On Hereditarily Codiskcyclic Operators

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Abstract:
Many codiskcyclic operators on infinite-dimensional separable Hilbert space do not satisfy the criterion of codiskcyclic operators. In this paper, a kind of codiskcyclic operators satisfying the criterion has been characterized, the equivalence between them has been discussed and the class of codiskcyclic operators satisfying their direct summand is codiskcyclic. Finally, this kind of operators is used to prove that every codiskcyclic operator satisfies the criterion if the general kernel is dense in the space.

Keywords: Codiskcyclic operators, Criterion for codiskcyclic operators, Generalize kernel, Hereditarily codiskcyclic operators.

Introduction:
Let $H$ be an infinite – dimensional separable Hilbert space, the unit disk is denoted by $𝔻$, $𝔹$ is the unit ball, and $B(H)$ is the set of all linear bounded operators onto $H$. Hilden and Wallen in 1974 introduced the concept of supercyclic operator $T \in B(H)$, as there exists a non-zero vector $y$ in $H$ such that $\text{orb} (T, y) := \{ \alpha T^n y : n \geq 0, \alpha \in 𝔹 \}$ is dense in $H$. After that in 2002, Jamil in her thesis divided $𝔻$ into three areas according to the unit circle:
- The interior of the unit circle: thus, an operator is called diskcyclic, if $\text{orb} (T, y) := \{ \alpha T^n y : n \geq 0, \alpha \in 𝔹 \}$ is dense in $H$.
- The unit circle: then the operator is called circle cyclic, if $\text{orb} (T, y) := \{ T^n y : n \geq 0, |\alpha| = 1 \}$ is dense in $H$.
- The exterior of the unit circle: hence the operator is called codiskcyclic, if $\text{orb} (T, y) := \{ \alpha T^n y : n \geq 0, \alpha \in 𝔹^c \}$ is dense in $H$.
She studied some of their properties like the range of them, some necessarily and sufficient conditions to be.

In 2004, Leon-Saavedra and Muller, proved that every circle cyclic operator is hypercyclic, which mean $\text{orb} (T, y) := \{ T^n y : n \geq 0 \}$ is dense in $H$, while the other kinds have been gaining importance in recent years such as Liang and Zhou, Wong and Zeng, etc.

In 2002, Jamil introduced a criterion for codiskcyclic operators. She showed that there are codiskcyclic operators that do not satisfy this criterion. Thus, the natural question arises which kind of codiskcyclic operators can satisfy the criterion?

This paper offered a partial solution to this problem by presenting a new kind of codiskcyclic operator that is called hereditarily codiskcyclic. By using the concept of hereditarily codiskcyclic, codiskcyclic operators have been proved to satisfy the criterion whenever the generalize kernel is dense in the space.

Codiskcyclic vectors:
In this section, the focus of attention is on studying the properties of the set of codiskcyclic vectors of $\{T^k\}$ where $\{k\}$ is a sequence of non-negative integers.

Definition (1):
Let $T \in B(H)$, and $\{i\}$ be a non-negative integer, then $\{T^i\}$ is called codiskcyclic if there is non-zero vector $y$ in $H$ such that $\{\beta T^i y : \beta \in 𝔹^c, 0 \leq i \}$ is dense in $H$ in this case $y$ is called codiskcyclic vector for $\{T^i\}$

Remark (1):
If $< i >= < n >$, then $\{T^i\}$ is codiskcyclic, if and only if $T$ is a codiskcyclic operator.
Starting point is discussing how much the set of codiskcyclic vectors for \(\{T^k\}\) is

**Proposition (1):**
Let \(k > k\) be a non-negative integer sequence and \(T \in B(H)\), then for all \(m \geq 0\), \(T^m z\) is codiskcyclic vector for \(\{T^k\}\) whenever \(z\) is

**Proof:**
Let \(z\) be a codiskcyclic vector for \(\{T^k\}\), then for all \(m \geq 0\),
\[
\mathbb{D}^c \text{Orbit } \{T^k, T^m z\} = \{\beta T^k (T^m z); k \geq 1, \beta \in \mathbb{B}^c\}
\]

In addition to discussing how big the set of codiskcyclic vectors is for \(\{T^k\}\), the following proposition studying the relation between \(\{T^k\}\) is codiskcyclic and topologically transitive.

Since the proof of \((3) \Rightarrow (1)\) is trivial, thus it is omitted.

**Proposition (2):**
Let \(H\) be a separable infinite dimensional complex Hilbert space and \(T \in B(H)\). Let \(\{k\}\) be a non-negative integer sequence. Then the following statements are equivalent.

1. \(\{T^k\}\) is codiskcyclic.
2. For all \(V, U\) non-empty open subsets of \(H\), there is \(\ell \in \{k\}\) large enough and \(\lambda \in \mathbb{B}^c\) such that \(T^{\ell}(\lambda V) \cap U \neq \emptyset\).
3. \(\{T^k\}\) has a dense \(G_\delta\)-set of codiskcyclic vectors.

