A simple quantum oblivious transfer protocol

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Abstract

A simple and efficient protocol for quantum oblivious transfer is proposed. The protocol can easily be implemented with present technology and is secure against cheaters with unlimited computing power provided the receiver does not have the technology to store the particles for an arbitrarily long period of time. The proposed protocol is a significant improvement over the previous protocols. Unlike the protocol of Crépeau and Kilian which is secure if only if the spin of the particle is measured along the $x$ or the $y$ axis, the present protocol is perfectly secure no matter along which axes the spin of the particles are measured, and unlike the protocol of Bennett et al. which requires tens of thousand of particles, the present protocol requires only two particles.

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In 1970, Wiesner [1] wrote a highly innovative paper about quantum cryptography [2], introducing a new branch of Physics and computation. In his original paper, he also introduced the concept of Multiplexing, which was later rediscovered by Rabin [3], and is now usually called Oblivious Transfer (OT) [4]. The concept of OT has turned out to be a very useful tool in designing cryptographic protocols, and has been used for quite a while as a standard primitive tool for constructing more complex protocols. Let us briefly describe the OT protocol:

1 - Alice knows one bit $\lambda$, where $\lambda$ is either 1 or $-1$ [5].

2 - Bob obtains bit $\lambda$ from Alice with probability 0.5.

3 - Bob knows whether or not he obtained bit $\lambda$.

4 - Alice does not learn whether or not Bob obtained bit $\lambda$.

Crépeau and Kilian [6] have suggested that OT can simply be achieved by having Alice send Bob a single spin $\frac{1}{2}$ particle, for example an electron, encoding the OT bit into the spin of the particle along the horizontal or vertical axis (or encoding the OT bit in polarization of the photon along the horizontal or diagonal axis). Bob then randomly chooses the $x$ or the $y$ axis, and measures the spin of the particle along that axis. Finally Alice tells Bob the correct axis. This simple protocol is secure if and only if Bob measures the spin of the particle along the horizontal or the vertical axis. For example if Bob measures the spin of the particle along the diagonal axis, he will then obtain a considerable amount of partial information about Alice’s bit [6]. Bennett et al. [7] have proposed a protocol for quantum OT which is free from this disadvantage: but their protocol is rather inefficient, requiring tens of thousands of particles to be sent and received for a simple decision making. It is worth noting that all previous quantum oblivious protocols are
insecure against EPR attack.

In this paper, we propose a simple and efficient protocol for quantum OT. The protocol is a considerable improvement over the previous protocols. Unlike the protocol of Bennett et al., the present protocol requires only two particles to be sent for a simple decision making, and unlike the protocol of Crépeau and Kilian, the proposed protocol is perfectly secure no matter along which axes the spin of the particles are measured.

The proposed protocol consists of the following steps:

1. Alice and Bob agree that $\lambda$ is encoded in the product of the spin of the two particles along the horizontal axis or along the vertical axis, i.e., $\lambda$ is encoded in $b_1b_2$, where $b_1$ and $b_2$ are spins of the first and the second particles along the horizontal axis, or $b_1$ and $b_2$ are spins of the first and the second particles along the vertical axis (here horizontal axis refers to $x$ or $-x$ axes and vertical axis refers to $y$ or $-y$ axes). They also agree that $b_1 = 1$ ($b_2 = 1$) indicates that spin of the first (second) particle is along $0$ or $\frac{\pi}{2}$ axis, and $b_1 = -1$ ($b_2 = -1$) indicates that spin of the first (second) particle is along $\pi$ or $\frac{3\pi}{2}$ axis. For example if $\lambda = 1$, and if Alice decides to encode $\lambda$ in the product of the spin of the two particles along the horizontal axis, then she prepares two particles with their spins along the $x$ axis, or two particles with their spins along the $-x$ axis. Similarly if $\lambda = -1$, and if Alice decides to encode $\lambda$ in the product of the spin of the two particles along the vertical axis, then she prepares two particles with spin of the first particle along the $y$ axis and spin of the second particle along the $-y$ axis or spin of the the first particle along the $-y$ axis and spin of the second particle along the $y$ axis.

2. Alice encodes $\lambda$ in $b_1b_2$ and sends Bob the two particles.
(3) Bob measures the spin of both particles randomly along the $x$ axis or along the $y$ axis.

