POLYNOMIAL CONTROL ON STABILITY, INVERSION AND POWERS OF MATRICES ON SIMPLE GRAPHS

CHANG EON SHIN AND QIYU SUN

Abstract. Spatially distributed networks of large size arise in a variety of science and engineering problems, such as wireless sensor networks and smart power grids. Most of their features can be described by properties of their state-space matrices whose entries have indices in the vertex set of a graph. In this paper, we introduce novel algebras of Beurling type that contain matrices on a connected simple graph having polynomial off-diagonal decay, and we show that they are Banach subalgebras of $B(ℓ^p)$, $1 \leq p \leq ∞$, the space of all bounded operators on the space $ℓ^p$ of all $p$-summable sequences. The $ℓ^p$-stability of state-space matrices is an essential hypothesis for the robustness of spatially distributed networks. In this paper, we establish the equivalence among $ℓ^p$-stabilities of matrices in Beurling algebras for different exponents $1 \leq p \leq ∞$, with quantitative analysis for the lower stability bounds. Admission of norm-control inversion plays a crucial role in some engineering practice. In this paper, we prove that matrices in Beurling subalgebras of $B(ℓ^2)$ have norm-controlled inversion and we find a norm-controlled polynomial with close to optimal degree. Polynomial estimate to powers of matrices is important for numerical implementation of spatially distributed networks. In this paper, we apply our results on norm-controlled inversion to obtain a polynomial estimate to powers of matrices in Beurling algebras. The polynomial estimate is a noncommutative extension about convolution powers of a complex function and is applicable to estimate the probability of hopping from one agent to another agent in a stationary Markov chain on a spatially distributed network.

1. Introduction

A spatially distributed network (SDN) contains a large amount of agents with limited sensing, data processing, and communication capabilities for information transmission. It arises in a variety of science

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and engineering problems ([11][13][28][62]). The topology of an SDN can be described by a graph

\[ G := (V, E) \]

of large size, where a vertex in \( V \) represents an agent and an edge \((\lambda, \lambda') \in E\) between two vertices \( \lambda \) and \( \lambda' \in V \) means that a direct communication link exists. In this paper, we always assume that \( G \) is connected and simple. Here a simple graph means that it is an unweighted, undirected graph containing no graph loops or multiple edges. Our motivating examples are 1) circular graphs \( \mathbb{Z}_N := (\mathbb{Z}/N\mathbb{Z}, E_N) \) of order \( N \geq 3 \), where \((m, n) \in E_N\) means \( m - n \in N\mathbb{Z} \pm 1; \) and 2) lattice graphs \( \mathbb{Z}^d := (\mathbb{Z}^d, E^d), d \geq 1, \) where \((m, n) \in E^d\) implies that \( m \) and \( n \in \mathbb{Z}^d \) have distance one. For the graph to describe an SDN, the assumption on its connectivity and simplicity can be understood as that agents in the SDN can communicate across the entire network, direct communication links between agents are bidirectional, agents have the same communication specification, communication components are not used for data transmission within an agent, and no multiple direct communication channels exist between agents [12].

SDNs could give extraordinary capabilities especially when creating a data exchange network requires significant efforts or when establishing a centralized facility to process and store all the information is formidable. A comprehensive mathematical analysis of SDNs does not appear to exist yet, and there is a huge research gap between mathematical theory and engineering practice [6, 12, 19, 35, 37, 48]. This inspires us to consider various properties of state-space matrices

\[ A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \]

of SDNs with indices in the vertex set \( V \) of a graph. This work is also motivated by the emerging field of signal processing on graphs, where matrices of the form (1.2) are used for linear processing such as filtering, translation, modulation, dilation and downsampling [11, 39, 42, 43, 47].

An abundant family of SDNs is spatially decaying linear systems whose state-space matrices have off-diagonal decay. Examples of such systems include smart power grids with sparse interconnection topologies, multi-agent systems with nearest-neighbor coupling structures, and (wireless) sensor networks for environment monitoring ([1][12][13][19][28][35][62]). To describe off-diagonal decay property of matrices of the form (1.2), we introduce Banach algebras \( \mathcal{B}_{r,\alpha}(G) \) of Beurling type for \( 1 \leq r \leq \infty \) and \( \alpha \geq 0 \), see (3.1) and (3.2) in Section 3. Matrices \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \) in \( \mathcal{B}_{r,\alpha}(G) \) have their entries dominated by a...
positive decreasing function $h_A$ with polynomial decay,
\begin{equation}
|a(\lambda, \lambda')| \leq h_A(\rho(\lambda, \lambda')) \quad \text{for all } \lambda, \lambda' \in V,
\end{equation}
where $\rho(\lambda, \lambda')$ is the geodesic distance between vertices $\lambda, \lambda' \in V$. For the lattice graph $\mathbb{Z}^d$, Banach algebras $\mathcal{B}_{r,\alpha}(\mathbb{Z}^d)$ are introduced by Beurling in [9] for $r = d = 1$ and $\alpha = 0$, Jaffard in [29] for $r = \infty$ and Sun in [51] for $1 \leq r \leq \infty$.

Let $\ell^p := \ell^p(V), 1 \leq p \leq \infty$, be Banach spaces of $p$-summable sequences on $V$ with standard norm $\| \cdot \|_{\ell^p}$, and $\mathcal{B}(\ell^p)$ be the Banach algebra of all bounded operators on $\ell^p$ with norm $\| \cdot \|_{\mathcal{B}(\ell^p)}$. We say that a matrix $A \in \mathcal{B}(\ell^p)$ has $\ell^p$-stability if there exists a positive constant $A_p$ such that
\begin{equation}
\|Ac\|_p \geq A_p\|c\|_p \quad \text{for all } c \in \ell^p.
\end{equation}
The optimal lower $\ell^p$-stability bound of a matrix $A$ is the maximal number $A_p$ for (1.4) to hold. The $\ell^p$-stability is an essential hypothesis for some matrices arising in time-frequency analysis, sampling theory, wavelet analysis and many other applied mathematical fields [3, 15, 23, 53, 56]. For the robustness against bounded noises, the sensing matrix arisen in the sampling and reconstruction procedure of signals on a SDN is required in [12] to have $\ell^\infty$-stability, however there are some difficulties to verify $\ell^p$-stability of a matrix in a distributed manner for $p \neq 2$.

For a finite graph $\mathcal{G} = (V, E)$ and a matrix $A$ with indices in its vertex set $V$, its $\ell^p$-stability and $\ell^q$-stability are equivalent to each other for any $1 \leq p, q \leq \infty$, and its optimal lower stability bounds satisfy
\begin{equation}
M^{-|1/p - 1/q|} \leq \frac{A_q}{A_p} \leq M^{1/p - 1/q},
\end{equation}
where $M = \#V$ is the number of vertices of the graph $\mathcal{G}$. The above estimation on lower stability bounds is unfavorable for matrices of large size but it cannot be improved if there is no restriction on the matrix $A$. Let $d$ be the Beurling dimension of the graph $\mathcal{G}$. Matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ are bounded operators on $\ell^p, 1 \leq p \leq \infty$, and there exists a positive constant $C$ such that
\begin{equation}
\|A\|_{\mathcal{B}(\ell^p)} \leq C\|A\|_{\mathcal{B}_{r,\alpha}} \quad \text{for all } A \in \mathcal{B}_{r,\alpha}(\mathcal{G}) \text{ and } 1 \leq p \leq \infty.
\end{equation}
For their lower stability bounds, it is proved that
\begin{align*}
A_q > 0 & \text{ if and only if } A_p > 0 \\
& \text{ in [2, 51, 57] for the infinite lattice graph } \mathbb{Z}^d, \text{ and that } \\
A_q > 0 & \text{ if } A_2 > 0
\end{align*}
in [12] for any infinite graph \( G \) with finite Beurling dimension and \( r = \infty \), where \( 1 \leq p, q \leq \infty \). In Theorem 4.1 of this paper, we provide a quantitative version of \( \ell^p \)-stability for different \( 1 \leq p \leq \infty \) and prove the following result,

\[
D_1 \left( \frac{\|A\|_{B_{r,\alpha}}}{A_p} \right)^{-D_0[1/p-1/q]} \leq \frac{A_q}{A_p} \leq D_2 \left( \frac{\|A\|_{B_{r,\alpha}}}{A_p} \right)^{D_0[1/p-1/q]},
\]

for any matrix \( A \in B_{r,\alpha}(G) \) with \( 1 \leq r \leq \infty \) and \( \alpha > d(1 - 1/r) \), where \( D_0, D_1, D_2 \) are absolute constants independent of matrices \( A \in B_{r,\alpha}(G) \), exponents \( 1 \leq p, q \leq \infty \) and the size \( M \) of the graph \( G \), cf. (1.5). The proof of Theorem 4.1 depends on an important estimate to the commutator between a matrix in the Beurling algebra \( B_{r,\alpha}(G) \) and a truncation operator. Similar estimate has been used by Sjöstrand in [46] to establish invertibility of infinite matrices in the Baskakov-Gohberg-Sjöstrand class.

A Banach subalgebra \( \mathcal{A} \) of \( B \) is said to be inverse-closed if an element in \( \mathcal{A} \), that is invertible in \( B \), is also invertible in \( \mathcal{A} \). The inverse-closed subalgebras have numerous applications in time-frequency analysis, sampling theory, numerical analysis and optimization, and it has been established for matrices, integral operators, pseudo-differential operators satisfying various off-diagonal decay conditions. The reader may refer to [5, 7, 17, 20, 21, 22, 24, 25, 29, 33, 36, 46, 51, 52] and the survey papers [21, 32, 45] for historical remarks and recent advances.

A quantitative version of inverse-closedness is the norm-controlled inversion [26, 27, 38, 41, 49]. Here an inverse-closed Banach subalgebra \( \mathcal{A} \) of \( B \) is said to admit norm control in \( B \) if there exists a continuous function \( h \) from \([0, \infty) \times [0, \infty)\) to \([0, \infty)\) such that

\[
\|A^{-1}\|_\mathcal{A} \leq h(\|A\|_\mathcal{A}, \|A^{-1}\|_B)
\]

for all \( A \in \mathcal{A} \) with \( A^{-1} \in B \). Admission of norm-control inversion plays a crucial role in [50] to solve nonlinear sampling problems termed with instantaneous companding and local identification of signals with finite rate of innovation. Norm-controlled inversion was first studied by Stafney in [49], where it is shown that \( B_{1,0}(Z) \) does not admit a norm-controlled inversion in \( B(\ell^2) \). The polynomial norm-control inversion is established in [26] for matrices in differential algebras and [27] for matrices in Besov algebras, Bessel algebras, Dales-Davie algebras and Jaffard algebra. In Theorem 5.1 of this paper, we show that Banach algebra \( B_{r,\alpha}(G) \) with \( 1 \leq r \leq \infty \) and \( \alpha > d(1 - 1/r) + 1 \) admit norm-controlled inversion in \( B(\ell^2) \), and there exists an absolute constant \( C \) such that

\[
\|A^{-1}\|_{B_{r,\alpha}} \leq C\|A^{-1}\|_{B(\ell^2)}(\|A^{-1}\|_{B(\ell^2)}\|A\|_{B_{r,\alpha}})^{\alpha + d/r}
\]
hold for all $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $A^{-1} \in \mathcal{B}(\ell^2)$. Moreover, the above polynomial norm-control inversion is close to optimal, as shown in Example 5.2 that the exponent $\alpha + d/r$ in (1.8) cannot be replaced by $\alpha + d/r - 1 - \epsilon$ for any $\epsilon > 0$. We remark that a weak version of the norm-controlled estimate (1.8), with the exponent $\alpha + d/r$ in (5.2) replaced by a larger exponent $2\alpha + 2 + 2/(\alpha - 2)$, is established in [27] for the Jaffard algebra $\mathcal{J}_\alpha(\mathcal{Z}) = \mathcal{B}_{\infty,\alpha}(\mathcal{Z})$.

