On the decomposition of connected graphs into their biconnected components

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Abstract

We give a recursion formula to generate all equivalence classes of biconnected graphs with coefficients given by the inverses of the orders of their groups of automorphisms. We give a linear map to produce a connected graph with say, $\mu$, biconnected components from one with $\mu - 1$ biconnected components. We use such map to extend the aforesaid result to connected or 2-edge connected graphs. The underlying algorithms are amenable to computer implementation.

1 Introduction

As pointed out in [12], generating graphs may be useful for numerous reasons. These include giving more insight into enumerative problems or the study of some properties of graphs. Problems of graph generation may also suggest conjectures or point out counterexamples.

In particular, the problem of generating graphs taking into account their symmetries was considered as early as the 19th-century [5]. Sometimes such problem is so that graphs are weighted by scalars given by the inverses of the orders of their groups of automorphisms. One instance is [11] (page 209). Many

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other examples may be found in mathematical physics (see for instance [4] and references therein). In this context, the problem is traditionally dealt with via generating functions and functional derivatives. However, any method to straightforwardly manipulate graphs may actually be used. In particular, the main results of [9] and [10] are recursion formulas to generate all equivalence classes of trees and connected graphs (with multiple edges and loops allowed), respectively, via Hopf algebra. The key feature is that the sum of the coefficients of all graphs in the same equivalence class is given by the inverse of the order of their automorphism group.

Furthermore, in [7] the algorithm underlying the main result of [10] was translated to the language of graph theory. To this end, basic graph transformations whose action mirrors that of the Hopf algebra structures considered in the latter paper, were given. The result was also extended to further classes of connected graphs, namely, 2-edge connected, simple and loopless graphs.

In the following, let graphs be loopless graphs with external edges allowed. That is, edges which are connected to vertices only at one end. In the present paper, we generalize formula (5) of [7] to biconnected graphs. Moreover, we give a linear map to produce connected graphs from connected ones by increasing the number of their biconnected components by one unit. We use this map to give an algorithm to generate all equivalence classes of connected or 2-edge connected graphs with the exact coefficients. This is so that generated graphs are automatically decomposed into their biconnected components. The proof proceeds as suggested in [9]. That is, given an arbitrary equivalence class whose representative is a graph on \( m \) internal edges, say, \( G \), we show that every one of the \( m \) internal edges of the graph \( G \) adds \( 1/(m \cdot |\text{Aut}(G)|) \) to the sum of the coefficients of all graphs isomorphic to \( G \). To this end, we use the fact that vertices carrying (labeled) external edges are held fixed under any automorphism.

This paper is organized as follows: Section 2 reviews the basic concepts of graph theory underlying much of the paper. Section 3 contains the definitions of the basic linear maps to be used in the following sections. Section 4.1 presents an algorithm to generate biconnected graphs and gives some examples. Section 4.2 extends the result to connected and 2-edge connected graphs.

2 Basics

We briefly review the basic concepts of graph theory that are relevant for the following sections. More details may be found in any standard textbook on graph theory such as [2], or in [7] for the treatment of graphs with external edges allowed. This section overlaps Section 2 of [7] except for the concept of biconnected graph. However, as the present paper only considers loopless graphs, the definition of graph given in that paper specializes here for loopless graphs.

Let \( A \) and \( B \) denote sets. By \([A, B]\), we denote the set of all unordered pairs of elements of \( A \) and \( B \), \( \{\{a, b\} | a \in A, b \in B\} \). In particular, by \([A]^2 := [A, A]\),
we denote the set of all 2-element subsets of \( A \). Also, by \( 2^A \), we denote the power set of \( A \), i.e., the set of all subsets of \( A \). By \( \text{card}(A) \), we denote the cardinality of the set \( A \). Furthermore, we recall that the symmetric difference of the sets \( A \) and \( B \) is given by \( A \symdif B := (A \cup B) \setminus (A \cap B) \). Finally, given a graph \( G \), by \( |\text{Aut}(G)| \), we denote the order of its automorphism group \( \text{Aut}(G) \).

Let \( V = \{v_i\}_{i \in \mathbb{N}} \) and \( K = \{e_a\}_{a \in \mathbb{N}} \) be infinite sets so that \( V \cap K = \emptyset \). Let \( V \subset \mathcal{V} \); \( V \neq \emptyset \) and \( K \subset \mathcal{K} \) be finite sets. Let \( E = E_{\text{int}} \cup E_{\text{ext}} \subseteq \mathbb{Z}^2 \) and \( E_{\text{int}} \cap E_{\text{ext}} = \emptyset \). Also, let the elements of \( E \) satisfy \( \{e_a, e_{a'}\} \cap \{e_b, e_{b'}\} = \emptyset \). That is, \( e_a, e_{a'} \neq e_b, e_{b'} \). In this context, a graph is a triple \( G = (V, K, E) \) together with the following maps:

\[
\begin{align*}
(\text{a}) \quad & \varphi_{\text{int}} : E_{\text{int}} \to [V]^2; \{e_a, e_{a'}\} \mapsto \{v_i, v_{i'}\}; \\
(\text{b}) \quad & \varphi_{\text{ext}} : E_{\text{ext}} \to [V, K]; \{e_a, e_{a'}\} \mapsto \{v_i, e_{a'}\}.
\end{align*}
\]

