MINIMAL GRAPHS IN $\mathbb{R}^4$ WITH BOUNDED JACOBIANS

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Abstract. We obtain a Bernstein type result for entire two dimensional minimal graphs in $\mathbb{R}^4$, which extends a previous one due to L. Ni. Moreover, we provide a characterization for complex analytic curves.

1. Introduction

The famous theorem of Bernstein states that the only entire minimal graphs in the Euclidean space $\mathbb{R}^3$ are planes. More precisely, if $f : \mathbb{R}^2 \to \mathbb{R}$ is an entire (i.e., defined over all of $\mathbb{R}^2$) smooth function whose graph

$$G_f := \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\}$$

is a minimal surface, then it is an affine function, and the graph is a plane.

This type of result has been generalized in higher dimension and codimension under various conditions. See [1], [3], [9] and the references therein for the codimension one case and [5], [10], [12] for the higher codimension case.

The aim of this paper is to study the following special case. Let $M$ be a minimal surface in $\mathbb{R}^4$ that can be described as the graph of an entire and smooth vector valued function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (f_1(x, y), f_2(x, y))$, that is

$$M = G_f := \{(x, y, f_1(x, y), f_2(x, y)) : (x, y) \in \mathbb{R}^2\}.$$

The following question arises in a natural way: Is it true that the graph $G_f$ of $f$ is a plane in $\mathbb{R}^4$? In general, the answer is no. An easy counterexample is given by the function $f(x, y) = (x^2 - y^2, 2xy)$. Actually, the graph of any holomorphic function $\Phi : \mathbb{C} \to \mathbb{C}$ is a minimal surface. So, the problem of finding geometric conditions in order to have a result of Bernstein type is reasonable. In this direction, R. Schoen [8] obtained a Bernstein type result by imposing the assumption that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Moreover, L. Ni [6] by using the result of R. Schoen [8] and results due to J. Wolfson [11] on minimal Lagrangian surfaces has derived a result of Bernstein type under the assumption that $f$ is an area-preserving map, that is the Jacobian $J_f := \det (df)$ satisfies $J_f = 1$, where $df$ denotes the differential of $f$.

In this paper we prove, firstly, the following result of Bernstein type, which generalizes the result due to L. Ni.

Theorem 1.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an entire smooth vector valued function such that its graph $G_f$ is a minimal surface in $\mathbb{R}^4$. If the Jacobian $J_f$ of $f$ is bounded, then $G_f$ is a plane.

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As a consequence we derive an easy alternative proof of the following well known result of Jörgens [4]:

**Jörgens’ Theorem.** The only entire solutions \( f : \mathbb{R}^2 \to \mathbb{R} \) of the Monge-Ampere equation \( f_{xx}f_{yy} - f_{xy}^2 = 1 \) are the quadratic polynomials.

There are plenty of entire minimal graphs in \( \mathbb{R}^4 \), the so called complex analytic curves. More precisely, if \( \Phi : \mathbb{C} \to \mathbb{C} \) is any entire holomorphic or anti-holomorphic function, then the graph
\[
G_{\Phi} = \{(z, \Phi(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\}
\]
of \( \Phi \) in \( \mathbb{C}^2 = \mathbb{R}^4 \) is a minimal surface and is called a complex analytic curve. Such surfaces are locally characterized (see for example L.P Eisenhart [2]) by the relation
\[
|K| = |K_N|
\]
where \( K \) and \( K_N \) stand for the Gauss and normal curvature of the surface, respectively. The following result is in valid.

**Theorem 1.2.** Let \( G_f \) be the graph of an entire smooth vector valued function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with Gaussian curvature \( K \) and normal curvature \( K_N \). Assume that \( G_f \) is minimal in \( \mathbb{R}^4 \). Then,
\[
\inf_{K < 0} \frac{|K_N|}{|K|} = 0,
\]
unless \( G_f \) is a complex analytic curve.

**Corollary 1.3.** Let \( G_f \) be the graph of an entire smooth vector valued function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) with Gaussian curvature \( K \) and normal curvature \( K_N \). If \( K_N = cK \), where \( c \) is a constant, then \( G_f \) is a complex analytic curve. More precisely, \( K_N = K = 0 \) and \( G_f \) is a plane or \( |c| = 1 \) and \( G_f \) is a non-trivial complex analytic curve.

