Observational constraints on interacting Tsallis holographic dark energy model

Ehsan Sadri*
Azad University Central Tehran Branch, Tehran 34353-17117, Iran

In this paper, we investigate a recent proposed model - so called the Tsallis holographic dark energy (THDE) model. In this case, we consider the non-gravitational and phenomenological interaction between dark sectors. We fit the free parameters of the model using Pantheon Supernovae Type Ia data, Baryon Acoustic Oscillations, Cosmic Microwave Background, Gamma-Ray burst and the the local value of the Hubble constant. We examine the THDE model to check its compatibility with observational data using objective Information Criterion (IC). We find that the THDE model cannot be supported by observational data. Using the Alcock-Pacynski (AP) test we check the deviation of the model compared to ΛCDM. Surveying the evolution of squared of sound speed $v_s^2$ as an another test we check the stability of the interacting and non-interacting THDE models. In addition, using the modified version of the CAMB package, we observe the suppressing the CMB spectrum at small K-modes and large scale.

**I. INTRODUCTION**

Dark energy - raised in 1998 - with strong negative pressure is the main component of the acceleration of the universe[1–9]. The nature of dark energy has become one of the main unknown issues in modern cosmology and many models have been proposed to understand this new concept[10–16]. The cosmological constant as the simplest dark energy fluid is a good candidate to study the universe acceleration and behavior of the dark energy. However, the cosmological constant suffers from some problem[17–22]. In order to alleviate these problems a holographic model of dark energy has been suggested and has drawn many attentions in recent two decades[23–30]. As the name of holographic dark energy model (HDE) suggests, this model is originated from holographic principle and its energy density can be expressed by $\rho_D = 3c^2 M_P^2/L^2$ where $c^2$ is a numerical constant, $M_P$ is the reduced Planck mass and $L$ denotes the size of the current universe such as the Hubble scale[31, 32]. In addition, the HDE has some problems and cannot explain the timeline of a flat FRW universe[33, 34]. One of the proposed solutions for the HDE problems is the consideration of different entropies. In the recent work, using the concept of holography in the Tsallis entropy a new holographic dark energy model[THDE][35]. It is stated that by applying the Tsallis statics[36–39] to the system horizon the Bekenstein entropy can be achieved and leads to stable models[40]. The authors studied different aspects of the THDE model without consideration of interaction between dark sectors[35]. The model has been studied to be checked if it can satisfy the condition of FRW universe and it has been found that the non-interacting model is unstable.

In this work, in the direction of the main work, we would like to investigate the behavior of the Tsallis Holographic Dark Energy model (THDE) with consideration of a non-gravitational and phenomenological interaction. This will be done by the use of the latest observational data, namely the Pantheon Supernovae type Ia, Baryon Acoustic Oscillations (BAO), Cosmic Microwave Background (CMB), Gamma-Ray burst and the the local value of the Hubble constant. We check the compatibility of the model with observational data employing AIC and BIC model selection tools. In addition, using Alcock-Pecinski (AP) test we survey the deviation of the THDE from the ΛCDM as the reference model and we make a comparison with the well-known holographic dark energy (HDE) model. Using the best values of the model’s free parameters we check the stability of the interacting THDE. We also study the behavior of the model in CMB angular power spectrum.

The structure of this paper is as follows. In the next section (section 2) we introduce the background physics of the THDE with consideration of an interaction between dark sectors. In section 3, we introduce the data and method used in this work. In section 4, we discuss the results of data analysis. In section 5, we study the AP test for measuring the deviation of THDE compared to the HDE and ΛCDM models. In section 6, we study the evolution of stability of the model within the different redshift values. The section 7 is allocated to the behavior of the THDE in the CMB angular power spectrum. The last section is dedicated to some concluding remarks.

**II. BACKGROUND OF THDE**

The description of a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FRW) universe can be introduce by,

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (1)$$

*ehsan@sadri.id.ir
in which \(a\) is the scale factor and \(k = 0, 1, -1\) denote a flat, closed and open universe, respectively. For a spatially flat FRW universe the first Friedmann equations can be written as

\[
H^2 = \frac{1}{3M_p^2}(\rho_D + \rho_m), \tag{2}
\]

where \(\rho_D\) is the THDE energy density, \(\rho_m\) is the energy density of pressure-less matter which must contain all the material constituents of the Universe and \(M_p\) denotes the reduced Planck mass. We may also write the dark energy and dark matter density with respect to the critical density \(\rho_c = 3M_p^2H^2\) as

\[
\Omega_D = \frac{\rho_D}{3M_p^2H^2}, \tag{3}
\]

\[
\Omega_m = \frac{\rho_m}{3M_p^2H^2}. \tag{4}
\]

