On the forking topology of a reduct of a simple theory

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Abstract

Let $T$ be simple and $T^-$ a reduct of $T$. For variables $x$, we call an $\emptyset$-invariant set $\Gamma(x)$ of $C$ with the property that for every formula $\phi^-(x, y) \in L^-$: for every $a, \phi^-(x, a) L^-$-forks over $\emptyset$ iff $\Gamma(x) \land \phi^-(x, a)$ $L$-forks over $\emptyset$, a universal transducer. We show that there is a greatest universal transducer $\tilde{\Gamma}_x$ (for any $x$) and it is type-definable. In particular, the forking topology on $S_y(T)$ refines the forking topology on $S_y(T^-)$. Moreover, we describe the set of universal transducers in terms of certain topology on the Stone space and show that $\tilde{\Gamma}_x$ is the unique universal transducer that is $L^-$-type-definable with parameters. In the case where $T^-$ is a theory with the wnfcp (the weak nfcpcp) and $T$ is the theory of its lovely pairs we show $\tilde{\Gamma}_x = (x = x)$ and give a more precise description of all its universal transducers in case $T^-$ has the nfcpcp.

1 Introduction

The forking topology introduced in [S] is a generalization of topologies introduced by Hrushovski [H0] and Pillay [P]. It is the minimal topology on $S_x(A)$ such that all the relations $\Gamma_F(x)$ defined by $\Gamma_F(x) = \exists y (F(x, y) \land y \downarrow_A x)$ are closed for any type-definable relation $F(x, y)$ over $A$. Originally, it has been introduced (around 1984) by Hrushovski in [H0] for an intermediate
step in the (unpublished) proof of supersimplicity of countable unidimensional stable theories namely, in the proof an unbounded open set of finite $SU$-rank produced from which the existence of a definable set of finite $SU$-rank followed. In [P], where supersimplicity of any countable unidimensional wnfcps hypersimple theory is established, the topology has been modified to work for theories with the wnfcps (Shortly after, the proof has been extended by Pillay to the general low case using the elimination of the "there exists infinitely many" quantifier in unidimensional simple theories[S0]). In [S] we modified the definition in [P] of the topology in such a way that one can prove more general theorems suitable to the simple case rather than the wnfcps case. A new application of the forking topology was the finite length analysis of any type in a forking open set provided that it is analyzable in it (possibly by infinitely many steps). This required the assumption that the forking topologies are closed under projections. In [S1], supersimplicity of any countable unidimensional hypersimple theory is proved. One of the major steps in that proof applies the forking topology to get the existence of an unbounded open set of finite $SU_{sc}$-rank (i.e. $SU$-rank with respect to stable formulas); this was established via introducing more complicated sets related to the forking topology. In the proof of [S2], where a generalization of Buechler’s dichotomy for $D$-rank 1 types in simple theories is proved, it seems the forking topology is essential for getting the required definable set.

In this paper we investigate the behaviour of the forking topology in a reduct, in particular we show that the natural projection from $S_x(T)$ to $S_x(T^{-})$, where $T$ is simple and $T^{-}$ is a reduct of $T$, is continuous with respect to the forking topologies on the Stone spaces. Moreover, for a given theory and a reduct of it, we define the notion of a universal transducer (for any given variables) as an invariant set that transfers forking open sets in the reduct to open sets in the original theory as indicated in the abstract and characterize the set of universal transducers for the given variables. We conclude the uniqueness of a universal transducer that is type-definable with parameters in the reduct. The results are proved in a more general setting (we fix an invariant set $F$ and define universal $F$-transducers). Lastly, we get a more precise information in the lovely-pair case.
2 Preliminaries

We assume basic knowledge of simple theories as in [K],[KP],[HKP]. A good textbook on simple theories is [W]. Here we fix the notations related to a theory and a reduct of it and recall the definition of the forking topology on the Stone space. In this paper, unless otherwise stated, $T$ will denote a complete simple theory in an arbitrary language $L$ and we work in a $\lambda$-big model $C$ (i.e. any expansion of it by less than $\lambda$ constants is splendid) for some large $\lambda$. We call $C$ the monster model. Note that any $\lambda$-big model (of any theory) is $\lambda$-saturated and $\lambda$-strongly homogeneous and that $\lambda$-bigness is preserved under reducts (by Robinson consistency theorem). We use standard notations. For a small subset $A \subseteq C$, $T_A$ will denote the theory of $(C, A)$ ($C$ expanded by constants for each $a \in A$). Partial types are usually identified with the set of its solutions in the monster model. For invariant set of a fixed sort (or finitely many) we write (e.g.) $U(x)$ where $x$ is a finite tuple of variables suitable for these sorts. For variables $x$, $C^x$ denotes the set of tuples in $C$ whose sort is the sort of $x$. An invariant set of possibly some distinct sorts will be denoted by (e.g.) $U$ (with no variables added). If $U$ is a set we denote by $U^{<\omega}$ the set of all finite sequences of elements in $U$. For a partial type $p$ over a model, $Cl(p)$ denotes the set of formulas $\phi(x, y) \in L$ that are represented in $p$.

2.1 Reducts

In this subsection we fix some conventions and notations regarding $T$ and a reduct of it. A theory $T^-$ is a reduct of $T$ to a sublanguage $L^-$ of $L$ if $T^-$ is the set of $L^-$-sentences in $T$. In this paper, we will assume for simplicity of notations that $L^-$ has the same set of sorts as the sorts of $L$. We say $T^-$ is a reduct of $T$ if $T^-$ is a reduct of $T$ to some sublanguage of $L$. For a reduct $T^-$ of $T$ we let $C^- = C|L^-$. As mentioned previously, we know that both $C$ and $C^-$ are highly saturated and highly strongly-homogeneous. $C^{heq}$ ($C^{eq}$) denotes the set of hyperimaginaries of small ($< \lambda$) length (imaginaries) of $C$ and $C^{heq^-}$ ($C^{eq^-}$) denotes the set of hyperimaginaries of small length (imaginaries) of $C^-$. 