## 2. Basic Notation and Definitions

A surface \( M \) in the Euclidean space \( \mathbb{R}^n \) is represented, locally, by a transformation \( X : D \subset \mathbb{R}^2 \to \mathbb{R}^n \) of rank 2, given by
\[
X(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_n(x, y)), \quad (x, y) \in D,
\]
where \( D \) is an open subset of \( \mathbb{R}^2 \) and \( f_i : D \to \mathbb{R}, \ i \in \{1, \ldots, n\}, \) are smooth functions. Following the standard notation of differential geometry, we denote by \( \langle \ , \ \rangle \) the Euclidean inner product on \( \mathbb{R}^n \) and by \( E, F, G \) the coefficients of the first fundamental form, which are given by
\[
E = \sum_{i=1}^n \left( \frac{\partial f_i}{\partial x} \right)^2, \ F = \sum_{i=1}^n \frac{\partial f_i}{\partial x} \frac{\partial f_i}{\partial y}, \ G = \sum_{i=1}^n \left( \frac{\partial f_i}{\partial y} \right)^2.
\]
We recall that the parameters \( (x, y) \) are called isothermal if and only if \( E = G \) and \( F = 0 \), everywhere on \( D \).

Consider a local orthonormal frame field \( \{e_1, e_2; \xi_3, \ldots, \xi_n\} \) in \( \mathbb{R}^n \) such that, restricted to \( M \), the vectors \( e_1, e_2 \) are tangent to \( M \) and, consequently, \( \xi_3, \ldots, \xi_n \) are normal to \( M \). Denote by \( \nabla \) the usual linear connection on \( \mathbb{R}^n \) and let
\[
h_{ij}^\alpha = \langle \nabla_{e_i} \xi_\alpha, e_j \rangle, \quad i, j \in \{1, 2\}, \quad \alpha \in \{3, \ldots, n\},
\]
be the coefficients of the second fundamental form.
The mean curvature vector $H$ and the Gauss curvature $K$ of $M$ are given, respectively, by

$$H = \frac{1}{2} \sum_{\alpha=3}^{n} (h_{11}^{\alpha} + h_{22}^{\alpha}) \xi_{\alpha},$$

$$K = \sum_{\alpha=3}^{n} \left( h_{11}^{\alpha} h_{22}^{\alpha} - (h_{12}^{\alpha})^2 \right).$$

Moreover, if $|h|^2 = \sum_{i,j=1}^{2} \sum_{\alpha=3}^{n} (h_{ij}^{\alpha})^2$ is the square of the length of the second fundamental form $h$, then the Gauss equation implies

$$2K = 4H^2 - |h|^2.$$

In the case where $M$ is minimal, i.e., $H = 0$, the above become

(2.1) \hspace{1cm} K = - \sum_{\alpha=3}^{n} \left\{ (h_{11}^{\alpha})^2 + (h_{12}^{\alpha})^2 \right\},

(2.2) \hspace{1cm} 2K = - |h|^2.

Another geometric invariant which plays an important role in the theory of surfaces in $\mathbb{R}^4$ is the normal curvature $K_N$ of $M$ which is given by

$$K_N = \sum_{i=1}^{2} \left( h_{11}^{4} h_{22}^{4} - h_{12}^{4} h_{11}^{4} \right).$$

In particular, for minimal surfaces we have

(2.3) \hspace{1cm} K_N = 2 \left( h_{11}^{4} h_{12}^{4} - h_{12}^{4} h_{11}^{4} \right).

One of the simplest ways to express a surface in $\mathbb{R}^{n+2}$ is in non-parametric form, that is to say, as the graph

$$G_f = \{ (x, y, f_1(x, y), \cdots, f_n(x, y)) \in \mathbb{R}^{n+2} : (x, y) \in D \}$$

of a vector valued map $f : D \to \mathbb{R}^n$, $f(x, y) = (f_1(x, y), \cdots, f_n(x, y))$, where $D$ is an open subset of $\mathbb{R}^2$. Of course, any surface can be locally described in this manner. By computing the Euler-Lagrange equations for the area integral we see that the surface $G_f$ is minimal if and only if $f$ satisfies the following equation,

(2.4) \hspace{1cm} \left( 1 + |f_y|^2 \right) f_{xx} - 2 \langle f_x, f_y \rangle f_{xy} + \left( 1 + |f_x|^2 \right) f_{yy} = 0.

