Great Circle Route and Its Plotting on Chart Projection

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Abstract. The great circle route is often used for the navigation on the sea and in the air. In this paper, the great circle route and its plotting on the four main chart projections including the polar stereographic projection, the Gauss projection, the Mercator projection and the gnomonic projection is proposed. The equations of the great circle route on the chart are derived from the equations of the projection and the great circle on the earth. By making the longitude or the latitude coordinate as the parameter, the formulae for the evaluation of the curvature and the extreme distance is derived. The numerical implementations are also provided. The polar stereographic chart is proved to be suitable for the great circle route navigation, also for the gauss chart.

1. Introduction
The shortest distance between two arbitrary points on the spherical earth is the minor arc of the great circle and the shortest route for the navigation on the sea and in the air is the the great circle route [1-3]. The great circle route is also known as the cost-optimal route. For the navigation, the computations of the great circle route may includes the initial course angle, the navigational distance and the coordinates of the divided points [4, 5]. We must need a nautical chart for routes plotting when sailing on the sea. The chart projections are mainly the polar stereographic projection, the Gauss projection (the transverse Mercator projection), the Mercator projection and the gnomonic projection. By the properties of projection, the Gauss and the Mercator projections are cylindrical map projections as well as conformal projections while the other two projections are azimuthal map projections [6-9]. Moreover, the polar stereographic projection is also conformal.

The investigations for the great circle route navigation with vectorial methods has been proposed by [10-12]. The great circle route on the Mercator and the gnomonic projections and its applications for the determination of the orientation and the distance have been studied by [13]. In this paper, we focused on the great circle route and its plotting on the four main chart projections. We investigated not only the basic navigation parameters of the route projection but also the detailed analysis including the curvature of the route and the extreme distance for drawing curve by straight line. We also gave the numerical implementations for the algorithm of this paper.

2. Great Circle Route and Its Projection
2.1. Great Circle Route
The great circle route on the spherical earth from the starting point \( P_1(\varphi_1, \lambda_1) \) to the ending point \( P_2(\varphi_2, \lambda_2) \) is shown in Fig. 1, where the symbols \( \varphi \) and \( \lambda \) denote the latitude and the longitude.
The initial course angle $A_{GC}$ can be obtained from the geographic coordinates [14, 15]:

$$\tan A_{GC} = \frac{\sin(\lambda_2 - \lambda_1)}{\cos \phi_1 \tan \phi_1 - \sin \phi_1 \cos(\lambda_2 - \lambda_1)}$$  \hspace{1cm} (1)

Further, the equation for the great circle route on the earth can be written as:

$$\tan \phi = \tan \phi_1 \cos(\lambda - \lambda_1) + \sec \phi_1 \cot A_{GC} \sin(\lambda - \lambda_1)$$  \hspace{1cm} (2)

The navigation distance is the product of the radius of the earth $R$ and the central angle crossed from $P_1$ to $P_2$:

$$S_{GC} = R \arccos(\cos \phi_1 \cos \phi_2 \cos(\lambda - \lambda_2) + \sin \phi_1 \sin \phi_2)$$  \hspace{1cm} (3)

2.2. Projection on the Chart

The equations for the four main chart projections can be written as [5, 6]:

$$x = 2k_p R \tan(\frac{\pi}{4} - \frac{\phi}{2}) \sin \lambda, \quad y = -2k_p R \tan(\frac{\pi}{4} - \frac{\phi}{2}) \cos \lambda$$  \hspace{1cm} (4)

$$x = k_G R \arctan(\tan \phi \sec \lambda), \quad y = k_G R \arctanh(\cos \phi \sin \lambda)$$  \hspace{1cm} (5)

$$x = k_M R \arctan(\tan \phi \sec \lambda), \quad y = k_M R \arctanh(\cos \phi \sin \lambda)$$  \hspace{1cm} (6)

$$x = R \cot \phi \sin \lambda, \quad y = -R \cot \phi \cos \lambda$$  \hspace{1cm} (7)

where $k_p$, $k_G$ and $k_M$ are the proportionality coefficients. In detail the coefficient $k_p = \cos^2(\pi/4 - \phi_0/2)$, $\phi_0$ being the standard latitude. It may be $k_p = 1$ because the pole is usually chose as the standard parallel. The coefficients $k_G$ and $k_M$ are also often the units 1. In this paper, we take $k_p = k_G = k_M = 1$.

