Statistical indicators useful in real spectrum location

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ABSTRACT

In this paper some new results are described, useful for locating the spectrum of a matrix \( A \) through mean, standard deviation and third centered moment of the spectrum distribution which can be expressed in terms of traces. Sufficient conditions for a graph to be connected are given. Numerical examples are also provided, showing how former results in the literature are improved through these methods.

Key Words: Eigenvalues, Inequalities, Moments, Trace

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1. INTRODUCTION

Very few information on the localization of eigenvalues of a matrix are generally needed in application problems. As an example, we quote the well-known conditions for (i) the stability of linear, or linearizable, systems of differential equations, (ii) the stability of linear, or linearizable, systems of difference equations, and (iii) the productivity of an economic linear system (see e.g. [1], [7], [12]). In multidimensional data analysis, the position of the first two eigenvalues (in decreasing order) of the correlation matrix, whether they are “well separated”
from the others or not, gives a first indication of the goodness of the representation on the two factorial axes [8].

For the existence of limiting intervals of the whole spectrum, some classical results are known, such as the relationship between the absolute value of eigenvalues and the matrix norm or the Geršgorin Theorem. Concerning the maximum eigenvalue of a positive matrix, lower and upper bounds are found in [11]; the spectral radius of a non-negative matrix has a lower bound given by the spectral radius of its geometrical symmetrization [15].

Such methodologies are better known as "localization theorems" and are particularly interesting both from theoretical and practical standpoint. It is well known that the numerical computation of the eigenvalues through the characteristic equation is an ill-conditioned problem and, even using numerical techniques, considerable errors, due to truncation or round-off, may occur. Therefore it is significant to accompany to each eigenvalue approximation an inclusion region [6].

Recently, bounds for eigenvalues have been determined through the statistical properties of the spectrum distribution, using in particular the mean and the second centered moment of the distribution. Since those moments can be computed in terms of the traces of the matrix itself and of its second power, \( \text{tr}(A) \) and \( \text{tr}(A^2) \) respectively, eigenvalues can be localized easily by simple functions of the elements of \( A \) and \( A^2 \) ([10], [16], [17], [18], [19]).

In this paper, we give necessary and/or sufficient conditions for some eigenvalues of real-spectrum matrices being below or above a given threshold. In Section 2 we discuss the real spectrum case and new insights on maximum asymmetry and maximum dispersion distributions are given.

In Section 3 sufficient conditions for the connectedness of an undirected graph are provided, which are based on the use of statistical indicators of the adjacency matrix spectrum distribution.

Note that the results we obtained for real spectrum matrices can be easily extended to the general case of a complex matrix with complex spectrum using the decomposition in real and imaginary parts. Bounds for the modulus, the real and the imaginary part of the complex eigenvalues can be found by splitting the matrix in the Hermitian real part and the Hermitian imaginary part ([13], [17]).

2. LIMITING INTERVALS FOR REAL EIGENVALUES

Let \( A \in \mathbb{C}^{n \times n} \) have real ordered eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). From now on we will denote with \( m^a(\lambda) \) the algebraic multiplicity of
the \( \lambda \) eigenvalue. We first recall the relationships existing between the trace of \( A \) and its powers and the mean, the second and third centered moments of the spectrum distribution:

\[
\mu = \frac{\text{tr}(A)}{n}, \quad \sigma^2 = \frac{\sum_{i=1}^{n}(\lambda_i - \mu)^2}{n} = \frac{\text{tr}(A^2)}{n} - \left( \frac{\text{tr}(A)}{n} \right)^2,
\]

\[
\sum_{i=1}^{n}(\lambda_i - \mu)^3 = \frac{\text{tr}(A^3)}{n} + 2 \left( \frac{\text{tr}(A)}{n} \right)^3 - \frac{\text{tr}(A^2)\text{tr}(A)}{n^2}.
\] (2.1)

The following inequalities are well known ([18], [19]).

\[
\mu - \sigma \sqrt{\frac{n-1}{n-i+1}} \leq \lambda_i \leq \mu + \sigma \sqrt{\frac{n-i}{i}}, \quad (i = 1, \ldots, n). \quad (2.2)
\]

In particular, more binding inequalities hold:

\[
\mu + \frac{\sigma}{\sqrt{n-1}} \leq \lambda_1 \quad (2.3)
\]

and

\[
\lambda_n \leq \mu - \frac{\sigma}{\sqrt{n-1}}. \quad (2.4)
\]

Dispersion and skewness indicators, i.e. the variance and the Pearson coefficient respectively, provide information on the spectrum distribution.

