A theory of tensor products for module categories for a vertex operator algebra, I

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Abstract

This is the first part in a series of papers developing a tensor product theory for modules for a vertex operator algebra. The goal of this theory is to construct a “vertex tensor category” structure on the category of modules for a suitable vertex operator algebra. The notion of vertex tensor category is essentially a “complex analogue” of the notion of symmetric tensor category, and in fact a vertex tensor category produces a braided tensor category in a natural way. The theory applies in particular to many familiar “rational” vertex operator algebras, including those associated with WZNW models, minimal models and the moonshine module. In this paper (Part I), we introduce the notions of P(z)- and Q(z)-tensor product, where P(z) and Q(z) are...
two special elements of the moduli space of spheres with punctures and local coordinates, and we present the fundamental properties and constructions of $Q(z)$-tensor products.

1 Introduction

In this paper, we begin the detailed development of the theory of tensor products of modules for a vertex operator algebra initiated in [HL1] and described in [HL5]. Because this work is based on a new point of view and new techniques, we start with some fairly extensive motivation for and explanation of this point of view before presenting the main content of this paper.

It has become more and more evident that the theory of vertex operator algebras and their representations provides the natural foundation and context for the deeper study of a wide range of structures and concepts, including the Fischer-Griess Monster sporadic finite simple group and its relations with “monstrous moonshine”; conformal field theory as it occurs in both mathematics and physics; representation theory of several classes of infinite-dimensional Lie algebras, including affine Kac-Moody algebras and the Virasoro algebra; and topological invariants such as the Jones [J], HOMFLY and Jones-Witten [W2] invariants. The theory, which has been extensively developed in many directions and under different guises, has exhibited a striking philosophical paradox, which may be described as follows (cf. [Le]):

One of the main motivations for the development of the theory in [B1] and [FLM2] was the need for a natural infinite-dimensional structure on which the Monster would act in a natural way as the automorphism group, and whose properties would provide a proof of the Conway-Norton conjectures [CN] relating the Monster to modular functions. In fact, a certain part of these conjectures was proved in [FLM1] and [FLM2] by means of the construction of the “moonshine module” and of the action of the Monster on it, and the rest of the main conjecture was established in [B2], where the McKay-Thompson series for the moonshine module not determined in [FLM1], [FLM2] were all determined and found to agree with those proposed in [CN].

The Monster is arguably the most “exceptional” finite symmetry group that nature allows, and while there is no claim that it is easy to define, the moonshine module is arguably the most natural structure on which the Mon-
ster acts as the automorphism group. On the other hand, in a certain precise sense, this structure which has the richest possible symmetry group exhibits the most trivial possible monodromy: In conformal field theory or vertex operator algebra theory, the monodromy of the correlation functions formed from intertwining operators among modules for a suitable vertex operator algebra provides representations of braid groups and other mapping class groups. These in turn lead to the construction of certain braided monoidal (tensor) categories (see [JS]) and to knot and 3-manifold invariants, for such conformal field theories as the Wess-Zumino-Novikov-Witten model [W1]. One can also obtain these braided monoidal categories from representations of certain quantum groups. For important aspects of this extensive program, see in particular [BPZ], [Dr1], [Dr2], [KZ], [K1], [K2], [MS], [R1], [SV], [IK], [Va]. But as conjectured in [FLM2] and proved by Dong [Do], the moonshine module vertex operator algebra admits only itself as an irreducible module, and this implies the monodromy-triviality; that is, in terms of monodromy, the moonshine module is no different from the trivial conformal field theory. The subtlety of the Monster and the moonshine module is “orthogonal” to the subtlety of braiding and fusing in conformal field theory (cf. [MS]). Thus we have the fundamental question of “lifting” both the theory of the symmetry-rich but monodromy-free model, the moonshine module, and also the theory of the monodromy-rich models, to a common theory incorporating the deepest features of both.

On the other hand, in recent years, there has been much effort by mathematicians to develop conformal field theory as a serious mathematical theory. Certain works by physicists already suggested that if the solid foundation of conformal field theory could be laid down, there would be many interesting and powerful mathematical consequences and physical applications. Segal’s definition of conformal field theory [Se] was essentially the first work by a mathematician in this direction. Though his definition is beautiful and rigorous, the problem of constructing even a single complete nontrivial example satisfying the definition has remained open. Most of the mathematical works on conformal field theories center on “modular functors” or “conformal blocks” (see for example [Se], [TUY]), including monodromy and representations of braid groups and other mapping class groups, rather than on conformal fields themselves. The difficulties involve not just some new formalism needed to rigorize physicists’ work, but more importantly, certain powerful machinery enabling one to construct a conformal field theory from
some relatively elementary mathematical data. The theory of vertex operator algebras provides exactly the formalism we need, and the appropriate powerful machinery is being developed within its framework.

In the present series of papers, an essential part of the new formalism and machinery, a tensor product theory for modules for a vertex operator algebra, will be developed, with the problems and requirements above as its most basic goals and features. The main idea is to exploit fully the conformal structure of conformal field theory rather than just the topological structure upon which the familiar considerations of monodromy and braided monoidal categories are based. The precise formulation of this idea is accomplished with the help of the notion of operad, which was originally introduced in a topological context ([St1], [St2], [M]) in connection with the homotopy-theoretic characterization of loop spaces. Operads are systems which abstract the notion and properties of such operations as substitution operations in algebra or certain sewing operations in geometry. They can be found “everywhere”: Classical algebraic structures like groups, algebras and Lie algebras are in fact always implicitly based on operads defined using one-dimensional geometric objects such as punctured circles and binary trees. In [HL2] and [HL3], the geometric interpretation given by the first-named author ([H1], [H2], [H5]) of the notion of vertex operator algebra is used to define a certain operadic structure, but this time two-dimensional (or rather, one-complex-dimensional), based on spheres with tubes (punctures and local coordinates), in such a way that vertex operator algebras can be viewed as based on these structures by analogy with the way that say associative algebras can be viewed as based on the circle with punctures and local coordinates. This amounts essentially to a precise formulation of the well-known philosophy of the physical and geometric foundation of conformal field theory at genus zero. For the exact definition and the detailed study of this particular operadic structure, the “vertex (partial) operad,” the reader is referred to [HL2], [HL3] and [H3].

This operadic formulation of the notion of vertex operator algebra enables us to raise questions at a fundamental level: For instance, the very notion of tensor category can be viewed as based on the circle operad. Is there a reasonable “vertex” analogue of the notion of tensor category, based on the “vertex operad” of [HL2], [HL3]? An answer to this question is given in [HL3], an overview of the theory being developed in this series of papers, in the form of a new notion of “vertex tensor category”—an abelian category equipped with suitably symmetric “tensor product” operations and associa-
tivity isomorphisms and constraints, etc., which are essentially parametrized and controlled by spheres with punctures and local coordinates rather than by the traditional natural isomorphisms and commutative diagrams of coherence theory for monoidal categories. This is not merely a curious abstraction; the goal of the present series of papers is to show that the category of modules for a suitable vertex operator algebra is a vertex tensor category in a natural way. Vertex tensor categories contain canonical conformal information, not just the much more restricted topological (monodromy) information on which the braided tensor categories are based. In general, vertex tensor categories naturally give rise to braided tensor categories, when one forgets the conformal structure and keeps only the topological information.

In the representation theory of Lie algebras, we have the classical notion of tensor product of two modules, providing the conceptual foundation of the Clebsch-Gordan coefficients. The tensor product operation is an operation on the category of modules for a Lie algebra, giving a classical example of a tensor category satisfying an additional symmetry axiom. For quantum groups (Hopf algebras), the module categories are also tensor categories but in general do not satisfy the symmetry axiom, corresponding to the fact that the Hopf algebra need not be cocommutative. Instead, such tensor categories satisfy weaker conditions — braiding conditions. From the resulting braid group representations, one can construct knot and link invariants. See in particular [J], [K1], [K2], [Dr1], [Dr2], [R T].

Vertex operator algebras ([B1], [FLM2], [FHL]) are “complex analogues” of both Lie algebras and commutative associative algebras. They are essentially equivalent to “chiral algebras” in conformal field theory (see in particular [BPZ] and [MS]). For vertex operator algebras, we also have the notions of modules, intertwining operators among triples of modules and fusion rules (dimensions of spaces of intertwining operators) analogous to those for Lie algebras. We need to use the versions of these notions given in [FLM2] and [FHL], and recalled below. In particular, the appropriate notion of intertwining operator is the one defined in [FHL], based on the Jacobi identity axiom for vertex operator algebras ([FLM2], [FHL]). In the study of rational conformal field theories ([BPZ], [FS]), intertwining operators (or chiral vertex operators) are fundamental tools. Many important concepts and results, for example, representations of braid groups, the relationship between modular transformations and fusion rules, and duality relations, are obtained through the study of intertwining operators; see for example [KZ], [TK], [Ve] and
The fusion rules for a vertex operator algebra being the analogues of the Clebsch-Gordan coefficients for a Lie algebra, we have the natural question whether there exists a conceptual notion of tensor product for modules for a vertex operator algebra which would naturally provide a conceptual foundation for fusion rules and intertwining operators.

We noticed a few years ago that the Jacobi identity axiom (see [FLM2], [FHL]) for vertex operator algebras suggests a kind of “complex analogue” of the coalgebra diagonal map for primitive elements of a Hopf algebra, but it turned out that a considerable amount of work was needed to make this idea precise and useful. In this paper we begin this program in detail. Given two modules \( W_1 \) and \( W_2 \) for a vertex operator algebra \( V \), when one tries to define a tensor product module, the first serious problem is that the tensor product vector space \( W_1 \otimes W_2 \) is not a \( V \)-module in any natural way (although it is a module for the vertex operator algebra \( V \otimes V \)), and so the underlying vector space of a tensor product module would not be expected to be the tensor product vector space. Another serious problem is that a vertex operator algebra is not a Hopf algebra in any natural sense. We need a new way to define and construct a tensor product module — both the underlying vector space and the action of the vertex operator algebra. As we shall see, the analogy between vertex operator algebras and Lie algebras, centered on the Jacobi identity axiom, provides an analogue of a Hopf algebra diagonal map for a construction of a tensor product module, under appropriate hypotheses. In addition, the analogy between vertex operator algebras and commutative associative algebras, via the geometric and operadic formulation of the notion of vertex operator algebra (\([H1],[H2],[HL2],[HL3],[H5]\)), provides the geometric foundation for the construction.

One important class of examples of vertex operator algebras is constructed from certain modules for affine Lie algebras (see for example [FZ], [DL]). There are interesting relations between representations of affine Lie algebras and of quantum groups discussed in several of the works mentioned above, for example, and to understand these relations on a deeper level, one natural strategy is to compare categories of modules for affine Lie algebras with a fixed nonzero central charge and categories of modules for associated quantum groups. While the category of modules for a quantum group is a tensor category, a category of modules of a fixed nonzero level for an affine Lie algebras does not close under the classical tensor product of Lie algebra modules. Thus an appropriate tensor product module, if it exists, could not
be the ordinary one.

Recently, Kazhdan and Lusztig ([KL1]–[KL5]) have found and constructed such a tensor product operation for certain module categories of a fixed but (mostly) negative level for an affine Lie algebra and have shown that these module categories can in fact be made into braided tensor categories. Moreover, they have shown that these tensor categories are equivalent to suitable categories of modules for corresponding quantum groups. On the other hand, from the viewpoint of conformal field theory, the more relevant cases involve positive integral levels, including the case of the category whose objects are finite direct sums of modules isomorphic to standard (integrable highest weight) modules of a fixed positive integral level for an affine Lie algebra. Such “fusion categories” have been discussed on a physical level of rigor in many works, including [MS]. Under the assumption, which is a consequence of the theory developed in the present series of papers, that these categories have natural braided tensor category structure, Finkelberg [F] has shown that these braided tensor categories are related to corresponding braided tensor categories constructed by Kazhdan and Lusztig, and that they are equivalent to certain categories of representations of quantum groups. (See also [Va]). The construction of Kazhdan-Lusztig was in fact motivated by conformal field theory, and we expected that their tensor product operation should come from more general and natural structures in conformal field theory.

In [HL1], partly motivated by the analogy between vertex operator algebras and Lie algebras and partly motivated by the announcement [KL1], a project toward a theory of tensor products for modules for a vertex operator algebra was initiated. In the present series of papers, we shall present this theory of tensor products for suitable module categories for a vertex operator algebra. Our methods are independent of the methods of [KL1]–[KL5], even in the case in which our vertex operator algebra is associated to an affine Lie algebra. In place of the braided monoidal categories that arise from the Kazhdan-Lusztig construction ([KL1]–[KL5]), the result is instead the “vertex tensor categories” mentioned above. What we have is a conceptual “complexification” of the notion of symmetric tensor category. Moreover, as we have mentioned, a systematic specialization process yields an ordinary braided monoidal category from the vertex tensor category, for suitable vertex operator algebras. This category is the usual braided monoidal category giving the connection with the representation theory of quantum groups (see [KL1]–[KL5], [F]) and knot and link invariants. That is, the familiar and
fundamental topological information generated by conformal field theory at genus zero now becomes a specialization of a theory systematically “complexified” starting from its foundations. In the case that the vertex operator algebra is constructed from an affine Lie algebra, the braided tensor category obtained from the vertex tensor category of modules for this vertex operator algebra gives us in particular a braided tensor category structure on the relevant category of modules for the corresponding affine Lie algebra; thus the present theory of tensor products for module categories for a vertex operator algebra yields expected but nontrivial conformal-field-theoretical properties of affine Lie algebras from a general viewpoint.

We emphasize that the theory being developed in the present series of papers is based on the concepts of vertex operator algebra theory rather than on the methods of [KL1]–[KL5], which use special properties of affine Lie algebras, and that an important feature of this distinction in viewpoints is that the notion of intertwining operator that is the starting point of this work is the notion based on the Jacobi identity rather than the notion of intertwining operator used by many researchers in conformal field theory, based on certain Lie algebra coinvariants. The two notions are indeed equivalent for the WZNW model and related models, but at every stage in the analysis it is much more natural for us to work with the Jacobi identity for vertex operator algebras. Using this Jacobi identity, a canonical notion of tensor product of modules for a suitable vertex operator algebra is introduced, defined in terms of an appropriate universal property and depending on a given element of the moduli space of spheres with three tubes. One aspect of the subtle nature of this construction is that, as was also the case in [KL1]–[KL5], the underlying vector space of the tensor product module is not at all the tensor product vector space of the given modules.

However, the theory does in fact provide analogues of the concrete elements of classical tensor product modules (the usual linear combinations of tensors of elements of the given modules). Let \( W_1 \) and \( W_2 \) be modules for a suitable vertex operator algebra. For \( w_{(1)} \in W_1 \) and \( w_{(2)} \in W_2 \), we have the element \( w_{(1)} \boxtimes w_{(2)} \) (more precisely, \( w_{(1)} \boxtimes_{P(z)} w_{(2)} \) or \( w_{(1)} \boxtimes_{Q(z)} w_{(2)} \) for \( z \in \mathbb{C}^\times \) or more generally, \( w_{(1)} \boxtimes_{\tilde{Q}} w_{(2)} \) for any \( \tilde{Q} \) in the determinant line bundle over the moduli space of spheres with punctures and local coordinates). But these analogues of classical tensor product elements are elements of the algebraic completion of the tensor product module \( W_1 \boxtimes W_2 \) (more precisely,
$W_1 \boxtimes_{P(z)} W_2$ or $W_1 \boxtimes W_2$ or more generally, $W_1 \boxtimes_{\tilde{Q}} W_2$), not of the tensor product module itself. This necessary feature makes the theory much more difficult than the classical theory. But this difficulty is overcome by a characterization of a certain abstractly-defined subspace of the dual space of the tensor product vector space of two modules by means of a certain list of conditions, the most important of which is what we call the “compatibility condition,” which is motivated by the Jacobi identity and which allows the abstract machinery to work. In fact, one of the main theorems—Theorem 6.3—of the present paper says that this space of vectors is in fact a module in a certain generalized sense; the proof of this result requires intricate formal calculations based on the Jacobi identity. This general theorem enables us to establish the conceptual vertex-tensor-categorical properties of the tensor product operation, by analogy with the way in which one’s ability to write down concrete tensor product vectors in a classical tensor product module enables one to establish the properties of classical tensor categories (such as the associativity or commutativity properties). Even though the tensor product of two modules can still be constructed without Theorem 6.3, this result is crucial for proving all the important theorems, including the associativity (see Part IV [H6]) and the vertex tensor category structure. The methods in the present theory are necessarily based heavily on both the machinery of the purely algebraic formal calculus (see [FLM2] and [FHL]) and the machinery of the geometric interpretation of the notion of vertex operator algebra (see [H1], [H2], [H5]).

It is important to note that this theory is both valid and nontrivial for such vertex operator algebras as the moonshine module. Even though the moonshine module exhibits no monodromy, it is expected to possess rich vertex-tensor-categorical structure coming from the conformal geometry. In particular, the present theory is expected to provide a resolution of the philosophical paradox described above.

