Membranes and Three-form Supergravity

Burt A. Ovrut$^1$ $^2$ and Daniel Waldram$^3$

$^1$ Department of Physics, University of Pennsylvania
Philadelphia, PA 19104–6396, USA

$^2$ School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

$^3$ Department of Physics, Joseph Henry Laboratories
Princeton University, Princeton, NJ 08544, USA

Abstract

We discuss membranes in four-dimensional $N = 1$ superspace. The $\kappa$-invariance of the Green-Schwarz action implies that there is a dual version of $N = 1$ supergravity with a three-form potential. We formulate this new supergravity in terms of a three-form superfield in curved superspace, giving the relevant constraints on the field strength. We find the corresponding membrane soliton in the new supergravity and discuss how the extended supersymmetry algebra emerges from the symmetries of the flat superspace background.
1 Introduction

As was first pointed out some years ago [1], there is a nexus of relations connecting the existence of consistent Green-Schwarz super $p$-brane actions and of supergravity theories containing $(p + 1)$-form potentials. The purpose of the present paper is to discuss the $p = 2$ membrane and the corresponding supergravity in four dimensions. The membrane naturally couples to a three-form potential. We derive a corresponding three-form version of $N = 1$ supergravity in superfields. It is a dualized version of old minimal supergravity where one of the scalar auxiliary fields is replaced by the four-form field strength of the three-form potential. The theory is a toy model of the theory of a membrane in eleven dimensions, the three-form supergravity sharing some of the structure of eleven-dimensional supergravity. One advantage is that the four-dimensional theory is an off-shell description, with all necessary auxiliary fields. One notes that it is also strictly this three-form version of $N = 1$ supergravity that appears in the reduction of higher-dimensional supergravities containing form potentials of degree three or higher, as was discussed in [2] in the context of $N = 8$ supergravity.

Consider starting with the theory of a fundamental Green-Schwarz $p$-brane, which has manifest spacetime supersymmetry. Consistent actions are $\kappa$-symmetric. That is, they have a local fermionic world-volume symmetry which can be used to gauge away half of the fermionic degrees of freedom. When such a symmetry exists it is possible to go to a gauge where the action exhibits world-volume supersymmetry. In this sense, the presence of $\kappa$-symmetry allows the supersymmetry to be projected down from the spacetime onto the world volume.

The requirement of $\kappa$-symmetry puts strong constraints on the structure of the target superspace. First, it implies that there is a non-trivial background super $(p + 1)$-form potential $B_{p+1}$ [3, 4, 5]. The simplest action, proportional to the volume the $p$-brane sweeps out as it moves through the target superspace, is not $\kappa$-symmetric. A second, Wess-Zumino term, describing the coupling of the $p$-brane to $B_{p+1}$, must be included. The field strength $\Sigma = dB_{p+1}$ is required to satisfy a set of constraints [3] so that, even in flat space, $B_{p+1}$ is not zero. These constraints cannot be solved in a general $D$-dimensional spacetime. One finds, for instance, that the membrane is only consistent in $D = 4, 5, 7$ and 11 dimensions and always has $N = 1$ target-space supersymmetry. In addition, the existence of a $\kappa$-symmetry implies that the torsion, given in terms of the supervielbein of the target superspace, is also constrained.

Thus $\kappa$-symmetric $p$-brane actions exist only in superspace backgrounds with a non-trivial super $(p + 1)$-form. One might ask if the constrained superform and the supervielbein of the background superspace together form some supergravity multiplet. In various cases, this has been shown to be true. For a membrane in eleven dimensions, the constraints on $\Sigma$ and the torsions reproduce
the equations of motion of $N = 1$, $D = 11$ supergravity \cite{5, 6}. The five-brane in ten dimensions gives the constraints of $N = 1$, $D = 10$ supergravity formulated in terms of a six-form, the dual of the usual anti-symmetric tensor \cite{4}. The authors of \cite{1} also point out that the three-form coupling to the membrane in four dimensions can be considered as a special type of auxiliary field for $D = 4$, $N = 1$ supergravity but do not further discuss the structure of this non-standard off-shell supermultiplet.

Alternatively one can start with a supergravity theory with a $(p + 1)$-form potential. As is now well-known, $p$-brane states then appear as topological defects, solitons carrying either electric or magnetic $(p + 1)$-form charge (see for instance \cite{7}). When the bosonic zero modes of the defect are all world-volume scalars, the fluctuations of the $p$-brane are described by the Green-Schwarz action.

An important property of the simplest $p$-brane solitons is that they preserve one-half of the supersymmetries. As such, they are not representations of the usual super-Poincaré algebra. Instead, they are described by an algebra extended by a topological charge, proportional to the $(p + 1)$-form charge of the $p$-brane. Again this extension can be seen either in terms of the supergravity solution (see for instance \cite{8, 9}) or as a topological charge in the supersymmetry algebra of the Green-Schwarz action \cite{10}.

The membrane we will consider in four dimensions is the analog of the eight-brane in ten dimensions. In each case the $p$-brane is a domain wall dividing the spacetime into two parts. The introduction of a nine-form potential for the eight-brane \cite{11} required the dualization of the massive form of supergravity due to Romans \cite{12}. In the same way, we will find that the dualization of old minimal to three-form supergravity naturally arises in the ‘massive’ form of the theory: that is, the action with a cosmological constant. In fact, the supergravity solution corresponding to the four-dimensional membrane is simply the patching of two pieces of anti-de Sitter space across a domain wall. (Similar solutions with matter are familiar in the domain wall literature \cite{13}.) Domain-wall, $(D - 2)$-branes were discussed in six and seven dimensions in \cite{14}, and then more generally in \cite{15, 16}. The latter papers include the solution corresponding to a membrane in $D = 4$, $N = 1$ supergravity but do not discuss what form of the corresponding dualized supergravity might take.

The paper is arranged as follows. In section two we discuss the matter three-form supermultiplet in flat superspace, first in components, as the dualization of a chiral superfield, and then as a constrained super three-form field, as was first given in \cite{17}. The fields of the dualized chiral multiplet correspond to the full three-form multiplet in a particular Wess-Zumino gauge. In section three we give a self-contained derivation of the three-form version of $N = 1$ supergravity multiplet which naturally couples to the membrane in four dimensions. We start with old minimal
supergravity multiplet in components. The three-form theory can be obtained by simply dualizing one of the scalar auxiliary fields in a way very similar to the chiral-field dualization of section two. It is interesting to note that a version of supergravity with both scalar auxiliary fields replaced by three-forms was given in components in early studies of supergravity [18, 19]. A deeper understanding of the three-form theory and its gauge symmetries can only be obtained in superspace. To this end, we then discuss the three-form superfield $B$ coupled to old minimal supergravity in curved superspace, following a recent paper by Binétruy et al [20]. (These authors treat the three-form supermultiplet as matter, rather than try to construct an irreducible supergravity multiplet involving the three-form.) We present and solve the additional constraints on $B$ required to merge the old minimal supergravity multiplet with the matter three-form multiplet to produce irreducible off-shell three-form supergravity. This is the three-form analog of the super two-form formalism used in [21] to construct new-minimal supergravity. We discuss the general gauge structure of this theory and Wess-Zumino gauge. The local supersymmetry algebra on the component fields corresponds to the usual superdiffeomorphism transformations together with a WZ-gauge-restoring transformation. The superdiffeomorphisms and gauge transformations are distinct symmetries, so that, in this superspace formulation, the three-form supergravity in no way corresponds to gauging the extended form of the global superalgebra. The next two sections make the connection to supermembranes in four dimensions. In section four we show that the three-form theory naturally arises in dualizing the massive action of old minimal supergravity, namely the theory with a cosmological constant. We briefly present the elementary membrane solution, match it to a delta-function membrane source and show it preserves half the supersymmetries. In section five we consider the fundamental Green-Schwarz membrane action. The requirement of $\kappa$-symmetry in four dimensions leads to exactly the super three-form and torsion constraints we used in section three to construct the three-form supergravity. Section six describes how the extended supersymmetry algebra appears from the symmetries of a flat supergravity background. We stress how the result depends on the choice of background by also investigating the algebra in an anti-de Sitter background.

Throughout the paper we will use the conventions of Wess and Bagger [22] and the authors of [20], except that we define the Ricci tensor as $R_{mn} = \partial_p \Gamma^p_{mn} + \ldots$.

2 The three-form supermultiplet in flat superspace

The supermembrane naturally couples to a three-form potential. We start, in this section, therefore by simply describing the structure of the three-form matter supermultiplet in flat space. It is most easily derived in components by dualizing one of the auxiliary fields of a chiral multiplet. The superspace formulation, on the other hand, requires a full three-form superfield [17, 21].
Consider a chiral superfield,

$$
\Phi = Y + \sqrt{2} \theta \eta + \theta^2 F + \ldots
$$

(2.1)

The component fields $Y$, $\eta_\alpha$ and $F$ are known to transform under supersymmetry as

$$
\delta_\xi Y = \sqrt{2} \xi \eta \\
\delta_\xi \eta = i \sqrt{2} \sigma^m \xi \partial_m Y + \sqrt{2} \xi F \\
\delta_\xi F = i \sqrt{2} \bar{\sigma}^m \partial_m \eta
$$

(2.2)

We now want to write the auxiliary field $F$ as $F = H + i F_2$, where both $H$ and $F_2$ are real fields. Then the last equation in (2.2) becomes

$$
\delta_\xi H = \frac{i}{\sqrt{2}} (\bar{\xi} \bar{\sigma}^m \partial_m \eta + \xi \sigma^m \partial_m \bar{\eta})
$$

(2.3)

and

$$
\delta_\xi F_2 = \frac{1}{\sqrt{2}} (\bar{\xi} \bar{\sigma}^m \partial_m \eta - \xi \sigma^m \partial_m \bar{\eta})
$$

(2.4)

Now, we can always choose to express the real field $F_2$ as $\epsilon^{mnpq} F_{mnpq}$ where $F$ is a four-form. Since a four-form in four dimensions is always closed, one can write locally $F = dC$ where $C$ is a three-form. That is, we can always write

$$
F_2 = - \frac{4}{3} \epsilon^{mnpq} \partial_m C_{npq}
$$

(2.5)

where we have chosen the normalization $-4/3$ to conform to the conventions of reference [20]. Inserting this expression in (2.4), we find that this equation can be completely integrated. The result is that

$$
\delta C_{mnp} = - \frac{\sqrt{2}}{16} \epsilon_{mnpq} \left( \bar{\xi} \bar{\sigma}^q \eta - \xi \sigma^q \bar{\eta} \right)
$$

(2.6)

Putting everything together we have

$$
\delta Y = \sqrt{2} \xi \eta \\
\delta \eta = i \sqrt{2} \sigma^m \xi \partial_m Y + \sqrt{2} \xi F \\
\delta H = \frac{i}{\sqrt{2}} \left( \bar{\xi} \bar{\sigma}^m \partial_m \eta - \partial_m \bar{\eta} \bar{\sigma}^m \xi \right) \\
\delta C_{mnp} = - \frac{\sqrt{2}}{16} \epsilon_{mnpq} \left( \bar{\xi} \bar{\sigma}^q \eta - \xi \sigma^q \bar{\eta} \right)
$$

(2.7)

Note that we have dropped the subscript $\xi$ in writing these equations. The reason for this will become clear below. It follows that the component fields $Y$, $\eta_\alpha$, $H$ and $C_{mnp}$ form an irreducible
supermultiplet, a three-form matter multiplet. One could, of course, have dualized the other auxiliary field $H$, or some linear combination of $H$ and $F_2$. However, as we will see, all these dualizations are simply reparameterizations of the same underlying multiplet. Dualizing both auxiliary field does lead to a completely new multiplet \cite{23}. However it is not relevant to the coupling of the membrane to supergravity.

