BRIEF COMMUNICATION. AN INDEFINITE STURM THEORY.

ALESSANDRO PORTALURI

INTRODUCTION

Sturm theory for second order differential equations was generalized to systems and higher order equations with positive leading coefficient by several authors. (See for instance [1], [2] and [6] among the others). Here we propose a Sturm oscillation theorem for indefinite systems with Dirichlet boundary conditions of the form

$$l(x, D) v := p_{2m} \frac{d^{2m} v}{dx^{2m}} + p_{2m-2}(x) \frac{d^{2m-2} v}{dx^{2m-2}} + \cdots + p_1(x) \frac{dv}{dx} + p_0(x) v = 0,$$

where $p_i$ is a smooth path of matrices on the complex $n$-dimensional vector space $\mathbb{C}^n$, $p_{2m}$ is the symmetry of the form $\text{diag}(I_{-\nu}, -I_{\nu})$ for some $\nu \geq 0$. This generalization was obtained along the lines of [4] and [5] in which the second order equations were considered. Full proofs and the relation with Maslov index will appear elsewhere.

1. VARIATIONAL SET UP AND MAIN RESULTS

We will use the variational approach to (0.1) as described in [2] and we will stick to the notations of that paper. Given the complex $n$-dimensional Hermitian space $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$, for any $m \in \mathbb{N}$ let $\mathcal{H}^m := H^m(J, \mathbb{C}^n)$ be the Sobolev space of all $H^m$-maps from $J := [0, 1]$ into $\mathbb{C}^n$.

A derivative dependent Hermitian form is the form $\Omega(x)[u] = \sum_{i,j=0}^m \langle D^i u(x), \omega_{i,j}(x) D^j u(x) \rangle$, where each $\omega_{i,j}$ is a smooth path of $x$-dependent Hermitian matrices with constant leading coefficient $\omega_{m,m} := p_{2m}$ and such that $\omega_{i,2m-1-i} = 0$ for each $i = 0, \ldots, m$. Each derivative dependent Hermitian form, defines a Hermitian form $q: \mathcal{H}^m \to \mathbb{R}$ by setting $q(u) := \int_J \Omega(x)[u] dx$. If $v \in \mathcal{H}^m$ and $u \in \mathcal{H}^{2m}$ then, using integration by parts, the corresponding sesquilinear form $q(u, v)$ can be written as

$$q(u, v) = \int_J \langle v(x), l(x, D) u(x) \rangle dx + \phi(u, v),$$

where $l(x, D)$ is a differential operator of the form of (0.1) and $\phi(u, v)$ is a bilinear form depending only on the $(m-1)$-jet, $j^{m-1} v(x) := (v(x), \ldots, v^{(m-1)}(x))$ and on the $(2m-1)$-jet $j^{2m} u(x)$ at the boundary $x = 0, 1$.

Let $\mathcal{H}^m_0 := \mathcal{H}^m_0(J) := \{ u \in \mathcal{H}^m : j_{-1} u(0) = 0 = j_{m-1} u(1) \}$ and let $q_0$ be the restriction of the Hermitian form $q$ to $\mathcal{H}^m_0$. For each $\lambda \in J$, let us consider the space $\mathcal{H}^m_0([0, \lambda])$ with the form $\int_{[0, \lambda]} \Omega(x) dx$.

Via the substitution $x \mapsto \lambda x$, we transfer this form to $\mathcal{H}^m_0(J)$, so, we come to the forms $\Omega_\lambda$ and $q_\lambda$ defined respectively by

$$\Omega_\lambda(x)[u] := \sum_{i,j=0}^m \langle D^i u(x), \lambda^{2m-(i+j)} \omega_{i,j}(\lambda x) D^j u(x) \rangle$$

and we let $q_\lambda(u) := \int_J \Omega_\lambda(x)[u] dx$.

Then $\lambda \mapsto q_\lambda$ is a smooth path of Hermitian forms acting on $\mathcal{H}^m_0$ with $q_1 = q_0$ and with $q_0(u) = \int_J \langle p_{2m} D^{2m} u, D^m u \rangle dx$. Now we introduce the following definition.

DEFINITION 1.1. A conjugate instant for $q_0$ is any point $\lambda \in (0, 1]$ such that $\ker q_\lambda \neq \{0\}$.