This theory also reduces the construction of the genus-zero chiral parts of “rational conformal field theories” to the construction of the corresponding vertex operator algebras, the proof of the rationality of these vertex operator algebras and the proof of certain properties of products of intertwining operators for the vertex operator algebras; these are typically much easier problems. For some familiar conformal field theories, for example, the WZNW models and the minimal models, these problems concerning the corresponding vertex operator algebras and intertwining operators can be solved easily.
using existing results. For some other interesting conformal field theories, for example, those constructed from certain $\mathcal{W}$-algebras and orbifold theories, these problems can also be solved.

In particular, in an application related to both monstrous moonshine and the construction of conformal field theories, in the case of the $\mathbb{Z}_2$-orbifold conformal field theory which produces the moonshine module for the Monster (as constructed in \cite{FLM2}) the whole genus-zero chiral part of the theory has recently been shown by the first-named author \cite{H7}, heavily using the present new tensor product theory, to be an abelian intertwining algebra, in the sense of \cite{DL}. (Abelian intertwining algebras are certain generalizations of vertex operator algebras for which one-dimensional braid group representations are incorporated naturally into the structure of the algebra.) Without the tensor product theory, this would be an exceedingly difficult problem, since, for example, even the proof by Frenkel, Meurman and the second-named author \cite{FLM2} that the moonshine module is a vertex operator algebra is already so involved.

It should be emphasized that our theory applies (at least) to an arbitrary rational vertex operator algebra satisfying certain additional conditions including some convergence conditions. It applies in particular to the WZNW models, minimal models and the moonshine module vertex operator algebra (whose rationality has been proved by Dong \cite{Do}). Many of the notions, constructions and techniques also apply to more general vertex operator algebras.

In the remainder of this Introduction, we focus more technically on the material in this paper. We use the analogy between vertex operator algebras and Lie algebras as a guide. In the theory of Lie algebras we have the following standard notion of intertwining map (of type $\left(\mathcal{W}_3\mathcal{W}_1\mathcal{W}_2\right)$) among modules $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ for a Lie algebra $\mathcal{V}$, with corresponding actions $\pi_1, \pi_2, \pi_3$ of $\mathcal{V}$: a linear map $I$ from the tensor product vector space $\mathcal{W}_1 \otimes \mathcal{W}_2$ to $\mathcal{W}_3$, satisfying the identity

$$\pi_3(v)I(w_{(1)} \otimes w_{(2)}) = I(\pi_1(v)w_{(1)} \otimes w_{(2)}) + I(w_{(1)} \otimes \pi_2(v)w_{(2)})$$

(1.1)

for $v \in \mathcal{V}$, $w_{(1)} \in \mathcal{W}_1$, $w_{(2)} \in \mathcal{W}_2$. This “Jacobi identity for intertwining maps” agrees with the Jacobi identity for $\mathcal{V}$ when all three modules are the adjoint module. Let us call a product of $\mathcal{W}_1$ and $\mathcal{W}_2$ a third module $\mathcal{W}_3$ equipped with an intertwining map $I$ of type $\left(\mathcal{W}_3\mathcal{W}_1\mathcal{W}_2\right)$; we denote this by
(W_3, I). Then a tensor product of W_1 and W_2 is a product (W_1 \otimes W_2, \otimes) such that given any product (W_3, I), there exists a unique module map \eta from W_1 \otimes W_2 to W_3 such that

\[ I = \eta \circ \otimes. \]  

Thus any tensor product of two given modules has the following property: The intertwining maps from the tensor product vector space of the two modules to a third module correspond naturally to the module maps from the tensor product module to the third module. Moreover, this universal property characterizes the tensor product module up to unique isomorphism.

In this paper (Part I), we analogously define notions of P(z)-tensor product and Q(z)-tensor product of two modules for a vertex operator algebra, where z is a nonzero complex number and P(z) and Q(z) are two particular elements, depending on z, of a certain moduli space of spheres with punctures and local coordinates (see [H1], [H2], [H3], [H4] or [H5]). We give two constructions of a Q(z)-tensor product when the vertex operator algebra that we consider is such that its module category (or some fixed subcategory) is closed under a certain operation. This occurs in particular if the module category of a vertex operator algebra satisfies certain finiteness and semisimplicity conditions, and so the Q(z)-tensor product of two modules exists in this case. Such vertex operator algebras are said to be “rational.” (For such algebras we also give a “tautological” construction of a tensor product module; this in fact provides an existence proof.) The construction of a P(z)-tensor product will be given in Part III [HL6] using the results in the constructions of the Q(z)-tensor product constructed in this paper (Part I).

The first of our two constructions of a Q(z)-tensor product is straightforward and conceptually simple, but it is difficult to use. Our second, much more useful, construction, presents the Q(z)-tensor product module of two modules W_1 and W_2 (when it exists) in terms of the subspace of the dual (W_1 \otimes W_2)^* of the vector space tensor product consisting of the elements satisfying a certain set of conditions, the most important of which is the “compatibility condition.”

The dependence of the tensor product operation on the nonzero complex number z is a fundamental feature of our theory. The number z actually represents an element in the moduli space of spheres with punctures and local coordinates mentioned above. In one of the papers in this series, we
shall see that for every element of this moduli space (more precisely, for any element of the determinant line bundle over this moduli space (see [H5])), we have a tensor product operation. The associativity, commutativity and coherence properties of this tensor product depend on (the determinant line bundle over) this moduli space and the sewing operation in a natural way. Such properties are in fact the data and axioms in the definition of vertex tensor category (see [HL3]).

Our approach is based on the formal calculus developed in [FLM2], and also (in later papers in this series) on the geometric methods developed in [H1], [H5]. Our use of formal calculus (see [FLM2], [FHL]) is equivalent to the use of contour integral methods, but is far more natural and appropriate for our formulations and arguments. For example, in Section 3, the space of rational functions whose action we must define is described conceptually by means of the formal $\delta$-function.

The formal-calculus techniques that we use in the present series of papers might seem unfamiliar at first but we would like to reassure the reader that these methods are as natural as any used in classical mathematics, and are the right ones for the task. A reader who spends a bit of time to become familiar with the use of the basic properties of the formal $\delta$-function will have little difficulty reading the papers in this series.

Results in the present series of papers were announced in [HL1] and in talks presented by both authors at the June, 1992 AMS-IMS-SIAM Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups at Mount Holyoke College.

A notion of tensor product related to but different from ours has been studied in [Li]. Also, related ideas, on a physical level of rigor, based on suggestions of Borcherds, are discussed in [G].

Part I is organized as follows: Section 2 reviews basic concepts in the representation theory of vertex operator algebras. Section 3 discusses affinizations of vertex operator algebras, the opposite module structure on a module for a vertex operator algebra and a related $*$-operation. In [B1], in fact Borcherds placed a vertex algebra structure on a certain affinization of a vertex algebra (in his sense), while in this paper we are using more general affinizations of a vertex operator algebra, but in a simpler way. Section 4 gives the definitions of $P(z)$- and $Q(z)$-tensor product of two modules for a vertex operator algebra and establishes some straightforward consequences, including relations among intertwining operators, “intertwining maps” and
tensor products, and the existence of a $Q(z)$-tensor product of the two modules for a rational vertex operator algebra. In this section, we formulate and use a result (Proposition 4.9) giving an isomorphism between certain spaces of intertwining operators and we defer its proof to Part II [HL4]. Sections 5 and 6 present the first and second constructions of the $Q(z)$-tensor product of two modules, respectively. In the course of these constructions, we formulate and use three results, Proposition 5.2, Theorem 6.1 and Proposition 6.2, whose proofs will form the main content of Part II. Proposition 5.2 is a commutator formula for vertex operators acting on the dual space of the vector space $W_1 \otimes W_2$, where $W_1$ and $W_2$ are given modules whose vertex-operator-algebra-theoretic tensor product we are studying. Theorem 6.1 states that the Jacobi identity holds when we restrict to the subspace of $(W_1 \otimes W_2)^*$ consisting of the elements satisfying compatibility condition. Proposition 6.2 asserts that this subspace is stable under the action of our operators. Theorem 6.1 and Proposition 6.2 are combined in Theorem 6.3 to give a characterization of the tensor product of two modules. Theorem 6.1 and Proposition 6.2 and their extensions constitute the foundation of the whole theory.

**Historical note** Since the time that the present paper and Part II [HL4] were circulated as preprints starting in 1993 and were submitted to the high-energy theoretical physics electronic preprint archive hep-th (as numbers 9309076 and 9309159, respectively), we have expanded and updated the introduction and references and corrected a few misprints; otherwise, Parts I and II are the same as the original preprints. Since then, the main technical part of the theory, building on Theorem 6.3 of the present paper, has been carried out, including the associativity. See Part III [HL5], Part IV [H6] and the overview [HL5] for details. The theory has also been applied to the study of the moonshine module in [H7] and to the study of the (nonmeromorphic) operator product expansion for intertwining operators in [H8].

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2 Review of basic concepts

In this section, we review some basic definitions and concepts in the representation theory of vertex operator algebras. Except for Definition 2.11, everything in this section can be found in [FLM2] and [FHL].

In this paper, all the variables \( x, x_0, \ldots \) are independent commuting formal variables, and all expressions involving these variables are to be understood as formal Laurent series or, when explicitly so designated, as formal rational functions. (Later, we shall also use the symbols \( z, z_0, \ldots \), which will denote complex numbers, not formal variables.) We use the formal expansion

\[
\delta(x) = \sum_{n \in \mathbb{Z}} x^n. \tag{2.1}
\]

This “formal \( \delta \)-function” has the following simple and fundamental property: For any \( f(x) \in \mathbb{C}[x, x^{-1}] \),

\[
f(x)\delta(x) = f(1)\delta(x). \tag{2.2}
\]

This property has many important variants. For example, for any

\[ X(x_1, x_2) \in \langle \text{End } W \rangle[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \]

(where \( W \) is a vector space) such that

\[
\lim_{x_1 \to x_2} X(x_1, x_2) = X(x_1, x_2) \bigg|_{x_1 = x_2} \tag{2.3}
\]

exists, we have

\[
X(x_1, x_2)\delta \left( \frac{x_1}{x_2} \right) = X(x_2, x_2)\delta \left( \frac{x_1}{x_2} \right). \tag{2.4}
\]

The existence of the “algebraic limit” defined in (2.3) means that for an arbitrary vector \( w \in W \), the coefficient of each power of \( x_2 \) in the formal
expansion \( X(x_1, x_2)w \bigg|_{x_1=x_2} \) is a finite sum. We use the convention that negative powers of a binomial are to be expanded in nonnegative powers of the second summand. For example,

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N}, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^n x_2^m.
\]

We have the following identities:

\[
x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) = x_2^{-1} \left( \frac{x_1 - x_0}{x_2} \right), \quad (2.6)
\]

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right). \quad (2.7)
\]

We shall use these properties and identities extensively later on without explicit comment. See \[FLM2\] and \[FHL\] for further discussion and many examples of their use.

We now quote the definition and basic "duality" properties of vertex operator algebras from \[FLM2\] or \[FHL\]:

**Definition 2.1** A vertex operator algebra (over \( \mathbb{C} \)) is a \( \mathbb{Z} \)-graded vector space (graded by weights)

\[
V = \coprod_{n \in \mathbb{Z}} V(n); \text{ for } v \in V(n), \ n = \text{wt } v; \quad (2.8)
\]

such that

\[
\dim V(n) < \infty \text{ for } n \in \mathbb{Z}, \quad (2.9)
\]

\[
V(n) = 0 \text{ for } n \text{ sufficiently small}, \quad (2.10)
\]

equipped with a linear map \( V \otimes V \to V[[x, x^{-1}]] \), or equivalently,

\[
\begin{align*}
V & \to (\text{End } V)[[x, x^{-1}]] \\
v & \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End } V), \quad (2.11)
\end{align*}
\]

\( Y(v, x) \) denoting the vertex operator associated with \( v \), and equipped also with two distinguished homogeneous vectors \( 1 \in V(0) \) (the vacuum) and \( \omega \in V(2) \).
The following conditions are assumed for $u, v \in V$: the *lower truncation condition* holds:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \quad (2.12)$$

(or equivalently, $Y(u, x)v \in V((x)))$;

$$Y(1, x) = 1 \quad (1 \text{ on the right being the identity operator); \quad (2.13)$$

the *creation property* holds:

$$Y(v, x) 1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x) 1 = v \quad (2.14)$$

(that is, $Y(v, x) 1$ involves only nonnegative integral powers of $x$ and the constant term is $v$); the *Jacobi identity* (the main axiom) holds:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0) v, x_2) \quad (2.15)$$

(note that when each expression in (2.15) is applied to any element of $V$, the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in the obvious way to $V[[x_0, x_0^{-1}]]$; the Virasoro algebra relations hold:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \quad (2.16)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2} \quad (2.17)$$

and

$$c \in \mathbb{C}; \quad (2.18)$$

$$L(0)v = nv = (\text{wt } v)v \quad \text{for } n \in \mathbb{Z} \text{ and } v \in V(n); \quad (2.19)$$

$$\frac{d}{dx} Y(v, x) = Y(L(-1)v, x) \quad (2.20)$$

(the *$L(-1)$-derivative property*).
The vertex operator algebra just defined is denoted by \((V, Y, 1, \omega)\) (or simply by \(V\)). The complex number \(c\) is called the \textit{central charge} or \textit{rank} of \(V\). Homomorphisms of vertex operator algebras are defined in the obvious way.

\textbf{Remark 2.2} The axioms above imply that if \(v \in V\) is homogeneous and \(n \in \mathbb{Z}\),
\[
\text{wt } v_n = \text{wt } v - n - 1 \quad (2.21)
\]
as an operator. We shall also use the fact that in the presence of other axioms, the Virasoro algebra commutator relations (2.16) are equivalent to the relation
\[
Y(\omega, x)\omega = \frac{1}{2}c 1 x^{-4} + 2\omega x^{-2} + L(-1)\omega x^{-1} + v \quad (2.22)
\]
where \(v \in V[[x]]\).

Vertex operator algebras have important “rationality,” “commutativity” and “associativity” properties, collectively called “duality” properties. These properties can in fact be used as axioms replacing the Jacobi identity in the definition of vertex operator algebra, as we now recall.

In the propositions below, \(\mathbb{C}[x_1, x_2]_S\) is the ring of rational functions obtained by inverting (localizing with respect to) the products of (zero or more) elements of the set \(S\) of nonzero homogeneous linear polynomials in \(x_1\) and \(x_2\). Also, \(\iota_{12}\) (which might also be written as \(\iota_{x_1x_2}\)) is the operation of expanding an element of \(\mathbb{C}[x_1, x_2]_S\), that is, a polynomial in \(x_1\) and \(x_2\) divided by a product of homogeneous linear polynomials in \(x_1\) and \(x_2\), as a formal series containing at most finitely many negative powers of \(x_2\) (using binomial expansions for negative powers of linear polynomials involving both \(x_1\) and \(x_2\)); similarly for \(\iota_{21}\) and so on. (The distinction between rational functions and formal Laurent series is crucial.)

For any \(\mathbb{Z}\)-graded, or more generally, \(\mathbb{C}\)-graded, vector space \(W = \bigsqcup W_{(n)}\), we use the notation
\[
W' = \bigsqcup W_{(n)^*} \quad (2.23)
\]
for its graded dual.

