UNIQUENESS IN THE CALDERÓN PROBLEM AND BILINEAR RESTRICTION ESTIMATES

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Abstract. Uniqueness in the Calderón problem in dimension bigger than two was usually studied under the assumption that conductivity has bounded gradient. For conductivities with unbounded gradients uniqueness results have not been known until recent years. The latest result due to Haberman basically relies on the optimal $L^2$ restriction estimate for hypersurface which is known as the Tomas-Stein restriction theorem. In the course of developments of the Fourier restriction problem bilinear and multilinear generalizations of the (adjoint) restriction estimates under suitable transversality condition between surfaces have played important roles. Since such advanced machineries usually provide strengthened estimates, it seems natural to attempt to utilize these estimates to improve the known results. In this paper, we make use of the sharp bilinear restriction estimates, which is due to Tao, and relax the regularity assumption on conductivity. We also consider the inverse problem for the Schrödinger operator with potentials contained in the Sobolev spaces of negative orders.

1. Introduction

For $d \geq 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, and let $A(\Omega)$ denote the set of all functions $\gamma \in L^\infty(\Omega)$ satisfying $\gamma \geq c$ in $\Omega$ for some $c > 0$. Throughout the paper, we assume $\gamma \in A(\Omega)$. For $f \in H^{1/2}(\partial \Omega)$ and $\gamma \in A(\Omega)$, we consider the Dirichlet problem:

$$\begin{cases}
\text{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega, \\
u = f \quad \text{on } \partial \Omega.
\end{cases}$$

Let $\partial/\partial \nu$ denote the outward normal derivative on the boundary $\partial \Omega$. The Dirichlet-to-Neumann map $\Lambda_\gamma$ is formally defined by $\Lambda_\gamma(f) = \gamma \partial u_f / \partial \nu |_{\partial \Omega}$. Since the boundary value problem (1.1) has a unique solution $u_f \in H^1(\Omega)$ (for example, see [14, Theorem 2.52]), by the trace theorem and Green’s formula, the operator can be formulated in the weak sense. Precisely, for $f \in H^{1/2}(\partial \Omega)$ and $g \in H^{1/2}(\partial \Omega)$,

$$\langle \Lambda_\gamma(f), g \rangle = \int_\Omega \gamma \nabla u_f \cdot \nabla v dx$$

where $v \in H^1(\Omega)$ and $v|_{\partial \Omega} = g$. It is well known that $\Lambda_\gamma$ is well defined and $\Lambda_\gamma$ is continuous from $H^{1/2}(\partial \Omega)$ to $H^{-1/2}(\partial \Omega)$.

Calderón’s problem. Calderón’s inverse conductivity problem concerns whether $\gamma$ can be uniquely determined from $\Lambda_\gamma$, that is to say, whether the map $\gamma \mapsto \Lambda_\gamma$ is injective. The problem was introduced by Calderón [9] who showed uniqueness for the linearized problem. Afterwards, numerous works have been devoted to extending the function class $X(\Omega) \subset A(\Omega)$ for which the map $X(\Omega) \ni \gamma \mapsto \Lambda_\gamma$ is injective [47]. Kohn and Vogelius [26] showed that if $\partial \Omega$ is smooth and $\Lambda_\gamma = 0$ then $\gamma$ vanishes to infinite order at $\partial \Omega$ provided that $\gamma \in C^\infty(\Omega)$ (also, see [40]). Consequently, the mapping $\gamma \mapsto \Lambda_\gamma$ is injective if we choose $X(\Omega)$ to be the space of analytic functions on $\overline{\Omega}$. Sylvester and Uhlmann in their influential work [39] proved that $\gamma$ is completely determined by $\Lambda_\gamma$ if $\gamma \in C^2(\Omega)$ for $d \geq 2$. They made use of the complex geometrical optics solutions which become most predominant tool not only in the Calderón problem but also in various

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related problems. Afterward, it has been shown that regularity on conductivity can be lowered further. The $C^2$ regularity assumption was relaxed to $C^{3/2 + \epsilon}$ by Brown [5]. Päivärinta, Panchenko, and Uhlmann [37] showed global uniqueness of conductivities in $W^{3/2, \infty}$ and results with conductivities in $W^{3/2, p}$, $p > 2d$ were obtained by Brown and Torres [7]. Nguyen and Spira [36] obtained a result with conductivities in $W^{s, 3/2}$ for $3/2 < s < 2$ when $d = 3$. In two dimensions, the problem has different nature and uniqueness of $L^\infty$ conductivity was established by Astala and Päivärinta [2]. Their result is an extension of the previous ones in [35, 8]. Recently, Čárstea and Wang [11] obtained uniqueness of unbounded conductivities. (See [11] and references therein for related results.) For $d \geq 3$, the regularity condition was remarkably improved by Haberman and Tataru [21]. By making use of Bourgain’s $X^{s,b}$ type spaces, they proved uniqueness when $\gamma \in C^1(\overline{\Omega})$, or $\gamma \in W^{1,\infty}(\Omega)$ with the assumption that $\|\nabla \log \gamma\|_{L^\infty(\overline{\Omega})}$ is small. This smallness assumption was later removed by Caro and Rogers [10].

As already mentioned before, for $d \geq 3$, most of the previous results were obtained under the assumption that $\gamma$ has bounded gradient. Since the equation $\nabla (\gamma \nabla u) = 0$ can be rewritten as $\Delta u + W \cdot \nabla u = 0$ with $W = \nabla \log \gamma$, it naturally relates to the unique continuation problem for $u$ satisfying $|\Delta u| \leq W|\nabla u|$. Meanwhile, it is known that the unique continuation property holds with $W \in L^p_{loc}$ [18] and generally fails if $W \in L^p_{loc}$ for $p < d$ [25]. In this regard, Brown [7] proposed a conjecture that uniqueness should be valid for $\gamma \in W^{1,d}(\Omega)$, but no counterexample which shows the optimality of this conjecture has been known yet.

Recently, Brown’s conjecture was verified by Haberman [19] for $d = 3, 4$, and he also showed that uniqueness remains valid even if $\nabla \gamma$ is unbounded when $d = 5, 6$. More precisely, he showed that $\gamma \to \Lambda_\gamma$ is injective if $\gamma$ belongs to $W^{s,p}(\Omega)$ with $d \leq p \leq \infty$ and $s = 1$ for $d = 3, 4$, and $d \leq p < \infty$ and $s = 1 + \frac{d-4}{2p}$ for $d = 5, 6$.

For a given function $f$, let $M_q$ be the multiplication operator $f \mapsto qf$ and let $O_d$ be the orthonormal group in $\mathbb{R}^d$. Most important part of the argument in Haberman [19] (21) is to show that there are sequences $\{U_j\}$ in $O_d$ and $\{\tau_j\}$ in $(0, \infty)$ such that

$$
\lim_{j \to \infty} \|M_1(\nabla f)_{U_j}\|_{X^{1/2}_{(\tau_j)}\to X^{-1/2}_{(\tau_j)}} = 0
$$

and $\tau_j \to \infty$ as $j \to \infty$. We refer the reader forward to Section 2 for the definition of the spaces $X^{1/2}_{(\tau)}$ and $X^{-1/2}_{(\tau)}$. If $f \in L^2$ and has compact support, it is not difficult to show $\lim_{\tau \to \infty} \|M_f\|_{X^{1/2}_{(\tau)}\to X^{-1/2}_{(\tau)}} = 0$, see Remark 5. However, $\|M_1(\nabla f)\|_{X^{1/2}_{(\tau)}\to X^{-1/2}_{(\tau)}}$ does not behave as nicely as $\|M_f\|_{X^{1/2}_{(\tau)}\to X^{-1/2}_{(\tau)}} = 0$. This is also related to the failure of the Carleman estimate of the form $\|e^{\tau \cdot x} \nabla u\|_{L^q} \leq C \|e^{\tau \cdot x} \Delta u\|_{L^p}$ when $d \geq 3$. See [24, 3] [38, 22]. To get around the difficulty averaged estimates were considered ([21, 36, 19]). In view of Wolff’s work [48] it still seems plausible to expect (1.2) or its variant holds with $\gamma \in C^1(\overline{\Omega})$ or $\gamma \in W^{1,\infty}(\Omega)$ with the assumption that $\|\nabla \log \gamma\|_{L^\infty(\overline{\Omega})}$ is small. This smallness assumption was later removed by Caro and Rogers [10].

*Restriction estimate.* Let $S \subset \mathbb{R}^{d-1}$ be a smooth compact hypersurface with nonvanishing Gaussian curvature and let $d\mu$ be the surface measure on $S$. The estimate $\|f\delta\|_{L^2(d\mu)} \lesssim \|f\|_{L^r(\mathbb{R}^{d-1})}$, $r \leq 2d/(d+2)$ is known as the Stein-Tomas theorem. The range of $r$ is optimal since the estimate fails if $r > 2d/(d+2)$. The restriction estimate can be written in its adjoint form:

$$
\|f\delta d\mu\|_{L^r(\mathbb{R}^{d-1})} \leq C(d, p, q, S) \|f\|_{L^p(S, d\mu)}.
$$

The restriction conjecture is to determine $(p,q)$ for which (1.3) holds. Even for most typical surfaces such as the sphere and the paraboloid, the conjecture is left open when $d \geq 4$. We refer the reader to [17, 48] for the most recent progress. There have been bilinear and multilinear generalizations of the linear estimate (1.3) under additional transversality conditions between surfaces (33, 11, 4), and these estimates played important roles in development of the restriction problem. To be precise, let $S_1, S_2 \subset S$ be hypersurfaces in $\mathbb{R}^{d-1}$ and let $d\mu_1, d\mu_2$ be the surface measures on $S_1, S_2$, respectively. The following form of estimate is called bilinear (adjoint) restriction estimate:

$$
\|f \delta d\mu_1 g d\mu_2\|_{L^{r/2}(\mathbb{R}^{d-1})} \leq C\|f\|_{L^2(S_1, d\mu_1)} \|g\|_{L^2(S_2, d\mu_2)}.
$$

1We use $d-1$ instead of $d$ to avoid confusion in the subsequent discussion.
Under certain condition between $S_1$ and $S_2$ the estimate (1.4) remains valid for some $q < \frac{2d}{d-2}$ with which (1.3) fails if $p = 2$. (See Theorem 2.5 and [42, 29] for detailed discussion.)

By duality, in order to get estimate for $\|M\nabla f\|_{X^{1/2}_\xi(\tau) \to X^{-1/2}_\xi(\tau)}$, we consider the bilinear operator $B\nabla f$ which is given by

$$X^{1/2}_\xi(\tau) \times X^{1/2}_\xi(\tau) \ni (u, v) \mapsto B\nabla f (u, v) = \langle \nabla f u, v \rangle.$$

Compared with the previous results the main new input in [19] was the $L^2$-Fourier restriction theorem for the sphere which is due to Tomas [45] and Stein [38] (Theorem 2.3). This is natural in that the multiplier which defines $X^{-1/2}_\xi(\tau)$ has mass concentrated near the surface $\Sigma$ given by (2.1) while the restriction estimate provides estimates for functions of which Fourier transform concentrates near hypersurface. The use of the bilinear restriction estimate instead of the linear one has a couple of obvious advantages. The bilinear restriction estimate not only has a wider range of boundedness but also naturally fits with the bilinear operator $B\nabla f$.

In this paper we aim to improve Haberman’s results by making use of the bilinear restriction estimate (1.4) for the elliptic surfaces (see Definition 2.2 and Theorem 2.5). However, the bilinear estimates outside of the range of the $L^2$ restriction estimate are only true under the extra separation condition between the supports of Fourier transforms of the functions (see Corollary 2.6). Such estimates cannot be put in use directly. This leads to considerable technical involvement. The following is our main result.

**Theorem 1.1.** Let $d = 5, 6$ and $\Omega$ be a bounded domain with Lipschitz boundary. Then the map $\gamma \mapsto \Lambda_\gamma$ is injective if $\gamma \in W^{s,p}(\Omega) \cap A(\Omega)$ for $s > s_d(p)$, where

$$s_d(p) = \begin{cases} 
1 + \frac{d-5}{2p} & \text{if } d + 1 \leq p < \infty, \\
1 + \frac{d^2-5d+6-p}{2p(d-1)} & \text{if } d \leq p < d + 1.
\end{cases}$$

Here, $W^{s,p}(\Omega)$ is the Sobolev-Slobodeckij space.

In particular, uniqueness holds for $\gamma \in W^{s,5}(\Omega) \cap A(\Omega)$ if $d = 5$ and $s > \frac{41}{30}$, and for $\gamma \in W^{s,6}(\Omega) \cap A(\Omega)$ if $d = 6$ and $s > \frac{11}{10}$. Since $W^{s_d(p)+\epsilon,p} \not\hookrightarrow W^{1,\infty}$ if $\epsilon > 0$ is small enough, this result is not covered by the result in [10].

Even though our estimates are stronger than those in [19], the estimates do not immediately yield improved results in every dimensions. As is to be seen later in the paper, our estimates for the low frequency part are especially improved but this is not the case for the high frequency part since we rely on the argument based on the properties of $X^b_k$ spaces ([21, 19]).

The argument based on the complex geometrical optics solutions shows that the Fourier transforms of $q_i = \gamma_i^{-1/2} \Delta \gamma_i^{1/2}$, $i = 1, 2$, are identical as long as $\Lambda_\gamma_1 = \Lambda_\gamma_2$. As was indicated in [19] this approach has a drawback when we deal with less regular conductivity. In order to use the Fourier transform one has to
extend $\gamma_1 - \gamma_2 \in W_0^{s,p}(\Omega)$ to the whole space $\mathbb{R}^d$ such that $\gamma_1 - \gamma_2 = 0$ on $\Omega^c$. Such extension is possible by exploiting the trace theorem ([39] Theorem 1) but only under the condition $s - \frac{1}{p} \leq 1$. This additional restriction allows new results only for $d = 5, 6$ in Theorem 1.4. The same was also true with the result in [19]. However, as is mentioned in [19], if we additionally impose the condition $\partial \gamma_1/\partial \nu = \partial \gamma_2/\partial \nu$ on the boundary $\partial \Omega$, then Theorem 1.4 can be extended to higher dimensions $d \geq 7$. In fact, by [39] Theorem 1, the restriction $s - \frac{1}{p} \leq 1$ can be relaxed so that $s - \frac{1}{p} \leq 2$, which is valid for $s > s_d(p)$ for $d \geq 7$ and $p \geq d$.

See Remark 1 for the value of $s_d(p)$. However, the additional condition on the boundary is not known to be true under the assumption $\lambda_{\gamma_1} = \lambda_{\gamma_2}$ for $\gamma_1, \gamma_2$ as in Theorem 1.1. In [7] Brown and Torres proved that if $\lambda_{\gamma_1} = \lambda_{\gamma_2}$, then $\partial \gamma_1/\partial \nu = \partial \gamma_2/\partial \nu$ on $\partial \Omega$ for $\gamma_1, \gamma_2 \in W^{3/2,p}$, $p > 2d$, and $d \geq 3$.

If we had the endpoint bilinear restriction estimate (i.e., the estimate (1.4) with $q = \frac{2(d+1)}{d-1}$, see Remark 1), the argument in this paper would allow us to obtain the uniqueness result with $s = 1$ and $p \geq 6$ when $d = 5$, and with $s = 1 + 1/p$ and $p \geq 8$ when $d = 7$. Unfortunately the endpoint bilinear restriction estimate is still left open.

**Inverse problem for the Schrödinger operator.** For $d \geq 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $C^\infty$ boundary. We now consider the Dirichlet problem:

$$
\begin{align*}
\Delta u - qu &= 0 \quad \text{in } \Omega, \\
u u &= f \quad \text{on } \partial \Omega.
\end{align*}
$$

Let us set

$$
H^{s,p}_c(\Omega) = \{ q \in H^{s,p}(\mathbb{R}^d) : \text{supp } q \subset \Omega \}.
$$

Here $H^{s,p}_c$ is the Bessel potential space, see Notations for its definition. Since $H^{s,p}_c$ is defined by Fourier multipliers, the space is more convenient for dealing with various operators which are defined by Fourier transform. If we disregard $\epsilon$-loss of the regularity, the spaces $H^{s,p}_c$ and $W^{s,p}$ are essentially equivalent because $W^{s_1,p} \hookrightarrow H^{s_2,p}_c$ and $H^{s_1,p}_c \hookrightarrow W^{s_2,p}$ provided $s_1 > s_2$. (See [40] Section 2.3 for more details.) Thus, the statement of Theorem 1.2 does not change if $H^{s,p}_c$ is replaced by $W^{s,p}$.

Let $q \in H^{s,p}_c(\Omega)$ with $s, p$ satisfying (1.8). We assume that zero is not a Dirichlet eigenvalue of $\Delta - q$. Then, by the standard argument ([31]) we see that there is a unique solution $u_f \in H^1(\Omega)$ for every $f \in H^{+}_c(\partial \Omega)$. In fact, this can be shown by a slight modification of the argument in [27] Appendix A (also see the proof of Lemma 6.4 where $\int q u v \, dx$ is controlled while $q \in H^{s,p}_c(\Omega)$ and $u, v \in H^1(\Omega)$).

For $q \in H^{s,p}_c(\Omega)$ let $L_q$ denote the Dirichlet-to-Neumann map given by

$$
L_q f, g = \int_\Omega \nabla u \cdot \nabla v + q u v \, dx,
$$

where $u$ is the unique solution to (1.6) and $v \in H^1(\Omega)$ with $v|_{\partial \Omega} = q$. As in the Calderón problem, one may ask whether $q \mapsto L_q$ is injective. As is well known the problem is closely related to the Calderón problem. In fact, the Calderón problem can be reduced to the inverse problem for $\Delta - q$ with $q = \gamma^{-1/2} \Delta \gamma^{1/2}$ (see [39]). The problem of injectivity of $q \mapsto L_q$ was originally considered with $q \in H^{s,p}(\Omega)$, but it is not difficult to see that we may consider $q_1, q_2 \in H^{s,p}_c(\Omega)$ with $s > 0$. (See, for example, Brown-Torres [7].) Since $u, v \in H^1(\Omega)$, it is natural to impose $s \geq -1$. In fact, $L_q f$ is well defined provided that $q \in H^{s,p}_c(\Omega)$ with

$$
\max \left\{ -2 + \frac{d}{p}, -1 \right\} \leq s.
$$

The standard argument shows that $L_q : H^{+}_c(\partial \Omega) \rightarrow H^{-\frac{d}{p}}(\partial \Omega)$ is continuous.

The injectivity of the mapping

$$
H^{s,p}_c(\Omega) \ni q \mapsto L_q
$$

2By the inclusion $W^{s_1,p} \subset W^{s_2,p}$ for $s_1 \geq s_2$ and $1 < p < \infty$, it is enough to show Theorem 1.1 for $(s, p)$ satisfying $s_d(p) < s \leq \frac{1}{p} + 1$. 

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was shown with \( s = 0, p = \infty \) by Sylvester and Uhlmann [39]. The result was extended to include unbounded potential \( q \in L^{d+\epsilon} \) by Jerison and Kenig (see Chanillo [12]). The injectivity for \( q \) contained in the Fefferman-Phong class with small norm was shown by Chanillo [12] and the result for \( q \in L^\frac{d}{2} \) was announced by Lavine and Nachman in [34]. Their result was recently extended to compact Riemannian manifolds by Dos Santos Ferreira, Kenig, and Salo [13]. Also see [27] for extensions to the polyharmonic operators.

The regularity requirement for \( q \) can be relaxed. Results in this direction were obtained by Brown [5], Päivärinta, Panchenko, and Uhlmann [37], Brown and Torres [7] in connection with the Calderón problem. Those results can be improved to less regular \( q \). In fact, Haberman’s result implies that the injectivity holds with \( q \in H^{-1,d} \) when \( d = 3, 4 \) (see [20]). It seems natural to conjecture that the same is true in any higher dimensions. Interpolating this conjecture with the result due to Lavine and Nachman [34] \( (q \in L^\frac{d}{2}) \) leads to the following:

**Conjecture 1.** Let \( \frac{d}{2} \leq p < \infty \), and \( \Omega \) be a bounded domain with Lipschitz boundary. Suppose \( q_1, q_2 \in H^{s,p}_{c}(\Omega) \) and \( L_{q_1} = L_{q_2} \). If \( s > s_d(p) := \max\{-1, -2 + \frac{d}{p}\} \), then \( q_1 = q_2 \).

We define \( r_d : [\frac{d}{2}, \infty) \rightarrow \mathbb{R} \). For \( 3 \leq d \leq 6 \), set

\[
\begin{align*}
r_d(p) &= \begin{cases} 
-1 + \frac{d-5}{2p} & \text{if } p \geq d+1, \\
-\frac{d}{2} + \frac{d-2}{p} & \text{if } d+1 > p \geq 4, \\
-2 + \frac{d}{p} & \text{if } 4 > p \geq \frac{d}{2},
\end{cases}
\end{align*}
\]

and, for \( d \geq 7 \), set

\[
\begin{align*}
r_d(p) &= \begin{cases} 
-1 + \frac{d-5}{2p} & \text{if } p \geq \frac{d+9}{2}, \\
-\frac{d}{2} + \frac{3d-1}{4p} & \text{if } \frac{d+9}{2} > p \geq \frac{3d+7}{6}, \\
-\frac{12}{7} + \frac{6d}{7p} & \text{if } \frac{3d+7}{6} > p \geq \frac{d}{2}.
\end{cases}
\end{align*}
\]

The following is a partial result concerning Conjecture 1 when \( d \geq 5 \).

**Theorem 1.2.** Let \( d \geq 3 \) and \( d/2 \leq p < \infty \). The mapping (1.9) is injective if \( s > \max\{-1, r_d(p)\} \).

\[\text{3Their result can also be recovered by the argument in this paper. See Remark 4.}\]
When $d = 5$, Theorem 1.2 shows that $s > s^*_5(p)$ for $\frac{d}{2} \leq p < 4$ or $p \geq d + 1$, where Conjecture 1 is verified except for the critical case $s = s^*_5(p)$. Similarly, when $d = 6$ injectivity of (1.9) holds if $s > s^*_6(p)$ and $\frac{d}{2} \leq p < 4$. We illustrate our result in Figure 2.

Organization of the paper. In Section 2, we recall basic properties of the spaces $X^b$, and obtain estimates which rely on $L^2$ linear and $L^2$ bilinear restriction estimates. In Section 3 which is the most technical part of the paper, we make use of the bilinear restriction estimates and a Whitney type decomposition to get the crucial case $a > s^*_5(p)$, For a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ and a matrix $U \in O_d$ we define $f_U(x) := f(\tau^{-d} f(\tau^{-1} U x))$. If $U$ is the identity matrix, we denote $f_U$ by $f$.

For a function $f : \mathbb{R}^d \to \mathbb{C}$, we write $\mathcal{F} u(\xi) = \tilde{u}((\xi) = \int_{\mathbb{R}^d} e^{-ix \xi} u(x) dx$ and $\mathcal{F}^{-1} u(\xi) = (2\pi)^{-d} \mathcal{F} u(-\xi)$.

Notations. We list notations which are used throughout the paper.

- For $A, B > 0$ we write $A \lesssim B$ if $A \leq CB$ with some constant $C > 0$. We also use the notation $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.
- The orthogonal group in $\mathbb{R}^d$ is denoted by $O_d$.
- Let $\tau > 0$. For a function $f : \mathbb{R}^d \to \mathbb{C}$ and a matrix $U \in O_d$ we define $f_U(x) := f(\tau^{-d} f(\tau^{-1} U x))$. If $U$ is the identity matrix, we denote $f_U$ by $f$.
- The Fourier and inverse Fourier transforms: For an integrable function $u : \mathbb{R}^d \to \mathbb{C}$, we write $\mathcal{F} u(\xi) = \tilde{u}((\xi) = \int_{\mathbb{R}^d} e^{-ix \xi} u(x) dx$ and $\mathcal{F}^{-1} u(\xi) = (2\pi)^{-d} \mathcal{F} u(-\xi)$.
- For a measurable function $a$ with polynomial growth, let $a(D)f = F^{-1}(aFf)$.
- For $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we write $\text{dist}(x, E) = \inf\{|x - y| : y \in E\}$.
- For $E \subset \mathbb{R}^d$ and $\delta > 0$, we denote by $E + O(\delta)$ the $\delta$-neighborhood of $E$ in $\mathbb{R}^d$, i.e., $E + O(\delta) = \{x \in \mathbb{R}^d : \text{dist}(x, E) < \delta\}$.
- We set $S^{d-1}_k = \{x \in \mathbb{R}^d : |x| = 1\}$. Also, for $a \in \mathbb{R}^d$ and $r > 0$, $B_k(a, r) = \{x \in \mathbb{R}^d : |x - a| < r\}$.
- If $e_1$ and $e_2$ are a pair of orthonormal vectors in $\mathbb{R}^d$, we write $\xi_1 = \xi \cdot e_1$, $\xi_2 = \xi \cdot e_2$.
- For $\xi \in \mathbb{R}^d$ we sometimes write $\xi = (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2}$.
- For $s \geq 0$ and $p \in [1, \infty]$, we denote by $H^{s,p}$ the Bessel potential space $\{\varphi \in S' : (1 + |D|^2)^{\frac{s}{2}} \varphi \in L^p\}$ which is endowed with the norm $\|\varphi\|_{H^{s,p}} = \|(1 + |D|^2)^{\frac{s}{2}} \varphi\|_{L^p}$.
- For $s < 0$ and $p \in (1, \infty)$, we denote by $H^{s,p}((\Omega), p' = \frac{p}{p-1}$, the dual space of $H^{s,p}_0((\Omega))$.
- We use $\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle$ to denote the inner product and the bilinear pairing between distribution and function, respectively.

2. $X^b_{\xi}$ spaces and $L^2$ linear and bilinear restriction estimates

In this section, we recall basic properties of the $X^b_{\xi}$ spaces and linear and bilinear restriction estimates, which are to be used later.

