Vibration of Generalized Double Well Oscillators

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We have applied the Melnikov criterion to examine a global homoclinic bifurcation and transition to chaos in a case of a double well dynamical system with a nonlinear fractional damping term and external excitation. The usual double well Duffing potential having a negative square term and positive quartic term has been generalized to a double well potential with a negative square term and a positive one with an arbitrary real exponent \( q > 2 \). We have also used a fractional damping term with an arbitrary power \( p \) applied to velocity which enables one to cover a wide range of realistic damping factors: from dry friction \( p \rightarrow 0 \) to turbulent resistance phenomena \( p = 2 \). Using perturbation methods we have found a critical forcing amplitude \( \mu_c \) above which the system may behave chaotically. Our results show that the vibrating system is less stable in transition to chaos for smaller \( p \) satisfying an exponential scaling law. The critical amplitude \( \mu_c \) as an exponential function of \( p \). The analytical results have been illustrated by numerical simulations using standard nonlinear tools such as Poincare maps and the maximal Lyapunov exponent. As usual for chosen system parameters we have identified a chaotic motion above the critical Melnikov amplitude \( \mu_c \).

1 Introduction

A nonlinear oscillator with single or double well potentials of the Duffing type and linear damping is one of the simplest systems leading to chaotic motion studied by [1,2,3,4]. The problem of its nonlinear vibrations has attracted researchers from various fields of research across natural science and physics [4,5,6], mathematics [8] mechanical engineering [10,11,14,12,13], and finally electrical engineering [1,2,3]. This system, for a negative linear part of stiffness, shows homoclinic orbits, and the transition to chaotic vibration can be treated analytically via the Melnikov method [7]. Such a treatment has been already performed successfully to selected problems with various potentials [8,9,14]. Vibrations of a single Duffing oscillator have got a large bibliography [1,2,3,5,8,9,10,11,12,13,14,15,16,17,18]. In the last decade coupled Duffing oscillators [19,20,21,22,23] with numerous modifications to potential and forcing parts have been studied. On the other hand the problem of nonlinear damping in chaotically vibrating system has not been discussed in detail. Some insight into this problem can also be found in the context of self excitation effects [15,19,22,25,24,26] and dry friction effects [33,34,35,36]. In the paper by Trueba \textit{et al.} [16], the systematic discussion on square and cubic damping effects on global homoclinic bifurcations in the Duffing system has been given. Recently Trueba \textit{et al.} [17] and Borowiec \textit{et al.} [18] have analyzed a single degree of freedom nonlinear oscillator with the Duffing potential and fractional damping. Different aspects fractionally damped systems have been studied recently by Mickens \textit{et al.}, Gottlieb, and Mickens [27,28,29]. On the other hand Maia \textit{et al.} and Padovan and Sawicki [30,32,31] analyzed similar systems where fractional damping have been introduced in different way through a fractional derivative. Awrejcewicz and Holicke [37] and more recently Awrejcewicz and Pyryev [38] applied Melnikov’s method in the presence of dry friction for a stick-slip oscillator. More general introduction to the problem of non-smooth or discontinuous mechanical systems can be found in [39,40].

In the present paper we revisit this problem looking for a global homoclinic bifurcation and transition to chaotic vibrations in a system described by a more general double well potential where its usual positive quartic term has been generalized to term with an arbitrary real exponent grater than 2. Below we would also apply a nonlinear damping term with a fractional exponent covering the gap between viscous, dry friction and turbulent damping phenomena.

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The equation of motion has the following form:

$$\ddot{x} + \alpha \dot{x} |\dot{x}|^{p-1} + \delta x + \gamma \text{sgn}(x)|x|^{q-1} = \mu \cos \omega t,$$

(1)

where $x$ is displacement and $\dot{x}$ velocity, respectively, while the external force $F_x$:

$$F_x = -\delta x - \gamma \text{sgn}(x)|x|^{q-1},$$

(2)

and corresponding potential $V(x)$ (Fig. 1a) is defined as:

$$V(x) = \frac{\delta x^2}{2} + \frac{\gamma |x|^q}{q},$$

(3)

where $q > 2$ is a real number. In spite of the definition $V(x)$ (Eq. 3) in terms of absolute value $|x|$ it is still a function of $C^2$ class if only $q > 2$ (see Appendix A).

The non-linear damping term is defined by the exponent $p$:

$$d_{pt}(\dot{x}) = \alpha \dot{x} |\dot{x}|^{p-1}.$$

(4)

In Fig. 1b we have plotted the above function versus velocity ($v = \dot{x}$) for few values of $p$. Note that, the case $p \to 0$ (see $p = 0.1$ in Fig. 1b for a relatively small velocity) mimics the dry friction phenomenon [33, 34].