**Proof:**
1) \(\Rightarrow 2)\): Let \(V, U\) be open subsets of \(H\). By (2), there is a codiskcyclic vector \(z\) for \(\{T^k\}\). Thus, there exist \(k_1 \in \{k\}\) and \(\beta_1 \in \mathbb{B}^c\) such that \(\beta_1 T^{k_1} z \in V\). By proposition (1.3), \(z_0 = \beta_1 T^{k_1} z\) is a codiskcyclic vector for \(\{T^k\}\), hence there is \(k_2 \in \{k\}\) and \(\beta_2 \in \mathbb{B}^c\) such that \(\beta_2 T^{k_2} z_0 \in U\). Therefore \(\beta_2 T^{k_2} \cap V \cap U \neq \emptyset\).

2) \(\Rightarrow 3)\): Let \(\{B_i\}\) be a countable bases for the topology of \(H\). It is easy to see that
\[
\mathbb{D}^C (\{T^k\}) = \bigcap_r \left( \bigcup_{\beta \in \mathbb{B}^c} \bigcup_{k \in \mathbb{N}} T^{-k}(\beta B_r) \right)
\]

So, by the continuity of \(T\), \(\bigcup_{\beta \in \mathbb{B}^c} \bigcup_{k \in \mathbb{N}} T^{-k}(\beta B_r)\) is open set.

Now, let \(V\) be a non-empty open set in \(H\). Thus by (4), there is \(\ell \in \{k\}\) and \(\lambda \in \mathbb{B}^c\) such that \(V \cap T^{-\ell} (\frac{1}{\beta} B_r) \neq \emptyset\). Therefore, by Bair theorem, \(\mathbb{D}^C (\{T^k\})\) is dense \(G_\delta\)-set.

**Hereditarily Codiskcyclic Operators:**
It is well known that not every codiskcyclic operators satisfies codiskcyclic criterion, so this section introduces the following concept and argue the relation between these operators with codiskcyclic criterion.

**Definition (2):**
A bounded linear operator \(T\) is called here ditarility codiskcyclic if there is a sequence \(\{n_k\}\) such that for all subsequence \(\{m_k\}\) of \(\{n_k\}\), \(\{T^{m_k}\}\) is codiskcyclic.

Every hereditarily codiskcyclic is a codiskcyclic operator. One question raises is the converse true. The following definition and proposition are needed to answer this question.

**Definition (3):**
A bounded linear operator \(T\) onto \(H\), satisfies codiskcyclic criterion, if there are increasing sequences of positive integer \(\{n_k\}\) and \(\{\alpha_{n_k}\}\) in \((1, \infty)\), for which there are dense sets \(Y, X\) in \(H\) and a sequence of mappings \(S_{n_k}: Y \to H\) such that for all \(y \in Y, x \in X\) and when \(k \to \infty\):

1. \(\alpha_{n_k} T^{n_k} x \to 0\)
2. \(\frac{1}{\alpha_{n_k}} S_{n_k} y \to 0\)
3. \(T^{n_k} S_{n_k} y \to y\)

**Example (1)**: Let \(T \in B(\ell^2(\mathbb{Z}))\) be the forward weighted shift with weight sequence,
\[
w_n = \begin{cases} 
1 & \text{if } n > 0 \\
\frac{1}{2} & \text{otherwise}
\end{cases}
\]
Then \(T\) satisfies codiskcyclic criterion.

**Proposition (3):**
\(T \in B(H)\) is a codiskcyclic operator whenever \(T\) satisfies codiskcyclic criterion

**Proof:**
Let \(V, U\) be non-empty open sets in \(H\). By the density of \(Y\) and \(X\), there are \(x \in X \cap V\) and \(y \in Y \cap U\) such that \(x + \frac{1}{\alpha_{n_k}} S^{n_k} y \to x\) and \(\alpha_{n_k} T^{n_k} y \to y\).

Thus, for suitable \(k \in \mathbb{N}\), there are \(\alpha = \alpha_{n_k} \in (1, \infty)\) and \(n = n_k \in \mathbb{N}\) such that \(T^{n_k} V \cap U \neq \emptyset\). Therefore, by proposition (2) \(T\) is a codiskcyclic operator.

The result now discusses the relation between hereditarily codiskcyclic operator and operator which satisfies codiskcyclic criterion.

Jamal in \(^2\) proved that if \(\bigoplus_{i=1}^n T_i \in B(\bigoplus_{i=1}^n H)\) is a codiskcyclic, then \(T_i\) is a codiskcyclic operator for all \(i\).

