Characterization of measurement uncertainties using the correlations between local outcomes obtained from maximally entangled pairs

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Joint measurements of non-commuting observables are characterized by unavoidable measurement uncertainties that can be described in terms of the error statistics for input states with well-defined values for the target observables. However, a complete characterization of measurement errors must include the correlations between the errors of the two observables. Here, we show that these correlations appear in the experimentally observable measurement statistics obtained by performing the joint measurement on maximally entangled pairs. For two-level systems, the results indicate that quantum theory requires imaginary correlations between the measurement errors of \( \hat{X} \) and \( \hat{Y} \) since these correlations are represented by the operator product \( \hat{X}\hat{Y} = i\hat{Z} \) in the measurement operators. Our analysis thus reveals a directly observable consequence of non-commutativity in the statistics of quantum measurements.

I. INTRODUCTION

Recently, advances in experimental possibilities have renewed the interest in the physics of measurement uncertainties [1–9]. This topic of research actually has a long history going all the way back to Heisenberg’s justification of the quantum formalism by the definition of uncertainty limits for the simultaneous determination of position and momentum. Consequently, most of the recent work has focused on the quantitative evaluation of uncertainties using either the original evaluation by variances [10–12] or the more recent concept of entropic uncertainties [13–15]. Crucially, all of these approaches are based on the statistical limitations imposed by the mathematics of the quantum formalism, following Heisenberg’s implicit suggestion that it is impossible to obtain any direct experimental evidence of the physics that causes the appearance of uncertainties in the first place. However, the study and characterization of entanglement shows that quantum correlations can exceed the local limits of uncertainties [16–18]. It is therefore possible to observe otherwise hidden details of the measurement error statistics by using the uncertainty-free correlations of entangled states as a reference. In the present paper, we show how the correlations between the errors that characterize a joint measurement of two non-commuting observables can be obtained by analyzing the experimental results from maximally entangled inputs. For two-level systems, the complete statistics of measurement errors is obtained. Significantly, the error statistics for quantum measurements consistent with the standard formalism of measurement theory exceed the bounds for real-valued measurement errors, effectively resulting in a derivation of complex-valued error probabilities from the experimentally observed data.

The starting point of our discussion is a specific form of a joint measurement of the non-commuting observables \( \hat{A} \) and \( \hat{B} \), where the results of the joint measurement correspond to the \( d^2 \) combinations of eigenvalues observed in separate projective measurements of the two observables. It is then possible to describe measurement errors in one of the observables in terms of the conditional probabilities experimentally obtained for eigenstate inputs of that observable. We are then able to discuss measurement errors in qualitative terms, based on the notion that the outcome is either correct or incorrect. We apply this method of analysis to the orthogonal Bloch vector components \( \hat{X} \) and \( \hat{Y} \) of a two level system and show that the only missing element in the description of the measurement process is the correlation between the errors in \( \hat{X} \) and \( \hat{Y} \). To obtain this correlation experimentally, we then introduce a maximally entangled state of two systems. Since the correlations of \( \hat{X} \) and \( \hat{Y} \) between the two systems are known, the application of joint measurements to both systems should ideally produce the same correlations. From the actual results, we can judge whether the occurrence of an error in only one of the two measurements changed the correlation or not. Since the changes in correlations are obtained for both \( \hat{X} \) and \( \hat{Y} \) in a single measurement, the measurement outcomes reveal the correlation between the occurrence of errors in the two local measurements, and the missing element of the error statistics can be derived from the experimental results.

We can use the standard formalism of quantum mechanics to predict the results of joint measurements on maximally entangled states. Interestingly, the predicted experimental results violate the limits that apply to any positive valued
(and hence classical) selection of error probabilities. In fact, the quantum result indicates that the error correlation must be given by an imaginary part in the error probabilities. The experimentally observable statistics thus indicates that there is a qualitative difference between the quantum correlations of measurement uncertainties and classical correlations described by real-valued probabilities.

