Branes and BPS Configurations of Non-Commutative/Commutative Gauge Theories

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Abstract

We study BPS Dirac monopole in $U(1)$ gauge theory on non-commutative space-time. The corresponding brane configuration is obtained in the equivalent ordinary gauge theory through the map proposed by Seiberg and Witten. This configuration coincides exactly with a tilted D-string as predicted. This study provides an interesting check of the equivalence of the non-commutative and ordinary gauge theories.

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1 Introduction

In these five years, string theory has provided various interesting tools for analyzing field theories [1, 2, 3]. This owes to the fact that effective theories on D-branes, non-perturbative solitons in string theory, are supersymmetric gauge theories in various dimensions [4]. Since string theory contains lots of perturbative and non-perturbative dualities [5], consequently various field theories are related by the string dualities. Through string theory, one obtains non-trivial equivalence between different field theories.

One of the most intriguing examples is the equivalence between gauge theories on non-commutative (NC) spacetime (non-commutative gauge theories) and ordinary gauge theories in the background of constant NS-NS two-form field \( B \) [6, 7]. To be concrete, let us consider a D3-brane in the background \( B \)-field in type IIB string theory. When the \( B \)-field is polarized along the D3-brane, then using T-duality and Fourier transformation the theory on the D3-brane is shown to be equivalent with 4-dimensional \( N = 4 \) supersymmetric gauge theory on the non-commutative spacetime defined by

\[
[x_i, x_j] = i\theta_{ij},
\]  

where the parameter \( \theta \) specifies the extent of the non-commutativity. Seiberg and Witten [8] showed that these two (non-commutative and ordinary) descriptions are actually stemmed from the different methods of regularization when derived from string theory. According to them, the fields in each description are related by some field redefinition, and the actions in two descriptions are the same under this redefinition (Seiberg-Witten transformation). In this paper, we study small \( \theta \) expansion. In order for the calculation in both descriptions to be reliable, we choose the region \( \alpha' \gg \theta \). In this region, metrics in both descriptions are almost flat, \( \eta_{ij} + \mathcal{O}((\theta/2\pi\alpha')^2) \). The small \( \theta \) limit is equivalent with the small \( B \) limit,

\[
2\pi\alpha' B_{ij} = -\frac{\theta_{ij}}{2\pi\alpha'} + \mathcal{O}\left((\theta/2\pi\alpha')^3\right).
\]  

In this small \( \theta \) and \( B \) expansion, the effective actions in both sides were shown to be the same [8].

So as to investigate theories on the non-commutative spacetime, the first step is to study the properties of the solitons existing in those theories. Using the above equivalence, monopoles and dyons in 4-dimensional non-commutative gauge theory have been analyzed [9, 10, 11, 12]. In ref. [9], using brane configurations in the background \( B \)-field, the ‘non-commutative monopoles’ were analyzed through the brane interpretation of ref. [13]. The key observation of ref. [9] was that the stuck D-string tilts in the \( B \)-field background. The existence of the \( B \)-field is effectively the same as the existence of the magnetic field on the
D3-brane, and the magnetic force acting on the end of the D-string is compensated by the tension of the tilted D-string [14] (see Fig. 1).

It is very easy to see that the tilt of the D-string in the background $B$-field is actually given by $\theta$. Let us consider a D3-brane in this background. In the world volume $U(1)$ gauge theory on this D3-brane, the usual BPS equation is

$$F_{ij} + B_{ij} = \epsilon_{ijk}\partial_k\Phi,$$

where we turn on only a single scalar field $\Phi$. A point magnetic charge preserving half of the supersymmetries of the theory is described by the solution

$$\Phi = -\frac{g}{r} + \frac{1}{2}\epsilon_{ijk}B_{ij}x_k.$$  

(1.4)

This solution is depicted in fig. 1. The first singular term in the right hand side represents the stuck D-string. The linear behavior of the second term indicates the tilt of the D3-brane. The relative angle between the D3-brane and the D-string is given by

$$2\pi\alpha’\frac{1}{2}\epsilon_{ijk}B_{ij} = -\frac{1}{2}\epsilon_{ijk}\theta_{ij}/2\pi\alpha’.$$  

(1.5)

where we have introduced the parameter $2\pi\alpha’$ for defining the dimensionless slope in the target space. In eq. (1.5), we have adopted the limit $\theta \ll \alpha’$ and used eq. (1.2).

In this paper, we concentrate on monopoles in the non-commutative $U(1)$ gauge theory. In the $U(1)$ case, there is a clear understanding between the ordinary and non-commutative gauge theories [8], compared to the non-Abelian case. It is possible to investigate the correspondence of the BPS equations in both sides. From the viewpoint of the brane interpretation, monopoles are more suitable than instantons whose non-commutative version were studied in refs. [8, 15, 16, 17, 18, 19].
This paper is organized as follows. In sec. 2, we solve the BPS equation in the non-commutative $U(1)$ gauge theory. Then in sec. 3, we perform the Seiberg-Witten transformation on the solution obtained in sec. 2, and show that this exhibits an expected brane configuration of the tilted D-string. In sec. 4, we analyze the relation between the BPS equations in the non-commutative and ordinary theories. In the commutative spacetime description, the non-linearly realized supertransformation plays a crucial role. In sec. 5, we study the target space rotation which relates the solution in sec. 3 with the simple solution (1.4). Finally in sec. 6, we conclude with future directions. In addition to the $U(1)$ case of our main interest, the non-commutative $U(2)$ monopole and the non-commutative 1/4 BPS dyon are briefly studied in the commutative description in sec. 3.2 and Appendix A.

## 2 Dirac monopole in non-commutative $U(1)$ gauge theory

The soliton in the gauge theory is suitable for checking correspondence between the Dirac-Born-Infeld (DBI) action [20] on the non-commutative spacetime and the DBI action with constant NS-NS two-form background. We consider the simple situations, i.e., the Dirac monopole and electrically charged particle (with source) solutions in the $U(1)$ gauge theory.

In this paper we concentrate on the effect of the non-commutativity. Then the leading effect on non-commutativity to the configurations is $\theta/r^2$. Therefore we do not take the DBI theory but the Maxwell theory on the non-commutative spacetime. Before we write the action, we comment on the justification we use the Maxwell theory. In this approximation one may wonder higher derivative generalization of the DBI action introduces the $\alpha'$ corrections

$$\frac{\alpha'}{r^2}, \frac{\alpha'^2}{r^4}, \frac{\alpha'\theta}{r^4}, \cdots,$$

and the correction $\alpha'/r^2$ is larger than $\theta/r^2$. However the corrections $(\alpha'/r^2)^k$ do not exist since by taking $\theta \to 0$ limit BPS solutions in the Maxwell theory are also the ones in the DBI action [24]. The correction $\alpha'\theta/r^4$ may exist, but this is sub-leading compared to $\theta/r^2$. Hence the $\theta/r^2$ effect is accurately reproduced from the Maxwell theory.

