Isomorphisms of fusion subcategories on permutation algebras

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Abstract

We prove two isomorphisms of some subcategories of fusion systems defined on $N$-interior $p$-permutation $G$-algebras by extending the main results obtained by M. E. Harris in [5].

Keywords: finite group, group algebra, block, defect group, $G$-algebra, $p$-permutation, fusion system.

2010 MSC: 20C20

1. Introduction

There are published results obtained in the case of blocks of group algebras that are extended to the general situation of $p$-permutation $G$-algebras ($p$ is a prime and $G$ is a finite group), for instance this is the case of the Brauer pairs of $p$-blocks, that were introduced in this general situation in [2]. The first author introduced the definition of covering points 5.1 on certain $p$-permutation algebras (see Section 5), which was used to extend the Harris-Knörr correspondence. M. E. Harris obtained in [3] two isomorphisms of categories involving the generalized Brauer pairs determined by a $G$-invariant block of $kN$, where $N$ is normal in $G$. The main purpose of this paper is to extend these results to a class of $p$-permutation algebras.

Throughout the paper $k$ is a field of characteristic $p$, not necessarily algebraically closed and, $N$ is a normal subgroup of the finite group $G$ such that $p$ divides $|N|$. Section 2 contains a brief discussion involving saturated triples and states the main result

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February 27, 2023
of this section, Proposition 2.2, which is important for the following sections. Section 3 proves an extension of a useful result, that will be used to show an important ingredient (Corollary 4.3) for the proof of our first main result. Section 4 contains the first main result of the paper, that is Theorem 4.8. This theorem is an extension of [5, Theorem 6] to a class of \( p \)-permutation algebras. Sections 5 and 6 gather information on covering points on \( p \)-permutation algebras. Finally, in Section 7 we prove the last two main results of the paper, namely Theorem 7.5 and Theorem 7.6. In Theorem 7.5 we obtain a bijective correspondence involving covering points of some group acted \( p \)- permutation algebras and Theorem 7.6 is a generalization of [5, Corollary 8].

Extending results from block algebras to \( p \)-permutation algebras is still of interest. For example [1, Part IV, Section 7, Question 5] contains an open question about the realizability of any saturated fusion system: given a saturated fusion system \( \mathcal{F} \) on a finite \( p \)-group \( D \), are there a finite group \( G \), a \( p \)-permutation algebra \( A \), and a primitive idempotent \( b \) of \( A^G \) such that \((A,b,G)\) is a saturated triple (see Section 2) whose corresponding fusion system is isomorphic to \( \mathcal{F} \)? A different motivation for the results in this paper is the fact that we are able to generalize from blocks of group algebras over finite groups to points of some \( G \)-algebras, using heavily techniques different than those of Harris in [5], the result [7, Lemma 8.6.4], which is an important ingredient of [7, Theorem 8.6.3].

For basic facts regarding points on \( G \)-algebras we follow [10] and for the language of Brauer pairs in the context of \( p \)-permutation algebras we refer to [1] and [2]. All algebras that appear in this paper are assumed to be finite dimensional over \( k \). We use the lifting idempotents techniques as presented in [8, Section 3]. \( A^* \) denotes the group of units of a ring \( A \) and we use conjugation on the left: \( a f := a f a^{-1} \) for any \( f \in A, a \in A^* \). Depending on the situation, we sometimes let ”\( \cdot \)” denote the usual multiplication between two elements of a \( k \)-algebra. Most frequently we only use juxtaposition.

We recall the basic facts about points, pointed groups and defect groups attached to pointed groups, for details see [10, Section 13 and Section 18]. For any \( G \)-algebra \( A \) and for any subgroup \( H \) of \( G \), a point of \( A^H \) is the \((A^H)^*\)-conjugacy class of a primitive idempotent of \( A^H \). If \( \alpha \) is a point of \( A^H \), that is \( \alpha = \{aia^{-1} \mid a \in (A^H)^* \} \), the pair \((H,\alpha)\) is called a pointed group. We denote this pair by \( H_{\alpha} \).

A \( p \)-subgroup \( D \) of \( H \) is a defect group of \( H_{\alpha} \), or a defect group of \( \alpha \), if \( \alpha \subseteq A^H_D \) and \( Br^A_D(\alpha) \neq 0 \), here \( Br^A_D \) is the Brauer homomorphism determined by \( D \). If \( \alpha \subseteq A^H \) is a point admitting defect group \( D \) then any \( i \in \alpha \) may be regarded as a point of \( iA^Hi = (iAi)^H \) consisting of a unique element, namely \( \{i\} \). It is clear that \( \{i\} \) has defect group \( D \). In this context we say that \( i \) has defect group \( D \). Note that \( D \) is a defect group of \( \alpha \) if and only if it is a defect group for any \( i \in \alpha \).

Any \( G \)-algebra embedding \( \mathcal{F} : A \to B \) induces an injective map between the points of \( A^H \) into the points of \( B^H \), for any subgroup \( H \) of \( G \). This injective map preserves the
defect (pointed groups) between the corresponding points. For details see [10, p. 58] and [10, Proposition 15.1 (a), (c) ]. We freely use various characterizations of defect groups, see also [10, 18.3 (iv)]. Even though the main results work only in special cases of interior $G$-algebras, for a finite group $G$, we emphasize that each section has its own set of assumptions representing the most general setting for the results developed there.

2. On the definition of saturated triples

First we prove a technical lemma regarding idempotents in a $k$-algebra $A$, which is a consequence of [8, Proposition 3.19].

**Lemma 2.1.** Let $A$ be a unital $k$-algebra. Let $e, f$ be two idempotents of $A$ such that $f \in Z(A)$. The following statements are true.

a) If $f$ is primitive in $A$ and $fe \neq 0$ then $fe = ef = f$;

b) If $e$ is primitive in $A$ and $fe \neq 0$ then $fe = ef = e$.

