EQUIDISTRIBUTION, ERGODICITY AND IRREDUCIBILITY IN CAT(-1) SPACES

ADRIEN BOYER

Abstract. We prove an equidistribution theorem à la Bader-Muchnik ([3]) for operator-valued measures associated with boundary representations in the context of discrete groups of isometries of CAT(-1) spaces thanks to an equidistribution theorem of T. Roblin ([22]). This result can be viewed as a generalization of Birkhoff’s ergodic theorem for quasi-invariant measures. In particular, this approach gives a dynamical proof of the fact that boundary representations are irreducible. Moreover, we prove some equidistribution results for conformal densities using elementary techniques from harmonic analysis.

1. Introduction

Any action of a locally compact group $G$ on a measure space $(X, \mu)$ where $\mu$ is a $G$-quasi-invariant measure gives rise to a unitary representation, after renormalization with the square root of the Radon-Nikodym derivative of the action of $G$ on $(X, \mu)$. This unitary representation is called a quasi-regular representation, and generalizes the standard notion of quasi-regular representations given by $G \curvearrowright G/H$ where $H$ is a closed subgroup of $G$, and $G/H$ carries a $G$-quasi-invariant measure.

The dynamical properties of the action $G \curvearrowright (X, \mu)$ can be reflected in a such representation.

In the context of fundamental groups of compact negatively curved manifolds, U. Bader and R. Muchnik prove in [3, Theorem 3] an equidistribution theorem for some operator-valued measures. This theorem can be thought of a generalization of Birkhoff’s ergodic theorem for quasi-invariant measures for fundamental groups acting on the Gromov boundary of universal covers of compact negatively curved manifolds endowed with the Patterson-Sullivan measures. These quasi-regular representations are called boundary representations. It turns out that the irreducibility of boundary representations follows from this generalization of Birkhoff’s ergodic theorem. We mention the works of [4, 3, 11, 12, 14] and [19] for examples of natural irreducible quasi-regular representations which are related to the following conjecture:

Conjecture. For a locally compact group $G$ and a spread-out probability measure $\mu$ on $G$, the quasi-regular representation associated to a $\mu$-boundary of $G$ is irreducible.
In this paper, we generalize the work of U. Bader and R. Muchnik to convex cocompact groups of isometries of a CAT(-1) with a non-arithmetic spectrum and to (non-uniform) lattices of a non-compact connected semisimple Lie group of rank one. Our results are based on the fundamental work of T. Roblin in [22]. The main tool of this paper is an equidistribution theorem of T. Roblin (see Subsection 5.3) which is inspired by the ideas of G. Margulis (see [20]). More specifically, Roblin’s equidistribution theorem is based on the mixing property of the geodesic flow. Following the technical ideas developed in [3] and using Roblin’s equidistribution theorem, we obtain a dynamical explanation of irreducibility of boundary representations in the context of CAT(-1) spaces: it comes from the mixing property of the geodesic flow. Nevertheless, due to lack of equidistribution theorems, this approach does not work in the context of general hyperbolic groups and we refer to [11], [12], [19] and more recently [14] for different approaches.

Moreover, we prove two equidistribution results for densities associated to the Poisson kernel and the square root of the Poisson Kernel in CAT(-1) spaces with respect to the weak* convergence of the dual space $L^1$ functions on the boundary.

**Main Results.** The Banach space of finite signed measures on a topological compact space $Z$ is, by the Riesz representation theorem, the dual of the continuous functions $C(Z)$ on $Z$. The Banach space of bounded linear operators from the Banach space of continuous functions to the Banach space of bounded operators on a Hilbert space will be denoted by $\mathcal{L}(C(Z), B(\mathcal{H}))$. Observe that $\mathcal{L}(C(Z), B(\mathcal{H}))$ is isomorphic as Banach spaces to the dual of the Banach space $C(Z) \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}$ where $\mathcal{H}$ denotes the conjugate Hilbert space of the complex Hilbert space $\mathcal{H}$, and $\hat{\otimes}$ denotes the projective tensor product (see Subsection 3.1). Thus $\mathcal{L}(C(Z), B(\mathcal{H}))$ will be called the space of operator-valued measures.

Let $\Gamma$ be a non-elementary discrete group of isometries of $(X,d)$ a proper CAT(-1) metric space (i.e. the balls are relatively compact). We denote by $\partial X$ its Gromov boundary, and let $\overline{X}$ be the topological space $X \cup \partial X$ endowed with its usual topology that makes $\overline{X}$ compact. Recall the critical exponent $\alpha(\Gamma)$ of $\Gamma$:

$$\alpha(\Gamma) := \inf \left\{ s \in \mathbb{R}_+^* \mid \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < \infty \right\}.$$  

Notice that the definition of $\alpha(\Gamma)$ does not depend on $x$. We assume from here on now that $\alpha(\Gamma) < \infty$.

The limit set of $\Gamma$ denoted by $\Lambda_{\Gamma}$ is the set of all accumulation points in $\partial X$ of an orbit. Namely $\Lambda_{\Gamma} := \overline{\gamma x} \cap \partial X$, with the closure in $\overline{X}$. Notice that the limit set does not depend on the choice of $x \in X$. Following the notations in [8], define the geodesic hull $GH(\Lambda_{\Gamma})$ as the union of all geodesics in $X$ with both endpoints in $\Lambda_{\Gamma}$. The convex hull of $\Lambda_{\Gamma}$ denoted by $CH(\Lambda_{\Gamma})$, is the smallest subset of $X$ containing $GH(\Lambda_{\Gamma})$ with the property that every geodesic segment between any pair of points $x, y \in CH(\Lambda_{\Gamma})$ also lies in $CH(\Lambda_{\Gamma})$. We say that $\Gamma$ is convex cocompact if it acts cocompactly on $CH(\Lambda_{\Gamma})$.

The translation length of an element $\gamma \in \Gamma$ is defined as $t(\gamma) := \inf \{ d(x, \gamma x), x \in X \}$. The spectrum of $\Gamma$ is defined as the subgroup of $\mathbb{R}$ generated by $t(\gamma)$ where $\gamma$ ranges over the hyperbolic isometries in $\Gamma$. We say that $\Gamma$ has an arithmetic spectrum if its
spectrum is a discrete subgroup of $\mathbb{R}$. We are interested in discrete groups with a *non-arithmetic spectrum* because they guarantee the mixing property of the geodesic flow (see Subsection 2.2), and this condition is verified in the following cases: for isometries group of Riemannian surfaces, hyperbolic spaces and isometries groups of a CAT(-1) space such that the limit set has a non-trivial connected component. We refer to [10] and to [22, Proposition 1.6, Chapitre 1] for more details.

A Riemannian symmetric space $X$ of non-compact type of rank one endowed with its natural Riemannian metric is a particular case of CAT(-1) space. The space $X$ as well as its boundary $\partial X$ can be described by the quotients $X = G/K$ and $\partial X = G/Q$ where $G$ is a non-compact connected semisimple Lie group of real rank one, $K$ a maximal compact subgroup and $Q$ a minimal parabolic subgroup of $G$. A lattice $\Gamma$ is a discrete subgroup of $G$ such that the quotient $\Gamma \backslash G$ has finite volume w.r.t. the Haar measure. In this case $\Lambda_{\Gamma} = \partial X$ and $\text{CH}(\Lambda_{\Gamma}) = X$. If $\Gamma \backslash G$ is a compact, we say that $\Gamma$ is a uniform lattice and this is a particular case of convex compact groups. Otherwise we say that $\Gamma$ is a *non-uniform lattice*.

The foundations of Patterson-Sullivan measures theory are in the important papers [21], [26]. See [6], [7], and [22] for more general results in the context of CAT(-1) spaces. These measures are also called *conformal densities*.

We denote by $M(Z)$ the Banach space of Radon measures on a locally compact space $Z$, which is identified to the dual space of compactly supported functions denoted by $C_c(Z)^*$, endowed with the norm $\|\mu\| = \sup\{|\int_Z f d\mu|, \|f\|_\infty \leq 1, f \in C_c(Z)\}$ where $\|f\|_\infty = \sup_{z \in Z} |f(z)|$. Recall that $\gamma_*\mu$ means $\gamma_*\mu(B) = \mu(\gamma^{-1}B)$ where $\gamma$ is in $\Gamma$ and $B$ is a borel subset of $Z$.

We say that $\mu$ is a $\Gamma$-invariant conformal density of dimension $\alpha \geq 0$, if $\mu$ is a map which satisfies the following conditions:

- $\mu$ is a map from $x \in X \mapsto \mu_x \in M(X)$, i.e. $\mu_x$ is a positive finite measure (density).
- For all $x$ and $y$ in $X$, $\mu_x$ and $\mu_y$ are equivalent, and we have
  \[
  \frac{d\mu_y}{d\mu_x}(v) = \exp (\alpha \beta_v(x, y))
  \]
  (conformal of dimension $\alpha$).
- For all $\gamma \in \Gamma$, and for all $x \in X$ we have $\gamma_*\mu_x = \mu_{\gamma x}$ (invariant),

where $\beta_v(x, y)$ denotes the horoshperical distance from $x$ to $y$ relative to $v$ (see Subsection 2.1).

If $X$ is a CAT(-1) space and if $\Gamma$ is a discrete group of isometries of $X$, then there exists a $\Gamma$-invariant conformal density of dimension $\alpha(\Gamma)$ whose the support is $\Lambda_{\Gamma}$. This is a theorem and a proof can be found in [21] and [20] for the case of hyperbolic spaces and see [3] and [10] for the case of CAT(-1) spaces.

A conformal density $\mu$ gives rise to unitary representations $(\pi_x)_{x \in X}$ defined for $x \in X$ as:

\[
\pi_x : \Gamma \to \mathcal{U}(L^2(\partial X, \mu_x))
\]

(1)

\[
(\pi_x(\gamma)\xi)(v) = \xi(\gamma^{-1}v) \exp \left(\frac{\alpha}{2} \beta_v(x, \gamma x)\right),
\]
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where $\xi \in L^2(\partial X, \mu_x)$ and $v \in \partial X$.
These representations are unitarily equivalent: the multiplication operator
\[ U_{xy} : L^2(\partial X, \mu_x) \to L^2(\partial X, \mu_y) \]
defined by the function
\[ m_{xy}(v) = \exp \left( -\frac{\alpha}{2} \beta_v(x, y) \right), \]
is a unitary operator which intertwines the unitary representations $\pi_x$ and $\pi_y$.

The matrix coefficient
\[ \phi_x : \Gamma \to \langle \pi_x(\gamma)1_{\partial X}, 1_{\partial X} \rangle \in \mathbb{R}^+, \]
where $1_{\partial X}$ is the characteristic function of $\partial X$, is called the Harish-Chandra function.

Pick $x$ in $X$, and a positive real number $\rho$ and define for all integers $n$ such that $n \geq \rho$ the annulus
\[ C_n(x, \rho) = \{ \gamma \in \Gamma | n - \rho \leq d(\gamma x, x) < n + \rho \}. \]
Assume that $C_n(x, \rho)$ is not empty for all $n \geq N_{x,\rho}$ for some integer $N_{x,\rho}$. Denote by $|C_n(x, \rho)|$ the cardinality of $C_n(x, \rho)$. Let $D_y$ be the unit Dirac mass centered at a point $y \in X$. Consider the sequence of operator-valued measures defined for all $n \geq N_{x,\rho}$ as:
\[ M_n^{x,\rho} : f \in C(X) \mapsto \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma x}(f) \frac{\pi_x(\gamma)}{\phi_x(\gamma)} \in B(L^2(\partial X, \mu_x)). \]
If $f \in C(X)$, we denote by $f|_{\partial X}$ its continuous restriction to the space $\partial X$. Consider also the operator-valued measure $M_x$ defined as:
\[ M_x : f \in C(X) \mapsto \left( M_x(f) : \xi \mapsto \left( \int_{\partial X} \xi d\mu_x \right) f|_{\partial X} \right) \in B(L^2(\partial X, \mu_x)). \]

The main result of this paper is the following theorem:

**Theorem A.** (Equidistribution à la Bader-Muchnik)
Let $\Gamma$ be a convex cocompact discrete group of isometries of a CAT(-1) space $X$ with a non-arithmetic spectrum or a non-uniform lattice acting by isometries on a Riemannian symmetric space of non-compact type of rank one denoted also by $X$. Let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\alpha(\Gamma)$. Then for each $x$ in $X$ there exists $\rho > 0$ such that
\[ M_n^{x,\rho} \rightrightarrows M_x \]
as $n \to +\infty$ w.r.t. the weak* topology of the Banach space $L(C(X), B(L^2(\partial X, \mu_x)))$.

**Remark 1.1.** In the case of lattices acting by isometries on rank one symmetric spaces or of fundamental groups acting on the universal cover of compact negatively curved manifolds the above theorem holds for all $\rho > 0$ (more generally when $\Lambda\Gamma = \partial X$).

With the same hypothesis of the above theorem, we deduce immediately an ergodic theorem à la Birkhoff for the $\Gamma$-quasi-invariant measures $\mu_x$ on $\partial X$.
Let $x \in X$, and denote by $Q_x$ the orthogonal projection onto the subspace of constant functions of $L^2(\partial X, \mu_x)$. 

