Using Cauchy theorem to compute complicated real integrals

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Abstract. As the central object in the theory of complex analysis, Holomorphic functions have many elegant mathematical properties. Holomorphic functions are extremely valuable because, on the one hand, they are unexpectedly common, and on the other hand, they may be used to establish extremely powerful theorems. For example, To establish theorems like the prime number theorem, analytic number theorists commonly create holomorphic or meromorphic functions that hold number-theoretic information, such as the Riemann zeta function. Given knowledge about a holomorphic function in a relatively small part of its domain, one may extract information about the function's behavior in other a priori unrelated sections of its domain, according to Cauchy's integral formula and the identity theorem (and this is what allows things like contour integration to work). Due to difficulties in obtaining primitives of some real function, the fundamental theorem of Calculus does not work in most cases. In this article, we review the Cauchy theorem and use it as a tool to compute several real integrals.

1. Introduction

Complex analysis, branches of analysis, mainly focus on the analytic function on an open region of the complex plane. However, complex analysis has an intertwined deep connection with other fields of mathematics and physics, such as number theory, algebraic geometry, quantum mechanics \([1-10]\). Complex analysis revolves around holomorphic functions, which are functions defined on an open subset of the complex number plane \(C\) with values in \(C\) that are complex-differentiable at all points. This is a considerably more stringent criterion than real differentiability, implying that the function is infinitely often differentiable and may be represented by its Taylor series. We say a function \(f\) is entire if and only if it is homlomorphic on the whole complex plane. There are another two equivalent definitions for entire functions. One is that a function \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) is entire if and only if it converges for every \(z \in C\). Another equivalent definition for entire functions is that a function \(f\) is entire if and only if it is everywhere analytic, that is for every \(z_0 \in C\) there is a positive number \(r\) such that

\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{for} \quad |z-z_0| < r. \]

Because entire functions permit power series expansions that converge everywhere, they can be considered as a complex-valued version of generalization of polynomials. For physical phenomena with finite spectra, such as significant classes of time-changing signals and spatially variable fields, entire functions are suitable mathematical models.
In this paper, our main purpose is to investigate several challenging real integrals by using tools from complex analysis. In the first section, we review a nice property of the holomorphic function. In the second section, we give a modified proof of the Cauchy theorem with full details. In the last section, we use the Cauchy theorem as a tool to investigate several important real integrals in analysis.

2. Main work

Proposition 2.1.1[10]

The function \( f \) is holomorphic in an open set \( \Omega \). Then if one of the following conditions hold,
1. \( \text{Re}(f) \) is constant;
2. \( \text{Im}(f) \) is constant;
3. \( |f| \) is constant;
Then the function \( f \) is a constant function.

Proof. Suppose that \( \text{Re}(f) \) is constant. Thus, \( f(z) = a + i \cdot v(x, y) \) for \( z = x + i \cdot y \), where \( v(x, y) \) is a real-valued function. Then by the Cauchy-Riemann equations \( 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \), we have that the derivative of \( v \) with regard to variable \( y \), \( \frac{\partial v}{\partial y} \), is zero. Thus \( v \) depends only on the variable \( x \). However, using another part of the Cauchy-Riemann equations again, we obtain that \( 0 = \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}. \)

So \( \frac{\partial v}{\partial x} \) is zero, and thus \( v \) depends only on \( y \). Since \( v \) depends on neither \( x \) nor \( y \), it follows that \( v \) is a constant.

The similar argument works if \( \text{Im}(f) \) is constant. Alternatively, if \( \text{Im}(f) \) is constant, then \( \text{Re}(f) \) is constant, and thus \( f \) is constant.

Now suppose \( |f| \) is constant. Writing \( f(z) = u(x, y) + i \cdot v(x, y) \) for \( z = x + iy \), we then have that \( u(x, y)^2 + v(x, y)^2 \) is constant. In particular, this implies that \( \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \) and that \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \) Thus, we can again use the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}
\]

and

\[
\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x}.
\]

Using a similar argument as above, we can get to the conclusion that \( f \) is a constant.

Theorem 2.1[10]

Suppose \( f \) is continuously complex differentiable on a region \( \Omega \), then

\[
\int_{\gamma} f(z)dz = 0,
\]

where \( \gamma \) is any closed curve inside the region \( \Omega \).

