CENTERS OF CUNTZ-KRIEGER $C^*$-ALGEBRAS

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Abstract. For a finite directed graph $\Gamma$ we determine the center of the Cuntz-Krieger $C^*$-algebra $CK(\Gamma)$.

1. Definitions and Main Results

Let $\Gamma = (V, E, s, r)$ be a directed graph that consists of a set $V$ of vertices and a set $E$ of edges, and two maps $r : E \to V$ and $s : E \to V$ identifying the range and the source of each edge. A graph is finite if both sets $V$ and $E$ are finite. A graph is row-finite if $|s^{-1}(v)|<\infty$ for an arbitrary vertex $v \in V$.

Definition 1. If $\Gamma$ is a row-finite graph, the Cuntz-Krieger $C^*$-algebra $CK(\Gamma)$ is the universal $C^*$-algebra generated (as a $C^*$-algebra) by $V \cup E$ and satisfying the relations (1) $s(e)e = er(e) = e$ for all $e \in E$, (2) $e^*f = \delta_{e,f}r(f)$ for all $e, f \in E$, (3) $v = \sum_{e \in s^{-1}(v)} ee^*$ whenever $s^{-1}(v) \neq \emptyset$.

The discrete $C$-subalgebra of $CK(\Gamma)$ generated by $V, E, E^*$ is isomorphic to the Leavitt path algebra $L(\Gamma)$ of the graph $\Gamma$ (see [T2]). We will identify $L(\Gamma)$ with its image in $CK(\Gamma)$. Clearly $L(\Gamma)$ is dense in $CK(\Gamma)$ in the $C^*$-topology.

A path is a finite sequence $p = e_1 \cdots e_n$ of edges with $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. We consider the vertices to be paths of length zero. We let $\text{Path}(\Gamma)$ denote the set of all paths in the graph $\Gamma$ and extend the maps $r, s$ to $\text{Path}(\Gamma)$ as follows: for $p = e_1 \cdots e_n$ we set $s(p) = s(e_1), r(p) = r(e_n)$. For $v \in V$ viewed as a path we set $s(v) = r(v) = v$. A vertex $w$ is a descendant of a vertex $v$ if there exists a path $p \in \text{Path}(\Gamma)$ such that $s(p) = v, r(p) = w$.

A cycle is a path $C = e_1 \cdots e_n, n \geq 1$ such that $s(e_1) = r(e_n)$ and all vertices $s(e_1), \ldots, s(e_n)$ are distinct. An edge $e \in E$ is called an exit from the cycle $C$ if $s(e) \in \{s(e_1), \ldots, s(e_n)\}$, but $e \notin \{e_1, \ldots, e_n\}$. A cycle without an exit is called a

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A subset $W \subset V$ is hereditary if all descendants of an arbitrary vertex $w \in W$ also lie in $W$. For two nonempty subsets $W_1, W_2 \subset V$ let $E(W_1, W_2)$ denote the set of edges $\{e \in E \mid s(e) \in W_1, r(e) \in W_2\}$.

For a hereditary subset $W \subset V$ the $C^*$-subalgebra of $CK(\Gamma)$ generated by $W, E(W, W)$ is isomorphic to the Cuntz-Krieger algebra of the graph $(W, E(W, W), s|_{E(W, W)}, r|_{E(W, W)})$, (see [BPRS], [BHRS]). We will denote it as $CK(W)$.

**Example 1.** Let the graph $\Gamma$ be a cycle, $V = \{v_1, \ldots, v_n\}$, $E = \{e_1, \ldots, e_n\}$, $s(e_i) = v_i, 1 \leq i \leq n; r(e_i) = v_{i+1}$ for $1 \leq i \leq n-1$, $r(e_n) = v_1$. The algebra $CK(\Gamma)$ in this case is isomorphic to the matrix algebra $M_n(T)$, where $T$ is the $C^*$-algebra of continuous functions on the unit circle. The center of $CK(\Gamma)$ is generated by the element $e_1 \cdots e_n + e_2e_3 \cdots e_1 + \cdots + e_ne_1 \cdots e_1$ and is isomorphic $T$.

