PSO-KS Algorithm for Fitting Lognormal Distribution: Simulation and Empirical Implementation to Women’s Age at First Marriage Data

Ari Purwanto Sarwo Prasojo*1 and Puguh Prasetyoputra1
1Research Center for Population, Indonesian Institute of Sciences, Jakarta, Indonesia

*Corresponding author’s e-mail: arip002@lipi.go.id

Abstract. Lognormal distribution plays an essential role in the distribution modeling of right-skewed data in many areas. In social sciences, for instance, it can be used to model women’s age at first marriage pattern, a key indicator in studying fertility patterns. Distribution fitting is a fundamental and essential stage of data modeling before doing advancing the analysis. Kolmogorov-Smirnov (KS) distance is applicable as maximum goodness-of-fit (GOF) estimators for distribution parameters. Minimizing KS distance is optimization problem. Particle swarm optimization (PSO) algorithm is a general optimizer that can handle various optimization problems. This study assesses the characteristics of minimum KS distance estimator for lognormal distribution parameters. KS distance estimators were obtained via optimization using the PSO algorithm, so the combination of these is called the PSO-KS algorithm. We conducted a simulation to assess the performance of PSO-KS, Maximum Likelihood (MLE), Method of Moment (MME). The bias and mean square error (MSE) of point estimators were used in simulation to assess the characteristics of estimators. Meanwhile, MSE of distribution fitting, KS distance, and log-likelihood value were used to evaluate the GOF characteristics. Moreover, we demonstrated the performance of the algorithm by implementing it to women’s age at first marriage data in Indonesia. The results show that based on the bias and MSE properties, the PSO-KS point estimators yield similar characteristics with MLE, but better than MME. From the GOF perspective, PSO-KS outperforms in MSE of distribution fitting and KS distance, but not in log-likelihood value. We also observed these patterns in the women’s age at first marriage data. The contributions of this study are two-fold, first to assess the PSO-KS algorithm in the lognormal distribution case. Second, it implements the algorithm on women’s age at first marriage data, which has broad social, economic, and public health implications.

Keywords: Particle Swarm Optimization, Kolmogorov-Smirnov, Lognormal Distribution, Age at First Marriage

1. Introduction
Knowing the distribution of data is one of the prominent steps in choosing the appropriate statistical approach. One example is the lognormal distribution that plays an essential role in various areas [1]. It is essential to modeling a right-skewed univariate variable before stepping in more advanced analysis. Simple analysis example when use lognormal distribution modeling is obtaining mode, the median or expectation of a variable according to the appropriate distribution properties. Lognormal distribution classified as non-negative distribution. One example of its use is in time data. So the lognormal
distribution usually used in reliability, survival analysis and called as life distribution. In social sciences, lognormal distribution can use to model the distribution pattern of women’s age at first marriage data [2-4]. This variable, which is a vital base employed in analyzing fertility patterns, has paramount social, economic and public health implications [5-11]. Hence, lognormal distribution fitting important as such.

Talking about distribution fitting is related to parameter estimation method. The methods commonly employed to estimate distribution parameter are method of moment estimation (MME), maximum likelihood estimation (MLE). MME estimate parameters by solving equating of population and sample moment, while MLE estimate via maximizing likelihood function [12]. When fitting a sample to a specific distribution, appropriate distribution can be assessed using a measure called goodness-of-fit (GOF) statistics. Hence, many scholars have proposed the parameter estimation method through maximizing GOF. First, a study by Weber, Leemis, and Kincaid [13] presented and implemented an algorithm to distribution fitting by minimizing Kolmogorov-Smirnov (KS) distance. The algorithm was applied to several distributions include lognormal, but they did not assess the characteristics of estimators. Second, Luceño [14] studied the maximum GOF estimate using a variety of GOF statistics, including KS distance. Assessing estimator’s characteristics regarding bias and root mean square error (RMSE), but not include lognormal distribution.

