Insight into neutrino mass phenomenology
by exploring the non-relativistic regime in quantum field theory

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Abstract

We present a novel formulation, based on quantum field theory, for neutrino oscillation phenomenology that can be applied to nonrelativistic and relativistic energies for neutrinos. The formulation is constructed as the time evolution of a lepton family number density operator. We also introduce a Gaussian momentum distribution for the initial state that incorporates wave packet-like decoherence effects. At time \( t = 0 \), we assign either a Dirac or Majorana mass to the neutrino. Then, the time evolution of the lepton family number density operator becomes dependent on the mass and new features appear. We show in the nonrelativistic regime, by taking the expectation value of the density operator, the absolute mass hierarchy and the type of neutrino mass are distinguishable even under the presence of decoherence effects.
I. INTRODUCTION

Neutrinos were originally formulated, within the weak interactions of the Standard Model (SM), to be massless fermions. However, hints of massive neutrinos appeared in the second half of the 20th century with the solar neutrino problem[1]. Eventually, updated solar models and experiments from the Kamiokande laboratory and the Sudbury Neutrino Observatory (SNO) proved the disappearance is caused by flavor oscillations[2, 3]. For neutrinos the existence of flavor oscillations mean there is a mixing between mass and flavor eigenstates. The mixing between mass and flavor is governed by the unitary PMNS matrix, which has six (four) free parameters in the $3 \times 3$ instance[4, 5]. Thus, the mixing caused by flavor oscillations is direct evidence that neutrinos are massive and, consequently, require physics beyond the SM.

In this decade, the six parameters related to the PMNS matrix; the oscillation angles $\theta_{12}, \theta_{23},$ and $\theta_{13},$ the mass squared differences $\Delta m^2_{21}$ and $\Delta m^2_{31},$ and the Dirac CP violating phase $\delta_{\text{cp}},$ are expected to be measured within a few percentages by neutrino oscillation experiments[6–11]. In addition, the absolute mass of the lightest neutrino, $m_1$ or $m_3$, is expected to be directly limited down to the sub-eV range by the Karlsruhe Tritium Neutrino Experiment (KATRIN)[12]. However, even with these precision measurements, questions remain in neutrino phenomenology. Specifically, the question of what is the absolute mass hierarchy for neutrinos, normal $m_1 < m_2 \ll m_3$ or inverted $m_3 \ll m_1 < m_2,$ and the question of what type of mass neutrinos possess, Dirac or Majorana. Those questions could be answered by future neutrinoless double-$\beta$ ($0\nu\beta\beta$) decay experiments[13–17]. Arguably, additional approaches to answer those questions would be ideal.

For this work, we present a unified description of neutrino phenomenology that leads to an additional approach for investigations of the previously mentioned questions. We define unified to mean a description that includes neutrino flavor oscillations and, particle-antiparticle or chiral oscillations. In addition, our description is different from the usual neutrino oscillation theory that assumes neutrinos are relativistic. The relativistic assumption is often taken, because cosmological limits on the neutrinos absolute masses place them in the sub-eV range[13]; whereas, experiments are performed on neutrinos with energies in the KeV, MeV, GeV, and recently the PeV ranges[19]. Nonrelativistic neutrinos are predicted to exist in nature as the cosmic neutrino background (C$\nu$B). Experiments to detect the C$\nu$B are under
preparation (see for example, [20]), and we plan future studies of the CνB as an application of our description.

For studies of the CνB we need to build a framework that uses a momentum distribution [21]. For oscillation experiments, if the neutrino is produced with a localized space a momentum distribution description is necessary. Descriptions of that type have been done using wave packet formulations [22–26]. To incorporate a momentum distribution in our framework, we consider densities of the lepton family numbers that are localized in space. We have previously discussed details of the formulation for the Majorana mass case using plane waves [27].

II. LEPTON NUMBER DENSITY

A. Majorana neutrino formulation

We consider the following Lagrangian \( L^M \) for Majorana neutrinos,

\[
L^M = \bar{\nu}_{L\alpha} \gamma^\mu \partial_\mu \nu_{L\alpha} - \theta(t) \left( \frac{m_{\alpha\beta}}{2} \bar{\nu}_{L\alpha} \nu_{L\beta} \right) + h.c,
\]

in which the first term is kinetic one and the second term is the Majorana mass one. A step-function controls the second term, and initially guarantees a pure lepton family number. A weak interaction produces the neutrinos at \( t = -\epsilon \) for \( \epsilon \to 0^+ \), and we treat them as part of a SU(2)\(_L\) doublet. The SU(2)\(_L\) doublet is used to assign a lepton family number of \( L_e, L_\mu, \) or \( L_\tau \) to the produced neutrino [28]. We then formulate a Heisenberg operator for \( L_\alpha \), where \( \alpha = (e, \mu, \tau) \), that is distributed over space.

