8π-periodic dissipationless ac Josephson effect on a quantum spin-Hall edge via a Quantum magnetic impurity

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Time-reversal invariance places strong constraints on the properties of the quantum spin Hall edge. One such restriction is the inevitability of dissipation in a Josephson junction between two superconductors formed on such an edge without the presence of interaction. Interactions and spin-conservation breaking are key ingredients for the realization of dissipationless ac Josephson effect on such quantum spin Hall edges. We present a simple quantum impurity model that allows to create a dissipationless fractional Josephson effect on a quantum spin Hall edge. We then use this model to substantiate a general argument that shows that any such non-dissipative Josephson effect must necessarily be 8π-periodic.

I. INTRODUCTION

The Josephson effect, which was originally a direct manifestation of macroscopic quantum coherence in superconductors, has turned out to be one of the most reliable ways of diagnosing the topological properties of a junction. Topological superconductors (TSCs) supporting Majorana modes have been shown to demonstrate a 4π-periodic Josephson effect, which is doubled period compared to the conventional Josephson effect. This phenomenon, which is known as the fractional Josephson effect, has been observed in quite a few devices, including the quantum spin Hall edge. At first this is quite counter-intuitive given that the Hamiltonian itself is 2π-periodic. The fractional Josephson effect in this case arises because the topological property of the superconductor forces the local fermion parity (FP) of the junction to change with each rotation of the phase by 2π. If one assumes contact with a bath that equilibrates the system to the ground state of the appropriate FP Hamiltonian then the topological nature of the superconductor is precisely reflected in the fractional Josephson effect. Interestingly, the addition of interaction can often modify the topological classification qualitatively. In fact, some of the states such as parafermion states in superconductors are already known to be characterized by exotic Josephson effects.

While several topological superconducting phases involving interactions have been proposed, not many of them are within experimental reach. On the other hand, an interesting 8π-periodic Josephson effect, which relies on the combination of interaction and topology that has recently been proposed, certainly appears to be within the realm of experimental possibility. Ideally, one could just obtain this effect by studying a Josephson junction (JJ) on an interacting two dimensional topological insulator (TI) edge.

However, as we discuss in this paper, the 8π Josephson effect turns out to be the only possible non-dissipative Josephson effect that a quantum spin Hall edge can support. To understand what we mean by non-dissipative Josephson effect consider a finite voltage biased Josephson junction whose Andreev state is tunnel-coupled to a quantum dot. The quantum dot is coupled to another localized spin. At first this is quite counter-intuitive given that the Hamiltonian itself is 2π-periodic. The fractional Josephson effect in this case arises because the topological property of the superconductor forces the local fermion parity (FP) of the junction to change with each rotation of the phase by 2π. If one assumes contact with a bath that equilibrates the system to the ground state of the appropriate FP Hamiltonian then the topological nature of the superconductor is precisely reflected in the fractional Josephson effect. Interestingly, the addition of interaction can often modify the topological classification qualitatively. In fact, some of the states such as parafermion states in superconductors are already known to be characterized by exotic Josephson effects.

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![Diagram](Figure 1. (a) The system with a short Josephson junction whose Andreev state is tunnel-coupled to a quantum dot. The quantum dot is coupled to another localized spin. [Eq. (1)] (b) Schematic diagram showing the 8π cycle of states. represent the states of the spin and represent the electron in the quantum dot. represents the spin symmetric and asymmetric exchange interactions inside the dot respectively. In each 2π cycle of phase shown by the bold arrow a spin up electron is pumped into or a spin down hole leaves the edge.)
tion of a relaxation mechanism also leads to a dissipative but 4π-periodic fractional Josephson effect. Thus, one can say that the Josephson effect on a non-interacting TI edge with TRS is always dissipative and fundamentally accompanied by a shunt resistance. As shown by Zhang and Kane, the addition of interaction qualitatively changes this story and introduces a topologically protected 8π-periodic fractional Josephson effect, which is non-dissipative (i.e. free of the shunt resistance). It is worth noting that this effect, unlike the 4π, 8π and 12π-periodic Josephson effect that can arise from fine-tuning in conventional systems, is indeed topologically protected in the sense that it is completely robust to all perturbations of the Hamiltonian that preserve time-reversal symmetry.