**Proposition (4):**
Let \(T \in B(H)\). Then the following statements are equivalent:

1. For all \(n \in \mathbb{N}\), \(\bigoplus_{i=1}^n T_i \in \mathbb{D}^c C (\bigoplus_{i=1}^n H)\).
2. \(T\) satisfies codiskcyclic criterion.
3. \(T\) is hereditarily codiskcyclic operator.
Proof:
1) ⇒ 2): By Jamil’s proposition 2, it is enough to prove when \( n = 2 \).
Since \( T \ominus T \in \mathbb{D}^c C(H \ominus H) \), then there is
\((x, y) \in \mathbb{D}^c C(T \ominus T) \).
\( y = X = \mathbb{D}^c \text{orb} (T, x) \), for all \( \alpha \in \mathbb{B}^c \) and
\( m \in \mathbb{N} \) since \( I_\ominus \alpha T^m \) has dense range and
commutes with \( T \ominus T \). Thus \((x, \alpha T^m y) \in
\mathbb{D}^c C(T \ominus T) \), but \( y \in \mathbb{D}^c C(T) \), therefore for all
zero-neighbourhood \( V_0 \), there exists \( v_k \in V_0 \) closed
enough to zero, and \( \alpha_k \in \mathbb{B}^c \), \( m \in \mathbb{N} \) such that
\( v_k = \alpha_k T^m y \). Thus \((x, v_k) \in \mathbb{D}^c C(T \ominus T) \) for all
\( k \geq 1 \). Then there exist sequences: \( \{n_k\} \in \mathbb{N} \), \( \{\lambda_{n_k}\} \)
in \( \mathbb{B}^c \) such that when \( k \to \infty \),
a) \( T^{n_k} v_k \to 0 \)
b) \( \lambda_{n_k} T^{n_k} x \to 0 \)
c) \( \lambda_{n_k} T^{n_k} v_k \to x \)

Now define a sequence of operator \( S_{n_k} : Y \to H \) as
\( S_{n_k}(\alpha T^x) = \sum_{i=1}^{n_k} T^i u_k \) for all \( \alpha \in \mathbb{B}^c \) and \( k \in \mathbb{N} \). It
is easy to prove that \( T \) and \( \{S_{n_k}\} \) satisfy the
conditions of codiskcyclic criterion (3) thus the result
is done by (3).
2) ⇒ 3): Let \((U, V)\) be a pair of non-empty open
sets in \( H \), since \( T \) satisfies codiskcyclic criterion,
there is a pair of dense sets, say \((X, Y)\), sequence \( \{n_k\} \in \mathbb{N} \), \( \{\alpha_k\} \in \mathbb{B}^c \), and \( S_{n_k} : Y \to H \)
such that for all \( x \in X \) and \( y \in Y \),
a) \( \alpha_{n_k} T^{n_k} x \to 0 \)
b) \( \frac{1}{\alpha_{n_k}} S_{n_k} y \to 0 \)
c) \( T^{n_k} S_{n_k} y \to y \)

which is true for all subsequence<br><m>\( \alpha_{n_k} \)\><br><m>\( \alpha_{n_k} T^{n_k} (V) \cup U \neq \emptyset \). Thus,
by proposition (2) \( T \) is a hereditarily codiskcyclic.

3) ⇒ 1) Let \( V, U \) be non-empty set in \( H \) where
\( i = 1, \ldots, n \); \( n \geq 1 \). Since \( T \) is a hereditarily
codiskcyclic operator, then there is a sequence \( \{n_k\} \)
in \( \mathbb{N} \) such that for all subsequence \( \{m_k\} \) of \( \{n_k\} \),
\( \{T^{m_k}\} \) is codiskcyclic. By proposition (2), there are
\( m \in \{m_k\} \) and \( \alpha_i \in \mathbb{B}^c \) such that
\( \alpha_i T^{m_i} V \cup U \neq \emptyset \). But \( \{T^{m}\} \) is codiskcyclic, thus
there exist subsequences \( \{m_i\} \) and \( \{\alpha_{i,j}\} \) such that
\( \alpha_{i,j} T^{m_i} V \cup U \neq \emptyset \), and so on there are subsequence
\( \{m_p\} \) and \( \{\alpha_{p}\} \) such that
\( \alpha_{p} T^{m_p} V_{n-1} \cap V_{n-1} \neq \emptyset \). Because of \( \{T^{m_i}\} \)

Conclusions:
The open problem is “Which kind of
codiskcyclic operators satisfy codiskcyclic
criterion?” This paper introduced a hereditarily
codiskcyclic operators, studied some
characterization, and proved that every
codiskcyclic operator satisfies the codiskcyclic
criterion if the space contains dense general kernel set.

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خواص المؤثرات القرصية الدوراية المشاركة الوراثية

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الخلاصة:
العديد من المؤثرات القرصية المشاركة المعرفة على فضاء هلبرت منفصل وغير نهائي الابعاد لا تحقق معيارية المؤثرات القرصية الدوراية المشاركة. في هذا البحث شخصننا نوع من أنواع المؤثرات القرصية الدوراية المشاركة التي تحقق معيارية المؤثرات القرصية الدوراية المشاركة واستخدمنا لبرهان أنه أي مؤثر قرصي دوري المشارك يحقق معيارية المؤثرات القرصية الدوراية المشاركة إذا كان نواة العمومية كثيفة في الفضاء.

الكلمات المفتاحية: المؤثرات القرصية الدوراية المشاركة، معيار المؤثرات القرصية الدوراية المشاركة، المؤثرات القرصية الدوراية المشتركة، النواة العمومية.