1D Hyperbolic Systems with Nonlinear Boundary Conditions II: Criteria for Finite Time Stability

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Abstract

We investigate the finite time stability property of one-dimensional nonautonomous initial boundary value problems for linear decoupled hyperbolic systems with nonlinear boundary conditions. We establish sufficient and necessary conditions under which continuous or $L^2$-generalized solutions stabilize to zero in a finite time. Our criteria are expressed in terms of a propagation operator along characteristic curves.

1 Introduction

1.1 Problem

Established in the middle of the 50th, the Finite Time Stability (FTS) concept attracts growing attention in view of its applications in control and system engineering [4, 5, 13, 14, 17, 18], output-feedback stabilization [6, 7, 8, 19], inverse problems [15, 16], ATM networks [1], car suspension systems [2], and robot manipulators [3]. This concept is used in two ways. Quantitatively, it describes a restrained behavior of the dynamical system over a specified time interval. Qualitatively, it characterizes asymptotically stable dynamical systems whose trajectories reach an equilibrium point in a finite time. In this paper we characterize FTS hyperbolic systems using the qualitative notion of FTS.

In [10] we gave a comprehensive FTS analysis of a class of linear initial-boundary value problems with reflection boundary conditions for decoupled nonautonomous hyperbolic systems, providing algebraic and combinatorial criteria. In the autonomous setting, we provided also a spectral criterion. Asymptotic properties of solutions to perturbed FTS problems were

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studied in [12]. In the present paper, we establish FTS criteria for a class of nonlinear boundary value problems. These results can be applied to solving inverse problems for hyperbolic systems with FTS boundary conditions (as we demonstrate in Subsection 3.1).

Let $n \geq 2$. Our stability results concern the decoupled nonautonomous hyperbolic system

$$\partial_t u + A(x, t) \partial_x u + B(x, t) u = 0, \quad 0 < x < 1, t > 0,$$

where $u = (u_1, \ldots, u_n)$ is a vector of real-valued functions and the diagonal matrices $A = \text{diag}(a_1, \ldots, a_n)$ and $B = \text{diag}(b_1, \ldots, b_n)$ have real entries.

Set $\Pi = \{ (x, t) : 0 \leq x \leq 1, t \geq 0 \}$. Suppose that

$$\inf_{(x, t) \in \Pi} a_j \geq a \text{ for all } j \leq m \text{ and } \sup_{(x, t) \in \Pi} a_j \leq -a \text{ for all } j > m$$

for some $a > 0$ and $0 \leq m \leq n$. The system (1.1) is subjected to the initial conditions

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1,$$

and the homogeneous nonlinear boundary conditions

$$u^{\text{out}}(t) = h(t, u^{\text{in}}(t)), \quad t \geq 0,$$

where $h = h(t, \xi) = (h_1(t, \xi), \ldots, h_n(t, \xi))$, with $\xi \in \mathbb{R}^n$, is a real valued function,

$$h(t, 0) = 0 \text{ for all } t \geq 0,$$

and

$$u^{\text{out}}(t) = (u_1(0, t), \ldots, u_m(0, t), u_{m+1}(1, t), \ldots, u_n(1, t)),$$

$$u^{\text{in}}(t) = (u_1(1, t), \ldots, u_m(1, t), u_{m+1}(0, t), \ldots, u_n(0, t)).$$

1.2 Preliminaries on continuous and $L^2$-generalized solutions

Let

$$\varphi^{\text{out}} = (\varphi_1(0), \ldots, \varphi_m(0), \varphi_{m+1}(1), \ldots, \varphi_n(1)),
\varphi^{\text{in}} = (\varphi_1(1), \ldots, \varphi_m(1), \varphi_{m+1}(0), \ldots, \varphi_n(0)).$$

We say that a function $\varphi$ satisfies the zero order compatibility conditions between (1.3) and (1.4) if

$$\varphi^{\text{out}} = h(0, \varphi^{\text{in}}).$$

We consider the set $C_h(\Pi)^n$ of functions $u \in C(\Pi)^n$ such that $u^{\text{out}}(0) = h(0, u^{\text{in}}(0))$. Note that, if $u \in C_h(\Pi)^n$, then $u(x, 0)$ satisfies the zero order compatibility conditions between (1.3) and (1.4) with $\varphi = u(x, 0)$. Let $C_h([0, 1])^n$ be a closed subset of a Banach space.
that consists of functions $\varphi \in C([0, 1])^n$ fulfilling the condition (1.7). Furthermore, $C_h([0, 1])^n = C_h([0, 1])^n \cap C^1([0, 1])^n$.