**Proof.** Fix a decomposition of $e$ into pairwise orthogonal primitive idempotents $e = \sum_{i \in I} i$. According to [8, Proposition 3.19] for any idempotent $f$ (central or non-central) there is an $A$-conjugate $af$ such that $e \cdot af = af \cdot e$. Further, if

$$e \cdot af = af \cdot e \neq 0$$

then

$$e \cdot af = af \cdot e = \sum_{i \in S} i,$$

where $S \subseteq I$ such that

$$\sum_{i \in S} i = e \cdot \left( \sum_{i \in S} i \right) = \left( \sum_{i \in S} i \right) \cdot e,$$

and

$$\sum_{i \in S} i = af \cdot \left( \sum_{i \in S} i \right) = \left( \sum_{i \in S} i \right) \cdot af.$$

Moreover, for any $i \in I \setminus S$ we have $i \cdot af = af \cdot i = 0$. Clearly, if either $e$ or $f$ is primitive in $A$ then $|S| = 1$. Since $f \in Z(A)$ the rest of the proof is immediate.
In this section we let $A$ denote a $p$-permutation $G$-algebra and $c$ denote a primitive idempotent of the subalgebra $A^G$ of $A$. We recall that $(Q,e)$ is an $(A,c,G)$-Brauer pair if $Q$ is a $p$-subgroup of $G$ and $e$ is a primitive idempotent of $Z(A(Q))$ satisfying $\text{Br}_Q^A(c)e \neq 0$. The set of $(A,c,G)$-Brauer pairs is a $G$-poset with the order relation denoted as usual "$\leq$", see [1, Part IV, Definition 2.9]. The definition of a saturated triple $(A,c,G)$, that was first introduced in [6, Introduction p.92], requires $c$ to be a central idempotent of $A$ and, for any $(A,c,G)$-Brauer pair $(Q,e)$, the idempotent $e$ to be primitive in $A(Q)^{C_G(Q,e)}$. The reason for calling $(A,c,G)$ a saturated triple is a consequence of [6, Theorem 1.6]. There, the authors show that if, $(A,c,G)$ is a triple which satisfy conditions (i) and (ii) of [6, Theorem 1.6], then the poset of subpairs, of any maximal $(A,c,G)$-Brauer pair, is a saturated fusion system. Verifying the proof of [6, Theorem 1.6], we noticed that the condition (i) of this theorem, the fact that $c$ is a central idempotent of $A$, is needed only in [6, Lemma 3.2]. However, by Proposition 2.2 below we can now state that the conclusion of [6, Lemma 3.2] still hold by replacing condition (i) of [6, Theorem 1.6] with condition (ii) of [6, Theorem 1.6].

**Proposition 2.2.** Let $A$ be a $p$-permutation $G$-algebra and $c$ be a primitive idempotent of $A^G$. Let $Q$ be a $p$-subgroup of $G$ and $e$ a block of $A(Q)$. If $e$ is a primitive idempotent of $(A(Q))^{C_G(Q,e)}$ then $(Q,e)$ is a $(A,c,G)$-Brauer pair if and only if $\text{Br}_Q^A(c)e = e$.

**Proof.** If $(Q,e)$ is an $(A,c,G)$-Brauer pair then $\text{Br}_Q^A(c)e \neq 0$. The $N_G(Q)$-algebra surjective homomorphism $\text{Br}_Q^A:A^Q \to A(Q)$ gives $\text{Br}_Q^A(A^G) \subseteq A(Q)^{N_G(Q)}$, hence $\text{Br}_Q^A(c) \in (A(Q))^{C_G(Q,e)}$. At this point Lemma 2.1 a) establishes the result.

In conclusion, in our opinion, we can reformulate the definition of a saturated triple by eliminating condition (i) of [6, Theorem 1.6]. The above proposition is also necessary for other results in the sequel of this paper.

3. Generalized Brauer pairs and $p$-permutation $N$-interior $G$-algebras

In this section $A$ is an $N$-interior $p$-permutation $G$-algebra, where $N \leq G$ and $c$ is a primitive idempotent of $A^G$.

Note that, if $(Q,f_Q)$ is an $(A,c,G)$-Brauer pair, since $f_Q$ is a block idempotent of $A(Q)$ and $A(Q)$ in a $C_N(Q)$-interior algebra, we have $f_Q \in A(Q)_{C_N(Q)}$ and that $\text{Br}_Q^A(c)f_Q$ is an idempotent of $(A(Q))^{C_N(Q)}$. The next proposition is an extension of [7, Lemma 8.6.4] to the case of $N$-interior $p$-permutation $G$-algebras. Although the proof is similar to that of [6, Lemma 3.4], for completeness we present it here.
Proposition 3.1. Let \((Q, f_Q)\) denote an \((A, c, G)\)-Brauer pair verifying that \(f_Q\) is primitive in \(A(Q)^{C_N(Q)}\). Consider a subgroup \(H\) of \(G\) with \(C_N(Q) \leq H \leq N_G(Q, f_Q)\), and a \(p\)-subgroup \(S\) of \(G\) such that \(Q \leq S \leq H\). Also let \(f_S\) be a block of \(A(S)\). The following are equivalent:

i) \((S, f_S)\) is an \((A, c, G)\)-Brauer pair with \(Q, f_Q) \leq (S, f_S)\);

ii) \((S, f_S)\) is an \((A(Q), f_Q, H)\)-Brauer pair.

Proof. First we prove "i) \(\Rightarrow\) ii)". Since \(A(Q)^{H} \subseteq A(Q)^{C_N(Q)}\) it follows that \(f_Q\) remains a primitive idempotent in \(A(Q)^{H}\). In [6, Theorem 2.1] it is stated that \((Q, f_Q)\) is the unique Brauer pair included in \((S, f_S)\) and that \(f_Q\) is invariant with respect to the \(S\)-action. That is \(\text{Br}^A_Q(c) f_S = f_S\).

Conversely for "ii) \(\Rightarrow\) i)", by our assumptions we have that \(\text{Br}^A_Q(c) f_Q = f_Q\) and that \(\text{Br}^A_Q(c) f_S = f_S\), see Proposition 2.2. Indeed, \(\text{Br}^A_Q : A(Q)^S \to A(S)\) is a surjective homomorphism of \(k\)-algebras, hence \(\text{Br}^A_Q(Z(A(Q)^S)) \subseteq Z(A(S))\).

The commutativity of the diagram

\[
\begin{array}{ccc}
A^S & \xrightarrow{\text{Br}_Q} & A(Q)^S \\
\text{Br}_S & \\ A(S) & \xleftarrow{\text{Br}^A_S} & A(S) \\
\end{array}
\]

gives

\[
\text{Br}^A_S(c) f_S = \text{Br}^A_S(\text{Br}^A_Q(c)) \cdot \text{Br}^A_S(f_Q) f_S = \text{Br}^A_S(f_Q) f_S = \text{Br}^A_S(f_Q) f_S = f_S
\]

This shows that \((S, f_S)\) is an \((A, c, G)\)-Brauer pair.

We apply [6, Theorem 2.1] again to obtain an \(S\)-invariant block of \(A(Q)\), say \(f'_Q\), such that \((Q, f'_Q) \leq (S, f_S)\) and \(\text{Br}^A_S(f'_Q) f_S = f_S\). If \(f_Q \neq f'_Q\) then

\[
f_S = \text{Br}^A_S(f_Q) f_S = \text{Br}^A_S(f_Q) f'_S f_S = 0,
\]

a contradiction. \(\square\)
4. Isomorphisms of categories of Brauer pairs

In this section we continue to assume that $A$ is a $p$-permutation $N$-interior $G$-algebra, where $N \leq G$. Let $c$ be primitive idempotent of $A^N$ such that $c \in A^G$. In particular $c$ remains a primitive idempotent of $A^G$. For shortness, we say that a $p$-subgroup $Q$ of $N$ is a defect group of $c$ in $N$ if $Q$ is a defect group of $N(c)$, considered as a pointed group on the $G$-algebra $cAc$, see [10, Proposition 18.5]. Similarly we say that a $p$-subgroup $P$ of $G$ is defect group of $c$ in $G$.

