Complete ideals defined by sign conditions
and the real spectrum of a two-dimensional local ring

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0. Introduction. Let \((A, m)\) be a regular local ring and let \(\alpha\) and \(\beta\) be points of the real spectrum of \(A\) centered at \(m\). \(\alpha\) and \(\beta\) may be viewed as total orderings on quotients of \(A\). Associated with \(\alpha\) and \(\beta\), there is the so-called “separating ideal”, \(\langle \alpha, \beta \rangle \subseteq A\), which is generated by all \(a \in A\) such that \(a\) is non-negative with respect to \(\alpha\) and \(-a\) is non-negative with respect to \(\beta\). \(\langle \alpha, \beta \rangle\) is a valuation ideal for the valuation canonically associated with \(\alpha\) (or \(\beta\)), and hence is complete. It is known that a thorough understanding of \(\langle \alpha, \beta \rangle\) would contribute greatly to a solution of the long-standing Pierce-Birkhoff conjecture (see [M]) but up to now no good techniques for working with it have been known. In this paper, we investigate \(\langle \alpha, \beta \rangle\) by applying quadratic transforms to \((A, m)\) and using Zariski’s theory of complete ideals in two-dimensional regular local rings (see [Z]) to analyze how \(\langle \alpha, \beta \rangle\) is affected. Suppose that \((A', m')\) is a quadratic transform of \(A\). Under natural hypotheses, \(\alpha\) and \(\beta\) induce points \(\alpha'\) and \(\beta'\) in the real spectrum of \(A'\) which are centered at \(m'\). Our main result, Theorem 4.7, is a formula which relates \(\langle \alpha', \beta' \rangle \subseteq A'\) with the ideal transform \(\langle \alpha, \beta \rangle'\) of \(\langle \alpha, \beta \rangle\) in \(A'\). It says that if \(A\) is two-dimensional and has real closed residue field, and if \(\langle \alpha, \beta \rangle\) is not the maximal ideal, then \(\langle \alpha', \beta' \rangle = \langle \alpha, \beta \rangle'\). Applications of this result are presented in detail elsewhere (see [MR] and [MS]). The applications show that the transformation formula provides an essentially complete understanding of separating ideals in two dimensional regular algebras over real closed fields.

Here is a summary of the contents. In Section 1, we review notation for valuations and for the real spectrum, and then make some observations on valuations induced by points of the real spectrum. In Section 2, we prove (Proposition 2.2) that separating ideals are simple if a certain technical condition is satisfied. In section 3, we examine the behavior of the real spectrum under quadratic transformation, and we prove some general facts about the effect of quadratic transformations on separating ideals. We also give an example which indicates that in dimensions 3 and higher the behavior of separating ideals under quadratic transformations will be very difficult to analyze. In section 4, we consider 2-dimensional regular local rings, and we prove the transformation formula mentioned above. With Zariski’s theory and the results of sections 1 through 3 at hand, the hardest part of the proof of the transformation formula is Theorem 4.4. A sampling of some of the results which will be presented in [MR] and [MS] is given in the last section.

The present paper had a rather lengthy gestation. Alvis and Madden worked together on the type of ideals studied here in 1989, but were unaware of Zariski’s work on complete ideals at that time. Alvis wrote several computer programs which searched for generators for separating ideals. Without the wealth of examples found this way, connections to quadratic transforms would not have been noticed later on. Madden learned about complete ideals in two-dimensional rings from Johnston in 1990-91, and together they conjectured a version of Theorem 4.7 in early 1991. The proof of 4.7 was completed
1. Valuations and orderings. We describe first the notation we use when discussing valuations. Let $A$ be a noetherian ring and let $v$ be an additive valuation. In other words, $v$ is a map from $A$ to $\Gamma \cup \{+\infty\}$, where $\Gamma$ is a totally ordered abelian group, and $v$ satisfies $v(a + b) \geq \min\{v(a), v(b)\}$, $v(ab) = v(a) + v(b)$, $v(0) = +\infty$ and $v(1) = 0$. An ideal $I \subseteq A$ is a $v$-ideal if $a \in I$ and $v(b) \geq v(a)$ imply $b \in I$. The collection of all $v$-ideals in $A$ is totally ordered by inclusion. The smallest $v$-ideal in $A$ (namely, $v^{-1}(+\infty)$) is called the support of $v$ in $A$ and is denoted $\text{supp}_A v$, or just $\text{supp} v$ if no confusion is possible. The largest proper $v$-ideal in $A$ is called the center of $v$ in $A$, and is denoted $\text{cntr}_A v$, or just $\text{cntr} v$. We say that $v$ is non-trivial if $\text{cntr} v \neq \text{supp} v$. The largest $v$-ideal properly contained in the $v$-ideal $I$—this exists because $A$ is noetherian—is called the successor of $I$ and is denoted $I^v$. The valuation $v$ induces a valuation $\overline{v}$ on the fraction field of $A/\text{supp} v$. The valuation ring of this valuation is denoted $V_v$ and its maximal ideal is denoted $m_v$. The residue field of $v$ is $K_v := V_v/m_v$. Let $d_v := A/\text{cntr} v$, and let $k_v$ denote the fraction field of $d_v$. We have the containments $d_v \subseteq k_v \subseteq K_v$. The dimension of $v$ is the transcendence degree of $K_v$ over $k_v$, $\dim v := \text{tr.deg}(K_v/k_v)$.

We now describe the notation we use when dealing with the real spectrum. It is close to [BCR], Chapter 7. Spec$_v A$ is the set of prime cones in $A$. Each $\alpha \in \text{Spec}_v A$ determines a pair $(\text{supp} \alpha, \leq \alpha)$, where $\text{supp} \alpha = \alpha \cap -\alpha$ is a real prime ideal of $A$ and $\leq \alpha$ is a total ordering of $A/\text{supp} \alpha$ satisfying the usual conditions of compatibility with the ring structure. Given a real prime ideal $I$ and a total order $\leq_0$ on $A/I$, there is a prime cone $\alpha = \{a \in A \mid 0 \leq_0 a + I\}$, with supp $\alpha = I$ and $\leq_\alpha = \leq_0$. We let $A[\alpha]$ denote $A/\text{supp} \alpha$ endowed with the ordering $\leq_\alpha$. If $a \in A$, then $a(\alpha)$ denotes the residue of $a$ in $A[\alpha]$. Also, $A(\alpha)$ denotes the ordered fraction field of $A[\alpha]$. If $\alpha \supseteq \beta$, we say $\alpha$ is a specialization of $\beta$. In this case, there is an order-preserving homomorphism $A[\beta] \to A[\alpha]$.