According to Eqs. (2), (4) to (7), we can get the equations of the great circle routes for the four projections:

$$\sinh(x/R) = \tan \phi_1 \cos(y/R - \lambda_1) + \sec \phi_1 \cot A_{GC} \sin(y/R - \lambda_1)$$  \hspace{1cm} (8)

$$\frac{\sin(x/R)}{\sqrt{1 - \sin^2(x/R) \sech^2(y/R)}} = \tan \phi_1 \cos \lambda_1 + \sec \phi_1 \cot A_{GC} \sin \lambda_1$$  \hspace{1cm} (9)

$$(x - x_0)^2 + (y - y_0)^2 = r_{GC}^2$$  \hspace{1cm} (10)
\( \beta_{GC} x - \alpha_{GC} y = R \) \hspace{1cm} (11)

where  \( \lambda_0 = \arctan (\sec(x/R) \sinh(y/R)) - \lambda_1 \),  \( r_{GC} = 2R \sec\phi_1 \csc A_{GC} \),  \( \alpha_{GC} = \tan\phi_1 \cos\lambda_1 - \sec\phi_1 \cot A_{GC} \sin\lambda_1 \) and  \( \beta_{GC} = \tan\phi_1 \sin\lambda_1 + \sec\phi_1 \cot A_{GC} \cos\lambda_1 \).  \( x_0 \) and  \( y_0 \) are the coordinates of the circle center.

3. Curvature of the Route and Extreme Distance

3.1. Polar Stereographic Projection

For convenience of the calculations, we take the longitude coordinate as the great circle route parameter. Hence, the equation for the route can be written as:

\[
\varphi = \varphi(\lambda) = \arctan \left( \tan \varphi_1 \cos(\lambda - \lambda_1) + \sec \varphi_1 \cot A_{GC} \sin(\lambda - \lambda_1) \right)
\]

The first- and second order derivatives of the latitude with respect to the longitude are:

\[
\varphi'_x = \left( -\tan \varphi_1 \sin(\lambda - \lambda_1) + \sec \varphi_1 \cot A_{GC} \cos(\lambda - \lambda_1) \right) \cos^2 \varphi
\]

\[
\varphi''_x = -2\varphi_x^3 \tan \varphi - \sin \varphi \cos \varphi
\]

According to the formulae for computing the curvature  \( \kappa \) and the curvature radius  \( M \):

\[
\kappa = \frac{x'_x y'_y - y'_x x'_y}{(x'_x + y'_x)^2} \!
\]

\[
M = \frac{1}{|\kappa|}
\]

we can obtain the curvature and the curvature radius of the great circle route on the polar stereographic projection:

\[
\kappa = \frac{\cos \phi_1 \sin A_{GC}}{2R}
\]

\[
M = 2R \sec\phi_1 \csc A_{GC}
\]

The extreme distance for drawing a curve by straight line on the chart is determined by the map scale denominator  \( C_0 \) and the curvature or the curvature radius \([5, 6]\):

\[
S_{\text{\tiny E}} = 89.44 \sqrt{\frac{M}{1000C_0}}
\]

where the unit of the extreme distance and the curvature radius is the centimeter and the meter, respectively. We remove here the infinitesimal of 4 order and higher order.

Hence, the extreme distance of the route on the polar stereographic projection is:

\[
S_{\text{\tiny E}} = 89.44 \sqrt{\frac{2R \sec \phi_1 \csc A_{GC}}{1000C_0}}
\]

When the great circle route is coincide with the meridian, the curvature is zero and the curvature radius and the extreme distance both are infinite.