![Fig. 1](image_url)

**Fig. 1** External potential $V(x) = \frac{\delta x^2}{2} + \frac{\gamma |x|^q}{q}$ (Eq. 3) for $\delta = -2$ for a few values of $q$ ($q = \gamma > 2$ in Fig. 1a), Damping term for various $p$ (Fig. 1b).

## 2 Melnikov Analysis

We start our analysis with the unperturbed Hamiltonian: $H^0$

$$H^0 = \frac{v^2}{2} + V(x).$$

(5)

Note that for our choice of potentials $\delta = -2$ and $\gamma = q$ (Fig. 1a) $V(x)$ has the three nodal points ($x = -1, 0, 1$) where the middle one ($x = 0$) corresponds to the local peak at the saddle point. The existence of this point with a horizontal tangent enables occurrence of homoclinic bifurcations. This includes transitions from regular to chaotic solutions. To study the effects of damping and excitation on the saddle point bifurcations, we apply small perturbations around the homoclinic orbits. Our strategy is to use a small parameter $\epsilon$ to the Eq. 4 with perturbation terms. Uncoupling Eq. 4 into two differential equations of the first order we obtain

$$\dot{x} = v$$

$$\dot{v} = -\epsilon \alpha v |v|^{p-1} - \delta x - \gamma \text{sgn}(x)|x|^{q-1} + \epsilon \tilde{\mu} \cos \omega t,$$

(6)

where $\epsilon \alpha = \alpha$ and $\epsilon \tilde{\mu} = \mu$, respectively.
Fig. 2 Left hand side homoclinic orbits for unperturbed Hamiltonian (Eq. 5). Note, in our case the potential has reflection symmetry over 0-y axis so the orbits appear in pairs for corresponding regions \( x > 0 \) and \( x < 0 \).

At the saddle point \( x = 0 \), for an unperturbed system (Fig. 1a), the system velocity reaches zero \( v = 0 \) (for infinite time \( t = \pm \infty \)) so the total energy has only its potential part which has been gauged out to zero too. Thus transforming Eqs. 3 and 5 for a nodal energy (\( E = 0 \)) and for \( \delta < 0, \gamma > 0 \) we get the following expression for velocity:

\[
v = \frac{dx}{dt} = \sqrt{2 \left( -\frac{\delta x^2}{2} - \frac{\gamma |x|^q}{q} \right)}.
\]

Performing integration over \( x \) we get

\[
t - t_0 = \pm \int \frac{dx}{x \sqrt{\delta - \frac{2\gamma |x|^{q-2}}{q}}},
\]

where \( t_0 \) represents here a time like integration constant.

Integration in Eq. 8 has been performed analytically. For \( q > 2 \), one can write \( x^* \) as:

\[
x^* = x^*(t - t_0) = \pm \left( -\frac{\delta q}{2\gamma} \right)^{\frac{1}{q-2}} \frac{1}{\cosh \frac{\pi}{2} \frac{1}{\sqrt{\delta}} \left( \frac{(q-2)}{2} \sqrt{-\delta} (t - t_0) \right)}.
\]

The corresponding velocity \( v^* \) reads:

\[
v^* = v^*(t - t_0) = \mp \sqrt{-\delta} \left( -\frac{\delta q}{2\gamma} \right)^{\frac{1}{q-2}} \frac{\tanh \frac{\pi}{2} \frac{1}{\sqrt{\delta}} \left( \frac{(q-2)}{2} \sqrt{-\delta} (t - t_0) \right)}{\cosh \frac{\pi}{2} \frac{1}{\sqrt{\delta}} \left( \frac{(q-2)}{2} \sqrt{-\delta} (t - t_0) \right)}.
\]

Due to the reflection symmetry of potential \( V(-x) = V(x) \) (Eq. 3) there are two symmetric solutions for unperturbed homoclinic orbits with ‘+’ and ‘−’ signs in Eqs. 9-10. A family of right hand side homoclinic orbits \((x^*, v^*)\) has been plotted in Fig. 2. In unperturbed case both stable and unstable manifolds (Poincare sections of the orbits are usually denoted by \( W_s \) and \( W_u \)) can be identified with the orbits discussed above while perturbations would influence them in a different way [12]. Existence of cross-sections of between \( W_S \) and \( W_U \) manifolds signals Smale’s horseshoe scenario of transition to chaos.

