The de Broglie-Bohm Interpretation of Evaporating Black-Holes

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Abstract

In this work we apply the de Broglie-Bohm interpretation of quantum mechanics to the quantized spherically symmetric black-hole coupled to a massless scalar field. The wave-functional used was first obtained by Tomimatsu using the standard ADM quantization and a gauge that places the observer close to the black-hole horizon. Using the causal interpretation, we compute quantum trajectories determined by the initial conditions. We show that the quantum trajectories for the black-hole mass can either increase or decrease with time. The quantum trajectories that show increasing mass represent the usual black-hole behavior of continuous energy absorption. The mass-decreasing quantum trajectories are a purely quantum mechanical phenomena. They can be physically interpreted as describing a black-hole that evaporates.

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I. INTRODUCTION.

Since the fundamental discovery, made by S. W. Hawking, that black-holes may emit radiation [1], many studies have been made in order to better understand this process. Initially, most of the works were concentrated in the area of quantum field theory in curved space-time [2]. More recently, some physicists have started studying the Hawking radiation with the aid of a quantum gravity theory [3], [4], [5], [6]. Most of these works deal with the theory of quantum general relativity. In this theory, the standard probabilistic Copenhagen interpretation of quantum mechanics cannot be applied. New interpretations of quantum mechanics have been proposed over the years to deal with quantum general relativity [7], [8]. Another interpretation of quantum mechanics that may be applied to quantum general relativity is the causal interpretation.

The causal interpretation of quantum mechanics was first proposed by de Broglie, and later on it was extended by Bohm to include many-particle systems and fields [9]. In this interpretation, variables corresponding to observable physical quantities have an ontological meaning regardless of whether they are observed or not, contrary to the standard Copenhagen interpretation of quantum mechanics.

The problems of applying Copenhagen’s interpretation of quantum mechanics to quantum cosmology has raised recent interest on the causal interpretation of quantum mechanics in quantum cosmology [10], as this interpretation does not need an external observer to bring a observable into reality. The causal interpretation has been applied with success, by several authors, to quantum general relativity [10], [11].

In the present work we would like to study the Hawking radiation process using the theory of quantum general relativity and the causal interpretation. We shall use the wave-functionals derived in Tomimatsu’s work [3]. There, he considered a spherically symmetric space-time, minimally coupled to a massless scalar field. He wrote the Hamiltonian form of the theory and derived from it the supermomentum and the superhamiltonian constraints. In order to derive these constraints he used a particular gauge that places the observer close
to the black-hole horizon. Tomimatsu found two wave-functionals by solving the operatorial version of the superhamiltonian constraint. The first wave-functional was interpreted as representing the classical black-hole behavior, mainly because the expected value of the time derivative of the black-hole mass is positive. The second wave-functional was interpreted as representing the quantum black-hole behavior, mainly because the expected value of the time derivative of the black-hole mass is negative. Furthermore, the mass loss rate is in agreement with the one derived directly from the Hawking emission process [12].

In the next section II, we re-write the Hamiltonian form of the theory of general relativity for a spherically symmetric space-time, minimally coupled to a massless scalar field. We use the notation introduced by K. Kuchař in Ref. [13]. We obtain the total supermomentum and superhamiltonian for the gravitational and matter sectors.

In Section III, we show that using Kuchař’s notation, in the gauge proposed by Tomimatsu, the constraints are still proportional to each other, although the proportionality constant is different from Tomimatsu’s. We present Tomimatsu’s solutions to the Wheeler-DeWitt equation and apply the causal interpretation for both wave-functionals. We explicitly solve the dynamical equations for the mass variable coming from both wave-functionals, to find the time dependence of this variable. For the first wave-functional the mass increases with time, representing the classical behavior, and for the second wave-functional the mass decreases with time, representing the quantum behavior. The time dependency for the mass loss rate is in agreement with predictions on how the evaporation should take place, if one considers the elementary particle picture of black-hole emission [12]. We also compute the quantum potential for both wave-functionals and confirm the classical and quantum behavior of them.

Finally, in Section IV, we summarize the main points and results of the paper.
II. CLASSICAL FORMALISM.

As pointed out in the introduction we would like to study the quantum general relativity theory of spherically symmetric, massless scalar field minimally coupled to gravity. Therefore, our starting point must be the Hamiltonian formalism for neutral, spherically symmetric space-times, minimally coupled to a massless scalar field.

We start with the general, spherically symmetric metric, written in the Arnowitt-Deser-Misner (ADM) form,

\[ ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 d\Omega^2, \tag{1} \]

where \( d\Omega^2 \) is the metric on the unit two-sphere, and \( N, N^r, \Lambda, \) and \( R \) are functions of \( t \) and \( r \) only. Here, we are using a unit system in which all physical constants are set to the identity.

