HOMOMORPHISMS INTO A SIMPLE $\mathcal{Z}$-STABLE $C^*$-ALGEBRAS

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Abstract. Let $A$ and $B$ be unital separable simple amenable $C^*$-algebras which satisfy the Universal Coefficient Theorem. Suppose that $A$ and $B$ are $\mathcal{Z}$-stable and are of rationally tracial rank no more than one. We prove the following: Suppose that $\phi, \psi : A \to B$ are unital monomorphisms. There exists a sequence of unitaries $\{u_n\} \subset B$ such that

$$\lim_{n \to \infty} u_n^* \phi(a) u_n = \psi(a) \text{ for all } a \in A,$$

if and only if $[\phi] = [\psi]$ in $KL(A, B), \phi_\sharp = \psi_\sharp$ and $\phi^\dagger = \psi^\dagger$,

where $\phi_\sharp, \psi_\sharp : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ and $\phi^\dagger, \psi^\dagger : U(A)/CU(A) \to U(B)/CU(B)$ are the induced maps and where $T(A)$ and $T(B)$ are tracial state spaces of $A$ and $B$, and $CU(A)$ and $CU(B)$ are the closures of the commutator subgroups of the unitary groups of $A$ and $B$, respectively. We also show that this holds for some AH-algebra $A$ which is not necessarily simple.

Moreover, for any $\kappa \in KL(A, B)$ preserving the order and the identity, any $\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ a continuous affine map and any $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ a homomorphism which are compatible, we also show that there is a unital homomorphism $\phi : A \to B$ so that $([\phi], \phi_\sharp, \phi^\dagger) = (\kappa, \lambda, \gamma)$, at least in the case that $K_1(A)$ is a free group.

1. Introduction

For topological spaces, it is important to understand continuous maps from one topological space to another. Let $X$ and $Y$ be two compact Hausdorff spaces, and denote by $C(X)$ (or $C(Y)$) the $C^*$-algebra of complex valued continuous functions on $X$ (or $Y$). Any continuous map $\lambda : Y \to X$ induces a homomorphism $\phi$ from the commutative $C^*$-algebra $C(X)$ into the commutative $C^*$-algebra $C(Y)$ by $\phi(f) = f \circ \lambda$, and any $C^*$-homomorphism from $C(X)$ to $C(Y)$ arises this way. It should be noted, by the Gelfand theorem, every unital commutative $C^*$-algebra has the form $C(X)$ as above.

For non-commutative $C^*$-algebras, one also studies homomorphisms. Let $A$ and $B$ be two unital $C^*$-algebras and let $\phi, \psi : A \to B$ be two homomorphisms. An important problem in the study of $C^*$-algebras is to determine when $\phi$ and $\psi$ are (approximately) unitarily equivalent. One of the important aspects of $C^*$-algebra theory is the program of classification of amenable $C^*$-algebras by the Elliott invariant, or, otherwise known as the Elliott program. What are called the uniqueness theorems in the Elliott program is, in fact, a sufficient condition for two homomorphisms $\phi$ and $\psi$ being approximately unitarily equivalent, which plays an extremely important role in the study of Elliott program.

The last two decades saw the rapid development in the Elliott program. For instance, $C^*$-algebras that can be classified by the Elliott invariant include all unital simple inductive limits of homogeneous $C^*$-algebras (AH-algebras for short) with no dimension growth (\cite{2}). In fact,
classifiable simple C*-algebras includes all unital separable amenable simple C*-algebras with the tracial rank at most one which satisfy the Universal Coefficient Theorem (the UCT) (see [8]).

Recently, W. Winter’s method ([25]) greatly advances the Elliott classification program. The class of amenable separable simple C*-algebras that can be classified by the Elliott invariant has been enlarged so that it contains simple C*-algebras which no longer have finite tracial rank. In fact, with [25], [12], [18] and [15], the classifiable C*-algebras now include any unital separable simple Z-stable C*-algebra A satisfying the UCT such that A ⊗ U has the tracial rank no more than one for some UHF-algebra U. This class of C*-algebras is strictly larger than the class of AH-algebras without dimension growth. For example, it contains the Jiang-Su algebra Z which is projectionless (with the same K theory as that of C) and all simple unital inductive limits of so-called generalized dimension drop algebras (see [17]).

Recall that the Elliott invariant for a stably finite unital simple separable C*-algebra A is

$$\text{Ell}(A) := ((K_0(A), K_0(A)_, [1_A], T(A)), K_1(A)),$$

where \((K_0(A), K_0(A)_, [1_A], T(A))\) is the quadruple consisting of the \(K_0\)-group, its positive cone, an order unit and tracial simplex together with their pairing, and \(K_1(A)\) is the \(K_1\)-group.

Denote by \(\mathcal{C}\) the class of all unital simple C*-algebras A for which \(A \otimes U\) has tracial rank no more than one for some UHF-algebra \(U\) of infinite type. Suppose that \(A\) and \(B\) are two unital separable amenable C*-algebras in \(\mathcal{C}\) which satisfy the UCT. The classification theorem in [15] states that if the Elliott invariants of \(A\) and \(B\) are isomorphic, i.e.,

$$\text{Ell}(A) \cong \text{Ell}(B),$$

then, there is an isomorphism \(\phi : A \to B\) which carries the isomorphism above.

However, if one has two isomorphisms \(\phi, \psi : A \to B\), when are they approximately unitarily equivalent? A more general question is: for any two such C*-algebras \(A\) and \(B\), and, for any two homomorphisms \(\phi, \psi : A \to B\), when are they approximately unitarily equivalent?

If \(\phi\) and \(\psi\) are approximately unitarily equivalent, then one must have, as the classification theorem suggests,

$$[\phi] = [\psi] \text{ in } KL(A,B) \text{ and } \phi_* = \psi_*,$$

where \(\phi_*, \psi_* : \text{Aff}(T(A)) \to \text{Aff}(T(B))\) are the affine maps induced by \(\phi\) and \(\psi\), respectively. Moreover, as shown in [13], one also has

$$\phi^\dagger = \psi^\dagger,$$

where \(\phi^\dagger, \psi^\dagger : U(A)/CU(A) \to U(B)/CU(B)\) are homomorphisms induced by \(\phi\), \(\psi\), and \(CU(A)\) and \(CU(B)\) are the closures of the commutator subgroups of the unitary groups of \(A\) and \(B\), respectively.

In this paper, we will show that the above conditions are also sufficient, that is, \(\phi\) and \(\psi\) are approximately unitarily equivalent if and only if \([\phi] = [\psi] \text{ in } KL(A,B),\ \phi_* = \psi_*\) and \(\phi^\dagger = \psi^\dagger\). This type of results are often called the uniqueness theorem.

Not surprisingly, the proof of this uniqueness theorem is based on the methods developed in the proof of the classification result mentioned above, which can be found in [15, 14, 13, 18].
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and [11]. Most technical tools are stated in those papers, either directly or implicitly. In the present paper, we will collect them and then assemble them into production.

It should be noted that the above-mentioned uniqueness theorem also works in a more general setting where the source algebra $A$ is not necessary in the class $\mathcal{C}$. For example, the uniqueness theorem is also valid for certain AH-algebras $A$ which are not necessarily simple (which are called AH-algebras with Property (J) in this paper), Moreover, in the case that $B \otimes U$ has tracial rank zero for some UHF-algebra $U$ of infinite type, the uniqueness theorem holds for general AH-algebra $A$. In particular, $A$ could be just $C(X)$ for some compact metric space $X$. In that situation, the first version of this kind of uniqueness theorem was stated in [4], where $A = C(X)$ and $B$ is a unital simple $C^*$-algebra with the unique tracial state and with real rank zero, stable rank one and weakly unperforated $K_0(B)$. In [7], it was shown that, if $A = C(X)$ for some compact metric space $X$ and $B$ is a unital simple $C^*$-algebra with tracial rank zero, then any unital monomorphisms $\phi$ and $\psi$ from $A$ to $B$ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$ and $\phi_\sharp = \psi_\sharp$. This result was then generalized to the case that $B$ has tracial rank no more than one, with the additional assumption $\phi^\dagger = \psi^\dagger$. From this point of view, the main result in this paper may also be regarded as a further generalization of these so-called uniqueness theorems. One should also realize that these uniqueness theorems have a common root: The Brown-Douglass-Fillmore theorem for essentially normal operators. One version of it can be stated as follows: Two monomorphisms $\phi, \psi : C(X) \to B(H)/K$—the Calkin algebra, which is a unital simple $C^*$-algebra with real rank zero—are unitarily equivalent if and only if $[\phi] = [\psi]$ in $KK(C(X), B(H)/K)$.

As this research was under way, we learned that H. Matui was conducting his own investigation on the same problems. In fact, he proved the same uniqueness theorems mentioned under the assumption that $B \otimes U$ has tracial rank zero. Moreover, he actually showed the same result holds if the assumption that $B \otimes U$ has tracial rank zero is weaken to the assumption that $B \otimes U$ has real rank zero, stable rank one and weakly unperforated $K_0(B \otimes U)$, at least for the case that quasi-traces are traces and there are only finitely many of extremal tracial states.

In [19], it is shown that, for any partially ordered simple weakly unperforated rationally Riesz group $G_0$ with order unit $u$, any countable abelian group $G_1$, any metrizable Choquet simple $S$, and any surjective affine continuous map $r : S \to S_u(G_0)$ (the state space of $G_0$) which preserves extremal points, there exists one (and only one up to isomorphism) unital separable simple amenable $C^*$-algebra $A \in \mathcal{C}$ which satisfies the UCT so that $\text{Ell}(A) = (G_0, (G_0)_+, u, G_1, S, r)$.

A natural question is: Given two unital separable simple amenable $C^*$-algebras $A, B \in \mathcal{C}$ which satisfy the UCT, and a homomorphism $\Gamma$ from $\text{Ell}(A)$ to $\text{Ell}(B)$, does there exist a unital homomorphism $\phi : A \to B$? We will give an answer to this question. Related to the uniqueness theorem discussed earlier and also related to the above question, one may also ask the following: Given an element $\kappa \in KL(A, B)$ which preserves the unit and order, an affine map $\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ and a homomorphism $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ which are compatible, does there exist a unital homomorphism $\phi : A \to B$ so that $[\phi] = \kappa$, $\phi_\sharp = \lambda$ and $\phi^\dagger = \gamma$? We will, at least, partially answer this question.
2. Preliminaries

2.1. Let $A$ be a unital stably finite C*-algebra. Denote by $T(A)$ the simplex of tracial states of $A$ and denote by $\text{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. Suppose that $\tau \in T(A)$ is a tracial state. We will also use $\tau$ for the trace $\tau \otimes \text{Tr}$ on $M_k(A) = A \otimes M_k(\mathbb{C})$ (for every integer $k \geq 1$), where $\text{Tr}$ is the standard trace on $M_k(\mathbb{C})$. A trace $\tau$ is faithful if $\tau(a) > 0$ for any $a \in A_+ \setminus \{0\}$. Denote by $T_+(A)$ the convex subset of $T(A)$ consisting of all faithful tracial states.

Denote by $M_{\infty}(A)$ the set $\bigcup_{k=1}^{\infty} M_k(A)$, where $M_k(A)$ is regarded as a C*-subalgebra of $M_{k+1}(A)$ by the embedding

$$M_k(A) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{k+1}(A).$$

Define the positive homomorphism $\rho_A : K_0(A) \to \text{Aff}(T(A))$ by $\rho_A([p])(\tau) = \tau(p)$ for any projection $p$ in $M_k(A)$. Denote by $S(A) = C_0((0,1)) \otimes A$ the suspension of $A$, by $U(A)$ the unitary group of $A$, and by $U(A)_0$ the connected component of $U(A)$ containing the identity.

Suppose that $C$ is another unital C*-algebra and $\phi : C \to A$ is a unital *-homomorphism. Denote by $\phi_T : T(A) \to T(C)$ the continuous affine map induced by $\phi$, i.e., $\phi_T(\tau)(c) = \tau \circ \phi(c)$ for all $c \in C$ and $\tau \in T(A)$. Also denote by $\phi^\# : \text{Aff}(T(C)) \to \text{Aff}(T(A))$ by $\phi^\#(f)(\tau) = f(\phi_T(\tau))$, for all $\tau \in T(A)$.

**Definition 2.2.** Let $A$ be a unital C*-algebra. Denote by $CU(A)$ the closure of the subgroup generated by commutators. If $u \in U(A)$, the image of $u$ in $U(A)/CU(A)$ will be denoted by $\overline{u}$.

Let $B$ be another unital C*-algebra and let $\phi : A \to B$ be a unital homomorphism. Then $\phi$ maps $CU(A)$ into $CU(B)$. The induced homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$ is denoted by $\phi^\dagger$.

Let $n \geq 1$ be an integer and denote by $U_n(A)$ the unitary group of $M_n(A)$. Denote by $CU_n(A)$ the closure of commutator subgroup of $U_n(A)$. One views $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the embedding in $[2,1]$. Denote by $U_{\infty}(A)$ the union of $U_n(A)$. Denote by $U(A)_\infty/CU_{\infty}(A)$ the inductive limits of $U_n(A)/CU_n(A)$. We will also use $\phi^\dagger$ for the homomorphism from $U_{\infty}(A)/CU_{\infty}(A)$ into $U_{\infty}(B)/CU_{\infty}(B)$ induced by $\phi$.

**Definition 2.3.** Let $A$ be a unital C*-algebra and let $T(A)$ be the tracial state space. Suppose that $u \in U(A)_0$. Suppose that $u(t) \in C([0,1], A)$ is a piecewise smooth path of unitaries such that $u(0) = u$ and $u(1) = 1$. The de la Harpe and Skandalis determinant is defined by

$$\text{Det}(u(t))(\tau) = \frac{1}{2\pi i} \int_{0}^{1} \tau(\frac{du(t)}{dt} u(t)^*) dt \text{ for all } \tau \in T(A).$$

This gives a homomorphism $\text{Det} : U(A)_0 \to \text{Aff}(T(A))/\rho_A(K_0(A))$. We also extend this map on $U_{\infty}(A)_0$ into $\text{Aff}(T(A))/\rho_A(K_0(A))$. The homomorphism $\text{Det}$ vanishes on the commutator subgroup of $U_\infty(A)$. In fact, this induces an isomorphism $\overline{\text{Det}} : U_{\infty}(A)_0/CU_{\infty}(A) \to \text{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ Moreover, by a result of K. Thomsen $[2,1]$, one has the following short exact sequence

$$0 \to \text{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \overset{\eta}{\to} K_1(A) \to 0$$
which splits. We will fix a splitting map \( s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A) \). The notation \( I2 \) and \( s_1 \) will be used late without further warning. In the case that \( U(A)/U_0(A) = K_1(A) \), one has the following splitting short exact sequence:

\[
0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(A)/CU(A) \to K_1(A) \to 0.
\]

**Definition 2.4.** Let \( A \) be a unital C*-algebra and let \( C \) be a separable C*-algebra which satisfies the Universal Coefficient Theorem. By [1] of Dădărlat and Loring, 

\[
(2.3) \quad KL(C, A) = \text{Hom}_\Lambda(K(C), K(A)),
\]

where, for any C*-algebra \( B \),

\[
\overline{K}(B) = \left( K_0(B) \oplus K_1(B) \right) \oplus \left( \bigoplus_{n=2}^\infty (K_0(B, \mathbb{Z}/n\mathbb{Z}) \oplus K_1(B, \mathbb{Z}/n\mathbb{Z})) \right).
\]

We will identify the two objects in (2.3). Note that one may view \( KL(C, A) \) as a quotient of \( KK(C, A) \).

Denote by \( KL(C, A)^{++} \) the set of those \( \kappa \in \text{Hom}_\Lambda(K(C), K(A)) \) such that 

\[
\kappa(K_0(H) \setminus \{0\}) \subseteq K_0(A) \setminus \{0\}.
\]

Denote by \( \kappa_i : K_i(C) \to K_i(A) \) the homomorphism given by \( \kappa \), \( i = 0, 1 \). Denote by \( KL_0(C, A)^{++} \) the set of those elements \( \kappa \in KL(C, A)^{++} \) such that \( \kappa([1_C]) = [1_A] \).

Suppose that both \( A \) and \( C \) are unital and \( T(C) \neq \emptyset \) and \( T(A) \neq \emptyset \).

Let \( \lambda_T : T(A) \to T(C) \) be a continuous affine map. Let \( h_0 : K_0(C) \to K_0(A) \) be a positive homomorphism. We say \( \lambda_T \) is compatible with \( h_0 \) if for every projection \( p \in M_\infty(C) \), \( \lambda_T(\tau)(p) = \tau(h_0([p])) \) for all \( \tau \in T(A) \). Let \( \lambda : \text{Aff}(T_1(C)) \to \text{Aff}(T(A)) \) be an affine continuous map. We say \( \lambda \) and \( h_0 \) is compatible if \( h_0 \) is compatible to \( \lambda_T \), where \( \lambda_T : T(A) \to T_1(C) \) is the map induced by \( \lambda \). We say \( \kappa \) and \( \lambda \) (or \( \lambda_T \)) are compatible, if \( \kappa_0 \) is positive and \( \kappa_0 \) and \( \lambda \) is compatible.

Denote by \( KLT(e)(C, A)^{++} \) the set of those pairs \( (\kappa, \lambda_T) \), (or, \( (\kappa, \lambda) \)), where \( \kappa \in KL(C, A)^{++} \) and \( \lambda_T : T(A) \to T_1(C) \) (or, \( \lambda : \text{Aff}(T_1(C)) \to \text{Aff}(T(A)) \)) is a continuous affine map which is compatible with \( \kappa \). If \( \lambda \) is compatible with \( \kappa \), then \( \lambda \) maps \( \rho_A(K_0(C)) \) into \( \rho_B(K_0(A)) \). Therefore \( \lambda \) induces a continuous homomorphism \( \overline{\lambda} : \text{Aff}(T_1(C))/\rho_A(K_0(C)) \to \text{Aff}(T(A))/\rho_B(K_0(A)) \).

Suppose that \( \gamma : U_\infty(C)/CU_\infty(C) \to U_\infty(A)/CU_\infty(A) \) is a continuous homomorphism and \( h_1 : K_1(C) \to K_1(A) \) are homomorphisms for which \( h_0 \) is positive. We say \( \gamma \) and \( h_1 \) are compatible if \( \gamma(U_\infty(C)/CU_\infty(C)) \subseteq U_\infty(A)/CU_\infty(A) \) and \( \gamma \circ s_1 = s_1 \circ h_1 \). We say \( h_0, h_1, \lambda \) and \( \gamma \) are compatible, if \( \lambda \) and \( h_0 \) are compatible, \( \gamma \) and \( h_1 \) are compatible and 

\[
\overline{\text{Det}_B} \circ (\gamma|_{U_\infty(A)/CU_\infty(A)}) = \overline{\lambda} \circ \overline{\text{Det}_A}.
\]

We say \( \kappa, \lambda \) and \( \gamma \) are compatible, if \( \kappa_0, \kappa_1, \lambda \) and \( \gamma \) are compatible.

**2.5.** For each prime number, let \( p \) be a number in \( \{0, 1, 2, \ldots, +\infty\} \). Then a supernatural number is the formal product \( p = \prod_p p^\epsilon_p \). Here we insist that there are either infinitely many \( p \) in the product, or, one of \( \epsilon_p \) is infinite. Two supernatural numbers \( p = \prod_p p^\epsilon_p \) and \( q = \prod_p p^\epsilon_q \) are relatively prime if for any prime number \( p \), at most one of \( \epsilon_p(p) \) and \( \epsilon_p(q) \) is nonzero. A supernatural number \( p \) is called of infinite type if for any prime number, either \( \epsilon_p(p) = 0 \) or
\( \epsilon_p(p) = +\infty \). For each supernatural number \( p \), there is a UHF-algebra \( M_p \) associated to it, and the UHF-algebra is unique up to isomorphism.

