On the Inverse Ultrahyperbolic Klein-Gordon Kernel

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Abstract: In this work, we define the ultrahyperbolic Klein-Gordon operator of order \( \alpha \) on the function \( f \) by \( T^\alpha p f \), where \( \alpha \in \mathbb{C} \), \( W^\alpha \) is the ultrahyperbolic Klein-Gordon kernel, the symbol \( \ast \) denotes the convolution, and \( f \in S, S \) is the Schwartz space of functions. Our purpose of this work is to study the convolution of \( W^\alpha \) and obtain the operator \( L^\alpha = [T^\alpha]^{-1} \) such that if \( T^\alpha f = \varphi \), then \( L^\alpha \varphi = f \).

Keywords: ultrahyperbolic Klein-Gordon operator; ultrahyperbolic Klein-Gordon kernel; ultrahyperbolic kernel of Marcel Riesz; ultrahyperbolic operator; Dirac delta function

1. Introduction

Consider the linear differential equation of the form

\[ \vartriangle^k u(x) = f(x), \tag{1} \]

where \( u(x) \) and \( f(x) \) are generalized functions, \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( \vartriangle k \) is the \( n \)-dimensional ultra-hyperbolic operator iterated \( k \) times, which is defined by

\[ \vartriangle k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{2} \]

where \( p + q = n \) is the dimension of \( \mathbb{R}^n \), and \( k \) is non-negative integer.

The fundamental solution of Equation (1) was first introduced by Gelfand and Shilov [1] but the form is complicated and Trione [2] showed that the generalized function \( R^\gamma(x) \), defined by Equation (22) with \( \gamma = 2k \), is the fundamental solution of Equation (1). Later, Tellez [3] also proved that \( R^\gamma(x) \) exists only when \( p \) is odd with \( p + q = n \).

In 1997, Kananthai [4] introduced the diamond operator \( \diamond^k \) iterated \( k \) times, which is defined by

\[ \diamond^k = \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \frac{p+q}{\sum_{j=p+1} \frac{\partial^2}{\partial x_j^2}} \right)^2 \right)^k, \tag{3} \]

where \( k \) is a non-negative integer and \( p + q = n \) is the dimension of \( \mathbb{R}^n \). The operator \( \diamond^k \) can be expressed as the product of the operators \( \vartriangle^k \) and \( \vartriangle k \), that is

\[ \diamond^k = \vartriangle^k \vartriangle k = \vartriangle k \vartriangle^k, \tag{4} \]
where $\alpha^k$ is defined by Equation (2), and $\Delta^k$ is the Laplace operator iterated $k$ times, which is defined by

$$
\Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k.
$$

On finding the fundamental solution of diamond operator iterated $k$ times, Kananthai applied the convolution of functions which are fundamental solutions of the operators $\alpha^k$ and $\Delta^k$. He showed that $(-1)^k S_{2k}(x) \ast R_{2k}(x)$ is the fundamental solution of the operator $\bigotimes^k$. That is,

$$
\bigotimes^k \left( (-1)^k S_{2k}(x) \ast R_{2k}(x) \right) = \delta,
$$

where $R_{2k}(x)$ and $S_{2k}(x)$ are defined by Equations (22) and (29), respectively, with $\gamma = 2k$, and $\delta$ is the Dirac delta function. The solution $(-1)^k S_{2k}(x) \ast R_{2k}(x)$ is called the diamond kernel of Marcel Riesz. Interested readers are referred to [5–13] for some advance in the property of the diamond kernel of Marcel Riesz.