Proof. For any close curve \( \gamma \), (Let \( R \) be the region inside \( \gamma \))

\[
\int_{\gamma} Fdx + Gdy = \int_{R} \left( F_{x} - G_{y} \right) dx dy.
\]

Now, suppose that \( f = u + i \cdot v \) and that \( u(x, y) \) and \( v(x, y) \) have continuous partial derivatives. Then \( \int_{\gamma} f dz = \int_{\gamma} (udx - vdy) + i \int_{\gamma} (udy + vdx). \)

Consider the vector fields \((F_1, G_1) = (u, -v)\) and \((F_2, G_2) = (v, u).\) Then by the Cauchy Riemann equations

\[
\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial y},
\]

And thus by Green’s theorem, the line integral is zero.

Theorem 2.2 (Goursat’s theorem)[5]
Let $S$ be a triangle in $\mathbb{C}$ which is fully contained in an open $\Omega \subseteq \mathbb{C}$ (including the boundary of $S$). Denote by $T$ the boundary curve of $T$ with counterclockwise orientation. Then for every analytic $f : \Omega \to \mathbb{C}$ we have

$$\int_T f(z)dz = 0.$$ 

Proof. Let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$. Denote the corners of the triangle $S$ by $z_0, z_1, z_2$. Let $z'_0 = \frac{z_0 + z_1}{2}$, $z'_1 = \frac{z_1 + z_2}{2}$, and $z'_2 = \frac{z_2 + z_0}{2}$ be the midpoints between these corners.

We would cut the $S$ into four smaller triangles which are $S_0^{(1)}, S_1^{(1)}, S_2^{(1)}$ and $S_3^{(1)}$, where $S_0^{(1)}$ has corners $z_0, z'_0, z'_2$, $S_1^{(1)}$ has corners $z_0, z_1, z'_1$, $S_2^{(1)}$ has corners $z'_1, z_2, z'_2$, and $S_3^{(1)}$ has corners $z_0, z'_1, z'_2$. Let $T_0^{(1)}, \ldots, T_3^{(1)}$ be the boundary curves of $S_0^{(1)}, \ldots, S_3^{(1)}$. Then we have

$$\int_T f(z)dz = \int_{T_0^{(1)}} f(z)dz + \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz,$$

because the integrals over the inner edges cancel. In particular, there exists at least one of the $T_i^{(1)}$ such that

$$\left|\int_T f(z)dz\right| \leq 4 \left|\int_{T_i^{(1)}} f(z)dz\right|.$$ We pick the corresponding triangle and denote it by $S_i^{(1)}$.

We iterate this procedure and obtain a sequence $S^{(n)}$ of triangles with boundary curves $T^{(n)}$ such that for every $n$ we have $\left|\int_T f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right|$. Furthermore, for every $n$ we have

$$d^{(n)} = 2^{-n}d^{(0)}, p^{(n)} = 2^{-n}p^{(0)}.$$ 

For every $n$ let $z_n$ be an arbitrary point in $S_n$. By construction the $z_n$ form a Cauchy sequence, and hence they converge to a point $z_0 \in S$.

By assumption $f$ is differentiable in $z_0$. Hence for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $|z - z_0| < \delta$ we have

$$f(z) = f(z_0) + f'(z)(z - z_0) + R(z, z_0),$$

where the remainder satisfies $|z, z_0| \leq \varepsilon |z - z_0|$. We can choose $n$ large enough such that for all $z_0 \in S_n$ we have $|z - z_0| < \delta$. Then we get

$$\left|\int_{T_n} f(z)dz\right| = \left|\int_{T_n} R(z, z_0)dz\right| \leq \varepsilon 2^{-n}d^{(n)} 2^{-n}p^{(n)}.$$ 

Hence we can conclude that $\left|\int_T f(z)dz\right| \leq \varepsilon d^{(n)} p^{(n)}$. As $\varepsilon \to 0$ was arbitrary, this finishes the proof.

Next, with the knowledge of complex analysis stated above, some important integrals are elaborated. Proposition 2.3 For all $\xi \in \mathbb{C}$

$$e^{-\pi \xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx.$$ 

Proof. Expressing that $e^{-\pi x^2}$ equals its Fourier transform up to a constant factor.