For any defect group $Q$ of $c$ in $N$ there is a defect group $P$ of $c$ in $G$ such that $Q = P \cap N$. This claim is well known but, for consistency, we give the justifying details in the next lines. Indeed, if $Q$ is a defect group in $N$ of $c$, then $c \in A^N_Q$ and $Q$ is minimal with this property. Similarly, if $R$ is a defect group of $c$ in $G$ then $R$ is minimal with the property $c \in A^G_R$. Given the decomposition:

$$c = \sum_{x \in [N \setminus G/R]} \text{Tr}^N_{N(\gamma)^R}(x),$$

where $a \in A^R$ (see for details [10, Proposition 11.4]), we obtain $c \in A^N_{N(\gamma)^R}$, for some $x \in G$. Since $\text{Br}^A_R(c) \neq 0$ it follows $\text{Br}^A_{N(\gamma)^R}(c) \neq 0$, hence $\text{Br}^A_{N(\gamma)^R}(c) \neq 0$. So, there is $y \in N$ with $\gamma N = N \cap \gamma R$. It follows $Q = N \cap \gamma R$, $z = y^{-1}x$. We set $P = \gamma R$ and the claim is proved.

4.1. For the rest of this section we fix a $(A,c,G)$-Brauer pair $(Q,f_Q)$ such that $Q$ is a defect group of $c$ in $N$. By the above discussion we choose a maximal $(A,c,G)$-Brauer pair $(P,f_P)$ such that $(Q,f_Q) \leq (P,f_P)$. We assume that $f_Q$ remains a primitive idempotent of $A(Q)^{\mathcal{C}_N(Q)}$ and denote by $H$ the group $N_G(Q,f_Q)$.

Definition 4.2. Define the sets:

$$F_1 := \{(R,f_R)|(R,f_R) \text{ is a } (A,c,G)\text{-Brauer pair such that}$$

$$(Q,f_Q) \leq (R,f_R) \text{ as } (A,c,G)\text{-Brauer pairs}\}$$

$$F_2 := \{(R,f_R)|(R,f_R) \text{ is a } (A(Q),f_Q,H)\text{-Brauer pair such}$$

$$(Q,f_Q) \leq (R,f_R) \text{ as } (A(Q),f_Q,H)\text{-Brauer pairs}\}$$

Corollary 4.3. Let $S$ be a $p$-subgroup of $G$ such that $Q \leq S$ and let $f_S$ be a block of $A(S)$. The following are equivalent:

a) $(S,f_S) \in F_1$;

b) $(S,f_S) \in F_2$. 

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Proof. We first show that $Q \subseteq S$. If b) holds then $S \leq H$ and since $H = N_G(Q, f_Q)$ it follows that $Q$ is normal in $S$. If a) holds, there is $x \in G$ such that 

$$(Q, f_Q) \leq (S, f_S) \leq x(P, f_P),$$

hence $Q \leq S \leq xP$. We have $Q = P \cap N$, thus $Q \leq xP \cap N = xQ$. It follows that $x \in N_G(Q)$ and then $Q \leq x^{-1}S \leq P$. Since $Q$ is normal in $P$ it follows that $x^{-1}Q$ is normal in $S$, thus $Q \subseteq S$.

The rest of the proof follows from Proposition 3.1.

\[\square\]

Proposition 4.4. Let $S, T$ be two p-subgroups of $G$ such that $S \leq T$ and let $f_S, f_T$ denote blocks of $A(S), A(T)$, respectively.

The following are equivalent:

1) $(S, f_S) \in F_1$ such that $(S, f_S) \leq (T, f_T)$;

2) $(S, f_S) \in F_2$ such that $(S, f_S) \leq (T, f_T)$.

Proof. According to Corollary 4.3 any pair $(S, f_S)$ is in $F_1$ in and only if it is in $F_2$, hence we only need to prove that $(S, f_S) \leq (T, f_T)$ in $F_1$ if and only if $(S, f_S) \leq (T, f_T)$ in $F_2$.

Let $e$ denote a primitive idempotent appearing in a pairwise orthogonal decomposition of $f_T$ into primitive idempotents in $A(T)$. By lifting idempotents, there exists a primitive idempotent $i \in A^T$ such that $e = Br^A_T(i)$, hence $Br^A_T(i) f_T = Br^A_T(i)$. By our assumption we also have $Br^A_S(i) f_S \neq 0$. Since $i$ is primitive in $A^T$, we obtain that $j := Br^A_Q(i)$ is also a primitive idempotent of $A(Q)^T$. We have

$$0 \neq Br^A_T(i) f_T = Br^A_T(Q)(j) f_T.$$ 

Similarly we obtain

$$0 \neq Br^A_S(i) f_S = Br^A_S(Q)(Br^A_Q(i)) f_S.$$ 

This proves that $(S, f_S) \leq (T, f_T)$ as elements of $F_2$, according to [2, Corollary 1.9].

Conversely, considering a similar decomposition of $f_T$ in $A(T)$, we find a primitive idempotent $e$ with $f_T e = f_T$. Let $j$ be such that $Br^A_T(Q)(j) = e$ and let $i$ be such that $Br_T(i) = e$. Then we may assume $i$ and $j$ to be primitive idempotents in $A^T, A(Q)^T$, respectively. Further we get $Br^A_Q(i) = a j$, for some $a \in (A(Q)^T)^*$. We obtain:

$$Br^A_T(i) f_T = Br^A_T(Q)(a j) f_T = f_T \neq 0.$$ 

Again $Br^A_T(i) \neq 0$ implies $Br^A_S(i) \neq 0$ and then

$$0 \neq Br^A_S(Q)(a j) f_S = Br^A_S(i) f_S.$$ 

This concludes the proof.
Corollary 4.5. Let \((P, f_P)\) be an \((A, c, G)\)-Brauer pair such that \((Q, f_Q) \leq (P, f_P)\). Then \((P, f_P)\) is a maximal \((A, c, G)\)-Brauer pair if and only if it is a maximal \((A(Q), f_Q, H)\)-Brauer pair, where \(H = N_G(Q, f_Q)\).

