Exact approaches to charged particle motion in a time-dependent flux-driven ring

Pi-Gang Luan\textsuperscript{1} and Chii-Shung Tang\textsuperscript{2}

\textsuperscript{1}Institute of Optical Sciences, National Central University, Chung-Li 32054, Taiwan
\textsuperscript{2}Physics Division, National Center for Theoretical Sciences, P.O. Box 2-131, Hsinchu 30013, Taiwan

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We consider a charged particle which is driven by a time-dependent flux threading a circular ring system. Various approaches including classical treatment, Fourier expansion method, time-evolution method, and Lewis-Riesenfeld method are used and compared to solve the time-dependent problem. By properly managing the boundary condition of the system, a time-dependent wave function of the charged particle can be obtained by using a non-Hermitian time-dependent invariant, which is specific to the linear combination of initial angular-momentum and azimuthal-angle operators. The eigenfunction of the linear invariant can be realized as a Gaussian-type wave packet with a peak moving along the classical angular trajectory, while the distribution of the wave packet is determined by the ratio of the coefficient of the initial angle to that of the initial canonical angular momentum. In this topologically nontrivial system, we find that although the classical trajectory and angular momentum can determine the motion of the wave packet; however, the peak position is no longer an expectation value of the angle operator. Therefore, in such a system, the Ehrenfest theorem is not directly applicable.

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I. INTRODUCTION

Charged particle driven by a time-dependent perturbation in a quantum system is a nontrivial fundamental issue.\cite{1,2,3,4,5,6}. One can access the charged particle wave function by placing it in a quantum ring threaded by a time-dependent magnetic flux. The vector potential $A(t)$ times the charge $q$ leading to a phase shift proportional to the number of flux quanta penetrating the ring, is known as Aharonov-Bohm (AB) effect.\cite{7,8,9,10}. In adiabatic cyclic evolution, Berry\cite{11} first discovered that there exists a geometric phase. Later on, Aharonov and Anandan (AA)\cite{12,13} removed the adiabatic restriction to explore the geometric phase for any cyclic evolution.\cite{14}.

Time-dependent fields are also used to deal with field driven Zener tunneling, in which nonadiabaticity plays a crucial role.\cite{15,16,17}.\cite{18,19}.

In mesoscopic systems, a number of manifestations of the AB effect have been predicted and verified experimentally.\cite{20,21} On the other hand, Stern demonstrated that the Berry phase affects the particle motion in the ring similar to the AB effect, and a time-dependent Berry phase induces a motive force.\cite{22} Recently, AA phases have been found to play the role of classical canonical actions and are conserved in the adiabatic evolution of non-eigenstates.\cite{23}.

In the present work, we consider a noninteracting spinless charged particle moving cyclically in a quantum ring in the presence of a time-dependent vector potential. Such a particle motion can be described by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(t)\psi,$$

where the Hamiltonian $\hat{H}(t)$ is induced by an external time-dependent vector potential $A(t)$, given by

$$\hat{H}(t) = \frac{1}{2m} \left[ \hat{P} - qA(t) \right]^2 = \frac{1}{2\hbar} \left[ \hat{L} - qRA(t) \right]^2. \quad (2)$$

Here $\hat{P} = e_\theta \hat{P}_\theta$ is the canonical momentum operator with $e_\theta$ being the unit vector along the azimuthal angle $\theta$; $\hat{L} = \hat{L}_z = (\hat{r} \times \hat{\mathbf{P}})_z$ is the canonical angular momentum operator in the $z$ direction; $I = mR^2$ is the moment of inertia of the particle; $A(t) = A(t)e_\theta$ is the vector potential; and $R$ is the radius of the circular ring. This time-dependent dynamical problem can be solved by taking into account the Fourier expansion, time evolution operator, and Lewis-Riesenfeld (LR) method.\cite{24,25,26,27}.

