RESEARCH ARTICLE

Synthetic construction of the Hopf fibration in a double orthogonal projection of 4-space

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Abstract

The Hopf fibration mapping circles on a 3-sphere to points on a 2-sphere is well known to topologists. While the 2-sphere is embedded in 3-space, four-dimensional Euclidean space is needed to visualize the 3-sphere. Visualizing objects in 4-space using computer graphics based on their analytical representations has become popular in recent decades. For purely synthetic constructions, we apply the recently introduced method of visualization of 4-space by its double orthogonal projection onto two mutually perpendicular 3-spaces to investigate the Hopf fibration as a four-dimensional relation without analogy in lower dimensions. In this paper, the method of double orthogonal projection is used for a direct synthetic construction of the fibers of a 3-sphere from the corresponding points on a 2-sphere. The fibers of great circles on the 2-sphere create nested tori visualized in a stereographic projection onto the modeling 3-space. The step-by-step construction is supplemented by dynamic three-dimensional models showing simultaneously the 3-sphere, 2-sphere, and stereographic images of the fibers and mutual interrelations. Each step of the synthetic construction is supported by its analytical representation to highlight connections between the two interpretations.

Keywords: Hopf fibration; Hopf tori; four-dimensional visualization; stereographic projection; synthetic construction

1 Introduction

Mathematical visualization is an important instrument for understanding mathematical concepts. While analytical representations are convenient for proofs and analyses of properties, visualizations are essential for intuitive exploration and hypothesis making. Four-dimensional (4D) mathematical objects may lie beyond the reach of our three-dimensional (3D) imagination, but this is not an obstacle to their mathematical description and study. Furthermore, using the modeling tools of computer graphics, we are able to construct image representations of 4D objects to enhance their broader understanding. While many higher dimensional mathematical objects are natural generalizations of lower dimensional ones, the object of our study – the Hopf fibration – does not have this property. The Hopf fibration, introduced by Hopf (1931, 1935), defines a mapping between spheres of different dimensions. In this paper, we restrict ourselves to the correspondence it gives between spheres embedded in 4D and 3D spaces. To each point on a 2-sphere in 3D space is assigned a circular fiber on a 3-sphere in 4D space. The standard method of visualizing the fibers on the 3-sphere is to project them onto a 3D space via stereographic projection, by which we can also grasp the topological properties of the Hopf fibration. Two distinct points on the 2-sphere correspond to disjoint circular fibers on the 3-sphere and their stereographic images are linked circles (Fig. 1). To the points of a circle on the 2-sphere correspond circles on the 3-sphere that form a torus (Fig. 2).

Another facet of our interpretation is the method of visualization itself. Instead of visualizing an analytical representation of a given object, we create the object directly in a graphical environment constructively point by point. To do so, we use
1.1 Related work

Construction of the Hopf fibration is usually performed in abstract algebraic language as it involves the 3-sphere embedded in \( \mathbb{R}^6 \) or \( \mathbb{C}^2 \). However, recent research in computational geometry and graphics based on analytical representations has made possible partial video animations of the Hopf fibration along with models of stereographic images and visualizations in various software. The front covers of *Mathematical Intelligencer*, vol. 8, no. 3 and vol. 9, no. 1 featured stereographic images of the Hopf fibration by Koçak and Laidlaw (1987), taken from their pioneering computer-generated film projects with Banchoff, Bisschop, and Margolis. Banchoff (1990) wrote a comprehensive illustrated book on the fourth dimension, and its front cover has another inspiring picture of the Hopf fibration from his film. Moreover, moving toward the computer visualization of the Hopf fibration, Banchoff (1988) constructed stereographic images of Pinkall’s tori of given conformal type. The primary inspiration for writing this paper was another film – “Dimensions” by Alvarez et al. (2008), in which the Hopf fibration is well explained and visualized in a variety of separate models. Coincidentally, another front cover – of *Notices of the AMS*, vol. 44, no. 5 – was inspired by explanatory illustrations created in Wolfram Mathematica published by Kreminski (1997) in the context of the structure of the projective extension of real 3-space, \( \mathbb{RP}^3 \). A popular visualization of the Hopf fibration showing points on the base 2-sphere and the corresponding stereographic images of the fibers was created by Johnson (2011) in the mathematics software Sage. Johnson’s code was modified by Chinyere (2012) to visualize a similar fibration with trefoil knots as fibers instead of circles.

The visual aspect is foregrounded by Hanson (2006) (pp. 80–85 and 386–392) in visualizing quaternions on a 3-sphere, and the author also describes them using Hopf fibrations. Visualizations of Hopf fibrations have been beneficial in topology in relation to the Heegaard splitting of a 3-sphere by Canlubo (2017), in the use of quaternions in physics by O’Sullivan (2015), and in the description of motion in robotics by Yershova et al. (2010). Based on analytical representations, Black (2010) in chapter 6 of his dissertation depicts similar orthogonal projections of tori as do we, and supplemented them with animations.

Interactive tools for the visualization of 2D images of 4D objects from different viewpoints in 4-space were developed and applied by Zhou (1991). In another early-stage thesis in the area, Heng (1992) (supervised by Hanson) wrote on the use of interactive mathematical visualization techniques in computer graphics applied to the exploration of 3-manifolds. The continuous development of interactive frameworks and methods of 4D visualization is also apparent in subsequent work co-authored by Hanson (e.g. Hanson et al., 1999; Thakur & Hanson, 2007; Zhang & Hanson, 2007; Chu et al., 2008, the last two papers including visualizations of a flat torus embedded in 4D from different viewpoints).

In this exposition, we use the language and elementary constructions of the double orthogonal projection described in Zamboj (2018a), and the constructions of sections of 4D polytopes, cones, and spheres published in the series of articles in Zamboj (2018b, 2019a, b).

1.2 Contribution

We present an application of a method of visualization using computer graphics for constructing and examining a phenomenon of 4-space that has no analogy in lower dimensions. We use the method of double orthogonal projection to create
a purely synthetic graphical construction of the Hopf fibration. This paper contributes to the field in two directions: a novel graphical construction of the Hopf fibration and an application of the double orthogonal projection. In contrast to previous work on visualization of the Hopf fibration, in which separate illustrations created from analytical representations were graphical results or explanatory additions, we use mathematical visualization via the double orthogonal projection as a tool to synthetically construct the fibration. For this purpose, we revisit the analytical definition of the Hopf fibration (points on a circle on a 3-sphere map to a point on a 2-sphere, given by expression 5), which gives us the solved puzzle, and initiate our synthetic approach by decomposing the fibration into pieces. After this, we propose an elementary step-by-step construction of the inverse process. From points on the 2-sphere, we construct fibers of the 3-sphere with the use of only elementary (constructive) geometric tools. On top of that, we construct the resulting stereographic images of the fibers in one complex graphical interpretation. Even though stereographic images are common in visualization of the Hopf fibration, the difference here lies in our synthetic construction of them in order to confirm our results and observations. Finally, the method of visualization is applied to provide a graphical analysis of the properties of cyclic surfaces on a 3-sphere, 4D modulations, and filament packings. Our constructions (Figs 8–12, 14–17) are supplemented by interactive 3D models in GeoGebra 5 [see the online GeoGebra Book; Zamboj, 2019c; the software GeoGebra 5 is used due to its general accessibility, but any other 3D interactive (dynamic) geometry software may be used with the same outcomes. For overall understanding, we strongly recommend following the 3D models simultaneously with the text]. The final visualizations (Figs 18–24) and videos (Suppl. Files 8 and 9) are created in Wolfram Mathematica 11 for better graphical results. Throughout the paper, we give the relevant analytical; background to our synthetic visual approach to enrich overall understanding of the Hopf fibration.

1.3 Paper organization

In Section 2, we introduce the analytical definition and basic properties of the Hopf fibration, which is followed in Section 3 by a brief description of how to depict a 3-sphere in its double orthogonal projection onto two mutually perpendicular 3-spaces and how to construct its stereographic image. In the main part of the paper, Section 4, we give a synthetic construction of a Hopf fiber on a 3-sphere, corresponding to a point on a 2-sphere, using only elementary tools. The resulting double orthogonal projection and stereographic images of tori on the 3-sphere, corresponding to two families of circles on the 2-sphere, are given in Section 5. In Section 6, we apply the method to constructions of cyclic surfaces, visualization of 4D modulations with respect to polyhedral arrangements of vertices on the 2-sphere, and filament packings of the 3-sphere. After concluding with perspectives on future work, an appendix gives the parametrizations used in the figures.