The energy density of the Tsallis holographic dark energy (THDE) is given by the following relation

\[
\rho_D = BL^{2\delta-4}, \tag{5}
\]

where \(B\) is an unknown parameter, \(L\) can be considered as the size of the current Universe such as the Hubble scale and \(\delta\) is a free parameter reduces the THDE to the HDE model at \(\delta = 1\). In this case we are able to use the Hubble horizon as the IR cutoff of the system \(L = H^{-1}\) and the Eq.5 takes the form

\[
\rho_D = BH^{4-2\delta}, \tag{6}
\]

taking time derivative of the equation above and using Eq. 3, we reach

\[
\dot{\Omega}_D = (-2\delta + 2) \Omega_D \frac{\dot{H}}{H}. \tag{7}
\]

Considering an interaction term between dark sectors, one can write the conservation equations for THDE as

\[
\dot{\rho}_m + 3H\rho_m = Q, \tag{8}
\]

\[
\dot{\rho}_D + 3H(\rho_D + P_D) = -Q, \tag{9}
\]

where \(Q\) indicates the interaction term explaining energy flow between the components. Regarding the usual option for \(Q\)-term as \(3H(b_1\rho_D + b_2\rho_m)\) which \(b_{1,2}\) are the coupling constant, a single coupling constant can be used properly (e.g. \(Q_1 = 3Hb\rho_D\), \(Q_2 = 3Hb\rho_m\) and \(Q_3 = 3Hb(\rho_D + \rho_m)\)). In our recent work, we compared different phenomenological linear and non-linear interaction cases in the framework of the holographic ricci dark energy model \[41\] and we found that the linear interaction \(Q = 3Hb\rho_D\) is the best case among the others. Accordingly, in this work we take \(Q = 3Hb\rho_D\) in our calculations.

Taking time derivative of Eq.2 and using Eqs. 2, 8, 9 we find a definition for dark energy pressure

\[
P_D = -\frac{2}{3H^2}(\rho_D + \rho_m) - \rho_D - \rho_m, \tag{10}
\]

combining the Eqs.10 and 9 yields

\[
\dot{\Omega}_D + 3H\left(-\frac{2}{3H^2}\right) - 3H\Omega_m = 0, \tag{11}
\]

Inserting the Eq.7 into Eq.11 leads to

\[
\frac{\dot{H}}{H^2} = 3\left(1 - \Omega_D + 3b\Omega_D\right)\left(\frac{2\Omega_D(2 - \delta) - 2}{\Omega_D(4 - 2\delta) - 2}\right), \tag{12}
\]

and combining the Eqs. 12 and 7 we have

\[
\dot{\Omega}_D = 6(1 - \delta)\Omega_D\left(\frac{1 - \Omega_D + 3b\Omega_D}{\Omega_D(4 - 2\delta) - 2}\right), \tag{13}
\]

in which \(\dot{\Omega}_D = \Omega'_{\delta}H\) and \(\dot{H} = H'H\) where the prime denotes the derivate respect to \(x = \ln a\) and \(a = (1 + z)^{-1}\). Then the evolution of the density of dark energy and the Hubble parameter for THDE in terms of redshift can be written as

\[
\frac{d\Omega_D}{dz} = -\left(\frac{1}{1 + z}\right)\left(6(1 - \delta)\Omega_D\left(1 - \Omega_D + 3b\Omega_D\right)\left(\Omega_D(4 - 2\delta) - 2\right)ight), \tag{14}
\]

\[
\frac{dH}{dz} = \left(\frac{H}{1 + z}\right)\left(3\left(1 - \Omega_D + 3b\Omega_D\right)\left(\Omega_D(4 - 2\delta) - 2\right)\right). \tag{15}
\]

### III. OBSERVATIONAL DATA

To analyze the models and to obtain the best fit values for the model parameters, in this paper we combine the latest observational data including BAO, CMB, SNIa, \(H_0\) and GRB. For this purpose, we employed the public codes EMCEE \[43\] for implementing the MCMC method and GetDist Python package \[82\] for analyzing and plotting the contours.

