The Fermion Mass Problem

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Abstract

I review the ‘puzzles’ associated with the fermion mass matrices and describe some recent attempts to resolve them, at least partially. Models which attempt to explain the observed mass hierarchy as arising from radiative corrections are discussed. I then scrutinize possible inter–relations among quark and lepton masses and the mixing angles in the context of grand unified theories. It is argued that the absence of CP violation in the strong interaction sector (the strong CP problem) may also have its origin in the structure of the quark mass matrices; such a resolution does not invoke approximate global $U(1)$ symmetries resulting in the axion. Arguments in favor of tiny neutrino masses are summarized (the solar neutrino puzzle, atmospheric neutrino problem) and ways to accommodate them naturally are described.

1. Introduction

The standard $SU(2)_L \times U(1)_Y$ model of electro–weak interactions is in remarkably good shape, it has successfully confronted a wealth of precision data accumulated at the LEP $e^+e^-$ collider over the past several years. The model is beginning to be tested at the quantum level. Despite its enormous success and its internal consistency all the way up to Planckian energies, it is the widely held view among theorists that the model will be replaced by a more fundamental theory at higher energies. The arguments stem from questions of naturalness in the symmetry–breaking sector where the theory is untested. There is the well–known fine–tuning problem associated with fundamental scalars employed for symmetry breaking, there are other puzzles as well. The focus of my talk will be the problems associated with the fermion mass matrices. There are several facets to the fermion mass puzzle; in a nutshell, they are our lack of understanding of (i) family replication and the resulting proliferation of couplings, (ii) the observed hierarchy in the fermion masses and mixing angles, (iii) the origin of CP violation in weak interactions and its absence in strong interactions, and (iv) the origin of tiny neutrino masses. I shall review these problems and describe some recent proposals to cure one or more of them.

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The origin of the mass problem can be traced to family replication, for with just one generation of fermions most of the puzzles listed above would not exist. We know now from the precision measurement of the $Z^0$ width that there are 3 families with light neutrinos. All three families share the same gauge quantum numbers, they are distinguished only by their Yukawa interactions with the Higgs scalar. There seems to be no simple explanation, at least based on the assumption that quarks and leptons and the Higgs bosons are elementary, for the multiplicity of generations. Ideas based on horizontal symmetries such as $SU(3)_H$ only accommodate and do not explain the observed family structure. Moreover, generating realistic fermion masses and mixing angles in such theories is a non-trivial and often complicated issue. In what follows, I shall simply accept the existence of three families and pursue an understanding of the puzzles it creates.

2. Mass and mixing angle hierarchy

The observed masses of the three families of quarks and leptons span over five orders of magnitude, from the electron mass (0.5 MeV) to the top–quark mass ($\geq 91$ GeV). If neutrinos have masses, as some indirect observations indicate, then $\nu_e$ should weigh less than 9 eV, which would further enhance the hierarchy. The approximate masses of quarks and leptons (or bounds in the case of top–quark and the neutrinos) are listed in Table 1 where the light quark masses ($u,d,s$) are the ones evaluated at 1 GeV using current algebra.\(^1\)

Table 1. Masses of quarks and leptons in MeV.

|     | $u$   | $c$   | $t$   |
|-----|-------|-------|-------|
| $d$ | $5.1$ | 1270  | $\geq 91000$ |
| $s$ | 175   | $b$   |       |
| $c$ | 0.5   | 106   | 1780  |
| $\nu_e$ | $\leq 9 \times 10^{-6}$ | $\leq 0.27$ | $\leq 35$ |

Even if the neutrino masses are ignored, the Yukawa couplings that are needed to generate the masses in Table 1 should span from $h_e \simeq 10^{-6}$ to $h_t \simeq 1$. The natural value of all couplings is order $g$, the gauge coupling, which is of order 1. The puzzle is then why some of the Yukawa’s are as small as $10^{-6}$.

A few features are worth noting regarding the masses in Table 1. Observe that the charge 2/3 quark of a given family is heavier than the charge $-1/3$ quark, which in turn is heavier than the corresponding charged lepton. This feature is violated
only by the first family where \(m_u < m_d\). Note also that all charged fermions of a given family are heavier than those of the preceding family—that is, there is no family overlap. These features will be helpful in attempts to explain the hierarchy in a natural manner.