Let \((U, f_U)\) be a maximal \((A, c, G)\)-Brauer pair. Then \(\mathcal{F}(U, f_U)(A, c, G)\) is a finite category, called fusion system, introduced in [6, Definition 2.3]. It has as objects the \((A, c, G)\)-Brauer pairs \((R, f_R)\) such that \((R, f_R) \leq (U, f_U)\) and the morphisms are described by

\[
\text{Hom}_{\mathcal{F}(U, f_U)(A, c, G)}(S, T) = \{c_x : S \to T | x \in G, \ x(S, f_S) \leq (T, f_T)\},
\]

with \(c_x(u) = xux^{-1}\) for any \(u \in S\) and \(S, T \leq U\). Similarly, we introduce the fusion system \(\mathcal{F}(U, f_U)(A(Q), f_Q, H)\).

Proposition 4.6. Let \((P, f_P)\) be a maximal \((A, c, G)\)-Brauer pair such that \((Q, f_Q) \leq (P, f_P)\). If \((S, f_S) \in F_1\) and \((T, f_T)\) is a Brauer pair belonging to \(\mathcal{F}(P, f_P)(A, c, G)\) then

\[
\text{Hom}_{\mathcal{F}(P, f_P)(A, c, G)}(S, T) = \text{Hom}_{\mathcal{F}(P, f_P)(A(Q), f_Q, H)}(S, T).
\]

Proof. Similar to the proof of [5, Proposition 5].

We define two subcategories of the above fusion systems and, in the first main result of this paper, we show that these two categories are isomorphic.

Definition 4.7. The full subcategory of \(\mathcal{F}(P, f_P)(A, c, G)\) whose objects are those of \(F_1\) is denoted by \(\mathcal{C}\).

The full subcategory of \(\mathcal{F}(P, f_P)(A(Q), f_Q, H)\) whose objects are those of \(F_2\) is is denoted by \(\mathcal{D}\).

The proof of the first main result is immediate by collecting the results of this section, applied to the categories given in Definition 4.7.

Theorem 4.8. Let \(A\) be an \(N\)-interior \(p\)-permutation \(G\)-algebra, where \(N \leq G\) and let \(c\) be primitive idempotent of \(A^N\) such that \(c \in A^G\). Let \((Q, f_Q)\) be an \((A, c, G)\)-Brauer pair such that \(Q\) is a defect group of \(c\) in \(N\). We fix a maximal \((A, c, G)\)-Brauer pair \((P, f_P)\) such that \((Q, f_Q) \leq (P, f_P)\) and we assume that \(f_Q\) is primitive in \(A(Q)^{C_N(Q)}\). Then, the categories \(\mathcal{D}\) and \(\mathcal{C}\) are isomorphic.

5. Covering points revisited

As in Section 4, \(N\) is a normal subgroup of the finite group \(G\). We consider \(A\), a \(p\)-permutation \(G\)-algebra, and we assume that \(A\) contains a \(G\)-invariant \(k\)-subalgebra \(C\) such that \(1_A \in C\). We recall the definition of covering points that was introduced in [3].
Definition 5.1. [3, Definition 3.2] Let \( \alpha \subseteq A^G \) be a point and let \( \beta \subseteq C^N \) be a point with defect group \( Q \). We say that the point \( \alpha \) covers the point \( \beta \), if the following conditions are satisfied:

a) the point \( \alpha \) admits a defect group \( D \) such that \( D \cap N = Q \);

b) for any \( i \in \alpha \) there is an idempotent \( j_1 \) of \( A^N \), such that \( j_1 = a j \) for some \( j \in \beta, a \in (A^N)^* \), and there is a primitive idempotent \( f \in A^N \) belonging to a point with defect group \( Q \) such that \( j_1 f = fj_1 = f \) and \( if = fi = f \).

The following are useful characterizations of the above definition.

Proposition 5.2. The point \( \alpha \subseteq A^G \) covers the point \( \beta \subseteq C^N \), \( (Q \) is a defect group of \( \beta \)) if and only if for any idempotents \( i \in \alpha, j \in \beta \) the following conditions are satisfied:

a') the idempotent \( i \) admits a defect group \( D \) such that \( D \cap N = Q \);

b') there is a primitive idempotent \( f_1 \in A^N \) with defect group \( Q \) such that

\[
i \cdot a f_1 = a f_1 \cdot i = a f_1 \text{ and } j f_1 = f_1 j = f_1 \text{ for some } a \in (A^N)^*.
\]

Proof. Assume that \( \alpha \) covers \( \beta \). Let \( i \in \alpha, j \in \beta \). Then condition a') is verified. For the chosen \( i \in \alpha, j \in \beta \) there is \( j_1 = a j \in \beta, a_1 \in (A^N)^* \) such that \( j_1 f = fj_1 = f \) and \( if = fi = f \) for some \( f \in \delta \), where \( \delta \subseteq A^N \) is a point of defect \( Q \). We obtain

\[
j \cdot a^{-1} f = a^{-1} f \cdot j = a^{-1} f, \quad if = fi = f,
\]

hence, if we denote by \( f_1 \) the primitive idempotent \( a^{-1} f \) of \( A^N \) we obtain

\[
jj_1 = f_1 j = f, \quad i \cdot a f_1 = a f_1 \cdot i = a f_1.
\]

Assume a') and b') are satisfied for some \( i \in \alpha, j \in \beta \), hence condition a) of Definition 5.1 is verified. By b') there is a point \( \delta \subseteq A^N \) with defect \( Q \) and \( f_1 \in \delta \) such that

\[
i \cdot a f_1 = a f_1 \cdot i = a f_1, \quad jj_1 = f_1 j = f_1,
\]

for some \( a \in (A^N)^* \). But that means \( a j \cdot a f_1 = a f_1 \cdot a j = f_1 \). We denote by \( f \) the idempotent \( a f_1 \in \delta \).

Note that \( a j \) is in the \( (A^N)^* \)-conjugacy class of \( \beta \) so there is \( j_1 \), a primitive idempotent of \( A^N \), such that \( j_1 f = fj_1 = f \).
We introduce the next definition of covering between points, which is a particular case of Definition 5.1. Note that this definition is also equivalent to the usual cover in case of blocks of group algebras that cover a $G$-invariant block of $kN$.

**Definition 5.3.** Let $\alpha$ denote a point of $A^G$ and $\beta \subseteq C^N$ be a point with defect group $Q$ such that $\beta \cap C^G \neq \emptyset$. We say that $\alpha$ *strongly covers* $\beta$ if, for any $j \in \beta \cap C^G$ there is $i \in \alpha$ and there is a point $\varepsilon \subseteq A^N$ with defect group $Q$ such that:

$$a'') \quad ji = ij = i;$$

$$b'') \quad if = fi = f, \quad jf_1 = f_1j = f_1 \text{ for some } f, f_1 \in \varepsilon.$$

**Proposition 5.4.** Let $\alpha$ denote a point of $A^G$ and $\beta \subseteq C^N$ be a point with defect group $Q$ such that $\beta \cap C^G \neq \emptyset$. The following statements are true:

(i) if $\alpha$ strongly covers $\beta$ then $\alpha$ covers $\beta$;

(ii) $\alpha$ strongly covers $\beta$ if and only if for any $j \in \beta \cap C^G$ there is $i \in \alpha$ such that $i \in (jA^G_j)$ and there is a point $\varepsilon \subseteq (jA^G_j)^N$ with defect group $Q$ such that $if = fi = f$, for some $f \in \varepsilon$.