\[
L^M_{\alpha}(t) = \int d^3x \, l^M_{\alpha}(t, \mathbf{x}),
\]

where the colon : denotes normal ordering. For \( t = -\epsilon \) with \( \epsilon \to 0^+ \) the neutrinos are massless spinors that follow the usual Fourier expansion,

\[
\lim_{\epsilon \to 0^+} \nu_{L\alpha}(-\epsilon, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|} \left( a_{\alpha}(\mathbf{p}) u_L(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}} + b_{\alpha}^\dagger(\mathbf{p}) v_L(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}} \right).
\]
mass. We rewrite Eq.(3) using Eq.(4):

\[
l^M_\alpha(t, x) = \int' \frac{d^3k}{(2\pi)^3} \int' \frac{d^3p}{(2\pi)^3} \bigg[ a_\alpha^\dagger(k, t) a_\alpha(p, t) \bar{u}_L(k) \gamma^0 u_L(p) e^{-i(k-p)\cdot x} + b_\alpha(k, t) a_\alpha(p, t) \bar{u}_L(k) \gamma^0 u_L(p) e^{i(k+p)\cdot x} \\
+ a_\alpha^\dagger(k, t) b_\alpha^\dagger(p, t) \bar{u}_L(k) \gamma^0 v_L(p) e^{-i(k+p)\cdot x} - b_\alpha^\dagger(p, t) b_\alpha(k, t) \bar{u}_L(k) \gamma^0 v_L(p) e^{i(k-p)\cdot x} \bigg],
\]  

(5)

where the integration region \( \int' \) does not include the zero momentum mode.

We take the expectation value of the Majorana density operator to study the spacetime evolution \( \langle \psi_\sigma(q^0; \sigma_q) | l^M_\alpha(t, x) | \psi_\sigma(q^0; \sigma_q) \rangle \). The initial momentum state of \( | \psi_\sigma(q^0; \sigma_q) \rangle \) is equally distributed in the first and third dimensions, but has a Gaussian distribution in the second, i.e., \( q = (0, q, 0) \). This leads to a shape similar to a 1-D Gaussian wave packet,

\[
| \psi_\sigma(q^0; \sigma_q) \rangle = \frac{1}{\sqrt{\sigma_q(2\pi)^{3/2}\delta(0)}} \int' \frac{dq}{2\pi\sqrt{2q}} e^{-\frac{(q-q^0)^2}{4\sigma_q^2}} a_\sigma^\dagger(0, q, 0) |0\rangle;
\]

(6)

where \( \sigma_q \) is the width of the Gaussian distribution in the second component of the momentum. The second component of the mean momentum \( q^0 \) is positive for the Gaussian distribution.

Sandwiching the Majorana density operator of Eq.(5) with Eq.(6) and taking the integration over \( k \) and \( p \) results in,

\[
\langle \psi_\sigma(q^0; \sigma_q) | l^M_\alpha(t, x) | \psi_\sigma(q^0; \sigma_q) \rangle = \frac{1}{\sigma_q(2\pi)^{3/2}\delta(0)^2} \int' \int' \frac{dq'dq}{(2\pi)^2} e^{-\frac{(q'-q^0)^2+(q-q^0)^2}{4\sigma_q^2}} -i(q'-q)e_2\cdot x \\
\times \left[ \sum_{i,j} V_{\alpha i} V_{\sigma i} V_{\alpha j} V_{\sigma j} \left( \cos E_i(q') t + i \frac{|q|}{E_i(q')} \sin E_i(q') t \right) \left( \cos E_j(q) t - i \frac{|q|}{E_j(q)} \sin E_j(q) t \right) \\
- \sum_{i,j} V_{\alpha i} V_{\sigma i} V_{\alpha j} V_{\sigma j} \frac{m_j}{E_j(q')} \sin E_j(q') t \frac{m_i}{E_i(q)} \sin E_i(q) t \right],
\]

(7)

where \( e_2^T = (0, 1, 0) \) is the unit vector. To perform the integration over \( q' \) and \( q \) we must assume two properties about the Gaussian distributions. First, the distributions are sharply peaked around the mean momentum value \( q^0 \), i.e., \( \sigma_q \ll q^0 \). Second, the width (variance) of the distribution \( \sigma_q \) does not change in spacetime. Those two assumptions allow us to approximate the \( q' \) and \( q \) integration as Gaussian; because,

\[
E_{i,j}(q'^{\prime}) \approx E_{i,j}(q^0) + \frac{q^0}{E_{i,j}(q^0)} (q'^{\prime} - q^0).
\]

(8)
The Gaussian integration of Eq. (7), and integration over density of Eq. (9) over all space, results in a linear density,

\[
\lambda^M_{\sigma \to \alpha}(t, x_2) \simeq \frac{\sigma_q}{(2\pi)^{1/2}} \sum_{i,j} V^*_{\sigma i} V_{\alpha i} V_{\alpha j} V^*_{\sigma j} \\
	imes \frac{1}{2} \left[ (v_{i0} + v_{j0} + 1 + v_{i0}v_{j0}) e^{i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2[(x_2-v_{i0}t)^2+(x_2-v_{j0}t)^2]} \\
- (v_{i0} + v_{j0} - 1 - v_{i0}v_{j0}) e^{-i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2[(x_2+v_{i0}t)^2+(x_2+v_{j0}t)^2]} \\
+ (v_{i0} - v_{j0} + 1 - v_{i0}v_{j0}) e^{i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2[(x_2-v_{i0}t)^2+(x_2+v_{j0}t)^2]} \\
- (v_{i0} - v_{j0} - 1 + v_{i0}v_{j0}) e^{-i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2[(x_2+v_{i0}t)^2+(x_2-v_{j0}t)^2]} \right] \\
- \frac{\sigma_q}{(2\pi)^{1/2}} \sum_{i,j} V^*_{\sigma i} V^*_{\alpha i} V_{\alpha j} V_{\sigma j} \sqrt{1 - v_{i0}^2} \sqrt{1 - v_{j0}^2} \\
\times \frac{1}{2} \left[ e^{i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2[(x_2-v_{i0}t)^2+(x_2-v_{j0}t)^2]} + e^{-i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2[(x_2+v_{i0}t)^2+(x_2+v_{j0}t)^2]} \\
- e^{i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2[(x_2-v_{i0}t)^2+(x_2+v_{j0}t)^2]} - e^{-i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2[(x_2+v_{i0}t)^2+(x_2-v_{j0}t)^2]} \right], \tag{9}
\]

