Operators Associated with Soft and Hard Spectral Edges from Unitary Ensembles

Gordon Blower

Department of Mathematics and Statistics, Lancaster University
Lancaster, LA1 4YF, England, UK. E-mail: g.blower@lancaster.ac.uk

27th April 2006

Abstract. Using Hankel operators and shift-invariant subspaces on Hilbert space, this paper develops the theory of the operators associated with soft and hard edges of eigenvalue distributions of random matrices. Tracy and Widom introduced a projection operator $W$ to describe the soft edge of the spectrum of the Gaussian unitary ensemble. The subspace $WL^2$ is simply invariant under the translation semigroup $e^{itD}$ ($t \geq 0$) and invariant under the Schrödinger semigroup $e^{it(D^2+x)}$ ($t \geq 0$); these properties characterize $WL^2$ via Beurling’s theorem. The Jacobi ensemble of random matrices has positive eigenvalues which tend to accumulate near to the hard edge at zero. This paper identifies a pair of unitary groups that satisfy the von Neumann–Weyl anti-commutation relations and leave invariant certain subspaces of $L^2(0,\infty)$ which are invariant for operators with Jacobi kernels. Such Tracy–Widom operators are reproducing kernels for weighted Hardy spaces, known as Sonine spaces. Periodic solutions of Hill’s equation give a new family of Tracy–Widom type operators.

MSC 2000: 15A52 (47B35, 60E15)

Keywords: Random matrices; GUE; Hankel operators; Sonine spaces, Hill’s equation

1 Introduction

This paper concerns the spectral theory and invariant subspaces of operators that arise in random matrix theory, particularly the soft and hard edges that occur on the limiting eigenvalue distributions of the Gaussian and Jacobi unitary ensembles. Tracy and Widom [28, 29, 30] introduced various operators to describe the soft edge of the spectrum of the Gaussian unitary ensemble; that is, the eigenvalues near to the supremum of the support of the equilibrium distribution. Burnol proposed that the theory of random matrices should be expressed in terms of Sonine spaces [7, p 692]. Here we develop this theory in a systematic manner to show that the Tracy–Widom calculations are instances of more general results on Hankel operators, and introduce new settings where the theory applies.
In section 2 we consider operators on $L^2(\mathbb{R})$ with kernels

$$W(x, y) = \frac{A(x)B(y) - B(x)A(y)}{x - y}$$

(1.1)

where $\text{col } [A, B]$ satisfies a first-order linear differential equation, and give sufficient conditions for $W$ to be the square of a Hankel operator. Further, we show that the determinants $\det(I - zW)$ are related to the solutions of Marchenko integral equations. As we show in section 3, kernels such as $W$ arise as reproducing kernels for weighted Hardy spaces on the upper half-plane $\mathbb{C}_+ = \{ z : \Re z > 0 \}$ as in [2, 6].

Let $J$ be the ‘flip’ map $Jf(x) = f(-x)$, and $M_u$ the multiplication operator $f \mapsto uf$. The classical Hardy space $H^2$ consists of the holomorphic functions $F$ on $\mathbb{C}_+$ such that $\sup_{y > 0} \int_{-\infty}^{\infty} |F(x + iy)|^2 dx < \infty$, and we identify such a function with its $L^2$ boundary values. The Fourier transform is $\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \frac{dx}{\sqrt{2\pi}}$. Given $u \in L^\infty$, the bounded linear operator $\sqrt{2\pi}\mathcal{F}^* M_u \mathcal{F}^*$ is the Hankel operator $\Gamma_u$ on $L^2(\mathbb{R}_+)$ that has distributional kernel $\mathcal{F}^* u(x + y)$ as in [23].

We recall how examples of such operators appear in the theory of the Gaussian unitary ensemble. Let $x_{j,k}$ and $y_{j,k}$ ($1 \leq j \leq k \leq n$) be a family of mutually independent $N(0, 1/n)$ random variables. We let $X_n$ be the $n \times n$ Hermitian matrix that has entries $[X_n]_{jk} = (x_{j,k} + iy_{j,k})/\sqrt{2}$ for $j < k$, $[X_n]_{jj} = x_{jj}$ for $1 \leq j \leq n$ and $[X_n]_{kj} = (x_{j,k} - iy_{j,k})/\sqrt{2}$ for $j < k$; the space of all such matrices with the probability measure $\sigma_n^{(2)}$ forms the Gaussian unitary ensemble.

**Bulk of the spectrum.** The eigenvalues of $X_n$ are real and may be ordered as $\lambda_1 \leq \ldots \leq \lambda_n$, so their positions are specified by the empirical distribution $\mu_n = (1/n) \sum_{j=1}^n \delta_{\lambda_j}$. As $n \to \infty$, the empirical distributions converge weakly to the equilibrium distribution, namely the Wigner semicircle law

$$\rho(dx) = \frac{1}{2\pi} I_{[-2, 2]}(x) \sqrt{4 - x^2} dx,$$

(1.2)

for almost all sequences $(X_n)$ of matrices under $\otimes_n \sigma_n^{(2)}$. The bulk of the spectrum consists of those eigenvalues in $[-2, 2]$. See [22, p 93].

Let $B_t$ be the operator on $L^2(\mathbb{R})$ that has kernel

$$B_t(x, y) = \frac{\sin t\pi(x - y)}{\pi(x - y)},$$

(1.3)

let $I_S$ be the indicator function of a set $S$, and let $P_{(\alpha, \beta)}$ be the orthogonal projection on $L^2(\mathbb{R})$ given by $P_{(\alpha, \beta)}f(x) = I_{(\alpha, \beta)}(x)f(x)$; for brevity we write $P_+ = P_{(0, \infty)}$ and $P_- = P_{(-\infty, 0)}$. 


Let $E_{\sigma_n}(k; \alpha, \beta)$ be the probability with respect to $\sigma_n^{(2)}$ that $(\alpha, \beta)$ includes exactly $k$ eigenvalues. Mehta and Gaudin [22] showed that

$$E_{\sigma_n}(k; \alpha, \beta) \to \frac{(-1)^k}{k!} \left(\frac{d^k}{dz^k}\right)_{z=1} \det \left[ I - zP_{(\alpha, \beta)}B_1P_{(\alpha, \beta)} \right].$$

(1.4)

This determinant can alternatively be expressed in terms of the operator $\Psi_a$:

$$L^2(-a,a) \to L^2$$

that has kernel $\Psi_a(x,y) = e^{ixy}\mathbb{I}_{[-a,a]}(y)/\sqrt{2\pi}$ and satisfies $\Psi_a\Psi_a^* = B_{a/\pi}$.

Hard edges. Let $Y_n$ random $n \times n$ matrices with independent $N(0, 1/n)$ entries, and let $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of the positive operator $Y_n^*Y_n$. Then $\nu_n = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j}$ is the empirical eigenvalue distribution, and $\nu_n$ converges weakly almost surely to the Marchenko–Pastur distribution; so that

$$\int_0^\infty f(x)\nu_n(dx) \to \int_0^\infty f(x)\sqrt{\frac{4-x}{2\pi}} dx$$

(1.5)

almost surely as $n \to \infty$ for all continuous and bounded real functions $f$. Thus the $\lambda_j$ tend to accumulate near to their minimum possible value of zero, where the density of the limiting distribution is unbounded; this is the hard edge effect.

Hard edges also arise from random matrices of the Jacobi and Laguerre ensembles. The Jacobi ensemble of order $N$ with parameters $\nu, \gamma > -1/2$ at inverse temperature $\beta > 0$ is the joint distribution function

$$\sigma_{N,J}^{(\beta)}(dx) = \frac{1}{Z_N} \prod_{j=1}^N (1 + x_j)^{\beta\gamma} (1 - x_j)^{\beta\nu} \prod_{1 \leq j < k \leq N} (x_k - x_j)^{\beta} dx_1 \ldots dx_N$$

(1.6)

where $-1 \leq x_1 \leq \ldots \leq x_N \leq 1$ are the eigenvalues.

Forrester [12] showed that the integral operator $F^{a,b}$ on $L^2((0,1), dx)$ with kernel

$$F^{a,b}(x,y) = \mathbb{I}_{(a,b)}(x)J_\nu(\sqrt{x})\sqrt{\gamma}J_\nu'(\sqrt{y}) - \sqrt{x}J_\nu'(\sqrt{x})J_\nu(\sqrt{y}) \mathbb{I}_{(a,b)}(y)$$

(1.7)

determines the limiting distribution of scaled eigenvalues $x_j/(4n)$ from the Laguerre ensemble near to the hard edge, and conjectured that a similar result holds for the Jacobi ensemble. Using the orthogonal polynomial technique, Forrester and Rains [13] have verified the cases of $\beta = 1, 2$ and 4 following earlier work by Borodin [5] and Dueñez.

We introduce the scaled eigenvalues $\xi_j$ by $x_j = \cos \xi_j/\sqrt{N}$, to ensure that the mean spacing of the $\xi_j$ is of order $O(1)$ near to the hard edge at $x_j \approx 1$. One can show that

$$\sigma_{N,J}^{(2)}((a,b) \text{ contains no } \xi_j) \to \det(I - F^{a,b}) (N \to \infty).$$

(1.8)
For subsequent analysis we change variables by writing $x = e^{-2\xi}$ and $y = e^{-2\eta}$ so that $\xi, \eta \in (0, \infty)$ for $x, y \in (0, 1)$. Let $G_\ell$ be the unitary integral operator on $L^2(\mathbb{R})$ that has kernel $e^{-\ell - \xi - \eta}J_\nu(e^{-\ell - \xi - \eta})$; let $Q_\ell = G_\ell P_+ G_\ell$ ($\ell \in \mathbb{R}$), which gives a strongly continuous family of orthogonal projections. For compact operators $S$ and $T$ on Hilbert space, the spectrum of $ST$ equals the spectrum of $TS$. The integral operator $\Phi_\ell = P_+ G_\ell P_+ + G_\ell$ on $L^2(0, \infty)$ is Hilbert–Schmidt, and when $0 < a < 1$ and $\alpha = -(1/2) \log a$ satisfies

$$\det(I - z \Phi_{\alpha}^0) = \det(I - z \Phi_{\alpha}^2).$$

In section 4 we interpret these operators on the Sonine spaces $u_\nu H^2$ where $u_\nu(x) = 2^{ix} \Gamma((1 + \nu + ix)/2)/\Gamma((1 + \nu - ix)/2)$. 

**Soft edge of the spectrum.** We recall some results of Tracy and Widom [12, 28] concerning the largest few eigenvalues. The Airy function $\text{Ai}(x)$, as defined by the oscillatory integral

$$\text{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz^3/3} dt,$$

satisfies the Airy differential equation [27, page 18] $y'' - xy = 0$. Let $W_{1/3}$ be the integral operator on $L^2(\mathbb{R})$ defined by the Airy kernel

$$W_{1/3}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}.$$

We scale the eigenvalues of the Gaussian ensemble by introducing

$$\xi_j = n^{2/3} \left( \frac{\sqrt{2}}{\sqrt{n}} \lambda_j - 2 \right),$$

and let $E_{\sigma_n}(k; \xi; \alpha, \beta)$ be the probability with respect to $\sigma_n^{(2)}$ that $(\alpha, \beta)$ contains exactly $k$ of the $\xi_j$ ($j = 1, \ldots, n$); see [22, page 116, A7]. Aubrun [3] proved that the operator $W_{1/3}^{\alpha, \beta} = P_{(\alpha, \beta)} W_{1/3} P_{(\alpha, \beta)}$ on $L^2(\mathbb{R}_+)$ is of trace class for $0 < \alpha < \beta \leq \infty$, and

$$E_{\sigma_n}(k; \xi; \alpha, \beta) \to \frac{(-1)^k}{k!} \left( \frac{d^k}{dz^k} \right)_{z=1} \det(I - z W_{1/3}^{\alpha, \beta}) \quad (n \to \infty).$$

The compression of $W_{1/3}^{\alpha, \infty}$ to $L^2(\alpha, \infty)$ may be identified, under the change of variables $s \mapsto \alpha + s$, with $\Gamma_{(\alpha)}^2$ where the Hankel integral operator $\Gamma_{(\alpha)}$ on $L^2[0, \infty)$ satisfies

$$\Gamma_{(\alpha)} f(s) = \int_0^{\infty} \text{Ai}(\alpha + s + t)f(t) \, dt \quad (f \in L^2(0, \infty)).$$
The spectrum of $P(\alpha,\beta)\Gamma^2(0)P(\alpha,\beta)$ equals the spectrum of $\Gamma(0)P(\alpha,\beta)\Gamma(0)$, and hence
\[
\det(I - zP(\alpha,\infty)W_{1/3}P(\alpha,\infty)) = \det(I - z\Gamma^2_\alpha).
\] (1.15)

**Edge distributions and KdV.** For $0 \leq t \leq 1$ let $w(x; t)$ be the unique solution to the Painlevé II equation $w'' = 2w^3 + xw$ that satisfies $w(x; t) \approx -\sqrt{t}\text{Ai}(x)$ as $x \to \infty$. By the theory of inverse scattering for the concentric Korteweg–de Vries equation, this solution is given by the Fredholm determinant
\[ w(x; t)^2 = -\frac{\partial^2}{\partial x^2} \log \det(I - t\Gamma^2_\alpha); \] (1.16)
see [10, p. 86, 174]. Tracy and Widom [28] introduced the cumulative distribution function $F(x; t) = \det(I - t\Gamma^2_\alpha)$ so that
\[ F(x; t) = \exp\left(-\int_x^\infty (y - x)w(y; t)^2 dy\right); \] (1.17)
in particular, $F(x; 1)$ is the Tracy–Widom distribution.