Next, we must write the action for the space-times given by the metric (1). The action \( S \), for space-times with generic boundary properties is given by the sum of an hypersurface term for the gravitational sector \( S^G_\Sigma \), plus a boundary term for the gravitational sector \( S^G_{\partial \Sigma} \), plus a term for the matter sector [14],

\[ S = \frac{1}{16\pi} \int_M R (-g)^{1/2} dx^4 + \frac{1}{8\pi} \int_{\partial M} K h^{1/2} dx^3 - \frac{1}{8\pi} \int_M (-g)^{1/2} g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} dx^4, \tag{2} \]

where \( R \) is the curvature scalar, \( g \) is the determinant of the four-dimensional metric, \( K \) is the trace of the second fundamental form of the boundary, \( h \) is the determinant of the three-dimensional metric induced on the boundary, and \( \Phi \) is the scalar field. Since the boundary action \( S^G_{\partial \Sigma} \) will not modify the equations of motion, in the present case, we shall not consider it here. The action of the gravitational sector will be entirely represented by \( S^G_\Sigma \).

Let us write down \( S \) in terms of the fields describing the gravitational degrees of freedom, \( R \) and \( \Lambda \), and their conjugate momenta. In order to identify these momenta one must first cast the hypersurface action in its ADM Lagrangian form [15]. Then, using the results obtained in Ref. [13], we may easily write \( S \) Eq. (2), without \( S^G_{\partial \Sigma} \), for the present situation,
\[ S[R, \Lambda, \Phi; N, N'] = \int dt \int_{-\infty}^{\infty} dr \left\{ \frac{1}{N} \left\{ R \left[ (\Lambda N')' - \dot{\Lambda} \right] (N' R' - \dot{R}) + \frac{\Lambda}{2} \left( N' R' - \dot{R} \right)^2 \right\} + N \left( \frac{\Lambda}{2} - \frac{RR''}{\Lambda} + \frac{RR' \Lambda'}{\Lambda^2} - \frac{R^2}{2\Lambda} \right) + \frac{1}{2} \left[ N^{-1} \Lambda R^2 \left( \dot{\Phi} - N' \Phi' \right)^2 - N \Lambda^{-1} R^2 \Phi'^2 \right]\}, \]  

where the over-dots and primes mean differentiations in the time and radial parameters, respectively.

By functional differentiation of the above action Eq. (3), with respect to the velocities \( \dot{\Lambda}, \dot{R} \) and \( \dot{\Phi} \), we obtain the momenta \( P_\Lambda, P_R \) and \( P_\Phi \),

\[ P_\Lambda = -\frac{R}{N} (\dot{R} - N' R') , \quad (4) \]
\[ P_R = -\frac{1}{N} \left\{ \Lambda (\dot{R} - N' R') + R [\dot{\Lambda} - (N' \Lambda')] \right\} , \quad (5) \]
\[ P_\Phi = \frac{\Lambda R^2}{N} (\dot{\Phi} - N' \Phi') . \quad (6) \]

Now, we are prepared to write the canonical Hamiltonian which has the explicit form,

\[ H_c = N H + N' H_r , \quad (7) \]

such that

\[ H = \frac{\Lambda P_\Lambda^2}{2R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{RR' \Lambda'}{\Lambda^2} + \frac{R^2}{2\Lambda} - \frac{\Lambda}{2} + \frac{1}{2\Lambda} (R^{-2} P_\Phi^2 + R^2 \Phi'^2) , \quad (8) \]
\[ H_r = P_R R' - \Lambda P_\Lambda' + P_\Phi \Phi' , \quad (9) \]

where \( H_r \) and \( H \) are, respectively, the supermomentum and the superhamiltonian constraints of the model.

The hypersurface action is promptly written in terms of the canonical Hamiltonian as

\[ S[\Lambda, R, \Phi, P_\Lambda, P_R, P_\Phi; N, N'] = \int dt \int_{-\infty}^{\infty} dr \left( P_\Lambda \dot{\Lambda} + P_R \dot{R} + P_\Phi \dot{\Phi} - NH - N' H_r \right) . \quad (10) \]
III. QUANTUM FORMALISM.

A. Tomimatsu’s Gauge.

In this section, we quantize the model described by the superhamiltonian (8). Since we want to study black-hole emission, we use the gauge proposed by Tomimatsu in Ref. [3]. In this gauge, we have,

\[ N^{-2} = \Lambda^2 = 1 + \frac{2M}{r}, \quad N_r = \frac{2M}{r}, \]

(11)

where \( M = M(r, t) \) plays the role of a mass function. The apparent horizon is located at,

\[ r = 2M(r, t). \]

(12)

If one imposes equation (12) in the equation that determines the apparent horizon,

\[ g^{\alpha\beta} R_{\alpha\beta}, \]

(13)

one finds that \( R \) has to satisfy \( R_{,t} = 0 \).

