Low-Rank Approximation from Communication Complexity

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Abstract

In the low-rank approximation with missing entries problem, one is given data matrix $A \in \mathbb{R}^{n \times n}$ and binary weight matrix $W \in \{0,1\}^{n \times n}$. The goal is to find a rank-$k$ matrix $L$ for which:

$$\text{cost}(L) \overset{\text{def}}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{i,j} \cdot (A_{i,j} - L_{i,j})^2 \leq \text{OPT} + \epsilon \|A\|_F^2,$$

where $\text{OPT} = \min_{\text{rank}-k \mathbf{L}} \text{cost}(\mathbf{L})$ and $\epsilon$ is a given error parameter. The above problem is also known as matrix completion and, depending on the choice of $W$, captures factor analysis, low-rank plus diagonal decomposition, robust PCA, low-rank plus block matrix approximation, low-rank recovery from monotone missing data, and a number of other important problems.

Many of these problems are NP-hard, and while algorithms with provable guarantees are known in some cases, they either 1) run in time $n^{O(rk^2/\epsilon)}$, where $r \geq 1$ is a parameter of the weight matrix, or 2) make strong assumptions, for example, that $A$ is incoherent or that the entries in $W$ are chosen independently and uniformly at random.

In this work, we consider bicriteria algorithms, which output a rank-$k'$ matrix $L$, with $k' > k$, for which $\text{cost}(L) \leq \text{OPT} + \epsilon \|A\|_F^2$. We show, rather surprisingly, that a common heuristic, which simply sets $A$ to 0 where $W$ is 0, and then finds a standard low-rank approximation of this matrix, achieves this approximation bound with rank $k'$ depending on the communication complexity of $W$. Namely, interpreting $W$ as the two-player communication matrix of a Boolean function $f(x, y)$ with $x, y \in \{0,1\}^{\log n}$, we show that it suffices to set $k' = O(k \cdot 2^{R_{1-\text{sided}}(f)})$, where $R_{1-\text{sided}}(f)$ is the randomized communication complexity of $f$ with 1-sided error probability $\epsilon$. For many important problems, this yields bicriteria algorithms with $k' = k \cdot \text{poly}(\log n/\epsilon)$. We further show that $k' = O(k \cdot 2^{R_{2-\text{sided}}(f)})$, where $R_{2-\text{sided}}(f)$ is the randomized communication with 2-sided error, suffices to obtain an approximation of $\text{OPT} + \epsilon \|A\|_F^2 + \epsilon \|L_{\text{opt}}\|_F^2$, where $L_{\text{opt}}$ is any rank-$k$ matrix achieving $\text{OPT}$.

Beyond these results, we show that different models of communication yield bicriteria algorithms for natural variants of the low-rank approximation problem with missing entries. For example, multi-player number-in-hand communication complexity connects to tensor decomposition and non-deterministic communication complexity to Boolean low-rank factorization. We note that most of our results extend beyond the Frobenius norm, to low-rank approximation in any entrywise norm.
1 Introduction

The goal of low-rank approximation is to approximate a given \( n \times n \) data matrix \( A \) with a rank-\( k \) matrix \( L \). \( L \) can be written as the product \( L = U \cdot V \) of a “tall-and-thin” matrix \( U \) and a “wide-and-fat” matrix \( V \) with \( k \) columns and rows respectively. For \( k \ll n \) this approximation can lead to computational speedups: one can store the factors \( U \) and \( V \) with considerably less memory than storing \( A \) itself, and can compute the product \( U \cdot V \cdot x \) with a vector \( x \) much faster than computing \( A \cdot x \). Additionally, low-rank approximation is useful for denoising and can reveal low-dimensional structure in high-dimensional data (it is e.g., the basis behind principal component analysis). It thus serves as a useful preprocessing step in many applications, including clustering, data mining, recommendation systems, and many others. The optimal low-rank approximation to \( A \) with distance measured in the Frobenius, spectral, or any unitarily invariant norm can be computed in polynomial time using a singular value decomposition (SVD). There are also extremely efficient approximation algorithms for finding a near optimal \( L \) under different measures, including the Frobenius norm, spectral norm, and various entrywise norms. For a comprehensive treatment, we refer the reader to the surveys \[ KV09, \text{Mah11, Woo14} \].

Despite its wide applicability, in many situations standard low-rank approximation algorithms do not suffice. For example, it is common for entries to be missing in \( A \). Relatedly, certain entries may not obey underlying low-rank structure. For example, \( A \) may be close to low-rank but with a small number of corrupted entries, or may be a mixture of a low-rank matrix plus a high-rank, but still efficiently representable, diagonal or block diagonal matrix. In both these cases, one must compute a low-rank approximation of \( A \) ignoring the missing or corrupted entries. One can formalize this problem, considering a binary weight matrix \( W \) with \( W_{i,j} = 0 \) for each missing entry \((i,j)\) of \( A \) and \( W_{i,j} = 1 \) otherwise.

**Problem 1** (Low-Rank Approximation with Missing Entries). Given \( A \in \mathbb{R}^{n \times n} \), binary \( W \in \{0,1\}^{n \times n} \), and rank parameter \( k \), find rank-\( k \) \( L \) minimizing:

\[
\|W \odot (A - L)\|_F^2 = \sum_{i,j \in [n]} W_{i,j} \cdot (A_{i,j} - L_{i,j})^2,
\]

where for two matrices \( M \) and \( N \) of the same size, \( M \odot N \) denotes the entrywise (Hadamard product): with \((M \odot N)_{i,j} = M_{i,j} \cdot N_{i,j}\) and for integer \( n \), \([n]\) denotes \( \{1, \ldots, n\} \).

As stated, Problem 1 minimizes the squared Frobenius norm of \( W \odot (A - L) \). However any matrix norm can be used. In any case, is unclear how to extend standard low-rank approximation algorithms to solving Problem 1, since they optimize over the full matrix \( A \), without the ability to take into account \( W \) encoding entries that are missing or should be ignored. We note that Problem 1 is equivalent to minimizing \( \|A - (L + S)\|_F^2 \) where \( L \) is rank-\( k \) and \( S \) is any matrix with support restricted to the 0 entries of \( W \). If these zeros are on the diagonal, then \( S \) is diagonal. If they are sparse, then \( S \) is sparse, etc. This is how Problem 1 is traditionally stated in many applications.

1.1 Existing Work

The problem of missing entries is the focus of the entire field of data imputation, see, e.g., \[ vB12 \] and references therein. Various heuristics are used in the data mining community, such as replacing missing entries in a row with the average of the remaining entries or fitting them via regression
techniques. After such entries have been filled in, one can use existing low-rank approximation algorithms. Another common approach is to apply EM or alternating minimization. In fact, factor analysis, a slight variant of Problem 1 when $W$ is 0 on its diagonal and 1 off the diagonal, was one of the original motivations of the EM algorithm [DLR77, RT82]. Much recent work in the matrix completion literature studies when alternating minimization for Problem 1 converges in polynomial time under the assumptions that (1) there is a solution $L = U \cdot V \approx A$ which is incoherent, meaning that the squared row norms of $U$ and column norms of $V$ are small and (2) the entries of $W$ are selected at random or have nice pseudorandom properties [ZWG18, KC12, NUNS+14]. Under similar assumptions it can be shown that the Problem 1 and the related problem of robust PCA can be solved via polynomial time convex relaxation methods [CR09, WGR+09, CLMW11]. In many cases, these algorithms perform well in practice even when the above assumptions do not hold. Additionally, they can be proven to run in polynomial time in some common settings when the entries of $W$ are not random — e.g., when $W$ is zero only on its diagonal or only at a few arbitrary locations. That is, when we want to approximate $A$ as a low-rank plus diagonal component, or a low-rank matrix with arbitrary sparse corruptions respectively. However, these results still require assuming the existence of $U \cdot V$ that is incoherent and further that is exact — with $U \cdot V$ equal to $A$ on all non-missing entries [CSPW11, SCPW12, JO14, SW15].

A natural question is if for common missing entry patterns, one can obtain provable algorithms without incoherence or other strong assumptions. This approach was taken in [RSW16] in the context of weighted low-rank approximation, where $W$ is a nonnegative matrix and the objective is still to minimize $\|W \circ (A - L)\|_F^2$. When $W$ is binary, this reduces to Problem 1. In [RSW16] it was shown that if $W$ has at most $r$ distinct columns, then it is possible to obtain a relative error guarantee in $2^{\text{poly}(rk/\epsilon)} \cdot \text{poly}(n)$ time. More generally, if the rank of $W$ over the reals is at most $r$, then $n^{\text{poly}(rk/\epsilon)}$ time is achievable. Note that such algorithms are only polynomial time if $k, r,$ and $1/\epsilon$ are very small. In many common use cases, such as when $W$ is all 0s on the diagonal and 1 off-diagonal, (corresponding to low-rank plus diagonal decomposition), or when $W$ is all 0s above the diagonal and 1s on or beneath the diagonal, $r$ is large: in fact $\text{rank}(W) = r = n$ in these cases.

When $A$ is low-rank with sparse corruptions, specifically when $W$ has at most $t$ nonzero entries per row and column, the algorithms of [RSW16] can be applied if there is an exact solution (with $A = L$ on all non-corrupted entries). [RSW16] referred to this problem as adversarial matrix completion and gave an $n^{O(k^3)}$ time algorithm. This is only polynomial time for constant values of $t$ and $k$, and even for constant $t$ and $k$ is a very large polynomial running time. Moreover, their method cannot be used in the approximate case since it requires creating a low-rank weight matrix $W'$ whose support matches that of $W$. Since the binary matrix $W$ may be far from low-rank, the non-zero entries of $W$ and $W'$ necessarily have very different values. This introduces significant error, unless $A = L$ exactly on the support of $W$.

### 1.2 Our Contributions

With the goal of obtaining fast algorithms the low-rank approximation problem with missing entries, we consider bicriteria algorithms with additive error. These are algorithms for which one allows the rank $k'$ of the output matrix $L$ to be slightly larger than $k$, but one still compares to the best rank-$k$ approximation. Formally, given $A \in \mathbb{R}^{n \times n}$, binary $W \in \{0, 1\}^{n \times n}$, and an error parameter $\epsilon$, we would like to find a rank-$k'$ matrix $L$ for which:

$$\|W \circ (A - L)\|_F^2 \leq OPT + \epsilon\|A\|_F^2,$$

(1)
where $OPT = \min_{\text{rank-}k \hat{L}} \| W \circ (A - \hat{L}) \|_F^2$ is the optimal value of Problem 1.

Assuming a variant of the Exponential Time Hypothesis, [RSW16] shows a lower bound of $2^{\Omega(r)}$ time for finding rank-$k L$ achieving (1) with constant $\epsilon$ when $W$ is rank-$r$. Thus the relaxation to bicriteria approximation seems necessary. In a number of applications it is not essential for the output rank $k'$ to be exactly $k$ – as long as $k'$ is small, one still obtains significant time and memory savings. Indeed, bicriteria algorithms are a common method for coping with hardness [DV07, FFSS07, CW15a, SWZ19, MMSW15, HT16]. The starting point of our work is the following question:

For which patterns $W \in \{0, 1\}^{n \times n}$ can one obtain efficient bicriteria low-rank approximation algorithms with $k' \leq k \cdot \text{poly}(\log n/\epsilon)$ satisfying (1)?

Main Results: We show rather surprisingly that the answer to this question is related to the randomized communication complexity of the matrix $W$. If the rows and columns of $W \in \{0, 1\}^{n \times n}$ are indexed by strings $x \in \{0, 1\}^{\log n}$ and $y \in \{0, 1\}^{\log n}$, respectively, we can think of the entries of $W$ as the entries of a two-player communication matrix for a Boolean function $f$, where $f(x, y) = W_{x,y}$. Here Alice has $x$ and Bob has $y$ and the two parties want to exchange messages with as few bits as possible to compute $f(x, y)$ with probability at least $1 - \delta$. The number of bits required is the randomized communication complexity $R_\delta(f)$. If we further require that the protocol never errs when $f(x, y) = 1$, but for any fixed $(x, y)$ for which $f(x, y) = 0$, the protocol errs with probability at most $\delta$, then the number of bits required is the 1-sided randomized communication complexity $R_\delta^{1-sided}(f)$. Informally, we show:

**Theorem 1.** Letting $f$ be the function computed by $W$ and $\neg f$ be its negation, there is a bicriteria low-rank approximation $L$ with rank $k' = k \cdot 2^{R_\delta^{1-sided}(-f)}$ achieving:

$$\| W \circ (A - L) \|_F^2 \leq OPT + 2\epsilon\|A\|_F^2,$$

where $OPT = \min_{\text{rank-}k \hat{L}} \| W \circ (A - \hat{L}) \|_F^2$. $L$ is computable in $O(\text{nnz}(A) + n \cdot \text{poly}(k'/\epsilon))$ time.

As we will see, for many common $W$, $R_\delta^{1-sided}(-f)$ is very small – giving $2^{R_\delta^{1-sided}(-f)}$ at most $\text{poly}(\log n/\epsilon)$. We also show a bound in terms of the communication complexity with 2-sided error.

**Theorem 2.** Letting $f$ be the function computed by $W$, there is a bicriteria low-rank approximation $L$ with rank $k' = k \cdot 2^{R_\epsilon(f)}$ achieving:

$$\| W \circ (A - L) \|_F^2 \leq OPT + 2\epsilon\|A\|_F^2 + \epsilon\|L_{\text{opt}}\|_F^2,$$

where $OPT = \min_{\text{rank-}k \hat{L}} \| W \circ (A - \hat{L}) \|_F^2$ and $L_{\text{opt}}$ is any rank-$k$ matrix achieving $OPT$. $L$ is computable in $O(\text{nnz}(A) + n \cdot \text{poly}(k'/\epsilon))$ time.

Further, the algorithm achieving Theorems 1 and 2 is extremely simple and efficient: just zero out the entries in $A$ corresponding to entries in $W$ that are 0 (i.e., compute $A \circ W$), and then output a standard rank-$k'$ of the resulting matrix! This is already a widely-used heuristic for solving Problem 1 [AFK⁺01, ZLG16], and we obtain the first provable guarantees. An optimal low-rank approximation of $A \circ W$ can be computed in polynomial time via an SVD. An approximation achieving relative error $(1 + \epsilon)$ can be computed with high probability in $O(\text{nnz}(A) + n \cdot \text{poly}(k'/\epsilon))$ time, giving the runtime bounds of Theorems 1 and 2 [CW15b].

Theorems 1 and 2 provide the first bicriteria approximation algorithms for Problem 1 with small $k'$ for a number of important special cases of the weight matrix $W$:
1. If $W$ has at most $t$ zero entries in each row, this is the Low-Rank Plus Sparse (LRPS) matrix approximation, which captures the challenge of finding a low-rank approximation when a few entries are not known, or do not obey underlying low-rank structure. It has been studied in the context of adversarial matrix completion [SW15], robust matrix decomposition [HKZ11, CLMW11], optics, system identification [BGS90], and more [CSPW11].

2. If $W$ is zero exactly on the diagonal entries, this is the Low-Rank Plus Diagonal (LRPD) matrix approximation problem. This problem arises from the observation that, in practice, many matrices that are not close to low-rank, are close to diagonal, or contain a mixture of diagonal and low-rank components [CSPW11]. This observation has been used, for example, to construct compact representations of kernel matrices in machine learning [WZQZ14], weight matrices in deep neural networks [MB18, ZLG16], and covariance matrices [TBK91, Ste14]. The LRPD problem also arises in applications related to source separation [LY17] and variational inference [MFA17] and is closely related to the factor analysis problem [Spe04, SCPW12].

3. If $W$ is the negation of a block-diagonal matrix with blocks of varying sizes, meaning that $W$ is 0 on entries in the blocks and 1 on entries outside of the blocks, we call this the Low-Rank Plus Block-Diagonal (LRPBD) matrix approximation problem. This is a natural generalization of the LRPD problem and has been studied e.g., in the context of anomaly detection in networks [AS15], foreground detection [GBhZ12], and robust principal component analysis [LTN16]. We also consider the natural generalization of LRPS approximation discussed above, which we call the Low-Rank Plus Block-Sparse (LRPBS) matrix approximation problem.

4. If each row of $W$ has a prefix of an arbitrary number of ones, followed by a suffix of zeros, this is the Monotone Missing Data Pattern (MMDP) problem. This is a very common missing data pattern, corresponding to the event that when a variable is missing from an individual, all subsequent variables are also missing. Methods for handling this data problem are, e.g., included in the SAS/STAT software package for statistical analyses [SAS].

We refer the reader to [vB18] for more examples of common missing patterns, such as “connected” and “file matching” patterns.

5. If $W$ is the negation of a banded matrix where $W_{i,j} = 0$ iff $|i - j| < p$ for some distance $p$, we call this the Low-Rank Plus Banded (LRPBand) matrix approximation problem. Variants of this problem arise commonly in scientific computing and machine learning applications, in particular in the approximation of kernel matrices via fast multipole methods, such as the fast Gauss transform [Rok85, GS91, YDGD03]. Roughly, these methods approximate a kernel matrix using a low-rank ‘far-field’ component, which approximates the kernel function between far away points, and a second ‘near-field’ component, which explicitly represents the kernel function between close points. If points are in one dimension and sorted, this corresponds to approximating $A$ with a low-rank matrix plus a banded matrix. While the mentioned methods typically compute the low-rank component analytically (using polynomial expansions of the kernel function), a natural alternative is to seek an optimal decomposition via Problem 1. In many of these applications, it is common to work in higher dimensions. For example, in the two-dimensional case, each $i \in [n]$ can be mapped to $(i_1, i_2) \in [\sqrt{n}] \times [\sqrt{n}]$ where $i_1, i_2$ correspond to the first and second halves of $i$’s binary expansion. $W_{i,j} = 0$ iff $|i_1 - j_1| + |i_2 - j_2| < p$. We give similar bounds for this multidimensional problem.
We summarize our results for the above weight patterns in Table 1. We give more detail on the specific functions \( f \) used in these applications in Sections 2 and 3, but note that (1), (2), and (3) use variants of the Equality problem, which has \( O(\log(1/\epsilon)) \) randomized 1-sided error communication complexity, (4) and (5) use a variant of the Greater-Than problem with \( O(\log \log n + \log(1/\epsilon)) \) randomized 2-sided error communication complexity for \( \log n \) bit strings.

| Pattern                                    | \( k' \)                      | Communication Problem          | Ref.         |
|--------------------------------------------|-------------------------------|--------------------------------|--------------|
| LRPD/LRPBD                                 | \( O(k/\epsilon) \)          | Equality                       | Cors. 18 & 19|
| LRPS/LRPBS (w/ sparsity \( t \))          | \( O(kt/\epsilon) \)         | Variant of equality            | Cors. 20 & 21|
| MMDP                                       | \( k \cdot \text{poly}\left(\frac{\log n}{\epsilon}\right) \) | Greater-Than                   | Cor. 26      |
| LRPBand/Multi-dim. banded                  | \( k \cdot \text{poly}\left(\frac{\log n}{\epsilon}\right) \) | Variant of Greater-Than        | Cors. 24 & 25|
| Subsampled Toeplitz                        | \( O(\min(pk, k/\epsilon)) \) | Equality mod \( p \)          | Cor. 23      |

Table 1: Summary of applications of Theorems 1 and 2.