There is a valuation on $A$ naturally associated with $\alpha$. Let

$$V_\alpha := \{a \in A(\alpha) \mid \exists x \in A[\alpha] \ |a| \leq |x| \}.$$ 

This is a valuation ring in $A(\alpha)$. Let $\overline{v}_\alpha : A(\alpha) \to \Gamma_\alpha \cup \{+\infty\}$ be the valuation associated with $V_\alpha$, and let $v_\alpha : A \to \Gamma_\alpha \cup \{+\infty\}$ be the valuation obtained by mapping $A$ to $A(\alpha)$ and then applying $v_\alpha$. Note that $v_\alpha$ is non-negative on $A$. It seems worth explicit mention that $v_\alpha$ may be trivial even in cases where the ordering $\alpha$ is in no sense trivial, viz., when $A$ is a field. This is also possible when $A$ is not a field, for example, let $A = \mathbb{Q}[x_1, \ldots, x_n]$ and let $\alpha$ be induced by the embedding of $A$ in $\mathbb{R}$ which identifies the indeterminates with $n$ algebraically independent real transcendental numbers.

The $v_\alpha$-ideals have a nice description in terms of the ordering of $A[\alpha]$. An ideal $I$ in $A[\alpha]$ is convex if $a \in I$ and $|b| \leq |a|$ implies $b \in I$. An ideal of $A$ is called an $\alpha$-ideal if it is the pull-back of a convex ideal of $A[\alpha]$.

**Lemma 1.1:** $I \subseteq A$ is an $\alpha$-ideal if and only if it is a $v_\alpha$-ideal.

**Proof.** For all $a, b \in A[\alpha]$, $0 \neq a$, we have the following equivalences:

$$\exists x \in A[\alpha] \ |b| \leq |xa| \Leftrightarrow \exists x \in A[\alpha] \ |b|/|a| \leq |x| \Leftrightarrow |b|/|a| \in V_\alpha \Leftrightarrow v_\alpha(|b|/|a|) \geq 0 \Leftrightarrow v_\alpha(b) \geq v_\alpha(a).$$
Suppose $I \subseteq A[\alpha]$ is a $v_\alpha$-ideal. If $a \in I$, $0 \neq a$, and $|b| \leq |a|$, then $v_\alpha(b) \geq v_\alpha(a)$, so $b \in I$. Thus, $I$ is convex.

On the other hand, if $I$ is convex and $a \in I$ and $v_\alpha(b) \geq v_\alpha(a)$, then $|b|$ is less than or equal to some element of $I$, so $I$ is a $v_\alpha$-ideal.

Some important concepts related to $v_\alpha$ can be defined in terms of the ordering $\alpha$. For example, $\text{cntr}_A v_\alpha$ is the largest proper $\alpha$-ideal of $A$. We shall denote this by $\text{cntr} \alpha$. Let $m_\alpha$ be the maximal ideal of $V_\alpha$ and let $K_\alpha := V_\alpha/m_\alpha$. Note that $K_\alpha$ is naturally an ordered field. Let $f$ be the map $A \to V_\alpha \to K_\alpha$. The kernel of $f$ is $\text{cntr} \alpha$. The image of $f$ is a sub-domain $d_\alpha \subseteq K_\alpha$. This notation is summarized by the following diagram, in which the rows are exact the vertical arrows are inclusions:

$$
\begin{array}{cccc}
0 & \to & \text{cntr} \alpha/\text{supp} \alpha & \to & A[\alpha] & \to & d_\alpha & \to & 0 \\
0 & \to & m_\alpha & \to & V_\alpha & \to & K_\alpha & \to & 0
\end{array}
$$

The fraction field of $d_\alpha$ is denoted $k_\alpha$; $\dim \alpha$ is defined to be $\text{tr.deg.}(K_\alpha/k_\alpha)$. (Caution: $\dim \alpha$ should not be confused with $\dim \text{supp} \alpha$, by which is meant the Krull dimension of $A/\text{supp} \alpha$.)

Since every element of $V_\alpha$ is between two elements of $A[\alpha]$, every element of $K_\alpha$ is between two elements of $d_\alpha$, i.e., $d_\alpha$ is cofinal in $K_\alpha$. This has an important consequence when $d_\alpha$ happens to be the field of real numbers (as, for example, when $A$ is an $R$-algebra). In this case, $d_\alpha = K_\alpha$, since $R$ is not cofinal in any ordered field which is a proper extension. In general, of course, one must deal with valuations centered at maximal ideals which are not zero-dimensional, e.g., the order valuation on a local ring.

**Examples 1.** We first describe a convenient method of producing examples, which will be used repeatedly. Suppose $B$ is a totally-ordered domain and $\phi : A \to B$ is any ring homomorphism. Let $\leq_\phi$ be the ordering on $A/\ker \phi$ induced by its inclusion in $B$. Then $(\ker \phi, \leq_\phi)$ is a point of $\text{Spec}_r A$. We shall simply denote this point by the letter $\phi$. For any $a \in A$, we may identify $A[\phi]$ with the image of $A$ in $B$ under the map $\phi$ and $a(\phi)$ with the image of $a$.

Let $A = k[x, y]$, where $k \subseteq R$ is the field of real algebraic numbers and $R$ is the field of real numbers. We shall give examples of points $\phi$ and $\psi$ in $\text{Spec}_r A$ both with zero support, both centered at $(x, y)$ and both with the same value group but with $\dim \phi = 0$ and $\dim \psi = 1$. Let $B$ denote $R[[t]]$, ordered so that $0 < t < \rho$ for all positive $\rho \in R$. The point $\phi$ is defined by setting $x(\phi) = t$ and $y(\phi) = e^t - 1$; $\psi$ is defined by setting $x(\psi) = t$ and $y(\psi) = \pi t$. It is easy to see that $\Gamma_{v_\phi} = Z = \Gamma_{v_\psi}$ and that $v_\phi(a) = \text{ord}_t(a(\phi))$ and $v_\psi(a) = \text{ord}_t(a(\psi))$. We have $\text{supp} \phi = \{0\} = \text{supp} \psi$. Now $K_\phi = k$ and so $\dim \phi = 0$. In contrast, $K_\psi = k(\pi)$ and $\dim \psi = 1$.

**2. Separating ideals.** The main result of this section is Proposition 2.2, which states a sufficient condition for the simplicity of a separating ideal. Before turning to this, we recall the definition and chief properties of separating ideals and some facts about simple ideals and successor ideals.
Let \( \alpha, \beta \in \text{Spec}_r A \). We say \( a \in A \) changes sign between \( \alpha \) and \( \beta \) if either: i) \( a(\alpha) \geq 0 \) and \( a(\beta) \leq 0 \), or ii) \( a(\alpha) \leq 0 \) and \( a(\beta) \geq 0 \). The separating ideal determined by \( \alpha \) and \( \beta \), denoted \( \langle \alpha, \beta \rangle \), is the ideal of \( A \) generated by the elements of \( A \) which change sign between \( \alpha \) and \( \beta \). In \([M]\), the following is shown:

**Proposition 2.1:** If \( A \) is any ring and \( \alpha, \beta \in \text{Spec}_r A \), then:

a) \( \langle \alpha, \beta \rangle \) is both an \( \alpha \)-ideal and a \( \beta \)-ideal. The orderings induced on \( A/\langle \alpha, \beta \rangle \) by \( \alpha \) and \( \beta \) are the same (and hence, the set of \( \alpha \)-ideals containing \( \langle \alpha, \beta \rangle \) is equal to the set of \( \beta \)-ideals containing \( \langle \alpha, \beta \rangle \)).

b) \( \langle \alpha, \beta \rangle \) is the smallest ideal of \( A \) with the properties in (1).

c) \( \sqrt{\langle \alpha, \beta \rangle} \) is the support of the least common specialization of \( \alpha \) and \( \beta \) in \( \text{Spec}_r A \).

