A Unifying Framework for Testing Shape Restrictions*

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August 1, 2021

Abstract

This paper makes the following original contributions. First, we develop a unifying framework for testing shape restrictions based on the Wald principle. The test has asymptotic uniform size control and is uniformly consistent. Second, we examine the applicability and usefulness of some prominent shape enforcing operators in implementing our framework. In particular, in stark contrast to its use in point and interval estimation, the rearrangement operator is inapplicable due to a lack of convexity. The greatest convex minorization and the least concave majorization are shown to enjoy the analytic properties required to employ our framework. Third, we show that, despite that the projection operator may not be well-defined/behaved in general parameter spaces such as those defined by uniform norms, one may nonetheless employ a powerful distance-based test by applying our framework. Monte Carlo simulations confirm that our test works well. We further showcase the empirical relevance by investigating the relationship between weekly working hours and the annual wage growth in the high-end labor market.

Keywords: Shape restrictions, Distance function, Projection, Rearrangement, Greatest convex minorant, Least concave majorant.

*I thank Dora Gicheva for helping me make sense of her data set. Advanced computing resources provided by Texas A&M High Performance Research Computing are gratefully acknowledged.
1 Introduction

Shape restrictions are ubiquitous in economics, often arising as characterizations or implications of economic theory; see Matzkin (1994), Mas-Colell et al. (1995), and Chetverikov et al. (2018) for surveys and textbook treatment, as well as Couprie et al. (2010), Ellison and Ellison (2011), Card et al. (2012), Gicheva (2013), Chandra et al. (2014), Boffa et al. (2016), Scheuer and Werning (2017), and Elliott et al. (2021) for some recent concrete examples. Contrary to their extensive roles in economics and despite the sizable literature, the formal statistical analysis of shape restrictions appears to be relatively scant in empirical work. One possible explanation is that many existing inferential procedures are either problem-specific or lack computational tractability without sacrificing statistical power (Fang and Seo, 2021).

In this paper, we develop a unifying framework for testing shape restrictions concerning a parameter of interest $\theta_0$, where the hypotheses are formulated as:

$$
H_0 : \phi(\theta_0) = 0 \quad \text{vs.} \quad H_1 : \phi(\theta_0) > 0 ,
$$

for $\phi$ some known map taking nonnegative values. In the spirit of the Wald test, we base our framework on the plug-in statistic $\phi(\hat{\theta}_n)$ for an unconstrained estimator $\hat{\theta}_n$ of $\theta_0$. Statistical properties of the resulting test then depend on the choice of the map $\phi$, which we shall call the Wald functional in what follows. As an example, for $\Lambda$ the family of all elements in the parameter space satisfying the shape restriction (hereafter), Fang and Seo (2021) opt for $\phi(\theta) = \|\theta - \Pi_\Lambda(\theta)\|_H$, where $\Pi_\Lambda(\theta)$ is the closest element in $\Lambda$ to $\theta$ (i.e., the projection of $\theta$ onto $\Lambda$), and $\| \cdot \|_H$ is some $L^2$-norm. While the fruitful analytic properties of the projection operator enable Fang and Seo (2021) to develop a powerful test for a class of shape restrictions, there are regrettably other prominent choices of $\phi$ that fall beyond the scope of their framework.

First, there are other shape enforcing operators that have received increasing attention recently, notably the rearrangement (Chernozhukov et al., 2009, 2010, 2018; Chen et al., 2021) and the greatest convex minorization (GCM) or the least concave majorization (LCM) (Carolan and Tebbs, 2005; Delgado and Escanciano, 2012; Beare and Moon, 2015; Seo, 2018; Chen et al., 2021). The use of these two operators have been, however, confined to the construction of point and interval estimates or testing problems where the parameter $\theta_0$ is $\sqrt{n}$-estimable. Empirical settings of shape restrictions (e.g., non-parametric regression and density estimation), on the other hand, are often concerned with $\theta_0$ that is only estimable at slower than the $\sqrt{n}$-rate. Second, one may be interested in projection defined by non-$L^2$-norms such as the sup norm. Analogous to the comparison of the Cramér-von Mises ($L^2$-type) test vs. the Kolmogorov-Smirnov (sup-type) test, different norms lead to tests that are powerful against different classes of alternatives, and there are settings where it may be desirable to employ a particular test—see Chap-
ter 14 in Lehmann and Romano (2005), Andrews and Shi (2013), Armstrong (2018), and references therein for related discussions. Unfortunately, the projection operator in general may not be well-defined/behaved with respect to non-$L^2$-norms, rendering the framework of Fang and Seo (2021) not directly applicable.

As the first contribution of this paper, we show that the attractive statistical properties of Fang and Seo (2021)’s test are not unique to the $L^2$-projection operator. Indeed, as far as forming a suitable Wald functional $\phi$ is concerned, it suffices to require convexity, positive homogeneity, and Lipschitz continuity. The importance of convexity has been noted in Fang and Santos (2019), while positive homogeneity and Lipschitz continuity are mild and automatically fulfilled for the class of Wald functionals we consider. To accommodate non-$\sqrt{n}$ estimable parameters $\theta_0$, we appeal to strong approximations as in Fang and Seo (2021). We show that our test has asymptotic uniform size control and is uniformly consistent (under regularity conditions).

In turn, with the testing framework in hand, we examine some concrete choices of the Wald functional. First, for a shape enforcing operator $\Upsilon$, we consider

$$\phi(\theta) = \| \theta - \Upsilon(\theta) \|, \quad (2)$$

where $\| \cdot \|$ is a norm that may not be $L^2$. We show that this strategy works for GCM and LCM, but not for the rearrangement as the resulting $\phi$ is not convex. Second, since projection may not be well-defined, one may instead employ

$$\phi(\theta) = \inf_{\lambda \in \Lambda} \| \theta - \lambda \|. \quad (3)$$

The aforementioned analytic properties are then met whenever $\Lambda$ is a nonempty closed convex cone. Devising a powerful test, however, further demands that the restriction under the null be incorporated into the construction of critical values, as well understood in the literature. This may be accomplished by using $\Upsilon(\hat{\theta}_n)$ for (2), but using $\Pi_\Lambda(\hat{\theta}_n)$ for (3) as in Fang and Seo (2021) is problematic as $\Pi_\Lambda$ may not be well-defined/behaved if $\| \cdot \|$ is not a $L^2$-norm. To circumvent this challenge, we make use of $\Upsilon(\hat{\theta}_n)$ for (3) as well, provided $\Upsilon$ is a well-defined shape enforcing operator. In particular, the rearrangement is well suited to this end, despite that it is not suitable in forming (2).

Together, these results broaden the use of GCM/LCM and rearrangement to a wide range of nonparametric settings, and extend the $L^2$-projection test in Fang and Seo (2021) to a distance-based test (defined by a general norm).

We conduct Monte Carlo simulations to evaluate the performance of our test. The simulation results show that our framework yields tests that are competitive (in terms of size and power) compared to the prominent sup tests of Chernozhukov et al. (2013) and Chetverikov (2019). To showcase the empirical relevance of our framework, we test the shape restrictions on the relationship between weakly working hours and the annual

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wage growth in the high-end labor market. Overall, our empirical findings support the theoretical predictions of Gicheva (2013) for both men and women. The application also highlights the importance of conducting formal statistical tests when it comes to shape restrictions, rather than just relying on eyeball inspection.

The literature on shape restrictions dates back to Hildreth (1954), Ayer et al. (1955), Brunk (1955), van Eeden (1956), and Grenander (1956). We contribute to the problem of testing shape restrictions. Our paper extends Fang and Seo (2021) who employ (3) defined by a $L^2$-norm. We also build upon some of the analytic results and computational algorithms in Chen et al. (2021). These authors study the use of shape enforcing operators for point and interval estimation, which is different from our focus on testing. We note that recent tests of concavity based on LCM are limited to univariate settings where $\theta_0$ is $\sqrt{n}$-estimable—see aforementioned references as well as Delgado and Escanciano (2013, 2016), Beare and Schmidt (2016), Beare and Shi (2019), and Fang (2019).

Other recent work includes Chernozhukov et al. (2015) and Zhu (2020) who study moment restriction models with partial identification, Freyberger and Reeves (2018) and Chiang et al. (2021) who construct shape-constrained confidence bands, Breunig and Chen (2020) who develop adaptive and rate-optimal tests in nonparametric instrumental variable models, Komarova and Hidalgo (2020) who devise a pivotal test based on the Khmaladze’s transformation for univariate nonparametric regression models, and Kostyshak and Luo (2021) who propose the partial monotonicity parameter as a generalization of regression monotonicity. For brevity, we refer the reader to Fang and Seo (2021) for additional references.

We now introduce some notation and concepts. Set $\mathbb{R}_+ = \{a \in \mathbb{R} : a \geq 0\}$. For a vector $a \in \mathbb{R}^k$, we let $a^{(j)}$ be its $j$th entry and set $\|a\|_p$ to equal $\{\sum_{j=1}^{k} |a^{(j)}|^p\}^{1/p}$ if $p \in [1, \infty)$ and $\max_{j=1}^{k} |a^{(j)}|$ if $p = \infty$. We denote by $\mathbf{M}^{m \times k}$ the space of $m \times k$ matrices. For a function $f : \mathcal{Z} \rightarrow \mathbb{R}$ with $\mathcal{Z} \subset \mathbb{R}^d$, we set its $L^p$ norm $\|f\|_p$ to be $\{\int_{\mathcal{Z}} |f(z)|^p \, dz\}^{1/p}$ if $p \in [1, \infty)$ and $\sup_{z \in \mathcal{Z}} |f(z)|$ if $p = \infty$. In turn, we define $L^p(\mathcal{Z}) \equiv \{f : \mathcal{Z} \rightarrow \mathbb{R} : \|f\|_p < \infty\}$ for $p \in [1, \infty)$ and $\ell^\infty(\mathcal{Z}) \equiv \{f : \mathcal{Z} \rightarrow \mathbb{R} : \|f\|_\infty < \infty\}$. For a vector space $\mathbf{B}$, we recall that a map $\phi : \mathbf{B} \rightarrow \mathbb{R}$ is said to be positively homogeneous if $\phi(a\theta) = a\phi(\theta)$ for all $a \geq 0$ and $\theta \in \mathbf{B}$. For generic families of distributions $\mathbf{P}_n$, a sequence $\{a_n\}$ of positive scalars, and a sequence $\{X_n\}$ of random elements in a normed space $\mathbf{B}$ with norm $\|\cdot\|_\mathbf{B}$, write $X_n = o_p(a_n)$ uniformly in $P \in \mathbf{P}_n$ if $\lim_{n \to \infty} \sup_{P \in \mathbf{P}_n} P(\|X_n\|_\mathbf{B} > a_n\epsilon) = 0$ for any $\epsilon > 0$, and $X_n = O_p(a_n)$ uniformly in $P \in \mathbf{P}_n$ if $\lim_{n \to \infty} \limsup_{n \to \infty} \sup_{P \in \mathbf{P}_n} P(\|X_n\|_\mathbf{B} > Ma_n) = 0$.

The remainder of the paper is structured as follows. In Section 2, we develop the testing framework, investigates a number of possible Wald functionals, and provide some implementation guidance. Section 3 conducts Monte Carlo simulation studies, while Section 4 tests some shape restrictions concerning labor supply. Section 5 concludes. All proofs are relegated to the appendix.
2 The Testing Framework

As well understood in the literature (Imbens and Manski, 2004; Mikusheva, 2007; Andrews and Guggenberger, 2009), it is imperative to ensure that testing procedures in non-standard settings (such as the present one) be uniformly valid. To this end, we shall make explicit the dependence of the parameter interest \( \theta_0 \) on the underlying distribution \( P \) by instead writing \( \theta_P \) whenever appropriate. Moreover, we denote by \( P_0 \) the model under the null and by \( P_1 \) the model under the alternative. That is, \( P_0 = \{ P : \phi(\theta_P) = 0 \} \) and \( P_1 = P \setminus P_0 \), where \( P \) is the posited family of distributions possibly generating the data. We allow \( P_0 \), \( P_1 \) and \( P \) to depend on the sample size \( n \), but such dependence is suppressed for notational simplicity.

2.1 Size and Power

In order to present a unifying treatment, we assume throughout that the parameter of interest \( \theta_0 \) lives in an abstract Banach space \( B \) (i.e., a complete normed space) with a known norm \( \| \cdot \|_B \). The following assumption formalizes our main restrictions on the Wald functional \( \phi : B \to \mathbb{R}_+ \).