#### A. Supernovae Type Ia

The compilation of Pantheon sample including 1048 data points embrace the redshift range \(0.01 < z < 2.3\) \[44\]. This sample contains 276 SNIa case from PanSTARRS1 Medium Deep Survey, SDSS, Low–z and HST samples. We use the systematic covariance \(C_{\text{sys}}\) for a vector of binned distances

\[
C_{ij,\text{sys}} = \sum_{n=1}^{i} \frac{\partial \mu_i}{\partial \sigma_n} \frac{\partial \mu_j}{\partial \sigma_n} \sigma_k, \tag{16}
\]
in which the summation is over the $n$ systematic with $S_n$ and its magnitude of its error $\sigma_{S_n}$. The $\chi^2$ relation for Pantheon SNIa data is

$$\chi^2_{\text{Pantheon}} = \Delta \mu^T \cdot C_{\text{Pantheon}}^{-1} \cdot \Delta \mu$$  \hspace{1cm} (17)

in which $\Delta \mu = \mu_{\text{data}} - \mu_{\text{obs}}$ and $M$ is a nuisance parameter. It should be noted that the $C_{\text{Pantheon}}$ is the summation of the systematic covariance and statistical matrix $D_{\text{stat}}$ having a diagonal component. The complete version of full and binned Pantheon supernova data are provided in the online source [83]

### B. Baryon Acoustic Oscillations

The combination of the extended Baryon Oscillation Spectroscopic Survey (eBOSS) quasar clustering at $z = 1.52$ [45], isotropic BAO measurements of 6dF survey at an effective redshift ($z = 0.106$) [46] and the BOSS DR12 [47] including six data points of Baryon Oscillations as the latest observational data for BAO makes the total data used for BAO in this section. The $\chi^2_{\text{BAO}}$ of BOSS DR12 can be explained as

$$\chi^2_{\text{BOSS DR12}} = X^T C_{\text{BAO}}^{-1} X,$$

where $X$ for six data points is

$$X = \left( \begin{array}{c} \frac{D_M(0.38)r_{s,fid}}{H(0.38) r_s(z_0)} - 1512.39 \\ \frac{D_M(0.51)r_{s,fid}}{H(0.51) r_s(z_0)} - 1975.22 \\ \frac{D_M(0.61)r_{s,fid}}{H(0.61) r_s(z_0)} - 90.9 \\ \frac{D_M(0.81)r_{s,fid}}{H(0.81) r_s(z_0)} - 2306.68 \\ \frac{D_M(0.81) r_{s,fid}}{H(0.81) r_s(z_0)} - 98.694 \end{array} \right),$$  \hspace{1cm} (19)

and $r_{s,fid} = 147.78$ Mpc is the sound horizon of fiducial model, $D_M(z) = (1 + z) D_A(z)$ is the comoving angular diameter distance. The sound horizon at the decoupling time $r_s(z_d)$ is defined as

$$r_s(z_d) = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz,$$

in which $c_s = 1/\sqrt{3 (1 + R_b/(1 + z))}$ is the sound speed with $R_b = 315003 h^2 (2.726/2.7)^{-4}$. The covariance matrix $Cov_{\text{BAO}}$ [47] is:

The $\chi^2$ for combined data is

$$\chi^2_{\text{BAO}} = \chi^2_{\text{BOSS DR12}} + \chi^2_{\text{6dF}} + \chi^2_{\text{eBOSS}},$$

\hspace{1cm} (20)

### C. Cosmic Microwave Background

Surveying the evolution of the expansion history of the universe leads us to check the Cosmic Microwave Background (CMB). For this, we use the data of Planck 2015 [48]. The $\chi^2_{\text{CMB}}$ function may be explained as

$$\chi^2_{\text{CMB}} = q_i - q_i^{\text{data}} Cov^{-1} (q_i, q_j),$$

where $q_1 = R(z_*)$, $q_2 = l_A(z_*)$ and $q_3 = \omega_b$ and CovCMB is the covariance matrix [48]. The data of Planck 2015 are

$$q_{1}^{\text{data}} = 1.7382, q_{2}^{\text{data}} = 301.63, q_{3}^{\text{data}} = 0.02262.$$  \hspace{1cm} (21)

The acoustic scale $l_A$ is

$$l_A = \frac{3.14d_L(z_*)}{(1 + z) r_s(z_*)},$$

in which $r_s(z_*)$ is the comoving sound horizon at the drag epoch ($z_*$). The function of redshift at the drag epoch is [49]

$$z_* = 1048 \left[1 + 0.00124 \left(\Omega_b h^2 \right)^{-0.738}\right] \left[1 + g_1 \left(\Omega_m h^2 \right)^{g_2}\right],$$  \hspace{1cm} (22)

and

$$g_1 = \frac{0.0783 \left(\Omega_b h^2 \right)^{-0.238}}{1 + 39.5 \left(\Omega_b h^2 \right)^{-0.763}}, \quad g_2 = \frac{0.560}{1 + 21.1 \left(\Omega_b h^2 \right)^{1.81}}.$$  \hspace{1cm} (23)

The CMB shift parameter is [50]

$$R = \sqrt{\Omega_m h^2} \frac{c}{C} r_s(z_*).$$

(24)

The reader should notice that the usage of CMB data does not provide the full Planck information but it is an optimum way of studying wide range of dark energy models.