For an arbitrary row-finite graph $\Gamma$ if a path $C = e_1 \cdots e_n$ is a $NE$-cycle with the hereditary set of vertices $V(C) = \{s(e_1), \ldots, s(e_n)\}$ then we denote $CK(C) = CK(V(C)) \cong M_n(T)$. The center $Z(C)$ of the subalgebra $CK(C)$ of $CK(\Gamma)$ is generated by the element $z(C) = e_1 \cdots e_n + e_2e_3 \cdots e_ne_1 + \cdots + e_ne_1 \cdots e_{n-1}$.

We will need some definitions and some results from [AA1], [AA2].

**Definition 2.** Let $W \subset V$ be a nonempty subset. We say that a path $p = e_1 \cdots e_n$, $e_i \in E$, is an **arrival path** in $W$ if $r(p) \in W$, and $\{s(e_1), \ldots, s(e_n)\} \not\subseteq W$. In other words, $r(p)$ is the first vertex on $p$ that lies in $W$. In particular, every vertex $w \in W$, viewed as a path of zero length, is an arrival path in $W$. Let $Arr(W)$ be the set of all arrival paths in $W$.

**Definition 3.** A hereditary set $W \subseteq V$ is called finitary if $|Arr(W)| < \infty$.

If $W$ is a hereditary finitary subset of $V$ then $e(W) = \sum_{p \in Arr(W)} pp^*$ is a central idempotent in $L(\Gamma)$ and, hence in $CK(\Gamma)$, see [AA1],[AA2]. If $C$ is a $NE$-cycle in $\Gamma$ with the hereditary set of vertices $V(C)$ then we will denote $Arr(C) = Arr(V(C))$. If the set $V(C)$ is finitary then we will say that the cycle is finitary. In this case for an arbitrary element $z \in Z(C)$ the sum $\sum_{p \in Arr(C)} pzp^*$ lies in the center of $L(\Gamma)$ and $CK(\Gamma)$, see [AA2].
Theorem 1. Let $\Gamma$ be a finite graph. The center $Z(CK(\Gamma))$ is spanned by:

(i) central idempotents $e(W)$, where $W$ runs over all nonempty hereditary finitary subsets of $V$;

(ii) subspaces $\{ \sum_{p \in \text{Arr}(C)} pzp^* \mid z \in Z(C) \}$, where $C$ runs over all finitary NE-cycles of $\Gamma$.

Corollary 1. $Z(CK(\Gamma))$ is the closure of $Z(L(\Gamma))$.

Corollary 2. The center $Z(CK(\Gamma))$ is isomorphic to a finite direct sum $\mathbb{C} \oplus \cdots \oplus \mathbb{C} \oplus T \oplus \cdots \oplus T$.

In [CGBMGSMSH] it was shown that the center of a prime Cuntz-Krieger $C^*$-algebra is equal to $\mathbb{C} \cdot 1$ (for a finite graph) or to $\langle 0 \rangle$ (for infinite graph).

We remark that for two distinct hereditary subsets $W_1 \subseteq W_2 \subseteq V$ the central idempotents $e(W_1), e(W_2)$ may be equal.

To make the statement of the Theorem more precise we will consider annihilator hereditary subsets.

Let $W$ be a nonempty subset of $V$. Consider the subset $W^\perp = \{ v \in V \mid$ the vertex $v$ does not have descendants in $W \}$.

For empty set we let $\emptyset^\perp = V$. It is easy to see that $W^\perp$ is always a hereditary subset of $V$. If $W_1 \subseteq W_2 \subseteq V$ then $W_1^\perp \supseteq W_2^\perp$, $(W^\perp)^\perp \supseteq W$, $((W^\perp)^\perp)^\perp = W^\perp$.

Definition 4. We will refer to $W^\perp$, $W \subseteq V$ as annihilator hereditary subsets.

In [AA1] it was proved that $(W^\perp)^\perp$ is the largest hereditary subset of $V$ such that every vertex in it has a descendant in $W$ and that $e(W) = e((W^\perp)^\perp)$.

Hence in the part (i) of the Theorem we can let $W$ run over nonempty finitary hereditary annihilator subsets of $V$.

In fact there is a 1-1 correspondence (and a Boolean algebra isomorphism) between finitary hereditary annihilator subsets of $V$ and central idempotents of $L(\Gamma)$ and $CK(\Gamma)$.