**2.6.** Let \( Q \) be the UHF-algebra with \( (K_0(Q), K_0(Q)_+, [1_A]) = (\mathbb{Q}, \mathbb{Q}, 1) \) (then the supernatural number associated to \( Q \) is \( \prod p^{\infty} \)), and let \( M_p \) and \( M_q \) be two UHF-algebras with \( M_p \otimes M_q \cong Q \), and \( p = \prod p^{r_p(p)} \) and \( q = \prod p^{r_q(q)} \) relatively primes. Then, it follows that \( p \) and \( q \) are of infinite type. Denote by

\[
\mathbb{Q}_p = \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_n}] \subseteq \mathbb{Q}, \quad \text{where } \epsilon_{p_n}(p) = +\infty \text{ and }
\]

\[
\mathbb{Q}_q = \mathbb{Z}[\frac{1}{q_1}, \ldots, \frac{1}{q_m}] \subseteq \mathbb{Q}, \quad \text{where } \epsilon_{q_m}(q) = +\infty.
\]

Note that \( (K_0(M_p), K_0(M_p)_+, [1_{M_p}]) = (\mathbb{Q}_p, (\mathbb{Q}_p)_+, 1) \) and \( (K_0(M_q), K_0(N_q)_+, [1_{M_q}]) = (\mathbb{Q}_q, (\mathbb{Q}_q)_+, 1) \). Moreover, \( \mathbb{Q}_p \cap \mathbb{Q}_q = \mathbb{Z} \) and \( \mathbb{Q} = \mathbb{Q}_p + \mathbb{Q}_q \).

**2.7.** For any pair of relatively prime supernatural numbers \( p \) and \( q \), define the C*-algebra \( \mathcal{Z}_{p,q} \) by

\[
\mathcal{Z}_{p,q} = \{ f : [0, 1] \to M_p \otimes M_q ; f(0) \in M_p \otimes 1_{M_q} \text{ and } f(0) \in 1_{M_p} \otimes M_q \}.
\]

The Jiang-Su algebra \( \mathcal{Z} \) is the unital inductive limit of dimension drop interval algebras with unique trace, and \( (K_0(\mathcal{Z}), K_0(\mathcal{Z}), [1]) = (\mathbb{Z}, \mathbb{Z}^+, 1) \) (see [D]). By Theorem 3.4 of [22], for any pair of relatively prime supernatural numbers \( p \) and \( q \) of infinite type, the Jiang-Su algebra \( \mathcal{Z} \) has a stationary inductive limit decomposition:

\[
\mathcal{Z}_{p,q} \xrightarrow{\mathcal{Z}_{p,q}} \mathcal{Z}_{p,q} \xrightarrow{\cdots} \mathcal{Z}_{p,q} \xrightarrow{\cdots} \mathcal{Z}.
\]

By Corollary 3.2 of [22], the C*-algebra \( \mathcal{Z}_{p,q} \) absorbs the Jiang-Su algebra: \( \mathcal{Z}_{p,q} \otimes \mathcal{Z} \cong \mathcal{Z}_{p,q} \). A C*-algebra \( A \) is said to be \( \mathcal{Z} \)-stable, if \( A \otimes \mathcal{Z} \cong A \).

**Definition 2.8.** A unital simple C*-algebra \( A \) has tracial rank at most one, denoted by \( \text{TR}(A) \leq 1 \), if for any finite subset \( F \subseteq A \), any \( \epsilon > 0 \), and nonzero \( a \in A^+ \), there exist nonzero projection \( p \in A \) and a sub-C*-algebra \( I \cong \bigoplus_{i=1}^m C(X_i) \otimes M_{r(i)} \) with \( 1_I = p \) for some finite CW complexes \( X_i \) with dimension at most one such that

1. \( \| [x, p] \| \leq \epsilon \) for any \( x \in F \),
2. for any \( x \in F \), there is \( x' \in I \) such that \( \| p x p - x' \| \leq \epsilon \), and
3. \( 1 - p \) is Murray-von Neumann equivalent to a projection in \( a A a \).

In the above, if \( I \) can be chosen to be a finite dimensional C*-algebra, then \( A \) is said to have tracial rank zero, and in such case, we write \( \text{TR}(A) = 0 \). It is a theorem of Guihua Gong [3] that every unital simple AH-algebra with no dimension growth has tracial rank at most one. It has been proved in [13] that every \( \mathcal{Z} \)-stable unital simple AH-algebra has tracial rank at most one.

**Definition 2.9.** Denote by \( \mathcal{N} \) the class of all separable amenable C*-algebras which satisfy the Universal Coefficient Theorem. Denote by \( \mathcal{C} \) the class of all simple C*-algebras \( A \) for which \( \text{TR}(A \otimes M_p) \leq 1 \) for some UHF-algebra \( M_p \), where \( p \) is a supernatural number of infinite type. Note, by [19], that, if \( \text{TR}(A \otimes M_p) \leq 1 \) for some supernatural number \( p \) then \( \text{TR}(A \otimes M_p) \leq 1 \) for all supernatural number \( p \).
Denote by $\mathcal{C}_0$ the class of all simple $C^*$-algebras $A$ for which $\text{TR}(A \otimes M_p) = 0$ for some supernatural number $p$ of infinite type (and hence for all supernatural number $p$ of infinite type).

**Definition 2.10** (7.4 of [13]). Let $X$ be a compact metric space. It is said to satisfy the property (H) if the following holds. For any finite subset $\mathcal{F} \subset C(X)$, for any $\rho > 0$ and any $\sigma > 0$, there exists $\eta > 0$ (which depends on $\rho$ and $\mathcal{F}$ but not $\sigma$) and there exists a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ (which depend on $X, \mathcal{F}, \rho$ as well as $\sigma$) satisfying the following: Suppose that $\phi, \psi : C(X) \to M_n$ (for any integer $n$) are two unital homomorphisms such that

$$\|\phi(f) - \psi(f)\| < \delta, \quad \forall f \in \mathcal{G},$$

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma \eta \quad \text{and} \quad \mu_{\tau \circ \psi}(O_\eta) \geq \sigma \eta$$

for any open ball $O_\eta$ of $X$ with radius $\eta$, where $\tau$ is the normalized trace on $M_n$, and

$$\text{ad}(u) \circ \phi = \psi, \quad \text{for some unitary } u \in A.$$

Then, there exists a homomorphism $\Phi : C(X) \to C([0, 1], M_n)$ such that

$$\pi_0 \circ \Phi = \phi, \quad \pi_1 \circ \Phi = \psi \quad \text{and} \quad \|\psi(f) - \pi_t \circ \Phi(f)\| < \rho, \quad \forall f \in \mathcal{F},$$

where $\pi_t$ is the evaluation map at $t$.

**Definition 2.11** (8.2 of [13]). Let $X_0$ be the family of finite CW complexes which consists of all those with dimension no more than one and all those which have property (H). Note that $X_0$ contains all finite CW complex $X$ with finite $K_1(C(X)), I \times \mathbb{T}^n$-dimensional tori and those with the form $\mathbb{T} \vee \cdots \vee \mathbb{T} \vee Y$ with some finite CW complex $Y$ with $K_1(C(Y))$ being torsion.

Let $X$ be the family of finite CW complexes which contains all those in $X_0$ and those finite CW complexes with torsion $K_1$.

**Definition 2.12** (11.4 of [13]). An AH-algebra $C$ is said to have property (J) if $C$ is isomorphic to an inductive limit $\lim_{j \to \infty} (C_n, \phi_j)$, where $\bigoplus_{j=1}^{\mathbb{R}(i)} P_{n,j} M_{r(n,j)}(C(X_{n,j})) P_{n,j}$, $X_{n,j} \in X$ and where $P_{n,j} \in M_{r(n,j)}(C(X_{n,j}))$ is a projection, and each $\phi_j$ is injective.

Any unital simple AH-algebra without dimension growth has property (J). Moreover, for AH-algebras with property (J), one has the following uniqueness theorem:

**Theorem 2.13** (11.7 of [13]). Let $C$ be a unital AH-algebra with property (J) and let $A$ be a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms.

Then $\phi$ and $\psi$ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \quad \text{in } KL(C, A),$$

$$\phi^* = \psi^* \quad \text{and} \quad \phi^\dagger = \psi^\dagger.$$

**Remark 2.14.** One of the main purposes of this paper is to generalize this result so that $A$ can be allowed in the class $C$ mentioned in 2.9.
2.15. Let $A$ and $B$ be two unital C*-algebras. Let $h : A \to B$ be a homomorphism and $v \in U(B)$ be such that

$$[h(g), v] = 0 \quad \text{for any } g \in A.$$  

We then have a homomorphism $\overline{h} : A \otimes C(\mathbb{T}) \to B$ defined by $f \otimes g \mapsto h(f)g(v)$ for any $f \in A$ and $g \in C(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \to K_1(A \otimes C(\mathbb{T})) \quad \text{and} \quad \beta^{(1)} : K_1(A) \to K_0(A \otimes C(\mathbb{T})).$$

The second one is the usual Bott map. Note that, in this way, one writes $K_i = K_i(A \otimes C(\mathbb{T}))$ and $K_0 = K_0(A \otimes C(\mathbb{T}))$.

Let us use $\beta^{(0)} : K_0(A) \to K_1(A \otimes C(\mathbb{T}))$ and $\beta^{(1)} : K_1(A) \to K_0(A \otimes C(\mathbb{T}))$ to denote the projection.

For each integer $k \geq 2$, one also has the following injective homomorphisms:

$$\beta^{(i)}_k : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_{i-1}(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1.$$  

Thus, we write

$$K_1(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) = K_1(A, \mathbb{Z}/k\mathbb{Z}) \oplus \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z}).$$  

Denote by $\widehat{\beta}^{(i)}_k : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z})$ the map analogous to that of $\widehat{\beta}^{(i)}$.

If $x \in K(A)$, we use $\beta(x)$ for $\beta^{(i-1)}(x)$ if $x \in K_i(A)$, and $\beta^{(i)}_k(x)$ if $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\beta : K(A) \to K(A \otimes C(\mathbb{T}))$ as well as $\beta : K(A) \to \beta(K(A))$. Therefore, we may write $K(A \otimes C(\mathbb{T})) = K(A) \oplus \beta(K(A))$.

On the other hand, $h$ induces homomorphisms

$$\overline{h}_{s_i,k} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}),$$  

$k = 0, 2, \ldots, $ and $i = 0, 1$.

We use Bott($h,v$) for all homomorphisms $\overline{h}_{s_i,k} \circ \beta^{(i)}_k$, and we use bott($h,v$) for the homomorphism $\overline{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B)$, and bott($h,v$) for the homomorphism $\overline{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)$. Bott($h,v$) as well as bott($h,v$) ($i = 0, 1$) may be defined for a unitary $v$ which only approximately commutes $h$. In fact, given a finite subset $\mathcal{P} \subset K(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$\text{Bott}(h,v)|_{\mathcal{P}}$$  

is well defined if

$$\|[h(a), v]\| < \delta_0$$  

for all $a \in \mathcal{F}$. See 2.11 of [11] and 2.10 of [10] for more details.

We have the following generalized Exel’s formula for the traces of Bott elements.

**Theorem 2.16** (Theorem 3.5 of [15]). There is $\delta > 0$ satisfying the following: Let $A$ be a unital separable simple C*-algebra with $\text{TR}(A) \leq 1$ and let $u, v \in U(A)$ be two unitaries such that $\|uv - vu\| < \delta$. Then bott$_1(u,v)$ is well defined and

$$\tau(\text{bott}_1(u,v)) = \frac{1}{2\pi i}(\tau(\log(vuv^*u^*)))$$  

for all $\tau \in T(A)$.  


Definition 2.17. Let $A$ be a unital C*-algebra with $T(A) \neq \emptyset$. Let us say that $A$ has Property (B1) if the following holds: For any unitary $z \in U(M_k(A))$ (for some integer $k \geq 1$) with $\text{sp}(z) = \mathbb{T}$, there is a non-decreasing function $1/4 > \delta_z(t) > 0$ on $[0, 1]$ with $\delta_z(0) = 0$ such that for any $x \in K_0(A)$ with $|\tau(x)| \leq \delta_z(\varepsilon)$ for all $\tau \in T(A)$, there exists a unitary $u \in M_k(A)$ such that

$$\|u, z\| < \min\{\varepsilon, \frac{1}{4}\} \text{ and } \text{bott}_1(u, z) = x.$$  

Let $C$ be a unital separable C*-algebra. Let $1/4 > \Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) > 0$ be a function defined on $t \in [0, 1]$, the family of all finite subsets $\mathcal{F} \subset C$, and the family of all finite subsets $\mathcal{P}_0 \subset K_0(C)$, and family of all finite subsets $\mathcal{P}_1 \subset K_1(C)$, and the set of all unital monomorphisms $h : C \to A$. Let us say that $A$ has Property (B2) associated with $C$ and $\Delta_c$ if the following holds:

For any unital monomorphism $h : C \to A$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C$, any finite subset $\mathcal{P}_0 \subset K_0(C)$, and any finite subset $\mathcal{P}_1 \subset K_1(C)$, there are finitely generated subgroups $G_0 \subset K_0(C)$ and $G_1 \subset K_1(C)$ with finite sets of generators $G_0$ and $G_1$ respectively, such that $\mathcal{P}_0 \subset G_0$ and $\mathcal{P}_1 \subset G_1$, satisfying the following: For any homomorphisms $b_0 : G_0 \to K_1(A)$ and $b_1 : G_1 \to K_0(A)$ such that

$$|\tau \circ b_1(g)| < \Delta_c(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h)$$

for any $g \in G_1$ and any $\tau \in T(A)$, there exists a unitary $u \in U(A)$ such that

$$\text{bott}_0(h, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \quad \text{bott}_1(h, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad \text{and}$$

$$\|h(c), u\| < \varepsilon \quad \text{for all } c \in \mathcal{F}.$$  

Remark 2.18. By Theorem 6.3 of [15], any unital simple separable C*-algebra with tracial rank at most one (in particular any simple separable C*-algebra with tracial rank zero) has Property (B1). Also by Theorem 6.3 of [15] (see also Remark 6.4), any unital simple separable C*-algebra $A$ with tracial rank no more than one has Property (B2) with respect to any unital AH-algebra $C$ with Property (J) (see Definition 2.12 below) and any monomorphism from $C$ to $A$.

By Lemma 7.5 of [11], any simple unital separable C*-algebra $A$ with tracial rank at most one has Property (B2) with respect to any unital AH-algebra $C$ and any monomorphism from $C$ to $A$.

3. Rotation maps

In this section, we collect several facts on the rotation map which are going to be used frequently in the rest of the paper. Most of them are known.

Definition 3.1. Let $A$ and $B$ be two unital C*-algebras, and let $\psi$ and $\phi$ be two unital monomorphisms from $B$ to $A$. Then the mapping torus $M_{\phi, \psi}$ is the C*-algebra defined by

$$M_{\phi, \psi} := \{f \in C([0, 1], A); \ f(0) = \phi(b) \text{ and } f(1) = \psi(b) \text{ for some } b \in B\}.$$  

For any $\psi, \phi \in \text{Hom}(B, A)$, denoting by $\pi_0$ the evaluation of $M_{\phi, \psi}$ at 0, we have the short exact sequence

$$0 \longrightarrow S(A) \xrightarrow{1} M_{\phi, \psi} \xrightarrow{\pi_0} B \longrightarrow 0.$$
If $\phi_{s_i} = \psi_{s_i}$ ($i = 0, 1$), then the corresponding six-term exact sequence breaks down to the following two extensions:

\[
\eta_h(M_{\phi,\psi}) : 0 \longrightarrow K_{i+1}(A) \longrightarrow K_i(M_{\phi,\psi}) \longrightarrow K_i(B) \longrightarrow 0 \quad (i = 0, 1).
\]

3.2. Suppose that, in addition, for any continuous piecewise smooth path of unitaries $\phi_{t} = w(t)u(t)$ in $A$.

(3.1)

\[
\tau \circ \phi = \tau \circ \psi \quad \text{for all} \quad \tau \in T(A).
\]

For any continuous piecewise smooth path of unitaries $u(t) \in M_{\phi,\psi}$, consider the path of unitaries $w(t) = u^*(0)u(t)$ in $A$. Then it is a continuous and piecewise smooth path with $w(0) = 1$ and $w(1) = u^*(0)u(1)$. Denote by $R_{\phi,\psi}(u) = \text{Det}(w)$ the determinant of $w(t)$. It is clear with the assumption of (3.1) that $R_{\phi,\psi}(u)$ depends only on the homotopy class of $u(t)$. Therefore, it induces a homomorphism, denoted by $R_{\phi,\psi}$, from $K_1(M_{\phi,\psi})$ to $\text{Aff}(T(A))$. One has the following lemma.

**Lemma 3.3** (3.3 of [11]). When (3.1) holds, the following diagram commutes:

\[
\begin{array}{ccc}
K_0(A) & \rightarrow & K_1(M_{\phi,\psi}) \\
\rho_A & \downarrow & \vee \\
\text{Aff}(T(A)) & \rightarrow & R_{\phi,\psi}
\end{array}
\]

**Definition 3.4.** Fix two unital C*-algebras $A$ and $B$ with $T(A) \neq \emptyset$. Define $\mathcal{R}_0$ to be the subset of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$ consisting of those homomorphisms $h \in \text{Hom}(K_1(B), \text{Aff}(T(A)))$ for which there exists a homomorphism $d : K_1(B) \rightarrow K_0(A)$ such that

\[h = \rho_A \circ d.\]

It is clear that $\mathcal{R}_0$ is a subgroup of $\text{Hom}(K_1(B), \text{Aff}(T(A)))$.

3.5. If $[\phi] = [\psi]$ in $KK(B, A)$, then the exact sequences $\eta_i(M_{\phi,\psi})$ ($i = 0, 1$) split. In particular, there is a lifting $\theta : K_1(B) \rightarrow K_1(M_{\phi,\psi})$. Consider the map

\[R_{\phi,\psi} \circ \theta : K_1(B) \rightarrow \text{Aff}(T(A)).\]

If a different lifting $\theta'$ is chosen, then, $\theta - \theta'$ maps $K_1(B)$ into $K_0(A)$. Therefore

\[R_{\phi,\psi} \circ \theta - R_{\phi,\psi} \circ \theta' \in \mathcal{R}_0.\]

Then define

\[\overline{R}_{\phi,\psi} = [R_{\phi,\psi} \circ \theta] \in \text{Hom}(K_1(B), \text{Aff}(T(A)))/\mathcal{R}_0.\]

If $[\phi] = [\psi]$ in $KL(B, A)$, then the exact sequences $\eta_i(M_{\phi,\psi})$ ($i = 0, 1$) are pure, i.e., any finitely generated subgroup in the quotient groups has a lifting. In particular, for any finitely generated subgroup $G \subseteq K_1(B)$, one has a map

\[R_{\phi,\psi} \circ \theta : G \rightarrow \text{Aff}(T(A)),\]

where $\theta : G \rightarrow K_1(M_{\phi,\psi})$ is a lifting. With a similar definition of $\mathcal{R}_0$ of 3.4 with $K_1(B)$ replaced by $G$, one shows that if a different lifting $\theta'$ is chosen, then,

\[R_{\phi,\psi} \circ \theta - R_{\phi,\psi} \circ \theta' \in \mathcal{R}_0.\]
One defines
\[ R_{\phi,\psi}|_G = [R_{\phi,\psi} \circ \theta]|_G \in \Hom(G, \text{Aff}(\text{T}(A)))/R_0. \]

Let \( \phi_1, \psi_1 : B \to A \) be another pair of unital monomorphisms. We will write
\[ R_{\phi,\psi} = R_{\phi_1,\psi_1} \]
if \( R_{\phi,\psi}|_G = R_{\phi_1,\psi_1}|_G \) for every finitely generated subgroup \( G \) of \( K_1(B) \). In particular, we write
\[ R_{\phi,\psi} = 0 \]
if \( R_{\phi,\psi}|_G = 0 \) for every finitely generated subgroup \( G \) of \( K_1(B) \). This is equivalent to the condition that
\[ [\phi] = [\psi] \text{ in } KL(B,A) \text{ and } R_{\phi,\psi}(K_1(M_{\phi,\psi})) \subseteq \rho_A(K_0(A)). \]

See 3.4 of [15] for more details.

