Prediction of vortex precession in the draft tube of a model hydro turbine using mean field stability theory and stochastic modelling

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Abstract. In this work we employ mean field stability theory (MFST) to predict the onset of the precessing vortex core (PVC) in the draft tube of Francis turbines. MFST is based on the linear stability analysis of the mean field of turbulent flows. Recent work shows that MFST very accurately predicts the formation of coherent structures in turbulent shear flows, such as the PVC. MFST may further reveal the flow regions that are most susceptible to flow actuation to suppress the PVC, which is of great practical relevance. In this work, MFST is accompanied by a data-driven approach to predict the linear growth rate of the PVC based on pointwise wall pressure measurements. The method is based on statistical evaluation of the probability density function of the PVC amplitude at limit cycle. It makes use of the intense noise induced by the background turbulence, which is expected to be a major driver of hydrodynamic instabilities. The empirical and analytic results are compared to phase-locked LDV measurements conducted inside the draft tube at various operating conditions, to assess the quantitative accuracy of the approach. The methodologies outlined in this work will be of relevance for future design of hydro turbines to run stable over a wide range of operating conditions.

1. Introduction
To compensate for the fluctuating power production of weather-dependent renewables, conventional power plants are required to operate flexibly over a wide operational range. Hydropower represents a reliable and cost-effective method to respond to these fluctuations without producing environmental pollutants. Most hydropower plants are running Francis turbines which can be operated apart from the best efficiency point to cover varying energy demands. However, at part load conditions with smaller flow rates, the flow exiting the turbine still contains a large swirling component with dramatic effect on the turbine performance.

When the swirl intensity exceeds the critical swirl number the flow undergoes vortex breakdown, which then triggers a strong hydrodynamic instability. It is of helical shape and known as the precessing vortex core (PVC) or rope. First attempts to describe the PVC in Francis turbines were made in the 1980s by Nishi et al. [1], who observed the generation of the PVC at stationary part load conditions. More recent studies describe the PVC in much more detail based on numerical [2] and experimental data [3, 4, 5]. The PVC itself generates
synchronous (plunging) pressure oscillations that may resonate with the entire tube system causing serious damage to the runner, casing and other parts of the turbine [6].

Hence, a Francis turbine that reliably works over a wide operational range must obey design rules that mitigate the hydrodynamic instability leading to the PVC. This requires a deeper understanding of the physics driving this instability. Furthermore, a tailored instability control approach may be required for a complete suppression. Mean field stability theory (MFST) is a well suited analytic framework to meet these goals.

MFST is based on linear stability analysis (LSA), which is essentially an eigenmode analysis of the Navier–Stokes equations linearised around a mean flow [7]. This mean flow can be multi-dimensional, turbulent and multiphase. It has been shown in various applications that the resulting eigenmodes accurately predict the vortical structures that are driven by the hydrodynamic instability. Moreover, the adjoints of the eigenmodes reveal the physical cause for these instabilities and pinpoint regions of high receptivity to flow control. Recent experimental and analytic investigations have proven the robustness of this framework for complex flows in swirl-stabilised gas turbine combustors [8, 9] and in Francis turbines [2].

In this work, we conduct first steps towards the control of the PVC based on MFST. A global LSA is conducted in the draft tube of a Francis turbine based on experimental mean flow data acquired over a wide range of operating conditions [10]. It is shown that the analytic framework reliably predicts the occurrence of the PVC at part load conditions. We further discuss the structural sensitivity that reveals the origin of the PVC. Based on a stochastic model (SM), the occurrence and absence of the PVC is predicted and the results are compared to the MFST.

2. Theory
In this section we present the theoretical methods used for modelling and prediction of the PVC with global LSA and an SM.

2.1. Global linear stability analysis
Coherent structures can be described by a triple decomposition [11]. Let \( \mathbf{q}(x, t) = [\mathbf{u}, p]^\top \) be the summarised vector of the flow velocity vector \( \mathbf{u} \) and the pressure \( p \). \( \mathbf{q} \) can then be decomposed by

\[
\mathbf{q}(x, t) = \bar{\mathbf{q}}(x) + \tilde{\mathbf{q}}(x, t) + \mathbf{q}''(x, t),
\]

where \( \bar{\mathbf{q}} \) is the time-averaged mean part, \( \tilde{\mathbf{q}} \) is the coherent part and \( \mathbf{q}'' \) is the stochastic part.

