Fitting of the Rutherford Equation for Neoclassical Tearing Mode stabilisation in ASDEX Upgrade

L. Urso\textsuperscript{1}, M. Maraschek\textsuperscript{2}, H. Zohm\textsuperscript{2} and ASDEX Upgrade Team

\textsuperscript{1} Università degli studi di Padova, Italy
\textsuperscript{2} MPI für Plasmaphysik, Garching

E-mail: \textsuperscript{1}laura.urso@ipp.mpg.de, \textsuperscript{2}marc.maraschek@ipp.mpg.de, \textsuperscript{2}hartmut.zohm@ipp.mpg.de

Abstract. This paper describes a form of the generalised Rutherford Equation for Neoclassical Tearing Modes suited to fit experiments. All the relevant terms contributing to the perturbed current parallel to the magnetic field have been included both in the cases of small or large island limit; we present a fit to ASDEX Upgrade experimental data.

1. Introduction
Magnetic islands due to Neoclassical Tearing Modes limit the maximum achievable $\beta$ which is a measure of the efficiency of the confinement in tokamak operations; therefore, it is important to study possible ways to avoid NTMs or to stabilise them. The Rutherford Equation describes the temporal evolution of the island width $W$ taking into account the physical processes causing the appearance and the growth of NTMs. It has the following general form:

$$\tau_s \frac{dW}{r_s} = \Delta_{cl} + \Delta_{bs} + \Delta_{GGJ} + \Delta_{ECCD} + \Delta_{pol}$$

where $\tau_s = \mu_0 r_s^2 / \eta_r$ is the resistive time and $r_s$ is the radius at the resonant surface.

It is analytically derived by integrating Ampère’s law across the island region and using Ohm’s law to express the various contributions to the current density [1].

For a large island, the tearing mode stability index $\Delta'$ and the perturbed bootstrap current dominate the evolution while for a sufficiently small island the Glasser-Greene-Johnson term, the finite ratio between the parallel and perpendicular heat transport coefficients and possibly the ion polarisation current also contribute to the dynamical behaviour. In section 2, a derivation of the form for the Rutherford Equation will be provided and the mechanisms which contribute to the island evolution will be addressed. In section 3, (3,2) NTMs in ASDEX Upgrade discharges will be considered and for these the Rutherford Equation at island saturation’s values will be analysed.

2. Derivation of a form of the Rutherford Equation suited to fit experimental data
The first term on the rhs of eq. 1, is expressed [2] by $\Delta_{cl} = r_s \Delta'$. In addition, since the tearing mode stability index $\Delta'$ has been found to depend on the island width in the nonlinear stage...
of the mode growth, the following fitting [3] for a typical (3, 2) in ASDEX Upgrade at island saturation is considered $\Delta' = -(1.97 + 45.5 \frac{W}{a})$ so that the classical term for the Rutherford Equation becomes:

$$\Delta_{cl} = -1.97 \frac{r_s}{a}(1 + 23.1 \frac{W}{a})$$  \hspace{1cm} (2)

The second term of eq. 1 describes the main destabilising effect for the NTMs’ evolution which is the flattening of the pressure profile and the consequent formation of a hole in the bootstrap current. The bootstrap current is expressed in a basic model with $j_{bs} = -1.46 \frac{e^{1/2} \frac{d\rho}{dt}}{B_p}$ but a better fit [4] gives $j_{bs} = -1.4^{1/2} \frac{e^{227}}{B_p} \left( \frac{W}{L_m} + \frac{0.6}{L_T} \right)$ which considers the effect of both a density gradient and a temperature gradient [5]. In the small island limit, there are two mechanisms which prevent the pressure profile from becoming completely flattened:

1. the presence of a finite perpendicular transport;
2. the size of the magnetic island being comparable to the size of the banana orbit.

