Abstract. We study (support) \(\tau\)-tilting modules over the trivial extensions of finite dimensional algebras. More precisely, we construct two classes of (support) \(\tau\)-tilting modules in terms of the adjoint functors which extend and generalize the results on (support) \(\tau\)-tilting modules over triangular matrix rings given by Gao-Huang.

1. Introduction

In 2014, Adachi, Iyama and Reiten [AIR] introduced the \(\tau\)-tilting theory as a generalization of the classical tilting theory in terms of mutations. It is shown that \(\tau\)-tilting theory are closed related to cluster tilting theory and silting theory. In \(\tau\)-tilting theory, support \(\tau\)-tilting modules are very essential. Therefore it is interesting to classify support \(\tau\)-tilting modules for given algebras. Many scholars focus on this topics. Mizuno [Mi] classified support \(\tau\)-tilting modules over preprojective algebras; Adachi [A1] classified support \(\tau\)-tilting modules over Nakayama algebras; Adachi [A2] and Zhang [Z] studied \(\tau\)-rigid modules over algebras of radical square zero; F. Eisele, G. Janssens and T. Raedschelders [EJR] studied the \(\tau\)-rigid modules over special biserial algebras; Iyama and Zhang [IZ] classified the support \(\tau\)-tilting modules over the Auslander algebra of \(K[x]/(x^n)\); Zito [Zi] studied the support \(\tau\)-tilting modules over cluster-tilted algebras. In particular, Gao and Huang [GH] studied support \(\tau\)-tilting modules over lower triangular matrix rings; Peng, Ma and Huang [PMH] also considered support \(\tau\)-tilting modules over lower triangular matrix rings. Moreover, Zhang [Zh] studied the support \(\tau\)-tilting modules over lower triangular matrix rings and generalized the results of Gao and Huang in terms of recollement of abelian categories.

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On the other hand, the trivial extension of algebras can be back to Fossum-Griffith-Reiten’s trivial extension of abelian categories [FGR]. Later it has gained attention of algebraists. Tachikawa [T] classified the indecomposable representation of the trivial extension by $\mathbb{D}A$ over hereditary algebras of finite representation type. Later Yamagata [Y] studied the trivial extension by $\mathbb{D}A$ over tilted algebras of Dynkin type. Moreover, Happel [H] built a connection between the bounded derived category and the trivial extensions of algebras. Recently Assem, Brustle, Schiffler and Todrov [ABST] showed that some special trivial extension of algebras are closed related to cluster categories. For applications of trivial extension of abelian categories in ring theory, we refer to [AnF, M].

In the present paper, we study the support $\tau$-tilting modules over trivial extensions of finite dimensional algebras in terms of adjoint pairs. We have the following main result (See subsection 2.3 for the definitions of the functors $T$ and $Z$).

**Theorem 1.1.** (Theorem 3.8) Let $A$ be a finite dimensional algebra over a field $K$ and let $\Lambda$ be the trivial extension of $A$ by a finitely generated $A$-bimodule $M$. For a pair $(X, P) \in A$-mod, we have

1. $(T(X), T(P))$ is a support $\tau$-tilting pair in $\Lambda$-mod if and only if $(X, P)$ is a support $\tau$-tilting pair in $A$-mod, $\text{Hom}_A(P, M \otimes_A X) = 0$ and $\text{Hom}_A(M \otimes_A X, \tau X) = 0$.

2. $(Z(X), T(P))$ is a support $\tau$-tilting pair in $\Lambda$-mod if and only if $(X, P)$ is a support $\tau$-tilting pair $A$-mod and $\text{Hom}_A(Q, X) = 0$, where $Q$ is a projective cover of $M \otimes_A X$.

As a result of Theorem 1.1, we can generalize and extend the results on support $\tau$-tilting modules over lower triangular matrix rings by Gao-Huang [GH, Theorem 4.3]. Moreover, we have the following results on $\tau$-rigid modules over trivial extensions and lower triangular matrix algebras.

**Theorem 1.2.** (Proposition 3.3 Corollary 3.4) (1) Let $\Lambda$ be the trivial extension of a finite dimensional algebra $A$ by a finitely generated bimodule $M$. If $M \otimes_R X \xrightarrow{\alpha} X$ is a $\tau$-rigid module in $\Lambda$-mod, then both $X$ and $\text{coker} \alpha$ are $\tau$-rigid modules in $A$-mod.

