ABSOLUTELY CONTINUOUS SPECTRUM OF
MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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Abstract. We prove that 3-dimensional Schrödinger operator with slowly
decaying potential has an a.c. spectrum that fills $\mathbb{R}^+$. Asymptotics of Green’s
functions is obtained as well.

Consider the Schrödinger operator

$$H = -\Delta + V, x \in \mathbb{R}^d$$

(1)

We are interested in finding the support of an a.c. spectrum of $H$ for the slowly
decaying potential $V$. The following conjecture is due to B. Simon [21].

Conjecture. If $V(x)$ is such that

$$\int_{\mathbb{R}^d} \frac{V^2(x)}{1 + |x|^{d-1}} \, dx < \infty$$

(2)

then $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$. It was proved for the one-dimensional case by Deift and Killip [5] (see also [11] [13] [6]). For some Dirac operators, this conjecture was shown to be true for

d = 1 by M. Krein [14] and for $d = 3$ by the author [7].

The spatial asymptotics of the Green function is a classical subject [1, 2]. In the
current paper we deal with $d = 3$ and prove the preservation of an a.c. spectrum under more restrictive conditions on the potentials rather than (2).

Methods of the paper can be generalized to other $d$ and perhaps to the discrete case too. We take $d = 3$ for simplicity only. The structure of the paper is as follows. In the
introduction, we obtain different results that serve as a motivation to the main
theorem of the paper. In the second section, we prove the spatial asymptotics of the Green kernel. Then, in the third part, the main result on the preservation of the a.c. spectrum is obtained. The last section contains different applications.

Let us introduce some notations we will be using later. We denote the integral kernel of $R_z = (H - z)^{-1}$ by $G_z(x, y)$. Recall that for the Green’s kernel of the free
Laplacian, we have

$$G^0_z(x, y, z) = \frac{\exp(ik|x - y|)}{4\pi|x - y|}, z = k^2, k \in \mathbb{C}^+$$

(3)

The symbol $\Sigma$ stands for the unit sphere in $\mathbb{R}^3$. The inner product of two vectors $\xi, \zeta$ in $\mathbb{R}^3$ is denoted by $\langle \xi, \zeta \rangle$. 

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Acknowledgement. We are grateful to B. Simon for the useful discussions.

1. Introduction

We will start with the following simple result.

Proposition. Consider the $C^1(\mathbb{R}^3)$ vector-field $Q(x)$ such that

$$\int_{\mathbb{R}^3} \frac{|Q(x)|^2}{1 + |x|^2} \, dx < \infty$$

and $\text{div} \, Q(x) \in L^\infty(\mathbb{R}^3)$. Let $V = \gamma \cdot \text{div} \, Q + |Q|^2, |\gamma| \leq 1$ and $H = -\Delta + V$. Then, $\sigma_{ac}(H) = \mathbb{R}^+$. The proof of this fact follows immediately from the arguments given in [15].

Integrating by parts in the quadratic form for $H$, one can easily see that $H \geq 0$ for any $|\gamma| \leq 1$. Then, since

$$\left| \int_{\mathbb{R}^3} \frac{V(x)}{1 + |x|^2} \, dx \right| < \infty$$

the first trace-inequality (6.8) from [15] yields $\sigma_{ac}(H) = \mathbb{R}^+$. Since $H \geq 0$, one does not have to worry about the negative eigenvalues and an analysis in [14] is now easy.

Analogous argument works for any $d$ including $d = 1$. But in the one-dimensional case one then can argue that $Q^2$ is a relative trace-class perturbation and the Rosenblum-Kato theorem would yield $\sigma_{ac}(-d^2/dx^2 + Q') = \mathbb{R}^+$ for $Q$ being any $L^2(\mathbb{R})$ function (see [8]). In the meantime, this argument does not work in the multidimensional case. Even assuming $|Q(x)| \leq C/(1 + |x|)^{0.5+\varepsilon}, \varepsilon > 0$, one has $|Q(x)|^2$– short-range only and the trace-class argument does not work (see [22], p.22, Problem 2.12). Still, we will consider this case and apply different technique to show the preservation of the a.c. spectrum. But first we want to discuss the following problem. In the one-dimensional case, the positivity of the operators $H = -d^2/dx^2 \pm Q' + Q^2$ on $\mathbb{R}$ follows from the following factorization identity

$$D = \begin{bmatrix} 0 & d/dx + Q \\ -d/dx + Q & 0 \end{bmatrix}, \quad D^2 = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}$$

Operator $D$ corresponds to a certain Krein system [14], which simply makes the one-dimensional scattering theory a branch of the approximation theory, in particular, the theory of orthogonal polynomials. In $d > 1$ case, we don’t know analogous result. Still one can come up with the following substitute. Consider the following operators

$$L = \begin{bmatrix} 0 & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_3} & \partial_{x_2} & \partial_{x_1} & 0 \end{bmatrix}, \quad M_v = \begin{bmatrix} 0 & -v_1 & -v_2 & -v_3 \\ v_1 & 0 & v_3 & -v_2 \\ v_2 & -v_3 & 0 & v_1 \\ v_3 & v_2 & v_1 & 0 \end{bmatrix}$$

acting in, say, $[S(\mathbb{R}^3)]^4$ and $v(x)$ is some real, smooth vector-field. We introduce

$$\mathcal{D} = \begin{bmatrix} 0 & L + M_v \\ L - M_v & 0 \end{bmatrix}$$

The straightforward calculations show that $\mathcal{D}^2$ has $(1,1)$ component in the block representation equal to $H = -\Delta + |v|^2 + \text{div} \, v$. Thus, $H \geq 0$. If $v = \nabla \nu$, then
Lemma 2.1. Assume that \( V = \Delta \nu + |\nabla \nu|^2 \) and the other elements in the first raw/column of \( \mathcal{D}^2 \) are all zero, i.e. \( \mathcal{D}^2 \) is a direct sum of the scalar Schrödinger operator and another operator whose characterization is rather complicated. In fact, the operator \( \mathcal{D} \) has a very nice structure. Consider the unitary matrix \( Y \)

\[
Y = \begin{bmatrix} 0 & U \\ U & 0 \end{bmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & -i & 1 & 0 \\ 0 & -i & -1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix}
\]

Then, \( Y \mathcal{D} Y^{-1} \) has the “Dirac operator” form that allows one to apply the methods of the paper [7]. In particular, it shows that the Schrödinger operator with the potential \( V = \Delta \nu + |\nabla \nu|^2 \) has the Green kernel with certain spatial asymptotics as long as \( |\nabla \nu(x)| < C/(1 + |x|)^{0.5 + \varepsilon}, \varepsilon > 0 \). In this paper, we choose a direct method to study Schrödinger operators with potentials being the divergence of a slowly-decaying vector-field. Notice here that this type of potentials was studied earlier in the papers [17, 9].