The quark mixing matrix, the Cabibbo-Kobayashi-Maskawa (CKM) matrix, also exhibits a certain hierarchy, which is likely to be linked to the mass hierarchy since both the mass eigenvalues and the mixing angles are obtained from the diagonalization of the same primordial mass matrices. The approximate magnitudes of the elements of the CKM matrix can be displayed in the basis \((u, c, t)^T_L \cdot (d, s, b)_L\) as

\[
V = \begin{pmatrix}
0.975 & 0.22 & 0.005 \\
0.22 & 0.974 & 0.043 \\
0.01 & 0.04 & 0.99
\end{pmatrix}.
\]  

(1)

Unitarity of the \(3 \times 3\) matrix has been used to write down the yet to be measured elements of the third row. Notice that \(V\) is approximately identity, with the leading non–diagonal entry being the \((12)\) and \((21)\) elements. The \((23)\) entry is somewhat smaller and \((13)\) is the smallest. Indeed, such a hierarchy has prompted a particularly simple perturbative parameterization of \(V\) due to Wolfenstein:

\[
V = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4),
\]  

(2)

where \(\lambda \approx 0.22\) is the Cabibbo angle. The other parameters are \(A \approx 1\) and \(\sqrt{\rho^2 + \eta^2} \approx 0.5\) with \(\eta \geq 0.1\).

In the standard model, both the fermion masses and the quark mixing angles are free parameters adjusted to their observed values by hand. This is clearly unsatisfactory, since it leads to a proliferation of parameters. Although a consistent procedure, it does not explain why certain Yukawa couplings are orders of magnitude different from others. A more complete theory, it is hoped, will address these issues and provide some answers. Suppose the quark mass matrices have the form prescribed as follows:

\[
M_{u,d} = m_{t,b} \begin{pmatrix}
\epsilon^2 & \epsilon & \epsilon^2 \\
\epsilon^2 & \epsilon & \epsilon \\
\epsilon^2 & \epsilon & 1
\end{pmatrix}
\]  

(3)

where \(\epsilon \ll 1\). Numbers of order unity multiplying various terms with \(\epsilon\) in eq. (3) are not displayed. It is easy to see that the eigen–values of these matrices obey the
hierarchy
\[ m_t : m_c : m_u = 1 : \epsilon : \epsilon^2 \quad ; \quad m_b : m_s : m_d = 1 : \epsilon : \epsilon^2 \]  
\[ (4) \]

The mixing angles have the pattern
\[ V_{ud} \sim V_{cs} \sim V_{tb} \sim 1 \]
\[ V_{us} \sim V_{cb} \sim \epsilon \quad ; \quad V_{ub} \sim \epsilon^2 . \]  
\[ (5) \]

It is apparent that the form of the matrices in eq. (3) will lead to the desired hierarchy in the masses as well as in the mixing angles. The “average” value of \( \epsilon \) should be around \( 10^{-2} \) to accommodate both the mixing and the mass hierarchy. It is then tempting to postulate that \( \epsilon \) is a loop expansion parameter, \( \epsilon \sim (h^2/16\pi^2) \) with \( h \) a typical Yukawa coupling of order 1. The third generation masses will then arise at tree–level, the second family masses arise out of one–loop radiative corrections and the first family masses are two–loop effects. There will be no need to assume any of the couplings to be unnaturally small. In the next section, I shall describe a model which illustrates these features.

3. Supersymmetric model for fermion mass hierarchy

The idea that the second and first family masses arise from radiative corrections to some tree–level “bare” masses is not new.\(^3\) The implementation of this idea has gone through a revival lately.\(^4\) Several models have been proposed in the past five years or so which attempt to achieve this goal. In this section, I shall describe one such attempt. This is based on work done in collaboration with B.S. Balakrishna and R.N. Mohapatra.\(^5\)

The model is based on the supersymmetric left-right gauge group \( SU(2)_L \times SU(2)_R \times SU(4)_C. \)\(^6\) There is a reason for choosing this specific gauge structure, aside from other well–known motivations for left–right symmetry and supersymmetry. The \( SU(4)_C \) of color contains a lepto–quark gauge boson which has an electric charge \( 2/3 \). Its fermionic superpartner, denoted by \( \lambda \) can mix with the up–type quarks. This is an attractive feature which can explain why the top–quark is special, it gets its mass via mixing with the \( \lambda \) gaugino.\(^7\)

The chiral supermultiplets of the model belong to the following representations \((a = 1 − 3 \) is the generation index):

\[ \psi_a(2,1,4) = \left( \begin{array}{cccc} u_1 & u_2 & u_3 & \nu \\ d_1 & d_2 & d_3 & e \end{array} \right)_a \quad ; \quad \psi^c_a(1,2,\overline{4}) = \left( \begin{array}{ccc} u^c_1 & u^c_2 & u^c_3 \\ d^c_1 & d^c_2 & d^c_3 \end{array} \right)_a \]
\[ N(1,1,6) \quad ; \quad N^c(1,1,6) \]  
\[ (6) \]

\( N \) and \( N^c \) each contain a color triplet and a color anti–triplet fields. The di–quark (color triplet) and the lepto–quark (color anti–triplet) components of \( N \) will be denoted by \( N_1 \) and \( N_2 \). \( N^c_1 \) and \( N^c_2 \) components of \( N^c \) have the opposite color properties. Under left–right symmetry,

\[ \psi_a \leftrightarrow \psi^c_a, \quad N \leftrightarrow N^c . \]  
\[ (7) \]
Baryon number conservation will be imposed on the model which prevents $NN$ and $N^cN^c$ terms. The superpotential of the model has then a very simple form:

$$W = \frac{1}{4} f_{ab} \left( \psi_a^T i\tau_2 C^{-1} \psi_b N + \psi_a^T i\tau_2 \psi_b^c N^c \right) + \frac{1}{2} M_N N N^c$$

Here in the second line, we have expanded the fields $\psi_a$ and $\psi_a^c$ in their component form. Color indices have been suppressed and summation over the generation indices $a, b$ is implied.