Let us introduce solution concepts, that will be used in the paper. To this end, we first define characteristics of (1.1) as follows. For given $j \leq n$, $x \in [0, 1]$, and $t > 0$, the $j$-th characteristic of (1.1) passing through the point $(x, t) \in \Pi$ is the solution $\omega_j(x, t) = \omega_j(x, x, t) : [0, 1] \to \mathbb{R}$ to the initial value problem

$$
\partial_t \omega_j(x, t) = \frac{1}{a_j(x, \omega_j(x, x, t))} \quad \omega_j(x, x, t) = t.
$$

Let a continuous function $u : \Pi \to \mathbb{R}^n$ be continuously differentiable in $\Pi$ excepting at most a countable number of characteristic curves of (1.1). If $u$ satisfies (1.1), (1.3), and (1.4) in $\Pi$ except the aforementioned characteristic curves, then it is called a piecewise continuously differentiable solution to the problem (1.1), (1.3), (1.4).

If the initial function $\varphi$ is sufficiently smooth, then using integration along characteristics, we can transform the problem (1.1), (1.3), (1.4) to a system of integral equations. The characteristic curve $\tau = \omega_j(x, x, t)$ reaches the boundary of $\Pi$ in two points with distinct ordinates. Let $x_j(x, t)$ denote the abscissa of that point whose ordinate is smaller. Note that the value of $x_j(x, t)$ does not depend on $x$ and $t$ if $t > 1/a$, where $a > 0$ satisfies (1.2). More precisely, if $t > 1/a$, then

$$
x_j(x, t) = x = \begin{cases} 
0 & \text{if } 1 \leq j \leq m \\
1 & \text{if } m < j \leq n.
\end{cases}
$$

Set

$$
c_j(x, x, t) = \exp \int_x^t \left( \frac{b_j}{a_j} \right) \left( \eta, \omega_j(\eta, x, t) \right) d\eta.
$$

Define a linear operator $S : C(R_+)^n \to C(\Pi)^n$ by

$$
[Sv]_j(x, t) = c_j(x_j(x, t), x, t)v_j(\omega_j(x(x, t), x, t)), \quad j \leq n,
$$

and a nonlinear operator $R : C(\Pi)^n \to C(R_+)^n$ by

$$
[Ru]_j(t) = h_j(t, u^{\infty}(t)), \quad j \leq n.
$$

As it follows from the method of characteristics, any piecewise continuously differentiable solution $u$ to the problem (1.1), (1.3), (1.4) satisfies the following system of functional equations:

$$
u_j(x, t) = [Qu]_j(x, t)
$$

where the affine operator $Q : D(Q) \subset C_h(\Pi)^n \to C_h(\Pi)^n$ is defined by

$$
[Qu]_j(x, t) = \begin{cases} 
[SRu]_j(x, t) & \text{if } x_j(x, t) = 0 \text{ or } x_j(x, t) = 1 \\
c_j(x_j(x, t), x, t)\varphi_j(x_j(x, t)) & \text{if } x_j(x, t) \in (0, 1),
\end{cases}
$$

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and
\[ D(Q) = \{ u \in C_h(\Pi)^n : u(x, 0) = \varphi(x) \} . \]
Note that the definition of \( Q \) depends on the choice of the function \( \varphi \). We will write \( Q = Q_\varphi \) when we want to specify this dependence explicitly.

Vice versa, if a \( C \)-map \( u : \Pi \to \mathbb{R}^n \) is piecewise continuously differentiable excepting at most a countable number of characteristic curves of (1.1) and satisfies (1.8) pointwise, then it is a piecewise continuously differentiable solution to (1.1), (1.3), (1.4). This motivates the following definition.

**Definition 1.1** A continuous function \( u : \Pi \to \mathbb{R}^n \) satisfying (1.8) in \( \Pi \) is called a continuous solution to (1.1), (1.3), (1.4).

For a Banach space \( X \), the \( n \)-th Cartesian power \( X^n \) is considered to be a Banach space of vectors \( u = (u_1, \ldots, u_n) \) normed by \( \|u\|_X^n = \max_{i \leq n} \|u_i\|_X \). Let \( \|\cdot\|_{\text{max}} = \max_{jk} |m_{jk}| \) denote the max-matrix norm of \( M = (m_{jk}) \) in the space of matrices \( M_n \).

Below we will use our result from [9, Theorem 3.1] about the existence and uniqueness of global regular solutions.

**Theorem 1.2** Let the condition (1.2) be fulfilled. Moreover, assume that for all \( j, k \leq n \) the functions \( a_j, b_j, \) and \( h_j \) are continuously differentiable in all their arguments (1.10)
\[ \{\|\nabla h(t, \xi)\|_{\text{max}} : 0 \leq t \leq T, \; \xi \in \mathbb{R}^n \} \leq C(T) (\log \log (\|\xi\|))^{1/4}. \] (1.11)
Then the following is true.