In 1978, Dominguez and Trione [14] introduced the distributional functions $H_a(P \pm i0, n)$, which is defined by

$$
H_a(P \pm i0, n) = \frac{e^{\pm \pi i / 2} e^{\pm \pi i / 2} \Gamma((n - a) / 2)(P \pm i0)^{(n-a)/2}}{2^{n/2} \pi^{n/2} \Gamma(n/2)},
$$

where

$$
P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,
$$

$p + q = n$, and $q$ is the number of negative terms of the quadratic form $P$. The distributions $(P \pm i0)^\lambda$ are defined by

$$
(P \pm i0)^\lambda = \lim_{\epsilon \to 0} (P \pm i\epsilon|x|^2)^\lambda,
$$

where $\lambda \in \mathbb{C}, \epsilon > 0$, and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, see [1]. They also showed the distributional functions $H_a(P \pm i0, n)$ are causal (anticausal) analogues of the elliptic kernel of Marcel Riesz [15]. Next, Cerutti and Trione [16] defined the causal (anticausal) generalized Marcel Riesz potentials of order $a, a \in \mathbb{C}$, by

$$
R^a \varphi = H_a(P \pm i0, n) \ast \varphi,
$$

where $\varphi \in \mathfrak{S}, \mathfrak{S}$ is the Schwartz space of functions [17], and $H_a(P \pm i0, n)$ is defined by Equation (7). They also studied the operator $(R^a)^{-1}$, that is the inverse operator of $R^a$, such that $f = R^a \varphi$ implies $(R^a)^{-1}f = \varphi$.

In 1999, Aguirre [18] defined the ultra-hyperbolic Marcel Riesz operator $M^a$ of the function $f$ by

$$
M^a(f) = R_\alpha \ast f,
$$

where $\alpha \in \mathbb{C}, R_\alpha$ is defined by (22), and $f \in \mathfrak{S}$. He also studied the operator $N^a = (M^a)^{-1}$ such that $M^a(f) = \varphi$ implies $N^a \varphi = f$.

In 2000, Kananthai [8] introduced the diamond kernel of Marcel Riesz $K_{a,b}$, which is given by

$$
K_{a,b} = S_\alpha \ast R_\beta,
$$

where $R_\beta$ and $S_\alpha$ are defined by Equations (22) and (29), respectively. Next, Tellez and Kananthai [13] proved that $K_{a,b}$ exists and is in the space of tempered distributions. In addition, they also showed the relationship between the convolution of the distributional families $K_{a,b}$ and diamond operator iterated $k$ times.

In 2011, Maneetus and Nonlaopon [19] defined the Bessel ultra-hyperbolic Marcel Riesz operator of order $a$ on the function $f$ by

$$
U^a(f) = R_\alpha^B \ast f,
$$

where $\alpha \in \mathbb{C}, R_\alpha^B$ is defined by Equations (22) and (29), respectively, with $\gamma = 2k$, and $\delta$ is the Dirac delta function.
where $\alpha \in \mathbb{C}, R_{n}^{B}$ is the Bessel ultra-hyperbolic kernel of Marcel Riesz, and $f \in S$. In addition, they studied the operator $E_{n}^{\alpha} = (U_{n})^{-1}$ such that $U_{n}^{\alpha}(f) = \varphi$ implies $E_{n}^{\alpha}\varphi = f$. Moreover, they defined the diamond Marcel Riesz operator of order $(\alpha, \beta)$ of the function $f$ by

$$M^{(\alpha, \beta)}(f) = K_{\alpha, \beta} \ast f,$$

where $\alpha, \beta \in \mathbb{C}, K_{\alpha, \beta}$ is defined by (12), and $f \in S$; see [20], for more details. In addition, they have also studied the operator $N^{(\alpha, \beta)} = \left[M^{(\alpha, \beta)}\right]^{-1}$ such that $M^{(\alpha, \beta)}(f) = \varphi$ implies $N^{(\alpha, \beta)}\varphi = f$.

In 2013, Salao and Nonlaopon [21] defined the Bessel diamond kernel of Marcel Riesz by

$$K_{\alpha, \beta}^{B}(x) = S_{\alpha}^{B}(x) * R_{\beta}^{B}(x),$$

where $S_{\alpha}^{B}(x)$ and $R_{\beta}^{B}(x)$ are the Bessel elliptic kernel of Marcel Riesz and the Bessel ultra-hyperbolic kernel of Marcel Riesz, respectively. They also defined the Bessel diamond Marcel Riesz operator of order $(\alpha, \beta)$ on the function $f$ by

$$U^{(\alpha, \beta)}(f) = K_{\alpha, \beta}^{B} \ast f,$$

where $\alpha, \beta \in \mathbb{C}, K_{\alpha, \beta}^{B}$ is defined by (15), and $f \in S$. In addition, they studied the operator $E^{(\alpha, \beta)} = \left[U^{(\alpha, \beta)}\right]^{-1}$ such that $U^{(\alpha, \beta)}(f) = \varphi$ implies $E^{(\alpha, \beta)}\varphi = f$.