Let $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}$. We say that $\mathcal{A}$ is its differential subalgebra of order $\theta \in (0, 1]$ ([10, 14, 31, 50, 54]) if there exists a positive constant $C$ such that

$$\|AB\|_A \leq C\|A\|_A\|B\|_A\left(\left(\frac{\|A\|_B}{\|A\|_A}\right)^\theta + \left(\frac{\|B\|_B}{\|B\|_A}\right)^\theta\right) \quad \text{for all } A, B \in \mathcal{A}. \quad (1.9)$$

The differential subalgebras was introduced in [10, 31] for $\theta = 1$ and [14, 50, 54] for $\theta \in (0, 1)$. In [14, 24], it is shown that a $C^*$-subalgebras $\mathcal{A}$ of $\mathcal{B}$ with a common unit admits norm controlled inversion if $\mathcal{A}$ is also a differential subalgebra. The reader may refer to [10, 14, 31, 36, 45, 50, 55] and references therein for historical remarks and recent advances in operator theory, harmonic analysis, non-commutative geometry, numerical analysis and optimization. The differential norm inequality (1.9) is satisfied by many Banach subalgebras of $\mathcal{B}(\ell^2)$ [24, 51, 52, 54]. For $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$, it is shown in Proposition 3.2 that Banach algebras $\mathcal{B}_{r,\alpha}(\mathcal{G})$ are differential subalgebra of $\mathcal{B}(\ell^2)$ with order $\theta_{r,\alpha} = 2(\alpha - d + d/r)/(1 + 2\alpha - 2d + 2d/r)$. Applying the differential property (1.9) repeatedly and using the argument in [50, Proposition 2.4], we have the following subexponential estimate for the norms of powers $A^n, n \geq 1$, in $\mathcal{B}_{r,\alpha}(\mathcal{G})$,

$$\|A^n\|_{\mathcal{B}_{r,\alpha}} \leq \|A\|_{\mathcal{B}(\ell^2)}^n \left(\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{\theta_{r,\alpha}} \left(1 + \log_2(1 + \theta_{r,\alpha})\right)^n, \quad n \geq 1,$$

where $C$ is an absolute constant independent of integers $n \geq 1$ and matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$. In Theorem 6.1 of this paper, we refine the above estimate to show that powers of a matrix in $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r) + 1$ have polynomial growth,

$$\|A^n\|_{\mathcal{B}_{r,\alpha}} \leq Cn \left(\frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{\alpha + d/r} \|A\|_{\mathcal{B}(\ell^2)}^n, \quad n \geq 1. \quad (1.10)$$

Moreover, the above estimate is close to optimal, as shown in (6.3) that the exponent $\alpha + d/r$ in (1.10) cannot be replaced by $\alpha + d/r - 1 - \epsilon$.
for any $\epsilon > 0$. The polynomial norm estimate in (1.10) is a noncommutative extension about convolution powers of a complex function on $\mathbb{Z}^d$, cf. [18, 40, 58, 59] and (6.4). The power estimate in (1.10) is also applicable to estimate the probability $\Pr(X_n = \lambda | X_1 = \lambda')$ of hopping from one agent $\lambda$ to another agent $\lambda'$ in a stationary Markov chain $X_n, n \geq 1$, on a spatially distributed network, see Corollary 6.2.

The paper is organized as follows. In Section 2, we recall some preliminaries on connected simple graphs $\mathcal{G}$ and provide two basic estimates about their geometry. In Section 3, we introduce novel algebras of matrices, $B_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha \geq 0$, and we prove that they are differential and inverse-closed subalgebra of $\mathcal{B}(\ell^2)$. In Section 4, we establish the equivalence among $\ell^p$-stabilities of matrices in the Beurling algebra $B_{r,\alpha}(\mathcal{G})$ for different exponents $1 \leq p \leq \infty$, and we further show that their lower stability bounds are controlled by some polynomials. In Section 5, we prove that the Beurling algebra $B_{r,\alpha}(\mathcal{G})$ admit norm-controlled inversion and a polynomial can be selected to be the norm-controlled function $h$ in (1.7). In Section 6, we consider noncommutative extension of convolution powers and show that norms of powers $A^n, n \geq 1$, of a matrix $A \in B_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ are dominated by a polynomial.

Notation: $\mathbb{Z}_+$ contains all nonnegative integers, $\lfloor t \rfloor, \lceil t \rceil$ and $t_+ = \max(t, 0)$ of a number $t$ are the greatest preceding integer, the least succeeding integers and the positive part respectively, and for a set $F$ denote its cardinality and characteristic function by $\#F$ and $\chi_F$ respectively. In this paper, the capital letter $C$ is an absolute constant which is not necessarily the same at each occurrence.

2. PRELIMINARIES ON CONNECTED SIMPLE GRAPHS

In this section, we recall some concepts on connected simple graphs, and we establish some estimates about their geometry.

Let $\mathcal{G} := (V, E)$ be a connected simple graph. Denote by $\rho$ the geodesic distance on $\mathcal{G}$, which is the nonnegative function on $V \times V$ such that $\rho(\lambda, \lambda) = 0$ for all vertices $\lambda \in V$, and $\rho(\lambda, \lambda')$ is the number of edges in a shortest path connecting distinct vertices $\lambda, \lambda' \in V$ (16). For some real-world applications of SDNs, communication between two distinct agents happens by transmitting information through the chain of intermediate agents connecting them using a shortest path, and the geodesic distance is widely used to measure the communication cost to data exchange.
The geodesic distance $\rho$ on $\mathcal{G}$ is a metric on $V$. For the simple graph $\mathbb{Z}^d$, the geodesic distance between $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ is given by $\rho(m, n) = \sum_{i=1}^{d} |m_i - n_i|$. With the geodesic distance $\rho$ on $\mathcal{G} := (V, E)$, we denote the closed ball with center $\lambda \in V$ and radius $R$ by

$$B(\lambda, R) := \{ \lambda' \in V : \rho(\lambda, \lambda') \leq R \},$$

and the counting measure on $V$ by $\mu$, where $\mu(F)$ is the number of vertices in $F$ for any $F \subset V$. The counting measure $\mu$ is said to be a doubling measure \cite{[34, 61]} if there exists a positive constant $D_0(\mathcal{G})$ such that

$$\mu(B(\lambda, 2R)) \leq D_0(\mathcal{G})\mu(B(\lambda, R)) \quad \text{for all } \lambda \in V \text{ and } R \geq 0. \quad (2.1)$$

The minimal constant $D_0(\mathcal{G})$ in (2.1) is known as the doubling constant of the measure $\mu$. Under the doubling assumption to the measure $\mu$, the triple $(V, \rho, \mu)$ is a space of homogeneous type. The reader may refer to \cite{[34, 61]} for harmonic analysis on spaces of homogeneous type.

We say that the counting measure $\mu$ on the graph $\mathcal{G}$ has polynomial growth if there exist positive constants $D_1(\mathcal{G})$ and $d := d(\mathcal{G})$ such that

$$\mu(B(\lambda, R)) \leq D_1(\mathcal{G})(R + 1)^d \quad \text{for all } \lambda \in V \text{ and } R \geq 0. \quad (2.2)$$

The minimal constants $d$ and $D_1(\mathcal{G})$ in (2.2) are called as the Beurling dimension and density of the graph $\mathcal{G}$ \cite{[12]}. For the simple graph $\mathbb{Z}^d$, its Beurling dimension is the same as the Euclidean dimension $d$. We remark that a simple graph $\mathcal{G}$ with its counting measure $\mu$ satisfying the doubling condition (2.1) has finite Beurling dimension,

$$\mu(B(\lambda, R)) \leq D_0(\mathcal{G})(R + 1)^{\log_2 D_0(\mathcal{G})} \quad \text{for all } \lambda \in V \text{ and } R \geq 0,$$

where $D_0(\mathcal{G})$ is the doubling constant of the measure $\mu$.

We say that the counting measure $\mu$ on the graph $\mathcal{G}$ is normal if there exist $D_1(\mathcal{G})$ and $D_2(\mathcal{G})$ such that

$$D_2(\mathcal{G})(R + 1)^d \leq \mu(B(\lambda, R)) \leq D_1(\mathcal{G})(R + 1)^d \quad (2.3)$$

for all $\lambda \in V$ and $0 \leq R \leq \text{diam}(\mathcal{G})$, the diameter of the graph $\mathcal{G}$. One may verify that the counting measures on $\mathbb{Z}^d$ and $\mathbb{Z}_N$, $N \geq 1$, are normal, and a normal measure has the doubling property (2.1) and the polynomial growth property (2.2). The reader may refer to \cite{[30, 34, 60, 61]} and references therein for normal measures.

We conclude this section with a proposition on geometry of a connected simple graph with finite Beurling dimension.
**Proposition 2.1.** Let $G := (V,E)$ be a connected simple graph with Beurling dimension $d$, and $h := \{h(n)\}^\infty_{n=0}$ be a positive decreasing sequence. Then the following statements hold.

(i) For any vertex $\lambda \in V$ and integer $s \geq 0$,

$$(2.4) \quad \sum_{\rho(\lambda, \lambda') \leq s} \rho(\lambda, \lambda') h(\rho(\lambda, \lambda')) \leq (d + 1)D_1(G) \sum_{n=0}^{s} h(n)(n + 1)^d.$$ 

(ii) If $\sum_{n=0}^{\infty} h(n)(n + 1)^{d-1} < \infty$, then

$$(2.5) \quad \sum_{\rho(\lambda, \lambda') \geq s} h(\rho(\lambda, \lambda')) \leq D_1(G) \left((s + 1)^d h(s) + d \sum_{n=s+1}^{\infty} h(n)(n + 1)^{d-1}\right)$$

for any vertex $\lambda \in V$ and integer $s \geq 0$.

**Proof.** (i). Given a vertex $\lambda \in V$ and an integer $s \geq 0$, we obtain

$$\sum_{\rho(\lambda, \lambda') \leq s} \rho(\lambda, \lambda') h(\rho(\lambda, \lambda')) = \sum_{n=0}^{s} nh(n) (\mu(B(\lambda, n)) - \mu(B(\lambda, n - 1))) = sh(s)\mu(B(\lambda, s)) + \sum_{n=0}^{s-1} \mu(B(\lambda, n)) (nh(n) - (n + 1)h(n + 1))$$

$$\leq D_1(G) \left(sh(s)(s + 1)^d + \sum_{n=0}^{s-1} n(h(n) - h(n + 1))(n + 1)^d\right) = D_1(G) \sum_{n=1}^{s} h(n)(n + 1)^d - (n - 1)n^d,$$

where the inequality follows from the polynomial growth property (2.2) and the monotonic assumption on the nonnegative sequence $\{h(n)\}^\infty_{n=0}$. Hence (2.4) follows.
(ii). Take a vertex $\lambda \in V$ and an integer $s \geq 0$. Similar to the first argument, we have

$$\sum_{\rho(\lambda, \lambda') \geq s} h(\rho(\lambda, \lambda')) \leq \lim_{N \to \infty} h(N) \mu(B(\lambda, N)) + \sum_{n=s}^{N-1} \mu(B(\lambda, n))(h(n) - h(n + 1))$$

$$\leq D_1(G) \lim_{N \to \infty} \left(h(N)(N + 1)^d + \sum_{n=s}^{N-1} (n + 1)^d(h(n) - h(n + 1))\right)$$

$$= D_1(G) \left( (s + 1)^d h(s) + \sum_{n=s+1}^{\infty} h(n)((n + 1)^d - n^d) \right).$$

This proves (2.5). $\square$

3. Matrices with polynomial off-diagonal decay

Let $G$ be a connected simple graph with Beurling dimension $d$. For $1 \leq r \leq \infty$ and $\alpha \geq 0$, define

$$B_{r,\alpha}(G) := \left\{ A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} : \|A\|_{B_{r,\alpha}} < \infty \right\},$$

where $h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')|, n \geq 0$, and

$$\|A\|_{B_{r,\alpha}} := \left\{ \begin{array}{ll} \left( \sum_{n=0}^{\infty} |h_A(n)|(n + 1)^{\alpha r + d - 1} \right)^{1/r} & \text{if } 1 \leq r < \infty \\ \sup_{n \geq 0} h_A(n)(n + 1)^\alpha & \text{if } r = \infty \end{array} \right.$$

[8, 9, 12, 29, 51]. We will use the abbreviated notation $B_{r,\alpha}$ instead of $B_{r,\alpha}(G)$ if there is no confusion. The commutative subalgebra

$$A^* := \left\{ (a(k - k'))_{k, k' \in \mathbb{Z}} : \sum_{n=0}^{\infty} \sup_{|k| \geq n} |a(k)| < \infty \right\}$$

of the class $B_{r,\alpha}(G)$ with $r = 1, \alpha = 0$ and $G = \mathbb{Z}$ was introduced by Beurling to study contraction of periodic functions [9]. The set $B_{r,\alpha}(G)$ with $r = +\infty, \alpha \geq 0$ and $G = \mathbb{Z}$ is the Jaffard class $J_\alpha(\mathbb{Z})$ of matrices with polynomial off-diagonal decay ([12, 29]), since

$$\|A\|_{J_\alpha} := \sup_{i,j \in \mathbb{Z}} |a(i, j)|(1 + |i - j|)^\alpha = \|A\|_{B_{\infty,\alpha}}$$

for $A := (a(i, j))_{i, j \in \mathbb{Z}}$.

