Finite-difference representations of the degenerate affine Hecke algebra

D. Uglov

Department of Physics, State University of New York at Stony Brook
Stony Brook, NY 11794-3800
e-mail: denis@max.physics.sunysb.edu

September 26, 1994

Abstract

The representations of the degenerate affine Hecke algebra in which the analogues of the Dunkl operators are given by finite-difference operators are introduced. The non-selfadjoint lattice analogues of the spin Calogero-Sutherland hamiltonians are analysed by Bethe-Ansatz. The $sl(m)$-Yangian representations arising from the finite-difference representations of the degenerate affine Hecke algebra are shown to be related to the Yangian representation of the 1-d Hubbard Model.
1 Introduction

The Calogero-Sutherland model and its spin generalizations discovered in [1] have recently come to occupy a prominent place among exactly solvable 1+1 dimensional many-particle dynamical systems. The chief interest of these models for solid-state physics lies in the fractional character of their quasi-particle excitations. On the other hand remarkable relations were found between these models and the long-range interacting Haldane-Shastry spin chains and Conformal Field Theory [1, 2].

One of the attractive features of the Calogero-Sutherland model is its relative simplicity as compared to, for example, the models solvable by QISM such as the Heisenberg spin chain. In particular the wave functions are sufficiently simple as to permit explicit computation of some correlation functions [9, 10]. This simplicity is ultimately due to the close relationship that exists between these models and the representation of the degenerate affine Hecke algebra given by the differential Dunkl operators [1, 5, 11]. This algebra also gives rise to the Yangian symmetry in the higher-spin Calogero-Sutherland model and the Haldane-Shastry spin chain [1].

The main aim of the present letter is to report upon the finite-difference representations of the degenerate affine Hecke algebra. In these representations the analogues of the differential Dunkl operators are difference operators that are acting on the functions of several integer variables running through all integer numbers. There are a pair of such representations - one contains the left finite-differences, another - the right ones. These representations are referred to as the “left” and the “right” in what follows. The left and right operators do not commute. Like in the continuum case the representations of the degenerate affine Hecke algebra generate representations of the $sl(m)$-Yangian in the space of wave functions of the lattice bosons or fermions with $m$ internal degrees of freedom. The generators of such Yangian representations can in principle be expressed in terms of the creation and annihilation operators for lattice bosons or fermions. In the case of the fermions of spin
1/2 the expressions obtained this way are closely related to the fermionic representation of $sl(2)$-Yangian described in [12].

As for the spin Calogero-Sutherland model [1] the quantum determinant of the Yangian can be used to obtain a family of mutually commuting operators that also commute with the Yangian. These operators then are natural candidates for the discrete counterparts of the Calogero-Sutherland hamiltonian. The simplest nontrivial operators that one obtains in this fashion from the left and right underlying representations of the degenerate affine Hecke algebra are found to be the left- and the right-hopping parts (denote them by $h_L$ and $h_R$) of the higher spin Hubbard hamiltonians (denote such a hamiltonian by $H_H$) or of their bosonic analogues: $H_H = h_L + h_R$; $(h_L)^\dagger = h_R$. The operators $h_L, h_R$ are mutually adjoint and do not commute i.e. they are not normal and cannot be brought to the diagonal form like the Calogero-Sutherland hamiltonian. Still one can analyse these operators separately and find their characteristic numbers (that are in general complex) and corresponding eigenfunctions. This is done with the aid of the coordinate or algebraic Bethe-Ansatz. The continuum limit taken in the explicit eigenfunctions of the operators $h_L, h_R$ then yields the eigenfunctions of the continuum Hamiltonians that describe particles with $\delta$-interaction [13].

