QUANTITATIVE LONG RANGE CURVATURE ESTIMATE FOR MEAN CURVATURE FLOW

JINGZE ZHU

ABSTRACT. We prove that smooth convex $\alpha$-noncollapsed ancient mean curvature flow satisfies a quantitative curvature estimate $H(y, t) \leq CH(x, t)(H(x, t)|x - y| + 1)^2$ for any pair of $x, y$. In other words, the rescaled curvature grows at most quadratically in terms of the rescaled extrinsic distance.

1. INTRODUCTION

In the study of mean convex mean curvature flow, the smooth convex $\alpha$-noncollapsed ancient solutions play a crucial role, as they model the singularities of the mean convex flow. Such results were first proved by the seminal work of White [Whi00, Whi03]. Sheng-Wang [SW09] later gave an alternative proof.

In 2013, Haslhofer-Kleiner [HK17] found an interesting way to significantly simplify the theory. They made the noncollapsedness an assumption, not the consequence. This is a crucial difference from the earlier work. This assumption is reasonable because of the direct proof of noncollapsing property by Andrews [And12] using maximum principle. One of the important results in [HK17] is the long range curvature estimate. For our purpose, we state a simplified version of this estimate:

**Theorem 1.1** (c.f [HK17] Theorem 1.10, Corollary 3.2). Given $\alpha \in (0, 1]$, there is an increasing function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_+$ such that for any smooth convex ancient $\alpha$-noncollapsed mean curvature flow $M_t^\alpha \subset \mathbb{R}^{n+1}$ and $x, y \in M_t$ we have

$$H(y, t) \leq H(x, t)\varphi(H(x, t)|x - y|)$$

Roughly speaking, Haslhofer-Kleiner’s theorem allows one to compare the curvature uniformly in terms of their rescaled extrinsic distance. However, the control function $\varphi$ might be growing very quickly near infinity.

The goal of this note is to show that $\varphi$ grows at most quadratically, i.e, we can take $\varphi(s) = C(s + 1)^2$. Here is our main theorem:

**Theorem 1.2.** Given $\alpha \in (0, 1]$, there is a constant $C = C(\alpha, n)$ such that for any smooth convex ancient $\alpha$-noncollapsed mean curvature flow $M_t^\alpha \subset \mathbb{R}^{n+1}$ and $x, y \in M_t$ we have

$$H(y, t) \leq CH(x, t)(1 + H(x, t)|x - y|)^2$$

By switching \(x, y\), We can also get a quantitative curvature decay rate estimate:

**Corollary 1.3.** Setting up as in the Theorem 1.2, we can find \(c = c(\alpha, n) > 0\) such that

\[
H(y, t) \geq cH(x, t) (1 + H(x, t)|x - y|)^{-\frac{2}{3}}
\]

The proof of Theorem 1.2 relies on the Ecker-Huisken’s interior estimate [EH91]. The convexity and noncollapsing property gives that the surface is graphical in a small ambient ball \(B_r(y)\). We need to figure out the graphical radius \(r\) carefully (it could be very small when \(|x - y|\) is large) and then use Ecker-Huisken’s interior estimate.

We want to mention that part of the argument is similar to those in [BH17] (Section 5).

**Acknowledgements** The author would like to thank his advisor Simon Brendle for giving insightful ideas.

### 2. Proof of the results

Let \(M^n_t\) be a smooth ancient solution of mean curvature flow which bounds the convex domain \(K_t\) in \(\mathbb{R}^{n+1}\) and is \(\alpha\)-noncollapsed.

We recall the definition of the noncollapsing property:

**Definition 2.1** (c.f [SW09], [And12]). Let \(M\) be a smooth mean convex hypersurface bounding a domain \(K\) in \(\mathbb{R}^{n+1}\). Then \(M\) is \(\alpha\)-noncollapsed, if for each \(x \in M\) there is an interior ball \(B \subset K\) and an exterior open ball \(B' \subset K^c\) of radius \(\alpha H(x)^{-1}\) with \(x \in \partial B\) and \(x \in \partial B'\).

We need an elementary lemma for convex set:

**Lemma 2.2.** Suppose that \(K\) is a convex domain in \(\mathbb{R}^{n+1}\) containing a ball \(B_r(0)\) and \(x \in \partial K\). Let \(\omega = \frac{x}{|x|}\), then for any supporting plane \(P\) at \(x\) we have \(\langle \nu, \omega \rangle \geq \frac{r}{|x|}\), where \(\nu\) is the outward unit normal vector of \(P\).

**Proof.** By the definition of the supporting hyperplane, \(K\) lies in one side of \(P\), consequently we have \(\langle x - y, \nu \rangle \geq 0\) for any \(y \in K\). Since \(B_r(0) \subset K\), we can take \(y = r\nu\), then the result follows. \(\square\)

**Remark.** If \(\partial K\) is smooth, then \(\nu\) is the outward normal of the tangent plane of \(\partial K\) at \(x\).

**Proof of Theorem 1.2.** We may assume that the mean curvature is positive, for otherwise Theorem 1.2 is equivalent to Theorem 1.1 (or by strong maximum principle the solution is flat)

Since (1) is scale invariant, we may assume without loss of generality that \(H(x, t) = 1\).

