Fault-Tolerant Postselected Quantum Computation: Schemes

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Postselected quantum computation is distinguished from regular quantum computation by accepting the output only if measurement outcomes satisfy predetermined conditions. The output must be accepted with nonzero probability. Methods for implementing postselected quantum computation with noisy gates are proposed. These methods are based on error-detecting codes. Conditionally on detecting no errors, it is expected that the encoded computation can be made to be arbitrarily accurate. Although the probability of success of the encoded computation decreases dramatically with accuracy, it is possible to apply the proposed methods to the problem of preparing arbitrary stabilizer states in large error-correcting codes with local residual errors. Together with teleported error-correction, this may improve the error tolerance of non-postselected quantum computation.

I. INTRODUCTION

An important problem in the theory of scalable quantum computation is to improve the maximum gate error rate for quantum gates at which it is possible to quantum compute fault tolerantly. In [1], it was shown that if the errors are all detected, then it is possible to quantum compute with up to 50% error probability per Bell measurement. The techniques used in [1] show that for depolarizing errors that are not detected, high error rates can be tolerated if certain entangled states can be prepared with sufficiently small, effectively independent errors. To prepare the required states, it suffices to use postselected quantum computation. Postselected quantum computation has the property that computations are accepted only if predetermined conditions are satisfied. The conditions must be satisfied with nonzero probability. Here, methods are proposed for implementing postselected quantum computation in the presence of noisy gates. The basic idea is to use a simple error-detecting code to encode a computation so that errors at a sufficiently low rate are detected. If the output is accepted only if no error is detected (that is, if the output is postselected), then the conditional probability of error can be reduced arbitrarily. The states required for (non-postselected) fault-tolerant quantum computation can then be prepared in an encoded form with some probability of success. To use them, the underlying error-detecting code is decoded with further error-detection to eliminate remaining errors. If the error-detecting code is obtained by concatenation, the final state can be obtained so that residual error is local with error-rate determined by the error model and the complexity of the decoding process.

The description of the methods in the next sections assumes the notation and knowledge of the teleportation techniques and stabilizer codes discussed in [1]. The basic ideas presented here are based on the large body of work on fault-tolerant methods using stabilizer codes that have been developed by the quantum information science community. [2, 3, 4, 8, 9, 10, 11, 12, 13].
The main task of this paper is to describe codes and networks for fault-tolerant postselected quantum computation. That they may be expected to have good fault-tolerance properties can be seen qualitatively. A detailed analysis is still required to determine these properties quantitatively and verify that they are as good as one would expect.

II. OVERVIEW

Fault-tolerant postselected quantum computation is easier to achieve than regular fault-tolerant quantum computation because there is no need to correct errors: it suffices to detect them. The disadvantage is that the probabilities of success can be very small. For theoretical purposes, this is not a problem, provided that only states of a bounded number of qubits need to be prepared.

To achieve fault-tolerant postselected quantum computation, a four-qubit error-detecting code is combined with purification, teleported error detection and transversal operations. Transversal operations involve applying gates only between pairs of corresponding qubits in two blocks of four qubits used for encoding a state. By concatenating this code, it is shown how to implement the accurate operations needed to encode CSS stabilizer codes. With these operations and the ability to encode an additional state with constant encoded error, purification of this state leads to a universal set of accurate gates and hence the full power of postselected quantum computation. An interesting feature of the technique is that all computations have finite depth. That postselected quantum computation can be performed at constant depth has been pointed out in the context of quantum complexity theory in [22].

To obtain stabilizer states with bounded, local errors, postselected encoded quantum computation is used to obtain an accurate stabilizer state, encoded in the concatenated error-detecting code. The concatenated error-detecting code is decoded bottom-up. If the state is accepted only if no errors are detected during decoding, then the output state is as desired.

