Reflected BSDE with monotonicity and general increasing in $y$, and non-Lipschitz conditions in $z$

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Abstract In this paper, we study the reflected BSDE with one continuous barrier, under the monotonicity and general increasing condition on $y$ and non Lipschitz condition on $z$. We prove the existence and uniqueness of the solution to these equation by approximation method.

Keywords Reflected backward stochastic differential equation, monotonicity, non-Lipschitz

1 Introduction

Nonlinear backward stochastic differential equations (BSDE in short) were firstly introduced by Pardoux and Peng in 1990, [12]. They proved that there exists a unique solution $(Y, Z)$ to this equation if the terminal condition $\xi$ and coefficient $f$ satisfy smooth square-integrability assumptions and $f(t, \omega, y, z)$ is Lipschitz in $(y, z)$ uniformy in $(t, \omega)$. Later many assumptions have been made to relax the Lipschitz condition on $f$. Pardoux (1999, [11]) and Briand et al. (2003, [1]) studied the solution of a BSDE with a coefficient

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\( f(t, \omega, y, z) \), which still satisfies the Lipschitz condition on \( z \), but only monotonicity, continuity and generalized increasing on \( y \), i.e. for some continuous increasing function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \), real number \( \mu > 0 \):

\[
|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(|y|), \forall (t, y) \in [0, T] \times \mathbb{R}, \text{ a.s.};
\]

\[
(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2, \forall (t, z) \in [0, T] \times \mathbb{R}^d, y, y' \in \mathbb{R}, \text{ a.s.}.
\]

The case when \( f \) is quadratic on \( z \) and \( \xi \) is bounded was firstly studied by Kobylanski in [6]. She proved an existence result when the coefficient is only linear growth in \( y \), and quadratic in \( z \). In [9], Lepeltier and San Martín generalized to a superlinear case in \( y \). More recently, in [2], they and Briand considered the BSDE whose coefficient \( f \) satisfies only monotonicity, continuity and generalized increasing on \( y \), and quadratic or linear increasing in \( z \), i.e.

\[
(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2, \forall (t, z) \in [0, T] \times \mathbb{R}^d, y, y' \in \mathbb{R}, \text{ a.s.};
\]

\[
|f(t, y, z)| \leq \varphi(|y|) + A|z|^2, \forall (t, y) \in [0, T] \times \mathbb{R}, \text{ a.s.};
\]

or

\[
|f(t, y, z)| \leq g_t + \varphi(|y|) + A|z|, \forall (t, y) \in [0, T] \times \mathbb{R}, \text{ a.s.}.
\]

In the same paper, they studied the case \( f(t, y, z) = |z|^p \), for \( p \in (1, 2] \), and gave some sufficient and necessary conditions on \( \xi \) for the existence of solutions.

El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced the notion of reflected BSDE (RBSDE in short) on one lower barrier in 1997, [4]: the solution is forced to remain above a continuous process, which is considered as the lower barrier. More precisely, a solution for such equation associated to a coefficient \( f(t, \omega, y, z) \), a terminal value \( \xi \), a continuous barrier \( L \), is a triple \((Y_t, Z_t, K_t)_{0 \leq t \leq T}\) of adapted processes valued on \( \mathbb{R}^{1+d+1} \), which satisfies a square integrability condition,

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, 0 \leq t \leq T, \text{ a.s.,}
\]

and \( Y_t \geq L_t \), \( 0 \leq t \leq T \), a.s.. Furthermore, the process \((K_t)_{0 \leq t \leq T}\) is non decreasing, continuous, and the role of \( K_t \) is to push upward the state process in a minimal way, to keep it above \( L \). In this sense it satisfies \( \int_0^T (Y_s - L_s) dK_s = 0 \). They proved the existence and uniqueness of the solution when
Let \( f(t, \omega, y, z) \) be Lipschitz in \((y, z)\) uniformly in \((t, \omega)\). Then Matoussi (1997, [10]) consider RBSDE’s where the coefficient \( f \) is continuous and at most linear growth in \( y, z \). In this case, he proved the existence of maximal solution for the RBSDE.

In [7], Kobylanski, Lepeltier, Quenez and Torres proved the existence of a maximal and minimal bounded solution for the RBSDE when the coefficient \( f(t, \omega, y, z) \) is super linear increasing in \( y \) and quadratic in \( z \), i.e. there exists a function \( l \) strictly positive such that

\[
|f(t, y, z)| \leq l(y) + A |z|^2, \text{ with } \int_0^\infty \frac{dx}{l(x)} = +\infty.
\]

In this case, \( \xi \) and \( L \) are required to be bounded, and \( L \) is a continuous process. Recently, in [8] Lepeltier, Matoussi and Xu considered the case when \( f(t, \omega, y, z) \) satisfies \( \Pi \) and is Lipschitz in \( z \). They proved the existence and uniqueness of the solution by an approximation procedure.

In this paper, we study the RBSDEs whose the coefficient \( f \) satisfies the conditions \( \Pi \) or \( \Pi \), when the lower barrier \( L \) is uniformly bounded. We prove the existence of a solution, following the methods in [2], and we give a necessary and sufficient condition for the case when \( f(t, \omega, y, z) = |z|^2 \), and its explicit solution.

The paper is organized as follows: in Section 2, we present the basic assumptions and the definition of the RBSDE; then in Section 3, we prove the existence of a solution when \( f(t, \omega, y, z) \) satisfies the conditions \( \Pi \), \( \xi \) and \( L \) are bounded; in the following section, we consider the case when \( f(t, \omega, y, z) = |z|^2 \), and \( \xi \) is not necessarily bounded. In this section, we give a necessary and sufficient condition on the terminal condition \( \xi \) for \( p = 2 \) and its explicit solution. Finally, in section 5, we study the RBSDE with the condition \( \Pi \), and prove the existence of a solution. At last, in Appendix, we generalize the comparison theorem in [7], and get some comparison theorems, which help us to pass to the limit in the approximations.

\section{Notations}

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, and \((B_t)_{0 \leq t \leq T} = (B^1_t, B^2_t, \ldots, B^d_t)_{0 \leq t \leq T}\) be a \( d \)-dimensional Brownian motion defined on a finite interval \([0, T]\), \( 0 < T < +\infty \). Denote by \( \{\mathcal{F}_t; 0 \leq t \leq T\} \) the standard filtration generated by
the Brownian motion $B$, i.e. $\mathcal{F}_t$ is the completion of

$$\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\},$$

with respect to $(\mathcal{F}, P)$. We denote by $\mathcal{P}$ the $\sigma$-algebra of predictable sets on $[0, T] \times \Omega$.

We will need the following spaces:

$\text{L}^2(\mathcal{F}_t) = \{\eta: \mathcal{F}_t\text{-measurable random real-valued variable, s.t. } E(|\eta|^2) < +\infty\};$

$\text{H}^2(0, T) = \{\psi_t \downarrow_0 \leq t \leq T: \psi_t,  \text{predictable process valued in } \mathbb{R}, \text{s.t. } E(\int_0^T |\psi(t)|^2 dt < +\infty)\};$

$\text{S}^2(0, T) = \{(\psi_t)_{0 \leq t \leq T}: \text{progressively measurable, continuous, real-valued process, s.t. } E(\sup_{0 \leq t \leq T} |\psi(t)|^2) < +\infty\};$

$\text{A}^2(0, T) = \{(K_t)_{0 \leq t \leq T}: \text{adapted continuous increasing process, s.t. } K(0) = 0, E(K(T)^2) < +\infty\}.$

Now we introduce the definition of the solution of reflected backward stochastic differential equation with a terminal condition $\xi$, a coefficient $f$ and a continuous reflecting lower barrier $L$ (in short RBSDE($\xi, f, L$)), which is the same as in El Karoui et al. (1997, [4]).

**Definition 2.1** We say that the triple $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ of progressively measurable processes is a solution of RBSDE($\xi, f, L$), if the followings hold:

(i) $(Y_t)_{0 \leq t \leq T} \in S^2(0, T), (Z_t)_{0 \leq t \leq T} \in H^2(0, T), \text{and } (K_t)_{0 \leq t \leq T} \in A^2(0, T)$.

