Aharonov-Bohm instability in fermionic $\mathbb{Z}_2$ gauge theories: topological order and soliton-induced deconfinement

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The notion of gauge invariance plays a predominant role in the most fundamental theories of Nature. However, understanding non-perturbative effects is a challenging open problem with implications in other disciplines, such as condensed matter and quantum information. Here, motivated by the recent experimental progress in the field of quantum simulations with ultracold atoms, and using numerical tools developed in quantum information, we explore the interplay of topology with local and global symmetries in a fermionic Ising gauge theory. We find an Aharonov-Bohm instability, resulting in the spontaneous generation of a topologically-ordered $\pi$-flux phase that can coexist with a symmetry-protected topological phase and, interestingly, intertwine with it by a topological flux-threading phenomenon. This instability leads to deconfined phases in lattice gauge theories without the need of plaquette interactions, identifying a promising route for future quantum-simulation experiments. We show that, at finite chemical potentials, the deconfined phase occurs via the generation of topological solitons in the gauge-field configuration, in which fermions can localise forming $\mathbb{Z}_2$-charged quasi-particles.

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I. INTRODUCTION

Understanding quantum many-body systems is generally a hard problem, as their complexity increases exponentially with the number of constituents. From this large complexity, exotic collective phenomena may arise, as occurs for the so-called spin liquids. These phases of matter evade spontaneous symmetry breaking, and thus long-range order [1], down to the lowest possible temperatures [2–4]. In spite of this, spin liquids can be characterized by a different notion of order: topological order [5]. Systems with topological order have degenerate ground-states, the number of which depends on the underlying topology. Each ground-state is a strongly-correlated state, as witnessed by the multipartite long-range entanglement among the constituents [6, 7]. Besides, the ground-state manifold is separated from the rest of the spectrum by a finite energy gap and, more importantly, only non-local perturbations can act non-trivially within it. It is thus a natural subspace to encode quantum information, and a promising route for fault-tolerant quantum computers [8, 9].

The absence of spontaneous symmetry breaking is not only a footprint of topological phases but, actually, a generic feature of gauge theories. These theories, used to describe strong, weak and electromagnetic interactions [10], have local symmetries that cannot be broken spontaneously [11], evading thus the standard form of ordering. For this reason [12], emergent gauge theories also play an important role in long-wavelength descriptions of non-standard phases of matter, such as high-$T_c$ superconductors [13] and frustrated magnets [14]. Formally, gauge theories can be described through Hamiltonians that commute with an extensive number of local symmetry operators forming a group, the gauge group [15].

Pure gauge theories describe the physics of gauge bosons, the generalization of photons to arbitrary gauge groups [16], and host different phases that can be characterized by the potential that the bosons mediate between test charges [10, 17].
In a deconfined phase, particles generated in pairs of opposite charge can be separated arbitrarily far away with a finite energy cost. Conversely, there can also exist confined phases where this potential energy increases linearly with the distance. The simplest gauge theory on the lattice [18], the so-called $Z_2$ or Ising gauge theory (IGT), already gives rise to a confined-deconfined phase transition without spontaneous symmetry breaking [19]. We note that the very nature of this deconfined phase is the key underlying Kitaev’s toric code [8], a spin-liquid phase allowing for topological quantum error correction and fault-tolerant quantum computing [20]. It is thus important, both from fundamental and applied perspectives, to study the fate of the IGT deconfined phase and, more generally, its full phase diagram as perturbations are introduced [8, 21–24]. Understanding such phase diagrams when the gauge fields interact with matter fields, either bosonic or fermionic, is generally a very hard problem with longstanding open questions [10]. In the simplest case, the deconfined phase of the IGT coupled to dynamical $Z_2$ matter can be understood through the toric code perturbed by both parallel and transverse fields [8, 25–27]. While the corresponding phase diagram is known since the late 70s [25], exchanging $Z_2$ for fermionic matter leads to a much richer scenario, which is only beginning to be explored [28–31].

These connections have fuelled a multi-disciplinary effort towards, not only improving our understanding of these lattice gauge theories (LGTs), but also realizing them experimentally, either in natural or in synthetic quantum materials, such as cold atoms in optical lattices [32]. These are systems where atoms are very dilute and, thus, primarily interact by s-wave scattering. Trapping the atoms by an optical lattice allows to reach the strongly-interacting regime, but the interactions are still limited to be on-site [33]. This fact constitutes a major hurdle when trying to realize lattice gauge theories with ultra-cold atoms [34–37], as they require interactions between all the atoms connected through elementary loops of the lattice (i.e. plaquettes) [38–42]. Aside from this point, the implementation of the tunnelling of matter dressed by the gauge fields is also far from trivial. Floquet engineering in strongly-interacting gases [43–45], and spin-changing collisions in atomic mixtures [46–49], have identified neat directions towards this goal, which are particularly promising in light of recent experiments [50–52]. Since the realization of plaquette terms is currently the major experimental bottleneck, a timely question would be: is it possible to find characteristic features, such as deconfinement and topological order, in lattice gauge theories without plaquette terms?

In this work, we show that this is indeed possible through a novel mechanism: the Aharonov-Bohm instability. This allows for the spontaneous generation of a $\pi$-flux for the Ising fields which, in analogy to the Peierls instability in metals [53], leads to a gap opening for the fermions. We show that magnetic-flux dominated phases appear even in the absence of leading plaquette terms and, moreover, exhibit both topological order and deconfinement for large values of the transverse electric field. By studying a cross-linked lattice connectivity (see Fig. 1), we identify a new avenue for the interplay of symmetry and topology in LGTs, as the Aharonov-Bohm instability can induce a symmetry-protected topological (SPT) phase [54] that coexists with topological order.

The paper is organized as follows. In Sec. II, we introduce the Creutz-Ising ladder, a quasi-1D $Z_2$ LGT where the Ising fields are coupled to spinless fermions hopping in a cross-linked ladder, and summarise our main findings. In Sec. III, we describe the Aharonov-Bohm instability and the emergence of an SPT phase. We study this phenomenon in the presence of quantum fluctuations of the gauge fields, and provide a full discussion of the phase diagram. In Sec. IV, we demonstrate that the cross-linked ladder can be understood as the thin-cylinder limit of a 2D LGT, providing a practical scenario where the ground-state degeneracy is related to the topology of the underlying manifold. In Sec. V, we explore the mechanism of fermionic deconfinement mediated by topological solitons, which can be neatly understood in the limit of large quantum fluctuations. Finally, we present our conclusions and outlook in Sec. VI.