**Proof.** (i) We assume that $\alpha$ strongly covers $\beta$ and let $i_1 \in \alpha, j_1 \in \beta$. By assumption we can choose $j \in \beta \cap C^G$ and $i \in \alpha$ such that statements $a'')$ and $b'')$ of Definition 5.3 hold for some point $\varepsilon \subseteq A^N$ with defect group $Q$. Thus, there exist $f, f_1 \in \varepsilon$ such that

$$if = fi = f, \quad jf_1 = f_1j = f_1.$$

Since there is $a_1 \in (A^G)^*$ (which is included in $(A^N)^*$) and there is $b_1 \in (C^N)^*$ (which is included in $(A^N)^*$) such that $i_1 = a_1i, j_1 = b_1j$, we obtain

$$i_1(a_1f) = (a_1f)i_1 = a_1f, \quad j_1(b_1f_1) = (b_1f_1)j_1 = f_1.$$

We claim that $f' := b_1f_1$ is the primitive idempotent (with defect group $Q$) which satisfies condition $b'$) of Proposition 5.2. Let $a_2 \in (A^N)^*$ such that $f = a_2f_1$. The claim is true because

$$a_1a_2b_1^{-1} f' = a_1a_2f_1 = a_1f \quad \text{and} \quad j_1f' = f'j_1 = f',$$

where $a_1a_2b_1^{-1} \in (A^N)^*$.

Since clearly $ij = ji = i$, we show next that there is a defect group $D$ of $i$ (hence of $\alpha$) which satisfies $D \cap N = Q$, i.e. we show that condition $a'$) of Proposition 5.2 is verified. Let $R$ be a defect group of $\alpha$. There is $a \in A^R$ such that

$$i = Tr^G_R(a) = \sum_{x \in [N \setminus (G/R)]} Tr^N_{N \setminus R}(x a) \in \sum_{x \in [N \setminus (G/R)]} A^N_{N \setminus R}.$$
Since $if = fi = f$ the idempotent $f$ belongs to some ideal $A^N_{N \cap \gamma R}$, implying $^yQ \leq N \cap \gamma R$ for some $y \in N$. Then $Q \leq N \cap \gamma R$ for some $g \in G$. Let $T$ denote a defect group of $j$ regarded as an idempotent of $A^G$. Then $j \in C^G_j \subseteq A^G_j$. We also have $ij = ji = i$ in $A^G$, hence we may assume $R \leq T$. Next, there is $b \in C^T$ with

$$j = Tr^G_T(b) = \sum_{y \in [N \setminus G/T]} Tr^N_{N \cap \gamma T}(^yb) \in \sum_{y \in [N \setminus G/T]} C^N_{N \cap \gamma T}$$

By our assumption $j \in C^N_{N \cap \gamma T}$ for some $y \in G$. Since $Br_T(j) \neq 0$ we get $Br_{N \cap \gamma T}(j) \neq 0$. This means $^tQ = N \cap ^yT$ for some $t \in N$, equivalently $Q = N \cap \gamma T$ for some $z \in G$. Finally

$$Q \leq N \cap \gamma R \leq N \cap \gamma T = \gamma z^{-1}Q,$$

forcing $gz^{-1} \in N_G(Q)$ and

$$Q = N \cap \gamma R = N \cap \gamma T.$$

(ii) First we assume that $\alpha$ strongly covers $\beta$ and let $j \in \beta \cap C^G$. Then, by $a''$) of Definition 5.3 there is $i \in \alpha$ such that $ij = ji = i$, which is equivalent to $i \in (jA_j)^G$. Using Definition 5.3, $b''$) there is a point $\varepsilon' \subseteq A^N$ with defect group $Q$ such that for some $f, f_1 \in \varepsilon'$ we have

$$jf_1 = f_1j = f_1 \quad if = fi = f.$$

It follows

$$jf = j(if) = (ji)f = if = (fi)j = fj = f,$$

which is equivalent to the fact that $f \in (jA_j)^N$ remains a primitive idempotent in $(jA_j)^N$. Similarly $jf_1 = f_1j = f_1$ is equivalent to the fact that $f_1$ is a, primitive idempotent, in $(jA_j)^N$. Let $\varepsilon \subseteq (jA_j)^N$ be the point corresponding to $f$ (i.e. $\varepsilon := \{afar^{-1} | a \in ((jA_j)^N)^*\}$) and let

$$F : (jA_j)^N \to A^N$$

be the embedding determined by the inclusion. Then $\varepsilon'$ is the unique point of $A^N$ corresponding to $\varepsilon$ through $F$. Since $f, f_1 \in \varepsilon'$ and $f \in \varepsilon$ it follows that $f_1 \in \varepsilon$. Clearly $\varepsilon$ has the same defect $Q$ as $f$.

For the converse implication, fix $j \in \beta \cap C^G$. Then there is $i \in \alpha$ such that $i \in (jA_j)^G$ and there is a point $\varepsilon \subseteq (jA_j)^N$ with defect group $Q$ satisfying $if = fi = f$ for some $f \in \varepsilon$. Then statement $a''$ of Definition 5.3 is verified. Let $F$ be the same embedding as above and set $\varepsilon' = F(\varepsilon)$. Then $\varepsilon'$ is a point of $A^N$ with defect group $Q$ and $f \in \varepsilon'$ satisfying $jf = fj = f$. 

\[\Box\]
6. On pairs determined by covering points

In this section we assume that $N$ is a normal subgroup of $G$, that $A$ is a $p$-permutation $G$-algebra which contains $C$ and $A$ is also an $N$-interior $G$-subalgebra such that $1_A \in C$. Let $c$ be primitive idempotent of $C^N$ such that $c \in C^G$ and let $\beta \subseteq C^N$ be the point containing $c$. We fix $Q$, a defect group of $N_\beta$. Since $C$ remains a $p$-permutation $G$-algebra, see [10, Corollary 27.2], we choose a maximal $(C,c,N)$-Brauer pair $(Q,f_Q)$ such that $f_Q$ remains primitive in $C(Q)^{C_N(Q)}$. It follows that we are in the setup of Section 3 and of Section 4, hence we can introduce $F_1$ and $F_2$ and the conclusion of Theorem 4.8 holds for $(C,c,G)$.