**Operators for parts of the spectrum.** In section 5 we show how $W_{1/3}$ and $\Gamma(0)$ arise from the Airy group $e^{itD^3}$ on $L^2(\mathbb{R})$ where $D = -i\frac{\partial}{\partial x}$. By suitable changes of variable we arrange that the edge of the support of the equilibrium distribution is at zero and we consider the operators on $L^2(\alpha, \infty)$ that describe the probability that scaled eigenvalues lie in $(\alpha, \infty)$.

The relative positions of $L^2(\mathbb{R}_+)$ and $W_{1/3}L^2(\mathbb{R})$ are described by $P_+W_{1/3}P_+$. Generally we have $W_t = e^{itD^3}P_-e^{-itD^3}$ and the complementary orthogonal projection is $W_t^{\perp} = I - W_t = e^{itD^3}P_+e^{-itD^3}$. For comparison, $R_+ = \mathcal{F}^*P_+\mathcal{F}$ and $R_- = \mathcal{F}^*P_-\mathcal{F}$ are the Riesz projections on $L^2$ that have images $H^2$ and $\overline{H^2}$ respectively. These formulæ suggests an analogue of prediction theory such that the subspace $W_t^{\perp}L^2(\mathbb{R})$ corresponds to the subspace $L^2(\mathbb{R}_+)$ which represents the future, and such that the unitary operator $e^{itD^3}$ plays the rôle corresponding to the inverse Fourier transform $\mathcal{F}^*$ in the Hardy space theory of [17]. The projections $\tau_\alpha W_t^{\perp}\tau_{-\alpha} = e^{itD^3}P(\alpha,\infty)e^{-itD^3}$ form a decreasing nest as $\alpha$ increases. The spectrum of the Hankel operator determines the limiting eigenvalue distribution via (1.13) and (1.15).

The following table describes the analogy between the operators and subspaces in the various cases.
Huang Hui 

we start by making unitary transformations to identify $uH$ positions of uniquely determined up to a unimodular constant factor, such that $vH$ operators sometimes write $uH$.

Definition. Let $(U_t) (t \in \mathbb{R})$ be a $C_0$ (strongly continuous) group of unitary operators on an infinite-dimensional separable Hilbert space $H$, and let $K$ be a closed linear subspace of $H$. We say that $K$ is doubly invariant for $(U_t)$ when $U_tK \subseteq K$ for all $t \in \mathbb{R}$. Further, $K$ is simply invariant for $(U_t) (t \geq 0)$ when $U_tK \subseteq K$ for $t \geq 0$ and moreover $\cap_{t\geq 0} U_tK = \{0\}$.

Beurling and Lax characterized the subspaces of $L^2$ that are invariant for the shift operators $S_s : f(x) \mapsto e^{isx} f(x)$; see [15, 17 page 114]. For notational simplicity, we sometimes write $e^{itx} H^2 = \{e^{itx} f(x) : f \in H^2\}$.

A closed linear subspace $\mathcal{T}$ is simply invariant for the semigroup $\{S_s : s \geq 0\}$, if and only if there exists a unimodular measurable function $u$ such that $\mathcal{T} = uH^2 = \{uf : f \in H^2\}$; such a $u$ is uniquely determined up to a unimodular constant factor. In each case we start by making unitary transformations to identify $u$ and to determine the relative positions of $uH^2$ and $H^2$. Either $uH^2 \cap H^2 = 0$ or there exist inner functions $v$ and $w$, uniquely determined up to a unimodular constant factor, such that $u = vw, uH^2 \cap H^2 = vH^2$ and $vwH^2 = vH^2 \cap wH^2$. In sections 4 and 5, we find $uH^2 \cap H^2 = 0$, so we factorize $u(z) = E(z)/E(z)$ where $E$ is a meromorphic function on $\mathbb{C}$ that has no zeros. By using de Branges’s version of Beurling’s theory [6], we are able to show that $W$ from (1.1) is unitarily equivalent to $\Gamma_u^* \Gamma_u$ and hence that $W$ is the reproducing kernel of some weighted Hardy spaces of holomorphic functions inside $\mathbb{C}_+$.

Definition. A Weyl pair $(U_s, V_t)$ consists of a pair of $C_0$ unitary groups $(U_s) (s \in \mathbb{R})$ and $(V_t) (t \in \mathbb{R})$ on $H$ that satisfy $U_s V_t = e^{ist} V_t U_s$ for all $(s, t \in \mathbb{R})$.

The shifts $S_s (s \in \mathbb{R})$ and the translations $\tau_t = e^{-itD} (t \in \mathbb{R})$ give a Weyl pair on $L^2$; moreover, this is the unique representation of the Weyl relations of multiplicity one on $L^2$, up to unitary equivalence; see [32]. Katavolos and Power [16] obtained the following description of the invariant subspaces for a Weyl pair of multiplicity one. Let $\mathcal{L}$ be the space of orthogonal projections $P$ onto closed linear subspaces $K$ of $L^2$ that are invariant under $(S_t) (t \geq 0)$ and $(\tau_{-t}) (t \geq 0)$, where $\mathcal{L}$ has the strong operator topology. There is a homeomorphism $\rho : \{z : |z| \leq 1\} \to \mathcal{L}$ such that:

| Future Projection | Classical | Bulk | Hard Edge | Soft Edge |
|-------------------|----------|------|-----------|-----------|
| Future space      | $\mathcal{F}^* P_+ \mathcal{F}$ | $\mathcal{F}^* P_{(-a,a)} \mathcal{F}$ | $G_\ell P_+ G_\ell$ | $e^{iD^3/3} P_+ e^{-iD^3/3}$ |
| Subspace position | $e^{i2ax} H^2 \subset H^2$ | $\sigma_P \mathcal{V}$ | $u \geq H^2 \cap H^2 \neq 0$ | $e^{itx^3} H^2 \cap H^2 = 0$ |
| Painlevé Equation | $\Psi_a$ | $P_{III}$ | $P_{II}$ | $\Gamma_{(0)}$ |

Weyl Relations and Invariant Subspaces.
(1) $\rho(-i) = 0$, and $\rho(i) = I$;

(2) $\rho(z)L^2$ with $|z| < 1$ is simply invariant for both $(S_s)$ ($s \geq 0$) and $(\tau_{-t})$ ($t \geq 0$);

(3) $\rho(e^{i\theta})L^2$ with $-\pi/2 < \theta < \pi/2$ is simply invariant for $(S_s)$ ($s \geq 0$) and doubly invariant for $(\tau_{-t})$ ($t \in \mathbb{R}$);

(4) $\rho(e^{i\theta})L^2$ with $-\pi/2 < \pi - \theta < \pi/2$ is doubly invariant for $(S_s)$ ($s \in \mathbb{R}$) and simply invariant for $(\tau_{-t})$ ($t \geq 0$).

In section 4 we introduce for the Jacobi ensemble an appropriate Weyl pair for the subspaces $Q_{\ell}L^2$. On account of the natural ordering of the subspaces $Q_{\ell}L^2$, and the probabilistic interpretation of (1.9), we naturally take the translations to be one of the groups in the Weyl pair; whereas we need to hunt down the other one. For the soft edge ensemble, the Weyl pair consists of the translations $e^{isD}$ and the Schrödinger group $e^{it(D^2+x)}$, as we discuss in section 5.

In section 6 we extend these ideas to a new context, namely the Mathieu functions. Here the KdV equation is $2\pi$ periodic and associated with an infinite-dimensional manifold. Whereas we do not propose that this corresponds to a natural random matrix ensemble, the results illustrate the scope of the theory of Tracy–Widom operators.

2. Kernels from differential equations and the Marchenko integral equation

In this section we prove some results concerning Tracy–Widom operators which are already known in specific cases from [8, 28, 29, 30]. Here $B(H)$, $c^2$ and $c^1$ respectively denote the bounded, Hilbert–Schmidt and trace-class linear operators on Hilbert space $H$.

**Lemma 2.1.** Suppose that $A$ and $B$ are bounded, measurable and real functions. Then

$$W(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y}$$

defines a self-adjoint and bounded linear operator on $L^2(\mathbb{R})$.

**Proof.** This follows from the fact that $M_A$, $M_B$ and $R_+$ are bounded on $L^2$.

**Proposition 2.2.** Suppose that $A$ and $B$ are bounded, continuous and integrable functions such that

$$\frac{d}{dx} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} = \begin{bmatrix} \alpha(x) & \beta(x) \\ -\gamma(x) & -\alpha(x) \end{bmatrix} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix},$$

where $\alpha$, $\beta$ and $\gamma$ are linear functions such that

$$C = \frac{1}{x - y} \begin{bmatrix} \gamma(x) - \gamma(y) & \alpha(x) - \alpha(y) \\ \alpha(x) - \alpha(y) & \beta(x) - \beta(y) \end{bmatrix} = \begin{bmatrix} c & a \\ a & b \end{bmatrix}$$
is a negative (semi-) definite constant matrix.

(i) Then there exist continuous real functions \( F \) and \( G \) such that

\[
\frac{A(x)B(y) - A(y)B(x)}{x - y} = \int_0^\infty (F(x + t)F(t + y) + G(x + t)G(t + y)) \, dt
\]  

(2.4)

so \( P_+ WP_+ \) is a sum of two squares of Hankel operators.

(ii) In particular when \( C \) has rank one, the operator \( P_+ WP_+ \) on \( L^2(0, \infty) \) is the square of a self-adjoint Hankel operator.

**Proof.** We take a real bilinear pairing and write

\[
A(x)B(y) - A(y)B(x) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \rangle,
\]

(2.5)

so by a short calculation

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{A(x)B(y) - A(y)B(x)}{x - y} = \langle \begin{bmatrix} c & a \\ a & b \end{bmatrix} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \begin{bmatrix} A(y) \\ B(y) \end{bmatrix} \rangle. 
\]

(2.6)

We introduce a real symmetric matrix \( X \) such that \( X^2 = -C \), let \( \col[\cos \theta, \sin \theta] \) and \( \col[-\sin \theta, \cos \theta] \) be the unit eigenvectors of \( X \) corresponding to eigenvalues \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) respectively. Then

\[
F(x) = \lambda_1 (A(x) \cos \theta + B(x) \sin \theta), \quad G(x) = \lambda_2 (-A(x) \sin \theta + B(x) \cos \theta)
\]

(2.7)

are bounded, continuous and integrable functions that satisfy

\[
\left( \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \right) \frac{A(x)B(y) - A(y)B(x)}{x - y} = -F(x)F(y) - G(x)G(y)
\]

\[= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \right) \int_0^\infty (F(x + t)F(y + t) + G(x + t)G(y + t)) \, dt.
\]

(2.8)

Hence both sides of (2.4) differ by \( f(x - y) \) for some differentiable function \( f \); but both sides converge to zero as \( x \) or \( y \) tend to \( \infty \); so \( f = 0 \), and equality holds.