The coherent fluctuations of a global instability can be modelled by a global LSA. It is based on the Navier–Stokes equations of the coherent fluctuation \( \tilde{\mathbf{q}} \) linearised around the mean flow \( \bar{\mathbf{q}} \) [11]:

\[
\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \tilde{\mathbf{u}} = -\frac{\nabla \rho}{\rho} + \nabla \cdot \left( \nu (\nabla + \nabla^\top) \tilde{\mathbf{u}} \right) - \nabla \cdot \mathbf{u}^\prime \mathbf{u}''
\]

\[
\nabla \cdot \tilde{\mathbf{u}} = 0.
\]

For the solution only normal modes are used as an ansatz of the form

\[
\tilde{\mathbf{q}}(x, t) \propto \mathbf{q}(x, r)e^{i(m\theta-2\pi ft)} + \text{c.c.}
\]

Here, \( x \) is the axial, \( r \) the radial and \( \theta \) the azimuthal coordinate. \( f \) denotes the complex frequency, with \( f = f_r + if_i \), and \( \mathbf{q} \) being the complex amplitude. A biglobal LSA is conducted where the azimuthal direction is assumed to be homogeneous. For the PVC \( m = 1 \), with \( m \) denoting the azimuthal wavenumber. This leads to a generalised eigenvalue problem. Its solution provides the eigenmodes with their respective frequencies \( f_r \) and growth rates \( f_i \). In MFST, an
oscillator mode at limit cycle such as the PVC is ideally marginally stable, i.e. \( f_i = 0 \), since the instability neither grows nor decays [12]. With that criterion and the known frequency from the experiment, the PVC eigenmode can be identified. The adjoint eigenmode is obtained from the corresponding adjoint eigenvalue problem. The structural sensitivity, which localises the region of strongest internal feedback, is then calculated by the product of direct and adjoint mode [13].

A Fortran code [14] is employed for the solution of the eigenvalue problem inferred from equations 2 and 3 with an iterative Arnoldi algorithm. A high-order finite difference scheme with nonuniformly distributed Chebyshev nodes is used for discretisation. The boundary conditions are set to homogeneous Dirichlet conditions at inlet, outlet and outer wall for velocity and pressure. On the axis, homogeneous Dirichlet conditions are set for axial velocity and pressure, and homogeneous Neumann conditions are set for radial and azimuthal velocity according to [15]. For further details on the numerical procedures the interested reader is referred to [9].

Equations 2 and 3 are not closed since \( \overline{u''u''} \) is unknown. This term describes the interaction between the coherent and stochastic field due to the perturbation of the stochastic Reynolds stresses by a coherent wave. This interaction can be modelled with a Boussinesq approximation,

\[
\overline{u''u''} = -\nu_t (\nabla + \nabla^\top) \overline{\dot{u}},
\]

in which only the unknown eddy viscosity \( \nu_t \) needs to be determined. In the context of the PVC in swirling jets a very effective approach is a least-squares fit over all measured stochastic Reynolds stresses [16, 9]. Since these were not measured, we will follow a different approach and use a mixing length model [17] to estimate the eddy viscosity field. The eddy viscosity is then set to a globally constant value that is calculated by taking the average of the eddy viscosity field obtained from the mixing length model. In the future, time-resolved stereoscopic particle image velocimetry measurements will make this approach obsolete and provide an improved model.

### 2.2. Stochastic model

In addition to the analytical description, the flow dynamics are investigated from a stochastic analysis of measured time-series. The analysis is based on the statistical properties of the dynamics which are used to calibrate a stochastic reduced order model of the flow. The approach is comprehensively introduced in the work of Noiray et al. [18, 19], where the analysis is applied to thermo-acoustic oscillations in a combustor. Departing from this approach, we use the Landau equation [20] to describe the dynamics of the flow close to the bifurcation point and add a stochastic forcing term to account for unresolved turbulent perturbations. Separation of variables and stochastic averaging in analogy to [19] leads to the following reduced model for the oscillation envelope

\[
\frac{d|A|}{dt} = \sigma |A| - \alpha |A|^3 + \frac{\Gamma}{|A|} + \xi
\]

where \( \sigma \) is the amplification rate, \( \alpha \) the Landau constant and \( \Gamma \) is the variance of the white noise forcing \( \xi \). Solving the corresponding stationary Kolmogorov equation of this Langevin type equation gives an analytical expression for the stationary probability density function of the envelope [19]. This is calibrated against the measurement data to obtain the model parameters of equation (6). The focus is put on amplification rate \( \sigma \) which is comparable to the \( f_i \) from the MFST in the linearly stable or subcritical regime.

The analysis relies on processing of the measurement data to get reliable values of the model parameters. In the following analysis we use the ‘method 3’ as described in [21] where band-pass filtering relative to the PVC frequency \( f_{PVC} \) is set to the range from \( f_{PVC}/2.5 \) to \( f_{PVC} \cdot 2.5 \).

