A NOTE ON THE HYPER–CR EQUATION, AND GAUGED $N = 2$ SUPERGRAVITY

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Abstract. We construct a new class of solutions to the dispersionless hyper–CR equation, and show how any solution to this equation gives rise to a supersymmetric Einstein–Maxwell cosmological space–time in (3 + 1)–dimensions.

1. Introduction

Let $H = H(x, y, t)$ satisfy the partial–differential equation

$$H_{xt} - H_{yy} + H_yH_{xx} - H_xH_{xy} = 0. \tag{1}$$

This equation is integrable by twistor transform [3], and by the Manakov–Santini type inverse–scattering procedure [13, 9]. It arises in $(2 + 1)$–dimensional Einstein–Weyl geometry, where its solutions characterise the Einstein–Weyl spaces of the hyper–CR type. It also appears in several other integrable constructions [1, 2, 4, 5, 7, 15, 16, 17].

In this note we shall construct all solutions to (1), where the linear and nonlinear terms in (1) vanish separately. This will be done in §2. In §3 we shall show how solutions to (1) lift to supersymmetric solutions to $N = 2$ pseudo–supergravity in $3 + 1$ space–time dimensions.

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2. Hyper–CR equation

Consider three one–forms on a three–dimensional manifold $B$

$e^1 = dx -udy + wdt, \quad e^2 = dy - wdt, \quad e^3 = dt,$

where $(x, y, t)$ is a local coordinate system on $B$, and $u, w$ are two functions of $(x, y, t)$. Let

$\omega = u_xdy + (uu_x + 2u_y)dt, \quad V = \frac{u_x}{2}$

be another one–form, and a function on $B$. The Gauduchon–Tod system of equations [8]

$$de^i = \frac{1}{2} \omega \wedge e^i - V \ast e^i, \quad i = 1, 2, 3 \tag{2}$$

holds where $\ast$ is the Hodge operator $1$ of a Lorentzian metric on $B$

$h = e^2 \otimes e^2 - 4e^1 \otimes e^3$

if the functions $(u, w)$ satisfy a system of integrable equations of hydrodynamic type (the hyper–CR system)

$$u_t + w_y + uw_x - wu_x = 0, \quad u_y + w_x = 0. \tag{3}$$
Conversely, it has been shown in [3] that a pair \((h, \omega)\) gives rise to an Einstein–Weyl structure: there exists a torsion–free connection \(D\) such that \(Dh = \omega \otimes h\) and the symmetrised Ricci tensor of \(D\) is proportional to \(h\) if and only if the system (3) holds. The integrability conditions for (2) are given by the monopole equation

\[ * \left( dV + \frac{1}{2} V \omega \right) = \frac{1}{2} d\omega. \]

This equation holds as a consequence of (3) – it becomes a derivative of the first equation in (3). The Bianchi identity implies that no other integrability conditions arise.

2.1. **Conformal invariance.** Both the Einstein–Weyl condition, and the Gauduchon–Tod system (2) are conformally invariant if

\[ e^i \rightarrow e^f e^i, \quad \omega \rightarrow \omega + 2 df, \quad V \rightarrow e^{-f} V. \]  

In particular, it is possible to fix a conformal gauge such that \(V \equiv -2 \ell^{-1}\) is a constant so that the monopole equation reduces to

\[ d\omega + 2 \ell^{-1} * \omega = 0. \]

In this gauge \(\omega\) is divergence-free (so this is the Gauduchon gauge - note that the converse is not true. There is a residual gauge freedom if the Gauduchon gauge has been fixed which allows for non–constant \(V\)).

2.2. **The \(\psi\)–equation.** Let \(\psi\) be any \(p\)–form of conformal weight \(m\), so that \(\psi \rightarrow e^{mf} \psi\) under (4). The weighted exterior derivative

\[ D\psi \equiv d\psi - \frac{m}{2} \omega \wedge \psi \]

is a \((p + 1)\)–form of weight \(m\). Let us assume that \(\psi\) is a one–form. Using the conformal properties of the Hodge operator we verify that the equation

\[ D\psi = V * \psi \]  

is conformally invariant as the weight of \(V\) is \(-1\). In [11] this equation has arisen in a gauge where \(V = -2 \ell^{-1}\) is a constant, and \(m = -1\) where it becomes

\[ d\psi + \frac{1}{2} \omega \wedge \psi = -2 \ell^{-1} * \psi. \]

There is a particular solution to this equation given by \(\psi = c\omega\), where \(c\) is a constant.