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Let $x \in X$, and denote by $Q_x$ the orthogonal projection onto the subspace of constant functions of $L^2(\partial X, \mu_x)$.
Corollary B. (Ergodicity à la Birkhoff)
For all $x \in X$ there exists $\rho > 0$ such that
\[
\frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \frac{\pi_x(\gamma)}{\phi_x(\gamma)} \to Q_x
\]
as $n \to +\infty$ w.r.t. the weak operator topology in $\mathcal{B}(L^2(\partial X, \mu_x))$.

Remark 1.2. Same remark as Remark 1.1 with the above corollary.

Remark 1.3. Consider an action of $\mathbb{Z}$ on a finite measure space $(X,\mu)$ by measure
preserving transformations. Birkhoff’s very well-known ergodic theorem states, in the
functional analytic framework, that the ergodicity of the action is equivalent to the con-
vergence
\[
\frac{1}{2n+1} \sum_{k=-n}^{n} \pi(k) \to Q
\]
w.r.t. the weak operator topology, where $\pi$ is the quasi-regular representation obtained
from the action of $\mathbb{Z}$, and where $Q$ is the orthogonal projection onto the space of constant
functions of the space $L^2(X,\mu)$. This theorem belongs to the foundation of ergodic theory
and stays an important source of inspiration (see for example [18]).

With the same hypothesis of Theorem A we have:

Corollary C. (Irreducibility)
For all $x \in X$, the representations $\pi_x : \Gamma \to U(L^2(\partial X, \mu_x))$ are irreducible.

Notice that Corollary C for lattices is well known, see [9].

The Poisson kernel. Recall the definition of the Poisson kernel in the context of CAT(-1)
spaces. Let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\alpha$. Fix $x \in X$ a base point
and define the Poisson kernel associated to the measure $\mu_x$ as:
\[
P : (y,v) \in X \times \partial X \mapsto P(y,v) = \exp \left( \alpha \beta_v(x,y) \right) \in \mathbb{R}^+.
\]
We follow the notations of Sjögren ([24]) and we define for $\lambda \in \mathbb{R}$ and $f \in L^1(\partial X, \mu_x)$:
\[
P_\lambda f(y) = \int_{\partial X} P(y,v)^{\lambda+1/2} f(v) d\mu_x(v).
\]
Furthermore we denote by $\nu_y$ the measure associated to $P_0$ defined as
\[
d\nu_y(v) = \frac{P(y,v)^{1/2}}{P_0\partial X(y)} d\mu_x(v).
\]
Observe that the measure $\nu_y$ is a probability measure. We prove:

Proposition D. Let $\Gamma$ be a discrete group of isometries of a CAT(-1) space $X$ with
a non-arithmetic spectrum. Let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\alpha(\Gamma)$
the critical exponent of the group. Assume that $\Gamma$ has a finite Bowen-Margulis-Sullivan
measure. We have for all $x \in X$ and for all $\rho > 0$ that
\[
\frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \mu_{\gamma x} \to \frac{\mu_x}{\|\mu_x\|}
\]
w.r.t. the weak* convergence of $L^1(\partial X, \mu_x)^\ast$.

Moreover if $\Gamma$ is convex cocompact with a non-arithmetic spectrum or if $\Gamma$ is a non-uniform lattice acting by isometries on a Riemannian symmetric space of non-compact type of rank one, then for all $x$ in $X$ there exists $\rho$ such that

$$\frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \nu_{\gamma x} \rightharpoonup \nu_x$$

w.r.t. the weak* convergence of $L^1(\partial X, \nu_x)^\ast$.

See Subsection 2.2 for the definition of the Bowen-Margulis-Sullivan measure.

The method of proof consists in two steps: given a sequence of functionals of the dual of a separable Banach space, we shall prove

- **Step 1:** The sequence is uniformly bounded: existence of accumulation points (by the Banach-Alaoglu theorem).

- **Step 2:** Identification of the limit using equidistribution theorems (only one accumulation point).

Structure of the paper. In Section 2 we remind the reader of some standard facts about CAT(-1) spaces, and we recall the definition of Bowen-Margulis-Sullivan measures as well as the Roblin’s equidistribution theorem.

In Section 3 we recall some general facts about Banach spaces and projective tensor products, and we give a general construction of operator-valued measures that we deal with in the context of CAT(-1) spaces.

In Section 4 we prove uniform boundedness for two sequences of functions, and we deduce Step 1 of our results.

In Section 5 we use Roblin’s equidistribution theorem to achieve Step 2 of our main result.

In Section 6 we prove our main theorem and its corollaries.

In Section 7 we prove two equidistribution results w.r.t. the weak* convergence of the dual space of $L^1$ functions on the boundary, dealing with the Poisson kernel and the square root of the Poisson kernel using the dual inequality established in Section 4.

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2. Preliminaries

2.1. CAT(-1) spaces. A CAT(-1) space is a metric geodesic space such that every geodesic triangle is thinner than its comparison triangle in the hyperbolic plane, see [5, Introduction]. Let $(X, d)$ be a proper CAT(-1) space. A geodesic ray of $(X, d)$ is an isometry:

$$r : I \to X,$$
where $I = [0, +\infty) \subset \mathbb{R}$. Two geodesic rays are equivalent if the Hausdorff distance between their images are bounded, equivalently $\sup_{t \in I} d(r_1(t), r_2(t)) < +\infty$. If $r$ is a geodesic ray, $r(+\infty)$ denotes its equivalence class. The boundary $\partial X$ is defined as the set of equivalence classes of geodesic rays.

A geodesic segment of $(X, d)$ is an isometry:

$$r : I \to X,$$

where $I = [0, a]$ with $a < \infty$.

Fix a base point $x$. We denote by $R(x)$ the set of geodesic rays and of geodesic segments starting at $x$ with the following convention: if $r$ is a geodesic segment defined on $[0, a]$, we set $r(t) = r(a)$ for all $t > a$. Hence we have a natural map

$$R(x) \to \overline{X} = X \cup \partial X$$

$$r \mapsto r(+\infty),$$

which is surjective. The set $R(x)$ is endowed with the topology of uniform convergence on compact subsets of $[0, +\infty)$. By the Arzela-Ascoli theorem, $R(x)$ is a compact space. Hence, endowed with the quotient topology, $X$ is compact. Notice that the topology on $X$ does not depend on the choice of $x$, see [5, 3.7 Proposition (1), p. 429].

Let $x$ be in $X$, and let $r$ be a geodesic ray. By the triangle inequality the function $t \mapsto d(x, r(t)) - t$ is decreasing and bounded below. Recall that the Busemann function associated to a geodesic ray $r$, is defined as the function

$$b_r(x) = \lim_{t \to \infty} d(x, r(t)) - d(y, r(t)).$$

Let $x$ and $y$ be in $X$, and let $v$ be in $\partial X$. Let $r$ be a geodesic ray whose extremity is $v$, namely $r(+\infty) = v$. The limit $\lim_{t \to +\infty} d(x, r(t)) - d(y, r(t))$ exists, is equal to $b_r(x) - b_r(y)$, and is independent of the choice of $r$. The horospherical distance from $x$ to $y$ relative to $v$ is defined as

$$\beta_v(x, y) = \lim_{t \to \infty} d(x, r(t)) - d(y, r(t)).$$

It satisfies for all $v \in \partial X$, and for all $x, y \in X$ that

$$\beta_v(x, y) = -\beta_v(y, x)$$

$$\beta_v(x, y) + \beta_v(y, z) = \beta_v(x, z)$$

$$\beta_v(x, y) \leq d(x, y).$$

If $\gamma$ is an isometry of $X$ we have

$$\beta_{\gamma v}(\gamma x, \gamma y) = \beta_v(x, y).$$

Recall that the Gromov product of two points $a, b \in X$ relative to $x \in X$ is

$$(a, b)_x = \frac{1}{2}(d(x, a) + d(x, b) - d(a, b)).$$

Let $v, w$ be in $\partial X$ such that $v \neq w$. If $a_n \to v \in \partial X$, $b_n \to w \in \partial X$, then

$$(v, w)_x = \lim_{n \to \infty} (a_n, b_n)_x$$

exists and does not depend on $v$ and $w$. 
Let \( r \) be a geodesic ray which represents \( v \). We have
\[
(v, y)_x = \lim_{t \to +\infty} \frac{1}{2} (d(x, r(t)) + d(x, y) - d(r(t), y)),
\]
then we obtain:
\[
(12) \quad \beta_v(x, y) = 2(v, y)_x - d(x, y).
\]
Besides, if \( q \in X \) is a point of the geodesic defined by \( v \) and \( w \), then we also have:
\[
(v, w)_x = \frac{1}{2}(\beta_v(x, q) + \beta_w(x, q)).
\]

The formula
\[
(13) \quad d_x(v, w) = \exp \left( - (v, w)_x \right)
\]
defines a distance on \( \partial X \) (we set \( d_x(v, v) = 0 \)). This is due to M. Bourdon, we refer to \([6, 2.5.1 \text{ Théorème}]\) for more details. We have the following comparison formula:
\[
d_y(v, w) = \exp \left( \frac{1}{2}(\beta_v(x, y) + \beta_w(x, y)) \right) d_x(v, w).
\]

We say that \((d_x)_{x \in X}\) is a family of visual metrics. A ball of radius \( r \) centered at \( v \in \partial X \) w.r.t. \( d_x \) is denoted by \( B(v, r) \). A ball of radius \( r \) centered at \( y \in X \) is denoted by \( B_X(y, r) \).

If \( \gamma \) is an isometry of \((X, d)\), its conformal factor at \( v \in \partial X \) is:
\[
\lim_{w \to v} \frac{d_x(\gamma v, \gamma w)}{d_x(v, w)} = \exp \left( \beta_v(x, \gamma^{-1}x) \right),
\]
(see \([6, 2.6.3 \text{ Corollaire}]\)).

If \( x \) and \( y \) are points of \( X \) and \( R \) is a positive real number, we define the shadow
\[
\mathcal{O}_R(x, y)
\]
to be the set of \( v \) in \( \partial X \) such that the geodesic ray issued from \( x \) with limit point \( v \) hits the closed ball of center \( y \) with radius \( R > 0 \).

The Sullivan shadow lemma is a very useful tool in ergodic theory of discrete groups acting on a CAT(-1) space. See for example \([22, \text{Lemma 1.3}]\) for a proof.

**Lemma 2.1.** (D. Sullivan) Let \( \Gamma \) be a discrete group of isometries of \( X \). Let \( \mu = (\mu_x)_{x \in X} \) a \( \Gamma \)-invariant conformal density of dimension \( \alpha \). Let \( x \) be in \( X \). Then for \( R \) large enough there exists \( C > 0 \) such that for all \( \gamma \in \Gamma \):
\[
\frac{1}{C} \exp \left( -\alpha d(x, \gamma x) \right) \leq \mu_x(\mathcal{O}_R(x, \gamma x)) \leq C \exp \left( -\alpha d(x, \gamma x) \right).
\]

In a \( \delta \)-hyperbolic space we have the following inequality: for all \( x, y, z, t \in X \)
\[
(14) \quad (x, z)_t \geq \min\{(x, y)_t, (y, z)_t\} - \delta,
\]
see \([5, 3.17 \text{ Remarks (4), p. 433}]\).
2.2. Bowen-Margulis-Sullivan measures and Roblin’s equidistribution theorem. We follow [22] Chapitre 1 Préluminaires, 1C. Flot géodésique].

In [25], D. Sullivan constructs measures on the tangent bundle of $X$ where $X$ is the $n$-dimensional real hyperbolic space, and proves striking results for this new class of measures. We refer to [26] for more details about these measures. We recall the definitions of these analogous measures in CAT(-1) spaces.

Let $SX$ be the set of isometries from $\mathbb{R}$ to $(X, d)$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. In other words, $SX$ is the set of geodesics of $X$ parametrized by $\mathbb{R}$. We have a projection $p$, from $SX$ to $X$, which associates to $u \in SX$ a base point “o” in $X$. We will write $\beta_v(x, u)$ with $x \in X$ and $u \in SX$ and $v \in \partial X$, instead of $\beta_v(x, p(u))$. The trivial flow on $\mathbb{R}$ induces a continuous flow $(g_t)_{t \in \mathbb{R}}$ on $SX$, called the geodesic flow. For $u \in SX$, we will denote by $g_{+\infty}(u)$ the end of the geodesic determined by $u$ for the positive time and $g_{-\infty}(u)$ the end of the geodesic for the negative time. We denote by $\partial^2 X$ the set: $\partial X \times \partial X - \{(x, x) | x \in \partial X\}$. We have an identification of $SX$ with $\partial^2 X \times \mathbb{R}$ via $u \mapsto (g_{-\infty}(u), g_{+\infty}(u), \beta_{g_{-\infty}(u)}(u, o))$.

Observe that $\Gamma$ acts on $\partial^2 X \times \mathbb{R}$ by $\gamma \cdot (v, w, s) = (\gamma v, \gamma w, s + \beta_v(o, \gamma^{-1} o))$. We have $\mathbb{R}$ acts on $\partial^2 X \times \mathbb{R}$ by $t \cdot (v, w, s) = g_t((v, w, s)) = (v, w, s + t)$. Notice these actions commute on $SX$.

