On Computational Power of Quantum Read-Once Branching Programs

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In this paper we review our current results concerning the computational power of quantum read-once branching programs. First of all, based on the circuit presentation of quantum branching programs and our variant of quantum fingerprinting technique, we show that any Boolean function with linear polynomial presentation can be computed by a quantum read-once branching program using a relatively small (usually logarithmic in the size of input) number of qubits. Then we show that the described class of Boolean functions is closed under the polynomial projections.

1 Introduction

One notable thing about the recent realizations of a quantum computer (say, the one based on multiatomic ensembles in resonator [1]) is that they abide the isolation of a quantum system, e.g. in [1] the transformation of a quantum state is performed by an external magnetic field. Thus it is quite adequate to describe such computations by quantum models with classical control. Here we consider one of such models – the model of quantum branching programs introduced by Ablayev, Gainutdinova, Karpinski [1] (leveled programs), and by Nakanishi, Hamaguchi, Kashiwabara [14] (non-leveled programs). Later it was shown by Sauerhoff [16] that these two models are polynomially equivalent.

It is also worth noting that in spite of constant progress in experimental quantum computation all of the physical implementations of a quantum computer are still rather weak in a sense that they are suffering from fast decoherence of the quantum states and are able to organize the interaction of a small number of qubits. This naturally leads to the restricted variants of a quantum computer – the idea first proposed by Ambainis and Freivalds in 1998 [6]. Considering one-way quantum finite automata, they suggested that the first quantum-mechanical computers would consist of a comparatively simple and fast quantum-mechanical part connected to a classical computer.

In this paper we consider a restricted model of computation known as Quantum Read-Once Branching Programs of polynomial width. The classical variant of this model is also known in computer science as Ordered Binary Decision Diagrams (OBDDs) and that is why we will also use the notion of quantum OBDDs (QOBDDs) for the considered model. The small coherence time is formalized in this model by

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allowing only a single test of each variable. The restriction of polynomial width of quantum OBDDs leads to the number of qubits which is logarithmic in the size of the input. On the other hand, the generalized lower bound on the width of quantum OBDDs \[2\] leads to logarithmic number of qubits as a lower bound for almost all Boolean functions in the OBDD setting.

For the model of quantum OBDDs we develop the fingerprinting technique introduced in \[3\]. The basic ideas of this approach are due to Freivalds (e.g. see the book \[13\]). These ideas were later successfully applied in the quantum automata setting by Ambainis and Freivalds in 1998 \[6\] (later improved in \[7\]). Subsequently, the same technique was adapted for the quantum branching programs by Ablayev, Gainutdinova and Karpinski in 2001 \[11\], and was later generalized in \[3\].

For our technique we use a variant of polynomial presentation for Boolean functions, which we call characteristic. The polynomial presentations of Boolean functions are widely used in theoretical computer science. For instance, an algebraic transformation of Boolean functions has been applied in \[10\] and \[5\] for verification of Boolean functions. In the quantum setting polynomial representations were used for proving lower bounds on communication complexity in \[8\] as well as for investigating query complexity in \[18\]. Our approach combines the ideas similar to the definition of characteristic polynomial from \[10\], \[5\] and to the notion of zero-error polynomial (see, e.g. \[18\]).

Finally, we use the technique of polynomial projections to outline the limits of our fingerprinting method. This technique was intensively applied by Sauerhoff in the model of classical branching programs (see, e.g., \[15\]). In this paper we apply this approach for the quantum OBDD model by showing that the class of functions effectively computable via the quantum fingerprinting technique is closed under polynomial projections.

2 Preliminaries

We use the notation \(|i\rangle\) for the vector from \(\mathcal{H}^d\), which has a 1 on the \(i\)-th position and 0 elsewhere. An orthonormal basis \(|1\rangle, \ldots, |d\rangle\) is usually referred to as the standard computational basis. In this paper we consider all quantum transformations and measurements with respect to this basis.