2.3. **Example.** **The Heisenberg group.** Let us consider a particular solution of (3) given by \(u = 4\ell^{-1} x, v = 0\), where \(\ell\) is a constant. The resulting Lorentizian Einstein–Weyl structure is defined on the nilpotent Lie group:

\[ h = (dy + 4\ell^{-1} x dt)^2 - 4 dx dt, \quad \omega = 4\ell^{-1} (dy + 4\ell^{-1} x dt), \]

and \(V = 2 \ell^{-1}\) is a constant. A MAPLE–aided computation shows that the most general solution to (5) which does not depend on \(y\) is of the form

\[ \psi = c\omega + d(c + k), \]

where \(c = c(x)\) and \(k = k(t)\) are arbitrary functions.
2.4. **Differential constraints.** Let us look for a class of solutions to (3) where both the linear and nonlinear parts of the first PDE in (3) vanish separately. The second equation in (3) can be solved in general to give $u = H_x, w = -H_y$, where $H = H(x,y,t)$ satisfies (1) and then the special constraint resulting from $u_t + w_y = 0$ is that $H$ is a solution to the wave equation on the flat background $dy^2 - 4dxdt$, and additionally $H_{xx}H_y - H_{xy}H_x = 0$.

An example is provided by the fundamental solution

$$H = \frac{1}{\sqrt{y^2 - 4xt}}. \quad (7)$$

In general we can establish the following

**Proposition 2.1.** Let $(h, \omega)$ be a hyper-CR Einstein–Weyl structure arising from the equation (1) such that

$$H_{xx}H_y - H_{xy}H_x = 0,$$

and $H_{xt} - H_{yy} = 0$. \quad (8)

Then there exists a local coordinate system $(p,y,t)$ on $B$ such that $(h, \omega)$ takes one of the following three forms

**Class A**

$$h = (dy + pdt)^2 - 4\left(\frac{dp}{p} - \frac{\beta_t}{\beta}(dy + pdt) - \frac{\beta_y}{\beta} dt\right) dt, \quad \omega = -p(dy + pdt) + 2p\beta_y dt, \quad (9)$$

where $\beta = \beta(y,t)$ satisfies $\beta_t + \beta_{yy} = 0$.

**Class B**

$$h = (dy + pdt)^2 + 4Fdpdt, \quad \omega = -F^{-1}(dy + pdt), \quad (10)$$

where $F = F(p)$ is an arbitrary function.

**Class C**

$$h = (dy + pdt)^2 + 4\left(2K\frac{dp}{p} - \frac{y}{2t}dy + \frac{1}{t}\left(\frac{y^2}{4t} + K - \frac{dy}{2}\right) dt\right) dt, \quad \omega = -\frac{p}{2K}\left((dy + pdt) - \frac{y}{t} dt\right), \quad (11)$$

where $K = K(tp^2)$ is an arbitrary function.

**Proof.** We rewrite the non–linear constraint in (8) as $dH \wedge dH_x \wedge dt = 0$, and perform a Legendre transform $G(p,t,y) = H - px$, where $H_x = p$ and $x = -G_p$, and the constraint can be solved as

$$G(p,y,t) = A(p,t) + pB(y,t). \quad (12)$$

Imposing the wave equation (the linear constraint in (8)) yields

$$G_{yy} - G_{y}G_{pp} - G_{pt} = 0. \quad (13)$$

Substituting (12) and differentiating the resulting expression with respect to $y$ yields

$$2B_yB_{yy} - pB_{yyy}A_{pp} - B_{yt} = 0. \quad (14)$$

There are two cases to consider.