Let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\alpha$. The Bowen-Margulis-Sullivan measure which is referred to as the BMS measure $m$ on $SX$ is defined as:

$$dm(u) = \frac{d\mu_x(v) d\mu_x(w) ds}{d_x(v, w)^{2\alpha}}.$$

The measure $m$ is invariant by the action of the geodesic flow, and observe also that $m$ is a $\Gamma$-invariant measure. We denote by $m_\Gamma$ quotient measure on $SX/\Gamma$. We say that $\Gamma$ admits a BMS finite measure if $m_\Gamma$ is finite. We denote by $g^\Gamma_t$ the geodesic flow on $SX/\Gamma$. We say that $g^\Gamma_t$ is mixing on $SX/\Gamma$ w.r.t. $m_\Gamma$ if for all bounded Borel subsets $A, B \subset SX/\Gamma$ we have $\lim_{t \to +\infty} m_\Gamma(A \cap g^\Gamma_t(B)) = m_\Gamma(A) m_\Gamma(B)$.

The assumption of non-arithmeticity of the spectrum of $\Gamma$ guarantees that the geodesic flow on $SX$ satisfies the mixing property w.r.t. BMS measures. We refer to [22] Proposition 7.7] for a proof of this fact in the case of negatively curved manifold. We refer to [22] Chapitre 3] for a general proof in CAT(-1) spaces.

In [22] Théorème 4.1.1, Chapitre 4, T. Roblin proves the following theorem based on the mixing property of the geodesic flow on $SX/\Gamma$ w.r.t. BMS measures:

**Theorem 2.2.** (T. Roblin) Let $\Gamma$ be a discrete group of isometries of $X$ with a non-arithmetic spectrum. Assume that $\Gamma$ admits a finite BMS measure associated to a $\Gamma$-invariant conformal density $\mu$ of dimension $\alpha = \alpha(\Gamma)$. Then for all $x, y \in X$ we have:

$$\alpha e^{-\alpha n} \|m_\Gamma\| \sum_{\{\gamma \in \Gamma | d(x, \gamma y) < n\}} D_{\gamma^{-1} x} \otimes D_{\gamma y} \to \mu_x \otimes \mu_y$$

as $n \to +\infty$ w.r.t. the weak* convergence of $C(\overline{X} \times \overline{X})^*$.
3. Functional Analytic setting

3.1. The space of operator-valued measures. We shall explain why the Banach space $\mathcal{L}(C(Z), B(\mathcal{H}))$ is naturally isomorphic to the dual of the Banach space $C(\partial X) \hat{\otimes} \mathcal{H} \hat{\otimes} \mathcal{H}$ where $\hat{\otimes}$ denotes the projective tensor product:

3.1.1. Projective tensor product. We recall now the definition of the projective tensor product of Banach spaces. Let $E$ and $F$ be Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$.

We consider the algebraic tensor product $E \otimes_{alg} F$. The projective norm of an element $g$ in $E \otimes_{alg} F$ is defined by

$$\|g\|_p := \inf \left\{ \sum_{\text{finite}} \|e_i\|_E \|f_i\|_F, \text{ such that } g = \sum_{\text{finite}} e_i \otimes f_i \right\}.$$  

The projective tensor product is defined as the completion of the algebraic tensor product for the projective norm $\| \cdot \|_p$, and it is denoted by $E \hat{\otimes} F := E \otimes_{alg} F \|\|_p$.

Recall also that we have the Banach isomorphism

$$\mathcal{L}(E, F^*) \rightarrow (E \hat{\otimes} F)^*$$

given by:

$$\mathcal{M} \mapsto (e \otimes f \mapsto \mathcal{M}(e)f).$$

See [23, p. 24] for more details.

3.1.2. Standard facts about the Banach space of bounded operators on Hilbert space. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{H}$ which is antilinear on the second variable. Define for $\xi \in \mathcal{H}$ the map $\xi^* \in \mathcal{H}^*$ which satisfies $\xi^*(\zeta) = \langle \zeta, \xi \rangle$ for $\zeta \in \mathcal{H}$. Recall the canonical isomorphism between a conjugate Hilbert space and its dual:

$$\xi \in \mathcal{H} \mapsto \xi^* \in \mathcal{H}^*$$

Define the map

$$\xi \otimes \eta^* \in \mathcal{H} \otimes \mathcal{H}^* \mapsto t_{\xi, \eta} \in B(\mathcal{H})$$

where

$$\forall \zeta \in \mathcal{H}, t_{\xi, \eta}(\zeta) = \eta^*(\zeta)\xi = \langle \zeta, \eta \rangle \xi.$$  

Let $Tr$ be the usual semi-finite trace on $B(\mathcal{H})$ and let $T$ be an operator in $B(\mathcal{H})$. Notice that for all $\xi$ and $\eta$ in $\mathcal{H}$:

$$\langle T\xi, \eta \rangle = Tr(T^*t_{\xi, \eta}).$$

It is well known that we have the isomorphism

$$\xi \otimes \eta \in \mathcal{H} \hat{\otimes} \mathcal{H} \mapsto t_{\xi, \eta} \in L^1(\mathcal{H}),$$

where $L^1(\mathcal{H})$ denotes the space of Trace class operators.

Recall that we have also an isomorphism

$$T \in B(\mathcal{H}) \mapsto TrT \in L^1(\mathcal{H})^*,$$

where $TrT(S) = Tr(TS)$ for all $S \in L^1(\mathcal{H})$. 

Recall that $T_n \rightarrow T$ w.r.t. the weak operator topology if for all $\xi$ and $\eta$ in $\mathcal{H}$ we have $\langle T_n \xi, \eta \rangle \rightarrow \langle T \xi, \eta \rangle$ as $n \rightarrow +\infty$.

3.1.3. An explicit isomorphism. Let $Z$ be a compact space. The space $\mathcal{L}(C(Z), B(\mathcal{H}))$ is a Banach space with the norm $\|M\| = \sup\{|M(f)| \|\mathcal{B}(\mathcal{H})\|, \text{with} \|f\|_\infty \leq 1\}$.

**Proposition 3.1.** The map $\mathcal{M} \in \mathcal{L}(C(Z), B(\mathcal{H})) \mapsto \tilde{\mathcal{M}} \in (C(Z) \hat{\otimes} \mathcal{H} \hat{\otimes} \overline{\mathcal{H}})^*$ is a Banach isomorphism and satisfies for all $(f, \xi, \eta) \in (C(Z) \times \mathcal{H} \times \overline{\mathcal{H}})$:

$$\tilde{\mathcal{M}}(f \otimes \xi \otimes \eta) = \text{Tr}(\mathcal{M}(f) t_{\xi, \eta}) = \langle \mathcal{M}(f) \xi, \eta \rangle.$$ 

**Proof.** Combining isomorphisms (15), (17), and (18) with the observation (16) we obtain the required isomorphism. \hfill \square

3.2. Operator-valued measures: general setting. We give in this section a general construction of “ergodic” operator-valued measures that we are interested in.

3.2.1. Quasi-regular representations. Let $(Y, \mu)$ be a measure space. Consider an action $\Gamma \curvearrowright (Y, \mu)$ such that $\mu$ is a finite $\Gamma$-quasi-invariant measure (i.e. $\mu$ and $\gamma_*\mu$ are in the same measure class). We denote by

$$\frac{d\gamma_*\mu}{d\mu}(y)$$

the Radon-Nikodym derivative of $\gamma_*\mu$ w.r.t. $\mu$ at a point $y$, with $\gamma$ in $\Gamma$. Consider $\mathcal{H} = L^2(Y, \mu)$. For all $\xi \in \mathcal{H}$ and for all $\gamma$ in $\Gamma$ define $\pi$ to be:

$$(\pi(\gamma)\xi)(y) = \left(\frac{d\gamma_*\mu}{d\mu}\right)^\frac{1}{2}(y)\xi(\gamma^{-1}y).$$

The representation $\pi : \Gamma \rightarrow U(\mathcal{H})$ is a unitary representation on the Hilbert space $\mathcal{H}$, and is called a quasi-regular representation. Observe that $\pi$ is a positive representation in the sense that $\pi$ preserves the cone of positive functions.

Notice that $\pi$ extends to a representation of the group algebra denoted $\mathbb{C}\Gamma$ by

$$\pi : \sum c_\gamma \gamma \in \mathbb{C}\Gamma \mapsto \sum c_\gamma \pi(\gamma) \in B(\mathcal{H}).$$

Define also the following matrix coefficient

$$\phi : \gamma \in \Gamma \mapsto \langle \pi(\gamma)1_Y, 1_Y \rangle \in \mathbb{R}^+, \qquad \text{where} \quad 1_Y \text{ denotes the characteristic function of the measure space } Y.$$ 

3.2.2. An ergodic operator-valued measure. Let $Z$ be a topological space and consider the space of continuous functions on $Z$ denoted by $C(Z)$. Consider a family of functional $(\ell_\gamma)_{\gamma \in \Gamma}$ in $C(Z)^*$. Assume that $\Gamma$ acts isometrically on a metric space $(X, d)$. Let $x \in X$ and $\rho > 0$. Define for all $n \geq \rho$ the annulus

$$C_n(x, \rho) := \{n - \rho \leq d(\gamma x, x) < n + \rho\}.$$ 

Assume for all $n \geq N_{x, \rho}$ that $C_n(x, \rho)$ is not empty (for some integer $N_{x, \rho}$). Define the sequence of operator-valued measures $(\mathcal{M}^n_{x, \rho})_{n \geq N_{x, \rho}}$ as:

$$\mathcal{M}^n_{x, \rho} : f \in C(Z) \mapsto \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n} \ell_\gamma(f) \frac{\pi(\gamma)}{\phi(\gamma)}$$
and observe
\[ M_{n,\rho}^n(f) = \pi \left( \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \ell_\gamma(f) \frac{\gamma}{\phi(\gamma)} \right) \in B(H). \]

3.2.3. Properties. Let \( T \) be a bounded operator on a Hilbert space \( T^* \) is its adjoint. Let \( 1_Z \) and \( 1_Y \) be the constant functions which are equal to 1 on \( Z \) and on \( Y \). The Banach space \( L^\infty(Y) \) is a Banach space with its usual norm \( \| \cdot \|_\infty \). We denote by \( L(L^\infty(Y), L^\infty(Y)) \) the Banach space of operators from \( L^\infty(Y) \) to itself with the norm \( \| \cdot \|_{L(L^\infty(Y), L^\infty(Y))} \).

We state some fundamental properties of the sequence \( (M_{n,\rho}^n)_{n \geq N_{x,\rho}} \) and the proofs are easy and left to the reader.

**Proposition 3.2.** Let \( n \) be in \( \mathbb{N} \). Assume that \( \ell_\gamma \) is positive (i.e. \( f \geq 0 \) implies \( \ell_\gamma(f) \geq 0 \)). We have:

1. For all \( f \in C(Z) \), we have
\[ (M_{x,\rho}^n(f))^* = \left( \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \ell_\gamma(f) \frac{\rho(\gamma)}{\phi(\gamma)} \right)^* = \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \ell_{-1}(f) \frac{\rho(\gamma)}{\phi(\gamma)}. \]

2. \( \|M_{x,\rho}^n\|_{L(C(Z), B(H))} \leq \|M_{x,\rho}^n(1_Z)\|_{B(H)}. \)

3. \( \|M_{x,\rho}^n(1_Z)\|_{L(L^\infty(Y), L^\infty(Y))} \leq \|M_{y,\rho}^n(1_Z)1_Y\|_\infty. \)

4. **Uniform boundedness**

4.1. **Useful functions.** Let \( \mu \) be a \( \Gamma \)-invariant conformal density of dimension \( \alpha \). We denote by \( L^\infty(\mu) \) the Banach space of essentially bounded functions w.r.t. to the measure class given by \( \mu \) with the usual norm \( \| \cdot \|_\infty \). Fix \( x \) in \( X \) and \( \rho > 0 \). Assume that there exists \( N_{x,\rho} \) such that \( |C_n(x,\rho)| > 0 \) for all \( n \geq N_{x,\rho} \). We consider the sequence of positive functions \( F_{x,\rho}^n \) defined for all \( n \geq N_{x,\rho} \) as:

\[ F_{x,\rho}^n : \gamma \in \partial X \mapsto \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \exp \left( \frac{\beta_\rho(x, \gamma x)}{\phi_\rho(x)} \right) \in \mathbb{R}^+. \]

where \( \phi_\rho \) is the Harish-Chandra function defined in the introduction (2). Observe that \( F_{x,\rho}^n \) is nothing else than
\[ F_{x,\rho}^n = M_{x,\rho}^n(1_X)1_{\partial X}. \]

Consider also the sequence of positive functions \( H_{x,\rho}^n \) defined for all \( n \geq N_{x,\rho} \) as:

\[ H_{x,\rho}^n : \gamma \in \partial X \mapsto \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} \exp (\alpha \beta_\rho(x, \gamma x)) \in \mathbb{R}^+. \]

We will prove that \( F_{x,\rho}^n \) and \( H_{x,\rho}^n \) are uniformly bounded in the \( L^\infty(\mu) \) norm. The fact that \( F_{x,\rho}^n \) is uniformly bounded is the first step in the proof of Theorem A.

The proof of uniform boundedness for \( (F_{x,\rho}^n)_{n \geq N_{x,\rho}} \) consists in two parts: we shall obtain sharp estimates of Busemann functions on the shadows, then use Ahlfors regularity condition to estimate the Harish-Chandra function \( \phi_\rho \).
The proof of the uniform boundedness of \((F_{x,\rho}^n)_{n \geq N_{x,\rho}}\) will also show that \((H_{x,\rho}^n)_{n \geq N_{x,\rho}}\) is uniformly bounded.