(2) Let $R$ and $S$ be finite dimensional algebras and $S M_R$ a finitely generated bimodule. Let $\Lambda$ be the lower triangular matrix algebra $(\frac{R}{M} \frac{0}{S})$. If $(0, M \otimes_R X) \xrightarrow{(0, \alpha)} (X, Y)$ is a $\tau$-rigid module in $\Lambda$-mod, then $X$ is a $\tau$-rigid module in $R$-mod and $\text{coker} \alpha$ is a $\tau$-rigid module in $S$-mod.

Now we state the organization of the paper as follows:
In Section 2, we recall the basic results on trivial extensions of rings. In Section 3, we study the \( \tau \)-rigid modules and support \( \tau \)-tilting modules over trivial extensions of algebras and prove Theorems 1.1 and 1.2. In Section 4, we give examples to illustrate our main results.

Throughout this paper, \( \tau \) is theAuslander-Reiten translation functor. For an algebra \( \Lambda \), we use \( \Lambda \)-mod to denote the category of finitely generated left \( \Lambda \)-modules.

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2. THE CATEGORIES OF MODULES OVER TRIVIAL EXTENSIONS OF RINGS

In this section, we recall some basic results on modules over trivial extensions of rings.

2.1. Trivial extensions of rings. Let \( R \) be a ring with 1 and \( M \) an \( R \)-bimodule. Denote by \( R \ltimes M \), called the trivial extension of \( R \) by \( M \), to be the ring whose additive group is the direct sum \( R \times M \) with multiplication given by \((r,m)(r',m') = (rr', rm' + mr')\).

It is well-known that a left \( R \ltimes M \)-module is a morphism \( M \otimes_R X \xrightarrow{\alpha} X \) such that the composition \( M \otimes_R M \otimes_R X \xrightarrow{M \otimes_R \alpha} M \otimes_R X \xrightarrow{\alpha} X \) is zero. If \( M \otimes_R X \xrightarrow{\alpha} X \) and \( M \otimes_R Y \xrightarrow{\beta} Y \) are two left \( R \ltimes M \)-modules, then a morphism \( f: \alpha \to \beta \) is a morphism \( f: X \to Y \) such that the diagram

\[
\begin{array}{ccc}
M \otimes_R X & \xrightarrow{\alpha} & X \\
\downarrow{1_M \otimes f} & & \downarrow{f} \\
M \otimes_R Y & \xrightarrow{\beta} & Y
\end{array}
\]

is commutative. The composition of morphisms coincides with the composition in the category of left \( R \)-modules.

In the sequel, sometimes we use a pair \((X, \alpha)\) to denote the left \( R \ltimes M \)-module \( M \otimes_R X \xrightarrow{\alpha} X \).
2.2. The lower triangular matrix rings. Let $R$ and $S$ be rings and let $SM_R$ be a bimodule. Then the lower triangular matrix ring \( \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \) is the ring with addition
\[
(\begin{pmatrix} r & 0 \\ m & s \end{pmatrix}) + (\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix}) = (\begin{pmatrix} r + r' & 0 \\ m + m' & s + s' \end{pmatrix})
\]
and product
\[
(\begin{pmatrix} r & 0 \\ m & s \end{pmatrix})(\begin{pmatrix} r' & 0 \\ m' & s' \end{pmatrix}) = (\begin{pmatrix} rr' & 0 \\ m'r + sm' & ss' \end{pmatrix}).
\]
Recall that the lower triangular matrix ring \( \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \) is isomorphic to the trivial extension \( R \times S \), where the right \( R \times S \)-module structure of \( M \) is given via \( R \times S \rightarrow R \) and the left \( R \times S \)-module structure of \( M \) is given via \( R \times S \rightarrow S \). A left \( R \times S \)-module is an ordered pair \( (X, Y) \) with \( X \in R\text{-Mod} \) and \( Y \in S\text{-Mod} \). In fact, if \( L \) is a left \( R \times S \)-module, then \( X = (1, 0)L \) is a left \( R \)-module and \( Y = (0, 1)L \) is a left \( S \)-module. Conversely, given a pair \( (X, Y) \) in \( (R\text{-Mod}, S\text{-Mod}) \), define \( (r, s)(x, y) = (rx, sy) \), then \( (X, Y) \) obtains a structure of left \( R \times S \)-module. Therefore, a left \( \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \)-module is of the form \( M \otimes_{R \times S} (X, Y) = (0, M \otimes_{R} X) \xrightarrow{(0,\alpha)} (X, Y) \).

Hence, a left \( \begin{pmatrix} R & 0 \\ M & S \end{pmatrix} \)-module is determined uniquely by a morphism \( M \otimes_{R} X \xrightarrow{\alpha} Y \) of left \( S \)-modules.

