The many faces of cyclic branched coverings of 2-bridge knots and links

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Abstract

We discuss 3-manifolds which are cyclic coverings of the 3-sphere, branched over 2-bridge knots and links. Different descriptions of these manifolds are presented: polyhedral, Heegaard diagram, Dehn surgery and coloured graph constructions. Using these descriptions, we give presentations for their fundamental groups, which are cyclic presentations in the case of 2-bridge knots. The homology groups are given for a wide class of cases. Moreover, we prove that each singly-cyclic branched covering of a 2-bridge link is the composition of a meridian-cyclic branched covering of a determined link and a cyclic branched covering of a trivial knot.

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1 Introduction and preliminaries

The family of 2-bridge knots/links is a well-studied subject, starting from
the classical papers of Listing, Dehn, Alexander, Reidemeister, Schubert and
others. Almost everything is known about the symmetries, invariants, and
geometric properties of these links, of their complements, and of manifolds
connected with them. The aim of the present paper is to provide a review of
the properties of 3-manifolds which are cyclic branched coverings of 2-bridge
knots/links. We recall that there are a lot of ways of describing a 3-manifold:
by fundamental polyhedra [78], by Heegaard diagrams [31], by surgery on
links in $S^3$ [74], by branched coverings of $S^3$ [91] and by gems/crystallizations
[48, 71]. So, each manifold has many “faces”, which depend on the type of
the description we choose. This choice depends on the nature of the problem
one studies. In this paper we are going to present these aspects of cyclic
branched coverings of 2-bridge knots/links. As an example, we will describe
the different faces of the classical Hantzsche–Wendt manifold [85].

By the term manifold we always mean a compact, connected, orientable
PL-manifold without boundary. Let $M, N$ be triangulated $m$-dimensional
manifolds and let $L$ be an $(m - 2)$-subcomplex of $N$. A non-degenerate
map $f : M \to N$ is an $n$-fold covering map branched over $L$, with $n > 1$, if:
(i) $f|_{M - f^{-1}(L)} : M - f^{-1}(L) \to N - L$ is an ordinary $n$-fold covering,
and (ii) $L = \{x \in N \mid \# f^{-1}(x) < n\}$. The manifold $M$ is said to be a
branched covering of $N$, and $L$ is called the branching set of the covering.
Two branched coverings $f' : M' \to N'$ and $f'' : M'' \to N''$ are equivalent if
there exist two homeomorphisms $\psi : M' \to M''$ and $\phi : N' \to N''$ such that
$\psi \circ f'' = f' \circ \phi$.

A remarkable result by R. H. Fox [24] states that a branched covering is
uniquely determined by the ordinary covering induced by restriction. This
proves the existence of a one-to-one correspondence between the \( n \)-fold coverings of \( N \) branched over \( L \) and the equivalence classes of monodromies (i.e. transitive representations) \( \omega : \pi_1(N - L, x_0) \to \Sigma_n \), where \( \Sigma_n \) is the symmetric group on \( n \) elements and \( x_0 \in N - L \) is an arbitrary base point.

If \( N \) is an \( m \)-sphere, then a covering of \( N \) branched over \( L \) is simply called a branched covering of \( L \).

Two \( n \)-fold branched coverings \( f' : M' \to N' \), \( f'' : M'' \to N'' \), with branching sets \( L' \subset N' \), \( L'' \subset N'' \) and monodromy maps \( \omega_f, \omega_{f''} \) respectively, are equivalent if and only if there exists an inner automorphism \( \lambda \) of \( \Sigma_n \) and a homeomorphism \( \phi : N' \to N'' \), such that \( \phi(L') = L'' \) and \( \lambda \circ \omega_f = \omega_{f''} \circ \phi \), where \( \phi_* : \pi_1(N' - L', x_0') \to \pi_1(N'' - L'', \phi(x_0')) \) is the homomorphism induced by \( \phi|_{N' - L'} \) on the fundamental groups.

Branched coverings of spheres are of great interest, in particular as a method for representing manifolds. A classical result concerning this point was obtained by J. Alexander \[1\] and states that every \( m \)-manifold is a covering of \( S^m \), branched over the \((m - 2)\)-skeleton of a standard \( m \)-simplex.

Fox’s result gives the possibility of extending the concept of cyclic coverings from ordinary coverings to branched coverings. Thus, a branched covering is called cyclic if its associated ordinary covering is cyclic. Similar extension also applies to regular and abelian coverings.

Since a cyclic covering is abelian, it is determined up to equivalence by an epimorphism
\[
\bar{\omega}_f : H_1(N - L) \to \mathbb{Z}_n,
\]
where \( \mathbb{Z}_n \) is the cyclic group of order \( n \), embedded into \( \Sigma_n \) through the monomorphism sending 1 \( \in \mathbb{Z}_n \) to the standard cyclic permutation \((1 2 \cdots n) \in \Sigma_n \). If \( N = S^3 \) and \( L' = \bigcup_{j=1}^\nu L_j \) is a \( \nu \)-component link in \( S^3 \), then \( H_1(N - L') \cong \mathbb{Z}^\nu \) and a basis is given by any set of homology classes of meridian loops around the components of \( L \). Therefore, an \( n \)-fold cyclic branched covering \( f \) of \( L \) is defined by orienting \( L \) and assigning an integer \( k_j \in \mathbb{Z}_n - \{0\} \) to each component \( L_j \), such that the set \( \{k_1, \ldots, k_\nu\} \) generates \( \mathbb{Z}_n \). If \( m_j \) is a meridian around \( L_j \), coherently oriented with the chosen orientations of \( L \) and \( S^3 \), we define \( \omega_f[m_j] = k_j \in \mathbb{Z}_n \) and therefore \( \omega_f[m_j] = (1 2 \cdots n)^{k_j} \). We will denote this manifold by \( M_{n,k_1,\ldots,k_\nu}(L) \). By multiplying each \( k_j \) by the same invertible element \( u \) of \( \mathbb{Z}_n \), we obtain an equivalent covering. More precisely, two \( n \)-fold cyclic branched coverings \( f' : M' \to N' \) and \( f'' : M'' \to N'' \), with associated maps \( \bar{\omega}_{f'} : H_1(N' - L') \to \mathbb{Z}_n \) and \( \bar{\omega}_{f''} : H_1(N'' - L'') \to \mathbb{Z}_n \), respec-
tively, are equivalent if and only if there exist \(u \in \mathbb{Z}_n\) and a homeomorphism \(\phi : N' \to N''\), such that \(\gcd(u, n) = 1\), \(\phi(L') = L''\) and \(\bar{\omega}_{f'} \circ \phi_\# = u\bar{\omega}_{f'}\), where \(\phi_\# : H_1(N' - L') \to H_1(N'' - L'')\) is the homomorphism induced by \(\phi|_{N' - L'}\) on the first homology groups and \(u\bar{\omega}_{f'}\) is the multiplication of \(\bar{\omega}_{f'}\) by \(u\). General references on cyclic branched coverings of knots/links are the interesting books \([4]\), \([40]\), and \([74]\).

Following \([52]\) we shall say that a cyclic branched covering \(M_{n,k_1,...,k_\nu}(L)\) is:

(i) strictly-cyclic if \(k_{j'} = k_{j''}\), for every \(j', j'' \in \{1, \ldots, \nu\}\);

(ii) almost-strictly-cyclic if \(k_{j'} = \pm k_{j''}\), for every \(j', j'' \in \{1, \ldots, \nu\}\);

(iii) meridian-cyclic if \(\gcd(n, k_j) = 1\), for every \(j \in \{1, \ldots, \nu\}\);

(iv) singly-cyclic if there exists \(j \in \{1, \ldots, \nu\}\) such that \(\gcd(n, k_j) = 1\);

(v) monodromy-cyclic if it is cyclic.

The following implications are straightforward: (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) \(\Rightarrow\) (v). Moreover, all five definitions are equivalent when either \(L\) is a knot or \(n = 2\). Note that strictly-cyclic coverings are also called uniform in \([88]\) and meridian-cyclic coverings are also called strongly cyclic in \([89]\). By a suitable reorientation of the link, an almost-strictly-cyclic covering becomes a strictly-cyclic one. For a singly-cyclic covering we can always assume \(k_1 = 1\), up to equivalence and possible reordering of the components of \(L\). Therefore, when \(\nu = 2\) the covering is completely determined by an integer \(k = k_2 \in \mathbb{Z}_n - \{0\}\).

In order to simplify the notations, the \(n\)-fold strictly-cyclic branched covering \(M_{n,1,...,1}(L)\) will be denoted by \(M_n(L)\) and the \(n\)-fold singly-cyclic branched covering of a 2-component link \(M_{n,1,k}(L)\) will be denoted by \(M_{n,k}(L)\). In particular, \(M_{n,k}(L)\) is meridian-cyclic when \(\gcd(n, k) = 1\) and strictly-cyclic when \(k = 1\).

The class of 2-bridge knots/links (also called “rational knots/links”) is of great interest and the main properties of its elements can be found in \([1]\), \([10]\), \([74]\). We denote by \(b(\alpha, \beta)\) the 2-bridge knot or link of type \((\alpha, \beta)\), with integers \(\alpha > 1\) and \(\beta \in \mathbb{Z}_{2\alpha}\) such that \(\gcd(\alpha, \beta) = 1\). It is well known that \(b(\alpha, \beta)\) is a knot when \(\alpha\) is odd and a two-component link when \(\alpha\) is even. In our discussion we will skip the “singular” case of the 2-component trivial link, which is a 2-bridge link and is often indicated by \(b(0,1)\). A
standard diagram presentation for 2-bridge knots/links is Schubert’s normal form [4, Ch. 12]. Actually, Schubert’s normal form is only defined when \( \beta \) is odd, but this is not a restriction, up to equivalence (see classification of 2-bridge knots/links below). Figure 1 gives two examples of such diagrams for the figure-eight knot \( b(5,3) \) and for the Whitehead link \( b(8,3) \).