**Lemma 3.6 (Lemma 9.2 of [15]).** Let \( C \) and \( A \) be unital \( C^* \)-algebras with \( \text{T}(A) \neq \emptyset \). Suppose that \( \phi, \psi : C \to A \) are two unital homomorphisms such that
\[ [\phi] = [\psi] \text{ in } KL(C,A), \phi_z = \psi_z \text{ and } \phi^\dagger = \psi^\dagger. \]

Then the image of \( R_{\phi,\psi} \) is in the \( \rho_A(K_0(A)) \subseteq \text{Aff}(\text{T}(A)) \).

**Proof.** Let \( z \in K_1(C) \). Suppose that \( u \in U_n(C) \) for some integer \( n \geq 1 \) such that \([u] = z\). Note that \( \psi(u)^* \phi(u) \in CU_n(A) \). Thus, by [23] for any continuous and piecewise smooth path of unitaries \( \{w(t) : t \in [0,1]\} \subseteq U(A) \) with \( w(0) = \psi(u)^* \phi(u) \) and \( w(1) = 1 \),
\[ (3.2) \quad \text{Det}(w)(\tau) = \int_0^1 \tau(\frac{dw(t)}{dt}w(t)^*)dt \in \rho_A(K_0(A)). \]

Suppose that \( \{(v(t) : t \in [0,1]\} \) is a continuous and piecewise smooth path of unitaries in \( U_n(A) \) with \( v(0) = \phi(u) \) and \( v(1) = \psi(u) \). Define \( w(t) = \psi(u)^*v(t) \). Then \( w(0) = \psi^*(u)\phi(u) \) and \( w(1) = 1 \). Thus, by (3.2),
\[ (3.3) \quad R_{\phi,\psi}(z)(\tau) = \int_0^1 \tau(\frac{dv(t)}{dt}v(t)^*)dt \]
\[ (3.4) = \int_0^1 \tau(\psi(u)^*\frac{dv(t)}{dt}v(t)^*)dt \]
\[ (3.5) = \int_0^1 \tau(\frac{dw(t)}{dt}w(t)^*)dt \in \rho_A(K_0(A)). \]

\[ \Box \]

**3.7.** Let \( A \) be a unital \( C^* \)-algebra and let \( u \) and \( v \) be two unitaries with \( \|u^*v - 1\| < 2 \). Then \( h = \frac{1}{2\pi i} \log(u^*v) \) is a well defined self-adjoint element of \( A \), and \( w(t) := u \exp(2\pi i ht) \) is a smooth path of unitaries connecting \( u \) and \( v \). It is a straightforward calculation that for any \( \tau \in \text{T}(A) \),
\[ \text{Det}(w(t))(\tau) = \frac{1}{2\pi i} \tau(\log(u^*v)). \]
3.8. Let $A$ be a unital C*-algebra, and let $u$ and $w$ be two unitaries. Suppose that $w \in U_0(A)$. Then $w = \prod_{k=0}^{m} \exp(2\pi ih_k)$ for some self-adjoint elements $h_0, \ldots, h_m$. Define the path

$$w(t) = \left(\prod_{k=0}^{l-1} \exp(2\pi ih_k)\right) \exp(2\pi ih_l t), \quad \text{if } t \in [(l-1)/m, l/m],$$

and define $u(t) = w^*(t)uw(t)$ for $t \in [0,1]$. Then, $u(t)$ is continuous and piecewise smooth, and $u(0) = u$ and $u(1) = w^*uw$. A straightforward calculation shows that $\det(u(t)) = 0$.

In general, if $w$ is not in the path connected component, one can consider unitaries $\diag(u,1)$ and $\diag(w,w^*)$. Then, the same argument as above shows that there is a piecewise smooth path $u(t)$ of unitaries in $M_2(A)$ such that $u(0) = \diag(u,1)$, $u(1) = \diag(w^*uw,1)$, and

$$\det(u(t)) = 0.$$

**Lemma 3.9** (Lemma 3.5 of [11]). Let $B$ and $C$ be two unital C*-algebras with $T(B) \neq 0$. Suppose that $\phi, \psi : C \to B$ are two unital monomorphisms such that $[\phi] = [\psi]$ in $KL(C,B)$ and $\tau \circ \phi = \tau \circ \psi$ for all $\tau \in T(B)$. Suppose that $u \in U_1(C)$ is a unitary and $w \in U_1(B)$ such that

$$\|((\phi \otimes \id_{M_l})(u)w^*(\psi \otimes \id_{M_l})(u^*)w - 1\| < 2.$$

Then, for any unitary $U \in U_1(M_{\phi,\psi})$ with $U(0) = (\phi \otimes \id_{M_l})(u)$ and $U(1) = (\psi \otimes \id_{M_l})(u)$ such that

$$\frac{1}{2\pi i} \tau(\log(((\phi \otimes \id_{M_l})(u^*)w^*(\psi \otimes \id_{M_l})(u)w)) - R_{\phi,\psi}([U])(\tau) \in \rho_B(K_0(B))$$

**Proof.** Without loss of generality, one may assume that $u \in C$. Moreover, to prove the theorem, it is enough to show that (3.6) holds for one path of unitaries $U(t)$ in $M_2(B)$ with $U(0) = \diag(\phi(u),1)$ and $U(1) = \diag(\psi(u),1)$.

Let $U_1$ be the path of unitaries specified in 3.7 with $U_1(0) = \diag(\phi(u),1)$ and $U_1(1/2) = \diag(w^*\psi(u)w,1)$, and let $U_2$ be the path specified in 3.8 with $U_2(1/2) = \diag(w^*\psi(u)w,1)$ and $U_2(1) = \diag(\psi(u),1)$.

Set $U$ the path of unitaries by connecting $U_1$ and $U_2$. Then $U(0) = \diag(\phi(u),1)$ and $U(1) = \diag(\psi(u),1)$. By applying 3.7 and 3.8 for any $\tau \in T(B)$, one computes that

$$R_{\phi,\psi}([U]) = \det(U(t))(\tau) = \det(U_1(t))(\tau) + \det(U_2(t))(\tau) = \frac{1}{2\pi i} \tau(\phi(u^*)w^*\psi(u)w),$$

as desired. \qed

4. APPROXIMATELY UNITARILY EQUIVALENT

First we begin with the following lemma which is a simple combination of the uniqueness theorem 2.13 and the proof of Theorem 4.2 in [18].

**Lemma 4.1.** Let $A$ be a C*-algebras with $\TR(A) \leq 1$, and let $C$ be a unital AH-algebra with property (J). If there are monomorphisms $\phi, \psi : C \to A$ such that

$[\phi] = [\psi]$ in $KL(C,A)$, $\phi_2 = \psi_2$, and $\phi^\dagger = \psi^\dagger$. 


then, for any $2 > \epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$, any finite subset $\mathcal{P} \subseteq U_n(C)$ for some $n \geq 1$, there exist a finite subset $\mathcal{G} \subseteq K_1(C)$ with $\overline{\mathcal{P}} \subseteq \mathcal{G}$ (where $\overline{\mathcal{P}}$ is the image of $\mathcal{P}$ in $K_1(C)$) and $\delta > 0$ such that, for any map $\eta : \langle \mathcal{G} \rangle \to \text{Aff}(T(A))$ with $|\eta(z)(\tau)| < \delta$ for all $\tau \in T(A)$ and $\eta(z) - \overline{R}_{\phi,\psi}(z) \in \rho_A(K_0(A))$ for all $z \in \mathcal{G}$, there is a unitary $u \in A$ such that

$$\| \phi(x) - u^* \psi(x) u \| < \epsilon \quad \forall x \in \mathcal{F},$$

and $\tau(\frac{1}{2\pi i} \log((\phi \otimes \text{id}_{M_n}(z^*))((u \otimes 1_{M_n})^*(\psi \otimes \text{id}_{M_n}(z))((u \otimes 1_{M_n})))) = \tau(\eta([z]))$ for all $z \in \mathcal{P}$ and for all $\tau \in T(A)$.

**Proof.** Without loss of generality, one may assume that any element in $\mathcal{F}$ has norm at most one. Let $\epsilon > 0$. Choose $\epsilon > \eta > 0$ and a finite subset $\mathcal{F} \subseteq \mathcal{F}_0 \subseteq C$ satisfying the following: For all $z \in \mathcal{P}$, $\tau(\frac{1}{2\pi i} \log(\phi(z^* w_j^* \psi(z) w_j)))$ is well defined and

$$\tau(\frac{1}{2\pi i} \log(\phi(z^*) w_j^* \psi(z) w_j))$$

whenever

$$\| \phi(f) - v_j^* \psi(f) v_j \| < \eta \quad \text{for all} \quad f \in \mathcal{F}_0,$$

where $v_j$ are unitaries in $A$ and $w_j = v_1 \cdots v_j$, $j = 1, 2, 3$. In the above, if $z \in U_n(C)$, we denote by $\phi$ and $\psi$ the extended maps $\phi \otimes \text{id}_{M_n}$ and $\psi \otimes \text{id}_{M_n}$, and replace $w_j$, and $v_j$ by $\text{diag}(w_j, \ldots, w_j)$ and $\text{diag}(v_j, \ldots, v_j)$, respectively.

Note that $A$ has Property (B2) associated to $C$ (see [2,18] and some $\Delta_C$. Set $\delta = \frac{1}{2} \Delta_C(\eta/2, \mathcal{F}_0, \{0\}, \mathcal{P}, \psi)$, and $\mathcal{G}$ the subset corresponding to $\mathcal{P}$. Let $\mathcal{G}' \subseteq U_m(C)$ be a finite subset containing a representative for each element in $\mathcal{G}$. Without loss of generality, one may assume that $\mathcal{P} \subseteq \mathcal{G}'$.

By Corollary 11.7 of [13], the maps $\phi$ and $\psi$ are approximately unitary equivalent. Hence, for any finite subset $\mathcal{Q}$ and any $\delta_1$, there is a unitary $v \in A$ such that

$$\| \phi(x) - v^* \psi(x) v \| < \delta_1 \quad \forall x \in \mathcal{Q}.$$

By choosing $\mathcal{F}_0 \subseteq \mathcal{Q}$ sufficiently large and $\delta_1 < \eta/2$ sufficiently small, the map

$$[z] \mapsto \tau(\frac{1}{2\pi i} \log(\phi^*(z) v^* \psi(z) v)), \quad z \in \mathcal{G'},$$

induces a homomorphism $\eta_1 : \langle \mathcal{G} \rangle \to \text{Aff}(T(A))$, and moreover, $\|\eta_1([z])\| < \delta$ for all $z \in \mathcal{G}'$.

By Lemma 3.9 the image of $\eta_1 - \overline{R}_{\phi,\psi}$ is in $\rho(K_0(A))$. Since $\eta([z]) - \overline{R}_{\phi,\psi}([z]) \in \rho_A(K_0(A))$ for all $z \in \mathcal{G}'$, the image of $\eta - \eta_1$ is also in $\rho_A(K_0(A))$. Since $\rho_A(K_0(A))$ is torsion free and $\langle \mathcal{G} \rangle$ is finitely generated, there is a map $h : \langle \mathcal{G} \rangle \to K_0(A)$ such that

$$\eta - \eta_1 = \rho_A \circ h.$$

Note that $|\tau(h(z))| < 2\delta = \Delta_C(\eta/2, \mathcal{F}_0, \{0\}, \mathcal{P}, \psi)$ for all $\tau \in T(A)$ and $z \in \mathcal{G}$.

Since $A$ has property (B2), there is a unitary $w$ such that

$$\| [w, \psi(x)] \| < \eta/2, \quad \forall x \in \mathcal{F}_0,$$

and $\text{bott}_1(w, \psi)(z) = h([z])$ for all $z \in \mathcal{P}$. 


Set $u = wv$. One then has
\[
\|\phi(x) - u^*\psi(x)u\| < \eta, \quad \forall x \in F_0,
\]
and for any $z \in P$ and any $\tau \in T(A)$,
\[
\tau(\frac{1}{2\pi i}\log(\phi(z^*)u^*\psi(z)u)) = \tau(\frac{1}{2\pi i}\log(\phi(z)v^*w^*\psi(z)wv))
\]
\[
= \tau(\frac{1}{2\pi i}\log(\phi(z^*)v^*\psi(z)vv^*\psi(z^*)w^*\psi(z)wv))
\]
\[
= \tau(\frac{1}{2\pi i}\log(\phi(z^*)v^*\psi(z)v) + \tau(\frac{1}{2\pi i}\log(\psi(z^*)w^*\psi(z)w))
\]
\[
= \eta_1([z]) + h([z]) = \eta([z])(\tau).
\]

\[\square\]

**Remark 4.2.** The lemma also holds for the case $C$ is any unital AH-algebra if $\text{TR}(A) = 0$. The assumption that $C$ has Property (J) is used so that the uniqueness theorem 2.13 can be applied. In the case that $\text{TR}(A) = 0$, then one can apply Theorem 3.6 of [9] in which case the condition $\phi^\dagger = \psi^\dagger$ is not needed. This case of the lemma is also observed by H. Matui in [20].

**Theorem 4.3.** Let $A$ be a $C^*$-algebra with $\text{TR}(A \otimes Q) \leq 1$, and let $C$ be a unital AH-algebra with property (J). Suppose that there are two unital monomorphisms $\phi, \psi : C \to A$ with
\[
[\phi] = [\psi] \text{ in } KL(C, A), \quad \phi_\sharp = \psi_\sharp \text{ and } \phi^\dagger = \psi^\dagger.
\]
Then, for any finite subset $F \subseteq C$, there exists a unitary $u \in A \otimes Z$ such that
\[
\|\phi(x) \otimes 1 - u^*(\psi(x) \otimes 1)u\| < \epsilon \quad \forall x \in F.
\]

**Proof.** Assume that any element in $F$ has norm at most one. Let $p$ and $q$ be a pair of relatively prime supernatural numbers of infinite type with $\mathbb{Q}_p + \mathbb{Q}_q = \mathbb{Q}$. Denote by $M_p$ and $M_q$ the UHF-algebras associated to $p$ and $q$.

Denote by $\delta, \mathcal{G} \subseteq C$ and $\mathcal{P} \subseteq \mathcal{K}(C)$ the finite subsets of Theorem 8.4 of [14] corresponding to $\mathcal{F}$ and $\epsilon/2$. Without loss of generality, we may assume that $\delta < \epsilon/2$ and $\mathcal{G}$ is large enough so that for any homomorphism $h : C \to A$, the map $\text{Bott}(h, u_j)$ and $\text{Bott}(h, w_j)$ are well defined and
\[
\text{Bott}(h, w_j) = \text{Bott}(h, u_1) + \cdots + \text{Bott}(h, u_j)
\]
on the subgroup generated by $\mathcal{P}$, if $u_j$ is any unitaries with $\|[h(x), u_j]\| < \delta$ for all $x \in \mathcal{G}$, where $w_j = u_1 \cdots u_j$, $j = 1, 2, 3, 4$.

Let $\iota : A \to A \otimes M_\tau$ be the embedding defined by $\iota(a) = a \otimes 1$ for all $a \in A$, where $\tau$ is a supernatural number. Define $\phi_{\tau} = \iota \circ \phi$ and $\psi_{\tau} = \iota \circ \psi$.

For any supernatural number $\tau$, note that $A \otimes M_\tau$ has Property (B2). Denote by $\delta_\tau$ and $\mathcal{H}_\tau \subseteq \mathcal{K}(C)$ the constant and finite subset corresponding to $\delta/4$, $\mathcal{G}$, $\mathcal{P}$ and $\psi_\tau$. Pick $0 < \delta_2 < \min\{\delta_\tau, \delta_q\}$, and set $\mathcal{H} = \mathcal{H}_p \cup \mathcal{H}_q$. Denoted by $\mathcal{H}_1 = \mathcal{H} \cap K_1(C)$, and pick a finite subset $\mathcal{U} \subset U_n(C)$ for some integer $n \geq 1$ such that any element in $\mathcal{H}_1$ has a representative in $\mathcal{U}$. Let $S \subseteq C$ be a finite subset such that, if $u = (a_{ij}) \in \mathcal{U}$, then $a_{ij} \in S$. 


Furthermore, one may assume that $\delta_2$ is sufficiently small such that for any unitaries $z_1, z_2$ in a C*-algebra with tracial states, $\tau(\frac{1}{2\pi i} \log(z_i z_j^*)) (i, j = 1, 2, 3)$ is well defined and

$$
\tau(\frac{1}{2\pi i} \log(z_1 z_2^*)) = \tau(\frac{1}{2\pi i} \log(z_1 z_3^*)) + \tau(\frac{1}{2\pi i} \log(z_3 z_2^*))
$$

for any tracial state $\tau$, whenever $\|z_1 - z_3\| < \delta_2$ and $\|z_2 - z_3\| < \delta_2$.

Let $Q \subset K_1(C)$ and $\delta_3$ be the finite subset and constant of Lemma 4.1 with respect to $G \cup S, U$ and $\delta_2$. By Lemma 5.1, the image of $R_{\phi, \psi}$ is in the closure of $\rho_A(K_0(A))$. Note that kernel of $R_{\phi, \psi}$ contains Tor$(\langle Q \rangle)$ and $(\langle Q \rangle)$ is finitely generated. There exists a homomorphism $\eta: \langle Q \rangle \to \text{Aff}(T(A))$ such that $\eta(z) - \overline{R}_{\phi, \psi}(z) \in \rho_A(K_0(A))$ and $\|\eta(z)\| < \delta_3$ for all $z \in Q$.

Then the image of $\tau_p \circ \eta - \overline{R}_{\phi, \psi}$ is in $\rho_A \otimes \rho_p(K_0(A \otimes M_p))$. The same holds for $q$. By Lemma 1.1 there exists unitaries $u_p$ and $u_q$ such that

$$
\|\phi_p(x) - u_p^* \psi_p(x) u_p\| < \delta_2/n^2 \quad \text{and} \quad \|\phi_q(x) - u_q^* \psi_q(x) u_q\| < \delta_2/n^2, \quad \forall x \in G \cup S.
$$

Moreover,

$$
\tau(\frac{1}{2\pi i} \log(\phi_p(z^*) u_p^* \psi_p(z) u_p)) = (\tau_p)_z \circ \eta([z]) (\tau) \quad \text{for all} \quad \tau \in T(A_p)
$$

(4.3)

$$
\tau(\frac{1}{2\pi i} \log(\phi_q(z^*) u_q^* \psi_q(z) u_q)) = (\tau_q)_z \circ \eta([z]) (\tau) \quad \text{for all} \quad \tau \in T(A_q)
$$

and for all $z \in U$, where we identify $\phi$ and $\psi$ with $\phi \otimes \text{id}_{M_n}$ and $\phi \otimes \text{id}_{M_n}$, and $u$ with $u \otimes \text{id}_{M_n}$, respectively.