It is a classical result [4] that for a given mean, i.e. for a given trace, the distribution of maximum dispersion is found when all eigenvalues, except one, are equal to the minimum and one is equal to the maximum, that is \( m^q(\lambda_n) = n - 1 \) and \( m^q(\lambda_1) = 1 \). The corresponding standard deviation is:

\[
\sigma_{\text{MAX}} = \frac{\sqrt{n-1}}{n} (\lambda_1 - \lambda_n).
\]

If the matrix is positive semidefinite and singular, \( \lambda_n = 0 \), thus \( \lambda_1 - \lambda_n = \text{tr}(A) \).

We recall that the Pearson coefficient \( \gamma = \frac{c^3}{\sigma^3} \) increases, in absolute value, as the asymmetry of the distribution does. As a consequence,
the maximum value is attained when all eigenvalues, except one, are equal to the minimum and one is equal to the maximum, (note that the same distribution gives the maximum dispersion). In this case, the third centered moment is

\[ c^3 = \frac{(n-1)(n-2)}{n^3} (\lambda_1 - \lambda_n)^3 \]

so that

\[ \gamma_{\text{MAX}} = \frac{c^3}{\sigma^2_{\text{MAX}}} = \frac{n-2}{\sqrt{n-1}}. \]

On the other hand, the spectrum distribution, which is “specular” to the former one (that is \( m'(\lambda_i) = n-1, \ m'(\lambda_n) = 1 \)), has also the maximum (in absolute value) Pearson coefficient, but negative in sign. This distribution exhibits the following position and dispersion parameters respectively:

\[ \mu = \frac{\lambda_n + (n-1)\lambda_1}{n} \]  \[ \sigma^2 = \frac{n-1}{n^2} (\lambda_1 - \lambda_n)^2. \]

Its (negative) third centered moment is

\[ c^3 = \frac{(n-1)(2-n)}{n^3} (\lambda_1 - \lambda_n)^3 \]

so that the corresponding Pearson coefficient is

\[ \frac{2-n}{\sqrt{n-1}}. \]

We shall consider now matrices with a non negative spectrum. It should be noted that this hypothesis is not restrictive. Without loss of generality, our results can be applied to any real spectrum matrix by performing a suitable transformation on the matrix \( A \). Given \( \tilde{A} = A - \min(0, \mu - \sigma \sqrt{n} - 1)I \), then \( \tilde{\lambda}_i = \lambda_i - \min(0, \mu - \sigma \sqrt{n} - 1) \), where \( \{ \tilde{\lambda}_i, i = 1, ..., n \} \) is the spectrum of \( \tilde{A} \). By virtue of (2.4), it immediately follows that \( \tilde{\lambda}_i \in [0, 2\sigma \sqrt{n} - 1], i = 1, ..., n \), since \( \tilde{\mu} = \sigma \sqrt{n} - 1 \) (if \( \mu - \sigma \sqrt{n} - 1 < 0 \)) and \( \tilde{\sigma} = \sigma \) where \( \tilde{\mu} \) and \( \tilde{\sigma} \) are the mean and the standard deviation of the spectrum of \( \tilde{A} \) respectively.

The results that follow are based on some inequalities derived by [16], [17], [19]. Through our conditions, that are simple and powerful, it is possible to check how many eigenvalues are located above a fixed value.

Before stating the main result of this section (Theorem 2.1), let us introduce a preliminary result whose proof is based on a result shown by Tarazaga [16] who proved that, when \( A \) is positive semidefinite and
\[ \frac{(\text{tr}(A))^2}{\text{tr}(A^2)} < \frac{\alpha^2}{(\alpha - k + 1)^2 + k - 1}, \quad \alpha \geq 1 \]  

(2.5)

then A has at most \( k - 1 \) eigenvalues greater than \( \frac{\text{tr}(A)}{\alpha} \). As can be proved easily, the inequality (2.5) holds, more generally, for non-negative real spectrum matrices.

**Lemma 2.1.** Let \( A \) be a real spectrum matrix with nonnegative eigenvalues (one at least different from zero). If (2.5) holds, then the number of eigenvalues exceeding \( \mu = \frac{\text{tr}(A)}{n} \) is not greater than \( t \) where

\[ t = \left\lfloor \frac{(2n + 1)g - \sqrt{g^2 - 4ng^2 + 4n^2}}{2g} \right\rfloor, \quad g = \frac{\text{tr}(A)}{\sqrt{\text{tr}(A^2)}} \text{ and } \lfloor \cdot \rfloor \text{ integer part}, \]

or equivalently

\[ t = \left\lfloor \frac{(2n + 1) - \sqrt{1 + 4n\nu^2}}{2} \right\rfloor, \]  

being \( \nu = \frac{\sigma}{\mu} \) and \( \lfloor \cdot \rfloor \) integer part.