In 2007, Tariboon and Kananthai [22] introduced the diamond Klein-Gordon operator $(\hat{\gamma} + m^2)^k$ iterated $k$ times, which is defined by

$$(\hat{\gamma} + m^2)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k,$$

where $m \geq 0, k$ is non-negative integer, $p + q = n$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Next, Nonlaopon et al. [23] studied the fundamental solution of diamond Klein-Gordon operator iterated $k$ times, which is called the diamond Klein-Gordon kernel, and studied the Fourier transform of the diamond Klein-Gordon kernel and its convolution [24].

In 2011, Liangprom and Nonlaopon [25] studied some properties of the distribution $e^{ax}(\hat{\gamma} + m^2)^k\delta$ and showed the boundedness property of the distribution $e^{ax}(\hat{\gamma} + m^2)^k\delta$, where $(\hat{\gamma} + m^2)^k$ is defined by Equation (17), $\alpha \in \mathbb{C}$, and $\delta$ is Dirac delta function.

In 2013, Sattaso and Nonlaopon [26] defined the diamond Klein-Gordon operator of order $\alpha$ on the function $f$ by

$$D^{\alpha}(f) = T_{\alpha} \ast f,$$

where $\alpha \in \mathbb{C}$, and $T_{\alpha}$ is the diamond Klein-Gordon kernel. They also studied the convolution of $T_{\alpha}$ and obtain the operator $L^{\alpha} = [D^{\alpha}]^{-1}$ such that $D^{\alpha}(f) = \varphi$ implies $L^{\alpha}\varphi = f$.

In 1988, Trione [27] studied the fundamental solution of the ultrahyperbolic Klein-Gordon operator $(\hat{\gamma} + m^2)^k$ iterated $k$ times, which is defined by

$$(\hat{\gamma} + m^2)^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2\right)^k.$$

She showed that $W_{2k}(x, m)$, defined by Equation (37) with $\alpha = 2k$, is the fundamental solution of the operator $(\hat{\gamma} + m^2)^k$, which is called the ultra-hyperbolic Klein-Gordon kernel. Next, Tellez [28] studied the convolution product of $W_{\alpha}(x, m) \ast W_{\beta}(x, m)$, where $\alpha, \beta \in \mathbb{C}$. In addition, Trione [29] has studied the fundamental $(P \pm i0)^k$-ultrahyperbolic solution of the Klein-Gordon operator iterated $k$
times and the convolution of such fundamental solution. She also studied the integral representation of the kernel \( W_n(x, m) \), see [30] for more details.

In this paper, we define the Klein-Gordon operator of order \( \alpha \) of the function \( f \) by

\[
T^\alpha (f) = W_n * f,
\]

where \( \alpha \in \mathbb{C}, W_n \) is the ultra-hyperbolic Klein-Gordon kernel defined by Equation (37), and \( f \in \mathcal{S} \). Our aim of this paper is to obtain the operator \( L^\alpha = [T^\alpha]^{-1} \) such that if \( T^\alpha (f) = \varphi \) then \( L^\alpha \varphi = f \).

Before we proceed to that point, we clarify some concepts and definitions.