4.2. Estimates for Busemann functions. These techniques using the hyperbolic inequality \((14)\) extended to the whole space \(\overline{X}\) are very powerful. See for example [8] where the methods used in [3] have been influenced.

Lemma 4.1. Let \(x \in X\), let \(R > 0\) and let \(v \in \partial X\). We have for all \(y \in X\) and for all \(w \in \mathcal{O}_R(x, y)\):

\[
\min\{(w, v)_x, d(x, y)\} - R - \delta \leq (v, y)_x \leq (v, w)_x + R + \delta.
\]

Proof. Recall that \(d(x, y) - 2R \leq \beta_w(x, y) \leq d(x, y)\) for all \(w \in \mathcal{O}_R(x, y)\). Hence, by equation \((12)\), we have

\[
\begin{align*}
d(x, y) - R &\leq (w, y)_x \leq d(x, y) - R + \delta \\
&\leq \beta_w(x, y) + 2(R + \rho) + 4\delta.
\end{align*}
\]

On one hand, using first the hyperbolic property \((14)\), then the observation \((22)\) and \((v, y)_x \leq d(x, y)\) we have

\[
\begin{align*}
(v, y)_x &\geq \min\{(v, w)_x, (w, y)_x\} - \delta \\
&\geq \min\{(w, v)_x, d(x, y) - R\} - \delta \\
&\geq \min\{(w, v)_x, d(x, y)\} - R - \delta
\end{align*}
\]

On the other hand we have

\[
\begin{align*}
(v, w)_x &\geq \min\{(v, y)_x, (y, w)_x\} - \delta \\
&\geq \min\{(v, y)_x, d(x, y) - R\} - \delta \\
&\geq (v, y)_x - R - \delta
\end{align*}
\]

\(\square\)

Proposition 4.2. Fix \(x \in X\) and \(R > 0\). Let \(n \in \mathbb{N}^*\) such that \(n \geq \rho\) and let \(v \in \partial X\). There exists \(q\) in \(X\) such that for all \(y \in X\) satisfying \(n - \rho \leq d(x, y) < n + \rho\), and for all \(w \in \mathcal{O}_R(x, y)\) we have

\[
\beta_v(x, y) \leq \beta_w(x, q) + 2(R + \rho) + 4\delta.
\]

Proof. Define \(q\) as the point on the unique geodesic passing through \(v\) and \(x\) such that \(d(x, q) = n + \rho\).

Since \((v, y)_x \leq d(x, y)\), the right hand side inequality of Lemma 4.1 and the definition of \(q\) combined with the hyperbolic inequality \((14)\) imply for all \(w \in \mathcal{O}_R(x, y)\) that

\[
\begin{align*}
(v, y)_x &\leq \min\{(v, w)_x, d(x, y)\} + R + \delta \\
&\leq \min\{(v, w)_x, d(x, q)\} + R + \delta \\
&= \min\{(w, v)_x, (v, q)_x\} + R + \delta \\
&\leq (w, q)_x + R + 2\delta.
\end{align*}
\]

Since \(d(x, y) \geq n - \rho\) and \(d(x, q) = n + \rho\) it follows that:

\[
\beta_v(x, y) = 2(v, y)_x - d(x, y) \\
\leq 2(w, q)_x + 2R + 4\delta - d(x, y) \\
\leq 2(w, q)_x + 2R + 4\delta - n + \rho \\
\leq 2(w, q)_x - d(x, q) + 2\rho + 2R + 4\delta \\
= \beta_w(x, q) + 2(\rho + R) + 4\delta.
\]

□

4.3. Ahlfors regularity and Harish-Chandra functions. Let \((Z, d, m)\) be a compact metric measure space with a metric \(d\) and a measure \(m\). We denote by \(\text{Diam}(Z)\) the diameter of \(Z\). We say that the metric measure space \(Z\) is Ahlfors \(\alpha\)-regular if there exists a positive constant \(C > 0\) such that for all \(z\) in \(Z\) and \(0 < r < \text{Diam}(Z)\) we have

\[
\frac{1}{C} r^\alpha \leq m(B(z, r)) \leq Cr^\alpha.
\]

Let \(\Gamma\) be a discrete group of isometries of a CAT(-1) space \(X\) and let \(\mu\) be a \(\Gamma\)-invariant conformal density of dimension \(\alpha\). Fix a point \(x\) in \(X\) and define the function

\[
\phi_x : y \in X \mapsto \int_{\partial X} \exp \left( \frac{\alpha}{2} \beta_v(x, y) \right) d\mu_x(v).
\]

Observe that \(\phi_x\) is the restriction of \(\phi_x\) to the orbit \(\Gamma x\).

Let \(\mathcal{Y}\) be a subset of \(X\). We say that \(\phi_x\) satisfies the Harish-Chandra estimates on \(\mathcal{Y}\) if there exist two polynomials \(Q_1\) and \(Q_2\) of degree one such that for all \(y \in \mathcal{Y}\) we have

\[
Q_1(d(x, y)) \exp \left( - \frac{\alpha}{2} d(x, y) \right) \leq \phi_x(y) \leq Q_2(d(x, y)) \exp \left( - \frac{\alpha}{2} d(x, y) \right).
\]

Let \(R > 0\) and such that for all \(x\) and \(y\) in \(X\) the shadows \(O_R(x, y)\) are not empty. Pick a point \(w^y_x\) in \(O_R(x, y)\). In the context of negatively curved manifold, we can think \(w^y_x\) as the ending point of the geodesic passing through \(x\) and \(y\), oriented from \(x\) to \(y\).

**Lemma 4.3.** Let \(v \in \partial X\) and \(y \in X\). Let \(w^y_x\) be a point in \(O_R(x, y)\). Then, we have

\[
\exp \left( \frac{\alpha}{2} \beta_v(x, y) \right) \leq \exp \left( \alpha(\delta + R) \right) \frac{\exp \left( - \frac{\alpha}{2} d(x, y) \right)}{d^\alpha_x(v, w^y_x)}.
\]

and

\[
\exp \left( \frac{\alpha}{2} \beta_v(x, y) \right) \geq \exp \left( - \alpha(\delta + R) - \frac{\alpha}{2} d(x, y) \right) \left( \min \left\{ \frac{1}{d^\alpha_x(v, w^y_x)}, \exp \left( - \alpha d(x, y) \right) \right\} \right)
\]

**Proof.** We prove the first inequality. The right hand side inequality of Lemma 4.1 leads to

\[
(v, y)_x \leq (v, w^y_x)_x + R + \delta.
\]
Combining this inequality with equation \([12]\), we have
\[
\exp \left( \frac{\alpha}{2} \beta_x(x, y) \right) \leq \exp \left( \alpha (\delta + R) \right) \exp \left( \alpha (v, w_x^y) - \frac{\alpha}{2} d(x, y) \right) \\
\leq \exp \left( \alpha (\delta + R) \right) \exp \left( \alpha (v, w_x^y) \right) \exp \left( - \frac{\alpha}{2} d(x, y) \right).
\]
The definition \([13]\) of the visual metric completes the proof.
The left hand side of the inequality of Lemma \([4.3]\) gives the other inequality. \(\square\)

**Proposition 4.4.** Let \(\mu = (\mu_x)_{x \in X}\) be a \(\Gamma\)-invariant conformal density of dimension \(\alpha\). Assume that \((\Lambda_\Gamma, d_x, \mu_x)\) is Ahlfors \(\alpha\)-regular for all \(x \in X\). Then for each \(x \in X\) the function \(\varphi_x\) satisfies the Harish-Chandra estimates on \(\Gamma x\).

Moreover, if \(\Gamma\) is convex cocompact the function \(\varphi_x\) satisfies, for each \(x \in X\), the Harish-Chandra estimates on \(CH(\Lambda_\Gamma)\).

**Proof.** We first prove the right hand side inequality of \((24)\) on \(\mathcal{Y} = \Gamma x\). Let \(\gamma \in \Gamma\). Consider a point \(w_x^\gamma \in \mathcal{O}_R(x, \gamma x) \cap \Lambda_\Gamma\). Consider the ball \(\partial X\) of radius \(\exp(-d(x, \gamma x))\) w.r.t. \(d_x\) centered at \(w_x^\gamma\) denoted by
\[
B_\gamma := B\left(w_x^\gamma, \exp(-d(x, \gamma x))\right).
\]
\[
\phi_x(\gamma) = \int_{\partial X} \exp \left( \frac{\alpha}{2} \beta_v(x, \gamma x) \right) d\mu_x(v) \\
= \int_{B_\gamma} \exp \left( \frac{\alpha}{2} \beta_v(x, \gamma x) \right) d\mu_x(v) + \int_{\partial X \setminus B_\gamma} \exp \left( \frac{\alpha}{2} \beta_v(x, \gamma x) \right) d\mu_x(v).
\]

Ahlfors \(\alpha\)-regularity implies for the first term that there exists \(C > 0\) such that
\[
\int_{B_\gamma} \exp \left( \frac{\alpha}{2} \beta_v(x, \gamma x) \right) d\mu_x(v) \leq \mu_x(B_\gamma) \exp \left( \frac{\alpha}{2} d(x, \gamma x) \right) \\
\leq C \exp \left( - \frac{\alpha}{2} d(x, \gamma x) \right).
\]
The right hand side inequality of Lemma \([4.3]\) implies that
\[
\int_{\partial X \setminus B_\gamma} \exp \left( \frac{\alpha}{2} \beta_v(x, \gamma x) \right) d\mu_x(v) \leq C_{\alpha, \delta, R} \exp \left( - \frac{\alpha}{2} d(x, \gamma x) \right) \int_{\partial X \setminus B_\gamma} \frac{1}{d_x^2(v, w_x^\gamma)} d\mu_x,
\]
for some positive constant \(C_{\alpha, \delta, R} > 0\).

Write now
\[
\int_{\partial X \setminus B_\gamma} \frac{1}{d_x^2(v, w_x^\gamma)} = \int_{\partial \Gamma} d_x^2 \left( v \in \partial X \mid \frac{1}{d_x^2(v, w_x^\gamma)} > t \right) dt \\
= \int_{1/D} \exp d(x, \gamma x) \mu_x \left( \left\{ v \in \Lambda_\Gamma \mid d_x^2(v, w_x^\gamma) < \frac{1}{t} \right\} \right) dt
\]
where \(D\) denotes \(\text{Diam}(\partial X)\). A standard computation based on Ahlfors regularity (see for example \([8]\) Lemma 3.3)) provides constants \(C', C'' > 0\) such that
\[
\int_{\partial X \setminus B_\gamma} \frac{1}{d_x^2(v, w_x^\gamma)} \leq C' d(x, \gamma x) + C''.
\]
Hence, we have
\[
\int_{\partial X \setminus B_x} \exp\left(\frac{\alpha}{2} \beta_v(x, \gamma x)\right) d\mu_x(v) \leq C_{\alpha, \beta, \mu} C'' \exp\left(-\frac{\alpha}{2} d(x, \gamma x)\right) d(x, \gamma x) + C'''.
\]

We have found a polynomial of degree one such that \(\varphi_x\) satisfies the (right hand side) Harish-Chandra estimates on \(\Gamma x\). The left hand side of Harish-Chandra estimates on \(\Gamma x\) is analogous by the second inequality of Lemma 4.3.

Assume that \(\Gamma\) is convex cocompact and fix \(x \in X\). We shall estimate \(\varphi_x\) on \(CH(\Lambda \Gamma)\). Let \(y \in CH(\Lambda \Gamma)\) and pick a fundamental domain \(D_\Gamma \subset CH(\Lambda \Gamma)\) relatively compact, and consider \(D'_\Gamma\) a relatively compact neighborhood of \(x\) which contains \(D_\Gamma\). Then there exists \(\gamma \in \Gamma\) such that \(y \in \gamma D'_\Gamma\). Thanks to the cocycle identity (9) we have
\[
\varphi_x(y) = \int_{\partial X} \exp\left(\frac{\alpha}{2} \beta_v(x, \gamma x)\right) \exp\left(\frac{\alpha}{2} \beta_v(\gamma x, y)\right) d\mu_x(v).
\]

Thanks to the properties of Busemann functions (8) and (10), observe that
\[
\exp\left(-\frac{\alpha}{2} \text{Diam}(D'_\Gamma)\right) \leq \exp\left(\frac{\alpha}{2} \beta_v(\gamma x, y)\right) \leq \exp\left(\frac{\alpha}{2} \text{Diam}(D'_\Gamma)\right)
\]
and so
\[
\exp\left(-\frac{\alpha}{2} \text{Diam}(D'_\Gamma)\right) \phi_x(\gamma) \leq \varphi_x(y) \leq \phi_x(\gamma) \exp\left(\frac{\alpha}{2} \text{Diam}(D'_\Gamma)\right)
\]

Observe also that
\[
d(x, y) - \text{Diam}(D'_\Gamma) \leq d(\gamma x, x) \leq d(x, y) + \text{Diam}(D'_\Gamma)
\]
for all \(y \in X\) such that \(d(x, y) \geq \text{Diam}(D'_\Gamma)\). Since \(\varphi_x\) satisfies the Harish-Chandra estimates on \(\Gamma x\) we have the Harish-Chandra estimates of \(\varphi_x\) on \(CH(\Lambda \Gamma) \setminus B_X(x, \text{Diam}(D'_\Gamma))\). Furthermore, since \(\varphi_x\) is a positive continuous function, \(\varphi_x\) is bounded above and below on \(B_X(x, \text{Diam}(D'_\Gamma))\). Hence the Harish-Chandra estimates of \(\varphi_x\) on \(CH(\Lambda \Gamma)\) for all \(x \in X\) follow.