What is the superfield for which $Y, \eta_\alpha, H$ and $C_{mnp}$ are the component fields? To answer this, we must consider the theory of a constrained three-form supermultiplet \cite{17}. We will use the notation and results from reference \cite{20}. One begins by introducing a real three-form superfield

$$B = \frac{1}{3!} e^A e^B e^C B_{CBA} \tag{2.8}$$

where $e^A$ denotes the frame of flat superspace. The invariant field strength is a four-form

$$\Sigma = dB \tag{2.9}$$

Its component superfields, defined by

$$\Sigma = \frac{1}{4!} e^A e^B e^C e^D \Sigma_{DCBA} \tag{2.10}$$

are then subject to the constraints

$$\Sigma_{\delta \gamma \beta A} = 0 \tag{2.11}$$

where $\alpha \sim \alpha, \dot{\alpha}$, together with the conventional constraint

$$\Sigma^{\delta \gamma \beta a} = 0 \tag{2.12}$$

The Bianchi identity $d\Sigma = 0$ can be solved subject to these constraints. One finds that all the coefficients of the four-form $\Sigma$ either vanish, or can be expressed in terms of a complex superfield $Y$ and and its conjugate $\bar{Y}$ as follows

$$\Sigma_{\delta \gamma ba} = \frac{1}{2} (\sigma_{ba} \epsilon_{\delta \gamma}) \bar{Y}, \tag{2.13}$$

$$\Sigma^{\dot{\delta} \dot{\gamma} \beta a} = \frac{1}{2} (\bar{\sigma}_{ba} \epsilon^{\dot{\delta} \dot{\gamma}}) Y \tag{2.14}$$

and

$$\Sigma_{\delta cba} = -\frac{1}{16} \sigma^{\delta \dot{\delta}} \epsilon_{dcba} D_{\dot{\delta}} Y, \tag{2.15}$$

$$\Sigma_{\delta cba} = \frac{1}{16} \sigma^{\delta \dot{\delta}} \epsilon_{dcba} D_\delta Y \tag{2.16}$$
while
\[ \Sigma_{dcba} = \frac{1}{12} \epsilon_{dcba} (D^2 Y - \bar{D}^2 \bar{Y}) \]  \hspace{1cm} (2.17)

In addition, the superfield \( Y \) and its conjugate \( \bar{Y} \) are subject to the chirality conditions
\[ \bar{D}^{\dot{\alpha}} Y = 0, \quad D^{\alpha} \bar{Y} = 0 \]  \hspace{1cm} (2.18)

One can also solve the constraints (2.11) directly to find the coefficients of the super three-form in terms of a real, but otherwise unconstrained, prepotential, \( \Omega \). The result is that there is a gauge where all coefficients vanish with the exception of
\[ B^{\dot{\gamma}}_\gamma^\alpha = -2i(\sigma_a \epsilon)^\dot{\gamma}^\beta \Omega \]  \hspace{1cm} (2.19)
\[ B_{\gamma ba} = 2(\sigma_{ba})^{\gamma\beta} D^{\beta} \Omega, \quad B^{\dot{\gamma}}_{\dot{\beta} ba} = 2(\bar{\sigma}_{ba})^{\dot{\gamma}^\dot{\beta}} \bar{D}^{\dot{\beta}} \Omega \]  \hspace{1cm} (2.20)

and
\[ B_{cba} = \frac{3}{8} \epsilon_{dcba} \bar{\sigma}^{d\dot{\alpha}} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] \Omega \]  \hspace{1cm} (2.21)

Substituting these expressions back into the field strength yields the expressions
\[ \bar{Y} = -D^2 \Omega \]  \hspace{1cm} (2.22)
\[ Y = -\bar{D}^2 \Omega \]  \hspace{1cm} (2.23)

reflecting the fact that \( Y \) is chiral. It is important to note that there is a residual gauge freedom in \( B \). The fields \( Y \) and \( \bar{Y} \) remain unchanged under the transformation
\[ \delta_L \Omega = L \]  \hspace{1cm} (2.24)

where \( L \) is a real linear multiplet satisfying the constraints
\[ D^2 L = \bar{D}^2 L = 0 \]  \hspace{1cm} (2.25)

If we now define
\[ B_{cba} \big|_{\theta = \bar{\theta} = 0} = C_{cba} \]  \hspace{1cm} (2.26)

as well as
\[ Y \big|_{\theta = \bar{\theta} = 0} = Y, \quad \bar{Y} \big|_{\theta = \bar{\theta} = 0} = \bar{Y}, \quad D^{\alpha} Y \big|_{\theta = \bar{\theta} = 0} = \sqrt{2} \eta_{\alpha} \]  \hspace{1cm} (2.27)
and
\[ D^2 Y|_{\theta=\bar{\theta}=0} + \bar{D}^2 \bar{Y}|_{\theta=\bar{\theta}=0} = -8H \] (2.28)

then, using (2.17), we can identify \( Y \) with the dualized chiral superfield \( \Phi \) where
\[ F = H - i\frac{4}{3} \epsilon^{mnpq} \partial_m C_{npq} \] (2.29)

We recall that we could have dualized any linear combination of the auxiliary fields in \( \Phi \). We now see that the resulting multiplet corresponds to a parameterization where we identify \( Y \) with \( e^{i\alpha} \Phi \) for some constant phase \( \alpha \). Henceforth, we will assume that the simple identification \( Y = \Phi \) is made. In this case, the general structure of an unconstrained real superfield \( \Omega \) satisfying equations (2.22) and (2.23) is given by
\[
\Omega = C + i\theta \chi - i\bar{\theta} \bar{\chi} + \frac{1}{16} \theta^2 Y^* + \frac{1}{16} \bar{\theta}^2 \bar{Y} + \frac{1}{6} \theta \sigma^m \bar{\theta} \epsilon_{mnpq} C^{npq} + \frac{1}{2} \theta^2 \bar{\theta} \left( \frac{\sqrt{2}}{8} \bar{\eta} + \bar{\sigma}^m \partial_m \bar{\chi} \right) \\
+ \frac{1}{2} \bar{\theta}^2 \theta \left( \frac{\sqrt{2}}{8} \eta - \sigma^m \partial_m \chi \right) + \frac{1}{4} \theta^2 \bar{\theta}^2 \left( \frac{1}{4} H - \theta^2 C \right)
\] (2.30)

We see that, in addition to \( Y, \eta_\alpha, H \) and \( C_{mnp} \), the prepotential generically contains the two extra fields \( C \) and \( \chi \). However, by an appropriate choice of \( L \) these last two fields can be transformed away to give
\[
\Omega = \frac{1}{16} \theta^2 Y^* + \frac{1}{16} \bar{\theta}^2 \bar{Y} + \frac{1}{6} \theta \sigma^m \bar{\theta} \epsilon_{mnpq} C^{npq} + \frac{1}{8\sqrt{2}} \theta^2 \bar{\theta} \bar{\eta} + \frac{1}{8\sqrt{2}} \theta^2 \bar{\theta} \bar{\eta} + \frac{1}{16} \theta^2 \bar{\theta}^2 H
\] (2.31)

This is the WZ gauge form for the superfield \( \Omega \).

We can now answer our original question. The fields \( Y, \eta_\alpha, H \) and \( C_{mnp} \) are the component fields of the superfield \( \Omega \) in WZ gauge. Let us now consider the supersymmetry transformation of this superfield. It is a simple exercise to show that \( \delta \Omega \) contains terms proportional to \( \theta \) and \( \bar{\theta} \) and, hence, is no longer in WZ gauge. However, WZ gauge can be restored by a gauge transformation \( \delta_L \Omega \) where the component fields \( B, \lambda_\alpha \) and \( B_{mn} \) of \( L \) are chosen to satisfy
\[
B = 0 \\
\lambda = \frac{1}{6} i \sigma^m \bar{\xi} \epsilon_{mnpq} C^{npq} + \frac{i}{8} \bar{\xi} Y^* \\
\partial_{[m} B_{np]} = 0
\] (2.32)

It is not hard to verify that, under supersymmetry plus the WZ gauge-restoring transformation,
the component fields transform as

\[(\delta_\xi + \delta_L) Y = \sqrt{2}\xi \eta\]
\[(\delta_\xi + \delta_L) \eta = i\sqrt{2}\sigma^m \bar{\xi} \partial_m Y + \sqrt{2}\xi F\]
\[(\delta_\xi + \delta_L) H = \frac{i}{\sqrt{2}}(\bar{\xi} \sigma^m \partial_m \eta + \xi \sigma^m \partial_m \bar{\eta})\]
\[(\delta_\xi + \delta_L) C_{mnp} = -\frac{\sqrt{2}}{16} \epsilon_{mnpq} (\bar{\xi} \sigma^q \eta - \xi \sigma^q \bar{\eta})\]  

(2.33)

Comparing these results with equation (2.7), we see that

\[\delta = \delta_\xi + \delta_L\]  

(2.34)

This explains why we dropped the subscript \(\xi\) in (2.7).

We conclude that the supermultiplet obtained by integrating the supersymmetry transformations of the component fields of \(\Phi\) is precisely the vector prepotential \(\Omega\) in the WZ gauge. Furthermore, the transformation law of the component fields after this integration correspond to the combined supersymmetry and WZ gauge-restoring transformations of the prepotential supermultiplet. It is useful to note, using (2.19)–(2.21), that for \(\Omega\) in WZ gauge, the lowest components of the superfield coefficients \(B_{CBA}\) satisfy

\[B_{\gamma a}^{\beta} \big|_{\theta = \bar{\theta} = 0} = B_{\gamma ba}^{\beta} \big|_{\theta = \bar{\theta} = 0} = B_{\gamma ba}^{\beta} \big|_{\theta = \bar{\theta} = 0} = 0\]  

(2.35)

\[B_{cba}^{\gamma} \big|_{\theta = \bar{\theta} = 0} = C_{cba}\]  

(2.36)

If \(\Omega\) is transformed outside of its WZ gauge, non-zero components begin to appear in (2.33). Thus, WZ gauge for the prepotential corresponds to the gauge one would naturally define for the coefficients \(B_{CBA}\).

We conclude this section by computing the commutator of the transformations \(\delta\) acting on the component field \(C_{mnp}\). We find that

\[(\delta \delta' - \delta' \delta) C_{mnp} = 2\eta^q \partial_q C_{mnp} + 3\partial_{[m} \Lambda_{np]}\]  

(2.37)

where

\[\Lambda_{mn} = -2C_{mnp} \eta^p - \frac{1}{2} (\eta_{pq} Y_1 + \eta_{pq} Y_2)\]  

(2.38)

with

\[\eta^m = -i(\xi \sigma^m \xi' - \xi' \sigma^m \xi)\]
\[\eta_{pq}^+ = \xi \sigma_{pq} \xi' + \bar{\xi} \sigma_{pq} \xi'\]
\[\eta_{pq}^- = -i(\bar{\xi} \sigma_{pq} \xi' - \bar{\xi} \sigma_{pq} \xi')\]  

(2.39)
and $Y = Y_1 + iY_2$. What is the interpretation of the two terms on the right hand side of equation (2.37)? Substituting $\delta = \delta_\xi + \delta_L$ on the left side of equation (2.37), and using the fact that $\delta_L\delta_L' C_{mnp} = \delta_L\delta_\xi C_{mnp} = 0$, we find

$$\left(\delta_\xi\delta_\xi' - \delta_\xi'\delta_\xi\right) C_{mnp} = 2\eta^q \partial_q C_{mnp} \quad (2.40)$$

and

$$\left(\delta_\xi\delta_L' - \delta_L'\delta_\xi\right) C_{mnp} = 3\partial_{[m} \Lambda_{np]} \quad (2.41)$$

The first of these equations is simply the representation of the supersymmetry algebra $\{Q_\alpha, Q_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m$ acting on the component field $C_{mnp}$. Hence, the first term on the right side of equation (2.37) is the translation part of the superalgebra. If WZ gauge restoration was unnecessary, then this would be the only term to appear. However, WZ gauge restoration is necessary and (2.41) tells us that the effect of this gauge restoration is the second term on the right hand side of (2.37). The main point is this. One might try to interpret the second term in (2.37) as indicating the presence of central charges in the supersymmetry algebra associated with the supermultiplet $\Omega$. These central charges would be Lorentz indexed, $Z_{mn}$, and would extend the supersymmetry algebra to

$$\{Q_\alpha, Q_\beta\} = 2i (\sigma^m_{\alpha\beta}) Z_{mn} \quad (2.42)$$

The second term in (2.37) would then be viewed as a (field-dependent) gauge transformation

$$\delta_{\text{gauge}} C_{mnp} = 3\partial_{[m} \Lambda_{np]} \quad (2.43)$$

associated with these extended charges. One problem is that the algebra does not then close simply. However, it is clear from the above analysis that this interpretation is unwarranted, that the extra term is an artifact of the WZ gauge and is perfectly consistent with the usual, unextended supersymmetry algebra.