2. Preliminaries

Definition 1. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and

\[
u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2
\]

be the non-degenerated quadratic form, where \( p + q = n \) is the dimension of \( \mathbb{R}^n \). Let \( \Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0 \} \) be the interior of a forward cone and let \( \Gamma_+ \) denote its closure. For any complex number \( \gamma \), we define

\[
R_\gamma(x) = \begin{cases} \frac{u^{(\gamma-n)/2} K_\gamma(y)}{K_n(\gamma)}, & \text{for } x \in \Gamma_+; \\ 0, & \text{for } x \notin \Gamma_+, \end{cases}
\]

where

\[
K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma((2 + \gamma - n)/2) \Gamma((1 - \gamma)/2) \Gamma(\gamma)}{\Gamma((2 + \gamma - p)/2) \Gamma((p - \gamma)/2)}.
\]

The function \( R_\gamma(x) \), which was introduced by Y. Nozaki [31], is called the ultra-hyperbolic kernel of Marcel Riesz. It is well known that \( R_\gamma(x) \) is an ordinary function when \( \text{Re}(\gamma) \geq n \) and is a distribution of \( \gamma \) otherwise. The support of \( R_\gamma(x) \) is denoted by \( \text{supp} \Gamma_\gamma(x) \) and suppose that \( \text{supp} R_\gamma(x) \subset \Gamma_+ \), that is, \( \text{supp} R_\gamma(x) \) is compact.

By putting \( p = 1 \) in \( R_\gamma(x) \) and taking into the Legendre’s duplication formula

\[
\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2),
\]

we obtain

\[
I_\gamma(x) = \frac{\nu^{(\gamma-n)/2}}{H_n(\gamma)},
\]

and \( \nu = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2 \), where

\[
H_n(\gamma) = \pi^{(n-2)/2} 2^\gamma - 1 \Gamma((\gamma + 2 - n)/2) \Gamma(\gamma/2).
\]

The function \( I_\gamma(x) \) is called the hyperbolic kernel of Marcel Riesz. From [2], the generalized function \( R_{2k}(x) \) is the fundamental solution of the operator \( \psi^k \), that is

\[
\psi^k (R_{2k}(x)) = \delta,
\]

In addition, it can be shown that

\[
R_{-2k}(x) = \psi^k \delta
\]

for \( k \) is a nonnegative integer, see [2,13].
**Definition 2.** Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of \( \mathbb{R}^n \) and \( \omega = x_1^2 + x_2^2 + \cdots + x_n^2. \) The elliptic kernel of Marcel Riesz is defined by

\[
S_\gamma(x) = \frac{\omega^{(\gamma-n)/2}}{U_n(\gamma)},
\]

where \( \gamma \in \mathbb{C}, n \) is the dimension of \( \mathbb{R}^n, \) and

\[
U_n(\gamma) = \frac{\pi^{n/2} \Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}.
\]

Note that \( n = p + q. \) By putting \( q = 0 \) (i.e., \( n = p) \) in (22) and (23), we can reduce \( u(\gamma-n)/2 \) to \( \omega_p^{(\gamma-p)/2}, \)
where \( \omega_p = x_1^2 + x_2^2 + \cdots + x_p^2, \) and reduce \( K_\gamma(\gamma) \) to

\[
K_p(\gamma) = \frac{\pi^{(p-1)/2} \Gamma((1 - \gamma)/2) \Gamma(\gamma)}{\Gamma((p - \gamma)/2)}.
\]

Using the Legendre’s duplication formula (Equation (24)) and

\[
\Gamma(1/2 + z) \Gamma(1/2 - z) = \pi \sec(\pi z),
\]

we obtain

\[
K_p(\gamma) = \frac{1}{2} \sec(\gamma \pi/2) U_p(\gamma).
\]

Thus, for \( q = 0, \) we have

\[
R_\gamma(x) = \frac{u^{(\gamma-p)/2}}{K_p(\gamma)} = 2 \cos(\gamma \pi/2) \frac{u^{(\gamma-p)/2}}{U_p(\gamma)} = 2 \cos(\gamma \pi/2) S_\gamma(x).
\]

In addition, if \( \gamma = 2k \) for some non-negative integer \( k, \) then

\[
R_{2k}(x) = 2(-1)^k S_{2k}(x).
\]