2.1. Basic properties of $X^b_{\xi}$ spaces. For a fixed pair of orthonormal vectors $e_1, e_2$ in $\mathbb{R}^d$, let us set

$$\zeta(\tau) = \tau(e_1 - i e_2), \quad \tau > 0.$$  

For $\zeta \in \mathbb{C}^d$ with $\zeta \cdot \zeta = 0$, we denote the symbol of $e^{-x \cdot \zeta} \Delta e^{-x \cdot \zeta} = \Delta + 2 \zeta \cdot \nabla$ by $p_{\zeta}$, i.e., 

$$p_{\zeta}(\xi) = -|\xi|^2 + 2i \zeta \cdot \xi.$$  

By $\Sigma^\tau$ we denote the zero set of the polynomial $p_{\zeta(\tau)}$, i.e.,

$$\Sigma^\tau = \{\xi \in \mathbb{R}^d : p_{\zeta(\tau)}(\xi) = 0\} = \{\xi \in \mathbb{R}^d : \xi_1 = 0, |\xi - \tau e_2| = \tau\}. $$
Clearly, $\tau^{-1}\Sigma^\tau = \Sigma^1$ and it is easy to check that

\begin{equation}
|p_{\xi}(\tau)| \simeq \begin{cases} 
\tau \text{dist}(\xi, \Sigma^\tau) & \text{if dist}(\xi, \Sigma^\tau) \leq 2^{-7}\tau, \\
\tau^2 + |\xi|^2 & \text{if dist}(\xi, \Sigma^\tau) > 2^{-7}\tau.
\end{cases}
\end{equation}

For $\sigma, \tau > 0$ and $b \in \mathbb{R}$, we denote by $X_{\xi(\tau), \sigma}^b$ and $\hat{X}_{\xi(\tau)}^b$ the function spaces which were introduced in [21][14][19]:

\begin{align*}
X_{\xi(\tau), \sigma}^b &:= \{ u \in S'(\mathbb{R}^d) : \|u\|_{X_{\xi(\tau), \sigma}^b} = \|(p_{\xi(\tau)} + \sigma)\mathring{h}u\|_{L^2(\mathbb{R}^d)} < \infty \}, \\
\hat{X}_{\xi(\tau)}^b &:= \{ u \in S'(\mathbb{R}^d) : \|u\|_{\hat{X}_{\xi(\tau)}^b} = \|p_{\xi(\tau)}\mathring{h}u\|_{L^2(\mathbb{R}^d)} < \infty \},
\end{align*}

and for simplicity we also set $X_{\xi(\tau)}^1 = X_{\xi(\tau), \sigma}^1$.

Immediately, from the definition of $X_{\xi(\tau)}^{1/2}$ we have

\begin{equation}
\|u\|_{L^2(\mathbb{R}^d)} \leq C\tau^{-1/2}\|u\|_{X_{\xi(\tau)}^{1/2}}
\end{equation}

with $C$ independent of $\tau > 0$.

**Lemma 2.1** ([21] Lemma 2.2), [19]. For $\phi \in S(\mathbb{R}^d)$ the estimates

\begin{align*}
\|\phi u\|_{\hat{X}_{\xi(\tau)}^{-1/2}} &\leq C\|u\|_{X_{\xi(\tau)}^{-1/2}}, \\
\|\phi u\|_{X_{\xi(\tau)}^{1/2}} &\leq C\|u\|_{\hat{X}_{\xi(\tau)}^{1/2}}
\end{align*}

hold, where $C$ depends on $\phi$, but is independent of $\tau > 0$. Consequently, for a compactly supported function $q$, there is a constant $C > 0$ such that

\begin{equation}
\|M_q\|_{\hat{X}_{\xi(\tau)}^{1/2} \to \hat{X}_{\xi(\tau)}^{-1/2}} \leq C\|M_q\|_{X_{\xi(\tau)}^{1/2} \to X_{\xi(\tau)}^{-1/2}}, \\
\|\phi u\|_{L^2(\mathbb{R}^d)} \leq C\tau^{-1/2}\|u\|_{\hat{X}_{\xi(\tau)}^{1/2}}.
\end{equation}

By dilation $\xi \to \tau\xi$, we see that

\begin{equation}
\|u\|_{\hat{X}_{\xi(\tau)}^b} = \tau^{2b - \frac{d}{2}}\|u(\tau^{-1} \cdot)\|_{\hat{X}_{\xi(1)}^b}, \quad \|u\|_{X_{\xi(\tau)}^b} = \tau^{2b - \frac{d}{2}}\|u(\tau^{-1} \cdot)\|_{X_{\xi(1)}^{b, 1/2}}.
\end{equation}

For any $b \in \mathbb{R}$, $\tau \geq 1$, and $u$ with dist(supp $\mathring{h}$, $\Sigma^\tau$) $\geq 2^{-\tau}$, it is easy to check by (2.2) that

\begin{equation}
\|u\|_{X_{\xi(\tau)}^b} \simeq \|u\|_{\hat{X}_{\xi(\tau)}^b}
\end{equation}

uniformly in $\tau \geq 1$. Equivalently, using (2.6), we have $\|u\|_{X_{\xi(\tau), 1/2}^b} \simeq \|u\|_{\hat{X}_{\xi(\tau)}^b}$ whenever dist(supp $\mathring{h}$, $\Sigma^1$) $\geq 2^{-\tau}$.

**Definition 2.1.** Let $\kappa \geq 0$. We denote by $m^{\kappa}$ any (scalar or vector-valued) function which is smooth on $\mathbb{R}^d \setminus \{0\}$ and satisfy

\begin{equation}
|\partial^\alpha m^{\kappa}(\xi)| \lesssim \begin{cases}
|\xi|^{\kappa - |\alpha|} & \text{if } |\xi| \geq 1, \\
1 & \text{if } |\xi| < 1,
\end{cases}
\end{equation}

for all multi-indices $\alpha$ with $|\alpha| \leq d + 1$. For $\tau > 0$ we also set $m^{\kappa}(\xi) := \tau^{-\kappa}m^{\kappa}(\tau\xi)$.

Particular examples of $m^{\kappa}(\xi)$ include $(1 + |\xi|^2)^{\frac{d}{2}}$ and $\xi$ (when $\kappa = 1$), and it is easy to see that

\begin{equation}
|\partial^\alpha m^{\kappa}(\xi)| \lesssim \begin{cases}
|\xi|^{\kappa - |\alpha|} & \text{if } |\xi| \geq \tau^{-1}, \\
\tau^{- \kappa + |\alpha|} & \text{if } |\xi| < \tau^{-1}.
\end{cases}
\end{equation}
Lemma 2.2. Let $\tau > 0$. The following are equivalent:

\begin{align}
(2.9) & \quad \|\langle (m^\kappa(D)f)u, v \rangle\| \leq B\|f\|_{L^p(\mathbb{R}^d)}\|u\|_{X^{1/2}_{\xi(\tau)}}, \\
(2.10) & \quad \|\langle (m^\kappa(D)f)u, v \rangle\| \leq B\tau^{p-2+\kappa}\|f\|_{L^p(\mathbb{R}^d)}\|u\|_{X^{1/2}_{\xi(\tau)}}, \\
\end{align}

In particular, if we take $\kappa = 1$ and $m^1(D) = \nabla$, Lemma 2.2 shows that the condition $p \geq d$ is necessary for (2.10) to hold uniformly in $\tau \geq 1$.

**Proof of Lemma 2.2.** First, we show (2.10) assuming (2.9). By Plancherel’s theorem and dilation $\xi \rightarrow \tau \xi$ we have

\[\langle (m^\kappa(D)f)u, v \rangle = \int \mathcal{F}(m^\kappa(D)f)\mathcal{F}^{-1}(u\tau)d\xi = \frac{1}{(2\pi)^{2d}} \int m^\kappa(\xi)\hat{f}(\xi) \int \hat{u}(\eta - \xi)\overline{\hat{v}(\eta)}d\eta d\xi = \tau^{p-2d}\|\langle f, \hat{u}, \hat{v} \rangle\|_{L^p(\mathbb{R}^d)}\|u\|_{X^{1/2}_{\xi(\tau)}}, \]

Thus, from the assumption (2.9) it follows that

\[\|\langle (m^\kappa(D)f)u, v \rangle\| \leq B\tau^{p-2d}\|f\|_{L^p(\mathbb{R}^d)}\|u\|_{X^{1/2}_{\xi(\tau)}},\]

This gives the bound (2.10) via (2.9). The same argument shows the reverse implication from (2.10) to (2.9).

We omit the details.

### 2.2. Linear and bilinear restriction estimates.

The following is (the dual form of) the Stein-Tomas restriction theorem. The same estimate holds for any compact smooth surfaces with nonvanishing Gaussian curvature.

**Theorem 2.3 ([45, 38]).** Let $d \geq 3$, and let $S^{d-2}$ be the unit sphere in $\mathbb{R}^{d-1}$ with the surface measure $d\sigma$. Then

\[\left\| \int_{S^{d-2}} e^{i\xi \cdot \omega} g(\omega)d\sigma(\omega) \right\|_{L_{\mathbb{R}^d}} \lesssim \|g\|_{L^2(S^{d-2})}.\]

By the standard argument and Plancherel’s theorem, Theorem 2.3 implies the following (see [19, Corollary 3.2]), which played a key role in proving the result in [19].

**Corollary 2.4.** Let $d \geq 3$ and $0 < \delta \ll 1$. If $\text{supp} \hat{f} \subset S^{d-2} + O(\delta)$, then

\[\|f\|_{L^2(\mathbb{R}^{d-1})} \lesssim \delta^{\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^{d-1})}.\]

Conversely, by a limiting argument it is easy to see that Corollary 2.4 implies Theorem 2.3.

**Bilinear restriction estimate for the elliptic surfaces.** For $\varepsilon > 0$ and $N \in \mathbb{N}$ we say $\psi : [-1, 1]^{d-2} \rightarrow \mathbb{R}$ is of elliptic type $(\varepsilon, N)$ if $\psi$ satisfies

(i) $\psi(0) = 0$ and $\nabla \psi(0) = 0$;

(ii) if $w(\xi') = \psi(\xi') - |\xi'|^2/2$, then

\[\sup_{\xi' \in [-1, 1]^{d-2}} \max_{0 \leq |\alpha| \leq N} |\partial^\alpha w(\xi')| \leq \varepsilon.\]

**Definition 2.2.** We say that $S$ is an **elliptic surface of type $(\varepsilon, N)$** if $S$ is given by $S = \{ (\xi', \xi_{d-1}) \in \mathbb{R}^{d-2} \times \mathbb{R} : \xi_{d-1} = \psi(\xi'), |\xi'| \leq 1/2 \}$, where $\psi$ is of elliptic type $(\varepsilon, N)$.
Most typical examples are the paraboloid and the surface which is given by parabolic rescaling of a small subset of the sphere. In general, any convex hypersurface with nonvanishing Gaussian curvature can be rescaled (after being decomposed into sufficiently small pieces and then translated and rotated) so that the resulting surfaces are of elliptic type \((\varepsilon, N)\). The following sharp bilinear restriction estimate for elliptic surfaces is due to Tao [41, Theorem 1.1].

**Theorem 2.5 (III).** Let \(d \geq 3\) and let \(q > \frac{2(d+1)}{d-1}\). There are \(\varepsilon > 0\) and \(N \in \mathbb{N}\) such that the estimate \((1.4)\) holds (with \(C\) independent of \(S, S_1, \) and \(S_2\)) whenever \(S \subset \mathbb{R}^{d-1}\) is of type \((\varepsilon, N)\) and \(S_1, S_2 \subset S\) are hypersurfaces with \(\text{dist}(S_1, S_2) \simeq 1\).

When \(d = 3\), the estimate \((1.4)\) is true with \(q = 4\). This is an easy consequence of Plancherel’s theorem. Unlike the Stein-Tomas theorem, the bilinear restriction estimate for a surface with nonvanishing Gaussian curvature exhibits different natures depending whether the surface is elliptic or not. If the surface with nonvanishing Gaussian curvature is not elliptic, the separation condition \(\text{dist}(S_1, S_2) \simeq 1\) is not sufficient in order for \((1.4)\) to hold for \(q < \frac{2d}{d-1}\) (for example, see [29] for more details).

**Remark 1.** The constant \(C\) in Theorem 2.5 is clearly stable under small smooth perturbation of \(S\). It is known that the estimate \((1.4)\) fails if \(q < \frac{2d+1}{d-1}\) but the endpoint case \(q = \frac{2(d+1)}{d-1}\) is still open when \(d \geq 4\). In this case, under the assumption of Theorem 2.5 the following local estimate

\[
\|f d\mu_1 g d\mu_2\|_{L^2(B_{\delta}(0, R))} \leq C R^{d-1}\|f\|_{L^2(S_1, d\mu_1)}\|g\|_{L^2(S_2, d\mu_2)}
\]

holds for any \(\varepsilon > 0\) and \(R \geq 1\) (see [31]) provided that \(S\) is an elliptic surface of type \((\varepsilon, N)\) with small enough \(\varepsilon > 0\) and large enough \(N\).

Making use of the bilinear estimate \((1.4)\) from Theorem 2.5 and interpolation, we obtain the following.

**Corollary 2.6.** Let \(d, S, S_1, \) and \(S_2\) be as in Theorem 2.5 and let \(0 < \delta_1, \delta_2 \ll 1\). Suppose that \(\text{supp} \tilde{u}_i \subset S_i + O(\delta_i), i = 1, 2\). Then, for any \(\varepsilon > 0\), there exists a constant \(C = C(\varepsilon, N, \varepsilon, d)\) such that

\[
\|u_1 u_2\|_{L^\frac{d+1}{d+1}(\mathbb{R}^{d-1})} \leq C(\delta_1 \delta_2)^{\frac{1}{2} - \varepsilon}\|u_1\|_{L^2(\mathbb{R}^{d-1})}\|u_2\|_{L^2(\mathbb{R}^{d-1})}.
\]

If \(\varepsilon = 0\), then \((2.13)\) is equivalent to the endpoint bilinear restriction estimate \((1.4)\) with \(q = \frac{2(d+1)}{d-1}\).

**Proof.** By Theorem 2.5 we have (see [28] Proof of Lemma 2.4) for \(p > \frac{d+1}{d+1}\),

\[
\|u_1 u_2\|_{L^p(\mathbb{R}^{d-1})} \leq C(\delta_1 \delta_2)^{\frac{1}{2}}\|u_1\|_{L^2(\mathbb{R}^{d-1})}\|u_2\|_{L^2(\mathbb{R}^{d-1})}.
\]

Interpolating this with the trivial estimate \(\|u_1 u_2\|_{L^1} \leq \|u_1\|_{L^2}\|u_2\|_{L^2}\) and taking \(p\) arbitrarily close to \(\frac{d+1}{d-1}\) give \((2.13)\) for any \(\varepsilon > 0\).

\[\square\]

### 2.3. Frequency localized estimates

We use Corollary 2.4 and Corollary 2.6 to show additional estimates which we need for proving our main estimates in Section 3. We begin with introducing additional notations.

**Linear estimates.** Recalling \((2.1)\), for \(\tau > 0\), we define \(\Sigma_\tau^\mu\) and \(\Sigma_{\leq \mu}\) by

\[
\Sigma_\tau^\mu = \{\xi \in \mathbb{R}^d : \mu/2 < \text{dist}(\xi, \Sigma^\mu) \leq \mu\}, \quad \Sigma_{\leq \mu}^\tau = \{\xi \in \mathbb{R}^d : \text{dist}(\xi, \Sigma^\mu) \leq \mu\}.
\]

By \(Q_\tau^\mu\) and \(Q_{\leq \mu}^\tau\) we denote the multiplier operators given by

\[
Q_\tau^\mu f = \chi_{\Sigma_\tau^\mu} f, \quad Q_{\leq \mu}^\tau f = \chi_{\Sigma_{\leq \mu}^\tau} f.
\]

For an orthonormal basis \(\{e_i\}_{i=1}^d\) for \(\mathbb{R}^d\), the \(i\)-th coordinate \(\xi_i\) of \(\xi\) with respect to \(\{e_i\}\) is given by \(\xi_i = \xi \cdot e_i\). We write

\[
\xi = (\xi_1, \xi_2, \xi_3), \quad \xi \in \mathbb{R}^{d-1}, \quad \xi_3 \in \mathbb{R}^{d-2}.
\]
For $0 < \delta < \tau$ and $h > 0$, we also set

$$
\Sigma_{\leq \delta} = \{ \xi \in \mathbb{R}^d : |\xi_1| \leq h, \ \tau - \delta \leq \left| \frac{\xi}{\tau \delta^2} \right| \leq \tau + \delta \},
$$

and let $Q_{\leq \delta}^{\tau, h}$ be the multiplier operator given by

$$
Q_{\leq \delta}^{\tau, h} f = \chi_{\Sigma_{\leq \delta}}(D)f.
$$

**Lemma 2.7.** Let $d \geq 3$, $1 \leq \tau \leq 2$ and $0 < \delta, h \leq 1$. For $2 \leq p \leq 2d/(d - 2)$ there exists a constant $C > 0$, independent of $\tau$, $\delta$ and $h$, such that

$$
\|Q_{\leq \delta}^{\tau, h} u\|_{L^p(\mathbb{R}^d)} \leq C h^{\frac{d}{2}} \frac{1}{\tau \delta^2} \|u\|_{L^2(\mathbb{R}^d)}.
$$

**Proof.** When $p = 2$, (2.14) is obvious by Plancherel’s theorem. Thus, in view of interpolation, it suffices to show

$$
\|Q_{\leq \delta}^{\tau, h} u\|_{L^p(\mathbb{R}^d)} \leq h^{\frac{d}{2}} \frac{1}{\tau \delta^2} \|u\|_{L^2(\mathbb{R}^d)}.
$$

Since $\mathcal{F}(Q_{\leq \delta}^{\tau, h} u)$ is supported in $\{ \xi : |\xi_1| \leq h \}$, one may use Bernstein’s inequality to get

$$
\|Q_{\leq \delta}^{\tau, h} u(\cdot, \bar{x})\|_{L^{\frac{2d}{d-2}}(\mathbb{R})} \leq h^{\frac{d}{2}} \|Q_{\leq \delta}^{\tau, h} u(\cdot, \bar{x})\|_{L^2(\mathbb{R})}
$$

uniformly in $\bar{x} \in \mathbb{R}^{d-1}$. Applying Minkowski’s inequality and Corollary [2.4] we obtain

$$
\|Q_{\leq \delta}^{\tau, h} u\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq h^{\frac{d}{2}} \left\| Q_{\leq \delta}^{\tau, h} u(x_1, \cdot) \right\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^{d-1})} \lesssim h^{\frac{d}{2}} \frac{1}{\tau \delta^2} \|Q_{\leq \delta}^{\tau, h} u\|_{L^2(\mathbb{R}^d)}. \quad \square
$$

From Lemma 2.7 the following is easy to show.

**Lemma 2.8** ([19 Lemma 3.3]). For $1 < \mu \leq \tau$, we have

$$
\|Q_{\leq \delta}^\tau f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim (\mu/\tau)^{1/d} \|f\|_{X^{1/2}_{\xi(\tau)}},
$$

$$
\|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \|f\|_{X^{1/2}_{\xi(\tau)}}.
$$

**Proof.** By rescaling the estimate (2.16) is equivalent to

$$
\|Q_{\leq \delta}^\tau f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \frac{1}{\tau \delta^2} \|f\|_{X^{1/2}_{\xi(\tau)}}, \quad 1/\tau < \delta \leq 1.
$$

To show this we decompose $Q_{\leq \delta}^\tau$ dyadically as follows:

$$
\|Q_{\leq \delta}^\tau f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq \|Q_{\leq 1/\tau}^\tau f\|_{L^{\frac{2d}{d-2}}} + \sum_{1/\tau < 2^j < \delta} \|Q_{\leq 2^j}^\tau f\|_{L^{\frac{2d}{d-2}}}.
$$

Application of Lemma 2.7 gives the bound $\|Q_{\leq 1/\tau}^\tau f\|_{L^{\frac{2d}{d-2}}} \lesssim \tau^{-\frac{d}{d-2}} \|Q_{\leq 1/\tau}^\tau f\|_{L^2}$, and the definition of $X^{1/2}_{\xi(1,1/\tau)}$ gives $\|Q_{\leq 1/\tau}^\tau f\|_{L^2} \lesssim \tau^{-\frac{d}{2}} \|f\|_{X^{1/2}_{\xi(1,1/\tau)}}$. Combining these we get

$$
\|Q_{\leq 1/\tau}^\tau f\|_{L^{\frac{2d}{d-2}}} \lesssim \tau^{-\frac{d}{4}} \|f\|_{X^{1/2}_{\xi(1,1/\tau)}}.
$$

Utilizing Lemma 2.7 and (2.2), the similar argument gives $\|Q_{\leq 2^j}^\tau f\|_{L^{\frac{2d}{d-2}}} \lesssim 2^{\frac{d}{2}} \|f\|_{X^{1/2}_{\xi(1,1/\tau)}}$. Now summation of the estimates over $j$ gives the desired bound (2.18).

The estimate (2.17) is even easier to prove once we have (2.16) since $|p_{\xi(\tau)}(\xi)| \simeq \tau^2 + |\xi|^2$ if $\text{dist}(\xi, \Sigma^\tau) > 2^{-\tau} \tau$. In fact, $\|f - Q_{\leq 2^{-2\tau}} f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \|f\|_{X^{1/2}_{\xi(\tau)}}$ follows by the Hardy-Littlewood-Sobolev inequality and Plancherel’s theorem. Combining this and (2.16) with $\mu = 2^{-7} \tau$ yields (2.17). \(\square\)
Bilinear estimates. We obtain some bilinear estimates which are consequences of the bilinear restriction estimate (2.13).

**Lemma 2.9.** Let \( d \geq 3, 0 < \delta_2 \leq \delta_1 \leq 1, 0 < h_2 \leq h_1 \leq 1, \) and let \( S, S_1, \) and \( S_2 \) be as in Theorem 2.5. Suppose that

\[
\supp \tilde{u}_j \subset \{ (\xi_1, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} : |\xi_1| \leq h_j, \xi \in S_j + O(\delta_j) \}, \quad j = 1, 2.
\]

Then, for any \( \epsilon > 0 \) and \( \frac{d+1}{2} \leq p \leq \infty, \)

\[
(2.19) \quad \|u_1 u_2\|_{L^{p'}(\mathbb{R}^d)} \lesssim \delta_2^{-\epsilon} h_1^3 (\delta_1 \delta_2)^{\frac{d+1}{2}} \|u_1\|_{L^2(\mathbb{R}^d)} \|u_2\|_{L^2(\mathbb{R}^d)},
\]

\[
(2.20) \quad \|u_1 u_2\|_{L^{p'}(\mathbb{R}^d)} \lesssim \delta_2^{-\epsilon} (h_1/h_2)^{\delta_2} \|u_1\|_{L^2(\mathbb{R}^d)} \|u_2\|_{L^2(\mathbb{R}^d)}.
\]

When \( d = 7 \) the estimates (2.19) and (2.20) are identical. If \( d < 7, \) (2.20) gives a bound better than (2.19). When \( d > 7, \) the bound from (2.19) is stronger. The bounds in (2.19) and (2.20) are sharp in that the exponents of \( \delta_1, \delta_2 \) cannot be improved except for the \( \delta_2^{-\epsilon} \) factor. This can be shown without difficulty by modifying the (squashed cap) example in [43], especially with \( \delta_1 \simeq \delta_2 \) and \( h_1 \simeq h_2. \)

**Remark 2.** There are linear counterparts of (2.19) and (2.20). Let \( \tau \simeq 1, 0 < h_2 \leq h_1, \) and \( 0 < \delta_2 \leq \delta_1. \) Suppose \( \supp \tilde{u}_i \) is contained in \( \Sigma^\tau_{\leq h_i}, i = 1, 2. \) Then (2.15) implies that

\[
(2.21) \quad \frac{\|u_1 u_2\|_{L^{p'}(\mathbb{R}^d)}}{\|u_1\|_{L^2} \|u_2\|_{L^2}} \lesssim h_1^{\frac{1}{2}} \|u_1\|_{L^2} \|u_2\|_{L^2}.
\]

We may compare this estimate with the estimate (2.19). In particular, with \( p = d \) in (2.19) and the assumption in Lemma 2.9 we have

\[
(2.22) \quad \frac{\|u_1 u_2\|_{L^{p'}(\mathbb{R}^d)}}{\|u_1\|_{L^2} \|u_2\|_{L^2}} \lesssim \delta_2^{-\epsilon} h_1^3 (\delta_1 \delta_2)^{\frac{d+1}{2}} \|u_1\|_{L^2} \|u_2\|_{L^2}.
\]

If \( \delta_1 \simeq \delta_2 \simeq h_1 \simeq h_2 \simeq \delta, \) the bound of (2.22) is roughly better than that of (2.21) by a factor of \( \delta^\frac{\delta}{\delta_1}. \) However, since the estimate (2.22) is only possible under additional assumption on the supports of \( \tilde{u}_1, \tilde{u}_2, \) we cannot directly exploit this improvement. Nevertheless, this will be made possible via the bilinear method which has been used in the study of restriction problem (13).

