Low energy properties of the $SU(m|n)$ supersymmetric Haldane-Shastry spin chain

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Abstract

The ground state and low energy excitations of the $SU(m|n)$ supersymmetric Haldane-Shastry spin chain are analyzed. In the thermodynamic limit, it is found that the ground state degeneracy is finite only for the $SU(m|0)$ and $SU(m|1)$ spin chains, while the dispersion relation for the low energy and low momentum excitations is linear for all values of $m$ and $n$. We show that the low energy excitations of the $SU(m|1)$ spin chain are described by a conformal field theory of $m$ non-interacting Dirac fermions which have only positive energies; the central charge of this theory is $m/2$. Finally, for $n \geq 1$, the partition functions of the $SU(m|n)$ Haldane-Shastry spin chain and the $SU(m|n)$ Polychronakos spin chain are shown to be related in a simple way in the thermodynamic limit at low temperatures.

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1 Introduction

The Haldane-Shastry (HS) spin-1/2 chain is an integrable model in which equally spaced spins on a circle interact with each other through pairwise interactions which are inversely proportional to the square of their chord distances [1, 2]. Interestingly, the HS spin-1/2 chain is easier to study in some respects than the integrable spin-1/2 chain with nearest-neighbor interactions; it has a Yangian quantum group symmetry, and the low energy excitations form an ideal gas with fractional statistics [3, 4, 5].

The HS spin-1/2 chain has an $SU(2)$ symmetry, with the spin at each site forming the fundamental representation of $SU(2)$. This model can be generalized to an $SU(m)$ symmetric model whose Hamiltonian is given, with $N$ lattice sites, by

$$H_{HS} = \frac{1}{2} \sum_{1 \leq j < k \leq N} \frac{1 + P_{jk}}{\sin^2(\xi_j - \xi_k)}, \quad (1.1)$$

where $\xi_j = j\pi/N$, and $P_{jk}$ is the exchange operator which interchanges the ‘spins’ (which can take $m$ possible values) on the $j$-th and $k$-th lattice sites.

One can find the complete energy spectrum of the HS spin chain in (1.1), including the degeneracy of each energy level, through the motif representations of its $Y(gl_m)$ Yangian symmetry [6, 7, 8]. By using this information about the spectrum, it is possible in principle to construct the partition function of this spin chain. However, there is a simpler method for calculating the partition function of the $SU(m)$ HS spin chain [9] which uses the so-called freezing technique [10, 11, 12]. The freezing technique consists of taking the strong coupling limit of the spin Calogero-Sutherland (CS) Hamiltonian; then the coordinates of the particles freeze at the classical equilibrium positions of the scalar part of the potential, and the spins get decoupled from the coordinate degrees of freedom. As a result, one can derive the partition function of the HS spin chain by ‘modding out’ the partition function of the spinless CS model from that of the spin CS model.

There exists an $SU(m|n)$ supersymmetric extension of the HS spin chain [3], where each site is occupied by either one of $m$ type of bosonic states or one of $n$ type of fermionic states. Such supersymmetric spin chains play a role in describing some correlated systems in condensed matter physics, where holes moving in the dynamical background of spins behave as bosons, and spin-1/2 electrons behave as fermions [13, 14, 15]. The $SU(m|n)$ supersymmetric HS spin chain exhibits the $Y(gl_{(m|n)})$ super-Yangian symmetry [3]; this is also the quantum group symmetry of the $SU(m|n)$.
supersymmetric Polychronakos spin chain. So it is expected that the spectra and partition functions of these two spin chains would share some common features. The freezing technique has been used to compute the partition function of the $SU(m|n)$ Polychronakos spin chain [16, 17], after mapping the corresponding supersymmetric exchange operators to a representation of the permutation algebra containing ‘anyon like’ spin dependent interactions [18, 19]. In Ref. [20], this technique has been used to compute the exact partition function of the $SU(m|n)$ HS spin chain. Subsequently, it was shown that the partition function of the $SU(m|n)$ HS spin chain can be expressed through the Schur polynomials associated with the motif representations, and an exact duality relation has been established between the partition functions of the $SU(m|n)$ and $SU(n|m)$ HS spin chains [21].

In this paper, our main aim is to study the low energy spectrum of the $SU(m|n)$ HS spin chain in the thermodynamic limit $N \to \infty$. To this end, in Sec. 2 we review some of the results known for this model. We subsequently use its exact partition function to compute the complete spectrum for finite values of $N$. In particular, we give explicit expressions for the degeneracies of all the energy levels by taking a limit of the Schur polynomials corresponding to the motif representations. In Sec. 3, we discuss the momentum eigenvalues associated with the motifs of the $SU(m|n)$ HS spin chain. In Sec. 4, we focus on the ground state and low energy excitations of the $SU(m|n)$ HS spin chain for all possible values of $m$ and $n$. In particular, we study the degeneracy of the ground state and the relation between the energy and momentum of the low energy excitations in the thermodynamic limit. In Sec. 5, we discuss whether the low energy excitations can be described by conformal field theories [22] for certain values of $m$ and $n$. In Sec. 6, we explicitly prove the equivalence at low temperatures of the partition functions of the $SU(m|1)$ HS spin chain and a model of $m$ non-interacting fermions with a particular kind of energy dispersion. We also derive an interesting relation between the partition functions of the $SU(m|n)$ HS spin chain and the $SU(m|n)$ Polychronakos spin chain at low temperatures, for any value of $n \geq 1$. We summarize our results in Sec. 7.

2 Energy spectrum of the $SU(m|n)$ HS spin chain

For the purpose of defining the Hamiltonian of the $SU(m|n)$ supersymmetric HS spin chain, let us consider operators like $C^\dagger_{j\alpha}$ ($C_{j\alpha}$) which create (annihilate) a particle of species $\alpha$ on the $j$-th lattice site. These creation (annihilation) operators are assumed
to be bosonic when $\alpha \in \{1, 2, \ldots, m\}$, and fermionic when $\alpha \in \{m+1, m+2, \ldots, m+n\}$. Thus, the parity of $C_{j\alpha}^\dagger (C_{j\alpha})$ is defined as

$$p(C_{j\alpha}) = p(C_{j\alpha}^\dagger) = 0 \text{ for } \alpha \in \{1, 2, \ldots, m\},$$

and

$$p(C_{j\alpha}) = p(C_{j\alpha}^\dagger) = 1 \text{ for } \alpha \in \{m+1, m+2, \ldots, m+n\}. \tag{2.1}$$

These operators satisfy the commutation (anticommutation) relations

$$[C_{j\alpha}, C_{k\beta}]_{\pm} = 0, \quad [C_{j\alpha}^\dagger, C_{k\beta}^\dagger]_{\pm} = 0, \quad [C_{j\alpha}, C_{k\beta}^\dagger]_{\pm} = \delta_{jk}\delta_{\alpha\beta}, \tag{2.2}$$

where $[A, B]_{\pm} \equiv AB - (-1)^{p(A)p(B)}BA$. We now consider a subspace of the related Fock space in which the number of particles on each site is exactly 1, namely,

$$\sum_{\alpha=1}^{m+n} C_{j\alpha}^\dagger C_{j\alpha} = 1 \tag{2.3}$$

for all $j$. On this subspace, we define the supersymmetric exchange operators

$$\hat{P}_{jk}^{(m|n)} \equiv \sum_{\alpha, \beta=1}^{m+n} C_{j\alpha}^\dagger C_{k\beta}^\dagger C_{j\beta} C_{k\alpha}, \tag{2.4}$$

where $1 \leq j < k \leq N$. These $\hat{P}_{jk}^{(m|n)}$s yield a realization of the permutation algebra given by

$$P_{jk}^2 = 1, \quad P_{jk}P_{kl} = P_{jl}P_{jk} = P_{kl}P_{jl}, \quad [P_{jk}, P_{lm}] = 0, \tag{2.5}$$

where $j, k, l, m$ are all distinct indices. Replacing $P_{jk}$ by $\hat{P}_{jk}^{(m|n)}$ in Eq. (1.1), we obtain the Hamiltonian of the $SU(m|n)$ supersymmetric HS model as [3]

$$H_{HS}^{(m|n)} = \frac{1}{2} \sum_{1 \leq j < k \leq N} \frac{1 + \hat{P}_{jk}^{(m|n)}}{\sin^2(\xi_j - \xi_k)}. \tag{2.6}$$

As shown in Ref. [20], the $SU(m|n)$ supersymmetric HS model in (2.6) can be transformed to a spin chain. We consider a representation of the permutation algebra (2.5), which acts on a spin state like $|\alpha_1\alpha_2\ldots\alpha_N\rangle$, with $\alpha_j \in \{1, 2, \ldots, m+n\}$, as
\[ \tilde{P}_{jk}^{(m|n)} |\alpha_1 \ldots \alpha_j \ldots \alpha_k \ldots \alpha_N \rangle = e^{i\Phi(\alpha_j,\alpha_{j+1}, \ldots, \alpha_k)} |\alpha_1 \ldots \alpha_k \ldots \alpha_j \ldots \alpha_N \rangle. \]  

(2.7)

Here \( e^{i\Phi(\alpha_j,\alpha_{j+1}, \ldots, \alpha_k)} = 1 \) if \( \alpha_j, \alpha_k \in \{1, 2, \ldots, m\} \), \( e^{i\Phi(\alpha_j,\alpha_{j+1}, \ldots, \alpha_k)} = -1 \) if \( \alpha_j, \alpha_k \in \{m+1, m+2, \ldots, m+n\} \), and \( e^{i\Phi(\alpha_j,\alpha_{j+1}, \ldots, \alpha_k)} = (-1)^\pi \sum_{p=j+1}^{k-1} \sum_{\tau=m+1}^{m+n} \delta_{\alpha_p, \tau} \) if \( \alpha_j \in \{1, 2, \ldots, m\} \) and \( \alpha_k \in \{m+1, m+2, \ldots, m+n\} \) or vice versa. We will call \( \alpha_i \) a ‘bosonic’ spin if \( \alpha_i \in \{1, 2, \ldots, m\} \), and a ‘fermionic’ spin if \( \alpha_i \in \{m+1, m+2, \ldots, m+n\} \). From Eq. (2.7), it follows that the exchange of two bosonic (fermionic) spins produces a phase factor of \( 1(-1) \) irrespective of the nature of the spins situated in between the \( j \)-th and \( k \)-th lattice sites. However, if we exchange one bosonic spin with one fermionic spin, the phase factor is \( (-1)^\rho \) where \( \rho \) is the total number of fermionic spins situated in between the \( j \)-th and \( k \)-th lattice sites. The constraint in Eq. (2.3) implies that the Hilbert space associated with the \( SU(m|n) \) HS Hamiltonian in (2.6) can be spanned through the following orthonormal basis vectors:

\[ C_{1\alpha_1}^{\dagger} C_{2\alpha_2}^{\dagger} \ldots C_{N\alpha_N}^{\dagger} |0\rangle, \]

where \( |0\rangle \) is the vacuum state and \( \alpha_j \in \{1, 2, \ldots, m+n\} \). We define a one-to-one mapping between these basis vectors and those of the above mentioned spin chain as

\[ |\alpha_1 \alpha_2 \ldots \alpha_N \rangle \leftrightarrow C_{1\alpha_1}^{\dagger} C_{2\alpha_2}^{\dagger} \ldots C_{N\alpha_N}^{\dagger} |0\rangle. \]  

(2.8)

Using the commutation (anticommutation) relations in (2.2), we can verify that

\[ \tilde{P}_{jk}^{(m|n)} C_{1\alpha_1}^{\dagger} \ldots C_{j\alpha_j}^{\dagger} \ldots C_{k\alpha_k}^{\dagger} \ldots C_{N\alpha_N}^{\dagger} |0\rangle = e^{i\Phi(\alpha_j, \ldots, \alpha_k)} C_{1\alpha_1}^{\dagger} \ldots C_{j\alpha_j}^{\dagger} \ldots C_{k\alpha_k}^{\dagger} \ldots C_{N\alpha_N}^{\dagger} |0\rangle, \]  

(2.9)

where \( e^{i\Phi(\alpha_j, \ldots, \alpha_k)} \) is the same phase factor which appeared in Eq. (2.7). A comparison of Eq. (2.9) with Eq. (2.7) through the mapping in (2.8) shows that the representation \( \tilde{P}_{jk}^{(m|n)} \) is equivalent to the supersymmetric exchange operator \( \hat{P}_{jk}^{(m|n)} \). Hence, if we define a spin chain Hamiltonian through \( \tilde{P}_{jk}^{(m|n)} \) as

\[ H_{HS}^{(m|n)} = \frac{1}{2} \sum_{1 \leq j < k \leq N} \frac{1 + \tilde{P}_{jk}^{(m|n)}}{\sin^2(\xi_j - \xi_k)}, \]  

(2.10)

it would be completely equivalent to the \( SU(m|n) \) supersymmetric HS model in (2.6) [18]. For the special case \( n = 0 \), \( \tilde{P}_{jk}^{(m|n)} \) reproduces the original spin exchange operator
\( p_{jk} \) and \( H_{\text{HS}}^{(m|n)} \) in (2.10) reduces to the Hamiltonian of the \( SU(m) \) HS spin chain in (1.1). We will henceforth study the \( SU(m|n) \) supersymmetric HS model defined in (2.10) instead of its original form in (2.6).

