ON POINCARÉ BUNDLES OF VECTOR BUNDLES ON CURVES

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Abstract. Let \( M \) denote the moduli space of stable vector bundles of rank \( n \) and fixed determinant of degree coprime to \( n \) on a non-singular projective curve \( X \) of genus \( g \geq 2 \). Denote by \( \mathcal{U} \) a universal bundle on \( X \times M \). We show that, for \( x, y \in X, x \neq y \), the restrictions \( \mathcal{U}|\{x\} \times M \) and \( \mathcal{U}|\{y\} \times M \) are stable and non-isomorphic when considered as bundles on \( X \).

1. Introduction

Let \( X \) be a non-singular projective curve of genus \( g \geq 2 \) over the field of complex numbers. We denote by \( M = M(n, L) \) the moduli space of stable vector bundles of rank \( n \) with determinant \( L \) of degree \( d \) on \( X \), where \( \gcd(n, d) = 1 \). We denote by \( \mathcal{U} \) a universal bundle on \( X \times M \). For any \( x \in X \) we denote by \( \mathcal{U}_x \) the bundle \( \mathcal{U}|\{x\} \times M \) considered as a bundle on \( M \).

In a paper of M. S. Narasimhan and S. Ramanan \( \text{(3)} \) it was shown that \( \mathcal{U}_x \) is a simple bundle and that the infinitesimal deformation map

\[
T_{X,x} \to H^1(M, \text{End}(\mathcal{U}_x))
\]

is bijective for all \( x \in X \). In \( \text{(1)} \) Proposition 2.4 \( \text{[1]} \) it is shown that \( \mathcal{U}_x \) is semistable with respect to the unique polarization of \( M \). In fact, \( \mathcal{U}_x \) is stable; since we could not locate a proof of this in the literature, we include one here.

Let \( \mathcal{M} \) denote the moduli space of stable bundles on \( M \) having the same Hilbert polynomial as \( \mathcal{U}_x \). Then \( \text{(1)} \) implies that the natural morphism

\[
X \to \mathcal{M}
\]

is étale and surjective onto a component \( \mathcal{M}_0 \) of \( \mathcal{M} \).

It is stated in \( \text{(3)} \) that it can be easily deduced from the results of that paper that the map \( X \to \mathcal{M}_0 \) is also injective. This would imply

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that the curve $X$ can be identified with $\mathcal{M}_0$. However no proof of this fact seems to be given. There is a proof in a paper of A. N. Tyurin [5, Theorem 2], but this seems to us to be incomplete. We offer here a proof which is in the spirit of [5]. To be more precise, our main result is the following theorem.

**Theorem** Let $X$ be a non-singular projective curve of genus $g \geq 2$. If $x, y \in X, x \neq y$, then $\mathcal{U}_x \not\cong \mathcal{U}_y$.

Note that if $X$ is a general curve of genus $g \geq 3$ or any curve of genus 2, then $X$ does not admit étale coverings $X \to \mathcal{M}_0$ of degree $> 1$. So for such curves the theorem is immediate. For the proof we can therefore assume that $g \geq 3$. In fact, our proof fails for $g = 2$.

In Section 2 we prove the stability of $\mathcal{U}_x$. In Sections 3 and 4 we make some cohomological computations, from which a family of stable bundles on $X$ can be constructed. This construction is carried out in Section 5, where we also use the morphism to $M$ given by this family in order to prove the theorem.

### 2. Stability of $\mathcal{U}_x$

Let $X$ be a non-singular projective curve of genus $g \geq 2$. Let $n \geq 2$ and $d$ be integers with $\gcd(n, d) = 1$. There are uniquely determined integers $l$ and $e$ with $0 < l < n$ and $0 \leq e < d$ such that

$$
ld - en = 1.
$$

The bundles $\mathcal{U}_x$ were shown to be semistable in [1, Proposition 2.4], but the proof does not seem to imply stability directly, even though we know also by [3] that $\mathcal{U}_x$ is simple.