The distance \( d \) (Fig. 3) between them can be estimated by the Melnikov function \( M(t_0) \):

\[
M(t_0) = \int_{-\infty}^{+\infty} h(x^*, v^*) \wedge g(x^*, v^*) dt
\]

where the corresponding differential form \( h \) means the gradient of unperturbed Hamiltonian (Eq. 3):

\[
h = \left( \delta x^* + \gamma \text{sgn}(x)|x|^{q-1} \right) dx + v^* dv,
\]

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Fig. 3  Schematic plot of stable and unstable manifolds (\(W_s\) and \(W_u\)) of perturbed system Eq. 5. \(d\) denotes the distance between manifolds given by Melnikov function \(M(t_0)\) Eq. 11.

while \(g\) is a perturbation form (Eq. 5) to the same Hamiltonian:

\[
g = \left( \tilde{\mu} \cos \omega t - \tilde{\alpha} v^* |v^*|^{p-1} \right) dx. \tag{13}\]

All differential forms are defined on homoclinic orbits \((x, v) = (x^*, v^*)\) (Eqs. 9-10). Thus the Melnikov function \(M(t_0)\):

\[
M(t_0) = \int_{-\infty}^{+\infty} v^*(t) \left( \tilde{\mu} \cos (\omega (t + t_0)) - \tilde{\alpha} v^*(t) |v^*(t)|^{p-1} \right) dt
\]

\[
= -\sin \omega t_0 \int_{-\infty}^{+\infty} v^*(t) \tilde{\mu} \sin \omega dt - \int_{-\infty}^{+\infty} \tilde{\alpha} v^2(t) |v^*(t)|^{p-1} dt \tag{14}
\]

where \(I_1\) and \(I_2\) are integrals to be evaluated. \(\sin \omega t_0\) appears because of the odd parity of the function \(v^*(t)\) under the above integral where

\[
\cos(\omega (t + t_0)) = \cos(\omega t) \cos(\omega t_0) - \sin(\omega t) \sin(\omega t_0). \tag{15}\]

Thus a condition for a global homoclinic transition, corresponding to a horseshoe type of stable and unstable manifolds cross-section (Fig. 2), can be written as:

\[
\bigvee_{t_0} M(t_0) = 0 \quad \text{and} \quad \frac{\partial M(t_0)}{\partial t_0} \neq 0. \tag{16}\]

For a perturbed system the above constraint together with the explicit form of Melnikov function Eq. 14 gives the critical amplitude \(\mu_c\):

\[
\frac{\mu_c}{\alpha} = \left| \frac{I_2}{I_1} \right|, \tag{17}\]

where \(I_1\) and \(I_2\) are corresponding integrals given in Eq. 14. In case of \(I_1\) we have the following integral

\[
I_1 = \sqrt{-\delta} \left( \frac{-\delta q}{2\gamma} \right)^{\frac{p+1}{2}} \int_{-\infty}^{+\infty} \frac{\tanh \left( \frac{(q-2)\sqrt{-\delta}}{2} \right)}{\cosh \left( \frac{(q-2)\sqrt{-\delta}}{2} \right)} \sin(\omega t) \ dt \tag{18}\]

to be evaluated numerically in general but for some cases can be easily performed numerically (see Appendixes B and C) while, in analogy to [16, 17, 18], \(I_2\) can be expressed as

\[
I_2 = (-\delta)^{\frac{p+1}{2}} \left( \frac{-\delta q}{2\gamma} \right)^{\frac{p+1}{2}} \int_{-\infty}^{+\infty} \frac{\sinh^{p+1} \left( \frac{(q-2)\sqrt{-\delta}}{2} \right)}{\cosh \left( \frac{(q-2)\sqrt{-\delta}}{2} \right)} \ dt
\]

\[
= (-\delta)^{\frac{p+1}{2}} \left( \frac{-\delta q}{2\gamma} \right)^{\frac{p+1}{2}} B \left( \frac{p+2}{2}, \frac{p+1}{q-2} \right), \tag{19}\]

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Fig. 4  Critical amplitude $\mu_c/\alpha$ versus frequency $\omega$ for few values of $p$ ($p = 0.1, 0.5, 1.0, 2.0$) and different $q$ ($q = 2.5$ in Fig. 4a, $q = 3.0$ in Fig. 4b, $q = 3.5$ in Fig. 4c, $p = 4.0$ in Fig. 4d). Dependence of the slope $\kappa$ ($\kappa = \tan \phi$ in Fig. 4e) on the exponent $q$ in Fig. 4e (‘1’ corresponds to present calculations for $\omega = 1$ while ‘2’ is a fitting trail $\kappa = 1/(q/1.3 + 1)$).

where $B(r, s)$ is the Euler Beta function dependent of arbitrary complex arguments with real parts ($\text{Re } r > 0$ and $\text{Re } s > 0$) defined as

