OUTPUT FEEDBACK OVERLAPPING CONTROL DESIGN
OF INTERCONNECTED SYSTEMS WITH INPUT SATURATION

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ABSTRACT. In this paper, we establish new results to the problem of output feedback control design for a class of nonlinear interconnected continuous-time systems subject to input saturation. New schemes based on overlapping design methodology are developed for both static and dynamic output feedback control structures. The theoretical developments are illustrated by numerical simulations of a linearized nuclear power plant model.

1. Introduction. Many of contemporary technological problems involve complex and interconnected systems. For such systems, it is usually necessary to avoid centralized controllers due to a number of reasons such as the cost of implementation, complexity of on-line computations as well as complexity of controller design and reliability [1]. The development of techniques to design output feedback controllers for interconnected systems has been of great interest since the past few years [2]–[10]. Various theorems were developed for calculating the unknowns of the dynamic output feedback controller gain matrix. Applications of multi-agent control designs subject to feedback have been generalized and looked upon. In the area of decentralized control designs it has been implied that the system with local feedback closed around the sub-systems is generally stable. Several methods were used for the development of static and dynamic output feedback designs as in [36, 12, 32, 24, 35, 9, 26, 17, 34]. LMI solution to the decentralized output feedback control problem for the interconnected non-linear systems was developed in [2], where the interacting non-linearity of each subsystem was considered to be bounded by a quadratic form of states of the overall system. Local output signals from each subsystems were used to generate the local control inputs. The robust stabilization problem of a class of nonlinear interconnected systems was considered in [3] and a decentralized dynamic output feedback controller was proposed. The authors formulated the controller design in the LMI framework, and used local sliding mode observers for the subsystems state estimation. However, the problem of designing local observers that are robust with respect to measurement noise is still unresolved. In [9], the authors developed a new LMI-based procedure for the design of

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decentralized dynamic output controllers for systems composed of linear subsystems coupled by uncertain nonlinear interconnections satisfying quadratic constraints. The scheme utilizes the general linear dynamic output feedback structure. The design procedure consists of two steps, the first providing the local Lyapunov matrices together with the corresponding robustness degrees, and the second the controller parameters providing a robustly stable overall system. A comprehensive review of decentralized control design techniques was provided in [10]. The synthesis of output feedback controllers with saturating inputs was studied in [16] where an observer based controller and a dynamic output feedback controller based on the circle criterion was developed via LMI formulation. In [17], the authors presented a method for designing an output feedback law that stabilizes a linear system subject to actuator saturation with a large domain of attraction. This method applies to general linear systems including strictly unstable ones. A nonlinear output feedback controller is first expressed in the form of a quasi-LPV system. Conditions under which the closed-loop system is locally asymptotically stable are then established in terms of the coefficient matrices of the controller. The design of the controller (gain matrices) that maximizes an estimate of the domain of attraction is then formulated and solved as an optimization problem with LMI constraints. On another research front, the behavior of linear, time-invariant (LTI) systems subject to saturation has been extensively studied for several decades. It is known that saturation usually degrades the performance of a system and leads to instability. Over the last years systems subject to saturation has attracted a lot of researchers and a considerable amount of work has been done. Most of the study has been done on systems subject to actuator saturation, which involves problems as global, semi-global stabilization and local stabilization, anti-windup compensation, null controllable regions [2]–[27], to mention a few.

More recently, some systematic design procedures based on rigorous theoretical analysis have been proposed through various frameworks, see [19] for a nice overview of application cases requiring a formal treatment of the saturation constraints. Most of the research efforts geared toward constructive linear or nonlinear control for saturated plants. By a major approach, called anti-windup design, a pre-designed controller is given, so that its closed-loop with the plant without input saturation is well behaved (at least asymptotically stable but possibly inducing desirable unconstrained closed-loop performance). The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention, see [20, 21, 22] and the references therein. One approach to dealing with actuator saturation has been to take control constraints into account at the outset of control design. A low-and-high gain method was presented in [22] to design linear semi-globally stabilizing controllers. The overlapping decomposition principle has been used extensively for interconnected systems for the design of the feedback controller [10]. In [30], a robust decentralized fixed structure $H_\infty$ frequency stabilizer design based on overlapping decomposition for frequency stabilization in interconnected power systems was presented. The control parameters of frequency stabilizer are optimized by a tabu search.

In this paper we use the principle of overlapping decomposition to design output feedback controllers for each of the interconnected subsystems. This method helps us to differentiate each subsystem and design a local output feedback controller for the same and finally contract them to the original system comprising of all the subsystems. Controlling the interconnected systems with saturating inputs
we have used the Overlapping design methodology and applied two types of controllers for the design, that is, static and dynamic output feedback schemes. For the design of controllers we firstly expand the system and carry on with the control design methods and after the controller has been designed we utilize the overlapping decomposition technique of contracting the expanded system to the original form.

2. Problem Statement. Consider a nonlinear interconnected system composed of a finite number $N$ of coupled subsystems subject to input saturation and represented by:

$$\dot{x}(t) = Ax(t) + B\text{sat}(u(t)) + h(t, x(t)), \quad y(t) = Cx(t)$$  \hspace{1cm} (1)

where $x = [x_1^T, ..., x_N^T]^T \in \mathbb{R}^n$, $n = \sum_{j=1}^N n_j$ is the overall system state, $\text{sat}(u) = [\text{sat}(u_1)^T, ..., \text{sat}(u_N)^T]^T \in \mathbb{R}^m$, $m = \sum_{j=1}^N m_j$ is the saturated input of the overall system and $y = [y_1^T, ..., y_N^T]^T \in \mathbb{R}^p$, $p = \sum_{j=1}^N p_j$ is the measured output of the overall system. The model matrices are $A = \text{diag}[A_{11}, ..., A_{NN}]$, $A_{jj} \in \mathbb{R}^{n_j \times n_j}$, $B = \text{diag}[B_1, ..., B_N]$, $B_j \in \mathbb{R}^{n_j \times m_j}$ and $C = \text{diag}[C_1, ..., C_N]$, $C_j \in \mathbb{R}^{p_j \times n_j}$. The function

$$h(t, x(t)) = [h_1^T(t, x(t)), ..., h_N^T(t, x(t))]^T$$

is a vector function piecewise-continuous in its arguments. In the sequel, we assume that this function is uncertain and the available information is that, in the domain of continuity $G$, is satisfies the global quadratic inequality

$$h^T(t, x(t))h(t, x(t)) \leq x^T(t)\tilde{R}^{\text{T}}\tilde{\Phi}^{-1}\tilde{R}x(t)$$  \hspace{1cm} (2)

where $\tilde{R} = \text{diag}[\tilde{R}_1^T, ..., \tilde{R}_N^T]^T$, $\tilde{R}_j \in \mathbb{R}^{r_j \times n}$ are constant matrices such that $h(t, 0) = 0$ and $x = 0$ is an equilibrium state of system (1). The $j$th subsystem model of system (1) can be described by

$$\begin{align*}
\dot{x}_j(t) &= A_{jj}x_j(t) + B_j\text{sat}(u_j(t)) + h_j(t, x) \\
y_j(t) &= C_jx_j(t)
\end{align*}$$  \hspace{1cm} (3)

