CLASSIFYING SINGULARITIES UP TO ANALYTIC EXTENSIONS OF SCALARS

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ABSTRACT. The singularity space consists of all germs \((X, x)\), with \(X\) a Noetherian scheme and \(x\) a point, where we identify two such germs if they become the same after an analytic extension of scalars. This is a Polish space for the metric given by the order to which infinitesimal neighborhoods, or jets, agree after base change. In other words, the classification of singularities up to analytic extensions of scalars is a smooth problem in the sense of descriptive set-theory. Over \(\mathbb{C}\), the following two classification problems up to isomorphism are now smooth: (i) analytic germs; and (ii) polarized schemes.

1. INTRODUCTION

Roughly speaking, a classification problem consists of a class of objects together with an equivalence relation telling us which objects to identify; a solution to this problem is then an ‘effective’ or ‘concrete’ description of the quotient, preferably by a ‘system of complete invariants’. What constitutes a reasonably concrete or effective solution to a classification problem, however, might depend on one’s purposes or even one’s taste. Descriptive set-theory proposes smoothness to be the decisive indication that a classification is explicit and/or concrete (see for instance [7, 9] for a discussion). More precisely, recall that a Polish space is a complete metric space containing a countable dense subset. Considering a Polish space to be concrete is justified by the fact that its underlying Borel structure is in essence equal to the standard Borel space \(\mathcal{R}\). With this in mind, an equivalence relation on a Polish space, and by extension, the classification problem it encodes, is called smooth if there is a Borel map to a Polish space which factors through the quotient. A more suggestive, albeit slightly less precise formulation is that, up to a Borel isomorphism, equivalence classes are completely classified by real numbers.

Most classification problems in algebraic geometry, like classifying varieties over a fixed algebraically closed field up to isomorphism or up to bi-rational equivalence, are not known to be smooth. Of course, this is in no way preventing geometers to seriously, and often successfully, work on these classification problems. It would be nice to know though what their descriptive set-theoretical status is. In this paper, I will propose a local classification problem, which will fall at the right side of the dividing line: one can ‘concretely’, that is to say, smoothly, classify germs of points on arbitrary Noetherian schemes up to similarity (a slightly weaker equivalence relation than the isomorphism relation). Using this general result, we can also deduce some smoothness results for certain isomorphism problems. For analytic germs, that is to say, formal completions of germs (in the sense of [8, II.9]), we have:
1.1. **Theorem.** The classification, up to isomorphism, of analytic germs over an algebraically closed field of size the continuum, is smooth.

This also enables us to obtain a smooth classification problem of a more global nature, namely for projective schemes together with a choice of a very ample line bundle, the so-called polarized schemes.

1.2. **Theorem.** The classification, up to isomorphism, of polarized schemes over an algebraically closed field of size the continuum, is smooth.

For the proof of our main smoothness result, we associate to a point its local ring, thus reducing the problem to the study of the category of all Noetherian local rings. If we were to classify these only up to isomorphism, then as part of this problem, we would have to classify already all fields, and even for countable fields [5] or fields of finite transcendence degree [22] these are non-smooth problems. Hence to circumvent this arithmetical obstruction, we can either fix the residue field—the route taken for the two isomorphism problems stated above—or, otherwise, allow for ‘extensions of scalars’, resulting in the identification of any two fields of the same characteristic. Even after taking the latter modification, the local classification problem is probably still not smooth. We introduce one further identification, inspired in part by Grothendieck’s suggestion that one should consider working in the etale topos instead of the (classical) Zariski topos. A down-to-earth interpretation of this point of view is that two local rings can be considered identical if they have a common etale extension, or more generally, if they have the same completion. In summary, we say that two Noetherian local rings are similar if they can be made isomorphic by an analytic extension of scalars, that is to say, by the process of extending scalars and taking completion. To also make sense of this in mixed characteristic, we subsume these types of extensions under the larger class consisting of all formally etale (=unramified and faithfully flat) extensions. We will show that similar points (meaning that their corresponding local rings are similar) have the same type of singularity (see Theorem 4.1). As a spinoff of this investigation, we obtain a flatness criterion generalizing a result of Kollár:

**Theorem 3.12.** Let \( R \to S \) be a local homomorphism between Noetherian local rings and suppose \( R \) is an excellent normal domain with perfect residue field. If \( \dim(R) = \dim(S) \) and \( R \to S \) is unramified, then \( R \to S \) is faithfully flat.

Our assertion that classifying points up to similarity is smooth is established by effectively putting a metric on the space of similarity classes \( \mathbb{S}_\text{lim} \), called the jet metric. We will prove that the induced topology is complete, and that the collection of similarity classes of Artinian local rings with a finitely generated residue field is a countable dense subset. This shows that \( \mathbb{S}_\text{lim} \) is a Polish space and hence classification up to similarity is a smooth problem. The jet metric on \( \mathbb{S}_\text{lim} \) is induced by a semi-metric on the class of all Noetherian local rings. In terms of (germs of) points, this semi-metric measures to which order the jets of two points agree. In fact, the proof yields that for classification up to similarity, the collection of all jets of a point form a complete set of invariants.

So far, all concepts are algebraic-geometric in nature, but the existence of limits relies on a tool from model-theory, to wit, the ultraproduct construction. Of course, the ultraproduct of Noetherian local rings is in general no longer Noetherian. However, if we have a Cauchy sequence of Noetherian local rings, then their cataproduct, obtained by killing all infinitesimals in the ultraproduct, yields a complete Noetherian local ring, which, up to similarity, is the limit of the sequence.
2. LIMITS AND ULTRAPRODUCTS

Let $(\Sigma, d)$ be a semi-metric space. In this paper, we understand this to mean that the semi-metric is non-archimedean, that is to say, $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ for all $x, y, z \in \Sigma$, and bounded, that is to say, after possibly normalizing the metric, $d(x, y) \leq 1$ for all $x, y \in \Sigma$. We call $d$ a metric, if $d(x, y) = 0$ if and only if $x = y$. To include the jet metric in our treatment, we allow for $\Sigma$ to be merely a class. We say that two elements $r, s \in \Sigma$ are $d$-equivalent, written $r \sim_d s$, if $d(r, s) = 0$. The quotient space $\Sigma/ \sim_d$ has an induced semi-metric which is in fact a metric; we therefore call this quotient the metrization of $(\Sigma, d)$.

Let $(\Sigma_w, d_w)$ be semi-metric spaces, for $w \in \mathbb{N}$. We will identify the elements of the product $\Pi := \prod_w \Sigma_w$ with the sequences $r: \mathbb{N} \to \Pi$ such that $r(w) \in \Sigma_w$ for each $w$. The product semi-metric on $\Pi$ is given by letting the distance $d(r, s)$ between two sequences $r$ and $s$ be the lim-inf of the distances $d_w(r(w), s(w))(\leq 1)$ of their respective components. Below, we will introduce weaker semi-metrics on $\Pi$, induced by ultrafilters.

Cauchy sequences. Let $r$ be a sequence in $\Sigma$ (meaning that all $r(w) \in \Sigma$) and let $r^+$ be its twist, given as the sequence whose $w$-th element is $r(w + 1)$. We call $r$ a Cauchy sequence if $r \sim r^+$ (with respect to the product semi-metric). One verifies that $r$ is a Cauchy sequence, if for each $\varepsilon > 0$, there exists an $N$ such that $d(r(w), r(w')) < \varepsilon$ for all $v, w > N$, and that two Cauchy sequences $r$ and $s$ are equivalent if for each $\varepsilon > 0$, there exists an $N$ such that $d(r(w), s(w)) < \varepsilon$ for all $w > N$. Let $\text{Cau}(\Sigma, d)$, or simply, $\text{Cau}(\Sigma)$, denote the set of all Cauchy sequences in $\Sigma$ with the induced product semi-metric. There is a natural isometry $\Sigma \to \text{Cau}(\Sigma)$ sending $x$ to the constant sequence $x$ given as $x(w) := x$; we will identify the element $x$ with its constant sequence in $\text{Cau}(\Sigma)$.

A limit of a sequence $r$ is an element $x \in \Sigma$ such that $r \sim x$. It is easy to see that if $r$ has a limit, then it must be Cauchy. We call $(\Sigma, d)$ complete if every Cauchy sequence has a unique limit. This implies in particular that $d$ is a metric. We define the completion of $(\Sigma, d)$ as the metrization $\hat{\Sigma} := \text{Cau}(\Sigma)/ \sim$ of the semi-metric space $\text{Cau}(\Sigma)$; it is a complete metric space containing $\Sigma$ as a dense subspace.

Adic metric. A local ring $(R, m)$ comes with a canonical semi-metric, its $m$-adic semi-metric defined as follows: the order of an element $x \in R$ is the supremum of all $n$ for which $x \in m^n$; the distance $d_R(x, y)$ between two elements is then equal to $2^{-n}$ where $n$ is the order of $x - y$ (we allow the case $n = \infty$, with the convention that $2^{-\infty} = 0$). The subset of all elements which are $d_R$-equivalent to zero forms an ideal, equal to the intersection of all the powers $m^n$; it is called the ideal of infinitesimals of $R$ and is denoted $\text{Inf}(R)$. By Krull’s intersection theorem, if $R$ is Noetherian, then $\text{Inf}(R) = 0$. The completion of $R$ in the $m$-adic semi-metric will be denoted $\hat{R}$. If $R$ has finite embedding dimension, then $\hat{R}$ is a complete Noetherian local ring by [21, Theorem 2.2].

Below, we will define a semi-metric on the class of all Noetherian local rings, not to be confused with the adic metric on a single Noetherian local ring. To calculate limits in the former semi-metric, we need a notion from model-theory: the ultraproduct construction (some references for ultraproducts are [4, 11, 16, 20], or the brief review in [17, §2]).

Ultraproducts and cataproducts. Let $(R_w, m_w)$, for $w \in \mathbb{N}$, be a sequence of Noetherian local rings. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which we always assume to be non-principal. The ultraproduct of the $R_w$ with respect to $\mathcal{U}$, denoted $R_\mathcal{U}$, is obtained from the product $\Pi := \prod_w R_w$ by modding out the ideal of all sequences almost all of whose entries are zero (it is customary to use the expression “almost all” to mean “all indices belonging to a
member of the ultrafilter’"). The particular choice\(^1\) of ultrafilter \(\mathcal{U}\) does not matter for our purposes, and hence we do not include it in our notation. Although not useful for proving results, let me recall an alternative construction from [20, Theorem 2.5.4]: there exists a minimal prime ideal \(g\) of the Cartesian power \(\mathbb{Z}^\mathcal{N}\), containing the direct sum ideal \(\mathbb{Z}^\mathcal{N}\), such that \(R_\mathcal{N} = \Pi/g\Pi\), where we view the Cartesian product \(\Pi\) as an algebra over \(\mathbb{Z}^\mathcal{N}\) in the natural way; and conversely, any such prime ideal determines in this way an ultraproduct of the \(R_w\).

The ultraproduct \(R_\mathcal{N}\) is again a local ring, with maximal ideal \(m_\mathcal{N}\) given as the ultraproduct of the \(m_w\). In general, however, \(R_\mathcal{N}\) will no longer be Noetherian. If almost all \(R_w\) have embedding dimension at most \(n\), then so does \(R_\mathcal{N}\). A key role will played by the homomorphic image of \(R_\mathcal{N}\) modulo its ideal of infinitesimals \(\text{Inf}(R_\mathcal{N})\), which we call the cataproduct of the \(R_w\) and which we denote by \(R_\mathcal{N}\). A more direct way for defining the cataproduct, although less useful in proofs, is as follows: on the product \(\Pi\), the ultrafilter \(\mathcal{U}\) induces a semi-metric \(d_\mathcal{U}\) by the condition that \(d_\mathcal{U}(r,s) \leq \varepsilon\) for some \(\varepsilon\) if and only if \(d_{R_w}(r(w),s(w)) \leq \varepsilon\) for almost all \(w\). The cataproduct is then the metrization of \((\Pi, d_\mathcal{U})\), that is to say, the residue ring of the product modulo the ideal of all sequences which are \(d_\mathcal{U}\)-equivalent to zero.