## 3 $N = 1$ three-form supergravity

In this section we give a self-contained description of the three-form version of $N = 1$ supergravity which naturally couples to the membrane in four dimensions. We first give the multiplet in components, by dualizing one of the auxiliary fields of the old minimal supergravity multiplet. We then turn to the full description in terms of superfields.

A geometrical construction of any $N = 1$ supergravity multiplet starts by introducing a supervielbein, $E_M{}^A$, and a superconnection one-from $\phi_A{}^B$ to gauge local Lorentz transformations
One then forms the superfield curvature, \( R^B_{AB} = d\phi^B_A + \phi^C_A \wedge \phi^B_C \), and the supertorsion, \( T^A = dE^A + E^B \wedge \phi^B_A \). The usual, so-called old minimal, form of the supergravity multiplet is obtained by constraining particular components of the torsion superfield. These constraints are summarized in equation (3.41) later in this section. One then solves the Bianchi identities for \( R^B_{AB} \) and \( T^A \) subject to these constraints. The result is that all superfield components of the curvature and torsion that do not vanish can be constructed from three fundamental superfields, \( R \), \( G_{\alpha\dot{\alpha}} \) and \( W_{\alpha\beta\gamma} \). The component fields of the old minimal gravity supermultiplet are obtained as follows. First, one uses part of the local Lorentz and diffeomorphic symmetries to rotate the lowest component of the supervielbein into the form

\[
E^{A|1}_{M}|_{\theta = \bar{\theta} = 0} = \begin{pmatrix}
 e_{m}^{a} & \frac{1}{2}\psi_{m}^{\alpha} & \frac{1}{2}\bar{\psi}_{m}\dot{\alpha} \\
 0 & \delta_{\mu}^{\alpha} & 0 \\
 0 & 0 & \delta^{\mu}_{\dot{\alpha}} \\
\end{pmatrix}
\]

(3.1)

A vielbein in this form is said to be in WZ gauge. One also defines a complex scalar field

\[
R|_{\theta = \bar{\theta} = 0} = -\frac{1}{6}M
\]

(3.2)

and a real vector field

\[
G_{a}|_{\theta = \bar{\theta} = 0} = -\frac{1}{3}b_{a}
\]

(3.3)

The component fields of the old minimal supergravity multiplet are then \( e_{m}^{a} \), \( \psi_{m}^{\alpha} \), \( b_{a} \) and \( M \). Generic fermionic superdiffeomorphisms, \( \delta_{\xi} \), parameterized by a supervector field \( \xi^{\alpha} \), take one outside of the WZ gauge. However, the gauge can be re-established using a restoring transformation, \( \delta_{WZ} \), as discussed, for example, in \([22]\). It is customary to define, \( \delta = \delta_{\xi} + \delta_{WZ} \), and to refer to these variations as supersymmetry transformations. The variation of the component fields under supersymmetry are

\[
\delta e_{m}^{a} = i \left( \psi_{m}\sigma^{a}\xi + \bar{\psi}_{m}\bar{\sigma}^{a}\xi \right)
\]

\[
\delta \psi_{m}^{\alpha} = -2D_{m}\xi^{\alpha} + ie_{m}^{c} \left\{ \frac{1}{3}M \left( \epsilon\sigma_{\xi}\xi^{\alpha} + b_{c}\xi^{\alpha} + \frac{1}{3}\delta^{d} \left( \xi\sigma_{d}\bar{\sigma}_{c}\right)^{\alpha} \right) \right\}
\]

\[
\delta b_{\alpha\dot{\alpha}} = \xi^{\dot{\beta}} \left\{ \frac{3}{4}\bar{\psi}_{\alpha}\gamma^{\gamma}\psi_{\dot{\alpha}}\gamma^{\dot{\alpha}} - \frac{i}{2}\bar{M^{\ast}}\psi_{\alpha\dot{\alpha}}\delta + \frac{i}{4} \left( \bar{\psi}_{\alpha}\gamma^{\beta}b_{\beta\dot{\alpha}} + \bar{\psi}_{\dot{\alpha}}\gamma^{\beta}b_{\alpha\dot{\alpha}} - \bar{\psi}_{\beta}\gamma^{\alpha}b_{\alpha\dot{\alpha}} \right) \right\} + \text{h.c.}
\]

\[
\delta M = -\xi \left( \sigma^{a}\delta_{b} \psi_{ab} + ib_{a}\psi_{a} - i\sigma^{a}\bar{\psi}_{a}M \right)
\]

(3.4)

We have included the variation of \( b_{a} \) for completeness, though it is not required in the following. The spinor notation used for \( \delta b_{a} \) is explained in \([22]\). Writing the auxiliary field \( M \) as \( M = M_{1} + iM_{2} \),
the last equation in (3.4) becomes
\[
\delta M_1 = -\xi\sigma^{ab}\psi_{ab} - \bar{\xi}\bar{\sigma}^{ab}\bar{\psi}_{ab} - \frac{i}{2}(\xi\bar{\psi}_a - \bar{\xi}\bar{\psi}_a) b^a + \frac{i}{2}(\xi\sigma^a\bar{\psi}_a + \bar{\xi}\bar{\sigma}^a\bar{\psi}_a) M_1 - \frac{1}{2}(\xi\sigma^a\bar{\psi}_a - \bar{\xi}\bar{\sigma}^a\bar{\psi}_a) M_2
\]
(3.5)
and
\[
\delta M_2 = i\left(\xi\sigma^{ab}\psi_{ab} - \bar{\xi}\bar{\sigma}^{ab}\bar{\psi}_{ab}\right) - \frac{1}{2}(\xi\psi_a + \bar{\xi}\bar{\psi}_a) b^a + \frac{1}{2}(\xi\sigma^a\bar{\psi}_a - \bar{\xi}\bar{\sigma}^a\bar{\psi}_a) M_1 + \frac{i}{2}(\xi\sigma^a\bar{\psi}_a + \bar{\xi}\bar{\sigma}^a\bar{\psi}_a) M_2
\]
(3.6)

In analogy with the previous section, we will try to introduce a three-form field into the multiplet by dualizing the real field \(M_2\). We first write \(M_2 \propto \epsilon^{mnpq}F_{mnpq}\) where \(F\) is a four-form. Since a four-form in four-dimensions is always closed, one has locally \(F = dC\) where \(C\) is a three-form. We can then hope to integrate the supersymmetry transformation to give the variation of \(C\). Here, however, the exact analogy with the flat superspace chiral multiplet case comes to an end. It turns out that, in the supergravity case, it is necessary to add a second term to \(dC\) in order to insure integrability. This term naturally arises when one tries to dualize the old minimal supergravity action with a cosmological constant, as we shall see in the next section. The appropriate expression is
\[
M_2 = \epsilon^{mnpq}\partial_mC_{npq} + \frac{i}{2}\left(\psi_a\sigma^{ab}\psi_b - \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b\right)
\]
(3.7)
where we have chosen unit normalization of the first term on the right hand side for convenience. Inserting this in (3.6), we find that the expression can be completely integrated. The result is that
\[
\delta C_{mnp} = \frac{i}{3}\epsilon_{mnpq}\left(\xi\sigma^r\psi_r - \bar{\xi}\bar{\sigma}^r\bar{\psi}_r\right)
\]
(3.8)
This expression was actually first given by Stelle and West in the one of the original formulations of off-shell \(N = 1\) supergravity [18]. However in that case both the components of \(M\) were dualized. The proof that equation (3.6) can be integrated is considerably more involved than in the flat superspace case. In particular, there are several of terms cubic in the gravitino field that arise which, at first sight, seem to obstruct the integrability. However, by prodigious use of Fierz rearrangement, one can show that these terms all cancel. Putting everything together, we find the variations
\[
\begin{align*}
\delta e_m^a &= i (\psi_m \sigma^a \bar{\xi} + \bar{\psi}_m \bar{\sigma}^a \bar{\xi}) \\
\delta \psi_m^\alpha &= -2D_m \xi^\alpha + i e_m^c \left\{ \frac{1}{3} M (\epsilon \sigma_c \bar{\xi})^\alpha + b_c \xi^\alpha + \frac{1}{3} b^d (\xi \sigma_d \bar{\sigma}_c)^\alpha \right\} \\
\delta b_{a\dot{a}} &= \xi^\delta \left\{ \frac{3}{4} \bar{\psi}_a \gamma^\delta \bar{\gamma} \gamma^{\dot{a}} \gamma - \frac{i}{2} M^* \psi_{a\dot{a} \dot{\delta}} + \frac{i}{4} \left( \bar{\psi}_{a\dot{b}} \bar{\psi}_{b\dot{a}} + \bar{\psi}_{\dot{b}} \bar{\psi}_{b\dot{a} \dot{a}} - \bar{\psi}_{\dot{a}} \bar{\psi}_{\dot{a} \dot{b}} \right) \right\} + \text{h.c.} \\
\delta M_1 &= -\xi \sigma^{ab} \psi_{ab} - \xi \tilde{\sigma}^{ab} \bar{\psi}_{ab} - \frac{i}{2} (\xi \psi_{a} - \xi \bar{\psi}_{a}) b^a \\
&\quad + \frac{i}{2} (\xi \sigma^a \bar{\psi}_a + \xi \tilde{\sigma}^a \bar{\psi}_a) M_1 - \frac{1}{2} (\xi \sigma^a \bar{\psi}_a - \xi \tilde{\sigma}^a \bar{\psi}_a) M_2 \\
\delta C_{mnp} &= \frac{i}{3} \epsilon_{mnpq} (\xi \sigma^{qr} \bar{\psi}_r - \xi \tilde{\sigma}^{qr} \bar{\psi}_r)
\end{align*}
\]

(3.9)

It follows that the component fields \(e_m^a, \psi_m^\alpha, b_a, M_1\) and \(C_{mnp}\) form an irreducible supergravity multiplet.

This fact is somewhat remarkable and deserves further discussion. As stated above, the old-minimal form of the \(N = 1\) supergravity multiplet consists of the component fields \(e_m^a, \psi_m^\alpha, b_a\) and \(M\). It has long been known that there exists another irreducible \(N = 1\) supergravity multiplet, the so-called new minimal supermultiplet, consisting of the component fields \(e_m^a\) and \(\psi_m^\alpha\) along with a real vector field \(a_m\) and a real two-form \(B_{mn}\). That is, the auxiliary fields \(M\) and \(b_a\) of the old minimal multiplet are replaced by auxiliary fields \(a_m\) and \(B_{mn}\) in the new minimal multiplet. Both the new and old minimal forms of supergravity have 12 bosonic and 12 fermionic off-shell degrees of freedom. There is one other irreducible \(N = 1\) supergravity multiplet known, the so-called 16-16 multiplet, which, as the name implies, has 16 bosonic and 16 fermionic off-shell degrees of freedom. It also contains a real two-form. However, none of these known multiplets contains a real three-form. We seem, therefore, by writing \(M_2\) as the curl of a three-form and by integrating transformation equation (3.6), to have constructed a new irreducible \(N = 1\) supergravity multiplet containing a three-form \(C_{mnp}\). This multiplet also has 12 bosonic and 12 fermionic off-shell degrees of freedom. As we have mentioned, a component form of the supergravity multiplet with two three-form fields was derived some time ago in [18, 19]. The case of a single three-form multiplet is easily extracted from these papers, but was not explicitly discussed. This multiplet is interesting in that it is similar in form, albeit in four dimensions, to eleven-dimensional supergravity. Unlike eleven-dimensional supergravity, however, we can construct the off-shell theory and completely study its behaviour. In particular, and this is our main point, one can ask for what superfields are \(e_m^a, \psi_m^\alpha, b_a, M_1\) and \(C_{npq}\) the component fields?

To answer this, we must consider the theory of a constrained three-form supermultiplet coupled to supergravity in curved superspace. This construct is the analog of the super two-form formalism.
used to \[21\] to derive the new minimal multiplet. The supergravity torsion constraints, and consequently the solution of the geometrical Bianchi identities, remain the same as above. It follows that the component fields arising from the geometrical supergravity sector are just those defined in equations (3.1)–(3.3). Similarly, the transformations of these component fields under supersymmetry are still given by (3.4)–(3.6). The theory of the super three-form, however, requires some modification over the flat superspace case discussed in the previous section. This theory was provided in detail in reference \[20\], and we will use their notation and some of their results in our construction. It is worth emphasizing that these authors treated the three-form as a matter supermultiplet so did not use their formalism to construct the new three-form irreducible gravity supermultiplet found above. That construction requires imposing further constraints on the three-form multiplet, as we will now describe.

