Supersymmetry in Quantum Mechanics: Inverted and Non-Inverted Operators.

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Supersymmetry in quantum mechanics has been generalised using inverted and non-inverted operators. In fact the appropriate energy conditions have been derived. Mathematically, Hamiltonians satisfy either (i) SUSY conditions i.e $E_n^{(+)} = E_{n+1}^{(-)}$; $E_0^{(-)} = 0$ or (ii) Iso-spectral condition i.e $E_n^{(+)} = E_n^{(-)}$.

PACS 11.30.Pb, 03.65Db, 11.30.Er, 03.65.Ge.

Key words—Superpotential, SUSY condition, Iso-spectral condition, real Hamiltonians, complex Hamiltonians, exactly solvable systems, numerical results.
I. Introduction

Generation of Quantum systems has been a subject of main interest since the early development of Quantum Mechanics. In fact it gained momentum after the work of Witten [1], who proposed that similar quantum systems can be generated through “Supersymmetry”. However, Gendeshtein [2] proposed that when generated Hamiltonians (\( H_+, H_- \)) reflect similar shape in potentials (commonly known as shape invariant) then energy eigenvalues can be related to each other obey only SUSY conditions i.e.

\begin{align*}
E_{n}^{(+)} &= E_{n+1}^{(-)} \quad (1)
\end{align*}

and

\begin{align*}
E_{0}^{(-)} &= 0 \quad (2)
\end{align*}

where \( E_{n}^{(+)} \rightarrow H_+ \) and \( E_{n}^{(-)} \rightarrow H_- \). Interestingly, not only a report but also a book on supersymmetry has been published by Cooper, Khare and Sukhatme [3] where the authors strongly believe in the validity of relations in Eqs(1,2) and
proposed many exactly solvable Hamiltonians reflecting shape invariants potentials i.e

\[ V_+(x, a_0) = V_-(x, a_1) + R(a_1) \]  \hspace{1cm} (3)

to justify the same. Meanwhile \( PT \) symmetric Quantum Mechanics has gained momentum particularly after the work of Bender and Boettecher [4]. In order to develop the supersymmetry in \( PT \) systems Bazeia, Das, Greenwood and Lanso [5] proposed that for shape invariant potentials (as in Eq(3)) only Iso-spectral condition

\[ E_n^{(+)} = E_n^{(-)} \]  \hspace{1cm} (4)

is the most appropriate one. Of course authors [5] cite examples to justify their stand. Here we notice that when the generated Hamiltonians are Hermitian in nature they obey SUSY condition [3], however when generated Hamiltonians determined using similarity type of transformations [5] in \( PT \) symmetric systems, obey Iso-spectral condition. Hence we feel there is lacking co-operation between two groups of authors [3,5]. Further from the literature on Supersymmetry we notice
that Hamiltonian of the type

\[ H = (p + ig)^2 + V(x) \] (5)

proposed much earlier by Hatano and Nelson [6], has not been considered [5,3]. Hence from the above discussions we feel that existing work on Supersymmetry needs development. Here we propose a new method for the development of Supersymmetry in Quantum Systems which can be visualized in real or complex Hamiltonians. Further interesting examples (including exactly solvable shape invariant potentials) are presented in addition to numerical calculation.

II. Generators, Superpotentials and Hamiltonians.

Let us define two generators using operator theory as

\[ A = -\frac{d}{dx} + iW(x) \] (6)

\[ B = \frac{d}{dx} + iW^*(-x) \] (7)

where superpotential \( W(x) \) can be real or complex. It may not be practically possible to generate either \( A \) from \( B \) or
vice-versa. Now using $A$ and $B$ we define two Hamiltonians

$$H_+ = AB = p^2 - [W(x) - W^*(-x)]p - i\frac{dW^*(-x)}{dx} - W(x)W^*(-x)$$

(8)

and

$$H_- = BA = p^2 - [W(x) - W^*(-x)]p + i\frac{dW(x)}{dx} - W(x)W^*(-x)$$

(9)

One can notice that $AB \neq BA$ i.e $H_+ \neq H_-$. This type of generated Supersymmetric Hamiltonians are new to the literature and can not be visualised by any of the previous work[1,2,3,5].