If almost all \(R_w\) have embedding dimension at most \(n\), then so does the cataproduct \(R_\mathcal{N}\). Moreover, by the saturatedness property of ultraproducts, the cataproduct is a complete local ring, whence Noetherian by \([13, \text{Theorem 29.4}]\) (for more details see \([21, \text{Lemma 5.6}]\) or \([20, \text{Theorem 12.1.4}])\). The same argument also shows that the \(R_w\) and their completions \(\hat{R}_w\) have the same cataproduct.

We will only consider cataproducts of Noetherian local rings of bounded embedding dimension, so that we tacitly may assume that they are complete and Noetherian. In case all \(R_w\) are equal to a fixed Noetherian local ring \(R\), then their ultraproduct \(R_\mathcal{N}\) and cataproduct \(R_\mathcal{N}\) are called, respectively, the ultrapower and catapower of \(R\). By Łos’ Theorem, ultrapowers commute with base change, that is to say \((R/I)_\mathcal{N} \cong R_\mathcal{N}/IR_\mathcal{N}\); the same is true for catapowers by \([21, \text{Corollary 5.7}]\):

2.1. Lemma. If \(R\) is a Noetherian local ring and \(I\) an ideal in \(R\), then \((R/I)_\mathcal{N} = R_\mathcal{N}/IR_\mathcal{N}\).

3. Scalar extensions

Cohen’s structure theorems for complete Noetherian local rings will play an essential role in this paper, so we quickly review the relevant properties; a good reference for all this is \([13, \S 29]\).

Cohen’s structure theorems. For each field \(\kappa\) of prime characteristic \(p\), there exists a unique complete discrete valuation ring \(V\) of characteristic zero whose residue field is \(\kappa\) and whose maximal ideal is \(pV\); we call \(V\) the complete \(p\)-ring over \(\kappa\). Let \(R\) be a Noetherian local ring with residue field \(\kappa\). We say that \(R\) has equal characteristic if \(R\) and \(\kappa\) have the same characteristic; in the remaining case, we say that \(R\) has mixed characteristic. Assume \(R\) is moreover complete and let \(X\) be a finite tuple of indeterminates. Cohen’s structure theorems now claim, among other things, the following:

- if \(R\) has equal characteristic, then it is a homomorphic image of \(\kappa[[X]]\);

\(^1\)There is really no reason to restrict only to ultraproducts on a countable index set, although it is the only type we will use in this paper. However, for the cataproduct (see below) to be Noetherian and complete, we do have to impose that the ultrafilter is countably incomplete, which automatically holds on countable index sets and always exists on arbitrary index sets.
• if $R$ has mixed characteristic, then it is a homomorphic image of $V[[X]]$, where $V$ is the complete $p$-ring over $\kappa$.

Let $(R, m)$ be a Noetherian local with residue field $\kappa$ and let $\lambda$ be a field extension of $\kappa$. With a scalar extension of $R$ over $\lambda$ we mean a local $R$-algebra $(S, n)$ with residue field $\lambda$ such that $R \to S$ is faithfully flat, $n = mS$ and $R \to S$ induces the embedding $\kappa \subseteq \lambda$ on the residue fields. A scalar extension of a local ring $R$ is then a scalar extension of $R$ over some field extension of its residue field. The condition that $n = mS$ is also expressed by saying that $R \to S$ has trivial closed fiber or that it is unramified. In other words, a scalar extension is the same as an unramified, faithfully flat homomorphism (also called a formally etale extension). By [6, 011I 10.3.1], for any Noetherian local ring $R$ and any extension $\lambda$ of its residue field, at least one scalar extension of $R$ over $\lambda$ exists; we will reprove this in Corollary 3.3 below.

3.1. Proposition. Consider the following commutative triangle of local homomorphisms between Noetherian local rings

\[
\begin{array}{ccc}
(R, m) & \xrightarrow{f} & (S, n) \\
\downarrow & & \downarrow g \\
(S, n) & \xrightarrow{h} & (T, p)
\end{array}
\]

If any two are scalar extensions, then so is the third.

Proof. It is clear that the composition of two scalar extensions is again scalar. Assume $g$ and $h$ are scalar extensions. Then $f$ is faithfully flat and $mT = p = nT$. Since $g$ is faithfully flat, we get $mS = mT \cap S = nT \cap S = n$, showing that $f$ is also a scalar extension. Finally, assume $f$ and $h$ are scalar extensions. Let

\[
\ldots R^{b_2} \to R^{b_1} \to R \to R/m \to 0
\]

be a free resolution of $R/m$. Since $S$ is flat over $R$, tensoring yields a free resolution

\[
\ldots S^{b_2} \to S^{b_1} \to S \to S/mS \to 0.
\]

By assumption $S/mS$ is the residue field $\lambda$ of $S$. Therefore, $\text{Tor}_\lambda^S (T, \lambda)$ can be calculated as the homology of the complex

\[
\ldots T^{b_2} \to T^{b_1} \to T \to T/mT \to 0
\]

obtained from (3) by the base change $S \to T$. However, (4) can also be obtained by tensoring (2) over $R$ with $T$. Since $T$ is flat over $R$, the sequence (4) is exact, whence, in particular, $\text{Tor}_\lambda^S (T, \lambda) = 0$. By the local flatness criterion, $T$ is flat over $S$. Since $n = mS$ and $p = mT$, we get $p = nT$, showing that $g$, too, is a scalar extension. \qed

Three important examples of scalar extensions are given by the following proposition.

3.2. Proposition. Let $R$ be a Noetherian local ring.

(1) The natural map $R \to \hat{R}$ is a scalar extension.

(2) Any etale map is a scalar extension.

(3) The natural map $R \to R^\sharp$ is a scalar extension, where $R^\sharp$ is any catapower of $R$.
Proof. The first two assertions are well-known, so remains to show the last. Let \( m \) be the maximal ideal of \( R \). It is easy to show that \( mR_\lambda \) is the maximal ideal of \( R_\lambda \). So remains to prove that \( R \to R_\lambda \) is flat. Since \( R_\lambda \) is complete, and in fact equal to the catapower of \( \hat{R} \), we may assume without loss of generality that \( R \) is already complete. In particular, \( R \) is a homomorphic image of a regular local ring and if we prove the corresponding result for this regular local ring, then base change yields the desired result by Lemma 2.1. Therefore, we may moreover assume that \( R \) is regular. Since \( mR_\lambda \) is the maximal ideal of \( R_\lambda \) and since \( R_\lambda \) is also regular by [21, Corollary 5.15], of the same dimension as \( R \), the flatness of \( R \to R_\lambda \) then follows from [13, Theorem 23.1].

In fact, 3.2(2) has the following converse: if \( R \to S \) is essentially of finite type inducing a finite separable extension on the residue fields, then \( R \to S \) is a scalar extension if and only if it is etale. In this sense, scalar extensions are generalizations of etale maps (whence the alternative terminology ‘formally etale’ for them). This shows already that a scalar extension of complete Noetherian local rings inducing an isomorphism on their residue fields is itself an isomorphism; see [13, Theorem 8.4]. Hence it is of interest to generate scalar extensions \( R \to S \) with \( S \) complete. We will see that there exists a canonical choice over any field.

**Completions along a residual extension.** Let \((R, m)\) be a Noetherian local ring with residue field \( \kappa \), and let \( \lambda \) be a field extension of \( \kappa \). The completion of \( R \) along \( \lambda \) is the (unique) local \( R \)-algebra \( \hat{R}^\lambda \) solving the following universal problem: given an arbitrary Noetherian local \( R \)-algebra \( S \) with residue field \( \lambda \), if \( S \) is complete, then there exists a unique local \( R \)-algebra homomorphism \( \hat{R}^\lambda \to S \). When \( \kappa = \lambda \), we recover the usual completion \( \hat{R}^\kappa = \hat{R} \) of \( R \). Here and elsewhere, we say that there is a unique homomorphism with certain properties, when we actually mean that there exists a unique homomorphism up to isomorphism; this is consistent with our practice of identifying two local rings when they are isomorphic.

**Proof of the existence of a completion along \( \lambda \).** We have to treat the equal and mixed characteristic cases separately. Assume first that \( R \) has equal characteristic (this case is also discussed in [10, (6.3)]). By Cohen’s structure theorems, there exists an embedding \( \kappa \to \hat{R} \). Let \( \hat{R}^\lambda \) be the \( m(\hat{R} \otimes_\kappa \lambda) \)-adic completion of \( \hat{R} \otimes_\kappa \lambda \). To see that this is a completion along \( \lambda \), let \( S \) be a Noetherian local \( R \)-algebra with residue field \( \lambda \) and assume \( S \) is complete. By the universal property of ordinary completions, we get a unique homomorphism \( \hat{R} \to S \). Since \( S \) is complete, we can find an embedding \( \lambda \to S \) which agrees on the subfield \( \kappa \) of \( \lambda \) with the composition \( \kappa \to \hat{R} \to S \). By the universal property of tensor products, the two maps \( \hat{R} \to S \) and \( \lambda \to S \) combine to a unique local \( R \)-algebra homomorphism \( \hat{R} \otimes_\kappa \lambda \to S \), and using once more the universal property of completion, this then yields a unique \( R \)-algebra homomorphism \( \hat{R}^\lambda \to S \).

In the mixed characteristic case, coefficient fields no longer exist and we now proceed as follows. Let \( V \) be the (unique) complete \( p \)-ring over \( \kappa \), where \( p \) is the characteristic of \( \kappa \). We first define the completion of \( V \) along \( \lambda \), that is to say, \( \hat{V}^\lambda \), as the unique complete \( p \)-ring over \( \lambda \). That the latter satisfies the universal property of a completion along \( \lambda \) is proven in [13, Theorem 29.2]. To define \( \hat{R}^\lambda \), let \( S \) be any Noetherian local \( R \)-algebra with residue field \( \lambda \) extending \( \kappa \), and assume \( S \) is complete. As before, we have a unique
local $R$-algebra homomorphism $\hat{R} \to S$. By Cohen's structure theorems, there exists a commutative diagram of local homomorphisms

$$
\begin{array}{ccc}
V & \rightarrow & \hat{V}^\lambda \\
\downarrow & & \downarrow \\
\hat{R} & \rightarrow & S.
\end{array}
$$

By the universal property of tensor products, we get a unique $R$-algebra homomorphism $\hat{R} \otimes_V \hat{V}^\lambda \to S$. Define $\hat{R}^\lambda$ now as the $(\hat{R} \otimes_V \hat{V}^\lambda)$-adic completion of $\hat{R} \otimes_V \hat{V}^\lambda$, so that we get a unique local $R$-algebra homomorphism $\hat{R}^\lambda \to S$, as required. \hfill $\square$

### 3.3. Corollary

For every Noetherian local ring $R$ and every extension field $\lambda$ of its residue field, $\hat{R}^\lambda$, the completion of $R$ along $\lambda$, exists and is unique. For every ideal $I$ in $R$, the completion of $R/I$ along $\lambda$ is equal to $\hat{R}^\lambda/I\hat{R}^\lambda$.

Moreover, the natural map $R \to \hat{R}^\lambda$ is a scalar extension over $\lambda$.

**Proof.** Existence was proven above; uniqueness then follows formally from being a solution to a universal problem. To prove the second assertion, assume $R/I \to S$ is a local homomorphism with $S$ a complete Noetherian local ring with residue field $\lambda$. The composition $R \to R/I \to S$ yields by definition a unique local $R$-algebra homomorphism $\hat{R}^\lambda \to S$. Since $IS = 0$, the latter homomorphism factors through $\hat{R}^\lambda/I\hat{R}^\lambda$, showing that $\hat{R}^\lambda/I\hat{R}^\lambda$ satisfies the universal property of completions along $\lambda$. As for the last assertion, in the equal characteristic case, the base change $\hat{R} \to \hat{R} \otimes_\kappa \lambda$ of $\kappa \subseteq \lambda$ is faithfully flat. Since completion is exact, each map in

$$
R \to \hat{R} \to \hat{R} \otimes_\kappa \lambda \to \hat{R}^\lambda
$$

is faithfully flat, whence so is their composition. In the mixed characteristic case, $\hat{V}^\lambda$ is torsion-free whence flat over $V$. Hence by the same argument as in the equal characteristic case, the composite map

$$
R \to \hat{R} \to \hat{R} \otimes_V \hat{V}^\lambda \to \hat{R}^\lambda
$$

is faithfully flat. By our second assertion, $\hat{R}^\lambda/m\hat{R}^\lambda$ is the completion of $R/m \cong \kappa$ along $\lambda$ in either characteristic. In other words, $\hat{R}^\lambda/m\hat{R}^\lambda \cong \lambda$ and hence in particular, $m\hat{R}^\lambda$ is the maximal ideal of $\hat{R}^\lambda$. This proves that $R \to \hat{R}^\lambda$ is a scalar extension. \hfill $\square$

### 3.4. Proposition

Let $R \to S$ be a scalar extension over $\lambda$. If $S$ is complete, then $S \cong \hat{R}^\lambda$.

