GAUSSIAN RANDOM MATRIX MODELS FOR q–DEFORMED
GAUSSIAN VARIABLES

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Abstract. We construct a family of random matrix models for the q–
deformed Gaussian random variables $G_\mu = a_\mu + a_\mu^*$ where the annihilation
operators $a_\mu$ and creation operators $a_\mu^*$ fulfil the q–deformed commutation re-
lation $a_\mu a_\nu^* - qa_\nu^* a_\mu = \Gamma_{\nu\mu}$, $\Gamma_{\nu\mu}$ is the covariance and $0 < q < 1$ is a given
number. Important feature of considered random matrices is that the joint
distribution of their entries is Gaussian.

1. Introduction

1.1. The deformed Gaussian variables. The q-deformed Gaussian random
variables $G_\mu = a_\mu + a_\mu^*$, where operators $a_\mu$ and their adjoints $a_\mu^*$ fulfil deformed
commutation relations

$$a_\mu a_\nu^* - qa_\nu^* a_\mu = \Gamma_{\nu\mu}1,$$ (1)

were introduced by Bourret and Frisch [FB]. These operators act on a Hilbert space
$\mathcal{K}$ which has a unital vector $\Omega$, called a vacuum, with the property that

$$a_\mu \Omega = 0,$$ (2)

for every value of the index $\mu$.

With the help of the vector $\Omega$ one can introduce a state $\tau$ on the algebra of
operators acting on $\mathcal{K}$ as follows $\tau(X) = \langle \Omega, X\Omega \rangle$. The state $\tau$ plays a role of the
non commutative expectation value. From (1) and (2) follows [BS1] that for any
$m \in \mathbb{N}$ and any indexes $\mu_1, \ldots, \mu_{2m}$ we have that

$$\tau(G_{\mu_1} \cdots G_{\mu_{2m-1}}) = 0,$$ (3)

$$\tau(G_{\mu_1} \cdots G_{\mu_{2m}}) = \sum_\pi q^{i(\pi)} \Gamma_{c_1 d_1} \cdots \Gamma_{c_m d_m},$$ (4)

where the sum is taken over all pair partitions $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the
set $\{1, \ldots, 2m\}$ and $i(\pi)$ is the number of crossings of the partition $\pi$. For the
reader’s convenience we shall recall definitions of a pair partition and of its number
of crossings in Sect. 3.

From the quantum probability point of view all the information about non com-
mutative random variables $G_\mu$ is encoded in their moments $\tau(G_{\mu_1} \cdots G_{\mu_m})$ and
therefore Eq. (3) and (4) can be treated as an alternative definition of q-deformed
Gaussian variables $G_\mu$. 

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1.1.1. Applications of deformed Gaussian variables. Eq. (3) and (4) show that for \( q = 1 \) operators \( G_\mu \) have the same moments as classical Gaussian variables with the mean zero and the covariance \( \Gamma_{\mu\nu} \), what should explain why do we call \( G_\mu \) deformed Gaussian variables. Eq. (1) is for \( q = 1 \) called the canonical (or bosonic) commutation relation.

For other special choices of the deformation parameter \( q \) variables \( G_\mu \) also have natural probabilistic interpretations \([FB]\), namely as increments of a dichotomic Markov process (for \( q = -1 \)) or as Wigner’s large random matrices (for \( q = 0 \)). Voiculescu \([V]\) has made a remarkable observation that for \( q = 0 \) random variables \( G_\mu \) are free semicircular elements (an analogue of independent Gaussian variables in the free probability theory of Voiculescu \([V3, VDN]\)). Eq. (4) is for \( q = -1 \) called the canonical anticommutation relation (or fermionic relation) and for \( q = 0 \) is called the free relation.

Therefore it was natural to expect that the relations (3) and (4) which are a simple generalisation of the three mentioned above: bosonic, fermionic and free cases would give rise to interesting probabilistic objects.

Indeed, it was observed by Bożejko and Speicher \([BS1]\) that a related to Eq. (1) Brownian motion is a one component of an \( n \)-dimensional Brownian motion which is invariant under the quantum group \( S_{\sqrt{q}U(n)} \) of Woronowicz for \( 0 < q < 1 \).

Another application of \( q \)-deformed Gaussian variables, this time as generalised quantum statistics was proposed by Greenberg \([Gr]\) and Speicher \([Sp2]\).

The existence of operators \( a_\mu \) and \( a^*_\mu \) fulfilling deformed commutation relations (1) was proven by Bożejko and Speicher \([BS2]\). Later it was proven by Bożejko, Kümmerer, and Speicher \([BKS]\) that the von Neumann algebra generated by \( q \)-deformed Gaussian variables \( G_1, G_2, \ldots \) \((-1 < q < 1\) is a \( \mathcal{II}_1 \) factor. There are today many open questions concerning these factors, particularly if they are different from the free group factors.

In this paper we present a natural probabilistic representation of the \( q \)-deformed Gaussian variables for all \( q \in [0, 1] \) as some random matrices, what was one of the open questions posed in the paper \([FB]\). A remarkable property of our model is that the joint distribution of entries of our matrices is Gaussian.

Recently a related problem of finding a random matrix model for so called \( q \)-deformed circular system was solved by Mingo and Nica \([MN]\).

1.1.2. The covariance \( \Gamma_{\mu\nu} \). Indexes \( \mu, \nu \) are elements of a certain set \( \mathfrak{M} \). A necessary and sufficient condition for operators \( G_\mu \) to exist is that the function \( \Gamma_{\mu\nu} \) is positive definite \([BS2]\), i.e.

\[
\sum_{1 \leq i, j \leq n} \alpha_i \alpha_j \Gamma_{\mu_i \mu_j} \geq 0
\]

for all \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and \( \mu_1, \ldots, \mu_n \in \mathfrak{M} \). Typical examples of sets \( \mathfrak{M} \) and covariance functions are:

- \( \mathfrak{M} = \mathbb{N} \) and \( \Gamma_{i,j} = \delta_{ij} \). For \( q = 1 \) we have that \( G_1, G_2, \ldots \) is a sequence of independent, standard Gaussian variables while for \( q = 0 \) we have that \( G_1, G_2, \ldots \) is a sequence of free semicircular elements \([VDN]\).
- \( \mathfrak{M} = \mathbb{R}_+ \) and \( \Gamma_{t,s} = \min(t,s) \). For \( q = 1 \) we have that \( G_t \) is a Brownian motion, for \( q = 0 \) we have that \( G_t \) is a noncommutative stochastic process with free increments.
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- $\mathcal{M}$ is a real Hilbert space and the covariance is defined by the scalar product: $\Gamma_{\phi\psi} = \langle \phi, \psi \rangle$. The case $\mathcal{M} = L^2(\mathbb{R}^+)$ is often used in the white noise calculus.

1.1.3. The distribution of a deformed Gaussian variable. A distribution of a random variable corresponding to the bounded selfadjoint operator $G$ is a measure $\nu$ supported on the real line $\mathbb{R}$ such that $\tau(G^n) = \int x^n d\nu(x)$ for all $n \in \mathbb{N}$.

It can be shown that the distribution $\nu_q$ of a $q$-deformed Gaussian variable with the variance equal to 1 is given by a measure $\nu_q$ supported on the interval $\left[-\sqrt{2(1-q)}, \sqrt{2(1-q)}\right]$ with a density

$$\nu_q(dx) = \frac{1}{\pi} \sqrt{1-q} \sin \theta \prod_{n=1}^{\infty} \left|1-q^n(1-\sqrt{1-q}) \right|^2 dx,$$

where $x = \frac{2}{\sqrt{1-q}} \cos \theta$ with $\theta \in [0, \pi]$.

1.1.4. Canonical commutation relations and Itô’s formula. It is not merely an accident that for $q = 1$ there is a correspondence between the commutation relation (2) and Gaussian random variables.

If we consider a probability space generated by a Brownian motion $B(t)$ then every real random variable $X$ with a finite second moment can be uniquely expressed as a series of iterated Itô integrals

$$X = X^{(0)} + \sum_{i=1}^{\infty} \int_{0 \leq t_1 \leq \cdots \leq t_i < \infty} X^{(i)}(t_1, \ldots, t_i) dB(t_1) \cdots dB(t_i),$$

where $X^{(0)} \in \mathbb{R}$ and for each $i \in \mathbb{N}$ we have that $X^{(i)} : (\mathbb{R}^+)^i \to \mathbb{R}$ is a symmetric function of its $i$ arguments. By the bosonic Fock space we call the space of sequences $(X^{(i)})_{i \geq 0}$ such that $X^{(i)} : (\mathbb{R}^+)^i \to \mathbb{R}$ is a symmetric function.

