Electrically Charged Sphalerons.

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(March 26, 2022)

Abstract

We investigate the possibility that the Higgs sector of the Weinberg-Salam model admits the existence of electrically charged, sphaleron states. Evidence is provided through an asymptotic and numerical perturbative analysis about the uncharged sphaleron. By introducing a toy model in two dimensions we demonstrate that such electrically charged, unstable states can exist. Crucially, they can have a comparable mass to their uncharged counterparts and so may also play a role in electroweak baryogenesis, by opening up new channels for baryon number violating processes.

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I. INTRODUCTION

There has for some time been great interest in the mechanism which generated the baryon anti-baryon asymmetry in the early Universe. An important observation was made by t’Hooft [1], when he noted that the Adler-Bell-Jackiw anomalies [2] in the standard model meant that baryon number was not conserved quantum mechanically. This opened up the possibility that baryogenesis could occur naturally at the electroweak scale. Moreover, the fact that sphalerons formed in the early Universe possessed the ability to wash out any effect of primordial baryogenesis [3], focussed attention on the electroweak transition as the source for the observed baryon–asymmetry. The field configuration now known as the sphaleron first appeared in the work of Daschen et al [4], but its potential role in baryon number violating transitions was only later clarified by Manton [5]. It is a saddle point solution between field configurations of baryon number zero and one, whilst itself has a baryon number of one half.

The energy of the sphaleron is of the order of 10Tev such that at comparable temperatures sphaleron configurations may be thermally excited, leading to baryon number violation. Interest in a charged state with fractional baryon number arises from the possibility that a new class of decay channels could emerge which would lead to baryon number violation if they were mediated by such a charged sphaleron, namely electrically charged initial states. One reason why we might expect such a solution on physical grounds comes from an interpretation of the SU(2) sphaleron. This state may be considered as a pair of oppositely charged magnetic monopoles in a special unstable equilibrium, where the attractive magnetic forces are balanced by repulsive topological ones. Vachaspati [6] has indicated the existence of dyon states in the standard model and it seems a natural extension to consider an unstable equilibrium of two dyons rather than monopoles.

If we are to investigate the existence of a charged sphaleron then we cannot work in the symmetry breaking scheme SU(2)→1, where the sphaleron was first found, as there is no electric charge symmetry in the vacuum of this model. We must therefore consider the full SU(2)×U(1)→U(1) of the standard model. This case is complicated by the presence of currents which reduce the sphaleron symmetry from being spherically to axially symmetric, leading to a magnetic dipole moment [7]. As far as we are aware, such states have not yet been observed in numerical simulations of the non abelian phase transition. Two reasons for this are that most dynamical simulations take a vanishing Weinberg angle [8] or, in the case of static calculations, force the temporal components of the gauge fields to vanish [9]. Both of which effectively forbid the existence of such states.

II. TOY MODEL

To investigate the charge on the sphaleron we start by introducing a toy model which has an electromagnetic symmetry generator in the vacuum. The main motivation for introducing such a model is that the equations obtained are ordinary rather than partial differential equations, allowing a simpler numerical treatment. In three spatial dimensions the symmetry breaking pattern SU(2)→U(1) leads to the topologically stable t’Hooft-Polyakov monopoles. In two dimensions however this admits static, unstable states which are the analog of the sphalerons in the standard model. Our lagrangian for a gauged SU(2) theory is,
\[ \mathcal{L} = \frac{1}{2}(D_\mu \Phi^a)(D^\mu \Phi^a) - \frac{1}{4} W^{a\mu \nu} W_{\mu \nu}^a - \mathcal{V}(\Phi^a \Phi^a), \]  

where

\[ \Phi^a = (\Phi^1, \Phi^2, \Phi^3) \]

\[ D_\mu \Phi^a = \left( \partial_\mu \Phi^a + g \epsilon^{abc} W^b_\mu \Phi^c \right) \]

\[ W^{a\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon^{abc} W^b_\mu W^c_\nu \]

with \( g \) as the gauge coupling constant. The two dimensional aspect of the theory means that the single winding, uncharged sphaleron is found using a Nielsen-Olesen string ansatz where the relevant gauge field is \( A_\mu = W^3_\mu \) with \( W^1_\mu = W^2_\mu = 0 \) and only the upper two components of the triplet higgs field are used: \( \phi^{(2)} = \Phi^{(2)}, \Phi^3 = 0 \). It is convenient to introduce the unit vector \( n^a \) and to re-express the Higgs and gauge fields,

\[ n^i = (x/r, y/r), \]

\[ \phi^a = \eta h(r)n^a/g \]

\[ A^i = f(r) \epsilon^{ij} n^j / (gr). \]

