OPTIMAL PENSION DECISION UNDER HETEROGENEOUS HEALTH STATUS AND BEQUEST MOTIVES

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ABSTRACT. In this paper, we study the optimal decision between ELA (Equity Linked Annuity) and ELID (Equity Linked Income Drawdown) pension plans under heterogeneous personal health statuses and bequest motives. In the ELA pension plan, the survival member receives the mortality credit, and leaves no bequest at the time of death, while the member receives no mortality credit and receives the fund wealth as bequest at the time of death in the ELID pension plan. The pension member controls the asset allocation and benefit outgo policies to achieve the objectives. We explore the square deviations between the actual benefit outgo and the pre-set target, and the square and negative linear deviations between the actual bequest and the pre-set target as the disutility function. The minimization of the disutility function is the objective of the stochastic optimal control problem. Using HJB (Hamilton-Jacobi-Bellman) equations and variational inequality methods, the closed-form optimal policies of the ELA and ELID pension plans are derived. Furthermore, the optimal decision boundary between the ELA and ELID plans is established. It is the first time to study the impacts of heterogeneous personal health status and bequest motive on the optimal choice between the ELA and ELID pension plans under the original performance criterions. The worse health status and higher bequest motive result in the higher utility of the ELID pension plan, and vice versa. The worse heath status increases the proportion allocated in the risky asset and increases the benefit outgo in both pension plans. The bequest motive has positive impacts on the proportion in the risky asset and negative impacts on the benefit outgo in the ELID pension plan.

1. Introduction. In this paper, we study the optimal asset allocation and benefit outgo policies during the distribution phases of the ELA and ELID pension plans under heterogeneous personal health statuses and bequest motives.

At the early stage of the DC (Defined Contribution) pension development, the fund wealth is converted into annuities at the time of retirement in order to avoid the risk of being exhausted. As the fund wealth is not adequate to provide high standard old-care utility in most of the DC plans, the investment in the risky asset is allowed during some period of the distribution phase. In the ELA and ELID pension plans, the pension member chooses the optimal proportion allocated in the risky asset as a
control variable to achieve the objectives. The stochastic optimal control theory has been extensively applied in the asset allocation problems of the pension plan under the framework of Merton\cite{16}. Cairns et al.\cite{8} and Josa-Fombellida and Rincón-Zapatero\cite{13} assume that the risky asset follows geometric Brownian motions and study the optimal asset allocation policies. Therefore, these give the possibility of studying the optimal asset allocation and benefit outgo policies in the ELA and ELID pension plans.

In the ELA plan, the fund wealth of the dead members is equally distributed by the survival members as mortality credit, and the member receives no bequest at the time of death. While the member receives no morality credit and receives the fund wealth as bequest at the time of death in the ELID plan. In this paper, we extend the model of He and Liang\cite{11} to describe the dynamics of the fund wealth using constant force of mortality. In the practice of the ELA pension management, as in Milevsky and Robinson\cite{17}, and Albrecht and Maurer\cite{1}, the benefit outgo is distributed as life annuities by recalculating the fund wealth at every time interval. Here the pension members are given more flexibility in choosing the optimal benefit outgo, which is taken as a control variable in the model. In order to avoid the exhaustion of the fund wealth, we suppose the existence of compulsory conversion time to annuities as in Milevsky and Young\cite{18}.

For the performance criterion aspects, the pension member chooses appropriate control policies to maximize the old-care utility after retirement. The objectives of the pension member are divided into two categories: the maximization of the CRRA (constant relative risk aversion) and CARA (constant absolute risk aversion) utilities of the benefit outgo, and the minimization of the disutility as the deviations between the actual benefit outgo and a pre-set target. The literatures on the former objectives include Battocchio and Menoncin\cite{2}, and Milevsky and Young\cite{19}. They choose power utility and exponential utility of the benefit outgo as the performance criterions, respectively. The idea of the latter objectives originates from the work of Bordley and Li Calzi\cite{5}, which establishes the deviations as the cost function, i.e., the disutility function. Cairns\cite{8}, and Vigna and Haberman\cite{21} choose the deviations of the actual benefit outgo and a pre-set target as the disutility function in the DC pension management. The latter objective is used in the model of the ELA plan. In the ELID plan, in order to consider the disutility provided by bequest in the model, we integrate the square and negative linear deviations between the bequest and the pre-set target in the objective function. The minimization of the disutility function represents the preference of bequest, i.e., the bequest motive. The integrated objective function is aspired by Chang et al.\cite{9} for integrating the negative linear deviation in the objective function as the preference of positive deviation. We aim at establishing the dynamical control policies under the integrated performance criterion with the disutilities provided both by the benefit outgo and the bequest.

The heterogeneous personal health status and the bequest motive have influence on the optimal choice between the ELA and ELID pension plans, as well as the optimal control policies of the two pension plans. Empirical studies give the following findings. Bernheim\cite{3} finds that the ELID plan is the optimal choice of members with strong bequest motive. The conclusions in Brown\cite{7}, and Finkelstain and Poterba\cite{10} confirm the theoretical model in Brugiavini\cite{6} that the members with worse health status prefer the ELID plan to the ELA plan. In this paper, we suppose the pension members have heterogeneous personal health statuses and
bequest motives, as in Blake, et al.[4], and establish the impacts of health status and bequest motive on the optimal choice between the ELA and ELID plans.

It is the first time to study the stochastic optimal control problems of the ELA and ELID pension plans under the integrated disutility functions with heterogeneous personal health statuses and bequest motives. Using HJB equations and variational inequality methods, as similar procedures in Ngwira and Gerrard[20], we solve the optimal stochastic control problems and derive the closed-form optimal asset allocation and benefit outgo policies. The results of Lions and Sznitman[15] guarantee the dynamics of the fund wealth is uniquely determined by the stochastic differential equation with respect to the optimal feedback functions. Furthermore, we use MCM (Monte Carlo Methods) to investigate the impacts of the heterogeneous personal health status and the bequest motive on the optimal control policies.

The paper will be organized as follows. In Section 2, we establish stochastic differential equations to model the dynamics of the fund wealth in the ELA and ELID plans, respectively. The minimization of the integrated disutilities provided by the benefit outgo and bequest is the objective function. In Section 3, using HJB equations and variational inequality methods, we establish the closed-form optimal asset allocation and benefit outgo policies of the ELA and ELID plans. In Section 4, we use MCM to study the impacts of the heterogeneous personal health status and bequest motive on the optimal control policies. The conclusions of this paper are given in Section 5.

