Communication Cost for Non-Markovianity of Tripartite Quantum States: A Resource Theoretic Approach

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Abstract—To quantify non-Markovianity of tripartite quantum states from an operational viewpoint, we introduce a class $\Omega^*$ of operations performed by three distant parties. A tripartite quantum state is a free state under $\Omega^*$ if and only if it is a quantum Markov chain. Among the sets of operations composed of local operations and classical and quantum communication, $\Omega^*$ is maximal in the sense that any operation that does not belong to $\Omega^*$ can generate non-Markovianity. We introduce a function of tripartite quantum states that we call the non-Markovianity of formation, and prove that it is a faithful measure of non-Markovianity, which is continuous and monotonically nonincreasing under a subclass $\Omega$ of $\Omega^*$. We consider a task in which the three parties generate a non-Markov state from scratch by operations in $\Omega$, assisted with quantum communication from the third party to the others, which does not belong to $\Omega$. We prove that the minimum cost of quantum communication required therein is asymptotically equal to the regularized non-Markovianity of formation. Based on this result, we provide a direct operational meaning to a measure of bipartite entanglement called the c-squashed entanglement.

Index Terms—Quantum Markov chains, Operational Resource Theory

I. INTRODUCTION

The conditional quantum mutual information (CQMI) is defined for a tripartite quantum state and quantifies the amount of correlation between two subsystems that exists when conditioned by the third one. CQMI has operational meanings in the context of quantum state redistribution [1], [2], conditional decoupling [3], [4] and recoverability [5]. States for which CQMI is zero are called quantum Markov chains [6], and the others are called non-Markov states. All Non-Markov states can be exploited as a resource for conditional quantum one-time pad [7], which provides another operational meaning to CQMI. Refs. [8], [9] showed that entanglement of a bipartite quantum state is equivalent to non-Markovianity of its tripartite extensions. However, operational understanding of non-Markovianity of quantum states is relatively limited [10]–[12], compared to those of quantum Markov chains (see e.g. [13]), entanglement [14], [15] and that of classical ones [16]–[23].

The concept of operational resource theory (ORT) has been applied to various notions in quantum information theory, such as coherence, asymmetry, athermality and non-Gaussianity (see [24] for a review). The approaches based on ORT do not only provide understanding of these notions from an operational viewpoint, but also lead to findings of tasks for which these properties can be exploited as resources. In every ORT, states of the system are classified into free states and resource states, and operations thereon are into free operations and the others. It is required that (i) any free state is generated from scratch by a free operation, and that (ii) any free operation keeps the set of free states invariant. The main goal of an ORT is to obtain conditions under which a resource state is convertible to another by free operations.

In this paper, we develop an approach in [25] to analyze non-Markovianity of tripartite quantum states from the viewpoint of ORT. We introduce a class of operations performed by three distant parties, say Alice, Bob and Eve, which we denote by $\Omega^*$. The class $\Omega^*$ consists of public communication among the parties, quantum communication from Alice and Bob to Eve, local operations by each of Alice and Bob, and local reversible operations by Eve. We prove that the set of quantum Markov chains and $\Omega^*$ satisfy the conditions for free states and free operations, namely, Conditions (i) and (ii) presented above. Among the sets of operations composed of local operations, classical communication and quantum communication by the three parties, $\Omega^*$ is maximal in the sense that any operation that does not belong to $\Omega^*$ can convert a quantum Markov chain to a non-Markov state. Thereby we provide a groundwork for an ORT of non-Markovianity.

For evaluating non-Markovianity of tripartite quantum states, we introduce a function that we call the non-Markovianity of formation (nMF). We prove that nMF is a faithful measure of non-Markovianity, which is asymptotically continuous and monotonically nonincreasing under $\Omega$, a subclass of $\Omega^*$ that was introduced in [25]. An operational meaning of nMF is investigated in terms of a task that we call non-Markovianity generation. The task is for the three parties to generate a non-Markov quantum state from scratch by operations in $\Omega$ and quantum communication from Eve to the others, which does not belong to $\Omega$. We consider an asymptotic limit of infinitely many copies and vanishingly small error. We analyze the non-Markovianity cost, namely, the minimum cost of quantum communication per copy required for non-Markovianity generation. We prove that the non-Markovianity cost is equal to the regularized nMF.

A measure of entanglement of a bipartite quantum state called the c-squashed entanglement [26]–[28] is obtained from nMF by taking the infimum over all tripartite extensions, anal-
Hilbert space is denoted by \( \mathcal{H} \), and its dimension is denoted by \( d_A \). A system composed of two subsystems \( A \) and \( B \) is denoted by \( AB \). When \( M \) and \( N \) are linear operators on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, we denote \( M \otimes N \) as \( M^A \otimes N^B \) for clarity. We abbreviate \( |\psi\rangle^A \otimes |\phi\rangle^B \) as \( |\psi\rangle A |\phi\rangle B \). The identity operator on a Hilbert space is denoted by \( \mathbb{I} \). We denote \( (M^A \otimes I^B)|\psi\rangle^{AB} \) as \( M^A |\psi\rangle^A \), and \( (M^A \otimes I^B)\rho^{AB} (M^A \otimes I^B)^\dagger \) as \( M^A \rho^{AB} M^A \). The identity operation on a system is denoted by \( \mathbb{I}_S \). We denote \( (\mathbb{I}_A \otimes \mathbb{I}_B)|\psi\rangle^{AB} \) as \( \mathbb{I}_A |\psi\rangle^A \), and \( (\mathbb{I}_A \otimes \mathbb{I}_B)\rho^{AB} (\mathbb{I}_A \otimes \mathbb{I}_B)^\dagger \) as \( \mathbb{I}_A \rho^{AB} \mathbb{I}_A \). The Shannon entropy of a probability distribution is defined as \( \sum_i p_i \log p_i \). We denote \( |\psi\rangle \langle \psi| \) as \( \rho \). The von Neumann entropy of a state \( \rho \) is interchangeably denoted by \( S(\rho) \) and \( S(A, \rho) \). log \( x \) represents the base 2 logarithm of \( x \). For the properties of quantum entropies and mutual informations, see e.g. [29].

### II. Operational Framework

In this section, after reviewing the concept of operational resource theory (ORT), we introduce a class \( \Omega^* \) of operations performed by three distant parties. We prove that the conditional quantum mutual information is monotonicity non-increasing under \( \Omega^* \), and that a tripartite quantum state is a quantum Markov chain if and only if it can be generated from scratch by an operation in \( \Omega^* \). We also prove that \( \Omega^* \) is maximal in the set of LOCC operations. That is, among the set of operations composed of local operations by each parties, and classical and quantum communications among the parties, any operation that does not belong to \( \Omega^* \) can generate and increase non-Markovianity. We introduce a subclass \( \Omega \) of \( \Omega^* \), that will be considered in the remaining sections.

#### A. General Concepts of Operational Resource Theory

We briefly review the concept of ORT (see e.g. [24] for the details). In an ORT, we consider a system equipped with a certain structure. For example, one may consider a quantum system composed of several subsystems, a quantum system with a fixed Hamiltonian, and one associated with a symmetry group. The minimal assumptions that any ORT must satisfy are as follows:

(a) All operations on the system are classified into free operations and non-free operations.

(b) All states of the system are classified into free states and resource states.

(c) The set of free states are closed under free operations.

(d) Any free state can be generated from scratch by free operations.

The main interest in an ORT is on conditions under which a resource state is convertible to another by a free operation.

Depending on which of (a) and (b) is determined prior to the other, there are mainly two approaches for constructing an ORT. The (a)-first approach is logically straightforward, because the classification in (b) is uniquely determined from (a) due to Assumptions (c) and (d). The (b)-first approach is heuristic in general, because the classification in (a) is not necessary unique for a given classification in (b). For example, in entanglement theory in which separable states are regarded as free states, we usually adopt the set of LOCC (local operations and classical communication) for the set of free operations. However, LOCC is not the only set of operations that satisfies Conditions (c) and (d). It is known that the class of operations called separable operations also satisfies the two conditions (see e.g. [15] and the references therein).

Our approach in this paper is the (b)-first one, because we aim at constructing an ORT in which quantum Markov chains are regarded as free states. However, for the logical clarity, we adopt the (a)-first approach in presenting the obtained results, in which case the free states are defined as the states that satisfy Condition (d).

#### B. Free Operations

Consider three distant parties, say Alice, Bob and Eve. We introduce classes of operations performed by the parties.
In order that the obtained ORT has operational significance, we only consider classes of \textit{LOCQC} operations, i.e., those composed of local operations by each parties, classical communication and quantum communication among the parties. The classes are as follows:

- \( L_A \): local operations by Alice
- \( L_B \): local operations by Bob
- \( R_E \): local reversible operations by Eve
- \( Q_{A \rightarrow E} \): quantum communication from Alice to Eve
- \( Q_{B \rightarrow E} \): quantum communication from Bob to Eve
- \( P_A \): broadcasting of classical messages by Alice
- \( P_B \): broadcasting of classical messages by Bob
- \( C_{A \rightarrow E} \): classical communication from Alice to Eve
- \( C_{E \rightarrow A} \): classical communication from Eve to Alice
- \( C_{B \rightarrow E} \): classical communication from Bob to Eve
- \( C_{E \rightarrow B} \): classical communication from Eve to Bob

Here, an operation \( V : E \rightarrow \hat{E} \) is said to be reversible if there exists an operation \( V^* : \hat{E} \rightarrow E \) such that \( V^* \circ V = \text{id}_E \). We require that Eve cannot refuse to receive classical messages sent by Alice in \( P_A \) and one by Bob in \( P_B \). This implies that secret communication channels between Alice and Bob are not available. We denote by \( \Omega^s \) the set of operations that can be represented as a composition of operations in the above classes (see Figure 1).

In the rest of this paper, we analyze convertibility of tripartite quantum states shared among Alice, Bob and Eve, under \( \Omega^s \) and a subclass thereof. The set of all tripartite quantum states is denoted by \( \mathcal{S}_{\text{all}} \).

### C. Monotonicity of The Conditional Quantum Mutual Information

For a tripartite quantum state \( \rho \) on system \( ABE \), the conditional quantum mutual information (CQMI) is defined by

\[
I(A : B|E)_\rho := S(AE)_\rho + S(BE)_\rho - S(ABE)_\rho - S(E)_\rho.
\]

Here, \( S \) is the von Neumann entropy of the reduced state of \( \rho \) on each subsystem, e.g.,

\[
S(AE)_\rho = S(\rho^{AE}) = -\text{Tr}[\rho^{AE} \log \rho^{AE}] .
\]

For simplicity, we denote \( I(A : B|E)_\rho \) by \( M_1(\rho) \). The strong subadditivity of the von Neumann entropy implies that CQMI is nonnegative [30]. The following lemma states that CQMI is monotonically nonincreasing under \( \Omega^s \).

\begin{lemma}
For any \( \rho \in \mathcal{S}_{\text{all}} \) and \( F \in \Omega^s \), it holds that \( M_1(\rho) \geq M_1(F(\rho)) \).
\end{lemma}

\textbf{Proof}: It suffices to prove that \( M_1 \) is monotonically nonincreasing under any class of operations that comprises \( \Omega^s \). Monotonicity of functions under \( C_{A \rightarrow E} \) immediately follows from one under \( Q_{A \rightarrow E} \), because communicating a classical message is equivalent to sending a quantum system, under the condition that the state is diagonal with respect to a given basis. Monotonicity under \( P_A \) follows from one under \( C_{A \rightarrow E} \) and \( C_{E \rightarrow B} \). This is because broadcasting of a classical message by Alice is equivalent to Alice’s communicating a classical message to Eve, followed by Eve’s transferring it to Bob while keeping the copy thereof. Monotonicity under \( L_A \), \( P_A \), \( C_{A \rightarrow E} \), \( C_{E \rightarrow A} \), and \( Q_{A \rightarrow E} \) are equivalent to those under \( L_B \), \( P_B \), \( C_{B \rightarrow E} \), \( C_{E \rightarrow B} \), and \( Q_{B \rightarrow E} \), respectively, due to symmetry of CQMI in \( A \) and \( B \). Therefore, we only need to prove that \( M_1 \) is monotonically nonincreasing under \( L_A \), \( R_E \), \( C_{E \rightarrow B} \) and \( Q_{A \rightarrow E} \).

1. \textbf{Monotonicity under} \( L_A \): Due to the data processing inequality for CQMI, for any \( \rho \in \mathcal{S}_{\text{all}} \) and \( F \in L_A \), it holds that \( I(A : B|E)_{\rho} \geq I(A : B|E)_{F(\rho)} \).

2. \textbf{Monotonicity under} \( R_E \): Since the quantum mutual information is invariant under local reversible operations, we have \( I(A : BE)_{\rho} = I(A : BE)_{V(\rho)} \) and \( I(A : E)_{\rho} = I(A : E)_{V(\rho)} \). Thus, by the chain rule of CQMI, we have \( I(A : BE)_{\rho} = I(A : BE)_{\rho} - I(A : E)_{\rho} = I(A : BE)_{V(\rho)} - I(A : E)_{V(\rho)} = I(A : B|E)_{V(\rho)} \).

3. \textbf{Monotonicity under} \( C_{E \rightarrow B} \): The states before and after classical communication from Eve to Bob is represented by density operators

\[
\rho_i = \sum_m r_m |m \rangle \langle m|^{ME} \otimes \rho_m^{ABE} \tag{2}
\]

and

\[
\rho_f = \sum_m r_m |m \rangle \langle m|^{MB} \otimes |m \rangle \langle m|^{ME} \otimes \rho_m^{ABE}, \tag{3}
\]

respectively. Here, \( \{r_m\}_m \) is a probability distribution, \( \{\rho_m\}_m \) is a set of quantum states on \( ABE \), \( \{|m\}_m \) is a set of orthonormal pure states, and \( M_B \) and \( M_E \) are “classical” systems in which the messages are stored. It follows that \( M_1(\rho_i) = I(A : B|EME)_{\rho_i} = \sum_m r_m I(A : B|E)_{\rho_m} = I(A : BM_B|ME)_{\rho_f} = M_1(\rho_f) \).