We say that an ideal is simple if it is proper and cannot be expressed as a product of proper ideals. Using the notation at the beginning of section 1, if \( J \subseteq A \) is any ideal, then \( \overline{J} := \{ b \in A \mid \exists a \in J \ v(b) \geq v(a) \} \) is a \( v \)-ideal. This is proper if and only if \( J \) is contained in the center of \( v \). If \( I \) is a \( v \)-ideal and \( I = J_1 J_2 \), then also \( I = \overline{J_1} \overline{J_2} \). It follows that if \( A \) is local and the center of \( v \) is the maximal ideal of \( A \), then a \( v \)-ideal \( I \subseteq A \) is simple if and only if it is proper and cannot be expressed as a product of proper \( v \)-ideals.

If \( I \) is an \( \alpha \)-ideal (in a noetherian ring), its successor (with respect to the valuation \( v_\alpha \)) is denoted \( I^\alpha \). This is the largest \( \alpha \)-ideal properly contained in \( I \). In the proof of proposition 2.2 which we are about to give, we shall consider the successors of \( \langle \alpha, \beta \rangle \) with respect to the orderings \( \alpha \) and \( \beta \). It is possible for \( \langle \alpha, \beta \rangle^\alpha \) to equal \( \langle \alpha, \beta \rangle^\beta \) or for the two ideals to be distinct. If, for example, \( v_\alpha = v_\beta \) then the successors are equal, but equality may occur even when \( v_\alpha \) and \( v_\beta \) are distinct. Examples are given at the end of this section.

**Proposition 2.2:**

Let \( A \) be a noetherian ring, and suppose \( \alpha, \beta \in \text{Spec}_r A \) are centered at a maximal ideal \( m \subseteq A \) and \( \langle \alpha, \beta \rangle \subseteq m \). Let \( k := A/m \). If

\[ I/I^\alpha \cong k \text{ for all } \alpha\text{-ideals } I \text{ properly containing } \langle \alpha, \beta \rangle, \tag{*} \]

then \( \langle \alpha, \beta \rangle \) is simple.

**Proof.**

Pick \( x \in \langle \alpha, \beta \rangle \) satisfying the following conditions:

i) \( x(\alpha) \geq 0 \) and \( x(\beta) \leq 0 \),

ii) \( v_\alpha(x) = v_\alpha(\langle \alpha, \beta \rangle) \),

iii) \( v_\beta(x) = v_\beta(\langle \alpha, \beta \rangle) \).

Such an \( x \) exists because surely we can find \( x_\alpha \) satisfying (i) and (ii) and \( x_\beta \) satisfying (i) and (iii). Then \( x = x_\alpha + x_\beta \) satisfies all three conditions. Suppose \( \langle \alpha, \beta \rangle = GH \). Without loss of generality, \( G \) and \( H \) are \( \alpha \)-ideals. Since they contain \( \langle \alpha, \beta \rangle \), \( G \) and \( H \) are also \( \beta \)-ideals. Now it is possible to write \( x = \sum g_i h_i + e \) where \( g_i \in G \setminus G^\alpha = G \setminus G^\beta \), \( h_i \in H \setminus H^\alpha = H \setminus H^\beta \) and \( e \in \langle \alpha, \beta \rangle^\alpha \cap \langle \alpha, \beta \rangle^\beta \). Pick any \( g \in G \setminus G^\alpha \) and \( h \in H \setminus H^\alpha \). By (\*), \( g_i = u_i g + g'_i \) and \( h_i = v_i h + h'_i \) for some \( u_i, v_i \notin m \), \( g'_i \in G^\alpha \), \( h'_i \in H^\alpha \). Then we have
\[ x = \sum_i (u_i g + g'_i)(v_i h + h'_i) + e = gh \sum_i u_i v_i + E \]

where \( v_\alpha(E) > v_\alpha(\langle \alpha, \beta \rangle) \) and \( v_\beta(E) > v_\beta(\langle \alpha, \beta \rangle) \). (This implies that \( \sum_i u_i v_i \not\in \mathfrak{m} \).

We can conclude that \((x - E)(\alpha) > 0\) and \((x - E)(\beta) < 0\). Hence, one of the factors \( \sum_i u_i v_i, g \) or \( h \) must change sign between \( \alpha \) and \( \beta \). But this is impossible, because none of these elements is in \( \langle \alpha, \beta \rangle \).

Under the hypotheses in the first sentence of the proposition, \( k \subseteq K_\alpha \) and \( k \subseteq K_\beta \). (Recall that \( K_\alpha \) is the residue field of the valuation associated with \( \alpha \).) Sometimes the containments are proper. If \( I \) is an \( \alpha \)-ideal properly containing \( \langle \alpha, \beta \rangle \), then \( I/I^\alpha = I/I^\beta \) is a finitely generated sub-\( k \)-vector space of \( K_\alpha \) and of \( K_\beta \). Obviously, therefore, if \( k \)

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**Corollary 2.3:** If \( A \) is the real coordinate ring of a compact algebraic subset of \( \mathbb{R}^n \) (e.g. \( A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1) \)) then every proper separating ideal defined by points of Spec, \( A \) is simple.

**Proof.** In this case, every point of Spec, \( A \) is centered at a maximal ideal. Points centered at different maximal ideals have the unit ideal as separating ideal. /////

**Examples 2.** Define \( V(\alpha, \beta) := \langle \alpha, \beta \rangle / (\langle \alpha, \beta \rangle^\alpha \cap \langle \alpha, \beta \rangle^\beta) \). Under the assumptions occurring in proposition 2.2, \( V(\alpha, \beta) \) is a vector space over \( k \). The residues in this space of the elements of \( A \) satisfying \( a(\alpha) \geq 0 \) and \( a(\beta) \leq 0 \) make up a convex cone. We shall give some concrete illustrations. Let \( A = \mathbb{R}[x, y] \). We consider points of Spec, \( A \) induced by homomorphisms \( \alpha, \beta : A \rightarrow \mathbb{R}[t] \), where the latter ring is ordered so that \( 0 < t < \rho \) for all positive \( \rho \in \mathbb{R} \). The image of \( a \) under the map corresponding to \( \alpha \) is denoted \( a(\alpha) \), and \( v_\alpha(a) = \text{ord}_a(a(\alpha)) \).

- Suppose \( \alpha, \beta : A \rightarrow \mathbb{R}[t] \) are induced by letting \( x(\alpha) = t^2 \), \( y(\alpha) = t^4 + 2t^5 \), \( x(\beta) = t^2 \) and \( y(\beta) = t^4 - t^5 \). Let \( f = y - x^2 \). Then \( f(\alpha) = 2t^5 \) and \( f(\beta) = -t^5 \). It is easy to check that no polynomial \( g \) with \( v_\alpha(g) < 5 \) changes sign, so \( v_\alpha(\langle \alpha, \beta \rangle) = 5 \), and \( \langle \alpha, \beta \rangle = (f, xy, y^2) \). Moreover, if \( g \) changes sign and \( v_\alpha(g) = 5 \), then \( g = f + h \), where \( v_\alpha(h) > 5 \), as one can easily check. Hence \( \langle \alpha, \beta \rangle^\alpha = (x^3, xy, y^2) \) and \( V(\alpha, \beta) \) is one-dimensional. In this example \( \langle \alpha, \beta \rangle^\alpha = \langle \alpha, \beta \rangle^\beta \).