Let us now discuss the partition function of the \( SU(m|n) \) supersymmetric HS model, which has been derived by using the freezing technique \[20\]. Consider a set of positive integers \( k_1, k_2, \ldots, k_r \), where \( \sum_{i=1}^r k_i = N \), and \( r \) is an integer which can take any value from 1 to \( N \). The vector \( \mathbf{k} \equiv \{k_1, \ldots, k_r\} \) belongs to the set \( \mathcal{P}_N \) of ordered partitions of \( N \). Associated with each \( \mathbf{k} \), we attach a dimensionality given by

\[
d^{(m|n)}(\mathbf{k}) = \prod_{i=1}^r d^{(m|n)}(k_i),
\]

where \( d^{(m|n)}(k_i) \) is a function of \( m, n \) and \( k_i \). In the case of the supersymmetric HS spin chain, for which both \( m \) and \( n \) are positive integers, \( d^{(m|n)}(k_i) \) is expressed as

\[
d^{(m|n)}(k_i) = \sum_{j=0}^{\min(m,k_i)} mC_{j_{k_i-j+n-1}}c_{k_i-j},
\]

with \( pC_l = \frac{p!}{l!(p-l)!} \) for \( l \leq p \) and \( pC_l = 0 \) for \( l > p \). In the case of the \( SU(n) \) fermionic model, \( d^{(0|n)}(k_i) \) is obtained by putting \( m = 0 \) in Eq. (2.12),

\[
d^{(0|n)}(k_i) = k_i+n-1C_{k_i}.
\]

The dimensionality of the \( SU(m) \) bosonic case can also be obtained from Eq. (2.11) by taking \[9\]

\[
d^{(m|0)}(k_i) = mC_{k_i}.
\]

We note that, while the dimensionality appearing in Eq. (2.11) can take a non-zero value for the bosonic case only if \( k_i \leq m \) for all \( i \), it takes a non-zero value for any \( \mathbf{k} \in \mathcal{P}_N \) for both the supersymmetric as well as the fermionic case.

Next we define the quantities \( K_i = \sum_{j=1}^i k_j \) which denote the partial sums corresponding to the partition \( \mathbf{k} \in \mathcal{P}_N \). The partition function of the \( SU(m|n) \) HS spin chain, obtained through the freezing technique, is then given by \[20\]

\[
Z_{\text{HS}}^{(m|n)}(q) = \prod_{l=1}^{N-1} (1-q^{E(l)}) \sum_{\mathbf{k} \in \mathcal{P}_N} d^{(m|n)}(\mathbf{k}) \prod_{j=1}^{r-1} \frac{q^{E(K_j)}}{(1-q^{E(K_j)})},
\]
where $E(l) = l(N - l)$ and $q = e^{-1/T}$; here $T$ is the temperature, and we have set the Boltzmann constant $k_B = 1$. Note that the dimension of the summation variable $k$ (i.e., $r$) in Eq. (2.15) takes all possible values within the range 1 to $N$. Since the partial sums $K_1, K_2, \ldots, K_r$ associated with $k$ are natural numbers obeying $1 \leq K_1 < K_2 < \cdots < K_{r-1} < K_r = N$, one can define their complements as elements of the set: $\{1, 2, \ldots, N - 1\} - \{K_1, K_2, \ldots, K_{r-1}\}$. Let $K_j$’s with $j \in \{r + 1, r + 2, \ldots, N\}$ denote these conjugate partial sums. Hence one can rearrange the product $\prod_{l=1}^{N-1}(1 - q^{E(l)})$ into two terms as [9]

$$
\prod_{l=1}^{N-1}(1 - q^{E(l)}) = \prod_{j=1}^{r-1}(1 - q^{E(K_j)}) \prod_{i=r+1}^{N}(1 - q^{E(K_i)}).
$$

(2.16)

By substituting this relation to Eq. (2.15), we get a simplified expression for the partition function of the $SU(m|n)$ HS model as

$$
Z_{HS}^{(m|n)}(q) = \sum_{k \in \mathcal{P}_N} d^{(m|n)}(k) q^{\sum_{j=1}^{r-1} E(K_j)} \prod_{i=r+1}^{N}(1 - q^{E(K_i)}).
$$

(2.17)

Even though the partition function given in Eq. (2.17) is useful for studying various global properties of the spectrum like the level density distribution [20], it is not very suitable for analyzing the degeneracy of energy levels associated with the super-Yangian symmetry of the $SU(m|n)$ HS model. However, it has been found recently that the partition function appearing in Eq. (2.17) can also be expressed as [21]

$$
Z_{HS}^{(m|n)}(q) = \sum_{k \in \mathcal{P}_N} q^{\sum_{j=1}^{r-1} E(K_j)} S_{(k_1,k_2,\ldots,k_r)}(x,y)|_{x=1,y=1},
$$

(2.18)

where $x \equiv \{x_1, x_2, \ldots, x_m\}$, $y \equiv \{y_1, y_2, \ldots, y_n\}$, $\langle k_1, k_2, \ldots, k_r \rangle$ denotes a ‘border strip’ which is drawn in Fig. 1, and $S_{(k_1,k_2,\ldots,k_r)}(x,y)$ is the Schur polynomial corresponding to such a border strip.

These border strips represent a class of irreducible representations of the $Y(gl_{m|n})$ Yangian algebra, and they span the Fock space of Yangian invariant spin systems. These border strips can equivalently be described by motifs, which for an $N$-site spin chain is given by a sequence of $N - 1$ number of 0’s and 1’s, $\delta = (\delta_1, \delta_2, \ldots, \delta_{N-1})$ with
\( \delta_j \in \{0, 1\} \). There exists a one-to-one map from a border strip to a motif as

\[
\langle k_1, k_2, \ldots, k_r \rangle \implies \delta = (1, \ldots, 1, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1) .
\]

(2.19)

Thus the elements of this motif \( \delta \) satisfy the following rule: \( \delta_j = 0 \) if \( j \) coincides with one of the partial sums \( K_i \), and \( \delta_j = 1 \) otherwise. The dimensionality of the irreducible representation associated with a border strip or motif is obtained by setting \( x = 1, y = 1 \) in the corresponding Schur polynomial \( S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) \).

There exist several alternative expressions for the Schur polynomial in the literature. For \( x = 1, y = 1 \), one such expression for the Schur polynomial \([21]\) is given by

\[
S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) |_{x = 1, y = 1} = \sum_{\ell \in P_r} (-1)^{r-s} \prod_{i=1}^{s} d^{(m|n)} \left( \sum_{j=1}^{\ell_i} k_{\ell_1 + \ell_2 + \ldots + \ell_{i-1} + j} \right) ,
\]

(2.20)

where the summation variable \( \ell \equiv \{ \ell_1, \ell_2, \ldots, \ell_s \} \) belongs to the set \( P_r \) of ordered partitions of \( r \) (thus \( s \) is an integer which runs from 1 to \( r \)), and we assume that \( l_0 = 0 \). As an illustration, let us consider the Schur polynomial \( S_{\langle k_1, k_2, k_3 \rangle}(x, y) \), for which \( r = 3 \) and \( P_3 \) is given by \{ \{3\}, \{2, 1\}, \{1, 2\}, \{1, 1, 1\} \}. For \( x = 1, y = 1 \), Eq. (2.20) gives the value of this Schur polynomial to be

\[
S_{\langle k_1, k_2, k_3 \rangle}(x, y) |_{x = 1, y = 1} = d^{(m|n)}(k_1 + k_2 + k_3) - d^{(m|n)}(k_1 + k_2) d^{(m|n)}(k_3) - d^{(m|n)}(k_1) d^{(m|n)}(k_2 + k_3) + d^{(m|n)}(k_1) d^{(m|n)}(k_2) d^{(m|n)}(k_3) .
\]

It may be noted that, by using another combinatorial expression \([21]\) for the Schur polynomial \( S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) \), we find its value for \( x = 1, y = 1 \) to be

\[
S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) |_{x = 1, y = 1} = N_{\langle k_1, k_2, \ldots, k_r \rangle} ,
\]

(2.21)

where \( N_{\langle k_1, k_2, \ldots, k_r \rangle} \) denotes the number of all possible allowed tableaux corresponding to the border strip \( \langle k_1, k_2, \ldots, k_r \rangle \). An allowed tableau is obtained by filling the numbers \( 1, 2, \ldots, m + n \) in a given border strip \( \langle k_1, k_2, \ldots, k_r \rangle \) following the rules:

\begin{itemize}
  \item The entries in each row are increasing, allowing the repetition of elements of the set \( \{1, 2, \ldots, m\} \), but not permitting the repetition of elements of the set \( \{m + 1, m + 2, \ldots, m + n\} \),
\end{itemize}
• The entries in each column are increasing, allowing the repetition of elements of the set \( \{m + 1, m + 2, \ldots, m + n\} \), but not permitting the repetition of elements of the set \( \{1, 2, \ldots, m\} \).

For example, in the case of the \( SU(2|1) \) spin chain, it is possible to construct the following tableaux corresponding to the border strip \( \langle 2, 1 \rangle \):

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 3 \\
2 & 1 & 3 & 2 \\
2 & 2 & 3 & 1 \\
2 & 3 & 1 & 2 \\
3 & 3 & 1 & 2 \\
3 & 3 & 2 & 1 \\
\end{array}
\]

which gives \( S_{(2,1)}(x, y)|_{x=1, y=1} = 8 \). In the following, we shall use both of the expressions in Eqs. (2.20) and (2.21) according to our convenience.

Now we are in a position to find the energy levels and their degeneracies in the case of the \( SU(m|n) \) HS spin chain. The energy level associated with a border strip \( \langle k_1, k_2, \ldots, k_r \rangle \) or corresponding motif \( \delta \) is obtained from the power of \( q \) appearing on the right hand side of Eq. (2.18):

\[
E(\delta) = \sum_{i=1}^{r-1} \mathcal{E}(K_i) = \frac{N(N^2 - 1)}{6} + \sum_{j=1}^{N-1} \delta_j j(j - N).
\]  

(2.22)

From Eq. (2.18), it also follows that the degeneracy of the energy level \( E(\delta) \) associated with motif \( \delta \) is given by setting \( x = 1, y = 1 \) in the Schur polynomial \( S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) \).

We have already presented some expressions for this limit of the Schur polynomial in Eqs. (2.20) and (2.21). It is interesting to observe that setting \( x = 1, y = 1 \) in the Schur polynomial \( S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y) \) also gives the degeneracy of the energy level associated with motif \( \delta \) in the case of the \( SU(m|n) \) supersymmetric Polychronakos spin chain [17].

