Correlated noise induced control of prey extinction

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Abstract

We study the steady state properties of a phenomenological two-state predator model in presence of correlated Gaussian white noise. Based on the corresponding Fokker-Planck equation for probability distribution function the steady state solution of the probability distribution function and its extrema have been investigated. We show for a typical value of noise correlation there is a giant loss of bistability which in turn prevents the prey population from going into extinction.

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I. INTRODUCTION

The subject of noise-induced transition has got wide applications in the field of physics, chemistry and biology [1]. In most of these theories the noise affects the dynamics through system variable, i.e., the noise is multiplicative in nature. The focal theme of these investigations is the study of steady state properties of the system where the fluctuations, in general, are applied from outside and are independent of the system’s characteristic dissipation. Such systems are generally termed as open systems [2], since they lack the principle of detailed balance which ensures fluctuation-dissipation relation to hold for the thermodynamically closed systems. However, it may also happen that the external fluctuations instead of affecting only some system’s parameters affect the system directly, i.e., they drive system dynamics multiplicatively as well as additively. Because the two noise processes owe a common origin they get correlated in the relevant timescale of the problem [3,4]. Correlated noise processes have found applications in studying steady state properties of a single mode laser [5], in analyzing bistable kinetics [6], in giant suppression of activation rate [7], in producing directed motion in spatially symmetric periodic potentials [8], in studying stochastic resonance in linear systems [9], in steady state entropy production [10], etc. In this brief communication we investigate a simple noise-driven two-state predator model [1] and show how noise correlation can dynamically prevent the prey population from extinction.

II. THE MODEL

To start with we consider an environment of the prey which in absence of predation grows logistically and at the same time its density in a territory depends linearly on a constant source of migration. We also consider a population of predators in the given territory which lives by feeding on prey. The characteristic time scale over which the population of prey and predator varies are very much different, so one can consider the predator population to be constant within the generation time of prey. The predators are engaged in two types
of activities, viz, hunting or resting. The time scale of predator’s two activities are very
short compared to the generation time of prey, i.e., \( \tau_R, \tau_H \ll \mu^{-1} \) where \( \tau_R \) and \( \tau_H \) are the
characteristic average time of resting and hunting, respectively and \( \mu \) is the birth rate of
prey. The activity of the predator in the territory resembles the mode of action of enzymes or
catalysts in a chemical reaction. The enzymes or catalysts in a chemical reaction transform
substrates in a continuous manner without destroying themselves. The constant predator
population acts in a similar way by feeding on the prey. To put this ideas in a quantitative
way we write the evolution equations for the predator and prey 1,

\[
\dot{X} = A + \mu X \left(1 - \frac{X}{K}\right) - \frac{1}{\tau_H} XY ,
\]

\[
\dot{Y} = -\frac{1}{\tau_H} XY + \frac{1}{\tau_R} Z
\]

where \( X \) is the density of prey in a given territory. The constant \( A \) in Eq.(1) is due to a
constant source of prey through immigration. The second term in (1) is the Fisher logistic
growth term with birth rate \( \mu \) and carrying capacity \( K \). \( Y \) and \( Z \) are the numbers of
predators in the hunting and resting state, respectively. \( E \) is the total constant population
of the predators, i.e., \( E \equiv Y(t) + Z(t) = \) constant. The last term in (1) describes the decay
rate of prey. The model is hybrid in nature in the sense that it has virtue of taking into
consideration of the logistic growth model as well as of the predator-prey model.