II. A CLASSICAL TREATMENT

We first analyze the time-dependent problem in a classical manner. The time-varying magnetic flux induces an electric field $\mathbf{E} = Ee_\theta$ such that $E = -\partial A/\partial t$. The charged particle thus obtain a kinetic momentum increment during the time interval from 0 to $t$, namely

$$\Delta p_e = (mv) = m[v(t) - v(0)] = -q[A(t) - A(0)], \quad (3)$$

where $p_e = mv$ is the kinetic momentum. It should be noted that both $p_e$ and $qA$ are not conservative quantities, while from Eq. \(3\) we see that the canonical momentum $P_e$ is a constant of motion:

$$P_e(t) = mv(t) + qA(t) = mv(0) + qA(0) = P_e(0). \quad (4)$$

Comparing the two identities in Eq. \(3\), we see that the result of Eq. \(3\) is equivalent to

$$\Delta \omega = I[\omega(t) - \omega(0)] = -\frac{q}{2\pi} [\Phi(t) - \Phi(0)], \quad (5)$$

$$\Delta \mathbf{P}_\theta = I[\mathbf{P}_\theta(t) - \mathbf{P}_\theta(0)] = -\frac{q}{2\pi} [\Phi(t) - \Phi(0)], \quad (6)$$

$$\Delta \mathbf{P} = \hat{P} = \hat{L} = \hat{L}_z = \hat{r} \times \hat{P} = -i\hbar \left[ \hat{L}, \hat{P} \right] = \hat{L} \times \hat{P} = \hat{L} \times \hat{r} \times \hat{P} = -qA(t) \hat{r} \times \hat{P} = -qA(t) \mathbf{E} = -qA(t) \mathbf{E}.$$
where \( l_c = (\mathbf{r} \times \mathbf{p}_c)_z = I \omega \) indicates the kinetic angular momentum, \( \omega \) is the angular velocity, \( \Phi \) is the magnetic flux threading the ring, and the fact
\[
\Phi(t) = 2\pi RA(t)
\]
has been used. Also, from Eq. \( \text{(4)} \) we have
\[
L_c(t) = l_c(t) + \frac{q}{2\pi} \Phi(t) = l_c(0) + \frac{q}{2\pi} \Phi(0) = L_c(0).
\]
Thus the canonical angular momentum \( L_c = (\mathbf{r} \times \mathbf{p}_c)_z \) is also a constant of motion.

Now we define the writhing number as
\[
n_{\Phi}(t) \equiv \frac{\Phi(t)}{\Phi_0},
\]
where \( \Phi_0 = h/q \) is a flux quantum. We also define
\[
L_c \equiv n_0 \hbar, \quad l_c(t) \equiv n_c(t) \hbar,
\]
then we have
\[
n_c(t) = n_0 - n_{\Phi}(t).
\]
All of these \( n \)'s are real numbers.

Now the angular position of the driven particle is given by
\[
\theta_c(t) = \theta_0 + \int_0^t \frac{n_c(\tau) \hbar}{I} d\tau = \theta_0 + \omega_0 t - \int_0^t n_{\Phi}(\tau) \hbar \frac{1}{I} d\tau.
\]
Here \( \theta_0 \) indicates the initial azimuthal angle; and \( \omega_0 = \omega(0) = n_0 \hbar/I \) stands for the initial angular velocity. Below we denote the initial kinetic angular momentum \( l_0 \equiv l_c(0) \) for simplicity.

### III. A FOURIER EXPANSION METHOD

The simplest method for solving the time-dependent flux-driven problem is the Fourier expansion method. The first thing about the system we discuss is that the wave function satisfies the periodic boundary condition:
\[
\psi(\theta, t) = \psi(\theta + 2\pi, t).
\]
The most general form of \( \psi \) for the present problem is thus written as
\[
\psi(\theta, t) = \sum_{n=-\infty}^{\infty} c_n f_n(t) e^{i n \theta},
\]
where the \( c_n \)'s are appropriate coefficients to be determined by the initial and the boundary conditions.

Substituting Eq. \( \text{(13)} \) into Eq. \( \text{(1)} \), we can find
\[
\sum_{n=-\infty}^{\infty} i \hbar c_n \dot{f}_n(t) e^{i n \theta} = \sum_{n=-\infty}^{\infty} c_n f_n(t) \left( \frac{\hbar n - q RA(t)}{2I} \right) e^{i n \theta}.
\]
Solving Eq. \( \text{(14)} \), after some procedures we obtain
\[
f_n(t) = \exp \left\{ -\frac{i}{2I} \int_0^t [n \hbar - q RA(t)]^2 dt \right\},
\]
and thus
\[
\psi(\theta, t) = \sum_{n=-\infty}^{\infty} c_n \exp \left\{ -\frac{i}{2I} \int_0^t [n - n_{\Phi}(\tau)]^2 d\tau + i n \theta \right\}.
\]
As an simple example, let us choose
\[
c_n = N \exp \left\{ -\sigma^2 (n - n_0)^2 - i \theta_0 (n - n_0) \right\},
\]
where \( N \) indicates an appropriate normalization constant; \( \sigma, \theta_0 \) and \( n_0 \) are real numbers, and \( n \) is an integer.

