Connection between Wigner distribution of nonclassical fields and collapse-revival of atomic dynamics

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The transient evolution of nonclassical radiation fields interacting with an atom in a cavity is correlated to the atomic dynamics. Connection between the atomic phases of collapse and revival, and the Wigner function pattern is explored. Initial classical coherent field can evolve into nonclassical field. Schrodinger cat state field is generated during the atomic collapse phase. For initial Schrodinger cat field state, the state dissolves with time but revives during the atomic phase that corresponds with the initial phase. More intricate but orderly characteristics are found between the collapse-revival dynamics and the patterns of the Wigner function for initial thermal field.

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Introduction

Schrödinger’s cat state $|\alpha\rangle \pm e^{i\phi} - |\alpha\rangle$, a superposition of classical (coherent) states, is a nonclassical state. One of the challenges in quantum communication science is to produce the state with large $|\alpha|^2$. Six atoms atomic cat state has also been produced. The cat state for fields may be produced from squeezed sources, linear processes, simple photon counting scheme, and also by conditional measurement. Single photon subtraction from a squeezed vacuum state can generate Schrödinger’s kitten (small $|\alpha|^2$) via a technique which involves homodyne detection. Recently, large-amplitude coherent-state superposition has been generated via photon subtraction. These are promising developments toward macroscopic quantum superposition of states for quantum information processing.

Photon subtraction via conditional measurement can improve the quality of teleportation with continuous variables. Photon subtraction of squeezed vacuum gives coherent superposition of odd photon number states with nonclassicality, such as negativity of the Wigner function and sub-Poissonian statistics. Entanglement between two pulses that are Gaussian quadrature-entangled can also be increased by coherent subtraction of single photons.

Conversely, $m$-photon addition of coherent state produces phase squeezing, sub-Poissonian statistics and negativity of the Wigner function. Similar nonclassical character is found in photon-added and even coherent states (positive and negative Schrödinger’s cat). For time dependent electromagnetic field frequency, photon addition of coherent state gives enhanced dynamical squeezing. In general, both the subtraction and addition of single photon give nonclassical results.

Since photon addition (subtraction) correspond to downward (upward) transition in atomic dynamics, the atom and the field are closely correlated. Thus, the distinct mechanisms between photon addition and subtraction can be better understood by looking at the atomic dynamics of a single atom-single mode cavity field system. Such cavity system has displayed quantum effects like Rabi oscillations, collapse-revival dynamics and atom-field entangled states.

In this letter, we explore the nature of nonclassical fields by studying the atomic dynamics, particularly the connection between the nonclassical behavior of the Wigner function and the phenomena of collapse-revival or atomic inversion. Although such system was studied by ref. [16], it was confined only photon addition process on single mode thermal field and weak coupling (short time) regime $gt << 1$. Our present work is valid for arbitrary time. One interesting result is that the Schrödinger cat state can be generated from coherent state during the collapse in the atomic inversion. The results in this work provide insights into the subtle effect of atomic quantum interference on nonclassical light, photon subtraction and addition processes; and lead to the possibility of controlling the quantum state of the field by controlling the atomic dynamics.

Atom-field dynamics in cavity

The single mode-single atom dynamics is governed by the interaction Hamiltonian $V = \hbar g (\sigma_+ e^{i\Delta t} + \sigma_- e^{-i\Delta t})$ (in interaction picture) with $\sigma_+ = |a\rangle \langle b|$, and $\sigma_- = |b\rangle \langle a|$. Consider that initially the atomic state $\rho(0)$ is uncorrelated to the field $\rho(0)$, so $\rho(0) = \rho_f(0) \otimes \rho_a(0)$. The evolution of the atom-field system evolves according to $|\psi(t)\rangle = \sum_n |C_{a,n}(t)a,n + C_{b,n}(t)b,n]\rangle$ with the coefficients given by $|C_{a,n}(t)\rangle = e^{-i\Delta t/2}|C_{a,n}(0)r_n - iC_{b,n+1}(0)q_n|$, $|C_{b,n}(t)\rangle = e^{-i\Delta t/2}|C_{b,n}(0)r^*_{n-1} - iC_{a,n-1}(0)q_n|$, where $r_n = \cos\left(\frac{\Omega_n^2}{2}\right) - i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n^2}{2}\right)$, $q_n = \frac{2g\sqrt{\Delta^2 + 4g^2}}{\Omega_n}\sin\left(\frac{\Omega_n^2}{2}\right)$ and $\Omega_n^2 = \Delta^2 + 4g^2(n + 1)$. The initial coefficients are $C_{a,n}(0) = C_{a0}(0)C_n(0)$, $C_{b,n+1}(0) = C_{b0}(0)C_{n+1}(0)$ satisfy $|C_{a,n}(0)|^2 + |C_{b,n}(0)|^2 = 1$.