This result enables us to calculate a determinant as in (1.4), (1.9) and (1.15).

**Theorem 2.3.** Let \( A : (0, \infty) \to \mathbb{R} \) be a continuous function such that
\[ \int_{0}^{\infty} uA(u)^2 \, du \leq 1. \] Then
\[
W(u, v) = \int_{0}^{\infty} A(u + t)A(t + v) \, dt \tag{2.9}
\]
is the kernel of a trace-class operator on \( L^2(0, \infty) \) such that, when \( |\kappa| < 1 \),
\[
K(x, z) - \kappa^2 \int_{x}^{\infty} K(x, y)W(y, z) \, dy = \kappa W(x, z) \tag{2.10}
\]
has a solution \( K(x, z) \), which is a trace-class kernel, such that
\[
\frac{\partial}{\partial x} \log \det(I - \kappa^2 P_{(x, \infty)} WP_{(x, \infty)}) = \kappa K(x, x) \quad (x > 0). \tag{2.11}
\]

In the subsequent proof, we shall use the self-adjoint Hankel operator \( \Gamma(x) \) as in
\[
\Gamma(x)f(t) = \int_{0}^{\infty} A(x + t + u)f(u) \, du \quad (f \in L^2(0, \infty)), \tag{2.12}
\]
where \( \Gamma^2(x) \) on \( L^2(0, \infty) \) is unitarily equivalent to \( P_{(x, \infty)} WP_{(x, \infty)} \) on \( L^2(x, \infty) \).

**Lemma 2.4.** For \( |\kappa|^2 \int_{0}^{\infty} uA(u)^2 \, du < 1 \) there exists a solution \( L \in L^2((0, \infty)^2) \) to the integral equation
\[
L(x, s) - \kappa^2 \int_{0}^{\infty} L(x, y) \int_{x}^{\infty} A(y + u)A(u + s) \, du \, dy = \kappa A(x + s). \tag{2.13}
\]

**Proof of Lemma 2.4.** By the Hilbert–Schmidt theorem applied to \( \Gamma(0) \), there exist \((\varphi_j)\), an orthonormal basis of \( L^2(0, \infty) \), and real \( \gamma_j \) such that \( A(x + y) = \sum_{j=1}^{\infty} \gamma_j \varphi_j(x)\varphi_j(y) \) and
\[
\sum_{j=1}^{\infty} \gamma_j^2 = \int_{0}^{\infty} \int_{0}^{\infty} A(u + v)^2 \, du \, dv = \int_{0}^{\infty} uA(u)^2 \, du < \infty. \tag{2.14}
\]
Our solution separates into series \( L(x, s) = \sum_{j=1}^{\infty} \chi_j(x)\varphi_j(s) \), where \( \text{col} [\chi_j(x)] \) satisfies the equation with column vectors in \( \ell^2 \)
\[
[I - \kappa^2 \Phi(x)] \text{col} [\chi_j(x)] = \kappa \text{col} [\gamma_j \varphi_j(x)] \tag{2.15}
\]
and with matrix
\[
\Phi(x) = \left[ \gamma_j \gamma_k \int_{x}^{\infty} \varphi_j(y)\varphi_k(y) \, dy \right]_{1 \leq j, k < \infty}. \tag{2.16}
\]
By the Cauchy–Schwarz inequality, $\Phi(x)$ defines a Hilbert–Schmidt operator on $\ell^2$ with norm
\[
\|\Phi(x)\|_{c^2}^2 \leq \sum_{j,k=1}^{\infty} \gamma_j^2 \gamma_k^2 \left( \int_{x}^{\infty} |\varphi_j(y)|^2 \, dy \right) \left( \int_{x}^{\infty} |\varphi_k(y)|^2 \, dy \right) \leq \left( \sum_{j=1}^{\infty} \gamma_j^2 \right)^2;
\] (2.17)
hence, $I - \kappa^2 \Phi(x)$ is invertible whenever $|\kappa|^2 \|\Phi(x)\|_{c^2} < 1$. We deduce that (2.15) has a unique solution and by orthogonality
\[
\int_{0}^{\infty} \int_{0}^{\infty} |L(x, y)|^2 \, dx \, ds \leq \int_{0}^{\infty} \sum_{j=1}^{\infty} |\chi_j(x)|^2 \, dx
\leq \int_{0}^{\infty} \| (I - \kappa^2 \Phi(x))^{-1} \|_{B(\ell^2)}^2 \| (\kappa \gamma_j \varphi_j(x)) \|_{\ell^2}^2 \, dx
\leq \left( 1 - |\kappa|^2 \sum_{j=1}^{\infty} \gamma_j^2 \right)^{-2} |\kappa|^2 \sum_{j=1}^{\infty} \gamma_j^2.
\] (2.18)

**Remark.** When $\sum_{j=1}^{\infty} |\gamma_j| < \infty$, the operator $\Phi(x)$ is trace class and the determinant $\det(I - \kappa^2 \Phi(x))$ defines an entire function of $\kappa$.

**Proof of Theorem 2.3.** By Lemma 2.4, the integral kernels
\[
W(x, y) = \int_{0}^{\infty} A(x + s)A(s + y) \, ds
\] (2.19)
and
\[
K(x, z) = \int_{0}^{\infty} L(x, s)A(s + z) \, ds
\] (2.20)
are trace class and satisfy the Marchenko integral equation
\[
K(x, z) - \kappa^2 \int_{x}^{\infty} K(x, s)W(s, z) \, ds = \kappa W(x, z).
\] (2.21)

By iterated substitution, we deduce that
\[
K(x, z) = \kappa W(x, z) + \kappa^3 W(x, \cdot)P_{(x, \infty)}(I - \kappa^2 P_{(x, \infty)} WP_{(x, \infty)})^{-1}P_{(x, \infty)} W(\cdot, z).
\]

We deduce, by simplifying the integral kernels, that
\[
\kappa K(x, x) = \kappa^2 \langle (I - \kappa^2 \Gamma^2_{(x)})^{-1}A(x + \cdot), A(x + \cdot) \rangle_{L^2(0, \infty)}.
\] (2.22)

By unitary equivalence as in (2.12), we have
\[
\det(I - \kappa^2 P_{(x, \infty)} WP_{(x, \infty)}) = \det(I - \kappa^2 \Gamma^2_{(x)}).
\]
and hence
\[
\frac{\partial}{\partial x} \log \det \left( I - \kappa^2 P(x, \infty) WP(x, \infty) \right) = -\kappa^2 \text{trace} \left( (I - \kappa^2 \Gamma^2(x))^{-1} \frac{\partial}{\partial x} \Gamma^2(x) \right),
\]
(2.23)

where \(\frac{\partial}{\partial x} \Gamma^2(x)\) is the rank-one operator that has kernel
\[
\frac{\partial}{\partial x} \Gamma^2(x)(u, v) = \frac{\partial}{\partial x} \int_0^\infty A(s + u + x)A(s + v + x) \, ds
\]
\[
= -A(u + x)A(v + x);
\]
(2.24)

hence
\[
\frac{\partial}{\partial x} \log \det (I - \kappa^2 \Gamma^2(x)) = -\kappa^2 \langle (I - \kappa^2 \Gamma^2(x))^{-1}A(x + .), A(x + .) \rangle_{L^2(0, \infty)}.
\]
(2.25)

By comparing the terms in the power series in \(\kappa\), we deduce that
\[
\kappa K(x, x) = \frac{\partial}{\partial x} \log \det (I - \kappa^2 P(x, \infty) WP(x, \infty)).
\]
(2.26)

**Corollary 2.5.** Suppose further that \(A\) is an entire function such that \(\int_0^\infty u|A(z + u)|^2 \, du < \infty\) for each \(z \in \mathbb{C}\), and let \(\Gamma(z)\) be the Hankel operator on \(L^2(0, \infty)\) that has kernel \(A(z + s + t)\). Then
\[
\frac{d}{dz} \log \det (I - \kappa^2 \Gamma^2(z))
\]
(2.27)

defines a meromorphic function on \(\mathbb{C}\), extending \(\kappa K(x, x)\).

**Proof.** By Morera’s theorem, \(z \mapsto \Gamma(z)\) defines an entire function with values in \(e^2\), and hence \(\det (I - \kappa^2 \Gamma^2(z))\) defines an entire function. The formula (2.27) defines a holomorphic function, except at those isolated points where the determinant vanishes, and these give rise to poles.

In some cases \(y = \frac{d}{dx} K(x, x)\) satisfies a Painlevé equation as in [14, p 344]; that is, \(y'' = F(y', y, x)\) where \(F\) is rational in \(y\) and \(y'\), and analytic in \(x\), and such that the only movable singularities of \(y\) in \(\mathbb{C}\) are poles. In particular, the bulk kernel gives rise to the \(\sigma\) form of \(P_V\), the hard edge ensemble gives rise to the \(P_{III}\) equation and the soft edge to \(P_{II}\) as in [28, 29, 30, 12].

The Painlevé ODE test asserts that every ordinary differential equation that arises from a partial differential equation via a Marchenko linear integral equation may be transformed to a Painlevé equation; see [1].
A special feature of (2.10) is that $W$ is the square of a self-adjoint Hankel operator $\Gamma$ on $L^2(0, \infty)$, and Proposition 2.2 gives an explicit construction of the symbol for $\Gamma$. Up to unitary equivalence, this holds under some general spectral conditions which we list below. In section 3 we give a means for verifying (iii) for operators of the form (1.1).

**Proposition 2.6.** Suppose that $W$ is a linear operator on $H$ such that

(i) the nullspace of $W$ is either trivial or infinite-dimensional;
(ii) $W$ is not invertible;
(iii) $W$ is bounded and self-adjoint, and $W \geq 0$;
(iv) $W$ has a simple discrete spectrum.

Then there exists a self-adjoint Hankel operator $\Gamma$ on $L^2(0, \infty)$ and a unitary operator $U : H \rightarrow L^2(0, \infty)$ such that $W = U^* \Gamma^2 U$.

**Proof.** Megretskii, Peller and Treil [21, p 257] have obtained sufficient conditions for a self-adjoint operator to be unitarily equivalent to the modulus of a Hankel operator. Under the more stringent condition (iv), their construction gives a Hilbert space $K$, a bounded linear operator $X : K \rightarrow K$, and vectors $\xi, \eta \in K$ such that the Hankel operator

$$
\Gamma f(t) = \int_0^\infty h(s+t)f(s)\,ds \quad (f \in L^2(0, \infty))
$$

(2.29)

with symbol $h(t) = \langle e^{tX}\xi, \eta \rangle_K$ is unitarily equivalent to $W^{1/2}$; thus $\Gamma$ is realized from a balanced linear system in continuous time with one-dimensional input and output spaces.

**3. Reproducing kernels and the bulk of the spectrum**

In this section we recover the bulk kernel as a reproducing kernel, and show more generally why operators of the form (1.1) are positive on weighted Hardy spaces. Let $E$ be a meromorphic and zero-free function on $\mathbb{C}$ and let $E^*(z) = \overline{E(z)}$, which has similar properties. We also introduce the meromorphic functions $A(z) = (E(z) + E^*(z))/2$ and $B(z) = (E^*(z) - E(z))/(2i)$, which have $A(x)$ and $B(x)$ real for real $x$.