4. \textbf{Monotonicity under} \( Q_{A \rightarrow E} \): Let \( Q \) be a quantum system transmitted from Alice to Eve. By the chain rule of CQMI, we have \( I(QA : B|E)_{\rho} = I(Q : B|E)_{\rho} + I(A : B|EQ)_{\rho} \geq I(A : B|E)_{\rho} \).

\[\blacksquare\]

### D. Quantum Markov Chains are Free States

Tripartite quantum states for which CQMI is zero are called quantum Markov chains [6]. For a tripartite system composed of systems \( A, B \) and \( E \), the condition is represented as

\[
I(A : B|E)_\rho = 0. \tag{4}
\]

We denote the set of quantum Markov chains satisfying (4) by \( \mathcal{S}_{\text{Markov}} \). Ref. [6] proved that Equality (4) is equivalent to the condition that there exists a linear isometry \( \Gamma \) from \( E \) to \( E_0E_LE_R \) such that

\[
\Gamma \rho^{ABE} \Gamma^* = \sum_{j \in J} p_j |j \rangle \langle j|^{E_0} \otimes \sigma_j^{AE_L} \otimes \tau_j^{BE_R}. \tag{5}
\]
Here, $E_0$, $E_L$ and $E_R$ are finite-dimensional quantum systems, $\{p_j\}_{j \in J}$ is a probability distribution, $\{|j\rangle\}_{j \in J}$ is an orthonormal basis of $E_0$, and $\sigma_j$ and $\tau_j$ are quantum states on systems $AE_L$ and $BE_R$, respectively, for each $j$. The following proposition implies that a state is a free state under $\Omega^*$ if and only if it is a quantum Markov chain. Thus $\Omega^*$ provides a groundwork for an ORT of non-Markovianity.

**Proposition 2** A state $\sigma^{ABE}$ can be generated from scratch by an operation in $\Omega^*$ if and only if $\sigma \in \mathcal{S}_{\text{Markov}}$.

**Proof:** To prove the “if” part, consider the following procedure: 1. Alice generates a random variable $J$ which takes values in $\mathcal{J}$ according to a probability distribution $\{p_j\}_{j \in \mathcal{J}}$. 2. Alice broadcasts $\mathcal{J}$ values in $\Omega$ to Bob and Eve. 3. Eve records $J$ on her register $E_0$. 3. Alice locally prepares a state $\sigma^{AE_L}$ and sends $E_L$ to Eve. 4. Bob locally prepares a state $\sigma^{BE_R}$ and sends $E_R$ to Eve. 5. Eve applies $\Gamma^J$, and 6. Alice and Bob discards $J$. It is straightforward to verify that any state in the form of (5) can be generated by this protocol.

To prove the “only if” part, suppose that $\sigma \in \mathcal{S}_{\text{all}}$ is a free state. By definition, for a “dummy” state $\sigma_0 := |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \in \mathcal{S}_{\text{Markov}}$, there exists an operation $\mathcal{F} \in \Omega^*$ such that $\mathcal{F}(\sigma_0) = \sigma$. From the monotonicity of CQMI under $\Omega^*$, it follows that $0 = M_1(\sigma_0) \geq M_1(\sigma)$, which yields $M_1(\sigma) = 0$ and thus $\sigma \in \mathcal{S}_{\text{Markov}}$. □

**E. Maximality**

Among all classes of operations composed of LOCQc (local operations and classical and quantum communication), the following classes are not included in $\Omega^*$:

- $\mathbb{L}_{E}\setminus\mathbb{R}_E$: local irreversible operations by Eve
- $\mathcal{Q}_{E\rightarrow A}$: quantum communication from Eve to Alice
- $\mathcal{Q}_{E\rightarrow B}$: quantum communication from Eve to Bob
- $\mathcal{S}_{AB}$: secret communication between Alice and Bob
- $\mathcal{Q}_{AB}$: quantum communication between Alice and Bob

The following examples show that any operation that does not belong to $\Omega^*$ can generate non-Markov state from scratch. Thus $\Omega^*$ is the maximal set of free operations for an ORT of non-Markovianity.

- $\mathbb{L}_{E}\setminus\mathbb{R}_E$: Suppose that the three parties initially share a quantum state
  \[
  \rho := \frac{1}{2}(|000\rangle\langle 000|^{AEB} + |111\rangle\langle 111|^{AEB}) \in \mathcal{S}_{\text{Markov}}.
  \]
  By Eve randomly performing $I$ are $\sigma_x$ with probability 1/2 on her qubit, the above state is transformed to
  \[
  \rho' := \frac{1}{2}(|00\rangle\langle 00|^{AB} + |11\rangle\langle 11|^{AB}) \otimes \frac{1}{2}I_E,
  \]
  which is not a quantum Markov chain because $M_1(\rho') = 1$.
- $\mathcal{Q}_{E\rightarrow A}$: Suppose that the initial state is
  \[
  \rho := |0\rangle\langle 0|^{A'} \otimes |\Phi_2\rangle\langle \Phi_2|^{BE'} \otimes |0\rangle\langle 0|^{E} \in \mathcal{S}_{\text{Markov}},
  \]
  where $|\Phi_2\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ is a Bell state. Eve sends $E'$ to Alice, and then Alice discards $A'$. The obtained state is
  \[
  \rho' := |\Phi_2\rangle\langle \Phi_2|^{AB} \otimes |0\rangle\langle 0|^{E},
  \]
  which is not a quantum Markov chain because $M_1(\rho') = 2$.
- $\mathcal{Q}_{E\rightarrow B}$: By exchanging the roles of Alice and Bob in the above example, it can be shown that quantum communication from Eve to Bob can generate non-Markov state.
- $\mathcal{S}_{AB}$: Suppose that the three parties initially share a state $\rho := |000\rangle\langle 000|^{AEB} \in \mathcal{S}_{\text{Markov}}$. Alice flips a fair coin and sends the result through a secret communication channel to Bob. Depending on the result, Alice and Bob performs $I$ or $\sigma_x$ to their qubits, after which the state is
  \[
  \rho' := \frac{1}{2}(|000\rangle\langle 000|^{AB} + |11\rangle\langle 11|^{AB}) \otimes |0\rangle\langle 0|^{E}.
  \]
  This is not a quantum Markov chain because $M_1(\rho') = 1$.
- $\mathcal{Q}_{AB}$: It is straightforward to verify that, by quantum communication from Alice to Bob, the three parties can obtain a state of the form (7) from scratch.

**F. A Subclass of $\Omega^*$**

Let $\Omega$ be the class of operations composed of operations in all classes comprising $\Omega^*$ but $\mathbb{C}_{E\rightarrow A}$ and $\mathbb{C}_{E\rightarrow B}$. It immediately follows that $\Omega \subset \Omega^*$. In the rest of this paper, we consider convertibility of tripartite quantum states under operations in $\Omega$ assisted by non-free operations. Ideally, it would be desirable to adopt $\Omega^*$ as a class of free operations, instead of $\Omega$. However, at this point, we have not succeeded in proving monotonicity of a function (the non-Markovianity of formation) under $\Omega^*$, which plays a central role in analyzing non-Markovianity generation protocols (see Section III for the detail). Thus we adopt $\Omega$ for free operations in the this paper. It should be noted that a tripartite quantum state is a quantum Markov chain if and only if it is a free state under $\Omega$ (see the proof of Proposition 2 in Section II-D). The structure of an ORT under $\Omega$ have been analyzed in [25].

**III. THE NON-MARKOVIANITY OF FORMATION**

In this section, we introduce a function of tripartite quantum states that we call the non-Markovianity of formation (nMF). We prove its properties such as faithfulness, continuity and monotonicity under $\Omega$. An operational meaning of the (regularized) nMF will be provided in Section IV in the context of a non-Markovianity generation protocol.

**A. Definition**

Consider a tripartite quantum state $\rho$ on system $ABE$. Suppose that $\rho$ is decomposed in the form of $\rho = \sum_k p_k \rho_k$, where $\{p_k, \rho_k\}_{k}$ is an ensemble of quantum states on $ABE$. For each $k$, define

\[
\lambda(\rho_k) := \inf_{A'B'} \inf_{\tilde{\rho}_k} \{S(AA')\tilde{\rho}_k + S(BB')\tilde{\rho}_k - S(A'B')\tilde{\rho}_k - S(A'B)\tilde{\rho}_k\},
\]

where $A'$, $B'$, $A$, $B$, $\tilde{\rho}_k$, and $\rho_k$ are quantum states on $AB$ and $E$, respectively.
where the infimum is taken over all finite dimensional quantum systems \( A', B' \) and all quantum states \( \tilde{\rho}_k \) on \( AA'BB'E' \) such that \( \text{Tr}_{A'B'}[\tilde{\rho}_k] = \rho_k \). Using this function, we define \( \Lambda \) by

\[
\Lambda(\rho) := \inf_{k} \inf_{\{p_k, \rho_k\} \in \mathbb{K}} \left[ \sum_{k \in \mathbb{K}} p_k \lambda(\rho_k) \right],
\]

where the infimum is taken over all finite sets \( \mathbb{K} \) and ensembles \( \{p_k, \rho_k\} \in \mathbb{K} \) such that \( \rho = \sum_{k \in \mathbb{K}} p_k \rho_k \). The non-Markovianity of formation (nMF) is defined as

\[
M_F(\rho) := S(AB|E)_\rho + \Lambda(\rho),
\]

in which the first term is the conditional entropy defined by \( S(AB|E)_\rho := S(ABE)_\rho - S(E)_\rho \).

For later convenience, we present an alternative (but equivalent) expression for nMF. Let \( \{p_k, |\phi_k\rangle\} \) be an arbitrary ensemble of pure states on \( AA'BB'E'EE' \) such that

\[
\rho^{ABE} = \sum_k p_k \text{Tr}_{A'B'}[|\phi_k\rangle\langle\phi_k|].
\]

Consider a state of the form

\[
\varrho^{AA'BB'E'K} := \sum_k p_k |\phi_k\rangle\langle\phi_k|^{AA'BB'E'K} \otimes |k\rangle\langle k|^{K},
\]

where \( K \) is a quantum system with an associated orthonormal basis \( \{|k\rangle\}_k \). nMF is represented as

\[
M_F(\rho) = \inf_{A'B'E', K} \inf_{\{p_k, |\phi_k\rangle\} \in \mathbb{K}} \left[ I(AA' : BB'|K)_\varrho + I(AB : E'K|E)_\varrho \right],
\]

where the infimum is taken over all finite dimensional quantum systems \( A', B', E', \) finite sets \( \mathbb{K} \) and ensembles \( \{p_k, |\phi_k\rangle\} \in \mathbb{K} \) that satisfy (12). Due to the nonnegativity of the (conditional) quantum mutual information, it immediately follows that \( M_F \) is nonnegative. For clarity, we interchangeably denote nMF by \( M_F(\rho) \) and \( M_F(A : B|E)_\rho \). By definition, nMF is symmetric in \( A \) and \( B \), namely, it holds that

\[
M_F(A : B|E)_\rho = M_F(B : A|E)_\rho.
\]

**B. Properties**

The following lemma states that nMF is a measure of non-Markovianity of quantum states, which has similar properties as the conditional quantum mutual information. Proofs are provided in Appendix [A] and [B].

**Lemma 3** The non-Markovianity of formation satisfies the following properties:

(P1) **lower bound:** \( M_F(\rho) \geq M_I(\rho) \) for any \( \rho \in \mathcal{S}_{\text{all}} \).

(P2) **upper bound:** \( M_F(\rho) \leq 2 \min \{S(A|\rho), S(B|\rho)\} \) for any \( \rho \in \mathcal{S}_{\text{all}} \).

(P3) **faithfulness:** \( M_F(\rho) = 0 \) if and only if \( \rho \in \mathcal{S}_{\text{Markov}} \).

(P4) **pure states:** \( M_F(\psi) = I(A : B)_\psi \) for any pure state \( \psi \) on \( AB'E \).

(P5) **subadditivity:** \( M_F(\rho \otimes \sigma) \leq M_F(\rho) + M_F(\sigma) \) for any \( \rho, \sigma \in \mathcal{S}_{\text{all}} \).

(P6) **weak chain rule:** For any state on system \( ABCE \), it holds that

\[
M_F(AC : B|E)_\rho \geq M_F(A : B|EC)_\rho.
\]

(P7) **conditional convexity:** For any ensemble \( \{r_m, \rho_m\}_m \) of states on \( ABE \), and for the state

\[
\rho = \sum_m r_m \rho_m^{ABE} \otimes |m\rangle\langle m|^M, \langle m|m'\rangle = \delta_{m,m'},
\]

it holds that

\[
M_F(A : B|EM)_\rho \leq \sum_m r_m M_F(A : B|E)_{\rho_m}.
\]

(P8) **isometric invariance:** For any \( \rho \in \mathcal{S}_{\text{all}} \) and \( \forall V \in \mathbb{R}_E \), it holds that \( M_F(V(\rho)) = M_F(\rho) \).

(P9) **average monotonicity:** For any state \( \rho \in \mathcal{S}_{\text{all}} \) and any measurement on \( A \), it holds that

\[
M_F(A : B|E)_\rho \geq \sum_m \nu_m M_F(A : B|E)_{\rho_m},
\]

where \( \nu_m \) is the probability of obtaining the outcome \( m \), and \( \rho_m \) is the state after the measurement corresponding to the outcome \( m \).

(P10) **\( \Omega \)-monotonicity:** For any \( \rho \in \mathcal{S}_{\text{all}} \) and \( F \in \Omega \), it holds that \( M_F(\rho) \geq M_F(F(\rho)) \).

(P11) **asymptotic continuity:** For any \( \rho, \sigma \in \mathcal{S}_{\text{all}} \) satisfying \( \|\rho - \sigma\|_1 \leq \epsilon \), it holds that \( |M_F(\rho) - M_F(\sigma)| \leq \eta(\epsilon) \log d_A d_B \), where \( \eta(\epsilon) \) is a function that satisfies \( \lim_{\epsilon \to 0} \eta(\epsilon) = 0 \) and is independent of dimensions of the systems.

Due to the subadditivity (P5), the following limit exists:

\[
M_F^\infty(\rho) := \lim_{n \to \infty} \frac{1}{n} M_F(\rho^\otimes n).
\]

We refer to this function as the regularized non-Markovianity of formation. It should be noted that \( M_I \) also satisfies Properties (P1)-(P11), saturating inequalities in (P1), (P5) and (P7).