2. **The stochastic optimal control problems.** In this paper, we study the optimal asset allocation and benefit outgo policies of the ELA and ELID pension plans during the distribution phase under heterogeneous personal health statuses and bequest motives. The pension member controls the asset allocation and the benefit outgo policies to achieve the optimal performance criterion. In order to avoid the fund wealth of being exhausted, the fund wealth is compulsorily converted into annuities after the conversion time.

As receiving stable and sustainable benefit outgo is the objective of the pension member, minimizing the square deviations between the actual benefit outgo and the pre-set target is the objective function in the ELA plan. In order to consider the bequest motive effects, the square and negative linear deviations between the bequest and the pre-set target are integrated in the disutility function in the ELID plan. The weight of the negative linear deviation represents the degree of bequest motive. The larger weight represents the higher preference for the larger bequest, and the lower degree of risk aversion.

In this paper, we suppose the pension member has heterogeneous personal health status. The premiums of the pension plans are calculated by the actuarial equations according to the law of large members, and the mortality credit is calculated by the statistical mortality results of all the members in the plan. Meanwhile, pension members have heterogeneous personal health statuses and mortality probabilities, and the information is better known by themselves. The different expectations of the mortality probabilities result in different disutility functions, optimal control policies of the pension members, and the optimal choice between the ELA and ELID pension plans. Now we extend the model of He and Liang[11] to establish the dynamics of the fund wealth of the two pension plans by continuous-time models.

2.1. **Stochastic model of the ELA pension plan.** In this section, we establish the fund wealth dynamics of the two pension plan. First, the following stochastic
differential equations are used to describe the dynamics of the risk-free asset and the risky asset. The risk-free asset and the risky asset are governed by the following SDE (stochastic differential equations) on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), respectively:

\[
\begin{align*}
    dS^0(t) &= rS^0(t)dt, \\
    dS^1(t) &= S^1(t)\left\{cdt + \sigma dB(t)\right\},
\end{align*}
\]

where \(S^0_t\) and \(S^1_t\) are the values of the risk-free asset and the risky asset at time \(t\). \(r\) is the risk-free interest rate. \(c\) and \(\sigma\) are the expected return and the volatility of the risky asset, respectively. \(\{B(t), t \geq 0\}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual conditions(cf.[14]). \(\mathcal{F}_t\) represents the information available at time \(t\) and any decision made up to time \(t\) is based on this information.

During the distribution phase of the ELA pension plan, the fund wealth is affected by investment return, benefit outgo and mortality credit. In the traditional ELA pension plan, the benefit outgo is calculated by dividing the current fund wealth by the actuarial present value of one unit life annuities to avoid the fund wealth of being exhausted. According to the new trend in pension management, the members are given more flexibilities to choose the appropriate benefit outgo to achieve optimal objectives. So, the asset allocation and benefit outgo policies are both control variables in the model.

We use similar methods in He and Liang[11, 12] to model the mortality credit. The dynamics of the fund wealth can be written by the following SDE:

\[
\begin{align*}
    dY_1(t) &= \left\{[\pi_1(c - r) + r]Y_1(t) + \mu^OY_1(t) - p_1(t)\right\}dt \\
    &\quad + \pi_1\sigma Y_1(t)dB(t), \\
    Y_1(0) &= y_0, \quad (2.1)
\end{align*}
\]

where \(Y_1(t)\) is the fund wealth at time \(t\), \(\pi_1(t)\) is the proportion allocated in the risky asset and \(1 - \pi_1(t)\) is the proportion allocated in the risk-free asset. \(p_1(t)\) is the amount of the fund wealth on the benefit outgo. The admissible domain of the control variable \((\pi_1(t), p_1(t))\) are \((-\infty, +\infty) \times (-\infty, +\infty)\). These hypotheses are used to avoid the time-inconsistent problem, and it is the compromise between the closed-form solution and the coordination with the pension management practice. In practice, the actual admissible domain of the control variable \((\pi_1(t), p_1(t))\) are \([0, 1] \times [0, Y_1(t)]\), i.e., the short sales of the risky asset and the risk-free asset are not allowed, and the benefit outgo should not exceed the fund wealth at that time. Fortunately, according to empirical data, the probability that the optimal control policies exceed the actual admissible domain is very small. The time-inconsistent problem is studied in Zhao et al.[23]. \(\mu^O\) is the force of mortality of all the members. The mortality credit is that the fund wealth of the dead members is equally distributed by the survival members. In this paper, we use constant force of mortality model to establish the mortality credit. \(y_0\) is the initial fund wealth, i.e., the fund wealth at the time of retirement. The equation (2.1) represents the impacts of investment return, mortality credit and benefit outgo on fund wealth dynamics, respectively.

As the benefit outgo is a control variable in the model, methods should be implemented to avoid the fund wealth of being exhausted. We suppose that the conversion time is \(T\). At that time, the fund wealth is all invested in the risk-free asset and
where \( E \) is distributed as life annuities, i.e.,

\[
\pi_1(t) = 0, \quad p_1(t) = \frac{Y_1(T)}{\ddot{a}_{x_0+T}^O}, \quad \forall \ T < +\infty,
\]

where \( \ddot{a}_{x_0+T}^O \) is the actuarial present value of one unit life annuities at the age of \( x_0 + T \) from all the members’ perspective, and it is calculated by

\[
\ddot{a}_{x_0+T}^O = \int_0^{+\infty} e^{-rs} s \dot{p}_{x_0+T}^O ds = \int_0^{+\infty} e^{-rs} e^{-\int_0^t \mu^S du} ds = \frac{1}{\mu^O + r},
\]

where \( s \dot{p}_{x_0+T}^O \) is the conditional probability that a person is alive at the age of \( x_0 + T \), will be still alive at the age of \( x_0 + T + s \). As the actuarial present value of the life annuities is calculated according to the statistical survival results of all the members, we use \( \mu^O \) as the force of mortality in the calculations.

According to the economics studies, the objectives of the pension members are the maximization of the old-care utility after retirement. The widely recognized performance criterion in the optimal pension management problem is to minimize the deviations between the actual benefit outgo and a pre-set target, which is an exogenous variable. The actual benefit outgo should be close to the pre-set target, and both the positive and negative deviations are penalized.