\textbf{Proposition 2.3 (a) (rationality of products)} For \(v, v_1, v_2 \in V\) and \(v' \in V'\), the formal series \(\langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle\), which involves only finitely
many negative powers of $x_2$ and only finitely many positive powers of $x_1$, lies in the image of the map $\iota_{12}$:

$$\langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = \iota_{12}f(x_1, x_2), \quad (2.24)$$

where the (uniquely determined) element $f \in \mathbb{C}[x_1, x_2]_S$ is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \quad (2.25)$$

for some $g \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) (commutativity) We also have

$$\langle v', Y(v_2, x_2)Y(v_1, x_1)v \rangle = \iota_{21}f(x_1, x_2). \quad \square \quad (2.26)$$

**Proposition 2.4 (a) (rationality of iterates)** For $v, v_1, v_2 \in V$ and $v' \in V'$, the formal series $\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle$, which involves only finitely many negative powers of $x_0$ and only finitely many positive powers of $x_2$, lies in the image of the map $\iota_{20}$:

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle = \iota_{20}h(x_0, x_2), \quad (2.27)$$

where the (uniquely determined) element $h \in \mathbb{C}[x_0, x_2]_S$ is of the form

$$h(x_0, x_2) = \frac{k(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t} \quad (2.28)$$

for some $k \in \mathbb{C}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) The formal series $\langle v', Y(Y(v_1, x_0 + x_2)v_2, x_2)v \rangle$, which involves only finitely many negative powers of $x_2$ and only finitely many positive powers of $x_0$, lies in the image of $\iota_{02}$, and in fact

$$\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle = \iota_{02}h(x_0, x_2). \quad \square \quad (2.29)$$

**Proposition 2.5 (associativity)** We have the following equality of rational functions:

$$\iota_{12}^{-1} \langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = (\iota_{20}^{-1} \langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle) \bigg|_{x_0 = x_1 - x_2}. \quad \square \quad (2.30)$$
Proposition 2.6 In the presence of the other axioms, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of vertex operator algebra, the Jacobi identity may be replaced by these properties. □

We have the following notions of module and of intertwining operator for vertex operator algebras:

**Definition 2.7** Given a vertex operator algebra \((V, Y, 1, \omega)\), a module for \(V\) (or \(V\)-module or representation space) is a \(\mathbb{C}\)-graded vector space (graded by weights)

\[
W = \prod_{n \in \mathbb{C}} W_{(n)}; \quad \text{for } w \in W_{(n)}, \quad n = \text{wt } w;
\]  

(2.31)
such that

\[
\text{dim } W_{(n)} < \infty \quad \text{for } n \in \mathbb{C},
\]  

(2.32)

\[
W_{(n)} = 0 \quad \text{for } n \text{ whose real part is sufficiently small},
\]  

(2.33)
equipped with a linear map \(V \otimes W \to W[[x, x^{-1}]]\), or equivalently,

\[
V \to (\text{End } W)[[x, x^{-1}]]
\]

\[
v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad \text{(where } v_n \in \text{End } W)\]

(2.34)

(note that the sum is over \(\mathbb{Z}\), not \(\mathbb{C}\)), \(Y(v, x)\) denoting the vertex operator associated with \(v\), such that “all the defining properties of a vertex operator algebra that make sense hold.” That is, for \(u, v \in V\) and \(w \in W\),

\[
v_n w = 0 \quad \text{for } n \text{ sufficiently large}
\]  

(2.35)

(the lower truncation condition);

\[
Y(1, z) = 1;
\]  

(2.36)

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)
\]  

(2.37)
(the Jacobi identity for operators on $W$); note that on the right-hand side, $Y(u, x_0)$ is the operator associated with $V$; the Virasoro algebra relations hold on $W$ with scalar $c$ equal to the central charge of $V$:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$  \hspace{1cm} (2.38)

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \text{ for } n \in \mathbb{Z}, \text{ i.e., } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2};$$  \hspace{1cm} (2.39)

$$L(0)w = nw = (wt)w \text{ for } n \in \mathbb{C} \text{ and } w \in W;$$  \hspace{1cm} (2.40)

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x),$$  \hspace{1cm} (2.41)

where $L(-1)$ is the operator on $V$.

This completes the definition of module. We may denote the module just defined by $(W, Y)$ (or simply by $W$). If necessary, we shall use $Y_W$ or similar notation to indicate that the vertex operators concerned act on $W$. Homomorphisms (or maps) of $V$-modules are defined in the obvious way. For $V$-modules $W_1$ and $W_2$, we shall denote the space of module maps from $W_1$ to $W_2$ by $\text{Hom}_V(W_1, W_2)$.

**Remark 2.8** Formula (2.21) holds for modules. Also note that the Virasoro algebra commutator relations (2.38) are in fact consequences of the other axioms, in view of (2.22).

For any vector space $W$ and any formal variable $x$, we use the notation

$$W\{x\} = \left\{\sum_{n \in \mathbb{C}} a_n x^n | a_n \in W, \ n \in \mathbb{C}\right\}.$$  \hspace{1cm} (2.42)

In particular, we shall allow complex powers of our commuting formal variables.

**Definition 2.9** Let $V$ be a vertex operator algebra and let $(W_1, Y_1)$, $(W_2, Y_2)$ and $(W_3, Y_3)$ be three $V$-modules (not necessarily distinct, and possibly equal
to $V$). An intertwining operator of type $\left(\frac{W_3}{W_1, W_2}\right)$ is a linear map $W_1 \otimes W_2 \to W_3\{x\}$, or equivalently,

$$W_1 \to (\text{Hom}(W_2, W_3))\{x\}$$
$$w \mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1} \quad \text{(where } w_n \in \text{Hom}(W_2, W_3)) \quad (2.43)$$

such that “all the defining properties of a module action that make sense hold.” That is, for $v \in V$, $w^{(1)} \in W_1$ and $w^{(2)} \in W_2$, we have the lower truncation condition

$$(w^{(1)})_n w^{(2)} = 0 \text{ for } n \text{ whose real part is sufficiently large;} \quad (2.44)$$

the following Jacobi identity holds for the operators $Y_1(v, \cdot)$, $Y_2(v, \cdot)$, $Y_3(v, \cdot)$ and $\mathcal{Y}(\cdot, x_2)$ acting on the element $w^{(2)}$:

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_2 - x_0}\right) Y_3(v, x_1) \mathcal{Y}(w^{(1)}, x_2) w^{(2)}$$
$$-x_0^{-1} \delta \left(\frac{x_2 - x_1}{x_2 - x_0}\right) \mathcal{Y}(w^{(1)}, x_2) Y_2(v, x_1) w^{(2)}$$
$$= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(v, x_0) w^{(1)}, x_2) w^{(2)} \quad (2.45)$$

(note that the first term on the left-hand side is algebraically meaningful because of the condition (2.44), and the other terms are meaningful by the usual properties of modules; also note that this Jacobi identity involves integral powers of $x_0$ and $x_1$ and complex powers of $x_2$);

$$\frac{d}{dx} \mathcal{Y}(w^{(1)}, x) = \mathcal{Y}(L(-1)w^{(1)}, x), \quad (2.46)$$

where $L(-1)$ is the operator acting on $W_1$.

The intertwining operators of the same type $\left(\frac{W_3}{W_1, W_2}\right)$ form a vector space, which we denote by $\mathcal{V}_{W_1W_2}$. The dimension of this vector space is called the fusion rule for $W_1$, $W_2$ and $W_3$ and is denoted by $N_{W_1W_2}^{W_3} (\leq \infty)$. Formula (2.21) holds for intertwining operators, with $v_n$ replaced by $w_n$ ($n \in \mathbb{C}$).

There are also duality properties for modules and intertwining operators. See [FHL] and [DL] for details.
Let \((W, Y)\), with
\[
W = \prod_{n \in \mathbb{C}} W_{(n)},
\] (2.47)
be a module for a vertex operator algebra \((V, Y, 1, \omega)\), and consider its graded dual space \(W'\) (recall (2.23)). We define the \textit{contragredient vertex operators} (called “adjoint vertex operators” in [FHL]) \(Y'(v, x)\) \((v \in V)\) by means of the linear map
\[
V \rightarrow (\text{End } W')[[x, x^{-1}]]
\]
\[
v \mapsto Y'(v, x) = \sum_{n \in \mathbb{Z}} v'_{n} x^{-n-1}
\] (where \(v'_{n} \in \text{End } W'\)), (2.48)
determined by the condition
\[
\langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^{-2})L(0)v, x^{-1})w \rangle
\] (2.49)
for \(v \in V, w' \in W', w \in W\). The operator \((-x^{-2})L(0)\) has the obvious meaning; it acts on a vector of weight \(n \in \mathbb{Z}\) as multiplication by \((-x^{-2})^{n}\). Also note that \(e^{xL(1)}(-x^{-2})L(0)v\) involves only finitely many (integral) powers of \(x\), that the right-hand side of (2.49) is a Laurent polynomial in \(x\), and that the components \(v'_{n}\) of the formal Laurent series \(Y'(v, x)\) indeed preserve \(W'\).

We give the space \(W'\) a \(\mathbb{C}\)-grading by setting
\[
W'_{(n)} = W_{(n)}^{*} \text{ for } n \in \mathbb{C}
\] (2.50)
(cf. (2.23)). The following theorem defines the \(V\)-module \(W'\) \textit{contragredient to} \(W\) (see [FHL], Theorem 5.2.1 and Proposition 5.3.1):

\textbf{Theorem 2.10} The pair \((W', Y')\) carries the structure of a \(V\)-module and \((W'', Y'') = (W, Y)\). \(\Box\)

Given a module map \(\eta : W_{1} \rightarrow W_{2}\), there is a unique module map \(\eta' : W'_{2} \rightarrow W'_{1}\), the \textit{adjoint} map, such that
\[
\langle \eta'(w'_{(2)}), w_{(1)} \rangle = \langle w'_{(2)}, \eta(w_{(1)} \rangle
\] (2.51)
for \(w_{(1)} \in W_{1}\) and \(w'_{(2)} \in W'_{2}\). (Here the pairings \(\langle \cdot, \cdot \rangle\) on the two sides refer to two different modules.) Note that
\[
\eta'' = \eta.
\] (2.52)

In the construction of the tensor product module of two modules for a vertex operator algebra, we shall need the following generalization of the notion of module recalled above:
Definition 2.11 A generalized $V$-module is a $\mathbb{C}$-graded vector space $W$ equipped with a linear map of the form (2.34) satisfying all the axioms for a $V$-module except that the homogeneous subspaces need not be finite-dimensional and that they need not be zero even for $n$ whose real part is sufficiently small; that is, we omit (2.32) and (2.33) from the definition.

3 Affinizations of vertex operator algebras and the $*$-operation

In order to use the Jacobi identity to construct a tensor product of modules for a vertex operator algebra, we shall study various “affinizations” of a vertex operator algebra with respect to certain algebras and vector spaces of formal Laurent series and formal rational functions.

Let $V$ be a vertex operator algebra and $W$ a $V$-module. We adjoin the formal variable $t$ to our list of commuting formal variables. This variable will play a special role. Consider the vector spaces

$$V[t, t^{-1}] = V \otimes \mathbb{C}[t, t^{-1}] \subset V \otimes \mathbb{C}((t)) \subset V \otimes \mathbb{C}[t, t^{-1}] \subset V[[t, t^{-1}]]$$

(note carefully the distinction between the last two, since $V$ is typically infinite-dimensional) and $W \otimes \mathbb{C}\{t\} \subset W\{t\}$ (recall (2.42)). The linear map

$$\tau_W : V[t, t^{-1}] \to \text{End } W$$
$$v \otimes t^n \mapsto v_n \quad (3.1)$$

$(v \in V, n \in \mathbb{Z})$ extends canonically to

$$\tau_W : V \otimes \mathbb{C}((t)) \to \text{End } W$$
$$v \otimes \sum_{n>N} a_n t^n \mapsto \sum_{n>N} a_n v_n \quad (3.2)$$

(but not to $V((t))$, in view of (2.21). It further extends canonically to

$$\tau_W : (V \otimes \mathbb{C}((t)))[x, x^{-1}] \to (\text{End } W)[x, x^{-1}]$$

where of course $(V \otimes \mathbb{C}((t)))[x, x^{-1}]$ can be viewed as the subspace of $V[[t, t^{-1}, x, x^{-1}]]$ such that the coefficient of each power of $x$ lies in $V \otimes \mathbb{C}((t))$. 

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Let \( v \in V \) and define the “generic vertex operator”
\[
Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n)x^{-n-1} \in (V \otimes \mathbb{C}[t, t^{-1}])[x, x^{-1}].
\] (3.4)

Then
\[
Y_t(v, x) = v \otimes x^{-1}\delta\left(\frac{t}{x}\right)
= v \otimes t^{-1}\delta\left(\frac{x}{t}\right)
\in \ V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]]
(\subset \ V[[t, t^{-1}, x, x^{-1}]])
\] (3.5)

and the linear map
\[
V \rightarrow V \otimes \mathbb{C}[[t, t^{-1}, x, x^{-1}]]
\]
\[
v \mapsto Y_t(v, x)
\] (3.6)
is simply the map given by tensoring by the “universal element” \( x^{-1}\delta\left(\frac{t}{x}\right) \).

We have
\[
\tau_W(Y_t(v, x)) = Y_W(v, x). \quad (3.7)
\]

For all \( f(x) \in \mathbb{C}[[x, x^{-1}]] \), \( f(x)Y_t(v, x) \) is defined and
\[
f(x)Y_t(v, x) = f(t)Y_t(v, x). \quad (3.8)
\]

In case \( f(x) \in \mathbb{C}((x)) \), then \( \tau_W(f(x)Y_t(v, x)) \) is also defined, and
\[
f(x)Y_W(v, x) = f(x)\tau_W(Y_t(v, x)) = \tau_W(f(x)Y_t(v, x)) = \tau_W(f(t)Y_t(v, x)). \quad (3.9)
\]
The expansion coefficients, in powers of \( x \), of \( Y_t(v, x) \) span \( v \otimes \mathbb{C}[t, t^{-1}] \), the \( x \)-expansion coefficients of \( Y_W(v, x) \) span \( \tau_W(v \otimes \mathbb{C}[t, t^{-1}]) \) and the \( x \)-expansion coefficients of \( f(x)Y_t(v, x) \) span \( v \otimes f(t)\mathbb{C}[t, t^{-1}] \). In case \( f(x) \in \mathbb{C}((x)) \), the \( x \)-expansion coefficients of \( f(x)Y_W(v, x) \) span \( \tau_W(v \otimes f(t)\mathbb{C}[t, t^{-1}]) \).

Using this viewpoint, we shall examine each of the three terms in the Jacobi identity (2.45) in the definition of intertwining operator. First we consider the formal Laurent series in \( x_0, x_1, x_2 \) and \( t \) given by
\[
x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_t(v, x_0) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y_t(v, x_0)
= v \otimes x_1^{-1}\delta\left(\frac{x_2 + t}{x_1}\right)x_0^{-1}\delta\left(\frac{t}{x_0}\right)
\] (3.10)

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(cf. the right-hand side of (2.45)). The expansion coefficients in powers of $x_0$, $x_1$ and $x_2$ of (3.10) span just the space $v \otimes \mathbb{C}[t, t^{-1}]$. However, the expansion coefficients in $x_0$ and $x_1$ only (but not in $x_2$) of

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_t(v, x_0) =$$

$$= v \otimes x_1^{-1}\delta\left(\frac{x_2 + t}{x_1}\right) x_0^{-1}\delta\left(\frac{t}{x_0}\right)$$

$$= v \otimes \left(\sum_{m \in \mathbb{Z}} (x_2 + t)^m x_1^{-m-1}\right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1}\right)$$

(3.11)

span

$$v \otimes \iota_{x_2,t} \mathbb{C}[t, t^{-1}, x_2 + t, (x_2 + t)^{-1}] \subset v \otimes \mathbb{C}[x_2, x_2^{-1}](t),$$

where $\iota_{x_2,t}$ is the operation of expanding a formal rational function in the indicated algebra as a formal Laurent series involving only finitely many negative powers of $t$ (cf. the notation $\iota_{12}$, etc., above). We shall use similar $\iota$-notations below. Specifically, the coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (3.11) is $v \otimes (x_2 + t)^m t^n$.

We may specialize $x_2 \mapsto z \in \mathbb{C}^\times$, and (3.11) becomes

$$z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) Y_t(v, x_0) =$$

$$= x_1^{-1}\delta\left(\frac{z + x_0}{x_1}\right) Y_t(v, x_0)$$

$$= v \otimes x_1^{-1}\delta\left(\frac{z + t}{x_1}\right) x_0^{-1}\delta\left(\frac{t}{x_0}\right)$$

$$= v \otimes \left(\sum_{m \in \mathbb{Z}} (z + t)^m x_1^{-m-1}\right) \left(\sum_{n \in \mathbb{Z}} t^n x_0^{-n-1}\right).$$

(3.12)

The coefficient of $x_0^{-n-1}x_1^{-m-1}$ ($m, n \in \mathbb{Z}$) in (3.12) is $v \otimes (z + t)^m t^n \in V \otimes \mathbb{C}(t)$, and these coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \subset v \otimes \mathbb{C}(t).$$

(3.13)

Our tensor product construction will be based on a certain action of the space $V \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$, and the description of this space as the span of the coefficients of the expression (3.12) (as $v \in V$ varies) will be very useful.
Now consider
\[ x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_t(v, x_1) = \]
\[ = v \otimes x_0^{-1} \delta \left( \frac{-x_2 + t}{x_0} \right) x_1^{-1} \delta \left( \frac{t}{x_1} \right) \]
\[ = v \otimes \left( \sum_{n \in \mathbb{Z}} (-x_2 + t)^n x_0^{-n-1} \right) \left( \sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right) \] (3.14)

(cf. the second term on the left-hand side of (2.45)). The expansion coefficients in powers of \( x_0 \) and \( x_1 \) (but not \( x_2 \)) span
\[ v \otimes \iota_{x_2,t} \mathbb{C}[t, t^{-1}, -x_2 + t, (-x_2 + t)^{-1}] , \]
and in fact the coefficient of \( x_0^{-n-1} x_1^{-m-1} \) \((m, n \in \mathbb{Z})\) in (3.14) is \( v \otimes (-x_2 + t)^n t^m \). Again specializing \( x_2 \mapsto z \in \mathbb{C}^{\times} \), we obtain
\[ x_0^{-1} \delta \left( \frac{-z + x_1}{x_0} \right) Y_t(v, x_1) = \]
\[ = v \otimes x_0^{-1} \delta \left( \frac{-z + t}{x_0} \right) x_1^{-1} \delta \left( \frac{t}{x_1} \right) \]
\[ = v \otimes \left( \sum_{n \in \mathbb{Z}} (-z + t)^n x_0^{-n-1} \right) \left( \sum_{m \in \mathbb{Z}} t^m x_1^{-m-1} \right) . \] (3.15)
The coefficient of \( x_0^{-n-1} x_1^{-m-1} \) \((m, n \in \mathbb{Z})\) in (3.15) is \( v \otimes (-z + t)^n t^m \), and these coefficients span
\[ v \otimes \mathbb{C}[t, t^{-1}, (-z + t)^{-1}] \subset v \otimes \mathbb{C}((t)). \] (3.16)

Finally, consider
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(v, x_1) = \]
\[ = v \otimes x_0^{-1} \delta \left( \frac{t - x_2}{x_0} \right) x_1^{-1} \delta \left( \frac{t}{x_1} \right) . \] (3.17)
The coefficient of \( x_0^{-n-1} x_1^{-m-1} \) \((m, n \in \mathbb{Z})\) is \( v \otimes (t - x_2)^n t^m \), and these expansion coefficients span
\[ v \otimes \iota_{t,x_2} \mathbb{C}[t, t^{-1}, t - x_2, (t - x_2)^{-1}] . \]
If we again specialize $x_2 \mapsto z$, we get

$$x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y_t(v, x_1) = v \otimes x_0^{-1} \delta \left( \frac{t - z}{x_0} \right) x_1^{-1} \delta \left( \frac{t}{x_1} \right), \quad (3.18)$$

whose coefficient of $x_0^{-n-1}x_1^{-m-1}$ is $v \otimes (t - z)^n t^m$. These coefficients span

$$v \otimes \mathbb{C}[t, t^{-1}, (t - z)^{-1}] \subset v \otimes \mathbb{C}((t^{-1})) \quad (3.19)$$

(cf. (3.13), (3.16)).