Let us suppose that there exists a unique motif \( \delta \) which minimizes \( E(\delta) \) and, therefore, represents the ground state of the \( SU(m|n) \) HS spin chain. It may be noted that, if \( Z_{HS}^{(m|n)}(q) \) in Eq. (2.17) is expressed in a polynomial form, then the term with the lowest power of \( q \) is generated only from the partition \( k \) corresponding to the motif \( \delta \), and the coefficient of this power of \( q \) is given by \( d^{(m|n)}(k) \). Consequently, the degeneracy of the motif \( \delta \) representing the ground state is obtained as

\[
S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y)|_{x=1, y=1} = d^{(m|n)}(k) = \prod_{i=1}^{r} d^{(m|n)}(k_i).
\]  

(2.23)

This relation implies that only the vector \( l = \{1, 1, \ldots, 1\} \) contributes to the right
hand side of Eq. (2.20) for the case of the ground state. If there exist more than one motif corresponding to the minimum energy of the system, then the degeneracy of the ground state is obtained by summing over all the $d^{(m|n)}(k)$ associated with such motifs.

An interesting aspect of the HS models is a duality between the $SU(m|n)$ and $SU(n|m)$ spin chains of the form [21]

$$U \tilde{P}^{(m|n)}_{jk} U^\dagger = -\tilde{P}^{(n|m)}_{jk},$$

and

$$U H_{HS}^{(m|n)} U^\dagger = \frac{N(N^2-1)}{6} - H_{HS}^{(n|m)},$$

where $U$ is a unitary operator. It should be observed that, one can obtain the relation $\tilde{P}^{(m|0)}_{jk} = -\tilde{P}^{(0|m)}_{jk}$ by using Eq. (2.7). Hence, for the particular case $n = 0$, $U$ in Eqs. (2.24) and (2.25) acts like an unit operator between the Fock spaces of the $SU(m|0)$ bosonic and $SU(0|m)$ fermionic spin chains. [Note that the $SU(2)$ bosonic and fermionic spin chains correspond to antiferromagnetic and ferromagnetic spin-1/2 chains respectively. This is because $\tilde{P}^{(2|0)}_{jk} = -\tilde{P}^{(0|2)}_{jk} = 2\vec{s}_j \cdot \vec{s}_k + 1/2$, where $\vec{s}_j$ denotes a spin-1/2 operator at site $j$.]

In general, Eq. (2.25) implies that if $|\psi_i\rangle$ is an eigenfunction of $H_{HS}^{(m|n)}$ with eigenvalue $E_i$, then $U|\psi_i\rangle$ is an eigenfunction of $H_{HS}^{(n|m)}$ with eigenvalue $E_i'$, where

$$E_i' = \frac{N(N^2-1)}{6} - E_i.$$  

Furthermore, the degeneracy of the eigenvalue $E_i$ in the spectrum of the $SU(m|n)$ spin chain coincides with that of the eigenvalue $E_i'$ in the spectrum of the $SU(n|m)$ spin chain. Consequently, the partition functions of the $SU(m|n)$ and $SU(n|m)$ spin chains satisfy the duality relation

$$Z_{HS}^{(m|n)}(q) = q^{N(N^2-1)/6} Z_{HS}^{(n|m)}(q^{-1}).$$

For the special case of a ‘self dual’ model with $m = n$, Eq. (2.26) implies that the spectrum would be symmetric under reflection around the energy value $N(N^2-1)/12$.

Let us now try to find out how the unitary operator $U$ connects the motif representations appearing in the Fock spaces of the $SU(m|n)$ and $SU(n|m)$ HS spin chains. For this purpose, we need to define the conjugate of a border strip or corresponding motif. While the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ has $r$ elements, its conjugate has $N-r+1$ elements which lead to a partition of $N$ [21]. Thus one can write this conjugate border
strip as \(\langle k'_1, k'_2, \ldots, k'_{N-r+1}\rangle\), where \(\{k'_1, k'_2, \ldots, k'_{N-r+1}\} \in \mathcal{P}_N\), and denote the partial sums corresponding to this conjugate border strip as \(K'_i\) with \(i \in \{1, 2, \ldots, N-r+1\}\). The first \(N-r\) such partial sums form a set, which is related to the complementary partial sums associated with the original border strip \(\langle k_1, k_2, \ldots, k_r\rangle\) as [21]

\[
\{K'_1, K'_2, \ldots, K'_{N-r}\} = \{N - K_{r+1}, N - K_{r+2}, \ldots, N - K_N\}. \tag{2.28}
\]

From the above equation, it follows that the conjugate of a motif can be obtained from the original motif by replacing 0’s with 1’s (and vice versa) and rewriting all binary digits in the opposite order. For example, the conjugate of the motif \((10110)\) is obtained as \((10110) \rightarrow (01001) \rightarrow (10010)\). With the help of Eq. (2.22), we find that the eigenvalue of the Hamiltonian \(H_{HS}^{(n|m)}\) for the conjugate border strip \(\langle k'_1, k'_2, \ldots, k'_{N-r+1}\rangle\) is given by \(\sum_{i=1}^{N-r} \mathcal{E}(K'_i)\). Using Eq. (2.28) along with the relations \(\mathcal{E}(N - K_i) = \mathcal{E}(K_i)\) and \(\sum_{i=1}^{N} \mathcal{E}(K_i) = N(N^2 - 1)/6\), we obtain

\[
\sum_{i=1}^{N-r} \mathcal{E}(K'_i) = \frac{N(N^2 - 1)}{6} - \sum_{i=1}^{r-1} \mathcal{E}(K_i). \tag{2.29}
\]

Comparing this equation with Eq. (2.26), we find that the unitary operator \(U\) maps the border strip \(\langle k_1, k_2, \ldots, k_r\rangle\) appearing in the Fock space of the \(SU(m|n)\) spin chain to the conjugate border strip \(\langle k'_1, k'_2, \ldots, k'_{N-r+1}\rangle\) appearing in the Fock space of the \(SU(n|m)\) spin chain. Furthermore, it is known that the Schur polynomials associated with a border strip and its conjugate border strip satisfy a duality relation like [21]

\[
S_{(k_1,k_2,\ldots,k_r)}(x,y) = S_{(k'_1,k'_2,\ldots,k'_{N-r+1})}(y,x). \tag{2.30}
\]

For \(x = 1, y = 1\), this equation implies that the number of degenerate eigenfunctions of the \(SU(m|n)\) HS spin chain associated with the motif \(\langle k_1, k_2, \ldots, k_r\rangle\) coincides with that of the \(SU(n|m)\) HS spin chain associated with the conjugate motif \(\langle k'_1, k'_2, \ldots, k'_{N-r+1}\rangle\). This result is also consistent with our observation that a motif and its conjugate motif are related to each other through the unitary operator \(U\).

It may be noted that the expression in (2.22) for the energy levels of the \(SU(m|n)\) HS spin chain does not explicitly depend on the values of \(m\) and \(n\), and apparently coincides with that of the \(SU(m|0)\) bosonic case [6]. However, as we shall see shortly, the degeneracy corresponding to some of these energy levels vanishes in the pure bosonic case or
pure fermionic case. As a result, the spectrum of a supersymmetric spin chain admits many more energy levels in comparison with the spectrum of a bosonic or fermionic spin chain with the same number of lattice sites. For the case of the $SU(m|0)$ bosonic spin chain, let us assume that the value of some $k_i$ in the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ exceeds $m$. Since it is not possible to construct a tableau corresponding to this border strip without the repetition of any number (from the set $\{1, 2, \ldots, m\}$) in the column which has the length $k_i$, we get $S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y)|_{x=1, y=1} = 0$ by using Eq. (2.21). The same conclusion can also be drawn from Eq. (2.20) by observing that $d^{m|0}(l) = 0$ for $l > m$. Consequently, by using Eq. (2.19), one finds a selection rule for the $SU(m|0)$ bosonic HS spin chain, which prohibits the occurrence of $m$ or more consecutive 1’s in a motif. Note that if a motif contains a sequence of $m$ or more consecutive 1’s, then its conjugate motif would contain a sequence of $m$ or more consecutive 0’s. So, by using the duality relation in (2.30), we find a complementary selection rule for the $SU(0|m)$ fermionic HS spin chain, which prohibits the occurrence of $m$ or more consecutive 0’s in a motif. In the case of the $SU(m|n)$ supersymmetric spin chain, however, we can construct at least one tableau corresponding to the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ for arbitrary values of $k_i$. The form of such a tableau has been shown in Fig. 2.

Consequently, by using Eq. (2.21), we find that $S_{\langle k_1, k_2, \ldots, k_r \rangle}(x, y)|_{x=1, y=1}$ has a nonzero value for an arbitrary border strip $\langle k_1, k_2, \ldots, k_r \rangle$. Thus, the selection rules occurring in the bosonic and fermionic case are lifted for the case of the supersymmetric HS spin chain; this was observed by Haldane on the basis of numerical calculations [3].

Due to the absence of any selection rule, we can easily evaluate the maximum and minimum energy eigenvalues for the case of a supersymmetric HS spin chain. From the expression for $E(\delta)$ in Eq. (2.22) it is evident that, in the case of the $SU(m|n)$ supersymmetric as well as the $SU(0|n)$ fermionic spin chain, the motif $\delta \equiv (1, 1, \ldots, 1)$ corresponding to the border strip $\langle N \rangle$ gives the minimum energy of the system as $E_{\text{min}} = \frac{N(N^2-1)}{6} + \sum_{j=1}^{N-1} j(j - N) = 0$. Due to the selection rule which prohibits the occurrence of $m$ or more consecutive 1’s in a motif, the $SU(m|0)$ bosonic model has a nonzero ground state energy which will be discussed in Sec. 4. On the other hand, the motif $\delta \equiv (0, 0, \ldots, 0)$ corresponding to the border strip $\langle 1, 1, \ldots, 1 \rangle$ gives the maximum energy $E_{\text{max}} = \frac{N(N^2-1)}{6}$ for the $SU(m|n)$ supersymmetric as well as the $SU(m|0)$ bosonic spin chain. Due to the presence of the selection rule which prohibits the occurrence of $m$ or more consecutive 0’s in a motif, the maximum energy of the $SU(0|m)$ fermionic spin chain is lower than $N(N^2 - 1)/6$. The maximum energy of the $SU(0|m)$ fermionic spin chain can be obtained from the ground state energy of the $SU(m|0)$ bosonic spin chain by using the relation in (2.26).
3 Definition of momentum for the $SU(m|n)$ HS spin chain

In order to study the energy-momentum dispersion relation, we need to find the momentum eigenvalues corresponding to the energy eigenstate $s$ of the $SU(m|n)$ HS spin chain. In this section, we will study how the momentum operator can be defined for the $SU(m|n)$ HS spin chain, and we will find the eigenvalues of this operator when it acts on the motif eigenstates.

To begin with, we consider the case of the $SU(m|0)$ bosonic spin chain, for which the momentum eigenvalues associated with the motif eigenstates are already known [6]. The action of the translation operator $T$ is defined on the spin space of this bosonic model as

$$T |\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N \rangle = |\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 \rangle.$$ (3.1)

$T$ is an unitary operator which can be related to the momentum operator $P$ as

$$T = e^{i P}.$$ (3.2)

It is well known that all degenerate energy eigenstates associated with the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ or corresponding motif $\delta$ are also degenerate eigenstates of the translation operator $T$ with eigenvalue $e^{i P(\delta)}$, where $P(\delta)$ is given by [6]

$$P(\delta) = \left[ \pi (N - 1) - \frac{2 \pi}{N} \sum_{i=1}^{r-1} K_i \right] \mod 2 \pi = \frac{2 \pi}{N} \sum_{j=1}^{N-1} \delta_j \mod 2 \pi. \quad \text{(3.3)}$$

Hence, due to the relation in (3.2), the momentum eigenvalue of all eigenstates associated with the motif $\delta$ is given by Eq. (3.3) for the case of a bosonic spin chain.