2. ASYMPTOTICS OF THE GREEN’S FUNCTION

Let \( 0 < \delta < C \) be fixed. We begin with the following auxiliary results

**Lemma 2.1.** Assume that \( 1 < \rho < 2|x|/3 \). Then

\[
\int_{|y|=\rho} e^{-\delta(|x-y|+|y|)} d\tau_y < C\delta^{-1} \rho e^{-\delta |x|} \tag{5}
\]

\[
\int_{|y|=\rho} e^{-\delta(|x-y|+|y|)} \zeta(x,y) d\tau_y < C\delta^{-1.5} \rho^{0.5} e^{-\delta |x|} \tag{6}
\]

\[
\int_{|y|=\rho} e^{-\delta(|x-y|+|y|)} \zeta^2(x,y) d\tau_y < C\delta^{-2} e^{-\delta |x|} \tag{7}
\]

**Proof.** Without loss of generality, assume \( x = (0,0,|x|) \). Introducing the spherical coordinates \( y_1 = \rho \cos \theta \cos \varphi, y_2 = \rho \cos \theta \sin \varphi, y_3 = \rho \sin \theta \), we get

\[
\rho^2 \int_{-\pi}^{\pi} d\varphi \int_{-\pi/2}^{\pi/2} d\theta \cos \theta \exp \left(-\delta \left[ \rho + \sqrt{|x|^2 + \rho^2 - 2|x|\rho \sin \theta} \right] \right)
\]

\[
< C \rho^2 e^{-|x|} \int_{-\pi/2}^{\pi/2} d\theta \cos \theta \exp \left[-c\delta |x|\rho (|x| - \rho)^{-1} (1 - \sin \theta) \right] < Ce^{-|x|} |x|^{-\rho} \delta^{-\rho}
\]

The estimate (5) is now straightforward. To prove (6) and (7), it suffices to notice that \( \zeta(x,y) \sim \sin(x,y) \) for small \( \zeta(x,y) \).

Let \( |x| > 1 \) and \( \Upsilon = \{ y : |y| > 2|x|/3, |x-y| > 2|x|/3 \} \).

**Lemma 2.2.** The following estimate holds

\[
\int_{\Upsilon} \exp (-\delta (|x-y|+|y|)) dy \leq C\delta^{-3} \exp(-\gamma \delta |x|) \tag{8}
\]

with \( \gamma > 1 \).
The proof repeats the one of lemma 3.4 from [7] and is elementary.

Consider the following class of functions in $\mathbb{R}^3$.

**Definition 2.1.** We say that $\psi(x) \in Cl(k)$, $k \in \mathbb{C}^+$ if $\psi(x) = \exp(ik|x|) [\psi_1(x) + \psi_2(x)]$, where $\psi_{1,2}$ measurable, and

$$|\psi_1(x)| < \frac{1}{|x|^{1.5} + 1}, \quad \text{Group (A)}$$

$$|\psi_2(x)| < \frac{1}{|x| + 1}, |\nabla \psi_2(x)| < \frac{1}{|x|^{1.5} + 1}, \quad \text{Group (B)}$$

Introduce an operator

$$B(k)f(x) = \int_{\mathbb{R}^3} \frac{\exp(ik|x-y|)}{4\pi|x-y|} \text{div}(Q(y))f(y)dy$$

(9)

**Theorem 2.1.** Assume that $Q(x)$ is a vector-field such that

$$|Q(x)| < \frac{m(Q)}{1 + |x|^{0.5}}$$

Then, for

$$|\text{Re} k| < a, 0 < \text{Im} k < b, m(Q) < C(\varepsilon, a, b)|\text{Im} k|^3$$

$B(k)$ acts within the class $Cl(k)$.

Proof. We will always assume that $k : |\text{Re} k| < a, 0 < \text{Im} k < b$. Denote $\text{Im} k = \delta$.

Group (A). Consider $\psi(x) = \exp(ikx)\psi_1(x)$. Let us show that $B(k)\psi$ belongs in group (B). For $x : |x| < 1$, all estimates are easy. Assume that $|x| > 1$. By lemma [22] we have

$$\left| \int_{y \in \mathbb{R}^3} \frac{\exp(ik|x-y|)}{|x-y|} \text{div}(Q(y)) \exp(ik|y|)\psi_1(y)dy \right| < C(a, b)\delta^{-3} m(Q)(1+|x|)^{-3}$$

Then, using (6), we get

$$\left| \int_{|y| < 2|x|/3} \frac{\exp(ik|x-y|)}{|x-y|} \text{div}(Q(y)) \exp(ik|y|)\psi_1(y)dy \right| < C(a, b)\delta^{-3} m(Q)(1+|x|)^{-3}$$

Similarly, making change of variables $y - x = t$, we have

$$\left| \int_{|y-x| < 2|x|/3} \frac{\exp(ik|y-x|)}{|x-y|} \text{div}(Q(t+x)) \exp(ik|t|)\psi_1(t+x)dt \right|$$
$$\left| \int_{|y|<|x|^{1/3}} \frac{x-y}{|x-y|^3} \exp(ik|x-y|) \cdot \text{div}[Q(y)] \cdot \exp(ik|y|) \cdot \psi_1(y)dy \right|$$

$$< \frac{C(a,b)m(Q)}{\delta(1+|x|^2)} \int_0^{2|x|/3} (1+\rho)^{1+\varepsilon}d\rho < \frac{C(a,b,\varepsilon)m(Q)}{\delta(1+|x|^2)};$$

$$\left| \int_{|y-x|<|x|^{1/3}} \frac{\exp(ik|x-y|)\text{div}[Q(y)] \exp(ik|y|)\psi_1(y)dy}{|x-y|^2} \right|$$

$$< \frac{C(a,b)m(Q)}{\delta(1+|x|^2)} \int_0^{2|x|/3} (1+\rho)^{-1}d\rho < \frac{C(a,b,\varepsilon)m(Q)}{\delta(1+|x|^2)}$$

The integral over Υ is less than $C(a,b)m(Q)\delta^{-3}(1+|x|^{-4})$.