* If $B_{yy} \neq 0$ then we can solve (13) for $A_{pp}$, and find

$$A = \mu(p \ln(p) - p) + \kappa(t)p + \rho(t),$$

where $\mu$ is a constant. Setting $B = -\mu \ln \beta(y,t) - \kappa(t)$ removes $\kappa(t)$ from $G$ and reduces the equation (14) to $\beta_t + \mu \beta_{yy} = 0$. Rescaling $t$ and $p$ in the resulting EW structure can be used to set $\mu = 1$. The function $\rho(t)$ does not appear in the EW structure, so can be set to zero. This yields (9).
If \( B_{yy} = 0 \) then \( B = c_1(t)y^2 + c_2(t)y + c_3(t) \) and (13) implies that
\[
\dot{c}_1 = 4(c_1)^2.
\]
The classification now branches. If \( c_1 = 0 \) then \( B \) has to be linear in \( y \), and take the form \( B = cy + \gamma(t) \), where \( c \) is a constant. The equation (13) becomes
\[
\dot{c}_1 = 4c_2^2.
\]
so that \( A = c^2t - p\gamma(t) + a(p) + b(t) \), where \( a, b \) are some arbitrary functions of one variable. Substituting this into \( G \) shows that \( \gamma(t) \) disappears from the EW structure, and \( b(t) \) can be set to zero. The constant \( c \) can also be set to zero by an affine transformation of the coordinate \( p \) in the EW structure. This yields (10), where \( F = a_{pp} \) is an arbitrary function of \( p \). The nilpotent example (6) belongs to this class, and corresponds to \( F = -\ell/4 \), where \( \ell \) is a constant.

Next consider the case where \( c_1 = -1/(4t) \) (the constant of integration in the denominator has been set to zero by shifting \( t \)). The function \( c_3(t) \) can be absorbed into \( A(t,p) \), and (13) gives \( c_2(t) = ct \), where \( c \) is a constant. The resulting equation for \( A \) is
\[
 tpA_{pp} - 2t^2A_{pt} + 2c^2 = 0,
\]
which can be solved in terms of an arbitrary function of \( tp^2 \). The constant \( c \) can be set to zero by shifting \( y \rightarrow y - 2c \). The final expression for the EW structure takes the form (11). The solution (7) belongs to this class with \( A_p = 2^{-4/3}(tp^2)^{-1/3} \).

3. Einstein–Maxwell cosmological space–times

It is known \([10, 6, 12]\) that Riemannian solutions to the hyper–CR Einstein–Weyl equations lift to supersymmetric solutions to the minimal gauged supergravity in four dimensions (see also \([14]\) where the EW geometry appears in supergravity in a rather different context). Here, following \([11]\), we present an analytic continuation of these constructions where the underlying base Einstein–Weyl manifold is Lorentzian, and the resulting four–dimensional theory admits pseudo–supersymmetry. The bosonic content of the theory consist of a metric, and a one–form which satisfy the cosmological Einstein–Maxwell equations with non–standard coupling between the Maxwell and the Einstein terms. The metric and the one–form given by
\[
 g = \left( \frac{\ell}{\sin \alpha} - \frac{\ell}{2} \cos \alpha \omega + \sqrt{2} \sin \alpha \psi \right)^2 + \frac{1}{\sin \alpha^2} h
\]
\[
 A = \frac{\sqrt{2}}{2} \sin 2\alpha \psi - \frac{\ell}{4} \cos 2\alpha \omega
\]
satisfy the Einstein–Maxwell equations
\[
 R_{ab} + 3\ell^{-2}g_{ab} + 2F_{ac}F_b^c - \frac{1}{2} |F|^2 g_{ab} = 0, \quad d\ast_g F = 0
\]
iff equations (2) and (3) hold with \( V = -2\ell^{-1} \).

Computing the curvature invariants \( |F|^2 \) and \( |\text{Riemann}|^2 \) suggests that \( \alpha = 0, \pi, 2\pi, \ldots \) is just a coordinate singularity. A coordinate transformation
\[
 \sin \alpha = (\cosh \ell^{-1} p)^{-1}, \quad \cos \alpha = \tanh \ell^{-1} p
\]
brings the metric and the Maxwell potential to a regular form
\[
 g = \left( dp - \frac{\ell}{2} \tanh (\ell^{-1} p) \omega + \frac{\sqrt{2}}{\cosh (\ell^{-1} p)} \psi \right)^2 + \cosh^2 (\ell^{-1} p) h.
\]
We can attempt to take a limit where the cosmological constant vanishes, or equivalently $\ell \to \infty$. In this limit $\cosh^2 (\ell^{-1} p) \to 1$, and $\ell \tanh (\ell^{-1} p) \to p$ so that

$$g \to \left( dp - \frac{1}{2} \rho \omega \right)^2 + h, \quad F \to -\frac{\ell}{4} d\omega.$$  

The limit exists if $\omega$ depends explicitly on $\ell$, and $F$ does not blow up. This is the case for the Heisenberg group example where $g$ becomes the Minkowski metric, and $F = 0$ in the limit.

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