Next, we consider the function

\[
W_\alpha(x, m) = \begin{cases} \frac{(m/2)^{1/2} \Gamma(\alpha/2)}{\Gamma(\alpha/2) \Gamma(\alpha/2)} \int_{\mathbb{R}^n} (m^2 u^{1/2}), & \text{for } x \in \Gamma_+; \\ 0, & \text{for } x \notin \Gamma_+,
\end{cases}
\]

where \( \alpha \in \mathbb{C}, n \) is defined by Equation (21), \( m \) a real non-negative number, \( n \) is the dimension of \( \mathbb{R}^n, \)
and \( J_\nu(z) \) is Bessel function of the first kind, which is defined by

\[
J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k! \Gamma(k + \nu + 1)}.
\]

It is well known that \( W_\alpha(x, m) \) is an ordinary function when \( \Re(\alpha) \geq n \) and is a distribution otherwise. In addition, \( W_\alpha(x, m) \) can be expressed as an infinitely linear combination of \( R_\alpha(x) \) of different orders, that is

\[
W_\alpha(x, m) = \sum_{\nu=0}^{\infty} \frac{(-\alpha/2)^{\nu} \Gamma(\alpha/2)}{\Gamma(\nu + 1)} m^{2\nu} R_{\alpha+2\nu}(x),
\]

where \( \alpha \in \mathbb{C}, R_\alpha(x) \) is defined by Equation (22), see [27,29,30], for more details.

From Equation (37) and by putting \( \alpha = -2k, \) for \( k \) is non-negative integer, we have

\[
W_{-2k}(x, m) = \sum_{\nu=0}^{\infty} \frac{k \nu}{{\nu \choose k} m^{2\nu} R_{-2(k-\nu)}(x)}.
\]
Since the operator \( (\sigma + m^2)^k \) defined by Equation (19) is linearly continuous and injective mapping of this possess its own inverse. From Equation (28), we obtain
\[
W_{-2k}(x, m) = \sum_{v=0}^{\infty} \binom{k}{v} m^{2v} \sigma^{k-v} \delta = (\sigma + m^2)^k \delta. \tag{39}
\]

Substituting \( k = 0 \) in Equation (39), yields \( W_0(x, m) = \delta \). On the other hand, by putting \( \alpha = 2k \) in Equation (37), yields
\[
W_{2k}(x, m) = \left(\frac{-k}{0}\right) m^{2(0)} R_{2k+0}(x) + \sum_{v=1}^{\infty} \binom{-k}{v} m^{2v} R_{2k+2v}(x). \tag{40}
\]

The second summand of the right-hand side of Equation (40) vanishes when \( m^2 = 0 \). Therefore, we obtain
\[
W_{2k}(x, m = 0) = R_{2k}(x),
\]
is the fundamental solution of the ultra-hyperbolic operator \( \sigma^k \). For the convenience, we will denote \( W_{\alpha}(x, m) \) by \( W_{\alpha} \).

The proof of Lemmas 1 and 2 are given in [28].

**Lemma 1.** The function \( W_{\alpha} \) has the following properties:

(i) \( W_0 = \delta \);
(ii) \( W_{-2k} = (\sigma + m^2)^k \delta \);
(iii) \( (\sigma + m^2)^k W_{\alpha} = W_{-2k} \);
(iv) \( (\sigma + m^2)^k W_{2k} = \delta \);
(v) \( W_{\alpha} \ast W_{-2k} = (\sigma + m^2)^k W_{\alpha} \).

**Lemma 2.** (The convolutions of \( W_{\alpha} \))

(i) If \( p \) is odd, then
\[
W_{\alpha} \ast W_{\beta} = W_{\alpha + \beta} + A_{\alpha, \beta}, \tag{41}
\]
where
\[
A_{\alpha, \beta} = \frac{i \sin(\alpha \pi/2) \sin(\beta \pi/2)}{2 \sin((\alpha + \beta)\pi/2)} \sum_{v=0}^{\infty} (m^2)^v \binom{-\alpha - \beta/2}{v} \left[ H^+_{\alpha + \beta + 2v} - H^-_{\alpha + \beta + 2v} \right] \tag{42}
\]
and
\[
H^\pm_{\alpha + \beta + 2v} = H_{\alpha + \beta + 2v}(P \pm i0, n) \tag{43}
\]
as defined by Equation (7) with \( \gamma = \alpha + \beta + 2v \).