where \(v_{i,j} = q^0/E_{i,j}(q^0)\) are the group velocities of the distributions, \(x_2 = e_2 \cdot x\), and \(\lambda^M_{\sigma \to \alpha}(t, x_2) = \int dx_1 dx_3 \langle \psi_\sigma(q^0; \sigma_q)|\lambda^M(t, x)|\psi_\alpha(q^0; \sigma_q) \rangle\). The terms with the PMNS matrix combination \(V^*_{\sigma i} V^*_{\alpha i} V_{\alpha j} V_{\sigma j}\) are dependent on the Majorana phases and are suppressed by the small masses of neutrinos \(\sqrt{1 - v_{i,j0}^2} = m_{i,j}/E_{i,j}(q^0)\). Lastly, to compare with our previous work’s result that considered only the time evolution of plane waves, we integrate the linear density of Eq. (9) over all space,

\[
\int \lambda^M_{\sigma \to \alpha}(t, x_2) dx_2 \simeq \frac{1}{4} \sum_{i,j} V^*_{\sigma i} V_{\alpha i} V^*_{\sigma j} V_{\alpha j} \\
\times \left[ (v_{i0} + v_{j0} + 1 + v_{i0}v_{j0}) e^{i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \\
- (v_{i0} + v_{j0} - 1 - v_{i0}v_{j0}) e^{-i(E_i(q^0) - E_j(q^0))t} e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \\
+ (v_{i0} - v_{j0} + 1 - v_{i0}v_{j0}) e^{i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \\
- (v_{i0} - v_{j0} - 1 + v_{i0}v_{j0}) e^{-i(E_i(q^0) + E_j(q^0))t} e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \right] \\
- \frac{1}{4} \sum_{i,j} V^*_{\sigma i} V^*_{\alpha i} V_{\alpha j} V_{\sigma j} \sqrt{1 - v_{i0}^2} \sqrt{1 - v_{j0}^2} \\
\times \left[ (e^{i(E_i(q^0) - E_j(q^0))t} + e^{-i(E_i(q^0) - E_j(q^0))t}) e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \\
- (e^{i(E_i(q^0) + E_j(q^0))t} + e^{-i(E_i(q^0) + E_j(q^0))t}) e^{-\sigma_q^2\frac{(v_{i0}v_{j0})^2}{2}} \right]. \tag{10}
\]
The result of our previous work [27] is recovered in the plane wave limit, for which we take the width $\sigma_q$ of the density in momentum space to zero.

B. Dirac neutrino formulation

We consider the following Lagrangian $\mathcal{L}^D$ for Dirac neutrinos,

$$\mathcal{L}^D = \overline{\nu_{La}} i \gamma^\mu \partial_\mu \nu_{La} + \overline{\nu_{Ra}} i \gamma^\mu \partial_\mu \nu_{Ra} - \theta(t) (\overline{\nu_{Ra}} m_{\alpha\beta} \nu_{L\beta} + \text{h.c.}),$$

(11)

The first and second terms are kinetic ones and third term is Dirac mass one. Similar to the Majorana case, a time dependent step function controls the third term. In contrast to the Majorana case, the Dirac mass matrix is diagonalized by two mixing matrices,

$$\nu_{L\beta} = V_{\beta j} \nu_{Lj},$$

(12)

$$\nu_{Ra} = U_{\alpha i} \nu_{Ri},$$

(13)

$$(U^\dagger)_{i\alpha} m_{\alpha\beta} V_{\beta j} = m_i \delta_{ij}.$$  

(14)

We relate the two regions of the step-function in Eq.(11) with the continuity condition at $t = 0$:

$$\lim_{\epsilon \to 0^+} \nu_{Li}(t = +\epsilon) = \lim_{\epsilon \to 0^+} \sum_{\beta = \epsilon}^\tau \nu_{L\beta}(t = -\epsilon)$$

(15)

$$\lim_{\epsilon \to 0^+} \nu_{Ra}(t = +\epsilon) = \lim_{\epsilon \to 0^+} \sum_{\alpha = \epsilon}^\tau \nu_{Ra}(t = -\epsilon).$$

(16)

The Weyl fields of $\nu_{Ra}$ and $\nu_{L\beta}$ are Fourier expanded,

$$\lim_{\epsilon \to 0^+} \nu_{La}(-\epsilon, x) = \int \frac{d^3 p}{(2\pi)^3 2|p|} \left( u_L(p) a_{La}(p) e^{ip \cdot x} + v_L(p) b_{La}^\dagger(p) e^{-ip \cdot x} \right),$$

(17)

$$\lim_{\epsilon \to 0^+} \nu_{Ra}(-\epsilon, x) = \int \frac{d^3 p}{(2\pi)^3 2|p|} \left( u_R(p) a_{Ra}(p) e^{ip \cdot x} + v_R(p) b_{Ra}^\dagger(p) e^{-ip \cdot x} \right).$$

(18)

Then the operators obey the usual anti-commutation relations of,

$$\{ a_{La}(\pm p), a_{L\beta}^\dagger(\pm q) \} = 2|p|(2\pi)^3 \delta^{(3)}(p-q) \delta_{\alpha\beta},$$