The set $B_{r,\alpha}(G)$ with $1 \leq r < \infty, \alpha \geq 0$ and $G = \mathbb{Z}$ is defined in [51] to contain all matrices $A = (a(i, j))_{i, j \in \mathbb{Z}}$ with

$$\|A\|_{B_{r,\alpha}} := \left( \sum_{n=0}^{\infty} \left( \sup_{|i-j| \geq n} |a(i, j)|(1 + |i - j|)^\alpha \right)^r \right)^{1/r} < \infty.$$
We remark that norms in (3.2) and (3.4) are equivalent to each other,

\[
\|A\|_{B_{r,\alpha}} \leq \|A\|_{B_{r,\alpha}}^* \leq 2^{2(\alpha+1/r)} \|A\|_{B_{r,\alpha}} \quad \text{for all } A \in B_{r,\alpha}(\mathcal{Z}).
\]

The first inequality in (3.5) follows immediately from (3.2) and (3.4), while the second estimate holds because for any \( A := (a(i, j))_{i,j \in \mathbb{Z}} \in B_{r,\alpha}(\mathcal{Z}) \) we have

\[
\left( \|A\|_{B_{r,\alpha}}^* \right)^r \leq \sup_{i,j \in \mathbb{Z}} |a(i, j)|^r (1 + |i - j|)^{\alpha r}
+ \sum_{l=0}^{\infty} 2^l \left( \sup_{|i-j|\geq 2^l} |a(i, j)|^r (1 + |i - j|)^{\alpha r} \right)
\leq (h_A(0))^r + \sum_{m=0}^{\infty} (h_A(2^m))^r 2^{(m+1)\alpha r}
+ \sum_{l=0}^{\infty} 2^l \sum_{m=l}^{\infty} (h_A(2^m))^r 2^{(m+1)\alpha r}
\leq (h_A(0))^r + 2^{2\alpha r+2} \sum_{m=0}^{\infty} (h_A(2^m))^r 2^{(m-1)(\alpha r+1)}
\leq 2^{2\alpha r+2} \left( \|A\|_{B_{r,\alpha}}^* \right)^r.
\]

Due to the above equivalence (3.5), we follow the terminology in [51] to call \( B_{r,\alpha}(\mathcal{G}) \) as a Beurling class of matrices with polynomial off-diagonal decay.

Define the Schur norm \( \|A\|_S \) of a matrix \( A := (a(\lambda, \lambda'))_{\lambda,\lambda' \in V} \) by

\[
\|A\|_S = \max \left( \sup_{\lambda \in V} \sum_{\lambda' \in V} |a(\lambda, \lambda')|, \sup_{\lambda' \in V} \sum_{\lambda \in V} |a(\lambda, \lambda')| \right).
\]

Shown in the proposition below are some elementary properties of the Beurling class \( B_{r,\alpha}(\mathcal{G}) \), with their proofs postponed to the end of this section.

**Proposition 3.1.** Let \( 1 \leq q, r \leq \infty, \alpha \geq 0 \), \( \mathcal{G} := (V, E) \) be a connected simple graph with Beurling dimension \( d \). Then the following statements hold.

(i) \( B_{1,0}(\mathcal{G}) \subset \mathcal{S} \subset B(\ell^q) \) and

\[
\|A\|_{B(\ell^q)} \leq \|A\|_S \leq dD_1(\mathcal{G}) \|A\|_{B_{1,0}} \quad \text{for all } A \in B_{1,0}(\mathcal{G}).
\]
Proposition 3.2. Let $\mathcal{G}$ be a connected simple graph with Beurling dimension $d$, $1 \leq r \leq \infty$, $\alpha > d(1 - 1/r)$, and set
\[
\theta_{r,\alpha} = \frac{2(\alpha - d + d/r)}{1 + 2\alpha - 2d + 2d/r}.
\]

(ii) $\mathcal{B}_{r,\beta}(\mathcal{G}) \subset \mathcal{B}_{r,\gamma}(\mathcal{G}) \subset \mathcal{B}_{r,\alpha}(\mathcal{G}) \subset \mathcal{B}_{r,\alpha}(\mathcal{G})$ for all $r'' \geq r$, $\gamma \geq \alpha$ and $\beta > \gamma + d(1/r - 1/r'')$. Moreover, following the argument in [51, 54] and applying (2.5), we approximate by matrices with finite bandwidth,
\[
\|A\|_{\mathcal{B}_{r,\alpha}} \leq \|A\|_{\mathcal{B}_{r,\gamma}} \leq \|A\|_{\mathcal{B}_{r,\beta}} \leq \left( \frac{\beta - \gamma - (d - 1)(1/r - 1/r'')}{\beta - \gamma - d(1/r - 1/r'')} \right)^{1/r - 1/r''} \|A\|_{\mathcal{B}_{r,\beta}}
\]
for all $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$.

(iii) $\mathcal{B}_{r,\alpha}(\mathcal{G})$ is a Banach algebra if $\alpha > d(1 - 1/r)$, and
\[
\|AB\|_{\mathcal{B}_{r,\alpha}} \leq 2^{\alpha+d/r}(\|B\|_{\mathcal{B}_{r,\alpha}} + \|A\| \|B\|_{\mathcal{B}_{r,\alpha}})
\]
\[
\leq 2^{\alpha+1+d/r} dD_1(\mathcal{G}) \left( \frac{\alpha - (d - 1)(1 - 1/r)}{\alpha - d(1 - 1/r)} \right)^{1 - 1/r} \|A\|_{\mathcal{B}_{r,\alpha}} \|B\|_{\mathcal{B}_{r,\alpha}}
\]
for all $A, B \in \mathcal{B}_{r,\alpha}(\mathcal{G})$.

(iv) $\mathcal{B}_{r,\alpha}(\mathcal{G})$ is solid if $\alpha \geq 0$, i.e., if $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ and $B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in V}$ satisfies $|a(\lambda, \lambda')| \leq |b(\lambda, \lambda')|$ for all $\lambda, \lambda' \in V$, then $\|A\|_{\mathcal{B}_{r,\alpha}} \leq \|B\|_{\mathcal{B}_{r,\alpha}}$.

A matrix $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in \mathcal{B}_{1,0}(\mathcal{G})$ (and hence in $\mathcal{B}_{r,\alpha}(\mathcal{G})$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ by Proposition 3.1) can be well approximated by matrices with finite bandwidth,
\[
A_N = (a(\lambda, \lambda')\chi_{\left[0,1]\left(\rho(\lambda, \lambda')/N\right)\right])_{\lambda, \lambda' \in \mathcal{G}}, \quad N \geq 0,
\]
in the Schur norm. In particular, it follows from (2.5) and (3.7) that
\[
\|A - A_N\|_S \leq D_1(\mathcal{G}) \left( (N + 2)^d h_A(N + 1) + d \sum_{n=N+2}^{\infty} h_A(n)(n + 1)^{d - 1} \right),
\]
where $h_A(n) = \sup_{\rho(\lambda, \lambda) \geq n} |a(\lambda, \lambda')|, n \geq 0$.

By (3.7) and (3.8) in Proposition 3.1, $\mathcal{B}_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ are Banach algebras, and they are subalgebras of $\mathcal{B}(\ell^p)$, $1 \leq p \leq \infty$,
\[
\|A\|_{\mathcal{B}(\ell^p)} \leq \frac{\alpha - (d - 1)(1 - 1/r)}{\alpha - d(1 - 1/r)} dD_1(\mathcal{G}) \|A\|_{\mathcal{B}_{r,\alpha}} \quad \text{for all } A \in \mathcal{B}_{r,\alpha}(\mathcal{G}).
\]
Moreover, following the argument in [51, 54] and applying (2.5), we obtain that $\mathcal{B}_{r,\alpha}$ is differential subalgebras of $\mathcal{B}(\ell^2)$.
Then there exists an absolute constant $C$ such that
\begin{equation}
\|AB\|_{B_{r,\alpha}} \leq C\|A\|_{B_{r,\alpha}}\|B\|_{B_{r,\alpha}} \left( \left( \frac{\|A\|_{B(\ell^2)}}{\|B\|_{B(\ell^2)}} \right)^{\theta_{r,\alpha}} + \left( \frac{\|B\|_{B(\ell^2)}}{\|A\|_{B(\ell^2)}} \right)^{\theta_{r,\alpha}} \right)
\end{equation}

hold for all $A, B \in B_{r,\alpha}(G)$.

Applying (3.13) repeatedly and using the argument in [50, Proposition 2.4], we can find an absolute constant $C$ such that
\begin{equation}
\|A_1 \ldots A_n\|_{B_{r,\alpha}} \leq \max_{1 \leq k \leq n} \|A_k\|_{B(\ell^2)}^n \times \left( \frac{C\max_{1 \leq k \leq n} \|A_k\|_{B_{r,\alpha}}}{\max_{1 \leq k \leq n} \|A_k\|_{B(\ell^2)}} \right)^{\theta_{r,\alpha} \log_2(1+\theta_{r,\alpha})^{1+\theta_{r,\alpha}}} \leq \infty,
\end{equation}

for all $A_1, \ldots, A_n \in B_{r,\alpha}(G), n \geq 1$. This together with Proposition 3.1 implies that Banach algebras $B_{r,\alpha}$ admit norm-controlled inversions in $B(\ell^2)$.

**Corollary 3.3.** Let $G$ be a connected simple graph with Beurling dimension $d$, and let $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$. Then matrices in the Banach algebra $B_{r,\alpha}(G)$ admit norm-controlled inversions in $B(\ell^2)$.

