Bistability, autowaves and dissipative structures in semiconductor fibers with anomalous resistivity properties

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(Received 25 August 2011; final version received 22 November 2011)

This work provides a discussion of bistability conditions, switching autowave properties and emergence of dissipative structures in semiconducting fibers with anomalous positive dependence of electrical resistivity on temperature of sigmoid type, \((1 + e^{-\frac{T}{C_0}})^{-1}\). An open system thermodynamics approach is utilized for the analysis of this dissipative solid-state system. The approach aims to represent the structure of the solution space of its governing equation in the form of physical phase diagrams, known as non-equilibrium phase diagrams, and two specific binary diagrams have been obtained here. One of the diagrams, where the electrical power density and ambient temperature represent external parameters, shows a wide region with dissipative structures as non-uniform steady-state temperature profiles on the fiber. The possibility of efficient external control over the dissipative structure geometry is also demonstrated.

Keywords: open system; autowave; dissipative structure; non-equilibrium phase diagram; barium titanate

1. Introduction

A significant volume of recent research publications describes the ability of inorganic systems to demonstrate some generic forms of emerging synergetic behavior that were earlier attributed to living organisms only. This behavior is represented by the processes of temporal evolution, self-organization and complication in highly non-equilibrium physical systems given a sufficient inflow of energy and building material. According to the open systems theory [1,2], these processes are accompanied by entropy exchange between the system and the ambient media. Highly non-equilibrium conditions in the system created by the inward and outward energy flux may trigger autonomous formation of some stable non-uniform patterns on the spatial or temporal profiles of system parameters. In a closed system, such patterns could only be attributed to transient, rather than stationary processes. Examples of such stable patterns in open systems can be very broad, from current oscillations in electric generator circuits and oscillatory chemical reactions, to Benard cell vortices in convective flow of heated liquids. With the formation of stable non-uniform patterns, known as dissipative structures [2,3], the open system undergoes a transition

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ISSN 1478–6435 print/ISSN 1478–6443 online
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http://dx.doi.org/10.1080/14786435.2011.644814
http://www.tandfonline.com
to a new physical phase often allowing for a greater energy throughput compared to the states with uniform system parameters. Such transitions are sometimes called non-equilibrium phase transitions and the system demonstrating these is called the active system or active medium. In the context of dissipative structures, the phenomenon of autonomous organization and complication is not a unique feature of the biological world; it may also be attributed, possibly in simpler forms, to various inorganic systems.

One basic type of active system is the bistable or trigger system [2–9]. Constitutive elements of a bistable system are found in one of two possible states or phases which are stable with respect to small changes of external conditions. An external effect of a magnitude greater than some finite threshold value may trigger a non-equilibrium phase transformation in one or several constitutive elements; that in turn may lead to a synergistic transition of the entire bistable system from one global stable phase to another. In this paper we explore the conditions on bistability, non-equilibrium phase transitions, and emergence of dissipative structures in a simple type of material systems: resistively heated prismatic semiconductor fibers with a highly nonlinear positive dependence of electrical resistivity on temperature. Our discussion utilizes an open system thermodynamics approach. Here, the final goal is to represent the solution space structure of the nonlinear governing equations for this open dissipative system in the form of physical phase diagrams, using both analytical and numerical techniques.

2. Anomalous resistivity of some semiconducting ceramics

Barium titanate based ceramics can be given as one example of semiconducting materials that demonstrate the very interesting property of an anomalous increase of electrical resistivity for temperatures near the Curie point, which is 120°C for pure BaTiO₃, see Figure 1. This effect is uncommon for semiconducting materials and is called the positive temperature coefficient of resistivity (PTCR). Pure barium titanate is electrically insulating, but can be transformed into a semiconductor with Verwey’s controlled valency method [10]. The method is based on a partial (0.1–0.3%) replacement of barium Ba⁺² ions with trivalent donor or pentavalent acceptor ions. For example, addition of La⁺³ donors leads to Ba₁₋ₓLaₓTiO₃ semiconducting ceramics.