**Proof.** By the universal property, we have a local $R$-algebra homomorphism $\hat{R}^\lambda \to S$. It follows from [13, Theorem 8.4] that $\hat{R}^\lambda \to S$ is surjective. Since $\hat{R} \to \hat{R}^\lambda$ and $R \to S$ are scalar extensions by Corollary 3.3 and by assumption respectively, $\hat{R}^\lambda \to S$ is faithfully flat by Proposition 3.1, whence injective. \hfill $\square$

### 3.5. Corollary ((Lifting of scalar extensions))

Let $R \to S$ be a scalar extension with $S$ complete. If $R$ is the homomorphic image of a Noetherian local ring $A$, then there exists a scalar extension $A \to B$ whose base change is $R \to S$, that is to say, $S = B \otimes_A R$. 
Proof. We leave it to the reader to verify that, after taking completions, we may assume that also $A$ and $R$ are complete. By Cohen’s structure theorems, $A$ and $R$ are the homomorphic images of $V[[X]]$ modulo some ideals $J \subseteq I$ respectively, where $V$ is either their common residue field or otherwise a complete $p$-ring over that residue field, and where $X$ is a finite tuple of indeterminates. Moreover, $S \cong \hat{R}^\lambda$ by Proposition 3.4, where $\lambda$ is the residue field of $S$. In particular, $S \cong \hat{V}^\lambda[[X]]/\hat{J}\hat{V}^\lambda[[X]]$. Hence putting $B := \hat{V}^\lambda[[X]]/\hat{J}\hat{V}^\lambda[[X]]$ yields a scalar extension $A \to B$ with $A/IA = R \to B/IB = S$, as required.

The following result is a sharpening of [14, Theorem 2.4].

3.6. Corollary. Let $R$ be a Noetherian local ring with residue field $\kappa$. If $\kappa^\sharp$ is the ultrapower of $\kappa$, then the catapower $R^\sharp$ of $R$ is equal to the completion $\hat{R}^{\kappa^\sharp}$ along $\kappa^\sharp$.

Proof. By Lemma 2.1, the residue field of $R^\sharp$ is $\kappa^\sharp$. Since $R \to R^\sharp$ is a scalar extension by 3.23, and since $R^\sharp$ is complete, $R \cong \hat{R}^{\kappa^\sharp}$ by Proposition 3.4.

3.7. Corollary. Let $R \to S$ be a finite local homomorphism inducing a trivial extension on the residue fields. For every extension $\lambda$ of this common residue field, $\hat{S}^\lambda \cong \hat{R}^\lambda \otimes_R S$.

Proof. The base change $S \to \hat{R}^\lambda \otimes_R S$ is faithfully flat. Let $m$ and $n$ be the maximal ideals of $R$ and $S$ respectively. Since

$$((\hat{R}^\lambda \otimes_R S)/n(\hat{R}^\lambda \otimes_R S)) \cong ((\hat{R}^\lambda/m\hat{R}^\lambda) \otimes_{R/m}(S/n) \cong \lambda \otimes_{\kappa} \kappa = \lambda$$

the ideal $n(\hat{R}^\lambda \otimes_R S)$ is a maximal ideal. Since the base change $\hat{R}^\lambda \to \hat{R}^\lambda \otimes_R S$ is finite with trivial residue field extension and since $\hat{R}^\lambda$ is complete whence Henselian, $\hat{R}^\lambda \otimes_R S$ is a complete local ring. Hence we showed that $S \to \hat{R}^\lambda \otimes_R S$ is a scalar extension and since the latter ring is complete with residue field equal to $\lambda$, it is isomorphic to $\hat{S}^\lambda$ by Proposition 3.4.

3.8. Corollary. Suppose $R$ is an excellent local ring. If $R \to S$ is a scalar extension inducing a separable extension on the residue fields, then $R \to S$ is a regular homomorphism.

Proof. By [13, Theorem 28.10], the scalar extension $R \to S$ is formally smooth, since it is unramified and the residue field extension is separable. The assertion now follows from a result by André in [1] (see also [13, p. 260]).

In fact, with aid of Proposition 3.4, Corollary 3.5 and Cohen’s structure theorems, one reduces to proving that $V[[X]] \to \hat{V}^\lambda[[X]]$ is regular, where $V$ is either a field or a complete $p$-ring, and where $\lambda$ is a separable extension of the residue field of $V$. This approach circumvents the use of André’s deep result.

3.9. Definition. A Noetherian local ring $R$ is called analytically irreducible, if $\hat{R}$ is a domain; it is called absolutely analytically irreducible, if $\hat{R}^{\text{alg}}$ is a domain, where $\kappa^{\text{alg}}$ is the algebraic closure of the residue field $\kappa$ of $R$; and it is called universally irreducible, if any scalar extension of $R$ is a domain.

3.10. Corollary. If $R$ is an excellent normal local domain with perfect residue field, then $R$ is universally irreducible.

Proof. Let $S$ be a scalar extension of $R$. By Corollary 3.8, the map $R \to S$ is regular and hence $S$ is again normal by [13, Theorem 32.2], whence a domain.
3.11. **Proposition.** A Noetherian local ring is absolutely analytically irreducible if and only if it is universally irreducible.

*Proof.* Since we will make no essential use of this result, we only give a sketch of a proof. One direction is obvious. For the other, we may reduce to the case that $R$ is a complete Noetherian local domain with algebraically closed residue field $k$. We need to show that $\hat{R}$ is a domain, where $\lambda$ is an arbitrary extension field of $k$. By Cohen’s structure theorems, there exists a finite extension $S := V[[X]] \subseteq R$, where $V$ is either $k$ or the complete $p$-ring over $k$, and $X$ is a tuple of indeterminates. Write $R = S[Y]/p$ for some finite tuple of indeterminates $Y$, so that $p$ is in particular a prime ideal. Since the fraction field of $\hat{S} = \hat{V}[[X]]$ is a regular extension of the fraction field of $S = V[[X]]$, the same argument as in the proof of [2, Lemma 5.21] then shows that $p\hat{S}^\lambda[Y]$ is a prime ideal. Hence we are done, since $\hat{R}^\lambda = \hat{S}^\lambda[Y]/p\hat{S}^\lambda[Y]$ by Corollary 3.7. □

We are ready to formulate a flatness criterion generalizing [12, Theorem 8]; we prove a slightly stronger version than the one quoted in the introduction.

3.12. **Theorem.** Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume $R$ is universally irreducible, e.g., an excellent normal local domain with perfect residue field, or a complete local domain with algebraically closed residue field. If $\hat{R} \to S$ is unramified and $\dim(R) = \dim(S)$, then $R \to S$ is faithfully flat, whence a scalar extension.

*Proof.* Recall that $(R, m) \to (S, n)$ being unramified means that $n = mS$. It suffices to prove the assertion under the additional assumption that both $R$ and $S$ are complete. Indeed, if $R \to S$ is arbitrary, then $\hat{R} \to \hat{S}$ satisfies again the hypotheses of the theorem and therefore would be faithfully flat. By an easy descent argument, $R \to S$ is then also faithfully flat.

So assume $R$ and $S$ are complete and let $\lambda$ be the residue field of $S$. By assumption, $\hat{R}^\lambda$ is a domain, of the same dimension as $R$. By the universal property of the completion along $\lambda$, we get a local $R$-algebra homomorphism $\hat{R}^\lambda \to S$. By [13, Theorem 8.4], this homomorphism is surjective. It is also injective, since $\hat{R}^\lambda$ and $S$ have the same dimension and $\hat{R}^\lambda$ is a domain. Hence $\hat{R}^\lambda \cong S$, so that $R \to S$ is a scalar extension. □

4. **Similarity relation**

Next, we introduce an equivalence relation on the class of Noetherian local rings which, although coarser than the isomorphism relation, preserves most local singularity properties (see for instance Theorem 4.1 below). Namely, we say that two Noetherian local rings $R$ and $S$ are *similar*, denoted $R \approx S$, if they admit a common scalar extension. Let $T$ be this common scalar extension. Its completion is again a scalar extension and by Proposition 3.4, it is therefore isomorphic to both $\hat{R}^\lambda$ and $\hat{S}^\lambda$, where $\lambda$ is the residue field of $T$. In other words, we showed that $R \approx S$ if and only if $\hat{R}^\lambda \cong \hat{S}^\lambda$ for some sufficiently large common extension $\lambda$ of their respective residue fields. It follows easily from this that $\approx$ is an equivalence relation. The collection of all local rings similar to a given Noetherian local ring $R$ is called the *similarity class* of $R$ and is denoted $[R]$. Immediately from the results in [13, §23] and [18, Proposition 9.3] (where the notion of a *singularity defect* is introduced), we get:

4.1. **Theorem.** If two Noetherian local rings are similar, then they have the same dimension, depth and Hilbert series, and one is regular (respectively, Cohen-Macaulay, Gorenstein, complete intersection) if and only if the other is. More generally, any two similar local rings have the same singularity defects. □
Using Corollary 3.8, other properties, such as being reduced or normal, are also invariant under the similarity relation, provided the rings are excellent with perfect residue field. Note that being a domain is not preserved under the similarity relation, necessitating definitions 3.9.

4.2. **Proposition.** Any two catapowers of a Noetherian local ring, or more generally, any two Noetherian local rings which are elementary equivalent, are similar.

More generally, let $R_w$ and $S_w$ be sequences of Noetherian local rings of embedding dimension at most $d$. If almost each $R_w$ is similar to $S_w$, then the respective cataproducts $R_\#$ and $S_\#$ are also similar.

**Proof.** Suppose $R$ and $S$ are elementary equivalent Noetherian local rings. By the Keisler-Shelah theorem (see [11, Theorem 9.5.7]), some ultrapower of $R$ and $S$ are isomorphic, whence so are their corresponding catapowers (strictly speaking, the underlying index set will in general no longer be countable, so that we have to make some minor modifications alluded to in footnote (1); details are left to the reader). By Proposition 3.2, these are scalar extensions of $R$ and $S$, respectively, proving the first assertion.

To prove the second assertion, we may without loss of generality assume that all rings are complete. By our discussion above, we may further reduce to the case that $S_w$ is a scalar extension of $R_w$. Since $R_w$ is a homomorphic image of a $d$-dimensional regular local ring by Cohen’s structure theorems, and since the property we seek to prove is preserved under homomorphic images by Lemma 2.1 and Corollary 3.3, we may moreover assume by Corollary 3.5 that each $R_w$ is regular, of dimension $d$. By Theorem 4.1, almost all $S_w$ are then also regular of dimension $d$. By [21, Corollary 5.15], the cataproducts $R_\#$ and $S_\#$ are therefore again $d$-dimensional regular local rings. The induced homomorphism $R_\# \to S_\#$ is unramified by Lemma 2.1. Hence, it is faithfully flat by [13, Theorem 23.1], whence a scalar extension, as we wanted to show. □

We denote the collection of all similarity classes of Noetherian local rings by $\mathcal{S}_{\text{fin}}$. Although the class of Noetherian local rings is not a set, we do no longer have this complication for its quotient:

4.3. **Proposition.** The quotient $\mathcal{S}_{\text{fin}}$ is a set.

