NON-MARKOVIAN LIMITS OF ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

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Abstract. In this paper we consider an additive functional of an observable $V(x)$ of a Markov jump process. We assume that the law of the expected jump time $t(x)$ under the invariant probability measure $\pi$ of the skeleton chain belongs to the domain of attraction of a subordinator. Then, the scaled limit of the functional is a Mittag-Leffler process, provided that $\Psi(x) := V(x)t(x)$ is square integrable w.r.t. $\pi$. When the law of $\Psi(x)$ belongs to a domain of attraction of a stable law the resulting process can be described by a composition of a stable process and the inverse of a subordinator and these processes are not necessarily independent. On the other hand when the singularities of $\Psi(x)$ and $t(x)$ do not overlap with large probability the law of the resulting process has some scaling invariance property. We provide an application of the results to a process that arises in quantum transport theory.

1. Introduction

Consider a Markovian jump process \{\text{\textit{K}}_t, t \geq 0\} taking values in a Polish space \( (E,d) \) whose generator is given by $Lf(x) = t^{-1}(x) \int P(x,dy)[f(y) - f(x)]$, where $P(x,\cdot)$ is a family of probability measures and $t(x) > 0$. Suppose that $V : E \rightarrow \mathbb{R}$ is a measurable function. We are concerned in the behavior of an additive functional of the process given by $Y_t := \int_0^t V(\text{\textit{K}}_s)ds$. If $\mu_*$ is an invariant and ergodic probability measure, $V \in L^2(\mu_*)$ and is centered then, under some additional assumptions concerning dissipative properties of the process (e.g. the spectral gap estimate) one can prove the central limit theorem, i.e. the law of $N^{-1/2}Y_{Nt}$ converges, as $N \rightarrow +\infty$ to a Brownian motion. The situation changes when $\mu_*$ is no longer finite. In that case the laws of $\{N^{-1/\alpha}Y_{Nt}, t \geq 0\}$ for an appropriate $\alpha$ may converge to the law of a non-Markovian process \{\zeta_t, t \geq 0\} that can be described as follows. Suppose that $\{T_t, t \geq 0\}$ is a stable subordinator, see e.g. [21], Example 24.12. Since the trajectory of the process is in fact a.s. strictly increasing, see [21] Theorem 21.3 p. 136, its right-continuous inverse $T_s^{-1} := \inf\{t : T_t > s\}$ (called the first passage time) has a.s. continuous trajectories. Suppose that $\{B_t, t \geq 0\}$ is a Brownian motion with diffusion coefficient $\sigma > 0$. The process $\{\zeta_t := B_{T_t^{-1}}, t \geq 0\}$ is called Mittag-Leffler, see e.g. [14]. One can show, see Theorem 2 of [12], that its one-point statistics $u(s,x) := \mathbb{E}u_0(x + \zeta_s)$

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satisfies a fractional heat equation, i.e. the Volterra-type equation
\[
    u(s, x) = u_0(x) + \frac{\sigma^2}{2c_\alpha \Gamma(\alpha)} \int_0^s (s - r)^{\alpha - 1} (\partial_r^2 u)(r, x) dr.
\]
When \(u_0(x) = \delta_0(x)\) its Fourier transform in \(x\) is therefore given by
\[
    \hat{u}(s, \xi) = \int_\mathbb{R} e^{ix\xi} u(s, x) dx = E_\alpha(-\sigma^2 s^{\alpha} \xi^2/(2c_\alpha)),
\]
where
\[
    E_\alpha(z) := \sum_{m=0}^{+\infty} \frac{z^m}{\Gamma(1+m\alpha)}
\]
is a Mittag-Leffler function and \(\pi(t(x) \geq \lambda) \sim c_\alpha \lambda^{-\alpha}\) for \(\lambda \gg 1\). The convergence of additive functionals of Markov processes to Mittag-Leffler processes is a subject that can be traced back to the paper of Darling and Kac, see [7] (also [1]). We refer the reader to e.g. [14] or [23] and the references therein for a review of respectively mathematical and physical results concerning the subject.

In fact when \(V\) is no longer square integrable one can expect that the limiting process shall be of the form \(\{\zeta_s := B_{T_s^{-1}}, s \geq 0\}\), where \(\{T_s^{-1}, s \geq 0\}\) is an inverse of a subordinator and \(\{B_t, t \geq 0\}\) is a stable process, the cases of random walks with independent increments and partial sum process of arrays of independent random variables has been considered respectively in [2] and [19].

In the present article we consider the limiting behavior of an additive functional \(Y_t := \int_0^t V(K_s) ds\) of a Markovian jump processes \(\{K_t, t \geq 0\}\). This process is given by \(K_t = X_n\) for \(t \in [t_n, t_{n+1})\), where \(\{X_n, n \geq 0\}\) is a Markov chain (called skeleton) with transition probabilities \(P(x, \cdot), t_n := \sum_{k=0}^{n-1} \tau_k t(X_k)\) with \(\{\tau_n, n \geq 0\}\) is a sequence of i.i.d. mean one, exponentially distributed random variables independent of the chain and \(t : E \to (0, +\infty)\) is a measurable function. We can interpret \(t^{-1}(x)\) as the mean jump rate of the process at \(x\).

The skeleton chain is assumed to have the spectral gap property with respect to an invariant probability measure \(\pi\), see [2.1] below. On the other hand, the law of \(t(x)\) belongs to the normal domain of attraction of a certain \(\alpha\)-stable subordinator. We show in Section 3 see Theorem 3.1 that the processes \(\{N^{-\alpha/2}Y_{Nt}, t \geq 0\}\) converge, as \(N \to +\infty\), weakly over \(D(0, +\infty)\) to a Mittag-Leffler process, provided that \(\Psi(x) := V(x)t(x)\) belongs to \(L^2(\pi)\).

When \(\Psi\) is no longer square integrable but belongs to the normal domain of attraction of another \(\beta\)-stable law the situation becomes more complex. If the singularities of \(\Psi(x)\) and \(t(x)\) occur at the same points, see condition [2.13] below for a precise definition, then the processes \(\{N^{-\alpha/\beta}Y_{Nt}, t \geq 0\}\) weakly converge to a process of the form
\[
    \zeta_s := B_{T_s^{-1}}, s \geq 0,
\]
where \(\{(B_t, T_t), t \geq 0\}\) is a Levy process whose jump measure, in our case, is supported on a certain curve with a cusp, see Theorem 3.2. On the other hand, if we insist that the \(\pi\)-probability of \(\Psi(x)\) and \(t(x)\) being large together is negligible w.r.t. to the tails of each of
these functions, see condition (2.11), then the limiting process is also of the form (1.1) but with the processes \( \{B_t, t \geq 0\} \) and \( \{T_t, t \geq 0\} \) independent of each other. In the case when \( \{B_t, t \geq 0\} \) is symmetric \( \beta \)-stable, the one point statistics \( u(s, x) = \mathbb{E}u_0(x + \zeta_s) \) satisfies the fractional, both in time and space, heat equation \( \partial_t^s u(s, x) = -c_s(-\partial_x^2)^{\beta/2}u(s, x) \) for some constant \( c_s > 0 \), i.e.

\[
u(s, x) = u_0(x) - c_s \int_0^s (s - r)^{a-1}(-\partial_x^2)^{\beta/2}u(r, x)dr \quad (1.2)
\]

and \( \mathcal{F}((-\partial_x^2)^{\beta/2}f)(\xi) = -|\xi|^\beta \mathcal{F}(f)(\xi) \). Here \( \mathcal{F}(f) \) denotes the Fourier transform of \( f \).

The cases discussed above are in some sense extreme. If we can write the observable \( V(x) \) as a sum \( V_1(x) + V_2(x) \), where corresponding \( \Psi_i(x) = V_i(x)t(x), i = 1, 2 \) are of the above types respectively and belong to the normal domain of attraction of the same \( \beta \)-stable law then it can be concluded from the argument presented in the proofs of Theorems 3.2 and 3.4 that the limiting process equals \( B^{(1)}_{T^\alpha} + B^{(2)}_{T^\beta} \), with the joint law of \( \{(B_t^{(1)}, B_t^{(2)}, T_t) t \geq 0\} \) described by a Levy process with the first two components being \( \beta \)-stable and the last one \( \alpha \)-stable. Moreover \( \{B_t^{(1)} t \geq 0\} \) is then independent of the other two components, while \( \{(B_t^{(2)}, T_t) t \geq 0\} \) is as in the description following (1.1).

We remark also that the law of the limiting process \( \{\zeta_s, s \geq 0\} \) when (2.11) holds has scaling invariance property with exponent \( \alpha/\beta \), i.e. the laws of \( \{\zeta_{as}, s \geq 0\} \) and \( \{a^{\alpha/\beta}\zeta_s, s \geq 0\} \) are identical for each \( a > 0 \). In the particular case when \( \beta = 2\alpha \) we call such processes fake diffusions.

In Section 4 we illustrate our results taking as an example a jump process \( \{K_s, s \geq 0\} \) on a onedimensional torus, see (1.1) below. Such process arises in quantum transport theory, see [6] and describes the projection onto a 0-fiber of the solution of a translation invariant Lindblad equation. It possesses a unique \( \sigma \)-finite invariant measure that is absolutely continuous with respect to Lebesgue measure, see Proposition 4.1. Its dynamics is completely mixing and its one dimensional statistics converges to a mixture of delta type measures supported on the set \( [k : t(k) = +\infty] \), see Theorem 4.2. As an application of Theorems 2.6 and 3.2 we conclude also, see Corollary 4.3 convergence results for additive functionals of the type \( N^{-\alpha/\beta} \int_0^N V(K_s)ds \). In the particular case considered in [6] we have \( t(k) \sim |k + \pi/2|^{-2} \), as \( |k + \pi/2| \ll 1 \), so \( \alpha = 1/2 \). When \( V(k) \sim |k + \pi/2|^{-\gamma} \), where \( \gamma > 1 \), and \( V(k) \sim |k - k_0|^{-\gamma} \) for \( k_0 \notin \{-\pi/2, \pi/2\} \), the law of \( \Psi(k) \) belongs to the normal domain of attraction of a Cauchy law and the scaling properties of the limiting process are the same as those of the Brownian motion, so the limiting process \( \{\zeta_s, s \geq 0\} \) is in this case a fake diffusion.

2. Preliminaries and statements of the main results

2.1. A Markov chain. Let \((E,d)\) be a Polish metric space and let \( \mathcal{E} \) be its Borel \( \sigma \)-algebra. Assume that \( \{X_n, n \geq 0\} \) is a Markov chain with state space \( E \) and \( \pi \) - the law of \( X_0 \) - is an invariant and ergodic measure for the chain. We suppose that the chain satisfies:
Condition 2.1. Spectral gap condition:

\[ \sup \{ \| Pf \|_{L^2(\pi)} : f \perp 1, \| f \|_{L^2(\pi)} = 1 \} = a < 1. \] (2.1)

Since \( P \) is also a contraction in \( L^1(\pi) \) and \( L^\infty(\pi) \) we conclude, via Riesz-Thorin interpolation theorem, that for any \( p \in [1, +\infty) \):

\[ \| Pf \|_{L^p(\pi)} \leq a^{1-\vert 2/p-1 \vert} \| f \|_{L^p(\pi)}, \] (2.2)

for all \( f \in L^p(\pi) \), such that \( \int f d\pi = 0 \).

Suppose that \( t : E \to [0, +\infty) \) is measurable over \((E, \mathcal{E})\) and satisfies:

Condition 2.2. There exist \( \alpha \in (0, 1) \) and \( c_\alpha > 0 \) such that

\[ \lim_{\lambda \to +\infty} \lambda^\alpha \pi(\{ t \geq \lambda \}) = c_\alpha \] (2.3)

and there exists \( t_* > 0 \) such that

\[ t(x) \geq t_* > 0, \quad \forall x \in E. \]

We assume this condition in order to avoid the issue of explosions or accumulation points. In addition we suppose that the absolute continuous part of the transition probability function has some regularity property and the tails of \( t(x) \) under this measure is heavier than under the singular part. Namely, we assume that:

Condition 2.3. There exist a measurable family of Borel measures \( Q(x, dy) \) and a measurable, non-negative function \( p(x, y) \) such that

\[ P(x, dy) = p(x, y)\pi(dy) + Q(x, dy), \quad \text{for all } x \in E, \]

\[ C(2) := \sup_{y \in E} \int p^2(x, y)\pi(dx) < +\infty \]

and

\[ \sup_{x \in E} \frac{Q(x, [t(y) \geq \lambda])}{\int_{[t(y) \geq \lambda]} p(x, y)\pi(dy)} \leq C < +\infty, \quad \forall \lambda \geq 0. \] (2.6)

A simple consequence of (2.4) and the fact that \( \pi \) is invariant is that

\[ \int p(x, y)\pi(dy) \leq 1 \quad \text{and} \quad \int p(y, x)\pi(dy) \leq 1, \quad \forall x \in E. \]

Assume that \( \{k_N, N \geq 1\} \) is an arbitrary increasing sequence that converges to infinity. Let

\[ T^{(N)}(t) := \frac{1}{k_N^{1/\alpha}} \sum_{n=0}^{[k_N t]} t(X_n). \] (2.7)

It has been shown, see part ii) of Theorem 2.3 in [17], that the laws of the processes \( \{T^{(N)}(t), t \geq 0\} \) converge weakly over \( \mathcal{D} := D[0, +\infty) \) to the law of an \( \alpha \)-stable subordinator \( \{T_t, t \geq 0\} \).
2.2. The case when the observable is square integrable. Assume that $\Psi \in L^2(\pi)$ and $\int \Psi d\pi = 0$. Let

$$B_t^{(N)} := k_N^{-1/2} \sum_{n=0}^{[k_N t]} \Psi(X_n). \quad (2.8)$$

Since the generator of the Markov chain $\{X_n, n \geq 0\}$ has a spectral gap the processes $\{B_t^{(N)}, t \geq 0\}$ converge weakly over $\mathcal{D}$ to the law of a Brownian motion $\{B_t, t \geq 0\}$, see e.g. [20] Chapter VII. The characteristic function of the limiting process is given by

$$\psi(t) := e^{it \Psi} = \exp\{\sigma^2 t \xi^2\},$$

where $\sigma^2 = 2\langle (I - P)^{-1}\Psi, \Psi \rangle_\pi$. Our first theorem concerns the convergence of the joint law of $\{(B_t^{(N)}, T_t^{(N)}), t \geq 0\}$.

**Theorem 2.4.** With the assumptions made above, the joint laws of processes $\{(B_t^{(N)}, t \geq 0), \{T_t^{(N)}, t \geq 0\}\}$ converge, as $N \to +\infty$, in the product $J_1$ topology on $\mathcal{D} \times \mathcal{D}$ to the joint law of independent Lévy processes: $\{B_t, t \geq 0\}$ - a Brownian motion and $\{T_t, t \geq 0\}$ - an $\alpha$-stable subordinator.

2.3. The case when the observable is not square integrable. We do not assume here that $\Psi \in L^2(\pi)$. Instead, we suppose that the law of $\Psi$ under $\pi$ belongs to the normal domain of attraction of a stable law. Namely, suppose that $\Psi : E \to \mathbb{R}$ is Borel measurable such that there exist $\beta \in (0, 2)$ and two constants $c^+_\beta, c^-_\beta$ satisfying $c^+_\beta + c^-_\beta > 0$ and

$$\pi(\Psi \geq \lambda) = \frac{c^+_\beta}{\lambda^\beta} (1 + o(1)), \quad (2.9)$$

$$\pi(\Psi \leq -\lambda) = \frac{c^-_\beta}{\lambda^\beta} (1 + o(1)),$$

as $\lambda \to +\infty$. We let $c_\beta(\lambda) = c^+_\beta$ for $\lambda > 0$ and $c_\beta(\lambda) = c^-_\beta$ for $\lambda < 0$. The above assumption guarantees in particular that $\Psi \in L^\gamma(\pi)$ for any $\gamma < \beta$. It has been shown, see [17] Theorem 2.3, that if $\beta \neq 1$ then the laws of $\{B_t^{(N)}, t \geq 0\}$ converge weakly over $\mathcal{D}$ to the law of a stable process. In case $\beta = 1$ the result still holds but we have to center $B_t^{(N)}$. Let $c_N := \int |\Psi| d\pi$. Then, the laws of $\{B_t^{(N)} - c_N t, t \geq 0\}$ converge weakly to the law of a stable process. In both cases the limiting process is described by the characteristic functional $\mathbb{E}e^{it \xi B_t} = e^{i\psi(\xi)}$, where

$$\psi(\xi) := \int \psi(\xi, \lambda) d\nu_\beta(\lambda), \quad (2.10)$$

$$e_\beta(\xi, \lambda) := \begin{cases} e^{i\beta \lambda}, & \beta \in (0, 1), \\ e^{i\xi \lambda} - 1 - i \xi \lambda |1 - 1|_{\beta = 1}, & \beta = 1, \\ e^{i\beta \lambda} - 1 - i \beta \lambda, & \beta \in (1, 2) \end{cases}$$

and $\nu_\beta(\lambda) := \beta c_\beta(\lambda) |\lambda|^{-1-\beta} d\lambda$.

In our first result we adopt a hypothesis that $t(x)$ and $|\Psi(x)|$ cannot be large together. Namely,

$$\pi \{ x : t(x) \geq \lambda, |\Psi(x)| \geq \lambda \} \leq \frac{C_\ast}{\lambda^\gamma} \quad (2.11)$$
for some \( C_\ast > 0 \) and \( \gamma > \alpha \lor \beta \). Our principal result in this case is the following.