Assumption 2.1. For a Banach space \( B \) with norm \( \| \cdot \|_B \), a known map \( \phi : B \to \mathbb{R}_+ \) is (i) positively homogeneous; (ii) convex; (iii) Lipschitz continuous.

Assumptions 2.1(i)(ii) effectively demand that the null parameter space \( \Lambda \equiv \{ \theta \in B : \phi(\theta) = 0 \} \) be a nonempty convex cone, which must be closed under Assumption 2.1(iii). As illustrated in Fang and Seo (2021), this is satisfied by a number of common shape restrictions such as nonnegativity, monotonicity, convexity/concavity, Slutsky restriction, supermodularity and any intersections of these restrictions. Nonetheless, some other prominent restrictions such as quasi-convexity/concavity are excluded. We note that, given Assumption 2.1(i), Assumption 2.1(ii) is equivalent to \( \phi \) being subadditive, i.e., \( \phi(\theta_1 + \theta_2) \leq \phi(\theta_1) + \phi(\theta_2) \) for all \( \theta_1, \theta_2 \in B \). Assumption 2.1(iii) is a convenient mild condition for our distributional and bootstrap approximations. As shall be discussed in Section 2.2, Assumption 2.1 is satisfied for the distance function generated by a closed convex cone and the map \( \phi \) generated by GCM/LCM, but is violated for the map \( \phi \) generated by the rearrangement operator.

Assumption 2.1 implies that the map \( a \mapsto \phi(h + a\theta_0) \) is weakly decreasing on \([0, \infty)\) for any \( \theta_0 \in \Lambda \) and \( h \in B \), a property shared by the \( L^2 \)-distance map in Fang and Seo (2021, Lemma D.1). This in turn yields the following key relation that is fundamental in the development of our test: for an estimator \( \hat{\theta}_n \) of \( \theta_0 \) and \( 0 \leq \kappa_n \leq r_n \),

\[
r_n \phi(\hat{\theta}_n) = \phi(r_n \{ \hat{\theta}_n - \theta_0 \} + r_n \theta_0) \leq \phi(r_n \{ \hat{\theta}_n - \theta_0 \} + \kappa_n \theta_0).
\] (4)
The display (4) reveals both challenges and opportunities in devising a test based on $r_n\phi(\hat{\theta}_n)$. On one hand, while the law of $r_n\{\hat{\theta}_n - \theta_P\}$ may be consistently bootstrapped, the generic impossibility of consistently estimating $r_n\theta_P$ poses challenges for estimating the distribution of $r_n\phi(\hat{\theta}_n)$, an issue prevalent in other nonstandard settings (Andrews and Soares, 2010; Chernozhukov et al., 2015). On the other hand, consistently estimating the law of the upper bound in (4) is possible if $\kappa_n$ is suitably small, in view of the identity $\kappa_n\hat{\theta} - \kappa_n\theta_P = \kappa_n/r_n \cdot r_n\{\hat{\theta}_n - \theta_P\}$. Taken together, the inequality in (4) thus suggests that a test with $r_n\phi(\hat{\theta}_n)$ as the test statistic and critical values from the upper bound may control size in large samples.

To formalize the above idea, we introduce our second main assumption.

**Assumption 2.2.** i) $\hat{\theta}_n : \{X_i\}_{i=1}^n \rightarrow \mathcal{B}$ satisfies $\|r_n\{\hat{\theta}_n - \theta_P\} - Z_{n,P}\|_\mathcal{B} = o_p(c_n)$ uniformly in $P \in \mathcal{P}$ for some $r_n \rightarrow \infty$, $Z_{n,P} \in \mathcal{B}$ and $c_n > 0$ with $c_n = O(1)$; (ii) $\hat{G}_n \in \mathcal{B}$ is a bootstrap estimator satisfying: $\|\hat{G}_n - \tilde{Z}_{n,P}\|_\mathcal{B} = o_p(c_n)$ uniformly in $P \in \mathcal{P}$, for $\tilde{Z}_{n,P}$ a copy of $Z_{n,P}$ that is independent of $\{X_i\}_{i=1}^n$.

Assumption 2.2(i) requires a uniform (in $P \in \mathcal{P}$) distributional approximation of $r_n\{\hat{\theta}_n - \theta_P\}$ by a sequence of coupling variables $Z_{n,P}$, in lieu of an asymptotic distribution. While one may work with the latter if $r_n\{\hat{\theta}_n - \theta_P\}$ does converge in distribution, this becomes problematic in general nonparametric settings in which $r_n\{\hat{\theta}_n - \theta_P\}$ fails to converge as a process. Nonetheless, Chernozhukov et al. (2013) and subsequently, Belloni et al. (2015), Chernozhukov et al. (2015), Chen and Christensen (2018), Belloni et al. (2019), Cattaneo et al. (2020), and Li and Liao (2020) show that approximations as in Assumption 2.2(i) can be established under regularity conditions, thereby greatly facilitating inference on global constraints (e.g., shape) of nonparametric functions. Assumption 2.2(ii) simply says that the law of $\tilde{Z}_{n,P}$ (or equivalently the law of $Z_{n,P}$) can be consistently bootstrapped by $\hat{G}_n$, which may be verified by the same machineries developed in the aforementioned work. We stress that Assumption 2.2 places no restrictions on the particular schemes underlying the estimator $\hat{\theta}_n$ or the bootstrap $\hat{G}_n$. Thus, one may resort to sieve or kernel estimation in constructing $\hat{\theta}_n$ and various bootstrap/simulation methods for $\hat{G}_n$. Finally, Assumption 2.2 implicitly entails certain smoothness on $\theta_P$, as typically required for nonparametric estimation.

Given the estimator $\hat{\theta}_n$ and the bootstrap $\hat{G}_n$, we may employ $\phi(\hat{G}_n + \kappa_n\hat{\theta}_n)$ as a bootstrap estimator for the upper bound in (4). As well understood in analogous nonstandard settings (Andrews and Soares, 2010; Romano et al., 2014), however, this may cause loss of power in finite samples because $\hat{\theta}_n$ (in the term $\kappa_n\hat{\theta}_n$) does not reflect the restriction $\theta_P \in \Lambda$ under the null. This motivates us to use the restricted estimator $\Gamma(\hat{\theta}_n)$, provided a suitable shape enforcing operator $\Gamma : \mathcal{B} \rightarrow \Lambda$ is available, an issue we shall revisit in Section 2.2. In turn, for a given significance level $\alpha \in (0,1)$, we may then
obtain the critical value $\hat{c}_{n,1-\alpha}$ as:

$$\hat{c}_{n,1-\alpha} \equiv \inf\{c \in \mathbb{R} : P(\phi(\hat{\theta}_n) + \kappa_n \Gamma(\hat{\theta}_n)) \leq c|\{X_i\}_{i=1}^n \geq 1 - \alpha\}.$$  \hfill (5)

By construction, $\hat{c}_{n,1-\alpha}$ is an estimator of the $1 - \alpha$ quantile, denoted $c_{n,P}(1 - \alpha)$, of $\phi(Z_n|\theta_P)$ (a distributional approximation of the upper bound in (4)). To justify the construction of $\hat{c}_{n,1-\alpha}$, we further impose:

**Assumption 2.3.** An operator $\Gamma : B \to \Lambda$ (i) satisfies $\Gamma(h) = h$ for all $h \in \Lambda$ where $\Lambda \equiv \{\theta \in B : \phi(\theta) = 0\}$; (ii) is Lipschitz continuous.

**Assumption 2.4.** (i) $Z_{n,P}$ is tight and centered Gaussian in $B$ for each $n \in \mathbb{N}$ and $P \in \mathcal{P}$; (ii) $\sup_{P \in \mathcal{P}} E[\|\hat{\theta}_n\|_B] \leq \zeta_n$ for some $\zeta_n \geq 1$; (iii) $c_{n,P}(1 - \alpha - \varpi) \geq c_{n,P}(0.5) + \zeta_n$ for some constants $\varpi, \zeta_n > 0$, each $n$ and $P \in \mathcal{P}_0$; (iv) $c_n \zeta_n/\zeta_n^2 = O(1)$ as $n \to \infty$.

Assumption 2.3 demands mild requirements on $\Gamma$, which are satisfied by, e.g., the rearrangement for monotonicity, GCM for convexity, and a suitable composition of these two for monotonicity jointly with convexity—see Section 2.2. Assumption 2.4(i) is a mild technical restriction, while Assumption 2.4(iii)(iv) imply that the cdfs of $\psi_{\kappa_n,P}(Z_{n,P})$ are suitably continuous around $c_{n,P}(1 - \alpha)$. Assumption 2.4(ii) differs from Fang and Seo (2021) who assume uniform boundedness of $E[\|\hat{\theta}_n\|_B]$, because $E[\|\hat{\theta}_n\|_B]$ may grow with $n$ in our setting (e.g., when $B = \ell^\infty([0,1])$).

Assumptions 2.1, 2.2, 2.3, and 2.4 together deliver our first main result.

**Theorem 2.1.** Let Assumptions 2.1 and 2.2 hold.

(i) (Size) If Assumptions 2.3 and 2.4 hold and $0 < \kappa_n \zeta_n/r_n = o(c_n)$, then

$$\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P\{r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha}\} \leq \alpha ,$$  \hfill (6)

and, for $\mathcal{P}_0 \equiv \{P \in \mathcal{P}_0 : \phi(h + \theta_P) = \phi(h), \forall h \in \mathcal{B}\}$,

$$\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} |P\{r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha}\} - \alpha| = 0 .$$  \hfill (7)

(ii) (Power) If Assumptions 2.3(i) and 2.4(ii) hold and $\kappa_n \geq 0$, then

$$\liminf_{n \to \infty} \liminf_{\Delta \to \infty} \inf_{P \in \mathcal{P}_1^n} P\{r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha}\} = 1 ,$$  \hfill (8)

where $\mathcal{P}_1^n \equiv \{P \in \mathcal{P}_1 : \phi(\theta_P) \geq \Delta \zeta_n/r_n\}$ for $\Delta > 0$.

Theorem 2.1 formally establishes the asymptotic size control of our test, under a proper choice of $\kappa_n$. While one may ignore $\zeta_n$ from $\kappa_n \zeta_n/r_n = o(c_n)$ if $B$ is a Hilbert
space (Fang and Seo, 2021), it should be taken into account when \( \mathbf{B} \) is endowed with, e.g., a uniform norm. In the nonparametric regression settings of Chernozhukov et al. (2013), one may take \( \zeta_n = \sqrt{\log n} \).\(^1\) Theorem 2.1 further show that our test uniformly attains the nominal rejection rates in the limit over a class of null distributions that may heuristically be thought of as the “boundary”. We also note that the classes of alternatives against which our test is uniformly consistent depends on the Wald functional \( \phi \). In general, there does not exist a choice that leads to the most powerful test, and a different \( \phi \) “distributes” power over a different region of the parameter space.

A key element in implementing our test is the choice of the tuning parameter \( \kappa_n \). Intuitively, while \( \kappa_n \) should be small as dictated in Theorem 2.1, it should not be “too small” in the sense of causing the upper bound in (4) overly crude. Following Fang and Seo (2021), we provide a data-driven choice of \( \kappa_n \) as follows. For some small \( \gamma_n \in (0, 1) \), set \( \hat{\kappa}_n \equiv r_n c_n / \hat{\tau}_{n, 1 - \gamma_n} \) where

\[
\hat{\tau}_{n, 1 - \gamma_n} \equiv \inf \{ c \in \mathbb{R} : P(\|\hat{\mathbf{G}}_n\|_{\mathbf{B}} \leq c | \{X_i\}_{i=1}^n) \geq 1 - \gamma_n \}. \quad (9)
\]

In order to validate the use of \( \hat{\kappa}_n \), we need to introduce our final assumption where

\[
\bar{\sigma}_{n, \mathbf{P}}^2 \equiv \sup_{b^* \in \mathbf{B}^* : \|b^*\|_{\mathbf{B}^*} \leq 1} E[(b^*, Z_n, \mathbf{P})^2], \quad (10)
\]

for \( \mathbf{B}^* \) the space of continuous linear functions \( b^* : \mathbf{B} \to \mathbb{R} \) endowed with the norm \( \|b^*\|_{\mathbf{B}^*} \equiv \sup_{b \in \mathbf{B} : \|b\|_{\mathbf{B}} \leq 1} |\langle b^*, b \rangle| \), and \( \langle b^*, b \rangle \equiv \hat{b}^* (b) \).