IIA.SUSY conditions : Real Hamiltonian

Let us consider that

$$W(x) = ix$$

(10)

in this case we find

$$PTW(x)PT = W(x) = ix$$

(11)

The generated Hamiltonians are

$$H_- = p^2 + x^2 - 1$$

(12)
and

\[ H_+ = p^2 + x^2 + 1 \]  \hspace{1cm} (13)

It is easy to see that the corresponding energy levels are

\[ E_n^{(-)} = 2n \]  \hspace{1cm} (14)

\[ E_n^{(+)} = 2n + 2 = E_{n+1}^{(-)} \]  \hspace{1cm} (15)

with

\[ E_0^{(-)} = 0 \]  \hspace{1cm} (16)

where \( E_n^{(+)} \rightarrow H_+ \) and \( E_n^{(-)} \rightarrow H_- \).

**IIB.SUSY conditions : Complex Hamiltonian**

Now consider that

\[ W(x) = ix - ig \]  \hspace{1cm} (17)

In this case for \( g \neq 0 \) we find

\[ PTW(x)PT \neq W(x) = ix - ig \]  \hspace{1cm} (18)

The generated Hamiltonians are

\[ H_- = (p + ig)^2 + x^2 - 1 \]  \hspace{1cm} (19)
and

\[ H_+ = (p + ig)^2 + x^2 + 1 \]  \hspace{1cm} (20)

In order to calculate eigenvalues of the above Hamiltonians, we use \( SU(11) \) algebra [6]. Let us consider the Hamiltonian of the type [6]

\[ H = h_{11}p^2 + h_{22}x^2 + ih_{12}(xp + px) + ih_1p + h_2x \]  \hspace{1cm} (21)

having energy eigenvalue

\[ \epsilon_n = \sqrt{h_{11}h_{22} + h_{12}^2(2n + 1)} + \frac{(h_{12}^2h_{22} - h_{22}^2h_{11} - 2h_1h_2h_{12})}{4(h_{11}h_{22} + h_{12}^2)} \]  \hspace{1cm} (22)

Now using the above formula we find for \( H_- ; H_+ \) one has

(i) \( h_1 = h_2 = 1 \) (ii) \( h_{12} = h_2 = 0 \) (iii) \( h_1 = 2g \). Hence it is easy to show that

\[ E_n^{(-)} = 2n \]  \hspace{1cm} (23)

and

\[ E_n^{(+) = 2n + 2 = E_{n+1}^{(-)}} \]  \hspace{1cm} (24)
with

\[ E_0^{(-)} = 0 \]  \hspace{1cm} (25)

where \( E_n^{(+)} \rightarrow H_+ \) and \( E_n^{(-)} \rightarrow H_- \). This type of Hamiltonian was suggested earlier [7]. Interestingly we realise it in a systematic way in the context of Supersymmetry and notice that SUSY is not broken. Interestingly the above Hamiltonian is non-\( \mathcal{PT} \) invariant and non-Hermiticity in nature. Further to support the above example on SUSY condition, we consider another interesting form of \( W(x) \) as

\[ W(x) = ikx^3 - ix^2g \]  \hspace{1cm} (26)

The corresponding Hamiltonians are the following

\[ H_- = p^2 + 2igx^2p - 3kx^2 + 2gx - g^2x^4 + k^2x^6 \]  \hspace{1cm} (27)

and

\[ H_+ = p^2 + 2igx^2p + 3kx^2 + 2gx - g^2x^4 + k^2x^6 \]  \hspace{1cm} (28)

Now using matrix diagonalisation method [8] we calculate the eigenvalues and reflect the same in table-1 Table -1

**First five eigenvalues of SUSY Hamiltonians**.
In the previous examples we have considered superpotentials having complex in nature. Now we consider superpotentials having real in nature i.e

$$W(x) = x^2$$  \hspace{1cm} (29)
in this case we find

\[ PTW(x)PT = W(x) = x^2 \]  

(30)

The generated Hamiltonians are

\[ H_- = p^2 - x^4 - 2ix \]  

(31)

and

\[ H_+ = p^2 - x^4 + 2ix \]  

(32)

The energy eigenvalues of these Hamiltonians are obtained by solving Schroedinger’s equation numerically [9] and are tabulated in Table-2. From the table one can see that energy levels satisfy the iso-spectral condition [7,8] i.e

\[ E_n^{(+)} = E_n^{(-)} \]  

(33)

Table -2

First five eigenvalues of Iso-spectral Hamiltonians.
| Level | $E_n^{(+)}$     | $E_n^{(-)}$     |
|-------|----------------|----------------|
| 0     | 0              | 0              |
| 1     | 3.398 150      | 3.398 150      |
| 2     | 8.700 453      | 8.700 453      |
| 3     | 14.977 808     | 14.977 808     |
| 4     | 21.999 001     | 21.999 001     |

IID.SUSY conditions: Exactly solvable shape invariant Potentials.

