Closed Billiards Trajectories with Prescribed Bounces

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Abstract. We give the asymptotic growth of the number of primitive periodic trajectories of a two-dimensional dispersive billiard, when we prescribe their number of bounces on one of the obstacles.

Introduction

Consider $D_0, D_1, \ldots, D_r \subset \mathbb{R}^2$ ($r \geq 3$) some compact and strictly convex open sets, with smooth boundaries $\partial D_0, \ldots, \partial D_r$. We assume that $D_i \cap D_j = \emptyset$ whenever $i \neq j$. We, moreover, assume that the billiard $B' = \{D_0, D_1, \ldots, D_r\}$ satisfies the non-eclipse condition, that is,

$$\text{conv}(D_i \cup D_j) \cap D_k = \emptyset, \quad k \neq i, j,$$

where $\text{conv}(A)$ denotes the convex hull of a set $A$. We will denote $D = \bigcup_j D_j$.

A billiard trajectory is a piecewise Euclidean trajectory $\gamma : I \to \mathbb{R}^2 \setminus D^o$ (here $I \subset \mathbb{R}$ is an interval) which rebounds on each $\partial D_j$ according to Fresnel Descartes’ law (see Fig. 1).

A trajectory $\gamma : [0, \tau] \to \mathbb{R}^2 \setminus D^o$ will be said to be closed if $\gamma(0) = \gamma(\tau)$ and $\gamma'(0) = \gamma'(\tau)$; a closed trajectory will be said to be primitive if $\gamma|_{[0, \tau']} \neq \emptyset$ for every $\tau' < \tau$. We will identify two closed trajectories $\gamma_j : \mathbb{R}/\tau_j \mathbb{Z} \to \mathbb{R}^2 \setminus D^o$ ($j = 1, 2$) whenever $\tau_1 = \tau_2$ and $\gamma_1(\cdot) = \gamma_2(\cdot + \tau)$ for some $\tau \in \mathbb{R}$. Denote by $P_{B'}$ the set of primitive closed trajectories of the billiard table $B'$. Then, a result of Morita [13] states that there is $h_{B'} > 0$ such that

$$\sharp\{\gamma \in P_{B'} : \tau(\gamma) \leq t\} \sim \frac{e^{h_{B'} t}}{h_{B'} t}, \quad t \to \infty,$$  \hfill (0.1)

where $\tau(\gamma)$ denotes the period of a periodic trajectory $\gamma$.

The purpose of the present paper is to give the asymptotic growth of the number of primitive closed trajectories of $B'$ when we additionally prescribe

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1By “Euclidean,” we mean trajectories going in a straight line with constant speed 1.
Figure 1. A billiard trajectory $\gamma$ bouncing twice on $D_0$, so that $r(\gamma) = 2$

their number of rebounds on $D_0$. More precisely, for $\gamma \in \mathcal{PB}^\prime$ we denote by $r(\gamma)$ the number of rebounds of $\gamma$ on $D_0$; we have the following result.

**Theorem 1.** There are $c, h_B > 0$ such that for every $n \geq 1$, it holds

$$
\sharp\{\gamma \in \mathcal{PB}^\prime : \tau(\gamma) \leq t, \ r(\gamma) = n\} \sim \frac{(ct)^n e^{h_B t}}{n! h_B t}, \quad t \to \infty.
$$  

(0.2)

Moreover, $h_B$ depends only on the billiard table $B = \{D_1, \ldots, D_r\}$.

As discussed in Sect. 4, by using the symbolic representation of the billiard flow and (0.1), one can prove that for some constants $a, b > 0$ we have

$$
at^{n-1} \exp(h_B t) \leq \sharp\{\gamma \in \mathcal{PB}^\prime : \tau(\gamma) \leq t, \ r(\gamma) = n\} \leq bt^{n-1} \exp(h_B t)
$$

provided $t$ is large enough; yet this method does not a priori provide the more precise asymptotics (0.2).

Our approach for proving (0.2) is reminiscent of a previous work [5] about the asymptotic growth of the number of closed geodesics on negatively curved surfaces for which certain intersection numbers are prescribed. In particular, we make use of the work of Dyatlov–Guillarmou [9] about the existence of Pollicott–Ruelle resonances for open hyperbolic systems (the recent work of Küster–Schütte–Weich [12] details how a hyperbolic billiard flow can be described by the framework of [9]). This allows to obtain a microlocal description of the transfer operator $T(s)$ associated with the first return map (of the billiard flow) to $\pi^{-1}(\partial D_0)$ (here $\pi : \mathbb{S}^2 \to \mathbb{R}^2$ is the natural projection), weighted by $\exp(-st_0(\cdot))$ where $t_0(\cdot)$ is the first return time to $\pi^{-1}(\partial D_0)$ (see Sect. 2), and to apply a Tauberian theorem of Delange to the (transversal)
trace of the composition $^2 T(s)^n$ (which is linked to some dynamical zeta function involving the periodic orbits rebounding $n$ times on $\partial D_0$).

Similar asymptotics for open dispersive billiards in $\mathbb{R}^d$ ($d \geq 3$) could also be obtained with our methods; however, here we restrict ourselves to the case $d = 2$ for the sake of simplicity.

**Related Works**

In [13], Morita proves the asymptotics (0.1) by constructing a symbolic coding of the billiard flow and by using the work of Parry–Pollicott [15]. Later, Stoyanov [18] proved the more precise asymptotics

$$\sharp \{ \gamma \in P_{B'} : \tau(\gamma) \leq t \} = \int_2^{\exp(h_{B'}t)} \frac{du}{\log u} + O(e^c t), \quad t \to +\infty,$$

for some $c \in ]0, h_{B'}[$, by proving some non-integrability condition over the non-wandering set and by using Dolgopyat-type estimates (see also [17] for an asymptotics of the number of primitive closed trajectories with periods lying in exponentially shrinking intervals. We finally mention the book of Pektov–Stoyanov [16].

**Organization of the Paper**

The paper is organized as follows. In Sect. 1, we present some geometrical and dynamical tools. In Sect. 2, we introduce the weighted transfer operator associated with the first return map to $\partial D_0$ and we compute its Attiyah–Bott transversal trace. In Sect. 3, we make use of a Tauberian argument. In Sect. 4, we prove some a priori estimates on $\sharp \{ \gamma \in P_{B'} : \tau(\gamma) \leq t, \ r(\gamma) = n \}$. Finally, in Sect. 5 we combine the results of Sects. 3, 4 to prove Theorem 1.

1. Preliminaries

In this section, we expose some well-known facts about open dispersive billiards.

1.1. The Billiard Flow

Let $D_1, \ldots, D_r \subset \mathbb{R}^2$ be pairwise disjoint compact convex obstacles, where $r \in \mathbb{N}_{\geq 3}$. We denote by $S\mathbb{R}^2$ the unit tangent bundle of $\mathbb{R}^2$ and $\pi : S\mathbb{R}^2 \to \mathbb{R}^2$ the natural projection. For $x \in \partial D_j$, we denote by $n_j(x)$ the outward unit normal vector to $\partial D_j$ at the point $x$. We define the (nonglancing) billiard table $M$ as

$$M = N/ \sim, \quad N = S\mathbb{R}^2 \setminus (\pi^{-1}(D^o) \cup G),$$

where $G = T\partial D$ and $D = \bigcup_{j=1}^r D_j$, and where $(x,v) \sim (y,w)$ if and only if $x = y \in \partial D_j, \quad w = v - 2\langle v, n_j(x) \rangle n_j(x), \quad j = 1, \ldots, r$.

The set $M$ is endowed with the quotient topology. We denote by $\varphi = (\varphi_t)_{t \in \mathbb{R}}$ the billiard flow (which is defined on an open subset of $\mathbb{R} \times M$). The manifold $M$
can be endowed with a differential structure by declaring that flow charts near \( \pi^{-1}(\partial D) \) are smooth charts for \( M \); we refer to the work of Küster–Schütte–Weich [12] for a detailed exposition of the construction of this differential structure. The flow \( \varphi \) becomes a smooth flow on \( M \) and we denote by \( X \) the associated vector field. Define the Liouville one form \( \alpha \) structure. The flow be a parametrization of \( \partial D \) 

\[
\text{where } \pi : \R^2 \times S^1 \to \R^2 \text{ is the projection over the first factor.}
\]

Lemma 1.1. The form \( \alpha \) gives rise to a one form on \( M \) which is smooth outside the glancing set \( G \). Moreover, \( \alpha \) is of contact type and \( X \) is its associated Reeb vector field, that is, \( \alpha \wedge d\alpha \) is a volume form and

\[
i_X \alpha = 1, \quad i_X d\alpha = 0,
\]

where \( i_X \) denotes the interior product.

Proof. The assertion is clear far from \( \pi^{-1}(D) \). Thus, we take \( j \in \{1, \ldots, r\} \) and some \((x, v) \in \pi^{-1}(\partial D_j) \setminus G_j \) (here \( G_j = T\partial D_j \)). Let \( \psi : [-\varepsilon, \varepsilon[ \to \R^2 \) be a parametrization of \( \partial D_j \) near \( x \) such that \( |\psi'(s)| = 1 \) for \( s \in ]-\varepsilon, \varepsilon[ \) and \( \psi(0) = x \). Let \( \varphi_0 \) be the (counterclockwise) angle between \( n(x) \) and \( v \). Let \( U \) be a small neighborhood of \((0, \varphi_0, 0) \in \R^3 \), and set

\[
V = U \cup U', \quad U' = \{(s, \pi - \varphi, \tau) : (s, \varphi, \tau) \in U\}.
\]

As \( \varphi_0 \notin \pi/2 + \pi \Z \), we may assume (up to shrinking \( U \)) that the map

\[
\Psi : V \to \R^2 \times S^1 \text{ defined by}
\]

\[
\Psi : (s, \varphi, \tau) \mapsto \left(\psi(s) + \tau R_\varphi n(s), \ R_\varphi n(s) \right)
\]

is a smooth local embedding, where \( R_\varphi : S^1 \to S^1 \) is the rotation of angle \( \varphi \) and where we set \( n(s) = n(\psi(s)) \). From (1.1), it is then immediate to check that, in the coordinates \((s, \varphi, \tau)\), we have

\[
\alpha = \sin(\varphi) ds + d\tau.
\]

(1.2)

In particular, we have \( Q^*\alpha = \alpha \) where \( Q \) denotes the map \((s, \varphi, \tau) \mapsto (s, \pi - \varphi, \tau)\). Let \( p : N \to M \) be the natural projection. We have a map \( \Phi : U \to M \) defined by \( \Phi(z) = p(z) \) if \( \Psi(z) \in N \) and \( \Phi(z) = p(Q(z)) \) otherwise. Up to shrinking \( U \), this map realizes a local diffeomorphism from \( U \) to a neighborhood of \( p(x, v) \) in \( M \) (this map is a chart for \( M \) near \( p(x, v) \), see [12, §4]). Since \( Q^*\alpha = \alpha \), it follows that \( \alpha \) gives rise to a smooth one-form on \( \Phi(U) \). The fact that \( \alpha \) is a contact form follows from the expression (1.2). In the coordinates \((s, \varphi, \tau)\), \( X \) is represented by \( \partial_\tau \); thus, \( X \) satisfies the announced Reeb conditions. \( \square \)