The Fock space carries a structure of a Hilbert space with a scalar product

$$\langle X, Y \rangle = \mathbb{E}[XY] = X^{(0)}Y^{(0)} + \sum_{n \geq 1} \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty X^{(n)}(t_1, \ldots, t_n) Y^{(n)}(t_1, \ldots, t_n) dt_1 \cdots dt_n.$$  

The Fock space representative of the random variable constantly equal to 1 will be denoted by $\Omega$. We have $\Omega^{(0)} = 1$, $\Omega^{(n)}(t_1, \ldots, t_n) = 0$ for every $n \geq 1$. Note that for every random variable we have

$$\mathbb{E}[X] = \mathbb{E}[X^1] = \langle \Omega, X \rangle.$$  

It is often convenient to work with the Fock space representations than with random variables themselves. An interesting question is to determine the Fock space representation of a product $\int_0^\infty \phi(t) dB(t)|X$ if the Fock space representation of $X$ is given and the integral is taken in the Itô sense. It is a simple corollary from the Itô’s theorem that the multiplication by $\int_0^\infty \phi(t) dB(t)$ can be expressed as a
The sum of two operators acting on the Fock space: so called annihilation operator $a_\phi$ and its adjoint, creation operator $a^*_\phi$:

$$\int_0^\infty \phi(t)dB(t) = a_\phi + a^*_\phi.$$  

They have the following properties:

(5)  

$$a_\phi a^*_\psi - a^*_\psi a_\phi = \langle \phi, \psi \rangle,$$

(6)  

$$a_\phi \Omega = 0.$$  

We see that the commutation relations (5) are equivalent to the statement of the Itô’s theorem. Conversely, postulating commutation relations (6) is equivalent to saying that the non commutative stochastic process $G_t$ fulfils some non commutative Itô’s formula. Such a stochastic calculus for $q$–deformed operators was considered by the author \cite{Sn1}.

1.2. Random matrices. For a general introduction to random matrices and their applications in mathematics and physics we refer to the monographs of Mehta \cite{M}, Hiai and Petz \cite{HP} and to overview articles \cite{Br, GMGW}.

There are essentially two kinds of questions concerning eigenvalues of a random matrix one can ask. Global questions are of the type: what is the asymptotic distribution of eigenvalues if the size of a matrix is large enough, while local questions concern, for example, the distribution of the spacings between consecutive eigenvalues. The global questions are much easier to answer and free probability theory has provided powerful tools to answer such questions for many random matrix model \cite{V2, V3, VDN, Sh}.

Recently random matrices were used as a powerful tool in the theory of $C^*$–algebras by Haagerup and Thorbjoersen \cite{HT}.

1.3. Overview of the paper. In Sect. 2 (which is independent of the rest of this article) we present heuristic motivations for some matrix models considered in this article.

In Sect. 3 we introduce the notations and construct a family we construct an auxiliary family of Gaussian random matrices $R^{(N), A, \mu}$.  

In Sect. 4 we construct a central object of this paper, namely a family of Gaussian random matrices $S^{(N)}, \mu$. Since the structure of matrices $S$ is a bit complicated, it is convenient to think about them as some weighted sums of the auxiliary matrices $R$, which have a much easier structure.

As the index $N$ tends to infinity, the size of our matrices grows exponentially. We show that matrices $S$ asymptotically have the same expectation values as $q$–deformed Gaussian random variables.

Our construction bases on the observation that for a finite dimensional Hilbert space $\mathcal{H}$ there are $2^N$ decompositions of a tensor power into two factors $\mathcal{H}^{\otimes N} = \mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes (N-k)}$ which correspond to ways of decomposing a set $\{1, \ldots, N\} = A \cup (\{1, \ldots, N\} \setminus A)$ into two subsets. The appropriate isomorphisms $j_A : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes |A|} \otimes \mathcal{H}^{\otimes (N-|A|)}$ give rise to isomorphisms of matrix algebras $j_A : \text{End}(\mathcal{H}^{\otimes |A|}) \otimes \text{End}(\mathcal{H}^{\otimes (N-|A|)}) \to \text{End}(\mathcal{H}^{\otimes N})$.

Each auxiliary matrix $R^{(N), A, \mu}$ is obtained by embedding a small standard hermitian random matrix $R^{(N), A, \mu} \in \text{End}(\mathcal{H}^{A})$ into a bigger algebra $\text{End}(\mathcal{H}^{N})$. The
embedding is implemented by the isomorphism $\tilde{j}_A$. We have that

$$S^{(N),\mu} = \sum_{A \subseteq \{1, \ldots, N\}} \sigma_A^{(N)} R^{(N),A,\mu} = \sum_{A \subseteq \{1, \ldots, N\}} \sigma_A^{(N)} \tilde{j}_A[R^{(N),A,\mu} \otimes 1],$$

where $\sigma_A^{(N)}$ is a certain weight function.

It turns out that the commutation properties of two random matrices $R^{(N),A,\mu}$ corresponding to two decompositions given by sets $A_1, A_2$ depend on the number of common elements of $A_1$ and $A_2$. For example, if $A_1 \cap A_2 = \emptyset$ then these two matrices commute and, informally speaking, the more elements $A_1$ and $A_2$ have in common, the more they behave like a pair of free random variables. Therefore the expectation value of a product of many matrices $R$ can be evaluated from the number of elements of $A_1 \cap A_j$ and in this way we are able to calculate the mixed moments of matrices $S$.

The choice of a normed weight $\sigma_A^{(N)}$ is equivalent to a choice of a probabilistic measure $\rho$ on the set of all subsets of $\{1, \ldots, N\}$ defined such that the measure of a singleton $\{A\}$ is equal to $(\sigma_A^{(N)})^2$ for any $A \subseteq \{1, \ldots, N\}$. If this measure is, loosely speaking, concentrated on the sets of order $c\sqrt{N}$ then if $A$ and $B$ are independent random variables with distribution given by the measure $\rho$ then $|A \cap B|$ is asymptotically Poisson distributed with the parameter $\lambda = c^2$. For appropriate choice of $c$ we are able in this way to obtain a random matrix model for $q$–deformed Gaussian random variables for all $0 \leq q \leq 1$.

In Sect. 5 we show that our matrices converge to $q$–deformed Gaussian variables not only in the sense of expectation values mixed moments, but that that mix moments converge almost surely.

Sect. 6 is devoted to proofs of technical lemmas.

The random matrices $S$ considered in this article are complex hermitian. However there are no difficulties to extend these results to real symmetric or symplectic hermitian.

This paper shows that the $q$–deformed probability theory, which was regarded until today as purely abstract and algebraic, in fact has natural probabilistic models just like the free probability theory has.

## 2. Heuristics of Random Commutation Relations and Random Gaussian Matrices

This section an an independent part of the article and notations used here will not be used in the subsequent considerations. We have to warn the reader that this section is very informal, however by such informal considerations it is much easier to get an insight to the nature of the problem.

### 2.1. $q$–deformed Gaussian variables

Our motivations how to find random matrices which asymptotically behave like $q$–deformed Gaussian variables were inspired by a careful study of the article of Speicher [Sp1].

In this paper he shows a certain non commutative central limit theorem that if a suitably normalised family of centered non commutative random variables $K_1, K_2, \ldots$ has a property that each pair of them either commutes (with probability $p$) or anticommutes (with probability $1 - p$) and if for each pair the choice
of one of these possibilities is made independently, then the distribution of a normalized mean $\frac{K_n}{\sqrt{n}}$ converges (as $n \to \infty$) to the distribution of a $q$–deformed Gaussian random variable with $q = 2p - 1$.

Now we shall construct a family of random (non Gaussian) matrices which almost fulfills the assumptions of the Speicher’s theorem.

Let us fix a real number $0 < q < 1$. For any $N \in \mathbb{N}$ let us consider a family of $2^N \times 2^N$ matrices

$$K_s = K_s^1 \otimes \cdots \otimes K_s^N$$

indexed by $s \in \mathbb{N}$, where for each $s \in \mathbb{N}$ and $1 \leq n \leq N$ we have that $K^n_s$ is a $2 \times 2$ matrix chosen randomly according to the following table:

| matrix probability | $\sigma_0$ | $-\sigma_0$ | $\sigma_1$ | $-\sigma_1$ | $\sigma_2$ | $-\sigma_2$ | $\sigma_3$ | $-\sigma_3$ |
|-------------------|------------|--------------|------------|--------------|------------|--------------|------------|--------------|
| $\frac{1-3r}{2}$ | $\frac{1-3r}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

where $0 \leq r \leq \frac{1}{2}$ is a real number to be specified later and

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are Pauli matrices. The random choices of matrices $K^n_s$ should be made independently.

These eight hermitian matrices $\pm \sigma_i$ have a property that each pair of them either commutes or anticommutes. It is a simple calculation that the probability that two independent $2 \times 2$ matrices (chosen according to the above table) anticommute is equal to $6r^2$.

