Determining the whole pure symmetric $N$-qubit state from its parts

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The Majorana representation of symmetric $N$-qubit states is employed here to investigate how correlation information of the whole pure symmetric state gets imprinted in its parts. It is shown that reduced states of $(N-1)$ qubits uniquely specify the entire class of pure $N$ qubit states containing two distinct spinors.

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I. INTRODUCTION

Knowing if higher order correlations in a multiparticle system follow entirely from lower order ones involving fewer parties is one of the basic issues of interest in quantum information science [1], as also in many body physics [2]. Construction of the many electron state with the knowledge of its two particle reduced density matrices has been discussed in Ref. [3]. On the other hand, it is also well-known that $N$-party entanglement cannot, in general, be reversibly transformed into entanglement of two parties [4]. In this context, Linden et. al. [5] proved an illuminating result that reduced states of a smaller fraction of the parties uniquely specify most of the generic multiparty pure states. Walck and Lyons [6] showed that the $N$ party GHZ (Greenberger-Horne-Zeilinger) states and their local unitary equivalents are the only exceptions to this result. More recently, Preeti Parashar and Swapan Rana have shown that $N$ qubit W class states can be uniquely determined by their bipartite marginals [7]. They have also extended their arguments to determine generalized Dicke class states from their marginal density matrices [8].

In the present paper, we employ the Majorana representation [4] to determine the whole symmetric state in terms of its parts. We show that only two of the $N-1$ qubit reduced states determine the pure symmetric states characterized by two distinct Majorana spinors. We further show that our method offers a natural way of determining a more general class of non-symmetric states too from their $N-1$ qubit reduced density matrices.

The paper is organized as follows: Majorana representation [4] of $N$-qubit pure states obeying permutation symmetry and a classification [9] of symmetric states based on the distinct Majorana spinors constituting the state are discussed in Sec. II. In Sec. III we focus on pure symmetric states of $N$ qubits, containing two distinct spinors and show explicitly that the state is uniquely determined by specifying only two of their $N-1$ party reduced states. We extend this method, in Sec. IV, to prove that generalized states of the family of two distinct Majorana spinors are the unique whole pure states consistent with their $N-1$ qubit subsystems. Sec. V contains a summary of our results.

II. MAJORANA REPRESENTATION FOR $N$-QUBIT PURE SYMMETRIC STATES

Majorana [4] expressed an arbitrary pure symmetric state $|\Psi_{\text{sym}}\rangle$ of spin $N/2$ systems as a symmetrized combination of $N$ spinors as follows:

$$|\Psi_{\text{sym}}\rangle = N \sum_P \hat{P} \{|\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$$ (1)

where

$$|\epsilon_l\rangle = \cos(\beta_l/2)e^{-i\alpha_l/2}|0\rangle + \sin(\beta_l/2)e^{i\alpha_l/2}|1\rangle,$$

$l = 1, 2, \ldots, N$ (2)

denote the spinors constituting the symmetric state $|\Psi_{\text{sym}}\rangle$; $\hat{P}$ denotes the set of all $N!$ permutations and $N$ corresponds to an overall normalization factor. Thus, $N$ complex parameters

$$z_l = \tan \frac{\beta_l}{2}e^{i\alpha_l}$$ (3)

parametrize the pure $N$ qubit symmetric state – allowing for a geometric representation of $N$ qubit pure symmetric states in terms of $N$ points on the Bloch sphere.

On the other hand, states of $N$-qubits obeying exchange symmetry get restricted to a $(N+1)$ dimensional Hilbert space spanned by the collective basis vectors $\{|\frac{N}{2}, k, \frac{N}{2}\rangle, k = 0, 1, 2, \ldots, N\}$ where,

$$|\frac{N}{2}, k, \frac{N}{2}\rangle = \frac{1}{\sqrt{N!C_k}}(|0, \ldots, 0, 1, \ldots, 1, \ldots, 0, \ldots, 0\rangle_{N-k} + \text{Permutations}$$ (4)

are the $N+1$ Dicke states – expressed in the standard qubit basis $|0\rangle$, $|1\rangle$. (Here, $N!C_k = \frac{N!}{k!(N-k)!}$ denotes the...
binomial coefficient). In other words, an arbitrary pure symmetric state,

\[ |\Psi_{\text{sym}}\rangle = \sum_{k=0}^{N} c_k \left| \frac{N}{2}, k - \frac{N}{2} \right\rangle, \]

is specified by the \((N + 1)\) complex coefficients \(c_k\). Eliminating an overall phase and normalizing the state implies that \(N\) complex parameters are required to completely characterize a pure symmetric state of \(N\) qubits.