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r + s)}, \quad (20)$$

while $\Gamma(r)$ denotes the Euler Gamma function:

$$\Gamma(z) = \int_{0}^{\infty} e^{-s} s^{z-1} \text{d}s \quad \text{for} \quad \text{Re } z > 0. \quad (21)$$

In Fig. 4a–d we plotted the results of Melnikov analysis for a critical amplitude $\mu_c/\alpha$ for few values of $p$ ($p = 0.1, 0.5, 1.0, 2.0$) and different $q$ ($q = 2.5$ in Fig. 4a, $q = 3.0$ in Fig. 4b, $q = 3.5$ in Fig. 4c, $p = 4.0$ in Fig. 4d). For $\mu > \mu_c$, the
system can transit to chaotic vibrations. Note, in spite of some quantitative changes all four figures (Fig. 4a-d) have similar shape and the sequence of corresponding curves with particular exponents $p = 0.1, 0.5, 1.0, 2.0$ is preserved for any $\omega$ and $q$. This gives us a conviction that $p$ may play some independent role. In fact plotting $\ln(\mu_c/\alpha)$ in Fig. 4a versus $p$ and we have got straight lines with characteristic slope independent on $\omega$ but changing with $q$. Dependence of the slope $\kappa$:

$$\kappa = \tan \phi$$

(22)
defined in Fig. 4b, versus $q$ has been plotted in Fig. 4c for $\omega = 1$. Note, the curve '1' corresponds to present calculations for while '2' is a fitting curve:

$$\kappa = \frac{1}{q^{1/3} - 1}$$

(23)

The above scaling is not a surprise taking into account the structure of $M(t_0)$ (Eq. 12). In this expression the exponent $p$ is entering to the second integral independent of $\omega$. On can also look into the analytic formulae for $\mu_c$ in cases of $q = 4$ and 3 in the Appendix B (Eq. B.5-B.8) where the $p$ appears as an exponent.

### 3 Results of Numerical Simulations

To illustrate the dynamical behaviour of the system it is necessary to simulate the proper equations. Here we have used the Runge-Kutta method of the forth order and Wolf algorithm [41] to identify the chaotic motion. In our numerical code we started calculations from the same initial conditions $(x_0, v_0) = (0.45, 0.1)$ for any new examined value of $\mu$. The system parameters $\delta = -2$ and $\gamma = q$ have been chosen the same as for analytic calculations. We have performed numerical calculations for different choices of system parameters: $\alpha$, $\omega$, $p$ and $q$, but here, for or technical reasons, we limited our discussion to $\alpha = 0.1$, $\omega = 1.1$, $p = 0.5$ and $q = 2.5$.

In Fig. 5a we have plotted the maximal Lyapunov exponent versus external forcing amplitude $\mu$. Here one can clearly see points of $\lambda_1$ sing changes. For $\mu \in [0.23, 0.27]$ and $[0.33, 0.38]$ we have got $\lambda_1 > 0$ indicating chaotic vibrations. In Fig. 5b we have plotted the corresponding bifurcation diagram. Thus a black region means chaotic motion. This result, as
Fig. 6 Critical amplitude $\mu_c/\alpha$ versus $\omega$ the dashed line corresponds to $\omega = 1.1$ (Fig. 6a). Phase portrait and Poincare maps for $\omega = 1.1$, $\alpha = 0.1$ and two different $\mu$ ($\mu = 0.05$ in Fig. 6b while $\mu = 0.24$ in Fig. 6c). The results have been obtained for $p = 0.5$ and $q = 2.5$.

well as others, calculated for different sets of system parameters, is consistent with the Melnikov results. For comparison we have plotted the Melnikov curve again (Fig. 6a) with two trial points $\mu = 0.05$ and $\mu = 0.24$ (for $\alpha = 0.1$). There is no doubt that Fig. 6b shows the regular synchronized motion represented by a single loop on a phase portrait and a singular point on Poincare stroboscopic map. On the other hand Fig. 6c shows clearly a strange attractor of chaotic vibration with complex structure of the Poincare map.

4 Summary and Conclusions

We have examined criteria for transition to chaotic vibrations in the double well system with a damping term $dpt(v) = |v|^{p-1}$ described by a fractional exponent $p$ and nonlinear potential with negative square term (related negative stiffness) and a positive term with higher exponent $|x|^q$ where $q > 2$. In spite of non-smoothness of corresponding vector-fields ($h$ and $g$ – Eqs. 11 and 12 respectively) it has been proven in the Appendix A that extra terms to the Melnikov integral
projected out. Thus the critical value of excitation amplitude $\mu$ above which the system vibrate chaotically has been estimated, in by means of the Melnikov theory [7]. For some selected values of the exponent $q (q = 4.3, 8/3, 2/5)$ it was possible to derive a final formula for $\mu_c$ while for other cases one of Melnikov’ integrals has be calculated numerically.

