Wave dispersion curves in discrete lattices derived through asymptotic multi-scale method

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Abstract. This paper falls within the study of dispersion feature of elastic periodic media. In most cases, no analytic description is reachable and the problem is solved via numerical computations of the dispersion curves. We propose in this paper an analytic method dedicated to lattice systems that enables to reconstruct part by part the dispersion curves via an asymptotic multi-scale method. This method is illustrated on periodic reticulated beams. At low frequency, when there is a large scale separation between the length of the cell and the characteristic size of the vibrations, the classical homogenization method allows efficiently to establish the continuous equivalent dynamic description and the associated wave propagation properties. This scale separation is lost for frequencies of the order or higher than the diffraction frequency. However, instead of considering the amplitude of the mean displacement in a unit cell, the concept of scale separation can still be used by considering the amplitude of periodic eigenmodes defined on (multi-)cells. Thus, similar principles of asymptotic multi-scale method enables to describe the large scale modulations around the eigenfrequencies of the mono- and/or multi-cells period. Finally, the properties of the modulation are straightforwardly related to the dispersion curves at the considered frequencies.

Introduction
This paper deals with the study of dispersion feature of elastic periodic media. The dispersion curve, linking the frequency and the wave number, provides essential information on the dynamic behaviour of an elastic media. The feature of the wave propagation is useful in numerous applications, including structural monitoring. It also enables to evidence the presence of atypical behaviour with band gap in phononic crystals or acoustic metamaterials.

The present asymptotic multi-scale approach applies at frequencies lower or higher than the diffraction frequency. It take advantage of the concept of scale separation expressed on the amplitude of periodic eigenmodes defined on (multi-)cells. The procedure is splitted into the following key steps. 1. Reduce the studied structure to an equivalent discrete model. This is obtained by condensing the balance equations of the beam elements within the period at the nodes of the period. 2. Establish an exact recurrence equation by expressing the balance of the edges nodes of the considered (multi-)cells period. 3. Identify the periodic eigenmodes and eigenfrequencies of the (multi-)cell. 4. Consider frequencies close to a given eigenfrequency and assume a large scale modulation. Then rewrite the recurrence equation in terms of Taylor expansions of the continuous variables that describe the amplitude of the eigenmode. 5. Introduce the asymptotic expansions of the variables and rescale the physical
parameters of the element constituting the period. 6. Deduce the various envelope models that apply close to the eigenfrequencies of the considered (multi-)cells. This method is illustrated on the case of periodic reticulated beams.

1. Exact dispersion relations in reticulated beams

1.1. Dynamics of periodic framed beams.

The studied periodic structure (figure 1) is made of the 1D-repetition of an irreducible cell \( \Omega \) (a frame). Each frame is constituted by two parallel identical beams (called walls, index "w") oriented along the structure length and a beam (called floor, index "f") oriented transversally to the structure length. The mechanical properties of the two type of beams \((i = w, f)\) are the elastic modulus \( E_i \) and the mass density \( \rho_i \); and their geometrical parameters are, the length \( \ell_i \), width \( a_i \), depth \( h_i \), cross section \( A_i = a_i h_i \) and the inertia regarding their longitudinal axis \( I_i = a_i^3 h_i / 12 \). The connections are assumed perfectly rigid and mass-less. Thus, the motions of the extremities of the beams connected to a same node are identical and define the discrete kinematic variables. Integrating the dynamic balance of a beam taking the motions at its two extremities as boundary conditions provides the expressions of the forces at the two extremities. The latter forces involves the motions at the extremities (i.e. the discrete kinematic variables) and two dimensionless parameters describing the local regime of the beam in compression \((\chi_i = \sqrt{A_i \rho_i E_i} / \ell_i \) \( \ell_i = 2 \pi f_i \)) and in bending \((\gamma_i = \sqrt{A_i \rho_i E_i I_i} / \ell_i \) \( \ell_i = 2 \pi f_i \)). Each period comprises two nodes then six degrees of freedom. Due to the symmetry of the considered structures, the local variables can be reorganized in two sets of uncoupled variables linked to (i) the transversal kinematic of the reticulated beam (namely the average transversal displacement \( V \) and the differential rotation \( \theta \) of the nodes). Denoting by the vectors

\[
\mathbf{U}_n = \begin{bmatrix} (U_n, \Delta_n, \Phi_n) \end{bmatrix}^T \quad \text{and} \quad \mathbf{\Sigma}_n = \begin{bmatrix} (1/2 \alpha_n, \ell_n, V_n, \theta_n) \end{bmatrix}^T
\]

the longitudinal and transversal generalized displacements of the \( n^{th} \) cell, the force and momentum balances of the \( n^{th} \) cell involves the generalized displacements of the neighboring cell \( n - 1 \) and \( n + 1 \) and take the matrix form:

\[
\mathbf{H}_L = \begin{bmatrix} a_L & 0 & b_L \\ 0 & c_L & 0 \\ b_L & 0 & d_L \end{bmatrix} \quad \text{with}
\]

\[
\begin{align*}
\mathbf{U}_n &= \begin{bmatrix} \mathbf{H}_L^T \mathbf{U}_{n-1} + \mathbf{H}_L \mathbf{U}_n + \mathbf{H}_L^T \mathbf{U}_{n+1} = 0 \\
\mathbf{\Sigma}_n &= \begin{bmatrix} \mathbf{H}_T^T \mathbf{\Sigma}_{n-1} + \mathbf{H}_T \mathbf{\Sigma}_n + \mathbf{H}_T^T \mathbf{\Sigma}_{n+1} = 0 \end{bmatrix}
\end{align*}
\]

Figure 1. Structure.

