The almost Schur Lemma in quaternionic contact geometry

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Abstract
We establish quaternionic contact (qc) versions of the so called Almost Schur Lemma, which give estimations of the qc scalar curvature on a compact qc manifold to be a constant in terms of the norm of the [−1]-component and the norm of the trace-free part of the [3]-component of the horizontal qc Ricci tensor and the torsion endomorphism, under certain positivity conditions.

Keywords Quaternionic contact structure · QC Lichnerowicz condition · QC Cordes estimate · P-function

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1 Introduction

For a Riemannian manifold \((M^n, g)\) of dimension \(n \geq 3\) the famous Schur Lemma states that if \((M^n, g)\) is Einstein, then it has constant scalar curvature, \(S = \text{Const}\). The metric \(g\) is said to be Einstein, if the Ricci tensor is proportional to the metric, \(\text{Ric} = \frac{S}{n} g\). A generalization of the Schur Lemma is a recent result of De Lellis and Topping [9] that states as follows.

**Theorem 1.1** [9, Almost Schur Lemma] Let \((M^n, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\) with non–negative Ricci tensor, \(\text{Ric} \geq 0\). Then the following inequality holds

\[
\int_M (S - \bar{S})^2 \ vol_g \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left| \text{Ric} - \frac{S}{n} g \right|^2_g \ vol_g, \tag{1.1}
\]

where \(\bar{S}\) means the average value of the scalar curvature \(S\) of \(g\).

The equality holds if and only if the manifold is Einstein.

It is also shown in [9] that the positivity condition \(\text{Ric} \geq 0\) assumed on the Ricci tensor is essential and cannot be dropped.

Under the condition that the Ricci tensor is positive, (1.1) has been established earlier by Andrews [7, Theorem B.18 and Corollary B.20].

In the CR case there are known two positivity conditions written in terms of the Webster Ricci curvature and the pseudohermitian torsion; one is used for obtaining a lower bound of the first eigenvalue of the sub-Laplacian (see e.g. [12]), while the other one appears in the CR Cordes type estimate (see [5]). A CR version of the almost Schur Lemma was first established in [6] under the Greenleaf’s positivity condition. Recently, a CR version of the almost Schur Lemma was presented in [16] under the CR Cordes positivity condition which gives better estimate in the torsion-free (Sasakian) case.

The aim of this note is to present a quaternionic contact (qc) version of the Almost Schur Lemma. Quaternionic contact geometry is an example of sub–Riemannian geometry and any quaternionic contact manifold admits a canonical connection \(\nabla\), namely the Biquard connection, whose role in qc geometry is similar to that of Levi–Civita connection in Riemannian geometry and Tanaka–Webster connection in CR geometry. We present in terms of the Biquard connection two versions of quaternionic contact almost Schur Lemma depending on the positivity assumption, see Theorems 3.1 and 3.3 below.

**Convention 1.2** a) We shall use \(X, Y, Z, U\) to denote horizontal vector fields, i.e. \(X, Y, Z, U \in H\).

b) \(\{e_1, \ldots, e_{4n}\}\) denotes a local orthonormal basis of the horizontal space \(H\).

c) The triple \((i, j, k)\) denotes any cyclic permutation of \(1, 2, 3\).

d) \(s\) will be any number from the set \(\{1, 2, 3\}\), \(s \in \{1, 2, 3\}\).

2 Quaternionic contact manifolds

It is well known that the sphere at infinity of a non-compact symmetric space of rank one carries a natural Carnot-Carathéodory structure, see [22, 23]. A quaternionic contact (qc) structure [3], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space.

Quaternionic contact manifolds were introduced in [3]. We also refer to [14] and [20] for further results and background.
2.1 Quaternionic contact structures and the Biquard connection

A quaternionic contact (qc) manifold \((M, g, \mathbb{Q})\) is a \(4n+3\)-dimensional manifold \(M\) with a codimension three distribution \(H\) equipped with an \(Sp(n)Sp(1)\)-structure. Explicitly, \(H\) is the kernel of a local 1-form \(\eta = (\eta_1, \eta_2, \eta_3)\) with values in \(\mathbb{R}^3\) together with a compatible Riemannian metric \(g\) and a rank-three bundle \(\mathbb{Q}\) consisting of endomorphisms of \(H\), locally generated by three almost complex structures \(I_1, I_2, I_3\) on \(H\), satisfying the identities of the imaginary unit quaternions. Thus, we have \(I_1 I_2 = -I_2 I_1 = I_3, \quad I_1 I_2 I_3 = -i d_{|H}\), which are hermitian compatible with the metric \(g(I_3, \cdot, \cdot) = g(\cdot, \cdot)\), and the following compatibility conditions hold \(2g(I_3 X, Y) = d\eta_3(X, Y)\).

On a qc manifold of dimension \((4n+3) > 7\) with a fixed metric \(g\) on \(H\) there exists a canonical connection defined in [3]. Biquard also showed that there is a unique connection \(\nabla\) with torsion \(T\) and a unique supplementary subspace \(V\) to \(H\) in \(TM\), such that:

(i) \(\nabla\) preserves the splitting \(H \oplus V\) and the \(Sp(n)Sp(1)\)-structure on \(H\), i.e., \(\nabla g = 0\).

(ii) for \(\xi \in V\), the endomorphism \(T(\xi, .)|_H\) of \(H\) lies in \((sp(n) \oplus sp(1)) \perp \subset gl(4n)\).

(iii) the connection on \(V\) is induced by the natural identification \(\varphi\) of \(V\) with \(\mathbb{Q}\), \(\nabla\varphi = 0\).

If the dimension of \(M\) is at least eleven [3] also described the supplementary vertical distribution \(V\), which is (locally) generated by the so called Reeb vector fields \(\{\xi_1, \xi_2, \xi_3\}\) determined by

\[
\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \, d \eta_s)|_H = 0, \quad (\xi_s \, d \eta_k)|_H = -(\xi_k \, d \eta_s)|_H, \quad (2.1)
\]

where \(\_\) denotes the interior multiplication.

If the dimension of \(M\) is seven Duchemin shows in [10] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1). This implies the existence of the connection with properties (i), (ii) and (iii) above.

The fundamental 2-forms \(\omega_s\) of the quaternionic structure are defined by

\[
2\omega_s|_H = d\eta_s|_H, \quad \xi \, \omega_s = 0, \quad \xi \in V.
\]

The torsion restricted to \(H\) has the form \(T(X, Y) = -[X, Y]|_V = 2\sum_{s=1}^{3} \omega_s(X, Y)\xi_s\).

