BOUNDENESS OF SOLUTIONS TO FRACTIONAL LAPLACIAN GINZBURG-LANDAU EQUATION

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Abstract. In this paper, we give the boundeness of solutions to Fractional Laplacian Ginzburg-Landau equation, which extends the Brezis theorem into the nonlinear Fractional Laplacian equation. A related linear fractional Schrodinger equation is also studied.

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1. Introduction

In this paper, we continue our study of nonlocal nonlinear elliptic problem with the fractional Laplacian [7]. We give the boundeness of solutions to Fractional Laplacian Ginzburg-Landau equation, which extends the Brezis theorem [1] [8] [9] [10] into the nonlinear Fractional Laplacian equation. The proof of our result depends on a Liouville type theorem for $L^p$ non-negative solutions to a nonlinear fractional Laplacian inequality.

We begin with the definition of fractional Laplacian on $\mathbb{R}^n$. Let $0 < \alpha < 2$. Following [2] we define

$$E = C^{1,1}_{loc}(\mathbb{R}^n) \cap L_\alpha,$$

where

$$L_\alpha = \{u \in L^1_{loc}(\mathbb{R}^n); \int_{\mathbb{R}^n} \frac{|u(x)|dx}{1 + |x|^{n+\alpha}} < \infty \}.$$

For $u \in E$, we define the fractional Laplacian operator $(-\Delta)^{\alpha/2}$ by

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy,$$

where $C_{n,\alpha}$ is the uniform constant [6]. The fractional Ginzburg-Landau equation is

$$(1) \quad (-\Delta)^{\alpha/2}u = u(1 - |u|^2), \quad \text{in} \quad \mathbb{R}^n.$$
We consider the physical meaningful solutions and our main result is below.

**Theorem 1.** Let \( u \in E \) is a solution to (1) such that

\[
\int_{\mathbb{R}^n} (1 - |u|^2)^2 dx < \infty.
\]

Then, we have

\[ |u(x)| \leq 1, \quad \text{in } \mathbb{R}^n \]

This is an extension of Brezis theorem [10] about the Ginzburg-Landau equation/system. The result is also true for the corresponding vector-valued solution \( u : \mathbb{R}^n \to \mathbb{R}^N \). It is quiet possible to remove the condition \( 1 - u^2 \in L^2 \). However, we can give an example of linear fractional Schrodinger equation, which shows that the behavior of solutions to linear equation is also very subtle.

Assume \( k(x) \geq 0 \) is non-negative smooth function on \( \mathbb{R}^n \). Let \( g_\alpha(x,y) = C_{n,-\alpha} \frac{1}{|x-y|^\alpha} \). We also consider non-negative solutions to the following linear fractional Laplacian equation

(2) \[ (-\Delta)^{\alpha/2} u + k(x) u = 0, \quad \text{in } \mathbb{R}^n. \]

We have the below

**Theorem 2.** Assume \( k(x) \geq 0 \) is a nontrivial non-negative smooth function on \( \mathbb{R}^n \). Assume that for each \( x \in \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} g_\alpha(x,y)k(x)dy < \infty.
\]

Then there is a non-trivial non-negative solution to (2).

One remark is given now. We actually only need to assume (2) is true at some point \( x \).

The plan of this note is below. The proof of Theorem 2 is given in section 2 and Theorem 1 is proven in section 3.

### 2. Linear Equation

We now prove Theorem 2. Assume (2). We let, for each \( x \in \mathbb{R}^n \),

\[
V(x) = \int_{\mathbb{R}^n} g_\alpha(x,y)k(x)dy < \infty.
\]

Then \( V(x) \) is the minimum non-negative solution to the Poisson equation

\[ (-\Delta)^{\alpha/2} u = k(x), \quad \text{in } \mathbb{R}^n. \]
and we have $\inf_{R^n} V(0) = 0$. Let $R^n = \bigcup_{j \geq 1} B_j(0)$ be a ball exhaustion of $R^n$. We denote by $B_j = B_j(0)$. We solve $u_j(x) \geq 0$ such that

$$(-\Delta)^{\alpha/2} u_j + k(x)u_j = 0, \quad \text{in } B_j$$

with the boundary condition $u_j = 1$ on $B_j^c = R^n - B_j$. By the Maximum principle [3] [4] we have $0 \leq u_j(x) \leq 1$ on $R^n$. This solution can be obtained by the variation method or the monotone method. By the comparison lemma we know that $(u_j(x))$ is monotone non-increasing sequence and we may let

$$U(x) = \lim_{j \to \infty} u_j(x).$$

Note that $0 \leq U(x) \leq 1$ on $R^n$. We now show that $U$ is non-trivial. Let $\tilde{u}_j = 1 - u_j$. Then

$$(-\Delta)^{\alpha/2} \tilde{u}_j = k(x)u_j(x) \leq k(x), \quad \text{in } B_j$$

and $\tilde{u}_j(x) = 0$ on $B_j^c$. By the Maximum principle we have $\tilde{u}_j(x) \leq V(x)$ on $R^n$. Passing to limit we have

$$1 - U(x) \leq V(x), \quad \text{on } R^n.$$ 

Since $\inf V(x) = 0$, we know that $U(x)$ is a non-trivial non-negative solution to (2). This completes the proof of Theorem 2.

3. Proof of Theorem 1

Recall that we have Kato’s inequality of the form [5]

$$(-\Delta)^{\alpha/2} |f(x)| \leq \text{sgn}(f)(-\Delta)^{\alpha/2} f(x), \quad \text{a.e. } R^n.$$ 

By this we have for any $f \in E$, we have

$$(-\Delta)^{\alpha/2} f_+(x) \leq \text{sgn}(f_+)(-\Delta)^{\alpha/2} f(x), \quad \text{a.e. } R^n,$$

where $f_+(x) = \sup(f(x), 0)$.