**Proof.** Take $A \in B_{r,\alpha}(G)$ with $A^{-1} \in B(\ell^2)$. Set $B = I - A^*A/\|A\|_{B(\ell^2)}^2$. One may verify that
\begin{equation}
\|B\|_{B(\ell^2)} \leq 1 - (\kappa(A))^{-2} < 1 \quad \text{and} \quad \|B\|_{B_{r,\alpha}} \leq 1 + \|A\|_{B(\ell^2)}^{-2} \|A^*A\|_{B_{r,\alpha}},
\end{equation}

where $\kappa(A) = \|A\|_{B(\ell^2)}\|A^{-1}\|_{B(\ell^2)}$. Therefore by (3.9), (3.14) and (3.15), we obtain
\begin{align*}
\|A^{-1}\|_{B_{r,\alpha}} &= \|(A^*A)^{-1}A^*\|_{B_{r,\alpha}} \leq C\|A^*\|_{B_{r,\alpha}}\|A\|_{B(\ell^2)}^{-2} \sum_{n=0}^{\infty} \|B^n\|_{B_{r,\alpha}} \\
&\leq C\|A\|_{B_{r,\alpha}}\|A\|_{B(\ell^2)}^{-2} \sum_{n=0}^{\infty} (1 - (\kappa(A))^{-2})^n \times \left( \frac{C\|A\|_{B(\ell^2)}^{-2}\|A\|_{B_{r,\alpha}}^2}{1 - (\kappa(A))^{-2}} \right)^{\theta_{r,\alpha} \log_2(1+\theta_{r,\alpha})^{1+\theta_{r,\alpha}}} < \infty,
\end{align*}

where $C$ is an absolute constant independent of the matrix $A$. \hfill $\square$

We conclude this section with a proof of Proposition 3.1.
Proof of Proposition 3.1. (i). The first inequality in (3.7) is well known. Take \( A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in B_{1,0}(G) \). Then it follows from Proposition 2.1 that

\[
\sum_{\lambda' \in V} |a(\lambda, \lambda')| \leq \sum_{\lambda' \in V} h_A(\rho(\lambda, \lambda'))
\leq D_1(G)(h_A(0) + d \sum_{n=1}^{\infty} h_A(n)(n + 1)^{d-1}) \leq dD_1(G)\|A\|_{B_{1,0}},
\]

where \( h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')|, n \geq 0 \). This proves the second estimate in (3.7).

(ii). The conclusion is obvious for \( r = r'' \). Then it remains to prove (3.8) for \( r < r'' \). The first inequality in (3.8) follows from the embedding property for weighted sequence spaces, and the second one is obvious. Now we prove the third inequality in (3.8). For any \( A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in B_{r''}(G) \) with \( r'' = \infty \), we have

\[
\|A\|_{B_{r''}} \leq \|A\|_{B_{\infty,0}} \sum_{n=0}^{\infty} (n + 1)^{(\gamma - \beta)r + d - 1} \leq \frac{\beta - \gamma - d/r + 1/r}{\beta - \gamma - d/r} \|A\|_{B_{\infty,0}}.
\]

This proves the third inequality in (3.8) with \( r'' = \infty \). We can use similar argument to prove the third inequality in (3.8) with \( 1 \leq r'' < \infty \).

(iii). We follow the argument in [51] where the conclusion with \( G = \mathbb{Z}^d \) is proved. Clearly \( \|\cdot\|_{B_{r,a}} \) is a norm. Then it suffices to prove (3.9). Take \( A := (a(\lambda, \lambda'))_{\lambda, \lambda' \in V} \) and \( B := (b(\lambda, \lambda'))_{\lambda, \lambda' \in V} \in B_{r',\beta}(G) \), and write \( AB := (c(\lambda, \lambda'))_{\lambda, \lambda' \in V} \). Then for all \( \lambda, \lambda' \in V \) we have

\[
|c(\lambda, \lambda')| \leq \sum_{\lambda'' \in V} |a(\lambda, \lambda'')| |b(\lambda'', \lambda')| \\
\leq h_A([\rho(\lambda, \lambda')/2]) \sum_{\lambda'' \in V} |b(\lambda'', \lambda')| \\
+ h_B([\rho(\lambda, \lambda')/2]) \sum_{\lambda'' \in V} |a(\lambda, \lambda'')| \\
\leq \|B\|_S h_A([\rho(\lambda, \lambda')/2]) + \|A\|_S h_B([\rho(\lambda, \lambda')/2]),
\]

where \( h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')| \) and \( h_B(n) = \sup_{\rho(\lambda, \lambda') \geq n} |b(\lambda, \lambda')|, n \in \mathbb{Z}_+ \). Therefore

\[
\|AB\|_{B_{\infty,0}} \leq \|B\|_S \sup_{n \geq 0} h_A([n/2])(n + 1)^{\alpha} + \|A\|_S \sup_{n \geq 0} h_B([n/2])(n + 1)^{\alpha}
\leq 2^{\alpha}(\|B\|_S \|A\|_{B_{\infty,0}} + \|A\|_S \|B\|_{B_{\infty,0}})
\]

(3.16)
for \( r = \infty \), and
\[
\|AB\|_{\mathcal{B}_{r,\alpha}} \leq \|B\|_s \left( \sum_{n=0}^{\infty} |h_A([n/2])|^r (n+1)^{\alpha r + d - 1} \right)^{1/r} \\
+ \|A\|_s \left( \sum_{n=0}^{\infty} |h_B([n/2])|^r (n+1)^{\alpha r + d - 1} \right)^{1/r}
\]
(3.17)
\[
\leq 2^{\alpha + d/r} \left( \|B\|_s \|A\|_{\mathcal{B}_{r,\alpha}} + \|A\|_s \|B\|_{\mathcal{B}_{r,\alpha}} \right)
\]
for \( 1 \leq r < \infty \). This proves the first inequality in (3.9). The second inequality in (3.9) follows from (3.7), (3.8) and the first estimate in (3.9).

(iv). The solidness follows immediately from the definition (3.1) of the Beurling class \( \mathcal{B}_{r,\alpha}(G) \).

\[ \square \]

4. \( \ell^p \)-STABILITY BOUND CONTROL

In this section, we prove the following result on lower \( \ell^p \)-stability bounds of matrices in the Beurling class for different exponent \( 1 \leq p \leq \infty \).

**Theorem 4.1.** Let \( 1 \leq p, q, r \leq \infty \), \( r' = r/(r-1) \), \( G \) be a connected simple graph with Beurling dimension \( d \), the counting measure \( \mu \) on \( G \) have the doubling property (2.7), and let \( A \in \mathcal{B}_{r,\alpha}(G) \) for some \( \alpha > d/r' \). If there exists a positive constant \( A_p \) such that
\[
\|Ac\|_p \geq A_p \|c\|_p \quad \text{for all } c \in \ell^p,
\]
then there exists a positive constant \( A_q \) such that
\[
\|Ad\|_q \geq A_q \|d\|_q \quad \text{for all } d \in \ell^q.
\]
Moreover, there exists an absolute constant \( C \), independent of matrices \( A \in \mathcal{B}_{r,\alpha}(G) \) and exponents \( 1 \leq p, q \leq \infty \), such that the lower \( \ell^q \)-stability bound \( A_q \) in (4.2) satisfies
\[
\frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_q} \leq C \begin{cases} \left( \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \right)^{(1+\theta(p,q))K_0} & \text{if } \alpha \neq 1 + d/r' \\ \left( \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \ln \left( 1 + \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{A_p} \right) \right)^{(1+\theta(p,q))K_0} & \text{if } \alpha = 1 + d/r', \end{cases}
\]
where
\[
\theta(p, q) = \frac{d[1/p - 1/q]}{K_0 \min(\alpha - d/r', 1) - d[1/p - 1/q]}
\]
and $K_0$ is a positive integer with

$$K_0 > \frac{d}{\min(\alpha - d/r', 1)}.$$  

To prove Theorem 4.1, we introduce a truncation operator $\chi^N$ and its smooth version $\Psi^N$ by

\begin{equation}
\chi^N : \left( c(\lambda') \right)_{\lambda' \in V} \mapsto \left( \chi_{[0,1]}(\rho(\lambda, \lambda')/N)c(\lambda') \right)_{\lambda' \in V}
\end{equation}

and

\begin{equation}
\Psi^N : \left( c(\lambda') \right)_{\lambda' \in V} \mapsto \left( \psi_0(\rho(\lambda, \lambda')/N)c(\lambda') \right)_{\lambda' \in V},
\end{equation}

where $\psi_0$ is the trapezoid function given by

$$\psi_0(t) = \begin{cases} 
1 & \text{if } |t| \leq 1/2 \\
2 - 2|t| & \text{if } 1/2 < |t| \leq 1 \\
0 & \text{if } |t| > 1.
\end{cases}$$

The truncation operator $\chi^N$ and its smooth version $\Psi^N$ localize a vector to the $N$-neighborhood of the vertex $\lambda$, and it can also be considered as diagonal matrices with diagonal entries $\chi_{[0,1]}(\rho(\lambda, \lambda')/N) = \chi_{B(\lambda,N)}(\lambda')$ and $\psi_0(\rho(\lambda, \lambda')/N)$, $\lambda' \in V$, respectively. Our proof of Theorem 4.1 depends on the estimate (4.19) for the commutator between a matrix in the Beurling algebra and the truncation operator $\Psi^N$. Similar estimate has been used by Sjöstrand in [46] to establish inverse-closedness of the Baskakov-Gohberg-Sjöstrand subalgebra in $B(\ell^2)$.

To prove Theorem 4.1, we recall maximal $N$-disjoint subsets $V_N \subset V$, $N \geq 1$, which means that

\begin{equation}
B(\lambda, N) \cap \left( \cup_{\lambda_m \in V_N} B(\lambda_m, N) \right) \neq \emptyset \quad \text{for all } \lambda \in V
\end{equation}

and

\begin{equation}
B(\lambda_m, N) \cap B(\lambda_n, N) = \emptyset \quad \text{for all distinct vertices } \lambda_m, \lambda_n \in V_N.
\end{equation}

We call vertices in a maximal $N$-disjoint set as fusion vertices [12]. For a maximal $N$-disjoint set $V_N$, the $N$-neighborhoods $B(\lambda_m, N)$, $\lambda_m \in V_N$, centered at fusion vertices have no common vertices by (4.7). It is shown in [12] that the $(2N)$-neighborhood $B(\lambda_m, 2N)$, $\lambda_m \in V_N$, is a covering of the set $V$.

**Proposition 4.2.** Let $G := (V, E)$ be a connected simple graph and $\mu$ have the doubling property (2.1). If $V_N$ is a maximal $N$-disjoint subset...
af $V$, then

$$1 \leq \inf_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda) \leq \sup_{\lambda \in V} \sum_{\lambda_m \in V_N} \chi_{B(\lambda_m, N')}(\lambda) \leq (D_0(G))^{\lceil \log_2(2N'/N+1) \rceil}$$

(4.8)

for all $N' \geq 2N$.

To prove Theorem 4.1, we first establish its weak version, the equivalence between $\ell^p$ and $\ell^q$-stabilities of a matrix with small $|1/p - 1/q|$.