**Theorem 2.5.** Suppose that \((\ref{eq:2.11})\) holds and either of the three sets of conditions are satisfied:

i) when \( \beta \in (1,2) \) we suppose that \( \Psi \) is centered. Furthermore, assume that for some \( \beta' > \alpha \lor \beta \) we have

\[
\| P\Psi \|_{L^{\beta'}(\pi)} < +\infty. \tag{2.12}
\]

ii) if \( \beta \in (0,1) \) then, we no longer assume that \( \Psi \) is centered.

iii) when \( \beta = 1 \), assume that for some \( \beta' > 1 \) we have

\[
\sup_{N \geq 1} \| P\Psi_N \|_{L^{\beta'}(\pi)} < +\infty,
\]

where \( \Psi_N := \Psi 1_{[\Psi \leq N]} \). Define \( B^{(N)}_t := k^{-1} \sum_{n=0}^{\lfloor kNt \rfloor} (\Psi(X_n) - c_N) \).

Then, the joint laws of \( \{B^{(N)}_t, t \geq 0\}, \{T^{(N)}_t, t \geq 0\} \) converge, as \( N \to +\infty \), weakly in the product \( J_1 \) topology over \( D \times D \) to the joint law of the independent Lévy process: \( \{B_t, t \geq 0\} \) - a \( \beta \)-stable and \( \{T_t, t \geq 0\} \) an \( \alpha \)-stable subordinator.

Let \( e(\lambda) := C_{\alpha,\beta}|\lambda|^{\beta/\alpha} \), where \( C_{\alpha,\beta} := c_\alpha (c_\beta + c_\beta^+)^{-1} \). In our next result we assume that \( t \) is large only when \( \Psi \) is such, i.e.

\[
\pi \left[ x : |t(x) - e \circ \Psi(x)| \geq \lambda \right] \leq \frac{C_\ast}{\lambda^\gamma} \tag{2.13}
\]

for some \( C_\ast > 0 \) and \( \gamma > \alpha \). Define

\[
\Lambda_{\theta_1,\theta_2}(\lambda) := \theta_1 \lambda + \theta_2 e(\lambda).
\]

Consider now a Lévy process \( \{B_t, T_t, t \geq 0\} \) such that \( \{T_t, t \geq 0\} \) is an \( \alpha \)-stable subordinator and \( \{B_t, t \geq 0\} \) is a \( \beta \)-stable process as described above and whose jump measure is given by \( \nu_\ast(d\lambda_1, d\lambda_2) := \delta(\lambda_2 - e(\lambda_1))\nu_\ast(d\lambda_1)d\lambda_2 \).

**Theorem 2.6.** Suppose that \((\ref{eq:2.13})\) holds. Then, in any of the three cases i), ii) and iii) considered in Theorem 2.5 the laws of \( \{(B^{(N)}_t, T^{(N)}_t), t \geq 0\} \) converge, as \( N \to +\infty \), weakly in the \( J_1 \) topology over \( D_2 := D([0, +\infty), \mathbb{R}^2) \) to the law of the Lévy process described above.

3. **An Application to an Additive Functional of a Jump Process**

Consider now a Markovian jump process constructed in the following way. Let \( \{\tau_n, n \geq 0\} \) be a sequence of i.i.d. exponentially distributed with intensity 1 random variables. Let \( \{X_n, n \geq 0\} \) be a Markov chain as in the previous section, \( t_0 := 0 \) and

\[
t_n := \sum_{k=0}^{n-1} t(X_k)\tau_k \tag{3.1}
\]

for \( n \geq 1 \). Define \( K_t := X_n \) for \( t \in [t_n, t_{n+1}), n = 0, 1, \ldots \)

Suppose now that

\[
\Psi(x) := V(x)t(x). \tag{3.2}
\]
Let
\[ Y_t^{(N)} := \frac{1}{N^{\alpha/2}} \int_0^{Nt} V(K_s) ds, \quad t \geq 0, \]

3.1. **The case when the limit is a Mittag-Leffler process.** In this section we assume that \( \Psi \in L^2(\pi) \). As a corollary of Theorem 2.4, we obtain the following result.

**Theorem 3.1.** The processes \( \{Y_t^{(N)}, t \geq 0\} \) converge, as \( N \to +\infty \), weakly in \( C[0, +\infty) \), to the limit that is a Mittag-Leffler process corresponding to an \( \alpha \)-stable subordinator \( \{T_t, t \geq 0\} \).

**Proof.** We consider the Markov chain \( \{\hat{X}_n := (X_n, \tau_n), n \geq 0\} \). It satisfies all the assumptions made in Section 2.1 with \( \pi \otimes \lambda \) as an invariant measure, where \( \lambda(d\tau) := 1_{[0, +\infty)}(\tau)e^{-\tau} d\tau \). Consider the observables \( \tilde{\Psi}(x, \tau) := \Psi(x)\tau \) and \( \tilde{t}(x, \tau) := t(x)\tau \). The first observable is of zero mean and belongs to \( L^2(\pi \otimes \lambda) \). The tails of \( \tilde{t}(x, \tau) \) under \( \pi \otimes \lambda \) have the same exponent \( \alpha \) as the respective tails of \( t(x) \) under \( \pi \), cf. (2.3). The constant \( c_\alpha \) appearing there should be replaced by \( c_\alpha \Gamma(\alpha + 1) \).

Define \( n(Nt) \) as the (random) integer that satisfies the following condition
\[ t_{n(Nt)+1} > Nt \geq t_{n(Nt)}, \quad t > 0. \]

With \( K_N := N^\alpha \) define \( B_t^{(N)} := K_N^{-1/2} \sum_{n=0}^{[K_N t] - 1} \Psi(X_n) \tau_n, T_t^{(N)} := K_N^{-1/\alpha} t_{[K_N t]} \) and \( s_n(t) \) the right-continuous inverse of \( T_u^{(N)} \), i.e. \( s_n(t) := \inf[u : T_u^{(N)} > t] \). We have
\[ t_{[K_N s_n(t)]} = K_N^{1/\alpha} T_{s_n(t)}^{(N)} \geq K_N^{1/\alpha} t \geq K_N^{-1/\alpha} T_{s_n(t)-}^{(N)} \geq t_{[K_N s_n(t)-1]}. \]

We conclude therefore that \( n(Nt) = [K_N s_n(t)] - 1 \). From the definitions of the processes \( Y_t^{(N)}, T_t^{(N)} \) and \( B_t^{(N)} \) we obtain
\[ Y_t^{(N)} = B_{s_n(t)}^{(N)} + K_N^{-1/2}(t - t_{n(Nt)}) \Psi(X_n). \]

Tightness of the family of linear interpolation processes that corresponds to \( \{B_t^{(N)}, t \geq 0\} \) implies that
\[ \lim_{N \to +\infty} \mathbb{P}\left[ \sup_{t \in [0,T]} |B_{s_n(t)}^{(N)} - Y_t^{(N)}| \geq \varepsilon \right] = 0 \]
for any \( \varepsilon, T > 0 \). To prove the theorem it suffices therefore to show that the laws of \( \{B_{s_n(t)}^{(N)}, t \geq 0\} \) converge, as \( N \to +\infty \), weakly over \( C[0, +\infty) \). By Theorem 2.4 and the aforementioned Skorochod’s embedding theorem there exists a family of pairs of processes \( (\{\hat{B}_t^{(N)}, t \geq 0\}, \{\hat{T}_t^{(N)}, t \geq 0\}) \) such that
1) the law of \( (\{\hat{B}_t^{(N)}, t \geq 0\}, \{\hat{T}_t^{(N)}, t \geq 0\}) \) is identical with that of \( (\{B_t^{(N)}, t \geq 0\}, \{T_t^{(N)}, t \geq 0\}) \) for each \( N \geq 1 \),
2) \( \{\hat{B}_t^{(N)}, t \geq 0\}, \{\hat{T}_t^{(N)}, t \geq 0\} \) converges a.s. in \( \mathcal{D} \times \mathcal{D} \) equipped with the product of \( J_1 \) topologies, to \( (\{B_t, t \geq 0\}, \{T_t, t \geq 0\}) \). Here \( \hat{B}_t \) is a Brownian motion, \( \hat{T}_t \) is a subordinator process and they are independent.
As a result the law of \( \{ \hat{Y}_t^{(N)} := \hat{B}_{\hat{s}_N(t)}^{(N)}, t \geq 0 \} \) is identical with that of \( \{ Y_t^{(N)}, t \geq 0 \} \). Here \( \hat{s}_N(t) \) is the right inverse of \( \hat{T}_u^{(N)} \). According to Proposition 3.5.3 p. 119 of \cite{9} the a.s. convergence in the product \( J_1 \) topology means that for each \( T > 0 \) there exist sequences of increasing homeomorphisms \( \lambda_N, \rho_N : [0, T] \to [0, T] \) such that
\[
\lim_{N \to +\infty} \gamma(\lambda_N) = \lim_{N \to +\infty} \gamma(\rho_N) = 0,
\]
where
\[
\gamma(\lambda) := \sup_{0 < s < t < T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| = 0,
\]
and
\[
\sup_{t \in [0, T]} |\hat{B}_{\lambda_N(t)}^{(N)} - \hat{B}_t| = 0 \quad \text{and} \quad \sup_{t \in [0, T]} |\hat{T}_{\rho_N(t)}^{(N)} - \hat{T}_t| = 0.
\]
As a consequence of (3.3) we have
\[
\lim_{N \to +\infty} \sup_{t \in [0, T]} |\lambda_N(t) - t| = 0,
\]
which in turn implies that \( \{ s_N(\cdot), N \geq 1 \} \) converge, as \( N \to +\infty \), uniformly on compact intervals to \( s(\cdot) \), see also the proof of Lemma 3.6. Hence, \( \hat{Y}_t^{(N)} \to \hat{Y}_t \) a.s. in the \( C[0, +\infty) \) topology, where \( \hat{Y}_t \) is a Mittag-Leffler process. In consequence, also the processes \( \{ Y_t^{(N)}, t \geq 0 \} \) converge weakly over \( C[0, +\infty) \), as \( N \to +\infty \), to a Mittag-Leffler process. \( \square \)

3.2. The case when the limit is a self-similar process. Suppose that the law of \( \Psi(x) \) satisfies (2.9). We assume furthermore that for \( \beta \in (1, 2) \) it is centered. When \( \beta = 1 \) we let
\[
V_N(x) := V(x) - c_N x^{-1}(x),
\]
where \( c_N = \int \Psi(x) 1_{|\Psi(x)| \leq N} d\pi \). We shall also suppose that \( \Psi(x) \) and \( t(x) \) satisfy the assumption (2.11).

**Theorem 3.2.** Suppose that \( \beta \neq 1 \). Under the assumptions of Theorem 2.5 the laws of the processes
\[
Y_t^{(N)} := \frac{1}{N^{\alpha/\beta}} \int_0^{Nt} V(K_s) ds
\]
converge, as \( N \to +\infty \), weakly in the \( M_1 \) topology on \( D \) to the law of \( \{ \zeta_t := B_{s(t)}, t \geq 0 \} \), where \( \{ B_t, t \geq 0 \} \) and \( \{ T_t, t \geq 0 \} \) are independent stable processes as described in Theorem 2.5 and \( \{ s(t), t \geq 0 \} \) is the right inverse of \( \{ T_t, t \geq 0 \} \).

The result also holds when \( \beta = 1 \) but then the scaled process should be defined by
\[
Y_t^{(N)} := \frac{1}{N^\alpha} \int_0^{Nt} V_N(K_s) ds.
\]

**Proof.** It suffices to show the weak convergence statement on any \( D[0, T] \), where \( T > 0 \) is arbitrary. Define
\[
B_t^{(N)} := \frac{1}{R_N^{1/\beta}} \sum_{n=0}^{[K_N t^{-1}]} \Psi(X_n) \tau_n.
\]
Processes \( \{T_t^{(N)}, t \geq 0\} \) and \( \{s_N(t), t \geq 0\} \) are as in the proof of Theorem 3.1. Denote by \( \tilde{B}_t^{(N)} \) the process whose paths are obtained by the linear interpolation between the points \((mK_N^{-1}, B_{mK_N^{-1}})\), where \( m \geq 0 \) is a non-negative integer. The following result allows us to replace the first coordinate process in the statement of Theorem 2.5 by the linear interpolation \( \{\tilde{B}_t^{(N)}, t \geq 0\} \).

**Lemma 3.3.** Under the assumptions of Theorem 2.5 the joint laws of \( \{\tilde{B}_t^{(N)}, t \geq 0\}, \{T_t^{(N)}, t \geq 0\} \) converge, as \( N \to +\infty \), weakly in the product of the \( M_1 \) topologies on \( D \times D \) to the joint law of the independent Lévy process: \( \{B_t, t \geq 0\} \) - a \( \beta \)-stable and \( \{T_t, t \geq 0\} \) an \( \alpha \)-stable subordinator.

With the help of this lemma we can end the proof using the same argument as in [2], see the proof of Theorem 3.1. By the Skorohod’s embedding theorem we define a family of processes \( \{(\tilde{B}_t^{(N)}, \tilde{T}_t^{(N)}), t \geq 0\} \) such that

1) the law of \( \{\tilde{B}_t^{(N)}, t \geq 0\}, \{\tilde{T}_t^{(N)}, t \geq 0\} \) is identical with that of \( \{\tilde{B}_t^{(N)}, t \geq 0\}, \{\tilde{T}_t^{(N)}, t \geq 0\} \) for each \( N \geq 1 \),

2) \( \{\tilde{B}_t^{(N)}, t \geq 0\}, \{\tilde{T}_t^{(N)}, t \geq 0\} \) converges a.s., in \( D \times D \) equipped with the product of the \( M_1 \) topologies, to \( \{B_t, t \geq 0\}, \{\tilde{T}_t, t \geq 0\} \). Here \( \tilde{B}_t \) and \( \tilde{T}_t \) are independents stable processes as in Theorem 2.5.

We show that \( \{\tilde{Y}_t^{(N)} := \tilde{B}_{s_N(t)}^{(N)}, t \geq 0\} \) converge a.s. in the \( M_1 \) topology. Here \( \{s_N(t), t \geq 0\} \) is the right inverse of \( \{\tilde{T}_t^{(N)}, t \geq 0\} \). These processes converge in the \( M_1 \) topology (in fact even in the uniform topology) to \( \{\tilde{s}(t), t \geq 0\} \), the right inverse of \( \{\tilde{T}_t, t \geq 0\} \). By Theorem 13.2.4 of [22] we obtain therefore that \( \tilde{Y}_t^{(N)} \) converge in the \( M_1 \) topology to \( \tilde{Y}_t \) a.s., provided that the sets of discontinuities of \( \tilde{B}_t, t \geq 0 \) and \( \tilde{T}_t, t \geq 0 \) are a.s. disjoint. This however is a simple consequence of the independence of these processes.

**The proof of Lemma 3.3.** We recall below how the \( M_1 \) topology on \( D[0,T] \) can be metrized, see [22], p. 476 for details. For a given \( X \in D[0,T] \) we define by \( \Gamma_X \) the graph of \( X \), i.e. the subset of \( \mathbb{R}^2 \) given by

\[
\Gamma_X := \{(t,z) : t \in [0,T], z = cX(t-) + (1-c)X(t), \text{ for some } c \in [0,1]\}.
\]

On \( \Gamma_X \) we define an order by letting \((t_1, z_1) \leq (t_2, z_2)\) iff \( t_1 < t_2 \), or \( t_1 = t_2 \) and \(|X(t_1-) - z_1| \leq |X(t_1-) - z_2|\). Denote by \( \Pi(X) \) the set of all continuous mappings \( \gamma = (\gamma^{(1)}, \gamma^{(2)}) : [0,1] \to \Gamma_X \) that are non-decreasing, i.e. \( t_1 \leq t_2 \) implies that \( \gamma(t_1) \leq \gamma(t_2) \). The metric \( d(\cdot, \cdot) \) is defined as follows:

\[
d(X_1, X_2) := \inf \|\gamma^{(1)}_1 - \gamma^{(1)}_2\|_{\infty} \lor \|\gamma^{(2)}_1 - \gamma^{(2)}_2\|_{\infty}, \gamma_i \in \Pi(X_i), i = 1, 2\].
\]

This metric provides a metrization of the \( M_1 \) topology, see [22], Theorem 13.2.1.

For any \( \gamma_1 \in \Gamma_{\tilde{B}_t^{(N)}} \) we define \( \gamma_2 \in \Gamma_{\tilde{B}_t^{(N)}} \) as follows. Suppose \( \gamma_1(t) \) belongs to the graph corresponding to \((t, B_t^{(N)}), t \in [mK_N^{-1}, (m+1)K_N^{-1})\) for an integer \( m \geq 0 \). Let \( \gamma_2(t) \) be the
Let \( \hat{\text{Theorem 3.5.}} \)

\[
\hat{\text{Theorem 3.5.}} \{ \hat{\text{Theorem 3.5.}} \}
\]

Using the Skorohod's embedding theorem we define a family of processes \( \{ \hat{X}_n \} \). Proof.

\[ \gamma \]

\[ \hat{\text{Theorem 2.5.}} \]

Then, using the notation as in Theorem 3.2, we can formulate the following result.

Theorem 3.4.

\[ \hat{\text{Theorem 3.4.}} \]

Under the assumptions of Theorem 2.6 we have the laws of the processes \( \{ \hat{B}_n, t \geq 0 \} \) converge, as \( N \to +\infty \), weakly in the \( J_1 \) topology of \( \mathcal{D} \) to the law of \( \{ \hat{B}_t, t \geq 0 \} \).

Proof. Using the Skorochod's embedding theorem we define a family of processes \( \{ (\hat{B}_n, \hat{T}_n), t \geq 0 \} \) such that:

1) the law of each \( \{ (\hat{B}_n, \hat{T}_n), t \geq 0 \} \) is identical with that of \( \{ (\hat{B}_t, \hat{T}_t), t \geq 0 \} \) for each \( N \geq 1 \),

2) \( \{ (\hat{B}_n, \hat{T}_n), t \geq 0 \} \) converges a.s., in the \( J_1 \) topology of \( \mathcal{D}_2 \), to \( \{ (\hat{B}_t, \hat{T}_t), t \geq 0 \} \).