Maximizing GOF for parameter estimation is a part of the optimization problem which needs a flexible optimization technique. Also, if the probability density function is a nonlinear function, there exist several optimization techniques available to solve this problem. Particle swarm optimization (PSO), proposed by Kennedy and Eberhart in 1995 [15], is one of them. The concept of the PSO algorithm was derived from the social behavior of swarming observed in animals like bird and fish. PSO and its variant are applicable to solve optimization in many areas. Extant studies in the optimization problem of statistical distributions are by Wang and Huang [16], Orkcu et al. [17], and Jiang et al. [18].

These studies show that PSO is effective in solving the distribution parameter estimation problem, but they were not based on GOF fitness. The lognormal distribution is important, and KS distance is popular GOF statistics use [4, 19]. Hence, in this paper, we study about lognormal distribution parameter estimation via minimizing KS distance using PSO then called a PSO-KS algorithm. Then we assessed the PSO-KS lognormal estimator characteristics and the GOF measure. Lastly, we implemented the distribution on the women’s age at first marriage data in Indonesia.

2. Literature Review

2.1. Lognormal Distribution

The lognormal distribution is widely used in the case of a right-skewed univariate continuous random variable with non-negative values. It has two parameters, scale (or meanlog) and shape (sdlog) parameter. Theoretically, if is a normally distributed continuous random variable with mean \( \mu_N \) and standard deviation \( \sigma_N \), then the lognormal distribution has meanlog and sdlog. Meanlog and sdlog parameter are not the location and spread as in a normal distribution. The probability density function (p.d.f) and cumulative distribution function (c.d.f) of lognormal distribution are defined in equations (1) and (2).

\[
f(t; \theta) = \frac{1}{\sigma_N \sqrt{2\pi t}} \exp \left[ -\frac{1}{2} \left( \frac{\ln(t) - \mu_N}{\sigma_N} \right)^2 \right]
\]

\[
F(t; \theta) = \Phi \left( \frac{\ln(t) - \mu_N}{\sigma_N} \right)
\]

where \( \theta = (\mu_N, \sigma_N) \), \( -\infty < \mu_N < \infty \), \( \sigma_N > 0 \), \( t > 0 \), \( \Phi(.) \) is standard normal c.d.f.. Properties of lognormal distribution are displayed in table 1.
| Table 1. | Properties of lognormal distribution |
|---------|--------------------------------------|
| Median  | \( \exp(\mu_N) \)                    |
| Mode    | \( \exp(\mu_N - \sigma_N^2) \)      |
| Mean (1st raw moment) | \( \exp\left(\mu_N + \frac{\sigma_N^2}{2}\right) \) |

Source: Andrew N. O’Connor [20].

A popular method to estimate the parameter of the lognormal distribution is the maximum likelihood estimation (MLE). The likelihood function of lognormal distribution for a set of variable value \( t_i (i=1, 2, \ldots, n) \) is

\[
L(\theta; t_i (i=1, 2, \ldots, n)) = \frac{1}{\sigma_N \sqrt{2\pi}} \prod_{i=1}^{n} \frac{1}{t_i} \exp\left[-\frac{1}{2}\left(\frac{\ln(t_i) - \mu_N}{\sigma_N}\right)^2\right]
\]

(3)

taking natural logarithm, we then obtain the following log-likelihood function

\[
l(\theta; t_i (i=1, 2, \ldots, n)) = -\frac{n}{2} \ln(2\pi\sigma_N^2) - \sum_{i=1}^{n} \ln(t_i) - \frac{1}{2\sigma_N^2} \sum_{i=1}^{n} (\ln(t_i) - \mu_N)^2
\]

(4)

MLE estimators \( \hat{\theta} = (\hat{\mu}_N, \hat{\sigma}_N) \) can be obtained via optimizing its function. While the method of moment estimation (MME) is another popular estimation method which simple than MLE. MME estimators will be obtained by equating the population and sample moment [12].