Near the apparent horizon, the gauge assumes that the massless scalar field \( \Phi \) becomes ingoing and null. By introducing an advanced time \( v \equiv t + r \), one has the approximate forms

\[ \Phi = \Phi(v), \quad M = M(v), \quad R = r, \]

(14)

which will be valid near the apparent horizon.

Introducing all the above information coming from Tomimatsu’s gauge in the equations (4), (5), (6) and (11), we obtain, respectively,

\[ P_\Lambda = \frac{R}{\sqrt{2}}, \quad P_R = \frac{1}{2} \dot{M} + \frac{1}{4}, \quad P_\Phi = R^2 \dot{\Phi}, \quad \Lambda = \sqrt{2}, \]

(15)

where over-dot means derivative with respect to \( v \).

We notice that, from equation (15), \( \Lambda \) cannot be considered a dynamical variable. It means that the variables describing the model after the imposition of Tomimatsu’s gauge will be \( R \) and \( \Phi \). \( R \) is a variable because, from equations (12) and (14), it determines the position of the apparent horizon through the relationship
\[ R = 2M(v). \] \hfill (16)

Now, we may compute the value of \( H \) and \( H_r \), from equations (8) and (9), respectively, in Tomimatsu’s gauge. With the aid of equation (15), we obtain that,

\[ H = \frac{H_r}{\sqrt{2}} = \sqrt{2} \left( \frac{P_R^2}{2R^2} - P_R + \frac{1}{4} \right). \] \hfill (17)

The constraints are still proportional to each other, although the proportionality constant is different from Tomimatsu’s [3].

**B. Wave-function.**

We would like to quantize the theory using Dirac’s formalism for quantizing constrained systems [16]. First, we introduce a wave-function which is a functional of the canonical fields in their operatorial form \( \hat{R} \) and \( \hat{\Phi} \),

\[ \Psi = \Psi[\hat{R}, \hat{\Phi}]. \] \hfill (18)

Then, we impose the appropriated commutators between the fields operators and their conjugated momenta \( \hat{P}_R, \hat{P}_\Phi \). Finally, we demand that the operatorial form of the constraints, equation (17), annihilate the wave-function equation (18).

Working in the fields representation the operators \( \hat{R} \) and \( \hat{\Phi} \) are replaced by the fields themselves, and the conjugate momenta are defined as the following functional derivatives,

\[ \hat{P}_R = -i \frac{\delta}{\delta R}, \] \hfill (19)

\[ \hat{P}_\Phi = -i \frac{\delta}{\delta \Phi}, \] \hfill (20)

where we are using units where \( \bar{\hbar} = 1 \).

The most important motivation for Tomimatsu’s gauge is the result that \( H \) is proportional to \( H_r \). It means that one has to consider only one of the constraints. We may write the operatorial expression of the constraint equation (17) in terms of the new variable \( T \equiv 1/R \),
and the operatorial expressions for the momenta $\hat{P}_R$ (19), and $\hat{P}_\Phi$ (20). If we demand that $\Psi$ in equation (18) satisfies the operatorial constraint equation, we obtain,

$$-rac{1}{2} \frac{\partial^2 \Psi}{\partial \Phi^2} - i \frac{\partial \Psi}{\partial T} + \frac{1}{4T^2} \Psi = 0.$$  \hspace{1cm} (21)

Now, from Ref. [3] we have the following solutions to equation (21),

$$\Psi_c = C \exp \left[ i \left( \frac{1}{4T} - \frac{1}{2} k^2 T + k \Phi \right) \right],$$  \hspace{1cm} (22)

and

$$\Psi_q = C \exp \left[ i \left( \frac{1}{4T} + \frac{1}{2} k^2 T - |k \Phi| \right) \right],$$  \hspace{1cm} (23)

where $k$ and $C$ are arbitrary real and complex parameters, respectively.

Tomimatsu concluded that $\Psi_c$ equation (22) represents the classical black-hole behavior [3]. If one computes the expectation value of $\dot{M}$: $\langle \dot{M} \rangle = \langle 2P_R - 1/2 \rangle$, one finds a positive value. It means that the apparent horizon increases and the black-hole can only absorb. Also the scalar field sector is described by scalar waves penetrating the apparent horizon from the exterior region.