Construction of the three-form matter multiplet in curved superspace follows closely the flat-space case given in section two. One begins by introducing a real three-form superfield

\[
B = \frac{1}{3!} E^A E^B E^C B_{CBA} \tag{3.10}
\]

where now \(E^A\) denotes the frame of curved superspace. The invariant field strength is a four-form

\[
\Sigma = dB = \frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} \tag{3.11}
\]

Again the components of \(\Sigma\) are constrained to satisfy

\[
\Sigma_{\gamma \beta A} = \Sigma_{\dot{\gamma} \beta a} = 0 \tag{3.12}
\]

where \(a \sim \alpha, \dot{\alpha}\). Solving the Bianchi identity \(d\Sigma = 0\) one finds that all coefficients of the four-form \(\Sigma\) either vanish, or can be expressed in terms of an complex superfields \(Y\) and and its conjugate \(\bar{Y}\) as follows

\[
\Sigma_{\delta \gamma ba} = \frac{1}{2} (\sigma_{ba} \epsilon)_{\delta \gamma} \bar{Y}, \tag{3.13}
\]

\[
\Sigma_{\dot{\delta} \dot{\gamma} ba} = \frac{1}{2} (\bar{\sigma}_{ba} \bar{\epsilon})_{\dot{\delta} \dot{\gamma}} Y \tag{3.14}
\]

and

\[
\Sigma_{dcb a} = -\frac{1}{16} \sigma^d_{\delta \delta} \epsilon_{dcb a} D^\delta \bar{Y}, \tag{3.15}
\]

\[
\Sigma_{\dot{d} \dot{c} ba} = \frac{1}{16} \sigma^{\dot{d} \dot{d}} \epsilon_{dcb a} D_{\dot{d}} Y \tag{3.16}
\]

while

\[
\Sigma_{dcba} = \frac{i}{12} \epsilon_{dcba} \left[ (D^2 - 24 R^i) Y - (\bar{D}^2 - 24 R) \bar{Y} \right] \tag{3.17}
\]
Furthermore, superfields $Y$ and $\bar{Y}$ are subject to the curved-space chirality conditions

$$\bar{D}^i Y = 0, \quad D_\alpha \bar{Y} = 0$$

(3.18)

Also as before, one can solve the constraints (3.12) directly to find the coefficients of the super three-form in terms of a real but otherwise unconstrained prepotential, $\Omega$. One finds that there is a gauge where all coefficients vanish with the exception of

$$B_{\gamma}{}^\beta{}^a = -2i(\sigma_a)_{\gamma}{}^\beta \Omega$$

(3.19)

as well as

$$B_{\gamma ba} = 2(\sigma_{ba})_{\gamma}{}^\beta D_{\beta} \Omega, \quad B^{\gamma}{}_{ba} = 2(\bar{\sigma}_{ba})_{\gamma}{}^\beta \bar{D}^\beta \Omega$$

(3.20)

and

$$B_{\gamma a} = 1/4 \epsilon_{dcba} \bar{\sigma}^{d\alpha} \left( [D_\alpha, \bar{D}_\alpha] - 4G_{\alpha\bar{\alpha}} \right) \Omega$$

(3.21)

Substituting these expressions back into the field strength yields the expressions

$$\bar{Y} = -4 \left( D^2 - 8R^i \right) \Omega$$

(3.22)

$$Y = -4 \left( \bar{D}^2 - 8R \right) \Omega$$

(3.23)

The residual gauge freedom is reflected in the fact that the fields $Y$ and $\bar{Y}$ remain unchanged under the transformation

$$\delta_L \Omega = L$$

(3.24)

where $L$ is a real linear multiplet satisfying the constraints

$$\left( D^2 - 8R^i \right) L = (\bar{D}^2 - 8R) L = 0$$

(3.25)

The discussion of the WZ gauge for the three-form supermultiplet in curved superspace is much the same as it was in the flat superspace case. The prepotential $\Omega$ can be put into WZ gauge using transformation (3.24) with an appropriate choice of linear superfield $L$. In this gauge the lowest two components of $\Omega$ vanish. It follows from (3.13)–(3.21) that in WZ gauge

$$B_{\alpha a} |_{\theta = \bar{\theta} = 0} = B_{\alpha ba} |_{\theta = \bar{\theta} = 0} = B_{\beta a} |_{\theta = \bar{\theta} = 0} = 0$$

(3.26)

$$B_{\gamma a} |_{\theta = \bar{\theta} = 0} = C_{\gamma a}$$

(3.27)
If $\Omega$ is transformed outside of its WZ gauge, non-zero components begin to occur in (3.26). Thus, WZ gauge for the prepotential corresponds to the gauge one would naturally define for the coefficients $B_{CBA}$. We will assume, henceforth, that we are working in the WZ gauge.

The component fields of the three-form supermultiplet are defined as follows. We write

$$B_{mnp}\big|_{\theta=\bar{\theta}=0} = C_{mnp}$$  \hspace{1cm} (3.28)

while

$$Y\big|_{\theta=\bar{\theta}=0} = Y, \quad \bar{Y}\big|_{\theta=\bar{\theta}=0} = \bar{Y}$$  \hspace{1cm} (3.29)

and

$$\mathcal{D}_\alpha Y\big|_{\theta=\bar{\theta}=0} = \sqrt{2}\eta_\alpha, \quad \bar{\mathcal{D}}_\dot{\alpha} Y\big|_{\theta=\bar{\theta}=0} = \sqrt{2}\bar{\eta}_{\dot{\alpha}}$$

Equation (3.17) means that the orthogonal combination is not an independent component field. It is given by the expression

$$\mathcal{D}^2 Y\big|_{\theta=\bar{\theta}=0} - \bar{\mathcal{D}}^2 \bar{Y}\big|_{\theta=\bar{\theta}=0} = \frac{32i}{3} \varepsilon^{mnpq} \partial_m C_{npq} + 2\sqrt{2}i\bar{\psi}_m \bar{\sigma}^m \eta - 2\sqrt{2}i\psi_m \sigma^m \bar{\eta} - 4 (M + \bar{\psi}_m \bar{\sigma}^m \bar{\psi}_n) Y + 4 (M + \psi_m \sigma^m \psi_n) \bar{Y}$$  \hspace{1cm} (3.31)

Under a combination of a general superdiffeomorphism, $\delta_\xi$, and an arbitrary three-form supergauge transformation, $\delta_\Lambda$, the super three-form transforms as

$$(\delta_\xi + \delta_\Lambda) B = \iota_\xi \Sigma + d (\Lambda + \iota_\xi B)$$  \hspace{1cm} (3.32)

where $\iota_\xi$ is the super inner derivative. The superdiffeomorphism is parameterized by a general supervector field $\xi^A$, while the gauge transformation parameter is a super two-form $\Lambda$. Clearly, such a transformation generically takes $B$ out of the WZ gauge. However, as in the flat-space case, it is possible to find a compensating gauge transformation $\Lambda_\xi$, dependent on the parameter $\xi^A$, such that the combined transformation leaves $B$ in WZ gauge. We now restrict ourselves to the diffeomorphisms corresponding to the supersymmetry transformations of the supergravity multiplet (3.4). It is possible to show from the results of [20] that the corresponding transformation of the super three-form, restricted to WZ gauge, gives for the component $C_{mnp}$,

$$(\delta_\xi + \delta_\Lambda_\xi) C_{mnp} = \frac{\sqrt{2}}{16} (\bar{\xi} \bar{\sigma}^q \eta - \xi \sigma^q \bar{\eta}) \varepsilon_{qmn} - \frac{3}{4} \left[(\psi_m \sigma_{np} \xi) \bar{Y} + (\bar{\psi}_m \bar{\sigma}_{np} \bar{\xi}) Y\right]$$  \hspace{1cm} (3.33)
Up to this point, we have been discussing the theory of an independent three-form superfield coupled to old minimal supergravity. As we now show, by the application of two additional constraints, these two multiplets can be merged into one irreducible supergravity multiplet containing the real three-form component field. To do this, we impose two new constraints

\[ \Sigma_{\delta cba} = \Sigma_{\delta cba}^\delta = 0 \]  
(3.34)

which imply

\[ \bar{D}^\delta Y = 0, \quad D_\alpha Y = 0 \]  
(3.35)

These constraints, in addition to the original chirality constraints in (3.18), mean that superfield \( Y \) must be a constant. We will choose superfield \( Y = y \). It follows immediately from expressions (3.29) and (3.30) that component fields are

\[ Y = y \]  
(3.36)

and

\[ \eta_\alpha = H = 0 \]  
(3.37)

Thus, the additional constraints (3.35) eliminate all the component fields coming from the three-form supermultiplet except for \( C_{mnp} \). However, these constraints do something more interesting. It follows from expression (3.31) that now

\[ M_2 = -\frac{3}{4y} \epsilon^{mnpq} \partial_m C_{npq} + i \frac{1}{2} \left( \psi_m \sigma^{mn} \psi_n - \bar{\psi}_m \sigma^{mn} \bar{\psi}_n \right) \]  
(3.38)

That is, component field \( M_2 \) of the original supergravity multiplet is also eliminated in terms of the three-form and gravitino component fields. Furthermore, one should note that if we choose constant \( y = -4/3 \), then the normalization of this expression matches that of equation (3.7) which was found to be the appropriate relation to insure integrability of the supersymmetry transformation law. Henceforth, we will take \( y \) to have this value. The remaining component fields of the original supergravity multiplet, \( e_m^a, \psi_m^\alpha, b_a \) and \( M_1 \) are unaffected by the new constraints. Their supersymmetry transformations are still given by the first three equations in (3.4) and equation (3.3) where, however, \( M_2 \) must be written in terms of \( C_{mnp} \) using (3.38). Transformation (3.6) is no longer relevant. It is now replaced by supersymmetry transformation (3.33). Using (3.36) and (3.37), this transformation becomes

\[ (\delta \xi + \delta \Lambda) C_{mnp} = \frac{i}{3} \epsilon_{mnpq} \left( \xi \sigma^{qr} \psi_r - \bar{\xi} \bar{\sigma}^{qr} \bar{\psi}_r \right) \]  
(3.39)
Note that this is identical to the integrated supersymmetry transformation for $C_{mnp}$ given in expression (3.8) if we identify 
\[ \delta = \delta_\xi + \delta_\Lambda \xi \] (3.40)

The new constraints are superfield equations. Thus the supersymmetry transformations for the, now eliminated, component fields $Y, \eta, b, M_1$ and $C_{mnp}$. These fields transform irreducibly under the supersymmetry transformations given in equations (3.39). Thus, we have found a complete, off-shell superfield representation of the three-form supergravity multiplet.

For completeness let us summarize the full set of superfield constraints we used to derive this multiplet. We started in curved superspace with a supervielbein $E_M^A$ and a Lorentz-algebra-valued connection $\phi_A^B$, together with a super three-form potential $B$. For the vielbein we adopted the usual torsion constraints of old minimal supergravity, as given, for instance, in [22],

\[
T_{\alpha\beta}^c = T_{\dot{\alpha}\dot{\beta}}^c = T_{\dot{\alpha}^c\dot{\beta}}^c = T_{\alpha^c\beta}^c = 0
\]
\[
T_{\alpha\beta}^{c} = T_{\beta\alpha}^{c} = 2i\sigma^{c}_{\alpha\beta}
\]
(3.41)

Now we turn to the super form-form field strength $\Sigma = dB$. From equations (3.12)–(3.16) and (3.34), with the normalization $y = -4/3$, we see that all the components of $\Sigma$ are set to zero except for

\[
\Sigma_{\alpha\beta ba} = -\frac{2}{3}(\sigma_{ba}c)_{\alpha\beta} \quad \Sigma^{\dot{\alpha}\dot{\beta}}_{ba} = -\frac{2}{3}(\sigma_{ba}c)^{\dot{\alpha}\dot{\beta}}
\]
(3.42)

and the component $\Sigma_{abcd}$, which is unconstrained. Solving the $d\Sigma = 0$ Bianchi identity does, however, determine $\Sigma_{abcd}$ in terms of the curvature superfield $R$. From (3.17) we see

\[
\Sigma_{abcd} = \frac{i}{2} \left( R^i - R \right) \epsilon_{abcd}
\]
(3.43)

which links the vielbein and three-form field into a single irreducible supergravity multiplet.

