The Algebra of the Pseudo-Observables II: The Measurement Problem

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Abstract. In this second paper, we develop the full mathematical structure of the algebra of the pseudo-observables, in order to solve the quantum measurement problem. Quantum state vectors are recovered but as auxiliary pseudo-observables storing the information acquired in a set of observations. The whole process of measurement is deeply reanalyzed in the conclusive section, evidencing original aspects. The relation of the theory with some popular interpretations of Quantum Mechanics is also discussed, showing that both Relational Quantum Mechanics and Quantum Bayesianism may be regarded as compatible interpretations of the theory. A final discussion on reality, tries to bring a new insight on it.

Keywords: Quantum measurement problem, interpretation of quantum mechanics, relational quantum mechanics, quantum bayesianism

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1. Introduction

1.1. Measurement in Quantum Mechanics

Quantum measurement theory in standard textbooks is expressed in term of the Copenhagen interpretation, that can be summarized in the following essential points:

(i) A state vector gives a complete description of the state of a physical system. It determines the probability distribution of the measure outcomes of any observable quantity.

(ii) Our knowledge of the physical reality cannot be expressed but by means of the language of the “Classical Physics”. But the complete description of the physical phenomena requires the use of contrasting classical concepts, such as those of wave and particle. No contradiction, however, arises from their use, since they describe phenomena that occur in incompatible experimental situations (complementarity principle).

(iii) The uncertainty principle states the impossibility of the simultaneous existence of exactly defined values for two incompatible observables.

(iv) The observation results in a discontinuous change, that cannot be described in terms of the time evolution equations, of the state vector that causes a system state collapse into one of the eigenstates of the measured observable. This final state can be forecast only probabilistically.

The measuring apparatus must be described classically.

The two key concepts of the interpretation are the completeness of the quantum state, as expressing all that can be known about the physical system, and the the ontic probabilism: the need in the use of probability is not due to our lack of knowledge but it’s the measurement process itself to be intrinsically undeterministic.

To be honest, this is only one interpretation of the Copenhagen interpretation, since, according to Peres[1]:

“There seems to be at least as many different Copenhagen Interpretations as people who use that term, probably there are more.”

Generally speaking, despite the wide-most popularity of such interpretation, as witnessed by a poll executed by Schlosshauer et al.[2], most physicists, as a matter of fact, have subscribed an instrumentalist interpretation of quantum mechanics, a position, often equated with eschewing all interpretations, summarized by the sentence “Shut up and calculate!”, using a slogan sometimes attributed to Paul Dirac or Richard Feynman, but that seems to be due to David Mermin[3].

One way or another, the quantum measurement presents several conceptual problematic aspects, as it will be discussed in the next subsection.

1.2. The measurement problem

Pykacz, in his short survey on the main interpretations of Quantum Mechanics[4], identifies, as drawbacks of the Copenhagen interpretation, the artificial division of the physical world into the quantum world and the classical world and the “objectification problem”, i.e., the problem of how the “potential” properties become “actual” in the course of a measurement. The latter is strictly tied to the so-called measurement problem in quantum mechanics, that is the problem of how (or whether) wave function collapse occurs. The critical point is that wave functions evolve deterministically according to the Schrödinger equation, whereas the collapse is of a stocastical nature.
The problem gave rise to a plethora of interpretations of quantum mechanics, all of them more or less unsatisfying for some aspect, as evidenced, for instance, by the analysis of Pykacz above cited.

In order to better frame the question, we will report as the problem was fronted by a few of these interpretations.

The Copenhagen interpretation says nothing about the cause of the collapse. However, in its original formulations, wave functions aren’t regarded as ontologically real entities. Bohr did not think of quantum measurement in terms of a collapse of the wave function\(^5\), whereas Heisenberg considered wave functions representing a probability, not an objective reality itself\(^5, 6\).

According to the Von Neumann–Wigner interpretation\(^7, 8\) the collapse is caused by consciousness. Henry Stapp so summarizes this perspective: “In short, orthodox quantum mechanics is Cartesian dualistic at the pragmatic/operational level, but mentalistic on the ontological level”\(^9\).

The many-worlds interpretation\(^10, 11\) asserts the objective reality of an universal wave function and denies the actuality of wave function collapse: all possible alternative histories and futures are real, each representing an actual “world” (or “universe”).

In the De Broglie–Bohm theory\(^12, 13\), in addition to a wave function on the space of all possible configurations, it is also postulated an actual configuration that exists even when unobserved. The time evolution of the configuration is defined by the wave function via a guiding equation, while the wave function evolves over time according the Schrödinger equation. According to this interpretation the interaction with the environment during a measurement procedure separates the wave packets in configuration space, giving rise to an apparent wave function collapse. In such interpretation wave functions are ontologically real.

The objective collapse theories regard both the wave functions and the process of collapse as ontologically objective. In such theories, collapse occurs randomly (“spontaneous localization”) or when some physical threshold is reached, giving the observer no special role. The mechanism of collapse, that is not specified by standard quantum mechanics, needs to be introduced \textit{ex novo}, if this approach is correct. For this reason, Objective Collapse is more a new theory than an interpretation. Examples include the Ghirardi-Rimini-Weber theory\(^14\) and the Penrose interpretation\(^15\).

1.3. What it has been done in this paper

The above picture is objectively discouraging and justify the widespread attitude of physicists to simply ignore such issues. This is understandable and it is the right thing to do when a conceptual node is beyond the comprehension capabilities of the human of the time. When Isaac Newton had to front the properties of gravity, in particular for what regards distant action between material bodies, he was forced to a similar withdrawn, stigmatized by his famous quote “\textit{Hypotheses non fingo}”\(^16\). Notwithstanding this, the theory of universal gravitation allowed the most accurate description of the motion of celestial bodies for almost two centuries. In this long period, the Newton doubt was completely forgot. But in 1915, Einstein showed to the world, with his general theory of gravitation\(^17\), a much deeper insight of the Cosmos, giving an answer to the unsolvable question of Newton!

In a similar manner, it is not inconceivable that also the measurement problem and the plethora of interpretations to which it gave rise is the symptom of more fundamental issues in the deeper structure of the quantum mechanical theory. But what is the disease?

In my opinion, the problem sinks its roots in the heterogeneity of the entities assumed as \textit{fundamental} and \textit{primitive} in the formulation of Quantum Mechanics. As a matter of fact, they
are three and all of them ill-defined: quantum states (wave functions), observables and observers. In particular, whereas, on the one hand, it is recognized the fundamental role of the observer in the definition itself of the measurement outcomes of a physical quantities; on the other, observers seem to have no special role in the determining the expectation values of the quantities themselves. Besides, in order to define the one expectation value of a physical quantity, the theory requires two entities of different nature and mathematical representation: a quantum state, often represented by a wave function, and an observable, represented by an operator. This leads to an ill definition of both the entities and to severe ambiguities about their degrees of reality and objectivity. So in a group of interpretations wave function are ontologically real and not in another; in a group quantum state collapse is objective, in another is apparent and so on...

In this paper, after an introductory mathematical part, I will show how it is possible to define quantum states, observables and expectation values in term of an only mathematical entity: the algebra of pseudo-observables, introduced in the first paper[18] to justify the necessity of the Quantum Mechanics to describe the measurable properties of a physical system. In doing this, it will be also clearly stated the derived and partly subjective character of the quantum states. In the conclusion I will digress about what we have to mean with the term “reality” and in what a measure quantum states are “real”.

2. Quantum states

2.1. Eigenvectors and eigenvalues

After having indicated with \( Z \) a generic element of the space of the pseudo-observables \( \mathfrak{P} \), one is allowed to associate to each observable \( O \) the two linear mappings:

\[
\varphi_R(Z) := OZ
\]

\[
\varphi_L(Z) := ZO
\]

in correspondence of which we can consider a right eigenvector and a right eigenvalue relative to the observable \( O \), i.e. a non-null pseudo-observable \( \Phi' \) and a complex number \( \omega' \) such that it results:

\[ O \Phi' = \omega' \Phi' \quad (1) \]

and a left eigenvector and a left eigenvalue relative to the observable \( O \), i.e. a non-null pseudo-observable \( \Phi'' \) and a complex number \( \omega'' \) such that it results:

\[ \Phi'' O = \omega'' \Phi'' \quad (2) \]

A pseudo-observable \( \Psi \) that is both a right eigenvector, relative to the eigenvalue \( \omega' \), and left eigenvector, relative to the eigenvalue \( \omega'' \), of the same observable \( O \) will be called a bilateral eigenvector relative to the pair of eigenvalues \( (\omega', \omega'') \). If \( \Phi_1 \) and \( \Phi_2 \) are two bilateral eigenvectors of an observable \( O \) relative to the same pair of eigenvalues \( (\omega', \omega'') \), it is immediately verified that every their linear combination, with complex coefficients, is a bilateral eigenvector relative to the pair of eigenvalues \( (\omega', \omega'') \). The set of the bilateral eigenvectors of a given observable relative to the same pair of eigenvalues \( (\omega', \omega'') \) constitutes, therefore, a vector subspace of \( \mathfrak{P} \), that we will call bilateral eigenspace relative to the pair of eigenvalues \( (\omega', \omega'') \) and that we will denote with \( \mathfrak{A} (\omega', \omega'') \).
Note that the equations (1) and (2) certainly admit non-zero solutions, since they are satisfied by the projectors of the basis associated to $O$ and by its spectral coefficients, as shown by equation (10) in the first paper, of which (1) and (2) represent a generalization.