In this paper, we study the effect of a strongly interacting quantum dot (QD) in a quantum spin-Hall Josephson junction. By considering a simple model of such a QD that acts like a spin coupled to Andreev bound states (ABS), we show that such a junction would show an 8π-periodic fractional Josephson effect, in contrast to the 4π-periodicity expected from time-reversal breaking topological junction. We then argue that the generic low voltage periodicity of the quantum spin-Hall Josephson junction is 8π as opposed to the 4π periodicity for time-reversal breaking topological junctions. In fact, while higher frequencies could lead to 2π or 4π-periodic Josephson effects, such as Josephson effects are necessarily dissipative i.e. accompanied by a finite dc current.

II. JUNCTION WITH A QUANTUM DOT

While the non-dissipative 8π-periodic Josephson effect is generic, we start by demonstrating it’s origin through the simple device shown in Fig. [1a]. This model incorporates the key ingredients for a non-dissipative Josephson effect - namely, a topological quantum spin-Hall edge, spin-conservation breaking and interaction. The device in Fig. [1a] consists of a JJ on a QSH edge laterally coupled to a strongly-interacting multi-orbital quantum dot. The interaction is chosen to be large enough so that the quantum dot admits at most two electrons. Furthermore, the Hund’s coupling is also assumed to be strong so that one of the levels is always occupied by one electron, which can thus be considered to be a local moment with spin $S$. The resulting spin $-1/2$ is exchanged coupled to the dot electron with spin $s$ via the Hund’s coupling through a Hamiltonian

$$H_{d-s} = JS \cdot s + J_A (S^+ s^+ + S^- s^-) + \varepsilon_d (d^+_\uparrow d^\downarrow + d^\downarrow d^\uparrow),$$

where the spin of the dot electron $d^\sigma_{\uparrow \downarrow}$ is written as $s = \frac{1}{2} \sum_{\alpha \beta} d^\sigma_{\alpha \beta} \sigma_{\alpha \beta} d_{\beta \sigma}$. In addition a projection constraint ensures no-double occupancy of the electron level $d^\sigma_{\alpha \beta}$. The term proportional to $J$ is the Heisenberg interaction between the dot and the spin, while $J_A$ represents the process in which spin-conservation is broken.

The QD with Hamiltonian Eq. [1] is tunnel-coupled to the quantum spin Hall (QSH) edge through a Hamiltonian $H_{J-d}$

$$H_{J-d} = t \left[a^\dagger_s (x = 0) d^\uparrow + a^\dagger_d (x = 0) d^\downarrow + h.c.\right],$$

where $a_{\alpha}^\dagger (x = 0)$ creates electrons on the QSH edge. A time-reversal breaking impurity on the QSH edge is expected to produce a 4π-periodic Josephson effect because of the flip of fermion parity with each $2\pi$ shift of the phase $\phi$. While the spin in the QD acts as a magnetic impurity, as illustrated in Fig. [1b], this only works in the case of odd fermion parity of the JJ where by Kramers theorem the ground state is two-fold degenerate. As will be shown, while the QD returns to odd FP state each 4π period (as in the time-reversal breaking case), the spin in the QD is flipped over each such period. This leads to the generic 8π-periodicity of the current as a function of phase.