II. THE CREUTZ-ISING LADDER

A. The model

The Creutz ladder, which describes spinless fermions on a cross-linked ladder [55], is a lattice model hosting an SPT phase. The tunnelling of fermions is dressed by a static magnetic field that pierces the ladder, which is described by a gauge-invariant flux that pierces the elementary plaquettes. For a static $\pi$ flux, the ground-state of this model may correspond to either the BDI, or the AII class of topological insulators [56], a free-fermion insulating SPT phase. To go beyond this free-fermion scenario, a natural possibility is to include Hubbard-type interactions [57], which leads to correlated SPT phases with interesting connections to relativistic quantum field theories of self-interacting fermions [58, 59].

We hereby follow a different, and yet unexplored, route: we upgrade the background magnetic fields to a $Z_2$ LGT by introducing Ising fields on the links (see Fig. 1). This IGT is
described by the following Hamiltonian
\[ H_{\text{CI}}(t, \Delta, h) = \sum_{i,j} \left( -i c_i^\dagger \sigma_{(i,j)} c_j - h \sigma_{(i,j)} \right) + \frac{\Delta}{2} \sum_i s_i c_i^\dagger c_i, \]

where \( c_i^\dagger (c_i) \) creates (annihilates) a fermion at site \( i = (i_1, i_2) \). Here, \( i_2 \in Z_2 = \{0,1\} \) labels the lower and upper legs of the ladder, and \( i_1 \in Z_{N_{\text{lat}}} = \{0, \cdots, N_{\text{lat}}-1\} \) labels the sites of each of these legs. At the horizontal or diagonal links \( (i,j) \) adjacent to \( i \), we introduce the Pauli matrices \( \sigma_{(i,j)}^{x,y,z} \) as the corresponding Ising link operators. The first term of Eq. (1) describes the tunnelling of fermions dressed by the Ising gauge fields, which has tunnelling strength \( t \). The second term introduces an electric transverse field of strength \( h \). Finally, the third term describes an energy imbalance of magnitude \( \Delta \) for the fermions sitting on the upper \( s_i = +1 \) or lower leg \( s_i = -1 \).

The above Hamiltonian (1) displays a local \( Z_2 \) symmetry \( [H_{\text{CI}}, G_i] = 0, \forall i \in Z_{N_{\text{lat}}} \times Z_2 \), with the generators
\[ G_{i,x} = (-1)^{\delta_{i+y}} \prod_{(i,j) \in i \cdot c_l} \sigma_{(i,j)}^x, \quad G_{i,y} = (-1)^{\delta_{i+x}} \prod_{(i,j) \in i \cdot c_l} \sigma_{(i,j)}^y, \quad G_{i,z} = (-1)^{\delta_{i+x+y}} \prod_{(i,j) \in i \cdot c_l} \sigma_{(i,j)}^z, \]

displayed in Fig. 1. In addition, we also depict in this figure the smallest Wegner-Wilson loops, corresponding to gauge-invariant magnetic fluxes across two types of trapezia
\[ B_{i \cdot c_l} = \prod_{(i,j) \in i \cdot c_l} \sigma_{(i,j)}^z, \quad B_{i \cdot \triangle} = \prod_{(i,j) \in i \cdot \triangle} \sigma_{(i,j)}^z. \]

We note that, in the standard formulation of IGTs [60], one also introduces an additional magnetic-flux term
\[ \tilde{H}_{\text{CI}}(t, \Delta, h, J) = H_{\text{CI}}(t, \Delta, h) - J \sum_i \left( B_{i \cdot c_l} + B_{i \cdot \triangle} \right), \]
such that the magnetic plaquette coupling \( J \) competes with the electric transverse field \( h \). In the \( (2+1) \) pure IGT, this competition leads to a quantum phase transition between deconfined \( h/J < h/J_\text{c} \) and confined \( h/J > h/J_\text{c} \) phases [19]. These phases are not characterised by a local order parameter, but instead display Wegner-Wilson loops that scale either with the perimeter \( (h/J < h/J_\text{c}) \) or with the encircled area \( (h/J < h/J_\text{c}) \) of a closed loop, i.e. perimeter or area law.

The \( Z_2 \) symmetry generators (2) can be used to define different charge sectors of the Hilbert space, as the eigenstates of the Hamiltonian \( |\psi\rangle \) must also fulfill
\[ G_i |\psi\rangle = (-1)^{q_i} |\psi\rangle, \]

where \( q_i \in \{0,1\} \) are the so-called static \( Z_2 \) charges. Typically, one considers the vacuum/even sector \( \{q_i\} = \{0,0, \cdots, 0\} \), introducing a few static charges on top of it. For instance, \( \{q_i\} = \{\delta_{i,(x,0)}, \delta_{i,(y,L+L,0)}\} \) describes a pair of static \( Z_2 \) charges separated by a distance \( L \). In the \( (2+1) \) pure IGT [10], these test charges are subjected to a potential \( V(L) = E_{\text{gs}}(L) - E_{\text{gs}}(0) \) that either remains constant in the deconfined phase \( V(L) \propto V_0 \), or increases with the distance in the confined phase \( V(L) \propto L \). We note that \( (2+1) \) is the lower critical dimension, since the \( (1+1) \) IGT can only display an area law [10], hosting solely a confined phase. In the presence of fermionic matter, rather than through the aforementioned area law, the \( (1+1) \) confined phase can be characterised through the appearance of chargeless bound dimers [61].

In this work, we argue that fermionic \( Z_2 \) gauge theories in quasi-1D geometries, such as the ladder structure of Fig. 1, lead to a much richer landscape in comparison to the strict 1D limit. Let us summarise our main findings.

\section{B. Summary of our results}

In the pure gauge sector, which is obtained from Eq. (4) by setting \( t = \Delta = 0 \), we show that \( \tilde{H}_{\text{CI}}(0,0,h,J) \) still hosts a quantum phase transition at a critical \( h/J_\text{c} \), separating confined and deconfined phases. We characterise this phase transition quantitatively using matrix-product-state (MPS) numerical simulations [62], which allow us to extract the critical behaviour of the Ising magnetic fluxes, and their susceptibilities. We note that this is a generic feature of IGTs in the particular charge sectors considered in this work. To the best of our knowledge, this study provides the first quantitative analysis of such deconfinement mechanism.