**Proposition 2.1.** For all $x \in X$, the vector bundle $\mathcal{U}_x$ is stable with respect to the unique polarization of $M$.

**Proof.** By [1, Proposition 2.4] the bundle $\mathcal{U}_x$ is semistable. By [4, Remark 2.9] and possibly after tensoring $\mathcal{U}$ by a line bundle on $M$,

$$
c_1(\mathcal{U}_x) = l\alpha,
$$

where $\alpha$ is the positive generator of $H^2(M)$. By (2), $l$ and $n$ are coprime. It follows that $\mathcal{U}_x$ is stable. \qed

### 3. Cohomological constructions

Let $l$ and $n$ be as in [2]. Let $V$ be a semistable vector bundle of rank $l$ and degree $l(n - l) + e$ and $W$ a semistable bundle of rank $n - l$ and degree $d - e - l(n - l)$ on $X$. Then

$$
\deg(W^* \otimes V) = nl(n - l) - 1.
$$
Let $q_i$, $i = 1, 2$, denote the projections of $X \times X$ on the two factors, $\Delta$ the diagonal of $X \times X$ and write for brevity

$$U = q_i^*(W^* \otimes V).$$

**Lemma 3.1.** For $n \geq 2$ and $1 \leq i \leq n$,

(a) $h^0(U(-i\Delta)|\Delta) = (n + (2i-1)(g-1))l(n-l) - 1$;

(b) $h^1(U(-i\Delta)|\Delta) = 0$.

**Proof.** Identifying $\Delta$ with $X$, we have $U(-i\Delta)|\Delta = W^* \otimes V \otimes K_X^i$. Since

$$\deg(W^* \otimes V \otimes K_X^i) = (n + (2g-2)i)l(n-l) - 1 > l(n-l)(2g-2)$$

and $W^* \otimes V$ is semistable, (b) holds and Riemann-Roch gives (a). □

**Lemma 3.2.** For $n \geq 2$,

$$h^1(U(-n\Delta)) = gh^0(W^* \otimes V) + l(n-l)(n-1)(g(n-1) + 1) - (n-1).$$

**Proof.** For $0 \leq i \leq n$, consider the exact sequence

(3) $$0 \to U(-(i+1)\Delta) \to U(-i\Delta) \to U(-i\Delta)|\Delta \to 0$$
on $X \times X$. For $i = 0$, this sequence gives

$$0 \to H^1(U(-\Delta)) \to H^1(U) \xrightarrow{\psi} H^1(U|\Delta),$$
since the restriction map $H^0(U) \to H^0(U|\Delta)$ is an isomorphism. The map $\psi$ is surjective, since its restriction to the Künneth component $H^1(W^* \otimes V) \otimes H^0(\mathcal{O}) \subset H^1(U)$ is an isomorphism. Hence

$$h^1(U(-\Delta)) = h^1(U) - h^1(U|\Delta)$$

$$= h^1(W^* \otimes V)h^0(\mathcal{O}) + h^0(W^* \otimes V)h^1(\mathcal{O}) - h^1(W^* \otimes V)$$

$$= g \cdot h^0(W^* \otimes V).$$

For $1 \leq i \leq n-1$, the sequence (3) gives, by Lemma 3.1 (b),

$$0 \to H^0(U(-i\Delta)|\Delta) \to H^1(U(-(i+1)\Delta)) \to H^1(U(-i\Delta)) \to 0.$$

This gives, by Lemma 3.1 (a) and the above computation,

$$h^1(U(-n\Delta)) = h^1(U(-\Delta)) + \sum_{i=1}^{n-1} h^0(U(-i\Delta)|\Delta)$$

$$= gh^0(W^* \otimes V) + \sum_{i=1}^{n-1}((n + (2i-1)(g-1))l(n-l) - 1)$$

$$= gh^0(W^* \otimes V) + l(n-l)(n-1)(g(n-1) + 1) - (n-1).$$

□

**Lemma 3.3.** Let $n \geq 2$ and $x \in X$. Then, except in the case when $n = 2$ and $W^* \otimes V \simeq \mathcal{O}(x)$,

$$h^1(U(-n\Delta - X \times \{x\}) = h^1(U(-\Delta - X \times \{x\}) + l(n-l)(n-1)^2 g - (n-1).$$