**Assumption 2.5.** \( \lim \inf_{n \to \infty} \inf_{\mathbf{P} \in \mathbf{P}_0} \{\bar{\sigma}_{n, \mathbf{P}} / \zeta_n\} > 0 \).

Assumption 2.5 is in line with Assumption 2.4(ii) which allows \( \{\|Z_n, \mathbf{P}\|_{\mathbf{B}}\} \) to diverge. Intuitively, Assumption 2.5 requires that there be enough variations in \( \{Z_n, \mathbf{P}\} \) even if the sequence \( \{E[\|Z_n, \mathbf{P}\|_{\mathbf{B}}]\} \) of expected norms diverges.

**Proposition 2.1.** Let Assumptions 2.2, 2.4(i)(ii) and 2.5 hold, and set \( \hat{\kappa}_n \equiv r_n c_n / \hat{\tau}_{n, 1 - \gamma_n} \) with \( \gamma_n \in (0, 1) \) and \( \hat{\tau}_{n, 1 - \gamma_n} \) as in (9). If \( \gamma_n \to 0 \), then \( \hat{\kappa}_n \zeta_n / r_n = o_p(c_n) \) uniformly in \( P \in \mathbf{P}_0 \). If \( (r_n c_n)^{-2} \zeta_n^2 \log \gamma_n \to 0 \), then \( \hat{\kappa}_n / \zeta_n \overset{p}{\to} \infty \) uniformly in \( P \in \mathbf{P}_0 \).

The condition \( \gamma_n \to 0 \) ensures that our data driven \( \hat{\kappa}_n \) satisfies the rate condition in Theorem 2.1 in probability, while \( (r_n c_n)^{-2} \zeta_n^2 \log \gamma_n \to 0 \) formalizes the precise sense in which \( \gamma_n \) and \( \hat{\kappa}_n \) should not be “too small.” In line with prior findings in the literature (Fang and Santos, 2019; Chernozhukov et al., 2015; Fang and Seo, 2021), our Monte Carlo simulations show that the testing results are quite insensitive to the choice of \( \gamma_n \). We recommend \( \gamma_n = 0.01 / \log n \) or 1/n for practical implementations.

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\(^1\)Thus, the rate \( r_n \) may not necessarily be the convergence rate of \( \hat{\theta}_n \) as \( Z_n, \mathbf{P} \) may diverge.
2.2 The Wald Functional

We next introduce a number of concrete Wald functionals, and investigate their suitability in applying the previous framework.

2.2.1 Rearrangement

Rearrangement is an operation that enforces a specific shape, namely monotonicity—throughout monotonicity means “weakly increasing.” For \( \theta : Z \rightarrow \mathbb{R} \) a potentially non-monotonic function with some bounded set \( Z \subset \mathbb{R} \), then the rearrangement operator \( \Upsilon \) monotonizes \( \theta \) as follows: for any \( z \in Z \),

\[
\Upsilon(\theta)(z) = \inf \{ u \in \mathbb{R} : \int_{Z} 1\{ \theta(y) \leq u \} \, dy \geq z \}.
\] (11)

If \( \theta \) is multivariate, then one may monotonize \( \theta \) by applying (11) along each argument of \( \theta \)—see Section 2.3 for more details. While rearrangement has proven particularly convenient in obtaining restricted point and interval estimates (Fougères, 1997; Dette et al., 2006; Chernozhukov et al., 2009, 2010, 2018; Chen et al., 2021), it does not generate a convex \( \phi \) as in (2), a property that is crucial for implementing our test. This can be easily seen when \( \theta_0 \in \mathcal{B} \) is finite dimensional. For example, if \( \mathcal{B} = \mathbb{R}^4 \) is endowed with the max norm and \( \phi(\theta) = \| \theta - \Upsilon(\theta) \|_{\infty} \) with \( \Upsilon \) the rearrangement operator, then, for \( \theta_1 = [40, 54, 42, 69]^\top \) and \( \theta_2 = [21, 88, 3, 68]^\top \), simple calculations yield

\[
\phi(\theta_1 + \theta_2) = 92 > 12 + 67 = \phi(\theta_1) + \phi(\theta_2),
\] (12)

implying that \( \phi \) is not convex. The lack of convexity remains true for alternative (e.g., \( L^1 \) or \( L^2 \)) norms. Nonetheless, rearrangement satisfies Assumption 2.3 (Lieb and Loss, 2001; Chernozhukov et al., 2009; Chen et al., 2021), and may thus be utilized to construct constrained estimators for the purpose of power improvement.

2.2.2 Greatest Convex Minorization

GCM is also specific to a particular shape restriction, namely, convexity. The greatest convex minorant (also abbreviated GCM) of a function \( \theta \) is the pointwise supremum of convex functions lying below \( \theta \). Analogously, the least concave majorant (LCM) of a function \( \theta \) is the pointwise infimum of concave functions lying above \( \theta \). In what follows, we shall focus on GCM, as the analysis of LCM is similar. Following Chen et al. (2021), we define GCM directly through the Legendre-Fenchel transform. Concretely, let \( Z \subset \mathbb{R}^{d_Z} \) be a nonempty convex set, and \( \theta : Z \rightarrow \mathbb{R} \). Then the conjugate \( Z^* \) of \( Z \) is

\[
Z^* \equiv \{ y \in \mathbb{R}^{d_z} : \sup_{z \in Z} \{ \langle y, z \rangle - \theta(z) \} < \infty \}
\]

which is a nonempty convex set, and the
convex conjugate of $\theta$ is a map $\theta^* : Z^* \to \mathbb{R}$ defined by
\[
\theta^*(y) = \sup_{z \in Z} \{ \langle y, z \rangle - \theta(z) \}, \tag{13}
\]
for all $y \in Z^*$. In turn, the biconjugate $\theta^{**} \equiv (\theta^*)^* : Z \to \mathbb{R}$ of $\theta$ is
\[
\theta^{**}(z) = \sup_{y \in Z^*} \{ \langle y, z \rangle - \theta^*(y) \}. \tag{14}
\]
Thus, the shape enforcing operator $\Upsilon$ in this case assigns each $\theta$ with its biconjugate $\theta^{**}$.

While well understood in mathematics (see, e.g., Zeidler (1985, p.494)), the connection of GCM to the Legendre-Fenchel transform appears to be largely unnoticed in econometrics and statistics until the recent work by Chen et al. (2021). This connection enables Chen et al. (2021) to propose a linear programming algorithm for the computation of $\theta^{**}$—see Section 2.3 for more details.

Following the literature, notably Beare and Moon (2015), we may construct the Wald functional based on the GCM operator $\Upsilon$ as: for all $\theta \in \ell^\infty(Z)$ and $p \in [1, \infty]$,
\[
\phi(\theta) = \| \theta - \Upsilon(\theta) \|_p. \tag{15}
\]
Our next theorem establishes the analytic properties of $\phi$.

Theorem 2.2. Let $Z \subset \mathbb{R}^d$ be a nonempty bounded and convex set, and $p \in [1, \infty]$. Then map $\phi$ in (15) satisfies Assumption 2.1.

Positive homogeneity and Lipschitz continuity are well understood (Beare and Fang, 2017; Chen et al., 2021), though proving convexity of $\phi$ is nontrivial (to us). Theorem 2.2 remains true for concavity if we set $\Upsilon(\theta) = -(\theta)^{**}$ in the construction (15). We note that the boundedness of $Z$ may be dispensed with at the cost of introducing a suitable weighting function in the definition of the $L^p$ norm. Theorems 2.1 and 2.2 together extend the use of GCM/LCM to nonparametric settings where $\theta_0$ may be convex/concave with respect to multiple variables and/or may not be $\sqrt{n}$-estimable. As GCM/LCM may be obtained through linear programming, the test based on (15) may be more desirable than one based on $L^2$-projection (which requires quadratic programming), if computation cost is a binding constraint.

2.2.3 Distance and Projection

If $\Lambda$ is the class of all elements in $\mathcal{B}$ satisfying the shape restriction in question, then it is natural to form $\phi$ as the distance function defined in (3) with $\| \cdot \|$ being $\| \cdot \|_B$. By construction, statistical properties of the resulting test heavily depends on the shape of $\Lambda$ and the norm $\| \cdot \|_B$. If $\Lambda$ is a nonempty closed convex set and $\mathcal{B}$ is a Hilbert space (a
complete inner product space), then every $\theta \in \mathbf{B}$ admits a unique element in $\Lambda$, denoted $\Pi_\Lambda(\theta)$ and called the projection of $\theta$ onto $\Lambda$, which is closest to $\theta$. Thus, the map $\phi$ in (3) reduces to a particular instance of (2): for any $\theta \in \mathbf{B}$,

$$\phi(\theta) = \|\theta - \Pi_\Lambda(\theta)\|_B.$$ 

(16)

This is pursued in Fang and Santos (2019), and further in Fang and Seo (2021) who additionally exploit $\Lambda$ being a cone. Projection onto closed convex sets/cones in Hilbert spaces enjoy elegant analytic properties that these authors extensively utilize to establish the statistical properties of their tests.

In particular settings, however, practitioners may wish to work with a different norm, such as the uniform norm if he/she wants to guard against alternatives uniformly deviating from the null. Such an extension is nontrivial for two reasons. First, the projection operator $\Pi_\Lambda$ in non-Hilbert spaces is in general not well-defined/behaved in the sense that it may be empty-valued, set-valued, discontinuous, or continuous but not uniformly so—see Barbu and Precupanu (2012, pp.211-2), Dontchev and Zolezzi (1993, p.45), and Alber (1996). Second, employing the constrained estimator $\Pi_\Lambda(\hat{\theta}_n)$ as in Fang and Seo (2021) (to improve power) is also problematic in view of the erratic behaviors of $\Pi_\Lambda$. The first challenge prompts us to take a step back and work directly with the distance function (3), which remains well-defined. Importantly, $\phi$ enjoys attractive analytic properties, as summarized in the following well known lemma.

**Lemma 2.1.** If $\Lambda$ is a nonempty closed convex cone in a Banach space $\mathbf{B}$, then the map $\theta \mapsto \phi(\theta) = \inf_{\lambda \in \Lambda} \|\theta - \lambda\|_\mathbf{B}$ satisfies Assumption 2.1.

To circumvent the second challenge, we employ shape enforcing operators designed for specific restrictions, such as the rearrangement for monotonicity and GCM/LCM for convexity/concavity. For the joint restriction of monotonicity and convexity/concavity, if $\Upsilon_1$ denotes rearrangement and $\Upsilon_2$ GCM, then the composition $\Upsilon_2 \circ \Upsilon_1$ satisfies Assumption 2.3 but not $\Upsilon_1 \circ \Upsilon_2$—see Remark 17 in Chen et al. (2021). Together with Theorem 2.1, Lemma 2.1 thus extends Fang and Seo (2021) to a distance-based test under alternative norms. In particular, the sup-norm yields a competing test to existing sup-type tests such as Chernozhukov et al. (2013) and Chetverikov (2019).

### 2.3 Implementation

For the convenience of practitioners, we now provide an implementation guide.

**Step 1:** Compute the test statistic $r_n \phi(\hat{\theta}_n)$.

There are two aspects involved: obtain the estimator $\hat{\theta}_n$ and its rate $r_n$, and compute $r_n \phi(\hat{\theta}_n)$ for a given $\hat{\theta}_n$. The former may be obtained by standard procedures such as
kernel or sieve estimation—see Fang and Seo (2021) for more details. Given \( \hat{\theta}_n \) and \( r_n \), the key remains to compute \( \phi(\hat{\theta}_n) \) which we demonstrate through two examples. Let \( B = \ell^\infty([0,1]) \) and set \( \vartheta = (\hat{\theta}_n(z_0), \ldots, \hat{\theta}_n(z_N-1))^\top \) for a large enough \( N \) and \( z_j \equiv j/(N-1) \). Consider first the set \( \Lambda \) of weakly increasing functions in \( \ell^\infty([0,1]) \), and \( \phi \) being the distance function. Then we may approximate \( \phi(\hat{\theta}_n) \) by solving
\[
\max_{h \in \mathbb{R}^N: D_N h \geq 0} \| h - \vartheta \|_\infty ,
\]
where \( D_N \in \mathbb{M}^{(N-1) \times N} \) is the matrix such that \( D_N h = [h^{(2)} - h^{(1)}, \ldots, h^{(N)} - h^{(N-1)}]^\top \).
Note that (17) is a linear programming problem (Boyd and Vandenberghe, 2004, p.293).
Consider now the set \( \Lambda \) consisting of convex functions in \( \ell^\infty([0,1]) \), and the Wald functional \( \theta \mapsto \phi(\theta) = \| \theta - \vartheta^{**} \|_\infty \). Then we may approximate \( \phi(\hat{\theta}_n) \) by \( \| \theta - \vartheta^{**} \|_\infty \) where \( l \)th entry of \( \vartheta^{**} \) is given by (see, e.g., Carolan (2002, Lemma 1)):
\[
\max_{v \in \mathbb{R}, \xi \in \mathbb{R}^{d_z}} v
\text{ s.t. } v + \xi^\top (z_j - z_l) \leq \hat{\theta}_n(z_j), \ j = 1, \ldots, N .
\]
\[
\text{Step 2: Construct the critical value } \hat{c}_{n,1-\alpha} \text{ with } \alpha \in (0,1). \]
In this step, we presume that the practitioner is capable of computing/approximating \( \phi(\theta) \) for any given \( \theta \in B \) (as described above).