1.2. The Anosov Property

The trapped set \( \Lambda \) is defined as the set of points of \( z \in M \) which satisfy

\[
\sup T(z) = -\inf T(z) = +\infty \quad \text{where} \quad T(z) = \{ t \in \R : \pi(\varphi_t(z)) \in \partial D \},
\]

We define the first (future and the past) return times to \( \partial D \) by

\[
t_\pm(z) = \inf \{ t > 0 : \pi(\varphi_{\pm t}(z)) \in \partial D \}, \quad z \in M.
\]
Let \( K = \Lambda \cap \partial D \). The (future and past) billiard maps \( B_{\pm} : K \to K \) are then defined as

\[
B_{\pm}(z) = \varphi_{\pm t_{\pm}}(z), \quad z \in K.
\]

Throughout this paper, we will assume that we have the non-eclipse condition

\[
\text{conv}(D_i \cup D_j) \cap D_k = \emptyset, \quad k \neq i, j,
\]

where \( \text{conv}(A) \) is the convex hull of a set \( A \). This condition implies that \( \Lambda \cap G = \emptyset \), and by [13] (see also [6, §4.4], or the appendix of [7] for a proof of the Anosov property in every dimension), this implies that the billiard flow is uniformly hyperbolic, meaning that for each \( z \in \Lambda \) there is a \( \varphi_t \)-invariant decomposition

\[
T_z \Lambda = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z),
\]

which depends continuously on \( z \), and such that for some \( C, \nu > 0 \) independent of \( z \in \Lambda \), we have for some smooth norm \( |\cdot| \) on \( TM \),

\[
|d\varphi_t(z)v| \leq \begin{cases} 
Ce^{-\nu |t|}|v|, & v \in E_s(z), t \geq 0, \\
Ce^{-\nu |t|}|v|, & v \in E_u(z), t \leq 0.
\end{cases}
\]

### 1.3. The Non-eclipse Condition

By the non-eclipse condition (1.3) and [13], the billiard map \( B_{\pm} : K \to K \) is Hölder conjugated to a subshift of finite type. More precisely, let \( A = \{1, \ldots, r\} \) and

\[
\Sigma = \{(u_n) \in A^\mathbb{Z} : u_n \neq u_{n+1}, n \in \mathbb{Z}\}.
\]

Let \( \sigma_{\pm} : \Sigma \to \Sigma \) be the map \((u_n) \mapsto (u_{n\pm1})\). We endow \( \Sigma \) with the topology coming from the distance

\[
d_\Sigma(u, v) = \sum_{n \in \mathbb{Z}} 2^{-|n|}|u_n - v_n|.
\]

Then, there is a homeomorphism \( \psi : K \to \Sigma \), which is Hölder continuous, such that

\[
\sigma_{\pm} \circ \psi = \psi \circ B_{\pm}.
\]

In fact, \( \psi \) is simply given by

\[
\psi(z)_n = j, \quad B_{\pm}^n(z) \in \partial D_j, \quad z \in K, \quad n \in \mathbb{Z}.
\]

Of course, the billiard flow \( (\varphi_t) \) is conjugated to the suspension of \( \sigma_{\pm} \) associated with the time return map \( t_{\pm} \circ \psi^{-1} \). This means that we have a Hölder homeomorphism

\[
\Psi : \Lambda \to (K \times \mathbb{R}_+)/\sim
\]

where \((z, t_{\pm}(z)) \sim (B_{\pm}(z), 0)\) for \( z \in K \). In the coordinates \((z, t)\), \( X \) is simply represented by \( \partial_t \). In what follows, we will denote by \( \psi_n(z) \) the \( n \)-th term of the sequence \( \psi(z) \). An immediate consequence of the existence of a conjugacy \( \Psi \) as above is the following
Lemma 1.2. There is $C > 0$ and $\beta > 1$ such that the following holds. Assume that $z, z' \in K$ satisfy

$$\psi_n(z) = \psi_n(z'), \quad |n| \leq N.$$ 

Then $d(z, z') \leq C\beta^{-N}$.

1.4. Isolating Blocks

In this subsection, we show that we can work with the framework of [9] (see also [12, §5] for a more detailed exposition). We have

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(V \setminus TD)$$

where $V = \{ z \in M : T_-(z) \neq \emptyset \text{ and } T_+(z) \neq \emptyset \} \subset M$. Here we set

$$T_{\pm}(z) = \{ t \in T(z) \ : \ \pm t > 0 \}.$$

By [4, Theorem 1.5], $\Lambda$ is the maximal invariant set in some isolating block. More precisely, there exists a relatively compact neighborhood $U \subset M$ of $K$ such that $\partial U$ is smooth and

$$\partial_0 U = \{ (x, v) \in \partial U \ : \ v \in T_x \partial U \}$$

is a smooth submanifold of $\partial U$ of codimension 1, and with the property that for some $\varepsilon > 0$ one has

$$z \in \partial_0 U \implies \forall |t| \in [0, \varepsilon[, \varphi_t(z) \notin U.$$ 

By proceeding as in [10, Lemma 2.3], we may find a vector field $\tilde{X}$ on $M \setminus TD$ such that $X - \tilde{X}$ is supported in an arbitrary small neighborhood of $\partial_0 U$, which is arbitrarily small in the $C^\infty$ topology and such that for any boundary defining function $\rho : U \to \mathbb{R}_{\geq 0}$ of $\partial U$, we have, for any $z \in \partial U$,

$$\tilde{X}\rho(z) = 0 \implies \tilde{X}^2 \rho(z) < 0.$$ 

Moreover, we have $\Gamma_\pm(U) = \tilde{\Gamma}_\pm(U)$ where $\Gamma_\pm(U) = \{ z \in U \ : \ \varphi_t(z) \in U, \ \forall t \geq 0 \}$, $\Gamma_\pm(U) = \{ z \in U \ : \ \tilde{\varphi}_t(z) \in U, \ \forall t \geq 0 \}$, where $\tilde{\varphi}_t$ denotes the flow of $\tilde{X}$. Note also that $\text{dist}(\Gamma_\pm(U), \partial_0 U) > 0$. For simplicity, we will denote $\Gamma_\pm = \Gamma_\pm(U)$.

By [9, Lemma 2.10], there are two vector subbundles $E_\pm \subset T_{\Gamma_\pm} U$ with the following properties:

1. $E_+|_K = E_u$, $E_-|_K = E_s$ and $E_\pm(z)$ depends continuously on $z \in \Gamma_\pm$.
2. For some constants $C', \nu' > 0$, we have

$$\|d\tilde{\varphi}_{\pm t}(v)\| \leq C'e^{-\nu't}\|v\|, \quad v \in E_\pm(z), \quad z \in \Gamma_\pm, \quad t \geq 0;$$

3. If $z \in \Gamma_\pm$ and $v \in T_z U$ satisfy $\langle \alpha(z), v \rangle = 0$ and $v \notin E_\pm(z)$, then as $t \to \mp\infty$

$$\|d\tilde{\varphi}_t(z)v\| \to \infty, \quad \frac{d\tilde{\varphi}_t(z)v}{\|d\tilde{\varphi}_t(z)v\|} \to E_\mp|_K.$$

\[ \text{This means that } \rho > 0 \text{ on } U, \rho = 0 \text{ on } \partial U \text{ and } d\rho \neq 0 \text{ on } \partial U. \]
1.5. The Resolvent of the Billiard Flow

For \( \text{Re}(s) \gg 1 \), we define the (future and past) resolvents \( \tilde{R}_\pm(s) : \Omega^\bullet_c(U) \to \mathcal{D}'^\bullet(U) \) by

\[
\tilde{R}_\pm(s)\omega(z) = \pm \int_0^{\tilde{t}_\pm,U(z)} \tilde{\varphi}_\pm^*\omega(z)e^{-ts}dt, \quad \omega \in \Omega^\bullet_c(U), \quad z \in U,
\]

where we set

\[
\tilde{t}_\pm,U(z) = \inf\{t > 0 : \tilde{\varphi}_\pm(t) \in \partial U\}, \quad z \in U.
\]

Here \( \Omega^\bullet_c(U) \) denotes the space of smooth differential forms which are compactly supported in \( U \) while \( \mathcal{D}'^\bullet(U) \) denotes the space of currents in \( U \) (that is \( \mathcal{D}'^k(U) \) is the dual space of \( \Omega^{3-k}_c(U) \) for \( k = 0, \ldots, 3 \)). Note that

\[
(\mathcal{L}_X \pm s)\tilde{R}_\pm(s) = \tilde{R}_\pm(s)(\mathcal{L}_X \pm s) = \text{Id}_{\Omega^\bullet_c(U)}.
\]

Then, by [9], the family \( s \mapsto \tilde{R}_\pm(s) \) extends to a family of operators meromorphic in the parameter \( s \in \mathbb{C} \), whose poles have residues of finite rank. Denote by \( \text{Res}(\tilde{X}) \) the set of those poles. Near any \( s_0 \in \text{Res}(\tilde{X}) \), we have for some finite rank projector \( \Pi_\pm(s_0) : \Omega^\bullet \to \mathcal{D}'^\bullet \)

\[
R_\pm(s) = H_\pm(s) + \sum_{j=1}^{J(s_0)} \frac{(X \pm s)^{j-1}\Pi_\pm(s_0)}{(s - s_0)^j}
\]

where \( s \mapsto H_\pm(s) \) is holomorphic near \( s_0 \). Moreover, we have \( \text{supp}(\Pi_\pm(s_0)) \subset \Gamma_\pm \times \Gamma_\mp \) and

\[
\text{WF}'(H_\pm(s)) \subset \Delta(T^*U) \cup \Upsilon_\pm \cup (E^*_\pm \times E^*_\mp), \quad \text{WF}'(\Pi_\pm(s_0)) \subset E^*_\pm \times E^*_\mp,
\]

(1.4)

where \( \Delta(T^*U) = \{(\xi, \xi), \xi \in T^*U\} \subset T^*(U \times U) \) and

\[
\Upsilon_\pm = \{(\Phi_t(z, \xi), (z, \xi)), \pm t \geq 0, (\xi, X(z)) = 0, \ z \in U \: \tilde{\varphi}_t(z) \in U\}.
\]

Here \( \Phi_t \) denotes the symplectic lift of \( \varphi_t \) on \( T^*U \), that is,

\[
\Phi_t(z, \xi) = (\varphi_t(z), (d_z\varphi_t)^{-T}\xi), \quad (z, \xi) \in T^*U, \quad \varphi_t(z) \in U,
\]

and the subbundles \( E^*_\pm \subset T^*_{\Gamma_\pm}U \) are defined by \( E^*_\pm(\mathbb{R}X(z) \oplus E_\pm) = 0 \). Also we denoted

\[
\text{WF}'(\tilde{R}_\pm(s)) = \{(z, \xi, z', \xi') \in T^*(U \times U), (z, \xi, z', -\xi') \in \text{WF}(\tilde{R}_\pm(s))\},
\]

where \( \text{WF}(\tilde{R}_\pm(s)) \subset T^*(U \times U) \) is the Hörmander wavefront set of (the Schwartz kernel of) \( \tilde{R}_\pm(s) \), see [11, §8], and

\[
\text{WF}'(\tilde{R}_\pm(s)) = \{(z, \xi, z', \xi') : (z, \xi, z, -\xi') \in \text{WF}(\tilde{R}_\pm(s))\}.
\]
1.6. The Scattering Operator