Another type of diagram for 2-bridge knots/links is introduced by J. Conway [17] (see Figure 2). It comes from the representation of a rational number by a continued fraction. If

\[
\frac{\alpha}{\beta} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \cdots + \frac{1}{c_m}}},
\]

we will say that \([c_1, c_2, \ldots, c_m]\) are Conway parameters for \( b(\alpha, \beta) \) and the notation \( \alpha/\beta = [c_1, c_2, \ldots, c_m] \) will be used.

2-bridge knots/links are assumed to be oriented in a standard way, as indicated in Figures 1 and 2. By the Schubert construction of \( b(\alpha, \beta) \), there is an obvious orientation preserving involution \( \rho : S^3 \rightarrow S^3 \) exchanging the two bridges of \( b(\alpha, \beta) \) and preserving the orientation of the component(s).

Recall that two oriented links \( L \) and \( L' \) with \( \nu \) components with a given order \( (L_i)_{1 \leq i \leq \nu} \) and \( (L'_i)_{1 \leq i \leq \nu} \) are said to be equivalent if there exists an orientation preserving homeomorphism \( \phi : S^3 \rightarrow S^3 \) sending \( L_i \) to \( L'_i \), for \( i = 1, 2, \ldots, \nu \), and preserving their orientations. For 2-bridge links the equivalence does not depend on the ordering of the components because of the homeomorphism \( \rho \) previously described. H. Schubert in [77] classified 2-bridge knots/links (see also [40, Theorem 2.1.3]):

(i) two oriented two-bridge knots \( b(\alpha, \beta), b(\alpha', \beta') \) are equivalent if and only if \( \alpha' = \alpha \) and \( \beta' \equiv \beta \pm 1 \) mod \( \alpha \);

(ii) two oriented two-bridge links \( b(\alpha, \beta), b(\alpha', \beta') \) are equivalent if and only if \( \alpha' = \alpha \) and \( \beta' = \beta \pm 1 \).

If we consider unoriented links, then the condition of (ii) is reduced to that of (i).

Below we will use the following properties: \( b(\alpha, -\beta) \) is equivalent to the mirror image of \( b(\alpha, \beta) \) and the link \( b(\alpha, \beta - \alpha) \) is equivalent to the link \( b(\alpha, \beta) \) with the opposite orientation on one of the two components [4].

\(^{1}\)Here and in the following the second coordinate of a 2-bridge knot or link \( b(\alpha, \beta) \) will be considered mod \( 2\alpha \), except when otherwise specified.
Figure 1: Schubert’s normal form for $b(5, 3)$ and $b(8, 3)$. 
Figure 2: Conway's normal form for 2-bridge knots and links.
From now on, we use the notation $M_{n,k'}(\alpha/\beta) = M_{n,k'}(b(\alpha, \beta))$ for $n$-fold cyclic branched coverings of 2-bridge links (i.e., $\alpha$ is even). In particular, $M_{n,k}(\alpha/\beta)$ will denote a singly-cyclic branched covering and $M_n(\alpha/\beta)$ will denote a strictly-cyclic one. The last notation will always be used even in the case when $b(\alpha, \beta)$ is a knot (i.e., $\alpha$ is odd).

The class of cyclic branched coverings of 2-bridge knots/links has been intensively studied by many authors, and lot of their properties were discovered. In particular, the two-fold coverings are homeomorphic to lens spaces.

In Section 2 we introduce our class of manifolds, discuss homeomorphisms, geometric structures, and present them as 2-fold branched coverings of $S^3$.

In Section 3 we will describe the polyhedral construction of these manifolds, according to J. Minkus [58].

In Section 4 we will give symmetric Heegaard diagrams for cyclic branched coverings of 2-bridge knots. Following from a result obtained in [28], this description arises from a construction by M. Dunwoody [21]. In the same section, some estimates for the genus are given.

In Section 5 we will give a surgery description of these manifolds, for the case of knots. It was pointed out in [13] and [7] that some cyclic branched coverings of 2-bridge knots are Takahashi manifolds [83] and therefore they can be obtained by Dehn surgery on a certain chain of circles in $S^3$. The Takahashi construction has been generalized in [65] in order to obtain all cyclic branched coverings of 2-bridge knots.

In Section 6 we will deal with the coloured graph representation of manifolds introduced by M. Pezzana [74] and his school. A wide class of coloured graph encoding 3-dimensional (possibly singular) manifolds was defined by S. Lins and A. Mandel in [13], and intensively studied by many authors. It was shown in [12] that all Lins-Mandel manifolds are cyclic branched coverings of 2-bridge knots/links.

In Section 7 we will describe the presentation of the fundamental group of these manifolds obtained in [58] and [2]. Moreover, in the case of coverings of 2-bridge knots, we also describe the two different cyclic presentations obtained in [58] and [63].

In Section 8 we give the first integer homology groups for a large class of cases.

In Section 9 we will prove that each singly-cyclic branched covering of a 2-bridge link is the composition of a meridian-cyclic branched covering of a determined link and a standard cyclic covering of $S^3$ branched over a trivial knot. The particular case of singly-cyclic coverings of the Whitehead link
was studied in [14].

## 2 Cyclic branched coverings

Cyclic branched coverings of 2-bridge knots/links form a very important class of 3-manifolds, that is a natural generalization of the class of lens spaces. Indeed $M_{2,1}(\alpha/\beta)$ is the lens space $L(\alpha, \beta)$ and, when $\gcd(n, k) = 1$, $M_{n,k}(2/1)$ is the lens space $L(n, k)$ [74]. This class appears to be fairly rich, since it contains several interesting 3-manifolds, such as the Poincaré homology sphere, the Seifert–Weber hyperbolic dodecahedron space, the Euclidean Hantzsche–Wendt manifold, the hyperbolic Fomenko–Matveev–Weeks manifold and also an infinite family of Brieskorn manifolds.

More precisely, the Poincaré homology sphere [16] is the 5-fold cyclic branched covering of the trefoil knot $b(3,1)$ and the 3-fold cyclic branched covering of $b(5,1)$, i.e., $M_5(3/1) \cong M_3(5/1)$; the Seifert–Weber hyperbolic dodecahedron space [78] is the 5-fold singly-cyclic branched covering of the Whitehead link $b(8,3)$ defined by $k_1 = 1$ and $k_2 = 2$, i.e., $M_{5,2}(8/3)$ (its generalizations $M_{n,k}(8/3)$ were discussed in [14, 29, 87]); the Euclidean Hantzsche–Wendt manifold [85] is the 3-fold cyclic branched covering of the figure-eight knot $b(5,3)$, i.e., $M_3(5/3)$ (its generalizations $M_n(5/3)$, known as the Fibonacci manifolds, were studied in [16, 30, 32, 51, 53, 54, 73]); the hyperbolic Fomenko–Matveev–Weeks manifold [23, 37], which is the hyperbolic 3-manifold with the smallest known volume, is the 3-fold cyclic branched covering of the knot $b(7,3)$, i.e., $M_3(7/3)$ [50] (its generalizations $M_n(7/3)$ were studied in [2, 44]); the Brieskorn manifold $M(n, \alpha, 2)$ [57] is the $n$-fold strictly-cyclic branched coverings of the torus knots or links $b(\alpha, 1)$, i.e., $M_n(\alpha/1)$ (see also [11, 12, 79]).

Note that, as a consequence of the positive solution of the Smith conjecture [60], this family of manifolds contains no sphere, since $b(\alpha, \beta)$ is never a trivial knot.

Now we list some sufficient homeomorphism conditions.

**Proposition 2.1** Let $b(\alpha, \beta)$ be a 2-bridge link and $k, k' \in \mathbb{Z}_n - \{0\}$. Then:

(i) If $kk' = 1$, then $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,k'}(\alpha/\beta)$;

(ii) $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,k}(-\alpha/\beta)$;

(iii) $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,-k}(\alpha/(\beta - \alpha))$;
(iv) if the links $b(\alpha, \beta)$ and $b(\alpha', \beta')$ are equivalent, then $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,k}(\alpha'/\beta')$.