In this paper, we use the minimization of the square deviations between the actual benefit outgo and the pre-set target as objectives, as in Ngwira and Gerrard [20]. Therefore we define the value function \( V_1(y; t) \) as follows:

\[
\begin{align*}
V_1(y; t) &= \min_{(\tau_1, p_1) \in \Pi_t} \left\{ J_1(\pi_1, p_1, y; t) \right\} \\
&= \min_{(\tau_1, p_1) \in \Pi_t} \mathbb{E}_{(\pi_1, p_1, y; t)} \left\{ \int_t^T e^{-rs} s \dot{p}_{x_0}^S (p_1(s) - NP)^2 ds \right. \\
&\quad + \left. \int_T^{+\infty} e^{-rs} s \dot{p}_{x_0}^S \frac{Y_1(T)}{\ddot{a}_{x_0+T}^O} - NP)^2 ds \right\},
\end{align*}
\]

(2.2)

where \( \mathbb{E}_{(\pi_1, p_1, y; t)} \) is the conditional expectation given the initial value \( V_1(t) = y, \pi_1(t) = p_1, \pi_1(t) = \pi_1 \) at time \( t \). The performance criterion is \( J_1(\pi_1, p_1, y; t) \). \( \Pi_t \) is the set of all admissible policies whose initial values are replaced by \( p_1(1) = p_1, \pi_1(t) = \pi_1 \). \( NP \) is the pre-set target of the expected benefit outgo. It is the required benefit outgo to maintain a high standard retirement life. It is an exogenous variable and \( NP > 0 \). \( s \dot{p}_{x_0}^S \) is the conditional probability that a person is alive at the age of \( x_0 \), will be still alive at the age of \( x_0 + s \). \( s \dot{p}_{x_0}^S = e^{-\int_0^t \mu^S ds} \), where \( \mu^S \) is the personal force of mortality. Since the pension members have heterogeneous personal health statuses, they have different survival/mortality probabilities. We use personal survival/mortality probabilities in the performance criterion of the individual pension member. The items in (2.2) represent the disutilities provided by the benefit outgo before and after the conversion time, respectively. The objective function is to minimize the disutility function.

The asset allocation policy \( \pi_1^* \) and the benefit outgo policy \( p_1^* \) are called optimal (or optimal control processes) if \( (\pi_1^*, p_1^*) \in \Pi_t \) and satisfies

\[
V_1(y; t) = J_1(\pi_1^*, p_1^*, y; t).
\]

Thus, the continuous time stochastic optimal control problem (2.1)-(2.2) of the ELA pension plan has been established. We aim at deriving closed-form optimal policies \((\pi_1^*, p_1^*)\).
2.2. Stochastic model of the ELID pension plan. In the ELID pension plan, the pension members control the asset allocation and benefit outgo policies to achieve the objectives. Being different from the ELA pension plan, the member does not receive mortality credit, but receives the fund wealth as bequest at the time of death in the ELID pension plan. The dynamics of the fund wealth in the ELID pension plan is given by the following SDE:

\[
\begin{align*}
\frac{dY_2(t)}{dt} &= \{[\pi_2(c - r) + r]Y_2(t) - p_2(t)\}dt + \pi_2 \sigma Y_2(t)dB(t), \\
D_2(\tau_d) &= Y_2(\tau_d), \\
Y_2(0) &= y_0,
\end{align*}
\]

where \(Y_2(t)\) is the fund wealth at time \(t\). \(\pi_2(t)\) is the proportion allocated in the risky asset, \(1 - \pi_2(t)\) is the proportion allocated in the risk-free asset. \(p_2(t)\) is the amount of the fund wealth on the benefit outgo at time \(t\). They are control variables with \(\pi_2(t) \in (-\infty, +\infty)\) and \(p_2(t) \in (-\infty, +\infty)\) for any \(t \geq 0\), respectively. \(D_2(\tau_d)\) is the bequest at time \(\tau_d\), and it equals the fund wealth at that time, \(\tau_d\) is the time of death and it is a random variable.

For the performance criterion aspects, besides the disutility provided by the benefit outgo, the disutility provided by the bequest is also integrated in the performance criterion. The single square deviation represents the scenario of the members with no bequest motive. The scenario is fit for the younger members who have young children to raise, and they have little bequest motive. The negative linear deviation represents the scenario of the members with bequest motive. The scenario is fit for the older members who have young children to raise, and they have bequest motive. As the objective is to minimize the disutility function, the negative linear deviation represents the preference for the larger bequest, i.e., the bequest motive. Hence, we define the value function \(V_2(y; t)\) as follows:

\[
\begin{align*}
V_2(y; t) &= \min_{(\pi_2, p_2) \in \Pi_t} \left\{ J_2(\pi_2, p_2, y; t) \right\} \\
&= \min_{(\pi_2, p_2) \in \Pi_t} \left\{ \mathbb{E}_{(\pi_2, p_2, y; t)} \left[ \int_t^T e^{-r_s} s p_{x_0} (p_2(s) - NP)^2 ds \right. \\
&\quad + \int_t^T e^{-r_s} s p_{x_0} \left( \frac{Y_2(T)}{e^{\alpha (\tau_d) - N P}} - NP \right)^2 ds \right] \\
&\quad + \mathbb{E}_{(\pi_2, p_2, y; t)} \left[ \mathbb{E}_{(\tau_d)}]\left( [\alpha(D_2(T) - NP) - \beta(D_2(\tau_d) - NP)] \right) \right], \right. \\
Y_2(0) &= y_0,
\end{align*}
\]

where \(\mathbb{E}_{(\pi_2, p_2, y; t)}[\cdot]\) is the conditional expectation given the initial value \(Y_2(t) = y, p_2(t) = p_2, \pi_2(t) = \pi_2\) at time \(t\). The performance criterion is \(J_2(\pi_2, p_2, y; t)\). \(\Pi_t\) is the set of all admissible policies \((\pi_2, p_2)\) with admissible domains \(( -\infty, +\infty) \times (-\infty, +\infty)\). \(\tau_d\) is the pension member’s time of death. It is a random variable, and is determined by the member’s personal health status. \(\mathbb{E}_{(\tau_d)}[\cdot]\) is the expectation with respect to the random time of death. \(\alpha\) is the weight variable which measures the importance of the disutility provided by the last distribution (bequest) of the member with no bequest motive, and \(\alpha > 0\). \(\beta\) is another weight variable which measures the importance of the disutility provided by the bequest of the member with bequest motive, and \(\beta > 0\).
The value function has the following forms by some simple probability transforma-

tions:

\[
V_2(y; t) = \min_{(\pi_2, p_2) \in \Pi_t} \left\{ J_2(\pi_2, p_2, y; t) \right\}
\]

\[
= \min_{(\pi_2, p_2) \in \Pi_t} \mathbb{E}_{(\pi_2, p_2, y; t)} \left\{ J_2^{T} e^{-rs} s p_{x_0}^{\bar{S}} (p_2(s) - NP)^2 ds + \int_{T}^{+\infty} e^{-rs} s p_{x_0}^{\bar{S}} \frac{\partial^2}{\partial \pi_2^2} (Y_2(T)) - NP)^2 ds + \int_{T}^{T} e^{-rs} s p_{x_0}^{\bar{S}} \mu^{S} \left[ \alpha(Y_2(s) - NP)^2 - \beta(Y_2(s) - NP) \right] ds \right\},
\]

\[
Y_2(0) = y_0.
\]