(ii) If \( p \) is even, then
\[
W_{\alpha} \ast W_{\beta} = B_{\alpha, \beta} W_{\alpha + \beta}, \tag{44}
\]
where
\[
B_{\alpha, \beta} = \frac{\cos(\alpha \pi/2) \cos(\beta \pi/2)}{\cos((\alpha + \beta)\pi/2)}. \tag{45}
\]

3. The Convolution of \( W_{\alpha} \ast W_{\beta} \) when \( \beta = -\alpha \)

In this section, we will consider the property of \( W_{\alpha} \ast W_{\beta} \) when \( \beta = -\alpha \).

From Equations (41) and (44), we immediately obtain the following properties:

1. If \( p \) is odd and \( q \) is even, then
\[
W_{\alpha} \ast W_{\beta} = W_{\alpha + \beta} + A_{\alpha, \beta}, \tag{46}
\]
where \( A_{\alpha, \beta} \) is defined by Equation (42).
2. If $p$ and $q$ are both odd, then
\[ W_\alpha \ast W_\beta = W_{\alpha+\beta} + A_{\alpha,\beta}. \]  
\[ \text{Equation (46)} \]

3. If $p$ is even and $q$ is odd, then
\[ W_\alpha \ast W_\beta = \frac{\cos(\alpha \pi/2) \cos(\beta \pi/2)}{\cos((\alpha + \beta) \pi/2)} W_{\alpha+\beta}. \]  
\[ \text{Equation (47)} \]

4. If $p$ and $q$ are both even, then
\[ W_\alpha \ast W_\beta = \frac{\cos(\alpha \pi/2) \cos(\beta \pi/2)}{\cos((\alpha + \beta) \pi/2)} W_{\alpha+\beta}. \]  
\[ \text{Equation (48)} \]

Moreover, it follows from Equation (42) that
\[ A_{\alpha,\gamma} = \lim_{\beta \to -\alpha} A_{\alpha,\beta} \]
\[ = \frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\alpha \pi/2) \sin((\gamma - \alpha) \pi/2)}{\sin(\gamma \pi/2)} \sum_{v=0}^{\infty} (m^2)^v \left( -\frac{\gamma}{2} \right)^v \left[ H_{\gamma+2v}^+ - H_{\gamma+2v}^- \right] \]
\[ = \frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\alpha \pi/2) \sin((\gamma - \alpha) \pi/2)}{\sin(\gamma \pi/2)} \times \lim_{\gamma \to 0} \left\{ H_{\gamma+2v}^+ - H_{\gamma+2v}^- \right\} + \sum_{v=1}^{\infty} \frac{(m^2)^v}{k} \left[ H_{\gamma+2v}^+ - H_{\gamma+2v}^- \right] \]
\[ = \frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\alpha \pi/2) \sin((\gamma - \alpha) \pi/2)}{\sin(\gamma \pi/2)} \lim_{\gamma \to 0} \left[ H_{\gamma}^+ - H_{\gamma}^- \right], \]
\[ \text{Equation (49)} \]

where $\gamma = \alpha + \beta$.

On the other hand, using Equations (43) and (7), we have
\[ \lim_{\gamma \to 0} [H_{\gamma}^+ - H_{\gamma}^-] = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \lim_{\gamma \to 0} e^{-\gamma n i/2} e^{\gamma n i/2} \frac{(P + i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} - \lim_{\gamma \to 0} e^{\gamma n i/2} e^{-\gamma n i/2} \frac{(P - i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \right] \]
\[ = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \lim_{\gamma \to 0} e^{-\gamma n i/2} e^{\gamma n i/2} \frac{\text{Res}_{\beta=-n/2} (P + i0)^{\beta}}{\Gamma(\beta + n/2)} - \lim_{\gamma \to 0} e^{\gamma n i/2} e^{-\gamma n i/2} \frac{\text{Res}_{\beta=-n/2} (P - i0)^{\beta}}{\Gamma(\beta + n/2)} \right] \]
\[ \text{Equation (50)} \]