The contents of the letter are as follows: In sec. 2 the definition of the degenerate affine Hecke algebra is recalled and its finite-difference representations are introduced. In sec. 3 the Yangian representations arising from the difference representations of the degenerate affine Hecke algebra are discussed and compared to the Yangian representation of the Hubbard model. Then the operators $h_L$ and $h_R$ are introduced. Finally these operators are analysed by algebraic and coordinate Bethe-Ansatz in sec. 4.
2 The finite-difference representations of the degenerate affine Hecke algebra

The degenerate affine Hecke algebra is an extension of the symmetric group $S_N$ by $N$ additional generators $d_i$, $1 \leq i \leq N$. Denote the $N-1$ generators of $S_N$ by $K_{ii+1}$, $1 \leq i \leq N-1$. The defining relations of the degenerate affine Hecke algebra then read [1, 10]:

\[
K_{ii+1}K_{i+1i}K_{ii+1} = K_{i+1i}K_{ii+1}K_{ii+1};
\]

\[
K_{ii+1}^2 = 1;
\]

\[
[K_{ii+1}, d_k] = 0, \ k \neq i, i + 1;
\]

\[
K_{ii+1}d_i - d_{i+1}K_{ii+1} = \lambda;
\]

\[
[d_i, d_j] = 0.
\]

(1)

Where $\lambda$ is a complex number - parameter of the algebra.

The relations (1) have a well-known representation [3, 11] in which the generators act on the functions of $N$ variables $\{z_i\}_{1 \leq i \leq N}$. The $K_{ii+1}$ are represented by permutation of the arguments: $(K_{ij}f)(z_1, \cdots, z_i, \cdots, z_j, \cdots, z_N) = f(z_1, \cdots, z_j, \cdots, z_i, \cdots, z_N)$; and the $d_i$ are represented by the differential operators (Dunkl operators):

\[
d_i \rightarrow \partial z_i + \lambda \sum_{j \neq i} \frac{z_i}{z_i - z_j} K_{ij} - \lambda \sum_{j < i} K_{ij}.
\]

(2)

Let now $f(x_1, \cdots, x_N)$ be a function of variables $x_i$, $1 \leq i \leq N$ and each of the $x_i$ runs through all integer numbers. In the linear space of such functions one has two representations of the degenerate affine Hecke algebra. Introduce some notations. Let $\Delta_i^\pm$ denote the finite-differences: $(\Delta_i^\pm f)(\cdots, x_i, \cdots) = f(\cdots, x_i \pm 1, \cdots)$, and $\theta_i^\pm$ - the step-functions: $\theta_i^\pm \equiv \theta^\pm(x_i - x_j)$ where:

\[
\theta^+(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}
\]

\[
\theta^-(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}
\]
Let $K_{ij}$ be as before the permutation of the $i$-th and $j$-th coordinates. Then the operators:

$$d_i = \Delta_i^+ + \lambda \sum_{j \neq i} \theta_{ij}^+ K_{ij} - \lambda \sum_{j < i} K_{ij}$$

(3)

satisfy the defining relations of the degenerate affine Hecke algebra (1). In what follows this representation will be referred to as the “left” representation. There is another - “right” representation in which the $d_i$-s are given by:

$$d_i = \Delta_i^- + \lambda \sum_{j \neq i} \theta_{ij}^- K_{ij} - \lambda \sum_{j < i} K_{ij}$$

(4)

The proof that the difference operators (3,4) satisfy the relations (1) is straightforward and uses the following easily verified properties of the step-functions of integer arguments and finite-differences:

$$\theta_{ij}^+ \theta_{ik}^+ + \theta_{ik}^+ \theta_{jk}^- - \theta_{ij}^+ \theta_{jk}^+ = \theta_{ik}^+ ,$$

$$(\Delta_i^+ \theta_{ji}^+) - \theta_{ji}^+ = \mp \delta_{x_i x_j} ,$$

$$\theta_{ij}^+ + \theta_{ji}^+ = 1 \pm \delta_{x_i x_j} ,$$

here $\delta_{xy}$ is the Kronecker delta. Note, that for the second equation above to hold it is essential that the $x_i$-s run through all integer numbers.

3 The Yangian representations generated by the finite-difference representations of the affine Hecke algebra

Once the representations of the degenerate affine Hecke algebra are known it is straightforward to obtain out of them representations of $sl(m)$-Yangians. The way to do this for the finite-difference representations is completely analogous to the continuum case considered in detail in [1]. The Yangian transfer-matrices then can be in principle expressed in terms of the lattice bosons or fermions. In the case of the $sl(2)$-Yangian the fermionic transfer-matrices that one gets from the left and the right representations of the difference analogs of
the Dunkl operators (3,4) yield the fermionic representation of $Y(sl(2))$ found earlier in the Hubbard model [12]. The operators $d_i$ can also be used to derive the (left or right) finite difference analogs of the spin Calogero-Sutherland hamiltonians.