The PTCR effect is very pronounced for yttrium and strontium modified barium titanate ceramics [11–16], for samples doped with niobium [10,17,18], holmium [19], strontium, lanthanum [14], niobium pentoxide, yttrium hexaboride [15,16], some oxygen-free compounds [20], and for other types of barium titanate ceramics [21] obtained with various synthesis procedures [15,16,21,22]. According to reference [10] this phenomenon can be explained by the formation of barrier layers at the grain boundaries at higher temperatures, and was more recently demonstrated by direct observations of the temperature dependence of these potential barriers [18]. Positive temperature coefficient thermistors called posistors utilizing barium titanate based semiconductors are used in practice as temperature sensors, heating elements and current-protection devices.
3. Analytical model of anomalous resistivity

The resistivity of a typical barium titanate ceramic with the PTCR property has a temperature dependence of the type shown in Figure 1. At temperatures below 100°C, the resistivity ρ is almost constant or has a very weak dependence on temperature, while in the interval 100–200°C, the value of ρ increases dramatically by three to five orders of the magnitude, followed by another region of weak temperature dependence above 200°C. An example experimental curve ρ(T) adopted from [15] is shown in Figure 1, while an entire family of reported experimental dependencies [10,13–16,18,19,22–24] is quite diverse, and its approximate boundaries are shown on the same plot with dashed lines.

For the purpose of the analytical and numerical studies presented later in this paper, we approximate an experimental curve from [15] with a smooth analytical function of the logistic sigmoid type,

\[
\rho(T) = \frac{1}{(\rho_{\text{max}} - \rho_{\text{min}})^{-1} + e^{-a(T - T_0)}} + \rho_{\text{min}}. \tag{1}
\]

Here, \(a = 0.12 \text{ K}^{-1}\) is the steepness parameter; the value \(T_0 = 95^\circ\text{C}\) determines the onset of the transition; and \(\rho_{\text{min}} = 2 \Omega \text{ m}\) and \(\rho_{\text{max}} = 10^4 \Omega \text{ m}\) are asymptotic values of the function at low and high temperatures, respectively. This function has a single inflexion point at 171.8°C for the selected values of parameters. In Figure 1, analytical model (1) is plotted in comparison to the experimental data.

4. Energy balance and governing equations

We consider a long prismatic fiber made of a barium titanate ceramic type material whose electrical resistivity dependence on temperature is described by Equation (1).
Assume a circular cross-section of area $A$ and perimeter $p = 2\sqrt{\pi}A$, see Figure 2 (inset). Two ends of the fiber are connected to an electric power source to maintain a uniform current density $j = I/A$ in the longitudinal direction $x$ of the fiber. Consider a differential element of the fiber of length $dx$. During an infinitesimal time interval $dt$, resistive Joule heating of this element yields volume distributed thermal energy of magnitude $j^2 A \rho(T) dx dt$. This received energy is partially stored in the element volume, $\mu c_p A \frac{dT}{dx} dx dt$, where $\mu$ and $c_p$ are the mass density and specific heat capacity of the fiber material, and partially released convectively through the side surface of the element. The latter can be expressed using Newton’s formula for convective heat transfer, $(T - T_\infty)ph dx dt$, where $T_\infty$ is the ambient temperature. Finally, assuming the Fourier law for conductive heat flux in the longitudinal direction of the fiber, $-k A \nabla T$, where $k$ is the heat conduction coefficient, the elemental energy balance equation can be written as

$$j^2 A \rho(T) dx dt - kA \frac{\partial^2 T}{\partial x^2} |_{x} dt = \mu c_p A \frac{dT}{dx} dx + (T - T_\infty)ph dx dt - kA \frac{\partial T}{\partial x} |_{x+dx} dx.$$  \hspace{1cm} (2)

This equation can be rearranged in terms of the energies per unit time and unit fiber length to give a heat transfer PDE for the temperature function $T = T(x,t)$ with an essentially nonlinear right-hand side:

$$\mu c_p A \dot{T} - kA \frac{\partial^2 T}{\partial x^2} = f(T), \quad f(T) = Q_1(T) + Q_2(T)$$

$$Q_1(T) = j^2 A \rho(T), \quad Q_2(T) = ph \cdot (T_\infty - T)$$  \hspace{1cm} (3)

where $\rho$ is the resistivity function (1). Reported values of the mass density for barium titanate are in the range 5.83–6.02 g cm$^{-3}$ [25,26], thermal conductivity...
2.59–2.89 W (m K)\(^{-1}\) [26] and specific heat 400–450 J (kg \(\cdot\) K)\(^{-1}\) [26,27] at room temperature. These parameters are quite uniform in the temperature range 20–300°C, though the specific heat shows an increase of about 10–20% near the Curie point [26,27]. For the analysis below, we average these data and select \(\mu = 5.92\, \text{g cm}^{-3}\), \(k = 2.74\, \text{W (m K)}^{-1}\), and \(c_p = 480\, \text{J (kg \(\cdot\) K)}^{-1}\). The temperature dependence of the specific heat is ignored as insignificant compared to that of the electrical resistivity. The convection heat transfer coefficient \(h\) and ambient temperature \(T_1\) are set at 7.0 W (m\(^2\) K)\(^{-1}\) and 25°C, respectively. The cross-sectional area of the wire is \(A = 1.0\, \text{mm}^2\), and the current density \(j\) will be varied. In some cases we will also consider fibers with cross-sections 0.5 and 0.25 mm\(^2\) for comparison.