To illustrate the mechanism in Fig. [1b] quantitatively, we consider the limit of weak tunnel coupling $t$, the quantum dot electron can only tunnel to a low-energy ABS in a Josephson junction on the edge written as $\gamma = \sum_{\sigma} \int dx (u_{\sigma} a^\dagger_{\sigma} + v_{\sigma} a_{\sigma})$ with an energy $E(\phi)$, which depends on the phase difference across the Josephson junction. The effective Hamiltonian of the edge ABS is written as

$$H_j = E(\phi) (\gamma^\dagger \gamma - 1/2).$$

The wavefunction of the ABS $\gamma$ and its energy are solved from the BdG Hamiltonian of the JJ

$$H_{BdG} = \tau_x (-i v_F s_z \partial_x - \mu) + \Delta \cos \phi(x) \tau_x + \Delta \sin \phi(x) \tau_y,$$

where $s$ and $\tau$ are Pauli matrices on spin and Nambu spaces, respectively, and $\phi(x) = \phi(x)$. Since $[H_{BdG}, s_z] = 0$, the solutions are labeled with the eigenvalues of $s_z$, where the solution with $s_z = +1$ have $E(\phi) = -\Delta \cos \phi/2$, $u_{\uparrow} (x = 0) = v_{\downarrow} (x = 0)^\dagger = \sqrt{\sin \phi/\Delta} e^{-i\phi/4}$ and $u_\downarrow = v_\uparrow = 0$ in the interval $0 \leq \phi \leq 2\pi$, while its particle-hole-conjugated partner is the other branch of solution with $s_z = -1$. With this explicit form of quasiparticle solution, we can therefore rewrite $H_{j-d}$ as

$$H_{j-d} = t \left(u_{\uparrow} d^\dagger_{\uparrow} \gamma + v_{\downarrow} \gamma d^\downarrow + h.c.\right),$$

where we have dropped the position arguments of $u_{\uparrow}$ and $v_{\downarrow}$.

The simplified form of the effective coupling allows us to describe the cycle of the QD shown in Fig. [1b] as the phase $\phi$ is varied. During each cycle of advancing $\phi$ by $2\pi$ forward, a quasiparticle is “pumped” from the bulk occupied states towards the conduction states through the
The excitation of a bulk conduction band electron must be avoided to prevent dissipation. This can be accomplished by adding a spin-up electron or removing a spin-down electron from the QD (and releasing a Cooper pair). Starting with the state $|\downarrow\rangle$ at $\phi = 0$, i.e. with the dot empty and localized spin at $S_z = -1$. During an increment of $2\pi$ of $\phi$, a spin-up electron is added to the dot from the bulk, which due to the hybridization term Eq. (11) forms a singlet with the localized spin $\frac{d^{\dagger}|\uparrow\rangle - d|\downarrow\rangle}{\sqrt{2}}$. In the next cycle a spin-down electron is removed from the dot, leaving the localized spin at $S_z = +1$ ($|\uparrow\rangle$). The electron leaving the dot can combine with the next electron coming from the bulk and exit as a Cooper pair. The next two cycles proceed similarly, with $J_A$ breaking spin conservation to result in a triplet state after the third cycle $\frac{d^{\dagger}|\uparrow\rangle - d|\downarrow\rangle}{\sqrt{2}}$ and returning to the original state $|\downarrow\rangle$ at $\phi = 8\pi$.

The above process can be put on quantitative footing by projecting the Hamiltonian into the low-energy Hilbert space and solving for the energy-phase relation (EPR), as detailed in Appendix A. Typical results with $J_A$ being zero or non-zero are shown respectively in Fig. 2(a) and 2(b). In the absence the term proportional to $J_A$, the full $8\pi$-cycle could not be completed non-dissipatively because the dot-spin triplet state could not be formed. Fig. 2(a) illustrates that the state is eventually driven into continuum, thereby dissipating away the Josephson current. Alternatively, dissipation of the excess energy into a phonon might lead to a dissipative $4\pi$-periodic process. With non-zero $J_A$ [Fig. 2(b)], a full cycle of states fully gapped from other excited states can be obtained. We also note that $J_A$ breaks spin conservation along $z$-direction. The connection of the absence of spin conservation with the prevention of dissipation will be elaborated below.