To properly appreciate the significance of the results, it is necessary to keep in mind that the physics of measurement can only be understood in terms of experimentally observable evidence. The essential insight of the following discussion is that entanglement provides us with the necessary tool for the analysis of correlations between non-commuting properties. If the claim that non-positive probabilities can arise from experimental data seems to be somewhat odd, it should be kept in mind that the uncertainty principle absolutely forbids the observation or preparation of a joint reality for the non-commuting properties in question. Much of the following discussion is therefore concerned with the problem of connecting the experimental evidence directly with the fundamental physics, while avoiding any pre-judgment of the results based on the mathematical formalism of quantum theory or its interpretation.

## II. ERROR STATISTICS OF JOINT MEASUREMENTS

The established formalism of quantum mechanics represents physical properties by operators in Hilbert space. The precise measurement of a physical property \( \hat{A} \) is represented by a projection onto an eigenstate \(| a \rangle\), where the probability of an outcome \( a \) is given by the product trace of the projection operator \(| a \rangle \langle a |\) and the density matrix \( \hat{\rho} = | \psi \rangle \langle \psi |\) that described the state \(| \psi \rangle\) of the system before the measurement. Importantly, a joint assignment of outcomes \( a \) and \( b \) for two physical properties \( \hat{A} \) and \( \hat{B} \) is only possible if the two physical properties have shared eigenstates. It is therefore impossible to define a joint measurement of \( a \) and \( b \) without introducing some form of measurement uncertainty, such that the probability \( P(a, b) \) of a joint outcome is not given by an intrinsic joint probability \( \rho(a, b) \) of the initial state \( \hat{\rho} \). Nevertheless, we can design a joint measurement that yields a complete set of joint outcomes \((a, b)\) if we accept statistical errors in the joint measurement.

In standard quantum theory, such a joint measurement is described by a positive operator-valued measure, \( \hat{\Pi}_{a, b} \), so that the joint probability of the experimental measurement outcome is given by

\[
P_{\text{exp}}(a, b) = Tr\{\hat{\Pi}_{a, b} \hat{\rho}\}. \tag{1}
\]

Mathematically, this is a bilinear relation between the measurement and the state defined in the Hilbert space of the system. However, it is the purpose of the measurement to evaluate the physical properties \( \hat{A} \) and \( \hat{B} \) in terms of their precise measurement outcomes \( a \) and \( b \) that would have been obtained from the original input state \( \hat{\rho} \). For this purpose, we should express the input state in terms of the set of outcomes \((a, b)\) associated with precise error-free measurements. Here, it is interesting to observe that the quantum state \( \hat{\rho} \) in a \( d \)-dimensional Hilbert space is described by \( d^2 \) independent parameters, corresponding to the number of possible combinations of measurement outcomes \( a \) and \( b \). Therefore, the only possible form of a bilinear relation of the measurement \( \hat{\Pi}_{a, b} \) and the state \( \hat{\rho} \) expressed in terms of the outcomes \( a \) and \( b \) is given by

\[
P_{\text{exp}}(a, b) = \sum_{a', b'} P(a, b|a', b') \rho(a', b'), \tag{2}
\]

where the representation of the quantum state correspond to a joint probability \( \rho(a, b) \) of the outcomes \( a \) and \( b \), and the description of the measurement process corresponds to a conditional probability relating the input combination \((a', b')\) to the measurement result \((a, b)\). Importantly, Eq. (2) is not based on a hidden variable model and does not require the assumption that the combination of \( a \) and \( b \) in \( \rho(a, b) \) represent a joint reality of \( a \) and \( b \) before the measurement is performed. The purpose of jointly assigning \( a \) and \( b \) to the input is merely to obtain a formulation of the joint measurement \( \hat{\Pi}_{a, b} \) in terms of the target observables \( \hat{A} \) and \( \hat{B} \) that is equally valid for any input state, including eigenstates of either \( \hat{A} \) or \( \hat{B} \).