This ansatz may then be extended to include charged states by allowing for some of the temporal components of the gauge fields to be non-zero,

\[ W^{a0} = J(r)n^a/g \]

\[ W^{30} = 0. \]

With this, the profile functions \( h(r), f(r), J(r) \) are found to satisfy,

\[ h'' + h'/r - h/r^2 = 2hf(1 + f/2)/r^2 + 2h \frac{\partial \mathcal{V}}{\partial \phi^2} \]

\[ f'' - f'/r = (1 + f)(\eta^2 h^2 - J^2) \]

\[ J'' + J'/r - J/r^2 = 2Jf(1 + f/2)/r^2. \]

These equations are solved using a relaxation technique, where the profiles have the following boundary conditions. At the origin, \( h(r \to 0) \to ar, f(r \to 0) \to br^2, J(r \to 0) \to cr, \) whereas asymptotically we find, \( h(r \to \infty) \to 1, f(r \to \infty) \to -1, J(r \to \infty) \to A + B \ln(r). \) The presence of a logarithmic term should come as no surprise since Laplace’s equation for electrostatics in two dimensions has a logarithmic solution. The electromagnetic field tensor is defined by \( F_{\mu \nu} = \phi^a W^{a\mu}_{\nu}/|\phi|, \) and the electric field is then \( E^i = F^{ai} = \partial_i J(r)/g. \) To find the charge enclosed in a circle of radius \( r, \) the electric field is integrated over the circumference to give \( Q(r) = \int d\sigma E^i. \) The asymptotic form of \( J(r) \) then shows that, \( Q(r \to \infty) \to 2\pi B/g = Bg/\alpha, \) where \( \alpha = g^2/(2\pi) \) is the fine structure constant in two dimensions. In particular the total charge enclosed is finite. Typical profiles are shown below in Fig1.

An important property of sphalerons is their mass. Since they are thermally activated, their creation probability is suppressed by a Boltzmann factor. If the charged states are excessively massive then this exponential suppression would render their formation rate
negligible. Using the canonical definition of the stress-energy-momentum tensor, \( T^{\mu\nu} = -2/(\sqrt{-g}) \delta S/\delta g_{\mu\nu} \), with \( S \) being the action, we define the mass as \( m = \int d^2x T^{00} \). The variation of mass as a function of charge is shown in Fig 2. For small charges the mass rises slowly as the charge is increased. This behavior is reminiscent of the energy-charge relation for the dyon in the BPS limit \([10]\), \( E_{\text{dyon}} \propto \sqrt{1 + q^2} \) where \( q \propto Q \), the charge. From the presence of the fine structure constant we see that for a sphaleron to have a charge comparable to the elementary charge then the mass cost is small and so the Boltzmann factor for these charged sphaleron states is not significantly greater than the uncharged states, making them physically interesting and worth investigating in the full three dimensional picture.

III. ASYMPTOTIC ANALYSIS IN THE STANDARD MODEL.

We now turn our attention to the full symmetry breaking pattern of the standard model. Although it is too difficult to obtain complete analytic solutions for the charged state sphaleron, we intend to demonstrate the existence of the solution through a combination of numerical and analytical means. We start by determining the asymptotics of a static, compact field configuration. This technique was used by Vachaspati \([6]\) when looking at the construction of dyons in the standard model, but will be included here for completeness. The lagrangian of the higgs sector of the standard model is written as,

\[
\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{4} W^{\alpha \mu \nu} W^\alpha_{\mu \nu} - \frac{1}{4} B^{\mu \nu} B_{\mu \nu} - \mathcal{V}(\Phi^\dagger \Phi). \tag{13}
\]

where,

\[
D_\mu \Phi = \left( \partial_\mu - g W^{a}_{\mu} t^a - i/2 g' B_\mu \right) \Phi \tag{14}
\]

\[
[t^a, t^b] = -\epsilon^{abc} t^c \tag{15}
\]

\[
[D_\mu, D_\nu] \Phi = - \left(g W^{\mu \nu} + ig'/2 B_{\mu \nu}\right) \Phi \tag{16}
\]

\[
D_\mu W^{a \sigma \mu} = \partial_\mu W^{a \sigma \mu} + g \epsilon^{abc} W^b_{\mu} W^{c \sigma \mu}, \tag{17}
\]

with \( B_\mu \) an abelian gauge field with coupling constant \( g' \). The corresponding field equations are,

\[
D^\mu D_\mu \Phi = - \frac{\partial \mathcal{V}}{\partial \Phi^\dagger} \tag{18}
\]

\[
\partial_\mu B^{\sigma \mu} = i/2 g' \left[ \Phi^\dagger D^\sigma \Phi - (D^\sigma \Phi)^\dagger \Phi \right] \tag{19}
\]

\[
D_\mu W^{a \sigma \mu} = g \left[ \Phi^\dagger t^a D^\sigma \Phi - (D^\sigma \Phi)^\dagger t^a \Phi \right]. \tag{20}
\]