(4) Alice asks Bob if his measurements have been successful. If Bob says no, then Alice goes to step 2. If Bob says yes, then Alice tells him only one of the following two alternatives:

(i) $\lambda$ is encoded in the product of the spin of the two particles along the horizontal axis,

(ii) $\lambda$ is encoded in the product of the spin of the two particles along the vertical axis,

We now show that if Bob does not have the technology to store the particles until step 4, then the above oblivious transfer protocol is secure against cheaters with unlimited computing power. First note that if Bob is honest, then the oblivious transfer protocol can succeed without any difficulty. For example, assume $\lambda$ is encoded in the product of the spin of the two particles along the horizontal axis. If Bob measures the spins of the particles along the $x$ axis, then he learns the value of $\lambda$; but if he measures the spins of the particles along the $y$ axis, then he does not gain any information about $\lambda$.

Now consider a cheating Bob who measures the spin of the first particle along axis $\vec{a}$ at angle $\alpha$ with respect to the $x$ axis, and measures the spin of the second particle along axis $\vec{b}$ at angle $\beta$ with respect to the $x$ axis. The result of his measurement on the first photon can be represented by a random variable $b_1'$ which takes values in the set $\{1, -1\}$. According to the standard rules of quantum theory

$$p (b_1' = b_1) = \cos^2 \left( \frac{\alpha - \theta}{2} \right), \quad p (b_1' = -b_1) = \sin^2 \left( \frac{\alpha - \theta}{2} \right), \quad (1)$$

where $\theta$ is the angle at which the spin of the first particle is measured by
Alice and is in the set \( \left\{ 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\} \). Similarly the result of his measurement on the second photon can be represented by a random variable \( b_2' \) taking values in the set \( \{1, -1\} \). Again according to quantum theory

\[
p (b_2' = 1 \mid b_2 = 1) = \cos^2 \left( \frac{\beta - \phi}{2} \right), \quad p (b_2' = -1 \mid b_2 = -1) = \cos^2 \left( \frac{\beta - \phi}{2} \right),
\]

\[
p (b_2' = 1 \mid b_2 = -1) = \sin^2 \left( \frac{\beta - \phi}{2} \right), \quad p (b_2' = -1 \mid b_2 = 1) = \sin^2 \left( \frac{\beta - \phi}{2} \right),
\]

where \( \phi \) is the angle at which the spin of the first particle is measured by Alice and is in the set \( \left\{ 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\} \).

Without loss of generality, first we assume \( \lambda = 1 \). We now asks the following question: Given that \( \lambda = 1 \), what is the probability that Bob will obtain \( \lambda' = 1 \), i.e., what is \( p (\lambda' = 1 \mid \lambda = 1) \)? To answer this question, we note that \( \lambda \) is encoded along the horizontal and vertical axes with equal probability, i.e.,

\[
p (\lambda' = 1 \mid \lambda = 1) = p (b'_1 b'_2 = 1 \mid b_1 b_2 = 1)
\]

\[
= p (H) p (b'_1 b'_2 = 1 \mid b_1 b_2 = 1, H)
\]

\[
+ p (V) p (b'_1 b'_2 = 1 \mid b_1 b_2 = 1, V), \tag{3}
\]

where \( H \) means \( \lambda \) is encoded along the horizontal axis and \( V \) means \( \lambda \) is encoded along the vertical axis. We now note that

\[
p (b'_1 b'_2 = 1 \mid b_1 b_2 = 1, H) = p (b_1 = 1, b_2 = 1) p (b'_1 b'_2 = 1 \mid b_1 = 1, b_2 = 1, H)
\]

\[
+ p (b_1 = -1, b_2 = -1) p (b'_1 b'_2 = 1 \mid b_1 = -1, b_2 = -1, H),
\]

\[
p (b'_1 b'_2 = 1 \mid b_1 b_2 = 1, V) = p (b_1 = 1, b_2 = 1) p (b'_1 b'_2 = 1 \mid b_1 = 1, b_2 = 1, V)
\]

\[
+ p (b_1 = -1, b_2 = -1) p (b'_1 b'_2 = 1 \mid b_1 = -1, b_2 = -1, V). \tag{4}
\]
Since Alice encodes $\lambda$ along $x$ and $-x$ with equal probability,

$$p (b_1 = -1, b_2 = -1) = \frac{1}{2}, \quad p (b_1 = 1, b_2 = 1) = \frac{1}{2},$$  \hspace{1cm} (5)

we thus have

$$p (b_1' b_2' = 1 \mid b_1 b_2 = 1, H) = \frac{1}{2} p (b_1' = 1 \mid b_1 = 1, H) p (b_2' = 1 \mid b_2 = 1, H)$$

$$+ \frac{1}{2} p (b_1' = 1 \mid b_1 = -1, H) p (b_2' = 1 \mid b_2 = -1, H)$$

$$+ \frac{1}{2} p (b_1' = -1 \mid b_1 = 1, H) p (b_2' = -1 \mid b_2 = 1, H)$$

$$+ \frac{1}{2} p (b_1' = -1 \mid b_1 = -1, H) p (b_2' = -1 \mid b_2 = -1, H)$$

$$= \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) + \sin^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) \right. $$

$$+ \sin^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) + \cos^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) \right].$$  \hspace{1cm} (6)