**Proof of Lemma 2.9.** Let us first assume \( d \geq 4. \) By interpolation with a trivial estimate

\[
\|u_1 u_2\|_{L^1} \leq \|u_1\|_{L^2} \|u_2\|_{L^2},
\]

it suffices to prove (2.49) and (2.20) for \( p' = \frac{d+4}{d-4}. \) Since the one-dimensional Fourier transform \( \mathcal{F}(\cdot, \cdot, \cdot) \) is supported in the interval \([-2h_1, 2h_1]\), Bernstein’s inequality gives

\[
\|u_1(\cdot, \cdot, \cdot) u_2(\cdot, \cdot, \cdot)\|_{L^{\frac{d+4}{d-4}}(\mathbb{R})} \lesssim h_1^{1-\frac{d+4}{d+4}} \|u_1(\cdot, \cdot, \cdot) u_2(\cdot, \cdot, \cdot)\|_{L^1(\mathbb{R})}.
\]

Since \( \supp \mathcal{F}(u_1(x_1, \cdot)) \subset S_1 + O(\delta_i) \) for \( i = 1, 2, \) we have

\[
\|u_1(x_1, \cdot) u_2(x_1, \cdot)\|_{L^{\frac{d+4}{d-4}}(\mathbb{R}^{d-1})} \lesssim \delta_2^{-\epsilon} h_1 \|u_1(x_1, \cdot) u_2(x_1, \cdot)\|_{L^2(\mathbb{R}^{d-1})},
\]

which follows from Corollary 2.6. Thus, by Minkowski’s inequality and the Cauchy-Schwarz inequality, we obtain

\[
\|u_1 u_2\|_{L^{\frac{d+4}{d-4}}(\mathbb{R}^d)} \lesssim h_1^{1-\frac{d+4}{d+4}} \int \|u_1(x_1, \cdot) u_2(x_1, \cdot)\|_{L^{\frac{d+4}{d-4}}(\mathbb{R}^{d-1})} \, dx_1 \lesssim \delta_2^{-\epsilon} h_1^{\frac{1}{2}} \|u_1\|_{L^2} \|u_2\|_{L^2}.
\]

This completes the proof of (2.19).

Now we prove (2.20) for \( p' = \frac{d+4}{d-4}, \) which can be deduced from (2.19). Let \( \{I_\ell\} \) be a collection of disjoint subintervals of \([-h_1, h_1]\) each of which has side-length \( \simeq h_2 \) and \([-h_1, h_1] = \bigcup_{\ell} I_\ell. \) It is clear that

\[
u_1 u_2 = \sum_{\ell} \mathcal{F}^{-1}(\chi_{I_\ell}(\xi_1) \tilde{u}_1(\xi)) u_2 = : \sum_{\ell} u_{1\ell} u_2,
\]
and $\mathcal{F}(u_1^t u_2)$ is supported in $(I_\ell + [-h_2, h_2]) \times \mathbb{R}^{d-1}$. Since $p' \in [1, 2]$, from the well-known orthogonality argument ([43, Lemma 6.1]) we see that

$$\left\| \sum_{\ell} u_1^\ell u_2 \right\|_{L^{p'}(\mathbb{R}^d)} \lesssim \left( \sum_{\ell} \| u_1^\ell u_2 \|_{L^{p'}(\mathbb{R}^d)} \right)^{\frac{1}{p'}}.$$  

The length of the interval $I_\ell + [-h_2, h_2]$ is $\approx h_2$. Hence, (2.19) gives

$$\| u_1^\ell u_2 \|_{L^{p'}(\mathbb{R}^d)} \lesssim \delta_2^{-\varepsilon} h_2^\frac{1}{2} (\delta_1 \delta_2)^{\frac{d+1}{2p}} \| u_1^\ell \|_{L^2} \| u_2 \|_{L^2}.$$  

Combining the above two inequalities and using H"older’s inequality, we get

$$\| u_1 u_2 \|_{L^{p'}(\mathbb{R}^d)} \lesssim \delta_2^{-\varepsilon} h_2^\frac{1}{2} (\delta_1 \delta_2)^{\frac{d+1}{2p}} \| u_2 \|_{L^2} \left( \sum_{\ell} \| u_1^\ell \|_{L^2} \right)^{\frac{1}{2}},$$  

which gives (2.20) with $p' = \frac{d+1}{d-1}$.

When $d = 3$ we have the endpoint bilinear restriction estimate (2.20) with $d = 3$ and $q = 4$. Hence the estimate (2.13) in Corollary 2.6 is true without $\delta_2^{-\varepsilon}$. The same argument gives (2.19) and (2.20) without $\delta_2^{-\varepsilon}$.

### 3. Bilinear $X^{1/2}_{\zeta(1),1/\tau}$ estimates

As mentioned in the introduction, we regard $\langle (\nabla f) u, v \rangle$ as a bilinear operator and attempt to obtain estimates while $u, v \in X^{1/2}_{\zeta(1),1/\tau}$. In order to make use of the restriction estimates and its variants we work in frequency local setting after rescaling $\xi \to \tau \xi$. This enables us to deal with $\Sigma^1$ instead of $\Sigma^\tau$ which varies along $\tau$. In this section we use the estimates in the previous section to obtain estimates for $\langle (\nabla f) u, v \rangle$ in terms of $X^{1/2}_{\zeta(1),1/\tau}$.

#### 3.1. Localization near $\Sigma^1$

Throughout this section (Section 3) we assume that

$$\text{supp } \hat{u}, \text{ supp } \hat{v} \subset B_d(0,4),$$  

and obtain bounds on $\langle (m_\zeta^\varepsilon(D)f) u, v \rangle$ while $u, v$ are in $X^{1/2}_{\zeta(1),1/\tau}$. Note that $\hat{u}(\cdot) \ast \overline{v}$ is supported in $B_d(0,8)$.

Since $u, v$ are in $X^{1/2}_{\zeta(1),1/\tau}$, $\hat{u}$ and $\hat{v}$ exhibit singular behavior near the set $\Sigma^1$. Meanwhile, the desired estimates are easy to show if $\hat{u}$ or $\hat{v}$ is supported away from $\Sigma^1$ (see Section 5). Thus, for the rest of this section, we assume that

$$\text{supp } \hat{u}, \text{ supp } \hat{v} \subset \Sigma^1 + O(2^{-2} \varepsilon_c)$$  

with a fixed small number $\varepsilon_c \in (0, 2^{-7}/\sqrt{d}]$.

Let $\beta \in C_c^\infty((1, 2))$ be such that $\sum_{j \in \mathbb{Z}} \beta(2^{-j} t) = 1$ for $t > 0$. For a dyadic number $\lambda$, we define a Littlewood-Paley projection operator $P_\lambda$ by $\mathcal{F}(P_\lambda f)(\xi) = \beta(|\xi|/\lambda) \hat{f}(\xi)$ and write

$$\langle (m_\zeta^\varepsilon(D)f) u, v \rangle = \sum_{\lambda \leq 8} \langle (m_\zeta^\varepsilon(D)P_\lambda f) u, v \rangle$$  

(3.2)  

$$= (2\pi)^{-2d} \sum_{\lambda \leq 8} \int_{\mathbb{R}^d} m_\zeta^\varepsilon(\xi) \beta \left( \frac{|\xi|}{\lambda} \right) \hat{f}(\xi)(\hat{u}(\cdot) \ast \overline{v})(\xi) d\xi.$$  

In order to get estimate for $\langle (m_\zeta^\varepsilon(D)f) u, v \rangle$, we first obtain estimate for $\langle (m_\zeta^\varepsilon(D)P_\lambda f) u, v \rangle$. 

12
Primary decomposition. Before breaking the bilinear operator (3.2) into fine scales by a Whitney type de-
composition, we first decompose the unit sphere $\Sigma^1 - e_2$ (see (2.1)) into small $\epsilon_\circ$-caps. Let \{$S_\ell$\} be a collection of essentially disjoint subsets of $S^{d-2} \subset \mathbb{R}^{d-1}$ such that diam($S_\ell$) $\in [\frac{\epsilon_\circ}{10}, \epsilon_\circ]$ and $S^{d-2} = \bigcup_\ell S_\ell$. Thus,

$$(\Sigma^1 - e_2) \times (\Sigma^1 - e_2) = \bigcup_{\ell, \ell'} (\{0\} \times S_\ell) \times (\{0\} \times S_{\ell'}).$$

For the products $S_\ell \times S_{\ell'}$, we distinguish the following three cases:

(3.3) \quad transversal: $\text{dist}(S_\ell, S_{\ell'}) \geq \epsilon_\circ$ and $\text{dist}(-S_\ell, S_{\ell'}) \geq \epsilon_\circ$,

(3.4) \quad neighboring: $\text{dist}(S_\ell, S_{\ell'}) < \epsilon_\circ$,

(3.5) \quad antipodal: $\text{dist}(-S_\ell, S_{\ell'}) < \epsilon_\circ$.

This leads us to the primary decomposition

$$
\langle (m^\varphi_\lambda(D)P_\lambda f)u, v \rangle = \sum_{(S_\ell, S_{\ell'}) \text{transversal}} \langle (m^\varphi_\lambda(D)P_\lambda f)u_\ell, v_{\ell'} \rangle + \sum_{(S_\ell, S_{\ell'}) \text{neighboring}} \langle (m^\varphi_\lambda(D)P_\lambda f)u_\ell, v_{\ell'} \rangle + \sum_{(S_\ell, S_{\ell'}) \text{antipodal}} \langle (m^\varphi_\lambda(D)P_\lambda f)u_\ell, v_{\ell'} \rangle,
$$

where

(3.7) \quad $\text{supp } \hat{u}_\ell - e_2 \subset \{0\} \times S_\ell + O(2^{-2}\epsilon_\circ)$, $\text{supp } \hat{v}_{\ell'} - e_2 \subset \{0\} \times S_{\ell'} + O(2^{-2}\epsilon_\circ)$.

For derivation of linear restriction estimate from bilinear restriction estimate, it is enough to consider the
neighboring case (3.4) only, since we can decompose a single function into functions with smaller frequency
pieces. However, in our situation the functions $u$ and $v$ are completely independent. So, we cannot localize
the supports of $\hat{u}$, $\hat{v}$ in such a favorable manner.

In transversal case (3.3) we can apply the bilinear restriction estimate directly since the separation (transvers-
ality) condition is guaranteed. For the other two cases (3.4 and 3.5), the separation (transversality) con-
dition fails. To apply the bilinear estimates (2.19) and (2.20), we need to decompose further the sets $S_\ell$
and $S_{\ell'}$ by making use of a Whitney type decomposition (for example, see [43]). This is to be done in the
following section.

3.2. Estimates with dyadic localization. To get control over the functions of which Fourier transforms
are confined in a narrow neighborhood of $\Sigma^1$ (recall (3.1)), we first decompose the functions $\hat{u}$ and $\hat{v}$ in
(3.7) dyadically away from $\Sigma^1$, and then break the resulting pieces in the angular directions via the Whitney
type decomposition. We obtain estimates for each piece (Lemma 3.1) and combine those estimates together
to get the estimate (3.4) in Proposition 3.8.

Let $0 < \delta_2 \leq \delta_1 \leq 2\epsilon_\circ$. In this section (Section 3.2), we work with $u$, $v$ satisfying

$$
\text{supp } \hat{u} - e_2 \subset \{0\} \times S_\ell + O(\delta_1), \quad \text{supp } \hat{v} - e_2 \subset \{0\} \times S_{\ell'} + O(\delta_2),
$$

under the assumption (3.4) or (3.5).

Whitney type decomposition. Let $j_0$ be the smallest integer such that $2^{-j_0+3} \leq 1/\sqrt{d}$, and set $I_0 = [-2^{-j_0+3}, 2^{-j_0}]$. For each $j \geq j_0$, we denote by \{$I_0^j$\} the collection of the dyadic cubes of side length $2^{-j}$
which are contained in $I_0^{d-2}$. Fix $j_0 > j_0 + 3$. For $j_0 < j < j_0$, $k \sim k'$ means that $I_k^j$ and $I_{k'}^j$
are not adjacent but have adjacent parent dyadic cubes. If $j = j_0$, by $k \sim k'$ we mean dist($I_k^j, I_{k'}^j$) $\lesssim 2^{-j}$. By a Whitney type decomposition of $I_0^{d-2} \times I_0^{d-2}$ away from its diagonal, we have

$$
I_0^{d-2} \times I_0^{d-2} = \bigcup_{j_0 < j \leq j_0} \left( \bigcup_{k \sim k'} I_k^j \times I_{k'}^j \right).
$$

4 $S_\ell \cap S_{\ell'}$ is of measure zero if $\ell \neq \ell'$.

5 It should be noted that the index $k$ is subject to $j$. 
The cubes $I_k^d \times I_{k'}^d$ appearing in the above are essentially disjoint. Thus we may write
\[ \chi_{I_k^d \times I_{k'}^d} = \sum_{j_0 < j \leq j_*} \sum_{k \sim k'} \chi_{I_k^d \times I_{k'}^d}, \]
where $\chi_A$ denotes the indicator function of a set $A$.

Since $S_\ell \cap S_{\ell'}$ or $(-S_\ell) \cup S_{\ell'}$ is contained in $B_{d-1}(\theta, 3 \epsilon_\ell)$ for some $\theta \in S^{d-2}$, considering $I_0^{d-2}$ to be placed in the hyperplane $H$ in $\mathbb{R}^{d-1}$ that is orthogonal to $\theta$ and contains the origin (sharing the origin), there is obviously a smooth diffeomorphism $G : I_0^{d-2} \to S^{d-2}$ such that $S_\ell \cup S_{\ell'} \subset G(I_0^{d-2})$ in the case of (3.3) and $(-S_\ell) \cup S_{\ell'} \subset G(I_0^{d-2})$ in the case of (3.5). We now set
\[ S_k^d := G(I_k^d). \]

Then, it follows that
\[ \chi_{G(I_k^{d-2})} = \sum_{j_0 < j \leq j_*} \sum_{k \sim k'} \chi_{S_k^d \times S_{k'}^d}, \]

If $j < j_*$ we have $\text{dist}(S_k^d, S_{k'}^d) \approx 2^{-j}$ and, for $j = j_*$, $\text{dist}(S_k^d, S_{k'}^d) \lesssim 2^{-j}$.

The following are rather direct consequences of Lemma 2.9 via scaling.

**Lemma 3.1.** Assume $d \geq 3$, $0 < h_2 \leq h_1 \ll 1$, and $0 < \delta_2 \leq \delta_1 \leq 2^{-2j} \leq c$ with a constant $c$ small enough. Suppose that $\text{dist}(S_k^d, S_{k'}^d) \approx 2^{-j}$, and suppose
\[ \text{supp } \widehat{\nu}_1 \cap \{ \xi \in \mathbb{R}^d : |\xi| \leq h_1, \xi \in \pm S_k^d + O(\delta_2) \}, \]
\[ \text{supp } \widehat{\nu}_2 \cap \{ \xi \in \mathbb{R}^d : |\xi| \leq h_2, \xi \in S_{k'}^d + O(\delta_2) \}. \]

Then, the following estimates hold for any $\epsilon > 0$ and $\frac{d+1}{2} \leq p \leq \infty$:
\[ \|u_1 u_2\|_{L^p(R^d)} \lesssim \delta_2^{-\epsilon} h_1^\frac{1}{2} (\delta_1 \delta_2)^\frac{d+1}{p} \|u_1\|_{L^2(R^d)} \|u_2\|_{L^2(R^d)}, \]
\[ \|u_1 u_2\|_{L^p(R^d)} \lesssim \delta_2^{-\epsilon} h_2^\frac{1}{2} (h_1/h_2)^\frac{d+1}{p} (\delta_1 \delta_2)^\frac{d+1}{p} \|u_1\|_{L^2(R^d)} \|u_2\|_{L^2(R^d)}. \]

**Proof.** Note that if $\text{supp } \widehat{\nu}_1 \subset [-h_1, h_1] \times (-S_k^d + O(\delta_2))$, then $\text{supp } \widehat{\nu}_2 \subset [-h_2, h_2] \times (S_{k'}^d + O(\delta_2))$. Since $\|u_1 u_2\|_{L^p} = \|\pi_1 u_2\|_{L^p}$, we need only to consider the case
\[ \text{supp } \widehat{\nu}_1 \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq h_1, \xi \in S_k^d + O(\delta_2) \}. \]

We first observe that the supports of $\widehat{\nu}_1$ and $\widehat{\nu}_2$ are contained in the set $\{ \xi \in \mathbb{R}^d : |\xi| \leq h_1, |(\xi - \theta) \cdot \theta| \lesssim 2^{-2j}, |(\xi - \theta) \cdot (\xi - \theta) - \theta| \lesssim 2^{-j} \}$ for some $\theta \in S^{d-2}$. After rotation in $\xi$ we may assume that $\theta = -\bar{\epsilon}_2$, and by translation $\xi \to \xi + \bar{\epsilon}_2$ (since these changes of variables do not affect the estimates) we may assume that
\[ \text{supp } \widehat{\nu}_1 \subset \{ (\xi_1, \xi_2, \xi) : |\xi_1| \leq h_1, \xi_2 = \psi(\xi_1) + O(\delta_1), |\xi| \lesssim 2^{-j} \}, \quad i = 1, 2, \]
where $\psi(\xi) = 1 - (1 - |\xi|^2)^{1/2}$. Note that $\text{supp } \widehat{\nu}_1$ and $\text{supp } \widehat{\nu}_2$ are separated by $\lesssim 2^{-j}$. By an anisotropic dilation $(x_2, \bar{x}) \to (2^{2j} x_2, 2^{2j} \bar{x})$ we see that
\[ \|u_1 u_2\|_{L^p(R^d)} \approx 2^{2j \epsilon} \|u_1 u_2\|_{L^p(R^d)}, \]
where $u_i(x_1, x_2, \bar{x}) = u_i(x_1, 2^{2j} x_2, 2^{2j} \bar{x})$ for $i = 1, 2$. Then it follows that, for $i = 1, 2$,
\[ \text{supp } \widehat{\nu}_1 \subset \{ (\xi_1, \xi_2, \xi) : |\xi_1| \leq h_1, \xi_2 = 2^{2j} \psi(2^{-j} \xi_1) + O(\delta_2 2^{2j}), |\xi| \lesssim 1 \} \]
while $\text{supp } \widehat{\nu}_1$ and $\text{supp } \widehat{\nu}_2$ are separated by $\lesssim 1$. Since $2^{2j} \psi(2^{-j} \xi_1) = \frac{1}{2} |\xi_1|^2 + O(2^{-j})$ by the Taylor expansion, we see $2^{2j} \psi(2^{-j} \xi_1)$ is of elliptic type $(C2^{-2j}, N)$ for some $N$. Thus, for $j$ large enough we may apply Lemma 2.9. Rescaling gives
\[ \|u_1 u_2\|_{L^p(R^d)} \lesssim 2^{-2cj} \delta_2^{-\frac{d}{2}} h_1 \delta_1 (2^{2j} \delta_1 2^{2j} \delta_2)^\frac{d+1}{p} 2^{-dj} \|u_1\|_{L^2} \|u_2\|_{L^2}. \]

Footnote 6: the inverse of the projection from $S^{d-2}$ to the plane $H$

Footnote 7: Otherwise, we may directly use the bilinear estimate without rescaling since the surfaces are well separated.
Combining this with (3.12) we get the desired estimate (3.10). The same argument with (2.20) gives (3.11). So we omit the details.

\[\square\]

**Location of \( \pm S_j^k + S_j^{k'} \) for \( k \sim k' \).** Let us denote by \( c(S_j^k) \) the barycenter of \( S_j^k \) and set

\[ c_{j,k'}^{\pm} = \mp c(S_j^k) + c(S_j^{k'}). \]

Note that every \( S_j^k \) is contained in a rectangle of dimensions about \( 2^{-2j} \times 2^{-j} \times \cdots \times 2^{-j} \).

For each \( j \) we observe that, if \( k \sim k' \),

\[ -S_j^k + S_j^{k'} \subset R_{k,k'}^{j,+}, \]

where

\[ R_{k,k'}^{j,+} = \left\{ \tilde{\xi} : \left| \tilde{\xi} - c_{j,k'}^{j,+} \right| \leq 2^{-2j}, \left| \tilde{\xi} - c_{j,k'}^{j,+} - \left( \tilde{\xi} - c_{j,k'}^{j,+} - \frac{c_{j,k'}^{j,+}}{|c_{j,k'}^{j,+}|} \right) \frac{c_{j,k'}^{j,-}}{|c_{j,k'}^{j,-}|} \right| \leq 2^{-j} \right\}. \]

Thus, for \( j_0 < j < j_* \), we have

\[ \bigcup_{k \sim k'} (-S_j^k + S_j^{k'}) \subset B_{d-1}(0, C_1 2^{-j}) \setminus B_{d-1}(0, C_2 2^{-j}) \]

with some \( C_1, C_2 > 0 \) (see Figure 3). For \( j = j_* \), since there is no separation between \( S_j^k \) and \( S_j^{k'} \), we just have

\[ \bigcup_{k \sim k'} (-S_j^k + S_j^{k'}) \subset B_{d-1}(0, C 2^{-j_*}) \]

for some \( C > 0 \).

On the other hand, since \( \text{dist}(S_j^k, S_j^{k'}) \leq 2^{-j} \) whenever \( k \sim k' \), we note that if \( k \sim k' \)

\[ S_j^k + S_j^{k'} \subset R_{k,k'}^{j,-}, \]

where

\[ R_{k,k'}^{j,-} = \left\{ \tilde{\xi} : \left| \tilde{\xi} - c_{j,k'}^{j,-} \right| \leq 2^{-2j}, \left| \tilde{\xi} - c_{j,k'}^{j,-} - \left( \tilde{\xi} - c_{j,k'}^{j,-} - \frac{c_{j,k'}^{j,-}}{|c_{j,k'}^{j,-}|} \right) \frac{c_{j,k'}^{j,-}}{|c_{j,k'}^{j,-}|} \right| \leq 2^{-j} \right\}. \]
Figure 4. The antipodal case (3.5): the point $c^+_{k,k'} = c(S^j_k) + c(S^j_{k'})$ and the set $R^{j-}_{k,k'} \subset \mathbb{R}^{d-1}$ (the orange rectangle).

Clearly, $R^{j-}_{k,k'}$ is contained in a $C^{-2j}$-neighborhood of $2S^{j-2}$ provided that $k \sim k'$ (see Figure 4). We also see that for every $j$ with $j_o < j \leq j^*$,

$$\bigcup_{k \sim k'} (S^j_k + S^j_{k'}) \subset 2S^{j-2} + O(C^{-2j}).$$

Let us denote by $\phi^+_{j,k,k'}$ a smooth function adapted to $R^{j,\pm}_{k,k'}$ such that $\phi^+_{j,k,k'}$ is supported in the rectangle given by dilating $R^{j,\pm}_{k,k'}$ twice from its center and $\phi^+_{j,k,k'} = 1$ on $R^{j,\pm}_{k,k'}$. We also define the projection operator $P^{j,\pm}_{k,k'}$ by

$$\mathcal{F}(P^{j,\pm}_{k,k'} g)(\xi) = \phi^+_{j,k,k'}(\tilde{\xi}) \hat{g}(\xi).$$

In what follows we prove bilinear estimates which are the key ingredients for the main estimates. It will be done by considering the three cases (3.4), (3.5), and (3.3), separately. For a unit vector $e \in S^{d-1}$ and $\delta > 0$ let $P^{e \leq \delta}$ be the Littlewood-Paley projection in the $e$-direction which is defined by

$$\mathcal{F}(P^{e \leq \delta} g)(\xi) = \beta_0 \left( \frac{\xi \cdot e}{\delta} \right) \hat{g}(\xi)$$

with $\beta_0 \in C^\infty((-4,4))$ satisfying $\beta_0 = 1$ on $[-2,2]$.

3.3. Estimates for the neighboring case. In this section (Section 3.3) we consider the neighboring case in the primary decomposition (3.6). So, $S_\ell, S_{\ell'}$ satisfy (3.4), and $u, v$ satisfy (3.7) in place of $u_\ell, v_{\ell'}$.