Moreover, as a result of the cross-linked geometry, we also show that the matter sector may lie in an SPT phase characterised by a non-zero topological invariant. From the perspective of the fermions, the corresponding topological edge states can be understood as domain-wall fermions [63, 64] with the novelty that, instead of requiring fine tuning to incorporate chiral symmetry on the lattice, they are spontaneously generated by the Ising-matter coupling. The fact that a plaquette term \( J \neq 0 \) is not required to host this exotic behaviour is particularly interesting in light of current developments in cold-atom quantum simulations. Interestingly, the interplay between geometry and gauge-invariant interactions allows us to obtain a topological phase for gauge fields without introducing four-body plaquette terms, simplifying enormously the experimental implementation. This point is important since the main building blocks of the model have already been realized in cold-atom experiments [50]. Therefore, future quantum simulations of this fermionic IGT will be capable of testing the non-trivial equilibrium properties described in this work.
III. AHARONOV-BOHM INSTABILITY

We start by exploring the limit of zero electric-field strength \( \hbar = 0 \). Here, the Ising fields have vanishing quantum fluctuations, and the fermions tunnel in a classical \( \mathbb{Z}_2 \) background \( \{ \sigma_{ij} \} \), where \( \sigma_{ij} = \pm 1 \) are the eigenvalues of the \( \sigma^z \)-link operator. In this limit, there are only two translationally-invariant ground-states corresponding to the 0- or \( \pi \)-flux configurations, namely \( \langle B_{i\sigma} \rangle = \langle B_{i\pi+} \rangle = \pm 1 \). The fermions minimise their energy in these backgrounds by partially filling the corresponding energy bands \( E^0_{\pm}(k) \) or \( E^\pi_{\pm}(k) \).

For vanishing imbalance \( \Delta = 0 \), and considering periodic boundary conditions, these bands read

\[
e^0_{\pm}(k) = -2t \cos k \pm 2|\cos k|, \quad e^\pi_{\pm}(k) = \pm 2t,
\]

where \( k \in [-\pi, \pi] \). As depicted in Fig. 2(a), for magnetic flux \( \Phi_B = 0 \), the half-filled ground-state corresponds to a gapless state. Conversely, for \( \Phi_B = \pi \) flux (Fig. 2(b)), the band structure consists of two flat bands, such that the half-filled ground-state is a single gapped state with a fully-occupied lowest band. By direct inspection of Fig. 2, it is apparent that the \( \pi \)-flux case is energetically favourable. This is indeed the case, as one finds that \( E^\pi_{\pm} = -2\xi N_{\pi} < -\left(4t/\pi\right)N_{\pi} = E^0_{\pm} \). Recalling the Peierls instability in 1D metals [53], where the underlying lattice adopts a dimerized configuration and a gap is opened in the metallic band; here, it is the Ising fields which adopt a \( \pi \)-flux configuration leading to a gap opening in the fermionic sector. This spontaneous generation of a \( \pi \)-flux is in accordance with Lieb’s result for bipartite lattices [68] but, in contrast to the square lattice [28–30], it does not lead to a semi-metallic phase with emergent Dirac fermions [69]. In this case, it is an insulator with complete band flattening caused by destructive Aharonov-Bohm interference at \( \Phi_B = \pi \) [70]. Due to the remarkable similarities with the Peierls effect, we call this effect the Aharonov-Bohm instability.

This flux instability is actually generic for any imbalance \( \Delta > 0 \), in spite of the fact that the bands gain curvature. In this case, the corresponding ground-state energies are

\[
E^0_{gs} = -\frac{4t}{\pi} \left(1 + \frac{\Delta^2}{4t^2}\right) E(0) N_{\pi},
E^\pi_{gs} = -\frac{2t}{\pi} \left|1 + \xi\right| E(\pi) + \left|1 - \xi\right| E(\pi) N_{\pi},
\]

where we have introduced the parameters \( \xi = \Delta/4t, \theta_0 = 1/(1 + \xi^2), \theta_\pi = 4\xi/(1 + \xi^2)^2 \), and \( \theta_0 = -4\xi/(1 - \xi^2)^2 \). Additionally, we have used the complete elliptic integral of the second kind \( E(x) = \int_0^{\pi/2} dx \left(1 - \sin^2 x\right)^{1/2} \). Once again, one can readily confirm that \( E^\pi_{gs} < E^0_{gs} \), such that it is energetically favourable for the ground-state to lie in the \( \pi \)-flux phase, which is generally gapped except for \( \xi = 1 \), namely \( \Delta = 4t \).

A. Emerging Wilson fermions and SPT phases

By exploring the imbalanced case at long wavelengths, we can understand the insulating \( \pi \)-flux phase from a different perspective. Rather than the massless Dirac fermions that emerge in the square-lattice \( \pi \)-flux phase [28–30], we get the following long-wavelength dispersion around \( k_\pm = \pm \pi/2 \)

\[
e^\pi_{\pm}(k_\pm + p) \approx \pm \sqrt{(m_\pm^2)^2 + (cp)^2}, \quad m_\pm = (\xi \pm 1)/2t
\]

where \( c = 2t \) is the propagation speed, and \( m_\pm \) are two mass parameters. Except for \( \xi = 1 \), we get two massive relativistic fermions characterised by a different mass, which are known as Wilson fermions in a LGT context [71].

The fact that the Wilson masses are different \( m_+ \neq m_- \) turns out to be crucial in connection to the spontaneous generation of an SPT phase. The Chern-Simons form \( Q_{\pi} = \frac{1}{8\pi} \langle e^\pi_{\pm}(k) \rangle \partial_k [e^\pi_{\pm}(k)] dk \) [72], leads to a Chern-Simons invariant after integrating over all occupied quasi-momenta

\[
CS_1^\pi = \int_{-\pi}^{\pi} Q_{\pi} = \frac{1}{4} \left[ \text{sgn}(m_+) - \text{sgn}(m_-) \right].
\]

One can define a gauge-invariant Wilson loop \( W_1^\pi = e^{i2\pi CS_1^\pi} \) that detects the non-trivial topology when \( W_1^\pi = -1 \). This occurs when the pair of Wilson fermions have masses with opposite signs. Accordingly, if \( |\xi| < 1 \) (i.e. \( -4t < \Delta < 4t \)), the topological Wilson loop is non trivial \( W_1^\pi = -1 \), and the emerging \( \pi \)-flux phase is an insulating SPT phase.