The latter process is as in the statement of Theorem 2.6. The above means that for any \( L > 0 \) one can find a sequence \( \{ \lambda_n; n \geq 1 \} \) of increasing homeomorphisms in \( [0, L] \) such that \( \lambda_N(0) = 0, \lambda_N(L) = L \) and

\[
\sup_{t \in [0, L]} |\lambda_N(t) - t| \to 0, \tag{3.8}
\]

\[
\sup_{t \in [0, L]} |\hat{B}_{\lambda_N(t)} - \hat{B}_t| \to 0 \quad \text{and} \quad \sup_{t \in [0, L]} |\hat{T}_{\lambda_N(t)} - \hat{T}_t| \to 0, \tag{3.9}
\]

as \( N \to +\infty \).

Let \( \hat{s}_N(t), \hat{s}(t) \) be the right inverses of \( \hat{T}_n, \hat{T}_t \) respectively.

Repeating the argument made in the proof of Lemma 3.3 we can conclude the theorem from the following.

Theorem 3.5.

Under the assumptions of Theorem 2.6 we have the laws of the processes \( \{ \hat{B}_{\hat{s}_N(t)}, t \geq 0 \} \) converge, as \( N \to +\infty \), weakly in the \( J_1 \) topology of \( \mathcal{D} \) to the law of \( \{ \hat{B}_{\hat{s}(t)}, t \geq 0 \} \).
Before presenting the proof of the theorem we shall need some auxiliary results. For each $\delta > 0$, we denote by $A_\delta = A_\delta(\hat{s})$ the set of "plateau" points of $\hat{s}_t$ of size at least $\delta$, i.e. $t \in A_\delta$ iff $\hat{s}_t$ is constant in the interval $(t - \delta, t + \delta) \cap [0, \hat{T}_L]$. (3.10)

Observe that $A_\delta \subset A_{\delta'}$ if $\delta' \leq \delta$.

**Lemma 3.6.** For a fixed $L > 0$ the sequence $\{\hat{s}_N, N \geq 1\}$ converges to $\hat{s}$ in the following sense:

i) $\|\lambda_N^{-1} \circ \hat{s}_N - \hat{s}\|_\infty \to 0$, as $N \to +\infty$,

ii) there exists a decreasing sequence $\delta_N \to 0$, as $N \to +\infty$, such that $\lambda_N^{-1} \circ \hat{s}_n(t) = \hat{s}(t)$ for $t \in A_{\delta_N}$, $n \geq N$ and all $N \geq 1$.

The supremum norms appearing above are taken over the interval $[0, L]$.

**Proof.** Notice that this notion of convergence is stronger than the uniform convergence. In fact, since $\lambda_N$ tends to the identity, convergence in the above sense means the uniform convergence plus also the convergence of plateaus. Let

$$\delta_N := \sup_{n \geq N} \left\{ \|\hat{T}^{(n)} \circ \lambda_n - \hat{T}\|_\infty, \|\hat{B}^{(n)} \circ \lambda_n - \hat{B}\|_\infty, \|\lambda_n - \text{id}\|_\infty, \|\lambda_n^{-1} - \text{id}\|_\infty \right\}. \quad (3.11)$$

For any $t \in A_{\delta_N}$ and $n \geq N$ we have:

$$\hat{T}_{\lambda_n \circ \hat{s}_N}(t) - \hat{T}_{\lambda_n \circ \hat{s}_N}(t) \leq t - \delta_N \leq t \leq \hat{T}_{\lambda_n \circ \hat{s}_N}(t),$$

$$|\hat{T}_{\lambda_n \circ \hat{s}_N}(t) - \hat{T}_{\hat{s}_N}(t)| \leq \delta_N \quad \text{and} \quad |\hat{T}_{\lambda_n \circ \hat{s}_N}(t) - \hat{T}_{\hat{s}_N}(t)| \leq \delta_N.$$ 

Therefore, $\hat{T}_{\lambda_n \circ \hat{s}_N}(t) - \hat{T}_{\lambda_n \circ \hat{s}_N}(t) \leq t \leq \hat{T}_{\lambda_n \circ \hat{s}_N}(t)$, which proves that $\hat{s}_n(t) = \lambda_n \circ \hat{s}(t)$.

Let

$$a_N := \max\{\|\hat{T}^{(N)} \circ \lambda_N - \hat{T}\|_\infty, \|\hat{B}^{(N)} \circ \lambda_N - \hat{B}\|_\infty, \|\lambda_N - \text{id}\|_\infty, \|\lambda_N^{-1} - \text{id}\|_\infty\}. \quad (3.12)$$

Since all the functions $\hat{s}_N(t)$ are monotonic in order to prove i) it is enough to prove pointwise convergence of $\hat{s}_N(t)$ to $\hat{s}(t)$. We prove that

$$\hat{s}_N(t) \leq \hat{s}(t + a_N) + a_N. \quad (3.13)$$

Assume that $\hat{s}(t + a_N) \leq s - a_N$. Then, $\hat{s}(t + a_N) \leq \lambda_N^{-1}(s)$, which implies that $t + a_N \leq \hat{T}_{\lambda_N^{-1}(s)}$. In light of (3.11) we get $t \leq \hat{T}^{(N)}_{\lambda_N^{-1}(s)}$ and (3.13) follows. Likewise, we prove that $\hat{s}_N(t) \geq \hat{s}(t - a_N) - a_N$ and, as a result, we conclude the proof of part i). \(\square\)

For each $t \geq 0$ define $\Delta \hat{B}_t = \hat{B}_t - \hat{B}_{t-}$. Let us define $\{t_i; i \geq 1\}$ as the set of jumps of $\hat{B}_t$, ordered by its magnitude, i.e.: $|\Delta \hat{B}_{t_i}| \geq |\Delta \hat{B}_{t_{i+1}}|$ for any $i \geq 1$. Let also $T_n := \{t_i; i = 1, \ldots, n\}$.

For a fixed $L > 0$ and set $F \subseteq [0, L]$ we define

$$\gamma_{\hat{B}}(F) := \sup_{s, t \in [0, L]} \sup_{s \sim t} |\hat{B}_t - \hat{B}_s|,$$

where we say that $s \sim t$ if $F \cap [s, t] = \emptyset$. We say that $\gamma_{\hat{B}}(F) := 0$ if the supremum is taken over an empty set.
Lemma 3.7. We have
\[
\lim_{n \to \infty} \gamma_B(T_n) = 0. \tag{3.14}
\]

Proof. Let us define \( \omega'_B(\delta) \) as the modified modulus of continuity of \( \hat{B}_t \):
\[
\omega'_B(\delta; L) = \inf_{\Pi} \sup_i \sup_{s_{i-1} \leq y < s_i} |\hat{B}_y - \hat{B}_x|,
\]
where the infimum is taken over partitions \( \Pi = \{0 = s_0 < \cdots < s_l = L\} \) that are \( \delta \)-sparse, i.e. such that \( s_i - s_{i-1} > \delta \) for all \( i \)-s. We have \( \lim_{\delta \to 0^+} \omega'_B(\delta; L) = 0 \) a.s., see p. 123 of [3]. For each \( n \), define \( \epsilon_n \) to be the diameter of the partition of \( [0, L] \) generated by the points belonging to \( T_n \). Suppose that \( \epsilon > 0 \) is arbitrary and \( \Pi = \{0 = s_0 < \cdots < s_l = L\} \) is a partition \( \epsilon_n \)-sparse and such that
\[
\sup_{s_{i-1} \leq y < s_i} |\hat{B}_y - \hat{B}_x| \leq \omega'_B(\epsilon_n; L) + \epsilon.
\]
On each connected component \( \mathcal{I} \) of \( [0, L] \setminus T_n \) there is at most one \( s_i \). If there is no \( s_i \) that belongs to \( \mathcal{I} \) we have \( \sup_{x,y \in \mathcal{I}} |\hat{B}_y - \hat{B}_x| \leq \omega'_B(\epsilon_n; L) + \epsilon \). When, on the other hand some \( s_i \) belongs to \( \mathcal{I} \) for any \( x, y \in \mathcal{I} \) and \( x < s_i \leq y \) we can estimate
\[
|\hat{B}_y - \hat{B}_x| \leq |\hat{B}_y - \hat{B}_{s_i}| + |\Delta \hat{B}_{s_i}| + |\hat{B}_{s_i} - \hat{B}_x| \leq 2[\omega'_B(\epsilon_n; L) + \epsilon] + |\Delta \hat{B}_{s_i}|
\]
thus,
\[
\gamma_B(T_n) \leq 2\omega'_B(\epsilon_n; L) + |\Delta \hat{B}_{s_i}|.
\]
Since \( \hat{B}_t \) is càdlàg, \( \lim_n \Delta \hat{B}_{s_i} = 0 \). Combining this with the fact that \( \lim_n \epsilon_n = 0 \) we conclude (3.14).

Lemma 3.8. Let \( \{\delta_m, m \geq 1\} \) be as in the statement of Lemma 3.6. Under the assumptions of Theorem 2.6 we have
\[
\lim_{m \to \infty} \sup_{t \notin \delta(\lambda_m)} |\Delta \hat{B}_t| = 0 \text{ in probability} \tag{3.15}
\]

Proof. Let
\[
A_m := \{ \exists t \in [0, L] : |\Delta \hat{B}_t| \geq e^{-1}(4\delta_m) \text{ and } \Delta \hat{T}_t \leq \delta_m \}.
\]
We shall show that
\[
\mathbb{P}[A_m] = 0, \quad \forall m \geq 1. \tag{3.16}
\]
We suppose first that \( \beta < 1 \). Consider the jump process \( \{\hat{Z}^{(\rho)}_t := (\hat{B}^{(\rho)}_t, \hat{T}^{(\rho)}_t), t \geq 0\} \) corresponding to the jump measure
\[
\nu^{(\rho)}_s(d\lambda_1, d\lambda_2) := 1_B^{(\rho)}(\lambda_1, \lambda_2) \nu_s(d\lambda_1, d\lambda_2).
\]
Let \( Z := \nu^{(\rho)}_s(\mathbb{R}^2) \). This process can be realized as follows: \( \hat{Z}^{(\rho)}_t = Z^{(\rho)}_{N(t)} \), where \( Z^{(\rho)}_n \) is a sum of \( n \) independent random variables distributed according to \( Z^{-1}\nu^{(\rho)}_s(d\lambda_1, d\lambda_2) \) and \( N(t) \) is an independent Poisson process with intensity \( Z \). Therefore
\[
\mathbb{P}[A^{(\rho)}_m] = 0, \tag{3.17}
\]
where

\[ A^{(p)}_m := \left[ \exists t \in [0, L] : |\Delta \hat{B}_t^{(p)}| \geq e^{-1}(3\delta_m) \text{ and } \Delta \hat{T}_t^{(p)} \leq 2\delta_m \right]. \]

Let \( \hat{Z}_t := (\hat{B}_t, \hat{T}_t), t \geq 0 \). It is well known, see e.g. [4] Theorem 14.27, that

\[
\lim_{\rho \to 0+} \sup_{t \in [0, L]} |\hat{Z}_t^{(\rho)} - \hat{Z}_t| = 0, \text{ in probability.} \tag{3.18}
\]

Combining (3.17) and (3.18) we conclude (3.15).

The case when \( \beta \in [1, 2) \) can be concluded similarly. However, in that case the approximating processes should be of the form \( \{\hat{Z}_t^{(\rho)} - c^{(\rho)}t, t \geq 0\} \) for some \( c^{(\rho)} = (c_1^{(\rho)}, 0) \), where, in general, \( c_1^{(\rho)} \) may diverge, as \( \rho \to 0+ \).

**Proof of Theorem 3.5.** Suppose, with no loss of generality, that the sequence \( \{\delta_n, n \geq 1\} \) is strictly decreasing. Writing \( \hat{B}_{\delta_N}(t) = \hat{B}_{\lambda_N \circ \delta_N}(t) \), we notice that it is enough to show convergence in the \( J_1 \)-Skorohod topology of \( \hat{B}_{\lambda_N \circ \delta_N}(t) \) to \( \hat{B}_\delta(t) \). For any \( L > 0 \) we exhibit increasing homeomorphisms \( \Lambda_N : [0, \hat{T}_L] \to [0, \hat{T}_L^{(N)}] \) such that

\[
\lim_{N \to +\infty} B_{\lambda_N \circ \delta_N \circ \Lambda_N}(t) = \hat{B}_\delta(t)
\]

and

\[
\lim_{N \to +\infty} \Lambda_N(t) = t,
\]

uniformly on \([0, \hat{T}_L]\). Let \( \{\ell_k, k \geq 1\} \) be an increasing sequence of positive integers such that \( T_{\ell_k} = \hat{s}(A_{\delta_k}) \). Define \( C_k := \hat{s}^{-1}(T_{\ell_k}) \). This set is a union of \( \ell_k \) disjoint closed intervals that constitute the \( \ell_k \) plateaus of \( \hat{s}(t) \), each of length greater than \( 2\delta_k \). The open set \( C_k^c \) is a union of a finite number of open intervals (relative to \([0, \hat{T}_L]\)). Let \( \kappa_k \) be the minimum of the lengths of these intervals. Of course \( \kappa_k \) decreases to 0, as \( k \to +\infty \).

Let \( \{M_k, k \geq 1\} \) be an increasing sequence of positive integers such that \( \delta_N < \min[\kappa_k/2, \delta_k - \delta_{k+1}] \) for all \( N \geq M_k \). Recall that then \( |\hat{T}_{\lambda_N(t_k)}^{(N)} - \hat{T}_{\ell_k}^{(N)}| \) and \( |\hat{T}_{\lambda_N(t_k)}^{(N)} - \hat{T}_{\ell_k}^{(N)}| \), \( i = 1, \ldots, \ell_k \) are less than, or equal to \( \delta_{M_k} \) for \( N \geq M_k \). Therefore, for each \( N \geq M_k \) the intervals \( [\hat{T}_{\lambda_N(t_k)}^{(N)}, \hat{T}_{\lambda_N(t_k)}^{(N)}] \) (plateaus of \( \hat{s}_N \)) are mutually disjoint for \( i = 1, \ldots, \ell_k \). The same holds also of course for the respective \( [\hat{T}_{\ell_k}^{(N)}, \hat{T}_{\ell_k}^{(N)}] \) (plateaus of \( \hat{s} \)).

We say that the interval \([c, d] \) follows \([a, b] \) if \( c > b \). Let us take \( i, j \) such that their corresponding plateaus \( [\hat{T}_{i-1}, \hat{T}_i] \) and \( [\hat{T}_{j-1}, \hat{T}_j] \) are consecutive (in this order). Then \([\hat{T}_{\lambda_N(t_k)}^{(N)} - \hat{T}_{\lambda_N(t_k)}^{(N)}] \) follows \([\hat{T}_{\lambda_N(t_k)}^{(N)} - \hat{T}_{\lambda_N(t_k)}^{(N)}] \) for \( M_k+1 > N \geq M_k \). For these \( N \)-s we define \( \Lambda_N(\hat{T}_{i-1}) := \hat{T}_{\lambda_N(t_k)}^{(N)} \), and \( \Lambda_N(\hat{T}_i) = \hat{T}_{\lambda_N(t_k)}^{(N)} \) and elsewhere \( \Lambda_N(t) \) is defined by a linear interpolation. It is obvious from the construction that \( \Lambda_N(t) \) converges uniformly to \( t \) on \([0, \hat{T}_L]\), as \( N \to +\infty \).

Observe that for any \( t \in A_{\delta_k} \) we have \( \Lambda_N(t) \in A_{\delta_{k+1}} \) and, thanks to Lemma 3.6, we have then for all \( N \geq M_{k+1} \)

\[
B_{\lambda_N \circ \delta_N \circ \Lambda_N}(t) = B_{\hat{s} \circ \Lambda_N}(t) = B_{\hat{s}(t)}.
\]
The last equality is a consequence of the fact that both \( \Lambda_N(t) \) and \( t \) belong to the same component of \( C_k \). This proves that \( \lim_{N \to +\infty} B_{\lambda_N^{-1} \circ \hat{s}_N \circ \Lambda_N(t)} = B_{\hat{s}(t)} \) uniformly on \( \hat{s}(A_{\delta_k}) \).