Prior to supersymmetry breaking, gauge symmetry remains intact. Supersymmetry is broken softly with the soft breaking terms given as in supergravity models.

$$V_{soft} = \sum_i \mu_i^2 \phi_i^* \phi_i + m_{3/2} \int d^2 \theta^2 W + \sum_{i=1}^{i=3} M_i \lambda_i^T C^{-1} \lambda_i \ .$$

Here the sum over $\phi_i$ goes over all the spin zero components of the chiral superfields. The second term in eq. (9) stands for the soft SUSY breaking terms which have the same structure as the superpotential eq. (8). The last term is the SUSY–breaking Majorana mass terms for the gauginos.

$\mu_i^2$ will be chosen negative for $\tilde{\nu}_a$ and $\tilde{\nu}_a^c$ which will lead to spontaneous gauge symmetry breaking. The vacuum expectation values (vev’s) are denoted by

$$\langle \tilde{\nu}_a \rangle = (v_R)_a ; \quad \langle \tilde{\nu}_a^c \rangle = (v_L)_a \ .$$

By performing an orthogonal transformation on the fields $\psi_a$, the vev’s $\langle \tilde{\nu}_a \rangle$ can be brought to the form

$$\begin{pmatrix}
\langle \tilde{\nu}_1 \rangle \\
\langle \tilde{\nu}_2 \rangle \\
\langle \tilde{\nu}_3 \rangle
\end{pmatrix} = v_L \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} = v_L |h\rangle$$

A similar rotation on $\psi_a^c$ fields will bring those vev’s to $v_R \langle h\rangle$. The quark and lepton masses will be expressed in terms of the input parameters $f$, which is a $3 \times 3$ Yukawa coupling matrix, the vector $|h\rangle$ along with $M_N$ and the gaugino mass. After symmetry breaking the fields $\tilde{\nu}$ and $\tilde{\nu}^c$ will mix, $\tilde{u}$ will mix with $\tilde{u}^c$ (through $SU(4)_C$ $D$–term); $\tilde{d}$ mixes with $\tilde{N}_1^c$ (from the $F$–term) and with $\tilde{N}_2$ (from the soft SUSY breaking term).
Let us turn to the structure of fermion mass matrices in the model. In the charge $2/3$ sector, the vev’s of $\tilde{\nu}$ and $\tilde{\nu}^c$ generate mixing of the three generations with the charge $2/3$ gaugino $\lambda$. The mixing matrix is given in the basis $(u_a, \lambda)$ as

$$M_{\text{up}} = \begin{pmatrix} 0 & g_c |h\rangle v_L \\ \langle h | v_R g_c & M_\lambda \end{pmatrix}, \quad (12)$$

where $g_c$ is the strong gauge coupling. The matrix of eq. (12) has rank 2. That is, two of its eigenvalues are zero. These two massless states are identified with the charm and up quarks, the two non–zero masses are given by

$$M_\lambda' \simeq (M_\lambda^2 + g_c^2 v_R^2)^{1/2}; \quad m_t \simeq \frac{g^2 v_L v_R}{M_\lambda'} (|t\rangle \simeq |h\rangle). \quad (13)$$

Note that all three generations were treated on equal footing, yet only the top–quark acquires a tree–level mass.

In the charge $-1/3$ sector, the mass matrix that mixes the three generations with the fermions contained in $N, N^c$ is given by

$$M_{\text{down}} = \begin{pmatrix} 0 & f |h\rangle v_L \\ \langle h | f v_R & M_N \end{pmatrix}. \quad (14)$$

Again, this matrix has rank two, meaning that the $d$ and $s$ quark masses are zero at tree–level. The $b$ and $N$ masses are given by

$$M_{{N}_2}' = (M_N^2 + v^2_R \langle f^2 \rangle)^{1/2}; \quad m_b \simeq \langle f^2 \rangle \frac{v_L v_R}{M_{{N}_2}'} (|b\rangle \simeq f |h\rangle). \quad (15)$$

Here $\langle f^2 \rangle \equiv \langle h | f^2 |h\rangle$. Note that the $b$–quark mass scales inversely with $M_N$ which is a supersymmetry preserving mass, while $m_t$ scales inversely with the SUSY–breaking gaugino mass, which is likely to be smaller than $M_N$. This provides, at least qualitatively, a reason why top quark is heavier than the bottom.