Therefore all matrices $K_s$ are hermitian and furthermore each pair $K_s$ and $K_t$ ($s \neq t$) either commutes (if the number of indexes $n$ such that $K^n_s$ anticommutes with $K^n_t$ is even) or anticommutes (if this number is odd). We see that the probability of the event that $K_s$ commutes with $K_t$ is equal to

$$p = \sum_{0 \leq m \leq \frac{N}{2}} \binom{N}{2m}(1 - 6r^2)^{N-2m}(6r^2)^{2m},$$

while the probability that $K_s$ anticommutes with $K_t$ is equal to

$$1 - p = \sum_{0 \leq m \leq \frac{N-1}{2}} \binom{N}{2m+1}(1 - 6r^2)^{N-(2m+1)}(6r^2)^{2m+1}.$$ 

The difference of these two probabilities is equal to $2p - 1 = (1 - 12r^2)^N$.

We would like to apply now the Speicher’s theorem to the family $\{K_n\}$ by choosing $r$ as a function of $N$ such that $q = [1 - 12r^2]^N$. For large $N$ one can take the approximate value $r_N = \sqrt{-\frac{\ln q}{12N}}$.

Unfortunately the Speicher’s theorem cannot be applied directly since it turns out that the events “$K_s$ commutes with $K_t$” are not independent for different pairs $\{s, t\}$. However, if $N$ tends to infinity it can be justified that they are, loosely speaking, more and more independent.

By classical central limit theorem we have that the limit distribution of $K = \frac{1}{\sqrt{n}}(K_1 + \cdots + K_n)$ (regarded as a classical random variable with values in a vector space of matrices) is Gaussian. In order to characterise this distribution uniquely we have to give the mean and the covariance of entries. The mean value of each entry of the matrix $K$ is equal to $0$ and the covariance factorises as follows

$$\mathbb{E}[K(i_1, \ldots, i_N), (j_1, \ldots, j_N) K(k_1, \ldots, k_N), (l_1, \ldots, l_N)] =$$
3. Random Matrix Model for Deformed Gaussian Variables

3.1. Pair partitions.

Definition 1. A pair partition of a given finite set $M$ is any decomposition of $M$ into a family $\pi = \{\{c_1,d_1\}, \ldots, \{c_m,d_m\}\}$ of disjoint sets each having exactly two elements:

$$c_i \neq d_i, \quad \text{for } 1 \leq i \leq m,$$

$$\{c_i,d_i\} \cap \{c_j,d_j\} = \emptyset, \quad \text{for } i \neq j,$$

$$\{c_1,d_1\} \cup \cdots \cup \{c_m,d_m\} = M.$$  

The sets $\{c_1,d_1\}, \ldots, \{c_m,d_m\}$ are called lines of the pair partition $\pi$.

If $M$ is an ordered set we say that two distinct lines $\{a,b\}, \{u,v\}$, $a < b$, $u < v$ cross if $a < u < b < v$ or $u < a < v < b$.

For a given pair partition $\pi$ we will denote by $i(\pi)$ the number of crossings in $\pi$, i.e. number of all unordered pairs of lines $\{a,b\}, \{u,v\} \in \pi$ such that the lines $\{a,b\}, \{u,v\}$ cross.

Example. There is only one pair partition of a two element set $\{1,2\}$ and there are three pair partitions of a four element set $\{1,2,3,4\}$, namely $\{\{1,2\}, \{3,4\}\}$, $\{\{1,4\}, \{2,3\}\}$, $\{\{1,3\}, \{2,4\}\}$.

We have $i(\{\{1,2\}, \{3,4\}\}) = 0$, $i(\{\{1,3\}, \{2,4\}\}) = 1$.

Sets having an odd number of elements do not have any pair partitions at all.

3.2. Notations. Let us fix a natural number $d \geq 2$. For any $N \in \mathbb{N}$ we define $\mathcal{H}_r = \mathbb{C}^d$ for $1 \leq r \leq N$ and $\mathcal{H}^{(N)} = \bigotimes_{1 \leq r \leq N} \mathcal{H}_r$. In the following we shall often omit the index $(N)$ standing at various objects, however we have to remember about their dependence on $N$.

If $f_0, \ldots, f_{d-1}$ is an orthonormal basis of $\mathbb{C}^d$ then $e_i = f_{i_1} \otimes \cdots \otimes f_{i_N}$ is an orthonormal basis of $\mathcal{H}^{(N)}$, where $i = (i_1, \ldots, i_N)$ and $0 \leq i_1, \ldots, i_N \leq d - 1$.

Here and in the following by bold letters $i,j,k, \ldots$ we shall denote variables which index the basis of $\mathcal{H}^{(N)}$ and always we have $i = (i_1, \ldots, i_N)$, $k = (k_1, \ldots, k_N)$, etc.

For any set $A$ where $A = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, N\}$, $a_1 < \cdots < a_k$ and $A' = \{1,2, \ldots, N\} \setminus A = \{b_1, \ldots, b_{N-k}\}$, $b_1 < \cdots < b_{N-k}$ we consider Hilbert spaces $\mathcal{H}^A = \bigotimes_{r \in A} \mathcal{H}_r$ and $\mathcal{H}^{A'} = \bigotimes_{r \in A'} \mathcal{H}_r$ and an isomorphism $j_A : \mathcal{H}^{(N)} \to \mathcal{H}^A \otimes \mathcal{H}^{A'}$ given by a grouping of factors:

$$v_1 \otimes \cdots \otimes v_N \mapsto (v_{a_1} \otimes \cdots \otimes v_{a_k}) \otimes (v_{b_1} \otimes \cdots \otimes v_{b_{N-k}}).$$
This isomorphism induces an isomorphism of matrices \( \tilde{j}_A : \text{End}(\mathcal{H}^A) \otimes \text{End}(\mathcal{H}^A) \to \text{End}(\mathcal{H}^N) \).

In the following we shall denote by \( \text{tr} \) the normalised trace on \( \text{End}(\mathcal{H}^N) \) defined by \( \text{tr}(M) = \frac{1}{d} \text{Tr}(M) \), where \( \text{Tr} \) denotes the standard trace.

### 3.3. Iverson’s notation

In the following we shall use sometimes Iverson’s notation [GKP] as an alternative to the Kronecker’s notation:

\[
[x = y] = \delta_{xy} = \begin{cases} 
0 & \text{if } x \neq y \\
1 & \text{if } x = y
\end{cases}
\]

Of course the Iverson’s symbol \([x = y]\) is always equal to the Kronecker’s delta \(\delta_{xy}\) but it has some typographic advantages if \(x\) and \(y\) are complicated expressions with many upper and lower indexes.

### 3.4. Random matrices \( R \)

**Definition 2.** If \( V \) is a finite dimensional Hilbert space with an orthonormal basis \( e_1, \ldots, e_{\dim V} \) then a hermitian standard random matrix over \( V \) is a random variable \( M \) with values in \( \text{End}(V) \) such that the joint distribution of the complex matrix coefficients \( M_{ij} = (e_i, Me_j) \) is Gaussian, \( \bar{M}_{ij} = \overline{M_{ji}} \), all \( M_{ij} \) have mean zero and the covariance is given by

\[
\mathbb{E}[M_{ij}M_{kl}] = \mathbb{E}[M_{ij}\overline{M_{lk}}] = \frac{1}{\dim V} \delta_{il}[j = k].
\]

Alternatively one can define a hermitian standard random matrix \((M_{ij})\) by saying that the following random variables: \( M_{ii} \) for all indexes \( i \), \( \Re M_{ij}, \Im M_{ij} \) for \( i < j \) are independent real Gaussian variables with \( \mathbb{E}(M_{ij}) = 0 \) for all indexes \( i, j \) and \( \mathbb{E}[\Re M_{ii}^2] = \frac{1}{\dim V} \) for all values of index \( i \) and \( \mathbb{E}(\Re M_{ij})^2 = \mathbb{E}(\Im M_{ij})^2 = \frac{1}{2\dim V} \) for all \( i < j \). The entries \( M_{ij} \), where \( i > j \) are defined by the hermitianity condition \( M_{ij} = \overline{M_{ji}} \).

One can show that both definitions do not depend on the choice of the orthonormal basis of \( V \).

For each \( A \subseteq \{1, \ldots, N\} \) let us consider a family of hermitian standard random matrices \( R_i^{(A),\mu} \in \text{End}(\mathcal{H}_A) \) indexed by \( \mu \in \mathfrak{X} \) such that the entries of different matrices are independent. We define a family of random matrices \( R^{(N),A,\mu} \) by

\[
R^{(N),A,\mu} = \tilde{j}_A(R_1^{(N),A,\mu} \otimes 1_{\mathcal{H}^A}) \in \text{End}(\mathcal{H}^N)
\]

where \( 1_{\mathcal{H}^A} : \mathcal{H}^A \to \mathcal{H}^A \) denotes the identity operator.

Intuitively speaking, a matrix \( R^{(N),A,\mu} \) consists of \( d^{N-|A|} \) copies of a \( d^{|A|} \times d^{|A|} \) standard hermitian random matrix.

