ON THE JORDAN–CHEVALLEY DECOMPOSITION OF A MATRIX

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Abstract. The purpose of this note is to advertise an elegant algorithmic proof for the Jordan–Chevalley decomposition of a matrix, following and (slightly) revising the discussion of Couty, Esterle und Zarouf (2011). The basic idea of that method goes back to Chevalley (1951).

1. Introduction

Let $K$ be a field and $M_n(K)$ be the vector space of $n \times n$-matrices with entries in $K$. Let $A \in M_n(K)$ be such that the characteristic polynomial of $A$ splits into linear factors over $K$. Then we can write uniquely $A = D + N$ where $D, N \in M_n(K)$ are such that $D$ is diagonalisable, $N$ is nilpotent and $D \cdot N = N \cdot D$. This is called the Jordan–Chevalley decomposition of $A$. Usually, this is deduced from the Jordan normal form of $A$ (which is a stronger result). However, there is a direct, short algorithmic proof for the existence of $D, N$ which is inspired by the Newton iteration for finding roots of a function. This goes back to Chevalley and is discussed in Couty et al. [1] but, as the authors write, it does not seem to be widely known (especially not in teaching normal forms of matrices in Linear Algebra).

The purpose of this note is, firstly, to advertise that proof which yields an elegant procedure for producing $D$ and $N$, without even knowing the eigenvalues of $A$ (which may be a considerable advantage in some situations). Secondly, we somewhat reduce the required prerequisites about polynomials. The algorithm in [1] uses the “square-free part” of a non-constant polynomial $f \in K[X]$, the usual definition of which relies on the prime factorisation of $f$. But in basic courses on Linear Algebra, the prime factorisation of polynomials is often not yet available. And even if it is, then its use in concrete examples is not obvious because, in general, the prime factorisation of a polynomial is difficult to compute. Our aim is to present the algorithm in a way that avoids that completely; all we shall need is the formal derivative and the Euclidean Algorithm for polynomials in $K[X]$.

Thirdly, a GAP [4] program for computing the Jordan–Chevalley decomposition, based on the algorithm in this note and extending the functionality already developed in [3], is now available in the file frobenius.g at

https://pnp.mathematik.uni-stuttgart.de/idsr/idsrl/geckmf/

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This program appears to work quite well even for large matrices, especially when combined with the efficient algorithm for computing the Frobenius normal form of matrices in [3]; see Example 8 at the very end of this note.

2. Square-free polynomials

Let $K[X]$ be the polynomial ring over $K$ in the indeterminate $X$. We define the “formal derivative” as the linear map $D: K[X] \to K[X]$ such that $D(1) = 0$ and $D(X^i) = iX^{i-1}$ for all $i \geq 1$. The usual product rule holds in this case as well (as one easily checks):

$$
D(fg) = D(f)g + fD(g) \quad \text{for all } f, g \in K[X].
$$

Furthermore, let $f, g \in K[X]$ be such that $f \neq 0$ or $g \neq 0$. Then the Euclidean Algorithm yields the existence of a unique monic greatest common divisor $d = \gcd(f, g) \in K[X]$, as well as polynomials $r, s \in K[X]$ such that $d = rf + sg$. We say that $f, g$ are coprime if $\gcd(f, g) = 1$. — This is all we shall require in the following discussion.

Definition 1. Let $f \in K[X]$ be non-zero. We say that $f$ is square-free if $f$ and $D(f)$ are coprime, that is, $\gcd(f, D(f)) = 1$. In particular, a non-zero constant polynomial is square-free. (See also Remark 5 below.)

Lemma 2. Let $f, g \in K[X]$ be non-zero.

(a) If $f$ is square-free and $g \mid f$, then $g$ is also square-free.

(b) If $f, g$ are both square-free and coprime, then $fg$ is also square-free.

(c) If $f, g$ are square-free and $m \in K[X]$ is a least common multiple of $f$ and $g$, then $m$ is also square-free.