5. Qualitative discussion: monostability and bistability

In this section we discuss interesting qualitative properties of solutions to the governing Equation (3). This equation is written in units of thermal power per unit length of the fiber. Therefore, the nonlinear function \(Q_1(T)\) is the axial density of power generated in the fiber due to Joule heating, and the linear function \(Q_2(T)\) represents the power dissipated by the fiber to the ambient due to convection. Assume the convection coefficient \(h\), the fiber cross-sectional area \(A\) and the perimeter \(p\) are set constant as the internal (system) parameters, while the ambient temperature \(T_1\) and electric current density \(j\) are viewed as the controlled external parameters. Depending on the specific realization of these two parameters, the transcendental equation

\[
f(T) = Q_1(T) + Q_2(T) = 0
\]  

may normally have one \((T_1)\) or three \((T_{1-3})\) roots; see Figure 2, which shows the fiber temperatures at which the generated Joule heat is balanced by convective heat dissipation. Since heat exchange between the fiber and external bodies is absent at each of these temperatures, an initial condition of the type \(T(x,0) = T_i (i = 1, 2\) or 3\) for the governing PDE (3) will yield a stationary solution \(T(x,t) = T_i\) for any \(x\) and \(t\), which will be stable for \(T_1\) or \(T_3\) and unstable for \(T_2\). The slope of the function \(Q_1(T)\) is smaller than that of \(-Q_2(T)\) at the points \(T_1\) and \(T_3\), but greater at the point \(T_2\); see Figure 2. Thus, for a small increase in temperature about the value \(T_1\) or \(T_3\), heat dissipation \((-Q_2)\) prevails over heat generation \((Q_1)\), while on the other hand for the value \(T_2\) heat generation prevails. In other words, any small fluctuation of temperature in a fiber segment about \(T_1\) or \(T_3\) tends to be mitigated, while any small deviation about the point \(T_2\) tends to be magnified. Note that the equilibrium temperatures \(T_{1-3}\) are not altered for a constant ratio \(A/ph\), or \(A/h^2\) for circular cross-sections, which represents the compensation of the surface effect due to a linear increase in the fiber cross-sectional area with a quadratic increase of heat transfer through the side surface of the fiber.

**Monostability.** The usual situation of resistive fiber heating in convective media is observed for small enough currents, such that \(Q_1\) and \(-Q_2\) intersect at one point only; see Figure 2. This point represents the only possible thermal equilibrium of the system with the ambient, as in the case of usual fibers with a constant or monotonically varying resistivity. This standard situation is referred to as **monostability**.
According to (4), the system is monostable at currents smaller than some critical value $j_{cr}$, which depends on the heat transfer coefficient and the fiber cross-sectional area as $j_{cr} \sim h^{1/2}$ and $j_{cr} \sim A^{-1/4}$, respectively. In particular, $j_{cr} = 21.15 \text{ A m}^{-2}$ for the selected value $T_\infty = 25^\circ C$. Furthermore, there exists a minimal current, $j_{min} = 9.11 \text{ A m}^{-2}$, such that the slope of $Q_1$ at the saddle point is equal to the slope of $-Q_2$. If the supply current is smaller than $j_{min}$, the system is monostable for any ambient temperature. Since the ambient temperature $T_\infty$ gives a vertical shift to the $-Q_2$ line, the latter will intersect with $Q_1$ at one point only if $T < T_{\infty}^{\min}$ or $T > T_{\infty}^{\max}$, where $T_{\infty}^{\min}$ and $T_{\infty}^{\max}$ are some critical ambient temperatures for each given $j > j_{min}$. For example, the values $T_{\infty}^{\min} = 25.0^\circ C$ and $T_{\infty}^{\max} = 138.2^\circ C$ correspond to the current magnitude 21.15 A m$^{-2}$ mentioned earlier. In summary, the system is monostable if the ambient temperature is outside the interval $[T_{\infty}^{\min}, T_{\infty}^{\max}]$ or the current density drops below 9.11 A m$^{-2}$.