2.3. Adjoint pairs. For simplicity, we use \( F \) to denote the tensor functor \( M \otimes_{R} - \). There are pairs of adjoint pairs \( (T, U) \) and \( (C, Z) \), see [FGR, Proposition 1.3].

The functor \( T \) is defined on objects by
\[
T(N) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : F(N) \oplus F^2(N) \rightarrow N \oplus F(N)
\]
and on morphisms by
\[
T(f) = \begin{pmatrix} f & 0 \\ 0 & F(f) \end{pmatrix}, f : N \rightarrow L.
\]
Thus, given a left \( R \times M \)-module \( (X, \alpha) \), it is easy to get that
\[
\text{Hom}_{R \times M}(T(P), (X, \alpha)) = \{(f, \alpha F(f)) \mid f \in \text{Hom}_{R}(P, X)\}
\]
for any left \( R \)-module \( P \).

The functor \( U \) is defined by \( U(F(N) \xrightarrow{\alpha} N) = N \) and \( U(f) = f \). The functor \( Z \) is given by \( Z(N) = F(N) \xrightarrow{\alpha} N \) and \( Z(f) = f \). The functor \( C \) is defined by \( C(F(N) \xrightarrow{\alpha} N) = \text{cok} \alpha \) and \( C(f) \) is the induced morphism.

Recall that an epimorphism \( f : N \rightarrow L \) in an abelian category is called minimal if for any morphism \( g : N' \rightarrow N \), \( fg \) is surjective implies \( g \) is an epimorphism.

Lemma 2.1. [FGR, Proposition 1.5] The functors \( T \) and \( C \) are right exact and \( U \) and \( Z \) are exact. Moreover
• $CT = \text{Id}$;
• $UZ = \text{Id}$;
• the unit $\eta: \text{Id} \to ZC$ is a minimal epimorphism.

Remark 2.2. Let $R$ be a finite dimensional algebra and $M$ a finitely generated $R$-bimodule. For any $N \in R$-$\text{mod}$, the morphism $(1, 0): T(N) \to Z(N)$ is a minimal epimorphism. So the projective covers of $T(N)$ and $Z(N)$ coincide.

2.4. The projective modules of trivial extensions of rings. By [FGR, Corollary 1.6], a left $R \ltimes M$-module $M \otimes_R X \xrightarrow{\rho} X$ is projective if and only if $C(\rho)$ is a projective $R$-module and $\rho \cong T(C(\rho))$.

Consider the exact sequence

$$F(X) \xrightarrow{\alpha} X \xrightarrow{\pi} C(\alpha) \to 0.$$ 

Since $P$ is projective, there is a morphism $q: P \to X$ such that $\pi q = p$. Let $\omega$ denote the morphism $(q, \alpha F(q)): P \oplus F(P) \to X$. Since $\alpha F(\alpha) = 0$, we have that $(q, \alpha F(q)) \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} = \alpha(F(q), F(\alpha)F^2(q))$. Hence $\omega$ is a morphism from $T(P)$ to $(X, \alpha)$ in $R \ltimes M$-$\text{Mod}$. Now we recall the following lemmas on the projective presentation of a finitely generated module over trivial extensions.

Lemma 2.3. [FGR, Corollary 1.7(1)] Let $R$ be a finite dimensional algebra and $M$ a finitely generated $R$-bimodule. Let $(X, \alpha)$ be a module in $R \ltimes M$-$\text{mod}$. If $P$ is the projective cover of $C(\alpha)$, then $T(P)$ is the projective cover of $(X, \alpha)$.

Lemma 2.4. [M, Proposition 2.8] Let $R$ be a finite dimensional algebra with $X \in R$-$\text{mod}$ and $M$ a finitely generated $R$-bimodule. If $Q \xrightarrow{\xi} P \xrightarrow{\zeta} X \to 0$ is a minimal projective presentation of $X$, then $T(Q) \xrightarrow{T(\xi)} T(P) \xrightarrow{T(\zeta)} T(X) \to 0$ is a minimal projective presentation of $T(X)$ in $R \ltimes M$-$\text{mod}$.

3. Support $\tau$-tilting modules

In this section, we study the support $\tau$-tilting modules over trivial extensions of algebras in terms of adjoint functors. As a result, we can give two construction methods for support $\tau$-tilting modules over lower triangular matrix algebras.

In what follows, we assume that $A$ is a finite dimensional algebra and all modules are finitely generated. Denote by $A \ltimes M$ the trivial extension of $A$ by a bimodule $M$. Let the functors $F, T, C, U, Z$ be as in Section 2. For a left $A$-module $N$, denote by $\vert N \vert$ the number of non-isomorphic indecomposable direct summands of $N$.