We define
\[ \tilde{\partial}_\pm = \{ z \in \partial U : \mp \tilde{X} \rho(z) > 0 \} \text{ and } \tilde{\partial}_0 = \{ z \in \partial U : \tilde{X} \rho(z) = 0 \}. \]

The scattering map \( \tilde{S}_\pm : \tilde{\partial}_\mp \setminus \Gamma_\mp \to \tilde{\partial}_\pm \setminus \Gamma_\pm \) is defined by
\[ \tilde{S}_\pm(z) = \varphi_{t_\pm,U(z)}(z), \quad z \in \tilde{\partial}_\mp \setminus \Gamma_\mp \]
(see Fig. 2). The Scattering operator \( \tilde{S}_\pm(s) : \Omega_c^*(\tilde{\partial}_\mp \setminus \Gamma_\mp) \to \Omega_c^*(\tilde{\partial}_\pm \setminus \Gamma_\pm) \) is then defined by
\[ \tilde{S}_\pm(s) \omega = \left( \tilde{S}_\pm \omega \right) e^{-st_{\mp,U}^c(\cdot)}, \quad \omega \in \Omega_c^*(\tilde{\partial}_\mp \setminus \Gamma_\mp). \]

Note that for \( \text{Re}(s) \gg 1 \), \( \tilde{S}_\pm(s) \) extends as an operator \( C_c(\tilde{\partial}_\mp, \cdot^*T^*\tilde{\partial}_\mp) \to C_c(\tilde{\partial}_\pm, \cdot^*T^*\tilde{\partial}_\pm) \), where \( C_c(\tilde{\partial}_\pm, \cdot^*T^*\tilde{\partial}_\pm) \) is the space of compactly supported continuous forms on \( \tilde{\partial}_\pm \), since for any \( w \in \Omega^*(U) \) and \( t \in \mathbb{R} \) we have
\[ \| \varphi_t^s w \|_\infty \leq C e^{C|t|} \| w \|_\infty. \]

In what follows, we let \( \iota_\pm : \tilde{\partial}_\pm \to U \) be the inclusion and \( (\iota_\pm)_* : \Omega_c^*(\tilde{\partial}_\pm) \to \mathcal{D}^{\ast+1}(U) \) be the pushforward operator, which is defined by
\[ \int_U (\iota_\pm)_* u \wedge v = \int_{\tilde{\partial}_\pm} u \wedge \iota_\pm^* v, \quad u \in \Omega_c^*(\tilde{\partial}_\pm), \quad v \in \Omega^*(U). \]

**Proposition 1.3.** We have
\[ \text{WF}(\tilde{R}_\pm(s)) \cap N^*(\tilde{\partial}_\pm \times \tilde{\partial}_\mp) = \emptyset. \]

In particular, by [11, Theorem 8.2.4], the operator \( (\iota_\pm)_* \tilde{X} \tilde{R}_\pm(s)(\iota_\mp)_* \) is well defined. Moreover, for \( \text{Re}(s) \gg 1 \) large enough, we have
\[ \tilde{S}_\pm(s) = (-1)^N (\iota_\pm)_* \tilde{X} \tilde{R}_\pm(s)(\iota_\mp)_* : \Omega_c^*(\tilde{\partial}_\mp) \to \mathcal{D}^{\ast}(\tilde{\partial}_\pm), \tag{1.6} \]
where \( N : \mathcal{D}^{\ast} \to \mathcal{D}^{\ast} \) is the number operator.\(^4\)

**Proof.** By definition, \( \tilde{X}(z) \) is transverse to \( T_z \tilde{\partial}_\pm \) for \( z \in \tilde{\partial}_\pm \). In particular, if \( (z, \xi) \in T^* \tilde{\partial}_\pm \) satisfies \( \langle \xi, \tilde{X}(z) \rangle = 0 \) and \( \langle \xi, T_z \tilde{\partial}_\pm \rangle = 0 \) then \( \xi = 0 \). As \( \tilde{\partial}_\pm \cap \tilde{\partial}_\mp = \emptyset \) we obtain (1.5) by (1.4).

Now let \( W_\pm \subset \tilde{\partial}_\pm \) be open sets such that \( \overline{W}_\pm \subset \tilde{\partial}_\pm \). As \( \overline{W}_\pm \cap \tilde{\partial}_\mp = \emptyset \), there is \( \varepsilon > 0 \) such that \( \tilde{t}_{\mp,U}(z) > \varepsilon \) for every \( z \in \overline{W}_\pm \). In particular, the proof of [5, Lemma 3.3] applies and leads to the fact that (1.6) holds when \( \tilde{S}_\pm(s) \) is seen as an operator \( \Omega_c^*(\tilde{\partial}_\mp \setminus \Gamma_\mp) \to \mathcal{D}^{\ast}(\tilde{\partial}_\pm \setminus \Gamma_\pm) \). By [3, Theorem 5.6], as \( \Lambda \) is not an attractor, we have \( \mu(\Gamma_\pm) = 0 \) where \( \mu \) is the measure \( |\alpha \wedge d\alpha| \). Take \( U_\pm \subset \tilde{\partial}_\pm \) a small neighborhood of \( \Gamma_\pm \) in \( \tilde{\partial}_\pm \) and \( \delta > 0 \) small enough. Since \( \Gamma_\pm \cap \tilde{\partial}_0 = \emptyset \), we may assume that the map
\[ U_\pm \times [0, \delta] \to U, \quad (y, t) \mapsto \varphi_{\mp t}(y) \]

\(^4\)That is, \( N(\omega) = k\omega \) for \( \omega \in \Omega_c^k(\tilde{\partial}_\pm) \).
realizes a smooth diffeomorphism onto its image. In particular, because \( \varphi_{\pm t}(\Gamma_{\pm}) \subset \Gamma_{\pm} \) for \( t > 0 \), we have \( \mu_{\tilde{\partial}_{\pm}}(\Gamma_{\pm} \cap \tilde{\partial}_{\pm}) = 0 \) where \( \mu_{\tilde{\partial}_{\pm}} \) corresponds to the measure \( |t_{\pm}^{*}| \) da. Thus, we may proceed by similar arguments given in the proof of [5, Proposition 3.2] we obtain that (1.6) holds when \( \tilde{S}_{\pm}(s) \) is seen as an operator \( \Omega_{c}^{*}(\tilde{\partial}_{\pm}) \to \mathcal{D}'(\tilde{\partial}_{\pm}) \). \( \square \)

2. Adding an Obstacle

In this section, we add another obstacle \( D_{0} \) and we will consider some weighted transfer operator associated with the first return map to \( \pi^{-1}(\partial D_{0}) \); we will use the description of its microlocal structure to define and compute its flat trace.

2.1. Notations

We add another convex obstacle \( D_{0} \), and we assume that the billiard table \((D_{0},D_{1},\ldots,D_{r})\) satisfies the non-eclipse condition. We define 
\[ M',N',K',(\varphi_{t}'),T'_{\pm},t'_{\pm},B'_{\pm} \]
in the same way we defined \( M,\Lambda,K,(\varphi_{t}),T_{\pm},t_{\pm},B_{\pm} \) (see Sect. 1) by replacing the billiard table \( B = \{ D_{1},\ldots,D_{r} \} \) by the billiard table \( B' = \{ D_{0},D_{1},\ldots,D_{r} \} \).

Let 
\[ P'_{\pm} : \Lambda'_{\pm} \to \pi^{-1}(\partial D'), \quad z \mapsto \varphi_{\pm t'_{\pm}}(z), \]
where 
\[ \Lambda'_{\pm} = \{ z \in M' : t'_{\pm}(z) < \infty \}. \]
Let \( V_{0} \subset \pi^{-1}(\partial D_{0}) \) be a relatively compact neighborhood of \( K' \cap \pi^{-1}(\partial D_{0}) \) such that \( V_{0} \cap T\partial D_{0} = \emptyset \), and set 
\[ V_{\pm} = \{ z \in \partial U \cap \Lambda'_{\pm} : P'_{\pm}(z) \in V_{0} \}. \]
Note that \( U \) is a subset of \( M \). However, we may see \( U \) as a subset of \( M' \) since \( U \) does not intersect \( \pi^{-1}(D_{0}) \). We also let \( W_{\pm} \) be a neighborhood of \( \Lambda_{\pm} \cap \partial U \) in \( \partial U \) such that \( W_{\pm} \cap \text{supp}(\tilde{X} - X) = \emptyset \) and we set \( Y_{\pm} = W_{\pm} \cap V_{\pm} \). We take \( \phi_{\pm} \in C_{c}^{\infty}(V_{\pm},[0,1]) \) (resp. \( \psi_{\pm} \in C_{c}^{\infty}(W_{\pm},[0,1]) \) such that \( \phi_{\pm} \equiv 1 \) near \( (B'_{\pm})^{-1}(K') \) (resp. \( \psi_{\pm} \equiv 1 \) near \( \Lambda_{\pm} \)); we define 
\[ \chi_{\pm} = \phi_{\pm} \psi_{\pm} \in C_{c}^{\infty}(Y_{\pm}). \]
Note that \( P'_{\pm} \) realizes a diffeomorphism \( V_{\pm} \to P'_{\pm}(V_{\pm}) \subset \pi^{-1}(\partial D_{0}) \) which we denote by \( Q_{\pm} \). We define \( Q_{\pm}(s) : \mathcal{D}'_{c}(Y_{\pm}) \to \mathcal{D}'_{c}(Z_{\pm}) \), where \( Z_{\pm} = Q_{\pm}(Y_{\pm}) \), by
\[ Q_{\pm}(s)w = e^{-st'_{\pm}}(Q_{\pm}^{-1})^{*}w, \quad w \in \Omega_{c}(Y_{\pm}) \]
(see Fig. 2). We finally set with \( Z_{\pm} = Q_{\pm}(Y_{\pm}) \subset \pi^{-1}(\partial D_{0}) \),
\[ T_{\pm}(s) = Q_{\pm}(s)\chi_{\pm} \tilde{S}_{\pm}(s)\chi_{\mp} Q_{\mp}(s)^{\top} : \Omega_{c}(Z_{\pm}) \to \mathcal{D}'(Z_{\pm}) \quad (2.1) \]
The operator \( T_{\pm}(s) \) is the transfer operator associated with the first return map to \( \pi^{-1}(\partial D_{0}) \) weighed by \( e^{-st_{0,\pm}}(s) \), where \( t_{0,\pm}(z) = \inf \{ t > 0 : \pi(\varphi_{\pm t}(z)) \in \} \)
\( \partial D_0 \) \} are the first (future are past) return times to \( \partial D_0 \) of a point \( z \in \pi^{-1}(\partial D_0) \).

### 2.2. Composing the Scattering Maps

Let \( Z = Z_+ \cap Z_- \) and

\[
\varrho = (\chi_+ \circ Q_+^{-1})(\chi_- \circ Q_-^{-1}) \in C_c^\infty(Z).
\]

We have the following result.

