Going Up and Lying Over in Congruence–modular Algebras

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May 23, 2018

Abstract

In this paper, we extend properties Going Up and Lying Over from ring theory to the general setting of congruence–modular equational classes, using the notion of prime congruence defined through the commutator. We show how these two properties relate to each other, prove that they are preserved by finite direct products and quotients and provide algebraic and topological characterizations for them. We also point out many kinds of varieties in which these properties always hold.

2010 Mathematics Subject Classification: Primary: 08A30; secondary: 08B10, 03G10, 06F35.

Keywords: congruence–modular varieties, commutator, prime congruence, Going Up, Lying Over.

1 Introduction

Properties Going Up (GU) and Lying Over (LO) reflect the behaviour of commutative ring extensions with respect to finite chains of prime ideals. An extension of commutative rings $A \subseteq B$ fulfills GU iff, for any prime ideals $P, Q$ of $A$ and $P'$ of $B$, if $P \subseteq Q$ and $P' \cap A = P$, then there exists a prime ideal $Q'$ of $B$ such that $P' \subseteq Q'$ and $Q' \cap A = Q$; the extension $A \subseteq B$ fulfills LO iff, for any prime ideal $P$ of $A$, there exists a prime ideal $P'$ of $B$ such that $P' \cap A = P$. These two conditions have been generalized from ring embeddings to arbitrary morphisms of commutative rings: a morphism $f : A \rightarrow B$ between two commutative rings shall fulfill GU, respectively LO, iff the extension $f(A) \subseteq B$ fulfills GU, respectively LO. By the Cohen–Seidenberg Theorem [3, Theorem 5.11], integral ring extensions fulfill GU and LO.

GU and LO–type conditions have been studied for the prime ideals of some algebraic structures related to logic: bounded distributive lattices [21, p. 773], MV–algebras [5] and BL–algebras [24]. By [5], respectively [24], any MV-algebra, respectively BL–algebra morphism fulfills GU and LO.

In the present paper, we study properties GU and LO for morphisms in certain kinds of varieties of universal algebras, relating them to congruences instead of ideals. In order to define GU and LO in this general setting, we need a notion of prime congruence; we have chosen the prime congruences introduced through the notion of commutator, which can be defined in congruence–modular varieties [10, p. 82]. [1] shows that the prime spectra of algebras in semi–degenerate congruence–modular varieties have rich enough properties for developing an interesting mathematical theory concerning GU and LO.

While the inverse images of prime ideals through morphisms of commutative rings, bounded distributive lattices, MV–algebras and BL–algebras are again prime ideals, the same does not go for prime congruences in algebras from congruence–modular varieties, in general, and, since this property makes the theory of conditions GU and LO work for these particular kinds of algebras, we have had to restrict our research to morphisms that fulfill this property for prime congruences, which we have called admissible morphisms. However, in many kinds of varieties, all morphisms are admissible; we list such varieties in the final section of this paper.

In Section 2 of our paper, we recall some notions from universal algebra and commutator theory, including properties of prime congruences. The results in the following sections are new and original, excepting those that we cite from other papers.

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Section 3 has an introductory purpose; here we present the notion of admissible morphism, along with some properties of this kind of morphisms, most of which we cite from [22]. We also give some examples.

In Section 4, we define properties GU and LO for admissible morphisms in congruence–modular varieties, we provide examples, along with some simple characterizations that we need in what follows, and obtain the main results on these properties, such as the fact that their study can be reduced to embeddings, that surjectivity implies GU, GU implies LO in semi–degenerate congruence–modular equational classes, but the converses of these implications do not hold, GU is preserved by composition, while LO needs enforcing surjectivity on one of the morphisms or injectivity on the other for it to be preserved in the composition of those morphisms.

In some particular cases, concerning the structures of the posets of the prime congruences of the algebras in question, GU always holds or LO implies GU. Such cases are pointed out in Section 5. Here we notice that any morphism in the class of bounded distributive lattices is admissible and fulfills GU, a result which we also generalize both in this section and in Section 8.

In Section 6, we prove that GU and LO are preserved by finite direct products in semi–degenerate congruence–modular varieties and in congruence–distributive varieties, as well as by finite ordinal sums in the class of bounded lattices and in any congruence–modular variety of bounded ordered structures that fulfills certain conditions.

In Section 7, we prove that GU and LO are preserved by quotients, and obtain a series of algebraic and topological characterizations for GU and LO, which lead to further results on the relationships between these two properties.

In Section 8, we study admissibility, GU and LO in different kinds of congruence–modular equational classes. We prove that all morphisms are admissible in varieties having a system of congruence intersection terms without parameters, among which there are congruence–distributive varieties with the compact intersection property, which in turn include filtral varieties, discriminator varieties, bounded distributive lattices and residuated lattices. As for varieties in which all admissible morphisms fulfill GU and LO, it turns out that they include semi–degenerate filtral varieties, semi–degenerate discriminator varieties, bounded distributive lattices and residuated lattices and many other varieties which are important in the algebra of logic. If we put together these results, we obtain a set of varieties in which all morphisms are admissible and fulfill GU and LO, a fact which generalizes the results on MV–algebras and BL–algebras from [7] and [24], respectively, but also includes many other interesting cases, such as bounded distributive lattices, residuated lattices, semi–degenerate filtral varieties and semi–degenerate discriminator varieties.

2 Preliminaries

In this section, we recall some properties on congruences in universal algebras and a series of results from commutator theory; we shall provide short proofs for those which are least commonly used. For the notions on universal algebras that we use in the sequel, we refer the reader to [1], [3], [13], [18]. For those on lattices, see [4], [7], [9], [12], [25]. For a further study of commutator theory, we recommend [1], [10], [13], [23].

We shall denote by $N$ the set of the natural numbers and by $N^* = N \setminus \{0\}$. For any set $M$, we shall denote by $|M|$ the cardinality of $M$, by $P(M)$ the set of the subsets of $M$, by $Eq(M)$ the set of the equivalence relations on $M$, by $\Delta_M = \{(x, x) \mid x \in M\} \in Eq(M)$ and by $\nabla_M = M^2 \in Eq(M)$. If $M$ is non–empty and $\pi$ is a partition of $M$, then we shall denote by $eq(\pi)$ the equivalence on $M$ that corresponds to $\pi$; if $n \in N^*$ and $\{M_1, \ldots, M_n\}$ is a finite partition of $M$, $eq(\{M_1, \ldots, M_n\})$ shall also be denoted by $eq(M_1, \ldots, M_n)$. For any $\theta \in Eq(M)$, any $a \in M$, $V \subseteq M$ and $W \subseteq M^2$, $a/\theta$ will denote the equivalence class of $a$ with respect to $\theta$, $V/\theta = \{x/\theta \mid x \in V\}$, $W/\theta = \{(x/\theta, y/\theta) \mid x, y \in M, (x, y) \in W\}$ and $p_\theta : M \to M/\theta$ shall be the canonical surjection.

Let $I$ be a non–empty set and $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ be families of sets. If $X \subseteq \prod_{i \in I} X_i$, then by $a = (a_i)_{i \in I} \in X$ we mean $a_i \in X_i$ for all $i \in I$, such that $a \in X$. If $f_i : X_i \to Y_i$ for all $i \in I$, then $\prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ shall have the usual componentwise definition and, in the particular case when $I = \{1, \ldots, n\}$ for some $n \in N^*$ and $f_1 = \ldots = f_n = f$, then we denote $\prod_{i \in I} f_i = f^n$. If $\theta_i \in Eq(X_i)$ for all $i \in I$, then we denote $\prod_{i \in I} \theta_i = \{(x_i)_{i \in I}, (y_i)_{i \in I}\} \mid \forall i \in I \exists \theta_i \in Eq(X_i) \forall (x_i, y_i) \in \theta_i\}$. If $M$ and $N$ are sets and $f : M \to N$, then the direct image of $f^2$ shall simply be denoted by $f$, and $(f^2)^{-1}$, the inverse image of $f^2$, shall be denoted by $f^*$. Clearly, if $f$ is
injective, then so is \( f^2 \), thus \( f^*: \mathcal{P}(N^2) \rightarrow \mathcal{P}(M^2) \) is surjective, and, if \( f \) is surjective, then so is \( f^2 \), thus \( f^* \) is injective. Trivially, \( f^*(N_N) = N_M. \) We shall denote by \( \text{Ker}(f) = f^*(\Delta_N) \) the kernel of \( f. \) Notice that, if \( Q \) is a set and \( g: N \rightarrow Q, \) then \( (g \circ f)^* = ((g \circ f)^2)^{-1} = (g^2 \circ f^2)^{-1} = (f^2)^{-1} \circ (g^2)^{-1} = f^* \circ g^*. \)

Whenever there is no danger of confusion, any algebra shall be designated by its support set. All algebras shall be considered non–empty; by trivial algebra we mean one–element algebra, and by non–trivial algebra we mean algebra with at least two distinct elements. Any quotient algebra and any direct product of algebras shall be considered with the operations defined canonically. Sometimes, for brevity, we shall denote by \( A \cong B \) the fact that two algebras \( A \) and \( B \) of the same type are isomorphic.

For any \( n \in \mathbb{N}^+ \), we shall denote the \( n \)–element chain by \( L_n. \) We shall denote by \( D \) the diamond and by \( \mathcal{P} \) the pentagon. We shall abbreviate the ascending chain condition for lattices by \( ACC. \) For any lattice \( L \) and any \( x \in L, \) we shall denote by \( [x] \) the principal filter of \( L \) generated by \( x: [x] = \{ y \in L | x \leq y \}. \)

Throughout this paper, \( \tau \) shall be a type of universal algebras, \( C \) shall be an equational class of algebras of type \( \tau \) and \( L_\tau \) shall be the first order language associated to \( \tau. \) For any term \( t \) in \( L_\tau \) and any member \( M \) of \( C, \) we shall denote by \( t^M \) the derivative operation of \( M \) associated to \( t. \)

Reverting to the rest of this section, \( A \) shall be an algebra from \( C. \) We shall denote by \( \text{Con}(A) \) the set of the congruences of \( A; \) for each \( X \subseteq A^2, \) we shall denote by \( C_{GA}(X) \) the congruence of \( A \) generated by \( X; \) for every \( a, b \in A, \) \( C_{GA}(\{(a, b)\}) \) is also denoted by \( C_{GA}(a, b) \) and called the principal congruence of \( A \) generated by \( (a, b). \) Let \( \phi \in \text{Con}(A); \) \( \phi \) is said to be finitely generated if \( \phi = C_{GA}(X) \) for some finite subset \( X \) of \( A^2; \phi \) is called a proper congruence of \( A \) iff \( \phi \neq \nabla_A. \) The maximal congruences of \( A \) are the maximal elements of \( (\text{Con}(A) \setminus \{\nabla_A\}, \subseteq) \); the set of the maximal congruences of \( A \) is denoted by \( \text{Max}(A) \) and called the maximal spectrum of \( A. \) \( (\text{Con}(A), \vee, \wedge, \nabla_A, \Delta_A) \) is a bounded lattice, ordered by set inclusion, where, for all \( \phi, \psi \in \text{Con}(A), \phi \vee \psi = C_{GA}(\phi \cup \psi); \) moreover, \( \text{Con}(A) \) is a complete lattice, in which, for any family \( \{\phi_i\} \subseteq \text{Con}(A), \bigvee \phi_i = C_{GA}(\bigcup \phi_i). \) Obviously, \( A \) is non–trivial iff \( \Delta_A \neq \nabla_A. \) The algebra \( A \) is said to be congruence–modular, respectively congruence–distributive, iff the lattice \( \text{Con}(A) \) is modular, respectively distributive. The equational class \( C \) is said to be congruence–modular, respectively congruence–distributive, iff all algebras from \( C \) are congruence–modular, respectively congruence–distributive. The class of lattices and that of residuated lattices are congruence–distributive; that of commutative rings is congruence–modular and it is not congruence–distributive.

If \( J \) is a non–empty set, \( (A_i)_{i \in I} \) and \( (B_i)_{i \in I} \) are families of algebras in \( C \) and, for all \( i \in I, \theta_i \in \text{Con}(A_i) \) and \( f_i: A_i \rightarrow B_i \) is a morphism in \( C, \) then, clearly: \( \prod_{i \in I} \theta_i \in \text{Con}((\prod_{i \in I} A_i)) \) and \( \prod_{i \in I} f_i \) is a morphism in \( C. \)

Now let \( B \) be an algebra from \( C \) and \( f: A \rightarrow B \) be a morphism in \( C \) and let \( \phi \in \text{Con}(A) \) and \( \psi \in \text{Con}(B). \) Then \( f^*(\psi) \in \text{Con}(A), \) in particular \( \text{Ker}(f) \in \text{Con}(A), \) and, clearly, \( f(f^*(\psi)) = \psi \cap f(A^2), \) so, if \( f \) is surjective, then \( f(f^*(\psi)) = \psi. \) Also, \( f(\phi) \in \text{Con}(f(A)), \) thus, if \( f \) is surjective, then \( f(\phi) \in \text{Con}(B) \) and \( f^*(f(\phi)) = \phi \) since, in this case, \( f^* \) is injective.

For any \( \theta \in \text{Con}(A), p_{\theta} \) is a surjective morphism in \( C \) and the mapping \( \gamma \mapsto p_{\theta}(\gamma) = \gamma/\theta \) sets a bounded lattice isomorphism from \( \{\theta\} \) to \( \text{Con}(A/\theta), \) so \( \text{Con}(A/\theta) = \{\gamma/\theta \ | \ \gamma \in \{\theta\}\} \) and, for all \( \gamma \in \{\theta\}, \) \( p_{\theta}(\gamma/\theta) = p_{\theta}(\gamma/\theta) = \gamma, \) thus \( \text{Ker}(p_{\theta}) = p_{\theta}^{\ast}(\Delta_A/\theta) = p_{\theta}^{\ast}(\theta/\theta) = \theta. \) Notice that, for any \( \gamma \in \{\theta\} \) and any \( a, b \in A, \) the following hold: \( (a/\theta, b/\theta) \in \gamma/\theta \) iff \( (p_{\theta}(a), p_{\theta}(b)) \in p_{\theta}(\gamma) \) iff \( (a, b) \in p_{\theta}(\gamma) \) iff \( (a, b) \in \gamma; \) from this or directly from the fact that the map above is a lattice isomorphism, it follows that, for any \( \alpha, \beta \in \{\theta\}, \) \( \alpha/\theta = \beta/\theta \) iff \( \alpha = \beta, \) and, \( \alpha/\theta \subseteq \beta/\theta \) iff \( \alpha \subseteq \beta. \) From the above it follows that \( \text{Max}(A/\theta) = \{\mu/\theta \ | \ \mu \in \text{Max}(A), \theta \subseteq \mu\} = p_{\theta}(\theta) \cap \text{Max}(A). \)

**Theorem 2.1.** [13] If \( C \) is congruence–modular, then, for each member \( M \) of \( C, \) there exists a unique binary operation \( \cdot, \cdot \_M \) on \( \text{Con}(M) \) such that, for all \( \alpha, \beta \in \text{Con}(M), \) \( [\alpha, \beta]_M = \min\{\mu \in \text{Con}(M) | \mu \subseteq \alpha \cap \beta \} \) and, for any member \( N \) of \( C \) and any surjective morphism \( f: M \rightarrow N \in C, \mu \vee \text{Ker}(f) = f^*(\min(\mu \vee \text{Ker}(f), \beta \vee \text{Ker}(f)))_N \).

**Remark 2.2.** Notice that, since it refers to surjective functions \( f, \) the last equality from Theorem 2.1 implies: \( f(\mu \vee \text{Ker}(f)) = f(\min(\mu \vee \text{Ker}(f), \beta \vee \text{Ker}(f)))_N. \)

**Definition 2.3.** If \( C \) is congruence–modular, then, for any member \( M \) of \( C, \) the operation \( \cdot, \cdot \_M : \text{Con}(M) \times \text{Con}(M) \rightarrow \text{Con}(M) \) from Theorem 2.1 is called the commutator of \( M, \) and, for any \( \alpha, \beta \in \text{Con}(M), [\alpha, \beta]_M \) is called the commutator of \( \alpha \) and \( \beta. \)

**Theorem 2.4.** [15] If \( C \) is congruence–distributive, then, in each member of \( C, \) the commutator coincides to the intersection of congruences.
Throughout the rest of this section, $\mathcal{C}$ shall be congruence–modular.

**Remark 2.5.** For any $\alpha, \beta \in \text{Con}(A)$, we have:

- $[\alpha, \beta]_A \subseteq \alpha \cap \beta$;
- for any algebra $B$ from $\mathcal{C}$ and any surjective morphism $f : A \to B$, $[\alpha, \beta]_A \lor \text{Ker}(f) = f^*([\alpha \lor \text{Ker}(f)], f(\beta \lor \text{Ker}(f))]_B)$, which implies: $f([\alpha, \beta]_A \lor \text{Ker}(f)) = [f(\alpha \lor \text{Ker}(f)), f(\beta \lor \text{Ker}(f))]_B$;
- for all $\theta \in \text{Con}(A)$, $p_\theta : A \to A/\theta$ is a surjective morphism, so we may take $B = A/\theta$ and $f = p_\theta$ in the above, and we obtain: $([\alpha, \beta]_A \lor \theta)/\theta = [(\alpha \lor \theta)/(\beta \lor \theta)]_{A/\theta}$; in particular, if $\alpha, \beta \in \theta$, then $([\alpha, \beta]_A \lor \theta)/\theta = [\alpha/\theta, \beta/\theta]_{A/\theta}$.