**Other Communication Models:** Theorems 1 and 2 show a strong connection between communication complexity and low-rank approximation with missing entries. A natural question is:

*Can other notions of communication complexity, such as multi-party communication complexity, non-deterministic communication complexity, and communication complexity of non-Boolean functions yield algorithms for low-rank approximation with missing entries?*

We answer this question in the affirmative. We first look at multi-party communication complexity, which we show corresponds to tensor low-rank approximation with missing entries. Here we focus on order-3 tensors, though our results are proven for arbitrary order-\( t \) tensors. A tensor is just an array \( A \in \mathbb{R}^{n \times n \times n} \). In the low-rank tensor approximation problem with missing entries we are given such an \( A \) and a weight tensor \( W \in \{0,1\}^{n \times n \times n} \) and the goal is to find rank-\( k \) tensor \( L \) minimizing \( \|W \circ (A - L)\|_F \). This problem has been widely studied in the context of low-rank tensor completion [GRY11, LMWY13, MHWG14] and robust tensor PCA [LWQ+15, LFC+16], which corresponds to the setting where the zeros in \( W \) represent sparse corruptions of an otherwise low-rank tensor. Applications include, for example, color image and video reconstruction. Other applications include low-rank plus diagonal tensor approximations [BKK16], where \( W \) is zero on its diagonal and one everywhere else. We show:

**Theorem 3** (Multiparty Communication Complexity \( \rightarrow \) Tensor Low-Rank Approximation). Let \( f \) be the function computed by \( W \in \{0,1\}^{n \times n \times n} \), \( \neg f \) be its negation, and \( R_{3,1-sided}^3(\neg f) \) be the randomized 3-party communication complexity of \( \neg f \) in the number-in-hand blackboard model with 1-sided error. A bicriteria low-rank approximation \( L \) with rank \( k' = O\left((k/\epsilon)^2 \cdot 4^{R_{3,1-sided}^3(\neg f)}\right) \) achieving:

\[
\|W \circ (A - L)\|_F^2 \leq OPT + 2\epsilon\|A\|_F^2,
\]

where \( OPT = \inf_{\text{rank-}k \ L} \|W \circ (A - L)\|_F^2 \) can be computed in \( O(\text{nnz}(A)) + n \cdot \text{poly}(k/\epsilon) \) time.
We give applications of Theorem 3 to the low-rank plus diagonal tensor approximation problem, achieving $k' = O(k^2/\epsilon^4)$ (Cor. 29) and the low-rank plus sparse tensor approximation problem achieving $k' = O\left(\frac{k^2n^4}{\epsilon^6}\right)$ (Cor. 30), where $t$ is the maximum number of zeros on any face of $W$.

We also consider a common variant of low-rank approximation studied in data mining and information retrieval: Boolean low-rank approximation (also called binary low-rank approximation). Here one is given binary $A \in \{0,1\}^{n \times n}$ and seeks to find $U \in \{0,1\}^{n \times k}$ and $V \in \{0,1\}^{k \times n}$ minimizing $\|A - U \cdot V\|_0$ where $U \cdot V$ denotes Boolean matrix multiplication and $\| \cdot \|_0$ is the entrywise $\ell_0$ norm, equal to the squared Frobenius norm in this case. While Boolean low-rank approximation is NP-hard in general [DHJ15, GV18], there is a large body of work studying heuristic algorithms and approximation schemes for the problem, when no entries of $A$ are missing [LVA08, SJY09, Vai12, BKW17, FGP18]. We show that any black-box algorithm for standard Boolean low-rank approximation yields a bicriteria algorithm for Boolean low-rank approximation with missing entries, with rank depending on the nondeterministic communication complexity of the weight matrix $W$.

**Theorem 4** (Nondeterministic Communication Complexity $\rightarrow$ Boolean Low-Rank Approximation). Let $f$ be the function computed by $W$ and $N(f)$ be the nondeterministic communication complexity of $f$. For any $k' \geq k \cdot 2^{N(f)}$, if one computes $U, V \in \{0,1\}^{n \times k'}$ satisfying $\|A \circ W - U \cdot V\|_0 \leq \min_{U, V \in \{0,1\}^{n \times k'}} \|A \circ W - U \cdot V\|_0 + \Delta$ then:

$$\|W \circ (A - U \cdot V)\|_0 \leq 2^{N(f)} \cdot OPT + \Delta,$$

where $OPT = \min_{U, V \in \{0,1\}^{n \times k \times n}} \|W \circ (A - U \cdot V)\|_0$ and $U \cdot V$ denotes Boolean matrix multiplication.

We can apply Theorem 4 for example, to the low-rank plus diagonal Boolean matrix approximation problem, where $W$ is zero on its diagonal and one everywhere else. In this case we have $2^{N(f)} = \log n$ and correspondingly $k' = k \log n$ (Cor. 31).

1.3 Our Techniques

**Basic idea.** The key ideas behind Theorems 1 and 2 are similar. We focus on Theorem 1 for exposition. We want to argue that any near optimal rank-$k'$ approximation of $A \circ W$, gives a good bicriteria solution to the rank-$k$ approximation problem with missing entries. For simplicity, here we focus on showing this for the best rank-$k'$ approximation, $L = \min_{k' \leq k} \|A \circ W - \hat{L}\|_F^2$. We will show that $\|W \circ (A - L)\|_F^2 \leq OPT + O(\epsilon)\|A\|_F^2$. We do this via a comparison. Namely, we exhibit a rank $k'$ matrix $\hat{L}$ that:

1. Nearly matches how well the optimum rank-$k$ solution $L_{opt}$ to Problem 1 approximates $A$ on the support of $W$. In particular, $\|(A - \hat{L}) \circ W\|_F^2 \leq \|(A - L_{opt}) \circ W\|_F^2 + O(\epsilon)\|A\|_F^2$.

2. Places no mass outside the support of $W$. In particular, $\|\hat{L} \circ (1 - W)\|_F^2 = 0$.

Since $L$ minimizes the distance to $(A \circ W)$ among all rank-$k'$ matrices, we have $\|A \circ W - L\|_F^2 \leq \|(A \circ W) - \hat{L}\|_F^2$. However, by (2), $\hat{L}$ exactly matches $A \circ W$ outside the support of $W$ – both matrices are 0 there. Thus $L$ must have at least as large error outside the support of $W$, and in turn cannot have larger error on the support of $W$. That is, we must have $\|(A - L) \circ W\|_F^2 \leq \|(A - \hat{L}) \circ W\|_F^2$. Then by (1), $L$ satisfies the desired bound.
From accurate communication protocols to good low-rank approximations. The key question becomes how to exhibit $\bar{L}$, which we do using communication complexity. We view $W$ as the communication matrix of some function $f : \{0,1\}^{\log n} \times \{0,1\}^{\log n} \to \{0,1\}$, with $W_{x,y} = f(x,y)$, where in $f$ we interpret $x,y \in [n]$ as their binary representations. It is well-known that the existence of a deterministic communication protocol $\Pi$ that computes $f$ with $D(f)$ total bits of communication implies the existence of a partition of $W$ into $2^{D(f)}$ monochromatic combinatorial rectangles. That is, there are $2^{D(f)}$ non-overlapping sets $R_i = S \times T$ for $S, T \in [n]$ that partition $W$ and that satisfy $W(R_i)$ is either all 1 or all 0. We could construct $\bar{L}$ by taking the best $k$-rank approximation of each $A(R_i)$ where $R_i$ is colored 1 (i.e., contains inputs with $f(x,y) = 1$). We could then sum up these approximations to produce $\bar{L}$ with rank $\leq k \cdot 2^{D(f)}$. Note that $\bar{L}$ is 0 outside the rectangles colored 1 – i.e., outside the support of $W$. Thus condition (2) above is satisfied. Further, $\bar{L}$ matches the optimal rank-$k$ approximation on each $R_i$ colored 1. So it must approximate $A$ at least as well as $L_{opt}$ on these rectangles. And since these rectangles fully cover the support of $W$ we have $\|(A - \bar{L}) \circ W\|_F^2 \leq \|(A - L_{opt}) \circ W\|_F^2$, giving the requirement of (1).

Unfortunately, essentially none of the $W$ that are of interest in applications admit efficient deterministic communication protocols. In fact, $k' = k \cdot 2^{D(f)}$ will typically be larger than $n$, giving a vacuous bound. Thus we turn to randomized communication complexity with error probability $\epsilon$. $R_{\epsilon}(f)$, which is much lower in these cases. A randomized protocol $\Pi$ achieving this complexity corresponds to a distribution over partitions of $W$ into $2^{R_{\epsilon}(f)}$ rectangles. These rectangles are not monochromatic but are close to it – letting $W_{\Pi}$ be the communication matrix of the (random) function computed by the protocol, $W_{\Pi}$ is partitioned into $2^{R_{\epsilon}(f)}$ monochromatic rectangles and further matches $W$ on each $(x,y)$ with probability at least $1 - \epsilon$. We prove that, even with this small error, constructing $\bar{L}$ as above using the partition of $W_{\Pi}$ instead of $W$ itself gives a solution nearly matching $L_{opt}$ up to small additive error. This additive error will involve $\|A\|_F^2$ and $\|L_{opt}\|_F^2$, depending on whether the protocol makes 1 or 2-sided error, as can be seen in Theorems 1 and 2.

Other communication models. In extending our results to other communication models, we first consider the connection between multiparty communication and tensor low-rank approximation. We consider protocols in the multiparty number-in-hand model, which, as in the two-player case, correspond to a partition of the communication tensor $W \in \{0,1\}^{n \times n \times n}$ into $2^{R(f)}$ monochromatic (or nearly monochromatic) rectangles of the form $R_i = S \times T \times U$ for $S, T, U \subseteq [n]$, where $R_{\epsilon}^3(f)$ is the randomized 3-player communication complexity of $W$. We can again argue the existence of a rank $k' = k \cdot 2^{R_{\epsilon}(f)}$ tensor $\bar{L}$, obtained by taking a near optimal low-rank approximation to each rectangle colored 1 in $W_{\Pi}$, which is mostly 0 outside the support of $W$ and at the same time competes with the best rank-$k$ tensor approximation $L_{opt}$ on the support of $W$. This lets us argue, as in the two player case, that the best rank-$k'$ approximation of $A \circ W$ also competes with $L_{opt}$. It is not known how to find this best rank-$k'$ approximation efficiently, however using an algorithm of [SWZ19] we can find a rank $k'' = O((k'/\epsilon)^2)$ bicriteria approximation achieving relative error $1 + \epsilon$. Overall we have $k'' = O((k/\epsilon)^2 \cdot 2^{R_{\epsilon}(f)})$, giving Theorem 3.

We next consider the nondeterministic communication complexity. In a nondeterministic communication protocol for a function $f$, players can make “guesses” at any point during the protocol $\Pi$. The only requirement is that, (1) for every $x, y$ with $f(x,y) = 1$, for some set of guesses made by the players, the protocol outputs $\Pi(x,y) = 1$ and (2) the protocol never outputs $\Pi(x,y) = 1$ for $x, y$ with $f(x,y) = 0$. Such a protocol using $N(f)$ bits of communication corresponds to covering the communication matrix $W$ with $2^N(f)$ possibly overlapping monochromatic rectangles. In
many cases, the nondeterministic complexity is much lower than the randomized communication complexity. However, for low-rank approximation in the Frobenius norm, the overlap is a problem. We cannot construct $\bar{L}$ simply by approximating each rectangle and adding these approximations together. $\bar{L}$ will be too “heavy” where the rectangles overlap. However, for the Boolean low-rank approximation problem, the overlap is less of a problem. We simply construct $\bar{L}$ in the same way, letting it be the AND of the approximations on each rectangle. In the end, we obtain an error bound of roughly $2^{N(f)} \cdot OPT$, owing to the fact that error may still build up on the overlapping sections. Since there are $2^{N(f)}$ rectangles total, each entry is overlapped by at most $2^{N(f)}$ of them. However, since $N(f)$ can be very small, this result gives a tradeoff with Theorems 1 and 2 (which can also be extended to the Boolean case). For example, in Corollary 31 we show how to obtain error $\approx O(\log n \cdot OPT)$ for the Boolean low-rank plus diagonal approximation problem, with rank $k' = O(k \log n)$. This is smaller than the $O(k/\epsilon)$ achieved by Theorem 1 for small $\epsilon$, which may be required to achieve good error if, e.g., $\|A\|_F^2$ is large.

An alternative approach. Interestingly, in the case when $W$ is zero on its diagonal and one elsewhere or has a few non-zeros per row (the low-rank plus diagonal and low-rank plus sparse approximation problems, respectively) the existence of $\bar{L}$ satisfying the necessary conditions (1) and (2) above can be proven via a very different technique. The key idea is a structural result: that any low-rank matrix cannot concentrate too much weight on more than a few entries of its diagonal, or more generally, on a sparse support outside a few rows. Thus we can obtain $\bar{L}$ from $L_{opt}$ by explicitly zero-ing out these few large entries falling outside the support of $W$ (e.g., on its diagonal when $W = 1 - I$). We detail this approach in Section 6, giving a bound matching Theorem 1 in this case. We show that the same structural result can also be used to obtain a fixed-parameter-tractable, relative error, non-bicriteria approximation algorithm for Problem 1 in the low-rank plus diagonal case, as well as for the closely related factor analysis problem. We are unaware of any formal connection between this structural result and our communication complexity framework; however, establishing one would be very interesting.

1.4 Road Map

Section 2 We give preliminaries, defining the communication models we use and giving communication complexity bounds for common weight matrices in these models.

Section 3 We prove our main results, Theorems 1 and 2 connecting randomized communication complexity to low-rank approximation with missing entries. We instantiate these results for the common weight matrices shown in Table 1.

Section 4 We prove Theorem 3, connecting tensor low-rank approximation with missing entries to multiparty communication complexity. We give examples instantiating the theorem.

Section 5 We prove Theorem 4, connecting Boolean low-rank approximation with missing entries to nondeterministic communication complexity. We give examples instantiating the theorem.

Section 6 We give an alternative approach to proving Theorem 1 in the low-rank plus diagonal and low-rank plus sparse case. We show that this approach also yields an FPT, relative error, non-bicriteria algorithm for Problem 1 for the low-rank plus diagonal problem.
2 Preliminaries

2.1 Notation and Conventions

Throughout we use \( \log z \) to denote the base-2 logarithm of \( z \). For simplicity, so that we can associate any \( W \in \mathbb{R}^{n \times n} \) with a function \( f : \{0,1\}^{\log n} \times \{0,1\}^{\log n} \rightarrow \{0,1\} \) we will assume that \( n \) is a power of 2 and so \( \log n \) is an integer. Our results can be easily extended to general \( n \). Given a matrix \( M \in \mathbb{R}^{n \times n} \) and a combinatorial rectangle \( R = S \times T \) for \( S, T \subseteq [n] \), we let \( M_R \) denote the submatrix of \( M \) indexed by \( R \). For matrix \( M \) we let \( 1 - M \) denote the matrix \( N \) with \( N_{i,j} = 1 - M_{i,j} \). E.g., \( 1 - I \) is the matrix with all zeros on diagonal and all ones off diagonal.

While in the introduction we focus on low-rank approximation in the Frobenius norm, many of our results will apply to any entrywise matrix norm of the form:

**Definition 5.** An entrywise matrix norm \( \| \cdot \|_* : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \) is a function of the form:

\[
\| M \|_* = \sum_{i=1}^{n} \sum_{j=1}^{n} g(M_{i,j}),
\]

where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is some nonnegative function.

\( g(x) = x^2 \) gives the squared Frobenius norm, \( g(x) = |x|^p \) gives the entrywise \( \ell_p \) norm, \( g(x) = 1 \) iff \( x \neq 0 \) gives the entrywise \( \ell_0 \) norm, etc. See [SWZ17, BKW17, CGK+17, BBB+19] for a discussion of standard low-rank approximation algorithms for these norms. As discussed, our bicriteria results will simply require applying one of these algorithms to compute a near-optimal low-rank approximation to \( A \circ W \) (i.e., \( A \) with the missing or corrupted entries zeroed out).

2.2 Communication Complexity Models

We give a brief introduction to the communication models we consider, and refer the reader to the textbooks [KN97, RY19] for more background. We mostly consider two-party communication of Boolean functions, though will discuss extensions to more than two parties below.

Consider two parties, Alice and Bob, holding inputs \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \) respectively. They exchange messages in order to compute a function \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0,1\} \) evaluated at \((x, y)\). They would like to do this while minimizing the total number of bits exchanged. The communication between the parties is determined by a possibly randomized protocol, which specifies the message of the next player to speak as a function of previous messages received by that player and that player’s input. For a given protocol \( \Pi \), we let \( |\Pi(x, y)| \) denote the number of bits transmitted by the players on inputs \( x \) and \( y \), and we let \( |\Pi| = \max_{x,y} |\Pi(x, y)| \).

Let \( M \) be the communication matrix of \( f \), that is, the matrix whose rows are indexed by elements of \( \mathcal{X} \) and columns by elements of \( \mathcal{Y} \), and for which \( M_{x,y} = f(x, y) \). A well known and useful property is that \( \Pi \) partitions \( M \) into rectangles \( R = S \times T \), where \( S \subseteq \mathcal{X} \) and \( T \subseteq \mathcal{Y} \), and every pair \((x, y)\) of inputs with \((x, y) \in S \times T \) has the same output when running protocol \( \Pi \). The number of rectangles in the partition is equal to \( 2^{|\Pi|} \). We call the unique output of \( \Pi \) on a rectangle \( S \times T \) the label of the rectangle.

**Definition 6** (Deterministic Communication Complexity). The deterministic communication complexity \( D(f) = \min_{\Pi} |\Pi| \), where the minimum is taken over all protocols \( \Pi \) for which \( \Pi(x, y) = f(x, y) \) for every pair \((x, y)\) of inputs. Equivalently, \( D(f) \) is the minimum number so that \( M \) can...
be partitioned via a protocol $\Pi$ into $2^{D(f)}$ rectangles for which for every rectangle $R$ and $b \in \{0, 1\}$, if $R$ is labeled $b$, then for all $(x, y) \in R$, $f(x, y) = b$.

Another notion we need is non-deterministic communication complexity, which can be smaller than the deterministic communication complexity. In a nondeterministic protocol, Alice and Bob are each allowed to make arbitrary ‘guesses’. If $f(x, y) = 1$ the protocol is required to output 1 for at least some set of guesses. If $f(x, y) = 0$, the protocol should never output 1, no matter the guesses. Rather than partitioning $M$ into rectangles, a non-deterministic protocol $\Pi$ covers the support of $M$ with a set of at most $2^{\Pi}$ possibly overlapping rectangles.

**Definition 7** (Non-deterministic Communication Complexity). The non-deterministic communication complexity $N(f) = \min_{\Pi} |\Pi|$, where the minimum is taken over all all non-deterministic protocols $\Pi$ computing $f$. Equivalently, $N(f)$ is the minimum number so that $M$ can be covered via a protocol by $2^{N(f)}$ possibly overlapping rectangles such that (1) for every input $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with $f(x, y) = 1$, we have that $(x, y)$ occurs in at least one of these rectangles, and (2) there is no input $(x, y)$ with $f(x, y) = 0$ which occurs in any of these rectangles.

We next turn to randomized communication complexity. For the purposes of this paper, we will consider public coin randomized communication complexity, i.e., there is a shared random string $r$ that both Alice and Bob have access to. In a randomized protocol $\Pi$, Alice and Bob see $r$ and then run a deterministic protocol $\Pi_r$. We say a protocol $\Pi$ is a $(\delta_1, \delta_2)$-error protocol if for all $x, y \in \mathcal{X} \times \mathcal{Y}$, with $f(x, y) = 1$, $\Pr[\pi_r(x, y) = f(x, y)] \geq 1 - \delta_1$ and for all $x, y \in \mathcal{X} \times \mathcal{Y}$ with $f(x, y) = 0$, $\Pr[\pi_r(x, y) = f(x, y)] \geq 1 - \delta_2$. We can then define a general notion of randomized communication complexity:

**Definition 8** (Randomized Communication Complexity – General). The $(\delta_1, \delta_2)$-error randomized communication complexity $R_{\delta_1, \delta_2}(f) = \min_{\Pi} |\Pi|$, where the minimum is taken over all $(\delta_1, \delta_2)$-error protocols $\Pi$. Equivalently, $R_{\delta_1, \delta_2}(f)$ is the minimum number so that there is a distribution over protocols inducing partitions of $M$, each containing at most $2^{R_{\delta_1, \delta_2}(f)}$ rectangles, such that (1) for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with $f(x, y) = 1$, with probability at least $1 - \delta_1$, $(x, y)$ lands in a rectangle which is labeled 1 and (2) for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with $f(x, y) = 0$, with probability at least $1 - \delta_2$, $(x, y)$ lands in a rectangle which is labeled 0.