Finite energy considerations imply that asymptotically \( D_\mu \Phi = 0 \). From Eq. (14) and using the appendix we find

\[
-g W^{\mu}_{\mu} + g' B_\mu n^a = -i n^a \left( \Phi^\dagger \partial_\nu \Phi \right) + \epsilon^{abc} n^b \partial n^c \tag{21}
\]

\[
n^a = 2 i \Phi^\dagger t^a \Phi / \Phi^\dagger \Phi, \tag{22}
\]
where \( \mathbf{n} \) is a unit vector. It proves useful to split Eq. (21) into parts parallel and perpendicular to \( \mathbf{n} \), with the general solution,

\[
-gW^a_\mu = e^{abc}n^b\partial_\mu n^c - i \cos^2(\alpha)n^a \left( \Phi^\dagger \frac{\partial}{\partial \mu} \Phi \right) + n^a a_\mu \\
g'B_\mu = -i \sin^2(\alpha) \left( \Phi^\dagger \frac{\partial}{\partial \mu} \Phi \right) - a_\mu,
\]

(23)

(24)

where \( a_\mu \) is arbitrary and represents the gauge freedom of the statement \( D_\mu \Phi = 0 \) \([11]\), and \( \alpha \) is determined by the dynamics of the particular field configuration we are investigating.

As we are looking to charge the sphaleron then we are particularly interested in \( a_0 \). We now suppose that there is a static solution, denoted by barred fields, representing the usual uncharged sphaleron from which we may construct a new, charged state. Now choose the lorentz vector \( a_\mu = \gamma \delta^0_\mu \) such that,

\[
\Phi = \overline{\Phi} \\
W^a_\mu = \overline{W}^a_\mu - \frac{1}{g} \gamma n^a \delta^0_\mu \\
B_\mu = \overline{B}_\mu - \frac{1}{g} \gamma \delta^0_\mu.
\]

(25)

(26)

(27)

By construction \( D_\mu \Phi = \overline{D}_\mu \overline{\Phi} = 0 \) so \( \Phi \) satisfies the usual scalar field equation. The space-space components of the field strengths remain unchanged and the space-time components are found to be,

\[
B_{0i} = \frac{1}{g'} \partial_i \gamma \\
W_{0a} = - \left( \partial_i W_{0a} + g e^{abc} \overline{W}_i^b \overline{W}_0^c \right) \\
= \frac{1}{g} \partial_i (\gamma n^a).
\]

(28)

(29)

(30)

As we are in the vacuum, \( D_\mu \Phi = 0 \), then clearly \( D_i (n^a) = 0 \) which means that the field equations for \( B_\mu \) and \( W^a_\mu \) each lead to the relations,

\[
\partial_i \partial^i \gamma = 0 \\
\partial_0 \partial^0 \gamma = 0.
\]

(31)

(32)

A separation of variables then shows \( \gamma(\mathbf{x}, t) = \gamma(\mathbf{x}) \) and \( \nabla^2 \gamma = 0 \). With \( \gamma \) satisfying Laplace’s equation we therefore know that asymptotically the time component of the gauge fields take the form,

\[
W_{0a} = \frac{1}{g} n^a \sum_{l=0}^{l=\infty} \sum_{m=-l}^{m=l} C_{l,m} P_l^m (\cos(\theta)) / r^{l+1} \\
B_0 = \frac{1}{g'} \sum_{l=0}^{l=\infty} \sum_{m=-l}^{m=l} C_{l,m} P_l^m (\cos(\theta)) / r^{l+1}.
\]

(33)

(34)

Now, the electromagnetic and Z potentials are given by,
\[ A_\mu = \sin(\theta_w) n^a W^a_\mu + \cos(\theta_w) B_\mu \]  
\[ Z_\mu = \cos(\theta_w) n^a W^a_\mu - \sin(\theta_w) B_\mu, \]  

where \( \theta_w \) is the Weinberg angle. From eq. (36) we see that the \( Z_\mu \) potential vanishes at infinity, as is fitting for a massive field. However the electromagnetic potential, \( A_0 \), being a solution to Laplace’s equation is able to support all multipole moments, including the zeroth monopole term giving rise to electric charge – the charge that we are interested in. As for the baryonic charge of the state, this is found to be \[ Q_B = \frac{g^2}{32\pi^2} \int d^3 x K^0 \]  
\[ K^\mu = \varepsilon^{\mu\nu\rho\sigma} \left( W^a_\nu W^a_\rho - \frac{2}{3} g \varepsilon_{abc} W^a_\nu W^b_\rho W^c_\sigma \right). \]

When we look at the zero component of \( K \) to find the sphaleron’s baryonic charge, only the space components of the gauge field are used. These have remained unaltered from the uncharged sphaleron, consequently the charged and uncharged sphaleron have the same baryonic charge.