The growth rate \( \sigma \) characterises the growth of a linear instability with regard to the base flow. In the linearly stable or subcritical regime, global modes are absent and base flow and
mean flow coincide. In this case, the MFST and the SM are expected to produce the same results. In the supercritical regime, base flow and mean flow are not equal. The base flow describes the initial state of the flow after transition from the subcritical to supercritical state in which the base flow is linearly unstable. The global instability starts to grow exponentially, the production of coherent Reynolds stresses modifies the base flow state and the global mode saturates nonlinearly to a finite amplitude. A new state, the marginally stable mean flow, is reached. Therefore, the MFST is expected to predict growth rates of zero, whereas the SM is expected to predict growth rates larger than zero.

3. Experimental techniques

Measurements are conducted in the draft tube cone of a Francis turbine model based on the geometry Francis-99 (figure 1). The working fluid is air. The initial swirl is provided to the flow at the guide vanes and then at the runner, thereby modelling the desired velocity distribution at the inlet of the draft tube. The operating parameters, the range of volumetric flow rates from $Q = 70 \text{ m}^3/\text{h}$ to $250 \text{ m}^3/\text{h}$ and rotation speed $n = 2432 \text{ rpm}$ are set with an accuracy of 1.5 and 0.5%, respectively. This corresponds to turbine operating regimes from deep part load $(Q = 70 \text{ m}^3/\text{h})$ over best efficiency point $(Q = 175 \text{ m}^3/\text{h})$ to overload $(Q = 250 \text{ m}^3/\text{h})$. The mean axial and tangential velocity components, which are utilised for the global stability analysis, are measured using laser Doppler velocimetry ‘LAD-06i’ in the centreline cross-section. The part load condition at $Q = 90 \text{ m}^3/\text{h}$ is of particular interest and spatially resolved with $\Delta(x, y, z)/D = 0.001$. All other cases are resolved with $\Delta(x, y, z)/D = 0.05$. Two acoustic sensors are mounted diametrically flush with the cone walls for time-resolved pressure measurements in order to detect the frequency of the PVC.

![Figure 1](image)

**Figure 1.** Experimental setup and measurement area (ROI, red dashed line) inside the test section; flow goes from top to bottom

4. Results

In this section, we first examine an operating point of the model turbine where the PVC occurs most dominantly in the experiment. Global LSA predicts this PVC and the results are compared to the experimental data. Subsequently, the global LSA and the SM are applied to a large range of different operating points in order to study the validity of both models for predicting the onset of the PVC instability.
4.1. The global PVC mode and its wavemaker

The eigenspectrum obtained by the global LSA is shown in figure 2 where each marker represents a separate eigenmode. When the Francis turbine operates at part load conditions with $Q = 90 \text{ m}^3/\text{h}$, a mode at the frequency of 17 Hz exists (shown as a filled circle) which is close to the measured experimental frequency of 17.5 Hz. The proximity to the temporal growth rate of zero reflects the PVC oscillating stably at its limit cycle. The domain truncation up- and downstream introduces uncertainties regarding the ‘true’ frequency and growth rate as well as spurious modes close to the identified PVC eigenmode, as demonstrated for example in [22].

![Figure 2. Spectrum of the eigenfrequencies $f_r$ and temporal growth rates $f_i$ at $Q = 90 \text{ m}^3/\text{h}$ (filled circle ●: identified PVC mode, vertical solid line: measured experimental frequency, horizontal dotted line: stability limit)](image)

Figure 3 shows the real part of the transversal component of the PVC mode for an arbitrary phase angle of the oscillation cycle as derived from the LSA. The strong fluctuations along the centreline represent the precession motion. Negative values indicate a displacement of the vortex core in negative $x$-direction whereas positive values indicate a displacement in positive $x$-direction. The LSA is clearly able to reproduce the precession motion of the PVC.

![Figure 3. Transversal fluctuation of the PVC mode for an arbitrary phase angle of the oscillation cycle superposed with mean flow streamlines (nondimensionalised with maximum magnitude)](image)

![Figure 4. Structural sensitivity of the PVC mode superposed with mean flow streamlines (nondimensionalised with maximum magnitude)](image)

The structural sensitivity is plotted in figure 4. The maximum values around $z/D = 0.27$ reveal the region of strongest internal feedback or self-excitation where the adjoint mode,
quantifying the receptivity, and the direct mode, quantifying the coherent fluctuations, couple at maximum. From here, the instability waves propagate into the flow field, acting as an internal clock-work. For this reason, this location is commonly named ‘wavemaker’ [13]. Contrary to previous works concerning swirling flows [16, 8, 9], the wavemaker is located farther downstream than usual. This may either indicate a different instability mechanisms being active in the wake of the central recirculation zone or it may be related to the domain truncation upstream. To exclude the second option, additional mean flow data upstream of the draft tube are necessary. This additional data is going to be provided by future numerical simulations.