Let $C_\lambda$ be the path of bounded selfadjoint Fredholm operators associated to $q_\lambda$ via the Riesz representation theorem.

LEMMA 1.2. The Hermitian form $q_0$ is non-degenerate. Moreover each $q_\lambda$ is a Fredholm Hermitian form. (i.e. $C_\lambda$ is a Fredholm operator). In particular dim ker $q_\lambda < +\infty$ and $q_\lambda$ is non degenerate if and only if $\ker q_\lambda = \{0\}$.

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Proof. That the operator $C_0$ is an isomorphism can be proven exactly as in [4, Proposition 3.1]. On the other hand each $q_0$ is a weakly continuous perturbation of $q_0$ since it differs from $q_0$ only by derivatives of $u$ of order less than $m$. Therefore $C_\lambda - C_0$ is compact for all $\lambda \in J$ and hence $C_\lambda$ is Fredholm of index $0$. The last assertion follows from this. \hfill \Box

When the form $q_3$ is non degenerate, i.e. when 1 is not a conjugate instant, we introduce the following definition.

**Definition 1.3.** The (regularized) Morse index of $q_3$ is defined by

$$\mu_{\text{Mor}}(q_3) := -\text{sf}(q_3, J),$$

where sf denotes the spectral flow of the path $q_3$, i.e., the number of positive eigenvalues of $C_\lambda$ at $\lambda = 0$ which become negative at $\lambda = 1$ minus the number of negative eigenvalues of $C_\lambda$ which become positive. (See, for instance [4], for a more detailed exposition.)

Now we define the conjugate index. By (1.1) $q(u, v) = \int_J \langle v(x), l(x, D)u(x) \rangle dx$ for $u \in \mathcal{H}'_0^m(J)$ and $v \in \mathcal{H}_0^m$ and therefore elements of the kernel of $q_3$ are weak solutions of the Dirichlet boundary value problem

$$\begin{cases}
l(x, D)u = 0 \\
 j^{m-1}u(0) = 0 = j^{m-1}u(1).
\end{cases}$$

Let $\mathcal{O} = \{ z := \lambda + is \in \mathbb{C} : 0 < \lambda < 1, -1 < s < 1 \}$. For each $z \in \mathcal{O}$ let us consider the closed unbounded Fredholm operator $A_z$ on $L^2(\mathcal{O}, \mathbb{C}^m)$ with domain $\mathcal{D}(A_z) = \{ u \in \mathcal{H}^2_{\text{loc}} : j^{m-1}u(0) = 0 = j^{m-1}u(1) \}$ defined by $A_zu := l_z(x, D)u + i su$. Since $A_z = A_1$, it follows that $A_z$ has a continuous inverse for $s = 2 \pi \mathbb{Z}(z) \neq 0$. Let us consider the splitting $\mathbb{C}^{2m} := (\mathbb{C}^n)^m \times (\mathbb{C}^n)^m$. For each $w = (w_1, \ldots, w_m) \in (\mathbb{C}^n)^m$, let $u_z(x, w)$ be the unique solution of the Cauchy problem

$$\begin{cases}
l(x, D)u + i su = 0 \\
 u(0) = w'(0) = \ldots = u^{m-1}(0) = 0, \quad u^m(0) = w_1, \ldots, u^{2m-1}(0) = w_m,
\end{cases}$$

and let $\mathcal{A}_z : \mathbb{C}^m \to \mathbb{C}^m$ be the endomorphism defined by $\mathcal{A}_z(w) := j^{m-1}(u_z(w, 1))$. Clearly $u \in \ker A_z$ if and only if $(u^m(0), \ldots, u^{2m-1}(0)) \in \ker \mathcal{A}_z$. From this and by using regularity of the weak solutions, we get that the following three statements are equivalent:

(i) $\ker A_z \neq \{0\}$; (ii) $\exists(z) = 0$ and $\lambda = \mathfrak{R}(z)$ is a conjugate instant; (iii) $\det \mathcal{A}_z = 0$. Because of these three equivalent statements, the function $\rho(z) := \det \mathcal{A}_z$ does not vanish on the boundary $\partial \mathcal{O}$.

