ON THE PICARD GROUP OF MODULI SPACES

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Abstract. We study the Picard groups of moduli spaces in positive characteristics and we give a “$p$-adic” proof that the Picard group of moduli of vector bundles of fixed determinant is isomorphic to the group of integers. Along the way we prove that the local fundamental group scheme of a normal unirational projective variety is trivial. This is reminiscent of results of Serre and Nygaard who studied the fundamental groups of smooth, projective, unirational varieties.

1. Introduction

1.1. Notations. Throughout this paper, we work over an algebraically closed field $k$ and we will assume that $k$ has characteristic $p > 0$ unless stated otherwise (the letter $p$ will be reserved for the characteristic of $k$). The letter $\ell$ will denote a prime distinct from characteristic of $k$ (so $\ell \neq p$).

Let $C/k$ be a smooth, projective curve over $k$ of genus $g \geq 2$. Let $L \in \text{Pic}(C)$ be a line bundle on $C$, and let $\mathcal{M}(r, L)$ be the moduli of semistable vector bundles on $C$ of rank $r$ and determinant $L$.

It is well-known, that if the degree $\text{deg}(L) = d$ is coprime to the rank $r$, then $\mathcal{M}(r, L)$ is a smooth, projective variety; if the degree and the rank are not coprime, then $\mathcal{M}(r, d)$ is singular in general (the exception is genus $g = 2$, $L = \mathcal{O}_C$, $r = 2$).

When $k$ has characteristic zero, the Picard groups of $\mathcal{M}(r, L)$ have been studied in [DN89] where it has been shown that $\text{Pic}(\mathcal{M}(r, L)) = \mathbb{Z}$. In characteristic $p > 0$, one expects this result but as far as we are aware, the result has not been established in literature. The proof of [DN89] does not seem to adapt readily to positive characteristics because of the failure of “Kempf’s Lemma” which is crucial in the proof of [DN89].

In this note we prove that $\text{Pic}(\mathcal{M}(r, L)) = \mathbb{Z}$ in the following cases (1) the coprime case and (2) in the case when $L = \mathcal{O}_C$. Our proof is, in some sense, “a $p$-adic” proof and has the merit of working uniformly in all cases and may also work in the case of moduli of $G$-bundles ($G$ semi-simple) though we have not verified this at the moment.

As was pointed out to us by Norbert Hoffmann, that the fact that the Picard group of the moduli space of vector bundles on a curve is $\mathbb{Z}$ can also be established by using the relationship between moduli spaces and moduli stacks especially the corresponding results for moduli stacks (see [BLS98] and its references for characteristic zero case and [Fal03, BH08] for positive characteristic) and the methods of [BLS98]. This is, of course, of independent interest but we do not pursue this point of view here. Instead we show how the result on the Picard group may be established by a study of the local fundamental group scheme of the moduli space. In the course of this we establish that the local fundamental group scheme of a normal unirational variety $X$ is trivial (a result of independent interest) and in particular, if its étale...
fundamental group is also trivial, then we deduce that $H^1(X, \mathcal{O}_X)$ is zero for such a variety. This should be thought of as a complement to results of [Ser59], [Nyg78] and [Nor82] on the étale fundamental group (scheme) of smooth, unirational varieties.

To summarize our strategy in brief: we prove that the local fundamental group scheme (which classifies $F$-trivial vector bundles) is trivial. Using this we prove that Pic ($\mathcal{M}(r, L)$) is reduced and discrete. Next we show that Pic ($\mathcal{M}(r, L)$) is torsion-free so we reduce to computing the rank of the Néron-Severi group. This in turn is easily reduced to showing that the second betti number of $\mathcal{M}(r, L)$ is one (in the cases of interest).

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2. Unirationality of $\mathcal{M}(r, L)$

2.1. Let $X/k$ be a projective variety, we will say that $X$ is unirational if there is a dominant, separable morphism $\mathbb{P}^n \to X$ for some $n \geq 1$.

2.2. In this section we recall that $\mathcal{M}(r, L)$ is a unirational variety. In characteristic zero this is proved in [Ses82] and in positive characteristic this is proved in [Hei].

Theorem 2.2.1. Let $C$ be a smooth, projective curve and let $L$ be a line bundle on $C$, let $\mathcal{M}(r, L)$ be the moduli space of semistable vector bundles on $C$ of rank $r$ and with determinant $L$. Then $\mathcal{M}(r, L)$ is a unirational variety.