- Now let \( \gamma \) be induced by letting \( x(\gamma) = t^2 \) and \( y(\gamma) = t^4 + t^5 \). Let \( f_\lambda = (y - x^2)^2 - \lambda x^5 \). One may easily check that \( f_\lambda(\alpha) \geq 0 \) and \( f_\lambda(\gamma) \leq 0 \) if and only if \( \lambda \in [1, 4] \). In this case, \( v_\alpha(\langle \alpha, \gamma \rangle) = 10 \); \( \langle \alpha, \gamma \rangle^\alpha \) contains \( f_4 \), but \( \langle \alpha, \gamma \rangle^\gamma \) does not. \( V(\alpha, \gamma) \) is spanned by the residues of \( f_1 \) and \( f_4 \). The cone \( \{ \bar{a} \in V(\alpha, \gamma) \mid a \in \langle \alpha, \gamma \rangle, a(\alpha) \geq 0, a(\gamma) \leq 0 \} \) is equal to the set of non-negative-linear combinations of \( f_1 \) and \( f_4 \), i.e., \( \{ \mu f_\lambda \mid \mu, \lambda \in \mathbb{R}, 0 \leq \mu, 1 \leq \lambda \leq 4 \} \).

**3. Quadratic transforms.** We recall some information about quadratic transforms. Let \( A \) be a regular local (noetherian) domain with maximal ideal \( \mathfrak{m} \) and residue field \( k \). A **quadratic transform** of \( A \) is a local ring \( B = (A[x^\mathfrak{m}])_p \), where \( \text{ord}_A(x) = 1 \) (here, and below, \( \text{ord}_A(x) = \max\{ n \mid x \in \mathfrak{m}^n \} \)) and \( p \) is a prime ideal of \( A[x^\mathfrak{m}] \) such that
\( \mathfrak{m} \subseteq \mathfrak{p} \). If \( B \) is a quadratic transform of \( A \), we write \( A \prec B \). (Geometrically, if \( A \) is the local ring of a non-singular point \( M \) on a variety \( X \) and \( \tilde{X} \to X \) is the blow-up with center \( M \), then \( B \) is the local ring of a point \( P \) (possibly not closed) in the inverse image of \( \{M\} \).) Of course, if \( A \) is one-dimensional, than \( B = A \).

Let \( v \) be a valuation on \( A \). We say that \( v \) dominates \( A \) if \( v \) is non-negative on \( A \) and \( \text{ctr}_A v = \mathfrak{m} \). If \( v \) is non-trivial (recall that this means \( \text{ctr}_A v \neq \supp_A v \)), then the transform of \( A \) along \( v \) is defined to be the ring \( B = S^{-1}A[x^{-1}\mathfrak{m}] \), where \( x \) is an element of \( \mathfrak{m} \) of minimum value and \( S = \{ a \in A[x^{-1}\mathfrak{m}] \mid v(a) = 0 \} \). It can be shown that \( B \) is a regular local domain independent of the choice of \( x \) and that \( v \) dominates \( B \). Hence the process may be repeated indefinitely, and it continues to produce proper extensions until a discrete valuation ring is obtained, if this ever occurs. We frequently use the notation \( A = A^{(0)} \prec A^{(1)} \prec \cdots \) to denote the sequence of transforms along a given valuation. For more details, see [A]. If \( \alpha \in \text{Spec}_r A \) is centered at \( \mathfrak{m} \) and \( \supp \alpha \neq \mathfrak{m} \), then the quadratic transformation along \( \alpha \) is defined by the valuation \( v_\alpha \).

Transforms of ideals in \( A \) are defined as follows. Suppose \( I \subseteq A \) is an ideal with \( \text{ord}_A(I) = r \). If \( a \in I \), then \( x^{-r}a \in A[x^{-1}\mathfrak{m}] \). Hence, \( IA[x^{-1}\mathfrak{m}] = x^rI' \) for some ideal \( I' \subseteq A[x^{-1}\mathfrak{m}] \). This ideal is called the transform of \( I \) in \( A[x^{-1}\mathfrak{m}] \) and \( I'B \) is called the transform of \( I \) in \( B \). The transform of \( I \) in \( B \) is also denoted \( T_B(I) \), or \( T(B) \) if \( B \) is clear from context.

Certain facts regarding the real spectrum of a quadratic transform of a local ring will be needed. Suppose \( A \subseteq B \) are any rings. There is a functorial map \( \pi : \text{Spec}_r B \to \text{Spec}_r A \), where \( \pi(\gamma) := \gamma \cap A \) and \( \supp \pi(\gamma) = A \cap \supp \gamma \). In general, this is neither injective nor surjective, even when \( B \) is a quadratic transform of \( A \).

**Lemma 3.1:** Let \( A \) be a regular local domain with maximal ideal \( \mathfrak{m} \), and let \( \alpha \in \text{Spec}_r A \). Let \( B = (A[x^{-1}\mathfrak{m}])_\mathfrak{p} \) be a quadratic transform of \( A \). If \( \supp \alpha \subseteq \mathfrak{p} \cap A \) and \( \supp \alpha \neq \mathfrak{m} \), then there is a unique \( \gamma \in \text{Spec}_r B \) with \( \gamma \cap A = \alpha \).

**Proof.** By standard commutative algebra, if \( \mathfrak{q} \subseteq \mathfrak{p} \cap A \) is any prime ideal in \( A \) and \( \mathfrak{q} \neq \mathfrak{m} \), then there is a unique prime \( \mathfrak{p}' \subseteq B \) such that \( \mathfrak{q} = \mathfrak{p}' \cap A \). Note that \( x \notin \mathfrak{p}' \), and thus \( A_{\mathfrak{q}} = B_{\mathfrak{p}'} \). This applies in particular to \( \mathfrak{q} = \supp \alpha \), provided that \( \alpha \) satisfies the hypotheses of the lemma. If \( A_{\supp \alpha} = B_{\mathfrak{p}'} \) then the fraction fields of the domains \( A/\supp \alpha \subseteq B/\mathfrak{p}' \) are equal. Because the orderings of a domain are in one-to-one correspondence with the orderings of its fraction field, this shows that any \( \alpha \) satisfying the hypotheses of the lemma has a unique lift to \( B \).

If \( \alpha \in \text{Spec}_r A \) satisfies the conditions of the lemma, we say that \( \alpha \) lifts to \( B \). The unique preimage of \( \alpha \) in \( \text{Spec}_r B \), when such exists, is denoted \( T_B(\alpha) \), or simply \( T(\alpha) \) if \( \alpha \) is clear from context.