In the charged lepton sector, since there is no analogous see–saw partner for the $\tau$ lepton, it remains massless at tree-level. The relevant Lagrangian is

$$L_{\text{lepton}} = \sqrt{2} g v_L e_3^T C^{-1} \bar{W}^+_L + M_{\tilde{W}_L} \tilde{W}^-_L C^{-1} \tilde{W}^+_L + (L \leftrightarrow R) + h.c. \quad (16)$$

Since there is no tree–level mixing between $\bar{W}_L$ and $\bar{W}_R$, we concentrate only on the left sector. It is clear from eq. (16) that the linear combination defined as

$$\tau = \cos \theta e_3 - \sin \theta \tilde{W}^+_L \quad (17)$$
with \( \tan \theta = \sqrt{2} g v_L / M_{W^+} \) remains massless, which will be identified with the \( \tau \) lepton.

Note that we have not imposed any flavor symmetry on the model. So nothing will protect the massless states from acquiring small (and finite) masses once higher order radiative corrections are included. The one–loop diagrams that generate the charm quark and the \( \tau \)–lepton masses are shown in fig. 1. From fig. 1a, one sees

\[
M_{\text{up}}^{\text{1–loop}} \sim f |h \rangle \langle h| f .
\]  

(18)

This is a unit–rank matrix, so when added to the rank 2 tree–level matrix, the rank of the up–quark matrix increases to three. That is, the charm quark picks up a mass, but the up quark still remains massless. Similarly, both fig. 1b and 1c have the matrix structure \( f |h \rangle \langle h| f \), which generates non–zero \( \tau \)–lepton mass. \( \mu \) and \( e \) masses are zero at this stage. \( m_c \) and \( m_\tau \) can be estimated from fig. 1:

\[
m_c \simeq \frac{4}{16\pi^2} \left( \langle f^4 \rangle - \langle f^2 \rangle^2 \right) \frac{v_L v_R}{M_N} \left( |c \rangle \simeq f^2 |h \rangle - \langle f^2 \rangle |h \rangle \right)
\]

\[
m_\tau \simeq \frac{9}{16\pi^2} \left( \langle f^2 \rangle g^2 v_L v_R \right) \frac{M_\lambda}{M_N^2} \left( |\tau \rangle \simeq f |h \rangle \right) .
\]

(19)

The eigen–states given above are not normalized.
In the down sector, all the one–loop graphs have the matrix structure given by $f|h\rangle \langle h|f$, which is the same as the tree-level down quark matrix. This implies that one loop corrections do not increase the rank of the matrix, it only corrects the tree–level $b$–quark mass. $d$ and $s$ masses remain zero.

Two–loop graphs of fig. 2 will induce non–zero masses for $s$ and $d$ quarks.

Fig. 2. Two–loop diagrams generating (a) strange quark and (b) $d$–quark mass.

Fig. 2 has the matrix structure given by

$$M_{\text{down}}^{2-\text{loop}} \simeq m_s f^3 |h\rangle \langle h| f^3 + m_d |h\rangle \langle h|$$

(20)

where

$$m_s \simeq \frac{m_c}{16\pi^2} ; \quad m_d \simeq \frac{\alpha^2}{16\pi^2 \sin^4 \theta_W} \frac{M_T^2}{M_{W_L}^2 M_{W_R}^2} m_t^2 m_b .$$

(21)

Although $d$ and $s$ quark masses are generated at the same loop–level, there is a clear distinction between them, each arises from a separate rank–one graph. The magnitudes of these graphs are in the right range for acceptable masses even with all Yukawa couplings being order 1.

In the up–quark sector, there is a diagram analogous to fig. 2b with $u \leftrightarrow d$ interchange which brings in a new matrix structure $f|h\rangle \langle h|f$ in this sector. That generates non–zero up–quark mass given by the second formula of eq. (21) but with $t$ and $b$ masses interchanged. That means that $m_u \propto m_b$ while $m_d \propto m_t$, explaining why $m_u < m_d$ without further assumption.

In the charged lepton sector, muon and electron masses are generated at two–loop level via diagrams analogous to fig 2. The analog of fig. 2a creates muon mass, and fig 2b the electron mass. If the masses of $\tilde{N}_1$ and $\tilde{N}_2$ are not split by much, one sees that $m_{\mu} \simeq m_s$, which is in good agreement with observations.
Analysis of the neutrino sector at tree–level involves a $5 \times 5$ matrix which mixes the gauginos and the neutrinos $\nu_3$ and $\nu_{3}^{c}$. In the basis $(\nu_3, \nu_{3}^{c}, \tilde{W}_{3L}, \tilde{W}_{3R}, \tilde{B})$, the matrix reads as

$$
\begin{pmatrix}
0 & 0 & g_{V_L}/\sqrt{2} & 0 & -g'_{V_L}/\sqrt{2} \\
0 & 0 & 0 & g_{V_R}/\sqrt{2} & -g'_{V_R}/\sqrt{2} \\
g_{V_L}/\sqrt{2} & 0 & M_L & 0 & 0 \\
0 & g_{V_R}/\sqrt{2} & 0 & M_R & 0 \\
-g'_{V_L}/\sqrt{2} & -g'_{V_R}/\sqrt{2} & 0 & 0 & M_B
\end{pmatrix}.
$$