(ii) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \text{ a.s.}$

(iii) $Y_t \geq L_t, \quad 0 \leq t \leq T.$

(iv) $\int_0^T (Y_s - L_s) dK_s = 0, \text{ a.s.}$

3 The general case of $f$ quadratic increasing

In this section, we work under the following assumptions:

**Assumption 1.** $\xi$ is an $\mathcal{F}_T$-adapted and bounded random variable;

**Assumption 2.** a coefficient $f: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R},$ is such that for some continuous increasing function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, real numbers $\mu$ and $A > 0$ and $\forall(t, y, y') \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

(i) $f(\cdot, y, z)$ is progressively measurable;

(ii) $|f(t, y, z)| \leq \varphi(|y|) + A |z|^2$;

(iii) $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2$;

(iv) $y \rightarrow f(t, y, z)$ is continuous, a.s.
**Assumption 3.** A barrier \((L_t)_{0 \leq t \leq T}\), is a bounded continuous progressively measurable real-valued process, \(b := \sup_{0 \leq t \leq T} |L_t| < +\infty, L_T \leq \xi\), a.s.

Then we present our main result in this section.

**Theorem 3.1** Under the **Assumptions 1, 2 and 3**, RBSDE\((\xi, f, L)\) admits a maximal bounded solution.

**Proof.** First, notice that \((Y, Z, K)\) is the solution of RBSDE\((\xi, f, L)\) if and only if \((Y^b, Z^b, K^b)\) is the solution of the RBSDE\((\xi^b, f^b, L^b)\), where

\[
(Y^b, Z^b, K^b) = (Y - b, Z, K),
\]

and

\[
(\xi^b, f^b(t, y, z), L^b) = (\xi - b, f(s, y + b, z), L - b).
\]

Notice that \((\xi^b, f^b, L^b)\) satisfies **Assumption 1, 2** and \(-2b \leq L^b \leq 0\). So in the following, we assume that the barrier \(L\) is a negative bounded process.

For \(C > 0\), set \(g^C : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function, such that

\[
g^C(y) = 1, \text{ if } |y| \leq C, \\
g^C(y) = 0, \text{ if } |y| \geq 2C. \tag{4}
\]

Denote \(f^C(t, y, z) = g^C(y)f(t, y, z)\); then

\[
|f^C(t, y, z)| \leq g^C(y)(\varphi(|y|) + A|z|^2) \\
\leq 1_{[-2C, 2C]}(\varphi(|y|) + A|z|^2) \\
\leq \varphi(2C) + A|z|^2.
\]

From the theorem 1 in [7], there exists a maximal solution \((Y^C, Z^C, K^C)\) to the RBSDE\((\xi, f^C, L)\)

\[
Y^C_t = \xi + \int_t^T g^C(Y^C_s)f(s, Y^C_s, Z^C_s)ds - \int_t^T Z^C_s dB_s + K^C_T - K^C_t, \tag{5}
\]

\[
Y^C_t \geq L_t, \int_0^T (Y^C_t - L_t)dK^C_t = 0, \text{ a.e.}
\]

5
We choose $n \geq 2$ even, and $a \in \mathbb{R}$; applying Itô’s formula to $e^{at}(Y_t^C)^n$, we have

$$e^{at}(Y_t^C)^n = e^{aT}e^n + n \int_t^T e^{as}(Y_s^C)^{n-1}g^C(Y_s^C)f(s,Y_s^C,Z_s^C)ds - n \int_t^T e^{as}(Y_s^C)^{n-1}Z_s^CdB_s \tag{6}$$

From Assumption 2 and the fact that $n$ is even, we have

$$yf(s,y,z) \leq yf(s,0,z) + \mu y^2,$$

$$y^{n-1}f(s,y,z) \leq y^{n-1}f(s,0,z) + \mu y^n.$$

With $0 \leq g^C(y) \leq 1$, we get

$$g^C(y)y^{n-1}f(s,y,z) \leq g^C(y)|y|^{n-1}f(s,0,z) + \mu y^n$$

$$\leq g^C(y)|y|^{n-1}(\varphi(0) + A|z|^2) + \mu y^n$$

$$\leq (\frac{1}{n} + \frac{n-1}{n})|y|^n\varphi(0) + A|z|^2g^C(y)|y|^{n-1} + \mu y^n$$

$$\leq (1 + y^n)\varphi(0) + 2CA|z|^2y^{n-2} + \mu y^n.$$

Substitute it into (6), then

$$e^{at}(Y_t^C)^n \leq e^{aT}e^n + \frac{n\varphi(0)}{a}(e^{aT} - e^{at}) + (n\varphi(0) + n\mu - a)\int_t^T e^{as}(Y_s^C)^n ds$$

$$+ (2nCA - \frac{n(n-1)}{2})\int_t^T e^{as}(Y_s^C)^{n-2}Z_s^C ds + n\int_t^T e^{as}(L_s)^n dK^C_s$$

$$- n\int_t^T e^{as}(Y_s^C)^{n-1}Z_s^C dB_s.$$

Notice that since $K^C$ is an increasing process, $n$ is even and $L \leq 0$, we get immediately

$$\int_t^T e^{as}(L_s)^n dK^C_s \leq 0.$$

If we choose $n$ and $a$ satisfying

$$n - 1 \geq 4CA, a = n(\varphi(0) + \mu),$$

6
then
\[ e^{at}(Y_t^C)^n \leq e^{aT} \xi^n + \frac{n\varphi(0)}{a}(e^{aT} - e^{at}) - n \int_t^T e^{as}(Y_s^C)^{n-1}Z_s dB_s. \]

It follows that
\[ e^{at}(Y_t^C)^n \leq E[e^{aT}(\xi^n + \frac{n\varphi(0)}{a})|\mathcal{F}_t] \leq e^{aT}(\|\xi\|^n + 1), \]
at last we get
\[ (Y_t^C)^n \leq e^{a(T-t)}(\|\xi\|^n + 1) \leq (e^{aT} \vee 1)(\|\xi\|^n + 1). \]

Since \( a = n(\varphi(0) + \mu) \), it follows that
\[
|Y_t^C| \leq (e^{(\varphi(0)+\mu)T} \vee 1)(\|\xi\|^n + 1)^{\frac{1}{n}} \leq (e^{(\varphi(0)+\mu)T} \vee 1)(\|\xi\|_\infty + 1).
\]

If \( C \) is chosen to satisfy \( C \geq (e^{(\varphi(0)+\mu)T} \vee 1)(\|\xi\|_\infty + 1) \), then we have \( |Y_t^C| \leq C \), which implies \( g^C(Y_t^C) = 1 \), for \( 0 \leq t \leq T \). So, \((Y^C, Z^C, K^C)\) is the solution of the RBSDE \((\xi, f, L)\). \( \square \)

### 4 The case \( f(t, y, z) = |z|^2 \)

In this section we consider the case \( f(t, y, z) = |z|^2 \), which corresponds to the RBSDE
\[
Y_t = \xi + \int_t^T |Z_s|^2 ds + K_T - K_t - \int_t^T Z_s dB_s, \quad (7)
\]
\[
Y_t \geq L_t, \int_0^T (Y_t - L_t)dK_t = 0.
\]

Then we have

**Theorem 4.1** Under the assumption \( E(\sup_{0 \leq t \leq T} e^{2L_t}) < +\infty \), the RBSDE \((\xi, f, L)\) (7) admits a solution if and only if \( E(e^{2\xi}) < +\infty \).

**Proof.** For the necessary part, let \((Y, Z, K)\) be a solution of the RBSDE (7). By Itô's formula, we get
\[
e^{2Y_t} = e^{2\xi} + 2 \int_t^T e^{2Y_s}dK_s - 2 \int_t^T e^{Y_s}Z_s dB_s.
\]
\[
e^{2Y_t} = e^{2\xi} + 2 \int_0^t e^{2Y_s}Z_s dB_s - 2 \int_0^t e^{2Y_s}dK_s.
\]
Let for all \( n \), \( \tau_n = \inf \{ t : Y_t \geq n \} \wedge T \), then \( M_{t \wedge \tau_n} = 2 \int_0^{t \wedge \tau_n} e^{2Y_s} Z_s dB_s \) is a martingale, and we have
\[
E[e^{2Y_n}] = E[e^{2Y_0} - 2 \int_0^t e^{2Y_s} dK_s] \leq E[e^{2Y_0}],
\]
in view of \( 2 \int_0^t e^{2Y_s} dK_s \geq 0 \). Finally, since \( \tau_n \nearrow T \), when \( n \to \infty \):
\[
E[\lim_{n \to \infty} e^{2Y_n}] = E[e^{2\xi}] \leq E[e^{2Y_0}] < \infty,
\]
follows from Fatou’s Lemma.