2 Mathematical Background

In algebraic topology, a fibration is a certain type of projection from one topological space (the total space) onto another (the base space) that decomposes the total space into fibers. We proceed to formally define these and other terms that are needed in the sequel.
Villarceau circles on a torus in the 3-sphere when stereographically projected (Fig. 2).

An analytical definition of the Hopf fibration in coordinate geometry (expression 5) is given after we have made some preliminary remarks. For an elementary introduction to the Hopf fibration with visualizations, see Lyons (2003), Treisman (2009), and Ozols (2007), and for further details and proofs in modern topological language, see the textbook by Hatcher (2002) (chapter 4).

Let \( \{X_i\}_{i \in I} \) be a collection of sets indexed by a set \( I \). The set of functions \( f : I \rightarrow \bigcup_{i \in I} X_i \) such that \( f(i) \in X_i \) for each \( i \in I \) is called the Cartesian (or direct) product of the family of sets \( \{X_i\}_{i \in I} \). The Cartesian product is denoted by \( \prod_{i \in I} X_i \) or \( X_1 \times X_2 \times \ldots \).

For the purposes of visualization, we follow the direct construction of the Hopf fibration in real 4D space, \( \mathbb{R}^4 \), given for example in Treisman (2009); we also give an alternative construction in complex space useful for simple calculations. As a valuable by-product, we thus obtain visualizations of objects embedded not only in \( \mathbb{R}^4 \) but also in \( \mathbb{C}^2 \), where \( \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} \) is the Cartesian product of two complex coordinate systems.

**Analytical definition of the Hopf fibration.** At first, we construct a mapping that projects a point on the 3-sphere \( S_3 \) to a point on the 2-sphere \( S^2 \). The unit 2-sphere \( S^2 \) in \( \mathbb{R}^3 \) is the set of points given by the equation

\[
x^2 + y^2 + z^2 = 1.
\]

The unit 3-sphere \( S_3 \) in \( \mathbb{R}^4 \) is given by the analogous equation

\[
x^2 + y^2 + z^2 + w^2 = 1.
\]

Given complex numbers \( z_1 = x_0 + iy_0 \) and \( z_2 = x_2 + iy_2 \), the definition of the Hopf fibration as a ratio of complex numbers (see Hatcher, 2002) or via quaternions (see Hansson, 2005) would require a much broader theoretical discourse and would not fit our perspective any better.

**Proposition 1.** The preimage \( \{z_1, z_2\} \) of the point \( \{z, \bar{z}\} \) on \( \mathbb{C} \times \mathbb{R} \) under the Hopf fibration (5) is on \( S_3 \subset \mathbb{C}^2 \).

Using equation (4) yields

\[
|z|^2 + |\bar{z}|^2 = 1
\]

\[
4(\overline{z_1 z_2}) = |z_1|^2 - |z_2|^2 = 1
\]

\[
4(|z_1|^2 - |z_2|^2) = |z_1|^2 - |z_2|^2 = 1
\]

and since \( |z_1|^2 + |z_2|^2 \geq 0 \), we have \( |z_1|^2 + |z_2|^2 = 1 \).

According to equation (3), the point \( \{z_1, z_2\} \) lies on \( T^3 \).

Next, we describe how to create a circle from a point on \( T^3 \) such that its image is a fixed point on \( S^2 \).

**Proposition 2.** The Hopf fibration (5) maps all points of a circular fiber on \( T^3 \subset \mathbb{C}^2 \) to one point on \( S^2 \subset \mathbb{R}^2 \).

Let \( \{z_1, z_2\} \) be a point on the 3-sphere \( T^3 \subset \mathbb{C}^2 \). Let \( \lambda = \lambda_1 + i \lambda_2 \) be such that \( \lambda_1^2 = 1 \); i.e., \( \lambda \) represents a point on a unit circle embedded in \( \mathbb{C} \). From equation (6), for the point \( \{\lambda z_1, \lambda z_2\} \subset \mathbb{C}^2 \) it holds that

\[
|\lambda z_1|^2 + |\lambda z_2|^2 = |\lambda|^2(|z_1|^2 + |z_2|^2) = |z_1|^2 + |z_2|^2 = 1.
\]

Hence, the point \( \{\lambda z_1, \lambda z_2\} \) lies on \( S^3 \subset \mathbb{C}^2 \).

Rewriting the point \( \{\lambda z_1, \lambda z_2\} \subset \mathbb{C}^2 \) in its parametric representation in \( \mathbb{R}^4 \) with \( \lambda = \lambda_1 + i \lambda_2, z_1 = x_1 + iy_1 \), and \( z_2 = x_2 + iy_2 \), we have

\[
\begin{align*}
\text{Re}(\lambda z_1) &= \lambda_2 x_1 - \lambda_1 y_1 \\
\text{Im}(\lambda z_1) &= \lambda_1 x_1 + \lambda_2 y_1 \\
\text{Re}(\lambda z_2) &= \lambda_2 x_2 - \lambda_1 y_2 \\
\text{Im}(\lambda z_2) &= \lambda_1 x_2 + \lambda_2 y_2
\end{align*}
\]

For each \( \{1, l, b\} \subset \mathbb{R}^2 \), the last expression defines a set of points \( l_1 \overline{v} + l_2 \overline{w} \) in a plane in \( \mathbb{R}^4 \) through the origin \( (0, 0, 0) \) with the directional vectors \( \overline{v} = (x_0, y_0, z_0, w_0) \) and \( \overline{w} = (-y_0, x_0, -w_0, z_0) \). By equation (7), the set of all points \( \{\lambda z_1, \lambda z_2\} \) for each \( \lambda \) is the intersection of \( T^3 \) and a plane through its center. Hence, the set of all points \( \{\lambda z_1, \lambda z_2\} \) for \( \lambda \in \mathbb{C} \) is a unit circle on \( T^3 \). By equation (8) defining the Hopf fibration,

\[
(\lambda z_1, \lambda z_2) = (2\lambda z_1, \lambda z_2, |\lambda|^2 - |z_1|^2) = (2|\lambda|^2 z_1, \lambda z_2, |\lambda|^2(|z_1|^2 - |z_2|^2)) = (2|\lambda|^2 z_1, \lambda z_2, |z_1|^2 - |z_2|^2) = (2|\lambda|^2 z_1, \lambda z_2, |z_1|^2 - |z_2|^2) = \lambda z_1, \lambda z_2.
\]

so that \( f \) maps \( \{\lambda z_1, \lambda z_2\} \) for \( |\lambda|^2 = 1 \) to the same point on the unit 2-sphere \( S^2 \) as \( \{z_1, z_2\} \).

**Hopf coordinates.** In our synthetic reconstruction, we construct the inverse mapping – for a point on \( S^2 \), we find the circle on \( S^3 \). The point on \( S^2 \) will be constructed in terms of its polar and azimuthal angles in the representation of \( S^2 \) in spherical coordinates. We thus seek a relation between the spherical coordinates of a point on \( S^2 \) and a point \( \{z_1, z_2\} \) on \( S^3 \) in its trigonometric representation \( z_1 = r_1 \cos \alpha \cos \beta + i \sin \alpha \cos \beta \) and \( z_2 = r_2 \cos \sin \alpha \cos \beta \) for \( r_1, r_2 \neq 0 \) and \( \alpha, \beta \in \mathbb{R} \). From equation (6), for a point \( \{z_1, z_2\} \) on \( S^3 \), it holds that \( |z_1|^2 + |z_2|^2 = r_1^2 + r_2^2 = 1 \), and so there exists a unique \( \gamma \in (0, \pi) \) such that \( r_1 \cos \gamma = r_1 \) and \( r_2 \sin \gamma = r_2 \). Then a point on \( S^3 \) has the following coordinates in \( \mathbb{R}^4 \):

\[
\begin{align*}
\cos \gamma \cos \alpha \\
\sin \gamma \cos \beta \\
\sin \gamma \sin \beta
\end{align*}
\]