**Proof.** For $1 \leq i \leq n-1$ consider the exact sequence

$$0 \to U(-(i+1)\Delta - X \times \{x\}) \to U(-i\Delta - X \times \{x\})$$

$$\to U(-i\Delta - X \times \{x\})|\Delta \to 0$$

□
on $X \times X$. Identifying $\Delta$ with $X$, we have
\[ U(-i\Delta - X \times \{x\})|\Delta \cong K_X^i \otimes W^* \otimes V(-x). \]
If either $i \geq 2$ or $n \geq 3$,
\[ \deg(K_X^i \otimes W^* \otimes V(-x)) > l(n-l)(2g-2). \]
So semistability implies
\[ (4) \quad h^1(K_X^i \otimes W^* \otimes V(-x)) = 0. \]
If $n = 2$ and $i = 1$, then $W^* \otimes V$ has rank 1 and
\[ \deg(K_X \otimes W^* \otimes V(-x)) = 2g-2. \]
So (4) is still true, unless $W^* \otimes V \cong \mathcal{O}(x)$.

Now Riemann-Roch implies
\[ h^0(K_X^i \otimes W^* \otimes V(-x)) = ((2g-2)i + n - g)l(n-l) - 1. \]
Hence applying the above sequence $n - 1$ times, we get
\[
\begin{align*}
    h^1(U(-n\Delta - X \times \{x\}) &= h^1(U(-\Delta - X \times \{x\})) + \sum_{i=1}^{n-1} h^0(K_X^i \otimes W^* \otimes V(-x)) \\
    &= h^1(U(-\Delta - X \times \{x\})) + \sum_{i=1}^{n-1} \{(2g-2)i + n - g\}l(n-l) - 1 \\
    &= h^1(U(-\Delta - X \times \{x\})) + l(n-l)(n-1)^2g - (n-1). 
\end{align*}
\]
\[ \square \]

Now suppose $(V, W)$ is a general pair of bundles on $X$ with the given ranks and degrees. Here by “general” we mean that the theorem of Hirschowitz (see [2]) is true, which says that either $H^0(W^* \otimes V) = 0$ or $H^1(W^* \otimes V) = 0$.

**Proposition 3.4.** For $n \geq 3$, $g \geq 3$ and $(V, W)$ general, there is a 2-dimensional vector subspace $T_0 \subset H^1(U(-n\Delta))$ such that the restriction map
\[ (5) \quad H^1(U(-n\Delta)) \to H^1(W^* \otimes V(-nx)) \]
is injective on $T_0$ for all $x \in X$.