Definition 2.1 Fix a reduct $T^-$ of $T$.

1) For a small set $A \subseteq C^{heq}$, $DCL^{heq}(A)$ ($DCL^{eq}(A)$) denotes the set of countable hyperimaginaries in $C^{heq}$ (imaginaries in $C^{eq}$) that are in the defin-
able closure of $A$ in the sense of $\mathcal{C}$.

2) For a small set $A \subseteq \mathcal{C}^{eq}$, $dcl^{eq}(A)$ denotes the set of countable hyperimaginaries in $\mathcal{C}^{eq}$ (imaginaries in $\mathcal{C}^{eq}$) that are in the definable closure of $A$ in the sense of $\mathcal{C}$.

3) For a small set $A \subseteq \mathcal{C}^{eq}$, $BDD(A)$ denotes the set of countable hyperimaginaries in $\mathcal{C}^{eq}$ (imaginaries in $\mathcal{C}^{eq}$) that are in the bounded (algebraic) closure of $A$ in the sense of $\mathcal{C}$.

4) For a small set $A \subseteq \mathcal{C}^{eq}$, $bdd(A)$ denotes the set of countable hyperimaginaries in $\mathcal{C}^{eq}$ (imaginaries in $\mathcal{C}^{eq}$) that are in the bounded (algebraic) closure of $A$ in the sense of $\mathcal{C}$.

5) For an $\emptyset$-invariant set $F \subseteq \mathcal{C}^{eq}$ in $\mathcal{C}$ let $bdd(F)$ denote the set of all countable (length) hyperimaginaries in $\mathcal{C}^{eq}$ that are in the bounded (definable) closure in the sense of $\mathcal{C}$ of some small subset of $F$.

6) For a small set $X \subseteq \mathcal{C}^{eq}$, let $X^{-} = X \cap \mathcal{C}^{eq}$.

Notation 2.2
1) $\Downarrow$ denotes independence in $\mathcal{C}$, and $\Downarrow^{-}$ denotes independence in $\mathcal{C}$.
2) $Cb$ denotes the canonical base of an amalgamation base in $\mathcal{C}^{eq}$, and $Cb^{-}$ denotes the canonical base of an amalgamation base in $\mathcal{C}^{eq}$.

2.2 The forking topology

Definition 2.3 Let $A \subseteq \mathcal{C}$ and let $x$ be a finite tuple of variables. An invariant set $U$ over $A$ is said to be a basic $\tau^{f}$-open set over $A$ if there is a $\phi(x, y) \in L(A)$ such that

$$U = \{a | \phi(a, y) \text{ forks over } A\}.$$ 

Note that the family of basic $\tau^{f}$-open sets over $A$ is closed under finite intersections, thus form a basis for a unique topology on $S_{x}(A)$ which we call the $\tau^{f}$-topology or the forking-topology.

Remark 2.4 Note that the forking-topology on $S_{x}(A)$ refines the Stone-topology (for every $x$ and $A$) and that $\{a \in \mathcal{C}^{eq} | a \notin acl(A)\} = \{a \in \mathcal{C}^{eq} | x = a \text{ forks over } A\}$ is a forking-open subset of $S_{x}(A)$ (when we identify $A$-invariant sets with subsets of $S_{x}(A)$).
3 Transducers

In this section we prove a generalization of the results stated in the title regarding a general simple theory and a reduct of it and related results. In this section $T$ is assumed to be a simple theory and $T^-$ denotes any reduct of $T$. We start with some terminology.

In the following, if $\Gamma(x)$ is an invariant set in $C$ over some small set $B$ and $A$ is any small set then we say $\Gamma(x)$ $L$-doesn’t fork over $A$ if for some $c \models \Gamma(x)$, $\downarrow_A B$.

Definition 3.1 Let $\Gamma(x), F$ be $\emptyset$-invariant sets in $C$.
1) We say that $\Gamma(x)$ is an upper universal $F$-transducer if for every $\bar{a} \in F^{<\omega}$ and $\phi^-(x, \bar{y}) \in L^-$, if $\Gamma(x) \land \phi^-(x, \bar{a})$ $L$-doesn’t fork over $\emptyset$, then $\phi^-(x, \bar{a})$ $L^-$-doesn’t fork over $\emptyset$.
2) We say that $\Gamma(x)$ is a lower universal $F$-transducer if for every $\bar{a} \in F^{<\omega}$ and $\phi^-(x, \bar{y}) \in L^-$, if $\phi^-(x, \bar{a})$ $L^-$-doesn’t fork over $\emptyset$, then $\Gamma(x) \land \phi^-(x, \bar{a})$ $L^-$-doesn’t fork over $\emptyset$.
3) We say that $\Gamma(x)$ is a universal $F$-transducer if $\Gamma(x)$ is both an upper universal $F$-transducer and a lower universal $F$-transducer.
4) In case $F$ is omitted in 1)-3) in the current definition, it means $F = C$.

Definition 3.2 For variables $x$ and $\emptyset$-invariant set $F$ in $C$ we define the following $\emptyset$-invariant sets in $C$:
1) $\tilde{\Gamma}_x,F = \{ b \in C^x \mid \forall \bar{a} \in F^{<\omega} \exists a' : tp_L(\bar{a}) \land (b \downarrow \bar{a}') \}$. $\tilde{\Gamma}_x$ denotes $\tilde{\Gamma}_x,C$.
2) $\Gamma^*_x,F = \{ b \in C^x \mid \forall \bar{a} \in F^{<\omega} (b \downarrow \bar{a} \rightarrow b \downarrow \bar{a}') \}$. $\Gamma^*_x$ denotes $\Gamma^*_x,C$.
3) $B_x,F = \{ b \in C^x \mid b \downarrow bdd(F) \cap BDD(\emptyset)^- \}$. $B_x$ denotes $B_x,C$.