**Lemma 3.2:** Suppose \( \alpha, \beta \in \text{Spec}_r A \) are non-trivial and centered at \( \mathfrak{m} \) and that \( \langle \alpha, \beta \rangle \) is properly contained in \( \mathfrak{m} \). Let \( B \) be the first quadratic transform of \( A \) along \( \alpha \). Then:

a) \( \alpha \) and \( \beta \) both lift to \( B \),

b) \( T(\langle \alpha, \beta \rangle) \subseteq \langle T(\alpha), T(\beta) \rangle \).

c) If \( A \) is of dimension 3, the containment in b) may be proper.
Proof. Because $\langle \alpha, \beta \rangle \neq m$, neither supp $\alpha$ nor supp $\beta$ is $m$. Moreover, an element of $A$ has minimal non-zero $v_\alpha$-value if and only if it has minimal non-zero $v_\beta$-value. Pick $x \in A$ with minimal non-zero value with respect to both valuations, so $B = S^{-1}A[x^{-1}m]$, where $S = \{ a \in A[x^{-1}m] \mid v_\alpha(a) = 0 \}$. By 3.1, to prove (a) it is enough to note that $v_\alpha(a) \neq 0$ for all $a \in \text{supp } \beta$; this is clear, since $\text{supp } \beta$ contains no units.

Note that $\alpha$ and $\beta$ both lift to points of $\text{Spec}_r A[x^{-1}m]$, call them $\alpha'$ and $\beta'$. (Elements of $A[x^{-1}m]$ are of the form $f/x^n$, where $f \in m^a$.) To prove (b), let $\langle \alpha, \beta \rangle'$ denote the transform of $\langle \alpha, \beta \rangle$ in $A[x^{-1}m]$. Then $\langle \alpha, \beta \rangle'$ is generated by elements of $A[x^{-1}m]$ of the form $x^{-n}a$, where $a(\alpha) \geq 0$ and $a(\beta) \leq 0$. Any such element belongs to $\langle \alpha', \beta' \rangle$, since $x$ does not change sign between $\alpha$ and $\beta$. Hence, $\langle \alpha, \beta \rangle' \subseteq \langle \alpha', \beta' \rangle$, so $T(\langle \alpha, \beta \rangle') = \langle \alpha, \beta \rangle' B \subseteq \langle T(\alpha), T(\beta) \rangle$.

The example needed for the last assertion is given below (3.e).

The proof just given shows, in fact, that if $B = (A[x^{-1}m])_p$ is any first quadratic transform of $A$ and if $x \notin \langle \alpha, \beta \rangle$, then $T(\langle \alpha, \beta \rangle) \subseteq \langle T(\alpha), T(\beta) \rangle$ provided the lifts $T(\alpha)$ and $T(\beta)$ exist.

In dimension two, the transforms of separating ideals behave very well, as we show in the next section.

Examples 3. In these examples, we use $A'$ to denote a quadratic transform of $A$. Also, we write $I'$ in place of $T(I)$ to denote the ideal transform, $\alpha'$ to denote the lift of $\alpha$, and so forth. This saves space and is easier to read.

a. Let $A$ be the localization of $R[x, y]$ at the origin. We compute the transforms $A \prec A' \prec A'' \prec \cdots$ along the order $\alpha$, where $\alpha$ is determined as in the examples at the end of Section 2 by letting $x(\alpha) = t^2$ and $y(\alpha) = t^4 + 2t^5$. We shall also examine the lifts of the points $\beta$ and $\gamma$, which, as in the previous example, are defined by $x(\beta) = t^2$, $y(\beta) = t^4 - t^5$ and $x(\gamma) = t^2$, $y(\gamma) = t^4 + t^5$.

If we let $x' := x$ and $y' := y/x$, then $A'$ is the localization of $R[x', y']$ at the maximal ideal $(x', y')$. Then $\alpha'$, $\beta'$, and $\gamma'$ are defined by

$$x'(\alpha') = x'(\beta') = x'(\gamma') = t^2, \quad y'(\alpha') = t^2 + 2t^3, \quad y'(\beta') = t^2 - t^3 \quad \text{and} \quad y'(\gamma') = t^2 + t^3.$$ 

It is easy to check that $v_\alpha(\langle \alpha', \beta' \rangle) = 3$ and hence $\langle \alpha', \beta' \rangle = (y' - x', x^2) = \langle \alpha, \beta \rangle'$. Also, $v_\alpha(\langle \alpha', \gamma' \rangle) = 6$, and hence $\langle \alpha', \gamma' \rangle = (x', y'/x') = (\alpha, \gamma)'$. The transform $A''$ of $A'$ along $\alpha'$ is the localization of $R[x'', y'']$ at $(x'', y'')$ where $x'' := x'$ and $y'' := y'/x' - 1$. The lifts of $\alpha'$, $\beta'$ and $\gamma'$ are defined by $x''(\alpha'') = x''(\beta'') = x''(\gamma'') = t^2$, and $y''(\alpha'') = 2t$,

$$y''(\beta'') = -t \quad \text{and} \quad y''(\gamma'') = t.$$ 

Now, $v_\alpha(\langle \alpha'', \beta'' \rangle) = 1$ and hence $\langle \alpha'', \beta'' \rangle = (a'', y'') = \langle \alpha, \beta \rangle''$. Also, $v_\alpha(\langle \alpha'', \gamma'' \rangle) = 2$ and from this one easily deduces $\langle \alpha'', \gamma'' \rangle = (y'', x''/t) = \langle \alpha, \gamma \rangle''$.

$A'''$ is the localization of $R[x''', y''']$ at $(x''', y''')$ where $x''' := x''/y''$ and $y''' := y''$. One shows easily that $\langle \alpha''', \gamma''' \rangle = (x''', y''') = \langle \alpha, \gamma \rangle'''$.

b. We illustrate the remark about the non-uniqueness of points of $\text{Spec}_r A'$ which contract to a given point of $\text{Spec}_r A$ with support $m$. Let $\delta^+, \delta^- \in \text{Spec}_r A'$ be given by $x'(\delta^+) = x'(\delta^-) = 0$, $y'(\delta^+) = t$ and $y'(\delta^-) = -t$. Then $\pi(\delta^+) = \pi(\delta^-)$ is the unique point of $\text{Spec}_r A$ supported at $m$.

c. An example in which $A$ is a two-dimensional regular local ring and $A'$ is a discrete valuation ring may be obtained as follows. Let $k$ be the real algebraic numbers, and let $A$
be the localization of \( k[x, y] \) at the origin. Suppose \( \alpha \) is determined by a homomorphism to \( R[[t]] \) (ordered as in the example at the end of Section 1) with \( x(\alpha) = t \) and \( y(\alpha) = \pi t \). In this case, \( v_\alpha = \text{ord}_A : A' \) is the localization of \( k(\pi)[x] \) at \( x = 0 \).