where $x_j \in \mathbb{R}^{n_j}$, $u_j \in \mathbb{R}^{m_j}$, $y_j \in \mathbb{R}^{p_j}$ are the subsystem state, input and measured output, respectively. The function $h_j \in \mathbb{R}^{p_j}$ is a piecewise-continuous vector function in its arguments and in line of (2) it satisfies the quadratic inequality

$$h_j^T(t, x(t))h_j(t, x(t)) \leq \phi_j^2 x^T(t)\tilde{R}_j^T\tilde{R}_jx(t)$$  \hspace{1cm} (4)

where $\phi_i > 0$, $j \in \{1, ..., N\}$ are bounding parameters such that

$$\tilde{\Phi} = \text{diag}[\phi_1^{-2}I_{r_1}, ..., \phi_N^{-2}I_{r_N}]$$

with $I_{r_j} \in \mathbb{R}^{r_j \times r_j}$ representing the identity matrix. From (2) and (4), it is always possible to find a matrix $\Phi$ such that

$$h^T(t, x(t))h(t, x(t)) \leq x^T(t)R^T\Phi^{-1}Rx(t)$$  \hspace{1cm} (5)

where $R = \text{diag}[R_1^T, ..., R_N^T]^T$, $\Phi = \text{diag}[\delta_1^{-2}I_{r_1}, ..., \delta_N^{-2}I_{r_N}]$ and $\delta_j = \phi_j^{-2}$. The saturation function $\text{sat}(u_j)$ is for $u_j \in \mathbb{R}^{m_j}$ and is defined as,

$$\text{sat}(u_j) = \begin{cases} u_{j_{\text{max}}} & u_j \geq u_{j_{\text{max}}} \\ u_j & u_{j_{\text{min}}} < u_j < u_{j_{\text{max}}} \\ u_{j_{\text{min}}} & u_j \leq u_{j_{\text{min}}} \end{cases}, \quad \text{sat}(u) = \begin{bmatrix} \text{sat}(u_1) \\ \text{sat}(u_2) \\ \vdots \\ \text{sat}(u_m) \end{bmatrix}$$  \hspace{1cm} (6)
for all \( u \in \mathbb{R}^m \), where \( u_{j\text{min}} \) and \( u_{j\text{max}} \) are chosen to correspond to actual input limits either by measurement or by estimation. Input saturation can also be applied as upper and lower limits of input constraints as \( u_{j\text{min}} \) and \( u_{j\text{max}} \) respectively. It is also assumed that the N-pairs \( A_{jj}, B_j \) are controllable and the N-pairs \( C_j, A_{jj} \) are observable for all \( j = 1, 2, \ldots, N \).

**Remark 1.** It is significant to observe that the local function \( h_j(\cdot, \cdot) \) depends on the full state vector \( x(t) \) and therefore inequality (4) for \( j = 1, 2, \ldots, N \) represents a set of coupling relations that has to be manipulated simultaneously in order to achieve the desired objective.

Our objective in this paper is to design a constant output feedback controller and a dynamic output feedback controller that stabilizes the interconnected continuous system subject to input saturation using the overlapping decomposition technique.

### 3. Overlapping Decomposition

In most of the systems, the subsystems share common parts. It is advantageous to use this fact for building and designing of a decentralized control using overlapping information from these subsystems. On using the overlapping technique it is clear that the overlapping subsystems appear as disjoint. Using each subsystem as different systems a decentralized control can be designed in the expanded space. These designed controls are later contracted for their implementation in the original system. For these kind of systems a mathematical framework known as Inclusion Principle is used. The Inclusion Principle was proposed in the early 1980s in the context of analysis and control of complex and systems [10, 23, 24].

The main idea of the Inclusion Principle is to expand an initial system, with shared components, into higher dimensions in which overlapped subsystem appear as disjoint. Under certain conditions the expanded system contains the essential information about the initial system. The relation between the initial and the expanded system is constructed on the basis of appropriate linear transformations. These include a set of complementary matrices which have to satisfy well established necessary and sufficient conditions to ensure the Inclusion Principle. The conditions given in previous works [23, 24] on the complementary matrices to ensure the Inclusion Principle have a fundamental, implicit nature, in the sense that they have the form of matrix products from which it is not easy to select specific values for the matrices. The selection of these matrices helps in obtaining the expanded system and studying their properties. These matrices influence on properties like stability, controllability or even observability.

### 4. Controller Designs

In this section we will consider the two designs for the interconnected system subject to input saturation, one being the observer design and the other as dynamic output feedback design using the overlapping decomposition technique. The overlapping decomposition technique, in general, is

The system under consideration is described by the following state-space model:

\[
S : \quad \dot{x} = Ax + Bu
\]  

(7)

In order to successfully apply the decentralized control methodology, the input matrix \( B \) of a certain system should be in block-diagonal form. The system can then be decomposed into multiple subsystems with orders equal to the rows of the
corresponding block of the input matrix $B$. After performing the permutations, the system described by (7) can be described as

$$
\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u
$$

(8)

$$
\bar{A} = \begin{bmatrix}
\bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\
\bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\
\bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & \bar{A}_{34} \\
\bar{A}_{41} & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44}
\end{bmatrix}, \quad \bar{B} = \text{diag}([\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4])
$$

(9)

where the sub-matrices have appropriate dimensions. The seven components of the state vector $\bar{x}$ are arranged to four overlapping components as follows:

$$
\bar{x}_1 = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^T, \quad \bar{x}_2 = \begin{bmatrix} \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \end{bmatrix}^T
$$

(11)

$$
\bar{x}_3 = \begin{bmatrix} \bar{x}_4 & \bar{x}_5 & \bar{x}_6 \end{bmatrix}^T, \quad \bar{x}_4 = \begin{bmatrix} \bar{x}_6 & \bar{x}_7 \end{bmatrix}^T
$$

(12)

These four overlapping state vectors components constitute a new state vector

$$
\dot{\hat{x}} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 \end{bmatrix}^T, \quad \hat{x} = V\bar{x}, \quad V \in \mathbb{R}^{\hat{n} \times n}
$$

(14)

$$
V = \begin{bmatrix}
I_5 & 0_{5 \times 2} & 0_{5 \times 3} & 0_{5 \times 2} & 0_{5 \times 3} \\
0_{2 \times 5} & I_2 & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 3} \\
0_{2 \times 5} & I_2 & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 3} \\
0_{3 \times 5} & 0_{3 \times 2} & I_3 & 0_{3 \times 2} & 0_{3 \times 2} & 0_{3 \times 3} \\
0_{2 \times 5} & 0_{2 \times 2} & 0_{2 \times 3} & I_2 & 0_{2 \times 2} & 0_{2 \times 3} \\
0_{2 \times 5} & 0_{2 \times 2} & 0_{2 \times 3} & I_2 & 0_{2 \times 2} & 0_{2 \times 3} \\
0_{3 \times 5} & 0_{3 \times 2} & 0_{3 \times 3} & 0_{3 \times 2} & I_3 & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{2 \times 5} & 0_{2 \times 2} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 3} & I_2 & 0_{2 \times 3} \\
0_{2 \times 5} & 0_{2 \times 2} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 3} & I_3 & 0_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix}
$$

(15)

and $\hat{n} = 26$, hence $V$ is a $26 \times 20$ matrix. $I_i$ represent an identity matrix of order $i$ and $0_{i \times j}$ represent a $i \times j$ zero matrix, with the state $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$. If we define the following relations

$$
\bar{A} = U\bar{A}V + M, \quad \bar{B} = VB + N, \quad \bar{Q} = U^TQU + M_Q, \quad \bar{R} = R + N_R
$$

(16)

where we assume $UV = I$, and $\bar{Q}$, $Q$, $\bar{R}$ and $R$ are the appropriate weighting matrices in LQR designs. $M$, $N$, $M_R$ and $N_R$ are real matrices of appropriate dimensions.