Figure 2. Cell \( \Omega_0 \) of the reticulated beam.
Consider for instance the propagation of harmonic transversal waves. In order to get their dispersion equation of the transversal waves in the reticulated structure:

\[ D = \begin{bmatrix} a_T & 0 & b_T \\ 0 & c_T & 0 \\ b_T & 0 & d_T \end{bmatrix} \]

with

\[ \begin{align*}
  a_T &= -\frac{12 E_I I_t}{\ell_t^2} (f_1(\gamma_t) + f_4(\gamma_t)) - \frac{6 E_w I_w}{\ell_w^2} f_6(\gamma_w) \\
  b_T &= \frac{6 E_I I_t}{\ell_t^2} (f_3(\gamma_t) + f_6(\gamma_t)) \\
  c_T &= \frac{E_I A_t}{\ell_t} (g_2(\chi_t) - g_1(\chi_t)) - \frac{24 E_I I_w}{\ell_w^3} f_1(\gamma_w) \\
  d_T &= -\frac{2 E_I I_t}{\ell_t^2} (f_5(\gamma_t) + 2 f_2(\gamma_t)) - \frac{8 E_w I_w}{\ell_w^2} f_2(\gamma_w)
\end{align*} \]

The elasto-inertial functions \( f_i \) and \( g_i \) are defined by:

\[
\begin{align*}
  & g_1(x) = x \cot(x) \\
  & g_2(x) = \frac{x}{\sin(x)} \\
  & f_1(x) = \frac{\cosh(x) \sin(x) + \sinh(x) \cos(x)}{1 - \cos(x) \cosh(x)} x^2 \\
  & f_2(x) = \frac{\cosh(x) \sin(x) - \sinh(x) \cos(x)}{1 - \cos(x) \cosh(x)} 4 \\
  & f_3(x) = -\sinh(x) \sin(x) x^2 \\
  & f_4(x) = \sinh(x) \sin(x) x^3 \\
  & f_5(x) = -\cosh(x) \cos(x) x^2 \\
  & f_6(x) = \cosh(x) \cos(x) 6 
\end{align*}
\]

It should be noted that \( f_i \) and \( g_i \to 1 \) as \( x \to 0 \) (and furthermore \( \forall x \), \( g_2(x) \), \( f_4(x) \), \( f_5(x) \), \( f_6(x) \neq 0 \)). Hence, the elasto-inertial matrices \( \mathcal{H}_T \) tend to the usual static stiffness matrices when the frequency tends to zero.

1.2. Dispersion equations

Consider for instance the propagation of harmonic transversal waves. In order to get their dispersion relation \( \mathcal{E}(\omega, k_\omega) = 0 \) that relates the wavenumber \( k_\omega \) to the angular frequency \( \omega \), and look for the kinematic variable \( \mathcal{V}_n = \epsilon^{T} \left( \frac{1}{2} a_n \, \ell_t, \mathcal{V}_n, \theta_n \right) \) in the form (the time dependence \( e^{i \omega t} \) is omitted):

\[ \mathcal{V}_n = \mathcal{V}_n e^{i k_\omega n \ell_w} \quad k_\omega \in [0, \pi / \ell_w] \quad (4) \]

Introducing (4) in (2) yields:

\[
\mathbb{D}_T \mathcal{V}_0 = \left[ \begin{array}{ccc}
  \tilde{\alpha}_T & 0 & b_T \\
  0 & \tilde{c}_T & \tilde{e}_T \\
  b_T & -\tilde{e}_T & \tilde{d}_T
\end{array} \right] \mathcal{V}_0 = 0 \quad \text{where}
\]

\[
\begin{align*}
  \tilde{\alpha}_T &= \alpha_T + \frac{2 E A_w}{\ell_w} g_2(\chi_w) \cos(k_\omega \ell_w) \\
  \tilde{c}_T &= c_T + \frac{24 E I_w}{\ell_w^3} f_1(\gamma_w) \cos(k_\omega \ell_w) \\
  \tilde{d}_T &= d_T - \frac{4 E I_w}{\ell_w^2} f_5(\gamma_w) \cos(k_\omega \ell_w) \\
  \tilde{e}_T &= \frac{12 E I_w}{\ell_w^2} f_6(\gamma_w) i \sin(k_\omega \ell_w)
\end{align*}
\]

For non trivial solution the determinant of \( \mathbb{D}_T \) must vanish. This condition provides the dispersion equation of the transversal waves in the reticulated structure:

\[ \det \left( \mathbb{D}_T \right) = \left( \tilde{\alpha}_T \tilde{d}_T - \tilde{b}_T \right) \tilde{c}_T + \tilde{\alpha}_T \tilde{e}_T^2 = \mathcal{E}_T (\omega, k_\omega) = 0 \quad (5) \]
For the longitudinal kinematics the dispersion equation \( E_L(\omega, k_\omega) = 0 \) is established in the same way, replacing \( T \) by \( L \). The dispersion curves relative to the transversal and longitudinal kinematics are provided by solving the dispersion equations \( E_T(\omega, k_\omega) = 0 \) and \( E_L(\omega, k_\omega) = 0 \).