Let $EH^2$ be the weighted Hardy space of meromorphic functions $g$ on $\mathbb{C}_+$ such that $g/E$ belongs to the usual Hardy space $H^2$, and with the inner product

$$
\langle g_1, g_2 \rangle_{EH^2} = \langle g_1/E, g_2/E \rangle_{H^2} = \int_{-\infty}^\infty g_1(t)\bar{g}_2(t)\frac{dt}{|E(t)|^2}.
$$

(3.1)

Similarly we can introduce $E^*H^2$. When $\zeta \in \mathbb{C}_+$ is not a pole of $E$, the linear functional $g \mapsto g(\zeta)$ is bounded on $EH^2$, and hence given by $g(\zeta) = \langle g, k_\zeta \rangle_{EH^2}$, where the reproducing kernel is

$$
k_\zeta(z) = \frac{E(z)\overline{E(\zeta)}}{2\pi i(\zeta - z)}.
$$

(3.2)
We introduce $\Omega$ as the domain consisting of points $z \in \mathbb{C}_+$, that are not poles of $E$ or $E^*$. Let $u(z) = E^*(z)/E(z)$, which is meromorphic and unimodular on the real line, let $M_u : EH^2 \to E^*H^2$ be the isometry $M_u f = uf$, and let $T_{\bar{u}} : H^2 \to H^2$ be the Toeplitz operator $T_{\bar{u}} = R_+M_{\bar{u}}R_+$.

**Theorem 3.1.** (i) The operator $W$ on $EH^2$ that has kernel

$$W(z, w) = \frac{E^*(z)E^*(w) - E(z)E(w)}{2\pi i(z - \bar{w})} \quad (z, w \in \Omega) \quad (3.3)$$

compresses to an operator $EH^2 \to EH^2$ that is unitarily equivalent to $\Gamma_{\bar{u}}^*\Gamma_{\bar{u}}$, where $\Gamma_{\bar{u}} : H^2 \to H^2$ is the Hankel operator $\Gamma_{\bar{u}} = R_-M_uR_+$.

(ii) There exists a unique Hilbert space $H(W)$ of holomorphic functions on $\Omega$ such that $W(z, w)$ is the reproducing kernel for $H(W)$.

(iii) Suppose that $T_{\bar{u}}$ has a non-zero nullspace $K$. Then $\Gamma_{\bar{u}}$ restricts to an isometry $K \to H^2$.

**Proof.** (i) We write

$$\int_{-\infty}^{\infty} \frac{E^*(z)E(t) - E(z)E(t)}{2\pi i(z - t)} \frac{f(t)dt}{E(t)E(t)} = \frac{E(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)/E(t)}{t - z} dt$$

$$- \frac{E^*(z)}{2\pi i} \int_{-\infty}^{\infty} \frac{u^*(t)f(t)/E(t)}{t - z} dt, \quad (3.4)$$

and hence by Cauchy’s integral formula we have

$$Wf(z) = E(f/E - M_uR_+M_{\bar{u}}^*(f/E))$$

$$= EM_uR_-M_{\bar{u}}^*(f/E). \quad (3.5)$$

The map $V : EH^2 \to H^2 : f \mapsto f/E$ is a unitary equivalence with adjoint $V^* : g \mapsto Eg$, and $\Gamma_{\bar{u}}^*\Gamma_{\bar{u}} : H^2 \to H^2$ reduces to $\Gamma_{\bar{u}}^*\Gamma_{\bar{u}} = R_+M_uR_-M_{\bar{u}}R_+$, so $\langle Wf, g \rangle_{EH^2} = \langle V^*\Gamma_{\bar{u}}^*\Gamma_{\bar{u}}Vf, g \rangle_{H^2}$ for all $f, g \in EH^2$.

(ii) By (i), $W$ is a positive operator on $EH^2$, so the kernel $W(z, w)$ is of positive type on $\Omega$; further, $z \mapsto W(z, w)$ and $w \mapsto W(z, \bar{w})$ are holomorphic on $\Omega$. Hence we can apply [2, Theorem 2.3.5] to obtain the Hilbert space of holomorphic functions such that $W(z, w)$ is the reproducing kernel.

(iii) By [23, p 89], we have $T_{\bar{u}}^*T_{\bar{u}} = I - \Gamma_{\bar{u}}^*\Gamma_{\bar{u}}$, which leads directly to the identity $K = \{f \in H^2 : \|\Gamma_{\bar{u}}f\| = \|f\|\}$. 

\[ \square \]
Corollary 3.2. Suppose that $u$ belongs to $H^\infty$ so that $E^*H^2$ is a closed linear subspace of $EH^2$, and let $K = EH^2 \ominus E^*H^2$ be the orthogonal complement of the range of $M_u : EH^2 \to EH^2$. Then $K$ equals $H(W)$ and has reproducing kernel
\[ K_w(z) = \frac{A(z)B(\bar{w}) - B(z)A(\bar{w})}{\pi(\bar{w} - z)} \quad (z, w \in \Omega). \tag{3.6} \]

Proof. First, one can check by calculation that
\[ K_w(z) = W(z, w) = \frac{E^*(z)\overline{E^*(w)} - E(z)\overline{E(w)}}{2\pi i(z - \bar{w})}. \tag{3.7} \]

Then we observe that
\[ \frac{E^*(z)\overline{E^*(w)}}{2\pi i(z - \bar{w})} = u(z) \frac{E(z)\overline{E^*(w)}}{2\pi i(z - \bar{w})} \tag{3.8} \]
lies in the range of $M_u$; so for $g \in K$ the proof of Theorem 3.1(i) simplifies to give
\[ \langle g, K_w \rangle_{EH^2} = \langle g, k_w \rangle_{EH^2} = g(w) \quad (w \in \Omega). \]

\[ \square \]

Bulk of the spectrum. Thus when $u$ is an inner function we can identify $H(W)$ explicitly as the orthogonal complement of a shift-invariant subspace of $EH^2$. In particular, by taking the entire function $E(z) = e^{-i\alpha z}$, we find $u(z) = e^{2i\alpha z}$ and the reproducing kernel for $K = EH^2 \ominus E^*H^2$ to be
\[ K_w(z) = \frac{\sin \alpha(z - \bar{w})}{\pi(z - \bar{w})}, \tag{3.9} \]
as in the bulk kernel $B_{\alpha/\pi}(z, w)$ of (1.3). Here we have $EH^2 = \mathcal{F}^*L^2[-\alpha, \infty)$, and $\Psi_a = \mathcal{F}^*|L^2[-\alpha, \alpha]$ gives a unitary isomorphism $L^2[-\alpha, \alpha] \to K$ with $\Psi_a \Psi_a^* = B_{\alpha/\pi}$. The Hankel operator $\Gamma_a$ is isometric on $H^2 \ominus e^{2i\alpha x}H^2 \simeq K$.

The Paley–Wiener theorem [17, p. 179] characterizes $K$ as the space of functions $f \in L^2(\mathbb{R})$ that are entire and of exponential type with
\[ \lim \sup_{y \to \pm\infty} |y|^{-1} \log |f(iy)| \leq \alpha. \tag{3.10} \]

Alternatively, we can characterize the subspaces by their scaling properties. Let $(\delta_t) \ (t \in \mathbb{R})$ be the unitary dilatation group on $L^2(\mathbb{R})$ with $\delta_t f(x) = e^{t/2}f(e^t x)$. In [16], Katavolos and Power characterize the lattice of closed linear subspaces of $L^2$ that are simply invariant for both $S_s \ (s \geq 0)$ and $\delta_s \ (s \geq 0)$. 

G.Blower: Operators for soft and hard edges 14
Proposition 3.3. The closed linear subspace $B_a L^2$ is simply invariant for $(\delta_s) (s \leq 0)$, doubly invariant for $(\tau_s) (s \in \mathbb{R})$ and invariant under $J$. Conversely, if $\hat{K}$ is any closed linear subspace of $L^2$ that is simply invariant for $(\delta_t) (t \leq 0)$, doubly invariant for $(\tau_s) (s \in \mathbb{R})$ and invariant under $J$, then $\hat{K} = B_a L^2$ for some $a > 0$.

Proof. We have $\delta_{-s} = F^* \delta_s F$ and $\tau_s = F^* S_{-s} F$, so we shall characterize the subspaces $L^2[-\pi t, \pi t]$ under the operation of $\delta_s, S_s$ and $J$. Now $L^2[-\pi t, \pi t]$ is clearly doubly invariant for $(S_s)$ $(s \in \mathbb{R})$, and $\delta_s L^2[-\pi t, \pi t] = L^2[-\pi te^{-s}, \pi te^{-s}]$; so $L^2[-\pi t, \pi t]$ is simply invariant for $(\delta_s) (s \geq 0)$. Conversely, all closed linear subspaces $\hat{K}$ of $L^2$ that are simply invariant under $(\delta_s) (s \geq 0)$ and doubly invariant under $(S_s) (s \in \mathbb{R})$ have the form $\hat{K} = L^2(-a, b)$ for some $a, b \in \mathbb{R} \cup \{\infty\}$ by a simple case of Beurling’s theorem. When $\hat{K}$ is additionally invariant under $J$, we need to have $a = b$; hence $\hat{K} = L^2[-a, a]$. 

4. Hard-edge operators and Sonine spaces

In this section we consider the operators for the hard edge case and associated subspaces. Let $J_\nu$ be the Bessel function of the first kind for real $\nu > -1/2$, and let

$$h(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(1+2ik)z^k}{2^{\nu+2k}\Gamma(\nu+k+1)k!} = z^{-\nu/2} J_\nu(\sqrt{z}) + 2iz \frac{d}{dz} (z^{-\nu/2} J_\nu(\sqrt{z}))$$

which is entire and of order $1/2$ as in [11, p 190]. Then $E(z) = 1/h(z)$ is a meromorphic function, with no zeros, such that

$$\frac{E^*(z)E^*(w) - E(z)E(w)}{2\pi i (z - w)}$$

$$= \left( \frac{J_\nu(z^{1/2})w^{1/2}J'_\nu(w^{1/2}) - z^{1/2}J'_\nu(z^{1/2})J_\nu(w^{1/2})}{\pi(z - w)} \right) \left( \frac{E(z)E^*(z)E^*(w)E(w)}{z^{\nu/2}\overline{w}^{\nu/2}} \right).$$

We recognise the first factor on the right-hand side from (1.8), and the left-hand side from (4.4); but Corollary 3.2 does not apply directly to $E^*(z)/E(z)$; so we introduce operators that correspond to these kernels indirectly by means of the Hankel transform as in [25, p 298]. The Hankel transform of $f \in L^2(xdx; (0, \infty))$ is

$$\mathcal{H}_\nu f(x) = \int_0^\infty J_\nu(xy)f(y) \, dy.$$  

On $L^2(xdx; (0, \infty))$ we introduce the unitary dilatation group $\tilde{\delta}_t$ by $\tilde{\delta}_t g(x) = e^t g(e^t x)$ and the unitary operator $U : L^2(xdx, (0, \infty)) \to L^2(\mathbb{R})$ by $U g(\xi) = e^{-\xi} g(e^{-\xi})$ such that $U^* \tau_t U = \tilde{\delta}_t$. 

G.Blower: Operators for soft and hard edges 15
Lemma 4.1. Let $G_\ell$ be the integral operator on $L^2(\mathbb{R})$ that has kernel function

$$e^{-\ell-\xi-\eta}J_\nu(e^{-\ell-\xi-\eta}).$$

(4.4)

Then $G_\ell$ is a self-adjoint and unitary operator such that $G_\ell^2 = I$, and $G_\ell\tau_\ell = \tau_\ell G_\ell$.

Proof. From the shape of the integral kernel, the identity $G_\ell U = \tau_{-\ell} U \mathcal{H}_\nu$ is evident. Further, Hankel’s inversion formula leads to the identity $\mathcal{H}_\nu^2 = I$, whence to

$$G_\ell U U^* G_\ell = \tau_{-\ell} U \mathcal{H}_\nu \mathcal{H}_\nu U^* \tau_\ell = I.$$  

(4.5)

The identity (4.4) is evident from the definitions, and by (4.5) is equivalent to the scaling property $\mathcal{H}_\nu \delta_\ell = \tilde{\delta}_{-\ell} \mathcal{H}_\nu$ of the Hankel transform as in [25, p. 299].

The following result on position of subspaces contrasts with Corollary 3.2. Here $\Gamma$ denotes Euler’s gamma function.

Theorem 4.2. (i) The operator $Q_\ell = G_\ell P_+ G_\ell$ on $L^2(\mathbb{R})$ is an orthogonal projection.