Decomposition of $u$ and $v$. Let us define $u^j_k$ and $v^j_k$ by

$$\mathcal{F}(u^j_k)(\xi) = \chi_{S^j_k} \left( \frac{\xi - e_2}{|\xi - e_2|} \right) \tilde{u}(\xi), \quad \mathcal{F}(v^j_k)(\xi) = \chi_{S^j_k} \left( \frac{\xi - e_2}{|\xi - e_2|} \right) \tilde{v}(\xi).$$

Then by (3.10) it follows that

$$\langle (m^j_\ell(D)P_\lambda f) u, v \rangle = \sum_{j_o < j \leq j^*} \sum_{k \sim k'} \langle (m^j_\ell(D)P_\lambda f) u^j_k, v^j_{k'} \rangle = \sum_{j_o < j \leq j^*} \mathcal{I}_j,$$

where

$$\mathcal{I}_j = \sum_{k \sim k'} \langle (m^j_\ell(D)P_\lambda f) u^j_k, v^j_{k'} \rangle, \quad j_o < j \leq j^*.$$
For $0 < \delta \leq 2\epsilon_0$, we denote by $j_\ast(\delta)$ the largest integer $j_\ast$ satisfying $\sqrt{\delta} \leq 2^{-j_\ast}$. For $\lambda, \delta > 0$ and $f \in L^p(\mathbb{R}^d)$, let us set

$$\Gamma_{\lambda, \delta}^+(f) = \sup_{j_\ast < j \leq j_\ast(\delta)} \sup_{k \sim k'} \left\{ \lambda \frac{d}{2} \frac{d+1}{2} \left\| P_{k,k'}^\delta \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)} \right\}.$$  

**Lemma 3.2.** Let $d \geq 3$, $p > \frac{d+1}{2}$, $0 \leq k \leq 1$, $0 < \delta_2 \leq \delta_1 \leq 2\epsilon_0$, and let $f \in L^p(\mathbb{R}^d)$. Assume that $S_\ell$, $S_\nu$ satisfy (3.4) and $u$, $v$ satisfy (3.8). Then, for any $\epsilon > 0$,

$$\left| \langle (m_\epsilon^c(D)P_{\lambda} f)u, v \rangle \right| \lesssim \delta_2^{-\epsilon} \lambda^{\frac{d}{2}} \frac{d+1}{2} \left\| \Gamma_{\lambda, \delta}^+(f) \right\|_{L^2(\mathbb{R}^d)} \| u \|_{L^2(\mathbb{R}^d)} \| v \|_{L^2(\mathbb{R}^d)},$$

(3.23)

$$\left| \langle (m_\epsilon^c(D)P_{\lambda} f)u, v \rangle \right| \lesssim \delta_2^{-\epsilon} \lambda^{\frac{d}{2}} \frac{d+1}{2} \left\| \Gamma_{\lambda, \delta}^+(f) \right\|_{L^2(\mathbb{R}^d)} \| u \|_{L^2(\mathbb{R}^d)} \| v \|_{L^2(\mathbb{R}^d)}.$$  

(3.24)

**Proof.** In the decomposition (3.21) we take $j_\ast = j_\ast(\delta_1)$. Since $u$, $v$ satisfy (3.8), we see that $\text{supp} F u_k^\epsilon \subset \{(0) \times S_k^\epsilon + e_2\} + O(\delta_1)$ and $\text{supp} F v_k^\epsilon \subset \{(0) \times S_k^\epsilon + e_2\} + O(\delta_2)$. Hence, we have

$$\text{supp} (F u_k^\epsilon(-\cdot) * F v_k^\epsilon) \subset \{(0) \times (-S_k^\epsilon + S_k^\epsilon)\} + O(\delta_1) \subset \{\xi_1 \leq \delta_1\} \times R_{k,k'}^\epsilon,$$

where the last inclusion follows from (3.13) since $2^{-j} \geq \delta_1$ for $j \leq j_\ast$. This observation enables us to insert the projection operator $P_{k,k'}^\delta$ to write

$$I_j = \sum_{k \sim k'} \langle (P_{k,k'}^\delta m_\epsilon^c(D)P_{\lambda} f)u_k^\epsilon, v_k^\epsilon \rangle.$$  

(3.25)

We first obtain estimates for each $I_j$. By (3.4) and (3.15), it follows that $I_j \neq 0$ only if $\lambda \lesssim 2^{-j}$. From (3.8) we see

$$\langle (P_{k,k'}^\delta m_\epsilon^c(D)P_{\lambda} f)u_k^\epsilon, v_k^\epsilon \rangle = \langle (m_\epsilon^c(D)P_{k,k'}^\delta P_{\lambda} f)u_k^\epsilon, v_k^\epsilon \rangle.$$  

Thus, Hölder’s inequality and Lemma 3.3 below give us that for every $j_0 < j < j_\ast$ and $k \sim k'$,

$$\left| \langle (P_{k,k'}^\delta m_\epsilon^c(D)P_{\lambda} f)u_k^\epsilon, v_k^\epsilon \rangle \right| \lesssim \lambda^\epsilon \left\| P_{k,k'}^\delta P_{\lambda} f \right\|_{L^p(\mathbb{R}^d)} \| u_k^\epsilon \|_{L^2(\mathbb{R}^d)} \| v_k^\epsilon \|_{L^2(\mathbb{R}^d)}.$$  

As already mentioned, by (3.13) we may assume $\lambda \lesssim 2^{-j}$ since $P_{k,k'}^\delta P_{\lambda} f = 0$ otherwise. Translation in the frequency space does not have any effect on the estimates, so we may apply (3.10) in Lemma 3.1 (with $h_1 = \delta_1$ and $h_2 = \delta_2$) to get, for $j_0 < j < j_\ast$,

$$\| u_k^\epsilon v_k^\epsilon \|_{L^p(\mathbb{R}^d)} \lesssim \delta_2^{-\epsilon} \lambda^{\frac{d}{2}} \frac{d+1}{2} \| u_k^\epsilon \|_{L^2(\mathbb{R}^d)} \| v_k^\epsilon \|_{L^2(\mathbb{R}^d)}.$$  

(3.26)

Combining these estimates with (3.25), we have, for $j_0 < j < j_\ast$,

$$|I_j| \lesssim \delta_2^{-\epsilon} \lambda^{\frac{d}{2}} \frac{d+1}{2} \sup_{k \sim k'} \left\| P_{k,k'}^\delta P_{\lambda} f \right\|_{L^p(\mathbb{R}^d)} \sum_{k \sim k'} \| u_k^\epsilon \|_{L^2(\mathbb{R}^d)} \| v_k^\epsilon \|_{L^2(\mathbb{R}^d)}.$$  

(3.27)

Since $\lambda \lesssim 2^{-j}$ and $\sum_{k \sim k'} \| u_k^\epsilon \|_{L^2(\mathbb{R}^d)} \| v_k^\epsilon \|_{L^2(\mathbb{R}^d)} \lesssim \| u \|_{L^2(\mathbb{R}^d)} \| v \|_{L^2(\mathbb{R}^d)}$ for $j_0 < j < j_\ast$ it follows that

$$|I_j| \lesssim \delta_2^{-\epsilon} \lambda^{\frac{d}{2}} \frac{d+1}{2} \left\| \Gamma_{\lambda, \delta}^+(f) \right\|_{L^2(\mathbb{R}^d)} \| u \|_{L^2(\mathbb{R}^d)} \| v \|_{L^2(\mathbb{R}^d)}.$$  

(3.28)

Now we consider $I_{j_\ast}$ in (3.21). Since there is no separation between the supports of $F(u_k^\epsilon)$, $F(v_k^\epsilon)$, the bilinear restriction estimates are no longer available. Instead, we use more elementary argument which relies on Lemma 3.7. In fact, as is clear for the argument in the above, it is sufficient to show

$$\| u_k^\epsilon v_k^\epsilon \|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{\epsilon} \left\| \Gamma_{\lambda, \delta}^+(f) \right\|_{L^2(\mathbb{R}^d)} \| u_k^\epsilon \|_{L^2(\mathbb{R}^d)} \| v_k^\epsilon \|_{L^2(\mathbb{R}^d)},$$  

(3.29)
which plays the role of (3.29). For the purpose we may assume \( \lambda \lesssim 2^{-j^*} \), otherwise \( P_{k,k'}^{j^*} \) is 0. Since \( 2^{-j^*} \simeq \sqrt{\delta_1} \) we have \( 1 \lesssim \lambda^{-\frac{p}{2}} \delta_1^{\frac{1}{p}} \). Thus, we only need to show that, for \( p \geq \frac{d+1}{2} \),

\[
(3.29) \quad \| u_k^{j} v_k^{j'} \|_{L^p((R^d)^d)} \lesssim \delta_1^{-\frac{d+1}{2}} \delta_2^{-\frac{d+1}{2}} \| u_k^{j} \|_{L^2} \| v_k^{j'} \|_{L^2}.
\]

The estimate is trivial with \( p = \infty \). By interpolation it is sufficient to show (3.29) with \( p = \frac{d+1}{2} \). We note that \( \mathcal{F} u_k^{j} \) is supported in a rectangle of dimensions approximately

\[
\delta_1 \times \delta_1 \times \sqrt{\delta_1} \times \cdots \times \sqrt{\delta_1}.
\]

Let \( r \) be the number such that \( \frac{d+1}{2} \). By Hölder’s inequality \( \| u_k^{j} v_k^{j'} \|_{L^p((R^d)^d)} \leq \| u_k^{j} \|_{L^r} \| v_k^{j'} \|_{L^{q'}} \),

then, applying Bernstein’s inequality and Lemma 2.7 to \( u_k^{j} \) and \( v_k^{j'} \), respectively, we have

\[
(3.30) \quad \| u_k^{j} v_k^{j'} \|_{L^p((R^d)^d)} \lesssim \delta_1^{-\frac{d+1}{2}} \delta_2^{-\frac{d+1}{2}} \| u_k^{j} \|_{L^2} \| v_k^{j'} \|_{L^2}.
\]

Since \( \delta_2 \leq \delta_1 \), this gives the desired (3.29) with \( p = \frac{d+1}{2} \).

We now prove (3.29). We claim that, if \( j_0 < j \leq j_* \) and \( k \sim k' \),

\[
(3.31) \quad \| u_k^{j} v_k^{j'} \|_{L^p((R^d)^d)} \lesssim \delta_2^{-\frac{d+1}{2}} \| u_k^{j} \|_{L^2} \| v_k^{j'} \|_{L^2}.
\]

Once we have (3.31), by repeating the same argument as in the above we get the bound

\[
\sum_{j_0 < j \leq j_*} |I_j| \lesssim \delta_2^{-\frac{d+1}{2}} \| u_k^{j} \|_{L^2} \| v_k^{j'} \|_{L^2}.
\]

So, it remains to show (3.31) for \( j_0 < j \leq j_* \). If \( j_0 < j < j_* \), (3.31) follows by the estimate (3.11) (in Lemma 3.1) with \( h_2 = \delta_2, h_1 = \delta_1 \). Thus, it remains to show (3.31) with \( j = j_* \).

As before, the case \( j = j_* \) is handled differently because there is no separation between the supports of \( \mathcal{F} u_k^{j} \), \( \mathcal{F} v_k^{j'} \). In fact, we claim that

\[
(3.32) \quad \| u_k^{j} v_k^{j'} \|_{L^p((R^d)^d)} \lesssim \delta_1^{-\frac{d+1}{2}} \| u_k^{j} \|_{L^2} \| v_k^{j'} \|_{L^2}.
\]

Since the estimate is trivial with \( p = \infty \), by interpolation it is enough to show (3.32) with \( p = \frac{d+1}{2} \). For simplicity we set

\[
\mathbf{u} = u_k^{j}, \quad \mathbf{v} = v_k^{j'},
\]

and we use the argument for (2.20). Let \( \{I_\ell\} \) be a collection of essentially disjoint intervals of length \( \simeq \delta_2 \) such that \( I_\ell \subset [-\delta_1, \delta_1] \), and \([-\delta_1, \delta_1] = \bigcup \ell I_\ell \). Then, we have

\[
\mathbf{u} \mathbf{v} = \sum_{\ell} \mathbf{u}^\ell \mathbf{v}^\ell, \quad \mathbf{u}^\ell := \mathcal{F}^{-1}(\chi_{I_\ell}(\xi_1)\tilde{\mathbf{u}}(\xi)).
\]

Since \( \mathcal{F} (\mathbf{u} \mathbf{v}) \) is supported in \((I_\ell + [-\delta_2, \delta_2]) \times (\mathbb{R}^{d-1})\), by [43] Lemma 6.1, and successively applying Bernstein’s inequality, Minkowski’s inequality and Hölder’s inequality we see that

\[
(3.33) \quad \left\| \sum_{\ell} \| \mathbf{u}^\ell \mathbf{v}^\ell \|_{L^p((R^d)^d)} \right\| \lesssim \left( \sum_{\ell} \| \mathbf{u}^\ell \mathbf{v}^\ell \|_{L^p((R^d)^d)} \right)^\frac{1}{p} \lesssim \delta_2^\frac{d+1}{2} \left( \sum_{\ell} \| \mathbf{u}^\ell(x_1, \cdot) \mathbf{v}(x_1, \cdot) \|_{L_{x_1}^1 L_{x'}^p} \right)^\frac{1}{p}.
\]

Let \( r \) be the number such that \( \frac{d+1}{2} \). By Hölder’s inequality

\[
\left\| \mathbf{u}^\ell(x_1, \cdot) \mathbf{v}(x_1, \cdot) \|_{L^\frac{d+1}{2}((R^d)^d)} \right\| \lesssim \left\| \mathbf{u}^\ell(x_1, \cdot) \|_{L^{\frac{d+1}{2}((R^d)^d)}} \| \mathbf{v}(x_1, \cdot) \right\|_{L^{\frac{d+1}{2}((R^d)^d)}}.
\]
Since $\mathcal{F}(u'(x_1,\cdot))$ is supported in a rectangle of dimensions about $\delta_1 \leq \sqrt{\delta_1} \times \cdots \times \sqrt{\delta_1}$ and $\mathcal{F}(v(x_1,\cdot))$ is supported in $\mathcal{S}_d + \mathbb{R}^{d-2} + O(\delta_2)$, applying Bernstein's inequality to $\|u'(x_1,\cdot)\|_{L^p}$ and Corollary 2.3 to $\|v(x_1,\cdot)\|_{L^{q(\ast)}}$, we have

$$\|u'(x_1,\cdot)v(x_1,\cdot)\|_{L^{q(\ast)}} \leq \delta_1^{\frac{2}{p} - \frac{1}{2}} \delta_2^{\frac{1}{p}} \|u'(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1})} \|v(x_1,\cdot)\|_{L^q(\mathbb{R}^{d-1})}.$$

Putting this in (3.33) with $p = \frac{d+1}{2}$, we have

$$\|uv\|_{L^{q(\ast)}} \leq \delta_1^{\frac{d-2}{p} + \frac{1}{2}} \left( \sum_{\ell} \|u'(x_1,\cdot)\|_{L^p} \|v(x_1,\cdot)\|_{L^q} \right)^{\frac{1}{2}} \leq \delta_1^{\frac{d-2}{p} + \frac{1}{2}} \|u\|_{L^p} \|v\|_{L^q}.$$

Thus we get (3.32) with $p = \frac{d+1}{2}$.

**Lemma 3.3.** If $\kappa \geq 0$ and $\lambda \geq 1/\tau$, then

$$\|m_\kappa^\tau(D)P_\lambda f\|_{L^p(\mathbb{R}^d)} \leq C\lambda^\kappa \|P_\lambda f\|_{L^p(\mathbb{R}^d)}$$

with $C$ independent of $\lambda$ and $\tau$.

**Proof.** By scaling and Young’s inequality, it suffices to show that

$$\left\| \int_{\mathbb{R}^d} m_\kappa^\tau(\xi)\tilde{\beta}(\xi)e^{i\xi \cdot \xi} d\xi \right\|_{L^1(\mathbb{R}^d)} \lesssim 1,$$

where $\tilde{\beta} \in C_c^\infty((1/4, 4))$ such that $\tilde{\beta}\beta = \beta$. Since $(\lambda \tau)^{-1} \lesssim 1$, this readily follows from integration by parts and (2.8). \hfill \square

**Estimates in $X^{b}_{\kappa(1), \lambda(1/\tau)}$.** We define a function $\beta_d : [\frac{d+1}{2}, \infty] \to [0, 1]$ as follows. For $3 \leq d \leq 6$,

$$\beta_d(p) = \begin{cases} 
1 - \frac{d+5}{2p} & \text{if } p \geq d + 1, \\
\frac{1}{2} - \frac{2}{p} & \text{if } d + 1 > p \geq 4, \\
0 & \text{if } 4 > p \geq \frac{d+1}{2},
\end{cases}$$

and, for $d \geq 7$,

$$\beta_d(p) = \begin{cases} 
1 - \frac{d+5}{2p} & \text{if } p \geq \frac{d+9}{2}, \\
\frac{1}{2} - \frac{d+1}{4p} & \text{if } \frac{d+9}{2} > p \geq \frac{d+1}{2}.
\end{cases}$$

**Proposition 3.4.** Let $d \geq 3$, $\tau \geq \frac{1}{\epsilon} - 1$, $\frac{1}{\tau} \leq \lambda \leq 1$, $p \geq \frac{d+1}{2}$, and let $f \in L^p(\mathbb{R}^d)$. Suppose that $S_\ell$ and $S_{\ell'}$ satisfy (3.24) and (3.27), and that $u$, $v$ satisfy (3.27) in place of $u_\ell$, $v_{\ell'}$, respectively. Then, for any $\epsilon > 0$, there is a constant $C = C(\epsilon, p, d) > 0$, independent of $\tau$, $\lambda$, $f$, $u$, $v$, $S_\ell$, and $S_{\ell'}$, such that

$$\|\langle m_\kappa^\tau(D)P_\lambda f, u \rangle\|_{L^p(\mathbb{R}^d)} \leq C\lambda^{\kappa - \frac{2}{p} + \beta_d(p) + \epsilon} \sup_{\frac{1}{\lambda} \leq \delta \leq 1} \Gamma^p_{\lambda, \delta}(f) \|u\|_{X^{\kappa/2}_{b(1)}(\mathbb{R}^d)} \|v\|_{X^{\epsilon/2}_{b(1)}(\mathbb{R}^d)}.$$

**Proof.** Let $\delta_*$ be the dyadic number such that $\tau^{-1} \leq \delta_* < 2\tau^{-1}$. We begin with decomposing $u$ and $v$ as follows:

(3.35) \quad u = \sum_{\delta_* \leq \delta \text{dyadic}} u_{\delta} := Q_{\leq \delta_*} u + \sum_{\delta_* < \delta \text{dyadic}} Q_1 u,

(3.36) \quad v = \sum_{\delta_* \leq \delta \text{dyadic}} v_{\delta} := Q_{\leq \delta_*} v + \sum_{\delta_* < \delta \text{dyadic}} Q_1 v.

Thus, we may write

(3.37) \quad \langle m_\kappa^\tau(D)P_\lambda f, u, v \rangle = I + II,
where
\[
I = \sum_{\delta_s \leq \delta_s \leq \delta_1} \langle (m_\nu(D^\nu P_\lambda f)u_{\delta_1}, v_{\delta_2} \rangle, \quad II = \sum_{\delta_s \leq \delta_1 < \delta_2} \langle (m_\nu(D^\nu P_\lambda f)u_{\delta_1}, v_{\delta_2} \rangle.
\]

We first consider the case \( d \geq 7 \). Since \( \delta_s \approx 1/\tau \), it is easy to see that
\[
\|u_{\delta_s}\|_{L^2} \lesssim \min\{\delta, 1/\tau\}^{-\frac{d}{2}} \|u_{\delta_s}\|_{X^{1/2}_{\zeta(1)}(1/\tau)} \lesssim \delta^{-\frac{d}{2}} \|u\|_{X^{1/2}_{\zeta(1)}},
\]
for any \( \delta \geq \delta_s \). Hence, utilizing (3.23) we have
\[
|I| \leq \sum \langle (m_\nu(D^\nu P_\lambda f)u_{\delta_1}, v_{\delta_2} \rangle \lesssim \delta_2^{-\frac{d+9}{2}} \delta_1^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{1}{2}} \delta_1^{-\frac{1}{2}} \|v\|_{X^{1/2}_{\zeta(1)}(1/\tau)}
\]
for \( p \geq \frac{d+9}{2} \) and \( \delta_s \leq \delta_2 \leq \delta_1 \). Considering the cases \( p \geq \frac{d+9}{2} \) and \( p < \frac{d+9}{2} \) separately and taking summation along the dyadic numbers \( \delta_1, \delta_2 \), we have that, for \( p > \frac{d+1}{2} \),
\[
|I| \leq \sum \langle (m_\nu(D^\nu P_\lambda f)u_{\delta_1}, v_{\delta_2} \rangle \lesssim \lambda^{\frac{d+9}{2}} \tau^{\frac{d+1}{2}} \|v\|_{X^{1/2}_{\zeta(1)}(1/\tau)}
\]
Symmetrically, interchanging the roles of \( \delta_1, \delta_2 \) and repeating the argument for \( II \) give the same bound as for \( I \). Thus we get (3.31) for \( d \geq 7 \).

We now consider the case \( 3 \leq d \leq 6 \). If \( \delta_s \leq \delta_2 \leq \delta_1 \), we use (3.24) instead of (3.23) to get
\[
|I| \lesssim \delta_2^{-\frac{d+9}{2}} \delta_1^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{1}{2}} \delta_1^{-\frac{1}{2}} \|v\|_{X^{1/2}_{\zeta(1)}(1/\tau)}
\]
Similarly we also have the estimate for \( \delta_s \leq \delta_1 < \delta_2 \). Recalling (3.37) and summing along \( \delta_1, \delta_2 \), we obtain (3.34) for \( 3 \leq d \leq 6 \).

3.4. Estimates for the antipodal case. We now consider the case in which \( u, v \) satisfy (3.7) in place of \( u_\ell, u_\nu \), respectively, while \( S_\ell \) and \( S_\nu \) satisfy (3.5). In this case \( S_\ell \) and \( S_\nu \) are not close to each other, but so are \( -S_\ell \) and \( S_\nu \). To use the decomposition (3.9) we need to modify the definition of \( u^j_\ell \) as follows:
\[
F_{u^j_\ell}(\xi) = \chi_{S_\ell}^{\frac{j}{\zeta_2}} (\frac{\xi - \xi_2}{\xi - \xi_2}) \hat{u}(\xi).
\]
But we keep the definition of \( v^j_\ell \) the same as in (3.20). As before, by (3.9) we have (3.21) and (3.22). Now, for \( \lambda, \delta > 0 \) and \( f \in L^p(\mathbb{R}^d) \), we set
\[
\Gamma_{\lambda, \delta}^{\nu, \nu}(f) = \sup_{j_{\nu} < j \leq j_{\nu}(x)} \sup_{k - k'} \left\{ 2^{-j_{\nu}} \|P_{k, k'}^- f\|_{L^p(\mathbb{R}^d)} \right\}.
\]

Lemma 3.5. Let \( d \geq 3, p \geq \frac{d+1}{2}, \frac{1}{\lambda} \leq 1, 0 < \delta_s \leq \delta_2 \leq \delta_1 \leq 2\epsilon_0 \), and let \( f \in L^p(\mathbb{R}^d) \). Suppose that \( S_\ell, S_\nu \) satisfy (3.5), and \( u, v \) satisfy (3.8). Then, for any \( \epsilon > 0 \),
\[
|\langle (m_\nu(D^\nu P_\lambda f)u, v) \rangle \lesssim \delta_s^{-\frac{d+9}{2}} \delta_1^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{1}{2}} \delta_1^{-\frac{1}{2}} \|v\|_{L^2(\mathbb{R}^d)},
\]
\[
|\langle (m_\nu(D^\nu P_\lambda f)u, v) \rangle \lesssim \delta_s^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{d+1}{2}} \delta_1^{-\frac{1}{2}} \delta_2^{-\frac{1}{2}} \delta_1^{-\frac{1}{2}} \|v\|_{L^2(\mathbb{R}^d)},
\]
when \( \lambda \approx 1 \). If \( \lambda \leq 1/2 \), \( \langle (m_\nu(D^\nu P_\lambda f)u, v) \rangle = 0 \).

The proof is similar to that of Lemma 3.2 except for different support property of \( \phi_{k, k'}^{\nu, \nu} \) in the frequency domain. So, we shall be brief.

Proof. As before, we choose the stopping step \( j_s = j_s(\delta_1) \) in (3.21). Noting supp \( F_{u^j_\ell} \subset (\{0\} \times (-S_\nu) + e_2) + O(\delta_1) \) and supp \( F_{v^j_\ell} \subset (\{0\} \times S_\nu' + e_2) + O(\delta_2) \), we see from (3.16) that
\[
\text{supp} \ (F_{u^j_k}(-) \cdot F_{v^j_k}) \subset \{ |\xi_1| \leq \delta_1 \} \times \mathcal{R}_{k, k'}^-,
\]
where \( j_0 < j \leq j_s \).
since $2^{-2j_0} \geq \delta_1$. This is the main difference from the previous neighboring case. See (3.17), (3.18), and Figure 4. Hence, we can insert the harmless projection operator $P_{k,k'}^{l,-}$ and $P_{l}^{e_1}$ to write

$$\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle = \langle (P_{k,k'}^{l,-}P_{l}^{e_1} m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle, \quad j_0 < j < j_*. $$

From (3.18), it follows that $\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle \neq 0$ only if $\lambda \approx 1$. Thus, for the rest of this proof we may assume that $\lambda \approx 1$, and, for $j_0 < j < j_*$ and $k \approx k'$, we have

$$\|\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle\| \lesssim \|P_{k,k'}^{l,-}P_{l}^{e_1} P_{\lambda} f \|_{L^p(\mathbb{R}^d)} \|u^i_{k_1}\|_{L^2}\|v^j_{k_2}\|_{L^2}$$

by Lemma 3.3. Applying (3.10) with $h_1 = \delta_1, h_2 = \delta_2$ we get (3.26) for $j_0 < j < j_*$. Hence, as in the proof of Lemma 3.2 from (3.22) and (3.33) we see that

$$|I_j| \lesssim \delta_2^{-2}\delta_1^{-\frac{d+9}{2}} \delta_2^{-\frac{d+9}{2}} \Gamma_{\lambda, \delta_1}^{e_1}(f)\|u\|_{L^2}\|v\|_{L^2},$$

and, consequently, we get

$$\sum_{j_0 < j < j_*} |I_j| \lesssim \delta_2^{-\epsilon} \delta_1^{-\frac{d+9}{2}} \delta_2^{-\frac{d+9}{2}} \Gamma_{\lambda, \delta_1}^{e_1}(f)\|u\|_{L^p}\|v\|_{L^p}.$$ 

The same estimates for $|I_{j_*}|$ can be obtained exactly in the same way as in the proof of Lemma 3.2 since (3.29) holds with $j = j_*$. This is easy to show using (3.30). So, we omit the details.

On the other hand, applying (3.11) instead of (3.10) and using (3.32), we have (3.31) for $j_0 < j < j_*$. Thus, following the same argument as in the above we obtain

$$\|\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle\| \lesssim \delta_2^{-\epsilon} \delta_1^{-\frac{d+9}{2}} \delta_2^{-\frac{d+9}{2}} \Gamma_{\lambda, \delta_1}^{e_1}(f)\|u\|_{L^2}\|v\|_{L^2}.$$ 

The following can be shown in the same way as in the proof of Proposition 3.4 exploiting Lemma 3.5 instead of Lemma 3.2. So, we omit its proof.

**Proposition 3.6.** Let $d \geq 3$, $\tau \geq \varepsilon_\gamma^{-1}$, $\lambda \approx 1$, $p > \frac{d+1}{\tau}$, and let $f \in L^p(\mathbb{R}^d)$. Suppose that $S_\ell$ and $S_{\ell'}$ satisfy (3.5), and that $u, v$ satisfy (3.7) in place of $u_\ell, v_\ell$, respectively. Then, for any $\epsilon > 0$ and there is a constant $C = C(\epsilon, p, d_0)$ such that

$$\|\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle\| \leq C\tau^{-d(p+\epsilon)} \sup_{\frac{d}{2} \leq \delta \leq 1} \Gamma_{\lambda, \delta}^{e_1}(f)\|u\|_{X^{1/2}_{\chi(1,1/\tau)}}\|v\|_{X^{1/2}_{\chi(1,1/\tau)}}.$$ 

If $0 < \lambda \leq 1/2$, the left side is zero.