**Proposition 2.1.** For any \( n \geq 2 \), the composition \((\varrho T_\pm(s))^n : \Omega_\cdot(Z) \rightarrow \mathcal{D}'_\cdot(Z)\) is well defined.

**Proof.** By [11, Theorem 8.2.4] and Proposition 1.3, we have

\[
WF'((\tilde{S}_\pm(s)) \subset \left\{ (\varrho T_\pm(s))^2 \right\}.
\]

where \( \varrho T_\pm(s) : \tilde{\partial}_\pm \times \tilde{\partial}_\pm \rightarrow U \times U \) is the inclusion. To prove that \((\varrho T_\pm(s))^2\)

is well defined, it suffices to show by [11, Theorem 8.2.14] that \( A \cap B_1 = \emptyset \) where

\[
A = \left\{ (z, \xi) \in T^*Z : \exists \xi' \in Z, (z', 0, z, \xi) \in WF'((\varrho T_\pm(s))) \right\}
\]

and

\[
B_1 = \left\{ (z, \xi) \in T^*Z : \exists \xi' \in Z, (z, \xi, z', 0) \in WF((\varrho T_\pm(s))) \right\}.
\]

Note that \((d_x(z))^\top \mid_{\ker X(z)} : \ker X(z) \rightarrow T^*_z \tilde{\partial}_\pm \) is injective for any \( z \in \tilde{\partial}_\pm \),

since \( X(z) \) is transverse to \( T_z \tilde{\partial}_\pm \). Moreover, \( Q_\pm : Y_\pm \rightarrow Z_\pm \) is a diffeomorphism, and thus, \( dQ_\pm^{-1}(z)^\top : T^*_z \tilde{\partial}_\pm \rightarrow T^*_z \pi^{-1}(\partial D_0) \) is injective for any \( z \in Y_\pm \).
Now by (2.1), we have
\[ \WF'(qT_\pm(s)) \subset d(Q_\pm \times Q_\mp)^T \left( \WF'(\tilde{S}_\pm(s)) \cap \supp(\chi_\pm \times \chi_\mp) \right). \]
Moreover, by (1.5) and (2.2) we have
\[ \WF'(\tilde{S}_\pm(s)) \subset d(\iota_\pm \times \iota_\mp)^T \left( Y_\pm \cup (E_\pm^* \times E_\mp^*) \right), \]
where \( \Delta(T^*U) \cap \pi^{-1}(\partial_\pm \times \partial_\mp) = \emptyset. \) By injectivity of \( dQ_\pm^{-1}(z)^T : T_z^* \tilde{\partial}_\pm \to T_z^* \pi^{-1}(\partial D_0), \) we obtain
\[ A \subset d(Q_\pm^{-1})^T d(\iota_\pm)^T E_\pm^* \quad \text{and} \quad B_1 \subset d(Q_\pm^{-1})^T d(\iota_\mp)^T E_\mp^*. \]
We claim that this implies \( A \cap B_1 = \emptyset. \) Indeed, let \( (z, \xi) \in T^*Z_\pm \) which lies in \( d(Q_\pm^{-1})^T d(\iota_\pm)^T E_\pm^* \cap (d(Q_\pm^{-1})^T d(\iota_\mp)^T E_\mp^*). \) Thus, \( z \) lies in \( \Lambda' \) and there exists \( (z_\pm, \xi_\pm) \) in \( E_\pm^* \) such that
\[ (z, \xi) = d(Q_\pm^{-1})^T d(\iota_\pm)^T (z_\pm, \xi_\pm). \]
There are neighborhoods \( U_\pm \) of \( z_\pm \) in \( M' \) and smooth functions \( s_\pm : U_\pm \to \mathbb{R} \) such that \( Q_\pm(z_\pm') = \varphi_{s_\pm(z_\pm')}(z_\pm') \) for \( z_\pm' \in U_\pm \) and \( \varphi_{s_\pm(z_\pm')}(z_\pm') \in Z \) for \( z_\pm' \in U_\pm. \) Because \( \xi_\pm \in \ker X(z_\pm), \) we see that
\[ d(Q_\pm^{-1})^T d(\iota_\pm)^T (z_\pm, \xi_\pm) = d\iota_\pm d\xi_\pm (\varphi_{s_\pm(z_\pm')})^T \xi_\pm, \]
where \( \iota : Z \hookrightarrow M' \) is the inclusion. Because \( d\iota^T : \ker X \to T^*Z \) is injective, we obtain
\[ \xi_- = d\left[ u \mapsto \varphi_{s_-(z_-) - s_+(z_+)(u)}(z_-)^T \xi_+. \right] \]
Now we have \( z_\pm \in \Lambda' \) and since \( \xi_\pm \in E_\pm^* \) we obtain \( \xi_+ \in E_\pm^*(z_+) \) and \( \xi_- \in E_\pm^*(z_-). \) Thus, \( \xi \in E_\pm^*(z) \cap E_\pm^*(z) = \{0\}. \) Here we denoted by
\[ T_2M' = \mathbb{R}E_0^*(z') \oplus \mathbb{R}E_u^*(z') \oplus \mathbb{R}X(z'), \]
the hyperbolic decomposition of \( TM' \) over \( \Lambda'. \) We conclude that \( A \cap B_1 = \emptyset, \) which concludes the case \( n = 2. \)
By [11, Theorem 8.2.14], we also have the bound
\[ \WF((qT_\pm(s))^2) \subset (\WF'(qT_\pm(s)) \cap \WF'(qT_\pm(s))) \cup (B_1 \times 0) \cup (0 \times A), \]
where \( 0 \subset T^*M' \) denotes the zero section. Therefore, the set \( B_2, \) which is defined by
\[ B_2 = \{ (z, \xi) \in T^*Z : \exists z' \in Z, (z, \xi, z', 0) \in \WF((qT_\pm(s))^2) \}, \]
can be written
\[ \{ (z, \xi) \in T^*Z : \exists z', z'' \in Z, \exists \eta \in T_{z'}^*Z, (z, \xi, z' - \eta) \in \WF(qT_\pm(s)) \}
\text{and} \quad (z', \eta, z'', 0) \in \WF(qT_\pm(s)) \} \cup B_1. \]
As \( d(Q_\pm^{-1})^T d(\iota_\pm)^T E_\pm^* \cap d(Q_\pm^{-1})^T d(\iota_\pm)^T E_\mp^* = 0 \) (as shown above), we obtain
\[ B_2 \subset \left\{ (z, \xi) : (z, \xi, z', \eta) \in d(Q_\pm^{-1} \times Q_\mp^{-1})^T d(\iota_\pm \times \iota_\mp)^T \right\}, \]
for some \( \eta \in d(Q_\pm^{-1})^T d(\iota_\pm)^T (E_\mp^*). \)
This leads to
\[ B_2 \subset \left\{ d(Q_\pm^{-1})^\top d(\nu_\pm)^\top \Phi_t(z, \zeta) : (z, \zeta) \in T^*Y_T, \langle X(z), \zeta \rangle = 0, \right. \\
\left. d(\nu_\pm)^\top (z, \zeta) \in d(Q_\pm \circ Q_\pm^{-1})^\top d(\nu_\pm)^\top E^*_\pm, \varphi_t(z) \in \tilde{\partial}_\pm U, \ t \geq 0 \right\}. \]

As before, this set cannot intersect \( d(Q_\pm^{-1})^\top E^*_\pm \) since otherwise we would have \( z' \in A' \) and \( \ell' \in T^*_z M' \) contracted in the past and in the future by \( d\varphi_t^\top \).

Thus, \( B_2 \cap A = \emptyset \) and we obtain that \((\varphi T_\pm(s))^3\) is well defined. By iterating this process, we obtain that \((\varphi T_\pm(s))^n\) is well defined for every \( n \geq 2 \), which concludes the proof. \( \square \)

2.3. The Flat Trace of \( T_\pm(s) \)

Let \( A : \Omega_c^\bullet(\partial) \to \mathcal{D}'^\bullet(Z) \) be an operator such that \( \text{WF}'(A) \cap \Delta = \emptyset \), where \( \Delta \) is the diagonal in \( T^*(Z \times Z) \). Then, the flat trace of \( A \) is defined as
\[
\text{tr}^\flat_A = \langle \nu_\Delta A, 1 \rangle,
\]
where \( \nu_\Delta : z \mapsto (z, z) \) is the diagonal inclusion and \( A \in \mathcal{D}'^m(Z \times Z) \) is the Schwartz kernel of \( A \), i.e.,
\[
\int Z A(u) \wedge v = \int Z \times Z A(\pi^*_1 u \wedge \pi^*_2 v), \ \ u, v \in \Omega_c^\bullet(Z),
\]
where \( \pi_j : Z \times Z \to Z \) is the projection on the \( j \)-th factor \( (j = 1, 2) \). In fact, we have
\[
\text{tr}^\flat_A = \sum_{k=0}^2 (-1)^{k+1} \text{tr}^\flat(A_k), \tag{2.3}
\]
where \( \text{tr}^\flat \) is the transversal trace of Atiyah–Bott [1] and where we denoted by \( A_k \) the operator \( C_c^\infty(Z, \wedge^k T^* Z) \to \mathcal{D}'(Z, \wedge^k T^* Z) \) induced by \( A \) on the space of \( k \)-forms. The purpose of this subsection is to prove the following result.

**Proposition 2.2.** For \( n \geq 1 \), the flat trace of \((\varphi T_\pm(s))^n\) is well defined and we have
\[
\text{tr}^\flat_A ((\varphi T_\pm(s))^n) = n \sum_{r(\gamma)=n} (-1)^{1+m(\gamma)} \frac{\tau^\sharp(\gamma)}{\tau(\gamma)} e^{-s r(\gamma)} \left( \prod_{z \in R(\gamma)} g^2(z) \right)^{\tau(\gamma)/\tau^\sharp(\gamma)} \tag{2.4}
\]
whenever \( \text{Re}(s) \gg 1 \), where the sum runs over all periodic trajectories \( \gamma \) rebounding \( n \) times on \( \partial D_0 \). Here \( \tau^\sharp(\gamma) \) is the primitive length of \( \gamma \) and
\[
R(\gamma) = \{ (\gamma(\tau), \dot{\gamma}(\tau)) : \tau \in \mathbb{R} \} \cap \pi^{-1}(\partial D_0)
\]
is the set of incidence vectors of \( \gamma \) along \( D_0 \).