(19)

$$\{ b_{La}(\pm p), b_{L\beta}^\dagger(\pm q) \} = 2|p|(2\pi)^3 \delta^{(3)}(p-q) \delta_{\alpha\beta},$$

(20)

where we denote the three-dimensional momenta $+p$ and $-p$. $+p$ belongs to a hemisphere region where the azimuthal angle $\phi$ in the momentum direction is $\phi \in [0, \pi)$ and $-p$ belongs
to the other hemisphere field that corresponds to $\phi \in [\pi, 2\pi]$. We take the usual definition of a Dirac field $\psi_i \equiv \nu_{Ri} + \nu_{Li}$ with the on-shell Fourier expansion,

$$\lim_{\epsilon \to 0^+} \psi_i(+\epsilon, x) = \int \frac{d^4p}{(2\pi)^32E_i(p)} \sum_h \left(u_i(p, h)a_i(p, h)e^{ip\cdot x} + v_i(p, h)b_i^\dagger(p, h)e^{-ip\cdot x}\right),$$

and $h$ is the helicity. The mass operators $a_i(p, h)$ and $b_i(p, h)$ obey the usual anti-commutation relations,

$$\{a_i(p, h), a_j^\dagger(q, h)\} = (2\pi)^32E_i(p)\delta_{ij}\delta^{(3)}(p - q),$$

$$\{b_i(p, h), b_j^\dagger(q, h)\} = (2\pi)^32E_i(p)\delta_{ij}\delta^{(3)}(p - q).$$

All other anti-commutation relations are zero. The continuity conditions of Eqs.\((15, 16)\) are used to derive relations between the operators of Eqs.\((19, 20)\) and Eqs.\((22, 23)\);

$$\frac{1}{\sqrt{2|p|}} a_{La}(\pm p) = \sum_1^3 V_{a_i} \sqrt{E_i(p) + |p|} \left(a_i(\pm p, -) \pm i \frac{m_i}{E_i(p) + |p|} b_i^\dagger(\mp p, -)\right),$$

$$\frac{1}{\sqrt{2|p|}} b_{La}(\pm p) = \sum_1^3 V_{a_i} \sqrt{E_i(p) + |p|} \left(b_i^\dagger(\pm p, +) \mp i \frac{m_i}{E_i(p) + |p|} a_i(\mp p, +)\right).$$

Where the relations for the right-handed operators $a_{Ra}$, $b_{Ra}$ are found by replacing the PMNS matrices $V \to U$ and flipping the operator helicity $a_i(\pm p, \pm) \to a_i(\pm p, \mp)$, $b_i(\pm p, \pm) \to b_i(\pm p, \mp)$. The time evolution of the operators from Eqs.\((24, 25)\) are given as,

$$a_{La}(\pm p, t) = \sum_1^3 \sum_1^\tau \left[V_{a_i}^* \left(\cos E_i(p)t - i \frac{|p|}{E_i(p)} \sin E_i(p)t\right) a_{L\beta}(\pm p)\right]
+ V_{a_i}^* U_{\beta i} \frac{m_i}{E_i(p)} \sin E_i(p)t b_{R\beta}(\mp p),$$

$$a_{La}^\dagger(\pm p, t) = \sum_1^3 \sum_1^\tau \left[V_{a_i}^* V_{\gamma i} \left(\cos E_i(p)t + i \frac{|p|}{E_i(p)} \sin E_i(p)t\right) a_{L\gamma}(\pm p)\right]
+ V_{a_i}^* U_{\gamma i} \frac{m_i}{E_i(p)} \sin E_i(p)t b_{R\gamma}(\mp p),$$

$$b_{La}(\pm p, t) = \sum_1^3 \sum_1^\tau \left[V_{a_i}^* \left(\cos E_i(p)t - i \frac{|p|}{E_i(p)} \sin E_i(p)t\right) b_{L\beta}(\pm p)\right]
+ V_{a_i}^* U_{\beta i} \frac{m_i}{E_i(p)} \sin E_i(p)t a_{R\beta}(\mp p),$$

$$b_{La}^\dagger(\pm p, t) = \sum_1^3 \sum_1^\tau \left[V_{a_i}^* V_{\gamma i} \left(\cos E_i(p)t + i \frac{|p|}{E_i(p)} \sin E_i(p)t\right) b_{L\gamma}(\pm p)\right]
+ V_{a_i}^* U_{\gamma i} \frac{m_i}{E_i(p)} \sin E_i(p)t a_{R\gamma}(\mp p),$$
where the right-handed operators are just replacements of the PMNS matrices $U \rightarrow V$, $V \rightarrow U$ and the handedness $a_{(L,R)} \rightarrow a_{(R,L)}$, $b_{(L,R)} \rightarrow b_{(R,L)}$. We define the lepton family number density, for both the left- and right-handed Dirac neutrinos,

$$l^L_\alpha(t, x) = : \nu_{L\alpha}(t, x) \gamma^0 \nu_{L\alpha}(t, x) : ,$$

$$l^R_\alpha(t, x) = : \bar{\nu}_{R\alpha}(t, x) \gamma^0 \nu_{R\alpha}(t, x) : .$$

The Fourier expansion of $\nu_{L\alpha}(t, x)$ follows the massless form of Eq.(17):

$$\nu_{L\alpha}(t, x) = \int \frac{d^3 p}{(2\pi)^3 2|p|} \left( u_L(p)a_{L\alpha}(p, t) e^{i p \cdot x} + v_L(p)b^\dagger_{L\alpha}(p, t) e^{-i p \cdot x} \right) .$$