Remark 3.3 $\tilde{\Gamma}_x = \{ b \in C^x \mid \forall \phi(y) \in L : [\exists y \phi(y) \rightarrow \exists a : \phi(y) (b \downarrow a)] \}$. Moreover, for every model $M \models T$, $\tilde{\Gamma}_x = \{ b \in C^x \mid \exists M' : tp_L(M) (b \downarrow M') \}$

Proof: Just compactness. □

Lemma 3.4 For any $\emptyset$-invariant set $F$ in $C$ we have $\tilde{\Gamma}_{x,F} = \Gamma^*_{x,F} = B_{x,F}$.

Proof: To show $\tilde{\Gamma}_{x,F} \subseteq B_{x,F}$ we observe:
Claim 3.5 Let $M$ be a sufficiently saturated model of $T$. Then
\[ \text{bdd}(F) \cap BDD(\emptyset)^- = \text{bdd}(F^M) \cap BDD(\emptyset)^-. \]

Proof: Let $e \in \text{bdd}(F) \cap BDD(\emptyset)^-$. Then there exists a small subset $F_e \subseteq F$ (in fact of size at most $|T|$) such that $e \in \text{bdd}(F_e) \cap BDD(\emptyset)^-$. Since $M$ is sufficiently saturated, $e \in M^{\text{heq}^-}$ (if $e = a/E$ then on $tp_L(a)$ there are at most $2^{|T|^+}$ many $E$-classes). By saturation $M$, there exists $F'_e \subseteq M$ such that $tp_L(F'_e/e) = tp_L(F_e/e)$ and so $e \in \text{bdd}(F^M)$. \hfill $\Box$

Now, let $b \in \tilde{\Gamma}_{x,F}$. By compactness, there exists a sufficiently saturated model $M'$ of $T$ such that $b \not\subseteq F^{M'}$, so $b \not\subseteq \text{bdd}(F^{M'})$. By Claim 3.3 we are done. To show $B_{x,F} \subseteq \Gamma^*_{x,F}$ recall the following.

Fact 3.6 [HN, Theorem 2.2] Let $A, C \subseteq C^{\text{heq}^-}$ and let $B \subseteq C^{\text{heq}}$ be boundedly closed in $C^{\text{heq}}$. Assume $A \Downarrow C \hspace{1cm} B \Downarrow C$. Then $A \Downarrow B$. \hfill $\Box$

Now, let $b \in B_{x,F}$ and assume $b \Downarrow a$ for some $a \in F^{<\omega}$. By Fact 3.6, $b \Downarrow BDD(\emptyset)^- \Downarrow \tilde{a}$ (\#). From now on work in $C^-$. Let $e^- = Cb^- (Lstp(a/BDD(\emptyset)^-, b))$. $e^-$ is in the definable closure of a Morley sequence of $Lstp(a/BDD(\emptyset)^-, b)$, since $a \in F^{<\omega}$, we conclude $e^- \in \text{bdd}(F)$. By (\#), $e^- \in BDD(\emptyset)^-$ (note that $BDD(\emptyset)^-$ boundedly closed in $C^{\text{heq}^-}$). Thus $\tilde{a} \Downarrow BDD(\emptyset)^- b$. As $b \in B_{x,F}$, transitivity yields $b \Downarrow \tilde{a}$. The inclusion $\Gamma^*_{x,F} \subseteq \tilde{\Gamma}_{x,F}$ is immediate by extension. \hfill $\Box$

Proposition 3.7 For variables $x$ and $\emptyset$-invariant set $F$ in $C$ there exists a greatest (with respect to inclusion) set $\Gamma_{x,F}$ that is $\emptyset$-invariant in $C$, a subset of $C^x$ and is a universal $F$-transducer ($\Gamma_{x,F}$ is also such greatest upper universal $F$-transducer). Moreover, $\Gamma_{x,F} = \tilde{\Gamma}_{x,F} = \Gamma^*_{x,F}$ and $\Gamma_{x,F}$ is type-definable. In particular, the forking-topology of $T$ on $S_y(T)$ refines the forking-topology of $T^-$ on $S_y(T^-)$ for every $y$.

Proof: First, we show that $\tilde{\Gamma}_{x,F}$ is a universal $F$-transducer. Let $\phi^-(x, \bar{y}) \in L^-$ be arbitrary and let $\bar{a} \in F^{<\omega}$ be suitable for $\bar{y}$.

Claim 3.8 If $\tilde{\Gamma}_{x,F}(x) \land \phi^-(x, \bar{a}) \text{ L-doesn't fork over } \emptyset$, then $\phi^-(x, \bar{a}) \text{ L}^-$ doesn't fork over $\emptyset$. 

6
**Definition 3.11** Given a finite tuple of variables $y$, a set $U = U(y)$ is a basic open set in the $NI_F$-topology (or basic $NI_F$-open) on $S_y(T)$ iff there exists a type $p(x) \in S_x(T)$ with $p(x) \vdash F^{<\omega}$ and $\phi^{-}(x,y) \in L^-$ such that

$$U = U_{p,\phi^-} = \{b \mid p(x) \land \phi^{-}(x,b) \land \phi^{-}(x,y) \in L^- \land \phi^{-}(x,y) \in L^- \}. $$

**Proof:** If $\bar{\Gamma}_{x,F}(x) \land \phi^{-}(x,\bar{a})$ $L$-doesn’t fork over $\emptyset$ there exists $b \models \bar{\Gamma}_{x,F}(x) \land \phi^{-}(x,\bar{a})$ such that $b \models \neg \phi^{-}(x,\bar{a})$. By Lemma 3.4, $b \models \neg \phi^{-}(x,\bar{a})$ thus $\phi^{-}(x,\bar{a}) L^-$-doesn’t fork over $\emptyset$.