(i) Generate a collection of bootstrap estimates \( \{ \hat{G}_{n,k} \}_{k=1}^B \) for \( \hat{G}_n \) (e.g., \( B = 200 \) or 1000)—see Fang and Seo (2021) for more details.

(ii) Set \( \hat{c}_n = r_n c_n / \hat{r}_{n,1-\gamma_n} \) where \( \hat{r}_{n,1-\gamma_n} \) is the \( (1-\gamma_n) \)-quantile of \( \| \hat{G}_{n,1} \|, \ldots, \| \hat{G}_{n,B} \| \) in nonparametric regression models. We recommend \( \gamma_n = 0.01 / \log n \) or \( 1/n \) as in Fang and Seo (2021) and note that \( c_n = 1/\log n \) in nonparametric regression models. In the examples of Step 1, the \( \| \cdot \|_B \) norms may be approximated based on the grid points.

(iii) Compute \( \Gamma(\hat{\theta}_n) \). For the convexity examples in Step 1, this requires no additional
computation as one may simply let $\Gamma(\hat{\theta}_n) = \varphi^{**}$. For the monotonicity example with $B = \ell^\infty([0, 1])$, one may let $\Gamma$ be the rearrangement operator so $\Gamma(\hat{\theta}_n)$ is just the sorted version of $\hat{\theta}_n$. If instead $B = \ell^\infty([0, 1]^d z)$, then rearrangement may be implemented as follows (Chernozhukov et al., 2009):

$$\Gamma(\hat{\theta}_n) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} M_\pi \hat{\theta}_n,$$

(20)

where $\Pi$ consists of all permutations of $1, \ldots, d_z$, $|\Pi|$ is the cardinality of $\Pi$, and, for each permutation $\pi \equiv (\pi_1, \ldots, \pi_{d_z})$, $M_\pi \hat{\theta}_n \equiv M_{\pi_1} \circ \cdots \circ M_{\pi_{d_z}} \hat{\theta}_n$ with $M_j \hat{\theta}_n$ the sorted version of $\hat{\theta}_n$ viewed as a function of its $j$th argument (holding others fixed). The averaging in (20) is to eliminate the ambiguity caused by the order in which rearrangement is implemented.

(iv) Approximate $\hat{c}_{n, 1-\alpha}$ by the $(1 - \alpha)$-quantile of the $B$ numbers

$$\phi(\hat{G}_{n, 1} + \hat{c}_n \Gamma(\hat{\theta}_n)), \ldots, \phi(\hat{G}_{n, B} + \hat{c}_n \Gamma(\hat{\theta}_n)).$$

(21)

**Step 3:** Reject $H_0$ if and only if $r_n \phi(\hat{\theta}_n) > \hat{c}_{n, 1-\alpha}$.

3 Simulation Studies

We next conduct Monte Carlo simulations to examine the finite sample performance of our test, based on both univariate and bivariate designs of nonparametric regression models. Throughout, the significance level is 5%, the number of Monte Carlo replications is 3000, and the number of bootstrap samples for each replication is 200. The tuning parameter $\kappa_n$ will be selected as in Proposition 2.1, which entails a choice of $\gamma_n$. Since prior studies have repeatedly shown that testing results are quite insensitive to $\gamma_n$, we choose three values for $\gamma_n$: 0.01, 0.01/\log n, and 1/n.

3.1 Testing Monotonicity

In this section, we are concerned with monotonicity. For the univariate designs, the regression function $\theta_0 : [-1, 1] \to \mathbf{R}$ under the null is of the form:

$$\theta_0(z) = az - b \varphi(cz),$$

(22)

where $\varphi$ is the standard normal pdf, and $(a, b, c)$ equals $(0, 0, 0)$, $(0.1, 0.5, 0.5)$ or $(0.5, 2, 1)$, labeled D1, D2 and D3 respectively. We then consider two sets of alternatives, in order to showcase the relative advantages of our test. The first set of alternatives consists of functions as in (22) but with $\{(a, b, c) : a = c = -\Delta \delta, b = 0.2 \delta, \delta = 1, \ldots, 10\}$ for $\Delta = 0.05$. 
The second set of alternatives are defined by: for \( b = 0.5 \delta \) with \( \delta = 1, 2, \ldots, 10 \),

\[
\theta_0(z) = -b \varphi(|z|^{1.5}).
\]

The specification in (23) is inspired by Andrews and Shi (2013) and Chernozhukov et al. (2013, Section L.1), but modified so that the empirical power is close to one for \( \delta \) close to 10. Figure 1 depicts the curves of \( \theta_P \) based on these designs. Importantly, \( \theta_0 \) in Figure 1-(b) becomes sharp V-shaped as \( \delta \) increases, while \( \theta_0 \) in Figure 1-(c) has a visually “flat” bottom for each \( \delta \). Finally, we draw i.i.d. samples \( \{Z_i^*, u_i\}_{i=1}^n \) with \( n \in \{500, 750, 1000\} \) from the standard normal distribution in \( \mathbb{R}^2 \), and set \( Z_i = -1 + 2\Phi(Z_i^*) \in [-1, 1] \) and \( Y_i = \theta_0(Z_i) + u_i \), with \( \Phi \) the standard normal cdf.

![Designs in (22) under H0](image1.png)

(a) Designs in (22) under H0.

![Designs in (22) under H1](image2.png)

(b) Designs in (22) under H1.

![Designs in (23) under H1](image3.png)

(c) Designs in (23) under H1.

Figure 1. In the left figure, D1, D2 and D3 are solid, dashed and dotted respectively. In the middle and right figures, from top to bottom are curves corresponding to \( \delta = 1, 2, \ldots, 10 \).

For the bivariate designs, the regression function \( \theta_0 : [0, 1]^2 \to \mathbb{R} \) is specified as in Fang and Seo (2021): for some \( (a, b, c) \in \mathbb{R}^3 \),

\[
\theta_0(z_1, z_2) = a \left( \frac{1}{2} z_1^b + \frac{1}{2} z_2^b \right)^{1/b} + c \log(1 + z_1 + z_2),
\]

where the first term on the right-hand side of (24) equals \( a\sqrt{z_1z_2} \) when \( b = 0 \). We consider three choices of \((a, b, c)\) under \( H_0 \), namely \((0, 0, 0)\), \((0.2, 1, 0)\) and \((0.5, 0, 0.5)\) (labeled D1, D2, and D3, respectively), and, for the alternative, the collection \{ \((a, b, c) : a = c = -\Delta \delta, b = 0.2 \delta, \delta = 1, \ldots, 10 \) \} for \( \Delta = 0.05 \). We draw i.i.d. samples \( \{Z_{1i}^*, Z_{2i}^*, u_i\}_{i=1}^n \) with \( n \in \{500, 750, 1000\} \) from the standard normal distribution in \( \mathbb{R}^3 \), and set \( Y_i = \theta_0(Z_i) + u_i \) where \( Z_i \equiv (Z_{1i}, Z_{2i}) \) with \( Z_{ji} = \Phi(Z_{ji}^*) \in [0, 1] \) for all \( i \) and \( j = 1, 2 \).

The implementation of our test is based on series least squares estimation using B-splines. Specifically, we employ cubic B-splines with 3, 5, or 7 interior knots for the univariate designs, and quadratic as well as cubic B-splines each with one or zero knots for the bivariate designs. The knots in both cases are placed at the equispaced empirical quantiles of the regressors. We choose the test statistics based on the supremum distance as described in Section 2.3. In turn, \( \hat{G}_{n,b} \) is obtained by score bootstrap with i.i.d. weights from the standard normal distribution, and the coupling rate \( c_n = 1/\log n \). For ease of reference, we label our test with quadratic B-splines and \( j \) knots as F-Qj; similarly,
F-Cj is the implementation with cubic B-splines and j knots.

For the sake of fair comparisons, we implement the sup-test of Chernozhukov et al. (2013) and the one-step test of Chetverikov (2019) (labelled as C-OS) which is also of sup-type. For the former, we closely follow the steps articulated in Section 6.1 of Chernozhukov et al. (2013), and label the resulting test as CLR-Cj if the estimation is based on cubic B-splines with j interior knots—CLR-Qj is similarly defined. The C-OS test is implemented as in Fang and Seo (2021)—see also Chetverikov (2019, p.749) and its working paper version for more details.

Table 1. Empirical Size of Monotonicity Tests for \( \theta_0 \) in (22) at \( \alpha = 5\% \)

| n   | \( \gamma_n \) | D1   | D2   | D3   | D1   | D2   | D3   | D1   | D2   | D3   |
|-----|----------------|------|------|------|------|------|------|------|------|------|
| 500 | 1/n            | 0.059| 0.017| 0.010| 0.064| 0.032| 0.016| 0.065| 0.033| 0.014|
| 750 | 0.01/\log n   | 0.059| 0.017| 0.010| 0.064| 0.032| 0.016| 0.065| 0.033| 0.014|
| 1000| 0.01/\log n   | 0.060| 0.017| 0.010| 0.065| 0.033| 0.016| 0.065| 0.033| 0.014|
| 500 | 1/n            | 0.061| 0.020| 0.008| 0.061| 0.021| 0.011| 0.063| 0.027| 0.013|
| 750 | 0.01/\log n   | 0.061| 0.020| 0.008| 0.061| 0.021| 0.011| 0.063| 0.027| 0.013|
| 1000| 0.01/\log n   | 0.065| 0.015| 0.006| 0.069| 0.025| 0.011| 0.059| 0.028| 0.013|
| 500 | 1/n            | 0.066| 0.015| 0.006| 0.069| 0.025| 0.011| 0.059| 0.028| 0.013|
| 750 | 0.01/\log n   | 0.060| 0.039| 0.017| 0.065| 0.049| 0.023| 0.070| 0.050| 0.026|
| 1000| 0.01/\log n   | 0.062| 0.040| 0.017| 0.063| 0.043| 0.017| 0.068| 0.049| 0.021|
| 500 | 0.059 0.023 0.010| 0.063| 0.043| 0.017| 0.061| 0.043| 0.020|          |      |
| 750 | 0.059 0.043 0.014| 0.054| 0.041| 0.013| 0.056| 0.038| 0.010|          |      |
| 1000| 0.055 0.043 0.014| 0.054| 0.041| 0.013| 0.056| 0.038| 0.010|          |      |

Note: The parameter \( \gamma_n \) determines \( \kappa_n \) as in Proposition 2.1 with \( c_m = 1/\log n \) and \( r_m = (n/k_m)^{1/2} \).

Tables 1 and 2 present the empirical sizes. In the univariate designs, all tests seem to control size reasonably well, though our tests and the CLR tests slightly over-reject. In the bivariate designs, while the rejections rates of our tests and the C-OS test are close to the nominal level (at least in large samples), the CLR tests are notably oversized even in large samples. One possible explanation is that they are derivative-based tests and so the slow rate of convergence (Stone, 1982) is exacerbated in the bivariate designs. Figure 2 in turn depicts the power curves, where we only present our test with \( \gamma_n = 0.01/\log n \) as other choices lead to very similar results (here and below). Overall, these curves demonstrate that our test is competitive to the tests of Chernozhukov et al. (2013) and Chetverikov (2019) in terms of power as well. We note that, while the C-OS test enjoys some adaptivity and optimality properties in univariate settings, its bivariate version only captures part of the discordance between the outcome and the regressors so that its power in our bivariate designs is relatively low.
Chernozhukov et al. shows that our test based on the GCM/LCM operators (i.e., based on (23)) is replaced by

$$\theta_P(z) = b\varphi(|z|^{1.5}).$$

The null hypothesis in this case is that $\theta_0([-1,1] \to \mathbb{R}$ is convex. In bivariate designs, we instead test the concavity of $\theta_0$ by employing designs that are slight variations of (24) (so that the power curves get close to one as $\delta$ increases):

$$\theta_0(z_1, z_2) = a\left(\frac{1}{2}z_1^b + \frac{1}{2}z_2^b\right)^{1/b} + c\log(1 + 5(z_1 + z_2)),$$

where $(a, b, c)$ is chosen to be the same as for (24) except $\Delta = 0.2$. We implement our test based on the GCM/LCM operators (i.e., based on (15) with $p = \infty$), and compare it to the sup-test of Chernozhukov et al. (2013). The estimation and bootstrap steps are the same as those in Section 3.1.