**Proof.** Let $[R]$ be a similarity class and let $\kappa$ be the residue field of $R$. Since $R \approx \hat{R}$, we may assume that $R$ is complete, whence, by Cohen’s structure theorems, the homomorphic image of $S := V[[X]]$ with $V$ either equal to $\kappa$ or to the complete $p$-ring over $\kappa$, and with $X$ a finite tuple of indeterminates. Suppose $R = S/I$ with $I = (f_1, \ldots, f_s)S$. We may choose a subring $W$ of $V$ of size at most the continuum so that it contains all coefficients of the $f_i$ and so that $W$ is again a field or a complete $p$-ring. Let $T := W[[X]]$ and $J := (f_1, \ldots, f_s)T$, so that $S \cong \hat{T}_\kappa$ and $I = JS$. Hence, by base change, $R \cong S/I$ is a scalar extension of $T/J$, showing that $T/J \approx R$. In conclusion, we showed that every similarity class contains a ring of size at most the continuum, and therefore $\mathcal{S}_{\text{fin}}$ is a set. □

5. **JET METRIC**

Our next goal is to define a metric on the space $\mathcal{S}_{\text{fin}}$. We will first define a semi-metric on the space of all Noetherian local rings.
Jet semi-metric. Let \((R, m)\) be a Noetherian local ring. The \(n\)-th jet of \(R\) (also called the \(n\)-th infinitesimal neighborhood) is by definition the (Artinian) residue ring \(R/m^n\) and will be denoted \(J^nR\). Recall that the \((m\)-adic) completion \(\hat{R}\) of \(R\) is the inverse limit of all \(n\)-th jets of \(R\), and that \(J^nR \cong \hat{R}/J^n\hat{R}\). We define a semi-metric on the class of all Noetherian local rings, called the jet metric, as follows. Given two Noetherian local rings \(R\) and \(S\), let \(d(R, S)\) be the infimum of the numbers \(2^{-n}\) for which \(J^nR \cong J^nS\). In words, the distance between two local rings is at most \(2^{-n}\) if their \(n\)-th jets agree. One easily verifies that this distance function satisfies all the axioms of a metric, except that two distinct elements can be at distance zero, so that \(d(\cdot, \cdot)\) is only a semi-metric. It is an interesting problem to determine all local rings that are \(d\)-equivalent to a given local ring; a partial answer is provided in [23]. It is clear that any two Noetherian local rings with the same completion have this property. For our purposes, the following partial solution to this question suffices:

5.1. Proposition. Given two Noetherian local rings \(R\) and \(S\), if \(R \sim_d S\), that is to say, if \(d(R, S) = 0\), then \(R \approx S\).

Proof. By definition, there exists for each \(n\) an Artinian local ring \(T_n\) isomorphic to both \(J^nR\) and \(J^nS\). Let \(R_T\) and \(R_S\) be the respective ultrapower and catapower of \(R\), and let \(T_T\) and \(T_S\) be the respective ultraproduct and cataproduct of the \(T_n\). Taking ultraproducts of the surjections \(R \to T_n\) yields a surjection \(\hat{R}_T \to \hat{T}_T\) whence a surjection \(\hat{R}_S \to \hat{T}_S\). Let \(r \in \hat{R}_T\) be an element whose image in \(R_T\) lies in the kernel of \(\hat{R}_S \to \hat{T}_S\), that is to say, \(r \in \text{Inf}(\hat{T}_S)\). Let \(r_n\) be elements in \(R\) with ultraproduct equal to \(r\). Fix some \(N\). Since \(r \in m^NT_T\), Łos’ Theorem yields \(r_n \in m^NT_n\) for almost all \(n\). For \(n \geq N\), this implies \(r_n \in m^N\) and hence by Łos’ Theorem, \(r \in m^N\hat{R}_T\). Since this holds for all \(N\), the image of \(r\) in \(R_T\) is zero, showing that \(R_T \to \hat{T}_S\) is an isomorphism. Applying the same argument to the cataproduct \(S_T\) of \(S\), we also get \(S_T \approx \hat{T}_S\) and hence \(R_T \approx S_T\). Therefore, \(R \approx S\) by Proposition 3.2(3).

The jet semi-metric is non-archimedean, and hence the induced topology, called the jet topology, is totally disconnected. By convention, the zero-th jet of a ring is zero (since we think of \(m^0\) as the unit ideal). It follows that the distance between any two local rings is at most one, that is to say, \(d\) is bounded. Immediately from the definitions we also get:

5.2. Lemma. If \(d(R, S) < 1\), then \(R\) and \(S\) have the same residue field; if \(d(R, S) < 1/2\), then \(R\) and \(S\) have the same embedding dimension. \(\square\)

In particular, if, in this metric, \(R_w\) is a Cauchy sequence of Noetherian local rings, then almost of all \(R_w\) have the same residue field, called the residue field of the sequence, and the same embedding dimension. By the above discussion, the cataproduct \(\hat{R}_T\) is therefore a complete Noetherian local ring. By Lemma 5.2, the embedding dimension is a continuous map onto the discrete space \(\mathcal{Z}\). This is no longer true for dimension: for instance \(R := k[[X]]\) and \(\hat{R}_n := R/X^nR\) lie at distance \(2^{-n}\), yet their dimensions are not the same. One can show, however, that dimension is upper-semicontinuous.

By an \((open)\) ball \(B\) with \((center)\) \(R\) and \((radius)\) \(0 < \delta \leq 1\), we mean the collection of all Noetherian local rings \(S\) such that \(d(R, S) < \delta\). Since the metric is non-archimedean, any member of a ball is its center and every ball is both open and closed in the jet topology, that is to say, is a clopen. Because the distance function only takes discrete values (the powers of \(1/2\)), any two radii which lie between two consecutive powers of \(1/2\) yield the same ball. Therefore, by the \((radius)\) of a ball \(B\), we mean twice the largest distance between two
members of \( \mathbb{B} \); this is always a power of 1/2. (We need to take twice the distance since we used a strict inequality in the definition of a ball.)

A unit ball is a ball \( \mathbb{B} \) with radius 1 and hence consists of all local rings with the same residue field. We call this common residue field the residue field of \( \mathbb{B} \). This gives a one-one correspondence between unit balls and fields. More generally, to every ball \( \mathbb{B} \), we associate an Artinian local ring \( R_\mathbb{B} \), called the residue ring of \( \mathbb{B} \), given as the unique local ring such that \( J^n R \cong R_\mathbb{B} \), for all \( R \in \mathbb{B} \), where \( 2^{-n+1} \) is the radius of \( \mathbb{B} \). Note that \( R_\mathbb{B} \) is a center of \( \mathbb{B} \) and, moreover, the radius of \( \mathbb{B} \) is determined by \( R_\mathbb{B} \): it is equal to \( 2^{-n+1} \) where \( n \) is the nilpotency index of \( R_\mathbb{B} \). In conclusion, there is a one-one correspondence between balls \( \mathbb{B} \) and Artinian local rings.

5.3. Proposition. Every ball is a set.

Proof. It suffices to prove this for a unit ball \( \mathbb{B} \). The result will follow if we can show that there is a cardinal number so that every member of \( \mathbb{B} \) has size at most this cardinal. Let \( \kappa \) be the residue field of \( \mathbb{B} \) and let \( R \in \mathbb{B} \). Since the cardinality of a Noetherian local ring is at most the cardinality of its completion, we may assume that \( R \) is complete. By Cohen’s structure theorems, \( R \) is a homomorphic image of \( V[[X]] \), where \( X \) is a finite tuple of indeterminates and \( V \) is equal to \( \kappa \) in the equal characteristic case, and equal to the complete \( p \)-ring over \( \kappa \) in the mixed characteristic case. It is clear that in either case, the cardinality of \( V[[X]] \) is bounded in terms of the cardinality of \( \kappa \), whence so is its homomorphic image \( R \).

Note that each ball \( \mathbb{B} \) is infinite: if \( R_\mathbb{B} \) is its residue ring, then the latter is of the form \( S/I \), where \( (S, n) \) is a power series ring \( V[[X]] \). If \( n \) is the nilpotency index of \( R_\mathbb{B} \), then \( S/J \in \mathbb{B} \) for any ideal \( J \subseteq S \) such that \( J + n^n = I \).

5.4. Corollary. Let \( \kappa \subseteq \lambda \) be an extension of fields and let \( \mathbb{B}_\kappa \) and \( \mathbb{B}_\lambda \) be the unique unit balls with residue field \( \kappa \) and \( \lambda \), respectively. The map sending a ring in \( \mathbb{B}_\kappa \) to its completion along \( \lambda \) is an isometry \( \mathbb{B}_\kappa \to \mathbb{B}_\lambda \).

Proof. Take \( R, S \in \mathbb{B}_\kappa \). Clearly, the completions \( \hat{R}^\lambda \) and \( \hat{S}^\lambda \) along \( \lambda \) belong both to \( \mathbb{B}_\lambda \). Suppose \( d(R, S) \leq 2^{-n} \), that is to say, their \( n \)-th jets \( J^n R \) and \( J^n S \) are isomorphic. By Corollary 3.3, the completions of \( J^n R \) and \( J^n S \) along \( \lambda \) are respectively \( J^n \hat{R}^\lambda \) and \( J^n \hat{S}^\lambda \), and therefore are isomorphic, showing that \( d(\hat{R}^\lambda, \hat{S}^\lambda) \leq 2^{-n} \).

5.5. Proposition. If \( r \) and \( s \) are Cauchy sequences of Noetherian local rings, say, \( r(w) := R_w \) and \( s(w) := S_w \), with respective cataproducts \( R_\mathbb{L} \) and \( S_\mathbb{L} \), then \( d(R_\mathbb{L}, S_\mathbb{L}) \leq d(r, s) \). In particular, if \( r \sim s \), then \( R_\mathbb{L} \approx S_\mathbb{L} \).

Proof. The last assertion is immediate by the first and Proposition 5.1. Suppose \( d(r, s) \leq 2^{-n} \). This means that for some \( w_0 \) and all \( w > w_0 \), we have \( J^n R_w \cong J^n S_w \). By Lemma 2.1, the \( n \)-th jets \( J^n R_\mathbb{L} \) and \( J^n S_\mathbb{L} \) are isomorphic, showing that \( d(R_\mathbb{L}, S_\mathbb{L}) \leq 2^{-n} \).

The next result shows that cataproducts act as limits up to similarity. To formulate it, we extend our previous notation: let \( r \) be a sequence of Noetherian local rings with the same residue field \( \kappa \) (e.g., a Cauchy sequence) and let \( \lambda \) be an extension field of \( \kappa \). Then we let \( \hat{R}_\mathbb{L} \) denote the sequence of rings obtained by taking the completions along \( \lambda \) of all members of \( r \), that is to say, \( \hat{R}_\mathbb{L}(w) := \hat{R}_w \), if \( r(w) = R_w \).
5.6. Theorem. Let \( r \) be a Cauchy sequence of Noetherian local rings with residue field \( \kappa \). Let \( \lambda \) be any extension field of the ultrapower \( \kappa_\mu \) of \( \kappa \). Then \( \hat{R}^\lambda \) is a Cauchy sequence converging to \( \hat{R}^\lambda_\mu \). In particular, \( R_\mu \) is a limit of \( \hat{R}^{\kappa_\mu} \).

Proof. Let \( R_w := r(w) \). Fix \( n \) and choose \( w(n) \) so that all \( J^n R_w \) for \( w \geq w(n) \) are isomorphic, say, to \( T \). By Lemma 2.1, the \( n \)-th jet \( J^n R_\mu \) is isomorphic to the catapower \( T_\mu \); the latter is isomorphic to \( \hat{T}^{\kappa_\mu} \) by Corollary 3.6; and this in turn is isomorphic to \( J^n(\hat{R}_w^{\kappa_\mu}) \), for all \( w \geq w(n) \) by Corollary 3.3. In summary, we showed that

\[
\text{d}(\hat{R}_w^{\kappa_\mu} , R_\mu) \leq 2^{-n},
\]

for all \( w \geq w(n) \). By Corollary 5.4, taking completions along \( \lambda \) yields \( \text{d}(\hat{R}_w^{\kappa_\mu} , \hat{R}_\mu^{\kappa_\mu}) \leq 2^{-n} \), for all \( w \geq w(n) \). Since this holds for all \( n \), the assertion follows. \( \square \)

6. Similarity space

We are ready to define a metric on the similarity space \( \mathbb{S}_{\text{fin}} \). For two similarity classes \([R] \) and \([S] \), let \( \text{d}([R],[S]) \) be equal to the infimum of all \( \text{d}(R',S') \) with \( R' \approx R \) and \( S' \approx S \). Alternatively, recall that for a semi-metric space \((\Sigma, d)\), the distance \( d(U,V) \) between two subclasses \( U \) and \( V \) is defined to be the infimum of all \( d(x,y) \) with \( x \in U \) and \( y \in V \); hence \( \text{d}([R],[S]) \) is just the distance between \([R]\) and \([S]\) viewed as subclasses.