**Proof.** Let $m_1$ and $m_2$ be meridians corresponding to the components of $b(\alpha, \beta)$. (i) Let $f : M_{n,k}(\alpha/\beta) \to S^3$ and $f' : M_{n,k'}(\alpha/\beta) \to S^3$ the corresponding cyclic branched coverings of $b(\alpha, \beta)$. If $\bar{\omega}$ and $\bar{\omega}'$ are the associated monodromy maps, we have $\bar{\omega}[m_1] = 1$, $\bar{\omega}[m_2] = k$, $\bar{\omega}'[m_1] = 1$ and $\bar{\omega}'[m_2] = k'$. Therefore, $\bar{\omega}' \circ \rho_\# = k'\bar{\omega}$ and the two coverings are equivalent. (ii) The link $b(\alpha, -\beta)$ is equivalent to the mirror image of $b(\alpha, \beta)$. Let $\phi$ be an orientation reversing homeomorphism of $S^3$ sending $b(\alpha, \beta)$ to $b(\alpha, -\beta)$. Since $\phi$ preserve both the orientations of the two components of $b(\alpha, \beta)$, we have $\phi_\#[m_1] = -[m'_1]$ and $\phi_\#[m_2] = -[m'_2]$, where $m'_1$ and $m'_2$ are the generator meridians associated to $b(\alpha, -\beta)$. Therefore, $\bar{\omega}' \circ \phi_\# = -\bar{\omega}$ and the two coverings are equivalent. (iii) The link $b(\alpha, \beta - \alpha)$ is equivalent to the link $b(\alpha, \beta)$ with the opposite orientation in the second component. Let $\phi$ be the identity map on $S^3$, then $\phi_\#[m_1] = [m'_1]$ and $\phi_\#[m_2] = -[m'_2]$. Therefore, $\bar{\omega}' \circ \phi_\# = \bar{\omega}$ and the two coverings are equivalent. (iv) Let $\phi$ be the homeomorphism of $S^3$ realizing the equivalence. Without loss of generality we can assume that $\phi(K_1) = K'_1$ and $\phi(K_2) = K'_2$. Then $\phi_\#[m_1] = [m'_1]$ and $\phi_\#[m_2] = [m'_2]$. Therefore, $\bar{\omega}' \circ \phi_\# = \bar{\omega}$ and the two coverings are equivalent.

**Corollary 2.2** Let $b(\alpha, \beta)$ be an arbitrary 2-bridge link and $k \in \mathbb{Z}_n - \{0\}$ such that $\gcd(n, k) = 1$, then:

(i) $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,k\pm 1}(\alpha/\beta)$;

(ii) if $\beta^2 = \alpha \pm 1$, then $M_{n,k}(\alpha/\beta)$ is homeomorphic to $M_{n,\pm k\pm 1}(\alpha/\beta)$.

**Proof.** (i) See item (i) of the previous proposition. (ii) Assume $\beta^2 = \alpha + 1$. Since $\beta$ is odd, we have $\beta(\beta - \alpha) = \beta^2 - \beta\alpha = \beta^2 - \alpha = 1$. Therefore, $b(\alpha, \beta)$ is equivalent to $b(\alpha, \beta - \alpha)$ and from the previous proposition we obtain $M_{n,k}(\alpha/\beta) \cong M_{n,k}(\alpha/(\beta - \alpha)) \cong M_{n,-k}(\alpha/\beta)$. Now, let $\beta^2 = \alpha - 1$. Since $\beta$ is odd, we have $\beta(\alpha - \beta) = \beta\alpha - \beta^2 = \alpha - \beta^2 = 1$. Therefore, $b(\alpha, \beta)$ is equivalent to $b(\alpha, \alpha - \beta)$ and from the previous proposition we obtain $M_{n,k}(\alpha/\beta) \cong M_{n,k}(\alpha/(\alpha - \beta)) \cong M_{n,k}(\alpha/(\beta - \alpha)) \cong M_{n,-k}(\alpha/\beta)$.

When our manifolds have hyperbolic geometric structure, a partial converse to the previous results can be given (see Theorem 2.4 below). Recall that a 2-bridge knot or link $b(\alpha, \beta)$ is hyperbolic (i.e., the complement
$S^3 - b(\alpha, \beta)$ has hyperbolic structure) if and only if $b(\alpha, \beta)$ is non-toroidal, that is $\beta \not\equiv \pm 1 \mod \alpha$. Thus, the geometric structure of $M_{n,k}(\alpha/\beta)$ can be obtained in these cases from W. Thurston [84] and W. Dunbar [20] results. Moreover, when the branching set is toroidal, then $M_n(\alpha/\beta)$ turns out to be the Brieskorn manifold $M(n, \alpha, 2)$. Thus, we have the following result:

**Proposition 2.3 (i)** [33] If $\gcd(n, k) = 1$ and $\beta \not\equiv \pm 1 \mod \alpha$, then $M_{n,k}(\alpha/\beta)$ (or $M_n(\alpha/\beta)$ in the case of a knot) is hyperbolic for (i) $\alpha = 5$, $n \geq 4$ and (ii) $\alpha \neq 5$, $n \geq 3$. Moreover, $M_3(5/2)$ is Euclidean and $M_2(\alpha/\beta)$ is spherical for all $\alpha, \beta$.

(ii) [57] If $\beta \equiv \pm 1 \mod \alpha$, then $M_n(\alpha/\beta)$ is a spherical manifold for $1/n + 1/\alpha > 1/2$, a Nil-manifold for $1/n + 1/\alpha = 1/2$, and a $SL(2, \mathbb{R})$-manifold for $1/n + 1/\alpha < 1/2$.

From Theorem 1 of [77] (including the note (a) of page 293) and Theorem 4.1 of [13] (see tables of page 184), the next result holds.

**Theorem 2.4** Let $b(\alpha, \beta)$ be a 2-bridge hyperbolic link. If $\gcd(n, k) = 1$, then $M_{n,k'}(\alpha/\beta)$ and $M_{n,k}(\alpha/\beta)$ are homeomorphic if and only if

(i) $k' = k^{\pm 1}$, when $\beta^2 \neq \alpha \pm 1$;

(ii) $k' = \pm k^{\pm 1}$, when $\beta^2 = \alpha \pm 1$.

Volumes and Chern–Simons invariants of hyperbolic cyclic branched coverings of 2-bridge knots were obtained in [34], where the table for small values of parameters is given.

Now we will show that every $n$-fold strictly-cyclic covering of a 2-bridge knot or link can be obtained as the 2-fold branched covering of a certain $n$-periodic knot or link [18, 80]. If $\rho : S^3 \to S^3$ is the involution previously described, which leaves $b(\alpha, \beta)$ invariant and exchanges its bridges, we denote the quotient $b(\alpha, \beta)/\langle \rho \rangle$ by $b(1, \alpha, \beta)$, that is the trivial knot pictured in Figure 3. So $(S^3, b(\alpha, \beta))$ is a 2-fold covering of $(S^3, b(1, \alpha, \beta))$ branched over a trivial knot $B$ corresponding to the axis of the involution $\rho$. This covering gives us a natural way to construct a periodic generalization of 2-bridge knots/links. Denote by $b(n, \alpha, \beta)$ the preimage of $b(1, \alpha, \beta)$ under the $n$-fold covering of $S^3$ branched over $B$. This knot or link admits a natural $n$-bridge presentation (see an example in Figure 3) and will be called the $n$-cyclic extension of $b(\alpha, \beta)$. In particular, $b(2, \alpha, \beta) = b(\alpha, \beta)$ and $b(n, \alpha, 1)$ is the torus knot/link of type $(\alpha, n)$. 

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Figure 3: Diagrams of $b(1, 5, 3)$ and $b(3, 5, 3)$. 
Proposition 2.5 [18, 80] The manifold $M_n(\alpha/\beta)$ is the 2-fold branched covering of $b(n, \alpha, \beta)$.

Thus, as an example, the Hantzsche–Wendt manifold is the 3-fold branched covering of $b(2, 5, 3) = b(5, 3)$, pictured in Figure 1, and the 2-fold branched covering of $b(3, 5, 3)$, pictured in Figure 3, that is a 3-component link known as the Borromean rings (see also [54]).

Independently, a 2-fold covering description of $M_n(\alpha, \beta)$, when $\alpha$ is odd, arising from the generalized Takahashi manifolds (see Section 5), was obtained in [65].

3 Polyhedral construction

A classical method for constructing (orientable) closed 3-manifolds consists in the pairwise identification of (oppositely oriented) boundary faces of a triangulated 3-ball (see [78]). The resulting quotient complex triangulates a closed pseudomanifold, which is a manifold if and only if its Euler characteristic vanishes.

For strictly-cyclic branched coverings of 2-bridge knots/links $b(p, q)$ this construction was realized by J. Minkus [58]. The idea is based on the method of realizing lens spaces. Here we will briefly describe the construction.

Consider the boundary 2-sphere $S^2$ of the 3-ball $B^3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$. Draw $n$ equally spaced great semicircles joining the north pole $N = (0, 0, 1)$ to the south pole $S = (0, 0, -1)$. This decomposes $S^2$ into $n$ congruent lunes. Subdivide each semicircle into $p$ equal segments by drawing $p - 1$ equally spaced vertices on each semicircle. In this way, each lune can then be viewed as a curvilinear $2p$ sided polygon on $S^2$. Now bisect each lune by drawing a great circle arc inside the lune, joining the vertex which is $q$ segments down from $N$ (the point $P_i$ of Figure 4) on each semicircle with the vertex $q$ segments up from $S$ to the next clockwise semicircle. Figure 4 shows this decomposition of $\partial B^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$, where $S = \infty$.

The result is the decomposition of $\partial B^3$ into $2n$ regions $R_i, R'_i$, $i = 1, \ldots, n$. The regions $R_i$ are around $N$ and the regions $R'_i$ are around $S$, and each $R'_i$ can be reached from the corresponding $R_i$ by moving $R_i$ counterclockwise to the adjacent lune and then shifting from the northern to the southern hemisphere of $\partial B^3$. The cell 3-complex $\tilde{M}_n(p, q)$ is obtained from $B^3$ by identifying $R_i$ with $R'_i$ on $\partial B^3$ for each $i = 1, \ldots, n$ by an orientation reversing
Figure 4: Minkus polyhedral schemata for $\tilde{M}_n(p, q)$. 
Figure 5: Minkus polyhedral schemata for the Hantzsche–Wendt manifold.

homeomorphism which matches the vertex $P_i$ of $R_i$ with the vertex $P_{i-1}$ of $R'_i$.