III. BASIC GATES AND ERROR MODEL

The basic Clifford gate set used here consists of preparations of the +1 eigenstates of \( \sigma_z \) and \( \sigma_x \), the controlled-not (cnot) and \( \sigma_z \) and \( \sigma_x \) measurements. They suffice for encoding and decoding CSS [23] codes and states, and are used for the primary concatenation. These gates, together with the Hadamard gate, are referred to as the CSS gates. The Hadamard gate is needed only at the top level of the concatenation hierarchy, where, together with two prepared states, it is used to ensure universality. Since the error-detecting code used permits a transversal implementation of the Hadamard gate, it is possible to add this gate without significantly affecting error-tolerance. To enable all real quantum computations, preparation of the state \( |\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle \) (the +1 eigenstate of the Hadamard transform) is used. This is done fault tolerantly by encoding it (with postselection) into the top level with an encoded error related to the basic error rates. Accurate encoded cnot’s and Hadamards are then used to purify the encoded \( |\pi/8\rangle \). Using the CSS gates and \( |\pi/8\rangle \) preparation, it is possible to implement quantum computation without complex phases. Real-number quantum computation is equivalent in power to quantum computation [24]. But to complete the gate set, one can add preparation of the state \( |i\pi/4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \). The network symbols used for the gates and their definitions are shown in Fig. 1.
An error analysis is not given in this paper. Nevertheless, for making qualitative observations, it is necessary to specify an error model. The error model assumes that all errors are probabilistic Pauli products. A quantum computation is described in terms of its quantum network. Each instance of the quantum computation is modified by probabilistically inserting Pauli operators after each network element and before measurements. To describe the constraints on the probabilities, associate with each network element an error location. For a state preparation and the Hadamard gate, the error location is in the qubit line immediately following the state preparation gate. For the controlled not, it extends to both qubit lines immediately after the gate. For measurements, it is on the qubit to be measured just before the gate. For each Pauli operator $p$ (or product of two Pauli operators in the case of the cnot) that can act at an error location $l$, there is an associated error parameter $e_{p,l}$. The strongest assumption one can make on the error model is that $e_{p,l}$ is the probability that $p$ acts at location $l$, independently for different locations.

One of the distinguishing features of postselected quantum computation as implemented here is that memory error can be eliminated by optimistic precomputation. This feature requires that 1. there is no difficulty in using massively parallel processing to apply gates, and 2. cnots can be applied to any pair of qubits. But no assumption has to be made on the relative time taken by cnots and measurements. Memory error plays a role only when the final state is to be used for a standard (non-postselected) computation, where it is necessary to delay until all measurements have been completed before using the state. For the schemes used here, one measurement delay is sufficient.
IV. AN ERROR-DETECTING CODE FOR ONE QUBIT

A four-qubit code can be used for error detection. The simplest such error-detecting code’s stabilizer is generated by $[XXXX], [ZZZZ]$. However, this encodes two qubits. One of these can be used as the logical qubit. The other serves as a spectator qubit. It is convenient not to use a single state for the spectator qubit. Equivalently, two codes will be used for the logical qubit, depending on the state of the spectator qubit. The logical and spectator qubits’ operators are chosen to be

$$X^{(L)} = [XXII], \quad Z^{(L)} = [ZIZI]$$
$$X^{(S)} = [IXIX], \quad Z^{(S)} = [IIZZ],$$

where $L$ and $S$ are labels for the logical and the spectator qubit, respectively. The convention is that $X$ stands for $\sigma_x$, $Z$ for $\sigma_z$ and, for example, $[XXII]$ for the product $\sigma_x^{(1)}\sigma_x^{(2)}$ when referring to an operator on four ordered qubits 1, 2, 3, 4.

Every computation begins with preparations of one of two states defined by $|o+\rangle_{LS} = \Pi(Z^{(L)}, X^{(S)}, [XXXX], [ZZZZ])$ and $|+o\rangle_{LS} = \Pi(X^{(L)}, Z^{(S)}, [XXXX], [ZZZZ])$. The expression $\Pi(s_1, s_2, \ldots)$, where the $s_i$ are binary vectors associated with independent Pauli products $P(s_i)$, refers to the state or subspace of states with eigenvalue $+1\nu(s_i)$ for $P(s_i)$. The phase $\nu(s_i)$ is defined in Fig. 2 and is always 1 for the CSS codes used here. Since both states are pairs of Bell states involving different pairings of the four qubits, they can be encoded with the networks shown in Fig. 2.