2.2 Invariant decompositions

Any endomorphism \(\Psi\) of \(H\) can be decomposed with respect to the quaternionic contact structure \((\mathbb{Q}, g)\) uniquely into four \(Sp(n)\)-invariant parts \(\Psi = \Psi^{+++} + \Psi^{++-} + \Psi^{-++} + \Psi^{-+-}\), where \(\Psi^{+++}\) commutes with all three \(I_i\), \(\Psi^{++-}\) commutes with \(I_1\) and anti-commutes with the others two, etc. The two \(Sp(n)Sp(1)\)-invariant components are given by

\[
\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{+-+} + \Psi^{-+} + \Psi^{-+-}
\]

with the following characterizing equations

\[
\Psi = \Psi_{[3]} \iff 3\Psi + I_1 \Psi I_1 + I_2 \Psi I_2 + I_3 \Psi I_3 = 0,
\]

\[
\Psi = \Psi_{[-1]} \iff -I_1 \Psi I_1 - I_2 \Psi I_2 - I_3 \Psi I_3 = 0.
\]

These are the projections on the eigenspaces of the Casimir operator

\[
\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3,
\]
corresponding, respectively, to the eigenvalues 3 and −1, see [4]. Note here that each of the three 2-forms \( \omega_s \) belongs to the \([-1]\)-component, \( \omega_s = \omega_{s[-1]} \), and constitute a basis of the Lie algebra \( sp(1) \).

If \( n = 1 \) then the space of symmetric endomorphisms commuting with all \( I_\xi \) is 1-dimensional, i.e., the \([3]\)-component of any symmetric endomorphism \( \Psi \) on \( H \) is proportional to the identity, \( \Psi_{[3]} = -\frac{2}{3} \text{Id}_H \).

### 2.3 The torsion tensor

The torsion endomorphism \( T_\xi = T(\xi, \cdot) : H \to H \), \( \xi \in V \) will be decomposed into its symmetric part \( T_\xi^0 \) and skew-symmetric part \( b_\xi, T_\xi = T_\xi^0 + b_\xi \). Biquard showed in [3] that the torsion \( T_\xi^0 \) is completely trace-free, \( tr T_\xi^0 = tr (T_\xi \circ I_s) = 0 \), its symmetric part has the properties \( T_\xi^0 I_i = -I_i T_\xi^0 \), \( I_2(T_\xi^0)^{++--} = I_1(T_\xi^0)^{---+} = I_3(T_\xi^0)^{+++--} = I_2(T_\xi^0)^{----++} \). The skew-symmetric part can be represented as \( b_\xi = I_i U \), where \( U \) is a traceless symmetric (1,1)-tensor on \( H \) which commutes with \( I_1, I_2, I_3 \). Therefore we have \( T_\xi^0 = T_\xi^0 + I_i U \). When \( n = 1 \) the tensor \( U \) vanishes identically, \( U = 0 \), and the torsion is a symmetric tensor, \( T_\xi = T_\xi^0 \).

The two \( Sp(n)Sp(1) \)-invariant trace-free symmetric 2-tensors on \( H \)

\[
T^0(X, Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3) X, Y) \quad \text{and} \quad U(X, Y) = g(U X, Y)
\]

were introduced in [14] and enjoy the properties

\[
T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0,
U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y).
\]

From [20, Proposition 2.3] we have

\[
4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y), \tag{2.2}
\]

where, as usually, \( T^0(\xi_s, I_s X, Y) := g(T^0(\xi_s, I_s X), Y) \). Hence, taking into account (2.2) it follows

\[
T(\xi_s, I_s X, Y) = T^0(\xi_s, I_s X, Y) + g(I_s U I_s X, Y)
\]

\[
= \frac{1}{4} \left[ T^0(X, Y) - T^0(I_s X, I_s Y) \right] - U(X, Y).
\]

### 2.4 Torsion and curvature

Let \( R = [\nabla, \nabla] - \nabla_{[,]} \) be the curvature tensor of \( \nabla \) and the dimension is \( 4n + 3 \). We denote the curvature tensor of type \((0,4)\) and the torsion tensor of type \((0,3)\) by the same letter, \( R(A, B, C, D) := g(R(A, B)C, D), \quad T(A, B, C) := g(T(A, B), C), A, B, C, D \in \Gamma(TM) \). The Ricci tensor, the normalized scalar curvature and the Ricci 2-forms of the Biquard connection, called \( qc\)-Ricci tensor \( Ric \), normalized \( qc\)-scalar curvature \( S \) and \( qc\)-Ricci forms \( \rho_s \), respectively, are defined by

\[
Ric(A, B) = R(e_a, A, B, e_a), \quad S = \frac{1}{8n(n + 2)} Ric(e_a, e_a),
\]

\[
\rho_s(A, B) = \frac{1}{4n} R(A, B, e_a, I_s e_a).
\]
Definition 2.1 A qc structure is said to be qc-Einstein if the horizontal qc-Ricci tensor is a scalar multiple of the metric,
\[ \text{Ric}(X, Y) = 2(n + 2) Sg(X, Y). \]

The horizontal qc-Ricci tensor and the horizontal qc-Ricci 2-forms can be expressed in terms of the torsion of the Biquard connection [14] (also [20]). We collect the necessary facts from [14, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modification presented in [20].

Theorem 2.2 [14] On a \((4n + 3)\)-dimensional qc manifold \((M, \eta, Q)\) with a normalized qc scalar curvature \(S\) we have the following relations
\[
\text{Ric}(X, Y) = (2n + 2) T^0(X, Y) + (4n + 10) U(X, Y) + 2(n + 2) Sg(X, Y),
\]
\[
\rho_s(X, Y) = -\frac{1}{2} \left[ T^0(X, Y) + T^0(I_sX, I_sY) \right] - 2U(X, Y) - Sg(X, Y),
\]
\[
T(\xi_i, \xi_j) = -S\xi_k - [\xi_i, \xi_j]|_H, \quad S = -g(T(\xi_1, \xi_2), \xi_3). \tag{2.3}
\]

For \(n = 1\) the above formulas hold with \(U = 0\).

The qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case \(S\) is constant and the vertical distribution is integrable (see [15] for \(n = 1\)).

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc-scalar curvature is a positive constant [14].