Proof. (a) Write $f = gh$ with $h \in K[X]$. Let $0 \neq d \in K[X]$ be monic such that $d \mid g$ and $d \mid D(g)$. Since $g \mid f$, we have $d \mid f$. Since $D(f) = D(g)h + gD(h)$, we also have $d \mid D(f)$. Hence, $d \mid D(f)$ and $d \mid f$; so $d = 1$.

(b) Let $0 \neq d \in K[X]$ be monic such that $d \mid fg$ and $d \mid D(fg)$. Let $d_1 = \gcd(d, f)$ and $d_2 := \gcd(d, g)$. Now $d_1 \mid d, d \mid D(f)g$ and $d_1 \mid f$, so $d_1 \mid D(f)g$. Since $d, g$ are coprime it is clear that $d_1, g$ are coprime. So we must have $d_1 \mid D(f)$ (see [3, Remark 4.1(c)]) and, hence, $d_1 = 1$; that is, $d$ and $f$ are coprime. Similarly, one sees that $d$ and $g$ are coprime. So $d$ and $fg$ are coprime by [3, Remark 4.1(a)].

(c) By [3, Lemma 4.3], there exist $f_1, g_1 \in K[X]$ such that $f_1 \mid f$, $g_1 \mid g$, $m = f_1g_1$ and $f_1, g_1$ are coprime. So $m$ is square-free by (a) and (b). \qed

The field $K$ is called perfect if either $\text{char}(K) = 0$ or $p := \text{char}(K) > 0$ and the map $K \to K$, $x \mapsto x^p$, is surjective. In addition to fields of characteristic 0, all finite fields are known to be perfect; every algebraically closed field is, of course, perfect.

Proposition 3. Assume that $K$ is perfect. Let $f \in K[X]$ be non-constant. Then there exists a square-free $g \in K[X]$ such that $g \mid f$ and $f \mid g^m$ for some $m \in \mathbb{N}$.

Proof. We proceed by induction on $\deg(f) \geq 1$. If $\deg(f) = 1$, then the assertion holds with $g := f$ and $m := 1$. Now let $\deg(f) \geq 1$. 


First assume that $D(f) \neq 0$; let $f_1 := \gcd(f, D(f)) \in K[X]$. If $f_1$ is constant, then $f$ is square-free and the assertion holds again with $g := f$ and $m := 1$. Otherwise write $f = f_1f_2$ with $f_2 \in K[X]$. Since $D(f) \neq 0$, we certainly have $\deg(f_1) \leq \deg(D(f)) < \deg(f)$. Since $\deg(f_1) \geq 1$, we also have $\deg(f_2) < \deg(f)$. Hence, by induction, there exist square-free $g_i \in K[X]$ such that $g_i | f_1$ and $f_1 | g_i^{m_i}$ for $i = 1, 2$, where $m_i \in \mathbb{N}$. Let $g \in K[X]$ be a least common multiple of $g_1, g_2$. Then Lemma 2 shows that $g$ is square-free. Since $g_i | f$ for $i = 1, 2$, we also have $g | f$. Since $f_i | g_i^{m_i}$ for $i = 1, 2$, we have $f | g_1^{m_1}g_2^{m_2}$ and so $f | g^m$ for $m := m_1 + m_2$.

Now assume that $D(f) = 0$. Write $f = \sum_{i=0}^n a_iX^i$ where $n = \deg(f) \geq 1$ and $a_i \in K$ for all $i$. Then $p := \text{char}(K) > 0$ and $a_i = 0$ for all $i$ with $p \mid i$. So $n = pn'$ for some $n' \in \mathbb{N}$ and $f = \sum_{j=0}^{n'} p a_j X^{jp}$. Since $K$ is perfect, we can write $a_{pj} = b_j^p$ with $b_j \in K$. Then $f = h^p$ where $h := \sum_{j=0}^{n'} b_j X^j \in K[X]$. By induction, there exists a square-free $g \in K[X]$ such that $g \mid h$ and $g \mid g^{m'}$ for some $m' \in K[X]$. Then $g \mid f$ and $f \mid g^m$ with $m := m'p$. \hfill \Box