Later we shall evaluate the identity (2.45) on the elements of the contragredient module $W'_3$. This will allow us to convert the expansion (3.19) into an expansion in positive powers of $t$. It will be useful to examine the notion of contragredient vertex operator ((2.48), (2.49)) more closely. For a $V$-module $W$, we define the opposite vertex operator associated to $v \in V$ by

$$Y^*_W(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \quad (3.20)$$

and we define its components by:

$$Y^*_W(v, x) = \sum_{n \in \mathbb{Z}} v^*_n x^{-n-1}. \quad (3.21)$$

Then $v^*_n \in \text{End } W$ and $v \mapsto Y^*_W(v, x)$ is a linear map $V \rightarrow (\text{End } W)[[x, x^{-1}]]$ such that $V \otimes W \rightarrow W((x^{-1}))$ ($v \otimes w \mapsto Y^*_W(v)w$). Note that the contragredient vertex operators are the adjoints of the opposite vertex operators:

$$\langle w', Y^*_W(v, x)w \rangle = \langle Y'(v, x)w', w \rangle \quad (3.22)$$

and that if $v$ is homogeneous, the weight of the operator $v^*_n$ is $n + 1 - \text{wt } v$, by (2.21). The proof of Theorem 5.2.1 in [FHL], which asserts that $(W', Y')$ is a $V$-module, in fact proves the following opposite Jacobi identity for $Y^*_W$:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y^*_W(v_2, x_2)Y^*_W(v_1, x_1)$$

$$-x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y^*_W(v_1, x_1)Y^*_W(v_2, x_2)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y^*_W(Y(v_1, x_0)v_2, x_2) \quad (3.23)$$

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(which of course also follows from the assertion that \((W', Y')\) is a \(V\)-module).

The pair \((W, Y^*)\) should be thought of as a “right module” for \(V\).

We shall interpret the operator \(Y^*_W\) by means of a \(*\)-operation on \(V \otimes \mathbb{C}[t, t^{-1}]\). This operation will be an involution. We proceed as follows: First we generalize \(Y^*\) in the following way: Given any vector space \(U\) and any linear map \(Z(\cdot, x) : V \to U[[x, x^{-1}]] \quad (\text{with } x = \prod_{n \in \mathbb{Z}} U \otimes x^n)\)

\[
Z(\cdot, x) : V \to U[[x, x^{-1}]] \quad (\text{with } x = \prod_{n \in \mathbb{Z}} U \otimes x^n)
\]

\[
v \mapsto Z(v, x)
\]

(3.24)

from \(V\) into \(U[[x, x^{-1}]]\) (i.e., given any family of linear maps from \(V\) into the spaces \(U \otimes x^n\), we define \(Z^*(\cdot, x) : V \to U[[x, x^{-1}]]\) by

\[
Z^*(v, x) = Z(e^{xL(1)}(-x^{-2})L(0)v, x^{-1}),
\]

(3.25)

where we use the obvious linear map \(Z(\cdot, x^{-1}) : V \to U[[x, x^{-1}]]\), and where we extend \(Z(\cdot, x^{-1})\) canonically to a linear map \(Z(\cdot, x^{-1}) : V[x, x^{-1}] \to U[[x, x^{-1}]]\). Then by formula (5.3.1) in [FHL] (the proof of Proposition 5.3.1), we have

\[
Z^{**}(v, x) = Z^*(e^{xL(1)}(-x^{-2})L(0)v, x^{-1})
\]

\[
= Z(e^{-x^{-1}L(1)}(-x^2)L(0)e^{xL(1)}(-x^{-2})L(0)v, x)
\]

\[
= Z(v, x).
\]

(3.26)

That is,

\[
Z^{**}(\cdot, x) = Z(\cdot, x).
\]

(3.27)

Moreover, if \(Z(v, x) \in U((x))\), then \(Z^*(v, x) \in U((x^{-1}))\) and vice versa.

Now we expand \(Z(v, x)\) and \(Z^*(v, x)\) in components. Write

\[
Z(v, z) = \sum_{n \in \mathbb{Z}} v(n)x^{-n-1},
\]

(3.28)

where for all \(n \in \mathbb{Z}\),

\[
V \to U
\]

\[
v \mapsto v(n)
\]

(3.29)
is a linear map depending on \( Z(\cdot, x) \) (and in fact, as \( Z(\cdot, x) \) varies, these linear maps are arbitrary). Also write

\[
Z^*(v, x) = \sum_{n \in \mathbb{Z}} v^*_n x^{-n-1} \tag{3.30}
\]

where

\[
V \rightarrow U \\
v \mapsto v^*_n \tag{3.31}
\]

is a linear map depending on \( Z(\cdot, x) \). We shall compute \( v^*_n \). First note that

\[
\sum_{n \in \mathbb{Z}} v^*_n x^{-n-1} = \sum_{n \in \mathbb{Z}} (e^{xL(1)}(-x^{-2})L(0) v)_n x^{n+1}. \tag{3.32}
\]

For convenience, suppose that \( v \in V(h) \), for \( h \in \mathbb{Z} \). Then the right-hand side of (3.32) is equal to

\[
(-1)^h \sum_{n \in \mathbb{Z}} (e^{xL(1)}v)_{(-n)} x^{-n+1-2h} \\
= (-1)^h \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n)} x^{m-n+1-2h} \\
= (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} \sum_{n \in \mathbb{Z}} (L(1)^m v)_{(-n-m+2+2h)} x^{-n-1}, \tag{3.33}
\]

that is,

\[
v^*_n = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{(-n-m+2+2h)}. \tag{3.34}
\]

For \( v \in V \) not necessarily homogeneous, \( v^*_n \) is given by the appropriate sum of such expressions.

Now consider the special case where \( U = V \otimes \mathbb{C}[t, t^{-1}] \) and where \( Z(\cdot, x) \) is the "generic" linear map

\[
Y_t(\cdot, x) : V \rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[x, x^{-1}] \\
v \mapsto Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1} \tag{3.35}
\]

(recall (3.4)), i.e.,

\[
v^*_n = v \otimes t^n. \tag{3.36}
\]
Then for \( v \in V(h) \),
\[
v^*_n = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} ((L(1))^m v) \otimes t^{-n-m-2+2h}
\] (3.37)
in this case.

This motivates defining a \( * \)-operation on \( V \otimes \mathbb{C}[t, t^{-1}] \) as follows: For any \( n, h \in \mathbb{Z} \) and \( v \in V(h) \), define
\[
(v \otimes t^n)^* = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v) \otimes t^{-n-m-2+2h} \in V \otimes \mathbb{C}[t, t^{-1}],
\] (3.38)
and extend by linearity to \( V \otimes \mathbb{C}[t, t^{-1}] \). That is, \( (v \otimes t^n)^* = v^*_n \) for the special case \( Z(\cdot, x) = Y_t(\cdot, x) \) discussed above. (Note that for general \( Z \), we cannot expect to be able to define an analogous \( * \)-operation on \( U \).) Also consider the map
\[
Y^*_t(\cdot, x) = (Y_t(\cdot, x))^* : V \rightarrow (V \otimes \mathbb{C}[t, t^{-1}])[x, x^{-1}]
\]
\[
v \mapsto Y^*_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n)^* x^{-n-1}
\] (3.39)
Then for general \( Z(\cdot, x) \) as above, we can define a linear map
\[
\varepsilon_Z : V \otimes \mathbb{C}[t, t^{-1}] \rightarrow U
\]
\[
v \otimes t^n \mapsto v^*_n
\] (3.40)
("evaluation with respect to \( Z \"), i.e.,
\[
\varepsilon_Z : Y_t(v, x) \mapsto Z(v, x),
\] (3.41)
and a linear map
\[
\varepsilon^*_Z : V \otimes \mathbb{C}[t, t^{-1}] \rightarrow U
\]
\[
v \otimes t^n \mapsto v^*_n,
\] (3.42)
i.e.,
\[
\varepsilon^*_Z : Y_t(v, x) \mapsto Z^*(v, x).
\] (3.43)
Then
\[
\varepsilon^*_Z = \varepsilon_Z \circ *,
\] (3.44)
that is,
\[ \varepsilon_Z(Y_t^*(v, x)) = Z^*(v, x), \quad (3.45) \]
or equivalently, the diagram
\[
\begin{array}{ccc}
Y_t(v, x) & \xleftarrow{\varepsilon_Z} & Z(v, x) \\
\downarrow \ast & & \downarrow (Z(\cdot, x) \mapsto Z^*(\cdot, x)) \\
Y_t^*(v, x) & \xleftarrow{\varepsilon_Z} & Z^*(v, x)
\end{array}
\quad (3.46)
\]
commutes. Note that the components \( v^*(n) \) of \( Z^*(v, x) \) depend on all the components \( v(n) \) of \( Z(v, z) \) (for arbitrary \( v \)), whereas the component \( (v \otimes t^n)^* \) of \( Y_t^*(v, z) \) can be defined generically and abstractly; \( (v \otimes t^n)^* \) depends linearly on \( v \in V \) alone.

Since in general \( Z^{**}(v, x) = Z(v, x) \), we know that
\[ Y_t^{**}(v, x) = Y_t(v, x) \quad (3.47) \]
as a special case, and in particular (and equivalently),
\[ (v \otimes t^n)^{**} = v \otimes t^n \quad (3.48) \]
for all \( v \in V \) and \( n \in \mathbb{Z} \). Thus \( \ast \) is an involution of \( V \otimes \mathbb{C}[t, t^{-1}] \).

Furthermore, the involution \( \ast \) of \( V \otimes \mathbb{C}[t, t^{-1}] \) extends canonically to a linear map
\[
V \otimes \mathbb{C}[[t, t^{-1}]] \xrightarrow{\ast} V \otimes \mathbb{C}[[t, t^{-1}]].
\]

In fact, consider the restriction of \( \ast \) to \( V = V \otimes t^0 \):
\[
V \xrightarrow{\sim} V \otimes \mathbb{C}[t, t^{-1}] \\
v \mapsto v^* = (-1)^h \sum_{m \in \mathbb{N}} \frac{1}{m!}(L(1)^m v) \otimes t^{m-2+2h}, \quad (3.49)
\]
extended by linearity from \( V(h) \) to \( V \). Then for \( v \in V \) and \( n \in \mathbb{Z} \),
\[ (v \otimes t^n)^* = v^* t^{-n}, \quad (3.50) \]
and it is clear that \( \ast \) extends to \( V \otimes \mathbb{C}[[t, t^{-1}]] \): For \( f(t) \in \mathbb{C}[[t, t^{-1}]] \),
\[ (v \otimes f(t))^* = v^* f(t^{-1}). \quad (3.51) \]
To see that \( \ast \) is an involution of this larger space, first note that

\[
v^{**} = v
\]  
(3.52)

(although \( v^* \notin V \) in general). (This could of course alternatively be proved by direct calculation using formula (3.38).) Also, for \( g(t) \in \mathbb{C}[t,t^{-1}] \) and \( f(t) \in \mathbb{C}[[t,t^{-1}]] \),

\[
(v \otimes g(t)f(t))^* = v^*g(t^{-1})f(t^{-1}) = (v \otimes g(t))^*f(t^{-1}).
\]  
(3.53)

Thus for all \( x \in V \otimes \mathbb{C}[t,t^{-1}] \) and \( f(t) \in \mathbb{C}[[t,t^{-1}]] \),

\[
(xf(t))^* = x^*f(t^{-1}).
\]  
(3.54)

It follows that

\[
(v \otimes f(t))^{**} = (v^*f(t^{-1}))^*
\]
\[
= v^{**}f(t)
\]
\[
= vf(t)
\]
\[
= v \otimes f(t),
\]  
(3.55)

and we have shown that \( \ast \) is an involution of \( V \otimes \mathbb{C}[[t,t^{-1}]] \). We have

\[
\ast : V \otimes \mathbb{C}((t)) \leftrightarrow V \otimes \mathbb{C}((t^{-1})).
\]  
(3.56)

Note that

\[
Y_t^*(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n)^* x^{n-1}
\]
\[
= v^* \sum_{n \in \mathbb{Z}} t^{-n} x^{n-1}
\]
\[
= v^* x^{-1} \delta(tx)
\]
\[
= v^* t \delta(tx)
\]
\[
\in V \otimes \mathbb{C}[[t,t^{-1}, x, x^{-1}]].
\]  
(3.57)

Thus the map \( v \mapsto Y_t^*(v, x) \) is the linear map given by multiplying \( v^* \) by the “universal element” \( t \delta(tx) \) (cf. the comment following (3.6)). For all \( f(x) \in \mathbb{C}[[x, x^{-1}]] \), \( f(x)Y_t^*(v, x) \) is defined and

\[
f(x)Y_t^*(v, x) = f(t^{-1})Y_t^*(v, x)
\]
\[
= v^* f(t^{-1}) t \delta(tx).
\]  
(3.58)
Now we return to the starting point — the original special case: \( U = \text{End } W \) and \( Z(t, z) = Y_W(t, z) : V \to (\text{End } W)[[x, x^{-1}]] \). The corresponding map

\[
\varepsilon_Z = \varepsilon_{Y_W} : V[t, t^{-1}] \to \text{End } W
\]

\[v \otimes t^n \mapsto v_{(n)}\]  

(3.59)

(recall (3.40)) is just the map \( \tau_W : v \otimes t^n \mapsto v_n \) (recall (3.1)), i.e., \( v_{(n)} = v_n \) in this case. Recall that this map extends canonically to \( V \otimes \mathbb{C}((t)) \). The map \( \varepsilon_Z^\ast = \tau_W \circ \ast : V \otimes \mathbb{C}[t, t^{-1}] \to \text{End } W \) and this map extends canonically to \( V \otimes \mathbb{C}((t^{-1})) \). In addition to (3.7), we have

\[
\tau_W(Y_t^*(v, z)) = Y_W^*(v, z) \quad \text{(3.60)}
\]

\((v_n^*) = v_n^* \) in this case; recall (3.21)). In case \( f(x) \in \mathbb{C}((x^{-1})) \),

\[
f(x)Y_t^*(v, x) = \tau_W(f(x)Y_t^*(v, x))
\]

is defined and is equal to \( \tau_W(f(t^{-1})Y_t^*(v, z)) \) (which is also defined).

The \( x \)-expansion coefficients of \( f(x)Y_t^*(v, x) \), for \( f(x) \in \mathbb{C}[[x, x^{-1}]] \), span

\[
v^*f(t^{-1})C[t, t^{-1}] = (vC[t, t^{-1}])^*f(t^{-1})
\]

\[
= (vf(t)C[t, t^{-1}])^* \quad \text{(3.61)}
\]

The \( x \)-expansion coefficients of \( Y_W^*(v, x) \) span

\[
\tau_W(v^*C[t, t^{-1}]) = \tau_W((v \otimes C[t, t^{-1}])^*)
\]

\[
= \tau_W(v \otimes C[t, t^{-1}]). \quad \text{(3.62)}
\]

In case \( f(x) \in \mathbb{C}((x^{-1})) \), the \( x \)-expansion coefficients of \( f(x)Y_W^*(v, x) \) span \( \tau_W(v^*f(t^{-1})C[t, t^{-1}]) = \tau_W(vf(t)C[t, t^{-1}]) \). (Cf. the comments after (3.9).)