3. The fundamental group scheme of a normal unirational variety

3.1. In [Ser59] it was shown that any smooth, projective unirational variety over an algebraically closed field of characteristic zero has trivial fundamental group. This result was subsequently extended to smooth, unirational threefolds in positive characteristic in [Nyg78].

The main result of this section is to prove that if $X$ is a normal unirational variety, then its local fundamental group scheme is trivial. We recall that the local fundamental group scheme, denoted here by $\pi_\text{loc}^\text{Nori}(X)$, was introduced in [MS02]. We also note that when $X$ is a smooth, unirational variety the result follows from [Nor82] as $\pi_\text{loc}^\text{Nori}$ is a quotient of $\pi_\text{Nori}(X)$ and it was shown in [Nor82] that the latter is trivial.

Theorem 3.1.1. Let $X/k$ be a normal, unirational variety. Then $\pi_\text{loc}^\text{Nori}(X) = 1$.

3.2. Let us record the following important corollary of Theorem 3.1.1.

Corollary 3.2.1. Let $X/k$ be a normal, unirational variety. Assume that $X$ is simply connected. Then $H^1(X, \mathcal{O}_X) = 0$.

Proof. As $X$ is simply connected, we see that every étale $\mathbb{Z}/p$-cover is trivial. By [SGA73] the semi-simple part of $F : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ is controlled by étale $\mathbb{Z}/p$-covers of $X$ so the semi-simple part is trivial by our hypothesis that $X$ is simply connected. Hence we conclude that the action of Frobenius on $H^1(X, \mathcal{O}_X)$ is purely nilpotent. Suppose, if possible that, $H^1(X, \mathcal{O}_X)$ is not trivial. Let $V$ be a non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. As $F$ is nilpotent on $H^1(X, \mathcal{O}_X)$, we see that $V$ is $F$-trivial. By Theorem 3.1.1 we see that $V$ is trivial. This is a contradiction as we had assumed that $V$ is non-trivial. This completes the proof.
3.3. The proof of Theorem 3.1.1 is somewhat delicate and we will be break it up into several steps. Before we begin with the proof, we need some notational detail. Recall that $X$ is unirational, so there exists a dominant, separable morphism $\mathbb{P}^n \to X$ for some $n \geq 1$. Blowing up and normalizing if required, we may assume that we have a morphisms $Z \to \mathbb{P}^n$ and $Z \to X$ which makes the diagram commute and the arrow $Z \to Y \to X$ is the Stein factorization of $Z \to X$.

\begin{equation}
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \longrightarrow & X
\end{array}
\end{equation}

**Proposition 3.3.2.** In the notation of the diagram (3.3.1) above we have $\pi_{\text{loc}}(Z) = 1$.

*Proof.* Suppose that $V$ is an $F$-trivial vector bundle on $Z$. Then we may assume that $F^*(V) = \mathcal{O}_Z$ for $r = \text{rk} (V)$. Let $B^1_Z$ be the sheaf of locally exact differentials. Alternately, we may define $B^1_Z$ be the short exact sequence of sheaves

\begin{equation}
0 \to \mathcal{O}_Z \to F^*(\mathcal{O}_Z) \to B^1_Z \to 0.
\end{equation}

We claim that $V \otimes B^1_Z$ has no sections and in fact $V$ has exactly $r = \text{rk} (V)$ sections. Suppose, if possible, that $H^0(B^1_Z \otimes V) \neq 0$. Then we have a non-zero map $V^* \to B^1_Z$. But the Harder-Narasimhan flag of $B^1_Z$ has negative slopes (this is so because the restriction of $B^1_Z$ to a suitable open set is isomorphic to the restriction of $B^1_{\mathbb{P}^n}$ to an open set whose complement has codimension at least two and the latter is easily seen to have negative slopes), while that of $V^*$ has non-negative. So such a map must be zero. Thus we have proved that $B^1_Z \otimes V$ has no sections. Now we show that this implies that $V$ has $r = \text{rk} (V)$ sections.

Tensoring (3.3.3) with $V$ gives

\begin{equation}
0 \to V \to F^*(\mathcal{O}_Z) \otimes V \to B^1_Z \otimes V \to 0.
\end{equation}

Hence we have from the cohomology long exact sequence:

\[ 0 \to H^0(V) \to H^0(F^*(V)) \to H^0(B^1 \otimes V) = 0 \]

and $H^0(F^*(V)) = H^0(\mathcal{O}_Z)^r$.