(22)

The lightest eigenstate has a mass given by

$$
m_{\nu_3} \approx \frac{m_{\tilde{W}_L}^2}{M_L} \tan \phi ; \quad \tan \phi = \frac{g^2 M_B + g'^2 (M_L + M_R)}{g^2 M_B + g'^2 M_R}.
$$

(23)

Although there is a see–saw type suppression, since the scale $M_L$ cannot be too much higher than a few TeV in this minimal scheme, the mass of $\nu_\tau$ turns out to be in the MeV range unless the angle $\phi$ is tuned to be small. Such a neutrino should decay rather fast in order to avoid cosmological mass density constraints. One possibility is that $\nu_3$ decays into $\rightarrow \nu_{2,1} + \gamma$.

I conclude this section with a few observations.

1. The idea of generating the mass hierarchy out of radiative corrections seems to be very promising. In such a scenario, there is no need to assume certain Yukawa couplings to be artificially small. All couplings can be of the same order and yet small masses can result due to the loop suppression factors.

2. It should be emphasized that the mechanism does not rely on any sort of horizontal symmetry. All three families of fermions are treated on par, yet a large mass hierarchy is generated.

3. From phenomenological constraints such as $g−2$ of the $\mu$ and $\mu \rightarrow e\gamma$ rate, the masses of the scalars should exceed about 5 TeV.

4. In the example provided above, the weak iso–spin of the scalar used for mass generation is zero. This is a desired feature. If it were non–zero, flavor-changing couplings of the $Z$ boson with light quarks will be induced at an unacceptably high level.\textsuperscript{8}
5. An unsatisfactory feature of the idea discussed above is that if an iso-spin zero scalar is used for generating the charm quark mass, it becomes proportional to the $b$-quark mass. Although there is a log attached to the loop factor which can be somewhat larger than unity, since $b$ mass is roughly the same as $c$ mass, $m_c \simeq \alpha m_b$ is not very desirable.

6. It has been pointed out \cite{9} that in models where there is the cascade mechanism operative with 1st, 2nd and 3rd family masses induced at 2, 1 and 0 loop level, the Cabibbo angle tends to be small. In the model described above, both $d$ and $s$ quark masses arise at two-loop, so this problem does not seem to be present.

4. Inter–relations among quark and lepton masses and mixing angles

An important aspect of the mass puzzle is the proliferation of Yukawa couplings. Perhaps not all parameters are independent, there are sum rules relating various masses among themselves and with the mixing angles. The objective of such an approach is to reduce the number of arbitrary parameters and make the model more predictive. In this section I shall review some popular attempts along this line, especially within the context of grand unified theories.

In grand unified theories based on simple gauge groups such as $SU(5)$ or $SO(10)$, quarks and leptons of a given generation belong to the same grand–unified multiplet. The Yukawa couplings of quarks and leptons will then be related. The best known example is the minimal $SU(5)$ model which predicts

$$M^0_{\text{down}} = M^0_{\text{lepton}} .$$  \hspace{1cm} (24)

The superscript $^0$ is to remind ourselves that the equality holds at the unification scale $\sim 10^{15}$ GeV. From eq. (24), we have the asymptotic equality of the eigen–values:

$$m^0_b = m^0_\tau ; \quad m^0_s = m^0_\mu ; \quad m^0_d = m^0_e .$$  \hspace{1cm} (25)

To make contact with low energy observables, we should take into account the evolution of these mass relations. If all Yukawa couplings are small, these relations will evolve linearly which can be integrated analytically. However, since we know that the top Yukawa is non–negligible, the full non–linear renormalization group equations should be used. Since measurement of the weak mixing angle $\sin^2 \theta_W$ and the present limits on proton life–time have excluded minimal $SU(5)$, but are in good agreement with supersymmetric (SUSY) $SU(5)$, we adhere to the latter. In the SUSY extension of the standard model, the evolution of the third family Yukawa couplings are given by

$$8 \pi^2 \frac{dh^2_i}{dt} = h^2_i \left( 6h^2_t + h^2_u - \frac{13}{9} g^2_1 - 3g^2_2 - \frac{16}{3} g^2_3 \right) .$$
\[8\pi^2 \frac{dh_b^2}{dt} = h_b^2 \left( h_r^2 + 6h_b^2 + h_r^2 - \frac{7}{9}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2 \right)\]

\[8\pi^2 \frac{dh_r^2}{dt} = h_r^2 \left( 3h_b^2 + 4h_r^2 - 3g_1^2 - 3g_2^2 \right) . \tag{26}\]