Similarly

$$p (b_1' b_2' = 1 \mid b_1 b_2 = 1, V) = \frac{1}{2} p (b_1' = 1 \mid b_1 = 1, V) p (b_2' = 1 \mid b_2 = 1, V)$$

$$+ \frac{1}{2} p (b_1' = 1 \mid b_1 = -1, V) p (b_2' = 1 \mid b_2 = -1, V)$$

$$+ \frac{1}{2} p (b_1' = -1 \mid b_1 = 1, V) p (b_2' = -1 \mid b_2 = 1, V)$$

$$+ \frac{1}{2} p (b_1' = -1 \mid b_1 = -1, V) p (b_2' = -1 \mid b_2 = -1, V)$$

$$= \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \right. $$

$$+ \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \right].$$  \hspace{1cm} (7)

Using Eqs. (3), (6) and (7), and noting that $p (H) = p (V) = \frac{1}{2}$, we obtain

$$p (\lambda' = 1 \mid \lambda = 1) = \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) + \sin^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) \right. $$

$$+ \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \right].$$
Using the following simple trigonometric relations,

\[
\cos (\gamma_1 - \gamma_2) = \cos \gamma_1 \cos \gamma_2 + \sin \gamma_1 \sin \gamma_2,
\]

\[
\sin (2\gamma) = 2 \sin \gamma \cos \gamma,
\]

\[
\cos^2 \gamma = \frac{1 + \cos(2\gamma)}{2}.
\]

Eq. (7) may be simplified to

\[
p (\lambda' = 1 | \lambda = 1) = \frac{1}{2} + \frac{1}{4} (\cos \alpha \cos \beta + \sin \alpha \sin \beta)
\]

\[
= \frac{1}{2} + \frac{1}{4} \cos (\alpha - \beta).
\]

Obviously

\[
p (\lambda' = -1 | \lambda = 1) = 1 - p (\lambda' = 1 | \lambda = 1)
\]

\[
= \frac{1}{2} - \frac{1}{4} \cos (\alpha - \beta).
\]

The probability that \(\lambda' = 1\) (\(\lambda' = -1\)) given that \(\lambda = 1\) is maximized (minimized) if \(\alpha = \beta\) and the maximum (minimum) value is \(\frac{3}{4} (\frac{1}{4})\). Thus the best strategy for a cheating Bob is to measure the spin of both particles along the same axis, in which case he would obtain as much information as an honest Bob who measures the spin of both particles along the \(x\) or along the \(y\) axis, i.e., a cheating Bob can not gain any more information about \(\lambda\) than an honest Bob.

Next we assume that \(\lambda = -1\). We now ask the following question: Given that \(\lambda = -1\), what is the probability that \(\lambda' = 1\), i.e., what is \(p (\lambda' = 1 | \lambda = -1)\) ? To answer this question, again we note that \(\lambda\) is encoded along the horizontal and vertical axes with equal probability, i.e.,

\[
p (\lambda' = 1 | \lambda = -1) = p (b_1' b_2' = 1 | b_1 b_2 = -1)
\]

\[
= p (H) p (b_1' b_2' = 1 | b_1 b_2 = -1, H)
\]

\[
+ p (V) p (b_1' b_2' = 1 | b_1 b_2 = -1, V).
\]
We now note that

\[ p \left( b'_1 b'_2 = 1 \mid b_1 b_2 = -1, H \right) = p \left( b_1 = 1, b_2 = -1 \right) p \left( b'_1 b'_2 = 1 \mid b_1 = 1, b_2 = -1, H \right) \]

\[ + p \left( b_1 = -1, b_2 = 1 \right) p \left( b'_1 b'_2 = 1 \mid b_1 = -1, b_2 = 1, H \right) \]

\[ p \left( b'_1 b'_2 = 1 \mid b_1 b_2 = -1, V \right) = p \left( b_1 = 1, b_2 = -1 \right) p \left( b'_1 b'_2 = 1 \mid b_1 = 1, b_2 = -1, V \right) \]

\[ + p \left( b_1 = -1, b_2 = 1 \right) p \left( b'_1 b'_2 = 1 \mid b_1 = -1, b_2 = 1, V \right) . \]