So $V$ has numerically trivial determinant and $H^0(V) = r = \text{rk} (V)$ and $F^*(V)$ is trivial and such a $V$ is semistable (with respect to any polarization of $Z$). By [Ein80, Lemma 3.4, page 59], such a $V$ is trivial.

This proves the assertion. \qed

**Proposition 3.3.4.** We claim that for $Y$ as in the diagram (3.3.1), we have $\pi_{\text{loc}}(Y) = 1$.

*Proof.* Observe that if $f : Z \to Y$ has connected fibres so $f_*(\mathcal{O}_Z) = \mathcal{O}_Y$. Suppose that $V$ is an $F$-trivial vector bundle on $Y$. Then $f^*(V)$ is $F$-trivial on $Z$ so by Proposition 3.3.2 we see that $f^*(V) = \mathcal{O}_Z^r$ for some $r \geq 1$ and as $f_*(\mathcal{O}_Z) = \mathcal{O}_Y$ we see that $V = \mathcal{O}_Y^r$. This proves the assertion. \qed

**Theorem 3.3.5.** Let $X$ be a normal unirational variety over an algebraically closed field of characteristic $p > 0$. Then we have $\pi_{\text{loc}}(X) = 1$. 
Proof. We may assume that we have a diagram \[3.3.1\]. Suppose \(V\) is an \(F\)-trivial vector bundle on \(X\). Suppose \(E \to X\) is a finite group scheme cover over which \(V\) is trivial. Now the pull-back, to \(Y\), of \(V\) arises from \(E \times_X Y \to Y\). But as the pull-back of \(V\) to \(Y\) is trivial by Proposition \[3.3.4\], so we have a section \(Y \to E \times_X Y \to E\), so by composition have a map \(Y \to E\). On the other hand, as \(Y\) is reduced, this map factors as \(Y \to E_{\text{red}} \to X\). But \(Y \to X\) is separable, so \(E_{\text{red}} \simeq X\), but this gives a section \(X \to E_{\text{red}} \to E\) which is a contradiction. \(\Box\)

4. \(\mathcal{M}(r, L)\) is simply connected

4.1. In this section we outline a proof that \(\mathcal{M}(r, L)\) is simply connected. When \(k\) has characteristic zero, this is immediate from [Ser59] by the unirationality of \(\mathcal{M}(r, L)\).

Proposition 4.1.1. We have \(\pi_1^{\text{et}}(\mathcal{M}(r, d)) = 1\), that is, \(\mathcal{M}(r, L)\) is simply connected.

Proof. This is proved using the fact that \(\mathcal{M}(r, L)\) lives in a family over \(W = W(k)\) the ring of Witt vectors of \(k\). Indeed by [VBM08], we know that there is a moduli scheme \(\mathcal{M}(r, L) \to \text{Spec}(W)\) whose special fibre is our \(\mathcal{M}(r, L)\). Moreover, in characteristic zero, it is well-known, that \(\mathcal{M}(r, L)\) is simply connected. Now we use the specialization theorem of [sga71, Sect. 2, Corollary 2.4, Exposée X] to deduce that \(\mathcal{M}(r, L)/k\) is also simply-connected. \(\Box\)

5. The coprime case

5.1. Let us first consider the case when the degree \(d = \deg(L)\) and the rank \(r\) are coprime. In this case \(\mathcal{M}(r, L)\) is a smooth, projective variety, and hence we will refer to this case (of coprime degree and rank) as the “smooth case”. This case also explains our strategy of proof. Let us recall what we want to prove.

Theorem 5.1.1. Assume that \(L\) is a line bundle on \(C\) of degree \(d\) and that \((r, d) = 1\), then we have \(\text{Pic}(\mathcal{M}(r, L)) = \mathbb{Z}\).

Lemma 5.1.2. Under the hypothesis at the beginning of this section, we have \(\text{Pic}(\mathcal{M}(r, L))\) has no \(\mathbb{Z}/\ell\)-torsion for \(\ell \neq p\).