Substituting Eq. \( \text{(17)} \) into Eq. \( \text{(16)} \), we get
\[
\psi(\theta, t) = N \exp \left\{ -\frac{i}{\hbar} \int_0^t \frac{\dot{f}_c(\tau)}{2I} d\tau + in_0 \theta_c(t) \right\} \times \sum_{n=-\infty}^{\infty} \exp \left\{ -\sigma^2 \left( 1 + \frac{i t}{T} \right) (n - n_0)^2 + in \left[ \theta - \theta_c(t) \right] \right\},
\]
where \( T = 2I\sigma^2/\hbar \). Applying the Poisson summation formula
\[
\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x) e^{i 2\pi n x} dx \right)
\]
on the function
\[
f(x) = \exp \left\{ -\sigma^2 \left( 1 + \frac{i t}{T} \right) (x - n_0)^2 + i (\theta - \theta_c(t)) x \right\},
\]
we can obtain an alternative expression
\[
\psi(\theta, t) = N \sqrt{\frac{\pi}{\sigma^2(1 + \frac{t^2}{T})}} \exp \left\{ -\frac{i}{\hbar} \int_0^t \frac{\dot{f}_c(\tau)}{2I} d\tau \right\} \times \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{(\theta - \theta_c(t) + 2n\pi)^2}{4\sigma^2 \left( 1 + \frac{t^2}{T} \right)} + in_0 (\theta + 2n\pi) \right\}.
\]

### IV. A TIME EVOLUTION METHOD

In this section, we shall present how to get the general solution shown in the previous section in terms of the
time evolution operator $\hat{U}(t)$. The state $|\psi(t)\rangle$ is connected with the initial state $|\psi(0)\rangle$ through

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle,$$  \hfill (22)

and the wave function $\psi(\theta, t)$ is given by

$$\psi(\theta, t) = \langle \theta | \hat{U}(t) | \psi(0) \rangle,$$  \hfill (23)

where $|\theta\rangle$ is the $\theta$-eigenket in the Schrödinger picture, which will be explained later. 

To begin with, we introduce the canonical commutator

$$[\hat{\theta}(0), \hat{L}(0)] = i\hbar.$$  \hfill (24)

From this identity, we have

$$[\hat{\theta}(t), \hat{L}(t)] = \hat{U}^\dagger(t)[\hat{\theta}(0), \hat{L}(0)]\hat{U}(t) = i\hbar.$$  \hfill (25)

Utilizing Eq. (24), we can derive

$$\frac{d\hat{L}(t)}{dt} = \frac{[\hat{L}(t), \hat{H}(t)]}{i\hbar} = 0,$$  \hfill (26)

we thus obtain the identity $\hat{L}(t) = \hat{L}(0)$. Following similar procedure it is easy to obtain

$$\frac{d\hat{\theta}(t)}{dt} = \frac{[\hat{\theta}(t), \hat{H}(t)]}{i\hbar} = \frac{\hat{L}(0) - n_\Phi(t)\hbar}{I},$$  \hfill (27)

which gives us

$$\hat{\theta}(t) = \hat{\theta}(0) + \frac{\hat{L}(0)t}{I} - \int_0^t \frac{n_\Phi(\tau)\hbar}{I} d\tau.$$  \hfill (28)

We can see that the canonical angular momentum is a constant of motion. This is consistent with the classical results discussed in Sec. II.