**Proposition 2.6.** [10] The commutator is:

- commutative, that is $[\alpha, \beta]_A = [\beta, \alpha]_A$ for all $\alpha, \beta \in \text{Con}(A)$;
- increasing in both arguments, that is, for all $\alpha, \beta, \phi, \psi \in \text{Con}(A)$, if $\alpha \subseteq \beta$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_A \subseteq [\beta, \psi]_A$;
- distributive in both arguments with respect to arbitrary joins, that is, for any families $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$ of congruences of $A$, $[\bigvee_{i \in I} \alpha_i, \bigvee_{j \in J} \beta_j]_A = \bigvee_{i \in I, j \in J} [\alpha_i, \beta_j]_A$.

**Lemma 2.7.** [10] If $B$ is a subalgebra of $A$, then, for any $\alpha, \beta \in \text{Con}(A)$, $[\alpha \cap B^2, \beta \cap B^2]_B \subseteq [\alpha, \beta]_A \cap B^2$.

**Proposition 2.8.** [23] Theorem 5.17, p. 48] Let $n \in \mathbb{N}^*$, $M_1, \ldots, M_n$ be algebras from $\mathcal{C}$, $M = \prod_{i=1}^n M_i$ and, for all $i \in \{1, \ldots, n\}$, $\alpha_i, \beta_i \in \text{Con}(M_i)$. Then: $[\prod_{i=1}^n \alpha_i, \prod_{i=1}^n \beta_i]_M = \prod_{i=1}^n [\alpha_i, \beta_i]_{M_i}$.

**Definition 2.9.** [10] A proper congruence $\phi$ of $A$ is said to be prime iff, for all $\alpha, \beta \in \text{Con}(A)$, $[\alpha, \beta]_A \subseteq \phi$ implies $\alpha \subseteq \phi$ or $\beta \subseteq \phi$.

The set of the prime congruences of $A$ shall be denoted by $\text{Spec}(A)$. $\text{Spec}(A)$ is called the (prime) spectrum of $A$. Note that not every algebra in a congruence–modular equational class has prime congruences. We shall denote by $\text{Min}(A)$ the set of the minimal prime congruences of $A$, that is the minimal elements of the poset $(\text{Spec}(A), \subseteq)$.

**Lemma 2.10.** [1] Theorem 5.3] If $\nabla_A$ is finitely generated, then:

- any proper congruence of $A$ is included in a maximal congruence of $A$;
- any maximal congruence of $A$ is prime.

Following [18], we say that $\mathcal{C}$ is semi–degenerate iff no non–trivial algebra in $\mathcal{C}$ has one–element subalgebras. For instance, obviously, the class of bounded lattices, that of residuated lattices and that of unitary rings are semi–degenerate.

**Proposition 2.11.** [18] The following are equivalent:

(i) $\mathcal{C}$ is semi–degenerate;

(ii) for all members $M$ of $\mathcal{C}$, the congruence $\nabla_M$ is finitely generated.

**Proposition 2.12.** [10] Theorem 8.5, p. 85] The following are equivalent:

- for any algebra $M$ from $\mathcal{C}$, $[\nabla_M, \nabla_M] = \nabla_M$;
- for any algebra $M$ from $\mathcal{C}$ and any $\theta \in \text{Con}(M)$, $[\theta, \nabla_M] = \theta$.
for any \( n \in \mathbb{N}^* \) and any algebras \( M_1, \ldots, M_n \) from \( C \),
\[
\text{Con}(\prod_{i=1}^{n} M_i) = \left\{ \prod_{i=1}^{n} \theta_i \mid (\forall i \in \overline{1,n}) \left( \theta_i \in \text{Con}(M_i) \right) \right\}.
\]

**Lemma 2.13.**
(i) If \( C \) is semi–degenerate, then \( C \) fulfills the equivalent conditions from Proposition 2.12.
(ii) If \( C \) is congruence–distributive, then \( C \) fulfills the equivalent conditions from Proposition 2.12.

**Proof.** (i) This is exactly [1] Lemma 5.2. (ii) Clear, from Theorem 2.4.

**Definition 2.14.** A non–empty subset \( S \subseteq A^2 \) is called an \( m \)–system iff, for any \( (a, b), (c, d) \in S \), we have \( \{Cg_A(a, b), Cg_A(c, d)|A \cap S \neq \emptyset \}. \)

**Lemma 2.15.** For any \( \phi \in \text{Spec}(A) \), \( \nabla_A \setminus \phi \) is an \( m \)–system.

**Proof.** Let \( \phi \in \text{Spec}(A) \) and \( S = \nabla_A \setminus \phi \), so that \( S \neq \emptyset \) since \( \phi \) is a proper congruence of \( A \). Let \( (a, b), (c, d) \in S \), so that \( (a, b), (c, d) \notin \phi \), thus \( Cg_A(a, b) \notin \phi \) and \( Cg_A(c, d) \notin \phi \), hence \( \{Cg_A(a, b), Cg_A(c, d)|A \notin \phi \) since \( \phi \) is a prime congruence. Thus \( \{Cg_A(a, b), Cg_A(c, d)|A \cap S = \emptyset \) since \( \phi \) is a prime congruence. Therefore \( S \) is an \( m \)–system.

**Lemma 2.16.** Assume that \( \nabla_A \) is finitely generated. Let \( \alpha \in \text{Con}(A) \) and \( S \subseteq A^2 \) an \( m \)–system. If \( \phi \) is a maximal element of the set \( \{ \theta \in \text{Con}(A) \mid \alpha \subseteq \theta, \theta \cap S = \emptyset \} \), then \( \phi \in \text{Spec}(A) \).

For all \( \theta \in \text{Con}(A) \), we shall denote by \( V_A(\theta) = \{ \phi \in \text{Spec}(A) \mid \theta \subseteq \phi \} \) and \( D_A(\theta) = \text{Spec}(A) \setminus V_A(\theta) \). For all \( a, b \in A \), we denote \( V_A(a, b) = V_A(Cg_A(a, b)) \) and \( D_A(a, b) = D_A(Cg_A(a, b)) \). It is well known that, if \( \nabla_A \) is finitely generated, then \( \{ D_A(\theta) \mid \theta \in \text{Con}(A) \} \) is a topology on \( \text{Spec}(A) \), called the Stone topology, having \( \{ D_A(a, b) \mid a, b \in A \} \) as a basis, \( \{ V_A(\theta) \mid \theta \in \text{Con}(A) \} \) as the set of closed sets, and \( \{ V_A(a, b) \mid a, b \in A \} \) as a basis of closed sets. For any \( M \subseteq \text{Spec}(A) \), we shall denote by \( M \) the closure of \( M \) in the topological space \( \text{Spec}(A) \) with the Stone topology. Clearly, for all \( \phi \in \text{Spec}(A) \), \( \{ \phi \} = V_A(\phi) \).

### 3 Admissible Morphisms

In this section, we study admissible morphisms, that is those morphisms \( f \) with the property that \( f^* \) takes prime congruences to prime congruences. We recall some of their properties from [22], among which: surjectivity implies admissibility, but the converse is not true, nor does admissibility always hold. Throughout this section, \( A, B, C \) shall be algebras from \( C \) and \( f : A \rightarrow B, g : B \rightarrow C \) shall be morphisms in \( C \).

**Remark 3.1.** For any subalgebra \( S \) of \( A \), if \( i : S \rightarrow A \) is the canonical embedding and \( \alpha \in \text{Con}(A) \), then \( i^*(\alpha) = \alpha \cap S^2 \in \text{Con}(S) \).

**Remark 3.2.** [12] Lemma 6, p. 19, and Lemma 7, p. 20 Any class of a congruence of a lattice \( L \) is a convex sublattice of \( L \), thus it has a unique writing as an intersection between a filter and an ideal of \( L \).

**Remark 3.3.** For any \( \beta \in \text{Con}(B) \), we have \( \Delta_B \subseteq \beta \), thus \( \text{Ker}(f) = f^*(\Delta_B) \subseteq f^*(\beta) \).

**Remark 3.4.** If \( \phi, \theta \in \text{Con}(A) \) and \( \psi \in \text{Con}(A/\theta) \) such that \( p_\theta^*(\psi) = \phi \), then, by Remark 3.3, \( \theta = \text{Ker}(p_\theta) \subseteq \phi \).

**Remark 3.5.** \( \text{Ker}(f) \subseteq \text{Ker}(g \circ f) \), because \( \text{Ker}(f) = f^*(\Delta_B) \subseteq f^*(g^*(\Delta_C)) = (g \circ f)^*(\Delta_C) = \text{Ker}(g \circ f) \).

Following [22], for any algebra \( M \), we shall denote by \( \text{Con}_2(M) \) the set of the two–class congruences of \( M \): \( \text{Con}_2(M) = \{ \theta \in \text{Con}(M) \mid |M/\theta| = 2 \} \).

**Remark 3.6.** [22] \( \text{Con}_2(A) \subseteq \text{Max}(A) \).

**Lemma 3.7.** [22]
(i) If \( C \) is semi–degenerate, then \( f^*([\nabla_A]) = [\nabla_B] \).
(ii) If \( f^*[\nabla_A] = \nabla_B \), then \( f^*(\text{Con}_2(B)) \subseteq \text{Con}_2(A) \).
(iii) If \( C \) is semi–degenerate, then \( f^*(\text{Con}_2(B)) \subseteq \text{Con}_2(A) \).
Throughout the rest of this section, $\mathcal{C}$ shall be congruence–modular.

Remark 3.8. 
Clearly, if $\mathcal{C}$ is congruence–distributive (see Theorem 2.4) or the commutator in $A$ equals the intersection of congruences, then the prime congruences of $A$ are exactly the prime elements of the lattice $\text{Con}(A)$. If, additionally, $\text{Con}(A)$ is finite, then the prime congruences of $A$ are exactly the elements of $\text{Con}(A)$ that have exactly one successor in the lattice $\text{Con}(A)$. Thus, if, moreover, $\text{Con}(A)$ is a non–trivial finite chain, then $\text{Spec}(A) = \text{Con}(A) \setminus \{\nabla_A\}$.

Remarks 3.1, 3.2 and 3.8 are easy to use for determining all congruences and the prime congruences in the examples we shall give. Sometimes, we shall use the remarks in this paper without referencing them.

Lemma 3.9.

(i) If $\mathcal{C}$ is semi–degenerate, then $\text{Max}(A) \subseteq \text{Spec}(A)$.

(ii) If $A$ is congruence–distributive and the commutator in $A$ equals the intersection, then $\text{Max}(A) \subseteq \text{Spec}(A)$.

(iii) If $\mathcal{C}$ is congruence–distributive, then $\text{Max}(A) \subseteq \text{Spec}(A)$.

(iv) If the commutator in $A$ equals the intersection of congruences and $\text{Con}(A)$ is a Boolean algebra, then $\text{Spec}(A) = \text{Max}(A)$.

(v) If $\mathcal{C}$ is congruence–distributive and $\text{Con}(A)$ is a Boolean algebra, then $\text{Spec}(A) = \text{Max}(A)$.

Remark 3.10. By Remark 3.9 and Lemmas 2.10 and 3.9, if $\nabla_A$ is finitely generated or $A$ is congruence–distributive and the commutator in $A$ equals the intersection, or $\mathcal{C}$ is semi–degenerate or congruence–distributive, then $\text{Con}_2(A) \subseteq \text{Max}(A) \subseteq \text{Spec}(A)$.

Remark 3.11. If we consider the conditions:

$(c_1)$ $\text{Spec}(A) = \text{Max}(A)$, $(c_2)$ $(\text{Spec}(A), \subseteq)$ is unordered, then: $(c_1) \Rightarrow (c_2)$ and, if $\text{Max}(A) \subseteq \text{Spec}(A)$, for instance if $\nabla_A$ is finitely generated or $A$ is congruence–distributive and the commutator in $A$ equals the intersection, or $\mathcal{C}$ is semi–degenerate or congruence–distributive (see Lemmas 2.10 and 3.9), then $(c_1) \Leftrightarrow (c_2)$.

Definition 3.12. We say that the morphism $f : A \rightarrow B$ is admissible iff, for all $\psi \in \text{Spec}(B)$, we have $f^*(\psi) \in \text{Spec}(A)$.

So $f$ is admissible iff $f^*(\text{Spec}(B)) \subseteq \text{Spec}(A)$, where by $f^*$ we denote the direct image of the function $f^* : \text{Con}(B) \rightarrow \text{Con}(A)$. If $f$ is admissible, then we may consider the restriction $f^* |_{\text{Spec}(B)} : \text{Spec}(B) \rightarrow \text{Spec}(A)$.

Remark 3.13. $f$ is admissible iff $f^*(\text{Spec}(B)) \subseteq V_A(\text{Ker}(f))$. Indeed, $f^*(\text{Spec}(B)) \subseteq f^*(\text{Con}(B)) \subseteq [\text{Ker}(f)]$ by Remark 3.9, and, if $f$ is admissible, then $f^*(\text{Spec}(B)) \subseteq \text{Spec}(A)$ by the above, hence $f^*(\text{Spec}(B)) \subseteq \text{Spec}(A) \cap [\text{Ker}(f)] = V_A(\text{Ker}(f))$. The converse implication is trivial.

If $f$ is admissible, then, for all $\theta \in \text{Con}(B)$, $f^*(V_B(\theta)) \subseteq V_A(f^*(\theta))$, because: $f^*(V_B(\theta)) \subseteq f^*(\text{Spec}(B)) \subseteq \text{Spec}(A)$ by the admissibility of $f$, and $f^*(V_B(\theta)) \subseteq f^*(\theta)$, hence $f^*(V_B(\theta)) \subseteq [f^*(\theta) \cap \text{Spec}(A)] = V_A(f^*(\theta))$.

Proposition 3.14. 

(i) Any morphism in the class of bounded distributive lattices is admissible.

(ii) Moreover: any bounded lattice morphism whose co–domain is distributive is admissible.

Example 3.15. Let $\mathcal{L}_2^2$, $\mathcal{D}$ and $\mathcal{P}$ have the elements denoted as in the following Hasse diagrams, $i : \mathcal{L}_2^2 \rightarrow \mathcal{D}$ and $j : \mathcal{L}_2^2 \rightarrow \mathcal{P}$ be the canonical bounded lattice embeddings and $h : \mathcal{D} \rightarrow \mathcal{D}$ and $k : \mathcal{P} \rightarrow \mathcal{P}$ be the bounded lattice morphisms given by the following table:

| h(u) | 0 | 0 | 1 | 1 |
|------|---|---|---|---|
| k(u) | 0 | 0 | 1 | 1 |

| Con(\mathcal{D}) | Con(\mathcal{L}_2^2) | Con(\mathcal{P}) |
|------------------|------------------|------------------|
| \nabla_D \Delta_D | \nabla_{\mathcal{L}_2^2} \Delta_{\mathcal{L}_2^2} | \nabla_P \Delta_P |

\[\begin{array}{cccccc}
0 & 1 & y & \text{Spec}(\mathcal{D}) & \text{Spec}(\mathcal{L}_2^2) & \text{Spec}(\mathcal{P}) \\
\hline
x & y & z & h & \mathcal{D} & \mathcal{L}_2^2 \\
\end{array}\]
Remark 3.22.

Lemma 3.23.

Lemma 3.21.

isomorphism, then:

f

Remark 3.19.

3.16.

\(\alpha\)

admissible; then, for all \(\alpha\) \(\in\) \(\{0,1\}\), \(\beta\) \(\in\) \(\{0,1\}\), \(\gamma\) \(\in\) \(\{0,1\}\), thus \(\text{Spec}(P) = \{\Delta_P, \alpha, \beta\}\). \(\gamma \notin \text{Spec}(P)\), because \(\gamma = \alpha \cap \beta = [\alpha, \beta] \supseteq [\alpha, \beta]_P\), but \(\alpha \notin \gamma\) and \(\beta \notin \gamma\).

\(\Delta_P \in \text{Spec}(D)\) and \(i^*(\Delta_P) = \Delta_{L_2} \notin \text{Spec}(L_2)\), thus \(i\) is not admissible.

\(\Delta_P \in \text{Spec}(P)\) and \(j^*(\Delta_P) = \Delta_{L_2} \notin \text{Spec}(L_2)\), thus \(j\) is not admissible.

\(\Delta_P \in \text{Spec}(D)\) and \(h^*(\Delta_P) = \gamma \notin \text{Spec}(P)\), therefore \(h\) is not admissible.

For all \(\theta \in \text{Con}(P) \setminus \{\Delta_P\}\), \(k^*(\theta) = \Delta_D\), thus \(k^*(\Delta_P) = k^*(\alpha) = k^*(\beta) = \Delta_D \in \text{Spec}(D)\), hence \(k\) is admissible.

Lemma 3.16. Any surjective morphism in \(C\) is admissible, but the converse is not true, not even when \(C\) is semi–degenerate and congruence–distributive.

Proof. Proposition 2.1, (i)] shows that every surjective morphism is admissible. Proposition 3.14 provides us with an infinity of examples of admissible morphisms that are not surjective. Also, the admissible morphism \(k\) from Example 3.15 is not surjective. These are examples of morphisms in the semi–degenerate congruence–distributive class of bounded lattices.