Let now $f(x_i, s_i; \cdots; x_N, s_N)$ be a function of integer coordinates $x_i$ and spins $s_i$, $1 \leq s_i \leq m$; and $P_{ij}$ - the spin permutation operator. The bosonic functions then are characterized by the condition $(K_{ij} - P_{ij})f = 0$; and the fermionic - by $(K_{ij} + P_{ij})f = 0$.

As it was shown in [1], the relations (1) guarantee that the transfer-matrices:

$$T_0^\pm(u) = (I \pm \frac{\lambda P_{01}}{u - d_1})(I \pm \frac{\lambda P_{02}}{u - d_2}) \cdots (I \pm \frac{\lambda P_{0N}}{u - d_N}),$$

where the subscript 0 refers to the auxiliary $m$-dimensional vector space and $u$ is the spectral parameter preserve the space of bosonic (+) or fermionic (-) wave functions. Since the transfer matrices satisfy the Yang-Baxter relations:

$$(u - v \pm \lambda P_{00'})T_0^\pm(u)T_0^\pm(v) = T_0^\pm(v)T_0^\pm(u)(u - v \pm \lambda P_{00'})$$

they provide representations of the $sl(m)$-Yangians in the spaces of bosonic (+) or fermionic (-) wave-functions. Somewhat freely we call the algebra (6) a Yangian even though the quantum determinant of $T$ is not assumed to be equal to one.

From the defining relations of the degenerate affine Hecke algebra (1) it also follows that the polynomial $C(u) = (u - d_1)(u - d_2) \cdots (u - d_N)$ commutes with the operators of coordinate permutations $K_{ij}$ and all the operators $d_1, \cdots, d_N$ [1]. Hence $C(u)$ preserves the spaces of bosonic and fermionic wave-functions and in each of these spaces it commutes with the corresponding Yangian transfer-matrix $T^+$ or $T^-$. When the algebra (1) is represented by the Dunkl operators (2) $C(u)$ is the generating function for the spin Calogero-Sutherland hamiltonians and their higher-derivative conserved charges [1]. In the case when the $d_i$-s are the finite-difference operators (3) or (4) one can consider the corresponding $C(u)$ as the generating function of the difference analogs of the Calogero-Sutherland hamiltonians.
The simplest non-trivial operator that is given by one of the coefficients of the polynomial $C(u)$ is just the sum of all $d_i$-s. Since on the lattice one has a left-right pair of representations of the degenerate affine Hecke algebra there are a pair of such operators: $h_L = \sum_{1 \leq i \leq N} d^l_i$ and $h_R = \sum_{1 \leq i \leq N} d^r_i$ where $d^l_i$ and $d^r_i$ mean the operators given by correspondingly (3) and (4). Explicit forms of the operators $h_L$ and $h_R$ look as follows:

$$h_L = \sum_{1 \leq i \leq N} \Delta^+_i + \lambda \sum_{1 \leq i < j \leq N} \delta_{x_i,x_j} K_{ij},$$

$$h_R = \sum_{1 \leq i \leq N} \Delta^-_i - \lambda \sum_{1 \leq i < j \leq N} \delta_{x_i,x_j} K_{ij}.$$  \hspace{1cm} (7,8)

The permutations $K_{ij}$ appearing in the above expressions after the $\delta$-symbols will from now on be dropped since $(\delta_{x_i,x_j} K_{ij} f)(x_1, \cdots) = (\delta_{x_i,x_j} f)(x_1, \cdots)$.