From the above results we have

$$[\hat{H}(t), \hat{H}(t')]=0$$  \hfill (29)

for any two times $t$ and $t'$. Hence the time evolution operator is simply given by

$$\hat{U}(t) = \exp \left[ -\frac{i}{\hbar} \int_0^t \hat{H}(\tau)d\tau \right]$$

$$= \exp \left[ -\frac{\hbar}{2I} \int_0^t \left( \frac{\hat{L}(0)}{\hbar} - n_\Phi(\tau) \right)^2 d\tau \right].$$  \hfill (30)

To proceed further, we define $|n\rangle$ as the eignket of $\hat{L}(0)$ obeying

$$\hat{L}(0)|n\rangle = n\hbar|n\rangle.$$  \hfill (31)

In addition, we also assume that $|\theta\rangle$ is an eignket of $e^{i\theta(0)}$ obeying

$$e^{i\theta(0)}|\theta\rangle = e^{i\theta}|\theta\rangle.$$  \hfill (32)

The orthogonal conditions of the two eignets are given by

$$\langle m|n\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \delta_{mn}$$  \hfill (33)

and

$$\langle \theta|\theta'\rangle = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')} = \delta(\theta - \theta'),$$  \hfill (34)

which can be easily derived from the closure relations

$$\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = 1, \quad \int_0^{2\pi} d\theta |\theta\rangle\langle \theta| = 1$$  \hfill (35)

and taking into account the definition

$$\langle \theta|n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta} = \langle n|\theta\rangle^*.$$  \hfill (36)

We note in passing that both $\theta$ and $\theta'$ are defined in the interval $[0, 2\pi)$. In the coordinate representation, the $e^{i\theta\theta}$ is an eignfunction of $\hat{L}_{\text{rep}} = -\hbar \partial / \partial \theta$ with corresponding eignvalue $\hbar n$. This result can be expressed as

$$\langle \theta|\hat{L}(0)|n\rangle = \hat{L}_{\text{rep}}(\theta)|n\rangle = n\hbar|\theta\rangle.$$  \hfill (37)

The wave function $\psi$ now can be calculated:

$$\psi(\theta, t) = \sum_{n=-\infty}^{\infty} \langle \theta|\hat{U}(t)|n\rangle \langle n|\psi(0)\rangle$$

$$= \sum_{n=-\infty}^{\infty} \frac{\langle n|\psi(0)\rangle}{\sqrt{2\pi}} e^{-\frac{\hbar}{2I} \int_0^t \left[ n - n_\Phi(\tau) \right] d\tau + in\theta}.$$  \hfill (38)

If we define

$$c_n = \frac{\langle n|\psi(0)\rangle}{\sqrt{2\pi}}$$  \hfill (39)

then the result of Eq. (38) becomes that of Eq. (16).

V. A LEWIS-RIESENFELD METHOD

In this section, we briefly review the LR method and then apply it to solve the present problem. We shall show that LR method is not directly applicable, however, a simple modification concerning about the boundary condition makes it applicable to solving the problems with periodic boundary condition.

Traditionally, to utilize the LR method [23] solving a time-dependent system, we have to find an operator $\hat{Q}(t)$ such that

$$i\hbar \frac{d\hat{Q}}{dt} = i\hbar \frac{\partial \hat{Q}}{\partial t} + [\hat{Q}, \hat{H}] = 0,$$  \hfill (40)
and then find its eigenfunction \( \varphi_\lambda(\theta, t) \) satisfying
\[
\hat{Q}(t) \varphi_\lambda(\theta, t) = \lambda \varphi_\lambda(\theta, t),
\]
with \( \lambda \) being the corresponding eigenvalue. A wave function \( \psi_\lambda(\theta, t) \) satisfying Eq. 11 is then obtained via the relation
\[
\psi_\lambda(\theta, t) = e^{i\alpha_\lambda(t)} \varphi_\lambda(\theta, t),
\]
where \( \alpha(t) \) is a function of time only, satisfying
\[
\dot{\alpha}_\lambda = \varphi_\lambda^{-1}(i\partial/\partial t - \hat{H}/\hbar)\varphi_\lambda.
\]
A general solution \( \psi \) of Eq. 11 is then given by
\[
\psi(\theta, t) = \sum_\lambda g(\lambda)\psi_\lambda(\theta, t),
\]
where \( g(\lambda) \) is a weight function for \( \lambda \).