**Lemma 4.3.** Let $p, r, r', d, \alpha, G, A$ be as in Theorem 4.1. If $1 \leq q \leq \infty$ satisfies

$$d|1/p - 1/q| < \min(\alpha - d/r', 1),$$

(4.9)

then $A$ has $\ell^q$-stability. Furthermore there exists an absolute constant $C$, independent of matrices $A \in B_{r, \alpha}(G)$ and exponents $1 \leq p, q \leq \infty$, such that the optimal lower $\ell^q$-stability bound $A_q$ of the matrix $A$ satisfies

$$A_q \geq CA_p \times \begin{cases} \left( \frac{\|A\|_{B_{r, \alpha}}}{A_p} \right)^{-\theta_1(p,q)} & \text{if } \alpha \neq d/r' + 1 \\ \left( \frac{\|A\|_{B_{r, \alpha}}}{A_p} \ln \left( 1 + \frac{\|A\|_{B_{r, \alpha}}}{A_p} \right) \right)^{-\theta_1(p,q)} & \text{if } \alpha = d/r' + 1, \end{cases}$$

(4.10)

where

$$\theta_1(p, q) = \frac{d|1/p - 1/q|}{\min(\alpha - d/r', 1) - d|1/p - 1/q|}.$$  

**Proof.** Let $N \geq 2$ be a positive integer chosen later, $V_N$ be a maximal $N$-disjoint set of fusion vertices satisfying (4.6) and (4.7), and let $\Psi_{\lambda}', \lambda \in V$, be the localization operators in (4.5). Take $c = (c(\lambda))_{\lambda \in V} \in \ell^q$. Applying the covering property (4.8) of $\{B(\lambda_m, 2N), \lambda_m \in V_N\}$, we have

$$\|c\|_q \leq \left\| \left( \|\Psi_{\lambda_m}^{4N}c\|_q \right)_{\lambda_m \in V_N} \right\|_q.$$  

Combining it with the polynomial growth property (2.2) and the norm equivalence between $\|\Psi_{\lambda_m}^{4N}c\|_p$ and $\|\Psi_{\lambda_m}^{4N}c\|_q$, we obtain

$$\|c\|_q \leq \left\| \left( (\mu(B(\lambda_m, 4N)))^{(1/q-1/p)+} \|\Psi_{\lambda_m}^{4N}c\|_p \right)_{\lambda_m \in V_N} \right\|_q \leq CN^{d(1/q-1/p)+} \left\| \left( \|\Psi_{\lambda_m}^{4N}c\|_p \right)_{\lambda_m \in V_N} \right\|_q.$$  

(4.11)

Here in the proof, the capital letter $C$ denotes an absolute constant independent of matrices $A$, sequences $c$, integers $N$, and exponents $p$ and $q$, which is not necessarily the same at each occurrence.
For $\lambda \in V$, it follows from the $\ell^p$-stability (4.1) for the matrix $A$ that
\begin{equation}
A_p\|\Psi_{\lambda}^{4N} c\|_p \leq \|A\Psi_{\lambda}^{4N} c\|_p.
\end{equation}

Let $A_N, N \geq 2$, be matrices with finite bandwidth in (3.10). Combining (4.11) and (4.12), we get
\begin{equation}
A_p\|c\|_q \leq CN^d(1/q-1/p)+\left(\|A\Psi_{\lambda}^{4N} c\|_p\right)_{\lambda m\in V_N}\|q
\leq CN^d(1/q-1/p)+\left(\|A-A_N\|\Psi_{\lambda}^{4N} c\|_p\right)_{\lambda m\in V_N}\|q
+\left(\|\Psi_{\lambda}^{4N}(A_N-A)c\|_p\right)_{\lambda m\in V_N}\|q
\leq CN^d(1/q-1/p)+\|A-A_N\|\|c\|_q.
\end{equation}

where $[A_N, \Psi_{\lambda}^{4N}] = A_N\Psi_{\lambda}^{4N} - \Psi_{\lambda}^{4N}A_N$ is the commutator between $A_N$ and $\Psi_{\lambda}^{4N}$ (46.51).

For any $d \in \ell^q$, we obtain from the support property for $\Psi_{\lambda}^{2N}$, the equivalence between two norms $\|\chi_{\lambda m}^{4N} d\|_p$ and $\|\chi_{\lambda m}^{4N} d\|_q$, the polynomial growth property (2.2) and the covering property in Proposition 4.12 that
\begin{equation}
\|\|\Psi_{\lambda m}^{4N} c\|_p\|_{\lambda m\in V_N}\|\|q
\leq \|\|\chi_{\lambda m}^{4N} d\|_p\|_{\lambda m\in V_N}\|\|q
\leq \|\|\chi_{\lambda m}^{4N} d\|_q(\mu(B(\lambda_m, 4N))^{(1/p-1/q)+})_{\lambda m\in V_N}\|\|q
\leq CN^d(1/p-1/q)+\|\|\chi_{\lambda m}^{4N} d\|_q\|_{\lambda m\in V_N}\|\|q
\leq CN^d(1/p-1/q)\|\|A-A_N\|\|c\|_q.
\end{equation}

This together with (3.17) yields the following three estimates:
\begin{equation}
\|\|\Psi_{\lambda m}^{4N} c\|_p\|_{\lambda m\in V_N}\|\|q \leq CN^d(1/p-1/q)+\|A-c\|_q,
\end{equation}

and
\begin{equation}
\|\|\chi_{\lambda m}^{4N} d\|_p\|_{\lambda m\in V_N}\|\|q \leq \|\|\chi_{\lambda m}^{4N} d\|_q\|_{\lambda m\in V_N}\|\|q
\leq CN^d(1/p-1/q)+\|A-A_N\|\|c\|_q.
\end{equation}

Applying similar argument, we obtain
\begin{equation}
\|\|\|\Psi_{\lambda m}^{4N} c\|_p\|_{\lambda m\in V_N}\|\|q
\leq \left(\sup_{\lambda \in V}\|\|\chi_{\lambda m}^{4N} d\|_q\|_{\lambda m\in V_N}\|\right)^{1/p-1/q}.
\end{equation}
Combining (4.13)–(4.17), we get
\[ A_p\|c\|_q \leq CN^{d(1/p-1/q)} (\|A - A_N\|_S + \sup_{\lambda \in V} \|[A_N, \Psi_\lambda^{4N}]\|_S)\|c\|_q \]
(4.18) \quad +CN^{d(1/p-1/q)} \|Ac\|_q.

For any \( \lambda \in V \), we have
\[ \|[A_N, \Psi_\lambda^{4N}]\|_S = \left\| \left( a(\lambda', \lambda'')\chi_{[0,1]} \left( \frac{\rho(\lambda', \lambda'')}{N} \right) \right. \right. \]
\[ \left. \left. \times \left( \psi_0 \left( \frac{\rho(\lambda', \lambda)}{4N} \right) - \psi_0 \left( \frac{\rho(\lambda', \lambda)}{4N} \right) \right) \right\|_S \]
\[ \leq \frac{1}{2N} \left\| \left( |a(\lambda', \lambda'')| \rho(\lambda', \lambda'')|\chi_{[0,1]} \left( \frac{\rho(\lambda', \lambda'')}{N} \right) \right) \right\|_S \]
\[ \leq CN^{-1} \sum_{n=0}^N h_A(n)(n+1)^d, \]
where the last inequality follows from (2.4). Therefore for any \( \lambda \in V \),
\[ \|[A_N, \Psi_\lambda^{4N}]\|_S \leq CN^{-1} \|A\|_{\mathcal{B}_{r,\alpha}} \left( \sum_{n=0}^N (n+1)^{-(\alpha-1)r'+d-1} \right)^{1/r'} \]
(4.20) \quad \leq C\|A\|_{\mathcal{B}_{r,\alpha}} \times \begin{cases} 
N^{-1} & \text{if } \alpha > 1 + d/r' \\
N^{-1}(\ln(N + 1))^{1-1/r} & \text{if } \alpha = 1 + d/r' \\
N^{-\alpha+d/r'} & \text{if } \alpha < 1 + d/r'. 
\end{cases}

For the Schur norm of \( A - A_N \), there exists an absolute constant \( C_0 \), independent of \( N \geq 1 \) and \( A \in \mathcal{B}_{r,\alpha} \), such that
\[ \|A - A_N\|_S \leq C \left( (N + 2)^d h_A(N + 1) + d \sum_{n=N+2}^{\infty} h_A(n)(n+1)^{d-1} \right) \]
(4.21) \quad \leq C_0\|A\|_{\mathcal{B}_{r,\alpha}} N^{-\alpha+d/r'},
where the first inequality follows from (3.11) and the second inequality is true because
\[ \sum_{n=0}^{N+1} (h_A(n))^r(n+1)^{\alpha r+d-1} \geq (h_A(N + 1))^r \int_{0}^{N+2} t^{\alpha r+d-1} dt \]
\[ = (\alpha r + d)^{-1} (h_A(N + 1))^r (N + 2)^{\alpha r+d} \]
and
\[ \sum_{n=N+2}^{\infty} (n+1)^{-\alpha r'+d-1} \leq \int_{N+2}^{\infty} t^{-\alpha r'+d-1} dt \leq \frac{1}{\alpha r' - d} (N + 2)^{-\alpha r'+d}. \]
Combining (4.18), (4.20) and (4.21), we obtain
\[ A_p \|c\|_q \leq C_1 \|A\|_B N^{d[1/p−1/q]−\min(\alpha−d/r',1)} \|c\|_q \]
\[ + C N^{d[1/p−1/q]} \|Ac\|_q \]
(4.22)
if \( \alpha \neq 1 + d/r' \), and
\[ A_p \|c\|_q \leq C_1 \|A\|_B N^{d[1/p−1/q]−1}(\ln N)^{1/r'} \|c\|_q \]
\[ + C N^{d[1/p−1/q]} \|Ac\|_q \]
(4.23)
if \( \alpha = 1 + d/r' \), where \( C_1 \) is an absolute constant independent of matrices \( A \), integers \( N \) and sequences \( c \).

For \( \alpha \neq 1 + d/r' \), replacing \( N \) in (4.22) by \( N_0 = \left( 2C_1 \|A\|_B A_p \right)^{\min(\alpha−d/r',1)−d[1/p−1/q]−1} \), we get from (4.9) and (4.23) that
\[ A_p \|c\|_q \leq \frac{A_p}{2} \|c\|_q + C \left( \frac{\|A\|_B}{A_p} \right)^{\theta_1(p,q)} \|Ac\|_q. \]
This proves (4.10) for \( \alpha \neq 1 + d/r' \).

For \( \alpha = 1 + d/r' \), set \( C_2 := \frac{8C_1 \|A\|_B}{(1−d[1/p−1/q])A_p} \geq 8 \)
and
\[ N_1 := \left( (C_2 \ln C_2)\right)^{(1−d[1/p−1/q])−1} \geq \frac{1}{2} (C_2 \ln C_2)^{(1−d[1/p−1/q])−1}. \]
Then
\[ C_1 \|A\|_B N_1^{d[1/p−1/q]−1} \leq \frac{(1−d[1/p−1/q])A_p}{4 \ln C_2} \]
and
\[ N_1 \leq (C_2 \ln C_2)^{(1−d[1/p−1/q])−1} \leq C_2^{2(1−d[1/p−1/q])−1}. \]
Replacing \( N \) in (4.23) by \( N_1 \) and applying (4.24) and (4.25), we obtain
\[ A_p \|c\|_q \leq \frac{(1−d[1/p−1/q])A_p}{4 \ln C_2} \left( 2(1−d[1/p−1/q]) \ln C_2 \right)^{1−1/r} \|c\|_q \]
\[ + C(C_2 \ln C_2)^{\theta_1(p,q)} \|Ac\|_q \]
\[ \leq \frac{A_p}{2} \|c\|_q + C(C_2 \ln C_2)^{\theta_1(p,q)} \|Ac\|_q. \]
This proves (4.10) for \( \alpha = 1 + d/r' \).

Having the above technical lemma, we use a bootstrap approach to prove Theorem 4.1, cf. [29, 44, 51].
Proof of Theorem 4.1. Let $K$ be a positive integer with
\[ d|1/p - 1/q|/K < \min(\alpha - d/r', 1). \]
Then $K \leq K_0$. Let $\{p_k\}_{k=0}^K$ be a monotone sequence such that
\[ p_0 = p, p_K = q \text{ and } |1/p_k - 1/p_{k+1}| = |1/p - 1/q|/K, 0 \leq k \leq K - 1. \]
Applying Lemma 4.3 repeatedly, we conclude that $A$ has $\ell^p$-stability for all $1 \leq k \leq K$. Moreover the lower $\ell^p$-stability bound $A_{p_k}$ satisfies
\begin{equation}
\frac{A_{p_k}}{A_{p_{k+1}}} \leq C \begin{cases}
(\|A\|_{B_{r', \alpha}}/A_{p_k})^{\theta_K(p, q)} & \text{if } \alpha \neq d/r' + 1 \\
(\|A\|_{B_{r', \alpha}}/A_{p_k}) \ln \left(1 + \|A\|_{B_{r', \alpha}}/A_{p_k}\right)^{\theta_K(p, q)} & \text{if } \alpha = d/r' + 1
\end{cases}
\end{equation}
for all $0 \leq k \leq K - 1$, where
\[ \theta_K(p, q) = \frac{d|1/p - 1/q|}{K \min(\alpha - d/r', 1) - d|1/p - 1/q|} \]
and $C$ is an absolute constant independent of $A \in B_{r, \alpha}$.