The gradient of the exponent yields

$$\left| \int_{|y|<|x|^{1/3}} \frac{\exp(ik|x-y|-ik|x|)}{|x-y|} \left( ik \frac{x-y}{|x-y|} - ik \frac{x}{|x|} \right) \text{div}[Q(y)] \exp(ik|y|)\psi_1(y)dy \right|$$

$$< C(a,b) \left\| \int_{|y|<|x|^{1/3}} \frac{\exp(ik|x-y|-ik|x|)}{|x-y|} \zeta(x-y,x)\text{div}[Q(y)] \exp(ik|y|)\psi_1(y)dy \right\|$$

For $\zeta(\xi_1,\xi_2) < \pi/2$, $\zeta(\xi_1,\xi_2) \sim \sin \zeta(\xi_1,\xi_2)$. Due to sin– theorem, $\sin \zeta(x-y,x) = \sin \zeta(x,y)|x-y|^{-1}|y|$. By (9), (10) is less than

$$\frac{C(a,b)m(Q)}{\delta^{1.5}(1+|x|^2)} \int_0^{2|x|/3} \rho^{-0.5-\varepsilon}d\rho < \frac{C(a,b)m(Q)}{\delta^{1.5}(1+|x|^{1.5+\varepsilon})}$$

Then, 

$$\left| \int_{|y-x|<|x|^{1/3}} \frac{\exp(ik|x-y|-ik|x|)}{|x-y|} \zeta(x-y,x)\text{div}[Q(y)] \exp(ik|y|)\psi_1(y)dy \right|$$

$$< \frac{C(a,b)m(Q)}{\delta^{1.5}(1+|x|^2+\varepsilon)} \int_0^{2|x|/3} \rho^{-0.5}d\rho < \frac{C(a,b)m(Q)}{\delta^{1.5}(1+|x|^{1.5+\varepsilon})}$$

The integral over Υ is smaller than $C(a,b)m(Q)\delta^{-3}(1+|x|^{-3})$. Thus, we showed that $B(k)\psi(x)$ falls into group (B) assuming the corresponding estimate on $m(Q)$. 

Let us show that the gradient of $\exp(-ik|x|)B(k)\psi$ is small. Differentiating $|x-y|^{-1}$ gives a strong decay.

$$\left| \int_{|y|<|x|^{1/3}} \frac{x-y}{|x-y|^3} \exp(ik|x-y|) \cdot \text{div}[Q(y)] \cdot \exp(ik|y|) \cdot \psi_1(y)dy \right|$$

$$< \frac{C(a,b)m(Q)}{\delta(1+|x|^2)} \int_0^{2|x|/3} (1+\rho)^{1+\varepsilon}d\rho < \frac{C(a,b,\varepsilon)m(Q)}{\delta(1+|x|^2)}.$$
Group (B). Consider $\psi(x) = \exp(ik|x|)\psi_2(x)$. Integrating by parts, we have $\exp(-ik|x|)B(k)\psi(x) = I_1 + I_2 + I_3$. $I_1$ is the term with the derivative falling onto $|x - y|^{-1}$, etc.

$$|I_1| < \int_{\mathbb{R}^3} |Q(y)| \frac{\exp[\delta(|x| - |x - y| - |y|)]}{|x - y|^2} |\nabla\psi_2(y)| dy$$

Simple estimates show that

$$|I_1(x)| < \frac{C(a, b, \varepsilon)m(Q)}{\delta^3(1 + |x|^{1.5})}$$

Therefore, $\exp(ik|x|)I_1(x)$ belongs to group (A) for $m(Q)$ sufficiently small. Let us show that $\exp(ik|x|)I_2(3)(x)$ are in the group (B). For $I_2$,

$$\int_{|y| < |x|/3} |Q(y)| \frac{\exp[\delta(|x| - |y| - |x - y|)]}{|x - y|} |\nabla\psi_2(y)| dy < \frac{C(a, b)m(Q)}{\delta(1 + |x|^{2+\varepsilon})} \int_0^{2|x|/3} (1 + \rho)^{-1-\varepsilon} d\rho < \frac{C(a, b)m(Q)}{\delta(1 + |x|^{1+\varepsilon})}$$

The integral over $\Upsilon$ is smaller than $C(a, b)m(Q)^{-3}(1 + |x|^{-3})$. Let us estimate $|\nabla I_2(x)|$. It has two terms. The one containing derivative of $|x - y|^{-1}$ is easy to deal with. It provides, again, the stronger decay at infinity. The other term with the derivative of the exponent can be bounded as follows

$$\int_{|y| < |x|/3} |Q(y)| \frac{|\nabla\psi_2(y)|}{|x - y|} \left(\nabla_x \exp[ik(|x - y| + |y| - |x|)]\right) dy < C(a, b) \int_{|y| < |x|/3} |Q(y)| \frac{1}{|x - y|} |\nabla\psi_2(y)| \zeta(x - y, x) \exp[-\delta(|x - y| + |y| - |x|)] dy$$

The integral over $\Upsilon$ is smaller than $C_2(1 + |x|^{-1})$. Thus, $\exp(ik|x|)I_2(x)$ is in the group (B).
For $I_3$,
\[
\int_{|y|<2|x|/3} |Q(y)| \cdot |\psi_2(y)| \cdot \zeta(y, x - y) \exp \frac{\delta (|x| - |y| - |x - y|)}{|x - y|} dy < \{\text{by the sin theorem}\} \frac{C(a, b)m(Q)}{\delta^{1.5}(1 + |x|)} \int_0^{2|x|/3} (1 + \rho)^{-1/2} d\rho < \frac{C(a, b, \varepsilon)m(Q)}{\delta^{1.5}(1 + |x|)};
\]
\[
\int_{|y - x|<2|x|/3} |Q(y)| \cdot |\psi_2(y)| \cdot \zeta(y, x - y) \exp \frac{\delta (|x| - |y| - |x - y|)}{|x - y|} dy < \frac{C(a, b)m(Q)}{\delta^{1.5}(1 + |x|^{1.5+\varepsilon})} \int_0^{2|x|/3} (1 + \rho)^{-1/2} d\rho < \frac{C(a, b)m(Q)}{\delta^{1.5}(1 + |x|^{1.5+\varepsilon})};
\]

The integral over $\Sigma$ is smaller than $C(a, b)m(Q)\delta^{-3}(1 + |x|)^{-2.5}$. Assume that $\varepsilon > 0$.