The statement on the uniform convergence on the entire \([0, T_L] \) follows from Lemma 3.8. Indeed, suppose that \( \varepsilon > 0 \) is arbitrary and \( \kappa_k > 0 \) is sufficiently small so that

\[
2 \omega_B'(\kappa_k; L) + \max_{t \not\in \hat{s}(A_{\delta_k})} \Delta \hat{B}_t < \varepsilon.
\]

Then, for \( N_0 \) sufficiently large, so that for \( N \geq N_0 \) we have \( |\lambda_N^{-1} \circ \hat{s}_N \circ \Lambda_N(t) - \hat{s}(t)| < \kappa_k/2 \), \( \forall t \in [0, T_L] \). Suppose that \( \rho > 0 \) is arbitrary, \( 0 = s_0 < \ldots < s_K = L \) are \( \kappa_k \)-sparse and such that

\[
\sup_{s_{i-1} \leq x, y < s_i} |\hat{B}_y - \hat{B}_x| \leq \omega_B'(\kappa_k; L) + \rho.
\]

For \( t \not\in \hat{s}(A_{\delta_k}) \) we have \( \hat{s}(t) \in \mathcal{I} \), a connected component of \( C_k \). Since \( \mathcal{I} \cap \hat{s}(A_{\delta_k}) = \emptyset \) and the diameter of \( \mathcal{I} \) is less than, or equal to \( \kappa_k \) we have

\[
|B_{\lambda_N^{-1} \circ \hat{s}_N \circ \Lambda_N(t)} - B_{\hat{s}(t)}| \leq 2[\omega_B'(\kappa_k; L) + \rho] + \max_{u \not\in \hat{s}(A_{\delta_k})} \Delta \hat{B}_u < 2\rho + \varepsilon.
\]

and the lemma can be concluded from Lemma 3.8.

\[
\square
\]

4. AN EXAMPLE OF A JUMP PROCESS WITH A \( \sigma \)-FINITE INVARIANT MEASURE

The one dimensional torus \( \mathbb{T} \) is an interval \([-\pi, \pi] \) with the endpoints identified. We apply the above results to a jump process \( \{K_t, t \geq 0\} \) on \( \mathbb{T} \) with a \( \sigma \)-finite but not probabilistic invariant measure. The generator of the process is given by

\[
Lf(k) = \gamma(k) \int_{\mathbb{T}} \hat{r}(\theta, k)[f(\theta) - f(k)]d\theta, \quad f \in B_b(\mathbb{T}).
\]  (4.1)

Here \( B_b(\mathbb{T}) \) denotes the space of bounded, Borel measurable functions on \( \mathbb{T} \). Function \( r_0^{-1} \geq \hat{r}(\theta, k) \geq r_0 > 0 \) is continuous on \( \mathbb{T} \), even, i.e. \( \hat{r}(-\theta, -k) = \hat{r}(\theta, k) \) and doubly stochastic i.e.:

\[
\int_{\mathbb{T}} \hat{r}(\theta, k) d\theta = \int_{\mathbb{T}} \hat{r}(k, \theta) d\theta = 1
\]

for all \( k \in \mathbb{T} \). On the other hand, we assume that \( \gamma(-k) = \gamma(k) \) satisfies \( \inf_{|k - k_0| \geq \delta} \gamma(k) > 0 \) for any \( \delta > 0 \) and \( \gamma(k_0) = 0 \). We suppose furthermore that \( \gamma(k) \) is bounded and \( \int_{\mathbb{T}} \gamma^{-1}(k) dk = +\infty \). This kind of processes appears while considering the transport of particles in quantum systems, see e.g. Section 4.3 of \([6]\). It is easy to see that \( m_* dk = \gamma^{-1}(k)m_1 dk \) is an infinite, reversible, invariant measure for the process. Here \( m_1 \) denotes the normalized Lebesgue measure on the torus. Indeed, for any \( f \in B_b(\mathbb{T}) \)

\[
\int_{\mathbb{T}} Lf(k)m_*(dk) = \int_{\mathbb{T}} \int_{\mathbb{T}} \hat{r}(\theta, k)[f(\theta) - f(k)]d\theta dk
\]

\[
= \int_{\mathbb{T}} f(\theta) d\theta \int_{\mathbb{T}} \hat{r}(\theta, k) dk - \int_{\mathbb{T}} \int_{\mathbb{T}} \hat{r}(\theta, k) f(k) d\theta dk = 0.
\]

The process \( \{K_t, t \geq 0\} \) can be constructed using a Markov chain and a renewal process that corresponds to the jump times. Consider a skeleton Markov chain \( \{X_n, n \geq 0\} \) defined
on \(\mathbb{T}_{k_0} := \mathbb{T} \setminus \{-k_0, k_0\}\) with transition probability \(r(\theta, k)d\theta\) and \(X_0 = K_0\). Let \(t(k) := \gamma^{-1}(k)\) and \(\{\rho_n, n \geq 0\}\) be a sequence of i.i.d. random variables exponentially distributed with intensity 1, independent of the skeleton chain. Let \(t_0 := 0\) and \(t_n := \sum_{k=0}^{n-1} t(X_k)\rho_k, n \geq 1\). We can take then \(K_t := X_n\) for \(t \in [t_n, t_{n+1})\).

Assume that \(t(k) \geq t_* > 0\) for all \(k \in \mathbb{T}_{k_0}\). Our first result concerns the Harris property of an embedded Markov chain.

**Proposition 4.1.** Suppose that \(h \in (0, t_*]\). Consider an embedded Markov chain \(\{K_{nh}, n \geq 0\}\). It is Harris recurrent w.r.t. measure \(m_1\), i.e. for any Borel subset \(B\) with \(m_1[B] > 0\) we have

\[
\mathbb{P}[\exists n \geq 0 : K_{nh} \in B] = 1. \tag{4.2}
\]

**Proof.** Our hypotheses on the skeleton chain guarantee that \(\mathbb{P}[D] = 1\), where \(D = [X_n \in B, i.o.]\). Let \(A_n := [X_n \in B, t_{n+1} - t_n \geq 2h]\). To see that \((4.2)\) holds it suffices only to prove that

\[
\mathbb{P}[C] = 1, \tag{4.3}
\]

where \(C := \bigcup_{n \geq 0} A_n\). Note that \(1_{C^c}(\omega) \leq f(\omega)\), where

\[
f(\omega) := \prod_{n \geq 0} \left[ 1_{B^c}(X_n) + 1_{B}(X_n)1_{[\tau_n \geq 2]} \right].
\]

However, denoting by \(\tilde{\mathbb{E}}\) the expectation over another copy of measure \(\mathbb{P}\), we get

\[
\mathbb{E}f = \mathbb{E} \tilde{\mathbb{E}} \left\{ \prod_{n \geq 0} \left[ 1_{B^c}(X_n) + 1_{B}(X_n)1_{[\tau_n \geq 2]} \right] \right\}
\]

\[
= \mathbb{E} \left\{ \prod_{n \geq 0} \left[ 1_{B^c}(X_n) + 1_{B}(X_n)\tilde{\mathbb{E}}1_{[\tau_n \geq 2]} \right] \right\}
= \mathbb{E} \left\{ \prod_{n \geq 0} \left[ 1_{B^c}(X_n) + 1_{B}(X_n)e^{-2} \right] \right\}
= \mathbb{E} \left\{ \prod_{n \geq 0} \left[ 1_{B^c}(X_n) + 1_{B}(X_n)e^{-2} \right], D \right\} = 0
\]

and \((4.3)\) follows. \(\square\)

As an immediate corollary to the above proposition and [13], Theorem 1, p. 116 we obtain that \(m_*\) is the unique \(\sigma\)-finite invariant measure under the process that is absolutely continuous w.r.t. \(m_1\).

Denote by \(\{P_t, t \geq 0\}\) the transition semigroup of the process. It satisfies the following integral equation

\[
P_tf(k) = e^{-t\gamma(k)}f(k) + \gamma(k) \int_0^t e^{-\tau\gamma(k)}d\tau \int_{\mathbb{T}} \hat{r}(k', k)P_{t-\tau}f(k')dk'. \tag{4.4}
\]

For any \(N \geq 1\) and \(T > 0\) denote by

\[
\Delta_N(T) := \{(\tau_0, \ldots, \tau_{N-1}) : \tau_i \geq 0, i = 0, \ldots, N - 1, \sum_{i=0}^{N-1} \tau_i \leq T\}.
\]
Iterating equation (4.4) we can easily show that
\[ P_t f(k) = e^{-t\gamma(k)} f(k) + \sum_{N=1}^{+\infty} \gamma(k) \int \ldots \int e^{-t\gamma(k_N)} \]
\[ \times \prod_{i=1}^{N} \{ \gamma(k_i) \exp \{-\tau_i(\gamma(k_i) - \gamma(k_N))\} \hat{r}(k_i, k_{i-1}) \} f(k_N) d\tau^{(N)} dk^{(N)}. \]

Here \( k_0 := k \), \( d\tau^{(N)} := d\tau_0 \ldots d\tau_{N-1} \), \( dk^{(N)} := dk_1 \ldots dk_N \). The component of transition probability function absolutely continuous w.r.t. \( m_1 \) equals therefore
\[ p_t(k, k') dk' = \sum_{N=1}^{+\infty} \gamma(k) \int \ldots \int e^{-t\gamma(k_N)} \]
\[ \times \prod_{i=1}^{N} \{ \gamma(k_i) \exp \{-\tau_i(\gamma(k_i) - \gamma(k'))\} \hat{r}(k_i, k_{i-1}) \} d\tau^{(N)} dk^{(N-1)} dk'. \]

Thus, for every \( h > 0 \), \( C \in \mathcal{B}(\mathbb{T}_{k_0}) \) with \( \text{dist}(C, \{-k_0, k_0\}) > 0 \) and \( m_1(C) > 0 \) we have \( \inf_{k,k' \in C} p_h(k, k') > 0 \). Thus, the transition probability function of any embedded chain \( \{K_{nh}, n \geq 0\} \) is aperiodic in the sense of [16]. Suppose that \( f_0 = d\nu_0/dm_* \in L^2(m_*) \) is a density. Thanks to reversibility of \( m_* \) we obtain that the density
\[ f_t = \frac{d\nu_0 P_t}{dm_*} = P_t f_0, \quad \forall t \geq 0. \]

4.1. **Mixing property of the process.** Our first observation concerning the process is contained in the following.

**Theorem 4.2.** Suppose that the initial law \( \nu_0 \) of the process is absolutely continuous w.r.t. the Lebesgue measure \( m_1 \) on the torus. Under the above assumptions \( \nu_0 P_t \) converge weakly, as \( t \to +\infty \), to the measure \( \mu_* := 1/2(\delta_{k_0} + \delta_{-k_0}) \). In addition, the process is completely mixing, i.e. if \( \nu_0, \nu'_0 \) are two initial laws on \( \mathbb{T}_{k_0} \) then
\[ \lim_{t \to +\infty} \| \nu_0 P_t - \nu'_0 P_t \|_{TV} = 0. \]

To prove the above result we first show the following.

**Proposition 4.3.** For any compact set \( K \subset \mathbb{T}_{k_0} \) and a measure \( \nu_0 \) as in Theorem 4.2 we have
\[ \lim_{t \to +\infty} \nu_0 P_t[K] = 0. \]

**Proof.** Using a density argument it suffices to show (4.6) for measures \( \nu_0 \) whose density belongs to \( L^2(m_*) \). Thanks to strong continuity of semigroup \( \{P_t, t \geq 0\} \) in \( L^1(m_*) \) in order to prove (4.6) it suffices only to show that for any \( h > 0 \)
\[ \lim_{n \to +\infty} \nu_0 P_{nh}[K] = 0. \]
From the Harris recurrence property we know that for any set \( A \subset \mathbb{T}_{k_0} \) with \( m_*[A] > 0 \) we have \( P_h 1_A(x) > 0 \), \( m_* \)-a.e. hence from [11], pp. 85-102, we have (4.7) for any \( K \) such that \( +\infty > m_*[K] > 0 \), cf. Theorem C, p. 91 of ibid. □

The proof of Theorem 4.2. Let \( h \in (0, t_\ast) \). Define by \( \mathcal{C} := \bigcap_{n \geq 0} \mathcal{C}_n \), where \( \mathcal{C}_n \) is the smallest \( \sigma \)-algebra generated by \( \{K_{mh}, m \geq n\} \). According to Theorem 1, p. 45 of [16] the tail \( \sigma \)-algebra of the chain that is Harris recurrent and aperiodic has to be trivial. Therefore, according to Lemma 3, p. 43 of ibid. (4.5) follows.

Observe that if \( u(k) \) is a density w.r.t. \( m_* \) such that \( -k < u(k) \) then
\[
u P_t(-k) = u P_t(k).
\]
This follows from the fact that \( v_t(k) := u P_t(-k) \) satisfies equation
\[
\frac{dv_t}{dt}(k) = v_t L(k),

v_0(k) = u(k).
\]
Since \( u P_t(k) \) satisfies the same equation from the uniqueness of solutions we obtain \( u P_t(k) = v_t(k) = u P_t(-k) \). Let \( \nu_0 dk := u(k)m_*(dk) \). Combining (4.8) with Proposition 4.3 and (4.3) we conclude that \( \nu_0 P_t \Rightarrow (1/2)[\delta_{-k_0} + \delta_{k_0}] \), as \( t \to +\infty \). From the (already shown) complete mixing property we conclude in particular that for any initial distribution \( \mu_0 \) we have \( \mu_0 P_t \Rightarrow (1/2)[\delta_{-k_0} + \delta_{k_0}] \), weakly over \( C(\mathbb{T}) \), as \( t \to +\infty \). □

4.2. The convergence of additive functionals. Suppose that \( \gamma(k) \sim c_* |k - k_0|^{c_*} \), when \( |k - k_0| \ll 1 \), for some \( k > 1 \) and \( c_* > 0 \). Then the law of \( t(k) \) under \( m_1 \) belongs to the domain of attraction of a stable subordinator with index \( \alpha = 1/k \). When \( \Psi \in L^2(m_1) \) and \( \int_\mathbb{T} \Psi(k) dk = 0 \) we have the following.

Corollary 4.4. Let \( \Psi(k) := V(k)t(k) \). The laws of \( Y_t^{(N)} := N^{-\alpha/2} \int_0^N V(K_s)ds \) over \( C[0, +\infty) \) converge, as \( N \to +\infty \), to the law of the Mittag-Leffler process that corresponds to an \( \alpha \)-stable subordinator.

Assume also that the law of \( \Psi(k) \) under \( m_1 \) belongs to the domain of attraction of a \( \beta \)-stable law. Since the skeleton chain \( \{X_n, n \geq 1\} \) considered here satisfies the assumptions made in Section 2.1, from Theorem 3.1 we conclude the following.

Corollary 4.5. Suppose that \( \beta 
\neq 1 \) and for \( \beta \in (1, 2) \) we have \( \int_\mathbb{T} \Psi(k) dk = 0 \). In addition, assume that \( t(k) := \gamma^{-1}(k) \) and \( \Psi(k) \) satisfy (2.11). Then, the laws of \( Y_t^{(N)} := N^{-\alpha/\beta} \int_0^N V(K_s)ds \) converge, as \( N \to +\infty \), weakly in the \( M_1 \) topology on \( D \) to the law of \( \{\xi_t := B_{s(t), t \geq 0}\} \), where \( \{B_t, t \geq 0\} \) is a \( \beta \)-stable process and \( s(t) \) is the right inverse of an independent, \( \alpha \)-stable subordinator. When \( \beta = 1 \) the theorem still holds, provided that \( Y_t^{(N)} := N^{-\alpha} \int_0^N V_N(K_s)ds \), where \( V_N(k) \) is given by (3.3).

Remark. The jump process considered in Section 4.3 of [16] has the generator given by
\[
L f(k) = c \cos^2 k \int_\mathbb{T} \hat{r}(k' - k)(f(k') - f(k))dk'
\]
for some constant \( c > 0 \) and a density function \( \hat{r}(k) \) satisfying \( r_* \leq \hat{r}(k) \leq r_*^{-1} \) for some \( r_* \in (0, 1) \). Theorem 4.2 allows to claim that \( \nu_0 P_t \Rightarrow 1/2(\delta_{-\pi/2} + \delta_{\pi/2}) \), as \( t \to +\infty \) for any initial measure \( \nu_0 \) absolutely continuous w.r.t.
Lebesgue measure. This answers in the affirmative the conjecture made in the discussed paper. For a square integrable and centered observable \( \Psi \), we have the convergence of \( N^{-1/4} \int_0^{Nt} V(K_s)ds \) to a Mittag-Leffler process. Note that when \( \Psi(-k) = -\Psi(k) \) belongs to the normal domain of attraction of a Cauchy law and it is not singular at \( \pi/2 \) then \( N^{-1/2} \int_0^{Nt} V(K_s)ds \) converge, as \( N \to +\infty \), to a fake diffusion.

5. The proof of Theorem 2.4

To simplify the notation we shall assume throughout the remainder of the paper that \( k_N = N \). Tightness of laws over \( D \times D \) is obvious in light of the fact that each coordinate corresponds to a family of weakly convergent processes so we deal only with the problem of identifying the limiting law.

Let \( 0 = t_0 < t_1 < \ldots < t_M \) and \( \theta^{(i)}_1, \theta^{(i)}_2 \in \mathbb{R}, \ i = 1, \ldots, M \). Let also \( \Delta B^{(N)}_{ti} := B^{(N)}_{ti} - B^{(N)}_{ti-1} \) and \( \Delta T^{(N)}_{ti} := T^{(N)}_{ti} - T^{(N)}_{ti-1}, i = 1, \ldots, M \). We prove that the laws of random vectors

\[
(\theta^{(1)}_1 \Delta B^{(N)}_{t_1} + \theta^{(1)}_2 \Delta T^{(N)}_{t_1}, \ldots, \theta^{(M)}_1 \Delta B^{(N)}_{t_M} + \theta^{(M)}_2 \Delta T^{(N)}_{t_M})
\]

converge to the respective finite dimensional distribution of corresponding to \( \{(B_t, T_t), t \geq 0\} \). Here \( \{B_t, t \geq 0\} \) and \( \{T_t, t \geq 0\} \) are the Brownian motion and the subordinator, described in Section 2.2, independent of \( \{B_t, t \geq 0\} \). To simplify the notation we shall consider only the case when \( M = 1 \) and we prove that the laws of \( \theta_1 B^{(N)}_{t_1} + \theta_2 T^{(N)}_{t_1} \) converge to the law of \( \theta_1 B_t + \theta_2 T_t \). The proof in the general case is analogous.

