REMARKS ON NEIGHBORHOOD STAR-MEMBERG SPACES

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Abstract. A space $X$ is said to be neighborhood star-Menger if for every sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$ one can choose finite $A_n \subseteq X$, $n \in \mathbb{N}$ such that for every open $O_n \supseteq A_n$, $n \in \mathbb{N}$, $\{\text{St}(O_n, U_n) : n \in \mathbb{N}\}$ is an open cover of $X$. We investigate the relationship between neighborhood star-Menger spaces and related spaces, and study the topological properties of neighborhood star-Menger spaces.

1. Introduction

By a space, we mean a topological space. We give definitions of terms which are used in this paper. Let $\mathbb{N}$ denote the set of positive integers. Let $X$ be a space and $U$ a collection of subsets of $X$. For $A \subseteq X$, let $\text{St}(A, U) = \bigcup \{U \in U : U \cap A \neq \emptyset\}$. As usually, we write $\text{St}(x, U)$ instead of $\text{St}(\{x\}, U)$.

Let $A$ and $B$ be collections of open covers of a space $X$. Then the symbol $S_1(A, B)$ denotes the selection hypothesis so that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in U_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $B$. The symbol $S_{\text{fin}}(A, B)$ denotes the selection hypothesis so that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n$ is a finite subset of $U_n$ and $\bigcup_{n \in \mathbb{N}} V_n$ is an element of $B$ (see [6, 11]).

Kočinac [7, 8] introduced a star selection hypothesis similar to the previous ones. Let $A$ and $B$ be collections of open covers of a space $X$. Then:

(A) The symbol $S_{\text{fin}}^*(A, B)$ denotes the selection hypothesis so that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n$ is a finite subset of $U_n$ and $\bigcup_{n \in \mathbb{N}} \{\text{St}(V, U_n) : V \in V_n\}$ is an element of $B$.

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The symbol $SS^*_{\text{fin}}(A, B)$ denotes the selection hypothesis so that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X$ such that $\{\text{St}(A_n, U_n) : n \in \mathbb{N}\} \in B$.

Bonanzinga et al. [2] introduced the following definition.

The symbol $NSM(A, B)$ denotes the selection hypothesis so that for each sequence $(U_n : n \in \mathbb{N})$ of elements of $A$ one can choose finite $A_n \subseteq X, n \in \mathbb{N}$, such that for every open $O_n \supseteq A_n, n \in \mathbb{N}$, $\{\text{St}(O_n, U_n) : n \in \mathbb{N}\} \in B$.

Let $O$ denote the collection of all open covers of $X$.

**Definition 1.1.** [7, 8] A space $X$ is said to be star-Menger if it satisfies the selection hypothesis $S^*_{\text{fin}}(O, O)$.

**Definition 1.2.** [7, 8] A space $X$ is said to be strongly star-Menger if it satisfies the selection hypothesis $SS^*_{\text{fin}}(O, O)$.

**Definition 1.3.** [2] A space $X$ is said to be neighborhood star-Menger if it satisfies the selection hypothesis $NSM(O, O)$.

From the above definitions, we have the following diagram

\[
\text{strongly star-Menger} \Rightarrow \text{neighborhood star-Menger} \Rightarrow \text{star-Menger}.
\]

The purpose of this paper is to investigate the relationships between neighborhood star-Menger spaces and related spaces, and also study topological properties of neighborhood star-Menger spaces.

Throughout this paper, let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $\kappa$ the cardinality of the set of all real numbers. For a cardinal $\kappa$, let $\kappa^+$ be the smallest cardinal greater than $\kappa$. For each pair of ordinals $\alpha, \beta$ with $\alpha < \beta$, we write $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usually, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow from [5].