The non-commutative $U(1)$ gauge theory with a Higgs field is described by the following action,

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} + \frac{1}{2} D_\mu \Phi \ast D^\mu \Phi \right),$$

where we have defined the field strength and the covariant derivative as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],$$

$$D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi].$$
\[ D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi] . \] (2.4)

We put the gauge coupling one for convenience. The commutator is defined through the star product: \([A, B] = A* B - B* A\) and the star product is

\[ (f* g)(x) \equiv f(x) \exp(i/2) \theta^{\mu\nu} \partial_\mu f(x) g(x) = f(x) g(x) + \frac{1}{2} \{ f, g \}_p(x) + \mathcal{O}(\theta^2) , \] (2.5)

where \( \{ f, g \}_p(x) \) is the Poisson bracket,

\[ \{ f, g \}_p(x) = i \theta^{\mu\nu} \partial_\mu f(x) \partial_\nu g(x) . \] (2.6)

From the action the equations of motion for \( A_\mu \) and \( \Phi \) are

\[ D^\mu F_{\mu\nu} = -i[\Phi, D_\nu \Phi] , \] (2.7)

\[ D^\mu D_\mu \Phi = 0 , \] (2.8)

and the Bianchi identity is

\[ \epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0 . \] (2.9)

The energy of this system is in the same form as for the ordinary gauge theory except for changing the product by the star product. Therefore the BPS equation for the static monopole is

\[ \frac{1}{2} \epsilon_{ij k} F^{ij k} \equiv B_i = D_i \Phi , \quad i = 1, 2, 3 . \] (2.10)

We calculate various quantities in the \( \theta \) expansion and solve the above equation to \( \mathcal{O}(\theta) \) for studying the non-commutative effect. The zero-th order solutions in \( \theta \) are

\[ A_1^{(0)} = -\frac{g}{r(r + x_3)} x_2 , \] (2.11)

\[ A_2^{(0)} = \frac{g}{r(r + x_3)} x_1 , \] (2.12)

\[ A_3^{(0)} = 0 , \] (2.13)

\[ \Phi^{(0)} = -\frac{g}{r} , \] (2.14)

where the superscript means the order in \( \theta \). We take the solution with the Dirac string spreading on the negative \( x_3 \) axis and the gauge is fixed by \( A_0 = \partial_0 A_1^{(0)} = 0 \).

By expanding the equation (2.7) to the first order in \( \theta \) we obtain

\[ \epsilon_{ij k} \partial_j B_k^{(0)} + \epsilon_{ij k} \partial_j B_k^{(1)} - i \epsilon_{ij k} \{ A_j^{(0)}, B_k^{(0)} \}_p = -i \{ \Phi^{(0)}, \partial_i \Phi^{(0)} \}_p . \] (2.15)
Using $\partial^i A_i^0 = 0$ and $\partial_i \Phi^0 = B_i^0$ we can easily solve this equation for $B_i^{(1)}$ as

$$B_i^{(1)} = -i \{ A_i^{(0)}, \Phi^{(0)} \}_P + \partial_i f ,$$

with an arbitrary function $f$. We substitute this solution into the Bianchi identity (2.9) and obtain the equation for $f$,

$$\Box f = 2i \partial^i \{ A_i^{(0)}, \Phi^{(0)} \}_P .$$

(2.17)

The non-commutative effect appears as the form of the Poisson bracket, to which $\theta^{0i}$ does not contribute. In the following we turn on only $\theta^{12} = \theta$. Then with the boundary condition that the value of $f$ goes to zero asymptotically we can solve $f$ as

$$f = -\theta g^2 \left( \frac{2x_3}{r^4} - \frac{1}{r^3} \right) .$$

(2.18)

In the same way we put $B_i^{(1)}$ into the BPS equation and obtain the $O(\theta)$ solution for $\Phi^{(1)}$ as

$$\Phi^{(1)} = \theta g^2 \left( \frac{1}{r^3} - \frac{2x_3}{r^4} \right) .$$

(2.19)

We summarize the BPS solutions in $O(\theta)$

$$\Phi = -\frac{g}{r} + \theta g^2 \left( \frac{1}{r^3} - \frac{2x_3}{r^4} \right) + O(\theta^2) ,$$

$$B_i = \frac{gx_i}{r^3} - i \{ A_i^{(0)}, \Phi^{(0)} \}_P + \partial_i f + O(\theta^2) .$$

(2.20)

(2.21)

The $\theta/r^3$ term in the Higgs field is not proportional to $\epsilon^{ijk} \theta_{ij} x_k (= \theta x_3)$ and is not invariant under the Lorentz transformation that $\theta^{\mu \nu}$ is also properly transformed. This seems strange. Since it is usually believed that the (eigen) value of the Higgs field represents the brane configuration, it should be invariant under the Lorentz transformation. To avoid this problem the authors of [10, 12] searched the BPS solutions which have the Lorentz invariant eigenvalues of the Higgs fields and discussed the brane tilting. In this Dirac monopole case we can argue in the same way. Let us take the zero-th order solution with the Dirac string spreading on the positive $x_3$ axis. We calculate the Higgs field as

$$\Phi = -\frac{g}{r} + \theta g^2 \left( -\frac{1}{r^3} - \frac{2x_3}{r^4} \right) + O(\theta^2) ,$$

(2.22)

then we see only the part $\theta/r^3$ which is not Lorentz invariant has an extra negative sign compared with the previous solution (2.20). Then if one wants a Lorentz invariant solution one merges these solutions and obtain

$$\Phi = -\frac{g}{r} + \theta g^2 \left( -\frac{2x_3}{r^4} \right) + O(\theta^2) .$$

(2.23)
However there is another problem: the tiling angle does not agree with the prediction from the brane interpretation. As said in the introduction the D-string spreads from the D3-brane with angle $\theta$. Therefore the configuration of the Higgs field must respect this fact. We can represent this fact by the equation

$$
\Phi = -\frac{g}{|x_i + \frac{1}{2} \epsilon_{ijk} \theta^{jk} \Phi|}.
$$

(2.24)

In the situation we consider, only $\theta^{12} = \theta$ is non-zero, the above equation says

$$
\Phi = -\frac{g}{r} - \frac{g}{r} \theta x_3 g^2 \frac{1}{r^4} + \mathcal{O}(\theta^2/r^4) .
$$

(2.25)

This is different from the Lorentz invariant part of the solution (2.20) by a factor 2.