This is the classical non-parametric minimal surface equation.

The following result due to R. Osserman [7, Theorem 5.1] is the main tool for the proofs of our results.

**Theorem 2.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an entire solution of the minimal surface equation. Then there exists real constants $a$, $b$, with $b > 0$, and a non-singular linear transformation

$$x = u, \hspace{1cm} y = au + bv,$$

such that $(u, v)$ are global isothermal parameters for the surface $G_f$.

Moreover the following identity is useful in the proofs.
Lagrange's Identity. For two vectors \( V = (v_1, \cdots, v_n) \), \( W = (\omega_1, \cdots, \omega_n) \) in \( \mathbb{R}^n \), we have

\[
\left( \sum_{i=1}^{n} v_i^2 \right) \left( \sum_{i=1}^{n} \omega_i^2 \right) - \left( \sum_{i=1}^{n} v_i \omega_i \right)^2 = \sum_{i<j} (v_i \omega_j - v_j \omega_i)^2.
\]

3. Proofs of the Results

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), \( f(x, y) = (f_1(x, y), f_2(x, y)) \), \((x, y) \in \mathbb{R}^2\), be an entire solution of the minimal surface equation. Then, the graph \( G_f = \{(x, y, f_1(x, y), f_2(x, y)) : (x, y) \in \mathbb{R}^2\} \) of \( f \) is a minimal surface in \( \mathbb{R}^4 \). By virtue of the Theorem 2.1, we can introduce global isothermal parameters \((u, v)\), via a non-singular transformation

\[
x = u, \quad y = au + bv,
\]

where \( a, b \) are real constants with \( b > 0 \). Now, the minimal surface \( G_f \) is parametrized via the map

\[
X(u, v) = (u, au + bv, \varphi(u, v), \psi(u, v)),
\]

where \( \varphi(u, v) := f_1(u, au + bv) \) and \( \psi(u, v) := f_2(u, au + bv) \). Since \((u, v)\) are isothermal parameters, the vectors

\[
X_u = (1, a, \varphi_u, \psi_u), \quad X_v = (0, b, \varphi_v, \psi_v)
\]

are orthogonal and of the same length, that is

\[
\varphi_u \varphi_v + \psi_u \psi_v = -ab,
\]

(3.2)

\[
E = 1 + a^2 + \varphi_u^2 + \psi_u^2 = b^2 + \varphi_v^2 + \psi_v^2.
\]

Moreover, the fact that the surface \( G_f \) is minimal, implies that the functions \( \varphi \) and \( \psi \) are harmonic, that is

\[
\varphi_{uu} + \varphi_{vv} = 0, \quad \psi_{uu} + \psi_{vv} = 0.
\]

(3.3)

Appealing to the Lagrange’s Identity and taking the relations (3.2) into account, we obtain

\[
E^2 = b^2 + \varphi_v^2 + \psi_v^2 + (a \varphi_v - b \varphi_u)^2
+ (a \psi_v - b \psi_u)^2 + (\varphi_u \psi_v - \varphi_v \psi_u)^2,
\]

or equivalently,

(3.4)

\[
E^2 = (1 + a^2 + b^2) E - b^2 + (\varphi_u \psi_v - \varphi_v \psi_u)^2.
\]

We set \( \Phi(u, v) = (\varphi(u, v), \psi(u, v)) \). Because of the relation

\[
\frac{\partial (\varphi, \psi)}{\partial (u, v)} = \frac{\partial (f_1, f_2)}{\partial (x, y)} \frac{\partial (x, y)}{\partial (u, v)}
\]

for the Jacobians, we have

\[
J_\Phi = b J_f,
\]

where \( J_f, J_\Phi \) stand for the Jacobians of \( f \) and \( \Phi \), respectively. So (3.4) becomes

(3.5)

\[
J_\Phi^2 = E^2 - (1 + a^2 + b^2) E + b^2,
\]
a useful identity for us.