2.2. PSO-KS Algorithm for Parameter Estimation of Lognormal

When data is fitted to a specific distribution, its suitability is shown by GOF measures. Kolmogorov-Smirnov (KS) distance is the most popular GOF measure. The concept of KS distance is comparing the theoretical and empirical distribution [21]. The statistic is the maximum vertical distance between them [22]. So smaller distance show that goodness-of-fit. Let \( t_1, t_2, \ldots, t_n \) is sample of \( n \) IID continuous random variable \( T \) with c.d.f \( F(t; \theta) \). Meanwhile \( S(t) \) is empirical c.d.f which obtain from corresponding order statistics \( t_{(i)} \leq t_{(i)} \leq \ldots \leq t_{(n)} \). KS distance is defined by

\[
D = \max_{1 \leq i \leq n} \left( |F(t_i; \theta) - S(t_{i-1})|, |S(t_i) - F(t_i; \theta)| \right)
\]

(5)

estimators \( \hat{\theta} = (\hat{\mu}_N, \hat{\sigma}_N) \) for lognormal distribution can be obtained via minimizing equation (5). Hence, in this case, \( \hat{\theta} \) is called the maximum GOF estimators for \( \theta = (\mu_N, \sigma_N) \).

Minimizing equation (5) to obtain estimators \( \hat{\theta} = (\hat{\mu}_N, \hat{\sigma}_N) \) is a complex optimization problem formulated by

\[
\hat{\theta} = \arg \min_{\theta} \max_{1 \leq i \leq n} \left( |F(t_i; \theta) - S(t_{i-1})|, |S(t_i) - F(t_i; \theta)| \right)
\]

(6)

A general or flexible optimization method is needed to solve this optimization problem. Particle Swarm Optimization (PSO) is a non-gradient, heuristic-population optimization method which can handle
multidimensional and complex optimization problem. The population or swarm consists of \( m \) particles which move within two-dimensional search space with position \( \theta_j = (\mu_j, \sigma_j) \) and velocity \( \omega_j = (\omega_1, \omega_2) \), where \( j = 1, 2, ..., m \). Each particle is evaluated by KS distance function as formulated in equation (5). Velocity and position of each particle is updated during the iteration process as defined by

\[
\begin{align*}
\omega^{k+1}_j &= w^k \omega^k_j + c_1 r_{\text{rand}1} (\text{pbest}_j^{k} - \theta^k_j) + c_2 r_{\text{rand}2} (\text{gbest}_j^{k} - \theta^k_j) \quad \text{(7)} \\
\theta^{k+1}_j &= \theta^k_j + \omega^{k+1}_j \quad \text{(8)}
\end{align*}
\]

where \( c_1 = c_2 = 2 \) are the standard cognitive and social factors which considered from Kennedy and Eberhart [15], \( r_{\text{rand}1}, r_{\text{rand}2} \) are \([0,1]\) uniform random number, \( \text{pbest}, \text{gbest} \) are personal and global best position, \( k \) is iteration step, and \( w \) is inertia weight. During this process, each particle will save personal best position to the memory. PSO optimizer takes the global best position from the overall personal best position of all particles. The maximum number of iteration can be used as a criterion for termination. While inertia weight is calculated by

\[
w^{k+1} = w_{\text{min}} + (w_{\text{max}} - w_{\text{min}})0.95^k
\]

It is a modification from initially inertia weight in the study by Shi and Eberhart [23] that was used by Orkcu and colleagues [17], with \( w_{\text{min}} = 0.4 \) and \( w_{\text{max}} = 0.9 \). This type of inertia work gives better in convergence speed. The pseudo-code in table 2 describes the structure and algorithm of the PSO-KS algorithm.