**Bistability.** The triple intersection of the functions $Q_1$ and $-Q_2$ at sufficiently low ambient temperatures and currents $j > j_{cr}$, Figure 2, represents a qualitatively different physical behavior of the system. Increase of the current density beyond $j_{cr}$ leads to the emergence of two additional states of thermal equilibrium at $T(x, t) = T_2$ and $T(x, t) = T_3$. The system is now bistable having two stable equilibrium states, $T_1$ and $T_3$, and one absolutely unstable equilibrium state, $T_2$. Any small perturbation of state $T_2$ will trigger a fast transition of the entire system into one of the stable states, $T(x, t) = T_1$ or $T(x, t) = T_3$. On the other hand, a transition between the states $T_1$ and $T_3$ would require a large perturbation of magnitude greater than the difference $T_2 - T_1$ or $T_2 - T_3$ and a significant longitudinal span. This type of transformation between two stable configurations of an open system is known as a non-equilibrium phase transition [1,2]. Insufficiently large perturbations will relax to the original stable state, see Figure 4. Furthermore, any initial profile of the type $T(x, 0) < T_2$ for all $x$ may only relax to $T(x, t) = T_1$, and a profile $T(x, 0) > T_2$ for all $x$ will always relax to $T(x, t) = T_3$ at $t \to \infty$.

For currents just above $j_{cr}$, the values $T_2$ and $T_3$ are close to each other, and the state $T_3$ can be viewed as metastable. In other words, transitions from the hot to the cold state are easier than the reverse in this case. Further increase of the current density $j$ beyond the value $j_0 = 26.98 \text{ A m}^{-2}$, corresponding to the condition $Z(j_0, T_\infty) = 0$, where

$$Z(j, T_\infty) = -\int_{T_1}^{T_3} f(T) \, dT,$$

(5)

will transform $T_3$ into an absolutely stable state, while $T_1$ will become a metastable state. The numerical studies in Section 6 prove that the value $j_0$ serves as a threshold current density, after which a cold-to-hot transition of the system is easier to initiate than the reverse one. Therefore, we may consider the $Z$-integral (5) as a function of the external parameters $j$ and $T_\infty$, which is indicative of whether the hot state is metastable in the system ($Z > 0$), or the cold state is metastable instead ($Z < 0$). The dependencies of the value $Z$ and equilibrium temperatures $T_{1-3}$ on the current density $j$ at $T_\infty = 25^\circ C$ are drawn in Figure 3. We may also see from (1) and (5) that, for a constant ratio $A/h^2$, the magnitude of $Z$ is directly proportional to $A$.

**Critical nucleus.** Consider the situation where the hot state $T_3$ is absolutely stable, and the cold state $T_1$ is metastable. Assume the initial condition that the fiber is
Figure 3. Dependencies of the $Z$-integral and equilibrium temperatures $T_{1-3}$ on current density $j$ at ambient temperature $T_\infty = 25^\circ C$. Special values: $j_{cr} = 21.15$ A m$^{-2}$, $j_0 = 26.98$ A m$^{-2}$.

Figure 4. Evolution of small (left) and large (right) perturbations of the equilibrium state $T_1$; the parameter $d$ is the fiber diameter.
found mostly in the $T_1$ state, except for a small region $\delta L$ at temperature $T_3$, a hot-phase “nucleus”. Small perturbations of this type will relax back to the metastable state $T_1$, while those with $\delta L$ greater than some critical value $\delta L_{cr}$ will trigger a transition of the entire system to the state $T_3$; see Figure 4 and the inset of Figure 5. If the external parameters $(j, T_1)$ are maintained constant, the left and right boundaries of the initial hot-phase nucleus will disperse somewhat due to conductive heat exchange in the fiber and then start moving relative to their original positions at a finite velocity, as illustrated in Figure 4.