The numerical resolution is performed point by point by sampling half of the first Brillouin zone for a sufficiently large number of points. Furthermore, for each solution \((k_\omega f, k_\omega, f)\), the kernel of the dispersion tensor \( D_T(\omega, k_\omega, f) \) is calculated and normalized (norm \(\|\cdot\|_1\)) to identify the associated kinematics.

As an illustration the dispersion of the transversal waves of a framed beam whose wall and floor elements are identical is studied. Their geometrical and mechanical parameters are given in Table 1 and the dispersion curves are displayed in figure 3. In addition, the kernel kinematics is also depicted, in attributing a given color for each component namely \(\{\alpha, \nu, V, \theta, \ell\}\), the color of the dot being the linear combination of the three colors weighted by the proportion of each component.

| \( E \) (GPa) | \( \nu \) | \( \rho \) (kg/m\(^3\)) | \( \ell \) (cm) | \( a \) (mm) |
|---|---|---|---|---|
| 210 | 0.2 | 7800 | 1 | 0.5 |

Table 1. Framed beam. Geometrical and mechanical parameters used for the numerical simulations. The wall and floor are identical, hence the indices \( w \) and \( f \) are dropped.

The first quasi-linear branch on figure 3 clearly shows that at low frequencies the transversal dynamics is driven by the average transversal displacement \( V \). The second strongly non-linear branch corresponds to low wavenumbers to the gyration mode. The latter, driven by the rotation of the section \( \alpha \ell \), is activated after a cut-off frequency (around 4.4 kHz). Note however that for wavenumbers greater than \( k_\omega = 0.2\pi/\ell \) the dynamics activated in this branch, moves from \( \alpha \ell \) to the average rotation of the nodes \( \theta \ell \). As for the third branch, that appears at higher frequencies (around 37kHz), the kinematics is firstly driven by \( V \), and by \( \theta \) at larger wavenumbers. The next two branches are dominated by nodes rotation \( \theta \), the rigid body motion of the two nodes being negligible, consistently with the inner resonance mechanism within the framed beams. This shows that a given branch is not associated with a specific kinematics. The transversal dispersion equations (5) (and similar for longitudinal) apply to the whole family of framed beams described in figure 1. Following the studies on framed beams [1], [2], the macroscopic behaviour, depends strongly on the relative magnitude of the local stiffness in bending and compression of the wall and floor elements. Therefore a large number of situations may occur depending on the frame morphology. The sensibility of the dispersion properties is illustrated in figure 4 where the slenderness of the constituting beams is varied by changing their thickness \( a_f = a_w \), while all the other parameters presented in table 1 are kept constant. Decreasing the slenderness induces a rigidification of the structure. This mostly results in a change of the frequency, while the qualitative features of the dispersion curves are kept.

1.3. Bloch modes and multi-cells periodic eigenmodes

Consider now a period \( \Omega_p = \bigcup_p \Omega \) made of \( p \) irreducible cells \( \Omega \). The eigenmodes of \( \Omega_p \) with periodic boundary conditions are arising for wavenumbers such \( k_\omega \ell /2\pi \in [0,1/2] \) take rational values i.e. \( k_\omega \ell /2\pi = q/p \). As a consequence the whole set of eigenfrequencies of the multi-cells \( \Omega_p \) is constituted by the solutions of the following \( q \) dispersion equations

\[ E \left( \omega, \frac{q2\pi}{p\ell} \right) = 0 \quad \text{where} \quad q = 0, ..., \text{Int}(p/2) \]

For instance,
Figure 3. Dispersion curve in the half on the first Brilloin zone for the transversal kinematics: \( \{ \alpha \ell, V, \theta \ell \} \). Black circles correspond to the Bloch modes for periods made of one cell (\( \Omega \)), two cells (\( \Omega^2 \)) and three cells (\( \Omega^3 \)). For each of them, the modal shape is depicted.

Figure 4. Sensibility regarding the slenderness \( \ell_w/a \).