1. For every \( \varphi \in C_h([0, 1])^n \), the problem (1.1), (1.3), (1.4) has a unique continuous solution in \( \Pi \).
2. For every \( \varphi \in C_h^1([0, 1])^n \), the problem (1.1), (1.3), (1.4) has a unique piecewise continuously differentiable solution in \( \Pi \).

We now define an \( L^2 \)-generalized solution to the problem (1.1), (1.3), (1.4) similarly to [11, Definition 2].

**Definition 1.3** Assume that the conditions of Theorem 1.2 are fulfilled. Let \( \varphi \in L^2(0, 1)^n \). A function \( u \in C([0, \infty), L^2(0, 1))^n \) is called an \( L^2 \)-generalized solution to the problem (1.1), (1.3), (1.4) if, for any sequence \( \varphi^l \in C_h^1([0, 1])^n \) with \( \varphi^l \) converging to \( \varphi \) in \( L^2(0, 1)^n \), the sequence of piecewise continuously differentiable solutions \( u^l(x, t) \) to the problem (1.1), (1.3), (1.4) with \( \varphi \) replaced by \( \varphi^l \) fulfills the convergence condition
\[ \|u^l(\cdot, t) - u(\cdot, t)\|_{L^2(0,1)^n} \to 0 \quad \text{as} \quad l \to \infty, \] (1.12)
uniformly in \( t \) varying in the range \( 0 \leq t \leq T \), for each \( T > 0 \).
Here the norm in $L^2(0,1)^n$ is defined as usual by $\|u\|_{L^2(0,1)^n}^2 = \int_0^1 (u, u) \, dx = \int_0^1 \sum_{i=1}^n u_i^2 \, dx$, where $(\cdot, \cdot)$ here and below denotes the scalar product in $\mathbb{R}^n$.

The following existence and uniqueness result is obtained in [11, Theorem 2].

**Theorem 1.4** Let the conditions (1.2), (1.5), and (1.10) be fulfilled. Moreover, assume that for each $T > 0$ there exists a positive real $C(T)$ such that

$$\sup \{ \| \nabla h(t, \xi) \|_{\text{max}} : 0 \leq t \leq T, \xi \in \mathbb{R}^n \} \leq C(T).$$

(1.13)

Then, for every $\varphi \in L^2(0,1)^n$, the problem (1.1), (1.3), (1.4) has a unique $L^2$-generalized solution.

### 1.3 Our results

If the problem (1.1), (1.3), (1.4), (1.11) has an $L^2$-generalized solution, then it is unique just by Definition 1.3. If this problem has a continuous solution, it is also unique as shown in [9] (see the proof of [9, Theorem 3.1]).

**Definition 1.5** Assume that, for every $\varphi \in L^2(0,1)^n$ (resp., $\varphi \in C_h([0,1]^n)$), the problem (1.1), (1.3), (1.4), (1.11) has an $L^2$-generalized solution (resp., a continuous solution). We say that this problem is Finite Time Stabilizable (FTS) if there exists a positive real $T$ such that, for every $\varphi \in L^2(0,1)^n$ (resp., $\varphi \in C_h([0,1]^n)$), the $L^2$-generalized solution (resp., a continuous solution) is a constant zero function for $t > T$. The infimum of all $T$ with the above property is called the optimal stabilization time and is denoted by $T_{opt}$.

Since the operator $Q$ operates with functions on shifted domains and, thus, captures propagation from the boundary $\partial \Pi$ into the domain $\Pi$, the stabilization properties heavily depend on the powers of the operator $Q$. We start with a useful property of the operator $Q$.

**Theorem 1.6** For every $T > 0$ there exists $k \in \mathbb{N}$ such that the following is true. If, for $w \in C_h(\Pi)^n$, the problem (1.1), (1.3), (1.4), (1.11) with $\varphi(x) = w(x,0)$ has a unique continuous solution $u$ in $\Pi$, then $u(x,t) = [Q^k w](x,t)$ in $\Pi^T$ where $Q = Q_\varphi$ for $\varphi(x) = w(x,0)$.

Now we formulate our stabilization criterion in the nonautonomous setting.

**Theorem 1.7** Let the condition (1.5) be fulfilled. Assume that, for every $\varphi \in L^2(0,1)^n$ (resp., $\varphi \in C_h([0,1]^n)$), the problem (1.1), (1.3), (1.4), (1.11) has an $L^2$-generalized solution (resp., a continuous solution). Then this problem is FTS if and only if

$$\text{there is } T > 0 \text{ and } k \in \mathbb{N} \text{ such that, for all } w \in C_h(\Pi)^n \text{ and } x \in [0,1],$$

$$[Q^k w](x,T) \equiv 0 \text{ where } Q = Q_\varphi \text{ for } \varphi(x) = w(x,0).$$

(1.14)
In the autonomous setting a stabilization criterion is formulated in a stronger form.