**Proof.** Let \( p_k = \frac{n^2}{(n - k + 1)^2 + k - 1} \) \((k = 1, \ldots, n)\), then \( g^2 < p_k \) if and only if \( \delta_1 < k < \delta_2 \), being

\[ \delta_1 = n + \frac{1}{2} - \sqrt{\frac{1}{4} - n + \frac{n^2}{g^2}} \quad \text{and} \quad \delta_2 = n + \frac{1}{2} + \sqrt{\frac{1}{4} - n + \frac{n^2}{g^2}}. \]

Taking into account that:

- the sequence \( \{p_k\} \) is monotone increasing;
- \( \delta_2 > n \geq k \) \((k = 1, \ldots, n)\);
- \( \frac{1}{4} - n + \frac{n^2}{g^2} > 0 \), being \( g^2 \geq 1 \)
- \( \frac{(\text{tr}(A))^2}{\text{tr}(A^2)} \leq \text{rank}(A) \) (see [5])

then

\[ n \geq \text{rank}(A) > g^2 > \frac{g^2}{2} + \frac{g^2 - 1}{2}. \]
We conclude that $g^2 < p_k$ is satisfied, being $k$ integer, by $k = \lceil \delta_1 \rceil > 1$. So that, by virtue of (2.5) and setting $a = n$, there will be at most $(k - 1)$ eigenvalues greater than $\frac{\text{tr}(A)}{n}$, i.e.

$$k - 1 = \left\lfloor \frac{(2n + 1)t - \sqrt{t^2 - 4nt + 4n^2}}{2t} \right\rfloor.$$ 

This last condition is equivalent to:

$$t = \left\lfloor \frac{(2n + 1)t - \sqrt{t^2 + 4n^2}}{2} \right\rfloor,$$

being $t = \frac{\sigma}{\mu}$ and $\lfloor \rfloor$ integer part. \(\square\)

The case $t = 1$ is of particular interest. In fact, it is straightforward to prove that if

$$g^2 < \frac{n^2}{n^2 - 2n + 2}$$

then

$$\lambda_1 > \frac{\text{tr}(A)}{n}$$

and $\lambda_i \leq \frac{\text{tr}(A)}{n}, \ i = 2, \ldots, n$

which in terms of $\nu = \frac{\sigma}{\mu}$ becomes:

$$\text{if } \nu^2 > \frac{(n-1)(n-2)}{n} \text{ then } \lambda_1 > \mu \text{ and } \lambda_i \leq \mu, \ i = 2, \ldots, n.$$

As an immediate consequence of Lemma 2.1, formulas (2.2) and (2.3), the theorem 2.1 follows, which gives the exact number of eigenvalues belonging to the interval $[\beta, \text{tr}(A)]$, where $\beta \in (0, \text{tr}(A))$ is the lower bound of (2.2) or (2.3).

**Theorem 2.1.** Let $A$ be a real spectrum matrix with nonnegative eigenvalues (one at least different from zero). If

$$\frac{1}{\mu^2 + \sigma^2} < \frac{n}{(n\mu - \beta k)^2 + \beta^2 k},$$

then $A$ has exactly $k$ eigenvalues greater than $\beta$, where

$$\beta = \begin{cases} \frac{\mu + \sigma / \sqrt{t} - 1}{\mu - \sigma (k - 1) / (n - k + 1)} & k = 1 \\ \frac{\mu + \sigma / \sqrt{t} - 1}{\mu - \sigma (k - 1) / (n - k + 1)} & k = 2, \ldots, n. \end{cases}$$
EXAMPLES

Example 2.1. Let

\[ A = \begin{bmatrix} 37.775 & 37.225 & 12.725 & -12.725 \\ 37.225 & 37.775 & 12.275 & -12.275 \\ 12.725 & 12.275 & 37.775 & -37.225 \\ -12.725 & -12.275 & -37.225 & 37.775 \end{bmatrix} \]

a fourth order definite positive matrix with \( \mu = 37.775 \) and \( \sigma = 41.21 \).