**IV. NUMERICAL ANALYSIS OF THE CHARGED SPHALERON.**

In the previous section we analyzed the asymptotic behaviour of the fields. To investigate the core of the charged sphaleron we first look at the general axisymmetric field configuration to see what form the temporal components of the gauge fields can take. Then we shall use the SU(2) sphaleron as a background in which we numerically investigate whether a charged sphaleron state exists for small \( \theta_w \). The question of spatial symmetry in gauge theories has a subtlety in as much as a gauge transformation is not a physical transformation and we are concerned only with physical symmetries. A configuration is axisymmetric if a rotation about an axis is equivalent to a gauge transformation \[ (G + \mathcal{O}) \Xi = 0, \]

where \( \Xi \) represents a generic field and \( G, O \) are the generators of gauge transformations and rotations respectively. To help with the generation of a suitable ansatz we introduce the set of orthonormal vectors

\[ u_x = \left( x/\rho, y/\rho, 0 \right) \]
\[ u_y = \left( -y/\rho, x/\rho, 0 \right) \]
\[ u_z = \left( 0, 0, 1 \right) \]
\[ R^2 = x^2 + y^2 + z^2 \]
\[ \rho^2 = x^2 + y^2. \]

It may then be shown \[ 3 \] that the following field configuration is annihilated by a sum of rotation and gauge transformation generators.
\[
W_i^\alpha(x) = u_i^\alpha u_i^\beta \delta^{\alpha\beta}(\rho, z) \\
B_i(x) = u_i^\alpha d_\alpha(\rho, z) \\
\Phi(x) = h_i(\rho, z) u_i^\alpha t_\alpha \Psi \\
\Psi = (0, \eta/\sqrt{2}),
\]
a result that guarantees the axisymmetric nature of the configurations. This system of fields may be extended to include possible zero components of the gauge fields thus,
\[
W_0^\alpha(x) = u_i^\alpha F_i(\rho, z) \\
B_0(x) = J(\rho, z).
\]
It is possible to restrict the situation further to solutions which are invariant under various discrete symmetries. We impose the condition that the combined transformation of charge conjugation, parity reflection and the SU(2) transformation -1 leaves our sphaleron unchanged. This boils down to,
\[
w_x^x = w_y^y = w_z^z = w_x^z = 0 \\
F_y = 0 \\
d_x = d_z = 0 \\
h_y = 0.
\]
The numerical investigation is based on a perturbative expansion for small \(g'/g\). In this case the \(w_\alpha^a\) and \(h_x, h_z\) are found from the \(g'/g = 0\) sphaleron, namely,
\[
W^{ai} = 2 f(R) \epsilon^{aij} \frac{x_j}{R^2} \\
\Phi = 2h(R) \frac{x_i}{R} t_i \Psi,
\]
where \(h(R)\) and \(f(R)\) satisfy,
\[
\frac{\partial^2 f}{\partial R^2} = 2 \frac{f}{R^2} (1 + gf)(1 + 2gf) + \frac{1}{4} g\eta^2 h^2 (1 + gf) \\
R^2 \nabla^2 h = 2h(1 + gf)^2 + R^2 h \frac{\partial \nabla}{\partial (\Phi^i \Phi)}
\]
The only non-vanishing part of \(d_\alpha, d_y\), has been investigated previously, and leads to a magnetic dipole moment. The form of \(B_i\) in the small \(g'/g\) limit is found to be
\[
B_i = -g' p(R) \epsilon_{ij3} x_j \\
\frac{\partial^2 p}{\partial R^2} + 4 \frac{\partial p}{\partial R} = -\frac{1}{2} \eta^2 h^2 (1 + gf)
\]
The profiles of the functions involved in the small \(g'/g\) uncharged sphaleron are shown below in Fig3.