For $\xi = 0$ the formula reads

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$ 

This will not prove. Then using that cos is an even function and that sin is odd, we find from Euler’s formulas

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi x \xi) dx,$$

showing that it is enough to prove the formula for $\xi > 0$.

Fixing $\xi > 0$, we integrate $f(z) = e^{-\pi x^2}$ along the boundary of the rectangle whose parameterisations (let $b = 2\pi \xi$) of the four sides of the rectangle are given as

$$y_1(x) = x, \quad x \in [-a, a],$$
$$y_2(x) = a + ix, \quad x \in [0, b],$$
$$y_3(x) = -x + ib, \quad x \in [-a, a],$$
$$y_4(x) = -a + i(b - x), \quad x \in [-a, b].$$

This integral is 0 by the Goursat’s theorem because $f$ is holomorphic in $\mathbb{C}$. 


and we get

\[
\begin{align*}
\int_{\gamma_1} f &= \int_{-a}^{a} e^{-\pi x^2} \, dx \\
\int_{\gamma_2} f &= i e^{-\pi a^2} \int_{0}^{b} e^{\pi x^2} e^{-iax} \, dx \\
\int_{\gamma_3} f &= -e^\pi b^2 \int_{\gamma_4} e^{-\pi x^2} e^{ibx} \, dx \\
\int_{\gamma_4} f &= -i e^{-\pi a^2} \int_{0}^{b} e^{\pi (b-x)^2} e^{i(a(b-x))} \, dx
\end{align*}
\]

Hence

\[
\left| \int_{\gamma_2} f \right| , \left| \int_{\gamma_4} f \right| \leq e^{-\pi a^2} b e^{\pi b^2} \to 0 \text{ for } a \to \infty.
\]

Moreover,

\[
\lim_{a \to \infty} \int_{\gamma_3} f = -e^\pi b^2 \int_{-\infty}^{\infty} e^{-\pi x^2} e^{ibx} \, dx,
\]

and

\[
\lim_{a \to \infty} \int_{\gamma_4} f = \int_{-\infty}^{\infty} e^{\pi (b-x)^2} e^{i(a(b-x))} \, dx = 1,
\]

so letting \(a \to \infty\) in \(\sum_{n=1}^{4} \int_{\gamma_n} f = 0\), we get

\[
1 + 0 - e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i \xi x} \, dx + 0 = 0
\]

showing the formula.

**Proposition 2.4**

\[
\int_{0}^{\infty} \frac{1 - \cos(x)}{x^2} \, dx = \frac{\pi}{2}
\]

**Proof.**

Consider the function \(f(x) = \frac{1 - e^{ix}}{x^2}\).

The path which \(f(x)\) is integrated along is shown as Figure 1.

\[
\begin{align*}
\lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{-R}^{-\varepsilon} f(x) \, dx &= \int_{-\infty}^{0} \frac{1 - e^{ix}}{x^2} \, dx, \\
\lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{R} f(x) \, dx &= \int_{0}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx.
\end{align*}
\]

According to (1) and (2), we can get \(\int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx\), so
\[
\int_C f(x) \, dx = \int_{-\infty}^\infty \frac{1 - e^{ix}}{x^2} \, dx + \int_\varphi + \int_\Gamma
\]

Then, we can use the substitution \( x = \text{Re}^\imath \theta \) to estimate \( \int_\Gamma f(x) \, dx \),

\[
\left| \int_\Gamma f(x) \, dx \right| = \left| \int_0^\pi \frac{1 - e^{i\text{Re}^\imath \theta}}{x^2} \text{i}xd\theta \right|
\]

\[
\leq \int_0^\pi \left| \frac{1 - e^{i\text{Re}^\imath \theta}}{x} \right| \, d\theta = \int_0^\pi 1 - e^{i\text{Re}^\imath \theta} \, d\theta
\]

\[
\leq \frac{1}{R} \int_0^\pi 1 + |e^{i\text{Re}^\imath \theta}| \, d\theta = \frac{1}{R} \int_0^\pi 1 + |e^{i\text{Re}^\imath \theta}| \, d\theta
\]

\[
= \frac{1}{R} \int_0^\pi 1 + |e^{i\text{Re}^\imath \theta}| \, d\theta = \frac{1}{R} \int_0^\pi e^{-Rs\theta} \, d\theta
\]