In order to express the coefficients \(c_k\) in terms of the Majorana spinor orientations \((\alpha_l, \beta_l)\), it may be first identified that an identical rotation \(R \otimes R \ldots \otimes R\) on the symmetric state \(|\Psi_{\text{sym}}\rangle\) transforms it into another symmetric state. Choosing \(R = R^{-1}_l \equiv R^{-1}(\alpha_l, \beta_l, 0)\) (where \((\alpha_l, \beta_l, 0)\) denote the Euler angles of rotation [13]) aligning one of the constituent spinors say, \(|\epsilon_l\rangle\), along the positive \(z\)-direction i.e., \(R^{-1}_l|\epsilon_l\rangle = |0\rangle\), results in the following identification,

\[ \langle 1, 1_2, \ldots, 1_N | R^{-1}_l \otimes R^{-1}_l \ldots \otimes R^{-1}_l | \Psi_{\text{sym}} \rangle \equiv 0 \] (6)

This is because the rotation \(R^{-1}_l \otimes R^{-1}_l \ldots \otimes R^{-1}_l\) takes one of the spinors \(|\epsilon_l\rangle\) with orientation angles \((\alpha_l, \beta_l)\) to \(|0\rangle\) i.e., it aligns the spinor \(|\epsilon_l\rangle\) in the positive \(z\)-direction. Then, every term in the superposition [10] of the rotated state has atleast one \(|0\rangle\) and so, the projection \((1, 1_2, \ldots, 1_N | R^{-1}_l \otimes R^{-1}_l \otimes \ldots \otimes R^{-1}_s | \Psi_{\text{sym}} \rangle\) of the rotated state in the 'all-down' direction vanishes. Eq. (6) holds good for any identical rotations \(R^{-1}_l \otimes R^{-1}_l \otimes \ldots \otimes R^{-1}_s\), \(s = 1, 2, \ldots, N\), orienting any one of the constituent qubits in the positive \(z\)-direction. In other words, there exist \(N\) rotations \(R_s^{-1} = R^{-1}(\alpha_s, \beta_s, 0)\), \(s = 1, 2, \ldots, N\), which lead to the same result as in (6).

In terms of the alternate representation [5], we obtain

\[ \langle N/2, -N/2 | R^{-1}_l \otimes R^{-1}_l \ldots \otimes R^{-1}_l | \Psi_{\text{sym}} \rangle = \langle N/2, -N/2 | R^{-1}_l \otimes R^{-1}_l \ldots \otimes R^{-1}_l \left\{ \sum_{k=0}^{N} c_k |N/2, k - N/2\rangle \right\} = 0 \]

or

\[ \sum_{k=0}^{N} c_k \langle N/2, -N/2 | R^{-1}_l | N/2, k - N/2\rangle = \sum_{k=0}^{N} c_k \left( D_{k-N/2-N/2}^{-N/2+} \right)^{(N-2)}(\alpha_l, \beta_l, 0) = 0, \]

where we have denoted \(R^{-1}_l \otimes R^{-1}_l \ldots \otimes R^{-1}_l = R^{-1}\) in the collective \((N + 1)\) dimensional symmetric subspace of \(N\) qubits and \(D_{k-N/2-N/2}^{N/2+}(\alpha_l, \beta_l, 0)\). Substituting the explicit form of the \(D\)-matrix [13],

\[ (-1)^k \sqrt{N\over 2} C_k \left( \cos \frac{\beta_l}{2} \right)^{(N-2)} \left( \sin \frac{\beta_l}{2} \right)^k e^{i(k-N/2)\alpha_l}, \]

in (7) and simplifying, we obtain

\[ \mathcal{A} \sum_{k=0}^{N} (-1)^k \sqrt{N\over 2} C_k c_k z^k = 0 \]

(8)

where \(z = \left( \tan \frac{\beta_l}{2} e^{i\alpha_l} \right)\), and \(\mathcal{A} = \cos^{N/2} \frac{\beta_l}{2} e^{-i\alpha_l} N/2\). In other words, given the parameters \(c_k\), the \(N\) roots \(z_l, l = 1, 2, \ldots N\) of the Majorana polynomial

\[ P(z) = \sum_{k=0}^{N} (-1)^k \sqrt{N\over 2} C_k c_k z^k \]

(9)

determine the orientations \((\alpha_l, \beta_l)\) of the spinors constituting the \(N\)-qubit symmetric state.
with the constituent spinors transforming as $|e'\rangle = A|e\rangle$. This forms the main basis of the SLOCC classification of symmetric pure states \[D_N\].