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**Fig. 2**: Fundamental encoding networks. The stabilizer of the output of a. is generated by $[XXXX], [ZZZZ], [ZIZI], [IXIX]$. For the output of b. it is generated by $[XXXX], [ZZZZ], [XXII], [IIZZ]$. In both cases, the syndrome is 0 (all the eigenvalues are $+1$) with respect to these generators. It is useful to think of the two states as states of a logical and a spectator qubit encoded in the stabilizer code $\Pi([XXXX], [ZZZZ])$ generated by $[XXXX], [ZZZZ]$. This code is one-error-detecting. The two encoded qubits are defined by their encoded Pauli matrices: $X^{(L)} = [XXII], Z^{(L)} = [ZIZI], X^{(S)} = [IXIX], Z^{(S)} = [IIZZ]$. Thus the networks produce the encoded $\Pi(Z^{(L)}, X^{(S)})$ and $\Pi(X^{(L)}, Z^{(S)})$ states, respectively. The logical qubit L is used for robustly encoding states after postselection. Encoded qubit S plays a spectator role and is always in an eigenstate of $X^{(S)}$ or $Z^{(S)}$. The eigenstate in which it is intended to be is known and will be used for additional error detection.

$|o+\rangle_{LS}$ and $|+o\rangle_{LS}$ are CSS states, which means that the stabilizer can be generated by Pauli products that consist of either all $X$ or all $Z$ operators. This property simplifies fault-tolerant
implementations of encoded operations. In this case, encoded cnots and Hadamard gates can be realized transversally and act on the logical and the spectator qubits in parallel. The code space \( \Pi([XXX], [ZZZ]) \) is preserved by parallel Hadamard gates of the four qubits. However, \( X^{(L)} \) is transformed to an operator equivalent to \( Z^{(S)} \) (equivalent with respect to the stabilizer). This can be fixed by exchanging the middle two qubits. Thus, to apply simultaneous Hadamard gates to the logical and the spectator qubits, apply Hadamard gates to each qubit and swap the middle two qubits. Note that qubit swaps can be done with no error by doing them only logically. That is, the qubit positions remain fixed, but their labels are changed. To make this work requires that there is negligible latency when applying gates between any two qubits, no matter where they are.

To apply cnots to two encoded logical/spectator qubit pairs, it suffices to apply four cnots to corresponding pairs of qubits in the two four-tuples. An example of this occurs in preparing an encoded Bell state for teleporting gates with error detection. The network is shown in Fig. 3.

![Fig. 3](image-url)  

**Fig. 3:** State used to teleport encoded qubits. Before the final four cnots, qubits 1, 2, 3, 4 are in the state \(|+\rangle_{LS}^+\). Qubits 5, 6, 7, 8 are in the state \(|0\rangle_{LS'}\). The cnots in the last time step act as encoded cnots on the logical and spectator qubits—a property of CSS codes. The state after the cnots is

\[
\Pi([XXXXXXX], [ZZZZZZZ], [IIIIIXIX], [IIIIXXIX], [IIIIIZZZ])
\]

This can be recognized as an encoded \( \Pi(X^{(L)}X^{(L')}, Z^{(L)}Z^{(L')}, Z^{(S)}, X^{(S)}) \) state. In other words, the logical qubits to be used are entangled in the standard Bell state. The spectator qubits remain in their original states and are independent of each other. The light-gray vertical lines separate the different time steps. The network consists of three steps of parallel operations, including the state preparation. The first prepares the qubits, and the second and third each apply four parallel cnots. The depth of this network is three.
V. PURIFYING STABILIZER STATES

The state preparations of the previous section can result in errors in the encoded states at a rate that is proportional to the gate error rates. For any given four qubits encoding a logical qubit, the stabilizer is generated either by $[XXXX]$, $[ZZZZ]$ and $[IIZZ]$ or by $[XXXX]$, $ZZZZ$ and $[IXIX]$, depending on the state of the spectator qubit. The syndrome for the logical qubit is said to be 0 if the eigenvalues for the appropriate three operators are all $+1$. The goal is to purify prepared states so as to ensure that conditionally on the syndromes of the logical qubits being 0, the error in the intended logical state is quadratically suppressed.

The purification method for Bell states extends to any CSS state [2]. The basic idea is to combine pairs of states with transversal cnots and measure the qubits of one member of the pair to determine whether the syndromes agree. There are two choices for the measurement, depending on whether $X$-type or $Z$-type syndromes are to be compared. To do both requires two purification stages using a total of four identically prepared states. Each four-tuple of corresponding qubits is subjected to one of the two networks shown in Fig 4.

![Fig. 4: Purification networks with one-qubit outputs](image)

FIG. 4: Purification networks with one-qubit outputs. The networks are applied to the corresponding qubits of four identically prepared CSS states. The measurement operations are postselected based on the outcome as shown. A subscript of $+$ means that only the $+1$ eigenstate is accepted. Network a. purifies $X$ syndromes on two pairs of states and then purifies the $Z$ syndromes in the resulting two states. Network b. purifies $Z$ before $X$.