We first recall the definition of support $\tau$-tilting modules in [AIR].
Definition 3.1. Let $N$ be a left $A$-module.
(1) $N$ is called $\tau$-rigid if $\text{Hom}_A(N, \tau N) = 0$.
(2) $N$ is called $\tau$-tilting if $N$ is $\tau$-rigid and $|N| = |A|$.
(3) $N$ is called support $\tau$-tilting if $N$ is $\tau$-tilting over the algebra $A/(e)$, where $e$ is an idempotent of $A$.

We also need the following characterization of $\tau$-rigid modules.

Lemma 3.2. [AIR, Proposition 2.4 (3)] Let $X$ be a left $A$-module and let $P_1 \xrightarrow{f} P_0 \to X \to 0$ be a minimal projective presentation. Then $X$ is a $\tau$-rigid module if and only if $\text{Hom}_A(f, X)$ is surjective.

We begin with a necessary condition on $\tau$-rigid $\Lambda$-modules.

Proposition 3.3. Let $(X, \alpha)$ be a left $\Lambda$-module. If $(X, \alpha)$ is a $\tau$-rigid module, then $\text{cok}\alpha$ is a $\tau$-rigid $A$-module.

Proof. For $(X, \alpha)$, by [FGR, Corollary 1.6] we may assume its minimal projective presentation to be of the form
\[
T(P_1) \xrightarrow{(x \ y \ F(x))} T(P_0) \xrightarrow{\zeta=(q,\alpha F(q))} (X, \alpha) \to 0.
\]
By Lemma 3.2, $(X, \alpha)$ is a $\tau$-rigid module if and only if $\text{Hom}_\Lambda((x \ y \ F(x)), (X, \alpha))$ is surjective if and only if for any $f \in \text{Hom}_A(P_1, X)$ there is $g \in \text{Hom}_A(P_0, X)$ such that $f = gx + \alpha F(g)y$ by the equality $(\ast)$ in Section 2.

Since the functor $C$ is right exact, it follows that $P_1 \xrightarrow{\pi x} P_0 \xrightarrow{C(\zeta)} \text{cok}\alpha \to 0$ is a projective presentation of $\text{cok}\alpha$. Moreover $p := C(\zeta)$ is right minimal by Lemma 2.3. We write $x = (r, 0): P_1 = P_1' \oplus P_1'' \to P_0$, where $r$ is right minimal. Then the sequence
\[
P_1' \xrightarrow{\pi r} P_0 \xrightarrow{p} \text{cok}\alpha \to 0
\]
is a minimal projective presentation of $\text{cok}\alpha$.

By Lemma 3.2, $\text{cok}\alpha$ is a $\tau$-rigid $A$-module if and only if $\text{Hom}_A(r, \text{cok}\alpha)$ is surjective. If $f$ is a morphism from $P_1'$ to $\text{cok}\alpha$, then there exists a morphism $f' \colon P_1' \to X$ such that $f = \pi f'$ because $P_1'$ is projective and $\pi \colon X \to \text{cok}\alpha$ is surjective. We extend $f'$ to a morphism $(f', 0) \colon P_1 = P_1' \oplus P_1'' \to X$. Since $(X, \alpha)$ is a $\tau$-rigid $\Lambda$-module, by the previous discussion, there is a morphism $g \colon P_0 \to X$ such that $(f', 0) = gx + \alpha F(g)y$. So we have $(f, 0) = \pi(f', 0) = \pi gx + \pi \alpha F(g)y = (\pi gr, 0)$. In particular, $f = \pi gr$, which completes the proof. \[\square\]
Let $R$ and $S$ be finite dimensional algebras and let $SM_R$ be a finitely generated $S$-$R$-bimodule. Let $\Lambda$ be the lower triangular matrix algebra $(\frac{R}{M} \otimes_S S)$. Then $\Lambda$ is isomorphic to the trivial extension $(R \times S) \ltimes M$ of $R \times S$ by $M$. By Proposition 3.3, we can give a necessary condition for $\tau$-rigid $\Lambda$-modules over lower triangular matrix algebras.

**Corollary 3.4.** Let $\Lambda$ be the lower triangular matrix algebra $(\frac{R}{M} \otimes_S S)$.

If $(0, M \otimes_R X) \xrightarrow{(0,\alpha)} (X,Y)$ is a $\tau$-rigid $\Lambda$-module, then $X$ is a $\tau$-rigid $R$-module and $\text{cok}\alpha$ is a $\tau$-rigid $S$-module.

We can now formulate the following results on $\tau$-rigid modules.