So we get,

\[
\left| \int_\Gamma f(x) \, dx \right| \leq \frac{2}{R} \int_0^\pi e^{-Rs\theta} \, d\theta \leq \frac{2}{R} \int_0^\pi e^{-R^2\theta/2} \, d\theta
\]

\[
= \frac{2}{R} \left[ -\frac{\pi}{2R} e^{-R^2\theta/2} \right]_0^{\infty} = -\frac{\pi}{R^2} (e^{-R} - 1)
\]

Thus,

\[
\lim_{R \to \infty} \left| \int_\Gamma f(x) \, dx \right| \leq \lim_{R \to \infty} -\frac{\pi}{R^2} (e^{-R} - 1) = 0
\]

Which is \( \int_\Gamma f(x) \, dx = 0 \).

We also can use the substitution \( x = \epsilon e^{\theta} \) to estimate \( \int_\varphi \)

\[
\int_\varphi f(x) \, dx = \int_0^\pi \frac{1 - e^{i\epsilon e^{\theta} \theta}}{\epsilon e^{\theta}} \text{i}e^{\theta} \, d\theta = -i \int_0^\pi \frac{1 - e^{i\epsilon e^{\theta} \theta}}{\epsilon e^{\theta}} \, d\theta
\]

\[
= -i \int_0^\pi \frac{1}{\epsilon e^{\theta}} \sum_{k=0}^\infty \frac{i^k (\epsilon e^{\theta})^k}{k!} \, d\theta = i \int_0^\pi \frac{1}{\epsilon e^{\theta}} \left( \sum_{k=1}^\infty \frac{i^k (\epsilon e^{\theta})^k}{k!} \right) \, d\theta
\]

\[
= i \int_0^\pi \frac{i^1}{1!} \, d\theta = -\pi
\]

Also, we can easily see \( \oint f(x) \, dx = 0 \), thus we can get

\[
\oint f(x) \, dx = \int_{-\infty}^{-\epsilon} + \int_\varphi + \int_{\epsilon} + \int_\Gamma
\]

\[
\Rightarrow 0 = \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx - \pi + 0
\]

As we know,

\[
\int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx = \int_{-\infty}^{\infty} \frac{1 - \cos(x)}{x^2} - \frac{\sin(x)}{x^2} \, dx
\]

And,

\[
\text{Re} \left( \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} \, dx \right) = \text{Re}(\pi)
\]

Thus,
\[
\int_{-\infty}^{\infty} \frac{1 - \cos(x)}{x^2} \, dx = \pi \Rightarrow \int_{0}^{\infty} \frac{1 - \cos(x)}{x^2} \, dx = \frac{\pi}{2}
\]

Proposition 2.5 (Fresnel integrals)
\[
\int_{0}^{\infty} \sin(x^2) \, dx = \int_{0}^{\infty} \cos(x^2) \, dx = \frac{\sqrt{2\pi}}{4}.
\]
Here, \( \int_{0}^{\infty} f(x) \, dx \) is interpreted as \( \lim_{R \to \infty} \int_{0}^{R} f(x) \, dx \).

Proof.
As we know \( e^{ix^2} = \cos(x^2) + isin(x^2) \), so
\[
1 = \int_{0}^{\infty} (\cos(x^2) + isin(x^2)) \, dx
= \int_{0}^{\infty} e^{ix^2} \, dx = \int_{0}^{\infty} e^{-(e^{-\pi x^2})^2} \, dx = \lim_{R \to \infty} \int_{0}^{R} e^{-(e^{-\pi x^2})^2} \, dx
\]
We can find,
\[
\text{Re}(I) = \int_{0}^{\infty} \cos(x^2) \, dx
\]
\[
\text{Im}(I) = \int_{0}^{\infty} \sin(x^2) \, dx
\]
Then we can use the substitution, let
\[
t = e^{-\frac{\pi}{4}x}
\]
\[
dt = e^{-\frac{\pi}{4}x} \, dx
\]
\[
dx = e^{\frac{\pi}{4}} \, dt
\]
Thus,
\[
1 = e^{\frac{\pi}{4}} \lim_{R \to \infty} \int_{0}^{\lim_{R \to \infty}} e^{-(t)^2} \, dt
\]

Figure 2. The path of the integration.