Interestingly, the eigenstates of \( \hat{A} \) and \( \hat{B} \) provide well-defined and unambiguous joint probabilities of \( a \) and \( b \), since the eigenstates assign a probability of zero to any outcome other than the one specified by the state. The joint probabilities for the correct outcome are then given by the marginal probabilities for the eigenstates of the other property, as given by the squared inner products \(|\langle b | a \rangle|^2\). For an eigenstate \(| a \rangle\), the joint probabilities \( \rho_a(a', b') \) are all zero for \( a' \neq a \) and correspond to the probabilities of \( b' \) in \(| a \rangle\) else,

\[
\rho_a(a', b') = \delta_{a, a'} |\langle b' | a \rangle|^2. \tag{3}
\]

Likewise, the eigenstates of \( \hat{B} \) are described by joint probabilities of

\[
\rho_b(a', b') = \delta_{b, b'} |\langle b | a' \rangle|^2. \tag{4}
\]
The conditional probabilities \( P(a, b|a', b') \) can then be characterized experimentally by measuring the experimental probabilities of the outcomes \((a, b)\) for input states of \(|a''\rangle\) and \(|b''\rangle\). The result is an intuitive description of measurement errors in terms of conditional probabilities that relate the correct input values to the actual output values. However, the fact that we have to select either eigenstates of \(\hat{A}\) or eigenstates of \(\hat{B}\) for the input means that we do not obtain information on the detailed correlations between the measurement errors \((a'' \rightarrow a)\) and the measurement errors \((b'' \rightarrow b)\). The experimental data only provides as with the input marginals of \(P(a, b|a', b')\) given by

\[
P_{\text{exp}}(a, b|a'') = \sum_{b'} P(a, b|a'', b') \langle b' | a'' \rangle^2, \\
P_{\text{exp}}(a, b|b'') = \sum_{a'} P(a, b|a', b'') \langle b' | a' \rangle^2.
\]

(Equation 5)

Eigenstate inputs therefore only provide us with an incomplete characterization of the measurement statistics. To characterize the complete set of correlated errors described by the measurement operator \(\Pi_{a,b}\), it is necessary to use input states with well-defined correlations between the non-commuting properties \(\hat{A}\) and \(\hat{B}\). Fortunately, such states are actually available in the form of maximally entangled states, and we will show in section IV how correlations between the measurement errors appear in the experimental statistics obtained from correlated measurement of maximally entangled pairs. Since the main results will be obtained for two-level systems, however, it may be good to take a closer look at this specific case, where it is sufficient to characterize the measurement outcome as either correct or incorrect.

### III. EVALUATION OF MEASUREMENT ERRORS IN TWO-LEVEL SYSTEMS

The most simple example of a pair of non-commuting observables is given by the orthogonal spin components \(\hat{X}\) and \(\hat{Y}\) of a two-level system, where the eigenvalues are given by \pm 1. A joint measurement of \(\hat{X}\) and \(\hat{Y}\) therefore has four possible outcomes given by \((x, y) = (+1, +1), (+1, -1), (-1, +1), (-1, -1)\). As explained above, the joint measurement is described by 16 conditional probabilities \(P(x, y|x', y')\) that relate the input values of \((x', y')\) to the output values \((x, y)\). However, the number of unknowns can be greatly reduced if we assume that the error probabilities are symmetric under exchanges of +1 and −1, so that the probability of obtaining a correct outcome does not depend on the actual spin value. The measurement statistics is then described by only four error probabilities defined by the relation between the input values and the output values. We can define these error probabilities as

\[
\begin{align*}
\eta(0, 0) &= P(x, y|x, y) \\
\eta(0, 1) &= P(x, -y|x, y) \\
\eta(1, 0) &= P(-x, y|x, y) \\
\eta(1, 1) &= P(-x, -y|x, y),
\end{align*}
\]

(Equation 6)

where the arguments of the error probabilities \(\eta(r_x, r_y)\) define whether an error occurs \((r_i = 1)\) or not \((r_i = 0)\).