For example, when all the $N$ solutions of the Majorana polynomial are identically equal, the symmetric state is given by

$$|D_N\rangle = |\epsilon, \epsilon, \ldots, \epsilon\rangle,$$

with $d = 1$; the states belong to the family of separable states denoted by $\{D_N\}$.

The states with two distinct spinors have the form,

$$|D_{N-k,k}\rangle = N \left\{ |\epsilon_1, \epsilon_1, \ldots, \epsilon_k, \epsilon_2, \ldots, \epsilon_k\rangle + \text{Permutations} \right\}$$

where $k = 0, 1, 2, \ldots, \lfloor N/2 \rfloor$. Dicke states \[D_N\] are representative states of the entanglement classes $\{D_{N-k,k}\}$ and clearly, they are all inequivalent under SLOCC.

Further, when the solutions are all distinct, the pure symmetric states constitute the class $\{D_{1,1,\ldots,1}\}$; the $N$ qubit GHZ state belongs to this entanglement class.

III. DETERMINING $N$ QUBIT STATES OF THE FAMILY $\{D_{N-k,k}\}$ FROM ITS PARTS

In this section we show that $N-1$ qubit reduced density matrices uniquely specify the $N$ qubit pure symmetric state belonging to the entanglement class $\{D_{N-k,k}\}$ i.e., no other pure or mixed $N$ qubit state can share the same set of subsystem density matrices.

We first cast the symmetric state \[D_N\] with two distinct Majorana spinors, belonging to the family $\{D_{N-k,k}\}$, in the collective Dicke basis \[D_N\] as follows:

$$|D_{N-k,k}\rangle = N \sum_p \hat{P}\left\{ |\epsilon_1, \epsilon_1, \ldots, \epsilon_k, \epsilon_2, \ldots, \epsilon_k\rangle \right\}$$

$$= N R_1^{\otimes N} \sum_p \hat{P}\left\{ |0, 0, \ldots, 0; \epsilon'_2, \epsilon'_2, \ldots, \epsilon'_k\rangle \right\},$$

where we have expressed $\epsilon_1 = R_1|0\rangle$ and $\epsilon_2 = R_2|0\rangle$, and

$$|\epsilon'_2\rangle = R_1^{-1} R_2|0\rangle = d_0|0\rangle + d_1|1\rangle, \quad d_0^2 + d_1^2 = 1.$$  \[13\]

Substituting \[13\] in \[12\] and upon simplification, we obtain,

$$|D_{N-k,k}\rangle = R_1^{\otimes N} \sum_{r=0}^{k} \sqrt{N} C_r \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle,$$

where $\alpha_r = N \frac{(N-r)!}{(N-k)!(k-r)!} d_0^{k-r} d_1^r$. \[14\]

In other words, all symmetric states $|D_{N-k,k}\rangle$, constituted by two distinct Majorana spinors are equivalent (under local unitary transformations) to

$$|D'_{N-k,k}\rangle = R_1^{-1} \otimes N |D_{N-k,k}\rangle$$

$$= \sum_{r=0}^{k} \sqrt{N} C_r \alpha_r \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle.$$  \[15\]

We now proceed to prove that parts belong uniquely to the whole $N$ qubit pure state $|D'_{N-k,k}\rangle$.

