A Note on the Convexity of $\log \det(I + KX^{-1})$ and its Constrained Optimization Representation

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This note provides another proof for the convexity (strict convexity) of $\log \det(I + KX^{-1})$ over the positive definite cone for any given positive semidefinite matrix $K \succeq 0$ (positive definite matrix $K \succ 0$) and the strictly convexity of $\log \det(K + X^{-1})$ over the positive definite cone for any given $K \succeq 0$. Equivalent optimization representation with linear matrix inequalities (LMIs) for the functions $\log \det(I + KX^{-1})$ and $\log \det(K + X^{-1})$ are presented. Their optimization representations with LMI constraints can be particularly useful for some related synthetic design problems.

Results

\textbf{Theorem 1.} For given $K \succeq 0$ the function

$$f(X) = \log \det(I + KX^{-1})$$

is convex over the positive semidefinite cone $S_{++}^n = \{X \in \mathbb{C}^{n \times n} : X = X^* \succ 0\}$ and strictly convex if $K \succ 0$.

The first proof for Theorem \textsuperscript{1} was presented by (DC01) and some other approaches to its proof were given in (MSX06, KBS04, KK06). The proof in (DC01) is an information theoretic
approach, (MSX06) gives a complicated proof to correct the incomplete proof in (KBS04), and (KK06) provides a simple proof based on the theory of spectral functions. We provide a different approach to show (strict) convexity of the above function $f : S^n_{++} \to \mathbb{R}$ for which the convexity of a companion function $\log \det (X^{-1})$ is exploited. As the convexity of $\log \det (X^{-1})$ itself is not trivial, our proof is not stand-alone and the purpose of this note is not to claim to fame but to draw attentions to some equivalent convex programs with linear matrix inequalities for $f$.

**Lemma 1.** For $K \succeq 0$, the value of function $f(X)$ is the same to the optimal value of a semidefinite program for every $X \succ 0$:

$$
\log \det (I + K X^{-1}) \equiv \min_{Z \succ 0} \log \det (Z^{-1})
$$

s.t. \[
\begin{bmatrix}
I - Z & K^{1/2} \\
K^{1/2} & X + K
\end{bmatrix} \succeq 0.
\]

**Proof of Lemma** From the Sylvester’s determinant theorem,

$$
\log \det (I + K X^{-1}) = \log \det (I + K^{1/2} X^{-1} K^{1/2}).
$$

Introducing a slack variable $Z \succ 0$ satisfying the inequality $(I + K^{1/2} X^{-1} K^{1/2}) \preceq Z^{-1}$, we have the following equivalent representations:

$$(I + K^{1/2} X^{-1} K^{1/2})^{-1} \succeq Z \iff I - K^{1/2} (X + K)^{-1} K^{1/2} \succeq Z \iff \begin{bmatrix}
I - Z & K^{1/2} \\
K^{1/2} & X + K
\end{bmatrix} \succeq 0
$$

where the last equivalence follows from the Schur complement. Due to the monotonicity of $\log \det (\cdot)$, the minimum $Z \succ 0$ satisfying the equality gives the value of $\log \det (I + K X^{-1})$ for all $X$. Since the constraint set is compact for all $X \succ 0$ and $K \succeq 0$ and $- \log \det : S^n_{++} \to \mathbb{R}$ is strictly convex, the minimum denoted by $Z^*(X)$ always exists and is unique for every $X \succ 0$. 

\[\Box\]

\[\text{1The convexity of } \log \det (X^{-1}) \text{ is not trivial, but nevertheless well known and can be found in many convex analysis textbooks.}\]