These relations can be integrated from a unification scale of \(10^{16}\) GeV to the \(b\)-quark mass scale to arrive at a prediction for \(m_b\). Results are shown in Table 2 for varying values of \(\tan\beta\) (the ratio of the two Higgs vev’s in the SUSY extension of the standard model) and \(m_t\). The input parameters used are \(\alpha_s(M_Z) = 0.105, \alpha(M_Z) = 1/127.8, \alpha_2(M_Z) = 0.0322\). The supersymmetric threshold is taken to be at 300 GeV. These results are to be compared with values of \(m_b\) estimated from quarkonium spectroscopy: \(m_b(m_b) = 4.25 \pm 0.1\) GeV. The predictions for \(m_b\) for all values of \(m_t\) and \(\tan\beta\) are in good agreement with the low energy determination. Note however that a heavier top is preferred.

Table 2. Predictions for \(m_b(m_b)\) from the asymptotic equality \(m_b^0 = m_r^0\) in SUSY \(SU(5)\) as functions of \(m_t\) and \(\tan\beta\).

| \(m_t \to\) | 100  | 130  | 160  | 190  |
|------------|------|------|------|------|
| \(\downarrow \tan\beta\) |      |      |      |      |
| 1          | 4.95 | 4.55 | --   | --   |
| 2          | 5.08 | 4.91 | 4.61 | --   |
| 3          | 5.08 | 4.95 | 4.72 | 3.78 |
| 4          | 5.10 | 4.97 | 4.75 | 4.13 |
| 5          | 5.10 | 4.97 | 4.77 | 4.22 |
| 6          | 5.10 | 4.97 | 4.77 | 4.26 |
| 7          | 5.10 | 4.99 | 4.79 | 4.28 |

How about the other mass relations \(m_s^0 = m_\mu^0\) and \(m_d^0 = m_e^0\)? The former would predict \(m_s(1\ GeV) \simeq 400\ MeV\), a factor of 3 larger than the value quoted in Table 1. The relation \(m_d^0 = m_e^0\) would lead to \(m_d(1\ GeV) \simeq 2\ MeV\), which is a factor of 3 too small. This problem can also be seen by noting that the mass ratios \((m_s/m_d)\) and \((m_\mu/m_e)\) are essentially independent of momentum scale and so will obey the asymptotic relation which is off by at least a factor of 10.

Modifications of eq. (24) have been proposed to correct the bad mass relations.
The Georgi–Jarlskog proposal\textsuperscript{10} assumes the mass matrices have the asymptotic form

\[
M_{\text{down}} = \begin{pmatrix}
0 & a & 0 \\
 a & b & 0 \\
0 & 0 & c
\end{pmatrix}; \quad M_{\text{lepton}} = \begin{pmatrix}
0 & a & 0 \\
 a & -3b & 0 \\
0 & 0 & c
\end{pmatrix}.
\]

(27)

Such mass matrices can arise if the Higgs sector consists of a 45 coupling to the second family in addition to the usual 5 and if there are some discrete symmetries distinguishing generations. The asymptotic relations implied by eq. (27) are

\[
m_b^0 = m_\tau^0; \quad m_s^0 = \frac{1}{3} m_s^0; \quad m_d^0 = -3m_e^0.
\]

(28)

These relations will preserve the successful prediction of $m_b$, in addition, the $d$ and $s$ quark masses will be predicted to be $m_d(1\ GeV) \simeq 7\ MeV$, $m_s(1\ GeV) \simeq 140\ MeV$.

It is also possible to predict the quark mixing angles in terms of the quark mass ratios. A popular ansatz\textsuperscript{11} that has received much attention recently assumes the down quark and charged lepton matrices to have the Georgi–Jarlskog texture of eq. (27), but the up–quark matrix has the Fritzsch form:\textsuperscript{12}

\[
M_{\text{up}} = \begin{pmatrix}
0 & a' & 0 \\
 a' & 0 & b' \\
0 & b' & c'
\end{pmatrix}.
\]

(29)

Although the elements are assumed to be complex, all but one phase can be rotated away from these matrices. It can be taken to be the phase of $a$ in eq. (27). Since there are only seven parameters describing thirteen observables, there will be six predictions. Three of them are the mass relations of eq. (28), the other three are for the quark mixing angles. These asymptotic relations are given by ($^0$ is dropped for convenience)

\[
|V_{us}| = |V_{cd}| = \left|\sqrt{\frac{d}{s}} - \sqrt{\frac{u}{c}} e^{i\phi}\right|; \quad |V_{cb}| = |V_{ts}| = \sqrt{\frac{c}{t}}
\]

\[
|V_{ub}| = \sqrt{\frac{u}{t}}; \quad |V_{td}| = \left|\sqrt{\frac{u}{t}} c e^{i\phi} + \sqrt{\frac{d}{s}} \sqrt{\frac{c}{t}} e^{i\phi}\right|.
\]

(30)