Theorem 3.1 \cite[Theorem 7]{58} The manifold $\tilde{M}_n(p,q)$ is the $n$-fold strictly-cyclic branched covering of the 2-bridge knot or link $b(p,q)$.

Therefore the manifolds $\tilde{M}_n(p,q)$ are homeomorphic to the above defined manifolds $M_n(p/q)$ and provide a polyhedral construction for them. As an example, Figure 5 gives the polyhedral construction of the Hantzsche–Wendt manifold, that is $M_3(5/3)$ in our notations.

The Minkus polyhedral construction only produces strictly-cyclic branched coverings of 2-bridge knots/links. The generalization to singly-cyclic coverings $M_{n,k}(p/q)$ is straightforward and is illustrated in Figure 6, which is the same as Figure 5 of \cite{62}.

In this case, the cell 3-complex $M_{n,k}(p/q)$ is obtained from $B^3$ by identifying $R_i$ with $R'_i$ on $\partial B^3$ for each $i = 1, \ldots, n$ by an orientation reversing homeomorphism which matches the vertex $P_i$ of $R_i$ with the vertex $P_{i-k}$ of $R'_i$.

We remark that another polyhedral construction for $M_n((2\ell - 1)/\ell)$ is presented in \cite{45}. 

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Figure 6: Polyhedral schemata for $M_{n,k}(p/q)$. 
4 Heegaard diagram construction and genus

In this section we will discuss Heegaard diagrams as well as cyclically symmetric Heegaard diagrams for our manifolds.

Heegaard diagrams for meridian-cyclic branched coverings of 2-bridge knots/links can be directly obtained from the coloured graph construction that will be discussed in Section 6 (see [62]). More precisely, a Heegaard diagram of genus \( n - 1 \) of \( M_{n,k}(\alpha/\beta) \), with \( \gcd(n, k) = 1 \), can be obtained from Figure 7 by identifying the disk \( C_i \) with the disk \( C'_i \), for \( i = 1, \ldots, n - 1 \), according to the numeration of the vertices, and removing one of the \( n \) closed curves arising from these identifications, illustrated by the dashed lines in Figure 7.

Figure 8 presents a Heegaard diagram of \( M_3(5/3) \) obtained in this way.

At the same time, it is natural to ask about Heegaard diagrams for our manifolds corresponding to cyclic coverings. The connections between cyclic branched coverings of knots and cyclic presentations of groups induced by suitable Heegaard diagrams have recently been discussed in several papers [2, 11, 12, 21, 29, 30, 41, 43, 44, 51]. In order to investigate these relations, M.J. Dunwoody introduced in [21] a class of Heegaard diagrams, depending on six integers, having a cyclic symmetry and encoding a cyclic presentation for the fundamental group of the represented manifold \( D(a,b,c,n,r,s) \). In [28] it has been shown that the 3-manifolds represented by these diagrams (called Dunwoody manifolds) are cyclic coverings of lens spaces branched over genus one 1-knots (also called (1,1)-knots). As a corollary, it has been demonstrated that, for particular values of the parameters, the Dunwoody manifolds turn out to be cyclic coverings of \( S^3 \) branched over some knots. This gives a positive answer to a conjecture made by Dunwoody, which has also been independently proved in [81].

It is interesting to note that the class of Dunwoody manifolds properly contains the class of cyclic branched coverings of 2-bridge knots, as stated in the following theorem. Note that in the statement \( \bar{s} \) is an integer only depending on \( a \) and \( r \) (see details in Section 3 of [28]).

**Theorem 4.1** [28] For all \( a, r > 0 \) and \( n > 1 \), the \( n \)-fold cyclic covering of \( S^3 \) branched over \( b(2a + 1, 2r) \) is the Dunwoody manifold \( D(a,0,1,n,r,\bar{s}) \). Thus, all branched cyclic coverings of 2-bridge knots are Dunwoody manifolds.

As a consequence of the previous theorem, each \( n \)-fold cyclic branched covering of a 2-bridge knot admits a Heegaard diagram of genus \( n \) with
Figure 7.
Figure 8: An Heegaard diagram of genus 2 for the Hantzsche–Wendt manifold.
Figure 9: A 3-symmetric Heegaard diagram of genus 3 for $D(2, 0, 1, 3, 1, 0)$, the Hantzsche–Wendt manifold.
a cyclic symmetry of order $n$. These diagrams induce “geometric” cyclic presentations for the corresponding fundamental groups (see Section 7).

Now we discuss some results on genus (classical, $p$-symmetric and equivariant) of our manifolds.

From the standard representation of a 2-bridge knot/link as a 4-plat $(S^3, b(\alpha, \beta)) = (B', A') \cup_f (B'', A'')$, where $B', B''$ are 3-balls and $A' \subset B'$, $A'' \subset B''$ are pairs of trivially properly embedded arcs, we get a Heegaard splitting of $M_{n,k',k''}(\alpha/\beta)$, where the splitting surface is an $n$-fold cyclic covering of $S^2$ branched over 4 points. From the Riemann-Hurwitz formula, the surface has genus $g = n + 1 - \gcd(n, k') - \gcd(n, k'')$ (see details in [4]).

Therefore we have the following:

**Proposition 4.2** Let $g(M)$ be the genus of a 3-manifold $M$. Then

$$g(M_{n,k',k''}(\alpha/\beta)) \leq n + 1 - \gcd(n, k') - \gcd(n, k'').$$

In particular, for singly-cyclic coverings $g(M_{n,k}(\alpha/\beta)) \leq n - \gcd(n, k)$ and for meridian-cyclic coverings $g(M_{n,k}(\alpha/\beta)) \leq n - 1$. Thus, for strictly-cyclic coverings $g(M_{n}(\alpha/\beta)) \leq n - 1$.

From Theorem 4 of [64], for the case of strictly-cyclic branched coverings the same estimation also holds for the $n$-symmetric genus introduced by J. Birman and H. Hilden in [3].

**Proposition 4.3** Let $g_n(M)$ be the $n$-symmetric genus of a 3-manifold $M$. Then

$$g_n(M_{n}(\alpha/\beta)) \leq n - 1.$$ 

In many interesting cases the result of Proposition 4.2 can be improved.

**Proposition 4.4 (i)** For all $n, \alpha > 1$ we have

$$g(M_{n}(\alpha/1)) \leq \min\{\alpha - 1, n - 1\}.$$

(ii) For all $n, c > 1$ we have

$$g(M_{n}((3c - 1)/3)) \leq \min\{c, n - 1\}.$$
Proof. (i) The \( n \)-fold strictly-cyclic covering of \( b(\alpha, 1) \) is the 2-fold covering of the torus knot/link of type \((\alpha, n) \) [57], which can be presented as the closure of a \( \alpha \)-string braid. So it has bridge number \( b \leq \alpha \) and then \( g \leq \alpha - 1 \) [3]. (ii) The \( n \)-fold strictly-cyclic covering of \( b(3c - 1, 3) \) is the 2-fold covering of \( b(n, 3c - 1, 3) \), which can be presented as the closure of a \((c + 1)\)-string braid. So it has bridge number \( b \leq c + 1 \) and then \( g \leq c \) [3]. □

In particular, from (ii) we see that the \( n \)-fold cyclic branched covering of the figure-eight knot \((c = 2)\) has genus \( g = 2 \) if \( n > 2 \) and \( g = 1 \) if \( n = 2 \) (see also [55]). Similarly, the \( n \)-fold strictly-cyclic branched coverings of the Whitehead link \((c = 3)\) have genus \( g \leq 3 \).

Recall that the maximally possible order of a finite group of orientation-preserving homeomorphisms of a handlebody \( V_g \) of genus \( g \geq 2 \) equals \( 12(g - 1) \) (see [86]), analogously to the classical Hurwitz \( 84(g - 1) \)-bound for a closed Riemann surfaces of genus \( g \geq 2 \).

According to [88], a closed 3-manifold \( M \) is said to be a \( G \)-manifold of genus \( g \) if it admits an action of the finite group \( G \) and \( g \) is the minimal genus of a Heegaard splitting of \( M \) for which both handlebodies are invariant under the \( G \)-action. A \( G \)-manifold of genus \( g > 1 \) is called minimal if the induced \( G \)-action on each of the two handlebodies of an invariant Heegaard splitting of genus \( g \) is a strong genus action (i.e., there is no action of \( G \) on a handlebody of genus \( g \), with \( 1 < \bar{g} < g \)). Moreover, if \( G \) has maximal positive order \( 12(g - 1) \) then the \( G \)-manifold \( M \) and the \( G \)-action are called maximally symmetric. For \( n > 2 \), every minimal \( \mathbb{Z}_n \)-manifold is a minimal \( \mathbb{D}_n \)-manifold, where \( \mathbb{D}_n \) is the dihedral group of order \( 2n \). If \( n > 2 \) is prime, every minimal \( \mathbb{Z}_n \)-manifold is a minimal \( (\mathbb{D}_n \times \mathbb{Z}_2) \)-manifold [88]. Each manifold of genus 2 is a \( G \)-manifold of genus 2, where \( G \) is one of the four groups \( \mathbb{Z}_2, \mathbb{D}_2, \mathbb{D}_4 \) or \( \mathbb{D}_6 \), and the \( \mathbb{D}_6 \)-manifolds are maximally symmetric [88].

The cyclic branched coverings of 2-bridge knots/links play an important role in this theory.