- The set \( \{ \omega_{0,j} \} \) of angular eigenfrequencies of the irreducible cell \( \Omega \) (hence \( p = 1, q = 0 \) then \( k_{\omega,\ell} = 0 \)) are the roots of the equation
  \[ E(\omega,0) = 0 \]

- For the double cells \( \Omega^2 \) (hence \( p = 2, q = \{0,1\} \) then \( k_{\omega,\ell} = \{0,\pi\} \)) the whole angular eigenfrequencies are \( \{ \omega_{0,j} \} \cup \{ \omega_{\frac{\pi}{2},k} \} \) where the angular frequencies \( \{ \omega_{\frac{\pi}{2},k} \} \) are the roots of the equation
  \[ E\left(\omega,\frac{\pi}{2}\right) = 0 \]

- Similarly for a triple cells \( \Omega^3 \) (hence \( p = 3, q = \{0,1\} \) then \( k_{\omega,\ell} = \{0,2\pi/3\} \)) the set of angular eigenfrequencies consists in \( \{ \omega_{0,j} \} \cup \{ \omega_{\frac{2\pi}{3},k} \} \) where \( \{ \omega_{\frac{2\pi}{3},k} \} \) are the roots of
  \[ E\left(\omega,\frac{2\pi}{3}\right) = 0 \]

For each couple \( (k,\omega_{k,l}) \) satisfying the dispersion equation \( E(\omega_{k,l},k) = 0 \) the Bloch mode is given by the kernel of the Hermitian matrix \( \mathbb{D} \). The multiplicity of the Bloch mode is given by the multiplicity of the root of the characteristic polynomial of \( \mathbb{D} \).
The matching between rational Bloch wavenumbers and periodic eigenmodes of multi-cell is evidenced on figure 3. The first eigenfrequencies corresponding periods made of one, two and three cells have been calculated independently and the results represented by black circles match those deduced by Bloch resolution of (5). Moreover, the corresponding modal shape and their multiplicity, calculated by finite element on these three periods with periodic boundary conditions are also displayed.

2. Approximating the dispersion relations
The asymptotic method for modulation analysis of framed beams is fully described in [3]. In this section a summary of the key points of this approach is proposed. Then the modulation equations are directly used to reconstruct the dispersion curves.

2.1. Overview of the modulation approach
This asymptotic approach is closely related to the HPDM method [4] and to the modulation analysis of elastic composites [5], [6]. At first, one focuses on $p$-cells period $\Omega_p$, whose angular eigenfrequencies for periodic boundary conditions are denoted by $\omega_J$. For frequencies close to $\omega_J$, the modal amplitude envelop varies at a large scale $L$ compared to the length $\ell_p = p \ell_w$ of $\Omega_p$. This assumption of scale separation allows us to define the small parameter $\varepsilon_p = \ell_p/L = p \ell_w/L \ll 1$.

Consider the discrete motion vector $\mathbf{X}_n$ ($\mathbf{X} = \mathbf{U}$ or $\mathbf{V}$) at the extremity of the $n^{th}$ period $\Omega_p$. The asymptotic up-scaling process consists into the three following steps : (i) introduce the continuous variables $\mathbf{X}(x)$, such that $\mathbf{X}(x_n) = \mathbf{X}_n$ for $x_n = n \ell_p$ (ii) expand the motions $\mathbf{X}_{n \pm 1}$ in Taylor’s series, since thanks to the scale separation the end motions of two successive periods are quite the same, and (iii) expand in powers of $\varepsilon_p$ the continuous variables $\mathbf{X}(x)$ and the angular frequencies around the periodic eigenfrequencies $\omega_J$:

$$\mathbf{X}_{n \pm 1} = \mathbf{X}(x_n \pm \ell_p) = \sum_{i \geq 0} (\pm \ell_p)^i \frac{i!}{i!} \partial_x^i \mathbf{X}(x_n) ; \quad \mathbf{X}(x) = \sum_{i \geq 0} \varepsilon_p^i \mathbf{X}^{(i)}$$

$$\omega = \omega^{(0)} + \varepsilon_p \omega^{(1)} + \varepsilon_p^2 \omega^{(2)} + \ldots$$

where $\omega^{(0)} = \omega_J$.

Furthermore, the different parameters involved in the elasto-inertial matrices also have to be scaled and expanded according to the powers of $\varepsilon_p$. This scaling allows to balance mechanical effects of the same order regardless of the value of $\varepsilon_p$. When doing so, the same physics occurring when $0 \neq \varepsilon_p \ll 1$ (i.e. real structures) is kept at the limit $\varepsilon_p \to 0$, which corresponds to the continuum macroscopic model.

As an illustration, the study is focused on a structure whose the floors and the walls have mechanical and geometrical properties of the same magnitude, i.e.:

$$\ell_f = O(\ell_w) ; \quad \frac{\alpha_w}{\ell_w} = O(\varepsilon_p) ; \quad \frac{\alpha_l}{\ell_f} = O(\varepsilon_p) ; \quad \Lambda_f = O(\Lambda_w) ; \quad E_f A_f = O(E_w A_w) \quad (6)$$

where $\{\Lambda_i\}_{i=1}^{2,3}$ is the contribution of the floor and of the walls to the macroscopic linear mass, such as $\Lambda_f = \rho_f A_f \ell_f / \ell_w$ and $\Lambda_w = 2 \rho_w A_w$. Moreover we introduce the macroscopic shear stiffnesses due to the local bending of the floor $K_l = 12 \frac{E_f I_f}{\ell_f^3}$ and wall $K_w = 24 \frac{E_w I_w}{\ell_w^3}$. It is convenient to define the effective inertia $E_l = E_w A_w \ell_f^2 / 2$ and the effective compression stiffness $C_l = 12 \frac{E_l A_l}{\ell_f^3}$.

