds and obtained several results.

Now, we study this axiom for larger classes of almost Hermitian manifolds.

**Lemma 4.1.** Let \(M\) be an almost Hermitian manifold of dimension 2\(m \geq 4\). If \(M\) satisfies the axiom of co-holomorphic \((2n + 1)\)-spheres, then we have
\begin{equation}
\lambda(X, Y) = K(X, Y),
\end{equation}
for all orthonormal vectors \(X, Y \in T_p(M)\) with \(g(X, JY) = 0\), where \(\lambda(X, Y) = R(X, Y, X, Y) - R(X, Y, JX, JY)\) and \(K\) denotes anti-holomorphic sectional curvature.

**Proof.** Let \(p\) be an arbitrary point of \(M\). Let \(X\) and \(Y\) be any orthonormal vectors in \(T_p(M)\) with \(g(X, JY) = 0\), that is, they span an anti-holomorphic plane. Consider the co-holomorphic \((2n + 1)\)-plane \(\sigma\) containing \(X, JX\), and \(Y\) such that \(JY\) is normal to \(\sigma\). By the axiom of co-holomorphic \((2n + 1)\)-spheres, there exists a \((2n + 1)\)-dimensional totally umbilical submanifold \(N\) such that \(p \in N\) and \(T_p(N) = \sigma\). Then, we have
\begin{equation}
(R(X, Y)JX)\perp = 0 .
\end{equation}
with the help of (2.1) and (2.3), from (2.2). We get
\begin{equation}
R(X, Y, JX, JY) = 0 ,
\end{equation}
from (4.2), since \(JY\) is normal to \(N\). Thus, our assertion follows from Definition 2.1 and (4.3).

Now, we are ready to prove our second main result.
Theorem 4.1. Let $M$ be an almost Hermitian manifold of dimension $2m \geq 6$. If $M$ satisfies the axiom of co-holomorphic $(2n + 1)$-spheres for some $n$, then $M$ has pointwise constant type $\alpha$ if and only if $M$ has pointwise constant anti-holomorphic sectional curvature $\alpha$.

Proof. Let $M$ be an almost Hermitian manifold of dimension $2m \geq 6$ satisfying the axiom of co-holomorphic $(2n + 1)$-spheres for some $n$. If $M$ has pointwise constant type, that is, $M$ has constant type at $p$ for all $p \in M$. Then, for all $X, Y, Z \in T_p(M)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic and $g(Y, Y) = g(Z, Z)$, we have

$$\lambda(X, Y) = \lambda(X, Z)$$ \hspace{1cm} (4.4)

Here, we can assume that $g(Y, Y) = g(Z, Z) = 1$. Thus, for all orthonormal vectors $X, Y, Z \in T_p(M)$ with $g(X, JY) = g(X, JZ) = 0$, we get

$$K(X, Y) = K(X, Z)$$ \hspace{1cm} (4.5)

from Lemma 4.1.

On the other hand, we can choose a unit vector $U$ in $(\text{span}\{X, JX\})^\perp \cap (\text{span}\{Z, JZ\})^\perp$, since the dimension of $M$ is greater than 6. Then, we have

$$K(X, U) = K(X, Z)$$ \hspace{1cm} (4.6)

from (4.5). This implies that the sectional curvature is same for all anti-holomorphic sections which contain any given vector $X$. Hence, we write

$$K(X, Y) = K(Y, Z) = K(Z, U)$$ \hspace{1cm} (4.7)

Therefore, we find

$$K(X, Y) = K(Z, U)$$ \hspace{1cm} (4.8)

for all $X, Y, Z, U \in T_p(M)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{Z, U\}$ are anti-holomorphic. It follows that the sectional curvature is same for all anti-holomorphic sections at $p \in M$. Namely, $M$ has pointwise constant anti-holomorphic sectional curvature.

Conversely, let $M$ be of pointwise constant anti-holomorphic sectional curvature and let $p$ be any point of $M$. Then for all orthonormal vectors $X, Y, Z \in T_p(M)$ with $g(X, JY) = g(X, JZ) = 0$, $(\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic planes and $g(X, X) = g(Y, Y) = g(Z, Z) = 1$), we have

$$K(X, Y) = K(X, Z)$$ \hspace{1cm} (4.9)

By Lemma 4.1, we get

$$\lambda(X, Y) = \lambda(X, Z)$$ \hspace{1cm} (4.10)

for all orthonormal vectors $X, Y, Z \in T_p(M)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic. It is not difficult to see that (4.10) also holds in the case $g(Y, Y) = g(Z, Z) \neq 1$. It follows that $M$ has constant type at $p$. Additionally, if the constant value of $\lambda(X, Y)$ equals $\alpha$, then the pointwise constant anti-holomorphic sectional curvature $K$ must be $\alpha$, because of Lemma 4.1. $\square$

We remark that above technical method was used also in Theorem 3.4([IS]).