**Corollary 2.** As \( s \mapsto (\varphi T_\pm(s))^n \) extends meromorphically to the whole complex plane, so does the right-hand side of (2.4).
Proof. For \( z \in Z \), we define the first (future and past) return times to \( \pi^{-1}(\partial D_0) \) by

\[
\tau_{\pm,0}(z) = \inf \{ t > 0 : \varphi'_{\pm,t}(z) \in \pi^{-1}(\partial D_0) \}.
\]

We set \( \Lambda_{\pm,0} = \{ z \in Z : \tau_{\pm,0}(z) < \infty \} \), and we define by \( B_{\pm,0} : Z \to \pi^{-1}(\partial D_0) \) the first (future and past) return maps to \( \pi^{-1}(\partial D_0) \) by

\[
B_{\pm,0}(z) = \varphi'_{\pm,0}(z), \quad z \in \Lambda_{\pm,0}.
\]

For \( n \geq 1 \), we define the sets \( \Lambda_{\pm,0}^{(n)} \subset Z \) by induction as follows. We set \( \Lambda_{\pm,0}^{(1)} = \Lambda_{\pm,0} \) and

\[
\Lambda_{\pm,0}^{(n+1)} = \left\{ z \in \Lambda_{\pm,0} : B_{\pm,0}(z) \in \Lambda_{\pm,0}^{(n)} \right\}, \quad n \geq 1.
\]

In particular, \((B_{\pm,0})^n(z)\) is well defined for \( z \in \Lambda_{\pm,0}^{(n)} \). We finally set

\[
t^{(n)}_{\pm,0}(z) = \sum_{k=0}^{n-1} \tau_{\pm,0} \left( (B_{\pm,0})^k(z) \right), \quad z \in \Lambda_{\pm,0}^{(n)},
\]

and \( t^{(n)}_{\pm,0}(z) = +\infty \) for \( z \in Z \setminus \Lambda_{\pm,0}^{(n)} \). We now fix \( n \geq 1 \). Let \( g \in C^\infty(\mathbb{R}, [0, 1]) \) such that \( g \equiv 1 \) on \( ]-\infty, 1[ \) and \( g \equiv 0 \) on \( [2, +\infty[ \). For \( L > 0 \), we define

\[
g_L(z) = g \left( \tau^{(n)}_{\pm,0}(z) - L \right), \quad z \in Z.
\]

Then, by definition of \( T_{\pm}(s) \), the operator \( g_L (\vartheta T_{\pm}(s)) : \Omega^\bullet_c(Z) \to \mathcal{D}'^\bullet(Z) \) coincides with the operator

\[
w \mapsto g_L(\cdot) \left( \prod_{k=0}^{n} \vartheta^2 ((B_{\pm,0})^k(\cdot)) \right) e^{-st^{(n)}_{\pm,0}(\cdot)} ((B_{\mp,0})^n)^* w.
\]

It now follows, from the Atiyah–Bott trace formula [1, Corollary 5.4] that\(^5\)

\[
\langle \tau_{\pm,0}^*(s), g_L \rangle = \sum_{\begin{subarray}{l} B_{\pm,0}(z) = z \cr z \in Z \end{subarray}} \text{sgn det}(1 - d(B_{\mp,0})^n(z)) e^{-t^{(n)}_{\pm,0}(z)} g_L(z) \left( \prod_{k=0}^{n-1} \vartheta^2 (B_{\mp,0}^k(z)) \right).
\]

(2.5)

Now it is not hard to see that

\[
\text{sgn det}(1 - d(B_{\mp,0})^n(z)) = \text{sgn det}(1 - p_\gamma) = \begin{cases} 1, & \text{if } m(\gamma) \text{ is odd}, \\ -1, & \text{if } m(\gamma) \text{ is even}, \end{cases}
\]

where \( \gamma \) is the closed orbit generated by \( z \). This yields \( \text{sgn det}(1 - d(B_{\mp,0})^n(z)) = (-1)^{1+m(\gamma)} \).

It is a classical fact that for every \( k, n \geq 1 \), there is \( C_k > 0 \) such that

\[
\|d^k ((B_{\pm,0})^n)(z)\| \leq C_k \exp \left( C_k t^{(n)}_{\pm,0}(z) \right), \quad z \in \Lambda_{\pm,0}^{(n)}.
\]

Thus, we may proceed exactly as in the proof of [5, Proposition 3.6] to take the limit in (2.5) when \( L \to +\infty \) to obtain (2.4).

\(^5\)See the proof of [5, Proposition 3.6] for more details.
3. A Tauberian Argument

In this section, we use a Tauberian theorem of Delange [8] to derive an asymptotic growth of a weighted sum of periodic trajectories rebounding a fixed number of times on \( \partial D_0 \). To that aim we wish to work with series having positive coefficients, and we first explain how Proposition 2.2 can be adapted to remove the sign \((-1)^{1+m(\gamma)}\). To that aim we wish to work with series having positive coefficients, and we first explain how Proposition 2.2 can be adapted to remove the sign \((-1)^{1+m(\gamma)}\).

3.1. Doubling the billiard table

Let

\[
\hat{M}' = (N' \times \{ -1, 1 \}) / \sim, \quad N' = S \mathbb{R}^2 \setminus (\pi^{-1}(D^\circ) \cup G'),
\]

where \( G' = T\partial D' \) and \( D' = \bigcup_{j=0}^r D_j \), and where \((x, v, a) \sim (y, w, b)\) if and only if, for some \( j \in \{0, \ldots, r\}\), it holds

\[
x = y \in \partial D_j, \quad w = v - 2\langle v, n_j(x) \rangle n_j(x) \quad \text{and} \quad a = -b.
\]

Let \( \hat{\pi} : \hat{M}' \to M' \) be the natural projection, which is a 2-fold covering, and denote by \( J : \hat{M}' \to \hat{M}' \) the involution induced by \((x, v, a) \mapsto (x, v, -a)\). Then there is a unique continuous flow \((\varphi'_t)\) acting on \( \hat{M}' \) such that \( \hat{\pi} \circ \varphi'_t = \varphi'_t \circ \hat{\pi} \).

Also, the flow \((\varphi'_t)\) is hyperbolic on \( \hat{\Lambda}' = \hat{\pi}^{-1}(\Lambda') \). Moreover, each periodic orbit \( \gamma : [0, \tau(\gamma)] \to \Lambda' \) of the flow \((\varphi'_t)\) with an even number of bounds on \( \partial D' \) gives rise to two periodic orbits of \((\varphi'_t)\) which are generated by the two points lying in \( \hat{\pi}^{-1}(\gamma(0)) \) and every periodic orbit of \((\varphi'_t)\) is obtained in this way.

Next, we define \( \tilde{R}_\pm(s), \tilde{S}_\pm(s), \tilde{Q}_\pm(s), \tilde{\chi}_\pm, \tilde{T}_\pm(s) \) and \( \hat{\varrho} \) in the same way we defined \( R_\pm(s), S_\pm(s), Q_\pm(s), \chi_\pm, T_\pm(s) \) and \( \varrho \), by using the flow \((\varphi'_t)\) instead of \((\varphi'_t)\). Clearly, Propositions 1.3 and 2.1 extend for those operators, if we replace \( Z \) by \( \hat{Z} = \hat{\pi}^{-1}(Z) \) and \( \hat{\partial}_\pm \) by \( \hat{\pi}^{-1}(\hat{\partial}_\pm) \). Moreover, thanks to the description of the periodic orbits of \((\varphi'_t)\) given above, we may redo the proof of Proposition 2.2 to obtain the formula

\[
\frac{1}{2} \text{tr}_s \left( (1 - J^*)(\hat{\varrho} \hat{T}_\pm(s))^n \right) = -n \sum_{\tau(\gamma) = n} \frac{\tau^\#(\gamma)}{\tau(\gamma)} e^{-s\tau(\gamma)} \left( \prod_{z \in R(\gamma)} \varrho^2(z) \right)^{\tau(\gamma)/\tau^\#(\gamma)},
\]

which is valid for \( \text{Re}(s) \gg 1 \), where the at trace is taken on \( Z \). Indeed, this follows from the fact that there is a 2 : 1 correspondence between fixed points of \( J \circ \varphi'_t \) and fixed points of \( \varphi'_t \) with an odd number of bounds on \( \partial D' \).

3.2. Zeta Functions

Let \( \mathcal{P}_B \) be the set of primitive periodic orbits of \((\varphi_t)\), for the billiard table \( B \). We define the Ruelle zeta function \( \zeta_B \) associated with the billiard flow \( B \) by

\[
\zeta_B(s) = \prod_{\gamma \in \mathcal{P}_B} \left( 1 - e^{-s\tau(\gamma)} \right)^{-1}, \quad s \in \mathbb{C},
\]
where the product converges whenever Re(s) is large enough. By [14, Theorem 1.3], there is \( h_B > 0 \) and \( c_B > 0 \) such that \( \zeta_B \) admits a meromorphic extension to the half plane \( \{ \text{Re}(s) > -c_B \} \); moreover, \( \zeta_B \) is analytic and nonvanishing on the line \( \{ \text{Re}(s) = h_B \} \) except for a simple pole at \( s = h_B \) (as it follows from [13, Remark 3.1] and [15, Proposition 9]); hence, \( \zeta_B/\zeta_B \) is analytic on \( \{ \text{Re}(s) = h_B \} \), except for a simple pole with residue \(-1\) at \( s = h_B \).

In what follows, we set \( \tilde{U} = \tilde{\pi}^{-1}(U) \)

\[
\Omega_0^k = \{ w \in \Omega_c^k(\tilde{U}) : \iota_X w = 0 \},
\]

where \( X \) is the generator of \( (\tilde{\varphi}_t') \) on \( \tilde{U} \). Then it follows from [9] (see also [2, §4]) that we may write, for \( \text{Re}(s) \) large enough, any \( \varepsilon > 0 \) small and \( \hat{\chi} \in C_c^\infty(\tilde{U}, [0, 1]) \) satisfying \( \hat{\chi} \equiv 1 \) on \( \Lambda \),

\[
\zeta'_B(s)/\zeta_B(s) = -\frac{1}{2} \sum_{k=0}^{2} (-1)^{k} e^{\tau \varepsilon s} \text{tr}^b \left( (1 - J^*) \hat{\chi} \varphi^\ast_{\tau \varepsilon} \hat{R}_\pm(s) \hat{\chi} |_{\Omega_0^k} \right); \tag{2.1}
\]

moreover, the residue of \( \zeta_B(s)/\zeta_B(s) \) at \( s = s_0 \) is given by

\[
-\frac{1}{2} \sum_{k=0}^{2} (-1)^{k} \text{tr}^b \left( (1 - J^*) \hat{\chi} \hat{\Pi}_\pm(h_B) \hat{\chi} |_{\Omega_0^k} \right). \tag{2.2}
\]

where \( \hat{\Pi}_\pm(s_0) \) is the residue of \( \hat{R}_\pm(s) \) at \( s = s_0 \) Next, we know that \( s \mapsto \hat{R}_\pm(s)|_{\Omega_0^k} \) is holomorphic on \( \{ \text{Re}(s) > 0 \} \), simply because the integral defining \( \hat{R}_\pm(s)|_{\Omega_0^k} \) converges absolutely in this region. This implies that

\[ \hat{\Pi}_\pm(s)|_{\Omega_0^2} = 0, \quad \text{Re}(s) > 0, \]

since the map \( u \mapsto u \wedge d\alpha \) realizes an isomorphism \( \text{ran} \hat{\Pi}_\pm(s_0)|_{\Omega_0^k} \rightarrow \text{ran} \hat{\Pi}_\pm(s_0)|_{\Omega_0^k} \), where we put \( \alpha = \pi^* \alpha \).