In Section [V] we will consider a task in which a tripartite quantum state is generated from scratch by operations in \( \Omega \) and quantum communication from Eve to Alice and Bob. We analyze the minimum cost of quantum communication required for the task. The amount of quantum communication from Eve to Alice and Bob is quantified simply by the number of qubits transmitted. That is, for \( G \in \mathcal{Q}_E := \mathcal{Q}^E_{\to A} \cup \mathcal{Q}^E_{\to B} \), we define

\[
\text{Qc}(G) := \log \dim Q_c.
\]

where \( Q \) is the quantum message transmitted in \( G \). The following lemma states that the increment of nMF under quantum communication from Eve to the others is bounded from above by the quantum communication cost. A proof will be given in Appendix [A]-L.

**Lemma 4** For any \( \rho \in \mathcal{S}_{\text{all}} \) and \( G \in \mathcal{Q}_E \), it holds that

\[
M_F(G(\rho)) - M_F(\rho) \leq 2Qc(\mathcal{G}).
\]

As discussed in Section [V], we have not succeeded in proving that \( M_F \) is monotonically nonincreasing under \( \Omega^* \).
The following lemma states that the $\Omega^*$-monotonicity of $M_F$ is equivalent to the property of “conditional linearity”.

**Lemma 5** The following conditions are equivalent:
1. $M_F$ is monotonically nonincreasing under $\Omega^*$.
2. $M_F$ is monotonically nonincreasing under $\mathbb{C}_{E\rightarrow B}$.
3. $M_F$ is invariant under $\mathbb{C}_{E\rightarrow B}$.
4. $M_F$ saturates Inequality [17].
5. For any state of the form (16), it holds that

$$\Lambda(\rho) = \sum_m r_m \Lambda(\rho_m).$$

\[ (21) \]

**IV. NON-MARKOVIANITY GENERATION**

In this section, we introduce a task in which a non-Markov state is generated from scratch by operations in $\Omega$ and quantum communication from Eve to Alice and Bob, which does not belong to $\Omega$. We refer to this task as non-Markovianity generation. We consider an asymptotic limit of infinitely many copies and vanishingly small error. The “non-Markovianity cost” of a tripartite quantum state is defined as the minimum cost of quantum communication per copy required for non-Markovianity generation. We prove that the non-Markovianity cost is equal to the regularized non-Markovianity of formation (nMF), by which we provide an operational meaning to nMF.

**A. Definitions and Results**

We first define convertibility of states under operations in $\Omega$. Due to the condition of reversibility of Eve’s operations, it is too restrictive to define convertibility of a state $\rho_1$ to $\rho_2$ by the existence of an operation in $\Omega$ that maps $\rho_1$ to $\rho_2$. Thus, we define the convertibility of states under $\Omega$ by taking the degree of freedom of reversible operations by Eve into account. A rigorous definition is as follows:

**Definition 6** A state $\rho_1$ is $\epsilon$-convertible to $\rho_2$ under $\Omega$ if there exist operations $\mathcal{F} \in \Omega$ and $\mathcal{V} \in \mathbb{R}_E$ such that

$$\|\mathcal{F}(\rho_1^{A E B}) - \mathcal{V}(\rho_2^{A E B})\|_1 \leq \epsilon,$$

where $\| \cdot \|_1$ is the trace norm defined by $\| A \|_1 = \text{Tr}|A|$ for an operator $A$.

Let $\Omega_Q$ be the set of operations that can be represented as compositions of operations in $\Omega$ and $Q_E$. We define convertibility of states under $\Omega_Q$ analogously to one under $\Omega$, by taking the quantum communication cost from Eve and the others into account. For an operation $\mathcal{G} \in \Omega_Q$, we quantify the total “downward” quantum communication cost by

$$Q_c(\mathcal{G}) := \inf_{\{\mathcal{F}_k\}_k, \{\mathcal{G}_k\}_k} \sum_k Q_c(\mathcal{G}_k).$$

Here, the infimum is taken over all operations $\{\mathcal{F}_k\}_k$ and $\{\mathcal{G}_k\}_k$ such that $\mathcal{F}_k \in \Omega$, $\mathcal{G}_k \in Q_E$ and

$$\mathcal{G} = \mathcal{G}_K \circ \mathcal{F}_K \circ \mathcal{G}_{K-1} \circ \mathcal{F}_{K-1} \circ \cdots \circ \mathcal{G}_1 \circ \mathcal{F}_1.$$  \[ (24) \]

The $\epsilon$-convertibility of states under $\Omega_Q$ is defined as follows:

**Definition 7** A state $\rho_1$ is $\epsilon$-convertible to $\rho_2$ under $\Omega_Q$ with the quantum communication cost $N$ if there exists operations $\mathcal{G} \in \Omega_Q$ and $\mathcal{V} \in \mathbb{R}_E$ such that

$$\|\mathcal{G}(\rho_1^{A E B}) - \mathcal{V}(\rho_2^{A E B})\|_1 \leq \epsilon$$

and

$$Q_c(\mathcal{G}) \leq N.$$ \[ (26) \]

A rigorous definition of the non-Markovianity cost is as follows:

**Definition 8** A rate $R$ is achievable in non-Markovianity generation of a state $\rho \in \mathcal{G}_{all}$ if, for any $\epsilon > 0$ and sufficiently large $n$, the dummy state $\sigma_0 := |0\rangle\langle 0|^A \otimes |0\rangle\langle 0|^B \otimes |0\rangle\langle 0|^E$ is $\epsilon$-convertible to $\rho^{A E B} \otimes n$ under $\Omega_Q$ with the quantum communication cost $nR$. The non-Markovianity cost of a state $\rho$, which we denote by $M_C(\rho)$, is defined as the infimum of rate $R$ that is achievable in non-Markovianity generation of $\rho$.

The following theorem states that the non-Markovianity cost of a state is equal to the regularized nMF. Proofs are provided in the following subsections.

**Theorem 9** For any state $\rho$ on $A E B$, it holds that

$$M_C(\rho) = \frac{1}{2} M_F^E(\rho).$$

\[ (27) \]

**B. Proof of the Converse Part**

The converse part of Theorem 9 is formulated as

$$M_C(\rho) \geq \frac{1}{2} M_F^E(\rho),$$

and is proved as follows. Suppose that a rate $R$ is achievable in non-Markovianity generation of $\rho$. By definition, for any $\epsilon$ and sufficiently large $n$, there exist operations $\mathcal{G} \in \Omega_Q$ and $\mathcal{V} \in \mathbb{R}_E$ such that

$$\|\mathcal{G}(\sigma_0) - \mathcal{V}(\rho^{A E B} \otimes n)\|_1 \leq \epsilon$$

and

$$Q_c(\mathcal{G}) \leq nR.$$ \[ (30) \]

Due to Properties (P8) and (P11) in Lemma 3, we have

$$M_F(\rho^{\otimes n}) = M_F(\mathcal{V}(\rho^{\otimes n})).$$

and

$$M_F(\mathcal{V}(\rho^{\otimes n})) \leq M_F(\mathcal{G}(\sigma_0)) + 5n\eta(\epsilon) \log d_A d_B,$$

respectively, where $\eta$ is a nonnegative function that satisfy $\lim_{x \downarrow 0} \eta(x) = 0$. In addition, Lemma 4 yields

$$M_F(\mathcal{G}(\sigma_0)) \leq 2Q_c(\mathcal{G}).$$ \[ (33) \]

Combining these all together, we arrive at

$$\frac{1}{n} M_F(\rho^{\otimes n}) \leq 2R + 5n\eta(\epsilon) \log d_A d_B.$$ \[ (34) \]

By taking the limit of $n \rightarrow \infty$ and $\epsilon \rightarrow 0$, we have $M_F^E(\rho) \leq 2R$. Since this relation holds for any $R > M_C(\rho)$, we obtain (28).
C. Proof of the Direct Part

We prove the direct part of Theorem 9 by constructing a non-Markovianity generation protocol. The protocol consists of two steps. In the first step, an ensemble of pure states is prepared by the quantum reverse Shannon protocol in the version of [31]. In the second step, unnecessary correlation in the state is destroyed by conditional decoupling [3]. The details of the protocol are presented as follows.

1) Quantum State Redistribution: We first review quantum state redistribution protocol [1], [2], from which direct part of the quantum reverse Shannon theorem and that of conditional decoupling are obtained by reduction. Quantum state redistribution is a task in which the S2 part of a four party pure state Ψ on system S1S2S3SR is transmitted from the sender to the receiver by sending a quantum message Q. Here, S R is the reference system that is inaccessible to the sender and the receiver, S1 is the side information at the sender and S3 is one at the receiver. They may exploit an entangled state ΦE1E3 shared in advance as a resource, where E1 and E3 are quantum registers at the sender and the receiver, respectively. At the end of the protocol, some part of the entanglement resource may be retrieved in the form of Φ′E′1E′3. We are particularly interested in the limit of many copies and vanishingly small error, and in the minimum cost of quantum communication required for the task.

For the block length n, the encoding operation and the decoding operation are represented by quantum operations \(E_n : S_1^n S_2^n E_1 \rightarrow S_1^n E_1^n Q\) and \(D_n : S_1^n E_1^n Q \rightarrow S_1^n S_2^n S_3^n E_3^n\), respectively. The quantum communication cost is quantified by \(\log \dim Q\). It was proved in [1], [2] that the minimum cost of quantum communication is given by the conditional quantum mutual information of the state Ψ (see also [32]).

\[\left\| D_n \circ E_n((\Psi^{S_1 S_2 S_3 S_R})^{\otimes n} \otimes \Phi E_1 E_3) - (\Psi^{S_1 S_2 S_3 S_R})^{\otimes n} \otimes \Phi' E_1 E_3'\right\|_1 \leq \epsilon\] (35)

and

\[\log \dim Q \leq nR.\] (36)

In addition, we may assume that the encoding operation \(E_n\) is a linear isometry.

The direct part of the quantum reverse Shannon theorem in the version of [31] is obtained as a corollary of Lemma 10 by assuming that \(S_3\) is a trivial (one-dimensional) system, i.e., that the receiver has no side information. By taking the partial trace over \(S_2^n S_3^n E_3^n\) in (35), we also have

\[\left\| \text{Tr}_Q \circ E_n((\Psi^{S_1 S_2 S_R})^{\otimes n} \otimes \pi E_1) - (\Psi^{S_1 S_R})^{\otimes n} \otimes \pi' E_1'\right\|_1 \leq \epsilon,\]

which is the direct part of conditional decoupling [3].

2) Direct Part of Theorem 9 for Pure States: Suppose that \(\rho\) is a pure state on ABE. Due to (P4) in Lemma 5, nMF is equal to the quantum mutual information. I.e., for \(\rho = |\psi\rangle|\psi\rangle^{ABE}\), we have

\[M_{F^n}(\psi) = M_F(\psi) = I(A : B|\psi).\] (38)

We construct a protocol based on the quantum reverse Shannon theorem. That is, we apply Lemma 10 under the following correspondence:

\[S_1 \rightarrow E, S_2 \rightarrow A, S_R \rightarrow B, |\Psi\rangle \rightarrow |\psi\rangle.\] (39)

We assume that \(S_3\) is a trivial system. From Lemma 5, it follows that for any \(\epsilon > 0, R > \frac{1}{2} I(A : B|E)_\psi = \frac{1}{2} I(A : B)\), and sufficiently large n, there exist an encoding linear isometry \(V_n : A^n E^n F_E \rightarrow E^n F_E^n Q\), a decoding operation \(D_n : F_A Q \rightarrow A^n F'_A\) and entangled states \(\Phi F_E F_A\) and \(\Phi' F_E' F'A'\), such that

\[\left\| D_n \circ V_n((\psi^{ABE})^{\otimes n} \otimes \Phi F_E F_A) - (\psi^{ABE})^{\otimes n} \otimes \Phi' F_E' F'_A\right\|_1 \leq \epsilon\] (40)

and that

\[\log \dim Q \leq nR.\] (41)

Consider the following protocol \(G_1\) that is an element of \(\Omega_Q\):

1) Alice locally prepares \(\Phi F_E F_A\) and sends the \(F_E\) part to Eve.
2) Bob locally prepares \((|\psi\rangle^{ABE})^{\otimes n}\) and sends the \(A^n E^n\) part to Eve.
3) Eve performs a reversible operation \(V_n\).
4) Eve sends the quantum system \(Q\) to Alice.
5) Alice performs an operation \(D_n\) and discards \(F'_A\).

It is straightforward to verify (40) that the state obtained by this protocol is equal to \(|\psi^{ABE})^{\otimes n} \otimes \pi' F_E' F_A'\) up to error \(\epsilon\). The quantum communication cost of the above operation is no greater than \(nR\). Hence a rate \(R\) is achievable if \(R > \frac{1}{2} I(A : B)\), which implies \(MC(\rho) < \frac{1}{2} I(A : B)\).

3) Direct Part of Theorem 9 for Mixed States: Let \(|\phi_k\rangle, 1 \leq k \leq K\) be an ensemble of quantum states on ABE satisfying \(\rho = \sum_k p_k |\phi_k\rangle\langle\phi_k|\) and let \(|\xi_k\rangle\) be a pure state on AA’BB’EE’ such that Tr\(_{A’B’E’E}|\langle\phi_k|\xi_k\rangle|\rho_k|\). For each k, in addition, let \(K\) be a quantum system with dimension \(|K|\), and define a state

\[\rho := \sum_{k \in K} p_k \text{Tr}_{A’B’}|\langle\phi_k|\xi_k\rangle|\rho_k|\otimes |k|\langle k|\] (42)

Fix arbitrary \(\epsilon, \delta > 0\) and choose sufficiently large \(n\). We introduce the following notations:

\[k^n := (k_1, \cdots, k_n),\]

\[p_{k^n} := p_{k_1} \cdots p_{k_n},\]

\[|\phi_{k^n}\rangle := |\phi_{k_1}\rangle \cdots |\phi_{k_n}\rangle.\] (43)

We denote by \(T_{n, \delta}\) be the set of sequences \(k^n\) that are \(\delta\)-strongly typical with respect to \(\{p_k\}_k\) (see e.g. [32]). I.e., \(T_{n, \delta}\) is the set of all sequences \(k^n\) such that

\[\sum_{k \in K} |p_k - f_k(k^n)| \leq \delta\]
for all \( k \in \mathbb{K} \), where \( f_k(k^n) \) is the empirical distribution defined by

\[
f_k(k^n) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[i|1 \leq i \leq n, k_i = k].
\] (44)

Consider the following protocol \( G_2 \) that is an element of \( \Omega_Q \):

1) Alice randomly chooses \( k^n \) according to a probability distribution \( \{p_k\}_{k^n} \), and broadcasts it to Bob and Eve.
2) If \( k^n \notin T_n,\delta \), Alice, Bob and Eve prepare the dummy state \( \sigma_0 \in \mathcal{S}_{\text{Markov}} \). If \( k^n \in T_n,\delta \), they prepare a state \( |\phi_k^n\rangle \), which is equal to \( |\phi_k^n\rangle \) up to error \( \epsilon \), by the protocol \( G_3 \) presented above.
3) Alice and Bob discard \( A^n \) and \( B^n \), respectively, and erase \( k^n \).