With the use of this representation, the image \( f(z_1, z_2) \) of the point \( \{z_1, z_2\} \) on the Hopf fibration (5) has first coordinate

\[
2r_1 z_1 = 2 \cos \gamma (\cos \alpha + i \sin \alpha) \sin \gamma (\cos \beta - \sin \beta) = 2 \cos \gamma \sin \gamma (\cos \alpha \cos \beta + \sin \alpha \sin \beta)
\]

\[
i (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = i (2 \sin \gamma) (\cos \alpha \cos \beta + \sin \alpha \sin \beta)
\]

and second coordinate

\[
|z_1|^2 - |z_2|^2 = r_1^2 - r_2^2 = \cos \gamma \sin \gamma = \cos (2\gamma).
\]

Let \( 2\gamma = \psi \in (0, \pi) \), and \( \alpha - \beta = \psi \in (0, 2\pi) \) (taken modulo \( 2\pi \), and likewise for all further operations with coordinates. Considering the
real and imaginary parts in (11) and the third coordinate in (12), the coordinates of the image \(f(z_1, z_2)\) in \(\mathbb{E}^3\) are

\[
\begin{align*}
\begin{pmatrix} \sin(2\psi) \cos(\alpha - \beta) \\ \sin(2\psi) \sin(\alpha - \beta) \\ \cos(2\psi) \end{pmatrix} &= \begin{pmatrix} \sin \psi \cos \psi \\ \sin \psi \sin \psi \\ \cos \psi \end{pmatrix}, \\
\psi &\in (0, \pi), \psi \in (0, 2\pi).
\end{align*}
\] (13)

We thus obtain the spherical coordinates of the image of a point on \(T^3\), which lies on the unit 2-sphere \(S^2\), and we have also reduced the number of parameters (from three \(\alpha, \beta, \gamma\) to two \(\psi, \beta\)). Finally, to construct the preimage of a point on \(S^2\), we substitute \(\psi, \beta\) into equation (10) parametrizing \(T^3\) in \(\mathbb{R}^3\), thereby obtaining the Hopf coordinates of a 3-sphere:

\[
\begin{align*}
\begin{pmatrix} \cos \frac{\psi}{2} (\cos(\beta + \alpha) + i \sin(\beta + \alpha)) \\ \sin \frac{\psi}{2} (\cos(\beta - i \alpha)) \\ \sin \frac{\psi}{2} \cos \beta \end{pmatrix}, \beta, \psi \in (0, 2\pi), \psi \in (0, \pi).
\end{align*}
\] (14)

Represented in \(\mathbb{C}^2\), we have

\[
\begin{align*}
z_1 &= \begin{pmatrix} \cos \frac{\psi}{2} (\cos(\beta + \alpha) + i \sin(\beta + \alpha)) \\ \sin \frac{\psi}{2} (\cos(\beta - i \alpha)) \\ \sin \frac{\psi}{2} \cos \beta \end{pmatrix}, \\
z_2 &= \begin{pmatrix} \cos \frac{\psi}{2} (\cos(\beta - i \alpha)) + i \sin(\beta - i \alpha) \\ \sin \frac{\psi}{2} (\cos(\beta + \alpha)) \\ \sin \frac{\psi}{2} \cos \beta \end{pmatrix}.
\end{align*}
\] (15)

Let us show that two fibers are disjoint (see also Triseman, 2009).

**Proposition 3.** Hopf fibers are disjoint circles.

Let \(T^3\) be a 3-sphere given by equation (2). Its intersection with the 3-space \(w = 0\) is the (equatorial) 2-sphere \(S^2\) with equation (1).

Consider a point \(A\) on \(T^3\) with coordinates \(A[1, 0, 0, 0]\) in \(\mathbb{E}^4\). The set of points \(c_\lambda, \lambda \in \mathbb{C}\) is the unit circle defined by the rotation of \(A\) by \(\lambda \in \mathbb{C}\) about the origin in the plane \((x, y)\). Now let \(B[x_0, y_0, z_0, w_0]\) be another point on \(T^3\) and not on the circle \(c_\lambda\) through \(A\) so that \((c_\lambda, w_0) \neq (0, 0)\). The set of points \(c_\lambda = \lambda \in \mathbb{C}\) is by equation (7) a unit circle on \(T^3\) with parametric representation

\[
\begin{align*}
\begin{pmatrix} \Re(\lambda \sqrt{x_0^2 + y_0^2} (x_0 + iy_0)) \\ \Im(\lambda \sqrt{x_0^2 + y_0^2} (x_0 + iy_0)) \\ \Re(\lambda \sqrt{z_0^2 + w_0^2} (z_0 + iw_0)) \\ \Im(\lambda \sqrt{z_0^2 + w_0^2} (z_0 + iw_0)) \end{pmatrix}.
\end{align*}
\] (16)

The unit circle \(c_\lambda\) intersects the equatorial unit 2-sphere \(S^2\) only if some point on \(c_\lambda\) is in the 3-space \(w = 0\). Let us again use the trigonometric representation to show that there are only two intersections of \(c_\lambda\) with \(S^2\) (just as for great circles on a 2-sphere intersecting its equator). Then, \(z_0 + iw_0 = \sqrt{z_0^2 + w_0^2} (\cos \delta + i \sin \delta)\) for \(\delta \in (0, 2\pi)\), and \(\lambda \in \mathbb{C}\) is such that \(\Re(\cos \alpha + i \sin \alpha\lambda)\) for \(\alpha \in (0, 2\pi)\). For the point on \(c_\lambda\) in the 3-space \(w = 0\), the equation

\[
\begin{align*}
\Re(\lambda \sqrt{z_0^2 + w_0^2} (z_0 + iw_0)) &= \Re(\lambda \sqrt{z_0^2 + w_0^2} (z_0 + iw_0)) \\
= \sqrt{z_0^2 + w_0^2} \cos(\alpha + \delta) = 0
\end{align*}
\] (17)

has, for \((z_0, w_0) \neq (0, 0)\), two solutions for \(\lambda \in \mathbb{C}\): \(\alpha = -\delta \mod 2\pi\) or \(\alpha = \pi - \delta \mod 2\pi\) for each \(\delta \in (0, 2\pi)\). Therefore, \(c_\lambda\) has only two antipodal points \(K_1\) and \(K_2\) in common with \(\mathbb{E}^3\).

Since for \(z\)-coordinates of \(c_\lambda\) we have

\[
\begin{align*}
\Re((\cos \alpha + i \sin \alpha\lambda) \sqrt{{z_0^2 + w_0^2}} (\cos \delta + i \sin \delta)) &= \sqrt{z_0^2 + w_0^2} \cos(\alpha + \delta) \neq 0
\end{align*}
\] (18)

for \(\alpha = -\delta\) or \(\alpha = \pi - \delta\).

The points \(K_1\) and \(K_2\) on the circle \(c_\lambda\) are not in the plane \((x, y)\). Hence, the circles \(c_\lambda\) and \(c_\lambda\), which are circular fibers in the Hopf fibration, are disjoint.

As we can rotate any circular fiber of the unit 3-sphere \(T^3\) to the position of \(c_\lambda\), it follows that all circular fibers on \(T^3\) are disjoint.

**Stereographic projection.** A reasonable choice to create a map of an ordinary 2-sphere (e.g. a map of the reference sphere of the Earth) is to take its stereographic projection from the North Pole onto a tangent plane at the South Pole. Despite the distortion of lengths, angles are preserved (the projection is a conformal mapping). As a result, circles on the 2-sphere not passing through the North Pole are projected to circles, and circles through the North Pole become straight lines. In our analogous 4D case, each point of \(T^3\) is projected via projecting rays through a fixed point of \(T^3\) – the center of projection – onto a 3-space touching \(T^3\) at the point antipodal to the center of projection. As this stereographic projection of \(T^3\) to a 3-space is conformal, all the circular Hopf fibers are projected to circles apart from the fiber through the center of projection, which projects to a line.

We have already shown that the inverse images of points on \(S^2\) in the Hopf fibration are disjoint circles. Moreover, under the stereographic projection, these disjoint fibers are projected onto linked circles (or a line; see Lyons, 2003). Given a circle on \(S^2\), its inverse image under the Hopf fibration projects to a family of disjoint circles on \(T^3\), and these are projected in the stereographic projection to a torus. Consequently, the stereographic images of all the fibers create nested tori; see Tsai (2006) for Lun-Yi Tsai’s exquisite artistic geometric illustrations.