The networks in Fig. 4 involve postselection on one measurement outcome for each qubit measurement. This suffices for analysis and ensures that if the input states are sufficiently close to the desired CSS states, the conditionally produced output state is improved except for local error introduced in the purification network itself. It is, however, unnecessarily inefficient. To ensure that if no error occurs, the probability of success is 1, the purification can be conditioned on correctness of appropriately chosen parities of the measurement outcomes. For example, the last pairwise combination in Fig. 4 checks the $Z$-type syndrome of the two remaining states after the first round of purification. If one of the $Z$-type stabilizers is $\sigma_z^{(1)}\sigma_z^{(3)}$, then the total parity of the $Z$ measurements on the first and third set of four corresponding qubits should be 0. That is, the outcomes should be both $+1$ or both $-1$, unless there was an error.

In the concatenated scheme for postselected quantum computation, the primary states to be prepared are instances of the encoded entangled state of Fig. 3. Its preparation with a schematic purification is shown in Fig 5.
FIG. 5: Typical encoded Bell state preparation with purification. The spectator qubit is in the state \( \Pi(Z^{(S)}) \) in the top half of the pair of encoded states. In the bottom half it is in the state \( \Pi(X^{(S)}) \). The state preparation for one of the four entangled states that are purified to the one used for teleportation is shown. Purification occurs in the solid boxes at the right end of the network. The state preparation network shown is denoted by \( B_{zx} \).

VI. ENCODED GATES WITH ERROR DETECTION

As can be seen, the four-qubit error-detecting code requires only state preparations, measurements in the \( Z \) and \( X \) basis, and \textsc{cnots} for the purpose of concatenation. The first task is therefore to show how to implement these gates in encoded form. One consideration is that to use prior knowledge of the state of the spectator qubit for error detection, it is necessary to make sure that it is not affected by encoded operations and that information about its state can be extracted from the Bell measurements used for teleportation. To do this, each teleportation step in an encoded network is labeled “even” or “odd”. The encoded Bell state in even steps (an “even” Bell state) has the property that the spectator qubit at the output is in the state \( \Pi(X^{(S)}) \). In odd steps, the spectator qubit at the output is in the state \( \Pi(Z^{(S)}) \). Thus, the spectator qubits on the two sides of a transversal Bell measurement used to teleport are both in the same state and any differences arising from errors are detected.

Encoded state preparations are shown in Fig. 6 and involve the use of \( B_{zx} \) in either orientation (producing even or odd encoded Bell states) depending on the parity of the step. Encoded measurements are applied directly to the destination qubits of the state produced by \( B_{zx} \) for teleporting the previous gate. Measuring \( X \) on each qubit and postselecting on \(+1\) eigenvalues has the effect of postselecting on the \(+1\) \( X^{(L)} \) eigenstate while rejecting errors affecting the other \( X \)-type syndromes. There are one or two such syndromes, depending on the current state of the spectator.
qubit, which in turn depends on the parity of the previous teleportation step. Measuring $Z$ on each qubit has an analogous effect. An encoded cnot is shown in Fig. 7.

**FIG. 6:** Encoded preparation of $\sigma_x$ and $\sigma_z$ eigenstates. Network **a.** is the $\Pi(\mathbb{X}^{(L)})$ eigenstate preparation. Network **b.** is the $\Pi(\mathbb{Z}^{(L)})$ eigenstate preparation. The networks are shown for preparations at the zeroth step. The encoded Bell state is therefore even. For a preparation requiring an odd Bell state, the measurements are moved to the bottom four qubits and the top four are the output qubits. The postselection ensures that some errors are rejected. In **a.**, any errors affecting the $[\mathbb{XXXXIII}]$-syndrome will be rejected. In **b.**, errors affecting the $[\mathbb{ZZZZIII}], [\mathbb{IIZZIII}]$-syndrome will be rejected.
FIG. 7: Encoded cnot. The top Bell pairs belong to the previous operation. Shown is the cnot with the two sets of destination qubits (bottom) constituting the second halves of two odd encoded Bell states. For the even case, it is necessary to interchange the preparation steps for the top and bottom Bell pairs. The cnot is from the foreground to the background encoded qubits. The idea is to apply a cnot transversally, then perform the error-detecting Bell measurements. The Bell measurements consist of cnots followed by $X$ and $Z$ measurements on the control and target qubit, respectively. To maximize the ability to optimistically precompute and parallelize, the cnots needed for the Bell measurements have been arranged so that they can be interchanged with the transversal cnots that realize the operation. This delays the eight measurements on the input (top) qubits by one step, but allows the other eight measurements to be performed immediately.