### III. CONDITIONS FOR DISSIPATIONLESS JOSEPHSON EFFECT

We now discuss in general the necessary conditions to realize a topological TRS Josephson junction without dissipation. We first review how this is accomplished in the case where time-reversal symmetry is broken by an FI element in a topological junction with only one ABS (see Fig. 3(b)). The EPR is shown in Fig. 3(c). At $\phi \neq 2\pi p$, where $p$ is an integer, a particle-hole pair of ABS is present in the junction. Without the FI which breaks TRS, these states are required to join the continuum modes ($|E| = \Delta$) at the time-reversal-invariant points $\phi = 2\pi p$ in order to satisfy the Kramers theorem (black lines). This requirement corresponds (see Fig. 3(d)) to the adiabatic change of the phase $\phi$ connecting the ground state at $\phi = 0$ with the continuum at $\phi = 2\pi$. This creates quasiparticle excitations in the bulk that lead to dissipation in the Josephson junction. On the other hand, with the FI that breaks TRS even at $\phi = 2\pi p$, the ABSs remain disconnected from the continuum modes [blue lines in Fig. 3(c)], and thus the many-body state remains gapped from the continuum [blue lines in Fig. 3(d)]. The full $4\pi$-cycle of Josephson current could then be completed without dissipation if the temperature and rate of change of $\phi$ are low enough.

As seen in Fig. 3(a), the ABS levels in the non-interacting case connect the valence bands to the conduction bands. This is a necessary consequence of Kramer’s degeneracy and the time-reversal (i.e., $\phi \rightarrow -\phi$) properties of the eigenvalues shown in Fig. 3(a). This leads us to conclude that interactions are necessary to avoid dissipation in TRS topological Josephson junctions. Next we argue that spin-conservation breaking is crucial to avoid dissipation in the TI Josephson junction. To understand this, consider a Josephson junction built from a Corbino geometry (shown at Fig. 3(b)), in which the phase difference $\phi$ between the two sides of the junction is controlled by threading a flux $\Phi_0$ through the center of the setup, where $\Phi_0 = \frac{h}{2e}$ is the SC flux quantum. For the sake of the argument, we first take out the SC and FI elements, leaving behind a Corbino disk made of TI. It is known that if spin is conserved along the polariza-
tion axis of the TI, the system exhibits quantized spin Hall conductance, and by Laughlin’s argument threading a flux quantum has the effect of pumping a pair of spins with \( s_z = \pm 1 \) to opposite edges. This effect still holds with the introduction of the SC, since \( s_z \)-wave superconductivity preserves \( s_z \) conservation. In this scenario, the portion of TI edge not in contact with the SC can provide a finite number of ABSs (say, \( n_A \)) to accommodate the pumped spins. After threading \( n_A \) SC flux quanta (corresponding to an incrementation of \( \phi \) by \( 2\pi n_A \)), the ABSs fail to accommodate all of the pumped spins, leading to the occupation of the other continuum modes on the edges, which corresponds to a dissipation in the Josephson junction. The way to avoid this is to break \( s_z \)-conservation which would then destroy the quantized spin Hall conductance of the system. Threading a flux quantum would then flip the Fermion parities (FPs) of the two edges.\(^2\)

Based on these two observations, we expect that a dissipationless Josephson junction in a TI requires an interaction term that breaks \( s_z \)-conservation to a junction. This agrees with what we found in the Josephson junction coupled with a quantum dot as described above, where the term \( J_A \) provides the necessary breaking of spin conservation, without which dissipation could not be avoided.

**Figure 3.** (a) A short Josephson junction on a one-dimensional TI edge. (b) Equivalent construction where now the phase difference is controlled by threading a flux through the center of the Corbino disk. (c) The single-particle Andreev spectrum for the junction with (blue thin lines) or without (black thick lines) the FI element. (d) The many-body \( \Phi R \) with (blue thin lines) or without (black thick lines) the FI element, where solid (dotted) lines indicate states with even (odd) parity.

**Figure 4.** (a) \( \Phi R \) for a junction with length \( L = \pi v_F / \Delta_0 \). The solid (dotted) lines are solutions with eigenvalues \( s_z = \pm 1 \) (b) Many-body spectrum for the same non-interacting junction. Black (dotted) lines are states with even (odd) parity. The labels indicate which of the ABSs are occupied. (c) Many-body spectrum for a junction with \( s_z \) conserving interactions. Detailed calculations are presented in Appendix B.