The items in equation (2.5) are the deviations between the actual benefit outgo
and the pre-set target before and after the conversion time, and the square and
the negative linear deviations between the bequest and the target. The objective
function is to minimize the disutility function, i.e., maximize the old-care utilities
provided by the benefit outgo and the bequest.

The asset allocation policy \(\pi_2^*\) and the benefit outgo policy \(p_2^*\) are called optimal
control processes if \((\pi_2^*, p_2^*) \in \Pi_t\) and satisfies

\[
V_2(y; t) = J_2(\pi_2^*, p_2^*, y; t).
\]

Thus, the continuous-time stochastic optimal control problem (2.3)-(2.5) for the
ELID pension plan has been established. We aim at deriving closed-form optimal
policies \((\pi_2^*, p_2^*)\) to solve the optimal control problem (2.3)-(2.5).

It is the first time to study the optimal control problem of the ELID pension
plan with the performance criterion integrating the disutilities of the benefit outgo
and the bequest. In the next section, we will derive the closed-form solutions of the
stochastic control problems of the ELA and ELID pension plans. Furthermore, we
study the impacts of heterogeneous personal health status and the bequest motive
on the optimal control policies, and the optimal choice between the ELA and ELID
pension plans.

3. Solutions of the stochastic optimal control problems. In this section,
we use Itô stochastic calculus and variational methods to solve the optimal
control problems (2.1)-(2.2) and (2.3)-(2.5). First, we establish the HJB equations that
value function and optimal feedback functions satisfy. Second, we derive the optimal
feedback functions from the HJB equations. Third, the results in Lions and
Sznitman\[15\] guarantee that the solution of the stochastic differential equation that
the fund wealth dynamics satisfies is uniquely determined by the optimal feedback
functions. The unique solution of the SDE is the so-called the optimal state(or
wealth) process(cf.\[22\]). Finally, we get closed-form optimal asset allocation and
benefit outgo policies by composing the optimal feedback functions and the optimal
state process.

First, we derive the associated HJB equations with the stochastic control prob-
lems (2.1)-(2.2) and (2.3)-(2.5). We synthesize the solutions of the stochastic optimal
control problems in the following HJB equations (3.1):

Denote the solutions of the equations by \(\varphi_i(\pi_i, p_i, y; t)\), for \(i = 1, 2\). \(\varphi_1(\pi_1, p_1, y; t)\)
and \(\varphi_2(\pi_2, p_2, y; t)\) are the solutions of the stochastic control problems with the
optimal feedback functions of asset allocation and benefit outgo policies in the ELA
and ELID pension plans, respectively.

Using variational methods and Itô formula, we know that \(\varphi_i(\pi_i, p_i, y; t), i = 1, 2\)
satisfies the following HJB equations:
0 = \min_{\pi, p_i} \{ \frac{\partial \varphi_i}{\partial y}(\pi(c-r)+r \delta_1 \mu^O - p_i(t)) \\
+ \frac{1}{2} \frac{\partial^2 \varphi_i}{\partial y^2}(\pi^2 \sigma^2 y^2 + \frac{1}{\mu^O} e^{-rt} e^{-\int_0^t \mu^O ds} (p_i(t) - NP)^2 \\
+ (1 - \delta_1) e^{-rt} e^{-\int_0^t \mu^O ds} mu^S [\alpha(y(t) - NP)^2 - \beta(y(t) - NP)]) \} \\
\text{with boundary condition,}
\varphi_i(\pi_i, p_i, Y_i(T); T) = e^{-(\mu^S + r)T} \frac{1}{\mu^S + r} \left( \frac{Y_i(T)}{\mu^S + r} - NP \right)^2, \tag{3.1}
\end{array}

where \( \delta_1 = 1 \) and \( \delta_2 = 0 \). Eq. (3.1) are the HJB equations of the stochastic optimal control problems of the ELA and ELID pension plans, respectively.

Differentiating Eq. (3.1) with respect to \( \pi_i \) and \( p_i \) separately, and then letting these derivatives be zero, respectively, we get the following optimal feedback functions \( \pi_i(y; t) \) and \( p_i(y; t) \) for \( i = 1, 2 \).

\[
\begin{align*}
\pi_i(y; t) &= \frac{\partial \varphi_i}{\partial y}(r-c), \\
p_i(y; t) &= \frac{\partial \varphi_i}{\partial y}(r-c) + NP.
\end{align*}
\]

According to the boundary condition (3.2), we guess \( \varphi_i(\pi_i, p_i, y; t) \) has the following forms:

\[
\varphi_i(\pi_i, p_i, y; t) = e^{-(\mu^S + r)t} Q_i(t)(y^2 - 2S_i(t)y + R_i(t)), \tag{3.3}
\]

where \( Q_1(t) \), \( S_1(t) \) and \( R_1(t) \) are undetermined functions of \( t \) in the ELA pension plan, and \( Q_2(t) \), \( S_2(t) \) and \( R_2(t) \) are undetermined functions in the ELID pension plan. The boundary conditions for \( Q_i(t) \), \( S_i(t) \) and \( R_i(t) \), \( i = 1, 2 \), are

\[
\begin{align*}
Q_i(T) &= \frac{\mu^O + r)^2}, \\
S_i(T) &= \frac{N}{\mu^O + r}, \\
R_i(T) &= \frac{N}{(\mu^O + r)^2}.
\end{align*} \tag{3.4}
\]

So the optimal feedback functions can be rewritten as follows:

\[
\begin{align*}
\pi_i(y; t) &= \frac{(y-S_i(t)(r-c)}{y^2}, \\
p_i(y; t) &= Q_i(t)(y - S_i(t)) + NP. \tag{3.5}
\end{align*}
\]

Thus we only need to prove that (3.1) holds for all \( 0 < t \leq T \).