This difference apparently appears when we consider the electrically charged particle in the NC $U(1)$ theory. The BPS equation for the electrically charged particle is

$$
F_{i0} \equiv E_i = D_i \Phi ,
$$

(2.26)

and the zero-th order solutions are

$$
A_0^{(0)} = \Phi^{(0)} = -\frac{g}{r} , \quad \text{other fields} = 0 .
$$

(2.27)

As in the case of the Dirac monopole, we can easily solve the equations of motion and the BPS condition in the NC theory and show the zero-th order solutions are also full order solution no matter how we turn on the non-commutativity $\theta^{\mu \nu}$. From the brane interpretation when $B_0^i \sim \theta^0 i$ is non-zero, the F-string is tilted with angle $B \sim \theta$. However we cannot see the informations of the tilt from the Higgs configuration.

This shows that the Higgs field in the non-commutative theory is not a good object when compared with the brane interpretation. In the next section we resolve this question.

### 3 Brane interpretation

#### 3.1 Seiberg-Witten transformation and brane interpretation

Callan and Maldacena revealed the BPS solution of the Higgs field corresponds to the string structure [13]. The solution solved in the previous section must be realized in the same way. In ref.[10, 11], the NC $U(2)$ monopole was discussed. The author of the ref. [11] discussed the Nahm equation in the NC gauge theory and the effect of the non-commutativity in the Nahm equation showed the D-strings slanted with slope $\theta$. In ref. [10], eigenvalues of the
Higgs field were investigated in the NC gauge theory and their brane interpretations were investigated. The eigenvalue equation of a matrix valued function $M$ in the non-commutative space takes the form

$$M \ast \vec{v} = \lambda \ast \vec{v},$$

(3.1)

where $\vec{v}$ and $\lambda$ are the eigenvector and its eigenvalue, respectively. In this form the eigenvalue is the same as the expected form, i.e., D-string is tilted. However $\lambda$ in (3.1) is not gauge invariant and we have not known other forms taking informations on the brane configurations in NC gauge theory more properly.

We have argued in the previous section that the configurations in the NC side does not match the brane interpretation. In the NC theory, since the coordinates do not commute with each other, functions written by only the coordinates are also operators. However we do not know the appropriate method for extracting the gauge singlet c-number quantities.* Moreover the tilted brane is expected in the commutative spacetime, not in the non-commutative spacetime. Therefore we insist that it is appropriate to study the brane interpretation in the ordinary gauge theory which is equivalent with the NC gauge theory.

Seiberg and Witten [8] showed that non-commutative and ordinary gauge theories are equivalent under the following relation of the gauge fields

$$\hat{A}_\mu = A_\mu - \frac{\theta\rho}{4} \{A_\rho, \partial_\delta A_\mu + F_{\delta \mu}\} + \mathcal{O}(\theta^2),$$

(3.2)

$$\hat{\Phi} = \Phi - \frac{\theta\rho}{4} \{A_\rho, \partial_\delta \Phi + D_\delta \Phi\} + \mathcal{O}(\theta^2),$$

(3.3)

$$\hat{F}_{\mu \nu} = F_{\mu \nu} + \frac{\theta\rho}{4} \{2 \{F_{\mu \rho}, F_{\nu \delta}\} - \{A_\rho, D_\delta F_{\mu \nu}\}\} + \mathcal{O}(\theta^2),$$

(3.4)

where $\{A, B\} = AB + BA$ is the anti-commutator. These relations are obtained by requiring

$$\hat{A}(A) + \hat{\delta}_\lambda \hat{A}(A) = \hat{A}(A + \delta_\lambda A),$$

(3.5)

with infinitesimal $\lambda$ and $\hat{\lambda}$. We denote $\hat{A}$ as the gauge field in the non-commutative side and $A$ is the one in the ordinary gauge theory. Using these mappings, we can easily obtain the configurations for the non-commutative Dirac monopole in the ordinary gauge theory,

$$\Phi = -\frac{g}{r} + 2g^2\frac{x_3}{r^4} + \mathcal{O}(\theta^2),$$

(3.6)

$$B_1 = g \frac{x_1}{r^3} \left(1 + 4\theta x_3 \frac{g}{r^3}\right) + \mathcal{O}(\theta^2),$$

(3.7)

$$B_2 = g \frac{x_2}{r^3} \left(1 + 4\theta x_3 \frac{g}{r^3}\right) + \mathcal{O}(\theta^2),$$

(3.8)

$$B_3 = g \frac{x_3}{r^3} \left(1 + 4\theta x_3 \frac{g}{r^3}\right) - 2\theta g^2 \frac{1}{r^4} + \mathcal{O}(\theta^2).$$

(3.9)

*The Higgs field in the NC $U(1)$ theory is not singlet.
Then the Higgs field is invariant under the Lorentz transformation and the same as (2.25). The problem that the Higgs field is not invariant under the Lorentz transformation which occurs in the non-commutative side disappears and the bending angle from D3-brane exactly matches with the expected one.

The above discussion also holds for the non-commutative electrically charged particle. The corresponding solution in the ordinary theory is easily obtained from the mappings as

$$\Phi = -\frac{g}{r} - \frac{g^2 \theta \theta_i x_i}{r^4} + O(\theta^2),$$  \hspace{1cm} (3.10)

which takes the expected form. In the NC gauge theory, the electrically charged particle does not receive the non-commutative effect. On the other hand, in the ordinary theory, the $B$-field coupling in the DBI action generates corrections to the configuration of the electrically charged particle.

In the DBI action the $B$-field always appears in the combination $F^{\mu\nu} + B^{\mu\nu}_{\text{NS}}$, and so $B^{ij}_{\text{NS}}$ behaves as the magnetic field and $B^{0i}_{\text{NS}}$ behaves as the electric field. We recognize this behavior from the forms of the solutions. The “electric field” $B^{0i}_{\text{NS}}$ changes the configuration of the electrically charged particle and the “magnetic field” $B^{ij}_{\text{NS}}$ alters that of the Dirac monopole.

So far we have concentrated on the solutions of the Higgs field. Later we reanalyze the DBI action for small $B$-field and confirm that the configurations considered in this section are the BPS solutions in the ordinary theory.

A comment is in order: the $\theta$ expansion is well defined for $\theta \ll r^2$ since the dimensionless parameter for the $\theta$ expansion is $\theta g/r^2$. In that region the value of the Higgs field is reliable and the D-string slants with angle $\theta$ (the equation (2.25) is reliable). Therefore we naturally regard the Dirac monopole in the NC theory as the D-string attached to D3-brane with uniform magnetic fields. We discuss this relation in section 5.