**Definition 1.** A Quantum Branching Program \(Q\) over the Hilbert space \(\mathcal{H}^d\) is defined as

\[
Q = \langle T, |\psi_0\rangle, \text{Accept} \rangle,
\]

where \(T\) is a sequence of \(l\) instructions: \(T_j = (x_j, U_j(0), U_j(1))\) is determined by the variable \(x_j\) tested on the step \(j\), and \(U_j(0), U_j(1)\) are unitary transformations in \(\mathcal{H}^d\).

Vectors \(|\psi\rangle \in \mathcal{H}^d\) are called states (state vectors) of \(Q\), \(|\psi_0\rangle \in \mathcal{H}^d\) is the initial state of \(Q\), and \(\text{Accept} \subseteq \{1, 2, \ldots, d\}\) is the set of indices of accepting basis states.

We define a computation of \(Q\) on an input \(\sigma = \sigma_1 \ldots \sigma_n \in \{0, 1\}^n\) as follows:

1. A computation of \(Q\) starts from the initial state \(|\psi_0\rangle\):

2. The \(j\)-th instruction of \(Q\) reads the input symbol \(\sigma_j\) (the value of \(x_j\)) and applies the transition matrix \(U_j = U_j(\sigma_j)\) to the current state \(|\psi\rangle\) to obtain the state \(|\psi'\rangle = U_j(\sigma_j) |\psi\rangle\):

3. The final state is

\[
|\psi_\sigma\rangle = \left( \prod_{j=1}^{l} U_j(\sigma_j) \right) |\psi_0\rangle.
\]
4. After the $l$-th (last) step of quantum transformation $Q$ measures its configuration $|\psi_\sigma\rangle = (\alpha_1, \ldots, \alpha_d)^T$, and the input $\sigma$ is accepted with probability

$$Pr_{\text{accept}}(\sigma) = \sum_{i \in \text{Accept}} |\alpha_i|^2.$$  

Note, that using the set $\text{Accept}$ we can construct $M_{\text{accept}}$ – a projector on the accepting subspace $H_{\text{accept}}^d$ (i.e. a diagonal zero-one projection matrix, which determines the final projective measurement). Thus, the accepting probability can be re-written as

$$Pr_{\text{accept}}(\sigma) = \langle \psi_\sigma | M_{\text{accept}}^\dagger M_{\text{accept}} | \psi_\sigma \rangle = ||M_{\text{accept}} | \psi_\sigma \rangle ||^2.$$  

Note also that this is a “measure-once” model analogous to the model of quantum finite automata in [12], in which the system evolves unitarily except for a single measurement at the end. We could also allow multiple measurements during the computation, by representing the state as a density matrix $\rho$, and by making the $U_j$ superoperators, but we do not consider this here.

**Circuit representation.** Quantum algorithms are usually given by using quantum circuit formalism [9][19], because this approach is quite straightforward for describing such algorithms.

We propose, that a QBP represents a classically-controlled quantum system. That is, a QBP can be viewed as a quantum circuit aided with an ability to read classical bits as control variables for unitary operations.

Here $x_{i_1}, \ldots, x_{i_l}$ is the sequence of (not necessarily distinct) variables denoting classical control bits. Using the common notation single wires carry quantum information and double wires denote classical information and control.

**Complexity measures.** The width of a QBP $Q$, denoted by $\text{width}(Q)$, is the dimension $d$ of the corresponding state space $H_d^d$, and the length of $Q$, denoted by $\text{length}(Q)$, is the number $l$ of instructions in the sequence $T$. There is one more commonly used complexity measure – the size of $Q$, which we define as $\text{size}(Q) = \text{width}(Q) \cdot \text{length}(Q)$.