### D. Gamma-Ray Burst

Constraining the free parameters using Gamma-Ray burst data can be obtained by fitting the distance modulus $\mu(z)$ similar to SNIa data (Sec. III A). In this work we use 109 data of Gamma-Ray Burst in the redshift range $0.3 < z < 8.1$ [57]. This data contains 50 low-z GRBs ($z < 1.4$) and high-z GRBs ($z > 1.4$). The 70 GRBs are obtained in [58], 25 GRBs are taken from [59] and the rest 14 GRBs data points are provided from [60]. The $\chi^2$ for GRB is given by

$$\chi^2_{\text{GRB}} = \sum_{i=1}^{109} \frac{\left[\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i)\right]^2}{\sigma^2(z_i)},$$

where $\mu_{\text{obs}}(z)$ is the observed distance modulus and $\mu_{\text{th}}(z)$ is the theoretical distance modulus. The $\sigma^2(z)$ is the statistical uncertainty in the distance modulus.
in which the theoretical distance modulus \( \mu_{th}(z_i) \) can be defined as

\[
\mu_{th}(z_i) = 5 \log_{10} D_L(z_i) + \mu_0.
\] (29)

where \( \mu_0 = 42.38 - 5 \log_{10} h \) and \( h = H_0/100 \) with unit of \( \text{km/s/Mpc} \).

E. Local Hubble Constant

As the last data point, we use the \( H_0 \) which can be locally measured by ladder distance observation. According to the reported result [61] we use \( H_0 = 73.24 \pm 1.74 \text{ km/s/Mpc} \) in our analysis.

The data for BAO and CMB could be found in the online source of latest version of MontePython [84]. Using minimized \( \chi^2_{\text{min}} \), we can constrain and obtain the best-fitting values of the free parameters.

\[
\chi^2_{\text{min}} = (\chi^2_{\text{BAO}} + \chi^2_{\text{CMB}}) + \chi^2_{\text{Pantheon}} + \chi^2_{\text{H}_0} + \chi^2_{\text{GRB}}
\] (30)

The best-fit values of \( \Omega_D, H_0, \Omega_{rc}, c \) and \( b \) by consideration of the 1\( \sigma \) confidence level are shown in the Table I and Figs 1 and 2. The \( \chi^2 \) is known as the effective way of understanding the best values of free parameters, but it cannot be only used to determine the best model between variety of models. Hence, for this issue Akaike Information Criterion (AIC) [51] and Bayesian Information Criterion (BIC) [52] have been proposed. For further information see [53], [54], [55], [56]. The AIC can be explained as

\[
AIC = -2 \ln \mathcal{L}_{\text{max}} + 2k,
\] (31)

where \(-2 \ln \mathcal{L}_{\text{max}} = \chi^2_{\text{min}} \) is the highest likelihood, \( k \) is the number of free parameters (2 for ACDM, 3 for THDE and 4 for ITHDE models in addition of one further parameter \( M \) for SNIa) and \( N \) is the number of data points used in the analysis. The BIC is similar to AIC with different second term

\[
BIC = -2 \ln \mathcal{L}_{\text{max}} + k \ln N.
\] (32)

Using these definitions, it is obvious that a model giving a small AIC and a small BIC is favored by the observations. Hence, we explain the levels of supporting the models from AIC and BIC in Tables II and III, respectively.

IV. RESULTS

In this section we discuss the implication of observational data for the Tsallis holographic dark energy model. Both interacting and non-interacting THDE models confronted initially with BAO and CMB and then combined...
Table I: The best value of free parameters for ΛCDM, HDE, THDE and ITHDE. The pantheon Supernovae data(s), Baryon acoustic oscillations(B), cosmic microwave background(C), the local Hubble constant \(H_0\)(H) and Gamma-Ray burst(G) have been used.