Let

\[
\tilde{B}^{(N)}_{t_1} := N^{-1/2} \sum_{n=0}^{[Nt]} R_0(X_{n+1}, X_n), \tag{5.1}
\]

where \( \chi \in L^2(\pi) \) is the unique zero mean solution of

\[
(I - P) \chi = \Psi \tag{5.2}
\]

and

\[
R_0(x, y) := \chi(x) - P \chi(y) \quad x, y \in E. \tag{5.3}
\]

Note that from (5.2) it follows that \( \|\Psi\|_{L^p(\pi)} \leq 2\|\chi\|_{L^p(\pi)} \) for all \( p \in [1, 2] \). For each \( T > 0 \) we have

\[
\lim_{N \to +\infty} \sup_{t \in [0, T]} |\tilde{B}^{(N)}_{t} - B^{(N)}_{t}| = 0 \quad \text{a.s.} \tag{5.4}
\]

Indeed,

\[
\tilde{B}^{(N)}_{t} - B^{(N)}_{t} = N^{-1/2}[\Psi(X_{[Nt]+1}) + P\chi(X_{[Nt]+1}) - P\chi(X_0)].
\]

To prove (5.4) it suffices only to use the fact that for any stationary sequence \( \{Z_n, n \geq 0\} \) such that \( \mathbb{E}|Z_0| < +\infty \) we have

\[
\lim_{N \to +\infty} N^{-1} \max\{Z_0, \ldots, Z_N\} = 0
\]

both a.s. and in the \( L^1 \) sense. In our case we can take \( Z_n := [\Psi(X_n) + P\chi(X_n)]^2 \).
Recall that \( \{T_t^{(N)}, t \geq 0\} \) is the process defined by (2.7). For any \( 1 < \Delta < +\infty \) consider also the processes

\[
T_t^{(N, \Delta)} := N^{-1/\alpha} \sum_{n=0}^{[Nt]} t(X_n) 1_{[t(X_n) < \Delta N]}.
\]

and

\[
\tilde{T}_t^{(N, \Delta)} := T_t^{(N, \Delta)} - T_t^{(N, \Delta)},
\]

where

\[
\tilde{T}_t^{(N, \Delta)} := N^{-1/\alpha} \sum_{n=0}^{[Nt]} \mathbb{E}[t(X_n) 1_{[t(X_n) < \Delta N]} | \mathcal{G}_{n-1}].
\]

Here \( \mathcal{G}_{-1} \) is the trivial \( \sigma \)-algebra and \( \Delta_N := \Delta N^{1/\alpha} \).

Let

\[
S_t^{(N, \Delta)} := N^{-1/\alpha} \sum_{n=0}^{[Nt]} t(X_n) 1_{[t(X_n) > \Delta N]}.
\]

The following result holds.

**Proposition 5.1.** For each \( t > 0 \) we have

\[
\lim_{N \to +\infty} \mathbb{E} \left| \tilde{T}_t^{(N, \Delta)} - t \int_0^\Delta \lambda \nu_\alpha(d\lambda) \right| = 0. \tag{5.6}
\]

In addition, for any \( t, \varepsilon, \eta > 0 \) there exists \( \Delta > 1 \) such that

\[
\limsup_{N \to +\infty} \mathbb{P} \left[ S_t^{(N, \Delta)} \geq \varepsilon \right] < \eta. \tag{5.7}
\]

The proof of (5.1). Observe that for \( \delta > 0 \) we have

\[
N^{-1/\alpha} \sum_{n=0}^{[Nt]} \mathbb{E} \left[ 1_{[0, \delta]}(N^{-1/\alpha} t(X_n+1)) | \mathcal{G}_n \right] \leq C(t+1) N^{1-1/\alpha} \int t(x) 1_{[t(x) < \delta N^{1/\alpha}]} \pi(dx)
\]

\[
\leq C(t+1) \int_0^{\delta N^{1/\alpha}} \pi(t(x) > \lambda) d\lambda \leq C(t+1) \delta^{1-\alpha}.
\]

On the other hand for any \( g \in C_\infty(\mathbb{R} \setminus \{0\}) \) we shall demonstrate that

\[
\lim_{N \to +\infty} \mathbb{E} \left[ \sum_{n=0}^{[Nt]} g(N^{-1/\alpha} t(X_n+1)) | \mathcal{G}_n \right] - t \int_0^\Delta g(\lambda) \nu_\alpha(d\lambda) = 0. \tag{5.8}
\]

Equality (7.1) can be then easily concluded by an approximation argument.
To prove (5.8) suppose that \( \text{supp} g \subset (m, M) \), where \( 0 < m < M \) and rewrite the expression under the expectation as \( I_N^{(1)} + I_N^{(2)} \), where

\[
I_N^{(1)} := \sum_{n=1}^{[Nt]} \int g(N^{-1/\alpha} t(y)) P(X_{n-1}, dy) - [tN] \int g(N^{-1/\alpha} t(y)) \pi(dy),
\]

\[
I_N^{(2)} := [Nt] \int g(N^{-1/\alpha} t(y)) \pi(dy).
\]

We have,

\[
\lim_{N \to +\infty} I_N^{(2)} = t \lim_{N \to +\infty} N^{1-1/\alpha} \int_0^\infty g'(N^{-1/\alpha} \lambda) \pi(t(y) > \lambda) d\lambda.
\]

Changing variables \( \lambda' := N^{-1/\alpha} \lambda \) and letting \( N \to +\infty \) we obtain that the right hand side of the above expression tends to \( t \int_0^\Delta g(\lambda) \nu_\alpha(d\lambda) \).

We show that

\[
\lim_{N \to +\infty} \mathbb{E}|I_N^{(1)}| = 0. \tag{5.9}
\]

The expression under the limit equals

\[
\mathbb{E} \left| \int_0^{+\infty} g'(\lambda) \sum_{n=1}^{[Nt]} G_N(X_{n-1}, \lambda) d\lambda \right|, \tag{5.10}
\]

where \( G_N(x, \lambda) := P(x, B_{N,\lambda}) - \pi(B_{N,\lambda}) \) and \( B_{N,\lambda} := [y : t(y) \geq N^{1/\alpha} \lambda] \). From the tail estimates of \( t(x) \) we get

\[
\sup_{\lambda \in [m, M]} \pi[B_{N,\lambda}] \leq \frac{C}{N}
\]

for some constant \( C \). We have \( \int G_N(y, \lambda) \pi(dy) = 0 \) and

\[
\int G_N^2(y, \lambda) \pi(dy) = \int P^2(y, B_{N,\lambda}) \pi(dy) - \pi^2(B_{N,\lambda}) \leq 2 \left( \int_{B_{N,\lambda}} p(y, x) \pi(dx) \right)^2 \pi(dy) + 2 \int Q^2(y, B_{N,\lambda}) \pi(dy)
\]

\[
\leq C \int \left( \int_{B_{N,\lambda}} p(x, y) \pi(dx) \right)^2 \pi(dy).
\]

To estimate the utmost right hand side we can use Cauchy-Schwartz inequality and conclude that for \( \lambda \geq m \) the integral is bounded from above by

\[
C \pi(B_{N,m}) \int \int_{B_{N,m}} p^2(x, y) \pi(dx) \pi(dy)
\]

\[
\leq \frac{1}{N} o(1), \quad \text{as } N \to \infty, \tag{5.11}
\]
since \( \int \int p^2(x, y) \pi(dx) \pi(dy) < +\infty \). Thus, we have shown that

\[
\sup_{\lambda \geq m} \int G_N^2(y, \lambda) \pi(dy) \to 0, \tag{5.12}
\]
as \( N \to \infty \). We will show now that (5.12) and the spectral gap together imply that

\[
\sup_{\lambda \in [m, M]} \mathbb{E} \left[ \sum_{n=1}^{[Nt]} G_N(X_{n-1}, \lambda) \right]^2 \to 0, \tag{5.13}
\]
as \( N \to \infty \). Since \( \text{supp} \, g' \subset [m, M] \) the expression in (5.10) can be then estimated by

\[
\sup_{\lambda \in [m, M]} \mathbb{E} \left\{ \sum_{n=1}^{[Nt]} G_N(X_{n-1}, \lambda) \right\} < \infty.
\]
as \( N \to +\infty \) and (5.9) follows.

To prove (5.13) let \( u_N(\cdot, \lambda) = (I - P)^{-1} G_N(\cdot, \lambda) \). By the spectral gap condition (2.2) we have

\[
\int u_N^2(y, \lambda) \pi(dy) \leq \frac{1}{1 - \alpha^2} \int G_N^2(y, \lambda) \pi(dy).
\]

We can rewrite then

\[
\sum_{n=1}^{[Nt]} G_N(X_{n-1}, \lambda) = u_N(X_0) - u_N(X_{[Nt]}) + \sum_{n=1}^{[Nt]-1} U_n,
\]

where \( U_n := u_N(X_n) - Pu_N(X_{n-1}) \), \( n \geq 1 \) is a stationary sequence of martingale differences with respect to the natural filtration corresponding to \( \{X_n, n \geq 0\} \). Consequently,

\[
\mathbb{E} \left[ \sum_{n=1}^{[Nt]} G_N(X_{n-1}, \lambda) \right]^2 \leq CN \int u_N^2(y, \lambda) \pi(dy) \to 0
\]

and (5.13) follows from (5.12) and (5).

The proof of (5.14). Observe that for any \( \alpha' \in (0, \alpha) \) we have

\[
\mathbb{E} \left[ S_N^{(N, \Delta)}(t) \right]^{\alpha'} \leq \frac{1}{N^{\alpha'/\alpha}} \mathbb{E} \left[ \sum_{n=0}^{[Nt]} t(X_n) 1_{[t(X_n) > \Delta_N]} \right]^{\alpha'}.
\]

We use the elementary inequality \( \sum_i a_i^{\alpha'} \leq \sum a_i^{\alpha''} \) that holds for arbitrary \( a_i > 0 \) and \( \alpha' \in (0, 1) \). By stationarity of \( \{X_n, n \geq 0\} \) we obtain that the left hand side of (5.14) can be estimated by

\[
C(t + 1)N^{1-\alpha'/\alpha} \int t^{\alpha'}(x) 1_{[t(x) \geq \Delta_N]} \pi(dx)
\]

\[
\leq C(T + 1)N^{1-\alpha'/\alpha} \int_{\Delta N^{1/\alpha}}^{\infty} \lambda^{\alpha'-1} \pi(t(x) > \lambda) d\lambda \leq C(t + 1) \Delta^{\alpha'-\alpha}
\]
and the conclusion of the lemma follows upon choosing suitably large $\Delta > 1$. \hfill \square

The proof of Theorem 2.4 reduces therefore to showing that \{${\theta_1 B_t}^{(N)} + {\theta_2 \tilde{T}_t}^{(N,\Delta)}$, $t \geq 0$\} converge in law, as $N \to +\infty$, to \{${\theta_1 B_t} + {\theta_2 \tilde{T}_t}^{(\Delta)}$, $t \geq 0$\}, where \{${\tilde{T}_t}^{(\Delta)}$, $t \geq 0$\} is a Lévy process, independent of the Brownian motion $B_t$, $t \geq 0$.

We use Theorem 1 p. 450 of [5]. Let $Z_{n,N} := \theta_1 Z_{n,N}^{(1)} + \theta_2 Z_{n,N}^{(2)}$, where $Z_{0,N}^{(1)} := 0$ and

\begin{align}
Z_{n,N}^{(1)} &:= \frac{1}{N^{1/2}} R_0(X_n, X_{n-1}), \quad n \geq 1, \\
Z_{n,N}^{(2)} &:= \frac{1}{N^{1/\alpha}} \{t(X_n) 1_{[t(X_n) < \Delta N]} - \mathbb{E}[t(X_n) 1_{[t(X_n) < \Delta N]} | \mathcal{G}_{n-1}]\}, \quad n \geq 0,
\end{align}

(5.15)

Recall that $\Delta_N = \Delta N^{1/\alpha}$. Note that \{${Z_{n,N}, n \geq 0}$\} constitute an array of martingale differences. The following result holds.

**Proposition 5.2.** There exists a bounded increasing function $G_{\theta_1, \theta_2} (\cdot)$ such that

\[
\lim_{N \to +\infty} \sum_{n=1}^{[Nt]-1} \mathbb{E} \left[ Z_{n,N}^{2} 1_{[a < Z_{n,N} \leq b]} \mid \mathcal{G}_{n-1} \right] = t[G_{\theta_1, \theta_2} (b) - G_{\theta_1, \theta_2} (a)] \quad \text{for any } a < b
\]

(5.16)

in probability. Here $\mathcal{G}_n$, $n \geq 0$ is the natural filtration corresponding to the sequence \{${X_n, n \geq 0}$\}.

In addition, the function $G_{\theta_1, \theta_2} (\cdot)$ satisfies

\[
\int_{\mathbb{R}} (e^{i\xi \lambda} - 1 - i\xi \lambda) \lambda^{-2} G_{\theta_1, \theta_2} (d\lambda) = -(\sigma^2 \theta_1^2 \xi^2 + c_\alpha |\theta_2|^\alpha |\xi|^{\alpha}).
\]

According to Theorem 1 of [5], the above proposition implies that the characteristic function of the limiting process equals $\exp \left\{ -t(\sigma^2 \theta_1^2 \xi^2 + c_\alpha |\theta_2|^\alpha |\xi|^{\alpha}) \right\}$. This concludes the proof of the convergence of finite dimensional distributions.

**The proof of Proposition 5.2.** The proof shall be divided in two parts. First we prove that for any interval $(a, b)$ that does not contain 0 and any $C^\infty$ function $g$ that is supported in that interval we have

\[
\lim_{N \to +\infty} \sum_{n=1}^{[Nt]-1} \mathbb{E} \left[ g(Z_{n,N}) \mid \mathcal{G}_{n-1} \right] = t|\theta_2|^{\alpha} \int_{0}^{\Delta} g(\lambda) \nu_\alpha (d\lambda)
\]

(5.17)

in probability. Secondly, we show that for any $c > 0$

\[
\limsup_{N \to +\infty} \left| \sum_{n=1}^{[Nt]-1} \mathbb{E} \left[ Z_{n,N}^{2} 1_{[Z_{n,N} < c]} \mid \mathcal{G}_{n-1} \right] - \frac{1}{2} \sigma^2 t \theta_1^2 \right| = h(c),
\]
where \( \lim_{c \to 0^+} h(c) = 0 \). For an arbitrary interval \((a, b)\), where \( a < 0 < b \) we divide it into a sum of three disjoint intervals \((a, -c), (-c, c)\) and \((c, b)\), where \( 0 < c < \min\{-a, b\} \) and conclude using the above results and a standard approximation argument that for any \( \varepsilon > 0 \) we have

\[
\lim_{N \to +\infty} \sup E \left| \sum_{n=1}^{[Nt]-1} E \left[ Z_{N}^{2} \mid x < Z_{n, N} < b \right] - t[G_{\theta_{1}, \theta_{2}}(b) - G_{\theta_{1}, \theta_{2}}(a)] \right| \leq \varepsilon,
\]

which, of course, implies \((5.16)\).

To start with the proof of \((5.17)\) we let \( \phi_{\Delta}(x) := x1_{|x|<\Delta} \) and suppose that \( g(x) = x^{2} \varphi(x) \), where \( \varphi \in C_{0}^{\infty}(\mathbb{R}) \) is such that \( 0 \leq \varphi \leq 1 \), \( \text{supp} \varphi \subset (a, b) \) and \( 0 < a < b \). We can expand \( g(Z_{n+1,N}) \) using Taylor formula, up to the second derivative, around \( z^{(N)}(X_{n+1}) \), where \( z^{(N)}(x) := N^{-1/\alpha} \theta_{2}^{2} \phi_{\Delta_{N}} \circ t(x) \), and obtain

\[
\sum_{n=0}^{[Nt]-1} E \left[ g(Z_{n,N}) \mid \mathcal{G}_{n} \right] = \sum_{n=0}^{[Nt]-1} E \left[ g \left( z^{(N)}(X_{n+1}) \right) \mid \mathcal{G}_{n} \right] + \sum_{n=0}^{[Nt]-1} E \left[ R(X_{n+1}, X_{n})g' \left( z^{(N)}(X_{n+1}) \right) \mid \mathcal{G}_{n} \right] + \sum_{n=0}^{[Nt]-1} \int_{0}^{\lambda} d\lambda \int_{\lambda}^{\lambda} d\lambda' E \left[ R^{2}(X_{n+1}, X_{n})g'' \left( z^{(N)}(\lambda') \right) \mid \mathcal{G}_{n} \right] d\lambda',
\]

where

\[
z_{n}^{(N)}(\lambda') := \lambda' R(X_{n+1}, X_{n}) + z(X_{n+1})
\]

and

\[
R(x, y) := \theta_{1} N^{1/2} R_{0}(x, y) - \frac{\theta_{2}}{N^{1/\alpha}} P(\phi_{\Delta_{N}} \circ t)(y).
\]

Denote the terms appearing on the right hand side of \((5.18)\) by \( I_{N}, II_{N} \) and \( III_{N} \) respectively. Calculating as in the proof of \((7.1)\) we obtain that

\[
\lim_{N \to +\infty} E \left| I_{N} - t|\theta_{2}|^{\alpha} \int_{0}^{\Delta} g(\lambda) \nu_{a}(d\lambda) \right| = 0.
\]

We show that the remaining terms \( II_{N} \) and \( III_{N} \) tend to 0, as \( N \to +\infty \), in the \( L^{1} \) sense. Note that

\[
E |II_{N}| \leq E_{N}^{(1)} + E_{N}^{(2)},
\]

where

\[
E_{N}^{(1)} := |\theta_{1}| N^{1/2} E \left| R_{0}(X_{1}, X_{0})g' \left( z(X_{1}) \right) \right|, \\
E_{N}^{(2)} := |\theta_{2}| N^{1-1/\alpha} E \left| P(\phi_{\Delta_{N}} \circ t)(X_{0})g' \left( z(X_{1}) \right) \right|.
\]
By an application of the Cauchy-Schwartz inequality we obtain

\[ E_N^{(1)} \leq CN^{1/2}\|g'\|_{\infty}\{\mathbb{E}[R_0^2(X_1, X_0), t(X_1) > a|\theta_2^{-1}|N^{1/\alpha}]\}^{1/2}\pi^{1/2}[t(x) > a|\theta_2^{-1}|N^{1/\alpha}] \]

\[ \leq C\|g'\|_{\infty}\{\mathbb{E}[R_0^2(X_1, X_0), t(X_1) > a|\theta_2^{-1}|N^{1/\alpha}]\}^{1/2} \rightarrow 0, \]
as \( N \rightarrow +\infty. \)

To estimate \( E_N^{(2)} \) we shall need the following result.