The analytical results have been confirmed by simulations. In this approach we used standard methods of analysis as Poincare maps, bifurcation diagrams and Lyapunov exponent.

The Melnikov method, is sensitive to a global homoclinic bifurcation and gives a necessary condition for excitation amplitude $\mu = \mu_c$ system in its transition to chaos [8,9]. On the other hand the largest Lyapunov exponent [41], measuring the local exponential divergences of particular phase portrait trajectories gives a sufficient condition $\mu = \mu_{c2}$ for this transition which has obviously a higher value of the excitation amplitude $\mu = \mu_{c2} > \mu_{c1}$.

Above the Melnikov transition predictions ($\mu > \mu_{c1}$) we have obtained transient chaotic vibrations [9, 10, 11, 12, 13] as we expected drifting to a regular steady state away the fractal attraction regions separation boundary. This is typical behaviour of the system which undergo global homoclinic bifurcation.

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Appendix A

Starting with the perturbation equation (Eq. 6) we write it in a two element vector form

$$\dot{q} = h + \epsilon g,$$
(A.1)

where

$$q = [x, v]$$
$$h = [v, -\delta x - \gamma \text{sgn}(x)|x|^{q-1}]$$
$$g = [0, -\tilde{\alpha}v |v|^{p-1} + \tilde{\mu} \cos \omega t].$$

On the other hand the homoclinic orbit

$$q^*(t - t_0) = [x^*(t - t_0), v^*(t - t_0)],$$
(A.3)

where $t_0$ is usually defined by simple zero of Melnikov integral [7]. In the limit of extreme time $t \to \pm \infty$ the system state $[x, v]$ reaches a saddle point $[x, v] = [0, 0]$ (see Figs. 1, 2). Consequently, in the aim to examine the Melnikov criterion for chaos appearance, the vector fields $h$ and $g$ are defined on the homoclinic orbit (Fig. 2) as:

$$h(q^*) = [v^*(t - t_0), -\delta x^*(t - t_0) - \gamma \text{sgn}(x^*(t - t_0)|x^*(t - t_0)|^{q-1})]$$
(A.4)

and

$$g(q^*, t) = [0, -\tilde{\alpha}v^*(t - t_0)|v^*(t - t_0)|^{p-1} + \tilde{\mu} \cos \omega t].$$
(A.5)

The perturbed stable and unstable manifolds $W^s$ and $W^u$ read [40]

$$q^{u,s}(t, t_0) = q^*(t - t_0) + \epsilon q^{u,s}_1(t, t_0) + O(\epsilon^2)$$
(A.6)

respectively.

Note the perturbation correction to the homoclinic orbit $q^{u,s}_1(t, t_0)$ in the above expression (Eq. A.6) should be found by solving the following linear differential equation about the examined time $t$ (or a system state $q = q^*(t - t_0)$):

$$\dot{q}^{u,s}_1(t, t_0) = \left(\frac{\partial h_1}{\partial v} - \frac{\partial h_2}{\partial x}\right)|_{q = q^*(t - t_0)} q^{u,s}_1(t, t_0) + g(q^*(t - t_0), t)$$
(A.7)

Note that the above vector fields $h(x, v)$ and $g(x, v, t)$ are not $C^2$ functions. Namely $h$ is of $C^2$ only if $q \geq 2$ and of $C^1$ if $1 \leq q < 2$. Similarly $g$ is of $C^2$ only if $p = 1$ or $p \geq 2$ and of $C$ if $0 < p < 1$ and $C^1$ $1 < p < 2$, respectively. In case

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The same can be applied to the line of manifold at expressions. This means automatically no extra terms to the Melnikov integral (Eq. A.8) caused by the described by Fig. A.1.

Let us non focus on the line $x = 0$ separates the whole phase space $(x, v)$ into two parts where $h(x, v)$ is enough smooth ($C^2$). The same can be applied to the line $v = 0$ and $g(x, v, t)$ as a possible set for $0 < p < 1$ and $1 < p < 2$. This line crosses manifold at $[x_d, v_d]$ for the specific time $t = t_d$, such that $x_d = x(t_d)$ and $v_d = v(t_d)$ (Fig. A.4).

According to Kunze and K"upper [40] the $(C^2)$ space separation includes additional terms to the Melnikov integral.