Definition 8 is typically specialized to two cases: the standard randomized communication complexity with 2-sided error and the randomized communication complexity with 1-sided error.

**Definition 9** (Randomized Communication Complexity – 2-sided). The $\delta$-error randomized communication complexity of $f$ is $R_{\delta}(f) \overset{\text{def}}{=} R_{\delta, \delta}(f)$.

**Definition 10** (Randomized Communication Complexity – 1-Sided). The $\delta$-error 1-sided randomized communication complexity of $f$ is $R_{\delta}^{1\text{-sided}}(f) \overset{\text{def}}{=} R_{0, \delta}(f)$.

In Theorem 1 we consider the 1-sided communication complexity of $\neg f$, which we can see is equal to $R_{\delta, 0}(f)$. In giving a number of communication bounds, it will be helpful to also consider the 1-way communication complexity:

**Definition 11** (1-way Randomized Communication Complexity). The $\delta$-error 1-way randomized communication complexity $R_{\delta}^{1\text{-way}}(f) = \min_{\Pi} |\Pi|$, where the minimum is taken over all $\delta$-error
protocols in which Alice sends a single message to Bob who then outputs the answer. Note that Bob’s output bit is included in the communication complexity. Equivalently, \( R_1^{\text{1-way}}(f) \) is the minimum number so that there is a distribution on 1-way protocols inducing partitions of \( M \) each containing at most \( 2^{R_1^{\text{1-way}}(f)} \) rectangles, such that (1) for every \((x, y) \in X \times Y\), with probability at least \(1 - \delta\), \((x, y)\) lands in a rectangle labeled \( f(x, y)\) (2) each partition is obtained by considering a partition \( P_X \) of \( X \) and choosing rectangles of the form \( S \times T \) where \( S \in P_X \) and \( T \subseteq Y \).

We note that one can combine these notions in various ways, so one could look at \( R_1^{\text{1-sided,1-way}} \), for example, with the corresponding natural definition. We also consider communication with more than 2 players in the number-in-hand blackboard model. In this setting there are \( t \) players and an underlying \( t \)-th order communication tensor \( M \) with entries corresponding to elements in \( X^1 \times X^2 \times \cdots \times X^t \). Again a deterministic protocol number-in-hand protocol partitions \( M \) into combinatorial rectangles \( A_1 \times A_2 \times \cdots \times A_t \), where \( A_i \subseteq X^i \) for \( i = 1, 2, \ldots, t \), and each rectangle is labeled either 0 or 1. See Section 4 for a formal definition.

2.3 Specific Communication Bounds

We discuss a few problems that will be particularly useful for our applications. We will only need communication upper bounds and in specific models. Note that in this section, as is standard, we state bounds for communication problems with \( n \)-bit inputs. In our applications to low-rank approximation with missing entries, we will typically apply the bounds when the input size is \( \log n \).

Equality In the Equality problem, denoted \( EQ \), there are two players Alice and Bob, holding strings \( x, y \in \{0, 1\}^n \) respectively, and the function \( EQ(x, y) = 1 \) if \( x = y \), and \( EQ(x, y) = 0 \) otherwise.

**Theorem 12** ([KN97], combining Corollaries 26 and 27 of [BCK+16]). \( R_1^{\text{1-way}}(EQ) \leq (1 - \delta) \log((1 - \delta)^2/\delta) + 5 \), and \( R_1^{\text{1-sided,1-way}}(EQ) \leq \log(1/\delta) + 5 \).

We also can bound the nondeterministic communication complexity of inequality, i.e., the function \( NEQ(x, y) \) with \( NEQ(x, y) = 1 \) iff \( x \neq y \).

**Theorem 13.** \( N(NEQ) \leq \lceil \log n \rceil + 2 \).

**Proof.** Alice simply guesses an index at which \( x \) and \( y \) differ and sends this index (using \( \lceil \log n \rceil \) bits) along with the value of \( x \) at this index to Bob. Bob sends the value of \( y \) at this index and the players check if \( x \) and \( y \) differ at the index. \( \square \)

Essentially the same protocol can be used to solve the negation of the disjointness problem, with \( \neg \text{DISJ}(x, y) = 1 \) only if there is some \( k \in [n] \) with \( x(k) = y(k) = 1 \). We thus have:

**Theorem 14** ([She12]). \( N(\neg \text{DISJ}) \leq \lceil \log n \rceil + 2 \).

Greater-Than In the Greater-Than problem, denoted \( GT \), there are two players Alice and Bob, holding integers \( x, y \in \{0, 1, 2, \ldots, n - 1\} \) respectively, and the function \( GT(x, y) = 1 \) if \( x > y \), and \( GT(x, y) = 0 \) otherwise.

**Theorem 15** ([Nis93]). \( R_\delta(GT) = O(\log(n/\delta)) \).
Equality-Modulo-$p$  In the Equality-Modulo-$p$ problem, denoted $EQ_p$, there are two players Alice and Bob, holding integers $x, y \in \{0, 1, 2, \ldots, n-1\}$ respectively, and the function $EQ_p(x, y) = 0$ if $x - y = 0 \mod p$, and otherwise $EQ_p(x, y) = 1$.

Theorem 16. $D(EQ_p) \leq \lfloor \log p \rfloor + 1$, $R_{\delta}^{1\text{-way}}(EQ_p) \leq (1 - \delta) \log((1 - \delta)^2/\delta) + 5$, and $R_{\delta}^{1\text{-way},1\text{-sided}}(EQ_p) \leq \log(1/\delta) + 5$.

Proof. Note that the players can replace their inputs with $x \mod p$ and $y \mod p$ without loss of generality, and the problem is now equivalent to testing if $x = y$ on $\lceil \log p \rceil$-length bit strings. The bounds now follow from Theorem 12. $D(EQ_p)$ follows since Alice can just send $x \mod p$ using $\lfloor \log p \rfloor$ bits and Bob can send the answer using 1 bit. \hfill \square

\section{Bicriteria Approximation from Communication Complexity}

In this section we prove our main results, Theorems 1 and 2, which connect the randomized communication complexity of the binary matrix $W$ to the rank required to solve Problem 1 efficiently up to small additive error. We prove a general theorem connecting the rank to both $R_{\delta}^1$ and $R_{\delta}^2$, both for the 1-sided and 2-sided error complexity. Theorem 17 follows since Alice can just send $x \mod p$ using $\lfloor \log p \rfloor$ bits and Bob can send the answer using 1 bit.

Theorem 17 (Randomized Communication Complexity $\rightarrow$ Bicriteria Approximation). Consider $W \in \{0, 1\}^{n \times n}$ and let $f$ be the function computed by it. For $k' \geq k \cdot 2^{R_{\delta,2}(f)}$, and any entrywise norm $\| \cdot \|_*$ (Def. 5), for any $L$ satisfying $\|A \circ W - L\|_* \leq \min_{\text{rank} - k' \cdot \hat{L}} \|A \circ W - \hat{L}\|_* + \Delta$:

$$\| (A - L) \circ W \|_* \leq OPT + \epsilon_1 \| A \|_* + \epsilon_2 \|L_{\text{opt}}\|_* + \Delta,$$

where $OPT = \min_{\text{rank} - k \cdot \hat{L}} \| (A - L) \circ W \|_*$ and $L_{\text{opt}}$ is any rank-$k$ matrix achieving $OPT$.

Proof. As discussed (Def. 8), $R_{\epsilon_1,\epsilon_2}(f)$ is the minimum number so that there is a distribution on protocols inducing partitions of $W$, each containing at most $2^{R_{\delta,2}(f)}$ rectangles, such that (1) for every $x, y \in \{0, 1\}^{\log n}$ with $f(x, y) = 1$, $(x, y)$ lands in a rectangle labeled 1 with probability $\geq 1 - \epsilon_1$ and (2) for every $x, y \in \{0, 1\}^{\log n}$ with $f(x, y) = 0$, $(x, y)$ lands in a rectangle labeled 0 with probability $\geq 1 - \epsilon_2$. In other words, letting $W_{\Pi}$ be the (random) matrix corresponding to the function computed by the protocol: (1) $W \circ (1 - W_{\Pi})$ has each entry equal to 1 with probability $\leq \epsilon_1$ and (2) $W_{\Pi} \circ (1 - W)$ has each entry equal to 1 with probability $\leq \epsilon_2$. Thus, fixing some $L_{\text{opt}}$:

$$\mathbb{E}_{\text{protocol } \Pi} \| A \circ W \circ (1 - W_{\Pi}) \|_* + \| L_{\text{opt}} \circ W_{\Pi} \circ (1 - W) \|_* \leq \epsilon_1 \| A \circ W - L_{\text{opt}} \|_* + \epsilon_2 \| (1 - W) \|_* \leq \epsilon_1 \| A \|_* + \epsilon_2 \| L_{\text{opt}} \|_*.$$

Thus, there is at least one protocol $\Pi$ (inducing a partition with $\leq 2^{R_{\delta,2}(f)}$ rectangles) with:

$$\| A \circ W \circ (1 - W_{\Pi}) \|_* + \| L_{\text{opt}} \circ W_{\Pi} \circ (1 - W) \|_* \leq \epsilon_1 \| A \|_* + \epsilon_2 \| L_{\text{opt}} \|_*.$$

Let $P_1$ be the set of rectangles on which the protocol achieving (2) returns 1 and $P_0$ be the set on which it returns 0. For any $R \in P_1$ let $L^R = \arg \min_{\text{rank} - k \cdot \hat{L}} \| A \circ W_{\Pi} - \hat{L} \|_*$ (note that $L^R$ is the size of $R$). Let $L^R$ be the $n \times n$ matrix equal to $L^R$ on $R$ and 0 elsewhere. Let $L = \sum_{R \in P_1} L^R$. 12
Note that $\bar{L}$ has rank at most $\sum_{R \in P_1} \text{rank}(\bar{L}^R) \leq k \cdot |P_1| \leq k \cdot 2^{R_{1/2}(f)}$. Thus, by the assumption that $L$ satisfies $\|A \circ W - L\|_* \leq \min_{\text{rank}=k'} \|A \circ W - \bar{L}\|_* + \Delta$:

$$
\|(A - L) \circ W\|_* \leq \|A \circ W - L\|_* \leq \|A \circ W - L\|_* + \Delta
= \|\left(A \circ W - L\right) \circ W_{1}\|_* + \|\left(A \circ W - L\right) \circ (1 - W_{1})\|_* + \Delta
= \|\left(A \circ W - L\right) \circ W_{1}\|_* + \|A \circ W \circ (1 - W_{1})\|_* + \Delta,
$$

where the third line follows since $L$ is 0 outside the support of $W_{1}$ (i.e., outside of the rectangles in $P_1$). Since $\bar{L}$ is equal to the best rank-$k$ approximation to $A_R \circ W_R$ on each rectangle $R$ in $P_1$, and since these rectangles partition the support of $W_{1}$:

$$
\|\left(A \circ W - L\right) \circ W_{1}\|_* \leq \|\left(A \circ W - L_{opt}\right) \circ W_{1}\|_*
= \|\left(A - L_{opt}\right) \circ W \circ W_{1}\|_* + \|L_{opt} \circ (1 - W) \circ W_{1}\|_*
\leq \text{OPT} + \|L_{opt} \circ (1 - W) \circ W_{1}\|_*.
$$

Plugging back into (3) and applying (2):

$$
\|(A - L) \circ W\|_* \leq \text{OPT} + \|L_{opt} \circ (1 - W) \circ W_{1}\|_* + \|A \circ W \circ (1 - W_{1})\|_* + \Delta
\leq \text{OPT} + \epsilon_1 \|A\|_* + \epsilon_2 \|L_{opt}\|_* + \Delta,
$$

which completes the theorem. 

\textbf{Proof of Theorems 1 and 2.} Theorems 1 and 2 follow by applying Theorem 17 with $\epsilon_1 = \epsilon_2 = \epsilon$ and $\epsilon_1 = \epsilon$, $\epsilon_2 = 0$ respectively. When $\|\cdot\|_*$ is the squared Frobenius norm, $L$ satisfying $\|A \circ W - L\|_* \leq \min_{\text{rank}=k'} \|A \circ W - \bar{L}\|_* + \Delta$ for $\Delta = \epsilon \|A - A_{k'}\|_{F}^2 \leq \epsilon \|A\|_{F}^2$ can be computed with high probability in $O(\text{nnz}(A)) + n \cdot \text{poly}(k'/\epsilon)$ time. 

\textbf{3.1 Applications of Main Theorem}

We now instantiate Theorem 17 for a number of common missing entry patterns. See Table 1 for a summary. We start with the case when $W$ is the negation of a diagonal matrix or a block diagonal matrix, corresponding to the Low-Rank Plus Diagonal (LRPD) and Low-Rank Plus Block Diagonal (LRPBD) matrix approximation problems.

\textbf{Corollary 18 (Low-Rank Plus Diagonal Approximation).} Let $W = 1 - I$ where $I$ is the $n \times n$ identity matrix. Then for $k' = O\left(\frac{k}{\epsilon}\right)$ and $L$ with $\|A \circ W - L\|_* \leq \min_{\text{rank}=k'} \|A \circ W - \bar{L}\|_* + \epsilon \|A\|_*$:

$$
\|(A - L) \circ W\|_* \leq \text{OPT} + 2\epsilon \|A\|_*.
$$

If $\|\cdot\|_* = \|\cdot\|_{F}^2$, such an $L$ can be computed with high probability in $O(\text{nnz}(A)) + n \cdot \text{poly}(k'/\epsilon)$ time.

\textbf{Proof.} The function $f$ corresponding to $W$ is the inequality function $\text{NEQ}$. We have $R_{1}^{\text{left-sided}}(\text{NEQ}) = R_{1}^{\text{left-sided}}(\text{EQ})$, which by Theorem 12 is bounded by $\log(1/\epsilon) + 5$. Thus $2R_{1}^{\text{left-sided}}(\text{NEQ}) \leq \frac{22}{\epsilon}$. The corollary then follows directly from Theorem 17.

\textbf{Corollary 19 (Low-Rank Plus Block Diagonal Approximation).} Consider any partition $B_1 \cup B_2 \cup \ldots \cup B_{b} = \{n\}$ and let $W$ be the matrix with $W_{i,j} = 0$ if $i, j \in B_{k}$ for some $k$ and $W_{i,j} = 1$ otherwise. Then for $k' = O\left(\frac{k}{\epsilon}\right)$ and $L$ with $\|A \circ W - L\|_* \leq \min_{\text{rank}=k'} \|A \circ W - \bar{L}\|_* + \epsilon \|A\|_*$:

$$
\|(A - L) \circ W\|_* \leq \text{OPT} + 2\epsilon \|A\|_*.
$$

If $\|\cdot\|_* = \|\cdot\|_{F}^2$, such an $L$ can be computed with high probability in $O(\text{nnz}(A)) + n \cdot \text{poly}(k'/\epsilon)$ time.
Proof. The function $f$ corresponding to $W$ is the inequality function $NEQ$ where $x, y \in [n]$ are identified with $j, k \in [b]$ if block $B_j$ contains $x$ and $B_k$ contains $y$. The randomized communication complexity $-f$ is thus bounded by the complexity of equality. By Theorem 12, $R_{\epsilon}^{1\text{-sided}}(EQ) \leq \log(1/\epsilon) + 5$ and so $2R_{\epsilon}^{1\text{-sided}}(f) \leq \frac{32}{\epsilon}$, which gives the corollary.

We next consider the Low-Rank Plus Sparse (LRPS) and Low-Rank Plus Block Sparse (LRPBS) approximation problems, where $W$ has at most $t$ nonzeros (or nonzero blocks) per row. Note that this setting strictly generalizes the Low-Rank Plus (Block) Diagonal Problem, and our most general Corollary 21 in fact directly implies Corollaries 18, 19, and 20.

**Corollary 20** (Low-Rank Plus Sparse Approximation). Let $W \in \{0, 1\}^{n \times n}$ have at most $t$ zeros in each row. Then for $k' = O\left(\frac{k}{\epsilon}\right)$ and $L$ with $A \circ W - L \|_* \leq \min_{\text{rank} - k'} \hat{L} \| A \circ W - \hat{L} \|_* + \epsilon \| A \|_*$:

$$\| (A - L) \circ W \|_* \leq \text{OPT} + 2\epsilon \| A \|_*.$$ 

If $\| \cdot \|_* = \| \cdot \|_F^2$, such an $L$ can be computed with high probability in $O(\text{nnz}(A)) + n \text{poly}(kt/\epsilon)$ time.

Proof. The function $f$ corresponding to $W$ is the negation of the problem where Alice is given $x \in [n]$ and must determine if Bob has input $y \in S_x$ where $S_x \subseteq [n]$ has at most $t$ entries, corresponding to the locations of the zero entries in the $x^{th}$ row of $W$. This problem can be solved by running a 1-way equality protocol with error parameter $\epsilon/t$, which by Theorem 12 requires $R_{\epsilon/t}^{1\text{-sided},1\text{-sided}}(EQ) \leq \log(t/\epsilon) + 5$ bits of communication (including Alice’s output bit). Alice can then check, with probability $\geq \epsilon/t$, whether Bob’s input is equal to each entry in $S_x$. By a union bound, she succeeds in checking if Bob’s input is in $S_x$ with probability $\geq 1 - \epsilon$. Thus we have $R_{\epsilon}^{1\text{-sided}}(\neg f) \leq \log(t/\epsilon) + 5$ and so $2R_{\epsilon}^{1\text{-sided}}(\neg f) \leq \frac{32}{\epsilon}$, which completes the corollary.

As with equality, the block result simply follows from considering the same communication problem as in Corollary 20 where $x, y$ are identified with their corresponding blocks. We obtain:

**Corollary 21** (Low-Rank Plus Block Sparse Approximation). Consider any pair of partitions $B_1^x \cup \ldots \cup B_t^x = [n]$, and $B_1^y \cup \ldots \cup B_b^y = [n]$. Let $W'$ be any $b \times b$ matrix with at most $t$ zeros in each row. Let $W$ be the $n \times n$ matrix where $W_{i,j} = W'_{k,\ell}$ for $k, \ell$ with $i \in B_k^x$ and $j \in B_{\ell}^y$. Then for $k' = O\left(\frac{k}{\epsilon}\right)$ and $L$ with $A \circ W - L \|_* \leq \min_{\text{rank} - k'} \hat{L} \| A \circ W - \hat{L} \|_* + \epsilon \| A \|_*$:

$$\| (A - L) \circ W \|_* \leq \text{OPT} + 2\epsilon \| A \|_*.$$ 

If $\| \cdot \|_* = \| \cdot \|_F^2$, such an $L$ can be computed with high probability in $O(\text{nnz}(A)) + n \text{poly}(kt/\epsilon)$ time.