**Definition 1.4.** The conjugate index $\mu_{\text{con}}(q_3)$ of $q_3$ is minus the winding number of $\rho|_{\partial \mathcal{O}} \to \mathbb{C} - \{0\}$ or equivalently minus the Brouwer degree, i.e. $-\deg(\rho, \mathcal{O}, 0)$.

With this said our main results are

**Theorem 1.** (Generalized Sturm Oscillation theorem). With the notation above, we have

$$\mu_{\text{con}}(q_3) = \mu_{\text{Mor}}(q_3).$$

**Theorem 2.** (Generalized Sturm comparison theorem). If $\Omega_0, \Omega_1$ are two derivative dependent Hermitian forms with $\Omega_0[u](x) \leq \Omega_1[u](x)$ for all $x \in J$, then we have

$$\mu_{\text{Mor}}(q_{\Omega_0}) \leq \mu_{\text{Mor}}(q_{\Omega_1}).$$

2. PROOFS

**Proof.** (of Theorem 1). We split the proof of this result into some steps.

**First step.** Notice that the family $A_z$ has the form $A + B_z$, where $A$ is a fixed unbounded closed operator and $B_z$ is an $A$-bounded perturbation of $A$ depending on $z$ and that the operator valued one-form $dA_zA_z^{-1}$ is well-defined on $\partial \mathcal{O}$. Exactly as in Proposition 4.4 in [4] one proves the following Lemma.

**Lemma 2.1.** The form $dA_zA_z^{-1}$ takes values in operators of the trace class and

$$\mu_{\text{con}}(q_3) = -\frac{1}{2\pi i} \int_{\partial \mathcal{O}} \text{Tr} dA_zA_z^{-1}.$$

Let us assume now that the path $\{A_\lambda\}_{\lambda \in J}$ has only regular crossing points with the variety of singular operators. This means that the quadratic form $\Gamma(A, \lambda)$ given by the restriction to the kernel of $A_\lambda$ of $(A_\lambda^*, \cdot)$ is nondegenerate. Since regular crossing points are isolated, there are only finite number of them. In what follows we will show that if $\mathcal{D}_j$ is a small enough neighborhood of a crossing point $\lambda_j \in \mathcal{O}$, then

$$\frac{1}{2\pi i} \int_{\partial \mathcal{D}_j} \text{Tr} dA_zA_z^{-1} = \text{sign} \Gamma(A, \lambda_j).$$

For a fixed $j$, choose a positive number $\mu > 0$ such that the only point in the spectrum of $A_{\lambda_j}$ in the interval $[-\mu, \mu]$ is 0 and then choose $\eta$ small enough such that neither $\mu$ nor $-\mu$ lies in the spectrum of $A_{\lambda_j}$ for $|\lambda - \lambda_j| < \eta$. For such a $\lambda$, let $P_\lambda$ be the orthogonal projection in $H$ onto the spectral subspace associated
to the part of the spectrum of $A_1$ lying in the interval $[-\mu, \mu]$. Then $A_\lambda P_\lambda = P_\lambda A_\lambda$ on the domain of $A_\lambda$. By [3, Chapter II, Section 6] there exist a smooth path $U$ of unitary operators of $H$ defined in $[\lambda_1 - \eta, \lambda_1 + \eta]$, such that $U_{\lambda_1} = I_dH$ and such that $P_\lambda U_{\lambda_1} = U_{\lambda_1} P_\lambda$. Let us consider the smooth operator valued function $N_\lambda = U_{\lambda_1}^{-1} A_\lambda U_{\lambda_1}$ defined on some open neighborhood of $\mathcal{D}_\lambda = [\lambda_1 - \eta, \lambda_1 + \eta] \times [-1, 1]$ together with the differential one-form $\theta = dN_\lambda N_\lambda^{-1}$. Clearly $\theta$ is in the trace class and $\text{Tr} dA_\lambda A_\lambda^{-1} = \text{Tr} dN_\lambda N_\lambda^{-1}$.