### Table 2. Pseudo-code of PSO-KS algorithm

| Step | Description |
|------|-------------|
| 1    | Let \( t_i (i = 1, 2, ..., n) \) is sample of non-negative univariate continuous random variable to fit lognormal distribution |
| 2    | Setting PSO parameters: particle size = 20, maximum iteration = 200, \( c_1 = c_2 = 2 \), \( w_{\text{min}} = 0.4 \), \( w_{\text{max}} = 0.9 \), search limit = \([0,100]\) |
| 3    | For each particle \( j \) |
| 4    | Initialization particle with position \( \theta_j \) using uniform distribution and velocity \( \omega_j \) set as zero vector |
| 5    | Compute \( D(\theta_j; t_i (i = 1, 2, ..., n)) \) # fitness function is based on equation (5) |
| 6    | \( \text{pbest}_j = \theta_j \) # initial personal best position of particle |
| 7    | \( \text{gbest} = \arg \min_{\theta_j} D(\theta_j; t_i (i = 1, 2, ..., n)) \) # global best particle |
| 8    | End |
| 9    | While not maximum iteration |
| 10   | For each particle \( j \) |
| 11   | Update velocity and position using equation (7) and (8) |
| 12   | Compute \( D(\theta_j; t_i (i = 1, 2, ..., n)) \) # fitness function is based on equation (5) |
| 13   | If \( D(\theta_j; t_i (i = 1, 2, ..., n)) < D(\text{pbest}_j; t_i (i = 1, 2, ..., n)) \) then \( \text{pbest}_j = \theta_j \) # update personal best position |
5

If \( D(\theta_j; i = 1, 2, \ldots, n) < D(\text{gbest}; i = 1, 2, \ldots, n) \) then gbest = \( \theta_j \) # update global best

End

End

Output gbest = \( (\hat{\mu}_N, \hat{\sigma}_N) \)

3. Simulation Study

Before implementing the PSO-KS algorithm to women’s age at first marriage data, we conducted a simulation study to assess its performance. Scenarios which are applied in the simulation study included following conditions: true parameter value \( \mu_N = 1 \) is set for constant mean scenario and two different true standard deviations \( \sigma_N = 0.5, 2 \) respectively, the true parameter value is \( \sigma_N = 1 \) set as constant standard deviation scenario, and the two different true means \( \mu_N = 0.5, 1 \) respectively; sample sizes are \( n = 50, 100, 200, 500 \). Each scenario was generated by 1000 replications, so there are \( 2 \times 2 \times 4 \times 1000 = 16000 \) independent experiments. When true parameters are known as scenario settings, \([0,3]\) is used for algorithm search limit for simulation study. The programme for simulation study was written in the R programming language [24] and was shared on https://github.com/aripurwantosp/psoks_fit_lognormal_wafm.

Five measures were used to assess the performance of PSO-KS. Two measures, bias and MSE of point estimator were used to assess estimator’s characteristics. KS distance, log-likelihood, mean square error of distribution fitting (MSE-distr) were used to assess GOF performance. PSO-KS will be compared with MLE and MME.

\[
\text{Bias}(\hat{\mu}_N) = \overline{\mu}_N - \mu_N; \quad \text{Bias}(\hat{\sigma}_N) = \overline{\sigma}_N - \sigma_N
\]

\[
\text{MSE}(\hat{\mu}_N) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\mu}_N - \mu_N)^2; \quad \text{MSE}(\hat{\sigma}_N) = \frac{1}{R} \sum_{i=1}^{R} (\hat{\sigma}_N - \sigma_N)^2
\]

\[
\text{MSE-distr} = \frac{1}{n} \sum_{i=1}^{n} \left( \hat{F}(t; \theta) - F(t; \theta) \right)^2
\]

where \( R \) is the number of repetition in simulation, \( \hat{F}(t; \theta) \) and \( F(t; \theta) \) are theoretical and empirical lognormal c.d.f.

By using a random selection of simulation scenario, the convergence speed of PSO-KS algorithm sample scenarios is visualized in figure 1. From this visualization, we can see that for sample size \( n = 50, 100, 200, 500 \), PSO-KS converges after 50 iterations, but based on time record, the average time execution for all sample sizes are 1.789, 1.736, 2.300, 3.397 seconds, respectively. Thus it is safe to say that the convergence speed of algorithm is relatively fast.