Let $\bar{J}$ represent the LQR performance index for system $\bar{S}$ and $(\bar{S}, \bar{J})$ represent the system in the expanded space and its corresponding LQR performance index, respectively, then the so called inclusion conditions for $(\bar{S}, \bar{J})$ are stated in the following theorem [21].
Theorem 4.1. \( \tilde{S}, \tilde{J} \supset (S, J) \) if either

\[
(i) \quad MV = 0, \quad N = 0, \quad V^T M_Q V = 0, \quad N_R = 0 \tag{17}
\]

or

\[
(ii) \quad U M_i V = 0, \quad M_Q M_i^{-1} N = 0, \quad M_Q M_i^{-1} V = 0

U M_i^{-1} N = 0, \quad N_R = 0, \quad \forall i \in \tilde{n} \tag{18}
\]

For proof see chapter 8 of [21]. We chose \( \tilde{Q}, Q, \tilde{R} \) and \( R \) as identity matrices as earlier and (though only one one of them suffice,) both of the conditions of the aforementioned theorem are verified by choosing

\[
M = 0, \quad M_Q = 0, \quad N = 0, \quad N_R = 0 \tag{19}
\]

With these conditions, the expanded system can be expressed as:

\[
\tilde{S} : \dot{x} = \tilde{A} \dot{x} + \tilde{B} U
\]

where

\[
\tilde{A} = V \hat{A} \tag{20}
\]

\[
\hat{B} = V \hat{B} \tag{21}
\]

The matrix \( U \in \mathbb{R}^{20 \times 26} \) is defined using (16) as

\[
U = \hat{A}^{-1} V^T \hat{A} \tag{23}
\]

where \( V^T \) is the pseudo inverse of \( V \). The \( \hat{A} \) is now expressed as

\[
\hat{A} = \begin{bmatrix}
A_{11} & \tilde{A}_{12} & 0 & 0 & 0 & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} & \tilde{A}_{16} & \tilde{A}_{17} \\
\tilde{A}_{21} & \tilde{A}_{22} & 0 & 0 & 0 & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{A}_{27} \\
\tilde{A}_{21} & 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & 0 & 0 & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{A}_{27} \\
\tilde{A}_{33} & 0 & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} & 0 & 0 & \tilde{A}_{35} & \tilde{A}_{36} & \tilde{A}_{37} \\
\tilde{A}_{41} & 0 & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} & 0 & 0 & \tilde{A}_{45} & \tilde{A}_{46} & \tilde{A}_{47} \\
\tilde{A}_{41} & 0 & \tilde{A}_{42} & \tilde{A}_{43} & 0 & \tilde{A}_{44} & \tilde{A}_{45} & \tilde{A}_{46} & 0 & \tilde{A}_{47} \\
\tilde{A}_{51} & 0 & \tilde{A}_{52} & \tilde{A}_{53} & 0 & \tilde{A}_{54} & \tilde{A}_{55} & \tilde{A}_{56} & 0 & \tilde{A}_{57} \\
\tilde{A}_{61} & 0 & \tilde{A}_{62} & \tilde{A}_{63} & 0 & \tilde{A}_{64} & \tilde{A}_{65} & \tilde{A}_{66} & 0 & \tilde{A}_{67} \\
\tilde{A}_{61} & 0 & \tilde{A}_{62} & \tilde{A}_{63} & 0 & \tilde{A}_{64} & \tilde{A}_{65} & 0 & \tilde{A}_{66} & \tilde{A}_{67} \\
\tilde{A}_{71} & 0 & \tilde{A}_{72} & \tilde{A}_{73} & 0 & \tilde{A}_{74} & \tilde{A}_{75} & 0 & \tilde{A}_{76} & \tilde{A}_{77}
\end{bmatrix} \tag{24}
\]

The overlapping subsystems \( \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \) and \( \tilde{A}_4 \) are now described in the followings:

\[
\tilde{A}_1 = \begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix}
\tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\
\tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\
\tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44}
\end{bmatrix} \tag{25}
\]

\[
\tilde{A}_3 = \begin{bmatrix}
\tilde{A}_{44} & \tilde{A}_{45} & \tilde{A}_{46} \\
\tilde{A}_{54} & \tilde{A}_{55} & \tilde{A}_{56} \\
\tilde{A}_{64} & \tilde{A}_{65} & \tilde{A}_{66}
\end{bmatrix}, \quad \tilde{A}_4 = \begin{bmatrix}
\tilde{A}_{66} & \tilde{A}_{67} \\
\tilde{A}_{76} & \tilde{A}_{77}
\end{bmatrix} \tag{26}
\]

The interconnections among the overlapped subsystems can be easily obtained by simple inspection of \( \hat{A} \) in (24). system. Interconnection matrices \( \hat{H}_i, i = 1, 2, 3, 4 \) associated with each of the subsystem is given by

\[
\hat{H}_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} & \tilde{A}_{16} & \tilde{A}_{17} \\
0 & 0 & 0 & 0 & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{A}_{27}
\end{bmatrix} \tag{27}
\]
Since we have chosen where $\bar{\varphi}$ is contractible to control law $u = -\bar{K}\hat{x}$ if and only if [21]:

$$FM^{i-1}V = 0, \quad FM^{i-1}N = 0 \quad \forall i \in \bar{n}$$

(32)

$$\bar{K} = KU + F$$

(33)

Since we have chosen $M = 0$ and $N = 0$, the aforementioned conditions are satisfied. By Corollary 8.14 in [21] if

$$MV = 0, \quad N = 0$$

(34)

then, the contracted $K$ can be obtained as:

$$K = \check{K}V$$

(35)

Furthermore, the stabilizability check was performed for all the four subsystems which showed that all the subsystems are stabilizable. and non of the subsystems is controllable.

4.1. Observer Based Design. We will consider the whole expanded system for the design of the Observer based controller, which is given as

$$\begin{align*}
\check{S}: \dot{\hat{x}}(t) &= \hat{A}_D\hat{x}(t) + \hat{B}_D sat(u(t)) + \hat{H}_D(t, \hat{x}) \\
y(t) &= \hat{C}_D\hat{x}(t)
\end{align*}$$

(36)

where, $\hat{A}_D = diag(A_1, \ldots, A_N), \hat{B}_D = diag(B_1, \ldots, B_N), \hat{C}_D = diag(C_1, \ldots, C_N)$ and $\hat{H}_D$ will be the diagonal interconnections as $\hat{H} = diag(H_1, H_2, \ldots, H_N)$. Also the interconnections bounds (2) will be as follows,

$$\tilde{h}_j(t, \hat{x})\tilde{h}_j(t, \hat{x}) \leq \hat{x}^T(\sum_{i=1}^{N} \alpha_i^2 \hat{H}_i^T \hat{H}_i)\hat{x} := \hat{x}^T \Gamma^T \Gamma \hat{x}$$

(37)