**Theorem 1.8** Assume that the coefficient matrices $A$ and $B$ do not depend on $t$ and the boundary function $h$ does not explicitly depend on $t$, that is, $h(t, \xi) \equiv h(\xi)$. Moreover, let the condition (1.3) be fulfilled. Assume also that, for every $\varphi \in L^2(0, 1)^n$ (resp., $\varphi \in C_h([0, 1]^n)$), the problem (1.1), (1.3), (1.4), (1.11) has an $L^2$-generalized solution (resp., a continuous solution). Then this problem is FTS if and only if

there is $T > 0$ and $q \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, $w \in C_h(\Pi)^n$, and $x \in [0, 1]$, \[ [Q^k w](x, kT) = 0 \] where $Q = Q_{\varphi}$ for $\varphi(x) = w(x, 0)$.

(1.15)

Theorems 1.6–1.8 assume the existence of $L^2$-generalized or continuous solutions (recall that those are always unique). While some sufficient conditions for the existence of solutions to the problem (1.1), (1.3), (1.4), (1.11) are given in Theorems 1.2 and 1.4, we want to emphasize that Theorems 1.6–1.8 are not restricted to these particular conditions and are more general.

The rest of the paper is organized as follows. The FTS-criteria of Theorems 1.7 and 1.8 are proved in Section 2. Discussion of our stabilization criteria are provided in Section 3, where we also show how our Theorem 1.6 can be applied to solving inverse hyperbolic problems.

## 2 Stabilization criteria

### 2.1 Proof of Theorem 1.6

Fix an arbitrary $T > 0$. Since $Q$ is a down-shift operator along characteristic curves up to the boundary of $\Pi$ in the direction of time decrease, there exists an integer $q = q(T)$ such that all iterations of the operator $Q$ starting from the $q$-th iteration stabilize, namely for every $w \in C_h(\Pi)^n$ it holds in $\Pi^T$ that

\[ [Q^q w](x, t) = [Q^{q+1} w](x, t), \]

where in the definition (1.9) of the operator $Q$ we set $\varphi(x) = w(x, 0)$.

Fix a function $w \in C_h(\Pi)^n$ fulfilling the conditions of Theorem 1.6. Then the problem (1.1), (1.3), (1.4), (1.11) with $\varphi = w(x, 0)$ has a unique continuous solution. Set $u = Q^q w$. Hence, $u \in C_h(\Pi)^n$, and (2.1) implies that in $\Pi^T$ we have

\[ [Qu](x, t) = [Q^{q+1} w](x, t) = [Q^q w](x, t) = u(x, t). \]

It follows that the function $u = Q^q w$ is the continuous solution in $\Pi^T$ to the problem (1.1), (1.3), (1.4), (1.11) with $\varphi = w(x, 0)$. The proof of Theorem 1.6 is complete.
2.2 Nonautonomous case: proof of Theorem 1.7

Sufficiency. Let $T > 0$ and $k \in \mathbb{N}$ be numbers satisfying the condition (1.14). Fix an arbitrary $\varphi \in L^2(0, 1)^n$. Suppose that the problem (1.1), (1.3), (1.4), (1.11) has a unique $L^2$-generalized solution $u$.

First note that $C^1_1([0, 1])^n$ is densely embedded into $L^2(0, 1)^n$. Indeed, since the boundary conditions (1.4) are homogeneous (see 1.5), $C^\infty_0([0, 1])^n$ is a subset of $C^1_1([0, 1])^n$. As usual, by $C^\infty_0([0, 1])$ we denote a subspace of $C^\infty([0, 1])$ that consists of functions having support within $(0, 1)$. Now, we fix an arbitrary sequence $\varphi_l \in C^1_1([0, 1])^n$ such that $\varphi_l$ converges to $\varphi$ in $L^2(0, 1)^n$ and let $u_l(x, t)$ be the piecewise continuously differentiable solution to the problem (1.1), (1.3), (1.4), (1.11) with $\varphi$ replaced by $\varphi_l$ (see Theorem 1.2).

By Definition 1.3, the sequence $u_l(x, t)$ converges as in (1.12). Using integration along characteristics, we see that

$$u_l(x, t) = [Qu_l](x, t)$$

for all $x \in [0, 1]$ and $t \in [0, T]$.

This means that the function $u_l(x, t)$ is a fixed point of the operator $Q$ and, hence, of any power of $Q$. Combining this with the condition (1.14), we conclude that

$$u_l(x, T) = [Q^k u_l](x, T) = 0$$

for all $x \in [0, 1]$ and $l \in \mathbb{N}$.