Let us now define the translation and momentum operator for the general case of the $SU(m|n)$ HS spin chain. The translation operator acts on the creation (annihilation) operators associated with the Fock space of the $SU(m|n)$ spin chain as

$$TC_{i\alpha}T^\dagger = C_{i'\alpha}, \quad TC_{i\alpha}^\dagger T^\dagger = C_{i'\alpha}^\dagger,$$ (3.4)

with $i' \equiv i + 1$ (assuming $N + 1 \equiv 1$ due to the circular configuration of the lattice sites), and on the vacuum state as $T|0\rangle = |0\rangle$. Similar to the bosonic case, this trans-
luation operator is related to the momentum operator through Eq. (3.2). It is easy
to check that the Hamiltonian in (2.6) of the $SU(m|n)$ HS spin chain commutes
with the translation and momentum operator defined in the above mentioned way. Indeed,
all conserved quantities [3] of the $SU(m|n)$ HS model, which lead to the $Y(gl_{(m|n)})$
super-Yangian symmetry of this spin chain, also commute with these translation and
momentum operators. This fact ensures that all degenerate energy eigenstates
associated with the motif $\delta$ give the same momentum eigenvalue. With the help of Eqs.
(3.4) and (2.2), we obtain

$$T C_{1\alpha_1}^\dagger C_{2\alpha_2}^\dagger \cdots C_{N-1,\alpha_{N-1}}^\dagger C_{N\alpha_N}^\dagger |0\rangle = C_{2\alpha_1}^\dagger C_{3\alpha_2}^\dagger \cdots C_{N\alpha_N}^\dagger C_{1\alpha_N}^\dagger |0\rangle = (-1)^{p(\alpha_N) \sum_{j=1}^{N-1} p(\alpha_j)} C_{1\alpha_N}^\dagger C_{2\alpha_1}^\dagger \cdots C_{N\alpha_{N-1}}^\dagger |0\rangle. \tag{3.5}$$

Applying the mapping (2.8) to the above relation, we find the action of the translation
operator $T$ on the spin state $|\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\rangle$ as

$$T|\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\rangle = (-1)^{p(\alpha_N) \sum_{j=1}^{N-1} p(\alpha_j)} |\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1\rangle. \tag{3.6}$$

It may be noted that this general relation reduces to Eq. (3.1) in the particular case
of the $SU(m|0)$ bosonic spin chain, for which $p(\alpha) = 0$ for all possible values of $\alpha$. On
the other hand, $p(\alpha) = 1$ for all values of $\alpha$ in the case of the $SU(0|n)$ fermionic spin
chain. Hence, in this case, Eq. (3.6) reduces to

$$T|\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N\rangle = e^{i\pi(N-1)} |\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1\rangle. \tag{3.7}$$

Now we will find an expression for the momentum eigenvalue associated with a
motif in the case of the $SU(0|n)$ fermionic spin chain, by using its duality relation with
the $SU(n|0)$ bosonic spin chain. It has been already established that, any motif (say $\delta$) occurring in the Fock space of the $SU(0|n)$ fermionic spin chain can be related to
its conjugate motif (say $\delta_c$) occurring in the Fock space of the $SU(n|0)$ bosonic spin
chain through an unitary operator $U$ which appears in Eqs. (2.24) and (2.25). More
precisely, if $|\psi(\delta)\rangle$ is a state vector associated with the motif $\delta$, then there exists a
state vector $|\psi(\delta_c)\rangle$ associated with the conjugate motif $\delta_c$ such that $|\psi(\delta)\rangle = U|\psi(\delta_c)\rangle$. Since $U$ acts like an unit operator in this case, we can express both $|\psi(\delta)\rangle$ and $|\psi(\delta_c)\rangle$
in exactly the same form through the corresponding basis vectors like \(|\alpha_1, \alpha_2, \ldots, \alpha_N \rangle\):

\[
\sum_{\alpha_1, \alpha_2, \ldots, \alpha_N} C_{\alpha_1, \alpha_2, \ldots, \alpha_N} |\alpha_1, \alpha_2, \ldots, \alpha_N \rangle, \tag{3.8}
\]

where \(C_{\alpha_1, \alpha_2, \ldots, \alpha_N}\)'s are some expansion coefficients. However, it should be kept in mind that while all \(\alpha_i\) represent fermionic spins in the expression for \(|\psi(\delta)\rangle\), they represent bosonic spins in the expression for \(|\psi(\delta_c)\rangle\). Let us assume that \(|\psi(\delta)\rangle\) and \(|\psi(\delta_c)\rangle\) are eigenstates of the translation operator \(T\) with eigenvalues given by \(e^{iP(\delta)}\) and \(e^{iP(\delta_c)}\) respectively. Acting with \(T\) on the state vector appearing in Eq. (3.8), and using Eq. (3.1) or Eq. (3.7) in the case of bosons or fermions respectively, we find that

\[
P(\delta) = \left[ \pi(N - 1) + P(\delta_c) \right] \mod 2\pi. \tag{3.9}
\]

With the help of Eqs. (3.3) and (2.28), one obtains the value of \(P(\delta_c)\) as

\[
P(\delta_c) = -\frac{2\pi}{N} \sum_{i=1}^{r-1} K_i \mod 2\pi = \left[ -\pi(N - 1) + \frac{2\pi}{N} \sum_{j=1}^{N-1} \delta_j j \right] \mod 2\pi, \tag{3.10}
\]

where \(K_i\)'s are the partial sums associated with the border strip \(\langle k_1, k_2, \ldots, k_r \rangle\), which has a one-to-one correspondence with the motif \(\delta\). By inserting the above expression of \(P(\delta_c)\) to Eq. (3.9), we find that the momentum eigenvalue of all eigenstates associated with the motif \(\delta\) is again given by Eq. (3.3) for the case of the \(SU(0|n)\) fermionic spin chain.

Our next aim is to find the momentum eigenvalues associated with the motifs of the \(SU(m|n)\) supersymmetric spin chain, for which both \(m\) and \(n\) take nonzero values. For this purpose, we first consider the simplest case of the \(SU(1|1)\) HS spin chain. It is well known that this spin chain can be mapped a model of non-interacting spinless fermions \([3, 30]\). Since the bosonic spin in the \(SU(1|1)\) model can equivalently be described as a vacuum state for the fermionic spin, the corresponding exchange operator may be expressed as

\[
\tilde{P}_{jk}^{(1|1)} = 1 - C^\dagger_j C_j - C^\dagger_k C_k + C^\dagger_j C_k + C^\dagger_k C_j, \tag{3.11}
\]

where \(C^\dagger_j(C_j)\) creates (annihilates) a spinless fermion at site \(j\). The Hamiltonian in
Eq. (2.10) can then be diagonalized in the form

\[ H_{HS}^{(1|1)} = \frac{N(N^2 - 1)}{6} - \sum_{u=0}^{N-1} E_u \tilde{C}_u \tilde{C}_u^\dagger, \]

(3.12)

where \( E_u = u(N - u) \), and \( \tilde{C}_u \) is the Fourier transform of \( C_j \):

\[ \tilde{C}_u = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2i\pi u j/N} C_j. \]

(3.13)

Let us consider a pure fermionic state of the following form:

\[ |\psi(u_1, u_2, \ldots, u_r)\rangle \equiv \prod_{i=1}^{r} \tilde{C}_{u_i}^\dagger |0\rangle, \]

(3.14)

where \( u_1 < u_2 < \cdots < u_r \), and \( u_i \in \{0, 1, 2, \ldots, N - 1\} \). It is easy to see that, this state is an eigenstate of the Hamiltonian in (3.12) with eigenvalue given by

\[ E(\{u_i\}) = \frac{N(N^2 - 1)}{6} + \sum_{i=1}^{r} u_i(u_i - N). \]

(3.15)

Evidently, the set consisting of all states of the form given in (3.14) is a complete set of eigenstates for the Hamiltonian in (3.12).

Now we take an eigenstate of the form \( |\psi(u_1, u_2, \ldots, u_r)\rangle \), where all \( u_i \)'s are positive integers. It may be noted that both \( |\psi(u_1, u_2, \ldots, u_r)\rangle \) and \( |\psi(0, u_1, u_2, \ldots, u_r)\rangle \) give rise to the same energy eigenvalue. So these eigenstates lead to a doubly degenerate energy level characterized by a set of fermionic quantum numbers like \( \{u_1, u_2, \ldots, u_r\} \). It is natural to expect that such sets of quantum numbers would be connected in some way with the motifs of the SU(1|1) HS spin chain. Indeed, by comparing two energy expressions given in Eqs. (2.22) and (3.15), we find that there exists a well defined mapping between the set \( \{u_1, u_2, \ldots, u_r\} \) and the corresponding motif \( \delta \). The mapping rules are as follows:

- When \( j \in \{u_1, u_2, \ldots, u_r\} \), we have \( \delta_j = 1 \).
- When \( j \not\in \{u_1, u_2, \ldots, u_r\} \), we have \( \delta_j = 0 \).

To illustrate this mapping, we take a set of fermionic quantum numbers like \( \{1, 3, 4\} \).
for a spin chain with $N = 6$. Due to the above mentioned rules, this set of quantum numbers is mapped to a motif of the form (10110). Consequently, the degenerate eigenstates given by $\tilde{C}_1^\dagger \tilde{C}_3^\dagger \tilde{C}_4^\dagger |0\rangle$ and $\tilde{C}_0^\dagger \tilde{C}_1^\dagger \tilde{C}_3^\dagger \tilde{C}_4^\dagger |0\rangle$ correspond to the motif (10110).

We have already defined the translation and momentum operators for a supersymmetric spin chain. As will be shown shortly, for the case of the $SU(1|1)$ spin chain, these operators can be expressed in simple forms through Fourier transformed modes like $\tilde{C}_u$ and $\tilde{C}_u^\dagger$. By using Eqs. (3.4) and (3.13), we find the action of translation operator on $\tilde{C}_u$ and $\tilde{C}_u^\dagger$ as

$$T \tilde{C}_u T^\dagger = e^{-i2\pi u/N} \tilde{C}_u, \quad T \tilde{C}_u^\dagger T^\dagger = e^{i2\pi u/N} \tilde{C}_u^\dagger. \quad (3.16)$$

With the help of the Baker-Hausdorff relation, it is easy to check that a translation operator of the form given in (3.2) satisfies Eq. (3.16) if the momentum operator $P$ is given by

$$P = \frac{2\pi}{N} \sum_{u=0}^{N-1} u \tilde{C}_u^\dagger \tilde{C}_u. \quad (3.17)$$

Acting on the states $|\psi(u_1, u_2, \ldots, u_r)\rangle$ and $|\psi(0, u_1, u_2, \ldots, u_r)\rangle$, this momentum operator evidently generates the same eigenvalue given by

$$P(\{u_i\}) = \frac{2\pi}{N} \sum_{i=1}^{r} u_i \mod 2\pi. \quad (3.18)$$

By utilizing the mapping between fermionic quantum numbers $\{u_1, u_2, \ldots, u_r\}$ and motif $\delta$, we can also express the momentum eigenvalue in Eq. (3.18) through the elements of motif $\delta$. Interestingly, we find that such an expression for the momentum eigenvalue for the motif $\delta$ is identical in form with the expression in (3.3), which was originally proposed for the case of $SU(m|0)$ bosonic spin chain.

Finally, let us discuss how the momentum eigenvalue relations obtained for the simplest case of the $SU(1|1)$ supersymmetric spin chain can be useful in the context of the general $SU(m|n)$ supersymmetric spin chain. It should be observed that, for a fixed number of lattice sites, all possible motifs of the $SU(m|n)$ supersymmetric spin chain also occur in the case of the $SU(1|1)$ spin chain. However, while all motifs are doubly degenerate in the case of the $SU(1|1)$ spin chain, the number of degenerate eigenstates associated with a motif is much higher in general in the case of the $SU(m|n)$ supersymmetric spin chain. In fact, the doubly degenerate eigenstates associated with
a motif δ of the SU(1|1) spin chain form a subset of the multiply degenerate eigenstates associated with the same motif δ of the SU(m|n) spin chain. We have already found that, the momentum eigenvalue of this subset of SU(1|1) doublets is given by \( P(\delta) \) in Eq. (3.3). Since all degenerate multiplets associated with the motif δ of the SU(m|n) supersymmetric spin chain must yield the same momentum eigenvalue, it is also given by Eq. (3.3). Thus we find that \( P(\delta) \) given in Eq. (3.3) represents a general expression for the momentum eigenvalue corresponding to the motif δ, which is valid for all possible cases like the bosonic, fermionic and supersymmetric HS spin chains.