This result draws a further analogy between the Peierls and Aharonov-Bohm instabilities. In the former, when the instability is triggered by an electron-lattice coupling that modulates the tunneling [73, 74], one of the dimerization patterns of the lattice leads to a non-zero topological invariant and an SPT phase [75]. In our case, there is no dimerization due to SSB since the local \( \mathbb{Z}_2 \) symmetry cannot be spontaneously broken. However, there are two gauge-invariant fluxes at \( \hbar = 0 \), and it is only the \( \pi \)-flux configuration that leads to a non-zero topological invariant when \( |\Delta| < 4t \). We can thus conclude that, as a consequence of the Aharonov-Bohm instability, the fermions intertwine with the Ising fields in such a way that a gap is opened in the fermion sector with non-trivial topology.
B. Gauge-matter edge states and fractionalisation

So far, our discussion has revolved around the zero electric field limit \( h = 0 \), and assumed periodic boundary conditions. From now onwards, we abandon this limit and explore the effect of quantum fluctuations in open Creutz-Ising ladders (i.e. Dirichlet/hard-wall boundary conditions). Due to the bulk-boundary correspondence, when the bulk of the spontaneously-generated \( \pi \)-flux phase is characterised by a non-zero topological invariant \( \langle \rho \rangle \), one expects that edge states will appear at the boundaries of the ladder. In the context of LGTs, these states are lower-dimensional domain-wall fermions [65] with the key difference that, in our case, they are generated via the Aharonov-Bohm instability.

In Fig. 3, we show the real-space configuration of both matter and Ising fields. We use a MPS-based algorithm [76] of the density-matrix renormalization group (DMRG) [77], setting the bond dimension to \( D = 200 \) for a ladder of leg length \( N = 20 \) at half-filling, and introduce quantum fluctuations through \( h = 0.1t \). In these figures, we display the Ising flux

\[
\Phi_B^i = \arccos\left(\frac{1}{2}\langle B_{\ell C} \rangle + \frac{1}{2}\langle B_{\ell} \rangle \right)
\]

averaged over the two trapezoidal plaquettes, and the normal-ordered fermionic occupation

\[
\langle \hat{n}_i \rangle = \langle c_i^+ c_i \rangle - \rho,
\]

where \( \rho = 1/2 \) at half-filling. As shown in Fig. 3(a), due to the quantum fluctuations, the Ising flux is no longer fixed at \( \pi \). As the transverse field increases, \( \Phi_B \rightarrow \pi/2 \), which amounts to an electric-field dominated phase with a vanishing expectation values of the magnetic plaquettes \( \langle B_{\ell C} \rangle = \langle B_\ell \rangle = 0 \).

In Fig. 3(b), we show that the corresponding fermion distribution is not translationally invariant, but displays an excess/deficit of charge around the boundaries of the ladder. This real-space distribution is consistent with the existence of two topological edge states in the SPT phase, one of them being filled while the other one remains empty at half-filling. We note that, in analogy with the phenomenon of charge-fractionalization put forth by Jackiw and Rebbi [78], when these zero modes are occupie/empty, an excess/deficit of 1/2 fermion is formed around the boundaries. This fractionalization can be readily observed in Fig. 3(b), where we also show that the excess/deficit of charge with respect to the bulk density on each leg of the ladder \( \rho_\ell \) follows

\[
\langle \hat{n}_i \rangle - \rho_\ell = \pm \frac{1}{4\xi}\text{sech}\left(\frac{j - j_0}{\xi}\right).
\]

Here, \( j = 2i_1 \) (resp. \( j = 2i_2 \)) is the sublattice index for the lower (resp. upper) leg of the ladder, with \( j_0 = L \) (resp. \( j_0 = 0 \)), and \( \xi \) is the localization length of the corresponding edge state. This behaviour is a universal feature of zero modes in relativistic quantum field theories and condensed-matter models [79], and we show that it also holds for LGTs.

The presence of these edge states points towards the robustness of the SPT \( \pi \)-flux phase described in the previous section, which thus persists as one introduces non-zero quantum fluctuations. Therefore, the SPT phase should extend to a larger region in parameter space. Let us also highlight that, by looking at the enlarged fluctuations of the Ising flux close to the boundaries (Fig. 3(a)), one realises that the edge states are indeed composite objects where both the matter and gauge degrees of freedom are intertwined. We will unveil a very interesting consequence of this intertwining below.

C. Topological phase transitions

We explore the extent of this SPT phase in parameter space \( (\Delta/t, h/t) \). The topological invariant \( \langle \rho \rangle \) is related to the Berry phase \( \gamma \) acquired by the ground-state \( |E_\text{gs}(\theta)\rangle \) along an adia-
batic Hamiltonian cycle \( H(\theta) = H(\theta + 2\pi) \) [80], namely

\[
\gamma = i \int_0^{2\pi} d\theta \langle E_{gs}(\theta) | \partial_\theta | E_{gs}(\theta) \rangle
\]  

(13)

For non-interacting fermions in a classical \( \mathbb{Z}_2 \) background, one can use quasi-momentum as the adiabatic parameter \( \theta = k \), such that \( \gamma = 2\pi CS^2 \) [9]. However, as the electric field is switched on, the Ising fields fluctuate quantum-mechanically mediating interactions between the fermions, and the quasi-momentum is no longer an appropriate adiabatic parameter. Building on ideas of quantized charge pumping [81] and Hall conduction [82], one can obtain a many-body Berry phase by twisting the tunnelling \( t \rightarrow te^{i\theta} \) that connects the boundaries, and integrating over the twisting angle \( \theta \). Interestingly, this concept can be generalized to systems with hard-wall boundary conditions [83], since the twisting can actually be placed locally in any link that respects the underlying symmetry that protects the SPT phase, e.g. inversion symmetry in this case.