**Proposition 3.5.** Let $X$ be a left $A$-module. Then

1. $T(X)$ is a $\tau$-rigid $\Lambda$-module if and only if $X$ is a $\tau$-rigid $A$-module and $\text{Hom}_A(M \otimes_A X, \tau X) = 0$.

2. $Z(X)$ is a $\tau$-rigid $\Lambda$-module if and only if $X$ is a $\tau$-rigid $A$-module and $\text{Hom}_A(Q, X) = 0$ where $Q$ is a projective cover of $M \otimes_A X$.

**Proof.** (1) Let $P_1 \xrightarrow{f} P_0 \rightarrow X \rightarrow 0$ be a minimal projective presentation of $X$. Then $T(Q) \xrightarrow{T(f)} T(P) \rightarrow T(X) \rightarrow 0$ is a minimal projective presentation of $T(X)$ in $\Lambda$-mod by Lemma 2.4. Therefore $T(X)$ is a $\tau$-rigid module if and only if $\text{Hom}_\Lambda(T(f), T(X))$ is surjective. Notice that $(T, U)$ is an adjoint pair, this is equivalent to that $\text{Hom}_\Lambda(f, X \oplus M \otimes X)$ is surjective. Since $\text{Hom}_\Lambda(f, X \oplus M \otimes X) = \text{Hom}_\Lambda(f, X) \oplus \text{Hom}_\Lambda(f, M \otimes X)$, it follows that $T(X)$ is a $\tau$-rigid $\Lambda$-module if and only if $X$ is a $\tau$-rigid $A$-module and $\text{Hom}_\Lambda(M \otimes_A X, \tau X) = 0$ by Lemma 3.2 and [AIR] Proposition 2.4 (2).

(2) Let $P_1 \xrightarrow{f} P_0 \xrightarrow{v} X \rightarrow 0$ be a minimal projective presentation of $X$. Suppose $q: Q \rightarrow F(X)$ is a projective cover. Since $F$ is right exact, the morphism $F(p)$ is an epimorphism. Therefore there is a morphism $r: Q \rightarrow F(P_0)$ such that $F(p)r = q$.

Since $CZ(X) = X$ and $p: P_0 \rightarrow X$ is a projective cover, we know that $T(P_0) \xrightarrow{(p,0)} Z(X)$ is a projective cover. Let $u: \ker p \rightarrow P_0$ be the kernel of $p$ and $v: P_1 \rightarrow \ker p$ the projective cover such that $f = uv$. Notice that

$$(\begin{smallmatrix} u & 0 \\ 0 & 0 \end{smallmatrix}) : (\ker p \oplus F(P_0), (\begin{smallmatrix} 0 \\ F(u) \end{smallmatrix} 0)) \rightarrow T(P_0)$$

is the kernel of $(p,0)$. Since $\begin{pmatrix} u & 0 \\ 0 & q \end{pmatrix} : P_1 \oplus Q \rightarrow \ker p \oplus F(X) = C(\begin{smallmatrix} 0 \\ F(u) \end{smallmatrix} 0)$ is a projective cover, we know that

$$(\begin{smallmatrix} u & 0 \\ 0 & F(f) \end{smallmatrix} 0) : T(P_1 \oplus Q) \rightarrow (\ker p \oplus F(P_0), (\begin{smallmatrix} 0 \\ F(u) \end{smallmatrix} 0))$$
is a projective cover. Thus
\[ T(P_1 \oplus Q) \xrightarrow{(f \, 0 \, 0 \, 0 \, 0 \, r \, F(f) \, 0)} T(P_0) \xrightarrow{(p,0)} Z(X) \rightarrow 0 \]
is a minimal projective presentation. Therefore \( Z(X) \) is a \( \tau \)-rigid module if and only if \( \text{Hom}_\Lambda((f,0),X) = \text{Hom}_\Lambda(f,X) \) is surjective. Notice that \((C,Z)\) is an adjoint pair, this is equivalent to that \( \text{Hom}_A((f,0),X) = \text{Hom}_A(f,X) \). Since \( \text{Hom}_A((f,0),X) = \text{Hom}_A(f,X) \), it follows that \( Z(X) \) is a \( \tau \)-rigid \( \Lambda \)-module if and only if \( X \) is a \( \tau \)-rigid \( A \)-module and \( \text{Hom}_A(Q,X) = 0 \) by Lemma 3.2.

Using Proposition 3.5, we can get the following result.