Proposition 3.17. Not all morphisms are admissible, not even when \(C\) is semi–degenerate and congruence–distributive.

Proof. In Example 3.15 the morphisms \(i, j\) and \(h\) in the semi–degenerate congruence–distributive class of bounded lattices are not admissible.

Remark 3.18. For any \(\theta \in \text{Con}(A)\), the morphism \(p_\theta : A \rightarrow A/\theta\) is surjective, and thus admissible by Lemma 3.16.

Remark 3.19. Any composition of admissible morphisms is admissible. Indeed, assume that \(f\) and \(g\) are admissible; then, for all \(\chi \in \text{Spec}(C)\), it follows that \(g^*(\chi) \in \text{Spec}(B)\), hence \((g \circ f)^*(\chi) = (f^* \circ g^*)(\chi) = f^*(g^*(\chi)) \in \text{Spec}(A)\), therefore \(g \circ f\) is admissible.

Remark 3.20. Clearly, if \(g\) is an isomorphism, then: \(f\) is admissible iff \(g \circ f\) is admissible. Similarly, if \(f\) is an isomorphism, then: \(g\) is admissible iff \(g \circ f\) is admissible.

Lemma 3.21. \(\boxed{\text{(i)}\ \text{If } C\text{ is semi–degenerate and } \text{Spec}(B) = \text{Con}_2(B)\text{, then } f\text{ is admissible.}}\)

\(\boxed{\text{(ii)}\ \text{If } \nabla_A\text{ is finitely generated, } f^*(\{\nabla_A\}) = \{\nabla_B\}\text{ and } \text{Spec}(B) = \text{Con}_2(B)\text{, then } f\text{ is admissible.}}\)

\(\boxed{\text{(iii)}\ \text{If } A\text{ is congruence–distributive and the commutator in } A\text{ equals the intersection, } f^*(\{\nabla_A\}) = \{\nabla_B\}\text{ and } \text{Spec}(B) = \text{Con}_2(B)\text{, then } f\text{ is admissible.}}\)

\(\boxed{\text{(iv)}\ \text{If } C\text{ is congruence–distributive, } f^*(\{\nabla_A\}) = \{\nabla_B\}\text{ and } \text{Spec}(B) = \text{Con}_2(B)\text{, then } f\text{ is admissible.}}\)

Remark 3.22. \(\boxed{\text{If } L\text{ and } M\text{ are bounded lattices and } h : L \rightarrow M\text{ is a bounded lattice morphism with } h(L) = \{0,1\}\text{, then } h^*(\text{Con}(M) \setminus \{\nabla_M\}) \subseteq \text{Con}_2(L)\text{, thus } h\text{ is admissible by Lemma 3.7. Notice, also, that, if } M\text{ is non–trivial, then } h^*(\text{Con}(M) \setminus \{\nabla_M\}) = h^*(\text{Spec}(M)) = h^*(\{\Delta_M\}) = \{\text{Ker}(h)\}\text{; in particular, } \text{Ker}(h) \in \text{Con}_2(L) \subseteq \text{Spec}(L)\text{. This is the case for the bounded lattice morphism } k\text{ from Example 3.15.}}\)

\(\boxed{\text{The above actually holds for any equational class of bounded ordered structures.}}\)

Lemma 3.23. \(\boxed{\text{Let } L\text{ be a bounded lattice.}}\)

\(\boxed{\text{(i)}\ \text{If } L\text{ is distributive, then } \text{Spec}(L) = \text{Max}(L) = \text{Con}_2(L)\text{.}}\)
(ii) If $L$ can be obtained through finite direct products and/or finite ordinal sums from bounded distributive lattices and/or finite modular lattices and/or relatively complemented bounded lattices with ACC, then $\text{Spec}(L) = \text{Max}(L)$.

Note that Proposition 3.14 above follows from Lemma 3.21, and Lemma 3.23, (i).

Lemma 3.24. Assume that $\text{Spec}(A) = \text{Con}(A) \setminus \{\nabla_A\}$.

(i) If $(f^*)^{-1}(\{\nabla_A\}) = \{\nabla_B\}$, then $f$ is admissible.

(ii) If $C$ is semi-degenerate, then $f$ is admissible.

Proof. By Lemma 3.21 and Lemma 3.7, (i).

Lemma 3.25. Assume that $C$ is congruence-distributive or the commutator in $A$ equals the intersection. If $\text{Con}(A)$ is a non-trivial finite chain, then:

(i) If $(f^*)^{-1}(\{\nabla_A\}) = \{\nabla_B\}$, then $f$ is admissible.

(ii) If $C$ is semi-degenerate, then $f$ is admissible.

Proof. By Lemma 3.24 and Remark 3.8.

4 Properties Going Up and Lying Over

In this section we define conditions Going Up and Lying Over on admissible morphisms and start investigating their properties. We prove that they are non-trivial, that their study can be reduced to embeddings, that surjectivity implies Going Up and Going Up implies Lying Over, but the converses of these implications do not hold. Throughout this section, $\mathcal{C}$ shall be congruence-modular, $A, B, C$ shall be members of $\mathcal{C}$ and $f : A \rightarrow B, g : B \rightarrow C$ shall be admissible morphisms in $\mathcal{C}$. Then $g \circ f$ is admissible by Remark 3.19. Also, $M, N$ shall be members of $\mathcal{C}$ and $h : M \rightarrow N$ shall be a morphism in $\mathcal{C}$, not necessarily admissible.

Definition 4.1. We say that $f$ fulfills property Going Up (abbreviated GU) iff, for any $\phi, \psi \in \text{Spec}(A)$ and any $\phi_1 \in \text{Spec}(B)$ such that $\phi \subseteq \psi$ and $f^*(\phi_1) = \phi$, there exists a $\psi_1 \in \text{Spec}(B)$ such that $\phi_1 \subseteq \psi_1$ and $f^*(\psi_1) = \psi$.

We say that $f$ fulfills property Lying Over (abbreviated LO) iff, for any $\phi \in \text{Spec}(A)$ such that $\text{Ker}(f) \subseteq \phi$, there exists a $\phi_1 \in \text{Spec}(B)$ such that $f^*(\phi_1) = \phi$.

Remark 4.2. Trivially, any isomorphism is admissible (see also Lemma 3.15) and fulfills GU and LO.

Clearly, if $g$ is an isomorphism, then: $f$ fulfills GU, respectively LO, iff $g \circ f$ fulfills GU, respectively LO. Similarly, if $f$ is an isomorphism, then: $g$ fulfills GU, respectively LO, iff $g \circ f$ fulfills GU, respectively LO.

Hence, if all canonical embeddings of $A$ into other algebras from $\mathcal{C}$ are admissible and fulfill GU, respectively LO, then all embeddings of $A$ into other algebras from $\mathcal{C}$ are admissible and fulfill GU, respectively LO.

Remark 4.3. If $A$ is a subalgebra of $B$ and $i : A \rightarrow B$ is the canonical embedding, then, for all $\beta \in \text{Con}(B)$, $i^*(\beta) = \beta \cap A^2$, thus $\text{Ker}(i) = i^*(\Delta_B) = \Delta_B \cap A^2 = \Delta_A$, therefore, if $i$ is admissible:

- $i$ fulfills GU iff, for any $\phi, \psi \in \text{Spec}(A)$ and any $\phi_1 \in \text{Spec}(B)$ such that $\phi \subseteq \psi$ and $\phi_1 \cap A^2 = \phi$, there exists a $\psi_1 \in \text{Spec}(B)$ such that $\phi_1 \subseteq \psi_1$ and $\psi_1 \cap A^2 = \psi$;
- $i$ fulfills LO iff, for any $\phi \in \text{Spec}(A)$, there exists a $\phi_1 \in \text{Spec}(B)$ such that $\phi_1 \cap A^2 = \phi$.

Lemma 4.4. (i) $f$ fulfills GU iff, for all $\psi \in \text{Spec}(B)$, $V_A(f^*(\psi)) \subseteq f^*(V_B(\psi))$.

(ii) $f$ fulfills LO iff $V_A(\text{Ker}(f)) \subseteq f^*(\text{Spec}(B))$ iff $V_A(\text{Ker}(f)) = f^*(\text{Spec}(B))$.

(iii) $h$ is admissible and fulfills LO iff $V_M(\text{Ker}(h)) = h^*(\text{Spec}(N))$. 
\textbf{Proposition 4.5.} \ \ (i) If $g \circ f$ fulfills GU, $g$ fulfills LO and $\text{Spec}(B) \subseteq \{\text{Ker}(g)\}$, then $f$ fulfills GU.

(ii) If $g \circ f$ fulfills GU and $g$ is injective and fulfills LO, then $f$ fulfills GU.

\textbf{Proof.} \ \ (i) By the definition of GU.

(ii) Since $f$ is admissible, these equivalences follow from the definition of LO and Remark 3.13.

\textbf{Proposition 4.6.} \ \ (i) If $\text{Spec}(A) \subseteq \{\text{Ker}(f)\}$, then: $f$ fulfills LO iff the map $f^*|_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A)$ is surjective.

(ii) If $f$ is injective, then: $f$ fulfills LO iff the map $f^*|_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A)$ is surjective.

\textbf{Proof.} \ \ (i) By Lemma 4.4 (ii).

(ii) By (i) and the fact that, if $g$ is injective, then $\text{Ker}(g) = g^*(\Delta_C) = \Delta_B$.

\textbf{Example 4.7.} The non-surjective bounded lattice morphism $k$ in Example 3.10 is admissible and fulfills GU and LO (in a trivial way, because $\text{Spec}(D) = \{\Delta_D\}$).

Here is an admissible bounded lattice morphism which does not fulfill GU, nor does it fulfill LO: let $H$ and $K$ be the bounded lattices given by the following Hasse diagrams, with $H$ a bounded sublattice of $K$, and $i : H \to K$ be the canonical bounded lattice embedding:

\begin{center}
\begin{tikzpicture}
\node (H1) at (0,0) {$H$};
\node (H2) at (0,2) {$1$};
\node (H3) at (-1,1) {$a$};
\node (H4) at (1,1) {$c$};
\node (H5) at (0,1) {$b$};
\node (H6) at (-1,0) {$0$};
\node (H7) at (1,0) {$0$};
\node (H8) at (0,-1) {$z$};
\node (H9) at (0,1) {$y$};
\node (H10) at (-1,1) {$x$};
\node (H11) at (1,1) {$x$};
\node (H12) at (0,0) {$a$};
\node (H13) at (1,0) {$c$};
\node (H14) at (-1,0) {$b$};
\node (H15) at (0,-1) {$z$};
\node (H16) at (0,1) {$y$};
\draw (H3) -- (H5);
\draw (H5) -- (H7);
\draw (H7) -- (H9);
\draw (H9) -- (H11);
\draw (H11) -- (H3);
\draw (H12) -- (H13);
\draw (H13) -- (H14);
\draw (H14) -- (H12);
\draw (H15) -- (H16);
\draw (H16) -- (H9);
\draw (H9) -- (H11);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node (K1) at (4,0) {$K$};
\node (K2) at (4,2) {$1$};
\node (K3) at (3,1) {$a$};
\node (K4) at (5,1) {$c$};
\node (K5) at (4,1) {$b$};
\node (K6) at (3,0) {$0$};
\node (K7) at (5,0) {$0$};
\node (K8) at (4,-1) {$z$};
\node (K9) at (4,1) {$y$};
\node (K10) at (3,1) {$x$};
\node (K11) at (5,1) {$x$};
\node (K12) at (3,0) {$a$};
\node (K13) at (5,0) {$c$};
\node (K14) at (3,0) {$b$};
\node (K15) at (4,-1) {$z$};
\node (K16) at (4,1) {$y$};
\draw (K3) -- (K5);
\draw (K5) -- (K7);
\draw (K7) -- (K9);
\draw (K9) -- (K11);
\draw (K11) -- (K3);
\draw (K12) -- (K13);
\draw (K13) -- (K14);
\draw (K14) -- (K12);
\draw (K15) -- (K16);
\draw (K16) -- (K9);
\draw (K9) -- (K11);
\end{tikzpicture}
\end{center}

Notice that $\text{Con}(K) = \{\Delta_K, \nabla_K\} \cong L_2$, so $\text{Spec}(K) = \{\Delta_K\}$.

$\text{Con}(H) = \{\Delta_H, \nabla_H\} \cong L_3$, where $\nabla = \text{eq}\{\{0\}, \{a, x\}, \{y, 1\}\}$, so $\text{Spec}(H) = \{\Delta_H, \nabla\}$.

$i^*(\Delta_K) = \Delta_H$, thus $i$ is admissible. $\text{Ker}(i) = i^*(\Delta_K) = \Delta_H \subseteq \nabla \in \text{Spec}(H)$ and there exists no $\phi \in \text{Spec}(K) = \{\Delta_K\}$ such that $i^*(\phi) = \nabla$, therefore $i$ does not fulfill GU and it does not fulfill LO.

\textbf{Proposition 4.8.} Not all admissible morphisms fulfill GU or LO, not even when $C$ is congruence–distributive and semi–degenerate.

\textbf{Proof.} The bounded lattice embedding $i$ in Example 4.7 is admissible and does not fulfill GU or LO.

\textbf{Remark 4.9.} Clearly, by Remark 3.8

- If $C$ is congruence–distributive and $h^*|_{\text{Con}(N)} : \text{Con}(N) \to \text{Con}(M)$ is a bounded lattice isomorphism, then $h$ is admissible, $h^*|_{\text{Spec}(N)} : \text{Spec}(N) \to \text{Spec}(M)$ is an order isomorphism and $h$ fulfills GU and LO.

- If the commutator in $M$ and $N$ equals the intersection and $h^*|_{\text{Con}(N)} : \text{Con}(N) \to \text{Con}(M)$ is a bounded lattice isomorphism, then $h$ is admissible, $h^*|_{\text{Spec}(N)} : \text{Spec}(N) \to \text{Spec}(M)$ is an order isomorphism and $h$ fulfills GU and LO.
Example 4.10. Let us see that the converses of the implications in Remark 4.9 do not hold, and let us see some more examples, which illustrate different cases that can appear, regarding admissibility, GU and LO. The following are examples of morphisms in the semi-degenerate congruence-distributive class of bounded lattices.

Let $H$, $K$ and the canonical embedding $i$ be as in Example 3.7. We have seen that $i$ is admissible and does not fulfill GU or LO. Note that $\text{Con}(H) \nsubseteq \text{Con}(K)$, $|\text{Spec}(H)| \neq |\text{Spec}(K)|$ and $i^*|_{\text{Con}(K)}: \text{Con}(K) \to \text{Con}(H)$ and $i^*|_{\text{Spec}(K)}: \text{Spec}(K) \to \text{Spec}(H)$ are injective and they are not surjective.

Let us also consider the following bounded lattices:

\[
\begin{array}{c}
\text{E} & 1 & \text{F} & 1 & \text{L} & 1 & \text{Q} & 0 & \text{R} & 1 & \text{S} & 1 & \text{T} & 1 \\
\end{array}
\]

$\text{Con}(E) = \{ \Delta_E, \varepsilon, \nabla_E \} \cong \mathcal{L}_3$, where $\varepsilon = \text{eq}\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{1\}\}$, so $\text{Spec}(E) = \{ \Delta_E, \varepsilon \}$.

$\text{Con}(F) = \{ \Delta_F, \phi_1, \phi_2, \nabla_F \} \cong \mathcal{L}_4$, where $\phi_1 = \text{eq}\{\{0\}, \{a\}, \{b, c\}, \{x, y, t\}, \{z\}, \{1\}\}$ and $\phi_2 = \text{eq}\{\{0\}, \{a, x\}, \{b, d\}, \{c, z\}, \{y, 1\}\}$, so $\text{Spec}(F) = \{ \Delta_F, \phi_1, \phi_2 \}$.

$\text{Con}(L) = \{ \Delta_L, \lambda_1, \lambda_2, \lambda_3, \nabla_L \}$, with the Hasse diagram below, where $\lambda_1 = \text{eq}\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{x\}, \{y\}, \{z\}, \{1\}\}$, $\lambda_2 = \text{eq}\{\{0\}, \{a, x\}, \{b\}, \{d\}, \{c, z\}, \{y, 1\}\}$ and $\lambda_3 = \text{eq}\{\{0\}, \{a, x\}, \{b, d\}, \{c, z\}, \{y, 1\}\}$, so $\text{Spec}(L) = \{ \lambda_1, \lambda_2, \lambda_3 \}$.

$\text{Con}(Q) = \{ \Delta_Q, \gamma_1, \gamma_2, \gamma_3, \nabla_Q \}$, with the Hasse diagram below, where $\gamma_1 = \text{eq}\{\{0\}, \{a, x\}, \{b\}, \{c\}, \{z\}, \{1\}\}$, $\gamma_2 = \text{eq}\{\{0\}, \{a\}, \{x\}, \{b\}, \{c, z\}, \{1\}\}$ and $\gamma_3 = \text{eq}\{\{0\}, \{a, x\}, \{b\}, \{c, z\}, \{y\}, \{u\}, \{z\}, \{1\}\}$, so $\text{Spec}(Q) = \{ \gamma_1, \gamma_2, \gamma_3 \}$.