Remark. Clearly, the estimates for the integrals over $\Sigma$ can be improved.

Take the gradient of $I_3(x)$. Differentiation of
\[
\frac{1}{|x - y|} \left( \frac{y}{|y|} - \frac{x - y}{|x - y|} \right)
\]
in $x$ gives the term which can be estimated in the standard way by $C(a, b, \varepsilon)\delta^{-3}(1 + |x|)^{-1.5}$.

Taking the derivative of the exponent, we have
\[
\int_{|y|<2|x|/3} |Q(y)| \cdot |\psi_2(y)| \cdot \zeta(y, x - y) \cdot \zeta(x, x - y) \exp \frac{\delta (|x| - |y| - |x - y|)}{|x - y|} dy
\]
\[
< \{\text{by the sin theorem}\} \frac{C(a, b)m(Q)}{\delta^{2}(1 + |x|^2)} \int_0^{2|x|/3} (1 + \rho)^{-0.5} d\rho < \frac{C(a, b)m(Q)}{\delta^{2}(1 + |x|^{1.5+\varepsilon})};
\]
\[
\int_{|y - x|<2|x|/3} |Q(y)| \cdot |\psi_2(y)| \cdot \zeta(y, x - y) \cdot \zeta(x, x - y) \exp \frac{\delta (|x| - |y| - |x - y|)}{|x - y|} dy
\]
\[
< \frac{C(a, b)m(Q)}{\delta^{2}(1 + |x|^{1.5+\varepsilon})} \int_0^{2|x|/3} (1 + \rho)^{-1} d\rho < \frac{C(a, b, \varepsilon)m(Q)}{\delta^{2}(1 + |x|^{1.5})};
\]

The integral over $\Sigma$ is smaller than $C(a, b)m(Q)\delta^{-3}(1 + |x|)^{-2.5}$. Thus, for small $m(Q)$, $\exp(ik|x|)I_3$ falls in group (B) and the proof is finished.

Remark. Clearly, the estimates for the integrals over $\Sigma$ can be improved.

The following two theorems provide an asymptotics of the Green function. Fix any $\varepsilon > 0$.

**Theorem 2.2.** Assume that $Q(x)$ is a vector-field such that
\[
|Q(x)| < \frac{m(Q)}{1 + |x|^{0.5+\varepsilon}}, \quad |\text{div}\{Q(x)\}| < \frac{m(Q)}{1 + |x|^{0.5+\varepsilon}}.
\]
Take $z = k^2, k = \tau + i\delta, 0 < a_1 < \tau < a_2, 0 < \delta < b$. Let $V = \text{div} \{Q\}$ in $|z| > 2$.

If
\[
\delta^3 > C(a_1, a_2, b)m(Q)
\]
then
\[
q(x, y) = \frac{\delta^{2.5}(1 - \delta)}{2\pi |x - y|^{3.5}} (|x| - |y| - |x - y|) \exp \left( \frac{|x| - |y| - |x - y|}{2|x - y|} \right) + O(\delta^3)\delta^{2.5}(1 - \delta)
\]

is a solution of the operator $P_k + \delta^3 Q(x)\partial_k$.
then
\[ |G_z(x, y) - G_0^0(x, y)| < \frac{C(a_1, a_2, b)m(Q)}{\delta^3 - C(a_1, a_2, b)m(Q)} \exp(-\delta|x|) \] (12)
uniformly for |y| < 1, |x| > 1.

Proof. Fix \( \varepsilon > 0 \) and then \( a_1, a_2, b \). Consider the second resolvent identity for \( H \):
\[ (H - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1}V(H - z)^{-1} \]
Therefore,
\[ G_z(x, y) = G_0^0(x, y) - B(k)G_z(\cdot, y) \] (13)
with \( B(k) \) introduced in (11). Consider \( y \) as a parameter, |y| < 1. Iterate (13). It is an easy exercise to show that \( B(k)G_0^0(\cdot, y) \in C\ell(k) \) for small \( m(Q) \). Therefore, (12) follows directly from the theorem 2.1 by summing up the geometric series. ■

**Theorem 2.3.** Assume, again, that \( Q(x) \) is a vector-field such that
\[ |Q(x)| < \frac{C}{1 + |x|^{0.5 + \varepsilon}}, |\text{div} \, Q(x)| < \frac{C}{1 + |x|^{0.5 + \varepsilon}} \]
Take \( z = k^2, k = \tau + i\delta, 0 < a_1 < \tau < a_2, 0 < \delta < b \). Let \( V = \text{div} \, Q \) in (7). Consider any \( f(x) \) with the support inside the unit ball and \( \|f\|_2 < 1 \). Let \( u(x, k) = (H - z)^{-1}f \). The following estimate is true
\[ \limsup_{|x| \to \infty} \left\{ |x| \exp(\delta|x|) \cdot |u(x, k)| \right\} \leq A(\delta) \] (14)
and
\[ A(\delta) \leq \exp \left[ C(a_1, a_2, b)\delta^{-\gamma(\varepsilon)} \right] \] (15)
with \( \gamma(\varepsilon) > 0 \).