**Remark 4.5.** Notice that a slight modification of the first part of this proof gives a geometrical proof of the Harish-Chandra estimates of rank one semisimple Lie groups (see [1] and [13]). It would be interesting to establish Harish-Chandra estimates on \(CH(\Lambda \Gamma)\) for Harish-Chandra functions associated with geometrically finite groups with parabolic elements. Notice that we can construct geometrically finite groups with parabolic elements even though they satisfy Ahlfors regularity condition ([27] Theorem 3.1)]).

### 4.4. Uniform boundedness.

**Proposition 4.6.** Let \(\mu\) be a \(\Gamma\)-invariant conformal density of dimension \(\alpha(\Gamma)\) where \(\Gamma\) is a convex cocompact group of isometries of a CAT(-1) space \(X\) with a non-arithmetic spectrum or a non-uniform lattice acting by isometries on a Riemannian symmetric space of non-compact type of rank one denoted also by \(X\). Then for all \(x \in X\), there exists \(\rho\) and an integer \(N\) such that for all \(n \geq N\), the sequence \(F^n_{x, \rho}\) is uniformly bounded w.r.t. the \(L^\infty(\mu)\) norm.
Proof. We shall prove first that \( C_n(x, \rho) \) is not empty, at least for \( n \) large enough. For a positive real number \( s \), set \( \Gamma_s(x) := \{ \gamma \in \Gamma | d(x, \gamma x) < s \} \). Applying Theorem 2.2 to the function \( 1_{\partial X} \otimes 1_{\partial X} \) we obtain as \( n \rightarrow +\infty \), that

\[
|\Gamma_n(x)| \sim \frac{\exp(\alpha n) \| \mu_x \|^2}{\alpha \| m_{\Gamma} \|}
\]

and thus as \( n \rightarrow +\infty \)

\[
|C_n(x, \rho)| \sim \frac{\exp(\alpha n) (2 \sinh(\alpha \rho)) \| \mu_x \|^2}{\alpha \| m_{\Gamma} \|}.
\]

Hence, for all \( x \in X \) and for all \( \rho \) there exists \( N_{x, \rho} \) such that for all \( n \geq N_{x, \rho} \) we have \( |C_n(x, \rho)| > 0 \).

There are two steps:

Step 1: Assume that \( x \) is in \( CH(\Lambda_{\Gamma}) \). Then for all \( \rho > 0 \), there exists \( N_{x, \rho}' \) and for all \( n \geq N_{x, \rho}' \) the sequence \( F_{x, \rho}^n \) is uniformly bounded w.r.t. the \( L^\infty(\mu) \) norm.

Let \( \rho > 0 \) and let \( n \) be an integer such that \( n \geq N_{x, \rho} \) and let \( \nu \) be in \( \Lambda_{\Gamma} \). Proposition 4.2 provides \( q \in X \), which is indeed in \( CH(\Lambda_{\Gamma}) \), such that:

\[
F_{x, \rho}^n(\nu) = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \frac{\exp \left( \frac{n}{2} \beta_x(x, \gamma x) \right)}{\phi_x(\gamma)} \leq \exp \left( \frac{n}{2} (R + \rho + 4\delta) \right) \sup_{\gamma \in C_n(x, \rho)} \frac{1}{\phi_x(\gamma)} \sum_{\gamma \in C_n(x, \rho)} \frac{\chi_{\mathcal{O}_R(x, \gamma x)}(w)}{\phi_x(\gamma)} \exp \left( \frac{n}{2} \beta_w(x, q) \right).
\]

Define \( \gamma' \in C_n(x, \rho) \) such that \( \mu_x(\mathcal{O}_R(x, \gamma' x)) = \min \{ \mu_x(\mathcal{O}_R(x, \gamma x)) | \gamma \in C_n(x, \rho) \} \).

Then for all \( w \in \partial X \):

\[
F_{x, \rho}^n(\nu) \leq \exp \left( \frac{n}{2} (R + \rho + 4\delta) \right) \left( \sup_{\gamma \in C_n(x, \rho)} \frac{1}{\phi_x(\gamma)} \right) \sum_{\gamma \in C_n(x, \rho)} \frac{\chi_{\mathcal{O}_R(x, \gamma x)}(w)}{\phi_x(\gamma)} \exp \left( \frac{n}{2} \beta_w(x, q) \right)
\]

\[
\leq \exp \left( \frac{n}{2} (R + \rho + 4\delta) \right) \left( \sup_{\gamma \in C_n(x, \rho)} \frac{1}{\phi_x(\gamma)} \right) \sum_{\gamma \in C_n(x, \rho)} \frac{\chi_{\mathcal{O}_R(x, \gamma x)}(w)}{\phi_x(\gamma)} \exp \left( \frac{n}{2} \beta_w(x, q) \right)
\]

where the last inequality follows from the fact that there exists an integer \( m \) such that for all \( w \in \partial X \) the cardinality of \( \{ \gamma \in C_n(x, \rho) | w \in \mathcal{O}_R(x, \gamma x) \} \) is bounded by \( m \).
Combining the estimation (25) with the Shadow’s lemma (for $R$ large enough) we can find $c' > 0$ such that for all $n$ big enough we have:

$$|C_n(x, \rho)| \mu_x(\mathcal{O}_R(x, \gamma' x)) \geq c'.$$

Since the hypothesis guarantee the Ahlfors regularity of the limit set (see [6], Théorème 2.7.5 although the case of lattices is well known), Proposition 4.3 implies that there exists $C' > 0$, such that for $q \in CH(\Lambda_\Gamma)$ we have

$$\left( \sup_{\gamma \in C_n(x, \rho)} \frac{1}{\phi_x(\gamma)} \right) \varphi_x(q) \leq C'.$$

Hence for all $x \in CH(\Lambda_\Gamma)$ and for all $\rho > 0$, there exists $K > 0$ and $N_{x, \rho}'$ such that for all $n \geq N_{x, \rho}'$ we have

$$\|F_{x, \rho}^n\|_{\infty} \leq K.$$

**Step 2:** Assume that $x$ is in $X$. There exists $\rho_x'$ and an integer $N_{x, \rho}'$ such that for all $n \geq N_{x, \rho}'$ the sequence $F_{x, \rho}^n$ is uniformly bounded w.r.t. the $L^\infty(\mu)$ norm.

Fix $\rho > 0$ and let $x_0$ be the projection of $x$ in $CH(\Lambda_\Gamma)$ and set

$$\kappa := d(x, CH(\Lambda_\Gamma)) = d(x, x_0).$$

Using the relations (9), (11), (8), and (10) we obtain

$$\phi_x(\gamma) \geq \exp (\alpha \kappa) \phi_{x_0}(\gamma).$$

Observe that $C_n(x, \rho) \subset C_n(x_0, \rho + 2\kappa)$. We have:

$$F_{x, \rho}^n(v) = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \exp \left( \frac{\alpha}{2} \beta_n(x, \gamma x) \right) \phi_x(\gamma)$$

$$= \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \exp \left( \frac{\alpha}{2} \beta_n(x, x_0) \right) \exp \left( \frac{\alpha}{2} \beta_n(x_0, \gamma x_0) \right) \exp \left( \frac{\alpha}{2} \beta_n(\gamma x_0, \gamma x) \right) \phi_x(\gamma)$$

$$\leq \exp (2\alpha \kappa) \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x_0, \rho + 2\kappa)} \phi_{x_0}(\gamma)$$

$$= \exp (2\alpha \kappa) \frac{|C_n(x_0, \rho + 2\kappa)|}{|C_n(x, \rho)|} F_{x_0, \rho + 2\kappa}^n,$$

where the third inequality comes from the relations (11) and (10). Since $\frac{|C_n(x_0, \rho + 2\kappa)|}{|C_n(x, \rho)|}$ is bounded above by some constant depending on $\rho$ and $\kappa$, we apply Step 1 to $F_{x_0, \rho + 2\kappa}^n$ with $x_0$ and $\rho + 2\kappa$ to complete the proof.

**Remark 4.7.** If for all $x$ the metric measure space $(\Lambda_\Gamma, d_x, \mu_x)$ is Ahlfors regular and if $CH(\Lambda_\Gamma) = X$, then the above proposition holds for all $\rho > 0$. These conditions include the case of lattices in rank one semisimple Lie groups and fundamental groups of compact negatively curved manifolds.
Remark 4.8. For a proof of this uniform boundedness in the context of hyperbolic groups we refer to [14, Proposition 5.2].

Proposition 4.9. Let $\mu$ be $\Gamma$-invariant conformal density of dimension $\alpha(\Gamma)$ the critical exponent of the group and let $\Gamma$ be a discrete group of isometries of a CAT(-1) space $X$ with a non-arithmetic spectrum with a finite BMS measure. For all $x$ in $X$ and for all $\rho > 0$, there exists an integer $N$ such that for all $n \geq N$, we have $H^n_{x,\rho}$ is uniformly bounded w.r.t. the $L^\infty(\mu)$ norm.

The proof for $H^n_{x,\rho}$ follows the same method and is left to the reader. Notice that the proof for $H^n_{x,\rho}$ is easier because it does not deal with the Harish-Chandra estimates.

5. Analysis of matrix coefficients

5.1. Notation. Let $\Gamma$ be a discrete group of isometries of $X$ and let $\mu$ be a $\Gamma$-invariant conformal density of dimension $\alpha$. Let $(d_x)_x \in X$ be a family of visual metrics. Fix $x \in X$. Let $A$ be a subset of $\partial X$ and $a > 0$ positive real number and define $A_x(a)$ the subset of $\partial X$ as

$$A_x(a) = \{v| \inf_{w \in A} d_x(v, w) < \exp(-a)\}.$$ 

We will write $A(a)$ instead of $A_x(a)$ once $x$ has been fixed. Recall that $\cap_{a > 0}A(a) = \emptyset$.

Let $R$ a positive real number and define the cone of base $A$ to be

$$C_R(x, A) := \{y \in X| \exists v \in A \text{ satisfying } [xv] \cap B(y, R) \neq \emptyset\},$$

where $[xv]$ represents the unique geodesic passing through $x$ with the ending point $v \in \partial X$. In other words we have:

$$C_R(x, A) := \{y \in X| O_R(x, y) \cap A \neq \emptyset\}.$$ (26)

Define $b_x(y)$ the function

$$b_x(y) : v \in \partial X \mapsto \exp\left(\frac{\alpha}{2} \beta_v(x, y)\right).$$ (27)

Notice that $\varphi_x(y) = \int_{\partial X} b_x(y)(v) d\mu_x(v)$.

5.2. Sharp estimates. Assume that $\varphi_x$ satisfies Harish-Chandra estimates on $Y$.

Lemma 5.1. Let $A$ be a Borel subset of $\partial X$ and let $a > 0$. There exists a constant $C_0$ such that for all $y$ in $Y$ satisfying $O_R(x, y) \cap A(a) = \emptyset$, we have

$$\frac{\langle b_x(y), \chi_A \rangle}{\varphi_x(y)} \leq C_0 \exp(a) \frac{d(x, y)}{d(x, y)}.$$ 

Proof. Let $y \in Y$ and assume that $d(x, y) < a$. It is easy to check that

$$\frac{\langle b_x(y), \chi_A \rangle}{\varphi_x(y)} \leq \frac{\exp(a)}{d(x, y)}.$$ 

Now assume that $d(x, y) \geq a$.

If $v \in A(a)$ and $w \in O_R(x, y)$, since $O_R(x, y) \cap A(a) = \emptyset$ we have $d_x(v, w) > \exp(-a)$. 


Using the first inequality Lemma 4.3 and the above observation we have for all \( w \in O_R(x, y) \):

\[
\langle b_x(y), \chi_A \rangle \leq \exp \left( -\frac{\alpha}{2}d(x, y) \right) \int_{A(a)} \frac{1}{d^+_x(v, w)} \exp \left( \alpha(R + \delta) \right) d\mu_x(v)
\]

\[
\leq \exp \left( \alpha(R + \delta + a) \right) \|\mu\|_x \exp \left( -\frac{\alpha}{2}d(x, y) \right)
\]

\[
\leq \exp \left( \alpha(R + \delta + a) \right) \|\mu\|_x \frac{\varphi_x(y)}{Q_1(d(x, y))}
\]

where the last inequality comes from the left hand side of Harish-Chandra estimates on \( \mathcal{Y} \). Since \( Q_1 \) is a polynomial of degree one, the proof is complete. \( \square \)

Assume now that \( \varphi_x \) satisfies the left hand side of Harish-Chandra estimates on \( \mathcal{Y} = \Gamma_x \).