Since the initial boundary value problem (1.1), (1.3), (1.11) with the zero initial data at $t = T$ has a unique piecewise continuously differentiable solution for $t \geq T$ (see Theorem 1.2), we conclude that $u_l \equiv 0$ for $t \geq T$. The identity $u \equiv 0$ for $t > T$ follows from the convergence (1.12). The FTS property is therewith proved.

If the problem (1.1), (1.3), (1.4), (1.11) is FTS and all continuous solutions stabilize to zero in a finite time, the argument is similar and even simpler than in the case we considered. The proof of Theorem 1.7 is complete.
2.3 Autonomous case: proof of Theorem 1.8

Sufficiency. Since the condition (1.15) implies (1.14), this part immediately follows from the sufficiency part of Theorem 1.7.

Necessity. Consider two cases.

Case 1: the problem (1.1), (1.3), (1.4), (1.11) is FTS and all continuous solutions stabilize to zero in a finite time. Fix $T > T_{opt}$ and $q \in \mathbb{N}$ fulfilling both the condition (1.14) with $k = q$ and the equality (2.1) in $\Pi^{2T}$. For any continuous solution $u$ we have

$$0 = [Q^q u](x, t) = ((SR)^q u)(x, t) \quad \text{for all } x \in [0, 1]$$

(2.2)

and for all $t \geq T$, where the second equality can be proved as follows. We first prove that this equality is fulfilled for all $t \in [T, 2T]$. By the way of contradiction, assume that this is not true for some continuous solution $u$. Then there exist $x \in [0, 1], t \in [T, 2T], \text{ and } j \leq n$ such that the value $[Q^q u]^j (x, t)$ can be expressed in terms of the values of $u$ at points lying on the initial axis. Straightforward calculations show that there exist positive integers $q_1, \ldots, q_n$ as well as $C_1$-functions $F : \mathbb{R}^{q_1 + \ldots + q_n} \mapsto \mathbb{R}$ and $\tilde{F} : \mathbb{R}^{q_1} \times \ldots \times \mathbb{R}^{q_n} \mapsto \mathbb{R}$, and pairwise distinct reals $x_{sr} \in [0, 1]$ such that

$$[Q^q u]^j (x, t) = \tilde{F}(\bar{v}^u_1, \ldots, \bar{v}^u_n),$$

(2.3)

where

$$\tilde{F}(\bar{v}^u_1, \ldots, \bar{v}^u_n) = F(v_1^u, v_2^u, \ldots, v_{q_1}^u, v_{q_1+1}^u, \ldots, v_{q_1+q_2}^u, v_{q_1+q_2+1}, \ldots, v_{q_1+\ldots+q_n}^u)$$

and the vector-function $\bar{v}^u_s$ for all $s \leq n$ is given by

$$\bar{v}^u_s = (v_{q_1+q_2+\ldots+q_{s-1}+1}^u, \ldots, v_{q_1+q_2+\ldots+q_s}^u) = (u_s(x_{s1}, 0), \ldots, u_s(x_{sq_s}, 0)).$$

(2.4)

Since $u$ is a solution, we have $\varphi(x) = u(x, 0)$. It follows that $\tilde{F}$ is a composition of two homogeneous operators, namely the multiplication-shift operator $S$ and the nonlinear boundary operator $R$. This implies that $\tilde{F}(0, \ldots, 0) = 0$. Note that, due to (2.1) in $\Pi^{2T}$, the representation (2.3) is unique.

Equality (2.1) considered in $\Pi^{2T}$ implies that $u(x, t) = [Q^q u](x, t)$. Combined with (2.3), this gives the equality

$$u_j(x, t) = [Q^q u]^j (x, t) = \tilde{F}(\bar{v}^u, \ldots, \bar{v}^u_n) - \tilde{F}(0, \ldots, 0) = \sum_{i=1}^{q_1+\ldots+q_n} v_i \int_0^1 \partial_1 F(\gamma v_1^u, \gamma v_2^u, \ldots, \gamma v_{q_1+\ldots+q_n}^u) \, d\gamma,$$