**Autowaves.** The process of the widening of a hot-phase nucleus can be viewed as two progressive autowaves moving in opposite directions, whose passage switches the system from the cold to the hot state. These autowaves represent a special transient solution to (3) of the type $T = T(x \pm ct)$, where $c$ is the velocity of the autowave front. A solution with $c = 0$ would correspond to a very interesting situation of long-term coexistence of both cold and hot regions in the fiber. The system will demonstrate similar behavior for the reverse situation, where $T_1$ is the absolutely stable state. In both cases, the determination of the size of the critical nucleus and the velocity of the autowave front are interesting tasks that cannot be performed analytically even for the present, relatively simple nonlinearity of the function $f(T)$ in Equation (3). Obviously, $\delta L_{cr}$ and $c$ should depend on $Z$ and therefore on the external parameters $T_\infty$ and $j$, while the specific quantitative dependencies may only be determined numerically, as discussed below in Section 6. Quantitative information about $\delta L_{cr}$ and $c$ will also be useful in summarizing the properties of the system in the form of *non-equilibrium phase diagrams*. Another type of system behavior, different from that described in Figure 3, can be expected when

![Figure 5](https://example.com/figure5.png)

Figure 5. Minimal nuclear size required to trigger a transition between the stable states at $T_\infty = 25^\circ C$ for various current densities and fiber cross-sectional areas. The inset shows the profile of the nucleus for the $T_1 \rightarrow T_3$ transition corresponding to the three curves on the right of $j = j_0$. The three curves on the left of $j_0$ correspond to the $T_3 \rightarrow T_1$ transition, where the nuclear profile is shown upside down.
only a limited amount of electric power is supplied to the system. Indeed, widening of
the $T_3$ zone results in a dramatic increase in Joule heat dissipation in the fiber that
may be mitigated with a limited or fixed power supply. This situation is also
investigated numerically in Section 7.

6. Numerical studies

*Euler method.* A standard finite element method procedure [28,29] can be utilized
to reduce the nonlinear heat transfer Equation (3) to a first-order ODE in time
written in matrix form,

$$\mu c_p A L T + K T = Q(T), \quad Q(T) = J^2 A L \rho(T) + \phi L (T_\infty - T)$$  
(6)

where $L$ is the length of each of the axial finite elements, or the distance between the
FEM nodes, $T$ is a vector of nodal temperatures, so that $T_n$, the $n$-th element of $T$, is
the temperature at node $n$, and $K$ is the FEM conductance matrix. For a periodic
boundary condition $T_0 = T_{Ne}$, where $N_e$ is the total number of elements, $K$ is a
circular $N_e \times N_e$ matrix of the type

$$K = \frac{k A}{L} \begin{pmatrix}
2 & -1 & 0 & \ldots & -1 \\
-1 & 2 & -1 & \ldots & 0 \\
0 & -1 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & 2 \\
\end{pmatrix}.$$  
(7)

Time integration of the nonlinear matrix Equation (6) can be performed numerically
using Euler’s method,

$$T^{(i+1)} = T^{(i)} + \frac{\Delta t}{\mu c_p A L} (Q(T^{(i)}) - K T^{(i)}), \quad i = 1, 2, \ldots, N_{\Delta t}$$  
(8)

where $\Delta t$ is a numeral time-step and $N_{\Delta t}$ is the total number of time-steps.

Consider a prismatic fiber of circular cross-section with all the physical parameters
listed in Section 4. Three different cross-sectional areas will be considered:
$A = 1.0$ mm$^2$, 0.5 mm$^2$ and 0.25 mm$^2$. When studying the 0.5 mm$^2$ and 0.25 mm$^2$
fibers in comparison with the 1.0 mm$^2$ case, a decrease in the side surface area is
compensated by a proportional decrease of the convection coefficient, $h \sim A^{1/2}$,
to elucidate a pure geometric effect of fiber cross-sectional area. Since transient
processes occur faster in thinner fibers, the time-step $\Delta t$ was set at 0.5 s for the 1.0 mm$^2$
case, and at 0.35 s and 0.25 s for the 0.5 mm$^2$ and 0.25 mm$^2$ cases, respectively.
20,000 time-steps and 160 finite elements of length $L = 3d$, where $d = 2\sqrt{A/\pi}$ is the
fiber diameter, were utilized in all simulations.

($j, T_\infty$)-parametrization. Consider the current density $j$ provided by an external
power supply and ambient temperature $T_\infty$ as controlled external parameters that
are kept constant during an entire simulation run. The system behavior is analyzed
with reference to various selected values of $j$ and $T_\infty$. 
The critical nuclear size $\delta L_{cr}$ was first investigated. The initial shape of the nucleus is shown in Figure 5(inset), where $\delta s$ is the transient zone width set at $3d$ in all cases. The current density $j$ varied from $j_{cr} = 21.15 \text{ A m}^{-2}$ to 40–50 $\text{ A m}^{-2}$, depending on $A$. The value $\delta L_{cr}$ was determined by first setting $\delta s$ and then varying $j$ with 0.01 $\text{ A m}^{-2}$ resolution to determine the minimal current that is still able to initiate the $T_1 \rightarrow T_3$ transition in the entire fiber for the selected $\delta s$. The latter was then regarded as $\delta L_{cr}$ corresponding to the current value $j$. For the $T_3 \rightarrow T_1$ transition the procedure was similar, except that the threshold current was defined as the maximal current at which the transition to the cold state still occurs for the selected $\delta s$. The results of the calculations for the three cross-sectional areas of the fiber and ambient temperature $25^\circ\text{C}$ are presented in Figure 5. Remarkably, all the curves in this plot have a common vertical asymptote at $j_0 = 26.98 \text{ A m}^{-2}$. Furthermore, $\delta L_{cr}$ appears to be independent of the total fiber length.