The former is considered through the critical island width $W_d = 5.1 \sqrt{\frac{R_0 L_m}{m} \left( \chi_1 \chi_\parallel \right)^{1/4}}$ [6] where $\chi_1$ and $\chi_\parallel$ are the perpendicular and parallel thermal diffusivity respectively, expressed in $m^2$. The latter term is described using the banana width $W_b = \sqrt{\epsilon \rho_0}$ where $\rho_0$ is the ion poloidal gyroradius and $\epsilon$ is the inverse-aspect ratio. Based on this, fitting results in the small and large island conditions [7] lead to:

$$\Delta_{bs} = -4.63 r_s \beta_p L_q \sqrt{\frac{r_p}{R_0}} \left( \frac{2.27}{L_m} + \frac{0.6}{L_T} \right) \left( \frac{W}{W^2 + W_d^2} + \frac{W}{W^2 + 28W_b^2} \right)$$  \hspace{1cm} (3)

The third term of eq. 1, the Glasser-Greene-Johnson term, describes the effect of curvature on NTMs. It is related to the stabilising contribution of the Pfirsch-Schlüter currents induced by toroidicity and shaping of the poloidal cross section and it has the same $W$-dependence as $\Delta_{bs}$ term. Its basic form is [8] $\Delta_{GGJ} = \mu_0 \frac{6.35 D_R}{\sqrt{W^2 + 0.65 W_d^2}}$ where $D_R = \frac{(q^2 - 1)\beta_p p}{r q^2 R_t}$ is the resistive interchange parametre in the form of Shafranov-Yurchenko. Since no $q >> 1$ approximation is used the GJJ term to be fitted becomes:

$$\Delta_{GGJ} = 6.35 r_s \beta_p (q^2 - 1) \frac{r_s L_q^2}{q R_0^3 L_p} \frac{1}{\sqrt{W^2 + 0.65 W_d^2}}$$  \hspace{1cm} (4)

The GJJ term, unlike the bootstrap one, is influenced by the total pressure gradient and this has been emphasised in the final form of the equation by keeping for it the dependence on the $L_p$. The similar functional dependence of the bootstrap and the GJJ term at large island width leads to the joining of the two terms into a unique one; the ratio between them, which is that from the cylindrical calculation, is kept but a new fit parameter $e^*$ is introduced in order to account for realistic shaping.

The fourth term in eq. 1 describes the application of the ECCD system in order to stabilise the mode. For the ECCD current a gaussian profile is assumed $e^{-\left(\frac{r-r_d}{d}\right)^2}$ where $d$ is the deposition width at $\frac{1}{2}$ peak value. The effect of a too broad ECCD profile is quantified through the function $\eta$ which has the following fitted values [9]:

$$\eta(W) = \begin{cases} 0.3 \frac{W}{T} \text{ for } d > \frac{W}{T} \\ e^{-\frac{W}{T}} \text{ for } d \leq \frac{W}{T} \end{cases}$$

The ECCD system affects the evolution of the NTMs both by changing the equilibrium current profile and by generating a current in the island which replaces the missing bootstrap current;
therefore assuming that equilibrating the current on the island flux surfaces only creates \((0, 0)\) and \((m, n)\) components, the \((0, 0)\) component is expressed by \(j_{ECCD,00} = j_{ECCD}(1 - \eta(W/d))\).

The form for the ECCD term is then

\[
\Delta_{ECCD} = -c_j \mu_0 \frac{L_q r_s j_{ECCD}}{B_{pol}} \left(a_{mn} \frac{d}{W^2} \eta\left(\frac{W}{d}\right) + a_{00} \frac{4\sqrt{\pi}}{d} (1 - \eta\left(\frac{W}{d}\right))\right)
\]

where the coefficients \(a_{mn}\) and \(a_{00}\) are equal to 34 [9] and 1 [10] respectively and the coefficient \(c_j\) is added as another fitting coefficient, to scale the ECCD current.

