ON A SPECIAL CASE OF WATKINS’ CONJECTURE

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Abstract. Watkins’ conjecture asserts that for a rational elliptic curve $E$ the degree of the modular parametrization is divisible by $2^r$, where $r$ is the rank of $E$. In this paper, we prove that if the modular degree is odd, then $E$ has rank zero. Moreover, we prove that the conjecture holds for all rank two rational elliptic curves of prime conductor and positive discriminant.

1. Introduction

Given a rational elliptic curve $E$ of conductor $N$, by the modularity theorem, there exists a morphism of a minimal degree

$$\phi : X_0(N) \rightarrow E,$$

that is defined over $\mathbb{Q}$, where $X_0(N)$ is the classical modular curve. Its degree, denoted by $m_E$, is called the modular degree. While analyzing experimental data, Watkins conjectured that for an elliptic curve of rank $r$, $m_E$ is divisible by $2^r$ [9, Conjecture 4.1]. In particular, if the modular degree is odd, the rank should be zero; the proof of this assertion is the main result of this work.

The study of elliptic curves with odd modular degree was first developed in [1] by Calegari and Emerton, where they showed that a rational elliptic curve with odd modular degree has to satisfy a series of very restrictive hypotheses. For a detailed list of conditions see [1, Theorem 1.1]. Later, building on this work, Yazdani [8] studied abelian varieties having odd modular degree. As a by-product of his work, he proves that if a rational elliptic curve has odd modular degree, then it has rank zero, except perhaps if it has prime conductor and even analytic rank (see [8, Theorem 3.8] for a more general statement). The main result of this paper is the following theorem:

Theorem 1.1. If $E/\mathbb{Q}$ is an elliptic curve of odd modular degree, then $E$ has rank zero.

By the aforementioned results it is enough to restrict ourselves to the case where $E$ has prime conductor $p$ and even analytic rank. Moreover, it is clear that we can assume that the curve $E$ is the strong Weil curve, that is, the kernel of the map $J_0(p) \rightarrow E$ is connected ($J_0(p)$ is the Jacobian of $X_0(p)$).

The elliptic curve $E$ gives rise to a normalized newform $f_E \in S_2(\Gamma_0(p))$ by the modularity theorem. The main idea of the article is to associate to $f_E$ (or $E$) an

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element $v_E$ of the Picard group $\mathcal{X}$ of a certain curve $X$ (which is a disjoint union of curves of genus zero) as in [3]. More precisely, $\mathcal{X}$ can be described as the free $\mathbb{Z}$-module of divisors supported on the isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$, denoted by $e_1, e_2, \ldots, e_n$, where $n - 1$ is the genus of $X_0(p)$. They are in bijection with the isomorphism classes of supersingular elliptic curves $E_i/\mathbb{F}_p$.

The action of Hecke correspondences on $X$ induces an action on $\mathcal{X}$. There is a correspondence between modular forms of level $p$ and weight 2 and elements of $\mathcal{X} \otimes \mathbb{C}$ that preserves the action of the Hecke operators ([3, Proposition 5.6]). Let $v_E = \sum v_E(e_i)e_i \in \mathcal{X}$ be an eigenvector for all Hecke operators $t_m$ corresponding to $f_E$, i.e. $t_m v_E = a(m)v_E$, where $f_E(\tau) = \sum_{m=1}^{\infty} a(m)q^m$. We normalize $v_E$ (up to sign) such that the greatest common divisor of all its entries is 1. We define a $\mathbb{Z}$-bilinear pairing

$$\langle -, - \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{Z},$$

by requiring $\langle e_i, e_j \rangle = w_i \delta_{i,j}$ for all $i, j \in \{1, \ldots, n\}$, where $w_i = \frac{1}{2} \#\text{Aut}(E_i)$.

We have the following key result of Mestre that relates the norm of $v_E$ to the modular degree $m_E$.

**Proposition 1.2** ([6 Theorem 3]).

$$\langle v_E, v_E \rangle = m_E t,$$

where $t$ is the size of $E(\mathbb{Q})_{\text{tors}}$.

The final ingredient we need is the Gross-Waldspurger formula on special values of $L$-series [3]. An alternative approach is to use the Gross-Kudla formula for the special values of triple products of $L$-functions [4].