The autowave traveling velocity was investigated by tracking the motion of the saddle point on the autowave front – see Figure 4 – in a similar series of simulations. The autowave velocity $c$ was constant during the entire simulation run, except for the short initial and final periods of time, when the autowave front is yet forming or has already vanished due to the periodic boundary conditions. Interestingly, the value of $c$ was independent of the total fiber length and the initial nuclear size $\delta s$. The dependence on the fiber cross-sectional area was also weak, while the current density was a major parameter affecting $c$; see Figure 6. For the $T_1 \rightarrow T_3$ transition ($j > j_0$), this dependence is close to linear at high current densities. For the reverse transition $T_3 \rightarrow T_1$ ($j < j_0$), $c$ is very large for small currents in the vicinity of $j_{cr}$ and then decays exponentially when the current grows toward the $j_0$ value. The most important result of the autowave studies is the fact that the front velocity $c$ is never zero, unless the

![Figure 6. Velocity of the autowave front. At $j > j_0$ the autowave is moving as shown in Figure 4(right), while at $j < j_0$ it is moving in the opposite direction corresponding to a reverse phase transition.](image-url)
current density is close to the value $j_0$. When the current magnitude passes this value, autowave propagation reverses its direction.

$(j, T_\infty)$-type phase diagram. We may conclude that for the $(j, T_\infty)$-parametrization, the cold and hot phases generally do not coexist in the system. The only exception is the current value $j_0$ associated with $Z = 0$ and leading to a zero autowave velocity. If the initial condition $T(x, 0)$ contains at least one critical nucleus of each phase and $j \neq j_0$, only a limited time is required for the entire system to take up one of the stationary configurations $T(x, t) = T_1$ or $T(x, t) = T_3$. A non-zero temperature gradient in the fiber may only be observed in the vicinity of physical boundaries at $t \to \infty$, and the entire system will take one of the uniform states: $T_1$ at $j < j_0$, or $T_3$ at $j > j_0$. Thus, we are able to draw a phase diagram of the system with $j$ and $T_\infty$ serving as external control parameters; see Figure 7. As discussed earlier in Section 5, the overall bistability region in this diagram is defined by the conditions $T_3 = T_2$ and $T_1 = T_2$ for the bottom and top $T_\infty$ limits, respectively. Numerical solution of the respective transcendental equations involving $f(T)$ from (3), where $T_\infty$ is viewed as an unknown parameter for a selected $j$, gives two converging curves depicted in Figure 7 with solid lines. The boundary between the two stable phases in the regime of bistability (dashed line) is defined by the condition (5). All three curves terminate on the single branching point $(9.11 \text{ A m}^{-2}, 155.1^\circ \text{C})$, which is somewhat analogous to a triple point in the usual equilibrium phase diagram. Since the boundaries between different regimes and phases are fully determined by the behavior of the function $f(T)$ in Equation (3), the entire $(j, T_\infty)$-diagram is invariant to the heat conduction coefficient $k$, the mass density $\mu$ and the specific heat capacity $c_p$ of the fiber material. These parameters may only affect the characteristics of transient processes.

Figure 7. $(j, T_\infty)$-type phase diagram of the system. A definite boundary between the phases $T_1$ and $T_3$ exists (dashed line), when the initial condition contains the seeds of the both phases. Upon relaxation of the transient processes, the entire bistable system is in either the $T_1$ or $T_3$ state.
in the system, such as the critical size of the nucleus, the autowave velocity and relaxation times. The present \((j, T_\infty)\) phase diagram is therefore a fundamental characteristic of semiconducting PTCR fibers defined by the resistivity function (1).

