Perfect colorings of the 12-cube that attain the bound on correlation immunity

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Abstract

We construct perfect 2-colorings of the 12-hypercube that attain our recent bound on the dimension of arbitrary correlation immune functions. We prove that such colorings with parameters \((x, 12-x, 4+x, 8-x)\) exist if \(x=0, 2, 3\) and do not exist if \(x=1\).

Let \(H_n\) be the hypercube of dimension \(n\). Its vertices are the binary vectors of length \(n\) (we will identify such a vector with the set of its nonzero coordinates); two vertices are adjacent if their vectors differ in exactly one coordinate. A coloring of the vertices into black and white colors is called a perfect coloring with parameters \((a, b, c, d)\) if every black vertex has \(a\) black and \(b\) white neighbors and every white vertex has \(c\) black and \(d\) white neighbors. (For a general definition of perfect coloring and main properties, see [1], [3].)

In [2], it is proved that for every perfect 2-coloring of \(H_n\) with \(b \neq c\), it holds

\[c - a \leq n/3\]

(the correlation-immunity bound). Two series of colorings attaining this bound are known. They are obtained from a perfect code in the three-dimensional cube and from the Taranikov coloring with parameters \((1, 5, 3, 3)\) (see [4]) by the product construction [1, Proposition 1(c)] and have parameters \((0, 3k, k, 2k)\) and \((k, 5k, 3k, 3k)\), respectively. Every perfect coloring that attain the correlation-immunity bound has parameters \((i, 3x-i, i+x, 2x-i)\), the hypercube dimension being \(3x\). Without loss of generality we will assume \(i < x\). In the current work, we determine for which \(i\) such colorings of the 12-dimensional hypercube \((x=4)\) exist.

If \(i = 0\) and \(i = 2\), then we get parameters that belong to the families mentioned above; hence colorings exist in these cases.

**Theorem 1.** There are no perfect colorings of \(H = H_{12}\) with parameters \((1, 11, 5, 7)\).

**Proof.** Assume, seeking a contradiction, that such a coloring exists. As in [2], we define the real-valued function \(q\) on \(H\) that equals 11 on the black vertices and 5 on the

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white vertices. It follows from the definition of a perfect coloring that $q$ is an eigenfunction of the adjacency matrix of $H$ with the eigenvalue 4.

We will use the approach from [2]. Let us remind the notation for the faces of $H$ and the basis $\{f^x\}$ from eigenfunctions.

For $x, y \in H$, $x \cap y = \emptyset$, we define the set $[x] + y = \{ z \cup y \mid z \subseteq x \}$. This set is called a $k$-face of the hypercube, where $k = |x|$.

For every $x \in H$, the function $f^x$ is defined as $f^x(z) = (-1)^{|z \setminus x|}$.

The collection $\{f^x \mid x \in H\}$ is an orthogonal basis of the space of real-valued functions on $H$. Consider the expansion of $q$ in the basis $\{f^x\}$:

$$q = \sum_x w_x f^x$$

where the sum is over all the vectors of weight 4. For any vectors $x, y$, it is easy to check that $\langle \chi^{[x]}, f^y \rangle = 2^{|x|}$ if $x \subseteq y$; otherwise $\langle \chi^{[x]}, f^y \rangle = 0$. (Recall that $[x]$ is the smallest face that contains both vertices $x$ and 0 and $\chi^{[x]}$ is its characteristic function.) From here, we can find the coefficients $w_x$:

$$\langle \chi^{[x]}, q \rangle = 16w_x = \sum_{v \in [x]} q(v) = 11m - 5(16 - m),$$

$$w_x = -5 + m,$$

where $m$ is the number of the black vertices in the face $[x]$. In particular, all the coefficients are integer.

The value $\langle q, q \rangle$ can be calculated in two ways. At first, it equals $2^{12} \sum w_x^2$; at second, from counting the number of black and white vertices, it equals $5\cdot 2^8 \cdot 11^2 + 11 \cdot 2^8 \cdot 5^2 = 55 \cdot 2^{12}$. It follows that $\sum w_x^2 = 55$.

Hence there are at most 55 nonzero coefficients. Denote $S = \{x \mid w_x \neq 0\}$; $|S| \leq 55$.

Now consider an arbitrary 3-face $[y]$. We have:

$$\langle \chi^{[y]}, q \rangle = \sum_{[y] \subseteq [x]} 8w_x = 11m - 5(8 - m),$$

$$\sum_{[y] \subseteq [x]} w_x = 2m - 5 \neq 0,$$

where $m$ is the number of the black vertices in the face $[y]$. In particular, this means that each vector $y$ of weight 3 is contained (as a subset) in at least one vector $x \in S$. Hence $|S| \geq (\binom{12}{3}) / 4 = 55$.

So, $|S| = 55$, and every weight-3 vector is contained in exactly one vector of $S$. But this is impossible, because each of 12 coordinates must belong to $55 \cdot 4/12$ vectors of $S$, which is not an integer. This contradiction proves the theorem. ▲

**Theorem 2.** There exist perfect colorings of 12-dimensional hypercube with parameters $(3, 9, 7, 5)$. 

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Proof. We will give an explicit construction of such colorings, which is, in concept, similar to the construction from [1].