Figure 2 and figure 3 presents the aggregate results of simulation study. Summary from simulation analysis are:

1. MLE estimators have the smallest bias and MSE of point estimators, but not different significant with PSO-KS estimators. Also, for all methods, when sample size increased, bias and MSE value decreased;

2. Interesting finding when the true parameter \( \sigma_N = 1, 2 \), MME estimators have larger bias and MSE than MLE and PSO-KS, but PSO-KS relatively same with MLE estimators. In line with the study by Ginos [25], MME seem best to perform when \( \sigma_N \) less than or equal to one;
(3) PSO-KS has smallest KS distance, and MSE of distribution fitting, but smaller than MLE in log-likelihood value. This difference is a tradeoff in minimizing KS distance. Besides, increasing the sample size will decrease the KS distance and log-likelihood value.

![Figure 1. PSO-KS’s behavior for meanlog = 0.5; sdlog = 1 scenario](image1.png)

| Sample size (n) | 50  | 100 | 200 | 500 |
|-----------------|-----|-----|-----|-----|
| KS Distance     |     |     |     |     |
| Iteration       | 0   | 50  | 100 | 150 |
| PSO-KS          |     |     |     |     |
| MSE of meanlog  |     |     |     |     |
| MLE             |     |     |     |     |
| MME             |     |     |     |     |
| PSO-KS          |     |     |     |     |

![Figure 2. Bias and MSE of point estimators result from the simulation](image2.png)

| Bias of meanlog | MLE | MME | PSO-KS |
|-----------------|-----|-----|--------|
| meanlog = 1; sdlog = 0.5; n = 50 | -0.001 | 0.001 | -0.002 |
| meanlog = 1; sdlog = 0.5; n = 100 | -0.002 | 0.000 | -0.001 |
| meanlog = 1; sdlog = 0.5; n = 200 | 0.000 | 0.000 | 0.000 |
| meanlog = 1; sdlog = 0.5; n = 500 | 0.000 | 0.000 | 0.000 |

| Bias of sdlog | MLE | MME | PSO-KS |
|---------------|-----|-----|-------|
| meanlog = 1; sdlog = 1; n = 50 | -0.002 | 0.000 | -0.003 |
| meanlog = 0.5; sdlog = 1; n = 100 | 0.001 | 0.000 | 0.002 |
| meanlog = 0.5; sdlog = 1; n = 200 | -0.002 | 0.000 | -0.001 |
| meanlog = 0.5; sdlog = 1; n = 500 | 0.000 | 0.000 | 0.000 |

| MSE of sdlog | MLE | MME | PSO-KS |
|--------------|-----|-----|-------|
| meanlog = 1; sdlog = 1; n = 50 | -0.005 | 0.002 | 0.006 |
| meanlog = 1; sdlog = 1; n = 100 | 0.001 | 0.000 | 0.001 |
| meanlog = 1; sdlog = 1; n = 200 | 0.000 | 0.000 | 0.000 |
| meanlog = 1; sdlog = 1; n = 500 | -0.002 | 0.000 | -0.002 |
Figure 3. GOF measures result from the simulation

4. Empirical Implementation to Women’s Age at First Marriage Data

We make use of the latest round of the Indonesian Demographic and Health Survey (IDHS) datasets. The 2017 IDHS datasets are publicly accessible through the IDHS website http://sdki.bkkbn.go.id/index.php?lang=id. This paper is a further analysis of these datasets; hence no additional ethical clearance is needed.

Details of the survey can be read elsewhere [26]. Briefly, the 2017 IDHS was conducted by the National Population and Family Planning Board (Badan Kependudukan dan Keluarga Berencana Nasional – BKKBN), Statistics Indonesia (Badan Pusat Statistik – BPS), Ministry of Health, and United States Agency for International Development (USAID). The survey collected information on a wide range of socio-demographic and health characteristics such as fertility, mortality, education, and maternal and child health. For this analysis, we employed the Individual Recode and selected ever-married women. This selection comprised of 36,926 women. We analyzed the sample at the national level and also disaggregated it by place of residence (urban vs. rural area). All of the experiment and analyses were performed using R software [24].