The last term in eq. 1 is the polarisation term which describes finite orbit effects. It is neglected in the final form of the Rutherford Equation as in previous works [11] it has been shown that in the limit \(W < W_b\) it is no longer important and in the large island limit it will be small due to the \(1/W^3\) dependence.

Consequently, the final form for the \((3,2)\) NTMs’ evolution equation which is to be analysed is

\[
\tau_s \frac{dW}{dt} = -1.97 r_s (1 + 23.1 \frac{W}{a})
\]

\[
+ r_s \beta_p c^* \left(-4.63 \sqrt{\frac{2}{R_0}} \frac{L_q}{L_n} \left(2.27 \frac{2.27}{L_n} + \frac{0.6}{L_T} + \frac{W}{W^2 + W_d^2} + \frac{W}{W^2 + 28W_b^2}\right) + 6.35(q^2 - 1) \frac{r_s L_q^2}{q R_0 L_p} \frac{1}{\sqrt{W^2 + 0.65W_d^2}}\right)
\]

\[
- c_j \mu_0 \frac{L_q r_s j_{ECCD}}{B_{pol}} \left(34 \frac{d}{W^2} \eta\left(\frac{W}{d}\right) + 4\sqrt{\pi} \frac{d}{W^2} (1 - \eta\left(\frac{W}{d}\right))\right)
\]

so that for \(dW/dt = 0\) and without the ECCD term, \(c^*\) is calculated as \(c^* = \frac{\Delta_{cs}}{\Delta_{bs} + \Delta_{GGJ}}\).

3. Analysis of ASDEX Upgrade discharges

The discharges #19232, #14991 for \((3,2)\) NTM stabilisation experiments with typical profiles as fig. 1 are analysed.

The main parameters for the discharges are resumed in the table 1 where \(L_q\) is taken from the Cliste equilibrium solver [12] profile and the shear is calculated as \(s = \frac{\tau_s}{L_q}\)

| shot     | \(I_p[MA]\) | \(B_t[T]\) | \(n_e10^{19}[m^{-3}]\) | \(\beta_N\) | \(q_{95}\) | \(L_q[m]\) | shear |
|----------|--------------|------------|--------------------------|-----------|--------|-----------|-------|
| 19232    | 0.8          | -1.98      | 4.84                     | 2.55      | 4.68   | 0.41      | 0.56  |
| 14991    | 0.8          | -2.07      | 5.50                     | 2.22      | 4.88   | 0.39      | 0.60  |

3.1. Calculation of \(c^*\)

At island saturation, all the various coefficients present in the Rutherford Equation are calculated in order to evaluate the \(c^*\) coefficient with the results shown in table 2.

The value of the normalised poloidal flux radius \(\rho_p = \sqrt{\psi_{\text{edge}} - \psi_{\text{center}}}/\psi_{\text{center}}\) at which the mode is sitting is established from equilibrium profile for #19232 and from Soft X-ray diagnostic for #14991. It is consistent with local deposition at full stabilisation from Torbeam ECRH code [13].

Establishing the island width is the most difficult assessment since for these shots no ECE data has been available because of density cut-off or misaligned channels therefore the island width is determined by the Mirnov coils \((W \sim \sqrt{B_\theta(t)})\) multiplied by a calibration factor [14] derived by
3.2. Fitting of the Rutherford Equation

Figure 1. Typical time traces of a (3,2) NTM stabilisation experiment. The NBI heating power is initially 17.5 MW to reliably trigger the NTM and then it is decreased to 12.5 MW when the ECCD is applied (trace 1). At the maximum $\beta_N$ the NTM is triggered (trace 2). During the ECCD phase with a magnetic field ramp-up the NTM gets stabilised once the ECCD hits the resonance surface.