3.2 Non-commutative $U(2)$ monopole and Seiberg-Witten transformation

As seen in sec. 2, for the $U(1)$ BPS Dirac monopole, the solution in the non-commutative side (named (I)) does not exhibit the appropriately slanted D-string. The configuration after the Seiberg-Witten map (3.3) gives the precise tilt of the D-string. Therefore, although the non-commutative $U(2)$ monopole was already considered in ref. [10], it is natural to study their commutative counterparts.

The monopole solution in the non-commutative super Yang-Mills theory obtained in ref.
\[ \hat{A}_i = \epsilon_{aij} x_j W(r) \frac{1}{2} \sigma_a + \theta_{ij} x_j \frac{1}{4 r^2} W(r) \left( W(r) + 2 F(r) \right) \frac{1}{2} \mathbf{1} + \mathcal{O}(\theta^2), \]  

(3.11)

\[ \hat{\Phi} = \frac{x_a}{r} F(r) \frac{1}{2} \sigma_a + \mathcal{O}(\theta^2), \]  

(3.12)

where we have defined

\[ F(r) \equiv C \coth(C r) - \frac{1}{r}, \quad W(r) \equiv \frac{1}{r} - \frac{C}{\sinh(C r)}, \]  

(3.13)

with a dimensionful parameter \( C \). Note that there is no \( \mathcal{O}(\theta^1) \) correction \( \Phi^{(1)} \) in the Higgs field. Performing the Seiberg-Witten transformation (3.3), we obtain the configuration for \( \Phi \) in the commutative description (named (II)) as

\[ \Phi = \frac{x_a}{r} F(r) \frac{1}{2} \sigma_a + \epsilon_{ijk} x_i \theta_{jk} \frac{1}{8 r^3} \left( W(r) F(r) \left( 2 - r W(r) \right) \frac{1}{2} \mathbf{1} + \mathcal{O}(\theta^2) \right). \]  

(3.14)

The eigenvalues of this matrix \( \Phi \) are of course gauge invariant. Near the infinity of the worldvolume we have the asymptotic expansion for the eigenvalues as

\[ \lambda = \pm \left( C - \frac{1}{r} \right) \frac{1}{2} - \frac{\epsilon_{ijk} x_i \theta_{jk}}{8 r^3} \left( C - \frac{1}{r} \right) + \mathcal{O}(\theta^2). \]  

(3.15)

Remarkably, this asymptotic expression is the same as the one obtained in ref. [10] where the \( \mathcal{O}(\theta) \) eigenvalues are generated using the ‘non-commutative eigenvalue equation’. Since in ref. [10] this expression was shown to match the tilted D-string configuration with the proper slope \( \theta \), we see that the Seiberg-Witten transformed configuration in (II) exhibits the correct configuration of the slanted D-string.

In addition, the configuration (3.14) has another nice property: the configuration is regular even at the origin \( r = 0 \). (The eigenvalues obtained in ref. [10] were singular at the origin.) Since we can prove the following relation

\[ \lambda = \pm \frac{1}{2} F \left( |x_i - \frac{1}{2} \epsilon_{ijk} \theta_{jk} \lambda| \right), \]  

(3.16)

up to \( \mathcal{O}(\theta^2) \), we understand that the D-string is suspended really along the line

\[ x_i = \frac{1}{2} \epsilon_{ijk} \theta_{jk} \lambda, \]  

(3.17)

and has a tilt \( \theta \). The interesting is that the tilted D-string can be read not only from the asymptotic region but also from everywhere. The D3-brane configuration in the commutative description (II) (the eigenvalues of eq. (3.14)) is depicted in fig. 2.
In a similar manner, we obtain the magnetic field in (II) using the Seiberg-Witten transformation. The result for the gauge field is

$$A_i = \tilde{A}_i^{(0)} + \theta_{ij} x_j \frac{1}{2r^2} W(r) \left( W(r) + F(r) + r \frac{\partial W(r)}{\partial r} \right) \frac{1}{2} + \mathcal{O}(\theta^2),$$

(3.18)

and it is straightforward to obtain the magnetic field from this expression. Expanding near the infinity, this also coincides with the result obtained in ref. [10]. The magnetic field configuration from eq. (3.18) is also regular at the origin.

In Appendix A, we apply this procedure also for the non-commutative $U(3)$ monopole and the non-commutative 1/4 BPS dyon which were studied in ref. [12].

4 BPS condition for ordinary gauge theory

In the previous section we have concentrated on the configurations of the field in the ordinary gauge theory and do not paid attention to the action. Seiberg and Witten showed that the DBI action for the small $B$-field is equal to the non-commutative DBI action for small $\theta$ with $\alpha'$ fixed, i.e. $\theta/\alpha' \ll 1$, by the redefinition of fields and couplings. From this equivalence,

†Note that the expression of the magnetic field written in ref. [10] contains a typo of a factor 2.
we can consider the BPS equation for the ordinary gauge theory and check whether the configuration (3.6) \cdots (3.9) satisfies the BPS equation or not.

We use the DBI action for the ordinary gauge theory with a scalar field,

$$\mathcal{L} = \sqrt{\det \left( -g_{\mu\nu} + 2\pi\alpha'(B_{\mu\nu} + F_{\mu\nu}) + (2\pi\alpha')^2 \partial_\mu \Phi \partial_\nu \Phi \right)}, \quad (4.1)$$

where the numerical factor is omitted. Now we consider the situation $B_{\mu\nu} = -\theta_{\mu\nu}/(2\pi\alpha')^2$ and the metric $g_{\mu\nu}$ is flat. We do not expand the lagrangian (4.1) under the small $B$-field, because the obtained action has many interactions and is not suitable for picking up physical meanings. Therefore we consider the DBI action itself.

In the work of Seiberg and Witten, the equivalence between the non-commutative and ordinary gauge theories was shown in the approximation of slowly varying fields. However the solutions of the Dirac monopole in the NC gauge theory do not vary slowly, we shall investigate whether (3.6) \cdots (3.9) are the BPS solution in the ordinary theory. For this purpose let us study the BPS condition and see the solutions satisfy the BPS equation.