Let $u \in E$ be a solution to (1) such that

$$\int_{R^n} (1 - |u|^2)^2 \, dx < \infty.$$ 

Let

$$Q(x) = |u(x)|^2 - 1.$$ 

Note that

$$(-\Delta)^{\alpha/2} u^2(x) = C_{n,\alpha} \int_{R^n} \frac{u^2(x) - u^2(y)}{|x - y|^{n+\alpha}} \, dy$$

$$= 2u(x)(-\Delta)^{\alpha/2} u(x) - C_{n,\alpha} \int_{R^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} \, dy.$$ 

Then

$$(-\Delta)^{\alpha/2} u^2(x) \leq 2u(x)(-\Delta)^{\alpha/2} u(x).$$
By the equation (1) we have
\((-\Delta)^{\alpha/2} u^2(x) \leq -2u^2(x)Q(x) = -2Q^2(x) - 2Q(x)\).
That implies that
\((-\Delta)^{\alpha/2} Q(x) \leq -2Q^2(x) - 2Q(x)\).
Using the Kato inequality above we have
\((-\Delta)^{\alpha/2} Q_+(x) \leq -2Q_+^2(x)\).
Invoking lemma 3 below (with \(q = 2\)) we can conclude that \(Q_+(x) = 0\) on \(\mathbb{R}^n\), which implies Theorem 1.

**Lemma 3.** Let \(1 \leq r < \infty\). Assume that \(0 \leq f \in L^q(\mathbb{R}^n)\) for some \(1 \leq q < \infty\) such that
\((-\Delta)^{\alpha/2} f + f^r \leq 0, \text{ in } \mathbb{R}^n,\)
in the distributional sense, i.e.,
\(\int_{\mathbb{R}^n} f(-\Delta)^{\alpha/2} v + \int_{\mathbb{R}^n} f^r v \leq 0,\)
for any \(v \in C^\infty_0(\mathbb{R}^n)\) with \(v \geq 0\), then we have \(f = 0\) on \(\mathbb{R}^n\).

**Proof.** Let \(\xi(x) \in C^{1,1}(B_2(0))\) be the cut-off function such that \(\xi(x) = 1\) on \(B_1(0)\). For any \(R > 1\), let \(\xi_R(x) = \xi(x/R)\). Then (3) implies that
\(\int_{\mathbb{R}^n} f(-\Delta)^{\alpha/2} v + \xi_R(x) \int_{\mathbb{R}^n} f^r v \leq 0,\)
for any \(v \in C^\infty_0(\mathbb{R}^n)\) with \(v \geq 0\).
Define
\[\phi(x) = \int_{\mathbb{R}^n} g_\alpha(x,y)\xi_R(x)dy.\]
Then \(0 \leq \phi(x) \leq C\) for some uniform constant \(C > 0\), \(\phi(x) \leq C|x|^{\alpha-n}\) at infinity and
\((-\Delta)^{\alpha/2} \phi(x) = \xi_R(x), \text{ on } \mathbb{R}^n.\)
Define, for any \(p > 1,\)
\[W^{\alpha,p} = \{f \in L^p, (-\Delta)^{\alpha/2} f \in L^p \cap L^\infty\}.\]
Then \(C^\infty_0(\mathbb{R}^n)\) is dense in \(W^{\alpha,p}\). If \(q = 1\), we choose any \(p > 1\). If \(q > 1\), we let \(p = \frac{q}{q-1}\). By passing to limit, we can take the test function \(v \in W^{\alpha,p}\) for the inequality (3). In particular, we may let \(v = \phi\) and we have
\(\int_{\mathbb{R}^n} f\xi_R(x) + \xi_R(x)f^r \phi(x) \leq 0.\)
Note that each term in the integration is non-negative. Then we have
\(f\xi_R(x) = 0, \text{ a.e. } \mathbb{R}^n.\)
Since $R > 1$ is arbitrary, we have $f(x) = 0$ a.e. in $\mathbb{R}^n$. □

References

[1] H. Brezis, Comments on Two Notes by L. Ma and X. Xu, C. R. Math. Acad. Sci. Paris 349 (2011), no. 5-6, 269-271
[2] L. Caffarelli, L. Silvestre, regularity Theory for fully non-linear integro-Differential Equations, Comm. Pure Appl. Math. Vol. LXXII 0597-0638 (2009).
[3] W. Chen, Y. Fang, and R. Yang, Liouville theorems involving the fractional Laplacian on a half space, Advances in Math. in press, 2014.
[4] W. Chen and C. Li, An integral system and the Lane-Emden conjecture, Disc. Cont. Dynamics Sys. 4 (2009), no. 24, 1167-1184.
[5] RL. Frank, E. Lenzmann, and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, arXiv: 1302.2652v1, 2013.
[6] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag Berlin Heidelberg, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
[7] Li Ma, On nonlocal nonlinear elliptic problem with the fractional Laplacian, arxiv.org, 2015
[8] Li Ma, Liouville type theorem and uniform bound for the Lichnerowicz equation and the Ginzburg-Landau equation, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 993-996
[9] Li Ma, Liouville Type Theorems for Lichnerowicz Equations and Ginzburg-Landau Equation: Survey, Advances in Pure Mathematics, 2011, 1, 99-104
[10] Li Ma, Xingwang Xu, uniform bound and a non-existence result for Lichnerowicz equation in the whole n-space, C.R.Mathematique, ser.I,347(2009)805-808

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