**Claim 3.9** If $\phi^{-}(x,\bar{a}) L^-$-doesn’t fork over $\emptyset$, then $\bar{\Gamma}_{x}(x) \land \phi^{-}(x,\bar{a}) L^-$-doesn’t fork over $\emptyset$, in particular $\bar{\Gamma}_{x,F}(x) \land \phi^{-}(x,\bar{a}) L^-$-doesn’t fork over $\emptyset$.

**Proof:** Assume $\phi^{-}(x,\bar{a})$ $L^-$-doesn’t fork over $\emptyset$. Let $b \models \phi^{-}(x,\bar{a})$ be such that $b \models \neg \phi^{-}(x,\bar{a})$. Let $M$ be a model of $T$. By extension in $C^-$, we may assume $b \models M\bar{a}$. In particular, $tp_L(b/M\bar{a})$ $L$-doesn’t fork over $\emptyset$, so there exists $b^*$ such that $tp_L(b^*/M\bar{a}) = tp_L(b/M\bar{a})$ and $b^* \models M\bar{a}$. By Remark 3.3, $b^* \models \bar{\Gamma}_{x}(x)$. By the choice of $b^*$, $\phi^{-}(b^*,\bar{a})$, thus $\bar{\Gamma}_{x}(x) \land \phi^{-}(x,\bar{a})$ $L$-doesn’t fork over $\emptyset$. $\square$

It remains to show:

**Claim 3.10** If $U = U(x)$ is an $\emptyset$-invariant set in $C$ that is an upper universal $F$-transducer, then $U \subseteq \Gamma^*_{x,F}$. Therefore $\bar{\Gamma}_{x,F} = \Gamma^*_{x,F}$ is the greatest (with respect to inclusion) $\emptyset$-invariant set in $C$ that is a subset of $C^x$ and is an universal $F$-transducer ($\bar{\Gamma}_{x,F}$ is also such greatest upper universal $F$-transducer). $\bar{\Gamma}_{x,F}$ is type-definable.

**Proof:** Let $U(x)$ be as given in the claim and assume $b \models U(x)$ and let $\bar{a} \models b$. Then for all $\phi^{-}(x,\bar{y}) \in L^-$, if $\models \phi^{-}(b,\bar{a})$ then $\phi^{-}(x,\bar{a}) L^-$-doesn’t fork over $\emptyset$ (since $U(x)$ is an upper universal $F$-transducer). Thus $b \models \neg \phi^{-}(x,\bar{a})$, so $b \in \Gamma^*_{x,F}$. By Lemma 3.4, $\bar{\Gamma}_{x,F} = \Gamma^*_{x,F}$, so by Claims 3.8, 3.9, $\bar{\Gamma}_{x,F}$ is the greatest $\emptyset$-invariant set in $C$ that is a subset of $C^x$ and is a universal $F$-transducer (as well as an upper universal $F$-transducer). $\bar{\Gamma}_{x,F}$ is type-definable as $\bar{\Gamma}_{x,F} \equiv \bigwedge_i \Gamma_{p_i}$, where $\{p_i\}$ is the set of all complete $L$-types over $\emptyset$ of elements in $F^{<\omega}$ and $\Gamma_{p_i}$ is the partial $L$-type such that $a \models \Gamma_{p_i}$ if there exists $b \models p_i$ that is $L^-$-independent from $a$ over $\emptyset$. $\square$

From now on $F$ will denote an arbitrary $\emptyset$-invariant set in $C$. In order to describe the set of universal $F$-transducers for some $\emptyset$-invariant set $F$ in $C$ we introduce another topology on the Stone space $S_y(T)$.

**Definition 3.11** Given a finite tuple of variables $y$, a set $U = U(y)$ is a basic open set in the $NI_F$-topology (or basic $NI_F$-open) on $S_y(T)$ iff there exists a type $p(x) \in S_x(T)$ with $p(x) \vdash F^{<\omega}$ and $\phi^{-}(x,y) \in L^-$ such that

$$U = U_{p,\phi^-} = \{b \mid p(x) \land \phi^{-}(x,b) \land \phi^{-}(x,y) \in L^- \}. $$
Remark 3.12 Note that the intersection of two basic $NI_F$-open sets is a union of basic $NI_F$-open open sets, so the family of basic $NI_F$-open sets forms a basis for a unique topology on $S_y(T)$. Indeed, by extension if $b \in U_{p_0,\phi_0^-} \cap U_{p_1,\phi_1^-}$ for some $p_i, \phi_i^-$ as in Definition 3.11 then $b \in U_{q,\phi^-}$ for some $q = q(x_0, x_1)$ where $q = tp_L(a_0, a_1)$ for some independent $a_i \models p_i$ and $\phi^- = \phi_0^-(x_0, y) \land \phi_1^-(x_1, y)$ (clearly, $U_{q,\phi^-} \subseteq U_{p_0,\phi_0^-} \cap U_{p_1,\phi_1^-}$ and it is a basic $NI_F$-open set). Note that since the type $p$ in Definition 3.11 is a complete $L$-type, each basic $NI_F$-open set is $L$-type-definable. Also, note that the $NI_F$-topology will not change if we allow $p(x)$ to be a type in infinitely many variables.