For discussing the BPS condition, it is more useful to consider the supersymmetrized theory and the condition for some supersymmetry remained unbroken than to study the minimal bound of the energy for the system. In the case of monopole we must consider the $N = 2$ supersymmetric DBI action in 4 dimensions and study the supersymmetric transformation for fermions. If $B$-field is zero, the linearly realized supertransformations for fermions are (in 6 dimensional notation),

$$\delta \lambda = F_{MN} \Sigma^{MN} \eta, \quad M, N = 0, \cdots 5,$$ \quad (4.2)

where $A_M$ for $M = 0, \cdots, 3$ are gauge fields in four dimensional spacetime and $A_4 = \Phi$ and $A_5$ is another scalar field. Then the BPS condition for the (Dirac) monopole becomes simple; $F_{MN} \Sigma^{MN}$ has some zero eigenvalues. On the other hand, if $B$ is nonzero, all the linearly realized supertransformations are broken, and the unbroken supertransformations are some combinations of the linearly and non-linearly realized ones. Thus we must see the non-linearly realized supertransformations. The $N = 2$ DBI action is obtained as the model of partial breaking of $N = 4$ supersymmetry down to $N = 2$ [21, 22] and we can read the non-linear transformations for the broken supersymmetries. However fortunately the nonzero fields are only one Higgs scalar and the space components of the gauge fields, and they are static. Then we need not to know the full non-linear ones but we only see the $N = 1$ part which generate shifts for fermions as

$$\delta \lambda_+ = (F^+_\mn + B^\pm_{mn}) \sigma^{mn} \eta_+, \quad \text{and} \quad (4.3)$$

$$\delta \lambda_- = (F^-_{mn} + B^-_{mn}) \sigma^{mn} \eta_-, \quad (4.4)$$
\[ \hat{\delta} \lambda_+ = \frac{1}{4\pi \alpha'} \left( 1 - (2\pi \alpha')^2 \text{Pf}(F_{mn} + B_{mn}) + \sqrt{\det(\delta_{mn} + 2\pi \alpha'(F_{mn} + B_{mn}))} \right) \chi_+ , \quad (4.5) \]

\[ \hat{\delta} \lambda_- = \frac{1}{4\pi \alpha'} \left( 1 + (2\pi \alpha')^2 \text{Pf}(F_{mn} + B_{mn}) + \sqrt{\det(\delta_{mn} + 2\pi \alpha'(F_{mn} + B_{mn}))} \right) \chi_- , \quad (4.6) \]

where (4.3) and (4.4) are linear ones and (4.5) and (4.6) are non-linear ones\(^\ddagger\), and we have defined \( 4\text{Pf}F = \text{Tr}(F^2) - \text{Tr}(F^-)^2 \). In this expression we have already put unnecessary fields to zero. We consider the \( B \)-field is non-zero for the space direction, namely \( B_1^2 = \theta / \alpha'^2 \), since \( B_0^i \) does not affect the monopole configuration to the first order in \( B \), and we take the metric flat. Notice that \( m = 4 \) is not the spacetime direction but \( F_m^4 = \partial_m \Phi \) and \( B_m^4 = 0 \). The above transformation is the same as the \( N = 1 \) linear and non-linear transformations \[23\] in the Euclidean 4 dimensional space except for replacing the fourth gauge field by the Higgs field \( \Phi \) and putting \( \partial_4 = 0 \). This fact is natural, since from the 6 dimensional aspect to set \( \partial_0 = 0 \) which means static, \( \partial_5 = 0 \), \( A_0 = A_5 = 0 \) and half of fermion to zero, the theory reduces to the Euclidean 4 dimensional \( N = 1 \) supersymmetric gauge theory and the linearly and non-linearly realized supertransformations also reduce to the \( N = 1 \) ones.

Now we have tools for studying the 1/2 BPS condition for the monopole with non-zero \( B \). In the situation we now consider \( B \) and \( F \) have the following forms,

\[
B_{mn} = \begin{pmatrix}
B & 0 \\
0 & 0 \\
-B & 0
\end{pmatrix}, \quad B = -\theta / 4\pi^2 \alpha'^2 , \quad (4.7)
\]

\[
F_{mn} = \begin{pmatrix}
B_3 & -B_2 & -\partial_1 \Phi \\
-B_3 & B_1 & -\partial_2 \Phi \\
\partial_1 \Phi & \partial_2 \Phi & \partial_3 \Phi
\end{pmatrix}, \quad B_i = \frac{1}{2} \epsilon_{ijk} F^{jk} , (i = 1, 2, 3), \quad (4.8)
\]

and then Pf \( B = 0 \) obeys. \( B^+ \) and \( F^+ \) are defined as

\[
B_{mn}^+ = \frac{1}{2} (B_{mn} + \bar{B}_{mn}) , \quad \bar{B}_{mn} = \frac{1}{2} \epsilon_{mnpq} B^{pq} , \quad (4.9)
\]

\[
F_{mn}^+ = \frac{1}{2} (F_{mn} + \bar{F}_{mn}) , \quad \bar{F}_{mn} = \frac{1}{2} \epsilon_{mnpq} F^{pq} . \quad (4.10)
\]

\(^\ddagger\)We use an unusual decomposition. We decompose a Weyl fermion in 6 dimensions into \( SO(4) = SU(2)_+ \times SU(2)_- \) fermions \( \lambda_+ (2, 1) \) and \( \lambda_- (1, 2) \). \( F^+ \) is the tensor transforming as \( (3, 1) \) and \( F^- \) is the tensor transforming as \( (1, 3) \). Notice that this \( SO(4) \) is not the Lorentz group in our spacetime \( (M = 0, 1, 2, 3) \) but the rotation on the plane \( M = 1, 2, 3, 4 \).
Since $F_{mn}$ goes to zero asymptotically, the combination $\delta(\eta_+) + \tilde{\delta}(\chi_+)$,

$$\chi_+ = -\frac{4\pi\alpha'}{1 - (2\pi\alpha')^2\operatorname{Pf}B_{mn} + \sqrt{\det(\delta_{mn} + 2\pi\alpha'B_{mn})}}B^+_{mn}\sigma^{mn}\eta_+$$ (4.11)

$$= -\frac{4\pi\alpha'}{\left(1 + \sqrt{1 + (2\pi\alpha')^2B^2}\right)}B^+_{mn}\sigma^{mn}\eta_+ ,$$ (4.12)

is the unbroken supertransformation. Other supertransformations are all broken and then $N = 1$ supersymmetry is unbroken\(^\S\). Therefore the unbroken supertransformation must be $\delta(\eta_+) + \tilde{\delta}(\chi_+)$ ($\chi'$ does not change, i.e. eq. (4.12)) everywhere and the BPS condition is,

$$(\delta(\eta_+) + \tilde{\delta}(\chi_+))\lambda_+ = 0 .$$ (4.13)

This is written explicitly as

$$0 = (F^+_{mn} + B^+_{mn})$$ (4.14)

$$-\frac{1 - (2\pi\alpha')^2\operatorname{Pf}(F_{mn} + B_{mn}) + \sqrt{\det(\delta_{mn} + 2\pi\alpha'(F_{mn} + B_{mn}))}}{1 + \sqrt{1 + (2\pi\alpha')^2B^2}}B^+_{mn} .$$ (4.15)