Following Ref. 1 we now consider that the predator population, \( E \) is small compared
to prey population \( X \). To study the overall dynamics within the timescale \( \mu^{-1} \) we make the
following transformation

\[
\tau_H = \epsilon \tau_H^* , \quad \tau_R = \epsilon \tau_R^* , \quad Y = \epsilon Y^* \quad \text{and} \quad Z = \epsilon Z^*
\]

where \( \epsilon \) is a small quantity, \( \tau_H^*, \tau_R^* \) are quantities of order \( \mu^{-1} \) and \( Y^*, Z^* \) are quantities of
order \( X \). Using (3) in (1) and (2) we arrive at

\[
\dot{X} = A + \mu X \left(1 - \frac{X}{K}\right) - \frac{1}{\tau_H^*} XY^* ,
\]

\[
\epsilon \dot{Y}^* = -\frac{1}{\tau_H^*} XY^* + \frac{1}{\tau_R^*} Z^*
\]
Now eliminating $Y^*$ from (4) and using the limit $\varepsilon \to 0$ we arrive at the following dimensionless evolution equation for prey

$$\dot{x} = \alpha + x(1 - \theta x) - \beta \frac{x}{1 + x}$$  \hspace{1cm} (6)

where

$$x = \frac{\tau_R^* X}{\tau_H^*}, \quad \alpha = \frac{A \tau_R^*}{\mu \tau_H^*}, \quad \beta = \frac{E}{\mu \tau_H} \quad \text{and} \quad \theta = \frac{\tau_H}{\tau_R K}.$$  \hspace{1cm} (7)

It is interesting to note that the third term in Eq.(6) is the predation term which essentially emerges from the two-state of predator activities. The steady state solution of Eq.(6) shows a cusp type of catastrophe. The corresponding critical point $(\alpha_c, \beta_c, x_c)$ is given by

$$\alpha_c = \frac{(1 - \theta)^2}{27 \theta^2}, \quad \beta_c = \frac{(1 + 2\theta)^3}{27 \theta^2} \quad \text{and} \quad x_c = \frac{1 - \theta}{3\theta}.$$  \hspace{1cm} (8)

The necessary condition to have a physically realizable critical point i.e., for $\alpha_c, x_c$ to be positive, is $\theta < 1$. Thus the steady state curve of $x$ as a function of $\beta$ always shows a bistable region for small values of $\theta$. The smallness condition may be maintained by increasing the carrying capacity $K$ or by decreasing the ratio $\tau_H/\tau_R$.

Eq.(8) is the starting point of our further analysis. It may be noted that $\alpha$ and $\beta$ are the two quantities which appear in the prey evolution equation as a constant and a multiplicative factor, respectively. Expressions for $\alpha$ and $\beta$ in (8) suggest that they are connected by a common parameter $\mu$, the birth rate of the prey. Now if due to some environmental external disturbance the birth rate of the prey fluctuates, it is likely to affect both $\alpha$ and $\beta$ in the form of additive and multiplicative noises which are connected through a correlation parameter. Or in other words the external fluctuations affect the parameter $\beta$ which fluctuates around a mean value, thus generating multiplicative noise and at the same time environmental fluctuations perturbs the dynamics directly which gives rise to additive noise. As a result we have the stochastic differential equation in Stratonovich prescription,

$$\dot{x} = \alpha + x(1 - \theta x) - \beta \frac{x}{1 + x} - \frac{x}{1 + x} \xi(t) + \eta(t)$$  \hspace{1cm} (9)
where $\xi(t)$ and $\eta(t)$ are the stationary Gaussian white noises with the following properties

\[
\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0 ,
\]
\[
\langle \xi(t) \xi(t') \rangle = 2\sigma \delta(t-t') ,
\]
\[
\langle \eta(t) \eta(t') \rangle = 2D \delta(t-t') \quad \text{and}
\]
\[
\langle \xi(t) \eta(t') \rangle = \langle \eta(t) \xi(t') \rangle = 2\lambda(\sigma D)^{1/2} \delta(t-t')
\]

where $\lambda$ denotes the degree of correlation between noise processes $\xi(t)$ and $\eta(t)$ with $0 \leq \lambda \leq 1$. Using the above mentioned noise properties we write down the corresponding Fokker-Planck equation (in Stratonovich prescription) for the evolution of probability distribution function [4,6],

\[
\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} A(x,t) P(x,t) + \frac{\partial^2}{\partial x^2} B(x,t) P(x,t)
\]

where

\[
A(x,t) = \alpha + x(1-\theta x) - \beta \frac{x}{1+x} + \sigma \frac{x}{(1+x)^3} - \lambda(\sigma D)^{1/2} \frac{1}{(1+x)^2}
\]

and

\[
B(x,t) = D + \sigma \frac{x^2}{(1+x)^2} - 2\lambda(\sigma D)^{1/2} \frac{x}{1+x} .
\]