### 3.5. Estimates for the transverse case.

When $S_\ell$ and $S_{\ell'}$ satisfy (3.3) and $u, v$ satisfy (3.7) in place of $u_\ell, v_\ell$, respectively, we can obtain bilinear estimates without invoking the decomposition (3.9) since the supports of $\widehat{u}, \widehat{v}$ are well separated. Also, note that

$$\langle -\text{supp} \widehat{u} \pm \text{supp} \widehat{v} \rangle \cap B_\delta(0, 2^{-\varepsilon_\gamma}) = \emptyset.$$ 

For $\lambda, \delta > 0$ and $f \in L^p(\mathbb{R}^d)$, we set

$$\Gamma_{\lambda, \delta}^{e}(f) = \lambda^\frac{d}{2} \delta^{-\frac{d}{2}} \|P_{\lambda} f \|_{L^p(\mathbb{R}^d)}.$$ 

**Proposition 3.7.** Let $d \geq 3$, $p \geq \frac{d+1}{\tau}$, $\tau \geq \varepsilon_\gamma^{-1}$, $\frac{1}{2} \leq \lambda \leq 1$, and let $f \in L^p(\mathbb{R}^d)$. Suppose that $S_\ell, S_{\ell'}$ satisfy (3.8) and $u, v$ satisfy (3.7) in place of $u_\ell, v_\ell$, respectively. Then, for any $\epsilon > 0$,

$$\|\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle\| \lesssim \lambda^{-\frac{d}{2}} \tau^{-d(p+\epsilon)} \sup_{\frac{d}{2} \leq \delta \leq 1} \Gamma_{\lambda, \delta}^{e}(f)\|u\|_{X^{1/2}_{\chi(1,1/\tau)}}\|v\|_{X^{1/2}_{\chi(1,1/\tau)}}.$$ 

It should be also noted that $\langle (m^e_\tau(D)P_{\lambda}f)u^i_{k_1}, v^j_{k_2} \rangle \neq 0$ only if $\lambda \geq \varepsilon_\gamma$ because of (3.45).
Proof. We follow the same way as in the proof of Proposition 3.4. Using (2.19) in Lemma 2.9, the dyadic decomposition (3.35), (3.36), and (3.37), we see that
\[
\|(m^\mu_\ell(D)P_\lambda f)u_{v_1}, v_2)\| \leq \|(m^\mu_\ell(D)P_\lambda P_{\Xi_1} f)u_{v_1}, v_2)\| \leq \lambda^{\tau}\|P_{\Xi_1}f\|_{L^p}\|u_{v_1}, v_2\|_{L^2}
\]
\[
\leq \delta_1^{\tau}\lambda^{\tau}\max_{\delta_1 \leq \delta_2 \leq \delta} \lambda^{\tau}\|P_{\Xi_1}f\|_{L^p}\|u_{v_1}, v_2\|_{L^2}
\]
\[
\leq \delta_2^{\tau}\lambda^{\tau}\max_{\delta_2 \leq \delta_1 \leq \delta} \lambda^{\tau}\|P_{\Xi_1}f\|_{L^p}\|u_{v_1}, v_2\|_{L^2}
\]
whenever \(\delta_1 \leq \delta_2 \leq \delta_1\). This plays the role of (3.39) in the proof of Proposition 3.4. As before summation over \(\delta_1 \leq \delta_2 \leq \delta_1\) gives the desired bound on \(I\). Interchanging the roles of \(\delta_1\) and \(\delta_2\) yields the estimate for \(II\). Thus, we get (3.46) for \(d \geq 7\). Applying (2.20) instead of (2.19) gives the estimate (3.46) for \(3 \leq d \leq 6\). \(\square\)

Combining the three (neighboring, antipodal, and transversal) cases and using Proposition 3.4, Proposition 3.6 and Proposition 3.7 we obtain the following.

Proposition 3.8. Let \(d \geq 3, \nu \geq \frac{d+1}{2}, \tau \geq c_0^{-1}, \frac{d}{d+1} \leq \lambda \leq 1\), and let \(f \in L^p(\mathbb{R}^d)\). Suppose \(u\) and \(v\) satisfy (3.1). Then
\[
(3.47) \quad \|(m^\mu_\ell(D)P_\lambda f)u, v\| \leq \lambda^\tau\nu\|P_{\Xi_1}f\|_{L^p}\|u\|_{X^{1/2}_{\lambda,1/\tau}}\|v\|_{X^{1/2}_{\lambda,1/\tau}},
\]
where \(\nu = \kappa - \frac{d}{d+1}\), \(\mu > |\beta_0(p)|\), and\n\[
A_\nu(f, \lambda, e_1) = \max_{(S_\ell, S_\ell'):\text{neighboring}} \Gamma^p_{\lambda,\delta}(f) + \max_{(S_\ell, S_\ell'):\text{antipodal}} \Gamma^p_{\lambda,\delta}(f).
\]

Proof. In the primary decomposition (3.5), the number of pairs \((S_\ell, S_\ell')\) is finite. Thus, from the estimates in Proposition 3.4, Proposition 3.6 and Proposition 3.7, it follows that
\[
\|(m^\mu_\ell(D)P_\lambda f)u, v\| \leq \tau^\nu\nu\|P_{\Xi_1}f\|_{L^p}\|u\|_{X^{1/2}_{\lambda,1/\tau}}\|v\|_{X^{1/2}_{\lambda,1/\tau}} \times \max_{(S_\ell, S_\ell'):\text{neighboring}} \Gamma^p_{\lambda,\delta}(f) + \max_{(S_\ell, S_\ell'):\text{antipodal}} \Gamma^p_{\lambda,\delta}(f) + \lambda^\tau\nu\|P_{\Xi_1}f\|_{L^p}\|u\|_{X^{1/2}_{\lambda,1/\tau}}\|v\|_{X^{1/2}_{\lambda,1/\tau}}.
\]
This gives (3.47) since \(\lambda \leq 1\). \(\square\)

Remark 3. When \(d = 3\), it is possible to remove \(\tau^\nu\nu\) by replacing \(\lambda^{\tau\nu}\) with \(\log(1/\lambda)\lambda^{\tau}\nu\) since the number of nonzero terms in the summation (3.28) is \(\approx \log(1/\lambda)\).

3.6. Strengthening the estimates (3.47) when \(3 \leq d \leq 8\) and \(p \geq d\). The estimates in Proposition 3.4, Proposition 3.6 and Proposition 3.7 can be improved if we restrict the range of \(p\) to the interval \([d, \infty]\), and combine them with the following which is a consequence of the linear estimate (2.21).

Lemma 3.9. Let \(p \geq d \geq 3, \frac{1}{d} \leq \delta_0 \leq \delta_1 \leq 1\), and \(\frac{1}{d} \leq \lambda \leq 1\). Suppose that \(f \in L^p(\mathbb{R}^d)\) and \(u, v\) satisfy (3.8). Then
\[
(3.48) \quad \|(m^\mu_\ell(D)P_\lambda f)u, v\| \leq \lambda^\tau\nu\|P_{\Xi_1}f\|_{L^p}\|u\|_{X^{1/2}_{\lambda,1/\tau}}\|v\|_{X^{1/2}_{\lambda,1/\tau}}.
\]

Proof. Interpolation between (2.21) (with \(h = \delta_2 = \delta_1\)) and the trivial estimate \(\|uv\|_{L^1} \leq \|u\|_{L^2}\|v\|_{L^2}\) gives
\[
\|uv\|_{L^p(\mathbb{R}^d)} \leq \delta_2^{\frac{dp}{2}}\|u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}
\]
for \(p \geq d\). Using this estimate and Lemma 3.8, we see that
\[
\|(m^\mu_\ell(D)P_\lambda f)u, v\| = \|(m^\mu_\ell(D)P_{\Xi_1}^\lambda f)u, v\| \leq \lambda^\tau\nu\|P_{\Xi_1}^\lambda f\|_{L^p}\|uv\|_{L^p(\mathbb{R}^d)}
\]
\[
\leq \lambda^\tau\nu\|u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}.
\]
Combining this with (3.38) we get (3.48). \(\square\)
For $3 \leq d \leq 8$, we define $\gamma_d : [d, \infty) \to (0,1]$. For $3 \leq d \leq 6$, we set

\begin{equation}
\gamma_d(p) = \begin{cases} 
1 - \frac{d+5}{2p} & \text{if } p \geq d + 1, \\
\frac{(2d-3)p-(d^2+3d-6)}{2(d-1)p} & \text{if } d + 1 > p \geq \max\{d, \frac{d^2+3d-6}{2d-3}\}, \\
0 & \text{if } \max\{d, \frac{d^2+3d-6}{2d-3}\} > p \geq d,
\end{cases}
\end{equation}

(3.49)

and, for $d = 7,8$, we set

\begin{equation}
\gamma_d(p) = \begin{cases} 
1 - \frac{d+5}{2p} & \text{if } p \geq \frac{d+9}{2}, \\
\frac{2(d+4)p-(d^2+9d+16)}{2(d+5)p} & \text{if } \frac{d+9}{2} > p \geq d.
\end{cases}
\end{equation}

(3.50)

Note that $\frac{d^2+3d-6}{2d-3} \leq d$ if and only if $d \geq 5$, so there is no $p$ that belongs to the range of third line in (3.49) when $d = 5, 6$. In higher dimensions $d \geq 9$ the bounds in Proposition 3.4, Proposition 3.6, Proposition 3.7 are already better than the estimates which we can deduce by combining the linear and bilinear estimates. Improved bounds are possible for all three cases (3.3), (3.4), and (3.5), by the similar argument. So we provide the details only for the case (3.4) and state the estimates for the other cases without providing the proof.

**Proposition 3.10.** Let $3 \leq d \leq 8$, $\tau \geq \varepsilon_1^{-1}, \frac{1}{2} \leq \lambda \leq 1$, $p \geq d$, and let $f \in L^p(\mathbb{R}^d)$. Suppose that $\mathcal{S}_i$ and $\mathcal{S}_{i'}$ satisfy (3.4), and $u, v$ satisfy (3.7) in place of $u_i, v_{i'}$, respectively. For $\nu = -\frac{2}{p}$ and $\mu > \gamma_d(p)$, there is a constant $C > 0$, independent of $\tau$ and $\lambda$, such that

\begin{equation}
|\langle (m^c(D)P_\lambda f)u, v \rangle| \leq C\lambda^{\tau}\mu \sup_{\frac{1}{2} \leq \delta \leq 1} \left( \Gamma_{\lambda, \delta}^p + \Gamma_{\lambda, \delta}^q \right) \|u\|_{X^{1/2}_{\xi(t), \theta}} \|v\|_{X^{1/2}_{\xi(t), \theta}}.
\end{equation}

(3.51)

**Proof.** We first consider the case $3 \leq d \leq 6$ under the assumption that $d \leq p < d + 1$. Recalling (3.37), it is sufficient to handle $I$ because $II$ can be handled symmetrically. So, we assume $\delta_2 \leq \delta_1$. From (3.40) and (3.48) we have, for $0 \leq \theta \leq 1$,

\begin{equation}
|\langle (m^c(D)P_\lambda f)u_\delta, v_\delta \rangle| \lesssim \delta_2^{-\varepsilon} \lambda^{\frac{2}{p} \delta_1 \delta_2} \Gamma_{\lambda, \delta}^p \|u\|_{X^{1/2}_{\xi(t), \theta}} \|v\|_{X^{1/2}_{\xi(t), \theta}}
\end{equation}

(3.52)

where $\Gamma_{\theta} = \left( \Gamma_{\lambda, \delta_1}^p + \Gamma_{\lambda, \delta_2}^p \right)^{1-\theta}$ and $b_1(\theta) = \frac{d+1}{2} - \frac{1}{2}$, $b_2(\theta) = \frac{d+2-(d-2)\theta}{2} - \frac{1}{2}$.

Note that $\Gamma_{\theta} \leq \theta \Gamma_{\lambda, \delta_1}^p + (1-\theta)\Gamma_{\lambda, \delta_2}^p$. Hence, in order to show (3.50) it is sufficient to find $\theta \in [0,1]$ such that

\begin{equation}
\Sigma(\theta) := \sum_{\frac{1}{2} \leq \delta_2 \leq \delta_1 \leq 1} b_1(\theta) b_2(\theta) \leq C(\log \tau) \tau^{-\gamma_d(p)}.
\end{equation}

(3.53)

For the purpose let $\theta_1$ and $\theta_2$ be such that $b_1(\theta_1) = 0$ and $b_2(\theta_2) = 0$. That is to say, $\theta_1 = \frac{d+2-p}{d-2}$ and $\theta_2 = \frac{d+2-p}{d-2}$. Since $d \leq p < d + 1$ we see that $0 < \theta_1 < 1$ and $\theta_2 > 0$. We consider the two cases

(i): $\theta_2 \leq \theta_1 < 1$,  
(ii): $\theta_1 < \theta_2$,

which are equivalent to

(i): $(d^2 + 3d - 6)/(2d - 3) \leq p < d + 1$,  
(ii): $d \leq p < (d^2 + 3d - 6)/(2d - 3)$,

respectively. If $d = 3$, the case (i) is void. If $d \geq 5$, since $d \geq (d^2 + 3d - 6)/(2d - 3)$, the case (ii) is void and the other (i) is equivalent to $d \leq p < d + 1$. For $d = 3, 4$, the case (ii) is easy to handle. Indeed, if we choose any $\theta \in (\theta_1, \theta_2)$, (3.52) holds with $\gamma_d(p) = 0$ since $b_1(\theta), b_2(\theta) > 0$.

Now it remains to consider the case (i) when $d \geq 4$. We separately consider the following three cases:

(A): $0 \leq \theta \leq \theta_2$,  
(B): $\theta_2 < \theta \leq \theta_1$,  
(C): $\theta_1 < \theta \leq 1$. 

()}
We note that $b_1(\theta) + b_2(\theta) = \frac{\theta + d + 1}{4p} - 1$ is increasing in $\theta$. Hence for $0 \leq \theta \leq \theta_1$, $b_1(\theta) + b_2(\theta) \leq b_2(\theta_1) \leq 0$.

In the case (A), $b_2(\theta) \geq 0$. Thus,

$$\Sigma(\theta) = \sum_{\frac{1}{4} \leq \delta_1 \leq \delta_1 \leq 1} \delta_1 b_1(\theta) + b_2(\theta) \left( \frac{\delta_2}{\delta_1} b_2(\theta) \right) \lesssim \sum_{\frac{1}{4} \leq \delta_1 \leq 1} \delta_1 b_1(\theta) + b_2(\theta) \lesssim (\log \tau)^{-\left(b_1(\theta) + b_2(\theta)\right)}.$$

In the case (B), $b_1(\theta) \leq 0$ and $b_2(\theta) < 0$. Thus, $\Sigma(\theta) \lesssim (\log \tau)^{-\left(b_1(\theta) + b_2(\theta)\right)}$. In the case (C), $b_1(\theta) > 0$ and $b_2(\theta) < 0$, so summation along $\delta_1$ is finite and we get $\Sigma(\theta) \lesssim \tau^{-b_2(\theta)}$. Therefore, recalling $b_1(\theta) + b_2(\theta)$ is increasing and non-positive when $0 \leq \theta \leq \theta_1$ and $b_2(\theta)$ is decreasing and non-positive when $\theta \geq \theta_1$, we choose $\theta = \theta_1$ that makes \(3.52\) true with the smallest exponent of $\tau$, and we have \(3.52\) with

$$\gamma_d(p) = -b_2(\theta_1) = \frac{(2d - 3)p - (d^2 + 3d - 6)}{2p(d - 1)}$$

provided that $\max\left\{d, \frac{d^2 + 3d - 6}{2d - 3}\right\} \leq p < d + 1$.

If $p \geq d + 1$, since $\theta_2 \leq 1$, the case (ii) is void and we need only to consider the cases (A) and (B).

From the above computation we take $\gamma_d(p) = -(b_1(1) + b_2(1)) = 1 - \frac{d + 1}{2p}$, which corresponds to \(3.51\) with $\theta = 1$.

We now turn to the case $d = 7, 8$. Combining \(3.39\) and \(3.48\), we have

$$|(m_\tau^\varepsilon(D)P_\lambda f)u_\delta, v_\delta)| \lesssim \tau^\varepsilon \lambda^{\varepsilon \frac{1}{2}} \delta_1^{b_1(\theta)} \delta_2^{b_2(\theta)} \Gamma_\theta \|u\|_{X_\lambda^{1/2},1/\tau} \|v\|_{X_\lambda^{1/2},1/\tau},$$

where $\theta \in [0, 1]$ and $b_1(\theta) = \frac{(d+\delta)\theta + 4}{4p} - \frac{1}{2}$, $b_2(\theta) = \frac{2(d+\delta) - (d+3)\theta}{4p} - \frac{1}{2}$. Once we have this estimate we can repeat the same argument to get the desired bound \(3.50\). So, we omit the details.

For the other cases \(3.5\) and \(3.3\), we apply the same argument to get improved estimates. In fact, for the antipodal case \(3.5\), we use \(3.41\), \(3.42\), and \(3.48\). Thus, we get

$$(3.53) \quad |(m_\tau^\varepsilon(D)P_\lambda f)u, v| \lesssim \tau^{\gamma_d(p) + \varepsilon} \sup_{\frac{1}{4} \leq \delta \leq 1} \left( \Gamma_\theta^K P_\lambda^K (f) + \Gamma_\theta^K P_\lambda^K (f) \right) \|u\|_{X_\lambda^{1/2},1/\tau} \|v\|_{X_\lambda^{1/2},1/\tau},$$

for any $\varepsilon > 0$. For the transversal case \(3.3\), we have, for $\varepsilon > 0$,

$$(3.54) \quad |(m_\tau^\varepsilon(D)P_\lambda f)u, v| \lesssim \lambda^{\gamma_d(p) + \varepsilon} \sup_{\frac{1}{4} \leq \delta \leq 1} \|u\|_{X_\lambda^{1/2},1/\tau} \|v\|_{X_\lambda^{1/2},1/\tau}.$$ 

As in Proposition \(3.8\) combining the estimates \(3.50\), \(3.53\), and \(3.54\) of the three cases \(3.1\), \(3.5\), and \(3.3\), we obtain the following.

**Proposition 3.11.** Let $3 \leq d \leq 8$, $\tau \geq \tau^* \geq 1$, $\frac{1}{4} \leq \lambda \leq 1$, $p \geq d$, and let $f \in L^p(\mathbb{R}^d)$. Suppose $u$ and $v$ satisfy \(3.1\). Then \(3.47\) holds for $\nu = \frac{\tau}{p} - \frac{1}{p}$ and $\mu > \gamma_d(p)$.

### 4. Average over rotation and dilation

In this section, we consider the average of $A_p(f, \tau, \lambda, \tau^{-1}Ue)$ over $U \in O_d$ and $\tau \in [1, 2]$. The projection operators engaged in the definition of $A_p(f, \tau, \lambda, \tau^{-1}Ue)$ break the Fourier support of $f$ into small pieces. Average over $U \in O_d$ and $\tau \in [1, 2]$ makes it possible to exploit such smallness of Fourier supports. This gives considerably better bounds which are not viable when one attempts to control $A_p(f, \lambda, e)$ for a fixed $e$ with $\|P_\lambda f\|_{L^p}$.

For an invertible $d \times d$ matrix $U$, let us define the projection operator $(P_{k,k'}^{\pm})^U$ by

$$(4.1) \quad F((P_{k,k'}^{\pm})^U g)(\xi) = \phi_{k,k'}^{\pm}((\Xi^U)\xi) \tilde{g}(\xi),$$

where $\phi_{k,k'}^{\pm}$ is the Fourier transform of $\phi_{k,k'}$.
where \( U^t \) is the transpose of \( U \). Let \( dm \) be the normalized Haar measure on \( O_d \). Then we have, for any \( \theta \in \mathbb{S}^{d-1} \) and \( f \in L^1(\mathbb{S}^{d-1}) \),

\[
\int_{O_d} f(U\theta)dm(U) = c_d \int_{\mathbb{S}^{d-1}} f(\omega)d\sigma(\omega)
\]

for some dimensional constant \( c_d \). Let \( P_{\lambda} \) denote the operator given by \( F(P_{\lambda}f) = \tilde{\beta}(|\cdot|/\lambda)\hat{f} \) with \( \tilde{\beta} \in C^\infty_c((2^{-2},2^3)) \) such that \( \tilde{\beta} = 1 \) on \([2^{-1},2^0]\). The following lemma can be obtained in the same manner as in the proof of [19, Lemma 5.1].

**Lemma 4.1.** Let \( \delta, \lambda > 0 \). If \( f \in L^p(\mathbb{R}^d) \), \( p \in [2, \infty) \), then

\[
\int_1^2 \int_{O_d} \| P_{\leq \delta}^{-1} U \tau \|_{L^p(\mathbb{R}^d)} dm(U) d\tau \lesssim (\delta/\lambda) \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)},
\]

where the implicit constant is independent of \( \delta, \lambda \).

The following lemma is a consequence of Lemma 4.1 and properties of the projection operator \( P_{k,k'}^{-1} \). Recall the definition of \( j_0 \) and \( j_* = j_*(\delta) \) from Section 3.2 and Section 3.3.

**Lemma 4.2.** Let \( 0 < \delta, \lambda \lesssim 1 \) and \( f \in L^p(\mathbb{R}^d) \), \( p \in [2, \infty) \). For \((S_{\ell}, S_{p'})\) satisfying (3.3) and \( j_0 < j < j_\ast = j_*(\delta) \), we have

\[
\int_1^2 \int_{O_d} \sup_{k, k'} \| (P_{k,k'}^{-1})^{-1} U \|_{L^p_{\tau} \rightarrow L^p} dm(U) d\tau \lesssim 2^j \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)},
\]

and

\[
\int_1^2 \int_{O_d} \sup_{k, k'} \| (P_{k,k'}^{-1})^{-1} U \|_{L^p_{\tau} \rightarrow L^p} dm(U) d\tau \lesssim (\delta/\lambda) \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)}.
\]

Here the implicit constants are independent of \( \delta, \lambda, j \).

**Proof.** For any \( p \), the norms \( \| (P_{k,k'}^{-1})^{-1} U \|_{L^p_{\tau} \rightarrow L^p} \) are bounded uniformly with respect to \( j_0 < j \leq j_\ast \), \( k \sim k', \tau \in [1, 2] \), and \( U \in O_d \). Hence

\[
\sup_{k \sim k', \tau} \| (P_{k,k'}^{-1})^{-1} U \|_{L^p_{\tau} \rightarrow L^p} \lesssim \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)}
\]

holds uniformly for all \( j, \tau, \) and \( U \). When \( j_0 < j < j_\ast \), we note from (3.4) that the support of the multiplier of \( (P_{k,k'}^{-1})^{-1} U \) is contained in the annulus \( \{ \xi \mid |U\tau| \xi \approx 2^{-j} \} \) since \( \tau \in [1, 2] \). Hence, we may assume \( \lambda \approx 2^{-j} \) as seen in the proof of Lemma 3.2 and (4.4) follows from (4.6) and (4.3). Also, (4.5) follows similarly by using (4.5) instead of (4.4). \( \square \)

In Lemma 4.2, the average in \( \tau \) does not have any significant role. However, in what follows, the average in dilations yields additional improvement. To show this we exploit the support properties of the multiplier of \( P_{k,k'}^{-1} \).

**Lemma 4.3.** Let \( 0 < \delta, \lambda \lesssim 1 \) and \( f \in L^p(\mathbb{R}^d) \) for \( p \in [2, \infty) \). For \((S_{\ell}, S_{p'})\) satisfying (3.3) and \( j_0 < j < j_\ast = j_*(\delta) \), we have

\[
\int_1^2 \int_{O_d} \sup_{k \sim k'} \| (P_{k,k'}^{-1})^{-1} U \|_{L^p_{\tau} \rightarrow L^p} dm(U) d\tau \lesssim 2^{-2j} \| P_{\lambda} f \|_{L^p(\mathbb{R}^d)},
\]

where the implicit constant is independent of \( \delta, \lambda, j \).