We have computed the many-body Berry phase (13) for an infinite Creutz-Ising ladder using the iDMRG algorithm with bond dimension \( D = 200 \) [76], yielding the phase diagram of Fig. 4(a). The SPT phase is characterized by \( \gamma = \pi \) in the red region, and is separated from a trivial band insulator (TBI) with \( \gamma = 0 \) in the blue region by a critical line that reaches \( \Delta \approx 4t \) for \( h = 0 \). This corroborates our previous interpretation (9) in terms of the mass-inversion point of the emergent Wilson fermions at \( \xi = \Delta/4t = 1 \). As the electric field \( h \) increases, this inversion point flows towards smaller values of the imbalance \( \Delta \), which can be interpreted as a renormalization of the Wilson masses due to the interactions mediated by the gauge fields.

In this figure, we also show that the numerical critical line can be fitted to an exponential \( \xi_c = \xi_0 \exp\{-h/h_z\} \), where \( h_z \) is a fitting parameter, and \( \xi_0 = 1 \) is fixed by setting the critical point at \( \Delta/4t = 1 \) for \( h = 0 \). Let us remark that this exponential behaviour is consistent with the claim that the SPT phase and, in general, the Aharonov-Bohm instability and the magnetic-field dominated phase, persists to arbitrarily-large values of the transverse \( h \) when the imbalance is \( \Delta = 0 \). For zero imbalance, the appearance of the flat bands described previously endows the SPT phase with an intrinsic robustness to the interactions mediated by the fluctuating gauge field.

Let us note that the critical line describes first-order topological phase transitions, as can be appreciated in Fig. 4(b), where we display the derivative of the ground-state energy \( \partial_\Delta E_{gs} \) for three different values of electric field strength (dotted lines of Fig. 4(a)). The discontinuous jumps account for the first-order nature of the phase transitions. A similar discontinuity can be observed in the average magnetic flux (10), evaluated at the bulk of the ladder (Fig. 4(c)).

IV. TOPOLOGY FROM CONNECTIVITY

In this section, we provide quantitative evidence supporting the equivalence between the cross-linked ladder and a cylindrical geometry. This allows us to interpret our model as the thin-cylinder limit of a 2D LGT, and to identify various topological properties such as the ground-state degeneracy or the presence of topological order. We also show that the intertwining of the matter and gauge fields in the SPT phase leads to a topological flux threading of the cylinder, and give further arguments for its survival to arbitrary transverse fields.

A. The effective Creutz-Ising cylinder

Let us, momentarily, switch off the gauge-matter coupling and focus on the pure gauge theory \( H_{CI}(0, 0, h, J) \) in Eq. (4). For \( h/J \rightarrow 0 \), and for the sake of the argument, we assume that \( g \) is the single ground-state in a magnetic-field dominated phase with zero flux per plaquette \( \Phi_B = 0 \). As shown in Fig. 5(a), flipping the Ising fields via \( \sigma^z_{i,j} \) creates a pair of \( \Phi_B = \pi \) excitations at neighbouring plaquettes, which can be separated at the expense of flipping additional Ising fields along a path \( \Gamma_f \). By extending this path towards the boundaries of the ladder, the \( \pi \) fluxes get expelled, and one recovers
a state $|\tilde{g}\rangle = D_{t}|g\rangle$ with vanishing flux a $\Phi_B = 0$, where

$$D_{t} = \prod_{(i,j) \in \Gamma_{t}} \sigma_{i,j}^z$$

(14)
is the so-called Dirac string. Similarly to those in Eq. (3), one can define a 4-point correlator involving Ising fields

$$B_{t} = \prod_{(i,j) \in \Gamma_{t}} \sigma_{i,j}^z,$$  

(15)

where $\Gamma_{t}$ is a vertical path that connects the two legs of the ladder. As demonstrated below, $\Gamma_{t}$ is equivalent to a path that wraps around a non-trivial cycle of a cylinder, such that the correlator (15) can be interpreted as a Wegner-Wilson loop operator measuring the flux threading the hole of the cylinder.

In the lowest panel of Fig. 5(a), one can see that the Dirac string shares only one common link with the 4-point correlator, and thus anti-commutes $\{D_{t}, B_{t}\} = 0$. Conversely, the Dirac string shares a pair of links with the trapezoidal plaquettes (3), and thus commutes with the Hamiltonian $[H_{C}, (0,0,h,J), D_{t}] = 0$. As a consequence, if we assume that $B_{t}|g\rangle = +|g\rangle$, we immediately obtain $B_{t}|\tilde{g}\rangle = -|\tilde{g}\rangle$, whereas $H_{C}(0,0,h,J)|\tilde{g}\rangle = E_{gs}|\tilde{g}\rangle$, $H_{C}(0,0,h,J)|g\rangle = E_{gs}|g\rangle$. Accordingly, the two states are orthogonal and have the same energy $E_{gs}$. Our original assumption of a single ground-state thus needs to be dropped in favour of the existence of a two dimensional ground-state manifold spanned by $\{|g\rangle, |\tilde{g}\rangle\}$.

The presence of such ground-state manifold can be a manifestation of topological order. As outlined in the introduction, spin-liquid states with topological order can be characterised by a ground-state degeneracy that depends on the genus of the manifold in which they are defined [5]. The deconfined phase by a ground-state degeneracy that depends on the genus of the spin-liquid states with topological order can be characterised.

Figure 6. $\mathbb{Z}_2$ magnetic fluxes in the infinite Creutz-Ising cylinder:

(a) $\mathbb{Z}_2$-flux susceptibility as a function of the electric field strength. We compare the pure-gauge case, setting $t = 0$ and $J = 1$ (red circles) with the case in which the gauge fields interact with the fermionic matter, setting $t = 1$, $\Delta = 5t$ and $J = 0$ (blue circles). In the first case, the susceptibility shows a diverging peak, signalling a quantum phase transition between deconfined and confined phases. In the presence of dynamical matter, however, there is no apparent divergence hinting at the absence of such a transition. (b) $\mathbb{Z}_2$ inner flux piercing the cylinder $\Phi_B$. In the pure-gauge case (red circles), this inner flux displays non-analytical behaviour across a critical transverse field $h_c$. For dynamical matter (blue circles), this tendency is not abrupt, and the inner flux only attains the value $\Phi_B \approx \pi/2$ asymptotically without any non-analyticity. (c) Topological flux threading relating the existence of edge states to a trapped vison inside the cylinder. (d) Similarly to the Berry phase $\gamma$, the inner flux $\Phi_B$ changes from 0 to $\pi$ as one crosses the critical point separating TBI and SPT. The deviations from those precise are due to quantum fluctuations.