Definition 3.13 1) A set $U \subseteq C$ is said to be $(L, L^-)_F$-definable over $\emptyset$ if $U = \phi^-(C, \bar{a})$ for some $\phi^- \in L^-$ and $\bar{a} \in F^{\infty}$ such that $\phi^-(x, \bar{a})$ is $\emptyset$-invariant in $C$. If $F = C$ we omit $F$.

2) A set $U \subseteq C$ is said to be $(L, L^-)_F$-$\infty$-definable over $\emptyset$ if $U = p^-\bar{a}$ for some $L^-$-partial type $p^-$ over $\emptyset$ and tuple $\bar{a}$ of realizations of $F$ such that $p^-(C, \bar{a})$ is $\emptyset$-invariant in $C$. If $F = C$ we omit $F$.

Remark 3.14 By compactness, $U \subseteq C$ is $(L, L^-)_F$-$\infty$-definable over $\emptyset$ iff $U = p^-\bar{a}$ for some $L^-$-partial type $p^-$ over $\emptyset$ and tuple $\bar{a}$ of realizations of $F$ and $U$ is the solution set of an $L$-partial type over $\emptyset$. Likewise for $(L, L^-)_F$-definable sets over $\emptyset$.

Lemma 3.15 1) If $U$ is $(L, L^-)_F$-$\infty$-definable over $\emptyset$, then $U$ is $NI_F$-closed. If $U$ is $(L, L^-)_F$-definable over $\emptyset$, then $U$ is a basic $NI_F$-open set.

2) If $T$ is stable, then $U$ is a basic $NI_F$-open set if and only if $U$ is $(L, L^-)_F$-definable over $\emptyset$.

Proof: 1) By the assumption, there exists an $L^-$-partial type $p^-(x, \bar{y})$ over $\emptyset$ and tuple $\bar{a}$ (possibly infinite) of realizations of $F$ such that $U = p^-(C, \bar{a})$ and is $\emptyset$-invariant in $C$. Let $q = tp_L(\bar{a})$. Then

$$p^-(C, \bar{a}) = \{b \mid q(\bar{y}) \land \lnot \phi^-(b, \bar{y}) \text{ $L$-forks over $\emptyset$ for all $\phi^- \in p^-$}\} \, (\ast).$$

Indeed, let $R$ denote the right hand side of $\ast$. If $b \in p^-(C, \bar{a})$ and $q(\bar{y}) \land \lnot \phi^-(b, \bar{y})$ $L$-doesn’t fork over $\emptyset$ for some $\phi^- \in p^-$ then we get contradiction to $\emptyset$-invariance of $p^-(C, \bar{a})$ in $C$, so $b \in R$. If $b \not\in p^-(C, \bar{a})$, then by $\emptyset$-invariance of $p^-(C, \bar{a})$ in $C$ and extension we may assume $b \downarrow \bar{a}$ . Thus $b \not\in R$. We conclude that $p^-(C, \bar{a})$ is the intersection of complements of
basic $NI_F$-open sets. Assume now $U = \phi^-(C, \tilde{a})$ is $(L, L^-)_F$-definable over $\emptyset$. Then by \((\ast)\) we get immediately that $U$ is a basic $NI_F$-open set (take $p^-(x, \tilde{a}) = \{\neg \phi^-(x, \tilde{a})\}$). 2) Assume now that $T$ stable, it remains to show if $U$ is a basic $NI_F$-open set, then it is $(L, L^-)_F$-definable over $\emptyset$. Indeed, if $U = \Gamma(\emptyset)$, then by Proposition 3.7, we know that $\tilde{\Gamma}$ is $\emptyset$-invariant set in $C$. To show this we start with the following.

**Claim 3.18** For every type $p(x) \in S_x(T)$ with $p(x) \vdash F^{<\omega}$ and $\phi^-(x, y) \in L^-$, $U_{p, \phi^-} \cap \Gamma_{y, F} \neq \emptyset$ iff $\phi^-(a, y)$ $L^-$-doesn't fork over $\emptyset$ for $a \models p$. 

**Corollary 3.16** In a stable theory, a set is $(L, L^-)_F$-$\infty$-definable over $\emptyset$ iff it is a conjunction of $(L, L^-)_F$-definable sets over $\emptyset$ iff it is $NI_F$-closed.

**Proof:** Assume $T$ is stable. By Lemma 3.15(1), if $U$ is $(L, L^-)_F$-$\infty$-definable over $\emptyset$ then it is $NI_F$-closed. By Lemma 3.15(2) an $NI_F$-closed set is the intersection of $(L, L^-)_F$-definable sets over $\emptyset$. Finally, it is immediate that the intersection of $(L, L^-)_F$-definable sets over $\emptyset$ is $(L, L^-)_F$-$\infty$-definable over $\emptyset$. 

We give now a description of the set of universal $F$-transducers via the $NI_F$-topology.

**Proposition 3.17** Let $\Gamma(y)$ be an $\emptyset$-invariant set in $C$. Then $\Gamma(y)$ is a universal $F$-transducer iff $\Gamma(y)$ is a dense subset of $\Gamma_{y, F}$ in the relative $NI_F$-topology on $\Gamma_{y, F}$.

**Proof:** By Proposition 3.7 we know that $\Gamma_{y, F}$ is a universal $F$-transducer and an $\emptyset$-invariant set $\Gamma = \Gamma(y)$ in $C$ is an upper universal $F$-transducer if and only if $\Gamma \subseteq \Gamma_{y, F}$. Thus it remains to show that an $\emptyset$-invariant set $\Gamma \subseteq \Gamma_{y, F}$ in $C$ is a lower universal $F$-transducer if and only if $\Gamma$ is a dense subset of $\Gamma_{y, F}$ in the relative $NI_F$-topology on $\Gamma_{y, F}$. To show this we start with the following.