The postselection shown in Fig. 7 ensures that single errors and some double errors are eliminated. It also ensures that the Pauli corrections normally needed in teleportation are not required, since the Bell measurements project onto the standard Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ given postselection. Efficiency can be gained by postselecting on parities of the measurement associated with the syndromes and keeping track of the Pauli corrections required. The actual implementation of the correction can typically be deferred to the very end of a large state preparation procedure or can just be kept track of by bookkeeping techniques (see [20]).

VII. ENCODING ARBITRARY ONE-QUBIT STATES

In order to enable non-Clifford gates at the top encoded level of the concatenated error-detecting code, it is necessary to start with (noisy) non-stabilizer one-qubit states and encode them into the top level. This must be done in such a way that excess error in the encoded state compared to the starting state is well bounded. This is accomplished by teleporting it into the code by use of a prepared entanglement between a qubit and a logical qubit as shown in Fig. 8. An alternative method for preparing the required entanglement, by decoding one half of a fully encoded Bell pair, is shown in Fig. 9.
FIG. 8: Encoding an arbitrary state to the next level. The full network on the right first entangles qubit 0 with the logical qubit encoded in qubits 1, 2, 3, 4. The spectator qubit is in the state $\Pi(Z(S))$, suitable for an even output in a large network. The prepared entanglement is purified. Since the resulting state is defined by $\Pi([IXXXX], [IZZZZ], [IIIZZ], [XIXII], [ZZIZI])$, it is one-error detecting. There is an unavoidable memory delay on qubit 4. It may be chosen to occur either before or after purification. The last step is to teleport the desired state $|\psi\rangle$ into the code. The state preparation before purification for having the spectator qubit in state $\Pi(X(S))$ is shown on the left without the subsequent purification and teleportation networks. This can be used for odd steps. In this case, the unavoidable memory delay is on qubit 1.
FIG. 9: Encoding an arbitrary state using an encoded Bell state. Here, one of the logical qubits in an encoded Bell state is first decoded, then used to teleport $|\psi\rangle$ into the code. The full procedure is shown for decoding a logical qubit where its spectator qubit is in the $\Pi(Z^{(S)})$ state. The decoding network is labeled $D_z$. Note that it is identical to a version of the purification network. To decode with the spectator qubit in the $\Pi(X^{(S)})$ state (the bottom half of the encoded Bell pair in the picture), use the network labeled $D_x$. The decoding network is designed so that the three measured qubits yield syndrome information. Any error that affects the syndrome is detected and results in rejection of the state obtained. Note that memory, given optimistic precomputation on other qubits, delays do not occur.

VIII. CONCATENATION

The techniques of the previous sections can be concatenated by the usual method of substituting qubits by logical qubits and gates by encoded gates based on the networks shown. Because of the pervasive use of teleported operations and postselection, it can be seen that the maximum depth of the networks obtained is determined by the depth of the network at the first level of concatenation. Counting state preparation and measurement, this depth is at most 8 for CSS gates and 9 for encoding an arbitrary state, which can be compared to the circuits of depth 5 for postselected quantum computation found in [22].

The concatenated circuits can be simplified by avoiding explicit concatenation of gates involved in the Bell state measurements and the encoded cnots. That is, instead of reimplementing the cnots
required for these two operations, they can be directly applied transversally to the concatenated error-detecting code, as can the subsequent measurements.

IX. ONE-QUBIT STATES FOR UNIVERSAL COMPUTATION

In order to enable universal quantum computation at the top encoded level, one can use a scheme for encoding a few additional one-qubit states with bounded error. They can then be purified by using the ability to implement nearly error-free CSS gates. For real quantum computation, it suffices to prepare the state $|\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$, the $+1$ eigenstate of the Hadamard gate. However, it is convenient to have the ability to use complex phases, and for that it suffices to use the state $|i\pi/4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, the $+i$ eigenstate of $\sigma_x$ followed by $\sigma_z$. That these suffice is due to the networks shown in Fig. 11. They make it possible to implement the rotations $e^{-i\sigma_x\pi/4}$ and $e^{-i\sigma_y\pi/8}$. Together with the CSS gates they form a universal set [9].