From (2.2), (2.3) and (2.4) we obtain the following bounds:

\[
\begin{align*}
&61.568 \leq \lambda_1 \leq 109.15, \quad 13.982 \leq \lambda_2 \leq 78.98, \\
&0 \leq \lambda_3 \leq 61.568, \quad 0 \leq \lambda_4 \leq 13.982.
\end{align*}
\]

Note that, being \( A \) positive definite, we set \( \text{Inf}(\lambda_3) = \text{Inf}(\lambda_4) = 0 \).

We can sensibly improve these bounds by using Lemma 2.1. Indeed, it follows that at least \( \lambda_3 \leq 37.775 \) and \( \lambda_4 \leq 37.775 \). Furthermore, from Theorem 2.1 only one eigenvalue belongs to \([61.568, 109.15]\). Thus we obtain the following bounds:

\[
\begin{align*}
&61.568 \leq \lambda_1 \leq 109.15, \quad 13.982 \leq \lambda_2 \leq 61.568, \\
&0 \leq \lambda_3 \leq 37.775, \quad 0 \leq \lambda_4 \leq 13.982.
\end{align*}
\]

The spectrum of the matrix is \([100.00, 50.00, 1.00, 0.10]\).

Example 2.2. Let \( A \) be a hermitian positive definite matrix of order 4 with \( \mu = 9.25, \sigma = 12.02 \).

\[ A = \begin{bmatrix} 9.25 & 7.75 & 6.75 & -6.25 \\ 7.75 & 9.25 & 6.25 & -6.75 \\ 6.75 & 6.25 & 9.25 & -7.75 \\ -6.25 & -6.75 & -7.75 & 9.25 \end{bmatrix} \]

From (2.3) and (2.2) we get

\[
\begin{align*}
&16.19 \leq \lambda_1 \leq 30.08, \quad 2.30 \leq \lambda_2 \leq 21.27, \\
&0 \leq \lambda_3 \leq 16.19, \quad 0 \leq \lambda_4 \leq 2.30.
\end{align*}
\]

Note that, being \( A \) positive definite, the lower bounds for \( \lambda_2 \) and \( \lambda_3 \) have been set both equal to zero. From Theorem 2.1, holding for \( k = 1 \), it follows that \( \lambda_1 \) is the unique eigenvalue in \([16.19, 30.08]\) and, since \( n^2 > \frac{(n-1)(n-2)}{n} \), Lemma 2.1 implies that \( \lambda_i \leq 9.25 \) \((i = 2, 3, 4)\).
so that $\lambda_1$ and $\lambda_2$ are “well separated”. Consequently, the above intervals can be successfully modified by setting the upper bounds of $\lambda_2$ and $\lambda_3$ both equal to 9.25. The spectrum of $A$ is $\{30, 4, 2, 1\}$.

It should be noted that in some applications the entries of matrix $A$, being not known numerically, rather depend on one or more parameters. In those cases the higher order statistical indicators can take complicated expressions and can be hard to handle.

3. AN APPLICATION TO GRAPH CONNECTIVITY

In this section we discuss the connectivity of an undirected simple graph by means of statistical indicators of the spectrum distribution.

Let us consider an undirected, simple graph $G(V, E)$, where $V$ is a non empty vertex set of cardinality $n \geq 1$ and $E \subseteq V \times V$ is the edge set of cardinality $m$, and its Laplacian matrix $Q = D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and $A$ is the adjacency matrix of $G$ ($d_i$ is the degree of vertex $i$). $Q$ is a semidefinite positive and singular matrix and $m^n(\lambda_n(Q))$, the algebraic multiplicity of $\lambda_n(Q)$, is equal to the number of connected components of $G$. Thus, $\lambda_n(Q) = 0$ and $\lambda_{n-1}(Q)$, called the algebraic connectivity of $G$, is positive if and only if $G$ is connected (i.e. there is a path joining each pair of vertices). Note that

$$\mu = \frac{\text{tr}(Q)}{n} = \frac{\text{tr}(D) - \text{tr}(A)}{n}$$

is the average degree of the graph, loops excluded (see [14] for a review of the properties of $Q$).