For every type $p(x) \in S_x(T)$ with $p(x) \vdash F^{<\omega}$ and $\phi^-(x, y) \in L^-$, $U_{p, \phi^-} \cap \Gamma_{y, F} \neq \emptyset$ iff $\phi^-(a, y)$ $L^-$-doesn't fork over $\emptyset$ for $a \models p$. 

9
Proof: For such \( p \) and \( \phi^- \), \( U_{p,\phi^-} \cap \tilde{\Gamma}_{y,F} \neq \emptyset \) iff there exists \( b \models \tilde{\Gamma}_{y,F} \) such that \( p(x) \land \phi^-(x, b) \) L-doesn’t fork over \( \emptyset \) iff \( \tilde{\Gamma}_{y,F}(y) \land \phi^-(a, y) \) L-doesn’t fork over \( \emptyset \) for \( a \models p \). Since \( \tilde{\Gamma}_{y,F} \) is a universal F-transducer, the latest is equivalent to \( \phi^- (a, y) \) L\(^-\)doesn’t fork over \( \emptyset \) for \( a \models p \). \( \square \)

Now, let \( \Gamma(y) \subseteq \tilde{\Gamma}_{y,F} \). Then \( \Gamma(y) \) is a dense subset of \( \tilde{\Gamma}_{y,F} \) in the relative \( NI_F \)-topology on \( \tilde{\Gamma}_{y,F} \) iff for every \( p(x) \in S_x(T) \) with \( p(x) \vdash F^{<\omega} \) and \( \phi^- (x, y) \in L^- \) such that \( U_{p,\phi^-} \cap \tilde{\Gamma}_{y,F} \neq \emptyset \) we have \( U_{p,\phi^-} \cap \Gamma(y) \neq \emptyset \). By Claim 3.18, the latest is equivalent to: for every \( p(x) \in S_x(T) \) with \( p(x) \vdash F^{<\omega} \) and \( \phi^- (x, y) \in L^- \) such that \( \phi^- (a, y) \) L\(^-\)doesn’t fork over \( \emptyset \) for \( a \models p \), there exists \( b \models \Gamma \) such that \( p(x) \land \phi^- (x, b) \) L-doesn’t fork over \( \emptyset \); equivalently, for every \( p(x) \in S_x(T) \) with \( p(x) \vdash F^{<\omega} \) and \( \phi^- (x, y) \in L^- \) such that \( \phi^- (a, y) \) L\(^-\)-forks over \( \emptyset \) for \( a \models p \), the partial type \( \Gamma(y) \land \phi^- (a, y) \) L-doesn’t fork over \( \emptyset \) for \( a \models p \); namely \( \Gamma(y) \) is a lower universal F-transducer. \( \square \)

Theorem 3.19 Assume \( bdd(F) = \text{de}c^{heq-}(F) \). Given variables \( y \), \( \tilde{\Gamma}_{y,F} \) is the unique universal F-transducer subset of \( C^y \) that is \( (L, L^-)_{F,-\infty} \)-definable over \( \emptyset \). Thus, if \( T \) is stable, \( \tilde{\Gamma}_{y,F} \) is the unique universal F-transducer subset of \( C^y \) that is a conjunction of \( (L, L^-)_{F,-\infty} \)-definable sets over \( \emptyset \).

Proof: First, we observe that \( \tilde{\Gamma}_{y,F} \) is \( (L, L^-)_{F,-\infty} \)-definable over \( \emptyset \). Indeed, by Lemma 3.4 \( \tilde{\Gamma}_{y,F} = \{ b \in C^y \mid b \downarrow bdd(F) \cap BDD(\emptyset^-) \} \). For every \( d \in bdd(F) \cap BDD(\emptyset^-) \), let \( p_d(x, \bar{f}_d) = t_{PL^-}(d/\bar{f}_d) \), where \( \bar{f}_d \) is a tuple of realizations of \( F \) such that \( d \) is the unique solution in \( C^{heq^-} \) of \( t_{PL^-}(d/\bar{f}_d) \) (using the assumption \( bdd(F) = \text{de}c^{heq-}(F) \)). Now, \( \tilde{\Gamma}_{y,F} = L_{d \in D} \Lambda_d(C) \) where

\[
\Lambda_d(y) = \exists x(p_d(x, \bar{f}_d) \land y \downarrow x), \quad D = bdd(F) \cap BDD(\emptyset^-).
\]

Since each \( \Lambda_d(y) \) is \( L^- \)-type-definable with parameters in \( F \) and clearly \( \tilde{\Gamma}_{y,F} \) is \( \emptyset \)-invariant in \( C \) we get it is \( (L, L^-)_{F,-\infty} \)-definable over \( \emptyset \). Now, let \( \Gamma(y) \) be any universal F-transducer that is \( (L, L^-)_{F,-\infty} \)-definable over \( \emptyset \). Then by Lemma 3.15(1), \( \Gamma(y) \) is an \( NI_F \)-closed set in \( S_y(T) \). By Proposition 3.17, \( \Gamma(y) \) is a dense subset of \( \tilde{\Gamma}_{y,F} \) in the relative \( NI_F \)-topology on \( \tilde{\Gamma}_{y,F} \). It follows that \( \Gamma(y) = \tilde{\Gamma}_{y,F} \). \( \square \)

4 The lovely pair case

Recall first the basic notions of lovely pairs. Given \( \kappa \geq |T|^+ \), an elementary pair \((N, M)\) of models \( M \subseteq N \) of a simple theory \( T \) is said to be \( \kappa \)-lovely if (i)
it has the extension property: for any $A \subseteq N$ of cardinality $\prec \kappa$ and finitary $p(x) \in S(A)$, some nonforking extension of $p(x)$ over $A \cup M$ is realized in $N$, and (ii) it has the coheir property: if $p$ as in (i) does not fork over $M$ then $p(x)$ is realized in $M$. By a lovely pair (of models of $T$) we mean a $|T|^+$-lovely pair.