Note that for a QBP $Q$ in the circuit setting another important complexity measure explicitly comes out – a number of quantum bits, denoted by $\text{qubits}(Q)$, physically needed to implement a corresponding quantum system with classical control. From definition it follows that $\log \text{width}(Q) \leq \text{qubits}(Q)$. 

\[ \begin{array}{cccccccccc} x_{i_1} & & & & & & & & & \cdots \\ x_{i_2} & & & & & & & & & \cdots \\ & \vdots & & & & & & & & \\ x_{i_l} & & & & & & & & & \cdots \\ |\psi_0\rangle & & & & & & & & & |\phi_1\rangle \\ & & |\phi_2\rangle & & & & & & & \cdots \\ & & & & \vdots & & & & & \cdots \\ & & & & |\phi_q\rangle & & & & & \cdots \\ U_1(1) & U_1(0) & U_2(1) & U_2(0) & & & & & \cdots & \cdots \\ U_l(1) & U_l(0) & & & & & & & & \end{array} \]
**Acceptance criteria.** A QBP $Q$ computes the Boolean function $f$ with bounded error if there exists an $\varepsilon \in (0, 1/2)$ (called margin) such that for all inputs the probability of error is bounded by $1/2 - \varepsilon$.

In particular, we say that a QBP $Q$ computes the Boolean function $f$ with one-sided error if there exists an $\varepsilon \in (0, 1)$ (called error) such that for all $\sigma \in f^{-1}(1)$ the probability of $Q$ accepting $\sigma$ is 1 and for all $\sigma \in f^{-1}(0)$ the probability of $Q$ erroneously accepting $\sigma$ is less than $\varepsilon$.

**Read-once branching programs.** Read-once BPs is a well-known restricted variant of branching programs [17].

**Definition 2.** We call a QBP $Q$ a quantum OBDD (QOBDD) or read-once QBP if each variable $x \in \{x_1, \ldots, x_n\}$ occurs in the sequence $T$ of transformations of $Q$ at most once.

For the rest of the paper we’re only interested in QOBDDs, i.e. the length of all programs would be $n$ (the number of input variables). Note that for OBDD model size($Q$) = $n \cdot$ width($Q$) and therefore we’re mostly interested in the width of quantum OBDDs.

**Generalized Lower Bound.** The following general lower bound on the width of QOBDDs was proven in [2].

**Theorem 1.** Let $f(x_1, \ldots, x_n)$ be a Boolean function computed by a quantum read-once branching program $Q$ with bounded error for some margin $\varepsilon$. Then

$$\text{width}(Q) \geq \frac{\log \text{width}(P)}{2\log \left(1 + \frac{1}{\varepsilon}\right)}$$

where $P$ is a deterministic OBDD of minimal width computing $f(x_1, \ldots, x_n)$.

That is, the width of a quantum OBDD cannot be asymptotically less than the logarithm of the width of the minimal deterministic OBDD computing the same function. And since the deterministic width of many “natural” functions is exponential [17], we obtain the linear lower bound for these functions.

Let bits($P$) be the number of bits (memory size) required to implement the minimal deterministic OBDD $P$ for $f$ and $Q$ is an arbitrary quantum OBDD computing the same function.

Then theorem [1] can be restated as the following corollary using the number of bits and qubits as the complexity measure.

**Corollary 1.**

$$\text{qubits}(Q) = \Omega(\log \text{bits}(P)).$$

### 3 Algorithms for QBPs Based on Fingerprinting

Generally [13], fingerprinting – is a technique that allows to present objects (words over some finite alphabet) by their fingerprints, which are significantly smaller than the originals. It is used in randomized and quantum algorithms to test equality of some objects (binary strings) with one-sided error by simply comparing their fingerprints.

In this paper we develop a variant of the fingerprinting technique adapted for quantum branching programs. At the heart of the method is the representation of Boolean functions by polynomials of special type, which we call characteristic.
3.1 Characteristic Polynomials for Quantum Fingerprinting

The following definition is similar to the algebraic transformation of Boolean function from [10] and [5], though it is adapted for the fingerprinting technique.