Let us first express the state $|D'_{N-k,k}\rangle$ in the qubit basis:

$$|D'_{N-k,k}\rangle = \alpha_0 |0_1, 0_2, \ldots, 0_N\rangle + \alpha_1 \sum_p \hat{P}\left\{ |1_1, 0_2, \ldots, 0_{N-1}, 0_N\rangle \right\} + \alpha_2 \sum_p \hat{P}\left\{ |1_1, 1_2, 0_3, \ldots, 0_N\rangle \right\} + \ldots$$

$$+ \alpha_k \sum_p \hat{P}\left\{ |1_1, 1_2, \ldots, 1_k, 0_{k+1}, \ldots, 0_N\rangle \right\} = |\phi_0\rangle |0_N\rangle + |\phi_1\rangle |1_N\rangle$$  \[16\]

where

$$|\phi_0\rangle = \alpha_0 |0_1, 0_2, \ldots, 0_{N-1}\rangle + \alpha_1 \sum_p \hat{P}\left\{ |1_1, 0_2, \ldots, 0_{N-1}\rangle \right\} + \alpha_2 \sum_p \hat{P}\left\{ |1_1, 1_2, 0_3, \ldots, 0_{N-1}\rangle \right\},$$

$$+ \ldots + \alpha_k \sum_p \hat{P}\left\{ |1_1, 1_2, 1_3, \ldots, 1_k, 0_{k+1}, \ldots, 0_{N-1}\rangle \right\},$$  \[17\]

$$|\phi_1\rangle = \alpha_1 |0_1, 0_2, \ldots, 0_{N-1}\rangle + \alpha_2 \sum_p \hat{P}\left\{ |1_1, 0_2, \ldots, 0_{N-1}\rangle \right\} + \alpha_3 \sum_p \hat{P}\left\{ |1_1, 1_2, 0_3, \ldots, 0_{N-1}\rangle \right\},$$

$$+ \ldots + \alpha_k \sum_p \hat{P}\left\{ |1_1, 1_2, 1_3, \ldots, 1_k, 0_{k+1}, \ldots, 0_{N-1}\rangle \right\}.$$  \[18\]

Clearly, the $N-1$ qubit reduced density matrix $\rho_{1,2,\ldots,N-1}$ obtained by tracing out the $N$th qubit from
the state $|D_{N-k,k}\rangle$ is a rank-2 mixed state given by,
\[ \rho_{1,2,...,N-1} = \text{Tr}_N[|D_{N-k,k}\rangle\langle D_{N-k,k}|] = |\phi_0\rangle\langle \phi_0| + |\phi_1\rangle\langle \phi_1|, \tag{19} \]