Example 2. Let $\Gamma = \begin{array}{c} v_1 \\ \overline{v_2} \end{array}$. The set $\{v_2\}$ is hereditary, but not finitary. Thus there are no proper finitary hereditary subsets and $Z(L(\Gamma)) = \mathbb{C} \cdot 1$. 
Example 3. Let $\Gamma = v_1 \xrightarrow{v_4} v_2 \xrightarrow{v_3}$ . The set $\{v_2, v_3, v_4\}$ is hereditary and finitary, but $(\{v_2, v_3, v_4\})^\perp = V$. Thus there are no proper finitary hereditary annihilator subsets and again $Z(L(\Gamma)) = \mathbb{C} \cdot 1$.

Example 4. Let $\Gamma = v_1 \xrightarrow{v_5} v_2 \xrightarrow{v_3} v_4 \xrightarrow{v_4}$ . The only finitary hereditary annihilator subsets are $\{v_5\}$ and $\{v_2, v_3, v_4\}$. Hence $Z(L(\Gamma)) = T \oplus \mathbb{C}$.

2. Closed Ideals

Let $W$ be a hereditary subset of $V$. The ideal $I(W)$ of the Leavitt path algebra generated by the set $W$ is the $\mathbb{C}$-span of the set $\{pq^* \mid p, q \in \text{Path}(\Gamma), r(p) = r(q) \in W\}$. The closure $\overline{I(W)}$ of the ideal $W$ is the closed ideal of the $C^*$-algebra $CK(\Gamma)$ generated by the set $W$.

In this section we will use induction on $|V|$ to prove that for a proper hereditary subset $W \subset V$ central elements from $Z(CK(\Gamma)) \cap \overline{I(W)}$ are of the form predicted by the Theorem.

Given paths $p, q \in \text{Path}(\Gamma)$ we say $q$ is a continuation of the path $p$ if there exists a path $p' \in \text{Path}(\Gamma)$ such that $q = pp'$. In this case the path $p$ is called a beginning of the path $q$. We will often use the following well known fact: if $p, q \in \text{Path}(\Gamma)$, $p^* q \neq 0$, then one of the paths $p, q$ is a continuation of the other one.

Remark that if $W$ is a hereditary subset of $V$ and $p, q \in \text{Arr}(W), p \neq q$, then none of the paths $p, q$ is a continuation of the other one.

For a path $p \in \text{Path}(\Gamma)$ we consider the idempotent $e_p = pp^*$.

Lemma 1. Let $W$ be a nonempty hereditary subset of $V$ let $a \in \overline{I(W)}$, let $\epsilon > 0$. Then the set $\{p \in \text{Arr}(W) \mid \|e_p a\| \geq \epsilon\}$ is finite.
Proof. The ideal \( I(W) \) generated by the set \( W \) in the Leavitt path algebra \( L(\Gamma) \) is dense in \( \overline{7}(W) \). Choose an element \( b \in I(W) \), \( b = \sum_{p,q \in \text{Arr}(W)} p b_{p,q} q^* \), \( b_{p,q} \in L(W) \), such that \( \|a - b\| < \epsilon \). If \( p' \in \text{Arr}(W) \) is an arrival path in \( W \) that is different from all the paths involved in the decomposition of the element \( b \) then \( p'^* b = 0 \) and \( e_{p'} b = 0 \).

Now
\[
\|e_{p'} a\| = \|e_{p'}(a - b)\| \leq \|e_{p'}\| \|a - b\| < \epsilon,
\]
which proves the Lemma. \( \square \)

Lemma 2. Let \( a \in CK(\Gamma), v \in V, a \in vCK(\Gamma)v; p \in \text{Path}(\Gamma), r(p) = v \). Then \( \|pap^*\| = \|a\| \).

Proof. We have
\[
\|pap^*\| \leq \|p\| \cdot \|a\| \cdot \|p^*\| = \|a\|.
\]
On the other hand, \( a = p'^*(pap^*)p \),
\[
\|a\| \leq \|p'^*\| \cdot \|pap^*\| \cdot \|p\| = \|pap^*\|,
\]
which proves the Lemma. \( \square \)

Lemma 3. Let \( W \) be a nonempty hereditary subset of \( V \) let \( z \in Z(CK(\Gamma) \cap \overline{7}(W)) \). For an arbitrary vertex \( w \in W \) if \( wz \neq 0 \) then the set \( \{p \in \text{Arr}(W) \mid r(p) = w\} \) is finite.