Let $\infty$ be the supernatural number associated with $Q$. Let $e_p : A \otimes M_p \to A \otimes Q$ and $e_q : A \otimes M_q \to A \otimes Q$ be the standard embeddings. Then, one computes that, for all $z \in U$, by the Exel’s formula (see 2.16),

$$
\tau(\text{bott}_1(\psi(\infty) \otimes 1, u_p u_q^*)) = \tau(\frac{1}{2\pi i} \log(u_p u_q^*(\psi(z) \otimes 1) u_q u_p^*(\psi(z^*) \otimes 1)))
$$

(4.4)

$$
= \tau(\frac{1}{2\pi i} \log(u_p^*(\psi(z) \otimes 1) u_q u_p^*(\psi(z^*) \otimes 1) u_q) u_p)
$$

(4.5)

$$
= \tau(\frac{1}{2\pi i} \log(u_q^*(\psi(z) \otimes 1) u_q u_p^*(\psi(z^*) \otimes 1))
$$

(4.6)

$$
+ \tau(\frac{1}{2\pi i} \log((\phi(z^*) \otimes 1) u_p^*(\psi(z) \otimes 1) u_p)
$$

(4.7)

$$
= -(\tau_p)_z \circ \eta([z]) \circ \eta([z]) (\tau) + (\tau_p)_z \circ \eta([z]) (\tau)
$$

(4.8)

$$
= -(\tau_q)_z \circ \eta([z]) \circ \eta([z]) (\tau) = 0
$$

(4.9)

for all $\tau \in T(A \otimes Q)$, where we identify $\phi$ and $\psi$ with $\phi \otimes \text{id}_{M_n}$ and $\psi \otimes \text{id}_{M_n}$, and $u_p$ and $u_q$ with $u_p \otimes \text{id}_{M_n}$ and $u_q \otimes \text{id}_{M_n}$, respectively. Therefore, the image of the map $\text{bott}_1(\psi(\infty) \otimes 1, u_p u_q^*)$ is in $\ker \rho_{A \otimes Q}$. Note that $K_0(A \otimes Q) \cong K_0(A) \otimes \mathbb{Q}$ is torsion free. Hence the map $\text{bott}_1(\psi(\infty) \otimes 1, u_p u_q^*)$ factors through the torsion free part of $\langle H_1 \rangle$. Also note that $\ker \rho_{A \otimes Q} = \ker \rho_A \otimes \mathbb{Q}$, and, since $p$ and $q$ are relative prime, any rational number $r$ can be written as $r = r_p + r_q$ with $r_p \in \mathbb{Q}_p$ and $r_q \in \mathbb{Q}_q$ (see 2.7). One has two maps $\theta_p : \langle H_1 \rangle \to \ker \rho_{A \otimes M_p}$ and $\theta_q : \langle H_1 \rangle \to \ker \rho_{A \otimes M_q}$ such that $\text{bott}_1(\psi(\infty) \otimes 1, u_p u_q^*) = \theta_p - \theta_q$. Moreover, the same argument shows there are maps $\alpha_p : \langle H_0 \rangle \to K_1(A \otimes M_p)$ and $\alpha_q : \langle H_0 \rangle \to K_1(A \otimes M_q)$ such that $\text{bott}_0(\psi(\infty) \otimes 1, u_p u_q^*) = \alpha_p - \alpha_q$. 


Then, there are unitaries $w_p \in B \otimes M_p$ and $w_q \in B \otimes M_q$ such that
\[ \|w_p, \psi_p(x)\| < \delta/4, \quad \|w_q, \psi_q(x)\| < \delta/4, \]
for any $x \in \mathcal{G}$, and
\[ \text{bott}_1(\psi_p, w_p)|_{\mathcal{P}_1} = -\theta_p, \quad \text{bott}_1(\psi_q, w_q)|_{\mathcal{P}_1} = \theta_q \]
\[ \text{bott}_0(\psi_p, w_p)|_{\mathcal{P}_0} = -\alpha_p \quad \text{and} \quad \text{bott}_0(\psi_q, w_q)|_{\mathcal{P}_0} = \alpha_q \]

Consider the unitaries $v_p = w_p u_p$ and $v_q = w_q u_q$. One then has that
\[ \|\phi(x) \otimes 1 - u_p^* w_p^*(\psi(x) \otimes 1) w_p u_p\| < \delta/2 \quad \text{and} \quad \|\phi(x) \otimes 1 - u_q^* w_q^*(\psi(x) \otimes 1) w_q u_q\| < \delta/2, \quad \forall x \in \mathcal{G}. \]
Hence
\[ \|\{w_u, \psi(x) \otimes 1\}\| < \delta, \quad \forall x \in \mathcal{G}. \]
Moreover,
\[ \text{bott}_1(\psi \otimes 1, w_p u_p u_q^* w_q^*) = \text{bott}_1(\psi \otimes 1, w_p) + \text{bott}(\psi \otimes 1, u_p u_q) + \text{bott}(\psi \otimes 1, w_q^*) \]
\[ = -\theta_p + \theta_p + \theta_q - \theta_q = 0 \]
on the subgroup generated by $\mathcal{P}_1$. The same argument shows that $\text{bott}_1(\psi \otimes 1, w_p u_p u_q^* w_q^*) = 0$.
Since $K_i(A \otimes Q, \mathbb{Z}/m\mathbb{Z}) = \{0\}$ for all $m \geq 2$ and $i = 0, 1$, one has that
\[ \text{Bott}(\psi \otimes 1, w_p u_p u_q^* w_q^*)|_{\mathcal{P}} = 0. \]
Therefore, by Theorem 8.4 of [14], there is a continuous path of unitaries $v(t)$ in $A \otimes Q$ such that $v(1) = 1$ and $v(0) = w_p u_p u_q^* w_q^*$, and
\[ \|[u(t), \psi(x) \otimes 1]\| < \epsilon/2, \quad \forall x \in \mathcal{F}, \text{forall} \ t \in [0, 1]. \]
Consider the unitary $u(t) = v(t) w_q u_q \in A \otimes \mathbb{Z}_{p,q}$, and it has the property
\[ \|\phi(x) \otimes 1 - u^*(\psi(x) \otimes 1) u\| < \epsilon, \quad \forall x \in \mathcal{F}. \]

One then embeds $\mathbb{Z}_{p,q}$ into $\mathbb{Z}$ to get the desired conclusion. \hfill \Box

**Corollary 4.4.** Let $C$ be a unital AH-algebra with property (J) and let $A$ be a unital separable simple $\mathbb{Z}$-stable $C^*$-algebra in $C$. Let $\phi, \psi : C \to A$ be two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that
\[ \lim_{n \to \infty} u_n^* \psi(c) u_n = \phi(c) \quad \text{for all} \ c \in C, \]
if and only if
\[ [\phi] = [\psi] \text{ in } KL(C, A), \ \phi_p = \psi_q \text{ and } \phi^\dagger = \psi^\dagger. \]

**Proof.** We will show the “if” part only. Suppose that $\phi$ and $\psi$ satisfy the condition. Let $\epsilon > 0$ and let $\mathcal{F} \subset C$ be a finite subset. Then, by [13], there exists a unitary $v \in A \otimes \mathbb{Z}$ such that
\[ \|v^*(\psi(a) \otimes 1) v - \phi(a) \otimes 1\| < \epsilon/3 \text{ for all } a \in \mathcal{F}. \]
Let $\iota : A \to A \otimes \mathbb{Z}$ be defined by $\iota(a) = a \otimes 1$ for $a \in A$. There exists an isomorphism $j : A \otimes \mathbb{Z} \to A$ such that $j \circ \iota$ is approximately inner. So there is a unitaries $w \in A$ such that
\[ \|j(\psi(a) \otimes 1) - w^* \psi(a) w\| < \epsilon/3 \quad \text{and} \quad \|w^* \phi(a) w - j(\phi(a) \otimes 1)\| < \epsilon/3. \]
for all $a \in \mathcal{F}$. Let $u = wj(v)w^* \in A$. Then, for $a \in \mathcal{F}$,

\begin{align}
(4.12) \quad \|u^*\psi(a)u - \phi(a)\| &= \|wj(v)^*w^*\psi(a)wj(v)w^* - \phi(a)\| \\
(4.13) \quad &\leq \|wj(v)^*w^*\psi(a)wj(v)w^* - wj(v)^* (j(\psi(a) \otimes 1) j(v)w^*)\| \\
(4.14) \quad &+ \|wj(v)^*(j(\psi(a) \otimes 1) j(v)w^* - w(j(\phi(a) \otimes 1)w^*)\| \\
(4.15) \quad &+ \|w(j(\phi(a) \otimes 1)w^* - \phi(a)\| \\
(4.16) \quad < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \text{ for all } a \in \mathcal{F}.
\end{align}

\hspace{1cm} \square

A version of the following is also obtained by H. Matui.

**Corollary 4.5.** Let $C$ be a unital AH-algebra and let $A$ be a unital separable simple $C^*$-algebra in $C_0$ which is $\mathcal{Z}$-stable. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that

$$\lim_{n \to \infty} u_n^* \phi(c) u_n = \psi(c) \text{ for all } c \in C,$$

if and only if

$$[\phi] = [\psi] \text{ in } KL(C, A), \ \phi_2 = \psi_2 \text{ and } \phi^+ = \psi^+.$$

**Proof.** The proof is exactly the same as that of (4.3) and (4.4). When Theorem 2.13 is applied, one applies Theorem 3.6 of [9] instead. One also uses Remark 4.2. \hspace{1cm} \square

**Lemma 4.6.** Let $A$ be a unital separable simple $C^*$-algebra in $\mathcal{N} \cap \mathcal{C}$ and $B \in \mathcal{C}$ be a unital separable $C^*$-algebra and let $\phi, \psi : A \to B$. Suppose that

\begin{align}
(4.17) \quad [\phi] = [\psi] \text{ in } KL(A, B), \\
(4.18) \quad \phi_2 = \psi_2 \text{ and } \phi^+ = \psi^+.
\end{align}

Let $p$ and $q$ be two relatively prime supernatural numbers of infinite type with $M_p \otimes M_q = Q$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes \mathcal{Z}_{p,q}$, there exists a unitary $v \in B \otimes \mathcal{Z}_{p,q}$ such that

$$\|v^*(\phi \otimes \text{id}(a)) v - \psi \otimes \text{id}(a)\| < \epsilon \text{ for all } a \in \mathcal{F}.$$

**Proof.** Let $r$ be a supernatural number. Denote by $\nu_t : A \to A \otimes M_t$ the embedding defined by $\nu_t(a) = a \otimes 1$ for all $a \in A$. Denote by $j_t : B \to B \otimes M_t$ the embedding defined by $j_t(b) = b \otimes 1$ for all $b \in B$. Without loss of generality, one may assume that $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, where $\mathcal{F}_1 \subset A$ and $\mathcal{F}_2 \subset \mathcal{Z}_{p,q}$ are finite subsets and $1_A \in \mathcal{F}$ and $1_{\mathcal{Z}_{p,q}} \in \mathcal{F}_2$. Moreover, one may assume that any element in $\mathcal{F}_1$ and $\mathcal{F}_2$ has norm at most one.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ such that

$$\|b(t) - b(t_i)\| < \epsilon/4 \ \forall b \in \mathcal{F}_2, \ \forall t \in [t_{i-1}, t_i], \ i = 1, \ldots, m.$$
Consider
\[ \mathcal{E} = \{a \otimes b(t_i); \ a \in \mathcal{F}_1, b \in \mathcal{F}_2, i = 0, \ldots, m\} \subseteq A \otimes Q, \]
(4.21)
\[ \mathcal{E}_p = \{a \otimes b(t_0); a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_p \subseteq A \otimes Q \]
and
\[ \mathcal{E}_q = \{a \otimes b(t_m); a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_q \subseteq A \otimes Q. \]
(4.22)

Let \( H \subseteq A \otimes Q, \ P \subseteq \mathbb{K}(A \otimes Q) \), and \( \delta > 0 \) be the constant of Theorem 8.4 of [14] with respect to \( \mathcal{E} \) and \( \epsilon/8 \). Denote by \( \infty \) the supernatural number associated with \( Q \). Let \( P_i = P \cap K_i(A \otimes Q), i = 0, 1 \). There is a finitely generated free subgroup \( G(P)_{i,0} \subseteq K_i(A) \) such that if one sets
(4.23)
\[ G(P)_{i,0,0} = \langle \{gr : g \in (t_\infty)_{si}(G(P)_{i,0}) \text{ and } r \in Q_0\} \rangle, \]
where \( 1 \in Q_0 \subset Q \) is a finite subset, then \( G(P)_{i,0,0} \) contains the subgroup generated by \( P_i, i = 0, 1 \). Without loss of generality, we may assume that \( G(P)_{i,0} \) is the subgroup generated by \( P_i, i = 0, 1 \). Moreover, we may assume that, if \( r = k/m \), where \( k \) and \( m \) are nonzero integers, and \( r \in Q_0 \), then \( 1/m \in Q_0 \). Let \( P'_i \subseteq K_i(A) \) be a finite subset which generates \( G(P)_{i,0} \), \( i = 0, 1 \).

Assume that \( H \) is sufficiently large and \( \delta \) is sufficiently small such that for any homomorphism \( h \) from \( A \otimes Q \) to \( B \otimes Q \) and any unitary \( z_j \) \( (j = 1, 2, 3, 4) \), the map \( \text{Bott}(h, z_j) \) and \( \text{Bott}(h, w_j) \) are well defined on the subgroup generated by \( P \) and
\[ \text{Bott}(h, w_j) = \text{Bott}(h, z_1) + \cdots + \text{Bott}(h, z_j) \]
on the subgroup generated by \( P \), if \( \| [h(x), z_j] \| < \delta \) for any \( x \in H \), where \( w_j = z_1 \cdots z_j, j = 1, 2, \ldots, 4 \). By choosing even smaller \( \delta \), without loss of generality, we may assume that
\[ H = H^0 \otimes H^p \otimes H^q, \]
where \( H^0 \subseteq A, H^p \subseteq M_p \) and \( H^q \subseteq M_q \) are finite subsets, and \( 1 \in H^0, 1 \in H^p \) and \( 1 \in H^q \).

Denote by \( \delta_2, G \subseteq \mathbb{K}(A \otimes Q) \) the constant and finite subset of Theorem 6.3 of [15] corresponding to \( \delta/4, H \) and \( P \). Let \( H' \subseteq A \otimes Q \) be a finite subset and assume that \( \delta_2 \) is small enough such that for any homomorphism \( h \) from \( A \otimes Q \) to \( B \otimes Q \) and any unitary \( z_j \) \( (j = 1, 2, 3, 4) \), the map \( \text{Bott}(h, z_j) \) and \( \text{Bott}(h, w_j) \) is well defined on the subgroup generated by \( G \) and
\[ \text{Bott}(h, w_j) = \text{Bott}(h, z_1) + \cdots + \text{Bott}(h, z_j) \]
on the subgroup generated by \( G \), if \( \| [h(x), z_j] \| < \delta_2 \) for any \( x \in H' \), where \( w_j = z_1 \cdots z_j, j = 1, 2, \ldots, 4 \). Moreover, one may assume that \( G \supseteq P \). Furthermore, as above, one may assume, without loss of generality, that
\[ H' = H'^0 \otimes H'^p \otimes H'^q, \]
where \( H'^0 \subseteq H'^0 \subseteq A, H'^p \subseteq H'^p \subseteq M_q \) and \( H'^q \subseteq H'^q \subseteq M_q \) are finite subsets.

Let \( \delta'_2 > 0 \) be a constant such that for any unitary with \( \| u - 1 \| < \delta'_2 \), one has that \( \| \log u \| < \delta'_2/4 \).

Without loss of generality, one may assume that \( \delta'_2 < \delta/4 < \epsilon/4 \).

Let \( R_i \subseteq K(A \otimes M_i) \) and \( \delta_i \) be the finite subset and constant of Property (B2) with respect to \( H'^i \otimes H'^r \) and \( (t_r)^{si}(P'_i) \) \( (i = 0, 1) \) and \( \delta'_2/8 \) \( (r = p \text{ or } r = q) \). Let \( R_i^{(i)} = R_i \cap K_i(A \otimes M_i), i = 0, 1 \). There is a finitely generated subgroup \( G_{i,0,r} \subseteq K_i(A) \) which is free so that
\[ G_{i,0,r} = \langle \{gr : (t_r)^{si}(G_{i,0,r}) \text{ and } r \in Q_{00}\} \rangle \]
contains the subgroup generated by $R^i$, $i = 0, 1$. We may assume that these two subgroups are the same (for $i = 0, 1$).

Denote by $R \subset K(A \otimes Q)$ a finite subset containing the union of $[I_p](R_p)$ and $[I_q](R_q)$. Without loss of generality, one may also assume that $R \supseteq G$.

Let $H \subset A \otimes M_t$ be a finite subset and $\delta_3 > 0$ such that for any homomorphism $h$ from $A \otimes M_t$ to $B \otimes M_t$ ($r = p$ or $r = q$) any unitary $z_j$ ($j = 1, 2, 3, 4$), the map $\text{Bott}(h, z_j)$ and $\text{Bott}(h, w_j)$ are well defined on the subgroup generated by $R$ and

$$\text{Bott}(h, w_j) = \text{Bott}(h, z_1) + \cdots + \text{Bott}(h, z_j)$$
onumber

on the subgroup generated by $R$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in H$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, \ldots, 4$. Without loss of generality, we assume that $H^0 \otimes H^p \subset H_p$ and $H^0 \otimes H^q \subset H_q$. Furthermore, we may also assume that

$$H_r = H_{0,0} \otimes H_{0,r}$$

for some finite subsets $H_{0,0}$ and $H_{0,r}$ with $H^{p'} \subset H_{0,0} \subset A$, $H^{q'} \subset H_{0,p} \subset M_p$ and $H^{q'} \subset H_{0,q}$. In addition, we may also assume that $\delta_3 < \delta_2/2$.

Furthermore, one may assume that $\delta_3$ is sufficiently small such that, for any unitaries $z_1, z_2, z_3$ in a $C^\ast$-algebra with tracial states, $\tau(\frac{1}{2\pi i} \log(z_1z_2^*)) = \tau(\frac{1}{2\pi i} \log(z_1z_3^*)) + \tau(\frac{1}{2\pi i} \log(z_2z_3^*))$ for any tracial state $\tau$, whenever $\|z_1 - z_3\| < \delta_3$ and $\|z_2 - z_3\| < \delta_3$.