(ii) The range of $\mathcal{F} Q_\ell \mathcal{F}^*$ equals $e^{i\ell x} u_\nu H^2$, where the meromorphic function

$$u_\nu(z) = 2iz \frac{\Gamma((1 + \nu + iz)/2)}{\Gamma((1 + \nu - iz)/2)}$$

(4.6)

is holomorphic on $\{z : \Im z < 0\}$, and unimodular and continuous on $\mathbb{R}$.

(iii) Whereas $u_\nu^* H^2 \cap H^2 = \{0\}$, for $\nu > 0$ the subspace $K = (u_\nu H^2) \cap H^2$ is non-zero, and $\Gamma_{\tilde{u}_\nu} : H^2 \to \overline{H^2}$ restricts to an isometry $K \to \overline{H^2}$.

Proof. (i) This follows directly from the Lemma.

(ii) Our aim is to show that the range of the orthogonal projection $\mathcal{F}^* Q_\ell \mathcal{F}$ is simply invariant under the $S_\lambda$ for $\lambda > 0$. By Plancherel’s theorem we have

$$S_\lambda \mathcal{F} Q_\ell L^2 = \mathcal{F} \tau_{-\lambda} G_\ell P_+ L^2 = \mathcal{F} G_0 \tau_{\lambda+\ell} P_+ L^2,$$

(4.7)

where $\tau_{\lambda+\ell} P_+ L^2 = L^2(\lambda + \ell, \infty) \subseteq L^2(\ell, \infty)$ and $\cap_{\lambda > 0} L^2(\lambda, \infty) = 0$. Consequently by Beurling’s theorem, there exists a unimodular and measurable function $u_\nu$ such that $\mathcal{F} Q_0 u_\nu L^2 = u_\nu H^2$, and $u_\nu$ is unique up to a unimodular constant factor. One can easily deduce that $\mathcal{F} Q_\ell L^2 = e^{i\ell x} u_\nu H^2$.

The Fourier conjugate of $Q_\ell$ is $\mathcal{F} Q_\ell \mathcal{F}^* = \mathcal{F} G_\ell \mathcal{F}^* \mathcal{F} P_+ \mathcal{F}^* \mathcal{F} G_\ell \mathcal{F}^*$, wherein we recognise $\mathcal{F} P_+ \mathcal{F}^*$ as $R_- : L^2 \to \overline{H^2}$. To determine the range of $\mathcal{F} Q_\ell \mathcal{F}^*$, or equivalently the subspace $\mathcal{F} G_\ell L^2(0, \infty)$, we write

$$\mathcal{F} G_\ell f(x) = \int_{-\infty}^{\infty} \int_0^\infty e^{-\ell-\xi-\eta} J_\nu(e^{-\ell-\xi-\eta}) f(\eta) d\eta \frac{d\xi}{\sqrt{2\pi}}.$$
for \( f \in L^2(0, \infty) \), and then reduce this integral by simple transformations to

\[
\mathcal{F}G_\ell f(x) = e^{ix\ell} \mathcal{F}^* f(x) \int_{-\infty}^{\infty} e^{-(1+\nu+ix)\xi} e^{\nu \xi} J_\nu(e^{-\xi}) \, d\xi. \tag{4.8}
\]

The substitution \( y = e^{-\xi} \) reduces the final integral in (4.8) to a standard Mellin transform [25, p. 263], and we identify \( u_\nu \) from

\[
\mathcal{F}G_\ell f(x) = e^{ix\ell} \frac{2iz \Gamma((1 + \nu + ix)/2)}{\Gamma((1 + \nu - ix)/2)} \mathcal{F}^* f(x). \tag{4.9}
\]

(iii) Let \( E_\nu(z) = e^{-iz \log \sqrt{2}^\Gamma((1 + \nu - iz)/2) \right) \) so that \( E_\nu \) is meromorphic and zero-free with simple poles at \(-i - \nu i - 2ki \) for \( k = 0, 1, \ldots \), and \( u_\nu(z) = E_\nu^*(z)/E_\nu(z) \) has simple zeros at \( z_k = -i - \nu i - 2ki \) for \( k = 0, 1, \ldots \) and simple poles at \( i + \nu i + 2ki \) for \( k = 0, 1, \ldots \). The function \( u_\nu(z) \) is holomorphic in the lower half plane, but does not define a bounded holomorphic function on \( \{ z : \Im z < 0 \} \) since the series \( \sum_{k=0}^{\infty} \Im z_k/(1 + |z_k|^2) \) diverges, violating Blaschke’s condition for the zeros of a non-trivial function in \( H^\infty \) or \( H^2 \) as in [17, p. 92]. Hence the equations \( h_1(z) = u_\nu^*(z)h_2(z) \) with \( h_1, h_2 \in H^2 \) has only the trivial solution \( h_1 = h_2 = 0 \); so \( u_\nu^*H^2 \cap H^2 = 0 \). Note that \( \log |u_\nu^*(z)| \) is subharmonic on the \( \mathbb{C}_+ \), but is not the Poisson integral of a measure on \( \mathbb{R} \).

We take \( a > 0 \) and \( \nu + 1/2 > \lambda > 1/2 \), and let

\[
f(x) = a^{\nu - \lambda + 3/2} x^{1/2 - \nu} (x^2 - a^2)^{(\lambda - 1)/2} J_{\lambda - 1}(a \sqrt{x^2 - a^2}) I(a, \infty)(x),
\]

with Hankel transform

\[
g(t) = t^{1/2} \mathcal{H}_\nu(x^{-1/2} f(x); t).
\]

Then by a result of Sonine [6 p. 301 , 24 p. 75, 26 p. 38], both \( f \) and \( g \) are supported on \((a, \infty)\) and we have

\[
\int_a^{\infty} g(t) t^{-1/2 + ix} \, dt = u_\nu(x) \int_a^{\infty} f(t) t^{-1/2 - ix} \, dt \quad (x \in \mathbb{R}). \tag{4.10}
\]

Hence when \( a = 1 \) there exist non-zero functions \( h_1, h_2 \in H^2 \) such that \( h_2(x) = u_\nu(x)h_1^*(x) \), so \( h_2 \in u_\nu H^2 \). Now we apply Theorem 3.1(iii) to deduce that \( \Gamma_{\nu} H^2 \cap u_\nu H^2 \) is an isometry.

\[\hfill\]

**Proposition 4.3.** (i) The Hankel operator \( \Phi_\ell = P_+ G_\ell P_+ \) on \( L^2(0, \infty) \) has \( \Phi_\ell^2 = P_+ Q_\ell P_+ \).

(ii) The operator \( \Phi_\ell \) on \( L^2(0, \infty) \) is Hilbert–Schmidt, and each non-zero
G. Blower: Operators for soft and hard edges

\[ f \in L^2(xdx, (0,1)) \] such that

\[ \lambda f(x) = \int_0^1 J_\nu(\sqrt{xy})f(y) \, dy \tag{4.11} \]

corresponds to an eigenfunction \( g \in L^2(0, \infty) \) of \( \Phi_\ell \) with eigenvalue \( \frac{1}{2}\lambda\sqrt{s} \).

(iii) The kernel of \( Q_\ell \) as an integral operator on \( L^2(\mathbb{R}) \) is

\[
\frac{e^{-\ell-x}J_\nu(e^{-\ell-x})e^{-2\ell-2\eta}J_\nu'(e^{-\ell-\eta}) - e^{-2\ell-2\xi}J_\nu'(e^{-\ell-\xi})e^{-\ell-\eta}J_\nu(e^{-\ell-\eta})}{e^{-2\ell-2\xi} - e^{-2\ell-2\eta}}. \tag{4.12}
\]

(iv) \( \det(I - zF^{0,a}) = \det(I - z\Phi_\ell^{2(a)} \Phi_\ell^2) \) for \( \alpha = -(1/2) \log a \) and \( a > 0 \).

**Proof.** (i) For \( t > 0 \) we have the Hankel condition \( \Phi_\ell \tau_t = \tau_t^* \Phi_\ell \), where here \( \tau_t \) denotes the semigroup of translation operators on \( L^2(0, \infty) \). Then one uses Theorem 4.2(i).

(ii) The kernel function is clearly symmetric, real-valued and square integrable, since

\[
\int_0^\infty\int_0^\infty e^{-2(\ell+\eta+\xi)}J_\nu(e^{-(\ell+\eta+\xi)})^2 \, d\xi \, d\eta = \int_0^\infty ue^{-2\ell-2u}J_\nu(e^{-\ell-u})^2 \, du < \infty \tag{4.13}
\]

due to the asymptotic formula \( J_\nu(x) \approx x^\nu/\Gamma(\nu+1) \) as \( x \to 0+ \). Hence \( \Phi_\ell \) gives a self-adjoint operator of Hilbert–Schmidt type. The operator \( U \) restricts to a unitary \( L^2(xdx; (0,1)) \to L^2(0, \infty) \), and under this transformation the eigenfunction equations correspond via \( g(\xi) = e^{-\xi}f(e^{-2\xi}) \).

(iii) We use the method of proof of Proposition 2.2 to verify the stated formula for \( Q_\ell = G_\ell P_\ell G_\ell \), which is essentially the square of a self-adjoint Hankel operator. With \( A(\xi) = e^{-\xi}J_\nu(e^{-\xi}) \) and \( B(\xi) = e^{-2\xi}J_\nu'(e^{-\xi}) \), we have

\[
\frac{d}{d\xi} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ (e^{-2\xi} - \nu^2) & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{4.14}
\]

where

\[
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ (e^{-2\xi} - \nu^2) \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu^2 \\ e^{-2\xi} \end{bmatrix} = \begin{bmatrix} e^{-2\eta} - e^{-2\xi} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \tag{4.15}
\]

hence

\[
\left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \frac{A(\xi)B(\eta) - A(\eta)B(\xi)}{e^{-2\xi} - e^{-2\eta}} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \int_0^\infty A(\xi + u)A(\eta + u) \, du.
\]

Hence we can obtain the stated identity by following the proof of Proposition 2.2. Alternatively, one can transform a formula in [25, p. 303].
Proof. 

(i) The simplest way of proving that the operator

\[ \alpha \]

some real

\[ V \]

that

\[ \tau \]

simply invariant for

\[ L \]

a complete spectral family in

\[ (\nu, (0, 1), dx) \]

and \( L^2(0, \infty) \) involves \( g(x) \mapsto \sqrt{2} e^{-\xi} g(e^{-\xi}) \), so \( F^{(0, 1)} \) is unitarily equivalent to the operator that has kernel

\[
2e^{-\xi - \eta} F^{(0, 1)}(e^{-2\xi}, e^{-2\eta}) = \frac{e^{-\xi} J_\nu(e^{-\xi})e^{-2\eta}J'_\nu(e^{-\eta}) - e^{-2\xi} J'_\nu(e^{-\xi})e^{-\eta} J_\nu(e^{-\eta})}{e^{-2\xi} - e^{-2\eta}},
\]

which we recognise as the kernel of \( \Phi^2_{(0)} \). Comparing the spectra of the compressions to \( L^2(0, a) \) and \( L^2(\alpha, \infty) \), we deduce that

\[
det(I - zF^{0,a}) = det(I - zP(\alpha, \infty)\Phi^2_{(0)} P(\alpha, \infty)) = det(I - z\Phi_{(0)} P(\alpha, \infty)\Phi_{(0)}).
\]

(4.16)

Finally, \( \Phi_{(0)} P(\alpha, \infty)\Phi_{(0)} \) equals \( \Phi^2_{(\alpha)} \) since they both have kernel

\[
\int_{\alpha}^{\infty} e^{-\xi - u} J_\nu(e^{-\xi - u})e^{-\eta - u} J_\nu(e^{-\eta - u}) du.
\]

(4.17)

\( \square \)

Theorem 4.4. Let \( T \) be the operator

\[
Tf(\xi) = -\frac{\partial}{\partial \xi} \left( e^{2\xi} \frac{\partial f}{\partial \xi} \right) + (\nu^2 - 1) e^{2\xi} f(\xi).
\]

(4.18)

(i) Then \( T \) is an essentially self-adjoint and positive operator on \( C^\infty_c(\mathbb{R}) \) in \( L^2(\mathbb{R}) \), so that \( V_t = e^{-itT} T^{-it/2} \) \((t \in \mathbb{R})\) defines a \( C_0 \) group of unitary operators on \( L^2(\mathbb{R}) \).