The tensor \(T^0\) determines the traceless \([-1]\)-component of the horizontal qc-Ricci tensor, while the tensor \(U\) determines the traceless part of the \([3]\)-component of the horizontal qc-Ricci tensor [21] (see also [14, 20]):
\[
T^0 = \frac{1}{2n + 2} \text{Ric}_{[-1]}, \quad U = \frac{1}{4n + 10} \text{Ric}_{[3][0]}. \tag{2.4}
\]

A weaker condition than the qc-Einstein one is contained in the following

Definition 2.3 [14, Definition 6.1] Let \((M, g, Q)\) be a quaternionic contact manifold. We call \(M\) qc-pseudo-Einstein if the trace-free part of the \([3]\)-component of the qc-Ricci tensor vanishes, \(U = 0\). In dimension seven any qc manifold is qc-pseudo-Einstein.

We also give the following

Definition 2.4 Let \((M, g, Q)\) be a quaternionic contact manifold. We call \(M\) qc-nearly-Einstein if the \([-1]\)-component of the qc-Ricci tensor vanishes, \(T^0 = 0\).

We note that qc-nearly-Einstein manifolds are characterized by the condition that the almost contact structure on the corresponding twistor space is normal, see [8]. Examples of qc-nearly-Einstein manifolds are provided by sub-Riemannian manifolds with transverse symmetry of qc type as well as by Riemannian foliations with totally geodesic fibres of qc type (see [1, 2] and references therein).

We use the contracted second Bianchi identity, established in [14], in the form presented in [20, 21]:
\[
(n - 1)(\nabla_{e_a} T^0)(e_a, X) + 2(n + 2)(\nabla_{e_a} U)(e_a, X) - (n - 1)(2n + 1)dS(X) = 0. \tag{2.5}
\]

Clearly, the contracted second Bianchi identity (2.5) shows that for \(n > 1\) any qc-Einstein space has constant qc scalar curvature but for qc-pseudo-Einstein and qc-nearly-Einstein spaces the qc scalar curvature could not be a constant.
2.5 The Ricci identities

We use repeatedly the Ricci identities of order two and three, see also [20]. Let \( f \) be a smooth function on the qc manifold \( M \) with horizontal gradient \( \nabla f \) defined by \( g(\nabla f, X) = df(X) \). The sub-Laplacian of \( f \) is, by definition, \( \Delta f = -\sum_{a=1}^{4n} \nabla^2 f(e_a, e_a) \). We have the following Ricci identities (see e.g. [14, 21])

\[
\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2 \sum_{s=1}^{3} \omega_s(X, Y) df(\xi_s),
\]

\[
\nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) = T(\xi_s, X, \nabla f),
\]

\[
\nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) = -R(X, Y, Z, \nabla f) - 2 \sum_{s=1}^{3} \omega_s(X, Y) \nabla^2 f(\xi_s, Z).
\]

(2.6)

In view of (2.6) we have the decompositions

\[
(\nabla^2 f)_{[3][0]}(X, Y) = (\nabla^2 f)_{[3]}(X, Y) + \frac{1}{4n} \Delta f g(X, Y),
\]

\[
|\!(\nabla^2 f)_{[3][0]}\!)|^2 = |\!(\nabla^2 f)_{[3]}\!)|^2 - \frac{1}{4n} |\!(\Delta f)\!)|^2,
\]

\[
(\nabla^2 f)_{[-1]}(X, Y) = (\nabla^2 f)_{[-1][sym]}(X, Y) + (\nabla^2 f)_{[-1][as]}(X, Y)
\]

\[
= (\nabla^2 f)_{[-1][sym]}(X, Y) - \sum_{s=1}^{3} \omega_s(X, Y) df(\xi_s),
\]

\[
|\!(\nabla^2 f)_{[-1]}\!)|^2 = |\!(\nabla^2 f)_{[-1][sym]}\!)|^2 + 4n \sum_{s=1}^{3} df(\xi_s)^2.
\]

(2.7)

We also need the qc-Bochner formula [17, (4.1)]

\[
-\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + 2(n + 2) S|\nabla f|^2 + 2(n + 2) T^0(\nabla f, \nabla f)
\]

\[
+ 2(n + 2) U(\nabla f, \nabla f) + 4 \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f).
\]

(2.8)

Let \((M, g, \mathbb{Q})\) be a qc manifold of dimension \(4n + 3 \geq 7\). For a fixed local 1-form \( \eta \) and a fixed \( s \in \{1, 2, 3\} \) the form \( Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_{2n}^2 \) is a locally defined volume form. Note that \( Vol_\eta \) is independent of \( s \) as well as the local one forms \( \eta_1, \eta_2, \eta_3 \). Hence, it is a globally defined volume form. The (horizontal) divergence of a horizontal vector field/one-form \( \sigma \in \Lambda^1(H) \), defined by \( \nabla^* \sigma = -tr|_H \nabla \sigma = -\nabla \sigma(e_a, e_a) \), supplies the integration by parts formula for compact \( M \) [14], see also [24],

\[
\int_M (\nabla^* \sigma) \ Vol_\eta = 0.
\]

2.6 The \( P \)-form

We recall the definition of the \( P \)-form from [18]. Let \((M, g, \mathbb{Q})\) be a compact quaternionic contact manifold of dimension \( 4n + 3 \).
For a smooth function $f$ on $M$ the $P-$form $P_f$ on $M$ is defined by [18]

$$P_f(X) = \nabla^3 f(X, e_a, e_a) + \sum_{t=1}^{3} \nabla^{3} f(I_t X, e_a, I_t e_a) - 4n Sdf(X)$$

$$+ 4n T^0(X, \nabla f) - \frac{8n(n - 2)}{n - 1} U(X, \nabla f), \quad \text{if } n > 1,$$

$$P_f(X) = \nabla^3 f(X, e_a, e_a) + \sum_{t=1}^{3} \nabla^{3} f(I_t X, e_a, I_t e_a) - 4Sdf(X) + 4T^0(X, \nabla f), \quad \text{if } n = 1.$$ 

The $P-$function of $f$ is defined by $P_f(\nabla f)$. We say that the $P-$function of $f$ is non-negative if its integral exists and is non-positive:

$$- \int_M P_f(\nabla f) \text{Vol}_\eta \geq 0.$$ 

If the upper inequality holds for any smooth function of compact support, we say that the $P-$function is non-negative. It turns out that the $P-$function is non-negative on any compact qc manifold of dimension at least eleven [18]. Indeed, [18, Theorem 3.3] asserts that on a compact qc manifold of dimension bigger than seven, the next formula holds

$$\int_M P_f(\nabla f) \text{Vol}_\eta = - \frac{4n}{n - 1} \int_M |(\nabla^2 f)_{[3][0]}|^2 \text{Vol}_\eta. \quad (2.9)$$