Furthermore, the frequency dependent functions (3) are expanded in Taylor series around $\Gamma_l$ and $\Gamma_w$ that denote the bending wavenumbers $\gamma_l$ and $\gamma_w$ at the angular frequency $\omega_J$ i.e.:

$$\Gamma_l^4 = \gamma_l^4(\omega_J) = \frac{\rho_f A_f \omega_f^2}{E_f I_f} \ell_f^3 = \frac{12 \Lambda_f}{K_l} \frac{\ell_f}{\ell_w} \omega_f^2 ; \quad \Gamma_w^4 = \gamma_w^4(\omega_J) = \frac{\rho_w A_w \omega_w^2}{E_w I_w} \ell_w^4 = \frac{12 \Lambda_w}{K_w} \ell_w^2 \omega_w^2$$
2.2. From the modulation equations to the dispersion curves
For each period \( \Omega_p \) (\( p = 1, 2, 3 \)) and for each modulation equation we look for the modulation number \( \kappa_\omega \) in considering the driving variable \( A \) on the form \( A(x) = A_0 e^{i\kappa_\omega x} \). Introducing this variable in the modulation equation leads to a characteristic equation, which approximates the dispersion relation in the vicinity of the periodic eigenmode. Finally, the corresponding dispersion relation is deduced in adding to the modulation number and the rational Bloch parameters of the period \( \Omega_p \), that is:

\[
\forall q \in \left[0; \text{Int}(p/2)\right], \ k_\omega = \frac{2\pi q}{\ell_w p} + \kappa_\omega
\]

**Figure 5.** Asymptotic description of the dispersion curves derived by modulation analysis. The full dispersion diagram is displayed on figure 3.

The figure 5 (to be compared with figure 3) displays the numerical results calculated form the dispersion relations derived by modulation analysis. The latter is developed in the next sections.

2.3. Modulation based on the irreducible period \( \Omega \)
Thanks to the modulation approach applied on the irreducible period \( \Omega \), two independent modal kinematics can be identified, (i) a transverse kinematics driven by the transverse motion \( V \), (ii) the coupled kinematics \{\( \alpha, \theta \)\}, corresponding to the gyration mode. Note that, to lighten the reading, the expressions of the cumbersome analytical parameters appearing in the sequel are reported in the Appendix.

2.3.1. Modulation of the "V"-mode For this kinematics driven by \( V \), the asymptotic process yields an identification (at the leading order) of the \( \Omega \)-periodic angular eigenfrequencies, roots of (7).

\[
\omega_j^2 \left[ \Lambda_I + \Lambda_w \frac{f_1(\Gamma_w) - f_1(\Gamma_w)}{\Gamma_w^4/24} \right] = 0
\]
As a consequence, either \( \omega_J = 0 \), that corresponds to the static mode, see [1], or the bracketed term vanishes for non zero frequencies \( \{\omega_J\} \) associated with the \( V \)-eigenmodes of \( \Omega \). These two cases are investigated herebelow.

**Quasi-static \(^*V^*\)-mode**  The modulation of the quasi-static \(^*V^*\)-mode (i.e. when \( \omega \to 0 \)) leads to the macroscopic behavior at the leading order, given by:

\[
-\Lambda \omega^2 V(x) + \Lambda \omega^2 \delta^2 V''(x) + EI_s V'''(x) = 0
\]

\[
\Lambda = \Lambda_f + \Lambda_w; \quad K = \frac{K_w K_f}{K_f + K_w}; \quad \delta^2 = \frac{EI_s}{K}
\]

Looking for \( V = V_0 e^{i\kappa_\omega x} \) in (8), the characteristic equation reads:

\[
-\Lambda \omega^2 \left( 1 + \kappa_\omega^2 \delta^2 \right) + EI_s \kappa_\omega^4 = 0
\]

As in \( \Omega = \Omega_{p=1} \) the wavenumber \( k_\omega \) and the modulation number \( \kappa_\omega \) are identical (since \( q = 0 \)), an approximation at the leading order of the dispersion curve at low frequencies is given by (10) rewritten as

\[
\omega^2 = \frac{EI_s \kappa_\omega^4}{\Lambda \left( 1 + \kappa_\omega^2 \delta^2 \right)} \quad \text{or} \quad \kappa_\omega^2 = \frac{\delta^2}{2} \left( -1 \pm \sqrt{1 + \frac{4EI_s}{\Lambda [\omega^2 - \omega_J^2] \delta^2}} \right)
\]

This expression has been plotted on figure 5 and corresponds to the lower branch issued from the origin (\( \omega = 0, k_\omega = 0 \)).