Then, the evolution of the lepton family number density for the left-handed case Eq.(30) is obtained by substituting Eq.(32),

$$l^L_\alpha(t, x) = \int' \frac{d^3 k}{(2\pi)^3 2|k|} \int' \frac{d^3 p}{(2\pi)^3 2|p|}$$

$$\times \left[ a^\dagger_{L\alpha}(k, t) a_{L\alpha}(p, t) u_L(k) \gamma^0 u_L(p) e^{-i(k-p) \cdot x} + b_{L\alpha}(k, t) a_{L\alpha}(p, t) \bar{u}_L(k) \gamma^0 u_L(p) e^{i(k+p) \cdot x} \right. $$

$$+ a^\dagger_{L\alpha}(k, t) b^\dagger_{L\alpha}(p, t) \bar{u}_L(k) \gamma^0 v_L(p) e^{-i(k+p) \cdot x} - b^\dagger_{L\alpha}(k, t) b_{L\alpha}(p, t) \bar{u}_L(k) \gamma^0 v_L(p) e^{i(k-p) \cdot x} \right] ,$$

whereas the right-handed form is a replacement of the spinors and operators from $L$ to $R$.

Next, we write the expectation value using a similar state $|\psi_\sigma(q^0; \sigma_q)\rangle$ as Eq.(6), but with the left-handed operator replaced as $\sigma^\dagger_\sigma(0, q, 0) \rightarrow a^\dagger_{L\sigma}(0, q, 0)$. We sandwich the left-handed density for the lepton family number of Eq.(33) with the left-handed state $|\psi^L_\sigma(q^0; \sigma_q)\rangle$ and integrate over $k, p$ to result in,

$$\langle \psi^L_\sigma(q^0; \sigma_q)|t^L_\alpha(t, x)|\psi^L_\sigma(q^0; \sigma_q)\rangle = \frac{1}{\sigma_q(2\pi)^3 \delta(0)^2} \int \int' \frac{dq dq'}{(2\pi)^2} e^{-i(q-q')^2 x + (q-q')^2 - i(q-q)e_2 x}$$

$$\times \left[ \sum_{i,j} V^\ast_{\alpha i} V_{\alpha j} V^\ast_{\sigma i} V_{\sigma j} \left( \cos E_i(q') t + i \frac{|q'|}{E_i(q')} \sin E_i(q') t \right) \right.$$

$$\left. \times \left( \cos E_j(q) t - i \frac{|q|}{E_j(q)} \sin E_j(q) t \right) \right] .$$

An initial right-handed state of Eq.(34), $\langle \psi^R_\sigma(q^0; \sigma_q)|t^R_\alpha(t, x)|\psi^R_\sigma(q^0; \sigma_q)\rangle$, is obtained by interchanging the matrices as $U \rightarrow V$, $V \rightarrow U$. Lastly, we perform the integration over $q$ and $q'$ using the approximation of Eq.(8) to write the integrand in Gaussian form. The result is
the linear density expectation value of the left-handed lepton family number,

\[ \lambda_{\sigma \rightarrow \alpha}^{L}(t, x_2) \simeq \sigma_q \left( \frac{1}{2} \right) \sum_{i,j} V^*_{\alpha i} V_{\sigma i} V^*_{\sigma j} \]

\[ \times \left[ (v_{i0} + v_{j0} + 1 - v_{i0} v_{j0}) e^{i(E_i(q^0) - E_j(q^0)) t} e^{-\sigma_q^2 [(x_2 - v_{i0} t)^2 + (x_2 - v_{j0} t)^2]} 
   - (v_{i0} + v_{j0} - 1 - v_{i0} v_{j0}) e^{-i(E_i(q^0) - E_j(q^0)) t} e^{-\sigma_q^2 [(x_2 + v_{i0} t)^2 + (x_2 + v_{j0} t)^2]} 
   + (v_{i0} - v_{j0} + 1 - v_{i0} v_{j0}) e^{i(E_i(q^0) + E_j(q^0)) t} e^{-\sigma_q^2 [(x_2 + v_{i0} t)^2 + (x_2 + v_{j0} t)^2]} 
   - (v_{i0} - v_{j0} - 1 - v_{i0} v_{j0}) e^{-i(E_i(q^0) + E_j(q^0)) t} e^{-\sigma_q^2 [(x_2 - v_{i0} t)^2 + (x_2 - v_{j0} t)^2]} \right] . \]  

(35)

Where we have defined \( \lambda_{\sigma \rightarrow \alpha}^{L}(t, x_2) = \iint dx_1 dx_3 \langle \psi^L_\sigma(q^0; \sigma_q)|U^L_\alpha(t, x)|\psi^L_\sigma(q^0; \sigma_q) \rangle \). The process for solving for the linear density of the right-handed lepton family number of Eq.(31) is the same. Thus, we only write the resulting linear density expectation value of the right-handed lepton family number,

\[ \lambda_{\sigma \rightarrow \alpha}^{R}(t, x_2) \simeq \sigma_q \left( \frac{1}{2} \right) \sum_{i,j} \sqrt{1 - v_{i0}^2} \sqrt{1 - v_{j0}^2} e^{-\sigma_q^2 [(v_{i0}^2 + v_{j0}^2)^2 + 2x_2^2]} \]

\[ \times \{ \Re \left( U^*_{\alpha i} V_{\sigma i} U_{\sigma j} V^*_{\sigma j} \right) \left[ \cosh \left( 2\sigma_q^2(v_{i0} + v_{j0})tx_2 \right) \cos \left( (E_i(q^0) - E_j(q^0))t \right) 
   - \cosh \left( 2\sigma_q^2(v_{i0} - v_{j0})tx_2 \right) \cos \left( (E_i(q^0) + E_j(q^0))t \right) \right] 
   - \Im \left( U^*_{\alpha i} V_{\sigma i} U_{\sigma j} V^*_{\sigma j} \right) \left[ \sinh \left( 2\sigma_q^2(v_{i0} + v_{j0})tx_2 \right) \sin \left( (E_i(q^0) - E_j(q^0))t \right) 
   - \sinh \left( 2\sigma_q^2(v_{i0} - v_{j0})tx_2 \right) \sin \left( (E_i(q^0) + E_j(q^0))t \right) \right] \} . \]  