Theorem 4.5 [88]

(i) The minimal \( (\mathbb{D}_n \times \mathbb{Z}_2) \)-manifolds, of genus \( g = n - 1 \), are exactly the \( n \)-fold strictly-cyclic branched coverings of 2-bridge knots/links. The minimal \( \mathbb{Z}_n \) or \( \mathbb{D}_n \)-manifolds that are not minimal \( (\mathbb{D}_n \times \mathbb{Z}_2) \)-manifolds are cyclic branched coverings of 2-bridge links with two components of different branching index.
The maximally symmetric $D_6$-manifolds are exactly the 3-fold cyclic branched coverings of 2-bridge knots/links.

5 Surgery construction

In this section we describe a symmetric surgery presentation of cyclic branched coverings of 2-bridge knots. This presentation comes from a more general construction of generalized Takahashi manifolds introduced in [65]. For any pair of positive integers $m$ and $n$, we consider the link $L_{n,m} \subset S^3$ with $2mn$ components drawn in Figure 10. All its components $c_{i,j}$, $1 \leq i \leq 2n$, $1 \leq j \leq m$, are unknotted circles and they form $2n$ subfamilies of $m$ unlinked circles $c_{i,j}$, $1 \leq j \leq m$, with a common center. The link $L_{n,m}$ has a cyclic symmetry of order $n$ which permutes these $2n$ subfamilies of circles.

Consider the manifold obtained by Dehn surgery on $S^3$, along the link $L_{n,m}$, such that the surgery coefficients $p_{k,j}/q_{k,j}$ correspond to the components $c_{2k,1-j}$, and $r_{k,j}/s_{k,j}$ correspond to the components $c_{2k,j}$, where $1 \leq k \leq n$ and $1 \leq j \leq m$. Without loss of generality, we can always suppose that $\gcd(p_{k,j},q_{k,j}) = 1$, $\gcd(r_{k,j},s_{k,j}) = 1$ and $p_{k,j}, r_{k,j} \geq 0$.

We will denote the resulting 3-manifold by $T_{n,m}(p_{k,j}/q_{k,j}; r_{k,j}/s_{k,j})$. This manifold will be called a generalized Takahashi manifold since for $m = 1$ we get the manifolds introduced by M. Takahashi in [83].

When the surgery coefficients are $n$-periodic, i.e. $p_{k,j} = p_j$, $q_{k,j} = q_j$, $r_{k,j} = r_j$, and $s_{k,j} = s_j$, the resulting manifold $T_{n,m}(p_1/q_1, \ldots, p_m/q_m; r_1/s_1, \ldots, r_m/s_m)$ is said to be a generalized periodic ($n$-periodic) Takahashi manifold.

The following theorem generalizes the result obtained in [83] for periodic Takahashi manifolds, and gives the relation between the generalized periodic Takahashi manifolds and the cyclic branched coverings of 2-bridge knots.

**Theorem 5.1** [65] The manifold $T_{n,m}(1/q_1, \ldots, 1/q_m; 1/s_1, \ldots, 1/s_m)$ is homeomorphic to $M_n(\alpha/\beta)$, where $\alpha/\beta$ is the rational number defined by the continued fraction $[-2q_1, 2s_1, \ldots, -2q_m, 2s_m]$.

Because every 2-bridge knot admits a Conway representation with an even number of even parameters (see Exercise 2.1.14 of [10]), we have the following property:

**Corollary 5.2** [65] The family of generalized periodic Takahashi manifolds contains all the cyclic branched coverings of 2-bridge knots.
As a result, a surgery presentation with cyclic symmetry is obtained for all cyclic branched coverings of 2-bridge knots (see Figure 10).

As an example, the surgery description of the Hantzsche–Wendt manifold (that is $M_3(5/2)$ in the above notation) is illustrated in Figure 11.

6 Coloured graph construction

We recall some facts about the coloured-graph presentation of 3-manifolds.

A 4-coloured graph is a pair $(\Gamma, \gamma)$, where $\Gamma$ is a finite connected 4-regular graph without loops, and $\gamma : E(\Gamma) \to \Delta_3 = \{0, 1, 2, 3\}$ is a proper edge-coloration (i.e., adjacent edges have different colours). Every 4-coloured graph represents a pseudosimplicial complex $K(\Gamma)$ defined in the following way: (i) take a 3-simplex $\sigma(x)$ for each vertex $x \in V(\Gamma)$ and label its vertices by the elements of $\Delta_3$; (ii) identify, for every pair $x, y \in V(\Gamma)$ of $c$-adjacent vertices, the 2-faces of $\sigma(x)$ and $\sigma(y)$ opposite the vertices labelled
The underlying space $|K(\Gamma)|$ is a connected 3-dimensional pseudomanifold, which is orientable if and only if $\Gamma$ is bipartite \cite{22}. Note that only isolated singular points may appear, and such spaces are said to be singular manifolds in \cite{59}. A 3-gem is a 4-coloured graph representing a 3-manifold; every manifold $M$ is representable by gems \cite{22,48,71}. A gem is called a crystallization if, for each colour $c \in \Delta_3$, the subgraph of $\Gamma$ obtained by removing all $c$-coloured edges is connected.

There is a strict connection between crystallizations and Heegaard diagrams of 3-manifolds \cite{71}. Let $c'$ and $c''$ be two colours from $\Delta_3$, and $A = \{c', c''\}$, $B = \Delta_3 - A$. A Heegaard diagram of the represented manifold can be obtained just by removing one $A$-cycle (i.e., formed by edges coloured by the colours of $A$) and one $B$-cycle. The remaining $A$-cycles and $B$-cycles give, respectively, the two systems of curves of the Heegaard diagram. In other words, the $A$-cycles and the $B$-cycles of a crystallization represent an extended Heegaard diagram of the manifold (see \cite{66}).

Also manifolds of arbitrary dimension can be represented by edge-coloured graphs, which give a combinatorial way of representing manifolds; for general references see \cite{22} and \cite{18}.

The family of Lins-Mandel 4-coloured graphs $G(n, p, q, c)$, with $n, p > 0$, $q \in \mathbb{Z}_{2p}$, $c \in \mathbb{Z}_n$ and gcd$(p, q) = 1$, has been defined in \cite{49}. The set of vertices of $G(n, p, q, c)$ is $V = \mathbb{Z}_n \times \mathbb{Z}_{2p}$ and the coloured edges are obtained...
by the following four fixed-point-free involutions on $V$:

$$
u_0(i, j) = (i + c\eta(j - q), 1 - j + 2q), \quad \nu_1(i, j) = (i + \eta(j), 1 - j),$$

$$\nu_2(i, j) = (i, j + (-1)^j), \quad \nu_3(i, j) = (i, j - (-1)^j);$$

where $\eta : \mathbb{Z}_{2p} \to \{-1, +1\}$ is the map defined by

$$\eta(j) = \begin{cases} +1 & \text{if } 1 \leq j \leq p - 1 \\ -1 & \text{otherwise} \end{cases}.$$

For each $k \in \{0, 1, 2, 3\}$, we join the vertices $v, w \in V$ by a $k$-coloured edge if and only if $\nu_k(v) = w$.

Roughly speaking, the graph $G(n, p, q, c)$ is obtained by taking $n$ copies $C_i, i \in \mathbb{Z}_n$, of the $\{2, 3\}$-cycle of length $2p$ (so that $V(C_i) = \{(i, j) \mid j \in \mathbb{Z}_{2p}\}$) joined with $C_{i-1}$ and $C_{i+1}$ by $p$ edges of colour 1, and with $C_{i-c}$ and $C_{i+c}$ by $p$ edges of colour 0.

Each graph $G(n, p, q, c)$ is connected and bipartite; hence, it represents a connected, orientable 3-dimensional pseudomanifold $S(n, p, q, c)$. This class of graphs and spaces have been intensively studied \[5, 6, 7, 8, 9, 10, 18, 27, 39, 48, 49, 50, 61, 62, 63\]. Remark that Lins-Mandel spaces have been introduced as a combinatorial generalization of the lens spaces, since $G(2, p, q, 1)$ is the standard graph representing the lens space $L(p, q)$ (see \[19\]).

The following characterization of the Lins-Mandel gems exists:

**Proposition 6.1** \[61\] A Lins-Mandel graph $G(n, p, q, c)$ represents a 3-manifold if and only if either $p$ is even or $c = 0, (-1)^q$.

Moreover, a Lins-Mandel gem $G(n, p, q, c)$ is a crystallization if and only if $\gcd(n, c) = 1$ \[6\].

**Theorem 6.2** \[62, 63\] Lins-Mandel spaces have the following topological structure:

(i) If $S(n, p, q, c)$ is a manifold with $c \neq 0$ and $p > 1$, then it is homeomorphic to $M_{n-c}(p/q)$, otherwise it is homeomorphic to $S^3$.

(ii) If $S(n, p, q, c)$ is not a manifold, then it is an $n$-fold cyclic covering of $S^3$ branched over a $\theta$-graph, which is embedded as the 2-bridge knot $b(p, q)$ with an unknotting tunnel.
Figure 12: The Lins–Mandel graph $G(n, p, q, c)$. 

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Thus, the manifold $S(n, p, q, c)$ is a singly-cyclic branched covering of $b(p, q)$. In particular the covering is strictly-cyclic when $c = (-1)^q$, is almost-strictly-cyclic when $c = \pm 1$, and is meridian-cyclic when $\gcd(n, c) = 1$.

Remark that by removing the cycle $C_n$ from the graph $G(n, p, q, c)$, with $\gcd(n, c) = 1$, we will get exactly the graph pictured in Figure 7, once the relative identifications of $C_i$ with $C_i'$, for $i = 1, \ldots, n-1$, have been performed, according to the numeration of the vertices.