### 2. Neighborhood star-Menger spaces and related spaces

In this section, first we give some examples to clarify the relationships between neighborhood star-Menger spaces and related spaces. Recall that a space is called Urysohn if every two distinct points have neighborhoods with disjoint closures. Clearly, the property is between the Hausdorff condition and regularity. First we give a consistent example showing that there exists a neighborhood star-Menger space that is not strongly star-Menger by using the following example from [2]. We make use of one of the cardinals defined in [4]. Define $\omega^*$ as the set of all functions from $\omega$ to itself. For all $f, g \in \omega^*$, we say $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n$. The dominating number, denoted by $d$, is the smallest cardinality of a cofinal subset of $(\omega^*, \leq^*)$ (see [4] for details). Recall that a space $X$ is strongly star-Lindelöf (see [3] or [9] under different names) if for every open cover $\mathcal{U}$ of $X$ there exists a countable subset $A$ of $X$ such that $X = \text{St}(A, \mathcal{U})$. Clearly every strongly star-Menger space is strongly star-Lindelöf.
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Example 2.1. [2] (ω₁ < ϱ) There is a Urysohn neighborhood star-Menger space that is not strongly star-Lindelöf (hence not strongly star-Menger).

Recall that a space X is strongly starcompact (see [3] or [9] under different names) if for every open cover U of X there exists a finite subset A of X such that \(X = St(A, U)\). It is well known that strongly starcompactness is equivalent to countable compactness for Hausdorff spaces (see [3] or [9]). Recall that a space X is weakly starcompact [1] if for every open cover U of X there exists a finite subset A of X such that for every open \(O \supseteq A\), \(X = St(O, U)\). It is clear that every weakly starcompact space is neighborhood star-Menger. For \(T₃\) spaces, we have the following example.

Example 2.2. There exists a \(T₃\) neighborhood star-Menger space that is not strongly star-Menger.

Proof. Let \(X = [0, \omega₁) \cup D\), where \(D = \{d_α : α < ω₁\}\) is a set of cardinality \(ω₁\). We topologize X as follows: \([0, ω₁)\) has the usual order topology and is an open subspace of X; a basic neighborhood of a point \(d_α \in D\) takes the form \(O_β(d_α) = \{d_α\} \cup (β, ω₁)\), where \(β < ω₁\).

Then \(X\) is a \(T₃\) space.

First we show that \(X\) is neighborhood star-Menger. We only show that \(X\) is weakly starcompact, since every weakly starcompact space is neighborhood star-Menger. To this end, let U be an open cover of X. Without loss of generality, we can assume that U consists of basic open subsets of X. Thus it is sufficient to show that there exists a finite subset A of X such that for every open \(O \supseteq A\), \(X = St(O, U)\). Since \([0, ω₁)\) is countably compact, it is strongly starcompact (see [3, 9]), then we can find a finite subset \(A₁\) of \([0, ω₁)\) such that \([0, ω₁) \subseteq St(A₁, U)\). On the other hand, if we pick \(α₀ < ω₁\), then for every open \(O \supseteq d_α, D \subseteq St(O, U)\).

In fact, for each \(α < ω₁\), if \(d_α \in U_α \subseteq U\), then \(U_α \cap O \neq \emptyset\) by the construction of the topology of X, thus \(d_α \in St(O, U)\). Therefore \(D \subseteq St(O, U)\). If we put \(A = A₁ \cup \{d_α\}\), then A is a finite subset of X and \(X = St(O, U)\) for every open \(O \supseteq A\), which shows that \(X\) is weakly starcompact.

Next we show that \(X\) is not strongly star-Menger. For each \(n \in \mathbb{N}\), let

\[U_n = \{O_α(d_α) : α < ω₁\} \cup \{[0, ω₁]\}\].