Thus the Melnikov function $M(t_0)$:

$$M(t_0) = M_0(t_0) + \sum_{t_d} (h_{1,+}(q^*(t_0^d)) \cdot q_{1,1}^{u,+}(t_0 + t_0^d, t_0) - h_{1,-}(q^*(t_0^d)) \cdot q_{1,1}^{u,-}(t_0 + t_0^d, t_0))$$

$$+ h_{1,-}(q^*(t_0^d)) \cdot q_{1,1}^{s,-}(t_0 + t_0^d, t_0) - h_{1,-}(q^*(t_0^d)) \cdot q_{1,1}^{s,+}(t_0 + t_0^d, t_0)),$$

where $q_{1,1}^{u,+}$ and $q_{1,1}^{u,-}$ are stable and unstable manifold perturbation solutions (Eq. A.7) for $t$ in the vicinity of $t_d$ but $t > t_d$ for $'+'$ sign and $t < t_d$ for $'−'$ sign, respectively.

$M_0(t_0)$ is defined as for smooth vector fields:

$$M_0(t_0) = \int_{-\infty}^{\infty} h_{1}(q(t - t_0) \cdot g(q(t + t_0), t)) \, dt$$

and $h_{1} = [-h_2, h_1]$.

Once first discontinuity are identified for $x = 0$, $t_d \rightarrow \pm \infty$, one have to examine $\phi_{1}^{1}(t, t_0) = q_{1,s}^{u,\pm}(t, t_0)$ and $\phi_{2}^{1}(t, t_0) = q_{1,s}^{u,\pm}(t, t_0)$:

$$\begin{cases}
\dot{\phi}_1 = (1 + \delta + \gamma(q - 1)x^{q-2})\phi_1 \\
\dot{\phi}_2 = (1 + \delta + \gamma(q - 1)x^{q-2})\phi_2 - \tilde{\alpha}v|v|^{p-1} + \tilde{\mu}\cos\omega t
\end{cases}$$

for $x > 0$ (A.10)

and

$$\begin{cases}
\dot{\phi}_1 = (1 + \delta + \gamma(q - 1)(-x)^{q-2})\phi_1 \\
\dot{\phi}_2 = (1 + \delta + \gamma(q - 1)(-x)^{q-2})\phi_2 - \tilde{\alpha}v|v|^{p-1} + \tilde{\mu}\cos\omega t
\end{cases}$$

for $x < 0$ (A.11)

Note substituting $x = 0$ for $q > 2$ to Eqs. A.10 and A.11 we get the same equations for $\phi_{1/2}$ and consequently the same expressions. This means automatically no extra terms to the Melnikov integral (Eq. A.8) caused by the $x = 0$ discontinuity. Interestingly $q \leq 2$ would lead to a different result but for such case the is no homoclinic orbit in the unperturbed system described by $H^0$ (Eqs. S13).

Let us non focus on $v = 0$ discontinuity. In this case

$$\begin{cases}
\dot{\phi}_1 = (1 + \delta + \gamma(q - 1)|x^*|^{q-2})\phi_1 \\
\dot{\phi}_2 = (1 + \delta + \gamma(q - 1)|x^*|^{q-2})\phi_2 - \tilde{\alpha}v^{*}|v^*|^{p-1} + \tilde{\mu}\cos\omega t
\end{cases}$$

for $v > 0$ (A.12)

and

$$\begin{cases}
\dot{\phi}_1 = (1 + \delta + \gamma(q - 1)|x^*|^{q-2})\phi_1 \\
\dot{\phi}_2 = (1 + \delta + \gamma(q - 1)|x^*|^{q-2})\phi_2 + \tilde{\alpha}v^{*}|v^*|^{p-1} + \tilde{\mu}\cos\omega t
\end{cases}$$

for $v < 0$ (A.13)
Note, excluding natural odd numbers for the $p$ exponent, the above equations (Eqs. A.12 and A.13) are usually different for any other $p \geq 0$. However both solutions $[\phi^+_1, \phi^-_2]$ and $[\phi^+_1, \phi^+_2]$ have to be projected into

$$h_{[v=0]}^+ = [-h_2, h_1]_{v=0} = [\delta x^*(t_d - t_0) + \gamma \text{sgn}(x^*(t - t_0))|x^*(t_d - t_0)|q^{-1}, 0]$$

(A.14)

and the differences in solutions in $\phi^-_2$ and $\phi^+_2$ are effectively projected out. Interestingly this is also valid for $p = 0$ (a dry friction case).

Finally for $q > 2$ and $p \geq 0$ the Mielnikov function $M(t_0)$ can be treated as a

$$M(t_0) = M_0(t_0).$$

(A.15)

**Appendix B**

In this appendix we show how to get homoclinic orbits and analytically for some specific cases of exponent $q$: $q = 4, 3, 2.67$ and 2.5.