\[ \textbf{d.} \] If \( \langle \alpha, \beta \rangle = m \), it is possible that \( \langle \alpha', \beta' \rangle \subseteq m_1 \) (the maximal ideal of \( A' \)), showing that \( \langle \alpha, \beta' \rangle \not\subseteq \langle \alpha', \beta' \rangle \) is possible when the hypothesis in 3.2 is not met. Let \( A \) be the localization of \( R[x, y] \) at the origin and let \( \alpha \) and \( \beta \) be determined by letting \( x(\alpha) = t \), \( y(\alpha) = t^2 \), \( x(\beta) = -t \) and \( y(\beta) = -t^2 \). Then \( \langle \alpha, \beta \rangle = m \). As \( x'(\alpha') = t \), \( y'(\alpha') = t \), \( x'(\beta') = -t \) and \( y'(\beta') = t \), we see that \( \langle \alpha', \beta' \rangle = m_1 \). The points \( \alpha \) and \( \beta \) may be thought of as having a common tangent, but as determining different directions along that tangent, which explains geometrically why these points retain a common center after a quadratic transform. These points do not retain a common center after another transform, however.

\[ \textbf{e.} \] The following example was found using a computer program by Alvis which searches systematically for polynomials which change sign between given orders. The details will appear elsewhere. We take \( A \) to be the localization of \( R[x, y, z] \) at the origin. For any real number \( u \), let \( \gamma_u \) be determined by letting

\[
\begin{align*}
x(\gamma_u) &= t^6 \\
y(\gamma_u) &= t^{10} + ut^{11} \\
z(\gamma_u) &= t^{14} + t^{15}.
\end{align*}
\]

We examine the ideal \( I := \langle \gamma_1, \gamma_3 \rangle \). Let \( f := z^2 - x^3 y \). Then \( v_{\gamma_1}(f) = 29 \). Exhaustive search shows that \( f \) has minimal \( v_{\gamma_1} \)-value among polynomials which change sign between \( \gamma_1 \) and \( \gamma_3 \). Knowing this, it is routine to find a list of generators for \( I \); we find \( I = \langle f, x^3, x^4 y, x^3 z, x^2 y^2, x y z, x z^2, y^3, y^2 z, y z^2, z^3 \rangle \) (with some obvious redundancy in the list of generators). Clearly \( \text{ord}_A(I) = 2 \). Let \( x' = x \), \( y' = y/x \) and \( z' = z/x \). The quadratic transform of \( A \) along \( \gamma_1 \), \( A' \), is the localization of \( R[x', y', z'] \) at the origin. The transform \( I' \) of \( I \) is not a valuation ideal, so it is clearly not a separating ideal. But the situation is even more complicated. If \( v \) is the valuation induced by \( \gamma_1 \), we have \( v(I') = 17 \), but \( v(\langle \gamma_1', \gamma_3' \rangle) = 13 \), since \( 2y'z' - x'^2 - y'^3 \) changes sign between \( \gamma_1' \) and \( \gamma_3' \). (This shows that it is not possible to obtain an equality by using the “complete transform” defined in [L] in place of the ideal transform we have used. The strange behavior of this ideal seems to be related to the fact that it is not finitely supported—see [L] for the meaning of this term.)

4. Two-dimensional rings. We begin this section by recalling the results from Zariski’s theory of valuation ideals in two-dimensional regular local rings which we will be using. We make no attempt to give any indication of the algebra required to derive these and refer the reader to [Z], [ZS] or [Hu] for details.

In this section, we make the following assumptions:

\[ \begin{align*}
i) & \ A = (A, m, k) \text{ is a two-dimensional regular local domain. (Zariski’s theory requires no special assumptions about the residue field } k, \text{ but for our purposes we shall eventually need to assume } k \text{ is real closed.)} \\
ni) & \ v \text{ is a valuation centered at } m \text{ which is non-trivial (i.e., } \text{cntr } v \neq \text{ supp } v). 
\end{align*} \]
The following notational conventions are used in this section. Assume that \( A = A^{(0)} \prec A^{(1)} \prec \ldots \) is the sequence of quadratic transforms along \( v \). The maximal ideal and residue field of \( A^{(i)} \) are denoted \( m^{(i)} \) and \( k^{(i)} \), respectively. Let \( \{ I_i \mid i = 0, 1, 2, \ldots \} \) be the initial segment of the sequence of \( v \)-ideals of \( A \) \((I_0 = A, I_1 = m)\). Similarly, let \( \{ J_i \mid i = 0, 1, 2, \ldots \} \) be the initial segment of the sequence of \( v \)-ideals of \( A^{(1)} \). Also, let \( \{ I_i = I_{n_i} \mid i = 0, 1, \ldots \} \) be the subsequence of \( \{ I_i \} \) consisting of the simple \( v \)-ideals of \( A \) (with \( I_0 = m \)), and let \( \{ J_i = J_{n_i} \mid i = 0, 1, \ldots \} \) be defined similarly.

Let \( T \) be the ideal transform operation corresponding to the passage from \( A \) to \( A^{(1)} \). The inverse transform, denoted \( W \), is defined as follows. Suppose \( B \) is a first quadratic transform of \( A \) and \( J \subseteq B \) is an ideal. Since \( J \) is finitely generated, there is an integer \( n \) such that \( x^n J = IB \) for some ideal \( I \subseteq A \). The inverse transform of \( J \) is the ideal \( W(J) = x^{n_0} J \cap A \), where \( n_0 \) is the least such integer. Observe that \( T(W(J)) = J \), but in general \( W(T(I)) \supseteq I \) only. (This definition for \( W \) seems to be appropriate only in dimension 2. For a generalization, see [L], proof of 2.3.)

4.1: (See [ZS], p. 390.) If \( v \neq \text{ord}_A \), then:

a) The transform in \( A^{(1)} \) of any \( I_i \) is a member of the sequence \( \{ J_i \} \).

b) The inverse transform of any \( J_i \) is a member of the sequence \( \{ I_i \} \).

c) Any \( I_i \) is of the form \( m^h J \), where \( J \) is the inverse transform of some \( J_j \).

d) If \( W(I_i) \subseteq W(J_j) \), then \( I_i \subseteq J_j \).

If \( \text{tr.deg.}(K_v/k) = 1 \), then \( v \) is said to be a prime divisor. For example, if \( B \) is obtained from \( A \) by a finite sequence of quadratic transforms, then \( \text{ord}_B \) is a prime divisor, whose center in \( A \) is \( m \).

4.2: (See [ZS], p. 391–2.) The sequence \( \{ I_i \} \) is finite if and only if \( v \) is a prime divisor. If there are exactly \( n \) \( m \)-primary simple \( v \)-ideals \( \{ I_0, \ldots, I_{n-1} \} \), then the following are true:

a) \( v = \text{ord}_{A^{(n-1)}} \).

b) \( A^{(n)} \) is a discrete valuation ring and \( K_v = k^{(n)} \).

c) (See [ZS], p. 363.) \( k^{(i)} \) is algebraic over \( k^{(i-1)} \) for \( 1 \leq i \leq n-1 \), while \( k^{(n)} \) is a simple transcendental extension of \( k^{(n-1)} \) (indeed, \( k^{(n)} = k^{(n-1)}(x^t) \) for any generators \( s \) and \( t \) of \( m^{(n-1)} \)).