The pair $(\hat{A}_D, \hat{B}_D)$ is controllable and the pair $(\hat{C}_D, \hat{A}_D)$ is observable, which is the direct result of each subsystem being controllable and observable. Using the above expanded system we will be designing a decentralized linear controller and a decentralized linear observer that will stabilize the system. We consider the following linear decentralized controller and observer,

$$\begin{align*}
\dot{\hat{x}}(t) &= \hat{A}_D\hat{x}(t) + \hat{B}_D sat(u(t)) + \check{L}_D(y - \check{C}_D)\hat{x}(t) \\
u(t) &= \check{K}_D\hat{x}(t)
\end{align*}$$

(38)

The expanded input matrix $\check{B}$ is given by:

$$\check{B} = \begin{bmatrix}
\check{B}_1^T & 0 & 0 & 0 & 0 & 0 \\
0 & \check{B}_2^T & 0 & 0 & 0 & 0 \\
0 & 0 & \check{B}_3^T & 0 & 0 & 0 \\
0 & 0 & 0 & \check{B}_4^T & 0 & 0 \\
\end{bmatrix}^T$$

(31)
where $\hat{K}_D = \text{diag}(\hat{K}_1, \ldots, \hat{K}_N)$ and $\hat{L}_D = \text{diag}(\hat{L}_1, \ldots, \hat{L}_N)$ are the controller gain matrix and the observer gain matrix, respectively. The closed loop dynamics of the expanded system is,

$$
\dot{\hat{x}}(t) = (\hat{A}_D + \hat{B}_D \hat{K}_D)\hat{x}(t) - \hat{B}_D \hat{K}_D \hat{\dot{x}}(t) + \hat{H}(t)$$

(39)

$$
\dot{\hat{x}}(t) = (\hat{A}_D - \hat{L}_D \hat{C}_D)\hat{x}(t) + \hat{H}(t)
$$

where, $\hat{H}(t)$ is the interconnection function for the expanded system. Let,

$$
\hat{A}_c = \hat{A}_D + \hat{B}_D \hat{K}_D,
\hat{A}_o = \hat{A}_D - \hat{L}_D \hat{C}_D
$$

(40)

The closed-loop dynamics will be in the form of,

$$
\dot{\hat{x}}(t) = \hat{A}_c \hat{x}(t) - \hat{B}_D \hat{K}_D \hat{\dot{x}}(t) + \hat{H}(t)
$$

(41)

$$
\dot{\hat{x}}(t) = \hat{A}_o \hat{x}(t) + \hat{H}(t)
$$

For each subsystem we find the controller and observer gain matrices $K_D$ and $L_D$. For simplicity we will, from here on, use the expanded system only and after the designing of the controller and observer gain matrices for the expanded system($\hat{S}$) contract it for the actual system($S$) using the overlapping decomposition technique.

**Theorem 4.2.** Consider the following optimization problem for finding the Controller $K_D$ and Observer $L_D$ gains of each subsystems. With $Y$, $P_o$, $K_D$, $L_D$ and $\gamma_i, i \in I$, from the following optimization problem,

$$
\min \sum_{i=1}^{N} \gamma_i \text{ subject to } Y > 0, \ P_o > 0 \begin{bmatrix}
F_c & S_1 & S_2 \\
S_1^T & W_o & P_o \\
S_2^T & P_o & -I
\end{bmatrix} < 0
$$

The optimization problem (42) has to be solved by two steps [22].

**Step1.** Maximize the interconnection bounds $\alpha_i(=\frac{1}{\gamma_i})$ by solving the following optimization problem,

$$
\min \sum_{i=1}^{N} \gamma_i \text{ subject to } Y > 0, \ F_c < 0
$$

(42)

The optimization problem (42) gives $Y$ and $\bar{M}_D$. The control gain can be calculated as,

$$
K_D = \bar{M}_DY^{-1}
$$

(43)

**Step2.** Using the $K_D$ obtained from Step1, find $P_o$ and $N_D$ by solving the following optimization problem:

$$
\min \sum_{i=1}^{N} \beta_i \text{ subject to } P_o > 0 \ \Lambda > 0 \begin{bmatrix}
\Lambda F_c & S_1 & S_2 \\
S_1^T & W_o & P_o \\
S_2^T & P_o & -I
\end{bmatrix} < 0
$$

(44)

where, $\Lambda = \text{diag}(\beta_1I_1, \ldots, \beta_NI_N)$, $I_i$ denotes the $n_i \times n_i$ identity matrix, and

$W_o = A_{D}^TP_o + P_oA_{D} - N_DC_D - (N_DC_D)^T$ and $N_D = P_oL_D$. The matrices $F_c$ and $S_1$ in Step2 are obtained from step1. The observer gain $L_D$ is obtained as,

$$
L_D = P_o^{-1}N_D
$$

(45)
Proof We look at the following Lyapunov function candidate,

$$V(x, \bar{x}) = x^T P_c x + \bar{x}^T P_o \bar{x}$$

(46)

where, $P_c > 0$ and $P_o > 0$. The time derivative of $V(x, \bar{x})$ along the trajectories of (42) is given by,

$$\dot{V}(x, \bar{x}) = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}^T \begin{bmatrix} A_c P_c + P_c A_c & -P_c B DK_c \\ -K_c^T B_D P_c & A_o P_o + P_o A_o \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} < 0.$$  

(47)

we consider the bounds on the interconnections (2) this condition will be equivalent to,

$$\begin{bmatrix} x \\ \bar{x} \end{bmatrix}^T \begin{bmatrix} -\Gamma^T \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \leq 0.$$  

(48)

The stabilization of system (42) requires that,

$$\dot{V}(x, \bar{x}) < 0$$

(49)

for all $x, \bar{x} \neq 0$; together with the condition (48), one can obtain that if

$$\begin{bmatrix} A_c^T P_c + P_c A_c & -P_c B DK_c & P_c \\ -K_c^T B_D P_c & A_o^T P_o + P_o A_o & P_o \end{bmatrix} - \tau \begin{bmatrix} -\Gamma^T \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} < 0.$$  

(50)

$\bar{P}_c > 0$, $\bar{P}_o > 0$, $\tau > 0$.

The inequality of (49) is satisfied,

$$P_c = \frac{\bar{P}_c}{\tau} P_o = \frac{\bar{P}_o}{\tau}$$

the above conditions (50) is equivalent to,

$$\begin{bmatrix} A_c^T P_c + P_c A_c + \Gamma^T \Gamma & -P_c B DK_c & P_c \\ -K_c^T B_D P_c & A_o^T P_o + P_o A_o & P_o \end{bmatrix} - \tau \begin{bmatrix} -\Gamma^T \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} < 0.$$  

(51)

$P_c > 0$, $P_o > 0$.