It is easy to see from (3.3) that

\[
\begin{align*}
\frac{\partial \varphi_i}{\partial y} &= 2e^{-(\mu^S + r)t} Q_i(t)(y - S_i(t)), \\
\frac{\partial^2 \varphi_i}{\partial y^2} &= 2e^{-(\mu^S + r)t} Q_i(t), \\
\frac{\partial \varphi_i}{\partial y} &= -re^{-(\mu^S + r)t} Q_i(t)(y^2 - 2S_i(t)y + R_i(t)) \\
&\quad - \mu^S e^{-(\mu^S + r)t} Q_i(t)(y^2 - 2S_i(t)y + R_i(t)) \\
&\quad + e^{-(\mu^S + r)t} Q_i(t)(y^2 - 2S_i(t)y + R_i(t)) \quad \tag{3.6}
\end{align*}
\]
Substituting (3.5) and (3.6) into (3.1), the associated HJB equations can be rewritten as follows:

\[
\begin{cases}
0 = rQ_i(t)[y^2 - 2S_i(t)y + R_i(t)] + \mu_S Q_i(t)[y^2 - 2S_i(t)y + R_i(t)] \\
- Q_i(t)[y^2 - 2S_i(t)y + R_i(t)] + Q_i(t)[2S_i(t)y - R_i(t)] \\
+ 2Q_i(t)[y - S_i(t)][(c - r)^2 - \delta_1(y - S_i(t))] - ry - \delta_1 \mu_S y \\
+ Q_i(t)(y - S_i(t)) + NP - Q_i(t)(c - r)^2(y - S_i(t))^2 \\
- [Q_i(t)(y - S_i(t))]^2 - (1 - \delta_1) \mu_S [\alpha (y - NP)^2 - \beta(y - NP)],
\end{cases}
\]  

(3.7)

for \( i = 1, 2 \).

In the stochastic optimal control problem of the ELA pension plan, \( i = 1 \) and \( \delta_1 = 1 \). Equating the coefficient of quadratic factor to zero in (3.7), we get the following Riccati ordinary differential equation:

\[
Q'_i + (2\mu^O - \mu^S + r - \frac{(c - r)^2}{\sigma^2})Q_i - Q'^2_i = 0. 
\]  

(3.8)

Let \( Q(t) \equiv u^{-1}(t) \). Then

\[
u'(t) - (2\mu^O - \mu^S + r - \frac{(c - r)^2}{\sigma^2})u + 1 = 0,  
\]  

(3.9)

with boundary condition

\[
u(T) = \frac{\mu^S + r}{(\mu^O + r)^2}.  
\]

The solution of (3.9) is

\[
u(t) = \left(\frac{\mu^S + r}{(\mu^O + r)^2} - \frac{1}{M}\right)e^{M(t-T)} + \frac{1}{M},  
\]

with

\[M = r + 2\mu^O - \mu^S - \frac{(c - r)^2}{\sigma^2}.\]

Thus, the solution of (3.8) is

\[
Q_i(t) = \frac{1}{\left(\frac{\mu^S + r}{(\mu^O + r)^2} - \frac{1}{M}\right)e^{M(t-T)} + \frac{1}{M}}. 
\]  

(3.10)

Equating the coefficient of linear factor to zero in (3.7), we have

\[
S'_i - (\mu^O + r)S_i + NP = 0,  
\]  

(3.11)

The solution of (3.11) is

\[
S_i(t) = \frac{NP}{\mu^O + r}.  
\]  

(3.12)

Let the constant item be zero in (3.7), we can get the ordinary differential equation that \( R_1(t) \) satisfies:

\[
R'_1 - (\mu^O + r - \frac{Q'_1}{Q_1})R_1 - \left(\frac{(c - r)^2}{\sigma^2} - S^2_1 + Q_1S^2_1 - 2NP \cdot S_1\right) = 0.  
\]  

(3.13)

As \( Q_1(t) \) are \( S_1(t) \) have been solved in (3.10) and (3.12), it is easy to solve the differential equation (3.13) with boundary conditions in (3.4). Since \( R_1(t) \) is not related to the optimal feedback functions of (3.5), we omit the calculations here.
Similarly, in the stochastic optimal control problem of the ELID pension plan, \( i = 2 \) and \( \delta_2 = 0 \). Equating the coefficient of quadratic factor to zero in (3.7), we get the following ordinary differential equation:

\[
Q_2' + (-\mu^S + r - \frac{(c-r)^2}{\sigma^2})Q_2 - Q_2^2 + \mu^S \cdot \alpha = 0.
\]  (3.14)

Making the following transformations:

\[
Q_2 = Q_2^2 - K \cdot Q_2 - \mu^S \cdot \alpha = (Q_2 - w_1)(Q_2 - w_2),
\]

where

\[
K = -\mu^S + r - \frac{(c-r)^2}{\sigma^2},
\]

and

\[
w_1 = \frac{K + \sqrt{K^2 + 4\mu^S \cdot \alpha}}{2}, w_2 = \frac{K - \sqrt{K^2 + 4\mu^S \cdot \alpha}}{2}.
\]

We get

\[
\frac{dQ_2}{(Q_2-w_1)(Q_2-w_2)} = dt.
\]  (3.15)

Integrating both sides of Eq.(3.15), we have

\[
\frac{1}{w_1-w_2} \int_t^T \left( \frac{1}{Q_2-w_1} - \frac{1}{Q_2-w_2} \right) dQ_2 = \int_t^T dt.
\]

The solution of (3.14) is then:

\[
Q_2(t) = \frac{w_2\left(\frac{\mu^S + r}{\mu^{S+r}} - w_1\right) - w_1\left(\frac{\mu^S + r}{\mu^{S+r}} - w_2\right)e^{(w_1-w_2)(T-t)}}{\left(\frac{\mu^S + r}{\mu^{S+r}} - w_1\right) - \left(\frac{\mu^S + r}{\mu^{S+r}} - w_2\right)e^{(w_1-w_2)(T-t)}}.
\]  (3.16)