**IV. ROLE OF SPIN-ANISOTROPY**

To gain further insight into the necessity of breaking spin conservation to obtain a dissipationless TRS topological junction, we look into another example, first studied in Ref.\(^2\) Consider an SC-N-SC junction on a TI edge where the N portion is long enough with multiple ABSs present. The Hamiltonian is almost identical to our previous example (Eq. (4)) except that \( \Delta \) is chosen to represent a long junction as \( \Delta(x) = \Delta_0 \theta (|x| - \frac{L}{2}) \). When \( L = \pi v_F / \Delta \), three ABSs are present for all values of \( \phi \), and the single-particle and many-body \( \Phi R \) are shown in Fig. 4(a) and 4(b), respectively. The key feature in Fig. 4(b), which is needed to understand the Josephson behavior, is the four-fold degeneracy at \( \phi = \pi \). As discussed in Ref.\(^2\), splitting this degeneracy by Coulomb interactions into two two-fold Kramer’s degenerate crossings. However, as seen in Fig. 4(c), the states still continue to reach the continuum in the absence of spin conservation breaking interactions. This forbids a dissipationless ac Josephson effect in this case. To understand this, we note that the four states have different numbers of quasiparticles: \( \{ |0 \rangle, \gamma_2^\dagger |0 \rangle, \gamma_3^\dagger \gamma_1^\dagger |0 \rangle, \gamma_3^\dagger \gamma_2^\dagger \gamma_1^\dagger |0 \rangle \} \), and therefore have different values of \( s_z \). Once the four-fold degeneracy is lifted we are left with two-fold degeneracies at level crossings. These crossings are however between states of different \( s_z \). In accordance with our general arguments presented above, we find that \( s_z \)-conserving interaction terms (e.g. \( \int dx \left( a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow \right)^2 \)) in the Hamiltonian cannot split these crossings and the ground state necessarily reaches the continuum simply by adiabatic evolution. Adding \( s_z \)-breaking interactions like \( \int dx \left( a_\uparrow^\dagger a_\uparrow \partial_x a_\uparrow - a_\downarrow^\dagger a_\downarrow \partial_x a_\downarrow \right) + \text{h.c.} \), which were assumed to be comparable to the Coulomb interactions...
Finally, we stipulate that the FP must switch as \( \phi \) is advanced by \( 2\pi \), i.e.,

\[
\lambda_n(\phi + 2\pi) = -\lambda_n(\phi),
\]

which describes the topological property of the TI. More specifically, threading a flux through the TI Corbino disk (see Fig. 5) changes the \( Z_2 \) “time-reversal polarization”[29], which can be identified with the FP of each of the edges[30]. We remark that this property is not captured by the local Hamiltonian Eq. (6) of the junction.

The proof of \( 8\pi \)-periodicity for a state with even FP is illustrated in Fig. 5 and the mathematically rigorous proof is given in the Appendix C. Let us start with a non-degenerate state \( |0\rangle_0 \). Tuning \( \phi \) forward and backward by \( 2\pi \) reaches the degenerate states \( |m\rangle_{2\pi} \) and \( |\tilde{m}\rangle_{-2\pi} \) (recalling that by Eq. (6) \( |m\rangle_0 \) and \( |\tilde{m}\rangle_0 \) are odd-parity states and is a Kramers pair at \( \phi = 0 \)). Further increasing or decreasing \( \phi \) by \( 2\pi \), the state \( |n\rangle_{\pm4\pi} \) is reached. From Eq. (7) we know that \( |n\rangle_{4\pi} \neq |0\rangle_0 \) because \( |0\rangle_0 \) cannot be adiabatically connected both \( |m\rangle_{2\pi} \) and \( |\tilde{m}\rangle_{2\pi} \) at the same time (compare, e.g., light red and deep blue lines in Fig. 5). The proof for a state with odd FP proceeds in a similar way and is discussed in the appendix.