(36)

C. Comparison to Wave packet formulations

The results we present in Eqs.(9 35 36) are based on the evolution of a density operator. We will now discuss an interpretation of those results based on wave packet formulations. Specifically, wave packets have been used in connection with neutrino oscillation phenomenology (see Ref. [25] for a review) as several forms; the quantum mechanical (QM) formulation [22], and the external wave packet QFT formulation [23 24]. We will focus on comparisons to interpretations in the quantum mechanical formulation, because it is the simplest in literature.

Often discussed in literature see [26 29 31], neutrino oscillations occur when three types of coherence are satisfied.

1. The different massive neutrino components are coherently produced.
2. Coherent propagation of massive neutrino components.

3. Coherent detection of the different massive components.

We do not consider any production or detection processes, so we can not discuss the possibility of coherent detection in our formulation. The initial density state of Eq.(6) captures the properties of coherent propagation, similar to the QM wave packet.

A prediction of the QM wave packet formulation is the coherence length $L_{i,j}^{\text{coh}}$ for oscillations [32]. The coherence length is a measure of the group velocities for the mass eigenstates that appears after integration over time. It acts as a damping factor on the oscillations in space. We find a similar damping in our density formulation. The damping appears as a real exponential component that depends quadratically on spacetime,

\[ e^{-\sigma^2_i[(x_2 \pm v_{i0}t)^2 + (x_2 \pm v_{j0}t)^2]} \]  (37)

\[ e^{-\sigma^2_j[(x_2 \pm v_{i0}t)^2 + (x_2 \pm v_{j0}t)^2]} \]  (38)

These real components are a non-linear damping that is applied to the density oscillation. An important distinction between our density formulation and the QM formulation are the types of damping. The QM formulation of Eq.(3) in Ref. [22] has a single type of damping to coincide with the single type of oscillation

\[ e^{-\sigma^2_i[(x_2 - v_{i0}t)^2 + (x_2 - v_{j0}t)^2]} \]  (39)

which we denote as $(-, -)$. Whereas, our density formulation has four different types damping, $(-, -), (-, +), (+, -), (+, +)$. Each damping is applied to a different oscillation in Eqs.(9, 35, 36). The $(+, +)$ is the strongest damping factor and $(-, -)$ is the weakest damping, which we illustrate by minimizing the polynomials inside the exponentials assuming $x_2 > 0$;

\[ P_1(x_2, t = \frac{2x_2}{v_{i0} + v_{j0}}) = 2x_2^2(v_{i0} - v_{j0})^2 (v_{i0} + v_{j0})^2 \]  from $(-, -),$  (40)

\[ P_2(x_2, t = \frac{2x_2}{v_{i0} + v_{j0}}) = 2x_2^2(3v_{i0} + v_{j0})^2 + 2(v_{i0}^2 + v_{j0}^2) (v_{i0} + v_{j0})^2 \]  from $(+, +),$  (41)

\[ P_3(x_2, t = \frac{2x_2}{v_{i0} + v_{j0}}) = 2x_2^2 + 8x_2^2 \frac{v_{j0}^2}{(v_{i0} + v_{j0})^2} \]  from $(-, +),$  (42)

\[ P_4(x_2, t = \frac{2x_2}{v_{i0} + v_{j0}}) = 2x_2^2 + 8x_2^2 \frac{v_{i0}^2}{(v_{i0} + v_{j0})^2} \]  from $(+,-).$  (43)
Notice the peak value of $(-,-)$ is damped at the peak by $P_1 < 2x^2_2$ causing it to be weak. Whereas the peak values of $(+,+), (-,+)$, and $(+,-)$ are damped with $P_{2,3,4} > 2x^2_2$ strengthening their suppression of the oscillations.

If we perturb slightly away from the peak time, the strength of the damping factors depend directly on the velocities. For example,

$$P_1 \left( x_2, t = \frac{2x^2_2}{v_{i0} + v_{j0}} + \Delta t \right) - P_1 \left( x_2, t = \frac{2x^2_2}{v_{i0} + v_{j0}} \right) = \frac{(v_{i0}^2 + v_{j0}^2)(\Delta t)^2}{v_{i0} + v_{j0}} + \frac{2x^2_2(v_{i0} - v_{j0})^2}{v_{i0} + v_{j0}} \Delta t. \quad (44)$$

So, a decrease in the velocities lead to longer damping times.

After we integrate over all space to arrive at Eq. (10), the damping factors appear as two types $\pm$. The coherence time $t_{coh}^{ij}$ is approximated by

$$t_{coh}^{ij} \sim \frac{\sqrt{2}}{\sigma_q(v_{i0} \pm v_{j0})}, \quad (45)$$

which we write as a coherence time $t_{coh}^{ij}$. In the usual QM case only a single damping factor appears for the oscillations $(v_{i0} + v_{j0})$. However, new oscillation terms appear in our formulation corresponding to $e^{\pm i(E_i(q^0) + E_j(q^0))t}$, to which a new damping factor $(v_{i0} - v_{j0})$ is applied.