**Definition 3.** We call a polynomial \( g(x_1, \ldots, x_n) \) over the ring \( \mathbb{Z}_m \) a characteristic polynomial of a Boolean function \( f(x_1, \ldots, x_n) \) and denote it \( g_f \) when for all \( \sigma \in \{0,1\}^n \) \( g_f(\sigma) = 0 \) iff \( f(\sigma) = 1 \).

**Lemma 1.** For any Boolean function \( f \) there exists a characteristic polynomial \( g_f \) over \( \mathbb{Z}_{2^n} \).

**Proof.** One way to construct such characteristic polynomial \( g_f \) is transforming a sum of products representation for \( \neg f \).

Let \( K_1 \vee \ldots \vee K_l \) be a sum of products for \( \neg f \) and let \( \bar{K}_i \) be a product of terms from \( K_i \) (negations \( \neg x_j \) are replaced by \( 1 - x_j \)). Then \( \bar{K}_1 + \ldots + \bar{K}_l \) is a characteristic polynomial over \( \mathbb{Z}_{2^n} \) for \( f \) since it equals 0 \( \iff \) all of \( \bar{K}_i \) (and thus \( K_i \)) equal 0. This happens only when the negation of \( f \) equals 0. \( \square \)

Generally, there are many polynomials for the same function, but we’re interested in having a linear polynomial if it exists.

For example, the function \( EQ_n \), which tests the equality of two \( n \)-bit binary strings, has the following polynomial over \( \mathbb{Z}_{2^n} \):

\[
\sum_{i=1}^{n} (x_i(1-y_i) + (1-x_i)y_i) = \sum_{i=1}^{n} (x_i + y_i - 2x_iy_i).
\]

On the other hand, the same function can be represented by the polynomial

\[
\sum_{i=1}^{n} x_i2^{i-1} - \sum_{i=1}^{n} y_i2^{i-1}.
\]

Some functions don’t have a linear characteristic polynomial at all (e.g. a disjunction of \( n \) variables \( f = x_1 \vee x_2 \vee \ldots \vee x_n \)), while their negations have a linear characteristic polynomial (e.g. \( g_{\neg f} = \sum_{i=1}^{n} x_i \) over \( \mathbb{Z}_{n+1} \)).

We use this presentation of Boolean functions for our fingerprinting technique.

3.2 Fingerprinting technique

For a Boolean function \( f \) we choose an error rate \( \varepsilon > 0 \) and pick a characteristic polynomial \( g \) over the ring \( \mathbb{Z}_m \). Then for arbitrary binary string \( \sigma = \sigma_1 \ldots \sigma_n \) we create its fingerprint \( |h_\sigma\rangle \) composing \( t = 2^{\lceil \log((2/\varepsilon)\ln 2m) \rceil} \) single qubit fingerprints \( |h_\sigma^i\rangle \):

\[
|h_\sigma^i\rangle = \cos \frac{2\pi k_i g(\sigma)}{m} |0\rangle + \sin \frac{2\pi k_i g(\sigma)}{m} |1\rangle
\]

into entangled state \( |h_\sigma\rangle \) of \( \log t + 1 \) qubits:

\[
|h_\sigma\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle |h_\sigma^i\rangle.
\]

Here the transformations of the last qubit in \( t \) different subspaces “simulate” the transformations of all of the \( |h_\sigma^i\rangle \) \( (i = 1, \ldots, t) \). That is, the last qubit is in parallel rotated by \( t \) different angles about the \( \hat{y} \) axis of the Bloch sphere.