**Proof.** Note that the multiplier of the operator \( (P_{k,k'}^{-1})^{-1} U \) is supported in a dilation (from its center) of the rectangle \( \tau U([-\delta, \delta] \times R_{k,k'}^{-1}) \) of dimensions about \( \delta \times 2^{-2j} \times 2^{-j} \times \cdots \times 2^{-j} \). From (3.18) we note that
\( \{\tau R_{k,k'}^{j,-}\}_{k \sim k'} \) are boundedly overlapping and contained in \( 2\tau \mathbb{S}^{d-2} + O(2^{-2j}) \). Thus we may assume \( \lambda \approx 1 \); otherwise the left side of (4.12) vanishes, so the estimate is trivial. For the proof of (4.12), it suffices to prove that

\[
(4.8) \quad \int_1^2 \int_{O_d} \left( \sum_{k \sim k'} \| (P_{j,k,k'}^{+,-})^{-1} U_{P_{\leq \delta}^T U_{e_1}} P_{\tau \lambda f} \|_{L^p(\mathbb{R}^d)} \right) dm(U) d\tau \lesssim 2^{-2j} \| P_{\tau \lambda f} \|_{L^p}. 
\]

It is easy to see that

\[
\| (P_{j,k,k'}^{+,-})^{-1} U_{P_{\leq \delta}^T U_{e_1}} P_{\tau \lambda f} \|_{L^\infty(\mathbb{R}^d)} \leq C \| P_{\tau \lambda f} \|_{L^\infty(\mathbb{R}^d)}
\]

with \( C \) independent of \( j, k, k' \), \( \tau \), and \( U \). Hence, by interpolation between \( \ell^\infty_\mathbb{R} L^\infty \) and \( \ell^2 \mathbb{R}^2 \), and by Plancherel’s theorem, to get (4.8) for \( 2 \leq p < \infty \), it is enough to show

\[
\int_1^2 \int_{O_d} \sum_{k \sim k'} \left| \beta_0 \left( \frac{U e_1 \cdot \xi}{2\delta} \right) \phi_{j,k,k'} \left( \frac{\tau U \xi}{\tau} \right) \beta \left( \frac{\xi}{\lambda \tau} \right) \tilde{f}(\xi) \right|^2 d\xi dm(U) d\tau \lesssim 2^{-2j} \| P_{\tau \lambda f} \|_{L^2}.
\]

Again, by interpolation with the trivial \( \ell^\infty_\mathbb{R} L^\infty U, \tau \) estimate, it is enough to show that for any \( j \),

\[
\int_1^2 \int_{O_d} \beta_0 \left( \frac{U e_1 \cdot \xi}{2\delta} \right) \sum_{k \sim k'} \phi_{j,k,k'} \left( \frac{U \tau U \xi}{\tau} \right) \left| \beta \left( \frac{\xi}{\lambda \tau} \right) g(\xi) \right| d\xi dm(U) d\tau \lesssim 2^{-2j} \| g \|_{L^1}.
\]

This follows if we show that, for \( 2^{-1} \lambda \leq |\xi| \leq 2^2 \lambda 

\[
(4.9) \quad \int_1^2 \int_{O_d} \beta_0 \left( \frac{U e_1 \cdot \xi}{2\delta} \right) \sum_{k \sim k'} \phi_{j,k,k'} \left( \frac{U \tau U \xi}{\tau} \right) dm(U) d\tau \lesssim 2^{-2j} \delta.
\]

By (4.2) and Fubini’s theorem, we have

\[
\int \int \sum_{k \sim k'} \phi_{j,k,k'} \left( \frac{U \tau U \xi}{\tau} \right) \beta_0 \left( \frac{U e_1 \cdot \xi}{2\delta} \right) dm(U) d\tau \simeq \int_{\mathbb{R}^{d-1}} \left( \int_1^2 \sum_{k \sim k'} \phi_{j,k,k'} \left( \frac{\xi \omega}{\tau} \right) d\tau \right) \beta_0 \left( \frac{U e_1 \cdot \xi \omega}{2\delta} \right) d\sigma(\omega).
\]

Since \( |\xi| \approx \lambda \approx 1 \) and \( \bigcup_{k \sim k'} R_{k,k'}^{j,-} \) is contained in \( 2\mathbb{S}^{d-2} + O(C2^{-2j}) \), we see that

\[
(4.10) \quad \int_1^2 \sum_{k \sim k'} \phi_{j,k,k'} \left( \frac{\xi \omega}{\tau} \right) d\tau \lesssim \int_1^2 \chi_{\bigcup_{k \sim k'} R_{k,k'}^{j,-}} \left( \frac{\xi \omega}{\tau} \right) d\tau \simeq \int_{|\xi|/2}^{|\xi|} \chi_{\bigcup_{k \sim k'} R_{k,k'}^{j,-}} \left( \xi \omega \right) \frac{d\omega}{t^2} \lesssim 2^{-2j}
\]

for \( \omega \in \mathbb{S}^{d-1} \). For \( |\xi| \geq 2^{-1} \lambda \), it is easy to see that

\[
(4.11) \quad \int_{|\xi|/2}^{|\xi|} \beta_0 \left( \frac{U e_1 \cdot \xi \omega}{2\delta} \right) d\sigma(\omega) \lesssim \min\{ \delta, 1 \}.
\]

Combining (4.10) and (4.11), we get (4.9).

Combining Lemma 4.1, Lemma 4.2 and Lemma 4.3 we obtain the following.

**Proposition 4.4.** Let \( 0 < \lambda \lesssim 1 \) and let \( f \in L^p(\mathbb{R}^d) \) with \( p \in [2, \infty) \). Then,

\[
(4.12) \quad \int_1^2 \int_{O_d} [A_p(f \tau U, \lambda, e_1)]^P dm(U) d\tau \leq C \| P_{\tau \lambda f} \|_{L^p(\mathbb{R}^d)},
\]

where \( f \tau U(x) = \tau^{-d} f(\tau^{-1} U x) \), and \( C \) is independent of \( \lambda \) and \( e_1 \in \mathbb{S}^{d-1} \).

**Proof.** Since \( \hat{f} \tau U(\xi) = \hat{f}(\tau U \xi) \), by changing variables \( \xi \rightarrow \tau^{-1} U \xi \) in the frequency side, we have

\[
\| P_{\leq \delta}^{\tau \Delta} P_{\leq \delta}^{\tau U_{e_1}} P_{\tau \lambda f} \|_{L^p} = \tau^{-d} \| P_{\leq \delta}^{\tau \Delta} P_{\leq \delta}^{\tau U_{e_1}} P_{\tau \lambda f} \|_{L^p},
\]

\[
\| P_{\leq \delta}^{\tau \Delta} P_{\tau \lambda f} \|_{L^p} = \tau^{-d} \| P_{\leq \delta}^{\tau \Delta} P_{\tau \lambda f} \|_{L^p}.
\]
Since $\tau \approx 1$, from the definition of $\Gamma_{\lambda,\delta}^{p,\gamma}(f)$ and $\Gamma_{\lambda,\delta}^{p,\gamma}(f)$, it follows that

$$A_p(f_{\tau U}, \lambda, \epsilon_1) \leq \max_{(S_{\tau'}, y': y'' \in \mathbb{S}^1)} \sup_{0 < \delta \leq 1} \sup_{0 < i < j \leq k'} \sup_{k''} \left\{ \left( \frac{\lambda}{\delta} \right)^{\frac{\nu}{2}} \left( P_{k',k''}^{\tau,\gamma} \right)^{1-\nu} \left( P_{\delta}^{\tau,\gamma} \right)^{1-\nu} u_{\tau \lambda f} \right\}$$

(4.13)

Hence, we get (4.12) by Lemma 4.1, Lemma 4.2, and Lemma 4.3.

5. Key estimates: asymptotically vanishing averages

In this section, we assemble the various estimates in the previous sections and obtain the estimates that are the key ingredients for the proofs of Theorem 1.1 and Theorem 1.2.

**Proposition 5.1.** Let $0 \leq \kappa \leq 1$, $\tau > 1$, $\frac{d}{2-\kappa} \leq p < \infty$, and let $g \in L^p(\mathbb{R}^d)$ with $\sup p \in B_\delta(0, \tau)$. Suppose (3.47) holds. Then, we have

$$\| \mathcal{M}_{m U} g \|_{X_{\xi(1),1/\tau}^{1/2} \rightarrow X_{\xi(1),1/\tau}^{1/2}} \leq \tau^{2-\kappa-\frac{\nu}{2}} \| g \|_{L^p} + \sum_{\frac{\epsilon}{\lambda} \leq \epsilon < 1} \lambda^{\nu} \tau^\mu A_p(g, \lambda, \epsilon_1).$$

(5.1)

Therefore, by Proposition 3.8, the estimate (5.1) holds provided that $d \geq 3$, $0 \leq \kappa \leq 1$, $\max \left\{ \frac{d+1}{2} - \frac{d}{\kappa} \right\} \leq p < \infty$, $\nu = \kappa - \frac{2}{p}$, and $\mu > \delta_\lambda(p)$. Moreover, when $3 \leq d \leq 8$ and $p \geq d$, we can exploit Proposition 5.1 to obtain better bounds (5.1) with $\mu > \gamma_\lambda(p)$.

Recalling the definitions of $Q^\epsilon_\mu$ and $Q^\epsilon_\mu$ in Section 2.3, we define the Fourier multiplier operator $Q^\epsilon_\mu$ by $Q^\epsilon_\mu u := u - Q^\epsilon_\mu u$.

**Proof of Proposition 5.1.** To begin with, we fix a small number $\delta_\epsilon \in [2^{-3\epsilon_0}, 2^{-2\epsilon_0}]$. It is easy to see that

$$\| DQ^1_{\delta_\epsilon} u \|_{L^2(\mathbb{R}^d)} \leq \| u \|_{X^{1/2}_{\xi(1),1/\tau}},$$

(5.2)

$$\| Q^1_{\delta_\epsilon} v \|_{L^2(\mathbb{R}^d)} \leq \| Q^1_{\delta_\epsilon} v \|_{H^1} \leq \| v \|_{X^{1/2}_{\xi(1),1/\tau}},$$

(5.3)

$$\| DQ^1_{\delta_\epsilon} u \|_{L^2(\mathbb{R}^d)} \leq \| u \|_{X^{1/2}_{\xi(1),1/\tau}}, \quad \| Q^1_{\delta_\epsilon} v \|_{L^2(\mathbb{R}^d)} \leq \| v \|_{X^{1/2}_{\xi(1),1/\tau}}.$$
where the last inequality follows from (5.2) and (5.3). Using (5.2), (5.3), and (5.4), the same argument gives
\[
II \lesssim \|g\|_{L^p} \left( \|D(Q_{>\kappa_0}^2u\|_{L^2} \|Q_{\leq \kappa_0}^1 v\|_{L^2} + \|Q_{>\kappa_0}^1 u\|_{L^2} \|D(Q_{\leq \kappa_0}^1 v\|_{L^2}) \right)
\lesssim \tau^{-2-\frac{\kappa}{2}} \|g\|_{L^p} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}.
\]
Similarly, we have \(III \lesssim \tau^{-2-\kappa-rac{2}{p}} \|g\|_{L^p} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}\) by interchanging the roles of \(u\) and \(v\). Therefore
\[
I + II + III \lesssim \tau^{-2-\kappa-rac{2}{p}} \|g\|_{L^p} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}.
\]

Now we consider \(IV\) that is given by the low frequency parts \(Q_{\leq \kappa_0}^1 u\) and \(Q_{\leq \kappa_0}^1 v\). By Littlewood-Paley decomposition, we have
\[
|IV| \leq \left| \left\langle (m_\tau^\omega(D)P_{\leq \kappa_0}^1 g)Q_{\leq \kappa_0}^1 u, Q_{\leq \kappa_0}^1 v \right\rangle \right| + \sum_{\frac{\kappa}{2} \leq \lambda \leq 1: \text{dyadic}} \left| \left\langle (m_\tau^\omega(D)P_{\kappa} g)Q_{\leq \kappa_0}^1 u, Q_{\leq \kappa_0}^1 v \right\rangle \right|,
\]
where
\[
F(P_{\leq r} u)(\xi) := \left( 1 - \sum_{j \geq 0} \beta(2^{-j} r^{-1} |\xi|) \right) \hat{u}(\xi), \quad P_{> r} u := u - P_{\leq r} u.
\]
By the definition of \(X^{1/2}_{\zeta(1),1/\tau}\) we see that
\[
\|u\|_{L^2} \lesssim \tau^{1/2} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}}.
\]
It follows from Bernstein’s inequality and Mikhlin’s multiplier theorem that
\[
\left| \left\langle (m_\tau^\omega(D)P_{\leq \kappa_0}^1 g)Q_{\leq \kappa_0}^1 u, Q_{\leq \kappa_0}^1 v \right\rangle \right| \leq \|m_\tau^\omega(D)P_{\leq \kappa_0}^1 g\|_{L^\infty} \|Q_{\leq \kappa_0}^1 u\|_{L^2} \|Q_{\leq \kappa_0}^1 v\|_{L^2}
\lesssim \tau^{1-\frac{2}{p}} \|m_\tau^\omega(D)P_{\leq \kappa_0}^1 g\|_{L^p} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}
\lesssim \tau^{-2+\frac{2}{p}} \|g\|_{L^p} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}.
\]
Finally, applying the assumption \((5.4)\), we obtain
\[
\sum_{\frac{\kappa}{2} \leq \lambda \leq 1} \left| \left\langle (m_\tau^\omega(D)P_{\kappa} g)Q_{\leq \kappa_0}^1 u, Q_{\leq \kappa_0}^1 v \right\rangle \right| \lesssim \sum_{\frac{\kappa}{2} \leq \lambda \leq 1} \lambda^{\nu} \tau^{\mu} \|A_p(g, \lambda, e_1)\|_{L^\infty} \|u\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v\|_{X^{1/2}_{\zeta(1),1/\tau}}.
\]
This completes the proof. \(\square\)

As seen in Lemma \((2.2)\) by scaling we can obtain an estimate in terms of \(X^{1/2}_{\zeta(\tau,U)}\) and \(X^{-1/2}_{\zeta(\tau,U)}\) that is equivalent to \((5.1)\). Here
\[
\zeta(\tau,U) = \tau U(e_1 - ie_2) \in \mathbb{C}^d, \quad U \in O_d.
\]

**Corollary 5.2.** Let \(\tau \gg 1\) and \(g \in L^p(\mathbb{R}^d)\) with supp \(g \subset B_d(0,C)\). Suppose \((5.1)\) holds. Then, with \(s = \mu + \frac{d}{p} - 2 + \kappa\) we have
\[
\|M_{m^\omega(D)g}\|_{X^{1/2}_{\zeta(\tau,U)} \rightarrow X^{-1/2}_{\zeta(\tau,U)}} \lesssim \|g\|_{L^p} + \sum_{\frac{\kappa}{2} \leq \lambda \leq 1: \text{dyadic}} \lambda^{\nu} \tau^{\mu} \|A_p(g_{\tau U}, \lambda, e_1)\|_{L^\infty}.
\]

**Proof.** By Parseval’s identity and change of variables \(\xi \rightarrow \tau U \xi\), we see
\[
\left| \left\langle (m_\tau^\omega(D)g) u, v \right\rangle \right| = \tau^{\nu+2d} \left| \left\langle (m_\tau^\omega(U D)g_{\tau U}) u_{\tau U}, v_{\tau U} \right\rangle \right|.
\]
Since \(g_{\tau U}\) is supported in a ball of radius \(\sim \tau\), applying \((5.1)\) to the right hand side of the above we have
\[
\left| \left\langle (m_\tau^\omega(D)g) u, v \right\rangle \right| \lesssim \tau^{\nu+2d} \left( \tau^{2-\kappa-\frac{2}{p}} \|u_{\tau U}\|_{L^p} + \sum_{\frac{\kappa}{2} \leq \lambda \leq 1} \lambda^{\nu} \tau^{\mu} \|A_p(g_{\tau U}, \lambda, e_1)\|_{L^\infty} \right) \|u_{\tau U}\|_{X^{1/2}_{\zeta(1),1/\tau}} \|v_{\tau U}\|_{X^{-1/2}_{\zeta(1),1/\tau}}.
\]
By \((2.6)\), we have \(\|u_{τ}v\|_{L^{\infty}_{\xi(t),1/r}}^{1/2} \leq \|v_{τ}u\|_{L^{\infty}_{\xi(t),1/r}}^{1/2} = \tau^{-d-2}\|u\|_{L^{\infty}_{\xi(t),1/r}} \leq \|v\|_{L^{\infty}_{\xi(t),1/r}}^{1/2} \) and \(\|g_{τ}u\|_{L^{p}} = \tau^{-d+\#}\|g\|_{L^{p}}\). Thus \((5.9)\) follows.

Now we extend Corollary \(5.2\) to \(g \in H^{s-p}_{c}(\mathbb{R}^{d})\) with \(r < 0\). Naturally, one may attempt to replace \(g\) with \((1 + |D|^{2})^{-\frac{s}{2}}g\) in \((5.1)\) while taking \(m^{s}(D) = (1 + |D|^{2})^{-\frac{s}{2}}\). However, this simple strategy does not work since compactness of the support of \((1 + |D|^{2})^{-\frac{s}{2}}g\) is not guaranteed. We need to slightly modify the argument using the following easy lemma.

**Lemma 5.3.** Let \(1 < p < q < \infty\). If \(s_{1} < s_{2}\), then \(H^{s_{2}-q}_{c} \subseteq H^{s_{1}-p}_{c}\).

Unlike the \(L^{p}\) spaces over a compact set the inclusion \(H^{s_{2}-q}_{c} \subseteq H^{s_{1}-p}_{c}\) with \(p < q\) does not seem to be true in general unless \(s\) is an integer. Failure of the embedding \(W^{s,q}_{c} \subseteq W^{s,p}_{c}\) with \(p < q\) and non-integer \(s\) was shown by Mironescu and Sickel \([32]\). However, if we sacrifice a little bit of regularity such embedding remains true. Though this is easy to show, we couldn’t find a proper reference, so we include a proof.

**Proof of Lemma 5.3.** If \(p = q\) the inclusion is clear by Mikhlin’s multiplier theorem, so it is enough to consider the case \(p < q\). Without loss of generality we may assume that \(f\) is supported in \(B_{d}(0,1)\). Let \(ψ\) be a smooth function supported in \(B_{d}(0,3/2)\) and \(ψ = 1\) on \(B_{d}(0,1)\). We consider the operator \(T(ψ) = ψ f\). It is sufficient to show \(\|\tau T(ψ)\|_{H^{1}} \leq \|T(ψ)\|_{H^{1}}\). Trivially \(\|\tau T(ψ)\|_{H^{s-r}} \leq \|T(ψ)\|_{H^{s-r}}\) for any \(s \leq 1 < r \leq \infty\). Thus by interpolation it is enough to show that, for any \(ψ > 0\),

\[
(5.10) \quad \|T(ψ)\|_{L^{1}} \leq \|T(ψ)\|_{L^{∞}}.
\]

Using the typical dyadic decomposition we write \((1 + |ξ|^{2})^{-\frac{s}{2}} = β_{0}(ξ) + \sum_{k \geq 1} 2^{-rk} β_{k}(ξ)\), where \(β_{0}\) is a smooth function supported in \(B_{d}(0,1)\), and \(β_{k}\) is a smooth function supported in \(\{ξ : 2^{k-2} \leq |ξ| \leq 2^{k}\}\) satisfying \(|β_{k}| \leq 2^{-|α|k}\) for any multi-index \(α\). Let us set \(P_{k}f = F^{-1}(β_{k}F f)\). Since \(ψ\) is supported in \(B_{d}(0,3/2)\), from the rapid decay of \(F^{-1}(β_{k}F f)\) we have, for any \(|x| \geq 2\) and \(N\),

\[
|P_{k}T(ψ)| \leq 2^{-Nk} \int \frac{\|ψ(y)f(y)\|}{(1 + 2^k|x - y|)^{N}} dy.
\]

Thus, it follows that

\[
\|P_{k}T(ψ)\|_{L^{1}} \leq \|P_{k}T(ψ)\|_{L^{1}(B_{d}(0,2))} + \|P_{k}T(ψ)\|_{L^{1}(\mathbb{R}^{d} \setminus B_{d}(0,2))} \lesssim (1 + 2^{-Nk}) \|f\|_{L^{∞}}.
\]

Clearly, \((1 + |D|^{2})^{-\frac{s}{2}}T(ψ) \leq \sum_{k} 2^{-rk} \|P_{k}T(ψ)\|_{L^{1}}\). So, summation along \(k\) gives the desired estimate \((5.10)\).

**Corollary 5.4.** Let \(-1 \leq r \leq 0\), \(\frac{1}{2 + \tau} \leq p < \infty\), \(τ \gg 1\), and let \(m^{r}(ξ) = (1 + |ξ|^{2})^{\frac{r}{2}}\). Suppose that \((3.17)\) holds and \(f\) is supported in a bounded set. Then, for any \(ε > 0\),

\[
(5.11) \quad \|\mathcal{M}_{f}\|_{X^{\frac{1}{2}}_{ξ(t),τ} \rightarrow X^{\frac{1}{2}}_{ξ(t),τ}} \lesssim \|f\|_{H^{r+\tau} + p} \sum_{\frac{1}{2} \leq \lambda \leq 1} \frac{\lambda^{e_{1}}}{[\tau - 2 + |ξ|^{2}]^{\frac{r}{2}}},
\]

where \(m^{r}_{c}(ξ) = τ^{-r}m^{r}(τξ) = (\tau^{-2} + |ξ|^{2})^{\frac{r}{2}}\) as in Definition \((2.1)\).

**Proof.** Let us set \(κ = -r\) and take \(m^{r}(D) = (1 + |D|^{2})^{\frac{r}{2}}\). \(g = (1 + |D|^{2})^{-\frac{r}{2}}f\) so that \(f = m^{r}(D)g\). Scaling shows that \(|(f,u,v)| = τ^{k+2d}(m^{r}(D)g_{τ}u_{τ}v_{τ})|\). As before in the proof of Proposition \(5.1\) we decompose frequencies of the bilinear operator to get

\[
|(f,u,v)| \leq \tau^{k+2d}(I + II + III + IV),
\]

where

\[
I = |\langle (m^{c}_{τ}(D)g_{τ}u_{τ})Q_{>δ_{τ}}^{1}u_{τ}, Q_{>δ_{τ}}^{1}v_{τ} \rangle|, \quad II = |\langle (m^{c}_{τ}(D)g_{τ}u_{τ})Q_{≥δ_{τ}}^{1}u_{τ}, Q_{<δ_{τ}}^{1}v_{τ} \rangle|,
\]

\[
III = |\langle (m^{c}_{τ}(D)g_{τ}u_{τ})Q_{≤δ_{τ}}^{1}u_{τ}, Q_{>δ_{τ}}^{1}v_{τ} \rangle|, \quad IV = |\langle (m^{c}_{τ}(D)g_{τ}u_{τ})Q_{<δ_{τ}}^{1}u_{τ}, Q_{≤δ_{τ}}^{1}v_{τ} \rangle|.
\]
Then, following the argument in the proof of Proposition 5.1 (then rescaling back), it is easy to see that, for \( r \in [-1, 0], \)
\[
I + II + III \lesssim \tau^{r-2d}(1 + |D|^2)^{\frac{5}{2}} f \| u\|_{X^{1/2}_{\tau(r, U)}} \| v\|_{X^{1/2}_{\tau(r, U)}}.
\]
Using Lemma 5.3 for any \( \varepsilon > 0, -1 \leq r \leq 0, \) and \( \frac{d}{2p} \leq p < \infty, \) we have
\[
I + II + III \lesssim \tau^{r-2d} \| f \|_{H^{r+p, \rho}} \| u\|_{X^{1/2}_{\tau(r, U)}} \| v\|_{X^{1/2}_{\tau(r, U)}}.
\]
For the remaining \( IV, \) we may routinely repeat the same argument as before making use of (3.47) with \( \kappa = -r \) to get
\[
IV \lesssim \tau^{r-2d} \left( \| f \|_{H^{r+p, \rho}} + \sum_{\frac{d}{2} \leq \lambda \leq 1} \lambda^{\nu} \tau^{\nu - \frac{d}{4} - 2} \| R_\tau (m_{\tau}(D)) f \|_{L^p_{\tau(U)}} \right) \| u\|_{X^{1/2}_{\tau(r, U)}} \| v\|_{X^{1/2}_{\tau(r, U)}}.
\]
Combining the estimates for \( I + II + III \) and \( IV \) yields (5.11).

5.1. Non-averaged estimates. We recall the following ((13), (15), and (16) in [19]) which are immediate consequences of (2.3) and (2.17):

\[
\begin{align*}
(5.12) & \quad \| M_q \|_{X^{1/2}_{\tau(r, U)} \rightarrow X^{-1/2}_{\tau(r, U)}} \lesssim \tau^{-1} \| q \|_{L^\infty}, \\
(5.13) & \quad \| M_q \|_{X^{1/2}_{\tau(r, U)} \rightarrow X^{-1/2}_{\tau(r, U)}} \lesssim \| q \|_{L^{\frac{4}{3}}(\mathbb{R}^d)}, \\
(5.14) & \quad \| f \|_{X^{-1/2}_{\tau(r, U)}} \lesssim \| f \|_{L^{\frac{4d}{3d-4}}(\mathbb{R}^d)}.
\end{align*}
\]

From [21] Lemma 2.3 we also have
\[
(5.15) \quad \| M \nabla_q f \|_{X^{1/2}_{\tau(r, U)} \rightarrow X^{-1/2}_{\tau(r, U)}} \lesssim \| q \|_{L^\infty}.
\]

The estimates in the following proposition correspond to the estimates which one can get by formally interpolating the estimates (5.13) and (5.15).

Proposition 5.5. Let \( d/2 \leq p \leq \infty, \kappa = 1 - \frac{d}{2p}, \tau \gg 1, \) and let \( f \in L^p_{\tau}(B_d(0, C)). \) Suppose that \( m^\kappa(\xi) = (1 + |\xi|^2)^{\frac{1}{2}} \) or \( m^\kappa(\xi) = |\xi|^\kappa. \) Then,
\[
(5.16) \quad \| M_{m^\kappa(D)} f \|_{X^{1/2}_{\tau(r, U)} \rightarrow X^{-1/2}_{\tau(r, U)}} \lesssim \| f \|_{L^p(d)}.
\]

Proof. When \( \kappa = 0 \) the estimate (5.16) is identical with (5.13), so it is enough to prove (5.16) with \( \kappa = 1. \) For the purpose, it is more convenient to work with the rescaled space \( X^{1/2}_{\tau(1, 1/\tau)}. \) We note that \( m^\kappa(\xi) \) takes the particular forms \( (\tau^{-2} + |\xi|^2)^{\frac{1}{2}}, |\xi|^\kappa. \) We regard \( m^\kappa(D) \) as operators embedded in an analytic family of operators \( m^\kappa(D) \) with complex parameter \( \tau. \) We claim that
\[
(5.17) \quad |\langle m^\kappa(D)(D) f u, v \rangle| \lesssim (1 + |t|)^{\frac{d+1}{2}} \tau^\kappa \| f \|_{L^p(d)} \| u\|_{X^{1/2}_{\tau(1, 1/\tau)}} \| v\|_{X^{1/2}_{\tau(1, 1/\tau)}}
\]
whenever \( f \) is supported in \( B_d(0, C\tau). \) This gives (5.16) by Lemma 2.22. By Stein’s interpolation along analytic family we only need to show (5.17) for the cases \( p = \infty \) (\( \kappa = 1 \)) and \( p = d/2 \) (\( \kappa = 0 \)). Hölder’s inequality, (2.14), and (2.8) yield
\[
|\langle m^\kappa(D)(D) f u, v \rangle| \lesssim \| m^\kappa(D)(D) f \|_{L^p(d)} \| u\|_{X^{1/2}_{\tau(1, 1/\tau)}} \| v\|_{X^{1/2}_{\tau(1, 1/\tau)}}.
\]
Thus, (5.17) with \( \kappa = 0 \) follows from Mikhlin’s multiplier theorem. It remains to show (5.17) with \( \kappa = 1. \)

For the purpose we decompose
\[
u = v_0 + v_1 := P_{\leq 8} u + P_{> 8} u, \quad v = v_0 + v_1 := P_{\leq 8} v + P_{> 8} v.
\]
Proof. Let $P_{>\tau}$ and $P_{\leq \tau}$ are given by (5.3). Since $\langle m_\tau^{\epsilon+it}(D) f u_0, v_0 \rangle = \langle f, m_\tau^{\epsilon+it}(D) \hat{\psi}(\tau,0) \rangle$ and $\hat{\psi}$ and $\hat{\psi}_0$ are compactly supported (so, we may disregard the multiplier operator $m_\tau^{\epsilon+it}(D)$ because $\|m_\tau^{\epsilon+it}(D)g\|_{L^1} \lesssim (1+|t|)^{d+1}\|g\|_{L^1}$ whenever $\hat{g}$ is supported in $B_d(0,C)$ and $\kappa > 0$), we have the estimate
\[
\|\langle m_\tau^{\epsilon+it}(D) f u_0, v_0 \rangle \| \lesssim (1+|t|)^{d+1}\|f\|_{L^p} \|\hat{u}_0\|_{L^q} \lesssim \tau(1+|t|)^{d+1}\|f\|_{L^p} \|u_0\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_0\|_{X^{1/2}_{\xi(1,1/\tau)}}.
\]
For the second inequality we used (5.6).