In all of the above theorems we applied the 1-sided error communication complexity bound for EQ of Theorem 12. We can also apply the 2-sided error bound to give a lower rank matrix satisfying a slightly weaker bound. Stating the result for the most general Low-Rank Plus Block Sparse Approximation case we have:

**Corollary 22** (Low-Rank Plus Block Sparse Approximation – Two Sided Error). Consider any pair of partitions $B_1^x \cup \ldots \cup B_t^x = [n]$, and $B_1^y \cup \ldots \cup B_b^y = [n]$. Let $W'$ be any $b \times b$ matrix with at most $t$ zeros in each row. Let $W$ be the $n \times n$ matrix where $W_{i,j} = W'_{k,\ell}$ for $k, \ell$ with $i \in B_k^x$ and $j \in B_{\ell}^y$. Then for $k' = O\left(k \cdot \left(\frac{1}{\epsilon}\right)^{1-\epsilon/2}\right)$ and $L$ with $A \circ W - L \|_* \leq \min_{\text{rank} - k'} \hat{L} \| A \circ W - \hat{L} \|_* + \epsilon \| A \|_*$:

$$\| (A - L) \circ W \|_* \leq \text{OPT} + 2\epsilon \| A \|_* + \epsilon \| L_{\text{opt}} \|_*.$$ 

If $\| \cdot \|_* = \| \cdot \|_F^2$, such an $L$ can be computed with high probability in $O(\text{nnz}(A)) + n \text{poly}(k'/\epsilon)$ time.
Proof. \( f \) corresponding to \( W \) is the negation of the function where Alice and Bob first identify their inputs \( x, y \) with the blocks \( B_j \) and \( B_k \) that they are in and then Alice must determine if \( k \in S_j \) where \( S_j \) is the set of at most \( t \) blocks that correspond to zeros in the \( j^{th} \) row of \( W' \). As described in Corollary 20, this problem can be solved by running a 1-way equality protocol with error \( \epsilon/t \). Alice can then check if Bob’s block index \( k \) is equal to each element of \( S_j \), succeeding with probability \( \geq 1 - \epsilon \) by a union bound. By Theorem 12, the randomized communication complexity of this problem with two-sided error is thus bounded by \( R_{\epsilon,t}^{1\text{-sided}}(EQ) \leq (1 - \epsilon/t) \log((1 - \epsilon/t)^2 \cdot t/\epsilon) + 5 \) and thus \( 2R_{\epsilon,t}^{1\text{-sided}}(f) \leq 32 \left(\frac{1}{\epsilon}\right)^{1-\epsilon/t} \), giving the corollary.

We now consider common missing data patterns that are not sparse – \( W \) may have a large number of nonzero entries in each row and column.

Corollary 23 (Subsampled Toeplitz). For any integer \( p \), let \( W \in \{0, 1\}^{n \times n} \) be the Toeplitz matrix with \( W_{i,j} = 0 \) iff \( i - j = 0 \mod p \). Then for \( k' = O\left(\frac{n}{\epsilon}\right) \) and \( L \) with \( \|A \circ W - L\|_\ast \leq \min_{\text{rank} - k'} \|A \circ W - \hat{L}\|_\ast + \epsilon\|A\|_\ast \):

\[
\|(A - L) \circ W\|_\ast \leq OPT + 2\epsilon\|A\|_\ast.
\]

For \( k' \geq 4pk, \|(A - L) \circ W\|_\ast \leq OPT + \epsilon\|A\|_\ast \). If \( \|\cdot\|_\ast = \|\cdot\|_F \), \( L \) satisfying the required guarantee can be computed with high probability in \( O(\text{nnz}(A)) + n \text{poly}(k'/\epsilon) \) time.

Proof. The function \( f \) corresponding to \( W \) is the negation of the equality function mod \( p \) with \( EQ_p(x, y) = 1 \) iff \( x - y = 0 \mod p \). By Theorem 16, \( R_{\epsilon,t}^{1\text{-sided}}(EQ_p) \leq \log(1/\epsilon) + 5 \) and \( R_{\epsilon,t}^{0\text{-sided}}(EQ_p) = D(EQ_p) \leq \lceil \log p \rceil + 1 \). Thus \( 2R_{\epsilon,t}^{1\text{-sided}}(EQ_p) \leq 32 \frac{1}{\epsilon} \) and \( 2R_{\epsilon,t}^{0\text{-sided}}(EQ_p) \leq 4p \), giving the corollary.

Beyond equality, a number of common sparsity patterns are related to the communication complexity of the Greater-Than (GT) function, which is bounded by Theorem 15.

Corollary 24 (Low-Rank Plus Banded Approximation). For any integer \( p \leq n \), let \( W \in \{0, 1\}^{n \times n} \) be the banded Toeplitz matrix with \( W_{i,j} = 0 \) iff \( |i - j| < p \). Then for \( k' = k \cdot \text{poly}\left(\frac{\log n}{\epsilon}\right) \) and \( L \) with \( \|A \circ W - L\|_\ast \leq \min_{\text{rank} - k'} \|A \circ W - \hat{L}\|_\ast + \epsilon\|A\|_\ast \):

\[
\|(A - L) \circ W\|_\ast \leq OPT + 2\epsilon\|A\|_\ast + \epsilon\|L_{\text{opt}}\|_\ast.
\]

If \( \|\cdot\|_\ast = \|\cdot\|_F \), such an \( L \) can be computed with high probability in \( O(\text{nnz}(A)) + n \text{poly}(k'/\epsilon) \) time.

Proof. The function \( f \) corresponding to \( W \) is the negation of the AND of \( i + p < j \) and \( j + p > i \). Thus, it can be solved with two calls to a protocol for Greater-Than (GT). By Theorem 15, for \( \log n \) bit inputs, \( R_{\epsilon}(GT) = O\left(\log\left(\frac{\log n}{\epsilon}\right)\right) \). Thus \( R_{\epsilon}(f) = O\left(\log\left(\frac{\log n}{\epsilon}\right)\right) \) and \( 2R_{\epsilon}(f) = \text{poly}\left(\frac{\log n}{\epsilon}\right) \), which gives the corollary.

We also consider a ‘multi-dimensional’ banded pattern. Here each \( i \in \{0, 1\}^{\log n} \) corresponds to a point \((i_1, i_2)\) in a \( \sqrt{n} \times \sqrt{n} \) grid (\( i_1 \) and \( i_2 \) are determined by the first \( \frac{\log n}{2} \) and last \( \frac{\log n}{2} \) bits of \( i \) respectively). We focus on the two-dimensional case, although this set up can easily be generalized to higher dimensions. We can also imagine generalizing to different distance measures over the points \((i_1, i_2)\) using efficient sketching methods (which yield efficient communication protocols) for various distances [BYJKK04, KNW10]. We have:
Corollary 25 (Multi-Dimensional Low-Rank Plus Banded Approximation). For any \( i \in [n] \) let \( i_1, i_2 \in [\sqrt{n}] \) be the integers corresponding to the first and last half of its binary expansion. For any integer \( p \leq n \), let \( W \in \{0,1\}^{n \times n} \) be binary matrix with \( W_{i,j} = 0 \) iff \( |i_1, i_2| - (j_1, j_2)| < p \). Then for \( k' = k \cdot \text{poly}\left(\frac{\log n}{\epsilon}\right) \) and \( L \) with \( \|A \circ W - L\|_* \leq \min_{\text{rank} - k'} \|A \circ W - \hat{L}\|_* + \epsilon\|A\|_* \):

\[
\|(A - L) \circ W\|_* \leq \text{OPT} + 2\epsilon\|A\|_* + \epsilon\|L_{\text{opt}}\|_* .
\]

If \( \|\cdot\|_* = \|\cdot\|_F^2 \), such an \( L \) can be computed with high probability in \( O(\text{nnz}(A) + n \text{ poly}(k'/\epsilon)) \) time.

Proof. The function \( f \) corresponding to \( W \) is the predicate \( |i_1 - j_1| + |i_2 - j_2| \geq p \). We can first run a greater-than protocol to determine with probability \( \geq 1 - \epsilon/3 \) if \( i_1 > j_1 \) and similarly with probability \( \geq 1 - \epsilon/3 \) if \( i_2 > j_2 \). Depending on the outputs of these checks we can evaluate \( |i_1 - j_1| + |i_2 - j_2| \geq p \) with a third greater-than protocol succeeding with probability at least \( 1 - \epsilon/3 \). For example, if both hold, we can check if \( i_1 + i_2 < j_1 + j_2 + p \). A union bound gives total success probability at least \( 1 - \epsilon \). By Theorem 15, for \( \log n \) bit inputs, \( R_{\epsilon}(GT) = O\left(\log\left(\frac{\log n}{\epsilon}\right)\right) \). Thus \( R_{\epsilon}(f) = O\left(\log\left(\frac{\log n}{\epsilon}\right)\right) \) and \( 2^{R_{\epsilon}(f)} = \text{poly}\left(\frac{\log n}{\epsilon}\right) \), which gives the corollary. \( \square \)

A similar result holds for low-rank approximation with monotone missing data.

Corollary 26 (Monotone Missing Data Problem (MMDP)). Let \( W \in \{0,1\}^{n \times n} \) be any matrix where each row of \( W \) has a prefix of an arbitrary number of ones, followed by a suffix of zeros. Then for \( k' = k \cdot \text{poly}\left(\frac{\log n}{\epsilon}\right) \) and \( L \) with \( \|A \circ W - L\|_* \leq \min_{\text{rank} - k'} \|A \circ W - \hat{L}\|_* + \epsilon\|A\|_* \):

\[
\|(A - L) \circ W\|_* \leq \text{OPT} + 2\epsilon\|A\|_* + \epsilon\|L_{\text{opt}}\|_* .
\]

If \( \|\cdot\|_* = \|\cdot\|_F^2 \), such an \( L \) can be computed with high probability in \( O(\text{nnz}(A) + n \text{ poly}(k'/\epsilon)) \) time.

Proof. Let \( p_x \) be the length of the prefix of ones in the \( x^{\text{th}} \) row of \( W \). Then the function \( f \) corresponding to \( W \) is \( f(x,y) = 1 \) iff \( p_x \geq y \). That is, it is just the Greater-Than function where Alice maps her input \( x \) to \( p_x \). Thus by Theorem 15, \( R_{\epsilon}(f) \leq R_{\epsilon}(GT) = O\left(\log\left(\frac{\log n}{\epsilon}\right)\right) \). So \( 2^{R_{\epsilon}(f)} = \text{poly}\left(\frac{\log n}{\epsilon}\right) \), which gives the corollary. \( \square \)

4 Tensor Low-Rank Approximation from Multiparty Communication Complexity

In this section we prove Theorem 3, extending Theorem 17 to the low-rank approximation of higher order tensors using the number-in-hand multiparty communication model with a shared blackboard. We give applications to natural tensor generalizations of the low-rank plus (block) diagonal and low-rank plus (block) sparse problems.

We first formally define the communication model we use. Consider \( t \) players \( P_1, \ldots, P_t \) each with access to an input \( x_1 \in \mathcal{X}_1 \). The players would like to compute a function \( f : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_t \to \{0,1\} \). \( f \) corresponds to \( t^{\text{th}} \) order communication tensor \( M \in \{0,1\}^{\mathcal{X}_1 \times \cdots \times \mathcal{X}_t} \) with \( M_{x_1,\ldots,x_t} = f(x_1,\ldots,x_t) \). Players exchange messages by writing them on a shared blackboard that all others can see. In a randomized communication protocol \( \Pi \), players view a string of public random bits \( r \). After seeing \( r \), the players run a deterministic protocol \( \Pi_r \), which specifies the next
player to speak as a function of the information written on the blackboard, as well as the message of that player, as a function of what is written on the blackboard and of their input.

We say a protocol $\Pi$ is a $(\delta_1, \delta_2)$-error protocol if for all $(x_1, \ldots, x_t) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_t$, with $f(x_1, \ldots, x_t) = 1$, $\mathbb{P}_\ell[\Pi_r(x_1, \ldots, x_t) = f(x_1, \ldots, x_t)] \geq 1 - \delta_1$ and for all $(x_1, \ldots, x_t) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_t$ with $f(x_1, \ldots, x_t) = 0$, $\mathbb{P}_\ell[\Pi_r(x_1, \ldots, x_t) = f(x_1, \ldots, x_t)] \geq 1 - \delta_2$. We can then define the multiparty randomized communication complexity:

**Definition 27 (Multiparty Number-in-Hand Randomized Communication Complexity).** The $(\delta_1, \delta_2)$-error $t$-party randomized communication complexity $R_{\delta_1, \delta_2}^t(f)$ is the minimum number so that there is a distribution on protocols inducing partitions of $\mathcal{X}$, each containing at most $2^{R_{\delta_1, \delta_2}^t(f)}$ rectangles, such that (1) for every $(x_1, \ldots, x_t) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_t$ with $f(x_1, \ldots, x_t) = 1$, with probability at least $1 - \delta_1$, $(x_1, \ldots, x_t)$ lands in a rectangle labeled 1 and (2) for every $(x_1, \ldots, x_t) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_t$ with $f(x_1, \ldots, x_t) = 0$, with probability at least $1 - \delta_2$, $(x_1, \ldots, x_t)$ lands in a rectangle labeled 0.

We now connect the notion of communication complexity given in Definition 27 to tensor low-rank approximation with missing entries. The rank of any $t$th order tensor $M$ is the minimum integer $k$ such that $M = \sum_{i=1}^k u_{i,1} \otimes \cdots \otimes u_{i,t}$ for vectors $u_{i,1}, \ldots, u_{i,t}, \ldots, u_{k,1}, \ldots, u_{k,t}$. All other notions, such as entrywise norm, entrywise product, etc., are generalized in the natural way to $t$th order tensors. We note that $OPT$ in this setting will be defined as the infimum of the error obtained by a low rank tensor approximation $L$. Due to issues of border-rank this infimum may not be achieved by any $L$ (see e.g., [DSL08]). However, we can to efficiently compute a tensor $L$ achieving within small additive error of this infimum, using the algorithms of [SWZ19]. These algorithms apply to the squared Frobenius norm, and other entrywise $\ell_p$ norms, although we focus on the squared Frobenius norm in our runtime bounds.

**Theorem 28 (Multiparty Communication Complexity $\rightarrow$ Tensor Low-Rank Approximation).** Consider $t$th order tensor $W \in \{0,1\}^{n \times \cdots \times n}$ and let $f : \{0,1\}^{\log_n n} \times \cdots \times \{0,1\}^{\log_n n} \times \{0,1\}^{\log_n n} \rightarrow \{0,1\}$ be the function computed by it. For $k' \geq k \cdot 2^{R_{\delta_1, \delta_2}^t(f)}$, and any entrywise norm $\| \cdot \|_*$ (Def. 5) for any $L$ satisfying $\|A \circ W - L\|_* \leq \inf_{\text{rank} - k'} \| A \circ W - \hat{L}\|_* + \epsilon_3 \| A\|_* + \epsilon_2 \| L\|_* + \gamma$, where $OPT = \inf_{\text{rank} - k} \| (A - L) \circ W\|_*$ and $L\gamma$ is any rank-$k$ $t$th order tensor achieving error $OPT + \gamma$ for $\gamma > 0$. When $\| \cdot \|_*$ is the squared Frobenius norm, for any $\epsilon_3 > 0$, $L$ with rank $O((k'/\epsilon)^{t-1})$ satisfying the required bound can be computed with high probability in $O(n \text{nnz}(A)) + n \text{poly}(k'/\epsilon)$ time [SWZ19].

Note that the final rank of the approximation $L$ that can be efficiently computed using the algorithms of [SWZ19] is $O((k'/\epsilon)^{t-1}) = O\left((k/\epsilon)^{t-1} \cdot 2^{(t-1)R_{\delta_1, \delta_2}^t(f)}\right)$.

**Proof.** The proof closely follows that of Theorem 17. As discussed in Def. 27, $R_{\delta_1, \delta_2}^t(f)$ is the minimum number so that there is a distribution on protocols inducing partitions of $W$, each containing at most $2^{R_{\delta_1, \delta_2}^t(f)}$ rectangles, such that (1) for every $x_1, \ldots, x_t \in \{0,1\}^{\log_n n}$ with $f(x_1, \ldots, x_t) = 1$, $(x_1, \ldots, x_t)$ lands in a rectangle labeled 1 with probability $\geq 1 - \epsilon_1$ and (2) for every $x_1, \ldots, x_t \in \{0,1\}^{\log_n n}$ with $f(x_1, \ldots, x_t) = 0$, $(x_1, \ldots, x_t)$ lands in a rectangle labeled 0 with probability $\geq 1 - \epsilon_2$. In other words, letting $\Psi$ be the (random) binary tensor corresponding to the function computed
by the protocol, \( W \circ (1 - W\Pi) \) has each entry equal to 1 with probability at most \( \epsilon_1 \) and \( W\Pi \circ (1 - W) \) has each entry equal to 1 with probability at most \( \epsilon_2 \). Thus, fixing some \( L_\gamma \):

\[
\mathbb{E}_{\text{protocol } \Pi} \left[ \|A \circ W \circ (1 - W\Pi)\|_* + \|L_\gamma \circ W\Pi \circ (1 - W)\|_* \right] \leq \epsilon_1 \|A \circ W\|_* + \epsilon_2 \|L_\gamma \circ (1 - W)\|_* \\
\leq \epsilon_1 \|A\|_* + \epsilon_2 \|L_\gamma\|_*.
\]

Thus, there is at least one protocol \( \Pi \) (inducing a partition with \( \leq 2^{R_{1, \epsilon_2}(f)} \) rectangles) with:

\[
\|A \circ W \circ (1 - W\Pi)\|_* + \|L_\gamma \circ W\Pi \circ (1 - W)\|_* \leq \epsilon_1 \|A\|_* + \epsilon_2 \|L_\gamma\|_*.
\]

Let \( P_1 \) be the set of rectangles (each a subset of \( \{0,1\}^{\log n} \times \ldots \times \{0,1\}^{\log n} \)) on which the protocol achieving (4) returns 1 and \( P_0 \) be the set on which it returns 0. For any \( R \in P_1 \) let \( L^R \) be any rank-\( k \) tensor satisfying \( \|A_R \circ W_R - L^R\|_* \leq \inf_{\text{rank}-k' \leq k} \|A_R \circ W_R - L'\|_* + \frac{\gamma}{2^{R_{1, \epsilon_2}(f)}} \). Let \( \bar{L}^R \) be equal to \( L^R \) on \( R \) and 0 elsewhere. Let \( \bar{L} = \sum_{R \in P_1} \bar{L}^R \). Note that \( \bar{L} \) has rank at most \( k \cdot |P_1| \leq k \cdot 2^{R_{1, \epsilon_2}(f)} \). So by the assumption that \( L \) satisfies \( \|A \circ W - L\|_* \leq \inf_{\text{rank}-k' \leq k} \|A \circ W - L'\|_* + \epsilon_3 \|A\|_* \),

\[
\|(A - L) \circ W\|_* \leq \|A \circ W - L\|_* \leq \|A \circ W - \bar{L}\|_* + \epsilon_3 \|A\|_* \\
= \|(A \circ W - \bar{L}) \circ W\Pi\|_* + \|(A \circ W - \bar{L}) \circ (1 - W\Pi)\|_* + \epsilon_3 \|A\|_* \\
= \|(A \circ W - \bar{L}) \circ W\Pi\|_* + \|A \circ W \circ (1 - W\Pi)\|_* + \epsilon_3 \|A\|_*,
\]

where the third line follows since \( \bar{L} \) is 0 outside the support of \( W\Pi \) (i.e., outside the rectangles in \( P_1 \)). Since \( \bar{L} \) is within additive error \( \frac{\gamma}{2^{R_{1, \epsilon_2}(f)}} \) of the best rank-\( k \) approximation to \( A_R \circ W_R \) on each rectangle \( R \) in the support of \( W\Pi \) (i.e., each \( R \in P_1 \)) and since \( \|(A - L_\gamma) \circ W\|_* = OPT + \gamma \),

\[
\|(A - L_\gamma) \circ W\Pi\|_* \leq \|(A \circ W - L_\gamma) \circ W\Pi\|_* - \gamma + \frac{\gamma}{2^{R_{1, \epsilon_2}(f)}} \cdot |P_1| \\
\leq \|(A - L_\gamma) \circ W \circ W\Pi\|_* + \|L_\gamma \circ (1 - W) \circ W\Pi\|_* \\
\leq OPT + \gamma + \|L_\gamma \circ (1 - W) \circ W\Pi\|_*
\]

Plugging back into (5) and applying (4):

\[
\|(A - L) \circ W\|_* \leq OPT + \gamma + \|L_\gamma \circ (1 - W) \circ W\Pi\|_* + \|A \circ W \circ (1 - W\Pi)\|_* + \epsilon_3 \|A\|_* \\
\leq OPT + \gamma + \epsilon_1 \|A\|_* + \epsilon_2 \|L_\gamma\|_* + \epsilon_3 \|A\|_*,
\]

which completes the theorem.

\[\square\]

Theorem 3 follows immediately from Theorem 28 by considering \( R_{\epsilon,1}^{1-sided}(-f) = R_{0, \epsilon}^{f}(f) \).