If $\{I_j\}$ is a basis of elementary projectors associated to a complete set of observables compatible one with each other and with $O$, the dyads associated with all possible pairs of such projectors, as shown in subsection 4.2 in the first paper, constitute a basis and are bilateral eigenvectors of $O$. In fact, by virtue of equation (10) in the first paper, one has:

$$O \Gamma_{jk} = O I_j C_{jk} I_k = o_j I_j C_{jk} I_k = o_j \Gamma_{jk} \quad (3)$$

and

$$\Gamma_{jk} O = I_j C_{jk} I_k O = I_j C_{jk} (O I_k) = o_k I_j C_{jk} I_k = o_k \Gamma_{jk} \quad (4)$$

that show as the dyad $\Gamma_{jk}$ is a bilateral eigenvector of $O$ relative to pair of eigenvalues $(o_j, o_k)$.

Let $\Phi$ be a right eigenvector of an observable $O$, relative to the right eigenvalue $\omega$. By expressing $\Phi$ as a linear combination of the dyads $\Gamma_{jk}$ associated to the pairs of elementary projectors of a basis $\{I_j\}$ compatibles with $O$, making use of equation (48) in the first paper, one has:

$$\Phi = \sum_{j,k} \varpi_{jk} \Gamma_{jk} \quad (5)$$

According to (3), besides, it results:

$$O \Phi = \sum_{j,k} \varpi_{jk} O \Gamma_{jk} = \sum_{j,k} o_j \varpi_{jk} \Gamma_{jk}$$

which substituted together with (5) in (1) provides the relation:

$$\sum_{j,k} (\omega - o_j) \varpi_{jk} \Gamma_{jk} = 0$$

that, for the linear independence of the dyads of the basis, is equivalent to the condition:

$$(\omega - o_j) \varpi_{jk} = 0 \Rightarrow \varpi_{jk} \neq 0 \rightarrow \omega = o_j \quad (6)$$

According to (3), the right eigenvalues of $O$ are all and only the terms of its spectrum. Each right eigenvector relative to a certain right eigenvalue $o$ of $O$, besides, will be given by a linear combination of the dyads that contain as first factor the elementary projectors of a basis compatibles with $O$ whose spectral coefficient in the decomposition (9) in the first paper is equal to $o$. Similarly, for the left eigenvectors and eigenvalues. It is, therefore, possible to affirm that, indicated with $o_j$ the spectral coefficient, in the $O$ decomposition (9) in the first paper, of the elementary projector $I_j$ belonging to a basis compatible with $O$, the bilateral eigenspace $A(o', o'')$ admits as a basis the set of dyads: $\{\Gamma_{jk} | o_j = o' \wedge o_k = o''\}$. If the multiplicity of $o'$ is $m_{o'}$ and the multiplicity of $o''$ is $m_{o''}$, the dimension of the bilateral eigenspace $A(o', o'')$, therefore, is:

$$\dim (A(o', o'')) = m_{o'} m_{o''} \quad (7)$$

If, in particular, one considers: $o' = o'' = o$, where $o$ is a whatever outcome of $O$, one has:

$$\dim (A(o, o)) = m_o^2 \quad (8)$$

Since the size of an autospace is an intrinsic feature of the observable $O$, independent from the choice of the basis $\{I_j\}$ of elementary projectors, it is concluded that also the multiplicities of the terms of the spectrum of $O$ are intrinsic characteristics of the observable.
2.2. The trace of a pseudo-observable

According to what demonstrated in subsection 2.3 in the first paper [18] and at the end of subsection 2.1, both the terms of the spectrum of an observable \( O \) and the relative multiplicities are intrinsic characteristics of the observable itself.

We now define the trace functional \( (\text{tr}) \) of an observable \( O \) as the sum of all its spectral coefficients, \( o_j \), with respect to a basis \( \{I_j\} \) of elementary projectors compatible with it, i.e.:

\[
\text{tr} \left( O \right) := \sum_j o_j \tag{9}
\]

The trace of an observable, that is evidently given also by the sum of the products of the distinct terms of the observable spectrum by their relative multiplicities, is, for what was exposed at the beginning of this subsection, uniquely defined, since it can be expressed in terms of intrinsic characteristics of the observable.

Consider now a dyad basis, \( \{\tilde{\Gamma}_{ll'}\} \), whose elements are generally not compatible with the projectors \( I_j \), and let be \( \{\Gamma_{jl}\} \) a dyad basis associated to the elementary projector basis \( \{I_j\} \). Accord to equation (62) in the first paper, one has

\[
I_j = \Gamma_{jj} = \sum_{l,l'} \omega_{lj} \omega^*_{l'j} \tilde{\Gamma}_{ll'} \tag{10}
\]

where the complex constants \( \omega_{lj} \) are the components of the pseudo-observable of the basis changing. By substitution of (10) in the decomposition of the observable \( O \) according the basis \( \{I_j\} \) of elementary projectors, equation (9) in the first paper, one obtains:

\[
O = \sum_{l,l'} \left( \sum_j \omega_{lj} \omega^*_{l'j} o_j \right) \tilde{\Gamma}_{ll'} = \sum_{l,l'} \hat{o}_{ll'} \tilde{\Gamma}_{ll'} \tag{11}
\]

where the coefficient:

\[
\hat{o}_{ll'} := \sum_j \omega_{lj} \omega^*_{l'j} o_j
\]

represents the component of \( O \) relative to the dyad \( \tilde{\Gamma}_{ll'} \). We now demonstrate that the trace of an observable is given by the sum of its diagonal components, i.e. it coincides with the trace of the matrix associated with \( O \). In fact, according to (11) and making use of equation (67) in the first paper, one has:

\[
\sum_l \hat{o}_{ll'} = \sum_{l,j} \omega_{lj} \omega^*_{lj} o_j = \sum_j o_j \sum_l \omega^*_{lj} \omega_{lj} = \sum_j o_j = \text{tr} \left( O \right).
\]

In general, therefore, we can define the trace functional of a pseudo-observable as the sum of its diagonal components with respect to a whatever dyad basis. The definition is well placed, since if \( P \) is a generic pseudo-observable, of real part \( P_R \) and imaginary part \( P_I \), and \( \{\Gamma_{jk}\} \) is a whatever dyad basis, called \( \varpi_{jk} \), \( \alpha_{jk} \) and \( \beta_{jk} \) the component respectively of \( P \), of its real part and of its imaginary part relative to the dyad \( \Gamma_{jk} \), on the basis of equation (40) in the first paper, one has:

\[
\varpi_{jk} := \alpha_{jk} + i \beta_{jk} \tag{12}
\]

by which it results:

\[
\sum_j \varpi_{jj} = \sum_j \alpha_{jj} + i \sum_j \beta_{jj} = \text{tr} \left( P_R \right) + i \text{tr} \left( P_I \right) \tag{13}
\]

The so defined trace functional satisfies the three following properties:
(i) For each pseudo-observable $P$ of real part $P_R$ and imaginary part $P_I$, one has:

$$\text{tr}(P) := \text{tr}(P_R) + i \text{tr}(P_I) .$$

(ii) For any two given pseudo-observables $P$ and $Q$, the trace of their sum is equal to the sum of their traces:

$$\text{tr}(P + Q) = \text{tr}(P) + \text{tr}(Q) .$$

(iii) If $P$ and $Q$ are two whatever pseudo-observables, one has:

$$\text{tr}(PQ) = \text{tr}(QP) .$$

The first property is an immediate consequence of the definition and of (13). The second property follows immediately by the definition making use of equation (51) in the first paper. It allows to state that the trace is a linear functional. For what concerns the third property, indicated respectively by $\varpi_{jk}$ and $\vartheta_{jk}$ the components of $P$ and $Q$ relative to the dyad $\Gamma_{jk}$ of a given basis, according to equation (52) in the first paper, the component $\zeta_{jk}$ of the product $PQ$ relative to the dyad $\Gamma_{jk}$ is given by:

$$\zeta_{jk} = \sum_l \varpi_{jl} \vartheta_{lk} . \quad (14)$$

whereas the component $\xi_{jk}$ of the product $QP$ relative to the dyad $\Gamma_{jk}$ is given by:

$$\xi_{jk} = \sum_l \vartheta_{jl} \varpi_{lk} . \quad (15)$$

According to (14) and (15), therefore, it results:

$$\text{tr}(PQ) = \sum_j \zeta_{jj} = \sum_{j,l} \varpi_{jl} \vartheta_{lj} = \sum_{j,l} \vartheta_{jl} \varpi_{lj} = \text{tr}(QP) .$$

A first important consequence of these properties is that for every pseudo-observable $Z$ it results:

$$\text{tr}(Z^\dagger) = \text{tr}(Z^*). \quad (16)$$

Moreover, if $\gamma$ is a complex constant and $Z$ a pseudo-observable, it is easily demonstrated that it results:

$$\text{tr}(\gamma Z) = \gamma \text{tr}(Z) . \quad (17)$$

An important consequence of the third property is that for each pair of pseudo-observables $X$ and $Y$ it results:

$$\text{tr}([X,Y]) = 0 . \quad (18)$$

Note that if $O$ is a non-negative observable ($O \geq 0$), then the following important properties apply:

$$\text{tr}(O) \geq 0 \quad (19)$$

and

$$\text{tr}(O) = 0 \iff O = 0 \quad (20)$$

which is immediate to verify on the basis of the definition of the trace of an observable.

Now consider the trace of a generic dyad $\Gamma_{jj'}$ of a basis $\{\Gamma_{jk}\}$. Since the only non-zero component of $\Gamma_{jj'}$ according to the dyad basis is that relative to itself, trivially equal to 1, one has:

$$\text{tr}(\Gamma_{jj'}) = \delta_{jj'} . \quad (21)$$
This relation allows one to characterize the elementary projectors. Let, in fact, $J$ be a projector. If $J$ is not elementary, it will be expressible as a sum of $n$ mutually exclusive elementary projectors $I_j$:

$$J = I_1 + \cdots + I_n.$$  \hfill (22)

By definition, therefore, it will be $\text{tr}(J) = n$. According to this, the necessary and sufficient condition for a projector $I$ to be elementary is that it results: $\text{tr}(I) = 1$.