We end this section by computing the commutator of the transformations $\delta$ acting on the component field $C_{npq}$. We find that

\[
(\delta \delta' - \delta' \delta) C_{mnp} = 2\delta_\eta C_{mnp} + \delta_\xi C_{mnp} + \delta_\Lambda C_{mnp}
\]
(3.44)

where we define

\[
\delta_\eta C_{mnp} = \eta^q \partial_q C_{mnp} + (\partial_m \eta^q) C_{qnp} + (\partial_n \eta^q) C_{mpq} + (\partial_p \eta^q) C_{mqn}
\]
(3.45)

\[
\delta_\xi C_{mnp} = \frac{i}{3} \epsilon_{mnpq} \left( \xi'' \sigma^{qr} \psi_r - \check{\xi''} \sigma^{qr} \check{\psi}_r \right)
\]
(3.46)

\[
\delta_\Lambda C_{mnp} = 3 \partial_q \Lambda^q_{mnp}
\]
(3.47)
The fermionic parameter $\xi''$ and the two-form $\Lambda''_{mn}$ are given by

$$\xi''^\alpha = \eta^m \psi^\alpha_m$$

(3.48)

$$\Lambda''_{mn} = -3C_{mnp}\eta^p - \frac{1}{3}\eta^{+}_{mn}$$

(3.49)

while the parameters $\eta^r$ and $\eta^{+}_{pq}$ are given in (2.39). What is the interpretation of the three terms on the right hand side of equation (3.44)? Substituting $\delta = \delta_\xi + \delta_\Lambda$ on the left side of (3.44), and using the fact that $\delta_\Lambda \delta_\xi C_{mnp} = \delta_\Lambda \delta_\xi C_{mnp} = 0$, we find

$$\left(\delta_\xi \delta_\xi - \delta_\xi' \delta_\xi\right) C_{mnp} = 2\delta_\eta C_{mnp} + \delta_\xi'' C_{mnp}$$

(3.50)

and

$$\left(\delta_\xi \delta_\Lambda - \delta_\xi' \delta_\Lambda\right) C_{mnp} = \delta_\Lambda'' C_{mnp}$$

(3.51)

From equation (3.50) we see that $\delta_\eta C_{mnp}$ and $\delta_\xi'' C_{mnp}$ come from the superdiffeomorphism algebra. They may be understood in the following way. The term $\delta_\xi C_{mnp}$ is just a usual bosonic diffeomorphism; $\delta_\xi'' C_{mnp}$ is, in fact, a local supersymmetry transformation with parameter $\xi''$. These are the restricted class of transformations which leave the supervielbein in the gravitational WZ gauge. Equation (3.50) is just reproducing the well-known algebra of these restricted diffeomorphisms. The fact that it is not possible to define a local supersymmetry transformation which closes just into a bosonic diffeomorphism is a result of the requirement that one remains in the gravitational WZ gauge.

Turning to equation (3.51) we see that the $\delta_\Lambda'' C_{mnp}$ term arises because we have also chosen a WZ gauge for the three-form $B$. If WZ gauge restoration was unnecessary, we would have $\delta = \delta_\xi$ and the first two terms in (3.44) would be the only terms to appear. However, WZ gauge restoration is necessary and (3.51) tells us that the effect of this gauge restoration is the third term on the right hand side of (3.44). Note that the term proportional to $\eta^{+}_{pq}$ in $\Lambda$ is field independent. Therefore, it would appear to be a very different situation than in the previous section. However, one should note that this term is, in fact, identical to the second term on the right side of equation (2.41) with $Y_1 = y = -4/3$ and $Y_2$ set to zero, as is required by (3.36).

As in the flat-space case, one might have been tempted to interpret this term as indicating the presence of central extension to the local supersymmetry algebra associated with supermultiplet $\Omega$. The suggestion is that, in general, two superdiffeomorphisms can close in to a gauge transformation $\delta_\Lambda''$. It is clear from the above analysis that this interpretation is unwarranted. The general symmetry of the multiplet is local superdiffeomorphisms, which form a closed subalgebra, together with independent gauge transformations. The closure of two supersymmetry transformations into a gauge transformation is an artifact of the three-form WZ gauge.
We have carried out this analysis by re-expressing the imaginary part of the auxiliary field $M$ in terms of the three-form $C_{npq}$. In terms of the component field calculation we could just as easily dualized the real part of $M$ or some linear combination of $M_1$ and $M_2$. Indeed, $M$ could have been completely replaced by two independent three-forms. From the superfield point of view, only one three-form is introduced. The freedom in choosing a particular linear combination of auxiliary fields to dualize is reflected in the arbitrariness in choosing the phase in the constant $y$. However, as we will see in section five, demanding the Green-Schwarz membrane action is $\kappa$-symmetric, picks out specifically the $M_2$ dualization.

4 The elementary membrane solution

We claim that the three-form version of $N = 1$ supergravity derived in the previous section is the version relevant to the description of fundamental membranes in four dimensions. In the next section, we show that $\kappa$-symmetry of the Green-Schwarz membrane action leads to exactly the superfield constraints used in section three. This section demonstrates how the fundamental membrane solution appears in the three-form supergravity theory.

Since we are now interested in solutions rather than supergravity multiplets, we must begin by giving the action for the three-form supergravity. As discussed above, the old minimal supermultiplet of $D = 4$, $N = 1$ supergravity consists of a graviton $g_{mn}$, a gravitino $\psi_m$, a real vector field $b_a$ and a complex scalar field $M$. The bosonic part of the pure supergravity Lagrangian is given by

$$\kappa^2 e^{-1} \mathcal{L} = \frac{1}{2} R + \frac{1}{3} b^2 - \frac{1}{3} M_1^2 - \frac{1}{3} M_2^2$$

(4.1)

where $M = M_1 + i M_2$. The three-form supergravity multiplet was derived by dualizing $M_2$ into a four-form field strength $F_{mnpq}$. If we continue to ignore fermion terms, the transformation is

$$M_2 = \frac{1}{4} \epsilon^{mnpq} F_{mnpq} = \epsilon^{mnpq} \partial_m C_{npq}$$

(4.2)

where we write $\Sigma_{mnpq} |_{\theta = \bar{\theta} = 0} = F_{mnpq}$. Substituting into the supergravity Lagrangian (4.1) gives

$$\kappa^2 e^{-1} \mathcal{L} = \frac{1}{2} R + \frac{1}{3} b^2 - \frac{1}{3} M_1^2 + 2F^2$$

(4.3)

This lagrangian is not quite equivalent to the original one. Varying the auxiliary fields in (4.1) leads to the field equations $M_1 = M_2 = b_a = 0$. In (1.3), $b_a$ and $M_1$ similarly get set to zero, but the $C_{mnp}$ field equation gives

$$\nabla^m F_{mnpq} = 0 \iff F_{mnpq} = \lambda \epsilon_{mnpq}$$

(4.4)
where \( \lambda \) is a constant of integration. Thus, from (4.2), we see that now \( M_2 = \frac{1}{4} \varepsilon^{mnpq} F_{mnpq} = -6\lambda \neq 0 \). In fact, the constant of integration \( \lambda \) acts as a cosmological constant. On shell, where \( b_a = M_1 = 0 \), the metric equation of motion reads

\[
R_{mn} - \frac{1}{2} g_{mn} R = -4 F_m^{\ pqr} F_{npqr} - \frac{1}{2} g_{mn} F^2 = 12\lambda^2 g_{mn} \tag{4.5}
\]

where we have substituted the solution of the \( C_{mnp} \) equation (4.4). We see that (4.5) is simply Einstein’s equation for empty space with a negative cosmological constant \( \Lambda = -\frac{12}{\lambda^2} \). Note that the value of \( \Lambda \) is determined dynamically by solving the \( C_{mnp} \) equation of motion.

The equivalence of a three-form field coupled to gravity and a dynamically determined cosmological constant was noted some time ago in [2, 24, 25]. In the present context, it provides a way of deriving the fermion terms that had to be added to \( M_2 \) in the definition (3.7) to make the supersymmetry variation of \( F_{mnpq} \) integrable. The full action for old-minimal supergravity with a cosmological constant is

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left\{ R + \varepsilon^{mnpq} \left( \dot{\psi}_m \tilde{\sigma}^n \dot{D}_p \psi_q + \psi_m \sigma_n \dot{D}_p \bar{\psi}_q \right) + \frac{2}{3} b^2 - \frac{2}{3} (M_1^2 + M_2^2) \right. \\
+ \mu_1 \left[ M_1 + \frac{1}{2} \left( \psi_a \sigma^{ab} \psi_b + \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \right) \right] + \mu_2 \left[ M_2 - \frac{i}{2} \left( \psi_a \sigma^{ab} \psi_b - \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \right) \right] \right\} \tag{4.6}
\]

Eliminating \( M_1 \) and \( M_2 \), we find the cosmological constant is \( \Lambda = -\frac{3}{16} (\mu_1^2 + \mu_2^2) \). We would like to replace \( M_2 \) by a three-form field such that the cosmological constant is fixed by the three-form equation of motion. Since \( M_1 \) is not dualized, we first take \( \mu_1 = 0 \). Defining

\[
\frac{1}{4} \varepsilon^{mnpq} F_{mnpq} = \varepsilon^{mnpq} \partial_m C_{npq} = M_2 - \frac{i}{2} \left( \psi_a \sigma^{ab} \psi_b - \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b \right) \tag{4.7}
\]

means that the last term in (4.6) is a total divergence. The parameter \( \mu_2 \) effectively drops from the action and the cosmological constant becomes fixed dynamically by solving the \( C_{mnp} \) equation of motion. The definition (4.7) is exactly the combination that appeared in the supersymmetric dualization of \( M_2 \) in the previous section (3.7). One notes that, since the action with \( \mu_1 = \mu_2 = 0 \) is supersymmetric, the last term in (4.7) must be invariant by itself. Thus the variation of \( \varepsilon^{mnpq} F_{mnpq} \) must be a total divergence and so can be integrated to give a variation of \( C_{mnp} \). This was just the property required in the previous section to form the three-form multiplet.

Since the total-derivative \( \mu_2 \) term can effectively be dropped, if we rewrite \( M_2 \) in terms of \( C_{mnp} \) and put the auxiliary \( b_a \) and \( M_1 \) fields on shell, the bosonic lagrangian for three-form supergravity becomes

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left( R + F^2 \right) \tag{4.8}
\]
As we have argued, this is classically equivalent to pure gravity with a negative cosmological constant \( \Lambda = -12 \lambda^2 \) where \( F_{mnq} = \lambda \epsilon_{mnq} \) is the solution of the \( C_{mnp} \) equation of motion.

It is interesting to note that the action has the same general structure as higher-dimensional supergravity actions, such as type IIA supergravity in \( D = 10 \) or \( D = 11 \) supergravity, with Einstein gravity coupled to Neveu-Schwarz and Ramond forms. There is, of course, no dilaton field in our \( D = 4 \) action. More importantly, the sign in front of the \( F^2 \) term in (4.8) positive, whereas the usual sign in front of the form-field kinetic terms in higher-dimensional supergravity actions is negative.

We would like to find a fundamental membrane solution which is a source of three-form charge. The three-dimensional membrane is a domain wall, dividing the spacetime into two regions. Domain wall solutions for general \( (D-2) \)-branes in \( D \)-dimensional spacetime have been discussed in [14, 15, 16]. Here we reproduce the solution in terms of a three-form and map to an explicit membrane source.

Let the spacetime coordinates be \( t, x, y \) and \( z \). If a flat membrane lies in the \( t-x-y \) plane, its Lorentz and translational symmetries imply that the spacetime metric has the form

\[
ds^2 = e^{2A(z)} (-dt^2 + dx^2 + dy^2) + dz^2 \tag{4.9}\]

Since the three-form action (4.8) is equivalent to the action for pure gravity with a negative cosmological constant, the symmetries of (4.9) are sufficient to determine the solution. The spacetime is simply anti-de Sitter space. The membrane solution then corresponds to patching of two pieces of anti-de Sitter space, as was first described in [15].