**Lemma 5.3.** For \( N \rightarrow +\infty \) we have

\[ \|\phi_{\Delta_N} \circ t\|_{L^1(x)} = O(N^{1/\alpha - 1}), \quad (5.19) \]

\[ \|P(\phi_{\Delta_N} \circ t)\|_{L^2(x)} = O(N^{1/\alpha - 1}) \quad (5.20) \]

and

\[ \|P(1_{\Delta_{N,+\infty}} \circ t)\|_{L^2(\pi)} = O(N^{-1}). \quad (5.21) \]

Taking this lemma for granted, its proof shall be presented momentarily, we show how to finish estimating \( \mathbb{E}|II_N| \). We have

\[ E_N^{(2)} \leq |\theta_2|N^{1-1/\alpha}\|g'\|_{\infty}\mathbb{E}\left[P(\phi_{\Delta_N} \circ t)(X_0), |\theta_2|t(X_1) > aN^{1/\alpha}\right] \]

\[ \leq CN^{1-1/\alpha}\|P(\phi_{\Delta_N} \circ t)\|_{L^2(\pi)}\|P(1_{\Delta_{N,+\infty}} \circ t)\|_{L^2(x)}, \]

where \( \Delta_N := N^{1/\alpha}\Delta' \) and \( \Delta' := a|\theta_2^{-1}|. \) Using Lemma 5.3 we estimate the right hand side of (5.22) by an expression of order \( o(N^{-1}) \), which shows that \( \mathbb{E}|II_N| \rightarrow 0, \) as \( N \rightarrow +\infty. \)

Note also that from (5.19) it follows that for any \( m > 0 \)

\[ \mathbb{P}[|z^{(N)}(X_1)| > m] = \pi[|z^{(N)}(x)| > m] \leq \frac{|\theta_2|\|\phi_{\Delta_N} \circ t\|_{L^1(x)}}{mN^{1/\alpha}} = CN^{-1} \rightarrow 0, \]
as \( N \rightarrow +\infty. \) Likewise, one obtains that for any \( \lambda' \in [0, 1] \)

\[ \mathbb{P}[|z_0^{(N)}(\lambda')| > m] \rightarrow 0, \quad \text{as } N \rightarrow +\infty. \quad (5.23) \]

We have therefore

\[ \mathbb{E}|III_N| \leq N(t + 1)\|g''\|_{\infty}\int_0^1 \mathbb{E}\left[R^2(X_1, X_0), |z_0^{(N)}(\lambda')| \geq a\right] d\lambda' \]

\[ \leq CN(t + 1)\|g''\|_{\infty}\left\{N^{-1}\int_0^1 \mathbb{E}\left[\lambda^2(X_1) + [P\chi(X_0)]^2, |z_0^{(N)}(\lambda')| \geq a\right] d\lambda' \right. \]

\[ \left. +N^{-2/\alpha}\|P(\phi_{\Delta_N} \circ t)\|_{L^2(x)}^2 \right\} \rightarrow 0, \quad \text{as } N \rightarrow +\infty \]

by virtue of (5.20) and (5.23).
The proof of Lemma 5.3. Note that
\[
\|P(\theta_{\Delta_N} \circ t)\|_{L^2(\pi)}^2 \leq 2 \left[ \int \left( \int p(x, y)\phi_{\Delta_N} \circ t(y)\pi(dy) \right)^2 \pi(dx) + \int \left( \int Q(x, dy)\phi_{\Delta_N} \circ t(y)\pi(dy) \right)^2 \pi(dx) \right].
\]
Denote the first and the second term appearing on the right hand side by \(I_N\) and \(II_N\) respectively. Thanks to (2.5) one can estimate \(I_N\) from above by
\[
C(2)\|\theta_{\Delta_N} \circ t\|_{L^1(\pi)}^2 \leq C(2)\|\phi_{\Delta_N} \circ t\|_{L^1(\pi)}^2 \leq C(2) \left( \int_{[\theta(y) < \Delta_N]} t(y)\pi(dy) \right)^2 = C(2) \left( \int_0^{\Delta_N} \pi[t(y) > \lambda]d\lambda \right)^2 \leq C \left( \int_0^{\Delta_N} \frac{d\lambda}{1 + \lambda^\alpha} \right)^2 \leq C'N^{2(1/\alpha - 1)}.
\]
Observe that this estimate also proves (5.19). Additionally, we have
\[
II_N = \int \left( \int_0^{\Delta_N} Q(x, [\lambda \leq t(y) \leq \Delta_N])d\lambda \right)^2 \pi(dx)
\leq C \int \left( \int_0^{\Delta_N} \int_{[\lambda \leq t(y)]} p(x, y)\pi(dy)d\lambda \right)^2 \pi(dx)
\leq C \int \left( \int (t(y) \wedge \Delta_N)p(x, y)\pi(dy) \right)^2 \pi(dx)
\leq \|t \wedge \Delta_N\|_{L^1(\pi)}^2 \leq C'N^{2(1/\alpha - 1)}.
\]
This concludes the proof of (5.20). The proof of (5.21) is similar. □

Suppose now that \(c > 0\). Note that
\[
\sum_{n=1}^{[N]} E \left[ (Z_{n,N})^2 1_{||Z_{n,N}|<c} \mid G_{n-1} \right] = \theta_1^2 \sum_{n=1}^{[N]} E \left[ (Z_{n,N})^2 1_{||Z_{n,N}|<c} \mid G_{n-1} \right] + \theta_2^2 \sum_{n=1}^{[N]} E \left[ (Z_{n,N})^2 1_{||Z_{n,N}|<c} \mid G_{n-1} \right] + 2\theta_1\theta_2 \sum_{n=1}^{[N]} E \left[ Z_{n,N}^1 Z_{n,N}^2 1_{||Z_{n,N}|<c} \mid G_{n-1} \right].
\]
Denote the terms appearing on the right hand side by \(U_N\), \(V_N\) and \(W_N\). For an appropriate constant \(C > 0\) we have
\[
E|W_N| \leq C \left\{ N \left\{ \mathbb{E}[|\theta_1 Z_{1,N}^{(1)}|^2] \right\}^{1/2} \left\{ \mathbb{E}[|\theta_2 Z_{1,N}^{(2)}|^2] \right\} \right\}^{1/2} + N|\theta_1\theta_2|E \left[ |Z_{1,N}^1 Z_{1,N}^2|, |\theta_1 Z_{1,N}^{(1)}| > 9c \right].
\]
Denote the first and second term appearing in the braces on the right hand side by $W_N^{(1)}$ and $W_N^{(2)}$ respectively. We have

$$
\mathbb{P}[|\theta_2Z_{n,N}^{(2)}| > \lambda] \leq \mathbb{P}[|t(X_n) > N^{1/\alpha}\lambda/(2|\theta_2|)] + \pi\mathbb{P}(\phi_{\Delta_N} \circ t) > N^{1/\alpha}\lambda/(2|\theta_2|)].
$$

The first term on the right hand side is clearly less than, or equal to $CN^{-1}\lambda^{-\alpha}$ for all $\lambda > 0$, $N \geq 1$ and a certain constant $C > 0$, independent of $n \geq 0$. By the Markov inequality the second term can be estimated by

$$
\lambda^{-1}N^{-1/\alpha}\|\phi_{\Delta_N} \circ t\|_{L^1(\pi)} \leq C(\lambda N)^{-1},
$$

by (5.19). One can easily see that

$$
\mathbb{P}[|\theta_2Z_{n,N}^{(2)}| > \lambda] \leq C\lambda^{-1}(1 + \lambda^{1-\alpha})N^{-1}
$$

for all $\lambda > 0$, $N \geq 1$ and a certain constant $C > 0$, independent of $n \geq 0$. Using (5.24) and the elementary estimate

$$
\left\{\mathbb{E}[|\theta_1Z_{1,N}^{(1)}|^2]\right\}^{1/2} \leq CN^{-1/2}\|\chi\|_{L^2(\pi)}
$$

we obtain

$$
W_N^{(1)} \leq C\|\chi\|_{L^2(\pi)} \left[\int_0^{10c} (1 + \lambda^{1-\alpha})d\lambda\right]^{1/2} \leq C\|\chi\|_{L^2(\pi)}[c(1 + c^{1-\alpha})]^{1/2}
$$

for some constants $C, C' > 0$.

On the other hand, using Chebyshev’s inequality we get

$$
\mathbb{P}[|\theta_1Z_{n,N}^{(1)}| > \lambda] \leq \frac{C\|\chi\|^2_{L^2(\pi)}}{N\lambda^2}
$$

for all $\lambda > 0$. The constant $C > 0$ appearing here does not depend on $N$, $n$ and $\lambda$. Thus, for some constants $C, C' > 0$, we have

$$
W_N^{(2)} \leq CN\Delta\mathbb{E}\left[|\theta_1Z_{1,N}^{(1)}|, |\theta_1Z_{1,N}^{(1)}| > 9c\right] \leq CN\Delta\left\{\mathbb{E}\left[|\theta_1Z_{1,N}^{(1)}|^2, |\theta_1Z_{1,N}^{(1)}| > 9c\right]\right\}^{1/2}\mathbb{P}[|\theta_1Z_{1,N}^{(1)}| > 9c] \leq C'\left\{\mathbb{E}\left[R_0^2(X_1, X_0), |\theta_1Z_{1,N}^{(1)}| > 9c\right]\right\}^{1/2} \rightarrow 0,
$$

as $N \rightarrow +\infty$, cf. (5.13). We have proved therefore that

$$
\limsup_{N \rightarrow +\infty} \mathbb{E}|W_N| \leq C[c(1 + c^{1-\alpha})]^{1/2},
$$

where $c > 0$ can be chosen to be as small as we wish. Thus, $\lim_{N \rightarrow +\infty} \mathbb{E}|W_N| = 0$.

Note that

$$
\mathbb{E}|V_N| \leq CNE[|\theta_2Z_{n,N}^{(2)}|^2, |\theta_2Z_{n,N}^{(2)}| < 10c] + C\lambda^{-1/\alpha}\|\phi_{\Delta_N} \circ t|_{L^2(\pi)}\| |\theta_1Z_{n,N}^{(1)}| > 9c]
$$

(5.26)
for some constant $C > 0$. Denote the first and the second terms on the right hand side of (5.23) by $V_N^{(1)}$, $V_N^{(2)}$ respectively. Using (5.24) we obtain

$$V_N^{(1)} \leq C \left[ \int_0^{10c} \lambda(1 + \lambda^{-c}) d\lambda \right]^{1/2} \leq C'[c(1 + c^{1-c})]^{1/2}$$

for some constant $C' > 0$. This term can be made arbitrarily small by choosing a sufficiently small $c > 0$. On the other hand, from Chebyshev’s inequality

$$V_N^{(2)} \leq CN^2 \mathbb{P}[|\theta_1 R_0(X_1, X_0)| \geq 9cN^{1/2}]$$

$$\leq C'N^2 \mathbb{E}[|\theta_1 R_0(X_1, X_0)|^2, |\theta_1 R_0(X_1, X_0)| \geq 9cN^{1/2}] \to 0,$$

both a.s. and in the $L^1$ sense, as $N \to +\infty$. Finally, we can write that $U_N = \tilde{U}_N - \hat{U}_N$, where

$$\tilde{U}_N := \theta_1^2 \sum_{n=1}^{[Nt]} \mathbb{E}\left[ (Z_{n,N}^{(1)})^2 | \mathcal{G}_{n-1} \right]$$

and

$$\hat{U}_N := \theta_1^2 \sum_{n=1}^{[Nt]} \mathbb{E}\left[ (Z_{n,N}^{(1)})^2 I_{|Z_{n,N}^{(1)}| > c} | \mathcal{G}_{n-1} \right].$$

We have, by the ergodic theorem,

$$\tilde{U}_N = \frac{\theta_1^2}{N} \sum_{n=1}^{[Nt]} \left[ P \Psi^2(X_{n-1}) + P \chi^2(X_{n-1}) - (P \chi)^2(X_{n-1}) \right] \to \frac{1}{2} \sigma^2 \theta_1^2 t, \quad \text{as } N \to +\infty,$$

both a.s. and in the $L^1$ sense. Here

$$\sigma^2 := 2 \left[ \|\Psi\|_{L^2(\pi)}^2 + \|\chi\|_{L^2(\pi)}^2 - \|P \chi\|_{L^2(\pi)}^2 \right].$$

We can also estimate, using stationarity of $\{X_n, n \geq 0\}$, that

$$\mathbb{E}[\tilde{U}_N] \leq C \theta_1^2 \mathbb{E}[R_0^2(X_1, X_0), A_N],$$

where $A_N$ is the event that either $|\theta_1 R_0(X_1, X_0)| > cN^{1/2}/2$, or $|\theta_2 Z_{1,N}^{(2)}| > c$ and $C > 0$ is a certain constant. The conclusion of the proposition follows therefore from the $L^2$-integrability of $R_0(X_1, X_0)$ and (5.24). In addition,

$$G_{\theta_1, \theta_2}(d\lambda) = c_*(\lambda)|\theta_1|^{\alpha} \nu_0(d\lambda) + \theta_2^2 \sigma^2 \delta(d\lambda).$$

$\square$

6. The proof of Theorem 2.5

Tightness of the laws of $\{(B_t^{(N)}, T_t^{(N)}), t \geq 0\}$, $N \geq 1$ follows from tightness of each coordinate, see [17]. We shall prove the convergence of the finite dimensional distributions of the process $\{\theta_1 B_t^{(N)} + \theta_2 T_t^{(N)}, t \geq 0\}$ for any $\theta_1 \neq 0, \theta_2 \neq 0$. To simplify the notation we shall only consider one dimensional marginals and the case when $\beta \in (1, 2)$.
We maintain the notation from Section 5. Suppose that $\chi$ is the solution of the Poisson equation (5.2) and $Z_{n,N} := \theta_1 Z_{n,N}^{(1)} + \theta_2 Z_{n,N}^{(2)}$, where $Z_{0,N}^{(1)} := 0$, and

$$Z_{n,N}^{(1)} := \frac{1}{N^{1/\beta}} R_0(X_n, X_{n-1}),$$

with $Z_{n,N}^{(2)}$, $n \geq 1$ as in (5.15). Function $R_0(x, y)$ is defined by (5.3). We demonstrate that the laws of $\sum_{n=0}^{[N]} Z_{n,N}$ converge, as $N \to +\infty$, to the law of a stable random variable whose characteristic function is given by $e^{-t\psi_{\alpha_1, \alpha_2}(\xi)}$ with $\psi_{\alpha_1, \alpha_2}(\xi) = \psi^{(\beta)}(\theta_1 \xi) \psi^{(\alpha)}(\theta_2 \xi)$, where $\psi^{(\beta)}(\cdot)$, $\psi^{(\alpha)}(\cdot)$ are the Lévy exponents defined in (2.10) and corresponding to indices of stability $\beta$ and $\alpha$. This result can be in fact easily generalized to multidimensional statistics.

To further simplify considerations we only deal with the case $\beta \in (1, 2)$. Generalization to an arbitrary $\beta \in (0, 2)$ is routine and we leave it to a reader. Since $\Psi = \chi - P\chi$ and $P\chi$ has lighter tails than $\Psi(x)$ and $t(x)$, thanks to (2.12) we conclude the following.

**Lemma 6.1.** There exists a constant $C_*$ such that

$$\pi \{ x : t(x) \geq \lambda, |\chi(x)| \geq \lambda \} \leq \frac{C_*}{\lambda^\gamma}.$$

The exponent $\gamma$ is the same as in (2.11).

Suppose that $g(x) = x^2 \varphi(x)$ where $\varphi \in C_0^\infty(\mathbb{R})$ is such that $0 \leq \varphi \leq 1$, supp $\varphi \subset (a, b)$ and $0 < a < b$. We can expand $g(Z_{n+1,N})$ using Taylor formula, up to the first derivative, around $z^{(N)}(X_{n+1})$, where

$$z^{(N)}(x) := N^{-1/\beta} \theta_1 \Psi(x) + N^{-1/\alpha} \theta_2 \phi \Delta_N \circ t(x),$$

and we obtain

$$\sum_{n=0}^{[N]} \mathbb{E} \left[ g(Z_{n+1,N}) \mid \mathcal{G}_n \right] = \sum_{n=0}^{[N]} \mathbb{E} \left[ g(z^{(N)}(X_{n+1})) \mid \mathcal{G}_n \right]$$

$$+ \sum_{n=0}^{[N]} \int_0^1 \mathbb{E} \left[ R(X_{n+1}, X_n) g'(z^{(N)}(\lambda)) \mid \mathcal{G}_n \right] d\lambda.$$ 

Here

$$z_n^{(N)}(\lambda) := \lambda R(X_{n+1}, X_n) + z^{(N)}(X_{n+1})$$

and

$$R(x, y) := \frac{\theta_1}{N^{1/\beta}} [P\chi(x) - P\chi(y)] - \frac{\theta_2}{N^{1/\alpha}} P(\phi \Delta_N \circ t)(y).$$

By virtue of (2.12) and (5.19) we conclude that there exists $C > 0$ such that for all $\lambda \in (0, 1]$

$$\mathbb{P} \left[ \left| z_0^{(N)}(\lambda) \right| > a \right] \leq \frac{C}{N}, \quad \forall N \geq 1.$$ 

Denote the terms appearing on the right hand side of (7.23) by $I_N$ and $II_N$ respectively.

To compute the limit $\lim_{N \to +\infty} I_N$ we follow closely the argument made in the proof of (5.8). It is based on the following lemma.
Lemma 6.2. Suppose that \( \text{supp} g \subset (a, b) \), where \( 0 < a < b \). Under assumption (2.11) we have

\[
\lim_{N \to +\infty} N \int g \left( z^{(N)}(x) \right) \pi(dx) = |\theta_1|^{\beta} \int g(\lambda) \nu_{\beta}(d\lambda) + |\theta_2|^{\alpha} \int_{0}^{+\infty} g(\lambda) \nu_{\alpha}(d\lambda).
\]

Proof. Suppose that \( \gamma > \kappa_1 > \alpha \lor \beta \), where \( \gamma \) is the same as in (2.11), and

\[
A_N := \{ |\theta_1|t(x) \geq (a/2)N^{1/\kappa_1}, |\theta_2|\Psi(x) \geq (a/2)N^{1/\beta} \},
B_N := \{ |\theta_1|t(x) \geq (a/2)N^{1/\alpha}, |\theta_2|\Psi(x) \geq (a/2)N^{1/\kappa_1} \}.
\]

Observe that \( \pi(A_N) \leq \frac{1}{N} o(1) \) and \( \pi(B_N) \leq \frac{1}{N} o(1) \).