The chosen parameters \( k_i \in \{1, \ldots, m-1\} \) for \( i \in \{1, \ldots, t\} \) are “good” following the notion of [6].
**Definition 4.** A set of parameters $K = \{k_1, \ldots, k_t\}$ is called “good” for some integer $b \neq 0 \mod m$ if

$$\frac{1}{t^2} \left( \sum_{j=1}^{t} \cos \frac{2\pi k_j b}{m} \right)^2 < \varepsilon.$$  

The left side of inequality is the squared amplitude of the basis state $|0\rangle^{\otimes \log t} |0\rangle$ if $b = g(\sigma)$ and the operator $H^{\otimes \log t} \otimes I$ has been applied to the fingerprint $|h_{\sigma}\rangle$. Informally, that kind of set guarantees, that the probability of error will be bounded by a constant below 1.

The following lemma proves the existence of a “good” set and generalizes the proof of the corresponding statement from [7].

**Lemma 2.** [3] There is a set $K$ with $|K| = t = 2^{\lceil \log((2/\varepsilon)\ln 2m) \rceil}$ which is “good” for all integer $b \neq 0 \mod m$.

We use this result for our fingerprinting technique choosing the set $K = \{k_1, \ldots, k_t\}$ which is “good” for all $b = g(\sigma) \neq 0$. That is, it allows to distinguish those inputs whose image is 0 modulo $m$ from the others.

### 3.3 Boolean Functions Computable via Fingerprinting Method

Let $f(x_1, \ldots, x_n)$ be a Boolean function and $g(x_1, \ldots, x_n)$ be its characteristic polynomial. The following theorem holds.

**Theorem 2.** Let $\varepsilon \in (0, 1)$. If $g$ is a linear polynomial over $\mathbb{Z}_m$, i.e. $g = c_1 x_1 + \ldots + c_n x_n + c_0$, then $f$ can be computed with one-sided error $\varepsilon$ by a quantum OBDD of width $O\left(\frac{\log m}{\varepsilon}\right)$.

**Proof.** Here is the algorithm in the circuit notation:

![Circuit Diagram](image)

Initially qubits $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_{\log t}\rangle \otimes |\phi_{\text{target}}\rangle$ are in the state $|\psi_0\rangle = |0\rangle^{\otimes \log t} |0\rangle$. For $i \in \{1, \ldots, t\}$, $j \in \{0, \ldots, n\}$ we define rotations $R_{i,j}$ as

$$R_{i,j} = R_\theta \left( \frac{4\pi k_i c_j}{m} \right),$$

where $c_j$ are the coefficients of the linear polynomial for $f$ and the set of parameters $K = \{k_1, \ldots, k_t\}$ is “good” according to the Definition 4 with $t = 2^{\lceil \log((2/\varepsilon)\ln 2m) \rceil}$.
Let $\sigma = \sigma_1 \ldots \sigma_n \in \{0, 1\}^n$ be an input string.

The first layer of Hadamard operators transforms the state $|\psi_0\rangle$ into

$$|\psi_1\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle |0\rangle.$$

Next, upon input symbol 0 identity transformation $I$ is applied. But if the value of $x_j$ is 1, then the state of the last qubit is transformed by the operator $R_{t,j}$, rotating it by the angle proportional to $c_j$. Moreover, the rotation is done in each of $t$ subspaces with the corresponding amplitude $1/\sqrt{t}$. Such a parallelism is implemented by the controlled operators $C_t(R_{t,j})$, which transform the states $|i\rangle \cdot \rangle$ into $|i\rangle R_{t,j} \cdot \rangle$, and leave others unchanged. For instance, having read the input symbol $x_1 = 1$, the system would evolve into state

$$|\psi_2\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} C_t(R_{t,1}) |i\rangle |0\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle R_{t,1} |0\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle \left( \cos \frac{2\pi k_i c_1}{m} |0\rangle + \sin \frac{2\pi k_i c_1}{m} |1\rangle \right).$$

Thus, after having read the input $\sigma$ the amplitudes would “collect” the sum $\sum_{j=1}^{n} c_j \sigma_j$

$$|\psi_3\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle \left( \cos \frac{2\pi k_i g(\sigma)}{m} |0\rangle + \sin \frac{2\pi k_i g(\sigma)}{m} |1\rangle \right).$$