Let us consider one interesting type of complex superpotential as

$$W(x) = ix - i\frac{\lambda}{x}$$  \hspace{1cm} (34)

in this case we find

$$PTW(x)PT = W(x) = ix - i\frac{\lambda}{x}$$ \hspace{1cm} (35)

The present superpotential contains a linear term and an inverse term, whose similar type was first reflected in the work of Supersymmetry by Rath[9] in the context of Hermitian.
Hamiltonian. Later on this type was used by Bazeia et.al [5] by making a modification as

\[ W(x) = ix + \epsilon - \frac{a_0}{(ix + \epsilon)} \]  \hspace{1cm} (36)

in the context of $PT$ symmetry of supersymmetric Quantum Mechanics to discuss iso-spectral conditions. Interestingly Bazeia et.al [5] have not reflected neither numerical nor analytical results to justify their stand. Here the generated Hamiltonians are obtained using slight modified form of superpotential as

\[ H_+ = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda + 1)}{2x^2} - \lambda + 0.5 \] \hspace{1cm} (37)

and

\[ H_- = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda - 1)}{2x^2} - \lambda - 0.5 \] \hspace{1cm} (38)

It is easy to see that the corresponding exact energy levels are

\[ E_n^{(-)} = 2n + 0.5 + 0.5\sqrt{1 + 4\lambda(\lambda - 1)} - \lambda \] \hspace{1cm} (39)

\[ E_n^{(+) = 2n + 1.5 + 0.5\sqrt{1 + 4\lambda(\lambda + 1)} - \lambda \] \hspace{1cm} (40)
One can see that for any value of $\lambda > 2$, the energy levels satisfy the SUSY conditions as

$$E_n^{(+)}(\lambda > 2) = 2n + 2$$

$$E_n^{(-)}(\lambda > 2) = 2n$$

with

$$E_0^{(-)}(\lambda > 2) = 0$$

Interesting point is that eigenvalues are independent of $\lambda$. The corresponding wave functions are the following:

$$\Psi_n^{(+)} = C_n^{+} \left[ (-\frac{d}{dx} + x)^2 - \frac{\lambda(\lambda + 1)}{x^2} \right] n^{0.5+\sqrt{0.25+\lambda(\lambda+1)}} e^{-x^2/2}$$

and

$$\Psi_n^{(-)} = C_n^{-} \left[ (-\frac{d}{dx} + x)^2 - \frac{\lambda(\lambda - 1)}{x^2} \right] n^{0.5+\sqrt{0.25+\lambda(\lambda-1)}} e^{-x^2/2}$$

IIIE. Iso-spectral condition: Exactly solvable Potentials.
Here we consider slight modification of above complex superpotential as

\[ W(x) = ix + i\frac{\lambda}{x} \]  \hspace{1cm} (46)

in this case we find

\[ PTW(x) P T = W(x) = ix + i\frac{\lambda}{x} \]  \hspace{1cm} (47)

Here the generated Hamiltonians are obtained using slight modified form of superpotential as

\[ H_+ = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda - 1)}{2x^2} + \lambda + 0.5 \]  \hspace{1cm} (48)

and

\[ H_- = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda(\lambda + 1)}{2x^2} + \lambda - 0.5 \]  \hspace{1cm} (49)

It is easy to see that the corresponding exact energy levels are

\[ E_n^{(-)} = 2n + 0.5 + 0.5\sqrt{1 + 4\lambda(\lambda + 1) + \lambda} \]  \hspace{1cm} (50)

\[ E_n^{(+)} = 2n + 1.5 + 0.5\sqrt{1 + 4\lambda(\lambda - 1) + \lambda} \]  \hspace{1cm} (51)
One can see that for any value of $\lambda > 2$, the energy levels satisfy the iso-spectral condition as

$$E_n^{(+)}(\lambda > 2) = E_n^{(-)}(\lambda > 2) = 2n + 2\lambda + 1 \quad (52)$$

Here energy levels are parameter dependent.