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\textit{Proof of Theorem} \[1\] From Lemma \[1\] and linearity of the constraint, for all $\lambda \in [0,1]$ and $X,Y \succ 0$

$$f(\lambda X + (1-\lambda)Y) = \min_{Z \succ 0} \log \det(Z^{-1})$$

s.t. $\lambda \begin{bmatrix}I & K^{1/2} \\ K^{1/2} & X + K \end{bmatrix} + (1-\lambda) \begin{bmatrix}I & K^{1/2} \\ K^{1/2} & Y + K \end{bmatrix} \succeq 0$

$$\leq -\log \det(\lambda Z^*(X) + (1-\lambda)Z^*(Y))$$

$$\leq -\lambda \log \det(Z^*(X)) - (1-\lambda) \log \det(Z^*(Y))$$

$$= \lambda f(X) + (1-\lambda)f(Y)$$

for which the second inequality is due to the (strict) convexity of $-\log \det(\cdot)$. Since $-\log \det : S^n_{++} \to \mathbb{R}$ is strictly convex, it is straightforward that $\log \det(I + KX^{-1})$ is strictly convex if $X \neq Y$ implies $Z^*(X) \neq Z^*(Y)$, which is equivalent to the condition $K \succ 0$. \hfill \Box

\textbf{Corollary 1.} \textit{For given $K \succeq 0$ the function}

$$g(X) = \log \det(K + X^{-1})$$

\textit{is strictly convex over $S^n_{++}$}.

Similar to Lemma \[1\] for $K \succeq 0$ the above function $g(X)$ has the following convex program with linear matrix inequalities:

$$\log \det(K + X^{-1}) = \min_{Z \succ 0} \log \det(Z^{-1})$$

s.t. $\begin{bmatrix}X - Z & X \cdot K^{1/2} \\ K^{1/2}(X) & I + K^{1/2} \cdot X \cdot K^{1/2} \end{bmatrix} \succeq 0$.

\textbf{The Use of a MaxDet-LMI Representation}

The convex optimization problems with linear matrix inequalities for the functions $f(X) = \log \det(I + KX^{-1})$ and $g(X) = \log \det(K + X^{-1})$ can be particularly useful for some optimization related to the variable $X$. For example, consider a constrained optimization of the
\[
\min_{X \succ 0} g(X) \\
\text{s.t. } H(X) \succeq 0
\]

where \( H : \mathbb{S}^n_{++} \rightarrow \mathbb{S}^m \) is a matrix-valued \textit{affine} function. Then this optimization can be equivalently rewritten by negative logarithmic determinant minimization over linear matrix inequalities:

\[
\min_{X \succ 0, Z \succ 0} - \log \det(Z) \\
\text{s.t. } H(X) \succeq 0, \\
\begin{bmatrix}
X - Z & X K^{1/2} \\
K^{1/2} X I + K^{1/2} X K^{1/2}
\end{bmatrix} \succeq 0,
\]

for which a slack variable \( Z \) is introduced. For the function \( f(X) = \log \det(I + K X^{-1}) \), the same approach can be used.

**Remark 1.** In (VBW98), an overview of the applications of the determinant maximization problem with linear matrix inequalities to the computations of the Gaussian channel capacity is provided. Similar to the results in (VBW98), additional structures on the matrix \( X \) can be straightforwardly imposed in our minimization problems with linear matrix inequalities for the functions \( f(X) = \log \det(I + K X^{-1}) \) and \( g(X) = \log \det(K + X^{-1}) \).

**Concluding Remarks**

In this note, we provide a matrix algebra approach to prove the convexity of \( \log \det(I + K X^{-1}) \) and \( \log \det(K + X^{-1}) \) for \( K \succeq 0 \) and the strict convexity counterpart. The proof does not stand alone, since it requires the convexity of \( \log \det(X^{-1}) \). The method introducing a slack variable gives equivalent convex programs with linear matrix inequalities for those functions, which can be used for constrained optimization with variable \( X \succ 0 \).

\(^2\)This form of optimization is presented in (TKPM14) for which the corresponding optimization is to compute the sequential rate distortion function for a stationary Gauss-Markov process with a linear sensing function.
References and Notes

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