Proof. This follows from Lemma \[4.1.1\] as \(\pi_1^{\text{et}}(\mathcal{M}(r, L)) = 1\), we see that \(\mathcal{M}(r, L)\) does not admit any \(\mathbb{Z}/\ell^m\)-covers for any \(m \geq 1\). On the other hand, if \(\text{Pic}(\mathcal{M}(r, L))\) has \(\ell\)-torsion for some \(\ell\), then certainly \(\mathcal{M}(r, L)\) has étale \(\mathbb{Z}/\ell^m\)-covers for this \(\ell\) and some \(m \geq 1\). \(\Box\)

Lemma 5.1.3. Under the hypothesis at the beginning of this section, we have the following

1. \(H^1_{\text{dR}}(\mathcal{M}(r, L)/k) = 0\),
2. \(H^2_{\text{cris}}(\mathcal{M}(r, L)/W)\) is torsion free,
3. \(\text{Pic}(\mathcal{M}(r, L))\) has no \(p\)-torsion.

Proof. By [Ill79] we have the implications \((1) \Rightarrow (2) \Rightarrow (3)\). But for the convenience of the reader we will prove all of them.

Since \(\mathcal{M}(r, L)\) is unirational by [Ser59], we have \(H^0(\mathcal{M}, \Omega^1_{\mathcal{M}}) = 0\) and we have already established that \(H^1(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) = 0\), so we have the vanishing of the \(H^1_{\text{dR}}\) as required for \((1)\).
To prove (2) we note that by the universal coefficient theorem (see [HN75]) we have
0 \rightarrow H^1_{\text{cris}}(\mathcal{M}(r, L)/W) \otimes W k \rightarrow H^1_{\text{dR}}(\mathcal{M}(r, L)/k) \rightarrow \text{Tor}_1(H^2_{\text{cris}}(\mathcal{M}(r, L)/W), k) \rightarrow 0.

By (1) the term in the middle is zero so we are done, and in particular we have also proved that
\[ H^1_{\text{cris}}(\mathcal{M}(r, L)/W) = 0. \]

From Corollary 3.2.1 we know that \( H^1(\mathcal{M}(r, L), \mathcal{O}_{\mathcal{M}(r, L)}) = 0 \), it follows that Pic(\mathcal{M}(r, L)) is reduced and discrete. Hence Pic(\mathcal{M}(r, L)) \simeq \text{NS}(\mathcal{M}(r, L)). \]

Following Harder and Narasimhan (see [HN75], we have
\[ \text{NS}(\mathcal{M}(r, L)) \otimes W \rightarrow H^2_{\text{cris}}(\mathcal{M}(r, L)/W), \]
so the former is torsion free follows from the fact that the latter is torsion free.

Thus we have proved all the assertions. \( \square \)

**Proposition 5.1.4.** Under the hypothesis that the degree and the rank are coprime, we have that the rank of Néron-Severi group of \( \mathcal{M}(r, L) \) is one.

**Proof.** Since \( \text{NS}(\mathcal{M}(r, d)) \hookrightarrow H^2_{\text{cris}}(\mathcal{M}(r, L)/W) \), it suffices to prove that the latter has rank one. We may compute the rank after tensoring with \( \mathbb{Q}_\ell \). So it suffices to compute the second Betti number \( b_2(\mathcal{M}(r, L)) \). By [KM74] this may be computed using \( \ell \)-adic cohomology instead of crystalline cohomology.

Now we may further reduce to the case when \( k \) is a finite field. So we will assume for the rest of the proof that \( k \) is a finite field. In this case the required Betti number may be extracted from the computation of the Poincaré polynomial of \( \mathcal{M}(r, L) \) (see [HN75]). We carry out this computation now.

Following Harder and Narasimhan (see [HN75]), let
\[ Z_m(T) = \frac{(1 + T^{1-2m})^{2g}}{(1 - T^{-2m})(1 - T^{2(1-m)})}, \]
and let
\[ P(T) = T^{2(r^2-1)(g-1)}Z_2(T) \cdots Z_r(T). \]

Then expand \( P(T) \) as a power series in \( T^{-1} \), say
\[ P(T) = T^{2(r^2-1)(g-1)} \sum_{v=0}^{\infty} b_v T^{-v}. \]