### 2.3. Inner product of two pseudo-observables

The trace functional allows one to define a Frobenius-like inner product into the space $\mathcal{P}$ of the pseudo-observables. If $X$ and $Y$ are two pseudo-observables, we define their inner product, $\langle X \mid Y \rangle$, as follows:

$$\langle X \mid Y \rangle := \text{tr}(X^\dagger Y).$$ \hfill (23)

It’s easy to verify that the so defined operation actually satisfies the characteristic properties of a (complex) inner product, i.e. if $\alpha$ and $\beta$ are two complex constants and $X$, $Y$ and $Z$ three pseudo-observables, it holds that:

(i) **Anti-linearity** in the first argument:
$$\langle \alpha X + \beta Z \mid Y \rangle = \alpha^* \langle X \mid Y \rangle + \beta^* \langle Z \mid Y \rangle.$$

(ii) **Linearity** in the second argument:
$$\langle X \mid \alpha Y + \beta Z \rangle = \alpha \langle X \mid Y \rangle + \beta \langle X \mid Z \rangle.$$

(iii) **Conjugate symmetry**:
$$\langle X \mid Y \rangle = \langle Y \mid X \rangle^*.$$

(iv) **Positive-definiteness**:
$$\langle X \mid X \rangle \geq 0,$$
$$\langle X \mid X \rangle = 0 \text{ iff } X = 0.$$

The verification of the first three properties immediately follows by the definition (23), while for the fourth one must use the fifth and sixth property of the transposition and, in the order, the properties (19) and (20) of the trace.

The fourth property of the inner product allow one to introduce the norm $\|X\|$ of a generic pseudo-observable $X$, defined as follows:

$$\|X\| := \sqrt{\langle X \mid X \rangle}$$ \hfill (24)

that results always positive or zero, being equal to zero if and only if $X = 0$.

A normed space is also a **metric space**, and we will assume the further hypothesis that the space $\mathcal{P}$ of the pseudo-observables is also **complete**, that’s to say that every Cauchy sequence of elements of $\mathcal{P}$ converges to an element of $\mathcal{P}$. With respect to such a metric the space $\mathcal{P}$ of the pseudo-observables is therefore an **Hilbert space** formed by every formal expression obtained by summing or multiplying observables, that we will suppose to be, at most, a countable set, and by the limits of Cauchy sequences of such expressions.

Further interesting properties of the the inner product, defined by the (23), are given by the following relations:

$$\langle X \mid Y \rangle = \text{tr}(X^\dagger Y) = \text{tr}\left((Y^\dagger)^\dagger X^\dagger\right) = \langle Y^\dagger \mid X^\dagger \rangle,$$ \hfill (25)
\( (X \mid YZ) = \text{tr} (X^\dagger YZ) = \text{tr} \left( (Y^\dagger X)^\dagger Z \right) = \langle Y^\dagger X \mid Z \rangle \) \hspace{1cm} (26)

that immediately follow by the trace properties.

Let now \( O \) be a whatever observable and \( \{ \Gamma_{jk} \} \) a dyad basis formed by bilateral eigenvectors of \( O \). We want to demonstrate that such dyads constitute an orthonormal basis for the space \( \Psi \) of the pseudo-observables, that is that for them it holds the following relation:

\[
\langle \Gamma_{jk} \mid \Gamma_{j'k'} \rangle = \delta_{j,j'} \delta_{k,k'} .
\] \hspace{1cm} (27)

In fact, one has:

\[
\langle \Gamma_{jk} \mid \Gamma_{j'k'} \rangle = \text{tr} (\Gamma_{kj} \Gamma_{j'k'}) = \delta_{j,j'} \text{tr} (\Gamma_{kj} \Gamma_{j'k'}) = \delta_{j,j'} \text{tr} (\Gamma_{kk'}) = \delta_{j,j'} \delta_{k,k'}
\]

where it was made use of the second and third properties of the dyads (see paper [18]) and of the relation (21).

We have so demonstrated that for each observable \( O \) there exists an orthonormal basis of the space \( \Psi \) of the pseudo-observables formed by (bilateral) eigenvectors of \( O \) \textit{(first spectral theorem)}. Due to our assumptions on the spectra of the observables, we will, therefore, assume that the space \( \Psi \) of the pseudo-observables is also separable, i.e. it admits countable orthonormal bases.

The inner product introduced, finally, allows one to express the component \( \varpi_{jk} \) of a pseudo-observable \( P \) relative the dyad \( \Gamma_{jk} \) of a given basis in the form:

\[
\varpi_{jk} = \langle \Gamma_{jk} \mid P \rangle
\] \hspace{1cm} (28)

which is easily demonstrated starting from the decomposition (48) in the first paper and making use of the orthonormality relations (27).

2.4. State vectors

Given a basis \( \{ I_j \} \) of elementary projectors of a certain complete space of compatible observables, it will be proved that there is a set \( \{ \Psi_j \} \) of pseudo-observables such that the products \( \Psi_j \Psi_k^\dagger \) form a dyad basis associated to the projector basis \( \{ I_j \} \). By indicating with \( \{ \Gamma_{jk} \} \) such a dyad basis, this is equivalent to saying that, for each pair of indexes \( j \) and \( k \), it must result:

\[
\Psi_j \Psi_k^\dagger = \Gamma_{jk} .
\] \hspace{1cm} (29)

The existence of the pseudo-observables satisfying the (29) is proved, as it is easy to verify, after arbitrarily choosing an index \( k_0 \), by setting:

\[
\Psi_j := \Gamma_{jk_0} .
\]

The decomposition (29) justifies the name of dyad adopted in section 4 of the first paper.

For a given set of pseudo-observables satisfying the (29), we will call each its element a state vector.

In the next subsection, it will be proved the fundamental \textit{theorem of characterization of the sets of the state vectors: each set of vector states is formed, for each state index \( j \), by the pseudo-observables of the form}:

\[
\Psi_j = \Gamma_{jk_0} K
\] \hspace{1cm} (30)
where \( k_0 \) and \( K \) are, respectively, an index and an unitary pseudo-observable arbitrarily chosen. It is worth observing that the choice of the index \( k_0 \) in the (30) is irrelevant. In fact, if one regards a whatever different index \( k_1 \), one has:

\[
\Psi_j = \Gamma_{jk_0} K = \Gamma_{jk_1} S_{k_1 k_0} K = \Gamma_{jk_1} K'
\]

being \( S_{k_0 k_1} \) the unitary pseudo-observable defined in equation (68) of the first paper, and having put \( K' := S_{k_0 k_1} K \), which is still an unitary pseudo-observable.

2.5. Characterization of the state vectors

We now prove the theorem of characterization of the sets of the state vectors. With the notations seen in the subsection 2.4, equation (29), and by making use of equation (48) in the first paper, it is possible to express a state vector \( \Psi_j \) as a linear combination of the dyads of the basis \( \{ \Gamma_{jk} \} \):

\[
\Psi_j = \sum_{j',k} \psi_{j'k} \Gamma_{j'k}
\]

(32)

where the coefficients \( \psi_{j'k} \) are the components of \( \Psi_j \) relative to the dyads of the basis. According to (29), considered for \( k = j \), one has:

\[
\Psi_j \Psi_j^\dagger = \Gamma_{jj} = I_j .
\]

(33)

By substituting the (32) in the (33) and remembering the uniqueness of the decomposition of a pseudo-observable, one has that in the (32) the only non-null coefficients are those for which it results \( j' = j \), that is:

\[
\Psi_j = \sum_k \psi_{jk} \Gamma_{jk} .
\]

(34)

According to the (34), the (33) and to the properties of the dyads, besides, one has:

\[
\Psi_j \Psi_j^\dagger \Psi_j = I_j \Psi_j = \Psi_j .
\]

(35)

Consider, now, the following expression:

\[
J_j := \Psi_j^\dagger \Psi_j .
\]

(36)

According to the definition of state vector, to the (34) and to the (35), it results:

\[
J_j J_j^\dagger = \Psi_j^\dagger \left( \Psi_j \Psi_j^\dagger \right) \Psi_j = \Psi_j^\dagger I_j \Psi_j = \Psi_j^\dagger \Psi_j = J_j
\]

which implies, according to the equation (15) in the first paper, that \( J_j \) is a projector. Since, on the basis of the third property of the trace and of the characterization of the elementary projectors, given at the end of subsection 2.2, it besides results:

\[
\text{tr} \left( J_j \right) = \text{tr} \left( \Psi_j^\dagger \Psi_j \right) = \text{tr} \left( \Psi_j \Psi_j^\dagger \right) = \text{tr} \left( I_j \right) = 1 ,
\]

the projector \( J_j \) is also elementary. Furthermore it is:

\[
\Psi_j J_j = \Psi_j \Psi_j^\dagger \Psi_j = I_j \Psi_j = \Psi_j
\]

(37)

which follows immediately by the definition (34) and by the (35).