As one can see, matrices \( R^{A,\mu} \) are hermitian and the joint distribution of their entries is Gaussian, but different entries need not to be independent. We have:

\[
R_{ij}^{A,\mu} = \overline{R_{ji}^{A,\mu}}, \quad \mathbb{E}[R_{ij}^{A,\mu}] = 0,
\]

and from (8) it follows that

\[
\mathbb{E}[R_{ij}^{A,\mu}R_{kl}^{B,\nu}] = \mathbb{E}[R_{ij}^{A,\mu}\overline{R_{kl}^{B,\nu}}] =
\]

\[
= \delta_{AB}\delta_{\mu\nu} \left( \prod_{r \in A} [i_r = l_r][j_r = k_r] / d \right) \left( \prod_{r \in A'} [i_r = j_r][k_r = l_r] / d \right).
\]
3.5. Tensors $T$. The formula (8) can be written shorter if we introduce for all $A \subseteq \{1, \ldots, N\}$ and $1 \leq r \leq N$ tensors $T_{ij,k_l}^{A,r}$ as follows:

$$T_{ij,k_l}^{A,r} = \begin{cases} \frac{1}{d} \delta[i = l][j = k] & \text{if } r \in A \\ \delta[i = j][k = l] & \text{if } r \notin A \end{cases}.$$  

We define

$$(9) \quad T_{ij,k_l}^A = \prod_r T_{ij,k_l}^{A,r},$$

what with a small abuse of notation can be written as

$$T^A = T^A,1 \otimes \cdots \otimes T^A,N.$$  

Then (8) can be written as

$$(11) \quad \mathbb{E}[R_{ij,\mu}^{A,\rho} R_{kl,\nu}^{B,\sigma}] = \mathbb{E}[R_{ij,\mu}^{A,\rho} R_{ik,\nu}^{B,\sigma}] = \delta_{AB} \delta_{\mu\nu} T_{ij,k_l}^A.$$

3.6. Examples. First of all note that for the trivial case $d = 1$ all Hilbert spaces are one dimensional and all random matrices $R^{A,\mu}$ are in fact scalar random Gaussian variables.

Furthermore, the random matrix $R^{(N),\mu,0}$ is simply a scalar real random Gaussian variable multiplied by an identity matrix. The random matrix $R^{(N),\mu,\{1,\ldots,N\}}$ is a hermitian standard random matrix from Definition 3.

There is a correspondence between sequences $i = (i_1, \ldots, i_N)$ such that $0 \leq i_1, \ldots, i_N \leq d - 1$ and the set of integer numbers $\{0, 1, \ldots, d^N - 1\}$ given by the digit representation of natural numbers in the system with base $d$:

$$i = (i_1, \ldots, i_N) \mapsto i_1 + d i_2 + \cdots + d^{N-1} i_N.$$

Therefore we can introduce an orthonormal basis $g_0, \ldots, g_{d^N-1}$ of $\mathcal{H}^{(N)}$ indexed by integer numbers:

$$g_{i_1 + d i_2 + \cdots + d^{N-1} i_N} = f_{i_1} \otimes \cdots \otimes f_{i_N} = e_{(i_1, \ldots, i_N)},$$

where $0 \leq i_1, \ldots, i_N \leq d - 1$. In the following, if we want to write an endomorphism $M \in \text{End}(\mathcal{H}^{(N)})$ as a matrix $(M_{ij})_{0 \leq i,j \leq d^N-1}$ we shall do it in the basis $(g_i)_{0 \leq i \leq d^N-1}$.

For $d = 2$ and $N = 2$ the matrices $R^{(N),\mu,A}$ are of the following form:

$$R^{(1)} = \begin{bmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ 0 & 0 & a_{00} & a_{01} \\ 0 & 0 & a_{10} & a_{11} \end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix} b_{00} & 0 & b_{01} & 0 \\ 0 & b_{00} & 0 & b_{01} \\ b_{10} & 0 & b_{11} & 0 \\ 0 & b_{10} & 0 & b_{11} \end{bmatrix},$$

where $\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$ and $\begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$ are standard hermitian random matrices. Entries of the first matrix are by definition independent of the entries of the second matrix. The index $\mu$ was omitted, however it should be understood that for different values $\mu$ the entries of matrices are independent.
For $d = 2$ and $N = 3$ we have:

$$R^{(1)} = \begin{bmatrix}
    c_{00} & c_{01} & 0 & 0 & 0 & 0 & 0 \\
    c_{10} & c_{11} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & c_{00} & c_{01} & 0 & 0 & 0 \\
    0 & 0 & c_{10} & c_{11} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & c_{00} & c_{01} & 0 \\
    0 & 0 & 0 & 0 & c_{10} & c_{11} & 0 \\
    0 & 0 & 0 & 0 & 0 & c_{00} & c_{01} \\
    0 & 0 & 0 & 0 & 0 & c_{10} & c_{11}
\end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix}
    d_{00} & 0 & d_{01} & 0 & 0 & 0 & 0 \\
    0 & d_{00} & 0 & d_{01} & 0 & 0 & 0 \\
    d_{10} & 0 & d_{11} & 0 & 0 & 0 & 0 \\
    0 & d_{10} & 0 & d_{11} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & d_{00} & 0 & d_{01} \\
    0 & 0 & 0 & 0 & d_{00} & 0 & d_{01} \\
    0 & 0 & 0 & 0 & d_{10} & 0 & d_{11} \\
    0 & 0 & 0 & 0 & d_{10} & 0 & d_{11}
\end{bmatrix},$$

$$R^{(3)} = \begin{bmatrix}
    e_{00} & 0 & 0 & 0 & e_{01} & 0 & 0 \\
    0 & e_{00} & 0 & 0 & 0 & e_{01} & 0 \\
    0 & 0 & e_{00} & 0 & 0 & 0 & e_{01} \\
    e_{10} & 0 & 0 & 0 & e_{11} & 0 & 0 \\
    0 & e_{10} & 0 & 0 & 0 & e_{11} & 0 \\
    0 & 0 & e_{10} & 0 & 0 & 0 & e_{11} \\
    0 & 0 & 0 & e_{10} & 0 & 0 & 0 & e_{11}
\end{bmatrix},$$

where again $(c_{pq})_{0 \leq p,q \leq 1}, (d_{pq})_{0 \leq p,q \leq 1}, (e_{pq})_{0 \leq p,q \leq 1}$ are standard hermitian random matrices.

Furthermore

$$R^{(1,2)} = \begin{bmatrix}
    f_{00} & f_{01} & f_{02} & f_{03} & 0 & 0 & 0 & 0 \\
    f_{10} & f_{11} & f_{12} & f_{13} & 0 & 0 & 0 & 0 \\
    f_{20} & f_{21} & f_{22} & f_{23} & 0 & 0 & 0 & 0 \\
    f_{30} & f_{31} & f_{32} & f_{33} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & f_{00} & f_{01} & f_{02} & f_{03} \\
    0 & 0 & 0 & 0 & f_{10} & f_{11} & f_{12} & f_{13} \\
    0 & 0 & 0 & 0 & f_{20} & f_{21} & f_{22} & f_{23} \\
    0 & 0 & 0 & 0 & f_{30} & f_{31} & f_{32} & f_{33}
\end{bmatrix},$$

$$R^{(1,3)} = \begin{bmatrix}
    g_{00} & g_{01} & 0 & 0 & g_{02} & g_{03} & 0 & 0 \\
    g_{10} & g_{11} & 0 & 0 & g_{12} & g_{13} & 0 & 0 \\
    0 & 0 & g_{00} & g_{01} & 0 & 0 & g_{02} & g_{03} \\
    0 & 0 & g_{10} & g_{11} & 0 & 0 & g_{12} & g_{13} \\
    g_{20} & g_{21} & 0 & 0 & g_{22} & g_{23} & 0 & 0 \\
    g_{30} & g_{31} & 0 & 0 & g_{32} & g_{33} & 0 & 0 \\
    0 & 0 & g_{20} & g_{21} & 0 & 0 & g_{22} & g_{23} \\
    0 & 0 & g_{30} & g_{31} & 0 & 0 & g_{32} & g_{33}
\end{bmatrix},$$
where \((f_{pq})_{0 \leq p,q \leq 3}, (g_{pq})_{0 \leq p,q \leq 3}, (h_{pq})_{0 \leq p,q \leq 3}\) are standard hermitian random matrices.

### 3.7. The case of a general covariance \(\Gamma_{\mu\nu}\)

By a small change of definition of the matrices \(R\) we obtain a more general case.