By \(\alpha\)-noncollapsing assumption, there is a ball \(B_\alpha(p) \subset K_t\) that is tangential to \(M_t\) at \(x\).
QUANTITATIVE LONG RANGE CURVATURE ESTIMATE FOR MEAN CURVATURE FLOW

Let \( L = |y - p| \geq \alpha \) and \( \omega = \frac{y - p}{|y - p|} \).

Since the flow is mean convex, \( K_t \) is a nonincreasing set. Consequently, \( B_s(p) \subset K_s \) for any \( s \leq t \). Hence, for any \( z \in M_s \cap B_\frac{\alpha}{6}(y) \) with \( s \leq t \), we can apply Lemma 2.2 to obtain

\[
\langle \frac{z - p}{|z - p|}, \nu(z) \rangle \geq \frac{\alpha}{|z - p|} \geq \frac{5\alpha}{6L}
\]

Moreover, we have the following elementary inequality:

\[
|z - p| - \omega = \left| \frac{z - p}{|z - p|} - \frac{y - p}{|y - p|} \right| \\
\leq \frac{|z - y|}{|y - p|} + \left| 1 - \frac{|z - p|}{|y - p|} \right| \\
\leq \frac{\alpha}{3L}
\]

Combining (2) and (3) we have

\[
\langle \omega, \nu(z) \rangle \geq \frac{\alpha}{2L}
\]

In particular, \( B_\frac{\alpha}{6}(y) \cap M_s \) is a graph over the plane perpendicular to \( \omega \), whenever \( s \leq t \).

Now we let \( \tilde{R} = \frac{\alpha}{6}, t_0 = t - \frac{\alpha^2}{400n} \) and

\[
K(y, s, R) = \{ z \in M_s \mid |z - y|^2 + 2n(s - t_0) \leq R^2 \}
\]

By Ecker-Huisken’s interior estimate (c.f [EH91] Corollary 3.2):

\[
\sup_{K(y,t,R/2)} |A|^2 \leq C(n)(R^{-2} + (t - t_0)^{-1}) \sup_{s \in [t_0, t]} \sup_{K(y,s,R)} \langle \omega, \nu \rangle^{-4}
\]

Note that in the left hand side, \( (y, t) \in K(y,t,R/2) \). In the right hand side, each time slice is contained in \( B_\frac{\alpha}{6}(y) \). Hence by (4) (5) we have:

\[
|A(y, t)| \leq C(n)\alpha^{-1} \left( \frac{2L}{\alpha} \right)^2 = C(n)\alpha^{-1} L^2
\]

Note that by convexity \( 0 \leq H \leq \sqrt{n} |A| \). Moreover, \( L = |y - p| \leq |x - y| + \alpha \). Consequently,

\[
H(y, t) \leq C(n, \alpha)H(x, t)(1 + H(x, t)|x - y|)^2
\]

The theorem is proved.

\[
\square
\]

We now describe how to derive Corollary 1.3: First by (1) we see that either \( H(y, t) \leq CH(x, t) \) or \( H(x, t)|x - y| \geq 1 \), hence:

\[
H(y, t) \leq CH(x, t) + H(x, t)H(y, t)|x - y|
\]
It follows that
\[
\frac{1 + H(x, t)|x - y|}{1 + H(y, t)|x - y|} H(y, t) \leq CH(x, t)
\] (8)

Here $C$ might be different from line to line, but only depends on $n, \alpha$.

Next, combining (1) and (8) we obtain:
\[
H(y, t) \left( \frac{H(y, t)}{1 + H(y, t)|x - y|} \right)^2 \leq CH(x, t) \left( \frac{1 + H(x, t)|x - y|}{1 + H(y, t)|x - y|} H(y, t) \right)^2
\] \leq CH(x, t)^3
\] (9)

The result follows from swapping $x, y$ in (9) and taking cubic root.

**References**

[And12] Ben Andrews, *Noncollapsing in mean-convex mean curvature flow*, Geom. Topol. **16** (2012), no. 3, 1413–1418. MR 2967056

[BH17] Simon Brendle and Gerhard Huisken, *A fully nonlinear flow for two-convex hypersurfaces in Riemannian manifolds*, Invent. Math. **210** (2017), no. 2, 559–613. MR 3714512

[EH91] Klaus Ecker and Gerhard Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569. MR 1117150

[HK17] Robert Haslhofer and Bruce Kleiner, *Mean curvature flow with surgery*, Duke Math. J. **166** (2017), no. 9, 1591–1626.

[SW09] Weinmin Sheng and Xu-Jia Wang, *Singularity profile in the mean curvature flow*, Methods Appl. Anal. **16** (2009), no. 2, 139–155. MR 2563745

[Whi00] Brian White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **13** (2000), no. 3, 665–695. MR 1758759

[Whi03] Brian White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), no. 1, 123–138. MR 1937202

Department of Mathematics, Columbia University, New York, NY, 10027

Email address: zhujz@math.columbia.edu