**Proposition 5.2.** Let \( \psi_t \in l^1(\Gamma) \) such that \( \|\psi_t\|_1 \leq 1 \), and which satisfies

\[
\lim_{t \to +\infty} \psi_t(\gamma) = 0,
\]

for all \( \gamma \in \Gamma \). Then for every Borel subset \( A \subset \partial X \) we have for all \( a > 0 \)

\[
\limsup_{t \to +\infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \pi_x(\gamma)1_{\partial X}, \chi_A \rangle}{\phi_x(\gamma)} \leq \limsup_{t \to +\infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) D_{\gamma x}(\chi_{C_R(x, A(a)))}.
\]

**Proof.** Let \( A \) be Borel subset of \( \partial X \) and let \( a \) be a positive number. Let \( t_0 \) be another positive real number. Consider the following partition of \( \Gamma \):

\[
\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3
\]

with

\[
\Gamma_1 = \{ \gamma \in \Gamma | d(x, \gamma x) \leq t_0 \}
\]

and

\[
\Gamma_2 = \{ \gamma \in \Gamma | O_R(x, \gamma x) \cap A(a) \neq \emptyset \} \cap \Gamma^c_1
\]

and

\[
\Gamma_3 = \{ \gamma \in \Gamma | O_R(x, \gamma x) \cap A(a) = \emptyset \} \cap \Gamma^c_1.
\]

Since \( \pi_x \) is positive, we have that

\[
\sum_{\Gamma_1} \psi_t(\gamma) \frac{\langle \pi_x(\gamma)1_{\partial X}, \chi_A \rangle}{\phi_x(\gamma)} \leq \sum_{\Gamma_1} \psi_t(\gamma).
\]

Observe that

\[
\gamma \in \Gamma_2 \iff D_{\gamma x}(\chi_{C_R(x, A(a)))} = 1.
\]

Thus

\[
\sum_{\gamma \in \Gamma_2} \psi_t(\gamma) \frac{\langle \pi_x(\gamma)1_{\partial X}, \chi_A \rangle}{\phi_x(\gamma)} \leq \sum_{\gamma \in \Gamma_2} \psi_t(\gamma) D_{\gamma x}(\chi_{C_R(x, A(a)))}.
\]

Observe that

\[
\langle b_x(\gamma x), \chi_A \rangle = \langle \pi_x(\gamma)1_{\partial X}, \chi_A \rangle.
\]
Since $\mathcal{Y} = \Gamma x$ we can apply Lemma 5.1 via the above observation and thus for all $t_0 > 0$:

$$\sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \pi_x(\gamma) 1_{\partial X}, x_A \rangle}{\phi_x(\gamma)} \leq \left( \sum_{\Gamma} \psi_t(\gamma) \right) C_0 \frac{\exp(a)}{t_0}.$$ 

Then, since $\|\psi_t\|_1 \leq 1$, we obtain for all $t_0 > 0$

$$\sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \pi_x(\gamma) 1_{\partial X}, x_A \rangle}{\phi_x(\gamma)} \leq C_0 \frac{\exp(a)}{t_0}.$$ 

It follows that for all $a > 0$ and for all $t > t_0$ we have

$$\sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \pi_x(\gamma) 1_{\partial X}, x_A \rangle}{\phi_x(\gamma)} \leq \sum_{\Gamma} \psi_t(\gamma) + \sum_{\Gamma} \psi_t(\gamma) D_{\gamma x}(\chi_{C_R(x,A(a))}) + C_0 \frac{\exp(a)}{t_0}.$$ 

Since $\psi_t(\gamma) \to 0$ as $t \to +\infty$, we obtain by taking the lim sup in the above inequality

$$\limsup_{t \to +\infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \pi_x(\gamma) 1_{\partial X}, x_A \rangle}{\phi_x(\gamma)} \leq \limsup_{t \to +\infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) D_{\gamma x}(\chi_{C_R(x,A(a))}) + C_0 \frac{\exp(a)}{t_0}.$$ 

This inequality holds for all $t_0 > 0$, so we take $t_0 \to +\infty$ and the proof is complete. $\square$

5.3. A consequence of Roblin’s Theorem. Let $x$ be in $X$. If $A \subset \partial X$, we denote by $\partial A$ its frontier. We need a consequence of Theorem 2.2 which counts the points of a $\Gamma$-orbit $y_0$ in $C_R(x,A)$ when $A$ is a Borel subset with $\mu_x(\partial A) = 0$. This is a standard corollary based on the regularity of the conformal densities. We recall that the topology of $\overline{X}$ is compatible with the metric topology defined on $\partial X$ by the visual metrics $(d_x)_{x \in X}$ (see [3, §1.5]). If $O \subset \overline{X}$, we denote by $\overline{O}$ its closure in $\overline{X}$.

The first thing to observe is the following:

Lemma 5.3. Let $A$ be a closed subset of $\partial X$. Then $\overline{C_R(x,A)} = C_R(x,A) \cup A$.

Proof. It is easy to check that $\overline{C_R(x,A)} \cup A \subset \overline{C_R(x,A)}$.

Now, assume that $v \in \overline{C_R(x,A)} \cap \partial X$ (otherwise there is nothing to do). We shall prove that $v \in A$. There exists a sequence of $y_n \in C_R(x,A)$ such that $y_n \to v$ w.r.t. the topology of $\overline{X}$. Since $y_n$ is in $C_R(x,A)$, there exists $v_n \in A \cap O_R(x,y_n)$ such that $(y_n,v_n)_x \geq d(x,y_n) - R$, for all integers $n$. Thus, we have

$$(v_n,v)_x \geq \min \{(v_n,y_n)_x, (y_n,v)_x\} - \delta \geq (y_n,v)_x - R - \delta.$$

where the last inequality follows from $(y_n,v)_x \leq d(x,y_n)$. Since $y_n \to v$, it follows that $(y_n,v)_x$ goes to $+\infty$, and so $v_n \to v$ w.r.t. $d_x$. Since $A$ is closed the proof is done. $\square$

Corollary 5.4. (Extracted from [22, Théorème 4.1.1, Chapitre 4]) Let $\Gamma$ be a discrete group of isometries of $X$ with a non-arithmetic spectrum. Assume that $\Gamma$ admits a finite BMS measure associated with a $\Gamma$-invariant conformal density $\mu$ of dimension $\alpha(\Gamma)$. Let $A, B$ be two Borel subsets such that $\mu_x(\partial A) = 0 = \mu_x(\partial B)$. Then for all $x$ in $X$ and for all $\rho > 0$ we have

$$\limsup_{n \to +\infty} \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} D_{\gamma^{-1}x} \otimes D_{\gamma x}(\chi_{C_R(x,A)} \otimes \chi_{C_R(x,B)}) \leq \frac{\mu_x(A) \mu_x(B)}{\|\mu\|_2^2}.$$
Proof. Let \( x \) be in \( X \) and \( \rho \) be a positive real number. We have for all \( n \) large enough:

\[
\frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma^{-1} x} \otimes D_{\gamma x} = \frac{\alpha \|m_\Gamma\| \exp(-\alpha(n + \rho))}{|C_n(x, \rho)| \alpha \|m_\Gamma\| \exp(-\alpha(n + \rho))} \sum_{\gamma \in \Gamma_{n+\rho}(x)} D_{\gamma^{-1} x} \otimes D_{\gamma x} - \frac{\alpha \|m_\Gamma\| \exp(-\alpha(n - \rho))}{|C_n(x, \rho)| \alpha \|m_\Gamma\| \exp(-\alpha(n - \rho))} \sum_{\gamma \in \Gamma_{n-\rho}(x)} D_{\gamma^{-1} x} \otimes D_{\gamma x}.
\]

The estimation (25) for annulii implies as \( n \to +\infty \)

\[
|C_n(x, \rho)| \alpha \|m_\Gamma\| \exp(-\alpha(n + \rho)) \sim 2 \sinh(\alpha \rho) \exp(-\alpha \rho) \|\mu\|_x^2
\]

and

\[
|C_n(x, \rho)| \alpha \|m_\Gamma\| \exp(-\alpha(n - \rho)) \sim 2 \sinh(\alpha \rho) \exp(\alpha \rho) \|\mu\|_x^2.
\]

Therefore Theorem 2.2 implies

\[
(28) \quad \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma^{-1} x} \otimes D_{\gamma x} \to \frac{1}{\|\mu_x\|^2} \mu_x \otimes \mu_x,
\]

w.r.t. the weak* topology of \( C(\overline{X} \times \overline{X})^* \).

Consider a Borel subset \( A \) of \( \partial X \) such that \( \mu_x(\partial A) = 0 \). We have \( \mu_x(\overline{A}) = \mu_x(A) \).

Thus, by Lemma 5.3 we obtain

\[
\mu_x(\overline{C_R(x, A)}) = \mu_x(A).
\]

Let \( \epsilon > 0 \). Since \( \mu_x \) is a regular measure there exists an open subset \( O_A \) of \( \overline{X} \) such that

\[
(29) \quad \overline{C_R(x, A)} \subset O_A \text{ and } \mu_x(O_A) \leq \mu_x(A) + \epsilon.
\]

The subset \( \overline{C_R(x, A)} \) is a compact subset of \( \overline{X} \). By Urysohn’s lemma, we can find a compactly supported function \( f_{O_A} \) such that

\[
\chi_{\overline{C_R(x,A)}} \leq f_{O_A} \leq \chi_{O_A}.
\]

Let \( B \) be another Borel subset such that \( \mu_x(\partial B) = 0 \). Let \( f_{O_B} \) be continuous function provided by the above construction. Notice that for all \( n \) we have:

\[
\sum_{\gamma \in C_n(x, \rho)} D_{\gamma x} \otimes D_{\gamma^{-1} x} (\chi_{C_R(x,A)} \otimes \chi_{C_R(x,B)}) \leq \sum_{\gamma \in C_n(x, \rho)} D_{\gamma x} \otimes D_{\gamma^{-1} x} (f_{O_A} \otimes f_{O_B}).
\]
Let Proposition 5.5.

We assume that \( \varphi \) for all \( n \)

Proof. \( M \) of operator-valued measures (29). The purpose of this section is to use Corollary 5.4 for computing the limit of the sequence \( N \) and let

\[ s \in \mathbb{N} \]

Lemma 5.6.

The consequence of Roblin’s theorem (28) implies:

\[
\limsup_{n \to \infty} \frac{\|\mu_x\|^2}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \otimes D_{\gamma^{-1}}(\chi_{C_R(x, A)} \otimes \chi_{C_R(x, B)}) \\
\leq \limsup_{n \to \infty} \frac{\|\mu_x\|^2}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \otimes D_{\gamma^{-1}}(f_{O_A} \otimes f_{O_B}) \\
= \lim_{n \to \infty} \frac{\|\mu_x\|^2}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \otimes D_{\gamma^{-1}}(f_{O_A} \otimes f_{O_B}) \\
= \int_{\partial X \times \partial X} (f_{O_A} \otimes f_{O_B}) d\mu_x \otimes d\mu_x \\
\leq \mu_x(A) \mu_x(B) + e(\mu_x(A) + \mu_x(B)) + \epsilon^2,
\]

where the last inequality follows from (29). The above inequality holds for all \( \epsilon > 0 \), and so the proof is done. \( \square \)

5.4. Application of Roblin’s equidistribution Theorem. Fix \( x \) in \( X \) and \( \rho > 0 \), and let \( N_{x, \rho} \) be an integer such that for all \( n \geq N_{x, \rho} \) the sequence \( M_{x, \rho}^n \) is well defined. The purpose of this section is to use Corollary 5.4 for computing the limit of the sequence of operator-valued measures \( (M_{x, \rho}^n)_{n \geq N_{x, \rho}} \).

We assume that \( \varphi_x \) satisfies the left hand side of Harish-Chandra estimates on \( \Gamma x \).

**Proposition 5.5.** Let \( U, A, B \subset X \) be Borel subsets such that \( \mu_x(\partial U) = \mu_x(\partial A) = \mu_x(\partial B) = 0 \), let \( \tilde{U} \) be a borel subset of \( \overline{X} \) such that \( \tilde{U} \cap \partial X = U \). Then we have:

\[
\lim_{n \to +\infty} \langle M_{x, \rho}^n(\chi_{\tilde{U}}) \chi_A, \chi_B \rangle = \frac{\mu_x(U \cap B) \mu_x(A)}{\|\mu_x\|^2}.
\]

We need some lemmas to prepare the proof of this proposition.