Proof. Fix any \( \delta > 0 \). We can control the Green function only if \( m(Q) \) in the theorem 2.2 is relatively small. The idea now is to cut out the big ball of radius \( R(\delta) \) to guarantee that \( m(Q) \) satisfies assumptions of the theorem 2.2 but relative to different \( \varepsilon \), say \( \varepsilon/2 \). Take \( R > 0 \), it will be assigned with the precise value later. Consider the radially-symmetric function \( \chi_R(x) \):
\[ \chi_R(x) = \begin{cases} 
1, & \text{if } |x| < R; \\
0, & \text{if } |x| > R + 1 
\end{cases} \]
We can always assume that \( |\nabla \chi_R| < C \), where \( C \) is independent of \( R \). Write \( V = V_1 + V_2 \) where
\[ V_1(2) = \text{div} \, Q_1(2), Q_1 = \chi_R Q, Q_2 = (1 - \chi_R)Q \]
For \( Q_2 \),
\[ |Q_2(x)| < \frac{CR^{-\varepsilon/2}}{1 + |x|^{0.5 + \varepsilon/2}}, |\text{div} \, Q_2(x)| < \frac{CR^{-\varepsilon/2}}{1 + |x|^{0.5 + \varepsilon/2}} \] (16)
Consider \( H_2 = -\Delta + V_2 \). We have \( u(x, k) = (H_2 - z)^{-1}[f - V_1u] \). Denote the resolvent kernel of \( H_2 \) by \( L_z(x, y) \). Then,
\[ |u(x, k)| < C \int_{|y| < R + 1} |L_z(x, y)| \cdot |u(y, k)| |dy| < C\|u(\cdot, k)\|_2 \left[ \int_{|y| < R + 1} |L_z(x, y)|^2 |dy| \right]^{0.5} \] (17)
Clearly, \( \|u(\cdot,k)\|_2 < C\delta^{-1} \).

Fix any \( y,|y| < R+1 \). The function \( L_z(x,y) = K_{z,y}(x-y,0) \) where \( K_{z,y}(s,t) \) is the resolvent kernel of the Schrödinger operator with the shifted potential \( V_y(s) = V_2(s+y) \). Notice that

\[
|V_y(s)| < \frac{CR^{-\varepsilon/2}}{(1 + |s|)^{0.5+\varepsilon/2}}
\]

where \( C \) is independent of \( y,|y| < R+1 \). Let \( \varepsilon_1 = \varepsilon/2 \). Now, fix \( R \) such that (see (11))

\[
C(a_1,a_2,b)R^{-\varepsilon_1} < \delta^3
\]

We can take \( R = [\delta^3/(2C(a_1,a_2,b))]^{-1/\varepsilon_1} \). Then, the theorem 2.2 is applicable and we have (see (12))

\[
|x| \exp(\delta|x|) \cdot |K_{z,y}(x,0)| < C(a_1,a_2,b),|y| < R+1
\]

Thus

\[
\limsup_{|x| \to \infty} \left\{ |x| \exp(\delta|x|) \cdot |L_z(x,y)| \right\} < C(a_1,a_2,b) \exp[C\delta R],|y| < R+1
\]

and, by (17),

\[
\limsup_{|x| \to \infty} \left\{ |x| \exp(\delta|x|) \cdot |u(x,k)| \right\} \leq C\delta^{-1}R^{1.5} \sup_{|y| < R+1} \limsup_{|x| \to \infty} \left\{ |x| \exp(\delta|x|) \cdot |L_z(x,y)| \right\}
\]

Consequently,

\[
\limsup_{|x| \to \infty} \left\{ |x| \exp(\delta|x|) \cdot |u(x,k)| \right\} < \exp[C\delta^{-\gamma(\varepsilon)}]
\]

with \( \gamma(\varepsilon) > 1 \). \( \blacksquare \)

Remark 1. Modifying slightly the proof of the theorem 2.1 one should be able to drop the condition \(|\text{div } Q(x)| < C(|x| + 1)^{-0.5-\varepsilon}\). The mere boundedness of \( V \) could be enough. The other results of the paper then should also follow.

Remark 2. Perhaps, one can work directly with the equation \(-\Delta u + Vu - zu = f\) to improve an estimate (13). We expect at most polynomial growth for \( A(\delta) \) as \( \delta \to 0 \). It is a reasonable guess that \( \limsup_{|x| \to \infty} |x| \exp(\delta|x|) \cdot \|G_z(|x|,\theta,0)|_{L^2(\theta \in \Sigma)} \) is finite under the condition (4).

3. Absolutely continuous spectrum

We will start with an easy but fundamental factorization identity. Consider \( H \) with compactly supported bounded potential \( V(x) \). Take any \( f(x) \in L^\infty(\mathbb{R}^3) \) with a compact support. Let \( \Pi \) be a rectangle in \( \mathbb{C}^+ \): \( k = r + i\delta, 0 < a_1 < \tau < a_2, 0 < \delta < b \), and \( u(x,k) = (H - z)^{-1}f, z = k^2 \). We have

\[
u(x,k) = \frac{\exp(ikr)}{r} (A(k,\theta) + o(1)),
\]

\[
\frac{\partial u(x,k)}{\partial r} = \frac{ikr}{r} \exp(ikr) (A(k,\theta) + o(1)), \quad (\text{Sommerfeld's radiation conditions})
\]

\[
r = |x|, \theta = \frac{x}{|x|}, |x| \to \infty
\]

The amplitude \( A(k,\theta) \) has the following properties:

- \( A(k,\theta) \) is an analytic in \( k \in \Pi \) vector-function.
- The absorption principle holds, i.e. \( A(k,\theta) \) is continuous on \( \Pi \).
• For the boundary value of the resolvent, we have (22, p.40–42)\[ \text{Im}(R_{\lambda}^2f, f) = k\|A(k, \theta)\|^2_{L^2(S)}; k > 0. \] Therefore, \[ \sigma'_f(E) = k\pi^{-1}\|A(k, \theta)\|^2_{L^2(S)}; E = k^2 \]

where \( \sigma_f(E) \) is the spectral measure of \( f \).

The main result of this section is the following multidimensional version of the corollary on p.181, [5].

**Theorem 3.1.** Let \( Q(x) \) be a vector-field in \( \mathbb{R}^3 \) and

\[ |Q(x)| < \frac{C}{1 + |x|^{0.5+\varepsilon}}, \quad |\text{div} \ Q(x)| < \frac{C}{1 + |x|^{0.5+\varepsilon}}, \quad \varepsilon > 0 \]

Then, \( H = -\Delta + \text{div} \ Q \) has an a.c. spectrum that fills \( \mathbb{R}^+ \).

Proof. Let us fix any interval \( I = [a_1, a_2] \subset \mathbb{R}^+ \) and \( b > 0 \). We will show that \( I \subset \sigma_{ac}(H) \). Following [11], consider an isosceles triangle \( T \) in \( \mathbb{II} \) with the base equal to \( I \) and the adjacent angles both equal to \( \pi/\gamma_1, \gamma_1 > \gamma(\varepsilon) \) with \( \gamma(\varepsilon) \) from [11].