The details of the solution are as follows. The source is described by the bosonic action for a fundamental membrane coupled to \( C_{mnp} \)

\[
S_3 = T_3 \int d^3 \xi \sqrt{-\gamma} \left( -\frac{1}{2} \gamma^{ij} \partial_i X^m \partial_j X^n g_{mn} + \frac{1}{2} - \frac{\mathcal{C}}{3} \epsilon^{ijk} \partial_i X^m \partial_j X^n \partial_k X^p C_{mnp} \right) \tag{4.10} \]

where \( T_3 \) is the membrane tension, \( \gamma_{ij} \) is the intrinsic membrane metric, \( \epsilon_{ijk} \) is the volume three-form on the membrane and the \( X^m \) fields describe the embedding of the membrane into the four-dimensional target space. The dimensionless constant \( \mathcal{C} \) is not known a priori but, for a consistent solution, will be fixed by the normalization of the \( F^2 \) term in (4.8) in what follows. The full action is then the sum of (4.8) and (4.10). The associated field equations are the following. The metric field equation is

\[
\mathcal{R}^{mn} - \frac{1}{2} g^{mn} \mathcal{R} + 4 F_{mpq} F^{mpq} - \frac{1}{2} g^{mn} F^2 = \kappa^2 T^{mn} \tag{4.11} \]

where \( T^{mn} \) is the membrane stress-energy tensor given by

\[
T^{mn} = -T_3 \int d^3 \xi \sqrt{-\gamma} \gamma^{ij} \partial_i X^m \partial_j X^n \frac{\delta^4(x^p - X^p)}{\sqrt{-g}} \tag{4.12} \]
The three-form field equation is found to be
\[ \partial_m (\sqrt{-g} F^{mnpq}) = -2\kappa^2 T_3 C \int d^3 \xi \sqrt{-\gamma} \epsilon^{ijk} \partial_i X^m \partial_j X^n \partial_k X^q \delta^4 (x^p - X^p) \] (4.13)

and, finally, the \( X^m \) equation of motion is
\[ \partial_i (\sqrt{-\gamma} \gamma^{ij} \partial_j X^n g_{qm}) - \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i X^m \partial_j X^n \partial_q g_{mn} \]
\[ - \frac{C}{6} \sqrt{-\gamma} \epsilon^{ijk} \partial_i X^m \partial_j X^n \partial_k X^q F_{qmn} = 0 \] (4.14)
while the membrane-metric equation gives \( \gamma_{ij} = \partial_i X^m \partial_j X^n g_{mn} \).

As discussed above, the metric Ansatz for a membrane lying in the plane \( z = 0 \) is given by (4.9). Adopting a gauge where the three-form shares the symmetries of the membrane, means the Ansatz for \( C_{mnp} \) is that all components vanish except for
\[ C_{txy} = e^{C(z)} \] (4.15)
For the flat membrane itself, we choose static gauge where \( T = \sigma^0, X = \sigma^1 \) and \( Y = \sigma^2 \). The Einstein equations imply that
\[ 2\partial_z^2 A + 3 (\partial_z A)^2 - 12 e^{-6A+2C} (\partial_z C)^2 = -\kappa^2 T_3 \delta(z) \] (4.16)
\[ \frac{1}{3} e^{-6A} \partial_z (e^{3A} - 6e^C) \partial_z (e^{3A} + 6e^C) = 0 \] (4.17)
whereas the three-form equation and the \( X^m \) equation give
\[ \partial_z (e^{-3A} \partial_z e^C) = -\frac{\kappa^2}{24} T_3 C \delta(z) \] (4.18)
and
\[ \partial_z (e^{3A} - Ce^C) = 0 \] (4.19)
respectively. From equation (4.17) and (4.19) we see that
\[ e^C = \pm \frac{1}{6} e^{3A} + \text{const} , \quad C = \pm 6 \] (4.20)
The constant can be set to zero by a gauge transformation. We see that consistency of the membrane and spacetime equations of motion fixes the magnitude of \( C \). The particular value is fixed by our choice of normalization of \( F^2 \) in (4.8). The sign of \( C \) will be related in the next section to a choice of \( \kappa \)-symmetry helicity on the membrane. To match the conventions there, we will henceforth take \( C = -6 \).
The above equations of motion now all reduce to
\[ \partial_z^2 A = -\frac{\kappa^2}{2} T_3 \delta(z) \] (4.21)
The solution symmetrical in \( z \) is
\[ A = -\alpha |z| \] (4.22)
where \( \alpha = \frac{1}{4} \kappa^2 T_3 \). Given the factor of \( \sqrt{-g} \) in \( \epsilon_{mnpq} \), we can write the final solution as
\[ ds^2 = e^{-\alpha |z|} (-dt^2 + dx^2 + dy^2) + dz^2 \]
\[ F_{txyz} = \pm \frac{1}{2} \epsilon_{txyz} \] (4.23)
where one takes the \(-\) or \(+\) sign in the expression for \( F_{txyz} \) depending on whether one is the right \((y > 0)\) or left \((y < 0)\) of the membrane. As expected, the four-form field is proportional to \( \epsilon_{mnpq} \). From (4.5) we expect the metric to correspond to anti-de Sitter space with a cosmological constant \( \Lambda = -3\alpha^2 = -3\kappa^4 T_3^2 / 16 \). This is the case; the metric is written in unconventional horospherical coordinates [26].

Thus there is a fundamental membrane solution of three-form supergravity which matches consistently onto a membrane source. The solution is the patching of two pieces of anti-de Sitter space across a singular plane. The four-form field strength is constant either side of the membrane, but with opposite sign. This corresponds to the field on each side ‘pointing away’ from the source. The four-form cannot fall off away from the membrane because the transverse space is only one-dimensional. Solutions for multiple parallel branes are, of course, easily constructed, as discussed in [15].

Finally, again following [15], let us briefly discuss the supersymmetry of the membrane solution. It is well known that anti-de Sitter space is supersymmetric, so that, away from the membrane source, we expect there to be four distinct Killing spinors. The supersymmetric variation of the gravitino is given by
\[ \delta \psi_m{}^\alpha = -2 \partial_m \xi^\alpha - \omega_{mnp} (\epsilon \sigma^{mp} \xi)^\alpha - \alpha (\epsilon \sigma_m \xi)^\alpha \] (4.24)
The spin connection \( \omega_{mnl} \) can be evaluated from the vielbein associated with metric (4.9) and vanishes except for
\[ \omega_{ttz} = -\omega_{txz} = -\omega_{yyz} = e^{2A} \partial_z A \] (4.25)
The Killing spinors are determined by setting the variation of the gravitino to zero for each value of index \( m \). First consider the \( m = z \) case. The spin-connection term vanishes in this equation.
The solution is a spinor of the form

\[ \xi^\alpha = e^{-A/2} \eta^+ \alpha + e^{A/2} \eta^- \alpha \]  

(4.26)

where we define

\[ \eta^\pm = \eta \pm \sigma_3 \bar{\eta} \]  

(4.27)

and the general spinor \( \eta \) is an arbitrary function of \( t, x \) and \( y \). Setting the remaining components of the gravitino variation to zero fixes the form of \( \eta \). After some Dirac matrix manipulation, we find the general solution

\[ \xi = \left( e^{-A/2} \eta_0^+ \pm \frac{\alpha}{2} e^{A/2} x^i \sigma_i \eta_0^+ \right) + e^{A/2} \eta_0^- \]  

(4.28)

where \( i = \{ t, x, y \} \) and the - or + sign depends on whether one is to the right or the left of the membrane. As before

\[ \eta_0^\pm = \eta_0 \pm \sigma_3 \bar{\eta}_0 \]  

(4.29)

where now \( \eta_0 \) is an arbitrary constant spinor. Since \( \eta_0 \) has four real components, there are four independent Killing spinors described by this expression. We see that the solution naturally decomposes with respect to the two chiralities of \( \eta_0 \) given by (4.29). While both chiralities are Killing spinors of the anti-de Sitter space, it is not clear that both are compatible with the singular membrane surface at \( z = 0 \).

In fact only the two \( \eta_0^- \) chirality spinors are compatible. There are several ways to see this. The simplest, and the one we employ here, is to note that the corresponding Killing spinors, given by

\[ \xi = e^{A/2} \eta_0^- \]  

(4.30)

are continuous at the membrane surface. We can can conclude, therefore, that these Killing spinors are globally defined. For the two spinors of the opposite chirality, the corresponding Killing spinors are given by

\[ \eta = e^{-A/2} \eta_0^+ \pm \frac{\alpha}{2} e^{A/2} x^i \sigma_i \bar{\eta}_0^+ \]  

(4.31)

Note that the sign of the second term changes as one passes across the membrane surface. Thus the Killing spinors are discontinuous and are not compatible with the membrane source. We conclude that the membrane preserves precisely half of the supersymmetries and, hence, is a BPS solution. Of the four Killing spinors given above, only the chiral spinors (4.30) are global Killing spinors of the complete membrane solution.
5 Membrane $\kappa$-symmetry and the supergravity constraints

In the previous section, we demonstrated that the dual form of supergravity, in terms of the three-form potential, allowed a fundamental BPS membrane solution. In this section, we strengthen this link between membranes and the dual form of supergravity by considering the fundamental supersymmetric Green-Schwarz membrane. We show that the theory is only consistent in curved superspace if there exists a super three-form field satisfying exactly the constraints of the three-form supergravity given in section two. The consistency condition arises by insisting that the membrane action has a $\kappa$-symmetry, implying that half the fermionic degrees of freedom decouple on shell, and that the membrane consequently has the same number of fermionic as bosonic excitations. The argument was first given by Bergshoeff, Sezgin and Townsend in \[5\]. These authors presented a general formalism for $n$-extended objects in $d$-dimensional supergravity backgrounds. They argued that, for membranes, the only relevant supergravities are $N = 1$, $D = 11$ and $N = 2$, $D = 7$. Since the superspace formalism of $N = 2$, $D = 7$ is unknown, they concentrated on the case of a supermembrane in eleven dimensions. However, the $N = 1$, $D = 4$ three-form supergravity presented in this paper gives yet another context in which a supermembrane can couple to supergravity. In this section, we apply the formalism of \[5\] to this four-dimensional case.

The supersymmetric Green-Schwarz action for a membrane moving in curved superspace is given by

$$S = T_3 \int d^3\xi \sqrt{-\gamma} \left( \frac{1}{2} \gamma^{ij} E_i^a E_j^b \eta_{ab} + \epsilon^{ijk} E_i^A E_j^B E_k^C B_{CBA} \right)$$

The target space is now the full four-dimensional superspace so that the membrane coordinates are $Z^M = (X^m(\xi^i), \theta^a(\xi^i), \bar{\theta}^\dot{a}(\xi^i))$. The world volume metric on the membrane is $\gamma_{ij}$ while the functions $E_i^A$ are given by

$$E_i^A = \partial_i Z^M E_M^A$$

where $E_M^A$ are the usual inverse supervielbeins. The action also includes a Wess-Zumino term describing the coupling of the membrane to some, as yet unconstrained, super three-form field $B_{CBA}$. The bosonic part of this action was given in equation \(4.10\) in the previous section, with the $C = -6$, as we derived for the consistent membrane solution.