**Lemma 5.6.** Let \( U \) be a Borel subset of \( \partial X \) with \( \mu_x(\partial U) = 0 \) and let \( \tilde{U} \) be a Borel subset of \( \overline{X} \) such that \( \tilde{U} \cap \partial X = U \). Let \( B \) be a Borel subset of \( \partial X \) such that \( \mu_x(\partial B) = 0 \), satisfying \( U \cap B(b) = \emptyset \), for some \( b > 0 \). Then we have

\[
\limsup_{n \to +\infty} \langle M_{x, \rho}^n(\chi_{\tilde{U}}) 1_{\partial X}, \chi_B \rangle = 0.
\]

**Proof.** For all \( n \geq N_{x, \rho} \) we have:

\[
\langle M_{x, \rho}^n(\chi_{\tilde{U}}) 1_{\partial X}, \chi_B \rangle = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma}(\chi_{\tilde{U}}) \frac{\langle \pi_x(\gamma) 1_{\partial X}, \chi_B \rangle}{\phi_x(\gamma)} \\
\leq \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma}(\chi_{\tilde{U}}) \frac{\langle \pi_x(\gamma) 1_{\partial X}, \chi_{B(b)} \rangle}{\phi_x(\gamma)} \\
= \sum_{\gamma \in \Gamma} \psi_n(\gamma) \frac{\langle \pi_x(\gamma) 1_{\partial X}, \chi_{B(b)} \rangle}{\phi_x(\gamma)}
\]
where the inequality follows from the fact that $\pi_x$ is positive, and where
\[
\psi_n(\gamma) := \frac{1}{|C_n(x, \rho)|} \chi_{C_n(x, \rho)} D_{\gamma} \chi_{\hat{U}}.
\]

Proposition 5.2 implies that
\[
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\hat{U}}), 1_{\partial X}, \chi_B \rangle \leq \limsup_{n \to +\infty} \sum_{\gamma \in \Gamma} \psi_n(\gamma) D_{\gamma} \chi_{C_n(x, B(\rho))} D_{\gamma} \chi_{C_n(x, B(\rho))}.
\]

Note the general fact $\partial (A \cap B) \subset \partial A \cup \partial B$. Corollary 5.4 implies that
\[
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\hat{U}}), 1_{\partial X}, \chi_B \rangle \leq \frac{\mu_x(U \cap B(b))}{\|\mu_x\|}.
\]

By hypothesis $U \cap B(b) = \emptyset$ thus we have
\[
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\hat{U}}), 1_{\partial X}, \chi_B \rangle = 0.
\]

Lemma 5.7. Let $U$ be a Borel subset of $\partial X$ and let $\hat{U}$ be a Borel subset of $X$ such that $\hat{U} \cap \partial X = U$ and let $A$ be a Borel subset of $\partial X$. We have
\[
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\hat{U}}), \chi_A, 1_{\partial X} \rangle \leq \limsup_{n \to +\infty} \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \chi_{C_n(x, A(a))}. 
\]

Proof. We have for all $n \geq N_{x,\rho}$:
\[
\langle M_{x,\rho}^n(\chi_{\hat{U}}), \chi_A, 1_{\partial X} \rangle = \langle \chi_A, M_{x,\rho}^n(\chi_{\hat{U}}) \rangle = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \chi_{\hat{U}} \langle \pi_x(\gamma) 1_{\partial X}, \chi_A \rangle \phi_x(\gamma). 
\]

with
\[
\psi_n(\gamma) = \frac{1}{|C_n(x, \rho)|} D_{\gamma} \chi_{\hat{U}}.
\]

Applying Proposition 5.2 to $\psi_n$ defined above we obtain that:
\[
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\hat{U}}), \chi_A, 1_{\partial X} \rangle \leq \limsup_{n \to +\infty} \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma} \chi_{\hat{U}} \chi_{C_n(x, A(a))}. 
\]
Proof. Let $a > 0$ and $b > 0$, and consider $A(a)$ and $B(b)$ such that $\mu_x(\partial B(b)) = 0 = \mu_x(\partial A(a))$. Let $B(b)^c = \partial X \setminus B(b)$. Set $U_1 = U \cap B(b)$ and $U_2 = U \cap B(b)^c$. Observe that $U_1 \cap B(b)^c = \emptyset = U_2 \cap B(b)$. Extend $U_1$ and $U_2$ to $\overline{X}$ by $\overline{U_1}$ and $\overline{U_2}$ such that $\overline{U} = \overline{U_1} \sqcup \overline{U_2}$. We have:

$$
\langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle = \langle M_{x,\rho}^n(\chi_{\overline{U}_1}) \chi_A, \chi_B \rangle + \langle M_{x,\rho}^n(\chi_{\overline{U}_2}) \chi_A, \chi_B \rangle
\leq \langle M_{x,\rho}^n(\chi_{\overline{U}_2}) \chi_A, 1_{\partial X} \rangle + \langle M_{x,\rho}^n(\chi_{\overline{U}_2}) \chi_B, \chi_B \rangle.
$$

Applying Lemma 5.6 to the second term and Lemma 5.7 to the first term of the right hand side above inequality, we obtain:

$$
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle \leq \limsup_{n \to +\infty} \frac{1}{|C_n(x,\rho)|} \sum_{\gamma \in C_n(x,\rho)} D_{\gamma^{-1}}(\chi_{C_n(x,A)}) D_{\gamma}(\chi_{C_n(x,A)}) \mu_x(\partial X \setminus B(b)) \mu_x(\partial X \setminus B(b)^c).
$$

Since $\mu_x(\partial U_1) = 0 = \mu_x(\partial A(a))$, Robin’s corollary 5.4 leads to

$$
\limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle \leq \mu_x(U \cap B(b)) \mu_x(A(a)).
$$

Because the above inequality holds for all $a, b > 0$ but at most countably many values of $a$ and $b$, we obtain the required inequality.

\[ \square \]

Proof of Proposition 6.5. By Lemma 5.8 it is sufficient to prove that

$$
\liminf_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle = \frac{\mu_x(U \cap B) \mu_x(A)}{\|\mu_x\|^2}.
$$

If $W$ is a Borel subset of $\partial X$ (or $\overline{X}$), we set $W^0 = W$ and $W^1 = \partial X \setminus W$ (or $W^1 = \overline{X} \setminus W$). We have

$$
1 = \langle M_{x,\rho}^n(1_{\overline{U}}) 1_{\partial X}, 1_{\partial X} \rangle = \langle M_{x,\rho}^n(\chi_{\overline{U}_0} + \chi_{\overline{U}_1}) \chi_A, \chi_B \rangle = \sum_{i,j,k} \langle M_{x,\rho}^n(\chi_{\overline{U}_i}) \chi_A, \chi_B \rangle = \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle + \sum_{i,j,k \neq (0,0,0)} \langle M_{x,\rho}^n(\chi_{\overline{U}_i}) \chi_A, \chi_B \rangle.
$$

Then

$$
1 \leq \liminf_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle + \sum_{i,j,k \neq (0,0,0)} \limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}_i}) \chi_A, \chi_B \rangle
\leq \limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}}) \chi_A, \chi_B \rangle + \sum_{i,j,k \neq (0,0,0)} \limsup_{n \to +\infty} \langle M_{x,\rho}^n(\chi_{\overline{U}_i}) \chi_A, \chi_B \rangle
\leq \frac{1}{\|\mu_x\|^2} \sum_{i,j,k} \mu_x(U \cap B^k) \mu_x(A^i)
= 1,
$$
where the last inequality comes from Lemma 5.8. Hence the inequalities of the above computation are equalities, so

$$\liminf_{n \to +\infty} \langle M_n x, \rho(x^U \chi_A, \chi_B) \rangle = \frac{\mu_x(U \cap B)\mu_x(A)}{\|\mu_x\|^2} = \limsup_{n \to +\infty} \langle M_n x, \rho(x^U \chi_A, \chi_B) \rangle$$

and the proof is done. □

6. Conclusion

6.1. Standard facts about Borel subsets of measure zero frontier. We give a proof of a standard fact of measure theory:

**Lemma 6.1.** Assume that \((Z,d,\mu)\) is a metric measure space. Then the \(\sigma\)-algebra generated by Borel subset with measure zero frontier generates the Borel \(\sigma\)-algebra.

**Proof.** Let \(z\) be in \(Z\). Consider the concentric balls \(B(z,r)\) for \(r > 0\), as well as the spheres of radius \(r\) centered at \(z\) denoted by \(S(z,r)\). Then at most countably many of the spheres have non-zero measure. Since \(\partial B(z,r) \subset S(z,r)\), at most countably many of the balls \(B(z,r)\) have non-zero measure frontier. Take \(r\) and \(B(z,r)\) such that \(\mu(S(z,r)) > 0\). There exists a sequence of positive real numbers \(r_n\) with \(r_n \to r\) such that \(B(z,r) = \bigcup_{n \in \mathbb{N}} B(z,r_n)\) where \(\mu(\partial B(z,r_n)) = 0\). Let \(O\) be an open subset of \(Z\). Then the Whitney covering lemma (see for example [25, Chapter 1, §3, Lemma 2]) asserting that \(O\) can be written as a countable union of balls completes the proof. □

Let \(\chi_A\) be the characteristic function of a Borel subset \(A\) of \(\partial X\). We state another useful lemma (see [3, Appendix B, Lemma B.2 (1)] for a proof):

**Lemma 6.2.** Assume that \((Z,d,\mu)\) is a metric measure space such that \(\mu\) is regular. Then the closure of the subspace spanned by the characteristic functions of Borel subset having zero measure frontier is

\[
\text{Span}\{\chi_A | \mu(\partial A) = 0\}^{L^2} = L^2(Z,\mu).
\]

6.2. Proofs.

**Proof of Theorem A.** Let \(\mu\) be a \(\Gamma\)-invariant conformal density of dimension \(\alpha(\Gamma)\), where \(\Gamma\) is convex cocompact with a non-arithmetic spectrum or a lattice in a rank one semisimple Lie group. Since for all \(x \in X\), the metric measure space \((\Lambda \Gamma, dx, \mu_X)\) is Ahlfors \(\alpha\)-regular Proposition 4.4 ensures that the Harish-Chandra estimates hold on \(\Gamma x\). Hence Proposition 4.6 and 5.5 are available. The sequence \(M^n_{x,\rho}\) is defined for \(n \geq N_{x,\rho}\) for some integer \(N_{x,\rho}\). There are two steps.

**Step 1:** \((M^n_{x,\rho})_{n \geq N_{x,\rho}}\) is uniformly bounded. First of all, observe that \(M^n_{x,\rho}(1_X)\) is self-adjoint (see Proposition 4.2 (1)). Note that \(M^n_{x,\rho}(1_X)\) preserves \(L^\infty(\partial X, \mu_x)\), and by duality it preserves also \(L^1(\partial X, \mu_x)\).

Combining Proposition 4.4 with the fact that \(M^n_{x,\rho}(1_X)1_{\partial X} = F^n_{x,\rho}\), we have that the sequence \((M^n_{x,\rho}(1_X))_{n \geq N_{x,\rho}}\), with \(M^n_{x,\rho}(1_X)\) viewed as operators from \(L^\infty(\partial X, \mu_x)\) to \(L^\infty(\partial X, \mu)\), is uniformly bounded. Riesz-Thorin interpolation theorem implies the
sequence \((M^n_{x,p}(1_{\overline{\mathcal{X}}}))_{n \geq N_{x,p}}\) with \(M^n_{x,p}(1_{\overline{\mathcal{X}}})\) viewed as operators in \(\mathcal{B}(L^2(\partial X, \mu_x))\), is uniformly bounded. Then Proposition 3.2 (2) completes Step 1.

Step 2: computation of the limit of \((M^n_{x,p})_{n \geq N_{x,p}}\). By the Banach-Alaoglu theorem, Step 1 implies that \((M^n_{x,p})_{n \geq N_{x,p}}\) has accumulation points. Let \(M^\infty_x\) be an accumulation point of \((M^n_{x,p})_{n \geq N_{x,p}}\) w.r.t. the weak* topology of \(\mathcal{L}(C(\overline{\mathcal{X}}), \mathcal{B}(L^2(\partial X, \mu_x)))\). If \(U\) is a Borel subset of \(\partial X\) such that \(\mu_x(\partial U) = 0\), \(\hat{U}\) denotes an extension of \(U\) to \(\overline{\mathcal{X}}\). It follows from Proposition 5.3 and from the definition (3) of \(M_x\) that for all Borel subsets \(U, A, B \subset \partial X\) satisfying \(\mu_x(\partial U) = \mu_x(\partial A) = \mu_x(\partial B) = 0\) we have that

\[
\langle M^\infty_x(\chi_U)\chi_A, \chi_B \rangle = \frac{\mu_x(U \cap B)\mu_x(A)}{||\mu_x||^2} = \langle M_x(\chi_U)\chi_A, \chi_B \rangle.
\]

Lemma 6.1 combined with Carathéodory’s extension theorem implies that for all \(f \in C(\overline{\mathcal{X}})\) and for all Borel subsets \(A, B \subset \partial X\) satisfying \(\mu_x(\partial A) = \mu_x(\partial B) = 0\) we have

\[
\langle M^\infty_x(f)\chi_A, \chi_B \rangle = \langle M_x(f)\chi_A, \chi_B \rangle.
\]

Lemma 6.2 combined with the above equality imply that the operators \(M^\infty_x\) and \(M_x\) regarded as functionals of \((C(\overline{\mathcal{X}}) \otimes L^2(\partial X, \mu_x) \otimes L^2(\partial X, \mu_x))^*\) (see Proposition 3.1) are equal on a dense subset of \(C(\overline{\mathcal{X}}) \otimes L^2(\partial X, \mu_x) \otimes L^2(\partial X, \mu_x)\). We deduce that \(M_x\) is the unique accumulation point of the sequence \((M^n_{x,p})_{n \geq N_{x,p}}\).

Proof of Corollary B. Apply the definition of weak* convergence to \(1 \otimes \xi \otimes \eta\) for all \(\xi, \eta \in L^2(\partial X, \mu_x)\), and observe that \(M_x(1_{\overline{\mathcal{X}}})\) is the orthogonal projection onto the space of constant functions.

Proof of Corollary C. Since \((\pi_x)_{x \in \mathcal{X}}\) are unitarily equivalent, it suffices to prove irreducibility for some \(\pi_x\) with \(x \in X\). Theorem A shows that the vector \(1_{\partial X}\) is cyclic for the representation \(\pi_x\). Moreover, Corollary B shows that the orthogonal projection onto the space of constant functions is in the von Neumann algebra associated with \(\pi_x\). Then, the argument of [14] Lemma 6.1] completes the proof.

