Some Methods of Minimization of Calculations in High Energy Physics*

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Abstract

Two approaches to calculations’ minimization in High Energy Physics are considered. The first one is the method of covariant calculations for the amplitudes of processes with polarized Dirac particles. The second one connects with the possibility to reduce the expressions for the traces of products of ten and more Dirac $\gamma$-matrices.

1 Introduction

It is well known that the high order calculation of the observables within the perturbative theory turns to the serious difficulties (especially, if we take into account the polarization effects). The reason is in necessity to evaluate the traces of products of great number of Dirac $\gamma$-matrices. So the problems arise in the both cases of analytical and numerical calculations of the different physical quantities (for cumbersomeness of their expressions). It is clear that

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direct calculation of the processes' amplitudes (see Section II) is one of the chances to turn over such a problem.

In Section III the possibility to reduce the expressions for the traces of products of ten and more Dirac $\gamma$-matrices is discussed.

Some details can be found in the papers [1] – [3].

2 The method of covariant calculation of the amplitudes of processes with the polarized Dirac particles

There were a lot of attempts to calculate the amplitudes covariantly (see, for instance, [4] – [14]). However, expressions that have been obtained there are unsuitable for calculations with interfering diagrams. The scheme extending the mentioned results will be presented below. It is concretized so as one needs in order to avoid the problems with interference.

There is even number ($2N$) of fermions in initial and final state for any reaction with Dirac particles. Therefore every diagram contains $N$ nonclosed fermion lines. The expression

$$M_{if} = \bar{u}_f Q u_i$$

(1)

corresponds to every line in amplitude of process, where $u_i, \ u_f$ are Dirac bispinors for free particles (for definiteness, we anticipate that fermions are particles. However, obtained results are true in case of both fermions are antiparticles or one fermion is particle and another fermion is antiparticle).

$$\bar{u} = u^+ \gamma^0 .$$

$Q$ is matrix operator which characterize interaction. Operator $Q$ is expressed as linear combination of products of Dirac $\gamma$-matrices (or of its contractions with 4-vectors) and can have any number of free Lorentz indexes.

For calculating $M_{if}$ we use the following scheme:

$$M_{if} = \bar{u}_f Q u_i = (\bar{u}_f Q u_i) \cdot \frac{\bar{u}_i Z u_f}{\bar{u}_i Z u_f} = \frac{Tr(Q u_i \bar{u}_i Z u_f \bar{u}_f)}{\bar{u}_i Z u_f}$$

$$\simeq \frac{Tr(Q u_i \bar{u}_i Z u_f \bar{u}_f)}{|\bar{u}_i Z u_f|} = \frac{Tr(Q u_i \bar{u}_i Z u_f \bar{u}_f)}{[Tr(Z u_i \bar{u}_i Z u_f \bar{u}_f)]^{1/2}} = M_{if}$$

(2)

where $Z$ is arbitrary $4 \times 4$ matrix ,

$$\bar{Z} = \gamma^0 Z + \gamma^0$$

(the symbol \(\simeq\) stands for "an equality to within a phase factor sign").
The projection operators are substituted for \( u\bar{u} \) in (2). For particle with mass \( m \):

\[
u(p, n)\bar{u}(p, n) = \frac{1}{4m}(\hat{p} + m)(1 + \gamma_5 \hat{n}) = \mathcal{P} \tag{3}
\]

where \( \hat{p} = \gamma_\mu p^\mu, \quad p^2 = m^2, \quad n^2 = -1, \quad pn = 0, \quad \bar{u}u = 1, \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 . \)

(We use the same metric as in the book [13]:

\[
a^\mu = (a_0, \vec{a}), \quad a_\mu = (a_0, -\vec{a}), \quad ab = a_\mu b^\mu = a_0 b_0 - \vec{a}\vec{b} .
\]

For massless particle projection operator is following:

\[
u_\pm(q)\bar{u}_\pm(q) = \frac{1}{2}(1 \pm \gamma_5)\hat{q} = \mathcal{P}_\pm \tag{4}
\]

where \( q^2 = 0, \quad \bar{u}_\pm \gamma_\mu u_\pm = 2q_\mu \) (signs \( \pm \) correspond to helicity of particle).