This protocol generates a state

\[
|\hat{\rho}_{n,\delta}\rangle = \sum_{k^n \in T_n,\delta} p_{k^n} |\phi_k^n\rangle \otimes |k^n\rangle_{\hat{R}} + p_0 \sigma_0
\] (45)

where

\[
p_0 := 1 - \sum_{k^n \in T_n,\delta} p_{k^n} \leq \epsilon, \quad \left\| |\phi_k^n\rangle_{ABE} - |\phi_k^n\rangle_{ABE}\right\|_1 \leq \epsilon,
\]

and consequently,

\[
\left\| |\hat{\rho}_{n,\delta}\rangle - |\hat{\rho}_{\infty}\rangle\right\|_1 \leq 2\epsilon.
\] (46)

The quantum communication cost of this protocol is equal to one in Step 2. Hence, due to the result in Sec. IV-C2 we may compose \( G_2 \) so that

\[
\text{Qc}(G_2) = \frac{1}{n} \left( \sum_k p_k I(AA' : BB')_{\phi_k} + \delta \right).
\] (47)

The remaining task is to decouple \( E'K \) from the other parts. We adopt the conditional decoupling protocol to address this task. That is, we apply Inequality (37) by the correspondence:

\[
S_1 \rightarrow E, \quad S_2 \rightarrow E'K, \quad R \rightarrow AB, \quad \Psi \rightarrow \hat{\rho}.
\] (48)

It follows that, for \( R = \frac{1}{2}(I(AB : E'K|E)_{\hat{\rho}} + \delta) \), there exist a quantum system \( Q \) and a linear isometry \( V_n : E^n E'^n K^n F_E \rightarrow E^n F_E Q \) such that

\[
\left\| \text{Tr}_Q V_n \left( (|\hat{\rho}_{ABE}E'K\rangle \otimes |\pi_{F_E}\rangle) - (|\hat{\rho}_{ABE}\rangle \otimes |\pi_{F_E}\rangle) \right) \right\|_1 \leq \epsilon
\] (49)

and that log \text{dim} \( Q \leq nR \). Consider the following protocol \( G_3 \) that is an element of \( \Omega_Q \):

1) Eve locally prepares \( \pi_{F_E} \), performs \( V_n \) and sends quantum system \( Q \) to Alice.
2) Alice discards \( Q \).

The quantum communication cost of \( G_3 \) is bounded as

\[
\text{Qc}(G_3) \leq nR = \frac{1}{2} n(I(AB : E'K|E)_{\hat{\rho}} + \delta).
\] (50)

The total error of the protocol is calculated from (46) and (49) as

\[
\left\| G_3(\hat{\rho}_{n,\delta}) - \rho_{\infty}^{\otimes n} \otimes \pi \right\|_1
\leq \left\| G_3(\hat{\rho}_{n,\delta}) - G_3(\rho_{\infty}^{\otimes n}) \right\|_1 + \left\| G_3(\rho_{\infty}^{\otimes n}) - \rho_{\infty}^{\otimes n} \otimes \pi \right\|_1
\leq \left\| \hat{\rho}_{n,\delta} - \rho_{\infty}^{\otimes n} \right\|_1 + \left\| G_3(\rho_{\infty}^{\otimes n}) - \rho_{\infty}^{\otimes n} \otimes \pi \right\|_1
\leq 3\epsilon.
\] (51)

From (47) and (50), the total cost of quantum communication is given by

\[
\frac{1}{n} \left( \text{Qc}(G_2) + \text{Qc}(G_3) \right)
= \frac{1}{2} \left( \sum_k p_k I(AA' : BB')_{\phi_k} + I(AB : E'K|E)_{\hat{\rho}} \right) + \delta.
\]

By taking the infimum over all ensembles \( \{p_k, |\phi_k\rangle\} \) and all purifications \( |\phi_k\rangle \), the R.H.S. of the above equality is equal to \( 1/2 M_F(\rho) \). Since \( \delta > 0 \) can be arbitrarily small, it follows that a rate \( R \) is achievable if \( R > 1/2 M_F(\rho) \). Hence we obtain \( M_C(\rho) \leq 1/2 M_F(\rho) \). By applying this result to the state \( \rho^{\otimes n} \), and by taking the limit of \( m \rightarrow \infty \), we complete the proof of Theorem 9. \( \square \)

V. THE C-SQUASHED ENTANGLEMENT

In this section, we apply the result on non-Markovianity generation (Theorem 9 in Section IV-A) to analyze a measure of bipartite entanglement called the c-squashed entanglement (c-SE). c-SE is defined for a state \( \omega \in \mathcal{S}(HA) \) by

\[
E_{\text{sq},c}(\omega^{AB}) := \inf_{\{p_k, \sigma_k\}_{k \in \mathbb{K}}} \sum_{k \in \mathbb{K}} p_k I(A : B)_{\sigma_k},
\]

where the infimum is taken over all finite sets \( \mathbb{K} \) and ensembles \( \{p_k, \sigma_k\}_{k \in \mathbb{K}} \) of states on \( AB \) such that

\[
\sum_{k \in \mathbb{K}} p_k \sigma_k^{AB} = \omega^{AB}.
\] (52)

c-SE is different from entanglement of formation \( [34] \), in that \( \sigma_k \) are not necessarily pure states. The regularized c-squashed entanglement is defined by

\[
E_{\text{sq},c}^\infty(\omega) := \lim_{n \rightarrow \infty} \frac{1}{n} E_{\text{sq},c}(\omega^{\otimes n}).
\] (53)

c-SE satisfies monotonicity under LOCC, convexity, subadditivity, asymptotic continuity and normalization for pure states \([28]\). However, a direct operational meaning of c-SE is yet unknown. In the following, we prove that c-SE has a clear operational meaning in terms of a task that we call assisted entanglement dilation.

A. An Alternative Expression

We prove that c-SE is represented in terms of the non-Markovianity of formation by “squashing” it, in the same way as the squashed entanglement \([8]\) is defined from the conditional quantum mutual information.

**Lemma 11** For any bipartite quantum state \( \omega^{AB} \), it holds that

\[
E_{\text{sq},c}(\omega^{AB}) = M_{sq}(\omega^{AB}) := \inf_{\rho} M_F(\rho^{ABE}),
\]
where the infimum is taken over all quantum states $\rho^{ABE}$ such that

$$\text{Tr}_E[\rho^{ABE}] = \omega^{AB}.$$  

(54)

**Proof:** First, we prove that $E_{sq,c}(\omega^{AB}) \leq M_{sq}(\omega^{AB})$. Consider an arbitrary state $\rho^{ABE}$ satisfying (54), and let $\{\phi_k \rangle \in \mathbb{K}$ be an ensemble of pure states on $AA'B'B'EE'$ that satisfies $\rho^{ABE} = \sum_k p_k \text{Tr}_{A'B'E'}[|\phi_k\rangle\langle\phi_k|]$. Defining a state $\rho^{ABK} := \text{Tr}_{A'B'E'E'}[\rho^{ABE}] = \sum_k p_k \sigma_k^{AB} \otimes |k\rangle\langle k|$, we have

$$\rho^{ABK} := \text{Tr}_{A'B'E'E'}[\rho^{ABE}] = \sum_k p_k \sigma_k^{AB} \otimes |k\rangle\langle k|.$$  

(55)

It is straightforward to verify that $\rho'$ is an extension of $\omega^{AB}$. i.e., $\text{Tr}_K[\rho^{ABK}] = \omega^{AB}$. Due to the nonnegativity and monotonicity of the conditional quantum mutual information under partial trace, we have $I(AA' : BB'|E) + I(AB : E'E|K) = I(A : B|K)$, so $I(AA' : BB'|E) + I(AB : E'E|K) \geq I(A : B|K)$, where $I(AA' : BB'|E) := I(AA' : BB'|E)$. Taking the infimum over all finite dimensional quantum systems $A', B', E'$, finite sets $\mathbb{K}$ and all ensembles $\{\phi_k \rangle \in \mathbb{K}$, we obtain $M_{sq}(\omega^{AB}) \geq E_{sq,c}(\omega^{AB})$.

Next, we prove that $E_{sq,c}(\omega^{AB}) \geq M_{sq}(\omega^{AB})$. Let $\{\phi_k \rangle \in \mathbb{K}$ be an arbitrary ensemble of states on $AB$ that satisfies $\sum_k p_k \sigma_k^{AB} = \omega^{AB}$. Let $|\tilde{\phi}_k\rangle_{AB}$ be an arbitrary purification of $\sigma_k^{AB}$ for each $k$, and define $|\tilde{\phi}_k\rangle_{ABE} := |\tilde{\phi}_k\rangle_{ABE}|k\rangle$. Then, $\{k\}$ is an orthonormal set and $E = EE'$. Assuming that $A', B'$ and $E'$ are trivial (one-dimensional) systems, the state corresponding to (13) is given by

$$\rho^{ABK} := \sum_k p_k |\tilde{\phi}_k\rangle_{ABE} \otimes |k\rangle\langle k| = \Phi^{ABK}.$$  

(56)

Noting that $I(AB : K|E) = I(AB : K|EE') = 0$, we obtain $\sum_k p_k I(A : B|\sigma_k) = I(A : B|\rho) = I(AA' : BB'|K) + I(AB : E'E|K) \geq M_F(\rho)$. By taking the infimum over all ensembles $\{p_k, \sigma_k\}$, we obtain $E_{sq,c}(\omega^{AB}) \geq M_{sq}(\omega^{AB})$ and completes the proof.

**B. Assisted Entanglement Dilution**

Consider a task in which Alice, Bob and Eve collaborate in order that Alice and Bob share copies of the state $\omega^{AB}$. They are allowed to perform arbitrary LOCC protocols. In addition, they can exploit bipartite quantum states $\varphi$ and $\zeta$, which are initially shared between Alice and Eve, and between Bob and Eve, respectively. We refer to this task as **assisted entanglement dilution**. Classical communication by Alice and Bob to the other parties is regarded as free, but one from Eve to Alice and Bob is not. Our interest is on the minimum cost of the latter for achieving the task. We consider a scenario of infinitely many copies and vanishingly small error. A rigorous definition is as follows.

**Definition 12** Let $\omega^{AB}$ be a bipartite quantum states on system $AB$. Let Alice, Bob and Eve have quantum registers $\bar{A}_0, \bar{B}_0$ and $\{|E_A, E_B\}$, respectively, and let $T$ be a quantum communication from $A_0 \otimes B_0 \otimes E_A E_B$ to $A \otimes B$. $T$ is called an $(n, \epsilon)$-protocol for generating $\omega^{AB}$, if $T_n$ is an LOCC and there exist quantum states $\xi_n^{A_0 E_A}$ and $\eta_n^{B_0 E_B}$ such that

$$\|T_n(\xi_n^{A_0 E_A} \otimes \eta_n^{B_0 E_B}) - (\omega^{AB})^n\|_1 \leq \epsilon.$$  

(57)

Let $\Gamma(T_n)$ be the number of times of classical communication from Eve to Alice and Bob in a protocol $T_n$. We denote by $C^{E \rightarrow A}(\gamma(T_n))$ the bit length of the classical messages transmitted in the $\gamma$-th communication from Eve to Alice, and by $C_{E \rightarrow B}(T_n)$ that from Eve to Bob, for $\gamma = 1, \cdots, \Gamma(T_n)$. The “downward” classical communication cost of $T_n$ is defined as the sum of numbers of classical bits transmitted from Eve to Alice and Bob in $T_n$. That is, we define

$$C^↓(T_n) := \sum_{\gamma=1}^{\Gamma(T_n)} C_{E \rightarrow A}(T_n) + C_{E \rightarrow B}(T_n).$$

As mentioned above, our interest is on the minimum cost of downward classical communication per copy for accomplishing this task, in an asymptotic limit of $n \to \infty$ and $\epsilon \to 0$.

**Definition 13** A rate $R$ is said to be achievable in assisted entanglement dilution of $\omega^{AB}$ if there exists a sequence of $(n, \epsilon_n)$-protocols $T_n$ for generating $\omega^{AB}$ ($n = 1, 2, \cdots$), with the downward classical communication cost $C^↓(T_n) \leq nR$ for each $n$, such that $\lim_{n \to \infty} \epsilon_n = 0$. The downward classical communication cost of a state $\omega^{AB}$, which we denote by $C^↓(\omega^{AB})$, is defined as the infimum of rate $R$ that is achievable in assisted entanglement dilution of $\omega^{AB}$.

The following theorem states that the downward classical communication cost of a bipartite quantum state is equal to the regularized c-SE. Proofs are provided in the following subsections.

**Theorem 14** For any bipartite quantum state $\omega^{AB}$, it holds that

$$C^↓(\omega^{AB}) = E^∞_{sq,c}(\omega^{AB}).$$  

(58)

**C. Proof of The Converse Part**

The converse part of Theorem 14 is represented as

$$C^↓(\omega^{AB}) \geq E^∞_{sq,c}(\omega^{AB}).$$  

(59)

We prove this by using the converse part of non-Markovianity generation (Theorem 9). To this end, we introduce the following lemma, which states that any protocol for assisted entanglement dilution can be converted to a non-Markovianity...
generation protocol, and that the downward classical communication cost of the former is equal to (half of) the quantum communication cost in the latter. A proof of this lemma will be given in Appendix \[C\].