Take \( f(x) \) as any nonzero \( L^\infty(\mathbb{R}^3) \) function supported on the unit ball. Then

\[ A_0(k, \theta) = \lim_{|x| \to \infty} |x| \exp(-ik|x|) \int_{\mathbb{R}^3} \frac{\exp[ik|x-y|]}{4\pi|x-y|} f(y) dy \]

\[ = (4\pi)^{-1} \int_{|y| < 1} \exp[-ik\langle \theta, y \rangle] f(y) dy \]

For the fixed \( \theta' \), \( A_0(k, \theta') \) is entire in \( k \). Therefore, we can find a point \( k_0 = \tau_0 + i\delta_0 \) inside the triangle \( T \) such that \( A_0(k_0, \theta') \neq 0 \). Since \( A_0(k_0, \theta) \) is continuous in \( \theta \),

\[ \|A_0(k_0, \theta)\|_{L^2(\theta \in \Sigma)} > 0 \]

(21)

Fix this \( k_0 \) for the rest of the proof. Consider \( R > 0 \) and the function \( \chi_R(x) \) introduced in the proof of the theorem 2.2. Let, again, \( Q_1 = \chi_R Q, Q_2 = (1 - \chi_R)Q \) and \( V_{1(2)} = \text{div} \ Q_{1(2)} \). Notice that \( V_1 \) is compactly supported. Therefore, by the trace-class argument, \( \sigma_{ac}(-\Delta + V) = \sigma_{ac}(-\Delta + V_2) \). Thus, we can restrict our attention to \( H_2 = -\Delta + V_2 \). For \( V_2 \), we have \( |Q_2(x)| < CR^{-\varepsilon/2}/(1 + |x|)^{0.5+\varepsilon/2} \) and \( |\text{div} Q_2(x)| < CR^{-\varepsilon/2}/(1 + |x|)^{0.5+\varepsilon/2} \). Take \( R \) big enough to have the following estimate

\[ \limsup_{|x| \to \infty} \left\| |x| \exp(-ik_0|x|) \cdot (H_2 - k_0^2)^{-1} f \right\|_{L^2(\Sigma)} > 0 \]

(22)

We can always do that due to theorem 2.2 and 21. Fix this \( R \) and the corresponding \( V_2 \).

Now, take any \( \rho > R + 1 \) and consider \( Q^{(\rho)} = \chi_\rho Q_2, V^{(\rho)} = \text{div} \ Q^{(\rho)} \). Since \( V^{(\rho)} \) is compactly supported, an amplitude \( A_\rho(k, \theta) \) of \( f(x) \) is well defined. Consider the following function \( \nu_\rho(k) = \ln\|A_\rho(k, \theta)\|_{L^2(\Sigma)} \). It is subharmonic in \( T \). Let \( \omega(k_0, s), s \in \partial T \) denote the value at \( k_0 \) of the Poisson kernel associated to \( T \). One can easily show that

\[ 0 \leq \omega(k_0, s) < C|s - s_{1(2)}|^{|\gamma_1|^{-1}}, \quad s \in \partial T \]

(23)
where $s_1(2)$ are endpoints of $I$. By subharmonicity,
\[
\int_{s \in \partial T} \nu_\rho(s) \omega(k_0, s) ds \geq \nu_\rho(k_0)
\]  
(24)

If we denote the edges of $\partial T$ by $I_1(2)$, i.e. $\partial T = I_1 \cup I_2$, then
\[
\int_{s \in I} \omega(k_0, s) \ln \|A_\rho(s, \theta)\|_{L^2(\Sigma)} ds \geq \nu_\rho(k_0) - \int_{s \in I_1 \cup I_2} \omega(k_0, s) \ln \|A_\rho(s, \theta)\|_{L^2(\Sigma)} ds
\]  
(25)

Notice now that (22), theorem 2.3, and (23) yield
\[
\int_{s \in I} \omega(k_0, s) \ln \|A_\rho(s, \theta)\|_{L^2(\Sigma)} ds > C
\]  
(26)

with the constant $C$ independent of $\rho$. Recall the factorization identity (20) to have
\[
\int_{k \in I} \omega(k_0, k) \ln \sigma'_{(f,\rho)}(k^2) dk > C
\]  
(27)

where $\sigma_{(f,\rho)}$ is the spectral measure of $f$ with respect to $-\Delta + V^{(\rho)}$. It is an easy exercise to show that $(-\Delta + V^{(\rho)} - z)^{-1} \rightarrow (-\Delta + V_2 - z)^{-1}$ in the strong sense as $\rho \rightarrow \infty$. Therefore, $d\sigma_{(f,\rho)} \rightarrow d\sigma_f$ in the weak-$\ast$ sense. The usual argument with the semicontinuity of the entropy [10] then implies
\[
\int \ln \sigma'_f(E) dE > -\infty
\]  
(28)

for any subinterval $J \subset I$. Since $I$ was an arbitrary interval in $\mathbb{R}^+$, we have $\sigma_{ac}(H) = \mathbb{R}^+$. ■

The case of radially-symmetric potential shows that a very rich singular spectrum is allowed under the conditions of the theorem [8, 13].

Remark. Consider the short-range potential $V(x)$:
\[
|V(x)| < C/(1 + |x|^{1+\varepsilon}), \varepsilon > 0
\]  
(29)

Then one can easily show that the analogs of theorem 2.1, 2.2, 2.3 hold. That allows us to show that $\sigma_{ac}(-\Delta + V_1 + V_2) = \mathbb{R}^+$ where $V_1$ is from the theorem 3.1 and $V_2$ satisfies 2.3.

4. Applications

In this section, we consider some concrete examples of the potentials.

Example 1. Consider
\[
V(x) = \frac{\sin x_1}{(1 + x_1^2 + x_2^2 + x_3^2)^{\gamma}}, \gamma > 1/4
\]

One can write
\[
V(x) = -\frac{\partial}{\partial x_1} \frac{\cos x_1}{(1 + x_1^2 + x_2^2 + x_3^2)^{\gamma}} + V_2(x)
\]

with $V_2(x)$– short-range. By remark after the theorem, $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$. 
Then, by Theorem 2.2, the asymptotics of the Green function $G_{-k^2}(x,0)$ is similar to the asymptotics of $G_{0-k^2}(x,0)$. Here $k$ is assumed to be sufficiently large. In the meantime, take a point $x$ on the hyperplane $x_1 = \text{const}$ very far from the origin. One can see that the Agmon distance (i.e., solution to the corresponding eikonal equation) from $x$ to the origin will have the WKB-type correction to the usual linear growth. Thus, it is not the solution to the eikonal equation that governs the phase of the Green function.