In the articles [4], [5], [7] one chooses

\[
Z = 1 .
\]

The results of article [6] come to the same approach for processes with massless particles. In [3] one proposes

\[
Z = \gamma_5
\]

too. The results of article [8] come to the same choice. The results of articles [9], [10] correspond to choice

\[
Z = 1 + \gamma^0 .
\]

The results of [11], [12] come to

\[
Z = 1 + \hat{r}
\]

(where \( r \) is arbitrary 4-momentum, such as \( r^2 = 1 \)). In these papers for 4-vectors, which determine axes of spin projections, one uses

\[
n_i = \frac{m_i^2 p_f - (p_i p_f)p_i}{m_i[(p_i p_f)^2 - m_i^2 m_f^2]^{1/2}}, \quad n_f = -\frac{m_f^2 p_i - (p_i p_f)p_f}{m_f[(p_i p_f)^2 - m_i^2 m_f^2]^{1/2}} .
\]

The results of article [13] come to

\[
Z = \hat{k}_1 \hat{k}_2
\]

(where \( k_1, k_2 \) are arbitrary 4-vectors, such as \( k_1^2 = k_2^2 = 0 \) for processes with massless particles. The results of article [14] come to

\[
Z = \hat{n}\hat{p}
\]
(where $n, p$ are arbitrary 4-vectors, such as $n^2 = -1$, $p^2 = 0$, $(pn) = 0$) for processes with massless particles.

However all expressions for amplitudes [as it follows from (2)] are known to within a phase factor:

$$\mathcal{M}_{if} = M_{if} \cdot \frac{\bar{u}_i Z u_f}{|\bar{u}_i Z u_f|}.$$  \hspace{1cm} (5)

It is obviously that this circumstance creates no problems, when we calculate amplitude for alone diagram. Really

$$\mathcal{M}_{if} (\mathcal{M}_{if})^* = \frac{[\text{Tr}(Qu_i \bar{u}_i Z u_f \bar{u}_f)]^*}{[\text{Tr}(Z u_i \bar{u}_i Z u_f \bar{u}_f)]^{1/2}} = \frac{[(\bar{u}_f Qu_i)(\bar{u}_i Z u_f)]^*}{[\text{Tr}(Z u_i \bar{u}_i Z u_f \bar{u}_f)]^{1/2}}.$$  \hspace{1cm} (6)

$$\mathcal{M}_{if} (\mathcal{M}_{if})^* = \frac{\text{Tr}(Qu_i \bar{u}_i Z u_f \bar{u}_f) \cdot \text{Tr}(Z u_i \bar{u}_i Q u_f \bar{u}_f)}{\text{Tr}(Z u_i \bar{u}_i Z u_f \bar{u}_f)} = \text{Tr}(Qu_i \bar{u}_i Q u_f \bar{u}_f) = (\bar{u}_f Qu_i)(\bar{u}_i Q u_f)^* = M_{if} (M_{if})^* = |M_{if}|^2.$$  

For calculating $|M_{if}|^2$ we used identity

$$\text{Tr}(Au_i \bar{u}_i Bu_f \bar{u}_f) \cdot \text{Tr}(Cu_i \bar{u}_i Du_f \bar{u}_f) = (\bar{u}_f Au_i)(\bar{u}_i Bu_f)(\bar{u}_f Cu_i)(\bar{u}_i Du_f)$$

$$\equiv (\bar{u}_f Au_i)(\bar{u}_i Du_f)(\bar{u}_f Cu_i)(\bar{u}_i Bu_f) = \text{Tr}(Au_i \bar{u}_i Du_f \bar{u}_f) \cdot \text{Tr}(Cu_i \bar{u}_i Bu_f \bar{u}_f)$$  \hspace{1cm} (7)

where $A, B, C, D$ are arbitrary $4 \times 4$-matrices.

However, in general case, presence of unknown phase factor does not enable formula (3) to be used for calculation of amplitudes of processes which proceed in a few channels since expressions for amplitudes which correspond to different channels are multiplied by different phase factors. Besides that, ambiguity of the type $0$ can appear during the calculations.