**Remark 4.** Given a non-constant $f \in K[X]$, the above proof provides an efficient algorithm for computing the required square-free $g \in K[X]$. It is a somewhat more precise version of the procedure described in [5, p. 130]. (If $\text{char}(K) = 0$, then it is known that one can just take $g = f/\gcd(f, D(f))$.) Also note that a least common multiple of two non-zero $f, g \in K[X]$ is obtained as $fg/\gcd(f, g)$.

**Remark 5.** Assume that $f \in K[X]$ splits into linear factors in $K$. That is, we can write $f = c(X-\lambda_1)^{n_1} \cdots (X-\lambda_r)^{n_r}$ where $0 \neq c \in K$ and $\lambda_1, \ldots, \lambda_r \in K$ are the distinct roots of $f$ in $K$; furthermore, $n_i \geq 1$ for all $i$. Then one does not need to assume that $K$ is perfect in order to produce $g \in K[X]$ and $m \in \mathbb{N}$ as in Proposition 3. Indeed, just set $g := (X-\lambda_1) \cdots (X-\lambda_r) \in K[X]$ and $m := \max\{n_1, \ldots, n_r\}$. One easily computes $D(g)$ and checks that $D(g)(\lambda_i) \neq 0$ for all $i$; hence, $\gcd(g, D(g)) = 1$.

### 3. The Jordan–Chevalley Decomposition

Let us now fix a matrix $A \in M_n(K)$. We say that $A$ is *nilpotent* if $A^m = 0$ for some $m \in \mathbb{N}$. We say that $A$ is *semisimple* if $A$ is diagonalisable (possibly only over some larger field $L \supseteq K$). Let $f \in K[X]$ be non-constant such that $f(A) = 0$. Standard candidates for $f$ are the characteristic polynomial $\chi_A \in K[X]$ of $A$ (by the Theorem of Cayley–Hamilton), or the minimal polynomial $\mu_A \in K[X]$ of $A$. For example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Q})$$

is semisimple, since it is diagonalisable over $\mathbb{C} \supseteq \mathbb{Q}$.

It will be convenient to set $\mathfrak{A} := \{r(A) \mid r \in K[X]\} \subseteq M_n(K)$; note that all elements of $\mathfrak{A}$ commute with each other. If $B \in \mathfrak{A}$ is fixed, we denote $B \cdot \mathfrak{A} = \{B \cdot r(A) \mid r \in K[X]\}$.

**Lemma 6.** Let $B, C \in \mathfrak{A}$ and $h \in K[X]$. Then we have:

$h(B + C) - h(B) \in C \cdot \mathfrak{A}$ and $h(B + C) - h(B) - D(h)(B) \cdot C \in C^2 \cdot \mathfrak{A}$. 

Proof. It is enough to prove this for \( h = X^m \) where \( m \geq 0 \). In this case, the assertion immediately follows from the binomial formula for \((B + C)^m\). \( \square \)

Given \( f \in K[X] \) as above with \( f(A) = 0 \), assume that there is a square-free \( g \in K[X] \) such that \( g \mid f \) and \( f \mid g^m \) for some \( m \in \mathbb{N} \). (If \( K \) is perfect, this holds by Lemma 2 if \( f \) splits into linear factors, see Remark 3.) Since \( g, D(g) \) are coprime, there exist \( \tilde{g}, \tilde{g}' \in K[X] \) such that \( 1 = \tilde{g}D(g) + \tilde{g}'g \); note that \( \tilde{g} \neq 0 \). We define a sequence of matrices \((A_k)_{k \geq 0}\) in \( M_n(K) \) by

\[
A_0 := A \quad \text{and} \quad A_{k+1} := A_k - g(A_k) \cdot \tilde{g}(A_k) \quad \text{for} \quad k \geq 0.
\]