On the other hand, we have
\[
\|\langle m_\tau^{\epsilon+it}(D) f u_1, v_1 \rangle \| \leq \|f\|_{L^p}(\|m_\tau^{\epsilon+it}(D) P_{\leq \tau}(\hat{\psi}(\tau,1)) \|_{L^{p,q}} + \|m_\tau^{\epsilon+it}(D) P_{>\tau}(\hat{\psi}(\tau,1)) \|_{L^{p,q}}).
\]
For the low frequency part $P_{\leq \tau}(\hat{\psi}(\tau,1))$, we take $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{\psi} = 1$ on the Fourier support of $P_{\leq \tau}(\hat{\psi}(\tau,1))$. By Young’s and Hölder’s inequalities and the estimate (5.3),
\[
\|m_\tau^{\epsilon+it}(D) P_{\leq \tau}(\hat{\psi}(\tau,1)) \|_{L^{p,q}} \leq \|m_\tau^{\epsilon+it}(D) \psi\|_{L^1} \|P_{\leq \tau}(\hat{\psi}(\tau,1)) \|_{L^{p,q}} \lesssim (1+|t|)^{d+1}\|\hat{\psi}(\tau,1)\|_{L^{p,q}} \lesssim (1+|t|)^{d+1}\|f\|_{L^p} \|u_1\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_1\|_{X^{1/2}_{\xi(1,1/\tau)}}.
\]
For the high frequency part $P_{>\tau}(\hat{\psi}(\tau,1))$, we see that $m_\tau^{\epsilon+it}(D)^{-1}$ satisfies the assumption in Mikhlin’s multiplier theorem. Hence, it follows from the fractional Leibniz rule and the estimate (2.17) (with (2.6)) that
\[
\|m_\tau^{\epsilon+it}(D) P_{>\tau}(\hat{\psi}(\tau,1)) \|_{L^{p,q}} \lesssim (1+|t|)^{d+1}\|D\|_{L^{p,q}} \|u_1\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_1\|_{X^{1/2}_{\xi(1,1/\tau)}}.
\]
Since $f$ is supported in $B_d(0,C\tau)$, we obtain
\[
\|\langle m_\tau^{\epsilon+it}(D) f u_1, v_1 \rangle \| \leq (1+|t|)^{d+1}\|f\|_{L^p} \|u_1\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_1\|_{X^{1/2}_{\xi(1,1/\tau)}} \lesssim \tau(1+|t|)^{d+1}\|f\|_{L^p} \|u_1\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_1\|_{X^{1/2}_{\xi(1,1/\tau)}}.
\]
Finally, it is enough to consider $\langle m_\tau^{\epsilon+it}(D) f u_0, v_0 \rangle$ since the remaining $\langle m_\tau^{\epsilon+it}(D) f u_1, v_1 \rangle$ can be handled similarly. Since $\|D\|_{L^{p,q}} \lesssim \|\hat{\psi}(\tau,1)\|_{L^{p,q}}$, repeating the above argument, we have
\[
\|\langle m_\tau^{\epsilon+it}(D) f u_0, v_0 \rangle \| \lesssim (1+|t|)^{d+1}\|f\|_{L^p} \|u_0\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_0\|_{X^{1/2}_{\xi(1,1/\tau)}} \lesssim \tau(1+|t|)^{d+1}\|f\|_{L^p} \|u_0\|_{X^{1/2}_{\xi(1,1/\tau)}} \|v_0\|_{X^{1/2}_{\xi(1,1/\tau)}}.
\]
Thus, combining all the estimates together, we see that (5.17) holds with $\kappa = 1$. □

**Corollary 5.6.** Let $p = \frac{d}{(1-\kappa)}$ for $0 \leq \kappa < 1$. If $f$ is supported in a bounded set, for any $\epsilon > 0$ and $\tau \gg 1$ (5.18)
\[
\|M_f\|_{X^{1/2}_{\xi(1/\tau)}}, Y^{1/2}_{\xi(1/\tau)} \lesssim \|f\|_{H^{-\kappa+\epsilon,p}(\mathbb{R}^d)}.
\]

**Proof.** Let $m_\epsilon^{\tau}(D) = (1+|D|)^{\frac{\epsilon}{2}}$. Using rescaling and following the argument in the proof of Proposition 5.5 we have $\|m_\epsilon^{\tau}(D) gu, v \| \lesssim (\|g\|_{L^\infty} + \|g\|_{L^d})\|u\|_{X^{1/2}_{\xi(1/\tau)}} \|v\|_{X^{1/2}_{\xi(1/\tau)}}$ for any $g$ in the Schwartz class. Obviously this gives
\[
\|\langle gu, v \rangle \| \lesssim (\|m_\tau^{\epsilon+it}(D) g\|_{L^\infty} + \|m_\tau^{\epsilon+it}(D) g\|_{L^d})\|u\|_{X^{1/2}_{\xi(1/\tau)}} \|v\|_{X^{1/2}_{\xi(1/\tau)}}.
\]
Since the Schwartz class is dense in $H^{\kappa+\epsilon,p}$, using Lemma 5.3 and the embedding $H^{\frac{\epsilon}{2}+\kappa,p} \hookrightarrow L^\infty$, we get
\[
\|\langle f, u \rangle \| \lesssim (\|m_\tau^{\epsilon+it}(D) f\|_{L^p} + \|f\|_{H^{-\kappa+\epsilon,p}})\|u\|_{X^{1/2}_{\xi(1/\tau)}} \|v\|_{X^{1/2}_{\xi(1/\tau)}}.
\]

\[\text{It is not difficult to show that } \|m_\tau^{\epsilon+it}(D)\|_{L^1} \lesssim (1+|t|)^{d+1} \text{ if } \psi \in \mathcal{S}(\mathbb{R}^d) \text{ provided that } \kappa > 0.\]
provided that $d \leq p < \infty$, $\epsilon > 0$, and $\delta > \frac{d}{p}$. Now, taking $p$ arbitrarily close to $\infty$ in the above and interpolating the estimate with

$$|\langle f, v \rangle| \leq \|f\|_{L^{2\delta}} \|u\|_{X^{-\frac{1}{2},\epsilon}(\xi;U)} \|v\|_{X^{\frac{1}{2},\epsilon}(\xi;U)},$$

which is equivalent to (5.18), we get the bound (5.18) on the desired range. $\square$

5.2. Convergence of the averages to zero. We now show that averages of $\|q\|_{X^{-\frac{1}{2},\epsilon}(\xi;U)}$, $\|M_q\|_{X^{\frac{1}{2},\epsilon}(\xi;U)}$, and $\|\tau\|_{X^{-\frac{1}{2},\epsilon}(\xi;U)}$ over $U$ and $\tau$ asymptotically vanish as $\tau \to \infty$. Compared with the non-averaged counterpart, averaged estimates allow a considerable amount of regularity gain.

Proposition 5.7. Let $d \geq 3$, $0 \leq \kappa \leq 1$, and $\tau \geq 1$. Then, we have

$$\int_{O_d} \|m^\kappa(D)g\|_{X_{\xi;U}^{-\frac{1}{2},\kappa}}^2 \, dm(U) \lesssim \|g\|_{L^{\kappa+2}(\mathbb{R}^d)}^2.$$  

Proof. In order to show (5.19) it is enough to consider the case $\kappa = 1$. If $0 \leq \kappa < 1$, from (5.19) with $\kappa = 1$ and the Plancherel theorem, we have

$$\int_{O_d} \|m^\kappa(D)g\|_{X_{\xi;U}^{-\frac{1}{2},\kappa}}^2 \, dm(U) \lesssim \|(1 + |D|^2)^{-\frac{\kappa}{2}} m^\kappa(D)g\|_{L^2}^2 \lesssim \|D|^{\kappa-1}g\|_{L^2}^2.$$  

Thus, the desired estimate follows by the Hardy-Littlewood-Sobolev inequality.

To show (5.19) with $\kappa = 1$, we break $g$ into $g = P_{\leq 8}\, g + P_{>8} \, g$ where $P_{\leq 8}$ and $P_{>8}$ are given by (5.5).

Since $|p_{\xi;U}^\kappa(\xi)| \gtrsim |\xi|^2$ and $|m^\kappa(\xi)| \lesssim |\xi|$ for $|\xi| \geq 4\tau$, we have

$$\int_{O_d} \|m^\kappa(D)P_{\leq 8}\, g\|_{X_{\xi;U}^{-\frac{1}{2},\kappa}}^2 \, dm(U) \lesssim \int_{O_d} \int_{|\xi| \leq 8\tau} \frac{1 + |\xi|^2}{|\xi|} \, |\tilde{g}(\xi)|^2 \, d\xi \, dm(U) \lesssim \left(1 + \sup_{|\xi| \leq 8\tau} \int_{|\xi| \leq 8\tau} F(|\xi|/2\tau, \omega) \, d\sigma(\omega) \right) \|g\|_{L^2},$$

where $F(r, \omega) = (|e_2 \cdot \omega - r| + |e_1 \cdot \omega|)^{-1}$. Taking into account symmetry of the sphere, it is clear that $\sup_{|\xi| \leq 16\tau} \int_{|\xi| \leq 16\tau} F(|\xi|/2\tau, \omega) \, d\sigma(\omega) \lesssim \int_{|\xi| \leq 2\tau} F(1, \omega) \, d\sigma(\omega)$. By change of variables $\tau \omega = \eta \in \mathbb{R}^d$, we see that $\int_{|\xi| \leq 2\tau} F(1, \omega) \, d\sigma(\omega) \leq C_d$ for $d \geq 3$. Thus, we get

$$\int_{O_d} \|m^\kappa(D)P_{\leq 8}\, g\|_{X_{\xi;U}^{-\frac{1}{2},\kappa}}^2 \, dm(U) \lesssim \|g\|_{L^2}^2.$$  

This and (5.20) yield (5.19) with $\kappa = 1$. $\square$

Corollary 5.8. Let $d \geq 3$, $\frac{2d}{d+2} \leq p < \infty$, and $s \geq \max\{-1, -\frac{d+2}{p} + \frac{d}{p}\}$. If $f \in H_{c}^{s,p}(\mathbb{R}^d)$, then

$$\lim_{\tau \to \infty} \int_{O_d} \|f\|_{X_{\xi;U}^{-\frac{1}{2},\kappa}}^2 \, dm(U) = 0.$$  

Proof. We may assume that $s = \max\{-1, -\frac{d+2}{p} + \frac{d}{p}\}$ since $H^{t,p} \hookrightarrow H^{s,p}$ for $t \geq s$ and $1 < p < \infty$. By (5.19), it is enough to show that (5.21) holds with $X_{\xi;U}^{-\frac{1}{2},\kappa}$ replaced by $X_{\xi;U}^{-\frac{1}{2},\kappa}$. Note that $\|h\|_{H^{-1,2}} \lesssim \|h\|_{H^{-1,r}}$ for $2 \leq r < \infty$ whenever $h$ is supported in a bounded set $\Omega$. Hence, it suffices to consider $s = -\frac{d+2}{p} + \frac{d}{p}$ and

9This is possible because the order is an integer. In fact, it follows from the embedding $H_{c}^{1,2}(\Omega) \hookrightarrow H_{c}^{1,r}(\Omega)$ for any bounded set $\Omega$ and duality.
Since lim where \(M\) holds with the implicit constant independent of

\[\frac{2d}{d+2} \leq p \leq 2,\] for 0 \(\leq \kappa \leq 1\). From \(6.19\), with \(m^c(D) = (1 + |D|^2)\frac{2}{\kappa}\), we have

\[\tag{5.22}\]
\[
\int_{O_\delta} \|f\|_{L^2(\mathbb{R}^d)}^2 \, dm(U) \lesssim \|f\|_{L^{p_\kappa}}^2, \quad p = \frac{2d}{d + 2 - 2\kappa}.
\]

Let \(\phi \in C_\infty^0(B_d(0,1))\) such that \(\int \phi \, dx = 1\). We write \(f = (f - f \ast \phi) + f \ast \phi\). By Young’s convolution inequality and the embedding \(H^{1/2} \hookrightarrow H^s\) for \(1 < q < \infty\), we have

\[
\|f \ast \phi_x\|_{X^{1/2}} \lesssim \tau^{-\frac{1}{2}} \|f \ast \phi\|_{L^2} \lesssim \tau^{-\frac{1}{2}} \|f\|_{H^{s,p}} \|\phi\|_{H^{1/2}} \lesssim \tau^{-\frac{1}{2}} \epsilon^{-1 - \frac{d(2 - p)}{2p}} \|f\|_{H^{s,p}}.
\]

Combining this with \(5.22\), we obtain

\[
\left(\int_{O_{\delta}} \|f\|_{X^{1/2}}^2 \, dm(U)\right)^{1/2} \lesssim \|f - f \ast \phi\|_{H^{s,p}} + \tau^{-\frac{1}{2}} \epsilon^{-1 - \frac{d(2 - p)}{2p}} \|f\|_{H^{s,p}}.
\]

Since \(\lim_{\epsilon \to 0} \|f - f \ast \phi\|_{H^{s,p}} = 0\), \(5.21\) follows if we take \(\epsilon = \epsilon(\tau) > 0\) such that \(\epsilon^{-1 - \frac{d(2 - p)}{2p}} = \epsilon^{s-2} < \tau^{\frac{1}{2}}\).

### 5.3. Average over \(\tau\) and \(U\)

As we have seen in the proof of Proposition 5.1, to get the desired estimate we do not have to use the averaged estimate for the high-high, low-high, high-low frequency interactions. However, in the case of low-low frequency interaction we get significantly improved bounds by means of average over \(\tau\) and \(U\).

For simplicity we define

\[
\int_M f(\tau) \, d\tau := \frac{1}{M} \int_M^2 f(\tau) \, d\tau.
\]

For \(M \geq 2\), we set

\[
\mathcal{A}^{p,s}_{M}(f) = \left(\int_M \int_{O_d} |\mathcal{M}m^c(D)f|_{X^{1/2}}^p \, dm(U) \, d\tau\right)^{\frac{1}{p}}.
\]

**Lemma 5.9.** Let \(2 \leq p < \infty\). Suppose we have \(5.9\). If \(\nu > s\), then for any \(f \in W^{s,p}_c(B_d(0,1))\) the estimate

\[
\tag{5.23}
\mathcal{A}^{p,s}_{M}(f) \lesssim \|f\|_{W^{s,p}}
\]

holds with the implicit constant independent of \(M\). The same remains valid with \(W^{s,p}\) replaced by \(H^{s,p}\).

**Proof.** It is well-known that if \(1 < p < \infty\), then \(W^{k,p} = H^{k,p}\) for any \(k = 0, 1, 2, \ldots\), and \(H^{s,p} \hookrightarrow W^{s-\epsilon,p}\) for any \(\epsilon > 0, s \in \mathbb{R}\) (see [46, pp. 168–180]). Hence, it suffices to show that \(5.23\) holds with \(f \in W^{s,p}\). Taking \(p\)-th power and integrating over \(U\) and \(\tau\) on both side of \(5.9\), by Minkowski’s inequality we get

\[
\mathcal{A}^{p,s}_{M}(f) \lesssim \|f\|_{L^p} + A_M(f),
\]

where

\[
A_M(f) = \sum_{\frac{1}{2} \tau \leq \lambda \leq 1 \text{dyadic}} \lambda^s M^s \left(\int_M \int_{O_d} \left[\tau^{-\frac{1}{2}} A_p(f_{\tau U}, \lambda, e_1)\right]^p \, dm(U) \, d\tau\right)^{\frac{1}{p}}.
\]

Thus, for \(5.23\) it suffices to show that

\[
\left(\sum_{M \geq 2 \text{dyadic}} (A_M(f))^p\right)^{\frac{1}{p}} \leq C\|f\|_{W^{s,p}}.
\]

By scaling \(\tau \to M\tau\) and applying \(4.12\), it follows that

\[
\int_M \int_{O_d} \left[\tau^{-\frac{1}{2}} A_p(f_{\tau U}, \lambda, e_1)\right]^p \, dm(U) \, d\tau \lesssim M^{(d - \frac{1}{2})p} \int_1^2 \int_{O_d} A_p \left(f_{M\tau U}, \lambda, e_1\right) \, dm(U) \, d\tau 
\]

\[
\lesssim M^{(d - \frac{1}{2})p} \|\mathcal{P} f M\|_{L^p} = \|\mathcal{P} M f\|_{L^p}.
\]
This yields $A_M(f) \lesssim \sum_{2M}^{1 + \lambda \leq 1} \lambda^\nu M^\nu \|P_M f\|_{L^p}$. Reindexing $\rho = \lambda M$, we see that $A_M(f) \lesssim \sum_{1,\rho \leq M, \delta \leq \rho \leq \lambda \rho} \rho^{\nu - s} \|P_{\rho} f\|_{L^p}$. Since $\nu > s$, sup $M$ $1,\rho \leq M, \delta \leq \rho \leq \lambda \rho$ $\rho^{\nu - s} \lesssim 1$ and sup $M \geq \rho \rho $ $\rho^{\nu - s} \lesssim 1$. So, by Schur’s test, 

$$\left( \sum_{M > 2} (A_M(f))^p \right)^{1/p} \lesssim \left( \sum_{p > 1} (\rho^p \|P_{\rho} f\|_{L^p})^p \right)^{1/p} \lesssim \|f\|_{W^{s,p}}.$$ 

Here the last inequality follows from the embedding of $W^{s,p}$ into the Besov space $B^{s,p}_{p,p}$ for $2 \leq p < \infty$ and $s \in \mathbb{R}$ (see [40] pp. 179–180). Hence we get (5.24). \hfill \Box

In a similar way, using Corollary 5.4 we obtain the following.

**Lemma 5.10.** Let $2 \leq p < \infty$. Suppose (5.11) holds with $\nu = -r - \frac{2}{p}$. If $\mu < 2 - \frac{d + 2}{p}$ and $\mu + \frac{d}{p} - 2 \leq r$, then, for every $\epsilon > 0$ and any $f \in H^{r+s,p}(B_d(0,1))$, we have

$$A_M(f) \lesssim \|f\|_{H^{r+s,p}} + \sum_{\frac{d}{p} < \nu \leq \nu} \lambda^\nu M^\nu \|P_M m^\nu(Df)\|_{L^p},$$

where we set $s = \mu + \frac{d}{p} - 2 - r$. By the assumption on $\mu$ and $\nu$ it is easy to check that $\nu > s$. Repeating the argument in the proof of Lemma 5.9 immediately yields

$$A_M(f) \lesssim \|f\|_{H^{r+s,p}} + \|m^\nu(Df)\|_{B_{p,p}^s} \lesssim \|f\|_{H^{r+s,p}} + \|f\|_{H^{r+s,p}},$$

where the last inequality follows from the embedding of $H^{r,p}$ into the Besov space $B_{p,p}^s$ for any $-\infty < s < \infty$ and $2 \leq p < \infty$ ([16] p. 179). Thus we get (5.25) since $s \leq 0$. \hfill \Box

Now, we combine Proposition 5.8, Proposition 5.1, Corollary 5.2, and Lemma 5.9 altogether to conclude the following: If $d \geq 3, 0 \leq \kappa \leq 1$, max$\left\{\frac{d+1}{d}, \frac{3}{2}\right\} \leq p < \infty$, and $\kappa - \frac{d}{p} > s > \beta_d(p) + \frac{d}{p} - 2 + \kappa$, then we have (5.23) for any $f \in W^{s,p}_c(B(0,C))$. Here, let us specify the range of $p$ for the estimate (5.23) in Lemma 5.9.

**When $3 \leq d \leq 6$.** Such $s$ exists only if $\kappa - \frac{d}{p} > \beta_d(p) + \frac{d}{p} - 2 + \kappa$, which is equivalent to $p > \frac{d+2}{2}$. Hence we need to consider the cases $\frac{d}{2 - \kappa} \leq \frac{d+2}{2}$ and $\frac{d}{2 - \kappa} > \frac{d+2}{2}$ separately.

- If $0 \leq \kappa \leq \frac{d}{d+2}$, then (5.23) holds whenever $\frac{d+2}{2} < p < \infty$ and $s > \beta_d(p) + \frac{d}{p} - 2 + \kappa$.
- If $\frac{d}{2 - \kappa} < \kappa \leq 1$, then (5.23) holds whenever $\frac{d}{2 - \kappa} \leq p < \infty$ and $s > \beta_d(p) + \frac{d}{p} - 2 + \kappa$.

**When $d \geq 7$.** In this case, such $s$ exists only if $-\frac{2}{d} > \beta_d(p) + \frac{d}{p} - 2$, i.e., $p > \frac{d+2}{d-2}$.

- If $0 \leq \kappa \leq \frac{d}{d+2}$, then (5.23) holds whenever $\frac{d+2}{d-2} < p < \infty$ and $s > \beta_d(p) + \frac{d}{p} - 2 + \kappa$.
- If $\frac{d}{2 - \kappa} < \kappa \leq 1$, then (5.23) holds whenever $\frac{d}{2 - \kappa} \leq p < \infty$ and $s > \beta_d(p) + \frac{d}{p} - 2 + \kappa$.

On the other hand, (5.13) gives

$$A_M^{\frac{d}{2}}(f) \lesssim \|m^\kappa(Df)\|_{L^2} \lesssim \|f\|_{H^{s,p}}$$

for any $\kappa \geq 0$. Interpolating this bound and the estimate (5.23) with endpoints $p$ in the above, we can extend the aforementioned ranges of $p$ to the range $p \geq \frac{d}{2}$ as the following.
When $3 \leq d \leq 6$. By the definition of $\beta_d$ we need to consider the cases $\frac{d}{2-\kappa} \leq 4$ and $\frac{d}{2-\kappa} > 4$ separately.

- If $0 \leq \kappa \leq \frac{d}{2-\kappa}$, (5.23) holds for $\frac{d}{2} \leq p \leq \frac{d+2}{p}$ and $s > \frac{d}{p} - 2 + \kappa$.

- If $\frac{3d+7}{d+2} < \kappa \leq \frac{d}{2-\kappa}$, (5.23) holds for $\frac{d}{2} \leq p \leq \frac{d}{2-\kappa}$ and $s > \frac{d}{p} - 2 + \kappa$.

- If $\frac{8-d}{4} < \kappa \leq 1$, (5.23) holds for $\frac{d}{2} \leq p \leq \frac{d}{2-\kappa}$ and $s > \kappa + \frac{1}{\kappa} \left( \frac{2}{2-\kappa} - \frac{2(2-\kappa)}{d} - \kappa \right)$.

When $d \geq 7$. We consider the cases $\frac{d}{2-\kappa} \leq \frac{d+9}{2}$ and $\frac{d}{2-\kappa} > \frac{d+9}{2}$ separately.

- If $0 \leq \kappa \leq \frac{14}{3d+7}$, (5.23) holds for $\frac{d}{2} \leq p \leq \frac{3d+7}{6}$ and $s > \kappa - \frac{6}{7}(2 - \frac{d}{p})$.

- If $\frac{14}{3d+7} < \kappa \leq \frac{8}{d+2}$, then (5.23) holds for $\frac{d}{2} \leq p \leq \frac{d}{2-\kappa}$ and $s > \kappa - \left( \frac{3d-1}{2} + \frac{1}{12} \right) \left( \frac{d}{p} - \frac{1}{p} \right)$.

- If $\frac{8-d}{4} < \kappa \leq 1$, then (5.23) holds for $\frac{d}{2} \leq p \leq \frac{d}{2-\kappa}$ and $s > \kappa - \left( \frac{d+7}{2} + \frac{1}{12} \right) \left( \frac{d}{p} - \frac{1}{p} \right)$.

Now, for simplicity, let us denote by $s_0(d, p, \kappa)$ the conditions on $s$ arranged above.

**Proposition 5.11.** Let $d \geq 3$, $0 \leq \kappa \leq 1$, and let $\frac{d}{2} \leq p < \infty$. Suppose that $s > s_0(d, p, \kappa)$ and $s \geq 0$. Then (5.23) holds for all $f \in W^{s,p}_c(B_d(0, R))$ for any fixed $R > 0$. Moreover, if $f \in W^{s,p}_c(\mathbb{R}^d)$

(5.27) \[
\lim_{M \to \infty} \mathcal{A}^p_{M}(f) = 0.
\]

When $p = \frac{d}{2}$ and $\kappa = 0$, (5.27) holds for any $f \in L^{\frac{d}{2}}(\mathbb{R}^d)$.

For $4 \leq d \leq 8$, we can make the exponent $s_0(d, p, \kappa)$ slightly smaller for $p \geq d$ if we use Proposition 5.11 instead of Proposition 5.8. For the case $\kappa = 1$, this will be done in Proposition 5.13 below.