But there are no such difficulties if we choose

$$Z = \mathcal{P} \quad \text{[see (3)]} \quad \text{or} \quad Z = \mathcal{P}_\pm \quad \text{[see (4)]}.$$  

Really, in this case the unknown phase factor for each line of diagram splits into two parts [see (3)]:

$$\mathcal{M}_{if} = M_{if} \cdot \frac{\bar{u}_i u \bar{u} u_f}{|\bar{u}_i u \bar{u} u_f|} = M_{if} \cdot \frac{\bar{u}_i u}{|\bar{u}_i u|} \cdot \frac{\bar{u} u_f}{|\bar{u} u_f|}.$$  

Let us consider the diagrams of the process, which proceeds in 2 different channels in general form (see Fig.4):
The expression
\[ M = (\bar{u}_3 Q u_1) \cdot (\bar{u}_4 R u_2) = M_{13} \cdot M_{24} \]
corresponds to the first diagram. The expression
\[ M' = (\bar{u}_4 S u_1) \cdot (\bar{u}_3 T u_2) = M_{14} \cdot M_{23} \]
corresponds to the second diagram, where \( Q, R, S, T \) are arbitrary matrix operators which characterize interaction.

Calculating the amplitudes we have
\[
\mathcal{M} = M_{13} \cdot M_{24} \cdot \left( \frac{\bar{u}_1 u}{\bar{u}_1 u} \right) \cdot \left( \frac{\bar{u}_3 u}{\bar{u}_3 u} \right) \cdot \left( \frac{\bar{u}_2 u}{\bar{u}_2 u} \right) \cdot \left( \frac{\bar{u}_4 u}{\bar{u}_4 u} \right),
\]
\[
\mathcal{M}' = M_{14} \cdot M_{23} \cdot \left( \frac{\bar{u}_1 u}{\bar{u}_1 u} \right) \cdot \left( \frac{\bar{u}_4 u}{\bar{u}_4 u} \right) \cdot \left( \frac{\bar{u}_2 u}{\bar{u}_2 u} \right) \cdot \left( \frac{\bar{u}_3 u}{\bar{u}_3 u} \right),
\]
that is the amplitudes of the different diagrams have the same phase factor.

Thus we must calculate amplitude of process with interfering diagrams in form
\[
\mathcal{M} + \mathcal{M}' = \frac{Tr(Q P_i P P_3) \cdot Tr(R P_2 P P_4) + Tr(S P_1 P P_4) \cdot Tr(T P_2 P P_3)}{[Tr(P P_1)Tr(P P_2)Tr(P P_3)Tr(P P_4)]^{1/2}}.
\]

This expression enable to calculate the amplitude numerically. Complex numbers being obtained under calculation are used for calculation of cross section of process.

For the denominator in the right-hand side of (8) we take into account the identity
\[
Tr(\bar{P} P_i P P_f) = Tr(\bar{P} P_i)Tr(P P_f),
\]
since projection operators have the following properties
\[
\bar{P} = P, \quad P A P = Tr[P A] \cdot P,
\]
$$\mathcal{P}_\pm = \mathcal{P}_\pm , \quad \mathcal{P} A \mathcal{P}_\pm = \text{Tr} [ \mathcal{P} A ] \cdot \mathcal{P}_\pm .$$

Really

$$\mathcal{P} = \gamma^0 \mathcal{P}^+ + \gamma^0 = \gamma^0 (u \bar{u})^+ + \gamma^0 = \gamma^0 (u u + \gamma^0) + \gamma^0$$

$$= \gamma^0 [(\gamma^0)^+ (u^+) + u^+] \gamma^0 = \gamma^0 [\gamma^0 u u^+] \gamma^0 = u u^+ \gamma^0 = u \bar{u} = \mathcal{P} ,$$

$$\mathcal{P} A \mathcal{P} = (u)_\alpha (\bar{u})_\beta (A)^{\beta \rho} (u)_\rho (\bar{u})_\delta = [ (\bar{u})_\beta (A)^{\beta \rho} (u)_\rho ] (u)_\alpha (\bar{u})_\delta$$

$$= [ (u)_\rho (\bar{u})_\beta (A)^{\beta \rho} ] (u)_\alpha (\bar{u})_\delta = \text{Tr} [ \mathcal{P} A ] \cdot \mathcal{P} .$$

If in individual cases under numerical calculations denominator in (8) is equal to 0, it is sufficiently to change arbitrary projection operator \( \mathcal{P} \), which appears in the expression (8).

Notice that calculation of amplitude for alone diagram is easier than calculation of squared amplitude, if operators, which characterize interaction, contain product of greater number \( \gamma \)-matrices, then projection operator.