Our action of the space \( V \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \) will be based on certain translation operations and on the \( \ast \)-operation. More precisely, it is the space \( V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \) whose action we shall define, where we use the notations

\[
\iota_+ : \mathbb{C}(t) \hookrightarrow \mathbb{C}((t)) \subset \mathbb{C}[[t, t^{-1}]]
\]

\[
\iota_- : \mathbb{C}(t) \hookrightarrow \mathbb{C}((t^{-1})) \subset \mathbb{C}[[t, t^{-1}]]
\]

(3.63)
to denote the operations of expanding a rational function of the variable \( t \) in the indicated directions (as in Section 8.1 of [FLM2]). For \( a \in \mathbb{C} \), we define the translation isomorphism

\[
T_a : \mathbb{C}(t) \xrightarrow{\sim} \mathbb{C}(t)
\]

\[
f(t) \mapsto f(t + a)
\]

and we set

\[
T_a^\pm = \iota_\pm \circ T_a \circ \iota_\pm^{-1} : \mathbb{C}(t) \hookrightarrow \mathbb{C}((t^{\pm 1})).
\]

(Note that the domains of these maps consist of certain series expansions of rational functions rather than of rational functions themselves.) We shall be interested in

\[
T_{\pm z}^\pm : \mathbb{C}(t, t^{-1}, (z + t)^{-1}) \hookrightarrow \mathbb{C}((t^{\pm 1})),
\]

where \( z \) is an arbitrary nonzero complex number. The images of these two maps are \( \iota_\pm \mathbb{C}[t, t^{-1}, (z - t)^{-1}] \).

Extend \( T_{\pm z}^\pm \) to linear isomorphisms

\[
T_{\pm z}^\pm : V \otimes \iota_\pm \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \sim V \otimes \iota_\pm \mathbb{C}[t, t^{-1}, (z - t)^{-1}]
\]

given by \( 1 \otimes T_{\pm z}^\pm \) with \( T_{\pm z}^\pm \) as defined above. Note that the domain of these two maps is described by (3.12)–(3.13), that the image of the map \( T_{-z}^+ \) is described by (3.15)–(3.16) and that the image of the map \( T_{-z}^- \) is described by (3.18)–(3.19).

We have the two mutually inverse maps

\[
V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z - t)^{-1}] \xrightarrow{\ast} V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]
\]

\[
v \otimes f(t) \mapsto v^* f(t^{-1})
\]

and

\[
V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \xrightarrow{\ast} V \otimes \iota_- \mathbb{C}[t, t^{-1}, (z - t)^{-1}]
\]

\[
v \otimes f(t) \mapsto v^* f(t^{-1}),
\]

which are both isomorphisms. We form the composition

\[
T_{-z}^* = \ast \circ T_{-z}^-
\]
to obtain another isomorphism

\[ T^*_z : V \otimes \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \cong V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \]

The maps \( T^\pm_z \) and \( T^*_z \) will be the main ingredients of our action. The following result asserts that \( T^\pm_z, T^-_z \) and \( T^*_z \) transform the expression (3.12) into (3.15), (3.18) and the \( \ast \)-transform of (3.18), respectively:

**Lemma 3.1** We have

\[ T^\pm_z \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y(t, x) \right) = x_0^{-1} \delta \left( \frac{z - x_1}{x_0} \right) Y(t, x_0), \]  
\[ T^-_z \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y(t, x) \right) = x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y(t, x_0), \]  
\[ T^*_z \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y(t, x) \right) = x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y^*(t, x). \]

**Proof** We prove (3.71): From (3.12), the coefficient of \( x_0^{-n}x_1^{-m-1} \) in the left-hand side of (3.71) is \( T^\pm_z (v \otimes (z + t)^m t^n) \). By the definitions,

\[ T^\pm_z (v \otimes (z + t)^m t^n) = v \otimes t^m (-z - t)^n. \]  

On the other hand, the right-hand side of (3.71) can be written as

\[ v \otimes x_0^{-1} \delta \left( \frac{z - x_1}{x_1} \right) x_0^{-1} \delta \left( \frac{t}{x_1} \right) = v \otimes x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) x_0^{-1} \delta \left( \frac{t}{x_1} \right) \]

where we have used (3.5) and the fundamental property (2.4) of the formal \( \delta \)-function. The coefficient of \( x_0^{-n}x_1^{-m-1} \) in the right-hand side of (3.75) is also \( v \otimes t^m (-z - t)^n \), proving (3.71). Formula (3.72) is proved similarly, and (3.73) is obtained from (3.72) by the application of the map \( * \).

\[ \square \]

**4 The notions of \( P(z) \)- and \( Q(z) \)-tensor product of two modules**

For any \( \mathbb{C} \)-graded vector space \( W = \prod W(n) \) such that \( \dim W(n) < \infty \) for each \( n \in \mathbb{C} \), we use the notation

\[ W = \prod_{n \in \mathbb{C}} W(n) = W^*, \]

(4.1)
where as usual \(^\prime\) denotes the graded dual space and \(\ast\) denotes the dual space of a vector space.

Let \(V\) be a vertex operator algebra and \(W\) a \(V\)-module. For any \(v \in V\) and \(n \in \mathbb{Z}\), \(v_n\) acts naturally on \(\overline{W}\) because of (2.21) for modules (recall Remark 2.8) and \(v_n^\ast\) also acts naturally on \(\overline{W}\), in view of (2.21) and (3.20). Moreover, because of (2.21) and (2.44), for fixed \(v \in V\), any infinite linear combination of the \(v_n\) of the form \(\sum_{n<N} a_n v_n\) \((a_n \in \mathbb{C})\) acts on \(\overline{W}\), and from (3.22), for example, we see that any infinite linear combination of the form \(\sum_{n>N} a_n v_n^\ast\) also acts on \(\overline{W}\).

Fix a nonzero complex number \(z\) and let \((W_1, Y_1)\) and \((W_2, Y_2)\) be two \(V\)-modules. In the present paper (Part I), we give the algebraic definitions and algebraic constructions of certain tensor products of \((W_1, Y_1)\) and \((W_2, Y_2)\), depending on \(z\), but these have geometric meanings as well, which will be discussed and studied in other papers in this series. Let \((W_3, Y_3)\) be another \(V\)-module. We call a \(P(z)\)-intertwining map of type \((W_3, W_1, W_2)\) (see Remark 4.3 below for the meaning of the symbol \(P(z)\)) a linear map \(F : W_1 \otimes W_2 \to W_3\) satisfying the condition

\[
x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y_3(v, x_1) F(w_1(1) \otimes w_2(2)) =
\]

\[
z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) F(Y_1(v, x_0) w_1(1) \otimes w_2(2)) + x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(w_1(1) \otimes Y_2(v, x_1) w_2(2))
\]

for \(v \in V\), \(w_1(1) \in W_1\), \(w_2(2) \in W_2\) (cf. the identity (1.1) and the Jacobi identity (2.45) for intertwining operators). Note that the left-hand side of (4.2) is well defined in view of the comments in the preceding paragraph. A \(P(z)\)-product of \(W_1\) and \(W_2\) is a \(V\)-module \((W_3, Y_3)\) equipped with a \(P(z)\)-intertwining map \(F\) of type \((W_3, W_1, W_2)\). We denote it by \((W_3, Y_3; F)\) (or simply by \((W_3, F)\)).

Let \((W_4, Y_4; G)\) be another \(P(z)\)-product of \(W_1\) and \(W_2\). A morphism from \((W_3, Y_3; F)\) to \((W_4, Y_4; G)\) is a module map \(\eta\) from \(W_3\) to \(W_4\) such that

\[
G = \overline{\eta} \circ F,
\]

where \(\overline{\eta}\) is the natural map from \(\overline{W_3}\) to \(\overline{W_4}\) uniquely extending \(\eta\). We define the notion of \(P(z)\)-tensor product using a universal property as follows:
**Definition 4.1** A $P(z)$-tensor product of $W_1$ and $W_2$ is a $P(z)$-product $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \mathcal{E}_{P(z)})$ such that for any $P(z)$-product $(W_3, Y_3; F)$, there is a unique morphism from $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \mathcal{E}_{P(z)})$ to $(W_3, Y_3; F)$. The $V$-module $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is called a $P(z)$-tensor product module of $W_1$ and $W_2$.

**Remark 4.2** As in the case of tensor products of modules for a Lie algebra, it is clear from the definition that if a $P(z)$-tensor product of $W_1$ and $W_2$ exists, then it is unique up to unique isomorphism.

**Remark 4.3** The symbol $P(z)$ in the definitions above in fact represents a geometric object. Geometrically, to define a tensor product of $W_1$ and $W_2$, we need to specify an element of the moduli space $K$ of spheres with punctures and local coordinates vanishing at these punctures. (In this remark and in Remark 4.6 below, we invoke the detailed discussion of the moduli space $K$ and its role in the geometric interpretation of the notion of vertex operator algebra given in [H1], [H2], [H3] or [H4]. The present remark and Remark 4.6 are not actually needed in the algebraic treatment in Part I.) More precisely, we need to specify an element of the determinant line bundle over $K$ raised to the power $c$, where $c$ is the central charge of the vertex operator algebra. The definitions of intertwining map, product and tensor product above are those associated to the element $P(z)$ of $K$ containing $\mathbb{C} \cup \{\infty\}$ with ordered punctures $\infty$, $z$, $0$ and standard local coordinates $1/w, w-z, w$, vanishing at $\infty$, $z$, $0$, respectively. Note that $P(z)$ is the geometric object corresponding to vertex operators or intertwining operators in the geometric interpretation of vertex operators and intertwining operators. The appropriate language describing tensor products defined using elements of $K$ is that of operads, or more precisely, partial operads (see [M], [HL2], [HL3] and [H5]). These different tensor products will play important roles in the formulations and constructions of the associativity and commutativity isomorphisms.

Though it is natural to first consider $P(z)$-tensor products of two modules as defined above, in this paper (Part I) we shall instead construct another type of tensor product, the $Q(z)$-tensor product (see below), since the calculations involved in the direct construction of $Q(z)$-tensor products are simpler than those for $P(z)$-tensor products. Moreover, $P(z)$-tensor products can be
obtained from $Q(z)$-tensor products by performing certain geometric transformations. We shall give the construction of a $P(z)$-tensor product in Part III using the construction of the $Q(z)$-tensor product presented in Sections 5 and 6 below. The reader should observe that many of the considerations below concerning concepts based on $Q(z)$ carry over immediately to the analogous considerations related to $P(z)$; in Part I we focus only on $Q(z)$.

A $Q(z)$-intertwining map of type $\left(\frac{W_3}{W_1W_2}\right)$ is a map $F : W_1 \otimes W_2 \to W_3$ such that

$$z^{-1}\delta \left(\frac{x_1 - x_0}{z}\right) Y_3^*(v, x_0) F(w(1) \otimes w(2)) =$$

$$= x_0^{-1}\delta \left(\frac{x_1 - z}{x_0}\right) F(Y_1(v, x_1)w(1) \otimes w(2)) - x_0^{-1}\delta \left(\frac{z - x_1}{-x_0}\right) F(w(1) \otimes Y_2(v, x_1)w(2)) \quad (4.4)$$

for $v \in V$, $w(1) \in W_1$, $w(2) \in W_2$. As in the definition of $P(z)$-intertwining map, note that the left-hand side and both terms on the right-hand side of (4.4) are well defined. First replacing $v$ by $(-x_0^2)^{-L(0)} e^{-x_0 L(1)} v$ and then replacing $x_0$ by $x_0^{-1}$ in (4.4), we see that (4.4) is equivalent to:

$$z^{-1}\delta \left(\frac{x_1 - x_0^{-1}}{z}\right) Y_3(v, x_0) F(w(1) \otimes w(2)) =$$

$$= x_0\delta \left(\frac{x_1 - z}{x_0^{-1}}\right) F(Y_1(e^{x_1 L(1)}(x_1x_0)^{-2L(0)} e^{-x_0^{-1} L(1)} v, x_0^{-1})w(1) \otimes w(2)) - x_0\delta \left(\frac{z - x_1}{-x_0^{-1}}\right) F(w(1) \otimes Y_2((-x_0^{-2})^{L(0)} e^{-x_0^{-1} L(1)} v, x_1)w(2)). \quad (4.5)$$

(The reverse procedure is given by first inverting $x_0$ and then replacing $v$ by $e^{x_0 L(1)}(-x_0^{-2}) e^{-L(0)} v$.)

We denote the vector space of $Q(z)$-intertwining maps of type $\left(\frac{W_3}{W_1W_2}\right)$ by $\mathcal{M}[^{W_3}_{W_1W_2}]$. We define a $Q(z)$-product of $W_1$ and $W_2$ to be a $V$-module $(W_3, Y_3)$ together with a $Q(z)$-intertwining map $F$ of type $\left(\frac{W_3}{W_1W_2}\right)$ and we denote it by $(W_3, Y_3; F)$ (or $(W_3, F)$). Let $(W_3, Y_3; F)$ and $(W_4, Y_4; G)$ be two $Q(z)$-products of $W_1$ and $W_2$. A morphism from $(W_3, Y_3; F)$ to $(W_4, Y_4; G)$ is a
module map \( \eta \) from \( W_3 \) to \( W_4 \) such that

\[
G = \overline{\eta} \circ F
\]

where, as in (4.3), \( \overline{\eta} \) is the natural map from \( W_3 \) to \( W_4 \) uniquely extending \( \eta \).

**Definition 4.4** A \( Q(z) \)-tensor product of \( W_1 \) and \( W_2 \) is a \( Q(z) \)-product \((W_1 \boxtimes Q(z) W_2, Y_{Q(z)}; \boxtimes Q(z))\) such that for any \( Q(z) \)-product \((W_3, Y_3; F)\), there is a unique morphism from \((W_1 \boxtimes Q(z) W_2, Y_{Q(z)}; \boxtimes Q(z))\) to \((W_3, Y_3; F)\). The \( V \)-module \((W_1 \boxtimes Q(z) W_2, Y_{Q(z)})\) is called a \( Q(z) \)-tensor product module of \( W_1 \) and \( W_2 \).

**Remark 4.5** As in the case of \( P(z) \)-tensor products, a \( Q(z) \)-tensor product is unique up to unique isomorphism if it exists.

**Remark 4.6** In the definitions above, \( Q(z) \) represents the element of \( K \) containing \( \mathbb{C} \cup \{ \infty \} \) with ordered punctures \( z, \infty, 0 \) and standard local coordinates vanishing at these punctures. (Recall Remark 4.3.) In fact, this is the same as the element of \( K \) containing \( \mathbb{C} \cup \{ \infty \} \) with ordered punctures \( \infty, 1/z, 0 \) and local coordinates \( z/(zw-1), (zw-1)/z^2w, z^2w/(zw-1) \) vanishing at \( \infty, 1/z, 0 \), respectively, and (4.5) corresponds to this canonical sphere with punctures and local coordinates.

The existence of a \( Q(z) \)-tensor product is not obvious. We shall prove the existence and give two constructions under certain assumptions on the vertex operator algebra in this and the next two sections. First we relate \( Q(z) \)-intertwining maps of type \((W_3, W_1, W_2)\) to intertwining operators of type \((W_1', W_1, W_2)\).

Let \( Y \) be an intertwining operator of type \((W_3, W_1, W_2)\). For any complex number \( \zeta \) and any \( w_{(1)} \in W_1 \), \( Y(w_{(1)}, x) \bigg|_{x^n = e^{n\zeta}, n \in \mathbb{C}} \) is a well-defined map from \( W_2 \) to \( \overline{W}_3 \), in view of formula (2.21) for intertwining operators. For brevity of notation, we shall write this map as \( Y(w_{(1)}, e^\zeta) \), but note that
\( \mathcal{Y}(w_{(1)}, e^\zeta) \) depends on \( \zeta \), not on just \( e^\zeta \), as the notation might suggest. In this paper we shall always choose \( \log z \) so that

\[
\log z = \log |z| + i \arg z \quad \text{with} \quad 0 \leq \arg z < 2\pi. \tag{4.7}
\]

Arbitrary values of the log function will be denoted

\[
l_p(z) = \log z + 2p\pi i \tag{4.8}
\]

for \( p \in \mathbb{Z} \).

We now describe the close connection between intertwining operators of type \( \left( \frac{W_1'}{W_3} \right) \) and \( Q(z) \)-intertwining maps of type \( \left( \frac{W_3}{W_1 W_2} \right) \). Fix an integer \( p \).