Table 2. Calculation of $c^*$

| shot  | $t_{sat}[s]$ | $W_{sat}[m]$ | $\rho_p$ | $c^*$ |
|-------|--------------|--------------|----------|-------|
| 19232 | 1.95         | 0.147        | 0.56     | 0.64  |
| 14991 | 2.17         | 0.097        | 0.6      | 0.50  |

direct island size measurements with the ECE diagnostic in similar discharges. The values for $c^*$ strongly depend on the measure of the island widths and on the determination of the gradient lengths ($L_n = \frac{n}{n'}$, $L_T = \frac{T}{T'}$), whose values have been measured averaging on the steepness of the profiles outside the flattened region, which are very sensible to the precision of the density and temperature profiles’ measurements; thus the uncertainties on these and on the deposition width measurements from TORBEAM [15] have to be improved to better evaluate $c^*$.

3.2. Fitting of the Rutherford Equation

A possible remedy to the loss of bootstrap current within the island is to inject current with ECCD. In ASDEX Upgrade, stabilisation of (3,2) NTMs by ECCD is done by injecting up to 1.2 MW of ECRH (140 GHz, second-harmonic X mode-absorption on the high field side) and by varying the toroidal injection angle to change the amount of driven current.

The values in table 3 for the ECCD injection are measured when the mode is seen to be completely stabilised in the magnetic signal; the local parameters inserted in the Rutherford Equation refer to $t_{ECCD}$ and the deposition width is taken to be the FWHM. The stabilisation process modeled using the Rutherford Equation is described in the fig. 2. The $c_j$ coefficient to be added in the $\Delta_{ECCD}$ term is $c_j = 0.31$ and its value coincides for both discharges. This represents only a lower limit to $c_j$, but since (3,2) NTMs are generally not stabilised with
Table 3. Torbeam values for ECCD injection

| shot | $t_{ECCD}$[s] | $P_{ecrb}$[MW] | $I_{ECCD}$[MA] | $j_{ECCD}$[MA/m^2] | dep[m] | launch. $\theta_p$ | launch. $\phi_t$ |
|------|---------------|----------------|---------------|-------------------|-------|-----------------|-----------------|
| 19232 | 3.50          | 1.11           | 0.0107        | 0.9063            | 0.0050 | −10°            | 5°              |
| 14991 | 2.70          | 1.17           | 0.0196        | 0.2920            | 0.0238 | −9°             | 15°             |

0.8MW of ECCD in this type of experiment, the upper limit is not too far from the value used.

The fit for the modeled Rutherford Equation shows that, with ECCD injection, the (3,2) mode in both experiments is stabilised and for #14991 the $\Delta_{ECCD}$ term is more effective than the requirement for minimum stabilisation due to the amount of current injected (table 3).

![Figure 2. Fitting results for the two discharges with the modeled Rutherford Equation. The upper curve describes the evolution when no ECCD term is applied whereas the second curve describes the behaviour of the NTM when ECCD is applied.](image-url)

3.3. Effect of $\Delta_{cl}$ on the seed island width

In the fitting, a very small seed island is predicted to trigger NTMs whereas in typical cases it has been estimated to be of the order of 1−2 cm from the experiments [9]. In the adopted model for the Rutherford Equation, two effects may influence the seed island width which results from the fitting: having neglected the polarisation term and having used $\Delta_{cl}$ being linearly dependent on the island width (eq. 2). As a first attempt, the same analysis as the previous one is carried setting $\Delta'$ to the constant value $-\frac{m_r}{r_s}$ or $-\frac{2m_r}{r_s}$ in the Rutherford equation and with $-\frac{3}{r_s}$, for example, the values for $c^*$ become for #14991 and #19232, $c^* = 1.25$ and $c^* = 1.06$ respectively.
Results of the fitting are shown in fig. 3. The seed island shifts towards higher values for both discharges and this quantitatively depends on the choice of the value \( \Delta' = \text{constant} \) used to fit the data (usually it is obtained as a rough estimate value from cylindrical calculations).