Now, taking $n$ as an odd integer, yields
\[ \text{Res}_{\lambda=-n/2-k} (P \pm i0)^{\lambda} = \frac{e^{\pm q n i/2} \pi^{n/2}}{2^{k} k! \Gamma(n/2 + k)} \delta^k. \]
\[ \text{Equation (51)} \]

where $\delta^k$ is defined by (2), $p + q = n$, and $k$ is a non-negative integer; see [32,33]. If $p$ and $q$ are both even, then
\[ \text{Res}_{\lambda=-n/2-k} (P \pm i0)^{\lambda} = \frac{e^{\pm q n i/2} \pi^{n/2}}{2^{k} k! \Gamma(n/2 + k)} \delta^k. \]
\[ \text{Equation (52)} \]
Nevertheless, if \( p \) and \( q \) are both odd, then
\[
\text{Res}_{\lambda=-n/2-k} (P \pm \imath 0)^\lambda = 0,
\]
(53)

Therefore, we have
\[
\lim_{\gamma \to 0} \left[ H^+_{\gamma} - H^-_{\gamma} \right] = \frac{\Gamma(n/2)}{\pi^{n/2}} \cdot \frac{\pi^{n/2}}{\Gamma(n/2)} \cdot \left[ \lim_{\gamma \to 0} e^{-\gamma \pi i/2} - \lim_{\gamma \to 0} e^{\gamma \pi i/2} \right] \delta
\]
\[
= \lim_{\gamma \to 0} \left[ -2i \sin (\gamma \pi/2) \right] \delta.
\]
(54)

From Equations (50) and (53), we have
\[
\lim_{\gamma \to 0} \left[ H^+_{\gamma} - H^-_{\gamma} \right] = 0
\]
(55)

if \( p \) and \( q \) are both odd \( (n \text{ even}) \).

Applying Equations (54) and (55) into Equation (49), we have
\[
A_{a,-a} = \frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(a \pi/2) \sin((\gamma - a) \pi/2)}{\sin(\gamma \pi/2)} \cdot \lim_{\gamma \to 0} \left[ -2i \sin (\gamma \pi/2) \right] \delta(x)
\]
\[
= \sin^2 \left( a \pi/2 \right) \delta
\]
(56)

if \( p \) is odd and \( q \) is even, and
\[
A_{a,-a} = 0
\]
(57)

if \( p \) and \( q \) are both odd.

From Equations (45)–(48) and using Lemmas 1, 2 and Equations (56) and (57), if \( p \) is odd and \( q \) is even, then we obtain
\[
W_a \ast W_{-a} = W_0 + A_{a,-a}
\]
\[
= \delta + \sin^2 \left( a \pi/2 \right) \delta
\]
\[
= \left[ 1 + \sin^2 \left( a \pi/2 \right) \right] \delta.
\]
(58)

If \( p \) and \( q \) are both odd, then
\[
W_a \ast W_{-a} = W_0 = \delta.
\]
(59)

If \( p \) is even and \( q \) is odd, then
\[
W_a \ast W_{-a} = \frac{\cos \left( a \pi/2 \right) \cos \left( -a \pi/2 \right)}{\cos \left( (a - a) \pi/2 \right)} W_0
\]
\[
= \cos^2 \left( a \pi/2 \right) \delta.
\]
(60)

Finally, if \( p \) and \( q \) are both even, then
\[
W_a \ast W_{-a} = \frac{\cos \left( a \pi/2 \right) \cos \left( -a \pi/2 \right)}{\cos \left( (a - a) \pi/2 \right)} W_0
\]
\[
= \cos^2 \left( a \pi/2 \right) \delta.
\]
(61)
4. The Main Theorem

Let \( T^\alpha (f) \) be the ultrahyperbolic Klein-Gordon operator of order \( \alpha \) on the function \( f \), which is defined by

\[
T^\alpha (f) = W_\alpha \ast f,
\]

where \( \alpha \in \mathbb{C} \), \( W_\alpha \) is defined by Equation (37), and \( f \in \mathcal{S} \).