Let us suppose that a mixed $N$ qubit state $\omega_N$ too shares the same $N-1$ qubit reduced system $\rho_{1,2,...,N}$, i.e.,
\[ \rho_{1,2,...,N-1} = \text{Tr}_N[|D_{N-k,k}\rangle\langle D_{N-k,k}|] = \text{Tr}_N[|\omega_N\rangle\langle \omega_N|] = |\phi_0\rangle\langle \phi_0| + |\phi_1\rangle\langle \phi_1|. \tag{20} \]
The mixed state $\omega_N$ may always be thought of as a reduced system of a pure state $|\Omega_{NE}\rangle$ of the $N$ qubits and an environment $E$ such that
\[ \text{Tr}_E[|\Omega_{NE}\rangle\langle \Omega_{NE}|] = \omega_N. \tag{21} \]
In order that the pure state $|\Omega_{NE}\rangle$ (or the mixed state $\omega_N$) too shares the same $N-1$ qubit reduced density matrix $\rho_{1,2,...,N}$, we must have
\[ |\Omega_{NE}\rangle = |\phi_0\rangle |E_0\rangle + |\phi_1\rangle |E_1\rangle, \tag{22} \]
\[ \langle E_1|E_1\rangle = \delta_{i,j}. \tag{23} \]
Here, the states $|E_0\rangle$, $|E_1\rangle$ are the ones containing the qubit labelled $N$, and the environment $E$. Expanding $|E_{0,1}\rangle$ in the basis states of the qubit $N$ as,
\[ |E_0\rangle = |0_N\rangle |e_{00}\rangle + |1_N\rangle |e_{01}\rangle \]
\[ |E_1\rangle = |0_N\rangle |e_{10}\rangle + |1_N\rangle |e_{11}\rangle, \tag{24} \]
we may re-express the state $|\Omega_{NE}\rangle$ using (22), (23):
\[ |\Omega_{NE}\rangle = |\phi_0\rangle |0_N\rangle |e_{00}\rangle + |\phi_0\rangle |1_N\rangle |e_{01}\rangle + |\phi_1\rangle |0_N\rangle |e_{10}\rangle + |\phi_1\rangle |1_N\rangle |e_{11}\rangle. \tag{25} \]
If now demand that yet another $N-1$ qubit reduced system $\rho_{2,3,...,N}$ of $|D_{N-k,k}\rangle$ too is shared by $|\Omega_{NE}\rangle$, it imposes further constraints on its structure. It may be readily seen that a comparison of (15) with (25), leads to the identification $|e_{00}\rangle = 0$, and $|e_{11}\rangle = |e_{00}\rangle$, which corresponds to a simplified structure, $|\Omega_{NE}\rangle = |D'_{N-k,k}\rangle |e_{00}\rangle + |\phi_1\rangle |0_N\rangle |e_{10}\rangle$. Now, the orthonormality (23) implies that $|e_{00}\rangle |e_{00}\rangle = 1$ and $|e_{10}\rangle = 0$ leading to,
\[ |\Omega_{NE}\rangle = |D'_{N-k,k}\rangle |e_{00}\rangle, \tag{26} \]
This verifies this result more explicitly by comparing the matrix elements of the $N-1$ qubit reduced density matrix $\rho_{2,...,N}$ of the state $|D'_{N-k,k}\rangle$ with that obtained from $|\Omega_{NE}\rangle$: We first evaluate the following matrix element of the reduced $N-1$ qubit density matrix $\rho_{2,...,N}$
\[ \langle 0,2,0,3\cdots,0,0,0,0\cdots,0,0,0,0\rangle = |\alpha_k|^2 |e_{00}\rangle, \tag{27} \]
\[ \langle 0,2,0,3\cdots,0,0,0,0\cdots,0,0,0,0\rangle = |\alpha_k|^2 |e_{11}\rangle. \tag{28} \]
and compare it with the same matrix element evaluated from $|D'_{N-k,k}\rangle$:
\[ \langle 0,2,0,3\cdots,0,0,0,0\cdots,0,0,0,0\rangle = |\alpha_k|^2 |e_{11}\rangle. \tag{29} \]
This leads to the identification, $\langle e_{11}|e_{00}\rangle = 1$, which in turn implies (see (25)) that $|e_{11}\rangle \equiv |e_{00}\rangle$. Moreover, the orthonormality relation $\langle E_{10}|E_{10}\rangle = \langle e_{10}|e_{10}\rangle + $ $|e_{00}|e_{00}\rangle = 1$ leads to $|e_{10}\rangle \equiv 0$.

In other words, imposing the requirement that the state $|\Omega_{NE}\rangle$ shares same $N-1$ qubit reduced density
matrices as that of $|D_{N-k,k}\rangle$, we finally obtain,

$$|\Omega_{NE} \rangle = |D'_{N-k,k}\rangle |\psi_0\rangle.$$  \hfill (31)

In other words, $|D'_{N-k,k}\rangle$ happens to be the unique whole pure state that is consistent with its $N-1$ qubit reduced density matrices. We have employed only two of the $N-1$ reduced density matrices $\rho_1,2,..,N-1,\rho_2,3,..,N$ to establish this result.

IV. UNIQUENESS OF $N$ QUBIT STATES OF THE GENERALIZED FAMILY $\{D^G_{N-k,k}\}$ WITH ITS PARTS

The approach illustrated in Sec. III suggests a natural extension to a generalized family of $N$ qubit non-symmetric states, obtained from the class $\{D_{N-k,k}\}$ of states constituted by two distinct Majorana spinors. The generalized class of states have the form,