### 3 Preliminary Constructions

#### 3.1 Double orthogonal projection

The double orthogonal projection of 4-space onto two mutually perpendicular 3-spaces is a generalization of Monge’s projection of an object onto two mutually perpendicular planes (see Zambo, 2018a). We briefly describe the orthogonal projection of a 2-sphere onto a plane in order to aid understanding of the double orthogonal projection of a 3-sphere onto two mutually perpendicular 3-spaces described subsequently.

In an orthogonal projection, the contour generator of a 2-sphere is a great circle – the intersection of the polar plane of the infinite viewpoint with respect to the 2-sphere (i.e. the plane perpendicular to the direction of the projection through the center of the 2-sphere) and the 2-sphere itself. The apparent contour of the 2-sphere is also a circle – the orthogonal projection of the contour generator onto the plane of projection (Fig. 4).

Therefore, in this 3D case of Monge’s projection, we project a 2-sphere \(\gamma_1\) and \(\gamma_2\) in two perpendicular planes \((x, z)\) and \((x, y)\). Then, the plane \((x, y)\) is rotated about the x-axis to form a y-axis that coincides with the z-axis but with opposite orientation.

Any point \(P\) in the 3-space \((x, y, z)\) is projected orthogonally via its projecting rays to the conjugated image points \(P_t\) (front
about the modeling 3-space) such that the z-axis coincides with the incident rays of projection after the rotation in the modeling 3-space – the plane perpendicular to the direction of the projection through the center of the 3-sphere. In Monge’s projection, planes parallel to the plane (x, z) are projected onto a disk. To locate the position of a point in the image, we can draw its circle of latitude (in a 3-space perpendicular to the direction of the projection) and extend it continuously to the great circle and shrinking back to a point. Analogously, the 3-spaces parallel to the 3-space intersect the plane (x, z) in lines parallel to x. The sections of a 2-sphere with planes parallel to (x, z) in Monge’s projection are shown in Fig. 7a; they create line segments in the top view and circles at their true size in the front view. In other words, if we imagine a 2-sphere passing orthogonally through a plane, their common intersection will be the tangent point extending continuously to the great circle and shrinking back to a point.

Figure 4: Double orthogonal projection of a 2-sphere γ onto the disk γ₁ in the 2-space π(x, z) and onto the disk γ₂ in the 2-space π(x, y) rotated to γ₂ in π(x, y) about the x-axis.

Figure 5: Apparent contours of a 3-sphere Γ in the double orthogonal projection. The 2-sphere Γ₁ in the 3-space Σ(x, y, z) and the 2-sphere Γ₂ in the 3-space Ω(x, z, w) rotated to Γ₂ in Ω(x, z, w) about the plane π(x, z).

3.2 Visualization of a point in \( \mathbb{R}^4 \) and \( \mathbb{C}^2 \)

A point in 4-space is given by its two conjugated images (2 and \( \Omega \)-image) lying on their common ordinal line, i.e. the line of coinciding rays of projection after the rotation in the modeling 3-space perpendicular to \( \pi(x, z) \). In the other direction, let \( P \) be a point with coordinates \([x, y, z, w]\) in \( \mathbb{R}^4 \). The \( \Sigma \)-image \( P_1[x, y, z, w] \) and \( \Omega \)-image \( P_2[x, z, w] \) are synthetically constructed in Fig. 6 using true lengths on the coordinate axes (concretely, in the modeling 3-space with the orthogonal coordinate system \((x, y, z)\) given in Fig. 6, we should say that the images of \( P \) have coordinates \( P_1[x, y, z, w] \) and \( P_2[x, z, w] \). In an implementation, such coordinates would naturally be used if we wanted to construct the images from the parametric representation). The ordinal line of the point \( P \) is perpendicular to \( \pi(x, z) \) through \( P_1 \) and \( P_2 \). Moreover, if we interpret the point \( P \) to be in \( \mathbb{C}^2 \) with coordinates \( P[a_1, b_1] \), then \( a_1 = [x, y] \) is on the complex line generated by the real axes \( \alpha, \beta \), and \( b_1 = [z, w] \) is on another complex line generated by \( z, w \). Using the trigonometric representation, we have \( a_1 = [r_1 \cos \alpha, r_1 \sin \alpha] \) and \( b_1 = [r_2 \cos \beta, r_2 \sin \beta] \) (cf. equation 10).

3.3 Visualization of a point on a 3-sphere

In an orthogonal projection of a 2-sphere onto a plane, points on the 2-sphere are projected onto a disk. To locate the position of a point in the image, we can draw its circle of latitude (in a plane perpendicular to the direction of the projection). Analogously, when we orthogonally project a 3-sphere onto a 3-space, we can locate a point on the 3-sphere by drawing its “2-sphere of latitude” (in a 3-space perpendicular to the direction of the projection). In Monge’s projection, planes parallel to the plane (x, z) intersect the plane (x, y) in lines parallel to x. The sections of a 2-sphere with planes parallel to (x, z) in Monge’s projection are shown in Fig. 7a; they create line segments in the top view and circles at their true size in the front view. In other words, if we imagine a 2-sphere passing orthogonally through a plane, their common intersection will be the tangent point extending continuously to the great circle and shrinking back to a point. Analogously, the 3-spaces parallel to the 3-space \( \Sigma(x, y, z) \) intersect the 3-space \( \Omega(x, z, w) \) in planes parallel to \( \pi(x, z) \). If a 3-sphere passes orthogonally through a 3-space, their common intersection is the tangent point, extending to the great 2-sphere and shrinking back to a point. Therefore, the intersections of the 3-sphere with a bundle of 3-spaces parallel to \( \Sigma(x, y, z) \) are 2-spheres (Fig. 7b). Their \( \Sigma \)-images are circles and \( \Omega \)-images are 2-spheres at their
true size. This construction should give us some insight into the visualization of points on tori inside a 3-sphere that will be carried out later.

Let $P$ be a point on a 3-sphere $\Gamma$ (Fig. 8, Suppl. File 1). The point $P$ lies on some 2-sphere $\sigma$ in a 3-space $\Sigma$ parallel to $\Xi(x, y, z)$. Since $\Sigma$ has the same $w$-coordinate as $P$, its $\Omega$-image appears as the plane $\omega^2$ through $P_2$ parallel to $\pi(x, z)$. The intersection of $\omega^2$ and $\Pi_1$ is a circle $\sigma_2$ that is the boundary of the $\Omega$-image of $\sigma$. The $\Sigma$-image $\sigma_1$ through $P_1$ is a 2-sphere concentric with $\Gamma_1$ and has radius equal to the radius of $\sigma_2$.

### 3.4 Stereographic projection of a point on a 3-sphere onto a 3-space

Let us have a unit 3-sphere $\Gamma$ with the center $S = [0, 1, 0, 1]$ (cf. equation 2):

$$x^2 + (y - 1)^2 + z^2 + (w - 1)^2 = 1.$$  

(19)

We project points of the 3-sphere $\Gamma$ from its point $N = [0, 2, 0, 1]$ onto the 3-space $\Omega(x, z, w)$: $y = 0$ that touches $\Gamma$ at the antipodal point $M = [0, 0, 0, 1]$. Let $P$ be a point on $\Gamma$ with images $P_1$ and $P_2$ (Fig. 9, Suppl. File 2). The stereographic image $P_1$ of the point $P$ in $\Omega(x, z, w)$ is the intersection of the line $NP$ and $\Omega(x, z, w)$. The line $N_1P_1$ intersects $\pi(x, z)$ in the point $P_{1s}$. The intersection of the ordinal line through $P_{1s}$ and the $\Xi$-image $N_2P_2$ is the desired point $P_{2s}$ that coincides with $P_2$ in 4-space.

Since stereographic projection is a conformal mapping, we can also conveniently project the 2-sphere $\sigma$ through the point $P$.