(2.5)
where \( \partial_i \) here and in what follows denotes the partial derivative with respect to the \( i \)-th argument. Define

\[
I = \left\{ (s, r) \in \mathbb{N}^2 : 1 \leq s \leq n, \quad 1 + \sum_{j=1}^{s-1} q_j \leq r \leq \sum_{j=1}^{s} q_j, \right. \\
\left. \int_0^1 \partial_s F(\gamma v_1^u, \gamma v_2^u, \ldots, \gamma v_{q_1+\ldots+q_n}^u) d\gamma \neq 0 \right\},
\]

where the sum over the empty set equals zero. Note that the set \( I \) is not empty, for else the representation (2.3)–(2.4) is impossible and we immediately get a contradiction to our assumption. Then, for an arbitrarily fixed \((s_0, r_0) \in I\), one can choose the initial function \( \varphi \) such that \( \varphi(s_0) (x_{s_0 r_0}) \neq 0 \) while \( \varphi(s) (x_{sr}) = 0 \) for all other \((s, r) \in I\). On account of (2.4), the equality (2.5) now reads

\[
u_j(x, t) = \varphi(s_0) (x_{s_0 r_0}) \int_0^1 \partial_{r_0} F(\gamma v_1^u, \gamma v_2^u, \ldots, \gamma v_{q_1+\ldots+q_n}^u) d\gamma \neq 0,
\]

contradicting the FTS property of our problem. We, therefore, proved that the condition (2.2) is true for all \( t \in [T, 2T]\).

Now we show that (2.2) is true for all \( t \geq 2T \). To this end, observe that in the autonomous case the following formulas are true:

\[
\omega_j(\xi, x, t+T) = \omega_j(\xi, x, t) + T, \quad t \geq 0, \\
[Sv]_j(x, t) = c_j(x_j, x, t) v_j(\omega_j(x_j, x, t) + t - T), \quad t \geq T,
\]

for all \( v \in C(\mathbb{R}_+)^n \). Given \( w \in C_h(\Pi)^n \), set \( z(x, t) = w(x, t+T) \). It follows that

\[
[(SR)^q z] (x, t) = [(SR)^q w] (x, t+T), \quad t \geq T.
\]

Using the above argument for (2.2) for \( t \in [T, 2T] \) once again, we see that \( T > T_{opt} > 1/a \). On account of (2.6), we then have \( \omega_j(x_j, x, t, x, t) = \omega_j(x_j, x, t, x, t-T) + T > T \) for all \( t > 2T \), \( x \in [0, 1] \), and \( j \leq n \). Combining this with the FTS property, we conclude that \( u(\cdot, t) = [Qu](\cdot, t) = [SRu](\cdot, t) \equiv 0 \) for all \( t > 2T \). Summarizing, the condition (2.2) stays true for all \( t \geq T \), as desired.

Let \( q \) be now chosen such that (2.2) holds for \( t \geq T \) and, additionally, the equality (2.4) is fulfilled in \( \Pi^{3T} \). Let \( w \in C_h(\Pi)^n \) be arbitrarily fixed. Similarly to the proof of Theorem 1.6 the function \( [Q^q w](x, t) \) is a continuous solution to (1.1), (1.3), (1.4), (1.11) with \( \varphi(x) = w(x, 0) \) in the domain \( \Pi^{3T} \). By (2.2), we have \( [Q^q w](\cdot, T) \equiv 0 \) and, hence the function \( z^1(x, t) = [Q^q w](x, t + T) = [(SR)^q w](x, t + T) \) belongs to \( C_h(\Pi)^n \) and is a continuous solution (1.1), (1.3), (1.4), (1.11) with \( \varphi(x) = 0 \) in \( \Pi^{2T} \). It follows from (2.2) that

\[
0 = [Q^q z^1] (x, t) = [(SR)^q z^1] (x, t) \quad \text{for} \quad t \in [T, 2T].
\]
Similarly to (2.7), we have

\[
[(SR)^q z^1](x, t) = [(SR)^q Q^q w](x, t + T) = [Q^{2q} w](x, t + T).
\]

Therefore, \([Q^{2q} w](\cdot, t) \equiv 0\) for \(t \in [2T, 3T]\). In the next step we set \(z^2(x, t) = [Q^q z^1](x, t) = [Q^{2q} w](x, t + 2T)\). Due to the previous step, \(z^2(\cdot, 0) \equiv 0\) and, therefore, \(z^2\) belongs to \(C_h(\Pi)^n\) and is a continuous solution to (1.1), (1.3), (1.4), (1.11) with \(\varphi(x) = 0\) in \(\Pi^{2T}\). Similarly, for \(t \in [T, 2T]\), it holds

\[
0 = [Q^q z^2](x, t) = [(SR)^q z^2](x, t) = [(SR)^q Q^{2q} w](x, t + 2T) = [Q^{3q} w](x, t + 2T)
\]

and, hence \([Q^{3q} w](\cdot, t) \equiv 0\) for \(t \in [3T, 4T]\). Proceeding further by induction, where on the \(k\)-th step we set \(z^k(x, t) = [Q^{kq} w](x, t + kT)\), \(k \geq 3\), we conclude that the desired condition (1.15) is true. The proof of Case 1 is therewith complete.