§ By fixing a whatever bra \( \langle \upsilon | \) normalized to one, the vector state \( \Psi_j \), relative to state of index \( j \), thus corresponds to the operator \( | j \rangle \langle \upsilon | \) in the Dirac formulation Dirac, where also the ket \( | j \rangle \) is normalized to one. It is important to point out that is not possible to individually define the state vectors, since the unitary pseudo-observable \( K \) which appears in their definition is the same for all of them.
Consider now a complete set of compatible observables containing $J_j$. This set identifies a certain basis $\{\tilde{I}_k\}$ of elementary projectors, containing, on the basis of the procedure of construction seen in subsection 2.7 of the first paper, $J_j$. By arbitrarily choosing an index $k_0$, with a suitable reordering of the basis elements, you can make sure that $\tilde{I}_{k_0}$ is the basis element of index $k_0$:

$$\tilde{I}_{k_0} := J_j .$$  \hspace{1cm} (38)

By indicating with $\{\tilde{\Gamma}_{jk}\}$ the dyad basis associated to $\{\tilde{I}_k\}$, let $\Omega_j$ be the pseudo-observable of the change from the basis $\{\tilde{\Gamma}_{jk}\}$ to the basis $\{\Gamma_{jk}\}$. So it results:

$$J_j = \tilde{I}_{k_0} = \Omega_j \tilde{I}_{k_0} \Omega_j^\dagger .$$  \hspace{1cm} (39)

According to such relation, to the (35) and to the (37), therefore one has:

$$\Psi_j = I_j \Psi_j J_j = (I_j \Psi_j \Omega_j \tilde{I}_{k_0}) \Omega_j^\dagger = \tilde{\psi}_j \Gamma_{j k_0} \Omega_j^\dagger .$$  \hspace{1cm} (40)

having substituted the expression among brackets according to the decomposition (50) in the first paper, where $\tilde{\psi}_j$ is a suitable complex coefficient (the dyadic component). By imposing the validity of the (33), besides one finds:

$$\left|\tilde{\psi}_j\right|^2 = 1 .$$  \hspace{1cm} (41)

Chosen a whatever index $k$, consider, now, the following expression:

$$J_j J_k J_j = \Psi_j^\dagger \Psi_j \Psi_k^\dagger \Psi_k \Psi_j^\dagger \Psi_j = \Psi_j^\dagger \Gamma_{jk} \Gamma_{kj} \Psi_j = \Psi_j^\dagger I_j \Psi_j = \Psi_j^\dagger \Psi_j = J_j$$

having made use of the (29), of the third property of the dyads and of the (35). According to the lemma proved in Appendix A for each pair of indexes $j$ and $k$, $J_j = J_k$, i.e. all the projectors $J_j$ are equals one each other. We can therefore assume that also the bases of which they are part coincide one with each other, so that the pseudo-observable of the change of the dyad basis too is the same for all of the values of the state index $j$:

$$\Omega_j = \Omega .$$  \hspace{1cm} (42)

By substitution of such relation into the (40) and imposing the validity of the (29) for each pair of indexes $j$ and $k$, according to the (41), one finds also that the dyadic components $\tilde{\psi}_j$ for all the states are equal to a constant phase factor:

$$\tilde{\psi}_j = e^{i\theta} .$$  \hspace{1cm} (43)

By substituting the (42) and the (43) into the (40), one lastly finds:

$$\Psi_j = \Gamma_{j k_0} (e^{i\theta} \Omega^\dagger)$$

That coincides with the (50) as soon as you put $K := e^{i\theta} \Omega^\dagger$, that is immediately verified to be an unitary pseudo-observable.

State vectors verifying the (29) for equivalent dyad bases, belonging to different sets, corresponds one with each other according an equivalence relation, as it is easy to verify, and are therefore called equivalent.

If $\Psi_j$ and $\tilde{\Psi}_j$ are two equivalent state vectors, according to what has been demonstrated and to the (58) in the first paper, it will be: $\Psi_j = \Gamma_{j k_0} K$ and $\tilde{\Psi}_j = e^{i\theta_j} \Gamma_{j k_0} \tilde{K}$, where $K$ and $\tilde{K}$ are two
unitary pseudo-observables and \( \vartheta_j \) are arbitrary phase factors. So, by the properties of the unitary pseudo-observables, one obtains:

\[
\tilde{\Psi}_j = e^{i \vartheta_j} \Gamma_{jk_0} \tilde{K} = e^{i \vartheta_j} \Gamma_{jk_0} \tilde{K} = e^{i \vartheta_j} \Psi_j (K^\dagger \tilde{K}) = e^{i \vartheta_j} \Psi_j Y
\]  

being \( Y := K^\dagger \tilde{K} \) a suitable unitary pseudo-observable. One can easily realize, by making use of the properties of the unitary pseudo-observables, that by multiplying the elements of a set of state vectors for arbitrary phase factors and, to the left, for a same unitary pseudo-observable, it is obtained a set of state vectors equivalent to the previous ones.

2.6. The eigenstate space

Let’s now look at some other important properties of the state vectors.

(i) The state vectors are right eigenvectors for the observables of the complete space of compatible observables having the basis \( \{ I_j \} \) of elementary projectors. If, in fact, \( O \) is an observable of this space, according to (30), one has:

\[
O \Psi_j = O \Gamma_{jk_0} K = o_j \Gamma_{jk_0} = o_j \Psi_j .
\]  

For this reason, these pseudo-observables will be called also eigenstates of the observable \( O \).

(ii) The state vectors form an orthonormal set. In fact, it results:

\[
\langle \Psi_j | \Psi_{j'} \rangle = \text{tr} (\Psi_j^\dagger \Psi_{j'}) = \text{tr} (\Psi_j \Psi_{j'}) = \text{tr} (\Gamma_{j'j}) = \delta_{j'j}
\]  

as follows from the (29) and the (21).

(iii) By multiplying a state vector to the left for a whatever pseudo-observable \( P \), one obtains a linear combination of state vectors belonging to the same set. Indicated with \( \varpi_{j'k'} \) the dyadic component of \( P \) relative to the generic dyad \( \Gamma_{j'k'} \), according to the equation (48) in the first paper and to the (30), in fact, one has:

\[
P \Psi_j = \sum_{j', k'} \varpi_{j'k'} \Gamma_{j'k'} \Gamma_{jk_0} K = \sum_{j'} \varpi_{j'j} \Gamma_{j'k_0} K = \sum_{j'} \varpi_{j'j} \Psi_{j'}
\]  

having exploited the third property of the dyads.

According to the first of the above demonstrated properties, the state vectors of a given set are also simultaneous eigenstates of every set of observable of the complete space of compatible observables spanned by the basis \( \{ I_j \} \) of elementary projectors. In particular, they are simultaneous eigenstates of a complete set of observables of the space. If we put these observables in a tuple \( O \), after having indicated with \( o_j \) the \( n \)-tuple of eigenvalues of the observables of \( O \) relative to the state index \( j \), it is:

\[
O \Psi_j = o_j \Psi_j
\]  

where the product of a \( n \)-tuple of observables by an observable, in the left-hand side, is meant as the \( n \)-tuple having as components the products of the various observables composing the \( n \)-tuple itself by the observable by which the \( n \)-tuple is multiplied. Note that to each \( n \)-tuple \( o_j \) of eigenvalues corresponds, to the except of an equivalence i.e., in \( \Psi \), of the product by a phase factor, an only eigenstate \( \Psi_j \) of the set and vice versa, so that the correspondence is biunivocal, apart of an equivalence. On the other side, if one considers a set \( \{ \Psi_j \} \) of state vectors, this can be always thought as formed by simultaneous eigenstates of a complete set of the space spanned by the
basis \{I_j\} of elementary projectors and therefore each state vector of the set will be in a biunivocal correspondence, apart from an equivalence, with the \(n\)-tuple of the observables of the complete set of the space spanned by \{I_j\}. For this reason, from now on, we will call \{\Psi_j\} an \textit{eigenstate set}.

It is worth observing that the first property is generalizable to the case of a whatever complex function \(\phi(\mathbf{O})\) of the tuple \(\mathbf{O}\) of a complete set of observables. In such a case, however, the eigenvalues \(\phi(o_j)\) will be complex.

According to the properties of the state vectors, we can besides therefore state that the space \(\mathfrak{U} := \text{span}\{\{\Psi_j\}\}\) of the linear combinations of the state vectors of a given set is \textit{closed} under the left multiplication by a pseudo-observable of the space \(\mathfrak{P}\) and admits as an orthonormal basis the eigenstate set \{\Psi_j\}. The space \(\mathfrak{U}\) will be so called the \textit{eigenstate space}.

We now demonstrate the important \textit{eigenstate space characterization theorem}: each eigenstate set is equivalent to an orthonormal basis of \(\mathfrak{U}\); each orthonormal basis of \(\mathfrak{U}\) is an eigenstate set.

In order to prove the first part of the theorem, let \(\{\tilde{\Psi}_j\}\) be an eigenstate set and \(\{\tilde{\Gamma}_{jk}\}\) the corresponding dyad basis. By indicating with \(\Omega\) the unitary pseudo-observable of the change from the basis \(\{\tilde{\Gamma}_{jk}\}\) to the basis \(\{\Gamma_{jk}\}\), according to the equation (64) in the first paper and to the (29), one has:

\[
\tilde{\Gamma}_{jk} = \Omega \Gamma_{jk} \Omega^\dagger = \Omega \Psi_j \Psi_k^\dagger \Omega^\dagger = (\Omega \Psi_j) (\Omega \Psi_k)^\dagger = \tilde{\Psi}_j \tilde{\Psi}_k^\dagger
\]

where it was put:

\[
\tilde{\Psi}_j := \Omega \Psi_j
\]

which is an element of \(\text{span}\{\{\Psi_k\}\}\) by virtue of the third property of the state vectors. The set \(\{\tilde{\Psi}_j\}\) of pseudo-observables is then, by definition, equation (29), an eigenstate set corresponding to the dyad basis \(\{\tilde{\Gamma}_{jk}\}\) and therefore equivalent to every other eigenstate set relative to the same dyad basis. Since \(\Omega\) is invertible, as unitary, the eigenstates \(\tilde{\Psi}_j\) span the space \(\mathfrak{U}\) and since they form an orthonormal set, according to the second property of the state vectors, and are therefore linearly independent, they form an orthonormal basis of \(\mathfrak{U}\). It is worth noting that the equation (49) may, therefore, be interpreted as expressing the change from the eigenstate basis \(\{\tilde{\Psi}_j\}\) to the eigenstate basis \(\{\Psi_j\}\).