4 Low energy excitations of the SU(m|n) HS spin chain

In this section, we will study the ground state and low energy excitations of the SU(m|n) HS spin chain for various values of \( m \) and \( n \). In particular, we will be interested in the thermodynamic limit \( N \to \infty \). We will therefore rescale the Hamiltonian to take the form

\[
H^{(m|n)}_{\text{HS}} = \frac{\pi^2}{2N^2} \sum_{1 \leq j < k \leq N} \frac{1 + \tilde{P}^{(m|n)}_{jk}}{\sin^2(\xi_j - \xi_k)}. \tag{4.1}
\]

The pre-factor of \( \pi^2/(2N^2) \) in (4.1) ensures that the nearest-neighbor interaction is of the form \( (1 + \tilde{P}^{(m|n)}_{j,j+1})/2 \), and also that the ground state energy per unit length will remain finite as \( N \to \infty \). (We have set the lattice spacing equal to 1). In that limit, Eq. (4.1) can be re-written as

\[
H^{(m|n)}_{\text{HS}} = \frac{1}{2} \sum_{j<k} \frac{1 + \tilde{P}^{(m|n)}_{jk}}{(j-k)^2} \tag{4.2}
\]

for \( |j - k| \ll N \). The partition function corresponding to the rescaled Hamiltonian in (4.1) may be obtained from the expressions in (2.17) or (2.18) after replacing \( q \) by \( \tilde{q} \), where \( \tilde{q} = e^{-\pi^2/N^2} \).

Using the results of the previous section for the case of the rescaled SU(m|n) spin chain, we can express its energy and momentum eigenvalues corresponding to the motif
$\delta$ as some functions of the integers $K_i$, namely,

$$E = \frac{\pi^2}{N^2} \sum_{i=1}^{r-1} K_i(N - K_i),$$

and

$$P = \left[ \pi(N - 1) - \frac{2\pi}{N} \sum_{i=1}^{r-1} K_i \right] \mod 2\pi. \quad (4.3)$$

For the Hamiltonian given in Eq. (4.1), the low energy modes are those for which $E$ is of order $1/N$, while the high energy modes are those for which $E$ is of order 1. Let us now consider three cases separately.

**Case I: $SU(m|0)$ bosonic spin chain, with $m \geq 2$.**

Although this bosonic spin chain has been discussed extensively in the literature, we will consider it briefly for the sake of completeness. Let us consider the simplest case when $N$ is a multiple of $m$. In this case, the border strip $(m, m, \ldots, m)$ minimizes the energy $E$ in Eq. (4.3) and represents the ground state of the system. For this border strip, we have $r = N/m, k_j = m$ for all $j \in \{1, 2, \ldots, r\}$, and $K_j = jm$ for all $j \in \{1, 2, \ldots, r - 1\}$. Due to Eqs. (2.23) and (2.14), the ground state is non-degenerate for this case. Eq. (4.3) gives the ground state energy and momentum to be

$$E_0 = \frac{\pi^2 m}{6N} \left[ \left( \frac{N}{m} \right)^2 - 1 \right],$$

and

$$P_0 = \pi N(1 - \frac{1}{m}) \mod 2\pi. \quad (4.4)$$

In the thermodynamic limit, the ground state energy per unit length is given by $\pi^2/(6m)$.

A band of excited states is obtained by taking $r = N/m + 1$ as follows. In addition to the $N/m - 1$ values of $K_j = jm$ that are present in the ground state, we introduce an additional value of $K_i = u$, where $u$ is not a multiple of $m$. The excitation energy and momentum of such a state, called $\Delta E$ and $\Delta P$ respectively, are given by the differences between the energy and momentum of this state and the ground state energy and momentum given in Eq. (4.4). We find that $\Delta E = (\pi^2/N^2)u(N - u)$, while $\Delta P = -2\pi u/N \mod 2\pi$. In the thermodynamic limit $u/N \to 0$ or $(N - u)/N \to 0$, these correspond to excitations with low energy and low momentum. The velocity corresponding to these excitations is given by $v = |\Delta E/\Delta P|$ which is equal to $\pi/2$. 

19
It is interesting to observe that the excitations described above do not have the lowest possible value of $\Delta E$, even for $u = 1$ or $N - 1$. Rather, the excitations with the lowest energy are given by $r = N/m + 1$ and $K_j = (j-1)m + u$ for all $j \in \{1, 2, \ldots, r-1\}$, where $u = 1$ or $m - 1$. These excitations have $\Delta E = (\pi^2/N^2)N(1 - 1/m)$, $\Delta P = \pm 2\pi/m \mod 2\pi$, and a degeneracy of $m^2$. For $N \to \infty$, these excitations have less energy than the lowest energy excitations described in the previous paragraph, although their momentum is large, i.e., $2\pi/m$ instead of $2\pi/N$. The momentum value of these excitations suggests that they may be related to the algebraic long-range order which is exhibited by the ground state of this model; as discussed in Eq. (5.1) below, the two-point correlation oscillates with a wave number $2\pi/m$ and decays as a power of the distance. Finally, the ratio of the energy of these high momentum excitations to the lowest energy of the low momentum excitations is $1 - 1/m$. We note that low energy states with momentum equal to $\pi$ are also known to exist in the spin-1/2 XXZ chain with nearest-neighbor couplings [23].

**Case II: SU(0|$n$) fermionic spin chain, with $n \geq 2$.**

We have already seen in Sec. 2 that the border strip $\langle N \rangle$ or corresponding motif $(11\ldots1)$ represents the ground state of this system. The ground state energy is zero and momentum is given by $\pi(N - 1)$. Eqs. (2.23) and (2.13) yield the ground state degeneracy as $N^{n-1}C_N$; this goes as $N^{n-1}/(n-1)!$ in the thermodynamic limit.

The ground state and the vanishing of its energy can be understood as follows. The simplest ground state is given by a state in which every site has a fermionic spin of the same type, say, $\alpha = 1$, using the notation given at the beginning of Sec. 2 and remembering that $m = 0$. The arguments given after Eq. (2.7) imply that when $\tilde{P}^{(0|n)}_{jk}$ acts on such a state, it gives $-1$ for all values of $j$ and $k$. Hence the state has zero eigenvalue for the Hamiltonian in Eq. (4.1). One can now see the form of the general ground states. Consider a state in which the first $n_1$ sites have fermionic spins of type $\alpha = 1$, the next $n_2$ sites have spins of type 2, and so on, with the condition that $\sum_{j=1}^n n_j = N$. One can then consider a superposition of states, all having the same amplitude, in which the same set of spins is distributed in all possible ways over all the lattice sites. This gives a ground state with zero energy since $1 + \tilde{P}^{(0|n)}_{jk}$ acting on such a superposition gives zero for all pairs $j$ and $k$. The number of ways of choosing $n$ ordered integers (some of which can be zero) which add up to $N$ is equal to $N^{n-1}C_N$; this is the degeneracy of the ground state.

A band of excited states is obtained by taking border strips like $\langle k_1, k_2 \rangle$, with $k_1 = N - u$, $k_2 = u$, and $K_1 = N - u$, where $1 \leq u \leq N - 1$. The excitation
energy and momentum of this state are given by $\Delta E = (\pi^2/N^2)u(N-u)$ and $\Delta P = P - \pi(N-1) = 2\pi u/N$ respectively. The velocity is given by $v = |dE/dP| = \pi/2$ in the thermodynamic limit $u/N \to 0$ or $(N-u)/N \to 0$.

The wave function of these excited states can be visualized by considering a state $|j\rangle$, in which the site labeled $j$ is occupied by a fermionic spin of type, say, $\alpha = 2$, while all the other sites are occupied by fermionic spins of type $\alpha = 1$. We form a state with wave number $u$ by superposing such states,

$$|u\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i2\pi u j/N} |j\rangle,$$

where $u \in \{1, 2, \ldots, N-1\}$. We then find that this state is an eigenstate of the Hamiltonian in Eq. (2.10) with the eigenvalue

$$E_u = \frac{\pi^2}{2N^2} \sum_{j=1}^{N-1} \frac{1 - \cos(2\pi u j/N)}{\sin^2(\pi j/N)} = \frac{\pi^2}{N^2} u (N-u),$$

where the last equality can be found in Ref. [1]. Since the energy of the ground state is zero, the excitation energy of the state $|u\rangle$ is also given by $E_u$. In the limit $N \to \infty$, the excitation momentum $\Delta p = 2\pi u/N$ becomes a continuous variable lying in the range $(0, 2\pi)$, and we obtain the dispersion

$$\Delta E = \frac{\pi \Delta p}{2} - \frac{(\Delta p)^2}{4}. \quad (4.7)$$

The excitation energy $\Delta E$ goes to zero linearly as $\Delta p \to 0$ or $2\pi$, with a slope given by the velocity $v = |dE/dp| = \pi/2$.

Note that the linear dispersions near $\Delta p = 0$ and $2\pi$ in Eq. (4.7) are due to the $1/j^2$ interaction between pairs of sites separated by a distance $j$. If the interaction was short-ranged, the dispersion would be quadratic near $\Delta p = 0$ and $2\pi$; this is known to be the case for a ferromagnetic spin chain with nearest-neighbor interactions.

**Case III:** $SU(m|n)$ spin chain, with $m, n \geq 1$.

Similar to case II, the border strip $\langle N \rangle$ or corresponding motif $(11\ldots1)$ represents the ground state of this system. The ground state energy is zero and momentum is given by $\pi(N-1)$. (Note that all the ground states of case II are also ground states of
case III). However, due to the presence of bosons, the degeneracy of the ground state would be different from that of the $SU(0|n)$ fermionic case. By using Eqs. (2.12) and (2.23), and also assuming $N \geq m$, we find that the ground state degeneracy of the $SU(m|n)$ spin chain is given by

$$\sum_{j=0}^{m} mC_j N^{n-j-1}C_{N-j},$$

which goes as $2^m N^{n-1}/(n-1)!$ in the thermodynamic limit.

It is interesting to observe that, for the special case $n = 1$, the ground state degeneracy given in Eq. (4.8) reduces to $2^m$, independent of $N$. This can be understood as follows. The ground states consist of each of the $m$ types of bosonic spins either not appearing at all, or appearing in only one site of the chain; this gives rise to $2^m$ possibilities. Thus $p$ sites of the chain have bosonic spins, where $0 \leq p \leq m$, while the remaining $N - p$ sites are occupied by the fermionic spins.

A band of low energy excitations are obtained by taking border strips like $\langle N-u, u \rangle$, where $1 \leq u \leq N-1$. Again, by following a method similar to Case II which leads to the state vector $|u\rangle$ in Eq. (4.5), the wave function of these excited states can be constructed explicitly. Consequently, the excitation energy and momentum of these states are obtained as $\Delta E = (\pi^2/N^2)u(N-u)$ and $\Delta P = 2\pi u/N$ respectively. Thus we find that the low lying energy levels have the same motif structure and momentum for the $SU(m|n)$ supersymmetric as well as the $SU(0|n)$ fermionic case. In the thermodynamic limit, these low lying excitations have a linear dispersion, with the velocity being given by $\pi/2$.

For the $SU(m|1)$ spin chain, it is worth noting that the degeneracy of both the ground state and all the low lying excited states contains a factor of $2^m$. According to Eqs. (2.12) and (2.20), the degeneracy of all states for which at least one of the $k_i \geq m$ will contain a factor of $2^m$. The states whose degeneracy is not a multiple of $2^m$ are the ones in which all the $k_i < m$; these states necessarily have energies of order 1, and are therefore high energy states.

It is interesting to consider what happens if the couplings of the $SU(m|n)$ spin chain are not of the inverse square form, i.e., not of the Haldane-Shastry type. Let us restrict our attention to the case $n \geq 1$ and take a rather general form of a $SU(m|n)$
symmetric Hamiltonian like

\[ \mathcal{H}^{(m|n)} = \sum_{j<k} w(|k - j|) (1 + \tilde{P}_{jk}^{(m|n)}) , \tag{4.9} \]

where all the \( w(j) \)'s are arbitrary real positive numbers. We have already seen that the ground states of the HS spin chain Hamiltonian in Eq. (4.1) have zero energy for \( n \geq 1 \); hence they satisfy \( \tilde{P}_{jk}^{(m|n)} = -1 \) for all pairs of sites \( j, k \). Clearly, such states will continue to be ground states of \( \mathcal{H}^{(m|n)} \) in Eq. (4.9), since \( w(j) > 0 \) and the minimum eigenvalue of \( \tilde{P}_{jk}^{(m|n)} \) is \(-1\). Consequently, we find that the ground states of the SU\((m|n)\) symmetric Hamiltonian \( \mathcal{H}^{(m|n)} \) and the degeneracy of these states remain the same even if \( w(j) \) is not of the inverse square form. This conclusion is not surprising because the degeneracy of the ground state of the SU\((m|n)\) supersymmetric HS model is governed by the border strip \( \langle N \rangle \), which coincides with the single column Young diagram \([1^N]\) associated with the SU\((m|n)\) super Lie algebra.