$\text{Con}(R) = \{ \Delta_R, \rho, \nabla_R \} \cong \mathcal{L}_3$, where $\rho = \text{eq}\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{x\}, \{y\}, \{u\}, \{z\}, \{1\}\}$, so $\text{Spec}(R) = \{ \Delta_R, \rho \}$.

$\text{Con}(S) = \{ \Delta_S, \sigma, \nabla_S \} \cong \mathcal{L}_3$, where $\sigma = \text{eq}\{\{0\}, \{a\}, \{b\}, \{c\}, \{x\}, \{y\}, \{u, v\}, \{z\}, \{1\}\}$, so $\text{Spec}(S) = \{ \Delta_S, \sigma \}$.

$\text{Con}(T) = \{ \Delta_T, \tau_1, \tau_2, \tau_3, \nabla_T \}$, with the Hasse diagram below, where $\tau_1 = \text{eq}\{\{0\}, \{a\}, \{b\}, \{c\}, \{d\}, \{x\}, \{y\}, \{u, v\}, \{z\}, \{1\}\}$, $\tau_2 = \text{eq}\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{x\}, \{y\}, \{u\}, \{v\}, \{z\}, \{1\}\}$ and $\tau_3 = \text{eq}\{\{0\}, \{a\}, \{b, d\}, \{c\}, \{x\}, \{y\}, \{u, v\}, \{z\}, \{1\}\}$, so $\text{Spec}(T) = \{ \tau_1, \tau_2, \tau_3 \}$.
Con(F) ≢ Con(L); |Spec(F)| = |Spec(L)|, but the posets (Spec(F), ⊆) and (Spec(L), ⊆) are not isomorphic.

Ker(q) = g^*(Δ_L) = g^*(λ_1) = φ_1 and g^*(λ_2) = g^*(λ_3) = φ_2, thus q is admissible and fulfills GU and LO.

φ|_{Con(L)}: Con(L) → Con(F) and φ^*|_{Spec(L)}: Spec(L) → Spec(F) are neither injective, nor surjective.

Lemma 4.11. (i) For all θ ∈ Con(M), Spec(M/θ) = p_θ(V_M(θ)) and the mapping γ → γ/θ sets an order isomorphism from (V_M(θ), ⊆) to (Spec(M/θ), ⊆).

(ii) Con(h(M)) = h(Ker(h))).

(iii) Spec(h(M)) = h(V_M(Ker(h))).

(iv) If h is surjective, then Spec(N) = h(V_M(Ker(h))) and V_M(Ker(h)) = h^*(Spec(N)).

(v) For all α, β ∈ Con(M), h(α) ⊆ h(β) iff α ⊆ β.

(vi) For all θ ∈ [Ker(h)), V_M(h(θ)) = h^*(V_M(Ker(h))).

Proof. (i) Let θ ∈ Con(M), and recall that the mapping γ → p_θ(γ) = γ/θ sets a bounded lattice isomorphism from {θ} to Con(M/θ). Let ψ, γ, δ ∈ Con(M/θ), so that ψ = φ/θ, γ = α/θ and δ = β/θ for some φ, α, β ∈ [θ]. We have the following equivalences, according to Remark 2.4: [γ, δ]_{M/θ} ⊆ ψ iff [α, β]_{M/θ} ⊆ φ/θ iff ([α, β]_M ∩ [θ] = [φ, ψ]_θ iff [α, β]_M ⊆ φ iff [α, β]_M ⊆ φ, since φ ⊆ mol. We also have: γ ⊆ ψ iff α/θ ⊆ ϕ/θ iff α ⊆ φ, and: δ ⊆ ψ iff β/θ ⊆ φ/θ iff β ⊆ φ. Therefore: φ/θ = ψ ∈ Spec(M/θ) iff [γ, δ]_{M/θ} ⊆ ψ implies γ ⊆ ψ or δ ⊆ ψ, if [α, β]_M ⊆ φ implies α ⊆ φ or β ⊆ φ if φ ∈ Spec(M). Hence Spec(M/θ) = {φ/θ | φ ∈ [θ]∩Spec(M)} = {φ/θ | φ ∈ V_M(θ)} = p_θ(V_M(θ)). Thus the mapping above sets a surjection from V_M(θ) to Spec(M/θ); since it sets a bounded lattice isomorphism, thus an order isomorphism, from {θ} to Con(M/θ), it follows this map is also injective, thus it is a bijection from V_M(θ) to Spec(M/θ), hence it is an order isomorphism between these ordered sets.

(ii) By the Fundamental Isomorphism Theorem, the map φ : M/Ker(h) → h(M), defined by φ(a/Ker(h)) = h(a) for all a ∈ A, is well defined and it is an isomorphism in C. Hence Con(h(M)) = {φ(γ) | γ ∈ Con(M/Ker(h))} = {φ(θ/Ker(h)) | θ ∈ Ker(h)} = h(Ker(h)).

(iii) By (i) and (ii) and its proof, Spec(h(M)) = {φ(ψ) | ψ ∈ Spec(M/Ker(h))} = {φ(φ/Ker(h)) | φ ∈ V_M(Ker(h))} = h(V_M(Ker(h))).

(iv) By (iii) and the surjectivity of h, Spec(N) = Spec(h(M)) = h(V_M(Ker(h))) and, for all θ ∈ V_M(Ker(h)), h^*(h(θ)) = θ, thus V_M(Ker(h)) = h^*(V_M(Ker(h))) = h^*(Spec(N)).

(v) By the proof of (iii), h(α) ⊆ h(β) iff φ(α/Ker(h)) ⊆ φ(β/Ker(h)) iff α/Ker(h) ⊆ β/Ker(h) iff α ⊆ β.

(vi) By (ii) and its proof, along with (iii) and (iv), for all θ ∈ [Ker(h)), V_M(h(θ)) = h^*(Spec(h(M))) = h(θ) ∩ Spec(h(M)) = {h(φ) | φ ∈ V_M(Ker(h))} = h(φ) | φ ∈ V_M(Ker(h)), h(θ) ≤ h(φ) = {h(φ) | φ ∈ V_M(Ker(h)), θ ⊆ φ} = {h(φ) | φ ∈ Spec(M) ∩ [Ker(h) ∩ θ]} = {h(φ) | φ ∈ Spec(M) ∩ [θ]} = h(φ) | φ ∈ Spec(M) = h^*(Spec(N)), since Ker(h) ⊆ θ.

Proposition 4.12. If h is surjective, then h is admissible and fulfills GU and LO. The converse is not true, not even when C is congruence-distributive and semi-degenerate.

Proof. By Lemma 4.11 (i) h is admissible. By Lemma 4.11 (ii), and Lemma 4.11 (iii), V_M(Ker(h)) = h^*(Spec(N)), thus h fulfills LO. Now let ϕ, ψ ∈ Spec(M) and ϕ_1 ∈ Spec(N) such that h^*(ϕ_1) = ϕ and ϕ ⊆ ψ. Then, by Remark 3.3 Ker(h) ⊆ θ, thus Ker(h) ⊆ ψ, so, since h fulfills LO, it follows that h^*(ψ_1) = ψ for some ψ_1 ∈ Spec(N). We have h^*(ϕ_1) = ϕ ⊆ ψ = h^*(ψ), hence, by the surjectivity of h, ϕ_1 = h(h^*(φ_1)) ⊆ h(h^*(ψ_1)) = ψ. Thus h fulfills GU.

The bounded lattice morphisms l, r, q and m from Example 4.10 are admissible and fulfill GU and LO, but they are not surjective.

Corollary 4.13. For every θ ∈ Con(A), p_θ is admissible and fulfills GU and LO.

Lemma 4.14. (i) If f and g fulfill GU, then g ◦ f fulfills GU.

(ii) If f is surjective and g fulfills LO, then g ◦ f fulfills LO.

(iii) If f and g fulfill LO and g is injective, then g ◦ f fulfills LO.
Remark 3.3, \[ \psi \]

Let \( \phi, \psi \in \text{Spec}(A) \) and \( \phi_2 \in \text{Spec}(C) \) such that \( \phi \subseteq \psi \) and \( \phi = (g \circ f)^* (\phi_2) = f^*(g^* (\phi_2)) \). Denote \( \phi_1 = g^* (\phi_2) \in \text{Spec}(B) \), so that \( f^*(\phi_1) = \phi \). Since \( f \) fulfills GU, it follows that there exists a \( \psi_1 \in \text{Spec}(B) \) such that \( \phi_1 \subseteq \psi_1 \) and \( f^*(\psi_1) = \psi \). Since \( g^* (\phi_2) = \phi_1 \) and \( g \) fulfills GU, it follows that there exists a \( \psi_2 \in \text{Spec}(C) \) such that \( \phi_2 \subseteq \psi_2 \) and \( g^* (\psi_2) = \psi_1 \). Then \( (g \circ f)^*(\psi_2) = f^* (g^* (\psi_2)) = f^*(\psi_1) = \psi \). Therefore \( g \circ f \) fulfills GU.

Since \( f \) is surjective, it follows that \( f \) fulfills LO by Proposition 4.12. Let \( \phi \in \text{Spec}(A) \) such that \( \text{Ker}(g \circ f) \subseteq \phi \). Then, by Remark 3.3, the fact that \( f \) fulfills LO and the surjectivity of \( f \), \( \text{Ker}(f) \subseteq \phi \), there exists a \( \phi_1 \in \text{Spec}(B) \) such that \( f^*(\phi_1) = \phi \supseteq \text{Ker}(g \circ f) = (g \circ f)^*(\Delta C) = (f^* \circ g^*) (\Delta C) = f^*(g^*(\Delta C)) = f^*(\text{Ker}(g)) \), therefore \( \phi_1 = f(f^*(\phi_1)) \supseteq f(f^*(\text{Ker}(g))) = \text{Ker}(g) \), hence there exists a \( \phi_2 \in \text{Spec}(C) \) such that \( g^*(\phi_2) = \phi_1 \), so \( (g \circ f)^*(\phi_2) = f^*(g^* (\phi_2)) = f^* (\phi_1) = \phi \). Thus \( g \circ f \) fulfills LO.

Let \( \phi \in \text{Spec}(A) \) such that \( \text{Ker}(g \circ f) \subseteq \phi \). Then, by Remark 3.3 and the fact that \( f \) fulfills LO, \( \text{Ker}(f) \subseteq \phi \), hence there exists a \( \phi_1 \in \text{Spec}(B) \) such that \( f^*(\phi_1) = \phi \). Since \( g \) is injective and fulfills LO, \( \text{Ker}(g) = g^*(\Delta C) = \Delta_B \subseteq \phi_1 \), hence there exists a \( \phi_2 \in \text{Spec}(C) \) such that \( g^*(\phi_2) = \phi_1 \), so \( (g \circ f)^*(\phi_2) = (f^* \circ g^*) (\phi_2) = f^*(g^* (\phi_2)) = f^*(\phi_1) = \phi \). Thus \( g \circ f \) fulfills LO.

Proposition 4.15. Let \( i : h(M) \to N \) be the canonical embedding. Then:

(i) \( h \) is admissible iff \( i \) is admissible;

(ii) if \( h \) is admissible, then: \( h \) fulfills GU iff \( i \) fulfills GU;

(iii) if \( h \) is admissible, then: \( h \) fulfills LO iff \( i \) fulfills LO.

Proof. Let \( s : M \to h(M) \), for all \( x \in M \), \( s(x) = h(x) \). Then \( s \) is surjective, thus \( s \) is admissible and fulfills GU and LO by Proposition 4.12. We have: \( h = i \circ s \), so \( h^* = s^* \circ i^* \). Since \( s \) is surjective, it follows that \( s^* \) is injective. For all \( \theta \in \text{Con}(N) \), \( i^*(\theta) = \theta \cap h(M)^2 = h(h^*(\theta)) = s(h^*(\theta)) \).

\[ \begin{array}{ccc} M & \xrightarrow{h} & N \\ s \downarrow & & \downarrow \text{h(M)} \\ \text{h(M)} & \xrightarrow{i} & N \end{array} \]

\( s \) is admissible, thus, by Remark 4.19, if \( i \) is admissible, then \( h = s \circ i \) is admissible. Now assume that \( h \) is admissible, and let \( \chi \in \text{Spec}(N) \), so that \( h^*(\chi) \in \text{Spec}(M) \) and, by Remark 3.3, \( \text{Ker}(h) \subseteq h^*(\chi) \), thus \( h^*(\chi) \in V_M(\text{Ker}(h)) \), so that \( i^*(\chi) = h(h^*(\chi)) \in \text{Spec}(h(M)) \) by Lemma 4.11 (iii), hence \( i \) is admissible.

From now until the end of this proof, \( h \) shall be admissible, so that, by (i), \( i \) shall be admissible, too.

(ii) \( s \) fulfills GU, thus, by Lemma 4.14 (iv), if \( i \) fulfills GU, then \( h = s \circ i \) fulfills GU. Now assume that \( h \) fulfills GU, and let \( \phi_1, \psi_1 \in \text{Spec}(h(M)) \) and \( \phi_2 \in \text{Spec}(N) \) such that \( \phi_1 \subseteq \psi_1 \) and \( \phi_1 = i^*(\phi_2) \). Let \( \phi = h^*(\phi_2) \in \text{Spec}(M) \), since \( h \) is admissible, and \( \psi = s^*(\psi_1) \in \text{Spec}(M) \), since \( s \) is admissible. Then \( \phi_1 = i^*(\phi_2) = h(h^*(\phi_2)) = h(\phi) \) and, since \( s \) is surjective, \( \psi_1 = s(s^*(\psi_1)) = s(\psi) = h(\psi) \). We have \( h(\phi) = \phi_1 \subseteq \psi_1 \subseteq h(\psi) \), hence \( \phi \subseteq \psi \) by Lemma 4.11 (vii), so, since \( h \) fulfills GU, it follows that there exists a \( \psi_2 \in \text{Spec}(N) \) such that \( \phi_2 \subseteq \psi_2 \) and \( \psi = h^*(\psi_2) \), so that \( i^*(\psi_2) = h(h^*(\psi_2)) = h(h(\psi)) = \psi_1 \). Therefore \( i \) fulfills GU.

(iii) \( s \) is surjective, thus, by Lemma 4.14 (iii), if \( i \) fulfills LO, then \( h = s \circ i \) fulfills LO. Now assume that \( h \) fulfills LO, and let \( \psi \in \text{Spec}(h(M)) \). Trivially, \( \psi \supseteq \text{Ker}(h) \subseteq \text{Ker}(i) \). Since \( s \) is admissible, \( s^*(\psi) \in \text{Spec}(M) \) and, by Remark 3.3 \( s^*(\psi) \supseteq \text{Ker}(s) = s^*(\Delta_M) = s^*(i^*(\Delta_N)) = h(h^*(\chi)) = \text{Ker}(h) \). Since \( h \) fulfills LO, it follows that there exists a \( \chi \in \text{Spec}(N) \) such that \( s^*(\chi) = h(h^*(\chi)) \), so \( i^*(\chi) = h(h^*(\chi)) = s(s^*(\psi)) = \psi \), by the surjectivity of \( s \). Hence \( i \) fulfills LO.

Corollary 4.16. (i) The following are equivalent:

- any morphism in \( \mathcal{C} \) is admissible;
- any canonical embedding in \( \mathcal{C} \) is admissible.

(ii) The following are equivalent:

- any admissible morphism in \( \mathcal{C} \) fulfills GU;
- any admissible canonical embedding in \( \mathcal{C} \) fulfills GU.

(iii) The following are equivalent:
• any admissible morphism in $\mathcal{C}$ fulfills LO;
• any admissible canonical embedding in $\mathcal{C}$ fulfills LO.

Lemma 4.17. Let $\alpha \in \text{Con}(\mathcal{A})$ and $a, b \in A$. If $f$ is surjective, then:

(i) Proposition 1.2.1,v] $\text{Cg}_B(f(a)) = f(\alpha \lor \text{Ker}(f));$

(ii) Proposition 1.2.2] $\text{Cg}_B(f(\text{Cg}_A(a,b))) = \text{Cg}_B(f(a), f(b)).$

Lemma 4.18. For all $a, b, c, d \in A$, $f([\text{Cg}_A(a,b), \text{Cg}_A(c,d)]_A) \subseteq [\text{Cg}_B(f(a), f(b)], \text{Cg}_B(f(c), f(d))]_B$.

Proof. Let $A \rightarrow f(A)$ is a surjective morphism. Let $a, b, c, d \in A$. By Remark 2.2, $f([\text{Cg}_A(a,b), \text{Cg}_A(c,d)]_A \lor \text{Ker}(f)) = \{f(\text{Cg}_A(a,b) \lor \text{Ker}(f)), f(\text{Cg}_A(c,d) \lor \text{Ker}(f))\}_f(A)$. By Lemma 4.17, $f(\text{Cg}_A(a,b) \lor \text{Ker}(f)) = \text{Cg}_B(f(\text{Cg}_A(a,b))) = \text{Cg}_B(f(a))$ and, analogously, $\text{Cg}_B(f(c)) \lor \text{Ker}(f)) \subseteq \text{Cg}_B(f(c), f(d)) \lor \text{Ker}(f))$. By Proposition 2.6 and Lemma 2.7, it follows that $\text{Cg}_B(f(a), f(b)), \text{Cg}_B(f(c), f(d)) \lor \text{Ker}(f))$. Trivially, $f([\text{Cg}_A(a,b), \text{Cg}_A(c,d)]_A) \subseteq f([\text{Cg}_A(a,b), \text{Cg}_A(c,d)]_A \lor \text{Ker}(f))$. From all the above, it follows that $f([\text{Cg}_A(a,b), \text{Cg}_A(c,d)]_A \subseteq [\text{Cg}_B(f(a), f(b)], \text{Cg}_B(f(c), f(d))]_B$.