### 4.1 Applications of Main Tensor Theorem

We now give some example applications of Theorem 28. We focus on a few common settings of \( W \), however note that essentially any of the data patterns considered for matrices in Section 3 can be generalized to the tensor case. We first consider the natural generalization of the low-rank plus diagonal matrix approximation problem to tensors.
Corollary 29 (Low-Rank Plus Diagonal Tensor Approximation). Let $W$ be the $t$th order tensor with $W_{i_1,\ldots,i_t} = 0$ when $i_1 = i_2 = \ldots = i_t$ and $W_{i_1,\ldots,i_t} = 1$ otherwise. Then for $k' = O \left( \frac{(kt)^{t-1}}{\epsilon^t} \cdot 2^{(t-1)} \right)$ there is an algorithm computing rank-$k'$ $L$ with high probability in $O(\text{nnz}(A)) + n \text{poly}(k'/\epsilon)$ time satisfying:

$$\|(A - L) \circ W\|_F^2 \leq \text{OPT} + \epsilon\|A\|_F^2.$$ 

Proof. The function $f$ corresponding to $W$ is the inequality function $\text{NEQ}$. By Theorem 12, $R_{\epsilon,0}(\text{NEQ}) = R_{\epsilon,1}^{\text{way},1\text{-sided}}(-\text{NEQ}) = R_{\epsilon,1}^{\text{way},1\text{-sided}}(\text{EQ})$ is bounded by $\log(1/\epsilon) + 5$. In the number-in-hand blackboard model, a single player can simply run this protocol, with error probability $\epsilon' = \epsilon/(t-1)$. The remaining $t-1$ players can then check equality, all succeeding via a union bound with probability $\geq 1 - \epsilon$. These players can then all send the results of their equality test and all players can output the solution based on these results. The total communication is $\log(t/\epsilon) + t + O(1)$ and thus we have $2R_{\epsilon,1\text{-sided}}(-f) = O \left( \frac{t^2}{\epsilon} \cdot 2^t \right)$. Applying Theorem 28, we can set $k' = O \left( \frac{k^t \cdot t^2}{\epsilon^{t-1}} \right)$. We can efficiently output $L$ with rank $O \left( (k'/\epsilon)^{t-1} \right) = O \left( \frac{(kt)^{t-1}}{\epsilon^t} \cdot 2^{(t-1)} \right)$ achieving within an additive $\epsilon\|A\|_F^2$ error of the best $k'$-rank approximation to $A \circ W$. Additionally, we can set $\gamma < \epsilon\|A\|_F^2$. Since our error is 1-sided we do not pay any error in terms of $\|L_{\gamma}\|_F^2$. Overall, we will have $\|(A - L) \circ W\|_F^2 \leq \text{OPT} + 3\epsilon\|A\|_F^2$, which completes the corollary after adjusting $\epsilon$ by a constant factor.

As in the case of matrices, Corollary 29 also immediately extends to the low-rank plus block diagonal tensor approximation problem where $W$ is zero in a block diagonal pattern. We can also use a similar technique to solve the low-rank plus sparse tensor approximation problem. We have:

Corollary 30 (Low-Rank Plus Sparse Tensor Approximation). Let $W$ be any $t$th order binary tensor such that for any fixed $i_1 \in [n]$, the $(t-1)$th order ‘face’ $W(i_1,\ldots,\cdot)$ has at most $s$ zero entries. Then for $k' = O \left( \frac{(kt)^{t-1}}{\epsilon^{t-1}} \cdot 2^{6(t-1)} \right)$ there is an algorithm computing rank-$k'$ $L$ with high probability in $O(\text{nnz}(A)) + n \text{poly}(k'/\epsilon)$ time satisfying:

$$\|(A - L) \circ W\|_* \leq \text{OPT} + \epsilon\|A\|_*.$$ 

Proof. To compute the function $f$ corresponding to $W$, for each of the $s$ entries $(i_2,\ldots,i_t)$ that are zero on the face $W(x_1,\ldots,\cdot)$, player 1 must output zero if $x_j = i_j$ for all $j \in 2,\ldots,t$. This corresponds to running $t-1$ equality tests for each of the $s$ entries. Each player aside from player 1 can run the 1-way equality protocol of Theorem 12 with error $\epsilon' = \frac{\epsilon}{t}$, requiring $(t-1) \cdot \log \left( \frac{2}{\epsilon} \right) + 5$ bits in total. Player 1 can then run the $t-1$ equality tests for the $s$ different zero entries on their face, outputting 0 if for one of the entries all $t-1$ tests succeed. For any one of the entries, if $(x_2,\ldots,x_t) \neq (i_2,\ldots,i_t)$ the strings differ in at least one location and so this equality test will fail with probability at least $1 - \epsilon/s$. If $(x_2,\ldots,x_t) = (i_2,\ldots,i_t)$, since the equality tests are 1-sided, the protocol will always be correct. Union bounding over all $s$ entries tested, the protocol has 1-sided error at most $\epsilon$. The total communication, including the players’ output bits is $R_{\epsilon,0}(f) = (t-1) \cdot \log \left( \frac{2}{\epsilon} \right) + 5 + t$ and thus $2R_{\epsilon,0}(f) = O \left( \left( \frac{2}{\epsilon} \right)^{t-1} \cdot 2^t \right)$.

Applying Theorem 28, we can set $k' = \left( k \cdot \left( \frac{2}{\epsilon} \right)^{t-1} \cdot 2^t \right)$ and can efficiently output $L$ with rank $O \left( (k'/\epsilon)^{t-1} \right) = O \left( \frac{(kt)^{t-1}}{\epsilon^{t-1}} \cdot 2^{6(t-1)} \right)$ achieving within an additive $\epsilon\|A\|_F^2$ error of the best $k'$-rank approximation to $A \circ W$. Additionally, we can set $\gamma < \epsilon\|A\|_F^2$. Since our error is 1-sided we
do not pay any error in terms of \( \|L_\gamma\|_F^2 \). Overall, we will have \( \|(A - L) \circ W\|_F^2 \leq OPT + 3\epsilon\|A\|_F^2 \), which completes the corollary after adjusting \( \epsilon \) by a constant factor.

\[ \square \]

5 Boolean Low-Rank Approximation from Nondeterministic Communication Complexity

In this section we use nondeterministic communication complexity to give a bicriteria approximation result for the Boolean low-rank approximation problem with missing entries.

**Theorem 4** (Nondeterministic communication complexity \( \rightarrow \) Boolean Low-Rank Approximation). Given \( A, W \in \{0,1\}^{n \times n} \), let \( f \) be the function computed by \( W \). For any \( k' \geq 2^{N(f)} \cdot k \), if one computes \( U, V \in \{0,1\}^{n \times k'} \) satisfying \( \|A \circ W - U \cdot V\|_0 \leq \min_{\hat{U}, \hat{V} \in \{0,1\}^{n \times k'}} \|A \circ W - \hat{U} \cdot \hat{V}\|_0 + \Delta \) then:

\[
\|W \circ (A - U \cdot V)\|_0 \leq 2^{N(f)} \cdot OPT + \Delta,
\]

where \( OPT = \min_{\hat{U}, \hat{V} \in \{0,1\}^{n \times n}} \|W \circ (A - U \cdot V)\|_0 \) and \( U \cdot V \) denotes Boolean matrix multiplication.

**Proof.** As discussed in Definition 7, \( N(f) \) is the minimum number so that there is a protocol inducing a set of \( t = 2^{N(f)} \) possibly overlapping rectangles \( R_1, \ldots, R_t \) such that for any \( x, y \in \{0,1\}^{\log n} \) with \( f(x, y) = 1 \), \( (x, y) \) is in at least one of these rectangles and for any \( x, y \) with \( f(x, y) = 0 \), \( (x, y) \) is in none of these rectangles. Let \( W_{R_i} \) be the binary matrix that is one on \( R_i \) and zero elsewhere. Equivalently, we have \( W_{R_1} + \ldots + W_{R_t} = W \) where \( + \) denotes Boolean addition. Let \( \hat{U}_i, \hat{V}_i = \arg\min_{\hat{U}, \hat{V} \in \{0,1\}^{n \times n}} \|A \circ W_{R_i} - \hat{U} \cdot \hat{V}\|_0 \). Note that \( \hat{U} \cdot \hat{V} \) only has support on \( R_i \) and is 0 elsewhere. Let \( \hat{U} = [U_1, \ldots, U_t] \) and \( \hat{V} = [V_1, \ldots, V_t] \). Note that \( \hat{U} \cdot \hat{V} \) only has support on \( R_1 \cup \ldots \cup R_t \) and is zero wherever \( W_{R_1} + \ldots + W_{R_t} = W \) is 0. Using this fact:

\[
\|(A - U \cdot V) \circ W\|_0 \leq \|A \circ W - U \cdot V\|_0 \leq \|A \circ W - \hat{U} \cdot \hat{V}\|_0 + \Delta
\]

\[
= \|(A \circ W - \hat{U} \cdot \hat{V}) \circ W\|_0 + \Delta.
\]

We can then bound via triangle inequality (critically using Booleanity here so that we can write \( A \circ W = \sum_{i = 1}^{t} A \circ W_{R_i} \)):

\[
\|(A \circ W - \hat{U} \cdot \hat{V}) \circ W\|_0 \leq \sum_{i = 1}^{t} \|A \circ W_{R_i} - \hat{U}_i \cdot \hat{V}_i\|_0 \leq t \cdot OPT,
\]

which gives the theorem since \( t = 2^{N(f)} \). \( \square \)

5.1 Applications of Boolean Low-Rank Approximate Theorem

Using Theorem 4 we can give, for example, a bicriteria result for Boolean low-rank plus (block) diagonal approximation. Note that we could also apply Corollary 19 here, which uses 1-sided randomized communication complexity. The two theorems gives different tradeoffs between rank and accuracy.

**Corollary 31** (Boolean Low-Rank Plus Block Diagonal Approximation). Consider any partition \( B_1 \cup B_2 \cup \ldots \cup B_b = [n] \) and let \( W \in \{0,1\}^{n \times n} \) be the block diagonal matrix with \( W_{i,j} = 0 \) if \( i \notin B_k \) and \( j \notin B_k \).
if \( i, j \in B_k \) for some \( k \) and \( W_{i,j} = 1 \) otherwise. Then for \( k' \geq 8k\log b \) and \( U, V \) satisfying
\[
\|A \circ W - U \cdot V\|_0 \leq \min_{\hat{U}, \hat{V} \in \{0,1\}^{k' \times n}} \|A \circ \hat{U} \cdot \hat{V}\|_0 + \Delta:
\]
\[
\|(A - U \cdot V) \circ W\|_0 \leq 8\log b \cdot OPT + \Delta,
\]
where \( OPT = \min_{\hat{U}, \hat{V} \in \{0,1\}^{k \times n}} \|(A - U \cdot V) \circ \hat{W}\|_0 \) and \( U \cdot V \) denotes Boolean matrix multiplication.

Note that if \( W \) is simply \( 1 - I \), corresponding to the standard low-rank plus diagonal approximation problem, \( k' = O(k \log n) \) and the approximation factor is \( O(\log n) \).

Proof. \( W \) is the communication matrix of the inequality problem \( \text{NEQ} \) where Alice and Bob first map their inputs to the index of the block containing them. By Theorem 13 we have \( N(f) \leq \lceil \log (\lceil \log b \rceil) \rceil + 2 \) and so \( 2^{N(f)} \leq 8\log b \), which gives the corollary.

We note that, as in Corollary 23, a similar bound can be given where \( W \) is the binary Toeplitz matrix corresponding to inequality \( \text{mod} \ p \).

In some cases, the nondeterministic communication complexity can be much lower than the randomized communication complexity, allowing us to obtain much tighter bicriteria bounds. For example we can consider a sparsity pattern corresponding to the disjointness function:

**Corollary 32.** Let \( W \in \{0,1\}^{n \times n} \) have \( W_{i,j} = 0 \) if, letting \( x, y \in \{0,1\}^{\log n} \) be the binary representations of \( i, j \) respectively, there is no \( k \) for which \( x(k) = y(k) = 1 \) (i.e., \( x \) and \( y \) are disjoint). Otherwise, let \( W_{i,j} = 1 \). Then for \( k' \geq 8k\log n \) and \( U, V \) satisfying \( \|A \circ W - U \cdot V\|_0 \leq \min_{\hat{U}, \hat{V} \in \{0,1\}^{k' \times n}} \|A \circ \hat{U} \cdot \hat{V}\|_0 + \Delta:
\]
\[
\|(A - U \cdot V) \circ W\|_0 \leq 8 \log n \cdot OPT + \Delta
\]
where \( OPT = \min_{\hat{U}, \hat{V} \in \{0,1\}^{k \times n}} \|(A - U \cdot V) \circ \hat{W}\|_0 \) and \( U \cdot V \) denotes Boolean matrix multiplication.

Proof. \( W \) is the communication matrix of the negation of the disjointness function \( \neg \text{DISJ} \) on \( \log n \) bit strings, which by Theorem 14 has communication complexity \( N(\neg \text{DISJ}) = \lceil \log (\log n) \rceil + 2 \). We thus have \( 2^{N(f)} \leq 8 \log n \), which yields the corollary.

The randomized communication complexity of \( W \) in Corollary 32 is the same as the randomized complexity for set disjointness, which is \( \Theta(\log n) \) \cite{KS92} on \( \log n \) bit inputs. Plugging this complexity e.g. into Theorem 17 would thus require rank \( k' = \text{poly}(n) \).

## 6 Alternate Approach via a Structural Result for Low-Rank Matrices

In this section, we present a different approach to giving efficient algorithms for the low-rank approximation problem with missing entries (Problem 1). Rather than considering the communication complexity of the weight matrix \( W \), we prove a simple structural result about low-rank matrices:

*Any low-rank matrix cannot have too many “heavy” entries on its diagonal, or more generally, on the support of a column sparse matrix \( W \).*
We use this structural result to give an alternative proof of our main bicriteria approximation bound (Theorem 1) in the Low-Rank Plus Diagonal (LRPD) and Low-Rank Plus Sparse (LRPS) setting. That is, when $W$ is either zero exactly on its diagonal, or only has a few zeros per column. In this setting, Theorem 1 yields Corollaries 18 and 20 which show that simply outputting a standard low-rank approximation of $A \circ W$ achieves error $OPT + \epsilon \|A\|_F^2$ with rank $k' = O(kt/\epsilon)$, where $t$ is the maximum number of zeros in a column of $W$ ($t = 1$ in the LRPD case). Our alternative proof applies to the same algorithm gives the same error bound with the same setting of $k'$. We are not aware of any formal connection between the two approaches or more generally the structural result and the communication complexity of $W$. Identifying such a connection would be very interesting.

We further show that the above structural result can be used to obtain a fixed-parameter-tractable, relative error, non-bicriteria algorithm for Problem 1 in the LRPD setting. We also give a fixed-parameter-tractable algorithm for the closely related Factor Analysis (FA) problem, where we want to decompose positive semidefinite $A$ as $L + D$ where $L$ is rank-$k$ and positive semidefinite and $D$ is diagonal and positive semidefinite. Note that removing the positive semidefinite constraints, is exactly equivalent to the LRPD problem. Minimizing $\|W \circ (A - L)\|_F^2$ is equivalent to minimizing $\|A - (L + D)\|_F^2$, where $D$ is diagonal, since given $L$ we can always set $D = \text{diag}(A - L)$.

**Application to Bicriteria Approximation:**

Consider Problem 1 with weight matrix $W$ that has at most $t$ zeros per column (the LRPS approximation problem). As in the proof of Theorem 17 (which yields Theorem 1 as a corollary), we will prove the bicriteria approximation bound via a comparison argument. Let $L_{opt}$ be any rank-$k$ matrix achieving error $OPT = \min_{\text{rank-}k} L \|W \circ (A - L)\|_F^2$. Since $L_{opt}$ is low-rank, by the structural result above (stated formally in Theorem 34), it cannot place significant mass on the entries in the sparse support of $1 - W$, outside a small subset of rows (size $O(kt/\epsilon)$). This in turn implies the existence of a rank-$O(kt/\epsilon) + k$ matrix $\tilde{L}$ that exactly matches $A \circ W$ on those rows and matches $L_{opt}$ on the rest of the matrix. Overall, $\tilde{L}$ places very little weight outside the support of $W$. This is analogous to how $\tilde{L}$ constructed in the proof of Theorem 17 places no weight outside the support of the protocol communication matrix $W_{11}$, which closely approximates $W$.

In Theorem 17 we compare the error $\tilde{L}$ to that of $L$ obtained by outputting a (near) optimal rank-$O(kt/\epsilon)$ approximation to $A \circ W$. Here we perform the same comparison. Since it is optimal, we have $\|A \circ W - L\|_F^2 \leq \|A \circ W - \tilde{L}\|_F^2$. On the entries outside the support of $W$, $\tilde{L}$ is already very small and so close to $A \circ W$ (which is 0 on these entries). Thus $L$ cannot give significantly smaller error than $\tilde{L}$ on these entries. In turn, it cannot give significantly larger error on the entries in the support of $W$. This means that $L$ matches the approximation of $\tilde{L}$, and in turn $L_{opt}$ on the support of $W$, yielding our bound. See Theorem 36 for a formal statement and proof.

**Application to FPT Algorithm:**

In designing a fixed-parameter-tractable algorithm for the LRPD and FA problems we apply a recursive approach: we split our matrix into four quadrants and compute a low-rank plus diagonal decomposition of the top left and bottom right quadrants. Consider the case when $OPT$ is 0: there is a rank-$k$ $L^*$ with $\|W \circ (A - L^*)\|_F^2 = 0$. Equivalently, $A$ can be exactly decomposed as $A = D^* + L^*$ where $D^*$ is diagonal. Note that this decomposition may not be unique. Letting $A_{11} = D_{11}^* + L_{11}^*$ denote the upper left quadrant, our recursively computed output $D_{11}', L_{11}'$ satisfies:

$$D_{11}' + L_{11}' = A_{11} = D_{11}^* + L_{11}^*.$$

Thus $D_{11}' - D_{11}^* = L_{11}' + L_{11}^*$. Since this is a diagonal matrix and since it has rank $\leq 2k$, we can see that it can have at most $2k$ nonzero entries. This is a special case of our main structural result.
(Theorem 34), which lets us make an analogous claim that $D_{11} - D_{11}^*$ does not have many large entries in the case when $A$ does not admit an exact decomposition. The same bound holds for the lower right quadrant and so overall, appending the recursively computed diagonal matrices, we have found $D^*$ up to at most $4k$ incorrect entries.

If we iterate over all possible locations of these incorrect entries (a total of $O(n^{O(k)})$ possibilities), it only remains to solve the LRPD problem where we know all but $O(k)$ of the diagonal entries. This problem can be solved in $\poly(n) \cdot 2^{\poly(k)}$ time using generic polynomial solvers. Polynomial solvers have been used numerous times in the past to solve constrained low-rank approximation problems (see e.g., [AGKM12, Moi12, RSW16, BDL16]). Overall, since we need $n^{O(k)}$ guesses to succeed in identifying the incorrect entries, we obtain runtime $n^{O(k)} \cdot 2^{\poly(k)}$.