We expand the right hand side to the first order in $B$ and second order in $F$ since we consider the linearized Maxwell theory. This approximation is equivalent to that we consider the Maxwell theory in the NC gauge theory and study the first order in $\theta$. Then we obtain the BPS condition,

$$F^+_{mn} + B^+_{mn}\frac{(2\pi\alpha')^2}{8}(\operatorname{Tr} F\tilde{F} - \operatorname{Tr} F^2) = 0 .$$ (4.16)

When we substitute (4.7) and (4.8) into this condition, we finally obtain the following condition to the first order in $\theta$,

$$B_1 = \partial_1 \Phi ,$$ (4.17)

$$B_2 = \partial_2 \Phi ,$$ (4.18)

$$B_3 = \partial_3 \Phi - \theta g^2 \frac{1}{r^4} .$$ (4.19)

We can easily show the Dirac monopole solution in the ordinary theory (3.6) $\cdots$ (3.9) satisfies these equations. In the end we have shown that, using the mappings (3.2) and (3.3), the BPS equation in the NC theory is transformed to the BPS equation in the ordinary theory.

\(^\S\)When we consider anti monopole $B_i = -\partial_i \Phi$, we must consider the combination of $\eta_-$ and $\chi_-$ for the unbroken supertransformations.
5 Target space rotation

5.1 Reproduction of the solution

The brane configuration obtained in sec. 2 in the ordinary gauge theory, (3.6), has the desired property that the D-string is slanted with the slope $\theta$. Now, a natural question arises — how this solution in (II) is related to the configuration (1.4) (named (III)) considered in the introduction? The difference between the two originates in only the way of putting the coordinate system in the target space: they are related by the target space rotation by the angle defined by $\theta$ (1.5).

The BPS equation of the ordinary gauge theory adopted in the introduction is

$$\ddot{F}_{ij} + B_{ij} = \epsilon_{ijk} \partial_k \Phi,$$

and using (1.2) its solution is

$$\ddot{\Phi} = -\frac{g}{\ell} - \frac{1}{(2\pi\alpha')^2} \theta_{12} \dot{x}_3, \quad \frac{1}{2} \epsilon_{ijk} \ddot{F}_{jk} = \frac{g \dot{x}_i}{\ell^3},$$

where the check indicates that the variables are in the description (III). We have turned on only the $\theta_{12}$ component, and therefore the configuration is slanted in the direction along $\dot{x}_3$. The target space rotation which may relate (II) and (III) is

$$\begin{pmatrix} 2\pi\alpha'\Phi \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} 2\pi\alpha' \ddot{\Phi} \\ \dot{x}_3 \end{pmatrix},$$

while the other coordinates are left invariant ($\dot{x}_1 = x_1, \dot{x}_2 = x_2$). Note that, as mentioned in the introduction, we must multiply the factor $2\pi\alpha'$ on the scalar field so as to adjust the

![Diagram](image)

Figure 3: Rotation in the target space from (III) to (II).
dimensions. It is easy to see that the rotation angle $\varphi$ should be given by $\varphi = -\theta_{12}/2\pi\alpha' + \mathcal{O}((\theta/2\pi\alpha')^3)$. By substituting the solution (5.2) in (III) into eq. (5.3), we have

$$\Phi = -\frac{g}{r} + \mathcal{O}(\theta^2),$$  \hspace{1cm} (5.4)

$$x_3 = \theta_{12} \frac{g}{r} + \tilde{x}_3 + \mathcal{O}(\theta^2).$$  \hspace{1cm} (5.5)

We have chosen the value of $\varphi$ so that $\Phi$ vanishes asymptotically. From eq. (5.5) we have a relation

$$\tilde{r} = r \left(1 - \theta_{12} \frac{g x_3}{r^3}\right) + \mathcal{O}(\theta^2).$$  \hspace{1cm} (5.6)

Therefore combining this with eq. (5.4), finally we obtain

$$\Phi = -\frac{g}{r} - \theta_{12} \frac{g^2 x_3}{r^4} + \mathcal{O}(\theta^2),$$  \hspace{1cm} (5.7)

which coincides with the solution in the previous section (3.6).

From the very naive argument presented in this subsection, we have seen that the solution in (II) is easily obtained through the target space rotation from (III). Since we have seen in sec. 3 that the solution in the NC theory (I) is related to (II) through the Seiberg-Witten map, hence we have three equivalent descriptions.

5.2 Reproduction of the BPS equation

We have seen that the BPS equation in the description (I) corresponds to the unusual BPS condition which preserves a certain combination of linearly and non-linearly realized supersymmetries in (II). Now, as seen in sec. 5.1, the solution of this unusual BPS equation is obtained by the target space rotation (5.3) from the solution in the description of (III). In (III) where the D3-brane is slanted, it is enough to consider linearly realized supersymmetries, and the story becomes considerably simple. Thus it might be natural to study how the rotation acts on the BPS equation. In this subsection, we shall see that the BPS conditions in (II) and (III) are related with each other under the rotation.

The BPS equation in (II) reads

$$B_i + \delta_{i3} \theta_{12} g^2 \frac{1}{r^4} = \partial_i \Phi.$$  \hspace{1cm} (5.8)

We want to derive this equation from the BPS equation (5.1) in (III) by the rotation (5.3). First, from the relation

$$\check{\Phi}(\check{x}) = \Phi(x) - \frac{\theta_{12}}{(2\pi\alpha')^2} x_3,$$  \hspace{1cm} (5.9)
using eq. (5.3), we have
\[ \Phi(\tilde{x}) = \Phi(x_1, x_2, \tilde{x}_3 - \theta_{12} \Phi) - \frac{\theta_{12}}{(2\pi \alpha')^2} \tilde{x}_3. \] (5.10)

Thus the derivative with respect to \( \tilde{x}_3 \) is
\[ \hat{\partial}_3 \Phi(\tilde{x}) = \frac{\partial(x_3 - \theta_{12} \Phi)}{\partial \tilde{x}_3} \partial_3 \Phi(x) - \frac{\theta_{12}}{(2\pi \alpha')^2} \]
\[ = \left( 1 - \theta_{12} \frac{\partial \Phi}{\partial \tilde{x}_3} \right) \partial_3 \Phi(x) - \frac{\theta_{12}}{(2\pi \alpha')^2}. \] (5.11)

Therefore for the right hand side of eq. (5.1) we have
\[ \hat{\partial}_i \Phi = \partial_i \Phi(x) - \delta_{3i} \frac{\theta_{12}}{(2\pi \alpha')^2} - \theta_{12} \partial_3 \Phi \partial_i \Phi. \] (5.12)