**Case 2:** The problem (1.1), (1.3), (1.4), (1.11) is FTS and all \(L^2\)-generalized solutions stabilize to zero in a finite time. Let \(q\) be as in Case 1. Using the same argument as in the proof of the necessity part of Theorem 1.7 in the same \(L^2\)-case, fix an arbitrary \(w \in C_h(\Pi)^n\), put \(\varphi(x) = w(x, 0)\), and conclude that the function \(u = Q^q w \in C_h(\Pi)^n\) is a continuous solution to the problem (1.1), (1.3), (1.4), (1.11) in the domain \(\Pi^{2T}\). Since any continuous solution is an \(L^2\)-generalized solution, then using the FTS property for the \(L^2\)-generalized solutions and (2.2), we conclude that \([Q^q w](x, T) = 0\) for all \(x \in [0, 1]\). The proof is completed by repeating the argument used at the end of Case 1.

### 3 Examples

#### 3.1 Solving inverse problems

Let the boundary conditions (1.4) be linear, namely

\[
u^{\text{out}}(t) = P u^{\text{in}}(t), \quad t \geq 0,
\]

where \(P = (p_{jk})\) is an \(n \times n\)-matrix with constant entries. We assume that the matrix \(P_{\text{abs}} = (|p_{jk}|)\) is nilpotent. Then, due to [10, Theorem 1.10], the problem (1.1), (1.3), (3.1) is robust FTS, with respect to perturbations of the coefficients \(a_j\) and \(b_j\).

Fix an arbitrary \(r > 0\) and consider the following abstract setting of the autonomous problem (1.1), (1.3), (3.1) on \(L^2(0, 1)^n\) (as studied, e.g., in [15, 16]):

\[
\frac{d}{dt} u(t) = Au(t) + f, \quad (0 \leq t \leq r) \\
u(0) = u_0, \quad u(r) = u_r,
\]

where

\[
P = (p_{jk}) = (p_{jk}^{(1)}) \quad \text{and} \quad P_{\text{abs}} = (p_{jk}^{(1)})^{\text{abs}} = (|p_{jk}^{(1)}|).
\]
where the operator \( A : D(A) \subset L^2(0, 1)^n \to L^2(0, 1)^n \) is defined by

\[
(Av)(x) = -A(x)v' - B(x)v, \\
D(A) = \{ v \in L^2(0, 1)^n : v' \in L^2(0, 1)^n, v^{\text{out}} = Pv^{\text{in}} \},
\]

and \( u_0, u_r \in D(A) \) are known functions. Here \( v^{\text{out}}, v^{\text{in}} \) are defined similarly to (1.6). Solving the inverse problem (3.2)–(3.3), we are looking for a couple of functions \((u, f)\) such that \( u \in C^1([0, r], L^2(0, 1)^n) \), \( u(t) \in D(A) \) for all \( t \in [0, r] \), and \( f \in L^2(0, 1)^n \).

Since the problem (3.2)–(3.3) is autonomous, then, due to [12, Theorem 2.3], the operator \( A \) generates a \( C_0 \)-semigroup \( S(t) \). Since the problem (3.2)–(3.3) is FTS, the semigroup \( S(t) \) is nilpotent. Hence, there exists \( T > 0 \) such that \( S(t) = 0 \) for all \( t \geq T \). Accordingly to [16, Theorem 4], for any \( u_0, u_r \in D(A) \), there is a unique function \( f \in L^2(0, 1)^n \) solving the inverse problem (3.2)–(3.3). Moreover, this function admits the representation

\[
f = \begin{cases} 
-Au_r & \text{if } r \geq T \\
-Au_r + A \sum_{k=1}^{n_0} S(kr)(u_0 - u_r) & \text{if } r < T, 
\end{cases}
\]

where \( n_0 = \lceil T/r \rceil - 1 \). Recall that \( \lceil x \rceil \) denotes the integer nearest to \( x \) from above. The unknown function \( u(t) \) is then given by the formula

\[
u(t) = S(t)u_0 + \int_0^t S(s)f \, ds, \quad 0 \leq t \leq r.
\]

Now, using Theorem [16] we conclude that there exists \( k = k(T) \in \mathbb{N} \) such that for all \( x \in [0, 1] \) it holds that

\[
[S(t)u_0](x) = \begin{cases} 
[Q^kw](x, t) & \text{if } t \leq T \\
0 & \text{if } t > T,
\end{cases}
\]

the formula being true for any \( w \in C_h(\Pi)^n \) such that \( w(x, 0) = u_0(x) \).