7. Stable dissipative structures

Particularly interesting is the situation of the long-term coexistence of two phases in the same system. This implies the intriguing possibility of creating non-uniform temperature patterns, while keeping the system geometry and ambient conditions uniform. As shown in the previous section, this can be achieved with \((j, T_\infty)\) control parameters under the special condition \(j = j_0\) leading to a zero propagation velocity of the autowave front. However, this situation is not practical due to the necessity of fixing one of the external parameters, so that control over the temperature profile is limited to the variation of the ambient temperature only. Furthermore, such a profile would have only neutral stability, a tendency to be eliminated with a small fluctuation of external parameters and a strong dependence on initial conditions.

\((P, T_\infty)\)-parametrization. Another type of system behavior can be realized for an alternative selection of external parameters, where non-equilibrium temperature patterns are stable with respect to small changes of these parameters and also invariant for a wide class of initial conditions. As mentioned earlier, such patterns are known as dissipative structures [2,3]. Consider the situation when the ambient temperature and the electric power, rather than the current density, delivered to the fiber are chosen as the fixed external parameters. If they are maintained constant, one may expect that hot-phase expansion by the motion of the autowave and the resultant increase of fiber resistance will be mitigated by the limited power supply. As the magnitude of the current self-adjusts to the supplied power level, the evolution of a hot phase zone will stabilize in the fiber. With the numerical studies below in this section we elucidate the details of this expected behavior and discuss a \((P, T_\infty)\)-type phase diagram of the system in comparison with the \((j, T_\infty)\) diagram developed earlier. The possibilities of external control over the geometric properties of dissipative structures are also demonstrated.

Current density. Numerical studies of the \((P, T_\infty)\)-parametrized system began with the investigation of the dependence of the steady-state current density \(j\) (as \(t \to \infty\)) on the value \(P\), the supplied electric power divided by the total fiber length. The \(j-P\) dependence appears in a very interesting form presented in Figure 8 for \(T_\infty = 25^\circ\text{C}\). For powers smaller than \(0.10 \text{ W m}^{-1}\), the stationary state current grows as \(P^{1/2}\), reaching a value of several hundred A m\(^{-2}\) depending on the fiber’s cross-sectional area \(A\). Despite such large currents, much exceeding \(j_{cr}\) and even \(j_0\), the stationary temperature profiles were uniform at \(T_1\) for any \(P < 0.1 \text{ W m}^{-1}\), even if the initial conditions contained the seeds of the hot phase \(T_3\). In other words, such powers are insufficient to maintain a non-uniform temperature pattern; the entire system remains in the cold phase, because the occurrence of any hot-phase island would lead to a dramatic drop in the total conductance of the fiber and a subsequent decrease in the current density below the value of \(j_0\). For a range of powers from about 0.15 to \(9.0 \text{ W m}^{-1}\), the current density has a plateau where \(j\) is close to \(j_0\) for all
such powers; a more specific position of this plateau depends on $A$. Beyond the 9.0 W m$^{-1}$ power level, the current density again grows in a standard $P^{1/2}$ manner.

**Dissipative structures.** Studies of the temperature profiles for the range of $P$ from 0.15 to 9.0 W m$^{-1}$ corresponding to the anomalous behavior of the current density shown in Figure 8 indicate that stationary non-uniform temperature patterns are generated in the system under these conditions. These patterns demonstrate all the basic features of stable dissipative structures. In particular, they always have the shapes of standing autowaves as shown in Figure 4, i.e. their presence is equivalent to the coexistence of both cold and hot phases in the same fiber. Furthermore, any small region in the fiber with initial temperature greater than $T_1$ (the seed of the hot phase) evolves into a dissipative structure, whose shape as $t \to \infty$ is determined by the values of the external parameters only. Typical shapes of stationary temperature profiles for this dissipative structure at various $P$ and $T_\infty = 25^\circ$C are shown in Figure 9. The variation of the parameter $P$ only changes the width of the structure, while its height (temperature $T_3$) remains close to a constant value. Figure 9(inset) shows that this dependence is very closely linear for the range of powers 1.2–5.2 W m$^{-1}$. In the absence of any boundary effects on a sufficiently long fiber, the shape of dissipative structures is also invariant with respect to the total fiber length.

When two identical seeds of the hot phase $T_3$ are provided by the initial conditions and the distance between them is smaller than the width of the transition zone between phases $T_1$ and $T_3$ on a stationary temperature profile, these seeds will evolve into a single dissipative structure of the same type as shown in Figure 9. If the distance between two seeds is very large compared to the width of the transition zone, they evolve into two narrower dissipative structures that remain stable indefinitely. The linear growth of the width of the hot zone with the supplied power $P$,
see Figure 9(inset), provides a means of efficient control over the evolution of the dissipative structure. Since the total resistance of the fiber also grows linearly with the peak width, voltage control over dissipative structures can be alternatively utilized.