In general, \( I_m \supseteq I_n \) does not imply that \( T(I_m) \supseteq T(I_n) \), but the following shows that \( T \) is a one-to-one order-preserving map from the initial sequence of simple \( v \)-ideals of \( A \) (other than \( m \)) to the initial sequence of simple \( v \)-ideals of \( A^{(1)} \). Let \( T^{(n)} \) denote the iterated transform along \( v \) (so, if \( I \subseteq A \), then \( T^{(n)}(I) \subseteq A^{(n)} \)).

4.3: (See Theorem 4.1 and [ZS], p. 388-9; also [Hu], Remark 3.8.)

a) If \( v \neq \text{ord}_A \), then for \( i \geq 0 \), \( T(I_{i+1}) = J_i \) and \( W(J_i) = I_{i+1} \). Thus \( T^{(n)}(I_n) = m^{(n)} \subseteq A^{(n)} \).

b) Let \( P \subseteq A \) be any \( m \)-primary simple complete ideal. Then \( P \) uniquely determines an integer \( h \) and a sequence of quadratic transforms \( A = B^{(0)} \prec \cdots \prec B^{(h)} \) such that \( T_B^{(h)}(P) \) is the maximal ideal of \( B^{(h)} \), and \( P \) is an \( \text{ord}_{B^{(h)}} \)-ideal. If \( v \) is any valuation
centered on \( A \) for which \( \mathcal{P} \) is a \( v \)-ideal, then the sequence \( A = B^{(0)} \prec \ldots \prec B^{(h)} \) is the initial part of the sequence of transforms along \( v \).

To our previous assumptions, we now add the following:

\( ii' \) \( \) \( \alpha, \beta \in \text{Spec}_v A \), \( \text{cntr} \alpha = m = \text{cntr} \beta \) and \( \langle \alpha, \beta \rangle \) is properly contained in \( m \). (This ensures that \( v_\alpha \) and \( v_\beta \) satisfy assumption (ii), above.) \( A = A^{(0)} \prec A^{(1)} \prec \ldots \) will denote the sequence of quadratic transforms along \( v_\alpha \).

**Theorem 4.4:** Suppose \( k \) is real closed and \( \dim \alpha = 1 \) (so there are only finitely many simple \( m \)-primary \( \alpha \)-ideals). If \( \alpha \neq \beta \), then \( \langle \alpha, \beta \rangle \) contains the smallest \( m \)-primary simple \( \alpha \)-ideal of \( A \).

**Proof.** Suppose the smallest \( m \)-primary simple \( \alpha \)-ideal of \( A \) is \( \mathcal{I}_{n-1} \) and suppose that \( \mathcal{I}_{n-1} \) is not subset of \( \langle \alpha, \beta \rangle \). We shall show that \( \alpha = \beta \). Since \( \langle \alpha, \beta \rangle \) is an \( \alpha \)-ideal, \( \langle \alpha, \beta \rangle \) is a proper subset of \( \mathcal{I}_{n-1} \). By 2.1.a, each of the ideals \( \mathcal{I}_0, \ldots, \mathcal{I}_{n-1} \) is a \( \beta \)-ideal. Hence, by 4.3.b, \( A = A^{(0)} \prec A^{(1)} \prec \ldots \prec A^{(n-1)} \) is the initial part of the sequence of quadratic transforms along \( v_\beta \). By 4.2, \( v_\alpha = \text{ord}_{A^{(n-1)}} \), and \( K_\alpha = k(\frac{\alpha}{\beta}) \) for any for any generators \( s, t \) of \( m^{(n-1)} \). We may choose \( s \) and \( t \) to be of the form \( s = \frac{a}{u} \) and \( t = \frac{b}{u} \) with \( s_1, t_1 \in \mathcal{I}_{n-1} \setminus (\mathcal{I}_{n-1})^\alpha \), and we can at same time, arrange that the transform of \( A^{(n-1)} \) along \( \beta \) is a localization of \( A^{(n-1)}[\frac{a}{t_1}] = A^{(n-1)}[\frac{a_1}{t_1}] \).

Let \( H_\alpha = \{ h \in A \mid \text{res}_\alpha h <_\alpha \text{res}_\alpha \frac{a}{t_1} \} \) and \( H_\beta = \{ h \in A \mid \text{res}_\beta h <_\beta \text{res}_\beta \frac{a}{t_1} \} \). (Here, \( \text{res}_\alpha : A \to k_\alpha \) is the natural map, and \( \text{res}_\beta \) is defined similarly; the definition of \( k_\alpha \) was given just after 1.1.) These sets, we claim, are equal. If not—and if \( h_0 \) belongs to the symmetric difference—then \( s_1 - h_0 t_1 \) changes sign between \( \alpha \) and \( \beta \), and therefore \( v_\alpha(s_1 - h_0 t_1) \geq v_\alpha(\langle \alpha, \beta \rangle) \). Now \( \frac{a}{t_1} - h_0 \not\in m_\alpha \) (since \( \text{res}_\alpha(\frac{a}{t_1}) \not\in k \)), so \( v_\alpha(\frac{a}{t_1} - h_0) = 0 \), and thus \( v_\alpha(s_1 - h_0 t_1) = v_\alpha(t_1) = v_\alpha(\mathcal{I}_{n-1}) \). Accordingly, \( v_\alpha(\mathcal{I}_{n-1}) \geq v_\alpha(\langle \alpha, \beta \rangle) \) and therefore \( \mathcal{I}_{n-1} \subseteq \langle \alpha, \beta \rangle \), contrary to assumption. Therefore, \( H_\alpha = H_\beta \). Since \( k \) is cofinal in \( K_\beta \), \( \text{res}_\beta \frac{a}{t_1} \) is in \( k \) and therefore \( \dim \beta = 1 \), indeed by 4.2.c, \( v_\beta = \text{ord}_{A^{(n-1)}} \). Since \( k \) is real closed, any ordering of a simple transcendental extension \( k(\xi) \) of \( k \) is completely determined by the set \( \{ b \in k \mid b < \xi \} \). From this it follows that \( K_\alpha \) and \( K_\beta \) are not only the same field but that the orderings induced on this field by \( \alpha \) and \( \beta \) are the same. Since \( \alpha \) and \( \beta \) induce the same valuation and the induced orderings of the residue field are identical and since the value group is the group of integers, there are exactly two possibilities (as shown, for instance, in [Br]): \( \alpha = \beta \) or \( \beta \) is induced by choosing any generator of \( m^{(n)} = m_\alpha \in A^{(n)} = V_\alpha \) and assigning it the opposite sign from that which it has in \( \alpha \). In the latter case, since \( s \) generates \( m_\alpha \), either \( s_1 \) or \( u \) must change sign between \( \alpha \) and \( \beta \). But this implies that \( \mathcal{I}_{n-1} \subseteq \langle \alpha, \beta \rangle \), contrary to assumption. The only possibility is \( \alpha = \beta \).