To see the validity of these relations, one has to extrapolate them to low energies. It was noted some time ago\textsuperscript{13} that the quark mixing angles will run appreciably with
momentum if any of the Yukawa coupling is comparable to the gauge coupling. The mass ratios also run. Fig. 3 exhibits the behavior of these running for the standard spectrum as well as for the SUSY spectrum (with $\tan \beta = 3$). The Cabibbo angle as well as the mass ratios involving the first two families do not run. The running factors for $|V_{ub}|, |V_{cb}|, |V_{td}|, |V_{ts}|$ are identical, same is true for $m_d/m_b$ and $m_s/m_b$ etc. Note that while in the standard model, $|V_{cb}|, |V_{ub}|$ increase with energy, the opposite is true for the SUSY model. One can infer the low energy prediction for the mixing angles from fig. 3. $|V_{cb}|$ turns out to be $\geq 0.052$ and the top mass should be near its fixed point value of 180 GeV.

Suppose both the up–quark and the down–quark matrices have the Fritzsch texture. Then some of the relations of eq. (30) will be modified. Of particular interest is the relation for $|V_{cb}|$ which now takes the form

$$|V_{cb}| = \left| \sqrt{\frac{s}{c}} - \sqrt{\frac{c}{t}} e^{i\phi} \right|.$$  (31)

Note that the magnitude of the first term is about 0.16, so to get agreement with observed $|V_{cb}| = 0.043 \pm 0.009$, there should be a strong cancellation between the two terms. That sets an upper limit of about 90 GeV on top mass, which is excluded. The renormalization properties of this relation have been studied recently, where it was shown that $m_t$ as large as 145 GeV is still admissible in the SUSY model provided $\tan \beta$ is large.

Fig. 3. Running factors for $|V_{cb}|, |m_s/m_b|, |m_c/m_t|$ defined as $f(M_X)/f(M_Z)$ for the standard model (SM) and SUSY model with $\tan \beta = 3$ (from Ref. 14).
5. The strong CP Problem

The QCD Lagrangian admits a term

\[ L'_{\text{QCD}} = \frac{\theta_{\text{QCD}}}{16\pi^2} G \tilde{G} \]  

(32)

where \( G \) is the gluon field strength tensor. Such a term violates both Parity and \( CP \) symmetries. However, both symmetries are broken in the weak interaction sector, so there is no reason why such a term should not exist. \( \theta_{\text{QCD}} \) by itself is not a physical observable, for a chiral rotation on the quark fields will change its value. However, the combination

\[ \bar{\theta} = \theta_{\text{QCD}} + \text{Arg}(\text{Det}M_q) \]  

(33)

is invariant under such rotations and is observable. Limits on the electric dipole moment of the neutron puts a severe constraint \( \bar{\theta} \leq 10^{-9} \). The strong CP problem is why this parameter, which a priori is of order one is so small.

Several solutions to the puzzle have been put forth. The Peccei-Quinn symmetry, which is an approximate axial symmetry broken only by non-perturbative instantons, can solve the problem, but at the expense of introducing a light degree of freedom, the axion. Here I wish to discuss another solution to the problem which does not result in the axion.

\( \bar{\theta} \) carries information about the structure of the quark mass matrix. In theories which respect \( P \) or \( CP \), the bare QCD contribution to \( \bar{\theta} \) is zero. If the determinant of the quark mass matrix is arranged to be real, the second term will also be zero. This does not mean that there is no weak CP violation, it could arise from complex CKM matrix elements. Finite and calculable \( \bar{\theta} \) will be induced at higher order. If these induced \( \bar{\theta} \) is less than the present limit, that would provide a solution to the strong CP problem.\(^{15}\) In what follows, I demonstrate this idea in the context of a Parity invariant theory.\(^{16}\)

The model is based on the gauge group \( SU(2)_L \times SU(2)_R \times U(1) \). The quarks and leptons belong to the left–right symmetric representations under the gauge group:

\[ q_L(2, 1, 1/6) ; \quad q_R(1, 2, 1/6) ; \quad \psi_L(2, 1, -1/2) ; \quad \psi_R(1, 2, -1/2) . \]  

(34)

In addition, there are these singlet quarks and leptons, one per generation:

\[ P(1, 1, 2/3) ; \quad N(1, 1, -1/3) ; \quad E(1, 1, -1) . \]  

(35)

The Higgs sector of the model is very simple, it consists of \( \chi_L(2, 1, 1/2) + \chi_R(1, 2, 1/2) \). With such a spectrum, the ordinary quark and lepton masses can arise only via its mixing with the exotic fermions.\(^{17}\) The mass matrix for the up–sector is given by

\[ M_{\text{up}} = \begin{pmatrix} 0 & h v_L \\ h^* v_R & M \end{pmatrix} \]  

(36)

14
where $h$ is the $3 \times 3$ Yukawa coupling matrix. Although $h$ is complex, note that the determinant of $M_{up}$ is real, similarly for $M_{down}$. So $\mathcal{F}$ is zero at tree level. Yet realistic weak CP arises via the KM mechanism. In this model it turns out that even at the one-loop level, there is no $\mathcal{F}$ induced. $\mathcal{F}$ arises at two-loop level, which is estimated to be $\approx 10^{-12}$, well below the present limit. Although there is no $\mathcal{F}$ at one-loop, the neutron edm is non-zero, the natural value of it is around $10^{-26} - 10^{-27}$ ecm, which should be accessible to experiments in the near future.