We recall also the following integral identities,

$$\int_M \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta = \int_M \left[ \frac{3}{4n} |(\nabla^2 f)_{[3]}|^2 - \frac{1}{4n} |(\nabla^2 f)_{[-1]}|^2ight.$$

$$- \frac{n + 2}{2n} T^0(\nabla f, \nabla f) - \frac{3}{2} S|\nabla f|^2 \text{Vol}_\eta,$$

$$\left. - \frac{n - 4}{n - 1} U(\nabla f, \nabla f) \right] \text{Vol}_\eta, \quad (2.10)$$

proved in [17, Lemma 3.3], and

$$\int_M |(\nabla^2 f)_{[-1][0]}|^2 \text{Vol}_\eta = 4n \int_M \sum_{s=1}^{3} (df(\xi_s))^2 \text{Vol}_\eta = \int_M \left[ \frac{1}{4n} P_f(\nabla f) + \frac{1}{4n} \Delta f)^2ight.$$

$$+ S|\nabla f|^2 - T^0(\nabla f, \nabla f) + \frac{2n - 4}{n - 1} U(\nabla f, \nabla f) \right] \text{Vol}_\eta, \quad (2.11)$$

established in [19, formula (4.11)].

**Proposition 2.5** For any smooth function $f$ on a compact qc manifold $(M, g, \mathcal{Q})$ of dimension $4n+3$ bigger than seven we have
where

\[ Lic(\nabla f, \nabla f) = S|\nabla f|^2 + \frac{2n+3}{2n+1} T^0(\nabla f, \nabla f) + \frac{2(n+2)(2n-1)}{(2n+1)(n-1)} U(\nabla f, \nabla f), \] (2.13)

and also

\[ 0 = \int_M \left[ \frac{4n+3}{4n} (\Delta f)^2 - |(\nabla^2 f)_{[-1]}|^2 - \frac{n+3}{n-1} |(\nabla^2 f)_{[3][0]}|^2 - 2n Cor(\nabla f, \nabla f) \right] Vol_\eta, \] (2.14)

where

\[ Cor(\nabla f, \nabla f) = S|\nabla f|^2 + \frac{n+2}{n} T^0(\nabla f, \nabla f) + \frac{2(n+1)}{n-1} U(\nabla f, \nabla f). \] (2.15)

**Proof** Integrating the qc Bochner formula (2.8) over the compact \( M \) and applying (2.3) and (2.10), we get

\[
0 = \int_M \left[ (\nabla^2 f)^2 - (\Delta f)^2 + Ric(\nabla f, \nabla f) + 2T^0(\nabla f, \nabla f) + 6U(\nabla f, \nabla f) \right] Vol_\eta
\]

\[
= \int_M \left[ |\nabla^2 f|^2 - (\Delta f)^2 + Ric(\nabla f, \nabla f) + 2T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f) \right] Vol_\eta
\]

\[
+ \int_M \left[ \frac{3}{n} |(\nabla^2 f)_{[3][0]}|^2 - \frac{1}{n} |(\nabla^2 f)_{[-1]}|^2 - \frac{2(n+2)}{n} T^0(\nabla f, \nabla f) - 6S|\nabla f|^2 \right] Vol_\eta
\]

\[
= \int_M \left[ |(\nabla^2 f)_{[3][0]}|^2 + \frac{1}{4n^2} (\Delta f)^2 + |(\nabla^2 f)_{[-1]}|^2 - (\Delta f)^2 + \frac{3}{4n^2} |(\nabla^2 f)_{[3][0]}|^2 
\]

\[
+ \frac{3}{4n^2} (\Delta f)^2 - \frac{1}{n} |(\nabla^2 f)_{[-1]}|^2 \right] Vol_\eta
\]

\[
+ \int_M \left[ Ric(\nabla f, \nabla f) - \frac{4}{n} T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f) - 6S|\nabla f|^2 \right] Vol_\eta
\]

\[
= \int_M \left[ \frac{n+3}{n} |(\nabla^2 f)_{[3][0]}|^2 - \frac{(n-1)(4n+3)}{4n^2} (\Delta f)^2 + \frac{n-1}{n} |(\nabla^2 f)_{[-1]}|^2 
\]

\[
+ 2(n-1) Cor(\nabla f, \nabla f) \right] Vol_\eta,
\]

which proves (2.14).

The substitution of the fourth equality of (2.7) and (2.11) into (2.14) leads to
\[
0 = \int_M \left[ \frac{4n+3}{4n} (\Delta f)^2 - \frac{n+3}{n-1} (\nabla^2 f)_{3\parallel [0]}^2 - 2n \text{Cor}(\nabla f, \nabla f) \right] \text{Vol}_\eta \\
= \int_M \left[ \frac{4n+3}{4n} (\Delta f)^2 - \frac{n+3}{n-1} (\nabla^2 f)_{3\parallel [0]}^2 - 2n \text{Cor}(\nabla f, \nabla f) \right] \\
- \left[ S|\nabla f|^2 - T^0(\nabla f, \nabla f) + \frac{2n-4}{n-1} U(\nabla f, \nabla f) \right] \\
- 2n \left[ \frac{n+2}{n} T^0(\nabla f, \nabla f) + 2(n+1) \left( \frac{n}{n-1} U(\nabla f, \nabla f) + S|\nabla f|^2 \right) \right] \text{Vol}_\eta, \\
\]
which proves (2.12) in view of (2.9).

\[\square\]

**Remark 2.6** Note that the expression in the formula (2.13) appears in the qc Lichnerowicz-type positivity condition used to find a lower bound of the first eigenvalue of the sub-Laplacian (see [17, formula (1.1)] and [18, formula (1.2)]). On the other hand, the expression in (2.15) appears in the qc Cordes-type a priori inequality between the (horizontal) Hessian and the sub-Laplacian of a function, derived in [17, formula (1.2)].

Indeed, in view of (2.7), the equality (2.14) can be written in the form
\[
\frac{n+1}{n} \int_M (\Delta f)^2 \text{Vol}_\eta = \int_M |\nabla f|^2 \text{Vol}_\eta \\
+ \int_M \left[ \frac{4}{n-1} (\nabla^2 f)_{3\parallel [0]}^2 + 2n \text{Cor}(\nabla f, \nabla f) \right] \text{Vol}_\eta. \\
\]

In this way we recovered the next result from [17], supplying more information in the equality case:

**Theorem 2.7** [17] On a compact qc manifold of dimension bigger than seven one has the inequality
\[
\int_M (\Delta f)^2 \text{Vol}_\eta \geq \frac{n}{n+1} \int_M |\nabla f|^2 \text{Vol}_\eta + \frac{2n^2}{n+1} \int_M \text{Cor}(\nabla f, \nabla f) \text{Vol}_\eta. \tag{2.16}
\]
The equality in (2.16) can be achieved only for functions with vanishing trace-free part of the [3]-component of the horizontal Hessian.