Let \( \mathcal{Y} \) be an intertwining operator of type \( \left( \frac{W_3'}{W_1 W_2} \right) \). Then we have an element of \((W_1 \otimes W_3' \otimes W_2)^*\) whose value at \( w_{(1)} \otimes w_{(3)}' \otimes w_{(2)} \) \((w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)}' \in W_3')\) is

\[
\langle w_{(1)}, \mathcal{Y}(w_{(3)}', e^p(z))w_{(2)} \rangle_{W_1'}
\]

where \( \langle \cdot, \cdot \rangle_{W_1'} \) is the pairing between \( W_1 \) and \( \overline{W_1} = W_1^* \). Since any element of \((W_1 \otimes W_3' \otimes W_2)^*\) amounts exactly to a linear map from \( W_1 \otimes W_2 \) to \( W_3^* = \overline{W_3} \), our element of \((W_1 \otimes W_3' \otimes W_2)^*\) obtained from the intertwining operator \( \mathcal{Y} \) gives us a linear map \( F_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W_3} \) such that

\[
\langle w_{(3)}', F_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) \rangle_{W_3} = \langle w_{(1)}, \mathcal{Y}(w_{(3)}', e^p(z))w_{(2)} \rangle_{W_1'} \tag{4.9}
\]

for all \( w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)}' \in W_3' \), where \( \langle \cdot, \cdot \rangle_{W_3} \) is the pairing between \( W_3' \) and \( \overline{W_3} \). (For any module \( W \), we shall use the analogous notation \( \langle \cdot, \cdot \rangle_{W} \) to denote the pairing between \( W' \) and \( \overline{W} \).) The Jacobi identity for \( \mathcal{Y} \) is equivalent to the identity

\[
x_0^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle w_{(1)}, \mathcal{Y}(Y_3'(v, x_0)w_{(3)}', x_2)w_{(2)} \rangle_{W_1'}
\]

\[
= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle w_{(1)}, Y_3'(v, x_1)\mathcal{Y}(w_{(3)}', x_2)w_{(2)} \rangle_{W_1'}
\]

\[
- x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w_{(1)}, \mathcal{Y}(w_{(3)}', x_2)Y_3(v, x_1)w_{(2)} \rangle_{W_1'} \tag{4.10}
\]

for all \( w_{(1)}, w_{(2)} \) and \( w_{(3)}' \) (recall the notation (2.48)). Substituting \( e^{n t_p(z)} \) for \( x_2^n, n \in \mathbb{C} \), in (4.10), and noting that in case \( n \in \mathbb{Z} \), we may simply write \( z^n \).
for \( e^{nl_p(z)} \), we obtain

\[
\begin{align*}
& z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle w(1), \mathcal{Y}(Y_3'(v, x_0)w(3), e^{l_p(z)}w(2)) \rangle w_1' \\
& = x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle w(1), Y_1'(v, x_1)\mathcal{Y}(w(3), e^{l_p(z)}w(2)) \rangle w_1' \\
& - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle w(1), \mathcal{Y}(w(3), e^{l_p(z)})Y_2(v, x_1)w(2) \rangle w_1'. 
\end{align*}
\]  

(4.11)

Using (3.22) and (4.9), we see that (4.11) can be written as

\[
\begin{align*}
& z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) \langle w(3), Y_3^*(v, x_0)F_{Y,p}(w(1) \otimes w(2)) \rangle w_3 \\
& = x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \langle w(3), F_{Y,p}(Y_1^*(v, x_1)w(1) \otimes w(2)) \rangle w_3 \\
& - x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \langle w(3), F_{Y,p}(w(1) \otimes Y_2(v, x_1)w(2)) \rangle w_3. 
\end{align*}
\]  

(4.12)

Thus \( F_{Y,p} \) is a \( Q(z) \)-intertwining map of type \( \left( \frac{w_3}{w_1, w_2} \right) \).

The only part of the definition of intertwining operator we have not yet used is the \( L(-1) \)-derivative property (2.46). (Recall that the lower truncation condition (2.44) has already been used in the formulation of the first term on the left-hand side of the Jacobi identity (2.45).) Since we have specialized \( x \) to \( z \) in \( \mathcal{Y}(\cdot, x) \), there is no property of \( F_{Y,p} \) corresponding the \( L(-1) \)-derivative property of \( \mathcal{Y} \). Instead, the \( L(-1) \)-derivative property will enable us to recover \( \mathcal{Y}(\cdot, x) \) from \( F_{Y,p} \). Specifically, the \( L(-1) \)-derivative property enters into the proof of the formula

\[
\begin{align*}
x^{L(0)}\mathcal{Y}(w(3), x_0)x^{-L(0)} &= \mathcal{Y}(x^{L(0)}w(3), x_0) 
\end{align*}
\]  

(4.13)

(recall [FHL], formula (5.4.22), and Lemma 5.2.3 and its proof), and this is equivalent to the formula

\[
\begin{align*}
\langle x^{L(0)}w(1) \rangle \mathcal{Y}(x^{-L(0)}w(3), x_0)x^{-L(0)}w(2) \langle w_1 \rangle = \langle w(1) \rangle \mathcal{Y}(w(3), x_0)x^{-L(0)}w(2) \langle w_1 \rangle
\end{align*}
\]  

(4.14)

for all \( w(1) \in W_1 \). Substituting \( e^{nl_p(z)} \) for \( x_0^n \) and \( e^{-nl_p(z)}x^n \) for \( x^n \), in (4.14), we obtain

\[
\begin{align*}
\langle e^{-l_p(z)L(0)}x^{L(0)}w(1), \mathcal{Y}(e^{l_p(z)L(0)}x^{-L(0)}w(3), e^{l_p(z)}x^{L(0)}w(2)) \rangle w_1' &= \langle w(1), \mathcal{Y}(w(3), x)w(2) \rangle w_1'.
\end{align*}
\]  

(4.15)
or equivalently, by (4.9),
\[
\langle e^{t_p(z)L(0)x^{-L(0)}w_{(3)}}, F_{Y,p}(e^{-t_p(z)L(0)x^{-L(0)}w_{(1)}} \otimes e^{t_p(z)L(0)x^{-L(0)}w_{(2)}}) \rangle_{W_3} = \langle w_{(1)}, Y(w'_{(3)}, x)w_{(2)} \rangle_{W_1}.
\]
(4.16)

Thus we have recovered $Y$ from $F_{Y,p}$.

We shall also need the following alternative way of recovering $Y$ from $F_{Y,p}$, using components. We write (4.9) as:
\[
\langle w_{(1)}, \sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} e^{-n-1}t_p(z) \rangle_{W_1} = \langle w'_{(3)}, F_{Y,p}(w_{(1)} \otimes w_{(2)}) \rangle_{W_3}.
\]
(4.17)

This formula enables us to recover the components $(w'_{(3)})_n w_{(2)}$, $n \in \mathbb{C}$, of $Y(w'_{(3)}, x)w_{(2)}$ from $F_{Y,p}$, assuming for convenience that $w_{(2)}$ and $w'_{(3)}$ are homogeneous vectors, in the following way: The map $F_{Y,p}$ gives an element of $(W_1 \otimes W_3 \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ is equal to the right-hand side of (4.17). This element amounts to a map from $W_3 \otimes W_2$ to $W_1^*$. By (4.17), the image of $w'_{(3)} \otimes w_{(2)}$ under this map is equal to $\sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} e^{-n-1}t_p(z)$.

Projecting this image to the homogeneous subspace of $W_1^*$ of weight equal to
\[\text{wt } w'_{(3)} - n - 1 + \text{wt } w_{(2)},\]
we obtain $(w'_{(3)})_n w_{(2)} e^{-n-1}t_p(z)$. Multiplying this by $e^{(n+1)l_p(z)}$, we recover the coefficient $(w'_{(3)})_n w_{(2)}$.

Motivated by this procedure, we would like to construct an intertwining operator of type $\left(\frac{W_3}{W_1 W_2}\right)$ from a $Q(z)$-intertwining map of type $\left(\frac{W_3}{W_1 W_2}\right)$. Let $F$ be a $Q(z)$-intertwining map of type $\left(\frac{W_3}{W_1 W_2}\right)$. This linear map from $W_1 \otimes W_2$ to $W_3$ gives us an element of $(W_1 \otimes W_3 \otimes W_2)^*$ whose value at $w_{(1)} \otimes w'_{(3)} \otimes w_{(2)}$ is
\[\langle w'_{(3)}, F(w_{(1)} \otimes w_{(2)}) \rangle_{W_3}.
\]
But since every element of $(W_1 \otimes W_3 \otimes W_2)^*$ also amounts to a linear map from $W_3 \otimes W_2$ to $W_1^*$, we have such a map as well. Let $w'_{(3)} \in W_3'$ and $w_{(2)} \in W_2$ be homogeneous elements. Since $W_1^* = \prod_{n \in \mathbb{C}} (W_1')_n$, the image of $w'_{(3)} \otimes w_{(2)}$ under our map can be written as $\sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} e^{-n-1}l_p(z)$ where for any $n \in \mathbb{C}$, $(w'_{(3)})_n w_{(2)} e^{-n-1}l_p(z)$ is the projection of the image to the homogeneous subspace of $W_1'$ of weight equal to
\[\text{wt } w'_{(3)} - n - 1 + \text{wt } w_{(2)}.
\]
(Here we are defining elements denoted \((w'_{(3)})_n w_{(2)}\) of \(W'_1\) for \(n \in \mathbb{C}\).) We define
\[
\mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)} = \sum_{n \in \mathbb{C}} (w'_{(3)})_n w_{(2)} x^{-n-1} \in W'_1\{x\}
\]
for all homogeneous elements \(w'_{(3)} \in W'_3\) and \(w_{(2)} \in W_2\). Using linearity, we extend \(\mathcal{Y}_{F,p}\) to a linear map
\[
W'_3 \otimes W_2 \rightarrow W'_1\{x\}\]
\[
w'_{(3)} \otimes w_{(2)} \mapsto \mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)}. \tag{4.18}
\]
The correspondence \(F \mapsto \mathcal{Y}_{F,p}\) is linear, and from the definitions and the discussion in the preceding paragraph, we have \(\mathcal{Y}_{F_{Y,p}} = \mathcal{Y}\) for an intertwining operator \(\mathcal{Y}\) of type \((W'_1 W'_3 W_2)\).

**Proposition 4.7** For \(p \in \mathbb{Z}\), the correspondence \(\mathcal{Y} \mapsto F_{\mathcal{Y},p}\) is a linear isomorphism from the space \(\mathcal{V}^{W'_1}_{W'_3 W_2}\) of intertwining operators of type \((W'_1 W'_3 W_2)\) to the space \(\mathcal{M}^{W_3}_{W'_1 W_2} = \mathcal{M}[Q(z)]_{W'_1 W_2}\) of \(Q(z)\)-intertwining maps of type \((W'_3 W'_1 W_2)\). Its inverse is given by \(F \mapsto \mathcal{Y}_{F,p}\).

**Proof** We need only show that for any \(Q(z)\)-intertwining map \(F\) of type \((W'_1 W'_3 W_2)\), \(\mathcal{Y}_{F,p}\) is an intertwining operator of type \((W'_3 W'_1 W_2)\). From the discussion above and the definition of \(\mathcal{Y}_{F,p}\), the lower truncation condition \((2.44)\) holds for \(\mathcal{Y}_{F,p}\) and we have the equality
\[
\langle w_{(1)}, \mathcal{Y}_{F,p}(w'_{(3)}, x)w_{(2)} \rangle_{W'_1} = 
\langle e^z L(0) x^{-L(0)} w'_{(3)}, F(e^{-z} L(0) x^{-L(0)} w_{(1)} \otimes e^z L(0) x^{-L(0)} w_{(2)}) \rangle_{W_3}\]
\[
= \langle w'_{(3)}, Y_{3}(v, x_0)F(w_{(1)} \otimes w_{(2)}) \rangle_{W'_3}\]
\[
(4.19)
\]
(cf. \((4.16)\)). Now \((4.4)\) gives
\[
z^{-1} \delta \left(\frac{x_1 - x_0}{z}\right) \langle w'_{(3)}, Y_{3}(v, x_0)F(w_{(1)} \otimes w_{(2)}) \rangle_{W_3} = 
\]
\[
x_0^{-1} \delta \left(\frac{x_1 - z}{x_0}\right) \langle w'_{(3)}, F(Y_{1}(v, x_1)w_{(1)} \otimes w_{(2)}) \rangle_{W_3} 
- x_0^{-1} \delta \left(\frac{z - x_1}{x_0}\right) \langle w'_{(3)}, F(w_{(1)} \otimes Y_{2}(v, x_1)w_{(2)}) \rangle_{W_3}. \tag{4.20}
\]
Changing the formal variables $x_0$ and $x_1$ in (4.20) to $zx_2^{-1}x_0$ and $zx_2^{-1}x_1$, respectively, and using (3.22), we obtain

$$x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle Y'_3(v, zx_2^{-1}x_0)w'_3, F(w(1) \otimes w(2)) \rangle w_3 =$$

$$= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle w'_3, F(Y'_1(v, zx_2^{-1}x_1)w(1) \otimes w(2)) \rangle w_3$$

$$- x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w'_3, F(w(1) \otimes Y_2(v, zx_2^{-1}x_1)w(2)) \rangle w_3. \quad (4.21)$$

(Note that all powers of $z$ occurring here are integral.) Using the formulas

$$Y'_3(v, zx_2^{-1}x_0) = e^{l_p(z)L(0)}x_2^{-L(0)}Y'_3(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_0), \quad (4.22)$$

$$Y'_1(v, zx_2^{-1}x_1) = e^{-l_p(z)L(0)}x_2^{L(0)}Y'_1(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1), \quad (4.23)$$

$$Y_2(v, zx_2^{-1}x_1) = e^{l_p(z)L(0)}x_2^{-L(0)}Y_2(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_1), \quad (4.24)$$

which follow from Lemma 5.2.3 together with formula (5.2.39) of [FHL] (note that the eigenvalues of $L(0)$ are not in general integral on the modules), we see that (4.21) becomes

$$x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle e^{l_p(z)L(0)}x_2^{-L(0)}Y'_3(e^{-l_p(z)L(0)}x_2^{L(0)}v, x_0) \rangle w_3$$

$$= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle w'_3, F(e^{-l_p(z)L(0)}x_2^{L(0)}v) \rangle w_3$$

$$- x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w'_3, F(e^{-l_p(z)L(0)}x_2^{L(0)}w(1) \otimes w(2)) \rangle w_3$$

$$- x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \langle w'_3, F(w(1) \otimes e^{l_p(z)L(0)}x_2^{-L(0)}v) \rangle w_3. \quad (4.25)$$

Replacing $v, w(1), w(2)$ and $w'_3$ in (4.25) by

$$e^{l_p(z)L(0)}x_2^{-L(0)}v,$$
and
\[e^{\ell_p(z) L(0)} x_2^{-L(0)} w^{(2)}\]
respectively, we obtain
\[x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle e^{\ell_p(z) L(0)} x_2^{-L(0)} Y_3(v, x_0) w^{(3)} \rangle_{W_3}\]
\[= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle e^{\ell_p(z) L(0)} x_2^{-L(0)} w^{(3)} \rangle_{W_3}\]
\[= x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \langle e^{\ell_p(z) L(0)} x_2^{-L(0)} w^{(3)} \rangle_{W_3}\]
\[= -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \langle e^{\ell_p(z) L(0)} x_2^{-L(0)} Y_2(v, x_1) w^{(2)} \rangle_{W_3}. \quad (4.26)\]

But using (4.19), we can write (4.26) as
\[x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \langle w^{(1)}, Y_{F,p}(Y_3(v, x_0) w^{(3)}, x_2) w^{(2)} \rangle_{W_1}\]
\[= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \langle Y_3(v, x_1) w^{(1)}, Y_{F,p}(w^{(3)}, x_2) w^{(2)} \rangle_{W_1}\]
\[= -x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \langle w^{(1)}, Y_{F,p}(w^{(3)}, x_2) Y_2(v, x_1) w^{(2)} \rangle_{W_1}, \quad (4.27)\]
and (4.27) is equivalent to the Jacobi identity for \(Y_{F,p}\).

Finally, the Jacobi identity implies that
\[\{L(0), Y_{F,p}(w^{(3)}, x)\} = Y_{F,p}(L(0) w^{(3)}, x) + x Y_{F,p}(L(-1) w^{(3)}, x),\]
and since by construction the weight of the operator \((w^{(3)})_n (n \in \mathbb{C})\) is \(\text{wt } w^{(3)} - n - 1\) if \(w^{(3)}\) is homogeneous, the \(L(-1)\)-derivative property follows.