In case of $q = 4$ we follow works by Trueba et al. [16] and Borowiec et al. [18] (and Eqs. 9-10)

$$x^* = x^*(t - t_0) = \pm \frac{3 \delta}{2 \gamma} \sqrt{\frac{1}{\cosh^2 \left(\frac{\sqrt{-\delta}(t-t_0)}{2}\right)}}$$

$$v^* = v^*(t - t_0) = \pm \frac{3 \delta}{2 \gamma} \sqrt{\frac{-\delta}{\cosh^2 \left(\frac{\sqrt{-\delta}(t-t_0)}{2}\right)}}$$

(B.1)

where ‘+$’ and ‘$-$’ signs are related to left– and right–hand side orbits, respectively, $t_0$ is a time like integration constant. On the other hand for $q = 3$ we have

$$x^* = x^*(t - t_0) = \pm \frac{3 \delta}{2 \gamma} \sqrt{\frac{1}{\cosh^2 \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)}}$$

$$v^* = v^*(t - t_0) = \pm \frac{3 \delta}{2 \gamma} \sqrt{-\delta} \frac{\tanh \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)}{\cosh^3 \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)}.$$

(B.2)

Consequently for $q = \frac{2}{3} = 8/3 \approx 2.67$ we have

$$x^* = x^*(t - t_0) = \pm \left(\frac{-4 \delta}{3 \gamma}\right)^{3/2} \frac{1}{\cosh^3 \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)}$$

$$v^* = v^*(t - t_0) = \pm \left(\frac{-4 \delta}{3 \gamma}\right)^{3/2} \sqrt{-\delta} \frac{\tanh \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)}{\cosh^3 \left(\frac{\sqrt{-\delta}(t-t_0)}{3}\right)},$$

(B.3)

And for $q = 2.5$

$$x^* = x^*(t - t_0) = \pm \left(\frac{5 \delta}{4 \gamma}\right)^2 \frac{1}{\cosh^4 \left(\frac{\sqrt{-\delta}(t-t_0)}{4}\right)}$$

$$v^* = v^*(t - t_0) = \pm \left(\frac{4 \delta}{3 \gamma}\right)^2 \sqrt{-\delta} \frac{\tanh \left(\frac{\sqrt{-\delta}(t-t_0)}{4}\right)}{\cosh^4 \left(\frac{\sqrt{-\delta}(t-t_0)}{4}\right)}.$$

(B.4)

The results for a Melnikov integral can be easily found in the above cases. Evaluating the corresponding integral (Eq. 11) after some algebra the last condition (Eq. 10) yields to a critical value of excitation amplitude $\mu_c$. Thus for $q = 4, 16, 17, 18$ we have:

$$\mu_c = a \frac{2^{p/2}(-\delta)^{p+1/2}}{\pi \omega^p \gamma^{p/2}} \text{B} \left(\frac{p+2}{2}, \frac{p+1}{2}\right) \text{cosh} \left(\frac{\pi \omega}{2 \sqrt{-\delta}}\right),$$

(B.5)
while in case of \( q = 3 \): 
\[
\mu_c = \frac{3^p(-\delta)^{3p/2+2}}{2^{p+1}\pi\gamma^{3p}} B \left( \frac{p + 2}{2}, p + 1 \right) \sinh \left( \frac{\pi\omega}{\sqrt{-\delta}} \right)
\]  
(B.6)

for \( q = 8/3 \):
\[
\mu_c = \frac{2^{2p+1}(p+1)(-\delta)^{11p/10-2/5}}{3^{2.2}(p+1/9\omega^2-\delta)\gamma^{3p/5+9/10}} B \left( \frac{p + 2}{2}, \frac{3(p + 1)}{2} \right) \cosh \left( \frac{3\pi\omega}{2\sqrt{\delta}} \right)
\]  
(B.7)

and finally for \( q = 5/2 \):
\[
\mu_c = \frac{5^{2p}(-\delta)^{5p/2+3/2}}{2^{4p+3}\pi(4\omega^2-\delta)\gamma^{2p}} B \left( \frac{p + 2}{2}, 2(p + 1) \right) \sinh \left( \frac{2\pi\omega}{\sqrt{-\delta}} \right).
\]  
(B.8)