For $\alpha \neq 1 + d/r'$, we obtain from (4.26) that
\begin{equation}
\frac{A_{p_{k+1}}}{\|A\|_{B_{r', \alpha}}} \geq C \left(\frac{A_p}{\|A\|_{B_{r', \alpha}}}\right)^{(1+\theta_K(p, q))^{k+1}}, 0 \leq k \leq K - 1.
\end{equation}
This proves (4.3) for $\alpha \neq 1 + d(1 - 1/r)$.

For $\alpha = 1 + d/r'$, it follows from (4.26) that
\[ \frac{\|A\|_{B_{r', \alpha}}}{A_{p_{k+1}}} \leq C \left(\frac{A_p}{\|A\|_{B_{r', \alpha}}}\right)^{1+\theta_K(p, q)} \left(\ln \left(1 + \frac{\|A\|_{B_{r', \alpha}}}{A_{p_k}}\right)\right)^{\theta_K(p, q)}, 0 \leq k \leq K - 1. \]
Applying the above estimate repeatedly, we obtain
\[ \frac{\|A\|_{B_{r', \alpha}}}{A_{p_k}} \leq C \left(\frac{A_p}{\|A\|_{B_{r', \alpha}}}\right)^{(1+\theta_K(p, q))^k} \left(\ln \left(1 + \frac{\|A\|_{B_{r', \alpha}}}{A_{p}}\right)\right)^{(1+\theta_K(p, q))^{k-1}} \]
by induction on $1 \leq k \leq K$. This proves (4.3) for $\alpha = 1 + d/r'$.

5. Norm-controlled inversion

By Corollary 3.3, matrices in Banach algebras $B_{r, \alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ admit norm-controlled inversions in $B(\ell^2)$. In this section, we show that a polynomial can be selected to be the norm-controlled function $h$ in (1.7) if the the counting measure $\mu$ on the graph $G$ is normal.
Theorem 5.1. Let $1 \leq r \leq \infty, r' = r/(r - 1)$, $\alpha > d/r'$, $\mathcal{G}$ be a connected simple graph with Beurling dimension $d$ and normal counting measure $\mu$, and let $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ be invertible in $\mathcal{B}(\ell^2)$. Then there exists an absolute constant $C$, independent of $A$, such that

\[ \|A^{-1}\|_{\mathcal{B}_{r,\alpha}} \leq C \|A^{-1}\|_{\mathcal{B}(\ell^2)} \left( \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}} \right)^{(\alpha + d/r)/\min(\alpha - d/r', 1)} \]

(5.1)

\[ \times \begin{cases} 1 & \text{if } \alpha \neq 1 + d/r' \\ \left( \ln \left( \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}} + 1 \right) \right)^{(d+1)/r'} & \text{if } \alpha = 1 + d/r'. \end{cases} \]

For invertible matrices $A \in \mathcal{B}_{r,\alpha}(\mathcal{G})$ with $\alpha > 1 + d/r'$, it follows from Theorem 5.1 that

\[ \|A^{-1}\|_{\mathcal{B}_{r,\alpha}} \leq C \|A^{-1}\|_{\mathcal{B}(\ell^2)} \left( \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}} \right)^{\alpha + d/r}. \]

(5.2)

A weak version of the above estimate, with the exponent $\alpha + d/r$ in (5.2) replaced by a larger exponent $2\alpha + 2 + 2/(\alpha - 2)$, is established in [27] for matrices in the Jaffard algebra $\mathcal{J}_\alpha(\mathbb{Z}) = \mathcal{B}_{\infty,\alpha}(\mathbb{Z})$, where $r = \infty$.

The estimate (5.2) on norm-controlled inversion is almost optimal, as shown in the following example that for any $\epsilon > 0$ there does not exist an absolute constant $C_\epsilon$ such that

\[ \|A^{-1}\|_{\mathcal{B}_{r,\alpha}} \leq C_\epsilon \|A^{-1}\|_{\mathcal{B}(\ell^2)} \left( \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha}} \right)^{\alpha + d/r - 1 + \epsilon}. \]

(5.3)

Example 5.2. Let $1 \leq r \leq \infty, \alpha > 1 - 1/r$ and $\mathcal{G} = \mathbb{Z}^d$ with $d = 1$. For sufficiently small $\gamma \in (0, 1)$, define $A_{\gamma} = (a_{\gamma}(i, j))_{i, j \in \mathbb{Z}}$ by

\[ a_{\gamma}(i, j) = \begin{cases} 1 & \text{if } j = i \\ -e^{-\gamma} & \text{if } j = i + 1 \\ 0 & \text{elsewhere}. \end{cases} \]

(5.4)

Then

\[ \|A_{\gamma}\|_{\mathcal{B}_{r,\alpha}} = (1 + 2^\alpha e^{-\gamma r})^{1/r} \in [2^{\alpha - 1}, 2^{\alpha + 1}]. \]

(5.5)

Observe that $A_{\gamma}$ is invertible in $\mathcal{B}(\ell^2)$ and its inverse is given by $B_{\gamma} = (b_{\gamma}(i, j))_{i, j \in \mathbb{Z}}$, where

\[ b_{\gamma}(i, j) = \begin{cases} e^{-(j-i)\gamma} & \text{if } j \geq i \\ 0 & \text{elsewhere}. \end{cases} \]

(5.6)

Therefore for sufficiently small $\gamma \in (0, 1)$, we have

\[ \|A_{\gamma}^{-1}\|_{\mathcal{B}(\ell^2)} = (1 - e^{-\gamma})^{-1} \in [\gamma^{-1}, 2\gamma^{-1}] \]
and
\[
\|A^{-1}\|_{B_r, \alpha} = \begin{cases} 
\left( \sum_{n=0}^{\infty} (n+1)^{\alpha} e^{-nr} \right)^{1/r} & \text{if } 1 \leq r < \infty \\
\sup_{n \geq 0} (n+1)^{\alpha} e^{-nr} & \text{if } r = \infty
\end{cases}
\]
(5.7)
\[
\in \gamma^{-\alpha-1/r}[1,2] \times \begin{cases} 
\left( \frac{\Gamma(\alpha r + 1)}{(\alpha/e)^{\alpha}} \right)^{1/r} & \text{if } 1 \leq r < \infty \\
\left( \frac{\Gamma(\alpha r + 1)}{\alpha} \right)^{1/r} & \text{if } r = \infty
\end{cases}
\]
where \( \Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \) is the Gamma function. Hence for sufficiently small \( \gamma \), the left hand side of (5.3) is of order \( \gamma^{-\alpha-1/r} \) and the right hand side of (5.3) is of order \( \gamma^{-\alpha-1/r+\epsilon} \) for \( \alpha > 1 + d/r' \). This proves (5.3).

To prove Theorem 5.1, we need a distribution property for fusion vertices of a maximal \( N \)-disjoint set.

**Proposition 5.3.** Let \( G := (V,E) \) be a connected simple graph with Beurling dimension \( d \) and normal counting measure \( \mu \), and let \( V_N, 1 \leq N \leq \text{diam}(G) \), be maximal \( N \)-disjoint sets of fusion vertices. Then for all \( \lambda \in V \),

(5.8)
\[
\#\{\lambda_m \in V_N, \rho(\lambda_m, \lambda) \leq NR\} \leq \frac{D_1(G)}{D_2(G)} (R+1)^d, 0 \leq R \leq \frac{\text{diam}(G)}{N} + 1,
\]
and

(5.9)
\[
\#\{\lambda_m \in V_N, \rho(\lambda_m, \lambda) \leq NR\} \geq \frac{D_2(G)}{D_1(G)} \left( \frac{R-2}{3} \right)^d, 3 \leq R \leq \frac{\text{diam}(G)}{N} + 1.
\]

**Proof.** Take a vertex \( \lambda \in V \) and a nonnegative integer \( R \). Set
\[
E = \{\lambda_m \in V_N : \rho(\lambda_m, \lambda) \leq NR\}.
\]
Then
\[
\cup_{\lambda_m \in E} B(\lambda_m, N) \subset B(\lambda, N(R+1)).
\]
This, together with (2.2), (2.3) and (4.7), implies that
\[
D_2(G)(N+1)^d \#E \leq \sum_{\lambda_m \in E} \mu(B(\lambda_m, N))
\]
\[
= \mu(\cup_{\lambda_m \in E} B(\lambda_m, N)) \leq D_1(G)(N(R+1)+1)^d.
\]
Hence the upper bound estimate in (5.8) follows.

Take a vertex \( \lambda \in V \) and an integer \( R \geq 3 \). Applying the covering property (1.8), we have
\[
B(\lambda, (R-2)N) \subset \cup_{\lambda_m \in E} B(\lambda_m, 2N).
\]
This together with (2.2) and (2.3) implies that
\[
D_2(G)((R-2)N+1)^d \leq D_1(G)(2N+1)^d \#E.
\]
Hence the lower bound estimate in (5.9) follows. \[ \square \]

Let \( V_N, N \geq 1 \), be maximal \( N \)-disjoint sets of fusion vertices. Let \( B_{r,\alpha;N} \) contain all matrices \( B := (b_{\lambda_m, \lambda_m'})_{\lambda_m, \lambda_m' \in V_N} \) with \( \|B\|_{B_{r,\alpha;N}} < \infty \), where \( h_{B,N}(n) = \sup_{\rho(\lambda_m, \lambda_m') \geq Nn} |b_{\lambda_m, \lambda_m'}|, n \geq 0 \), and

\[
\|B\|_{B_{r,\alpha;N}} = \left\{ \begin{array}{ll}
\left( \sum_{n=0}^{\infty} (h_{B,N}(n))^{r}(n + 1)^{\alpha r + d - 1} \right)^{1/r} & \text{if } 1 \leq r < \infty \\
\sup_{n \geq 0} h_{B,N}(n)(n + 1)^{\alpha} & \text{if } r = \infty,
\end{array} \right.
\]

cf. the Beurling class \( B_{r,\alpha}(G) \) in (3.2). Clearly \( \cdot \|_{B_{r,\alpha;N}} \) is a norm. The next proposition states that \( B_{r,\alpha;N} \) are Banach algebras.

**Proposition 5.4.** Let \( G, r, \alpha \) be as in Theorem 5.1, and let \( V_N, N \geq 1 \), be a maximal \( N \)-disjoint set of fusion vertices. Then there exists an absolute constant \( C \), independent of integers \( N \geq 1 \), such that

\[
(5.10) \quad \|AB\|_{B_{r,\alpha;N}} \leq C\|A\|_{B_{r,\alpha;N}} \|B\|_{B_{r,\alpha;N}} \quad \text{for all } A, B \in B_{r,\alpha;N}.
\]

The above lemma can be proved by following the argument used in Proposition 3.1. We omit the detailed proof here.

To prove Theorem 5.1, we also need a technical lemma.