6.1. Lemma. For any two Noetherian local rings \( R \) and \( S \) and for any \( n \in \mathbb{N} \), we have

\[
\text{d}([R],[S]) \leq 2^{-n} \text{ if and only if } J^n R \approx J^n S.
\]

Proof. Suppose \( \text{d}([R],[S]) \leq 2^{-n} \) and choose \( R' \approx R \) and \( S' \approx S \) so that \( \text{d}(R',S') \leq 2^{-n} \). In other words, \( J^n R' \cong J^n S' \) and therefore, \( J^n R \cong J^n S \) by Corollary 3.3. Conversely, assume \( J^n R \cong J^n S \) and let \( T \) be a common scalar extension of \( J^n R \) and \( J^n S \). Let \( \lambda \) be the residue field of \( T \). By Corollary 3.3, the \( n \)-th jets of \( \hat{R}^\lambda \) and \( \hat{S}^\lambda \) are equal to \( T \). In other words, \( \text{d}(\hat{R}^\lambda , \hat{S}^\lambda) \leq 2^{-n} \). Since \( \text{d}([R],[S]) \) is defined as an infimum, it is at most \( 2^{-n} \). \( \square \)

6.2. Corollary. The quotient \( (\mathbb{S}_{\text{fin}}, d) \) is a metric space.

Proof. Suppose \( \text{d}([R],[S]) = 0 \). By Lemma 6.1, the \( n \)-th jets \( J^n R \) and \( J^n S \) of \( R \) and \( S \) are similar for all \( n \). Hence there exists a common scalar extension \( T_n \) of \( J^n R \) and \( J^n S \). We may inductively choose \( T_{n+1} \) to have a residue field containing the residue field of \( T_n \) by Corollary 5.4, since scalar extensions can only make the distance smaller. Let \( \lambda \) be the union of all these residue fields. By Corollary 3.3, the \( n \)-th jets of \( \hat{R}^\lambda \) and \( \hat{S}^\lambda \) are equal to \( \hat{T}_n^\lambda \). Since this holds for all \( n \), we showed that \( \text{d}(\hat{R}^\lambda , \hat{S}^\lambda) = 0 \). By Proposition 5.1, we get \( \hat{R}^\lambda \approx \hat{S}^\lambda \) and hence \([R] = [\hat{R}^\lambda] = [\hat{S}^\lambda] = [S]\). \( \square \)

It follows from Theorem 5.6 that given a Cauchy sequence \( r \) of Noetherian local rings, the sequence \( \hat{P}^{\kappa_\mu} \) has a limit, where \( \kappa_\mu \) is the ultrapower of the residue field of \( r \). Since the corresponding members of \( r \) and \( \hat{P}^{\kappa_\mu} \) are similar, we showed that every Cauchy sequence becomes convergent after replacing each of its components by an appropriately chosen similar ring. Therefore, the next result should not come as a surprise:

6.3. Theorem. The metric space \( \mathbb{S}_{\text{fin}} \) is complete.
Proof. We will define an isometry \( \hat{\iota} : \mathcal{S}\text{Lim} \rightarrow \mathcal{S}\text{Lim} \) as follows. We start with defining a map \( \iota : \text{Cat}(\mathcal{S}\text{Lim}) \rightarrow \mathcal{S}\text{Lim} \). Let \( \mathbf{r} \) be a Cauchy sequence in \( \mathcal{S}\text{Lim} \). For each \( w \), let \( R_w \) be a representative in the similarity class \( r(w) \), and let \( S_\mathbf{r} \) be their cataproduct. Note that \( S_\mathbf{r} \) is a complete Noetherian local ring, since almost all \( R_w \) consists of finitely many residue classes and \( R_w \) has the same embedding dimension. Thus we can define \( \iota(\mathbf{r}) := [S_\mathbf{r}] \). By Proposition 4.2, the map \( \iota \) is well-defined, that is to say, does not depend on the choice of representatives \( R_w \). Suppose \( \mathbf{s} \) is a second Cauchy sequence which is equivalent to \( \mathbf{r} \) and let \( S_\mathbf{s} \) be the cataproduct of the representatives \( S_w \) of each \( s(w) \). For a fixed \( n \), we have \( d([R_w], [S_w]) \leq 2^{-n} \) for all sufficiently large \( w \). By Lemma 6.1, the \( n \)-th jets of \( R_w \) and \( S_w \) are therefore similar, for all sufficiently large \( w \). By Proposition 4.2, then so also are the \( n \)-th jets of \( S_\mathbf{r} \) and \( S_\mathbf{s} \). Therefore, \( d([R_w], [S_w]) \leq 2^{-n} \) by another application of Lemma 6.1. Since this holds for all \( n \), Corollary 6.2 yields \([R_\mathbf{r}] = [S_\mathbf{s}]\). By definition of completion, \( \iota \) therefore factors through a map \( \hat{\iota} : \mathcal{S}\text{Lim} \rightarrow \mathcal{S}\text{Lim} \).

We leave it to the reader to check that \( \hat{\iota} \) preserves the metric. Note that \( \hat{\iota} \) restricted to \( \mathcal{S}\text{Lim} \) is the identity, since a catapower is a scalar extension by Proposition 4.2. Hence \( \hat{\iota} \) must be surjective. To prove injectivity, assume \( \mathbf{r} \) and \( \mathbf{s} \) are Cauchy sequences of Noetherian local rings whose respective cataproducts \( S_\mathbf{r} \) and \( S_\mathbf{s} \) are similar. Let \( \lambda \) be a large enough field extension so that

\[
\tilde{R}_\mathbf{s}^\lambda \cong \tilde{S}_\mathbf{s}^\lambda.
\]

By Theorem 5.6, the (component-wise) completion \( \tilde{R}_\mathbf{r}^\lambda \) along \( \lambda \) converges to \( \tilde{R}_\mathbf{s}^\lambda \), and likewise \( \tilde{S}_\mathbf{r}^\lambda \) converges to \( \tilde{S}_\mathbf{s}^\lambda \). Therefore, \( \tilde{R}_\mathbf{r}^\lambda \) and \( \tilde{S}_\mathbf{r}^\lambda \), as they converge to the same limit, are equivalent, which proves that \( \hat{\iota} \) is injective. 

We have the following generalization of Proposition 4.2.

**6.4. Corollary.** If \( R_w \) is a Cauchy sequence, then any two cataproducts of \( R_w \) (with respect to different ultrafilters) are similar. In particular, if the common residue field \( \kappa \) of the \( R_w \) is an algebraically closed field, then the cataproduct of the \( R_w \) is, up to isomorphism, independent from the choice of ultrafilter.

**Proof.** According to the proof of Theorem 6.3, the similarity class of any cataproduct \( R_\mathbf{r} \) of the \( R_w \) is a limit of the sequence of similarity classes \([R_w]\), and therefore, is unique by Corollary 6.2.

Since any two ultrapowers of \( \kappa \) are algebraically closed and have the same (uncountable) cardinality, they are isomorphic by Leibniz’s theorem. Since any two cataproducts of the \( R_w \) are similar by the first assertion, and are complete with isomorphic residue fields, they must be isomorphic by Proposition 3.4. 

We introduce the following notation. Let \( \mathcal{S} \subseteq \mathcal{S}\text{Lim} \) be a subset, and let \( d \geq 0 \) and \( e \geq 1 \). We let \( \mathcal{S}_d \) (respectively, \( \mathcal{S}_{d,e} \)) be the set of similarity classes of Noetherian local rings in \( \mathcal{S} \) having dimension \( d \) (and parameter degree \( e \)). Recall that the parameter degree of \( \mathcal{R} \) is the minimal length of a residue ring \( R/I \), where \( I \) runs over all parameter ideals of \( R \). It is not hard to show that two similar rings with infinite residue field have the same parameter degree, and so we may speak of the parameter degree of a similarity class as the parameter degree of any of its members having infinite residue field.

**6.5. Corollary.** For each \( d \geq 0 \) and \( e \geq 1 \), the subset \( \mathcal{S}\text{Lim}_{d,e} \subseteq \mathcal{S}\text{Lim} \) is closed.
Proof. It suffices to show that $\mathcal{S}_m d, e$ is closed under limits. Hence let $r$ be a Cauchy sequence in $\mathcal{S}_m d, e$, and choose representatives $R_w$ in each $r(w)$, of dimension $d$ and parameter degree $e$. Let $R_q$ be the cataproduct of the $R_w$, so that its similarity class is the limit of $r$ by Theorem 6.3. Since the cataproduct $R_q$ has dimension $d$ and parameter degree $e$ by [21, Theorem 5.22], the claim follows. \hfill \qed

We can now state and prove the main theorem of this paper:

6.6. Theorem. The metric space $\mathcal{S}_m$ is a Polish space. In particular, the similarity relation is smooth.

Proof. In view of Theorem 6.3, it remains to show that $\mathcal{S}_m$ contains a countable dense subset. We already observed that there is a one-one correspondence between balls and Artinian local rings, so that $\mathcal{S}_m 0$, the similarity classes of Artinian local rings, form a dense subset of $\mathcal{S}_m$. Let $R$ be an Artinian local ring with residue field $κ$. By Cohen’s structure theorems, $R$ is of the form $V[[X]]/I$, where $V$ is either $κ$ or the complete $p$-ring over $κ$, and where $X$ is a tuple of indeterminates. Since $R$ is Artinian, it is in fact finitely generated over $V$. Hence, by an argument similar to the one in the proof of Proposition 4.3, there exists a finitely generated subfield $κ_0 \subseteq κ$ and an Artinian local ring $R_0$ with residue field $κ_0$, such that $R_0 \approx R$. Since there are only countably many finitely generated fields, the collection of all these $R_0$ is again countable. \hfill \qed

7. Variants

A first variant is simply obtained by working in the category of all Noetherian local $Z$-algebras, for $Z$ some Noetherian ring, so that the morphisms are now given by local $Z$-algebra homomorphisms. This leads to the notion of two $Z$-algebras being $Z$-similar, and the same argument shows that classifying Noetherian local $Z$-algebras up to $Z$-similarity is again a smooth problem.

We may also extend the definition to include modules. Namely, given an $R$-module $M$ and an $S$-module $N$, we say that $d(M, N) \leq 2^{-n}$, if $J^n R$ and $J^n S$ have a common scalar extension $T$ such that $M \otimes_R T \cong N \otimes_S T$. In particular, $d(R, S) \leq d(M, N)$. We will not study the similarity problem for modules—and at present, I do not know whether this is a smooth classification problem, even over a fixed ring. We will use this metric in the proof of Theorem 8.2; see also [21, §11] for some further applications.

We now turn to some other classification problems that can be reduced to the classification up to similarity.

Classification of analytic germs. Let $κ$ be a field. By an analytic germ over $κ$, we mean a complete Noetherian local ring with residue field $κ$; we denote the set of isomorphism classes of analytic germs by $\mathcal{S}_m A(κ)$. Note that if $κ$ has prime characteristic $p$, then the germ can either have equal or mixed characteristic. By the Cohen structure theorem, analytic germs are simply homomorphic images of power series rings $V[[X]]$, with $V$ either $κ$ (equal characteristic) or the complete $p$-ring over $κ$ (mixed characteristic). Assume, moreover, that $κ$ is algebraically closed and has size of the continuum. It follows that every (countable) ultrapower $κ_2$ of $κ$ is again algebraically closed and has the same cardinality as $κ$, whence by Leibniz’s theorem, is isomorphic with $κ$. In the mixed characteristic case, by uniqueness of $p$-rings, the catapower of $V$ is then also isomorphic to $V$. This shows that the set of analytic germs over such a field $κ$ is, up to isomorphism, closed under cataproducts, whence under limits. Moreover, there are, up to isomorphism, only countably many analytic germs of dimension zero, and they form a dense subset $\mathcal{S}_m A(κ_0)$ of $\mathcal{S}_m A(κ)$. In
conclusion, we showed Theorem 1.1 from the introduction. Note that in the above, we may replace the size of the continuum by any cardinal of the form $2^\gamma$, with $\gamma$ an infinite cardinal. Moreover, under the Generalized Continuum Hypothesis, this means any uncountable cardinal.