### 3 The inequalities, main results

Denote by \( \bar{S} \) the average value of the scalar curvature,
\[
\bar{S} = \int_M S \text{Vol}_\eta.
\]

For a smooth function \( f \) the sub-Laplacian \( \triangle f \) is a subelliptic operator. According to [11] and [13], let \( \phi \) be the unique solution of the following PDE:
\[
\Delta \phi = S - \bar{S}, \quad \int_M \phi \text{Vol}_\eta = 0. \tag{3.1}
\]

We have

**Theorem 3.1** Let \((M, g, \mathbb{Q})\) be a compact qc manifold of dimension \((4n + 3), \quad n > 1.\)
(a) Suppose the next positivity condition is satisfied

\[ \text{Lic}(X, X) = Sg(X, X) + \frac{2n + 3}{2n + 1} T^0(X, X) + \frac{2(n + 2)(2n - 1)}{(n - 1)(2n + 1)} U(X, X) \geq 0, \quad X \in H. \]  

(3.2)

Then we have

\[ \int_M (S - \bar{S})^2 Vol_\eta \leq \frac{n + 2}{2n(n - 1)(2n + 5)^2(2n + 1)} \int_M |\text{Ric}[3][0]|^2 Vol_\eta \]

\[ \quad - \frac{2}{2n + 1} \int_M T^0(e_a, e_b)(\nabla^2 \varphi)_{[-1][\text{sym}]}(e_a, e_b) Vol_\eta \]

\[ \quad = \frac{2(n + 2)}{n(n - 1)(2n + 1)} \int_M |U|^2 Vol_\eta \]

(3.3)

If the equality in (3.3) holds then

\[ \int_M (S - \bar{S})^2 Vol_\eta = \frac{n + 2}{2n(n - 1)(2n + 5)^2(2n + 1)} \int_M |\text{Ric}[3][0]|^2 Vol_\eta \]

\[ \quad = \frac{2(n + 2)}{n(n - 1)(2n + 1)} \int_M |U|^2 Vol_\eta \]

(3.4)

and the qc conformal structure \( \bar{\eta} = \frac{\eta}{2} e^{-2\varphi} \) will be qc-pseudo-Einstein.

(b) Suppose the next positivity condition is satisfied

\[ \text{Cor}(X, X) = Sg(X, X) + \frac{n + 2}{n} T^0(X, X) + \frac{2(n + 1)}{n - 1} U(X, X) \geq 0, \quad X \in H. \]  

(3.5)

Then we have

\[ \int_M (S - \bar{S})^2 Vol_\eta \leq \frac{(n + 2)^2(4n + 3)}{4n(n - 1)(n + 3)(2n + 5)^2(2n + 1)^2} \int_M |\text{Ric}[3][0]|^2 Vol_\eta \]

\[ \quad - \frac{2}{2n + 1} \int_M T^0(e_a, e_b)(\nabla^2 \varphi)_{[-1][\text{sym}]}(e_a, e_b) Vol_\eta \]

\[ \quad = \frac{(n + 2)^2(4n + 3)}{n(n - 1)(n + 3)(2n + 1)^2} \int_M |U|^2 Vol_\eta \]

(3.6)

If the equality in (3.6) holds then

\[ \int_M (S - \bar{S})^2 Vol_\eta = \frac{(n + 2)^2(4n + 3)}{4n(n - 1)(n + 3)(2n + 5)^2(2n + 1)^2} \int_M |\text{Ric}[3][0]|^2 Vol_\eta \]

\[ \quad = \frac{(n + 2)^2(4n + 3)}{n(n - 1)(n + 3)(2n + 1)^2} \int_M |U|^2 Vol_\eta \]

(3.7)

and the qc conformal structure \( \bar{\eta} = \frac{n(n + 3)(2n + 1)}{(n + 2)(4n + 3)} e^{-2\varphi} \) will be qc-pseudo-Einstein.
Proof The proof follows the approach of [9]. We have
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta = \int_M (S - \bar{S}) \Delta \varphi \text{Vol}_\eta = \int_M dS(\nabla \varphi) \text{Vol}_\eta, \tag{3.8}
\]
where we used (3.1) and an integration by parts to achieve the last equality.

The equalities (2.12), (3.1), and the positivity condition (3.2) help us to obtain from (3.12) the next inequality
\[
\int_M (\nabla e_a R_{\text{ic}[3][0]})(e_a, \nabla \varphi) \text{Vol}_\eta \geq \frac{1}{2(n + 1)} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta + \frac{2n + 4}{(n - 1)(2n + 1)} \int_M (\nabla e_a U)(e_a, X).
\]
\[
\tag{3.9}
\]
Using (3.9), we obtain from (3.8) and an integration by parts that
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta = \int_M dS(\nabla \varphi) \text{Vol}_\eta
\]
\[
= \frac{n + 2}{(n - 1)(2n + 5)(2n + 1)} \int_M (\nabla e_a R_{\text{ic}[3][0]})(e_a, \nabla \varphi) \text{Vol}_\eta
\]
\[
+ \frac{1}{2n + 1} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta
\]
\[
= -\frac{n + 2}{(n - 1)(2n + 5)(2n + 1)} \int_M (R_{\text{ic}[3][0]})(e_a, e_b)(\nabla^2 \varphi)_{[3][0]}
\]
\[
\times (e_a, e_b) \text{Vol}_\eta + \frac{1}{2n + 1} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta.
\]
\[
\tag{3.10}
\]
Applying the Young’s inequality
\[
2pq \leq \lambda p^2 + \lambda^{-1} q^2,
\]
for a constant \(\lambda > 0\), we get from (3.10) that
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{n + 2}{2(n - 1)(2n + 5)(2n + 1)} \int_M \left[\lambda |Ric_{[3][0]}|^2 + \lambda^{-1} |(\nabla^2 \varphi)_{[3][0]}|^2\right] \text{Vol}_\eta
\]
\[
+ \frac{1}{2n + 1} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta.
\]
\[
\tag{3.12}
\]
First we prove a). The equalities (2.12), (3.1) and the positivity condition (3.2) help us to obtain from (3.12) the next inequality
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{\lambda(n + 2)}{2(n - 1)(2n + 5)(2n + 1)} \int_M |Ric_{[3][0]}|^2 \text{Vol}_\eta + \frac{1}{4n\lambda(2n + 5)} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta.
\]
\[
\times \int_M (S - \bar{S})^2 \text{Vol}_\eta + \frac{1}{2n + 1} \int_M (\nabla e_a T^0)(e_a, \nabla \varphi) \text{Vol}_\eta.
\]
\[
\tag{3.13}
\]
Thus, (3.13) yields
\[
\left[1 - \frac{1}{4n\lambda(2n + 5)}\right] \int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{\lambda(n + 2)}{2(n - 1)(2n + 5)(2n + 1)} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta \\
+ \frac{1}{2n + 1} \int_M (\nabla_{e_a} T^0) (e_a, \nabla \varphi) \text{Vol}_\eta.
\] (3.14)

Set
\[
\lambda = \frac{1}{4n^2 + 10n}
\] (3.15)

into (3.14) to get (3.3), which proves the first part of a).