B. Magnetic fluxes and Ising susceptibility

As announced in Sec II B, the cross-linked ladder geometry allows for a confinement-deconfinement phase transition akin to the $(2+1)$ IGT [19]. This phase transition can be probed by the $\mathbb{Z}_2$-flux susceptibility $\chi_{\text{bulk}} = \partial^2 \Phi_B/\partial h$, evaluated through the magnetic flux (10) at the bulk of the ladder. We use iDMRG to obtain the approximation of the ground-state of the system defined on an infinitely-long ladder as an MPS. The system is thus equivalent to a $2 \times \infty$ cylinder: the thin-cylinder limit of a $2+1$ fermionic IGT. The maximum bond dimension we have used is $D = 200$, testing that it is sufficient to achieve a good convergence. As clearly evidenced by the iDMRG results of Fig. 6(a)(red circles) , there is a peak in the $\mathbb{Z}_2$ susceptibility, whose height actually diverges with the ladder size at the critical coupling $h/J_c$. In Fig. 6(b), we plot the value of the inner flux $\Phi_B = \arccos\left(\langle B_{t} \rangle \right)$ through the hole of the effective cylinder as a function of the transverse field $h$ (red circles). The plot shows that, in the $h/J \to 0$ limit, the...
The cylinder has zero inner flux $\Phi_{\text{cycl}} = 0$, and the ground-state is $|g\rangle$ as anticipated. By increasing the transverse field, quantum fluctuations change the inner flux, which acts as a non-local order parameter for the transition to the confined phase displaying a non-analytical behaviour as we cross $h/J_z$.

Let us now switch on the gauge-matter coupling, and see how this picture gets modified by the inclusion of dynamical fermions governed by $H_{\text{CI}}(t, \Delta, h, 0)$ in Eq. (4). First of all, we find that there is no peak in the $Z_2$ susceptibility for any value of $h$ (see blue circles of Fig. 6(a)), which suggests the absence of a phase transition. Furthermore, we plot the value of the inner flux in Fig. 6(b), which again attains the value $\Phi_{\text{cycl}} = 0$ in the $h/t \to 0$ limit (blue circles). The zero inner-flux state can be understood as the generalization of the $|g\rangle$ ground-state to a situation that encompasses dynamical fermions intertwining with the Ising fields. As neatly depicted, the $Z_2$ flux changes smoothly from $\Phi_{\text{cycl}} = 0 \to \pi/2$ as the electric field strength is increased. Therefore, the absence of non-analyticities again suggests that there is a single magnetic-field dominated (i.e. deconfined) phase for arbitrary transverse fields.

C. Trapped Visons from topological flux threading

As discussed above for the pure-gauge limit, topological order becomes manifest through the two-fold ground-state degeneracy $\{|g\rangle, |g\rangle\}$, and the absence/presence of a trapped vison. Yet, in the previous section (see Fig. 6(b)), we have only found the dynamical-fermion generalisation of $|g\rangle$. As described in Sec. III, the Aharonov-Bohm instability can lead to an SPT ground-state or to a trivial band insulator (see Fig. 4(a)). We now discuss the difference of the intertwining of the gauge and matter fields in these two cases, and unveil a very interesting interplay between the topological degeneracy and the existence of edge states in the SPT phase.

To understand this interplay, let us recall Laughlin’s argument for the quantum Hall effect [87], which states that a single charge is transferred between the edges of a quantum Hall cylinder when a magnetic flux quantum is threaded through its hole. In the Creutz-Ising ladder, one can move from the TBI onto the SPT ground-state by gradually decreasing the imbalance $\Delta$. As the system crosses the critical point, topological edge states will appear at the boundaries of the ladder, which can be seen as the result of charge being transferred from the bulk to the edges (see Fig. 6(c)). In contrast to Laughlin’s pumping, where it is the external variation of the flux which leads to charge transport, here it is the transition into a topological phase and the associated charge transfer which should generate a non-vanishing $Z_2$ inner flux. In Fig. 6(d), we confirm this behavior, and show that the $Z_2$ inner flux changes from $\Phi_{\text{cycl}} \approx 0$ (TBI) to $\Phi_{\text{cycl}} \approx \pi$ (SPT) at fixed $h = 0.01t$.

This effect can be understood as a topological flux threading, where the existence of edge states gets intertwined with the trapping of a vison through the cylinder’s hole, giving access to the dynamical-fermion generalization of $|g\rangle$. We note that this phenomenon cannot be observed with a background static field, such as the magnetic field of the quantum Hall effect, but is instead characteristic of LGTs with fermionic matter, unveiling an interesting interplay between the Berry phase and the inner $Z_2$ flux. This offers a neat alternative to the numerical demonstration of the two-fold ground-state degeneracy, typically hindered by finite-size effects. As the quantum fluctuations are increased by raising $h$, we see that one tends smoothly to the electric-field dominated phase $\Phi_{\text{cycl}} = \pi/2$, but the first-order topological phase transition between SPT and TBI, and the intertwining of the edge and vison states is still captured by the discontinuity of the inner flux.

D. Topological entanglement entropy

As argued in the previous section, the ground-state degeneracy and the flux threading are topological phenomena related to the underlying cylindrical manifold. This raises the possibility that this quasi-1D IGT (4) displays topological order, as occurs for Kitaev’s toric code [8]. In recent years, quantum-information tools that quantify the entanglement of the ground-state have turned out to be extremely useful to characterise various many-body properties [88]. In particular, the Von Neumann entanglement entropy for a bi-partition of the ground-state $|g\rangle$ into two blocks $A-B$ of equal sizes is defined as $S(\rho_A) = -\text{Tr}[\rho_A \log \rho_A]$, where $\rho_A = \text{Tr}_B(|g\rangle\langle g|)$ is the reduced density matrix. For a $(2+1)$ topologically-ordered ground-state, this entanglement entropy scales as