**Infinitesimal deformations.** By a deformator $R$, we mean a pair $(R, x)$, with $(R, m)$ a Noetherian local ring and $x := (x_1, \ldots, x_d)$ a tuple generating an $m$-primary ideal. To emphasize the maximal ideal, we may also represent the deformator $R$ as the triple $(R, m, x)$. We call $R$ and $x$ respectively the underlying ring and tuple of $R$, and we call the length of $R/xR$ the colength of $R$. We call $R$ parametric, if $x$ is a system of parameters. When we say that a deformator has a certain ring theoretic property, then we mean that its underlying ring has this property. Let $S := (S, y)$ be a second deformator, with $y = (y_1, \ldots, y_e)$. A morphism $R \to S$ of deformators, is a ring homomorphism $R \to S$ mapping $x$ to $y$. In particular, there are no morphisms between deformators with tuples of different length. It is easy to verify that these definitions make the class of deformators into a category. We call a morphism $R \to S$ flat, unramified, a scalar extension, etc., if and only if the underlying homomorphism $R \to S$ has this property. We say that $R$ and $S$ are similar, in symbols, $R \approx S$, if they have a common scalar extension $T$ (as deformators). As before, we denote the similarity class of a deformator $R$ by $[R]$.

The $n$-th infinitesimal deformation of a deformator $R := (R, x)$, denoted $J^nR$, is by definition the Artinian deformator $(R/x^{(n)}R, x)$, where for an arbitrary tuple $y := (y_1, \ldots, y_s)$, we write $y^{(n)}$ for the tuple $(y_1^n, \ldots, y_s^n)$. If $R \to S$ is a morphism of deformators, then it induces, for each $n$, a morphism $J^nR \to J^nS$.

**7.1. Lemma.** If $R$ and $S$ are similar deformators, then $J^nR \approx J^nS$, for all $n$.

**Proof.** Since the respective underlying rings $R$ and $S$ are similar, they have the same dimension. Without loss of generality, we may assume that $R \to S$ is a scalar extension. By definition of morphism, under the scalar extension $R \to S$, the tuple of $R$ is sent to that of $S$, and the assertion is now clear. \(\square\)

Let $\mathbb{SimDef}$ denote the set of similarity classes of deformators (the argument of this is indeed a set is analogous to the one for $\mathbb{Sim}$). We define the deformation metric on $\mathbb{SimDef}$ in analogy with the jet metric: given two similarity classes of deformators $[R] := ([R], [x])$ and $[S] := ([S], [y])$, we set $d([R], [S]) \leq 2^{-n}$, if $J^nR \approx J^nS$. By Lemmas 6.1 and 7.1, this definition is independent from the choice of representatives. Moreover, if $J^nR \approx J^nS$, then the definition of morphisms in the category of deformators implies that $J^iR \approx J^iS$, for all $i \leq n$. Indeed, we may reduce to the case that we have a scalar extension $J^nR \to J^nS$, which therefore maps $x$ to $y$, and the claim is now clear. If $d([R], [S]) < 1$, then $R$ and $S$ have in particular the same colength. As with rings, we will often identify a similarity class with any deformator contained in it, and so we will omit brackets in our notation and speak of the distance between deformators. The connection between the jet metric and the deformational metric is given by:

**7.2. Proposition.** If $R$ and $S$ are deformators with respective underlying rings $R$ and $S$, then $d([R], [S]) \leq d(R, S)$. Conversely, for every deformator $R$, if $T$ is a Noetherian local ring at distance $\varepsilon$ from $R$, then we can find a deformator $T$ with underlying ring $T$, such that $d(R, T) \leq \varepsilon^{1/(lm+1)}$, where $l$ is the colength of $R$ and $m$ the length of its tuple. If, moreover, $R$ is parametric, and $\dim(R) = \dim(T)$, then we may also choose $T$ to be parametric.
Proof. Let \((R, m)\) and \((S, n)\) be the respective underlying rings of \(R\) and \(S\), and let \(x\) and \(y\) be their respective tuples. If \(J^k R \approx J^k S\), for some \(k\), then clearly \(J^k R \approx J^k S\), since \(x^{(k)} R \subseteq m^k\) and \(y^{(k)} S \subseteq n^k\). This proves the first assertion.

To prove the second, observe that since \(m^l \subseteq I := xR\), we get

\[
m^{lmn} \subseteq I^{mn} \subseteq x^{(n)} R,
\]

for all \(n\). Hence \(\bar{R} := R/x^{(n)} R\) is a homomorphic image of \(J^{lmn} R\). Suppose \(d(R, T) \leq 2^{-k}\), so that \(J^k R \approx J^k T\). Without loss of generality, we may assume that \(J^k R \to J^k T\) is a scalar extension. Let \(z\) be a lifting in \(T\) of the image of \(x\) in \(J^k T\) under this scalar extension, and put \(T := (T, z)\). Let \(n\) be an integer strictly less than \(k/lm\), so that \(lmn < k\). We want to show that \(p^k \subseteq z^{(n)} T\), where \(p\) is the maximal ideal of \(T\). Put \(\bar{T} := T/z^{(n)} T\). The map \(J^k \bar{R} \to J^k T\) induces a scalar extension \(\bar{R} \to \bar{T}\). By (6), the latter is annihilated by \(p^{lnm}\). Hence \(p^{lnm} \bar{T} = p^k \bar{T}\), and since \(lmn < k\), Nakayama’s Lemma yields \(p^{lnm} \bar{T} = 0\), and the claim follows. In particular, base change induces a scalar extension \(\bar{R} \to \bar{T}\), and hence a scalar extension \(J^n R \to J^n T\) of deformators, showing that \(d(R, T) \leq 2^{-n}\), as we wanted to show.

\[
7.3. \textbf{Theorem.} \ \textit{Classification of deformators up to similarity is smooth, or, more precisely, } S_{lim} \text{Def} \textit{is a Polish space.}
\]

Proof. Let \(R_w\) be a Cauchy sequence in \(S_{lim} \text{Def}\), and let \(R_w\) be the corresponding sequence of underlying rings. By Proposition 7.2, this latter sequence is also Cauchy, whence has a limit in \(S_{lim}\) by Theorem 6.3. In fact, we may take the cataproduct \(R_0\) of the \(R_w\) as a representative of this limit. Since all tuples in \(R_w\) must have the same length, their ultraproduct yields a finite tuple in \(x\) in \(R_0\). Moreover, almost all \(R_w\) have the same colength, which, by Łos’ Theorem, is then also the length of \(R_2/xR_3\). In particular, \((R_2, x)\) is a deformator. The second part of Proposition 7.2 shows that it is the limit of the \(R_w\). This proves that \(S_{lim} \text{Def}\) is complete. Remains to show that the subset \(S_{lim} \text{Def}_{0}\) of Artinian deformators is countable and dense. However, we argued in the proof of Theorem 6.6 that each similarity class of an Artinian local ring \(R\) contains a representative \(R_0\) with a finitely generated residue field. Given any (finite) tuple \(x\), we may choose \(R_0\) so that it also contains \(x\). From this it is easy to see that \(S_{lim} \text{Def}_{0}\) is countable, and density is also immediate.

We denote the subset of similarity classes of parametric deformators by \(S_{lim} \text{FGr}\). Dimension, as this is equal to the length of the tuple, partitions this space in the pieces \(S_{lim} \text{FGr}_{d}\). It follows immediately from the above proof that each \(S_{lim} \text{FGr}_{d}\) is a complete subspace of \(S_{lim} \text{Def}\). In particular, \(S_{lim} 0\) is isometric with \(S_{lim} \text{FGr}_{0}\). However, for \(d > 0\), it is no longer clear whether \(S_{lim} \text{FGr}_{d}\) has a countable dense subset, and therefore, it might fail to be a Polish subspace.

\[
\textbf{Classification of polarized schemes up to isomorphism.} \ \textit{Our next application is to the classification of projective schemes. We will tacitly assume that a projective scheme } X \ \textit{is always of finite type over some field } \kappa. \ \textit{A polarization of } X \ \textit{is a choice of a very ample line bundle } \mathcal{L} \ \textit{on } X; \ \textit{we refer to this situation also by calling } \mathfrak{X} := (X, \mathcal{L}) \ \text{a polarized scheme over } \kappa, \ \text{and we say that } X \ \text{is the underlying projective scheme of } \mathfrak{X}. \ \text{In particular, a polarization } (X, \mathcal{L}) \ \text{corresponds to a closed immersion } i : X \to \mathbb{P}^n_\kappa, \ \text{for some } n, \ \text{such that } \mathcal{L} \cong i^\ast \mathcal{O}(1), \ \text{where } \mathcal{O}(1) \ \text{is the canonical twisting sheaf on } \mathbb{P}^n_\kappa.
\]
The section ring of a polarized scheme $\mathfrak{X} := (X, \mathcal{L})$ is defined as the graded $\kappa$-algebra

$$S(\mathfrak{X}) := \sum_{n=0}^{\infty} H^0(X, \mathcal{L}^n)$$

Note that, since $\mathcal{L}$ is very ample, $S(\mathfrak{X})$ is a standard graded $\kappa$-algebra, meaning that it has no homogeneous components of negative degree, its degree zero component is $\kappa$, and, as an algebra over $\kappa$, it is generated by its homogeneous elements of degree one.

The vertex algebra of $X$ is the localization of $S(\mathfrak{X})$ at the irrelevant ideal of all elements of positive degree, and will be denoted by $\text{Vert}(\mathfrak{X})$. If $X$ is irreducible and reduced, then the field of fractions of $S(\mathfrak{X})$ (and hence of $\text{Vert}(\mathfrak{X})$) is equal to the function field $\kappa(X)$. In particular, $\text{Vert}(\mathfrak{X})$ is a birational invariant of $X$. In fact, more is true: the polarized scheme $X := (X, \mathcal{L})$ can be recovered from its section ring $S := S(\mathfrak{X})$ as $X = \text{Proj}(S)$ and $\mathcal{L} = S(1)$, where $S(1)$ is the Serre twist of $S$. We therefore say that two polarized schemes $X := (X, \mathcal{L})$ and $Y := (Y, \mathcal{M})$ are isomorphic, if their section rings are isomorphic as graded $\kappa$-algebras, and this is then equivalent with the existence of an isomorphism $f : X \to Y$ of projective schemes, such that $f^*\mathcal{M} = \mathcal{L}$.

Let $\mathcal{P}_{\text{ol}}$ be the set of isomorphism classes of polarized schemes over $\kappa$. We metrize this space via pull-back along the vertex functor, that is to say,

$$d((X, \mathfrak{X}), (Y, \mathfrak{Y})) := d(\text{Vert}(X), \text{Vert}(Y)).$$

The following easy lemma allows us to calculate this distance function:

**7.4. Lemma.** Let $\mathfrak{X} := (X, \mathcal{L})$ be a polarized scheme over $\kappa$ with vertex algebra $R := \text{Vert}(\mathfrak{X})$. For each $n$, we have an isomorphism of graded Artinian $\kappa$-algebras

$$J^nR \cong \bigoplus_{i=0}^{n-1} H^0(X, \mathcal{L}^i).$$

**Proof.** Let $S := S(\mathfrak{X})$ be the section ring of $\mathfrak{X}$, and let $m$ be the irrelevant maximal ideal. Since $R = S_m$, we have $J^nR = S/m^n$, for all $n$. Since $S$ is a standard graded algebra, $m^n$ consists of all elements of degree at least $n$, that is to say,

$$m^n = \bigoplus_{i \geq n} H^0(X, \mathcal{L}^i),$$

from which the assertion follows immediately. \hfill \square

We can now show that we have indeed a metric on $\mathcal{P}_{\text{ol}}$:

**7.5. Corollary.** If two polarized schemes $\mathfrak{X}$ and $\mathfrak{Y}$ over $\kappa$ are at distance zero, then they are isomorphic.