Lemma 15 For any \(n, l \in \mathbb{N}\) and any dilution protocol \(T_n: A_0 \otimes B_0 \otimes E_A E_B \rightarrow A^n \otimes B^n\), there exist a quantum system \(E_{nl}\) and an operation \(T_{nl}: A_0 \otimes B_0 \otimes E_A E_B \rightarrow A_{nl} \otimes B_{nl} \otimes E_{nl}\), such that \(T_{nl} \in \Omega_Q\) and

\[
\begin{align*}
\mathcal{T}_{nl} & = Tr_{E_{nl}} \circ T_{nl}, \\
Qc(\mathcal{T}_{nl}) & \leq \left( \frac{l}{2} + 1 \right) C^4(\mathcal{T}_n).
\end{align*}
\]

Proof of Inequality (59): By definition, for any \(R > C^4(\omega^{AB})\), \(\epsilon > 0\), \(n, l \in \mathbb{N}\) and sufficiently large \(n\), there exists an \((n, \epsilon/l)\)-protocol \(T_n\) with the downward communication cost \(nR\) for assisted entanglement dilution of \(\omega^{AB}\). That is, for certain states \(\varphi_{nl}^{A_0 E_A}\) and \(\zeta_{nl}^{B_0 E_B}\), it holds that

\[
\| T_n(\varphi_{nl}^{A_0 E_A} \otimes \zeta_{nl}^{B_0 E_B}) - (\omega^{AB})^\otimes n \|_1 \leq \epsilon/l.
\] (62)

Due to Lemma 15 there exists an operation \(T_{nl} \in \Omega_Q\) from \(A_0 \otimes B_0 \otimes E_A E_B\) to \(A_{nl} \otimes B_{nl} \otimes E_{nl}\) that satisfies Condition (61). Consider a state

\[
\omega(\mathcal{T}_{nl})^{A_{nl} B_{nl} E_{nl}} := T_{nl}(\varphi_{nl}^{A_0 E_A} \otimes \zeta_{nl}^{B_0 E_B} \otimes I).
\] (63)

From Lemma 4 and (61), we have

\[
M_F(\omega(\mathcal{T}_{nl})) \leq 2Qc(\mathcal{T}_{nl}) \leq (l + 2) C^4(\mathcal{T}_n).
\] (64)

Lemma 11 yields

\[
M_F(\omega(\mathcal{T}_{nl})) \geq M_{sq}(\omega(\mathcal{T}_{nl})^{A_{nl} B_{nl}}) = E_{sq,c}(\omega(\mathcal{T}_{nl})^{A_{nl} B_{nl}}).
\] (65)

Due to (60) and (63), we have

\[
\omega(\mathcal{T}_{nl})^{A_{nl} B_{nl}} = (T_n(\varphi_{nl}^{A_0 E_A} \otimes \zeta_{nl}^{B_0 E_B} \otimes I)) \otimes I.
\] (66)

Combining this with (62), it follows that

\[
\| \omega(\mathcal{T}_{nl})^{A_{nl} B_{nl}} - (\omega^{AB})^\otimes n \|_1 \leq \epsilon.
\] (67)

Thus, due to the asymptotic continuity of \(E_{sq,c}\), we obtain

\[
| E_{sq,c}(\omega(\mathcal{T}_{nl})^{A_{nl} B_{nl}}) - E_{sq,c}(\omega^{AB}^\otimes n) | \leq n \eta(\epsilon) \log d_A d_B.
\] (68)

From (64), (65) and (68), we arrive at

\[
\frac{1}{n} E_{sq,c}(\omega^{AB}^\otimes n) \leq \left( 1 + \frac{2}{l} \right) \frac{C^4(\mathcal{T}_n)}{n} + \eta(\epsilon) \log d_A d_B.
\]

By taking the limit of \(n, l \rightarrow \infty\) and \(\epsilon \rightarrow 0\), we complete the proof of Ineq. (59).

D. Proof of The Direct Part

The direct part of Theorem \[14\] is represented by

\[
C^4(\omega^{AB}) \leq E_{sq,c}(\omega^{AB}),
\] (69)

and is proved as follows. Let \(\{p_k, \sigma_k\}\) be an arbitrary ensemble of states on \(AB\) that satisfies \(\{53\}\), and \(|\phi_k\rangle_{ABE}\) be a purification of \(\sigma_k^{AB}\) for each \(k\). We consider a non-Markovianity generation protocol for the state

\[
\rho_{ABEK} := \sum_k p_k |\phi_k\rangle \langle \phi_k| \otimes |k\rangle \langle k| K.
\] (70)

We apply the protocol \(G_2\) presented in Section \[IV-C3\] under the assumption that \(A', B'\) and \(E'\) are trivial (one-dimensional) system. For any \(\epsilon\) and sufficiently large \(n\), the protocol \(G_2\) generates a state \(\rho_{n,\delta}^{ABEK}\) such that

\[
\left\| \hat{\rho}_{n,\delta}^{ABEK} - (\rho_{ABEK})^\otimes n \right\|_1 \leq \epsilon.
\] (71)

By tracing out \(EK\), and noting that \(\rho^{AB} = \omega^{AB}\), we obtain

\[
\left\| \hat{\rho}_{n,\delta}^{AB} - (\omega^{AB})^\otimes n \right\|_1 \leq \epsilon.
\] (72)

The quantum communication cost of \(G_2\) is given by \(\{47\}\), i.e.,

\[
Qc(G_2) = \frac{1}{n} \sum_k p_k I(A : B)_{\sigma_k} + \delta.
\] (73)

We construct an assisted entanglement dilution protocol \(G_2'\) for \(\omega^{AB}\) from \(G_2\). In \(G_2'\), instead of communicating quantum messages directly to Alice, Eve performs quantum teleportation by using an entanglement resource shared between Alice and Eve in advance. The downward classical communication cost of this protocol is equal to \(n \sum_k p_k I(A : B)_{\sigma_k} + \delta\). By taking the limit of \(n \rightarrow \infty\) and \(\delta \rightarrow 0\), and by taking the infimum over all ensembles \(\{p_k, \sigma_k\}\), we obtain

\[
C^4(\omega^{AB}) \leq E_{sq,c}(\omega^{AB}).
\] (74)

The same argument also applies to \((\omega^{AB})^\otimes m\) for any \(m \in \mathbb{N}\), leading to

\[
C^4(\omega^{AB}) = \frac{1}{m} C^4((\omega^{AB})^\otimes m) \leq \frac{1}{m} E_{sq,c}((\omega^{AB})^\otimes m).
\] (75)

By taking the limit of \(m \rightarrow \infty\), we arrive at (69) and completes the proof.

VI. CONCLUSION

In this paper, we proposed a framework for analyzing non-Markovianity of tripartite quantum states from the viewpoint of operational resource theory. In particular, we introduced a measure of non-Markovianity that we call the non-Markovianity of formation, and provided its operational meaning in terms of a task of non-Markovianity generation. A future direction is to formulate distillation and dilution of non-Markovianity, and analyze the optimal conversion rate for those tasks. It would also be beneficial to address the resource theory of non-Markovianity in a general probabilistic scenario.
APPENDIX A

PROPERTIES OF THE NON-MARKOVIANITY OF FORMATION

In this Appendix, we prove properties of the non-Markovianity of formation (nMF), presented in Section III. We first prove an alternative expression \[14\] for nMF, and then prove Properties (P1)–(P10) in Lemma 4 in addition to Lemma 5. A proof of (P11) will be presented in Appendix B.

A. Proof of Equality \[14\]

Define

\[ \tilde{M}_F(\rho) := \inf_{A',B',E',K\{p_k,|\phi_k\rangle\}} \{I(AA':BB'|K)_{\#} + I(AB:E'K|E)_{\#}\}. \]

We prove \(M_F(\rho) = \tilde{M}_F(\rho)\) by showing that \(M_F(\rho) \geq \tilde{M}_F(\rho)\) and \(M_F(\rho) \leq M_F(\rho)\).

To prove the first inequality, let \(\{p_k,|\phi_k\rangle\}\) be an arbitrary ensemble of states on \(ABE\) such that \(\rho = \sum_k p_k |\phi_k\rangle \langle \phi_k|\). Let \(A'\) and \(B'\) be finite-dimensional quantum systems, and consider a quantum state \(\tilde{\rho}_k\) on \(AA'BB'E'\) such that \(\text{Tr}_{A'B'}[\tilde{\rho}_k] = \rho_k\) for each \(k\). In addition, let \(E'\) be a finite-dimensional quantum system, and \(|\phi_k\rangle^{AA'BB'E'E'}\) be a purification of \(\tilde{\rho}_k\) for each \(k\). It is straightforward to verify that the ensemble \(\{p_k,|\phi_k\rangle\}_{k}\) satisfies

\[ \rho^{ABE} = \sum_k p_k \text{Tr}_{A'B'E'}[|\phi_k\rangle \langle \phi_k|]. \quad (76) \]

Consider a state \(\varrho\) defined by

\[ \varrho^{AA'BB'E'E'} = \sum_k p_k |\phi_k\rangle \langle \phi_k|^{AA'BB'E'E'} \otimes |k\rangle \langle k|_. \quad (77) \]

We have

\[ \sum_k p_k \{S(AA')\rho_k + S(BB')\rho_k\} = \sum_k p_k \{S(AA')\phi_k + S(BB')\phi_k\} = S(AA'|K)_{\#} + S(BB'|K)_{\#} = I(AA':BB'|K)_{\#} + S(AB|EE'K)_{\#}, \]

in addition to

\[ \sum_k p_k S(A'B')\rho_k = \sum_k p_k S(A'B')\phi_k \]

\[ = \sum_k p_k S(ABBE')\phi_k = \sum_k p_k S(EE')\phi_k + \sum_k p_k S(AB|EE')\phi_k = \sum_k p_k S(AA'BB')\phi_k + \sum_k p_k S(AB|EE')\phi_k = S(AA'BB'|K)_{\#} + S(AB|EE'K)_{\#} \quad (78) \]

and

\[ S(AB|E)_{\#} - S(AB|EE'K)_{\#} = I(AB:E'K|E)_{\#}. \]

In the second and fourth lines of (78), we have used the fact that \(|\phi_k\rangle\) is a pure state on \(AA'BB'E'E\) for each \(k\). Combining all the above equalities, we obtain

\[ S(AB|E)_{\#} + \sum_{k \in K} p_k \{S(AA')\rho_k + S(BB')\rho_k - S(A'B')\rho_k\} = I(AA':BB'|K)_{\#} + I(AB:E'K|E)_{\#} \geq \tilde{M}_F(\rho). \]

By taking the infimum over all finite-dimensional quantum systems \(A', B', E',\) all finite sets \(K,\) all ensembles \(\{p_k,|\phi_k\rangle\}_{k}\), and over all extensions \(\tilde{\rho}_k\) in the L.H.S., we arrive at \(M_F(\rho) \geq \tilde{M}_F(\rho).\)

The converse inequality \(M_F(\rho) \leq \tilde{M}_F(\rho)\) is proved along the similar line. Note that, for any ensemble \(\{p_k,|\phi_k\rangle\}_{k}\) satisfying (77), we may define \(\tilde{\rho}_k^{AA'BB'E'E} := \text{Tr}_{E'}[|\phi_k\rangle \langle \phi_k|]\) and \(\rho_k^{ABE} := \text{Tr}_{A'B'}[|\phi_k^{AA'BB'E'}\rangle \langle \phi_k^{AA'BB'E'}|]\) to obtain \(\rho^{ABE} = \sum_k p_k \tilde{\rho}_k^{ABE} \).

B. Proof of (P1): lower bound

Let \(K\) be a finite set and \(\{p_k,\rho_k\}_{k \in K}\) be an ensemble of states on \(ABE\) such that \(\rho = \sum_{k \in K} p_k \rho_k\). Let \(A', B', E'\) be finite dimensional quantum systems. In addition, let \(\phi_k\) be a state on \(AA'BB'E'E\) satisfying \(\text{Tr}_{A'B'}[\rho_k] = \rho_k^{ABE}\), and \(\phi_k\) be a pure state on \(AA'BB'E'E\) such that \(\text{Tr}_{A'B'}[\phi_k]\) for each \(k\). Let \(K\) and \(K'\) be quantum systems with dimension \(|K|,\) and consider a pure state on \(AA'BB'E'E'K'\) defined by

\[ |\Phi_{\rho}\rangle := \sum_{k \in K} \sqrt{p_k} |\phi_k\rangle^{AA'BB'E'E'} |k\rangle |k|', \]

where \(\{|k\rangle\}_{k \in K}\) is a set of orthonormal pure states. It is straightforward to verify that the above state is a purification of \(\rho^{ABE}\), and that

\[ \text{Tr}_{K'}(\Phi_{\rho}) = \sum_{k \in K} p_k |\phi_k\rangle \langle \phi_k|^{AA'BB'E'E'} \otimes |k\rangle \langle k|'. \quad (79) \]

Consequently, we obtain

\[ \sum_{k \in K} p_k \{S(AA')\phi_k + S(BB')\phi_k\} = S(AA'|K)_{\phi} + S(BB'|K)_{\phi} = S(AB|EE'K')_{\phi} \quad (80) \]

\[ = S(AE)_{\phi} + S(AB|EE'K')_{\phi} = S(AB|E'E'K')_{\phi} \quad (81) \]

\[ \geq S(AE)_{\phi} + S(A'B'EE'K')_{\phi} + S(AB|EE'K')_{\phi} = S(AB)_{\phi} + S(AB|E'E'K')_{\phi} \quad (82) \]

\[ \sum_{k \in K} p_k \{S(AB|E)_{\phi} + S(AB|EE'K)_{\phi}\} = S(AB|E)_{\phi} + S(AB|EE'K)_{\phi} = S(AB)_{\phi} + S(AB|EE'K)_{\phi} \quad (83) \]

\[ \geq S(AB|E)_{\phi} + S(A'B'EE'K')_{\phi} + S(AB|EE'K')_{\phi} = S(AB)_{\phi} + S(A'B'EE'K')_{\phi} \quad (84) \]

\[ \sum_{k \in K} p_k S(AB|E)_{\phi} = S(AB|E)_{\phi} = S(AB)_{\phi} \quad (85) \]

\[ \sum_{k \in K} p_k S(AB|EE'K)_{\phi} = S(AB|EE'K)_{\phi} \quad (86) \]
\[ I(A : B|E)_{\phi_\rho} - S(AB|E)_{\phi_\rho} + I(A' : B'|ABEE')_{\phi_\rho} + S(A'B'|K)_{\phi_\rho} \geq I(A : B|E)_{\rho} - S(AB|E)_{\rho} + I(A' : B'|K)_{\rho} \]

and that
\[ \Gamma_\rho \Gamma^\dagger = \sum_k p_k \psi^A_{\sigma_k} \otimes \psi^B_{\tau_k} \otimes |k\rangle \langle k| E_o \otimes |k\rangle \langle k| K. \]