| Model  | Dataset | \(H_0\)  | \(\Omega_D\)  | \(\delta\)  | \(c\)  | \(b\)  | \(z_t\)  | Age/Gyr |
|--------|---------|----------|----------|----------|--------|--------|---------|---------|
| ACDE   | BC      | 66.55(\pm 1.75) | 0.629(\pm 0.056) | 0.064 | 0.064 | 0.064 | 0.064 | 13.64(\pm 0.40) |
| HDE    | BC      | 68.77(\pm 1.75) | 0.673(\pm 0.035) | 0.035 | 0.035 | 0.035 | 0.035 | 13.56(\pm 0.46) |
| THDE   | BC      | 67.30(\pm 2.101) | 0.692(\pm 0.049) | 0.049 | 0.049 | 0.049 | 0.049 | 13.97(\pm 0.41) |
| ITHDE  | BC      | 67.412(\pm 1.062) | 0.665(\pm 0.056) | 0.056 | 0.056 | 0.056 | 0.056 | 13.81(\pm 0.59) |

| Model  | Dataset | \(H_0\)  | \(\Omega_D\)  | \(\delta\)  | \(c\)  | \(b\)  | \(z_t\)  | Age/Gyr |
|--------|---------|----------|----------|----------|--------|--------|---------|---------|
| ACDE   | BCS     | 68.501(\pm 0.849) | 0.693(\pm 0.020) | 0.020 | 0.020 | 0.020 | 0.020 | 13.99(\pm 0.31) |
| HDE    | BCS     | 68.87(\pm 0.854) | 0.675(\pm 0.023) | 0.023 | 0.023 | 0.023 | 0.023 | 13.69(\pm 0.40) |
| THDE   | BCS     | 69.521(\pm 0.921) | 0.689(\pm 0.039) | 0.039 | 0.039 | 0.039 | 0.039 | 13.96(\pm 0.37) |
| ITHDE  | BCS     | 68.822(\pm 0.871) | 0.673(\pm 0.055) | 0.055 | 0.055 | 0.055 | 0.055 | 13.75(\pm 0.32) |

| Model  | Dataset | \(H_0\)  | \(\Omega_D\)  | \(\delta\)  | \(c\)  | \(b\)  | \(z_t\)  | Age/Gyr |
|--------|---------|----------|----------|----------|--------|--------|---------|---------|
| ACDE   | BCSH    | 68.498(\pm 0.829) | 0.694(\pm 0.019) | 0.019 | 0.019 | 0.019 | 0.019 | 14.00(\pm 0.29) |
| HDE    | BCSH    | 69.999(\pm 0.551) | 0.675(\pm 0.018) | 0.018 | 0.018 | 0.018 | 0.018 | 13.67(\pm 0.37) |
| THDE   | BCSH    | 68.752(\pm 0.820) | 0.691(\pm 0.038) | 0.038 | 0.038 | 0.038 | 0.038 | 13.90(\pm 0.33) |
| ITHDE  | BCSH    | 68.830(\pm 0.871) | 0.677(\pm 0.047) | 0.047 | 0.047 | 0.047 | 0.047 | 13.65(\pm 0.46) |

| Model  | Dataset | \(H_0\)  | \(\Omega_D\)  | \(\delta\)  | \(c\)  | \(b\)  | \(z_t\)  | Age/Gyr |
|--------|---------|----------|----------|----------|--------|--------|---------|---------|
| ACDE   | BCSHG   | 69.182(\pm 0.788) | 0.707(\pm 0.019) | 0.019 | 0.019 | 0.019 | 0.019 | 13.93(\pm 0.25) |
| HDE    | BCSHG   | 69.059(\pm 0.501) | 0.684(\pm 0.011) | 0.011 | 0.011 | 0.011 | 0.011 | 13.75(\pm 0.32) |
| THDE   | BCSHG   | 69.664(\pm 0.733) | 0.690(\pm 0.031) | 0.031 | 0.031 | 0.031 | 0.031 | 13.90(\pm 0.29) |
| ITHDE  | BCSHG   | 70.611(\pm 0.691) | 0.693(\pm 0.038) | 0.038 | 0.038 | 0.038 | 0.038 | 13.62(\pm 0.41) |

Table II: The level of support for each model from AIC.

| Measurement | Explanation       |
|-------------|-------------------|
| AIC < 2     | Strong support    |
| 2 < AIC < 4 | Average support   |
| 4 < AIC < 7 | Less support      |
| 8 < AIC     | No support        |

Table III: The level depiction of evidence against models from BIC.

| Measurement | Explanation       |
|-------------|-------------------|
| BIC < 2     | No significant evidence |
| 2 < BIC < 6 | Positive evidence  |
| 6 < BIC < 10| Strong evidence    |
| 10 < BIC    | Very strong evidence |

with SNIa, \(H_0\) and Gamma-Ray burst data. The values of cosmological parameters of the models are shown in Table I and Figs. 1 and 2. We present the analysis of data in two the following parts: the cosmological parameters and the AIC and BIC model selection.