**Proof.** By Lemma 7.4, their section rings are isomorphic, and we already argued that this means that the two polarized schemes are isomorphic. \hfill \square

**Proof of Theorem 1.2.** We will show that $\mathcal{P}_{\text{ol}}$ is a Polish space, and to this end, we need to show that it contains a countable dense subset and is closed under limits. By Lemma 7.4, the polarizations $(X, \mathcal{L})$ of zero-dimensional projective schemes are dense. Any such scheme is the base change of a zero-dimensional projective scheme $X_0$ over a finitely generated field, and since very ample line bundles are generated by their global sections, we may choose $X_0$ so that it admits a very ample line bundle $\mathcal{L}_0$ which induces the line bundle $\mathcal{L}$ on $X$ by base change. This shows that, up to isomorphism, there are only countably many polarizations of zero-dimensional projective schemes.
So remains to show that every Cauchy sequence \(X_w := (X_w, L_w)\) in \(\mathcal{F}_w\) has a limit. Let \(R_w := \text{Vert}(X)\), so that by definition, \(R_w\) is a Cauchy sequence of Noetherian local rings. Let \(R\) be the cataproduct of the \(R_w\). By the same argument as in the proof of Theorem 1.1, our assumption on the field \(\kappa\) implies that \(R\) has residue field isomorphic to \(\kappa\). Let \(R\) be an isomorphic copy of \(R\) having residue field \(\kappa\), and let \(f: R \to R\) be the corresponding isomorphism. Fix some \(n\). By Lemma 7.4, \(J^n R \cong J^n R_w\), for all \(w \gg 0\).

In particular, the \(n\)-th homogeneous piece \(S_n \coloneqq H^n(X_w, L_w)\) is independent from \(w\), for \(w\) sufficiently large. Since \(S_n\) has finite length, its ultrapower is equal to its catapower, and, therefore, via \(f\), isomorphic to itself. Let \(S := \oplus_n S_n\). One verifies that this is a standard graded \(\kappa\)-algebra. For instance, to define the ring structure on \(S\), it suffices to define the multiplication of two homogeneous elements, say \(a \in S_i\) and \(b \in S_j\). Take \(w\) large enough so that \(H^n(X_w, L_w)\) is equal to \(S_{i+j}\). Choose \(a_w, b_w \in S(X_w)\) so that their images in \(R_w\) have ultraproducts \(a_w, b_w \in R\) with \(f(a_w) = a\) and \(f(b_w) = b\). By Los’ Theorem, \(a_w\) and \(b_w\) are homogeneous of degree \(i\) and \(j\) respectively. We then define \(ab\) as the image under \(f\) of the ultraproduct of the \(a_w b_w \in S_{i+j}\). The other properties are checked similarly. In particular, \(J^n R \cong S_0 \oplus \cdots \oplus S_n\), showing that \(R\) is the localization of \(S\) at its irrelevant maximal ideal. Let \(X := (X, \mathcal{L})\) be the polarized scheme determined by \(S\), namely, \(X := \text{Proj}(S)\) and \(\mathcal{L} := S(1)\). Remains to show that \(X\) is the limit of the \(X_w\), and this is immediate form the fact that \(\text{Vert}(X) = R\).

### 8. Prolegomena to a complete set of invariants: slopes

Theorem 6.6, although promising, is far from an efficient classification up to similarity. In this final section, we will discuss some (albeit feasible) attempts to make it more concrete. As mentioned in the introduction, any two (uncountable) Polish spaces are Borel equivalent, namely to the standard Borel space on the reals. So, given any (concrete) Polish space \(\mathcal{B}\), we ask for a Borel bijection \(q: \mathcal{S} \to \mathcal{B}\).

Let us call a map \(q: \mathcal{S} \to \mathcal{B}\) a slope, if it is continuous. Of course, the identity map into \(\mathcal{S}\) itself is a slope, but we seek more concrete examples. A solution to the classification problem would, for instance, be provided by any real-valued, injective slope. Extending this terminology, let us say that for some subset \(\mathcal{S} \subseteq \mathcal{S}\) and a map \(q: \mathcal{S} \to \mathcal{B}\), that \(q\) is a slope on \(\mathcal{S}\) when its restriction to \(\mathcal{S}\) is continuous. A priori, the theory only predicts that we can find a real-valued, injective Borel map, which in general is only continuous outside a meagre subset, but perhaps we may venture to postulate the existence of a countable partition \(\{S_i\}\) of \(\mathcal{S}\), and an injective map \(q: \mathcal{S} \to \mathbb{R}\), such that \(q\) restricted to each piece \(S_i\) is a slope. Moreover, we want this partition to be indexed by some natural discrete invariants that are preserved under the similarity relation, like dimension and/or parameter degree. We start with some examples of non-injective slopes taking values into a concrete complete topological space (from now on, we will confuse a similarity class with any of its members):

#### 8.1. Proposition. Viewing \(\mathbb{Z}[\ell]\) in its \(t\)-adic metric, the map \(\text{Hilb}: \mathcal{S} \to \mathbb{Z}[\ell]\) induced by associating to a Noetherian local ring its Hilbert series \(\text{Hilb}(R)\), is a slope.

**Proof.** Recall that the Hilbert series of \((R, m)\) is defined to be the formal power series

\[
\text{Hilb}(R) := \sum_{n=0}^{\infty} \ell(m^n/m^{n+1})t^n.
\]

If \(R\) and \(S\) are similar, then they have the same Hilbert series, showing that \(\text{Hilb}\) is defined on \(\mathcal{S}\). By an easy calculation, \(\ell(m^n/m^{n+1}) = \ell(J^{n+1}R) - \ell(J^R)\). Hence, if \(R_w\)
converges to \( R \), then for each \( n \), we have \( J^n R = J^n R_w \), for all sufficiently large \( w \), showing that \( \text{Hilb} \) is continuous.

For a second example of a (non-injective) slope, we make the following definition. Let \( R \) be a local ring with residue field \( \kappa \), and let \( M \) be a finitely generated \( R \)-module. The \( n \)-th Betti number \( \beta_n(M) \) of \( M \) is defined as the vector space dimension of \( \text{Tor}^R_n(M, \kappa) \). Alternatively, at least in the Noetherian case, the Betti numbers are the ranks in a minimal free resolution of \( M \), and hence by Nakayama’s Lemma, the minimal number of generators of the syzygies of \( M \). The generating series of these Betti numbers, that is to say, the formal power series

\[
Poin(M) := \sum_n \beta_n(M)t^n
\]

is called the Poincare series of \( M \). We define the residual Poincare series of \( R \) to be the Poincare series of its residue field, and denote it \( \text{Poin}^\kappa(R) \). If \( R \to S \) is a scalar extension, and \( F_* \) a minimal free resolution of the residue field \( \kappa \) of \( R \), then by flatness, \( F_* \otimes_R S \) is a minimal free resolution of \( \kappa \otimes_R S \), and the latter is the residue field of \( S \), since \( R \to S \) is unramified. Hence, any two similar rings have the same residual Poincare series. Let \( \mathcal{S} \) be the subset of \( \mathcal{S} \) consisting of all similarity classes of local Cohen-Macaulay rings. By [21, Corollary 8.7], if we also fix dimension and multiplicity, then each \( \mathcal{S} \) is closed under limits.

8.2. Theorem. The residual Poincare series is a slope on each class \( \mathcal{S} \).

Proof. The continuity of the map associating to a \( d \)-dimensional local Cohen-Macaulay ring \( R \) of multiplicity \( e \) its residual Poincare series \( \text{Poin}^\kappa(R) \) is an immediate consequence of [21, Theorem 11.4]. Indeed, if \( R_w \) is a Cauchy sequence, then, for any fixed \( n \), the residue field of each \( R_w \) has the same \( n \)-th Betti number, for \( w \) sufficiently large, by the cited result. By [21, Proposition 8.9], this is then also the Betti number of the cataproduct \( R_t \), that is to say, up to similarity, the limit of the \( R_w \). Therefore, \( \text{Poin}^\kappa(R_w) \) converges, in the \( t \)-adic topology, to \( \text{Poin}^\kappa(R_t) \).

For a local Cohen-Macaulay ring \( R \), we also define its canonical Poincare series, denoted \( \text{Poin}^\text{can}(R) \), as the Poincare series \( \text{Poin}(\omega_R) \) of the canonical module \( \omega_R \) of its completion \( \hat{R} \) (note that the canonical module always exists when the ring is complete; see for instance [3, §3.3]). In particular, \( R \) is Gorenstein if and only if its canonical Poincare series is constant (equal to 1): indeed, \( R \) is Gorenstein if and only if \( \omega_R \cong R \). It is not hard to check that the canonical Poincare series is independent from the choice of representative of a similarity class of a Cohen-Macaulay local ring (by the same argument as in [3, Theorem 3.3.5(c)]). Let \( R \) and \( S \) be two rings in \( \mathcal{S} \) at distance at most \( 2^{−d_e−1} \). After a scalar extension, we may assume that \( R \) and \( S \) are complete, with infinite residue fields \( \kappa \) and \( \lambda \), respectively. In particular, there exists a system of parameters \( x \) in \( R \) such that \( R := R/\langle x \rangle \) has length \( e \). Proposition 7.2 then yields a system of parameters \( y \) in \( S \) such that \( S := S/\langle y \rangle \). Since the canonical module \( \omega_R \) is maximal Cohen-Macaulay, \( x \) is \( \omega_R \)-regular, and likewise, \( y \) is \( \omega_S \)-regular. Therefore,

\[
\begin{align*}
\text{Tor}^R_n(\omega_R/\kappa, \omega_R, \kappa) & \cong \text{Tor}^\hat{R}_n(\omega_R/\kappa, \omega_R, \kappa) \\
\text{Tor}^S_n(\omega_S/\lambda, \omega_S, \lambda) & \cong \text{Tor}^\hat{S}_n(\omega_S/\lambda, \omega_S, \lambda)
\end{align*}
\]

for all \( n \). Since \( \hat{R} \) and \( \hat{S} \) are similar, they have the same canonical Poincare series \( P(t) \). By [3, Theorem 3.3.5], the respective canonical modules of \( \hat{R} \) and \( \hat{S} \) are \( \omega_R/\langle x \omega_R \rangle \) and
ω_S/ω_S$. By (7), therefore, the canonical Poincare series of $R$ and $S$ are both equal to $P(t)$. In conclusion, we showed:

8.3. **Proposition.** On each ball of radius $2^{-d-1}$ in $\mathcal{S}_{\mathbb{R}}^d_{\mathbb{C},\alpha}$, the canonical Poincare series is constant. In particular, if one of its members is Gorenstein, then so is any. $\square$

**Quasi-slopes.** To find a slope, it is enough to have it defined on the countable dense open subset $\mathcal{S}_m$ given by Theorem 6.6. For any map $q_0: \mathcal{S}_m \to \mathcal{B}$ into a complete metric space (not necessarily continuous), define its extension $\hat{q}_0$ as the partial map $\mathcal{S}_m \to \mathcal{B}$ given as the limit of the $\hat{q}_0(J^n R)$, for $R$ a Noetherian local ring, whenever this limit exists. Note that if $R \approx S$, then $\hat{q}_0(J^n R)$ converges if and only if $\hat{q}_0(J^n S)$ does, and their limits are similar. In particular, $\hat{q}_0(R) = \hat{q}_0(R)$ whenever $R$ is Artinian. We call a map $q_0: \mathcal{S}_m \to \mathcal{B}$ a quasi-slope, if $\hat{q}_0$ is everywhere defined. By abuse of terminology, we then also refer to this extension $q := \hat{q}_0$ as a quasi-slope. In other words, $q: \mathcal{S}_m \to \mathcal{B}$ is a quasi-slope if, $q(J^n R)$ converges to $q(R)$, for every Noetherian local ring $R$. The following corollary is now immediate from Theorem 6.6:

8.4. **Corollary.** Any continuous map $q_0: \mathcal{S}_m \to \mathcal{B}$ is a quasi-slope and its extension $\hat{q}_0$ is a slope. $\square$

We next show how some of the usual invariants, although in general not slopes, become quasi-slopes when properly modified. Let $\delta_0$ be defined on $\mathcal{S}_m$ as follows. Given an Artinian local ring $(A, m)$, let $n$ be its degree of nilpotency (that is to say, the least $k$ such that $m^k = 0$). Put

$$\delta_0(A) := \log_2 \left( \frac{\ell(A)}{\ell(J^{n/2}A)} \right)$$

where for a positive real number $r$, we define $J^r A := J^2 A$ with $z := \text{int}(r)$ the largest integer less than or equal to $r$.