3.2. Gauss Projection

On the basis of the equation of the Gauss projection and the derivatives of the latitude, we can get the derivatives  \( x'_h \),  \( y'_h \),  \( x''_h \) and  \( y''_h \) where the  \( x \) -axis is the vertical axis. Hence, the curvature of the great circle route on the Gauss projection can be given as
\[ \kappa = \frac{f_1(a_1 \phi_1^2 + a_2 \phi_1 + a_3 \phi + a_4) \cos \phi \sin \lambda}{R \left( f_2 \cos^2 \phi + f_3 \phi^2 \right)^{3/2}} \]  

(21)

where the coefficients 

\[ f_1 = 1 - \cos^2 \phi \sin^2 \lambda, \quad f_2 = 1 - \sin^2 \phi \cos^2 \lambda, \quad a_1 = \cos \lambda, \quad a_2 = \sin \phi \cos \phi \sin \lambda, \]

\[ a_3 = \sin^2 \phi \cos \lambda \]  
and \[ a_4 = \sin \phi \cos^3 \phi \sin \lambda. \]

Then the curvature radius and the extreme distance of the great circle route can be obtained.

When the great circle route is coincide with the meridian, we can not choose the longitude coordinate as the parameter of curve but the latitude coordinate. The isometric latitude \( q \) can be used to express the great circle route and is more favourable than the spherical latitude \( \phi \). The relation between these two latitudes is

\[ \arcsinh(tan \phi) = \arcsinh(q). \]

Denoting the symbol \( \lambda_{GC} \) as the constant longitude, we get the derivatives of the Gauss projection coordinates with respect to the isometric latitude:

\[ x' = \frac{\cos \lambda_{GC} \sinh q}{\cosh^2 \lambda_{GC} + \sinh^2 q}, \]

\[ y' = -\frac{\sin \lambda_{GC} \sin q}{\cosh^2 \lambda_{GC} + \sinh^2 q}, \]

\[ x'' = \frac{\cos \lambda_{GC} \sinh q}{\cosh^2 \lambda_{GC} + \sinh^2 q} - \frac{\cos \lambda_{GC} \cosh q \sinh 2q}{\left( \cosh^2 \lambda_{GC} + \sinh^2 q \right)^2}, \]

\[ y'' = -\frac{\sin \lambda_{GC} \cosh q}{\cosh^2 \lambda_{GC} + \sinh^2 q} + \frac{\sin \lambda_{GC} \sinh q \sinh 2q}{\left( \cosh^2 \lambda_{GC} + \sinh^2 q \right)^2}. \]

From Eqs. (15), (16) and (19), we then can get the curvature, the curvature radius and the extreme distance, where the \( x \)- and \( y \)-axes for the formula of the curvature must be exchanged.

### 3.3. Mercator Projection

From Eqs (6), (13) and (14), we can obtain the curvature and the curvature radius of the great circle route on the Mercator projection where the \( x \)-axis is also the vertical axis

\[ \kappa = -\frac{1}{R} \cos \phi \sin A_{GC} \tan \phi, \]

\[ M = R \sec \phi \csc A_{GC} \cot \phi \]  

(27)

Then we can get the extreme distance of the route on the Mercator projection. The curvature is zero when the great circle route is the meridian. Therefore, the curvature radius and the extreme distance are infinite.

### 3.4. Gnomonic Projection

We rewrite the equation of the great circle route as

\[ \tan \phi = \alpha_{GC} \cos \lambda + \beta_{GC} \sin \lambda \]  

(28)

Then we get

\[ \alpha_{GC} \cot \phi \cos \lambda + \beta_{GC} \cot \phi \sin \lambda = 1 \]  

(29)

Hence, the equation of the great circle route on gnomonic projection can be obtained, i.e. Eq. (11). The great circle route projection is a straight line and its curvature is zero. The great circle route on the gnomonic projection is also a straight which is coincide with the meridian. 


4. Numerical Experiments

From Eqs. (8) to (11), we can plot the great circle route on the four chart projections, which also can be obtained from Eqs. (2), (4) to (7), i.e. from the great circle route equation and the projection equations.