Considering (37) and (40), and applying the Schur complement to the inequality (51) we get,

$$P_c > 0, P_o > 0$$

$$\begin{bmatrix} W_c & -P_c B DK_d & P_c & \alpha_1 H_1^T & \ldots & \alpha_N H_N^T \\ -(P_c B DK_d)^T & W_o & P_o & 0 & \ldots & 0 \\ P_c & P_o & -I & 0 & \ldots & 0 \\ \alpha_1 H_1 & 0 & 0 & -I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_N H_N & 0 & 0 & 0 & \ldots & -I \end{bmatrix} < 0.$$  

(52)

$$W_c = A_c^T P_c + P_c A_D + (P_c B DK_d)^T + (P_c B DK_d)$$

$$W_o = A_o^T P_o + P_o A_D - P_o L_D C_D - (P_o L_D C_D)^T.$$  

(53)
Rearranging and scaling columns and rows related to \( H_i, i \in I \) of (52) we obtain,

\[
\begin{bmatrix}
W_c & H_1^T & \cdots & H_N^T & -P_cB_DK_D & P_c \\
H_1 & -\gamma I & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & 0 \\
H_N & 0 & \cdots & -\gamma_N I & 0 & 0 \\
-(P_cB_DK_D)^T & 0 & 0 & 0 & W_o & P_o \\
P_c & 0 & 0 & 0 & P_o & -I \\
\end{bmatrix}
\]

where \( \gamma_i = \frac{1}{\sigma_i^2} > 0 \). The optimization problem now becomes as follows,

\[
\text{Minimize } \sum_{i=1}^{N} \gamma_i \text{ subject to (54)}
\]  

(54)

This selection of the control gain matrix \( K_D \) and the observer gain matrix \( L_D \) does not only stabilize the overall system (42) but also simultaneously maximizes the interconnection bounds \( \alpha_i \).

Since there are coupled terms of matrix variables, \( P_c \) and \( K_D \), and \( P_o \) and \( L_D \) in the inequality (54), the above inequality becomes a BMI. We will transform the inequality to a form which is affine in the unknown variables. To achieve this, we introduce variables,

\[
M_D = P_cB_DK_D, \quad N_D = P_oL_D.
\]  

(55)

Then the optimization problem (54) becomes,

\[
\text{Minimize } \sum_{i=1}^{N} \gamma_i \text{ subject to } P_c > 0, \ P_o > 0
\]  

(56)

The solution to the above optimization problem gives \( M_D \) and \( N_D \). The controller and observer gain matrices were obtained as \([9, 13]\) as,

\[
L_D = P_o^{-1}N_D
\]

The controller gain matrix \( K_D \) can be obtained if the matrix \( B_D \) is invertible,

\[
K_D = B_D^{-1}P_C^{-1}M_D
\]

If matrix \( B_D \) is not invertible, i.e. very restrictive, then it becomes very difficult to obtain the control gain matrix \( K_D \) from (57).
To overcome this we will pre and post multiply the (54) by $\text{diag}(P_c^{-1}, I)$ and define $Y = P_c^{-1}$ to obtain the following conditions which are equivalent to (54):

\[
Y > 0, \quad P_o > 0 \tag{57}
\]

\[
\begin{bmatrix}
W_c' & YH_1^T & \ldots & YH_N^T & -BDK_d & I \\
H_1Y & -\gamma_1I & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & 0 \\
H_NY & 0 & \ldots & -\gamma_NI & 0 & 0 \\
-(BDK_d)^T & 0 & 0 & 0 & W_o & P_o \\
I & 0 & 0 & 0 & P_o & -I
\end{bmatrix} < 0
\]

where,

\[
W_c' = YA_D^T + A_DY + (BDK_d)^T + (BDK_d) \tag{58}
\]

Let,

\[
\begin{align*}
\bar{M}_D &= KDY \\
[ S_1 & S_2 ] &= \begin{bmatrix}
-BDK_d & I \\
0 & 0 \\
\vdots & \vdots \\
0 & 0
\end{bmatrix}, \\
F_c &= \begin{bmatrix}
W_c' & YH_1^T & \ldots & YH_N^T \\
H_1Y & -\gamma_1I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
H_NY & 0 & \ldots & -\gamma_NI
\end{bmatrix} < 0, \\
F_o &= \begin{bmatrix}
W_o & P_o \\
P_o & -I
\end{bmatrix} < 0
\end{align*}
\]

with these definitions, the problem now is to find $Y, P_o, K_D, L_D$ and $\gamma_i, i \in I$, from the following optimization problem,

\[
\text{Minimize} \sum_{i=1}^{N} \gamma_i \text{ subject to } Y > 0, \quad P_o > 0 \tag{59}
\]

\[
\begin{bmatrix}
F_c & S_1 & S_2 \\
S_1^T & W_o & P_o \\
S_2^T & P_o & -I
\end{bmatrix} < 0
\]

The optimization problem (59) has to be solved by two steps [22],

**Step1.** Maximize the interconnection bounds $\alpha_i (= \frac{1}{\gamma_i})$ by solving the following optimization problem,

\[
\text{Minimize} \sum_{i=1}^{N} \gamma_i \text{ subject to } Y > 0, \quad F_c < 0 \tag{60}
\]

The optimization problem (60) gives $Y$ and $\bar{M}_D$. The control gain can be calculated as,

\[
K_D = \bar{M}_DY^{-1} \tag{61}
\]
Step 2. Using the $K_D$ obtained from Step 1, find $P_o$ and $N_D$ by solving the following optimization problem:

\[
\text{Minimize } \sum_{i=1}^{N} \beta_i \text{ subject to } P_o > 0 \Lambda > 0
\]

\[
\begin{bmatrix}
\Lambda F_e & S_1 & S_2 \\
S_1^T & W_o & P_o \\
S_2^T & P_o & -I
\end{bmatrix} < 0
\]

where, $\Lambda = \text{diag}(\beta_1 I_1, \ldots, \beta_N I_N)$, $I_i$ denotes the $n_i \times n_i$ identity matrix, and $W_o = A_D^T P_o + P_o A_D - N_D C_D - (N_D C_D)^T$ and $N_D = P_o L_D$. The matrices $F_e$ and $S_1$ in Step 2 are obtained from step 1. The observer gain $L_D$ is obtained as,

\[
L_D = P_o^{-1} N_D
\]

The observer and gain matrix calculated for the subsystems will then be contracted and formed back for the original system using the overlapping decomposition principle.

4.2. Dynamic output-feedback design. We will consider the whole expanded system $(\hat{S})$ for the design of the dynamic output feedback controller, which is given as,

\[
\dot{\hat{S}} : \dot{\hat{x}}(t) = \hat{A}_D \dot{\hat{x}}(t) + \hat{B}_D \text{sat}(u(t)) + \hat{H}_D(t, \hat{x})
\]

\[
y(t) = \hat{C}_D \dot{\hat{x}}(t)
\]

where, $\hat{A}_D = \text{diag}(A_1, \ldots, A_N)$, $\hat{B}_D = \text{diag}(B_1, \ldots, B_N)$, $\hat{C}_D = \text{diag}(C_1, \ldots, C_N)$ and $\hat{H}_D$ will be the diagonal interconnections as $\hat{H} = \text{diag}(H_1, H_2, \ldots, H_N)$.