**Dynamic \(^*V^*\)-mode**  The modulation of the dynamic \(^*V^*\)-mode (i.e. \( \omega \approx \omega_J \)) yields the following fourth degree differential equation of modulation that governs the large scale evolution of \( V \) at the leading order in the vicinity of the angular eigenfrequencies \( \omega_J \) of \( \Omega \).

\[
-\widetilde{\Lambda} \left[ \omega^2 - \omega_J^2 \right] V(x) + \widetilde{\Lambda} \left[ \omega^2 - \omega_J^2 \right] \delta^2 V''(x) + \widetilde{EI} V'''(x) = 0
\]

Then introducing \( V(x) = V_0 e^{i\kappa_\omega x} \) gives:

\[
-\widetilde{\Lambda} \left[ \omega^2 - \omega_J^2 \right] (1 + \delta^2 \kappa_\omega^2) + \widetilde{EI} \kappa_\omega^4 = 0
\]

As the tilded coefficients depends upon \( \omega \) the dispersion relation is given by the roots of the second order algebraic equation, such as:

\[
\kappa_\omega^2 = \frac{\delta^2}{2} \left( -1 \pm \sqrt{1 + \frac{4EI_s}{\Lambda [\omega^2 - \omega_J^2] \delta^2}} \right)
\]

This expression has been plotted on figure 5 and corresponds to the third branch issued from the point (\( \omega = \omega_1 \approx 2\pi \times 36kHz, k_\omega = 0 \)).

2.3.2. Modulation of the gyration mode  The gyration mode is driven by the coupled variables \( \alpha, \theta \), related at the leading order by:

\[
\tilde{\xi}_t \alpha^{(0)} = \tilde{\zeta}_\theta \theta^{(0)}
\]
The leading order gyration equation (see [7] for a more detailed analysis), governing \( \alpha \) (or \( \theta \)) reads:

\[
K_f \left( \tilde{\zeta}_f - \frac{\xi_f^2}{\zeta_f} \right) - \frac{\ell_f^2}{4} \Lambda_w \omega^2 \alpha(x) - EI_s \alpha''(x) = 0 \quad (15)
\]

For large scale evolution, the first must vanish. This defines the set of eigenfrequencies \( \{ \omega_J \} \) of periodic modes associated with gyration behavior that fulfills the following equation:

\[
K_f \left( \tilde{\zeta}_f - \frac{\xi_f^2}{\zeta_f} \right) - \frac{\ell_f^2}{4} \Lambda_w \omega_j^2 = 0
\]

Introducing \( \alpha(x) = \alpha_0 e^{i \kappa \omega x} \) in (15), provides the following dispersion relation(s) at the leading order around the \( \{ \omega_J \} \)

\[
\kappa^2_\omega = \frac{K_f \left( \tilde{\zeta}_f - \frac{\xi_f^2}{\zeta_f} \right) - \frac{\ell_f^2}{4} \Lambda_w \omega_j^2}{-EI_s}
\]

(16)

This dispersion relation is plotted on figure 5 and corresponds to the second, fourth and fifth branches issued from the three points \((2\pi \times 4.4kHZ, 2\pi \times 48.7kHz, 2\pi \times 78.0kHz)\) and \(k_\omega = 0\).

2.4. Modulation based on the double cells period \( \Omega_2 \)

Considering a period \( \Omega_2 = \Omega \cup \Omega \), the modes with periodic boundary conditions satisfy, following the Bloch modes description,

- either the \( \Omega \)-modes repeated two times, whose a phase shift between two successive cells of \( e^{i \frac{2\pi q}{p}} = 1 \) \((p = 2, q = 0)\),
- or the modes specific to \( \Omega_2 \), with a phase shift \( e^{i \frac{2\pi q}{p}} = -1 \) \((p = 2, q = 1)\).

To separate these two kinds of modes it is therefore necessary to introduce the half sum \( \Sigma_m = 1/2(\Sigma_{2m} + \Sigma_{2m+1}) \) and the half difference \( \Delta_m = 1/2(\Sigma_{2m} - \Sigma_{2m+1}) \) of the kinematic variables of two successive nodes \((2m, 2m + 1)\). In the following only the modes specific to \( \Omega_2 \) described by \( \Delta \)-variables are considered.

The asymptotic process yields the following \( \Omega_2 \)-eigenfrequency leading order equation:

\[
\Lambda_f - \Lambda_w \frac{f_4(\Gamma_w) + f_1(\Gamma_w)}{\Gamma_w^4/24} = 0
\]

The modulation method is applied in the vicinity if these periodic eigenfrequencies. The modulation equation at the leading order governed by large scale evolution of the \( \Delta_V \) is:

\[
\left[ \omega^2 - \bar{\omega}_J^2 \right] \tilde{\Lambda} \Delta_V(x) - \tilde{K} \Delta''_V(x) = 0
\]

(17)

From which one deduces the modulation dispersion relation:

\[
\kappa^2_\omega = -\frac{\tilde{\Lambda} \left[ \omega^2 - \bar{\omega}_J^2 \right]}{\tilde{K}}
\]

Then the approximation of dispersion relation is provided in adding the rational Bloch parameter, so that:

\[
k_\omega = \frac{2 \pi}{\ell_w} \frac{1}{2} - \kappa_\omega, \quad k_\omega \in [0, \frac{\pi}{\ell_w}]
\]

The curve corresponding to these dispersion relations are plotted on figure 5. They are calculated for the three first specific modes of \( \Omega_2 \). They provide the three branches issued from the three points \( (\omega = \{2\pi \times 8.5kHz, 2\pi \times 15.4kHz, 2\pi \times 60.4kHz\} \) and \( k_\omega = \pi/\ell_w \).
2.5. Modulation based on the triple cells period \( \Omega_3 \)