To compute the limit on the left hand side of (6.4) it suffices therefore to compute the limits \( \lim_{N \to +\infty} K_N^{(i)} \), \( i = 1, 2 \), where

\[
K_N^{(1)} := N \int g \left( z^{(N)}(x) \right) 1_{C_N} \pi(dx),
K_N^{(2)} := N \int g \left( z^{(N)}(x) \right) 1_{D_N} \pi(dx),
\]

where

\[
C_N := \{ |\theta_1|t(x) \leq (a/2)N^{1/\kappa_1}, |\theta_2|\Psi(x) \geq (a/2)N^{1/\beta} \},
D_N := \{ |\theta_1|t(x) \leq (a/2)N^{1/\alpha}, |\theta_2|\Psi(x) \leq (a/2)N^{1/\kappa_1} \}.
\]

Up to a term of order \( o(1) \) we have \( K_N^{(1)} = K_N^{(1)} \), where

\[
\tilde{K}_N^{(1)} := N \int g \left( \tilde{z}^{(N)}(x) \right) \pi(dx),
\]

and

\[
\tilde{z}^{(N)}(x) := \theta_2N^{-1/\beta}\Psi(x).
\]

Repeating the argument used in the proof of (7.1) we conclude that

\[
\lim_{N \to +\infty} \tilde{K}_N^{(1)} = |\theta_2|^{\beta} \int g(\lambda) \nu_{\beta}(d\lambda)
\]

and likewise

\[
\lim_{N \to +\infty} K_N^{(2)} = |\theta_1|^{\alpha} \int_{0}^{+\infty} g(\lambda) \nu_{\alpha}(d\lambda).
\]
We use the above lemma to calculate the limit \( \lim_{N \to +\infty} I_N \). We can write \( I_N = I_N^{(1)} + I_N^{(2)} \), where
\[
I_N^{(1)} := \sum_{n=1}^{[Nt]} \int g(z^{(N)}(y))P(X_{n-1}, dy) - [Nt] \int g(z^{(N)}(x)) \pi(dx),
\]
\[
I_N^{(2)} := [Nt] \int g(z^{(N)}(x)) \pi(dx).
\]
It can be shown that
\[
\lim_{N \to +\infty} \mathbb{E}|I_N^{(1)}| = 0
\]
precisely as the corresponding term in the proof of (5.8). From Lemma 6.2 we obtain that
\[
\lim_{N \to +\infty} \mathbb{E}\left| I_N - t|\theta_1|^\beta \int g(\lambda) \nu(\lambda) - t|\theta_2|^\alpha \int_0^{+\infty} g(\lambda) \nu(\lambda) d\lambda \right| = 0.
\]
On the other hand
\[
\mathbb{E}|II_N| \leq E_N^{(1)} + E_N^{(2)},
\]
where
\[
E_N^{(1)} := (t + 1)\|g'\|_{L^\infty(\pi)}|\theta_1|N^{1-1/\beta} \int_0^1 \mathbb{E}\left[ |P\chi(X_1) - P\chi(X_0)|, |z^{(N)}_0(\lambda)| > a \right] d\lambda,
\]
\[
E_N^{(2)} := (t + 1)\|g'\|_{L^\infty(\pi)}|\theta_2|N^{1-1/\alpha} \int_0^1 \mathbb{E}\left[ |P(\phi_{\Delta t} \circ t)(X_0)|, |z^{(N)}_0(\lambda)| > a \right] d\lambda.
\]
From Hölder inequality and (2.12) we have
\[
E_N^{(1)} \leq C|\theta_1|N^{1-1/\beta} \|P\chi\|_{L^\beta(\pi)} \|g'\|_{L^\infty(\pi)} \sup_{\lambda \in (0,1]} \mathbb{P}^{1-1/\beta}\left[ |z^{(N)}_0(\lambda)| > a \right] \leq \frac{C}{N^{1/\beta} - 1/\beta^2}
\]
for all \( N \geq 1 \) and some \( C > 0 \).

We also have
\[
E_N^{(2)} \leq CN^{1-1/\alpha} \|P(\phi_{\Delta t} \circ t)\|_{L^\alpha(\pi)} \sup_{\lambda \in (0,1]} \mathbb{P}^{1/\alpha}\left[ |z^{(N)}_0(\lambda)| > a \right] \leq \frac{C}{N^{1/2}} \to 0
\]
as \( N \to +\infty \). This proves that \( \mathbb{E}|II_N| \to 0 \), as \( N \to +\infty \).

7. The proof of Theorem 2.6

To simplify the notation we maintain the assumption that \( \beta \in (1, 2) \). The consideration of the other cases \( \beta \in (0, 1) \) and \( \beta = 1 \) can be done similarly. In fact the respective arguments are simpler than the one presented here. Denote \( \{Z_t^{(N)} := (B_t^{(N)}, T_t^{(N)}) \}, t \geq 0 \), where the respective processes are defined by (2.7) and (2.8). Let \( \{	ilde{Z}_t^{(N,\Delta)} := (\tilde{B}_t^{(N)}, \tilde{T}_t^{(N,\Delta)}) \}, t \geq 0 \), where \( \tilde{B}_t^{(N)} \) and \( \tilde{T}_t^{(N,\Delta)} \) are given by (5.1) and (5.5) respectively. Since
\[
\lim_{\Delta \to +\infty} \limsup_{N \to +\infty} \sup_{t \in [0, L]} |Z_t^{(N)} - \tilde{Z}_t^{(N)}| = 0,
\]
in probability,
it suffices only to prove the conclusion of the theorem for $\bar{Z}_t^{(N,\Delta)}$ for a fixed $\Delta$. The weak convergence can be concluded from the following.

**Proposition 7.1.** Under the assumptions of Theorem 2.6 for any $g \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$

$$
\lim_{N \to +\infty} E \left[ \sum_{n=1}^N \mathbb{E} \left[ g(Z_{n,N}) \mid G_{n-1} \right] - \int_{\mathbb{R}^2} g(\lambda_1, \lambda_2) \nu_*(d\lambda_1, d\lambda_2) \right] = 0 \quad (7.1)
$$

and

$$
\lim_{N \to +\infty} N \mathbb{E} \left\{ \mathbb{E} \left[ g(Z_{n,N}) \mid G_0 \right] \right\}^2 = 0. \quad (7.2)
$$

Here $Z_{n,N} := (Z_{n,N}^{(1)}, Z_{n,N}^{(2)})$, where $Z_{n,N}^{(1)}, Z_{n,N}^{(2)}$ are given by (6.1), and $G_n$ is the $\sigma$-algebra generated by $X_0, \ldots, X_n$.

Postponing for the moment the proof of the above proposition we show how to use it in order to finish the proof of Theorem 2.6.

**The proof of the weak convergence.** This argument follows closely [8]. Below, we formulate a certain estimate of the total variation distance for counting random variable and a suitable Poisson random variable taken directly from [10], see Theorem 5, p. 258 and also Proposition 4.3, p. 268 there. Before formulating the result let us introduce an auxiliary notation $K_1(q) := -q^{-1} \log(1 - q)$ and $K_2(q) := q^{-2}[\log(1 - q) - q]$ for $|q| < 1$. For any random variables $X, Y$ denote by $d(X, Y)$ the total variation distance between their laws, given by $d(X, Y) = \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|$.

**Theorem 7.2.** Suppose that:

1) $\{G_i', i \geq 0\}$ is a filtration of $\sigma$-algebras, with $G_0' := \{\emptyset, \Omega\}$,
2) $\tau$ is a stopping time,
3) we have a family of events $\{A_i, i \geq 1\}$ such that $A_i \in G_i', i \geq 1$,
4) $N = \sum_{i=1}^\tau 1_{A_i}$,
5) Let $p_i := \mathbb{P}(A_i \mid G_{i-1}')$, $i \geq 1$, $0 < a < b$ and $\varepsilon, \delta \in (0, 1)$. We shall assume that

$$
\mathbb{P} \left( a \leq \sum_{i=1}^\tau p_i \leq b, \sum_{i=1}^\tau p_i^2 \leq \varepsilon \right) \geq 1 - \delta.
$$

Then,

$$
d(N, N_a) \leq \alpha \varepsilon + b - a + 2\delta, \quad (7.3)
$$

where $\alpha := (1/2)[K_1(\sqrt{\varepsilon})]^2 + K_2(\sqrt{\varepsilon})$ and $N_a$ is a Poisson random variable with the parameter $a$.

Let $\mathbb{R}^3_{+,0} := [0, +\infty) \times (\mathbb{R}^2 \setminus \{0\})$ and let $\mathcal{M}_{loc}(\mathbb{R}^3_{+,0})$ be the space of all locally finite point measures on $\mathbb{R}^3_{+,0}$ equipped with the vague topology.

We define an $\mathcal{M}_{loc}(\mathbb{R}^3_{+,0})$-valued random element by letting

$$
\mathcal{N}_N([0, t] \times A) := \sum_{n=1}^{[Nt]} 1_A(Z_{n,N}) \quad (7.4)
$$
for a Borel $A \subset \mathbb{R}^3_{+}$. As a conclusion from Theorem 7.2 we obtain.

Lemma 7.3. Measures $\mathcal{N}_N$, considered as random $\mathcal{M}_{\text{loc}}(\mathbb{R}^3_{+},0)$-valued random elements, are weakly convergent in law, as $N \to +\infty$, to a Poisson measure $\mathcal{N}$ on $\mathbb{R}^3_{+}$, with intensity

$$\nu_*( dt, d\lambda ) := dt \nu_*( d\lambda ).$$

Proof. To abbreviate let us write $\bigcup_{i=1}^m A_i$ where $A_i := (a_i, b_i] \times I_i$. Here $(a_i, b_i], \ i = 1, \ldots, m$ are pairwise disjoint while $I_i$ are finite unions of disjoint two dimensional intervals. We shall show that for such a set

$$\mathcal{N}_N[A] \Rightarrow \mathcal{N}[A], \quad \text{as } N \to +\infty.$$  \hspace{1cm} (7.5)

According to [3], p. 209, this implies the convergence in question. We have

$$\mathcal{N}_N(A) = \sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} 1_{I_i}(Z_{n,N}).$$

As a consequence of Proposition 7.1 we conclude that

$$\sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} \mathbb{E} \left[ 1_{I_i}(Z_{n,N}) \mid \mathcal{G}_{n-1} \right] - \sum_{i=1}^m (b_i - a_i) \nu_*(I_i) = 0.$$  \hspace{1cm} (7.6)

We also have

$$\mathbb{E} \left[ \sum_{i=1}^m \sum_{k_N(a_i) < n \leq k_N(b_i)} \{ \mathbb{E} \left[ 1_{I_i}(Z_{n,N}) \mid \mathcal{G}_{n-1} \right] \}^2 \right]$$

$$\leq \sum_{i=1}^m N(b_i - a_i + 1) \mathbb{E} \left\{ \mathbb{E} \left[ 1_{I_i}(Z_{1,N}) \mid \mathcal{G}_0 \right] \right\}^2 \to 0,$$

as $N \to +\infty$. Theorem 7.2 implies then (7.5). \hspace{1cm} \Box

Define a random measure

$$\mathcal{N}_N'[0, t] \times A := \sum_{n=1}^{\lfloor Nt \rfloor} \mathbb{E} \left[ 1_A(Z_{n,N}) \mid \mathcal{G}_{n-1} \right].$$  \hspace{1cm} (7.7)

As a consequence of Proposition 7.1 we conclude that $\mathcal{N}_N'[dt, d\lambda_1, d\lambda_2] \Rightarrow dt \nu_*(d\lambda_1, d\lambda_2)$, as $N \to +\infty$ over $\mathcal{M}_{\text{loc}}(\mathbb{R}^3_{+})$. Let $\square_{\delta, \Delta} := [ (\lambda_1, \lambda_2) : \delta \leq |\lambda_i| < \Delta, \ i = 1, 2]$ and $\square_\Delta := [ (\lambda_1, \lambda_2) : |\lambda_i| < \Delta, \ i = 1, 2]$. The mapping $\mathcal{H} : \mathcal{M}_{\text{loc}}(\mathbb{R}^3_{+}) \to \mathcal{D}_2$ given by

$$\mathcal{H}(\mu)(t) := \int_{|\lambda| \in \square_{\delta, \Delta}} \lambda \mu([0, t] \times d\lambda), \quad t \geq 0, \ \mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^3_{+})$$  \hspace{1cm} (7.8)

is continuous at any $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^3_{+},0)$, for which $\mu(\partial \square_{\delta, \Delta}) = 0$. Applying the continuous mapping theorem, see Theorem 2.7, p. 21 of [3], for $\mathcal{H}$ we obtain that the processes $\{ \mathcal{H}(\mathcal{N}_N')(t), t \geq 0 \}$ converge weakly over $\mathcal{D}_2$ to the process $\{ t \int_{\lambda \in \square_{\delta, \Delta}} \lambda \nu_*(d\lambda), t \geq 0 \}$. Since the limit is a deterministic, continuous function the convergence holds also in probability in $\mathcal{D}_2$ in the topology coming from the supremum norm.
Denote by \( \mathcal{L}_\mathcal{N} \) the law of the random element \( \mathcal{N} : \Omega \to \mathcal{M}_{\text{loc}}(\mathbb{R}^2_+) \). Observe that \( \mathbb{E} \mathcal{N}([0, t] \times \partial \Delta_\delta, \mathcal{H}) = 0 \) therefore the set \( \mathcal{Z} \) that consists of possible discontinuity points of \( \mathcal{H} \) satisfies \( \mathcal{L}_{\mathcal{N}}(\mathcal{Z}) = 0 \). Using the continuous mapping theorem we obtain that the processes \( \{Z_t^{(N)}(\delta, \Delta') : = \mathcal{H}(\mathcal{N}_t)(t), t \geq 0\} \) converge weakly over \( \mathcal{D}_2 \) to the process \( \{\mathcal{H}(\mathcal{N})(t), t \geq 0\} \) hence the processes \( \left\{ Z_t^{(N)}(\delta, \Delta') := \sum_{n=0}^{[N]} Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}), \ t \geq 0 \right\} \) converge weakly, over \( \mathcal{D}_2 \) to the jump process \( \left\{ \int_{\partial \Delta_\delta, \mathcal{H}} \mathcal{N}([0, t] \times d\lambda), t \geq 0 \right\} \). Suppose that \( \Delta' > \Delta \) and let \( \tilde{Z}_t^{(N)}(\delta, \Delta') := (\tilde{B}_t^{(N)}(\delta, \Delta'), T_t^{(N)}(\delta, \Delta')) \) be given by

\[
\tilde{B}_t^{(N)}(\delta, \Delta') := \sum_{n=0}^{[N]} \left\{ Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) - \mathbb{E} \left[ Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) | \mathcal{G}_{n-1} \right] \right\}, \quad (7.9)
\]

\[
T_t^{(N)}(\delta, \Delta') := \sum_{n=0}^{[N]} \left\{ Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) - \mathbb{E} \left[ Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) | \mathcal{G}_{n-1} \right] \right\}.
\]

Process \( \tilde{Z}_t^{(N)}(\Delta') := (\tilde{B}_t^{(N)}(\Delta'), T_t^{(N)}(\Delta')) \) is defined analogously to (7.9). The only difference is that \( \partial \Delta_\delta \) is replaced, throughout the definition, by \( \partial \Delta_\Delta \).

Combining the convergence statements on \( Z_t^{(N)}(\delta, \Delta') \), \( \tilde{Z}_t^{(N)}(\delta, \Delta') \) with Lemma 7.4 we conclude that the processes \( \{\tilde{Z}_t^{(N)}(\delta, \Delta'), t \geq 0\} \) converge in law over \( \mathcal{D}_2 \) (equipped with the \( J_1 \)-topology) to

\[
\tilde{Z}_t(\delta, \Delta') := \int_{\partial \Delta_\delta, \mathcal{H}} \mathcal{N}([0, t] \times d\lambda) - t \int_{\partial \Delta_\Delta} \lambda \nu_d(d\lambda).
\]

The proof of the theorem can be obtained from the above and the following facts:

\[
\lim_{\delta \to 0, t \to +\infty} \sup_{\Delta' \to +\infty} |\tilde{Z}_t(\delta, \Delta') - \tilde{Z}_t| = 0, \quad \text{in probability for any } L > 0, \quad (7.10)
\]

that follows from Theorem 14.27, p. 312 of [4], and the two lemmas

**Lemma 7.4.** For any \( \varepsilon > 0 \) and \( T \geq 1 \) there exists \( \Delta' \geq 1 \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \sum_{n=0}^{[N]} Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) \right] < \varepsilon \quad (7.11)
\]

and

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \sum_{n=0}^{[N]} \mathbb{E} \left[ Z_{n,N} 1_{\partial \Delta_\delta}(Z_{n,N}) | \mathcal{G}_{n-1} \right] \right] < \varepsilon. \quad (7.12)
\]

**Lemma 7.5.** For any \( \Delta', \varepsilon, \eta > 0 \) and \( L \geq 1 \) there exists \( \delta \in (0, \Delta') \) sufficiently small so that

\[
\lim_{N \to +\infty} \mathbb{P} \left[ \rho_D([0, L], \mathbb{R}^2)(\tilde{Z}_t^{(N)}(\delta, \Delta'), \tilde{Z}_t^{(N)}(\Delta')) \geq \varepsilon \right] < \eta. \quad (7.13)
\]

Here \( \rho_D([0, L], \mathbb{R}^2) \) is the Skorohod metric on \( D([0, L], \mathbb{R}^2) \).
The proof of Lemma 7.4. We only prove (7.11). The proof of (7.12) is analogous. For \( \Delta' > \Delta \) we can estimate the expression on the left hand side of (7.11) by
\[
N^{1-1/\beta} T \mathbb{E} \left[ |\chi(X_1) - P\chi(X_0)|, |\chi(X_1) - P\chi(X_0)| \geq \Delta' N^{1/\beta} \right] 
\]
\[
\leq C N^{1-1/\beta} T \int_{\Delta' N^{1/\beta}}^{+\infty} \pi(|\Psi| > \lambda)d\lambda \leq C_1 N^{1-1/\beta} T \int_{\Delta' N^{1/\beta}}^{+\infty} \frac{d\lambda}{\lambda^\beta} 
\]
\[
= \frac{C_1}{\beta - 1} N^{1-1/\beta} T (\Delta' N^{1/\beta})^{1-\beta} \leq C_2 (\Delta')^{1-\beta} < \varepsilon,
\]
provided that \( \Delta' \) is sufficiently large (recall \( \beta \in (1, 2) \) in this case). Here we have also used the fact that \( P\chi \) has lighter tails than \( \Psi \).