In [5], while studying supersingular zeros of divisor polynomials of elliptic curves, the authors posed the following conjecture.

**Conjecture 1.3.** If $E$ is an elliptic curve of prime conductor $p$, root number 1, and rank$(E) > 0$, then $v_E(e_i)$ is an even number for all $e_i$ with $j(E_i) \in \mathbb{F}_p$.

The conclusion of the conjecture holds for any elliptic curve $E/\mathbb{Q}$ of prime conductor and root number $-1$, as well as for any curve of prime conductor that has positive discriminant and no rational points of order 2 (see [5 Thms. 1.1, 1.2, 1.4]).

In the last paragraph of this paper, we will show the connection between this conjecture and Watkins’ conjecture:

**Theorem 1.4.** Let $E/\mathbb{Q}$ be an elliptic curve of prime conductor such that rank$(E) > 0$. If $v_E(e_i)$ is even number for all $e_i$ with $j(E_i) \in \mathbb{F}_p$, then $4|m_E$.

In particular, as remarked before, this verifies Watkins’ conjecture if $E$ has prime conductor, $\text{disc}(E) > 0$, and rank$(E) = 2$.

2. Proof of the main theorem

We will give a series of propositions that will allow us to prove Theorem 1.1.

**Proposition 2.1.** If $E/\mathbb{Q}$ has non-zero rank, then $L(E, 1) = 0$.

**Proof.** This is a classical application of the Gross-Zagier and Kolyvagin theorems. For a reference see [2 Theorem 3.22].

**Proposition 2.2.** If $E/\mathbb{Q}$ has prime conductor and non-zero rank, then $E(\mathbb{Q})_{\text{tors}}$ is trivial.
Proof. This is a well-known result; for example in [6] it is shown that the isogeny classes of rational elliptic curves with conductor $p$ and non-trivial rational torsion subgroup are either $11.a, 17.a, 19.a$ and $37.b$, or the so-called Neumann-Setzer curves that have a 2-rational point. All these curves have rank zero [7]. □

Proposition 2.3. Let $v_{E} = \sum_{i=1}^{n} v_{E}(e_{i})e_{i} \in \mathcal{X}$ be the vector corresponding to $f_{E}$. We have that $\sum_{i=1}^{n} v_{E}(e_{i}) = 0$.

Proof. The vector $e_{0} = \sum_{i=1}^{n} e_{i} w_{i}$ corresponds to the Eisenstein series ([3, Formula 4.9]). Moreover, the pairing $\langle -, - \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{Z}$ is compatible with the Hecke operators. Since the space of cuspforms is orthogonal to the Eisenstein series, we obtain $\langle v_{E}, e_{0} \rangle = \sum_{i=1}^{n} v_{E}(e_{i}) = 0$. □

Proposition 2.4. The numbers $w_{k}$ are all equal to 1 unless $j(E_{k}) = 0$ (in which case $w_{k} = 3$) or $j(E_{k}) = 1728$ (in which case $w_{k} = 2$). The value $j = 0$ is a supersingular $j$-invariant precisely for $p \equiv 2 \text{ (mod 3)}$ and $j = 1728$ is a supersingular $j$-invariant for $p \equiv 3 \text{ (mod 4)}$.

Proof. See [3, Table 1.3 p. 117]. □

Given $-D$ a fundamental negative discriminant, Gross defines

$$b_{D} = \sum_{i=1}^{n} h_{i}(-D) u(-D) e_{i},$$

where $h_{i}(-D)$ is the number of optimal embeddings of the order of discriminant $-D$ into $\text{End}(E_{i})$ modulo conjugation by $\text{End}(E_{i})^{\times}$ and $u(-D)$ is the number of units of the order. We are in position to state (a special case of) the Gross-Waldspurger formula [3, Proposition 13.5].