Lemma 4.19. (i) If $S$ is an m–system in $M$, then $h(S)$ is an m–system in $N$.

(ii) If $M \subseteq N$ and $S$ is an m–system in $M$, then $S$ is an m–system in $N$.

Proof. Let $(x, y), (z, u) \in h(S)$, so that $x = h(a), y = h(b), z = h(c), u = h(d)$ for some $(a, b, c, d) \in S \subseteq M^2$. Since $S$ is an m-system, it follows that $[\text{Cg}_M(a,b), \text{Cg}_M(c,d)]_M \cap S \neq \emptyset$, thus, by Lemma 4.18, $\emptyset \neq h([\text{Cg}_M(a,b), \text{Cg}_M(c,d)]_M \cap S) \subseteq h([\text{Cg}_N(a,h(b), \text{Cg}_N(c,h(d)]_N \cap h(S) \subseteq [\text{Cg}_N(x,y), \text{Cg}_N(z,u)]_N \cap h(S) \neq \emptyset$. Therefore $h(S)$ is an m–system.

Let $i : M \rightarrow N$ be the canonical embedding. Then $i$ is a morphism, so, by (i), $i(S) = S$ is an m–system in $N$.

Lemma 4.20. If $A \subseteq B$, $\phi \in \text{Spec}(\mathcal{A})$, $\nabla_B$ is finitely generated and $\psi$ is a maximal element of the set $\{\theta \in \text{Con}(\mathcal{B}) \mid \theta \cap (\nabla_B \setminus \phi = \emptyset\}$, then $\psi \in \text{Spec}(\mathcal{B})$.

Proof. Since $\phi \in \text{Spec}(\mathcal{A})$, by Lemma 2.15 it follows that $\nabla_A \setminus \phi$ is an m–system in $A$. Then, by Lemma 4.19 (ii), $\nabla_A \setminus \phi$ is an m–system in $B$. By Lemma 2.10 it follows that $\psi \in \text{Spec}(\mathcal{B})$.

Proposition 4.21. Assume that $\nabla_B$ is finitely generated, $A \subseteq B$ and the canonical embedding $i : A \rightarrow B$ is admissible. Then the following are equivalent:

(i) $i$ fulfills GU;

(ii) for all $\phi \in \text{Spec}(\mathcal{A})$, if $\psi$ is a maximal element of the set $\{\theta \in \text{Con}(\mathcal{B}) \setminus \{\nabla_B\} \mid \theta \cap (\nabla_B \setminus \phi = \emptyset\}$, then $\psi \cap A^2 = \phi$.

Proof. (i) $\Rightarrow$ (ii): Let $\phi$ and $\psi$ be as in the enunciation. Then $\psi \in \text{Spec}(\mathcal{B})$ by Lemma 4.20. Let $\phi_0 = \psi \cap A^2 = i^*(\psi) \in \text{Spec}(\mathcal{A})$, because $\psi$ is admissible. We have: $\emptyset = \emptyset \cap (\nabla_A \setminus \phi) = (\psi \cap \nabla_A \setminus \phi) \setminus (\psi \cap A^2) \setminus (\psi \cap \phi)$, thus $\psi \cap A^2 \subseteq \psi \cap \phi \subseteq \psi \cap A^2$, because $\phi \subseteq A^2$. Hence $\phi_0 = \psi \cap A^2 = \psi \cap \phi \subseteq \phi$. Since $i$ fulfills GU, it follows that there exists a $\psi_1 \in \text{Spec}(\mathcal{B})$ such that $\psi_1 \cap A^2 = i^*(\psi_1) = \phi_0$ and $\psi \subseteq \psi_1$. Then $\psi_1 \not\subseteq \nabla_B$, $\psi_1 \cap \nabla_A \setminus \phi = \psi_1 \cap \nabla_A \setminus \nabla_A \setminus \phi = \phi_0 \cap \nabla_A \setminus \phi \subseteq \phi \cap (\nabla_A \setminus \phi) = \emptyset$, because $\phi_0 \subseteq \phi$; so $\psi_0 \cap (\nabla_A \setminus \phi) = \emptyset$. Then it is straightforward that $\psi$ is a maximal element of the set $\{\theta \in \text{Con}(\mathcal{A}) \setminus \{\nabla_A\} \mid \theta \cap (\nabla_A \setminus \phi = \emptyset\}$, so $\psi \in \text{Spec}(\mathcal{B})$ by Lemma 4.20, and $i^*(\psi) = \psi \cap A^2 = \psi$ by the hypothesis of this implication. Thus $i$ fulfills GU.

Proposition 4.22. Assume that $\nabla_B$ is finitely generated, $A \subseteq B$ and the canonical embedding $i : A \rightarrow B$ is admissible. Then: if $i$ fulfills GU, then $i$ fulfills LO.
Lemma 5.1. (i) If \( \phi \in \text{Spec}(A) \); of course, \( \nabla_A = \nabla_B \cap A^2 = i^*(\nabla_B) \subseteq \phi \). Then, by Proposition 4.21 and Lemma 4.20 there exists a \( \psi \in \text{Spec}(B) \) such that \( i^*(\psi) = \psi \cap A^2 = \phi \), therefore \( i \) fulfills LO.

Proposition 4.23. If \( \nabla_B \) is finitely generated and \( f \) fulfills GU, then \( f \) fulfills LO, but the converse is not true.

Proof. By Propositions 4.13 and 4.22 if \( f \) fulfills GU, then \( f \) fulfills LO.

See in [16, Exercise 3, p. 41] a type of ring extension which proves that not all admissible morphisms fulfilling LO from a semi–degenerate congruence–modular equational class also fulfill GU.

Corollary 4.24. If \( C \) is semi–degenerate, then, in \( C \), GU implies LO, but the converse is not true.

Proof. By Propositions 2.11 and 4.23.

5 Going Up and Lying Over in Particular Cases

In this section we list some cases in which admissibility and GU hold and we show how admissibility, GU and LO relate to each other in some particular cases. Throughout this section, \( C \) shall be congruence–modular, \( A, B, M, N \) shall be members of \( C \), \( f : A \to B \) shall be an admissible morphism in \( C \) and \( h : M \to N \) shall be a morphism in \( C \), not necessarily admissible.

Lemma 5.1. (i) If \( \text{Spec}(A) = \text{Max}(A) \), then \( f \) fulfills GU.

(ii) If the commutator in \( A \) equals the intersection of congruences and \( \text{Con}(A) \) is a Boolean algebra, then \( f \) fulfills GU.

(iii) If \( C \) is congruence–distributive and \( \text{Con}(A) \) is a Boolean algebra, then \( f \) fulfills GU.

Proof. Let \( \phi, \psi \in \text{Spec}(A) \) and \( \phi_1 \in \text{Spec}(B) \) such that \( f^*(\phi_1) = \phi \) and \( \phi \subseteq \psi \). Then, by Remark 3.11 \( \phi = \psi \), so we may take \( \psi_1 = \phi_1 \in \text{Spec}(B) \) and we have: \( \phi_1 = \psi_1 \subseteq \psi_1 \) and \( f^*(\psi_1) = f^*(\phi_1) = \phi = \psi \), hence \( f \) fulfills GU.

(i) By (i) and Lemma 3.9 (iv).

(ii) By (i) and Lemma 3.9 (iv).

Proposition 5.2. (i) If \( C \) is semi–degenerate, \( \text{Spec}(M) = \text{Max}(M) \) and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

(ii) If \( C \) is semi–degenerate, the commutator in \( M \) equals the intersection of congruences, \( \text{Con}(M) \) is a Boolean algebra and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

(iii) If \( C \) is semi–degenerate and congruence–distributive, \( \text{Con}(M) \) is a Boolean algebra and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

(iv) If \( h^*(\{\nabla_M\}) = \{\nabla_N\} \), \( \text{Spec}(M) = \text{Max}(M) \) and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

(v) If \( h^*(\{\nabla_M\}) = \{\nabla_N\} \), the commutator in \( M \) equals the intersection, \( \text{Con}(M) \) is a Boolean algebra and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

(vi) If \( h^*(\{\nabla_M\}) = \{\nabla_N\} \), \( C \) is congruence–distributive, \( \text{Con}(M) \) is a Boolean algebra and \( \text{Spec}(N) = \text{Con}_2(N) \), then \( h \) is admissible and fulfills GU.

Proof. By Lemmas 3.21 and 5.1.

Proposition 5.3. Any morphism in the class of bounded distributive lattices is admissible and fulfills GU.

Proof. By Proposition 3.14 (i), Lemma 5.23 (ii), and Lemma 5.1 (ii).
If \( m \) is admissible, then \( m \) fulfills GU.

(ii) If \( L' \) is distributive, then \( m \) is admissible and fulfills GU.

(iii) If \( m(L) = \{0, 1\} \), then \( m \) is admissible and fulfills GU.

Proof. (i) By Lemma 3.23 (i), andLemma 5.1.
(ii) By (i) and Proposition 5.14 (ii).
(iii) By (ii) and Remark 5.23.

Remark 5.5. Of course, Lemma 3.23 (i), and Lemma 5.1 (ii), show that, if \( L \) is a bounded lattice with \( \text{Spec}(L) = \text{Max}(L) \) and \( L' \) is a bounded distributive lattice, then any bounded lattice morphism \( m : L \to L' \) is admissible and fulfills GU. See in [22] examples of finite lattices whose lattice of congruences is Boolean, thus whose prime congruences coincide to their maximal ones, and which can not be obtained through direct products and/or ordinal sums from modular lattices and relatively complemented lattices.

Proposition 5.6. (i) If \( f \) fulfills LO and \( (\text{Spec}(B), \subseteq) \) is a chain, then \( f \) fulfills GU.

(ii) If \( f \) fulfills LO and \( \text{Con}(B) \) is a chain, then \( f \) fulfills GU.

Proof. (i) Let \( \phi, \psi \in \text{Spec}(A) \) and \( \phi_1 \in \text{Spec}(B) \) such that \( f^*(\phi_1) = \phi \) and \( \phi \subseteq \psi \). If \( \phi = \psi \), then we may take \( \psi_1 = \phi_1 \), as in the proof of Lemma 5.1. Now assume that \( \phi \neq \psi \). By Remark 3.6, we have \( \psi \supseteq \phi \supseteq \ker(f) \), thus, since \( f \) fulfills LO, there exists a \( \psi_1 \in \text{Spec}(B) \) such that \( f^*(\psi_1) = \psi \). Assume by absurdum that \( \phi \nsubseteq \psi_1 \), so that \( \psi_1 \in \phi_1 \) since \( (\text{Spec}(B), \subseteq) \) is totally ordered. Then \( \psi = f^*(\psi_1) \subseteq f^*(\phi_1) = \phi \), thus, since \( \phi \subseteq \psi \), it follows that \( \phi = \psi \), and we have a contradiction. Hence \( \phi_1 \subseteq \psi_1 \), therefore \( f \) fulfills GU.

(ii) By (i).

Proposition 5.7. (i) If \( \text{Con}(M) = \{\Delta_M, \nabla_M\} \), \( M \) is non-trivial and \( h^*(\{\nabla_M\}) = \{\nabla_N\} \), then \( h \) is admissible and fulfills GU.

(ii) If \( \text{Con}(M) = \{\Delta_M, \nabla_M\} \) and \( C \) is semi-degenerate, then \( h \) is admissible and fulfills GU.

Proof. (i) Since \( M \) is non-trivial, we have \( \Delta_M \neq \nabla_M \), hence \( \text{Con}(M) = \{\Delta_M, \nabla_M\} \cong \mathcal{L}_2 \), thus \( \text{Spec}(M) = \text{Max}(M) = \{\Delta_M\} = \text{Con}(M) \setminus \{\nabla_M\} \). By Lemma 3.24 (i), andLemma 5.1 (ii), it follows that \( h \) is admissible and fulfills GU.

(ii) If \( M \) is the trivial algebra, then so is \( h(M) \), thus so is \( N \), because \( h(M) \) is a subalgebra of \( N \) and \( C \) is semi-degenerate. In this case, \( h \) is an isomorphism, thus \( h \) is admissible and fulfills GU. Now assume that \( M \) is non-trivial. Then \( h \) is admissible and fulfills GU by (i) and Lemma 5.1 (ii).

Example 5.8. \( \text{Con}(\mathcal{L}_2) \cong \mathcal{L}_2 \), because \( \mathcal{L}_2 \) is a finite Boolean algebra, thus \( \text{Con}(\mathcal{L}_2) = \{\Delta_{\mathcal{L}_2}, \nabla_{\mathcal{L}_2}\} \). We have seen in Example 3.17 that \( \text{Con}(\mathcal{D}) = \{\Delta_{\mathcal{D}}, \nabla_{\mathcal{D}}\} \). Therefore, by Proposition 5.7 (ii), any bounded lattice morphism whose domain is \( \mathcal{L}_2 \) or \( \mathcal{D} \) is admissible and fulfills GU. Many examples of such bounded lattices can be given. See some in [24], including one that is finite and can not be obtained through direct products and/or ordinal sums from modular lattices and relatively complemented lattices.

6 Going Up and Lying Over in Direct Products of Algebras and Ordinal Sums of Bounded Ordered Structures

In this section, we prove that admissibility, GU and LO are preserved by finite direct products and, in the class of bounded lattices, also by finite ordinal sums; actually, the latter holds in any congruence–modular equational class of bounded ordered structures that fulfills a certain condition on congruences. Throughout this section, \( C \) shall be congruence–modular, \( n \in \mathbb{N}^* \), \( A_1, \ldots, A_n, B_1, \ldots, B_n \) shall be algebras from \( C \), \( f_i : A_i \to B_i \) shall be a morphism in \( C \) for all \( i \in [1, n] \), \( A = \prod_{i=1}^{n} A_i \), \( B = \prod_{i=1}^{n} B_i \) and \( f = \prod_{i=1}^{n} f_i : A \to B \). We shall also assume that \( C \) fulfills the equivalent conditions from Proposition 2.12. Recall from Lemma 2.13 that this is the case if \( C \) is semi-degenerate or congruence–distributive.
Remark 6.1. Under the assumptions above, every $\beta \in \text{Con}(B)$ is of the form $\beta = \prod_{i=1}^{n} \beta_i$ for some $\beta_1 \in \text{Con}(A_1), \ldots, \beta_n \in \text{Con}(A_n)$, so that $f^*(\beta) = (\prod_{i=1}^{n} f_i)^*(\prod_{i=1}^{n} \beta_i) = \prod_{i=1}^{n} f_i^*(\beta_i)$. Therefore $\text{Ker}(f) = f^*(\Delta_B) = f^*(\prod_{i=1}^{n} \Delta_{B_i}) = \prod_{i=1}^{n} f_i^*(\Delta_{B_i}) = \prod_{i=1}^{n} \text{Ker}(f_i)$.

Lemma 6.2. \cite{22} $\text{Spec}(A) = \bigcup_{i=1}^{n} \{ \phi \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j} \mid \phi \in \text{Spec}(A_i) \}$.

Proposition 6.3. (i) For any $i \in \mathbb{T}$, and any $\theta \in \text{Con}(A_i)$, $V_A(\theta) = \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j}$ is admissible iff $\alpha \in \text{Spec}(A)$ and $\alpha \geq \theta \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j}$ for some $\phi \in V_A(\theta)$.

(iii) If $\theta \in \text{Con}(A_i)$ for all $i \in \mathbb{T}$, then $V_A(\prod_{i=1}^{n} \theta_i) = \bigcup_{i=1}^{n} \{ \phi \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j} \mid \phi \in V_A(\theta_i) \}$.

Proof. Let $\alpha \in \text{Con}(A)$. By Lemma 6.2, $\alpha \in V_A(\prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j})$ iff $\alpha \in \text{Spec}(A)$ and $\alpha \geq \theta \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j}$ for some $\phi \in V_A(\theta)$.

Corollary 6.4. (i) $f$ is admissible iff $f_1, \ldots, f_n$ are admissible;

(ii) if $f$ is admissible, then $f$ fulfills $\text{GU}$ iff $f_1, \ldots, f_n$ fulfill $\text{GU}$;

(iii) if $f$ is admissible, then: $f$ fulfills $\text{LO}$ iff $f_1, \ldots, f_n$ fulfill $\text{LO}$.

Proof. This is a result in \cite{22}, which follows immediately from Remark 6.1, Lemma 6.2 and Proposition 6.3 (ii) By Lemma 4.4 (ii), Remark 6.1 Proposition 6.3 (ii), and the fact that $f_i^*(\nabla_{B_i}) = \nabla_{A_i}$ for all $i \in \mathbb{T}$).