Following the procedure outlined in section III we can now characterize the error probabilities \(\eta(r_x, r_y)\) from experimental data obtained with eigenstate inputs. For the \(\hat{X}\)-eigenstates, the experimental results are

\[
\begin{align*}
P_{\text{exp}}(x, y|x) &= \eta(0, 0) + \eta(0, 1) \\
P_{\text{exp}}(-x, y|x) &= \eta(1, 0) + \eta(1, 1)
\end{align*}
\]

(Equation 7)

and these two results can be summarized by a single resolution parameter,

\[
V_x = \sum_y (P_{\text{exp}}(x, y|x) - P_{\text{exp}}(-x, y|x)) = \eta(0, 0) + \eta(0, 1) - \eta(1, 0) - \eta(1, 1).
\]

(Equation 8)

Likewise, the results obtained from \(\hat{Y}\)-eigenstates can be summarized by

\[
V_y = \sum_x (P_{\text{exp}}(x, y|y) - P_{\text{exp}}(x, -y|y)) = \eta(0, 0) - \eta(0, 1) + \eta(1, 0) - \eta(1, 1).
\]

(Equation 9)
The complete measurement is characterized by four unknown error probabilities $\eta(r_x, r_y)$. By performing the measurement on eigenstate inputs, we can evaluate two resolution parameters, $V_x$ and $V_y$. In addition, normalization requires that

$$\eta(0, 0) + \eta(0, 1) + \eta(1, 0) + \eta(1, 1) = 1. \quad (10)$$

This leaves us with a single unknown parameter, which can be expressed by

$$C = \eta(0, 0) - \eta(0, 1) - \eta(1, 0) + \eta(1, 1). \quad (11)$$

IV. CORRELATIONS BETWEEN ENTANGLED PAIRS

Maximally entangled pairs are states with particularly strong correlation between the two quantum systems. Specifically, a precise measurement of one system will project the remote system into an eigenstate corresponding to the measurement result obtained locally. If the physical properties of the two systems are properly aligned, the value of $A$ in the remote system can be determined by an $A$-measurement on the local system, and the value of $B$ in the remote system is determined from the result of a $B$-measurement on the local system. Maximally entangled pairs are therefore characterized by correlations in both $A$ and $B$, providing possible experimental evidence about the quantum correlations between the non-commuting observables $A$ and $B$.

To analyze the measurement statistics in detail, we first need to formulate maximally entangled states as joint probabilities of $a$ and $b$ in system 1 and system 2. In general, this requires the definition of a map between the measurement outcomes obtained in system 1 and the measurement outcomes obtained in system 2. In the following, we will express this map by a line above the outcome, $a_2 = \overline{a}_1$ and $b_2 = \overline{b}_1$. The joint probabilities of precise measurements in the two systems for the maximally entangled state $| E \rangle$ can then be given by

$$P(a_1, a_2) = |\langle a_1, a_2 | E \rangle|^2 = \frac{1}{d} \delta_{\overline{a}_1, a_2}$$

$$P(b_1, b_2) = |\langle b_1, b_2 | E \rangle|^2 = \frac{1}{d} \delta_{\overline{b}_1, b_2}. \quad (12)$$

It is also possible to determine the probabilities for a measurement of $A$ in one system and a measurement of $B$ in the other,

$$P(a_1, b_2) = |\langle a_1, b_2 | E \rangle|^2 = \frac{1}{d} |\langle \overline{a}_1 | b_2 \rangle|^2$$

$$P(b_1, a_2) = |\langle b_1, a_2 | E \rangle|^2 = \frac{1}{d} |\langle a_2 | \overline{b}_1 \rangle|^2. \quad (13)$$

In close analogy to the derivation of joint probabilities for local eigenstates of $A$ and $B$, it is possible to derive a complete joint probability for the maximal entangled state simply by setting all joint probabilities to zero whenever they include a contribution that has a total probability of zero in the directly observable measurement statistics. The result is a joint probability given by

$$\rho_E(a_1, b_1; a_2, b_2) = \frac{1}{d} \delta_{\overline{a}_1, a_2} \delta_{\overline{b}_1, b_2} |\langle a_2 | b_2 \rangle|^2. \quad (14)$$

It is now possible to apply joint measurements of $A$ and $B$ to both systems. If it is possible to ensure that the two measurements are identical, this results in a quadratic function of the conditional probabilities $P(a, b|a', b')$ that characterize the measurement. Specifically,

$$P_{\exp.}(a_1, b_1; a_2, b_2|E) = \sum_{a', b'} P(a_1, b_1|a', b')P(a_2, b_2|a', b') \frac{1}{d} |\langle a' | b' \rangle|^2. \quad (15)$$

In general, this is a very different sum from the one that determines the errors for eigenstate inputs. In particular, the sum runs over all $a'$ and all $b'$ of the input, since the input values of $A$ and $B$ are equally unknown. Nevertheless the correlations between system 1 and system 2 provide a clear statistical structure to the output, thus revealing something about the correlations between the measurement errors in $A$ and in $B$ in the form of correlations between the output values of $(a_1, b_1)$ and $(a_2, b_2)$. 