In general, the matrix elements of the field are obtained by tracing out the atomic system ($a$), $\rho_{nm} = \langle n|\hat{\rho}|m\rangle = \sum_s \langle s|\hat{\rho}|s\rangle \rho_{ss} = \langle s|\hat{\rho}|s\rangle, \langle n|\hat{\rho}|s\rangle = C_{s,n}(t)C^*_{s,m}(t)$ for $s = a,b$. Field density matrix elements

From the full expressions for the coefficients $C_{a,n}(t)$ and $C_{b,n}(t)$, the transient density matrix elements of the
field can be obtained from
\[ \rho_{nm}(t) = \langle n| \{ \hat{\rho}_{aa}(t) + i\hat{\rho}_{ab}(t) \} |m \rangle = C_{a,n}(t)C_{a,m}^*(t) + C_{b,n}(t)C_{b,m}^*(t) \]  
(1)

\[ \rho_{nm}(t) = \{|C_a(0)|^2 r_m r_m^* + |C_b(0)|^2 r_{m-1} r_{m-1}^* \} \rho_{nm}(0) + \{C_b(0)\}^2 q_m q_{m+1} \rho_{n-1,m+1}(0) + |C_a(0)|^2 q_n q_{m+1} \rho_{n-1,m}(0) \]  
\[ + iC_a(0)C_b^*(0) \{ r_m q_m \rho_{n+1,m}(0) - q_{n-1} r_m \rho_{n-1,m}(0) \} + iC_b(0)C_a^*(0) \{ r_{m-1} q_m \rho_{n,m-1}(0) - q_n r_m \rho_{n+1,m}(0) \} \]  
(2)

where \( \rho_{nm}(0) = \langle n| \hat{\rho}_f(0) |m \rangle \) with \( \hat{\rho}_f(0) \) being the initial state of the field. This expression will be used to compute the Wigner function for the field.

**Atomic inversion**

\[ n_{ab} = \sum_m \left[ \{ |C_a(0)|r_m(t) \}^2 - |C_b(0)|r_{m-1}(t) \}^2 \} \rho_{m}(0) + |C_{b,m+1}(0)|q_m|p_{m+1}(0) - |C_a(0)|q_{m-1}p_{m-1}(0) \]  
(3)

Wigner function versus density matrix elements

Nonclassical states of light have been studied through various physical parameters; the most typical ones being squeezing, antibunching in \( G^{(2)} \), entanglement criteria, Mandel’s Q and Wigner’s function. The squeezing and sub-Poissonian may not serve reliably to quantifying nonclassicality\[^1\], \[^7\]. Negativity of the Wigner function is a more reliable quantity and has been used to study the concept of photon addition and subtraction processes for producing nonclassical states. The relation between the Wigner function and the density matrix elements for the field \( \hat{\rho}_{nm} \) can be derived\[^8\] by using the identities found in \[^18\],

\[ W(\alpha, \alpha^*, t) = \frac{2 e^{-2|\alpha|^2}}{\pi^2} \sum_{m,n} \rho_{nm}(t) \times \int \langle -\beta|n\rangle \langle m|e^{2(\beta^* - \beta^*)}d^2\beta \]

where \( \hat{\rho}_f(t) = \sum_m |n\rangle \rho_{nm}(t) \langle m| \) for the field in photon number basis. After some calculations using \( \langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{a_n^\dagger}{\sqrt{n!}} \) and \( \int \beta^m \beta^n e^{-|\beta|^2} e^{\beta^* \alpha - \alpha^* \beta} d^2\beta = \pi L_{n}^{m-n}(2|\alpha|^2) n! e^{-2|\alpha|^2} (2\alpha)^{m-n} \) we finally obtain

\[ W(\alpha, \alpha^*, t) = \frac{2 e^{-2|\alpha|^2}}{\pi} \left[ \sum_{m=0}^{\infty} (-1)^m L_m(2|\alpha|^2) \rho_{nm}(t) + \right. \]

\[ \left. \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} (-1)^n \sqrt{\frac{n!}{m!}} L_n^m(2|\alpha|^2) \{ (2\alpha)^k \rho_{nm}(t) + c.c. \} \right] \]  
(5)