**Proof of Proposition [5.11]** We have already shown all the statement for the estimate (5.23). So, it remains to prove (5.27) and the proof is similar to that of Corollary 5.8. Writing $f = f * \phi_\epsilon + (f - f * \phi_\epsilon)$ and using (5.23), we get

\[
\mathcal{A}^p_{M}(f - f * \phi_\epsilon) \lesssim \| f - f * \phi_\epsilon \|_{W^{s,p}}.
\]

On the other hand, the estimate (5.12) and the Hölder inequality give

\[
\| M_{m(\epsilon)}(f * \phi_\epsilon) \|_{X^{s,p}((0, R))} \lesssim \tau^{-1} \| f * (m(\epsilon)(D)\phi_\epsilon) \|_{L^\infty} \lesssim \tau^{-1} \epsilon^{-\kappa - \frac{d}{p}} \| f \|_{L^p}.
\]

Here we used the Mikhlin multiplier theorem to see that $\| m(\epsilon)(D)\phi_\epsilon \|_{L^r} \lesssim \| \phi_\epsilon \|_{H^{s-r}} \lesssim \epsilon^{-\kappa - d + \frac{d}{p}}$ for $1 < r < \infty$. This immediately yields $\mathcal{A}^p_{M}(f * \phi_\epsilon) \lesssim M^{-1} \epsilon^{-\kappa - \frac{d}{p}} \| f \|_{L^p}$. Thus,

\[
\mathcal{A}^p_{M}(f) \leq \mathcal{A}^p_{M}(f * \phi_\epsilon) + \mathcal{A}^p_{M}(f - f * \phi_\epsilon) \lesssim M^{-1} \epsilon^{-\kappa - \frac{d}{p}} \| f \|_{L^p} + \| f - f * \phi_\epsilon \|_{W^{s,p}}.
\]

Since $s \geq 0$, $\| f \|_{L^p} \lesssim \| f \|_{W^{s,p}}$. Taking $\epsilon = M^{-1/\beta}$ with $\beta > \kappa + \frac{d}{p}$, we get (5.27). The last statement follows from (5.20) in a similar manner. \[\square\]

We now turn to (5.24). By Proposition 3.3, Corollary 5.4, and Lemma 5.10, we have the following: If $d \geq 3$, $\epsilon > 0$, $\frac{d+1}{2} \leq p < \infty$, $\beta_d(p) < \mu < 2 - \frac{d+2}{p}$, and $r \geq \max\{-1, \mu + \frac{d-2}{2}\}$, then (5.24) holds for any $f \in H^{r,\mu,p}_c(B_d(0, C))$.

Such $\mu$ exists only if $\beta_d(p) < 2 - \frac{d+2}{p}$, which is equivalent to $p > \frac{d+2}{d-2}$ if $3 \leq d \leq 6$, and $p > \frac{3d+7}{6}$ if $d \geq 7$. As before, we interpolate (5.20) and (5.24) (with $\kappa = 0$) to extend the range of $p$ and obtain the following proposition. The proof is very similar to and even simpler than that of Proposition 5.11 so we omit it.

---

10This interval is empty when $d = 6$.
11This is void when $d = 3$ or 4.
12This is void if $d \leq 9$. 

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Lemma 6.1. due to Brown [6]. We close this section weakening the condition on $s$ as before, from Proposition 3.11, Proposition 5.1, and Corollary 5.2 we have (5.9) with the definition of $\gamma$. Let Proposition 5.13.

$$(5.28) \lim_{M \to \infty} \mathcal{A}_{M}^{p}(f) = 0.$$ 

We close this section weakening the condition on $s$ in Proposition 5.11 in the case of $p \geq d$ and $\kappa = 1$. For the definition of $\gamma_d(p)$, see Section 5.6.

Proposition 5.13. Let $3 \leq d \leq 6$, $d \leq p < \infty$, and $\kappa = 1$. Suppose $s > \max\{0, \gamma_d(p) + \frac{d}{p} - 1\}$. Then, for any $f \in W^{s,p}(B_d(0,R))$ (for any constant $R > 0$ fixed), (5.23) and (5.27) hold.

Proof. As before, from Proposition 3.11, Proposition 5.1, and Corollary 5.2 we have (5.9) with $\nu = 1 - \frac{2}{p}$ and $s > \gamma_d(p) + \frac{d}{p} - 1$ for $p \geq d$. Thus, by Lemma 5.9, we get (5.23) since $2 - \frac{d+2}{p} > \gamma_d(p)$ whenever $p \geq d$. Since we have (5.23), the same argument as in the proof of Proposition 5.11 gives (5.27). $\square$

6. Proof of Theorem 1.1 and Theorem 1.2

Once we have the key estimates in the previous sections, we can prove Theorem 1.1 and Theorem 1.2 following the argument in [19], which also relies on the basic strategy due to Sylvester-Uhlmann [39], and subsequent modifications due to Haberman-Tataru [21] and Nguyen-Spirn [36]. We begin with recalling several basic theorems which we need in what follows.

6.1. Proof of Theorem 1.1. Let $\Omega$, $s$, $p$, and $d$ be as in Theorem 1.1. We may assume that $s \leq 1 + \frac{1}{p}$ by the inclusion $W^{s_1,p} \subset W^{s_2,p}$ for $s_1 \geq s_2$ and $1 < p < \infty$. For $k = 1, 2$, assume that $\gamma_k \in W^{s,p}(\Omega) \cap A(\Omega)$ satisfy $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. It is clear that $W^{s,p}(\Omega) \subset W^{1,d}(\Omega) \subset W^{1,1}(\Omega)$ since $s \geq 1$ and $p \geq d$. The following is due to Brown [6].

Lemma 6.1. If $\gamma_1, \gamma_2 \in W^{1,1}(\Omega) \cap A(\Omega)$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ on $\partial \Omega$.

Hence $(\gamma_1 - \gamma_2)|_{\partial \Omega} = 0$ by Lemma 6.1. Since $s - \frac{1}{p} \leq 1$, using [30, Theorem 1], we have $\gamma_1, \gamma_2 \in W^{s,p}(\Omega)$ extended to the whole space $\mathbb{R}^d$ such that $\gamma_1 = \gamma_2$ outside of $\Omega$ and $\gamma_k$ is supported in a large ball $B_d(0,R)$ containing $\Omega$. Thus, from now on we assume that $\gamma_1$ and $\gamma_2$ are in $W^{s,p}(\mathbb{R}^d)$.

Recall that the conductivity equation $\text{div}(\gamma \nabla u) = 0$ is equivalent to the equation $(\Delta - q)v = 0$, where $v = \gamma^{1/2}u$ and $q = \gamma^{-1/2}\Delta \gamma^{1/2}$.

In what follows we also make use of the next two lemmas:

Lemma 6.2. [7, Proposition 2] Suppose that $\gamma_k \in W^{1,d}(\mathbb{R}^d) \cap A(\mathbb{R}^d)$, $\nabla \gamma_k^{1/2}$ is supported in a bounded set, and $\gamma_1 = \gamma_2$ outside of $\Omega$. If $v_k$ are solutions in $H^{1}_{loc}(\mathbb{R}^d)$ to $(\Delta - q_k)v_k = 0$ with $q_k = \gamma_k^{-1/2}\Delta \gamma_k^{1/2}$, then $(q_1, v_1v_2) = (q_2, v_1v_2)$.

Lemma 6.3. [19, Lemma 7.2] Let $\gamma_k$ and $q_k$ be given as in Lemma 6.2. If $q_1 = q_2$ in the sense of distributions, then $\gamma_1 = \gamma_2$.

\footnote{Note that the conditions on $r$ in (I), (II) can be written equivalently $r \geq \max\{-1, r_d(p)\}$ (see Section 1).}
In order to prove Theorem 1.1 by Lemma 6.3 it suffices to show that \( \hat{q}_1 = \hat{q}_2 \). This will be done by constructing the complex geometrical optics solutions \( v_k \) with a parameter \( \tau \) to the equations \((\Delta - q_k)v_k = 0\) such that \( v_1(x)v_2(x) \) converges to \( e^{i\xi \cdot x} \) for a fixed \( \xi \in \mathbb{R}^d \) as \( \tau \to \infty \). In fact, the solutions \( v_k \) take the form of \( e^{x \cdot \zeta_k(\tau,U)}(1 + \psi_k) \), where \( \psi_k \)'s are solutions to
\[
(\Delta \zeta_k(\tau,U) - q_k)\psi_k = q_k,
\]
where \( \Delta \zeta = \Delta + 2\zeta \cdot \nabla \).

Fix orthonormal vectors \( e_1, e_2, e_3 \in \mathbb{R}^d \) and \( r > 0 \). For \( U \in O_d \) and \( \tau \geq \max\{1,r\} \), we set, as in [19],
\[
\zeta_1(\tau,U) = \tau U(e_1 - ie_2), \quad \zeta_1(\tau,U) = \tau Ue_1 - i\sqrt{\tau^2 - r^2}e_2 + irU e_3,
\]
\[
\zeta_2(\tau,U) = -\tau U(e_1 - ie_2), \quad \zeta_2(\tau,U) = -\tau Ue_1 + i\sqrt{\tau^2 - r^2}e_2 + irU e_3.
\]

We may write
\[
q_k = \gamma_k^{-1/2} \Delta \gamma_k^{1/2} = \sum_j \partial_j f_{k,j} + h_k,
\]
where \( f_{k,j} = \frac{1}{2} \partial_{x_j} \log \gamma_k \) and \( h_k = \frac{1}{2} |\nabla \log \gamma_k|^2 \). It is clear that \( f_{k,j} \in W^{s-1,p}(\mathbb{R}^d) \) and \( h_k \in L^{p/2}(\mathbb{R}^d) \) since \( s \geq 1 \).

By Proposition 6.3 (applied to \( f_{k,j} \in W^{s-1,p}(\mathbb{R}^d) \) and \( q_k \in L^{s/2}(\mathbb{R}^d) \)), Proposition 5.11 (applied to \( h_k \in L^{s/2}(\mathbb{R}^d) \)), and Corollary 5.8 (applied to \( \partial f_{k,j} \in H^{s-2-\epsilon,p}(\mathbb{R}^d) \) and \( h_k \in L^{s/2}(\mathbb{R}^d) \)), we see that
\[
\lim_{M \to \infty} \int_M \int_{O_d} \sum_{k,\ell=1,2} \left( \|M_{q_k}\|_{X^{1/2}_{\zeta_k(\tau,U)}} \|X_{\zeta_\ell(\tau,U)}^{1/2}\| + \|q_k\|_{X_{\zeta_k(\tau,U)}^{1/2}} \right) \langle dm(U) \rangle d\tau = 0,
\]
if \( s - 1 > \max\{\gamma_d(p) + \frac{d}{2} - 1,0\} \), i.e., \( s > \max\{\gamma_d(p) + \frac{d}{2},1\} \) when \( d = 5,6 \). Hence, there exist sequences \( \tau \gg \tau_j > 0 \), \( U \gg U_j \subset O_d \), and \( \delta \gg \delta_j > 0 \) (in what follows we occasionally omit the subscript \( j \) for simplicity of notation) such that
\[
\lim_{j \to \infty} \tau_j = \infty, \quad \lim_{j \to \infty} \delta_j = 0,
\]
and, for \( k, \ell = 1,2 \),
\[
\|M_{q_k}\|_{X^{1/2}_{\zeta_k(\tau_j,U_j)}} \|X_{\zeta_\ell(\tau_j,U_j)}^{1/2}\| < \delta_j, \quad \|q_k\|_{X_{\zeta_k(\tau_j,U_j)}^{1/2}} < \delta_j.
\]
Since \( |\zeta_k(\tau,j) - \zeta_\ell(\tau,j)| \approx r \), we have
\[
\|u\|_{X^{b}_{\zeta_k(\tau,U)}} \approx \|u\|_{X^{b}_{\zeta_\ell(\tau,U)}}
\]
for any \( b \in \mathbb{R} \) with the implicit constant depending on \( r \) (see [19, Lemma 6.3]). It follows from (6.3) and Lemma 2.1 that for \( k, \ell = 1,2 \),
\[
\|M_{q_k}\|_{X^{1/2}_{\zeta_k(\tau,U)}} \|X^{1/2}_{\zeta_\ell(\tau,U)}\| \lesssim \delta, \quad \|q_k\|_{X^{1/2}_{\zeta_k(\tau,U)}} \lesssim \delta.
\]
Here and later on, the implicit constants depend on \( r \) but are independent of \( \tau \). The precise dependence is not important since \( \delta \to 0 \) while \( \tau \to \infty \).

With sufficiently large \( j \), by the contraction mapping principle (or the operator \( \Delta_{\zeta_k}^{-1}(\tau,U) - q_k \) is invertible since \( \|\Delta_{\zeta_k}^{-1}(\tau,U)\|_{M_{q_k}} \|X^{1/2}_{\zeta_k(\tau,U)} \|_{X^{1/2}_{\zeta_k(\tau,U)}} \) is small) we have solutions \( \psi_k \in \tilde{X}^{1/2}_{\zeta_k(\tau,U)} \) to the equations \((\Delta_{\zeta_k}^{-1}(\tau,U) - q_k)\psi_k = q_k, k = 1,2 \), such that
\[
\|\psi_k\|_{\tilde{X}^{1/2}_{\zeta_k(\tau,U)}} \lesssim \|q_k\|_{\tilde{X}^{-1/2}_{\zeta_k(\tau,U)}}.
\]
Indeed, since \( \psi_k = \Delta_{\zeta_k}^{-1}(\tau,U)(q_k\psi_k + q_k) \) and \( \|\Delta_{\zeta_k}^{-1}\|_{\tilde{X}^{-1/2}_{\zeta_k} \to \tilde{X}^{1/2}_{\zeta_k}} = 1 \), we have that
\[
\|\psi_k\|_{\tilde{X}^{1/2}_{\zeta_k(\tau,U)}} = \|\Delta_{\zeta_k}^{-1}(\tau,U)(q_k\psi_k + q_k)\|_{\tilde{X}^{1/2}_{\zeta_k(\tau,U)}} \leq \|M_{q_k}\|_{\tilde{X}^{1/2}_{\zeta_k(\tau,U)}} \lesssim \|q_k\|_{\tilde{X}^{1/2}_{\zeta_k(\tau,U)}} + \|q_k\|_{\tilde{X}^{-1/2}_{\zeta_k(\tau,U)}}.
\]
If \( j \) is large enough, \( \psi \) follows. Furthermore, since \( \|u\|_{H^{1,2}(\mathbb{R}^d)} \lesssim \tau^{1/2} \|u\|_{X^{1/2}_{\zeta(r)}} \), we have \( \psi_k \in H^1(\mathbb{R}^d) \), hence \( v_k = e^{\xi\cdot q_1} (1 + \psi_k) \in H^{1}_{\text{loc}}(\mathbb{R}^d) \).

Therefore, by the assumption \( \Lambda_{\gamma_2} = \Lambda_{\gamma_2} \) and Lemma \( \ref{lemma5.1} \) we can apply Lemma \( \ref{lemma5.2} \) to get \( (q_1 - q_2, v_1 v_2) = 0 \). Since \( v_1 v_2 = e^{\xi\cdot q_2} (1 + \psi_1) (1 + \psi_2) = e^{\xi\cdot 2rU_3} (1 + \psi_1) (1 + \psi_2) \), we now obtain that

\[
(q_1 - q_2, e^{\xi\cdot 2rU_3} e_3) = \sum_{k,n=1,2} (-1)^k (q_k, e^{\xi\cdot 2rU_3} \psi_n) + (q_2 - q_1, e^{\xi\cdot 2rU_3} \psi_1 \psi_2).
\]

By \( \ref{5.8} \), we have

\[
| (q_k, e^{\xi\cdot 2rU_3} \psi_n) | \lesssim \|q_k\|_{X^{1/2}_{\zeta(r), m}} \|\psi_n\|_{X^{1/2}_{\zeta(r), m}} \lesssim \|q_k\|_{X^{1/2}_{\zeta(r), m}} \|q_n\|_{X^{1/2}_{\zeta(r), m}}.
\]

Here we also use that \( \|e^{\xi\cdot 2rU_3} \phi\|_{X^{1/2}_{\zeta(r), m}} \approx \|\phi\|_{X^{1/2}_{\zeta(r), m}} \) with the implicit constant which depend on \( r \). This is easy to see since the modulation \( e^{\xi\cdot 2rU_3} \) is acting only on \( \xi \). Applying \( \ref{5.7} \), \( \ref{5.9} \), and \( \ref{5.8} \), successively, we get

\[
| (q_1, e^{\xi\cdot 2rU_3} \psi_1 \psi_2) | \lesssim \|M_{q_1} (e^{\xi\cdot 2rU_3} \psi_1)\|_{X^{1/2}_{\zeta(r), m}} \|\psi_2\|_{X^{1/2}_{\zeta(r), m}} \approx \|e^{\xi\cdot 2rU_3} \psi_1\|_{X^{1/2}_{\zeta(r), m}} \|\psi_2\|_{X^{1/2}_{\zeta(r), m}}.
\]

where we use the fact that \( \zeta_1(\tau, U) = -\zeta_2(\tau, U) \). We also obtain similar estimate for \( q_2 \) replacing \( q_1 \). Hence, combining all the above estimates we get

\[
| (\hat{q}_1 - \hat{q}_2)(-2rU_3) | \lesssim \delta \sum_{k,m,n} \|q_k\|_{X^{1/2}_{\zeta(r), m}} \|q_m\|_{X^{1/2}_{\zeta(r), m}} \lesssim \delta^3.
\]

This shows that \( \lim_{j \rightarrow \infty} (\hat{q}_j - \hat{q}_j)(-2rU_3) e_3 = 0 \) for the sequence \( \{U_j\} \). Meanwhile, since \( O_d \) is compact, we can pass to a subsequence so that \( U_j \) converges to a unitary matrix \( U_\ast \). Thus we have \( (\hat{q}_1 - \hat{q}_2)(-2rU_\ast) e_3 = 0 \). Therefore, we conclude that \( \hat{q}_1 - \hat{q}_2 = 0 \) as \( e_3 \) and \( r \) are arbitrary. This completes the proof of Theorem \( \ref{theo1.1} \).

**Remark 4.** From Proposition \( \ref{prop5.13} \) (for \( d = 7, 8 \)) and Proposition \( \ref{prop5.11} \) (for \( d \geq 9 \)), if we apply \( \ref{5.27} \) (with \( \kappa = 1 \)) to the potentials \( q_k = \gamma_k^{1/2} \Delta \gamma_k^{1/2} \) where \( \gamma_k \in W^{s,p}_c(\mathbb{R}^d) \), we see that \( \ref{5.3} \) holds whenever \( d \leq p < \infty \) and \( s > s_d(p) \) that is given by

\[
s_d(p) = \begin{cases} 
1 + \frac{d-5}{4p} & \text{if } \frac{d+9}{2} \leq p < \infty, \\
1 + \frac{d^2 + d^2 - 16 - 2p}{2p(p+5)} & \text{if } d \leq p < \frac{d+9}{2},
\end{cases}
\]

for \( d = 7, 8 \),

\[
s_d(p) = \begin{cases} 
1 + \frac{d-5}{4p} & \text{if } \frac{d+9}{2} \leq p < \infty, \\
\frac{1}{2} + \frac{3d-1}{4p} & \text{if } d \leq p < \frac{d+9}{2},
\end{cases}
\]

for \( d \geq 9 \).

As is mentioned in the introduction, if we have the additional condition \( \partial \gamma_1 / \partial \nu = \partial \gamma_2 / \partial \nu \) on the boundary, the zero-extension of \( \gamma_1 - \gamma_2 \) is valid if \( s - \frac{1}{2} \leq 2 \). Since \( s_d(p) - \frac{1}{2} \leq 2 \) for \( d \leq p < \infty \) and \( \ref{6.3} \) is valid whenever \( s > s_d(p) \), by the above argument, the injectivity of the mapping \( W^{s,p} \ni \gamma \mapsto \Lambda_\gamma \) follows whenever \( d \leq p < \infty \) and \( s > s_d(p) \).

### 6.2. Proof of Theorem \( \ref{theo1.2} \)

Before we prove Theorem \( \ref{theo1.2} \) we justify \( \mathcal{L}_q \) is well defined with \( q \in H^{s,p}_c(\Omega) \) while \( s, p \) satisfy \( \ref{1.8} \).

**Lemma 6.4.** Let \( p \geq \frac{d}{2} \) and \( \mathcal{L}_q \) be given by \( \ref{1.7} \). Suppose \( q \in H^{s,p}_c(\Omega) \) and \( s, p \) satisfy \( \ref{1.8} \). Then, \( \mathcal{L}_q \) is well defined and continuous from \( H^{1/2}(\partial \Omega) \) to \( H^{-1/2}(\partial \Omega) \).
Proof. We may assume $s = \max\{-2 + \frac{d}{p}, -1\}$ since $H^{s,p} \hookrightarrow H^{s_1,p}$ if $s_1 \leq s_2$. Let $u \in H^1(\Omega)$ be a solution to (1.6) and $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Then the quantity
\[
S_u(v) = \int_{\Omega} \nabla u \cdot \nabla v + quv \, dx
\]
is well defined. In fact, we note that $|(q, uv)| = |(1 + |D|^2)^{\frac{1}{2}} \tilde{z}_q, (1 + |D|^2)^{-\frac{1}{2}} \tilde{\tau}(uv))| \leq \|(1 + |D|^2)^{\frac{1}{2}} \tilde{z}_q\|_{L^p} \|(1 + |D|^2)^{-\frac{1}{2}} \tilde{\tau}(uv)\|_{L^p}$. By the Kato-Ponce inequality (23, 16) and the Hardy-Littlewood-Sobolev inequality we get
\[
|q, uv| \lesssim \|q\|_{H^{s,p}} \|((1 + |D|^2)^{\frac{1}{2}} \tilde{z}_u, (1 + |D|^2)^{-\frac{1}{2}} \tilde{\tau}(v))\|_{L^{2d}} + \|q\|_{L^{2d}} \|((1 + |D|^2)^{\frac{1}{2}} \tilde{z}_u, (1 + |D|^2)^{-\frac{1}{2}} \tilde{\tau}(v))\|_{L^1},
\]
where $s = \max\{-2 + \frac{d}{p}, -1\}$ and $\frac{1}{2} = \frac{2d^2}{2d} - \frac{1}{p}$. Thus we have
\[
|S_u(v)| \lesssim \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.
\]
Since $u \in H^1(\Omega)$ is a solution to (1.6), $S_u(v_0 - v) = 0$ for all $v_0 \in H^1(\Omega)$ with $v_0|_{\partial\Omega} = g$ because $v_0 - v \in H^1_0(\Omega)$. This shows $S_u(v)$ does not depend on particular choices of $v$, that is to say, $L_q$ is well defined.

To show $L_q : H^1(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous, by duality it is sufficient to show that
\[
|\langle L_q f, g \rangle| \lesssim \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{-1/2}(\partial\Omega)}.
\]
From (1.7) and (6.3) we have $|\langle L_q f, g \rangle| \lesssim \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$. Using the right inverse of the trace operator we have $\|u\|_{H^1(\Omega)} \lesssim \|f\|_{H^{1/2}(\partial\Omega)}$ and $\|v\|_{H^1(\Omega)} \leq \|g\|_{H^{-1/2}(\partial\Omega)}$. This gives the desired estimate. \hfill \Box

From the standard argument, similarly handling $(q, uv)$ as in the above, it is also easy to see the following. (See Section 2.7 in [14], for example.)

**Proposition 6.5.** Let $p \geq \frac{d}{2}$. Suppose $q_1, q_2 \in H^{s,p}(\Omega)$ and $s, p$ satisfy (6.8) and suppose $L_{q_1} = L_{q_2}$. Then, $(q_1 - q_2, u_1u_2) = 0$ whenever $u_i \in H^1(\Omega)$ is a solution to $\Delta u - q_iu = 0$ for each $i = 1, 2$.

Now we show Theorem 1.2 by constructing the complex geometrical optics solutions. We follow the lines of arguments in the proof of Theorem 1.1 Let $s, p$ be given as in Theorem 1.2 and $q_1, q_2 \in H^{s,p}(\Omega)$. By Lemma 5.3 it is enough to consider the case $s \leq 0$. Then, by Proposition 5.12 we have, for $k = 1, 2$,
\[
\lim_{M \to \infty} \left( \int_M \int_{O_d} \|M_{q_k}\|_{X_{\zeta_k}(\tau, U)} \to X_{\zeta_k}(\tau, U) d\tau \right)^{\frac{1}{k}} = 0
\]
for $0 \leq s > \max\{-1, r_d(p)\}$. From Corollary 5.8 we see that (5.21) holds for $s \geq \max\{-1, \frac{d}{p} - 2\}$ and $\frac{d}{2} \leq p < \infty$. We also have, for $k = 1, 2$,
\[
\lim_{\tau \to \infty} \int_{O_d} \|q_k\|_{X_{\zeta_k}(\tau, U)}^2 \, d\tau = 0.
\]
Let $\zeta_1(\tau, U), \zeta_2(\tau, U)$, and $\zeta_3(\tau, U)$ be given by (5.2). Combining these two, we have $\tau^k \tau > 0$, $U = U_j \in O_d$, and $\delta_j > 0$ such that (6.4) and (6.5) hold. Once we have $\tau = \tau_j > 0$, $U = U_j \in O_d$, then the rest of argument works without modification. So we omit the details.

**Remark.** When $d \geq 3$ the above argument provide a different proof of the uniqueness result for $q \in L^{d/2}$ (34) by using (5.13) instead of (5.25). As observed above, it is sufficient to show (6.10) and (6.11) for $q \in L^{d/2}$. Following the proof of Proposition 5.11 ((5.27)), we write $q = q \ast \phi_\epsilon + (q - q \ast \phi_\epsilon)$. By (5.12) and (6.13), we obtain
\[
\|q\|_{M_0} \lesssim \tau^{-1} \|q \ast \phi_\epsilon\|_{L^\infty} + \|q - (q \ast \phi_\epsilon)\|_{L^d} \lesssim \tau^{-1} \epsilon^{-2} \|q\|_{L^d} + \|q - (q \ast \phi_\epsilon)\|_{L^d}.
\]
Taking $\epsilon = \tau^{-\frac{1}{2}}$, we see that (6.10) holds for $k_\epsilon \in L^d$. Meanwhile (6.11) is immediate by (5.21) with $s = 0$ and $p = \frac{d}{2}$. The remaining is identical with the previous argument.
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