V. ERROR CORRELATIONS FOR TWO-LEVEL SYSTEMS

We can now take a closer look at the case of singlet entanglement between a pair of two-level systems. In that case, all of the spin components have opposite values, so that \( \mathcal{I}_1 = -x_1 \) and \( \mathcal{I}_2 = -y_1 \) and the joint probability of \( x_i \) and \( y_i \) can be written as

\[
\rho E(x_1, y_1; x_2, y_2) = \frac{1}{4} \delta_{x_1, x_2} \delta_{y_1, y_2}.
\]

The experimental statistics for the correlated outcomes of joint measurements of system 1 and system 2 is given by

\[
P_{\text{exp.}}(x_1, y_1; x_2, y_2|E) = \frac{1}{4} \sum_{x', y'} P(x_1, y_1|x', y') P(x_2, y_2|x', -y').
\]

These sums can be further simplified by using the error probabilities \( \eta(r_x, r_y) \) introduced in section III. Importantly, this means that the 16 experimental probabilities can also be summarized in terms of only four output patterns, depending on whether the correlations between \( (x_1, y_1) \) and \( (x_2, y_2) \) depend on the original correlations or not:

\[
\begin{align*}
E(0, 0) &= P_{\text{exp.}}(x, y; -x, -y) \\
E(0, 1) &= P_{\text{exp.}}(x, y; -x, y) \\
E(1, 0) &= P_{\text{exp.}}(x, y; x, -y) \\
E(1, 1) &= P_{\text{exp.}}(x, y; x, y).
\end{align*}
\]

Each of the four values \( E(r_x, r_y) \) can be determined experimentally by applying the joint measurement of \( \hat{x} \) and \( \hat{Y} \) to both systems of the entangled pair. According to Eq. (17), these experimental results are related to the error probabilities \( \eta(r_x, r_y) \) by

\[
\begin{align*}
V_x^2 &= (\eta(0, 0) + \eta(0, 1) - \eta(1, 0) - \eta(1, 1))^2 = 4 (E(0, 0) + E(0, 1) - E(1, 0) - E(1, 1)) \\
V_y^2 &= (\eta(0, 0) - \eta(0, 1) + \eta(1, 0) - \eta(1, 1))^2 = 4 (E(0, 0) - E(0, 1) + E(1, 0) - E(1, 1)) \\
C^2 &= (\eta(0, 0) + \eta(0, 1) - \eta(1, 0) - \eta(1, 1))^2 = 4 (E(0, 0) + E(0, 1) - E(1, 0) - E(1, 1)).
\end{align*}
\]

While Eq. (19) and Eq. (20) merely reproduce the resolutions obtained from the measurements of eigenstate inputs, Eq. (21) provides direct experimental evidence of the correlation between measurement errors in \( \hat{X} \) and \( \hat{Y} \).

VI. QUANTUM THEORY OF JOINT MEASUREMENTS

It is important to note that the results for the measurement resolutions \( V_x, V_y, \) and \( C \) obtained from the experimental data given by the outcome probabilities \( E(r_x, r_y) \) according to Eqs. (19-21) are completely independent of the measurement model used to explain the measurement process. In particular, Eq. (21) represents an operational definition of the correlation between measurement errors in \( \hat{X} \) and \( \hat{Y} \) that applies equally well to classical and to quantum models. However, quantum theory prevents an independent confirmation of the validity of Eq. (21) by individual measurements of appropriately prepared inputs, since there are no joint eigenstates of \( \hat{X} \) and \( \hat{Y} \) that could be used to define the relation between the two non-commuting properties in the input. In quantum mechanics, Eq. (21) thus represents the most fundamental operational definition of correlations between measurement errors in joint measurements of \( \hat{X} \) and \( \hat{Y} \).