Then
\[ b_v = \dim H^v_{\text{et}}(\mathcal{M}(r, L), \mathbb{Q}_\ell) \]
for \( 0 \leq v \leq 2((r-1)(g-1) + d \) (here \( d = \deg(L) \)). So to prove the assertion that \( b_2 = 1 \), it suffices to prove that coefficient of \( T^{-2} \) in the expansion of \( Z_2(T) \cdots Z_r(T) \) is one. Now writing this out we have
\[ Z_2(T) \cdots Z_r(T) = \prod_{j=2}^{r} \frac{(1 + T^{1-2j})^{2g}}{(1 - T^{-2j})(1 - T^{2(1-j)})}. \]

As \( j \geq 2 \), the only term which contributes as \( T^{-2} \) in \( \sum_v b_v T^{-v} \) is the term \( \frac{1}{(1 - T^{2(1-j)})} \) for \( j = 2 \), and as this is
\[ \frac{1}{1 - T^{-2}} = 1 + T^{-2} + T^{-4} + \cdots, \]
so this term contributes exactly one to \( b_2 \) and so we see that \( b_2(\mathcal{M}(r, L)) = 1 \).

This completes our proof in the coprime case. \( \square \)
6. The trivial determinant case

6.1. We now consider the case when $L = \mathcal{O}_C$. In this case the moduli space is singular and we do not know a simple formula for the Poincaré polynomial which is valid in arbitrary characteristics. Hence to compute the Betti number $b_2(\mathcal{M}(r, \mathcal{O}_C))$ we have to resort to calculating the Betti number in characteristic zero. This calculation is well-known, but we indicate a proof for completeness.

**Theorem 6.1.1.** Let $C$ be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic $p > 0$. Then we have $\text{Pic}(\mathcal{M}(r, \mathcal{O}_C)) = \mathbb{Z}$.

**Proof.** For simplicity of notation let us write $X = \mathcal{M}(r, \mathcal{O}_C)$ for the moduli space of semistable bundles of rank $r$ and trivial determinant. Then we have shown that $H^1(X, \mathcal{O}_X) = 0$ and hence we deduce that Pic($X$) is a discrete scheme. Next we observe that Pic($X$) has no $p$-torsion or $\ell$-torsion. Let us first consider $p$-torsion. Then as $\pi^{\text{loc}}(X) = 1$, we see that $X$ has no $\alpha_p$ or $\mu_p$ covers. Further as $H^1(X, \mathcal{O}_X) = 0$

and as $H^1(X, \mathbb{Z}/p)$ is the semisimple part of the Frobenius map $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ so $H^1(X, \mathbb{Z}/p) = 0$. Hence $X$ has no étale $p$-covers. So Pic($X$) has no $p$-torsion. Next as $\pi^{\text{et}}_1(X) = 1$ we see that $X$ does not have $\ell$-covers for any $\ell \neq p$. Thus Pic($X$) has no $\ell$-torsion. So Pic($X$) is torsion free as claimed. Thus Pic($X$) = $NS(X)$. Next we remark that by [Gro68, Page 145] that $NS(X) \otimes \mathbb{Q}_\ell \to H^2_{\text{et}}(X, \mathbb{Q}_\ell(1))$ is injective. This is a standard argument but we give a proof for completeness. Let $\mathbb{G}_m$ be the multiplicative group scheme. Then we have an exact sequence (for each $n \geq 1$) of sheaves for the étale topology:

$1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1,$

This induces a long exact sequence

$\cdots H^1(X, \mu_{\ell^n}) \to \text{Pic}(X) \to \text{Pic}(X) \to H^2(X, \mu_{\ell^n}) \to \cdots$

From this we extract

$0 \to \text{Pic}(X)/\ell^n \to H^2(X, \mu_{\ell^n}) \to Br'(X)[\ell^n] \to 0.$

where $Br'(X) = H^2(X, \mathbb{G}_m)$ is the cohomological Brauer group of $X$ and $Br'(X)[\ell^n]$ is its $\ell^n$ torsion. In the inverse limit we get (see [Gro68, Page 145])

$0 \to NS(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)) \to T_{\ell}Br'(X) \to 0,$

where $T_{\ell}Br'(X)$ is the $\ell$-adic Tate module of $Br'(X)$. At any rate this shows that the rank of $NS(X)$ is at most $b_2(X)$ (the second Betti number of $X$); since $X$ is projective the rank is at least one. So it remains to prove that $b_2(X) = 1$. This is the content of the next proposition. \qed