If we have an equality in (3.3), we put the expression of $|\nabla^2 f_{[3][0]}|^2$ from (2.12) into (3.12), taken with the constant $\lambda$ given by (3.15), to get
\[
0 \leq - \int_M \left[ \left| (\nabla^2 \varphi)_{[-1]_{\text{sym}}} \right|^2 + (2n + 1) \text{LiC} (\nabla \varphi, \nabla \varphi) \right] \text{Vol}_\eta,
\]
which, in view of (3.2), yields
\[
(\nabla^2 \varphi)_{[-1]_{\text{sym}}} = 0;
\]
\[
\text{LiC} (\nabla \varphi, \nabla \varphi) = S |\nabla \varphi|^2 + \frac{2n + 3}{2n + 1} T^0 (\nabla \varphi, \nabla \varphi) + \frac{2(n + 2)(2n - 1)}{(n - 1)(2n + 1)} U (\nabla \varphi, \nabla \varphi) = 0.
\] (3.16)

The first equality in (3.16) and the equality in (3.3) give (3.4). Now, the equalities (3.4), (2.12) and (3.16), together with (3.8), imply
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta = \frac{n + 2}{2n(n - 1)(2n + 5)^2(2n + 1)} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta;
\]
\[
\int_M |(\nabla^2 \varphi)_{[3][0]}|^2 \text{Vol}_\eta = \frac{(2n + 1)(n - 1)}{2n(n + 2)} \int_M (S - \bar{S})^2 \text{Vol}_\eta
\]
\[
= \frac{1}{4n^2(2n + 5)^2} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta.
\] (3.17)

On the other hand, (3.10) and (3.17) yield
\[
\frac{n + 2}{2n(n - 1)(2n + 5)^2(2n + 1)} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta
\]
\[
= - \frac{n + 2}{(n - 1)(2n + 5)(2n + 1)} \int_M (\text{Ric}_{[3][0]}) (e_a, e_b) (\nabla^2 \varphi)_{[3][0]} (e_a, e_b) \text{Vol}_\eta.
\] (3.18)

The equalities (3.17) and (3.18) imply
\[
(\nabla^2 \varphi)_{[3][0]} = - \frac{1}{2n(2n + 5)} \text{Ric}_{[3][0]} = - \frac{1}{n} U.
\] (3.19)

We take the qc conformal change $\tilde{\eta} = \frac{1}{4\lambda(2n + 5)} e^{-2\varphi} \eta = \frac{\eta}{2} e^{-2\varphi} \eta$ with $\lambda$, given by (3.15), to get $\tilde{U} = 0$, according to [14, formula (5.6)], taken with $2h = 4\lambda(2n + 5)e^{2\varphi} = \frac{2}{n} e^{2\varphi}$ (cf. also [21, formula (5.23)]), and (3.19). Hence, $\text{Ric}_{[3][0]} = 0$ and the qc conformal qc structure $\tilde{\eta}$ is qc-pseudo-Einstein, which completes the proof of a).

Now we prove b). The equalities (2.14), (3.1) and the positivity condition (3.5) help us to obtain from (3.12) the following inequality
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{\lambda(n + 2)}{2(n - 1)(2n + 5)(2n + 1)} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta
\]
Thus, (3.20) yields
\[
\left[ 1 - \frac{(n+2)(4n+3)}{8n\lambda(2n+5)(2n+1)(n+3)} \right] \int_M (S - \bar{S})^2 Vol_\eta \\
\leq \frac{\lambda(n+2)}{2(n-1)(2n+5)(2n+1)} \int_M |Ric_{[3][0]}|^2 Vol_\eta \\
+ \frac{1}{2n+1} \int_M (\nabla_{e_a} T^0)(e_a, \nabla \psi) Vol_\eta.
\] (3.21)

Set
\[
\lambda = \frac{(n+2)(4n+3)}{4n(2n+5)(2n+1)(n+3)}
\] (3.22)
into (3.21) to get (3.6), which proves the first part of b).

If we have an equality in (3.6), we put the expression of \(|(\nabla^2 f)_{[3][0]}|^2\) from (2.14) into (3.12), taken with the constant \(\lambda\) given by (3.22), to get
\[
0 \leq - \int_M \left[ |(\nabla^2 \varphi)_{[-1]}|^2 + 2n Cor(\nabla \varphi, \nabla \varphi) \right] Vol_\eta,
\]
which, in view of (3.5), yields
\[
(\nabla^2 \varphi)_{[-1]} = 0;
\]
\[
Cor(\nabla \varphi, \nabla \varphi) = Sg(\nabla \varphi, \nabla \varphi) + \frac{n+2}{n} T^0(\nabla \varphi, \nabla \varphi) + \frac{2(n+1)}{n-1} U(\nabla \varphi, \nabla \varphi) = 0.
\] (3.23)

The first equality in (3.23) and the equality in (3.6) imply (3.7). Now, the equalities (3.7), (2.14) and (3.23) together with (3.8) imply
\[
\int_M (S - \bar{S})^2 Vol_\eta = \frac{(n+2)^2(4n+3)}{4n(n-1)(n+3)(2n+5)^2(2n+1)^2} \int_M |Ric_{[3][0]}|^2 Vol_\eta \\
\times \int_M |(\nabla^2 \varphi)_{[3][0]}|^2 Vol_\eta = \frac{(4n+3)(n-1)}{4n(n+3)} \int_M (S - \bar{S})^2 Vol_\eta \\
= \frac{(n+2)^2(4n+3)}{16n^2(n+3)^2(2n+5)^2(2n+1)^2} \int_M |Ric_{[3][0]}|^2 Vol_\eta.
\] (3.24)