**Proposition 6.1.** Let $\alpha \subseteq A^G$ be a point that strongly covers the point $\beta$. Then there is a defect group $D$ of $\alpha$ such that $D \cap N = Q$ and a block $f_D$ in $C(D)$ such that $(D, f_D) \in F_1$.

**Proof.** By Definition 5.1 we have a defect group $F$ hence we can introduce $G$ that $Br_6$. On pairs determined by covering points.

We have the inclusion $C(Q)^{Ng(Q)} \subseteq C(Q)^{C_N(Q)}$, hence since $Br_Q(c) \in C(Q)^{Ng(Q)}$ it follows that $Br_Q(c)$ is an idempotent of $C(Q)^{C_N(Q)}$ satisfying $Br_Q(c)f_Q \neq 0$. Since $f_Q$ is primitive in $C(Q)^{C_N(Q)}$ we apply Lemma 2.1 to obtain (1).

For any $x \in Ng(Q)$, (1) yields

$$Br_Q^C(c) \cdot x f_Q = x f_Q \cdot Br_Q^C(c) = x f_Q,$$

an equation viewed in $C(Q)$. If we set $H = Ng(Q, f_Q)$ then $s := \sum_{x \in Ng(Q)/H} x f_Q$ is an idempotent lying in $C(Q)^{Ng(Q)} \cap Z(C(Q))$ such that

$$Br_Q^C(c) = Br_Q^C(c)s = s \quad Br_Q^C(c) = s$$

in $C(Q)^{Ng(Q)}$. The equality on the right follows from (2). The equality on the left holds since $Br_Q^C(c)$ is a primitive idempotent of $C(Q)^{Ng(Q)}$ and by applying Lemma 2.1 In particular $s = Br_Q^C(c)$ lies in

$$Z(C(Q)) \cap C(Q)^R \subseteq Z(C(Q)^R).$$

Since $\alpha$ strongly covers $\beta$, by Definition 5.3 there is $i \in \alpha$ with $ic = i = ci$. It follows that $Br^K_Q(c) \neq 0$. The idempotent $Br_Q^C(c)$ belongs to $Z(C(R))$, since $Br_Q^C(c)$ is in $Z(Q)$ and $Br_Q^C = Br_Q^C(c) \circ Br_Q^C |_{Ch}$ is a composition of surjective homomorphisms of $k$-algebras. Let
Consider $c = \sum_{j \in J} j$, a decomposition of $c$ into a sum of pairwise orthogonal primitive idempotents of $C^R$. There is $j \in J$ that satisfies the relations $Br_R^C(j) \neq 0$ and $c_j = jc = j$. The idempotent

$$Br_R^C(j) = Br_R^C(c_j) = \sum_{f \in \mathcal{B}} f Br_R(j)$$

is primitive in $C(R)$ and there is a unique block $\overline{f_R} \in \mathcal{B}$ such that

$$Br_R(j) = Br_R^C(j) \overline{f_R} = \overline{f_R} Br_R(c).$$

Since $Br_R^C(j) \neq 0$ then $Br_Q^C(j) \neq 0$ and since $j = jc$ we also have

$$Br_Q^C(j) Br_Q^C(c) = Br_Q^C(j)s \neq 0.$$

This last relation determines at least one block, say $x f_Q$ (which appears in $s$), with $Br_Q^C(j) x f_Q \neq 0$. According to [2, Corollary 1.9] the blocks $\overline{f_R}$ of $C(R)$ and $x f_Q$ of $C(Q)$ must verify $(Q, x f_Q) \leq (R, \overline{f_R})$. Because $x \in N_G(Q)$, this becomes $(Q, f_Q) \leq (x^{-1} R, x^{-1} \overline{f_R})$. We set $D := x^{-1} R$ and $f_D := x^{-1} \overline{f_R}$ to conclude the proof.

7. Isomorphism of fusion subcategories

Throughout this section we use the following setup.

**Setup 7.1.** We consider a $G$-interior $k$-algebra $A$, with structural map $G \to A^*$ such that

$$G \ni u \mapsto u \cdot 1_A = 1_A \cdot u \in A,$$

for any $u \in G$. This makes $A$ into a $k[G \times G]$-module, explicitly

$$(u, v) \cdot a = u \cdot a \cdot v^{-1} \in A,$$

for any $u, v \in G, a \in A$. In particular we assume that this action stabilizes a $k$-basis of $A$, so that $A$ is also a $p$-permutation $G$-algebra. We denote by $C$ an $N$-interior $G$-subalgebra of $A$ that contains the identity element $1_A$. Let $c$ denote a primitive idempotent of $C^N$ such that $c \in C^G$ and let $\beta \subseteq C^N$ be the point containing $c$. We fix $Q$, a defect group of $N_{\beta}$. As in Section 6, we consider $(Q, f_Q)$, a maximal $(C, c, N)$-Brauer pair such that $f_Q$ remains primitive in $C(Q)^{C_N(Q)}$. Further we set $H = N_G(Q, f_Q)$. 

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Let $X := N^\text{Aut}(Q)(Q)$ denote the extended Brauer quotient, the $N_G(Q)$-interior algebra constructed in [9]. Note that $C(Q)$ is an $N_N(Q)$-interior $N_G(Q)$-subalgebra of $X$ and if $A = kG$ then [4, Remark 4.5, 2]) states $X = kN_G(Q)$.

We add the following assumption on $f_Q$. We assume that $f_Q$ normalizes a $k$-basis $\mathcal{B}$ of $X^{N_G(Q)}$. Explicitly for any $s \in \mathcal{B}$ we have $f_Q \cdot s = s' \cdot f_Q$ for some $s' \in \mathcal{B}$. Since $f_Q \in Z(C(Q))$ the point of $C(Q)^{C_N(Q)}$ containing $f_Q$ has a unique element. We denote this point by $\{f_Q\}$. We also assume that $A$ is projective as a $k[G \times 1]$-module and as a $k[1 \times G]$-module.

**Lemma 7.2.** Any point of $X^H$ that strongly covers $\{f_Q\}$ has a defect group $D$ such that $D \cap N = Q$ and, there is a block $f_D$ of $C(D)$ such that $(Q, f_Q) \leq (D, f_D)$ as $(C(Q), f_Q, H)$-Brauer pairs.