Really, when number of \( \gamma \)-matrices in operator \( Q \) increase by \( I \) [see (2)], theirs number in numerator of (2) increase only by \( I \) (denominator does not change), but in construction \( \text{Tr} ( Q u_i \bar{u}_i Q u_f \bar{u}_f ) \), which appear, when we calculate squared matrix element, the number of \( \gamma \)-matrices increase by \( 2I \). We take into account that trace of product of \( 2J \) \( \gamma \)-matrices contains \( 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2J - 1) \) terms and we obtain that the more complicated is a process the bigger are advantageous being given under calculation of this process by method of directly calculation of amplitudes.

However, for processes with interfering diagrams method of calculation of amplitudes is easier in any case, because we need not calculate the interference terms.

As it was mentioned before, it is simply to make a generalization of this method for a reactions with participation of antiparticles. It is sufficiently for it to substitute projection operators of antiparticles in place of operators of particles. Let, for example, we are interested in value \( \bar{v}_f Q u_i \), where \( v_f \) is bispinor for free antiparticle. Then

$$\bar{v}_f Q u_i = \frac{\text{Tr} ( Q u_i \bar{u}_i Z v_f \bar{v}_f )}{\sqrt{\text{Tr} [ Q u_i \bar{u}_i Z v_f \bar{v}_f ]}} \quad (9)$$

where

$$v(p, n) \bar{v}(p, n) = \frac{1}{4m} (-m + \hat{p})(1 + \gamma_5 \hat{n})$$

for massive antiparticle, or

$$v_\pm (q) \bar{v}_\pm (q) = \frac{1}{2} (1 \mp \gamma_5) \hat{q}$$

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for massless antiparticle. As always, we use (3) or (4) instead of $Z$.

Notice that in (9) and in further consideration we shall use equality sign instead of symbol $\simeq$, since there exist not any trouble with phase factors already.

As an example, we give formulas for calculation of amplitudes of processes with massless Dirac particles. In this case formula (2) takes the next form

$$
\bar{u}_\pm(p_3) Qu_\pm(p_1) = \frac{Tr[Q\hat{p}_1\hat{q}\hat{p}_3(1 \mp \gamma_5)]}{4[(qp_1)(qp_3)]^{1/2}}.
$$

(10)

Here $Z = \frac{1}{2}(1 \mp \gamma_5)\hat{q} = \mathcal{P}_\mp$, $q^2 = 0$.

Massless 4-vector $q$ can be arbitrary, but it must be the same for all considered nonclosed fermion lines of diagrams.

$$
\bar{u}_\pm(p_3) Qu_\mp(p_1) = \frac{Tr[Q\hat{p}_1(m \pm \hat{n})\hat{p}_3(1 \mp \gamma_5)]}{4[(pp_1) \mp m(np_3)]^{1/2}}.
$$

(11)

Here $Z = \frac{1}{4m}(m + \hat{p})(1 + \gamma_5\hat{n}) = \mathcal{P}$, $p^2 = m^2$, $n^2 = -1$, $pn = 0$. As regards 4-vectors $p$ and $n$ the same observation as the one for vector $q$ in (9) is right.

In the last case we can not use easier operator $\mathcal{P}_\pm$ for $Z$, since in this case numerator and denominator are identical with 0. But we may to require for maximum simplicity of calculations $m = 0$, $p^2 = 0$ in (11):

$$
\bar{u}_\pm(p_3) Qu_\mp(p_1) = \pm \frac{Tr[Q\hat{p}_1\hat{n}\hat{p}_3(1 \mp \gamma_5)]}{4[(pp_1)(pp_3)]^{1/2}}.
$$

(12)

Formula (12) generalize method of calculation of amplitudes being offered in [14].

If under numerical calculations denominator in (10) – (12) is equal to 0 for some values $p_1$ and $p_3$, it is sufficiently to change values of arbitrary 4-vectors $q$ or $p$, $n$ being contained by these formulae (simultaneously for all lines of diagrams being considered).