![Figure 7: (a) Circular sections $c_1, \ldots, c_7$ of a 2-sphere with planes $\gamma_1, \ldots, \gamma_7$ parallel to $(x, z)$ in Monge's projection. The planes are given by their intersections $\rho^i$ with the plane $(x, y)$. The apparent contours of the sections are circles $c_{11}, \ldots, c_{71}$ in the front view and segments $c_{12}, \ldots, c_{72}$ in the top view. (b) Sections $\sigma_1, \ldots, \sigma_7$ of a 3-sphere with 3-spaces $\Sigma_1, \ldots, \Sigma_7$ parallel to the 3-space $\Xi_1(x, y, z)$ are 2-spheres. The 3-spaces are given by their intersections $\omega^i$ with the 3-space $\Omega$. The $\Sigma$-images $\sigma_1^1, \ldots, \sigma_7^1$ of the spherical sections are 2-spheres in their true shape and $\Omega$-images $\sigma_1^2, \ldots, \sigma_7^2$ are disks.](https://academic.oup.com/jcde/article/8/3/836/6275216)
described in the previous section. The stereographic image $\sigma_2$ of the 2-sphere $\sigma$ is a 2-sphere $\sigma_5$ with center $G_S$ on the ordinal line through the center $S_1$ of $\sigma_1$. Its equatorial circle $g_2$ is the image of the tangent circle $g_1$ on the sphere $\Gamma_1$ of the cone, thereby liberating it from its analytical description. For the purposes of computation, we used the equations of a 3-sphere $T^3$ with the center at the origin, but for the sake of visualization (to differentiate the $\Sigma$ and $\Omega$-images) it proves more suitable to use a 3-sphere $T^3$ with center [0, 1, 0, 1]. Then, the 2-sphere $B^2$ has center with coordinates [0, 1, 0] in $\Sigma(x, y, z)$. Such a translation does not influence the properties of the Hopf fibration. This applies for Figs 10–12 and 14–18; the parametric equations corresponding to the visualizations are in the appendix.

4 Synthetic Construction of a Hopf Fiber

In this section, we illustrate geometrically the mathematical properties of the Hopf fibration given in Section 2 in images under the double orthogonal projection. This way, we synthetically construct a circular fiber on a 3-sphere from a point on a 2-sphere by elementary geometric constructions, thereby liberating the user from the analytic description. For the purpose of construction, we used the equation of a 3-sphere $T^3$ with the center at the origin, but for the sake of visualization (to differentiate the $\Sigma$ and $\Omega$-images) it proves more suitable to use a 3-sphere $T^3$ with center [0, 1, 0]. Then, the 2-sphere $B^2$ has center with coordinates [0, 1, 0] in $\Sigma(x, y, z)$. Such a translation does not influence the properties of the Hopf fibration. This applies for Figs 10–12 and 14–18; the parametric equations corresponding to the visualizations are in the appendix.

Let $Q \in \Sigma(x, y, z)$ be an arbitrary point on $B^2$ (Fig. 10, Suppl. File 3). We will find its Hopf fiber – a circle $c$ on $T^3$. Equations (13), (14), and (15) give a relation between the spherical coordinates of the point $Q$ (with parameters $\psi$ and $\phi$) and the Hopf coordinates of the fiber $c$ (with parameters $\phi$, $\psi$, and $\beta$). The construction proceeds by the following steps [see the step-by-step construction https://www.geogebra.org/m/w2kugajz (or Suppl. File 3)]:

1. Construct any point $Q$ on $B^2$.
2. Find the angle $\psi$ (from equation 13): Construct the plane parallel to $(x, y)$ through $Q$ that cuts $B^2$ in a circle. The oriented angle between the radius parallel to the $x$-axis and the radius terminating in the point $Q$ is the angle $\psi$.
3. Construct an arbitrary angle $\beta$ such that we can graphically add it to $\psi$: First, translate $\psi$ to $\phi$ in the plane $(x, y)$ with its vertex in the center of $B^2$, and the initial side in the direction of the non-negative $x$-axis. Now choose $\beta$ with the same vertex and initial side in the terminal side of $\phi$. For implementation, it is enough to choose $\beta$ on the top semicircle, as the points $P$ and $P'$ constructed in step 7 dependent on $\beta$ will be antipodal. Additionally, construct $\alpha$ such that $\alpha = \phi + \beta$ (cf. equation 13). The angles $\alpha$ and $\beta$ are arguments of the
complex points $a_P = [r_A \cos \alpha, r_A \sin \alpha] \in (x, y)$ and $b_P = [r_B \cos \beta, r_B \sin \beta] \in (z, w)$ (see the derivation of $z_1$ and $z_2$ above equation 10).

4. Find the angle $\psi$ (from equation 13): Construct the great circle of $B^2$ through Q with a diameter parallel to the x-axis. Choose a radius in the $(y, z)$ plane to be the initial side of the angle $\psi$ with the terminal side being the radius through Q.

5. Construct the moduli of the points $a_P$ and $b_P$ (cf. equation 10): Let $\gamma$ be the half-angle of $\psi$ and find its cosine by dropping a perpendicular onto the initial side of $\psi$. The length $r_1 = |a_P|$ is the modulus $r_1$ of the point $a_P$. Similarly, find the length $r_2 = |b_P|$ on the radius in the direction of the x-axis, which is the modulus $r_2$ of $b_P$.

6. Construct points $a_P$ and $b_P$: Using the moduli and arguments of $a_P$ and $b_P$, construct them according to Fig. 6.

7. Construct the $\Sigma$ and $\Omega$-images of the point P: Having $a_P$ and $b_P$, we finalize the images $P_1$ and $P_2$ on the parallels to the reference axes (as in Fig. 6).

8. Construct the antipodal point $P'$ on $B^2$: Parallel projection preserves central symmetry, and the images $P_1'$ and $P_2'$ are the reflections of $P_1$ and $P_2$ about the centers $S_1$ and $S_2$ of the images of $T_3$.

9. Construct the Hopf fiber corresponding to the point Q: The Hopf fiber is a circle $c$ consisting of the locus of points $P$ dependent on $\beta$. We have a point construction of $P$, which can be repeated for different choices of $\beta$ even though, at this point, we comfortably use the GeoGebra tool to draw the locus. The orthogonal projections of $c$ will appear as ellipses (or circles, or line segments) $c_1$ and $c_2$.
Figure 13: A torus in a 3D space generated by revolution of Villarceau circles. All the generating circles are interlinked.

Figure 14: Construction of the stereographic image $c_S$ of the Hopf fiber $c$ (cf. Fig. 9). The fiber $c_S$ is constructed as the locus of points $P_S$ and $P'_S$ dependent on the angle $\beta$.

Stereographic images are constructed in steps 10 and 11 in the step-by-step construction https://www.geogebra.org/m/w2kugajz (or Suppl. File 3) described in Fig. 10.

Varying $\beta$ in the interactive applet, $P$ moves on its fiber $c$. Moving with $Q \in B^2$, the whole fiber $c$ moves on $T^3$.

To illustrate the construction, we consider the images of fibers corresponding to certain points on the base sphere $B^2$ on diameters parallel to the $x$-, $y$-, and $z$-axes (Fig. 11).

(With respect to the translated coordinates of the centers of $B^2$ and $T^3$)

(a), (b) The fibers of points $[0, 1, \pm 1]$ (i.e. $\varphi = 0$ and $\psi = 0, \pi$) are in the planes $(x, w)$ and $(y, z)$, respectively. Thus, their conjugated images are a line segment and a great circle. In particular, for $\psi = \pi$ the point on the base sphere lies on its fiber.

(c), (d) The fibers of points $[\pm 1, 1, 0]$ (i.e. $\varphi = 0, \pi$ and $\psi = \frac{\pi}{2}$) lie in the plane of symmetry of the $x$- and $z$-axes and their conjugated images are congruent.

(e), (f) The fibers of points $[0, 1 \pm 1, 0]$ (i.e. $\varphi = \frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\psi = \frac{\pi}{2}$) have their $\Xi$-images in the plane of symmetry of the $y$- and $z$-axes and $\Omega$-images in the plane of symmetry of the $x$- and $w$-axes.