Let \(\Gamma_{\mu\nu}\) be a real positive definite function. For every \(A \subseteq \{1, \ldots, N\}\) we consider a family of random (non hermitian) matrices \(R_0^{(N),A,\mu} \in \text{End}(\mathcal{H}_A)\) such that for each pair of indexes \(i, j\) we have that the joint distribution of \((R_0^{(N),A,\mu})_{ij}\) \(\mu \in \mathfrak{M}\) is Gaussian,

\[
E(R_0^{(N),A,\mu})_{ij} = 0,
\]

the covariance of real and imaginary parts are defined by the function \(\Gamma\):

\[
E[\Re(R_0^{(N),A,\mu})_{ij}] = E[\Im(R_0^{(N),A,\mu})_{ij}] = \frac{1}{2d^{|A|}} \Gamma_{\mu\nu}
\]

and the real and imaginary parts are independent:

\[
E[\Re(R_0^{(N),A,\mu})_{ij}] \Im(R_0^{(N),A,\mu})_{ij} = 0.
\]

For different choices of sets \(A\) or a pair of indexes \((i, j)\) the random variables \((R_0^{(N),A,\mu})_{ij}\) should be independent.

We define hermitian random matrices

\[
R_1^{(N),A,\mu} = R_0^{(N),A,\mu} + (R_0^{(N),A,\mu})^*.
\]

Note that for the simplest choice of a positive definite function \(\Gamma_{\mu\nu} = \delta_{\mu\nu}\) this definition of random matrices \(R_1^{(N),A,\mu}\) coincides with the definition from Subsect. 3.4.

Similarly as in Subsect. 3.4 we define

\[
R^{(N),A,\mu} = \gamma_A(R_1^{(N),A,\mu} \otimes 1_{\mathcal{H}_{A'}}).
\]

The joint distribution of entries of hermitian matrices \(R^{(N),A,\mu}\) is Gaussian and

\[
E[R^{A,\mu}]_{ij} = 0,
\]

\[
E[R^{A,\mu}_{ij} R^{B,\nu}_{kl}] = E[R^{A,\mu}_{ij} \overline{R^{B,\nu}_{lk}}] = \delta_{AB} \Gamma_{\mu\nu} T^{A}_{ij,kl}.
\]

### 4. The Main Theorem

We define a family of random matrices indexed by \(\mu \in \mathfrak{M}\)

\[
S^{(N),\mu} = \sum_{A \subseteq \{1, \ldots, N\}} \sigma_A^{(N)} R^{(N),A,\mu}
\]

where \(\sigma^{(N)}\) is a real-valued function on the set of all subsets of \(\{1, \ldots, N\}\).
Matrices $S^{(N)}$ are hermitian and the joint distribution of their entries is Gaussian. Alternatively one can define these matrices by giving the mean and the covariance of the entries: we have

$$\mathbb{E}[S_{ij}^\mu] = 0,$$

$$\mathbb{E}[S_{ij}^\mu S_{kl}^\nu] = \mathbb{E}[S_{ij}^\mu \overline{S}_{kl}^\nu] = \Gamma_{\mu\nu} \sum_A (\sigma_A)^2 T_{ij,kl}^A.$$

In the following theorem we show conditions which the sequence of functions $(\sigma^{(N)})$ need to fulfill. Since these conditions may seem quite disgusting, we would like to give some hope to the reader by pointing to the equation (15), which gives a simple example of a covariance function fulfilling all assumptions.

**Theorem 1.** If for each $N \in \mathbb{N}$ we have that $\sigma^{(N)}$ is a real–valued function on the set of all subsets of $\{1, \ldots, N\}$ such that:

1. (normalisation) for each $N \in \mathbb{N}$ we have
   $$\sum_{A \subseteq \{1, \ldots, N\}} (\sigma_A^{(N)})^2 = 1,$$

2. (triple coincidences are rare)
   $$\lim_{N \to \infty} \sum_{A_1, A_2, A_3 \subseteq \{1, \ldots, N\}, A_1 \cap A_2 \cap A_3 = \emptyset} (\sigma_{A_1}^{(N)})^2 (\sigma_{A_2}^{(N)})^2 (\sigma_{A_3}^{(N)})^2 = 0,$$

3. (distribution of coincidences) there exists a sequence $(p_i)_{i \geq 0}$ of nonnegative real numbers such that $\sum_{i \geq 0} p_i = 1$ and for any $k \in \mathbb{N}$ and any nonnegative integer numbers $n_{ij}$, $1 \leq i < j \leq k$ we have
   $$\lim_{N \to \infty} \sum_{A_1, \ldots, A_k \subseteq \{1, \ldots, N\}, |A_i \cap A_j| = n_{ij} \text{ for any } 1 \leq i < j \leq k} (\sigma_{A_1}^{(N)})^2 \cdots (\sigma_{A_k}^{(N)})^2 = \prod_{1 \leq i < j \leq k} p_{n_{ij}}.$$

4. for each $n \in \mathbb{N}$
   $$\lim_{N \to \infty} \sum_{A_1, \ldots, A_n \subseteq \{1, \ldots, N\}} (\sigma_{A_1}^{(N)})^2 \cdots (\sigma_{A_n}^{(N)})^2 \frac{1}{d\mathbb{E}[A_1 \setminus (A_2 \cup \cdots \cup A_n)]} = 0.$$

Then for $q = \sum_{i=0}^{\infty} p_i q^i$ we have

$$\lim_{N \to \infty} \mathbb{E}[\text{tr } S^{(N), \mu_1} \cdots S^{(N), \mu_n}] = \sum_{\pi} q^{(\pi)} \Gamma_{\mu_1 \mu_1} \cdots \Gamma_{\mu_m \mu_m} = \tau[G_{\mu_1} \cdots G_{\mu_n}]$$

for every $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in \mathbb{N}$, where $G_{\mu_1}, \ldots, G_{\mu_n}$ are $q$-deformed Gaussian variables with covariance $\Gamma$.

Before we prove this theorem we would like to make some remarks and state auxiliary lemmas.

**Remark 1.** For a given function $\sigma^{(N)}$ we define a measure $\rho^{(N)}$ on the set of all subsets of $\{1, \ldots, N\}$ by assigning to set $A$ the weight $(\sigma_A^{(N)})^2$. Then the first three assumptions of the theorem can be reformulated as follows:

1. for each $N \in \mathbb{N}$ the measure $\rho^{(N)}$ is probabilistic,

2. if for each $N \in \mathbb{N}$ we have that $A_1^{(N)}, A_2^{(N)}, A_3^{(N)}$ are independent random variables with distribution given by the measure $\rho^{(N)}$ then the probability of the event: $A_1^{(N)} \cap A_2^{(N)} \cap A_3^{(N)} \neq \emptyset$ tends to 0 as $N$ tends to infinity,
3. Let \( k \) be a fixed natural number. If for each \( N \in \mathbb{N} \) we have that \( A_1^{(N)}, \ldots, A_k^{(N)} \) are independent random variables with distribution given by the measure \( \rho^{(N)} \) then the joint distribution of the random variables \( |A_i \cap A_j|, 1 \leq i < j \leq k \) tends to a product distribution as \( N \) tends to infinity. The limit distribution of a single variable \( |A_i \cap A_j| \) is given by

\[
\lim_{N \to \infty} P(|A_i^{(N)} \cap A_j^{(N)}| = k) = p_k.
\]

**Remark 2.** The assumption \( 4 \) follows from other assumptions, however the proof of this is rather technical and we omit it.

**Remark 3.** We would like to point out an interesting informal connection between stochastical properties of \( G_i \) regarded as large matrices and their entries.

Let us consider \( \Gamma_{\mu \nu} = \delta_{\mu \nu} \). For \( q = 1 \) we have that \( G_\mu \) is a family of independent Gaussian variables and for \( q = 0 \) we have that \( G_\mu \) is a family of free non-commutative random variables. Freeness is an analogue of classical independence; we can expect therefore that for in a general case \( -1 \leq q \leq 1 \) variables \( G_\mu \) are independent in some generalised way.

However, it was proven by Speicher \([Sp3]\) that there are only three generalisations of the notion of independence of random variables to the non commutative setup which would satisfy certain natural properties. These three generalisations are: classical independence, free independence (freeness) and boolean independence. Therefore except the cases \( q \in \{0,1\} \) which correspond to the free and the classical situation respectively we cannot formally say that the non commutative random variables \( G_\mu \) are independent in some sense.

We can of course weaken Speicher’s axioms and treat this “independence” on a very informal level. It is worth pointing out that a family of “independent” variables \( G_i \) is asymptotically represented as random matrices such that the entries of different matrices are classically independent random variables.

Similarly, for the choice of the covariance function \( \Gamma_{ts} = \min(t,s) \) for \( t,s \geq 0 \) we obtain a non commutative stochastic process \( G_t \) which can be regarded as some kind of a Brownian motion \([BS1]\) and \( G_t \) can be asymptotically represented as a matrix valued stochastic process. Entries of this matrix are classical Brownian motions.