Corollary 6.4. (iii) $V_A(\text{Ker}(f)) \subseteq f^*(\text{Spec}(B))$ iff $V_A(\text{Ker}(f)) = f^*(\text{Spec}(B))$ iff $V_A(\prod_{i=1}^{n} \text{Ker}(f_i)) = f^*(\prod_{i=1}^{n} \{ \phi \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j} \mid \phi \in \text{Spec}(B_i) \}) = \bigcup_{i=1}^{n} \{ \phi \times \prod_{j \in \mathbb{T} \setminus \{i\}} \nabla_{A_j} \mid \phi \in \text{Spec}(B_i) \}$)

For any bounded lattices $L$ and $M$, we shall denote by $L \oplus M$ the ordinal sum of $L$ with $M$ and, if $\alpha \in \text{Con}(L)$ and $\beta \in \text{Con}(M)$, then we denote by $\alpha \oplus \beta = eq((L/\alpha \cap c/\alpha) \cup (M/\beta \cap c/\beta) \cup \{ c/\alpha \cup c/\beta \})$, where $c$ is the common element of $L$ and $M$ in $L \oplus M$. If $L'$ and $M'$ are bounded lattices and $h : L \to M$ and $h' : L' \to M'$ are bounded...
lattice morphisms, then we define $h \oplus h' : L \oplus L' \to M \oplus M'$ by: for all $x \in L \oplus L'$, $(h \oplus h')(x) = \begin{cases} h(x) & x \in L, \\ h'(x) & x \in L'. \end{cases}$

Then, clearly, $\alpha \oplus \beta \in \text{Con}(L \oplus M)$ and $h \oplus h'$ is a bounded lattice morphism.

Throughout the rest of this section, for all $i \in \mathbb{T}_{nn}, L_i, M_i$ shall be bounded lattices and $h_i : L_i \to M_i$ shall be a bounded lattice morphism. We shall denote by $L = \bigoplus_{i=1}^n L_i, M = \bigoplus_{i=1}^n M_i$ and $h = \bigoplus_{i=1}^n h_i : L \to M$.

Remark 6.5. Let $\beta_i \in \text{Con}(M_i)$ for all $i \in \mathbb{T}_{nn}$ and $\beta = \bigoplus_{i=1}^n \beta_i \in \text{Con}(M)$. Then $h^*(\beta) = \bigoplus_{i=1}^n h_i^*(\beta_i)$. Therefore $\text{Ker}(h) = h^*(\Delta_M) = h^*(\bigoplus_{i=1}^n \Delta_{M_i}) = \bigoplus_{i=1}^n h_i^*(\Delta_{M_i}) = \bigoplus_{i=1}^n \text{Ker}(h_i)$.

Lemma 6.6. \[\text{Con}(L) = \{ \bigoplus_{i=1}^n \theta_i \mid (\forall i \in \mathbb{T}_{nn}) (\theta_i \in \text{Con}(L_i)) \} \cong \prod_{i=1}^n \text{Con}(L_i).\]

Remark 6.9. If $h$ is admissible iff $h_1, \ldots, h_n$ are admissible; if $h$ is admissible, then: $h$ fulfills GU iff $h_1, \ldots, h_n$ fulfill GU; if $h$ is admissible, then: $h$ fulfills LO iff $h_1, \ldots, h_n$ fulfill LO.

Proof. Similar to that of Proposition 6.3 but using Lemma 6.6 instead of Lemma 6.2.

Corollary 6.8. Let $h_i \in \text{Con}(L_i)$ for all $i \in \mathbb{T}_{nn}$, then $h_i \in \text{Con}(L_i)$ for all $i \in \mathbb{T}_{nn}$. Then $h_i \in \text{Con}(L_i)$ for all $i \in \mathbb{T}_{nn}$.

Proposition 6.7. (i) For all $i \in \mathbb{T}_{nn}$ and all $\theta \in \text{Con}(L_i)$, $V_L(\bigoplus_{i=1}^n \theta) = \{ \bigoplus_{i=1}^n \theta \mid \theta \in \text{Con}(L_i) \}$. (ii) If $\theta_i \in \text{Con}(L_i)$ for all $i \in \mathbb{T}_{nn}$, $V_L(\bigoplus_{i=1}^n \theta_i) = \{ \bigoplus_{i=1}^n \theta_i \mid \theta_i \in \text{Con}(L_i) \}$.

Proof. Similarly as in Corollary 6.8.

Remark 6.9. Lemma 6.6 and Proposition 6.7 hold for any congruence–modular equational class of bounded ordered structures whose finite ordinal sums have the congruences of the form in Lemma 6.6 and Corollary 6.8.

7 Characterizations for Properties Going Up and Lying Over

In this section, we obtain several characterizations for properties GU and LO, including topological ones, and prove that GU and LO are preserved by quotients. Throughout this section, $C$ shall be congruence–modular, $A$ and $B$ shall be algebras from $C$ and $f : A \to B$ shall be a morphism in $C$.

For every $\beta \in \text{Con}(B)$, we define $f_\beta : A/f^*(\beta) \to B/\beta$, for all $a \in A$, $f_\beta(a/f^*(\beta)) = f(a)/\beta$.

Remark 7.1. For each $\beta \in \text{Con}(B)$, $f_\beta$ is well defined and injective, because, for all $a, b \in A$, $a/f^*(\beta) = b/f^*(\beta)$ iff $(a, b) \in f^*(\beta)$ iff $(f(a), f(b)) \in \beta$ iff $f(a)/\beta = f(b)/\beta$ iff $f_\beta(a/f^*(\beta)) = f_\beta(b/f^*(\beta))$. Clearly, $f_\beta$ is a morphism in C and, if f is surjective, then $f_\beta$ is surjective. Also, the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{p_{f^*(\beta)}} & & \downarrow{p_{\beta}} \\
A/f^*(\beta) & \xrightarrow{f_\beta} & B/\beta
\end{array}
\]
For all $\psi \in \beta$, $f_\beta^*(\psi/\beta) = f^*(\psi)/f^*(\beta)$, because: $f_\beta^*(\psi/\beta) = \{(a/f^*(\beta), b/f^*(\beta)) \mid a, b \in A, (f(a)/\beta, f(b)/\beta) \in \psi/\beta\} = \{(a/f^*(\beta), b/f^*(\beta)) \mid (a, b) \in A(f(a), f(b)) \in \psi\} = \{(a/f^*(\beta), b/f^*(\beta)) \mid (a, b) \in f^*(\psi)\} = f^*(\psi)/f^*(\beta)$.

**Lemma 7.2.** $f$ is admissible iff, for each $\beta \in \text{Con}(B)$, $f_\beta$ is admissible.

**Proof.** For the converse implication, take $\beta = \Delta_B$, so that $p_\beta = p_{\Delta_B} : B \to B/\Delta_B$ is an isomorphism. By Lemma 3.11(iii) and Remarks 3.19 and 3.20, $p_{f_*(\Delta_B)} = p_{\text{Ker}(f)}$ is surjective and thus admissible, thus, since $f_{\Delta_B}$ is admissible, $f \circ p_{\Delta_B} = f_{\Delta_B} \circ p_{\text{Ker}(f)}$ is admissible, hence, $f$ is admissible.

Now assume that $f$ is admissible and let $\beta \in \text{Con}(B)$. By Lemma 4.11(iii), Spec($A/f^*(\beta)$) = { $\phi/f^*(\beta) \mid \phi \in V_A(f^*(\beta))$, Spec($B/\beta$) = { $\chi/\beta \mid \chi \in V_B(\beta)$, and, by Remark 3.13, $f^*(V_B(\beta)) \subseteq V_A(f^*(\beta))$, hence, for all $\chi \in V_B(\beta)$, $f^*_\beta(\chi/\beta) = \{(a/f^*(\beta), b/f^*(\beta)) \mid (a, b) \in A^2, (f_\beta(a/f^*(\beta)), f_\beta(b/f^*(\beta))) \in \chi/\beta\} = \{(a/f^*(\beta), b/f^*(\beta)) \mid (a, b) \in A^2, (f(a), f(b)) \in \chi\} = \{(a/f^*(\beta), b/f^*(\beta)) \mid (a, b) \in f^*(\chi)\} = f^*(\chi)/f^*(\beta)$, therefore $f^*_\beta(\text{Spec}(B/\beta)) \subseteq \text{Spec}(A/f^*(\beta))$, that is $f_\beta$ is admissible.

**Proposition 7.3.** If $V_B$ is finitely generated and $f$ is admissible, then the following are equivalent:

(i) $f$ fulfills GU;

(ii) for all $\beta \in \text{Spec}(B)$, the map $f_\beta |_{\text{Spec}(B/\beta)} : \text{Spec}(B/\beta) \to \text{Spec}(A/f^*(\beta))$ is surjective;

(iii) the map $f^* |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A)$ is closed with respect to the Stone topologies;

(iv) for all $\beta \in \text{Con}(B)$, $f_\beta$ fulfills GU;

(v) for all $\beta \in \text{Con}(B)$, $f_\beta$ fulfills LO;

(vi) for all $\beta \in \text{Spec}(B)$, $f_\beta$ fulfills GU;

(vii) for all $\beta \in \text{Spec}(B)$, $f_\beta$ fulfills LO.

Moreover, (i), (ii), (iii) and (iv) are equivalent even if $V_B$ is not finitely generated.

**Proof.** Since $f$ is admissible, the map $f^* |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A)$ is well defined and, by Remark 3.13 so is the map $f^* |_{V_B(\beta)} : V_B(\beta) \to V_A(f^*(\beta))$, for any $\beta \in \text{Con}(B)$. By Lemma 3.13 for all $\beta \in \text{Con}(B)$, $f_\beta$ is admissible, so that the map $f_\beta |_{\text{Spec}(B/\beta)} : \text{Spec}(B/\beta) \to \text{Spec}(A/f^*(\beta))$ is well defined.

Let $\beta \in \text{Spec}(B)$. Let $g_A : V_A(f^*(\beta)) \to \text{Spec}(A/f^*(\beta))$ and $g_B : V_B(\beta) \to \text{Spec}(B/\beta)$ be the bijections established in Lemma 4.11(ii) for all $\phi \in V_A(f^*(\beta))$ and all $\psi \in V_B(\beta)$, $g_A(\phi) = \phi/f^*(\beta)$ and $g_B(\psi) = \psi/\beta$.

Then the following diagram is commutative:

\[
\begin{array}{ccc}
V_B(\beta) & \xrightarrow{f^*} & V_A(f^*(\beta)) \\
\downarrow{g_B} & & \downarrow{g_A} \\
\text{Spec}(B/\beta) & \xrightarrow{f^*} & \text{Spec}(A/f^*(\beta)) \\
\end{array}
\]

Indeed, for all $\chi \in V_B(\beta)$, $g_A(f^*(\chi)) = f^*(\chi)/f^*(\beta) = f^*(\chi/\beta) = f_\beta^*(g_B(\chi))$ by Remark 3.13, thus $f_\beta^* \circ g_B = g_A \circ f^*$. Since $g_A$ and $g_B$ are bijections, it follows that: $f^* |_{V_B(\beta)} : V_B(\beta) \to V_A(f^*(\beta))$ is surjective iff $f^* |_{\text{Spec}(B/\beta)} : \text{Spec}(B/\beta) \to \text{Spec}(A/f^*(\beta))$ is surjective, that is: $f^*(V_B(\beta)) = V_A(f^*(\beta))$ iff $f^*(\text{Spec}(B/\beta)) = \text{Spec}(A/f^*(\beta))$.

By Lemma 4.14(i), it follows that: $f$ fulfills GU iff, for all $\beta \in \text{Spec}(B)$, $f^*(V_B(\beta)) = V_A(f^*(\beta))$ iff, for all $\beta \in \text{Spec}(B)$, $f^*(\text{Spec}(B/\beta)) = \text{Spec}(A/f^*(\beta))$ iff, for all $\beta \in \text{Spec}(B)$, the map $f_\beta |_{\text{Spec}(B/\beta)} : \text{Spec}(B/\beta) \to \text{Spec}(A/f^*(\beta))$ is surjective.

Let $\beta \in \text{Con}(B)$ and $\psi \in V_B(\beta)$, arbitrary, so that $\psi/\beta \in \text{Spec}(B/\beta)$, arbitrary. Then $(f_\beta)(\psi/\beta) : (A/f^*(\beta))/f_\beta^*(\psi/\beta) \to (B/\beta)(\psi/\beta)$ is defined by: for all $a \in A$, $(f_\beta)(\psi/\beta)(a/f^*(\beta))/f_\beta^*(\psi/\beta) = f_\beta(a/f^*(\beta))/f_\beta^*(\psi/\beta)$, by Remark 7.1 and Lemma 7.2 a well-defined admissible injective morphism in $C$. By Remark 7.1, $f_\beta^*(\psi/\beta) = f^*(\psi)/f^*(\beta)$. Let $g : A(f^*(\psi)) \to (A/f^*(\beta))/f_\beta^*(\psi/\beta)$ and $h : B/\psi \to (B/\beta)(\psi/\beta)$ be the isomorphisms given by the Second Isomorphism Theorem: for all $a \in A$, $g(a/f^*(\psi)) = (a/f^*(\beta))/(f^*(\psi)/f^*(\beta)) = (a/f^*(\beta))/f_\beta^*(\psi/\beta)$, and, for all $b \in B$, $h(b/\psi) = (b/\beta)(\psi/\beta)$. Then, clearly, $g^* : \text{Spec}(A(f^*(\beta))/f_\beta^*(\psi/\beta)) \to \text{Spec}(A/f^*(\psi))$ and $h^* : \text{Spec}(B(\psi/\beta)) \to \text{Spec}(B/\beta)$ are bijections, and the following diagram is commutative:
Since $V_B(\Delta_B) = \text{Spec}(B)$, by the equivalence $\mathbf{1} \iff \mathbf{1}$ proven above it follows that: $f$ fulfills GU iff, for all $\beta \in \text{Con}(B)$ and all $\psi \in V_B(\beta)$, $f^*_\psi : \text{Spec}(B/\psi) \to \text{Spec}(A/f^*(\psi))$ is surjective; if, for all $\beta \in \text{Con}(B)$ and all $\psi \in V_B(\beta)$, $(f^*_\psi)^{-1} : \text{Spec}(A/f^*(\psi)) \to \text{Spec}(\text{Con}(B_B/\beta)/(\psi/\beta))$ is well defined. By Lemma 4.6, (i), and again Remark 7.1 it follows that $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is a closed function. Hence $V_A(f^*(\beta)) = \{f^*(\beta)\} \subseteq f^*(V_B(\beta))$. Therefore $f$ fulfills GU by Lemma 4.4, (ii).}

Let $\beta \in \text{Con}(B)$, so that $V_B(\beta)$ is an arbitrary closed set in $\text{Spec}(B)$ with the Stone topology. The equivalence $\mathbf{1} \iff \mathbf{1}$ follows from the above, so, since $f$ fulfills GU, $f^*_\psi$ fulfills GU, hence $f^*_\psi$ fulfills LO by Proposition 4.23. By Remark 7.1 $f^*_\psi$ is injective. By Proposition 4.6, (ii), and again Remark 7.1 it follows that $f^*_\psi : \text{Spec}(B_B/\psi) = \text{Spec}(A/\text{Con}(\text{Spec}(B_B/\psi)))$, that is $f^*(\psi) = \{f^*(\psi)\} \subseteq f^*(V_B(\beta))$, which is a closed set in $\text{Spec}(A)$ with respect to the Stone topology, since $V_B(\beta)$ is closed in $\text{Spec}(B)$ and $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is a closed function. Hence $V_A(f^*(\beta)) = \{f^*(\beta)\} \subseteq f^*(V_B(\beta))$. Therefore the map $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is closed with respect to the Stone topologies.

Let us define $\varphi_f : \text{Con}(B) \to \text{Con}(A/\text{Ker}(f))$, for all $\beta \in \text{Con}(B)$, $\varphi_f(\beta) = f^*(\beta)/\text{Ker}(f)$. Here is a generalization of Proposition 4.6.

**Lemma 7.4.** If $f$ is admissible, then the restriction $\varphi_f |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A/\text{Ker}(f))$ is well defined and the following are equivalent:

(i) $f$ fulfills LO;

(ii) the map $\varphi_f |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A/\text{Ker}(f))$ is surjective.

**Proof.** By Remark 8.1 and Lemma 4.11, $f^*(\text{Spec}(B)) \subseteq V_A(\text{Ker}(f))$, hence $\varphi_f(\text{Spec}(B)) \subseteq \text{Spec}(A/\text{Ker}(f))$, so the restriction $\varphi_f |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A/\text{Ker}(f))$ is well defined. By Lemma 4.3, (ii), and again Lemma 4.11, (i), $f$ fulfills LO iff $f^*(\text{Spec}(B)) = V_A(\text{Ker}(f))$ iff $\varphi_f(\text{Spec}(B)) = \text{Spec}(A/\text{Ker}(f))$ iff the map $\varphi_f |_{\text{Spec}(B)} : \text{Spec}(B) \to \text{Spec}(A/\text{Ker}(f))$ is surjective.

For every $\theta \in \text{Con}(A)$, we shall denote by $\rho(\theta) = \bigcap_{\phi \in V_A(\theta)} \phi$, that is the intersection of the prime congruences of $A$ which include $\theta$; $\rho(\theta)$ is called the radical of $\theta$. Clearly, if $\theta \in \text{Spec}(A)$, then $\rho(\theta) = \theta$. Actually, $\rho(\theta) = \theta$ iff $\theta$ is an intersection of prime congruences.