In summary, interactions and spin-conservation breaking are the two key ingredients that are required to permit a dissipationless ac Josephson effect in a TRS topological Josephson junction. In this paper, we have given a general proof that the EPR of such a dissipationless TRS topological Josephson junction is \( 8\pi \)-periodic in \( \phi \). The \( 8\pi \) periodicity arises from the combination of the flip of FP and spin over each \( 2\pi \) period. We have shown that these ingredients can be incorporated naturally in a model of a quantum dot coupled to a Josephson junction on a TI edge.

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Note added- During the preparation of the manuscript we became aware of a related recent-published work by Peng et al.[30]

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**Appendix A: Low-energy Hamiltonian for the Junction-Dot System**

Since the Hamiltonian (1) conserves the parity of electron number, we expand it in odd and even parity subspaces. We also take the limit \( U \to \infty \), which projects out the states where the quantum dot is doubly-occupied. The basis states for the odd-parity subspace are \( \left\{ \gamma^\dagger |\uparrow\rangle, \gamma^\dagger |\downarrow\rangle, d_1^\dagger |\uparrow\rangle, d_1^\dagger |\downarrow\rangle, d_2^\dagger |\uparrow\rangle, d_2^\dagger |\downarrow\rangle \right\} \), where \( |\uparrow\downarrow\rangle \) satisfies \( S^z |\uparrow\downarrow\rangle = \pm |\uparrow\downarrow\rangle \)
and $\gamma |\uparrow/\downarrow\rangle = d_{\sigma} |\uparrow/\downarrow\rangle = 0$. The Hamiltonian in this basis is

$$H^{(o)} = \begin{pmatrix} \frac{E(\phi)}{2} & 0 & 0 & 0 \\ 0 & \frac{E(\phi)}{2} & tu^* & 0 \\ 0 & tu & -\frac{E(\phi)}{2} - \frac{J}{2} + \varepsilon_d & 0 \\ tu^* & 0 & 0 & -\frac{E(\phi)}{2} + \frac{J}{2} + \varepsilon_d \end{pmatrix}, \quad (A1)$$

while in the even subspace with basis $\{ |\uparrow\rangle, |\downarrow\rangle, \gamma d^\dagger_1 |\downarrow\rangle, \gamma d^\dagger_1 |\uparrow\rangle\}$, the Hamiltonian is expanded as

$$H^{(e)} = \begin{pmatrix} -\frac{E(\phi)}{2} & 0 & 0 & -tv \uparrow \\ 0 & -\frac{E(\phi)}{2} & 0 & 0 \\ 0 & 0 & -\frac{E(\phi)}{2} + \frac{J}{2} + \varepsilon_d & 0 \\ -tv^* \uparrow & 0 & 0 & -\frac{E(\phi)}{2} + \frac{J}{2} + \varepsilon_d \end{pmatrix}. \quad (A2)$$

Finally, we note that at $\phi = 2n\pi$, $v_\uparrow = u_\uparrow = 0$ which enables us to reach the simple forms of eigenstates shown in Fig. 1(c,d).

**Appendix B: Bogoliubov-de Gennes Solution for a Long Topological Junction**

Since $s_z$ commutes with $\mathcal{H}_{\text{BdG}}$, the solutions to $\mathcal{H}_{\text{BdG}} \psi_n = E_n \psi_n$ are labeled by the “spin” index $s_z = \pm 1$. The $s_z = +1$ solutions, denoted by $\psi_n^{(+)}$, have $n_{\pm}^{(+)} = n_{\mp}^{(+)} = 0$ and

$$u_{n_\uparrow}^{(+)} = A_n e^{\text{sgn}(x)itE_n/2} e^{i\frac{\phi(x)}{2} + i\mu \tilde{x} - \sqrt{1-E^2_\Delta} |\tilde{x}| - \frac{\xi}{2}} \quad (B1a)$$