We have already mentioned that fibers corresponding to two distinct points on the base sphere $B^2$ create linked circles. Let $Q$ and $Q'$ be two points on $B^2$ and $c$ and $c'$ their fibers, respectively. We can easily observe from the conjugated images of $c$ and $c'$ that the fibers are disjoint, for if $c$ and $c'$ had a point of
intersection $R$, their conjugated images $c_1, c_1'$ and $c_2, c_2'$ would intersect in the conjugated images $R_1, R_2$ of the point $R$. In the case that the conjugated images of a fiber are in a plane perpendicular to $x$ (e.g. Fig. 12a), we must not swap the conjugated images of its points. Furthermore, we should note that the circles are linked on the 3-sphere $\mathbb{T}^3$, but this property cannot be validated in the embedding 4-space. Analogously, imagine a point in a circular region on a 2-sphere embedded in 3-space. On the 2-sphere, the point cannot escape the bounding circle. However, if we remove the 2-sphere and leave only the 3-space, the point is not bounded at all. Hence, to establish the fibers’ interlinkedness, we need to understand the topology of the underlying 3-sphere $\mathbb{T}^3$. Let us construct a great circle on $\mathbb{S}^2$ through $Q$ and $Q'$ and observe the motion of the fiber of a point $Q_m$ on the circle moving from $Q$ to $Q'$ (Fig. 12b). During this motion, the moving fiber twists along a surface. In fact, the generating fibers are always one of a pair of Villarchaeus circles around a torus (cf. Fig. 13 with a 3D parallel projection onto a plane of a torus generated by Villarchaeus circles). The toroidal structure of a 3-sphere will be seen more clearly in Section 5 with the use of stereographic projection.

4.1 Construction of the stereographic image of a Hopf fiber

To see the circular structure of the Hopf fibration, we construct the fibers in stereographic projection (Fig. 14). We use the same center of projection and antipodal tangent space $\Omega(x, z, w)$ as in Section 3.4. Continuing from the previous construction:

10. Construct a stereographic image of the point $P$: Let $N = [0, 2, 0, 1]$ be on $\mathbb{T}^3$ and let the 3-space $\Omega(x, z, w)$ be tangent to $\mathbb{T}^3$ at the point $[0, 0, 0, 1]$. The intersection of $N_1P_1$ with $x$ is $P_{01}$. Dropping a perpendicular from $P_{01}$ to the line $N_2P_2$ gives us the point $P_5$ that is also the true stereographic image $P_5$.

11. Construct a stereographic image of the fiber $c$: We use the locus tool from GeoGebra to construct a locus of points $P_5$ dependent on the angle $\beta$. The stereographic image $c_5$ of $c$ is a circle or a line segment (if $c$ passes through $N$).

Again, by varying $\beta$ the point $P_5$ moves on $c_5$, and by varying the position of $Q \in \mathbb{S}^2$ the circle $c_5$ changes in such a way that it may cover the whole modeling 3-space. In the following section, we show how to move $Q$ so as to obtain the Hopf tori.

5 Hopf Tori Corresponding to Circles on $\mathbb{S}^2$

The geometric nature of the Hopf fibration becomes fully apparent when we visualize the tori of Hopf fibers corresponding to circles on the base 2-sphere $\mathbb{S}^2$. With respect to the chosen stereographic projection, we divide the following constructions into two cases. First, we construct the tori on $\mathbb{T}^3$ in $\mathbb{E}^4(x, y, z, w)$ corresponding to circles on $\mathbb{S}^2$ parallel with the plane $(x, y)$ in the 3-space $\mathbb{E}(x, y, z)$, and then the tori on $\mathbb{T}^3$ corresponding to circles on $\mathbb{S}^2$ with diameter parallel to $z$. Instead of point-by-point constructions, in the following interactive demonstrations (Suppl. Files 5 and 6) the objects are defined by their parametric representations [see equations (A4) of the circle $k$ on $\mathbb{S}^2$, (A5) of the corresponding torus $k_\kappa$, and (A6) of the stereographic image of the torus $k$ in the appendix], and the user can manipulate the angles $\psi$ and $\psi'$ using the sliders.

5.1 Hopf torus of a circle parallel to $(x, y)$

Let $Q$ be a point on $\mathbb{S}^2$ in $\mathbb{E}(x, y, z)$ and $k$ a circle parallel to $(x, y)$ through the point $Q$ (Fig. 15a, Suppl. File 5). From equation (13), we have the parametric coordinates of points on the circle $k$ given by the angle $\psi'$ for a fixed $\psi$:

$$k(\psi') = \left( \sin \psi \cos \psi', \sin \psi \sin \psi', \cos \psi \right), \psi' \in (0, 2\pi).$$

(20)

Varying the angle $\beta'$ (positions of $P$ on $c$), from equation (14) and the angle $\psi'$ (positions of $Q$ on $k$ corresponding to distinct fibers $c$) we obtain the parametrization of a torus $k$ covered by the fibers on $\mathbb{T}^3$:

$$k(\beta', \psi') = \left( \cos \frac{\beta}{2} \cos(\psi + \beta'), \cos \frac{\beta}{2} \sin(\psi + \beta'), \sin \frac{\beta}{2} \cos(\psi'), \sin \frac{\beta}{2} \sin(\psi') \right), \beta', \psi' \in (0, 2\pi).$$

(21)

Figure 16a shows the double orthogonal projection of the torus $k$ and its generating circles corresponding to points on the circle $k$ with their stereographic images.

The conjugated images $k_1$ and $k_2$ of this torus are parts of cylindrical surfaces of revolution in $\mathbb{T}_1^3$ and $\mathbb{T}_2^3$. This is a straightforward consequence of the relationship between the point $Q$ and the angle $\psi$. From equation (21) with a fixed $\psi$, we obtain the $\Xi$-image $k_1$ as a part of a cylindrical surface of revolution about the axis parallel to $x$ in the 3-space $\mathbb{E}(x, y, z)$. Similarly, the $\Omega$-image $k_2$ is a part of a cylindrical surface of revolution about the axis parallel to $x$ in the 3-space $\Omega(x, z, w)$.

5.2 Hopf torus of a circle with diameter parallel to $z$

Let $Q$ be a point on $\mathbb{S}^2$ and $m$ a circle with a diameter parallel to $z$ through the point $Q$ (Fig. 15b, Suppl. File 6). The circle $m$ has a parametric representation for a fixed angle $\psi$ and variable $\psi$ (from equation (13)):

$$m(\psi) = \left( \sin \psi \cos \psi, \sin \psi \sin \psi, \cos \psi \right), \psi \in (0, \pi).$$

(22)

Figure 15: (a) Torus $k$ on $\mathbb{T}_1^3$ corresponding to a circle $k$ on $\mathbb{S}^2$ parallel to the $(x, y)$ plane. The torus is generated by fibers $c$ above points $Q$ along the circle $k$. (b) Torus $k$ on $\mathbb{T}_1^3$ corresponding to a circle $m$ on $\mathbb{S}^2$ with a diameter parallel to the $z$-axis. Again, the torus is generated by fibers $c$ above points $Q$ on $m$. 

Figure 16: (a) Torus $k$ on $\mathbb{T}_1^3$ corresponding to a circle $k$ on $\mathbb{S}^2$ parallel to the $(x, y)$ plane. The torus is generated by fibers $c$ above points $Q$ along the circle $k$. (b) Torus $k$ on $\mathbb{T}_1^3$ corresponding to a circle $m$ on $\mathbb{S}^2$ with a diameter parallel to the $z$-axis.
Varying the angle $\psi'$ (positions of Q on m) changes the moduli $r_A'$ and $r_B'$ of $P$ on the corresponding fiber $c$ (from equation (4)). More precisely, $r_A' = \cos \psi' = \cos \frac{\psi'}{2}$ and $r_B' = \sin \psi' = \sin \frac{\psi'}{2}$, with the fixed angle $\phi$ induce a family of non-intersecting circular fibers $c(\beta')$ generating a torus:

$$
\mu(\beta', \psi') = \left( \begin{array}{c} 
\cos \frac{\psi'}{2} \cos(\psi + \beta') \\
\cos \frac{\psi'}{2} \sin(\psi + \beta') \\
\sin \frac{\psi'}{2} \cos(\beta') \\
\sin \frac{\psi'}{2} \sin(\beta') 
\end{array} \right), \beta' \in (0, 2\pi), \psi' \in (0, \pi) .
$$