However, the energy-momentum relation for the low energy excitations of \( \mathcal{H}^{(m|n)} \) will not be linear in general. Following arguments similar to Eqs. (4.5-4.7), we see that the energy of a state with momentum \( p \) (measured with respect to a ground state) is given by

\[ E_p = \sum_{j=1}^{\infty} w(j) [1 - \cos(pj)]. \tag{4.10} \]

Although \( E_p \) will go to zero at \( p = 0 \) and \( 2\pi \), the dispersion \( E_p \) versus \( p \) near those two points is not linear in general. For instance, if \( w(j) \) decreases exponentially with \( j \) as \( j \to \infty \), the dispersion is quadratic. As we will discuss in the next section, the low-energy excitations of a one-dimensional model cannot be described by a conformal field theory if the dispersion is not linear; hence the low-energy excitations of the Hamiltonian \( \mathcal{H}^{(m|n)} \) will not be governed by a conformal field theory unless \( w(j) \) is chosen in some specific way like inverse square interaction.

5 Conformal field theory description of low energy excitations

Having discussed the low energy excitations of the SU\((m|n)\) HS spin chain for different values of \( m \) and \( n \), we will now examine if these excitations can be described by a con-
formal field theory (CFT) in the thermodynamic limit [22]. A CFT must have a finite number of ground states and a linear energy-momentum relation for the excitations. The first property implies that only the $SU(m|0)$ bosonic spin chain with $m \geq 2$ and the $SU(m|1)$ spin chain with $m \geq 1$ can possibly be governed by some CFTs. The central charges for the $SU(m|0)$ and $SU(m|1)$ Polychronakos spin chains were calculated in Ref. [17], and were found to be $m - 1$ and $m/2$ respectively; we should point out that the convention for $(m|n)$ followed in [17] is the reverse of the convention that we are following here.

Let us first consider the $SU(m|0)$ HS spin chain briefly. It is known [6, 24, 25] that the low energy excitations of this model are governed by the $SU(m|1)$ Wess-Zumino-Novikov-Witten (WZNW) model [26, 27, 28]. In particular, the central charge of the $SU(m|0)$ HS spin chain is given by $c = m - 1$. In fact, Ref. [29] had already identified the $SU(m|1)$ WZNW model as providing a description of the low energy excitations of a class of $SU(m)$ symmetric spin chains. An important property of such a CFT is that the two-point equal-time correlation function in the ground state $|G\rangle$ goes as

$$
\sum_{\alpha,\beta=1}^{m} \langle G|S_{\alpha\beta}(j_1)S_{\beta\alpha}(j_2)|G\rangle \sim \frac{\cos[2\pi(j_1-j_2)/m]}{|j_1-j_2|^{2-2/m}}\quad (5.1)
$$

for $|j_1 - j_2| \rightarrow \infty$, where $S_{\alpha\beta}(j) = C^j_{\alpha\beta}C_{j\beta} - \delta_{\alpha\beta}/m$ [29]. (Note that for the $SU(m|0)$ spin chain, the constraint in (2.3) implies that $\sum_{\alpha=1}^{m} \langle G|S_{\alpha\alpha}(j)|G\rangle = 0$ for all $j$). The period $m$ of the oscillations in (5.1) is consistent with the observation that the size of the system defined on a lattice must be a multiple of $m$ in order to have a unique ground state.

We now turn to the $SU(m|1)$ supersymmetric spin chain. As we saw earlier, the ground state degeneracy is $2^m$. In all the ground states, most of the sites are occupied by a fermionic spin, and only $p$ sites are occupied by bosonic spins, where $0 \leq p \leq m$. It is therefore convenient to perform a duality transformation given by Eq. (2.24) to obtain a $SU(1|m)$ spin chain governed by the Hamiltonian

$$
\bar{H}^{(1|m)} = \frac{\pi^2}{2N^2} \sum_{1 \leq j < k \leq N} \frac{1 - \bar{F}_{jk}^{(1|m)}}{\sin^2(\xi_j - \xi_k)}, \quad (5.2)
$$

so that the ground states will have most of the sites occupied by a bosonic spin. Further, since there is only one kind of bosonic spin, we can think of it instead as a vacuum
state for the fermions. We will now study the model defined in Eq. (5.2) for different values of $m \geq 1$.

We have already mentioned in Sec. 3 that the $SU(1|1)$ spin chain is equivalent to a model of non-interacting spinless fermions. This is because for $m = 1$, the exchange operator appearing in Eq. (5.2) can be written in the form

$$\tilde{P}_{jk}^{(1\!\!1)} = 1 - D_j^\dagger D_j - D_k^\dagger D_k + D_j^\dagger D_k + D_k^\dagger D_j,$$

(5.3)

where $D_j^\dagger$ creates (annihilates) a fermion at site $j$. The Hamiltonian in Eq. (5.2) can then be diagonalized; it takes the form

$$\tilde{H}^{(1\!\!1)} = \sum_{u=0}^{N-1} E_u \tilde{D}_u^\dagger \tilde{D}_u,$$

where $E_u = \frac{\pi^2}{N^2} u (N - u)$,

(5.4)

and $\tilde{D}_u$ is the Fourier transform of $D_j$. Note that the mode with $u = 0$ has zero energy, while all the other modes have positive energy. The ground state corresponds to all the positive energy states being empty. The zero energy state can be either filled or empty; this gives rise to a two-fold degeneracy of the ground state. In the thermodynamic limit, we define an excitation momentum $\Delta p = 2\pi u / N$ as usual. The low energy excitations have a dispersion which is linear near $\Delta p = 0$ and $2\pi$. Near these two points, the momentum, which is defined mod $2\pi$, is restricted to positive and negative values respectively, and the dispersions are given by $dE/dp = \pm v$ respectively.

It is interesting to note that the fermionic operators appearing in Eqs. (5.3) and (3.11) are related by a particle-hole symmetry which implements the duality given in Eq. (2.24) for $m = n = 1$. Consider the unitary operator $U = \exp[i(\pi/2)\sum_j(C_j D_j + D_j^\dagger C_j^\dagger)]$, where the $C_j$'s and $D_j$'s are independent fermion operators which anticommute with each other. We find that $UC_j U^\dagger = iD_j^\dagger$ and $UC_j^\dagger U^\dagger = -iD_j$. We can then verify that an unitary transformation by $U$ relates the exchange operators in Eqs. (5.3) and (3.11) in such a way as to satisfy Eq. (2.24).

Due to the exact equivalence of the $SU(1|1)$ HS spin chain to a system of fermions given in Eq. (5.4) for any value of $N$, the partition function is exactly given by $Z_{HS}^{(1\!\!1)}(\tilde{q}) = Z_1$, where

$$Z_1 = \prod_{u=0}^{N-1} (1 + \tilde{q}^u(N-u)),$$

(5.5)
and $\tilde{q} = e^{-\pi^2/(N^2T)}$. A different proof of this identity is given in Appendix A. Let us now consider the thermodynamic limit of the model. In this limit, we can define an excitation momentum $\Delta p = 2\pi u/N$ at $u/N \to 0$ or $\Delta p = -2\pi(N - u)/N$ at $(N - u)/N \to 0$. Taking these two kinds of low energy modes together, the partition function at low temperatures (i.e., $T \ll 1$ in the units we are using) is given by

$$
\ln Z_1 = 2 \int_0^\infty \frac{dp}{2\pi/N} \ln(1 + e^{-vp/T}),
$$

(5.6)

where $v = \pi/2$ is the velocity. The energy per unit length is given by $(T^2/N)\partial \ln Z_1/\partial T$, and we find that this is equal to $\pi T^2/(12v)$. The specific heat/length is obtained by differentiating this with respect to $T$, and therefore equals $\pi T/(6v)$. One can also evaluate the average number of fermions per unit length. We introduce a chemical potential $\mu$ in Eq. (5.6) by replacing $e^{-vp/T}$ by $e^{-(vp-\mu)/T}$; the number of fermions per unit length is then given by $(T/N)(\partial \ln Z_1/\partial \mu)_{\mu=0}$, and it turns out to be equal to $T \ln 2/(\pi v)$.

For a CFT with central charge $c$, the specific heat/length equals $\pi c T/(3v)$ [31, 32]. The central charge of the $SU(1|1)$ HS spin chain is therefore given by $c = 1/2$. Note that this is half the central charge $c = 1$ of a massless Dirac fermion; the latter has both positive and negative energy modes with two different linear dispersions $\Delta E = \pm v\Delta p$, where $\Delta p$ can go from $-\infty$ to $\infty$ in both cases. The low energy modes of the $SU(1|1)$ HS spin chain only have positive energies, and can therefore be thought of as the modes of half of a Dirac fermion.

We will now consider the system defined in Eq. (5.2) for $m \geq 2$. We will argue that in the limit $N \to \infty$ and temperatures $T \ll 1$, this system is equivalent to a model of $m$ species of non-interacting fermions. This can be physically understood as follows. By using Eq. (2.7) for the special case $m = 1$, and interpreting the only one kind of bosonic spin occurring in this case as a hole for the fermions, the exchange operator in Eq. (5.2) can be written as

$$
\tilde{P}^{(1|m)}_{jk} = 1 + \sum_{\alpha=1}^m \left[ - D^\dagger_{j\alpha} D_{j\alpha} - D^\dagger_{k\alpha} D_{k\alpha} + D^\dagger_{j\alpha} D_{k\alpha} + D^\dagger_{k\alpha} D_{j\alpha} \right] + \sum_{1 \leq \alpha \neq \beta \leq m} \left[ D^\dagger_{j\alpha} D^\dagger_{k\beta} D_{j\beta} D_{k\alpha} - D^\dagger_{j\alpha} D^\dagger_{k\beta} D_{j\alpha} D_{k\beta} \right],
$$

(5.7)

which must be followed by a projection on to the subspace of states satisfying the
constraint given in Eq. (2.3). The ground state of the system is the vacuum for the
fermions, apart from a degeneracy of $2^m$ due to the presence of $m$ zero energy modes.
At low temperatures, the system will be described by a dilute gas of fermions; as
we saw above, the density of fermions is of order $T$. For such a gas, the interaction
energy per unit length between pairs of fermions belonging to different species is of
order $T^3$, since the typical distance between two such fermions is of order $1/T$ and the
interaction is inversely proportional to the square of the distance. On the other hand,
the kinetic energy per unit length is proportional to $T^2$ as shown above. Hence, the
two-body interaction terms appearing in the second line of Eq. (5.7) can be ignored at
low temperatures, and the corresponding Hamiltonian is well approximated by

$$H = \sum_{\alpha=1}^{m} \sum_{u=0}^{N-1} E_u \tilde{D}_{u\alpha}^\dagger \tilde{D}_{u\alpha}, \quad (5.8)$$

where $E_u$ has the same form as in Eq. (5.4). Hence the low energy sector consists of
$m$ species of non-interacting fermions each of which has the form of a massless Dirac
fermion with only positive energy states. This is described by a CFT with central
charge $c = m/2$. Using Eq. (5.8) and arguments similar to the ones given after Eqs.
(5.5) and (5.6), one can show that the specific heat per unit length of this system goes
as $\pi m T/(6v)$ at low temperatures.