Let $L_P$ be $L$ together with a new unary predicate $P$. Any elementary pair $(N, M)$ of models of $T$ ($M \subseteq N$) can be considered as an $L_P$-structure by taking $M$ to be the interpretation of $P$. A basic property from [BPV] says that any two lovely pairs of models of $T$ are elementarily equivalent, as $L_P$-structures. So $T_P$, the common $L_P$-theory of lovely pairs, is complete. $T$ has the wnfcp if every $|T|^+$-saturated model of $T_P$ is a lovely pair (equivalently, for every $\kappa \geq |T|^+$, any $\kappa$-saturated model of $T_P$ is a $\kappa$-lovely pair). Every theory with the wnfcp is in particular low (a subclass of simple theories).

This situation is, of course, a special case of our general setting in this paper, where $T_P$ is the given theory ($T$ in the general setting) and $T$ is the reduct ($T^-$ in the general setting).

Thus in this section we assume $T$ has the wnfcp and we work in a $\lambda$-big model $\mathcal{M} = (\bar{M}, P(\bar{M}))$ of $T_P$ for some large $\lambda$ (so $P^\mathcal{M} = P(\bar{M})$), $\downarrow$ will denote independence in $\mathcal{M}$ and $\downarrow^-$ will denote independence in $\bar{M} = \mathcal{M}|L$. Recall the following notation: for $a \in \mathcal{M}^{beq}$, let $a^c = Cb^-(a/P(\bar{M}))$, where $Cb^-$ denotes the canonical base (as a hyperimaginary element) in the sense of $T$.

**Proposition 4.1** 1) For every finite tuple of variables $x$, $\bar{\Gamma}_x = (x = x)$, namely the greatest universal transducer in the variables $x$ is $(x = x)$.

2) $\bar{P}(\bar{x})$ and $(\neg \bar{P}(\bar{x})) \cup acl_x(\emptyset)$ are universal transducers (where $\bar{P}(\bar{x})$ is the conjunction $\bigwedge_i P(x_i)$, $\bar{x} = (x_i)_i$).

3) If $T$ is in addition stable (equivalently $T$ has nfcp), then the $NI$-topology on $S_y(T_P)$ is generated by the family of $L$-definable sets over $\emptyset$. Thus an $\emptyset$-invariant set in $\mathcal{M}$ is a universal transducer iff it intersect every non-empty $L$-definable set over $\emptyset$.

We start with an observation (for part 3).

**Lemma 4.2** $\mathcal{M}^{eq} \cap ACL^{eq}(\emptyset) = acl^{eq}(\emptyset)$.

**Proof:** Otherwise, there exists $a \in (\mathcal{M}^{eq} \cap ACL^{eq}(\emptyset)) \setminus acl^{eq}(\emptyset)$. If $a \in acl^{eq}(a^c)$, then $a \in P(\bar{M})^{eq}$, but for all $b \in P(\bar{M})^{eq}$ we have $tp_L(b) \equiv tp_{L_P}(b)$.
so \( b \in (\mathcal{M}^{eq} \cap \text{ACL}^{eq}(\emptyset)) \) implies \( b \in \text{acl}^{eq}(\emptyset) \), a contradiction. Therefore, we may assume \( a \notin \text{acl}^{eq}(a^c) \). By the extension property there exists a sequence \( \langle a_i \mid i < \omega \rangle \) of realizations of \( \text{tp}_{L}(a/a^c) \) such that \( a_0 = a \) and for every \( i < \omega \), \( a_{i+1} \downarrow_{a^c} \{ a_0, \ldots a_i \} \cup P(\bar{M}) \).

**Claim 4.3** \( \text{tp}_{L_p}(a_i) = \text{tp}_{L_p}(a) \) for every \( i < \omega \).

**Proof:** By the construction of \( \langle a_i \mid i < \omega \rangle \), for every \( i < \omega \), \( \phi(x, a_i) \) is realized in \( P(\bar{M}) \) (where \( x \) is a tuple of variables from the home sort of \( \bar{M} \) and \( \phi(x, y) \in L^{eq} \)) iff \( \phi(x, a^c) \) L-doesn’t fork over \( P(\bar{M}) \) iff \( \phi(x, a_i) \) L-doesn’t fork over \( a^c \) iff \( \phi(x, a) \) L-doesn’t fork over \( P(\bar{M}) \) iff \( \phi(x, a) \) is realized in \( P(\bar{M}) \). We conclude that \( \text{Cl}(\text{tp}_{L}(a/P(\bar{M}))) = \text{Cl}(\text{tp}_{L}(a_i/P(\bar{M}))) \) and thus \( \text{tp}_{L_p}(a_i) = \text{tp}_{L_p}(a) \) for all \( i < \omega \) (this implication is [BPV, Corollary 3.11] for real tuples but remains true for imaginary elements).

Now, since \( a \notin \text{acl}^{eq}(a^c) \), we conclude that \( a_{i+1} \notin \text{acl}^{eq}(\{ a_0, \ldots a_i \}) \) for all \( i < \omega \) and in particular, the \( a_i \)-s are distinct, so \( a \notin \text{ACL}^{eq}(\emptyset) \), a contradiction.