In contrast to the previous case, in which the torus $\kappa$ was generated by the parameter $\psi$, on which the first two coordinates depend, the torus $\mu$ is generated by the parameter $\psi'$, on which all four coordinates depend. Therefore, the conjugated images of the torus $\mu$ are more twisted. For the choice $\psi' = 0$ and $\beta' = \frac{\pi}{2} - \psi'$, we always obtain the point $[0, 1, 0, 0]$ lying on the torus $\mu$. This is the center N of the stereographic projection before applying the translation for visualization purposes, so the torus $\mu$ always contains a point that is stereographically projected to infinity. Moreover, it contains a fiber through this point, too. The stereographic image of this fiber is a line. See Fig. 16b for the full illustration of the double orthogonal projection of the torus covered by its circles and their stereographic images. Let $\mu$ and $\nu$ be tori generated by great circles $m$ and $n$ with a diameter parallel to the $z$-axis (Fig. 17). The circles $m$ and $n$ intersect in antipodal points $U$ and $W$, which correspond to fibers $u$ and $v$, respectively. These fibers were depicted earlier in Fig. 11a and b. Therefore, the tori $\mu$ and $\nu$ have two common fibers such that one conjugated image is always a segment and the second im-

---

**Figure 16:** (a) Blue and orange points on a circle on $S^2$ parallel to the ($x$, $y$) plane and the corresponding family of circular fibers on $T^2$ generating a torus. The stereographic projection of this torus onto $\Omega_2$ is a torus of revolution. (b) Blue and orange points on a circle on $S^2$ with a diameter parallel to the $z$-axis and their fibers. One of the fibers passes through the center of stereographic projection, so its image is a line. In the interactive model (a) https://www.geogebra.org/m/k94dvfpx (or Suppl. File 7), the user can vary the spherical coordinates of $Q$ in equation (A1) and interactively change its circle on $S^2$ and the corresponding torus. The user can also turn off the visibility of the objects on $S^2$, the conjugated images in the double orthogonal projection, or the stereographic images.

**Figure 17:** (a) Conjugated images of the tori $\mu$ and $\nu$ corresponding to the circles $m$ and $n$, respectively. The circles $m$ and $n$ pass through the points $U$ and $V$ on the diameter of $S^2$ parallel to the $z$-axis. The fibers $u$ and $v$ above points $U$ and $V$, respectively, lie on both tori. (b) Stereographic images $\mu_S$ and $\nu_S$ of the tori. The stereographic images $u_S$ and $v_S$ of the fibers $u$ and $v$ are the intersecting line and circle of $\mu_S$ and $\nu_S$. In the interactive model https://www.geogebra.org/m/k94dvfpx (or Suppl. File 7), the user can manipulate the circles $m$ and $n$ by varying the parameter $\nu$ in equation (A1). The projections of tori $\mu$ and $\nu$ vary dependently on $m$ and $n$. The visibility of the objects on $S^2$, the conjugated images in the double orthogonal projection, or the stereographic image can be turned off.
Figure 18: (a) A family of circles on $\mathbb{H}^2$ parallel to $(x, y)$ and with a diameter parallel to $z$, (b) the corresponding nested tori on $T^3$, and their stereographic projection onto $\Omega(x, z, w)$. Colors (shades) refer to mutually related objects; in (b) the images of the torus highlighted in red correspond to the white great circle on $S^2$. The visualizations are based on the parametrization in equation (A7).

5.3 Nested Hopf tori corresponding to families of circles on $S^2$

We summarize the toroidal structure of a 3-sphere in the following visualizations. For each circle $k$ parallel to the $(x, y)$ plane on
the 2-sphere $B^2$, we obtain a torus $\kappa$. Figure 18a gives a model (see Suppl. File 8 for a video animation) of nested tori $\kappa$ on $T^3$ corresponding to circles $k$ on $B^2$. This family of disjoint tori contains only one fiber through the center of the stereographic projection, and hence the tori appear in the stereographic projection as nested tori of revolution including one line, which is their axis.

In the second case (Fig. 18b, Suppl. File 9), the family of circles $m$ on $B^2$ with a diameter parallel to the z-axis forms nested tori $\mu$ on $T^3$. Each of these tori contains two common fibers in the special positions shown in Fig. 17. The stereographic image of one of the fibers is a line, which is the common line for all the stereographic images of the tori. These families of tori cover the 3-sphere $T^3$ reparametrized by variables $\beta', \phi'$, and $\psi'$ as

$$
T^3(\beta', \phi', \psi') = \left( \begin{array}{c}
\cos \frac{\pi}{2} \cos(\phi' + \beta') \\
\cos \frac{\pi}{2} \sin(\phi' + \beta') \\
\sin \frac{\pi}{2} \cos(\beta') \\
\sin \frac{\pi}{2} \sin(\beta')
\end{array} \right).
$$

(24)

$\beta', \phi' \in (0, 2\pi)$, $\psi' \in (0, \pi)$.

6 Further Applications

In the last section, we provide a few brief references to geometrically challenging applications of the Hopf fibration.

6.1 Cyclic surfaces

Using the above-mentioned method of visualization and construction of the Hopf fibration, we can study related properties of geometric surfaces and design shapes that are formed by disjoint circles. If we consider a point moving along an arbitrary curve on the 2-sphere $B^2$, the motion of the corresponding Hopf fiber creates a cyclic surface consisting of disjoint circles of variable radius on the 3-sphere $T^3$ embedded in $R^4$. Consequently, two curves intersecting in one point on a 2-sphere create two cyclic surfaces with only one common circle (the Hopf fiber of the point of intersection; Fig. 19a). Stereographic projection preserves circles (up to a circle through the center of projection) and so the stereographic images are cyclic surfaces intersecting in a circle in $R^3$, too (Fig. 19b).

Finally, we can construct orthogonal and stereographic images of shapes consisting of cyclic surfaces or their parts connected by common circles. The case in Fig. 20 shows a union of three circular arcs (the vertices are tangent points of the corresponding circles) stereographically projected from the $(x, z)$ plane onto a 2-sphere $B^2$. Then, we apply the inverse Hopf projection and construct conjugated images of the corresponding surfaces in $T^3$ in the double orthogonal projection. After the stereographic projection from $T^3$ to the 3-space $\Omega(x, z, w)$, we obtain a 3D model of the shape as the union of parts of cyclic surfaces. The common points of each pair of circular arcs become the common circles of each pair of parts of the cyclic surfaces.

6.2 4D modulations

The Hopf fibration was used as a constructive tool in classical optical communications to design 4D modulations in Rodrigues et al. (2018). From the geometric point of view, nPolSK-mPSK modulations are constructed by $n$ vertices of a polyhedron inscribed in the base 2-sphere generating $n$ fibers in the 3-sphere, and each fiber contains $m$ points. The authors demonstrated their main results on 14PolSK-8PSK modulation generated by tetrakis hexahedron, which is a union of a hexahedron and an octahedron with 14 vertices. The visualization of this arrangement supplemented with its double orthogonal projection is in Fig. 21.

6.3 Twisted filaments

Our final application is inspired by twisted toroidal structures appearing in biological and synthetic materials. In this context, Atkinson et al. (2019), Grason (2015), and Kléman (1985) studied the problem of twisted filament packings. Geometric models are constructed with the use of the Hopf fibration, in which fibers play the role of filament backbones. To construct equally spaced filaments in the 3-sphere $T^3$, we consider equally spaced disks on the base 2-sphere $B^2$. Visualizations of such arrangements...
Figure 20: A planar shape created by three circular arcs in the $(x, z)$ plane is stereographically projected onto the 2-sphere $B^2 \subset \mathbb{R}^3$. The corresponding fibers in the Hopf fibration into $T^3 \subset \mathbb{R}^4$ form three connected parts of cyclic surfaces, each two connected by a common circle. A part of the shape is depicted in (a) the double orthogonal projection and (b) the stereographic projection. The whole shape is visualized in (c) the double orthogonal projection and (d) the stereographic projection.

Figure 21: Visualization of the 14PolSK-8PSK modulation in the double orthogonal projection and stereographic projection given by a tetrakis hexahedron with 14 vertices based on the vertices of Platonic solids (and triangular case) projected to the 2-sphere $B^2$ in the double orthogonal projection and also stereographic projection are in Figs 22 and 23. Tangent points of the circles on $B^2$ correspond to tangent circles along the filaments in $T^3$. Consequently, the number of neighboring filaments in $T^3$ is the same as the number of neighboring disks in $B^2$.