Then \(U_n\) is an open cover of X. Let us consider the sequence \((U_n : n \in \mathbb{N})\) of open covers of X. It suffices to show that \(\bigcup_{n \in \mathbb{N}} St(A_n, U_n) \neq X\) for any sequence \((A_n : n \in \mathbb{N})\) of finite subsets of X. Let \((A_n : n \in \mathbb{N})\) be any sequence of finite subsets of X. For each \(n \in \mathbb{N}\), the set \(A_n \cap \{d_α : α < ω₁\}\) is finite, since \(A_n\) is finite. Then there exists \(α_n < ω₁\) such that \(A_n \cap \{d_α : α > α_n\} = \emptyset\). Let \(α' = \sup\{α_n : n \in \mathbb{N}\}\). Then \(α' < ω₁\) and \((\bigcup_{n \in \mathbb{N}} A_n) \cap \{d_α : α > α'\} = \emptyset\). For each \(n \in \mathbb{N}\), the set \(A_n \cap [0, ω₁)\) is finite suborder of the linear order \([0, ω₁)\) and thus has a maximum. Let \(α'' = \max(A_n \cap [0, ω₁))\). Then \(A_n \cap (α'', ω₁) = \emptyset\). Let \(α'' = \sup\{α'' : n \in \mathbb{N}\}\). Then \(α'' < ω₁\) and \((\bigcup_{n \in \mathbb{N}} A_n) \cap (α'', ω₁) = \emptyset\). If we pick \(β > \max\{α', α''\}\), then \(O_β(d_β) \cap A_n = \emptyset\) for each \(n \in \mathbb{N}\). Hence \(d_β \notin St(A_n, U_n)\).
for each $n \in \mathbb{N}$, since $O_\beta(d_\beta)$ is the only element of $\mathcal{U}_n$ containing the point $d_\beta$ for each $n \in \mathbb{N}$, which shows that $X$ is not strongly star-Menger. □

**Remark 2.1.** The author does not know if there exists a Hausdorff (Urysohn, regular or Tychonoff) neighborhood star-Menger space that is not strongly star-Menger.

Bonanzinga et al. [2] constructed an example showing that there exists a Tychonoff star-Menger space $X$ that is not strongly star-Lindelöf. In fact, the Example also shows that there exists a Tychonoff star-Menger space $X$ that is not neighborhood star-Menger. Here we give the construction roughly for the sake of completeness.

**Example 2.3.** [2] There exists a Tychonoff star-Menger space $X$ that is not neighborhood star-Menger.

**Proof.** Let $D = \{d_\alpha : \alpha < \kappa\}$ be a discrete space of cardinality $\kappa$ and let $Y = D \cup \{d^*\}$ be one-point compactification of $D$.

Let $X = (Y \times [0, \kappa^+)) \cup (D \times \{\kappa^+\})$ be the subspace of the product space $Y \times [0, \kappa^+]$. Then $X$ is star-Menger, but not strongly star-Lindelöf (see [2] Example 3.7 for detail). Hence it is not neighborhood star-Menger, since every neighborhood star-Menger space is strongly star-Lindelöf. □

Next we study the topological properties of neighborhood star-Menger spaces. First we give an example from [1] that we shall use it in the following text.

**Example 2.4.** [1] Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\omega$ (i.e., the intersection of every two distinct elements of $\mathcal{A}$ is finite) and let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space constructed from $\mathcal{A}$ [3, 5]. Then

1. $X$ is strongly star-Menger if and only if $|\mathcal{A}| < 2$;
2. If $|\mathcal{A}| = c$, then $X$ is not star-Menger.

If $\omega_1 < 2$, the space $X = \omega \cup \mathcal{A}$ with $|\mathcal{A}| = \omega_1$ is strongly star-Menger (hence neighborhood star-Menger) by Example 2.4. It shows that a closed subset of a Tychonoff strongly star-Menger (hence neighborhood star-Menger) space $X$ need not be neighborhood star-Menger, since $\mathcal{A}$ is a discrete closed subset of cardinality $\omega_1$. Now we give a stronger example showing that a regular-closed subset of a Tychonoff neighborhood star-Menger space $X$ need not be neighborhood star-Menger. Here a subset $A$ of a space $X$ is said to be regular-closed in $X$ if $\text{Int} A = A$. For the next example, we need the following Lemma.