From (94), we have
\[ I(AA' : BB'|K)_{\psi} = \sum_k p_k I(AA' : BB'|E_k)_{\phi_k} = 0. \]

In addition, due to (95) and the invariance of CQMI under local isometry, we have
\[ I(AB : K|E)_{\phi} = I(AB : K|E_oE_oE_o)_{\Gamma(\phi)} = 0. \]

Combining these equalities with (14), we complete the proof. ■

E. Proof of (P4): pure states

Let \(|\psi\rangle\) be a pure state on \(ABE\), and consider an ensemble \(\{p_k, \rho_k\}_k\) of states on \(ABE\) such that \(|\psi\rangle\langle\psi| = \sum_k p_k \rho_k\). It is straightforward to verify that \(p_k \geq 0\) for each \(k\). For each \(k\), let \(\tilde{\rho}_k\) be a state on \(AA'B'B'E\) satisfying \(\text{Tr}_{A'B'}[\tilde{\rho}_k] = \rho_k = |\psi\rangle\langle\psi|\). It follows that there exists a state \(\varsigma_k\) on \(A'B'\) such that \(p_k = |\langle \psi | \varsigma_k \rangle|^2\). Hence we have
\[ S(\rho_{A\varsigma_k}) + S(B\varsigma_k) - S(A'B')_{\tilde{\rho}_k} \geq S(\rho_{A\varphi}) + S(B\varphi), \]
with equality if and only if \(\varsigma_k\) is a product state between \(A'\) and \(B'\). This implies that \(\lambda(\rho_k) = S(\rho_{A\varphi}) + S(\rho_{B\varphi})\) for all such that \(p_k > 0\). Noting that \(S(ABE|E)_{\psi} = S(ABE|E)_{\psi} - S(E|\psi) = S(ABE|\psi)\), we obtain \(M_F(\psi) = I(A : B|\psi)\). ■

F. Proof of (P5): subadditivity

Consider quantum states \(\rho\) on \(A_1B_1E_1\) and \(\sigma\) on \(A_2B_2E_2\), and fix ensembles \(\{p_k, \rho_k\}_k\) and \(\{q_l, \sigma_l\}_l\) such that \(\rho = \sum_k p_k \rho_k\) and \(\sigma = \sum_l q_l \sigma_l\). For each \(k\) and \(l\), let \(\tilde{\rho}_k\) and \(\tilde{\sigma}_l\) be states on \(A_1B_1E_1\) and \(A_2B_2E_2\), respectively, such that \(\text{Tr}_{A'B'}[\tilde{\rho}_k] = p_k\) and \(\text{Tr}_{A'B'}[\tilde{\sigma}_l] = \sigma_l\). We denote \(A_1A_2B_1B_2B_1B_2E_1E_2\) by \(\Gamma\) and \(A_1A_2B_1B_2B_1B_2E_1E_2\) by \(\Gamma'\). Noting that \(\tilde{\rho}_k \otimes \tilde{\sigma}_l\) is an extension of \(\tilde{\rho}_k \otimes \tilde{\sigma}_l\), we have
\[ S(A_1A_1'\tilde{\rho}_k) + S(B_1B_1'\tilde{\rho}_k) - S(A_1'B_1')_{\tilde{\rho}_k} + S(A_2A_2'\tilde{\sigma}_l) + S(B_2B_2'\tilde{\sigma}_l) - S(A_2'B_2')_{\tilde{\sigma}_l} \geq \lambda(\rho_k \otimes \sigma_l). \]

By taking the infimum over all \(\tilde{\rho}_k, \tilde{\sigma}_l, A_1'B_1'\) and \(A_2'B_2'\), we obtain
\[ \lambda(\rho_k) + \lambda(\sigma_l) \geq \lambda(\rho_k \otimes \sigma_l). \]
Thus, using the fact that
\[ \rho \otimes \sigma = \sum_{k,l} p_k q_l \mathcal{T}_{A'B'}[\hat{\rho}_k \otimes \hat{\sigma}_l], \] (101)
we arrive at
\[ \Lambda(\rho) + \Lambda(\sigma) \]
\[ = \inf_{k_1 \in K_1} \inf_{\{p_k, p_{\tilde{k}}\} \in K} \left[ \sum_k p_k \mathcal{L}(\hat{\rho}_k) \right] \]
\[ + \inf_{k_2 \in K_2} \inf_{\{q_{\tilde{k}}, \sigma_{\tilde{k}}\} \in K_2} \left[ \sum_l q_l \mathcal{L}(\hat{\sigma}_l) \right] \]
\[ = \inf_{k_1 \in K_1} \inf_{k_2 \in K_2} \left[ \sum_{k,l} p_k q_l \left( \mathcal{L}(\hat{\rho}_k) + \mathcal{L}(\hat{\sigma}_l) \right) \right] \]
\[ \geq \Lambda(\rho \otimes \sigma). \] (102)
Combining this with
\[ S(A_1 B_1 | E_1)_\rho + S(A_2 B_2 | E_2)_\sigma = S(AB | E)_{\rho \otimes \sigma}, \] (103)
we obtain the desired result. \[ \square \]

G. Proof of (P6): weak chain rule
Let \( \{\phi_k, \rho_k\}_k \) be an ensemble of pure states on \( A', B', E' \), such that \( \rho = \sum_k p_k \mathcal{T}_{A'B'E'}[\rho_k] \), and define a state \( \rho \) by
\[ \rho = \sum_k p_k |\phi_k, \rho_k, \phi_k|^{A'C'B'B'E'E'} \]
\[ := \sum_k p_k |\phi_k, \rho_k|^{A'C'B'B'E'E'} \otimes |k,k|^{E}. \] (104)
Due to the chain rule and the monotonicity of CQMI, we have
\[ I(A'C' : B'B'|E)_\rho = I(A'C' : B'B'|E)_\rho' + I(C' : E|E')_\rho \]
\[ \geq I(A'C' : B'B'|E)_\rho + I(C' : E|E')_\rho \]
\[ \geq M_{\rho'}(A : B|E), \] (105)
By taking the infimum over all \( A', B', E' \), and \( \{\phi_k, \rho_k\}_k \), we obtain the desired result. \[ \square \]

H. Proof of (P7): conditional convexity
Let \( A' \) and \( B' \) be finite-dimensional quantum systems, and let \( K \) be a finite set. For each \( m \), let \( \{\phi_k, \rho_k\}_k \) be an ensemble of states on \( A'B'B'E' \) such that
\[ \rho_m^{A'B'} = \sum_{k \in K} p_k \mathcal{T}_{A'B'}[\hat{\rho}_k]. \] (106)
Define an ensemble \( \{\phi_k, \rho_k\}_k \) of states on \( A'B'B'E' \) by
\[ \tilde{\rho}_m^{A'B'B'E'} = M_{\rho_m^{A'B'}} = \sum_{m \in M} m \] (107)
and
\[ p_{k,m} := p_{k|m}r_m. \] (108)
It follows that
\[ \rho^{A'B'E} = \sum_{k \in K} p_{k,m} \mathcal{T}_{A'B'}[\tilde{\rho}_{k,m}]. \] (109)
Consequently, we obtain
\[ \sum_m r_m \sum_{k \in K} p_{k|m} \lambda(\tilde{\rho}_{k,m}) = \sum_{k \in K} p_{k,m} \lambda(\tilde{\rho}_{k,m}) \]
\[ = \sum_{k \in K, m} p_{k,m} \lambda(\tilde{\rho}_{k,m}) \geq \Lambda(\rho). \] (110)
By taking the infimum over all ensembles \( \{p_{k|m}, \tilde{\rho}_{k,m}\}_k \) for each \( m \), we arrive at
\[ \sum_m r_m \Lambda(\rho_m) \geq \Lambda(\rho). \] (111)
Combining this with
\[ S(AB | E)_{\rho_m} = \sum_m r_m S(AB | E)_{\rho_m}, \] (112)
we obtain the desired result. \[ \square \]

I. Proof of (P8): isometric invariance
We first prove that \( \Lambda(\rho) \) is monotonically nonincreasing under operations on \( E \), that is,
\[ \Lambda(\rho) \geq \Lambda(\mathcal{E}(\rho)), \forall \rho \in \mathcal{E}, \forall \mathcal{E} \in \mathcal{L}_E. \] (113)
Let \( \{p_k, \rho_k\}_k \) be an ensemble of states on \( A'B'E \) such that \( \rho = \sum_k p_k \rho_k \), which yields \( \mathcal{E}(\rho) = \sum_k p_k \mathcal{E}(\rho_k) \). For each \( k \), let \( \rho_k^{A'B'B'} \) be an extension of \( \rho_k^{A'B'} \). An extension of \( \mathcal{E}(\rho_k) \) is given by \( \mathcal{E}(\tilde{\rho}_k) = \mathcal{E}(\tilde{\rho}_k^{A'B'B'}) \). Noting that \( \tilde{\rho}_k^{A'B'B'} = \rho_k^{A'B'B'} \), we have
\[ S(A'B')_{\tilde{\rho}_k} + S(B'B')_{\tilde{\rho}_k} - S(A'B')_{\tilde{\rho}_k} \]
\[ = S(A'B')_{\rho_k} + S(B'B')_{\rho_k} - S(A'B')_{\rho_k} \geq \Lambda(\mathcal{E}(\rho_k)). \] (144)
By taking the infimum over all \( A', B', E' \) and \( \tilde{\rho}_k \), the above inequality yields \( \Lambda(\rho) \geq \Lambda(\mathcal{E}(\rho_k)) \). Consequently, we obtain
\[ \Lambda(\rho) \geq \Lambda(\mathcal{E}(\rho)) \geq \Lambda(\mathcal{E} \circ \mathcal{E}(\rho)) = \Lambda(\rho). \] (115)

\[ \square \]

[1, 2]
Due to the monotonicity of the conditional entropy, we also have
\[
S(AB|E)_\rho \geq S(AB|E)_{\mathcal{V}(\rho)} \geq S(AB|E)_{\mathcal{V}_0(\rho)} = S(AB|E)_\rho.
\] (116)
Combining these two equalities, we obtain the desired result.

J. Proof of (P9): average monotonicity

Any measurement \( \mathcal{M} \) on \( A \) is represented by
\[
\mathcal{M}(\cdot) = \sum_m |m\rangle\langle m|^A \otimes \mathcal{M}_m(\cdot),
\] (117)
where \( \mathcal{M}_m \) is a linear CP map from \( A \) to \( \hat{A} \) such that \( \mathcal{M} := \sum \mathcal{M}_m \) is trace-preserving. Hence, the state after the measurement is represented by
\[
\mathcal{M}(\rho^{ABE}) = \sum_m \nu_m |m\rangle\langle m|^A \otimes \rho^{ABE}_m,
\] (118)
where we have introduced notations
\[
\nu_m := \text{Tr}[\mathcal{M}_m(\rho)], \quad \rho^{ABE}_m := \nu_m^{-1} \mathcal{M}_m(\rho^{ABE}).
\] (119)
We denote by \( V \) a linear isometry from \( A \) to \( A_0A_0^\prime \) such that a Stinespring dilation of \( \mathcal{M} \) is given by \( \mathcal{M}(\cdot) = \text{Tr}_{A_0}[V(\cdot)V^\dagger] \).

Fix an arbitrary ensemble \( \{|p_k,|\phi_k\rangle\}_k \) of states on \( AA'BB'EE' \) such that \( \rho = \sum_k p_k \text{Tr}_{A'B'E'}[|\phi_k\rangle\langle\phi_k|] \).

It is straightforward to verify that
\[
\mathcal{M}(|\phi_k\rangle\langle\phi_k|) = \sum_m p_m |m\rangle\langle m|^A \otimes \text{Tr}_{A'}[|\hat{\phi}_{mk}\rangle\langle\hat{\phi}_{mk}|],
\]
and consequently, that
\[
\mathcal{M}(\rho^{ABE}) = \sum_k p_k \text{Tr}_{A'B'E'} \otimes \mathcal{M}(|\phi_k\rangle\langle\phi_k|) = \sum_{k,m} p_k \nu_m |m\rangle\langle m|^A \otimes \text{Tr}_{A'B'E'}[|\hat{\phi}_{mk}\rangle\langle\hat{\phi}_{mk}|]
\]
where we have defined \( q_{km} := p_k \nu_m \).

Define notations \( \hat{\varrho}, \hat{\varrho}_0 \) and \( \hat{\varrho}_m \) by
\[
\hat{\varrho}
\]
\[
:= \sum_k p_k |\phi_k\rangle\langle\phi_k|^{AA'BB'EE'} \otimes |k\rangle\langle k|^{K},
\]
\[
:= V\hat{\varrho}^{AA'BB'EE'}K V^\dagger
\] (120)
and
\[
:= \sum_k q_{km} |\hat{\phi}_{mk}\rangle\langle\hat{\phi}_{mk}|^{AA'BB'EE'} \otimes |k\rangle\langle k|^{K}.
\] (121)
Denoting by \( D \) the dephasing operation on \( A_0 \) with respect to the basis \( \{|m\}\), it is straightforward to verify that
\[
D(\hat{\varrho}_0^{AA_0A_0'B'B'E'E'K}) = \sum_m \nu_m |m\rangle\langle m|^A \otimes \hat{\varrho}_m^{AA_0A_0'B'B'E'E'K}.
\]
Due to the monotonicity and the chain rule for CQMI, we have
\[
I(AA' : BB'|K)_\varrho + I(AB : E'K|E)_\varrho
\]
\[
= I(A_0A_0A_0'A' : BB'|KA_0)_\varrho + I(A_0A_0A_0'B : E'K|EA_0)_\varrho
\]
\[
\geq I(A_0A_0A_0'A' : BB'|KA_0)_\varrho + I(A_0A_0A_0'B : E'K|EA_0)_\varrho
\]
\[
\geq I(A_0A_0A_0'A' : BB'|KA_0)_\varrho + I(A_0A_0A_0'B : E'K|EA_0)_\varrho
\]
\[
\geq \sum_m \nu_m I(AA'' : BB'|K)_\varrho + \sum_m \nu_m I(AB : E'K|E)_\varrho
\]
\[
\geq \sum_m \nu_m M_F(\rho_m),
\]
where we have denoted \( A_0A_0'A' \) by \( A'' \) in the ninth line. By taking the infimum over all \( A', B', E', K \) and \( \{|p_k,|\phi_k\rangle\}_k \in K \), we obtain the desired result.