The elements of \( V \), \( E \) and \( K \) are called vertices, edges, and ends of edges, respectively. In particular, the elements of \( E_{\text{int}} \) and \( E_{\text{ext}} \) are called internal edges and external edges, respectively. The degree of a vertex is the number of ends of edges assigned to the vertex. Two distinct vertices connected together by one or more internal edges, are said to be adjacent. Two or more internal edges connecting the same pair of distinct vertices together, are called multiple edges. Furthermore, let \( \text{card}(E_{\text{ext}}) = s \). Let \( L = \{x_1, \ldots, x_s\} \) be a label set. A labeling of the external edges of the graph \( G \), is an injective map \( l : E_{\text{ext}} \to [K, L]; \{e_a, e_{a'}\} \mapsto \{e_a, x_z\} \), where \( z \in \{1, \ldots, s\} \). A graph \( G^* = (V^*, K^*, E^*) \); \( E^* = E_{\text{int}} \cup E_{\text{ext}} \), together with the maps \( \varphi_{\text{int}}^* \) and \( \varphi_{\text{ext}}^* \) is called a subgraph of a graph \( G = (V, K, E) \); \( E = E_{\text{int}} \cup E_{\text{ext}} \), together with the maps \( \varphi_{\text{int}} \) and \( \varphi_{\text{ext}} \). If \( V^* \subseteq V \), \( K^* \subseteq K \), \( E^* \subseteq E \) and \( \varphi_{\text{int}}^* = \varphi_{\text{int}}|_{E^*_{\text{int}}}, \varphi_{\text{ext}}^* = \varphi_{\text{ext}}|_{E^*_{\text{ext}}} \), \( \varphi_{\text{int}}^* \) and \( \varphi_{\text{ext}}^* \) are connected to each other.

A path is a map \( P = (V, K, E_{\text{int}}); V = \{v_1, \ldots, v_n\}, n := \text{card}(V) > 1 \), together with the map \( \varphi_{\text{int}} \) so that \( \varphi_{\text{int}}(E_{\text{int}}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\} \) and the vertices \( v_1 \) and \( v_n \) have degree 1, while the vertices \( v_2, \ldots, v_{n-1} \) have degree 2. In this context, the vertices \( v_1 \) and \( v_n \) are called the end point vertices, while the vertices \( v_2, \ldots, v_{n-1} \) are called the inner vertices. A cycle is a graph \( C = (V', K', E'_{\text{int}}); V' = \{v_1, \ldots, v_n\}, \) together with the map \( \varphi'_{\text{int}} \) so that \( \varphi'_{\text{int}}(E'_{\text{int}}) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\} \) and every vertex has degree 2. A graph is said to be connected if every pair of vertices is joined by a path. Otherwise, it is disconnected.

Given a graph \( G = (V, K, E) \); \( E = E_{\text{int}} \cup E_{\text{ext}} \), together with the maps \( \varphi_{\text{int}} \) and \( \varphi_{\text{ext}} \), a maximal connected subgraph of the graph \( G \) is called a component. Moreover, the set \( 2^E_{\text{int}} \) is a vector space over the field \( \mathbb{Z}_2 \) so that vector addition is given by the symmetric difference. The cycle space \( C \) of the graph \( G \) is defined as the subspace of \( 2^E_{\text{int}} \) generated by all the cycles in \( G \). The dimension of \( C \) is called the cyclomatic number of the graph \( G \). Let \( k := \dim C, n := \text{card}(V) \) and \( m := \text{card}(E_{\text{int}}) \). Then, \( k = m - n + c \), where \( c \) denotes the number of connected components of the graph \( G \) [6].

Furthermore, given a connected graph, a vertex whose removal (together with attached edges) disconnects the graph is called a cut vertex. A graph is
said to be 2-connected (resp. 2-edge connected) if it remains connected after erasing any vertex (resp. any internal edge). A 2-connected graph (resp. 2-edge connected graph) is also called biconnected (resp. edge-biconnected). Furthermore, a biconnected component of a connected graph is a maximal biconnected subgraph (see [1] 6.4 for instance).

Now, let \( L = \{x_1, \ldots, x_s\} \) be a finite label set. Let \( G = (V, K, E); E = E_{\text{int}} \cup E_{\text{ext}}, \text{card}(E_{\text{ext}}) = s \), together with the maps \( \varphi_{\text{int}} \) and \( \varphi_{\text{ext}}, \) and \( G^* = (V^*, K^*, E^*); E^* = E_{\text{int}} \cup E_{\text{ext}}, \text{card}(E_{\text{ext}}^*) = s \), together with the maps \( \varphi_{\text{int}}^* \) and \( \varphi_{\text{ext}}^* \) denote two graphs. Let \( l : E_{\text{ext}} \to [K, L] \) and \( l^*: E_{\text{ext}}^* \to [K^*, L^*] \) be labelings of the elements of \( E_{\text{ext}} \) and \( E_{\text{ext}}^* \), respectively. An isomorphism between the graphs \( G \) and \( G^* \) is a bijection \( \psi_V : V \to V^* \) and a bijection \( \psi_K : K \to K^* \) which satisfy the following three conditions:

(a) \( \varphi_{\text{ext}}(\{e_a, e_a'\}) = \{v_i, v_i'\} \) iff \( \varphi_{\text{ext}}^*(\{\psi_K(e_a), \psi_K(e_a')\}) = \{\psi_V(v_i), \psi_V(v_i')\}; \)

(b) \( \varphi_{\text{ext}}(\{e_a, e_a'\}) = \{v_i, v_i'\} \) iff \( \varphi_{\text{ext}}^*(\{\psi_K(e_a), \psi_K(e_a')\}) = \{\psi_V(v_i), \psi_V(v_i')\}; \)