**Second step.** If $H_1 = \text{Im} P_\lambda = \ker A_{\lambda_1}$, under the splitting $H = H_1 \oplus H_1^\perp$, the one-form $\theta$ splits into $\theta_0 = dN_\lambda N_\lambda^{-1} |_{H_1}$ and $\theta_1 = dN_\lambda N_\lambda^{-1} |_{H_1^\perp}$. Thus by taking traces we have

$$
\frac{1}{2\pi i} \int_{\partial \mathcal{D}_\lambda} \text{Tr} \theta = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\lambda} \text{Tr} \theta_0 + \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\lambda} \text{Tr} \theta_1
$$

where the last term vanishes in (2.3) vanishes.

In fact, $N_\lambda |_{H_1^\perp}$ is invertible on $\mathcal{D}_\lambda$ and hence the one form $\theta_1$ is exact. Now, let us consider the path of symmetric endomorphisms $M: [\lambda_1 - \eta, \lambda_1 + \eta] \rightarrow \mathcal{L}(H)$ given by $M_\lambda = N_\lambda |_{H_1}$. By elementary calculations it follows that $\text{sign} \Gamma(M_\lambda, \lambda_1) = \text{sign} \Gamma(A_\lambda, \lambda_1)$.

**Third step.** If $M$ is the path defined above and if $M_\xi = M_\lambda + i\text{Id}$, for $\xi \in \mathcal{D}_\lambda$, then

$$
\text{sign} \Gamma(M, \lambda_1) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}_\lambda} \text{Tr} dM_\xi M_\xi^{-1}.
$$

Using [3, Chapter II, Theorem 6.8], one reduces $M_\lambda$ to a diagonal form and the result follows by direct integration. This together with (2.3) proves (2.2).

**Fourth step.** Using regularity it is easy to see that any regular crossing point for the path $A_\lambda$ is also a regular crossing point for the path $q_1$, i.e., the crossing form defined as the restriction $\Gamma(q_\lambda, \lambda)$ to ker $q_\lambda$ is non-degenerate. Moreover the crossing forms $\Gamma(A_\lambda, \lambda)$ and $\Gamma(q_\lambda, \lambda)$ coincide. Since for path with only regular crossings the spectral flow is given by the sum of the signatures of the crossing forms it follows from Lemma 2.1 and formula (2.2) that theorem 1 holds for paths with only regular crossings. In order to conclude remains to show that it is possible to extend the above calculation to general paths having only regular crossings. To do so we will apply a perturbation argument of Robbin and Salamon to the path of operators.

Using [3, Chapter II, Theorem 6.8], one reduces $M_\lambda$ to a diagonal form and the result follows by direct integration. This together with (2.3) proves (2.2).

**Proof.** of Theorem 2. Since the top order terms coincide, the difference between the two Fredholm Hermitian forms $q_1$ and $q_2$ is weakly semi-continuous. Thus in order to conclude it is enough to show that if $q_1$ and $q_2$ are two admissible families of Fredholm Hermitian forms on $\mathcal{H}_0^n$ whose difference is weakly semi-continuous and such that $q_1^0 = q_2^0$, $q_2^1 < q_1^1$, then $\text{sf}(q_2^1) \leq \text{sf}(q_1^1)$. The rest of the proof is devoted to show this.

It is easy to see that there exists a small $\eta > 0$ such that $q_1^1 = q_2^1 + \lambda \eta |.|^2$ is a family of Fredholm Hermitian forms that is homotopic to $q_1^1$ by $t \mapsto q_1^1 + (1 - t)\lambda \eta |.|^2$ and therefore $\text{sf}(q_1^1) = \text{sf}(q_2^1)$. But now $q_1^0 = q_2^0$ and $q_1^1 < q_2^1$. Given the family of Fredholm Hermitian forms defined on $T$ by $\phi(s, \lambda) = s q_1^1 + (1 - s) q_2^1$ and by using the free homotopy property of the spectral flow for closed paths, it follows that the restriction of $\phi$ to $\mathcal{T}$ has spectral flow zero. Furthermore, $\text{sf}(q_2^1) - \text{sf}(q_1^1) = \text{sf}(\rho)$, where $\rho(s) = (1 - s) q_2^1 + s q_1^1$. Since $\rho = q_2^1 - q_1^1$ is positive definite at each crossing point, all crossing points are regular, and each gives a positive contribute dim ker $\rho(s)$ to the spectral flow of $\rho$. Thus $\text{sf}(\rho) \geq 0$ and hence $\text{sf}(q_2^1) \leq \text{sf}(q_1^1)$.

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