To proceed let us assume the time-dependent invariant \( \hat{Q}(t) \) takes the linear form
\[
\hat{Q}(t) = a(t) \hat{L} + b(t) \dot{\theta} + c(t),
\]
in which \( a(t), b(t), \) and \( c(t) \) are time-dependent c-number functions to be determined.

Substituting Eq. 45 into Eq. 40 and solving these operator equations, we get
\[
a(t) = a_0 - \frac{b_0 t}{\tau}, \quad b(t) = b_0,
\]
\[
c(t) = c_0 + b_0 \int_0^t \frac{n_\Phi(\tau)\hbar}{I} \mathrm{d}\tau,
\]
where \( a_0, b_0, \) and \( c_0 \) are arbitrary complex constants. Furthermore, substituting Eqs. 16 and 17 into Eq. 13, we find
\[
\hat{Q}(t) = a_0 \hat{L}(0) + b_0 \dot{\theta}(0) + c_0 = \hat{Q}(0).
\]
In other words, the invariant \( \hat{Q} \) in the Heisenberg picture is precisely the linear combination of the initial canonical angular momentum \( \hat{L}(0) \) and the initial azimuthal angle \( \dot{\theta}(0) \) with an arbitrary constant \( c_0 \). Note that in our system the \( \hat{L} \) operator is also an invariant.

How does the eigenvalue \( \lambda \) evolve in time? Multiplying the factor \( e^{i\alpha(t)} \) on both sides of Eq. 11, we get
\[
\hat{Q}(t) \psi_\lambda(\theta, t) = \lambda \psi_\lambda(\theta, t).
\]
Partially differentiating the both sides of Eq. 19 with respect to time and using Eq. 40, we find
\[
\lambda(t) = \lambda(0),
\]
thus \( \lambda \) is a constant.

To find a solution of Eq. 11, we have to solve Eq. 11 first. By solving Eq. 11, we get
\[
\varphi_\lambda(\theta, t) = \exp \left[ i \frac{\hbar}{\mu(t)\theta - \frac{1}{2} \nu(t)\theta^2} \right],
\]
where
\[
\mu(t) = \frac{\lambda - c(t)}{a(t)}, \quad \nu(t) = \frac{b_0}{a(t)}.
\]
Substituting Eq. 51 into Eq. 43, we obtain
\[
\alpha_\lambda(t) = \alpha_\lambda(0) - \int_0^t \frac{[\eta^2(\tau) + i\hbar\nu(\tau)]}{2I\hbar} \mathrm{d}\tau,
\]
where
\[
\eta(\tau) \equiv \mu(\tau) - n_\Phi(\tau)\hbar.
\]
In deriving Eq. 53 we have used the two identities:
\[
\hat{\mu} = \frac{\nu (\mu - n_\Phi \hbar)}{I}, \quad \hat{\nu} = \frac{\nu^2}{I}.
\]
Here we see that in general \( \alpha_\lambda(t) \) is a complex function.

Although the form of \( \psi_\lambda(\theta, t) = e^{i\alpha_\lambda(t)}\varphi_\lambda(\theta, t) \) is indeed a solution of Eq. 11, however, it does not satisfy the periodic boundary condition [see Eq. 12]. This problem can be resolved by defining the total wave function \( \psi(\theta, t) \) as the summation of all \( \psi_\lambda(\theta + 2n\pi, t) \) terms:
\[
\psi(\theta, t) = \sum_{n=-\infty}^{\infty} \psi_\lambda(\theta + 2n\pi, t)
\]
\[
= \sum_{n=-\infty}^{\infty} \exp \left[ i\alpha_\lambda(t) + \frac{i}{\hbar} \mu(t)(\theta + 2n\pi) \right.
\]
\[
\left. - \frac{i}{2\hbar} \nu(t)(\theta + 2n\pi)^2 \right].
\]

It can also be transformed to the equivalent form below using the Poisson summation formula:
\[
\psi(\theta, t) = \sum_{n=-\infty}^{\infty} \exp \left[ i\alpha_\lambda(t) + \frac{i}{\hbar} \mu(t)(\theta + 2n\pi) \right.
\]
\[
\times \left. \exp \left[ i\hbar(n - n_\lambda)^2 - 2\nu(t) \right] + \exp \left( -i\theta - \theta_\lambda(t) \right) \right].
\]
From these derivations it should be noted that when using the LR method, the boundary conditions have to be carefully managed, otherwise one may get an incorrect result.