The corresponding wave functions are the following:

$$\Phi_n^{(+)} = D_n^+[-(\frac{d}{dx} + x)^2 - \frac{\lambda(\lambda - 1)}{x^2}]^n x^{0.5 + \sqrt{0.25 + \lambda(\lambda - 1)}} e^{-x^2/2}$$

$$\Phi_n^{(-)} = D_n^-[-(\frac{d}{dx} + x)^2 - \frac{\lambda(\lambda + 1)}{x^2}]^n x^{0.5 + \sqrt{0.25 + \lambda(\lambda + 1)}} e^{-x^2/2} \quad (54)$$

III. Generators, Superpotentials: Twins and Quadruplets Hamiltonians.

In the above, we have generated only a pair of SUSY or Iso-spectra using suitable choice of superpotential. However in this generation we try to generate twin SUSY and Quadruplet iso-spectra.
using the method given below. Let us define two
generators using operator theory as

\[ A = -ip - W_1 \] (55)

\[ B = ip - W_2 \] (56)

such that the product \( W_1W_2 \) is an even function of \( x \).

\[ H_+ = AB = p^2 + i[W_2 - W_1]p + \frac{dW_2}{dx} + W_1W_2 \] (57)

and

\[ H_- = BA = p^2 + i[W_2 - W_1]p - \frac{dW_1}{dx} + W_1W_2 \] (58)

One can notice that \( AB \neq BA \) i.e \( H_+ \neq H_- \). This
type of generated Supersymmetric Hamiltonians
are new to the literature and can not be visualised
by any of the previous work[1,2,3,5]. Here as per
the condition stated above one has to select \( W_1; W_2 \)
even or odd.
IIIA. Twins SUSY

Case-(a) :: $W_1 = x$ and $W_2 = x^3$

\[ H_1^+ = AB = p^2 + ix^3p - ixp + 3x^2 + x^4 \] (59)

\[ H_1^- = BA = p^2 + ix^3p - ixp - 1 + x^4 \] (60)

Case-(b) :: $W_2 = x$ and $W_1 = x^3$

\[ H_2^+ = AB = p^2 - ix^3p + ixp + 1 + x^4 \] (61)

\[ H_1^+ = AB = p^2 - ix^3p + ixp - 3x^2 + x^4 \] (62)

In order to show the twin behaviour, we calculate the energy eigenvalues and reflect the same in table-3 using the matrix diagonalisation method [8].

Table - 3 : First four eigenvalues of Twin-SUSY Hamiltonians.
From the table one can see that $H_1^+$ and $H_2^+$ have the same eigenspectra even though the corresponding Hamiltonians are not the same.

Similarly $H_1^-$ and $H_2^-$ pertain to same eigen spectra. Hence they are called twin-SUSY. It is worth to mention that the generated Hamiltonians are complex and satisfy $\mathcal{PT}$ - symmetry condition.

### IIIB- Quadruplet Iso-Spectra

**Case-(c)** :: $W_1 = x^2$ and $W_2 = x^4$

$$H_3^+ = AB = p^2 + ix^4p - ix^2p + 4x^3 + x^6$$  \(63\)
and

\[ H_3^- = BA = p^2 + ix^4p - ix^2p - 2x + x^6 \] (64)

Case-(d) :: \( W_2 = x^2 \) and \( W_1 = x^4 \)

\[ H_4^+ = AB = p^2 - ix^4p + ix^2p + 2x + x^6 \] (65)

and

\[ H_4^+ = AB = p^2 - ix^4p + ix^2p - 4x^3 + x^6 \] (66)

In order to show the quadruplet behaviour , we calculate the energy eigenvalues and reflect the same in table-4 using the matrix diagonalisation method [8].

Table - 4 :First four eigenvalues of Quadruplet Iso-spectral Hamiltonians .
In this case the generated Hamiltonians are non-$\mathcal{PT}$ invariant in nature.