(c) \( L \cap l(\{e_a, e_a'\}) = L \cap l^*(\{\psi_K(e_a), \psi_K(e_a')\}). \)

An isomorphism defines an equivalence relation on graphs. A vertex (resp. edge) isomorphism between the graphs \( G \) and \( G^* \) is an isomorphism so that \( \psi_K \) (resp. \( \psi_V \)) is the identity map. In this context, a symmetry of a graph \( G \) is an isomorphism of the graph onto itself (i.e., an automorphism). A vertex symmetry (resp. edge symmetry) of a graph \( G \) is a vertex (resp. edge) automorphism of the graph. Given a graph \( G \), let \( \text{Aut}_{\text{vertex}}(G) \) and \( \text{Aut}_{\text{edge}}(G) \) denote the groups of vertex and edge automorphisms, respectively. Then, \( |\text{Aut}(G)| = |\text{Aut}_{\text{vertex}}(G)| \cdot |\text{Aut}_{\text{edge}}(G)| \) (a proof is given in [10] for instance).

## 3 Elementary linear transformations

We introduce the elementary linear maps to be used in the following.

Given an arbitrary set \( X \), by \( \mathbb{Q}X \), we denote the free vector space on the set \( X \) over \( \mathbb{Q} \). By \( \text{id}_X : X \to X; x \mapsto x \), we denote the identity map. Given maps \( f : X \to X^* \) and \( g : Y \to Y^* \), by \([f, g] : [X, Y] \to [X^*, Y^*]; \{x, y\} \mapsto \{f(x), g(y)\} \) with \([f]^2 := [f, f]\).

Let \( V = \{v_i\}_{i \in \mathbb{N}} \) and \( K = \{e_a\}_{a \in \mathbb{N}} \) be infinite sets so that \( V \cap K = \emptyset \). Fix an integer \( s \geq 0 \). Let \( L = \{x_1, \ldots, x_s\} \) be a label set. For all integers \( n \geq 1 \) and \( k \geq 0 \), by \( W_{n,k,s}^{\text{conn}} \) (resp. \( W_{n,k,s}^{\text{disc}} \)), we denote the set of all (loopless) connected graphs (resp. disconnected graphs with two components) with \( n \) vertices, cyclomatic number \( k \) and \( s \) external edges whose free ends are labeled \( x_1, \ldots, x_s \). In all that follows, let \( V = \{v_1, \ldots, v_n\} \subset V, K = \{e_1, \ldots, e_l\} \subset K \), and \( E = E_{\text{int}} \cup E_{\text{ext}} \subseteq [K]^2 \) be the sets of vertices, ends of edges, and edges, respectively, of all elements of \( W_{n,k,s}^{\text{conn}} \) (resp. \( W_{n,k,s}^{\text{disc}} \)), so that \text{card}(E_{\text{int}}) = k + n - 1 \) (resp. \text{card}(E_{\text{int}}) = k + n - 2) and \text{card}(E_{\text{ext}}) = s. Also, let \( l : E_{\text{ext}} \to [K, L] \) be a labeling of their external edges. Finally, by \( W_{n,k,s}^{\text{biconn}} \) and \( W_{n,k,s}^{2\text{-edge}} \), we denote
the subsets of $W_{n,k,s}^{n,k,s}$ whose elements are biconnected and 2-edge connected graphs, respectively.

We begin by briefly recalling the linear maps $\xi_{E_{ext}}, l, i, j$ and $s_{i \geq 1}$ introduced in Sections 3 and 5.3.4 of [7]. We refer the reader to that paper for the precise definitions.

(i) Distributing external edges between all elements of a given vertex subset in all possible ways: Let $G = (V, K, E)$; $E = E_{int} \cup E_{ext}$, together with the maps $\varphi_{int}$ and $\varphi_{ext}$ denote a graph in $W_{n,k,s}^{n,k,s}$. Let $V' = \{v_{z_{1}}, \ldots, v_{z_{m}}\} \subseteq V$. Let $K'$ be a finite set so that $K \cap K' = \emptyset$. Also, let $E'_{ext} \subseteq [K']^{2}$; $s' := \text{card}(E'_{ext})$. Assume that the elements of $E'_{ext}$ satisfy $\{e_a, e_a'\} \cap \{e_b, e_b'\} = \emptyset$. Let $L' = \{x_{s+1}, \ldots, x_{s+\nu}\}$ be a label set so that $L \cap L' = \emptyset$. Let $l' : E'_{ext} \to [K', L']$ be a labeling of the elements of $E'_{ext}$. Finally, let $T_{E_{ext}}^{n'}$ denote the set of all ordered partitions of the set $E'_{ext}$ into $n'$ disjoint subsets: $T_{E_{ext}}^{n'} = \{(E'_{ext}(1), \ldots, E'_{ext}(n'))|E'_{ext}(1) \cup \ldots \cup E'_{ext}(n') = E'_{ext}\}$ and $E'_{ext} = 0, \forall i, j \in \{1, \ldots, n'\}$ with $i \neq j$. In this context, the maps

$$\xi_{E'_{ext}} : V' \times QW_{conn}^{n,k,s} \to QW_{conn}^{n,k,s} : G \mapsto \sum_{(E'_{ext}(1), \ldots, E'_{ext}(n')) \in T_{E_{ext}}^{n'}} G(E'_{ext}(1), \ldots, E'_{ext}(n'))$$

are defined to produce each of the graphs $G(E'_{ext}(1), \ldots, E'_{ext}(n'))$ from the graph $G$ by assigning all elements of $E'_{ext}$ to the vertex $v_{z_{j}}$ for all $j \in \{1, \ldots, n'\}$.