**Corollary 3.6.** Let \( \Lambda \) be the lower triangular matrix algebra \( \left( \begin{array}{cc} R & 0 \\ M & S \end{array} \right) \). Then

1. \([\text{GH \ Proposition 4.1}]\) \( (0,M \otimes_R X) \xrightarrow{(0,(\, 0 \))} (X,Y \oplus M \otimes_R X) \) is a \( \tau \)-rigid \( \Lambda \)-module if and only if \( X \) is a \( \tau \)-rigid \( R \)-module, \( Y \) is a \( \tau \)-rigid \( S \)-module and \( \text{Hom}_S(M \otimes_R X, \tau Y) = 0 \).

2. \((0,M \otimes_R X) \xrightarrow{(0,0)} (X,Y) \) is a \( \tau \)-rigid \( \Lambda \)-module if and only if \( X \) is a \( \tau \)-rigid \( R \)-module, \( Y \) is a \( \tau \)-rigid \( S \)-module and \( \text{Hom}_S(Q,Y) = 0 \), where \( Q \) is a projective cover of \( M \otimes_R X \).

In the following we focus on the support \( \tau \)-tilting modules over trivial extensions of algebras. We need the following lemma on the number of non-isomorphic indecomposable direct summands of modules.

**Lemma 3.7.** Let \( \Lambda \) be the trivial extension of \( A \) by \( M \) and \( X \) a left \( A \)-module.

1. The number of indecomposable direct summands of \( X \) is equal to that of \( T(X) \), that is, \( |T(X)| = |X| \).

2. The number of indecomposable direct summands of \( X \) is equal to that of \( Z(X) \), that is, \( |Z(X)| = |X| \).

3. \( \Lambda \simeq T(A) \) as a \( \Lambda \)-module, and hence \( |A| = |\Lambda| \) holds.

**Proof.** (1) Since \( CT = \text{Id} \), it follows that \( T \) induces an equivalence \( T : \text{add} X \rightarrow \text{add} T(X) \) with a quasi-inverse given by \( C : \text{add} T(X) \rightarrow \text{add} X \). In particular, this implies the number of indecomposable direct summands of \( T(X) \) equals that of \( X \).

(2) Since \( UZ = \text{Id} \), it follows that \( Z \) induces an equivalence \( Z : \text{add} X \rightarrow \text{add} Z(X) \) with a quasi-inverse given by \( U : \text{add} Z(X) \rightarrow \text{add} X \). In particular, this implies the number of indecomposable direct summands of \( Z(X) \) equals that of \( X \).

(3) This is a straight result of (1).
Recall from [AIR] that a pair \((X, P) \in A\text{-mod}\) is called a \(\tau\)-rigid pair if \(X\) is \(\tau\)-rigid, \(P\) is projective and \(\text{Hom}_A(P, X) = 0\). Moreover, a pair \((X, P) \in A\text{-mod}\) is called a support \(\tau\)-tilting pair if it is a \(\tau\)-rigid pair and \(|P| + |X| = |A|\). Now we are in a position to state the main results in this paper.

**Theorem 3.8.** Let \(\Lambda\) be the trivial extension of \(A\) by a bimodule \(M\) and let \((X, P)\) be a pair in \(A\text{-mod}\). Then

1. \((T(X), T(P))\) is a support \(\tau\)-tilting pair in \(A\text{-mod}\) if and only if \((X, P)\) is a support \(\tau\)-tilting pair in \(A\text{-mod}\), \(\text{Hom}_A(P, M \otimes_A X) = 0\) and \(\text{Hom}_A(M \otimes_A X, \tau X) = 0\).

2. \((Z(X), T(P))\) is a support \(\tau\)-tilting pair in \(A\text{-mod}\) if and only if \((X, P)\) is a support \(\tau\)-tilting pair in \(A\text{-mod}\) and \(\text{Hom}_A(Q, X) = 0\), where \(Q\) is a projective cover of \(M \otimes_A X\).

**Proof.** (1) \(\Leftarrow\) Since \(X\) is \(\tau\)-rigid and \(\text{Hom}_A(M \otimes_A X, \tau X) = 0\), by Proposition 3.5 one gets \(T(X)\) is \(\tau\)-rigid in \(A\text{-mod}\).

(a) We show \((T(X), T(P))\) is a \(\tau\)-rigid pair. Since \((X, P)\) is a support \(\tau\)-tilting pair, one gets \(\text{Hom}_A(P, X) = 0\) which implies \(\text{Hom}_A(T(P), T(X)) = 0\). The assertion holds.

(b) It remains to show that \(|T(P)| + |T(X)| = |A|\). By Lemma 3.7 one gets \(|T(P)| = |P|, |T(X)| = |X|\) and \(|A| = |A|\). Since \((X, P)\) is a support \(\tau\)-tilting pair, then \(|P| + |X| = |A|\) implies that \(|T(P)| + |T(X)| = |A|\).