Tables 3 and 4 summarize the empirical sizes. In the univariate designs, both our tests and the CLR tests have reasonable size control, while in the bivariate designs the CLR tests once again exhibit notable over-rejections across the sample sizes and the sieve spaces. Note that both D1 and D2 are at the “boundaries” of the parameter spaces, and thus the rejections rates are expected to be close to $\alpha = 5\%$. Figure 3 shows that our test remains competitive to the sup-test of Chernozhukov et al. (2013) as far as power
3.3 Testing Joint Restrictions

Finally, we test the joint restrictions of monotonicity and convexity/concavity, based on the same designs as those in Section 3.2. We implement our test based on the supremum distance, coupled with the $\Gamma$ operator (see Assumption 2.3) obtained by taking the composition $\Gamma = \Upsilon_2 \circ \Upsilon_1$, where $\Upsilon_1$ is the rearrangement operator and $\Upsilon_2$ is the GCM/LCM operator. That is, we apply rearrangement first and then the GCM/LCM operation. The remaining steps of our tests as well as the CLR tests are the same as before, beyond the need of incorporating the joint restrictions.
Table 3. Empirical Size of Convexity Tests for $\theta_0$ in (22) at $\alpha = 5\%$

| $n$  | $\gamma_n$ | F-C3: $k_n = 7$ | F-C5: $k_n = 9$ | F-C7: $k_n = 11$ |
|-----|-------------|----------------|----------------|----------------|
| 500 | $0.01/\log n$ | 0.056 0.048 0.012 | 0.052 0.048 0.019 | 0.055 0.049 0.022 |
| 1/$n$ | 0.056 0.048 0.012 | 0.052 0.048 0.019 | 0.055 0.049 0.022 |
| 1/1000 | 0.056 0.048 0.011 | 0.055 0.049 0.018 | 0.061 0.056 0.024 |
| 0.01 | 0.056 0.048 0.011 | 0.055 0.049 0.018 | 0.061 0.056 0.024 |
| 1/750 | 0.056 0.048 0.011 | 0.055 0.049 0.018 | 0.061 0.056 0.024 |
| 1/n | 0.062 0.053 0.007 | 0.063 0.056 0.015 | 0.061 0.056 0.022 |
| 1/1000 | 0.062 0.053 0.007 | 0.063 0.056 0.015 | 0.061 0.056 0.022 |

Note: The parameter $\gamma_n$ determines $\kappa_n$ as in Proposition 2.1 with $c_n = 1/\log n$ and $r_n = (n/k_n)^{1/2}$.

Table 4. Empirical Size of Concavity Tests for $\theta_0$ in (26) at $\alpha = 5\%$

| $n$  | $\gamma_n$ | F-Q0: $k_n = 9$ | F-Q1: $k_n = 16$ | F-C0: $k_n = 16$ | F-C1: $k_n = 25$ |
|-----|-------------|----------------|----------------|----------------|----------------|
| 500 | $0.01/\log n$ | 0.057 0.056 0.021 | 0.064 0.063 0.029 | 0.065 0.065 0.029 | 0.064 0.062 0.034 |
| 1/1000 | 0.057 0.056 0.021 | 0.064 0.063 0.029 | 0.065 0.065 0.029 | 0.064 0.062 0.034 |
| 0.01 | 0.057 0.056 0.021 | 0.064 0.064 0.029 | 0.065 0.066 0.029 | 0.065 0.062 0.034 |
| 1/n | 0.060 0.060 0.015 | 0.068 0.067 0.027 | 0.073 0.070 0.025 | 0.070 0.068 0.037 |
| 1/1000 | 0.060 0.060 0.015 | 0.068 0.067 0.027 | 0.073 0.070 0.025 | 0.070 0.068 0.037 |
| 0.01 | 0.061 0.060 0.015 | 0.069 0.067 0.027 | 0.073 0.072 0.025 | 0.071 0.069 0.037 |
| 1/n | 0.056 0.061 0.010 | 0.063 0.063 0.018 | 0.064 0.064 0.015 | 0.060 0.059 0.025 |
| 1/1000 | 0.056 0.061 0.010 | 0.063 0.063 0.018 | 0.064 0.064 0.015 | 0.060 0.059 0.025 |
| 0.01 | 0.056 0.061 0.010 | 0.063 0.063 0.018 | 0.064 0.064 0.015 | 0.060 0.059 0.025 |

Note: The parameter $\gamma_n$ determines $\kappa_n$ as in Proposition 2.1 with $c_n = 1/\log n$ and $r_n = (n/k_n)^{1/2}$.

Tables 5 and 6 summarize the empirical sizes, while Figure 3 presents the power curves. Consistent with our previous findings, our tests control size reasonably well in both univariate and bivariate designs, across sample sizes and sieve spaces, while the CLR tests tend to over-reject in bivariate designs (but otherwise perform well in terms of size). Moreover, our tests enjoy competitive power compared to the CLR tests, especially when $\theta_0$ has a flat region.

To conclude our simulations, we stress that our intention is not to show that our test is uniformly better than existing tests. As well understood in the literature, uniformly most powerful tests typically do not exist in nonparametric (and also many parametric)
settings, and thus no single test would dominate all others in terms of both size and power. Indeed, our numerical results show that, depending on the sample size, the functional form of $\theta_0$, and the sieve space, the CLR tests and the C-OS test may perform better than our tests (in terms of size or power). Instead, we hope to convey the message that our tests may serve as competitive alternatives, and may perform better in particular settings. In addition, compared to tests such as Chetverikov (2019) that are designed for specific shapes, our framework readily accommodates additional/joint restrictions. Our numerical exercises also confirm the applicability and usefulness of shape enforcing operators in either forming the Wald functional or enforcing the null restriction for the purpose of power improvement.
Table 5. Empirical Size of Monotonicity-Convexity Tests for $\theta_0$ in (22) at $\alpha = 5\%$

| $n$  | $\gamma_n$ | D1 | D2 | D3 | D1 | D2 | D3 | D1 | D2 | D3 |
|------|-------------|----|----|----|----|----|----|----|----|----|
| 1/n  | 0.055 0.045 0.012 | 0.052 0.046 0.016 | 0.053 0.044 0.024 |
| 500  | 0.01/$\log n$ 0.055 0.045 0.012 | 0.052 0.046 0.016 | 0.053 0.044 0.024 |
| 750  | 0.01/$\log n$ 0.055 0.046 0.010 | 0.055 0.048 0.018 | 0.058 0.053 0.022 |
| 1000 | 0.01/$\log n$ 0.060 0.051 0.006 | 0.060 0.048 0.014 | 0.060 0.052 0.022 |

Table 6. Empirical Size of Monotonicity-Concavity Tests for $\theta_0$ in (26) at $\alpha = 5\%$

| $n$  | $\gamma_n$ | D1 | D2 | D3 | D1 | D2 | D3 | D1 | D2 | D3 |
|------|-------------|----|----|----|----|----|----|----|----|----|
| 1/n  | 0.060 0.031 0.001 | 0.069 0.037 0.005 | 0.076 0.037 0.005 | 0.067 0.045 0.009 |
| 500  | 0.01/$\log n$ 0.060 0.031 0.001 | 0.069 0.037 0.005 | 0.076 0.037 0.005 | 0.067 0.045 0.009 |
| 750  | 0.01/$\log n$ 0.060 0.025 0.001 | 0.066 0.032 0.003 | 0.070 0.033 0.003 | 0.068 0.039 0.006 |
| 1000 | 0.01/$\log n$ 0.059 0.020 0.000 | 0.055 0.025 0.001 | 0.057 0.025 0.001 | 0.059 0.030 0.006 |

Note: The parameter $\gamma_n$ determines $k_n$ as in Proposition 2.1 with $c_n = 1/\log n$ and $r_n = (n/k_n)^{1/2}$.

4 Empirical Application

A central aspect of labor supply is the decision about working hours and its implications for earnings (Ehrenberg and Smith, 2016). By incorporating the worker’s labor supply decision and inherent ability into the dynamic complete information framework of Gibbons and Waldman (1999), Gicheva (2013) builds up a model that predicts a positive and convex intertemporal relationship between working hours and wage growth. In turn, Gicheva (2013) validates these shape predictions by analyzing the data set from a panel survey of registrants for the Graduate Management Admission Test (GMAT).
Specifically, Gicheva (2013) employs a partially linear model to estimate the relationship between working hours and wage growth, and then verifies the theoretical predictions through eyeball inspection. Below we complement her results by carrying out formal statistical tests of these restrictions based on the same data.

The GMAT sample involves individuals who registered to take the GMAT between June 1990 and March 1991, and were living in the United States at the time of registration. The survey was conducted in four waves of interviews: shortly after the registration, 15 months after registration, 3.5-4 years after registration, and 7 years registration. The sampled workers are concentrated in the high-end labor market: they are college educated, tend to work long hours, and have high earnings. The final cleaned sample consists of 1,911 respondents of the survey; see Section III in Gicheva (2013) for more details. Following Gicheva (2013), we base our analysis on the model:

\[ Y = \theta_0(Z) + W'\gamma + u, \tag{27} \]
where \( Y \) is the annual wage growth rate between the second and fourth interviews, \( Z \) indicates weekly working hours, \( W \) denotes a vector of demographic control variables (e.g., experience, education, gender, age, race, and family characteristics), and \( E[u|Z,W] = 0 \).

We then respectively test monotonicity, convexity, and monotonicity jointly with convexity of \( \theta_0 \), based on the full sample as in Gicheva (2013) as well as each gender group —there are 808 women and 1,103 men in the sample.

We employ the sup-distance test for monotonicity and the joint restriction, and the sup test based on GCM for convexity. For all three tests, the number of bootstrap samples is 5,000, while the grid points for \( Z \) is 3,...,90 (based on the realized weakly working hours). The estimation and bootstrap steps are otherwise the same as the univariate designs in Section 3. Figure 5 depicts the curves of \( \hat{\theta}_n \) obtained based on the full sample, women and men respectively, with the sieve dimension \( k_n = 9 \) (i.e., cubic B-splines with 5 interior knots). While the aforementioned shape restrictions are “overall” present in the data, there are nonetheless local violations. It is therefore crucial to conduct hypothesis test in order to formally quantify the sampling uncertainty.

![Figure 5](image_url)

Figure 5. The curves of \( \hat{\theta}_n \) based on the full sample, women, and men, respectively.

Table 7 reports the \( p \)-values of our tests based on various choices of \( k_n \) (the sieve dimension) and \( \gamma_n \) (the tuning parameter that determines \( \hat{\kappa}_n \)). Consistent with Gicheva (2013), the shape restrictions are significant in the full sample and in men, at all three conventional significance levels. The evidence is, however, less strong among women. Inspecting Figure 5-(b), we see that there are two segments of the working hours, namely \([18, 36]\) and \([53, 69]\), over which the wage growth rate declines. Following Gicheva (2013), we also re-compute the \( p \)-values by restricting to observations with working hours \( Z \in [35, 65] \). This eliminates 231 observations in total, 128 women, and 103 men. The presence of the shape restrictions remains strong, but more so for women. Overall, our findings are in line with the theoretical predictions in Gicheva (2013).
This paper develops a unifying framework for testing shape restrictions, and investigates the suitability of some Wald functionals in applying the framework. In particular, while the influential rearrangement operator has proven particularly useful in obtaining restricted point and interval estimates, it is inapplicable to our framework due to a lack of convexity. In contrast, the greatest convex minorization (resp. the least concave majorization) is shown to enjoy attractive analytic properties, and thus may be employed to test convexity (resp. concavity) in nonparametric settings that have not been explored previously. Finally, despite that the projection operator may not be well-defined/behaved in general Banach spaces, we show that one may nonetheless devise a powerful distance-based test by applying our framework.