As written, the action \(5.1\) is spacetime supersymmetric by construction. However, to describe equal numbers of fermionic and bosonic degrees of freedom it must also have a $\kappa$-symmetry. The existence of this symmetry requires a non-zero Wess-Zumino term in \(5.1\). Thus the membrane theory only exists in a non-zero three-form background. Following \[5\], $\kappa$-symmetry implies that
the action is invariant under a variation of the form

\[
\begin{align*}
\delta E^a &= 0 \\
\delta E^\alpha &= \kappa^\alpha + \bar{\kappa}_\alpha \bar{\Gamma}^{\dot{\alpha}\alpha} \\
\delta E_{\dot{\alpha}} &= \bar{\kappa}_{\dot{\alpha}} + \kappa^{\dot{\alpha}} \Gamma_{\dot{\alpha}\dot{\alpha}} \\
\delta \gamma_{ij} &= \bar{\gamma}_{ij} X_{ij} - \gamma_{ij} X_k^k
\end{align*}
\]

where \( \delta E^A = \delta Z^M E_M^A \). Here \( X_{ij} \) is a function of the \( E_i^A \) and is linear in \( \kappa \), while the matrices \( \Gamma \) and \( \bar{\Gamma} \) are given by

\[
\begin{align*}
\Gamma^{\dot{\alpha}\alpha} &= \frac{1}{6} \epsilon^{ijk} E_i^a E_j^b E_k^c \epsilon_{abc} \sigma^{\dot{d}\alpha} \\
\Gamma_{\dot{\alpha}\dot{\alpha}} &= -\frac{1}{6} \epsilon^{ijk} E_i^a E_j^b E_k^c \epsilon_{abc} \sigma_{\dot{d}\dot{\alpha}}
\end{align*}
\]

In Dirac four-spinor notation we have

\[
\Gamma = \begin{pmatrix}
0 & \bar{\Gamma}^{\dot{\alpha}\beta} \\
\bar{\Gamma}_{\dot{\alpha}\beta} & 0
\end{pmatrix}
\]

Using the the \( \gamma_{ij} \) equation of motion \( \gamma_{ij} = E_i^a E_j^b \eta_{ab} \) it is easy to show that \( 1 + \Gamma \) is a projector on Dirac spinors since \( \Gamma^2 = 1 \). One notes that, for a flat membrane, \( 1 + \Gamma \) is the same projector that appears in the Killing spinors of membrane solution given in the previous section. In static gauge a membrane in the plane \( z = 0 \) has \( T = \sigma^0 \), \( X = \sigma^1 \) and \( Y = \sigma^2 \), so that \( \Gamma_{\dot{\alpha}\dot{\alpha}} = \sigma_{\dot{\alpha}\dot{\alpha}} \). This gives \( \delta E_\alpha = (\kappa - \sigma^3 \bar{\kappa})^\alpha \) which is the same projection used for the preserved Killing spinors \((4.30)\). This is to be expected: it reflects the usual realization of partially broken supersymmetry in the flat membrane world-volume theory \[27\]. It is also the choice of chiral projector \( 1 \pm \Gamma \) in the \( \kappa \)-symmetry which fixes the sign of the \( B_{CBA} \) coupling \( C = \mp 6 \) discussed in the previous section.

For general supervielbein \( E_M^A \) and three-form potential \( B_{CBA} \), variations of the form \((5.3)\) do not leave the action \((5.1)\) invariant. Requiring invariance imposes conditions on the derivatives of the supervielbein and three-form potential. In particular, following \[2\], we find that certain components of the torsion, constructed from \( E_M^A \), are constrained to be

\[
\begin{align*}
T_{\alpha\beta}^c &= T_{\dot{\alpha}\dot{\beta}}^c = 0, \\
T_{\alpha\dot{\alpha}}^a &= T_{\dot{\alpha}\alpha}^a = 2i A \sigma^a_{\alpha\dot{\alpha}} \\
\eta_{c(\alpha} T_{\beta)}^a &= \eta_{ab} \Lambda_\alpha, \\
\eta_{c(\alpha} T_{\beta)}^c &= \eta_{ab} \bar{\Lambda}_{\dot{\alpha}}
\end{align*}
\]

where \( A \) is an arbitrary real number, while \( \Lambda_\alpha \) is any spinor superfield. Similarly, the components
of the four-form field strength constructed from \( B_{CBA} \) are also constrained. These constraints are

\[
\Sigma_{\delta \gamma A} = \Sigma_{\delta A} = 0 \quad (5.8)
\]

\[
\Sigma_{\delta \gamma ba} = -\frac{2}{3} A (\sigma_{ba} \epsilon) \delta \gamma \quad (5.9)
\]

\[
\Sigma_{\delta cba} = \frac{i}{2} \tilde{A} \delta \epsilon_{dcba} \quad \Sigma_{\delta \epsilon_{cba}} = -\frac{i}{2} \tilde{A} \delta \epsilon_{dcba} \quad (5.10)
\]

Let us now compare these constraints with those used in section three to derive the three-form version of supergravity, as given in (3.41) and (3.42). The conditions are the same if \( A = 1 \) and the superfield \( \Lambda_\alpha \) is zero, except that the torsion is further constrained in the supergravity. In particular, we are missing the constraints

\[
T_{\alpha \beta \gamma} = \eta_{\epsilon \delta} T_{\beta \gamma \delta} = 0 \quad (5.11)
\]

The gravitational multiplet is described in terms of the supervielbein \( E^A \) and a superconnection \( \phi_A \). However there is in general some ambiguity in how these are defined. We note first that \( A \) can be set to unity by a simple normalization condition on the vielbein. Furthermore, several of the missing constraints in (5.11) are conventional, for instance \( T_{\alpha \beta \gamma} = 0 \). That is to say, they can be imposed simply by redefining \( \phi_A \). Finally, one notes that the Green-Schwarz action (5.1) does not fix the definition of the bosonic and fermionic parts of \( E^A \). In general we can redefine

\[
E'^a = E^b L^a_b \\
E'^\alpha = E^b L^\alpha_b + E^\beta L^\alpha_\beta + E_\beta L^{\beta \hat{\alpha}} \\
E'_\hat{\alpha} = E^b L_b \hat{\alpha} + E^\beta L_{\beta \hat{\alpha}} + E_{\hat{\beta}} L^{\hat{\beta} \hat{\alpha}}
\]

and leave the form of the action invariant. Here \( L^a_b \) is the usual Lorentz transformation, but the other matrices correspond to a general linear transformation. As first described in the context of the string in ten dimensions \[28\], these transformations are in fact, sufficient to set \( \Lambda_\alpha \) and the remaining missing torsion constraints to zero. Thus, up to this redefinition, the constraints of \( \kappa \)-symmetry for the membrane action are exactly equivalent to those used to define \( N = 1 \) three-form supergravity. One notes that, with \( A = 1 \), it is precisely the \( M_2 \) auxiliary field which is dualized, and no other linear combination of \( M_1 \) and \( M_2 \).

Rather than give the details of the redefinitions, let us demonstrate that \( \Lambda_\alpha \) can consistently be taken to zero. Assume all the usual torsion constraints are satisfied, including \( \Lambda_\alpha = 0 \) and \( A = 1 \). Comparing (5.3) with (3.13) and (3.14) we see that the superfield \( Y \) is equal to \(-4/3\). Furthermore, solving the Bianchi identities implies that \( \Sigma_{\epsilon \delta \gamma \delta} \sim \tilde{D}^\epsilon \hat{Y} \) and \( \Sigma_{\epsilon \delta \gamma} \sim \tilde{D}_\alpha Y \) (from (3.13) and (3.14)). Since \( Y \) is constant these components are zero, and so, by (5.10), \( \Lambda_\alpha = 0 \), consistent with the original assumption in the torsion constraints.
We see that imposing $\kappa$-symmetry on the Green-Schwarz membrane action implies a set of torsion and four-form constraints that are exactly those used to define the three-form version of supergravity. We are forced to have a non-zero $B$. The membrane does not couple consistently to old-minimal supergravity, but instead couples only to the three-form version of minimal supergravity derived in this paper.

6 The Extended Membrane Superalgebra

It is well known that $p$-brane states in flat space are representations of an extended superalgebra. The extension is by a topological tensor central charge corresponding to the form-field charge of the $p$-brane state $[10]$. In the case of a membrane in flat four-dimensional space, we would expect the algebra to have the form

\[
\begin{align*}
\{Q_\alpha, Q_\dot{\alpha}\} &= 2\sigma^m\, \sigma^{\dot{\alpha}} P_m \\
\{Q_\alpha, Q_\beta\} &= 2i (\sigma^{mn})_{\alpha\beta} Z_{mn} \\
\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} &= 2i (\sigma^{mn})_{\dot{\alpha}\dot{\beta}} Z_{mn}
\end{align*}
\] (6.1)

where $Z_{mn}$ is the conserved charge for the three-form potential $C_{mnp}$.

Normally the extended algebra is derived in one of two ways. Either one looks at the supersymmetry algebra in the world-volume theory of a fundamental Green-Schwarz $p$-brane $[11]$, or one looks directly at the algebra of charges for a $p$-brane solitonic supergravity solution, as in, for instance $[8,9]$. In each case the extension appears as a topological charge.

In this section we will show that the algebra can also be derived simply as the algebra of symmetries which leave the flat-space supergravity background, including the three-form $B$, invariant. It will be crucial in this context that the constraints on the field strength $\Sigma$ imply that even in flat space there is a non-trivial potential $B$.

Let us start by reconsidering the algebra of general local invariances of the supergravity derived in section three. The theory has three quite distinct symmetries: general superdiffeomorphic invariance, local Lorentz symmetry and symmetry under local gauge transformations of the super three-form potential $B$. Superdiffeomorphisms are characterized by super Killing vectors $\xi^A$, while gauge transformations are parameterized by a super two-form $\Lambda_{MN}$. The commutator of two diffeomorphisms, $\xi^A$ and $\xi'^A$, simply leads to a third diffeomorphism $\xi''^A$ given by the super-Lie bracket of $\xi^A$ and $\xi'^A$. It is easy to show that the commutator of a superdiffeomorphism $\xi^A$ with a gauge transformation $\Lambda$ is a second gauge transformation with a parameter

\[
\Lambda' = i \xi d\Lambda + d i \xi \Lambda
\] (6.2)
The commutator of two gauge transformations is, of course, zero. It is always possible to define transformations as combinations of diffeomorphisms and gauge transformations. The supersymmetry transformations in WZ gauge defined in section three are such an example. Such transformations will have a more complicated algebra, but this is an artifact of their definition. The underlying algebra is always superdiffeomorphisms together with gauge transformations.

We now want to consider a specific solution of the supergravity theory, namely flat space, and ask which of the local symmetries leave the solution invariant. From the bosonic action (4.8) we see that flat space is a solution of three-form supergravity with \( F_{mnpq} = 0 \). The constraints on the other components of \( \Sigma \) (3.42) imply that the super four-form completely vanishes except for the components

\[
\Sigma_{\delta \gamma ba} = -\frac{2}{3} (\sigma_{ba} \epsilon)_{\delta \gamma} \quad \Sigma^{\dot{\delta} \dot{\gamma}}_{ba} = -\frac{2}{3} (\bar{\sigma}_{ba} \epsilon)^{\dot{\delta} \dot{\gamma}}
\]

Thus, the flat-space solution in fact corresponds to a constant super four-form background.

If we ignore the four-form, we know that the symmetries of flat superspace are simply ordinary translations and Lorentz rotations, and supertranslations, which are just the flat-space supersymmetry transformations. Together these form the usual super-Poincaré algebra. From the form of the flat-space field strength (6.3) it is clear that these symmetries also leave \( \Sigma \) invariant. We are also free to make local gauge transformations. Thus, requiring \( \Sigma \) to be invariant, the symmetries of flat space are simply the usual unextended super-Poincaré transformations together with local gauge transformations.

We now ask a more restrictive question: what is the algebra of symmetries of flat space which leave the super three-form potential itself invariant? We will argue shortly, that this is the algebra relevant to matter which carries form-field charge. Given that \( \Sigma \) is not zero, clearly the potential \( B \) is also non-zero. We are in flat space, so can use the discussion of section two to find an expression for \( B \). The constraints on \( \Sigma \) imply that the components of the chiral superfield \( Y \) are set to \( Y = -4/3 \) and \( \eta_\alpha = H = 0 \) (see (3.36) and (3.37)). Since here \( F_{mnpq} = 0 \), we can also choose a gauge where \( C_{mnp} = 0 \). Comparing with equation (2.31), we see that there is a gauge where \( \Omega \) takes the form

\[
\Omega = -\frac{1}{12} (\theta^2 + \bar{\theta}^2)
\]

Equations (2.17)–(2.21) imply that all the components of \( B \) vanish except for

\[
B_{\alpha \beta a} = \frac{i}{6} (\sigma_{ba})_{\alpha}^\beta (\sigma_a \epsilon)^{\dot{\beta} \dot{\gamma}} \\
B_{aba} = -\frac{1}{3} (\sigma_{ba})_{\alpha}^\beta \theta_\beta \\
B_{\dot{a} \dot{b} a} = \frac{1}{3} (\bar{\sigma}_{ba})_{\alpha}^{\dot{\beta}} \bar{\theta}_{\dot{\beta}}
\]

(6.5)
That $B$ does not vanish is to be expected, since we know that even in flat space, $\kappa$-symmetry requires a non-zero Wess-Zumino term in the Green-Schwarz membrane action. This non-trivial term is none other than the coupling of the membrane to a background $B$ field.