\(\Rightarrow\) Since \(T(X)\) is a \(\tau\)-rigid module, by Proposition 3.5 one gets that \(X\) is \(\tau\)-rigid and \(\text{Hom}_A(M \otimes_A X, \tau X) = 0\). Since \((T(X), T(P))\) is a support \(\tau\)-tilting pair, then \(\text{Hom}_A(T(P), T(X)) = 0\) implies \(\text{Hom}_A(P, X) = 0\). By Lemma 3.7 one gets \(|T(P)| = |P|, |T(X)| = |X|\) and \(|A| = |A|\). Then \(|T(P)| + |T(X)| = |A|\) implies that \(|P| + |X| = |A|\).

(2) \(\Leftarrow\) Since \(X\) is a \(\tau\)-rigid \(A\text{-module}\) and \(\text{Hom}_A(Q, X) = 0\), where \(Q\) is a projective cover of \(M \otimes_A X\), one gets \(Z(X)\) is \(\tau\)-rigid by Proposition 3.5. We show that \((Z(X), T(P))\) is a support \(\tau\)-tilting pair.

(a) We show \(\text{Hom}_A(T(P), Z(X)) = 0\). Assume that we have the following commutative diagram:

\[
\begin{array}{c}
F(P) \oplus F^2(P) \xrightarrow{\alpha} P \oplus F(P) \\
\downarrow (F(m), F(n)) \quad \downarrow (m, n) \\
F(X) \quad \xrightarrow{0} X
\end{array}
\]
where \( \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then \((m, n)\alpha = 0(F(m), F(n))\) implies \(n = 0\). Since \(\text{Hom}_A(P, X) = 0\), we get \(m = 0\) and hence \((m, n) = 0\). We are done.

(b) We show that \(|Z(X)| + |T(P)| = |\Lambda|\). By Lemma 3.7 one gets \(|T(P)| = |P|, |Z(X)| = |X|\) and \(|\Lambda| = |A|\). The assertion follows from the fact \(|X| + |P| = |A|\).

\[ \Rightarrow \text{Since } Z(X) \text{ is } \tau \text{-rigid, then by Proposition 3.5 } X \text{ is } \tau \text{-rigid and } \text{Hom}_A(Q, X) = 0, \text{ where } Q \text{ is the projective cover of } M \otimes_A X. \text{ Now it suffices to show that } (X, P) \text{ is the support } \tau \text{-tilting pair.} \]

(a) We show that \(\text{Hom}_A(P, X) = 0\). By the equation \((*)\) in Section 2, one gets that

\[ \text{Hom}_A(T(P), Z(X)) = \{(f, 0) | \forall f \in \text{Hom}_A(P, X)\}. \]

Since \((Z(X), T(P))\) is a support \(\tau\)-tilting pair, then \(\text{Hom}_A(T(P), Z(X)) = 0\) implies \(\text{Hom}_A(P, X) = 0\).

(b) We show that \(|P| + |X| = |A|\). By Lemma 3.7 one gets \(|T(P)| = |P|, |Z(X)| = |X|\) and \(|\Lambda| = |A|\). Since \((Z(X), T(P))\) is a support \(\tau\)-tilting pair, then \(|T(P)| + |Z(X)| = |\Lambda|\) implies that \(|P| + |X| = |A|\). \(\square\)

As a corollary, we get the following results on the support \(\tau\)-tilting modules over lower triangular matrix algebras which extend and generalize [GH, Theorem 4.3].

**Corollary 3.9.** Let \(\Lambda\) be the lower triangular matrix algebra \(\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}\). Then

1. \((0, M \otimes_R X) \xrightarrow{(0, \begin{pmatrix} 0 \\ 1 \end{pmatrix})} (X, Y \oplus M \otimes_R X)\) is a support \(\tau\)-tilting \(\Lambda\)-module if and only if \(X\) is a support \(\tau\)-tilting \(R\)-module, \(Y\) is a support \(\tau\)-tilting \(S\)-module, \(\text{Hom}_S(M \otimes_R X, \tau Y) = 0\) and \(\text{Hom}_S(P, M \otimes_R X) = 0\), where \((Y, P)\) is a support \(\tau\)-tilting pair.

2. \((0, M \otimes X) \xrightarrow{(0, 0)} (X, Y)\) is a support \(\tau\)-tilting \(\Lambda\)-module if and only if \(X\) is a support \(\tau\)-tilting \(R\)-module, \(Y\) is a support \(\tau\)-tilting \(S\)-module and \(\text{Hom}_S(Q, Y) = 0\), where \(Q\) is a projective cover of \(M \otimes_R X\).