\((P, T_1)\)-type phase diagram. As found earlier in this section, \((P, T_\infty)\)-type parametrization of external conditions leads to the formation of stable dissipative structures in the system with non-zero temperature gradients along the fiber length. The emergence of these structures is equivalent to the existence of a standing autowave, and therefore long-term coexistence of both the cold and hot phases in the system. At ambient temperature \(T_\infty = 25^\circ\text{C}\), dissipative structures are formed if the supplied electric power is in the range 0.15–9.0 W m\(^{-1}\). Further numerical studies of the system at various ambient temperatures showed the emergence of dissipative structures for \(T_\infty\) from at least \(-160^\circ\text{C}\) up to the value 155.1°C, see Figure 10. Beyond this temperature the system becomes monostable, similar to the case of the \((j, T_1)\) diagram in Figure 7. Power values corresponding to the region with dissipative structures are somewhat greater at lower temperatures, and their range rapidly narrows in the vicinity of the 155.1°C point. The overall boundary of the bistability region is still entirely determined by the behavior of the function \(f(T)\), as for the \((j, T_\infty)\) diagram, though in a somewhat indirect manner. Namely, the values of \(P\) on this boundary at given \(T_\infty\) were calculated as \(j^2 \rho(T)\), where \(j\) is equal to \(j_{cr}\) from the \((j, T_1)\) diagram, and \(T\) is one of the stable equilibrium temperatures \((T_1\) or \(T_3)\) corresponding to the selected ambient temperature and current. Thus, the overall bistability region is invariant with respect to the system parameters \(k, \mu\) and \(c_p\). Meanwhile, the region of mixed states shown with a dashed line in Figure 10 will depend on the thermal conductivity coefficient \(k\) in Equation (3). For instance, if \(k\) was smaller than the value 2.74 W (m K\(^{-1}\)) chosen here, the system would allow for narrower dissipative structures with shaper edges, and therefore the region of mixed states would span more widely on the power axes at all ambient temperatures.
8. Conclusion

We conclude that open systems involving PTCR semiconducting fibers with anomalous resistivity characteristic of a logistic type (1) depicted in Figure (1) demonstrate bistability for a wide range of external parameters, such as ambient temperature and supplied electric current or power. When the ambient temperature $T_1$ and current magnitude $j$ are fixed as external parameters, the entire system is found in one of the stable phases after all transient processes are relaxed. Special transient processes in the form of autowave propagation can be triggered by introducing to the system a nucleus of opposite phase whose critical size has been investigated. Propagation of the autowaves, whose front velocity has also been discussed, leads to switching of the system to an opposite stable state. A non-equilibrium phase diagram of the $(j, T_1)$-parametrized system has been developed.

Even more interesting behavior was observed when the ambient temperature $T_1$ and the supplied electric power $P$ were selected as controlled external parameters. In this case, spatial dissipative structures in the form of stable patterns with non-zero temperature gradients emerge in the system for a well-defined range of external parameters. As a result, the $(P, T_1)$-type phase diagram features a large region where cold and hot phases coexist in the same fiber for an indefinitely long time. The shape and size of the dissipative structures are invariant with respect to the initial conditions and most system parameters, and the size can be controlled in a linear manner by changing the parameter $P$. For powers large enough to maintain dissipative structures, the current density in the fiber is almost constant for all such powers. In view of these findings, progressive autowave motion in the $(j, T_1)$-controlled system may also be interpreted as a process of evolution of newly formed dissipative structures under varying electric power supplied to the fiber. Generally, the autonomous emergence of complexity in the form of dissipative structures
studied here is a non-equilibrium phase transition enabling the system to dissipate much greater powers at smaller currents.

There is also a basic analogy with biological systems where population growth cannot be initiated before some minimal amount of food is provided to the system per unit time. When this condition is met the population grows to a maximal size mostly defined by the food supply rate. Similarly, the thermal dissipative structures discussed here emerge after some threshold electric power is provided and then stabilize their growth on a stationary configuration reflecting the condition of the given power supply.

One interesting task for the future could be the demonstration of pulsating (spatiotemporal) dissipative structures in similar systems through the implementation of a nonlinear resistivity function with retarded response properties.

Acknowledgments
Support for this research by the National Science Foundation via grant #0900498 is gratefully acknowledged.

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