On the other hand, (3.10) and (3.24) yield
\[
\frac{(n+2)^2(4n+3)}{4n(n-1)(n+3)(2n+5)^2(2n+1)^2} \int_M |Ric_{[3][0]}|^2 Vol_\eta \\
= - \frac{n+2}{(n-1)(2n+5)(2n+1)} \int_M (Ric_{[3][0]})(e_a, e_b)(\nabla^2 \varphi)_{[3][0]}(e_a, e_b) Vol_\eta.
\] (3.25)
The equalities (3.24) and (3.25) imply
\[
(\nabla^2 \varphi)_{[3][0]} = -\frac{(n+2)(4n+3)}{4n(n+3)(2n+5)(2n+1)} \text{Ric}_{[3][0]} = -\frac{(n+2)(4n+3)}{2n(n+3)(2n+1)} U. \tag{3.26}
\]
We take the qc conformal change \( \tilde{\eta} = \frac{1}{4\lambda (2n+5)} e^{-2\varphi} \eta = \frac{n(n+3)(2n+1)}{(n+2)(4n+3)} e^{-2\varphi} \eta \) with \( \lambda \), given by (3.22), to get \( \tilde{U} = 0 \), according to [14, formula (5.6)], taken with \( 2h = 4\lambda (2n+5) e^{2\varphi} = \frac{(n+2)(4n+3)}{n(n+3)(2n+1)} e^{2\varphi} \) (cf. also [21, formula (5.23)]), and (3.26). Hence, \( \overline{\text{Ric}}_{[3][0]} = 0 \) and the qc conformal qc structure \( \tilde{\eta} \) is qc-pseudo-Einstein, which completes the proof of b). \( \square \)

An integration by parts of the last term in (3.3) yields the following

**Corollary 3.2** In addition to the same conditions as in Theorem 3.1, we assume
\[
(\nabla_{e_a} \nabla_{e_b} T^0)(e_a, e_b) = 0.
\]

a) If \( \text{Lic}(X, X) \geq 0 \) then
\[
\int_M (S - \tilde{S})^2 \text{Vol}_\eta \leq \frac{n+2}{2n(n-1)(2n+1)^2(2n+1)} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta = \frac{2(n+2)}{n(n-1)(2n+1)} \int_M |U|^2 \text{Vol}_\eta.
\]

b) If \( \text{Cor}(X, X) \geq 0 \) then
\[
\int_M (S - \tilde{S})^2 \text{Vol}_\eta \leq \frac{(n+2)^2(4n+3)}{4n(n-1)(n+3)(2n+5)^2(2n+1)^2} \int_M |\text{Ric}_{[3][0]}|^2 \text{Vol}_\eta = \frac{(n+2)^2(4n+3)}{n(n-1)(n+3)(2n+1)^2} \int_M |U|^2 \text{Vol}_\eta.
\]

In both cases, if \( M \) is qc-pseudo-Einstein then the qc scalar curvature \( S \) is a constant.

Further, in a similar way as the proof of Theorem 3.1, we obtain

**Theorem 3.3** Let \( (M, g, \mathcal{Q}) \) be a compact qc manifold of dimension \( (4n+3), \ n > 1 \).

a) Suppose the positivity condition (3.2) holds. Then we have
\[
\int_M (S - \tilde{S})^2 \text{Vol}_\eta \leq \frac{1}{8n(n+1)^2(2n+1)} \int_M |\text{Ric}_{[-1]}|^2 \text{Vol}_\eta + \frac{4(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla \varphi) \text{Vol}_\eta
\]
\[
= \frac{1}{2n(2n+1)} \int_M |T^0|^2 \text{Vol}_\eta - \frac{4(n+2)}{(n-1)(2n+1)} \int_M U(e_a, e_b)(\nabla^2 \varphi)_{[3][0]}(e_a, e_b) \text{Vol}_\eta.
\]
\tag{3.27}

If the equality holds then
\[
\int_M (S - \tilde{S})^2 \text{Vol}_\eta = \frac{1}{8n(n+1)^2(2n+1)} \int_M |\text{Ric}_{[-1]}|^2 \text{Vol}_\eta = \frac{1}{2n(2n+1)} \int_M |T^0|^2 \text{Vol}_\eta.
\]
\tag{3.28}
b) Suppose the positivity condition (3.5) holds. Then we have

\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{4n + 3}{16n(n + 1)^2(2n + 1)^2} \int_M |\text{Ric}|_{[-1]}^2 \text{Vol}_\eta \\
+ \frac{4(n + 2)}{(n - 1)(2n + 1)} \int_M (\nabla_{e_a} U)(e_a, \nabla \varphi) \text{Vol}_\eta \\
= \frac{4n + 3}{4n(2n + 1)^2} \int_M |T^0|^2 \text{Vol}_\eta \\
- \frac{4(n + 2)}{(n - 1)(2n + 1)} \int_M U(e_a, e_b)(\nabla^2 \varphi)_{[3][0]}(e_a, e_b) \text{Vol}_\eta.
\]

(3.29)

If the equality holds then

\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta = \frac{4n + 3}{16n(n + 1)^2(2n + 1)^2} \int_M |\text{Ric}|_{[-1]}^2 \text{Vol}_\eta \\
= \frac{4n + 3}{4n(2n + 1)^2} \int_M |T^0|^2 \text{Vol}_\eta.
\]

(3.30)

**Proof** Using (3.9), we obtain from (3.8) and an integration by parts that

\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta = \int_M dS(\nabla \varphi) \text{Vol}_\eta \\
= \frac{1}{2(n + 1)(2n + 1)} \int_M \left[ (\nabla_{e_a} \text{Ric}_{[-1]})(e_a, \nabla \varphi) \\
+ \frac{4(n + 1)(n + 2)}{n - 1} (\nabla_{e_a} U)(e_a, \nabla \varphi) \right] \text{Vol}_\eta \\
= -\frac{1}{2(n + 1)(2n + 1)} \int_M (\text{Ric}_{[-1]})(e_a, e_b)(\nabla^2 \varphi)^{s}[3][0](e_a, e_b) \text{Vol}_\eta \\
+ \frac{2(n + 2)}{(n - 1)(2n + 1)} \int_M (\nabla_{e_a} U)(e_a, \nabla \varphi) \text{Vol}_\eta.
\]

(3.31)

First we prove a). Applying the Young’s inequality (3.11) for a constant \(\lambda > 0\), together with (2.12), we get from (3.31)