Notice that, the integral can be viewed as three parts shown as Figure 2.
It is natural to make the following changes of variables:
\[
z = \text{Re} \, e^{i\theta},
\]
\[
\theta \in \left[-\frac{\pi}{4}, 0\right],
\]
\[
dz = i \text{Re} \, e^{i\theta} \, d\theta.
\]
Thus,
Γ: \int_{\frac{-\pi}{4}}^{0} e^{-R^2e^{2i\theta}}iRe^{i\theta} d\theta.

Then, we use the inequality,
\[
\left| \int_{\Gamma} e^{-R^2e^{2i\theta}}iRe^{i\theta} \right| \leq \int_{\frac{-\pi}{4}}^{0} \left| e^{-R^2e^{2i\theta}}iRe^{i\theta} \right| d\theta
\]
\[
= R \int_{\frac{-\pi}{4}}^{0} e^{-R^2e^{2i\theta}} d\theta = R \int_{\frac{-\pi}{4}}^{0} \left| e^{-R^2\cos(2\theta)e^{-iR^2\sin(2\theta)}} \right| d\theta.
\]

The \( e^{-iR^2\sin(2\theta)} \) is a point on unit circle, so its absolute value is 1, so we can get
\[
R \int_{\frac{-\pi}{4}}^{0} e^{-R^2\cos(2\theta)} d\theta \leq R \int_{\frac{-\pi}{4}}^{0} e^{-R^2} d\theta = R \int_{\frac{-\pi}{4}}^{0} e^{-R^2\pi^2} e^{-R^2} d\theta
\]
\[
= Re^{-R^2} \int_{\frac{-\pi}{4}}^{0} e^{-R^2\pi^2} d\theta = Re^{-R^2} \left[ -\frac{\pi}{4R^2} e^{-R^2\pi^2} \right]_{\frac{-\pi}{4}}^{0}
\]
\[
= Re^{-R^2} (\frac{\pi}{4R^2} (1 - e^{-R^2\pi^2})) = -\pi e^{-R^2} (1 - e^{-R^2})
\]
\[
= -\frac{\pi e^{-R^2}}{4R} + \frac{\pi}{4R}.
\]

Thus,
\[
\lim_{R \to \infty} \left| \int_{\Gamma} e^{-R^2e^{2i\theta}}iRe^{i\theta} \right| \leq \frac{\pi}{4R} - \frac{\pi e^{-R^2}}{4R}
\]
\[
\lim_{R \to \infty} \left| \int_{\Gamma} e^{-R^2e^{2i\theta}}iRe^{i\theta} \right| = 0
\]

Then we evaluate \( \int_{\Gamma} e^{-(-t)^2} dt \),
\[
\int_{\Gamma} e^{-(-t)^2} dt = -e^{\frac{-\pi}{4}} \int_{0}^{R} e^{-(-t)^2} dt
\]
\[
I = e^{\frac{-\pi}{4}} \lim_{R \to \infty} \int_{0}^{R} e^{-(-t)^2} dt = e^{\frac{-\pi}{4}} \lim_{R \to \infty} \int_{0}^{R} e^{-(-t)^2} dt
\]
\[
= e^{\frac{-\pi}{4}} \int_{0}^{\infty} e^{-(-t)^2} dt
\]
\[
= e^{\frac{-\pi}{4}} \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2} \sqrt{\frac{\pi}{2}} + \frac{\sqrt{\pi}}{2} \sqrt{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{4} + \frac{\sqrt{2\pi}}{4}.
\]

So, in \( I \), we can get \( \text{Re}(I) = \text{Im}(I) = \frac{\sqrt{2\pi}}{4} \), which is
\[
\int_{0}^{\infty} \sin(x^2) dx = \int_{0}^{\infty} \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.
\]

3. Conclusion
The Cauchy theorem, a basic theorem in Complex analysis, has a powerful function in computing some real integrals that the fundamental theorem of Calculus cannot compute. In the future, we will report how to use the Residue theorem to compute real integrals. Also, the proof of Jensen’s formula, growth order and infinite products will be discussed in detail. Understanding how these theorems and ideas are linked to show some interesting features of the entire function is important. Weierstrass infinite products
and the Hadamard factorisation theorem proceed deeper into the function theory, where additional fascinating constructions can be found. We will also try to present a self-contained overview of microlocal analysis, a branch of contemporary analysis that is used in a wide range of mathematical areas, from partial differential equations research to dynamical systems research.

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