HOMOTOPY CLASSES IN SOBOLEV SPACES
AND ENERGY MINIMIZING MAPS

BY BRIAN WHITE

Let $M$ and $N$ be compact Riemannian manifolds. The energy of a lipschitz map $f : M \to N$ is $\int_M |Df|^2$ (where $|Df(x)|^2 = \sum |\partial f/\partial x_i|^2$ if $x_1, \ldots, x_m$ are normal coordinates for $M$ at $x$). Mappings for which the first variation of energy vanishes are called harmonic. The identity map from $M$ to $M$ is always harmonic, but it may be homotopic to mappings of less energy. For instance, the identity map on $S^3$ is homotopic to mappings of arbitrarily small energy (namely, conformai maps that pull points from the North Pole toward the South Pole). That suggests the question: For which manifolds $M$ is the identity map homotopic to maps of arbitrarily small energy? In this paper we give the simple answer: Those $M$ such that $\pi_1(M)$ and $\pi_2(M)$ are both trivial. More generally, we consider energy functionals like $\Phi(f) = \int_M |Df|^p$ and ask:

(1) When is the infimum of $\Phi(f)$ in some homotopy class of mappings $f : M \to N$ nonzero?

(2) When is the infimum of $\Phi(g)$ (among maps satisfying some homotopy condition) actually attained?

To answer such questions, it is convenient to regard $N$ as isometrically embedded in a euclidean space $\mathbb{R}^\nu$ and to work with the Sobolev norm

$$\|f\|_{1,p} = \left(\int_M |f|^p\right)^{1/p} + \left(\int_M |Df|^p\right)^{1/p}$$

(where $f : M \to \mathbb{R}^\nu$ has distribution derivative $Df$) and with the associated Sobolev spaces,

$$L^{1,p}(M, N) = \{f : M \to \mathbb{R}^\nu \mid f(x) \in N \text{ for a.e. } x, \text{ and } \|f\|_{1,p} < \infty\}$$

and

$$W^{1,p}(M, N) = \text{the closure of } \{\text{lipschitz maps } f : M \to N\} \text{ in } L^{1,p}(M, N).$$

Say that two continuous maps $f, g : M \to N$ are $k$-homotopic (or have the same $k$-homotopy type) if their restrictions to the $k$-dimensional skeleton of some triangulation of $M$ are homotopic. We have the following theorem about $W^{1,p}(M, N)$ (where $[p]$ is the integer part of $p$).

**THEOREM 1.** Two lipschitz maps are in the same connected component of $W^{1,p}(M, N)$ if and only if they are $[p]$-homotopic. Consequently every map in $W^{1,p}(M, N)$ has a well-defined $[p]$-homotopy type. Furthermore, the set of
lipschitz maps homotopic to a given map \( f \) is dense (with respect to \( \| \cdot \|_{1,p} \)) in the connected component containing \( f \).

As a corollary we have the answer to (1).

**Corollary.** The infimum of \( \Phi(g) \) among lipschitz maps \( g: M \to N \) homotopic to a given lipschitz map \( f: M \to N \) is equal to the infimum of \( \Phi(g) \) among all lipschitz maps that are merely \([p]-\)homotopic to \( f \). In particular, the infimum is 0 if and only if the restriction of \( f \) to the \([p]-\)skeleton of \( M \) is homotopically trivial.

The space \( W^{1,p}(M,N) \) is not, however, suitable for studying existence questions such as (2) because it lacks nice compactness properties. In \( L^{1,p}(M,N) \), on the other hand, closed bounded sets are compact in the weak topology. We have

**Theorem 2.** Every \( f \in L^{1,p}(M,N) \) has a well-defined \([p - 1]-\)homotopy type. If \( f_i \in L^{1,p}(M,N) \) is a \( \| \cdot \|_{1,p} \)-bounded sequence of functions with a given \([p - 1]-\)homotopy type, and if \( f_i \) converges weakly to \( f \), then \( f \) has the same \([p - 1]-\)homotopy type.

This gives the answer to (2).

**Corollary.** The infimum of \( \Phi(g) \) among all maps \( g \in L^{1,p}(M,N) \) with a given \([p - 1]-\)homotopy type is attained.

In case \( p = 2 \), then \( \Phi(g) \) is the ordinary energy of \( g \), and the minimizing map \( g \) is locally energy minimizing in the sense studied by Schoen and Uhlenbeck [SU1,2]. By combining the above existence result with their regularity theorems, we obtain

**Theorem 3.** In every 1-homotopy class of mappings in \( L^{1,2}(M,N) \), there is a map \( g \) of least energy. Such a map is a smooth harmonic map except on a closed set \( K \subset M \) of Hausdorff dimension \( \leq \dim(M) - 3 \).

Furthermore, if \( N \) has negative sectional curvatures or if \( \dim M = 3 \) and \( N \) is any surface other than \( S^2 \) or \( \mathbb{R}P^2 \), then the map is completely regular. Since in these cases \( N \) has a contractible covering space, the homotopy type of \( g \) is determined by its 1-homotopy type. Consequently

**Theorem 4.** If (1) \( N \) has negative sectional curvatures, or
(2) \( \dim M = 3 \) and \( N \) is a surface other than \( S^2 \) or \( \mathbb{R}P^2 \), then every homotopy class of mappings from \( M \) to \( N \) contains a smooth map \( g \) of least energy.

The main tools in the proofs are: a deformation procedure analogous to the Federer-Fleming one [F, 4.2.9], versions of the Poincaré and Sobolev inequalities that hold for polyhedral complexes (such as the \( k \)-skeleton of \( M \)), and the homotopy extension theorem. All of the results generalize in the expected way to manifolds \( M \) with boundary.

Some special cases of these results were known previously: see [W2] and [W3] for details and references. Also, Theorem 4(1) was originally proved in a different way by Eells and Sampson [ES]. The analogous questions for
area instead of energy are studied in [SU, SY] (when dim M = 2) and [W1] (when dim M > 2).

In [EL, II.2.4-5] it is pointed out that Theorem 4(2) follows from the case p = 2 of Theorem 2. However, it seems that no proof (even in that case) has been published (though Schoen and Yau [SY] gave a proof when dim M = p).

I would like to thank R. Schoen for bringing these questions to my attention and for helpful conversations, and J. Eells for pointing out some errors.

REFERENCES

[EL] J. Eells and L. Lemaire, Selected topics in harmonic maps, CBMS Regional Conf. Ser. in Math., no. 50, Amer. Math. Soc., Providence, R. I., 1983.
[ES] J. Eells and J. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109–160.
[F] H. Federer, Geometric measure theory, Springer-Verlag, Berlin-Heidelberg-New York, 1969.
[SU] J. Sacks and K. Uhlenbeck, The existence of minimal 2-spheres, Ann. of Math. (2) 113 (1981), 1–24.
[SU1] R. Schoen and K. Uhlenbeck, A regularity theory for harmonic maps, J. Differential Geom. 17 (1982), 307–335.
[SU2] ______, Boundary regularity and the Dirichlet problem for harmonic maps, J. Differential Geom. 18 (1983), 253–268.
[SY] R. Schoen and S. T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with non-negative scalar curvature, Ann. of Math. (2) 110 (1979), 127–142.
[W1] B. White, Mappings that minimize area in their homotopy classes, J. Differential Geom. (to appear).
[W2] ______, Infima of energy functionals in homotopy classes of mappings, preprint.
[W3] ______, Mappings that minimize energy functionals in their homotopy classes (in preparation).