PSO-KS’s behavior of women’s age at first marriage lognormal fitting is depicted in figure 4. It can be seen that the convergence speed in empirical implementation is slower than simulation study result (in figure 1). This difference is caused by different search limit setting. In the simulation analysis, the search limit is adjusted to the true parameters scenario. While the true parameters in empirical implementation are unknown, the search limit is set wider. Thus a wider search limit causes a decline in convergence speed.

The estimation using PSO-KS compared with MLE and MME method is listed in table 3. p.d.f of lognormal based on PSO-KS estimation method is shown in figure 5. All of the cases show that the KS distance for PSO-KS algorithm smaller than MLE and MME method, but log-likelihood value smaller than two other methods. Like as results of simulation analysis, tradeoff parameters estimation via minimizing distance or maximizing GOF is not maximizing log-likelihood value. Analysis by properties of lognormal distribution (according to the equation in table 1) show that majority and median of age at first marriage Indonesia’s women is estimated around 19 and 20 years old respectively. However, when disaggregating by place of residence, Indonesia’s rural women tend to marry earlier than their urban counterparts, with an estimated almost 1.7-year difference. This rural-urban gap is common in developing countries like Indonesia [27]. This disparity is not just explained by place of residence itself, but also by differences in educational attainment, labor force participation, and sex ratio [28].
Table 3. Estimation of lognormal distribution for women's age at first marriage data

| Cases   | Method | \( \hat{\mu}_N \) | \( \hat{\sigma}_N \) | MSE-\( N \) | KS    | Log-lik | Mode | Median |
|---------|--------|-----------------|-----------------|-----------|-------|---------|------|--------|
| All     | PSO-KS | 2.994           | 0.212           | 0.001     | 0.049 | -107424.400 | 19.088 | 19.965 |
|         | MLE    | 2.997           | 0.221           | 0.001     | 0.056 | -107356.300 | *    | *      |
|         | MME    | 2.997           | 0.223           | 0.001     | 0.056 | -107358.100 | *    | *      |
| Rural   | PSO-KS | 2.949           | 0.201           | 0.001     | 0.053 | -52177.280 | 18.331 | 19.087 |
|         | MLE    | 2.956           | 0.220           | 0.001     | 0.067 | -52014.270 | *    | *      |
|         | MME    | 2.955           | 0.224           | 0.001     | 0.067 | -52021.060 | *    | *      |
| Urban   | PSO-KS | 3.036           | 0.207           | 0.001     | 0.048 | -54724.730 | 19.948 | 20.822 |
|         | MLE    | 3.038           | 0.215           | 0.001     | 0.050 | -54693.450 | *    | *      |
|         | MME    | 3.038           | 0.215           | 0.001     | 0.050 | -54693.460 | *    | *      |

*not calculated because focused on PSO-KS estimation method

Figure 4. PSO-KS’s behavior of women’s age at first marriage fitting to lognormal distribution

Figure 5. Lognormal p.d.f of women’s age at first marriage in Indonesia based on PSO-KS estimation method
5. Conclusion
This paper addresses the performance of PSO-KS algorithm, a method to estimate parameters of a univariate lognormal distribution. PSO-KS is a maximum goodness-of-fit estimation method which minimizing KS distance using PSO algorithm. Simulation results showed that PSO-KS estimators have similar characteristics with MLE estimators regarding the bias and MSE of point estimators. However, MLE estimators have the smallest bias and MSE. PSO-KS outperforms the MLE and MME regarding the KS distance and MSE of distribution fitting. The tradeoff when minimizing KS distance is not maximizing log-likelihood function. But, PSO-KS algorithm can be used for estimating parameters of lognormal distribution which best KS GOF. Empirical implementation to Indonesia women’s age at first marriage data also showed that outperforms in KS distance but not in log-likelihood value. Using properties of the lognormal distribution, we yield valuable information that majority and median of women’s first marriage is estimated at around 19 and 20 years old. Then women in rural areas tend to marry earlier than women in urban areas.

The limitation of this study is the constant scenario in PSO parameters. Future studies can explore the performance algorithm for various PSO parameters, implementing to the other distributions, or replacing and compared KS distance with other GOF measures like the Anderson-Darling distance or other similar measures.

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