### Table 7. Testing Shape Restrictions in Labor Supply: p-Values

| Sample | Shape | $\gamma_n = 0.01$ | $\gamma_n = 0.01/\log n$ | $\gamma_n = 1/n$ |
|--------|-------|-----------------|-----------------|-----------------|
|        |       | $k_n = 7$ | $k_n = 9$ | $k_n = 11$ | $k_n = 7$ | $k_n = 9$ | $k_n = 11$ | $k_n = 7$ | $k_n = 9$ | $k_n = 11$ |
|        |       | Untrimmed Samples |  |  |  |  |  |  |  |  |
|        | Mon   | 0.717 | 0.724 | 0.682 | 0.722 | 0.731 | 0.689 | 0.726 | 0.732 | 0.691 |
|        | Full  | 0.588 | 0.567 | 0.817 | 0.591 | 0.573 | 0.821 | 0.592 | 0.574 | 0.822 |
|        | Woman | 0.591 | 0.570 | 0.827 | 0.594 | 0.575 | 0.829 | 0.596 | 0.578 | 0.830 |
|        | Con   | 0.402 | 0.134 | 0.112 | 0.419 | 0.142 | 0.120 | 0.423 | 0.143 | 0.121 |
|        | Mon-Con | 0.107 | 0.071 | 0.237 | 0.111 | 0.076 | 0.253 | 0.113 | 0.076 | 0.253 |
|        | Mon   | 0.110 | 0.075 | 0.265 | 0.115 | 0.080 | 0.275 | 0.117 | 0.081 | 0.276 |
|        | Man   | 0.592 | 0.326 | 0.359 | 0.602 | 0.333 | 0.364 | 0.604 | 0.333 | 0.365 |
|        | Mon-Con | 0.329 | 0.714 | 0.902 | 0.331 | 0.720 | 0.905 | 0.333 | 0.721 | 0.905 |
|        | Man   | 0.336 | 0.741 | 0.915 | 0.338 | 0.742 | 0.915 | 0.339 | 0.742 | 0.915 |
|        | Mon-Con | 0.997 | 0.549 | 0.649 | 0.997 | 0.558 | 0.656 | 0.997 | 0.560 | 0.657 |
|        | Full  | 0.895 | 0.267 | 0.612 | 0.896 | 0.269 | 0.615 | 0.897 | 0.269 | 0.616 |
|        | Mon-Con | 0.895 | 0.274 | 0.624 | 0.896 | 0.275 | 0.626 | 0.897 | 0.275 | 0.626 |
|        | Mon   | 0.325 | 0.569 | 0.660 | 0.329 | 0.574 | 0.667 | 0.329 | 0.574 | 0.667 |
|        | Woman | 0.153 | 0.374 | 0.405 | 0.153 | 0.374 | 0.405 | 0.153 | 0.374 | 0.405 |
|        | Mon-Con | 0.153 | 0.374 | 0.406 | 0.153 | 0.374 | 0.406 | 0.153 | 0.374 | 0.406 |
|        | Man   | 0.643 | 0.535 | 0.722 | 0.653 | 0.543 | 0.729 | 0.653 | 0.543 | 0.730 |
|        | Mon-Con | 0.974 | 0.900 | 0.985 | 0.975 | 0.904 | 0.985 | 0.975 | 0.904 | 0.985 |
|        | Man   | 0.976 | 0.910 | 0.986 | 0.978 | 0.912 | 0.987 | 0.978 | 0.912 | 0.987 |

### 5 Conclusion

This paper develops a unifying framework for testing shape restrictions, and investigates the suitability of some Wald functionals in applying the framework. In particular, while the influential rearrangement operator has proven particularly useful in obtaining restricted point and interval estimates, it is inapplicable to our framework due to a lack of convexity. In contrast, the greatest convex minorization (resp. the least concave majorization) is shown to enjoy attractive analytic properties, and thus may be employed to test convexity (resp. concavity) in nonparametric settings that have not been explored previously. Finally, despite that the projection operator may not be well-defined/behaved in general Banach spaces, we show that one may nonetheless devise a powerful distance-based test by applying our framework.
Appendix A  Proofs

For ease of reference, we collect some notation in the table below.

| $a \lesssim b$ | For some constant $M$ that is universal in the proof, $a \leq Mb$. |
| --- | --- |
| $\mathbb{R}$ | The extended real number system, i.e., $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$. |
| $\|\theta\|_p$ | For a function $\theta : \mathcal{Z} \to \mathbb{R}$ and $p \in [1, \infty)$, $\|\theta\|_p = \{\int_{\mathcal{Z}} |\theta(z)|^p dz\}^{1/p}$. |
| $\|\theta\|_\infty$ | For a function $\theta : \mathcal{Z} \to \mathbb{R}$, $\|\theta\|_\infty = \sup_{z \in \mathcal{Z}} |\theta(z)|$. |
| $L^p(\mathcal{Z})$ | For $p \in [1, \infty)$, $L^p(\mathcal{Z}) = \{\theta : \mathcal{Z} \to \mathbb{R} : \|\theta\|_p < \infty\}$. |
| $\ell^\infty(\mathcal{Z})$ | The space of bounded functions, i.e., $\ell^\infty(\mathcal{Z}) = \{\theta : \mathcal{Z} \to \mathbb{R} : \|\theta\|_\infty < \infty\}$. |
| $\theta^*$ | The convex conjugate of $\theta : \mathcal{Z} \to \mathbb{R}$ (defined by (13) in the main text). |
| $\theta^{**}$ | The biconjugate of $\theta : \mathcal{Z} \to \mathbb{R}$, i.e., $\theta^{**} = (\theta^*)^*$. |
| $\mathbb{B}^*$ | The space of bounded linear operators $f : \mathbb{B} \to \mathbb{R}$ for a normed space $\mathbb{B}$. |
| $\langle b^*, b \rangle$ | For $b \in \mathbb{B}$ (a normed space) and $b^* \in \mathbb{B}^*$ (the dual), $\langle b^*, b \rangle = b^*(b)$. |
| $\| \cdot \|_{\mathbb{B}^*}$ | For the dual space $\mathbb{B}^*$ of $\mathbb{B}$ and $b^* \in \mathbb{B}^*$, $\|b^*\|_{\mathbb{B}^*} = \sup_{b \in \mathbb{B}} |\langle b^*, b \rangle|$. |

The following lemma plays a fundamental role in the proofs below.

Lemma A.1. If a lower-semicontinuous map $\phi : \mathbb{B} \to \mathbb{R}$ on a Banach space $\mathbb{B}$ satisfies Assumption 2.1(i)(ii), then it follows that, for all $\theta \in \mathbb{B}$,

$$
\phi(\theta) = \sup_{b^* \in \Lambda_\phi^0} \langle b^*, \theta \rangle,
$$

(A.1)

where $\Lambda_\phi^0 \equiv \{b^* \in \mathbb{B}^* : \langle b^*, \theta \rangle \leq \phi(\theta) \text{ for all } \theta \in \mathbb{B}\}$.

Proof: Let $\phi^* : \mathbb{B}^* \to \mathbb{R}$ be the convex conjugate of $\phi$, i.e., for any $b^* \in \mathbb{B}^*$,

$$
\phi^*(b^*) = \sup_{\theta \in \mathbb{B}} \{\langle b^*, \theta \rangle - \phi(\theta)\}.
$$

(A.2)

By Assumption 2.1(i), we must have $\phi(0) = 0$ and so, for any $b^* \in \mathbb{B}^*$,

$$
\phi^*(b^*) = \sup_{\theta \in \mathbb{B}} \{\langle b^*, \theta \rangle - \phi(\theta)\} \geq \langle b^*, 0 \rangle - \phi(0) = 0.
$$

(A.3)

Moreover, Assumption 2.1(i) also implies that, for any $t > 0$,

$$
t \phi^*(b^*) = \sup_{\theta \in \mathbb{B}} \{\langle b^*, t\theta \rangle - \phi(t\theta)\} = \sup_{\theta \in \mathbb{B}} \{\langle b^*, \theta \rangle - \phi(\theta)\} = \phi^*(b^*)
$$

(A.4)

We thus deduce from results (A.3) and (A.4) that $\phi^*(b^*) \in \{0, \infty\}$ for any $b^* \in \mathbb{B}^*$. By Assumptions 2.1(ii) and $\phi$ being lower-semicontinuous, we may invoke the Fenchel-Moreau theorem (see, e.g., Theorem 51.A in Zeidler (1985)) to conclude

$$
\phi(\theta) = \sup_{b^* \in \mathbb{B}^*} \{\langle b^*, \theta \rangle - \phi^*(b^*)\} = \sup_{b^* \in \mathbb{B}^*: \phi^*(b^*) = 0} \langle b^*, \theta \rangle,
$$

(A.5)

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for any $\theta \in \mathcal{B}$, where the second equality also exploits $\langle b^*, \theta \rangle \in \mathbb{R}$ for all $b^* \in \mathcal{B}$. By result (A.2) and $\phi(0) = 0$, the restriction $\phi^*(b^*) = 0$ is equivalent to, for all $\theta \in \mathcal{B}$,

$$\langle b^*, \theta \rangle \leq \phi(\theta) ,$$  \hspace{1cm} (A.6)

where the equality holds if $\theta = 0$. The lemma now follows from (A.5) and (A.6).

**Proof of Theorem 2.1**: We proceed by establishing the monotonicity of the map $a \mapsto \phi(h + a\theta_0)$ on $[0, \infty)$ for any $h \in \mathcal{B}$ and $\theta_0 \in \Lambda$. To this end, fix $h \in \mathcal{B}$ and $\theta_0 \in \Lambda$. By Lemma A.1, we obtain that, for $\Lambda_0 \equiv \{ b^* \in \mathcal{B} : \langle b^*, \theta \rangle \leq \phi(\theta) \text{ for all } \theta \in \mathcal{B} \}$,

$$\phi(h + a\theta_0) = \sup_{b^* \in \Lambda_0} \langle b^*, h + a\theta_0 \rangle = \sup_{b^* \in \Lambda_0} \{ \langle b^*, h \rangle + a\langle b^*, \theta_0 \rangle \} .$$  \hspace{1cm} (A.7)

Since $\phi(\theta_0) = 0$ by the definition of $\Lambda$, we have $\langle b^*, \theta_0 \rangle \leq 0$ for all $b^* \in \Lambda_0$. The conclusion of the lemma then immediately follows from the representation (A.7).

Next, for notational simplicity, let $\mathcal{G}_{n,P} \equiv r_n \{ \hat{\theta}_n - \theta_P \}$. By Assumptions 2.1(i)(iii) and 2.2(i), we obtain that, uniformly in $P \in \mathcal{P}$,

$$r_n \phi(\hat{\theta}_n) = \phi(\mathcal{G}_{n,P} + r_n \theta_P) = \phi(\mathcal{Z}_{n,P} - \mathcal{Z}_{n,P} \theta_P) + o_p(c_n) .$$  \hspace{1cm} (A.8)

By Assumptions 2.1(iii) and 2.3 and the triangle inequality, we have

$$|\phi(\hat{\mathcal{G}}_n + \kappa \Gamma(\hat{\theta}_n)) - \phi(\hat{\mathcal{G}}_n + \kappa \theta_P)| \lesssim \kappa \|\Gamma(\hat{\theta}_n) - \Gamma(\theta_P)\|_\mathcal{B}$$

$$\lesssim \kappa \|\hat{\theta}_n - \theta_P\|_\mathcal{B} \leq \frac{\kappa}{r_n} \{ \|\mathcal{G}_{n,P} - \mathcal{Z}_{n,P}\|_\mathcal{B} + \|\mathcal{Z}_{n,P}\|_\mathcal{B} \} = o_p(c_n) ,$$  \hspace{1cm} (A.9)

uniformly in $P \in \mathcal{P}_0$, where the final step is due to Assumptions 2.2(i) and 2.4(ii). Markov’s inequality, $\zeta_n \geq 1$ and $\kappa \zeta_n / r_n = o(c_n)$ by hypothesis. By Assumptions 2.1(iii) and 2.2(ii), we also have: uniformly in $P \in \mathcal{P}$,

$$|\phi(\hat{\mathcal{G}}_n + \kappa \theta_P) - \phi(\hat{\mathcal{Z}}_{n,P} + \kappa \theta_P)| \leq \|\hat{\mathcal{G}}_n - \hat{\mathcal{Z}}_{n,P}\|_\mathcal{B} = o_p(c_n) .$$  \hspace{1cm} (A.10)