(ii) The unitary groups \( (V_s)_{s \in \mathbb{R}} \) and \( (\tau_t)_{t \in \mathbb{R}} \) satisfy \( V_s \tau_t = e^{i(t-t_0)} V_s \) for \( s, t \in \mathbb{R} \).

(iii) The subspace \( Q_\alpha L^2 \) is doubly invariant for \( (V_s) \) with \( s \in \mathbb{R} \) and simply invariant for \( (\tau_t) \) for \( t \geq 0 \). Conversely, if \( K \) is a non-trivial closed linear subspace of \( L^2 \) that is simply invariant for \( \tau_{-t} \) \((t \geq 0)\) and doubly invariant for \( V_s \) \((s \in \mathbb{R})\), then \( K = Q_\alpha L^2 \) for some real \( \alpha \).

Proof. (i) The simplest way of proving that the operator \( T \) is self-adjoint is to compute its spectral resolution. By simple transformations of the Bessel equation [14, p 171], we have

\[
-e^{2\xi} \left( \frac{\partial^2}{\partial \xi^2} + 2 \frac{\partial}{\partial \xi} + \nu^2 - 1 \right) (e^{-\xi - \ell - \eta} J_\nu(e^{-\xi - \ell - \eta})) = e^{-2\ell - 2\eta} (e^{-\xi - \ell - \eta} J_\nu(e^{-\xi - \ell - \eta})),
\]

so that \( e^{-\xi - \ell - \eta} J_\nu(e^{-\xi - \ell - \eta}) \) is an eigenfunction of \( T \) corresponding to the eigenvalue

\( e^{-2\ell - 2\eta} > 0 \). By Hankel’s inversion theorem [25, p 299], the functions \( \lambda y J_\nu(\lambda xy) \) give a complete spectral family in \( L^2(xdx; (0, \infty)) \), and the unitary transformation \( U \) takes \( \lambda y J_\nu(\lambda xy) \) to \( e^{-\xi - \ell - \eta} J_\nu(e^{-\xi - \ell - \eta}) \) after an obvious change of variable. By Stone’s theorem, \((-i/2) \log T \) generates a \( C_0 \) unitary group \( T^{-is/2} \).
(ii) We have
\[
V_s G_{\ell} f(\xi) = e^{-i s \ell T^{-i s/2}} \int_{-\infty}^{\infty} e^{-\xi - \eta - \ell} J_{\nu}(e^{-\xi - \ell - \eta}) f(\eta) \, d\eta
\]
\[
= \int_{-\infty}^{\infty} e^{i s \eta} e^{-\xi - \ell - \eta} J_{\nu}(e^{-\xi - \ell - \eta}) f(\eta) \, d\eta
\]
\[
= G_{\ell} S_s f(\xi);
\]
(4.20)
hence \(G_{\ell} V_s G_{\ell} = S_s\). When we conjugate the familiar Weyl–von Neumann relation \(\tau_{-t} S_s = e^{i s t} S_s \tau_{-t}\) by \(G_{\ell} \otimes G_{\ell}^*\) we obtain \(G_{\ell} \tau_{-t} G_{\ell} G_{\ell} S_s G_{\ell} = e^{i s t} G_{\ell} S_s G_{\ell} \tau_{-t} G_{\ell}\) or \(\tau_t V_s = e^{i s t} V_s \tau_t\).

(iii) From earlier relations, we have
\[
V_s Q_{\ell} = V_s G_{\ell} P_+ G_{\ell} = G_{\ell} S_s P_+ = G_{\ell} P_+ S_s = G_{\ell} P_+ G_{\ell} S_s = Q_{\ell} G_{\ell} S_s,
\]
(4.21)
which shows that the range of \(Q_{\ell}\) is mapped onto itself by \(V_s\); further
\[
\tau_{-t} Q_{\ell} = \tau_{-t} G_{\ell} P_+ G_{\ell} = G_{\ell} \tau_t P_+ G_{\ell} = G_{\ell} P_{[t, \infty)} \tau_t G_{\ell} = G_{\ell} P_{[t, \infty)} G_{\ell} \tau_{-t};
\]
(4.22)
has range contained in the range of \(Q_{\ell}\) for \(t > 0\), so \(Q_{\ell} L^2\) is simply invariant.

To obtain the converse, we consider the Fourier transforms of the groups. On \(e^{i t x} u_\nu H^2\), the unitary semigroups operate as
\[
\hat{V}_s = \mathcal{F} V_s \mathcal{F}^*: f(x) \mapsto e^{i s \nu} u_\nu(x) u_\nu(s - x) f(x - s) \quad (s \in \mathbb{R});
\]
(4.23)
\(\mathcal{F} \tau_{-t} \mathcal{F}^* = S_t\). To verify (4.23), we recall the flip map \(J\) by \(J f(x) = f(-x)\), and observe that \(\mathcal{F} J = J\), and \(\mathcal{F}^* \mathcal{F}^* = J\). We have
\[
\mathcal{F} V_s \mathcal{F}^* = \mathcal{F} G_{\ell} S_s G_{\ell} \mathcal{F}^* = S_{\ell} M_{u_\nu} \mathcal{F}^* S_s \mathcal{F} \mathcal{F}^* G_{\ell} \mathcal{F}^*
\]
(4.24)
so that \(\mathcal{F} V_s \mathcal{F}^* = S_{\ell} M_{u_\nu} \tau_s J S_{\ell} M_{u_\nu} J\). Using the Weyl–von Neumann relation for \(\tau_s\) and \(S_{\ell}\), one can easily simplify this expression to obtain \(\hat{V}_s = \mathcal{F} V_s \mathcal{F}^* = e^{i s \nu} M_{u_\nu} N_s \tau_s\), where \(N_s f(x) = u_\nu(s - x) f(x)\). The functions \(u_\nu\) satisfy \(u_\nu(-x) u_\nu(x) = 1\) and
\[
e^{i s \nu} u_\nu(x) u_\nu(s - x) = e^{i s \nu} 2^i \frac{\Gamma((1 + \nu + i s)/2) \Gamma((1 + \nu + i s - i s)/2)}{\Gamma((1 + \nu - i s)/2) \Gamma((1 + \nu + i s - i s)/2)}.
\]
(4.25)
Suppose that \(K\) is such an invariant subspace. Then by Beurling’s theorem, there exists a unimodular and measurable function \(w\) such that \(\mathcal{F} K = w H^2\); further, this \(w\) is uniquely determined up to a unimodular constant multiple. We apply \(\mathcal{F} V_s \mathcal{F}^*\) to this identity, and deduce by double invariance and (4.23) that
\[
\{ e^{i s \nu} u_\nu(x) u_\nu(s - x) w(x - s) f(x - s) : f \in H^2 \} = w H^2;
\]
(4.26)
so that,

\[ e^{i\ell s} \nu(x)u_{\nu}(s-x)w(x-s) = c(s)w(x) \quad (s \in \mathbb{R}) \]

holds for some \( c(s) \). We re-arrange this to

\[ u_{\nu}(s-x)w(x-s) = c(s)w(x) \]

then solve to obtain \( w(x) = e^{i\alpha x}u_{\nu}(x) \) for some \( \alpha \in \mathbb{R} \). Hence \( \mathcal{F}K = e^{i\alpha x}u_{\nu}(x)H^2 \), so \( K = Q_{\alpha}L^2 \) by Theorem 4.2(ii).

\[ \square \]

**Corollary 4.5.** The unitary groups \((\hat{V}_s) (s \in \mathbb{R})\) and \((S_t) (t \in \mathbb{R})\) form a Weyl pair. The closed linear subspaces that are simply invariant for \( \hat{V}_s (s \geq 0) \) and \( S_t (t \geq 0) \) have the form

\[ K_{\alpha,\beta} = \{ e^{i\alpha x - i\beta x^2} u_{\nu}(x) f(x) : f \in H^2 \} \quad (\alpha \in \mathbb{R}, \beta > 0). \]  

(4.27)

**Proof.** We shall check simple invariance of \( K_{\alpha,\beta} \) under \( \hat{V}_s (s \in \mathbb{R}) \), since the other statements follow easily from the proof of Theorem 4.4 and the Katavolos–Power theorem. We have

\[ \hat{V}_s : e^{i\alpha x - i\beta x^2} u_{\nu}(x) f(x) \mapsto e^{i\ell s - i\alpha s - i\beta s^2} e^{i\alpha x - i\beta x^2} u_{\nu}(x) e^{2i\beta sx} f(x-s), \]  

(4.28)

where \( e^{2i\beta sx} f(x-s) \in S_{2\beta s} H^2 \subset H^2 \) for \( s, \beta > 0 \).

**5 Soft edge operators and the Airy group**

With \( D = -i \frac{\partial}{\partial x} \) the Airy group \( e^{itD^3} \) is a \( C_0 \) group of unitary operators, as defined by

\[ e^{itD^3} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\xi^3+i\xi x} \mathcal{F}f(\xi) d\xi. \]  

(5.1)

In this section we shall consider how the Airy group is related to the kernel function of (1.11). Here \( J_t \) denotes the operator \( e^{itD^3} J \) on \( L^2(\mathbb{R}) \), not a Bessel function, and we shall use a subscript \( t \) to indicate scaling of the space variables \( x \) and \( y \) with respect to time \( t \).

**Lemma 5.1.** The operator \( J_t = e^{itD^3} J \) is self-adjoint with \( J_t^2 = I \), and \( J_t \) as an integral operator on \( L^2(\mathbb{R}) \) has kernel

\[ \frac{1}{(3t)^{1/3}} \text{Ai}\left(\frac{x+y}{(3t)^{1/3}}\right). \]  

(5.2)

**Proof.** For a compactly supported and smooth function \( f \) we have

\[ J e^{itD^3} J f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\xi^3-i\xi x} \mathcal{F}f(\xi) d\xi = e^{-itD^3} f(x), \]  

(5.3)
so $J_t^2 = I$. Further, the kernel of $J_t$ is given by
\[
e^{itD^3} J f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi^3 + ix} \int_{-\infty}^{\infty} e^{i\xi y} f(y) dy d\xi
\]
\[
= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t + i\xi (x+y)} d\xi \right\} f(y) dy
\]
\[
= \int_{-\infty}^{\infty} \frac{1}{(3t)^{1/3}} \text{Ai}\left( \frac{x+y}{(3t)^{1/3}} \right) f(y) dy.
\]
(5.4)

Since the Airy function is real valued, it also follows that $J_t$ is self-adjoint.

Most of the next result is essentially contained in \cite[Lemma 2]{28}, but we include a proof for completeness.

**Proposition 5.2.** (i) The operator
\[
W_t = e^{itD^3} P_+ e^{-itD^3} = J_t P_+ J_t
\]
on $L^2(\mathbb{R})$ is an orthogonal projection and the range of $FW_tF^*$ equals $e^{it\xi^3} H^2$.

(ii) The Hankel operator $\Gamma_{0,t} = P_+ J_t P_+$ has square $\Gamma_{0,t}^2 = P_+ W_t P_+$.

(iii) The kernel of $W_t$ as an integral operator on $L^2(\mathbb{R})$ is
\[
W_t(x, y) = \frac{1}{(3t)^{1/3}} W_{1/3}\left( \frac{x}{(3t)^{1/3}}, \frac{y}{(3t)^{1/3}} \right)
\]
(5.6)

where $W_{1/3}$ is the Airy kernel as in (1.11).

**Proof.** (i) By Lemma 5.1, we have $W_t^2 = J_t P_+ J_t^2 P_+ J_t = J_t P_+ J_t = W_t$, so that $W_t$ is a projection; further $W_t^* = W_t$. The range of $W_t$ equals $\{W_t k : k \in L^2(\mathbb{R})\}$, or equivalently the range of $J_t P_+$.