### III. STEADY STATE ANALYSIS AND RESULTS

Using the zero current condition at the stationary state we derive the stationary probability distribution function (SPDF) with $0$ and $\infty$ as the natural boundaries,

\[
P_s(x) = N \frac{1}{B(x)} \exp \left[ \int^x \frac{A(x')}{B(x')} dx' \right]
\]

where $N$ is the normalization constant. Using the explicit forms of $A(x)$ and $B(x)$ we have the following explicit forms of SPDF

\[
P_s(x) = N(1+x)^{\nu-\frac{1}{2}}(x) \exp[q_1x^3 + q_2x^2 + q_3x + q_4f(x)]
\]
where
\[ g(x) = a + bx + cx^2 \]  \hspace{1cm} (18)

\[ f(x) = \frac{-2}{b + 2cx} \quad \text{for} \quad \lambda = 1 \]
\[ = (2/\sqrt{\Delta}) \arctan[(b + 2cx)/\sqrt{\Delta}] \quad \text{for} \quad 0 \leq \lambda < 1 \]  \hspace{1cm} (19)

with
\[ a = D, \quad b = 2[D - \lambda(\sigma D)^{1/2}], \]
\[ c = D + \sigma - 2\lambda(\sigma D)^{1/2}, \Delta = 4\sigma D(1 - \lambda^2) \]  \hspace{1cm} (20)

along with
\[
q_1 = -\frac{\theta}{3c},
q_2 = \frac{1 - 2\theta}{2c} + \frac{b\theta}{2c^2},
q_3 = \frac{\alpha - \beta - \theta + 2}{c} - \frac{b(1 - 2\theta) + a\theta}{c^2} - \frac{b^2\theta}{c^3},
q_4 = \frac{\alpha - b(2\alpha - \beta + 1)}{2c} + \frac{(b^2 - 2ac)(\alpha - \beta - \theta + 2)}{2c^2} - \frac{b^2(b^2 - 3ac)\theta}{2c^4}
\]
\[ + \frac{a(b^2 - 2ac)\theta - b(b^2 - 3ac)(1 - 2\theta)}{2c^3} \quad \text{and} \]
\[ \nu = \frac{2\alpha - \beta + 1}{2c} - \frac{b(\alpha - \beta\theta + 2)}{2c^2} + \frac{(b^2 - ac)(1 - 2\theta) - ab\theta}{2c^3}
\]
\[ + \frac{b(b^2 - ac)\theta}{2c^4}. \]  \hspace{1cm} (21)

The extrema of SPDF is calculated using the condition \( A(x) - B'(x) = 0, \)
\[ \alpha + x(1 - \theta x) - \frac{\beta x}{1 + x} - \frac{\sigma x}{(1 + x)^3} + \frac{\lambda(\sigma D)^{1/2}}{(1 + x)^2} = 0 \quad \text{for} \quad 0 \leq \lambda \leq 1. \]  \hspace{1cm} (22)

For zero noise correlation, i.e., for \( \lambda = 0 \) the last term of Eq.(22) vanishes and we have the extrema of SPDF for pure multiplicative noise processes [1]. For zero correlation the additive noise has no extra effects in the steady state dynamics. To illustrate this we have plotted extrema of SPDF as a function of \( \beta \) in Fig.(1) using the parameters given in [1]. For zero noise correlation the curve shows a sharp minima which decreases on increasing \( \lambda \). Similarly, in Fig.(2) we have plotted extrema of SPDF as a function of \( \beta \) for different
values of additive noise strength \( D \) with maximum correlation \( (\lambda = 1) \). As the additive noise strength increases the well gets flattened and almost vanishes for a large enough value of \( D \).