**Theorem 7** (Cf. Couty et al. [1, Théorème 1]). Let \( k_0 \geq 0 \) be such that \( 2^{k_0} \geq m \). Then \( g(A_{k_0}) = 0 \), \( A_{k_0} \) is semisimple and \( A - A_{k_0} \) is nilpotent; furthermore, \( A_{k_0} \in \mathfrak{A} \). Thus, \( A = D + N \) with \( D := A_{k_0} \) and \( N := A - A_{k_0} \) is the Jordan–Chevalley decomposition of \( A \).

Proof. We somewhat re-organise the proof in [1] and check that everything still works in our slightly different setting. First note that, by a simple induction on \( k \), we have \( A_k \in \mathfrak{A} \) for all \( k \geq 0 \). The crucial step is to show:

\[(*) \quad A - A_k \in g(A) \cdot \mathfrak{A} \quad \text{and} \quad g(A_k) \in g(A)^{2^k} \cdot \mathfrak{A} \quad \text{for all} \quad k \geq 0.\]

Use induction on \( k \). For \( k = 0 \) we have \( A = A_0 \) and \((*)\) is clear. Now let \( k \geq 0 \). We write \( A_{k+1} = A_k + B \) where \( B := -g(A_k) \cdot \tilde{g}(A_k) \in g(A_k) \cdot \mathfrak{A} \). By induction, \( g(A_k) \in g(A)^{2^k} \cdot \mathfrak{A} \subseteq g(A) \cdot \mathfrak{A} \) and so \( B \in g(A) \cdot \mathfrak{A} \). Also by induction, \( A - A_k \in g(A) \cdot \mathfrak{A} \) and, hence, \( A - A_{k+1} = (A - A_k) - B \in g(A) \cdot \mathfrak{A} \). Now consider \( g(A_k) \). By the second formula in Lemma 5, we have

\[g(A_{k+1}) - g(A_k) - D(g(A_k)) \cdot B \in B^2 \cdot \mathfrak{A}.
\]

Using now the identity \( 1 = \tilde{g}D(g) + \tilde{g}'g \), we obtain

\[-D(g(A_k)) \cdot B = D(g(A_k)) \cdot \tilde{g}(A_k) \cdot g(A_k)
\]

\[= (1 - \tilde{g}'g)(A_k) \cdot g(A_k) = g(A_k) - \tilde{g}'(A_k) \cdot g(A_k)^2.
\]

This yields \( g(A_{k+1}) - \tilde{g}'(A_k) \cdot g(A_k)^2 \in B^2 \cdot \mathfrak{A} \). Since \( g(A_k) \in g(A)^{2^k} \cdot \mathfrak{A} \) and \( B \in g(A_k) \cdot \mathfrak{A} \), we deduce that \( g(A_{k+1}) \in g(A)^{2^{k+1}} \cdot \mathfrak{A} \), as required.

Having established \((*)\), we can now conclude as follows. Since \( 2^{k_0} \geq m \), we have \( f \mid g^m \mid g^{2^{k_0}} \) and so \( g^m(A) = g^{2^{k_0}}(A) = f(A) = 0 \). Hence, \((*)\) implies that \( g(A_{k_0}) = 0 \) and \( (A - A_{k_0})^m = 0 \). Now it is known that we can always find a field \( L \supseteq K \) such that \( g \in K[X] \) splits into linear factors over \( L \). Since \( g \) is square-free, that is, \( \gcd(g, D(g)) = 1 \), it follows that \( g \) does not have any repeated roots in \( L \). (If \( \lambda \in L \) was such a repeated root, then \( g = (X - \lambda)^2h \) for some \( h \in K[X] \) and so \( D(g) = 2(X - \lambda)h + (X - \lambda)^2D(h) \) would also be divisible by \( X - \lambda \), contradiction.) Hence, \( A \) will be diagonalisable over \( L \). (This is a standard result in Linear Algebra and can be established directly, without knowledge of the Jordan normal form; see, for example, [2, Theorem 7.16].) Finally, we have already noted that \( A_{k_0} \in \mathfrak{A} \).