**Remark 4.** The assumptions of the theorem are fulfilled for the following important examples of the functions \( \sigma \):

**Proposition 1.** For every real \( c > 0 \) the sequence of functions defined by

\[
\sigma_{A}^{(N)} = \left( \frac{c}{\sqrt{N}} \right)^{|A|} \left( 1 - \frac{c}{\sqrt{N}} \right)^{N-|A|}
\]

fulfills the assumptions of Theorem 2 with the sequence \( p_k = \frac{1}{k!}e^{2k}e^{-c^2} \) being the Poisson distribution with parameter \( c^2 \) and

\[
q = e^{-(1-\frac{1}{d})c^2}.
\]

In this case the covariance \( (14) \) takes a beautiful form

\[
(15) \quad E[S_{ij}^{\mu}S_{kl}^{\nu}] = E[S_{ij}^{\mu}S_{kl}^{\nu}] = \Gamma_{\mu \nu} \prod_r \left( \frac{c}{\sqrt{N}} \frac{[i_r = l_r][j_r = k_r]}{d} + \left( 1 - \frac{c}{\sqrt{N}} \right) [i_r = j_r][k_r = l_r] \right).
\]

Proof of this proposition will be presented in Sect. I.

Proposition 2. For every real number $c > 0$ the sequence of functions $\sigma^{(N)}_A$ defined for $N$ sufficiently large by

$$\left(\sigma^{(N)}_A\right)^2 = \left\{ \begin{array}{ll} \frac{1}{\left\lfloor (c\sqrt{N}) \right\rfloor} & \text{if } |A| = \left\lfloor c\sqrt{N} \right\rfloor \\ 0 & \text{otherwise} \end{array} \right. ,$$

where $\lfloor x \rfloor$ denotes the integer part of a real number $x$, fulfills the assumptions of Theorem I with $p_k = \frac{1}{k!} 2^k e^{-c^2}$ and

$$q = e^{-(1-\frac{1}{2^k})c^2}.$$

Since proof of this proposition is similar to the proof of Proposition I we skip it.

Lemma 1. For any pair partition $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$ and any sets $A_1, \ldots, A_{2m} \subseteq \{1, \ldots, N\}$ we have

$$0 \leq \frac{1}{d^{N}} \sum_{\pi} \prod_{1 \leq v \leq m} T_{A_{c_v}}^{A_{c_v}} \prod_{i<j \leq m} \left( \prod_{l=1}^{c_i, d_i} \prod_{l=1}^{c_j, d_j} \frac{1}{d^{2|A_{c_i} \cap A_{c_j}|}} \right) \leq 1.$$

If furthermore $A_{c_i} \cap A_{c_j} \cap A_{c_k} = \emptyset$ for all $1 \leq i < j < k \leq m$ then

$$\frac{1}{d^{N}} \sum_{\pi} \prod_{1 \leq v \leq m} T_{A_{c_v}}^{A_{c_v}} \prod_{i<j \leq m} \left( \prod_{l=1}^{c_i, d_i} \prod_{l=1}^{c_j, d_j} \frac{1}{d^{2|A_{c_i} \cap A_{c_j}|}} \right) = 0.$$

The proof of this lemma will be presented in Sect. I.

The following lemma states a well known property of the Gaussian distribution.

Lemma 2. If the joint distribution of random variables $(X_k)$ is Gaussian and $E[X_k] = 0$ then

$$E(X_1 \cdots X_{2m-1}) = 0,$$

$$E(X_1 \cdots X_{2m}) = \sum_{\pi} E(X_{c_1} X_{d_1}) \cdots E(X_{c_m} X_{d_m}),$$

where the sum is taken over all pair partitions $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$.

With this preparation we are able to start the proof of the main theorem.

Proof of Theorem I. In the following the sums over $\pi$ are taken over all pair partitions $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$ and sums over $A_1, \ldots, A_n$ are taken over all subsets of the set $\{1, \ldots, N\}$. Products over $v$ are taken over $1 \leq v \leq m$.

From Lemma I follows that for any $m \in \mathbb{N}$ and indexes $\mu_1, \ldots, \mu_{2m} \in \mathcal{M}$ we have:

$$E \text{ tr}(S^{(N), \mu_1} \cdots S^{(N), \mu_{2m-1}}) = 0$$

and furthermore

$$U^{(N)} := E \text{ tr}(S^{(N), \mu_1} \cdots S^{(N), \mu_{2m}}) =$$

$$= \sum_{\pi} \sum_{A_1, \ldots, A_{2m}} \left( \sigma^{(N)}_{A_1} \cdots \sigma^{(N)}_{A_m} \right) \left( \prod_{1 \leq v \leq 2m} \delta_{A_{c_v} A_{d_v}} \Gamma_{\mu_{c_v} \mu_{d_v}} T_{A_{c_v}}^{A_{c_v}} \prod_{i<j \leq m} \frac{1}{d^{2|A_{c_i} \cap A_{c_j}|}} \right).$$
We define
\[
V^{(N)} := \sum_{v} \left( \prod_{\mu \epsilon v} \delta_{\mu v} \Gamma_{\mu v} \right) \sum_{A_1, \ldots, A_m} \left( \prod_{\mu \epsilon v} (\sigma_{A_1}^{(N)} \cdots \sigma_{A_m}^{(N)} \times \] 
\times \prod_{1 \leq i < j \leq m} \frac{1}{d^{2[A_i \cap A_j]}} \right)
\]

Lemma 3 shows that the corresponding summands in the definitions of $U$ and $V$ are equal unless there are some indexes $1 \leq p < q < r \leq m$ such that $A_{c_1} \cap A_{c_2} \cap A_{c_3} \neq \emptyset$. There are $\binom{m}{3}$ choices of these indexes and again from Lemma 3 and the assumption 3 we have
\[
|U^{(N)} - V^{(N)}| \leq C \binom{m}{3} \sum_{A_1, A_2, A_3 \subseteq \{1, \ldots, N\}} \sum_{A_1 \cap A_2 \cap A_3 \neq \emptyset} (\sigma_{A_1}^{(N)})^2 \cdots (\sigma_{A_m}^{(N)})^2 = \]
\[
= C \binom{m}{3} \sum_{A_1, A_2, A_3 \subseteq \{1, \ldots, N\}} (\sigma_{A_1}^{(N)})^2 (\sigma_{A_2}^{(N)})^2 (\sigma_{A_3}^{(N)})^2,
\]
where $C = \max |\Gamma_{\mu_v, \mu_v}|$ and therefore from the assumption 3 we have
\[
\lim_{N \to \infty} |U^{(N)} - V^{(N)}| = 0.
\]
We have
\[
V^{(N)} = \sum_{v} \left( \prod_{\mu \epsilon v} \Gamma_{\mu v} \right) \sum_{A_1, \ldots, A_m} (\sigma_{A_1}^{(N)})^2 \cdots (\sigma_{A_m}^{(N)})^2 \times \]
\[
\times \prod_{1 \leq i < j \leq m} \frac{1}{d^{2[A_i \cap A_j]}} \] 
\[
= \sum_{v} \left( \prod_{\mu \epsilon v} \Gamma_{\mu v} \right) \int \prod_{1 \leq i < j \leq m} \frac{1}{d^{2n_{ij}}} d\lambda^{(N)}(n_{ij}),
\]
where measures $\lambda^{(N)}$ are defined on the set of all sequences $(n_{ij})_{1 \leq i < j \leq m}$, $n_{ij} \in \{0, 1, 2, \ldots\}$ by condition
\[
\lambda^{(N)}\{\{n_{ij}\}\} = \sum_{A_1, \ldots, A_k \subseteq \{1, \ldots, N\}} (\sigma_{A_1}^{(N)})^2 \cdots (\sigma_{A_k}^{(N)})^2.
\]
From the assumption 3 follows that this sequence converges pointwise to the product measure defined on the atoms by
\[
\lambda\{\{n_{ij}\}\} = \prod_{1 \leq i < j \leq m} p_{ij}.
\]
Since measures $\lambda^{(N)}$ and the measure $\lambda$ are probabilistic, this convergence is uniform and the statement of the theorem follows.
5. The Almost Surely Convergence

As a simple corollary of Lemma 3 we have

**Lemma 3.** If the joint distribution of random variables $X_1, \ldots, X_{2m}$ is Gaussian and $\mathbb{E}[X_k] = 0$ for all $1 \leq k \leq 2m$ then

$$\mathbb{E}[X_1 \cdots X_{2m}] - \mathbb{E}[X_1 \cdots X_m] \mathbb{E}[X_{m+1} \cdots X_{2m}] = \sum_{\pi'} \prod_{1 \leq v \leq m} \mathbb{E}[X_{x_v} X_{y_v}],$$

where the sum is taken over pair partitions $\pi' = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$ which have additional property that there exist $x \in \{1, \ldots, m\}$ and $y \in \{m + 1, \ldots, 2m\}$ such that $\{x, y\} \in \pi'$.

We define a permutation $\sigma : \{1, \ldots, 2m\} \rightarrow \{1, \ldots, 2m\}$ by $\sigma(k) = k + 1$ for $k \not\in \{m, 2m\}$, $\sigma(m) = 1$, $\sigma(2m) = m + 1$.

**Lemma 4.** Let $\pi' = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ be a pair partition as in Lemma 3, i.e. for some $1 \leq k \leq m$ we have $c_k \in \{1, \ldots, m\}$, $d_k \in \{m + 1, \ldots, 2m\}$.

