THE TORSION GENERATING SET OF THE DEHN TWIST SUBGROUPS OF NON-ORIENTABLE SURFACES

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Abstract. Let \( N_g \) be the non-orientable surface with genus \( g \), \( \text{MCG}(N_g) \) be the mapping class group of \( N_g \), \( T(N_g) \) be the index 2 subgroup generated by all Dehn twists of \( \text{MCG}(N_g) \). We prove that for odd genus, \( T(N_g) \) can be generated by three elements of finite orders.

1. Introduction

Let \( N_g \) be the non-orientable surface with genus \( g \), \( \text{MCG}(N_g) \) be the mapping class group of \( N_g \). Lickorish was the first one to discover that all Dehn twists can only generate an index 2 subgroup of \( \text{MCG}(N_g) \) ([6]). We denote this subgroup as \( T(N_g) \). Outside \( T(N_g) \), there is a mapping class called “Y-homeomorphism” or “cross-cap slide”. A finite set of generators for \( \text{MCG}(N_g) \) and \( T(N_g) \) was given by Chillingworth ([2]). When \( g = 2 \), Lickorish found \( \text{MCG}(N_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), Chillingworth found \( T(N_2) \) can be generated by one Dehn twist. When \( g = 3 \), Birman and Chillingworth proved that \( \text{MCG}(N_3) \) can be generated by three involutions ([1]), Chillingworth found \( T(N_3) \) can be generated by two Dehn twists.

It is a natural question how to simplify the generating sets for \( \text{MCG}(N_g) \) and \( T(N_g) \) as much as possible when \( g \) is large. We want to reduce both the number and the orders of the generators. When \( g \geq 4 \), a generating set for \( \text{MCG}(N_g) \) consisting of four involutions was constructed by Szepietowski. Szepietowski also proved when \( g \geq 4 \), \( \text{MCG}(N_g) \) can be generated by three elements (See [3]). The first homology of \( \text{MCG}(N_g) \) has been calculated by Korkmaz. By Korkmaz’s result, when \( g = 4 \), the smallest number of generators for \( \text{MCG}(N_4) \) is at least 3, So the minimal number of the generators for \( \text{MCG}(N_3) \) is 3. About \( T(N_g) \), Stukow gave a finite presentation of \( T(N_3) \) in [3]. Omori reduced the number of Dehn twist generators for \( T(N_g) \) to \( g + 1 \) when \( g \geq 4 \) ([7]).

2010 Mathematics Subject Classification. 57N05, 57M20, 20F38.

Key words and phrases. mapping class group, non-orientable surface, generator, torsion.

The author would like to thank Szepietowski for telling him the generators of the index 2 Dehn twist subgroup and pointing out some generator was missing in the earlier version of the paper.
In [3], the author proved the following: when the genus \( g' \geq 5 \), the extended mapping class group \( \text{MCG}^\pm(S_{g'}) \) can be generated by two elements of finite order. One is of order 2 and the other is of order \( 4g' + 2 \). In [4] (preprint), the author proved that the above result is also true for \( g' = 3, 4 \). We found that the method in [3] [4] can be used in the case of \( \mathcal{T}(N_g) \). We have the following result:

**Theorem 1.1.** Let \( \mathcal{T}(N_g) \) be the index 2 Dehn twist subgroup of the mapping class group of a non-orientable surface. If \( g \geq 5 \) is odd, \( \mathcal{T}(N_g) \) can be generated by three elements of finite order. One of the generators is of order \( 2g \). The other two are of order 2.

2. Preliminary

**Notations.**

(a) We use the convention of functional notation, namely, elements of the mapping class group are applied right to left, i.e. the composition \( FG \) means that \( G \) is applied first.

(b) A Dehn twist means a right-hand Dehn twist.

(c) We denote the curves by lower case letters \( a, b, c, d \) (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters \( A, B, C, D \). Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

**Cross-cap slide.**

In [6], Lickorish proved that the Dehn twists of all the two-sided curves on the non-orientable surface generate \( \mathcal{T}(N_g) \) and \( [\text{MCG}(N_g) : \mathcal{T}(N_g)] = 2 \). As an example of the mapping classes which do not lie in \( \mathcal{T}(N_g) \), he described a mapping class so-called "Y-homeomorphism" or "cross-cap slide" as shown in figure 1.

![Figure 1.](image_url)

Two points of view for the Möbius band partition of a non-orientable surface of odd genus.
If $g$ is odd, we can decompose the non-orientable surface $N_g$ into $g$ Möbius bands. Figure 2 shows two points of view to do this.

(1) The left picture of figure 2 is a $2g$-gon, with a cross-cap in the middle and the opposite sides glued together pairwise. Under this gluing, the vertices of this $2g$-gon is divided into two equivalent classes. After the gluing, they form two points on $N_g$. We denote them as $N$ and $S$. There are $g$ arcs connecting pairs of antipodal vertices and passing the cross-cap in the middle of the $2g$-gon. They divide $N_g$ into $g$ Möbius bands. We call this is the $2g$-gon presentation of $N_g$.

![Figure 2](image1)

(2) The middle and the right picture of figure 2 show a 2-sphere with $g$ projective planes attached. This is also $N_g$. Suppose the $g$ projective plane sit on the equator. Denote the north pole and the south pole as $N, S$. There are $g$ arcs connecting $N$ and $S$. They divide $N_g$ into $g$ Möbius bands. We call this is the $g$-cross-cap presentation of $N_g$.