Recall that our objective is to obtain the operator \( L^\alpha = [T^\alpha]^{-1} \) such that if \( T^\alpha (f) = \varphi \), then \( L^\alpha \varphi = f \) for all \( \alpha \in \mathbb{C} \).

**Theorem 1.** If \( T^\alpha (f) = \varphi \), then \( L^\alpha \varphi = f \) such that

\[
L^\alpha = [T^\alpha]^{-1} = \begin{cases} 
1 + \sin^2(\alpha \pi/2) \n & \text{if } p \text{ is odd and } q \text{ is even;} \\
W_{-\alpha} & \text{if } p \text{ and } q \text{ are both odd;} \\
\sec^2(\alpha \pi/2) W_{-\alpha} & \text{if } p \text{ is even with } \alpha/2 + 2s + 1
\end{cases}
\]

for any non-negative integer \( s \).

**Proof.** By Equation (62), we have

\[
T^\alpha (f) = W_\alpha \ast f = \varphi,
\]

where \( \alpha \in \mathbb{C} \), \( W_\alpha \) is defined by Equation (37), and \( f \in \mathcal{S} \). If \( p \) is odd and \( q \) is even, then, in view of Equation (58), we obtain

\[
\left[1 + \sin^2(\alpha \pi/2)\right]^{-1} W_{-\alpha} \ast (W_\alpha \ast f) = \left[1 + \sin^2(\alpha \pi/2)\right]^{-1} (W_{-\alpha} \ast W_\alpha) \ast f
= \left[1 + \sin^2(\alpha \pi/2)\right]^{-1} \left\{1 + \sin^2(\alpha \pi/2)\delta\right\} \ast f
= \delta \ast f = f.
\]

Therefore,

\[
\left[1 + \sin^2(\alpha \pi/2)\right]^{-1} W_{-\alpha} = [T^\alpha]^{-1} = (W_\alpha)^{-1}
\]

for all \( \alpha \in \mathbb{C} \).

Similarly, if both \( p \) and \( q \) are odd, then by Equation (59), we obtain

\[
W_{-\alpha} \ast (W_\alpha \ast f) = (W_{-\alpha} \ast W_\alpha) \ast f = \delta \ast f = f.
\]

Therefore,

\[
W_{-\alpha} = [T^\alpha]^{-1} = (W_\alpha)^{-1}
\]

for all \( \alpha \in \mathbb{C} \).

Finally, if \( p \) is even, then by Equations (60) and (61), we have

\[
\sec^2(\alpha \pi/2) W_{-\alpha} \ast (W_\alpha \ast f) = \sec^2(\alpha \pi/2) (W_{-\alpha} \ast W_\alpha) \ast f
= \sec^2(\alpha \pi/2) \left\{\cos^2(\alpha \pi/2)\delta\right\} \ast f
= \delta \ast f = f,
\]

provided that \( \beta/2 + 2s + 1 \) for any non-negative integer \( s \). Therefore,

\[
\sec^2(\alpha \pi/2) W_{-\alpha} = [T^\alpha]^{-1} = (W_\alpha)^{-1}
\]

for all \( \alpha \in \mathbb{C} \) with \( \alpha/2 \neq 2s + 1 \) for any non-negative integer \( s \).
Therefore, we have the desired results in Equations (63)–(65).

5. Conclusions

In this work, we have considered the property of convolution of the ultrahyperbolic Klein-Gordon kernel in the form \( W_\alpha \ast W_\beta \) when \( \beta = -\alpha \). We have obtained the inverse ultrahyperbolic Klein-Gordon kernel, that is, the operator \( L^\alpha = [T^\alpha]^{-1} \) such that if \( T^\alpha(f) = \varphi \), then \( L^\alpha \varphi = f \) for all \( \alpha \in \mathbb{C} \). It is expected that this work may stimulate further research in this field.

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