**Lemma 2.1.** [2] A space $X$ is neighborhood star-Menger if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$ there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X$ such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\text{St}(x, \mathcal{U}) \cap A_n \neq \emptyset$.

**Example 2.5.** ($\omega_1 < 2$) There exists a Tychonoff strongly star-Menger (hence neighborhood star-Menger) space having a regular-closed subspace which is not neighborhood star-Menger.
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Proof. Let $S_1 = \omega \cup A$ be the Isbell-Mrówka space with $|A| = \omega_1$ in Example 2.4. Then $S_1$ is Tychonoff strongly star-Menger by Example 2.4. Hence $S_1$ is neighborhood star-Menger.

Let $D = \{d_n : \alpha < \omega_1\}$ be a discrete space of cardinality $\omega_1$ and $Y = D \cup \{d^*\}$ one-point compactification of $D$. Let $S_2 = (\mathbb{Y} \times [0, \omega)) \cup (D \times \{\omega\})$ be the subspace of the product space $\mathbb{Y} \times [0, \omega]$. We show that $S_2$ is not neighborhood star-Menger. For each $n \in \mathbb{N}$, let

$$U_n = \{Y \times \{m\} : m \in \omega\} \cup \{(d_n) \times [0, \omega] : \alpha < \omega_1\}.$$ 

Then $U_n$ is an open cover of $S_2$. Let us consider the sequence $(U_n : n \in \mathbb{N})$ of open covers of $S_2$. It suffices to show that for any sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $S_2$, there exists a point $a \in S_2$ such that $\text{St}(a, U_n) \cap A_n = \emptyset$ for all $n \in \mathbb{N}$ by Lemma 2.4. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of $S_2$. For each $n \in \mathbb{N}$, since $A_n$ is finite, there exists $\alpha_n < \omega_1$ such that $A_n \cap \{(d_n) \times [0, \omega]\} = \emptyset$ for each $\alpha > \alpha_n$. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap \{(d_\alpha) \times [0, \omega]\} = \emptyset$$

for each $\alpha > \beta$.

Let us pick $\alpha > \beta$. Since $\{(d_\alpha) \times [0, \omega]\}$ is the only element of $U_n$ containing the point $(d_\alpha, \omega)$ for each $n \in \mathbb{N}$, $\text{St}((d_\alpha, \omega), U_n) = \{(d_\alpha) \times [0, \omega]\}$ for each $n \in \mathbb{N}$. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap \text{St}((d_\alpha, \omega), U_n) = \emptyset.$$ 

By the constructions of the topology of $S_2$ and the open cover $U_n$, we have

$$\text{St}((d_\alpha, \omega), U_n) = \{(d_\alpha) \times [0, \omega]\}.$$ 

Hence $\text{St}((d_\alpha, \omega), U_n) \cap A_n = \emptyset$ for all $n \in \mathbb{N}$.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\omega\} \to A$ be a bijection. Let $X$ be the quotient image of the disjoint sum $S_1 \oplus S_2$ obtained by identifying $(d_\alpha, \omega)$ of $S_2$ with $\pi((d_\alpha, \omega))$ of $S_1$ for every $\alpha < c$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_2)$ is a regular-closed subspace of $X$ which is not neighborhood star-Menger, since it is homeomorphic to $S_2$.

Finally we show that $X$ is strongly star-Menger. To this end, let $(U_n : n \in \mathbb{N})$ be a sequence of open covers of $X$. Since $\varphi(S_1)$ is homeomorphic to $S_1$, then $\varphi(S_1)$ is strongly star-Menger, there exists a sequence $(A'_n : n \in \mathbb{N})$ of finite subsets of $\varphi(S_1)$ such that $\varphi(S_1) \subseteq \bigcup_{n \in \mathbb{N}} \text{St}(A'_n, U_n)$. On the other hand, for each $n \in \mathbb{N}$, since $\varphi(\mathbb{Y} \times \{\{\}}$ is homeomorphic to $\mathbb{Y} \times \{\{\}$, then $\varphi(\mathbb{Y} \times \{\{\}$ is compact, we can find a finite subset $A''_n \subseteq \varphi(\mathbb{Y} \times \{\{\}$ such that $\varphi(\mathbb{Y} \times \{\{\}) \subseteq \text{St}(A''_n, U_m)$. For each $n \in \mathbb{N}$, we put $A_n = A'_n \cup A''_{n-1}$. Then the sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X$ witnesses that $X$ is strongly star-Menger, which completes the proof. \[ \square \]

In the following, we give a positive result, which can be easily proved.