For what concerns the demonstration of the second part of the theorem, we firstly remember the following important property: if \(\Phi\) is an element of the eigenstate space \(\mathfrak{U} = \text{span}\{\{\Psi_j\}\}\), then it results:

\[
\Phi = \sum_j \langle \Psi_j | \Phi \rangle \Psi_j .
\]

This being premised, let \(\{\Phi_j\}\) an orthonormal basis of \(\mathfrak{V} = \text{span}\{\{\Psi_j\}\}\). According to the (50), for each index \(j\), one has:

\[
\Phi_j = \sum_k \varphi_{kj} \Psi_k \quad \text{with} \quad \varphi_{kj} := \langle \Psi_k | \Phi_j \rangle .
\]

But it is also:

\[
\Psi_k = \sum_l \langle \Phi_l | \Psi_k \rangle \Phi_l = \sum_l \varphi_{kl}^* \Phi_l
\]
where it was made use of the third property of the inner product. By substitution of the (52) into the (51) and vice versa and exploiting the linear independence of the elements of the basis, one obtains:

$$\sum_k \varphi_{kj}^* \varphi_{kl} = \sum_k \varphi_{jk} \varphi_{lk}^* = \delta_{j,l}$$

(53)

where, as usual, $\delta_{j,l}$ is the Kronecker symbol. Observe, now, that, according to the (51) and the (29), it results:

$$\Phi_j \Phi_{j'}^\dagger = \sum_{k,l} \varphi_{kj} \varphi_{lj'}^* \Gamma_{kl} = \Omega \Gamma_{jj'} \Omega^\dagger$$

(54)

having put:

$$\Omega := \sum_{k,k'} \varphi_{kk'} \Gamma_{kk'}$$

(55)

which, by virtue of the (53) and of the relation (67) in the first paper, results to be unitary. The (54) may therefore interpreted as expressing the change from a certain dyad basis $\{\tilde{\Gamma}_{jj'}\}$ to the dyad basis $\{\Gamma_{jj'}\}$, with:

$$\Phi_j \Phi_{j'}^\dagger = \tilde{\Gamma}_{jj'}$$

By definition, the elements of the orthonormal basis $\{\Phi_j\}$ so form an eigenstate set, that is quod erat demonstrandum.

An important consequence of what above demonstrated is the state superposition theorem: each non-null element $\Phi$, with finite norm, of the space $\mathcal{V}$ is proportional to an eigenstate. This theorem is equivalent to the fundamental superposition principle in the Dirac formulation of quantum mechanics. It is, in fact, always possible, for instance by following the well-known Gram-Schmidt process, to build an orthonormal basis of $\mathcal{V}$ whose first element, $\tilde{\Phi}_1$, is such that it results:

$$\Phi = \|\Phi\| \tilde{\Phi}_1$$

The assert follows, therefore, as an immediate consequence of the second part of the eigenstate space characterization theorem.

According to what above demonstrated, one may conclude that the eigenstate space $\mathcal{V}$ is spanned by an orthonormal set of right eigenvectors of every observable $O$ (second spectral theorem).

In any case, as it is well known, the inner product of two right eigenvectors of an observable relative to different eigenvalues is always equal to zero (the eigenvectors are orthogonal to each other). In fact, if $\Phi_1$ and $\Phi_2$ are two right eigenvectors of an observable $O$ respectively relative to the eigenvalues $o_1$ and $o_2$, one has $O\Phi_1 = o_1 \Phi_1$ and $O\Phi_2 = o_2 \Phi_2$. According to the hypotheses assumed and to the properties of the inner product, it is: $\langle \Phi_1 \mid O\Phi_2 \rangle = o_2 \langle \Phi_1, \Phi_2 \rangle$, whereas, by virtue of the (26), it results: $\langle \Phi_1 \mid O\Phi_2 \rangle = \langle O\Phi_1 \mid \Phi_2 \rangle = o_1 \langle \Phi_1, \Phi_2 \rangle$, that, being by hypothesis $o_1 \neq o_2$, imply: $\langle \Phi_1, \Phi_2 \rangle = 0$.

The state vectors introduced in this section corresponds, apart from an equivalence relation, to the analogue ones in the Dirac formulation of quantum mechanics. A remarkable difference is, however, that while in the Dirac formulation of quantum mechanics the state vectors are introduced axiomatically and not without problems for what regards the physical interpretation, especially for what concern the superposition principle, in our framework the eigenstates are merely auxiliary
pseudo-observables that synthesize the statistical properties of the physical system with respect to a given complete set of compatible observables and that allow to simplify the equations describing the relationships and the transformation laws of the observables. Particular attention should also be paid to the circumstance, already highlighted in the note 3 on page 10 that the eigenstates cannot be defined individually, but only as a whole relative to all the possible states of the physical system. In this context, the simplifications in the formalism introduced through the eigenstates are paid in terms of the interpretability of the results, being the link between eigenstates and observation rather indirect.

3. Expectation values and wave functions

3.1. Expectation value of an observable

By performing repeated measurements of an observable relative to a given system in a certain state, you will usually obtain a set of different outcomes. Therefore, it is necessary to sum up such measures into a single estimate: the expectation value \( \langle O \rangle \) of the observable \( O \). The result of the measurement process of a observable is precisely given by this estimate.

The expectation value must satisfy the following fundamental properties:

(i) For two given observables \( O_1 \) and \( O_2 \), the expectation value of their sum is equal to the sum of their expectation values:
\[
\langle O_1 + O_2 \rangle = \langle O_1 \rangle + \langle O_2 \rangle.
\]

(ii) Given an observable \( O \) and a real constant \( c \), the expectation value of the product of the constant by the observable is equal to the product of the constant by the expectation value of the observable:
\[
\langle cO \rangle = c \langle O \rangle.
\]

(iii) For two given comparable observables \( O_1 \) and \( O_2 \), with \( O_1 \leq O_2 \), the expectation values follow the same order of the two observables, so one has:
\[
\langle O_1 \rangle \leq \langle O_2 \rangle.
\]

(iv) If \( c \) is a constant observable, its expectation value coincides with the value of the constant itself:
\[
\langle c \rangle = c.
\]

The first two properties express the linearity of the expectation value.

If one considers an observable \( O \), belonging to a complete space of compatible observables that admits the basis \( \{ I_j \} \) of elementary projectors, and decomposes it according such a basis in the form of equation (9) of the first paper, with the spectral coefficients \( o_j \), the linearity of expectation value implies that it results:

\[
\langle O \rangle = \sum_j o_j p_j \tag{56}
\]

where:

\[
p_j := \langle I_j \rangle \ . \tag{57}
\]

By virtue of the third and the fourth properties of the expectation value and remembering the relation (1) seen in the first paper, one also has:

\[
0 \leq p_j \leq 1 \ . \tag{58}
\]
The linearity of expectation value and the closure relation (7) in the first paper, besides, imply that it results:

\[ \sum_j p_j = 1. \]  \hspace{1cm} (59)

In force of the relations (58) and (59), each coefficient \( p_j \) therefore represents the probability of the elementary event associated to the projector \( I_j \).

The probability distribution \( \{p_j\} \) depends on the system state and on the complete set of compatible observables one is considering. But, more subtly, it depends also on a personal choice of the observer. In fact, as de Finetti stated: “PROBABILITY DOES NOT EXIST”, in the sense of “objective” probability, “only subjective probabilities exist – that is, the degree of belief in the occurrence of an event attributed by a given person at a given instant and with a given set of information” \[20\].

Chosen a certain complete set of compatible observables, the state of the system is therefore perfectly identified only after the assignment of a such probability distribution. In this sense, we can agree also with Fuchs’ quote: “QUANTUM STATES DO NOT EXIST” \[21\].

The states for which all but one of the probabilities are zero, that is for which it holds: \( p_j = \delta_{j,j'} \), are pure states. The states for which this instead is not true, are mixed states. It should be pointed out that whereas a mixed state describes the situation of a physical system relative to the possible measurement of a observer belonging to a given complete space of compatible observables, a pure state describes the situation of a physical system after the measurement has occurred and is, therefore, determined both by the system and by the measurement process.

We will now demonstrate the following important property: if the expectation value of an observable is zero for every state of a physical system, then the observable is equal to \( 0 \). Let \( O \) be an observable whose expectation value is zero for every state of the physical system. As a consequence of the (56), it will therefore be:

\[ \sum_j o_j p_j = 0 \]  \hspace{1cm} (60)

for any possible choice of the probabilities \( p_j \), under the constrains given by (58) and by (59). Fixed a whatever index \( j' \), let’s consider the pure state for which it results: \( p_j = \delta_{j,j'} \). From (60), it therefore follows: \( o_{j'} = 0 \). By the arbitrariness of the index \( j' \), it follows that all possible outcomes of \( O \) are equal to zero and therefore, for the equivalence criterion between observables, that \( O \) is the null observable.

As a corollary of this property, one has that if the expectation values of two observables coincide for every state of a physical system, then the two observables are equivalent. In order to demonstrate this statement, it suffices to consider the difference between the two observable and to note that, by virtue of the linearity of the expectation value and of the property just demonstrated, it have necessarily to be the null observable.

It is possible to characterize the state of a physical system, relative to the observation of a given complete space of compatible observables having the basis \( \{I_j\} \) of elementary indicators, with obvious reference to the density matrix \[22\] and the density operator \[23\], through the density observable:

\[ D := \sum_j p_j I_j \]  \hspace{1cm} (61)

where the probabilities \( p_j \) are given by (57). The density observable is that observable whose spectrum is formed by the probabilities associated to the projectors of the basis. Let now \( O \) be a
generic observable of the considered space. By making use of the equations: (9), (2) and (6) of the first paper, one obtains the following identity:

\[ OD = \sum_j o_j p_j I_j . \]  

(62)

Exploiting the definition of the trace functional \( \langle \rangle \) and the equations (62) and (56), one has, for whatever observable \( O \) belonging to a given complete space of compatible observables:

\[ \text{tr} (OD) = \sum_j o_j p_j = \langle O \rangle . \]  

(63)

Equation (63) shows that in order to calculate the expectation values it is actually necessary the only knowledge of the density observable.