We note first that, in this gauge, $B$ is invariant under normal translations. However it is not invariant under supertranslations. This is easily seen from (6.5), which clearly changes under $\theta^\alpha \rightarrow \theta^\alpha + \xi^\alpha$. However we know that the field strength $\Sigma = dB$ is invariant under supertranslations. This implies that the variation of $B$ must simply be a gauge transformation. That is, it is possible to write it in the form

$$\delta_\xi B = d\Lambda_\xi$$

where $\Lambda_\xi$ is a particular two-form dependent on $\xi^\alpha$. Clearly if we want to leave $B$ invariant we must modify the supertranslation transformation. We define a new variation which combines the supertranslation with a gauge transformation,

$$\delta^{\text{new}}_\xi B = \delta_\xi B + \delta_{\Lambda_\xi} B$$

where $\delta_{\Lambda_\xi} B = -d\Lambda_\xi$, so that we have $\delta^{\text{new}}_\xi B = 0$ as required. The only remaining transformations which preserve $B$ are a trivial subset of the general gauge transformations, for which $\delta_\Lambda B = dB = 0$.

What is the algebra of these symmetries? The easiest way to find out is to use the supersymmetry transformations given in section two. We recall that the variation (2.6) did not correspond to simple supertranslation but included a gauge transformation in order to preserve the WZ gauge. In fact, it is exactly the modified transformation we defined above (6.7) in order to leave $B$ invariant. To see this, recall that the three-form in the supergravity theory was constrained so that $Y = -4/3$ and $\eta_\alpha = H = 0$ (see (3.36) and (3.37)). Thus in WZ gauge the only surviving component of $B$ is $C_{mnp}$. Its variation is given by (2.6), which vanishes since $\eta_\alpha = 0$. Thus the WZ-gauge supersymmetry variations of section two leave $B$ invariant, and correspond exactly to the modified supertranslation (6.7). The commutator of two of the modified supertranslations can be taken directly from (2.37),

$$\left(\delta_\delta' - \delta' \delta\right) C_{mnp} = 2\eta^q \partial_q C_{mnp} + 3\partial_{[m} \Lambda_{np]}$$

where we have dropped the label ‘new’ and

$$\Lambda_{mn} = -2C_{mnp} \eta^p - \frac{1}{2} \left(\eta^q_{p[q} Y_1 + \eta^q_{p[q} Y_2\right)$$

with

$$\eta^m = -i \left(\xi^{m\prime} \xi^\prime - \xi^m \xi^\prime\right)$$

$$\eta^+_{pq} = \xi_{pq} \xi^\prime + \bar{\xi}_{pq} \bar{\xi}^\prime$$

$$\eta^-_{pq} = -i \left(\xi_{pq} \xi^\prime - \bar{\xi}_{pq} \bar{\xi}^\prime\right)$$

(6.10)
Recalling that $Y = Y_1 + iY_2 = -4/3$ and that we have chosen the gauge $C_{mnp} = 0$, we find the gauge transformation parameter simplifies to $\Lambda_{mn} = \frac{2}{3} \eta^+_{mn}$, so that

$$
(\delta \delta' - \delta' \delta) C_{npq} = 2 \eta^r \partial_r C_{npq} + 2 \partial_{[n} \eta^+_{pq]}.
$$

(6.11)

Thus the commutator of two of the modified supertranslations closes into an ordinary translation, parameterized by $\eta^m$, together with a gauge transformation, parameterized by $\eta^+_{mn}$. This can now be compared with the extended algebra (6.1). Given the form of $\eta^r$ and $\eta^+_{pq}$ we see that, up to a normalization of the charge $Z_{mn}$, the two are identical. The extended superalgebra is none other than the algebra of the symmetries of flat space which leave $B$ invariant.

Note that we derived the extended algebra in a particular gauge. However, we would have in fact obtained the same form whatever gauge we had started in. The only modification would be in the definition of the modified translation and supertranslation symmetries. Suppose, for instance, we chose a gauge where $C_{mnp}$ was non-zero. Under an ordinary translation

$$
\delta \eta^r \partial_r C_{mnp} \neq 0
$$

(6.12)

Thus we must redefine a modified translation symmetry which includes a gauge transformation, such that

$$
\delta \new \eta^r \partial_r C_{mnp} = \eta^r \partial_r C_{mnp} - 6 \partial_{[m} (\eta^r C_{np]r}) = \eta^r \Sigma_{rmnp} = 0
$$

(6.13)

and so $C_{mnp}$ is invariant. Using this modified symmetry, one then finds the same extended algebra. It turns out that there is a gauge-invariant property of $\Sigma$ which is leading to the extension of the algebra [29]. The point is that while the field strength $\Sigma$ was invariant under the super-Poincaré group, it was impossible to find a gauge where the potential $B$ was also invariant. Formally, $\Sigma$ is an element of the fourth Chevalley-Eilenberg cohomology class of the super-translation group, where flat space is considered as the group manifold. The central extension of the algebra is then the analog of the usual Abelian extension of a Lie algebra with a non-trivial second cohomology class ([30] and references therein).

The question remains as to why the algebra of the symmetries of flat-space which leave $B$ invariant should correspond to the topologically-extended algebra which describes membrane states. We start by noting that a membrane is the archetypical object which carries three-form charge. Whether it is fundamental or a solitonic solution, a membrane in a flat background will be effectively described by a Green-Schwarz action. It will couple to the three-form via the Wess-Zumino term

$$
S_{WZ} = \int d^3 \sigma \sqrt{-\gamma} \epsilon^{ijk} E_i^A E_j^B E_k^C B_{CBA}
$$

(6.14)
Under what symmetries is this term invariant? The projected vielbeins $E_i^A = \partial_i Z^M E_M^A$ are invariant under the usual translations and supertranslations by construction, and also trivially under any gauge transformations. But the three-form $B$ is only invariant under the set of modified transformations we described above. Thus the symmetry algebra of the Green-Schwarz action is exactly the algebra of the symmetries which leave $B$ invariant.

This also allows us to make the connection to the calculation of the extended algebra given in [10]. There it was noted that the WZ term was invariant under supertranslations only up to a total divergence. In the present context we can understand this result as a reflection of (6.6). Namely the variation of $B$ under a supertranslation is just a gauge transformation. It is clear from the form of (6.14) that a gauge transformation in $B$ leads to a total divergence. The authors of [10] defined a conserved current by subtracting the total divergence term from the conventional supertranslation current. But this is exactly the current one would define for a transformation which included a compensating gauge transformation together with the supertranslation. That is, it is just the current corresponding to the modified supertranslation (6.7).

We recall from the solution of section four, that the actual extended membrane does not live in asymptotically flat space in four dimensions. Since there is no fall off in the gauge field away from the membrane, the asymptotic space is actually anti-de Sitter. Given this, one might ask how the supersymmetry algebra is extended in a anti-de Sitter background. Calculating the algebra from the solution itself is problematic since it is not clear how to define the asymptotic charges. The patching of two pieces of anti-de Sitter space means that there is no clear sense in which the membrane is a perturbation about an anti-de Sitter background. However, we can ask what symmetries leave the Wess-Zumino term in the membrane action invariant. That is, find the algebra of symmetries that leave $B$ invariant, as we did in the flat case.

The fermionic part of the anti-de Sitter superalgebra is given by

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}}J_{m4}$$

$$\{Q_\alpha, Q_\beta\} = 2i(\sigma^{mn})_{\alpha\beta}J_{mn} \quad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 2i(\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}J_{mn}$$

$$(6.15)$$

$$[J_{m4}, Q_\alpha] = \frac{i}{4}\sigma^m_{\alpha\dot{\alpha}}\bar{Q}_{\dot{\alpha}} \quad [J_{m4}, \bar{Q}_{\dot{\alpha}}] = \frac{i}{4}\bar{\sigma}^m_{\dot{\alpha}\dot{\beta}}Q_{\beta}$$

$$[J^{mn}, Q_\alpha] = -\frac{1}{2}(\sigma^{mn})_{\alpha\beta}Q_\beta \quad [J^{mn}, \bar{Q}_{\dot{\alpha}}] = -\frac{1}{2}(\bar{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\beta}}$$

where $J_{mn}$, $J^{m4}$ are the generators of the $SO(3,2)$ group of isometries of anti-de Sitter space. As in the previous example we can use the supersymmetry variations from section three to calculate the algebra, since these leave the three-form $B$ invariant. The full calculation requires explicit forms for the Killing vectors and spinors in anti-de Sitter space and we will not give the details here. The result is simply that there is no extension of the algebra. A membrane state in anti-de Sitter space is
just a representation of the usual anti-de Sitter supersymmetry algebra (6.15). In fact, oscillations of the flat membrane have been identified with the singleton representation [31]. While it can carry three-form charge, this charge does not enter the supersymmetry algebra. Algebraically, this is because there is no corresponding central extension of the super-anti-de Sitter algebra possible. The extension must be central because gauge transformations commute, and no such extension by a two-form $Z_{mn}$ exists.

References

[1] A. Achúcarro, J. M. Evans, P. K. Townsend and D. L. Wiltshire, Phys. Lett. B198B 441 (1987)

[2] A. Aurilia, H. Nicolai and P. K. Townsend, Nucl. Phys. B176 509 (1980)

[3] M. Henneaux and L Mezincescu, Phys. Lett. 152B 340 (1985)

[4] J. Hughes, J. Liu and J. Polchinski, Phys. Lett. B180B 370 (1986)

[5] E. Bergshoeff, E. Sezgin and P. K. Townsend, Phys. Lett. B189 75 (1987)

[6] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, Phys. Lett. 191B 70 (1987)

[7] M. J. Duff, R. R. Khuri, J. X. Lu, Phys. Rept. 259 213 (1995)

[8] A. Achúcarro, J. P. Gauntlett, K. Itoh, and P. K. Townsend Nucl. Phys. B314 129 (1989)

[9] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz Nucl. Phys. B340 33 (1990)

[10] J. A. de Azcárraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, Phys. Rev. Lett. 63 2443 (1989)

[11] E. Bergshoeff, M. de Roo, M. B. Green, G.D Papadopoulos, P. K. Townsend, Nucl. Phys. B470 113 (1996)

[12] L. J. Romans, Phys. Lett. B169 374 (1986)

[13] M. Cvetic and H. H. Soleng, ‘Supergravity Domain Walls’, IASSNS-HEP-96-25, hep-th/9604090

[14] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, Phys. Lett. B371 46 (1996)

[15] H. Lu, C. N, Pope and P. K. Townsend, Phys. Lett. B391 39 (1997)
[16] P. M. Cowdall, H. Lu, C. N. Pope, K. S. Stelle and P. K. Townsend, *Nucl. Phys.* **B486** 49 (1997)

[17] S. J. Gates, Jr., *Nucl. Phys.* **B184** 381 (1981)

[18] K. S. Stelle and P. C. West, *Phys. Lett.* **74B** 330 (1978)

[19] V. I. Ogievetskii and E. S. Sokatchev, *Yad. Fiz.* **32** 1142 (1980) [Sov. J. Nucl. Phys. **32** 589 (1980)]

[20] P. Binetruy, F. Pillon, G. Girardi and R. Grimm *Nucl. Phys.* **B477** 175 (1996)

[21] M. Müller, *Nucl. Phys.* **B264** 292 (1986)

[22] J. Wess and J. Bagger, *Supersymmetry and supergravity*, 2nd ed., Princeton University Press, Princeton, 1992

[23] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel *Superspace or one thousand and one lessons in supersymmetry*, Benjamin/Cummings, Reading, 1983

[24] M. J. Duff and P. van Nieuwenhuizen, *Phys. Lett.* **94B** 179 (1980)

[25] M. Henneaux and C. Teitelboim, *Phys. Lett.* **143B** 415 1984

[26] G. W. Gibbons, *Nucl. Phys.* **B394** 3 (1993)

[27] J. Hughes and J. Polchinski, *Nucl. Phys.* **B278** 147 (1986)

[28] J. A. Shapiro and C. C. Taylor *Phys. Lett.* **186B** 69 (1987)

J. A. Shapiro and C. C. Taylor, *Phys. Lett.* **181B** 67 (1986)

[29] J. A. De Azcárraga and P. K. Townsend, *Phys. Rev. Lett.* **62** 2579 (1989)

[30] V. Aldaya and J. A. de Azcarraga, *J. Math. Phys.* **26** 1818 (1985)

[31] E. Bergshoeff, M. J. Duff, C. N. Pope and E. Sezgin, *Phys. Lett.* **199B** 69 (1987)