To simply notation, we also assume that, for any unitary $z_j$ ($j = 1, 2, 3, 4$) the map $\text{Bott}(h, z_j)$ and $\text{Bott}(h, w_j)$ are well defined on the subgroup generated by $R$ and

$$\text{Bott}(h, w_j) = \text{Bott}(h, z_1) + \cdots + \text{Bott}(h, z_j)$$

on the subgroup generated by $R$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in H''$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, \ldots, 4$, and assume that

$$H'' = H_{0,0} \otimes H_{0,p} \otimes H_{0,q}.$$  

Let $R^i = R \cap K_i(A \otimes Q)$. There is a finitely generated subgroup $G_{i,0}$ of $K_1(A)$ which is free and there is a finite subset $Q_0 \subset Q$ such that

$$G_{i,\infty} = \langle \{gr : g \in (t_\infty)_n(G_{i,0}) \text{ and } r \in Q_0\} \rangle$$

contains the subgroup generated by $R^i$, $i = 0, 1$. Without loss of generality, we may assume that $G_{i,\infty}$ is the subgroup generated by $R^i$. Note that we may also assume that $G_{i,0} \supset G(P)_{i,0}$ and $1 \in Q_0 \supset Q_0$. Moreover, we may assume that, if $r = k/m$, where $m, k$ are relatively prime non-zero integers, and $r \in Q_0$, then $1/m \in Q_0$. We may also assume that $G_{i,0} \subset G_{i,0}$ for $r = p, q$ and $i = 0, 1$. Let $R'' \subset K_i(A)$ be a finite subset which generates $G_{i,0}$, $i = 0, 1$. Choose a finite subset $U \subset U_n(A)$ for some $n$ such that for any element of $R''$, there is a representative in $U$. Let $S$ be a finite subset of $A$ such that if $(z_{i,j}) \in U$, then $z_{i,j} \in S$.

Denote by $\delta_4$ and $\delta_5$ the constant and finite subset of Lemma 4.1 corresponding to $C_r \cap H \otimes 1_{U_n(S)}$ and $\nu(U)$ and $\frac{1}{\nu(U)} \min\{\delta_2/8, \delta_3/4\}$ ($r = p$ or $r = q$). We may assume that $Q_1 = \{x \otimes r : x \in Q' \text{ and } r \in Q'_{\delta_5}\}$, where $Q' \subset K_1(A)$ is a finite subset and
\( \mathcal{Q}' \subset \mathbb{Q}_t \) is also a finite subset. Let \( K = \max\{|r| : r \in \mathcal{Q}_p \cup \mathcal{Q}_q\} \). Since \( [\phi] = [\psi] \) in \( KL(A, B) \), \( \phi_t = \psi_t \) and \( \phi^t = \psi^t \), by Lemma 4.6, \( \overline{\rho}(\phi, \psi)(K_1(A)) \subseteq \overline{\rho}(K_0(B)) \subseteq \text{Aff}(T(B)) \). Therefore, there is a map \( \eta : (\mathcal{Q}') \to \rho_B(K_0(B)) \subset \text{Aff}(T(B)) \) such that the image of \( \eta - \overline{\rho} \phi, \psi \) is in \( \rho_B(K_0(B)) \) and \( \|\eta(z)\| < \frac{\delta_3}{1+K} \) for all \( z \in \mathcal{Q}' \).

Consider the map \( \phi_t = \phi \otimes \text{id}_M_t \) and \( \psi_t = \psi \otimes \text{id}_M_t \) (\( t = p \) or \( t = q \)). Since \( \eta \) vanishes on the torsion part of \( (\mathcal{Q}') \), there is a homomorphism \( \eta_t : ((t)_s 1(\mathcal{Q}')) \to \rho_B(K_0(B \otimes M_t) = \text{Aff}(T(B \otimes M_t)) \) such that

\[
\eta_t \circ (t)_s 1 = \eta.
\]

Since \( \overline{\rho}(K_0(B)) \rho_B(M_t)(K_0(B \otimes M_t) = \overline{\rho}(K_0(B)) \) is divisible, one can extend \( \eta_t \) so it defines on \( K_1(A) \otimes \mathbb{Q}_t \). We will use \( \eta_t \) for the extension. Note that the image of \( \eta_t - \overline{\rho} \phi_t, \psi_t \) is in \( \rho_B(M_t)(K_0(B \otimes M_t)) \), and \( \|\eta_t(z)\| < \delta_3 \) for all \( z \in \mathcal{Q} \). By Lemma 4.1, there exists a unitary \( u_p \in B \otimes M_p \) such that

\[
\|u_p^*(\phi \otimes \text{id}_M_p)(c)u_p - (\psi \otimes \text{id}_M_p)(c)\| < \frac{1}{n^2} \min\{\delta_2/8, \delta_3/4\}, \quad \forall c \in \mathcal{E}_p \cup \mathcal{H}_p \cup \mathcal{U}(S).
\]

Note that

\[
\|u_p^*(\phi \otimes \text{id}_M_p)(z)u_p - (\psi \otimes \text{id}_M_p)(z)\| < \delta_3 \quad \text{for any } z \in \mathcal{U}.
\]

Therefore \( \tau \left( \frac{1}{2\pi i} \log(u_p^*(\phi \otimes \text{id}_p)(z)u_p(\psi \otimes \text{id}_p)(z^*)) \right) = \eta_t([z]) \) for all \( z \in \mathcal{U}(U) \), where we identify \( \phi \) and \( \psi \) with \( \phi \otimes \text{id}_M_p \) and \( \psi \otimes \text{id}_M_p \), and \( u_p \) with \( u_p \otimes 1_{M_p} \), respectively.

The same argument shows that there is a unitary \( u_q \in B \otimes M_q \) such that

\[
\|u_q^*(\phi \otimes \text{id}_M_q)(c)u_q - (\psi \otimes \text{id}_M_q)(c)\| < \frac{1}{n^2} \min\{\delta_2/8, \delta_3/4\}, \quad \forall c \in \mathcal{E}_q \cup \mathcal{H}_q \cup \mathcal{U}(S),
\]

and \( \tau \left( \frac{1}{2\pi i} \log(u_q^*(\phi \otimes \text{id}_q)(z)u_q(\psi \otimes \text{id}_q)(z^*)) \right) = \eta_t([z]) \) for all \( z \in \mathcal{U}(U) \), where we identify \( \phi \) and \( \psi \) with \( \phi \otimes \text{id}_M_q \) and \( \psi \otimes \text{id}_M_q \), and \( u_q \) with \( u_q \otimes 1_{M_p} \), respectively. We will identify \( u_p \) with \( u_p \otimes 1_{M_p} \) and \( u_q \) with \( u_q \otimes 1_{M_q} \). Then \( u_p^* u_q^* \in A \otimes Q \) and one estimates that for any \( c \in \mathcal{H}_0 \otimes \mathcal{H}_{0,p} \otimes \mathcal{H}_q \),

\[
\|u_q u_p^*(\phi \otimes 1_Q(c)u_q^* - (\phi \otimes 1_Q)(c))\| < \delta_3,
\]

and hence Bott(\( \phi \otimes \text{id}_Q, u_p u_q^* \))(z) is well defined on the subgroup generated by \( \mathcal{R} \). Moreover, for any \( z \in \mathcal{U} \), by the Exel’s formula (see 2.16) and applying (4.24),

\[
\tau(\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*((t)_s 1([z])))
\]

\[
\tau(\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*)(t_\infty(z)))
\]

\[
\tau(\frac{1}{2\pi i} \log(u_p^* u_q^*(\phi \otimes \text{id}_Q)(t_\infty(z))) u_q u_p^* (\phi \otimes \text{id}_Q)(t_\infty(z))^*)
\]

\[
\tau(\frac{1}{2\pi i} \log(u_q^* (\phi \otimes \text{id}_Q)(t_\infty(z))) u_q (\psi \otimes \text{id}_Q)(t_\infty(z)))
\]

\[
\tau(\frac{1}{2\pi i} \log(u_p^* (\phi \otimes \text{id}_Q)(t_\infty(z))) u_p (\psi \otimes \text{id}_Q)(t_\infty(z)^*))
\]

\[
\eta_t((t)_s 1([z]))(\tau) - \eta_p((t)_s 1([z]))(\tau)
\]

\[
\eta([z])(\tau) - \eta([z])(\tau) = 0 \quad \text{for all } \tau \in T(B),
\]
where we identify $\phi$ and $\psi$ with $\phi \otimes \text{id}_{M_n}$ and $\psi \otimes \text{id}_{M_n}$, and $u_p$ and $u_q$ with $u_p \otimes 1_{M_n}$ and $u_q$ with $u_q \otimes 1_{M_n}$, respectively.

Now suppose that $g \in G_{1,\infty}$. Then there is $g = (k/m)(t_\infty)_*([z])$ for some $z \in \mathcal{U}$ and where $k, m$ are non-zero integers. It follows that

$$\tau(\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*)(mg)) = k \tau(\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*)(([z]))) = 0$$

for all $\tau \in T(B)$. Since $\text{Aff}(T(B))$ is torsion free, it follows that

$$\tau(\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*)(g)) = 0$$

for all $g \in G_{1,\infty}$ and $\tau \in T(B)$. Therefore, the image of $\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*)$ on $\mathcal{R}^1$ is in $\ker \rho_{B \otimes Q}$. Using the same argument as that of Theorem 13, one can choose maps $\theta_p : (t_p)_*(G_{1,0}) \to \ker \rho_{B \otimes M_p}$ and $\theta_q : (t_q)_*(G_{1,0}) \to \ker \rho_{B \otimes M_q}$ such that

$$\text{bott}_1(\phi \otimes \text{id}_Q, u_p u_q^*) \circ (t_\infty)_*1 = \theta_p \circ (t_p)_*1 + \theta_q \circ (t_q)_*1,$$

on $G_{1,0}$ and one can choose maps $\alpha_p : (t_p)_*(G_{0,0}) \to K_1(B \otimes M_p)$ and $\alpha_q : (t_q)_*(G_{0,0}) \to K_1(B \otimes M_q)$ such that

$$\text{bott}_0(\phi \otimes \text{id}_Q, u_p u_q^*) \circ (t_\infty)_*0 = \alpha_p \circ (t_p)_*0 + \alpha_q \circ (t_q)_*0$$

on $G_{0,0}$. Restrict the maps $\{\theta_p, \alpha_p\}$ and the maps $\{\theta_q, \alpha_q\}$ to the subgroups generated by the images of $G_{i,0, p}$ and $G_{i,0,q}$ respectively, and keep the same notation. Since $B \otimes M_p$ and $B \otimes M_q$ have Property (B2), there exist unitaries $w_p \in B \otimes M_p$ and $w_q \in B \otimes M_q$ such that

$$\|[w_p, (\phi \otimes \text{id}_{M_p})(x)]\| < \delta_2'/8, \quad \|[w_q, (\phi \otimes \text{id}_{M_q})(y)]\| < \delta_2'/8,$$

for any $x \in \mathcal{H}^{0'} \otimes \mathcal{H}^{p'}$ and $y \in \mathcal{H}^{0'} \otimes \mathcal{H}^{q'}$, and

$$\begin{align*}
\text{bott}_1(\phi \otimes \text{id}_M_p, w_p)|_{(t_p)_*(G(p))_{1,0}} &= -\theta_p|_{(t_p)_*(G(p))_{1,0}}, \\
\text{bott}_1(\phi \otimes \text{id}_M_q, w_q)|_{(t_q)_*(G(p))_{1,0}} &= \theta_q|_{(t_q)_*(G(p))_{1,0}} \\
\text{bott}_0(\phi \otimes \text{id}_M_p, w_p)|_{(t_p)_*(G(p))_{0,0}} &= -\alpha_p|_{(t_p)_*(G(p))_{0,0}} \quad \text{and} \\
\text{bott}_0(\phi \otimes \text{id}_M_q, w_q)|_{(t_q)_*(G(p))_{0,0}} &= \alpha_q|_{(t_q)_*(G(p))_{0,0}}.
\end{align*}$$

Put $v_p = w_p u_p$ and $v_q = w_q u_q$. One then has that

$$\|\psi \otimes \text{id}_Q(x) - v_p^*(\phi \otimes \text{id}_Q(x))v_p\| < \delta_2'/4, \quad \forall x \in \mathcal{H}^{0'} \otimes \mathcal{H}^{p'} \otimes \mathcal{H}^{q'}$$

and

$$\|\psi \otimes \text{id}_Q(x) - v_q^*(\phi \otimes \text{id}_Q)(x)v_q\| < \delta_2'/4, \quad \forall x \in \mathcal{H}^{0'} \otimes \mathcal{H}^{p'} \otimes \mathcal{H}^{q'}.$$

Hence

$$\|[v_p v_q^*, \phi(x) \otimes 1_Q]\| < \delta_2'/2 < \delta_2, \quad \forall x \in \mathcal{H}.'
Thus $\text{Bott}(\phi \otimes \text{id}_Q, v_p v_q^*)$ is well defined on the subgroup generated by $\mathcal{P}$. Moreover,

$$
\text{bott}_1(\phi \otimes \text{id}_Q, v_p v_q^*) \circ (t, \ast) = \\
\text{bott}_1(\phi \otimes \text{id}_Q, w_p) \circ (t, \ast) + \text{bott}(\phi \otimes \text{id}_Q, u_p u^*_q) \circ (t, \ast)
$$

By Theorem 6.3 of [15], there exists a unitary $y$ such that

$$
\text{bott}_{\phi}(\phi \otimes \text{id}_Q, v_p v_q^*) = 0 \\
\text{bott}(\phi \otimes \text{id}_Q, u_p u^*_q) \circ (t, \ast)
$$

By the choice of $\delta$, for all $z \in G(\mathcal{P})_{1.0}$.

The same argument shows that $\text{bott}_0(\phi \otimes \text{id}_Q, v_p v_q^*) = 0$ on $G(\mathcal{P})_{0.0}$. Now, for any $g \in G(\mathcal{P})_{1.0}$, there is $z \in G(\mathcal{P})_{1.0}$ and integer $k, m$ such that $(k/m)z = g$. From the above,

$$
bott_1(\phi \otimes \text{id}_Q, v_p v_q^*)(mg) = kbott_1(\phi \otimes \text{id}_Q, v_p v_q^*)(z) = 0.
$$

Since $K_0(B \otimes Q)$ is torsion free, it follows that

$$
bott_1(\phi \otimes \text{id}_Q, v_p v_q^*)(g) = 0
$$

for all $g \in G(\mathcal{P})_{1.0}$. So it vanishes on $\mathcal{P} \cap K_1(A \otimes Q)$. Similarly,

$$
\text{bott}_0(\phi \otimes \text{id}_Q, v_p v_q^*)(\mathcal{P} \cap K_0(A \otimes Q)) = 0.
$$

Since $K_1(B \otimes Q, \mathbb{Z}/m\mathbb{Z}) = \{0\}$ for all $m \geq 2$, we conclude that

$$
\text{Bott}(\phi \otimes \text{id}_Q, v_p v_q^*)|_{\mathcal{P}} = 0.
$$

Since $[\phi] = [\psi]$ in $KL(A, B)$, $\phi_\sharp = \psi_\sharp$ and $\phi^\sharp = \psi^\sharp$, one has that

$$
[\phi \otimes \text{id}_Q] = [\psi \otimes \text{id}_Q] \text{ in } KL(A \otimes Q, B \otimes Q),
$$

(4.43)

$$
(\phi \otimes \text{id}_Q)_\sharp = (\psi \otimes \text{id}_Q)_\sharp \text{ and } (\phi \otimes \text{id}_Q)^\sharp = (\psi \otimes \text{id}_Q)^\sharp,
$$

Therefore, for any finite subset $\mathcal{L} \subset A \otimes Q$, there exists a unitary $u \in B \otimes Q$ such that

$$
\|u^\ast(\phi \otimes \text{id}_Q)(c)u - (\psi \otimes \text{id}_Q)(c)\| < \delta^2, \forall c \in \mathcal{L}.
$$

Without loss of generality, one may assume that $\mathcal{E} \cup \mathcal{H}' \subseteq \mathcal{L}$. One then has that

$$
\|uv_p^*(\phi(c) \otimes 1_Q)v_p u^* - \psi(c) \otimes 1_Q\| < \delta^2/2 + \delta^2/8 \forall c \in \mathcal{G}'.
$$

By the choice of $\delta'$ and $\mathcal{H}'$, $\text{Bott}(\phi \otimes \text{id}_Q, v_p u^*)$ is well defined on $\mathcal{G}$, and

$$
|\tau(\text{bott}_1(\phi \otimes \text{id}_Q, v_p u^*)(z))| < \delta^2/2, \forall \tau \in T(B), \forall z \in \mathcal{G} \cap K_1(A \otimes Q).
$$

By Theorem 6.3 of [15], there exists a unitary $y_p \in B \otimes Q$ such that

$$
\|[y_p, (\phi \otimes \text{id}_Q)(c)]\| < \delta/2, \forall c \in \mathcal{H},
$$

and $\text{Bott}(\phi \otimes \text{id}_Q, y_p) = \text{Bott}(\phi \otimes \text{id}_Q, v_p u^*)$ on the subgroup generated by $\mathcal{P}$. Consider the unitary $v = y_p u$, one has that

$$
\|[v, (\phi \otimes \text{id}_Q)(c)]\| < \delta, \forall c \in \mathcal{H} \text{ and } \text{Bott}(\phi \otimes \text{id}_Q, v_p v_q^*) = 0.
on the subgroup generated by \( \mathcal{P} \). By Theorem 8.4 of \( [14] \), there is a path of unitaries \( z_p(t) : [0, t_1] \rightarrow U(A \otimes Q) \) such that \( z_p(0) = 1, z_p(t_1) = v v_p^* \), and

\[
\| [z_p(t), \phi \otimes \text{id}_Q(c)] \| < \varepsilon / 8, \quad \forall t \in [0, t_1], \forall c \in \mathcal{E}.
\]

Note that

\[
\begin{align*}
\text{Bott}(\phi \otimes \text{id}_Q, v v^*) &= \text{Bott}(\phi \otimes \text{id}_Q, v v^*_p v^* p) \\
&= \text{Bott}(\phi \otimes \text{id}_Q, v v^*_p) + \text{Bott}(\phi \otimes \text{id}_Q, v^*_p v^*) \\
&= 0 + 0 = 0
\end{align*}
\]

on the subgroup generated by \( \mathcal{P} \). Since

\[
\| [v v^*_q, (\phi \otimes \text{id}_Q)(c)] \| < \delta, \quad \forall c \in \mathcal{H}.
\]

By Lemma 8.4 of \( [14] \), there is a path of unitaries \( z_q(t) : [t_{m-1}, 1] \rightarrow U(A \otimes Q) \) such that \( z_q(t_{m-1}) = v v^*_q, z_q(1) = 1 \) and

\[
\| [z_q(t), \phi \otimes \text{id}_Q(c)] \| < \varepsilon / 8, \quad \forall t \in [t_{m-1}, 1], \forall c \in \mathcal{E}.
\]

Consider the unitary

\[
v(t) = \begin{cases} 
z_p(t) v_p, & \text{if } 0 \leq t \leq t_1; \\
v, & \text{if } t_1 \leq t \leq t_{m-1}; \\
z_q(t) v_q, & \text{if } t_{m-1} \leq t \leq t_m.
\end{cases}
\]

Then, for any \( t_i \), \( 0 \leq i \leq m \), one has that

\[
\| v^*(t_i)(\phi \otimes \text{id}_Q(c)v(t_i) - (\psi \otimes \text{id}_Q)(c)) \| < \varepsilon / 2, \quad \forall c \in \mathcal{E}.
\]

Then for any \( t \in [t_j, t_{j+1}] \) with \( 1 \leq j \leq m-2 \), one has

\[
\begin{align*}
\| v^*(t)(\phi \otimes \text{id}(a \otimes b(t)))v(t) - \psi \otimes \text{id}(a \otimes b(t)) \| \\
&= \| v^*(\phi(a) \otimes b(t))v - \psi(a) \otimes b(t) \| \\
&< \| v^*(\phi(a) \otimes b(t))v - \psi(a) \otimes b(t) \| + \varepsilon / 4 \\
&< \varepsilon / 4 + \varepsilon / 4 < \varepsilon / 2.
\end{align*}
\]

For any \( t \in [0, t_1] \), one has that for any \( a \in \mathcal{F}_1 \) and \( b \in \mathcal{F}_2 \),

\[
\begin{align*}
\| v^*(t)(\phi \otimes \text{id}(a \otimes b(t)))v(t) - \psi \otimes \text{id}(a \otimes b(t)) \| \\
&= \| v^*_p z_p^*(t)(\phi(a) \otimes b(t))z_p(t)v - \psi(a) \otimes b(t) \| \\
&< \| v^*_p z_p^*(t)(\phi(a) \otimes b(t))z_p(t)v - \psi(a) \otimes b(t) \| + \varepsilon / 2 \\
&< \| v^*_p \phi(a) \otimes b(t) \| v - \psi(a) \otimes b(t) \| + 3\varepsilon / 4 \\
&< 3\varepsilon / 4 + \varepsilon / 4 = \varepsilon.
\end{align*}
\]

Same argument shows that for any \( t \in [t_{m-1}, 1] \), one has that for any \( a \in \mathcal{F}_1 \) and \( b \in \mathcal{F}_2 \),

\[
\| v^*(t)(\phi \otimes \text{id}(a \otimes b(t)))v(t) - \psi \otimes \text{id}(a \otimes b(t)) \| < \varepsilon.
\]

Therefore, one has that

\[
\| v(\phi \otimes \text{id}(f))v - \psi \otimes \text{id}(f) \| < \varepsilon \quad \text{for all } f \in \mathcal{F}.
\]
Remark 4.7. In fact, using the same argument as the lemma above, one has the following: Let $A$ and $B$ be two unital stably finite $C^*$-algebras. Assume they satisfy the following: Let $U$ be any UHF-algebra of infinite type.