Now we are ready to give the proof of Theorem 1.1.
Proof of Theorem 1.1. Since the Jacobian $J_\Phi$ is bounded, we conclude from (3.5) that $E$, and thus $\log E$, is bounded from above. On the other hand, the Gaussian curvature $K$ of $G_f$ is given by

$$K = \frac{\Delta \log E}{2E},$$

where $\Delta$ is the usual Laplacian operator on the $(u,v)$-plane. Appealing to (2.1), we deduce that the Gaussian curvature $K$ is non-positive. Consequently, $\Delta \log E \geq 0$ and thus the function $\log E$ is a subharmonic function defined on the whole plane. Since $\log E$ is also bounded from above, we deduce that $E$ is constant and consequently $K$ is identically zero. Then it follows immediately from (2.2) that the graph $G_f$ of $f$ is totally geodesic and hence a plane. \hfill \Box

Remark 3.1. In a similar way, we can prove the following result: Let $f : \mathbb{R}^2 \to \mathbb{R}^n$, \n
$$f(x,y) = (f_1(x,y), f_2(x,y), \ldots, f_n(x,y)),$$

be a vector valued function, defined on the whole $\mathbb{R}^2$, which is a solution of the minimal surface equation. If the quantity \n
$$\sum_{i<j} \left( \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial y} - \frac{\partial f_j}{\partial x} \frac{\partial f_i}{\partial y} \right)^2$$

is bounded, then the graph $G_f$ of $f$ is a plane in $\mathbb{R}^{n+2}$.

Now, we show that one can get the well known Jörgens’ result [4] as a consequence of Theorem 1.1.

Proof of Jörgens’ Theorem. Obviously $f_{xx} + f_{yy} \neq 0$ everywhere on $\mathbb{R}^2$. We consider the function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\Theta = \frac{f_{xx}f_{yy} - f_{xy}^2 - 1}{f_{xx}f_{yy}}.$$

The function $\Theta$, thanks to our assumption, is identically zero, so $\Theta_x = \Theta_y = 0$. On the other hand, one can readily verify that the equations $\Theta_x = \Theta_y = 0$ are equivalent to the minimal surface equation for the function $g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(x,y) = (f_x(x,y), f_y(x,y))$. Moreover, we have $J_g = 1$. So, according to Theorem 1.1, the graph $G_g$ of $g$ is a plane and the result is immediate. \hfill \Box

For the proof of Theorem 1.2, we need the following auxiliary result.

Lemma 3.2. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(u,v) = (\varphi(u,v), \psi(u,v))$, be a map, where $\varphi$ and $\psi$ are harmonic functions on $\mathbb{R}^2$, i.e., a harmonic map. Then $\inf |J_\Phi| = 0$, unless $\Phi$ is an affine map.

Proof. Suppose in the contrary that $\Phi$ is not affine and $\inf |J_\Phi| = c > 0$. Hence $|J_\Phi| \geq c > 0$. Assume at first that $J_\Phi \geq c > 0$. We view $\Phi$ as a complex valued function $\Phi : \mathbb{C} \to \mathbb{C}$, $\Phi = \varphi + i\psi$. Then, for $z = u + iv$, we have

$$\Phi_z = \frac{1}{2} (\varphi_u + \psi_v) + \frac{i}{2} (\psi_u - \varphi_v)$$

and

$$\Phi_\bar{z} = \frac{1}{2} (\varphi_u - \psi_v) + \frac{i}{2} (\psi_u + \varphi_v).$$
A simple calculation shows that
\[ (3.6) \quad J_\Phi = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2. \]
Furthermore, since \( \varphi \) and \( \psi \) are harmonic functions it follows that the function \( \Phi_z \)
is holomorphic. From our assumption and (3.6) we get
\[ (3.7) \quad |\Phi_z|^2 \geq |\Phi_{\overline{z}}|^2 + c \geq c > 0. \]
Since \( \Phi_z \) is an entire holomorphic function, Picard’s Theorem implies that \( \Phi_z \) must be constant. Therefore, there are real constants \( \kappa, \lambda \) such that
\[ \varphi_u + \psi_v = 2\kappa \quad \text{and} \quad \psi_u - \varphi_v = 2\lambda. \]
Then from (3.7) we deduce that
\[ (\psi_v - 2\kappa)^2 + (\psi_u - 2\lambda)^2 \leq \kappa^2 + \lambda^2 - c. \]
By the harmonicity of the functions \( \psi_v - 2\kappa, \psi_u - 2\lambda \) and the Liouville’s Theorem, we deduce that \( \varphi \) and \( \psi \) are affine functions, which contradicts our assumptions.