Talking about the asymptotics of Green’s function, one can suggest the following approach which works sometimes. Let us try to find $u(x,k) = \exp(-k|x| + \mu(x))/|x|$ that solves $-\Delta u + Vu + k^2 u = 0$ for $|x| > 1$. We also assume that $k \in \mathbb{R}^+$ and $k >> 1$. The equation for $\mu$ now reads as follows

$$\Delta \mu + |\nabla \mu|^2 - 2k \frac{\partial \mu}{\partial r} = V + \frac{2}{r} \cdot \frac{\partial \mu}{\partial r}, \ r = |x|$$ (30)

It is an eikonal equation with viscosity, modified by the radial derivative term. Making the following substitution $\mu = r \exp(r)\psi$, one ends up with a simple equation, which yields

$$\mu = -GV + G[|\nabla \mu|^2]$$ (31)

where the operator $G$ is defined as follows

$$Gf(x) = |x| \exp(k|x|) \int_{\mathbb{R}^3} \frac{\exp[-k(|x-y| + |y|)]}{4\pi|x-y| \cdot |y|} f(y)dy$$

If $V$ is such that the gradient of $GV$ is decaying fast, then, one can hope to iterate (31). Then the leading term in the asymptotics of the phase $\mu$ would be $GV$. Unfortunately, this idea does not work unless we assume the strong decay of the derivatives of $V$. As the first example suggests, neither the nonlinear term, nor the viscosity can be discarded in (30). We do not know the right WKB correction to the asymptotics of the Green kernel for potentials $|V(x)| < C/(1 + |x|^{0.5+\varepsilon})$. That is a major problem that prohibits us from proving Simon’s conjecture in its original form. An advantage of the equation (31) is that it contains the potential in $GV$ form only. This function $GV$ is an integral and provides a lot of averaging for $V$. That averaging might be useful for studying the random Schrödinger operators.

Example 2. Consider a smooth vector-field $Q(x)$ supported on the unit ball. Take

$$V(x) = \sum_{j \in \mathbb{Z}^3} a_j v(x - x_j)$$

where $V(x) = \text{div} \ Q(x)$, points $x_j$ are scattered in $\mathbb{R}^3$ such that $|x_k - x_l| > 2, k \neq l$, and $a_j \to 0$ such that $|V(x)| < C/(1 + |x|^{0.5+\varepsilon})$. Then the theorem 3.1 is applicable and $\sigma_{ac}(-\Delta + V) = \mathbb{R}^+$.  

If $V$ is such that the gradient of $GV$ is decaying fast, then, one can hope to iterate (31). Then the leading term in the asymptotics of the phase $\mu$ would be $GV$. Unfortunately, this idea does not work unless we assume the strong decay of the derivatives of $V$. As the first example suggests, neither the nonlinear term, nor the viscosity can be discarded in (30). We do not know the right WKB correction to the asymptotics of the Green kernel for potentials $|V(x)| < C/(1 + |x|^{0.5+\varepsilon})$. That is a major problem that prohibits us from proving Simon’s conjecture in its original form. An advantage of the equation (31) is that it contains the potential in $GV$ form only. This function $GV$ is an integral and provides a lot of averaging for $V$. That averaging might be useful for studying the random Schrödinger operators.
So far, we were able to deal with potentials that can be written as the divergence of a slowly-decaying field. But on the formal level, any function can be written as a divergence of some vector-field, for instance

\[ V(x) = \Delta \Delta^{-1} V = -\text{div} \nabla_x \int_{\mathbb{R}^3} \frac{V(y)}{4\pi |x-y|} dy = \text{div} \int_{\mathbb{R}^3} \frac{x-y}{4\pi |x-y|^3} V(y) dy \]

One can easily show that for continuous \( V \) with compact support, this identity holds true. In the general situation, given \( V(x) \), one might consider the vector-field

\[ Q(x) = \int_{\mathbb{R}^3} \frac{x-y}{4\pi |x-y|^3} V(y) dy \quad (32) \]

try to show that \( Q(x) \) is well defined, satisfies the bound \(|Q(x)| < C/(1 + |x|^{0.5+\epsilon})\), and \( V = \text{div} Q \). Then, as long as \( V(x) \) itself is slowly-decaying, i.e. \(|V(x)| < C/(1 + |x|^{0.5+\epsilon})\), the theorem would yield \( \sigma_{ac}(-\Delta + V) = \mathbb{R}^+ \).

Example 3 (The Anderson model with slow decay). Consider the following model. Take a smooth function \( \phi(x) \) with the support inside the unit ball. Like in the second example, consider

\[ V_0(x) = \sum_{j \in \mathbb{Z}^+} a_j \phi(x - x_j) \]

where the points \( x_j \) are scattered in \( \mathbb{R}^3 \) such that \(|x_k - x_l| > 2, k \neq l\), and \( a_j \to 0 \) in a way that \(|V_0(x)| < C/(1 + |x|^{0.5+\epsilon})\). Let us now “randomize” \( V_0 \) as follows

\[ V(x) = \sum_{j \in \mathbb{Z}^+} a_j \xi_j \phi(x - x_j) \quad (33) \]

where \( \xi_j \) are real-valued, bounded, independent random variables with \( \mathbb{E}[\xi_j^{2k+1}] = 0 \), \( k \in \mathbb{Z}^+ \). Clearly, any even distribution satisfies the last condition.

**Theorem 4.1.** For \( V \) given by (33), we have \( \sigma_{ac}(-\Delta + V) = \mathbb{R}^+ \) almost surely.

**Proof.** Fix any \( x_0 \). By Kolmogorov’s one series theorem, the integral in (32) converges almost surely, i.e.

\[ \int_{|y| < R} \frac{x_0 - y}{4\pi|x_0 - y|^3} V(y)dy \to Q(x_0), R \to \infty \quad (34) \]

for \( \omega \in \Omega, \mathbb{P}[^{\Omega}] = 1 \).