Proof. Let \( p \in \text{Arr}(W), r(p) = w, zw \neq 0 \). We have \( e_{p} z = pp^* z = pz p^* = p(wzw)p^* \). By Lemma 2 \( \|e_{p} z\| = \|wz\| > 0 \). Now it remains to refer to Lemma 1. \( \square \)

Let \( \tilde{Z} \) denote the sum of all subspaces \( Ce(W) \), where \( W \) runs over nonempty finitary hereditary subsets of \( V \), and all subspaces \( \{ \sum_{p \in \text{Arr}(C)} p z p^* \mid z \in Z(C)\} \), where \( C \) runs over finitary \( NE \)-cycles of \( \Gamma \).

Our aim is to show that \( Z(CK(\Gamma)) = \tilde{Z} \).

Let’s use induction on the number of vertices. In other words, let’s assume that for a graph with \( < |V| \) vertices the assertion of the Theorem is true.

Lemma 4. Let \( W \) be a proper hereditary subset of \( V \). Then \( Z(CK(\Gamma) \cap \overline{7}(W)) \subseteq \tilde{Z} \).
Proof. Let \( 0 \neq z \in Z(CK(\Gamma) \cap \overline{T}(W)) \). Consider the element \( z_0 = z(\sum_{w \in W} w) \in Z(CK(W)) \). If \( z_0 = 0 \) then \( zW = (0), z\overline{T}(W) = (0), z^2 = 0 \), which contradicts semiprimeness of \( CK(\Gamma) \) (see [BPRS], [BHRS]).

By the induction assumption there exist disjoint hereditary finitary (in \( W \)) cycles \( C_1, \ldots, C_r \) and hereditary finitary (again in \( W \)) subsets \( W_1, \ldots, W_k \subset W \) such that

\[
z_0 = \sum_{i=1}^{r} \alpha_i \left( \sum_{p \in Arr_W(C_i)} pa_i p^* \right) + \sum_{j=1}^{k} \beta_j \left( \sum_{q \in Arr_W(W_j)} qq^* \right) ; \alpha_i, \beta_j \in \mathbb{C}, \alpha_i \in Z(C_i).
\]

The notations \( Arr_W(C_i), Arr_W(W_j) \) are used to stress that arrival paths are considered in the graph \((W, E(W, W))\).

The fact that the hereditary finitary subsets \( V(C_i), W_j \) can be assumed disjoint follows from the description of the Boolean algebra of finitary hereditary subsets in [AA2].

If \( \alpha_i \neq 0 \) then for arbitrary vertex \( w \in V(C_i) \) we have \( z_0 w = zw \neq 0 \). Hence by Lemma 3 there are only finitely many paths \( p \in Arr(W) \) such that \( r(p) = w \). Hence \( V(C_i) \) is a finitary subset of \( V \). Similarly, if \( \beta_j \neq 0 \) then \( W_j \) is a finitary subset in \( V \).

Consider the central element

\[
z' = \sum_{i=1}^{r} \alpha_i \left( \sum_{p \in Arr_W(C_i)} pa_i p^* \right) + \sum_{j=1}^{k} \beta_j e(W_j) \in Z(CK(\Gamma)).
\]

We have \( z'(\sum_{w \in W} w) = z_0 = z(\sum_{w \in W} w) \). Hence, \( (z - z')(\sum_{w \in W} w) = (0), (z - z')\overline{T}(W) = (0), (z - z')^2 = 0 \). Again by semiprimeness of \( CK(\Gamma) \) we conclude that \( z = z' \), which proves the Lemma.

\[\square\]

3. Proof of the Theorem

**Definition 5.** A vertex \( v \in V \) is called a sink if \( s^{-1}(v) = \emptyset \).

**Definition 6.** A hereditary subset \( W \subset V \) is called saturated if for an arbitrary non-sink vertex \( v \in V \) the inclusion \( r(s^{-1}(v)) \subseteq W \) implies \( v \in W \).

**Definition 7.** If \( W \) is hereditary subset then we define the saturation of \( W \) to be the smallest saturated hereditary subset \( \hat{W} \) that contains \( W \). In this case \( I(W) = I(\hat{W}) \).
Definition 8. If \( W \) is a hereditary saturated subset of \( V \) then the graph \( \Gamma/W = (V \setminus W, E(V, V \setminus W)) \) is called the factor graph of \( \Gamma \) modulo \( W \).