Since the operators $h_L, h_R$ preserve the space of the bosonic or fermionic wave functions one can rewrite them in the secondly quantized form:

$$h_L = \sum_{s \in \mathbb{Z}} \sum_{\beta=1}^m a^\beta_s a^{\beta+1}_s + \frac{\lambda}{2} \sum_{s \in \mathbb{Z}} \left( n^2_s - n_s \right),$$

$$h_R = \sum_{s \in \mathbb{Z}} \sum_{\beta=1}^m a^{\beta+1}_s a^\beta_s - \frac{\lambda}{2} \sum_{s \in \mathbb{Z}} \left( n^2_s - n_s \right).$$  \hspace{1cm} (9,10)

Here the $a^\beta_s, a^\beta_t$ are either bosonic or fermionic creation and annihilation operators on the 1-dimensional lattice: $[a^\beta_s, a^\gamma_t]_\mp = \delta_{st} \delta_{\beta\gamma}; [a^\beta_s, a^\gamma_t]_\mp = 0$; $s, t \in \mathbb{Z}$, $1 \leq \beta, \gamma \leq m$; and $n_s = \sum_{1 \leq \beta \leq m} a^\beta_s a^{\beta+1}_s$.

In the subspace with the fixed number of particles $N$ the operators (9,10) evidently reduce to (7,8). Notice that one can get the “+” sign in front of the coupling constant $\lambda$ in (10) and (8) by taking the $h_R$ generated by the right representation of the operators $d_i$ (4) in which $\lambda$ is replaced by $-\lambda$. These $d_i$-s then will satisfy the relations (1) with $\lambda$ replaced by $-\lambda$. Subsequently it is assumed that the operator denoted $h_R$ is given by (10) with $\lambda \rightarrow -\lambda$. Then if $H_H$ is the hamiltonian of the Hubbard model of spin $(m-1)/2$ or its bosonic counterpart one has: $H_H = h_L + h_R$.

The operators (9,10) commute with the transfer-matrices (5) which in principle can be
expressed in terms of the creation/annihilation operators. In particular if the transfer matrix
is represented as the series:

\[ T(u) = I + \lambda \sum_{k \geq 0} \frac{1}{u^{k+1}} t^{(n)} \]  

(11)

then the bosonisation(fermionization) of \( t^{(0)} \) and \( t^{(1)} \) yields the bosonic or fermionic representations of the Yangian Serre relations \[4\]. Below this is done for the case of the \( sl(2) \)-Yangian. This case corresponds to \( m = 2 \) in the operators (9,10).

The auxiliary linear space being 2-dimensional \(( m = 2 \) introduce the operators:

\[
J_{a}^{(0)} , \ J_{a}^{(1)} , \ a = 1, 2, 3
\]

\[
J_{a}^{(k)} = tr_{0}(s_{a}^{o} t^{(k)}) , \ k = 0, 1 .
\]

(12)

Where \( s_{0}^{a} \equiv \frac{1}{2} \sigma_{0}^{a} \) are acting in the auxiliary space as designated by the subindex 0, and \( \sigma^{a} \) are the Pauli matrices.

Explicitly these operators read (here \( \pm \) means that the transfer matrix in (11) comes
from the left(+) or the right(-) representation of the degenerate affine Hecke algebra; and \( \kappa = 1(-1) \) for the bosonic(fermionic) case):

\[
J_{a}^{(0)} = \sum_{1 \leq i \leq N} s_{i}^{a} , \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

In these expressions it is understood that the operators \( s_{i}^{a} \) act upon i-th the spin variable of
the wave-functions and the notation \( \varepsilon(x) \) is introduced for the step function: \( \varepsilon(x) = 1(-1) \),
if \( x > (<)0 \); and \( \varepsilon(0) = 0 \).

It follows from the equation (6) that the operators (13,14) satisfy the \( sl(2) \)-Yangian Serre
relations \[4\]:

\[
[J_{a}^{(0)}, J_{b}^{(0)}] = i\varepsilon_{abc} J_{c}^{(0)}
\]
\[ [J_a^{(0)}, J_b^{(1)}] = i\epsilon_{abc}J_c^{(1)} \]

\[ [J_a^{(2)}, J_b^{(1)}] + [J_b^{(2)}, J_a^{(1)}] = i\lambda^2(\epsilon_{acd}\{J_b^{(0)}, J_c^{(0)}, J_d^{(1)}\} + \epsilon_{bcd}\{J_a^{(0)}, J_c^{(0)}, J_d^{(1)}\}). \quad (15) \]

Where
\[ J_a^{(2)} = -\frac{i}{2}\epsilon_{abc}[J_b^{(1)}, J_c^{(1)}] \]

, and \{.,.,.\} stands for symmetrization. Notice that the last term in the right-hand side of (14) can be dropped without affecting the Serre relations, this is done below in the eq. (17).