\begin{equation*}
|D^G_{N-k,k}\rangle = \alpha_0 |0,1,2,\ldots,0_N\rangle + \sum_{r=1}^{k} \alpha_{N+1} \left\{ \sum_{i=1}^{N_C} a_{i}^{(r)} \left[ |1_{P_{(1)}},1_{P_{(2)}},\ldots,1_{P_{(r)}},0_{P_{(r+1)}},\ldots,0_{P_{(N)}}\rangle \right] \right\} \\
= \alpha_0 a_0^{(0)} |0,1,2,\ldots,0_N\rangle + \alpha_1 \left\{ a_1^{(1)} |1,0,2,\ldots,0_N\rangle + a_2^{(1)} |0,1,2,\ldots,0_N\rangle + \ldots + a_N^{(1)} |0,1,2,\ldots,0_N\rangle \right\} \\
+ \alpha_2 \left\{ a_1^{(2)} |1,1,2,0,3,\ldots,0_N\rangle + a_2^{(2)} |1,0,2,1,3,0,4,\ldots,0_N\rangle + \ldots + a_N^{(2)} |1,0,2,1,3,0,4,\ldots,0_N\rangle \right\} \\
+ \ldots \\
+ \alpha_k \left\{ a_1^{(k)} |1,1,2,\ldots,1,0_k+1,\ldots,0_N\rangle + \ldots + a_N^{(k)} |1,1,2,\ldots,1,0_k+1,\ldots,0_N\rangle \right\} \\
+ a_{N+1}^{(k)} |1,1,2,\ldots,1,0_k,\ldots,0_N,1,\ldots,1_N\rangle + \ldots + a_{N+1}^{(N_C,k)} |1,1,2,\ldots,1,0_k,\ldots,0_N,1,\ldots,1_N\rangle}. \hfill (32)
\end{equation*}

where $\alpha_r$ are as given in (14); and the states $\sum_i a_i^{(r)} \left[ |1_{P_{(1)}},1_{P_{(2)}},\ldots,1_{P_{(r)}},0_{P_{(r+1)}},\ldots,0_{P_{(N)}}\rangle \right]$ are the generalized Dicke class states with arbitrary coefficients $a_i^{(r)}$. We now proceed to show that there do not exist any other (pure or mixed) $N$ qubit state, which shares the same $N-1$ party subsystem density matrices.

Expressing (32) in terms of $(N-1,1)$ partition of first $N-1$ qubits and the last qubit i.e.,

$$|D^G_{N-k,k}\rangle = |\phi^G_0\rangle |0\rangle_N + |\phi^G_1\rangle |1\rangle_N,$$

where
\[ |\phi_0^G \rangle = a_0 |0_1,0_2,\cdots,0_{N-1} \rangle + \sum_{r=1}^{k} \alpha_r \left\{ \sum_{i=1}^{N-1C_r} a_i^{(r)} \left[ |1_{P(1)},1_{P(2)},\cdots,1_{P(r)},0_{P(r+1)},\ldots,0_{P(N-1)} \rangle \right] \right\} \]
\[ = a_0 |0_1,0_2,\cdots,0_{N-1} \rangle + \alpha_1 \left\{ a_{1}^{(1)} |1_1,0_2,\cdots,0_{N-1} \rangle + \cdots + a_{N-1}^{(1)} |0_1,\cdots,0_{N-2},1_{N-1} \rangle \right\} + \alpha_2 \left\{ a_{1}^{(2)} |1_1,1_2,0_3,\cdots,0_{N-1} \rangle + a_{2}^{(2)} |1_1,0_2,1_3,0_4,\cdots,0_{N-1} \rangle + \cdots + a_{N-1}^{(2)} |0_1,1_2,\cdots,0_{N-3},1_{N-2},1_{N-1} \rangle \right\} \]
\[ + \cdots + \alpha_k \left\{ a_{1}^{(k)} |1_1,1_2,\cdots,1_k,0_{k+1},\cdots,0_{N-1} \rangle + a_{2}^{(k)} |1_1,1_2,\cdots,1_k,1_{k+1},0_{k+2},\cdots,0_{N-1} \rangle + \cdots + a_{N-1}^{(k)} |0_1,1_2,\cdots,1_k,0_{k+2},\cdots,0_{N-1} \rangle \right\} \]
\[ + \cdots + a_{N-1}^{(k)} |0_1,0_2,\cdots,0_{N-k},1_{N-k},\cdots,1_{N-1} \rangle \right\}, \quad \text{(33)} \]