6. Neutrino masses

If neutrinos have masses, they are much smaller than their charged lepton counterparts. A well-known mechanism that explains this feature is the see-saw mechanism where the light neutrino mass scales inversely with the Majorana mass of $\nu_R$. In $SO(10)$ for example, they scale as

$$m_{\nu_i} \sim m^2_u/M,$$

where $M$ is the $\nu_R$ mass, which can be as large as the grand unification scale.

There are some indirect indications in favor of tiny neutrino masses. The flux of $\nu_e$'s from the sun detected on earth by several experiments seems to be smaller than theoretical expectations. This deficit, known as the solar neutrino puzzle, can be resolved if neutrinos have tiny masses and if different flavors mix. An elegant explanation is the matter enhanced resonant oscillation (the Mikheyev-Smirnov-Wolfenstein effect). The favored value of the neutrino mass splitting and mixing angle is around $\Delta m^2 \simeq 10^{-5}$ $eV^2$ and $\sin^2 2\theta \simeq 10^{-2}$. Such small values originate naturally in $SO(10)$.

Another puzzle has emerged in the last several years regarding the flux of neutrinos from the sky (atmospheric neutrinos) detected by the Kamiokande and IMB experiments. There seems to be some discrepancy between the observed and expected $\mu/e$ flux ratio, which can be interpreted as a deficit of $\nu_\mu$'s or an excess of $\nu_e$'s. One possible explanation which is not inconsistent with the MSW mechanism for solar neutrinos is in terms of $\nu_\mu - \nu_\tau$ oscillation. The relevant mixing angle should then satisfy $\sin^2 2\theta_{\mu\tau} = (0.42 - 01)$ which is surprisingly large. For comparison, the analog of quark mixing is only $3 \times 10^{-2}$. The mass difference should lie in the range $(10^{-3} - 0.3)$ $eV^2$.

Are the two independent observations compatible with predictions of grand unified theories such as $SO(10)$? If one tries to explain both simultaneously, two potential problems arise. Firstly, as noted above, a resolution of the atmospheric neutrino puzzle requires rather large mixing angles. The mass ratios also are not in the favored range. Note that the desired spectrum has $(\nu_e, \nu_\mu, \nu_\tau) \sim (\leq 10^{-3}, 10^{-3}, 10^{-1})$ $eV$. However, from eq. (37), the naive expectation is

$$m_{\nu_\mu}/m_{\nu_\tau} \sim m^2_e/m^2_\tau \sim 10^{-4}$$

That is, if solar neutrino puzzle is explained via MSW, $\nu_\tau$ mass turns out to be in the range of a few eV.
In models with non–minimal Higgs, both features can be accommodated simultaneously. Consider the mixing of second and third families alone for now. Suppose the mass matrices are given by

\[
M_{\text{down}} = \begin{pmatrix} D & B \\ B & C \end{pmatrix} ; \quad M_{\text{up}} = \begin{pmatrix} 0 & B' \\ B' & C' \end{pmatrix} ; \quad M_{\text{lepton}} = \begin{pmatrix} -3D & -3B \\ -3B & C \end{pmatrix} ;
\]

\[
M_{\nu}^{\text{Dirac}} = \begin{pmatrix} 0 & -3B' \\ -3B' & C' \end{pmatrix} ; \quad M_{\nu}^{\text{Majorana}} = \begin{pmatrix} M_2 & M_3 \\ M_3 & 0 \end{pmatrix} .
\]

(39)

Here the elements \(C, C'\) arise from 10 of Higgs, whereas all other entries arise from 126. Such matrices preserve the successful prediction \(m_{\nu_\tau}^0 \simeq m_{\nu_\tau}^0\). The see-saw formula now leads to

\[
m_{\nu_\mu}/m_{\nu_\tau} \simeq 81m_c^2/m_t^2 .
\]

(40)

Note that the factor of 3 in the off–diagonal element of eq. (39) appears as a factor of 81 in eq. (40), which makes the mass ratio come out right. Due to the same factor of 3, the (2-3) mixing also turns out to be large (of order 30 degrees or more), explaining the atmospheric puzzle. The solar neutrino deficit can be explained via \(\nu_e - \nu_\mu\) MSW oscillations.

7. Conclusion

It is necessary to go beyond the standard model to address the puzzles associated with the fermion mass matrices. Some of the new ideas that have emerged recently seem to be very promising. However, a fully consistent picture that explains satisfactorily all the puzzles is still lacking. More work is needed in these directions.

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