If $f \in L^2(\mathbb{R}_+)$, then $F f(\xi) = \hat{G}(\xi)$, where $G \in H^2$. Since $e^{-itD^3}$ is unitary, we have $W_t L^2 = e^{itD^3} P_- e^{-itD^3} L^2 = e^{itD^3} P_- L^2$, and hence the image of $WL^2$ under the Fourier transform $F$ is $FW_t L^2 = \{e^{it\xi^3} F(x) : F \in H^2\}$.

(ii) We have $\Gamma_{0,t} = P_+ e^{itD^3} J P_+$ and hence
\[
\Gamma_{0,t}^2 = P_+ e^{itD^3} J P_+ e^{itD^3} J P_+
\]
\[
= P_+ e^{itD^3} J P_+ J e^{itD^3} J P_+ = P_+ e^{itD^3} P_- e^{-itD^3} P_+ = P_+ W_t P_+.
\]
(5.7)

(iii) It also follows from the Lemma that the kernel function is
\[
W_t(x, y) = \frac{1}{(3t)^{2/3}} \int_0^\infty \text{Ai}\left( \frac{x+u}{(3t)^{1/3}} \right) \text{Ai}\left( \frac{u+y}{(3t)^{1/3}} \right) du,
\]
(5.8)
a formula which reduces to (5.6) and (5.7) on account of the identity

\[
\int_0^\infty \text{Ai}(x + u)\text{Ai}(u + y) \, du = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}.
\] (5.9)

This formula is presented by Tracy and Widom in [28], and follows from Proposition 2.2 since \(A(x) = \text{Ai}(x)\) satisfies

\[
\frac{d}{dx} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}.
\]

Evidently \(W_0 = P_-\), and the relative positions of the ranges of \(P_-\) and \(W_t\) are described in the following Proposition, with different conclusions from Theorem 4.2.

**Definition.** [9] A function \(G \in H^2\) is said to be cyclic (for the backward shifts) when \(\text{span}\{S_t^* G; t > 0\}\) is dense in \(H^2\). Likewise, \(f \in L^2(\mathbb{R}_+)\) is cyclic when \(\text{span}\{\tau_t^* f : t > 0\}\) is dense in \(L^2(\mathbb{R}_+)\); \(g \in L^2(\mathbb{R}_-)\) is cyclic when \(\text{span}\{\tau_t^* g : t < 0\}\) is dense in \(L^2(\mathbb{R}_-)\).

**Proposition 5.3.** (i) For each \(t \neq 0\), the subspaces \(W_t L^2 \cap L^2(\mathbb{R}_-)\) and \((W_t L^2)^\perp \cap L^2(\mathbb{R}_+)\) equal \(\{0\}\); while any non-zero vector in \(W_t L^2 \cap L^2(\mathbb{R}_+)\) or \((W_t L^2)^\perp \cap L^2(\mathbb{R}_-)\) is cyclic.

(ii) For each \(t > 0\), the operator \(W_t\) on \(L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)\) has block matrix form

\[
\begin{bmatrix}
P_- W_t P_- & P_- W_t P_+ \\
P_+ W_t P_- & P_+ W_t P_+
\end{bmatrix} \equiv \begin{bmatrix} B & c^2 \\
c^2 & c^1 \end{bmatrix}.
\] (5.10)

(iii) For any real \(t\), the operators \(P_+ W_t P_-\) and \(P_- W_t P_+\) are Hilbert–Schmidt.

**Proof.** (i) First we check that \(W_t L^2 \cap L^2(\mathbb{R}_-) = 0\), or equivalently by Proposition 5.2(i) that \(e^{it\xi^3} H^2 \cap H^2 = 0\). Suppose that \(F, G \in H^2\) are non-zero and satisfy \(e^{it\xi^3} F(\xi) = G(\xi)\) for almost all \(\xi \in \mathbb{R}\). Then \(K(\zeta) = e^{it\zeta^3} F(\zeta) - G(\zeta)\) is a holomorphic function with zero boundary values at almost all points of \(\mathbb{R}\); so by the Lusin–Privalov theorem, \(K(\zeta)\) is identically zero on \(C_+\). Now by Szegö’s Theorem [17, page 108], the integrals

\[
\int_{-\infty}^{\infty} \frac{\log |F(\xi + i\eta)|}{1 + \xi^2} \, d\xi \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\log |G(\xi + i\eta)|}{1 + \xi^2} \, d\xi
\] (5.11)

converge. But this contradicts the identity \(e^{it\xi^3} F(\xi) = G(\xi)\), since

\[
t \int_{-\infty}^{\infty} \frac{3(\xi + i\eta)^3}{1 + \xi^2} \, d\xi
\] (5.12)

diverges for \(\eta, t > 0\); so \(F = G = 0\). Likewise the only solution of the equation \(e^{it\xi^3} F(\xi) = G(\xi)\) with \(F, G \in H^2\) is \(F = G = 0\).
Next prove that all non-zero vectors in $W_t L^2 \cap L^2(\mathbb{R}_+)$ are cyclic. Suppose that $G \neq 0$ is a non-cyclic vector in $H^2 \cap e^{it\xi_3}H^2$ so that $\overline{G(\xi)} = e^{it\xi_3}F(\xi)$ for some $F \in H^2$, and where $G \perp uH^2$ for some inner function $u$. We have $u\overline{G} \in H^2$; so we introduce inner functions $v$ and $w$, and an outer function $\theta$, such that $u\overline{G} = v\theta$ and $F = w\theta$. Then, as in [9, Theorem 3.1.1],

$$e^{it\xi_3} = \frac{\overline{G}}{F} = \frac{v}{uw}$$

(5.14)

is a quotient of inner functions and hence is of finite Nevanlinna type, but the corresponding logarithmic integral (5.12) diverges, and we have a contradiction. (The author conjectures that $W_t L^2 \cap L^2(\mathbb{R}_+) = 0$ so that $W_t L^2$ and $L^2(\mathbb{R}_+)$ are in general position, since any non-zero elements in the intersecting subspaces would satisfy some implausible equations.)

(ii) The Hankel operator $\Gamma_{0,t} = P_+ J_t P_+ = P_+ e^{itD^3} J P_+$ has kernel

$$\frac{1}{(3t)^{1/3}} I_{(0,\infty)}(x) Ai\left(\frac{x+y}{(3t)^{1/3}}\right) I_{(0,\infty)}(y);$$

(5.15)

which is of Hilbert–Schmidt type; see [23, page 46]. Indeed, the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(3t)^{2/3}} Ai\left(\frac{x+y}{(3t)^{1/3}}\right)^2 dx dy$$

may be transformed by the substitution $u = x + y$ to the convergent integral

$$\frac{1}{(3t)^{2/3}} \int_0^\infty u Ai\left(\frac{u}{(3t)^{1/3}}\right)^2 du < \infty;$$

(5.16)

here we use the bounds from [11, page 43]

$$Ai(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}\left(1 + O(x^{-3/2})\right) \exp\left(-\frac{2}{3}x^{3/2}\right) \quad (x \to \infty).$$

(5.17)

Hence the off-diagonal operators $P_- W_t P_+ = P_- J_t (P_+ J_t P_+)$ and $P_+ W_t P_- = (P_+ J_t P_+) J_t P_-$ are Hilbert–Schmidt. For the bottom-right entry, we have a stronger conclusion, namely that $P_- W_t P_+ = (P_+ J_t P_+)(P_+ J_t P_+)$ is trace class.

(iii) When we replace $t \geq 0$ by $t \leq 0$, we need to switch the roles of $P_+$ and $P_-$ in the previous discussion and we deduce that $P_- W_t P_+$ and $P_+ W_t P_-$ are Hilbert–Schmidt, while $P_- W_t P_-$ is of trace class.

\[ \square \]

Theorem 5.4. (i) The $C_0$ unitary groups $S_s$ and $U_t = e^{-it(D-x^2)}$ satisfy the Weyl relations $S_s U_t = e^{ist} U_t S_s$ for $s, t \in \mathbb{R}$. 

(ii) For $\alpha \geq 0$ and real $\delta$, the subspace $e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} H^2$ is simply invariant for $S_s (s \geq 0)$ and $U_t (t \geq 0)$. Conversely, if $\mathcal{T}$ is a non-trivial simply invariant subspace for $S_s (s \geq 0)$ and for $U_s (s \geq 0)$, then $\mathcal{T} = e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} H^2$ for some $\alpha \geq 0$ and real $\delta$.

**Proof.** (i) One can prove directly that the operators $U_t$ defined by

$$ U_t f(x) = e^{i(x^2 t - xt^2 + t^3/3)} f(x - t) \tag{5.18} $$

define a $C_0$ unitary group on $L^2(\mathbb{R})$. This formula for the $U_t$ of (5.18) was obtained by the method of characteristics. Indeed, when $f$ is differentiable, the function $g(x, t) = e^{i(x^3 - (x-t)^3)/3} f(x - t)$ satisfies

$$ \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} = ix^2 g, $$
$$ g(x, 0) = f(x); \tag{5.19} $$

and so $U_t f(x) = e^{-it(D-x^2)} f(x) = g(x, t)$ gives the unique solution of the initial value problem (5.19).

Let $V$ be the unitary operator $V : f(x) \mapsto e^{ix^3/3} f(x)$ on $L^2(\mathbb{R})$, then clearly $S_s = V^* S_s V$. The generator of the unitary group $V^* U_t V$ equals

$$ -iV^*(D - x^2)V = -ie^{-ix^3/3}(D - x^2)e^{ix^3/3} = -\frac{\partial}{\partial x} = -iD; \tag{5.20} $$

so by the uniqueness of groups with given generator we have $V^* U_s V = e^{-isD} = \tau_s$ and hence $U_s = V e^{-isD} V^* = V \tau_s V^*$. By conjugating the Weyl relations $\tau_s S_t = e^{-ist} S_t \tau_s$ for $(s, t \in \mathbb{R})$ by $V$, we can deduce (5.19).

(ii) Clearly any $\mathcal{T} = e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} H^2$ is simply invariant under $S_s (s \geq 0)$, and we can use the preceding calculations to show that $\mathcal{T}$ is also invariant for $U_s (s \geq 0)$. Indeed, for $g \in \mathcal{T}$ we can take $f \in H^2$ such that $g(x) = e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} f(x)$ and we have

$$ U_s g = U_s (e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} f) $$
$$ = V \tau_s V^* V \{e^{-i\alpha x^2 + i\delta x} f\} $$
$$ = V \tau_s \{e^{-i\alpha x^2 + i\delta x} f\} $$
$$ = e^{2i\alpha x - i\alpha s^2 - i\delta s} e^{ix^3/3-\iota \alpha x^2 + \iota \delta x} f(x - s) \tag{5.21} $$

where $f(x - s)$ is an $H^2$ function, so $U_s g$ belongs to the subspace $S_{2\alpha s} \mathcal{T}$ of $\mathcal{T}$. This proves the forward implication.

To prove the converse, we take any $\mathcal{T}$ that is simply invariant as in the Theorem, and observe that $V^* \mathcal{T}$ is simply invariant under $S_s (s \geq 0)$ since $V^*$ commutes with $S_s$, and
\( V^* \mathcal{T} \) is also invariant under \( \tau_s \ (s \geq 0) \) since \( \tau_s V^* \mathcal{T} = V^* U_s \mathcal{T} \subseteq V^* \mathcal{T} \). By the Katavolos–Power Theorem [15], there exist \( \alpha > 0 \) and a real \( \delta \) such that \( V^* \mathcal{T} = e^{-i\alpha x^2 + i\delta x} H^2 \), and hence \( \mathcal{T} \) has the required form.

**Corollary 5.5.** The unitary groups \( \tau_{-s} = e^{isD} \) and \( \hat{U}_t = e^{it(D^2 + x)} \) form a Weyl pair on \( L^2 \), such that the lattice of subspaces \( \{ \tau_{-\delta} \hat{U}_{-\alpha} W_{1/3} L^2 : \delta \in \mathbb{R}, \alpha > 0 \} \) gives the set of simply invariant closed linear subspaces for \( (\tau_{-s}) \ (s \geq 0) \) and \( (\hat{U}_t) \ (t \geq 0) \).