On the other hand, in the left hand side of eq. (5.1), the term containing the \( B \)-field is changed to
\[ \frac{1}{2} \epsilon_{ijk} B_{jk} = -\frac{1}{(2\pi \alpha')^2} \delta_{3i} \theta_{12}, \] (5.13)

hence this cancels the constant term in eq. (5.12) in the right hand side of eq. (5.1). Now the magnetic field is expanded as
\[ \hat{B}(\tilde{x}) = B(x_1, x_2, x_3 + \theta_{12} \Phi) = B(x) + \theta_{12} \Phi \partial_3 B(x) + O(\theta^2). \] (5.14)

Note that, as seen in eq. (5.2), the magnetic field function \( \hat{B} \) is equal to the zero-th order solution \( B^{(0)} \). Thus we can rewrite the BPS equation (5.1) as
\[ B^{(0)}(x) + \theta_{12} \Phi^{(0)} \partial_3 B^{(0)}(x) = \partial_i \Phi - \theta_{12} \partial_3 \Phi^{(0)} \partial_i \Phi^{(0)}. \] (5.15)

This is the same as the BPS equation (5.8) in (II), using the explicit solution for \( B \) in (II).

6 Conclusion and discussion

In this paper, we have analyzed the non-commutative BPS Dirac monopole in the three different descriptions: (I) the solution of the BPS equation in the non-commutative \( U(1) \) gauge theory, (II) the solution of the BPS equation which preserves a combination of the linearly and non-linearly realized supersymmetries in the ordinary \( U(1) \) gauge theory in the \( B \)-field background, and (III) the solution of the usual BPS equation from linearly realized supersymmetries in the ordinary \( U(1) \) gauge theory in the constant magnetic field. For
small $\theta$ and small $B$ approximation, we have shown that these three descriptions are related with each other as follows: (I) and (II) are related by the Seiberg-Witten transformation, and (II) and (III) are by the target space rotation. We have confirmed that the non-commutative Dirac monopole matches perfectly with the tilted D-string configuration in the $B$-field background.

The solution for the scalar field $\hat{\Phi}$ obtained in (I) is not invariant under the simultaneous rotation of $x_i$ and $\theta_{ij}$. However, performing the Seiberg-Witten transformation on this solution, then we obtain a rotation-invariant solution for $\Phi$ in the commutative description (II). This solution exhibits precisely the configuration of the D-string tilted due to the existence of the $B$-field. This solution is also obtained by the target space rotation from the simple solution in (III). This target space rotation is concerning the plane spanned by the scalar field and the worldvolume coordinate along the non-commutativity.

Furthermore, also the BPS equations for each description have been shown to be related with each other by the above Seiberg-Witten transformation and the target space rotation. We have checked the non-trivial equivalence of the three different descriptions, with use of a simple example of the non-commutative BPS Dirac monopole, on the level of not only the solution but also the BPS equation. We summarize our results in the following table.

| description | (I) | (II) | (III) |
|-------------|-----|------|-------|
| scalar field | $\Phi(x)$ | $\Phi(x)$ | $\Phi(\hat{x})$ |
| BPS equation | from linear susy | from linear + non-linear susy | from linear susy |
| brane | cannot be tilted D-string | vertical D-string |
| configuration | understood & horizontal D3 | & tilted D3 |

Table 1: Three different but equivalent descriptions.

It would be interesting if the calculation performed in this paper can be extended to the all order in the perturbative expansion in $\theta$ and $\alpha'$. Though we have considered the region $r \gg \sqrt{\alpha'} \gg \sqrt{\theta}$ in this paper, it is expected that the BPS solution considered in this paper is also a solution of the equations of motion of all order in $\alpha'$. This expectation follows from the proof in the ordinary commutative case [24]. The approach using solitons has advantages when one wants to study the equivalence between non-commutative and commutative descriptions beyond the perturbation.

The extension of our analysis to the non-Abelian case is also important. If the non-linearly realized supertransformation of the non-Abelian DBI theory is available, then our strategy can be applied to the non-commutative $U(2)$ monopole whose commutative description has
been briefly considered in sec. 3.2. Then it can be shown that the configuration calculated in sec. 3.2 subjects to some BPS equations.

The meaning of the target space rotation adopted in sec. 5 is still vague. We have seen in sec. 5.2 that the BPS equations are related with each other by this rotation. It is to be clarified how this non-trivial rotation which mixes the fields and the worldvolume coordinates are consistent with the lagrangian formalism of the DBI action.

Our final comment is on the ‘non-commutative eigenvalue equation’. This shows a correct D-string configuration at least asymptotically. However, we insist that the eigenvalues to be examined are of the commutative description. Since these two apparently different methods give the same result, some relations must exist between them. The study of this may provide interesting information of the non-commutative theory.

Note added
While this work was in the final stage, we became aware of the paper [25] which shows an overlapping results.

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A Non-commutative 1/4 BPS dyon and Seiberg-Witten transformation

As studied in sec. 3.2, the non-commutative monopoles can be interpreted in the brane description well by the commutative description after being performed the Seiberg-Witten transformation (3.3). In this appendix, we investigate the NC $U(3)$ monopoles and the NC 1/4 dyons in a similar manner.
A.1 Non-commutative $U(3)$ monopole

In ref. [12], a solution for the NC $U(3)$ monopole was obtained. After performing the Seiberg-Witten transformation to the solution of ref. [12], we obtain the following configuration of the scalar field $Y$:

\[
Y = Y^{(0)} + Y^{(1)} + O(\theta^2), \quad (A.1)
\]
\[
Y^{(0)} = -\hat{x}_i T_i H(\xi)/r, \quad (A.2)
\]
\[
Y^{(1)} = \frac{1}{r^3} \left( \theta_i \hat{x}_i T_0 U(\xi) + \theta_i \hat{x}_j T_{ij} V(\xi) + \theta_i \hat{x}_i \hat{x}_j \hat{x}_k T_{jk} W(\xi) \right), \quad (A.3)
\]

where we have defined $\xi \equiv C r$, and all of the conventions follow from the ones adopted ref. [12]. The functions $U$, $V$ and $W$ which specify the solution reads (after the Seiberg-Witten transformation)

\[
U(\xi) = U_{\text{HM}}(\xi) + \frac{1}{6} H(1 - K)(1 + K),
\]
\[
V(\xi) = V_{\text{HM}}(\xi) + \frac{1}{2} (1 - K)(K^2 - 1), \quad (A.4)
\]
\[
W(\xi) = W_{\text{HM}}(\xi) - \frac{1}{4} (1 - K)(H - 2 + 2K^2 + HK).
\]