(ii) The maps $l, i, j : QW_{conn}^{n,k,s} \to QW_{conn}^{n,k+1,s} : G \mapsto G^*$ are defined to produce the graph $G^*$ from the graph $G$ by connecting (or reconnecting) the vertices $v_i$ and $v_j$ with an internal edge for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

(iii) (a) We define the maps $s_{i \geq 1}$ from the maps $s_i$ given in Section 3 of [7], by restricting the image of the latter to graphs without isolated vertices. More precisely, let $G = (V, K, E)$; $E = E_{int} \cup E_{ext}$, together with the maps $\varphi_{int}$ and $\varphi_{ext}$ denote a graph in $W_{n,k,s}^{n,k,s}$. Let $E_{int,i} \subset K$ be the set of ends of internal edges assigned to the vertex $v_i \in V$; $i \in \{1, \ldots, n\}$. Let $T_{E_{int}}^{2}$ denote the set of all ordered partitions of the set $E_{int,i}$ into two non-empty disjoint sets: $T_{E_{int}}^{2} = \{(E_{int,i}(1), E_{int,i}(2))|E_{int,i}(1) \neq \emptyset, E_{int,i}(2) \neq \emptyset, E_{int,i}(1) \cup E_{int,i}(2) = E_{int,i}\}$. Moreover, let $E_{ext,i} \subset E_{ext}$ denote the set of external edges assigned to the vertex $v_i$. In this context, for all $i \in \{1, \ldots, n\}$, define the maps

$$s_{i \geq 1} : QW_{conn}^{n,k,s} \to QW_{conn}^{n+1,k-1,s} \cup \{QW_{diseconn}^{n+1,k,s}\};$$

$$G \mapsto \begin{cases} \xi_{E_{ext,i}}(v_i, v_{n+1}) & \text{if } 0 \leq \text{card}(E_{int,i}) < 2; \\ \sum_{(E_{int,i}(1), E_{int,i}(2)) \in T_{E_{int}}^{2}} G(E_{int,i}(1), E_{int,i}(2)) & \text{otherwise}, \end{cases}$$

where each of the graphs $G(E_{int,i}(1), E_{int,i}(2))$ is produced from the graph $G$ as follows: (a) split the vertex $v_i$ into two vertices, namely, $v_i$ and
\( v_{n+1}: \) (b) assign the ends of edges in \( E^{(1)}_{\text{int},i} \) and \( E^{(2)}_{\text{int},i} \) to \( v_i \) and \( v_{n+1}, \) respectively.

(b) Let \( B_i = \{ G_1^n, \ldots, G^n_i \} \) with \( n_i \geq 1, \) be the set of biconnected components of the graph \( G \) containing the vertex \( v_i. \) That is, \( v_i \in V^j_i, \) where \( V^j_i \) denotes the vertex set of the graph \( G^j_i \) for all \( j \in \{1, \ldots, n_i\}. \) Let \( K^j \) denote the set of ends of edges of the graph \( G^j_i. \) Let \( \mathcal{E}^{(1)}_{\text{int},i} \subset K^j \) be the set of ends of edges assigned to the vertex \( v_i \) which belong to the graph \( G^j_i. \) Hence, \( E^{(1)}_{\text{int},i} = \bigcup_{j=1}^{n_i} E^{(1)}_{\text{int},i} \) and \( E^{(2)}_{\text{int},i} \cap E^{n_i}_{\text{int},i} = \emptyset \) for all \( j, m = 1, \ldots, n_i \) with \( j \neq m. \) In this context, we define the maps \( \hat{s}_i \) by restricting the image of \( s_i \) to graphs obtained by replacing in the above definition the set \( \mathcal{I}^{(1)}_{\text{int},i} \) by \( \mathcal{I}^{(2)}_{\text{int},i} = \{(E^{(1)}_{\text{int},i}, E^{(2)}_{\text{int},i}) | E^{(1)}_{\text{int},i}, E^{(2)}_{\text{int},i} \neq \emptyset, \ E^{(1)}_{\text{int},i} \cup E^{(2)}_{\text{int},i} = E_{\text{int},i}, E^{(1)}_{\text{int},i} \cap E^{(2)}_{\text{int},i} = \emptyset \) and \( E^{(1)}_{\text{int},i} \cap E^{(2)}_{\text{int},i} \neq \emptyset \} \) with \( s_i \) if there exists \( j \) so that \( \text{card}(\mathcal{E}^{(1)}_{\text{int},i}) < 2. \)

(c) We define the maps \( q_{l_i}^{(\rho)} \) and \( q_{l_i}^{(\rho)} \) in analogy with the maps \( q_{l_i}^{(\rho)} : \frac{1}{2} l_{i,n+1}^{\rho} \circ s_i \) given in [7] (see also [3] for \( \rho = 1): \)

\[
q_{l_i}^{(\rho)} := \frac{1}{2(\rho - 1)!} l_{i,n+1}^{\rho} \circ s_i : \mathcal{Q}W^{n,k,s}_{\text{conn}} \to \mathcal{Q}W^{n+1,k+\rho-1,s}_{\text{conn}}, \quad (1)
\]

\[
q_{l_i}^{(\rho)} := \frac{1}{2(\rho - 1)!} l_{i,n+1}^{\rho} \circ s_i : \mathcal{Q}W^{n,k,s}_{\text{conn}} \to \mathcal{Q}W^{n+1,k+\rho-1,s}_{\text{conn}}, \quad (2)
\]

where \( l_{i,n+1}^{\rho} \) denotes the \( \rho \)th iterate of \( l_{i,n+1} \) with \( l_{i,n+1}^{0} = \text{id}. \)