VI. A COMPARISON TO VARIOUS APPROACHES

In this section we shall show that Eqs. 56 and 57 can be cast into the forms of Eq. 18 and Eq. 21. To
proceed further, let us first borrow some parameters from Sec. II, and notice the simple result

\[ a(t)n_0\hbar + b(t)\theta_c(t) + c(t) = a_0n_0\hbar + b_0\theta_0 + c_0. \]  

(58)

 Comparing this result with Eq. (50), we find that they are very similar. Now let us define

\[ \lambda \equiv a(t)n_0\hbar + b(t)\theta_c(t) + c(t). \]  

(59)

 From Eq. (59), we get

\[ \mu(t) = n_0\hbar + \nu(t)\theta_c(t), \]  

(60)

\[ \eta(t) = l_c(t) + \nu(t)\theta_c(t). \]  

(61)

 Moreover, using the identity

\[ \frac{d\theta^2_c(t)}{dt} = \frac{2}{I}l_c(t)\theta_c(t) \]  

(62)

 and the identity of $\nu$ in Eq. (59), we have

\[ \eta^2 = l^2_c + I\frac{d}{dt}(\nu\theta^2_c). \]  

(63)

 Substituting Eq. (63) into Eq. (58), we get

\[ e^{i\alpha\lambda(t)} = \frac{e^{i\alpha\lambda(0)}}{\sqrt{1 - \frac{\nu^2_0}{T}}} \exp \left( -i\frac{\nu}{\hbar} \int_0^t \frac{l^2_c(\tau)}{2I} d\tau - \frac{i\nu^2_0}{2T} \right). \]  

(64)

 Moreover, defining $e^{i\alpha\lambda(0)}$ and $\nu_0$ as

\[ e^{i\alpha\lambda(0)} = \frac{N\sqrt{\pi}}{\sigma}, \quad \nu_0 = -i\frac{T}{\hbar} = -\frac{i\hbar}{2\sigma^2}. \]  

(65)

 we see clearly that Eqs. (58) and (57) become exactly the same as Eqs. (18) and (21). These two approaches are thus verified to be equivalent.

 It is now interesting to discuss the physical meanings of $l_c(t)$ and $\theta_c(t)$ we have obtained. Although they are originated from the classical treatment, however, in what sense do they play a role of dynamic variables in the corresponding classical system should be further clarified. We would like to bring attention that in the Schrödinger picture within coordinate representation, the $l_c(t)$ is the expectation value of $\hat{L} - qRA(t) = -i\hbar\partial/\partial \theta - qRA(t)$. However, $\theta_c(t)$ is not the expectation value of $\hat{\theta} = \theta$ with respect to the wave function obtained in Eq. (18), instead, it is merely the peak position of the wave packet [see Eq. (21)].

 In other words, the conventional Ehrenfest Theorem is not directly applicable in this topologically nontrivial system. This consequence is due to the fact that we are not able to distinguish the phase between the angle $\theta$ and $\theta + 2\pi n$. Hence the $\hat{\theta}$ operator is not well-defined, only the $e^{i\hat{\theta}}$ is a well-defined operator, as has been demonstrated in Sec. IV. These facts cause the $\lambda$ losing its meaning as an expectation value of the $\hat{Q}$ operator.

## VII. SUMMARY

In this article, we have studied the problem of a charged particle moving in a ring subject to a time-dependent flux threading it. After analyzing the problem in a classical manner, various approaches including Fourier expansion method, time-evolution method, and Lewis-Riesenfeld method are considered and compared. In the Lewis-Riesenfeld approach, by appropriately managing the periodic boundary condition of the system, a time-dependent wave function can be obtained by using a non-Hermitian time-dependent linear invariant. The eigenfunction of the invariant can be realized as a Gaussian-type wave packet with the peak moving along the classical angular trajectory, while the distribution of the wave packet is determined by the ratio of the coefficient of the initial angle to that of the initial canonical angular momentum. In this circular system, we find that although the classical trajectory and angular momentum can determine the motion of the wave packet; however, the peak position is no longer an expectation value of the angle operator, and the Ehrenfest theorem is not directly applicable.

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