We have \( CK(\Gamma)/I(W) \cong CK(\Gamma/W) \) (see [T1]). In [AAP], [T2] it was proved that the following 3 statements are equivalent:

1) the Cuntz-Krieger \( C^* \)-algebra \( CK(\Gamma) \) is simple,
2) the Leavitt path algebra \( L(\Gamma) \) is simple,
3) (i) \( V \) does not have proper hereditary saturated subsets, (ii) every cycle has an exit.

We call a graph satisfying the condition 3) simple.

The following lemma is well known. Still we prove it for the sake of completeness.

Lemma 5. Let \( \Gamma \) be a graph such that \( V \) does not have proper hereditary subsets. Then \( \Gamma \) is either simple or a cycle.

Proof. If \( \Gamma \) is not simple then \( \Gamma \) contains a \( NE \)-cycle \( C \). The set of vertices \( V(C) \) is hereditary subset of \( V \). In view of our assumption \( V(C) = V \), which proves the Lemma.

\[ \Box \]

Let \( U = \{ \alpha \in \mathbb{C} \mid |\alpha| = 1 \} \) be the unit circle in \( \mathbb{C} \). Let \( E' \) be a subset of the set \( E \) of edges. For an arbitrary \( \alpha \in U \) the mapping \( g_{E'}(\alpha) \) such that \( g_{E'}(\alpha) : v \mapsto v, v \in V; g_{E'}(\alpha) : e \mapsto \alpha e, e^* \mapsto \overline{\alpha} e, e \in E'; g_{E'}(\alpha) : e \mapsto e, e^* \mapsto e^*, e \in E \setminus E', \) extends to an automorphism \( g_{E'}(\alpha) \) of the \( C^* \)-algebra \( CK(\Gamma) \). Denote \( G_E = \{ g_{E'}(\alpha), \alpha \in U \} \leq Aut(CK(\Gamma)) \). The group \( G_E \) is called the gauge group of the \( C^* \)-algebra \( CK(\Gamma) \). An ideal of \( CK(\Gamma) \) is called gauge invariant if it is invariant with respect to the group \( G_E \).

In [BPRS], [BHRS] it is proved that a nonzero closed gauge invariant ideal of \( CK(\Gamma) \) has a nonempty intersection with \( V \).

Lemma 6. Let the graph \( \Gamma \) be a cycle, \( \Gamma = (V, E), V = \{ v_1, \ldots, v_d \}, E = \{ e_1, \ldots, e_d \}, s(e_i) = v_i \) for \( 1 \leq i \leq d \); \( r(e_i) = v_{i+1} \) for \( 1 \leq i \leq d-1 \), \( r(e_d) = v_1 \). Then the central elements from \( Z(CK(\Gamma)) \) that are fixed by all \( g_{E'}(\alpha), \alpha \in U \), are scalars.

Proof. The center of \( CK(\Gamma) \) is isomorphic to the algebra of continuous function \( T = \{ f : U \to \mathbb{C} \} \), the corresponding action of \( G_E \) on \( T \) is \( (g_{E'}(\alpha) f)(u) = f(\alpha^d u) \). Now, if \( f(u) = f(\alpha^d u) \) for all \( \alpha, u \in U \) then \( f \) is a constant function, which proves the Lemma.

\[ \Box \]
Lemma 7. Let \( W \) be a hereditary saturated subset of \( V \) such that \( \Gamma/W \) is a cycle, \( E(V \setminus W, W) \neq \emptyset \), \( \Gamma/W = \{v_1, \cdots, v_d\} \). Then \( \left( \sum_{i=1}^{d} v_i CK(\Gamma)v_i \right) \cap Z(CK(\Gamma)) = (0) \).

Proof. Consider the set
\[
W' = \{ w \in W \mid E(V \setminus W, W)CK(\Gamma)w = (0) \}.
\]
The set \( W' \) is hereditary and saturated. Moreover, \( \left( \sum_{i=1}^{d} v_i CK(\Gamma)v_i \right) \cap \mathcal{T}(W') = (0) \).
Indeed, we only need to notice that if \( p \in \text{Path}(\Gamma) \), \( s(p) = v_i \) and \( r(p) \in W' \) then \( p = 0 \). Let \( p = e_1 \cdots e_n, e_i \in E \). At least one edge \( e_j, 1 \leq j \leq n \), lies in \( E(V \setminus W, W) \).
This implies the claim. Factoring out \( \mathcal{T}(W') \) we can assume that \( W' = \emptyset \).