### 6.1 Additional Notation and Tools

Throughout this section, given an $n \times p$ matrix $M$ and $i \in [n], j \in [p]$ we let $M(i,j)$ denote its $(i,j)^{th}$ entry. For sets $R \subseteq [n], C \subseteq [p]$ we let $M(R,C)$ be the $|R| \times |C|$ matrix composed of the intersection of the rows and columns indexed by $R$ and $C$ respectively. $\text{supp}(M) \subseteq [n] \times [p]$ denotes the set of indices of $M$’s nonzero entries and $\text{nnz}(M) = |\text{supp}(M)|$ denotes the number of such entries. The above definitions all extend to vectors, except that we omit the index of the second dimension. So for a length-$n$ vector $m$, $m(i)$ is its $i^{th}$ entry and $\text{supp}(m) \subseteq [n]$ is the set of indices of its nonzero entries.

We let $\text{orth}(M)$ output $Q \in \mathbb{R}^{n \times \text{rank}(M)}$ with orthonormal columns that span $M$. For $M \in \mathbb{R}^{n \times n}$, $M$ is positive semidefinite (PSD) if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. We sometimes denote this by $M \succeq 0$, with $M \succeq N$ denoting that $M - N \succeq 0$. Any $M \in \mathbb{R}^{n \times p}$ can be written via singular value decomposition as $M = U \Sigma V^T$ where $U \in \mathbb{R}^{n \times \text{rank}(M)}, V \in \mathbb{R}^{p \times \text{rank}(M)}$ have orthonormal columns and $\Sigma$ is a positive diagonal matrix containing the singular values of $M$, $\sigma_1(M) \geq \ldots \geq \sigma_{\text{rank}(M)}(M) > 0$. We have $\sum_{i=1}^{\text{rank}(M)} \sigma_i^2(M) = \|M\|_F^2$. The matrix pseudoinverse $M^+$ is given by $M^+ = V \Sigma^{-1} U^T$, so that $M^+ M = V V^T$.

### Polynomial Solvers

In our relative error approximation algorithms we make black-box use of polynomial system verifiers [Ren92a, Ren92b, BPR96], which can determine if there exists a solution to any given polynomial system of equations. We combine these verifiers with binary search techniques to perform polynomial optimization under polynomial constraints. Polynomial solvers have been used numerous times in the past to solve constrained low-rank approximation problems (see e.g., [AGKM12, Moi12, RSW16, BDL16]).

**Theorem 33** (Polynomial Decision Problem [Ren92a, Ren92b, BPR96]). Suppose we are given $m$ polynomial constraints over $v$ variables: $p_i(x_1, x_2, \ldots, x_v) \Delta_i 0$, where $\Delta_i$ is any of the “standard relations”: $\{\geq, =, \leq\}$. Let $d$ denote the maximum degree of $p_i$ for $i \in [m]$ and let $H$ denote the maximum bit-length of the coefficients in all of the polynomial constraints. Then, in

$$(md)^{O(v)} \poly(H)$$

time, one can determine if there exist $x_1, \ldots, x_v$ satisfying all of the constraints. I.e., if

$$\{ x \in \mathbb{R}^v | \forall i p_i(x_1, x_2, \ldots, x_v) \Delta_i 0 \} \neq \emptyset.$$
6.2 Main Structural Result

We begin with our main structural result. Informally, if $W \in \{0,1\}^{n \times p}$ is a sparsity pattern with few zero entries in each column, then, ignoring a small subset of rows, no low-rank matrix can place significant mass outside the entries in $\text{supp}(W)$ in comparison to those in this support. When $W = 1 - I$ (corresponding to the LRPD problem), this implies that $L$ cannot place significant mass on all but a small subset of the diagonal. This special case generalizes a simple fact: if $L$ is low-rank and diagonal (i.e., has mass 0 on the support of $W = 1 - I$), then it has exactly $k$ non-zero diagonal entries.

**Theorem 34** (Main Structural Result). Consider any support matrix $W \in \{0,1\}^{n \times p}$ with at most $t$ zero entries per column. For any rank-$k$ $L \in \mathbb{R}^{n \times p}$ and $0 \leq \epsilon \leq 1$, there is some subset of row indices $S \subset [n]$ with $|S| \leq \frac{k}{\epsilon}$ such that, if we let $\bar{W} = W([n] \setminus S, [p])$ and $\bar{L} = L([n] \setminus S, [p])$, then:

$$
\|\bar{L} \circ (1 - \bar{W})\|_F^2 \leq \frac{\epsilon}{1 - \epsilon} \cdot \|\bar{L} \circ W\|_F^2.
$$

That is, if we exclude $tk/\epsilon$ rows from $L$, the Frobenius norm mass outside the entries in $\text{supp}(W)$ is at most an $\frac{\epsilon}{1 - \epsilon}$ fraction of the mass on these entries. In the special case when $W = 1 - I$, this gives that, excluding a subset of $O(k/\epsilon)$ entries, $L$ cannot have more than an $\epsilon$ fraction of its Frobenius norm mass on the diagonal.

**Proof.** Let $l_i$ denote the $i^{th}$ column of $L$. The $i^{th}$ row leverage score of $L$ is defined to be

$$
\tau_i(L) = \max_{y \in \mathbb{R}^p} \frac{[Ly](i)^2}{\|Ly\|_2^2}.
$$

If we set $y = e_j$ where $e_j$ is the $j^{th}$ standard basis vector in $\mathbb{R}^p$, we can see that $\tau_i(L)$ upper bounds how much mass the $i^{th}$ entry can have in column $j$, in comparison to the other entries of the column. It is well known (see e.g. [CLM+15]) that when $L$ has rank $k$, the sum of leverage scores $\sum_{i=1}^n \tau_i(L) = k$. Thus we can see that there are at most $k/\epsilon$ rows with leverage score $\geq \epsilon$. In fact, it is possible to prove something even stronger: if we reweight at most $k/\epsilon$ rows of our matrix appropriately, then we can reduce all leverage scores to be simultaneously bounded by $\epsilon$. Formally:

**Claim 35** (Coherence Reducing Reweighting – Lemma 1 of [CLM+15]). For any rank-$k$ $L \in \mathbb{R}^{n \times n}$ and $\beta > 0$, there exists a diagonal $D \in [0,1]^{n \times n}$ with at most $k/\beta$ entries not equal to 1 and

$$
\tau_i(DL) \leq \beta \text{ for all } i \in [n].
$$

Armed with Claim 35 we are ready to prove the theorem. Let $D$ be the diagonal matrix guaranteed to exist by Claim 35 with $\beta = \epsilon/t$. Let $S = \{i : D(i,i) \neq 1\}$. We have $|S| \leq \frac{k}{\epsilon}$.

Additionally, note that for all $i \in [n] \setminus S$ and all $j \in [p]$, $[DL](i,j) = L(i,j)$ since multiplying by $D$ does not reweight the $i^{th}$ row of $L$. Letting $z_i$ denote the $i^{th}$ column of $DL$, by the leverage score bound of Claim 35, for all $i \in [n]$ and $j \in [p]$ we have:

$$
\frac{[DL](i,j)^2}{\|z_i\|_2^2} = \frac{[DLe_j](i)^2}{\|DLe_j\|_2^2} \leq \tau_i(DL) \leq \frac{\epsilon}{t}.
$$

Thus, letting $\bar{z}_i$ denote the $i^{th}$ column of $DL \circ (1 - W)$, which has at most $t$ nonzero entries:

$$
\|\bar{z_i}\|_2^2 \leq \epsilon \cdot \|z_i\|_2^2.
$$
Since $\bar{z}_i$ is just a subset of the entries in $z_i$, $\|z_i\|_2^2 = (\|z_i - \bar{z}_i\|_2^2) + \|\bar{z}_i\|_2^2$ and so:

$$\|\bar{z}_i\|_2^2 \leq \frac{\epsilon}{1 - \epsilon} \|z_i - \bar{z}_i\|_2^2,$$

where $z_i - \bar{z}_i$ is the $i^{th}$ row of $DL \circ W$. This gives:

$$\|\bar{L} \circ (1 - W)\|_F^2 \leq \|DL \circ (1 - W)\|_F^2 \leq \frac{\epsilon}{1 - \epsilon} \|DL \circ W\|_F^2 \leq \frac{\epsilon}{1 - \epsilon} \|L \circ W\|_F^2,$$

where we use that all entries of $D$ are in $[0, 1]$. This completes the theorem.

\[ \square \]

### 6.3 Polynomial Time Bicriteria Approximation

We start by using Theorem 34 to give an alternative proof that performing a standard low-rank approximation of $A \circ W$ gives a strong bicriteria approximation bound for Problem 1 when $W = 1 - I$ or more generally has at most $t$ zeros per column (the LRPD and LRPS approximation problems). The bounds match those of Corollaries 18 and 20 up to constants.

**Theorem 36.** Let $W \in \{0, 1\}^{n \times p}$ have at most $t$ zeros in each column. Then for $k' = \frac{6kt}{\epsilon}$ and $L$ with $\|A \circ W - L\|_F^2 \leq \min_{\text{rank}-k'} L \|A \circ W - \hat{L}\|_F^2 + \epsilon_1 \|A\|_F^2$.

$$\|A \circ W - \hat{L}\|_F^2 \leq OPT + \epsilon \|A\|_F^2,$$

where $OPT = \min_{\text{rank}-k} (A \circ W) \|W\|_F^2$ is the optimal value of Problem 1.

**Proof.** Since $\hat{L}$ minimizes $\|A \circ W - L\|_F^2$ over all rank-$k'$ matrices up to a $\epsilon_1 \|A\|_F^2$ additive factor, for any rank-$k'$ $\hat{L}$:

$$\|A \circ W - \hat{L}\|_F^2 \leq \|A \circ W - L\|_F^2 \leq \|A \circ W - \hat{L}\|_F^2 + \epsilon_1 \|A\|_F^2.$$

Thus to prove the theorem it suffices to exhibit any rank-$\frac{6kt}{\epsilon}$ matrix $\hat{L}$ achieving

$$\|A \circ W - \hat{L}\|_F^2 \leq OPT + \epsilon \|A\|_F^2.$$

Consider $L_{opt}$ achieving $OPT$. Let $S \subseteq [n]$ be the set of rows given by applying Theorem 34 to $L_{opt}$ with error parameter $\epsilon/5$. Let $\tilde{L}$ be equal to $L_{opt}$ on the rows in $[n] \setminus S$ and be equal to $A \circ W$ on the rows in $S$. Note that $\tilde{L}$ has rank $\leq |S| + k \leq \frac{6kt}{\epsilon}$. Since $\tilde{L}$ exactly matches $A \circ W$ on the rows in $S$ and matches $L_{opt}$ everywhere else we can bound:

$$\|A \circ W - \tilde{L}\|_F^2 = \|A \circ W - L_{opt}\|(\[n] \setminus S, p)\|_F^2 = OPT + \|[L_{opt} \circ (1 - W)]((n] \setminus S, p)\|_F^2. \tag{7}$$

By the guarantee of Theorem 34 applied with error parameter $\epsilon/5$ we have:

$$\|[L_{opt} \circ (1 - W)]((n] \setminus S, p)\|_F^2 \leq \frac{\epsilon/5}{1 - \epsilon/5} \|L_{opt} \circ W\|_F^2 \leq \epsilon/4 \|L_{opt} \circ W\|_F^2. \tag{8}$$

We can bound the right hand side since we must have $\|L_{opt} \circ W\|_F^2 \leq 4 \|A\|_F^2$. Otherwise we would have by the triangle inequality:

$$\sqrt{OPT} = \|(A - L_{opt}) \circ W\|_F \geq \|L_{opt} \circ W\|_F - \|A \circ W\|_F > \|A\|_F,$$

which is not possible as we can always set $L = 0$ and obtain objective function value $\|A\|_F^2$. Plugging back into (7) and (8) we thus have

$$\|A \circ W - \tilde{L}\|_F^2 \leq OPT + \epsilon \|A\|_F^2,$$

which gives the theorem. \[ \square \]
6.4 Fixed-Parameter-Tractable Algorithms

In this section we leverage Theorem 34 in a different way: to give a fixed-parameter-tractable, relative error, non-bicriteria approximation algorithm for the LRPD approximation problem and the related constrained Factor Analysis (FA) problem using a simple recursive scheme.

6.4.1 Exact Decomposition

For exposition, we first consider both problems in the case when $A$ can be exactly decomposed as $D^* + L^*$ where $D^*$ is diagonal and $L^*$ is rank $k$ (i.e., when $\min_{\text{rank} - k} L \|W \circ (A - L)\|_F^2 = 0$). In Section 6.4.2 we extend our techniques to solve the problems in full generality. Our algorithm uses a very simple recursive approach: We split $A$ into four quadrants and compute LRPD (resp. Factor Analysis) decompositions of the upper left and lower right quadrants. Using Theorem 34 we can prove that the diagonal matrices returned by these decompositions match $D^*$ on all but $O(k)$ entries.\footnote{In general, the decomposition $A = D^* + L^*$ may not be unique, however this result holds for any $D^*, L^*$.}

Letting $A_{11} = D_{11}^* + L_{11}^*$ denote the upper left quadrant, our algorithm recursively computes $D_{11}' + L_{11}' = A_{11} = D_{11}' + L_{11}'$. So $D_{11} - D_{11}' = L_{11}' - L_{11}$. Since this is a diagonal matrix and since it has rank $\leq 2k$, by Theorem 34 applied with $t = 1$ and constant $\epsilon$, it can have at most $O(k)$ nonzero entries (since the norm of the remaining entries is bounded by the norm of the off-diagonal entries, which is 0). Of course, Theorem 34 is overkill here and we can see this fact directly. However, the more general theorem will be important in extending our result to the non-exact decomposition case. The same bound holds for the lower right quadrant and so overall, appending the recursively computed diagonal matrices, we have found $D^*$ up to $O(k)$ incorrect entries.

If we guess the locations of these incorrect entries (which we can do with $O(n^{O(k)})$ guesses), it only remains to solve the LRPD problem where we know all but $O(k)$ of the diagonal entries. This problem can be solved in $\text{poly}(n) \cdot 2^{\text{poly}(k)}$ time using generic polynomial solvers. Overall, since we need $n^{O(k)}$ guesses to succeed in identifying the incorrect entries, we obtain runtime $n^{O(k)} \cdot 2^{\text{poly}(k)}$.

**Theorem 37** (Low-Rank Plus Diagonal Exact Decomposition). There is an algorithm solving the LRPD problem (Problem 1 with $W = 1 - I$) up to additive error $1/2^{\text{poly}(n)}$ in $n^{O(k)} \cdot 2^{O(k^2)}$ time when there exists and rank-$k$ $L^*$ with $\|W \circ (A - L^*)\|_F^2 = 0$ and all entries of $A, L^*$ are bounded in magnitude by $2^{\text{poly}(n)}$. That is, the algorithm outputs rank-$k$ $L$ with:

$$\|W \circ (A - L)\|_F^2 \leq \frac{1}{2^{\text{poly}(n)}}.$$  

Note that $1/2^{\text{poly}(n)}$ additive error is on the order of error introduced by rounding an exact solution (with possibly irrational entries) to a poly($n$) bit representation, so can be regarded as negligible. The assumption that $A, L$ have entries bounded by $2^{\text{poly}(n)}$ is required to apply a polynomial solver is also very mild, as without this assumption these matrices could not in general be represented in poly($n$) bits.

**Proof.** As discussed, we use a recursive algorithm, cutting the size of $A$ in half in each step. In the base case, when $n \leq k$ we can solve the problem trivially, simply by returning $L = A$.

Fixing $L^*$ and writing $D^* = A - L^*$, consider splitting $A$ into 4 quadrants, each $n/2 \times n/2$:

$$\
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
= 
\begin{bmatrix}
D_{11}^* & \\
& D_{22}^*
\end{bmatrix}
+ 
\begin{bmatrix}
L_{11}^* & L_{12}^* \\
L_{21}^* & L_{22}^*
\end{bmatrix}.
$$

\[\]

We can prove that the diagonal matrices returned by these decompositions match $D^*$ on all but $O(k)$ entries.\footnote{In general, the decomposition $A = D^* + L^*$ may not be unique, however this result holds for any $D^*, L^*$.}
Since $A_{11} = D_{11}^1 + L_{11}^1$ and $A_{22} = D_{22}^* + L_{22}^*$, we know that Problem 1 with $W = 1 - I$ has optimum value 0 on these submatrices. Assume via induction that we compute rank-$k$ $L_{11}^r, L_{22}^r \in \mathbb{R}^{n/2 \times n/2}$ and diagonal $D_{11}^r, D_{22}^r \in \mathbb{R}^{n/2 \times n/2}$ with such that $D_{11}^r = A_{11} - L_{11}^r$ and $D_{22}^r = A_{22} - L_{22}^r$.

By Theorem 34, $D_{11}^r$ and $D_{11}^i$ (similarly, $D_{22}^r$ and $D_{22}^i$) differ on at most $O(k)$ entries. Thus, letting $D' = \begin{bmatrix} D_{11}^r \\ D_{22}^r \end{bmatrix}$, $D'$ and $D^*$ differ on at most $O(k)$ entries. In $O(n^{O(k)})$ time we can iterate through every set of $O(k)$ indices $I \subseteq [n]$. We can reorder the matrix so that the rows/columns corresponding to the indices in $I$ are the first $O(k)$ rows/columns and repartition $A$ into 4 quadrants:

$$\begin{bmatrix} A_{TT} & A_{TB} \\ A_{BT} & A_{BB} \end{bmatrix} = \begin{bmatrix} D_{TT}^r & D_{TB}^r \\ D_{BT}^r & D_{BB}^r \end{bmatrix} + \begin{bmatrix} L_{TT}^r & L_{TB}^r \\ L_{BT}^r & L_{BB}^r \end{bmatrix}.$$ 

Here $A_{TT}, L_{TT}^r$, and $D_{TT}^r$ are $O(k) \times O(k)$ matrices. $A_{TB} = L_{TB}^r$ is $O(k) \times (n-O(k))$. $A_{BT} = L_{BT}^r$ is $(n-O(k)) \times O(k)$. And finally, $A_{BB}, L_{BB}^r$, and $D_{BB}^r$ are $(n-O(k)) \times (n-O(k))$.

Assuming that $D_{BB}^r$ matches $D_{BB}^r$ on all entries (which will happen for at least one guess of $O(k)$ indices $I$), we can compute $L_{BB}^r$ explicitly by setting $L_{BB}^r = A_{BB} - D_{BB}^r$. Let $U_B \in \mathbb{R}^{(n-O(k)) \times k}$ be an orthonormal span for the columns of $[L_{BB}^r \ L_{BB}^r]$ (which has rank $\leq k$ since $L^r$ is rank-$k$) and let $V_B \in \mathbb{R}^{n-O(k) \times k}$ be an orthonormal span for the rows of $[L_{BB}^r \ L_{BB}^r]$. We know that $A = D^* + L^*$ can be written as:

$$\begin{bmatrix} A_{TT} & A_{TB} \\ A_{BT} & A_{BB} \end{bmatrix} = \begin{bmatrix} D_{TT} & D_{TB} \\ D_{BT} & D_{BB} \end{bmatrix} + \begin{bmatrix} U_T & U_B \end{bmatrix} \begin{bmatrix} V_T^T & R_{V_B}V_T^T \\ V_T^T & R_{V_B}V_T^T \end{bmatrix},$$

where $R_V, R_U \in \mathbb{R}^{k \times k}$, $U_T, V_T \in \mathbb{R}^{O(k) \times k}$, and $D_{TT} \in \mathbb{R}^{O(k) \times O(k)}$ are unknown. In particular, one satisfying solution sets $D_{TT} = D_{TT}^*$ and $R_U, R_V, U_T, V_T$ so that

$$\begin{bmatrix} U_T \\ U_B \end{bmatrix} \begin{bmatrix} V_T^T & R_{V_B}V_T^T \\ V_T^T & R_{V_B}V_T^T \end{bmatrix} = L^*.$$ 

Equation (9) is a degree-2 polynomial system in $O(k^2)$ unknown variables (the entries of $D_{TT}, U_T, V_T, R_U, R_V$). Under the assumption that $A$ and $L^*$ (and hence $D^* = A - L^*$) have entries bounded by $2^{\text{poly}(n)}$, we can solve for entries satisfying (9) up to $1/2^{\text{poly}(n)}$ error using generic polynomial system solvers in $\text{poly}(n) \cdot 2^{O(k^2)}$ time. We detail this process in Section 6.5. Our final runtime follows since at each layer of the recursion we require $n^{O(k)} \cdot 2^{O(k^2)}$ time, and since the problem size is cut in half in each layer, the total runtime is also bounded by $n^{O(k)} \cdot 2^{O(k^2)}$. \hfill $\square$

We can give an algorithm for factor analysis in the exact decomposition case using very similar techniques to Theorem 37.