**Proof.** Consider the exact sequence
\[ 0 \to U(-n\Delta - X \times \{x\}) \to U(-n\Delta) \to U(-n\Delta)|X \times \{x\} \to 0 \]
on $X \times X$. Since $U(-n\Delta)|X \times \{x\} \cong W^* \otimes V(-nx)$ is of degree $-1$ and $W^* \otimes V$ is semistable, this gives $h^0(W^* \otimes V(-nx)) = 0$ and thus
\[ 0 \to H^1(U(-n\Delta - X \times \{x\}) \to H^1(U(-n\Delta)) \to H^1(W^* \otimes V(-nx)). \]
We claim that
\[ (6) \quad C := h^1(U(-n\Delta)) - h^1(U(-n\Delta - X \times \{x\})) \geq 3. \]
According to Lemmas [32] and [33]
\[ C = gh^0(W^* \otimes V) + l(n-l)(n-1) - h^1(U(-\Delta - X \times \{x\})). \]
Now the exact sequence
\[ 0 \to U(-\Delta - X \times \{x\}) \to U(-X \times \{x\}) \to U(-X \times \{x\})|\Delta \to 0 \]
implies
\[ h^1(U(-\Delta - X \times \{x\})) \leq h^0(U(-X \times \{x\})|\Delta) + h^1(U(-X \times \{x\})) = h^0(W^* \otimes V(-x)) + gh^0(W^* \otimes V). \]
Hence
\[ C \geq l(n - l)(n - 1) - h^0(W^* \otimes V(-x)). \]
According to the above mentioned theorem of Hirschowitz, either \( H^0(W^* \otimes V) = 0 \) or \( H^1(W^* \otimes V) = 0 \). In the first case also \( H^0(W^* \otimes V(-x)) = 0 \) and thus
\[ C \geq l(n - l)(n - 1) \geq 3. \]
In the second case Riemann-Roch implies
\[ h^0(W^* \otimes V(-x)) \leq h^0(W^* \otimes V) = (n + 1 - g)l(n - l) - 1 \]
and thus, for \( g \geq 3 \),
\[ C \geq l(n - l)(g - 2) + 1 \geq 3. \]
We have thus proved (6) in all cases. This implies that the codimension of the union of the kernels of (5) for \( x \in X \) is at least 2. Hence there is a vector subspace \( T_0 \) of dimension 2 meeting this union in 0 only. □

4. The case \( n = 2 \)
Now suppose \( n = 2 \), which implies \( l = 1 \). So \( V \) and \( W \) are line bundles with \( \deg(W^* \otimes V) = 1 \). In this case the proof of Proposition 3.4 fails. In fact, we have to choose \( V \) and \( W \) such that
\[ W^* \otimes V \simeq O(x_0) \]
for some fixed \( x_0 \in X \). Then Lemmas 3.1 and 3.2 remain true and so does Lemma 3.3 except when \( x = x_0 \).

**Proposition 4.1.** For \( n = 2 \), there is a \((g - 1)\)-dimensional vector subspace \( T_1 \subset H^1(U(-2\Delta)) \) such that the restriction map
\[ H^1(U(-2\Delta)) \to H^1(W^* \otimes V(-2x)) \]
is injective on \( T_1 \) for all \( x \in X \).

**Proof.** Since \( h^0(W^* \otimes V) = 1 \), Lemma 3.3 says that
\[ h^1(U(-2\Delta)) = 2g. \]
Lemma 3.3 implies that, if \( x \neq x_0 \), then
\[ h^1(U(-2\Delta - X \times \{x\})) = h^1(U(-\Delta - X \times \{x\})) + g - 1. \]
If \( x = x_0 \), then the same proof gives
\[ h^1(U(-2\Delta - X \times \{x\})) \leq h^1(U(-\Delta - X \times \{x\})) + g. \]
Now consider the exact sequence

\[ 0 \to U(-\Delta - X \times \{x\}) \to U(-X \times \{x\}) \to U(-X \times \{x\})|\Delta \to 0 \]

on \( X \times X \). Since under the identification of \( \Delta \) with \( X \),

\[ U(-X \times \{x\})|\Delta \simeq \mathcal{O}(x_0 - x), \]

we get, for \( x \neq x_0 \),

\[ 0 \to H^1(U(-\Delta - X \times \{x\})) \to H^1(U(-X \times \{x\})) \xrightarrow{\varphi} H^1(\mathcal{O}(x_0 - x)). \]

The map \( \varphi \) is surjective, since its dual is the canonical injection

\[ H^0(K_X(x - x_0)) \to \text{Hom}(H^0(\mathcal{O}(x_0)), H^0(K_X(x))) = H^0(K_X(x)). \]

Hence

\[ h^1(U(-\Delta - X \times \{x\})) = h^1(U(-X \times \{x\})) - h^1(\mathcal{O}(x_0 - x)) = h^0(\mathcal{O}(x_0))h^1(\mathcal{O}(-x)) - h^1(\mathcal{O}(x_0 - x)) = g - (g - 1) = 1. \]