Consider now a period \( \Omega_3 = \Omega \cup \Omega \cup \Omega \), whose the periodic eigenmodes are either the \( \Omega \)-modes repeated three times, or the modes specific to \( \Omega_3 \), with a phase shift between two successive nodes of \( e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \) with \( p = 3, q = 1 \). Considering the real and imaginary parts of \( \exp(2i\pi/3) \) this corresponds, for specific modes to the triplets of values \((1,-1/2,-1/2)\) and \((0,\sqrt{3}/2,-\sqrt{3}/2)\) at the three consecutive nodes, and to \((1,1,1)\) for the triple \( \Omega \)-modes. To separate triple \( \Omega \)-modes and the specific \( \Omega_3 \)-modes it is therefore necessary to introduce the following set of variables :

\[
\begin{align*}
\mathcal{V}_{3m} &= \Sigma_m + R_m \\
\mathcal{V}_{3m+1} &= \Sigma_m - \frac{1}{2} R_m + \frac{\sqrt{3}}{2} L_m \\
\mathcal{V}_{3m+2} &= \Sigma_m - \frac{1}{2} R_m - \frac{\sqrt{3}}{2} L_m
\end{align*}
\]

where \( \Sigma \) corresponds to the triple \( \Omega \)-modes, while \( R \) and \( L \) correspond to the specific modes of \( \Omega_3 \). From the asymptotic process it is first shown that periodic angular eigenfrequencies specific to \( \Omega_3 \) are the roots of :

\[
\frac{1}{F_{\Omega_3}} \frac{2}{F_{\Omega_3}} + \frac{3}{4} \left( \frac{K_w \xi_w}{l_w} \right)^2 = 0
\]

Applying the asymptotic method around these eigenfrequencies, the macroscopic modulation equation expressed below with the variable \( R_V \) that also apply to \( I_V, R_\theta \), and \( I_\theta \) :

\[
3 B_{\Omega_3}^2 \omega_j^2 (\omega - \omega_j)^2 R_V + K_{\Omega_3}^2 R_V'' = 0 \quad (18)
\]

From which one extracts the dispersion relation :

\[
\kappa_\omega^2 = \frac{3 B_{\Omega_3}^2 \omega_j^2 (\omega - \omega_j)^2}{K_{\Omega_3}^2}
\]

Then the approximated dispersion relation is obtained by adding the rational Bloch parameter, so that :

\[
k_\omega = \frac{2\pi}{l_w} \frac{1}{3} + \kappa_\omega, \quad k_\omega \in \left[0, \frac{\pi}{l_w}\right]
\]

The latter dispersion relations are displayed on figure 5 for the three first specific modes of \( \Omega_3 \). They yield the three branches passing through the three points \( (\omega = \{2\pi \times 5.5kHz, 2\pi \times 20.2kHz, 2\pi \times 58.1kHz\} \) and \( k_\omega = 2\pi/(3l_w) \).

3. Conclusion

In the figure 6 are represented in parallel the dispersion curves determined from a direct calculus (see paragraph 1.2) and the asymptotic dispersion curves derived from the analytical expressions given by the modulation analysis.

The figure 6 clearly shows that the asymptotic model for quasi-static 'V'-mode fits the exact dispersion curve related to the bending of the reticulated beam in a large wave number range, namely up \( k_\omega = 0.5\pi/l_w \), which is much larger than its theoretical validity domain \( k_\omega l_w \ll 1 \).

On the contrary, the asymptotic model for the dynamic 'V'-mode fits the exact solution in a limited range of wavenumbers. This relies on the fact that the dominant kinematic variable of this branch change from variable \( V \) for \( k_\omega \leq 0.05\pi/l_w \), to \( \theta \) for \( k_\omega \geq 0.05\pi/l_w \). As for the the asymptotic dispersion curves identified considering \( \Omega_2 \) and \( \Omega_3 \), one notices a large range of good matching between the asymptotic and exact results.

The main outcomes of this paper are (i) a given dispersion branch may involves different
kinematics, (ii) the modulation analysis performed on multi-cell around rational Bloch number allows to build by parts the dispersion curves and to relate them to an effective analytical description, (iii) these approximations are reliable in a range such that the kinematics do not significantly change. These results established on a particular beam lattices may be extended to more complexe periodic structures and clarify the physics behind the dispersion curves.

4. Appendix : Frequency dependent parameters involved in the modulation analysis

4.1. Modulation based on the irreducible period $\Omega$

- The dimensionless frequency dependent coefficients :

\[
\begin{align*}
\tilde{\zeta}_{f \alpha} &= (f_1(\Gamma_f) + f_4(\Gamma_f))/2 ; \\
\tilde{\zeta}_{f \theta} &= (2f_2(\Gamma_f) + f_5(\Gamma_f))/3 \\
\tilde{\xi}_f &= (f_3(\Gamma_f) + f_6(\Gamma_f))/2 ; \\
\tilde{\zeta}_{w \theta} &= (2f_2(\Gamma_w) + f_5(\Gamma_w))/3 \\
\tilde{\xi}_w &= f_6(\Gamma_w) ; \\
\tilde{\zeta}_{w''} &= f_4(\Gamma_w) \\
\tilde{\sigma}_w &= 6(f'_4(\Gamma_w) - f'_1(\Gamma_w))/\Gamma^3_w ; \\
\tilde{\omega}_\theta &= \tilde{\zeta}_{f \theta} + K_f K_w \tilde{\zeta}_{w''} - \tilde{\zeta}_w
\end{align*}
\]