**Cosmological Parameters:** As a key factor in modern cosmology for calculating the age and the size of the Universe and consideration of this quantity for measuring the brightness and the mass of stars, the Hubble constant \(H_0\) is of utmost importance. The Hubble constant corresponds to the Hubble parameter at the observation time. According to the best fitted value of the Hubble parameter using latest observational data in this work, we found that the Hubble parameter for the interacting and non-interacting THDE models constrained from BAO and CMB is close to the obtained value of the Hubble parameter from the Planck mission \((H_0 = 67.66 \pm 0.42)\) [64], DES collaboration \((H_0 = 67.77 \pm 1.30)\) [65] and SDSSIII BOSS \((H_0 = 67.60 \pm 0.7)\) [66]. Adding SNIa, \(H_0\) and GRB result in the bigger value of Hubble parameter. It is observed that the error bars of the Hubble parameter using the CMB and BAO are remarkably large and adding each data set makes the constraints to be narrower. It can be seen that the value of the dark energy for interacting THDE is smaller than the non-interacting model once we imply the BAO, CMB, SNIa and \(H_0\) for fitting parameters. After adding Gamma-Ray burst the dark energy density of both interacting and non-interacting THDE are identical. Totally the obtained values of dark energy density for THDE at 68% confidence level is compatible latest obtained results [48, 64, 69]. The value of the coupling constant at 1\(\sigma\) is obtained less than 0.1 similar to the previous results of interacting HDE, RDE and NHDE models [70–73]. We obtain the transition redshift using Brent’s method. This method uses the inverse quadratic interpolation as a secured version of the secant algorithm and using three prior points can estimate the zero crossing [74]. The obtained values for transition redshift listed in Table I is in range \((0.5 < z_t < 0.7)\) and in good agreement with recent obtained result for the transition redshift [75–79](To mention few) at 1\(\sigma\) and 2\(\sigma\) interval level.

**AIC and BIC model selection:** We investigate the models according to the objective Information Criterion (IC) containing AIC (Eq. 31) and BIC (Eq. 32). We present the obtained results of AIC and BIC in the T-
have 1169 data points.

{fivenum}
give the results of AIC and BIC in the Table V with consideration of the AIC and BIC, the definition of AIC supporting area (Table. II) and BIC evidence against the models (Table. III), it can be seen that both ITHDE and THDE are ruled out and unsupported by observational data. The BIC imposes a strict penalty against the additional parameters more than AIC as we can see in Tables IV and V. In this case we may reject the model (THDE) as an disfavored model, but we should note that the reason of the existence of various holographic dark energy models is to alleviate the ΛCDM problems. Thus, in this work we used the holographic dark energy (HDE) model as another reference for making an accurate comparison. In this manner, we give the results of AIC and BIC in the Table V with consideration of the HDE as the referring model. It is evident that the values of χ^2, AIC and BIC for THDE and ITHDE are close to the values of the χ^2, AIC and BIC for HDE model and even smaller with additional GRB and H_0 data. Thus, taking the HDE as the main model for comparison, one can see that the observational data strongly favor and support the THDE and ITHDE models.

V. ALCOCK-PACZYNSKI TEST

The Alcock-Paczynski (AP) test is a thoroughly geometric investigator of the cosmic expansion using observed/measured tangential and radial dimensions of ob-

| Data set | Model | χ²_{min} | AIC | ∆AIC | BIC | ∆BIC |
|----------|-------|----------|-----|------|-----|------|
| BAO + CMB | ΛCDM | 2.037    | 6.037 | 0    | 6.431 | 0    |
| BAO + CMB + SNIa | ΛCDM | 1030.236 | 1036.236 | 0    | 1051.125 | 0    |
| BAO + CMB + SNIa + GRB | ΛCDM | 1097.615 | 1103.616 | 0    | 1118.800 | 0    |
| BAO + CMB + SNIa + GRB + H_0 | ΛCDM | 1097.918 | 1103.918 | 0    | 1119.105 | 0    |

| Data set | Model | χ²_{min} | AIC | ∆AIC | BIC | ∆BIC |
|----------|-------|----------|-----|------|-----|------|
| BAO + CMB | THDE | 4.833    | 12.833 | 6.796 | 13.833 | 7.402 |
| BAO + CMB + SNIa | THDE | 1032.933 | 1042.934 | 6.698 | 1067.749 | 16.624 |
| BAO + CMB + SNIa + GRB | THDE | 1100.731 | 1110.731 | 7.115 | 1136.038 | 17.238 |
| BAO + CMB + SNIa + GRB + H_0 | THDE | 1104.699 | 1114.699 | 10.781 | 1140.010 | 20.905 |

| Data set | Model | χ²_{min} | AIC | ∆AIC | BIC | ∆BIC |
|----------|-------|----------|-----|------|-----|------|
| BAO + CM | ITHDE | 4.420    | 12.420 | 6.383 | 13.209 | 6.778 |
| BAO + CMB + SNIa | ITHDE | 1032.100 | 1042.100 | 5.864 | 1066.916 | 15.791 |
| BAO + CMB + SNIa + GRB | ITHDE | 1100.492 | 1110.492 | 0.252 | 1135.799 | 20.649 |
| BAO + CMB + SNIa + GRB + H_0 | ITHDE | 1104.443 | 1114.443 | 10.525 | 1139.754 | 20.649 |