**Proof.** The strongly continuous unitary semigroups \( \tau_{-s} = F S_s F \ (s \geq 0) \) and \( F \hat{U}_t F \) and \( F S_t F \ (t \geq 0) \) have generators \( iD \) and \( i(D^2 + x) \) respectively, so \( (\tau_{-s}, \hat{U}_t) \) forms a Weyl pair by Theorem 5.4(ii) and Proposition 5.2(i). Under the Fourier transform, we have

\[
F \tau_{-\delta} \tau_{-\alpha} \hat{U}_{-\alpha} W_{1/3} L^2 = S_\delta S_\alpha U_{-\alpha} VH^2 = \{ e^{i\delta x - i\alpha x^2 + ix^3/3} f(x) : f \in H^2 \};
\]

hence the subspaces correspond as stated in the Corollary.

**6. Mathieu functions and periodic potentials**

One can construct Tracy–Widom operators over the circle by means of the differential equations of section 2. Kernels of the form (6.6) below arise in the theory of random unitary matrices with Haar probability measure as in [22, p. 195]. The purpose of this section is to introduce examples beyond the list in [30].

Let \( S \) be the \( 2 \times 2 \) fundamental solution matrix of Hill’s equation with smooth \( \pi \)-periodic potential \( q \), so that

\[
\frac{d}{dx} S = \begin{bmatrix} 0 & 1 \\ -(\lambda + q(x)) & 0 \end{bmatrix} S, \quad S(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};
\]

then \( \det S = 1 \), and \( \Delta(\lambda) = \text{trace } S(\pi) \) defines the discriminant. When \( \lambda \) is real, evidently \( \Delta(\lambda)^2 \geq 4 \) if and only if the eigenvalues of \( S(\pi) \) are real, and \( \Delta(\lambda)^2 = 4 \) occurs if and only if \( S \) is periodic with period \( \pi \) or \( 2\pi \). The periodic spectrum

\[
\Lambda = \{ \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \ldots < \lambda_n \nearrow \infty \}
\]

of Hill’s equation consists of those real \( \lambda \) such that

\[
y'' + (\lambda + q)y = 0
\]
G. Blower: Operators for soft and hard edges 27

has a non-trivial π or 2π periodic solution as in [18, p.11]. The discriminant satisfies

$$4 - \Delta^2(\lambda) = 4(\lambda - \lambda_0) \prod_{j=1}^{\infty} \frac{(\lambda_{2j-1} - \lambda)(\lambda_{2j} - \lambda)}{j^4}. \quad (6.4)$$

**Theorem 6.1** For each real \(\alpha\) there exists an infinite sequence of \(\lambda_n\) such that Hill’s equation (6.3) with potential \(\alpha \cos 2x\) has a non-trivial 2π-periodic and real solution \(A\).

For such \(A\), let \(W\) be the kernel

$$W(x, y) = \frac{A(x)A'(y) - A'(x)A(y)}{\sin(x - y)}, \quad (6.5)$$

which continuously differentiable and doubly periodic with period 2π.

(i) Then \(W\) defines a self-adjoint and Hilbert–Schmidt operator on \(L^2[0, 2\pi]\);

(ii) the eigenfunction corresponding to each non-zero simple eigenvalue of \(W\) is a 2π-periodic solution of (6.3).

**Proof.** (i) When \(\alpha = 0\) and \(\lambda = n^2\) with \(n = 1, 2, \ldots\), we can take \(A(x) = \sin nx\), and recover the kernel

$$W(x, y) = \frac{n \sin n(x - y)}{\sin(x - y)} \quad (6.6)$$

as in the circular ensemble.

When \(\alpha \neq 0\), there exists by Hochstadt’s theorem [18, page 40] an increasing sequence \((\lambda'_n)\) which satisfies the estimates

$$\lambda'_{2n-1} = (2n - 1)^2 + \frac{\alpha^2}{32n^2} + o(n^{-2}) \quad (n \to \infty) \quad (6.7)$$

$$0 < \lambda'_{2n} - \lambda'_{2n-1} = o(n^{-2}),$$

and such that (6.3) has a non-trivial solution \(A\). This is Mathieu’s function of the first kind. As in section 2, we can calculate

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)W(x, y) = -2\alpha(\sin x \cos y + \cos x \sin y)A(x)A(y), \quad (6.8)$$

hence

$$W(x, y) = -\alpha \int_0^{x+y} \sin \theta A(\frac{1}{2}(\theta + x - y))A(\frac{1}{2}(\theta - x + y)) d\theta + g(x - y), \quad (6.9)$$

where, by letting \(x = -y\), one finds

$$g(x) = \frac{A(x/2)A'(-x/2) - A'(x/2)A(-x/2)}{\sin x}. \quad (6.10)$$
Evidently $W$ is a real, symmetrical and continuous kernel, and hence determines a self-adjoint and Hilbert–Schmidt operator on $L^2[0, 2\pi]$.

(ii) By differentiating (6.8) and recalling the definition of $W$, one can easily deduce that
\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) W(x, y) = \alpha (\cos 2x - \cos 2y) W(x, y). \tag{6.10}
\]

For $\nu \neq 0$, any non-zero solution $f \in L^2[0, 2\pi]$ of the integral equation
\[
\nu f(x) = \int_0^{2\pi} W(x, y) f(y) \, dy \tag{6.11}
\]
extends to define a twice continuously differentiable and $2\pi$-periodic function on $\mathbb{R}$. Now $g(x) = f''(x) + \alpha \cos 2xf(x)$ also gives a $2\pi$ periodic and continuous solution of (6.11); this follows from (6.10) by an integration-by-parts argument. By simplicity of the eigenvalue, we deduce that $g$ is a constant multiple of $f$, and hence that $f$ is a $2\pi$ periodic solution of Mathieu’s equation.

Conversely, let $M_\Lambda$ be the space of potentials $q$ that have periodic spectrum equal to a given $\Lambda$. McKean, van Moerbeke and Trubowitz [19, 20] have shown that $M_\Lambda$ can be considered as a torus
\[
M_\Lambda = \left\{ \frac{1}{2} (\Delta(x_j) + \sqrt{\Delta(x_j)^2 - 4}) : \lambda_{2j-1} \leq x_j \leq \lambda_{2j}; j = 1, 2, \ldots \right\} \tag{6.12}
\]
over the product over the intervals of instability $(\lambda_{2j-1}, \lambda_{2j})$ where $\Delta(\lambda)^2 < 4$ and that $M_\Lambda$ is associated with the Jacobi manifold over the Riemann surface of $\sqrt{\Delta^2(\lambda) - 4}$. Hence $M_\Lambda$ can have dimension $n = 0, 1, \ldots, \infty$, equal to the number of simple zeros $\Delta(\lambda)^2 - 4$. The periodic spectrum $\Lambda$ is preserved by Hamiltonian flows; in particular, there is a $2\pi$ periodic Korteweg–de Vries flow on $M_\Lambda$ associated with
\[
\frac{\partial q}{\partial t} = 3q \frac{\partial q}{\partial x} - \frac{1}{2} \frac{\partial^3 q}{\partial x^3}. \tag{6.13}
\]
By Theorem 6.1, the potential $\alpha \cos 2x$ gives an infinite-dimensional $M_\Lambda$ on which there are solutions to $KdV$ that are $2\pi$ periodic in the space variable and almost periodic in time [4, Appendix]. This makes an interesting contrast with the concentric KdV equation $u_t + u/(2t) - 6uu_x + u_{xxx} = 0$ which is used via $\Gamma_2(x)$ in Theorem 2.3 to linearize $P_{II}$ for the soft edge ensemble [10, page 173].

Acknowledgements
I am grateful to Stephen Power and Sergei Treil for helpful conversations. This work was partially supported by EU Training Network Grants HPRN-CT-2000-00116 ‘Classical Analysis and Operators’ and MRTN-CT-2004-511953 ‘Phenomena in High Dimensions’.

References
[1] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[2] D. Alpay, The Schur algorithm, reproducing kernel Hilbert spaces, and system theory, American Mathematical Society/Société Mathématique de France, 2001.
[3] G. Aubrun, A sharp small deviation inequality for the largest eigenvalue of a random matrix, Séminaire de Probabilités XXXVIII 320–337, Lecture Notes in Math. 1857, Springer, Berlin, 2005.
[4] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations II: the KdV equation, Geom. Funct. Anal. 3 (1993), 209–262.
[5] A. Borodin, Biorthogonal ensembles, Nuclear Phys. B 536 (1999), 704–732.
[6] L. De Branges, Hilbert Spaces of Entire Functions, Prentice–Hall, Englewood Cliffs, 1968.
[7] J.–F. Burnol, Sur les ‘espaces de Sonine’ associés par de Branges à la transformation de Fourier, C.R. Math. Acad. Sci. Paris 335 (2002), 689–692.
[8] P.A. Deift, A.R. Its and X. Zhou, A Riemann–Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. of Math. (2) 146 (1997), 149–235.
[9] R.G. Douglas, H.S. Shapiro and A.L. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, Ann. Inst. Fourier (Grenoble) 20 (1970), 37–76.
[10] P.G. Drazin and R.S. Johnson, Solitons: an Introduction, Cambridge University Press, Cambridge, 1989.
[11] A. Erdélyi, Asymptotic Expansions, Dover, California, 1955.
[12] P.J. Forrester, The spectrum edge of random matrix ensembles, Nuclear Physics B 402 (1993), 709-728.
[13] P.J. Forrester and E.M. Peter J.; Rains, Correlations for superpositions and decimations of Laguerre and Jacobi orthogonal matrix ensembles with a parameter, Probab. Theory Related Fields 130 (2004), 518–576.
[14] E.L. Ince, Ordinary differential equations, Dover Publications, London, 1956.
[15] A. Katavolos and S.C. Power, The Fourier binest algebra, Math. Proc. Cambridge Philos. Soc. 122 (1997), 525–539.
[16] A. Katavolos and S.C. Power, Translation and dilation invariant subspaces of $L^2(\mathbb{R})$, J. reine angew. Math. 552 (2002), 101–129.
[17] P. Koosis, Introduction to $H_p$ Spaces, Cambridge University Press, Cambridge, 1980.
[18] W. Magnus and S. Winkler, Hill’s Equation, Dover, New York, 1966.
[19] H.P. McKea and P. van Moerbecke, The spectrum of Hill’s equation, Invent. Math. 30 (1975), 217–274.
[20] H.P. McKea and E. Trubowitz, Hill’s operator and hyperelliptic function theory in the presence of infinitely many branch points, Comm. Pure Appl. Math. 29 (1976), 143–226.
[21] A.N. Megretskiǐ, V.V. Peller and S.R. Treil, The inverse spectral problem for self-adjoint Hankel operators, Acta Math. 174 (1995), 241–309.
[22] M.L. Mehta, Random Matrices, 2nd ed., Academic Press, San Diego, 1991.
[23] V. Peller, Hankel Operators and Their Applications, Springer, New York, 2003.
[24] J. Rovnyak and V. Rovnyak, Sonine spaces of entire functions, J. Math. Anal. Appl. 27 (1969), 68–100.
[25] I. N. Sneddon, The use of Integral Transforms, McGraw–Hill, New Delhi, 1974.
[26] N. Sonine, Recherches sur les fonctions cylindriques et le développement des fonctions continue en séries, Math. Annal. 16 (1880), 1–80.
[27] G. Szegő, Orthogonal Polynomials, American Mathematical Society, New York, 1959.
[28] C.A. Tracy and H. Widom, Level spacing distribution and the Airy kernel, Comm. Math. Phys. 159 (1994), 151–174.
[29] C.A. Tracy and H. Widom, Level spacing distribution and the Bessel kernel, Comm. Math. Phys. 161 (1994), 289–309.
[30] C.A. Tracy and H. Widom, Fredholm determinants, differential equations and matrix models, Comm. Math. Phys. 163 (1994), 33–72.
[31] C.A. Tracy and H. Widom, A system of differential equations for the Airy process, Electron. Comm. Probab. 8 (2003), 93–98.
[32] J. von Neumann, Die Eindeutigkeit der Schrödinger Operatoren, Math. Ann. 104 (1931), 570–578.