Necessary and sufficient conditions for the isomorphism between Lins-Mandel gems are obtained in [50], when $n, p > 2$. As a consequence, sufficient conditions for the homeomorphism between Lins-Mandel manifolds not homeomorphic to the sphere or the lens spaces directly follow.

**Theorem 6.3** [50] For $n, p > 2$ the following isomorphisms of graphs hold:

(i') If $p$ is even and $\gcd(n, c) \neq 1$, then $G(n', p', q', c') \cong G(n, p, q, c)$ if and only if

$$(1) \ n' = n, \ (2) \ p' = p, \ (3) \text{ either } \begin{cases} q' = \pm q^\pm 1 \\ c' = c \end{cases} \text{ or } \begin{cases} q' = \pm q^\pm 1 + p \\ c' = -c \end{cases}.$$  

(i'') If $p$ is even and $\gcd(n, c) = 1$, then $G(n', p', q', c') \cong G(n, p, q, c)$ if and only if

$$(1) \ n' = n, \ (2) \ p' = p, \ (3) \text{ either } \begin{cases} q' = \pm q^\pm 1 \pm 1 \\ c' = c^\pm 1 \end{cases} \text{ or } \begin{cases} q' = \pm q^\pm 1 + p \\ c' = -c^\pm 1 \end{cases}.$$  

(ii) If $p$ is odd, then $G(n', p', q', (-1)^q) \cong G(n, p, q, (-1)^q)$ if and only if

$$(1) \ n' = n, \ (2) \ p' = p, \ (3) \ q' \equiv \pm q^\pm 1 \mod p.$$  

Hence, if one of the above conditions holds, then the manifolds $S(n, p, q, c)$ and $S(n', p', q', c')$ are homeomorphic.

Observe that the isomorphism conditions of part (ii) of the previous theorem are the same as the homeomorphism conditions for lens spaces. This is not true for part (i), since, in this case, the situation is complicated by the presence of the additional parameter $c$.

Cases where $p$ is even are particularly interesting because the graph always represents a manifold without any restriction on $c$. From Theorem 6.3 we get:

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Corollary 6.4 \cite{56} Let \( n, p, q \) be fixed, with \( n, p \geq 3 \) and \( p \) even. Then 
\[ G(n, p, q, c') \cong G(n, p, q, c) \] 
if and only if

(i) \( c' = c \), when \( \gcd(n, c) \neq 1 \) and \( q^2 \neq p \pm 1 \);
(ii) \( c' = \pm c \), when \( \gcd(n, c) \neq 1 \) and \( q^2 = p \pm 1 \);
(iii) \( c' = c^{\pm 1} \), when \( \gcd(n, c) = 1 \) and \( q^2 \neq p \pm 1 \);
(iv) \( c' = \pm c^{\pm 1} \), when \( \gcd(n, c) = 1 \) and \( q^2 = p \pm 1 \).

Due to Theorem 2.4 and Corollary 6.4, in many cases graphs distinguish manifolds:

Corollary 6.5 Assume \( n, p > 2 \) and \( q \neq \pm 1, p \pm 1 \). For \( \gcd(n, c) = 1 \), the manifolds \( S(n, p, q, c') \) and \( S(n, p, q, c) \) are homeomorphic if and only if the graphs \( G(n, p, q, c') \) and \( G(n, p, q, c) \) are isomorphic.

The Lins-Mandel family contains only singly-cyclic coverings. For this reason it has been extended in \cite{62}, in order to obtain the whole class of cyclic branched coverings of 2-bridge knots/links.

This new class of 4-coloured graphs \( \tilde{G}(n, p, q, c') \) depends on five integer parameters, with \( n, p > 0, q \in \mathbb{Z}_{2p}, c, c' \in \mathbb{Z}_n, \gcd(p, q) = 1 \) and \( \gcd(n, c, c') = 1 \). Each \( \tilde{G}(n, p, q, c, c') \) is defined by the following four fixed-point-free involutions on the set \( V = \mathbb{Z}_n \times \mathbb{Z}_{2p} \):

\[
\begin{align*}
\tilde{\iota}_0 &= \iota_0, & \tilde{\iota}_1(i, j) &= (i + c'\eta(j), 1 - j), \\
\tilde{\iota}_2 &= \iota_2, & \tilde{\iota}_3 &= \iota_3.
\end{align*}
\]

The graph \( \tilde{G}(n, p, q, c, c') \) represents a connected, orientable pseudomanifold \( \tilde{S}(n, p, q, c, c') \).

Proposition 6.6 \cite{62} The following properties hold:

(i) \( \tilde{G}(n, p, q, c, c') \) represents a 3-manifold if and only if either \( p \) is even or at least one of the conditions (1) \( c = 0 \), (2) \( c' = 0 \), (3) \( c = (-1)^q c' \) is satisfied.

(ii) \( \tilde{S}(n, p, q, c, 1) \cong S(n, p, q, c) \).

(iii) \( \tilde{S}(n, p, q, 0, c') \cong \tilde{S}(n, p, q, c, 0) \cong \tilde{S}(n, 1, 1, -c', c') \cong S^3 \).
We give the connection between generalized Lins-Mandel manifolds and cyclic branched coverings of 2-bridge knots/links.

**Theorem 6.7** [62] The 3-manifold $\tilde{S}(n, p, q, c, c')$, with $c \neq 0 \neq c'$, is homeomorphic to $M_{n, c', -c}(p/q)$. Therefore, the class of generalized Lins-Mandel manifolds $\tilde{S}(n, p, q, c, c')$, with $c \neq 0 \neq c'$ and $p > 1$, is precisely the class of all cyclic coverings of $S^3$ branched over the 2-bridge knots/links (with the exception of the trivial link with two components).

Another family of 4-coloured graphs representing cyclic branched coverings of 2-bridge knots is described in [47].

### 7 Fundamental groups

A presentation of the fundamental group of $M_{n, k}(\alpha, \beta)$ can be obtained from the coloured graph construction. Define the following words in $x_1, \ldots, x_n$ with subscripts mod $n$:

$$Q_i = \prod_{j=0}^{n'-1} x_{i-jk}, \quad 1 \leq i \leq \gcd(n, k); \quad Q_i' = \prod_{j=0}^{\alpha-1} x_{i+e_j}, \quad 1 \leq i \leq n;$$

with $n' = n/\gcd(n, k)$, $e_j = -\mu(2j\beta)$ and

$$s_j = \begin{cases} -k \sum_{h=1}^{j} \mu(\alpha + 2\beta - 2h\beta) - \sum_{h=1}^{j} \mu(\alpha + \beta - 2h\beta) & \text{if } e_j = +1 \\ -k \sum_{h=1}^{j} \mu(\alpha + 2\beta - 2h\beta) - \sum_{h=1}^{j} \mu(\alpha + \beta - 2h\beta) + k & \text{if } e_j = -1 \end{cases}$$

where $\mu : \mathbb{Z}_{2\alpha} \to \{-1, +1\}$ is the map defined in Section 6.

**Theorem 7.1** [62] The fundamental group of $M_{n, k}(\alpha, \beta)$ has the following presentation

$$\pi_1(M_{n, k}(\alpha, \beta)) = \langle x_1, \ldots, x_n \mid Q_i = 1, 1 \leq i \leq \gcd(n, k), \quad Q_i' = 1, 1 \leq i \leq n \rangle,$$

where $Q_i$ and $Q_i'$ are as above.
Since the cyclic branched coverings of 2-bridge knots have a cyclic homeomorphism, it is natural to wonder about some cyclic presentations of their fundamental groups. We recall that a finite balanced presentation of a group \( G \cong \langle x_1, \ldots, x_n | r_1, \ldots, r_n \rangle \) is said to be a cyclic presentation if there exists a word \( w \) in the free group \( F_n \) generated by \( x_1, \ldots, x_n \) such that the relators of the presentation are \( r_k = \theta_n^{k-1}(w), k = 1, \ldots, n \), where \( \theta_n : F_n \to F_n \) denotes the automorphism defined by \( \theta_n(x_i) = x_{i+1} \) (subscripts mod \( n \), \( i = 1, \ldots, n \)).

Let us denote this cyclic presentation (and the related group) by the symbol \( G_n(w) \), so that:

\[
G_n(w) = \langle x_1, x_2, \ldots, x_n | w, \theta_n(w), \ldots, \theta_n^{n-1}(w) \rangle.
\]

A group is said to be cyclically presented if it admits a cyclic presentation. The polynomial associated with the cyclic presentation \( G_n(w) \) is given by

\[
f_w(t) = \sum_{i=1}^{n} a_i t^{i-1},
\]

where \( a_i \) is the exponent sum of \( x_i \) in \( w \). For the theory of cyclically presented groups we refer to [38].

Two different cyclic presentations for the fundamental groups of the manifolds \( M_n(\alpha/\beta) \) are obtained in [58] and [65]. Remark that explicit cyclic presentations different from the above are listed in the Appendix of [14], for 2-bridge knots up to nine crossings.

From the polyhedral construction of \( M_n(\alpha/\beta) \) the following presentation of the fundamental group holds. Denote by

\[
R_{\alpha/\beta}(x_1, x_2, \ldots, x_n) = x_1x_1^{-1}x_1x_2x_1^{-1}x_1x_3x_1^{-1} \cdots x_1x_{s_\alpha-1},
\]

with

\[
s_j = s_j(\alpha, \beta) = \sum_{i=1}^{j} (-1)^{\lceil i\beta^{-1}/\alpha \rceil},
\]

where \( \beta^{-1} \) is the inverse of the element \( \beta \) in \( \mathbb{Z}_{2\alpha} \) and \( \lceil x \rceil \) denotes the integral part of \( x \).