Since initial atomic state is not correlated to the field, \( C_{x,n}(0)C_{y,n'}(0) = C_x(0)C_y(t) \rho_{nm}(0) \) we may write the expression for \( \rho_{nm}(t) \) that is general valid, even for mixed field states

The dynamics of atomic inversion \( n_{ab} = \sum_m \{|C_{a,m}(t)|^2 - |C_{b,m}(t)|^2 \} \) is computed by tracing over the photon number states, using\[^2\]

**Initial coherent state**

For coherent state, \( |\alpha_0\rangle = \sum_n f_n |n\rangle \) with \( f_n = e^{-|\alpha_0|^2/2} \frac{a_n^\dagger}{\sqrt{n!}} \). For initial Schrödinger’s cat states \( |\psi\rangle_{\text{cat}} = N(|\alpha_0\rangle \pm e^{i\phi} | -\alpha_0\rangle) = \sum_n f_n |n\rangle \) with \( N = \frac{1}{\sqrt{2(1+e^{-2|\alpha|^2} \cos \phi)}} \), we have \( f_n = e^{-|\alpha|^2/2} \frac{a_n^\dagger \pm (-\alpha_0)^n}{\sqrt{n!}} \). The states are also referred to as even/odd superposition of coherent states. Thus, in general \( \hat{\rho}_f = |\alpha_0\rangle \langle \alpha_0| = e^{-|\alpha|^2} \sum_n \frac{a_n a_n^\dagger}{\sqrt{n!}} |n\rangle \langle m| = \sum_n \rho_{nm}(0) |n\rangle \langle m| \) gives

\[ \rho_{nm}(0) = f_n f_m^\dagger \]  
(6)

where \( \rho_{nm}(0) = e^{-|\alpha|^2} \frac{a_n a_n^\dagger}{\sqrt{n!}} \) and \( e^{-|\alpha|^2} \frac{a_n^\dagger \pm (-\alpha_0)^n}{\sqrt{n!}} \) for coherent state and the Schrödinger’s cat states, respectively. Here, \( \alpha_0 = r e^{i\theta} \) and \( r \) determines the distance between the centroid of the Wigner function and the origin \( \alpha = 0 \) while \( \theta \) determines the angle with respect to Re(\( \alpha \)).

**Fields with diagonal photon number**

For fields with diagonal number states,
\[ \rho_{nm}(t) = \delta_{mn}\{|C_a(0)r_m|^2 + |C_b(0)r_{m-1}|^2\}p_m(0) + |C_a(0)q_{m-1}|^2p_{m-1}(0) + |C_b(0)q_m|^2p_{m+1}(0) \]

\[ + \delta_{n,m-1}C_a(0)C_b(0)\{r_{n-1}q_{m-1}\}p_{m-1}(0) - r_n^*q_np_m(0) \]

where we have abbreviated the diagonal matrix elements of the field by \( p_{m-1}(0) = p_{m-1,1}(0) \).

Note that at \( t = 0 \) we have \( W(\alpha, \alpha^*, 0) = \frac{2e^{-|\alpha|^2}}{\pi} \sum_n (-1)^n L_n^0(|2\alpha|^2)p_n(0) \). However, for finite \( t \) the emission and absorption processes develop coherences between photon numbers, which are accounted by the second line when \( m = n + 1 \).

When \( C_a(0) = 1 \) only the diagonal elements contribute \( \rho_{nm}(t) = \delta_{mn}\{|r_n(t)|^2p_n(0) + q_{n-1}(t)^2p_{n-1}(0)\} \) which gives

\[ W(\alpha, \alpha^*, t) = \frac{2e^{-|\alpha|^2}}{\pi} \sum_n (-1)^n L_n^0(|2\alpha|^2) \times \]

\[ [r_n(t)|^2p_n(0) + q_{n-1}(t)^2p_{n-1}(0)] \quad (8) \]