Finally the operators \( J_a^{(0)}, J_a^{(1)}, a = 1, 2, 3 \) can be expressed in terms of the creation/annihilation operators (either bosonic or fermionic). As before there are two representations (left/right) which are distinguished by the superscripts +/- over the generators:

\[
J_a^{(0)\pm} = \sum_{t \in \mathbb{Z}} a_t^{\beta\dagger} (s^a)_{\beta\gamma} a_t^\gamma, \\
J_a^{(1)+} = \sum_{t \in \mathbb{Z}} a_t^{\beta\dagger} (s^a)_{\beta\gamma} a_{t+1}^\gamma - \frac{i\kappa \lambda}{2} \sum_{r \neq t} \varepsilon(r - t)\epsilon_{abc} a_r^{\beta\dagger} (s^b)_{\beta\gamma} a_t^\gamma a_t^{\mu\dagger} (s^c)_{\mu\nu} a_r^\nu + \\
+ \frac{\kappa \lambda}{2} \sum_{r \in \mathbb{Z}} (n_r - 1) a_r^{\beta\dagger} (s^a)_{\beta\gamma} a_r^\gamma, \\
J_a^{(1)-} = \sum_{t \in \mathbb{Z}} a_t^{\beta\dagger} (s^a)_{\beta\gamma} a_{t+1}^\gamma - \frac{i\kappa \lambda}{2} \sum_{r \neq t} \varepsilon(r - t)\epsilon_{abc} a_r^{\beta\dagger} (s^b)_{\beta\gamma} a_t^\gamma a_t^{\mu\dagger} (s^c)_{\mu\nu} a_r^\nu - \\
- \frac{\kappa \lambda}{2} \sum_{r \in \mathbb{Z}} (n_r - 1) a_r^{\beta\dagger} (s^a)_{\beta\gamma} a_r^\gamma. \quad (17) \]

The \( \kappa \) above refers to the bosonic (\( \kappa = 1 \)) or fermionic (\( \kappa = -1 \)) cases. The operators \( J_a^{(0,1)+} \) commute with the operator \( h_L \) (9) and \( J_a^{(0,1)-} \) commute with the operator \( h_R \) for \( m = 2 \).

It is a special feature of the case of the fermions of the spin-1/2, that is the case relevant to the Hubbard model, that the operators \( J_a^{(0)}, J_a^{(1)} \) given by:

\[
J_a^{(0)} = J_a^{(0)\pm}, \\
J_a^{(1)} = J_a^{(1)+} - J_a^{(1)-}|_{\lambda \to -\lambda}. 
\]
still satisfy the Serre relations (15) with the $\lambda$ changed to $2\lambda$. These operators coincide with the generators of the “spin” Yangian in the Hubbard model \cite{12}.

4 Characteristic numbers and eigenvectors of the operators $h_L, h_R$

Since the finite-difference operators $h_L, h_R$ are not normal with respect to the usual inner product in the Fock space, one cannot diagonalize them. Nevertheless the characteristic numbers of these operators and corresponding eigenvectors can be found by Bethe-Ansatz. Such eigenvectors apparently are the “lowest vectors” in Jordan chains \cite{3} corresponding to the characteristic numbers.

First, consider the pair of operators $h_L$ and $h_R$:

$$h_L = \sum_{1 \leq s \leq M} \sum_{\beta=1}^{m} a_{s}^{\beta\dagger} a_{s+1}^{\beta} + \frac{\lambda}{2} \sum_{1 \leq s \leq M} (n_s^2 - n_s), \quad h_R = h_L^\dagger$$

as defined on a periodic chain of length $M$: $M + 1 \equiv 1$. Recall, that the creation/annihilation operators here can be either bosonic or fermionic. Besides the hermitian conjugation, the operators $h_L, h_R$ are related by the spatial reflexion: $\Pi: a^\beta_s \rightarrow a^\beta_{M-s+1} \Pi$; $h_L = \Pi h_R \Pi$. In this setting the eigenvalues and the eigenvectors of the operators (18) can be constructed by means of the Algebraic Bethe-Ansatz. The basic ingredients of the Algebraic Bethe-Ansatz for the operators (18) i.e. the $R$-matrices, $L$-operators and the trace identities will now be described.