3 Reducing of expressions for the traces of products of ten and more Dirac $\gamma$-matrices

Let us consider the determinant

$$
\begin{vmatrix}
  g_{\mu\nu} & g_{\mu\alpha} & g_{\mu\beta} & g_{\mu\lambda} & g_{\mu\rho} \\
  g_{\sigma\nu} & g_{\sigma\alpha} & g_{\sigma\beta} & g_{\sigma\lambda} & g_{\sigma\rho} \\
  g_{\tau\nu} & g_{\tau\alpha} & g_{\tau\beta} & g_{\tau\lambda} & g_{\tau\rho} \\
  g_{\kappa\nu} & g_{\kappa\alpha} & g_{\kappa\beta} & g_{\kappa\lambda} & g_{\kappa\rho} \\
  g_{\omega\nu} & g_{\omega\alpha} & g_{\omega\beta} & g_{\omega\lambda} & g_{\omega\rho}
\end{vmatrix} \equiv 0
$$

(13)
where
\[
g_{\mu\nu} = \begin{cases} 
1 & \text{if } \mu = \nu = 0 \\
-1 & \text{if } \mu = \nu = 1, 2, 3 \\
0 & \text{if } \mu \neq \nu
\end{cases}
\]

The validity of this identity follows from the fact that the expression on the left-hand side of (13) is completely antisymmetric with respect to each of five indices: \(\nu, \alpha, \beta, \lambda, \rho\) (and \(\mu, \sigma, \tau, \kappa, \omega\) too). In four-dimensional space, every tensor that is antisymmetric with respect to more than four indices vanishes identically, since the values of at least two of them must be equal.

By analogy with the Gram determinant, we introduce the notation (see [16])
\[
\begin{vmatrix}
g_{\mu\nu} & g_{\mu\alpha} & g_{\mu\beta} & g_{\mu\lambda} & g_{\mu\rho} \\
g_{\sigma\nu} & g_{\sigma\alpha} & g_{\sigma\beta} & g_{\sigma\lambda} & g_{\sigma\rho} \\
g_{\tau\nu} & g_{\tau\alpha} & g_{\tau\beta} & g_{\tau\lambda} & g_{\tau\rho} \\
g_{\kappa\nu} & g_{\kappa\alpha} & g_{\kappa\beta} & g_{\kappa\lambda} & g_{\kappa\rho} \\
g_{\omega\nu} & g_{\omega\alpha} & g_{\omega\beta} & g_{\omega\lambda} & g_{\omega\rho}
\end{vmatrix} = G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \kappa & \omega \\
\nu & \alpha & \beta & \lambda & \rho
\end{array}\right) .
\]

It follows from the properties of determinants that
\[
G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \kappa & \omega \\
\nu & \alpha & \beta & \lambda & \rho
\end{array}\right) \equiv 0 ,
\]
\[
G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \kappa & \omega & \chi \\
\nu & \alpha & \beta & \lambda & \rho & \varphi
\end{array}\right) \equiv 0
\]

etc. We note that the identity (13) is valid only in a space of dimension not higher than 4, the identity (14) is valid in a space of dimension not higher than 5, etc.

Identities of the type (13), (14), (15) and others like them can be used to simplify the expressions for the traces of a product of 10 and more Dirac \(\gamma\)-matrices. For example, the expression for
\[
\text{Tr}(\gamma_\mu \gamma_\sigma \gamma_\tau \gamma_\kappa \gamma_\omega \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho)
\]
calculated by means of the computer system REDUCE, contains 945 terms. However, the expression
\[
\begin{align*}
\text{Tr}(\gamma_\mu \gamma_\sigma \gamma_\tau \gamma_\kappa \gamma_\omega \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho) - 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \kappa & \omega \\
\nu & \alpha & \beta & \lambda & \rho
\end{array}\right) & - 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \kappa & \rho \\
\omega & \nu & \alpha & \beta & \lambda
\end{array}\right) \\
- 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \tau & \lambda & \rho \\
\kappa & \omega & \nu & \alpha & \beta
\end{array}\right) & - 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \beta & \lambda & \rho \\
\tau & \kappa & \omega & \nu & \alpha
\end{array}\right) \\
+ 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \kappa & \lambda & \rho \\
\tau & \omega & \nu & \beta & \alpha
\end{array}\right) & + 4 \cdot G\left(\begin{array}{ccccc}
\mu & \sigma & \kappa & \omega & \beta \\
\tau & \nu & \alpha & \lambda & \rho
\end{array}\right) \\
+ 4 \cdot G\left(\begin{array}{ccccc}
\mu & \tau & \omega & \alpha & \lambda \\
\kappa & \nu & \beta & \rho & \sigma
\end{array}\right) & + 4 \cdot G\left(\begin{array}{ccccc}
\mu & \tau & \kappa & \alpha & \rho \\
\omega & \nu & \beta & \lambda & \sigma
\end{array}\right)
\end{align*}
\]

which is identical to it, contains only 531 terms.
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