If \( x = x_0 \), the map \( \varphi \) is still surjective and thus an isomorphism. So \( \varphi \) implies

\[ h^1(U(-\Delta - X \times \{x\})) = h^0(\mathcal{O}(x_0 - x)) = 1. \]

Now \( \varphi \) and \( \zeta \) give

\[ h^1(U(-2\Delta - X \times \{x\})) \begin{cases} \leq g + 1 & \text{if } x = x_0, \\ = g & \text{if } x \neq x_0. \end{cases} \]

Now

\[ 0 \to U(-2\Delta - X \times \{x\}) \to U(-2\Delta) \to U(-2\Delta)|X \times \{x\} \to 0 \]

gives

\[ 0 \to H^1(U(-2\Delta - X \times \{x\})) \to H^1(U(-2\Delta)) \to H^1(W^* \otimes V(-2x)). \]

So the kernel of the restriction map is \( H^1(U(-2\Delta - X \times \{x\})) \) which, together with \( \varphi \), implies the assertion as in the proof of Proposition 3.4 \( \square \)

5. Proof of the Theorem for \( g \geq 3 \)

We want to consider extensions of the form

\[ 0 \to q_1 V(-(n - l)\Delta) \to E \to q_1 W(l\Delta) \to 0 \]

on \( X \times X \). The extension \((e)\) is classified by an element \( e \in H^1(U(-n\Delta)) \). The restriction of \((e)\) to \( X \times \{x\} \) is the extension

\[ 0 \to V(-(n - l)x) \to E_x \to W(lx) \to 0 \]

corresponding to the image of \( e \) in \( H^1(W^* \otimes V(-nx)) \). We can therefore choose a vector subspace \( T_0 \) of \( H^1(U(-n\Delta)) \) of dimension 2 such that,
for all $0 \neq e \in T_0$, the image of $e$ in $H^1(W^* \otimes V(-nx))$ is non-zero.

Note that
\[
\det E_x = \det(V(-(n-l)x)) \otimes \det(W(lx)) = \det V \otimes \mathcal{O}(-(n-l)x) \otimes \det W \otimes \mathcal{O}(l(n-l)x)
\]
for all $x$. On the other hand, by [4, Lemma 2.1], provided $V$ and $W$ are stable, the bundle $E_x$ is stable for all $0 \neq e \in T_0$ and all $x \in X$.

Let $\mathbb{P}^1 = P(T_0)$ and consider the product variety $X \times X \times \mathbb{P}^1$. Let $p_i$ and $p_{ij}$ denote the projections of $X \times X \times \mathbb{P}^1$. The non-trivial extensions of the form $(e)$ with $e \in T_0$ form a family parametrized by $\mathbb{P}^1$ which has the form (see for example [4, Lemma 2.4])
\[
(11) \quad 0 \to p_1^* V \otimes p_{12}^* \mathcal{O}(-(n-l)\Delta)) \to E \to p_3^* W \otimes \mathcal{O}(\Delta) \otimes p_{23}^*(\tau^*) \to 0,
\]
where $\tau$ is the tautological hyperplane bundle on $\mathbb{P}^1$.

**Proof of the Theorem.** By what we have said above, $E$ is a family of stable bundles on $X$ of fixed determinant $L = \det V \otimes \det W$ parametrized by $X \times \mathbb{P}^1$. This gives a morphism
\[
f : X \times \mathbb{P}^1 \to M
\]
such that
\[
(id \times f)^* \mathcal{U} \simeq E \otimes p_{23}^*(N)
\]
for some line bundle $N \in \text{Pic}(X \times \mathbb{P}^1)$. Considering
\[
E_x = E|\{x\} \times X \times \mathbb{P}^1
\]
as a bundle on $X \times \mathbb{P}^1$, we have
\[
f^* \mathcal{U}_x \simeq E_x \otimes N.
\]
Hence, in order to complete the proof of the theorem, it suffices to show that the bundle $E_x \otimes N$ determines the point $x$.