Now we suppose \( E(e^{2\xi}) < +\infty \), set \( \tilde{L}_t = L_{t1_{\{t<T\}}} + \xi 1_{\{t=T\}} \) and
\[
N_t = S_t(e^{2\tilde{L}}) = ess \sup_{\tau \in \tilde{T}_{t,T}} E[e^{2\tilde{L}_\tau} | \mathcal{F}_t],
\]
where \( S_t(\eta) \) denotes the Snell envelope of \( \eta \) (See El Karoui [3]), \( \tilde{T}_{t,T} \) is the set of all stopping times valued in \( [t, T] \). Since
\[
E[\sup_{0 \leq t \leq T} e^{2\tilde{L}_t}] \leq E[\sup_{0 \leq t \leq T} e^{2L_t} + e^{2\xi}] < +\infty,
\]
using the results of Snell envelope, we know that \( N \) is a supermartingale, so it admits the following decomposition: for an increasing integrable process \( \bar{K} \),
\[
N_t = N_0 + \int_0^t Z_s dB_s - \bar{K}_t.
\]
Applying Itô’s formula to \( \log N_t \), we get
\[
\frac{1}{2} \log N_t = \frac{1}{2} \log N_0 + \frac{1}{2} \int_0^t \frac{Z_s}{N_s} dB_s - \frac{1}{4} \int_0^t \left( \frac{Z_s}{N_s} \right)^2 ds - \frac{1}{2} \int_0^t \frac{1}{N_s} d\bar{K}_s.
\]
Set \( Y_t = \frac{1}{2} \log N_t, Z_t = \frac{Z_t}{2N_t}, K_t = \frac{1}{2} \int_0^t \frac{1}{N_s} d\bar{K}_s \), then the triple satisfies
\[
Y_t = \xi + \int_t^T Z_s^2 ds + K_T - K_t - \int_t^T Z_s dB_s. \tag{9}
\]
Thanks to the results on the Snell envelope, we know that \( N_t \geq e^{2\tilde{L}_t} \) and \( \int_0^T (N_t - e^{2\tilde{L}_t}) d\bar{K}_t = 0 \). The first implies
\[
Y_t \geq \tilde{L}_t \geq L_t.
\]
Obviously, \( N_t > 0 \), \( 0 \leq t \leq T \), so \( K \) is increasing. Consider the stopping time \( D_t := \inf\{t \leq u \leq T; Y_u = L_u\} \wedge T \), then it satisfies \( D_t = \inf\{t \leq u \leq T; N_u = e^{2L_u}\} \wedge T \). By the continuity of \( K \), we get \( K_{D_t} - K_t = 0 \), which implies \( K_{D_t} - K_t = 0 \). It follows that

\[
\int_0^T (Y_t - L_t) dK_t = 0.
\]

Now the rest is to prove \( Y_t \in S^2(0, T) \), \( Z_t \in H^2_d(0, T) \), and \( K_t \in A^2(0, T) \).

With Jensen’s inequality

\[
Y_t = \frac{1}{2} \log N_t = \frac{1}{2} \log(\text{ess sup}_{\tau \in T_t} E[e^{2\tilde{L}_\tau} | \mathcal{F}_t])
\]

\[
\geq \frac{1}{2} \log(\exp(\text{ess sup}_{\tau \in T_t} E[2\tilde{L}_\tau | \mathcal{F}_t]))
\]

\[
= \text{ess sup}_{\tau \in T_t} E[\tilde{L}_\tau | \mathcal{F}_t] \geq E[\xi | \mathcal{F}_t] \geq U_t,
\]

where \( U_t = -E[\xi^-] \). For all \( a > 0 \), define

\[
\tau_a = \inf\{t; |N_t| > a, \int_0^t \left( \frac{Z_s}{N_s} \right)^2 ds > a, \left| \int_0^t \frac{Z_s}{N_s} dB_s \right| > a\}.
\]

From (9), we get for \( 0 \leq t \leq T \)

\[
0 \leq \int_0^t Z_s^2 ds = Y_0 - Y_t + \int_0^t Z_s dB_s - K_t
\]

\[
\leq Y_0 - U_t + \int_0^t Z_s dB_s.
\]

Then

\[
(\int_0^{\tau_a} Z_s^2 ds)^2 \leq 3(Y_0)^2 + 3(U_{\tau_a})^2 + 3(\int_0^{\tau_a} Z_s dB_s)^2.
\]

Taking the expectation, using the Jensen’s inequality and \( 3x \leq \frac{x^2}{2} + \frac{9}{2} \), we obtain

\[
E(\int_0^{\tau_a} Z_s^2 ds)^2 \leq \frac{3}{4} (\log N_0)^2 + 3E(\xi^-)^2 + \frac{1}{2} (E(\int_0^{\tau_a} Z_s^2 ds))^2 + \frac{9}{2}
\]

\[
\leq \frac{3}{4} (\log N_0)^2 + 3E(\xi^-)^2 + \frac{1}{2} E(\int_0^{\tau_a} Z_s^2 ds)^2 + \frac{9}{2}.
\]
Since \( \tau_a \nearrow T \) when \( a \to +\infty \), we get to the limit, and with the Schwartz inequality
\[
E \int_0^T Z_s^2 ds \leq (E(\int_0^T Z_s^2 ds))^\frac{1}{2} \leq C.
\]
So \( Z \in H^2_a(0,T) \). From (9), we get for \( 0 \leq t \leq T \)
\[
0 \leq K_t = Y_0 - Y_t + \int_0^t Z_s dB_s - \int_0^t Z_s^2 ds
\leq Y_0 - Y_t + \int_0^t Z_s dB_s.
\]
Notice that \( K \) is increasing, so it’s sufficient to prove \( E[K_T^2] < +\infty \). Squaring the inequality on both sides and taking expectation, we obtain
\[
E[(K_T)^2] \leq 3Y_0^2 + 3E[\xi^2] + 3E \int_0^T Z_s^2 ds \leq C.
\]
We consider now \( Y \); again from (9),
\[
Y_t = Y_0 - K_t + \int_0^t Z_s dB_s - \int_0^t Z_s^2 ds,
\]
so
\[
(Y_t)^2 \leq 4(Y_0)^2 + 4(K_t)^2 + 4 \left( \int_0^t Z_s dB_s \right)^2 + 4 \left( \int_0^t Z_s^2 ds \right)^2.
\]
Then by the Bukholder-Davis-Gundy inequality, we get
\[
E[\sup_{0 \leq t \leq T} (Y_t)^2] \leq 4(Y_0)^2 + 4E[K_T^2] + 4E[\sup_{0 \leq t \leq T} \left( \int_0^t Z_s dB_s \right)^2] + 4E \left( \int_0^T Z_s^2 ds \right)^2 \leq 4(Y_0)^2 + 4E[K_T^2] + CE \left( \int_0^T Z_s^2 dB_s \right) + 4E \left( \int_0^T Z_s^2 ds \right)^2 \leq C,
\]
i.e. \( Y \in S^2(0,T) \). \( \square \)
5 The case when \( f \) is linear increasing in \( z \)

In this section, we assume that the coefficient \( f \) satisfies

**Assumption 6.** (i) \( f(\cdot, y, z) \) is progressively measurable, and \( E \int_0^T f^2(t,0,0)dt < +\infty \);

(ii) for \( \mu \in \mathbb{R} \), \( \forall (t, z) \in [0, T] \times \mathbb{R}^d \) and \( y, y' \in \mathbb{R} 

\[(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu(y - y')^2; \]

(iii) there exists a nonnegative, continuous, increasing function \( \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), with \( \varphi(0) = 0 \), s.t. \( \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \),

\[|f(t, y, z)| \leq |g_t| + \varphi(|y|) + \beta |z|, \]

where \( g_t \in H^2(0, T); \)

(iv) for \( t \in [0, T] \), \( (y, z) \rightarrow f(t, y, z) \) is continuous.