![Figure 10: Definition of controlled-Y gates.](image)

![Figure 11: Implementation of $e^{-i\sigma_x\pi/4}$ and $e^{-i\sigma_y\pi/8}$ rotations. Note that postselection based on measurement outcomes is used. The sign of the rotation depends on the measurement outcome, so it is possible, in principle, to implement the rotations without postselection by following with $\pm 180^\circ$ or $\pm 90^\circ$ rotations if the opposite measurement outcome is obtained.](image)

There are several ways in which $|i\pi/4\rangle$ and $|\pi/8\rangle$ can be purified by use of CSS gates. Consider $|i\pi/4\rangle$. This is the $-i$ eigenstate of $\sigma_x\sigma_z$. Thus a measurement to verify $|i\pi/4\rangle$ can be performed using two instances of $|i\pi/4\rangle$ and a controlled-$\sigma_x\sigma_z$ gate, as shown in Fig. 12.
FIG. 12: Purifying $|i\pi/4\rangle$ using cnots and Hadamard gates. The controlled gate shown is defined in Fig. 10 and involves two cnots and Hadamard gates. Any single error is detected by the $X$ measurement and rejected by postselection.

The conditional gate in Fig. 12 kicks back a phase of $-i$ on $|1\rangle_2$, which changes $|i\pi/4\rangle_2$ to $|+\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_2)$. The measurement therefore succeeds. By decohering the input states using random applications of $i\sigma_y$ (if necessary), the error on the two prepared states can be assumed to be random $\sigma_x$ noise. It can be seen that a $\sigma_x^{(1)}$ after the state preparation is equivalent to $\sigma_x^{(1)}\sigma_z^{(2)}$ before the measurement. The measurement outcome therefore changes. A $\sigma_x^{(2)}$ error is equivalent to $\sigma_y^{(1)}\sigma_y^{(2)}$ before the measurement and again, the measurement outcome changes. This shows that any single error is rejected by the network. Therefore, the error in the prepared $|i\pi/4\rangle$ is reduced quadratically given success.

To purify $|\pi/8\rangle$ using CSS gates is more difficult. First note that a measurement to verify $|\pi/8\rangle$ can be implemented with a conditional-Hadamard gate, which in turn requires two prepared $|\pi/8\rangle$ states as shown in Fig. 13.

FIG. 13: Measurement to project the input onto $|\pi/8\rangle$. If the $X$ measurement results in the $-1$ eigenstate, then the measurement projects the input onto the orthogonal state.

By randomly applying Hadamard gates to the prepared $|\pi/8\rangle$ states, the error model can be assumed to be probabilistic $\sigma_y$ errors. That is, after an error, the state prepared is the orthogonal state $|5\pi/8\rangle$. For the $e^{-i\sigma_y\pi/8}$ rotations implemented with prepared $|\pi/8\rangle$’s, the error causes an unintentional additional $\sigma_y$. In the context of the measurement circuit of Fig. 13, such an error in the second $Y$ rotation results in a $\sigma_y$ error at the output. In the first $Y$ rotation, its effect is to introduce a $\sigma_y$ error at the input.

A straightforward way to purify the $|\pi/8\rangle$ state is to use a self-dual CSS code for encoding one qubit with good error-detecting properties. An example is the 7-qubit Hamming code. Specifically, consider a CSS code for one qubit for which the stabilizer is invariant when Hadamard gates are applied to each qubit (transversal Hadamard) and such that the logical $X$ and $Z$ operators for the encoded qubit are exchanged under the transversal Hadamard. Such codes can be obtained from self-dual classical codes (see [3]). Then one can encode the logical $|0\rangle_L$ with CSS gates. A measurement of the encoded Hadamard operator is performed by using an ancilla qubit prepared in $|+\rangle$ and conditional Hadamard gates from the ancilla to each qubit of the code. An $X$ measurement
with postselection on $+1$ should project onto the encoded $|\pi/8\rangle$ state unless errors occur in the $|\pi/8\rangle$ state preparations. Errors in the conditional Hadamard gates cause syndrome changes via the $\sigma_y$ effects on the qubits of the code. These syndrome changes are detected when the state is decoded to extract the encoded $|\pi/8\rangle$. Conditionally on no detected errors, the $|\pi/8\rangle$ has much reduced error. Self-dual CSS codes with the ability to detect $\sigma_y$ errors at a rate above 10% exist. (This is because classical self-dual codes that meet the Gilbert-Varshamov bound exist, see [25], chapter 19, and [23]).