Then

$$\left| \frac{1}{d_2^N} \sum_{i_1, \ldots, i_{2m}} \prod_{1 \leq v \leq m} T_{\Pi(v)}^{\{c_v\}, \{d_v\}\sigma(v)} \right| \leq \frac{1}{d_2^{|A_1 \setminus (A_2 \cup A_m)|}},$$

This lemma follows directly from Lemmas 3 and 4 from Sect. 3.

**Proposition 3.**

$$\text{Var} \left[ \text{tr} \ S^{(N), \mu_1} \cdots S^{(N), \mu_m} \right] \leq C \sum_{A_1, \ldots, A_m \subseteq \{1, \ldots, N\}} (\sigma^{(N)}_{A_1})^2 \cdots (\sigma^{(N)}_{A_m})^2 \frac{1}{d_2^{|A_1 \setminus (A_2 \cup \cdots \cup A_m)|}},$$

where $C = (2m)! \max |\Gamma_{\mu, \mu}|$.

**Proof.** We define $\mu_{m+k} = \mu_k$. In the following sums over $\pi$ are taken over all pair partitions $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$ with additional property that there exist $x \in \{1, \ldots, m\}$ and $y \in \{m + 1, \ldots, 2m\}$ such that $\{x, y\} \in \pi'$.

From Lemma 3 we have that

$$\text{Var} \left[ \text{tr} \ S^{(N), \mu_1} \cdots S^{(N), \mu_m} \right] =$$

$$= \mathbb{E}[(\text{tr} \ S^{(N), \mu_1} \cdots S^{(N), \mu_m})^2] - [\mathbb{E}(\text{tr} \ S^{(N), \mu_1} \cdots S^{(N), \mu_m})]^2 =$$

$$= \frac{1}{d_2^N} \sum_{\pi' \subseteq \{1, \ldots, 2m\}} \left( \mathbb{E}[S_{12}^{\mu_1} \cdots S_{1m-1m}^{\mu_{m-1}} S_{1m+1}^{\mu_m}] \right) - \mathbb{E}[S_{12}^{\mu_1} \cdots S_{1m-1m}^{\mu_{m-1}} S_{1m+1}^{\mu_m}] =$$

$$= \frac{1}{d_2^N} \sum_{\pi'} \left( \sum_{A_1, \ldots, A_m \subseteq \{1, \ldots, 2m\}} \prod_{1 \leq v \leq m} \mathbb{E}[S_{\Pi(v)}^{\mu_v} S_{\Pi(v)}^{\mu_v}] \right) =$$

$$= \frac{1}{d_2^N} \sum_{\pi'} \left( \sum_{A_1, \ldots, A_m \subseteq \{1, \ldots, 2m\}} \sum_{\sigma^{(N)}_{A_1}, \ldots, \sigma^{(N)}_{A_m}} \right) \leq$$

$$\leq \prod_{1 \leq v \leq m} \delta_{A_v} \Gamma_{\mu_v, \mu_v} T_{\Pi(v)}^{\{c_v\}, \{d_v\}\sigma(v)} \leq \prod_{1 \leq v \leq m} \delta_{A_v} \Gamma_{\mu_v, \mu_v} T_{\Pi(v)}^{\{c_v\}, \{d_v\}\sigma(v)} \leq \prod_{1 \leq v \leq m} \delta_{A_v} \Gamma_{\mu_v, \mu_v} T_{\Pi(v)}^{\{c_v\}, \{d_v\}\sigma(v)} \leq \prod_{1 \leq v \leq m} \delta_{A_v} \Gamma_{\mu_v, \mu_v} T_{\Pi(v)}^{\{c_v\}, \{d_v\}\sigma(v)} \leq$$
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$$\leq C \sum_{A_1, \ldots, A_m \subseteq \{1, \ldots, N\}} \left( \sigma_{A_1}^{(N)} \right)^2 \cdots \left( \sigma_{A_m}^{(N)} \right)^2 \frac{1}{d^{[A_1 \cup \cdots \cup A_m]}}.$$ 

where in the last inequality we used Lemmas 3 and 5. □

If $(M_{ij})_{1 \leq i, j \leq K}$ is a hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_K$ we define a probabilistic measure $\nu_M$ on the real line $\mathbb{R}$ by the following

$$\nu_M = \frac{1}{K} \sum_{1 \leq n \leq K} \delta_{\lambda_n}.$$ 

**Theorem 2.** If the assumptions of Theorem 1 are fulfilled then there exists an increasing sequence of natural numbers $(N_i)$ such that the sequence of measures $\nu_{S(N_i)}$ almost surely converges weakly to the measure $\nu_q$.

Since for $0 \leq q < 1$ the support of the limit measure $\nu_q$ is compactly supported, the theorem follows from the following stronger statement.

**Theorem 3.** If the assumptions of Theorem 1 are fulfilled then there exists a sequence $(N_i)$ such that the limit

$$\lim_{m \to \infty} \text{tr} S^{(N_i), \mu_1} \cdots S^{(N_i), \mu_m} = \tau(G_{\mu_1} \cdots G_{\mu_m})$$

holds almost surely.

**Proof.** Our goal is to construct a sequence $(N_i)$ such that

$$\sum_i \mathbb{E} \left[ \left( \text{tr} S^{(N_i), \mu_1} \cdots S^{(N_i), \mu_m} - \tau(G_{\mu_1} \cdots G_{\mu_m}) \right)^2 \right] < \infty$$

holds.

However,

$$\mathbb{E} \left[ \left( \text{tr} S^{(N), \mu_1} \cdots S^{(N), \mu_m} - \tau(G_{\mu_1} \cdots G_{\mu_m}) \right)^2 \right] =$$

$$\left( \mathbb{E} \left[ \text{tr} S^{(N), \mu_1} \cdots S^{(N), \mu_m} - \tau(G_{\mu_1} \cdots G_{\mu_m}) \right] \right)^2 + \text{Var} \left[ \text{tr} S^{(N), \mu_1} \cdots S^{(N), \mu_m} \right].$$

The first summand converges to 0 by Theorem 3.

From Proposition 3 we have that

$$\text{Var[tr } S^{(N), \mu_1} \cdots S^{(N), \mu_m} ] \leq$$

$$\leq C \sum_{A_1, \ldots, A_m \subseteq \{1, \ldots, N\}} \left( \sigma_{A_1}^{(N)} \right)^2 \cdots \left( \sigma_{A_m}^{(N)} \right)^2 \frac{1}{d^{[A_1 \cup \cdots \cup A_m]}}.$$ 

From assumptions 1 and 5 it follows that this expression converges to 0. □

6. PROOFS OF TECHNICAL LEMMAS

**Lemma 5.** For every $n, M \in \mathbb{N}$ if for all $1 \leq v \leq M$ we have $A_v \subseteq \{1, \ldots, N\}$, $1 \leq p_v, q_v, r_v, s_v \leq n$ then

$$\sum_{i_1, \ldots, i_n} \prod_{1 \leq v \leq M} T_{i_v}^{A_v} = \prod_{1 \leq r \leq N} \sum_{0 \leq j_1, \ldots, j_n \leq d-1, 1 \leq v \leq M} \prod_{j_v} T_{j_r}^{A_v}.$$ 

**Proof.** This lemma is a direct consequence of Eq. (16) and (10). □
Lemma 6. For any $A_1, \ldots, A_{2m} \subseteq \{1, \ldots, N\}$ and a pair partition $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ of the set $\{1, \ldots, 2m\}$ we define

\begin{equation}
\Theta^{A_1, \ldots, A_{2m}, \pi}_r = \frac{1}{d} \sum_{0 \leq j^1, \ldots, j^{2m} \leq d-1} \prod_{u} T^{A_\pi}_{j^u j^{u+1}, j^{u}j^{u+1}}
= \frac{1}{d} \sum_{0 \leq j^1, \ldots, j^{2m} \leq d-1} \prod_{1 \leq u \leq m} \{ \begin{array}{ll}
\prod_{k \leq c_u} j^{c_u} = j^{c_u+1} & \text{if } r \not\in A_u \\
\prod_{k \leq d_u} j^{d_u} = j^{d_u+1} & \text{if } r \in A_u
\end{array} \}
\end{equation}

It should be understood that $j^{2m+1} = j^1$.