At the next step we perform the rotations by the angle $\frac{4\pi k_i c_0}{m}$ about the $\hat{y}$ axis of the Bloch sphere for each $i \in \{1, \ldots, t\}$. Therefore, the state of the system would be

$$|\psi_4\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle \left( \cos \frac{2\pi k_i g(\sigma)}{m} |0\rangle + \sin \frac{2\pi k_i g(\sigma)}{m} |1\rangle \right).$$

Applying $H^{\otimes \log t} \otimes I$ we obtain the state

$$|\psi_5\rangle = \left( \frac{1}{t} \sum_{i=1}^{t} \cos \frac{2\pi k_i g(\sigma)}{m} \right) |0\rangle^{\otimes \log t} |0\rangle + 
+ \gamma |0\rangle^{\otimes \log t} |1\rangle + \sum_{i=2}^{t} |i\rangle \left( \alpha_i |0\rangle + \beta_i |1\rangle \right),$$

where $\gamma$, $\alpha_i$, and $\beta_i$ are some unimportant amplitudes.

The input $\sigma$ is accepted if the measurement outcome is $|0\rangle^{\otimes \log t} |0\rangle$. Clearly, the accepting probability is

$$Pr_{\text{accept}}(\sigma) = \frac{1}{t^2} \left( \sum_{i=1}^{t} \cos \frac{2\pi k_i g(\sigma)}{2^n} \right)^2.$$

If $f(\sigma) = 1$ then $g(\sigma) = 0$ and the program accepts $\sigma$ with probability 1. Otherwise, the choice of the set $K = \{k_1, \ldots, k_t\}$ guarantees that

$$Pr_{\text{accept}}(\sigma) = \frac{1}{t^2} \left( \sum_{i=1}^{t} \cos \frac{2\pi k_i g(\sigma)}{2^n} \right)^2 < \varepsilon.$$

Thus, $f$ can be computed by a quantum OBDD $Q$, with $\text{qubits}(Q) = \log t + 1 = O(\log \left( \frac{\log m}{\varepsilon} \right))$. The width of the program is $2^{\text{qubits}(Q)} = O \left( \frac{\log m}{\varepsilon} \right)$. \qed
The following functions (for definitions see, e.g., \cite{4}) have the aforementioned linear polynomials and thus are effectively computed via the fingerprinting technique.

| Function       | Expression                                                                 | Domain          | Complexity     |
|----------------|----------------------------------------------------------------------------|-----------------|---------------|
| \( MOD_m \)   | \( \sum_{i=1}^{n} x_i \)                                                  | \( \mathbb{Z}_m \) | \( O(\log \log m) \) |
| \( MOD'_m \)  | \( \sum_{i=1}^{n} x_i 2^{i-1} \)                                          | \( \mathbb{Z}_m \) | \( O(\log \log m) \) |
| \( EQ_n \)    | \( \sum_{i=1}^{\lfloor n/2 \rfloor} x_i 2^{i-1} - \sum_{i=1}^{n} y_i 2^{i-1} \) | \( \mathbb{Z}_2^n \) | \( O(\log n) \) |
| \( Palindrome_n \) | \( \sum_{i=1}^{\lfloor n/2 \rfloor} x_i 2^{i-1} - \sum_{i=\lceil n/2 \rceil}^{n} x_i 2^{n-i} \) | \( \mathbb{Z}_{2^{\lfloor n/2 \rfloor}} \) | \( O(\log n) \) |
| \( PERM_n \)  | \( \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \left( (n+1)^{i-1} + (n+1)^{n+j-1} \right) \) | \( \mathbb{Z}_{(n+1)^{2n}} \) | \( O(\log n) \) |

4 Generalized Approach

The fingerprinting technique described in the previous section allows us to test a single property of the input encoded by a characteristic polynomial. Using the same ideas we can test the conjunction of several conditions encoded by a group of characteristic polynomials which we call a characteristic of a function.