**Lemma 7.5.** If $f$ is admissible, $\nabla_B$ is finitely generated and $\phi \in \text{Spec}(A)$, then the following are equivalent:

(i) there exists a $\psi \in \text{Spec}(B)$ such that $f^*(\psi) = \phi$;

(ii) $f^*(\text{Con}(g_B(f(\phi)))) = \phi$. 

\[ A/f^*(\psi) \xrightarrow{f^*_\psi} B/\psi \]
\[ h \]
\[(A/f^*(\beta))/f^*_\beta(\psi/\beta) \xrightarrow{(f^*_\beta)^{-1}(\psi/\beta)} (B/\beta)/\psi \]

From this, it follows that the following diagram is commutative, that is $g^* \circ (f^*_\beta)^{-1} = f^*_\psi \circ h^*$, therefore, since $g^*$ and $h^*$ are bijective: $f^*_\psi |_{\text{Spec}(B/\psi)} : \text{Spec}(B/\psi) \to \text{Spec}(A/f^*(\psi))$ is surjective iff $(f^*_\beta)^{-1} |_{\text{Spec}(B/\beta)/(\psi/\beta)} : \text{Spec}(B/\beta)/(\psi/\beta)) \to \text{Spec}((A/f^*(\beta))/f^*_\beta(\psi/\beta))$ is surjective.
Proof. \( \text{(i) } \Rightarrow \text{(ii)} \): Since \( \phi = f^*(\psi) \), it follows that \( f(\phi) = f(f^*(\psi)) = \psi \cap f(A^2) \subseteq \psi \), hence \( C_{GB}(f(\phi)) \subseteq \psi \), thus \( f^*(C_{GB}(f(\phi))) \subseteq f^*(\psi) = \phi \). We also have \( \phi \subseteq f^*(\psi) \subseteq f^*(C_{GB}(f(\phi))) \). Therefore \( f^*(C_{GB}(f(\phi))) = \phi \).

\( \text{(ii) } \Rightarrow \text{(iii)} \): By Lemma 2.15 and Lemma 4.19, \( \nabla_A \setminus \phi \) is an m-system in \( A \), hence \( f(\nabla_A \setminus \phi) = f(A^2 \setminus \phi) \) is an m-system in \( B \). Let us notice that \( f(A^2 \setminus \phi) \cap C_{GB}(f(\phi)) = \emptyset \). Indeed, assume by absurdum that there exists a \((u,v) \in f(A^2 \setminus \phi) \cap C_{GB}(f(\phi)) \), that is there exists an \((x,y) \in A^2 \setminus \phi \) such that \( (f(x),f(y)) \in C_{GB}(f(\phi)) \), which means that \((x,y) \in (A^2 \setminus \phi) \cap f^*(C_{GB}(f(\phi))) = (A^2 \setminus \phi) \cap \emptyset = \emptyset \); we have a contradiction. Now let \( \psi \in \text{Con}(B) \) be a maximal element of the set of congruences \( \theta \) of \( B \) which fulfill \( C_{GB}(f(\phi)) \subseteq \theta \) and \( \theta \cap f(A^2 \setminus \phi) = \emptyset \). Then \( \psi \in \text{Spec}(B) \) by Lemma 2.16. Let us prove that \( f^*(\psi) = \theta \). Since \( C_{GB}(f(\phi)) \subseteq \psi \), we have: \( \phi = f^*(C_{GB}(f(\phi))) \subseteq f^*(\psi) \). Now let \((x,y) \in f^*(\psi) \), so that \((f(x),f(y)) \in \psi \). Since \( \psi \cap f(A^2 \setminus \phi) = \emptyset \), it follows that \((f(x),f(y)) \notin f(A^2 \setminus \phi) \), thus \((x,y) \notin A^2 \setminus \phi \), which means that \((x,y) \in \phi \). Hence also have \( f^*(\psi) \subseteq \phi \), therefore \( f^*(\psi) = \phi \).

**Proposition 7.6.** If \( f \) is admissible and \( \nabla_B \) is finitely generated, then the following are equivalent:

(i) \( f \) fulfills LO;

(ii) for all \( \phi \in \text{Spec}(A) \) such that \( \ker(f) \subseteq \phi \), \( f^*(C_{GB}(f(\phi))) = \phi \);

(iii) for all \( \theta \in \text{Con}(A) \) such that \( \ker(f) \subseteq \theta \), \( f^*(\rho(C_{GB}(f(\theta)))) = \rho(\theta) \).

Proof. \( \text{(i) } \Leftrightarrow \text{(ii)} \): By Lemma 7.5 and the definition of LO.

\( \text{(i) } \Rightarrow \text{(ii)} \): Let \( \theta \in \text{Con}(A) \) such that \( \ker(f) \subseteq \theta \). We have: \( f^*(\rho(C_{GB}(f(\theta)))) = f^*(\bigcap_{\beta \in \text{Va}(C_{GB}(f(\theta)))} \beta) = \bigcap_{\beta \in \text{Va}(C_{GB}(f(\theta)))} f^*(\beta) \). Let \( \beta \in \text{Va}(C_{GB}(f(\theta))) \). Then \( \beta \in \text{Spec}(B) \), so \( f^*(\beta) \in \text{Spec}(A) \), and \( f(\theta) \subseteq \beta \), thus \( \theta \subseteq f^*(\beta) \), so \( \rho(\theta) \subseteq f^*(\beta) \). Hence \( \rho(\theta) \subseteq \bigcap_{\beta \in \text{Va}(C_{GB}(f(\theta)))} f^*(\beta) = f^*(\rho(C_{GB}(f(\theta)))) \). By the equivalence \( \text{(i) } \Rightarrow \text{(ii)} \) proved above, since \( \ker(f) \subseteq \theta \), we have: \( \rho(\theta) = \bigcap_{\alpha \in \text{Va}(\theta)} \bigcap_{\alpha \in \text{Va}(\theta)} f^*(C_{GB}(f(\alpha))) \).

Let \((a,b) \in f^*(\rho(C_{GB}(f(\theta)))) = \bigcap_{\beta \in \text{Va}(C_{GB}(f(\theta)))} f^*(\beta) \), so that, for all \( \beta \in \text{Va}(C_{GB}(f(\theta))) \), \((a,b) \in f^*(\beta) \). Since \( f \) fulfills LO, for every \( \alpha \in \text{Spec}(A) \) such that \( \ker(f) \supseteq \theta \), there exists a \( \beta \in \text{Spec}(B) \) such that \( f^*(\beta) = \alpha \), so that \( f(\theta) \subseteq f(\alpha) = f(f^*(\beta)) = \beta \cap f(A^2) \subseteq \beta \). Hence \((a,b) \in f^*(\beta) \subseteq f(\alpha) \subseteq C_{GB}(f(\alpha)) \), so \((a,b) \in f^*(C_{GB}(f(\alpha))) \). Therefore \((a,b) \in \bigcap_{\alpha \in \text{Va}(\theta)} f^*(C_{GB}(f(\alpha))) = \rho(\theta) \), hence \( f^*(\rho(C_{GB}(f(\theta)))) \subseteq \rho(\theta) \). Therefore \( f^*(\rho(C_{GB}(f(\theta)))) = \rho(\theta) \).

\( \text{(ii) } \Leftrightarrow \text{(iii)} \): Let \( \phi \in \text{Spec}(A) \) such that \( \ker(f) \subseteq \phi \). Then \( \rho(\phi) = \phi \), so we have \( f^*(\rho(C_{GB}(f(\phi)))) = \phi \). Since \( C_{GB}(f(\phi)) \subseteq \rho(C_{GB}(f(\phi))) \) it follows that \( f^*(C_{GB}(f(\phi))) \subseteq f^*(\rho(C_{GB}(f(\phi)))) = \phi \). But \( \phi \subseteq C_{GB}(f(\phi)) \), thus \( \phi \subseteq f^*(\phi) \subseteq f^*(C_{GB}(f(\phi))) \). Therefore \( f^*(C_{GB}(f(\phi))) = \phi \), hence \( f \) fulfills LO by the equivalence \( \text{(ii) } \Leftrightarrow \text{(iii)} \) proved above.

For any \( \theta \in \text{Con}(A) \), we shall denote by \( f_{[\theta]} : A/\theta \to B/C_{GB}(f(\theta)) \), for all \( a \in A \), \( f_{[\theta]}(a/\theta) = f(a)/C_{GB}(f(\theta)) \).

**Remark 7.7.** For any \( \theta \in \text{Con}(A) \) and any \( a,b \in A \), if \( a/\theta = b/\theta \), which means that \( (a,b) \in \theta \), then \( f(a)/C_{GB}(f(\theta)) = f(b)/C_{GB}(f(\theta)) \), so \( f_{[\theta]} \) is well defined. Clearly, \( f_{[\theta]} \) is a morphism and the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{p_\theta} & & \downarrow{p_{C_{GB}(f(\theta))}} \\
A/\theta & \xrightarrow{f_{[\theta]}} & B/C_{GB}(f(\theta))
\end{array}
\]

Clearly, if \( f^*(C_{GB}(f(\theta))) = \theta \), then \( f_{[\theta]} = f_{C_{GB}(f(\phi))} \) (see the definition of \( f_\beta \) at the beginning of this section), so, by Proposition 7.6 if \( f \) fulfills LO and \( \nabla_B \) is finitely generated, then, for every \( \phi \in \text{Va}(\ker(f)) \), \( f_{[\theta]} = f_{C_{GB}(f(\phi))} \). Since \( \ker(f) = \Delta_A \) if \( f \) is injective, we obtain: if \( f \) is injective and fulfills LO and \( \nabla_B \) is finitely generated, then, for every \( \phi \in \text{Spec}(A) \), \( f_{[\theta]} = f_{C_{GB}(f(\phi))} \).

**Lemma 7.8.** Let \( \theta \in \text{Con}(A) \) and \( \lambda \in \text{Con}(B) \). Then:
• \( \theta \subseteq f^*(\lambda) \iff C_{GB}(f(\theta)) \subseteq \lambda \);

• if \( \theta \subseteq f^*(\lambda) \), then \( f_{[\theta]}(\lambda/C_{GB}(f(\theta))) = f^*(\lambda)/\theta \).

Proof. If \( \theta \subseteq f^*(\lambda) \), then \( f(\theta) \subseteq f(f^*(\lambda)) \subseteq \lambda \), hence \( C_{GB}(f(\theta)) \subseteq \lambda \). If \( C_{GB}(f(\theta)) \subseteq \lambda \), then \( f(\theta) \subseteq \lambda \), thus \( \theta \subseteq f^*(f(\theta)) \subseteq f^*(\lambda) \).

Now assume that \( \theta \subseteq f^*(\lambda) \), so that \( C_{GB}(f(\theta)) \subseteq \lambda \) by the above. Then, for every \( a, b \in A \), the following equivalences hold: 

\((/i)\) for any \( \theta \) and \( \psi \) in \( [Ker(\theta)] \), then \( \phi = \psi/C_{GB}(f(\theta)) \) for some \( \psi \in V_B(C_{GB}(f(\theta))) \). Then \( \psi \in Spec(B) \) and \( C_{GB}(f(\theta)) \subseteq \psi \), thus \( \theta \subseteq f^*(\psi) \) by Lemma 7.8 thus \( f^*(\psi) \in Spec(A) \cap \theta = V_A(\theta) \) since \( f \) is admissible. Then, by Lemma 7.8 and 4.11, \( f_{[\theta]}(\phi) = f_{[\theta]}(\psi/C_{GB}(f(\theta))) = f^*(\psi)/\theta \in Spec(A/\theta) \).

Thus \( f_{[\theta]} \) is admissible.

(iii) \( f \mid_{Ker(f)} \) is admissible.

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\theta} \) is admissible:

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\phi} \) is admissible.

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\phi} \) is admissible.

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\phi} \) is admissible.

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\phi} \) is admissible.

Proof. \( \boxed{\text{as in a result in 22, but we provide a proof for it, for the sake of completeness.}} \)

(iii) \( f \mid_{\phi} \) is admissible.
\( \alpha_1 / Cg_B(f(\theta)) \in \text{Spec}(B/Cg_B(f(\theta))) \) by Lemma 4.11 \( \bullet \), and by Lemma 7.8 \( f_{[\theta]}^*(\alpha_1 / Cg_B(f(\theta))) = f^*(\alpha_1) / \theta = \alpha / \theta. \) Since \( f_{[\theta]} \) fulfills GU, it follows that there exists a \( \psi_1 \in \text{Spec}(B/Cg_B(f(\theta))) \) such that \( \alpha_1 / Cg_B(f(\theta)) \subseteq \psi_1 \) and \( f_{[\theta]}^*(\psi_1) = \beta / \theta. \) Again by Lemma 4.11 \( \bullet \), \( \psi_1 = \beta_1 / Cg_B(f(\theta)) \) for some \( \beta_1 \in V_B(Cg_B(f(\theta))) \), so that \( \alpha_1 / Cg_B(f(\theta)) \subseteq \beta_1 / Cg_B(f(\theta)) \), hence \( \alpha_1 \subseteq \beta_1. \) And, again by Lemma 7.8 \( f^*(\beta_1) / \theta = f_{[\theta]}^*(\beta_1 / Cg_B(f(\theta))) = f_{[\theta]}^*(\psi_1) = \beta / \theta, \) thus \( f^*(\beta_1) = \beta. \) Therefore \( f \) fulfills GU.

\( \triangleright \Rightarrow \triangleright \): By the proof of Lemma 7.9 \( f_{[\ker(f)]} \circ p_{\ker(f)} = p_{\Delta_B} \circ f \), with \( p_{\Delta_B} \) an isomorphism and \( p_{\ker(f)} \) surjective, thus admissible and with GU by Proposition 4.12. Since \( f_{[\ker(f)]} \) fulfills GU, by Lemma 4.14 \( \bullet \), and Remark 7.2 it follows that \( f_{[\ker(f)]} \circ p_{\ker(f)} \) fulfills GU, thus \( f \) fulfills GU.

**Proposition 7.11.** If \( f \) is admissible, then the following are equivalent:

(i) \( f \) fulfills LO;

(ii) for all \( \theta \in \text{Con}(A) \), \( f_{[\theta]} \) fulfills LO;

(iii) \( f_{[\ker(f)]} \) fulfills LO.

**Proof.** \( \triangleright \Rightarrow \triangleright \): As in the proof of Proposition 7.10 take \( \theta = \Delta_A \) and apply Remark 7.2. \( \triangleright \Rightarrow \triangleright \): Let \( \theta \in \text{Con}(A) \) and \( \phi \in \text{Spec}(A/\theta) \) such that \( \ker(f_{[\theta]}) \subseteq \phi. \) By Lemma 4.11 \( \bullet \), and Lemma 7.8 \( \phi = \alpha / \theta \) for some \( \alpha \in V_A(\theta), \) and \( \alpha/\theta \subseteq \ker(f_{[\theta]}) = f_{[\theta]}^*(\Delta_B/Cg_B(f(\theta))) = f_{[\theta]}^*(Cg_B(f(\theta))) / \theta, \) so that \( \alpha = f^*(\beta) \) for some \( \beta \in \text{Spec}(B), \) so that, by Lemmas 7.8 and 4.11 \( \bullet \), \( \theta \subseteq \alpha = f^*(\beta), \) hence \( Cg_B(f(\theta)) \subseteq \beta, \) thus \( \beta \in V_B(Cg_B(f(\theta))), \) hence \( \beta/Cg_B(f(\theta)) \subseteq \text{Spec}(B/Cg_B(f(\theta))), \) and \( \phi = \alpha / \theta = f^*(\beta)/\theta = f_{[\theta]}^*(\beta/Cg_B(f(\theta))). \) Therefore \( f_{[\theta]} \) fulfills LO.

\( \triangleright \Rightarrow \triangleright \): Trivial.

\( \triangleright \Rightarrow \triangleright \): By the proof of Lemma 7.9 \( f_{[\ker(f)]} \circ p_{\ker(f)} = p_{\Delta_B} \circ f, \) with \( p_{\Delta_B} \) an isomorphism. Since \( f_{[\ker(f)]} \) fulfills LO and \( p_{\ker(f)} \) surjective and thus admissible according to Lemma 5.10 by Lemma 4.14 \( \bullet \), it follows that \( f_{[\ker(f)]} \circ p_{\ker(f)} \) fulfills LO, thus \( f \) fulfills LO by Remark 7.2.

**Corollary 7.12.** Assume that \( f \) fulfills LO and \( \nabla_B \) is finitely generated. Then:

(i) if \( \{ Cg_B(f(\phi)) \mid \phi \in V_A(\ker(f)) \} \supseteq \text{Spec}(B), \) then \( f \) fulfills GU;

(ii) if \( f \) is injective and \( \{ Cg_B(f(\phi)) \mid \phi \in \text{Spec}(A) \} \supseteq \text{Spec}(B), \) then \( f \) fulfills GU.

**Proof.** \( \triangleright \) By Proposition 7.11 \( f_{[\theta]} \) fulfills LO for all \( \theta \in \text{Con}(A), \) thus for all \( \theta \in \text{Spec}(A), \) hence for all \( \theta \in V_A(\ker(f)). \) By Remark 7.7 this means that, for all \( \theta \in V_A(\ker(f)), \) \( f_{Cg_B(f(\theta))} \) fulfills LO, hence, for all \( \beta \in \text{Spec}(B), \) \( f_{[\beta]} \) fulfills LO, therefore \( f \) fulfills GU by Proposition 7.3.