Next, we give some applications of Theorem 4.1.
Theorem 4.2. Let $M$ be a $2m$-dimensional almost Hermitian manifold with pointwise constant type $\alpha$ and $m \geq 3$. If $M$ satisfies the axiom of co-holomorphic $(2n+1)$-spheres for some $n$, then

i) $M$ is a space of constant curvature $\alpha$ and $M$ has global constant type $\alpha$,

ii) $M$ is an $AH_2^3$-manifold.

Proof. Let $p$ be any point of $M$. Let $X$ and $Y$ be any orthonormal vectors in $T_p(M)$ with $g(X, JY) = 0$. Consider the co-holomorphic $(2n+1)$-plane $\sigma$ containing $X$, $JX$, and $JY$ such that $Y$ is normal to $\sigma$. By the axiom of co-holomorphic $(2n+1)$-spheres, there exists a $(2n+1)$-dimensional totally umbilical submanifold $N$ such that $p \in N$ and $T_p(N) = \sigma$. Then, we have

$$R(X, JX)JX = 0 \tag{4.11}$$

with the help of (2.1) and (2.3), from (2.2). Hence, we get

$$R(X, JX, JX, Y) = 0 \tag{4.12}$$

for all orthonormal vectors $X, Y \in T_p(M)$ with $g(X, JY) = 0$, since $Y$ is normal to $N$. It follows that $M$ is an $AH_3$-manifold with pointwise constant holomorphic sectional curvature using (4.12) and Lemma 1(12) together with Lemma 3(12). On the other hand, we have that $M$ has constant anti-holomorphic sectional curvature $\alpha$ at $p$, from Theorem 4.1. In this case, we obtain that the constant holomorphic sectional curvature $H$ of $M$ is $\alpha$ at $p$ from Theorem 5(19). By using Theorem 4(19) in Theorem 2(19), we obtain

$$K(X, Y) = \alpha \tag{4.13}$$

for all orthonormal vectors $X, Y \in T_p(M)$, where $K(X, Y) = R(X, Y, X, Y)$ is sectional curvature. It is not difficult to see that (4.13) is also true for all $X, Y \in T_p(M)$. By the well-known Schur’s theorem(23) it follows that $M$ is a space of constant curvature $\alpha$ and $M$ has global constant type.

Now, we prove the part ii). From the part i), automatically, both holomorphic and anti-holomorphic sectional curvature equal to $\alpha$. In which case, it follows from Theorem 3(11) that $M$ is an $AH_3$-manifold. \qed

Remark 4.1. Theorem 4.2 without part ii) was also obtained by O.T. Kassabov in (13) with different approach. It is a generalization of Theorem 1(20) concerning $AH_3$-manifolds.

By Theorem 4.1, Theorem 4.2 and Theorem 5(19), we have the following result which is a generalization of Corollary 1(20) concerning $AH_3$-manifolds.

Corollary 4.1. Let $M$ be a $2m$-dimensional almost Hermitian manifold with vanishing constant type and $m \geq 3$. If $M$ satisfies the axiom of co-holomorphic $(2n+1)$-spheres for some $n$, then $M$ is a flat $AH_2^3$-manifold.

We end this paper by giving a result related to the Bochner curvature tensor of a Kählerian manifold satisfying the axiom of co-holomorphic $(2n+1)$-spheres.

Theorem 4.3. Let $M$ be a Kählerian manifold of dimension $2m \geq 6$. If $M$ satisfies the axiom of co-holomorphic $(2n+1)$-spheres for some $n$, then $M$ has a vanishing Bochner curvature tensor.
Proof. Let $p$ be any point of $M$. Let $X, Y,$ and $Z$ be any unit vectors of $T_p(M)$, which span an anti-holomorphic 3-plane, that is, $g(X, Y) = g(X, Z) = g(Y, Z) = 0$, and $g(X, JY) = g(X, JZ) = g(Y, JZ) = 0$. Consider the co-holomorphic $(2n+1)$-plane $\sigma$ containing $X, JX,$ and $Y$ such that $Z$ is normal to $\sigma$. By the axiom of co-holomorphic $(2n+1)$-spheres, there exists a $(2n+1)$-dimensional totally umbilical submanifold $N$ such that $p \in N$ and $T_p(N) = \sigma$. Then, we have

$$(4.14) \quad (R(X, JX)Y)^\perp = 0,$$

and

$$(4.15) \quad (R(X, Y)JX)^\perp = 0,$$

with the help of (2.1) and (2.3) from (2.2). For all unit vectors $X, Y, Z \in T_p(M)$, which span an anti-holomorphic 3-plane, we respectively get

$$(4.16) \quad R(X, JX, Y, Z) = 0$$

and

$$(4.17) \quad R(X, JX, Z) = 0,$$

from (4.14) and (4.15), since $Z$ is normal to $N$. Thus, our assertion follows from (4.16), (4.17) and Lemma(14). \hfill \Box

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