(12)

Since Alice encodes \( \lambda \) along \( x \) and \(-x\) with equal probability,

\[ p \left( b_1 = 1, b_2 = -1 \right) = \frac{1}{2}, \quad p \left( b_1 = -1, b_2 = 1 \right) = \frac{1}{2}, \]

(13)

we thus have

\[ p \left( b'_1 b'_2 = 1 \mid b_1 b_2 = -1, H \right) = \frac{1}{2} p \left( b'_1 = 1 \mid b_1 = 1, H \right) p \left( b'_2 = 1 \mid b_2 = -1, H \right) \]

\[ + \frac{1}{2} p \left( b'_1 = -1 \mid b_1 = 1, H \right) p \left( b'_2 = -1 \mid b_2 = -1, H \right) \]

\[ + \frac{1}{2} p \left( b'_1 = 1 \mid b_1 = -1, H \right) p \left( b'_2 = 1 \mid b_2 = 1, H \right) \]

\[ + \frac{1}{2} p \left( b'_1 = -1 \mid b_1 = -1, H \right) p \left( b'_2 = -1 \mid b_2 = 1, H \right) \]

\[ = \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) + \sin^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) \right] \]

\[ + \sin^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) + \cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) \] .

(14)

Similarly

\[ p \left( b'_1 b'_2 = 1 \mid b_1 b_2 = -1, V \right) = \frac{1}{2} p \left( b'_1 = 1 \mid b_1 = 1, V \right) p \left( b'_2 = 1 \mid b_2 = -1, V \right) \]

\[ + \frac{1}{2} p \left( b'_1 = -1 \mid b_1 = 1, V \right) p \left( b'_2 = -1 \mid b_2 = -1, V \right) \]

\[ + \frac{1}{2} p \left( b'_1 = 1 \mid b_1 = -1, V \right) p \left( b'_2 = 1 \mid b_2 = 1, V \right) \]
\[ + \frac{1}{2} p \left( b_1' = -1 \mid b_1 = -1, V \right) p \left( b_2' = -1 \mid b_2 = 1, V \right) = \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \\
+ \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \right]. \] (15)

Again using the fact that \( p(H) = p(V) = \frac{1}{2} \) and referring to Eqs. (11), (14), and (15), we have

\[ p \left( \lambda' = -1 \mid \lambda = -1 \right) = \frac{1}{2} \left[ \cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\beta}{2} \right) + \sin^2 \left( \frac{\alpha}{2} \right) \cos^2 \left( \frac{\beta}{2} \right) \\
+ \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \sin^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) + \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right) \cos^2 \left( \frac{\beta}{2} - \frac{\pi}{4} \right) \right] \] (16)

Simplifying the above equation, we obtain

\[ p \left( \lambda' = 1 \mid \lambda = -1 \right) = \frac{1}{2} - \frac{1}{4} (\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\
= \frac{1}{2} - \frac{1}{4} \cos (\alpha - \beta). \] (17)

Obviously

\[ p \left( \lambda' = -1 \mid \lambda = -1 \right) = 1 - p \left( \lambda' = 1 \mid \lambda = -1 \right) \\
= \frac{1}{2} + \frac{1}{4} \cos (\alpha - \beta). \] (18)

The probability that \( \lambda' = 1 \) (\( \lambda' = -1 \)) given that \( \lambda = -1 \) is minimized (maximized) if \( \alpha = \beta \) and the minimum (maximum) value is \( \frac{3}{4} \). Thus the best strategy for a cheating Bob is to measure the spin of both particles along the same axis, in which case he would obtain as much information as an honest Bob who measures the spin of both particles along the \( x \) or along the \( y \) axis. Finally it should be noted that Bob can cheat by storing the particles until step 4 and then perform his measurements. This sophisticated
attack, which is in principle possible, is completely infeasible at present or in the foreseeable future.

To summarize, a cryptographic protocol for quantum OT is proposed. The protocol is a significant improvement over the previous protocols. Unlike the protocol of Kilian and Crépeau which is secure if and only if the spin of the particles are measured along the horizontal or vertical axis, the present protocol is secure no matter along which axis the spin of the particles are measured, and unlike the protocol of Bennett et al. which requires tens of thousand of photons, the present protocol requires only two photons. However, similar to previous protocols, the present protocol is not secure against EPR attack. The advantage of the present protocol is that it is extremely simple, highly economical and is secure against cheater with technology that is available today or in foreseeable future.
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