Figure 2. Great circle routes on chart projections. (a) the polar stereographic projection; (b) the Gauss projection; (c) the Mercator projection; (d) the gnomonic projection

In this paper we choose a section of the Atlantic route as the required route: the great circle route from New York (40°43′N,74°00′W) to London (51°30′N,00°05′E), and take the the earth radius $R = 6378137$ m in WGS84 coordinate system. Now we use the Mathematica software and its built-in data to verify this paper’s algorithm. Here we just consider the theoretical route and the route is shown in Fig. 2.

According to this paper’s algorithm, we can plot the figures of the curvatures, the curvature radius and the extreme distances of the great circle route on the four chart projections, see Figs. 3 and 4. We choose here the ratio 1:500000 as the map scale, i.e. $C_0 = 500000$. The maximal extreme distances and the corresponding curvatures and curvature radius are given in Table 1, also for the extreme point (i.e. the longitude coordinate), where the units of the curvature, the curvature radius and the maximal extreme distance are $\text{m}^{-1}$, m and cm, respectively.

Figure 3. Curvature of the great circle route
Table 1. The maximal extreme distances and the corresponding curvatures and curvature radius of the circle route

| Projection          | Longitude | Curvature  | Curvature radius | Maximal extreme distance |
|---------------------|-----------|------------|------------------|--------------------------|
| polar stereographic | /         | 4.6340E-08 | 2.1580E+0        | 1.8581E+01               |
| Gauss projection    | -74°0.0′  | 5.2703E-08 | 1.8974E+0        | 1.7423E+01               |
| Mercator projection | -23°1.5′  | -1.2268E-07| 7.8940E+0        | 1.1238E+01               |
| gnomonic projection | /         | 0          | infinity         | infinity                 |

From Fig. 2, the great circle route on the polar stereographic projection is a portion of the circle which the radius $r_{GCC}$ is large enough. The arc is closed to a straight line. The route on the Gauss projection may be a curve and also close to a straight line. Moreover, the route on the Mercator projection is bent and has significant difference from the straight line. Therefore, our algorithm has been proved to be correct. A correction $\varepsilon$ on the Mercator chart between the great circle route and the rhumb line is often used for amending the course angle of the great circle route or the rhumb line. In particular, the projection on the gnomonic chart becomes a rigorous straight line. It's worth noting that we assume the great circle route is not coincide with the meridian. When coincidence is happened, the coordinates of great circle route on the sphere is only depending on the latitude. In this case the projections on the polar stereographic chart and the gnomonic chart both are straight, while the projections on the other two charts are curves.

From Figs. 3 and 4, the curvature and the curvature radius of the great circle route from NewYork to London on the polar stereographic projection both are constant. The curvature on the Gauss projection decreases in value with the longitude coordinate increasing in value, while the curvature radius becomes larger. The curvature on the Mercator projection is negative and its value decreases and then increases with respect to the longitude, the same as the variation trend of the curvature radius. They both reach the maximum value at the point $\lambda = -23°1.5′$. Furthermore, the curvature on the gnomonic projection is always zero. From Table 1, the relations of the maximal extreme distance between the four chart projections are as follows: the gnomonic projection > the polar stereographic projection > the Gauss projection > the Mercator projection. Although the great circle route on the polar stereographic projection is a circular arc, its maximal extreme distance may be large than the other projections except the gnomonic projection. We can make sure the great circle route projection
on the polar stereographic chart gets close enough to the straight line. The maximal extreme distance on the Gauss projection is less than the value on the polar stereographic projection but the difference is not great. So we can use the these charts except the Mercator chart to process the great circle route measurement.

5. Conclusions
We have proposed the algorithm for the great circle route and its plotting on the four main chart projections in this paper. The equation of the great circle route on the chart has been obtained, describing the curve shape of the route on the chart. By the parameterization, we get the formulae of the curvature and the extreme distance. The polar stereographic projection can be used to the great circle route measurement on account of the greater maximal extreme distance. Furthermore, its deformation is small including the zero angular deformation. We can also use the gnomonic and the conformal Gauss charts to process the great circle route measurement. Although the maximal extreme distance of the great circle route on the Mercator chart is small and may not be suitable for the great circle route measurement, it can be applied for the rhumb line measurement as a section of the great circle route due to the straight line of the rhumb line on the map or chart.

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7. References
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