### III. COMPARISON OF THE DIRAC AND MAJORANA FORMULATIONS

From the expectation value equations of Eq. (9), Eq. (35), and Eq. (36) we find signatures of the Dirac and Majorana formulations. Firstly, the terms proportional to the PMNS matrix combination of $V_{\alpha i}^*V_{\alpha i}V_{\alpha j}V_{\sigma j}^*$ are common between the formulations. Those terms are the first summation from Eq. (9) of the Majorana formulation and the left-handed Dirac formulation Eq. (35). The difference between the formulations is the secondary term of the Majorana formulation and the right-handed Dirac formulation Eq. (36). The secondary term of the Majorana formulation has the PMNS combination $V_{\alpha i}^*V_{\sigma i}V_{\alpha j}V_{\sigma j}$ that is related to the Majorana phases and is subtracted from the common oscillation terms. Whereas, the second unitary matrix $U_{\alpha i}$ from the right-handed Dirac formulation is present in Eq. (36) and is added to Eq. (35) as $\lambda^D_{\sigma \rightarrow \alpha}(t, x_2) = \int dx_1 dx_3 \langle \psi^L_\alpha(q^0; \sigma_q)|l^L_\alpha(t, x_2) + t^R_\alpha(t, x_2)|\psi^L_\alpha(q^0; \sigma_q) \rangle$. The sign of the different terms, subtraction for Majorana and addition for Dirac, is responsible for total lepton number violation in the Majorana case. To help emphasize the affect of the sign difference we write the expectation value of the total lepton number for the Dirac...
formulation case,
\[
\sum_{\alpha} \lambda_{\sigma \rightarrow \alpha}^D(t, x_2) \simeq \frac{\sigma_q}{\sqrt{2\pi}} \sum_{i} |V_{\sigma i}|^2 \left[ (1 + v_{i0}) e^{-2\sigma_q^2(x_2-v_{i0}t)^2} + (1 - v_{i0}) e^{-2\sigma_q^2(x_2+v_{i0}t)^2} \right].
\]

(46)

where integration over \(x_2\) would result in total lepton number conservation for all times. Next, from Eq.(9), we also take the summation over the lepton family numbers \(\alpha\) as follows,
\[
\sum_{\alpha} \lambda_{\sigma \rightarrow \alpha}^M(t, x_2) \simeq \frac{\sigma_q}{\sqrt{2\pi}} \sum_{i} |V_{\sigma i}|^2 \left[ v_{i0}(1 + v_{i0}) e^{-2\sigma_q^2(x_2-v_{i0}t)^2} - v_{i0}(1 - v_{i0}) e^{-2\sigma_q^2(x_2+v_{i0}t)^2} + 2(1 - v_{i0}^2) e^{-2\sigma_q^2(v_{i0}^2t^2+x_2^2)} \cos 2E_i(q^0)t \right].
\]

(47)

In Eq.(46) for the Dirac case, there is no oscillation term and the linear density of the total lepton number is always positive. For a nonrelativistic Majorana neutrino \((v_{i0} \ll 1)\) in Eq.(47), the oscillation term can dominate and the sign of the linear density can change with respect to time.

IV. ILLUSTRATION OF NONRELATIVISTIC DIFFERENCES

From inspection of the expectation values Eq.(9), Eq.(35), and Eq.(36); and the discussions in sections II C and III we see two key results.

• Differences between our formulation and the usual QM formulation are negligible under the ultra-relativistic assumption.

• The neutrino masses \(\frac{m_i m_j}{E_i E_j} = \sqrt{1 - v_i^2} \sqrt{1 - v_j^2}\) suppress any modifications by the Dirac or Majorana mass formulations.

To illustrate those results we choose \(m_{\text{lightest}} = 0.01\) eV < \(q^0 = 0.2\) eV for the linear density expectations of Eqs.(9) and (35). Even for a momentum an order of magnitude greater than the mass, differences between the Majorana and Dirac cases are not visible (Fig. 1 and Fig. 2). We used the best fit values of the PMNS mixing angles \(\theta_{12}, \theta_{23},\) and \(\theta_{13},\) the CP violating phase \(\delta_{\text{CP}},\) and the mass squared differences \(\Delta m_{21}^2\) and \(\Delta m_{31}^2\) from the work of the NuFIT 5.0 (2020) collaboration[33]. Then, we are left to choose the absolute mass of the lightest neutrino \(m_1\) or \(m_3\), the neutrino mass hierarchy, normal \(m_1 < m_2 \ll m_3\) or inverted \(m_3 \ll m_1 < m_2\), the value of the mean momentum \(q\), the width of the initial Gaussian density \(\sigma_q\), the time \(t\), the distance \(x_2\), and the Majorana phases \(\alpha_{21}\) and \(\alpha_{31}\).
FIG. 1. Time evolution of the linear density for the expectation value of a lepton family number with Majorana, panel (a), or Dirac, panel (b), mass. We take an arbitrary distance slice at $x_2 = 2.5\text{cm}$. The initial momentum density is a Gaussian distribution with a width of $\sigma_q = 0.00001$ and a mean momentum of $q^0 = 0.2\text{eV}$. Normal mass hierarchy is considered. Oscillation parameters are the best fit values from the NuFIT 5.0 (2020) collaboration[33].