Proposition 2.5. If $-D$ is a fundamental negative discriminant with $\left(\frac{-D}{p}\right) = -1$, then

$$L(E, 1)L(E \otimes \varepsilon_{D}, 1) = \frac{(f_{E}, f_{E})}{\sqrt{D}} \frac{m_{D}^{2}}{\langle v_{E}, v_{E} \rangle},$$

where $\varepsilon_{D}$ is the quadratic character associated to $-D$, $(f_{E}, f_{E})$ is the Petersson inner product on $\Gamma_{0}(p)$ and

$$m_{D} = \langle v_{E}, b_{D} \rangle.$$

We will use the formula in the case that $-D = -4$ (and thus $p \equiv 3 \text{ mod 4}$). In this situation a rational elliptic curve of $j$-invariant equal to 1728 with complex multiplication by $\mathbb{Z}[i]$ reduces mod $p$ to the supersingular elliptic curve $E_{k}$ and this reduction induces two optimal embeddings of $\mathbb{Z}[i]$ into $\text{End}(E_{k})$. On the other hand, we know that $\sum_{i} h_{i}(-4) = 2h(-4) = 2$, where $h(-4)$ is the class number of the quadratic imaginary field $\mathbb{Q}(\sqrt{-4})$ ([3, Formula 1.12]); thus $h_{i} = 0$ unless $i = k$ in which case $h_{k}(-4) = 2$. Since $u(-4) = 4$, we obtain that $b_{4} = \frac{1}{2} e_{k}$.

Now we have the necessary ingredients in order to prove Theorem 1.1.
Proof of Theorem 1.1. As remarked in the introduction, it is enough to prove the theorem when $E$ has prime conductor $p$ and it is the strong Weil curve. Suppose on the contrary that $E$ has positive rank. In consequence, by Proposition 1.2 and Proposition 2.2 we know that $\langle v_E, v_E \rangle$ must be odd. Moreover, 

$$\langle v_E, v_E \rangle = \sum_{i=1}^{n} w_i v_E(e_i)^2 \equiv \sum_{i=1}^{n} w_i v_E(e_i) \pmod{2}.$$ 

Using Propositions 2.3 and 2.4 we obtain that if $p \equiv 1 \pmod{4}$, $\langle v_E, v_E \rangle$ is even and if $p \equiv 3 \pmod{4}$, then $\langle v_E, v_E \rangle \equiv v_E(e_k) \pmod{2}$, where $k$ is the only index such that $w_k = 2$. In that case, since $L(E,1) = 0$ (by Proposition 2.1), Proposition 2.5 implies that 

$$m_E = \langle v_E, b_4 \rangle = 0.$$ 

Since $b_4 = \frac{1}{2} e_k$, we get that 

$$m_E = v_E(e_k) = 0.$$ 

Therefore, $\langle v_E, v_E \rangle$ is even, leading to a contradiction.

Remark 2.6. Another proof along the same lines uses that if $L(E,1) = 0$, then $\sum_i w_i^2 v_E(e_i)^3 = 0$. This is proved in [4, Corollary 11.5], as a consequence of the Gross-Kudla formula of special values of triple product $L$-functions. The number $\sum_i w_i^2 v_E(e_i)^3$ clearly has the same parity as $\langle v_E, v_E \rangle$, leading to the desired contradiction.

3. The proof of Theorem 1.4

Proof of Theorem 1.4. For a given $e_i$, denote by $\tilde{i} \in \{1, 2, \ldots, n\}$ the unique index such that $e_{\tilde{i}}$ corresponds to the curve $E_{\tilde{i}}^p$. Then [3, Proposition 2.4] implies that $v(e_i) = v(e_{\tilde{i}})$. By Proposition 2.4 we have that $j(E_k) \in \mathbb{F}_p$ whenever $w_k \neq 1$, and thus $v_E(e_k)$ is even. Hence Proposition 2.2 implies that 

$$m_E \equiv \sum_i v_E(e_i)^2 \pmod{4}.$$ 

If $E_i$ is defined over $\mathbb{F}_p$ (i.e., $\tilde{i} = i$), then by the assumption 

$$v_E(e_i)^2 \equiv 0 \pmod{4}.$$ 

Hence 

$$m_E \equiv \sum_i' 2 v_E(e_i)^2 \pmod{4},$$ 

where we sum over the pairs $\{i, \tilde{i}\}$ with $i \neq \tilde{i}$. Using again Proposition 2.1 and the Gross-Kudla formula, we get that 

$$\sum_i v_E(e_i)^3 \equiv \sum_i' 2 v_E(e_i) \equiv 0 \pmod{4},$$ 

where the second sum is over the pairs $\{i, \tilde{i}\}$ for which $v_E(e_i)$ is odd. It follows that the number of such pairs is even, hence $m_E \equiv 0 \pmod{4}$. \hfill $\square$

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