3.2. Expectation values of pseudo-observables

We will now extend the concept of expectation value to the pseudo-observables too.

Let’s suppose having characterized the state of a physical system through a maximum observation, i.e. the measurement of a complete set of compatible observables. These observables identify a certain complete space \( \mathfrak{A} \) of compatible observables with a basis \( \{ I_{\mathfrak{A},j} \} \) of elementary indicators. As outlined in subsection 3.1, such a state is characterized by of the density observable:

\[ D_{\mathfrak{A}} = \sum_j p_{\mathfrak{A},j} I_{\mathfrak{A},j} \]  

(64)

where the probabilities \( p_{\mathfrak{A},j} \) are equal to the expectation values of the basis projectors:

\[ p_{\mathfrak{A},j} = \langle I_{\mathfrak{A},j} \rangle . \]  

(65)

If now one measures an observable \( B \) incompatible with at least one of the observables of \( \mathfrak{A} \), the information obtained in previous measurements must affect the expectation value of \( B \), that must, in some ways, be brought back to the complete set of information acquired in previous measurements. The expectation value of \( B \) will be, therefore, given by the expectation value of an its suitable «projection» on the space \( \mathfrak{A} \). To this end, after indicating with \( B_{\mathfrak{A},j} \) the projection of the observable \( B \) according the projector \( I_{\mathfrak{A},j} \) (see subsection 4.1 in the first paper), given by:

\[ B_{\mathfrak{A},j} := I_{\mathfrak{A},j} B I_{\mathfrak{A},j} = b_{\mathfrak{A},j} I_{\mathfrak{A},j} , \]  

(66)

where \( b_{\mathfrak{A},j} \) is a suitable constant, it is natural to define as the projection \( B_{\mathfrak{A}} \) of an observable \( B \) on a complete space \( \mathfrak{A} \) of compatible observables, the observable given by:

\[ B_{\mathfrak{A}} := \sum_j B_{\mathfrak{A},j} . \]  

(67)

The projection is an observable compatible with those of the space \( \mathfrak{A} \), whose expectation value gives those of the observable \( B \) conditioned by the previous maximum observation relative to the space \( \mathfrak{A} \). By indicating with \( \langle B \rangle_{\mathfrak{A}} \) such a conditioned expectation value and remembering the (63), on the basis of (64), (65), (66) and (67), therefore we will set:

\[ \langle B \rangle_{\mathfrak{A}} := \langle B_{\mathfrak{A}} \rangle = \sum_j b_{\mathfrak{A},j} p_j = \text{tr} (D_{\mathfrak{A}} B_{\mathfrak{A}}) . \]  

(68)

The calculation of the expectation value of \( B \) can, however, be led also in the following way. After having indicated with \( \mathfrak{B} \) a complete space of compatible observables, spanned by the basis
\{I_{B,k}\} of elementary projectors \{I_{B,k}\}, generated from the observable \(B\) and all the observables of \(A\) compatible with \(B\), let’s consider the projection \(D_{A,B}\) of the density observable \(D_A\) on the space \(B\):

\[
D_{A,B} = \sum_k I_{B,k} D_A I_{B,k} = \sum_{j,k} p_{A,j} I_{B,k} I_{A,j} I_{B,k}
\]

whose spectrum, for what seen above, is formed by the probabilities of getting the events associated with the projectors \(I_{B,k}\) conditioned by a previous maximum observation relative to the space \(A\). The observable \(D_{A,B}\) represents the new density describing the statistical properties of the physical system after the measuring of \(B\).

In order to prove this, we observe that, according the properties of the projection of an observable, one must have:

\[
I_{B,k} I_{A,j} I_{B,k} = p_{B,j,k} I_{B,k}
\]

where, again, \(p_{B,j,k}\) is a suitable constant dependent on both indexes \(j\) and \(k\). By making use of the definition and the properties of the trace functional, of the idempotency relation (2) in the first paper and of the positive-definiteness of the inner product, one has:

\[
p_{B,j,k} = \text{tr} (I_{B,k} I_{A,j} I_{B,k}) = \langle I_{A,j} I_{B,k} | I_{A,j} I_{B,k} \rangle \geq 0.
\]

By virtue of the well-known Cauchy-Schwarz inequality, it also results:

\[
p_{B,j,k} = \text{tr} (I_{B,k} I_{A,j} I_{B,k}) = \text{tr} (I_{A,j} (I_{A,j} I_{B,k})) = \text{tr} (I_{A,j} I_{B,k}) = \text{tr} (I_{A,j} I_{B,k}) = |\langle I_{A,j} | I_{B,k} \rangle| \leq \|I_{A,j}\| \|I_{B,k}\| = 1.
\]

Besides, by summation on the index \(k\), one obtains:

\[
\sum_k p_{B,j,k} = \sum_k \text{tr} (I_{A,j} I_{B,k}) = \text{tr} \left( I_{A,j} \sum_k I_{B,k} \right) = \text{tr} \left( I_{A,j} \right) = 1.
\]

The coefficients:

\[
p_{A,j,k} = \text{tr} (I_{A,j} I_{B,k}) = \langle I_{A,j} | I_{B,k} \rangle
\]

may be, therefore, interpreted as probabilities. They are the well-known transition probabilities from pure states in the initial space to pure states in the second one. If we now consider the the projection \(D_{A,B}\), we have:

\[
D_{A,B} = \sum_{j,k} p_{A,j} p_{B,j,k} I_{B,k} = \sum_k p_{A,B,k} I_{B,k}
\]

where the coefficients:

\[
p_{A,B,k} := \sum_j p_{A,j} p_{B,j,k}
\]

satisfy again, as it easily verified, all the properties of a probability distribution. The projection \(D_{A,B}\) has, therefore, indeed the structure of a density observable.

Based on this observation and on (63), the expectation value of \(B\) will be also given by:

\[
\langle B \rangle_A = \text{tr} (D_{A,B} B).
\]

Since the expectation value of the observable \(B\) must be a well-defined quantity, for consistency it must be:

\[
\text{tr} (D_A B_A) = \text{tr} (D_{A,B} B)
\]
independently upon the choice of the complete space containing $B$.

To prove the validity of (72), we observe that, if $b_k$ is the spectral coefficient of $B$ relative to the projector $I_{B,k}$, one has:

$$\text{tr} \left( D_A B A \right) = \sum_{j,k} p_{A,j} b_k \text{tr} \left( I_{A,j} I_{B,k} I_{A,j} \right)$$

and

$$\text{tr} \left( D_A B A \right) = \sum_{j,k} p_{A,j} b_k \text{tr} \left( I_{B,k} I_{A,j} I_{B,k} \right).$$

But, according to the third hypothesis on the trace, it is obtained:

$$\text{tr} \left( I_{A,j} I_{B,k} I_{A,j} \right) = \text{tr} \left( I_{A,j} I_{B,k} I_{A,j} \right) = \text{tr} \left( (I_{B,k} I_{A,j}) I_{A,j} \right) = \text{tr} \left( I_{B,k} I_{A,j} \right)$$

and

$$\text{tr} \left( I_{B,k} I_{A,j} I_{B,k} \right) = \text{tr} \left( (I_{B,k} I_{A,j}) I_{B,k} \right) = \text{tr} \left( (I_{B,k} I_{A,j}) I_{B,k} \right) = \text{tr} \left( I_{B,k} I_{A,j} \right)$$

that, substituted in (73) and in (74), prove the relation (72), demonstrating the physical consistency of the hypotheses taken on the trace of a pseudo-observable.

By these relations it also follows that:

$$\langle B \rangle_A = \text{tr} \left( D_A B \right).$$

Based on (75), it is therefore natural to associate to a pseudo-observable $Z$, for a given state of a certain physical system described by the density observable $D$, as its expectation value $\langle Z \rangle$, the complex number:

$$\langle Z \rangle := \text{tr} \left( D Z \right) = \langle D \mid Z \rangle.$$  

According to such definition and to the properties of the trace, it’s easy to prove the following properties of the expectation value of a pseudo-observable, generalizing those presented in subsection 3.1:

(i) For two given pseudo-observables $Z_1$ and $Z$, the expectation value of their sum is equal to the sum of their expectation values:

$$\langle Z_1 + Z_2 \rangle = \langle Z_1 \rangle + \langle Z_2 \rangle.$$

(ii) Given a pseudo-observable $Z$ and a complex constant $\gamma$, the expectation value of their product is equal to the product of the value of the complex constant by the expectation value of the pseudo-observable:

$$\langle \gamma Z \rangle = \gamma \langle Z \rangle.$$

(iii) For two given comparable observables $O_1$ and $O_2$, with $O_1 \leq O_2$, the expectation values follow the same order of the two observables, so one has:

$$\langle O_1 \rangle \leq \langle O_2 \rangle.$$

(iv) The expectation value of the transposition of a pseudo-observable $P$ is equal to the complex conjugate of the expectation value of the pseudo-observable:

$$\langle Z^\dagger \rangle = \langle Z \rangle^*.$$

(v) The expectation value of constant $\gamma = \alpha + i\beta$ is equal to the value of the constant itself:

if $\gamma = \alpha + i\beta$ then $\langle \gamma \rangle = \alpha + i\beta$. 
What proved in this section is coherent with the well-known Gleason’s theorem [24], of which what proved above plays an analogous role in the framework of the algebra of the pseudo-observables, even if the starting assumptions are different.