Assume now that \( J_\Phi \leq -c < 0 \). In this case, we consider the complex valued function \( \tilde{\Phi} = \psi + i\varphi \). Since \( J_{\tilde{\Phi}} = -J_\Phi \geq c > 0 \), proceeding as above we deduce that \( \tilde{\Phi} \) is affine, and consequently \( \Phi \) is affine. This is again a contradiction. Thus \( \inf |J_\Phi| = 0 \), and the proof is concluded. \( \square \)

**Proof of Theorem 1.2.** Assume that \( G_f \) is not a plane and that
\[ \inf_{K < 0} \frac{|K_N|}{|K|} > 0. \]
We introduce global isothermal parameters \( (u, v) \) such that the minimal surface \( G_f \) is parametrized via the map
\[ X(u, v) = (u, au + bv, \varphi(u, v), \psi(u, v)), \]
where \( a, b \) are real constants with \( b > 0 \).

We claim that \( (a, b) = (0, 1) \). Arguing indirectly, we assume that \( (a, b) \neq (0, 1) \). Differentiating (3.2) with respect to \( u, v \) and taking (3.3) into account, we find
\[ (3.8) \quad \varphi_{uu} \varphi_v + \varphi_u \varphi_{uv} = -\psi_{uv} \varphi_v - \psi_u \varphi_{uv}, \]
\[ \varphi_{uu} \varphi_u - \varphi_u \varphi_{uv} = -\psi_{uv} \psi_v + \psi_u \psi_{uv}. \]
Squaring both of them and summing we obtain
\[ (\varphi^2_v + \varphi^2_u) (\varphi^2_{uu} + \varphi^2_{uv}) = (\psi^2_u + \psi^2_v) (\psi^2_{uu} + \psi^2_{uv}). \]
Consider the following subset of \( \mathbb{R}^2 \)
\[ M_0 = \{(u, v) \in \mathbb{R}^2 : \omega(u, v) = 0 \}, \]
where
\[ \omega(u, v) := (\varphi^2_u + \varphi^2_v) (\varphi^2_{uu} + \varphi^2_{uv}), \]
or, equivalently, in view of (3.9)
\[ \omega(u, v) = (\psi^2_u + \psi^2_v) (\psi^2_{uu} + \psi^2_{uv}). \]
We claim that the complement \( M_1 = \mathbb{R}^2 - M_0 \) is dense in \( \mathbb{R}^2 \). To this purpose it is enough to show that the interior, \( \text{int}(M_0) \), of \( M_0 \) is empty. Assume in the contrary that \( \text{int}(M_0) \neq \emptyset \) and let \( U \) be a connected component of \( \text{int}(M_0) \). Then it follows
easily that the analytic functions $\varphi$ and $\psi$ are affine. Thus, by analyticity, \( G_f \) is a plane, which is a contradiction.

In the sequel, we work on \( M_1 \). By virtue of (3.8), we get

\[
\varphi_{uv} = -\left( \varphi_u \psi_u + \varphi_v \psi_v \right) \psi_{uv} - J_\Phi \psi_{uu},
\]

(3.10)

\[
\varphi_{uu} = \frac{J_\Phi \psi_{uv} - \left( \varphi_u \psi_u + \varphi_v \psi_v \right) \psi_{uu}}{\psi_u^2 + \psi_v^2}.
\]

The vector fields

\[
\xi = (-b\varphi_u + a\varphi_v, -\varphi_u, b, 0), \quad \eta = (-b\psi_u + a\psi_v, -\psi_u, 0, b)
\]

are normal to \( G_f \) and satisfy

\[
|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 = b^2 E^2.
\]