Finally, let \( \eta(s) = \sum_{\gamma \in \mathcal{P}_B} \tau^d(\gamma) e^{-s \tau(\gamma)} = \zeta'_B(s)/\zeta_B(s) \), where \( \mathcal{P}_B \) is the set of periodic trajectories of \( (\varphi_t) \). Also, let \( \eta_{\text{even}}(s) \) (resp. \( \eta_{\text{odd}}(s) \)) be the series defined similarly by summing over periodic \( \gamma \)'s with an even (resp. odd) number of bounces \( m(\gamma) \). Let \( \mathcal{P}^\text{even}_B \) (resp. \( \mathcal{P}^\text{odd}_B \)) be the set of primitive \( \gamma \in \mathcal{P}_B \) such that \( m(\gamma) \) is even (resp. odd). Using the symbolic coding and similar arguments to that used in the proof of Lemma 4.1 below, it is not hard to construct injective maps \( F_\pm : \mathcal{P}_B^\text{even/odd} \rightarrow \mathcal{P}_B^\text{odd/even} \) such that for some \( C > 0 \) it holds

\[ \tau(\gamma) - C \leq \tau(F_\pm(\gamma)) \leq \tau(\gamma) + C, \quad \gamma \in \mathcal{P}_B^\text{even/odd}. \]

This estimate implies that both \( \eta_{\text{even}}(s) \) and \( \eta_{\text{odd}}(s) \) have a simple pole at \( s = h_B \), since \( \eta(s) \) does and \( \eta(s) = \eta_{\text{even}}(s) + \eta_{\text{odd}}(s) \). Moreover, the residuals

\[ \text{In fact, } \zeta_B \text{ admits a meromorphic continuation to the whole complex plane by [12].} \]

\[ \text{Again, we use that the periodic orbits of } (\tilde{\varphi}_t) \text{ in } \tilde{\Lambda} = \pi^{-1}(\Lambda) \text{ are in 2:1 correspondence with the periodic orbits of } (\varphi_t) \text{ bouncing an even number of times on } \partial D, \text{ while the fixed points of } J \tilde{\varphi}_t \text{ are in 2:1 correspondence with fixed points of } \varphi_t \text{ bouncing an odd number of times on } \partial D. \]
of \( \eta_{\text{even}}(s) \) and \( \eta_{\text{odd}}(s) \) at \( s = h_B \) are given respectively by

\[
\frac{1}{2} \tr^b(\tilde{\chi} \Pi_\pm (h_B) \tilde{\chi}) \quad \text{and} \quad -\frac{1}{2} \tr^b(\tilde{\chi} J^* \Pi_\pm (h_B) \tilde{\chi}).
\]

The first one coincides with rank \( \Pi_\pm (h_B) \) (see for example [9]). Moreover, since \( \eta_{\text{even}}(s) \leq \eta(s) \), it follows that this number is equal to 1 or 2; however it cannot be equal to 2, because otherwise \( \eta_{\text{odd}}(s) \) would not have a pole at \( s = h_B \), since the residue of \( \eta(s) \) at \( s = h_B \) is equal to 1. Therefore

\[
\text{rank } \Pi_\pm (h_B) = 1,
\]

and hence both residues in Eq. (2.3) are equal to 1/2. Thus it follows that

\[
J^* \Pi_\pm (h_B) = -\Pi_\pm (h_B),
\]

since \( J^* \) preserves ran \( \Pi_\pm (h_B) \) and \( J^2 \).

### 3.3. A Tauberian Argument

Taking the notations of paragraphs 1.6, 2.1 and 2.2, we set

\[
A_\pm = -Q_\pm (h_B) \chi_\pm \iota_\pm \chi_\mp \Pi_\pm (h_B) (\iota_\mp) \chi_\mp Q_\mp (h_B)^T.
\]

where \( \iota_\pm : \hat{\gamma} \to \hat{U} \) is the inclusion. Then we have, as operators \( \Omega_c^* (\hat{Z}) \to D^*(\hat{Z}) \),

\[
(\varrho \Theta_\pm (s))^n = \frac{(A_\pm)^n}{(s - h_B)^n} + \mathcal{O}((s - h_B)^{-n+1}), \quad s \to h_B.
\]

Now note that

\[
\frac{1}{2} - \frac{J^*}{2} \hat{A}_\pm = \frac{1}{2} \hat{Q}_\pm (h_B) \hat{\chi}_\pm \iota_\pm \hat{\chi}_\mp (1 - J^*) \hat{\Pi}_\pm (h_B) (\iota_\mp) \hat{\chi}_\mp \hat{Q}_\mp (h_B)^T = \hat{A}_\pm
\]

where we used (2.5). As \( \hat{A}_\pm \) is of rank one by (2.4), we have

\[
\tr^b \left( (\hat{A}_\pm)^n |_{\Omega_B^\dagger} \right) = \tr^b (\hat{A}_\pm |_{\Omega_B^\dagger})^n
\]

Thus, letting \( c_\pm = \tr^b (A_\pm |_{\Omega_B^\dagger}) \), we have

\[
\frac{1}{2} \tr^b \left( (1 - J^*) (\varrho \Theta_\pm (s))^n \right) = -\frac{(c_\pm)^n}{(s - h_B)^n} + \mathcal{O}((s - h_B)^{-n+1}), \quad s \to h_B. \quad (2.6)
\]

Now we define

\[
N_\varrho(t, n) = \sum_{\gamma \in \mathcal{P}} I_\varrho(\gamma), \quad t \geq 0,
\]

where we set, for a closed trajectory \( \gamma : [0, \tau(\gamma)] \to M' \),

\[
I_\varrho(\gamma) = \prod_{\tau(\gamma) \leq t} g^2(\gamma) \quad \text{where} \quad R(\gamma) = \pi^{-1}(D_0) \cap \{ (\gamma(\tau), \dot{\gamma}(\tau)) : \tau \in [0, \tau(\gamma)] \}.
\]

Note that if \( r(\gamma) = n \) one has \( g R(\gamma) = n \tau(\gamma) / \tau^2(\gamma) \).
**Proposition 3.1.** Assume that \( c_{\pm} > 0 \). Then,

\[
N_\varrho(t, n) \sim \frac{(c_{\pm} t)^n}{n!} e^{h_B t}, \quad t \to +\infty.
\]

**Proof.** Here we follow the proof of [5, Lemma 5.1]. Define

\[
g_{n, \varrho}(t) = \sum_{\gamma \in \mathcal{P}} \tau(\gamma) \sum_{k \geq 1} I_\varrho(\gamma)^k, \quad t \geq 0.
\]

For \( \text{Re}(s) \) large enough, we set

\[
G_{n, \varrho}(s) = \int_0^\infty g_{n, \varrho}(t) e^{-ts} dt.
\]

Then, a simple computation starting from (3.1) shows that

\[
G_{n, \varrho}(s) = \frac{1}{s} \sum_{r(\gamma) = n} \tau(\gamma) I_\varrho(\gamma)^{\tau(\gamma)/\tau(\gamma)} e^{-s\tau(\gamma)} = \frac{\partial s \text{tr}^s (\hat{T}_\varrho^T(s))}{2ns},
\]

where the sum runs over all periodic orbits (not necessarily primitive) \( \gamma \) such that \( r(\gamma) = n \). By (2.6), we have

\[
G_{n, \varrho}(h_B s) = \frac{(c_{\pm})^n}{h_B^{n+2} (s - 1)^{n+1}} + \mathcal{O}((s - 1)^{-n}), \quad s \to 1.
\]

Because \( s = h_B \) is the only pole of \( G_{n, \varrho} \) on the line \( \{\text{Re}(s) = h_B\} \), we may apply a classical Tauberian theorem from Delange [8, Théorème III] to obtain

\[
\frac{1}{h_B} g_{n, \varrho}(t/h_B) \sim \frac{(c_{\pm})^n}{h_B^{n+2} n! t^n}, \quad t \to +\infty,
\]

which reads \( g_{n, \varrho}(t) \sim \frac{(c_{\pm} t)^n}{n! h_B} \exp(h_B t) \) as \( t \to +\infty \). Now note that

\[
g_{n, \varrho}(t) \leq \sum_{\gamma \in \mathcal{P}} \tau(\gamma) [t/\tau(\gamma)] I_\varrho(\gamma) \leq t N_\varrho(t)
\]

which gives \( \liminf_{t \to +\infty} \frac{N_\varrho(t)}{g_{n, \varrho}(t)/t} \geq 1 \). On the other hand, let

\[
\zeta_{n, \varrho}(s) = \prod_{\gamma \in \mathcal{P}} \left(1 - I_\varrho(\gamma) e^{-s\tau(\gamma)}\right)^{1}, \quad \text{Re}(s) \gg 1.
\]

Then, we have

\[
\zeta_{n, \varrho}(s) \geq \prod_{\gamma \in \mathcal{P}} \left(1 + I_\varrho(\gamma) e^{-s\tau(\gamma)}\right) \geq \prod_{\gamma \in \mathcal{P}} \left(1 + I_\varrho(\gamma) e^{-st}\right) \geq e^{-st} N_\varrho(t).
\]
As $\partial_s \log \zeta_{n,\varrho}(s) = -sG_{n,\varrho}(s)$, it follows that $\zeta_{n,\varrho}$ extends holomorphically on \{Re$(s) > h_B$\} (as $G_{n,\varrho}$ does). Let $\sigma > 1$, and $\varepsilon > 0$ such that $(h_B + \varepsilon)/\sigma < h_B$. Then, by (2.7) applied with $s = h_B + \varepsilon$ we have

$$N_{\varrho}(t/\sigma) \leq \zeta'_{n,\varrho}(h_B + \varepsilon) \exp \left(\frac{(h_B + \varepsilon)t}{\sigma}\right).$$

This implies that $N_{\varrho}(t/\sigma)/N_{\varrho}(t) \to 0$ as $t \to +\infty$. Now we write

$$g_{n,\varrho}(t) \geq \sum_{\gamma \in P} \tau(\gamma) I_{\varrho}(\gamma) \geq \sum_{\gamma \in P} \frac{t}{\sigma} I_{\varrho}(\gamma) = \frac{t}{\sigma} \left(N_{\varrho}(t) - N_{\varrho}(t/\sigma)\right).$$

This leads to

$$\limsup_{t \to +\infty} \frac{N_{\varrho}(t)}{g_{n,\varrho}(t)/t} \leq \sigma \limsup_{t \to +\infty} \left(1 - \frac{N_{\varrho}(t/\sigma)}{N_{\varrho}(t)}\right)^{-1} = \sigma.$$

As $\sigma > 1$ is arbitrary, the proof of the lemma is complete, since we have

$$g_{n,\varrho}(t)/t \sim \left(c_{\pm}t\right)^n \frac{e^{h_B t}}{n!}$$

as $t$ goes to infinity. \hfill \Box

4. A Priori Bounds

In this section, we derive some a priori bounds on $N(n, t)$ (the number of primitive periodic orbits bouncing $n$ times on $\partial D_0$ and of length not greater than $t$) by using the fact that the billiard flow is conjugated to a subshift of finite type. This will allow us to convert the asymptotics obtained in Sect. 3 into an asymptotics on $N(n, t)$.