It is now important to clarify some points that can cause a misleading comprehension about the role of the density observable. Effectively, this observable does not correspond to an actually measured physical quantity, but it is, instead, built up to incorporate every statistical property of the physical system relative to a given choice of a complete set of compatible observables. What happens then if, subsequently to a certain maximum observation relative to a certain complete set of compatible observables, one does a measurement of an observable incompatible with some observable of the set? You are allowed to calculate the expectation value of the new observable making use of the (68), but the measurement deeply alters the starting situation. In fact, one switches from a description in terms of the density observable relative to the starting complete set of compatible observables, to another in which the density observable is relative to a new complete set of compatible observables that includes the last measured one and excludes those which are not compatible with it. This change does not, however, correspond to any physical process: the density observable is, in fact, a summary of the state of the system, which incorporates all the statistical information available to the observer. Such information refers to a description of reality. But this description, as anticipated in subsection 2.1 of the first paper, represents the outcome of a posteriori process, based on the fundamental requirement of consistency. When to the information acquired through the maximum observation it is added what obtained by the measurement of an observable incompatible with some of the observable of the starting complete set, the picture is no longer coherent and the new information deletes the older one incompatible with it.

It must be, however, clear that the observation does alter the physical state of the observed system, due to the unavoidable physical interaction between the measurement probe and the system. Unlike the change in the density observable, this is, instead, a physical process that can give rise to physical effects. This point will be deepen in the third paper.

3.3. Deviations and uncertainty relations

The difference between an observable $A$ and its expectation value is given by its deviation $\Delta A$:

$$\Delta A := A - \langle A \rangle.$$  \hfill (77)

The deviation is an observable also, whose expectation value is equal to zero:

$$\langle \Delta A \rangle = \langle A - \langle A \rangle \rangle = \langle A \rangle - \langle A \rangle = 0.$$  \hfill (78)

The amount of variation or dispersion of an observable $A$ from its expected value is usually quantified by introducing firstly the variance $\sigma_A^2$, defined as the expected value of the square of its deviation:

$$\sigma_A^2 := \langle (\Delta A)^2 \rangle$$  \hfill (79)

and then the standard deviation $\sigma_A$:

$$\sigma_A := \sqrt{\sigma_A^2}.$$  \hfill (80)

A limitation above the values assumed for the standard deviations of two incompatible observables is fixed by the well-known Heisenberg uncertainty relations. In order to derive in the
new context these relations, one can follows a procedure analogous to that used by Born in the Appendix XXVI of his famous book\[25].

Let $A$ and $B$ be two pseudo-observables. For each value of a parameter $\xi$, of a suitable physical dimension, consider the following pseudo-observable:

$$Z = \xi \Delta A + i \Delta B \quad (81)$$

According to the fifth property of transposition (see first paper), one has:

$$ZZ^\dagger = (\xi \Delta A + i \Delta B) (\xi \Delta A - i \Delta B) = \xi^2 (\Delta A)^2 - i \xi [\Delta A, \Delta B] + (\Delta B)^2 \geq 0. \quad (82)$$

According to the third property of the expectation value and by the variance definition, one, then, obtains:

$$\sigma^2_A \xi^2 - \langle i [A, B] \rangle \xi + \sigma^2_B \geq 0 \quad (83)$$

where the following immediate identity has also been exploited:

$$[\Delta A, \Delta B] = [A, B]. \quad (84)$$

Since the (83) must hold for every value of the parameter $\xi$, the left-hand side second degree trinomial cannot have a positive discriminant. It, therefore, must be:

$$|\langle i [A, B] \rangle|^2 - 4 \sigma^2_A \sigma^2_B \leq 0$$

by which the Heisenberg uncertainty relation is obtained:

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle i [A, B] \rangle| \quad (85)$$

The uncertainty relations states the impossibility of simultaneously obtaining exact values from the measurement of two observable mutually incompatible and thus represents a further aspect of the incompatibility itself.

### 3.4. Matrix elements and wave functions

The introduction of an eigenstate space $\mathfrak{H} = \text{span} \{\{\Psi_j\}\}$ allows one to write down some convenient expression for the dyadic components of a pseudo-observable $P$. In fact, by making use of the (28) and the (29), with reference to the decomposition (48) in the first paper:

$$P = \sum_{j,k} \varpi_{jk} \Gamma_{jk}$$

one has:

$$\varpi_{jk} = \langle \Gamma_{jk} \mid P \rangle = \text{tr} \left( \Psi_k \Psi_j^\dagger P \right) = \text{tr} \left( \Psi_j^\dagger P \Psi_k \right) = \langle \Psi_j \mid P \Psi_k \rangle \quad (86)$$

by which, in particular, one obtains:

$$\langle P \rangle_j := \varpi_{jj} = \langle \Psi_j \mid P \Psi_j \rangle \quad (87)$$

that may be interpreted as the expectation value of $P$ in the pure state of index $j$.

More generally, we can write down a new expression for the expectation value of a whatever observable $O$. Observe, first of all, that each probability $p_j$, defined by equation (57), appearing in the expression of the expectation value of $O$, may be interpreted as the probability of occurrence...
of the state \( j \) after the measurement. By exploiting the linearity of the inner product, according to the equation (76) and to the (86), one has:

\[
\langle O \rangle = \langle D \mid O \rangle = \sum_j p_j \langle I_j \mid O \rangle = \sum_j p_j \langle \Psi_j \mid O \Psi_j \rangle
\] (88)

that allows one to express the expectation value of an observable in terms of inner products of state vectors.

By supposing to having identified the elements of an eigenstate basis as simultaneous eigenvectors of a complete set of compatible observables placed in a \( n \)-tuple \( O \), on the basis of what exposed in the subsection 2.6, there will be a biunivocal correspondence, apart from an equivalence, between each basis element and a \( n \)-tuple \( o \) of eigenvalues of the observables in \( O \).

According to the second spectral theorem, each element \( \Phi \in \mathcal{V} = \text{span} \{ \Psi_o \} \) may, therefore, be decomposed in the form:

\[
\Phi = \sum_o \phi(o) \Psi_o
\] (89)

where \( \phi(o) = \langle \Psi_o \mid \Phi \rangle \), by virtue of 54, is a function of the \( n \)-tuple \( o \) of eigenvalues, that will be called, with obvious reference to the Schrödinger formalism, wave function of \( \Phi \) relative to the basis \( \{ \Psi_o \} \).

If \( \Phi_1 \) and \( \Phi_2 \) are two elements of \( \mathcal{V} \), whose wave functions are, respectively, \( \phi_1(o) \) and \( \phi_2(o) \), due to the orthonormality of the basis eigenstates and according to the properties of the inner product, one has:

\[
\langle \Phi_1 \mid \Phi_2 \rangle = \sum_o \phi_1^*(o) \phi_2(o) .
\] (90)

Equation (90) implies, in particular, that the norm of an element \( \Phi \in \mathcal{V} \) is given by:

\[
\| \Phi \| = \sqrt{\langle \Phi \mid \Phi \rangle} = \left( \sum_o |\phi(o)|^2 \right)^{1/2} .
\] (91)

According to the state superposition theorem, it can, therefore, be affirmed that any of the non-null elements of \( \mathcal{V} \) for whose wave function it’s finite the sum in the right-hand side of the 91 (square-summable function) is proportional to a suitable eigenstate of some observable. In particular, for an eigenstate \( \Phi \) whose wave function is \( \phi(o) \) it holds the normalization condition:

\[
\sum_o |\phi(o)|^2 = 1 .
\] (92)

Since each term in the summation in the left-hand side of 92 is non-negative, the condition is the closure relation typical of probabilities. It is easy to verify, by making use of the 29, the 70 and of the 89 that:

\[
p_o := |\phi(o)|^2 = |\langle \Phi \mid \Psi_o \rangle|^2
\] (93)

is just the transition probability from the pure state described by the eigenstate \( \Phi \) to that described by the eigenvector \( \Psi_o \), i.e. that for which the measurement of each observable in the tuple \( O \) has as outcome the corresponding value in the tuple \( o \).

Suppose now that the state of a physical system is initially described by the eigenstate \( \Phi \) and then is performed a measurement of an observable \( O_r \) belonging to the complete set of observables in the \( n \)-tuple \( O \). According to the 87, the expectation value of the measurement will be:

\[
\langle O_r \rangle = \langle \Phi \mid O_r \Phi \rangle = \sum_{o,o'} \phi^*(o) \phi(o') \langle \Psi_o \mid O_r \Psi_{o'} \rangle
\]
where it was made use of the \( \Psi_{o'} \). But since \( \Psi_{o'} \) is an eigenvector of the observable \( O_r \) and due to the orthonormality of the eigenstates, one has:

\[
\langle O_r \rangle = \sum_{o,o'} \phi^* (o) \phi (o') a_r \delta_{o,o'} = \sum_{o} |\phi (o)|^2 a_r .
\] (94)

By comparison of (94) with (56), one finally rediscovers the famous Born Rule [26], that gives the probability \( p(O_r = o) \) of obtaining the outcome \( o \) in the measurement of the observable \( O_r \) of a physical system initially described by the eigenstate \( \Phi \):

\[
p(O_r = o) = \sum_{o} |\phi (o)|^2 \delta_{o_r,o} .
\] (95)

4. Conclusions

4.1. A solution for the measurement problem

On the basis of what stated in this paper, we can now try to propose a solution for the measurement problem:

(i) **State vectors**, in the framework of the algebra of the pseudo-observables, **aren’t fundamentals entities**, but only the elements of an useful pseudo-observable set characterizing the pure state resulting by a previous observation of a complete set of compatible observables. They cannot be defined individually, but only as a whole relative to all the possible pure states of the physical system.