**Proof of 4.1** To prove 1), recall the following fact (for convenience, we state it for a special case).

**Fact 4.4** [BPV, Proposition 7.3]

Let \( B \subseteq \mathcal{M} \) and \( a \) a tuple from \( \mathcal{M} \). Then
\[
a \downarrow B \quad \text{iff} \quad [a \downarrow_{P(\bar{M})} B \cup P(\bar{M}) \quad \text{and} \quad a^c \downarrow B^c ].
\]

\( \Gamma^*_x = \bar{\Gamma}_x \), so we need to show that for every finite tuples \( a, b \) from \( \mathcal{M} \), \( a \downarrow b \) implies \( a \downarrow b^c \). By Fact 14 it means we need to show that for every finite tuples \( a, b \) from \( \mathcal{M} \), if \( a \downarrow_{P(\bar{M})} b \cup P(\bar{M}) \) and \( a^c \downarrow b^c \), then \( a \downarrow b \). Indeed, as \( b \downarrow_{b^c} P(\bar{M}) \), our assumption implies \( b \downarrow_{b^c} aP(\bar{M}) \) and in particular \( b \downarrow_{b^c} a^c \) (\(*\)). As \( b^c \in \text{dcl}^{heq}(P(\bar{M})) \), \( a \downarrow_{a^c} b^c \). Our assumption \( a^c \downarrow b^c \), implies \( b^c \downarrow aa^c \). By (\(*\)), \( b \downarrow a \).

We prove 2). First we show \( \bar{P}(\bar{x})^\mathcal{M} \) is a universal transducer. Assume
\( \phi^-(\bar{x}, a) \) \( L \)-doesn’t fork over \( \emptyset \), where \( \phi^-(\bar{x}, y) \in L \). By the extension property, there exists \( \bar{b} \in \mathcal{M} \) such that \( \phi^-(\bar{b}, a) \) and \( \bar{b} \upharpoonright aP(M) \). In particular, \( tp_L(\bar{b}/aa^c) \) \( L \)-doesn’t fork over \( \emptyset \) and in particular it doesn’t fork over \( P(M) \). By the coherent property, \( tp_L(\bar{b}/aa^c) \) is realized in \( P(M) \). Let \( \bar{b}^* \in P(M) \) realize it. Then \( \phi^-(\bar{b}^*, a) \) and \( \bar{b}^* \upharpoonright a^c \). By Fact 4.4 as \( \bar{b}^* \in P(M) \), it follows that \( \bar{b}^* \downarrow a \). Thus \( P(\bar{x}) \land \phi^-(\bar{x}, a) \) \( L_P \)-doesn’t fork over \( \emptyset \). By 1), we conclude that \( P(\bar{x}) \) is a universal transducer.

To show that \( \Gamma(\bar{x}) = (\neg P(\bar{x})) \cup acl_\emptyset(\emptyset) \) is a universal transducer we assume \( \phi^-(\bar{x}, a) \) \( L \)-doesn’t fork over \( \emptyset \) for \( \phi^-(\bar{x}, y) \in L \). If some realization of \( \phi^-(\bar{x}, a) \) is in \( acl_\emptyset(\emptyset) \), we are done so we may assume any realization of it is not in \( acl_\emptyset(\emptyset) \). Therefore, there exists \( \bar{b}^* \in \phi^-(\mathcal{M}, a) \) such that \( \bar{b}^* \upharpoonright aP(M) \) and \( \bar{b}^* \notin acl(aP(M)) \). Let \( p^- = tp_L(\bar{b}^*/aP(M)) \). Let \( p \in S(T_{aP(M)}) \) be an extension of \( p^- \) that \( L_P \)-doesn’t fork over \( \emptyset \). Let \( p^* = p|a \). Then, \( p^*(\bar{x}) \vdash (\neg P(\bar{x})) \land \phi^-(\bar{x}, a) \), so we are done.

We prove 3). We need to show that for every \( p(x) \in S_\emptyset(T_P) \) and \( \phi^-(x, y) \in L \), the set \( U_{p, \phi^-} \) is \( L \)-definable over \( \emptyset \). We go back to the proof of Lemma 3.15 (2): Let \( \chi^-(y) \in L(\mathcal{M}) \) be the definition of the \( \phi^- \)-type of some \( L_P \)-non-forking extension of \( p \). Then \( \chi^-(y) \) is over \( ACL^{eq}(\emptyset) \). Let \( \mathcal{C} \in M^{eq^-} \) be the canonical parameter of \( \chi^-(y) \). Since \( c \in ACL^{eq}(\emptyset) \), by Lemma 4.2, \( c \in acl^{eq}(\emptyset) \). As in Lemma 3.15 (2), it follows that \( U_{p, \phi^-} = \bigvee_{i<n} \chi^-_i(C) \) where \( \{\chi^-_i(y)\}_{i<n} \) is the set of \( \emptyset \)-conjugates of \( \chi^-(y) \) in \( \mathcal{M} \), but since \( c \in acl^{eq}(\emptyset) \) and \( acl^{eq}(\emptyset) \subseteq P(M)^{eq} \), \( \{\chi^-_i(y)\}_{i<n} \) is also the set of \( \emptyset \)-conjugates of \( \chi^-(y) \) in \( M^{eq^-} = M^{eq} \), so \( U_{p, \phi^-} \) is \( L \)-definable over \( \emptyset \).

**Remark 4.5**

1) Note that by Proposition 4.7(1),(2) it follows that for a simple theory \( T \) in \( L \) and a reduct \( T^- \) of \( T \) a universal transducer is not necessarily unique as an \( L \)-type-definable set over \( \emptyset \).

2) A corollary of Theorem 3.19 and Proposition 4.7(1),(2) is that any \( (L_P, L) - \infty \)-definable set over \( \emptyset \) containing \( P(\bar{x}) \) must equal to \( \bar{x} = \bar{x} \).

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