**Lemma 5.5.** Let \( G, A, r, r', \alpha \) be as in Theorem 5.1, and let \( V_N, N \geq 2 \), be maximal \( N \)-disjoint sets of fusion vertices. Then there exists an absolute constant \( C_0 \) independent of \( A \) such that

\[
(5.11) \quad \|\Psi_{\lambda_m}^{4N} c\|_2 \leq 2\|A^{-1}\|_{B(\ell^2)} \left( \|\Psi_{\lambda_m}^{4N} Ac\|_{\ell^2} + \sum_{\lambda_m' \in V_N} \|\chi_{\lambda_m}[\Psi_{\lambda_m}^{4N} A] \chi_{\lambda_m'}\| \|\Psi_{\lambda_m'}^{4N} c\|_2 \right)
\]

for all vertices \( \lambda_m \in V_N \), sequences \( c \in \ell^2 \) and integers \( N \) satisfying

\[
(5.12) \quad N^{\alpha - d/r'} \geq 2C_0\|A^{-1}\|_{B(\ell^2)}\|A\|_{B_{r,\alpha}},
\]

where \( [\Psi_{\lambda_m}^{4N} A] = \Psi_{\lambda_m}^{4N} A - A\Psi_{\lambda_m}^{4N} \) and \( C_0 \) is the constant in (4.21). Moreover, there exists an absolute constant \( C \) independent of \( A \in B_{r,\alpha}(G) \) such that

\[
(5.13) \quad \left( \sum_{n=0}^{\infty} (\sum_{Nn}^{\infty} h_{A,N}(n))^{r}(n + 1)^{\alpha r + d - 1} \right)^{1/r} \leq C\|A\|_{B_{r,\alpha}} N^{-\min(\alpha - d/r', 1)} \times \left\{ \begin{array}{ll} 1 & \text{if } \alpha \neq 1 + d/r' \\
(\ln N)^{1/r'} & \text{if } \alpha = 1 + d/r'
\end{array} \right.
\]

if \( 1 \leq r < \infty \), and

\[
(5.14) \quad \sup_{n \geq 0} h_{A,N}(n)(n + 1)^{\alpha} \leq C\|A\|_{B_{r,\alpha}} N^{-\min(\alpha - d, 1)} \times \left\{ \begin{array}{ll} 1 & \text{if } \alpha \neq d + 1 \\
\ln N & \text{if } \alpha = d + 1
\end{array} \right.
\]

\[ \square \]
if \( r = \infty \), where

\[
(5.15) \quad h_{A,N}(n) = \sup_{\rho(\lambda_m, \lambda_{m'}) \geq Nn} ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}||_S, \; n \geq 0.
\]

**Proof.** We follow the argument in \([46, 51]\) where \( G = \mathbb{Z}^d \) with \( d = 1 \). Take \( \lambda_m \in V_N, c := (c(\lambda))_{\lambda \in V} \in \ell^2 \), and let \( A_N \) be as in (3.10). By the invertibility on \( A \), we have

\[
||\Psi_{\lambda_m}^{4N}c||_2 \leq ||A^{-1}||_{B(\ell^2)}||A\Psi_{\lambda_m}^{4N}c||_2
\]

(5.16)

\[
\leq ||A^{-1}||_{B(\ell^2)}||A\Psi_{\lambda_m}^{4N}Ac||_2 + ||A^{-1}||_{B(\ell^2)}||[\Psi_{\lambda_m}^{4N}, A]c||_2.
\]

By the covering property in Proposition 4.2, \( \Psi^{4N} := \sum_{\lambda_{m'} \in V_N} \Psi_{\lambda_{m'}}^{4N} \) is a diagonal matrix with bounded inverse, and

\[
(5.17) \quad ||(\Psi^{4N})^{-1}||_{B(\ell^2)} \leq 1.
\]

Therefore

\[
||[\Psi_{\lambda_m}^{4N}, A]c||_2 \leq ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]c||_2 + ||(I - \chi_{\lambda_m}^{5N})A\chi_{\lambda_m}^{4N}\Psi_{\lambda_m}^{4N}c||_2
\]

\[
\leq \sum_{\lambda_{m'} \in V_N} ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}(\Psi^{4N})^{-1}\Psi_{\lambda_m}^{4N}c||_2
\]

\[
+ ||(I - \chi_{\lambda_m}^{5N})A\Psi_{\lambda_m}^{4N}c||_2
\]

\[
\leq \sum_{\lambda_{m'} \in V_N} ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}||_{B(\ell^2)}||(\Psi^{4N})^{-1}||_{B(\ell^2)}||\Psi_{\lambda_m}^{4N}c||_2
\]

\[
+ ||(I - \chi_{\lambda_m}^{5N})A\chi_{\lambda_m}^{4N}c||_2
\]

\[
\leq \sum_{\lambda_{m'} \in V_N} ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}||_S||\Psi_{\lambda_m}^{4N}c||_2
\]

\[
+ ||A - A_N||_S||\Psi_{\lambda_m}^{4N}c||_2,
\]

(5.18)

where the last inequality follows from (5.17) and Proposition 3.1. Combining (5.16) and (5.18), and then using (4.21) and (5.12), we complete the proof of the upper bound estimate (5.11) for \( ||\Psi_{\lambda_m}^{4N}c||_2 \).

Write \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in E} \) and define \( h_A(n) = \sup_{\rho(\lambda, \lambda') \geq n} |a(\lambda, \lambda')| \). For \( \lambda_m, \lambda_{m'} \in V_N \) with \( \rho(\lambda_m, \lambda_{m'}) \geq 16N \), we obtain from (2.2) and the supporting property for \( \Psi_{\lambda_m}^{4N} \) that

\[
(5.19) \quad ||\chi_{\lambda_m}^{5N}[\Psi_{\lambda_m}^{4N}, A]\chi_{\lambda_{m'}}^{4N}||_S = ||\Psi_{\lambda_m}^{4N}A\chi_{\lambda_{m'}}^{4N}||_S \leq C h_A(\rho(\lambda_m, \lambda_{m'})/2) N^d.
\]
For $\lambda_m, \lambda_{m'} \in V_N$ with $\rho(\lambda_m, \lambda_{m'}) < 16N$,

$$
\|\chi^5_{\lambda_m}[\Psi^4_{\lambda_m}, A]\chi^4_{\lambda_{m'}}\|_S
= \left\| \left( \chi_{[0,5N]}(\rho(\lambda, \lambda_m))a(\lambda, \lambda')\chi_{[0,4N]}(\rho(\lambda', \lambda_{m'})) \times \left( \psi_0 \left( \frac{\rho(\lambda, \lambda_m)}{4N} \right) - \psi_0 \left( \frac{\rho(\lambda', \lambda_{m'})}{4N} \right) \right) \right\|_{S, \lambda, \lambda' \in V_N}
\leq CN^{-1} \sum_{n=0}^{25N} h_A(n)(n+1)^d
$$

(5.20)

where the first inequality follows from (2.4), and the last estimate is obtained by applying a Hölder inequality, cf. (4.20). Combining (5.19) and (5.20) proves (5.13) and (5.14). □

Now we start to the proof of Theorem 5.1.

**Proof of Theorem 5.1.** Let $N \geq 2$ be chosen later. Define $V_{A,N} := (V_{A,N}(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_N}$ and write $(V_{A,N})^l = (V_{A,N}^l(\lambda_m, \lambda_{m'}))_{\lambda_m, \lambda_{m'} \in V_N}$, where

$$
V_{A,N}(\lambda_m, \lambda_{m'}) = 2 \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|\chi^5_{\lambda_m}[\Psi^4_{\lambda_m}, A]\chi^4_{\lambda_{m'}}\|_S
$$

Then we obtain from Proposition 5.4 and Lemma 5.5 that

$$
\|V_{A,N}\|_{\mathcal{B}_{r,\alpha,N}} \leq D_3 \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha,N}} N^{-\min(\alpha-d/r',1)} \times \begin{cases} 
1 & \text{if } \alpha \neq 1 + d/r' \\
(\ln N)^{1/r'} & \text{if } \alpha = 1 + d/r',
\end{cases}
$$

(5.21)

and

$$
\|(V_{A,N})^l\|_{\mathcal{B}_{r,\alpha,N}} \leq D_4^{l-1} \|(V_{A,N})^l\|_{\mathcal{B}_{r,\alpha,N}}
$$

(5.22)

where $D_3, D_4$ are absolute constants independent of matrices $A$ and integers $N$ and $l$.

Let $N_2 \geq 2$ be the minimal integer satisfying (5.12) and

$$
1 \geq 4D_3D_4 \|A^{-1}\|_{\mathcal{B}(\ell^2)} \|A\|_{\mathcal{B}_{r,\alpha,N_2}} N_2^{-\min(\alpha-d/r',1)} \times \begin{cases} 
1 & \text{if } \alpha \neq 1 + d/r' \\
(\ln N_2)^{1/r'} & \text{if } \alpha = 1 + d/r',
\end{cases}
$$

(5.23)
Then
\[ N_2 \leq C \left( \| A^{-1} \|_{B(\ell^2)} \| A \|_{B_{r,A}} \right)^{1/\min(\alpha-d/r',1)} \]
(5.24)
\[ \times \left\{ \begin{array}{ll} \frac{1}{(\ln \left( \| A^{-1} \|_{B(\ell^2)} \| A \|_{B_{r,A}} + 1 \right))^{1/r'}} & \text{if } \alpha \neq 1 + d/r' \\
\frac{1}{2} & \text{if } \alpha = 1 + d/r'. \end{array} \right. \]

Let \( W_{A,N_2} = \sum_{l=1}^{\infty} (V_{A,N_2})^l \). By (5.21), (5.22) and (5.23), we have
\[ \| (V_{A,N_2})^l \|_{B_{r,A,N_2}} \leq C 2^{-l}, \quad l \geq 1, \]
which implies that
\[ \| W_{A,N_2} \|_{B_{r,A,N_2}} \leq C. \]

For any \( \lambda_m \in V_{N_2} \) and \( c \in \ell^2(G) \), applying (5.11) repeatedly we obtain
\[ \| \Psi_{\lambda_m}^{4N_2} c \|_2 \leq 2 \| A^{-1} \|_{B(\ell^2)} \| \Psi_{\lambda_m}^{N_2} A c \|_2 + \sum_{\lambda_m' \in V_{N_2}} V_{A,N_2}(\lambda_m, \lambda_m') \| \Psi_{\lambda_m'}^{N_2} c \|_2 \]
\[ \leq \cdots \]
\[ \leq 2 \| A^{-1} \|_{B(\ell^2)} \| \Psi_{\lambda_m}^{N_2} A c \|_2 + \sum_{\lambda_m' \in V_{N_2}} \sum_{l=1}^{k} V_{A,N_2}'(\lambda_m, \lambda_m') \| \Psi_{\lambda_m'}^{N_2} c \|_2 \]
(5.27)
\[ + 2 \| A^{-1} \|_{B(\ell^2)} \sum_{l=1}^{k} \sum_{\lambda_m' \in V_{N_2}} V_{A,N_2}'(\lambda_m, \lambda_m') \| \Psi_{\lambda_m'}^{N_2} A c \|_2, \quad k \geq 2. \]

Using the argument used to prove the first conclusion in Proposition 3.1, we have
\[ \sum_{\lambda_m' \in V_{N_2}} V_{A,N_2}'(\lambda_m, \lambda_m') \| \Psi_{\lambda_m'}^{N_2} c \|_2 \]
(5.28)
\[ \leq \sum_{\lambda_m' \in V_{N_2}} V_{A,N_2}'(\lambda_m, \lambda_m') \| c \|_2 \leq C \| (V_{A,N_2})^{k+1} \|_{B_{r,A,N_2}} \| c \|_2. \]

Taking limit in (5.27), we obtain from (5.25) and (5.28) that
\[ \| \Psi_{\lambda_m}^{4N_2} c \|_2 \leq 2 \| A^{-1} \|_{B(\ell^2)} \| \Psi_{\lambda_m}^{N_2} A c \|_2 \]
(5.29)
\[ + 2 \| A^{-1} \|_{B(\ell^2)} \sum_{\lambda_m' \in V_{N_2}} W_{A,N_2}(\lambda_m, \lambda_m') \| \Psi_{\lambda_m'}^{N_2} A c \|_2, \]
where \( W_{A,N_2} = \left( W_{A,N_2}(\lambda_m, \lambda_m') \right)_{\lambda_m, \lambda_m' \in V_{N_2}} \).