4.1. Coding

Let $\Sigma'_N$ be the set of finite sequences $u = u_1 \cdots u_N$ with $u_j \in \{0, 1, \ldots, r\}$ and $u_j \neq u_{j+1}$ (with $j \in \mathbb{Z}/N\mathbb{Z}$) and such that $u$ is distinct from its cyclic permutations. We also define $\Sigma_N$ as above by replacing $\{0, 1, \ldots, r\}$ by $\{1, \ldots, r\}$. By Sect. 1.2, we have a one-to-one correspondence

$$\mathcal{P}_{B'} \longleftrightarrow \left(\bigcup_{N=2}^{\infty} \Sigma'_N\right) / \sim \quad (4.1)$$

where $u \sim v$ if and only if $u$ is a cyclic permutation of $v$. For any $\gamma \in \mathcal{P}_{B'}$, we will denote by $\text{wl}(\gamma)$ its word length, that is, the length of (any) word which is associated with $\gamma$ via the above correspondence.

For any sequence $u \in \Sigma'_N$, we will denote by $\gamma_u : \mathbb{R} \to \Lambda'$ the closed billiard trajectory (parametrized by arc length) starting from the point $z_u \in K'$ which is associated with the sequence

$$(\cdots uu u \cdots) \in \Sigma'.$$
Its period is then defined by
\[ \tau(\gamma_u) = \sum_{k=0}^{N-1} t'_+(B'^k(z_u)), \]
where \( t'_+ \) is defined in Sect. 1.2. We have the following result.

**Lemma 4.1.** There is \( C > 0 \) such that the following holds. Let \( \gamma : [0, T] \to \Lambda' \) be a billiard trajectory (parametrized by arc length) such that \( \gamma(0), \gamma(T) \in \pi^{-1}(\partial D_0) \) and denote by \( 0 = t_0 < \cdots < t_N = T \) the times for which \( \gamma \) hits \( \partial D \) and assume that \( N \geq 2 \). Let \( u = u_1 \cdots u_{N-1} \in \{0, \ldots, r\}^{N-1} \) be the finite sequence such that it holds \( \pi(\gamma(t_k)) \in \partial D_{u_k} \) for \( k = 1, \ldots, N-1 \), and assume that \( u_1 \neq u_{N-1} \) so that \( \gamma_u \) is well defined. Then,
\[ \tau(\gamma_u) - C \leq T \leq \tau(\gamma_u) + C. \]

**Proof.** By Lemma 1.2, it holds, for some \( C > 0 \) and \( \beta > 1 \) which are independent of \( \gamma \),
\[ \text{dist}(B'^k(z_u), \gamma(t_k)) \leq C\beta^{-N/2+|k-N/2|}, \quad k = 1, \ldots, N-1. \]
Now note that \( t'_+ : \{z \in \pi^{-1}(\partial D) : t'_+(z) < +\infty\} \to \mathbb{R}_+ \) is locally Lipschitz continuous. As \( K' \) is compact, it follows that for some \( C' > 0 \) we have
\[ |t'_+(B'^k(z_u)) - t'_+(\gamma(t_k))| \leq C'\beta^{-N/2+|k-N/2|} \]
and thus
\[ |\tau(\gamma_u) - T| \leq 2L_m + C' \sum_{k=1}^{N-1} \beta^{-N/2+|k-N/2|} \leq 2L_m + \frac{C'}{\beta - 1}, \]
where \( L_m = \sup\{\text{dist}(x_i, x_j) : x_i \in D_i, x_j \in D_j, i \neq j\} \). This concludes the proof. \( \square \)

### 4.2. The Bounds

Let \( \mathcal{P}_B \) be the set of oriented primitive periodic orbits of the flow associated with the billiard \( B \), and set \( \mathcal{P}_B(t) = \{\gamma \in \mathcal{P}_B : \tau(\gamma) \leq t\} \). Then, by [13] we have
\[ \sharp\{\gamma \in \mathcal{P}_B : \tau(\gamma) \leq t\} \sim \frac{e^{h_B t}}{n_B t}, \quad t \to +\infty. \quad (4.2) \]
In what follows, we will denote by \( \mathcal{P}_B(n, t) \) the set of primitive periodic trajectories of the billiard \( B' \) of period less than \( t \) which make exactly \( n \) rebounds on \( \partial D_0 \), and \( N(n, t) = \#\mathcal{P}_B(n, t) \). Finally, we denote by \( \tilde{\mathcal{P}}_B(t) \) (resp. \( \tilde{\mathcal{P}}_B'(n, t) \)) the set of (not necessarily primitive) periodic orbits for the billiard \( B \) (resp. for the billiard \( B' \)) of period less or equal than \( t \) (resp. and making \( n \) rebounds on \( \partial D_0 \) ; we denote \( \tilde{N}(t) = \#\tilde{\mathcal{P}}_B(t) \) and \( \tilde{N}(n, t) = \#\tilde{\mathcal{P}}_B'(n, t) \). It is a classical fact that we have
\[ \tilde{N}(t) \sim N(t), \quad t \to +\infty, \quad (4.3) \]
as it can be seen from the equalities
\[ \tilde{N}(t) = \sum_{\tau(\gamma) \leq t} 1 = \sum_{\gamma \in P} \sum_{k \gamma(\tau) \leq t} 1 = \sum_{\gamma \in P} 1 + \sum_{\tau(\gamma) \leq t/2} \lfloor t/\tau(\gamma) \rfloor, \]
and the fact that \( \sum_{\tau(\gamma) \leq t/2} \lfloor t/\tau(\gamma) \rfloor \ll N(t) \) as \( t \to +\infty \) by (4.2).

**Proposition 4.2.** For each \( n \geq 1 \), there is \( C_n > 0 \) such that if \( t \) is large enough we have
\[ C_n^{-1} t^n \exp(h_B t) \leq N(n, t) \leq C_n t^n \exp(h_B t). \]  

**Proof.** We start with the case \( n = 1 \). Consider the map \( F : \Sigma_N \to \Sigma_{N+1}^\prime \) defined by \( F(u_1 \cdots u_N) = 0 u_1 \cdots u_N \) (note that for any word \( u \in \Sigma_N \), \( F(u) \) is still a primitive word as it contains exactly one zero in its letters). By Lemma 4.1, we have
\[ \tau(\gamma_u) - C \leq \tau(\gamma_{F(u)}) \leq \tau(\gamma_u) + C, \quad u \in \Sigma_N. \]

The map \( F \) is obviously injective. Recalling the correspondance (4.1) (for both billiards \( B \) and \( B^\prime \)), we thus have
\[ N(1, t) \geq \sum_{N = 2}^\infty \sum_{u \in \tilde{\Sigma}_N} 1 = \sum_{\gamma \in P_B(t-C)} \text{wl}(\gamma), \]
where the last equality comes from the fact that each \( \gamma \in P_B \) corresponds to exactly \( \text{wl}(\gamma) \) words in \( \Sigma \). Note that for some \( C > 0 \) it holds
\[ C^{-1} \tau(\gamma) \leq \text{wl}(\gamma) \leq C \tau(\gamma), \quad \gamma \in P_B. \]  

In particular, we obtain
\[ N(1, t) \geq \frac{t}{2C^2} \#(P_B(t) \setminus P_B(t/2)). \]

By (4.2), we obtain that the first inequality of (4.4) holds for \( n = 1 \). For the second one, consider the set \( \tilde{\Sigma}_N \) of finite words \( u_1 \cdots u_N \) with \( u_j \neq u_{j+1} \) for \( j \in \mathbb{Z}/N\mathbb{Z} \) (note that \( \Sigma_N \subset \tilde{\Sigma}_N \) is the set of primitive words within \( \tilde{\Sigma}_N \)). Consider the map \( G : \tilde{\Sigma}_N \to \Sigma_{N+2}^\prime \) defined by
\[ G(u_1 \cdots u_N) = 0 u_1 \cdots u_N u_1, \quad u_1 \cdots u_N \in \tilde{\Sigma}_N. \]

Every primitive periodic orbit bouncing exactly one time on \( \partial D_0 \) can be encoded by a finite word of the form \( F(u) \) or \( G(u) \) for some \( u \in \tilde{\Sigma}_N \) where \( N \geq 2 \) (note that \( F \) extends to a map \( F : \tilde{\Sigma}_N \to \Sigma_{N+1}^\prime \)). In particular, by Lemma 4.1, we have for some \( C > 0 \)
\[ P(1, t) \subset \bigcup_N \left[ F \left( \left\{ u \in \tilde{\Sigma}_N : \tau(\gamma_u) \leq t + C \right\} \right) \cup G \left( \left\{ u \in \tilde{\Sigma}_N : \tau(\gamma_u) \leq t + C \right\} \right) \right]. \]

With (4.5) in mind, this leads to
\[ N(1, t) \leq 2 \sum_{N = 2}^\infty \sum_{\gamma \in P_B} \text{wl}(\gamma) \leq 2(t + C) \tilde{N}(t + C) \leq C \exp(h_B t), \]
where the last inequality holds for $t$ large enough and comes from (4.3). The case $n = 1$ is proved.

We now proceed by induction and assume that (4.4) holds for every $n = 1, \ldots, m$, for some $m \geq 1$. Similarly to (4.3), the estimate (4.4) also holds if we replace $N(n, t)$ by $\tilde{N}(n, t)$. Every element of $P_{B'}(m + 1, t)$ can be represented by the concatenation of a word (starting from 0) representing an element of $\tilde{P}_B(m, t_1)$ and a word (starting from 0) representing an element of $\tilde{P}_B(1, t_2)$, where $t_1 + t_2 \leq t + 2C$ (for some constant $C$). More precisely, for $N, k \geq 1$, set

$$A(k) = \left\{ u_1 \cdots u_N \in \tilde{\Sigma}_N' : N \geq 2, u_1 = 0, u_N \neq 0, \#\{j : u_j = 0\} = k \right\}.$$

Then every element $\gamma$ of $P_{B'}(m + 1, t)$ can be represented by a word $uv$ (i.e., $\gamma = \gamma uv$) where $u \in A(m)$ and $v \in A(1)$. Moreover, by Lemma 4.1, we must have

$$\tau(\gamma) - 2C \leq \tau(\gamma_u) + \tau(\gamma_v) \leq \tau(\gamma) + 2C \quad (4.6)$$

for some $C$ which does not depend of $\gamma$. Note also that for each periodic trajectory making $k$ rebounds on $\partial D_0$, there are at most $k$ words in $A(k)$ representing it (since the words have to start by the letter 0). Summarizing the above facts, we have for $t$ large enough (in what follows $C$ is a constant depending only on $m$ that may change at each line)

$$\tilde{N}(m + 1, t) \leq \sum_{u \in A(m)} \sum_{v \in A(1)} 1_{\tau(\gamma_u) \leq t + C, \tau(\gamma_v) \leq t - \tau(\gamma_u) + C}$$

$$\leq \sum_{u \in A(m)} \tilde{N}(1, t - \tau(\gamma_u) + C)$$

$$\leq \sum_{u \in A(m)} C \exp(h_B(t - \tau(\gamma_u) + C))$$

$$\leq \sum_{k=1}^{t+C} m \tilde{N}(m, k) C \exp(h_B(t - k + C))$$

$$\leq C \sum_{k=1}^{t+C} k^{m-1} \exp(h_B k) \exp(h_B(t - k + C))$$

$$\leq C t^m \exp(h_B t),$$

where we used $\tilde{N}(m, t) \leq C t^{m-1} \exp(h_B)$ as it follows from the induction hypothesis. For the lower bound, we proceed as follows. The map $A(m) \times A(1) \to A(m + 1)$ defined by $(u, v) \mapsto uv$ is injective; moreover, every element of $P_{B'}(m + 1, t)$ is represented by exactly $m + 1$ elements of $A(m + 1)$. By
(4.6), we have
\[
\tilde{N}(m+1,t) \geq \frac{1}{m+1} \sum_{u \in \mathcal{A}(m)} \sum_{v \in \mathcal{A}(1)} 1. \\
\]
Let \( T > 0 \) large enough (it will be chosen later). By similar computations as above, we have
\[
\tilde{N}(m+1,t) \geq C \left( \sum_{k=1}^{(t-C)/T} \right) \left( \tilde{N}(m,(k+1)T) - \tilde{N}(m,kT) \right) \exp(h_B(t - (k+1)T - C)).
\]