We now introduce the following auxiliary map:

(iv) Fix integers \( n' > 0 \) and \( 1 \leq i \leq n'. \) Let \( G = (V, K, E_{\text{ext}}) \) together with the map \( \varphi_{\text{int}} \) denote a graph in \( W^{n,k,0}_{\text{conn}}. \) We define the map \( \Xi_i : G \to G^{*}, \) where the graph \( G^{*} \) satisfies the following conditions:

(a) \( V^{*} = \chi^{l_i}_i(V), \) where \( \chi^{l_i}_i : v_i \mapsto \left\{ \begin{array}{ll}
v_i & \text{if } l = 1 \\
v_{n' + l - 1} & \text{if } l \in \{2, \ldots, n\} \end{array} \right. \) is a bijection;

(b) \( K^{*} = \chi^{l_i}_i(K), \) where \( \chi^{l_i}_i : e_b \mapsto e_{l+b} \) is a bijection;

(c) \( E^{*}_{\text{int}} = [\chi^{l_i}_i]^2(E_{\text{int}}); \)

(d) \( \varphi^{*} \circ [\chi^{l_i}_i]^2 = [\chi^{l_i}_i]^2 \circ \varphi_{\text{int}}. \)

The maps \( \Xi_i \) are extended to the whole of \( \mathcal{Q}W^{n,k,0}_{\text{conn}} \) by linearity.

(v) (a) **Distributing the biconnected components of a connected graph sharing a vertex, between all the vertices of a given biconnected graph in all possible ways:** Let \( G = (V, K, E) \): \( E = E_{\text{int}} \cup E_{\text{ext}}, \) together with the maps \( \varphi_{\text{int}} \) and \( \varphi_{\text{ext}} \) denote a graph in \( W^{n,k,s}_{\text{conn}}. \) Let \( \mathcal{L}_{\text{int},i} \subseteq E_{\text{int}} \) and \( \mathcal{L}_{\text{ext},i} \subseteq E_{\text{ext}} \) be the sets of internal and external edges, respectively, assigned to the vertex \( v_i \in V \) with \( i \in \{1, \ldots, n\}. \) Let
$B_i = \{G_{i1}^1, \ldots, G_{i_n}^n\}$ be the set of biconnected components of the graph $G$ sharing the vertex $v_i$. That is, $v_i \in V_i^j$, where $V_i^j$ denotes the vertex set of the graph $G_j^i$ for all $j \in \{1, \ldots, n_i\}$. Let $\mathcal{T}^n_B$ denote the set of all ordered partitions of the set $B_i$ into $n'$ disjoint sets: $\mathcal{T}^n_B = \{(B_1 \cup \cdots \cup B_{n'}) | B_1 \cup \cdots \cup B_{n'} = B_i \}$ and $B_i \cap B_l = \emptyset$, $\forall l, l' \in \{1, \ldots, n_i\}$ with $l \neq l'$. For all $l \in \{1, \ldots, n_i\}$, let $B_i(l) = \{G_{i1}^l, \ldots, G_{in_i}^l\}$. Also, let $V_i(l) \subseteq V$ denote the vertex set of the graph $G_i(l)$ for all $a \in \{1, \ldots, n_i\}$. Let $L_{\text{int},i,a} \subseteq L_{\text{int},i}$ and $V_i(l) \subseteq V_i(l) \{v_i\}$ be so that $\varphi_{\text{int}}(L_{\text{int},i,a}) = [v_i, V_i(l)]$. Let $L_{\text{int},i} := \bigcup_{a=1}^{n_i} L_{\text{int},i,a}$ and $V_i(l) := \bigcup_{a=1}^{n_i} V_i(l)$. Finally, let $G'$ denote a graph in $W_{\text{int}G}$. Also, let $\Xi(G') = (\hat{V}, \hat{K}, \hat{E}_{\text{int}})$ together with the map $\hat{\varphi}_{\text{int}}$ be the graph obtained from $G'$ by applying the map $\Xi_1$ given above. In this context, for all $i \in \{1, \ldots, n\}$ define

$$r_i^{G'} : \mathbb{Q}W_{\text{conn}} \to \mathbb{Q}W_{\text{conn}}^{n+k'-1,k+k'};$$

$$G \mapsto \xi_{\text{ext},i} \hat{V} \left( \sum_{(B_1 \cup \cdots \cup B_{n'}) \in \mathcal{T}^n_B} G_{(B_1 \cup \cdots \cup B_{n'})} \right);$$

where the graphs $(G_{(B_1 \cup \cdots \cup B_{n'})}) = (V^*, K^*, E^*)$; $E^* = E_{\text{int}} \cup E_{\text{ext}}$, together with the maps $\varphi_{\text{int}}^*$ and $\varphi_{\text{ext}}^*$ satisfy the following conditions:

(a) $V^* = V \{v_i\} \hat{V}$;
(b) $K^* = K \cup \hat{K}$;
(c) $E^* = E_{\text{int}} \cup E_{\text{ext}}$, where $E_{\text{int}} = E_{\text{int}} \cup \hat{E}_{\text{int}}$, $E_{\text{ext}} = E_{\text{ext}} \setminus L_{\text{ext},i}$;
(d) $\varphi_{\text{int}}^*|_{E_{\text{int}}} = \varphi_{\text{int}}|_{E_{\text{int}}} \cup \hat{E}_{\text{int}}$, $\varphi_{\text{ext}}^*|_{E_{\text{ext}}} = \varphi_{\text{ext}}|_{E_{\text{ext}}}$;

$$\varphi_{\text{int}}^*(L_{\text{int},i}) = [v_i, V_i(l)] \text{ and } \varphi_{\text{ext}}^*(L_{\text{int},i}) = [v_i+n'-1, V_i(l)] \text{ for all } l \in \{2, \ldots, n\};$$
(e) $\varphi_{\text{ext}}^*|_{E_{\text{ext}}} \setminus L_{\text{ext},i} = \varphi_{\text{ext}}|_{E_{\text{ext}}} \setminus L_{\text{ext},i}$;
(f) $l^* = l|_{E_{\text{ext}}} \setminus L_{\text{ext},i}$ is a labeling of the elements of $E_{\text{ext}}$.

The maps $r_i^{G'}$ are extended to the whole of $\mathbb{Q}W_{\text{conn}}$ by linearity. For instance, let $C_4$ denote a cycle on four vertices. Figure 1 shows the result of applying the map $r^{C_4}_{\text{ext}}$ to the cut vertex of a 2-edge connected graph with two biconnected components.

\[ R_{3,4} \circ \Delta_3 \left( \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right) = 2 \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} + 2 \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \]

Figure 1: Linear combination of graphs obtained by applying the maps $r^{C_4}_{\text{ext}}$ to the cut vertex of the graph consisting of two triangles sharing one vertex.
(b) We define the maps $r_i^G$ by restricting the image of $r_i^G$ to graphs obtained by replacing in the definition of the latter the set $I_{b_i}^G$ by the following set of $n'$-tuples: \{(B_i, \emptyset, \ldots, \emptyset), (\emptyset, B_j, \ldots, \emptyset), \ldots, (\emptyset, \ldots, \emptyset, B_l)\}.

(c) Given a linear combination of graphs \( \vartheta = \sum_{G \in W_{\text{biconn}}^{n,k,s}} \alpha_G G; \alpha_G \in \mathbb{Q} \) in $QW_{\text{biconn}}^{n,k,s}$, define

\[
\hat{r}_i^\vartheta := \sum_{G \in W_{\text{biconn}}^{n,k,s}} \alpha_G \hat{r}_i^G.
\]

We proceed to generalize the edge contraction operation to the operation of contracting a biconnected component of a connected graph.

(vi) **Contracting a biconnected component of a connected graph:** Let $G = (V, K, E); E = E_{\text{int}} \cup E_{\text{ext}}$, together with the maps $\varphi_{\text{int}}$ and $\varphi_{\text{ext}}$ denote a graph in $W_{\text{conn}}^{n,k,s}$, where $n > 1$. Consider the following biconnected component of the graph $G$: $\hat{G} = (\hat{V}, \hat{K}, \hat{E}); \hat{V} = \{v_{i_1}, \ldots, v_{i_l}\} \subseteq V$, $\hat{K} = \{e_{j_1}, \ldots, e_{j_{n'}}\} \subseteq K; \hat{E} = \hat{E}_{\text{int}} \cup \hat{E}_{\text{ext}} \subseteq E$, together with the maps $\hat{\varphi}_{\text{int}}$ and $\hat{\varphi}_{\text{ext}}$. Let $i_1 < \ldots < i_l$ and $j_1 < \ldots < j_{n'}$. Also, let $k'$ denote the cyclical number of the graph $G$. Let $B = \{G_1, \ldots, G_{n_B}\}$ be the set of biconnected components of the graph $G$ so that $\hat{V} \cap V_l \neq \emptyset$, where $V_l \subseteq V$ denotes the vertex set of the graph $G_l$ for all $l \in \{1, \ldots, n_B\}$. Now, let $L_{\text{int},l} \subseteq E_{\text{int}}$ and $V'_l \subseteq \hat{V} \setminus (\hat{V} \cap V_l)$ be so that $\varphi_{\text{int}}(L_{\text{int},l}) := [V \cap V_l, V'_l]; l \in \{1, \ldots, n_B\}$. Also, let $L_{\text{int}} := \bigcup_{l=1}^{n_B} L_{\text{int},l}$ and $V'^I := \bigcup_{l=1}^{n_B} V'_l$. Finally, let $E_{\text{free}} \subset K$ denote the set of free ends of the external edges in $\hat{E}_{\text{ext}}$. In this context, define

\[
c_G : QW_{\text{conn}}^{n,k,s} \rightarrow QW_{\text{conn}}^{n-n'+1,k-k',s}; G \mapsto G^*,
\]

where the graph $G^* = (V^*, K^*, E^*); E^* = E_{\text{int}}^* \cup E_{\text{ext}}^*$, together with the maps $\varphi_{\text{int}}^*$ and $\varphi_{\text{ext}}^*$, satisfies the following conditions:

(a) $V^* = \chi_v^{n'}(V \setminus (\hat{V} \setminus \{v_{i_1}\}))$, where

\[
\chi_v^{n'} : v_l \mapsto \begin{cases} v_l & \text{if } l \in \{1, \ldots, i_2 - 1\} \\ v_{l-j+1} & \text{if } l \in \{i_j + 1, \ldots, i_{j+1} - 1\}, j \in \{2, \ldots, n'\} \\ v_{l-n'+1} & \text{if } l \in \{i_{n'} + 1, \ldots, n\} \end{cases}
\]

is a bijection;