The following immediate result relates module maps from a tensor product module with intertwining maps and intertwining operators:
Proposition 4.8 Suppose that $W_1 \boxtimes_{Q(z)} W_2$ exists. We have a natural isomorphism

$$\text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) \xrightarrow{\sim} \mathcal{M}_{W_1W_2}^{W_3}$$

$$\eta \mapsto \overline{\eta} \circ \boxtimes_{Q(z)}$$  \hspace{1cm} (4.28)

and for $p \in \mathbb{Z}$, a natural isomorphism

$$\text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) \xrightarrow{\sim} V^{W_3}_{W_1'}W_2'$$

$$\eta \mapsto \mathcal{V}_{\eta,p}$$  \hspace{1cm} (4.29)

where $\mathcal{V}_{\eta,p} = \mathcal{V}_{F,p}$ with $F = \overline{\eta} \circ \boxtimes_{Q(z)}$. \hfill \Box

In Part II we shall prove the following:

Proposition 4.9 For any integer $r$, there is a natural isomorphism

$$B_r : V^{W_3}_{W_1W_2} \rightarrow V^{W_3'}_{W_1'W_2}$$  \hspace{1cm} (4.30)

defined by the condition that for any intertwining operator $\mathcal{V}$ in $V^{W_3}_{W_1W_2}$ and

$$w(1) \in W_1, \ w(2) \in W_2, \ w'(3) \in W'_3,$$

$$\langle w(1), B_r(\mathcal{V})(w'(3), x)w(2) \rangle_{W'_3} =$$

$$= \langle e^{-x^{-1}L(1)}w(3), \mathcal{V}(e^{xL(1)}w(1), x^{-1}) \cdot$$

$$\cdot e^{-xL(1)}e^{(2r+1)\pi L(0)}x^{-2L(0)}w(2) \rangle_{W_3}.$$  \hspace{1cm} (4.31)

Combining the last two results, we obtain:

Corollary 4.10 For any $V$-modules $W_1$, $W_2$, $W_3$ such that $W_1 \boxtimes_{Q(z)} W_2$ exists and any integers $p$ and $r$, we have a natural isomorphism

$$\text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) \xrightarrow{\sim} V^{W_3}_{W_1W_2}$$

$$\eta \mapsto B_r^{-1}(\mathcal{V}_{\eta,p}).$$  \hspace{1cm} (4.32)

It is clear from Definition 4.4 that the tensor product operation distributes over direct sums in the following sense:
Proposition 4.11  For $V$-modules $U_1, \ldots, U_k, W_1, \ldots, W_l$, suppose that each $U_i \boxplus_{Q(z)} W_j$ exists. Then $(\coprod_i U_i) \boxplus_{Q(z)} (\coprod_j W_j)$ exists and there is a natural isomorphism

$$\left( \coprod_i U_i \right) \boxplus_{Q(z)} \left( \coprod_j W_j \right) \simeq \coprod_{i,j} U_i \boxplus_{Q(z)} W_j.$$  \hfill (4.33)

Now consider $V$-modules $W_1, W_2$ and $W_3$ and suppose that $\dim \mathcal{M}^{W_3}_{W_1W_2} < \infty$. The natural evaluation map

$$W_1 \otimes W_2 \otimes \mathcal{M}^{W_3}_{W_1W_2} \to \overline{W}_3$$
$$w(1) \otimes w(2) \otimes F \mapsto F(w(1) \otimes w(2))$$  \hfill (4.34)

gives a natural map

$$\mathcal{F}^{W_3}_{W_1W_2} : W_1 \otimes W_2 \to \text{Hom}(\mathcal{M}^{W_3}_{W_1W_2}, \overline{W}_3) = (\mathcal{M}^{W_3}_{W_1W_2})^* \otimes \overline{W}_3.$$  \hfill (4.35)

Also, $(\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3$ is a $V$-module (with finite-dimensional weight spaces) in the obvious way, and the map $\mathcal{F}^{W_3}_{W_1W_2}$ is clearly a $Q(z)$-intertwining map, where we make the identification

$$(\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3 = (\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3.$$  \hfill (4.36)

This gives us a natural $Q(z)$-product. Moreover, we have a natural linear injection

$$i : \mathcal{M}^{W_3}_{W_1W_2} \to \text{Hom}_V((\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3, W_3)$$
$$F \mapsto (f \otimes w(3) \mapsto f(F)w(3))$$  \hfill (4.37)

which is an isomorphism if $W_3$ is irreducible, since in this case,

$$\text{Hom}_V(W_3, W_3) \simeq \mathbb{C}$$
(see [FHL], Remark 4.7.1). On the other hand, the natural map

$$h : \text{Hom}_V((\mathcal{M}^{W_3}_{W_1W_2})^* \otimes W_3, W_3) \to \mathcal{M}^{W_3}_{W_1W_2}$$
$$\eta \mapsto \overline{\eta} \circ \mathcal{F}^{W_3}_{W_1W_2}$$  \hfill (4.38)

given by composition clearly satisfies the condition that

$$h(i(F)) = F,$$  \hfill (4.39)
so that if \( W_3 \) is irreducible, the maps \( h \) and \( i \) are mutually inverse isomorphisms and we have the universal property that for any \( F \in \mathcal{M}_{W_1W_2}^{W_3} \), there exists a unique \( \eta \) such that

\[
F = \eta \circ \mathcal{F}_{W_1W_2}^{W_3}
\]  
(4.40)

(cf. Definition 4.4).

Now we consider a special but important class of vertex operator algebras satisfying certain finiteness and semisimplicity conditions.

**Definition 4.12** A vertex operator algebra \( V \) is *rational* if it satisfies the following conditions:

1. There are only finitely many irreducible \( V \)-modules (up to equivalence).
2. Every \( V \)-module is completely reducible (and is in particular a finite direct sum of irreducible modules).
3. All the fusion rules for \( V \) are finite (for triples of irreducible modules and hence arbitrary modules).

The next result shows that tensor products exist for the category of modules for a rational vertex operator algebra. Note that there is no need to assume that \( W_1 \) and \( W_2 \) are irreducible in the formulation or proof, but by Proposition 4.11, the case in which \( W_1 \) and \( W_2 \) are irreducible is in fact sufficient, and the tensor product operation is canonically described using only the spaces of intertwining maps among triples of irreducible modules.

**Proposition 4.13** Let \( V \) be rational and let \( W_1, W_2 \) be \( V \)-modules. Then \((W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})\) exists, and in fact

\[
W_1 \boxtimes_{Q(z)} W_2 = \bigoplus_{i=1}^k (\mathcal{M}_{W_1W_2}^{M_i})^* \otimes M_i,
\]  
(4.41)

where \( \{M_1, \ldots, M_k\} \) is a set of representatives of the equivalence classes of irreducible \( V \)-modules, and the right-hand side of (4.41) is equipped with the \( V \)-module and \( Q(z) \)-product structure indicated above. That is,

\[
\boxtimes_{Q(z)} = \sum_{i=1}^k \mathcal{F}_{W_1W_2}^{M_i}.
\]  
(4.42)
Proof From the comments above and the definitions, it is clear that we have a $Q(z)$-product. Let $(W_3, Y_3; F)$ be any $Q(z)$-product. Then $W_3 = \bigsqcup_j U_j$ where $j$ ranges through a finite set and each $U_j$ is irreducible. Let $\pi_j : W_3 \rightarrow U_j$ denote the $j$-th projection. A module map $\eta : \bigoplus_{i=1}^k (\mathcal{M}_{W_i}^{M_i})^* \otimes M_i \rightarrow W_3$ amounts to module maps

$$\eta_{ij} : (\mathcal{M}_{W_i}^{M_i})^* \otimes M_i \rightarrow U_j$$

for each $i$ and $j$ such that $U_j \simeq M_i$, and $F = \overline{\eta} \circ \otimes_{Q(z)}$ if and only if

$$\overline{\pi}_j \circ F = \overline{\eta}_{ij} \circ \mathcal{F}_{W_i W_2}^{M_i}$$

for each $i$ and $j$, the bars having the obvious meaning. But $\pi_j \circ F$ is a $Q(z)$-intertwining map of type $\left( \frac{U_j}{W_i W_2} \right)$, and so $\tau \circ \pi_j \circ F \in \mathcal{M}_{W_i W_2}^{M_i}$, where $\iota : U_j \simeq M_i$ is a fixed isomorphism. Denote this map by $\tau$. Thus what we finally want is a unique module map

$$\theta : (\mathcal{M}_{W_1}^{M_i})^* \otimes M_i \rightarrow M_i$$

such that

$$\tau = \overline{\theta} \circ \mathcal{F}_{W_1 W_2}^{M_i}.$$ 

But we in fact have such a unique $\theta$, by (4.39)–(4.40). \qed

Remark 4.14 By combining Proposition 4.13 with Proposition 4.7 or Proposition 4.9, we can express $W_1 \otimes_{Q(z)} W_2$ in terms of $\mathcal{V}_{M_1}^{W_1}$ or $\mathcal{V}_{M_2}^{W_2}$ in place of $\mathcal{M}_{W_1 W_2}^{M_i}$.

The construction in Proposition 4.13 is tautological, and we view the argument as essentially an existence proof. In the next two sections, we present “first and second constructions” of a $Q(z)$-tensor product.

5 First construction of $Q(z)$-tensor product

Here and in the next section, we give two constructions of a $Q(z)$-tensor product of two modules for a vertex operator algebra $V$, in the presence of a certain hypothesis which holds in case $V$ is rational. In this section, we first define an action of $V \otimes \iota_+ \mathcal{C}[t, t^{-1}, (z + t)^{-1}]$ on $(W_1 \otimes W_2)^*$ motivated by the
definition (4.4) of $Q(z)$-intertwining map. We establish some basic properties of this action, deferring the proof of a commutator formula (Proposition 5.2) to Part II. Then we take the sum of all “compatible modules” in $(W_1 \otimes W_2)^*$. Under the assumption that this sum is again a module, we construct the $Q(z)$-tensor product as its contragredient module equipped with the restriction to $W_1 \otimes W_2$ of the adjoint of the embedding map of this sum in $(W_1 \otimes W_2)^*$. In the next section we observe that every element in the sum of compatible modules in $(W_1 \otimes W_2)^*$ satisfies a certain set of conditions, and we show that, modulo two important results stated there but whose proofs are deferred to Part II, the subspace of $(W_1 \otimes W_2)^*$ consisting of all the elements satisfying these conditions is equal to this sum of compatible modules. In this way we obtain another construction of the $Q(z)$-tensor product.

Fix a nonzero complex number $z$ and $V$-modules $(W_1, Y_1)$ and $(W_2, Y_2)$ as before. We first want to define a linear action of $V \otimes \iota + \mathbb{C}[(t, t^{-1}, (z + t)^{-1})]$ on $(W_1 \otimes W_2)^*$, that is, a linear map

$$\tau_{Q(z)} : V \otimes \iota + \mathbb{C}[(t, t^{-1}, (z + t)^{-1})] \to \text{End } (W_1 \otimes W_2)^*.$$

Recall the maps

$$\tau_{W_i} : V \otimes \mathbb{C}((t)) \to \text{End } W_i, \quad i = 1, 2,$$

from (3.2). We define $\tau_{Q(z)}$ by

$$(\tau_{Q(z)}(\xi) \lambda)(w_{(1)} \otimes w_{(2)}) = \lambda(\tau_{W_1}(T^*_{-z} \xi)w_{(1)} \otimes w_{(2)}) - \lambda(w_{(1)} \otimes \tau_{W_2}(T^*_z \xi)w_{(2)}),$$

(5.1)

for $\xi \in V \otimes \iota + \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. Using (3.12)–(3.13), (3.60) and Lemma 3.1, we see that the definition (5.1) can be written using generating functions as:

$$\left(\tau_{Q(z)} \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y_i(v, x_0) \right) \lambda \right)(w_{(1)} \otimes w_{(2)})$$

$$= x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \lambda(Y_i^*(v, x_1)w_{(1)} \otimes w_{(2)})$$

$$- x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \lambda(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}).$$

(5.2)

Write

$$Y_{Q(z)}'(v, x) = \tau_{Q(z)}(Y_i(v, x)).$$

(5.3)
Using (2.6) and the fundamental property of the formal $\delta$-function, we have

\[
(Y'_Q(z)(v, x_0)\lambda)(w(1) \otimes w(2)) = \\
= \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) \lambda(Y'_1(v, x_1)w(1) \otimes w(2)) \\
- \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \lambda(w(1) \otimes Y_2(v, x_1)w(2)) \\
= \lambda(Y'_1(v, x_0 + z)w(1) \otimes w(2)) \\
- \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) \lambda(w(1) \otimes Y_2(v, x_1)w(2)),
\]

(5.4)

where we have used the notation $\text{Res}_{x_1}$, which means taking the coefficient of $x_1$ in a formal series. We have the following results for $Y'_Q(z)$:

**Proposition 5.1** The action $Y'_Q(z)$ has the property

\[
Y'_Q(z)(1, x) = 1,
\]

(5.5)

where 1 on the right-hand side is the identity map of $(W_1 \otimes W_2)^*$, and the $L(-1)$-derivative property

\[
\frac{d}{dx} Y'_Q(z)(v, x) = Y'_Q(z)(L(-1)v, x)
\]

(5.6)

for $v \in V$.

**Proof** From (5.4), (3.20) and (2.7),

\[
(Y(1, x)\lambda)(w(1) \otimes w(2)) = \\
= \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{x_1 - z}{x} \right) \lambda(w(1) \otimes w(2)) \\
- \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{z - x_1}{-x} \right) \lambda(w(1) \otimes w(2)) \\
= \text{Res}_{x_1} x_1^{-1} \delta \left( \frac{z + x}{x_1} \right) \lambda(w(1) \otimes w(2)) \\
= \lambda(w(1) \otimes w(2)),
\]

(5.7)
proving (5.5). We now prove the $L(-1)$-derivative property. From (5.4),
\[
\left( \left( \frac{d}{dx} Y_{Q(z)}'(v, x) \right) \lambda \right) (w_1 \otimes w_2) =
\]
\[
= \frac{d}{dx} \lambda(Y_1^*(v, x + z)w_1 \otimes w_2)
- \text{Res}_{x_1} \left( \frac{d}{dx} z^{-1} \delta \left( \frac{-x + x_1}{z} \right) \right) \lambda(w_1 \otimes Y_2(v, x_1 w_2)).
\]
(5.8)

Note that for any formal Laurent series $f(x)$, we have
\[
\frac{d}{dx} f \left( \frac{-x + x_1}{z} \right) = - \frac{d}{dx_1} f \left( \frac{-x + x_1}{z} \right)
\]
and if $f(x)$ involves only finitely many negative powers of $x$,
\[
\text{Res}_{x_1} \left( \frac{d}{dx_1} z^{-1} \delta \left( \frac{-x + x_1}{z} \right) \right) f(x_1) = - \text{Res}_{x_1} z^{-1} \delta \left( \frac{-x + x_1}{z} \right) \frac{d}{dx_1} f(x_1)
\]
(since the residue of a derivative is 0). From (3.22) and the $L(-1)$-derivative property for the contragredient module $W'_1$, we have
\[
\frac{d}{dx} Y_{1}^*(v, x) = Y_{1}^*(L(-1)v, x).
\]
Thus the right-hand side of (5.8) is equal to
\[
\lambda(Y_1^*(L(-1)v, x + z)w_1 \otimes w_2)
- \text{Res}_{x_1} z^{-1} \delta \left( \frac{-x + x_1}{z} \right) \frac{d}{dx_1} \lambda(w_1 \otimes Y_2(v, x_1 w_2))
= \lambda(Y_1^*(L(-1)v, x + z)w_1 \otimes w_2)
- \text{Res}_{x_1} z^{-1} \delta \left( \frac{-x + x_1}{z} \right) \lambda(w_1 \otimes Y_2(L(-1)v, x_1 w_2))
= (Y_{Q(z)}'(L(-1)v, x)\lambda)(w_1 \otimes w_2),
\]
(5.11)
completing the proof. \qed

**Proposition 5.2** The action $Y_{Q(z)}'$ satisfies the commutator formula for vertex operators, that is, on $(W_1 \otimes W_2)^*$,
\[
[Y_{Q(z)}'(v_1, x_1), Y_{Q(z)}'(v_2, x_2)] =
\]
\[
= \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_{Q(z)}'(Y(v_1, x_0)v_2, x_2)
\]
(5.12)
for $v_1, v_2 \in V$.

The proof of this proposition will be given in Part II.

From these results and the relation (2.22), we see that the coefficient operators of $Y'_{Q(z)}(\omega, x)$ satisfy the Virasoro algebra commutator relations, that is, writing
\[ Y'_{Q(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{Q(z)}(n)x^{-n-2}, \] (5.13)
we have
\[ [L'_{Q(z)}(m), L'_{Q(z)}(n)] = (m - n)L'_{Q(z)}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c. \] (5.14)

We call the eigenspaces of the operator $L'_{Q(z)}(0)$ the weight subspaces or homogeneous subspaces of $(W_1 \otimes W_2)^*$, and we have the corresponding notions of weight vector (or homogeneous vector) and weight. When there is no confusion, we shall simply write $L'_{Q(z)}(n)$ as $L(n)$.