We, easily, check that the vector fields \( \{ e_1, e_2; \xi_3, \xi_4 \} \) given by

\[
e_1 = \frac{1}{\sqrt{E}} X_u, \quad e_2 = \frac{1}{\sqrt{E}} X_v, \\
\xi_3 = \frac{\xi}{|\xi|}, \quad \xi_4 = \frac{b}{|\xi|} E \left( |\xi|^2 \eta - \langle \xi, \eta \rangle \xi \right),
\]

constitute an orthonormal frame field along \( G_f \). Moreover, \( \xi_3 \) and \( \xi_4 \) are normal to \( G_f \). Then a straightforward computation shows that the coefficients of the second fundamental form are given by

\[
h_{11}^3 = \frac{-b\varphi_{uv}}{E |\xi|}, \quad h_{11}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uu} - |\xi|^2 \psi_{uu}}{E^2 |\xi|}, \\
h_{12}^3 = \frac{-b\varphi_{uv}}{E |\xi|}, \quad h_{12}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uv} - |\xi|^2 \psi_{uv}}{E^2 |\xi|}.
\]

So using (2.1) and (2.3) and (3.10), we find

(3.11)

\[
K = \frac{1}{E^3} \varphi_{uu}^2 + \psi_{uv}^2 \left( 2b^2 - \left( 1 + a^2 + b^2 \right) E \right)
\]

and

(3.12)

\[
K_N = \frac{2b}{E^3} \frac{\psi_{uu}^2 + \psi_{uv}^2}{\varphi_u^2 + \varphi_v^2} J_\Phi.
\]

The second equation of (3.2), yields

\[
E \geq \frac{1 + a^2 + b^2}{2}.
\]

Hence,

\[
2b^2 - \left( 1 + a^2 + b^2 \right) E \leq -\frac{1}{2} \left( a^2 + (b - 1)^2 \right) \left( a^2 + (b + 1)^2 \right) < 0.
\]

This shows that

\[
M_1 \subset \{(u, v) \in \mathbb{R}^2 : K(u, v) < 0 \}.
\]

Moreover,

\[
\frac{K_N^2}{K^2} = 4b^2 W(E),
\]
where \( W(t) \) is the increasing real valued function
\[
W(t) := \frac{t^2 - (1 + a^2 + b^2) t + b^2}{((1 + a^2 + b^2) t - 2b^2)^{1/2}}, \quad t \geq 1.
\]
From our assumption \( \inf_{K < 0} \left| \frac{K_N}{|K|} \right| > 0 \), we get
\[
\inf_{M_1} \frac{|K_N|}{|K|} > 0.
\]
Since \( W(t) \) is increasing, we have
\[
\inf_{M_1} \frac{K_N^2}{K^2} = 4b^2 W \left( \inf_{M_1} E \right).
\]
Hence \( W \left( \inf_{M_1} E \right) > 0 \) or, equivalently,
\[
\left( \inf_{M_1} E \right)^2 - \left( 1 + a^2 + b^2 \right) \inf_{M_1} E + b^2 > 0.
\]
Appealing to the identity (3.5), we deduce that \( \inf_{M_1} |J_\Phi| > 0 \). By continuity, and bearing in mind the fact that \( M_1 \) is dense in \( \mathbb{R}^2 \), we infer that \( |J_\Phi| \) is bounded from below away from zero. On the other hand \( \Phi(u, v) = (\varphi(u, v), \psi(u, v)) \) is a harmonic map. Therefore, according to Lemma 3.2, \( G_f \) is a plane which contradicts our assumptions. Thus \((a, b) = (0, 1)\) and the equations (3.2) become
\[
\varphi_u \varphi_v + \psi_u \psi_v = 0,
\]
\[
\varphi_u^2 + \psi_u^2 = \varphi_v^2 + \psi_v^2.
\]
So, \( \varphi_u = \pm \psi_v, \varphi_v = \mp \psi_u \) and \( G_f \) is a complex analytic curve. \( \square \)

**Proof of Corollary 1.3.** In the case where \( K \equiv 0 \), \( G_f \) is a plane. Consider now the case where \( K \) is not identically zero. According to Theorem 1.2 we have \( c = 0 \), unless \( G_f \) is a complex analytic curve. We claim that the case \( c = 0 \) does not occur. Indeed, arguing indirectly suppose that \( c = 0 \). As in the proof of Theorem 1.2, the set \( M_1 \) is dense in \( \mathbb{R}^2 \). From the assumption \( K_N = cK \), we conclude that \( K_N = 0 \) in \( M_1 \). Furthermore, the relation (3.12) yields that \( J_\Phi = 0 \). Taking into account the identity (3.5), we get that \( E \) is constant, which implies that \( K \) is identically zero, a contradiction. Therefore, \( G_f \) is a complex analytic curve. \( \square \)

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