(4.7)

If \( k \) is large enough, we have by the induction hypothesis
\[
\tilde{N}(m,(k+1)T) - \tilde{N}(m,kT) \\
\geq C^{-1}[(k+1)T]^{m-1}e^{h_B(k+1)T} - C_m[kT]^{m-1}e^{h_BkT} \\
\geq (kT)^{m-1}e^{h_BkT} \left( C^{-1} \left( 1 + \frac{1}{k} \right)^{m-1} e^{h_BT} - C_m \right).
\]

If \( T \) is large enough the last term of the above equation is bounded from below by \( C(kT)^{m-1}e^{h_BkT} \) for some \( C > 0 \) independent of \( k \). Injecting this in (4.7), we obtain
\[
\tilde{N}(m+1,t) \geq C \sum_{k=1}^{(t-C)/T} (kT)^{m-1} \exp(h_BkT) \exp(h_B(t - (k+1)T - C)) \\
\geq Ct^m \exp(h_Bt).
\]

Thus, we proved that (4.4) holds for \( \tilde{N}(m+1,t) \). We now show that this also holds for \( N(m+1,t) \), as follows. Because of Lemma 4.1 and the fact that any nonprimitive word in \( A(m+1) \) can be written as the concatenation of \( (m+1)/d \) identical words (where \( d < m+1 \) is a divisor of \( m+1 \)), we have, for \( t \) large enough,
\[
\tilde{N}(m+1,t) - N(m+1,t) \leq \sum_{d \mid m+1} \tilde{N} \left( d, \frac{td}{m+1} + C \right) \\
\leq C \sum_{d \mid m+1} \left( \frac{td}{m+1} \right)^{d-1} \exp \left( h_B \left( \frac{td}{m+1} + C \right) \right),
\]
where the sums run over the divisors \( d \) of \( m+1 \) which are strictly less than \( m+1 \). In particular, we have \( \tilde{N}(m+1,t) - N(m+1,t) \leq t^{(m+1)/2} \exp(h_B t/2) \) for \( t \) large, and thus, \( N(m+1,t) \) also satisfies (4.4). This concludes the proof. \( \square \)
5. Proof of the Main Result

In this section we prove the estimate announced in the introduction. In fact, we will prove that $N_\varrho(n, t) \sim N(n, t)$ as $t \to +\infty$, which will imply the sought result.

5.1. First Considerations

If $\gamma : \mathbb{R}/(\gamma)\mathbb{Z} \to \Lambda'$ is a periodic orbit rebounding exactly $n$ times on $\partial D_0$, we denote $I_1(\gamma), \ldots, I_n(\gamma) \subset \mathbb{R}/(\gamma)\mathbb{Z}$ the cyclically ordered sequence of intervals satisfying $\gamma(I_j^\circ) \notin \partial D_0$ for each $j$, where $I_j^\circ$ denotes the interior of $I_j$ (this sequence is unique modulo cyclic permutations). We start by the following easy result.

**Lemma 5.1.** There is $t_0 > 0$ such that the following holds. For every $\gamma \in \tilde{\mathcal{P}}_{B'}$ such that $\ell(I_j(\gamma)) \geq t_0$, $j = 1, \ldots, n$, we have $I_\varrho(\gamma) = 1$.

**Proof.** Let $\gamma$ as above (for some large $t_0 > 0$ which will be chosen later) and $z \in R(\gamma)$ (see Sect. 2.3). Let $z_\pm = B'_\pm(z)$. Then $z_\pm \in \Lambda^{(m)}_\pm$, where $m = m(t_0) \to +\infty$ as $t_0 \to +\infty$. Here we set

$$\Lambda^{(m)}_\pm = \{z \in M : \not\exists T(z) \geq m\}.$$

In particular, by the proof of Lemma 4.1 we have $\text{dist}(z_\pm, \Gamma_\pm) \leq C\beta^m$. Thus, if $t_0 > 0$ is big enough, we have $\chi_\pm(z_\pm) = 1$ since $\chi_\pm \equiv 1$ on $\Gamma_\pm$. As a consequence, $\varrho(z) = 1$, also by definition of $\varrho$. Finally, we have $I_\varrho(\gamma) = \Pi_{z \in R(\gamma)} \varrho(z)^2 = 1$. \hfill $\square$

For any $t_0 > 0$, we will denote $\tilde{N}(n, t_0, t) = \#\tilde{\mathcal{P}}_{B'}(n, t_0, t)$ where

$$\tilde{\mathcal{P}}_{B'}(n, t_0, t) = \{\gamma \in \tilde{\mathcal{P}}_{B'} : r(\gamma) = n, \ell(I_j(\gamma)) \leq t_0 \text{ for some } 1 \leq j \leq n\}.$$

**Lemma 5.2.** Let $t_0 > 0$ and $n \geq 2$. Then, for some $C > 0$ we have for $t$ large enough

$$\tilde{N}(n, t_0, t) \leq Ct^{n-2} \exp(h_{B'}t).$$

**Proof.** By Lemma 4.1, there is $C > 0$ such that the following holds. Every trajectory $\gamma \in \tilde{\mathcal{P}}_{B'}(n, t_0, t)$ can be represented by a word in $\tilde{\mathcal{S}}'_{N'}$ obtained by the concatenation of two words $u \in A(n - 1)$ and $v \in A(1)$ satisfying

$$\tau(\gamma_u) \leq t + C, \quad \tau(\gamma_v) \leq t_0 + C.$$

Now for $t$ large enough one has

$$\#\{u \in A(n - 1) : \tau(\gamma_u) \leq t + C\} \leq (n - 1)(t + C)^{n-2} \exp(h_{B'}t)$$

by Proposition 4.2. As $\{v \in A(1) : \tau(\gamma_v) \leq t_0 + C\}$ is finite, the lemma is proved. \hfill $\square$
5.2. Proof of Theorem 1

First, we note that the constants $c_{\pm}$ given in Sect. 3.3 is positive. Indeed, if $c_{\pm} = 0$, then $s \mapsto \text{tr}_t^{\sharp}(\varrho T_{\pm}(s))$ would be regular at $s = h_B$ by the proof of Proposition 3.1. In particular, we would have

$$N_{\varrho}(1, t) \ll \exp(h_B t), \quad t \to \infty.$$  

However, by Lemma 5.1, we have $I_{\varrho}(\gamma) = 1$ whenever $\tau(\gamma)$ is large enough and $r(\gamma) = 1$, which gives $N_{\varrho}(1, t) \sim N(1, t)$ as $t \to \infty$. Now $N(1, t) \geq C \exp(h_B t)$ for large $t$ by Proposition 4.2, which contradicts the fact that $N_{\varrho}(1, t) \ll \exp(h_B t)$. Thus, $c_{\pm} > 0$. By Lemmas 5.1 and 5.2, we have

$$N(n, t) - N_{\varrho}(n, t) \leq N(n, t, t_0) \leq C t^{n-2} \exp(h_B t).$$

Thus, by Propositions 3.1 and 4.2, we obtain $N_{\varrho}(n, t) \sim N(n, t)$ as $t \to \infty$, which reads

$$N(n, t) \sim \frac{(c_{\pm} t)^n e^{h_B t}}{n! h_B t}, \quad t \to \infty.$$  

This concludes the proof of Theorem 1.

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References

[1] Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes: I. Ann. Math. 86, 374–407 (1967)
[2] Barkhofen, S., Schütte, P., Weich, T.: Meromorphic continuation of weighted zeta functions on open hyperbolic systems. arXiv preprint arXiv:2112.05791, (2021)
[3] Bowen, R., Ruelle, D.: The ergodic theory of axiom a flows. In: The Theory of Chaotic Attractors, pp. 55–76. Springer (1975)
[4] Conley, C., Easton, R.: Isolated invariant sets and isolating blocks. Trans. Am. Math. Soc. 158(1), 35–61 (1971)

[5] Chaubet, Y.: Closed geodesics with prescribed intersection numbers. arXiv preprint arXiv:2103.16301 (2021)

[6] Chernov, N., Markarian, R.: Chaotic billiards. Number 127. American Mathematical Soc. (2006)

[7] Chaubet, Y., Petkov, V.: Dynamical zeta functions for billiards. arXiv preprint arXiv:2201.00683 (2022)

[8] Delange, H.: Généralisation du théorème de Ikehara. In: Annales scientifiques de l’École Normale Supérieure, vol. 71, pp. 213–242 (1954)

[9] Dyatlov, S., Guillarmou, C.: Pollicott–Ruelle resonances for open systems. In: Annales Henri Poincaré, vol. 17, pp. 3089–3146. Springer (2016)

[10] Guillarmou, C., Mazzucchelli, M., Tzou, L.: Boundary and lens rigidity for non-convex manifolds. arXiv preprint arXiv:1711.10059 (2017)

[11] Hörmander, L.: The Analysis of Linear Partial Differential Operators: Distribution Theory and Fourier Analysis. Springer, Berlin (1990)

[12] Küster, B., Schütte, P., Weich, T.: Ruelle resonances and weighted zeta functions for obstacle scattering. To appear on arXiv

[13] Morita, T.: The symbolic representation of billiards without boundary condition. Trans. Am. Math. Soc. 325(2), 819–828 (1991)

[14] Morita, T.: Meromorphic extensions of a class of zeta functions for two-dimensional billiards without eclipse. Tohoku Math. J. Second Ser. 59(2), 167–202 (2007)

[15] Parry, W., Pollicott, M.: An analogue of the prime number theorem for closed orbits of axiom a flows. Ann. Math. 118, 573–591 (1983)

[16] Petkov, V., Stoyanov, L.: Geometry of Reflecting Rays and Inverse Spectral Problems. Pure and Applied Mathematics. Wiley, New York (1992)

[17] Petkov, V., Stoyanov, L.: Distribution of periods of closed trajectories in exponentially shrinking intervals. Commun. Math. Phys. 310(3), 675–704 (2012)

[18] Stoyanov, L.: Non-integrability of open billiard flows and Dolgopyat-type estimates. Ergod. Theory Dyn. Syst. 32(1), 295–313 (2012)

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