Let \( 0 \neq z \in \left( \sum_{i=1}^{d} v_i CK(\Gamma)v_i \right) \cap Z(CK(\Gamma)) \). For an arbitrary edge \( e \in E(V \setminus W, W) \) we have \( ze = ez \). Hence \( ze = zer(e) = ezr(e) = 0 \). Consider the ideal
\[
J = \{ a \in CK(\Gamma) \mid E(V \setminus W, W)CK(\Gamma)a = aCK(\Gamma)E(V \setminus W, W) = (0) \}
\]
of the algebra \( CK(\Gamma) \). The element \( z \) lies in \( J \). The ideal \( J \) is gauge invariant. Hence by [BPRS], [BHRS] \( J \cap V \neq \emptyset \). A vertex \( v_i, 1 \leq i \leq d \), can not lie in \( J \) because \( v_iCK(\Gamma)E(V \setminus W, W) \neq (0) \). On the other hand \( J \cap W \subseteq W' \neq \emptyset \), a contradiction, which proves the Lemma.

\[
\square
\]

Lemma 8. Let \( W \) be a hereditary saturated subset of \( V \) such that \( \Gamma/W \) is a cycle and \( E(V \setminus W, W) \neq \emptyset \). Then \( Z(CK(\Gamma)) \subseteq C \cdot 1_{CK(\Gamma)} + \mathcal{T}(W) \).

Proof. As in the Lemma 7 we assume that \( V \setminus W = \{v_1, \cdots, v_d\} \), \( E' = E \setminus E(V, W) = \{e_1, \cdots, e_d\} \); \( s(e_i) = v_i, 1 \leq i \leq d, r(e_i) = v_{i+1} \) for \( 1 \leq i \leq d-1 \), \( r(e_d) = v_1 \). Consider the action of the group \( G_{E'} \) on \( CK(\Gamma) \). Let \( z \in Z(CK(\Gamma)) \), \( z = a + b, a = \sum_{i=1}^{d} v_i z v_i, b = \sum_{w \in W} w z w \). Since every element from \( G_{E'} \) fixes \( CK(\Gamma) \) it follows that \( g(b) = b \) for all \( g \in G_{E'} \). Now \( g(z) = g(a) + b, g(z) - z = g(a) - a \in \left( \sum_{i=1}^{d} v_i CK(\Gamma)v_i \right) \cap Z(CK(\Gamma)) = (0) \) by Lemma 7. we proved that an arbitrary element of \( Z(CK(\Gamma)) \) is fixed by \( G_{E'} \). The ideal \( \mathcal{T}(W) \) is invariant with respect to \( G_{E'} \). Hence the group \( G_{E'} \) acts on \( CK(\Gamma)/\mathcal{T}(W) \cong CK(\Gamma/W) \) as the full gauge group of automorphisms. The image of the central element \( z \) in \( Z(CK(\Gamma/W)) \) is fixed by \( G_{\Gamma/W} \). Hence by Lemma 6 it is scalar. This finishes the proof of the Lemma.

\[
\square
\]
Proof of Theorem 1. If $V$ does not contain proper hereditary subsets then by Lemma $5$ $\Gamma$ is either simple or a cycle. If the $C^*$-algebra $CK(\Gamma)$ is simple then $Z(CK(\Gamma)) = \mathbb{C} \cdot 1$ (see [D]) and the assertion of the Theorem is clearly true.

If $C$ is a cycle then $Z(CK(\Gamma)) \cong T$ and the assertion of the Theorem is again true.

Let now $W$ be a maximal proper hereditary subset of $V$. The saturation $\hat{W}$ is equal to $V$ or the set $W$ is saturated. In the first case $CK(\Gamma) = I(W)$ and it suffices to refer to Lemma 4. Suppose now that the set $W$ is saturated. The graph $\Gamma/W$ does not have proper hereditary subsets. Again by Lemma 5 the graph $\Gamma/W$ is either simple or a cycle. If $\Gamma/W$ is simple then $Z(CK(\Gamma/W)) = \mathbb{C} \cdot 1$, which implies $Z(CK(\Gamma) \subseteq \mathbb{C} \cdot 1 + I(W))$, which together with Lemma 4 implies the Theorem. If $\Gamma/W$ is a cycle then by Lemma 8 we again have $Z(CK(\Gamma) \subseteq \mathbb{C} \cdot 1 + I(W))$ and the Theorem follows.

\[ \square \]

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