**Theorem 38 (Factor Analysis Exact Decomposition).** There is an algorithm solving the Factor Analysis problem up to additive error $1/2^{\text{poly}(n)}$ in $n^{O(k)} \cdot 2^{O(k^2 \log k)}$ time when $A = D^* + L^*$ for PSD $A, D^*, L^*$ and all entries of $A$ are bounded in magnitude by $2^{\text{poly}(n)}$. That is, the algorithm outputs rank-$k$ $L \succeq 0$ and diagonal $D \succeq 0$ with:

$$\|A - (D + L)\|_F^2 \leq \frac{1}{2^{\text{poly}(n)}}.$$ 

---

2We actually only have these equalities up to $1/2^{\text{poly}(n)}$ additive error. For now we ignore this detail. It will be formally handled in our proof for the general case, when $A$ is not exactly equal to $D^* + L^*$ (Theorem 40).
Proof. Note that in the theorem statement we only assume the entries of $A$ are bounded by $2^{\text{poly}(n)}$ and do not require an additional assumption on the entries of $L^*$ as we did in Theorem 37. This is simply because, since $D^*, L^* \succeq 0$, both have positive diagonal entries, and we should never choose $\hat{D}$ such that the sum of these entries is greater than the corresponding diagonal entry of $A$. Thus, the entries of $D^*$ are bounded by $2^{\text{poly}(n)}$. In turn, since $L^*$ is an approximation of $A - D^*$, which has bounded magnitude entries, $L^*$ also has bounded magnitude entries (otherwise, a better approximation could be achieved by setting $L^* = 0$.)

We use the same recursive approach as Theorem 37. The only difference is that we need to ensure that $D$ and $L$ are PSD. We already have that $D'_{BB}$ is PSD (i.e., has all nonnegative entries) since it is composed of entries of $D'_{11}$ and $D'_{22}$, which are PSD by the recursive guarantee. When solving the final polynomial system we can ensure that $D_{TT}$ is PSD by adding $4k$ positivity constraints. Additionally since $A$ is PSD and symmetric, we have $U_B = V_B$. To ensure that $L$ is PSD we simple require that $R_U = R_V$ and that $U_T = V_T$. Thus, $L = \begin{bmatrix} U_T & R_T^T U_B^T \end{bmatrix}$, which is PSD. In solving the final polynomial system, this amounts to just using a single set of variables for the entries of $R_U = R_V$ and a set for the entries of $U_T = V_T$.

Overall, the runtime is similar to what is given in Theorem 37, except the polynomial system now involves $O(k)$ constraints (each still with $O(1)$ degree and $O(k)$ variables) and thus requires $2^{O(k^2 \log k)} \cdot \text{poly}(n)$ time to solve. See Section 6.5 for a detailed explanation. \hfill \square

### 6.4.2 Approximate Decomposition

We now consider LRPD and FA problems when $\min_{D,L} \| A - (D + L) \|_F^2 > 0$ and we wish to find $L, D$ achieving a $(1 + \epsilon)$ approximation to this minimum. Using Theorem 34 we prove an analogous result to what we used in the exact decomposition case, namely, that given two near optimal low-rank plus diagonal approximations to $A$, their diagonal entries cannot differ outside a small subset of entries.

**Lemma 39** (Sparse Difference of Approximate Solutions). Consider any $A$ with $\| A - (D + L) \|_F^2 = C$ and $\| A - (D' + L') \|_F^2 \leq C + \gamma$ for diagonal $D, D'$, rank-$k$ $L, L'$, and $\gamma \geq 0$. Let $\mathcal{D}$ be either the set of all diagonal matrices or of all nonnegative diagonal matrices. If $D = \arg \min_{D \in \mathcal{D}} \| A - (\hat{D} + L) \|_F^2$ and $D' = \arg \min_{D \in \mathcal{D}} \| A - (\hat{D} + L') \|_F^2$ then there is some set of $O(k/\epsilon^2)$ indices $S \subseteq [n]$ with:

$$\sum_{i \in S} |D(i,i) - D'(i,i)|^2 \leq \epsilon^2 C + \epsilon^2 \gamma.$$

For the LRPD problem, the set $\mathcal{D}$ will be the set of all diagonal matrices, in the factor analysis problem it will be all nonnegative diagonal matrices. If we solve one of these problems, fixing $L$, it is easy to compute $\arg \min_{D \in \mathcal{D}} \| A - (\hat{D} + L) \|_F^2$, which can only improve our error. When $\mathcal{D}$ is all diagonal matrices, we just set $\hat{D}$ to $\text{diag}(L - A)$. If $\mathcal{D}$ is all non-negative diagonal matrices we set $\hat{D}$ to $\max(0, \text{diag}(L - A))$, where max denotes the entrywise maximum.

**Proof.** By triangle inequality we have:

$$\|(D + L) - (D' + L')\|_F^2 \leq \left( \sqrt{C} + \sqrt{C + \gamma} \right)^2 \leq 4C + 2\gamma.$$  (10)
By (10) we also have (just ignoring any of the on-diagonal difference between \(D + L\) and \(D' + L\):
\[
\| (L - L') - \text{diag}(L - L') \|_F^2 = \| (D + L) - (D' + L') - \text{diag}((D + L) - (D' + L')) \|_F^2 \leq 4C + 2\gamma.
\]
Applying Theorem 34 with \(W = 1 - I\) and \(t = 1\), since \(L - L'\) has rank \(\leq 2k\), there is some set of \(O(k/\epsilon^2)\) indices \(S\) such that:
\[
\sum_{i \in [n]\setminus S} |L(i, i) - L'(i, i)|^2 = \sum_{i \in [n]\setminus S} |(A - L) - (A - L')|(i, i)^2 \leq \epsilon^2 C + \epsilon^2 \gamma. \quad (11)
\]
If \(D\) is the set of all diagonal matrices, then by our assumption that \(D\) is the closest diagonal matrix in \(D\) to \(\text{diag}(L - A)\) and \(D'\) is the closest to \(\text{diag}(L' - A)\) we have \(D(i, i) = -A(i, i) + L(i, i)\) and \(D'(i, i) = -A(i, i) + L'(i, i)\). Combined with (11) this gives:
\[
\sum_{i \in [n]\setminus S} |D(i, i) - D'(i, i)|^2 \leq \epsilon^2 C + \epsilon^2 \gamma
\]
completing the lemma in this case.

Alternatively, if \(D\) is the set of all non-negative diagonal matrices, \(D(i, i) = \max(0, -A(i, i) + L(i, i))\) and \(D'(i, i) = \max(0, -A(i, i) + L'(i, i))\). If \(A(i, i) - L(i, i)\) and \(A(i, i) - L'(i, i)\) have the same sign, then either \(D(i, i) = D'(i, i) = 0\) or \(D(i, i) = -A(i, i) + L(i, i)\) and \(D'(i, i) = -A(i, i) + L'(i, i)\). If they have opposite signs, then we have \(|D(i, i) - D'(i, i)| \leq |[A - L](i, i)| - [A - L'](i, i)|\). Overall using (11) we again have:
\[
\sum_{i \in [n]\setminus S} |D(i, i) - D'(i, i)|^2 \leq \epsilon^2 C + \epsilon^2 \gamma,
\]
which completes the lemma.

Using Lemma 39 we can now give analogous results to Theorems 37 and 42 using a similar recursive scheme. As in the exact case, we will split \(A\) into 4 quadrants and recursively approximate the top left and bottom right. This will yield \(D'\) that nearly matches \(D\) on all but \(O(k/\epsilon^2)\) indices by Lemma 39. We can guess what these indices are in \(n^{O(k/\epsilon^2)}\) time. We then subtract \(D'\) from \(A\) and have to solve LRPD/Factor Analysis with just \(O(k/\epsilon^2)\) unknown entries. This last step is more complicated in the approximate case since we no longer know an exact low-rank span for the submatrix corresponding to the correct diagonals in \(D'\). However, we can still solve the problem efficiently using ideas similar to the “projection-cost preserving” sketches of [CEM+15].

**Theorem 40 (Low-Rank Plus Diagonal Approximate Decomposition).** There is an algorithm solving the LRPD problem (Problem 1 with \(W = 1 - I\)) to relative error \((1 + \epsilon)\) and additive error \(1/2^{O(n)}\) in \(n^{O(k/\epsilon^2) \cdot 2^{O(k^2/\epsilon^2)}}\) time, assuming that all entries of \(A\) are bounded in magnitude by \(2^{O(n)}\) and there exists an optimal \(L^*\) with all entries bounded in magnitude by \(2^{O(n)}\). That is, the algorithm outputs rank-\(k\) \(L\) with:
\[
\|W \circ (A - L)\|_F^2 \leq (1 + \epsilon)\|W \circ (A - L^*)\|_F^2 + \frac{1}{2^{O(n)}}.
\]
Note that Theorem 37 follows as a special case, when \(\|W \circ (A - L^*)\|_F^2 = 0\) and we set \(\epsilon = \Omega(1)\).
Proof. As discussed, we use a recursive algorithm, cutting the size of $A$ in half in each step. In the base case, when $n \leq k$ we can solve the problem trivially, simply by returning $L = A$. Fix $L^*$ and $D^* = \text{diag}(A - L^*)$ with

$$L^* \in \arg \min_{\text{rank-}k \hat{L}} \| W \circ (A - \hat{L}) \|_F^2.$$  

and let $\Delta^* = A - (D^* + L^*)$. Split $A$ into 4 quadrants, each $n/2 \times n/2$:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} D_{11}^* & L_{12}^* \\ L_{21}^* & L_{22}^* \end{bmatrix} + \begin{bmatrix} \Delta_{11}^* & \Delta_{12}^* \\ \Delta_{21}^* & \Delta_{22}^* \end{bmatrix}.$$  

Assume that we recursively compute rank-$k$ $L_{11}', L_{22}' \in \mathbb{R}^{n/2 \times n/2}$ and diagonal $D_{11}', D_{22}' \in \mathbb{R}^{n/2 \times n/2}$ with $D_{ii}' = \text{diag}(A - L_{ii}')$ such that for $i \in \{1, 2\}$:

$$\| A_{ii} - (D_{ii}' + L_{ii}') \|_F^2 \leq (1 + \epsilon) \min_{\text{rank-}k \hat{L}} \| A_{ii} - (\hat{D} + \hat{L}) \|_F^2 + \frac{1}{2^{\text{poly}(n)}}.$$  

Note that this ensures in particular that for both $i \in \{1, 2\}$,

$$\| A_{ii} - (D_{ii}' + L_{ii}') \|_F^2 \leq (1 + \epsilon) \| \Delta_{ii}^* \|_F^2 + \frac{1}{2^{\text{poly}(n)}}.$$  

Applying Lemma 39, we then have that there is some set of $S_i$ indices with $|S_i| = O(k/\epsilon^2)$ such that, letting $\bar{D}_{ii}$ match $D_{ii}'$ on these indices and match $D_{ii}'$ everywhere else,

$$\| D_{ii}' - \bar{D}_{ii} \|_F^2 \leq \epsilon^2 \left( \| \Delta_{ii}^* \|_F^2 + \frac{1}{2^{\text{poly}(n)}} \right).$$  

Thus, letting $\bar{D} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{22} \end{bmatrix}$, we have

$$\| D^* - \bar{D} \|_F^2 \leq \epsilon^2 \left( \| \Delta_{11}^* \|_F^2 + \| \Delta_{22}^* \|_F^2 \right) + \frac{1}{2^{\text{poly}(n)}} \leq \epsilon^2 \| \Delta^* \|_F^2 + \frac{1}{2^{\text{poly}(n)}}. \quad (12)$$  

After computing $D_{11}'$ and $D_{22}'$ recursively, we know $\bar{D}$ up to $O(k/\epsilon^2)$ entries. We know the entries matching those of $D'$ and do not know those in $S_1 \cup S_2$ matching $D^*$. In $n^{O(k/\epsilon^2)}$ time we can iterate through every set of $O(k/\epsilon^2)$ indices $\mathcal{I} \subseteq [n]$. We can order the matrix so that the rows/columns corresponding to the indices in $\mathcal{I}$ are the first $O(k/\epsilon^2)$ rows/columns and repartition $A$ into 4 quadrants:

$$\begin{bmatrix} A_{TT} & A_{TB} \\ A_{BT} & A_{BB} \end{bmatrix} = \begin{bmatrix} D_{TT}^* & L_{TB}^* \\ L_{BT}^* & L_{BB}^* \end{bmatrix} + \begin{bmatrix} \Delta_{TT}^* & \Delta_{TB}^* \\ \Delta_{BT}^* & \Delta_{BB}^* \end{bmatrix}.$$  

Assume that $\bar{D}_{BB} = D_{BB}'$, which will happen for at least one guess of $O(k/\epsilon^2)$ indices $\mathcal{I}$. By the triangle inequality and (12) we have:

$$\min_{\text{rank-}k \hat{L}} \| A - \left( \begin{bmatrix} D_{TT} & L_{TB} \\ L_{BT} & L_{BB} \end{bmatrix} + \bar{L} \right) \|_F^2 \leq \| A - (\bar{D} + \bar{L}) \|_F^2 \leq (\| A - (D^* + L) \|_F + \| D^* - \bar{D} \|_F)^2 \leq (1 + \epsilon)^2 \| \Delta^* \|_F^2 + \frac{1}{2^{\text{poly}(n)}}. \quad (14)$$
where we make use of the fact that certainly $\|\Delta^*\|_F^2 \leq \|A\|_F^2 \leq 2^{\text{poly}(n)}$ by the assumption that $A$ has bounded entries. Thus to approximately solve LRPD, it suffices to solve the minimization problem in (13). Since $L$ is rank $k$, we can rewrite this problem as:

$$\min_{Z_T, W_T \in \mathbb{R}^{O(k/\epsilon^2) \times k}, \ Z_B, W_B \in \mathbb{R}^{n-O(k/\epsilon^2) \times k}} \left\| A - \begin{bmatrix} D_{TT} & D'_{BB} \\ Z_T & Z_B \end{bmatrix} \begin{bmatrix} W_T^T & W_B^T \end{bmatrix} \right\|_F^2. \quad (15)$$

Unfortunately, we cannot efficiently solve this minimization problem directly because it involves $O(nk)$ unknowns: $Z_T$, $Z_B$, $W_T$, and $W_B$ contain $2nk$ free variables in total, so applying Theorem 33 would take time exponential in $n$.

Recall that for the exact decomposition problem, we reduced this cost by writing $L$ exactly in a factored form, using the rank-$k$ column and row spans of $[L^*_{TT}, L^*_{BB}]$ and $[L^T_{BB}, L^*_{BB}]$. This reduced the number of free variables to $O(k^2)$ – see (9). Because $[A_{TT} \ A_{BB} - D'_{BB}]$ and $[A_{BB} - D'_{BB}]$ do not in general exactly match $L^*$ and so have rank $k$ in the approximate decomposition problem, this is no longer possible. However, it is possible to show that (15) can be solved approximately with an $L$ that is restricted to a particular factored form involving few variables. Specifically, let:

$U_B \in \mathbb{R}^{(n-O(k/\epsilon^2)) \times [k/\epsilon]}$ contain the top $[k/\epsilon]$ column singular vectors of $[A_{TT} \ A_{BB} - D'_{BB}]$;

$V_B \in \mathbb{R}^{(n-O(k/\epsilon^2)) \times [k/\epsilon]}$ contain the top $[k/\epsilon]$ row singular vectors of $[A_{TT} \ A_{BB} - D'_{BB}]$.

Claim 41. For $U_B$ and $V_B$ as described above,}

$$\min_{R_U, R_V \in \mathbb{R}^{O(k/\epsilon^2) \times k}, \ Z_T, W_T \in \mathbb{R}^{O(k/\epsilon^2) \times k}} \left\| A - \begin{bmatrix} D_{TT} & D'_{BB} \\ Z_T & U_B R_U \end{bmatrix} \begin{bmatrix} W_T^T & R_V^T V_B^T \end{bmatrix} \right\|_F^2 \leq (1 + \epsilon)^2 \min_{Z_T, W_T \in \mathbb{R}^{O(k/\epsilon^2) \times k}, \ Z_B, W_B \in \mathbb{R}^{n-O(k/\epsilon^2) \times k}} \left\| A - \begin{bmatrix} D_{TT} & D'_{BB} \\ Z_T & Z_B \end{bmatrix} \begin{bmatrix} W_T^T & W_B^T \end{bmatrix} \right\|_F^2. \quad (16)$$

It immediately follows from (14) and the equivalence of (13) and (15) that, if we solve (16), we will obtain an LRPD approximation with error $\leq (1 + \epsilon)^3 \|\Delta^*\|_F^2 + \frac{1}{2^{\text{poly}(n)}}$. This can be done using a generic polynomial solver because, in contrast to (15), (16) has just $O(k^2/\epsilon^2)$ unknowns, and can be turned into a polynomial system with degree $O(1)$ and $O(1)$ constraints. Thus it can be solved to $\frac{1}{2^{\text{poly}(n)}}$ error in $\text{poly}(n) \cdot 2^{O(k^2/\epsilon^2)}$. We describe this process in detail in Section 6.5. Our final runtime follows since at each layer of recursion we require $n^{O(k/\epsilon^2)} \cdot 2^{O(k^2/\epsilon^2)}$ time and since the problem size is cut in half at each layer (recall that the $n^{O(k/\epsilon^2)}$ term comes from the fact that we must guess which entries of $D'$ are close to those of $D^*$). Thus, to complete the proof of Theorem 40, it just remains to prove Claim 41.
**Proof of 41.** Let $Z_T^*, Z_B^*, W_T^*, W_B^*$, and $D_{TT}^*$ comprise an optimal solution for (15) and denote:

$$
\tilde{D}^* = \begin{bmatrix} D_{TT}^* & D_{BB}^* \end{bmatrix}
\quad \text{and} \quad
\tilde{L}^* = \begin{bmatrix} Z_T^* \\ Z_B^* \end{bmatrix} \begin{bmatrix} W_T^* & W_B^* \end{bmatrix}
$$

We will make the argument in two steps. First we consider a third minimization problem:

$$
\min_{R_U \in \mathbb{R}^{O(k/c^2) \times k}, \text{ \text{Z_T, W_T} \in} \mathbb{R}^{O(k/c^2) \times k}, W_B \in \mathbb{R}^{n-O(k/c^2) \times k} \text{ \text{diagonal} \ D_{TT}}}
\left\| A - \left( \begin{bmatrix} D_{TT} & D_{BB} \end{bmatrix} + \begin{bmatrix} Z_T \\ U_BR_U \end{bmatrix} \begin{bmatrix} W_T & W_B \end{bmatrix} \right) \right\|_F^2.
$$

(17)

Let $OPT_1$ be the optimum value of (15), $OPT_2$ be the optimum value of (17), and $OPT_3$ be the optimum value of (16). We will establish the claim by separately showing:

$$
OPT_2 \leq (1 + \epsilon)OPT_1 \quad \text{(18)}
$$

$$
OPT_3 \leq (1 + \epsilon)OPT_2. \quad \text{(19)}
$$

We prove (18) first. With $Z_T^*, Z_B^*, W_T^*$, and $W_B^*$ as defined above, let $\tilde{L}_1$ equal:

$$
\tilde{L}_1 = \begin{bmatrix} Z_T^* \\ (U_BU_B^T) Z_B^* \end{bmatrix} \begin{bmatrix} W_T^* & W_B^* \end{bmatrix}.
$$

Since $\| A - (\tilde{D}^* + \tilde{L}_1) \|_F^2 \geq OPT_2$, if we can show that $\| A - (\tilde{D}^* + \tilde{L}_1) \|_F^2 \leq (1 + \epsilon)OPT_1$, we prove (18). To do so, we introduce the additional notation:

$$
F = \begin{bmatrix} A_{BT} & A_{BB} - D_{BB}^* \end{bmatrix}
\quad \text{and} \quad
Q = \text{orth} \left( \begin{bmatrix} W_T^* \\ W_B^* \end{bmatrix} \right).
$$

We first notice that we always have $Z_B^* \begin{bmatrix} W_T^T & W_B^T \end{bmatrix} = FQQ^T$. This holds because, in choosing the minimal solution to (15), after fixing all variables besides $Z_B$, we can always obtain a better solution by choosing $Z_B$ so that $Z_B \begin{bmatrix} W_T^T & W_B^T \end{bmatrix}$ is the optimal approximation to $F$ in the row span of $\begin{bmatrix} W_T^T & W_B^T \end{bmatrix}$. This is obtained by setting $Z_B = \begin{bmatrix} W_T^T & W_B^T \end{bmatrix}^+$, in which case $Z_B = FQQ^T$ for $Q = \text{orth} \left( \begin{bmatrix} W_T^T & W_B^T \end{bmatrix} \right)$.