- $\tilde{\omega}_J$ slightly differs from $\omega_J$ by a corrective term accounting for the compression effects disregarded in the leading order estimation $\omega_J$.

\[
\tilde{\omega}_J^2 = \omega_J^2 \left( 1 - \frac{\Lambda_f^2 \omega_J^2}{(\Lambda_f + \Lambda_w \tilde{\sigma}_w) \tilde{C}_f} \right)
\]

- Finally :

\[
\tilde{K}_f^2 = K_f^2 (\tilde{\zeta}_{f \alpha} \tilde{\zeta}_\theta - \tilde{\xi}_f^2) ; \\
\tilde{K}_w^2 = K_w^2 \left( \frac{K_f}{K_w} \tilde{\zeta}_\theta \tilde{\zeta}_{w''} - \tilde{\xi}_w^2 \right)
\]

Figure 6. Overlap of the exact and the approximated dispersion branches.
\[
\begin{align*}
\tilde{\Lambda} &= \Lambda_f + \Lambda_w \tilde{\sigma}_w ; \quad \tilde{E}I = EI_s \frac{K_w^2}{K_f}
\end{align*}
\]

\[
\delta^2 = \frac{EI_s K_f}{K^2_f} \tilde{\zeta}_\theta - \frac{K_w}{\Lambda [\omega^2 - \tilde{\omega}_j^2]} \left( \zeta_{ww} + \frac{K_f K_w \tilde{\zeta}_w \tilde{\zeta}_\alpha}{K^2_f} \right)
\]

4.2. Modulation based on the double period \( \Omega_2 \)

\[
\begin{align*}
\tilde{\zeta}_{w\Delta_\theta} &= \frac{2 f_2(\Gamma_w) - f_5(\Gamma_w)}{3} \\
\tilde{\tilde{\sigma}}_w &= -6 \frac{f'_1(\Gamma_w) + f'_2(\Gamma_w)}{\Gamma_w^3} > 0 \\
\tilde{\omega}_j^2 &= \omega_j^2 \left( 1 - \frac{\Lambda^2 \omega_j^2}{(\Lambda_f + \Lambda_w \tilde{\sigma}_w) C_f} \right)
\end{align*}
\]

\[
\tilde{\Lambda} = \left( \Lambda_f + \Lambda_w \tilde{\sigma}_w \right) ; \quad \tilde{\tilde{K}} = K_w \left( \zeta_{ww} + \frac{K_w}{K_w \zeta_{\theta} + \zeta_{w\Delta_\theta}} \right)
\]

4.3. Modulation based on the triple period \( \Omega_3 \)

\[
\begin{align*}
\frac{1}{\bar{\Omega}_3} &= \Lambda_f \omega_j^2 - \frac{K_w}{\ell_w} (2 f_1(\Gamma_w) + f_4(\Gamma_w)) \\
\frac{2}{\bar{\Omega}_3} &= \frac{1}{6} \left[ K_w \left( 4 f_2(\Gamma_w) - f_5(\Gamma_w) \right) + 2 K_f \left( 2 f_2(\Gamma_f) + f_5(\Gamma_f) \right) \right]
\end{align*}
\]

\[
\begin{align*}
\bar{\Omega}_3 &= 3 \frac{K_w \tilde{\zeta}_w^2}{\ell_w} \left( \frac{1}{\bar{\Omega}_3} - \frac{3 K_w \tilde{\zeta}_{ww}^2}{2 \ell_w^2} \right) + \frac{K_w \ell_w f_w f_5(\Gamma_w)}{2} \frac{1}{\bar{\Omega}_3} \\
\bar{\Omega}_3 &= - \frac{3 K_w \tilde{\zeta}_w^2}{2 \ell_w} \bar{\Omega}_3 + \frac{1}{\bar{\Omega}_3} \bar{\Omega}_3^2 \bar{\Omega}_3
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\bar{\Omega}_3} &= \frac{K_w \tilde{\zeta}_w}{\ell_w} \left[ \Lambda_f - \frac{3 \Lambda_w}{\Gamma_w^3} (2 f'_1(\Gamma_w) + f'_4(\Gamma_w)) \right] - \frac{3 \Lambda_w \ell_w}{\Gamma_w^3} f'_6(\Gamma_w) \\
\frac{2}{\bar{\Omega}_3} &= \frac{\Lambda_w \ell_w^2}{\Gamma_w^3} \left( 4 f'_2(\Gamma_w) - f'_5(\Gamma_w) + \frac{9}{2} \frac{K_w \tilde{\zeta}_w}{\ell_w^2} f'_6(\Gamma_w) \right) + \frac{2 \Lambda_f \ell_f^2}{\Gamma_f^3} (2 f'_2(\Gamma_f) + f'_5(\Gamma_f))
\end{align*}
\]

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