(b) $K^* = \chi_v^{k'}(K \setminus \hat{K})$, where

\[
\chi_e^{k'} : e_b \mapsto \begin{cases} e_b & \text{if } b \in \{1, \ldots, j_1 - 1\} \\ e_{b-l} & \text{if } b \in \{j_l + 1, \ldots, j_{l+1} - 1\}, l \in \{2, \ldots, l' - 1\} \\ e_{b-t'} & \text{if } b \in \{j_{v} + 1, \ldots, t\} \end{cases}
\]

is a bijection.
Let $E^* = E_{\text{int}} \cup E_{\text{ext}}^*$, where $E_{\text{int}}^* = [\chi_e^t]^2(E_{\text{int}} \setminus \hat{E}_{\text{int}})$, $E_{\text{ext}}^* = [\chi_e^t]^2(E_{\text{ext}})$;

(d) $\varphi_{\text{int}}^* \circ [\chi_e^t]^2|_{E_{\text{ext}} \setminus (\hat{E}_{\text{int}} \cup E_{\text{int}})} = [\chi_v^n]^2 \circ \varphi_{\text{int}}|_{E_{\text{int}} \setminus (\hat{E}_{\text{int}} \cup E_{\text{int}})}$;

$\varphi_{\text{int}}^*([\chi_e^t]^2|_{E_{\text{int}}}) = [v_{i1}, \chi_e^t(V_{\text{int}})]$;

(e) $\varphi_{\text{ext}}^* \circ [\chi_e^t]^2|_{E_{\text{ext}} \setminus E_{\text{ext}}^*} = [\chi_v^n, \chi_e^t] \circ \varphi_{\text{ext}}|_{E_{\text{ext}} \setminus E_{\text{ext}}^*}$;

$\varphi_{\text{ext}}^*([\chi_e^t]^2|_{E_{\text{ext}}^*}) = [v_{i1}, \chi_e^t(E_{\text{free}})]$;

(f) $l^* = [\chi_e^t, \text{id}_L] \circ l \circ [\chi_e^t^{-1}]^2 : E_{\text{ext}}^* \to [K^*, L]$ is a labeling of the elements of $E_{\text{ext}}^*$.

(vii) Erasing all external edges of a connected graph: Let $G = (V, K, E)$; $E = E_{\text{int}} \cup E_{\text{ext}}$, together with the maps $\varphi_{\text{int}}$ and $\varphi_{\text{ext}}$ denote a graph in $W_{\text{conn}}^{n,k,s}$. Let $K_1, K_2 \subset K$ satisfy $K_1 \cap K_2 = \emptyset$ and $E_{\text{ext}} = [K_1, K_2]$. Let $K_1 \cup K_2 = \{e_i, \ldots, e_{s_2}\}$ with $i_1 < \ldots < i_{2s}$. In this context, define

$$\psi : W_{\text{conn}}^{n,k,s} \to W_{\text{conn}}^{n,k,0}; G \mapsto G^*,$$

where the graph $G^* = (V^*, K^*, E_{\text{int}}^*)$ together with the map $\varphi_{\text{int}}^*$ satisfies the following conditions:

(a) $V^* = V$;

(b) $K^* = [\chi_e^t]^2(K \setminus (K_1 \cup K_2))$, where the bijection $\chi_e^2$ is given in (vi);

(c) $E^* = [\chi_e^t]^2(E_{\text{int}})$;

(d) $\varphi_{\text{int}}^*([\chi_e^t]^2(E_{\text{int}})) = \varphi_{\text{int}}$.

The following lemmas are now established.

**Lemma 1.** Fix integers $s \geq 0$, $k > 0$ and $n > 1$. Then, for all $i \in \{1, \ldots, n\}$, $q_{i}^{(p)}(W_{\text{conn}}^{n,k,s}) \subseteq W_{\text{conn}}^{n+1,k+p-1,s}$.

**Proof.** Let $G$ denote a graph in $W_{\text{conn}}^{n,k,s}$. Apply the map $q_{i S_i}^{(p)} := \frac{1}{2^{(p-1)!}} S_i^{p} \circ s_{i} \geq 1$ to the graph $G$. $s_{i} \geq 1(G)$ is a linear combination of connected graphs. Therefore, by Lemma 5 of [7], the graphs in $q_{i}^{(p)}(G)$ are 2-edge connected. Clearly, they must be biconnected in particular.

**Lemma 2.** Fix integers $s \geq 0$, $k, > 0$ and $n > 1$. (a) Let $G$ denote a graph in $W_{2-\text{edge}}^{n,k,s}$ with only one cut vertex. Then, $q_{i}^{(p)}(G) \in W_{\text{conn}}^{n+1,k+p-1,s}$. (b) Let $G$ denote a graph in $W_{2-\text{edge}}^{n,k,s}$ with at least two cut vertices. Then, $q_{i}^{(p)}(G) - q_{i}^{(p)}(G) \in W_{2-\text{edge}}^{n+1,k+p-1,s} \setminus W_{\text{conn}}^{n+1,k+p-1,s}$.

**Proof.** The proof is trivial.

**Lemma 3.** Fix integers $s \geq 0$, $k, k' > 0$ and $n, n' > 1$. Let $G$ denote a graph in $W_{\text{conn}}^{n,k,s}$. Then, for all $i \in \{1, \ldots, n\}$, $q_i^{(p)}(\mathbb{Q}W_{2-\text{edge}}^{n,k,s}) \subseteq \mathbb{Q}W_{2-\text{edge}}^{n+1,k+p-1,s}$.