Due to the great similarity between the bosonic and fermionic cases they are considered in parallel. Both bosonic and fermionic local $L$-operators related to the site with the number $s$ are $(1+m) \times (1+m)$ matrices with operator coefficients expressed in terms of the creation/annihilation operators $a_{s}^{\beta\dagger}$, $1 \leq \beta \leq m$:

$$L(u)_s = \begin{pmatrix} u + \lambda n_s & (a_s)_m \\ \kappa \lambda (a_s)^\dagger_m & I_{m \times m} \end{pmatrix}$$

(19)
Where \((a_{s})_{m}\) is the row: \((a_{1}, \ldots, a_{m})\), \(I_{m \times m} - m \times m\) identity matrix, \(u\) -the spectral parameter and \(\kappa = +(-)\) for bosonic(fermionic) case. For the fermions the matrix (19) is graded: if \(L = (L_{ab})_{1 \leq a, b \leq 1+m}\) then \(p(a) = 0\), if \(a = 1\); \(p(a) = 1\), if \(2 \leq a \leq 1 + m\).

The \(L\)-operators (19) satisfy the Yang-Baxter relation with the rational \(R\)-matrix [8] which is \(sl(1+m)\)-symmetric for the bosons and \(sl(1|m)\)-symmetric for the fermions:

\[
(\lambda + (u - v)P)L(u)_{s} \otimes L(v)_{s} = L(v)_{s} \otimes L(u)_{s}(\lambda + (u - v)P)
\]

\(P\) here is the permutation(graded permutation) operator in \(C^{1+m} \otimes C^{1+m}\) and \(\otimes\) means tensor(graded tensor) product for correspondingly the bosonic(fermionic) case.

The \(L\)-operators are used to construct the transfer-matrices \(\tau(u)\):

\[
\tau_{Bose}(u) = \text{tr}(L_{M}(u)L_{M-1}(u) \cdots L_{1}(u)), \quad \tau_{Fermi}(u) = \text{str}(L_{M}(u)L_{M-1}(u) \cdots L_{1}(u)),
\]

(21)

And the operators \(h_{L}\) are then obtained from the transfer-matrices by means of the trace identities:

\[
\tau(u) = u^{M} + u^{M-1}\lambda N + u^{M-2}\lambda (h_{L} - \frac{\lambda}{2}N + \frac{\lambda}{2}N^{2}) + O(u^{M-3}),
\]

(22)

where \(N\) is the particle number. The trace identities are the same in form for the bosons and for the fermions.

The operators \(h_{R} = h_{L}^{\dagger}\) are obtained in the same way from the transfer-matrices \(\tau(\overline{u})^{\dagger} = \Pi \tau(u)\Pi = (s)tr(L_{1}(u)L_{2}(u), \cdots, L_{M}(u))\).

Since the \(L\)-operators (19) become lower-triangular when applied to the local Fock pseudovacuum, the well-known machinery of the nested Algebraic Bethe Ansatz [6] can be employed to construct a family of eigenvectors of the operators \(h_{L}, h_{R}\) and the corresponding eigenvalues. The structure of these eigenfunctions and eigenvalues, however, is more transparent if one uses the coordinate Bethe-Ansatz approach [3].