If \( \varphi(x) = |x| \), then \( f \) is linear increasing in \( y \) and \( z \). Matoussi proved in [10] that when \( \xi \in L^2(\mathcal{F}_T) \) and \( L \in S^2(0, T) \), there exists a triple \((Y, Z, K)\) which is solution of the RBSDE\((\xi, f, L)\).

Our result of this section is the following:

**Theorem 5.1** Suppose that \( \xi \in L^2(\mathcal{F}_T) \), \( f \) and \( L \) satisfy **Assumption 6** and 3, respectively, then the RBSDE\((\xi, f, L)\) has a minimal solution \((Y, Z, K) \in S^2(0, T) \times H^2_d(0, T) \times A^2(0, T)\), which satisfies

\[Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_sdB_s, \quad Y_t \geq L_t, \text{ and } \int_0^T (Y_s - L_s)dK_s = 0. \]

First we note that the triple \((Y, Z, K)\) solves the RBSDE\((\xi, f, L)\), if and only if the triple

\[(\overline{Y}_t, \overline{Z}_t, \overline{K}_t) := (e^{\lambda t}Y_t, e^{\lambda t}Z_t, e^{\lambda t}\int_0^t e^{\lambda s}dK_s) \quad (10)\]

solves the RBSDE\((\overline{\xi}, \overline{f}, \overline{L})\), where

\[(\overline{\xi}, \overline{f}(t, y, z), \overline{L}_t) = (\xi e^{\lambda T}, e^{\lambda t}f(t, e^{-\lambda t}y, e^{-\lambda t}z) - \lambda y, e^{\lambda t}L_t). \]
If we choose \( \lambda = \mu \), then the coefficient \( \bar{f} \) satisfies the same assumptions as in Assumption 6, with (ii) replaced by

(ii') \((y - y')(f(t, y, z) - f(t, y', z)) \leq 0\).

Since we are in the 1-dimensional case, (ii') means that \( f \) is decreasing on \( y \). From another part \( \xi \) still belongs to \( L^2(\mathcal{F}_T) \) and the barrier \( L \) still satisfy the assumptions Assumption 3. So in the following, we shall work under Assumption 6’ with (ii) replaced by (ii’).

Before proving this theorem, we consider an estimate result and a monotonic stability theorem for RBSDEs.

**Lemma 5.1** We consider RBSDE \((\xi, g, L)\), with \( \xi \in L^2(\mathcal{F}_T) \), \( g \) and \( L \) satisfy Assumption 6’ and 3. Moreover \( g(t, y, z) \) is Lipschitz in \( z \). Then we have the following estimation

\[
E[\sup_{0 \leq t \leq T} |y_t|^2 + \int_0^T |z_s|^2 ds + |k_T|^2] \leq C_\beta E[|\xi|^2 + \int_0^T g_s^2 ds + \varphi^2(b) + \varphi^2(2T) + 1]
\]

where \((y_t, z_t, k_t)_{0 \leq t \leq T}\) is the solution of RBSDE\((\xi, g, L)\). \( C_\beta \) is a constant only depends on \( \beta, T \) and \( b \).

**Remark 5.1** The constant \( C_\beta \) does not depend on Lipschitz coefficient of \( g \) on \( z \).

**Proof.** Since \( g \) is Lipschitz in \( z \), by the theorem 2 in [8], the RBSDE\((\xi, g, L)\) admits the unique solution \((y_t, z_t, k_t)_{0 \leq t \leq T}\). Apply Itô’s formula to \( |y_t|^2 \), in view of \( yg(t, y, z) \leq g(t, 0, 0) |y| + \beta |y||z| \) and \( \sup_{0 \leq t \leq T} |L_t| \leq b \), we get

\[
E[|y_t|^2 + \int_t^T |z_s|^2 ds] = E[|\xi|^2 + \int_t^T y_s g(s, y_s, z_s) ds + \int_t^T L_s dk_s] \\
\leq E[|\xi|^2 + \int_t^T y_s g_s ds + 2\beta \int_t^T y_s z_s ds + 2b(k_T - k_t)].
\]

It follows that

\[
E[|y_t|^2 + \frac{1}{2} \int_t^T |z_s|^2 ds] \leq E[|\xi|^2 + \int_t^T g_s^2 ds + (1 + 2\beta^2) \int_t^T |y_s|^2 ds + 2b(k_T - k_t)].
\]
By Gronwall’s inequality, we know there exists a constant $c_1$ depending on $\beta$ and $T$, such that for $t \in [0, T]$,

$$E[|y_t|^2] \leq c_1 E[|\xi|^2 + \int_0^T g_s^2 ds + b(k_T - k_t)].$$

(11)

It follows that

$$E[\int_t^T |z_s|^2 ds] \leq 2(1 + (1 + 2\beta^2)T)c_1 E[|\xi|^2 + \int_0^T g_s^2 ds + b(k_T - k_t)].$$

(12)

Now we estimate the increasing process $k$ by approximation. Take $z$ as a known process, without losing of generality, we write $g(t, y)$ for $g(t, y, z_t)$, here $g(t, 0) = g(t, 0, z_t)$ is a process in $H^2(0, T)$ in view of linear increasing property of $g$ on $z$.

For $m, p \in \mathbb{N}$, set $\xi^{m,p} = (\xi \lor (-p)) \land m$, $g^{m,p}(t, u) = g(t, u) - g_t + (g_t \lor (-p)) \land m$. We consider RBSDE$(\xi^{m,p}, g^{m,p}, L)$,

$$y_t^{m,p} = \xi^{m,p} + \int_t^T g^{m,p}(s, y^{m,p}_s)ds + k_T^{m,p} - k_t^{m,p} - \int_t^T z^{m,p}_s dB_s,$$

(13)

$$y_t^{m,p} \geq L_t, \int_0^T (y_t^{m,p} - L_t)dk_t^{m,p} = 0.$$

It is easy to check that $(y^{m,p}, z^{m,p}, k^{m,p})$ is the solution of RBSDE$(\xi^{m,p}, g^{m,p}, L)$, if and only if $(\hat{y}^{m,p}, \hat{z}^{m,p}, \hat{k}^{m,p})$ is the solution of RBSDE$(\hat{x}^{m,p}, \hat{g}^{m,p}, \hat{L})$, where

$$\left(\hat{y}^{m,p}_t, \hat{z}^{m,p}_t, \hat{k}^{m,p}_t\right) = (y_t^{m,p} + m(t - 2(T \lor 1)), z^{m,p}_t, k^{m,p}_t),$$

and

$$\hat{x}^{m,p} = x^{m,p} + mT - 2m(T \lor 1),$$
$$\hat{g}^{m,p}(t, y) = g^{m,p}(t, y - m(t - 2(T \lor 1))) - m,$$
$$\hat{L}_t = L_t + m(t - 2(T \lor 1)).$$

Without losing of generality, we set $T \geq 1$. Since $\xi^{m,p}$ and $g^{m,p}_t \leq m$, we have $\hat{\xi}^{m,p}$ and $\hat{\xi}^{m,p}_t \leq 0$. By (13),

$$\hat{k}_T^{m,p} - \hat{k}_t^{m,p} = \hat{y}_t^{m,p} - \hat{x}^{m,p} - \int_t^T \hat{g}^{m,p}(s, \hat{y}^{m,p}_s)ds + \int_t^T \hat{z}^{m,p}_s dB_s,$$
taking square and expectation on both sides, we get

\[
E[(\hat{k}_T^m-p-\hat{k}_t^m)^2] \leq 4E[(\hat{y}_t^m-p)^2 + (\hat{\xi}^m)^2] + \int_t^T \hat{g}^m_p(s, \hat{y}_s^m)p ds + \int_t^T |\hat{z}_s^m|^2 ds.
\]

(14)

In order to estimate the first and the last form on the left side, we apply Itô’s formula to \(|\hat{y}_t^m|^2\), and get the following with Gronwall inequality,

\[
E[|\hat{y}_t^m|^2 + \int_t^T |\hat{z}_s^m|^2 ds] \leq c_2 E[|\hat{\xi}_m|^2 + \int_t^T (\hat{g}_s^m)^2 ds + \int_t^T L_s d\hat{k}_s^m],
\]

where \(c_2\) is a constant only depends on \(T\). For the third term, let us recall a comparison result of \(\hat{y}_t^m\) in step 2 of the proof of theorem 2 in [8],

\[
\hat{y}_t^m \leq \hat{y}_t^m \leq \hat{y}_t^m,
\]

where \(\hat{y}_t^m\) is the solution of BSDE\((\hat{\xi}^m, \hat{g}^m)\), i.e.