$$v_{n_\downarrow}^{(+)} = A_n e^{-\text{sgn}(x)itE_n/2} e^{-i\frac{\phi(x)}{2} + i\mu \tilde{x} - \sqrt{1-E^2_\Delta} |\tilde{x}| - \frac{\xi}{2}} \quad (B1b)$$

for $|x| > \frac{\xi}{2}$, and

$$u_{n_\uparrow}^{(+)} = A_n e^{-itE_n/2} e^{i\mu \tilde{x} + i\tilde{E}_n(x + \frac{\xi}{2})} \quad (B1c)$$

$$v_{n_\downarrow}^{(+)} = A_n e^{i\mu \tilde{x} - i\tilde{E}_n(x + \frac{\xi}{2})} \quad (B1d)$$

otherwise. Here the normalization factor is $A_n = \left( 2L + 2\xi/\sqrt{1-E^2_\Delta} \right)^{-1/2}$, and $\tilde{E}_n = \frac{E_n}{\Delta_0}$, $\tilde{\mu} = \frac{\mu}{\Delta_0}$, $\tilde{x} = \frac{x}{v_F/\Delta_0}$, $\tilde{L} = \frac{L}{\xi}$, $e^{\pm i\theta E_n} = E_n \pm i \sqrt{1-E^2_\Delta}$, where $E_n$ satisfies

$$\sqrt{1 - \left( \frac{E_n}{\Delta_0} \right)^2} \cos \left( \frac{E_nL}{v_F} - \frac{\phi}{2} \right) - \frac{E_n}{\Delta_0} \sin \left( \frac{E_nL}{v_F} - \frac{\phi}{2} \right) = 0. \quad (B2)$$

The $s_z = -1$ solutions $\psi_n^{(-)}$ with energy $-E_n$ are related to $\psi_n^{(+)}$ by particle-hole conjugation, $\psi_n^{(-)} = \Xi \psi_n^{(+)}$, where $\Xi = s y \gamma_y K$. With the solutions to $\mathcal{H}_{\text{BdG}}$ we can expand the Hamiltonian in quasiparticle operators as

$$H_0(\phi) = \sum_n E_n(\phi) \left( \gamma_\uparrow n \gamma_n - \frac{1}{2} \right) \quad (B3)$$

where $E_n$ are determined from Eq. (B2) and $\gamma_n^\dagger = \sum_{\sigma=\uparrow,\downarrow} \int dx \left( u_{n\sigma}^{(+)} a_{\sigma}^\dagger + v_{n\sigma}^{(+)} a_{\sigma} \right)$. Since only the branch of solutions with $s_z = +1$ are summed, the number of quasiparticles in a many-body state coincide with the value of $s_z$ for that state.
Figure 6. For odd-parity state we define \( m \) and \( n \) as the states that Kramers pair \( \{ \ket{0}, \ket{\bar{0}} \} \) traverses as a flux quantum is inserted in either direction. Here again deep-colored and light-colored lines are time-reversed paths of each other. Use Eq. (7) to shift the light blue and light red lines by \( 4\pi \), we get the dashed path, completing the full \( 8\pi \)-periodic path that the state follows.

### Appendix C: Mathematically Rigorous Proof of \( 8\pi \)-Periodicity

Let \( U_{\phi_2,\phi_1} = e^{-i \int_{\phi_1}^{\phi_2} H(\phi) d\phi} \) be the operator that adiabatically changes the phase from \( \phi_1 \) to \( \phi_2 \). The conditions

\[
\langle m \rangle_{\phi_2} = U_{\phi_2,\phi_1} \ket{n}_{\phi_1} \iff \ket{n}_{\phi_2} = U_{\phi_1,\phi_2} \langle m \rangle_{\phi_2}
\]

\[
\iff \langle m \rangle_{2\pi + \phi_2} = U_{2\pi + \phi_2,\phi_1} \ket{n}_{2\pi + \phi_2},
\]

for all integers \( p \), follows directly from the unitarity of \( U \) and Eq. (7).