We can check the above two presentations of $N_g$ are equivalent. In the following, we will go back and forth between such presentations.

**Mapping classes supported on an Klein bottle with boundary**

For the non-orientable surface $N_g$, there is a subsurface homeomorphic to a one-holed Klein bottle, see figure 3. We use the notation as those in [9]. The one-holed Klein bottle contains two cross-caps in its interior. Suppose $U$ is the mapping class that exchanges the two cross-caps, which is like a half-twist generator of the braid group. There is a curve $a$ which is two-sided and passes both cross-caps. Let $A$ be the Dehn twist along $a$. The mapping class $Y$ is a $Y$-homeomorphism, sliding one cross-cap along the one-sided curve which passes the other cross-cap once.

Szepietowski showed $Y = AU$. In other words, both the cross-cap exchanging map $U$ and the cross-cap side $Y$ do not lie in $\mathcal{T}(N_g)$. Moreover, it is not hard to check that both $Y^2$ and $U^2$ equals the Dehn twist along the boundary curve of the one-holed Klein bottle.
The curves needed for generating $T(N_g)$.

Omori construct a generating set consist of $g + 1$ Dehn twists for $T(N_g)$ ([7]). When we use the $g$-cross-cap presentation of $N_g$, the curves for those Dehn twists are $a_1, a_2, \ldots, a_g, b_0, c$ shown in figure 4. We can check $A_1^{-1}(e) = c$. Hence the Dehn twists along $a_1, a_2, \ldots, a_g, b_0, c$ can also generate $T(N_g)$.

We can also use the $2g$-gon presentation to see what these curves are. See figure 5.
3. The proof of the main theorem

We now give a proof for Theorem 1.1.

Proof of Theorem 1.1. We first give the torsion generators. Suppose \( g \) is odd. See figure 6.

Let \( \{a_1, \ldots, a_g, b_0, c\} \) be the set of curves whose Dehn twists generate \( \mathcal{T}(N_g) \), \( \sigma \) be the rotation in the \( 2g \)-gon presentation, \( \tau_1 \) be a reflection of the \( 2g \)-gon presentation that preserves the curve \( b_0 \), \( \tau_2 \) be a reflection of the \( g \)-cross-cap presentation that preserves \( c \). We can easily see that \( \sigma^{2g} = 1, (\tau_1 \circ B_0)^2 = 1, (\tau_2 \circ C)^2 = 1 \).

Let \( G = \langle \sigma, \tau_1 \circ B_0, \tau_2 \circ C \rangle \) be the subgroup of \( \text{MC}G(N_g) \) generated by these three elements of finite orders. We claim that when \( g \) is odd, \( G = \mathcal{T}(N_g) \).

By the method in [3] and [4], the Dehn twists \( A_1, \ldots, A_g, B_0 \) are in \( G \). Then \( \tau_1 \) is also in \( G \).

We can interpret some of the torsion elements in more geometric ways. See figure 7. We can check that \( \tau_1 \) is not only a reflection in the \( 2g \)-gon presentation but also a reflection in the \( g \)-cross-cap presentation. Let \( \tau_3 \) be the north-south reflection of the \( g \)-cross-cap presentation of \( N_g \), \( t \) be an order \( g \) rotation. Then \( \sigma = t \circ \tau_3 \) and \( \tau_3 = \sigma^g \). Hence \( \tau_3 \) and \( t \) are also in \( G \).

Now \( \tau_2 \) is conjugated to \( \tau_1 \) by some power of \( t \). So \( \tau_2 \) also lies in \( G \). Hence \( C \) lies in \( G \). Since \( A_1, \ldots, A_g, B_0, C \) generate \( \mathcal{T}(N_g) \), \( \text{MC}G(N_g) \geq \mathcal{T}(N_g) \).
$G \geq \mathcal{T}(N_g)$. We want to prove $G$ is not $\text{MCG}(N_g)$. We need to verify all the generators lie in $\mathcal{T}(N_g)$.

Since $\sigma = A_{2g} A_{2g-1} \ldots A_2 A_1$, $\sigma$ is in $\mathcal{T}(N_g)$. So $\tau_3$ and $t$ also lies in $\mathcal{T}(N_g)$. When $g$ is odd, in the $g$-cross-cap presentation of $N_g$, the composition of $\tau_1$ and $\tau_3$ is a rotation of the 2-sphere, fixing one cross-cap. If we look every cross-cap as a punctured point, then $\tau_1 \circ \tau_3$ becomes an element in the spherical braid group. It can be written as a product of the standard half-twist generators of the braid group. We can check the number of the half-twists in the product is an even number. Each half-twist corresponds to a mapping class exchanging two cross-caps and supported on a Klein bottle with one boundary. This means the half-twist generators in the braid group correspond to the mapping classes outside $\mathcal{T}(N_g)$. The number of half-twists in the product is even means $\tau_1 \circ \tau_3$ lies in $\mathcal{T}(N_g)$, hence $\tau_1$ also lies in $\mathcal{T}(N_g)$. $\tau_2$ is in $\mathcal{T}(N_g)$ because it is conjugated to $\tau_1$. We get $G = \langle \sigma, \tau_2 \circ C, \tau_1 \circ B_0 \rangle = \mathcal{T}(N_g)$. \hfill $\square$

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