**Theorem 7.2** [58, Theorem 10] The fundamental group of \( M_n(\alpha, \beta) \) admits the presentation:

\[
\langle x_1, \ldots, x_n | R_{\alpha/\beta}(x_i, \ldots, x_{i+n-1}) = 1, \quad i = 1, \ldots, n \rangle
\]
if \( \alpha \) is odd,

\[
\langle x_1, \ldots, x_n, y | x_n = 1, R_{\alpha/\beta}(x_i, \ldots, x_{i+n-1}) = y, i = 1, \ldots, n \rangle \tag{2}
\]

if \( \alpha \) is even.

Remark that (1) is a cyclic presentation and (2) is a non-cyclic one.

There is a nice relation between the Alexander polynomial of a 2-bridge knot and the polynomial associated with the above cyclic presentation.

**Proposition 7.3** [58, Theorem 11] The Alexander polynomial of the knot \( b(\alpha, \beta) \) is equal to the polynomial associated with the cyclic presentation (1), up to units of \( \mathbb{Z}[t, t^{-1}] \).

According to [58, Remark 4], this property holds for a wider class of knots and cyclic presentations of their cyclic branched coverings.

The following cyclic presentation for \( \pi_1(M_n(\alpha/\beta)) \), when \( \alpha \) is odd, arises from the surgery description of the manifold as Takahashi manifold (see Section 5).

**Theorem 7.4** [53] Let \( M_n(\alpha/\beta) \) be the \( n \)-fold cyclic branched covering of the 2-bridge knot \( b(\alpha/\beta) \), with \( \alpha/\beta = [-2q_1, 2s_1, \ldots, -2q_m, 2s_m] \). Then its fundamental group has the following cyclic presentation:

\[
\pi_1(M_n(\alpha/\beta)) = \langle x_1, \ldots, x_n | w_{\alpha/\beta}(x_i, \ldots, x_{i+n-1}) = 1, \ i = 1, \ldots, n \rangle,
\]

where

\[
w_{\alpha/\beta}(x_i, \ldots, x_{i+n-1}) = b_{i+1,m}^{s_i}d_{i+1,m}^{s_i},
\]

for \( i = 1, \ldots, n \) (subscripts mod \( n \)). The right part of these formulas are defined by the recurrent rule

\[
d_{k,j} = b_{k,j-1}^{-s_j}d_{k,j-1}^{s_{j-1}}b_{k-1,j-1}, \quad b_{k,j} = d_{k,j}^{q_j}b_{k,j-1}d_{k+1,j}^{-q_j}, \quad j = 2, \ldots, m
\]

and

\[
b_{k,1} = d_{k,1}^{q_1}d_{k+1,1}^{-q_1},
\]

where \( x_k = d_{k,1} \), for \( k = 1, \ldots, n \).
We will illustrate the result obtained for $m = 1$ and $m = 2$.

If $m = 1$, then $\alpha/\beta = -2q + 1/(2s)$. Observe that this case corresponds to Takahashi manifolds and was discussed in [12, 13]. We get

$$\pi_1(M_n(-2q + \frac{1}{2s})) = \langle x_1, \ldots, x_n \mid (x_k^q x_{k+1}^{-q})^{-s} x_k (x_{k-1}^q x_k^{-q})^s = 1, \quad k = 1, \ldots, n \rangle.$$  

For example, if $q = -1$ and $s = 1$ then $\alpha/\beta = 5/2$, and $b(5/2)$ is the figure-eight knot $4_1$ [4]. So, its $n$-fold cyclic branched covering has the fundamental group with the following cyclic presentation

$$\pi_1(M_n(5/2)) = \langle x_1, \ldots, x_n \mid x_{k+1}^{-1} x_k^2 x_{k-1}^{-1} x_k = 1, \quad k = 1, \ldots, n \rangle$$

(compare with [13, 12, 13]).

If $m = 2$ then $b(\alpha/\beta)$ has Conway parameters $[-2q_1, 2s_1, -2q_2, 2s_2]$ and $\pi_1(M_n(\alpha/\beta))$ has the following presentation:

$$\langle x_1, \ldots, x_n \mid w_{\alpha/\beta}(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) = 1, \quad k = 1, \ldots, n \rangle,$$

where

$$w_{\alpha/\beta}(x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}) =
\left[\left(\left(x_{k-1}^{-q_1} x_k x_{k+1}^q\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right)^q x_k^q x_{k+1}^q \left(\left(x_{k+1}^q x_{k+2}^{-q_1}\right)^{-s_1} x_{k+1} (x_k^q x_{k+1}^{-q_1})^{s_1}\right)^q x_k^q x_{k+1}^q\right]^{-s_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_k^q x_{k+1}^{-q_1})^{s_1}\right]^{-q_2} x_k^q x_{k+1}^q\left[\left(x_k^q x_{k+1}^{-q_1}\right)^{-s_1} x_k (x_
We point out that the problem of determining whether a balanced presentation of a group is geometric – i.e., induced by a Heegaard diagram of a closed orientable 3-manifold – is of considerable interest within geometric topology [67, 68, 69, 70, 82].

Corollary 7.5 [28] The fundamental group of every branched cyclic covering of a 2-bridge knot admits a cyclic presentation which is geometric.

8 Homology

In this section we present the homology groups of some classes of cyclic branched coverings of 2-bridge knots/links. Recall (see, for example [40, p. 71]) that when \( H_1(M_n(\alpha/\beta)) \) is a finite abelian group, the order of this group is given by the absolute value \( \left| \prod_{k=1}^{n} \Delta(\alpha, \beta)\left(\zeta^k\right) \right| \) where \( \Delta(\alpha, \beta)(t) \) is the Alexander polynomial of \( b(\alpha, \beta) \) and \( \zeta \) is an \( n \)-th primitive root of unity.

As is well known, the manifold \( M_n(\alpha/1) \) is homeomorphic to the Brieskorn manifold \( M(n, \alpha, 2) \). Thus, in this case, the homology groups can be obtained from [72].

If \( \alpha \) is odd, then the \( n \)-fold cyclic branched covering is unique and the homology groups are the following.

**Proposition 8.1** [10] If \( \alpha \) is odd then:

\[
H_1(M_n(\alpha/1)) \cong \begin{cases} 
Z^{d-1} \oplus Z_{\alpha/d} & \text{if } n \text{ is even} \\
Z_{d-1}^2 & \text{if } n \text{ is odd}
\end{cases}
\]

where \( d = \gcd(n, \alpha) \).

If \( \alpha \) is even the situation is rather more complicated, since \( k \) can assume any value in \( \mathbb{Z}_n - \{0\} \).

**Proposition 8.2** [62] Let \( \alpha \) be even. Then:

\[
H_1(M_n,k(\alpha/1)) \cong \begin{cases} 
Z^{d-m} \oplus Z^m_a & \text{if } h = 1 \\
Z^{d-h+1-m} \oplus Z^{m-h+1}_a \oplus Z^{h-2}_{ab} \oplus Z_{hab} & \text{if } 1 < h < m + 1 \\
Z^{d-h+1-m} \oplus Z^{h-1-m}_b \oplus Z^{m-1}_{ab} \oplus Z_{hab} & \text{if } h \geq m + 1
\end{cases}
\]

where \( s = \gcd(n, k) \), \( d = \gcd(n, \alpha(k+1)/2) \), \( h = \gcd(n, k+1) \), \( m = \gcd(d, s) \), \( a = nm/(sd) \) and \( b = \alpha h/(2d) \).
As a consequence we have the homology groups in the interesting cases \( \gcd(n, k) = 1 \) (i.e., meridian-cyclic coverings).

**Corollary 8.3** Let \( \alpha \) be even and \( \gcd(n, k) = 1 \). Then:

\[
H_1(M_{n,k}(\alpha/1)) \cong \begin{cases} 
Z^{d-1} \oplus Z_b & \text{if } h = 1 \\
Z^{d-h} \oplus Z_b^{h-2} \oplus Z_{hab} & \text{if } h > 1 
\end{cases},
\]

where \( d = \gcd(n, \alpha(k+1)/2) \), \( h = \gcd(n, k+1) \), \( a = n/d \) and \( b = \alpha h/(2d) \).

The previous corollary contains, as a particular case, the homology groups of the Brieskorn manifolds \( M(n, \alpha, 2) \cong M_{n}(\alpha/1) \) (see also [10]).

**Corollary 8.4** If \( \alpha \) is even, then:

\[
H_1(M_{n}(\alpha/1)) \cong \begin{cases} 
Z^{d-1} \oplus Z_{n/d} & \text{if } n \text{ is odd} \\
Z^{d-2} \oplus Z_{2n\alpha/d^2} & \text{if } n \text{ is even} 
\end{cases},
\]

where \( d = \gcd(n, \alpha) \).

The homology of cyclic branched coverings of 2-bridge knots of genus one has been obtained in [62] and [15], using an algorithm of Fox [25]. Recall that the 2-bridge knot \( b(\alpha, \beta) \) has genus one if and only if \( \beta/2 \) divides \((\alpha - 1)/4\) when \( \alpha \equiv 1 \mod 4 \) and \( \beta/2 \) divides \((\alpha + 1)/4\) when \( \alpha \equiv 3 \mod 4 \) (up to equivalence we can assume that \( \beta \) is even for any 2-bridge knot \( b(\alpha, \beta) \)).