\[
\hat{y}_t^m = \hat{\xi}_m + \int_t^T \hat{g}_s^m(s, \hat{y}_s^m)ds - \int_t^T \hat{z}_s^m dB_s,
\]

(16)

and

\[
\hat{y}_t^m = ess \sup_{\tau \in \mathcal{T}_{t,T}} E[(\hat{L}_\tau)^+ 1_{\{\tau < T\}} + (\hat{\xi}_m)^+ 1_{\{\tau = T\}} |\mathcal{F}_t],
\]

where \(\mathcal{T}_{t,T}\) is the set of stopping times valued in \([t, T]\). Moreover, we have

\[
\sup_{0 \leq s \leq T} \hat{y}_s^m = \sup_{0 \leq s \leq T} \hat{L}_s.
\]

Since \(\hat{g}^m\) is decreasing in \(y\), we get

\[
\hat{g}^m(s, \hat{y}_s^m) \leq \hat{g}^m(s, \hat{y}_s^m) \leq \hat{g}^m(s, \hat{y}_s^m).
\]

So to estimate \(E[(\int_t^T \hat{g}^m_p(s, \hat{y}_s^m)ds)^2]\), it is sufficient to get the estimations of \(E[(\int_t^T \hat{g}^m_p(s, \hat{y}_s^m)ds)^2]\) and \(E[(\int_t^T \hat{g}^m_p(s, \bar{y}_s^m)ds)^2]\). First we know that

\[
E[(\int_t^T \hat{g}^m_p(s, \hat{y}_s^m)ds)^2] \leq 3E[(\hat{\xi}_m)^2 + |\hat{y}_t^m|^2 + \int_t^T |\hat{z}_s^m|^2 ds]
\]

\[
\leq c_3 E[(\hat{\xi}_m)^2 + \int_t^T (\hat{g}_s^m)^2 ds],
\]

14
in view of estimate result of BSDE (10). Here $c_3$ is a constant only depends on $T$. Then with the presentation of $\mathbf{y}_t^{m,p}$, we have

$$E[\int_t^T \hat{g}^{m,p}(s, \hat{\mathbf{y}}_s^{m,p}) ds]^2 \leq E[2T \int_0^T (\hat{g}_s^{m,p})^2 ds + 2T \varphi^2 (\sup_{0 \leq t \leq T} (\hat{\mathbf{L}}_t)^+)] \tag{18}$$

From (14), with (15), (17) and (18), we have

$$E[(k_T^{m,p} - k_t^{m,p})^2] = E[(\hat{k}_T^{m,p} - \hat{k}_t^{m,p})^2]$$

$$\leq c_4 E[|\xi^{m,p}|^2 + \int_t^T (g_s^{m,p})^2 ds + \int_t^T L_s \hat{d}^{m,p} + \varphi^2 (\sup_{0 \leq t \leq T} (\hat{\mathbf{L}}_t)^+)]$$

$$\leq c_4 E[2|\xi^{m,p}|^2 + 4 \int_t^T (g_s^{m,p})^2 ds + 2c_4b^2 + \varphi^2 (b)]$$

$$+ 4c_4(m^2T^2 + \varphi^2 (2mT)) + \frac{1}{2} E[(k_T^{m,p} - k_t^{m,p})^2],$$

where $c_4 = c_2 \vee c_3 \vee (2T)$, which only depends on $T$. It follows that

$$E[(k_T^{m,p} - k_t^{m,p})^2]$$

$$\leq c_5 E[|\xi^{m,p}|^2 + \int_t^T (g_s^{m,p})^2 ds + b^2 + \varphi^2 (b)] + c_5(m^2T^2 + \varphi^2 (2mT))$$

$$\leq c_5 E[|\xi|^2 + \int_t^T g_s^2 ds + b^2 + \varphi^2 (b)] + c_5(m^2T^2 + \varphi^2 (2mT)),$$

where $c_5 = 4c_4^2 \vee 8c_4$.

Now we consider the RBSDE $(\xi^{p}, g^{p}, L)$, where $\xi^{p} = \xi \vee (-p)$, $g^{p}(t, u) = g(t, u) - g_t + g_t \vee (-p)$. Thanks to the convergence result in [8], we know that

$$(y^{m,p}, z^{m,p}, k^{m,p}) \rightarrow (y^{p}, z^{p}, k^{p}) \text{ in } S^2(0, T) \times H^2_d(0, T) \times A^2(0, T),$$

where $(y^{p}, z^{p}, k^{p})$ is the solution of RBSDE$(\xi^{p}, g^{p}, L)$. Moreover, we have $dk_t^{p} \leq dk_t^{1,p}$, by comparison theorem. So

$$E[(k_T^{p} - k_t^{p})^2] \leq E[(k_T^{1,p} - k_t^{1,p})^2]$$

$$\leq c_5 E[|\xi|^2 + \int_t^T g_s^2 ds + b^2 + \varphi^2 (b)] + c_5(T^2 + \varphi^2 (2T)).$$

Then let $p \rightarrow \infty$, thanks to the convergence result in [8], we know

$$(y^{p}, z^{p}, k^{p}) \rightarrow (y, z, k) \text{ in } S^2(0, T) \times H^2_d(0, T) \times A^2(0, T).$$
In view Assumption 6-(iii), it follows that

\[
E[(k_T - k_t)^2] \leq c_5 E[|\xi|^2 + \int_t^T (g(s, 0, z_s))^2 ds + b^2 + \varphi^2(b)]
\]

\[
\leq c_5 E[|\xi|^2 + 2 \int_t^T g_s^2 ds + 2\beta^2 \int_t^T |z_s|^2 ds + b^2 + \varphi^2(b)]
\]

\[
+ c_5 (T^2 + \varphi^2(2T)).
\]

With (12), setting \(c_6 = c_5 \vee (4\beta^2(1 + (1 + 2\beta^2)T)c_1 + 2) \vee c_5(b^2 + T^2)\), we get

\[
E[(k_T - k_t)^2] \leq c_6 E[|\xi|^2 + \int_t^T g_s^2 ds + b(k_T - k_t) + \varphi^2(b) + \varphi^2(2T) + 1]
\]

\[
\leq c_6 E[|\xi|^2 + \int_t^T g_s^2 ds + 2c_0 b^2 + \varphi^2(b) + \varphi^2(2T) + 1]
\]

\[
+ \frac{1}{2} E[(k_T - k_t)^2]
\]

It follows that

\[
E[(k_T - k_t)^2] \leq 2c_6 E[|\xi|^2 + 2 \int_t^T g_s^2 ds + \varphi^2(b) + \varphi^2(2T) + 2c_0 b^2 + 1].
\]

Consequently, by (11) and (12), we obtain

\[
\sup_{0 \leq t \leq T} E[|y_t|^2] + E[\int_0^T |z_s|^2 ds + |k_T|^2]
\]

\[
\leq C_\beta E[|\xi|^2 + \int_0^T g_s^2 ds + \varphi^2(2T) + \varphi^2(b) + 1],
\]

where \(C_\beta\) is a constant only depends on \(\beta\), \(T\) and \(b\). The final result follows from BDG inequality. \(\square\)

The proof of this theorem is step 1 and step 2 of the proof of theorem 4 in [7], with comparison theorem. So we omit it.

With these preparations, we begin our main proof.

**Proof of theorem 5.1.** The proof consists 4 step.

**Step 1.** Approximation. For \(n \geq \beta\), we introduce the following functions

\[
f_n(t, y, z) = \inf_{q \in Q^\beta} \{f(t, y, q) + n |z - q|\};
\]
then we have
1. for all \((t, z), y \to f_n(t, y, z)\) is non-increasing;
2. for all \((t, y), z \to f_n(t, y, z)\) is \(n\)-Lipschitz;
3. for all \((t, y, z), |f_n(t, y, z)| \leq |g_t| + \varphi(|y|) + \beta |z|\).