For thermal state \( p_n(0) \) is \( \frac{(1 - e^{-\beta \hbar \nu})e^{-n\beta \hbar \nu}}{(1 + \beta \hbar \nu)^n} \) where \( \beta = \frac{\hbar \nu}{k_BT} \). When \( \beta \) is large, \( n_{-1} \approx q_n \) and \( p_n(0) \approx p_{n-1}(0) \) are good approximations, and since \( r_n(t) = \cos(\frac{\Omega t}{2}) \), \( q_n(t) = \sin(\frac{\Omega t}{2}) \) for \( \Delta = 0 \), the Wigner function becomes essentially time independent \( W \approx \frac{2e^{-|\alpha|^2}}{\pi(1 + \beta \hbar \nu)} \sum \infty (-\frac{\beta}{1 + \beta \hbar \nu})^n L_n^0(|2\alpha|^2) \).

**Results and Discussions**

We find some interesting properties by comparing the dynamics of the inversion \( n_{ab}(t) \) with the pattern in the Wigner function. For initial coherent state, Fig. 1 shows that the field evolves into the Schrödinger cat state \( \langle \Psi \rangle_{cat} = N(|\alpha_0 \rangle \pm e^{i\phi} - \alpha_0 \rangle) \) when the atomic inversion collapse. Points where \( n_{ab} = 0 \), the Wigner patterns show the features that only resemble the Wigner pattern for the cat state, i.e., two main peaks across \( \alpha = 0 \) with interference fringes in between. However, in the region of atomic inversion collapse, i.e., \( n_{ab} = 0 \) for a finite period of time \( (gt = 2.6 - 6.2) \) in Fig. 1, we find that the Wigner pattern is identical to the ideal Wigner pattern of the Schrödinger cat. This suggest that the atomic collapse phase is associated with photon entanglement.

Let us now consider the initial Schrödinger’s cat state \( |\Psi \rangle_{cat} \). Figure 2 shows that the two main peaks of the cat state "dissolve" initially with time as the inversion begins to collapse. The main peaks reappear as the result of rephasing, when the inversion revives, as shown for \( gt = 7.2 \). Even though \( n_{ab} = 0 \) at \( gt = 7.45 \), the Wigner pattern shows the cat state since the atom is in the revival state. This situation is reverse from the case with initial coherent state in Fig. 1. Thus, the cat state is not always associated with the collapse phase. Also, in general, the overall shape of the Wigner pattern depends on the envelope of the inversion while the detailed signs of the peaks in the oscillations depend on the sign of the inversion peaks.

We now consider the atom to be initially in the superposition of internal states, \( C_a(0) = C_b(0) = \frac{1}{\sqrt{2}} \) while the field is still in the initial cat state. As shown in Fig. 3 the cat state is found only during the collapse phase, which is the same as the initial phase. Base on these observations, it may be conjectured that, the state of the field returns to or close to the initial state, when the phase returns to the initial phase at \( t = 0 \). In other words, the revived field state corresponds to the initial atomic phase.

In the case of thermal field (Fig. 1), the inversion is connected to the Wigner function in a more sophisticated way and yet in an orderly manner. Although there seems to be no region of collapse, the points in \( n_{ab} \) can be categorized three groups. First, at points with \( n_{ab} = 0 \), the Wigner function shows an alternation between three distinct Wigner patterns which we designate as \( \nabla \), \( \square \) and \( \Delta \). Second, at the positive peaks of \( n_{ab} \), the Wigner pattern has only one typical form, the \( \Delta \) which looks like the "Borobodur". Third, the negative peaks of \( n_{ab} \) correspond to another typical Wigner pattern, the \( \nabla \).

From the results we may conjecture two points in summary. First, an entangled field state can be controlled by the atomic dynamics. Second, the atomic dynamics contain information about the nature the fields. Verification of these results may involve application of tomography technique 20 to map out the transient evolution of the Wigner function in connection with the collapse-revival dynamics.
**FIG. 2:** (Color online) Initial field in Schrodinger cat state while atom is excited. Parameters are $C_a(0) = 1, C_b(0) = 0$, $\alpha_0 = \sqrt{5}$, $\phi = 0$.

**FIG. 3:** (Color online) Initial field in Schrodinger cat state while atom is in superposition of excited and ground states. Parameters are $C_a(0) = C_b(0) = 1/\sqrt{2}$, $\alpha_0 = \sqrt{5}$, $\phi = 0$.

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FIG. 4: (Color online) Initial thermal state. Parameters are $C_a(0) = 1, \bar{n} = 1$. 