$$S(\rho_s) = \alpha |\partial A| - \gamma,$$

where $|\partial A|$ is the number of sites that belong to the boundary separating the $A-B$ regions, $\alpha$ is a constant that characterises this entanglement area law, while $\gamma$ is a universal sub-leading constant that quantifies the topological corrections [63, 64]. Although in a gapped phase the value of $\gamma$ is constant, it has already been observed that close to a QPT there are strong finite size effects, and from numerical simulation it is very hard to extract a reliable determination of it [89]. Furthermore, the value of $\gamma$ for bipartitions that are not contractible to a point depends on both the choice of the bipartition and the choice of the ground-state in the ground-state manifold [86].
In order to reliably extract $\chi$, we turn to study finite-size Creutz-Ising ladders, interpreted through the mapping to the thin cylinder of length $2 \times N_s$ of Fig. 5(b). We consider a bipartition separating the two legs, such that $|\partial A| = N_s$. In the effective manifold, this corresponds to a longitudinal bipartition of the cylinder (see Fig. 5(b)), such that the entropy (16) should scale with the length of the cylinder. In this finite-size regime, the ground-state is an eigenstate of $D$, and thus has minimal entropy. $\chi$ should thus get saturated at its maximum value, namely $\chi \approx \log(2)$. Our numerical analysis is limited to short ladders, as the particular bipartition limits the efficiency of the MPS routines. In Fig. 7(a), we plot the entanglement entropy as a function of the ladder length for two points deep in the SPT and the TBI. The fit of the data allows to confirm that $\chi \approx \log 2$ in both the SPT and TBI phases. After repeating the same analysis for several values of the imbalance, we obtain Fig. 7(b). In this figure, $\chi$ is constantly very close to the expected $\log(2)$ within both SPT and TBI phases. It only departs significantly from that value close to the phase transition, where the larger correlation length increases the finite-size effects [89].

The presence of a non-zero topological entropy is a further indication that the complete gauge-matter system is topologically ordered both in the SPT phase and in the TBI. Due to the lack of signature of criticality in our numerical results about the fluxes threading the cylinder and the bulk susceptibility of Figs. 6(a) and (b) (blue circles), we are thus confident that the topologically-ordered phase survives for large values of the electric-field.

**V. $\mathbb{Z}_2$ FERMIONIC DECONFINEMENT**

In this section, we argue that the topologically-ordered $\pi$-flux phase described above shows fermionic deconfinement for any value of the transverse field. We first introduce the notion of gauge frustration, and how it generates deconfined topological defects when the system is doped above or below half filling. We then quantitatively characterise the absence of confinement using static charges, and we compare it with the more standard case involving string breaking.

**A. Gauge frustration and topological defects**

Paralleling the situation in the standard (2+1) IGT [12], the existence of topological order in the Creutz-Ising ladder suggests that the ground-state lies in a deconfined phase despite the lack of plaquette interactions $J = 0$. As outlined above, the absence of criticality for large $h$ suggests that this deconfinement may survive to arbitrarily-large electric-field strengths, which contrasts to the standard IGT [10].

Let us start by discussing the half-filled regime of the Hamiltonian (4) for $h \gg t$ and $\Delta = 0$. In this case, the link...
Ising fields minimise their energy for \( |+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \). However, the presence of fermions can frustrate some links in order to satisfy the constraints (5), forcing the Ising fields to lie in \( |-\rangle = (|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2} \). We call this gauge frustration, namely the impossibility of simultaneously minimising all the individual Hamiltonian terms due to the Gauss constraint.

In contrast to pure gauge theories, this type of frustration can occur in the even sector \( q_i = 0 \), as some of the sites might be occupied by a dynamical fermion (see Fig. 8(a)). By plotting only the frustrated links/bonds, one understands that the ground-state corresponds to a partial covering of the ladder with a single restriction: each site can be touched by one bond at most (see Fig. 8(b)). This is precisely the definition of a dimer, with the peculiarity that dimer models typically consider the complete covering of the lattice [90, 91], whereas in our half-filled case the ground-state will be a linear superposition of all partial dimer coverings. We note that, in the absence of dynamical fermions, the original connection of an IGT to a quantum dimer model in the large-\( h \) limit was put forth by R. Moessner et al. by introducing a static background \( Z_2 \) charge \( q_i = 1 \) at every site (i.e. odd charge sector) [92]. In our case, the dynamical fermions allow for this dimer limit even in the absence of static charges, albeit only with partial coverings.

So far, the imbalance has been fixed to zero. If we now allow for \( \Delta > 0 \), the fermions will preferably occupy the lower leg, such that only two degenerate coverings are relevant for the large-\( h \) ground-state (see Fig. 8(c)). These two coverings, which we label as A and B, are related by a simple lattice translation and, yet, they are essential for the deconfinement of the Creutz-Ising ladder. If one adds a pair of fermions at a distance \( L \) above half-filling, these must be accommodated in the upper leg, such that the Ising fields change to comply with the Gauss constraints. As depicted in Fig. 8(d), if we insist on maintaining one of the dimer coverings, say A, an electric field string must connect the fermions in the upper leg, such that the energy is \( E(L) - E_0 = hL \), and the charges are confined \( V(L) \propto L \). This is the standard situation in the (2+1) IGT in the even sector [10]. In our case, however, the two-fold coverings allow for a different situation: one can interpolate between the A and B configurations, such that \( E(L) - E_0 = 3h \), and the charges are deconfined \( V(L) \propto V_0 \). The charges, which are no longer confined in pairs but localised at the topological soliton that interpolates between A and B, carry a non-zero \( Z_2 \) charge, which is at the very heart of the notion of deconfinement.

To assess the validity of these arguments, and extend them beyond the \( h \gg 1 \) limit, we explore the Creutz-Ising ladder for finite doping using the MPS numerics. In Fig. 9(a), we depict the occupation \( \pi_i = \langle n_{i,1}\rangle \) summed over the pair of sites in the upper and lower legs. This density displays an inversion-symmetric distribution of the extra doped charges, which are localised around distant centres maximising their corresponding distances. In Fig. 9(b), we represent the integrated \( Z_2 \) charge from the left boundary of the ladder \( Q_i = \sum_{j<i} \pi_j \). Comparing this profile to Fig. 9(c), where we represent the electric-field configuration, it becomes manifest that each of the doped fermions is localised within a topological soliton of the gauge fields. The bound fermion-soliton quasi-particles are deconfined, as they carry a unit \( Z_2 \) charge (Fig. 9(b)), and can interact among each other forming a crystalline structure (Fig. 9(a)). To be best of our knowledge, our results confirm quantitatively this mechanism for the first time, and show that it can also appear in fermionic LGTs that combine topological order and SPT phases. We note that a similar deconfinement mechanism has been suggested for the odd sector of a pure IGT in (1+1) dimensions [92], based on an analogy to the Peterls solitons in polymers [73]. Our detailed analysis shows that, in our case, this type of solitons characterised by charge fractionalization in polymers [78, 93] are not the underlying mechanism explaining the deconfinement for the \( h \gg 1 \) limit of IGTs. As discussed above, the integrated charge around the solitons is quantized in units of the \( Z_2 \) charge, but there is no signature of charge fractionalization (Fig. 9(b)). This result points to a different nature of topological defects in the magnetic- and electric-dominated phases, which will be the subject of detailed future studies.