In Fig.(3) we show the effect of correlation parameter \( \lambda \) on SPDF. For a low value of \( \lambda \) the SPDF shows the typical bistable region (see Fig.3(a)) which vanishes for higher values of \( \lambda \) (see Fig.3(b)). As the value of correlation parameter \( \lambda \) increases the peak on the lower values of \( x \) decreases while for a higher value of \( \lambda \) we have a single peak at a higher values of \( x \). Since \( x \) denotes the prey population, it is clear from Fig.(3) that with the increase of \( \lambda \) values the prey population recovers from going into extinction. In other words, the distribution of prey which was mainly peaked about zero (for a low value of \( \lambda \)) signifying high extinction rate, moves away from zero with the increase of correlation between noises thus favouring the prey’s survival. Though Gaussian white noise acting independently and multiplicatively favours the extinction of prey [1], the extinction rate decreases drastically for a simultaneous perturbation of additive and multiplicative white noise originating from a common source, hence connected through a correlation parameter.

From the expressions of \( f(x) \), \( b \) and \( c \) given in Eqs. (19) and (20) it is clear that for \( \lambda = 1.0 \) we have always a singular distribution for \( \sigma = D \), since it makes both the parameter \( b \) and \( c \) zero and eventually leads to the divergence of all the \( q \)’s and \( \nu \). However this divergence can be removed for appreciable difference between the \( \sigma \) and \( D \) values. In Fig.(4) we have plotted the typical behaviour of SPDF for maximum correlation \( \lambda = 1 \) which shows monotonic decreasing behaviour. In contrast to the behaviour shown in Fig.(3), Fig.(4), however, shows the hastening of prey’s extinction for a full correlation between additive and multiplicative noises.

In this brief communication we have studied the effect of environmental fluctuation of the birth rate of the prey in terms of external correlated noise processes which appreciably modify the macroscopic behaviour of a two-state predator model. We have shown how the correlation between the two noise processes which owe a common origin may drastically prevent the extinction of the prey.
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REFERENCES

[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).

[2] K. Lindenberg and B. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH, New York, 1990).

[3] A. Fulinski and T. Telejko, Phys. Lett. A **152**, 19 (1991).

[4] Li Cao and Da-jin Wu, Phys. Lett. A **185**, 59 (1994).

[5] S. Zhu, Phys. Rev. A **47**, 2405 (1993).

[6] Wu Da-jin, Cao Li and Ke Sheng-zhi, Phys. Rev. E **50**, 2496 (1994); Ya Jia and Jia-rong Li, Phys. Rev. E **53**, 5786 (1996).

[7] A. J. R. Madureira, P. Hänggi and H. S. Wio, Phys. Lett. A **217**, 248 (1996).

[8] J. H. Li and Z. Q. Huang, Phys. Rev. E **53**, 3315 (1996); *ibid* **57**, 3917 (1998).

[9] V. Berdichevsky and M. Gitterman, Phys. Rev. E **60**, 1494 (1999).

[10] B. C. Bag, S. K. Banik and D. S. Ray, Phys. Rev. E **64**, 026110 (2001).
FIGURES

FIG. 1. Plot of extrema of SPDF as a function of β for different values of noise correlation λ using α = 4.5, θ = 0.1, σ = 33.0 and D = 3.0.

FIG. 2. Same as in Fig.(1) but for different values of additive noise strength D. The other parameters are same except λ = 1.

FIG. 3. Plot of $P_s(x)$ against $x$ for different values of noise correlation λ using $\alpha = 4.5$, $\theta = 0.1$, $\beta = 7.5$, $\sigma = 3.0$ and $D = 0.3$. (a) For low values of λ and (b) for high values of λ.

FIG. 4. Same as in Fig.(3) but for $\lambda = 1.0$ and $D = 2.12$. 
Extrema of SPDF

Fig. (1)
Fig. (2)

D = 0.0
D = 1.0
D = 9.0
D = 18.0
D = 30.0

Extrema of SPDF
Fig. (3)
Fig.(4)