**Proposition 6.1.6.** Let $C$ be a smooth, projective curve over an algebraically closed field $k$ of characteristic $p > 0$. Let $\mathcal{M}(r, \mathcal{O}_C)$ be the moduli of semistable vector bundles on $C$ of rank $r$ and trivial determinant. Then we have

$b_2(\mathcal{M}(r, \mathcal{O}_C)) = \dim H^2_{\text{et}}(\mathcal{M}(r, \mathcal{O}_C), \mathbb{Q}_\ell) = 1.$

**Proof.** Let $W = W(k)$ be the ring of Witt vectors of $k$, let $K$ be the quotient field of $W$. Then we have a family $\mathcal{M} \to \text{Spec}(W)$ whose special is $\mathcal{M}(r, \mathcal{O}_C)$ and whose generic fibre is $\mathcal{M}(r, \mathcal{O}_{C_K})_K$ for some lift $C_K$ of $C$ to $K$. By the proper base change theorem, to prove our assertion it suffices to prove that $H^2_{\text{et}}(\mathcal{M}(r, \mathcal{O}_{C_K}), \mathbb{Q}_\ell)$ has
Thus we reduce to calculating the Betti number in characteristic zero. So we may assume that $k = \mathbb{C}$ and that $C/k$ is a smooth, projective curve of genus $g \geq 2$. In this case it is well-known, see for instance [BK05], that we have an isomorphism

$$\text{Pic} (\mathfrak{M}(r, \mathcal{O}_C)) \cong H^2(\mathfrak{M}(r, \mathcal{O}_C), \mathbb{Z}),$$

and hence by the main result of [DN89] we deduce that the Betti number

$$b_2(\mathfrak{M}(r, \mathcal{O}_C)) = 1.$$

This completes the proof of our assertion.

\[ \square \]

References

[BH08] I. Biswas and Norbert Hoffmann, The line bundles on moduli stacks of principal bundles on a curve, Preprint [http://arxiv.org/abs/0805.2915v2], 2008.
[BK05] Arzu Boysal and Shrawan Kumar, Explicit determination of the Picard group of moduli spaces of semistable $G$-bundles on curves, Math. Ann. 332 (2005), no. 4, 823–842.
[BLS98] Arnaud Beauville, Yves Laszlo, and Christoph Sorger, The Picard group of the moduli of $G$-bundles on a curve, Compositio Math. 112 (1998), no. 2, 183–216.
[DN89] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1989), no. 1, 53–94.
[Ein80] Lawrence Ein, Stable vector bundles on projective spaces in char $p > 0$, Math. Ann. 254 (1980), 53–72.
[Fal03] Gerd Faltings, Algebraic loop groups and moduli spaces of bundles, J. Eur. Math. Soc. (JEMS) 5 (2003), no. 1, 41–68.
[Gro68] Alexander Grothendieck, Le groupe de Brauer. III. Exemples et compléments, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188.
[Hei] Georg Hein, $SU(r, L)$ is separably unirational, [http://arxiv.org/abs/0810.4079].
[HN75] G. Harder and M. S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Ann. 212 (1974/75), 215–248.
[Ill79] Luc Illusie, Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 4, 501–661.
[KM74] Nicholas M. Katz and William Messing, Some consequences of the Riemann hypothesis for varieties over finite fields, Invent. Math. 23 (1974), 73–77.
[MS02] V. B. Mehta and S. Subramanian, On the fundamental group scheme, Invent. Math. 148 (2002), no. 1, 143–150.
[Nor82] Madhav V. Nori, The fundamental group scheme, Proceedings of Indian Academy of Science (Math. Sci.) 91 (1982), no. 2, 73–122.
[Nyg78] Niels Nygaard, On the fundamental group of a unirational 3-fold, Invent. Math. 44 (1978), no. 1, 75–86.
[Ser59] J.-P. Serre, On the fundamental group of a unirational variety, J. London Math. Soc. 34 (1959), 481–484.
[Ses82] C. S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque, vol. 96, Société Mathématique de France, Paris, 1982, Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.
[SGA71] Revêtements étales et groupe fondamental, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.
[SGA73] Groupes de monodromie en géométrie algébrique. II, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in Mathematics, Vol. 340.
[VBM08] T. E. Venkata Balaji and V. B. Mehta, Singularities of moduli spaces of vector bundles over curves in characteristic 0 and $p$, Michigan Math. J. 57 (2008), 37–42.
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