(ii) **The “collapse” of quantum states does not correspond to any physical process**. The density observable, in fact, does not correspond to an actually measured physical quantity, but it is, instead, a statistical summary of the state of the system, which incorporates all the information available to the observer. Such information refers to a description of reality. But this description represents the outcome of a posteriori process, based on the fundamental requirement of consistency. When to the information acquired through the maximum observation it is added what obtained by the measurement of an observable incompatible with some of the observable of the starting complete set, the picture is no longer coherent and **the new information deletes the older one incompatible with it**. On this question This last sentence may seem puzzling and requires a deeper discussion that will be presented in the next subsection.

(iii) **We agree with Fuchs’ quote: “QUANTUM STATES DO NOT EXIST”**. The quantum state of a physical system is **undefined** till the moment it is performed a measurement of some observables and it is **not completely defined** till the moment it is measured a complete set of compatible observable. However, even then, the density observable characterizing the quantum state depends on the assignment of a probability distribution, which can be done only on the basis of the degree of belief of the observer in the occurrence of the events associated to the elementary projectors of a basis, according to a given set of information.

(iv) **Quantum states are**, besides, **observer-relative**: different independent observers (i.e. not communicating each with other) may have different sets of information, deriving by different sets of observational outcomes. They will, therefore, assume different density observables, till to the extreme case in which for an observer the system is in a pure state whereas for the other is in a mixed one. The relational view of Rovelli [27] is so necessarily integrated in our vision.
(v) Whereas the “perceptions” of two independent observers may disagree, in the moment they communicate each with other, they acquire a new bunch of information and, consequently, they are forced to assume a new common description of the system. The exchange of information (communication) therefore causes a “collapse” not only of the quantum state of the physical system measured but also of the “cognitive” states of the two observers. This apparently paradoxical situation rises from the fact that each observer is a subject with respect to himself and an object with respect to the other.

(vi) The exchange of information between the two observers may, thus, be considered as a process of reciprocal observation, or, better, as a measurement performed by the meta-observer generated by the two observers of the physical system under study. Besides, since also the instrumental setups used by each of the observers to perform their measurements, according to the conclusive discussion made in the first paper, may also be regarded as “passive” observers, one may always think to measurement as a flow of information from the physical system to the observer, through the measurement instrument, giving rise to a logical higher order meta-observer.

(vii) Quantum states are, then, properties of the meta-observers generated by the information exchanges associated to the measurements. They are, therefore, “relational” entities.

(viii) Quantum states evolves through the time in a continuous manner, according the time evolution equations, because during this process no information exchange occurs and no new meta-observer arises or changes. We will return to this point in the next paper.

In order to clarify the last three points, we may think to the example of the detection of the two-slit interference pattern generated by single electrons, a remarkable experimental example of which is in [28]. For the electrons can generate the interference pattern, it is necessary that they have the same linear momentum, that must be therefore measured before each electron reaches the slits. The position of the electron, after it has traversed the slits, is then detected on a viewing screen. The point is that since moment and position measurements are incompatible one with each other, the initial meta-observer (experimenter+instrumentation for moment measurement) is different from the final one (experimenter+viewing screen), and so it will be also for relative quantum states!

What above outlined, in my opinion, remove every ambiguity and obscurity in the quantum process of measurement.

The picture, indeed, may be difficult to be accepted, but it is based on a deep analysis of the involved processes, it conforms to experimental evidence and it is a logical consequence of the hypotheses assumed to build the theory of measurement in the framework of the algebra of the pseudo-observables. So, I think, it is the only interpretation possible in this context.

One could object that quantum mechanics may be formulated in other different forms, but the construction of the algebra of the pseudo-observables is such to leave little space, if any, to nonequivalent alternatives. Moreover even the apparently equivalent ones, like that of Dirac-von Neumann, suffer for the introduction of unneeded extra concepts that cause ambiguities in the interpretation: the assumption of quantum states as the basis concept was, in my views, like putting the cart before the horse, since observations precede and not follow quantum states definition!

A remarkable final consideration is that Relational Quantum Mechanics [27, 29] and QBism [21, 30, 31] may both be considered as two compatible interpretations of the formulation of Quantum Mechanics in term of the algebra of the pseudo-observables outlined in this and my previous paper.
4.2. What is real?

In this closing subsection, we want to discuss about the deeper conceptual consequences of picture of the reality that results by our theory.

First of all, if it is assumed the validity of the new formulation and its embedded vision of the world, quantum states and, consequently, wave functions cannot be ontologically real, due to their relational nature. Therefore, de Broglie-Bohm theory, but also many world ones, based on the concept of an ontologically real universal wave function, are incompatibles with my vision. Moreover, since the quantum state “collapse” is a logical process and not a physical one, the theories of objective collapse would have also to be ruled out.

The final question we want to analyze is what picture of the reality emerges from our vision. The starting point has mandatory to be the so called EPR argument \[32\], conceived as an attack against the description of measurements according the Copenhagen interpretation and a criticism of the idea that Quantum Mechanics could be a complete description of reality. Instead of a detailed exposition of the argument, for which a convenient reference is made to the paper of Smerlak and Rovelli\[33\], here we want simply to show as the failure of the argument, experimentally demonstrated by Aspect et al.\[34, 35\], gives a new insight about reality.

The argument was based on the assumption of the three fundamental concepts of realism, locality and separability. According to Einstein, realism is meant as the assumption\[36\] that:

“There exists a physical reality independent of substantiation and perception”

The concept of separability is also well illustrated by Einstein’s words\[37\] :

“Without such an assumption of the mutually independent existence … of spatially distant things, as assumption which originates in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such a clean separation.”

The enunciation of the locality principle is, implicitly, reported in the EPR paper\[32\] :

“since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system.”

All of the three statements sound obvious to plain common sense, but, this, following the argumentation of the EPR paper, would imply that Quantum Mechanics is not a complete theory, i.e., according to the Einstein’s definition, that not every element in the physical reality has a counterpart in the theory.

John Bell, in 1966, showed that no complete local realistic theory can be compatible with Quantum Mechanics\[38\]. The result of Bell, however, left open the way to the existence of complete non-local hidden variable theories (such that of Bohm). But giving up locality is even worse than sacrificing realism. In fact non-locality implies that an event here may be instantly the cause of an effect in a distant point, and due to relativity, this would imply that for some observer the effect will precede the cause! So non-locality, at a fundamental level, destroys causality and so also Physics!

The Kochen-Specker theorem\[39\] gave what may considered a blow of grace to realism. In fact, they prove rigorously the impossibility of a value assignment to an observable before the measurement act, independently by the compatible observables measured with it.

The waiver to realism is a compelling issue. In a conversation with A. Pais, Einstein, to this regards, suddenly asked if he thought that the moon existed only when he looked at it\[40\].
The point is that the whole REALITY DOES NOT EXISTS, in the sense that it is only an emerging property, an a posteriori reconstruction, resulting by the exchange of information in a closed network of observer (that thus give rise to a single meta-observer). It is an hard truth to accept, but the logical and mathematical analyses and the experimental results points only in this direction!

Like in Plato’s Cave, we lived exchanging shadows for the truth... 

«Stat rosa pristina nomine, nomina nuda tenemus»
(The rose, which was, [now] exists only in the name, we only possess bare names)[41].

Appendix A. Demonstration of the lemma in subsection 2.5

We will prove the following lemma:

If $I$ and $J$ are two elementary projectors and it results: $IJI = I$ then: $I = J$.

To this end, consider a complete space of compatible observables containing $I$. Such space admits a basis of elementary projectors of which $I$ is an element, by virtue of the procedure seen in subsection 2.7 of the first paper, and, by means of a suitable reordering of the basis elements, one can make sure that the index of $I$ is equal to an arbitrarily chosen value $k_0$. If $\{\Gamma_{jk}\}$ is the dyad basis associated to such projector basis, according to equation (48) in the first paper, the projector $J$ may be decomposed in the form:

$$J = \sum_{j,k} t_{jk} \Gamma_{jk} . \quad \text{(A.1)}$$

Since $J$ is an observable its associated matrix will be Hermitian:

$$t_{jk} = t_{kj}^* \quad \text{(A.2)}$$

According to the characteristic property of the projectors, equation (2) in the first paper, and to the expression of the product of pseudo-observables, given by equation (52) in the first paper, besides one has:

$$J = J^2 \Rightarrow t_{jk} = \sum_{k'} t_{jk'} t_{k'k}^*$$

by which, in particular, for $k = j$, it follows:

$$t_{jj} = \sum_{k} |t_{jk}|^2 \quad \text{(A.3)}$$

The (A.3) implies, in particular, that the diagonal components of $J$ are all non-negative and that, for each index $k$, it results:

$$0 \leq |t_{jk}|^2 \leq t_{jj} . \quad \text{(A.4)}$$

Since $J$ is an elementary projector, its trace must be equal to 1, due to the characterization given at the end of the subsection 2.2 and therefore it will be:

$$\text{tr} (J) = \sum_{j} t_{jj} = 1 . \quad \text{(A.5)}$$

But, by virtue of the hypothesis and of the equation (50) in the first paper, and recalling that is $I = I_{k_0} = \Gamma_{k_0 k_0}$, one has:

$$I_{k_0}JI_{k_0} = \Gamma_{k_0 k_0} \Rightarrow t_{k_0 k_0} = 1 . \quad \text{(A.6)}$$
Since all the diagonal component of $J$ are positive or null, by substitution of the (A.6) into the (A.5), it is obtained:

$$t_{jj} = \delta_{j,k_0}$$

(A.7)

where $\delta_{j,k_0}$ is the Kronecker symbol. On the other side, the equations (A.3) and (A.7) imply that it results:

$$t_{jk} = \delta_{j,k_0}\delta_{k,k_0}$$

that is equivalent to assert:

$$J = \Gamma_{k_0k_0} = I$$

Q.E.D.

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