8.5. **Proposition.** The map $\delta_0$ is a quasi-slope. In fact, $\hat{\delta_0}(R)$ is equal to the dimension of $R$, whenever this dimension is non-zero.

**Proof.** Let $R$ be a Noetherian local ring of dimension $d > 0$. By the Hilbert-Samuel theory, there exists a polynomial $P_R \in \mathbb{Q}[t]$ of degree $d$, such that $\ell(J^n R) = P_R(n)$ for $n \gg 0$. Hence $\delta_0(J^n R) = \log_2 \left( P_R(n) / P_R(\text{int}(n/2)) \right)$, for $n \gg 0$. It is now an exercise to show that for any polynomial $P$ of degree $d$, the limit of $P(n)/P(\text{int}(n/2))$ is equal to $2^d$. $\square$

In view of this result, we call $\hat{\delta_0}(R)$ the quasi-dimension of $R$. So only Artinian local rings have a quasi-dimension which is different from their (Krull) dimension. Using the formula

$$\lim_{n \to \infty} \frac{P(\text{int}(\sqrt{\pi})))^2}{P(n)} = a_d$$

where $a_d$ is the leading coefficient of a polynomial $P$, we get by a similar argument that the map $\epsilon_0$ defined on $\mathcal{S}_m$ by the condition $\epsilon_0(A) := \ell(J^{\sqrt{\pi}A})^2/\ell(A)$ is a quasi-slope, and $\epsilon_0(R) = e/d!$ whenever $d > 0$, where $e$ is the multiplicity of $R$ and $d$ its dimension.

Several questions now arise naturally: what is the nature of the subset of $\mathcal{S}_m$ of all Noetherian local rings of a fixed quasi-slope? Can we break up (or even stratify) $\mathcal{S}_m$ in “natural” pieces on which a quasi-slope becomes continuous. I will conclude with an example of how one can answer the second question for quasi-dimension. For a Noetherian
local ring $R$, define $\rho(R)$ as the supremum over all $n$ of
\[
\left| \frac{d! \ell(J^n R)}{e n^d} - n \right|
\]
where $d := \dim(R)$ and $e := \mult(R)$ are respectively the dimension and multiplicity of $R$. In other words, $\rho(R)$ is the smallest real number $\rho \geq 0$ such that
\[
(8) \quad n^d - \rho n^{d-1} \leq \frac{d! \ell(J^n R)}{e} \leq n^d + \rho n^{d-1}
\]
for all $n > 0$. That this supremum exists is an easy consequence of the Hilbert-Samuel theory. For instance, if $R$ is Artinian of length $l$, then $1 \leq \rho(R) \leq l$, but these bounds are not sharp. Note that $\rho$ is not a quasi-slope (this is easily checked for $R := \kappa((t))$).

This new invariant determines the rate of convergence in the definition of the quasi-dimension, as the next result shows. To formulate it, we use $\lceil \cdot \rceil$ to denote the rounding to the nearest integer of a real number $\ell$, that is to say, $\lceil \ell \rceil$ is the unique integer inside the half open interval $[\ell - \frac{1}{2}, \ell + \frac{1}{2})$.

8.6. Lemma. For a Noetherian local ring $R$, if $n \geq 10 \rho(R)$, then $\lceil \delta_0(J^n R) \rceil$ is equal to its dimension.

Proof. Let $b := \rho(R)$, and let $d$ and $e$ be the respective dimension and multiplicity of $R$. Using (8), we get estimates
\[
(9) \quad 1 - \frac{b}{n} \leq \frac{d! \ell(J^n R)}{e n^d} \leq 1 + \frac{b}{n}
\]
for all $n$. In the convergence of $\delta_0$ we may take the limit over even $n$ only, so let us assume that $n = 2m$. Dividing inequalities (9) for $2m$ by those for $m$, we get estimates
\[
2^d \left(1 - \frac{b}{2m}\right) \leq \frac{\ell(J^{2m} R)}{\ell(J^m R)} = 2^d \delta_0(J^m R) \leq 2^d \left(1 + \frac{b}{2m}\right).
\]
Hence, if the ratio between the two outside fractions is strictly less than 2, then after taking the logarithm with base two, they become the endpoints of an interval $[\alpha, \beta]$ of length strictly less than one, containing $\delta_0(J^m R)$. Since $\alpha < d < \beta$, the only integer in $[\alpha, \beta]$ is $d$, showing that $\lceil \delta_0(J^n R) \rceil = d$.

For the ratio to be at most 2, we need
\[
(m + \frac{b}{2})(m + b) < 2(m - \frac{b}{2})(m - b)
\]
and a simple calculation shows that this is true whenever $m > 5b$. \hfill \Box

Immediately from this we get:

8.7. Corollary. For each $b \in \mathbb{N}$, let $\mathcal{S}_{\mathcal{L}} \subseteq b$ be the subset of $\mathcal{S}_{\mathcal{L}}$ consisting of all Noetherian local rings $R$ such that $\rho(R) \leq b$, thus yielding a filtration $\mathcal{S}_{\mathcal{L}} \subseteq \mathcal{S}_{\mathcal{L}} \subseteq \ldots$ of $\mathcal{S}_{\mathcal{L}}$. Then quasi-dimension is a slope on each $\mathcal{S}_{\mathcal{L}} \subseteq b$.

A similar argument can be used to show that $\epsilon_0$ is a slope on each $\mathcal{S}_{\mathcal{L}} \subseteq b$, by establishing an analogous bound for the convergence of $\epsilon_0$ to $e/d!$ which only depends on $d := \dim(R)$, $e := \mult(R)$ and $\rho := \rho(R)$. As far as bounding $\rho$ itself is concerned, if $R$ is Cohen-Macaulay, then it is bounded as a function of $e$ and $d$ only, but without the Cohen-Macaulay assumption this is probably false. In the latter case, we can use any “big degree” $D \la$ la Vasconcelos to arrive at such a bound in terms of $d$ and $D(R)$ (this is an easy consequence of [15, Theorem 4.1]).
Recall (see, for instance, [8, III. Ex. 5.1]) that the Euler characteristic of a projective scheme $X$ is defined by the formula
\[
\chi(X) := \sum_i (-1)^i h^i(X, \mathcal{O}_X)
\]
where $h^i(X, \mathcal{O}_X)$ is the vector space dimension of the sheaf cohomology $H^i(X, \mathcal{O}_X)$. In particular, if $X$ is a curve, then $\chi(X) - 1$ is the genus of $X$.

8.8. Proposition. The map $\mathbb{P}^n \to \mathbb{R}$ sending a polarized scheme $X = (X, \mathcal{L})$ to $\chi(X)$ is continuous.

Proof. We calculate the Euler characteristic by means of the Hilbert-Samuel polynomial $P_X(n)$ of $X$ as
\[
\chi(X) = P_X(0).
\]
In fact, as it is a birational invariant, we may calculate the Euler characteristic by means of any polarization $X = (X, \mathcal{L})$ of $X$. The Hilbert series of $X$ is defined as
\[
\text{Hilb}(X) := \sum_{n=0}^{\infty} h^0(X, \mathcal{L}^n) t^n.
\]
Let $(R, \mathfrak{m}) := \text{Vert}(X)$. It is not hard to see that $H^0(X, \mathcal{L}^n) \cong \mathfrak{m}^n / \mathfrak{m}^{n+1}$ and hence that $X$ and $\mathfrak{m}$ have the same Hilbert series and the same Hilbert-Samuel polynomial. Moreover, the connection between the Hilbert series $h(t)$ and the corresponding Hilbert-Samuel polynomial $P(n)$ is given by the formula
\[
P(n) = \sum_{j=0}^{d-1} \frac{(-1)^j}{j!} \binom{n + d - 1 - j}{n} \frac{\partial^j}{\partial t^j} ((1 - t)^d h(t)) \bigg|_{t=1},
\]
where $d$ is the degree of $P$ (that is to say, the dimension of $X$). Moreover, if $h_i$ are Hilbert series with corresponding Hilbert polynomial $P_i$, then from the fact that
\[
\frac{\partial^j}{\partial t^j} ((1 - t)^d t^n) \bigg|_{t=1} = 0
\]
for all $j < d$ and all $n$, we get from (11) that $P_1 = P_2$ whenever $h_1$ and $h_2$ are $t$-adically close. Therefore, by Proposition 8.1 and (10), the Euler characteristic is continuous.

Future work. In work in progress ([19]), we assign to any Artinian $\kappa$-algebra, with $\kappa$ an algebraically closed field of size the continuum, a first-order formula modulo the theory of Artinian $\kappa$-algebras. Associating to this theory its Grothendieck ring $K_0 := K_0(\kappa)$ on pp-formulae, we get a map $\text{SimAn}_0(\kappa) \to K_0: R \mapsto [R]$, which is compatible with direct sum and tensor product, and which becomes injective when we replace the isomorphism relation with a stable version of it. Hence, we may associate to any analytic germ $R$, its formal Hilbert series
\[
\text{Hilb}^{\text{form}}(R) := \sum_n [J^n R] t^n \in K_0[[t]]
\]
This would yield a complete invariant modulo the Grothendieck ring $K_0$ (this Grothendieck ring, however, admits the classical Grothendieck ring of $\kappa$ as a homomorphic image, whence is potentially a very complicated object).
References

[1] M. Andrè, *Localisation de la lissité formelle*, Manuscripta Math. **13** (1974), 297–307.

[2] M. Aschenbrenner and H. Schoutens, *Lefschetz extensions, tight closure and big Cohen-Macaulay algebras*, Israel J. Math. **161** (2007), 221–310.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

[4] P. Eklof, *Ultraproducts for algebraists*, Handbook of Mathematical Logic, North-Holland Publishing, 1977, pp. 105–137.

[5] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symbolic Logic **54** (1989), no. 3, 894–914.

[6] A. Grothendieck and J. Dieudonné, *Elements de géométrie algébrique I-IV*, Inst. Hautes Études Sci. Publ. Math., vol. 4, 8, 11, 17, 20, 24, 28, Presses universitaires de France, Paris, 1960-1965.

[7] L. Harrington, A. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928. MR MR1057041 (91h:28023)

[8] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.

[9] G. Hjorth, *When is an equivalence relation classifiable?*, Doc. Math. (1998), no. Extra Vol. II, 23–32.

[10] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.

[11] W. Hodges, *Model theory*, Cambridge University Press, Cambridge, 1993.

[12] J. Kollár, *Flatness criteria*, J. Algebra **175** (1995), 715–727.

[13] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[14] B. Olberding, S. Saydam, and J. Shapiro, *Completions, valuations and ultrapowers of Noetherian domains*, J. Pure Appl. Algebra **197** (2005), no. 1-3, 213–237.

[15] M. Rossi, N. Trung, and G. Valla, *Castelnuovo-Mumford regularity and extended degree*, Trans. Amer. Math. Soc. **355** (2003), no. 5, 1773–1786 (electronic).

[16] P. Rothmaler, *Introduction to model theory*, Algebra, Logic and Applications, vol. 15, Gordon and Breach Science Publishers, Amsterdam, 2000.

[17] H. Schoutens, *Non-standard tight closure for affine C-algebras*, Manuscripta Math. **111** (2003), 379–412.

[18] , *Constructible invariants*, J. Algebra **304** (2006), 1059–1089.

[19] , *The schemic Grothendieck ring*, in preparation, 2008.

[20] , *Use of ultraproducts in commutative algebra*, in preparation, 2008.

[21] , *Dimension theory for local rings of finite embedding dimension*, (2007) preprint, in preparation.

[22] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for fields of finite transcendence degree*, J. Pure Appl. Algebra **159** (2001), no. 2-3, 347–363.

[23] L. van den Dries, *Isomorphism of complete local Noetherian rings and strong approximation*, to appear in Proc. AMS.