1. The approximately unitarily equivalence classes of the monomorphisms from $A \otimes U$ to $B \otimes U$ is classified by the induced elements in $KL(A \otimes U, B \otimes U)$, the induced maps on traces, together with the induced maps from $U_\infty(A \otimes U)/CU_\infty(A \otimes U)$ to $U_\infty(B \otimes U)/CU_\infty(B \otimes U)$;
2. $B \otimes U$ has property (B2) with respect to any embedding of $A \otimes U$;
3. $B \otimes U$ satisfies the homotopy lemma as 8.4 of [14] for any embedding of $A \otimes U$ to $B \otimes U$.

Then, for any monomorphisms $\phi, \psi : A \to B$, one has that $\phi \otimes \text{id}$ and $\psi \otimes \text{id}$ from $A \otimes Z$ to $B \otimes Z$ are approximately unitarily equivalent if and only if

$$\square$$

Theorem 4.8. Let $A \in \mathcal{N} \cap \mathcal{C}$ and $B \in \mathcal{C}$ be two unital separable $\mathcal{Z}$-stable $C^*$-algebras. If $\phi$ and $\psi$ are two homomorphisms from $A$ to $B$ with

$$[\phi] = [\psi] \text{ in } KL(A, B), \quad \phi_\sharp = \psi_\sharp \text{ and } \phi^\dagger = \psi^\dagger.$$ 

Then, for any $\epsilon > 0$ and any finite subset $F \subseteq A$, there exists a unitary $u \in B$ such that

$$\|v^*\phi(a)v - \psi(a)\| < \epsilon \quad \text{for all } a \in F.$$ 

Proof. Let $\alpha : A \to A \otimes \mathcal{Z}$ and $\beta : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ be isomorphisms. Consider the map

$$\Gamma_A : A \xrightarrow{\alpha} A \otimes \mathcal{Z} \xrightarrow{id \otimes \beta} A \otimes \mathcal{Z} \otimes \mathcal{Z} \xrightarrow{\alpha^{-1} \otimes \text{id}} A \otimes \mathcal{Z}.$$ 

Then, $\Gamma$ is an isomorphism. However, since $\beta$ is approximately unitarily equivalent to the map $\mathcal{Z} \ni a \mapsto a \otimes 1 \in \mathcal{Z} \otimes \mathcal{Z}$,

the map $\Gamma_A$ is approximately unitarily equivalent to the map $A \ni a \mapsto a \otimes 1 \in A \otimes \mathcal{Z}$.

Hence the map $\Gamma_B \circ \phi \circ \Gamma_A$ is approximately unitarily equivalent to $\phi \otimes \text{id}_{\mathcal{Z}}$. The same argument shows that $\Gamma_B \circ \psi \circ \Gamma_A$ is approximately unitarily equivalent to $\psi \otimes \text{id}_{\mathcal{Z}}$. Thus, in order to prove the theorem, it is enough to show that $\phi \otimes \text{id}_{\mathcal{Z}}$ is approximately unitarily equivalent to $\psi \otimes \text{id}_{\mathcal{Z}}$.

Since $\mathcal{Z}$ is an inductive limit of $C^*$-algebras $\mathcal{Z}_{p,q}$, it is enough to show that $\phi \otimes \text{id}_{\mathcal{Z}_{p,q}}$ is approximately unitarily equivalent to $\psi \otimes \text{id}_{\mathcal{Z}_{p,q}}$, and this follows from Lemma 4.6.

5. The Range of Approximate Equivalence Classes of Homomorphisms

Now let $A$ and $B$ be two unital $C^*$-algebras in $\mathcal{N} \cap \mathcal{C}$. Theorem 4.8 states that two unital monomorphisms are approximately unitarily equivalent if they induce the same element in $KLT_e(A, B)^{++}$ and the same map on $U(A)/CU(A)$. This section will discuss the following problem: Suppose that $\kappa \in KLT_e(A, B)^{++}$ and a continuous homomorphism $\gamma : U(A)/CU(A) \to \ldots$
$U(B)/CU(B)$ which is compatible with $\kappa$ are given. Is there always a unital monomorphism $\phi : A \to B$ such that $\phi$ induces $\kappa$ and $\phi^\dagger = \gamma$? At least in the case that $K_1(A)$ is free, Theorem 5.9 states that such $\phi$ always exists.

**Lemma 5.1.** Let $A$ and $B$ be two unital infinite dimensional separable stably finite $C^*$-algebras whose tracial state spaces are non-empty. Let $\gamma : U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ be a continuous homomorphism, $h_i : K_i(A) \to K_i(B)$ ($i = 0, 1$) be homomorphisms for which $h_0$ is positive, and $\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ be an affine map so that $(h_0, h_1, \lambda, \gamma)$ are compatible. Let $p$ be a supernatural number. Then $\gamma$ induces a unique homomorphism $\gamma_p : U_\infty(A_p)/CU_\infty(A_p) \to U_\infty(B_p)/CU_\infty(B_p)$, which is compatible with $(h_p)_i$ ($i = 0, 1$) and $\gamma_p$, and moreover, the diagram

\[
\begin{array}{ccc}
U_\infty(A)/CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B)/CU_\infty(B) \\
\downarrow \gamma_p & & \downarrow (\gamma_p)^\dagger \\
U_\infty(A_p)/CU_\infty(A_p) & \xrightarrow{\gamma_p} & U_\infty(B_p)/CU_\infty(B_p),
\end{array}
\]

commutes, where $(h_p)_i : K_i(A) \otimes \mathbb{Q}_p \to K_i(B) \otimes \mathbb{Q}_p$ is induced by $h_i$ ($i = 0, 1$) and where $A_p = A \otimes M_p$, $\iota_p : A \to A_p$ and $\iota'_p : B \to B_p$ is the map induced by $a \mapsto a \otimes 1$ and $b \mapsto b \otimes 1$, respectively.

**Proof.** Denote by $A_0 = A$, $A_p = A \otimes M_p$ and $B_0 = B$ and $B_p = B \otimes M_p$. By a result of K. Thomsen ([24]), using the de la Harpe and Skandalis determinant, one has the following short exact sequences:

\[0 \to \text{Aff}(T(A_i))/\rho_A(K_0(A_i)) \to U_\infty(A_i)/CU_\infty(A_i) \to K_1(A_i) \to 0, \ i = 0, p,
\]

and

\[0 \to \text{Aff}(T(B_i))/\rho_A(K_0(B_i)) \to U_\infty(B_i)/CU_\infty(B_i) \to K_1(B_i) \to 0, \ i = 0, p,
\]

Note that, in all these cases, $\text{Aff}(T(A_i))/\rho_A(K_0(A_i))$ and $\text{Aff}(T(B_i))/\rho_A(K_0(B_i))$ are divisible groups, $i = 0$ and $i = p$. Therefore the exact sequences above splits. Fix splitting maps $s_i : K_1(A_i) \to U_\infty(A_i)/CU_\infty(A_i)$ and $s'_i : K_1(B_i) \to U_\infty(B_i)/CU_\infty(B_i)$, $i = 0, p$ for the above two splitting short exact sequences. Let $\iota_p : A \to A_p$ be the homomorphism defined by $\iota_p(a) = a \otimes 1$ for all $a \in A$ and $\iota'_p : B \to B_p$ be the homomorphism defined by $\iota'_p(b) = b \otimes 1$ for all $b \in B$. Let $\iota^*_p : U_\infty(A)/CU_\infty(A) \to U_\infty(A_p)/CU_\infty(A)$ and $(\iota'_p)^\dagger : U_\infty(B)/CU_\infty(B) \to U_\infty(B_p)/CU_\infty(B_p)$ be the induced maps. The map $\iota_p$ induces the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{Aff}(T(A))/\rho_A(K_0(A)) \xrightarrow{(\iota_p)_*} U_\infty(A)/CU_\infty(A) \xrightarrow{\iota^*_p} K_1(A) \to 0 \\
& \downarrow (\iota(p)_*) & \downarrow (\iota^*_p) & \downarrow (\iota(p)_*),_1 \\
0 & \to & \text{Aff}(T(A))/\rho_A(K_0(A_p)) \xrightarrow{(\iota'_p)_*} U_\infty(A_p)/CU_\infty(A_p) \xrightarrow{\iota'_p} K_1(A_p) \to 0.
\end{array}
\]

Since there is only one tracial state on $M_p$, one may identify $T(A)$ with $T(A_p)$ and $T(B)$ with $T(B_p)$. One may also identify $\rho_A(K_0(A_p))/\rho_A(K_0(A))$ with $\mathbb{R} \rho_A(K_0(A))$ which is the closure of those elements $r\hat{[p]}$ with $r \in \mathbb{R}$. Note that $(h_p)_i : K_i(A \otimes M_p) \to K_i(B \otimes M_p)$ ($i = 0, 1$) is given by the Künneth formula. Since $\gamma$ is compatible with $\lambda$, $\gamma$ maps $\mathbb{R} \rho_A(K_0(A))/\rho_A(K_0(A))$ into
This implies that
\[
\ker(t_p)_{+1} = \{x \in K_1(A) : px = 0 \text{ for some factor } p \} \text{ and}
\]
\[
\ker(t'_p)_{+1} = \{x \in K_1(B) : px = 0 \text{ for some factor } p \}.
\]

Therefore
\[
\ker(t_p) = \{x + s_0(y) : x \in \ker(t_p)_{+1} / \ker(t'_p)_{+1}, y \in \ker(t'_p)_{+1} \} \text{ and}
\]
\[
\ker(t'_p) = \{x + s'_0(y) : x \in \ker(t'_p)_{+1} / \ker(t_p)_{+1}, y \in \ker(t_p)_{+1} \}.
\]

If \(y \in \ker((t_p)_{+1})\), then, for some factor \(p\), \(py = 0\). It follows that \(p\gamma(s_0(y)) = 0\). Therefore \(\gamma(s_0(y))\) must be in \(\ker((t'_p)_{+1})\). It follows that
\[
\gamma(\ker(t_p)) \subset \ker((t'_p)_{+1}).
\]

This implies that \(\gamma\) induces a unique homomorphism \(\gamma_p\) such that the following diagram commutes:
\[
\begin{array}{ccc}
U_\infty(A) / CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B) / CU_\infty(B) \\
\downarrow_{t_p} & & \downarrow_{(t'_p)} \\
U_\infty(A_p) / CU_\infty(A_p) & \xrightarrow{\gamma_p} & U_\infty(B_p) / CU_\infty(B_p).
\end{array}
\]

The lemma follows.

Lemma 5.2. Let \(A\) and \(B\) be two unital infinite dimensional separable stably finite \(C^*\)-algebras whose tracial state spaces are non-empty. Let \(\gamma : U_\infty(A) / CU_\infty(A) \to U_\infty(B) / CU_\infty(B)\) be a continuous homomorphism, \(h_i : K_i(A) \to K_i(B) (i = 0, 1)\) be homomorphisms and \(\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))\) be an affine homomorphism which are compatible. Let \(p\) and \(q\) be two relatively prime supernatural numbers such that \(M_\infty \varotimes M_\infty = Q\). Denote by \(\infty\) the supernatural number associated with the product \(p\) and \(q\). Let \(E_B : B \to B \varotimes \mathbb{Z}_{p,q}\) be defined by \(E_B(b) = b \varotimes 1\) for all \(b \in B\). Then
\[
\begin{align}
(\pi_t \circ E_B)^{\dagger} \circ \gamma &= \gamma_\infty \circ (\pi_\infty)^{\dagger} \text{ for all } t \in (0, 1), \\
(\pi_0 \circ E_B)^{\dagger} \circ \gamma &= \gamma_p \circ (\pi_p)^{\dagger} \text{ and} \\
(\pi_1 \circ E_B)^{\dagger} \circ \gamma &= \gamma_q \circ (\pi_q)^{\dagger}
\end{align}
\]
in the notation of [5.7] where \(\pi_t : \mathbb{Z}_{p,q} \to Q\) is the point-evaluation at \(t\).

Proof. Fix \(z \in U_\infty(B) / CU_\infty(B)\). Let \(u \in U_n(B)\) for some integer \(n \geq 1\) such that \(\overline{u} = z\) in \(U_\infty(B) / CU_\infty(B)\). Then
\[
E_B^{\dagger}(z) = u \varotimes \overline{1}.
\]

In other words, \(E_B^{\dagger}(z)\) is represented by \(w(t) \in M_n(B \varotimes \mathbb{Z}_{p,q})\) for which
\[
w(t) = u \varotimes 1 \text{ for all } t \in [0, 1].
\]

Therefore, for any \(t \in (0, 1)\), \(\pi_t \circ E_B^{\dagger}(z)\) may be written as
\[
\pi_t \circ E_B^{\dagger}(z) = \overline{u} \varotimes \overline{1} \text{ in } U_\infty(B \varotimes Q) / CU_\infty(B \varotimes Q).
\]

This implies that
\[
\pi_t \circ E_B^{\dagger}(z) = (\pi_\infty)^{\dagger}(z) \text{ for all } z \in U_\infty(B) / CU_\infty(B),
\]
where \( \iota_\infty : B \to B \otimes Q \) is defined by \( \iota_\infty(b) = b \otimes 1 \) for all \( b \in B \). It follows from \ref{eq:5.1} that

\[
(\pi_t \circ E_B)^\dagger \circ \gamma = \gamma_\infty \circ \iota_\infty^t \quad \text{for all } t \in (0, 1).
\]

The identity \( \ref{eq:5.7} \) and \( \ref{eq:5.8} \) for end points follows exactly the same way. \( \square \)

The following is standard (see the proof of 9.6 of \cite{15}).

**Lemma 5.3.** Let \( C \) and \( A \) be two unital separable stably finite \( C^* \)-algebras, and let \( \phi_1, \phi_2, \phi_3 : C \to A \) be three unital homomorphisms. Suppose that

\[
[\phi_1] = [\phi_2] = [\phi_3] \quad \text{in} \quad KL(C, A),
\]

(\ref{eq:5.15})

\[
(\phi_1)_x = (\phi_2)_x = (\phi_3)_x.
\]

Then

(\ref{eq:5.16})

\[
\mathcal{R}_{\phi_1, \phi_3} = \mathcal{R}_{\phi_1, \phi_2} + \mathcal{R}_{\phi_2, \phi_3}.
\]

**Lemma 5.4.** Let \( A \) be a unital AH-algebra with property (J) or let \( A \) be a unital \( C^* \)-algebra in \( \mathcal{N} \) with \( TR(A) \leq 1 \) and let \( B \) be a unital separable simple amenable \( C^* \)-algebra with \( TR(B) \leq 1 \). Suppose that \( \phi_1, \phi_2 : A \to B \) are two monomorphisms such that

(\ref{eq:5.17})

\[
[\phi_1] = [\phi_2] \quad \text{in} \quad KK(A, B), \quad (\phi_1)_x = (\phi_2)_x \quad \text{and} \quad \phi_1^\dagger = \phi_2^\dagger.
\]

Then there exists a monomorphism \( \beta : \phi_2(A) \to B \) such that \([\beta \circ \phi_2] = [\phi_2] \) in \( KK(A, B) \),

(\ref{eq:5.18})

\[
(\beta \circ \phi_2)_x = (\phi_2)_x \quad \text{and} \quad (\beta \circ \phi_2)^\dagger = (\phi_2)^\dagger \quad \text{and} \quad \beta \circ \phi_2 \quad \text{is asymptotically unitarily equivalent to} \quad \phi_1.
\]

Moreover, if \( H_1(K_0(A), K_1(B)) = K_1(B) \), they are strongly asymptotically unitarily equivalent.

**Proof.** By Theorem 9.4 of \cite{15}, there is a monomorphism \( \beta \in \overline{\operatorname{im}}(\phi_2(A), B) \) such that \( [\beta] = [\iota] \) in \( KK(\phi_2(A), B) \) and

\[
\mathcal{R}_{\iota, \beta} = -\mathcal{R}_{\phi_1, \phi_2}
\]

where \( \iota \) is the embedding of \( \phi_2(A) \) to \( B \) and \( \mathcal{R}_{\iota, \beta} \) is viewed as a homomorphism from \( K_1(A) = K_1(\phi_2(A)) \) to \( \text{Aff}(T(B)) \). In other words

(\ref{eq:5.19})

\[
\mathcal{R}_{\phi_2, \beta \circ \phi_2} = -\mathcal{R}_{\phi_1, \phi_2}.
\]

One also has that

(\ref{eq:5.20})

\[
[\phi_2] = [\beta \circ \phi_2] \quad \text{in} \quad KK(A, B),
\]

\[
(\beta \circ \phi_2)_x = (\phi_2)_x \quad \text{and} \quad (\beta \circ \phi_2)^\dagger = (\phi_2)^\dagger.
\]

Thus

(\ref{eq:5.21})

\[
[\phi_1] = [\beta \circ \phi_2] \quad \text{in} \quad KK(A, B),
\]

(\ref{eq:5.22})

\[
(\phi_1)_x = (\beta \circ \phi_2)_x \quad \text{and} \quad \phi_1^\dagger = (\beta \circ \phi_2)^\dagger.
\]

It follows from \ref{lem:5.3} and \ref{eq:5.18} that

\[
\mathcal{R}_{\phi_1, \beta \circ \phi_2} = \mathcal{R}_{\phi_1, \phi_2} + \mathcal{R}_{\phi_2, \beta \circ \phi_2} = 0.
\]

Therefore, it follows from 7.2 of \cite{15} that the map \( \phi_1 \) and \( \beta \circ \phi_2 \) are asymptotically unitarily equivalent.
In the case that \( H_1(K_0(A), K_1(B)) = K_1(B) \), it follows from 10.5 of \([5]\) that \( \beta \circ \phi_2 \) and \( \phi_1 \) are strongly asymptotically unitarily equivalent. \( \square \)

**Lemma 5.5.** Let \( C \) and \( A \) be two unital separable stably finite \( C^* \)-algebras. Suppose that \( \phi, \psi : C \to A \) are two unital monomorphisms such that

\[
[\phi] = [\psi] \text{ in } KL(C, A), \quad \phi_\sharp = \psi_\sharp \text{ and } \overline{\mathcal{R}}_{\psi, \phi} = 0.
\]

Suppose that \( \{U(t) : t \in [0, 1]\} \) is a piecewise smooth and continuous path of unitaries in \( A \) with \( U(0) = 1 \) such that

\[
\lim_{t \to 1} U^*(t) \phi(u) U(t) = \psi(u)
\]

for some \( u \in U(C) \). Suppose there exists \( w \in U(A) \) such that \( \psi(u) w^* \in U_0(A) \). Let

\[
Z = Z(t) = U^*(t) \phi(u) U(t) w^* \quad \text{if } t \in [0, 1)
\]

and \( Z(1) = \psi(u) w^* \). Suppose also that there is a piecewise smooth continuous path of unitaries \( \{z(s) : s \in [0, 1]\} \) in \( A \) such that \( z(0) = \phi(u) w^* \) and \( z(1) = 1 \). Then, for any piecewise smooth continuous path \( \{Z(t, s) : s \in [0, 1]\} \subset C([0, 1], A) \) of unitaries such that \( Z(t, 0) = Z(t) \) and \( Z(t, 1) = 1 \), there is \( f \in \rho_A(K_0(A)) \) such that

\[
\frac{1}{2\pi \sqrt{-1}} \int_0^1 \tau \left( \frac{dZ(t, s)}{ds} Z(t, s)^* \right) ds = \frac{1}{2\pi \sqrt{-1}} \int_0^1 \tau \left( \frac{dz(s)}{ds} z(s)^* \right) ds + f(\tau)
\]

for all \( t \in [0, 1] \) and \( \tau \in T(A) \).