K. Proof of (P10): \( \Omega \)-monotonicity

Due to the symmetry of \( M_F(A : B|E) \) in \( A \) and \( B \), we only need to prove monotonicity of \( M_F \) under \( \mathbb{L}_A, \mathbb{R}_E, \mathbb{Q}_A \rightarrow \mathbb{E} \) and \( \mathbb{F}_A \). The first three cases directly follow from (P9), (P8) and (P6), respectively. To prove monotonicity under \( \mathbb{F}_A \), note that the state before and after broadcasting of classical message by Alice is represented by density operators
\[
\rho_i = \sum_m r_m |m\rangle\langle m|^A \otimes \rho_m^{ABE}
\] (122)
and
\[
\rho_f = \sum_m r_m |m\rangle\langle m|^A \otimes |m\rangle\langle m|^E \otimes \rho_m^{ABE},
\] (123)
respectively. Due to the average monotonicity (P9) and the conditional convexity (P7), it follows that
\[
M_F(AM_A : B|E)_{\rho_i} \geq \sum_m r_m M_F(A : B|E)_{\rho_m}
\]
\[
\geq M_F(AM_A : BM_B|EM_E)_{\rho_m},
\] (124)
which completes the proof.
L. Proof of Lemma 4

Let \{p_k, \tilde{\rho}_k\} be an arbitrary ensemble of states on \( AA'BB'E' \) such that \( \rho^{ABCE} = \sum_k p_k \text{Tr}_{AB}[\tilde{\rho}_k] \). We have

\[
M_F(AC : B|E)_\rho \\
\leq S(AB|EC)_\rho + \sum_k p_k \left[ S(AC|A')_{\tilde{\rho}_k} + S(BB'|B')_{\tilde{\rho}_k} - S(A'B')_{\tilde{\rho}_k} \right] \\
= S(AB|EC)_\rho + \sum_k p_k \left[ S(AA')_{\tilde{\rho}_k} + S(BB')_{\tilde{\rho}_k} - S(A'B')_{\tilde{\rho}_k} \right] \\
+ S(C|E)_\rho + \sum_k p_k S(C|AA')_{\tilde{\rho}_k}.
\]

Due to the monotonicity and the concavity of the conditional quantum entropy, we have

\[
\sum_k p_k S(C|AA')_{\tilde{\rho}_k} \leq \sum_k p_k S(C|A)_{\tilde{\rho}_k} \leq S(C|A)_\rho.
\]

Substituting this to (125), and noting that \( S(C|E)_\rho + S(C|A)_\rho \leq 2S(C) \leq 2\log d_C \), we obtain

\[
M_F(AC : B|E)_\rho \leq S(AB|EC)_\rho + 2\log d_C.
\]

By taking the infimum over all \( A', B' \) and \( \{p_k, \tilde{\rho}_k\} \), we obtain the desired result.

M. Proof of Lemma 5

(C1)⇒(C2) follows from the definition of \( \Omega^* \). (C2)⇒(C1) follows from the \( \Omega \)-monotonicity of \( M_F \) and symmetry of \( M_F(A : B|E) \) in \( A \) and \( B \). (C3)⇒(C2) immediately follows by definition. Equivalence between (C4) and (C5) follows from (11) and (112).

We compete the proof by showing that (C2)⇒(C4)⇒(C3). Recall that the states before and after classical communication from Eve to Bob is represented by density operators \( \tilde{\sigma} \) and \( \sigma \), respectively. Suppose that \( M_F \) is monotonically nonincreasing under \( C_{E \rightarrow B} \), i.e., \( M_F(\rho) \geq M_F(\rho_f) \). Due to the conditional convexity (P7), we have

\[
\sum_m r_m M_F(A : B|E)_{\rho_m} \geq M_F(\rho_i) \geq M_F(\rho_f).
\]

In addition, due to the average monotonicity (P9) under the measurement on \( M_B \) with respect to the basis \( \{|m\}_m \), we have

\[
M_F(\rho_f) \geq \sum_m r_m M_F(A : B|E)_{\rho_m}.
\]

Combining these two inequalities, we obtain \( M_F(\rho_i) = \sum_m r_m M_F(A : B|E)_{\rho_m} \), which implies (C2)⇒(C4). Next, suppose that \( M_F \) saturates Inequality (17), in which case we have

\[
M_F(\rho_i) = \sum_m r_m M_F(A : B|E)_{\rho_m} = M_F(\rho_f).
\]

This implies (C4)⇒(C3) and completes the proof.

APPENDIX B

ASYMPTOTIC CONTINUITY OF THE NON-MARKOVIANITY OF FORMATION

In this appendix, we prove Property (P11) in Lemma 3, i.e., asymptotic continuity of the non-Markovianity of formation. The proof proceeds along the same line as the proof of asymptotic continuity of entanglement of formation, in the version of Corollary 4 in [35].

Theorem 16 For any states \( \rho \) and \( \sigma \) on system \( ABE \) such that \( ||\rho - \sigma||_1 \leq \epsilon \), it holds that

\[
|M_F(\rho) - M_F(\sigma)| \leq 12\eta(\sqrt{2}\epsilon) \log d_A d_B,
\]

where \( \eta \) is a function defined by

\[
\eta(x) := \begin{cases} 
  x - x \log x & (x \leq 1/e) \\
  x + \frac{1}{x} & (x \geq 1/e) 
\end{cases}.
\]

Proof: Suppose that \( ||\rho - \sigma||_1 \leq \epsilon \). The Alicki-Fannes inequality [37] yields

\[
|S(AB|E)_\rho - S(AB|E)_\sigma| \leq 4\eta(\sqrt{2}\epsilon) \log d_A d_B.
\]

As we prove below, it also holds that

\[
|\Lambda(\rho) - \Lambda(\sigma)| \leq 4\eta(\sqrt{2}\epsilon) \log d_A d_B.
\]

Combining these two inequalities, we obtain (129).

To prove (130), we may, without loss of generality, assume that

\[
\Lambda(\sigma) \leq \Lambda(\rho).
\]

Due to the condition \( ||\rho - \sigma||_1 \leq \epsilon \) and Proposition 5 in [36], there exists a purification \( |\psi_\sigma\rangle^{ABER} \) of \( \sigma \) and a state \( \tilde{\rho}^{ABER} \) such that

\[
|||\psi_\sigma\rangle\langle\psi_\sigma| - \theta_\rho||_1 \leq \sqrt{2}\epsilon
\]

and that

\[
\theta_\rho^{ABE} = \rho^{ABE}, \quad \theta_\rho^R = |\psi_\sigma^R\rangle.
\]

Let \( A' \) and \( B' \) be finite dimensional quantum systems, \( K \) be a finite set and \( \{p_k, \tilde{\sigma}_k\}_{k \in K} \) be an ensemble of states on \( AA'BB'E' \) such that

\[
\sigma^{ABE} = \sum_{k \in K} p_k \text{Tr}_{A'B'}[\tilde{\sigma}_k].
\]

There exists a quantum operation \( M : R \rightarrow A'B'K \), which is in the form of

\[
M(\cdot) = \sum_{k \in K} M_k(\cdot) M_k^|k\rangle\langle k|^{K}, \quad \langle k|k'\rangle = \delta_{k,k'},
\]

such that

\[
\tilde{\sigma} := M(\psi_\sigma) = \sum_{k \in K} p_k \tilde{\sigma}_k^{AA'BB'E'} \otimes |k\rangle\langle k|^K.
\]
It holds that
\[
\sum_{k \in \mathbb{K}} p_k [S(AA')_{\bar{\sigma}_k} + S(BB')_{\bar{\sigma}_k} - S(A'B')_{\bar{\sigma}_k}]
\]
\[
= S(AA'|K)_{\bar{\sigma}} + S(BB'|K)_{\bar{\sigma}} - S(A'B'|K)_{\bar{\sigma}}
\]
\[
= S(A|A')_{\bar{\sigma}} + S(B|B')_{\bar{\sigma}} + I(A' : B'|K)_{\bar{\sigma}}.
\]  \hfill (137)

Applying the same map $\mathcal{M}$ to the state $\rho$, we obtain
\[
\tilde{\rho} := \mathcal{M}(\rho) = \sum_{k \in \mathbb{K}} \tilde{p}_k A_k^{BB'} E_k^{\text{op}} \otimes |k\rangle\langle k|.
\]  \hfill (138)

where
\[
\tilde{p}_k A_k^{BB'} E := p_k^{-1} M_k(\rho) M_k^\dagger.
\]  \hfill (139)

Note that, due to (134), it holds that
\[
\text{Tr} [\rho_k] = p_k^{-1} \text{Tr} [M_k(\rho) M_k^\dagger] = p_k^{-1} \text{Tr} [M_k(\psi_\sigma) M_k^\dagger] = 1.
\]

In addition, from (133), it follows that
\[
\sum_{k \in \mathbb{K}} p_k A_k^{BB'} E = \text{Tr} A_k^{BB'} K \circ \mathcal{M}(\rho) = \rho^{ABE} = \rho^{ABE}.
\]

Thus, similarly to (137), we have
\[
\Lambda(\rho) \leq \sum_{k \in \mathbb{K}} p_k [S(AA')_{\bar{\sigma}_k} + S(BB')_{\bar{\sigma}_k} - S(A'B')_{\bar{\sigma}_k}]
\]
\[
= S(A|A')_{\bar{\sigma}} + S(B|B')_{\bar{\sigma}} + I(A' : B'|K)_{\bar{\sigma}}.
\]  \hfill (140)

Since (134) implies $\tilde{\sigma}^{AB} K = \tilde{\rho}^{AB} K$, we also have
\[
I(A' : B'|K)_{\bar{\sigma}} = I(A' : B'|K)_{\bar{\rho}}.
\]  \hfill (141)

Combining (137), (140) and (141), we arrive at
\[
\Lambda(\rho) \leq \sum_{k \in \mathbb{K}} p_k [S(AA')_{\bar{\sigma}_k} + S(BB')_{\bar{\sigma}_k} - S(A'B')_{\bar{\sigma}_k}]
\]
\[
+ S(A|A')_{\bar{\sigma}} - S(A|A')_{\bar{\sigma}} + S(B|B')_{\bar{\sigma}} - S(B|B')_{\bar{\sigma}}.
\]  \hfill (142)

From (132), (136), (138) and the monotonicity of the trace distance, we have
\[
||\tilde{\rho} - \tilde{\sigma}||_1 \leq ||\rho_\sigma - \|\psi_\sigma\|\|_1 \leq \sqrt{2\epsilon}.
\]

Consequently, the Alicki-Fannes inequality yields
\[
|S(A|A')_{\bar{\sigma}} - S(A|A')_{\bar{\sigma}}| \leq 4\eta(\sqrt{2\epsilon}) \log d_A,
\]  \hfill (143)

\[
|S(B|B')_{\bar{\sigma}} - S(B|B')_{\bar{\sigma}}| \leq 4\eta(\sqrt{2\epsilon}) \log d_B.
\]  \hfill (144)

Hence, from (142), we arrive at
\[
\Lambda(\rho) \leq \sum_{k \in \mathbb{K}} p_k [S(AA')_{\bar{\sigma}_k} + S(BB')_{\bar{\sigma}_k} - S(A'B')_{\bar{\sigma}_k}]
\]
\[
+ 8\eta(\sqrt{2\epsilon}) \log d_A d_B.
\]

Taking the infimum over all $A'$, $B'$, $K$ and $\{p_k, \bar{\sigma}_k\}_{k \in \mathbb{K}}$, the above inequality yields
\[
\Lambda(\rho) \leq \Lambda(\sigma) + 8\eta(\sqrt{2\epsilon}) \log d_A d_B.
\]  \hfill (145)

Combining this with (131), we obtain (130) and complete the proof.

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**APPENDIX C**

**PROOF OF LEMMA [15]**

In this Appendix, we prove Lemma [15] in Section [V-C].

The lemma states that any protocol for assisted entanglement dilution can be converted to a non-Markovianity generation protocol, and that the downward classical communication cost of the former is equal to half of the quantum communication cost in the latter. To this end, we first introduce a description of an arbitrary dilution protocol. In particular, we investigate the description of operations by Eve in detail. Based on the obtained description, we construct a non-Markovianity generation protocol from an assisted entanglement dilution protocol, such that the communication costs satisfies the above condition (see Figure 3).

**A. Description of Dilution Protocols**

Without loss of generality, we may assume that any dilution protocol proceeds as follows. Here, $\Gamma$ is the number of communication rounds in the protocol. Symbols $K_\gamma$, $L_\gamma$, $M_\gamma$ and $\bar{M}_\gamma$ are for random variables that represent the classical messages communicated among the parties and the outcomes of the measurement by Eve, respectively. We use the same symbols for systems in which those variables are registered. These variables take values in certain finite sets $\mathcal{K}_\gamma$, $\mathcal{L}_\gamma$, $\mathcal{M}_\gamma$ and $\bar{M}_\gamma$, respectively.

1) Alice, Bob and Eve recursively apply the following operation from $\gamma = 1$ to $\gamma = \Gamma$:

a) Alice performs a measurement on her system and obtains an outcome.

b) Alice broadcasts a classical message $K_\gamma$ to Bob and Eve.

c) Bob performs a measurement on his system and obtains an outcome.

d) Bob broadcasts a classical message $L_\gamma$ to Alice and Eve.

e) Eve performs a measurement $M_\gamma$ on her system and obtains an outcome $\bar{M}_\gamma$.

f) Eve broadcasts a classical message $M_\gamma$ to Alice and Bob.

2) Alice and Bob perform local operations on their systems.

3) Eve discards all of her systems.

Denoting the cardinality of $\mathcal{M}_\gamma$ by $\mu_\gamma$, the total number of classical bits, broadcasted by Eve from Alice to Bob during
the protocol, is given by

\[ C^\gamma(T_n) := \sum_{\gamma=1}^{\Gamma} \log \mu_{\gamma}. \]

It should be noted that all operations in Step (a)-(d) above belong to \( \Omega \).