A more realistic approach is to adopt a model for the \( \Delta' \) which, in the small island width case, brings back to the assumption of \( \Delta' = -\frac{m}{r_s} \) or \( \Delta' = -\frac{2m}{r_s} \) whereas in the large island limit it agrees with the fit \( \Delta' = -(1.97 + 45.5W) \). For both discharges, the matching of these two conditions has been chosen at two different island widths \( W = 0.01 \) m and \( W = 0.02 \) m. The fitted curves used for \( W < 0.01 \) m and for \( W < 0.02 \) m are shown in table 4.

| shot  | \( W < 0.01 \) m     | \( W < 0.02 \) m     |
|-------|-----------------------|-----------------------|
| 19232 | \( r_s \Delta' = 166W - 3 \) | \( r_s \Delta' = 62W - 3 \) |
|       | \( r_s \Delta' = 466W - 6 \) | \( r_s \Delta' = 212W - 6 \) |
| 14991 | \( r_s \Delta' = 161W - 3 \) | \( r_s \Delta' = 60W - 3 \) |
|       | \( r_s \Delta' = 461W - 6 \) | \( r_s \Delta' = 209W - 6 \) |

In addition, the complete nonlinear-code result (referred to as \( r_s \Delta'(p_{nu}) \)) is also considered. The behaviour for these three cases plus the fit at island saturation \( r_s \Delta'(W) \) is shown in fig. 4.

The consequent effect of these three different scenarios for \( \Delta_{cl} \) is, as expected, to shift the seed island width at bigger values as resumed in table 5. Changing the matching island width between the initial behaviour of \( \Delta_{cl} \) and the saturated fit (eq. 2) from \( W = 0.01 \) m and \( W = 0.02 \) m determines only a slight change for \( W_{seed} \)'s values and increases the stabilising effect of the \( \Delta_{cl} \) in the Rutherford Equation as shown in fig. 5, whereas the use of the nonlinear description of
Figure 4. \( \Delta_{cl} \) chosen to match the cylindrical approximation at small island width and to recover the fit at the saturated island (eq. 2) at \( W = 0.01 \) m or \( W = 0.02 \) m for the two discharges analysed.

Table 5. Values for \( W_{seed} \) depending on the \( \Delta_{cl} \).

| shot   | \( W_{seed} \) for \( r_s\Delta'(W) = -3 \) | \( W_{seed} \) for \( r_s\Delta' = -6 \) | \( W_{seed} \) for \( r_s\Delta'_{yu} \) |
|--------|-------------------------------------------|-----------------------------------------|---------------------------------|
| 19232  | 1 mm                                      | 2.1 mm                                  | 4 mm                            |
| 14991  | 1.5 mm                                    | 2.7 mm                                  | 5.5 mm                          |

\( \Delta_{cl} \) results in a bigger \( W_{seed} \) as the curve for \( W < 0.01 \) m is much steeper than in previous cases.

4. Conclusion

A new modeled Rutherford Equation for the time evolution of NTMs has been presented and the calculated values for \( c^* \) allow a reasonable fitting for stabilisation experiments in ASDEX Upgrade. However, peculiar investigation is necessary to better highlight the role of the \( c_j \) coefficient which is introduced in the \( \Delta_{ECCD} \) term to account for the right shaping of the plasma in a non-cylindrical approximation. As the uncertainty on measurements for island widths, length scales and for the deposition width largely affect evaluation of \( c^* \), improvements in the measuring conditions should be applied (e.g. avoidance of density cut-offs for ECE profile, interpolation methods on temperature and density profiles to calculate the spatial derivatives).

To deeply understand the influence of \( \Delta'(W) \) on the seed island width, a better comparison with
Figure 5. Fitting results for the two discharges with different values for $r_s \Delta'$. 

the assumption of $\Delta' = constant$ is to be carried out focusing on the behaviour of the growth of the island at smaller values.

The application of the modeled Rutherford Equation to low-$q_{95}$ discharges and the consequent examination of the physical effects which influence NTMs’ evolution will be a proper basis for an ITER prediction.

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