Also the interconnections bounds (2) will be as follows,

\[
\hat{h}_i^j(t, \hat{x}) \hat{h}_j(t, \hat{x}) \leq \hat{x}^T \left( \sum_{i=1}^{N} \alpha_i^2 \hat{H}_i^T \hat{H}_i \right) \hat{x} := \hat{x}^T \Gamma^T \Gamma \hat{x}
\]

The pair $(\hat{A}_D, \hat{B}_D)$ is controllable and the pair $(\hat{C}_D, \hat{A}_D)$ is observable, which is the direct result of each subsystem being controllable and observable. Let,

\[
sat(u(t)) = \begin{bmatrix} sat(u_1(t))^T & sat(u_2(t))^T & \ldots & sat(u_N(t))^T \end{bmatrix}^T
\]

\[
x(t) = \begin{bmatrix} x_1(t)^T & x_2(t)^T & \ldots & x_N(t)^T \end{bmatrix}^T
\]

\[
y(t) = \begin{bmatrix} y_1(t)^T & y_2(t)^T & \ldots & y_N(t)^T \end{bmatrix}^T
\]

where, $x_i \in \mathbb{R}^{n_i}$, $sat(u_i) \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^n$, $i \in \tilde{v} := \{1, 2, \ldots, N\}$ are the states, inputs and outputs of the $i^{th}$ subsystem $S_i$ respectively. Let the dynamic output feedback controller be,

\[
K : \dot{z}(t) = A_k z(t) + B_k y(t)
\]

\[
u(t) = C_k z(t) + D_k y(t)
\]

where, $z \in \mathbb{R}^{n_c}$ is the controller state, $u(t) \in \mathbb{R}^m$ is the controller output and $D_k D = 0$. 
For the above expanded system (64) the dynamic output feedback controller of the form,
\[ \tilde{K}_D : \dot{z}_i(t) = A_k z_i(t) + B_k y_i(t) \]
\[ u_i(t) = C_k z_i(t) + D_k y_i(t) \]
with \( z_i(t) \in \mathbb{R}^{n_i} \) with appropriate dimensions. We will represent the above equations in matrix form as,
\[ \begin{bmatrix} \dot{x}_i(t) \\ \dot{z}_i(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{D_i} + \tilde{B}_{D_i} D_k C_k & \tilde{B}_{D_i} C_k \\ B_k \tilde{C}_{D_i} & A_k \end{bmatrix} \begin{bmatrix} x_i(t) \\ z_i(t) \end{bmatrix} + \sum_{j \neq i} \begin{bmatrix} \tilde{A}_{D_{ij}} \\ 0_{n \times n_p} \end{bmatrix} \begin{bmatrix} x_j(t) \\ z_j(t) \end{bmatrix} \]
(68)

The following theorem will be used for calculating the unknowns in the controller matrix \( \tilde{K}_D \).

**Theorem 4.3.** Given system (1), such that the pair \((A_j, B_j)\) is stabilizable and pair \((C_j, A_j)\) is detectable. If there exists a positive definite matrix \( K_{\min} < I_m \), \( K_{\max} > I_m \) and \( K = K_{\max} - K_{\min} \), matrices \((A_k, B_k, C_k, D_k)\) of suitable dimensions and such that sat\((D_k C x(t) + C_k z(t))\) is sector bounded in \((K_{\min}, K_{\max})\), a symmetric positive definite matrix \( P \in \mathbb{R}^{2n \times 2n} \) and a positive scalar \( \epsilon \) satisfying:
\[ A_k^T P + PA_k + \epsilon P + \frac{1}{2} (F^T K - PB)(F^T K - PB)^T < 0 \]
(69)

with:
\[ A_o = \begin{bmatrix} A - \epsilon B_i K_{\min} D_k C_i & -B_i K_{\min} C_k \\ B_k C_i & A_k - B_k D K_{\min} C_k \end{bmatrix} \]
(70)
\[ B = \begin{bmatrix} B_i \\ B_k D_i \end{bmatrix}, F^T = \begin{bmatrix} C_i D_k^T \\ C_k D_k^T \end{bmatrix} \]
then the saturated closed-loop system resulting from the interconnections of system (64) and the controller (68) is locally asymptotically stable.

**Proof** The open loop interconnection between systems (64) and the controller (68) is,
\[ \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ B_k C_i & A_k \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_i \\ B_k D \end{bmatrix} u \]
(71)
\[ u = \begin{bmatrix} D_k C_i & C_k \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \]

Then, if there exists a diagonal positive definite matrices \( K_{\min} < I_m \) and \( K_{\max} > I_m \), matrices \((A_k, B_k, C_k, D_k)\) of suitable dimensions and such that sat\((D_k C_i x(t) + C_k z(t))\) is sector bounded in \((K_{\min}, K_{\max})\), a symmetric positive definite matrix \( P \in \mathbb{R}^{2n \times 2n} \) and a positive scalar \( \epsilon \) satisfying inequality (69), then it follows the following proposition.

**Proposition 1.** Assume the existence of a triplet\((F, K_{\min}, K_{\max})\) with \( K_{\min} < I_m \) and \( K_{\max} \geq I_m \), and \( K = K_{\max} - K_{\min} \), such that the matrix \( A_i - B_i K_{\min} F_i \) is Hurwitz, pair \((F, A)\) is observable and sat\((F x)\) satisfies the following sector condition,
\[ (\psi(t, y) - K_{\min} y)^T (\psi(t, y) - K_{\max} y) \leq 0 \forall t \geq 0, y \in S \subset \mathbb{R}^m \]
(72)
then the system (64) is locally asymptotically stable in $S(P, \mu)$ defined by:

$$S(P, \mu) = \{ x \in \mathbb{R}^n; x^T P x \leq \mu \} \subset S(F, u_o^{K_{\min}}), \quad \mu > 0$$

where, $S(F, u_o^{K_{\min}}) = \{ x \in \mathbb{R}^n; -u_o \leq F_i \leq u_o \ K_{\max}, i = 1, \ldots, m \}$.

Now in order to find the unknowns of the dynamic output feedback matrix $K$, an LMI formulation using a linearizing change of variables is presented. Partition matrices $P$ and $P^{-1}$ are defined as,

$$P = \begin{bmatrix} \mathcal{Y} & N \\ N^T & \ast \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \mathcal{X} & M \\ M^T & \ast \end{bmatrix} \quad (73)$$

where $\mathcal{X}$ and $\mathcal{Y}$ belong to $\mathbb{R}^{n \times n}$ and are symmetric positive definite. By $\ast$ we denote terms which are not used in the linearizing change of variable, but which are, of course, depending on the matrices appearing in the partition of $P$ and $P^{-1}$. This decomposition is general because no specific structure is assigned to the matrices in partition. Thus it does not reduce the choice for matrix $P$. Now, we define matrices,

$$\Pi_1 = \begin{bmatrix} \mathcal{X} & I_n \\ M^T & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I_n & \mathcal{Y} \\ 0 & N^T \end{bmatrix} \quad (74)$$

It can be easily noticed that $\Pi_1^T \Pi_1 = \Pi_2$ whatever the $\ast$ terms are. Define the change of variables as:

$$\mathcal{A} = \mathcal{Y}(A - B K_{\min} D_k C)\mathcal{X} + N A_k M^T + N B_k C \mathcal{X}$$

$$\mathcal{B} = N B_k - \mathcal{Y} B K_{\min} D_k$$

$$\mathcal{C} = C_k M^T + D_k C \mathcal{X}, \quad \mathcal{D} = D_k \quad (75)$$