As an example of more complex structure of filaments packing, we chose buckminsterfullerene with 60 vertices, 12 pentagonal, and 20 hexagonal faces (Fig. 24).
7 Conclusion

By the use of elementary constructive tools in the double orthogonal projection of 4-space onto two mutually perpendicular 3-spaces rotated into one 3D modeling space, we have described a synthetic step-by-step construction of a Hopf fiber on a 3-sphere embedded in 4-space that corresponds to a point on a 2-sphere. The virtual modeling space is accessible in supplementary interactive models created in the interactive 3D geometric software GeoGebra 5, in which the reader can intuitively manipulate fundamental objects and achieve a sense of the fourth dimension through two interlinked 3D models. The choice of the method of visualization plays a significant role in several aspects. First, two conjugated 3D images of a 4D object carry all the necessary information to determine this object uniquely. For example, if we had only one image of parallel sections of a 3-sphere in Fig. 7b, we would miss important details for reconstruction. The interpretation of two 3D images as one object indeed assumes some training and experience. However, in the case of projection onto a plane, we would need at least three images for a visual representation. Another advantage of the double orthogonal projection is that synthetic constructions generalize constructions in Monge’s projection. Thus, for example, the localization of a point on a 3-sphere or constructions of stereographic images are elementary. Furthermore, with the use of this technique, we have visualized a torus formed by the Hopf fibers corresponding to a circle on the 2-sphere and shown it in two different positions of the circles on the 2-sphere. The Hopf fibration is usually visualized in the stereographic projection based on an analytical representation. Since intuition in 4D visualization, including the double orthogonal projection, is often misleading, stereographic images, in this case, support our reasoning. Therefore, the tori were projected to synthetically constructed stereographic images in the modeling 3-space, revealing the true nature of the Hopf fibration. The final visualizations show how the points of the 2-sphere cover the 3-sphere by their corresponding fibers on the nested tori, and the whole modeling 3-space when stereograph-
Figure 23: Octahedral, icosahedral, and dodecahedral arrangements of vertices on the base 2-sphere and their corresponding twisted filaments with backbones visualized in the double orthogonal projection and in the stereographic projection.

Figure 24: Vertices of a buckminsterfullerene projected to the base 2-sphere, their corresponding filament backbones in the double orthogonal projection, and a close-up to stereographic images of the corresponding twisted filaments.
ically projected. In this way, we have built a mathematical visualization of the Hopf fibration in which we have presented a constructive connection between the base space (a 2-sphere), fibers covering the total space (a 3-sphere), and the stereographic images, and through which we can study and explain properties of the Hopf fibration in a way that does not depend on its analytical description. However, for the purposes of verification and implementation, the objects are supported by their parametric representations used in the classical analytical approach.

We finished by giving a short application of the Hopf fibration for constructing shapes on the 3-sphere and in stereographic projection in 3-space. The method provided promises future applications in visualizing other curves on a 2-sphere and their corresponding surfaces generated by their Hopf fibers on a 3-sphere in 4-space, along with their corresponding stereographic images in the modeling 3-space. The double orthogonal projection method is also likely to be useful for visualizing and analyzing the properties of further 3-manifolds embedded in 4-space.

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Supplementary Data

Supplementary data is available at JCDENG Journal online.

Conflict of interest statement

None declared.

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A Parametrizations Relevant to the Figures

Parametrizations in \( \mathbb{R}^4 \) are given with \((x, y, z, w)\) coordinates. For the implementation of visualizations in the double orthogonal projection in the modeling space with \((x, y, z)\) coordinates, we decompose the images so that a 3-image has \((x, y, z)\) coordinates and an \(\Omega\)-image has \((x, -w, z)\) coordinates.

Figs 10 and 14
A point \( Q \) on the 2-sphere \( B^2 \) (cf. equation 13):

\[
Q = \left( \begin{array}{c} \sin \psi \cos \varphi \\ \sin \psi \sin \varphi + 1 \\ \cos \psi \end{array} \right), \quad \psi \in (0, \pi), \varphi \in (0, 2\pi). \tag{A1}
\]

Further on \( \psi \) and \( \varphi \) are fixed, and \( \varphi = \frac{\pi}{2}, \psi = \alpha - \beta, \) for \( \beta \in (0, 2\pi) \).

A point \( P \) on the 3-sphere \( T^3 \) (cf. equation 14):

\[
P = \left( \begin{array}{c} \cos \frac{\pi}{2} \cos(\psi + \beta) \\ \cos \frac{\pi}{2} \sin(\psi + \beta) + 1 \\ \sin \frac{\pi}{2} \cos \beta \\ \sin \frac{\pi}{2} \sin \beta + 1 \\ \end{array} \right), \tag{A2}
\]

and also the parametrization of the circle \( c(\beta) \) by the variable \( \beta \).

The point \( P_5 \) in \( \Omega(x, z, w) \) – the stereographic image of the point \( P \) from the center \( N = (0, 2, 0, 1) \):

\[
P_5 = \left( \begin{array}{c} 2 \cos \frac{\pi}{2} \cos(\psi + \beta) \\ 1 - \cos \frac{\pi}{2} \sin(\psi + \beta) \\ 2 \sin \frac{\pi}{2} \cos \beta \\ 2 \sin \frac{\pi}{2} \sin \beta \\ 1 - \cos \frac{\pi}{2} \sin(\psi + \beta) + 1 \end{array} \right). \tag{A3}
\]

and also the parametrization of the circle \( c_5(\beta) \) by the variable \( \beta \).

Figs 15 and 16
A circle \( k \) on \( B^2 \) for a fixed \( \psi \) (cf. equation 20):

\[
k(\psi) = \left( \begin{array}{c} \sin \psi \cos \varphi \\ \sin \psi \sin \varphi + 1 \\ \cos \psi \end{array} \right), \quad \varphi \in (0, 2\pi). \tag{A4}
\]

The torus \( \kappa \) corresponding to the circle \( k \) (cf. equation 21):

\[
\kappa(\beta', \psi) = \left( \begin{array}{c} \cos \frac{\pi}{2} \cos(\psi + \beta') \\ \cos \frac{\pi}{2} \sin(\psi + \beta') + 1 \\ \sin \frac{\pi}{2} \cos(\beta') \\ \sin \frac{\pi}{2} \sin(\beta') + 1 \end{array} \right), \quad \beta', \psi \in (0, 2\pi). \tag{A5}
\]

The stereographic image of the torus \( \kappa \) in \( \Omega(x, z, w) \) (cf. equation 24):

\[
k(\beta', \psi) = \left( \begin{array}{c} 2 \cos \frac{\pi}{2} \cos(\psi + \beta') \\ 1 - \cos \frac{\pi}{2} \sin(\psi + \beta') \\ 2 \sin \frac{\pi}{2} \cos(\beta') \\ 2 \sin \frac{\pi}{2} \sin(\beta') + 1 \end{array} \right), \quad \beta', \psi \in (0, 2\pi). \tag{A6}
\]

Analogously, a circle \( m \) and torus \( \mu \), along with their stereographic images, differ from \( k \) and \( \kappa \) only by fixing \( \psi \) and making the variable \( \psi' \).

Fig. 18
The stereographic images of the nested tori in \( \Omega(x, z, w) \) have the following parametric representation (cf. equation 24):

\[
T^3(\beta', \psi') = \left( \begin{array}{c} 2 \cos \frac{\pi}{2} \cos(\psi' + \beta') \\ 1 - \cos \frac{\pi}{2} \sin(\psi' + \beta') \\ 2 \sin \frac{\pi}{2} \cos(\beta') \\ 2 \sin \frac{\pi}{2} \sin(\beta') + 1 \end{array} \right). \tag{A7}
\]

The stereographic image in Fig. 18a is for the sake of clarity restricted to \( \beta' \in (\frac{\pi}{6}, \frac{\pi}{3}) \), and \( \psi' = k_5 \beta' \) for \( k \in \{0, 1, \ldots, 12\} \) is chosen to be the leading variable. In Fig. 18b, the \( \Sigma \) and \( \Omega \)-images, \( \mu_1 \) and \( \mu_2 \), are apart from the one highlighted in red, restricted to \( \beta' \in (\frac{\pi}{6}, \frac{\pi}{3}) \), \( \psi' \in (\frac{\pi}{2}, 2\pi) \), the stereographic images are restricted to \( \beta' \in (\frac{\pi}{6}, \frac{\pi}{3}) \), \( \psi' \in (\frac{\pi}{2}, 2\pi) \), and the leading variable is \( \psi' = k_5 \beta' \) for \( k \in \{0, 1, \ldots, 6\} \).