It should be pointed out that for finite values of $N$, the partition function of the
model defined in Eq. (5.2) does not agree with that of $m$ species of non-interacting
fermions, each with a dispersion relation given by Eq. (5.4). For instance, if we expand
the two partitions functions, we obtain

$$Z_{HS}^{(m)}(\tilde{q}) = 2^m \left[ 1 + 2m \tilde{q}^{N-1} + m^2 \tilde{q}^{2N-2} + m(m+1) \tilde{q}^{2N-4} + \ldots \right] \quad (5.9)$$

from Eq. (2.17), and

$$Z_1^m = 2^m \left[ 1 + 2m \tilde{q}^{N-1} + m(2m-1) \tilde{q}^{2N-2} + 2m \tilde{q}^{2N-4} + \ldots \right] \quad (5.10)$$

from Eq. (5.5). These two expressions do not agree at orders higher than $\tilde{q}^{N-1}$,
thereby showing the effect of two-particle interactions. However, since $\tilde{q} = e^{-\pi^2/(N^2T)}$,
the difference between $\tilde{q}^{2N-2}$ and $\tilde{q}^{2N-4}$ becomes negligible in the limit $N \to \infty$; hence
the total contribution from those terms becomes equal in Eqs. (5.9) and (5.10) since
their coefficients add up to $m(2m + 1)$ in both equations. Remarkably, we find that this kind of equality works up to all finite powers of $q^N$, although it fails for powers of $q^{N^2}$; the latter corresponds to contributions from high energy states whose energies are of order 1. If we now impose the condition that $T \ll 1$, the terms of order $q^{N^2}$ go to zero. Motivated by this observation, we will explicitly prove in Sec. 6 that for $N \to \infty$ and $T \ll 1$, the partition function of the $SU(m|1)$ HS spin chain is identical to that of a model of $m$ species of non-interacting fermions.

The simple ground state structure of the $SU(m|1)$ HS spin chain implies that the typical two-point equal-time correlation function in this model is trivial, in contrast to the correlation function of the $SU(m|0)$ spin chain given in Eq. (5.1). As discussed before Eq. (5.7), the $SU(m|1)$ HS spin chain is equivalent to a model in which there are $m$ species of fermionic spins and only species of bosonic spin which can be interpreted as a hole for the fermions. As in Eq. (5.7), we may define creation and annihilation operators for the $m$ species of fermions, $D_{j\alpha}^\dagger$ and $D_{j\alpha}$. We have seen that this system has $2^m$ ground states; for simplicity, let us first consider the ground state $|G\rangle$ in which there are no fermions and all the sites are occupied by holes. Namely, $D_{j\alpha}|G\rangle = 0$ for all values of $j$ and $\alpha$. We then obtain the following two-point correlation function

$$\langle G|D_{j\alpha}D_{k\beta}^\dagger|G\rangle = \delta_{jk} \delta_{\alpha\beta}.$$  

(5.11)

Even if we consider one of the other ground states in which there are $\ell$ fermions, where $1 \leq \ell \leq m$, the correlation function would have the same form as in Eq. (5.11) in the thermodynamic limit $N \to \infty$. This is because the factor of $1/\sqrt{N}$ in the definition of the Fourier transform as in Eq. (3.13) kills any contribution from the fermions if $\ell \ll N$. The simple form in Eq. (5.11) is in contrast to the correlation function for a system of non-interacting Dirac fermions in which the ground state has all one-particle states occupied up to some Fermi energy. For such a ground state, the two-point correlation function defined in Eq. (5.11) typically falls off as $1/|j - k|$ in one dimension.

Finally, we would like to mention that the $SU(1|2)$ and $SU(1|m)$ spin chains defined in Eq. (5.2) have been studied in Refs. [14] and [15] respectively. However, a chemical potential was implicitly introduced in those papers in order to consider ground states with a non-zero filling of the fermions. Hence the ground states and excitations considered in Refs. [14, 15] differ from the ones that we have studied here.
6 Equivalence of the $SU(m|1)$ HS spin chain and $m$ species of non-interacting fermions

In this section it will be shown that, in the thermodynamic limit and for low temperatures, the partition functions of the $SU(m|1)$ HS spin chain and a model of $m$ species of non-interacting fermions are equal to each other. As a by-product of this proof, we will derive a simple relation between the partition functions of the $SU(m|n)$ HS spin chain and the $SU(m|n)$ Polychronakos spin chain for any value of $n \geq 1$. While writing the above mentioned partition functions, we shall extensively use the notations introduced in Sec. 2. Let us begin by discussing the Polychronakos spin chain and its partition function.

In Ref. [17], the Hamiltonian of the $SU(m|n)$ supersymmetric Polychronakos spin chain is defined as

$$\tilde{H}_{P}^{(m|n)} = \frac{\pi^2}{N} \sum_{1 \leq j < k \leq N} \frac{1 - \tilde{P}_{jk}^{(m|n)}}{(z_j - z_k)^2},$$

where the $z_j$'s are the roots of the $N$-th order Hermite polynomial $H_N(z)$. We have introduced a pre-factor of $\pi^2/N$ in Eq. (6.1) for the following reason. While the distance between nearest neighbor sites is of order $1/N$ in the HS spin chain ($\xi_j - \xi_{j+1}$ in Eq. (4.1)), it is of order $1/\sqrt{N}$ in the Polychronakos spin chain ($z_j - z_{j+1}$ in Eq. (6.1)). The latter statement can be derived from the fact that the solution of the equation $d^2H_N/dz^2 - 2z dH_N/dz + 2NH_N = 0$ is given by $H_N \sim \cos(\sqrt{2N}z + N\pi/2)$ for $N \to \infty$ and $|z| \ll \sqrt{N}$; this region corresponds to sites near the middle of the chain. The zeros of this function have a spacing of $\pi/\sqrt{2N}$. Thus the Hamiltonian in (6.1) takes the form

$$\tilde{H}_{P}^{(m|n)} = 2 \sum_{j < k} \frac{1 - \tilde{P}_{jk}^{(m|n)}}{(j - k)^2}$$

for $j$ and $k$ lying close to $N/2$, as compared to the form given in Eq. (4.2). We thus see that the pre-factors in Eqs. (4.1) and (6.1) must differ by a factor of $N$ in order to ensure that the energy levels of the two Hamiltonians scale as the same power of $N$. We can then use the same variable $\tilde{q} = e^{-\pi^2/(N^2T)}$ when we compare the partition functions of the HS and Polychronakos spin chains.

According to Eq. (3.8) of Ref. [17], the partition function corresponding to the
Hamiltonian in Eq. (6.1) can be written in the form

$$Z_P^{(m|n)}(\tilde{q}) = \sum_{k \in \mathcal{P}_N} \tilde{q}^{\frac{x^2(N-1)}{2}} \frac{r-1}{2} K_l S_{(k_1,k_2,...,k_r)}(x,y) \bigg|_{x=1,y=1}. \quad (6.3)$$

From this expression of the partition function, one obtains the eigenvalue of $\tilde{H}_P^{(m|n)}$ in (6.1) corresponding to the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ as

$$\tilde{E}_{(k_1,k_2,...,k_r)} = \frac{\pi^2}{N} \left[ \frac{N(N-1)}{2} - \sum_{l=1}^{r-1} K_l \right]. \quad (6.4)$$

Let us now define the Hamiltonian of the $SU(m|n)$ Polychronakos spin chain in a slightly different form given by

$$H_P^{(m|n)} = \frac{\pi^2}{N} \sum_{1 \leq j < k \leq N} \frac{1 + \tilde{P}_{jk}^{(m|n)}}{(z_j - z_k)^2}. \quad (6.5)$$

Using Eqs. (6.1) and (6.5) along with an identity given by (see, for example, Ref. [33])

$$\sum_{1 \leq j < k \leq N} \frac{1}{(z_j - z_k)^2} = \frac{N(N-1)}{4}, \quad (6.6)$$

it is easy to see that $\tilde{H}_P^{(m|n)} = \pi^2 (N-1)/2 - H_P^{(m|n)}$. Comparing this operator relation along with the eigenvalue relation (6.4), we find the energy eigenvalue of $H_P^{(m|n)}$ corresponding to the border strip $\langle k_1, k_2, \ldots, k_r \rangle$ to be

$$E_{(k_1,k_2,...,k_r)} = \frac{\pi^2}{N} \sum_{l=1}^{r-1} K_l. \quad (6.7)$$

In analogy with Eq. (6.3), we can write down the partition function corresponding to the Hamiltonian in (6.5) as

$$Z_P^{(m|n)}(\tilde{q}) = \sum_{k \in \mathcal{P}_N} \tilde{q}^{\frac{x^2(N-1)}{2}} \frac{r-1}{2} K_l S_{(k_1,k_2,...,k_r)}(x,y) \bigg|_{x=1,y=1}. \quad (6.8)$$
In Ref. [21], it has been shown that

\[
\sum_{k \in \mathcal{P}_N} \left( \prod_{i=1}^{r} \hat{q}^{\mathcal{E}(K_i)} \right) \mathcal{S}_{(k_1,k_2,\ldots,k_r)}(x,y) \bigg|_{x=1,y=1} = \sum_{k \in \mathcal{P}_N} \left( \prod_{i=1}^{r} d_{k_i}^{(m|n)} \right) \mathcal{E}(K_j) \prod_{j=r+1}^{N} (1 - \hat{q}^{\mathcal{E}(K_j)}), \tag{6.9}
\]

where \( \mathcal{E}(K_j) = K_j(N - K_j) \). We now observe that the proof of the relation in Eq. (6.9), as described in Sec. 3 of Ref. [21], remains valid if we choose \( \mathcal{E}(K_j) = K_jN \), instead of \( \mathcal{E}(K_j) = K_j(N - K_j) \); in fact, the proof of this relation does not use any specific form of \( \mathcal{E}(K_j) \). By using Eq. (6.9) for the case \( \mathcal{E}(K_j) = K_jN \), we can express the partition function \( Z_P^{(m|n)}(\hat{q}) \) in Eq. (6.8) as

\[
Z_P^{(m|n)}(\hat{q}) = \left[ \prod_{j=1}^{N-1} (1 - \hat{q}^{jN}) \right] \sum_{r=1}^{N} \sum_{k_1 + \cdots + k_r = N, k_j \geq 1} \left( \prod_{i=1}^{r} d_{k_i}^{(m|n)} \right) \prod_{j=1}^{r-1} \hat{q}^{K_jN} \prod_{j=r+1}^{N} \frac{1}{1 - \hat{q}^{K_jN}}, \tag{6.10}
\]

where the summation over \( k \in \mathcal{P}_N \) is written explicitly through its components. Eq. (6.10) is a new expression for the partition function of the \( SU(m|n) \) Polychronakos spin chain; this expression is very similar in form to the partition function of the \( SU(m|n) \) HS spin chain.