X. PREPARING ARBITRARY STABILIZER STATES

Given a stabilizer state defined by generator matrix $Q$, there is an encoding network using CSS gates and $e^{-i\sigma_x\pi/4}$ rotations [26]. Using the above methods, it is possible to use postselected computation to prepare the state $\Pi(Q)$ at a level of concatenation where the postselected, encoded CSS gates are sufficiently accurate to also enable accurate 90° $\sigma_x$ rotations with $|i\pi/4\rangle$ preparations. Once the encoded $\Pi(Q)$ state has been obtained, one can apply the decoding networks $D_x$ and $D_z$ shown in Fig. 9 to collapse the concatenated code to one qubit. The collapse should be performed bottom up, not top down. That is, decoding is performed on physical qubits four at a time until only one qubit remains for each top-level logical qubit. Conditionally on no errors being detected during the decoding, one would expect that the resulting state is perturbed from $\Pi(Q)$ only by local errors that either had not been detected when the encoded state was prepared, or were introduced during decoding without being detected. Provided the local error-rate is small enough, the state can be used in a general scheme for fault-tolerant quantum computation [1].

It is worth noting again that with optimistic precomputation, memory errors do not play a role until the state is used elsewhere in a non-postselected computation. That is, every qubit is subject to an operation at every step. Before the state can be used in a non-postselected computation, there is one memory delay, which is required to wait for a positive outcome of the last measurement in the decoding circuit. The price of avoiding memory delays is massive parallelism.

XI. DISCUSSION

At this point, the only evidence that the methods above lead to fault-tolerant postselected quantum computation with reasonable error thresholds is qualitative. Inspection of the methods suggests that isolated errors within each code block will be detected so that, conditionally on success within one level of concatenation, the encoded error rate is expected to be proportional to the square of the base error rate. Compared to the concatenated codes typically used for regular quantum computation, the codes here are significantly simpler. The small depth and breadth of the relevant circuits suggests that the constant of proportionality should be relatively small. However, it is necessary to take into account that postselection requires conditioning on potentially rare events. Determining whether the relative probabilities of error combinations really behave well in this setting requires further analysis.

As shown, accurate postselected quantum computation is inefficient. The following can be used to improve the overall efficiency. First, it is a good idea to stop concatenation using short codes as soon as possible. In an application to state preparation, the main goal is to achieve sufficient accuracy for other, non-postselected techniques that work well at lower error rates can be used to scale up. This approach is used in [1] in the context of erasure errors to explicitly show efficiency of a fault-tolerant scheme. Second, wherever possible, the postselection should be
modified to have probability of success 1 if there are no errors. In the context of the methods used here, this requires using bookkeeping techniques to keep track of the syndromes and adjust the postselection accordingly. Finally, the small depth of the networks makes it possible to combine postselected subnetworks in a tree-like fashion. This avoids having to redo the entire preparation each time and, under ideal circumstances, makes the total overhead for a successful computation efficient. How efficiently one can prepare states with the schemes given in this paper using the improvements just mentioned remains to be determined.

The code and networks used here have the remarkable feature of involving little more than Bell state preparation followed by a small number of different ways of combining them and partial and complete Bell state measurements. It might be useful to better understand the role of Bell states and Bell measurements as used here. In this context, one could ask whether a smaller error-detecting code could be used. Unfortunately, there are no three-qubit error-detecting codes for qubits \[^{27}\]. However, there are such codes for qutrits (three-level quantum systems) \[^{28}\], suggesting that the error-tolerance for qutrits might be more favorable. This is worth investigating, but it is necessary to be cautious when comparing error rates for qubit and qutrit Clifford gates. At least some of the latter are physically more complex, meaning that a higher error rate is to be expected. For example, the analogue of the Hadamard gate for qutrits is not readily implemented in spin-1 systems by use of typically available control based on spin observables. Multiple individually addressed level-transitions are needed to implement it. They also involve a large-dimensional space, with more opportunities for errors to occur. Furthermore, the four-qubit error-detecting code is in a sense more compact then the three-qutrit error-detecting code. The former lives in a 16 dimensional state space, the latter in a 27 dimensional one.

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