### 4. Examples

In this section we give examples to show our main results.

**Example 4.1.** Let \(A = KQ\) with the quiver \(Q : 1 \rightarrow 2\) and \(M = DA\). Let \(\Lambda\) be the trivial extension of \(A\) by \(M\). Then

1. \(\Lambda\) is given by the quiver \(Q' : 1 \xrightarrow{a} 2\) with \(a^3 = 0\), see [C].

2. The indecomposable \(\tau\)-rigid modules in \(A\)-mod are \(P(1), P(2), S(1)\).

3. \(T(P(1)), T(P(2))\) are indecomposable projective \(\Lambda\)-modules and hence \(\tau\)-rigid.
(4) $T(S(1)) = Z(S(1))$ and $Z(S(2))$ are indecomposable non-projective $\tau$-rigid $\Lambda$-modules.

(5) The non-zero support $\tau$-tilting $\Lambda$-modules are: $P(1) \oplus P(2)$, $P(1) \oplus S(1)$, $S(1)$ and $S(2) = P(2)$.

(6) $T(P(1)) \oplus T(P(2))$, $T(P(1)) \oplus T(S(1))$, $T(S(1)) = Z(S(1))$, $Z(S(2))$ are non-zero support $\tau$-tilting $\Lambda$-modules with the form $T(X)$ or $Z(X)$.

(7) There is a support $\tau$-tilting $\Lambda$-module $T(P(2)) \oplus Z(S(2))$ which is neither of the form $T(X)$ nor of form $Z(X)$.

Now we give an example on the support $\tau$-tilting modules over a lower triangular matrix algebra.

**Example 4.2.** Let $A = KQ$ with the quiver $Q : 1 \to 2$ and $M = A$. Let $\Lambda$ be the lower triangular matrix algebra $\left( \begin{smallmatrix} A & 0 \\ \lambda & A \end{smallmatrix} \right)$. Then

1. $\Lambda$ is the trivial extension of $B = \left( \begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right)$ by $A$.

2. The non-zero support $\tau$-tilting $\Lambda$-modules are: $A_1 = P(1) \oplus P(2)$, $A_2 = P(1) \oplus S(1)$, $A_3 = S(1)$ and $A_4 = S(2)$.

3. $T((A_i, 0)) = (0, A_i) \xrightarrow{(0, \chi^i)} (A_i, 0 \oplus A_i)$ is not a support $\tau$-tilting $\Lambda$-module for $i = 1, 2, 3, 4$;

4. $T((A_i, A_i)) = (0, A_i) \xrightarrow{(0, \chi^i)} (A_i, A_i \oplus A_i)$ is a support $\tau$-tilting $\Lambda$-module for $i = 1, 2, 3, 4$;

5. $T((A_i, A_1)) = (0, A_i) \xrightarrow{(0, \chi^i)} (A_i, A_1 \oplus A_i)$ is a support $\tau$-tilting $\Lambda$-module for $i = 2, 3, 4$;

6. $T((A_3, A_2)) = (0, A_3) \xrightarrow{(0, \chi^i)} (A_3, A_2 \oplus A_3)$ is a support $\tau$-tilting $\Lambda$-module;

7. $T((A_2, A_3)) = (0, A_2) \xrightarrow{(0, \chi^i)} (A_2, A_3 \oplus A_2)$ is not a support $\tau$-tilting $\Lambda$-module;

8. $T((A_i, A_4)) = (0, A_i) \xrightarrow{(0, \chi^i)} (A_i, A_4 \oplus A_i)$ is not a support $\tau$-tilting $\Lambda$-module for $i = 1, 2, 3$;

9. $Z(A_i, 0) = (0, A_i) \xrightarrow{(0, 0)} (A_i, 0)$ is a support $\tau$-tilting $\Lambda$-module for $i = 1, 2, 3, 4$;

10. $Z(0, A_i) = (0, 0) \xrightarrow{(0, 0)} (0, A_i) = T(0, A_i)$ is a support $\tau$-tilting $\Lambda$-module for $i = 1, 2, 3, 4$;

11. $Z(A_i, A_3) = (0, A_i) \xrightarrow{(0, 0)} (A_i, A_3)$ is a support $\tau$-tilting $\Lambda$-module;

12. $Z(A_i, A_4) = (0, A_i) \xrightarrow{(0, 0)} (A_i, A_4)$ is a support $\tau$-tilting $\Lambda$-module for $i = 2, 3$. 
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