Figure 5 compares the coherent fluctuations obtained by LSA and the coherent fluctuations obtained by phase-averaging of the time-resolved laser Doppler velocimetry data for all three velocity components at $z/D = 0.35$ for the azimuthal wavenumber $m = 1$. The general PVC structure of the LSA mode matches the actual PVC structure of the phase-average qualitatively well. Slight deviations concern the radial extent of the PVC. The phase-averaged data reveals a radially more extended PVC compared to the LSA mode. This is probably owed to the simplified eddy viscosity model, as described in section 2.1. This will be improved with an enhanced eddy viscosity model based on stereoscopic particle image velocimetry data in the future. Further deviations concern the structure of the axial fluctuations. These are more distorted in the phase-averaged data.

4.2 Prediction of the PVC onset by global linear stability analysis and stochastic model

For a parametric study, the global LSA and the SM are applied to a range of different operating points of the model turbine from $Q = 70 \text{ m}^3/\text{h}$ to $250 \text{ m}^3/\text{h}$. The validity of both methods for
predicting occurrence and absence of the PVC are assessed.

Figure 6 presents the growth rates for different operating points of the model turbine. For supercritical conditions in which the PVC occurs in the experiment, the LSA of the mean flow predicts growth rates that are negative but very close to the theoretical value of zero. The slight deviations are owed to the limited size of the domain that typically lead to underestimated growth rates [22]. The SM of the base flow correctly predicts the growth rates to be positive in the supercritical regime. It also predicts the critical flow rate very reliably. Note that in the supercritical regime the discrepancy of the growth rates from LSA and SM is expected and consistent with the assumptions made earlier in section 2.2. Under subcritical conditions, both models agree very well for flow rates close to the critical limit. Both models predict negative growth rates that become lower with increasing flow rate. This decrease of the growth rates also agrees with the decreasing swirl number. For high flow rates (and low swirl number), the growth rates from the SM strongly deviate from the prediction by LSA. This is due to the fact that the contribution of the PVC in the signals of the pressure sensors vanishes in the background noise. The SM analysis is applied at the frequency where the PVC was observed for larger swirl numbers. However, the deviation indicates that the obtained growth rates are no longer reliable.

Figure 6. Temporal growth rate $f_i$ from mean flow predicted by linear stability analysis (LSA), temporal growth rate $\sigma$ from base flow predicted by the stochastic model (SM), and swirl number $S$ for different volumetric flow rates $Q$ (solid line: theoretical limit cycle growth rate of zero; dashed line: critical flow rate for onset of PVC obtained by inspection of power spectra)

5. Conclusions
Experiments in a model hydro turbine with air as a working fluid have been conducted. Mean flow and phase-average data of the velocity field within the draft tube as well as time-resolved pressure measurements at the tube walls were recorded. A wide range of operating points was traversed from deep part load over best efficiency point to overload. Mean field stability theory (MFST) in form of a global linear stability analysis (LSA) and a stochastic model (SM) were applied to the dataset in order to recover the frequency, growth rate, spatial shape and structural sensitivity of the precessing vortex core (PVC) mode. The model results were compared to the measured data from the experiments.

The global LSA results show an overall good agreement of the predicted frequency and mode shape compared to the experimental data. This demonstrates the validity of MFST for modelling the PVC in hydro turbines. This is promising for extended receptivity and sensitivity analyses in the future that will greatly help to develop and improve flow control applications. In the research field of gas turbine combustors, this approach has been successfully realised very recently [23, 24].

The SM also proves to be a valuable method. In the supercritical regime, the SM reliably predicts the critical flow rate, i.e. the stability limit, required for the onset of the PVC, showing
positive growth rates of the instability with regard to the base flow. Under subcritical conditions, both models correctly predict a linearly stable flow, which coincides with the absence of the PVC in the experiments. Contrary to the global LSA, the data-driven SM does not allow a sensitivity analysis of the mode for flow control applications. One key strength of the SM, however, is that it facilitates excellent predictions for the critical flow rate for the onset of the PVC, despite its low requirements regarding the data (time-resolved pressure measurements at a single location suffice). Additionally, it quantifies the influence of stochastic forcing on the mode. Furthermore, the growth rate can be used as a control parameter for flow optimisations. Owing to the turbulence, this is already utilisable in stable flow regimes, which can be of great practical relevance. It enables to characterise the PVC close to the critical flow rate limit without exceeding this limit.

At last, it needs to be noted critically that laser Doppler velocimetry is not well suited for thoroughly analysing the PVC. First, the lack of synchronous field measurements impedes the extraction of global coherent structures. Secondly, it prohibits the separation of coherent and stochastic fluctuations in the time series. This forces the LSA to fall back to simplified turbulence models giving rise to errors in the model. Further errors are introduced by the limited domain size up- and downstream. This affects the growth rate and the wavemaker position in particular. Numerical simulations giving full access to all spatio-temporal structures in a much larger domain promise to provide superior accuracy and insight to the sensitivities of the PVC.

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