Let us now consider the limit \( N \to \infty \) and \( T << 1 \), for which one can retain all terms with finite powers of \( \hat{q}^N \) and neglect terms of the order of \( \hat{q}^{2N} \). Consequently, the dominant contribution on the right hand side of Eq. (6.10) comes from terms in which \( k_1, \ldots, k_{r-1} \) are of order 1, and \( k_r \) is close to \( N \). For \( k_r \sim N \to \infty \), we have \( d_{k_r}^{(m|n)} \to 2^m N^{n-1}/(n-1)! \). We thus obtain

\[
\lim_{N \to \infty} Z_P^{(m|n)}(\hat{q}) = \frac{2^m N^{n-1}}{(n-1)!} \left[ \prod_{j=1}^{N} (1 - \hat{q}^{jN}) \right] \sum_{k_1, \ldots, k_l \geq 1} \prod_{j=1}^{l} d_{k_j}^{(m|n)} \left( \frac{\hat{q}^{K_jN}}{1 - \hat{q}^{K_jN}} \right). \tag{6.11}
\]

The Hamiltonian \( \tilde{H}_P^{(m|n)} \) in Eq. (6.1) can be related to \( H_P^{(m|n)} \) in Eq. (6.5) through a unitary transformation described in Eq. (2.24), \( U \tilde{H}_P^{(m|n)} U^\dagger = H_P^{(m|n)} \). Consequently, the corresponding partition functions satisfy the relation

\[
\tilde{Z}_P^{(m|n)}(\hat{q}) = Z_P^{(m|n)}(\hat{q}). \tag{6.12}
\]
Using Eqs. (6.11) and (6.12) for the special case \( n = 1 \), we obtain an expression for the partition function of the \( SU(1|m) \) Polychronakos spin chain

\[
\lim_{N \to \infty} \tilde{Z}^{(1|m)}_P(\tilde{q}) = 2^m \left[ \prod_{j=1}^{\infty} (1 - \tilde{q}^{jN}) \right] \sum_{k_1, \ldots, k_l \geq 1} l \prod_{j=1}^{l} \left( d_{k_j}^{(m|1)} \frac{\tilde{q}^{K_j N}}{1 - \tilde{q}^{K_j N}} \right). \tag{6.13}
\]

On the other hand, it is shown in Ref. [17] that

\[
\lim_{N \to \infty} \tilde{Z}^{(1|m)}_P(\tilde{q}) = \prod_{j=0}^{\infty} (1 + \tilde{q}^{jN})^m. \tag{6.14}
\]

Comparing the right hand sides of Eqs. (6.13) and (6.14), we get

\[
\left[ \prod_{j=1}^{\infty} (1 - \tilde{q}^{jN}) \right] \sum_{k_1, \ldots, k_l \geq 1} l \prod_{j=1}^{l} \left( d_{k_j}^{(m|1)} \frac{\tilde{q}^{K_j N}}{1 - \tilde{q}^{K_j N}} \right) = \prod_{j=1}^{\infty} (1 + \tilde{q}^{jN})^m. \tag{6.15}
\]

Squaring both sides of this equation and multiplying by \( 2^m \), we obtain

\[
2^m \left[ \prod_{j=1}^{\infty} (1 - \tilde{q}^{jN}) \right]^2 \left[ \sum_{k_1, \ldots, k_l \geq 1} l \prod_{j=1}^{l} \left( d_{k_j}^{(m|1)} \frac{\tilde{q}^{K_j N}}{1 - \tilde{q}^{K_j N}} \right) \right]^2 = 2^m \prod_{j=1}^{\infty} (1 + \tilde{q}^{jN})^{2m}. \tag{6.16}
\]

We will now prove the equivalence of the partition functions of the \( SU(m|1) \) HS spin chain and \( m \) species of non-interacting fermions in the limits \( N \to \infty \) and \( T \ll 1 \), by showing that the left hand side of (6.16) is equal to the partition function of the \( SU(m|1) \) HS spin chain, while the right hand side of (6.16) is the partition function of \( m \) species of non-interacting fermions. For \( N \to \infty \) and \( T \ll 1 \), the partition function of one fermion given in Eq. (5.5) only gets contributions from values of \( u \) close to either 0 or \( N \). The term with \( u = 0 \) contributes a factor of 2, while the terms with \( u \) non-zero and close to 0 and \( u \) close to \( N \) each contribute \( \prod_{j=1}^{\infty} (1 + \tilde{q}^{jN}) \). Putting these together, we see that the partition function of \( m \) non-interacting fermions is equal to the right hand side of Eq. (6.16). Similarly, for \( N \to \infty \) and \( T \ll 1 \), we find that the only terms which contribute in Eq. (2.15) are those partitions \( k = \{k_1, k_2, \ldots, k_s, \ldots, k_r\} \) in which each of the \( k_i \)’s is of order 1 except for one, say, \( k_s \) which is close to \( N \). In the limit
\[ k_s \sim N \to \infty, \text{ we have } d_{k_s}^{(m|1)} \to 2^m. \] Therefore, we can write \( d^{(m|1)}(k) \) in Eq. (2.11) as

\[ d^{(m|1)}(k) = \left( 2^{m-1} \prod_{a=1}^{s-1} d^{(m|1)}(k_a) \right) \left( 2^{m-1} \prod_{b=s+1}^{r} d^{(m|1)}(k_b) \right). \quad (6.17) \]

Further, for the above mentioned partitions, the value of partial sums \( K_1, K_2, \ldots, K_{s-1} \) are close to 0 and the value of partial sums \( K_s, K_{s+1}, \ldots, K_{r-1} \) are close to \( N \). One can approximate \( \tilde{q}^{K_j(N-K_j)} \) by \( \tilde{q}^{K_jN} \) if \( K_j \) is close to 0, and by \( q^{(N-K_j)N} \) if \( K_j \) is close to \( N \) in Eq. (2.15). Combining this result along with the form of \( d^{(m|1)}(k) \) given in Eq. (6.17), we find that the contributions of the terms with \( K_j \) close to 0 and the terms with \( K_j \) close to \( N \) have the same form for various partitions \( k \) in Eq. (2.15); each of them is given by

\[ 2^{m-1} \left[ \prod_{j=1}^{\infty} \left( 1 - \tilde{q}^{jN} \right) \right] \sum_{k_1, \ldots, k_l \geq 1} \prod_{j=1}^{l} \left( d_{k_j}^{(m|1)} \tilde{q}^{K_jN} \right), \quad (6.18) \]

where all the \( k_j \)'s and \( K_j \)'s are now of order 1. The partition function of the \( SU(m|1) \) HS spin chain is evidently obtained by taking the square of the expression in Eq. (6.18). Thus we find that, in the limit \( N \to \infty \) and \( T \ll 1 \), the partition function of the \( SU(m|1) \) HS spin chain coincides with left hand side of Eq. (6.16).

Finally, let us note that the derivation of Eqs. (6.17) and (6.18) given above for \( n = 1 \) can be generalized easily to any value of \( n \geq 1 \). Combining this with Eq. (6.11), we see that for \( N \to \infty \) and \( T \ll 1 \), the partition functions of the \( SU(m|n) \) HS and Polychronakos spin chains are related as

\[ \frac{2^m N^{n-1}}{(n-1)!} Z_{HS}^{(m|n)}(\tilde{q}) = \left[ Z_P^{(m|n)}(\tilde{q}) \right]^2. \quad (6.19) \]

7 Conclusions

In this paper, we have used the exact partition function of the \( SU(m|n) \) HS spin chain to find its complete spectrum, including the degeneracy of all energy levels, in terms of the motif representations. We have also obtained the momentum eigenvalue associated with different motifs. We have then studied the ground state and low energy excitations of the \( SU(m|n) \) HS spin chain with \( N \) sites, for various values of \( m, n \) and \( N \). In the
thermodynamic limit $N \to \infty$, the low energy, low momentum spectrum is always found to have a linear relation between the energy and momentum, with the velocity being independent of $m$ and $n$. The $SU(m|0)$ spin chain has some low energy, high momentum excitations which may be related to the algebraic long-range order of the system.

In the thermodynamic limit, the ground state degeneracy remains finite only for the $SU(m|0)$ and $SU(m|1)$ HS spin chains. Hence the low energy excitations of only these spin chains can possibly be described by conformal field theories. The $SU(m|0)$ spin chain is known to be described by the $SU(m)_1$ WZNW CFT with central charge $m - 1$. We have derived exact expressions for the partition function of the $SU(1|1)$ spin chain for any value of $N$, and of the $SU(m|1)$ spin chain for $m \geq 2$ in the limit $N \to \infty$ and the temperature $T \ll 1$. We have shown that for all $m \geq 1$, the low temperature properties of the $SU(m|1)$ HS spin chain are the same as those of a model of $m$ non-interacting Dirac fermions, each of which has only positive energy states. Such a theory has central charge $m/2$.

Finally, we have shown that in the thermodynamic limit and at low temperatures, the partition function of the $SU(m|n)$ HS spin chain is related to the square of the partition function of the $SU(m|n)$ Polychronakos spin chain for $n \geq 1$.

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Appendix A. Equivalence of the $SU(1|1)$ HS spin chain and one species of non-interacting fermions

By using Eq. (2.12) we find that $d_{k_i}^{(1|1)} = 2$ for any value of $k_i$. Substituting this value of $d_{k_i}^{(1|1)}$ in Eq. (2.20), and writing the corresponding summation variable $l (\in \mathcal{P}_r)$
through its components, we obtain

\[
S_{(k_1,k_2,\ldots,k_r)}(x,y)\mid_{x=1,y=1} = \sum_{s=1}^{r} \sum_{\ell_1+\ell_2+\cdots+\ell_s=N,\ell_s\geq 1} \prod_{i=1}^{s} 2 (-1)^{r-s} 2^s \sum_{\ell_1+\ell_2+\cdots+\ell_s=N,\ell_s\geq 1} 1
\]

\[
= \sum_{s=1}^{r} (-1)^{r-s} 2^s \sum_{\ell_1+\ell_2+\cdots+\ell_s=N,\ell_s\geq 1} 1
\]

\[
= \sum_{s=1}^{r} (-1)^{r-s} 2^s f(r,s). \quad (A1)
\]

Here \(f(r,s)\) indicates the number of possible ways of partitioning \(r\) into length \(s\), taking care of ordering. Clearly, this problem is equivalent to the problem of distributing \(r\) identical balls amongst \(s\) identical boxes, where each box contains at least one ball. After putting one ball in each box, there will be \(r-s\) balls remaining, which can be distributed freely amongst the \(s\) different boxes. This problem is the same as distributing \(r-s\) bosons amongst \(s\) states. Therefore the number of different distributions is

\[
f(r,s) = \frac{(r-s+s-1)!}{(s-1)!(r-j)!} = r^{-1}C_{s-1}. \quad (A2)
\]

Substituting this in Eq. (A1), we obtain

\[
S_{(k_1,k_2,\ldots,k_r)}(x,y)\mid_{x=1,y=1} = 2 \sum_{s=1}^{r} r^{-1}C_{s-1} 2^{s-1} (-1)^{r-s}
\]

\[
= 2 \sum_{s=0}^{r-1} r^{-1}C_s 2^s (-1)^{r-s+1}
\]

\[
= 2 (2-1)^{r-1} = 2. \quad (A3)
\]

Substituting this value of \(S_{(k_1,k_2,\ldots,k_r)}(x,y)\mid_{x=1,y=1}\) in Eq. (2.18), we find that

\[
Z^{(1|1)}_{HS}(\tilde{q}) = \sum_{r=1}^{N} \sum_{k_1+\cdots+k_r=N, k_j\geq 1} 2^{r-1} \sum_{\ell_i=1}^{r-1} \xi(K_i)
\]

\[
= 2 \sum_{r=1}^{N} \sum_{1\leq K_1<K_2<\cdots<K_{r-1}\leq N-1} 2^{r-1} \sum_{\ell_i=1}^{r-1} \xi(K_i), \quad (A4)
\]
where $\mathcal{E}(K_j) = K_j(N - K_j)$.

To compare $Z_{HS}^{(11)}(\tilde{q})$ with the partition function of one species of non-interacting fermions, let us expand the fermion partition function in Eq. (5.5) as follows:

$$
2 \prod_{j=1}^{N-1} (1 + \tilde{q}^{\mathcal{E}(j)}) = 2 \left(1 + \tilde{q}^{\mathcal{E}(1)}\right) \left(1 + \tilde{q}^{\mathcal{E}(2)}\right) \ldots \left(1 + \tilde{q}^{\mathcal{E}(N-1)}\right)
$$

$$
= 2 \left[ 1 + \sum_{l_1=1}^{N-1} \tilde{q}^{\mathcal{E}(l_1)} + \sum_{1 \leq l_1 < l_2 \leq N-1} \tilde{q}^{\mathcal{E}(l_1) + \mathcal{E}(l_2)} + \ldots \right]
$$

$$
= 2 \sum_{s=0}^{N-1} \sum_{1 \leq l_1 < l_2 \ldots < l_s \leq N-1} \tilde{q}^{\sum_{j=1}^{s} \mathcal{E}(l_j)}. \tag{A5}
$$

Comparing Eqs. (A4) and (A5), we find complete equivalence between the partition function of the $SU(1|1)$ HS spin chain and that of one species of non-interacting fermions for any value of $N$ and $T$. 

36
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\[ \langle k_1, k_2, \ldots, k_r \rangle = \]

Figure 1: Shape of the border strip \( \langle k_1, k_2, \ldots, k_r \rangle \).
Figure 2: Form of an allowed tableau corresponding to the border strip \(\langle k_1, k_2, \ldots, k_r \rangle\) (with arbitrary values of \(k_i\)) occurring in the Fock space of the \(SU(m|n)\) supersymmetric HS spin chain. Here \(\alpha\) is any number within the set \(\{1, 2, \ldots, m\}\), and \(\beta\) is any number within the set \(\{m + 1, m + 2, \ldots, m + n\}\).