It is easy to check that

$$\text{tr}(A) = 0, \quad \text{tr}(A^2) = 2m, \quad \text{tr}(A^3) = 6t$$ \hspace{1cm} (3.1)

$$\text{tr}(Q) = 2m, \quad \text{tr}(Q^2) = \sum_{i=1}^{n} d_i^2 + 2m,$$ \hspace{1cm} (3.2)

where $t$ is the number of triangles of $G$.

$$\mu_A = 0, \quad \sigma_A^2 = \frac{2m}{n}, \quad \gamma_A^3 = \frac{6t}{n}$$ \hspace{1cm} (3.3)

$$\mu_Q = \frac{2m}{n}, \quad \sigma_Q^2 = \frac{\sum_{i=1}^{n} d_i^2}{n} + \frac{2m}{n} \left(1 - \frac{2m}{n}\right)$$ \hspace{1cm} (3.4)

**Theorem 3.1.** A necessary condition for a graph to be a forest is $\gamma_A^3 = 0$. 
Proof. Proof is immediate from (3.3), recalling that a forest is a graph without circuits of degree higher than 2. □

Theorem 3.2. A sufficient condition for a graph to be connected is

\[
\frac{4m^2 + 4m - 2mn}{n - 2} > \sum_{i=1}^{n} d_i^2.
\]

Proof. G is connected if and only if \(\lambda_{n-1}(Q) > 0\). From (2.2)

\[
\lambda_{n-1}(Q) \geq \mu_Q - \sigma_Q \sqrt{\frac{n-2}{2}}.
\]

Thus, a sufficient condition for connectivity is \(\mu_Q - \sigma_Q \sqrt{\frac{n-2}{2}} > 0\), or, from (3.4),

\[
\frac{2}{n-2} [2m^2 - m(n-2)] > \sum_{i=1}^{n} d_i^2. \quad \square
\]

A preliminary inspection of the second and third moment of the \(Q\) spectrum distribution gives interesting qualitative information on graph connectivity and its general structure.

The Laplacian spectrum of a complete \(n\)-order graph is such that

\[
m^0(n) = n - 1, \quad m^0(0) = 1.
\]

\[
tr(Q) = n(n-1), \quad tr(Q^3) = n^2(n-1), \quad tr(Q^3) = n^3(n-1),
\]

\[
\mu = \sigma^2 = n - 1, \quad \sigma^3 = (n-1)(2-n), \quad \gamma = \frac{2-n}{\sqrt{n-1}}.
\]

The distribution is of maximum asymmetry, as previously shown, (absolute skewness percentage index \(\gamma/\gamma_{\text{MAX}} = 100\%\)). Note that the distribution is not of maximum variance (dispersion percentage index \(\sigma/\sigma_{\text{MAX}} = \frac{1}{n-1}\%\)).

A high dispersion percentage index is indicative of a non-connected graph.

Take for instance the following two graphs, whose Laplacian matrices are:
\[ Q_1 = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
0 & -1 & 3 & -1 & 0 & -1 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & -1 & 0 & -1 & 3 & -1 \\
0 & 0 & -1 & 0 & -1 & 2
\end{bmatrix} \]

\[ Q_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -1 & -1 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 2
\end{bmatrix} \]

Figure 1. A connected graph (Laplacian \( Q_1 \))

Figure 2. A non connected graph (Laplacian \( Q_2 \))

\[ tr(Q_1) = 14; \quad \sigma_{Q_1}^2 = 2.8889, \quad \sigma_{Q_1}^3 = 1.4075, \quad \sigma / \sigma_{\text{MAX}} = 32.576\% . \]

\[ tr(Q_2) = 8.0; \quad \sigma_{Q_2}^2 = 1.8889, \quad \sigma_{Q_2}^3 = 0.4074, \quad \sigma / \sigma_{\text{MAX}} = 46.098\% . \]
The conditions stated previously assume particularly meaningful expressions when graphs are regular, that is when $d_1 = d_2 = \ldots = d_n = r$. Consider a regular graph of order $r$; then

1. $2m = nr$ \hspace{1cm} (3.5)
2. $\mu_Q = r$, $\sigma_Q^2 = r$, $\gamma_Q^3 = -\frac{6r}{n}$ \hspace{1cm} (3.6)

**THEOREM 3.3.** A necessary condition for a regular graph to be a regular forest (if connected a regular tree) is $\gamma_Q^3 = 0$.

**PROOF.** Proof is immediate recalling that a forest is a graph without circuits of degree higher than 2 and a tree is a connected forest. \(\Box\)

**THEOREM 3.4.** A sufficient condition for a regular graph to be connected is

$$r > \frac{n - 2}{2}.$$ \hspace{1cm} (3.7)

**PROOF.** From $\lambda_{n-1}(Q) \geq \mu_Q - \sigma_Q \sqrt{\frac{n - 2}{2}}$ and (3.5), the result follows. \(\Box\)

Condition (3.7), very easy to assess, combined with (3.5), assures that if $m \geq \left\lceil \frac{n^2}{4} \right\rceil$ the graph is connected.

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