For \( x \) inside a fixed compact \( K \), we use the Lagrange theorem to have

\[ \left| \frac{x_0 - y}{|x_0 - y|^3} - \frac{x - y}{|x - y|^3} \right| < \frac{C(K)}{1 + |y|^3} |y| >> 1 \quad (35) \]

Therefore,

\[ Q(x) = \lim_{R \to \infty} \int_{|y| < R} \frac{x - y}{4\pi|x - y|^3} V(y)dy \]
exists for any $\omega \in \Omega$ and is continuous in $x \in \mathbb{R}^3$. One also has $\text{div } Q(x) = V(x)$. Let us now show that $|Q(x)| < C/(1 + |x|)^{0.5 + \varepsilon_1}$ with probability one for some $\varepsilon_1 > 0$. We can write

$$Q(x) = Q_1(x) + Q_2(x) = \int_{|x-y|<1} \frac{x-y}{4\pi|x-y|^3} V(y)dy + \int_{|x-y|>1} \frac{x-y}{4\pi|x-y|^3} V(y)dy$$

Clearly, for $Q_1(x): |Q_1(x)| < C/(1 + |x|)^{0.5 + \varepsilon}$. As about $Q_2(x)$, we don’t have singularity under the integral anymore and one can easily show that

$$|DQ_2(x)| < C \ln(1 + |x|)/(1 + |x|)^{0.5 + \varepsilon}$$

where $D$ means the differential of any component of $Q_2$. We introduce

$$S_j(x) = \int_{|x-y|>1} \frac{x-y}{4\pi|x-y|^3} \phi(y - x_j)dy$$

Then,

$$\mathbb{E}\left[|Q_2(x)|^2\right] \leq \sum_{j \in \mathbb{Z}^+} a_j^2 \cdot \mathbb{E}[\xi_j^2] \cdot |S_j(x)|^2$$

$$< C \sum_{j \in \mathbb{Z}^+} \frac{a_j^2}{1 + |x - x_j|^4} < C \int_{\mathbb{R}^3} \frac{dy}{(1 + |y|^{1+2\varepsilon})(1 + |x - y|^4)} < C$$

Now, consider the following sum

$$\sum_{k \in \mathbb{Z}^3} |k|^\gamma |Q_2(k)|^{2p}$$

where $|k|^2 = k_1^2 + k_2^2 + k_3^2$. Let us prove it converges almost surely for the suitable choice of $\gamma > 0$ and $p \in \mathbb{N}$. We calculate the expectation

$$\mathbb{E}\left[\sum_{k \in \mathbb{Z}^3} |k|^\gamma |Q_2(k)|^{2p}\right] = \sum_{k \in \mathbb{Z}^3} |k|^\gamma \cdot \mathbb{E}\left[|Q_2(k)|^{2p}\right]$$

Consider

$$\mathbb{E}\left[\langle Q_2(k), Q_2(k)\rangle^p\right]$$

$$\leq \sum_{j_1, \ldots, j_p, m_1, \ldots, m_p} a_{j_1} a_{m_1} \ldots a_{j_p} a_{m_p} \mathbb{E}\left[\xi_{j_1} \xi_{m_1} \ldots \xi_{j_p} \xi_{m_p}\right] \langle S_{j_1}(k), S_{m_1}(k)\rangle \ldots \langle S_{j_p}(k), S_{m_p}(k)\rangle$$

Since all odd moments of $\xi_j$ are zero, $\mathbb{E}\left[\xi_{j_1} \xi_{m_1} \ldots \xi_{j_p} \xi_{m_p}\right]$ is nonzero iff the indices $j_1, \ldots, j_p, m_1, \ldots, m_p$ coincide pairwise. Therefore,

$$\mathbb{E}\left[\langle Q_2(k), Q_2(k)\rangle^p\right] < C(p) \sum_{l_1, \ldots, l_p} a_{l_1}^2 \ldots a_{l_p}^2 |S_{l_1}(k)|^2 \ldots |S_{l_p}(k)|^2$$

where $C(p)$ is a combinatorial factor. Thus, just like in (37),

$$\mathbb{E}\left[|Q_2(k)|^{2p}\right] < C(p) \left[\sum_l a_l^2 |S_l(k)|^2\right]^p < C(p)/(1 + |k|)^p(1+2\varepsilon)$$
So,

\[ E \left[ \sum_{k \in \mathbb{Z}^3} |k|^{\gamma} |Q_2(k)|^{2p} \right] < \infty \]

as long as \( \gamma = p(1 + 2\varepsilon) - 3 - \delta, \delta > 0 \). So, the sequence \( |k|^{\gamma} |Q_2(k)|^{2p} \in \ell^1(\mathbb{Z}^3) \subset \ell^\infty(\mathbb{Z}^3) \) almost surely. That means

\[ |Q_2(k)| < C(1 + |k|)^{-\frac{\gamma}{2p}} = \frac{C}{1 + |k|^{0.5 + \varepsilon - \frac{2\delta}{4p}}} \]

Taking \( p \) big enough, we see that \( |Q_2(k)| < C(1 + |k|)^{-0.5 - \varepsilon_1} \) almost surely. Constant \( C \), of course, is random. But since \( Q_2 \) satisfies an estimate (38), we have

\[ |Q_2(x)| < C/(1 + |x|)^{0.5 + \varepsilon_2}, 0 < \varepsilon_2 < \varepsilon_1 \] (39)

for all \( x \in \mathbb{R}^3 \) with probability one. Thus, the theorem 3.1 is applicable. ■

Remark. The assumption that the odd moments of \( \xi_j \) are zeroes can probably be dropped. In this case, one might have nonzero contribution from factors like \( E \left[ \xi_j^3 \right] \), etc. in the sum (38). Perhaps, the corresponding terms can be estimated as well. We do not pursue it here. The representation of the potential \( V \) as a divergence of slowly-decaying vector-field is a multidimensional phenomena. In dimension one, the argument does not work. Notice that we not only found the support of an a.c. spectrum, but also proved an asymptotics of the Green function for the spectral parameter in the resolvent set. In the discrete setting, Bourgain [3, 4] obtains stronger results for the Anderson model with slow decay. See also the following paper [20].

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