\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{1}{4(n + 1)(2n + 1)} \left[ \lambda \int_M |\text{Ric}|_{[-1]}^2 \text{Vol}_\eta + \lambda^{-1} \int_M |(\nabla^2 \varphi)^{s}[3][0]|^2 \text{Vol}_\eta \right] \\
+ \frac{4(n + 2)}{(n - 1)(2n + 1)} \int_M (\nabla_{e_a} U)(e_a, \nabla \varphi) \text{Vol}_\eta \\
= \frac{\lambda}{4(n + 1)(2n + 1)} \int_M |\text{Ric}|_{[-1]}^2 \text{Vol}_\eta \\
+ \frac{4(n + 2)}{(n - 1)(2n + 1)} \int_M (\nabla_{e_a} U)(e_a, \nabla \varphi) \text{Vol}_\eta + \frac{1}{2\lambda} \int_M \left[ \frac{1}{4n(n + 1)} (\Delta \varphi)^2 \\
- \frac{n + 2}{2(n^2 - 1)(2n + 1)} |(\nabla^2 \varphi)^{s}[3][0]|^2 - \frac{1}{2(n + 1)} \text{Lic}(\nabla \varphi, \nabla \varphi) \right] \text{Vol}_\eta.
\]

(3.32)

The equalities (3.1), the positivity condition (3.2) and (3.32) yield
\[
\int_M (S - \bar{S})^2 \text{Vol}_g \leq \frac{\lambda}{4(n+1)(2n+1)} \int_M |Ric_{[-1]}|^2 \text{Vol}_g + \frac{1}{8n\lambda(n+1)} \left( \int_M (S - \bar{S})^2 \text{Vol}_g + \frac{2(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g \right).
\]

Thus, (3.33) yields
\[
\left[ 1 - \frac{1}{8n\lambda(n+1)} \right] \int_M (S - \bar{S})^2 \text{Vol}_g \leq \frac{\lambda}{4(n+1)(2n+1)} \int_M |Ric_{[-1]}|^2 \text{Vol}_g + \frac{2(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g.
\]

Set \( \lambda = \frac{1}{8n(n+1)} \) into (3.34) to get (3.27).

If we have an equality in (3.27), we use (2.12) to get from (3.32) that
\[
0 \leq -\int_M \left[ \frac{n+2}{n-1} |(\nabla^2 \varphi)_{[3][0]}|^2 + (2n+1)Lic(\nabla \varphi, \nabla \varphi) \right] \text{Vol}_g,
\]
which, in view of (3.2), yields

\( (\nabla^2 \varphi)_{[3][0]} = 0; \)

\( S|\nabla \varphi|^2 + \frac{2n+3}{2n+1} T^0(\nabla \varphi, \nabla \varphi) + \frac{2(n+2)(2n-1)}{(n-1)(2n+1)} U(\nabla \varphi, \nabla \varphi) = 0. \) (3.35)

Now, the equalities (3.27) and (3.35) imply (3.28), which proves a).

For b), using the Young’s inequality (3.11) for a constant \( \lambda > 0 \) together with (2.14), we get from (3.31)
\[
\int_M (S - \bar{S})^2 \text{Vol}_g \leq \frac{1}{4(n+1)(2n+1)} \left[ \lambda \int_M |Ric_{[-1]}|^2 \text{Vol}_g + \lambda^{-1} \int_M |(\nabla^2 \varphi)_{[-1]}|^2 \text{Vol}_g \right] + \frac{2(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g \\
= \frac{\lambda}{4(n+1)(2n+1)} \int_M |Ric_{[-1]}|^2 \text{Vol}_g + \frac{2(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g \\
\times \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g + \frac{1}{4\lambda(n+1)(2n+1)} \int_M \left[ \frac{4n+3}{4n} (\Delta \varphi)^2 - \frac{n+3}{n-1} |(\nabla^2 \varphi)_{[3][0]}|^2 - 2nCor(\nabla \varphi, \nabla \varphi) \right] \text{Vol}_g.
\]

The equality (3.1), the positivity condition (3.5) and (3.36) imply
\[
\left[ 1 - \frac{4n+3}{16n\lambda(n+1)(2n+1)} \right] \int_M (S - \bar{S})^2 \text{Vol}_g \leq \frac{\lambda}{4(n+1)(2n+1)} \int_M |Ric_{[-1]}|^2 \text{Vol}_g + \frac{2(n+2)}{(n-1)(2n+1)} \int_M (\nabla_{e_a} U)(e_a, \nabla) \text{Vol}_g.
\]

Set \( \lambda = \frac{4n+3}{8n(n+1)(2n+1)} \) into (3.37) to get (3.29).
If we have an equality in (3.29), we use (2.14) to get from (3.36) that
\[
0 \leq - \int_M \left[ \frac{n+3}{n-1} |(\nabla^2 \varphi)_{[3][0]}|^2 + 2n \text{Cor}(\nabla \varphi, \nabla \varphi) \right] \text{Vol}_\eta,
\]
which, in view of (3.5), yields
\[
(\nabla^2 \varphi)_{[3][0]} = 0; \quad \text{Cor}(\nabla \varphi, \nabla \varphi) = Sg(\nabla \varphi, \nabla \varphi) + \frac{n+2}{n} T^0(\nabla \varphi, \nabla \varphi) + \frac{2(n+1)}{n-1} U(\nabla \varphi, \nabla \varphi) = 0. \tag{3.38}
\]
Now, the equalities (3.29) and (3.38) imply (3.30), which completes the proof. \qed

An integration by parts of the last term in (3.27) and in (3.29), respectively, yields the following

**Corollary 3.4** In addition to the same conditions as in Theorem 3.3, we assume
\[
(\nabla_{e_a} \nabla_{e_b} U)(e_a, e_b) = 0.
\]

(a) If \( \text{Lic}(X, X) \geq 0 \) then
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{1}{8n(n+1)^2(2n+1)} \int_M |\text{Ric}_{[-1]}|^2 \text{Vol}_\eta = \frac{1}{2n(2n+1)} \int_M |T^0|^2 \text{Vol}_\eta.
\]

(b) If \( \text{Cor}(X, X) \geq 0 \) then
\[
\int_M (S - \bar{S})^2 \text{Vol}_\eta \leq \frac{4n+3}{16n(n+1)^2(2n+1)^2} \int_M |\text{Ric}_{[-1]}|^2 \text{Vol}_\eta = \frac{4n+3}{4n(2n+1)^2} \int_M |T^0|^2 \text{Vol}_\eta.
\]

In both cases, if \( M \) is qc-nearly-Einstein then the qc scalar curvature \( S \) is a constant.

**Remark 3.5** Note that the results of Theorems 3.1, 3.3, Corollaries 3.2 and 3.4 exclude the limiting 7-dimensional case. Apparently, this case needs a different approach, since our analysis relies heavily on the contracted second Bianchi identity (2.5) which is trivially satisfied in the 7-dimensional case.

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**Data Availability** The data that supports the findings of this study are available within the article (and the references therein).

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