and
\[ |\phi_1^G \rangle = \sum_{r=0}^{k-1} \alpha_{r+1} \left\{ \sum_{i=1}^{N-1C_{r+1}} a_i^{(r+1)} \left[ |1_{P(1)},1_{P(2)},\cdots,1_{P(r)},0_{P(r+1)},\ldots,0_{P(N-1)} \rangle \right] \right\} \]
\[ = \alpha_1 a_1^{(N)} |0_1,0_2,\cdots,0_{N-1} \rangle + \alpha_2 \left\{ a_{2}^{(N-1)} |1_1,0_2,0_3,\cdots,0_{N-1} \rangle + \cdots + a_{N-1}^{(N-1)} |0_1,0_2,\cdots,0_{N-2},1_{N-1} \rangle \right\} \]
\[ + \cdots + \alpha_k \left\{ a_{k}^{(N-1)} |1_1,1_2,\cdots,1_k,0_{k+1},\cdots,0_{N-1} \rangle + \cdots + a_{N-1}^{(N-1)} |0_1,0_2,\cdots,0_{N-k},1_{N-k},\cdots,1_{N-1} \rangle \right\}, \quad \text{(34)} \]
the N − 1 qubit reduced density matrix obtained by tracing the N-th qubit is readily found to be,
\[ \rho_{1,2,\ldots,N-1} = |\phi_0^G \rangle \langle \phi_0^G | + |\phi_1^G \rangle \langle \phi_1^G |. \quad \text{(35)} \]

Suppose that another N qubit mixed state \( \omega_0^G \) too shares the same N − 1 qubit reduced system \( \text{(35)} \). This requires that an extended pure state \( |\Omega_{NE}^G \rangle \) of N qubits – appended with an environment E in such a way that Tr
\[ E[|\Omega_{NE}^G \rangle \langle \Omega_{NE}^G |] = \omega_0^G \] should be expressible as,
\[ |\Omega_{NE}^G \rangle = |\phi_{01}^G \rangle |E_{01}^G \rangle + |\phi_{11}^G \rangle |E_{11}^G \rangle, \quad \text{(36)} \]
\[ \langle E_{i1}^G |E_{j1}^G \rangle = \delta_{i,j}, \quad \text{(37)} \]
in order to be consistent with the marginal state \( \rho_{1,2,\ldots,N-1} \). Here, \( |E_{01}^G \rangle = |0_N \rangle |e_{00}^G \rangle + |1_N \rangle |e_{01}^G \rangle \) and \( |E_{11}^G \rangle = |0_N \rangle |e_{10}^G \rangle + |1_N \rangle |e_{11}^G \rangle \) correspond to the states containing the N-th qubit and the environment E. Thus the extended pure state \( |\Omega_{NE}^G \rangle \) takes the following form,
\[ |\Omega_{NE}^G \rangle = |\phi_{01}^G \rangle |0_N \rangle |e_{00}^G \rangle + |\phi_{11}^G \rangle |1_N \rangle |e_{01}^G \rangle + |\phi_{11}^G \rangle |0_N \rangle |e_{10}^G \rangle + |\phi_{11}^G \rangle |1_N \rangle |e_{11}^G \rangle. \quad \text{(38)} \]

In order that \( |\Omega_{NE}^G \rangle \) shares one more N − 1 qubit reduced density matrix \( \rho_{2,3,\ldots,N}^G \) (obtained by tracing the first qubit of the original state \( |D_{N-k,k}^G \rangle \)) imposes further constraints which we discuss below.

We compare the following matrix element of \( \rho_{2,3,\ldots,N}^G \), evaluated from both \( |\Omega_{NE}^G \rangle \) and \( |D_{N-k,k}^G \rangle \):
\[ \langle 0_2,0_3,\cdots,0_{N-1-k},1_{N-k},\cdots,1_N |\text{Tr}_{1,E} [ |\Omega_{NE}^G \rangle \langle \Omega_{NE}^G |] |0_2,0_3,\cdots,0_{N-1-k},1_{N-k},\cdots,1_N \rangle = |\alpha_k|^2 |a_{(N-1)C_k}^{(k)}|^2 \langle e_{01}^G |e_{01}^G \rangle, \quad \text{(39)} \]
from which we infer that \( |e_{01}^G \rangle \equiv 0 \). The orthogonality relations \( \langle e_{00}^G |e_{00}^G \rangle = 1 \) would then lead to
\[ \langle e_{00}^G |e_{00}^G \rangle = 1, \quad \text{(40)} \]
Next, we consider (see Eqs. \( \text{(32)} \) – \( \text{(34)} \), \( \text{(35)} \), \( \text{(36)} \), \( \text{(37)} \), \( \text{(38)} \), \( \text{(39)} \), \( \text{(40)} \).
the condition
\[ \langle e_{11}^G | e_{00}^G \rangle = 1. \]  
(42)