**Definition 5.** We call a set \( \chi_f \) of polynomials over \( \mathbb{Z}_m \) a characteristic of a Boolean function \( f \) if for all polynomials \( g \in \chi_f \) and all \( \sigma \in \{0,1\}^n \) it holds that \( g(\sigma) = 0 \) iff \( \sigma \in f^{-1}(1) \).

We say that a characteristic is linear if all of its polynomials are linear.

From Lemma 1 it follows that for each Boolean function there is always a characteristic consisting of a single characteristic polynomial.

Now we can generalize the Fingerprinting technique from section 3.2.

**Generalized Fingerprinting technique** For a Boolean function \( f \) we choose an error rate \( \varepsilon > 0 \) and pick a characteristic \( \chi_f = \{g_1, \ldots, g_l\} \) over \( \mathbb{Z}_m \). Then for arbitrary binary string \( \sigma = \sigma_1 \ldots \sigma_n \) we create its fingerprint \( |h_\sigma\rangle \) composing \( t \cdot l \) (\( t = 2^{\lceil \log((2/\varepsilon)\ln2m) \rceil} \)) single qubit fingerprints \( |h_\sigma^i(j)\rangle\):

\[
|h_\sigma^i(j)\rangle = \cos \frac{\pi k_i g_{j}(\sigma)}{m} |0\rangle + \sin \frac{\pi k_i g_{j}(\sigma)}{m} |1\rangle
\]

\[
|h_\sigma\rangle = \frac{1}{\sqrt{t}} \sum_{i=1}^{t} |i\rangle |h_\sigma^i(1)\rangle |h_\sigma^i(2)\rangle \ldots |h_\sigma^i(t)\rangle
\]

**Theorem 3.** If \( \chi_f \) is a linear characteristic then \( f \) can be computed by a quantum OBDD of width \( O(2^{\vert \chi_f \vert} \log m) \).

**Proof.** The proof of this result somewhat generalizes the proof of Theorem 2 to the case of several target
qubits. Here is the circuit:

1. Upon the input $\sigma = \sigma_1 \ldots \sigma_n$, we create the fingerprint $|h_\sigma\rangle$.

2. We measure $|h_\sigma\rangle$ in the standard computational basis and accept the input if the outcome of the last $l$ qubits is the all-zero state. Thus, the probability of accepting $\sigma$ is

$$P_{accept}(\sigma) = \frac{1}{l} \sum_{i=1}^{l} \cos^2 \frac{\pi k_i g_1(\sigma)}{m} \cdots \cos^2 \frac{\pi k_i g_l(\sigma)}{m}.$$  

If $f(\sigma) = 1$ then all of $g_i(\sigma) = 0$ and we will always accept.

If $f(\sigma) = 0$ then there is at least one such $j$ that $g_j(\sigma) \neq 0$ and the choice of the “good” set $K$ guarantees that the probability of the erroneously accepting is bounded by

$$P_{accept}(\sigma) = \frac{1}{l} \sum_{i=1}^{l} \cos^2 \frac{\pi k_i g_1(\sigma)}{m} \cdots \cos^2 \frac{\pi k_i g_l(\sigma)}{m} \leq \frac{1}{l} \sum_{i=1}^{l} \cos^2 \frac{2 \pi k_i g_i(\sigma)}{m} = \frac{1}{l} \sum_{i=1}^{l} \left( 1 + \cos \frac{2 \pi k_i g_i(\sigma)}{m} \right) \leq \frac{1}{l} + \frac{\sqrt{l}}{2}.$$  

The number of qubits used by this QBP $Q$ is qubits($Q$) = $O(\log \log m + |\chi_f|)$. Therefore, the width of the program is $2^{\text{qubits}(Q)} = O(2^{|\chi_f| \log m}).$