Quantum theory actually makes very precise predictions about the measurement outcomes obtained from joint measurements of \( \hat{X} \) and \( \hat{Y} \). Since the measurement operators are defined by self-adjoint matrices in a two-dimensional Hilbert space, they can be expressed as linear combinations of the Pauli matrices \( \hat{X}, \hat{Y}, \hat{Z} \) and the identity \( \hat{I} \). The requirement of symmetry between results of \( +1 \) and results of \( -1 \) further limits the choice to equal coefficients with variable signs, so the only possible form of the positive operator-valued measure for a joint measurement of \( \hat{X} \) and \( \hat{Y} \) is

\[
\begin{align*}
\hat{\Pi}_{1+1} &= \frac{1}{4} (\hat{I} + V_x \hat{X} + V_y \hat{Y} + V_z \hat{Z}) \\
\hat{\Pi}_{1+1} &= \frac{1}{4} (\hat{I} + V_x \hat{X} - V_y \hat{Y} + V_z \hat{Z}) \\
\hat{\Pi}_{-1+1} &= \frac{1}{4} (\hat{I} - V_x \hat{X} + V_y \hat{Y} + V_z \hat{Z}) \\
\hat{\Pi}_{-1+1} &= \frac{1}{4} (\hat{I} - V_x \hat{X} - V_y \hat{Y} + V_z \hat{Z})
\end{align*}
\]
By convention, $V_x$ and $V_y$ should be positive numbers between zero and one. Since $V_z$ is not related to the measurement outcomes, it can be either positive or negative. In general, positivity limits the sum of the squares of $V_i$ to a maximal value of one, which corresponds to projections onto pure states.

It is now possible to predict the experimental results that can be obtained from correlated measurements of maximally entangled pairs characterized by expectation values of $\langle \hat{X}_1 \hat{X}_2 \rangle = -1$, $\langle \hat{Y}_1 \hat{Y}_2 \rangle = -1$ and $\langle \hat{Z}_1 \hat{Z}_2 \rangle = -1$. The probabilities of the outcomes are given by the expectation values of the measurement operators,

$$P(x_1, y_1; x_2, y_2) = \langle \hat{P}_{x_1,y_2} \hat{P}_{x_2,y_2} \rangle = \frac{1}{16} (1 - x_1 x_2 V_x^2 - y_1 y_2 V_y^2 - x_1 x_2 y_1 y_2 V_z^2) .$$

As expected, there are only four different results $E(r_x, r_y)$, where $r_x = 0$ for $x_1 = -x_2$ and $r_y = 0$ for $y_1 = -y_2$. Specifically, the four experimentally observable probabilities are given by

$$E(0,0) = \frac{1}{16} (1 + V_x^2 + V_y^2 - V_z^2)$$
$$E(0,1) = \frac{1}{16} (1 + V_x^2 - V_y^2 + V_z^2)$$
$$E(1,0) = \frac{1}{16} (1 - V_x^2 + V_y^2 + V_z^2)$$
$$E(1,1) = \frac{1}{16} (1 - V_x^2 - V_y^2 - V_z^2) .$$

This result confirms the identification of $V_x$ and $V_y$ with the measurement resolutions of $\hat{X}$ and $\hat{Y}$ according to Eq. (8) and Eq. (9), respectively. Interestingly, the sensitivity to $\hat{Z}$ now appears as a correlations between the measurement errors in $\hat{X}$ and in $\hat{Y}$. However, this correlation is anomalous, since the sign of $C^2$ becomes negative:

$$C^2 = (\eta(0,0) - \eta(0,1) - \eta(1,0) + \eta(1,1))^2$$
$$= 4(E(0,0) - E(1,0) - E(0,1) + E(1,1))$$
$$= -V_z^2 .$$

The experimentally observed correlations between measurement errors in $\hat{X}$ and $\hat{Y}$ therefore correspond to imaginary error probabilities defined by

$$C = (\eta(0,0) - \eta(0,1) - \eta(1,0) + \eta(1,1)) = \pm i V_z .$$

This result clearly describes a qualitative difference between the predictions of quantum theory and the predictions of any possible classical model of measurement errors. Specifically, all classical statistical models would require that $E(0,0) + E(1,1)$ is greater than $E(0,1) + E(1,0)$. Indeed, it is obvious that a precise measurement of $\hat{X}$ and $\hat{Y}$ would have to result in the correct correlations, so that only $E(0,0)$ would retain a non-zero value. The fact that all quantum measurements result in $E(0,1) + E(1,0) \geq E(0,0) + E(1,1)$ is therefore closely linked to the impossibility of uncertainty free joint measurements.