Table IV: Summary of the AIC and BIC values calculated for interacting and non-interacting THDE model with respect to the reference ΛCDM model. ∆AIC = AIC_i - AIC_{ΛCDM} and ∆BIC = BIC_i - BIC_{ΛCDM} in which i denotes the number of models \(i = 1, 2, ..., N\) with \(N = 5\) interacting THDE, \(N = 4\) for non-interacting THDE and \(N = 3\) for the ΛCDM models. Here we have 1169 data points.

Table V: Summary of the AIC and BIC values calculated for interacting and non-interacting THDE model with respect to the reference HDE model. ∆AIC = AIC_i - AIC_{HDE} and ∆BIC = BIC_i - BIC_{HDE} in which i denotes the number of models \(i = 1, 2, ..., N\) with \(N = 5\) interacting THDE, \(N = 4\) for non-interacting THDE and \(N = 4\) for the HDE models. Here we have 1169 data points.

Table VI: Summary of the AIC and BIC values calculated for interacting and non-interacting THDE model with respect to the reference HDE model. ∆AIC = AIC_i - AIC_{HDE} and ∆BIC = BIC_i - BIC_{HDE} in which i denotes the number of models \(i = 1, 2, ..., N\) with \(N = 5\) interacting THDE, \(N = 4\) for non-interacting THDE and \(N = 4\) for the HDE models. Here we have 1169 data points.
The significant advantage of this test is its independency on the galaxies’ evolution. In this paper we use this method as a test for the THDE cosmological model. We also carry out the AP test according to the best fitted results using BAO, CMB. We take the ΛCDM as the reference model to compare with THDE.

According to the radius of objects’ distribution along the line of sight

$$s_\parallel = \Delta z \frac{d}{dz} d_c(z),$$ \hspace{1cm} (33)

where $d_c$ is the comoving distance and the radius of objects’ distribution perpendicular to the line of sight

$$s_\perp = \Delta \theta (1 + z)^m d_A(z),$$ \hspace{1cm} (34)

in which $\Delta z$ is the redshift span, $\Delta \theta$ is the angular size and $m = 1, 0$ denote the expanding and static Universe respectively, one can find the following ratio

$$y = \frac{\Delta z}{z \Delta \theta} \frac{s_\parallel}{s_\perp},$$ \hspace{1cm} (35)

which using the definition of the diameter angular distance if the Universe is expanding, the Eq.35 can be written as

$$y(z) = \left(1 + \frac{1}{z}\right) \frac{d_A(z) H(z)}{c}.$$ \hspace{1cm} (36)

This relation is against the incorrect cosmological parameters and models. Using this relation one can check the deviation from the reference model which means the deviation from the correct measurement. In figure 3 we compare the THDE with HDE and ΛCDM model. This comparison has been performed using the fitted parameters of the models (See Table I). According to the $y(0.38) = 1.079 \pm 0.042$, $y(z = 0.61) = 1.248 \pm 0.044$ and $y(z = 2.34) = 1.706 \pm 0.083 \ [81]$ the ΛCDM is not favor by the BAO data while we choose the ΛCDM as the model of comparison. The evolution of Alcock-Paczynski for ΛCDM, HDE, THDE and ITHDE in terms of redshift is plotted in Fig.3. All models at $z = 0$ have identical values of $y$. It is observed that they behave similar to ΛCDM in low redshift while in higher redshift the deviation from the reference model can be seen. The deviation of THDE and ITHDE can be seen at $z = 0.7$ and $z = 1.2$ respectively.