//\ //

**Lemma 4.5:** Under hypotheses (i) and (ii') and assuming that \( k \) is real closed, the condition:

\[
I/I^\alpha \cong k \text{ for all } \alpha \text{-ideals } I \text{ properly containing } \langle \alpha, \beta \rangle,
\]

is satisfied. In particular, under these hypotheses, \( \langle \alpha, \beta \rangle \) is simple.
Proof. Since $k$ has no orderable algebraic extension, it follows from 4.2.c that $I/I^\alpha \cong k$ provided that $I$ contains a simple $\alpha$-ideal. This is clearly the case if $\alpha$ is not a prime divisor, since then there are simple $\alpha$-ideals of arbitrarily large value. The case when $\alpha$ is a prime divisor is handled by 4.4.\\ As in Section 3, let $T(\alpha)$ denote the lift of $\alpha$ to a point of $\text{Spec}_r A^{(1)}$. By Lemma 3.2, $T(\beta)$ exists.

**Lemma 4.6:** Under hypotheses (i) and (ii') and assuming the condition (\ast) of 2.2 and 4.5,
\[
W(\langle T(\alpha), T(\beta) \rangle) \subseteq \langle \alpha, \beta \rangle.
\]

**Proof.** Let $I = W(\langle T(\alpha), T(\beta) \rangle)$. If $I/I^\alpha \not\cong k$, then the conclusion follows from (\ast). Otherwise, argue as follows. Since $\langle \alpha, \beta \rangle \neq m$, $x$ (being of minimal non-zero $v_\alpha$-value) does not change sign between $\alpha$ and $\beta$. Pick $s \in \langle T(\alpha), T(\beta) \rangle$ which changes sign between $T(\alpha)$ and $T(\beta)$ and is of minimal $v_\alpha$-value with this property. Then (referring to the notation at the end of the paragraph preceding 4.1), $x^m s = wu$ for some $w \in I$ and some unit $u \in A^{(1)}$ (since $I/I^\alpha \cong k$). Because neither $x$ nor $u$ changes sign between $T(\alpha)$ and $T(\beta)$, $w$ must change sign. Hence $w \in \langle \alpha, \beta \rangle$. This shows that $v_\alpha(I) = v_\alpha(s) + n_0 \geq v_\alpha(\langle \alpha, \beta \rangle)$, which implies $I \subseteq \langle \alpha, \beta \rangle$.\\ **Theorem 4.7:** Suppose $A = (A, m, k)$ is a two-dimensional regular local domain with real closed residue field $k$. Also suppose $\alpha, \beta \in \text{Spec}_r A$, $\text{cntr} \alpha = m = \text{cntr} \beta$ and $\langle \alpha, \beta \rangle$ is properly contained in $m$. Then:

a) $T(\langle \alpha, \beta \rangle) = \langle T(\alpha), T(\beta) \rangle$,

b) $W(\langle T(\alpha), T(\beta) \rangle) = \langle \alpha, \beta \rangle$.

**Proof.** By Lemmas 3.2, 4.5 and 4.6,
\[
T(\langle \alpha, \beta \rangle) \subseteq \langle T(\alpha), T(\beta) \rangle \quad \text{and} \quad W(\langle T(\alpha), T(\beta) \rangle) \subseteq \langle \alpha, \beta \rangle.
\]
Our assumptions guarantee that $v_\alpha$ is not the order valuation. Applying $W$ and $T$ to the containments we have and using 2.2 and 4.3, we get
\[
\langle \alpha, \beta \rangle \subseteq W(\langle T(\alpha), T(\beta) \rangle) \quad \text{and} \quad \langle T(\alpha), T(\beta) \rangle \subseteq T(\langle \alpha, \beta \rangle).
\]

5. Applications. Throughout this section, $(A, m)$ is a 2-dimensional regular local ring with real closed residue field. As an immediate corollary of 4.7, we have

**Proposition 5.1:** An $m$-primary ideal of $A$ is a separating ideal if and only if it is a simple complete ideal.

**Proof.** One direction is immediate from 4.5 and 2.2. Suppose that $P$ is a simple complete ideal, and $A = B^{(0)} \prec \cdots \prec B^{(h)}$ is the sequence of transforms in 4.3.b. The maximal ideal of $B^{(h)}$ is a separating ideal—for example, let $u$ and $v$ be a pair of generators for this ideal and let $\alpha$ and $\beta$ have as their supports the ideals $(u)$ and $(v)$, respectively. Note that $\alpha$ and $\beta$ are the lifts of their restrictions to $B^{(i)}$ for any $0 \leq i < h$. Applying 4.7 repeatedly shows that $P = \langle \alpha \cap A, \beta \cap A \rangle$.\\
Two different proofs of the Pierce-Birkhoff conjecture for smooth surfaces defined over an arbitrary real closed field can be given based on the results of section 4. In [MR], a proof based on the following approximation theorem is given:

**Proposition 5.2:** Suppose \( \langle \alpha, \beta \rangle \) is \( m \)-primary. Let \( X, Y \subseteq \text{Spec}_r A \) be closed constructible sets with \( \alpha \in X \) and \( \beta \in Y \). Then there are points \( \alpha' \in X \) and \( \beta' \in Y \) with \( \dim \text{supp} \alpha' = \dim \text{supp} \beta' \leq 1 \) and \( \langle \alpha', \beta' \rangle = \langle \alpha, \beta \rangle \).

Actually, this is not difficult to prove directly when the residue field of \( A \) is the real numbers, in which case one is assured that \( K_\alpha = \mathbb{R} \). In general, however, \( K_\alpha \) may be a transcendental extension of the residue field of \( A \). [MR] first proves 5.2 in the special case that \( \langle \alpha, \beta \rangle \) is maximal. To prove the general case, quadratic transforms are applied until the separating ideal is maximal, the special case is invoked, and then the solution is transported back to the original ring via 4.7.

In [MS], a different and more abstract approach to the Pierce-Birkhoff conjecture for smooth surfaces is given. Here, 4.7 is used to prove:

**Proposition 5.3:** Suppose that \( \phi \) is an abstract semialgebraic function on \( \text{Spec}_r A \) with the property that for all \( \gamma \in \text{Spec}_r A \), there is \( a \in A \) with \( \phi(\gamma) = a(\gamma) \), i.e., \( \phi \) is an “abstract piecewise polynomial”. Let \( \alpha \) and \( \beta \) be any points of \( \text{Spec}_r A \). Suppose \( \phi(\alpha) = f(\alpha) \) and \( \phi(\beta) = g(\beta) \) for some \( f, g \in A \). Then \( f - g \in \langle \alpha, \beta \rangle \).

In the terminology of [M], 5.3 says that \( A \) is a “Pierce-Birkhoff ring”. In proving 5.3, the only cases that require any work are when \( \langle \alpha, \beta \rangle \) is \( m \)-primary. As with 5.2, this is handled applying quadratic transforms to simplify the problem, and then transporting information back to the original, untransformed ring by means of 4.7. From 5.3, [MS] deduces that any regular two-dimensional algebra over a real closed field is Pierce-Birkhoff. This result includes as a special case the Pierce-Birkhoff conjecture for smooth surfaces. Another result to be presented in [MS] depends upon defining abstractly the direction of \( \alpha \) in the Zariski tangent space at \( \text{cntr} \alpha \). It is shown, using 4.7, that \( \langle \alpha, \beta \rangle \) is the simple ideal corresponding to the first of the transforms along \( \alpha \) in which the abstract directions of \( \alpha \) and \( \beta \) are distinct.
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Suggested shortened version of title:
Complete ideals defined by sign conditions