**B. Deconfinement versus string breaking**

In this last section, we argue that the appearance of the mechanism of soliton deconfinement depends on the particular charge sector (5). So far, we have focused on the even sector, which is characterised by the absence of background \( Z_2 \) charges \( q_i = 0, \forall i \in \mathbb{N}_L \times \mathbb{Z}_2 \). We now make a full comparison with a different sector, hereby referred to as the imbalanced sector, where there is a static \( Z_2 \) charge at each site of the lower leg, namely \( q_{(i,0)} = 1, q_{(i,1)} = 0, \forall i \in \mathbb{N}_L \). In this case, there are neither frustrated bonds in the ground-state, nor partial dimer coverings or solitons as in Fig. 8. Accordingly, the situation and the confinement properties change completely. To quantify these differences, we introduce two additional background charges on the upper leg of the ladder that are separated by a distance \( L \), namely we add \( q_{(i,0)} = q_{(i-L,1)} = 1 \) to the two different charge sectors.

In Figs. 10(a)-(b), we represent the total integrated charge \( Q_i = Q_i + \sum_{j<i} (q_{(j,0)} + q_{(j,1)}) \), which includes both the dynamical fermions and the background charges above the even charge sector. We also depict the underlying averaged electric field, \( E_i = \sum_{j\neq i} \left( \sigma_{(j,1)(i,1)} + \sigma_{(j,1)(i-1,0)} \right) / 4 \). The situation is analogous to the soliton-induced deconfinement discussed in the previous section, but the location of each solitons is now pinned to the position of the extra static charge, while in Fig. 9 the unpinned solitons tend to maximise their spread and distance forming a soliton lattice. In Figs. 10(c)-(d), we depict the same observables for the imbalanced sector. It is clear that an electric field line connecting the static charges is established, which leads to the aforementioned confinement. The new aspect brought by the dynamical matter is that, by lowering the energy imbalance \( \Delta \), it can become energetically favorable to create a particle-antiparticle pair that breaks this string and screens the static \( Z_2 \) charges, as can be observed in Figs. 10 (e)-(f). In these figures, one sees that the electric fields are restricted to the regions around the static charges, and that the \( Z_2 \) charges are no longer unity, but get screened to \( Q \mod 2 = 0 \), signalling the aforementioned string breaking.

We can make a quantitative study of the difference in the
confinement properties of the two charge sectors by calculating the dependence of the effective potential with the distance between the static charges \(V(L)\). In Fig. 10 (g), we present the results for the even charge sector and different values of \(\Delta\). In all cases, apart from an even-odd effect due to the so-called Peierls-Nabarro barriers associated with the defects [94, 95], the energy does not grow with the distance, signaling deconfinement. In contrast, in the imbalanced sector of Fig. 10 (h), the potential grows linearly with the distance until the string breaks at a certain length, signalling confinement.

VI. CONCLUSIONS AND OUTLOOK

In this work, we have identified novel topological effects in finite-density fermionic gauge theories. In particular, we introduce a minimal fermionic \(\mathbb{Z}_2\) lattice gauge theory, the Creutz-Ising ladder, which allows investigating the interplay between topology and gauge invariance. We show how, even in the absence of a plaquette term, the system presents a magnetic-flux dominated phase, in which a dynamical \(\pi\) flux appears in the groundstate as a consequence of an Aharonov-Bohm instability. This phenomenon results from the interplay between gauge-invariant interactions and the particular connectivity of the model, which also gives rise to SPT phases in the fermionic sector. We characterize the properties of these phases, including the presence of protected gauge-matter edge states, through MPS-based numerical calculations, and use a topological invariant to find first-order phase transitions between the topological and trivial phases.

Our model can also be interpreted as a thin-cylinder limit of a \((2+1)\) \(\mathbb{Z}_2\) LGT. This equivalence allows us to uncover the presence of topological order by calculating the topological correction to the entanglement entropy. Topological order is also associated with the degeneracy of the ground-state, which can be characterised by two different fluxes threading the hole of the cylinder (i.e. presence or absence of a trapped vison). We have shown that, in the Creutz-Ising ladder, the topological order intertwines with topological symmetry protection, and this connection manifests in the change of the inner flux, and thus the trapping of a vison, when crossing phase transition lines towards the SPT phase such that edge states emerge from the bulk and localise within the ladder boundaries. This feature could facilitate the detection of topological order in future experiments.

Finally, we show how fermionic deconfinement, which accompanies the topologically-order phase, survives for the whole parameter space considered. This occurs due to the presence of deconfined topological defects associated to the fermionic quasi-particles, that appear on a frustrated background of electric fields imposed by gauge invariance. We investigate this mechanism using both static and dynamical charges, and compare it to the more standard confining case where string breaking usually takes place.

We believe that our results advance substantially the understanding of topological phenomena in lattice gauge theories. Moreover, we have shown that the inclusion of dynamical fermions can stabilise a magnetic-dominated deconfined phase even in the absence of plaquette interactions. Therefore, our work identifies a new avenue for the realization of spin-liquid physics in LGTs, relevant for both condensed matter and high-energy physics, in cold-atom experiments based on state-of-the-art building blocks that have already been used for quantum simulation purposes. This would allow to investigate complex phenomena, such as topological order and deconfinement, using minimal resources. The methods applied here can be used to further explore the static and dynamical properties of \(\mathbb{Z}_2\) fermionic gauge theories, including the phase diagram.
at different fillings or the non-equilibrium quench dynamics.

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