Equating the coefficient of linear factor to zero in (3.7), we have

\[
S_2' - \left(\frac{r + \mu^S \cdot \alpha}{Q}\right)S_2 + NP + \frac{\mu^S (NP \cdot \alpha + \frac{\beta}{2})}{Q} = 0.
\]  (3.17)

The solution of the ordinary differential equation (3.17) is

\[
S_2(t) = \frac{1}{w_2\left(\frac{\mu^S + r}{\mu^{S+r}} - w_1\right) - w_1\left(\frac{\mu^S + r}{\mu^{S+r}} - w_2\right)e^{(w_1-w_2)(T-t)}} \cdot \left\{ \left(\frac{\mu^S + r}{\mu^{S+r}}\right)NP \cdot \frac{\mu^S}{\mu^{S+r}} \cdot e^{(r + \frac{\mu^S}{\mu^{S+r}})(T-t)} - \left[ NP \cdot w_1 + \mu^S (NP \cdot \alpha + \frac{\beta}{2})\right]\left(\frac{\mu^S + r}{\mu^{S+r}} - w_2\right) e^{(w_1-w_2)(T-t)} - 1 \right\}.
\]  (3.18)
Let the constant item be zero in (3.7), we can get ordinary differential equation that $R_2(t)$ satisfies:

$$R_2' - \left(\mu^S + r - \frac{Q_2^2}{Q_2}\right)R_2 - \left[\frac{(c-r)^2}{\sigma^2}S_2^2 + Q_2S_2^2\right]$$

$$-2NP \cdot S_2 - \frac{\mu^S \cdot \alpha \cdot NP^2}{Q_2} - e^t \cdot \beta \cdot NP \cdot S_2 = 0.$$  

(3.19)

Similarly, as $Q_2(t)$ and $S_2(t)$ have been solved in (3.16) and (3.18), it is easy to solve the differential equation (3.19) with boundary conditions in (3.4). However, since $R_2(t)$ is not related to the optimal feedback functions of (3.5), the expression of $R_2(t)$ is omitted.

Therefore, we get the closed-form optimal feedback functions $\pi_i^*(y; t)$ and $p_i^*(y; t), i = 1, 2$ by (3.5), (3.10) and (3.12), as well as (3.5), (3.16) and (3.18), respectively. It is easy to see that $\pi_i^*$ and $p_i^*$ are the functions of time $t$ and fund wealth $y$ at time $t$.

Now, we turn to establishing the optimal asset allocation policy $\pi_i^*(t)$ and the optimal benefit outgo policy $p_i^*(t)$. Let $\pi_i^*(\cdot, \cdot)$ and $p_i^*(\cdot, \cdot), i = 1, 2$ be defined by (3.5). Then the following two SDEs

$$\begin{cases}
dY_i(t) = ([\pi_i^*(Y_i(t), t)\left(\sigma Y\right)] + \delta_i \mu^O Y_i(t) - p_i^*(Y_i(t), t)\left(\sigma Y\right))dt + \pi_i^*(Y_i(t), t)\sigma Y_i(t)dB(t), \\
D_i(t) = (1 - \delta_i)Y_i(t), \\
Y_i(0) = y_0
\end{cases}$$

(3.20)

have unique solutions $Y_{1i}(t)$ and $Y_{2i}(t)$, respectively. We use the similar approach in Øksendal and Sulem [24] to prove the verification theorem on the stochastic optimal control problems (2.1)-(2.2) and (2.3)-(2.5). The functions $\pi_i^*(y; t)$ and $p_i^*(y; t)$ defined by (3.5) and the function $\varphi_i(\pi_i^*(y; t), p_i^*(y; t), y; t)$ defined by (3.3) are the optimal feedback functions and the value function, respectively. The unique solution $\{Y_{i}(t), t \geq 0\}$ of (3.20) is the optimal state process. The optimal asset allocation and benefit outgo policies $(\pi_i^*(t), p_i^*(t))$ are the compositions of the optimal feedback functions $(\pi_i^*(\cdot, \cdot), p_i^*(\cdot, \cdot))$ and the optimal state process $Y_{i}(t)$, i.e., $\pi_i^*(t) = \pi_i(Y_{i}(t), t), p_i^*(t) = p_i(Y_{i}(t), t)$. We also have $V_i(y, t) = \varphi_i(\pi_i^*(y; t), p_i^*(y; t), y; t) = J_i(\pi_i^*, p_i^*, y; t)$.

4. Analysis of the optimal control policies. In this section, we use MCM to study the impacts of heterogeneous personal health status and bequest motive on the optimal asset allocation and benefit outgo policies of the ELA and ELID pension plans. Furthermore, we establish the optimal old-care utility boundary between the ELA and ELID pension plans.

In the numerical analysis procedures, we use MCM to randomly generate 10000 paths of the fund wealth processes. At each step, the fund wealth is re-calculated by the $\pi_i^*$ and $p_i^*$ of the last step and a randomly generated Brownian motion, as in (3.20). Then, $\pi_i^*$ and $p_i^*$ of the step are re-calculated by the current fund wealth according to (3.5). The procedures are periodically repeated 10000 times and we use the average of the optimal control policies $\pi_i^*$ and $p_i^*$ to study the behaviors of the optimal control policies.
According to the empirical data in the capital market and the pension management, we make the following assumptions for the parameters. The age at the beginning of the distribution phase is $x_0 = 60$, i.e., the retirement age is 60. The risk-free interest rate is $r = 0.02$ and it is the yield of the one year U.S. bond. The expected return and the volatility of the risky asset are $c = 0.1$ and $\sigma = 0.3$, respectively. They are estimated by the investment return of the U.S. stock index. Furthermore, we make some assumptions on the individual member’s pension account. The initial fund wealth is $y_0 = 35$ (ten thousand dollars), and the pre-set target for the expected benefit outgo is $NP = 3$ (ten thousand dollars). In most countries, the fund wealth is not adequate to maintain the high standard old-care utilities. The above assumptions represent the scenario of inadequate fund wealth. $\mu^O = 0.05$ is the force of mortality of all the pension members and it is the statistical mortality results of the large numbers. $\mu^S = 0.075$ is the personal force of mortality of the member with heterogeneous health status. The increase of $\mu^S$ represents the worse health status of the member. As the increase of the force of mortality with respect to age is not considered in the model, the results may have some flaws. It is also the compromise between the closed-form solution and the coordination with the practice.