**Proof.** Let $\mathcal{F} \subset X^H$ be a point that strongly covers $\{f_Q\} \subset C(Q)^{C_N(Q)}$. Let $D$ denote a defect group of $\mathcal{F}$, hence $D \cap QC_N(Q) = Q$. There is $\tilde{j} \in \mathcal{F}$ such that

$$\tilde{j} \in (f_Q X f_Q)^{H}_D \subseteq \sum_{x \in [N_N(Q, f_Q)]/H[D]} (f_Q X f_Q)^{N_N(Q, f_Q)}_{N_N(Q, f_Q) \cap H[D]}.$$

The block $f_Q$ of $C(Q)$ remains a primitive idempotent of $C(Q)^L$, for any subgroup $L$ of $H$ containing $QC_N(Q)$. In particular $\{f_Q\}$ is a point of $C(Q)^{N_N(Q, f_Q)}$ with defect group $Q$. Any decomposition of $\tilde{j}$ in $(f_Q X f_Q)^{N_N(Q, f_Q)}$ contains at least one idempotent that belongs to a point, say $\mathcal{E} \subset (f_Q X f_Q)^{N_N(Q, f_Q)}$, with defect group $N_N(Q, f_Q) \cap \mathcal{F} D$, for some $x \in H$. Since $\mathcal{E}$ is a subset of $(f_Q X f_Q)^{N_N(Q, f_Q)}$ and, since $Q \subseteq N_N(Q, f_Q) \cap \mathcal{F} D$ we must have $D \cap N_N(Q, f_Q) = Q$. It follows that $\mathcal{E}$ strongly covers $\{f_Q\} \subset C(Q)^{N_N(Q, f_Q)}$ and this forces $Q = D \cap N_N(Q, f_Q) = D \cap N_N(Q) = D \cap N$.

We apply Proposition [6.1] with $H, QC_N(Q), X, C(Q)$ and $\{f_Q\}$, in place of $G, N, A, C$ and $\beta$ respectively. \hfill $\Box$

**Lemma 7.3.** The following statements hold:

a) The $k$-algebras $\text{Br}_Q^C(c)X^{N_G(Q)} \text{Br}_Q^C(c)$ and $(f_Q X f_Q)^H$ are isomorphic;

b) The above isomorphism determines a bijective correspondence between the points $\mathcal{F} \subset X^{N_G(Q)}$ such that for some $\tilde{i} \in \mathcal{F}$ we have $\text{Br}_Q^C(c) \cdot \tilde{i} = \tilde{i}$. $\text{Br}_Q^C(c)$ and the points $\mathcal{F} \subset X^H$ such that for some $\tilde{j} \in \mathcal{F}$ we have $\tilde{j} \cdot f_Q = f_Q \cdot \tilde{j} = \tilde{j}$. This bijection preserves the defect groups in the sense that any defect group of $\mathcal{F}$ is also a defect group of $\tilde{\alpha}$ and conversely, some $N_G(Q)$-conjugate of any defect group of $\tilde{\alpha}$ is a defect group of $\mathcal{F}$.  

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Proof. a) The map
\[
\text{Tr}_{H}^{N_{G}(Q)} : (f_{Q}Xf_{Q})^H \to (f_{Q}Xf_{Q})_{H}^{N_{G}(Q)},
\]
\[
(f_{Q}Xf_{Q})^H \ni a \mapsto \text{Tr}_{H}^{N_{G}(Q)}(a) := \sum_{x \in [N_{G}(Q)/H]} x \alpha \in (f_{Q}Xf_{Q})_{H}^{N_{G}(Q)},
\]
is a k-algebra isomorphism. It is easy to verify that an inverse map of \( \text{Tr}_{H}^{N_{G}(Q)} \) maps b \( \in (f_{Q}Xf_{Q})_{H}^{N_{G}(Q)} \) to \( f_{Q}b \) \( \in (f_{Q}Xf_{Q})^H \) and that \( \text{Tr}_{H}^{N_{G}(Q)} \) is an algebra homomorphism. Next we show that \( (f_{Q}Xf_{Q})_{H}^{N_{G}(Q)} \) is an ideal of \( B_{Q}(c)X^{N_{G}(Q)}B_{Q}(c) \). Let \( a_{1} \in f_{Q}Xf_{Q} \) and let \( a_{2} \in B_{Q}(c)X^{N_{G}(Q)}B_{Q}(c) \). We have
\[
\text{Tr}_{H}^{N_{G}(Q)}(a_{1}) \cdot a_{2} = \text{Tr}_{H}^{N_{G}(Q)}(a_{1}a_{2}) = \text{Tr}_{H}^{N_{G}(Q)}(f_{Q}a_{1}f_{Q}a_{2}B_{Q}(c))
\]
\[
= \text{Tr}_{H}^{N_{G}(Q)}(f_{Q}a_{1}a_{2}f_{Q}B_{Q}(c)) = \text{Tr}_{H}^{N_{G}(Q)}(f_{Q}a_{1}a_{2}')f_{Q},
\]
where \( a_{2}' \) is an element of \( X^{N_{G}(Q)} \) such that \( f_{Q}a_{2} = a_{2}'f_{Q} \). Since
\[
B_{Q}(c) = \text{Tr}_{H}^{N_{G}(Q)}(f_{Q}) \in (f_{Q}Xf_{Q})_{H}^{N_{G}(Q)}
\]
we obtain
\[
(f_{Q}Xf_{Q})_{H}^{N_{G}(Q)} = B_{Q}(c)X^{N_{G}(Q)}B_{Q}(c).
\]
b) The \( N_{G}(Q)-\)interior algebra embedding \( B_{Q}(c)XB_{Q}(c) \to X \), the \( H-\)interior algebra embedding \( f_{Q}Xf_{Q} \to X \) and \( a \) determine the mentioned bijection. We only need to verify the second part of the assertion.

Consider a point \( \overline{x} \subseteq X^{N_{G}(Q)} \) and let \( \overline{y} \subseteq X^{H} \) be the point corresponding to \( \overline{x} \). The mentioned embeddings determine unique points \( \overline{c}_{1} \) and \( \overline{y}_{1} \), corresponding to \( \overline{x} \) and \( \overline{y} \) respectively, such that \( \text{Tr}_{H}^{N_{G}(Q)}(\overline{y}_{1}) = \overline{c}_{1} \) and \( f_{Q} \cdot \overline{c}_{1} = \overline{y}_{1} \).

It suffices to show that \( \overline{c}_{1} \) and \( \overline{y} \) verify the statement on defect groups. Let \( D \) be a defect group of \( \overline{y} \), then \( \overline{c}_{1} \subseteq X^{N_{G}(Q)} \). Assuming by contradiction that \( \overline{c}_{1} \subseteq \sum_{T < D} X_{T}^{D} \) we obtain \( f_{Q} \cdot \overline{c}_{1} \subseteq \sum_{T < D} X_{T}^{D} \). If \( R \leq N_{G}(Q) \) is a defect group of \( \overline{c}_{1} \) it is clear that \( xR = D \) for some \( x \in N_{G}(Q) \). \( \square \)

Lemma 7.4. The inclusion \( A(Q) \subseteq X \) of \( N_{G}(Q) \)-algebras determines a defect group preserving bijective correspondence between the points of \( B_{Q}(c)A(Q)^{N_{G}(Q)}B_{Q}(c) \) and the points of \( B_{Q}(c)X^{N_{G}(Q)}B_{Q}(c) \).
Proof. Since $A = cAc \oplus (1 - c)A(1 - c)$ it follows that $cAc$ is a projective $k[G \times 1]$-module (and $k[1 \times G]$-module). Also note that $(cAc)(Q) = Br^C_Q(c)A(1 - c)$ and $$N_{cAc}^{\text{Aut}(Q)}(Q) \cong Br^C_Q(c)X Br^C_Q(c),$$ as $N_G(Q)$-algebras and as $N_G(Q)$-interior algebras respectively. The conclusion now follows by applying [9, Proposition 3.3] to the $G$-interior algebra $cAc$.