Write \( A^{-1} = (d(\lambda', \lambda))_{\lambda', \lambda \in V} \) and set \( d_\lambda = (d(\lambda', \lambda))_{\lambda' \in V}, \lambda \in V \). Take \( \lambda, \lambda' \in V \) and let \( \lambda_m \in V_{N_2} \) be so chosen that
\[ \rho(\lambda', \lambda_m) \leq 2N_2. \]
The existence of such a fusion vertex $\lambda_m$ follows from the covering property in Proposition 4.2. Applying (5.29) with $c$ replaced by $d_\lambda$, we obtain

$$
|d(\lambda', \lambda)| \leq \|\Psi_{\lambda_m}^4d_\lambda\|_2 \leq 2\|A^{-1}\|_{B(\ell^2)}\left|\psi_0\left(\frac{\rho(\lambda_m, \lambda)}{4N_2}\right)\right|
$$

(5.31)

$$
+ 2\|A^{-1}\|_{B(\ell^2)} \sum_{\lambda_m' \in V_{N_2}} W_{A,N_2}(\lambda_m, \lambda_m')\left|\psi_0\left(\frac{\rho(\lambda_m', \lambda)}{4N_2}\right)\right|.
$$

Therefore

$$
\sup_{\lambda', \lambda \in V} |d(\lambda', \lambda)| \leq 2\|A^{-1}\|_{B(\ell^2)}\left(1 + \left(\sup_{\lambda_m, \lambda_m' \in V_{N_2}} |W_{A,N_2}(\lambda_m, \lambda_m')|\right)\right)
\times \left(\sup_{\lambda \in V} \sum_{\lambda_m' \in V_{N_2}} \chi_{B(\lambda_m', 4N_2)}(\lambda)\right)
$$

(5.32)

$$
\leq C\|A^{-1}\|_{B(\ell^2)},
$$

where the last inequality follows from (5.26) and Proposition 4.2. For $n \geq 12$, it follows from (5.30) and (5.31) that

$$
\sup_{\rho(\lambda', \lambda) \geq nN_2} |d(\lambda', \lambda)| \leq 2\|A^{-1}\|_{B(\ell^2)}g_{A,N_2}(n/2) \sup_{\lambda \in V} \sum_{\lambda_m' \in V_{N_2}} \chi_{B(\lambda_m', 4N_2)}(\lambda)
$$

(5.33)

$$
\leq C\|A^{-1}\|_{B(\ell^2)}g_{A,N_2}(n/2),
$$

where $g_{A,N_2}(n) = \sup_{\rho(\lambda_m, \lambda_m') \geq N_n} |W_{A,N_2}(\lambda_m, \lambda_m')|.$

Observe that

$$
\|A^{-1}\|_{B_{r,\alpha}} \leq C N_2^{\alpha + d/r} \left(\sum_{m \geq 0} \left(\sup_{\rho(\lambda, \lambda') \geq mN_2} |d(\lambda, \lambda')|\right)^r (m + 1)^{\alpha r + d - 1}\right)^{1/r}
$$

for $1 \leq r < \infty$, and

$$
\|A^{-1}\|_{B_{r,\alpha}} \leq C N_2^{\alpha} \sup_{m \geq 0} \left(\sup_{\rho(\lambda, \lambda') \geq mN_2} |d(\lambda, \lambda')|\right)(m + 1)^{\alpha}
$$

for $r = \infty$. Combining the above two estimates with (5.26), (5.32) and (5.33), we obtain

$$
\|A^{-1}\|_{B_{r,\alpha}} \leq C\|A^{-1}\|_{B(\ell^2)} N_2^{\alpha + d/r}.
$$

(5.34)

Hence the desired estimate (5.1) follows from (5.24) and (5.34). \qed

6. Norm-controlled powers

By (3.14), norms of powers $A^n, n \geq 1,$ of a matrix $A \in B_{r,\alpha}$ with $1 \leq r \leq \infty$ and $\alpha > d(1 - 1/r)$ are dominated by a subexponential function. In this section, we show that norms of powers $A^n, n \geq 1,$ are controlled by a polynomial when the counting measure is normal.
Theorem 6.1. Let \( 1 \leq r \leq \infty, r' = r/(r-1) \), \( G \) be a connected simple graph with Beurling dimension \( d \) and normal counting measure \( \mu \), and let \( A \in \mathcal{B}_{r,\alpha}(G) \) with \( \alpha > d/r' \). Then there exists an absolute positive constant \( C \), independent of matrices \( A \) and integers \( n \geq 1 \), such that

\[
\frac{\|A^n\|_{\mathcal{B}_{r,\alpha}}}{\|A\|^n_{B(\ell^2)}} \leq C \left( \frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{B(\ell^2)}} \right)^{(\alpha+d/r)/\min(\alpha-d/r',1)} \times \begin{cases} 
1 & \text{if } \alpha \neq d/r' + 1 \\
\left( \ln \left( \frac{n\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{B(\ell^2)}} + 1 \right) \right)^{(d+1)/r'} & \text{if } \alpha = d/r' + 1
\end{cases}
\]

(6.1)

hold for all integers \( n \geq 1 \).

For matrices \( A \in \mathcal{B}_{r,\alpha} \) with \( \alpha > d/r' + 1 \), we obtain from Theorem 6.1 that

\[
\|A^n\|_{\mathcal{B}_{r,\alpha}} \leq C \left( \frac{\|A\|_{\mathcal{B}_{r,\alpha}}}{\|A\|_{B(\ell^2)}} \right)^{\alpha+d/r} n^{\alpha+d/r+1} \|A\|_{B(\ell^2)}^n, \quad n \geq 1.
\]

(6.2)

As shown in (6.3) below, the estimate (6.2) on powers of matrices in the Beurling algebra \( \mathcal{B}_{r,\alpha} \) is almost optimal. Let \( \delta \) be the delta function with \( \delta(0) = 1 \) and \( \delta(k) \neq 0 \) for all nonzero integers \( k \). Then for the matrix \( A_1 = (\delta(i-j-1))_{i,j\in\mathbb{Z}} \), we have that \( (A_1)^n = (a(i-j-n))_{i,j\in\mathbb{Z}}, n \geq 1 \), and hence

\[
\|A_1^n\|_{\mathcal{B}_{r,\alpha}} = \begin{cases} 
\left( \sum_{k=0}^{n}(k+1)^{\alpha r} \right)^{1/r} & \text{if } 1 \leq r < \infty \\
(n+1)^{\alpha} & \text{if } r = \infty
\end{cases}
\]

(6.3)

\[
\geq C \left( \frac{\|A_1\|_{\mathcal{B}_{r,\alpha}}}{\|A_1\|_{B(\ell^2)}} \right)^{\alpha+d/r} n^{\alpha+d/r} \|A_1\|_{B(\ell^2)}^n, \quad n \geq 1,
\]

where \( C \) is an absolute constant.

Let \( \hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi} \) satisfy \( |\hat{a}(\xi)| \leq 1 \) for all \( \xi \in \mathbb{R} \) and \( \sup_{k\in\mathbb{Z}} |a(k)|(1+|k|)\alpha < \infty \) for some \( \alpha > 1 \), and write

\[
(\hat{a}(\xi))^n = \sum_{k\in\mathbb{Z}} a_n(k)e^{-ik\xi}, \quad n \geq 1.
\]

Then there exists a positive constant \( C \) independent of \( n \geq 1 \) such that

\[
|a_n(k)| \leq Cn^{\alpha+1}(1+|k|)^{-\alpha}, \quad k \in \mathbb{Z}
\]

by Theorem 6.1. Therefore for any \( \epsilon > 0 \), there exists a positive constant \( C_{\epsilon} \) such that

\[
\sum_{k\in\mathbb{Z}} |a_n(k)| \leq Cn^{1+\epsilon}, \quad n \geq 1,
\]

cf. [18, 40, 58, 59] and references therein for various estimates. We remark that the above estimate for the Wiener norm of \( (\hat{a}(\xi))^n, n \geq 1 \),
was established in [59], with the polynomial exponent $1 + \epsilon$ replaced by a smaller exponent $(1 - \mu/\nu)/2$, when
\[ \hat{a}(\xi) = e^{-i\alpha \xi + i\xi^\nu q(\xi) - \gamma \xi^\nu (1+o(1))} \]
near the origin for some real polynomial $q$ with $q(0) \neq 0$.

Let random variables $X_n, n \geq 1$, be a stationary Markov chain on a spatially distributed network, which is described by a connected simple graph $G = (V,E)$. Then the probabilities $\Pr(X_{n+1} = \lambda \mid X_n = \lambda')$ of going from one vertex $\lambda'$ at time $n$ to another vertex $\lambda$ at time $n + 1$ is independent of $n \geq 1$,
\[ \Pr(X_{n+1} = \lambda \mid X_n = \lambda') = p(\lambda, \lambda'), \lambda, \lambda' \in V \text{ and } n \geq 1. \]

Define the transition matrix of the above stationary Markov chain by $P = (p(\lambda, \lambda'))_{\lambda, \lambda' \in V}$. Then by Theorem 6.1, we have the following estimate on the probability $\Pr(X_m = \lambda \mid X_n = \lambda')$, $m > n \geq 1$, with the input vertex $\lambda'$ and output vertex $\lambda \in V$.

**Corollary 6.2.** Let $G := (V,E)$ be a connected simple graph with Beurling dimension $d$ and normal counting measure $\mu$, and let $X_n, n \geq 1$, be a stationary Markov chain on the graph $G$ with transition matrix $P \in B_{\infty, \alpha}$ for some $\alpha > d + 1$. Then there exists a positive constant $C_\alpha$ such that
\[ (6.5) \quad \Pr(X_m = \lambda \mid X_n = \lambda') \leq \min\left( C_\alpha (m - n)^{\alpha + 1}(\rho(\lambda, \lambda'))^{-\alpha}, 1 \right) \]
for all $\lambda, \lambda' \in V$ and $m > n \geq 1$.

We finish this section with the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Let $A \in B_{r, \alpha}$, and write
\[ A^n = \frac{1}{2\pi i} \int_{|z| = (1+1/n)\|A\|_{B(\ell^2)}} z^n (zI - A)^{-1}dz. \]
Then
\[ (6.6) \quad \|A^n\|_{B_{r, \alpha}} \leq C\|A\|_{B(\ell^2)}^n \int_{|z| = (1+1/n)\|A\|_{B(\ell^2)}} \|(zI - A)^{-1}\|_{B_{r, \alpha}} |dz|. \]

Observe that for $|z| = \|A\|_{B(\ell^2)}(1 + 1/n)$, we have
\[ (6.7) \quad \|(zI - A)^{-1}\|_{B(\ell^2)} \leq |z|^{-1} \sum_{l=0}^\infty |z|^{-l} \|A\|_{B(\ell^2)}^l \leq n(\|A\|_{B(\ell^2)})^{-1}, \]
and
\[ (6.8) \quad \|zI - A\|_{B_{r, \alpha}} \leq |z| + \|A\|_{B_{r, \alpha}} \leq C\|A\|_{B_{r, \alpha}}, \]
where the last inequality holds by (3.12). By (6.7), (6.8) and Theorem 5.1, we get
\[
\| (zI - A)^{-1} \|_{B_{r,\alpha}} \leq C n (\| A \|_{B(\ell^2)})^{-1} \left( \frac{n \| A \|_{B(\ell^2)}}{\| A \|_{B(\ell^2)}} \right)^{(\alpha+d/r)/\min(\alpha-d/r', 1)} 
\times \left\{ \begin{array}{ll}
1 & \text{if } \alpha \neq d/r' + 1 \\
\ln \left( \frac{n \| A \|_{B(\ell^2)}}{\| A \|_{B(\ell^2)}} + 1 \right)^{(d+1)/r'} & \text{if } \alpha = d/r' + 1.
\end{array} \right.
\]
(6.9)

This together with (6.6) proves (6.1).

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C. E. Shin: Department of Mathematics, Sogang University, Seoul, 04109, Korea.

E-mail address: shinc@sogang.ac.kr

Q. Sun: Department of Mathematics, University of Central Florida, Orlando, FL 32828, USA.

E-mail address: qiyu.sun@ucf.edu