Further, from the orthonormality relation \( \text{(10)} \) we obtain,
\[ |e_{11}^G \rangle \equiv |e_{00}^G \rangle. \]  
(43)

Consequently, we identify that
\[ \langle E_{10}^G \rangle |E_{10}^G \rangle = 1 \Rightarrow \langle e_{10}^G | e_{10}^G \rangle + \langle e_{00}^G | e_{00}^G \rangle = 1 \]
\[ \Rightarrow \langle e_{10}^G | e_{10}^G \rangle = 0 \text{ or } |e_{10}^G \rangle \equiv 0. \]  
(44)

We are thus led to our final result
\[ |\Omega_{NE}^G \rangle = |D_{N-k,k}^G \rangle |e_{00}^G \rangle \]  
(45)

i.e., the state of the environment is merely a multiple of a fixed basis state \( |e_{00}^G \rangle \) in order that the reduced density matrices of \( |D_{N-k,k}^G \rangle \) are shared by \( |\Omega_{NE}^G \rangle \) too. So, the only whole \( N \) qubit state consistent with its \( N-1 \) reduced systems is the starting state \( |D_{N-k,k}^G \rangle \) itself. Only two of the \( N-1 \) qubit reduced density matrices suffice to establish this result.

V. SUMMARY

We have explicitly shown that \( N-1 \) qubit reduced systems of a pure \( N \)-qubit symmetric state – constituted by two distinct spinors – determine the full state uniquely. No other pure or mixed state of \( N \) qubits is consistent with the same set of \( N-1 \) party marginals. We have employed only two of the \( N-1 \) qubit reduced states to establish the uniqueness of the whole pure state with its parts. The method developed is readily applied to a generalization family of (non-symmetric) \( N \) qubit states containing two distinct spinors and we have established that specifying only two of the \( N-1 \) qubit marginal systems leaves no freedom in the full state to which they belong to. From the point of view of parametrization of the \( N \) qubit state, this suggests that higher order tensor parameters of the whole pure state get completely specified in terms of the lower order tensors. It would be illuminating to explore how the higher order correlations in the whole pure state originate from the lower order ones. Also, a clear understanding on the asymptotic convertibility and reversibility of entanglement within smaller fractions in these special classes of \( N \) qubit states would shed light on different kinds of multiparty correlations.

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000)
[2] A. J. Coleman, Rev. Mod. Phys. 35, 668 (1963).
[3] F. Colmenero, C. Perez del Valle, and C. Valdemoro, Phys. Rev. A 47, 971 (1993); F. Colmenero and C. Valdemoro, Phys. Rev. A 47, 979 (1993); H. Nakatsuji and K. Yasuda, Phys. Rev. Lett. 76, 1039 (1996); K. Yasuda and H. Nakatsuji, Phys. Rev. A 56, 2648 (1997); D. A. Mazzioti, Phys. Rev. A 57, 4219 (1998); D. A. Mazzioti, Phys. Rev. A 60, 3618 (1999).
[4] N. Linden, S. Popescu, B. Schumacher, and M. Westmoreland, quant-ph/9912039; C.H. Bennett, S. Popescu, D. Rohrlich, J.A. Smolin, and A.V. Thapliyal, Phys. Rev. A 63, 012307 (2000).
[5] N. Linden, S. Popescu and W. K. Wootters, Phys. Rev. Lett. 89, 207901 (2002); N. Linden and W. K. Wootters, Phys. Rev. Lett. 89, 277906 (2002); N. S. Jones, N. Linden Phys. Rev. A 71, 012324 (2005).
[6] S. N. Walck and D. W. Lyons, Phys. Rev. Lett. 100, 050501 (2008); S. N. Walck and D. W. Lyons, Phys. Rev. A 79, 032326 (2009).
[7] Preeti Parashar and Swapan Rana, Phys. Rev. A 80, 012319 (2009).
[8] Preeti Parashar and Swapan Rana, arXiv:0910.1422 (To appear in J. Phys. A).
[9] E. Majorana, Nuovo Cimento 9, 43 (1932).
[10] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. 103, 070503 (2009).
[11] W. Dur, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[12] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, arXiv e-print: 0908.0886.
[13] M. E. Rose, *Elementary Theory of Angular Momentum,* (New York, Wiley) 1957.