We consider the set

$$\mathcal{D} = \{ D \leq G \mid D \text{ is a } p\text{-subgroup with } D \cap N = Q \}.$$  

**Theorem 7.5.** There is a bijective correspondence preserving the defect groups in $\mathcal{D}$ between the points of $A^G$ that strongly cover $\beta$ and the points of $X^H$ that strongly cover $\{f_Q\}$.

**Proof.** Let $\alpha \subseteq A^G$ be a point that strongly covers $\beta$. Let $D \in \mathcal{D}$ denote a defect group of $\alpha$. According to [3, Theorem 3.5], Lemma 7.2 and Lemma 7.3 there is a unique point $\gamma \subseteq X^H$ that corresponds to $\alpha$. We obtain that $j \in f_QX^Hf_Q$, for some $j \in \gamma$. We obtain

$$j \in (f_QXf_Q)_R^H \subseteq \sum_{x \in [QC_N(Q) \cap H]} (f_QXf_Q)^{QC_N(Q)}_{QC_N(Q) \cap H} = (f_QXf_Q)^{QC_N(Q)}_Q,$$

since for all $x \in H$ we have

$$Q \leq QC_N(Q) \cap xH = x(QC_N(Q) \cap H) \leq x(N \cap R) = xQ = Q.$$

It follows that $j \in (f_QXf_Q)^{QC_N(Q)}_Q$ and since $Br^A_D(j) \neq 0$ we have $Br^A_Q(j) \neq 0$. Consider $j = \sum f_Q$, a primitive decomposition of $j$ in $(f_QXf_Q)^{QC_N(Q)}_Q$. There is at least one idempotent in the above decomposition, say $f_Q$ such that $Br^A_Q(f_Q) \neq 0$. We denote by $\overline{e}$ the point of $(f_QXf_Q)^{QC_N(Q)}_Q$ that contains $f_Q$. Applying Proposition 5.4 ii) we obtain that $\gamma$ strongly covers $\{f_Q\}$.

Conversely, let $\gamma$ be a point of $X^H$ that strongly covers the point $\{f_Q\} \subseteq (C(Q))^N(Q)$. Denote by $D$ defect group of $\gamma$. Lemma 7.2 states that $D \in \mathcal{D}$. Proposition 5.4 ii) determines $j \in \gamma$ such that $\gamma \in (f_QXf_Q)^H$. Consider the point $\overline{\gamma}$ of $(f_QXf_Q)^H$ that contains $\overline{j}$. 

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By applying Lemma 7.3 and Lemma 7.4 there is a unique point \( \overline{\alpha} \subseteq A(Q)^{NG(Q)} \), admitting defect group \( D \), such that \( \overline{\alpha} \in Br^C_Q(c)A(Q)^{NG(Q)} Br^C_Q(c) \) for some \( \alpha \in \overline{\alpha} \). Since we have
\[
\overline{i} \in Br^C_Q(c) \cdot A(Q)^{NG(Q)} \cdot Br^C_Q(c) \subseteq \sum_{x \in [N_Q(N_G(Q))/D]} (Br^C_Q(c) \cdot A(Q)^{NG(Q)} \cdot Br^C_Q(c))^{N_Q(N_G(Q))}
\]
we obtain \( \overline{i} \in (Br^C_Q(c) \cdot A(Q) \cdot Br^C_Q(c))^{N_Q(N_G(Q))} \). Let \( \overline{i} = \overline{\sum f} \) be a primitive decomposition of \( \overline{i} \) in \( (Br^C_Q(c) \cdot A(Q) \cdot Br^C_Q(c))^{N_Q(N_G(Q))} \). Since \( Br_A^D(Q)(\overline{i}) \neq 0 \) it follows that there is a primitive idempotent
\[
\overline{f} \in (Br^C_Q(c) \cdot A(Q) \cdot Br^C_Q(c))^{N_Q(N_G(Q))}
\]
such that \( Br_A^D(Q)(\overline{f}) \neq 0 \). We denote by \( \overline{\epsilon'} \subseteq (Br^C_Q(c) \cdot A(Q) \cdot Br^C_Q(c))^{N_Q(N_G(Q))} \) the point that contains \( \overline{f} \). We obtained that \( \overline{\alpha} \) strongly covers \( \overline{\beta} \).

Let \( \alpha \subseteq A^G \) and \( \gamma \subseteq X^H \) be two points lying in the correspondence determined by Theorem 7.5. Hence \( \alpha \) strongly covers \( \beta \) and \( \gamma \) strongly covers \( \{f_Q\} \). Let \( D \in \mathfrak{D} \) be a defect group of \( \gamma \) that verifies the conclusion of Lemma 7.2. So, there is a block \( f_D \in C(D) \) such that \( (Q, f_D) \leq (D, f_D) \) as \( (C(Q), f_Q, H) \)-Brauer pairs and as \( (C, c, G) \)-Brauer pairs. We fix a maximal \( (C, c, G) \)-Brauer pair \( (P, f_P) \) such that \( (D, f_D) \leq (P, f_P) \). Let \( \mathfrak{C} \) and \( \mathfrak{D} \) be the categories defined as in Definition 4.7 for the triple \( (C, c, G) \). Let \( \mathfrak{C}(D, f_D) \) denote the full subcategory of \( \mathfrak{C} \) consisting of \( (C, c, G) \)-Brauer pairs \( (T, f_T) \) such that \( (T, f_T) \leq (D, f_D) \). Let \( \mathfrak{D}(D, f_D) \) denote the full subcategory of \( \mathfrak{D} \) consisting of \( (C(Q), f_Q, H) \)-Brauer pairs \( (T, f_T) \) with \( (T, f_T) \leq (D, f_D) \).

As a consequence of Theorem 7.5 and Theorem 4.8 we obtain the final main result of our paper.

**Theorem 7.6.** The isomorphism between the categories \( \mathfrak{C} \) and \( \mathfrak{D} \) determines an isomorphism between \( \mathfrak{C}(D, f_D) \) and \( \mathfrak{D}(D, f_D) \).

**Acknowledgments.** This work was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PN-III-P1-1.1-TE-2019-0136, within PNCDI III.

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