\( \triangleright \) By \( \triangleright \) and the fact that, if \( f \) is injective, then \( \ker(f) = \Delta_A, \) so \( V_A(\ker(f)) = \text{Spec}(A). \)

8 Admissibility, Going Up and Lying Over in Different Kinds of Congruence–modular Equational Classes

In this section, we point out certain kinds of congruence–modular equational classes in which all morphisms are admissible, and others in which all admissible morphisms fulfill GU and LO. From these results we obtain some classes in which all morphisms are admissible and fulfill GU and LO. Throughout this section, \( A, B \) shall be members of \( C \) and \( f : A \to B \) shall be a morphism in \( C \).

**Lemma 8.1.** If \( C \) is congruence–modular and \( \theta \in \text{Con}(A), \) then the following are equivalent:

(i) \( \theta \in \text{Spec}(A); \)

(ii) for all \( a, b, c, d \in A, [Cg_A(a, b), Cg_A(c, d)]_A \subseteq \theta \) implies \( Cg_A(a, b) \subseteq \theta \) or \( Cg_A(c, d) \subseteq \theta. \)
Proof. \( \square \)

\( \qquad \text{Proof.} \) Assume by absurdum that \( \square \) is satisfied, but there exist \( \alpha, \beta \in \text{Con}(A) \) that \( [\alpha, \beta] \subseteq \emptyset \), \( \alpha \nsubseteq \emptyset \) and \( \beta \nsubseteq \emptyset \). Then there exist \( (a, b) \in \alpha \) and \( (c, d) \in \beta \) with \( (a, b) \notin \emptyset \) and \( (c, d) \notin \emptyset \), so that \( Cg_A(a, b) \subseteq \alpha, Cg_A(c, d) \subseteq \beta, Cg_A(a, b) \nsubseteq \emptyset \) and \( Cg_A(c, d) \nsubseteq \emptyset \), which imply \([Cg_A(a, b), Cg_A(c, d)]_A \subseteq [\alpha, \beta] \subseteq \emptyset \) by \( \text{Proposition 4.3} \) and \([Cg_A(a, b), Cg_A(c, d)]_A \nsubseteq \emptyset \) by the hypothesis of this implication; we have a contradiction. So \( \emptyset \in \text{Spec}(A) \).

Let \( I \) be a non–empty set and, for each \( i \in I \), let \( p_i \) and \( q_i \) be terms of arity 4 from \( \mathsf{L}_\ast \). Following \( \text{Section 2} \), we call \( \{(p_i, q_i) \mid i \in I\} \) a system of congruence intersection terms without parameters for \( \mathcal{C} \) iff, for any member \( M \) of \( \mathcal{C} \) and all \( a, b, c, d \in M \), \( Cg_M(a, b) \cap Cg_M(c, d) = \bigvee_{i \in I} Cg_M(p_i^M(a, b, c, d), q_i^M(a, b, c, d)) \).

\textbf{Theorem 8.2.} \( \square \) \textbf{Theorem 2.4} \( \square \) If \( \mathcal{C} \) has a system of congruence intersection terms without parameters, then \( \mathcal{C} \) is congruence–distributive.

\textbf{Proposition 8.3.} If \( \mathcal{C} \) has a system of congruence intersection terms without parameters, then any morphism in \( \mathcal{C} \) is admissible.

\textbf{Proposition 8.4.} If \( \mathcal{C} \) has a discriminator variety \( \text{iff} \) there exists \( \alpha, \beta \in \text{Con}(A) \) that \( [\alpha, \beta] \subseteq \emptyset \), \( \alpha \nsubseteq \emptyset \) and \( \beta \nsubseteq \emptyset \). Then there exist \( (a, b) \in \alpha \) and \( (c, d) \in \beta \) with \( (a, b) \notin \emptyset \) and \( (c, d) \notin \emptyset \), so that \( Cg_A(a, b) \subseteq \alpha, Cg_A(c, d) \subseteq \beta, Cg_A(a, b) \nsubseteq \emptyset \) and \( Cg_A(c, d) \nsubseteq \emptyset \), which imply \([Cg_A(a, b), Cg_A(c, d)]_A \subseteq [\alpha, \beta] \subseteq \emptyset \) by \( \text{Proposition 4.3} \) and \([Cg_A(a, b), Cg_A(c, d)]_A \nsubseteq \emptyset \) by the hypothesis of this implication; we have a contradiction. So \( \emptyset \in \text{Spec}(A) \).

We recall that the compact elements of the lattice \( \text{Con}(A) \) are the exactly finitely generated congruences of \( A \). We shall denote by \( \text{Con}_\omega(A) \) the set of the finitely generated congruences of \( A \). Clearly, \( (\text{Con}_\omega(A), \lor, \Delta_A) \) is a lower bounded join–semilattice.

We say that \( \mathcal{C} \) has the \textbf{compact intersection property} (abbreviated \textit{CIP}) \text{iff}, for any algebra \( M \) from \( \mathcal{C} \), the intersection of every two compact congruences of \( M \) is a compact congruence of \( M \). We say that \( \mathcal{C} \) has the \textbf{principal intersection property} (abbreviated \textit{PIP}) \text{iff}, for any algebra \( M \) from \( \mathcal{C} \), the intersection of every two principal congruences of \( M \) is a principal congruence of \( M \).

\textbf{Remark 8.4.} If \( \mathcal{C} \) is congruence–distributive and has the \textit{CIP}, then \( \mathcal{C} \) has the \textit{CIP}, because, if \( A \) is congruence–distributive, then, for any \( n, k \in \mathbb{N}^+ \) and any \( a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_k, d_1, \ldots, d_k \in A \), \( Cg_A((a_1, b_1), \ldots, (a_n, b_n)) \cap Cg_A((c_1, d_1), \ldots, (c_k, d_k)) = \bigvee_{i=1}^n \bigvee_{j=1}^k Cg_A(a_i, b_i) \cap Cg_A(c_j, d_j) \).

See also \( \text{[2]} \) p. 109).

\textbf{Proposition 8.5.} \( \square \) \textbf{Any congruence–distributive equational class with CIP has a system of congruence intersection terms without parameters.}

\textbf{Corollary 8.6.} \textbf{Any congruence–distributive equational class with PIP has a system of congruence intersection terms without parameters.}

\textbf{Corollary 8.7.} \( (i) \) If \( \mathcal{C} \) is congruence–distributive and has the \textit{CIP}, then every morphism in \( \mathcal{C} \) is admissible.

\( (ii) \) If \( \mathcal{C} \) is congruence–distributive and has the \textit{PIP}, then every morphism in \( \mathcal{C} \) is admissible.

Following \( \text{[13]} \) Chapter 4 and \( \text{[8]} \) Chapter IV, Section 9, we call \( \mathcal{C} \) a \textit{discriminator variety} \text{iff} there exists a ternary term \( t \) from \( \mathsf{L}_\ast \) such that, for every subdirectly irreducible algebra \( M \) in \( \mathcal{C} \) and all \( a, b, c, e \in M \) :

\[
\begin{align*}
t^M(a, b, c) = & \begin{cases} 
a, & a \neq b, 
\emptyset, & a = b.
\end{cases}
\end{align*}
\]
Lemma 8.8. [8, Theorem 9.4, p. 166] If $C$ is a discriminator variety, then $C$ is congruence–distributive, and there exists a term $s$ of arity 4 in $L_r$ such that, for any member $M$ of $C$ and any $a, b, c, d \in M$, $C_{GM}(a, b) \cap C_{GM}(c, d) = C_{GM}(s^M(a, b, c, d), c)$.

• [2, Corollary 2.7] If $C$ is congruence–distributive, then $C$ has the PIP iff there exist terms $P$ and $q$ of arity 4 from $L_r$ such that, for any algebra $M$ from $C$ and any $a, b, c, d \in M$, $C_{GM}(a, b) \cap C_{GM}(c, d) = C_{GM}(P^M(a, b, c, d), q^M(a, b, c, d))$.

Following [2], we call $C$ a filtral variety iff, for any up–directed set $(I, \leq)$ and any family $(M_i)_{i \in I}$ of subdirectly irreducible algebras from $C$, if $S$ is a subdirect product of the family $(M_i)_{i \in I}$, then every congruence of $S$ is of the form $\{(a_i)_{i \in I}, (b_i)_{i \in I}\} \subseteq S \setminus \{j \in I \mid a_j = b_j\} \subseteq F$ for some filter $F$ of $(I, \leq)$.

We recall that a (commutative) residuated lattice is an algebra $(R, \lor, \land, \rightarrow, \lhd, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, in which $(R, \lor, \land, 0, 1)$ is a bounded lattice, $(R, \rightarrow, 0, 1)$ is a commutative monoid, and each $a, b, c \in R$ fulfill the law of residuation: $a \lhd b \rightarrow c$ iff $a \odot b \leq c$, where $\leq$ is the partial order of the underlying lattice of $R$. For the results on residuated lattices that we use in what follows, we refer the reader to [11, 14, 19]. Residuated lattices form a semi–degenerate congruence–distributive equationally class, which includes BL–algebras and MV–algebras.

Throughout the rest of this section, $R$ shall be a residuated lattice. For all $a, b \in R$ and all $n \in \mathbb{N}$, we denote by $a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a)$, $a^0 = 1$ and $a^{n+1} = a^n \odot a$. We shall denote by $\text{Filt}(R)$ the set of the filters of $R$, that is the non–empty subsets of $R$ which are closed with respect to $\lor$ and to upper bounds. Then $(\text{Filt}(R), \lor, \land, \{1\}, R)$ is a complete distributive lattice, with $\land$ defined as in the case of bounded lattices. The map $F \mapsto \sim_F = \{(a, b) \in R^2 \mid a \leftrightarrow b \in F\}$ is a bounded lattice isomorphism from $\text{Filt}(R)$ to $\text{Con}(R)$. For any $a \in R$, we shall denote by $[a]$ the principal filter of $R$ generated by $a$: $[a] = \{x \in [a] \in \mathbb{N} \mid a^n \leq x\}$. For all $a, b \in R$: $[a] \cap [b] = [a \odot b]$.

Example 8.9. By the above, the class of residuated lattices is congruence–distributive and has the PIP.

• The class of bounded distributive lattices is congruence–distributive and, by [2], it has the PIP.

• By [2, Example 2.11], any filtral variety has the CIP.

• By Lemma 8.8, any discriminator variety is congruence–distributive and has the PIP.

Following [15, p. 382], we say that $C$ has equationally definable principal congruences (abbreviated, EDPC) iff, for any algebra $M$ from $C$, there exist an $n \in \mathbb{N}^*$ and terms $p_1, q_1, \ldots, p_n, q_n$ of arity 4 from $L_r$ such that, for all $a, b \in M$, $C_{GM}(a, b) = \{(c, d) \in M^2 \mid (\forall i \in \mathbb{N}) (p_i(a, b, c, d) = q_i(a, b, c, d))\}$.

Theorem 8.10. [6] If $C$ has EDPC, then $C$ is congruence–distributive.

Example 8.11. Here are some examples of varieties with EDPC, from [6, 15, Theorem 2.8] and [20]:

• distributive lattices, residuated lattices;

• discriminator varieties, which include: Boolean algebras, $n$–valued Post algebras, $n$–valued Lukasiewicz algebras, $n$–valued MV–algebras, relation algebras, monadic algebras, $n$–dimensional cylindric algebras, Gödel residuated lattices;

• dual discriminator varieties;

• filtral varieties;

• implication algebras, de Morgan algebras, Hilbert algebras, Brouwerian semilattices, Heyting algebras, modal algebras.

Let $(L, \lor, 0)$ be a lower bounded join–semilattice. $L$ is said to be dually Browerian iff it has a binary derivative operation $\sim$ such that, for all $x \in L$, $a \sim b \leq x$ iff $a \leq b \lor x$.

Proposition 8.12. [17] $C$ has EDPC iff, for any algebra $M$ from $C$, $\text{Con}_{\sim}(M)$ is dually Browerian. In this case, if $M$ is a member of $C$ and $n \in \mathbb{N}^*$ and $p_1, q_1, \ldots, p_n, q_n$ are the terms of arity 4 which define the principal congruences of $M$ as above, then, for any $a, b, c, d \in M$, $C_{GM}(c, d) \sim C_{GM}(a, b) = \bigvee_{i=1}^n C_{GM}(p_i(a, b, c, d), q_i(a, b, c, d))$. 
Proposition 8.13. If $C$ is semi-degenerate and has EDPC, then every admissible morphism in $C$ fulfills GU.

Proof. By Theorem 8.10 $C$ is congruence-distributive. By Corollary 4.16, it is sufficient to prove that every admissible canonical embedding in $C$ fulfills GU. Let $B$ be a member of $C$, a subalgebra of $B$ and $i : A \to B$ be the canonical embedding. Let $\phi \in \text{Spec}(A)$ and $\psi$ be a maximal element of the set 

$$\{ \theta \in \text{Con}(B) \mid \theta \cap (\nabla_A \setminus \phi) = \emptyset \}.$$ 

Then $0 = \psi \cap (A^2 \setminus \phi) = (\psi \cap A^2) \setminus (\psi \cap \phi)$, thus there exists $(x, y) \in \phi \setminus (\psi \cap A^2)$, and $(x, y) \not\in \psi \setminus A^2$, hence $(x, y) \not\in \psi$, therefore $(\psi \cap Cg_B(x, y)) \cap (A^2 \setminus \phi) \neq \emptyset$ or $\psi \cap Cg_B(x, y) = \nabla_B$, by the maximality of $\psi$. Since $\phi \in \text{Spec}(A)$, so $\phi \subseteq \nabla_A$ and thus $\nabla_B \cap (A^2 \setminus \phi) = A^2 \setminus \phi \neq \emptyset$, if it follows that $(\psi \cap Cg_B) \cap (A^2 \setminus \phi) \neq \emptyset$. Let $(s, t) \in (\psi \cap Cg_B(x, y)) \cap (A^2 \setminus \phi)$, so that there exist an $n \in \mathbb{N}$ such that $(s, t) \in Cg_B((a_1, b_1), \ldots, (a_n, b_n)) \setminus Cg_B(x, y)$, hence $Cg_B(s, t) \subseteq Cg_B((a_1, b_1), \ldots, (a_n, b_n)) \setminus Cg_B(x, y)$. Since $Cg_B(s, t), Cg_B((a_1, b_1), \ldots, (a_n, b_n)), Cg_B(x, y) \in \text{Con}_w(A)$ and, by Proposition 8.12, $\text{Con}_w(A)$ is a dually Browerian join-semilattice, it follows that $Cg_B((s, t) \setminus Cg_B(x, y) \subseteq Cg_B((a_1, b_1), \ldots, (a_n, b_n)) \setminus Cg_B(x, y)$, hence $\nabla_B \cap (Cg_B(s, t) \setminus Cg_B(x, y)) \subseteq \nabla_B \cap \psi \subseteq \psi$. Let $n \in \mathbb{N}$ and $q_1, q_2, \ldots, q_n$ be the terms in the equations which define the principal congruences of $A$, as written above. $x, y, s, t \in A$, thus, for every $i \in \mathbb{N}$, we may write $Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) \subseteq Cg_B(p_i(x, y, s, t), q_i(x, y, s, t))$, so, by Proposition 8.12.

$$Cg_B(s, t)^{-1}Cg_B(x, y) = \bigvee_{i=1}^n Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) \subseteq \bigvee_{i=1}^n Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) = Cg_B(s, t)^{-1}Cg_B(x, y),$$

hence $Cg_B(s, t)^{-1}Cg_B(x, y) \subseteq \nabla_B \cap (Cg_B(s, t)^{-1}Cg_B(x, y)) \subseteq \phi$, so

$$\bigvee_{i=1}^n Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) \subseteq \phi,$$

thus, for all $i \in \mathbb{N}$, $Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) \subseteq \phi$, so, for all $i \in \mathbb{N}$, $Cg_B(p_i(x, y, s, t), q_i(x, y, s, t)) \subseteq \phi$, hence, for all $i \in \mathbb{N}$, $p_i(x, y, s, t) \subseteq \phi$, so $s/(s, t) \subseteq \phi$, which means that $(s/(s, t), t) \subseteq \phi$. But $(x, y) \subseteq \phi$, so $(x, y) \subseteq \phi$, hence $Cg_B(x, y) \subseteq \phi$. Thus $(s/(s, t), t) \subseteq \phi$. Hence $\phi \subseteq \psi \setminus A^2$, therefore $\psi \setminus A^2 = \phi$. By Proposition 4.27 it follows that $i$ fulfills GU, which concludes the proof.

Corollary 8.14. If $C$ is semi-degenerate and has EDPC, then every admissible morphism in $C$ fulfills LO.

Proof. By Proposition 8.13 and Corollary 4.24.

Corollary 8.15.

- If $C$ is semi-degenerate and has EDPC and CIP, then any morphism in $C$ is admissible and fulfills GU and LO.

- If $C$ is semi-degenerate and has EDPC and PIP, then any morphism in $C$ is admissible and fulfills GU and LO.

Proof. By Proposition 8.12, Theorem 8.10 and Corollary 4.24.

Corollary 8.16.

- Any morphism in the class of residuated lattices is admissible and fulfills GU and LO.

- Any morphism in the class of bounded distributive lattices is admissible and fulfills GU and LO.

- Any morphism in a semi-degenerate filtral variety is admissible and fulfills GU and LO.

Proof. By Corollary 8.15 and Examples 8.9 and 8.11.

The previous corollary implies the results from [5] and [24] which say that any morphism of MV-algebras or BL-algebras is admissible and fulfills GU and LO.

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