In this expression, the functions with “HM” mean the ones obtained in ref. [12], which give the solution of the NC BPS equations. The other terms in the right hand sides are produced from the Seiberg-Witten transformation. It is straightforward to evaluate the three eigenvalues of the scalar $Y$ as

\[
\lambda_Y = \frac{1}{r^3} \theta_i \hat{x}_i \left( 4U - \frac{4}{3}(V + W) \right), \quad -(\pm 1) \frac{H}{r} + \frac{1}{r^3} \theta_i \hat{x}_i \left( 4U + \frac{2}{3}(V + W) \right), \quad (A.5)
\]

where $\theta_i \equiv \epsilon_{ijk} \theta_{jk}/2$. Asymptotically, these become

\[
\lambda_Y = \frac{\theta_i \hat{x}_i}{4r^3} (1 + 4z) \xi + O(\theta^2), \quad \mp C \pm \frac{1}{r} + \frac{\theta_i \hat{x}_i}{4r^3} ((3 - 4z) \xi - 4) + O(\theta^2). \quad (A.6)
\]

Here the parameter $z$ is included in $V_{\text{HM}}$ and $W_{\text{HM}}$, and it indicates a moduli of the relative location of the two D-strings which suspend between the three D3-branes. Remarkably, this (A.6) is the same as the result of ref. [12] in which three eigenvalues are obtained by solving the ‘non-commutative eigenvalue equation’ in the non-commutative space. Therefore, asymptotically, we obtain the same configuration of slanted D-string as ref. [12]. However, as in the case of the NC $U(2)$ monopole in sec. 3.2, we have the agreement only in the asymptotic region. Our eigenvalues are regular even at the origin $r = 0$. What is more interesting, the latter two eigenvalues in (A.6) are arranged to the first order in $\theta$ in the following way:

\[
\lambda_Y = -(\pm 1) \frac{H \left( C(x_i + \theta_i \lambda_Y) \right)}{|x_i + \theta_i \lambda_Y|}, \quad (A.7)
\]
where we have chosen a special value $z = -1/4$ in that case the two D-strings are aligned\(^4\). This relation (A.7) indicates that the eigenvalues (A.6) really exhibits the tilted D-string configuration with the center on a straight line

$$x_i = \theta_i \lambda.$$  

(A.8)

### A.2 String junction in $B$-field and non-commutative $1/4$ BPS dyon

In the previous subsection, we have obtained a consistent brane picture of the NC $U(3)$ monopole, then let us proceed to the case of the NC $1/4$ BPS dyon studied in ref. [12].

The authors of ref. [12] solved the NC Gauss law for another scalar $X$ in the background of the NC $U(3)$ monopole. We perform the Seiberg-Witten transformation and obtain the configuration for $X$ in the commutative description as

$$X = X^{(0)} + X^{(1)} + \mathcal{O}(\theta^2),$$  

(A.9)

$$X^{(0)} = \frac{1}{r} \hat{x}_i \hat{x}_j T_{ij} \frac{Q(\xi)}{\xi},$$  

(A.10)

$$X^{(1)} = \frac{1}{\xi r^3} \left( \theta_i \hat{x}_i R(\xi) + \theta_i \hat{x}_i \hat{x}_j T_j S(\xi) \right).$$  

(A.11)

We choose $(\alpha, \beta) = (0, 1)$ and the zero-th order solution is specified by $Q(\xi) = -2H^2 - H + 1 - K^2$. The functions appearing in the above are given as

$$R(\xi) = R_{\text{HM}}(\xi) + \frac{1}{3}(1 - K)(2Q - \mathcal{D}Q),$$  

(A.12)

$$S(\xi) = S_{\text{HM}}(\xi) - \frac{1}{3}(1 - K)(\mathcal{D}Q - (5 + 3K)Q).$$  

(A.13)

When $X$ commutes with $Y$, these are simultaneously diagonalizable, hence the brane interpretation is possible. This requirement provides a condition

$$[X, Y] = i \frac{\theta_k \hat{x}_i \epsilon_{ikm} T_m}{\xi r^3} \left( 2V(\xi)Q(\xi) + R(\xi)H(\xi) \right) + \mathcal{O}(\theta^2) = 0.$$  

(A.14)

Interestingly enough, using the solution (A.4), (A.12) and (A.13), this condition is satisfied only when $z = -1/4$, in whose case the junction interpretation was possible in ref. [12].

We calculate three eigenvalues of the scalar field $X$ with $z = -1/4$ as

$$\lambda_X = \frac{8}{3} C - \frac{4}{r} + \mathcal{O}(\theta^2), \quad -\frac{4}{3} C + \frac{2}{r} \pm \frac{\theta_i \hat{x}_i}{r^3} (2\xi - 2) + \mathcal{O}(\theta^2).$$  

(A.15)

where we only write the asymptotic expression. Again, this shows a perfect agreement with

\(^4\)With this special value of $z$, another eigenvalue vanishes exactly.
A consistent string junction in the B-field whose brane configuration was previously studied in ref. [9], our commutative picture provides a consistent configuration of the string junction, too.

As in the case of $Y$ (A.7), it is possible to arrange the latter two eigenvalues of $X$ at finite $x$ into the form (to the first order in $\theta$)

$$\lambda_X = \frac{2Q(C(x_i + \theta_i h(\lambda_X)))}{3C|x_i + \theta_i h(\lambda_X)|^2}.$$  

(A.16)

Here the function $h(\lambda_X)$ is defined as

$$h(\lambda_X) = \mp \frac{H(Cs)}{s} \bigg|_{s=s(\lambda_X)},$$  

(A.17)

where the parameter $s$ is a solution of the equation

$$\lambda_X = \frac{2Q(Cs)}{3Cs^2}.$$  

(A.18)

The expression (A.16) indicates that the $(p,q)$-string locates on the line

$$x_i + \theta_i h(\lambda_X) = 0.$$  

(A.19)

The whole configuration of the NC string junction is given by the two equations (A.8) and (A.19). Eliminating $x_i$ from these two equations, then we obtain a relation between $\lambda_Y$.
and $\lambda_X$. It is easy to see that this relation is precisely the same as the one obtained in the usual commutative case ($\theta = 0$) of refs. [26, 27]. In ref. [27], the bending of the $(p, q)$-strings in the network was analyzed, and it was found that the bend of the strings are consistent with the effective charge defined at a finite distance $r$. Therefore, we conclude that the bending of the NC string junction is interpreted in the same way. The difference between the NC string junction and the previous usual string junction [26, 27] is only that now the junction is on a plane (A.8) tilted by the angle $\theta$. This fact was predicted in ref. [9], and we have given the proof of the prediction.

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