10
The coordinate Bethe-Ansatz gives the family of eigenfunctions $f(x_1, \cdots, x_N|\{z_i\})$ of the operator $h_L$ (7) in the form of Bethe sums parametrized by $N$ complex numbers $z_1, \cdots, z_N : h_L f(\{z_i\}) = (\sum_{1\leq i\leq N} z_i) f(\{z_i\})$. In the sector $x_1 \leq x_2 \leq \cdots \leq x_N$ the explicit expression for the eigenfunction is:

\[
f(x_1, \cdots, x_N|\{z_i\}) = \sum_{R \in S_N} \phi(R) z_1^{x_1} z_2^{x_2} \cdots z_N^{x_N},
\]

(23)

\[
\phi((s \leftrightarrow s + 1)R) = -\frac{\lambda - \kappa (z_{R_s} - z_{R_{s+1}}) P_{s,s+1}}{z_{R_s} - z_{R_{s+1}} + \lambda} \phi(R)
\]

(24)

Where it is understood that the $f$ and $\phi$ depend also on $N$ spins and the $f$ is either bosonic ($\kappa = 1$) or fermionic ($\kappa = -1$). On the infinite lattice the above expressions provide eigenfunctions of $h_L$ in the algebraic sense. If lattice is periodic than as usual the restrictions on the set of $z_i$ and the functions of spins $\phi(R)$ arise from the periodic boundary conditions.

The eigenfunctions (23) resemble very much the exact eigenfunctions for the continuum particles with $\delta$-interaction [13] and converge to them in the continuum limit:

\[
z_i = e^{i\epsilon k_i} , x_i = x_i/\epsilon , \lambda = \epsilon c , \epsilon \to 0.
\]

(25)

Here $k_i$ are the set of momenta, $x_i$ - continuous coordinates of the particles and $c$ - the coupling constant for the continuum particles with $\delta$ -interaction.

## 5 Conclusions

In this letter we described the finite-difference representations of the affine Hecke algebra and the lattice analogues of the spin Calogero-Sutherland hamiltonians. These lattice operators are not selfadjoint yet their explicit eigenfunctions, which can be constructed by means of Bethe-Ansatz, go over in the continuum limit into the eigenfunctions of the continuum particles with $\delta$-interaction. The Yangian representations that are generated by the finite-difference representations of the degenerate affine Hecke algebra were seen to produce the
Yangian representation of the 1-d Hubbard model. The structure of the Yangian generators in the lattice analogues of the spin Calogero-Sutherland Hamiltonians is very close to that one in the Hubbard model yet unlike the former models the Hubbard model is not known to be solvable by Algebraic Bethe-Ansatz, which is detrimental to the understanding of its correlation functions. It would be interesting then to see if the relation between the Yangians mentioned above cannot be used to develop Algebraic Bethe-Ansatz for the Hubbard model.

Acknowledgements

I am grateful to F.H.L. Essler and V.E. Korepin for stimulating discussions. This work was partially supported by the NSF grant 9309888.

References

[1] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier, J. Phys. A 26 (1993) 5219

[2] D. Bernard, V. Pasquier and D. Serban, “Spinons in Conformal Field Theory” preprint SPhT/94/039 (hep-th-9404050)

[3] H. Bethe, Z. Phys. 71 (1931) 205

[4] V.G. Drinfel’d, in: Proc. ICM (Berkeley Univ. Press, Berkeley, CA, 1986); A. Leclair and F.A. Smirnov, Int. J. Mod. Phys. A 7 (1992) 2997

[5] C.F. Dunkl, Trans. Am. Soc. 311 (1989) 167

[6] I. Gokhberg, P. Lancaster, L. Rodman, “Invariant subspaces of matrices with applications”, New York: Wiley 1986

[7] P. Kulish, Dokl. Akad. Nauk USSR 255 (1980) 323; O. Babelon, H.J. de Vega, C. M. Viallet, Nucl. Phys. B200 (FS4) (1982) 266

[8] P. Kulish, E. Sklyanin, J. Sov. Math. 19 (1982) 1596
[9] F.Lesage, V.Pasquier and D.Serban, “Dynamical correlation functions in the Calogero-Sutherland model” Saclay preprint, April-94 (hep-th-9405008)

[10] V.Pasquier, “A lecture on the Calogero-Sutherland models” preprint SPhT/94-060 (hep-th-9405104)

[11] A.P.Polychronakos, Phys.Rev.Lett. 69 (1992) 703

[12] D.Uglov and V.Korepin, Phys.Lett.A 190 (1994) 238

[13] C.N. Yang, Phys.Rev.Lett. 20 (1968) 1312