On the other hand, $\Psi_q$ in equation (23) represents the quantum-mechanical black-hole behavior [3]. In this case the value of $\langle \dot{M} \rangle$ is given by

$$\langle \dot{M} \rangle = -\frac{k^2}{4M^2}.$$  \hspace{1cm} (24)

The rhs of equation (24) is always negative, which means that the apparent horizon decreases and the black-hole can only emit. The scalar field cannot penetrate the horizon; it is exponentially suppressed. This can be interpreted as a classically forbidden state.

C. Causal Interpretation.

Let us see, in the present subsection, what the causal interpretation tell us about the states described by the wave-functionals in equations (22) and (23). Following the causal
interpretation formalism applied to quantum general relativity [10], if we write our wave-
functionals in equations (22) and (23) as,

\[ \Psi = \mathcal{R} \exp(iS), \]  

(25)

we may obtain a dynamical equation for the physical variables in the following way,

\[ P_{X_i} = \frac{\delta S}{\delta X_i}, \]  

(26)

where \( X_i \) stands for \( R \) and \( \Phi \). Also, there is a quantum potential \( Q \), which governs the
dynamics of the system. The expression for \( Q \) is given, in the present situation, by,

\[ Q = -\nabla^2 \mathcal{R} \frac{\mathcal{R}}{R}. \]  

(27)

The dynamical equations for \( \Phi \) for both wave-functionals (22) and (23) are trivial and do
not bring any contribution to the understanding of the system. Therefore, we shall restrict
our attention to the dynamical equation for \( R \).

Starting with \( \Psi_c \) equation (22), we may write the dynamical equation (26) for \( X_i = R \),

\[ P_R = \frac{1}{4} + \frac{k^2}{2R^2}. \]  

(28)

Now, introducing the expression of \( P_R \) given in equation (15) in equation (28), we obtain
the following equation for the evolution of \( M \),

\[ \dot{M} = \frac{k^2}{4M^2}. \]  

(29)

This equation is easily integrated to give,

\[ M^3 = \frac{3}{4}k^2(v - v_0) + M_0^3, \]  

(30)

where \( v_0 \) and \( M_0 \) are the initial values of \( v \) and \( M \), respectively.

Solution (30) tell us that the black-hole mass \( M \) increases continuously as the time,
measured by \( v \), increases. This wave-functional is associated with the classical behavior of
the black-hole. In particular, if we compute the value of the quantum potential \( Q \) from
equation (27) for $\Psi_c$ in equation (22), we find that it is zero, as expected for the classical situation.

Consider, now, the dynamical equation for $M$ coming from $\Psi_q$ equation (23). With the aid of $P_R$ in equation (26), which in this case is

$$P_R = \frac{1}{4} - \frac{k^2}{2R^2},$$

and $P_R$ from equation (15), we find the dynamical equation for $M$,

$$\dot{M} = -\frac{k^2}{4M^2}.$$  \hspace{1cm} (32)

Note that equation (32) is similar to equation (24) for the expectation value of $\dot{M}$. The difference is that equation (32) can be integrated to give the exact evolution of $M$ and not just the expectation value of this evolution. This equation is easily integrated to give

$$M^3 = -\frac{3}{4}k^2(v - v_0) + M_0^3,$$

where $v_0$ and $M_0$ are the initial values of $v$ and $M$, respectively. Equation (33) tell us that if the black-hole has an initial mass $M_0$ at $v_0$ after a time $v_e = 4M_0^3/3k^2 + v_0$, it will completely evaporate. This is in accordance with the qualitative predictions made by S. W. Hawking that, taking into account quantum properties, black-holes evaporate [1]. Equation (33) is also in accordance with predictions on how this evaporation should take place, if one considers the elementary particle picture of black-hole emission [12]. The quantum potential $Q$ in equation (27), computed for $\Psi_q$ in equation (23), is given by $-k^2/2R^2$. This may be interpreted as an attractive potential that pulls $R$ to zero. If we remember that $R = 2M$, this potential also pulls the mass towards zero.

IV. CONCLUSIONS.

In this work we applied the de Broglie-Bohm interpretation of quantum mechanics, also known as the causal interpretation, to the quantized spherically symmetric black-hole coupled to a massless scalar field. The wave-functional used was first obtained by Tomimatsu
using the standard ADM quantization and a gauge that places the observer close to the black-hole horizon. In Tomimatsu’s paper, he obtains two wave-functionals that are solutions to the Wheeler-DeWitt equation, one of which predicts a decreasing expected value for the black-hole mass and another that predicts the standard classical result. Using the causal interpretation, we computed the individual quantum trajectories determined by the initial conditions. We showed that the quantum trajectories for the black-hole mass could either increase or decrease with time. The quantum trajectories that show increasing mass represent the usual black-hole behavior of continuous energy absorption. The mass-decreasing quantum trajectories are a purely quantum mechanical phenomena. They can be physically interpreted as describing a black-hole that evaporates.

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