**Proof.** Define

\[
Z_1(t, s) = \begin{cases} 
U^*(t - 2s) \phi(u) U(t - 2s) w^* & \text{for } s \in [0, t/2) \\
\phi(u) w^* & \text{for } s \in [t/2, 1/2) \\
z(2s - 1) & \text{for } s \in [1/2, 1]
\end{cases}
\]

for \( t \in [0, 1) \) and define

\[
Z_1(1, s) = \begin{cases} 
\psi(u) w^* & \text{for } s = 0 \\
U^*(1 - 2s) \phi(u) U(1 - 2s) w^* & \text{for } s \in (0, 1/2) \\
z(2s - 1) & \text{for } s \in [1/2, 1].
\end{cases}
\]

Thus \( \{Z_1(t, s) : s \in [0, 1]\} \subset C([0, 1], A) \) is a piecewise smooth continuous path of unitaries such that \( Z_1(t, 0) = Z(t) \) and \( Z_1(t, 1) = 1 \). Thus, there is an element \( f_1 \in \rho_A(K_0(A)) \), such that

\[
f_1(\tau) = \frac{1}{2\pi \sqrt{-1}} \int_0^1 \tau \left( \frac{dZ_1(t, s)}{ds} Z_1(t, s)^* \right) ds = \frac{1}{2\pi \sqrt{-1}} \int_0^1 \tau \left( \frac{dZ_1(t, s)}{ds} Z_1(t, s)^* \right) ds
\]

for all \( \tau \in T(A) \) and for all \( t \in [0, 1] \).

On the other hand, let \( V(t) = U(t)^* \phi(u) U(t) \) for \( t \in [0, 1) \) and \( V(1) = \psi(u) \). For any \( s \in [0, 1) \), since \( U(0) = 1 \), \( U(t) \in U(C([0, s], A))_0 \) (for \( t \in [0, s] \)). There are \( a_1, a_2, ..., a_k \in...
$U([0, s], A)_{s.a.}$ such that
\[ U(t) = \prod_{j=1}^{k} \exp(ia_j(t)) \text{ for all } t \in [0, s] \]

Then a straightforward calculation shows that
\[ \int_0^s \frac{dV(t)}{dt} V^*(t)dt = 0. \]

We also have
\[ \int_0^1 \frac{1}{2\pi \sqrt{-1}} \int_0^1 \frac{dZ_1(1, s)}{ds} Z_1(1, s)^*ds = \int_0^{1/2} \frac{dV(2s - 1)}{ds} V(2s - 1)^*ds \]

Then
\[ \int_0^{1/2} \frac{dZ_1(t, s)}{ds} Z_1(t, s)^*ds = \int_0^{1/2} \frac{dV(t - 2s)}{ds} V(t - 2s)^*ds + \int_{1/2}^1 \frac{dz(s - 1)}{ds} z(2s - 1)^*ds \]

One computes that, for any $\tau \in T(A)$ and for any $t \in [0, 1)$, by applying (5.31),
\[ \int_0^{1/2} \frac{dZ_1(t, s)}{ds} Z_1(t, s)^*ds = \int_0^{1/2} \frac{dV(t - 2s)}{ds} V(t - 2s)^*ds + \int_{1/2}^1 \frac{dz(s - 1)}{ds} z(2s - 1)^*ds \]

It then follows from (5.31) that
\[ \int_0^{1/2} \frac{dZ_1(1, s)}{ds} Z_1(1, s)^*ds = \int_0^{1/2} \frac{dV(t - 2s)}{ds} V(t - 2s)^*ds + \int_{1/2}^1 \frac{dz(s - 1)}{ds} z(2s - 1)^*ds \]

The lemma follows. \qed
Remark 5.6. Note that the lemma 5.5 applies to $M_n(C)$ and $M_n(A)$ for all integer $n \geq 1$. So it works for $u \in U_n(C)$.

Lemma 5.7. Let $A$ be a unital AH-algebra with property (J), or let $A$ be a unital $C^*$-algebra in $\mathcal{N} \cap \mathcal{C}$ and let $B$ be a unital simple $C^*$-algebra in $\mathcal{N} \cap \mathcal{C}$. Let $\kappa \in KL_0(A,B)^{++}$ and $\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ be an affine homomorphism which are compatible. Then there exists a unital homomorphism $\phi : A \to B$ such that

$$[\phi] = \kappa \text{ and } (\phi)_2 = \lambda.$$ 

Moreover, if $\gamma \in U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ is a continuous homomorphism which is compatible with $\kappa$ and $\lambda$, then one may also require that

$$\phi|_{U_\infty(A)/CU_\infty(A)} = \gamma|_{U_\infty(A)/CU_\infty(A)} \text{ and } (\phi)\dagger \circ s_1 = \gamma \circ s_1 - \bar{h},$$

where $s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A)$ is a splitting map (see [2.3]), and

$$\bar{h} : K_1(A) \to \mathbb{R}\rho_B(K_0(B))/\rho_B(K_0(B))$$

is a homomorphism.

Moreover,

$$\phi \otimes \text{id}_{Z_p,q} \dagger \circ s_1 = E_B \circ \gamma \circ s_1 - \bar{h},$$

where $E_B$ is as defined in 5.2.

Proof. Let us first prove the lemma for $A \in \mathcal{N} \cap \mathcal{C}$. Let $p$ and $q$ be two relative prime supernatural numbers such that $Q = M_p \otimes M_q$. Let $A_p = A \otimes M_p$ and $A_q = A \otimes M_q$. Let $\kappa_p \in KL(A_p,B_p)$, $\kappa_q \in KL(A_p,B_p)$, $\lambda_p : \text{Aff}(T(A_p)) \to \text{Aff}(T(B_p))$, $\lambda_q : \text{Aff}(T(A_q)) \to \text{Aff}(T(B_q))$, $\gamma_p : U(A_p)/CU(A_p) \to U(B_p)/CU(B_p)$ and $\gamma_q : U(A_q)/CU(A_q) \to U(B_q)/CU(B_q)$ be induced by $\kappa$, $\lambda$ and $\gamma$, respectively. It follows from 8.6 of [15] that there is a unital homomorphism $\phi_p : A_p \to B_p$ such that

$$[\phi_p] = \kappa_p \text{ in } KL(A_p,B_p)(\phi_p)\dagger = \gamma_p \text{ and } (\phi_p)_2 = \lambda_p.$$ 

For the same reason, there is also a unital homomorphism $\psi_q : A_q \to B_q$ such that

$$[\psi_q] = \kappa_q \text{ in } KL(A_q,B_q)(\psi_q)\dagger = \gamma_q \text{ and } (\psi_q)_2 = \lambda_q.$$ 

Define $\phi = \phi_p \otimes \text{id}_{M_q}$ and $\psi = \psi_q \otimes \text{id}_{M_p}$. From above, one has that

$$[\phi] = [\psi] \text{ in } KL(A \otimes Q, B \otimes Q), \phi_2 = \psi_2 \text{ and } \phi\dagger = \psi\dagger.$$ 

Since both $K_i(B \otimes Q)$ are divisible ($i = 0, 1$), one actually has

$$[\phi] = [\psi] \text{ in } KK(A \otimes Q, B \otimes Q).$$

It follows from 5.4 that there is $\beta_0 \in \text{Inn}(\psi(A \otimes Q), B \otimes Q)$ such that if $\iota_{\psi(A \otimes Q)}$ denotes the embedding of $\psi(A \otimes Q)$ into $B \otimes Q$,

$$[\beta_0] = [\iota_{\psi(A \otimes Q)}] \text{ in } KK(\psi(A \otimes Q), B \otimes Q),$$

$$[\beta_0]_2 = (\iota_{\psi(A \otimes Q)})_2 \text{ and } (\beta_0)\dagger = (\iota_{\psi(A \otimes Q)})\dagger.$$
such that $\phi$ and $\beta_0 \circ \psi$ are strongly asymptotically unitarily equivalent (since in this case $H_1(K_0(A \otimes Q), K_1(B \otimes Q)) = K_1(B \otimes Q)$). Note that one may identify $T(B_q)$, $T(B_p)$ and $T(B \otimes Q)$. Moreover,

$$\overline{\rho_{B \otimes Q}(K_0(B \otimes Q))} = \mathbb{R} \rho_B(K_0(B)) = \rho_{B_q}(K_0(B_q)).$$

Denote by $i_p : B_q \to B \otimes Q$ the embedding $a \mapsto a \otimes 1_{M_p}$, and note that the image of $i_p \circ \psi_q$ is in the image of $\psi$. Thus, by \textbf{3.6} $R_{\beta_0 \circ i_p \circ \psi_q, \psi_q}$ is in $\text{Hom}((K_1(M_{\beta_0 \circ i_p \circ \psi_q, \psi_q}), \rho_{B_q}(K_0(B_q))))$. Note that

$$[\beta_0 \circ i_p \circ \psi_q] = [i_p \circ \psi_q] \text{ in } KK(A_q, B_q).$$

By Theorem 9.4 of [15], there exists $\alpha \in \text{Hom}(\psi_q(A_q), B_q)$ such that

$$[\alpha] = [i_p \circ \psi_q(A_q)] \text{ in } KK(B_q, B_q),$$

where $i_{\psi_q(A_q)}$ is the embedding of $\psi_q(A_q)$ into $B_q$, and

$$\overline{R} \alpha, i_{\psi_q(A_q)} = -\overline{R} \beta_0 \circ i_p \circ \psi_q.$$

As computed in the proof of \textbf{5.4} one has that

$$[i_p \circ \alpha \circ \psi_q] = [\beta_0 \circ i_p \circ \psi_q] \text{ in } KK(A_q, B \otimes Q),$$

and

$$([i_p \circ \alpha \circ \psi_q])^\dagger = ([\beta_0 \circ i_p \circ \psi_q])^\dagger;$$

and

$$\overline{R} (i_p \circ \alpha \circ \psi_q, \beta_0 \circ i_p \circ \psi_q) = 0.$$

It follows from 7.2 and 10.5 of [15] that $i_p \circ \alpha \circ \psi_q$ and $\beta_0 \circ i_p \circ \psi_q$ are strongly asymptotically unitarily equivalent.

Consider maps

$$(\beta_0 \circ i_p \circ \psi_q) \otimes \text{id}_{M_p}, \ i \circ \beta_0 \circ \psi : A \otimes M_q \otimes M_p \to (B \otimes M_q \otimes M_p) \otimes M_p,$$

where $i : B \otimes Q \to (B \otimes Q) \otimes M_p$ is the embedding $b \mapsto b \otimes 1_{M_p}$ for all $b \in B \otimes Q$.

Identify $\beta_0 \circ \psi(B \otimes M_q \otimes M_p) \otimes M_p$ with $\beta_0 \circ \psi(B) \otimes \beta_0 \circ \psi(M_q) \otimes \beta_0 \circ \psi(M_p) \otimes M_p$, and consider the automorphism $\theta$ on $\beta_0 \circ \psi(B) \otimes \beta_0 \circ \psi(M_q) \otimes \beta_0 \circ \psi(M_p) \otimes M_p$ defined by

$$\theta : a \otimes b \otimes c \otimes d \mapsto a \otimes b \otimes d \otimes c.$$

Then

$$[\theta |_{\beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p}] = [\text{id}_{\beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p}] \text{ in } KK(\beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p, \beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p).$$

Since $K_1(\beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p) = \{0\}$, it follows from 10.5 of [15] that $\theta |_{\beta_0(M_q) \otimes \beta_0(M_p) \otimes M_p}$ is strongly asymptotically unitarily equivalent to the identity map. Therefore $\theta$ is strongly asymptotically unitarily equivalent to the identity map. Note that for any $a \in A$, $b \in M_q$, and $c \in M_p$, one has

$$\theta(((\beta_0 \circ i_p \circ \psi_q) \otimes \text{id}_{M_p})(a \otimes b \otimes c)) = \theta(\beta_0(\psi_q(a \otimes b) \otimes 1_{M_p}) \otimes c)$$

$$= \beta_0(\psi_q(a \otimes b) \otimes c) \otimes 1_{M_p}$$

$$= i \circ \beta_0 \circ \psi(a \otimes b \otimes c).$$
Thus, the map $(\beta_0 \circ t_p \circ \psi_q) \otimes \text{id}_{M_p}$ is strongly asymptotically unitarily equivalent to $\iota \circ \beta_0 \circ \psi$.

Define a map $\Psi_q : A \otimes M_q \otimes M_p \to B \otimes M_q \otimes M_p \otimes M_p$ by

$$\Psi_q : a \otimes b \otimes c \mapsto \alpha(\psi_q(a \otimes b)) \otimes c \otimes 1_{M_p}. \tag{5.53}$$

Note that for all $a \otimes b \otimes c \in A \otimes M_q \otimes M_p$,

$$((t_p \circ \alpha \circ \psi_q) \otimes \text{id}_{M_p})(a \otimes b \otimes c) = \alpha(\psi_q(a \otimes b)) \otimes 1_{M_p} \otimes c \tag{5.54}$$

Then the same argument as above shows that $\Psi_q$ is strongly asymptotically unitarily equivalent to $(t_p \circ \alpha \circ \psi_q) \otimes \text{id}_{M_p}$.

Since $\phi$ and $\beta_0 \circ \psi$ are strongly asymptotically unitarily equivalent, one has that the map $\iota \circ \phi$ is strongly asymptotically unitarily equivalent to $\iota \circ \beta_0 \circ \psi$, and hence strongly asymptotically unitarily equivalent to $(\beta_0 \circ t_p \circ \psi_q) \otimes \text{id}_{M_p}$, and therefore strongly asymptotically unitarily equivalent to $(t_p \circ \alpha \circ \psi_q) \otimes \text{id}_{M_p}$. It follows that the map $\iota \circ \phi$ is strongly asymptotically unitarily equivalent to $\Psi_q$. Thus there is a continuous path of unitaries $\{w(t) : t \in [0,1]\}$ in $B \otimes M_q \otimes M_p \otimes M_p$ with $w(0) = 1$ such that

$$\lim_{t \to 1} w^*(t)(\iota \circ \phi)(a)w(t) = \Psi_q(a), \quad \forall a \in A \otimes Q. \tag{5.55}$$

Pick an isomorphism $\chi' : M_p \otimes M_p \to M_p$, and consider the induced isomorphism $\chi : B \otimes M_q \otimes M_p \otimes M_p \to B \otimes M_q \otimes M_p$. Note that $(\chi')^{-1}$ is strongly asymptotically unitarily equivalent to the map $\iota' : M_p \to M_p \otimes M_p$ defined by $a \mapsto 1 \otimes a$. Then, it is straightforward to verify that $\chi \circ \iota' \circ \phi$ is strongly asymptotically unitarily equivalent to $\phi$, and $\chi \circ \Psi_q$ is strongly asymptotically unitarily equivalent to $(\alpha \circ \psi_q) \otimes \text{id}_{M_p}$. Thus, there is a continuous path of unitaries $u(t)$ in $B \otimes M_p \otimes M_q$ (one can be made it into piecewise smooth—see Lemma 4.1 of [15]) such that $u(0) = 1$ and

$$\lim_{t \to 1} \text{ad} u(t) \circ \phi(a) = (\alpha \circ \psi_q) \otimes \text{id}_{M_p}(a) \quad \text{for all} \quad a \in A \otimes Q. \tag{5.56}$$

This provides a unital homomorphism $\Phi : A \otimes \mathcal{Z}_{p,q} \to B \otimes \mathcal{Z}_{p,q}$ such that, for each $t \in (0,1)$,

$$\pi_t \circ \Phi(a) = \text{ad} u(t) \circ \phi(a(t)) \quad \text{for all} \quad a \in A \otimes \mathcal{Z}_{p,q}. \tag{5.57}$$

Denote by $\vartheta$ a unital embedding $\mathcal{Z} \to \mathcal{Z}_{p,q}$, and let $j : \mathcal{Z}_{p,q} \to \mathcal{Z}$ be a unital homomorphism induced by the stationary inductive limit

$$\mathcal{Z}_{p,q} \xrightarrow{j} \mathcal{Z}_{p,q} \xrightarrow{j} \mathcal{Z}_{p,q} \xrightarrow{j} \cdots \to \mathcal{Z} \tag{5.57}$$

given by [22].

As in the proof of 7.1 of [25] (note that it follows from the same proof that Proposition 4.6 of [25] also works for homomorphisms which are not necessary being injective),

$$((\text{id}_B \otimes j) \circ \Phi \circ (\text{id}_A \otimes \vartheta))_{\kappa_i} = \kappa_i, \quad i = 0, 1, \tag{5.58}$$

$$((\text{id}_B \otimes j) \circ \Phi \circ (\text{id}_A \otimes \vartheta))_{\lambda} = \lambda. \tag{5.58}$$

In fact, one has that

$$\Phi_{\tau}(a \otimes b)(\tau \otimes \mu) = \gamma(a(\tau))\mu(b) \quad \text{for all} \quad a \in A_{s.a.} \quad \text{and} \quad b \in (\mathcal{Z}_{p,q})_{s.a.}. \tag{5.59}$$
By considering \((\text{id}_B \otimes j) \circ \Phi \circ (\text{id}_A \otimes i) \otimes \text{id}_{C(X_k)} : A \otimes C(X_k) \to B \otimes C(X_k)\) for some suitable compact metric spaces \(X_k\), the same argument shows that, in fact,

(5.60) \[
[(\text{id}_B \otimes j) \circ \Phi \circ (\text{id}_A \otimes \vartheta)] = \kappa.
\]

Define the map \(H = (\text{id}_B \otimes j) \circ \Phi \circ (\text{id}_A \otimes \vartheta)\). Then \([H] = \kappa\) in \(KL(A, B)\) and \(H_\vartheta = \lambda\).