Let us start the construction from the auxiliary 6-dimensional cube $X$. We will denote the coordinates in $X$ by the symbols from the set $\Omega = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. For the convenience, we will denote an element of $X$ by the list of its nonzero coordinates, omitting the sign “+” between them. We will represent the 12 pairwise disjoint 2-faces (their vertices will be called grey). The partition will be union of pairwise disjoint layers $L_x, x \in X$. The elements of every layer are also marked by vectors from $X$: $L_x = \{y_x|y \in X\}$ (where $y_x = (y, y + x) - \text{transl. rem.}$). The layers $L_x$ are independent sets in $H$. Two layers $L_x, L_x'$ with adjacent $x, x'$ induce a bipartite graph of order 2; explicitly, if $x' = x + a, a \in \Omega$, then $y_{x'}$ is adjacent with $y_x$ and $(y + a)_x$.

Let us partition all the vertices of $X$ into 4 black vertices, 12 white vertices, and 12 pairwise disjoint 2-faces (their vertices will be called grey). The partition will be invariant with respect to the group $A = \langle \alpha, \beta \rangle$ of automorphisms of $X$ generated by the automorphism of the coordinate permutation $\alpha = (a_1a_2a_3)(b_1b_2b_3)$ and the affine automorphism $\beta(x) = a_1a_2a_3 + \sigma(x)$ where $\sigma = (a_1b_1)(a_2b_2)(a_3b_3)$.

The black are the four vertices of the orbit $0^A$, namely: $0, a_1a_2a_3, a_1a_2a_3b_1b_2b_3, b_1b_2b_3$.

The white are the 12 vertices of the orbit $a_1^A$, namely: $0 + a_i, a_1a_2a_3 + a_i, a_1a_2a_3b_1b_2b_3 + a_i, b_1b_2b_3 + a_i$.

At last, the other vertices of $X$ are partitioned into the 2-faces that are the images under the action of $A$ of the face $b_1 + \langle a_2, a_3 \rangle$, namely:

\[ b_i + \langle a_j, a_k \rangle, \]
\[ a_ja_k + \langle b_j, b_k \rangle, \]
\[ a_1a_2a_3b_jb_k + \langle a_j, a_k \rangle, \]
\[ a_ib_1b_2b_3 + \langle b_j, b_k \rangle, \]

where $i, j, k$ is an arbitrary permutation of the indices 1, 2, 3.

Now, define a coloring $c : L_x \to \{0, 1\}$ for every layer $L_x$. If $x$ is black, then set $c(L_x) = 1$. Similarly, $c(L_x) = 0$ for a white $x$.

For every grey face $G = x + \langle p, q \rangle$ from our partition (where $x$ is the image of $b_1$ under some automorphism from $A$, and $p, q$ are the elements of $\Omega$ that specify the direction of the face), let $L_G = \{y_z|y \in X, z \in G\}$ be the union of the corresponding layers. We arbitrarily choose the value $c(G) = c(0_x)$ for one vertex from $L_x$. The other values $c(y_z)$ for $y_z \in L_G$ are defined, starting from this vertex and applying the following rules:

For $r \in \Omega$, set $c((y + r)_z) = c(y_z)$ if $r \in \{p, q\}$, and $c((y + r)_z) = 1 - c(y_z)$ otherwise; set $c(y_{z+r}) = 1 - c(y_z)$ for any $r \in \{p, q\}$.

It is easy to see that these rules uniquely determine the colors of all the vertices $c(y_z)$ for $y \in X, z \in G$, and that any two adjacent vertices from this set have different colors. Observe also that (*): any vertex out of $L_G$ adjacent with $L_G$ has exactly two neighbors in $L_G$, and the colors of these two neighbors are different.

Now, it is not difficult to calculate the number of neighbors of colors 0 and 1 for each vertex of $X$.

If $z \in X$ is a black vertex, then it has three white and three grey neighbors in $X$. Every vertex $y_z \in L_z$ has color 1 and, according to (*), has three color-1 neighbors and $3 + 3 \cdot 2 = 9$ neighbors of color 0.

If $z \in X$ is a white vertex, then it has one black and five grey neighbors in $X$. Every
vertex \( y_z \in L_z \) has color 0 and, according to (*), has five color-0 neighbors and \( 5 + 1 \cdot 2 = 7 \) neighbors of color 1.

If \( z \) belongs to a grey face \( G \), then it has two neighbors inside the face and either one black, two white and one grey, or one white and three grey neighbors outside the face. Consequently, each vertex \( y_z \) has four neighbors of color different from its color inside \( L_G \); and, calculated using (*) as above, three outside neighbors of color 1 and five, of color 0, independently of the color of \( y_z \).

So, each vertex of \( H \) has the required number of neighbors of each color, and a coloring with required parameters is constructed. ▲

Finally, we note that the arbitrary choice of the twelve values \( c(G) \) in the construction enables to construct non-isomorphic perfect colorings with parameters \((3, 9, 7, 5)\). Nevertheless, the number of pairwise non-isomorphic such colorings, as well as the existence of colorings with parameters \((3, 9, 7, 5)\) that cannot be obtained by our construction, remains unknown.

References

[0] D. G. Fon-Der-Flaass. Perfect colorings of the 12-cube that attain the bound on correlation immunity // Siberian Electronic Mathematical Reports 4, 2007, 292–295 [in Russian, with English Abstract].

[1] D. G. Fon-Der-Flaass. Perfect colorings of a hypercube // Siberian Math. J. 48(4), 2007, 740–745.

[2] D. G. Fon-Der-Flaass. A bound on correlation immunity // Siberian Electronic Mathematical Reports 4, 2007, 133–135.

[3] C. Godsil. Equitable partitions // In: Combinatorics, Paul Erdős is Eighty (Vol. 1). Keszthely (Hungary), 1993, 173–192.

[4] Yu. Tarannikov. On resilient Boolean functions with maximal possible nonlinearity. Cryptology ePrint archive (http://eprint.iacr.org), Report 2000/005, March 2000, 18p.