### VII. OPERATOR CORRELATIONS AND IDEAL MEASUREMENTS

In order to understand why quantum theory can produce results that contradict classical expectations in a qualitative way, it is necessary to remember that the uncertainty principle prevents any joint measurement of the non-commuting properties $\hat{X}$ and $\hat{Y}$. If experimentally observable relations between non-commuting probabilities violate expectations associated with a hypothetical joint reality, the most likely conclusion is that there is no such reality.

The present analysis allows us to identify the actual measurement errors in a quantum measurement, using a close analogy with classical error probabilities. The ideal classical measurement would be characterized by $\eta(0,0) = 1$ and $V_x = V_y = C = 1$. Interestingly, we can construct an operator that represents this ideal classical measurement by using $V_z = \pm i$ in Eq. (22). The resulting operators can then be factorized into a product of two projectors,

$$\hat{\Pi}_{\text{ideal}} = \frac{1}{4} \left( \hat{I} \pm \hat{X} \right) \left( \hat{I} \pm \hat{Y} \right) .$$

Quantum theory thus provides a well-defined theoretical form for uncertainty-free measurements, but this form is unphysical because it is non-hermitian and would therefore result in complex probabilities for the joint outcomes. Specifically, the correlations between $\hat{X}$ and $\hat{Y}$ are imaginary and satisfy the operator equation

$$\langle \hat{X} \hat{Y} \rangle = i \langle \hat{Z} \rangle .$$

Thus, our analysis suggests that (a) the correlation between $\hat{X}$ and $\hat{Y}$ is correctly represented by an ordered product of the operators, and hence has an imaginary value and (b) error free joint measurements are impossible because imaginary error probabilities are needed to convert the imaginary correlations into real-valued probabilities.

Importantly, our analysis shows that the non-classical error correlations of joint measurements can be observed directly in experiments using entangled state inputs, without any prior assumptions about non-commutative observables. The correlations between the outcomes of joint measurements performed on entangled pairs thus show that the imaginary correlations represented by operator products do have experimentally observable consequences and should be taken seriously as valid descriptions of non-classical correlations. This result is also consistent with recent progress on the analysis of quantum correlations using weak values and may therefore pave the way for a more complete understanding of quantum phenomena in terms of complex-valued correlations [19–24].

VIII. CONCLUSIONS

Although it is impossible to obtain uncertainty free values of non-commuting observables in a single joint measurement, it is possible to describe the statistical errors of joint measurements in terms of conditional probabilities relating the error-free results to the actual results. For two-level systems, the complete set of error probabilities can be obtained by comparing the correlations between the joint outcomes obtained for the two entangled systems with the known correlations of the initial entangled state. Since the analysis of the experimental results does not depend on any prior assumptions about the measurement statistics, it is possible to compare the predictions of quantum theory with classical models of measurement errors.

Significantly, the experimentally observable correlations predicted by quantum theory are qualitatively different from the predictions of classical statistical models, since the correlations between the errors in $\hat{X}$ and $\hat{Y}$ correspond to imaginary error probabilities. This experimentally observable difference between quantum theory and classical statistics is a direct consequence of non-commutativity, since the imaginary probability can be traced to the operator product $\hat{X}\hat{Y} = i\hat{Z}$. Uncertainty limited joint measurements of non-commuting observables are therefore sensitive to the non-classical correlations described by the non-commutativity of the operators that represent physical properties in the quantum formalism. By performing the same type of joint measurement on two maximally entangled systems, it is possible to make these non-classical correlations visible in the form of joint probability distributions that cannot be explained in terms of positive-valued statistics. The correlations between the outcomes of joint measurements observed with entangled input states thus provides direct experimental evidence for the non-classical correlations associated with non-commutativity.

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