Note that it follows from (5.59) that

(5.61) \[
\Phi^t|_{U(A)_0/CU(A)} = E^t_B \circ \gamma|_{U(A)_0/CU(A)}.
\]

Let \(z \in U(A)/CU(A)\). Then, one has

(5.62) \[
H^t = \gamma_\infty = \iota^t_\infty \circ \gamma.
\]

On the other hand, for each \(z \in U(A)/CU(A)\), there is a unitary \(w \in B \otimes \mathcal{Z}_{p,q}\) such that

(5.63) \[
\pi_t(w) = \pi_{t'}(w) \quad \text{for all } t, t' \in [0, 1] \quad \text{and} \quad E^t_B \circ \gamma(z) = \overline{w}.
\]

Since \(\pi_t(w) \in B\) is constant, one may use \(w\) for its evaluation at \(t\). Let \(v_0 \in U(A)\) be such that \(\overline{v}_0 = z\). For any \(t \in (0, 1)\), define

(5.64) \[
Z(t) = \pi_t \circ \Phi(v_0)w^* = u(t)^* \phi(v_0)u(t)w^*.
\]

Let \(Z(t, s)\) be a piecewise smooth continuous path of unitaries in \(B \otimes \mathcal{Z}_{p,q}\) such that \(Z(t, 0) = Z(t)\) and \(Z(t, 1) = 1\). Denote by \(\tau_0\) the unique tracial state in \(T(M_\tau)\), where \(\tau\) is a supernatural number. For each \(s_\mu \in T(\mathcal{Z}_{p,q})\), one may write

\[
s_\mu(a) = \int_0^1 \tau_0(a(t))d\mu(t),
\]

where \(\mu\) is a probability Borel measure on \([0, 1]\).

Then, for \(\tau \in T(B)\) and \(s_\mu \in T(\mathcal{Z}_{p,q})\), by applying (5.5),

(5.65) \[
\text{Det}(Z)(\tau \otimes s_\mu) = \frac{1}{2\pi\sqrt{-1}} \int_0^1 (\tau \otimes s_\mu)(\frac{dZ(t, s)}{ds}Z(t, s)^*)ds
\]

(5.66) \[
= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \int_0^1 (\tau \otimes \tau_0)(\frac{dZ(t, s)}{ds}Z(t, s)^*)d\mu(t)ds
\]

(5.67) \[
= \int_0^1 \frac{1}{2\pi\sqrt{-1}} \int_0^1 (\tau \otimes \tau_0)(\frac{dZ(t, s)}{ds}Z(t, s)^*))ds)d\mu(t)
\]

(5.68) \[
= \int_0^1 \text{Det}(\phi(v_0)w^*)(\tau)d\mu(t) + f(\tau) \quad \text{for some } f \in \rho_B(K_0(B)).
\]

By (5.2) and (5.60),

(5.69) \[
\text{Det}(Z)(\tau \otimes s_\mu) = \text{Det}(\phi(v_0)w^*)(\tau) + f(\tau) \in \mathbb{R}\rho_B(K_0(B)) \subseteq \text{Aff}(T(B \otimes \mathcal{Z}_{p,q}))
\]

Thus, \(\Phi^t(z)(E_B \circ \lambda(z)^*)\) defines a homomorphism from \(U(A)/CU(A)\) into \(\mathbb{R}\rho_B(K_0(B))/\rho_B(K_0(B))\), which will be denoted by \(h\). By (5.60),

(5.70) \[
h|_{U(A)_0/CU(A)} = 0.
\]

Thus \(h\) induces a homomorphism \(\tilde{h} : K_1(A) \to \mathbb{R}\rho_B(K_0(B))/\rho_B(K_0(B))\).
If $A$ is an AH-algebra with property (J), for any two relatively prime supernatural number $p$ and $q$ with $M_p \otimes M_q \cong Q$, by 8.6 of [13], there are homomorphisms $\phi_p : A \to B_p$ and $\psi_q : A \to B_q$ such that

$$[\phi_p] = \kappa_p \text{ in } KL(A, B_p), (\phi_p)^\sharp = \gamma_p \text{ and } (\phi_p)_z = \lambda_p,$$

and

$$[\psi_q] = \kappa_q \text{ in } KL(A, B_q), (\psi_q)^\sharp = \gamma_q \text{ and } (\psi_q)_z = \lambda_q,$$

where we consider $U_\infty(A)/CU_\infty(A)$ and $U_\infty(B)/CU_\infty(B)$ instead of $U(A)/CU(A)$ and $U(B)/CU(B)$, respectively.

Using the same argument as above, one has $\alpha \in \text{Aut}(B_q)$ such that

$$[\alpha] = \text{id}_{B_q} \text{ in } KK(B_q, B_q), \quad \alpha_z = (\text{id}_{B_q})_z, \quad \alpha^\sharp = \text{id}_{B_q}^\sharp$$

and $\phi_p$ is strongly asymptotically unitarily equivalent to $\psi_q$, and thus induces a homomorphism $\Phi : A \to B \otimes \mathbb{Z}_{p,q}$, and then, the same argument as above shows that the map $H = (\text{id} \otimes j) \circ \Phi : A \to B \cong B \otimes \mathbb{Z}$ is the desired homomorphism.

In [13], it was shown that, given two unital separable simple C*-algebras $A$ and $B$ in $\mathcal{N} \cap \mathcal{C}$, if there is an isomorphism on the Elliott invariant, i.e.,

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)_+, [1_B], T(B), r_B),$$

then $A \cong B$. The following corollary is a more general statement.

**Corollary 5.8.** Let $A$ and $B$ be two unital separable C*-algebras in $\mathcal{N} \cap \mathcal{C}$. Suppose that there is a homomorphism $\kappa_i : K_i(A) \to K_i(B)$ such that $\kappa_0$ is order preserving and $\kappa_0([1_A]) \leq [1_B]$ and there is a continuous affine map $\lambda : \text{Aff}(T(A)) \to \text{Aff}(T(B))$ which is compatible with $\kappa_0$. Then there is a homomorphism $\phi : A \to B$ such that

$$[\phi]_{\kappa_i} = \kappa_i, \quad i = 0, 1 \text{ and } \phi_z = \lambda.$$

**Proof.** Consider the splitting short exact sequence:

$$0 \to \text{Ext}_\mathbb{Z}(K_*(A), K_*(B)) \to KK(A, B) \to \text{Hom}(K_*(A), K_*(B)) \to 0.$$

There exists an element $\kappa \in KK(A, B)$ such that the image of $\kappa$ in $\text{Hom}(K_*(A), K_*(B))$ is exactly the same as that $\kappa_z$. Let $\tilde{\kappa}$ in $KL(A, B)$ be the image of $\kappa$. There is a projection $p \in B$ such that $[p] = \kappa_0([1_A])$. Let $B_1 = pBp$. Then $\tilde{\kappa} \in KL_c(A, B_1)^{++}$ and $\lambda$ and $\tilde{\kappa}$ are compatible. It follows from [5.7] that there is a unital homomorphism $\phi : A \to B_1 \subset B$ such that

$$[\phi] = \tilde{\kappa} \text{ and } \phi_z = \lambda.$$

**Theorem 5.9.** Let $C$ be a unital AH-algebra with property (J), or let $C$ be a unital C*-algebra in $\mathcal{N} \cap \mathcal{C}$ and let $A$ be a unital simple C*-algebra in $\mathcal{N} \cap \mathcal{C}$ which is Z-stable. Then, for any $\kappa \in KL_c(C, A)^{++}$ and a continuous homomorphism $\gamma : U_\infty(C)/CU_\infty(C) \to U_\infty(A)/CU_\infty(A)$ which are compatible, there is a unital monomorphism $\phi : C \to A$ such that

$$([\phi], \phi_z) = \kappa \text{ and } \phi^\sharp = \gamma,$$
provided that

1. \( K_1(C) \) is a free group, or
2. \( \mathbb{R}p_A(K_0(A))/\rho_A(K_0(A)) = \{0\} \), or
3. \( \mathbb{R}p_A(K_0(A))/\rho_A(K_0(A)) \) is torsion free and \( K_1(C) \) is finitely generated.

**Proof.** We will consider the case that \( C \in \mathcal{N} \cap C \). The case that \( C \) is an AH-algebra with property (J) is similar.

It follows from (5.7) that there is a unital monomorphism \( \psi : C \to A \) such that

\[
(\psi, \psi^\sharp) = \kappa, \quad \psi^\sharp|_{\mathfrak{u}(C)} = \lambda|_{\mathfrak{u}(C)} \quad \text{and} \quad (\psi \otimes \text{id}_{Z_p}) \circ s_1 = E_\beta^\sharp \circ \gamma \circ s_1 - \bar{h},
\]

where \( \bar{h} : K_1(C) \to \mathbb{R}p_A(K_0(A))/\rho_A(K_0(A)) \) is a homomorphism. If \( K_1(C) \) is free, there exists a homomorphism \( h_1 : K_1(C) \to \mathbb{R}p_A(K_0(A)) \) which induces \( h_1 \). In the case that \( \mathbb{R}p_A(K_0(A))/\rho_A(K_0(A)) \) is torsion free and \( K_1(C) \) is finitely generated, one also obtains a such \( h_1 \). Since \( \mathbb{R}p_A(K_0(A)) \) is torsion free, \( h_1 \) induces a homomorphism \( \bar{h}_1 : K_1(C)/(\text{Tor}(K_1(C))) \to \mathbb{R}p_A(K_0(A)) \). Since the map from \( K_1(C)/(\text{Tor}(K_1(C))) \to (K_1(A)/(\text{Tor}(K_1(A))) \otimes \mathbb{Q}_p \) is injective, one obtains a homomorphism \( h_{1,p} : K_1(C \otimes M_p) \to \mathbb{R}p_A(K_0(A)) \) such that

\[
h_1 = h_{1,p} \circ (\iota_p)_* s_1,
\]

where \( \iota_p : A \to A \otimes M_p \) is the embedding so that \( \iota_p(a) = a \otimes 1 \) for all \( a \in A \) (\( \iota \) is a supernatural number). Similarly, there is a homomorphism \( h_{1,q} : K_1(C \otimes M_q) \to \mathbb{R}p_A(K_0(A)) \) such that

\[
h_1 = h_{1,q} \circ (\iota_q)_* s_1.
\]

Put \( C'_p = (\psi \otimes \text{id}_{M_p})(C \otimes M_p) \), where \( \mathfrak{r} \) is a supernatural number. It follows from (9.4) that there is a monomorphism \( \beta_0 \in \text{Im}(C'_p, A_p) \) such that

\[
[\beta_0] = [\iota_{C'_p}] \quad \text{in} \quad KK(C'_p, A_p), \quad (\beta_0)^\sharp = \iota_{C'_p}^\sharp, \quad \beta_0^\sharp = \iota_{C'_p}^\sharp \quad \text{and} \quad R_{\psi \otimes \text{id}_{M_p}, \beta_0 \circ (\psi \otimes \text{id}_{M_p})} = h_{1,p},
\]

where \( \iota_{C'_p}^\sharp \) is the embedding of \( C'_p \).

Similarly, there is a monomorphism \( \beta_1 \in \text{Im}(C'_q, A_q) \) such that

\[
[\beta_1] = [\iota_{C'_q}] \quad \text{in} \quad KK(C'_q, A_q), \quad (\beta_1)^\sharp = \iota_{C'_q}^\sharp, \quad \beta_1^\sharp = \iota_{C'_q}^\sharp \quad \text{and} \quad R_{\psi \otimes \text{id}_{M_q}, \beta_1 \circ (\psi \otimes \text{id}_{M_q})} = h_{1,q},
\]

where \( \iota_{C'_q}^\sharp \) is the embedding of \( C'_q \).

As in the proof of (5.7) by applying (5.4) and its proof, one has a morphism \( \beta_2 \in \text{Im}(\beta_1 \circ (\psi \otimes \text{id}_{M_q}))(C_q, A_q) \) and a piecewise smooth continuous path of unitaries \( \{U(t) : t \in [0,1]\} \) of \( A \otimes Q \) such that \( U(0) = 1 \)

\[
[\beta_2 \circ \beta_1 \circ (\psi \otimes \text{id}_{M_q})] = [\beta_0 \circ (\psi \otimes \text{id}_{M_p})] \quad \text{in} \quad KK(C_q, A_q),
\]

\[
(\beta_2 \circ \beta_1 \circ (\psi \otimes \text{id}_{M_q}))^\sharp = (\beta_0 \circ (\psi \otimes \text{id}_{M_p}))^\sharp
\]

and

\[
(\beta_2 \circ \beta_1 \circ (\psi \otimes \text{id}_{M_q}))^\sharp = (\beta_0 \circ (\psi \otimes \text{id}_{M_p}))^\sharp.
\]
Moreover, if denote by $\psi_0 = \beta_0 \circ (\psi \circ \text{id}_{M_p})$ and $\psi_1 = \beta_1 \circ (\psi \circ \text{id}_{M_q})$, one has that
\begin{equation}
(5.82) \quad \lim_{t \to 1} U(t)^* (\psi_0 \otimes \text{id}_{M_q})(a) U(t) = (\psi_1 \otimes \text{id}_{M_p})(a)
\end{equation}
for all $a \in A \otimes Q$. In particular,
\begin{equation}
(5.83) \quad \mathcal{R}_{\psi_0 \otimes \text{id}_{M_q}, \psi_1 \otimes \text{id}_{M_p}} = 0.
\end{equation}

Let $\Phi : A \otimes \mathbb{Z}_{p,q} \to A \otimes \mathbb{Z}_{p,q}$ be defined by
\begin{equation}
(5.84) \quad \Phi(a \otimes b)(t) = U^*(t)((\psi_0 \otimes \text{id}_{M_q}(a \otimes b(t)))U(t) \quad \text{for all} \quad t \in [0, 1] \quad \text{and}
\end{equation}
\begin{equation}
(5.85) \quad \Phi(a \otimes b)(1) = \psi_1 \otimes \text{id}_{M_p}(a \otimes b(1)).
\end{equation}
for all $a \otimes b \in A \otimes \mathbb{Z}_{p,q}$.

We claim that
\begin{equation}
(5.86) \quad \Phi^\dagger \circ (E_A \circ \psi)^\dagger \circ s_1 = (E_A)^\dagger \circ \gamma \circ s_1.
\end{equation}

To compute $\Phi^\dagger$, let $x \in s_1(K_1(C))$ and $v_0 \in U(C)$ such that $\overline{v_0} = x$. There is $w \in U(A \otimes \mathbb{Z}_{p,q})/CU(A \otimes \mathbb{Z}_{p,q})$ such that $w(t) = w(t')$ for all $t, t' \in [0, 1]$ and
\begin{equation}
(5.87) \quad E_A^\dagger \circ \gamma \circ s_1(x) = \overline{w}.
\end{equation}

Let $Z = (\Phi \circ (\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0))w^* \in A \otimes \mathbb{Z}_{p,q}$. Note that $Z \in U(A \otimes \mathbb{Z}_{p,q})_0$. Suppose that there is a piecewise smooth continuous path $\{Z(t, s) : s \in [0, 1]\} \subset A \otimes \mathbb{Z}_{p,q}$ such that $Z(t, 0) = Z(t)$ and $Z(t, 1) = 1$. Then
\begin{equation}
(5.88) \quad \det(Z(t, s)) = \det(\Phi \circ ((\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0))(\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0)^*) + \det((\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0)w^*)
\end{equation}
\begin{equation}
(5.89) \quad = \det(\Phi \circ ((\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0))(\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0)^*) + \bar{h} \circ s_1(x).
\end{equation}

It follows from (5.89) that
\begin{equation}
(5.91) \quad \det(\Phi \circ ((\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0))(\psi \otimes \text{id}_{\mathbb{Z}_{p,q}})(v_0)^*) = \det(\beta_0 \circ \psi(v_0)) + \rho_A(K_0(A))
\end{equation}
\begin{equation}
(5.92) \quad = R_{\beta_0 \circ \psi}(v_0) + \rho_A(K_0(A))
\end{equation}
\begin{equation}
(5.93) \quad = -h_{1,p} \circ s_1(x) + \rho_A(K_0(A)).
\end{equation}

Therefore, by (5.75) and by (5.88),
\begin{equation}
\det(Z(t, s))(\tau \otimes s_\mu) \in \rho_A(K_0(A)).
\end{equation}

This proves the claim.

Regard $\psi$ as a map to $A \otimes \mathbb{Z}$. Denote by $j : \mathbb{Z}_{p,q} \to \mathbb{Z}$ the unital homomorphism induced by the stationary innuctive limit decomposition of $\mathbb{Z}$, and denote by $\vartheta : \mathbb{Z} \to \mathbb{Z}_{p,q}$ the unital embedding induced by tensoring $\mathbb{Z}$ ($\mathbb{Z}_{p,q}$ is $\mathbb{Z}$-stable). Consider
\[ \phi = (\text{id}_A \otimes j) \circ \Phi \circ (\text{id}_A \otimes \vartheta) \circ \psi. \]

One then checks that
\[ [\psi] = [\phi] \quad \text{in} \quad KL(C, A), \quad \phi^\ast = \psi^\ast \quad \text{and} \quad \phi^\dagger = \gamma. \]
Corollary 5.10. Let $C$ be a unital AH-algebra and let $A$ be a unital simple $C^*$-algebra in $N \cap C_0$ which is $\mathbb{Z}$-stable. Then, for any $\kappa \in KLT_e(C,A)^{++}$ and any continuous homomorphism $\gamma : U_\infty(C)/CU_\infty(C) \to U_\infty(A)/CU_\infty(A)$ which are compatible, there exists a unital monomorphism $\phi : C \to A$ such that

$$([\phi], \phi^\sharp) = \kappa \text{ and } \phi^\dagger = \gamma,$$

provided that

1. $K_1(C)$ is a free group, or
2. $\mathbb{R}\rho_A(K_0(A))/\rho_A(K_0(A)) = \{0\}$, or
3. $\mathbb{R}\rho_A(K_0(A))/\rho_A(K_0(A))$ is torsion free and $K_1(C)$ is finitely generated.

Remark 5.11. It follows from Proposition 3.6 of [16] that, if $TR(A) \leq 1$, then

$$\mathbb{R}\rho_A(K_0(A))/\rho_A(K_0(A)) = \{0\}.$$

So Theorem 5.9 recovers a version of Theorem 8.6 of [15].

Now suppose that in 5.9

$$U_\infty(C)/CU_\infty(C) = U_\infty(C)_0/CU_\infty(C) \oplus G_1 \oplus \text{Tor}(K_1(C)),$$

where $G_1$ is identified with a free subgroup of $K_1(C)$. From the proof of Theorem 5.9 we see that, if $\kappa \in KLT_e(C,A)^{++}$ and $\gamma : U_\infty(C)/CU_\infty(C) \to U(A)/CU_\infty(A)$ which is compatible to $\kappa$ are given, there is a unital monomorphism $\phi : C \to A$ such that $([\phi], \phi^\sharp) = \kappa$ and

$$\phi|_{U_\infty(C)_0/CU_\infty(C) \oplus G_1} = \gamma|_{U_\infty(C)_0/CU_\infty(C) \oplus G_1}$$

and

$$\phi^\dagger(z)\gamma(z) \in \mathbb{R}\rho_A(K_0(A))/\rho_A(K_0(A))$$

for all $z \in \text{Tor}(K_1(C))$.

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