Uniqueness also follows by a standard argument. Suppose that we also have \( A = D' + N' \) where \( D' \) is semisimple, \( N' \) is nilpotent and \( D' \cdot N' = N' \cdot D' \). Then consider the identity \( D - D' = N' - N \). Since \( D', N' \) commute with each other, they commute with \( A \). Since \( A_{k_0} \in \mathfrak{A} \), we also have \( D, N \in \mathfrak{A} \).
Hence, $D, D'$ commute with each other, and $N, N'$ commute with each other. But then $N' - N$ will also be nilpotent and so $(D - D')^d = 0$ for some $d \in \mathbb{N}$. Since $D, D'$ are diagonalisable (over some $L \supseteq K$) and commute, they can in fact be simultaneously diagonalised (see, for example, Exercise 39 in [2, §5.4]). Hence $D - D'$ can also be diagonalised. But then $(D - D')^d = 0$ implies $D - D' = 0$; so $D = D'$ and also $N = N'$. □

Example 8. The GAP package mentioned in the Introduction now also provides the function \texttt{JordanChevalleyDecMat}, which takes as input a matrix $A \in M_n(K)$ and a non-constant polynomial $f \in K[X]$ such that $f(A) = 0$; the output are $D, N$ as above.

For the example matrix $U \in M_{15}(\mathbb{C})$ in [1, §3] (for which the eigenvalues can only be computed approximately), we obtain $D, N$ in a few milli-seconds.

Now let $K = \mathbb{F}_2$ (the field with 2 elements) and $A = M_2 \in M_{4370}(\mathbb{F}_2)$ be the (challenging) test matrix already considered at the end of [3]. The GAP function \texttt{MinimalPolynomial} produces the minimal polynomial $\mu_A \in \mathbb{F}_2[X]$ in a few seconds; we have $\deg(\mu_A) = 2097$. The square-free part $g \in \mathbb{F}_2[X]$ of $\mu_A$ has degree 2087. The Euclidean Algorithm quickly yields $\tilde{g}, \tilde{g}' \in \mathbb{F}_2[X]$ such that $1 = \tilde{g}D(g) + \tilde{g}'g$; here, $\deg(\tilde{g}) = 2085$ and $\deg(\tilde{g}') = 2084$. The function \texttt{JordanChevalleyDecMat} then takes about 5 minutes to compute $D, N$; the matrix $N$ has minimal polynomial $X^5$.

Two further remarks:

1) One has to expect that the computation for $A = M_2$ as above takes a considerable time, because it requires the evaluation of $g(A_k)$ and $\tilde{g}(A_k)$ for $k = 1, 2, 3$, which involves thousands of matrix multiplications in $M_{4370}(\mathbb{F}_2)$. In order to perform this in 5 minutes, we actually first compute the Frobenius normal form of $A$ and then apply the function to each diagonal block in that normal form; these diagonal blocks are quite sparse and, hence, computations will be much faster. We provide a function \texttt{JordanChevalleyDecMatF} which does this automatically. (Applying \texttt{JordanChevalleyDecMat} directly to $A$ takes around 45 minutes.)

2) We do obtain the Jordan–Chevalley decomposition in this example, but it is practically impossible to compute the Jordan normal form. This is because the square-free polynomial $g \in \mathbb{F}_2[X]$ (which is the minimal polynomial of the semisimple part $D$ of $A$) has irreducible factors of degree 1, 2, 4, 6, 88, 197, 854, 934. So, although exact computations are possible in finite fields, one would need to go to an enormous field $L \supseteq \mathbb{F}_2$ in order to write down the eigenvalues of $A$.

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