It follows from (A.9), (A.10) and the triangle inequality that, uniformly in $P \in \mathcal{P}_0$,

$$\phi(\hat{\mathcal{G}}_n + \kappa \Gamma(\hat{\theta}_n)) = \phi(\hat{\mathcal{Z}}_{n,P} + \kappa \theta_P) + o_p(c_n) .$$  \hspace{1cm} (A.11)

Given results (A.8) and (A.11), we may select a sequence $\{ \epsilon_n \}$ of scalars such that $\epsilon_n = o(c_n)$ such that, as $n \to \infty$,

$$\sup_{P \in \mathcal{P}_0} P(|r_n \phi(\hat{\theta}_n) - \phi(\mathcal{Z}_{n,P} + r_n \theta_P)| > \epsilon_n) = o(1) ,$$  \hspace{1cm} (A.12)

$$\sup_{P \in \mathcal{P}_0} P(|\phi(\hat{\mathcal{G}}_n + \kappa \Gamma(\hat{\theta}_n)) - \phi(\hat{\mathcal{Z}}_{n,P} + \kappa \theta_P)| > \epsilon_n) = o(1) .$$  \hspace{1cm} (A.13)
By Markov’s inequality (see, e.g., Lemma 6.10 in Kosorok (2008)), Lemma 1.2.6 in van der Vaart and Wellner (1996) and (A.13), we have: for each $\eta > 0$,

$$
\sup_{P \in \mathcal{P}_0} P(\sup_{i} |\phi(\hat{\theta}_n) - \phi(\tilde{Z}_{n,P} + \kappa_n \theta_P)| > \epsilon_n | \{X_i\}_{i=1}^n > \eta)
\leq \sup_{P \in \mathcal{P}_0} \frac{1}{\eta} P(\sup_{i} |\phi(\hat{\theta}_n) - \phi(\tilde{Z}_{n,P} + \kappa_n \theta_P)| > \epsilon_n) = o(1). \quad (A.14)
$$

Thus, we may select positive scalars $\eta_n \downarrow 0$ such that: uniformly in $P \in \mathcal{P}_0$,

$$
P(\sup_{i} |\phi(\hat{\theta}_n) - \phi(\tilde{Z}_{n,P} + \kappa_n \theta_P)| > \epsilon_n | \{X_i\}_{i=1}^n = o(\eta_n). \quad (A.15)
$$

Since $\tilde{Z}_{n,P}$ is independent of $\{X_i\}_{i=1}^n$ by Assumption 2.2(ii), the conditional cdf of $\phi(\tilde{Z}_{n,P} + \kappa_n \theta_P)$ given $\{X_i\}_{i=1}^n$ is precisely its unconditional analog. Thus, we may conclude by Lemma 11 in Chernozhukov et al. (2013) and result (A.15) that

$$
\liminf_{n \to \infty} \inf_{P \in \mathcal{P}_0} P(\hat{c}_{n,1-\alpha} + \epsilon_n \geq c_n P(1 - \alpha - \eta_n)) = 1. \quad (A.16)
$$

By results (A.12) and (A.16), we have:

$$
\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha})
\leq \limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha}, |r_n \phi(\hat{\theta}_n) - \phi(Z_{n,P} + r_n \theta_P)| \leq \epsilon_n)
\leq \limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(\phi(Z_{n,P} + r_n \theta_P) > c_n P(1 - \alpha - \eta_n) - 2\epsilon_n)
\leq \limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(\phi(Z_{n,P} + \kappa_n \theta_P) > c_n P(1 - \alpha - \eta_n) - 2\epsilon_n), \quad (A.17)
$$

where the final step follows by the monotonicity established in the beginning and the fact $0 \leq \kappa_n \leq r_n$ for all large $n$ (due to $\kappa_n / r_n = o(c_n / \kappa_n)$, $\kappa_n \geq 1$ and $c_n = O(1)$). In turn, $\epsilon_n = o(c_n)$ (by construction), Assumption 2.4(iii), Proposition A.1, result (A.17) and $\eta_n = o(1)$ imply that

$$
\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(r_n \phi(\hat{\theta}_n) > \hat{c}_{n,1-\alpha}) \leq \liminf_{n \to \infty} \sup_{P \in \mathcal{P}_0} P(\phi(Z_{n,P} + \kappa_n \theta_P) > c_n P(1 - \alpha - \eta_n))
\leq \limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} \{\alpha + \eta_n\} = \alpha, \quad (A.18)
$$

as desired for the first claim of part (i). For the second claim, we note that

$$
\phi(Z_{n,P} + r_n \theta_P) = \phi(Z_{n,P} + \kappa_n \theta_P) = \phi(Z_{n,P}) \quad (A.19)
$$

for all $P \in \mathcal{P}_0$, by Assumption 2.1(i) and the definition of $\mathcal{P}_0$. The remaining arguments proceed as in Fang and Seo (2021), and we thus omit the details for brevity.
For part (ii), by Assumption 2.3(i) and the established monotonicity, we have

\[
\phi(\hat{\varphi}_n + \kappa_n \Gamma(\hat{\theta}_n)) \leq \phi(\hat{\varphi}_n) = \phi(\hat{\varphi}_n) - \phi(0) \\
\lesssim \|\hat{\varphi}_n\|_B \leq \|\hat{\varphi}_n - \bar{\varphi}_{n,P}\|_B + \|\bar{\varphi}_{n,P}\|_B , \quad (A.20)
\]

where the equality is by \(\phi(0) = 0\) (due to Assumption 2.1(i)), the second inequality is due to Assumption 2.1(iii), and the final step is by the triangle inequality. By Assumptions 2.2 and 2.4(ii) and Markov’s inequality, we in turn have from (A.20) that

\[
\phi(\hat{\varphi}_n + \kappa_n \Gamma(\hat{\theta}_n)) = o_p(c_n) + O_p(\zeta_n) = O_p(\zeta_n) , \quad (A.21)
\]

uniformly in \(P \in P\). By the definition of \(\hat{\varphi}_{n,1-\alpha}\), Markov’s inequality and Lemma 1.2.6 in van der Vaart and Wellner (1996), we note that, for any \(M > 0\),

\[
P(\hat{\varphi}_{n,1-\alpha} > M\zeta_n) \leq P(P(\phi(\hat{\varphi}_n + \kappa_n \Gamma(\hat{\theta}_n))) > M\zeta_n\{|X_i\}_{i=1}^n) > \alpha \leq \frac{1}{\alpha} P(\phi(\hat{\varphi}_n + \kappa_n \Gamma(\hat{\theta}_n)) > M\zeta_n) . \quad (A.22)
\]

Hence, results (A.21) and (A.22) yield \(\hat{\varphi}_{n,1-\alpha} = O_p(\zeta_n)\) uniformly in \(P \in P\).

Next, Assumption 2.1(iii) and the triangle inequality imply: uniformly in \(P \in P\),

\[
|r_n \phi(\hat{\theta}_n) - r_n \phi(\theta_P)| \lesssim \|r_n \{\hat{\theta}_n - \theta_P\}\|_B \\
\leq \|r_n \{\hat{\theta}_n - \theta_P\} - Z_{n,P}\|_B + \|Z_{n,P}\|_B \leq o_p(c_n) + O_p(\zeta_n) = O_p(\zeta_n) , \quad (A.23)
\]

where the third inequality follows by Assumptions 2.2(i) and 2.4(ii), and the last step is due to \(c_n = O(1)\) and \(\zeta_n \geq 1\). Result (A.23) and the definition of \(P_{\alpha,1,n}^\Delta\) thus imply

\[
r_n \phi(\hat{\theta}_n) = r_n \phi(\theta_P) + r_n \phi(\hat{\theta}_n) - r_n \phi(\theta_P) \geq \Delta \zeta_n + O_p(\zeta_n) , \quad (A.24)
\]

uniformly in \(P \in P_{\alpha,1,n}^\Delta\). The theorem thus follows from combining result (A.24) and the order \(\hat{\varphi}_{n,1-\alpha} = O_p(\zeta_n)\) (uniformly in \(P \in P\)) that we have established. \(\blacksquare\)

PROOF OF PROPOSITION 2.1: The proof is similar to the proof of Proposition 3.1 in Fang and Seo (2021) by working with \(\|\hat{\varphi}_n/\zeta_n\|_B\). We thus omit the details for brevity. \(\blacksquare\)

PROOF OF THEOREM 2.2: By Proposition 13.23(i)(ii) in Bauschke and Combettes (2017), we have that, for any \(a > 0\) and \(\theta \in B \equiv \ell^\infty(Z)\),

\[
\phi(a\theta) = \|a\theta - (a\theta)^*\|_p = \|a\theta - (a\theta^*(\cdot/a))^*\|_p = \|a\theta - a\theta^*\|_p = a\phi(\theta) . \quad (A.25)
\]

For \(a = 0\), note that \(\phi(0) = \phi(2 \cdot 0) = 2\phi(0)\) by (A.25) and hence

\[
\phi(0 \cdot \theta) = \phi(0) = 0 = 0\phi(\theta) . \quad (A.26)
\]
Together, results (A.25) and (A.26) imply positive homogeneity of $\phi$.

Next, since any $\theta \in B$ is real-valued and $Z$ is nonempty, $\theta^*$ and hence $\theta^{**}$ do not attain $-\infty$. Fix any $a \in (0, 1)$ and $\theta_1, \theta_2 \in B$. By Proposition 51.6(3) in Zeidler (1985) (the proposition also holds for maps defined on a subset of the entire space by inspecting the proof there), $a\theta_1^{**} + (1 - a)\theta_2^{**}$ is convex and lower semicontinuous such that

$$a\theta_1 + (1 - a)\theta_2 \geq a\theta_1^{**} + (1 - a)\theta_2^{**}.$$  \hfill (A.27)

By Proposition 51.6(5) in Zeidler (1985), it follows from (A.27) that

$$(a\theta_1 + (1 - a)\theta_2)^{**} \geq a\theta_1^{**} + (1 - a)\theta_2^{**}.$$  \hfill (A.28)

For notational simplicity, set $D = L^p(Z)$ if $p \in [1, \infty)$ and $D = \ell^\infty(Z)$ if $p = \infty$. Clearly, $B \subset D$ since $Z$ is bounded. Moreover, $\theta \geq 0$ with $\theta \in B$ means $\theta(z) \geq 0$ for all $z \in Z$. By Lemma A.1 in Fang (2019) and the fact $\theta - \theta^{**} \geq 0$ for any $\theta \in B$ (see, e.g., Proposition 51.6(3) in Zeidler (1985)), we obtain that, for all $\theta \in B$,

$$\phi(\theta) = \sup_{f \in D^*_1: f \geq 0} f(\theta - \theta^{**}) = \sup_{f \in D^*_1: f \geq 0} \{f(\theta) - f(\theta^{**})\},$$  \hfill (A.29)

where $f \geq 0$ means $f(\varphi) \geq 0$ whenever $\varphi \geq 0$, and $D^*_1 \equiv \{h \in D^*: \|h\|_{D^*} = 1\}$ is the unit sphere in the topological dual $D^*$ of $D$. Result (A.29) then implies

$$\phi(a\theta_1 + (1 - a)\theta_2) = \sup_{f \in D^*_1: f \geq 0} \{f(a\theta_1 + (1 - a)\theta_2) - f(a\theta_1^{**} + (1 - a)\theta_2^{**})\}$$

$$\leq \sup_{f \in D^*_1: f \geq 0} \{f(a\theta_1 + (1 - a)\theta_2) - f(a\theta_1^{**} + (1 - a)\theta_2^{**})\}$$

$$\leq a \sup_{f \in D^*_1: f \geq 0} \{f(\theta_1) - f(\theta_1^{**})\} + (1 - a) \sup_{f \in D^*_1: f \geq 0} \{f(\theta_2) - f(\theta_2^{**})\}$$

$$= a\phi(\theta_1) + (1 - a)\phi(\theta_2),$$  \hfill (A.30)

where the first inequality follows by $f \geq 0$ and (A.28), the second inequality by the linearity of $f \in D^*_1$ and the subadditivity of the supremum operator, and the last step by (A.29). It follows from result (A.30) that $\phi$ is convex.