Remark 6.3. The hypothesis: \(\Gamma\) is convex cocompact or a lattice in a rank one semisimple Lie group guarantees the Ahlfors regularity of the limit set, that implies the Harish-Chandra estimates of \(\varphi_x\) for each \(x \in X\) on \(CH(\Lambda_{\Gamma})\) and on \(\Gamma x\). In other words, the proof of irreducibility of boundary representations for a geometrically finite group with a non-arithmetic spectrum is reduced, by this approach, to the Harish-Chandra estimates of \(\varphi_x\) for each \(x \in X\) on \(CH(\Lambda_{\Gamma})\) and on the orbit \(\Gamma x\). And this approach applied to some geometrically finite groups (see [21]) which are neither convex cocompact nor lattices.

7. Some remarks about equidistribution results

7.1. Dirac-Weierstrass family. Let \(\Gamma\) be a discrete group of isometries of \(X\). Consider \((d_x)_{x \in \mathcal{X}}\) a visual metric on \(\partial X\), and let \(\mu\) be a \(\Gamma\)-invariant conformal density of dimension \(\alpha\). We follow [17] Chapter 2, §2.1, p 46], and adapt the definition of a Dirac-Weierstrass family to the density \(\mu\):
Definition 7.1. A Dirac-Weierstrass family \((K(y,\cdot))_{y \in X}\) w.r.t. \(\mu_x\), is a continuous map \(K : (y, v) \in X \times \partial X \mapsto K(y, v) \in \mathbb{R}\) satisfying

1. \(K(y, v) \geq 0\) for all \(v \in \partial X\) and \(y \in X\),
2. \(\int_{\partial X} K(y, v) d\mu_x(v) = 1\) for all \(y \in X\),
3. for all \(v_0 \in \partial X\) and for all \(\varepsilon > 0\) we have:
   \[
   \int_{\partial X \setminus B(v_0, \varepsilon)} K(y, v) d\mu_x(v) \to 0 \quad \text{as} \quad y \to v_0.
   \]

A Dirac-Weierstrass family yields an integral operator \(K:\)

\[
K : f \in L^1(\partial X, \mu_x) \mapsto Kf \in C(X)
\]

defined as:

\[
Kf : y \in X \mapsto \int_{\partial X} f(v) K(y, v) d\mu_x(v) \in \mathbb{C}.
\]

7.2. Continuity. Let \(f\) be a function on \(\partial X\). We define the function \(Kf\) on \(X\) as the following:

\[
Kf : y \in X \mapsto \int_{\partial X} f(v) K(y, v) d\mu_x(v) \in \mathbb{C}.
\]

Thus, \(K\) is an operator which assigns a function defined on \(X\) to a function defined on \(\partial X\).

Proposition 7.2. If \(f\) is a continuous functions on \(\partial X\), the function \(K(f)\) is a continuous function on \(X\).

Proof. Since \(K\) is a continuous function, by Lebesgue’s Theorem we have that \(Kf\) is continuous on \(X\). Let \(v_0\) be in \(\partial X\).

Let \(\epsilon > 0\). Since \(f\) is continuous, there exists \(r > 0\) such that

\[
|f(v_0) - f(v)| < \frac{\epsilon}{2},
\]

whenever \(v \in B(v_0, r)\). Besides, by (3) in Definition 7.1 there exists a neighborhood \(V\) of \(v_0\) such that for all \(x \in V\) we have:

\[
\int_{\partial X \setminus B(v_0, r)} K(y, v) d\mu_x(v) \leq \frac{\epsilon}{4\|f\|_{\infty}}.
\]

We have for all \(x \in V\):

\[
|Kf(v_0) - Kf(y)| = |f(v_0) - K(y)| = \left| \int_{\partial X} f(v_0) - f(v) K(y, v) d\mu_x(v) \right|
\]

\[
\leq \int_{\partial X} |f(v_0) - f(v)| K(y, v) d\mu_x(v)
\]

\[
= \int_{B(v_0, r)} |f(v_0) - f(v)| K(y, v) d\mu_x(v) + \int_{\partial X \setminus B(v_0, r)} |f(v_0) - f(v)| K(y, v) d\mu_x(v)
\]

\[
\leq \frac{\epsilon}{2} + 2\|f\|_{\infty} \int_{\partial X \setminus B(v_0, r)} K(y, v) d\mu_x(v)
\]

\[
\leq \epsilon.
\]
Hence, $\bar{K}f$ is a continuous function on $\bar{X}$. □

7.3. An example of Dirac-Weierstrass family. Let $R > 0$, and consider for each $y \in X$ a point $w_x^y \in \mathcal{O}_R(x, y)$. We start by a lemma:

**Lemma 7.3.** Let $v_0$ be in $\partial X$. Then $d_x(v_0, w_x^y) \to 0$ as $y \to v_0$.

**Proof.** Let $y_n$ be a sequence of points of $X$ such that $y_n \to v_0$. Apply the right hand side inequality of Lemma 4.1 to get

$$(v_0, w_x^{y_n})_x \geq (v_0, y_n)_x - R - \delta.$$ 

Since $y_n \to v_0$, we have $(v_0, y_n)_x$ goes to infinity, and thus $d_x(v_0, w_x^y) \to 0$ as $y \to v_0$. □

**Proposition 7.4.** Assume that there exists a polynomial $Q_1$ (at least of degree 1), such that for all $y \in X$ we have

$$Q_1(d(x, y)) \exp\left(-\frac{\alpha}{2}d(x, y)\right) \leq P_01_{\partial X}(y).$$

Then

$$\left(\frac{P(y, \cdot)^{1/2}}{P_01_{\partial X}(y)}\right)_{y \in X}$$

is a Dirac-Weierstrass family.

**Proof.** Let $B(v_0, \varepsilon)$ the ball of radius $\varepsilon$ at $v_0$ in $\partial X$ w.r.t. $d_x$. Let $\varepsilon > 0$. Since $Q_1$ is a polynomial at least of degree one, there exists $R' > 0$ such that for far all $y$ satisfying $d(x, y) > R'$ we have:

$$\frac{C_{\varepsilon, \alpha, \delta}||\mu_x||}{Q_1(d(x, y))} < \varepsilon$$

where $C_{\varepsilon, \alpha, \delta} = 2^\alpha \exp(\alpha(\delta+R))$ is a positive constant.

Lemma 7.3 yields a neighborhood $V$ of $v_0$ such that $d_x(v_0, w_x^y) \leq \frac{\varepsilon}{2}$ for all $y \in V$. We have for all $v$ in $\partial X \setminus B(v_0, \varepsilon)$:

$$d_x(v, w_x^y) \geq d_x(v, v_0) - d_x(v_0, w_x^y)$$

$$\geq \varepsilon - d_x(v_0, w_x^y)$$

$$\geq \frac{\varepsilon}{2}.$$
We set $V_{R'} = V \cap X \setminus B_X(x, R')$. Combining Lemma 4.3 with the above inequality we obtain for all $y \in V_{R'}$:

$$\int_{\partial X \setminus B(v_0, \varepsilon)} \frac{P^1_1(y, v)}{P_01_{\partial X}(v)} \, d\mu_x(v) \leq \int_{\partial X \setminus B(v_0, \varepsilon)} \frac{\exp \left( \alpha(\delta + R) \right) \exp \left( - \frac{\alpha}{2} d(x, y) \right)}{d_x^\alpha(v, w_x^y)(P_01_{\partial X}(y))} \, d\mu_x(v)$$

$$\leq C_{\varepsilon, \alpha, \delta} \int_{\partial X \setminus B(v_0, \varepsilon)} \frac{\exp \left( - \frac{\alpha}{2} d(x, y) \right)}{Q_1(d(x, y))} \, d\mu_x(v)$$

$$= C_{\varepsilon, \alpha, \delta} \int_{\partial X \setminus B(v_0, \varepsilon)} \frac{1}{Q_1(d(x, y))} \, d\mu_x(v)$$

$$\leq \frac{C_{\varepsilon, \alpha, \delta} \mu_x(\partial X)}{Q_1(d(x, y))} \leq \epsilon.$$  

It follows that

$$\int_{\partial X \setminus B(v_0, \varepsilon)} \frac{P^1_1(y, v)}{P_01_{\partial X}(v)} \, d\mu_x(v) \to 0 \text{ as } y \to v_0.$$

Besides, the same method proves the following proposition:

**Proposition 7.5.** The Poisson kernel $(P(y, \cdot))_{y \in X}$ is a Dirac-Weierstrass family.

7.4. Equidistribution theorems extended to $(L^1)^*$. Theorem 2.2 of T. Roblin has for immediate consequence:

**Theorem 7.6.** (T. Roblin) Let $\Gamma$ be a discrete subgroup of isometries of $X$ with a non-arithmetic spectrum. Assume that $\Gamma$ admits a finite BMS measure associated to a $\Gamma$-invariant conformal density $\mu$ of dimension $\alpha = \alpha(\Gamma)$. Then for each $x \in X$ and for all $\rho > 0$ we have as $n$ goes to infinity:

$$\frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma^{-1}x} \to \frac{\mu_x}{\| \mu_x \|}$$

w.r.t. the weak* topology of $C(X)^*$.

Thanks to the handling ability of CAT(-1) spaces the Poisson kernel, and the square root of Poisson kernel can be defined. Moreover, they enjoy some properties of the standard Poisson kernel in the unit disc. Thus, we can prove Proposition [1]

**Proof of Proposition D.** Let $x$ in $X$ and $\rho > 0$, and consider $N_{x, \rho}$ such that $n \geq N_{x, \rho}$ implies $|C_n(x, \rho)| > 0$. We give a proof for the densities $(\mu_x)_{x \in X}$. For all $n \geq N_{x, \rho}$, we denote by $\lambda_{n, x, \rho}$ the following measure

$$\lambda_{n, x, \rho} = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \mu_{\gamma x}.$$

**Step 1:** the sequence of measures $(\lambda_{n, x, \rho})_{n \geq N_{x, \rho}}$ is uniformly bounded.
Since the dual space of $L^1(\partial X, \mu_x)$ is $L^\infty(\partial X, \mu)$ we have for $n \geq N_{x,\rho}$:
\[
\|H^n_{x,\rho}\|_\infty = \sup_{\|f\|_1 \leq 1} \left\{ \left| \int_{\partial X} H^n_{x,\rho}(v) f(v) d\mu_x(v) \right| \right\} \\
= \sup_{\|f\|_1 \leq 1} \left\{ \left| \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \mu_{\gamma x}(f) \right| \right\} \\
= \|\lambda^n_{x,\rho}\|_{(L^1)'}.
\]

Hence Proposition 7.9 completes Step 1.

Step 2: computation of the limit of $(\lambda^n_{x,\rho})_{n \geq N_{x,\rho}}$.

By Banach-Alaoglu’s theorem, $(\lambda^n_{x,\rho})_{n \geq N_{x,\rho}}$ has accumulation points. Denote by $\lambda^\infty_{x,\rho}$ such accumulation point. Let $f \in C(\partial X)$, and as in (30) in Subsection 7.2 define $\overline{P}f$ as a continuous function on $X$. We have for all $n \geq N_{x,\rho}$ that:
\[
\lambda^n_{x,\rho}(f) = \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \mu_{\gamma x}(f) \\
= \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} P(f)(\gamma x) \\
= \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} D_{\gamma x}(\overline{P}(f)).
\]

Applying Roblin’s theorem [7.6] by taking the limit in the above inequality, we obtain for all $f \in C(\partial X)$
\[
\lambda^\infty_{x}(f) = \mu_x(\overline{P}(f)) = \mu_x(f).
\]

Since $C(\partial X)$ is dense $L^1(\partial X, \mu_x)$ w.r.t. the $L^1$ norm, we deduce that $(\lambda^n_{x,\rho})_{n \geq N_{x,\rho}}$ has only one accumulation point which is $\mu_x$, and the proof is done.

The proof concerning $(\nu_x)_{x \in X}$ follows the same method, and uses $\varphi_x = P_0$ in order to have available Proposition 7.4 for $\Gamma$ a convex cocompact group with a non-arithmetic spectrum or a lattice in a non-compact connected semisimple Lie group of rank one.

Remark 7.7. We may ask if an analogous theorem of Theorem 2.2 for $\mu_x$ instead of the Dirac mass, namely:
\[
\frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \mu_x \otimes \mu_{\gamma^{-1} x} \rightarrow \mu_x \otimes \mu_x
\]

w.r.t. the weak* convergence of $L^1(\partial X \times \partial X, \mu_x \otimes \mu_x)^*$ for some $\rho$ holds? The answer is negative because it would imply that the sequence function
\[
G_n : (v, w) \mapsto \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \exp(\alpha\beta_v(x, \gamma x)) \exp(\alpha\beta_w(x, \gamma^{-1} x))
\]

is uniformly bounded w.r.t. the $L^\infty(\mu)$ norm (by duality combined with Banach-Steinhaus theorem). It is easy to see that this is impossible by evaluating $G_n$ at $(v, w) \in O_R(x, \gamma x) \times O_R(x, \gamma^{-1} x)$.
$O_B(x, \gamma^{-1}x)$ for some $\gamma \in C_n(x, \rho)$. The same answer holds also for the same question about $(\nu_x)_{x \in X}$ by considering the sequence of functions
\[
(v, w) \mapsto \frac{1}{|C_n(x, \rho)|} \sum_{\gamma \in C_n(x, \rho)} \exp(\frac{\alpha}{2} \beta_v (x, \gamma x)) \exp(\frac{\alpha}{2} \beta_w (x, \gamma^{-1}x)) \cdot \phi_2^2(\gamma).\
\]

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