The terms \(J, F_x, F_z\) determine the sphaleron charge. We have solved for them numerically in the background of the sphaleron with vanishing Weinberg angle. The definition of the electromagnetic tensor \(F_{\mu\nu}\) and currents used here are the same as in 7.
\[ F_{\mu\nu} = \sin(\theta_W)n^aW^a_{\mu\nu} + \cos(\theta_W)B_{\mu\nu} \] (48)
\[ \partial^\nu F_{\mu\nu} = j_\mu. \] (49)

The resulting electric charge distribution, \( j_0 \), is displayed in Fig.4 where we see clear evidence that the electric charge density is localized at the centre of the system.

The issue of electric charge quantization is a tricky one. The charged sphaleron, having two displaced regions of magnetic charge density will develop an angular momentum, due to the Poynting vector. If this angular momentum is then quantized we shall find ourselves with sphalerons that occur with discrete values of electric charge. Such a calculation has successfully been performed in the case of a charged topological string in an abelian higgs model \cite{12}, where exact analytic expressions are obtained. Unfortunately, the precise nature of the quantization in our case is obscured by the complexity of the field equations and the fact that we do not have exact analytic results to use. The principle though still holds, evaluating the angular momentum for different charged states should lead to a determination of the fundamental electric charge.

V. CONCLUSION.

The aim of this paper has been to demonstrate that there may exist unstable charged states in the standard model. These could contribute to the process of electroweak baryogenesis. By analogy with a sphaleron in two dimensions we showed that such states do exist. Following Vachaspati \cite{6} we saw that asymptotically the fields can support a charged distribution and a perturbative numerical treatment also indicated that such charged states exist. Finally, it was shown how the axial symmetry of the sphaleron, which causes the computational problems leads to a natural way to quantize the electric charge.

When considering how such configurations would contribute to baryogenesis we need to know the mass of such states. It was shown that the mass-charge relation of the 2D sphaleron was similar to that of the BPS dyon in that the mass increases slowly for small, physical values of electric charge. If this relation is generic, which may be expected as the uncharged sphaleron is so massive, then the charged sphalerons in the standard model will be excited at temperatures similar to the uncharged states. Such states would open up a new channel for baryon violating processes in the standard model through their interactions with charged particles. This may prove to be a fruitful avenue of investigation.

ACKNOWLEDGMENTS

We are grateful to M. Hindmarsh, T. Barreiro, and J. Grant for useful conversations. EJC and PS are grateful to PPARC for financial support. Partial support was obtained from the European Commission under the Human Capital and Mobility program, contract no. CHRX-CT94-0423.
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FIGURES

Profile of Higgs Field

Profile of Spatial Gauge Field Component

Profile of Temporal Gauge Field Component

Charge Inside Radius $r$

Energy–Charge Dependence
Plot Showing the Density of Electric Charge
APPENDIX:

Consider the set of matrices $\Gamma^\mu = (1, 2it^a)$. These then satisfy $Tr (\Gamma^\mu \Gamma^\nu) = 2\delta^{\mu\nu}$. Using the fact that the $\Gamma^\mu$ form a linearly independent basis of 2x2 complex matrices, we see that

$$\Gamma_{\alpha\beta}^\mu \Gamma_{\gamma\delta}^\mu = 2\delta_{\alpha\gamma} \delta_{\beta\delta},$$

where summation over repeated indices is implied. Then by considering

- $(\Phi^\dagger \Gamma^\mu \Phi) (\Phi^\dagger \Gamma^\mu \Phi)$
- $(\Phi^\dagger \Gamma^\mu \Phi) \Gamma^\mu \Phi$
- $(\Phi^\dagger \Gamma^\mu \Phi) (\Phi^\dagger \Gamma^\mu \partial_\nu \Phi)$
- $(\Phi^\dagger \Gamma^\mu \Phi) \partial_\nu (\Phi^\dagger \Gamma^\sigma \Gamma^\mu \Phi)$

we find that

- $(\Phi^\dagger t^a \Phi) (\Phi^\dagger t^a \Phi) = -1/4 (\Phi^\dagger \Phi)^2$
- $[1 + 4(\Phi^\dagger t^a \Phi)/\Phi^\dagger \Phi] \Phi = 0$
- $(\Phi^\dagger t^a \Phi) (\Phi^\dagger t^a \partial_\nu \Phi) = -1/4 (\Phi^\dagger \Phi) (\Phi^\dagger \partial_\nu \Phi)$
- $(\Phi^\dagger \Phi) (\Phi^\dagger t^a \partial_\nu \Phi) = (\Phi^\dagger t^a \Phi) (\Phi^\dagger \partial_\nu \Phi) - 2\epsilon^{abc} (\Phi^\dagger t^b \Phi) \partial_\nu (\Phi^\dagger t^c \Phi)$,

expressions which prove useful in the asymptotic analysis of the charged sphaleron solutions.