We first consider the case where the ground state at \( \phi = 0, \) \( \ket{0} \), has even FP, i.e., \( \lambda_0(0) = 1 \) (Fig. 5). Starting with \( \ket{0} \), as the phase is adiabatically tuned to \( 2\pi \) and \( 4\pi \), the state is brought to the \( m^\text{th} \) and \( n^\text{th} \) excited states respectively, i.e.

\[
U_{2\pi,0} \ket{0} = \ket{m}_{2\pi},
\]

\[
U_{4\pi,2\pi} \ket{m}_{2\pi} = \ket{n}_{4\pi}.
\]

Now we know \( m \neq 0 \) because from Eqs. (8), \( \lambda_0(2\pi) = -\lambda_0(0) = -1 \), i.e. the FP of the ground state at \( \phi = 2\pi \) is odd. The state that \( \ket{0} \) transforms into must be even in FP, i.e. \( \lambda_m(2\pi) = \lambda_0(0) = 1 \), and from Eq. (8) we have \( \lambda_m(0) = -1 \), i.e. \( \ket{m}_{0} \) is odd in FP. Since \( H(0) \) is TRS, Kramers theorem guarantees that there is an orthogonal state \( \ket{\bar{m}}_{0} = \Theta \ket{m}_{0} \) with the same energy: \( E_{\bar{m}}(0) = E_m(0) \). Apply \( \Theta \) on Eqs. (C2) we have

\[
U_{-2\pi,0} \ket{0} = \ket{\bar{m}}_{-2\pi},
\]

\[
U_{-4\pi,2\pi} \ket{m}_{-2\pi} = \ket{n}_{-4\pi}.
\]

We have thus obtained an energy-phase relation (EΦR) schematically shown in Fig. 5 which is \( 8\pi \)-periodic if \( n \neq 0 \) (and assuming no accidental degeneracy \( E_n \neq E_0 \)). To establish this, we use Eq. (C1) to derive from Eq. (C3b)

\[
U_{2\pi,0} \ket{n}_{0} = \ket{\bar{m}}_{2\pi},
\]

which can be compared with Eq. (C2a). Since \( U_{0,2\pi} \ket{m}_{2\pi} = \ket{0} \), we know \( U_{0,2\pi} \ket{\bar{m}}_{2\pi} \neq \ket{0} \) (as \( \ket{0} \) cannot be adiabatically connected to two states at \( \phi = 2\pi \)) and from Eq. (C4) this means \( n \neq 0 \). The \( 8\pi \)-periodic EΦR is therefore established.

The case of \( \lambda_0(0) = -1 \) can be considered in a similar fashion (Fig. 6). Let \( \ket{0} \) and \( \ket{\bar{0}} \) be the Kramers pair of degenerate ground states at \( \phi = 0 \). Define \( \ket{m}_{2\pi} \) and \( \ket{n}_{-2\pi} \) be respectively the states that \( \ket{0} \) transforms into as \( \phi \) is tuned from 0 to \( \pm 2\pi \), respectively. We have

\[
U_{2\pi,0} \ket{0} = \ket{m}_{2\pi}, 
U_{-2\pi,0} \ket{0} = \ket{n}_{-2\pi},
\]

\[
U_{2\pi,4\pi} \ket{\bar{0}}_{4\pi} = \ket{m}_{2\pi},
U_{-2\pi,4\pi} \ket{\bar{0}}_{-4\pi} = \ket{n}_{-2\pi},
\]

where the second line is obtained by applying \( \Theta \) and Eq. (C1b) subsequently on the first line. Finally, we use Eq. (C1b) to derive, from the last relation above,

\[
U_{2\pi,0} \ket{\bar{0}}_{0} = \ket{n}_{2\pi}.
\]
Upon comparison with the first relation of Eq. (C5), this shows that $n \neq m$ and hence the full $8\pi$-periodic cycle of $E\Phi R$ shown in Fig. 6 (with $E_n \neq E_m$) is established.

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