**Appendix C**

The integral \( I_1 \) can be evaluated analytically in some specific cases of exponents \( q \) corresponding to homoclinic orbits Eqs. B.1-B.4 numbered by the corresponding power index \( m \) applied to hyperbolic \( \cos \) function in the denominators. Let us consider integrals \( I_1 \), for given \( m = 1, 2, 3 \) and 4 related to various \( q \) exponents \( q = 4, 3, 8/3 \) and \( 5/2 \), respectively. To better clarity we will use new notation \( I_1 \to I_1(m) \) for given \( m \): 
\[
I_1(m) = \int_{-\infty}^{+\infty} v^*(t) \hat{\mu} \sin \omega t dt = C_m \int_{-\infty}^{+\infty} \left( \frac{\tanh(\tau)}{\cosh^m(\tau)} \right) \sin(\omega_m \tau) d\tau
\]  
(C.1)

while constants \( C_m \) and \( \omega_m \) are defined as follows:
\[
C_m = \sqrt{-\delta} \left( \frac{-(m+1)\delta}{m\gamma} \right)^{m/2}, \quad \omega_m = \frac{m\omega}{\sqrt{-\delta}}
\]  
(C.2)

Evaluating the integral \( J_m(\omega_m) \), for positive integer \( m \), twice by parts we have got the following recurrence identity
\[
J_{m+2}(\omega_{m+2}) = \frac{\omega_{m+2}^2 + m^2}{m(m+1)} J_m(\omega_{m+2}) \quad \text{for} \quad m = 1, 2, 3, \ldots
\]  
(C.3)

Thus only \( J_1 \) and \( J_2 \) need to be calculated. Below we evaluate them on the complex plane by summing corresponding residue.

![Deformed contour integration schema and imaginary poles.](image)

\[
\oint f(z) dz = 2\pi i \sum_{k=1}^{N} \text{Res}[f(z), z_k],
\]  
(C.4)
where
\[ \text{Res}[f(z), z_k] = \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}} [(z - z_k)^m f(z)], \] (C.5)
for \( m = 1 \) or 2, in our case.

The examined function \( f(z) \) is defined as:
\[ f(z) = \frac{2^m}{(\exp(z) + \exp(-z))^m} \exp(i \omega_m z). \] (C.6)

Note that on the real axis (Fig. C.1) \( \text{Re} \, z = \tau \) it can be written as
\[ \text{Im} f(\tau) = \frac{\cos(\omega_m \tau)}{\cosh^m \tau}. \] (C.7)

The multiplicity of each pole of the complex function \( f(z) \) (Eq. C.6) is given by
\[ z_k = \left( \frac{\pi}{2} + \pi k \right) i \quad \text{for} \quad k = 1, 2, 3, ... \] (C.8)

Note \( J_m \) (Eq. C.4) can be easily determined for \( m = 1 \) or 2. Namely, after summation of all poles in the upper half-plane (Fig. C.1), we get for \( m = 1 \)
\[ J_1 = \int_{-\infty}^{+\infty} d\tau \frac{\cos(\omega_1 \tau)}{\cosh \tau} = \frac{\pi}{\cosh \left( \frac{\pi \omega_1}{2} \right)} \] (C.9)
while for \( m = 2 \) we obtain
\[ J_2 = \int_{-\infty}^{+\infty} d\tau \frac{\cos(\omega_2 \tau)}{\cosh^2 \tau} = \frac{\pi \omega_2}{\sinh \left( \frac{\pi \omega_2}{2} \right)}. \] (C.10)

On the other hand, in case of \( m = 3 \) and \( m = 4 \) (and also for any larger \( m \)), we can use the recurrence relation (Eq. C.3):
\[ J_3 = \frac{\pi (\omega_3^2 + 1)}{2 \cosh \left( \frac{\pi \omega_3}{2} \right)}, \quad J_4 = \frac{\pi \omega_4 (\omega_4^2 + 4)}{6 \sinh \left( \frac{\pi \omega_4}{2} \right)}. \] (C.11)

Consequently using Eq. C.4:
\[ I_1(1) = \left( \frac{-2\delta}{\gamma} \right)^{1/2} \frac{\pi \omega}{\cosh \left( \frac{\pi \omega}{2\sqrt{\gamma}} \right)}, \quad I_1(2) = \left( \frac{-3\delta}{2\gamma} \right)^{2} \frac{2\pi \omega^2}{\sqrt{\delta} \sinh \left( \frac{\pi \omega}{2\sqrt{\gamma}} \right)}, \] (C.12)
\[ I_1(3) = \left( \frac{-4\delta}{3\gamma} \right)^{3/2} \frac{\pi (9\omega^2 - \delta)}{2\sqrt{-\delta} \omega \cosh \left( \frac{3\pi \omega}{2\sqrt{-\delta}} \right)}, \quad I_1(4) = \left( \frac{-5\delta}{4\gamma} \right)^{2} \frac{8\pi (4\omega^2 - \delta)}{3(-\delta) \sinh \left( \frac{2\pi \omega}{2\sqrt{-\delta}} \right)}. \]
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