The proof of Lemma 7.5. Since \( \tilde{Z}_t^{(N)}(\Delta) = \tilde{Z}_t^{(N)}(\delta, \Delta) + \tilde{Z}_t^{(N)}(\delta) \), to prove (7.13) it suffices only to show that for any \( \eta > 0 \) there exists \( \delta \in (0, 1) \) such that
\[
\limsup_{N \to +\infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{Z}_t^{(N)}(\delta)| \right]^2 < \eta. \tag{7.15}
\]
Let \( F(\lambda) := \pi(|\Psi| \geq \lambda) \) and \( G(\lambda) := \pi(|t| \geq \lambda) \) By Doob’s inequality we can estimate
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{Z}_t^{(N)}(\delta)| \right]^2 \leq \frac{C}{N^{2/\alpha}} \sum_{n=1}^{[NT]} \mathbb{E} \left[ |Z_{n,N}|^2 1_{\delta(\lambda)} (Z_{n,N}) \right] \tag{7.16}
\]
\[
= C(T + 1) \left\{ N^{1-2/\beta} \mathbb{E} \left[ \Psi^2(X_0) 1_{[0, \delta]} (N^{-1/\beta} \Psi(X_0)) \right] \right. 
\]
\[
\left. + N^{1-2/\alpha} \mathbb{E} \left[ t^2(X_0) 1_{[0, \delta]} (N^{-1/\alpha} t(X_0)) \right] \right\} \tag{7.17}
\]
\[
= -C(T + 1) \left\{ N^{1-2/\beta} \int_0^{\delta N^{1/\beta}} \lambda^2 F(d\lambda) + N^{1-2/\alpha} \int_0^{\delta N^{1/\alpha}} \lambda^2 G(d\lambda) \right\}
\]
Using integration by parts we obtain that the utmost right hand side equals
\[
2C(T + 1) \left\{ N^{1-2/\beta} \int_0^{\delta N^{1/\beta}} \lambda F(\lambda)d\lambda - C(T + 1) N^{1-2/\beta} \lambda^2 F(\lambda) \big|_{\lambda=0}^{\delta N^{1/\beta}} \right\}
\]
\[
N^{1-2/\alpha} \int_0^{\delta N^{1/\alpha}} \lambda G(\lambda)d\lambda - C(T + 1) N^{1-2/\alpha} \lambda^2 G(\lambda) \big|_{\lambda=0}^{\delta N^{1/\alpha}} \right\}
\]
\[
\leq 2C(T + 1) \left\{ N^{1-2/\beta} \int_0^{\delta N^{1/\beta}} \lambda d\lambda + \frac{\delta^{2-\beta}}{\beta + 1} + \frac{\delta^2}{\delta^{2-\alpha}} \right\} \sim C''(\delta^{2-\beta} + \delta^{2-\alpha}),
\]
as \( N \to +\infty \), for some constants \( C, C', C'' > 0 \). Estimate (7.15) then follows, upon a suitable choice of \( \delta \). \( \square \)
The proof of Proposition 7.1. Since \( \Psi = \chi - P \chi \) and \( P \chi \) has lighter tails than \( \Psi \), thanks to (2.12) we conclude the following.

**Lemma 7.6.** Suppose that (2.13) holds. Then, there exists a constant \( C^{(1)}_\phi \) such that

\[
\pi \left[ x : |t(x) - e \circ \chi(x)| \geq \lambda \right] \leq \frac{C^{(1)}_\phi}{\lambda^\gamma}.
\] (7.18)

The exponent \( \gamma \) is the same as in (2.13).

**Proof.** The left hand side of (7.18) can be estimated by

\[
\pi \left[ x : |t(x) - e \circ \Psi(x)| \geq \lambda/2 \right] + \pi \left[ x : |e \circ \Psi(x) - e \circ \chi(x)| \geq \lambda/2 \right].
\] (7.19)

The first term can be estimated directly from (2.13). To estimate the second term recall that \( \Psi = \chi - P \chi \). Since \( P \chi = (I - P)^{-1} P \Psi \) we have \( \|P \chi\|_{L^{p'}(\pi)} < +\infty \), where \( p' \) is the same as in condition (2.12). When \( \beta \leq \alpha \) we have \( |e(\lambda_1) - e(\lambda_2)| \leq e(\lambda_1 - \lambda_2) \) for all \( \lambda_1, \lambda_2 \in \mathbb{R} \). Thus, the second term in (7.19) can be estimated by

\[
\pi \left[ x : |P \chi(x)| \geq (\lambda/2)^{\alpha/\beta} \right] \leq \frac{C\|P \chi\|_{L^{p'}(\pi)}}{\lambda^{\alpha\beta/\beta}}
\] (7.20)

and (7.18) follows for \( \beta' > \beta \).

When, on the other hand \( \beta > \alpha \) the second term in (7.19) can be estimated by

\[
\pi \left[ x : |e' \circ \Psi(x) P \chi(x)| \geq \lambda/2 \right] + \pi \left[ x : |e' \circ \chi(x) P \chi(x)| \geq \lambda/2 \right].
\] (7.21)

To estimate the first term we recall that according to Young’s inequality \( \lambda_1 \lambda_2 \leq \lambda_1^{p'/p} + \lambda_2^{q'/q} \) for any \( \lambda_1, \lambda_2 > 0 \) and \( p, q > 0 \) such that \( p^{-1} + q^{-1} = 1 \). Choose \( p \) such that \( p_1 := p(\beta/\alpha - 1) < \beta/\alpha \) and \( (\beta + \beta')/(2\alpha) > q > \beta/\alpha \). The first term in (7.21) can be estimated by

\[
\pi \left[ x : |\Psi(x)| \geq C_1 \lambda^{1/p_1} \right] + \pi \left[ x : |P \chi(x)| \geq C_2 \lambda^{1/q} \right]
\] (7.22)

for some constants \( C_1, C_2 > 0 \) independent of \( \lambda \). The first term can be estimated by \( C \lambda^{-\beta/p_1} \), while the second by \( C \lambda^{-\beta'/q} \|P \chi\|_{L^{p'}(\pi)} \). These together yield the desired bound on the first term in (7.21). The second term can be dealt with similarly.

Suppose that \( g \) is as in the statement of Proposition 7.1. We can expand \( g(Z_{n+1,N}) \) using Taylor formula, up to the first derivative, around \( z^{(N)}(X_{n+1}) \), where

\[
z^{(N)}(x) := (N^{-1/\beta} \Psi(x), N^{-1/\alpha} \phi_N \circ t(x)),
\]

and obtain that

\[
\sum_{n=0}^{[N]-1} \mathbb{E} \left[ g(Z_{n+1,N}) | \mathcal{G}_n \right] = \sum_{n=0}^{[N]-1} \mathbb{E} \left[ g \left( z^{(N)}(X_{n+1}) \right) | \mathcal{G}_n \right]
\] (7.23)

\[
+ \sum_{n=0}^{[N]-1} \int_0^1 \mathbb{E} \left[ R(X_{n+1}, X_n) \cdot \nabla g \left( z^{(N)}(\lambda) \right) | \mathcal{G}_n \right] d\lambda.
\]

Here

\[
z^{(N)}(\lambda) := \lambda R(X_{n+1}, X_n) + z^{(N)}(X_{n+1})
\]
and

\[ R(x, y) := (N^{-1/\beta}[P \chi(x) - P \chi(y)], -N^{-1/\alpha} P(\phi_{\Delta N} \circ t)(y)). \]

Let \( a_* := \text{dist}(0, \text{supp } g) \). By virtue of (2.12) and (5.19) we conclude that there exists \( C > 0 \) such that for all \( \lambda \in (0, 1] \)

\[
\mathbb{P} \left[ \left| z_0^{(N)}(\lambda) \right| > a_*/2 \right] \leq \frac{C}{N}, \quad \forall N \geq 1. \tag{7.24}
\]

Denote the terms appearing on the right hand side of (7.23) by \( I_N \) and \( II_N \) respectively.

To compute the limit appearing on the right hand side of (7.23) by \( I_N \) and \( II_N \) respectively. 

Let \( \Lambda^{(\Delta)}_{\theta_1, \theta_2}(\lambda) := (\lambda, \phi_\Delta \circ \epsilon(\lambda)) \). It is based on the following lemma.

**Lemma 7.7.** Under the above assumptions

\[
\lim_{N \to +\infty} N \int g \left( z^{(N)}(x) \right) \pi(dx) = \int g \left( \Lambda^{(\Delta)}_{\theta_1, \theta_2}(\lambda) \right) \nu(\beta)(d\lambda). \tag{7.25}
\]

Assuming this result for a moment we show how to calculate the limit \( \lim_{N \to +\infty} I_N \). We can write \( I_N = I_N^{(1)} + I_N^{(2)} \), where

\[
I_N^{(1)} := \sum_{n=1}^{[Nt]} \int g(z^{(N)}(y))P(X_{n-1}, dy) - [Nt] \int g(z^{(N)}(x)) \pi(dx),
\]

\[
I_N^{(2)} := [Nt] \int g(z^{(N)}(x)) \pi(dx).
\]

It can be shown that

\[
\lim_{N \to +\infty} \mathbb{E}|I_N^{(1)}| = 0 \tag{7.26}
\]

precisely as the corresponding term in the proof of (5.8). From the above lemma we obtain therefore that

\[
\lim_{N \to +\infty} \mathbb{E} \left| I_N - t \int g \left( \Lambda^{(\Delta)}_{\theta_1, \theta_2}(\lambda) \right) \nu(\beta)(d\lambda) \right| = 0. \tag{7.27}
\]

We write

\[
\mathbb{E}|II_N| \leq E_N^{(1)} + E_N^{(2)},
\]

where

\[
E_N^{(1)} := (t + 1)\|g\|_{\infty}|\theta_1|N^{1-1/\beta} \int_0^1 \mathbb{E} \left[ \left| P \chi(X_1) - P \chi(X_0) \right|, \left| z_0^{(N)}(\lambda) \right| > a_*/2 \right] d\lambda,
\]

\[
E_N^{(2)} := (t + 1)\|\nabla g\|_{\infty}|\theta_2|N^{1-1/\alpha} \int_0^1 \mathbb{E} \left[ P(\phi_{\Delta N} \circ t)(X_0), \left| z_0^{(N)}(\lambda) \right| > a_*/2 \right] d\lambda.
\]

From Hölder inequality and (2.12) we have

\[
E_N^{(1)} \leq C |\theta_1|N^{1-1/\beta} \left\| P \chi \right\|_{L^{\theta'}(\pi)} \|g\|_{\infty} \sup_{\lambda \in (0, 1]} \mathbb{E}^{1-1/\beta'} \left[ \left| z_0^{(N)}(\lambda) \right| > a_*/2 \right] \leq \frac{C}{N^{1/\beta - 1/\beta'}} \tag{7.26}
\]

for all \( N \geq 1 \) and some \( C > 0 \).
We also have

\[ E_N^{(2)} \leq CN^{1-1/\alpha}\|P(\phi_{N} \circ t)\|_{L^2(\pi)} \sup_{\lambda \in (0,1]} \mathbb{P}^{1/2} \left[ |z^{(N)}_0(\lambda)| > a_*/2 \right] \]

as \( N \to +\infty \). This proves that \( \mathbb{E}|II_N| \to 0 \), as \( N \to +\infty \).

The proof of Lemma 7.7. Let

\[ A_N := [||\Psi(x)|| \geq a_* N^{1/\beta}/2, \text{ or } t(x) \geq a_* N^{1/\alpha}/2] \]

and for some \( \gamma > \kappa > \alpha \) we let

\[ B_N := [||e(\Psi(x)) - t(x)|| \geq N^{1/\kappa}]. \]

Observe that

\[ \pi(A_N) \leq \frac{C}{N} \quad \text{and} \quad \pi(B_N) \leq \frac{C}{N^{\gamma/\kappa}}. \tag{7.28} \]

Let also

\[ z^{(N)}(x) := (N^{-1/\beta}\Psi(x), N^{-1/\alpha}\phi_{N} \circ e(\Psi(x))). \]

Note that \( z^{(N)}(x) = z^{(N)}(x) + r^{(N)}(x) \), where

\[ r^{(N)}(x) := (0, N^{-1/\alpha}[\phi_{N} \circ t(x) - \phi_{N} \circ e(\Psi(x))]). \]

Note that \( z^{(N)}(x) \) lies outside the support of \( g \) on \( A_N^c \). Therefore, the expression under the limit in (7.28) can be written as

\[ N \int g \left( z^{(N)}(x) \right) 1_{A_N} \pi(dx) = I_N + J_N, \]

where

\[ I_N := N \int g \left( z^{(N)}(x) \right) 1_{A_N} 1_{B_N} \pi(dx), \]

\[ J_N := N \int g \left( z^{(N)}(x) \right) 1_{A_N} 1_{B_N^c} \pi(dx). \]

Note that

\[ I_N \leq N\|g\|_\infty \pi(B_N) \leq CN^{1-\gamma/\kappa}\|g\|_\infty \to 0, \]

as \( N \to +\infty \).

Finally, \( J_N = J_N^{(1)} + J_N^{(2)} \), where

\[ J_N^{(1)} := N \int g \left( z^{(N)}(x) \right) 1_{A_N} 1_{B_N^c} \pi(dx), \]

\[ J_N^{(2)} := N \int \int_0^1 \nabla g \left( z^{(N)}(x) \right) \cdot r^{(N)}(x) 1_{A_N} 1_{B_N^c} \pi(dx) d\lambda. \]
Suppose that $\delta > 0$ is arbitrary and $N_0$ is such that for $N \geq N_0$ we have $N^{1/\kappa} < \delta N^{1/\alpha}$. Let
\begin{align*}
C_N^{(1)} := [e(\Psi(x)) < \Delta N^{1/\alpha}, (\Delta + \delta) N^{1/\alpha} \leq t(x)], \\
C_N^{(2)} := [t(x) < \Delta N^{1/\alpha}, (\Delta + \delta) N^{1/\alpha} \leq e(\Psi(x))]
\end{align*}
and $D_N := [N^{-1/\alpha}t(x) \in (\Delta - \delta, \Delta + \delta)]$. We have
\begin{align*}
\mathcal{J}_N^{(2)} &\leq N\|\nabla g\|_\infty \int r^{(N)}(x) 1_{A_N} 1_{B_N} \pi(dx) \\
&\leq C N^{1+1/\kappa-1/\alpha} \|\nabla g\|_\infty \pi(A_N) + N\|\nabla g\|_\infty (\pi(C_N^{(1)}) + \pi(C_N^{(2)}) + \pi(D_N)).
\end{align*}
The first term on the right hand side comes from the estimate $r^{(N)}(x) \leq N^{1/\kappa-1/\alpha}$ that holds on $B_N \cap [e(\Psi(x)), t(x) \leq \Delta_N] \cap [e(\Psi(x)), t(x) > \Delta_N]$. The remaining terms can be estimated by
\begin{align*}
c_\alpha (1 + o(1)) \|\nabla g\|_\infty [(\Delta - \delta)^{-\alpha} - (\Delta + \delta)^{-\alpha}],
\end{align*}
where $o(1) \to 0$, as $N \to +\infty$.

Concerning the term $\mathcal{J}_N^{(1)}$, repeating the above argument we can justify that it is equal, up to a term of order $o(1)$, to
\begin{align}
\tilde{\mathcal{J}}_N^{(1)} := N \int g(\tilde{z}^{(N)}(x)) \pi(dx) 
= N \int_0^{+\infty} \frac{d}{d\lambda} g \left( \Lambda_{\theta_1, \theta_2}^{(N)}(\lambda) \right) 1_{[0 < \lambda < \Psi(x)]} \pi(dx) d\lambda \\
- N \int_{-\infty}^0 \frac{d}{d\lambda} g \left( \Lambda_{\theta_1, \theta_2}^{(N)}(\lambda) \right) 1_{[0 > \lambda > \Psi(x)]} \pi(dx) d\lambda.
\end{align}
Here, $\Lambda_{\theta_1, \theta_2}^{(N)}(\lambda) := \Lambda_{\theta_1, \theta_2}^{(\Delta)}(\lambda N^{-1/\beta})$. Consider the first term on the utmost right hand side of (7.29). Integrating over $x$ we obtain that it equals
\begin{align*}
N \int_0^{+\infty} \frac{d}{d\lambda} g \left( \Lambda_{\theta_1, \theta_2}^{(N)}(\lambda) \right) \pi[\lambda < \Psi(x)] d\lambda.
\end{align*}
Changing variables $\lambda' := \lambda N^{1/\beta}$ and letting $N \to +\infty$ we obtain that the limit equals
\begin{align*}
c_\beta \int_0^{+\infty} \frac{d}{d\lambda} g \left( \Lambda_{\theta_1, \theta_2}^{(\Delta)}(\lambda') \right) \lambda^{-\beta} d\lambda = \int_0^{+\infty} g \left( \Lambda_{\theta_1, \theta_2}^{(\Delta)}(\lambda) \right) \nu_\beta(d\lambda).
\end{align*}
The limit for the second term is computed in the same way and we obtain (7.26). □

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