Now by pre-multiplying by $\Pi_1^T$ and by post-multiplying by $\Pi_1$ in (69) and by using change of variables define in (75) we get the following inequality in the variables $(\mathcal{X}, \mathcal{Y}, A, B, C, D)$, which is an LMI for a fixed $\epsilon$,

$$\begin{bmatrix} \mathcal{Q} & \mathcal{S} & -B + C^T K \\ \mathcal{S}^T & \mathcal{R} & -\mathcal{Y} B - \mathcal{D} D + C^T D K \\ K \mathcal{C} - B^T \mathcal{C} - B^T \mathcal{Y} - D^T B^T + K D \mathcal{C} - 2 I_m \end{bmatrix} < 0 \quad (76)$$

where,

$$\mathcal{Q} = A \mathcal{X} + \mathcal{X} A^T - B K_{\min} C - C^T K_{\min} B^T + \epsilon \mathcal{X}$$

$$\mathcal{R} = \mathcal{Y} A + A^T \mathcal{Y} - B C - C^T B + \epsilon \mathcal{Y}$$

$$\mathcal{S} = A^T + A - B K_{\min} D C + \epsilon I_n \quad (77)$$

In order to have $P$ as positive definite, the following LMI must be added,

$$\Pi_1^T P \Pi_1 = \Pi_1^T \Pi_2 = \begin{bmatrix} \mathcal{X} & I_n \mathcal{Y} \\ I_n \mathcal{Y} & \ast \end{bmatrix} \quad (78)$$

Finally, by using the change of variables $S(P, \mu)$ included in $S([D_k C], u_o^{K_{\min}})$,

$$\begin{bmatrix} \mathcal{X} & I_n & C^T \\ I_n & \mathcal{Y} & (DC)^T \\ \mathcal{C} & DC & \gamma \ u_o^{2 \ K_{\min}} \end{bmatrix} \geq 0, \forall i = 1, \ldots, m \quad (79)$$

Now, to compute the unknowns in the dynamic output feedback controller $(A_k, B_k, C_k, D_k)$ from $(\mathcal{X}, \mathcal{Y}, A, B, C, D)$ by the following steps,
Step1. Choose invertible matrices M and N such that \( MN^T = I_n - X\bar{Y} \) which is always possible by (78).

Step2. Compute \( \Pi_1 \) and \( \Pi_2 \) and finally \( P = \Pi_2\Pi_1^{-1} \).

Step3. Compute matrices \((\bar{A}_k, \bar{B}_k, \bar{C}_k, \bar{D}_k)\) as follows,

\[
\begin{align*}
\bar{D}_k &= \mathcal{D}, & \bar{C}_k &= (\mathcal{C} - \bar{D}_k\mathcal{C}\bar{X})M^{-T} \\
\bar{B}_k &= N^{-1}(\mathcal{B} + \mathcal{Y}\bar{B}K_{\text{min}}\mathcal{D}_k) \\
\bar{A}_k &= N^{-1}(\mathcal{A} + \mathcal{Y}(\mathcal{A} - \mathcal{B}K_{\text{min}}\mathcal{D}_k\mathcal{C})\mathcal{X})M^{-T} - \bar{B}_k\mathcal{C}M^{-T} \\
&+ N^{-1}\mathcal{Y}\bar{B}K_{\text{min}}\mathcal{C}_k + \bar{B}_kDK_{\text{min}}\mathcal{C}_k
\end{align*}
\] (80)

The matrix \((K)\), with \((A_k, B_k, C_k, D_k)\), calculated for the subsystems will then be contracted and formed back for the original system using the overlapping decomposition principle.

5. Simulation Results. Using the numerical data in the Appendix and the developed control design algorithms, we proceed to perform MATLAB simulation for the original system after all the subsystems were taken and the expanded system was contracted by the overlapping decomposition technique. State trajectories of the of the nuclear power plant after the static and dynamic feedback control designs were plotted and compared in Figs. (1)–(5). The corresponding input trajectories of the nuclear power plant after the static and dynamic control designs were compared in Fig. (6). Finally, the trajectory of outputs of the nuclear power plant after the static and dynamic control designs were compared in Fig. (7).

The simulation results clearly indicate that

- Smooth behavior of the closed-loop system trajectories is guaranteed under overlapping design by either observer-based or dynamic feedback control.
- In cases where observer-based feedback control is superior to dynamic feedback control, the associated control input has larger magnitudes.

6. Conclusions. In this paper, an LMI framework of design methodology was used for the observer based control and dynamic output feedback have been described which utilizes the overlapping decomposition methodology. The expanded systems were taken for the control design and after the design procedure was done for the interconnected systems formed by the expanded system, they were contracted using the overlapping decomposition method and finally the controllers were used for the original system.

7. Appendix.

7.1. Original data. Being of high dimensions, the system matrices \( A \) and \( B \) in (7) are given in partitioned forms as follows:

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\] (81)
Figure 1. State trajectories of state $x_1$ (left top), $x_2$ (right top), $x_3$ (left bottom) and $x_4$ (right bottom)

where

$$A_{11} = \begin{bmatrix}
-400 & 0.0125 & 0.0305 & 0.111 & 0.301 & 1.14 & 3.01 \\
13.125 & -0.0125 & 0 & 0 & 0 & 0 & 0 \\
87.5 & 0 & -0.0305 & 0 & 0 & 0 & 0 \\
78.125 & 0 & 0 & -0.111 & 0 & 0 & 0 \\
158.125 & 0 & 0 & 0 & -0.301 & 0 & 0 \\
46.25 & 0 & 0 & 0 & 0 & -1.14 & 0 \\
16.875 & 0 & 0 & 0 & 0 & 0 & -3.01
\end{bmatrix}$$

$$A_{12} = \begin{bmatrix}
-1781 & -13700 & -13700 & 0.111 & 411 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Figure 2. State trajectories of state $x_5$ (left top), $x_6$ (right top), $x_7$ (left bottom) and $x_8$ (right bottom)

$$A_{13} = [0]_{6 \times 7}, \quad A_{21} = \begin{bmatrix}
0.0756 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$A_{22} = \begin{bmatrix}
-0.1646 & 0.1646 & 0 & 0 & 0 & 0 & 0 \\
0.0570 & -24403 & 0 & 0 & 0 & 0 & 0 \\
0.0570 & 23262 & -23832 & 0 & 0 & 0 & 0 \\
0.0207 & -0.0207 & 0.0103 & 0.634 & -0.509 & 0 & 0 \\
0 & 0 & 0 & -53657 & 307017 & 0.3372 & 0 \\
0 & 0 & 0 & 0.53819 & -0.76642 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.349 & 0 & -0.2034
\end{bmatrix}$$
Figure 3. State trajectories of state $x_9$ (left top), $x_{10}$ (right top), $x_{11}$ (left bottom) and $x_{12}$ (right bottom)

$$A_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.240 & -0.279 & -0.130 & -0.116 & 0.0235 & 0.121 \\ 0 & 0 & 0.2238 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{31} = [0]_{6 \times 7}, \quad A_{32} = \begin{bmatrix} 0 & 0 & 0.33645 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.45 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Figure 4. State trajectories of state $x_{13}$ (left top), $x_{14}$ (right top), $x_{15}$ (left bottom) and $x_{16}$ (right bottom)

\[
A_{33} = \begin{bmatrix}
-0.33645 & 0 & 0 & 0 & 0 & 0 \\
2.5 & -2.5 & 0 & 0 & 0 & 0 \\
0 & 1.45 & -1.45 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.45 & 0 & 0 \\
0 & 0 & 0 & 1.48 & -1.48 & 0 \\
0 & 0 & 0 & 0 & 0.516 & -1.516
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
B_1 & B_2
\end{bmatrix}^\ell
\]

\[
B_1 = \begin{bmatrix}
10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^\ell
\]
\[ B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.03843 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0016 & 0 & 0 & 0 \\ -0.0062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t \]

\[ \rho_{rod_m} = 10^{-3}, \quad \rho_{rodM} = 1.2, \quad W_{FW_m} = 1.0, \quad \leq 8.0 \]
\[ W_{P_m} = -20, \quad W_{PM} = 0.01, \quad Q_m = -10, \quad Q_M = 0.01 \]