Note that though this upper bound is exponential our approach can be effectively used when the size of a characteristic is $O(\log \log m)$ and $m = 2^{o(1)}$. That is, in this case we will stay within the polynomial width. The Theorem[3] has two immediate consequences which might be useful for proving upper bounds.
Corollary 2. If a Boolean function \( f = f_1 \land f_2 \land \ldots \land f_s \) is a conjunction of \( s = O(\log n) \) Boolean functions \( f_i \), each having a linear characteristic polynomial over \( \mathbb{Z}_2^n \), then \( f \) can be computed by an \( O(\log n) \)-qubit quantum OBDD.

Corollary 3. If a Boolean function \( f = f_1 \lor f_2 \lor \ldots \lor f_s \) is a disjunction of \( s = O(\log n) \) Boolean functions \( f_i \) and the negation of each has a linear characteristic polynomial over \( \mathbb{Z}_2^n \), then \( f \) can be computed by an \( O(\log n) \)-qubit quantum OBDD.

The last corollary uses the fact that \( \neg f = \neg f_1 \land \neg f_2 \land \ldots \land \neg f_s \).

The generalized approach can be used to construct an effective quantum OBDD for the Boolean variant of the Hidden Subgroup Problem [4].

5 Using reductions for fingerprinting

Reduction is a well-known concept in complexity theory. The most investigated reduction for the model of branching programs is the technique of polynomial projections [17].

Definition 6. The sequence \((f_n)\) of Boolean functions is a polynomial projection of \((h_n)\), \( (f_n) \leq_{\text{proj}} (h_n) \), if \( f_n = h_{p(n)}(y_1, \ldots, y_{p(n)}) \) for some polynomial \( p \) and \( y_j \in \{0, 1, x_1, \ldots, x_n, \bar{x}_n\} \). The number of \( j \) such that \( y_j \in \{x_i, \bar{x}_i\} \) is called the multiplicity of \( x_i \).

Note that this type of reduction keep the linearity of the corresponding characteristic polynomials and thus the effective computability via the fingerprinting method.

Lemma 3. If \( (f_n) \leq_{\text{proj}} (h_n) \) and each \( h_n \) can be represented by a linear characteristic, then each \( f_n \) also has a linear characteristic.

Proof. The variable substitutions corresponding to projection are \( y_i \in \{0, 1, x_j, \bar{x}_j\} \). Obviously, if a characteristic is linear then substitutions of these types to polynomials \( g = c_1 y_1 + \ldots + c_n y_n + c_0 \) keep them linear.

On the other hand, the a special case of polynomial projections can be used to prove lower bounds in the OBDD model.

Definition 7. A projection is called a read-once projection, \( (f_n) \leq_{\text{rop}} (h_n) \), if the multiplicity of each variable is bounded by one.

The relation \( \leq_{\text{rop}} \) is reflexive and transitive, moreover if \( (f_n) \leq_{\text{rop}} (h_n) \) and the OBDD size of \( h_n \) is bounded by the polynomial \( q(n) \) then the OBDD size of \( f_n \) is bounded by the polynomial \( q(p(n)) \). The last property was proved for the classical OBDD model (see, e.g. [17]), but it can be proved for the quantum setting in an analogous way.

Thus the read-once projections may be used for proving lower bounds of the subclasses of Boolean functions whose projections are exponentially hard for the OBDD model. These are, for example, Set-Disjointness and Neighbored Ones Boolean functions which were proved to be exponentially hard in [16].

In particular these subclasses of “hard” functions cannot be effectively computed via quantum fingerprinting method in the model of read-once quantum branching programs.

Overall, the technique of polynomial projections outlines the limits of our fingerprinting method in the quantum OBDD model with upper bounds propagated by the general polynomial projections and lower bounds based on individual “hard” functions and read-once projections.
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