Thanks to the results of [8], we know that for each \(n \geq \beta\), there exits a unique triple \((Y^n, Z^n, K^n)\) satisfies the followings

\[
Y^n_t = \xi + \int_t^T f_n(s, Y^n_s, Z^n_s) ds + K^n_T - K^n_t - \int_t^T Z^n_s dB_s,
\]

\[
Y^n_T \geq L_t, \int_0^T (Y^n_s - L_t) dK^n_s = 0.
\]

**Step 2.** Estimates results. Let \(\alpha \geq 0\) be a real number to be chosen later. We set

\[
U^n_t = e^{\alpha t} Y^n_t, V^n_t = e^{\alpha t} Z^n_t, dJ^n_t = e^{\alpha t} dK^n_t.
\]

Then we know that \((U^n, V^n, J^n)\) is the solution of the RBSDE associated with \((\zeta, F_n, L^n)\), where

\[
\zeta = e^{\alpha T} \xi, F_n(t, u, v) = e^{\alpha t} f_n(t, e^{-\alpha t} u, e^{-\alpha t} v) - \alpha u, L^n_t = e^{\alpha t} L_t.
\]

It is easy to check

\[
|F_n(t, u, v)| \leq e^{\alpha t} |g_t| + e^{\alpha t} \varphi(|u|) + \alpha |u| + \beta |v|,
\]

setting \(\psi(u) = e^{\alpha T} \varphi(|u|) + \alpha |u|\), with \(\psi(u) = 0\), we get that \(F_n\) verifies **Assumption 6**-(iii). Moreover

\[
u F_n(t, u, v) = e^{\alpha t} u f_n(t, e^{-\alpha t} u, e^{-\alpha t} v) - \alpha u^2
\]

\[
\leq u e^{\alpha t} g_t + \beta |u| |v| - \alpha u^2.
\]

And \(\sup_{0 \leq t \leq T} L^n_t \leq e^{\alpha T} \sup_{0 \leq t \leq T} L_t \leq e^{\alpha T} b\). Now we apply Itô formula to \(|U^n|^2\) on \([0, T]\), and get

\[
|U^n_t|^2 + \int_t^T |V^n_s|^2 ds
= |\zeta|^2 + 2 \int_t^T U^n_s F_n(s, U^n_s, V^n_s) ds + 2 \int_t^T U^n_s dJ^n_s - 2 \int_t^T U^n_s V^n_s dB_s
\leq |\zeta|^2 + \int_t^T e^{2\alpha s} g^2_s ds + (1 + 2\beta^2 - \alpha) \int_t^T |U^n_s|^2 ds + \frac{1}{2} \int_t^T |V^n_s|^2 ds
+ \theta e^{2\alpha T} b^2 + \frac{1}{\theta} (J^n_T - J^n_t)^2 - 2 \int_t^T U^n_s V^n_s dB_s,
\]
where \( \theta \) is a constant to be decided later. By taking conditional expectation, we get

\[
|U_t^n|^2 + \frac{1}{2} E[\int_t^T |V_s^n|^2 \, ds | F_t] \leq E[|\zeta|^2 + \int_t^T e^{2\alpha s} g_s^2 \, ds + \theta e^{2\alpha T} b^2 | F_t] + (1 + 2\beta^2 - \alpha) E[\int_t^T |U_s^n|^2 \, ds | F_t] + \frac{1}{\theta} E[(J_T^n - J_t^n)^2 | F_t].
\] (19)

Since

\[
J_T^n - J_t^n = U_t^n - \zeta - \int_t^T F_n(s, U_s^n, V_s^n) \, ds - \int_t^T V_s^n \, dB_s,
\]

we have

\[
E[(J_T^n - J_t^n)^2 | F_t] \leq 4 |U_t^n|^2 + 4 E[|\zeta|^2] + (\int_t^T F_n(s, U_s^n, V_s^n) \, ds)^2 + \int_t^T |V_s^n|^2 \, ds | F_t].
\]

Using the same approximation as in Lemma 5.1 except considering conditional expectation \( E[\cdot | F_t] \) instead of expectation, we deduce

\[
E[(J_T^n - J_t^n)^2 | F_t] \leq c_\beta E[|\zeta|^2] + \int_t^T e^{2\alpha s} g_s^2 \, ds + \psi^2(e^{\alpha T} b) + \psi^2(2T) + 1 | F_t],
\]

where \( c_\beta \) is a constant which only depends on \( \beta, T, b \) and \( \alpha \). Substitute it into (19), set \( \alpha = 1 + 2\beta^2 \), \( \theta = c_\beta \), then we get,

\[
|U_t^n|^2 \leq 2E[|\zeta|^2] + \int_t^T F_n^2(s, 0, 0) \, ds | F_t] + e^{\alpha T} (\varphi(e^{\alpha T} b) + \varphi(2T)) + \alpha(e^{\alpha T} b + 2T) + 1 + c_\beta e^{2\alpha T} b^2.
\]

Recall the definition of \( U_t^n \), we get

\[
|Y_t^n|^2 \leq e^{-2\alpha t} (2E[e^{2\alpha T} | \xi|^2] + \int_t^T e^{2\alpha s} g_s^2 \, ds | F_t] + e^{\alpha T} (\varphi(e^{\alpha T} b) + \varphi(2T)) + \alpha(e^{\alpha T} b + 2T) + 1 + c_\beta e^{2\alpha T} b^2).
\]

If we set \( M_t = (e^{2\alpha T} 2E[|\xi|^2] + \int_t^T g_s^2 \, ds | F_t] + e^{\alpha T} (\varphi(e^{\alpha T} b) + \varphi(2T)) + c_\beta e^{2\alpha T} b^2 + \alpha(e^{\alpha T} b + 2T) + 1)^{\frac{1}{2}} \), then

\[
|Y_t^n| \leq M_t, \quad \forall t \in [0, T].
\] (20)

18
Step 3. Localisation.

First, we know that the sequence \( (f_n)_{n \geq \beta} \) is non-decreasing in \( n \), then from comparison theorem in [8], we get

\[
Y^n_t \leq Y^{n+1}_t, \forall t \in [0, T], \forall n \geq \beta.
\]

Define \( Y_t = \max_{n \geq \beta} Y^n_t \).

We now consider the localisation procedure. For \( m \in \mathbb{N} \), \( m \geq b \), let \( \tau_m \) be the following stopping time

\[
\tau_m = \inf \{ t \in [0, T] : M_t + g_t \geq m \} \wedge T,
\]

and we introduce the stopped process \( Y_{t,m} = Y_t^m \), together with \( Z_{t,m} = Z_t^m \mathbf{1}_{\{t \leq \tau_m\}} \) and \( K_{t,m}^n = K_t^m \mathbf{1}_{t \leq \tau_m} \). Then \((Y_{t,m}, Z_{t,m}, K_{t,m}^n)_{0 \leq t \leq T}\) solved the following RBSDE

\[
\begin{align*}
Y_{t,m} & = \xi_{t,m} + \int_t^T 1_{\{s \leq \tau_m\}} f_n(s, Y_{s,m}^n, Z_{s,m}^n) ds + K_{T,m}^n - K_{t,m}^n - \int_t^T Z_{s,m} dB_s, \\
Y_{t,m} & \geq L_t, \int_0^T (Y_{t,m}^n - L_t) dK_{t,m}^n = 0,
\end{align*}
\]

where \( \xi_{t,m} = Y_{t,m}^n = Y_{t,m}^n \).