7.2. **Permuted data.** The matrix \( \tilde{A} \) is now expressed as
Figure 6. Input trajectories of controls $u_1$ (left top), $u_2$ (right top), $u_3$ (left bottom) and $u_4$ (right bottom)

Figure 12. Trajectories of Input $U_3$ (left) and Input $U_4$ (right)

\[
\tilde{A} = \begin{bmatrix}
A_{11} & \tilde{A}_{12} & 0 & 0 & 0 & \tilde{A}_{13} & A_{14} & \tilde{A}_{15} & A_{16} & A_{17} \\
A_{21} & \tilde{A}_{22} & 0 & 0 & 0 & A_{23} & \tilde{A}_{24} & A_{25} & \tilde{A}_{26} & A_{27} \\
A_{21} & 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & 0 & 0 & \tilde{A}_{25} & \tilde{A}_{26} & A_{27} \\
A_{31} & 0 & A_{32} & A_{33} & A_{34} & 0 & 0 & A_{35} & \tilde{A}_{36} & A_{37} \\
A_{41} & 0 & A_{42} & A_{43} & A_{44} & 0 & 0 & A_{45} & \tilde{A}_{46} & A_{47} \\
A_{41} & 0 & A_{42} & A_{43} & 0 & A_{44} & \tilde{A}_{45} & A_{46} & 0 & A_{47} \\
A_{51} & 0 & A_{52} & A_{53} & 0 & A_{54} & A_{55} & \tilde{A}_{56} & A_{57} & 0 \\
A_{61} & 0 & A_{62} & A_{63} & 0 & A_{64} & A_{65} & A_{66} & 0 & A_{67} \\
A_{61} & 0 & A_{62} & A_{63} & 0 & A_{64} & A_{65} & 0 & A_{66} & A_{67} \\
A_{71} & 0 & A_{72} & A_{73} & 0 & A_{74} & A_{75} & 0 & A_{76} & A_{77}
\end{bmatrix}
\] 

(82)
Figure 7. Trajectories of outputs $y_1$ (left top), $y_2$ (right top), $y_3$ (left bottom) and $y_4$ (right bottom)

The overlapping subsystems $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ and $\tilde{A}_4$ are now described in the followings:

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

$$\tilde{A}_{11} = \begin{bmatrix} -400.0000 & 0.0125 & 0.0305 & 0.1110 \\ 13.1250 & -0.0125 & 0 & 0 \\ 87.5000 & 0 & -0.0305 & 0 \\ 78.1250 & 0 & 0 & -0.1110 \end{bmatrix},$$

$$\tilde{A}_{12} = \begin{bmatrix} 0.3010 & 0 & 3.0100 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_{21} = \begin{bmatrix} 158.1250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 16.8750 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_{22} = \begin{bmatrix} -0.3010 & 0 & 0 \\ 0 & -0.2034 & 0 \\ 0 & 0 & -3.0100 \end{bmatrix}.$$
The expanded input matrix $\tilde{A}_2$ is given by

$$\tilde{A}_2 = \begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\ \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -0.2034 & 0 & 0 & 0 & 0 & 0 & 1.3490 \\ 0 & -3.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1647 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0571 & -2.4403 & 0 & 0 & 0 \\ 0 & 0 & 2.3262 & -2.3832 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2.5 & 0 \\ 0.3372 & 0 & 0 & 0 & 0 & 0 & -5.3657 \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} \tilde{A}_{44} & \tilde{A}_{45} & \tilde{A}_{46} \\ \tilde{A}_{54} & \tilde{A}_{55} & \tilde{A}_{56} \\ \tilde{A}_{64} & \tilde{A}_{65} & \tilde{A}_{66} \end{bmatrix}$$

$$= \begin{bmatrix} -2.5000 & 0 & 0 & 0 & 2.5000 & 0 & 0 \\ 0 & -5.3657 & 3.0702 & 0 & 0 & 0 & 0.2238 \\ 0 & 0.5382 & -0.7664 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.1400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3365 & 0 & 0 \\ -0.2790 & 0.6340 & -0.5090 & 0 & 0.2400 & 0 & -0.1300 \\ 1.4500 & 0 & 0 & 0 & 0 & 0 & -1.4500 \end{bmatrix}$$

$$\tilde{A}_4 = \begin{bmatrix} \tilde{A}_{66} & \tilde{A}_{67} \\ \tilde{A}_{76} & \tilde{A}_{77} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.1300 & -0.1160 & 0.0235 & 0.1210 \\ 0 & -1.4500 & 0 & 0 & 0 \\ 0 & 0 & -1.4500 & 0 & 0 \\ 0 & 0 & 1.4800 & -1.48000 \\ 0 & 0 & 0 & 0.5160 & -0.5160 \end{bmatrix}$$

The interconnections among the overlapped subsystems can be easily obtained by simple inspection of $\tilde{A}$ in (24). Interconnection matrices $\tilde{H}_i$, $i = 1, 2, 3, 4$ associated with each of the subsystem is given by

$$\tilde{H}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} & \tilde{A}_{16} & \tilde{A}_{17} \\ 0 & 0 & 0 & 0 & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{A}_{27} \end{bmatrix}$$

$$\tilde{H}_2 = \begin{bmatrix} \tilde{A}_{21} & 0 & 0 & 0 & 0 & 0 & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{A}_{27} \\ \tilde{A}_{31} & 0 & 0 & 0 & 0 & 0 & \tilde{A}_{35} & \tilde{A}_{36} & \tilde{A}_{37} \\ \tilde{A}_{41} & 0 & 0 & 0 & 0 & 0 & \tilde{A}_{45} & \tilde{A}_{46} & \tilde{A}_{47} \end{bmatrix}$$

$$\tilde{H}_3 = \begin{bmatrix} \tilde{A}_{41} & 0 & \tilde{A}_{42} & \tilde{A}_{43} & 0 & 0 & 0 & 0 & \tilde{A}_{47} \\ \tilde{A}_{51} & 0 & \tilde{A}_{52} & \tilde{A}_{53} & 0 & 0 & 0 & 0 & \tilde{A}_{57} \\ \tilde{A}_{61} & 0 & \tilde{A}_{62} & \tilde{A}_{63} & 0 & 0 & 0 & 0 & \tilde{A}_{67} \end{bmatrix}$$

$$\tilde{H}_4 = \begin{bmatrix} \tilde{A}_{61} & 0 & \tilde{A}_{62} & \tilde{A}_{63} & 0 & \tilde{A}_{64} & \tilde{A}_{65} & 0 & 0 \\ \tilde{A}_{71} & 0 & \tilde{A}_{72} & \tilde{A}_{73} & 0 & \tilde{A}_{74} & \tilde{A}_{75} & 0 & 0 \end{bmatrix}$$

The expanded input matrix $\tilde{B}$ is given by:

$$\tilde{B} = diag[\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4]$$
where

\[
\bar{B}_1 = \begin{bmatrix} 10^6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} -0.03843 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_3 = \begin{bmatrix} 0.0016 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_4 = \begin{bmatrix} -0.00062 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

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