Furthermore, $T = 15$ is the compulsory conversion time. At that time, all the fund wealth is converted into annuities and fully invested in the risk-free asset. $\alpha = 0.005$ measures the importance of disutility provided by the last distribution (bequest) of the member without bequest motive. As the objective function is the minimization of the deviations, and the bequest of the member at younger age is several times of the pre-set target $NP$, $\alpha$ should be small to maintain the comparability. $\beta = 0.25$ measures the importance of disutility provided by the bequest of the member with bequest motive. As some members are at younger ages and have children to raise, they have bequest motives. The negative linear deviation in the disutility function represents the increase of the utility by the bequest received.

First, according to the above assumptions on the parameters, we simulate the optimal proportions allocated in the risky asset and optimal benefits outgo of the ELA and ELID pension plans. In Figure 1. and Figure 2., the optimal risky investment in the ELA plan is much lower than in the ELID plan, and the optimal benefit outgo in the ELA plan is much larger than in the ELID plan. In the
ELA pension plan, the survival member could receive the mortality credit and it increases the fund wealth. According to the counterintuitive effect between the risky investment and the fund wealth, as in Josa-Fombellida and Rincón-Zapatero[13], lower proportion is allocated in the risky asset in the ELA plan. Besides that, as bequest motive contributes part of the overall utility, the member in the ELID plan should increase the risky investment, and decrease the benefit outgo to enlarge the fund wealth, and increase the utility provided by the bequest. In Figure 1., as time passes by, the optimal proportions in risky asset reveal convergent effects. These are due to the following two reasons: the higher proportion of the risky investment increases the fund wealth, and the larger fund wealth results in the decrease in the risky investment due to the counterintuitive effects. As time approaches the compulsory conversion time, the utility provided by the bequest is less likely to be realized, the member in the ELID plan should reduce the risky investment in the latter time. In Figure 2., we find the optimal bequest process are almost at the same level over 15 years.

Second, we study the impacts of the heterogeneous personal health status $\mu^S$ on the optimal control policies. In Figure 3., the higher personal force of mortality increases the optimal proportion allocated in the risky asset in the ELA plan. The member with worse health status inclines to invest more in the risky asset to enlarge the fund wealth, and distribute adequate benefit outgo to decrease the disutility in the former time. As the survival probability of the member with worse health status is smaller at the higher age, the increase of the disutility due to the inadequate benefit outgo in the latter time has less impacts on the overall performance criterion. In Figure 4., for the members of normal health status such as those represented by $\mu^S = 0.025, 0.05, 0.075$, the overall investment trends in the risky asset are declining. As they are more confident to live a longer life, their investment strategy is more conservative and smaller proportion is allocated in the risky asset. When the health status is getting worse, the members would like to allocate larger proportion in the risky asset and expect a higher rate of return, as well as the larger bequest at the time of death. Similarly, they return to conservative strategies at later ages. This explains the trend of an increase first then followed by a decrease in the line for $\mu^S = 0.1$. 

Figure 3. The impacts of $\mu^S$ on optimal proportion in the risky asset $\pi^*_1$ in ELA plan.

Figure 4. The impacts of $\mu^S$ on optimal proportion in the risky asset $\pi^*_2$ in ELID plan.
In Figure 5 and Figure 6, the higher personal force of mortality increases the optimal benefit outgo in both of the ELA and ELID plans in the former time. As the member with worse health status has larger decline in survival probability, the disutility provided by the benefit outgo in the former time is more important in the performance criterion. The member with worse health status inclines to distribute more benefit outgo to decrease the disutility in the former time. As time passes by, the higher benefit outgo reduces the growth potential of the fund wealth in the latter time. The member with worse health status has to stand the lower annuities after the conversion, which are less important in the performance criterion. Besides that, in the ELID plan, the bequest motive decreases the optimal benefits outgo of the members under all health statuses compared with the counterparts in the ELA plan.

Third, we study the impacts of bequest motive $\beta$ on the optimal control policies. In the ELA pension plan, the pension member receives mortality credit and receives no bequest at the time of death, so the bequest motive has no impact on the utility, as well as the optimal control polices. In Figure 7 and Figure 8, in the ELID plan, the higher bequest motive increases the proportion allocated in the risky asset, and decreases the benefit outgo in the former time. In the former time, the optimal policy of the member with higher bequest motive is to enlarge the fund wealth, and the member could receive larger bequest at the time of death. So, the member with higher bequest motive increases the risky investment and decreases the benefit
outgo. The increase of the disutility by the inadequate former benefit outgo is compromised by the increase of the utility provided by the larger bequest. As time passes by, the fund wealth of the member with higher bequest motive increases more rapidly and it could afford the lower risky investment and the larger benefit outgo in the latter time. Besides that, the fund wealth will be converted into annuities and there is no bequest after the conversion. As time approaches the conversion time, the investment policy should be more conservative and the benefit outgo should be increased for the member with higher bequest motive to realize the utility of larger fund wealth.

Then, we study the impacts of health status $\mu^S$ and the bequest motive $\beta$ on the optimal objective functions in the ELA and ELID pension plans. Since we do not derive the explicit expressions of $R_1(t)$ and $R_2(t)$ in the objective functions, we use the following methods to calculate the optimal objective functions at time 0 of the two pension plans. Using MCM, we simulate 10000 paths of the optimal fund wealth processes and recalculate the optimal control policies at every time interval. We use the average of the simulated deviations to calculate the objective functions $V$ at time 0.

Since the objective function is to minimize the deviations, the smaller objective function represents lower disutility, i.e., higher utility. In Figure 9., the higher force of mortality increases the optimal utility as the deviations of the benefit outgo in the latter time are less important in the performance criterion and the member can increase the benefit outgo in the former time to decrease the disutility. Besides that, the utility is increased more rapidly in the ELID plan, as the utility increase provided by the larger bequest is realized at the younger age of the member with worse health status.

In Figure 10., the bequest motive has no impact on the optimal utility in the ELA plan. Furthermore, the higher bequest motive increases the utility enormously in the ELID plan. The higher bequest motive directly increases the the utility provided by bequest and it contributes a lot in the overall performance criterion.