**Proposition 8.5** [62, 15] Let \( b(\alpha, \beta) \) be a 2-bridge knot of genus one. Then:

\[
H_1(M_{n}(\alpha/\beta)) \cong \begin{cases} 
Z_{|A'(n)|} \oplus Z_{|A''(n)|} & \text{if } n \text{ is even} \\
Z_{|A''(n)|} \oplus Z_{|A''(n)|} & \text{if } n \text{ is odd} 
\end{cases},
\]

where \( A'(n), A''(n) \) are the integers defined by:

\[
A'(1) = 1, A'(2) = 1, A'(n+2) = A'(n+1) - hA'(n),
\]

\[
A''(1) = 1, A''(2) = 1 - 2h, A''(n+2) = A''(n+1) - hA''(n),
\]

and

\[
h = \begin{cases} 
(1 - \alpha)/4 & \text{if } \alpha \equiv 1 \mod 4 \\
(1 + \alpha)/4 & \text{if } \alpha \equiv 3 \mod 4 
\end{cases}.
\]

Thus, the homology does not depend on \( \beta \).
In some particular cases, explicit formulae are obtained by J. Minkus.

**Proposition 8.6** [58, Corollary 11.2]

(i) If $n > 1$ is even then $H_1(M_n((2n\beta \pm 1)/\beta)) \cong \mathbb{Z}_{2n\beta\pm1}$.

(ii) If $n > 0$ is odd then $M_n((2n\beta \pm 1)/\beta)$ is a homology sphere.

Cyclic branched coverings of the Whitehead link $b(8,3)$ have been intensively studied (see [14, 29, 85]).

**Proposition 8.7** [29] Let $n \geq 3$ and let $\gcd(n,k) = 1$, then:

$$H_1(M_{n,k}(8/3)) \cong \begin{cases} 
\mathbb{Z}_{n/6} \oplus \mathbb{Z}_{n/2} \oplus \mathbb{Z}_{n/2} & \text{if } n \equiv 0 \mod 6 \\
\mathbb{Z}_{n/2} \oplus \mathbb{Z}_{n/2} \oplus \mathbb{Z}_{3n} & \text{if } n \equiv \pm 2 \mod 6 \\
\mathbb{Z}_{n/3} \oplus \mathbb{Z}_{n} \oplus \mathbb{Z}_{3n} & \text{if } n \equiv 3 \mod 6 \\
\mathbb{Z}_{n} \oplus \mathbb{Z}_{n} \oplus \mathbb{Z}_{n} & \text{if } n \equiv \pm 1 \mod 6 
\end{cases}$$

For $n, \alpha \leq 9$ and $\gcd(n,k) = 1$, the homology groups of $M_{n,k}(\alpha/\beta) \cong S(n, \alpha, \beta, -k)$ are listed in [48] and [49].

9 Decomposition of singly-cyclic coverings

In this section we prove that each singly-cyclic branched covering of a 2-bridge link is the composition of a meridian-cyclic branched covering of a certain link $L(d, \alpha/\beta)$ described below and a cyclic branched covering of a trivial knot. This gives a generalization of a result obtained in [14] for the case of the Whitehead link.

We can always assume that $\alpha/\beta = [2a_1, -2b_1, \ldots, 2a_l, -2b_l]$ if $b(\alpha, \beta)$ is a knot, and $\alpha/\beta = [2a_1, -2b_1, \ldots, 2a_l]$ if $b(\alpha, \beta)$ is a link. For any $\alpha/\beta$ and $d \geq 1$, define a link $L(d, \alpha/\beta) \subset S^3$ as in Figure 13, where $a_i$ and $b_i$ denote numbers of half-twists (in the vertical direction) in the corresponding boxes, and the fragment “in degree” $d$ must be repeated $d$ times.

The link $L(d, \alpha/\beta)$ has $1 + \gcd(d,l)$ components, where

$$l = \text{lk}(b(\alpha, \beta)) = \sum_{h=1}^{\alpha/2} (-1)^{\lfloor{(2h-1)d/\alpha}\rfloor}$$

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Figure 13: The link $\mathcal{L}(d, \alpha/\beta)$. 

$\begin{array}{c}
\begin{array}{c}
  a \quad = \\
  \begin{array}{c}
    \begin{array}{c}
      \begin{array}{c}
        \vdots \\
        \vdots \\
        \vdots \\
        \vdots \\
      \end{array}
    \end{array}
  \end{array}
  \begin{array}{c}
    \begin{array}{c}
      \begin{array}{c}
        \vdots \\
        \vdots \\
        \vdots \\
        \vdots \\
      \end{array}
    \end{array}
  \end{array}

  a \text{ half twists}
\end{array}
\end{array}$
is the linking number of $b(\alpha, \beta)$ \cite{4}. Remark 12.7 \((\lfloor x \rfloor \text{ denotes the integral part of } x)\). For example \(\text{lk}(b(8, 3)) = 0\) and therefore \(L(d, 8/3)\) has \(d + 1\) components.

Observe that, roughly speaking, each 2-bridge link can be obtained as a quotient of a suitable 2-bridge knot/link by an involution whose axis does not intersect the knot/link (for example, see the 2-periodic presentations of 2-bridge knots/links in \cite{1}). Hence, any 2-bridge link $b(\alpha, \beta)$ can be presented in the form $L(1, \alpha_1/\beta)$, where $\alpha_1 = \alpha/2$. For example, if $b(8/3)$ is the Whitehead link, then $b(4/3)$ is equivalent to $b(4/1)$, and $4/1 = [2a_1]$ with $a_1 = 2$. Hence we get the presentation of the Whitehead link in the form $L(1, 4/1)$ (see, for example, Figure 3.4 in \cite{14}).

Denote by $L_m(d, \alpha/\beta)$ the orbifold with underlying space $\mathbb{S}^3$ and singular set $L(d, \alpha/\beta)$, with all branched indices equal to $m$. Moreover, denote by $b_{n,n/d}(\alpha, \beta)$ the orbifold having $\mathbb{S}^3$ as underlying space and the 2-bridge link $b(\alpha, \beta)$ as singular set, with branched indices $n$ and $n'$ respectively for the two components.

**Theorem 9.1** Let $M_{n,k}(\alpha/\beta)$ be a singly-cyclic covering of the 2-bridge link $b(\alpha, \beta)$, and $\alpha_1 = \alpha/2$. Suppose $d = \gcd(n, k)$. Then the following diagram

\[
\begin{array}{ccc}
M_{n,k}(\alpha/\beta) & \xrightarrow{n/d} & \mathcal{L}_{n/d}(d, \alpha_1/\beta) \\
\downarrow n & & \downarrow d \\
\mathcal{L}_{n/d}(d, \alpha_1/\beta) & \xrightarrow{n/d} & b_{n,n/d}(\alpha, \beta)
\end{array}
\]

is commutative. Moreover, the $n/d$–covering is a meridian-cyclic covering of the link $\mathcal{L}(d, \alpha_1/\beta)$ and the $d$-covering is the $d$-fold cyclic branched covering of a component of $b(\alpha, \beta)$ (which is a trivial knot).

**Proof.** The fundamental group $\Gamma$ of the orbifold $b_{n,n/d}(\alpha/\beta)$ admits the presentation

\[\Gamma = \langle \mu_1, \mu_2 \mid \mu_1^n = \mu_2^{n/d} = 1, w(\mu_1, \mu_2) = 1 \rangle\]

where $\mu_1$ and $\mu_2$ are the homotopy classes of two meridians $m_1$ and $m_2$ around the components $K_1$ and $K_2$ of the link, and $w(\mu_1, \mu_2)$ is the relation deriving
from the standard presentation of the group of $b(\alpha, \beta)$. From the definition of a singly-cyclic covering we have $\pi_1(M_{n,k}) = \ker(\varphi)$ for the epimorphism $\varphi : \Gamma \to \mathbb{Z}_n = \langle \gamma | \gamma^n = 1 \rangle$ defined by $\varphi(\mu_1) = \gamma$ and $\varphi(\mu_2) = \gamma^k$. Consider the subgroup $\mathbb{Z}_d \triangleleft \mathbb{Z}_n$ such that $\mathbb{Z}_d = \langle \delta | \delta^d = 1 \rangle$, where $\delta = \gamma^{n/d}$. Let $\theta : \Gamma \to \mathbb{Z}_d$ be the epimorphism defined by $\theta(\mu_1) = \delta$ and $\theta(\mu_2) = 1$. This epimorphism induces the $d$-fold cyclic covering $\Theta : \mathcal{O} \to b_{n,n/k}(\alpha, \beta)$ such that the axis of the cyclic group action is the component $K_1$ of $b(\alpha, \beta)$ corresponding to the meridian $\mu_1$. Therefore, the underlying space of the orbifold $\mathcal{O}$ is $S^3$ and the singular set is $\Theta^{-1}(K_1) \cup \Theta^{-1}(K_2)$. Obviously, $\Theta^{-1}(K_1)$ is a trivial knot in $S^3$ with singularity index $n/d$, and $\Theta^{-1}(K_2)$ is a $d$-periodic knot/link which also has singularity index $n/d$. Since $b(\alpha, \beta)$ is equivalent to $\mathcal{L}(1, \alpha_1/\beta)$ with $\alpha_1 = \alpha/2$, then we have $\Theta^{-1}(b(\alpha, \beta)) = \mathcal{L}(d, \alpha_1/\beta)$. Using $\pi_1(\mathcal{O}) = \ker(\theta) = \varphi^{-1}(\mathbb{Z}_d)$ and $\pi_1(M_{n,k}) = \ker(\varphi)$, the diagram of coverings is commutative. This completes our proof.

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