Since \((Y_{t,m}^n)_{n \geq \beta}\) is non-decreasing in \( n \), with (20), we get \( \sup_{n \geq \beta} \sup_{t \in [0, T]} |Y_{t,m}^n| \leq m \). Set \( \rho_m(y) = \frac{y_m}{\max \{ |y|, m \}} \), then it is easy to check that \((Y_{t,m}^n, Z_{t,m}^n, K_{t,m}^n)\) verifies

\[
\begin{align*}
Y_{t,m} & = \xi_{t,m} + \int_t^T 1_{\{s \leq \tau_m\}} f_n(s, \rho_m(Y_{s,m}^n), Z_{s,m}^n) ds + K_{T,m}^n - K_{t,m}^n - \int_t^T Z_{s,m} dB_s, \\
Y_{t,m} & \geq L_t, \int_0^T (Y_{t,m}^n - L_t) dK_{t,m}^n = 0.
\end{align*}
\]

Moreover, we have

\[
|1_{\{s \leq \tau_m\}} f_n(s, \rho_m(y), z)| \leq m + \varphi(m) + \beta |z|,
\]

and \( |\xi_{t,m}^{n,m}| \leq m \). From Dini’s theorem, we know that \( 1_{\{s \leq \tau_m\}} f_n(s, \rho_m(y), z) \) converge increasingly to \( 1_{\{s \leq \tau_m\}} f(s, \rho_m(y), z) \) uniformly on compact set of \( \mathbb{R} \times \mathbb{R}^d \), because \( f_n \) are continuous and \( f_n \) converge increasingly to \( f \). And \( \xi_{t,m}^{n,m} \) converge increasingly to \( \xi^m \) a.s., where \( \xi^m = \sup_{n \geq \beta} \xi_{t,m}^{n,m} \).
As in [10], we can prove that $Y^{n,m}$ converges increasingly to $Y^m$ in $S^2(0,T)$, and $Z^{n,m} \to Z^m$ in $H^2_2(0,T)$, $K^{n,m} \to K^m$ uniformly on $[0,T]$. Moreover, $(Y^m, Z^m, K^m)$ solves the following RBSDE

$$Y^m_t = \xi^m + \int_t^T 1_{\{s \leq \tau_m\}} f(s, \rho_m(Y^m_s), Z^m_s) ds + K^m_T - K^m_t - \int_t^T Z^m_s dB_s,$$

$$Y^m_t \geq L_t, \int_0^T (Y^m_t - L_t) dK^m_t = 0,$$

where $\xi^m = \sup_{n \geq \beta} Y^{n,m}$. Notice that $|Y^m_t| \leq m$, so we have

$$Y^m_t = \xi^m + \int_t^T 1_{\{s \leq \tau_m\}} f(s, Y^m_s, Z^m_s) ds + K^m_T - K^m_t - \int_t^T Z^m_s dB_s.$$

From the definition of $\{\tau_m\}$, it is easy to check that $\tau_m \leq \tau_{m+1}$, with the definition of $Y^m, Z^m, K^m$ and $Y$, we get

$$Y_{t \wedge \tau_m} = Y_{t \wedge \tau_{m+1}} = Y^m_t, Z^m_{t \wedge \tau_m} = Z^m_t, K^m_{t \wedge \tau_m} = K^m_t.$$

We define

$$Z_t := Z^1_t 1_{\{t \leq \tau_1\}} + \sum_{m \geq 2} Z^m_t 1_{\{\tau_{m-1}, \tau_m\}}(t), K_{t \wedge \tau_m} := K^m_t.$$

Processes $(Y^m)$ are continuous, and $P$-a.s. $\tau_m = T$, for $m$ large enough, so $Y$ is continuous on $[0,T]$. It follows that $K$ is also continuous on $[0,T]$. Furthermore, we have for $m \in \mathbb{N}$,

$$Y_{t \wedge \tau_m} = Y_{t \wedge \tau_m} + \int_{t \wedge \tau_m}^{\tau_m} f(s, Y_s, Z_s) ds + K_{t \wedge \tau_m} - K_t$$(21)

Finally, we have

$$P\left(\int_0^T |Z_s|^2 ds = \infty\right) = P\left(\int_0^T |Z_s|^2 ds = \infty, \tau_m = T\right) + P\left(\int_0^T |Z_s|^2 ds = \infty, \tau_m < T\right)$$

$$\leq P\left(\int_0^T |Z_s|^2 ds = \infty\right) + P(\tau_m < T),$$

in the same way,

$$P(|K_T|^2 = \infty) \leq P(|K_T|^2 = \infty) + P(\tau_m < T).$$
Since $\tau_m \not\rightarrow T$, $P$-a.s., we know that $\int_0^T |Z_s|^2 \, ds < \infty$ and $|K_T|^2 < \infty$, $P$-a.s. Let $m \to \infty$ in (21), we get $(Y, Z, K)$ verifies the equation.

**Step 4.** We want to prove that the triple $(Y, Z, K)$ is a solution of RBSDE($\xi, f, L$).

First, we consider the integrability of $(Y, Z, K)$. By (20), we know for $0 \leq t \leq T$,

$$|Y_t| \leq M_t.$$  \hspace{1cm} (22)

It follows immediately that

$$E[\sup_{0 \leq t \leq T} |Y_t|^2] \leq C \beta E[|\xi|^2] + \int_0^T g_s^2 \, ds + \varphi^2(b) + \varphi^2(2T) + 1],$$

where $C \beta$ is a constant only depends on $\beta$, $T$ and $b$. For $K$, notice that $K^{n, m} \not\rightarrow K^m$, then for each $m \in \mathbb{N}$, $0 \leq t \leq T$, we know $0 \leq K^{n, m}_t \leq K^{1, m}_t$. Obviously, the coefficient $1_{\{s \leq \tau_m\}} f_n(s, \rho_m(y), z)$ satisfies assumption 6', and Lipschitz in $z$, by Lemma 5.1

$$E[(K^{1, m}_T)^2] \leq C \beta E[|\xi^{1, m}|^2] + \int_0^T g_s^2 \, ds + \varphi^2(b) + \varphi^2(2T) + 1],$$

where $\xi^{1, m} = Y^{1, m}_{\tau_m}$. With (20), we have

$$E[(K^{1, m}_T)^2] \leq 2C \beta E[|\xi|^2] + \int_0^T g_s^2 \, ds + \varphi^2(b) + \varphi^2(2T) + 1],$$

which follows that for each $m \in \mathbb{N}$,

$$E[(K^{m}_T)^2] \leq 2C \beta E[|\xi|^2] + \int_0^T g_s^2 \, ds + \varphi^2(b) + \varphi^2(2T) + 1],$$

and so does for $K$, i.e. we get $E[(K_T)^2] < \infty$.

In order to estimate $Z$, we apply Itô’s formula to $|Y_t|^2$ on the interval $[0, T]$, then

$$|Y_0|^2 + \frac{1}{2} E \int_0^T |Z_s|^2 \, ds \leq E |\xi|^2 + E \int_0^T g_s^2 \, ds + (1 + 2\beta^2) E \int_0^T |Y_s|^2 \, ds + E[\sup_{0 \leq t \leq T} |Y_t|^2] + E[(K_T)^2].$$

21
Thanks to the estimates for $Y$ and $K$, there exists a constant $C$ only depends on $\beta$, $T$, and $b$, such that
\[
E \int_0^T |Z_s|^2 \, ds \leq CE[|\xi|^2 + \int_0^T g_s^2 \, ds + \varphi^2(b) + \varphi^2(2T) + 1].
\]
The last is to check the integral condition. Recall that $\int_0^T (Y_{t}^m - L_t) dK_t^m = 0$, then we have
\[
\int_0^{\tau_m} (Y_t - L_t) dK_t = 0, \text{ a.s.}
\]
Since $P$-a.s. $\tau_m = T$, for $m$ large enough, so
\[
\int_0^T (Y_t - L_t) dK_t = 0, \text{ a.s.}
\]
i.e. $(Y, Z, K)$ is a solution of $\text{RBSDE}(\xi, f, L)$ in $S^2(0, T) \times H^2_d(0, T) \times A^2(0, T)$.

\section{Appendix: Comparison theorems}

We first generalize the comparison theorem of RBSDE with superlinear quadratic coefficient, (in view to proposition 3.2 in \[7\]), to compare the increasing processes. Assume that Assumption 1 and 3 hold, and that the coefficient $f$ satisfies:

\textbf{Assumption 7.} For all $(t, \omega)$, $f(t, \omega, \cdot, \cdot)$ is continuous and there exists a function $l$ strictly positive such that
\[
f(t, y, z) \leq l(y) + A |z|^2, \text{ with } \int_0^\infty \frac{dx}{l(x)} = +\infty.
\]

\textbf{Proposition 6.1} Suppose that $\xi^i$ are $\mathcal{F}_T$-adapted and bounded, $f^i(s, y, z)$, $i = 1, 2$ satisfy the condition Assumption 7 and $L$ satisfies Assumption 3. The two triples $(Y^1, Z^1, K^1)$, $(Y^2, Z^2, K^2)$ are respectively solutions of the $\text{RBSDE}(\xi^1, f^1, L)$ and $\text{RBSDE}(\xi^2, f^2, L)$. If we have $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,
\[
\xi^1 \leq \xi^2, f^1(t, y, z) \leq f^2(t, y, z),
\]
then $Y^1_t \leq Y^2_t$, $K^1_t \geq K^2_t$ and $dK^1_t \geq dK^2_t$, for $t \in [0, T]$. 