We denote by \( E_\gamma \) and \( E_{\gamma+1} \) the input and output systems of the \( \gamma \)-th measurement by Eve, respectively. Let us introduce the following notations:

\[ K^\gamma := (K_1, \cdots, K_\gamma), \]  
\[ L^\gamma := (L_1, \cdots, L_\gamma), \]  
\[ M^\gamma := (M_1, \cdots, M_\gamma), \]  
\[ \hat{M}^\gamma := (\hat{M}_1, \cdots, \hat{M}_\gamma). \]

In general, Eve’s measurement in the protocol, as well as classical messages that she broadcasts, may depend on the previous measurement outcomes and messages in the following way (Figure 4):

- \( \mathcal{M}_\gamma \) depends on \( (K^\gamma, L^\gamma, \hat{M}^{\gamma-1}) \),
- \( \mathcal{M}_\gamma \) depends on \( (K^\gamma, L^\gamma, \hat{M}^\gamma) \).

Here, we have defined \( \mathcal{M}^0 = \emptyset \). In the following, we denote by \( \Theta^\gamma \) the triplet of random variables \( (K^\gamma, L^\gamma, \hat{M}^\gamma-1) \) and the system in which the variables are registered.

### B. Description of Eve’s Measurement

We prove that, without loss of generality, we may assume that the measurement by Eve in each step is a “noiseless” measurement. To be more precise, let \( T_n \) be an \((n, \epsilon)\)-protocol for assisted entanglement dilution of \( \omega^{AB} \). There exists an \((n, \epsilon)\)-protocol \( T'_n \) which induces the same map as \( T_n \), such that \( C^\chi(T_n) = C^\chi(T'_n) \) and that the \( \gamma \)-th measurement by Eve is represented by a CPTP map \( \mathcal{M}^\gamma \) from \( E_\gamma \) to \( E_{\gamma+1} \mathcal{M}_\gamma \), defined by

\[ \mathcal{M}^\gamma_\chi(\cdot) = \sum_{m \in \Omega_\gamma} \mathcal{M}^\gamma_\chi(m) \otimes |m\rangle\langle m|^{M^\gamma}, \]

where \( \{\mathcal{M}^\gamma_\chi\}_{m \in \Omega_\gamma} \) is the set of measurement operators.

The proof is as follows. In general, the \( \gamma \)-th measurement by Eve for a given value \( \theta^\gamma \) of \( \Theta^\gamma \) is represented by a CPTP map

\[ \mathcal{M}^\gamma_\chi(\cdot) = \sum_{\hat{m} \in \Omega_\gamma} \mathcal{M}^\gamma_\chi(\cdot) \otimes |\hat{m}\rangle\langle \hat{m}|^{M^\gamma}, \]

where \( \mathcal{M}^\gamma_\chi \) are completely positive maps from \( E_\gamma \) to \( E_{\gamma+1} \) such that \( \sum_{\hat{m}=1}^{M^\gamma} \mathcal{M}^\gamma_\chi(\cdot) \) is trace-preserving. The message \( \mathcal{M}_\gamma \) is obtained by applying a stochastic map on \( \Theta^\gamma \) and the measurement outcome \( \hat{M}^\gamma \). Thus, by incorporating the message, Eve’s operation is represented by a CPTP map

\[ \mathcal{M}^\gamma_\chi(\cdot) = \sum_{\hat{m} \in \Omega_\gamma} \sum_{m \in \Omega_\gamma} s_\gamma(m|\hat{m}, \theta^\gamma) \cdot \mathcal{M}^\gamma_\chi(\cdot) \otimes |\hat{m}\rangle\langle \hat{m}|^{M^\gamma} \otimes |m\rangle\langle m|^{M^\gamma}. \]

Here, \( \{s_\gamma(m|\hat{m}, \theta^\gamma)\}_{\gamma=1}^{\Omega_\gamma} \) is a probability distribution that represents the post-processing of the measurement outcome to obtain the message.

Define a map \( \mathcal{M}_\gamma : \Theta^\gamma \rightarrow \Theta^\gamma \mathcal{M}_\gamma \), defined by

\[ \mathcal{M}_\gamma(\cdot) := \sum_{\theta^\gamma} |\theta^\gamma\rangle\langle \theta^\gamma|^\gamma \otimes \mathcal{M}^\gamma_\chi(\cdot). \]

and let \( \mathcal{V}_\gamma : \Theta^\gamma \rightarrow \Theta^\gamma \mathcal{M}_\gamma \), defined by

\[ \mathcal{V}_\gamma(\cdot) := \sum_{m \in \Omega_\gamma} |m\rangle\langle m|^{M_\gamma} \otimes |m\rangle\langle m|^{M_\gamma}. \]

Eve discards all of her systems at the end, including the systems \( E_1 \cdots E_{\Gamma} \). It is straightforward to verify that the protocol \( T_n \) induces the same map as \( T_n \), i.e., \( T_n(\rho) = T_n(\rho) \) for any initial state \( \rho \).

### C. Simulation of Eve’s Measurement by Quantum Communication

Each measurement by Eve in the protocol \( T_n \), defined by [149], and the subsequent broadcasting of classical messages to Alice and Bob, can be converted to a reversible operation by Eve, followed by quantum communication from Eve to Alice and broadcasting of a classical message by Alice. To establish this, we adopt a protocol called coherent communication [38].

For each \( \gamma \), let \( A^\gamma_\chi \), and \( E^\gamma_\chi \) be a \( \sqrt{\Omega_\gamma} \)-dimensional quantum registers possessed by Alice and Eve, respectively, and let \( \phi^\gamma \) be the maximally entangled state thereon. Let \( \sigma_\gamma(m = 1, \cdots, \mu_\gamma) \) be the generalized Pauli operators on \( E^\gamma_\chi \). Based on \( \mathcal{M}_\gamma \), defined by (146), we introduce the following operation \( \mathcal{M}_\gamma \) that satisfies (149).

1. After receiving classical messages \( K_\gamma \) and \( L_\gamma \) from Alice and Bob, Eve performs an isometry \( \mathcal{V}_\gamma \) that satisfies (149).
2. Eve performs a controlled-unitary operation of the form

\[ U_\gamma E^\gamma_\chi := \sum_{m=1}^{\mu_\gamma} \sigma^\gamma_m \otimes |m\rangle\langle m|^{M^\gamma}. \]
3) Eve transmits the system $E_{\gamma}^*$ to Alice. Noting that $\sigma_{m|\Phi_{\gamma}}$ is orthogonal for $m \neq m'$, Alice can perfectly obtain $m$ by performing a measurement on $A_{\gamma}^*E_{\gamma}^*$. The quantum communication cost of this protocol is given by

$$\log \dim E_{\gamma}^* = \log \left( \sqrt{\gamma} \right).$$

Using $\mathcal{M}_{\gamma}$, we construct a protocol $\bar{T}_n \in \Omega^Q$ from $T_n$ as follows:

1) In the $\gamma$-th step, instead of performing a measurement $\mathcal{M}_{\gamma}$ and broadcasting a classical message, Eve performs $\mathcal{M}_{\gamma}$ and sends $E_{\gamma}^*$ to Alice.

2) In the $(\gamma+1)$-th step, before performing the $(\gamma+1)$-th operation in $T_n$, Alice performs a measurement on $A_{\gamma}^*E_{\gamma}^*$ to obtain $m$ and broadcasts it to Bob and Eve.

It is straightforward to verify that $T_n$ and $\bar{T}_n$ induce the same map.

D. Construction of $\bar{T}_{nl}$

Let $\bar{T}_{nl} \in \Omega^Q$ be an operation that is constructed from $T_{nl} := T_{nl}^{\otimes l}$, along the same line as we have constructed $\bar{T}_n$ from $T_n$. Without loss of generality, we may assume that

$$C^{E_{(\gamma)} \rightarrow A}(T_n) + C^{E_{(\gamma)} \rightarrow B}(T_n) \geq 1$$

for each $\gamma$, in which case we have

$$\Gamma(T_n) \leq C^J(T_n).$$

(151)

The quantum communication cost from Eve to Alice in the $\gamma$-th step of $\bar{T}_{nl}$ is given by

$$\log \left( \sqrt{\mu_{\gamma}} \right) \leq \log \left( \sqrt{\mu_{\gamma}} + 1 \right) = \log \mu_{\gamma} + \log \left( 1 + 1/\sqrt{\mu_{\gamma}} \right) \leq \frac{1}{2} \log \mu_{\gamma} + 1.$$  

(152)

Hence, the total quantum communication cost is calculated to be

$$Qc(\bar{T}_{nl}) \leq \sum_{\gamma=1}^{\Gamma(T_n)} \left( \frac{1}{2} \log \mu_{\gamma} + 1 \right) \leq \left( \frac{1}{2} + 1 \right) C^J(T_n),$$

where we have used (151) in the second inequality. This completes the proof of Lemma 15.

REFERENCES

[1] J. T. Yard and I. Devetak, “Optimal quantum source coding with quantum side information at the encoder and decoder,” IEEE Trans. Inf. Theory, vol. 55, no. 11, pp. 5339–5351, 2009.

[2] I. Devetak and J. Yard, “Exact cost of redistributing multipartite quantum states,” Phys. Rev. Lett., vol. 100, no. 23, p. 230501, 2008.

[3] M. Berta, F. G. Brandão, C. Majenz, and M. M. Wilde, “Conditional decopling of quantum information,” Phys. Rev. Lett., vol. 121, no. 4, p. 040504, 2018.

[4] M. Berta, F. G. Brandão, C. Majenz, and M. M. Wilde, “Deconstruction and conditional erasure of quantum correlations,” Phys. Rev. A, vol. 98, no. 4, p. 042320, 2018.

[5] O. Fawzi and R. Renner, “Quantum conditional mutual information and approximate markov chains,” Comm. Math. Phys., vol. 340, no. 2, pp. 575–611, 2015.

[6] P. Hayden, R. Jozsa, D. Petz, and A. Winter, “Structure of states which satisfy strong subadditivity of quantum entropy with equality,” Comm. Math. Phys., vol. 246, pp. 359–374, 2004.

[7] K. Sharma, E. Wakakuwa, and M. M. Wilde, “Conditional quantum one-time pad,” arXiv:1703.02903, 2017.

[8] M. Christandl and A. Winter, “‘squashed entanglement’: An additive entanglement measure,” J. Math. Phys., vol. 45.3. pp. 829–840, 2004.

[9] F. G. S. L. Brandão, M. Christandl, and J. Yard, “Faithful squashed entanglement,” Comm. Math. Phys., vol. 306, pp. 805–830, 2011.

[10] N. Gisin and S. Wolf, “Linking classical and quantum key agreement: is there “bound information”? in Ann. Int. Crypt. Conf. Springer, 2000, pp. 482–500.

[11] M. Christandl and R. Renner, “On intrinsic information,” in ISIT 2004. Proceedings IEEE, 2004, p. 135.

[12] M. Christandl, A. Ekert, M. Horodecki, P. Horodecki, J. Oppenheim, and R. Renner, “Unifying classical and quantum key distillation,” in Theory of Crypt. Conf. Springer, 2007, pp. 456–478.

[13] D. Sutter, “Approximate quantum markov chains,” in Approximate Quantum Markov Chains. Springer, 2018, pp. 75–100.

[14] M. B. Plenio and S. Virmani, “An introduction to entanglement measures,” Quant. Inf. Comput., vol. 7, pp. 1–51, 2007.

[15] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” Rev. Mod. Phys., vol. 81, no. 2, p. 865, 2009.

[16] U. Maurer and S. Wolf, “Towards characterizing when information-theoretic secret key agreement is possible,” in Int. Conf. Theo. App. Crypt. Inf. Sec. Springer, 1996, pp. 196–209.

[17] ——, “The intrinsic conditional mutual information and perfect secrecy,” in Proc. of 1997 IEEE Int. Symp. Info. Theory. IEEE, 1997, p. 88.

[18] U. M. Maurer and S. Wolf, “Unconditionally secure key agreement and the intrinsic conditional information,” IEEE Trans. Inf. Theory, vol. 45, no. 2, pp. 499–514, 1999.

[19] R. Renner and S. Wolf, “New bounds in secret-key agreement: The gap between formation and secrecy extraction,” in EUROCRYPT, vol. 2656. Springer, 2003, pp. 562–577.

[20] M. Wilde, Quantum Information Theory. Cambridge University Press, 2013.

[21] E. Chitambar and G. Gour, “Quantum resource theories,” arXiv preprint arXiv:1806.06107, 2018.

[22] E. Wakakuwa, “Operational resource theory of non-markovianity,” arXiv preprint arXiv:1709.07248, 2017.

[23] R. R. Tucci, “Entanglement of distillation and conditional mutual information,” arXiv preprint quant-ph/0202144, 2002.

[24] A. A. Nagel and G. A. Raggio, “Another state entanglement measure,” arXiv preprint quant-ph/0306024, 2003.

[25] ——, “Distributions attaining secret key at a rate of the conditional mutual information,” in Ann. Crypt. Conf. Springer, 2015, pp. 443–462.

[26] E. Chitambar and M. Gour, “Quantum resource theories,” J. Math. Phys., vol. 56, no. 3, p. 032202, 2015.

[27] M. Wilde, Quantum Information Theory. Cambridge University Press, 2013.

[28] E. H. Lieb and M. B. Ruskai, “Proof of the strong subadditivity of quantum-mechanical entropy,” J. Math. Phys., vol. 14, no. 12, pp. 1938–1941, 1973.

[29] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, “The mother of all protocols: Reconstructing quantum information’s family tree,” Proc. R. Soc. A, vol. 465, p. 2537, 2009.

[30] M.-Y. Ye, Y.-K. Bai, and Z. D. Wang, “Quantum state redistribution based on a generalized decoupling,” Phys. Rev. A, vol. 78, no. 3, p. 030302, 2008.

[31] T. M. Cover and J. A. Thomas, Elements of Information Theory (2nd ed.). Wiley-Interscience, 2005.

[32] C. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, “Mixed-state entanglement and quantum error correction,” Phys. Rev. A, vol. 54, p. 3824, 1996.

[33] M. Wilde, Quantum Computing and Quantum Information Theory. Cambridge University Press, 2017.

[34] A. Winter, “‘Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints,” Comm. Math. Phys., vol. 347, no. 1. pp. 291–313, 2016.
[37] R. Alicki and M. Fannes, “Continuity of quantum conditional information,” *J. Phys. A: Math. Gen.*, vol. 37, pp. L55–L57, 2004.

[38] A. Harrow, “Coherent communication of classical messages,” *Phys. Rev. Lett.*, vol. 92, p. 097902, 2004.