For this we compute the Chern class $c_2(E_x \otimes N)$ in the Chow group $\text{CH}^2(X \times \mathbb{P}^1)$.

From (11) we get
\[
(12) \quad c_1(E) = p_1^* \beta - (n-l)p_3^* h
\]
where $\beta$ is the class of $\det V \otimes \det W$ in $\text{CH}^1(X)$ and $h$ is the positive generator of $\text{CH}^1(\mathbb{P}^1)$.

For the computation of $c_2(E)$ we use the formula
\[
c_2(\mathcal{F} \otimes \mathcal{L}) = c_2(\mathcal{F}) + (r-1)c_1(\mathcal{F})c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2
\]
for any vector bundle $\mathcal{F}$ of rank $r$ and any line bundle $\mathcal{L}$.

The only terms in $c_2(E)$ which can possibly survive in $c_2(E_x)$ when restricting are those involving $[\Delta]h$. So $c_2(p_1^* V \otimes p_{12}^* \mathcal{O}(-(n-l)\Delta))$ does
not contribute. The coefficient of $[\Delta]h$ in $c_2(p_1^*W \otimes p_{12}^*O(l\Delta) \otimes p_3^*(\tau^*))$ is $(n-l)(-2l)$ and the coefficient of $[\Delta]h$ in

$$c_1(p_1^*V \otimes p_{12}^*O(-(n-l)\Delta)) \cdot c_1(p_1^*W \otimes p_{12}^*O(l\Delta) \otimes p_3^*(\tau^*))$$

is $-l(n-l)(-n-l) = l(n-l)^2$. This implies

$$c_2(E_x) = l(n-l)((n-l-1)+n-l)(x \times p) = l(n-l)(x \times p),$$

where $p$ is the class of a point in $\mathbb{P}^1$.

Hence, using (12), we get that

$$c_2(E_x \otimes N) = l(n-l)(x \times p) + \gamma$$

with $\gamma \in \text{CH}^2(X \times \mathbb{P}^1)$ independent of $x$.

If $U_x \cong U_y$, then $l(n-l)((x-y) \times p) = 0$ in $\text{CH}^2(X \times \mathbb{P}^1)$. This is equivalent to

$$l(n-l)(x-y) = 0 \quad \text{in} \quad \text{CH}^1(X) = \text{Pic}(X).$$

Hence $x-y$ is a point of finite order dividing $l(n-l)$ in $\text{Pic}^0(X)$. But there are only finitely many such points in $\text{Pic}^0(X)$ and any such point has at most 2 representations of the form $x-y$ (2 occurs only if $X$ is hyperelliptic). So, for general $x \in X$, there is no $y \in X$ such that $x-y$ is of finite order dividing $l(n-l)$ in $\text{Pic}^0(X)$.

Now, as stated in the introduction, the natural morphism $X \to \mathcal{M}_0$, $x \mapsto U_x$ is étale and surjective. We have now proved that this étale morphism has degree 1. Hence it is an isomorphism, which completes the proof of the theorem. $\square$

\section*{References}

[1] V. Balaji, L. Brambila-Paz and P. E. Newstead: \textit{Stability of the Poincaré Bundle}. Math. Nachr. 188 (1997), 5-15.

[2] A. Hirschowitz: \textit{Problème de Brill-Noether de Rang Supérieur}. Université de Nice, Prépublication Mathématiques No 91 (1986).

[3] M. S. Narasimhan and S. Ramanan: \textit{Deformations of the moduli space of vector bundles over an algebraic curve}. Ann. of Math. 101 (1975), 391-417.

[4] S. Ramanan: \textit{The moduli space of vector bundles over an algebraic curve}. Math. Ann. 200 (1973), 69-84.

[5] A. N. Tyurin: \textit{The geometry of moduli of vector bundles}. Usp. Mat. Nauk 29:6 (1974), 59-88. Russian Math. Surv. 29:6 (1974), 57-88.

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