THEOREM 2.1. If $X$ is a neighborhood star-Menger space, then every open and closed subset of $X$ is neighborhood star-Menger.
It is known that a continuous image of a strongly star-Menger space is strongly star-Menger. Similarly, we have the following result.

**Theorem 2.2.** A continuous image of a neighborhood star-Menger space is neighborhood star-Menger.

**Proof.** Let \( f : X \to Y \) be a continuous mapping from a neighborhood star-Menger space \( X \) onto a space \( Y \). Let \((U_n : n \in \mathbb{N})\) be a sequence of open covers of \( Y \). Then \( \{ \{ f^{-1}(U) : U \in U_n \} : n \in \mathbb{N} \} \) is a sequence of open covers of \( X \). Since \( X \) is neighborhood star-Menger, there exists a sequence \((A_n : n \in \mathbb{N})\) of finite subsets of \( X \) such that for every open \( O_n \supseteq A_n \), \( n \in \mathbb{N} \), \( \{ \text{St}(A_n, \{ f^{-1}(U) : U \in U_n \}) : n \in \mathbb{N} \} \) is an open cover of \( X \). Thus \( \{ f(A_n) : n \in \mathbb{N} \} \) is a sequence of finite subsets of \( Y \). We only show that for every open \( W_n \supseteq f(A_n), n \in \mathbb{N}, \{ \text{St}(W_n, U_n) : n \in \mathbb{N} \} \) is an open cover of \( Y \). In fact, let \( y \in Y \). Then there is \( x \in X \) such that \( f(x) = y \). Let \( W_n \) be an open subset of \( Y \) such that \( f(A_n) \subseteq W_n \) for \( n \in \mathbb{N} \). Then \( f^{-1}(W_n) \) is an open cover of \( X \). \( A_n \subseteq f^{-1}(W_n) \) for each \( n \in \mathbb{N} \) and
\[
\{ \text{St}(f^{-1}(W_n), \{ f^{-1}(U) : U \in U_n \}) : n \in \mathbb{N} \}
\]
is an open cover of \( X \). Hence there exist \( n \in \mathbb{N} \) and \( U \in U_n \) such that \( x \in f^{-1}(U) \) and \( f^{-1}(U) \cap f^{-1}(W_n) \neq \emptyset \). Thus \( y = f(x) \in f(f^{-1}(U)) = U \) and \( U \cap W_n \neq \emptyset \). This means that \( y \in \text{St}(W_n, U_n) \), which completes the proof. \( \square \)

Next we turn to considering preimages. To show that the preimage of a neighborhood star-Menger space under a closed 2-to-1 continuous map need not be neighborhood star-Menger, we use the Alexandorff duplicate \( A(X) \) of a space \( X \). The underlying set \( A(X) = X \times \{ 0, 1 \} \); each point of \( X \times \{ 1 \} \) is isolated and a basic neighborhood of \( \langle x, 0 \rangle \in X \times \{ 0 \} \) is a set of the form \( (U \times \{ 0 \}) \cup ((U \times \{ 1 \}) \setminus \{ \langle x, 0 \rangle \}) \), where \( U \) is a neighborhood of \( x \) in \( X \).

**Example 2.6.** \( \omega_1 < \omega \) There exists a closed 2-to-1 continuous map \( f : X \to Y \) such that \( Y \) is a neighborhood star-Menger space, but \( X \) is not neighborhood star-Menger.