The definition of the Majorana phases is given as $\alpha_{21} = 2 \arg(V_{e2})$ and $\alpha_{31} = 2 \arg(V_{\mu 3})$. Any changes caused by different Majorana phases are also hidden, because the differences between the Majorana and Dirac cases are not visible for $m_{lightest} = 0.01 \text{eV} < q^0 = 0.2 \text{eV}$.

The decoherence effects discussed in section II C from the real exponentials of Eq.(9), Eq.(35), and Eq.(36) cause the oscillations to be localized to a spacetime region. In Fig. 3, we show the 2-D spacetime contour for an electron number linear density for the $\sigma = e \rightarrow \alpha = e$ case. The region where the electron number linear density has a maximum peak of $\sim 0.4$ propagates from the spacetime origin $(t, x_2) = (0, 0)$ toward the upper right corner at a constant velocity. The initial Gaussian shape is maintained during propagation because we assumed no wave packet spreading in the approximation of Eq.(8).

We discussed in section III how the total lepton number becomes violated in the Majorana case, and we illustrate that by decreasing the momentum to $q^0 = 0.0002 < m_{lightest} = 0.01\text{eV}$ in Fig.4. This leads to the distinction between the Majorana and Dirac cases, in which the Majorana case has negative expectation values. Both cases feature the small period oscillations of the inset graph because of lines 4 and 5 from Eq.(35). Those lines are not
FIG. 2. Similar to Fig.1, time evolution of the linear density for the expectation value of a lepton family number at $x_2 = 2.5\text{cm}$. However, we now consider the $\sigma = e$, $\alpha = \tau$ lepton family number values. The initial momentum density is a Gaussian distribution with a width of $\sigma_q = 0.00001$ and a mean momentum of $q^0 = 0.2\text{eV}$. Normal mass hierarchy is considered. Oscillation parameters are the best fit values from the NuFIT 5.0 (2020) collaboration\[33].

present in the usual QM formulation, so these small period oscillations can distinguish our formulation.

By comparing Fig.1 and Fig.4, the oscillations occur for longer times when the average momentum is smaller. This was discussed in section II C, where Eq.(44) was given as an example. To recall, the strength of the damping factors depend directly on the velocities. So, a decrease in the velocities lead to longer damping times.

Next, in the distance plane of Fig.5 the smaller momenta suppress the peak value for the linear density of the lepton family numbers. This follows directly from Eq.(40), where for $q^0 = 0.2\text{ eV}$ the polynomial is larger than for $q^0 = 0.0002\text{ eV}$. Thus, the propagation coherence occurs at smaller distances, which is sometimes stated as a smaller coherence length. Even with this in mind, at smaller momenta the Dirac and Majorana cases are distinguishable. Similar to Fig.4, the Majorana case has negative expectation values and the Dirac case is always positive.
FIG. 3. 2-D Spacetime contour of the linear density for the expectation value of the electron family number with a Dirac mass. The initial momentum density is a Gaussian distribution with a width of $\sigma_q = 0.00001$ and a mean momentum of $q^0 = 0.2$eV. Normal mass hierarchy for neutrinos is considered. Lepton mixing angles, the Dirac CP phase, and the mass squared differences are the reported best fit values from the work of the NuFIT 5.0 (2020) collaboration[33].

V. CONCLUSION

We have proposed a novel formulation for neutrino oscillations from a QFT point of view. This formulation can be applied to both relativistic and nonrelativistic energies for neutrinos. The previous formulation was limited to Majorana neutrinos with a fixed momentum. We have extended the previous formulation to the Dirac neutrino case and have applied to the case with a momentum distribution. To study the oscillation of neutrino with a momentum distribution, we have considered the linear lepton number density and the initial state with a Gaussian distribution along a one spacial direction. After taking the expectation value of the density with the state, we find a result similar to the quantum mechanical wave packet approach used to describe neutrino flavor oscillations. Our result has additional terms that result in interesting behavior. In particular, we have studied type of decoherence ascribed to wave packet separation. We have found how the peak value of lepton number density is suppressed as it propagates. Even with that decoherence our formulation can
FIG. 4. Time evolution of the linear density for the expectation value of a lepton family number at $x_2 = 2.5$cm. We now consider the initial momentum density to have a mean momentum of $q^0 = 0.0002$eV for a Gaussian distribution with a width of $\sigma_q = 0.00001$. Compared to the preceding figures, differences between Majorana and Dirac are visible because $q^0 = 0.0002 < m_{\text{lightest}} = 0.01$eV. Normal mass hierarchy is considered. Oscillation parameters are the reported best fit values from the work of the NuFIT 5.0 (2020) collaboration. In addition, we choose the Majorana phases to be arbitrary values of $\alpha_{21} = \pi$ and $\alpha_{31} = 0.5\pi$.

distinguish between neutrino mass hierarchy and mass type at nonrelativistic energies. The mass hierarchies can be distinguished from the short period of the oscillation. The period is determined by the sum of the rest mass of neutrinos $m_i + m_j$ for nonrelativistic case. As for mass types, the expectation value of the lepton number density for Majorana neutrinos can take both positive and negative value while for Dirac neutrino, it does not change the sign.

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FIG. 5. Distance evolution of the linear density for the expectation value of a lepton family number. We have taken different time slices corresponding to when the peak of the linear densities are near $x_2 = 5.0$cm. Normal mass hierarchy is considered, and we choose the Majorana phases to be arbitrary values of $\alpha_{21} = \pi$ and $\alpha_{31} = 0.5\pi$. Oscillation parameters are the reported best fit values from the work of the NuFIT 5.0 (2020) collaboration.

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