VI. STABILITY

Surveying the stability of THDE can be performed by study the behavior of square sound speed ($v_s^2$) [42]. The sign of $v_s^2$ is important to specify the stability of background evolution which $v_s^2 > 0$ and $v_s^2 < 0$ denote a stable and unstable universe against perturbation respectively. The perturbed energy density of the background in a linear perturbation structure is

$$\rho(x, t) = \rho(t) + \delta \rho(x, t),$$ \hspace{1cm} (37)

in which $\rho(t)$ is unperturbed energy density of the background. The equation of energy conservation is [42]

$$\delta \dot{\rho} = v_s^2 \nabla^2 \delta \rho(x, t).$$ \hspace{1cm} (38)
For positive sign of squared sound speed the Eq.38 will be a regular wave equation which its solution can be obtained as $\delta \rho = \delta \rho_0 e^{-i\omega_0 t + ikx}$ indicating a propagation state for density perturbation. It is easy to see that the squared sound speed can be written as

$$v_s^2 = \frac{\dot{p}}{\dot{\rho}} = \frac{\omega_D}{\rho_D} + \omega_D.$$  

(39)

Taking time derivative of Eq.6 and again using the Eq.6 yields

$$\frac{\rho_D}{\dot{\rho}_D} = \frac{1}{3H} \frac{(2 - \delta)(\Omega_D - 1)}{(2 - \delta)(1 - \Omega_D + 3b\Omega_D)},$$  

(40)

Combining the Eqs.40 and 9 we have

$$\dot{\omega}_D = H \frac{(2 - \delta)(1 - \delta)\Omega_D^2 + 3b(2 - \delta)\Omega_D}{(2 - \delta)\Omega_D - 1},$$  

(41)

Now using Eqs.9, 14 and 15 and 40, we can plot the evolution of stability in terms of redshift for the THDE model. From the Fig.39 one can see that during the cosmic evolution, both interacting and non-interacting THDE are unstable against background perturbations in early time, present and late time.

VII. CMB POWER SPECTRUMS

In this section by the use of modified version of the Boltzmann code CAMB\cite{85} \cite{62, 63}, we compare the power spectrum of the cosmic microwave anisotropy in both interacting and non-interacting THDE models. Our results of the temperature power spectrum (TT) according to the fitted results in TableI are depicted in Figs. 5 and 6. From the figures, one can see that both the interacting and non-interacting THDE models show the trends of squeezing power spectrum of the cosmic microwave anisotropy to small $\ell$ or large angle scales. This squeezing can also be seen from the power spectrum of matter distribution in the Universe. Embodying on the large scale structure of matter distributions, both models exhibit approximately 20% suppressing in the peak of power spectrum which occur in small $k$ or large scale region. Another difference between the THDE and $\Lambda$CDM models lies before $\ell < 50$ where the amplitude of THDE is higher than the $\Lambda$CDM. The tendency of interacting THDE model is more than THDE towards $\Lambda$CDM.

VIII. CONCLUSION

In this work, we examined the Tsallis holographic dark energy model (THDE) using various cosmological tests. In this case we considered a phenomenological non-gravitational interaction between dark sectors. We used the Pantheon Supernovae type Ia, Baryon acoustic oscillation, cosmic microwave background, the local value of Hubble constant $H_0$ and the Gamma-Ray burst data as the observational data for constraining the free parameters of the models. For minimizing the $\chi^2$ we used MCMC method by employing the Cosmo Hammer (EMCEE) Python package. We observed that concerning the density of dark energy and the Hubble parameter, the THDE and ITHDE models have a good consistency with latest observational data. Both interacting and non-interacting THDE enter the accelerating universe within the $z_t = [0.5, 0.7]$. We investigated the models using the objective Information Criterion (IC) including AIC and
BIC. We found that the interacting and non-interacting THDE models are not supported by observational data. This result is obtained once the ΛCDM is chosen as the reference model. According to this case that the holographic dark energy models has been proposed to alleviate the problems of ΛCDM, one can compare the THDE with another holographic models (here HDE) as the reference rather than the ΛCDM. In this case, by choosing the HDE as the referring model, both interacting and non-interacting THDE models are strongly favored by AIC and BIC. Using Alcock-Paczynski (AP) we found that the HDE has the smallest deviation from the ΛCDM model. Accordingly both interacting and non-interacting THDE models behave similar to ΛCDM at low-z but the deviation compared to ΛCDM and HDE can be seen at z > 0.7 and 7 > 1.2 for THDE and ITHDE respectively. Using the squared of sound speed $v^2_s$, we found that the THDE model in non-interacting and even interacting form could not satisfy the condition of stability and remain as unstable model. Finally, using modified version of CAMB package, we observed that the 20% suppression of matter power spectrum from interacting and non-interacting THDE models in large scale region. It can be mentioned that for better revealing the deeper aspects of the THDE model more investigations should be done. For the future works, we would like to study the dynamical system methods for comprehension of the model’s behavior in the late time using different types of interaction. In addition, we are going to study the perturbation analysis in comparison to the Large Scale Structure (LSS) and the gravitational lenses.