Lemma 6. For any $A_1, \ldots, A_{2m} \subseteq \{1, \ldots, N\}$ and a pair partition $\pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\}$ we have

\begin{equation}
0 \leq \prod_{1 \leq r \leq N} \Theta^{A_1, \ldots, A_{2m}, \pi}_r \leq 1.
\end{equation}

If furthermore $A_{c_i} \cap A_{c_j} \cap A_{c_k} = \emptyset$ for all $1 \leq i < j < k \leq n$ then

\begin{equation}
\prod_{1 \leq r \leq N} \Theta^{A_1, \ldots, A_{2m}, \pi}_r = \prod_{\text{lines } (c_i, d_i), (c_j, d_j) \text{ cross}} \frac{1}{d^2|A_{c_i} \cap A_{c_j}|}.
\end{equation}

Proof. Let us consider an expression of the following type

\begin{equation}
\sum_{0 \leq j^1, \ldots, j^{2m} \leq d} [j^{c_1} = j^{f_1}] \ldots [j^{c_{2m}} = j^{f_{2m}}].
\end{equation}

We can represent this expression by a graph (see for example [Dd]) with $2m$ vertices which are labelled by variables $j^1, \ldots, j^{2m}$ and with vertices $j^{c_1}$ and $j^{f_1}$ connected by an edge for all $1 \leq i \leq 2m$. We see that the nonzero summands in (20) come from indexes $(j^1, \ldots, j^{2m})$ such that to all vertices in the same connected component of the graph is assigned the same value. Therefore the expression (20) is equal to $d^M$, where $M$ denotes the number of connected components of the graph.

Let us fix an index $r$. We shall apply the above observation to evaluate $\Theta_r^{A_1, \ldots, A_{2m}, \pi}$. Let $n$ be the number of indexes $v$ such that $r \in A_{c_v}$ and let $v_1, \ldots, v_n$ be all such indexes. We consider the following cases.

1. If $n = 0$ then we obtain a graph of type presented in Fig. 1. This graph has one connected component and therefore $\Theta_r^{A_1, \ldots, A_{2m}, \pi} = 1$.

2. If $n = 1$ then we obtain a graph of type presented in Fig. 2. This graph has two connected components and therefore $\Theta_r^{A_1, \ldots, A_{2m}, \pi} = 1$.

3. Suppose that $n = 2$. If the lines $(c_{v_1}, d_{v_1})$ and $(c_{v_2}, d_{v_2})$ do not cross (Fig. 3 and 4) then corresponding graphs have three components and and therefore $\Theta_r^{A_1, \ldots, A_{2m}, \pi} = 1$.

But if the lines $(c_{v_1}, d_{v_1})$ and $(c_{v_2}, d_{v_2})$ cross (Fig. 5) then the graph has only one connected component and therefore $\Theta_r^{A_1, \ldots, A_{2m}, \pi} = \frac{1}{d}$.

4. In the general case $n \geq 3$ the graph has at most $n + 1$ components and there is a factor $\frac{1}{d}$, therefore $0 < \Theta_r^{A_1, \ldots, A_{2m}, \pi} \leq 1$.

Therefore if for a given partition $\pi$ sets $A_{c_1}, \ldots, A_{c_k}$ have a property that the common part of each three of them is empty then

\begin{equation}
\prod_{1 \leq r \leq N} \Theta_r^{A_1, \ldots, A_{n}, \pi} = \prod_{\text{lines } (c_i, d_i), (c_j, d_j) \text{ cross}} \frac{1}{d^2|A_{c_i} \cap A_{c_j}|}.
\end{equation}
In the general situation the expression $\prod_{1 \leq r \leq N} \Theta_{r}^{A_{1}, \ldots, A_{2m}, \pi}$ is a real number from the interval $[0, 1]$. \hfill \Box
We recall that the permutation \( \sigma : \{1, \ldots, 2m\} \rightarrow \{1, \ldots, 2m\} \) was defined by \( \sigma(k) = k + 1 \) for \( k \not\in \{m, 2m\} \), \( \sigma(m) = 1 \), \( \sigma(2m) = m + 1 \).

For sets \( A_1, \ldots, A_{2m} \subseteq \{1, \ldots, N\} \) and a pair partition \( \pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\} \) of the set \( \{1, \ldots, 2m\} \) we define

\[
\gamma_r^{A_1, \ldots, A_{2m}, \pi} = \frac{1}{d} \sum_{0 \leq j_1, \ldots, j_{2m} \leq d-1} \prod_{1 \leq v \leq m} T_{\sigma(c_v), \sigma(d_v)}^\rho = \\
= \frac{1}{d} \sum_{0 \leq j_1, \ldots, j_{2m} \leq d-1} \prod_{1 \leq v \leq m} \left\{ \begin{array}{ll}
\frac{1}{d} j_{\sigma(c_v)} = j_{\sigma(d_v)} & \text{if } r \not\in A_{c_v} \\
\frac{1}{d} j_{\sigma(c_v)} = j_{\sigma(d_v)} & \text{if } r \in A_{c_v} . \end{array} \right.
\]

Lemma 7. For any \( A_1, \ldots, A_{2m} \subseteq \{1, \ldots, N\} \) and a pair partition \( \pi = \{\{c_1, d_1\}, \ldots, \{c_m, d_m\}\} \) we have

\[
0 \leq \gamma_r^{A_1, \ldots, A_{2m}, \pi} \leq 1.
\]

If furthermore \( r \in A_k \setminus \bigcup_{j \neq k} A_j \) then

\[
\gamma_r^{A_1, \ldots, A_{2m}, \pi} = \frac{1}{d^2}
\]

Proof of this lemma is very similar to the proof of Lemma 4 and we will skip it.

Proof of Prop. 1. The measure \( \rho^{(N)} \) on the set of all subsets of \( \{1, \ldots, N\} \) such that for any \( A \subseteq \{1, \ldots, N\} \) we have

\[
\rho^{(N)}(\{A\}) = \left( \frac{c}{\sqrt{N}} \right)^{|A|} \left( 1 - \frac{c}{\sqrt{N}} \right)^{N-|A|}
\]

is simply a Bernoulli distribution and the assumption 2 of Theorem 3 is obviously fulfilled.

Therefore if \( A_1, A_2, A_3 \) are independent random variables with distribution given by \( \rho^{(N)} \) then

\[
P\{\omega : A_1(\omega) \cap A_2(\omega) \cap A_3(\omega) \neq \emptyset\} \leq \\
\leq \sum_{1 \leq k \leq N} P\{\omega : k \in A_1(\omega) \cap A_2(\omega) \cap A_3(\omega)\} = N \left( \frac{c}{\sqrt{N}} \right)^3
\]

and the assumption 2 of Theorem 3 is fulfilled.

Let \( A_1, \ldots, A_k \) be independent random variables with distribution given by \( \rho^{(N)} \) and let us consider the probability of following event: \( |A_p \cap A_q| = n_{pq} \). The discussion from the previous paragraph shows that this probability is equal (up to an error of order \( O(N^{-\frac{1}{2}}) \)) to the probability of the event: \( |A_p \cap A_q| = n_{pq} \) and

\[
\sum_{1 \leq k \leq N} P\{\omega : k \in A_1(\omega) \cap A_2(\omega) \cap A_3(\omega)\} = N \left( \frac{c}{\sqrt{N}} \right)^3
\]
furthermore $A_p \cap A_q \cap A_r = \emptyset$ for all $p < q < r$. We shall now evaluate the latter probability.

There are \( \frac{N!}{\prod_{pq}(N-\sum n_{pq})!} \) choices of disjoint sets \((B_{pq})\) such that \( |B_{pq}| = n_{pq} \).

For \( i \in B_{pq} \) the probability of the event: \( i \in A_p \cap A_q \) and \( i \not\in A_r \) for \( r \not\in \{p, q\} \) is equal to

\[
\left( \frac{c}{\sqrt{N}} \right)^2 \left( 1 - \frac{c}{\sqrt{N}} \right)^{k-2}.
\]

For \( i \not\in \bigcup_{pq} B_{pq} \) the probability of the event: \( i \not\in A_p \cap A_q \) for all \( 1 \leq p < q \leq k \) is equal to

\[
\left( 1 - \frac{c}{\sqrt{N}} \right)^k + k \frac{c}{\sqrt{N}} \left( 1 - \frac{c}{\sqrt{N}} \right)^{k-1}.
\]

Therefore the probability of the considered event is equal to

\[
\frac{N!}{\prod_{pq}(N-\sum n_{pq})!} \left( \frac{c}{\sqrt{N}} \right)^2 \sum n_{pq} \left( 1 - \frac{c}{\sqrt{N}} \right)^{k-2} \sum n_{pq} \times \\
\times \left[ \left( 1 - \frac{c}{\sqrt{N}} \right)^k + k \frac{c}{\sqrt{N}} \left( 1 - \frac{c}{\sqrt{N}} \right)^{k-1} \right]^{N-\sum n_{pq}}
\]

where all sums and products are taken over \( 1 \leq p < q \leq k \).

After short calculations one can show that the limit of this expression as \( N \) tends to infinity is equal to

\[
\prod_{1 \leq p < q \leq k} \frac{1}{n_{pq}} e^{2n_{pq}e^{-c^2}}
\]

and therefore the assumption 3 is fulfilled.

The assumption 4 follows from the observation that the distribution of the random variable \(|A_1 \setminus (A_2 \cup \cdots \cup A_n)|\) is binomial and simple computations. \( \square \)

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