**Proof.** Let \( Y \) be the space \( S_1 \) in the proof of Example 2.5. Then \( Y \) is neighborhood star-Menger. Let \( X \) be the Alexandorff duplicate \( A(Y) \) of the space \( Y \). Then \( X \) is not neighborhood star-Menger. In fact, let \( A = \{ \langle a, 1 \rangle : a \in A \} \). Then \( A \) is an open and closed subset of \( X \) with \( |A| = \omega_1 \), and each point \( \langle a, 1 \rangle \) is isolated. Hence \( A(X) \) is not neighborhood star-Menger by Theorem 2.1, since \( A \) is not neighborhood star-Lindelöf. Let \( f : X \to Y \) be the projection. Then \( f \) is a closed 2-to-1 continuous map, which completes the proof. \( \square \)

Now we give a positive result:

**Theorem 2.3.** Let \( f \) be an open and closed, finite-to-one continuous map from a space \( X \) onto a neighborhood star-Menger space \( Y \). Then \( X \) is neighborhood star-Menger.

**Proof.** Let \((U_n : n \in \mathbb{N})\) be a sequence of open covers of \( X \) and let \( y \in Y \). For each \( n \in \mathbb{N} \), since \( f^{-1}(y) \) is finite, there exists a finite subcollection \( U_{n,y} \) of \( U_n \) such
that $f^{-1}(y) \subseteq \bigcup U_{n,y}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_{n,y}$. Since $f$ is closed, there exists an open neighborhood $V_{n,y}$ of $y$ in $Y$ such that $f^{-1}(V_{n,y}) \subseteq \bigcup \{U : U \in \mathcal{U}_{n,y}\}$. Since $f$ is open, we can assume that

$$V_{n,y} \subseteq \bigcap \{f(U) : U \in \mathcal{U}_{n,y}\}.$$  

For each $n \in \mathbb{N}$, taking such open set $V_{n,y}$ for each $y \in Y$, we have an open cover $V_n = \{V_{n,y} : y \in Y\}$ of $Y$. Thus $(V_n : n \in \mathbb{N})$ is a sequence of open covers of $Y$, so that there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $Y$ such that for very open $O_n \supseteq A_n$, $n \in \mathbb{N}$, $\{\text{St}(O_n, V_n) : n \in \mathbb{N}\}$ is an open cover of $Y$, since $Y$ is neighborhood star-Menger. Since $f$ is finite-to-one, $(f^{-1}(A_n) : n \in \mathbb{N})$ is a sequence of finite subsets of $X$. We show that for very open $O'_n \supseteq f^{-1}(A_n)$, $n \in \mathbb{N}$, $\{\text{St}(O'_n, U_n) : n \in \mathbb{N}\}$ is an open cover of $X$. Since $f$ is closed and $A_n$ is finite, there exists an open subset $O_n$ such that $A_n \subseteq O_n$ and $f^{-1}(O_n) \subseteq O'_n$ for each $n \in \mathbb{N}$, thus $\{\text{St}(O_n, V_n) : n \in \mathbb{N}\}$ is an open cover of $Y$. Let $x \in X$. Then there exist $n \in \mathbb{N}$ and $y \in Y$ such that $f(x) \in V_{n,y}$ and $V_{n,y} \cap O_n \neq \emptyset$. Since

$$x \in f^{-1}(V_{n,y}) \subseteq \bigcup \{U : U \in \mathcal{U}_{n,y}\},$$

we can choose $U \in \mathcal{U}_{n,y}$ with $x \in U$. Then $V_{n,y} \subseteq f(U)$ by (2.1), and hence $U \cap f^{-1}(O_n) \neq \emptyset$. Since $f^{-1}(O_n) \subseteq O'_n$ for each $n \in \mathbb{N}$, $U \cap O'_n \neq \emptyset$. Thus $x \in \text{St}(O'_n, U_n)$. Hence $\{\text{St}(O'_n, U_n) : n \in \mathbb{N}\}$ is an open cover of $X$, which shows that $X$ is neighborhood star-Menger. \hfill \Box

**Example 2.7.** ($\omega_1 < \emptyset$) There exist a neighborhood star-Menger space $X$ and a compact space $Y$ such that $X \times Y$ is not neighborhood star-Menger.