Finally, for any $\theta_1, \theta_2 \in B$, we have by the definition of conjugate that

$$\|\theta_1^* - \theta_2^*\|_\infty = \sup_{y \in Z^*} |\theta_1^*(y) - \theta_2^*(y)| \leq \sup_{z \in Z} |\theta_1(z) - \theta_2(z)| = \|\theta_1 - \theta_2\|_\infty.$$  \hfill (A.31)

Result (A.31) in turn implies that

$$\|\theta_1^{**} - \theta_2^{**}\|_\infty \leq \|\theta_1^* - \theta_2^*\|_\infty \leq \|\theta_1 - \theta_2\|_\infty.$$  \hfill (A.32)

Lipschitz continuity of $\phi$ then readily follows from (A.32).
Proposition A.1. Let Assumptions 2.1 and 2.4(i)(ii)(iii)(iv) hold. If \( \{ \epsilon_n \} \) is a sequence of scalars satisfying \( \epsilon_n = o(\epsilon_n) \), then
\[
\limsup_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{x \in [\epsilon_n P(0.5) + \epsilon_n \infty)} P(\| \phi(Z_{n,P} + \kappa_n \theta_P) - x \| \leq \epsilon_n) = 0. \tag{A.33}
\]

Proof: By Lemma A.1, we may rewrite: for \( e_{b^*}(n,P) \equiv \kappa_n \langle b^* , \theta_P \rangle \),
\[
\phi(Z_{n,P} + \kappa_n \theta_P) = \sup_{b^* \in \Lambda^\circ} \{ \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \}, \tag{A.34}
\]
where \( \Lambda^\circ \equiv \{ b^* \in \mathcal{B}^* : \langle b^* , \theta \rangle \leq \phi(\theta) \text{ for all } \theta \in \mathcal{B} \} \). Since \( \phi(0) = 0 \) by Assumption 2.1(i), we have by Assumption 2.1(iii) that, for all \( \theta \in \mathcal{B} \),
\[
\phi(\theta) = \phi(\theta) - \phi(0) \leq C \| \theta \|_{\mathcal{B}}, \tag{A.35}
\]
where \( C > 0 \) is an absolute constant throughout that may change at each appearance. Since \( \phi(Z_{n,P} + \kappa_n \theta_P) \) is nonnegative, the supremum in (A.34) may be restricted to the set of \( b^* \in \Lambda^\circ \) with \( \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \geq 0 \) (which is nonempty since \( 0 \in \Lambda^\circ \)). For any such \( b^* \), by the definition of \( \Lambda^\circ \) and (A.35), we must have \( e_{b^*}(n,P) \geq -CM\zeta_n \) whenever \( \| Z_{n,P} \|_{\mathcal{B}} \leq M\zeta_n \). Define \( \Lambda^\circ_{\phi,M}(n,P) \equiv \{ b^* \in \Lambda^\circ : e_{b^*}(n,P) \geq -CM\zeta_n \} \). It follows that, whenever \( \| Z_{n,P} \|_{\mathcal{B}} \leq M\zeta_n \),
\[
\phi(Z_{n,P} + \kappa_n \theta_P) = \sup_{b^* \in \Lambda^\circ_{\phi,M}(n,P)} \{ \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \}. \tag{A.36}
\]
Fix a sequence \( \{ \epsilon_n \} \) of scalars such that \( \epsilon_n = o(\epsilon_n) \). Without loss of generality, we may assume that \( \{ \epsilon_n \} \) are nonnegative. Then, for any \( x \in \mathbb{R} \),
\[
P(\| \phi(Z_{n,P} + \kappa_n \theta_P) - x \| \leq \epsilon_n)
\leq P(\| \sup_{b^* \in \Lambda^\circ_{\phi,M}(n,P)} \{ \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \} - x \| \leq \epsilon_n) + P(\| Z_{n,P} \|_{\mathcal{B}} > M\zeta_n)
\leq P(\| \sup_{b^* \in \Lambda^\circ_{\phi,M}(n,P)} \{ \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \} - x \| \leq \epsilon_n) + \frac{1}{M}, \tag{A.37}
\]
where the second inequality is due to Markov’s inequality and Assumption 2.4(ii).

Next, we proceed by following the steps in the proof of Proposition D.1 in Fang and Seo (2021), but keep the arguments concise whenever appropriate for brevity. Let \( F_{n,P,M} \) be the cdf of \( T_{n,P} \equiv \sup_{b^* \in \Lambda^\circ_{\phi,M}(n,P)} \{ \langle b^* , Z_{n,P} \rangle + e_{b^*}(n,P) \} \), and \( c_{n,P,M}(\tau) \) be its \( \tau \)-quantile with \( m_{n,P,M} \equiv c_{n,P,M}(0.5) \). Fix \( n \) and \( P \in \mathcal{P}_0 \). We claim that
\[
\sigma_{n,P,M}^2 = \sup_{b^* \in \Lambda^\circ_{\phi,M}(n,P)} E[\langle b^* , Z_{n,P} \rangle^2] > 0, \tag{A.38}
\]
for all large $M > 0$. To see this, we note that $\tilde{\sigma}_{n,P}^2 \equiv \sup_{t \in \Lambda^\circ} E[(b^*, Z_{n,P})^2] > 0$; otherwise $\phi(Z_{n,P} + \kappa_n\theta_P) = 0$ almost surely so that all quantiles of $\phi(Z_{n,P} + \kappa_n\theta_P)$ collapse to zero, contradicting Assumption 2.4(iii) (see Fang and Seo (2021) for more formal arguments). Fix $\eta > 0$. Then we may select some $b^*_{n,P} \in \Lambda^\circ_\phi$ such that

$$\tilde{\sigma}_{n,P}^2 \leq E[(b^*_{n,P}, Z_{n,P})^2] + \eta. \tag{A.39}$$

Since $n$ and $P \in P_0$ are being fixed and $\zeta_n \geq 1$ by Assumption 2.4(ii), it follows that $e_{b^*_{n,P}}(n, P) \geq -CM\zeta_n$ for all $M$ sufficiently large, so that $b^*_{n,P} \in \Lambda^\circ_{\phi,M}(n, P)$ and

$$E[(b^*_{n,P}, Z_{n,P})^2] \leq \tilde{\sigma}_{n,P,M}^2. \tag{A.40}$$

Combining results (A.39) and (A.40), we may then conclude that, for all $M$ large,

$$\tilde{\sigma}_{n,P}^2 - \eta \leq \tilde{\sigma}_{n,P,M}^2 \leq \tilde{\sigma}_{n,P}^2. \tag{A.41}$$

Since $\eta$ is arbitrary, result (A.41) implies that, as $M \to \infty$,

$$\tilde{\sigma}_{n,P,M}^2 \to \tilde{\sigma}_{n,P}^2. \tag{A.42}$$

The claim (A.38) then follows from result (A.42) and the fact $\tilde{\sigma}_{n,P}^2 > 0$.

With (A.38) in hand, we may proceed as in Fang and Seo (2021) in view of

$$F_{n,P,M}(r) \equiv P\left( \sup_{b^* \in \Lambda^\circ_{\phi,M}(n, P)} \left\{ (b^*, Z_{n,P}) + e_{b^*}(n, P) \right\} \leq r \right) = P\left( \sup_{b^* \in \Lambda^\circ_{\phi,M}(n, P)} \left\{ (b^*, Z_{n,P}) + e_{b^*,M}(n, P) \right\} \leq r + CM\zeta_n \right), \tag{A.43}$$

where $e_{b^*,M}(n, P) \equiv e_{b^*}(n, P) + CM\zeta_n \geq 0$ for all $b^* \in \Lambda^\circ_{\phi,M}(n, P)$ (by the definition of $\Lambda^\circ_{\phi,M}(n, P)$). In particular, Theorem 2.2.1 in Yurinsky (1995) implies that $F_{n,P,M}$ is absolutely continuous on an interval containing $(m_{n,P,M}, \infty)$, and Theorem 2.2.2 in Yurinsky (1995) in turn delivers (with $b = m_{n,P,M} + CM\zeta_n$ and $u = r + CM\zeta_n$ in Yurinsky’s notation) that, for all $r > m_{n,P,M},$

$$f_{n,P,M}(r) \equiv F'_{n,P,M}(r) \leq \frac{2(r + CM\zeta_n) - (m_{n,P,M} + CM\zeta_n)}{(r + CM\zeta_n) - (m_{n,P,M} + CM\zeta_n)} = \frac{2r - m_{n,P,M} + CM\zeta_n}{(r - m_{n,P,M})^2}. \tag{A.44}$$

Moreover, (A.38) and Borell’s inequality (see, e.g., Davydov et al. (1998, p.82)) imply

$$m_{n,P,M} + \tilde{\sigma}_{n,P,M}z_{1-\alpha} \geq c_{n,P,M}(1 - \alpha - \varpi), \tag{A.45}$$
where \( z_{1-\alpha-\varpi} \) is the \((1-\alpha-\varpi)\)-quantile of the standard normal distribution. Next, we note (1) \( m_{n,p,M} \leq c_n,p(0.5) \) due to \( T_{n,P} \leq \phi(Z_{n,P} + \kappa \theta_p) \), (2) \( c_n,p(0.5) \leq C' \varsigma_n \) by Kwapiéń (1994), (A.35) and Assumption 2.4(ii), and (3) \( \sigma_{n,P,M} \leq C\{E[|Z_{n,P}|^2_B]\}^{1/2} \) by the definition of \( \Lambda_0^c \) and result (A.35) so that \( \sigma_{n,P,M} \leq C\{E[|Z_{n,P}|^2_B]\}^{1/2} \) by Proposition A.2.4 in van der Vaart and Wellner (1996) and Assumption 2.4(i), with \( B^* \) the unit ball in \( B^* \); see the proof of Proposition 3.1 in Fang and Seo (2021) for formal arguments. These facts, together with (A.45) and Assumption 2.4(ii), yield

\[
c_{n,P,M}(1-\alpha-\varpi) \leq c_{n,P}(0.5) + C' \varsigma_n z_{1-\alpha-\varpi}. \tag{A.46}
\]

Note that \( c_{n,P,M}(1-\alpha-\varpi) \to c_{n,P}(1-\alpha-\varpi) \) as \( M \to \infty \) by \( T_{n,P} \to \phi(Z_{n,P} + \kappa \theta_p) \) (almost) surely and \( c_{n,P}(1-\alpha-\varpi) \) being a continuity point of the cdf of \( \phi(Z_{n,P} + \kappa \theta_p) \) (by an application of Theorem 2.2.1 in Yurinsky (1995)). Letting \( M \to \infty \), we may thus deduce from (A.46) and Assumption 2.4(iii) that \( \varsigma_n \leq C' \varsigma_{n} z_{1-\alpha-\varpi} \). In turn, by Assumption 2.4(iv) and \( \epsilon = o(c_n) \), we obtain that, as \( n \to \infty \),

\[
\epsilon_n = o\left(\frac{\varsigma_n^2}{\varsigma_n}\right) = o(\varsigma_n). \tag{A.47}
\]

Thus, \( \epsilon_n \leq \varsigma_n/2 \) for all \( n \) large, so that, whenever \( x \geq m_{n,p,M} + \varsigma_n \),

\[
x - \epsilon_n \geq m_{n,p,M} + \varsigma_n - \epsilon_n \geq m_{n,p,M} + \frac{\varsigma_n}{2}. \tag{A.48}
\]

By the fundamental theorem of calculus, results (A.44) and (A.48), and the fact that \( r \mapsto (2r - m_{n,p,M} + CM \varsigma_n)/(r - m_{n,p,M})^2 \) is decreasing on \((m_{n,p,M}, \infty)\), we may conclude that, for all \( x \geq m_{n,p,M} + \varsigma_n \) and \( n \) large,

\[
P\left( \sup_{b^* \in \Lambda_0^c, M(n,P)} \{\langle b^*, Z_{n,p} \rangle + e_{b^*}(n, P) \} - x \right| \leq \epsilon_n \right)
\]

\[
= \int_{x-\epsilon_n}^{x+\epsilon_n} f_{n,P,M}(r) \, dr \leq 2\epsilon_n \frac{2(m_{n,p,M} + \varsigma_n/2) - m_{n,p,M} + CM \varsigma_n}{(\varsigma_n/2)^2}. \tag{A.49}
\]

Since \( m_{n,p,M} \leq c_n,p(0.5) \leq C' \varsigma_n \) and \( \varsigma_n \leq C' \varsigma_n z_{1-\alpha-\varpi} \) as argued previously, we obtain from (A.49) that, for all large \( n \) and all \( M > 0 \),

\[
P\left( \sup_{b^* \in \Lambda_0^c, M(n,P)} \{\langle b^*, Z_{n,p} \rangle + e_{b^*}(n, P) \} - x \right| \leq \epsilon_n \right) \leq C\epsilon_n \frac{\varsigma_n + M \varsigma_n}{\varsigma_n^2}. \tag{A.50}
\]

The proposition then follows by (A.37), (A.50), \( \epsilon_n = o(c_n) \) and Assumption 2.4(iv).
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