IV. Inverted potentials

Here we generate inverted as follows using a slight different way as follows. Let

\[ A = \frac{d}{dx} + W_1(x) \]  
\[ B = \frac{d}{dx} + W_2(x) \]

The corresponding Hamiltonians are

\[ H_{3}^- = -BA \]  
\[ H_{4}^- = -AB \]
Considering $W_1 = x$ and $W_2 = x^K$, where $K=\text{odd} = 3, 5, 7, 9...$ We get the following Hamiltonians as for $K=3$

$$H^{(-)} = p^2 - 1 - ixp - ix^3p - x^4$$  

and

$$H^{(+)} = p^2 - ixp - 3x^2 - ix^3p - x^4$$

Similarly for $K=5$ we have

$$H^{(-)} = p^2 - 1 - ixp - ix^5p - x^6$$

and

$$H^{(+)} = p^2 - ixp - 5x^4 - ix^5p - x^6$$

Using matrix diagonalisation method, we present the first four levels in table given below.

Table - 5 : First four eigenvalues of Inverted Quartic , Sextic operators .


| $H^{-}$ | $H^{+}(K=3)$ | $H^{-}$ | $H^{+}(K=5)$ |
|---------|--------------|---------|---------------|
| 0       | 3.548 8      | 0       | 2.534 2       |
| 3.548 8 | 10.826 9     | 2.534 2 | 8.603 1       |
| 10.826 9| 20.539 5     | 8.603 1 | 16.832 8      |
| 20.539 5| 32.219 1     | 16.832 8| 27.152 4      |

Interested readers can see, energy levels in inverted operators satisfy the SUSY-EC.

\[ E_{n}^{(+)} = E_{n+1}^{(-)} \]  \hspace{1cm} (75)

and

\[ E_{0}^{(-)} = 0 \]  \hspace{1cm} (76)

In figs.1,2 we plot the corresponding wave function mod square.

V.Conclusion.

In this paper, we have developed a new method for supersymmetric Quantum System to realise it in real or complex operators. For shape invariant potentials we show that energy levels are
intimately related reflecting susy or iso-spectral behaviour only on selection of parameters. Hence we co-relate previously published two different concepts \([3,5,10]\). We also generate two-Twins satisfying SUSY behaviour and quadruplet satisfying iso-spectral behaviour. It should be remembered that present selections of \(A\) and \(B\) are not the only possibility. In fact one can follow this procedure to generate other types. In order to help the reader on new generation, we present another type as follows:

\[
A_1 = i \frac{d}{dx} + W(x) \tag{77}
\]

\[
B_1 = i \frac{d}{dx} + W^* (-x) \tag{78}
\]

where superpotential \(W(x)\) can be real or complex. It may not be practically possible to generate either \(A_1\) from \(B_1\) or vice-versa. Now using
A_1 and B_1 we define two Hamiltonians

\[ H_+ = A_1 B_1 = p^2 - [W(x) + W^*(-x)]p + i \frac{dW^*(-x)}{dx} + W(x)W^*(-x) \]

(79)

and

\[ H_- = B_1 A_1 = p^2 - [W(x) + W^*(-x)]p + i \frac{dW(x)}{dx} + W(x)W^*(-x) \]

(80)

This new form can easily generate new type of SUSY Hamiltonians discussed recently by Marques, Neg1eni and da Silva [12]. Here we select the form of \( W(x) \) as

\[ W(x) = i|x|^2 \]

(81)

and generate SUSY Hamiltonians as

\[ H_- = p^2 - 2|x| + x^4 \]

(82)

\[ H_+ = p^2 + 2|x| + x^4 \]

(83)

Further using matrix diagonalisation method, we present first five eigenvalues in table-5.

Table -6
First four eigenvalues of New SUSY Hamiltonians and previous results [12].

| $E_n^{(+)}$ | $E_n^{(-)}$ | $E_n^{(+)}$ Previous [12] | $E_n^{(-)}$ Previous [12] |
|------------|------------|---------------------------|---------------------------|
| 1.969 9    | 0          | 1.972 3                   | 0                         |
| 5.507 1    | 1.969 5    | 5.510 0                   | 1.969 6                   |
| 9.394 5    | 5.506 8    | 9.415 2                   | 5.508 4                   |
| 13.858 3   | 9.394 2    | 13.893 6                  | 9.398 6                   |

we believe that present development will enrich the existing knowledge on supersymmetry in the context of energy calculation.

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Figure 1: Quartic inverted case

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