Let $W_3$ be another $V$-module. Note that $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ acts on $W'_3$ in the obvious way. The following result, which follows immediately from the definitions (4.4) and (5.2), provides further motivation for the definition of our action on $(W_1 \otimes W_2)^*$:

**Proposition 5.3** Under the natural isomorphism
\[ \text{Hom}(W'_3, (W_1 \otimes W_2)^*) \cong \text{Hom}(W_1 \otimes W_2, W_3), \] (5.15)
the maps in Hom$(W'_3, (W_1 \otimes W_2)^*)$ intertwining the two actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ on $W'_3$ and $(W_1 \otimes W_2)^*$ correspond exactly to the $Q(z)$-intertwining maps of type $(W'_3)^{W_3} (W_1 \otimes W_2)$.
\[ \Box \]

**Remark 5.4** Combining the last result with Proposition 4.7, we see that the maps in Hom$(W'_3, (W_1 \otimes W_2)^*)$ intertwining the two actions on $W'_3$ and $(W_1 \otimes W_2)^*$ also correspond exactly to the intertwining operators of type $(W'_3)^{W_3} (W_1 \otimes W_2)$. In particular, given any integer $p$, the map $F'_{Y,p} : W'_3 \rightarrow (W_1 \otimes W_2)^*$ defined by
\[ F'_{Y,p}(w'_3)(w_{(1)} \otimes w_{(2)}) = \langle w_{(1)}, Y'(w'_{(3)}, e^{p(z)})w_{(2)} \rangle w_1 \] (5.16)
(recall (4.9)) intertwines the actions of $V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}]$ on $W'_3$ and $(W_1 \otimes W_2)^*$.
Suppose that $G \in \text{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions as in Proposition 5.3. Then for $w'_{(3)} \in W'_3$,

$$
\tau_{Q(z)} \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) G(w'_{(3)}) = \\
G \left( \tau_{W_3} \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) w'_{(3)} \right) \\
= G \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y'_3(v, x_0) w'_{(3)} \right) \\
= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y'_3(v, x_0) G(w'_{(3)}) \\
= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y'_Q(v, x_0) G(w'_{(3)}).
$$

(5.17)

Thus $G(w'_{(3)})$ satisfies the following nontrivial and subtle condition on $\lambda \in (W_1 \otimes W_2)^*$: The formal Laurent series $Y'_Q(v, x_0) \lambda$ involves only finitely many negative powers of $x_0$ and

$$
\tau_{Q(z)} \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda = \\
= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y'_Q(v, x_0) \lambda \quad \text{for all } v \in V.
$$

(5.18)

(Note that the two sides are not a priori equal for general $\lambda \in (W_1 \otimes W_2)^*$.) We call this the compatibility condition on $\lambda \in (W_1 \otimes W_2)^*$, for the action $\tau_{Q(z)}$.

Let $W$ be a subspace of $(W_1 \otimes W_2)^*$. We say that $W$ is compatible for $\tau_{Q(z)}$ if every element of $W$ satisfies the compatibility condition. Also, we say that $W$ is ($\mathbb{C}$-)graded if it is $\mathbb{C}$-graded by its weight subspaces, and that $W$ is a $V$-module (respectively, generalized module) if $W$ is graded and is a module (respectively, generalized module) when equipped with this grading and with the action of $Y'_Q(\cdot, x)$ (recall Definition 2.11). A sum of compatible modules or generalized modules is clearly a generalized module. The weight subspace of a subspace $W$ with weight $n \in \mathbb{C}$ will be denoted $W_n$.

Given $G$ as above, it is clear that $G(W'_3)$ is a $V$-module since $G$ intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$. We have in fact established that $G(W'_3)$ is in addition a compatible $V$-module since $G$ intertwines the full actions. Moreover, if $H \in \text{Hom}(W'_3, (W_1 \otimes W_2)^*)$ intertwines the two actions of $V \otimes$
\[ C[t, t^{-1}], \text{ then } H \text{ intertwines the two actions of } V \otimes \tau_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \text{ if and only if the } V\text{-module } H(W_3') \text{ is compatible.} \]

Define

\[ W_1 \boxtimes_{Q(z)} W_2 = \sum_{W \in W_{Q(z)}} W = \bigcup_{W \in W_{Q(z)}} W \subset (W_1 \otimes W_2)^*, \quad (5.19) \]

where \( W_{Q(z)} \) is the set all compatible modules for \( \tau_{Q(z)} \) in \( (W_1 \otimes W_2)^* \). Then \( W_1 \boxtimes_{Q(z)} W_2 \) is a compatible generalized module and coincides with the sum (or union) of the images \( G(W_3') \) of modules \( W_3' \) under the maps \( G \) as above. Moreover, for any \( V\)-module \( W_3 \) and any map \( H : W'_3 \to W_1 \boxtimes_{Q(z)} W_2 \) of generalized modules, \( H(W_3') \) is compatible and hence \( H \) intertwines the two actions of \( V \otimes \tau_+ \mathbb{C}[t, t^{-1}, (z + t)^{-1}] \). Thus we have:

**Proposition 5.5** The subspace \( W_1 \boxtimes_{Q(z)} W_2 \) of \( (W_1 \otimes W_2)^* \) is a generalized module with the following property: Given any \( V\)-module \( W_3 \), there is a natural linear isomorphism determined by (5.15) between the space of all \( Q(z)\)-intertwining maps of type \( (W_3' \ \wedge_{W_1 W_2}) \) and the space of all maps of generalized modules from \( W'_3 \) to \( W_1 \boxtimes_{Q(z)} W_2 \).

**Proposition 5.6** Let \( V \) be a rational vertex operator algebra and \( W_1, W_2 \) two \( V\)-modules. Then \( W_1 \boxtimes_{Q(z)} W_2 \) is a module.

*Proof* Since \( W_1 \boxtimes_{Q(z)} W_2 \) is the sum of all compatible modules for \( \tau_{Q(z)} \) in \( (W_1 \otimes W_2)^* \) and since by assumption every module is completely reducible, the generalized \( V\)-module \( W_1 \boxtimes_{Q(z)} W_2 \) is a direct sum of irreducible modules. If it is an infinite direct sum, it must include infinitely many copies of at least one irreducible \( V\)-module, say, \( W_3 \), since a rational vertex operator algebra has only finitely many irreducible modules. From Proposition 5.5, the space of \( Q(z)\)-intertwining maps of type \( (W_3' \ \wedge_{W_1 W_2}) \) must be infinite-dimensional, and by Proposition 4.7, this contradicts the assumed finiteness of the fusion rules. Thus \( W_1 \boxtimes_{Q(z)} W_2 \) is a finite direct sum of irreducible modules and hence is a module. \( \square \)

Now we assume that \( W_1 \boxtimes_{Q(z)} W_2 \) is a module (which occurs if \( V \) is rational, by the last proposition). In this case, we define a \( V\)-module \( W_1 \boxtimes_{Q(z)} W_2 \) by

\[ W_1 \boxtimes_{Q(z)} W_2 = (W_1 \boxtimes_{Q(z)} W_2)' \quad (5.20) \]
and we write the corresponding action as \( Y_{Q(z)} \). Applying Proposition 5.5 to the special module \( W_3 = W_1 \boxtimes_{Q(z)} W_2 \) and the identity map \( W_3' \to W_1 \boxtimes_{Q(z)} W_2 \) (recall Theorem 2.10), we obtain using (5.15) a canonical \( Q(z) \)-intertwining map of type \( (W_1 \boxtimes_{Q(z)} W_2) \), which we denote

\[
\boxtimes_{Q(z)} : W_1 \otimes W_2 \to \overline{W_1 \boxtimes_{Q(z)} W_2},
\]

\[
w(1) \otimes w(2) \mapsto w(1) \boxtimes_{Q(z)} w(2).
\] (5.21)

This is the unique linear map such that

\[
\langle \lambda, w(1) \boxtimes_{Q(z)} w(2) \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \lambda(w(1) \otimes w(2))
\] (5.22)

for all \( w(1) \in W_1, w(2) \in W_2 \) and \( \lambda \in W_1 \boxtimes_{Q(z)} W_2 \). Moreover, we have:

**Proposition 5.7** The \( Q(z) \)-product \((W_1 \boxtimes_{Q(z)} W_2, Y_{Q(z)}; \boxtimes_{Q(z)})\) is a \( Q(z) \)-tensor product of \( W_1 \) and \( W_2 \).

**Proof** Let \((W_3, Y_3; F)\) be a \( Q(z) \)-product of \( W_1 \) and \( W_2 \). By Proposition 5.5, there is a unique \( V \)-module map

\[
\eta' : W_3' \to W_1 \boxtimes_{Q(z)} W_2
\]

such that

\[
\langle w(3), F(w(1) \otimes w(2)) \rangle_{W_3} = \eta'(w(3))(w(1) \otimes w(2))
\]

for any \( w(1) \in W_1, w(2) \in W_2 \) and \( w(3) \in W_3' \). But by (5.22), this equals

\[
\langle \eta'(w(3)), w(1) \boxtimes_{Q(z)} w(2) \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \langle w(3), \bar{\eta}(w(1) \boxtimes_{Q(z)} w(2)) \rangle_{W_3},
\]

where \( \eta \in \text{Hom}_V(W_1 \boxtimes_{Q(z)} W_2, W_3) \) and \( \eta' \) are mutually adjoint maps. In particular, there is a unique \( \eta \) such that

\[
\langle w(3), F(w(1) \otimes w(2)) \rangle_{W_3} = \langle w(3), \bar{\eta}(w(1) \boxtimes_{Q(z)} w(2)) \rangle_{W_3},
\]

i.e., such that

\[
F = \bar{\eta} \circ \boxtimes_{Q(z)} : W_1 \otimes W_2 \to W_3,
\]

and this establishes the desired universal property. \( \Box \)

More generally, dropping the assumption that \( W_1 \boxtimes_{Q(z)} W_2 \) is a module, we have:
Proposition 5.8 The $Q(z)$-tensor product of $W_1$ and $W_2$ exists (and is given by (5.20)) if and only if $W_1 \boxtimes_{Q(z)} W_2$ is a module.

Proof It is sufficient to show that if the $Q(z)$-tensor product exists, then $W_1 \boxtimes_{Q(z)} W_2$ is a module. Consider the module

$$W_0 = (W_1 \boxtimes_{Q(z)} W_2)' .$$

Applying Proposition 5.5 to the $Q(z)$-product $W_1 \boxtimes_{Q(z)} W_2$, we have a unique map

$$i : W_0 \to W_1 \boxtimes_{Q(z)} W_2$$

of generalized modules such that

$$i(w(0))(w(1) \otimes w(2)) = \langle w(0), w(1) \boxtimes_{Q(z)} w(2) \rangle_{W_1 \boxtimes_{Q(z)} W_2}$$

for $w(0) \in W_0$, $w(1) \in W_1$ and $w(2) \in W_2$. It suffices to show that $i$ is a surjection.

Let $W \in W_{Q(z)}$ (recall (5.19)) and set $W_3 = W'$. By Proposition 5.5, the injection $W_3' \hookrightarrow W_1 \boxtimes_{Q(z)} W_2$ induces a unique $Q(z)$-intertwining map $F$ of type $(W_3' W_1 W_2)$ such that

$$w(w(1) \otimes w(2)) = \langle w, F(w(1) \otimes w(2)) \rangle_{W_3'}$$

for $w \in W'$, $w(1) \in W_1$ and $w(2) \in W_2$. But by the universal property of $W_1 \boxtimes_{Q(z)} W_2$, there is a unique module map $\eta' : W_1 \boxtimes_{Q(z)} W_2 \to W'$ such that $F = \eta' \circ \boxtimes_{Q(z)}$, and hence a unique module map $\eta : W \to W_0$ such that

$$\langle \eta(w), w_1 \boxtimes_{Q(z)} w_2 \rangle_{W_1 \boxtimes_{Q(z)} W_2} = \langle w, F(w_1 \otimes w_2) \rangle_{W_3'} .$$

Thus

$$w(w_1 \otimes w_2) = \langle \eta(w), w_1 \boxtimes_{Q(z)} w_2 \rangle_{W_1 \boxtimes_{Q(z)} W_2} = i(\eta(w))(w_1 \otimes w_2)$$

and so $w = i(\eta(w))$ for all $w \in W$, showing that $W$ lies in the image of the map $i$ and hence that $i$ is surjective. $\Box$
6 Second construction of $Q(z)$-tensor product

Let $V$ be a vertex operator algebra and $W_1, W_2$ two $V$-modules. From the definition (5.19) of $W_1 \Box_{Q(z)} W_2$, we see that any element of $W_1 \Box_{Q(z)} W_2$ is an element $\lambda$ of $(W_1 \otimes W_2)^*$ satisfying:

The compatibility condition (recall (5.18)): (a) The lower truncation condition: For all $v \in V$, the formal Laurent series $Y'_{Q(z)}(v, x)\lambda$ involves only finitely many negative powers of $x$.

(b) The following formula holds:

$$\tau_{Q(z)} \left( z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y_t(v, x_0) \right) \lambda = z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) Y'_{Q(z)}(v, x_0) \lambda \quad \text{for all} \quad v \in V. \quad (6.1)$$

The local grading-restriction condition: (a) The grading condition: $\lambda$ is a (finite) sum of weight vectors of $(W_1 \otimes W_2)^*$.

(b) Let $W_{\lambda}$ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing $\lambda$ and stable under the component operators $\tau_{Q(z)}(v \otimes t^n)$ of the operators $Y'_{Q(z)}(v, x)$ for $v \in V, n \in \mathbb{Z}$. Then the weight spaces $(W_{\lambda})_n, n \in \mathbb{C}$, of the (graded) space $W_{\lambda}$ have the properties

$$\dim (W_{\lambda})_n < \infty \quad \text{for} \quad n \in \mathbb{C}, \quad (6.2)$$

$$(W_{\lambda})_n = 0 \quad \text{for} \quad n \quad \text{whose real part is sufficiently small}. \quad (6.3)$$

Note that the set of the elements of $(W_1 \otimes W_2)^*$ satisfying any one of the lower truncation condition, the compatibility condition, the grading condition or the local grading-restriction condition forms a subspace.

In Part II, we shall prove the following two basic results:

**Theorem 6.1** Let $\lambda$ be an element of $(W_1 \otimes W_2)^*$ satisfying the compatibility condition. Then when acting on $\lambda$, the Jacobi identity for $Y'_{Q(z)}$ holds, that is,

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y'_{Q(z)}(u, x_1)Y'_{Q(z)}(v, x_2)\lambda$$

$$-x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y'_{Q(z)}(v, x_2)Y'_{Q(z)}(u, x_1)\lambda$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(u, x_0)v, x_2)\lambda$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y'_{Q(z)}(Y(u, x_0)v, x_2)\lambda \quad (6.4)$$
for \( u, v \in V \).

**Proposition 6.2** The subspace consisting of the elements of \((W_1 \otimes W_2)^*\) satisfying the compatibility condition is stable under the operators \( \tau_{Q(z)}(v \otimes t^n) \) for \( v \in V \) and \( n \in \mathbb{Z} \), and similarly for the subspace consisting of the elements satisfying the local grading-restriction condition.

These results give us another construction of \( W_1 \ltimes_{Q(z)} W_2 \):

**Theorem 6.3** The subspace of \((W_1 \otimes W_2)^*\) consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with \( Y'_{Q(z)} \), is a generalized module and is equal to \( W_1 \ltimes_{Q(z)} W_2 \).

**Proof** Let \( W_0 \) be the space of vectors satisfying the two conditions. We have already observed that \( W_1 \ltimes_{Q(z)} W_2 \subset W_0 \), and it suffices to show that \( W_0 \) is a generalized module which is a union of compatible modules. But \( W_0 \) is a compatible generalized module by Theorem 6.1 and Proposition 6.2, together with Proposition 5.1 and formula (5.14), and every element of \( W_0 \) generates a compatible module contained in \( W_0 \), by the local grading-restriction condition.

The following result follows immediately from Proposition 5.8, the theorem above and the definition of \( W_1 \ltimes_{Q(z)} W_2 \):

**Corollary 6.4** The \( Q(z) \)-tensor product of \( W_1 \) and \( W_2 \) exists if and only if the subspace of \((W_1 \otimes W_2)^*\) consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with \( Y'_{Q(z)} \), is a module. In this case, this module coincides with the module \( W_1 \ltimes_{Q(z)} W_2 \), and the contragredient module of this module, equipped with the \( Q(z) \)-intertwining map \( \ltimes_{Q(z)} \), is a \( Q(z) \)-tensor product of \( W_1 \) and \( W_2 \), equal to the structure \((W_1 \ltimes_{Q(z)} W_2, Y_{Q(z)}; \ltimes_{Q(z)})\) constructed in Section 5.

From this result and Propositions 5.6 and 5.7, we have:

**Corollary 6.5** Let \( V \) be a rational vertex operator algebra and \( W_1, W_2 \) two \( V \)-modules. Then the \( Q(z) \)-tensor product \((W_1 \ltimes_{Q(z)} W_2, Y_{Q(z)}; \ltimes_{Q(z)})\) may be constructed as described in Corollary 6.4.
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