### 3.2 Nonlinear boundary conditions and FTS property

In the domain \( \Pi \) we consider the \( 2 \times 2 \)-decoupled system

\[
\partial_t u_1 + \partial_x u_1 = 0, \quad \partial_t u_2 - \partial_x u_2 = 0
\]

(3.4)

with the nonlinear boundary conditions

\[
u_1(0, t) = r(t) \sin(u_2(0, t)), \quad u_2(1, t) = \sin^2(s(t)u_1(1, t))
\]

(3.5)
and the initial conditions
\[ u_1(x, 0) = \varphi_1(x), \quad u_2(x, 0) = \varphi_2(x). \] (3.6)

Here \( r \) and \( s \) are smooth and uniformly bounded functions for \( t \geq 0 \). Note that the boundary conditions are of the type (1.13). Our aim is, using Theorem 1.7, to find conditions on the functions \( r \) and \( s \) such that the problem (3.4)–(3.6) is FTS.

The operator \( Q \) defined by (1.9) is now specified to
\[
[Q u_1](x, t) = \begin{cases} 
\varphi_1(x - t) & \text{if } x > t \\
r(t - x) \sin(u_2(0, t - x)) & \text{if } t - x \geq 0,
\end{cases}
\]
\[
[Q u_2](x, t) = \begin{cases} 
\varphi_2(x + t) & \text{if } t + x < 1 \\
\sin^2(s(t + x - 1)u_1(1, t + x - 1)) & \text{if } t + x \geq 1.
\end{cases}
\]

The second power of \( Q \) is then given by
\[
[Q^2 u_1](x, t) = \begin{cases} 
\varphi_1(x - t) & \text{if } x > t \\
r(t - x) \sin(\varphi_2(t - x)) & \text{if } 0 \leq t - x < 1 \\
r(t - x) \sin^2(s(t - x - 1)u_1(1, t - x - 1)) & \text{if } 1 \leq t - x,
\end{cases}
\]
\[
[Q^2 u_2](x, t) = \begin{cases} 
\varphi_2(x + t) & \text{if } t + x < 1 \\
\sin^2(s(t + x - 1)\varphi_1(2 - (t + x))) & \text{if } 1 \leq t + x < 2 \\
\sin^2(s(t + x - 1)r(t + x - 2)\sin(u_2(0, t + x - 2))) & \text{if } 2 \leq t + x.
\end{cases}
\]

It follows that if there exist reals \( T_1 > 0 \) and \( T_2 > 0 \) with
\[ T_2 - T_1 \geq 1 \quad \text{and} \quad \left( r(t) = 0 \quad \text{and} \quad s(t) = 0 \quad \text{for} \quad T_1 \leq t \leq T_2 \right), \] (3.7)
then the condition (1.14) is true with \( k = 1 \). If there exist reals \( T_1 > 0 \) and \( T_2 > 0 \) with
\[ T_2 - T_1 \geq 2 \quad \text{and} \quad \left( r(t) = 0 \quad \text{or} \quad s(t) = 0 \quad \text{for} \quad T_1 \leq t \leq T_2 \right), \] (3.8)
then the condition (1.14) is true with \( k = 2 \). In other words, (3.7) and (3.8) are two sufficient conditions for the problem (3.4)–(3.6) to be FTS.

### 3.3 Theorem 1.7 does not extend for nonhomogeneous boundary conditions

In the domain \( \Pi \), we consider the \( 2 \times 2 \)-decoupled system (3.4) with the initial conditions (3.6) and the boundary conditions
\[ u_1(0, t) = g(t), \quad u_2(1, t) = u_1(1, t). \] (3.9)
Fix $g$ to be a smooth bounded function such that 

$$g(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq 4 \\
\neq 0 & \text{if } 4 < t.
\end{cases}$$

The formula (1.9) then reads 

$$[Q u]_1(x, t) = \begin{cases} 
\varphi_1(x - t) & \text{if } x > t \\
g(t - x) & \text{if } t - x \geq 0,
\end{cases} \quad [Q u]_2(x, t) = \begin{cases} 
\varphi_2(x + t) & \text{if } t + x < 1 \\
u_1(1, t + x - 1) & \text{if } t + x \geq 1,
\end{cases}$$

implying that 

$$[Q^2 u]_1(x, t) = [Q u]_1(x, t), \quad [Q^2 u]_2(x, t) = \begin{cases} 
\varphi_2(x + t) & \text{if } t + x < 1 \\
\varphi_1(2 - (t + x)) & \text{if } 1 \leq t + x < 2 \\
g(t + x - 2) & \text{if } 2 \leq t + x.
\end{cases}$$

It follows that $[Q^2 u](x, 3) \equiv 0$, while the problem (3.4), (3.6), (3.9) is not FTS.

**Acknowledgement**

This work was supported by the VolkswagenStiftung Project “From Modeling and Analysis to Approximation”.

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