Then, since $\tilde{L}^*$ and $\tilde{L}_1$ only differ on the lower block of their left factor, we can see that:

$$
\| A - (\tilde{D}^* + \tilde{L}_1) \|_F^2 - \| A - (\tilde{D}^* + \tilde{L}^*) \|_F^2 = \| F - FQQ^T \|_F^2 - \| F - U_BU_B^T FQQ^T \|_F^2.
$$

(20)

We have:

$$
\| F - (U_BU_B^T) FQQ^T \|_F^2 = \| F(I - QQ^T) + (F - U_BU_B^T F) QQ^T \|_F^2.
$$

$$
= \| F(I - QQ^T) \|_F^2 + \| (F - U_BU_B^T F) QQ^T \|_F^2.
$$

So plugging into (20), we have:

$$
\| A - (\tilde{D}^* + \tilde{L}_1) \|_F^2 - \| A - (\tilde{D}^* + \tilde{L}^*) \|_F^2 = \| (F - U_BU_B^T F) QQ^T \|_F^2.
$$

(21)

We just need to show that $\| (F - U_BU_B^T F) QQ^T \|_F^2$ is small.
To do so, let $\sigma_1, \ldots, \sigma_n$ denote the singular values of $F$. Since $U_B U_B^T$ was chosen to be the top $\lceil k/\epsilon \rceil$ singular vectors of $F$, $F - U_B U_B^T F$ has singular values:

$$\sigma_{\lceil k/\epsilon \rceil + 1}, \sigma_{\lceil k/\epsilon \rceil + 2}, \ldots, \sigma_n.$$ 

Since $Q$ has orthonormal columns, $QQ^T$ is a rank $k$ projection matrix and thus:

$$\|(F - U_B U_B^T F)QQ^T\|_F^2 \leq \sum_{i=\lfloor k/\epsilon \rfloor + 1}^{\lceil k/\epsilon \rceil + k} \sigma_i^2 \leq \epsilon \sum_{i=1}^{\lceil k/\epsilon \rceil + k} \sigma_i^2 \leq \epsilon \|F - F_k\|_F^2,$$

where $F_k$ is the optimal rank $k$ approximation to $F$ in Frobenius norm. Note that this argument is essentially identical to the proof in [CEM+15] that the top $\lceil k/\epsilon \rceil$ singular vectors of a matrix can be used to form a “projection-cost preserving sketch” of that matrix.

Finally, it of course holds that $\|F - F_k\|_F^2 \leq OPT_1$ since $OPT_1 \leq \|F - Z_B^* [W_B^T \ W_B^T] = \|F - FQQ^T\|_F^2$. Since $Q$ is rank $k$, this is a rank $k$ approximation to $F$, so $\|F - F_k\|_F^2 \leq \|F - FQQ^T\|_F^2$.

This concludes the proof of (18): returning to (21), we see that $\|A - (\bar{D}^* + \bar{L}_1)\|_F^2 \leq (1 + \epsilon)OPT_1$ and it follows that $OPT_2 \leq \|A - (\bar{D}^* + \bar{L}_1)\|_F^2 \leq (1 + \epsilon)OPT_1$.

The proof of (19) is essentially identical, so we omit it for brevity. Briefly, if we let $Z^*_T, R^*_U, W^*_T, W^*_B, D^*_{TT}$ comprise an optimal solution for (17) and $D^*, \bar{L}$ be defined as before. Denote:

$$F = \begin{bmatrix} A_{TB} \\ A_{BB} - D'_{BB} \end{bmatrix} \quad \text{and} \quad Q = \text{orth}\left( \begin{bmatrix} Z^*_T \\ U_B R^*_U \end{bmatrix} \right)$$

and then we have that $\begin{bmatrix} Z^*_T \\ U_B R^*_U \end{bmatrix} W^*_B = QQQ^T F$. This is all we need to prove that

$$\bar{L}_2 = \begin{bmatrix} Z^*_T \\ U_B R^*_U \end{bmatrix} \begin{bmatrix} W_T^T \\ W_B^T \end{bmatrix} (V_B V_B^T),$$

satisfies $\|A - (\bar{D}^* + \bar{L}_2)\|_F^2 \leq (1 + \epsilon)OPT_2$. We also have $\|A - (\bar{D}^* + \bar{L}_2)\|_F^2 \geq OPT_3$, which yields (19), and thus the proof of Claim 41. \(\square\)

We have an analogous result to Theorem 40 for the factor analysis problem, which is also proved in essentially the same way. As in the exact decomposition case, we just need to require that in the final optimization problem, $D_{TT}$ is positive, $Z_T = W_T$ and $R_U = R_V$. Overall we have:

**Theorem 42 (Factor Analysis Approximate Decomposition).** There is an algorithm solving the Factor Analysis problem to relative error $(1 + \epsilon)$ and additive error $1/\epsilon O\log(n)$ time, assuming that all entries of $A$ are bounded in magnitude by $1/\epsilon O\log(n)$. That is, the algorithm outputs rank-$k$ $L \succeq 0$ and diagonal $D \succeq 0$ with:

$$\|A - (D + L)\|_F^2 \leq (1 + \epsilon)\|A - (D^* + L^*)\|_F^2 + \frac{1}{\epsilon O\log(n)}.$$

Note that Theorem 38 follows as a special case when $\|A - (D^* + L^*)\|_F^2 = 0$ and we set $\epsilon = \Omega(1)$. 

33
6.5 Polynomial Optimization

In this section we detail our use of polynomial solvers in Theorems 37, 38, 40, and 42. We start with a technical lemma that we will use to bound the bit complexity of certain polynomial systems that arise in our algorithms.

**Lemma 43** (Bounded Low-Rank Factors). Consider $M \in \mathbb{R}^{n \times p}$ with $|M(i,j)| \leq \Delta$ for all $i,j$ and let $L \in \mathbb{R}^{n \times p}$ be an optimal rank-$k$ approximation:

$$L = \arg\min_{\text{rank-}k \ L} \| M - L \|_F.$$ 

We can write $L = UV^T$ for $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{p \times k}$ with $|U(i,j)| \leq \Delta$ and $|V(i,j)| \leq n^{1/4} \Delta^{1/2}$ for all $i,j$.

**Proof.** Assume that $\Delta = 1$. For general $\Delta$, we can first scale $M$ by $1/\Delta$ so that its entries are bounded in magnitude by 1, exhibit $U,V$ with entries bounded in magnitude by $n^{1/4}$, and then scale these matrices each up by a $\sqrt{\Delta}$ factor, yielding the result.

We can decompose $L = UV^T$ using e.g., a QR decomposition, where $U$ is orthonormal and so has entries bounded in magnitude by 1. Consider the $i$th row of $V$, $v_i \in \mathbb{R}^k$. If we project this row to the row span of $U$, $UV^T$ will be unchanged and so will still equal $L$. Additionally, after this projection, we can see that, letting $l_i, m_i \in \mathbb{R}^p$ be the $i$th columns of $L$ and $M$:

$$\|v_i\|_2 = \|UV_i\|_2 = \|l_i\|_2 \leq \|m_i\|_2 \leq \sqrt{n}. \quad (22)$$

where the second to last inequality follows from the fact that if $\|l_i\|_2 > \|m_i\|_2$, we could achieve a better low-rank approximation by replacing $\|l_i\|_2$ with the projection of $m_i$ onto this column, which has norm $\leq \|m_i\|_2$ and just rescales the column so does not change the rank of $L$.

By (22), $V$ has all entries bounded in magnitude by $\sqrt{n}$. Finally, if we multiply $U$ by $n^{1/4}$ and divide $V$ by $n^{1/4}$ we will still have $UV^T = L$ and all entries will be bounded in magnitude by $n^{1/4}$, completing the lemma. $\square$

We now discuss our use of polynomial solvers in the exact decomposition case, considered in Theorems 37 and 38.

6.5.1 Use of Polynomial Solvers in Theorems 37 and 38

In Theorem 37 we reduce solving the LRPD decomposition problem to finding matrices $R_V, R_U \in \mathbb{R}^{k \times k}$, $U_T, V_T \in \mathbb{R}^{4k \times k}$, and $D_{TT} \in \mathbb{R}^{4k \times 4k}$ satisfying (9). This equation defines a polynomial system of equations. This system has $O(k^2)$ variables: the entries of $R_V, R_U \in \mathbb{R}^{k \times k}$, $U_T, V_T \in \mathbb{R}^{4k \times k}$, and $D_{TT} \in \mathbb{R}^{4k \times 4k}$. It has $n^2$ constraints, each of the form

$$A(i,j) = \hat{D}(i,j) + \hat{L}(i,j)$$

where $\hat{D} = \begin{bmatrix} D_{TT} & D_{BB}^T \\ D_{BB} & \end{bmatrix}$ and $\hat{L} = \begin{bmatrix} U_T \\ U_B R_U \end{bmatrix} \begin{bmatrix} V_T^T & R_U^T V_B^T \end{bmatrix}$. Note that $A(i,j)$ is a known quantity and $\hat{D}(i,j) + \hat{L}(i,j)$ is a degree-2 polynomial in the unknown variables. We can combine all these equality constraints into a single constraint:

$$\sum_{i,j} \left( A(i,j) = \hat{D}(i,j) + \hat{L}(i,j) \right)^2 = 0. \quad (23)$$
Note that the left hand side of this constraint is a degree-4 polynomial in the unknown variables. We will identify entries of $R_T, R_U, U_T, V_T$ and $D_{TT}$ that satisfy (23) up to additive error $1/2^{\text{poly}(n)}$ via binary search. To apply binary search we must first bound the range that we must search over.

6.5.2 Bounding the Range of Binary Search

By the assumption of Theorem 37 that $A = D^* + L^*$ where $D^*$ and $L^*$ have entries bounded in magnitude by $2^{\text{poly}(n)}$, there is a solution to (23) where the entries of $\hat{D}$ (and hence $D_{TT}$) and $\hat{L}$ are bounded in magnitude by $2^{\text{poly}(n)}$. By Lemma 43, if this is the case, $\hat{L}$ admits a factorization with all entries bounded in magnitude by $2^{\text{poly}(n)}$. Thus there is a solution with all entries of $U_T, V_T, U_BR_U, V_BR_V$ bounded in magnitude by $2^{\text{poly}(n)}$. Finally, we note that this implies the existence of a solution with all entries of $R_U, R_V$ bounded in magnitude by $2^{\text{poly}(n)}$: we can assume that the columns of $R_U$ fall in the row span of $U_B$ (resp. for $R_V$ and $V_B$), thus their norms and hence entries are bounded by the column norms of $U_BR_U$, which are bounded by $2^{\text{poly}(n)}$. Thus, for all variables, we can restrict our search range to $[-2^{\text{poly}(n)}, 2^{\text{poly}(n)}]$.

6.5.3 Performing an Iteration of Binary Search

To perform binary search we will consider the polynomial system consisting of (23), along with a constraint that all variables are bounded in magnitude by $2^{\text{poly}(n)}$ (i.e., a degree-2 constraint that bounds the sum of squares of the variables by $2^{\text{poly}(n)}$). In each iteration of binary search, we will consider an entry $M(i, j)$ (where $M$ is $R_V, R_U, U_T, V_T$, or $D_{TT}$) and verify this polynomial system augmented with the search constraints $c_1 \leq M(i, j)$ and $M(i, j) \leq c_2$ for any $c_1, c_2 \in [-2^{\text{poly}(n)}, 2^{\text{poly}(n)}]$. The overall system has four constraints that are $O(1)$ degree polynomials in $O(k^2)$ variables. Its coefficients are all multiples of:

1. The entries of $D_{BB}'$, which are obtained via a recursive call to our algorithm and thus bounded in magnitude by $2^{\text{poly}(n)}$.
2. The entries of $U_B$ and $V_B$ which are bounded in magnitude by 1 since they are orthonormal.
3. The entries of $A$ and the values $c_1$, and $c_2$, which again are bounded in magnitude by $2^{\text{poly}(n)}$.

Thus, as long as we round all coefficients to additive error $1/2^{\text{poly}(n)}$, we can represent our full system using $H = \text{poly}(n)$ bits. Since, by our added magnitude constraint, any valid solution to the system has all variables bounded by $2^{\text{poly}(n)}$, rounding these coefficients can only affect the minimum value of (23) by $1/2^{\text{poly}(n)}$. We can verify that this minimum value is $\leq 1/2^{\text{poly}(n)}$ in $2^{O(k^2) \cdot \text{poly}(n)}$ time using Theorem 33. This certifies that there is some setting of $M(i, j) \in [c_1, c_2]$ that gives a solution with the left hand side of (23) bounded by $1/2^{\text{poly}(n)}$. By performing $\text{poly}(n)$ iterations in this way, we can identify $c_1, c_2$ that are $1/2^{\text{poly}(n)}$ apart such that some value of $M(i, j) \in [c_1, c_2]$ achieves such a value.

6.5.4 Fixing Variables

After identifying a small range $[c_1, c_2]$ in which some valid value for $M(i, j)$ lies, we will fix this variable. Specifically, we will chose an arbitrary value $c^*$ in this range and add the constraint $M(i, j) = c^*$ to our system by adding the term $(M(i, j) - c^*)^2$ to the left hand side of (23). $c^*$ is within $1/2^{\text{poly}(n)}$ of what ever the true valid value in $[c_1, c_2]$ is, so fixing $M(i, j) = c^*$ can only
increase the optimum of the left hand side of (23) by $1/2 \text{poly}(n)$ (this follows since all coefficients of the system are bounded by $2^{\text{poly}(n)}$, and additionally, by our magnitude constraints, in any feasible solution, all variables are bounded by $2^{\text{poly}(n)}$).

After fixing $M(i, j)$ we move on to the next variable. Note that after adding the constraint $M(i, j) = c^*$ our system still has coefficients bounded in magnitude by $2^{\text{poly}(n)}$ and we can still solve it in $2^{O(k^2)} \cdot \text{poly}(n)$ time. We perform $\text{poly}(n)$ such binary searches, each requiring $\text{poly}(n)$ polynomial system verifications, and thus requiring total runtime $\text{poly}(n) \cdot 2^{O(k^2)}$, as claimed in Theorem 37.

6.5.5 Extension to Factor Analysis

The same general technique described above can be used to solve the polynomial system that arises in the factor analysis algorithm of Theorem 38. The only difference is that we add $O(k)$ additional constraints to our original system, requiring that $D_{TT}(i,i) \geq 0$ for all $i$. Each system solve thus takes $\text{poly}(n) \cdot k^{O(k^2)} = \text{poly}(n) \cdot 2^{O(k^2 \log k)}$ time.

6.5.6 Use of Polynomial Solvers in Theorems 40 and 42

Our use of polynomials solvers in Theorems 40 and 42 is very similar to in Theorems 37 and 38, so we just discuss relevant modifications. Focusing on Theorem 40, we reduce solving the LRPD approximation problem to finding matrices $R_U, R_V \in \mathbb{R}^{O(k/\epsilon^2) \times k}$, $Z_T, W_T \in \mathbb{R}^{O(k/\epsilon^2) \times k}$, and $D_{TT} \in \mathbb{R}^{O(k/\epsilon^2) \times O(k/\epsilon^2)}$ minimizing the lefthand side of (16).

As in the exact case, the function to be minimized can be written as a degree four polynomial in $O(k^2/\epsilon^2)$ variables: the entries of the free matrices. We will first perform binary search to identify the minimum value of this polynomial up to $\frac{1}{2^{\text{poly}(n)}}$ error. We will then use binary search, as in Theorems 37 and 38 to find values for the free matrices that achieve within $\frac{1}{2^{\text{poly}(n)}}$ of this minimum. As before, to apply binary search we must first bound the range that we must search over.

6.5.7 Bounding the Range of Binary Search

In searching for the minimum value of (16), we know that it is bounded within $[0, \|A\| F^2]$ and thus is bounded in magnitude for $2^{\text{poly}(n)}$.

By the assumption of Theorem 40 that there exist optima for the LRPD problem $D^*, L^*$ with entries bounded in magnitude by $2^{\text{poly}(n)}$. As shown in (12) there is a near optimal solution to the alternative objective function (15) using a diagonal matrix that closely matches $D^*$ on all entries, and thus also has entries bounded in magnitude by $2^{\text{poly}(n)}$. Finally, we can see by examining the proof of Claim 41 that there is a near optimal solution to (16) with the same diagonal entries. Along with the assumption that $A$ has bounded entries and Lemma 43 this implies that there is a solution to (16) where the entries of all unknown matrices are bounded by $2^{\text{poly}(n)}$.

6.5.8 Performing an Iteration of Binary Search

To perform binary search for the minimum value of (16) we will consider the polynomial system that restricts the lefthand side $\geq c_1$ and $\leq c_2$ for $c_1, c_2 \in [0, 2^{\text{poly}(n)}]$. We will also require that all variables are bounded in magnitude by $2^{\text{poly}(n)}$ (i.e., include a degree-2 constraint that bounds the sum of squares of the variables by $2^{\text{poly}(n)}$). Overall this system has three constraints, $O(k^2/\epsilon^2)$ variables, and degree $O(1)$. Its bit complexity can be bounded in the same way as argued for
Theorem 37 and thus it can be verified in $\text{poly}(n) \cdot 2^{O(k^2/\varepsilon^2)}$ time. In $\text{poly}(n)$ iterations we can identify the optimum value up to $1/\text{poly}(n)$ error.

To perform binary search on the unknown matrix entries in (16) we will consider the polynomial system consisting of an inequality restricting that the lefthand side is upper bounded by the approximate minimum that we identify, along with the same magnitude constraints. In each iteration of binary search, we will consider an entry $M(i,j)$ (where $M$ is $R_V, R_U, Z_T, W_T$, or $D_{TT}$) and verify this polynomial system augmented with the search constraints $c_1 \leq M(i,j)$ and $M(i,j) \leq c_2$ for any $c_1, c_2 \in [-2^{\text{poly}(n)}, 2^{\text{poly}(n)}]$. The remainder of the argument exactly mirrors that used for Theorems 37 and 38.

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