**Proof.** Let $X$ be the space $S_1$ in the proof of Example 2.5. Then $X$ is neighborhood star-Menger. Let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality $\omega_1$ and $Y = D \cup \{d^*\}$ the one-point compactification of $D$. We show that $X \times Y$ is not neighborhood star-Menger. Since $|\mathcal{A}| = \omega_1$, we can enumerate $\mathcal{A}$ as $\{a_\alpha : \alpha < \omega_1\}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{\{a_\alpha \cup a_\alpha\} \times (Y \setminus \{d_\alpha\}) : \alpha < \omega_1\} \cup \{X \times \{d_\alpha\} : \alpha < \omega_1\} \cup \{\omega \times Y\}.$$ 

Then $\mathcal{U}_n$ is an open cover of $X \times Y$. Let us consider the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of $X \times Y$. It suffices to show that for any sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X \times Y$ there exists a point $a \in X \times Y$ such that $\text{St}(a, \mathcal{U}_n) \cap A_n = \emptyset$ for all $n \in \mathbb{N}$ by Lemma 2.4. Let $(A_n : n \in \mathbb{N})$ be any sequence of finite subsets of $X \times Y$. For each $n \in \mathbb{N}$, since $A_n$ is finite, there exists $\alpha_n < \omega_1$ such that $A_n \cap (X \times \{d_\alpha\}) = \emptyset$ for each $\alpha > \alpha_n$. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and $(\bigcup_{n \in \mathbb{N}} A_n) \cap (X \times \{d_\alpha\}) = \emptyset$ for each $\alpha > \beta$. Pick $\alpha > \beta$, since $X \times \{d_\alpha\}$ is the only element of $\mathcal{U}_n$ containing the point $\langle a_\alpha, d_\alpha \rangle$ for each $n \in \mathbb{N}$, $\text{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) = X \times \{d_\alpha\}$ for each $n \in \mathbb{N}$. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap \text{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) = \emptyset.$$ 

By the constructions of the topology of $X \times Y$ and the open cover $\mathcal{U}_n$, we have $\text{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) = X \times \{d_\alpha\}$. Hence $\text{St}(\langle a_\alpha, d_\alpha \rangle, \mathcal{U}_n) \cap A_n = \emptyset$ for all $n \in \mathbb{N}$. \hfill \Box
Remark 2.2. Example 2.7 also shows that Theorem 2.3 fails to be true if “open and closed, finite-to-one” is replaced by “open perfect”. The author does not know if there exists a ZFC example showing that the product of a neighborhood star-Menger space $X$ and a compact space $Y$ is not neighborhood star-Menger.

The following well-known example shows that the product of two countably compact (and hence neighborhood star-Menger) spaces need not be neighborhood star-Menger. Here we give the proof roughly for the sake of completeness. For a Tychonoff space $X$, let $\beta X$ denote the Čech-Stone compactification of $X$.

Example 2.8. There exist two countably compact (hence neighborhood star-Menger) spaces $X$ and $Y$ such that $X \times Y$ is not neighborhood star-Menger.

Proof. Let $D$ be a discrete space of cardinality $c$. We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$ and $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta D$ which are defined inductively so as to satisfy the following conditions 1, 2 and 3:

1. $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq c$ and $|F_\beta| \leq c$;
3. every infinite subset of $E_\alpha$ (resp., $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta D$ has cardinality $2^c$ (see [10]). Then $X \times Y$ is not neighborhood star-Menger, because the diagonal $\{(d, d) : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $c$ and the open and closed subsets of neighborhood star-Menger spaces are neighborhood star-Menger. □

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