Proof. From the demonstration of theorem 1 in [7], we know that for \( i = 1, 2 \), \((Y^i, Z^i, K^i)\) is the solution of RBSDE\((\xi^i, f^i, L)\) if and only if \((\theta^i, J^i, \Lambda^i)\) is the solution of RBSDE\((\eta^i, F^i, \mathcal{L})\) where

\[
(\theta^i, J^i, \Lambda^i) = \left( \exp(2AY^i), 2AZ^i\theta^i, 2 \int_0^\infty A \exp(2AY_s)dK^i_s \right)
\]

and

\[
\eta^i = \exp(2A\xi^i), \mathcal{L}_t = \exp(2AL_t)
\]

\[
F^i(t, x, \lambda) = 2Ax[f^i(s, \log x, \lambda) = \frac{\lambda^2}{4Ax^2}]
\]

Then we use the approximation to construct a solution. For \( p \in \mathbb{N} \), we consider the RBSDE\((\eta^i, \tilde{F}^i, \tilde{\mathcal{L}}), \theta^i, \Lambda^i)\), where

\[
\bar{F}^i_p(s, x, \lambda) = g(\rho(\theta))(1 - \kappa_p(\lambda)) + \kappa_p(\lambda)F^i(s, \rho(\theta), \lambda).
\]

Here \( g(x) = 2Ax[(\log x) \frac{1}{2A}, \kappa_p(\lambda) \text{ and } \rho(x) \text{ are smooth functions such that} \kappa_p(\lambda) = 1 \text{ if } |\lambda| \leq p, \kappa_p(\lambda) = 0 \text{ if } |\lambda| \geq p + 1, \text{ and } \rho(x) = x \text{ if } x \in [r, R], \rho(x) = \frac{x}{2} \text{ if } x \in (0, \frac{r}{2}), \rho(x) = R \text{ if } x \in (2R, +\infty), \text{ where } r \text{ and } R \text{ are two constants. Since } \bar{F}^i_p \text{ are bounded and continuous function of } (\theta, \lambda), \text{ the RBSDE}\((\eta^i, \tilde{F}^i, \tilde{\mathcal{L}})\) admits a bounded maximal solution \((\theta^{i,p}, J^{i,p}, \Lambda^{i,p})\), with \( m \leq \theta^{i,p} \leq V_0 \). Here \( m \) and \( V_0 \) are constants given in Theorem 2 in [7].

We know that \( \bar{F}^i_p \downarrow \bar{F}^i \), as \( p \to \infty \), where \( \bar{F}^i = F(s, \rho(\theta), \lambda) \). Thanks to the proof of theorem 1 in [7], it follows that \( \theta^{i,p}_t \downarrow \bar{\theta}^i_t, J^{i,p}_t \uparrow \bar{J}^i_t, 0 \leq t \leq T, \text{ and } \Lambda^{i,p} \to \bar{\Lambda}^i \) in \( \mathcal{H}_d^2(0, T) \) and \((\bar{\theta}^i, \bar{J}^i, \bar{\Lambda}^i)\) is a solution of the RBSDE\((\eta^i, \tilde{F}^i, \tilde{\mathcal{L}})\).

In addition, \( m \leq \bar{\theta}^i \leq V_0 \). So if we choose \( 0 < r < m \) and \( V_0 < R \), then \( \bar{F}^i = F^i \). It follows that \((\bar{\theta}^i, \bar{J}^i, \bar{\Lambda}^i)\) satisfies the RBSDE\((\eta^i, F^i, \mathcal{L})\), i.e. \((\bar{\theta}^i, \bar{J}^i, \bar{\Lambda}^i) = (\theta^i, J^i, \Lambda^i)\).

Since \( f^1(t, y, z) \leq f^2(t, y, z) \), for \((t, x, \lambda) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^d \), we have \( F^1(t, x, \lambda) \leq F^2(t, x, \lambda) \). Then for \( p \in \mathbb{N} \), \( \tilde{F}^i_p(s, x, \lambda) \leq \tilde{F}^i(s, x, \lambda) \). Notice that \( \tilde{F}^i \) is bounded and continuous in \((\theta, \lambda)\) and \( \theta^{i,p} > 0 \), by Lemma 2.1 in [7], it follows that \( \theta^{i,p}_t \leq \theta^2_t, J^{i,p}_t \geq J^2_t, dJ^{i,p}_t \geq dJ^2_t, 0 \leq t \leq T \). And it follows that for \( 0 \leq s \leq t \leq T, J^{i,p}_s \geq J^{2,p}_s \geq J^2_s, J^{1,p}_s \geq J^1_s \). Let \( p \to \infty \), thanks to the convergence results, we get that

\[
\theta^1_t \leq \theta^2_t, J^1_t \geq J^2_t, J^1_s - J^1_s \geq J^2_s - J^2_s.
\]
which implies $dJ^1_t \geq dJ^2_t$. From (10), we know that

$$Y^i_t = \frac{\log(\theta^i_t)}{2A}$$

$$Z^i_t = \frac{\Lambda^i}{2A\theta^i_t},$$

so $Y^1_t \leq Y^2_t$ and $dK^1_t \geq dK^2_t$, which implies that $K^1_t \geq K^2_t$ in view of $K^1_0 = K^2_0 = 0$. □

From this result, we prove the following comparison theorem when the coefficient $f$ satisfies monotonicity and general increasing condition in $y$, and quadratic increasing in $z$.

**Proposition 6.2** Suppose that $\xi^i$ and $f^i(s,y,z)$, $i = 1, 2$ satisfy the condition Assumption 1 and 2, $L$ satisfies Assumption 3. The two triples $(Y^1, Z^1, K^1)$, $(Y^2, Z^2, K^2)$ are respectively the solutions of the RBSDE$(\xi^1, f^1, L)$ and RBSDE$(\xi^2, f^2, L)$. If we have $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$f^1(t, y, z) \leq f^2(t, y, z), \quad \xi^1 \leq \xi^2,$$

then $Y^1_t \leq Y^2_t$, $K^1_t \geq K^2_t$ and $dK^1_t \geq dK^2_t$, for $t \in [0, T]$.

**Proof.** First with changement of $(Y, Z, K)$,

$$(Y^b, Z^b, K^b) = (Y - b, Z, K),$$

we work with $L^b \leq 0$. Since this transformation doesn’t change the monotonicity, then in the following, we assume that the barrier $L$ is a negative bounded process. As in the proof of theorem 3.1 for $C \in \mathbb{R}_+$, let $g^C : \mathbb{R} \to [0, 1]$ continuous which satisfies (11). Set $f^i_C(t, y, z) = g^C(x)f^i(t, y, z)$, $i = 1, 2$, which satisfies Assumption 7, with $l^i(y) = \varphi^i(|2C|)$. We consider solutions $(Y^{i,C}, Z^{i,C}, K^{i,C})$ of the RBSDE$(\xi^i, f^i_C, L^b)$ respectively. Using proposition 6.1 since

$$f^1_C(t, y, z) \leq f^2_C(t, y, z), \quad \xi^1 \leq \xi^2,$$

we get for $t \in [0, T]$,

$$Y^{1,C}_t \leq Y^{2,C}_t, \quad dK^{1,C}_t \geq dK^{2,C}_t.$$

Then by the bounded property of $Y^i$, we choose $C$ big enough like in the proof of theorem 3.1 which follows immediately

$$Y^1_t \leq Y^2_t, \quad dK^1_t \geq dK^2_t, \quad \forall t \in [0, T].$$

□
Proposition 6.3 Suppose that $\xi^i \in L^2(F_T)$, $f^i(s,y,z)$ satisfy the condition Assumption 6, and $L^i$ satisfies Assumption 3, $i = 1, 2$. The two triples $(Y^1, Z^1, K^1), (Y^2, Z^2, K^2)$ are respectively solutions of the RBSDE$(\xi^1, f^1, L)$ and RBSDE$(\xi^2, f^2, L)$. If we have for $\forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\xi^1 \leq \xi^2, \ f^1(t,y,z) \leq f^2(t,y,z), \ L^1_t \leq L^2_t,$$

then $Y^1_t \leq Y^2_t$, for $t \in [0,T]$.

The result comes from the comparison theorem in [8] and the approximation in the proof of theorem [5.1].

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