In Figure 11., we study the optimal choice between the ELA and ELID pension plans of the members under heterogeneous personal health statuses and bequest motives. As the optimal choice between the ELA and ELID plans is made at the beginning of the distribution phase, we compare the value functions of the ELA plan in (2.2) and the ELID plan in (2.4) at time 0 to get the priority. As the explicit expressions of the value functions are not derived, we use MCM to calculate the
cumulative deviations as the value functions. The results show that, the ELA plan is the optimal choice of the member with better health status and lower bequest motive. In this circumstance, the utility increase provided by the bequest is less likely to be realized and the member prefers to receive the mortality credit during the survival period. On the contrary, the ELID plan is the optimal choice of the member with worse health status and higher bequest motive. The utility increase provided by the bequest is more likely to be realized. The utility increase provided by the bequest exceeds the disutility increase provided by absence of mortality credit, and inadequate benefit outgo.

Finally, we explore two kinds of risk measures to study the risk of inadequate benefit outgo in the ELA and ELID pension plans. As the parameters are estimated according to the empirical data, they reveal the inadequate fund wealth in the pension management practice. The benefits outgo are less than the expected target during the whole distribution period in all scenarios.

In this part, we use VaR (Value at Risk) and ES (Expected Shortfall) as two kinds of risk measures. VaR(95%) is the threshold which a specific outcome of the benefits outgo will not be lower than under the 95% confidence level. VaR(95%) is the lowest 5th percentile of the simulated benefits outgo. ES is the conditional expectation of shortfall below the target, i.e.,

\[
(\text{ES (Expected Shortfall)} = \left| \sum_{j=1}^{k} \frac{1}{k} (\text{benefit outgo}_j - \text{target}) \right|, \]

where \((\text{benefit outgo}_j - \text{target}) < 0, \text{ for } j = 1, ..., k.

(4.1)

As the benefits outgo are distributed yearly during the whole distribution period, every benefit outgo is regarded as one outcome of the statistical samples. For simplicity, we calculate the benefits outgo from time \(T = 0\) to \(T = 20\). 100000 paths of fund wealth are generated by the MCM, and the optimal benefits outgo are calculated yearly. Then, we have 200000 benefit outgo samples.

Some studies criticize the admissible domain of the paper, as the optimal control policies are limited within the actual admissible domain \(\pi^* \in [0, 1], p^* \in [0, Y(t)]\) in practice. In order to avoid the time inconsistent optimization problem, we enlarge the admissible domain to \(\pi^* \in (-\infty, +\infty), p^* \in (-\infty, +\infty)\). In the simulation, we implement the sub-optimal policies which are more practical. In the sub-optimal policies, if the optimal control policies exceed the actual admissible domain, they
are modified as the boundary values of the actual admissible domain in sub-optimal policies.

The following two tables exhibit the risk of inadequate benefit outgo by the optimal and sub-optimal policies of the members under heterogeneous personal health statuses and bequest motives.

**Table 1. VaR(95%) of the benefits outgo under the optimal and sub-optimal policies.**

| VaR(95%) | ELA | ELID | ELA | ELID | ELA | ELID |
|----------|-----|------|-----|------|-----|------|
| $\mu^* = 0.025$ | 2.2219 | 2.218 | 1.0864 | 0.3212 | 1.0604 | 0.331 |
| $\mu^* = 0.05$ | 2.3046 | 2.2859 | 1.0864 | 0.079 | 1.0604 | 0.0719 |
| $\mu^* = 0.075$ | 2.3086 | 2.2663 | 1.2149 | 0.0085 | 1.2401 | 0.0042 |
| $\mu^* = 0.01$ | 2.3233 | 2.2675 | 1.0077 | 0.0072 | 1.1672 | 0.0033 |

**Table 2. ES of the benefits outgo under the optimal and sub-optimal policies.**

| ES | ELA | ELID | ELA | ELID | ELA | ELID | ELA | ELID |
|----|-----|------|-----|------|-----|------|-----|------|
| $\mu^* = 0.025$ | 0.299 | 0.3024 | 0.734 | 0.8892 | 0.7278 | 0.888 |
| $\mu^* = 0.05$ | 0.2644 | 0.2704 | 0.635 | 0.8238 | 0.6183 | 0.82 |
| $\mu^* = 0.075$ | 0.2405 | 0.2515 | 0.6034 | 0.8094 | 0.5865 | 0.8062 |
| $\mu^* = 0.01$ | 0.2196 | 0.2328 | 0.6125 | 0.8071 | 0.5645 | 0.7859 |

Table 1. and Table 2. are the lowest 5th percentiles and the expected shortfalls of the benefits outgo in the ELA and ELID plans, respectively. In the ELA pension plan, the result is not affected by the bequest motive $\beta$. The member with worse health status inclines to invest more in the risky asset and the aggressive policy increases the expectation and the volatility of the fund wealth, and the benefit outgo.

As the increase of the left tail is compromised by the increase of the expectation, the VaR(95%) increases and the ES decreases as the the member’s health status becomes worse. In the ELID plan, the member with worse health status invests less in the risky asset in the former time and invests more in the latter time. The former conservative policy increases the fund wealth limitedly, and the latter aggressive policy increases the expectation and volatility of the benefit outgo. Furthermore, the impacts of the investment policy in the latter time on volatility of the benefit outgo are enormous. So, VaR(95%) increases and ES decreases as the bequest motive becomes higher.

Furthermore, the performance of the sub-optimal policies is slightly worse than the optimal policies in the ELA plan. Meanwhile, the result of VaR(95%) of the sub-optimal policies in the ELID plan is unacceptable low. When the pension member experiences bad investment performance, the fund wealth could be quite small. In this circumstance, the member should increase the proportion allocated in the risky asset to gamble for the larger fund wealth. In the sub-optimal policy, the proportion of the risky investment is prohibited from being larger than one, and this limits the growth potential of the fund wealth. The member should be cautious on this issue when implementing the sub-optimal policy.

5. **Conclusions.** In this paper, we study the optimal asset allocation and benefit outgo polices during the distribution phase of the ELA and ELID pension plans under heterogeneous personal health statuses and bequest motives of the pension
members. It is the first time to study pension management problem under integrated performance criterions synthesizing disutilities provided by both the benefit outgo and the bequest. The results show that worse health status and higher bequest motive result in the higher utility of the ELID pension plan, while the ELA pension plan is the optimal choice of the member with better health status and lower bequest motive. In the former time of the distribution phase, the worse health status increases the proportion allocated in the risky asset and the benefit outgo in both pension plans. The bequest motive has positive influence on the optimal proportion in the risky asset and negative influence on the optimal benefit outgo in the ELID pension plan. As time passes by, the above optimal control policies change the fund wealth, and the utilities provided by the bequest are less likely to be realized, the optimal policies reveal convergent and reverse effects in the latter time.

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