**Proof.** The proof is trivial.
4  Recursion formulas

We give a recursion formula to generate all equivalence classes of biconnected graphs. In a recursion step, the formula yields a linear combination of biconnected graphs with the same vertex and cyclomatic numbers. The key feature is that the sum of the (rational) coefficients of all graphs in the same equivalence class corresponds to the inverse of the order of their group of automorphisms. We extend the result to connected or 2-edge connected graphs. The underlying algorithms are amenable to direct implementation via coalgebra in the sense of [8].

4.1 Biconnected graphs

We pick out the terms that generate biconnected graphs in formula (5) of [7].

**Theorem 4.** Fix an integer \( s \geq 0 \). For all integers \( k \geq 0 \) and \( n > 1 \), define \( \beta_{\text{biconn}}^{n,k,s} \in \mathbb{Q}W_{\text{biconn}}^{n,k,s} \) by the following recursion relation:

- \( \beta_{\text{biconn}}^{2,k,s} := \frac{1}{2(2k+1)} \xi_{\text{Ext},\{v_1,v_2\}}(1_{1,2}(P_2)) \), where \( P_2 \in W_{\text{biconn}}^{2,0,0} \) denotes a path on two vertices;

- \( \beta_{\text{biconn}}^{n,0,s} := 0 \), \( n > 2 \);

- \( \beta_{\text{biconn}}^{n,k,s} := \frac{1}{k+n-1} \left( \sum_{i=1}^{n-1} \sum_{\rho \geq 1} \left( \beta_{\text{biconn}}^{n-1,k+1-\rho,s} \right) + \sum_{j=2}^{n-2} \sum_{\rho=1}^{k-j+1} \hat{q}(\rho) \hat{q}_{\text{cutvertex}} \left( \beta_{\text{biconn}}^{n-1,k+1-\rho,s} \right) \right) \),

where for all integers \( j > 1 \), \( n \geq j+1 \) and \( k \geq j \), \( \beta_{j}^{n,k,s} \) is given by the following recursion relation:

- \( \beta_{2}^{n,k,s} := \frac{1}{k+n-1} \sum_{k' = 1}^{k-1} \sum_{n' = 2}^{n-1} \sum_{i=1}^{n'-1} \left( (k'+n'-1) \beta_{\text{biconn}}^{n',k',0} \beta_{\text{biconn}}^{n-n'+1,k-k',s} \right) ; \)

- \( \beta_{j}^{n,k,s} := \frac{1}{k+n-1} \sum_{k' = 1}^{k-1} \sum_{n' = 2}^{n-1} \left( (k'+n'-1) \beta_{\text{biconn}}^{n',k',0} \beta_{\text{biconn}}^{n-n'+1,k-k',s} \right) . \)
Then, for fixed values of $n$ and $k$, $\beta_{n,k,s}^{biconn} = \sum_{G \in W_{n,k,s}^{biconn}} \alpha_G G$; $\alpha_G \in \mathbb{Q}$ for all $G \in W_{n,k,s}^{biconn}$. Moreover, given an arbitrary equivalence class $\mathcal{C} \subseteq W_{n,k,s}^{biconn}$, the following holds: (i) There exists $G \in \mathcal{C}$ so that $\alpha_G > 0$; (ii) $\sum_{G \in \mathcal{C}} \alpha_G = 1/|\text{Aut}(\mathcal{C})|$.

In formula (5), the $q^{|\text{cutvertex} \geq 1}$ summand does not appear when $n < 4$ or $k < 2$.

**Proof.** Note that $\beta_{n,k,s}^{biconn}$ is the linear combination of all equivalence classes of 2-edge connected graphs with only one cut vertex, $j$ biconnected components, $n$ vertices, cyclomatic number $k$ and $s$ external edges, with coefficient given by the inverse of the order of their automorphism group. This is a particular case of Theorem 5 to be given later on. We refer the reader to the next section for the proof. Thus, by Lemmas 1 and 2, the proof of Theorem 4 follows straightforwardly from that of Theorem 18 of [7].

### 4.1.1 Examples

We show the result of computing all mutually non-isomorphic biconnected graphs without external edges as contributions to $\beta_{n,k,0}^{biconn}$ via formula (5) up to order $2 \leq n + k \leq 6$. The coefficients in front of graphs are the inverses of the orders of their groups of automorphisms.

- $n = 2, k = 0$  \[ \frac{1}{2} \]

- $n = 2, k = 1$  \[ \frac{1}{2^2} \]

- $n = 3, k = 1$  \[ \frac{1}{3!} \]

- $n = 2, k = 2$  \[ \frac{1}{2 \cdot 3!} \]

- $n = 4, k = 1$  \[ \frac{1}{8} \]
4.2 Connected graphs

We use the maps $r_{\beta_{\text{biconn}}}^{n,k,s}$ and the linear combination of graphs $\beta_{\text{conn}}^{n,k,s} \in \mathbb{Q} W_{\text{conn}}^{n,k,s}$ given by formulas (3) and (5), respectively, to generate all equivalence classes of connected graphs. The underlying algorithm is so that generated graphs are automatically decomposed into their biconnected components.

**Theorem 5.** Fix an integer $s \geq 0$. For all integers $k \geq 0$ and $n > 1$, define $\beta_{\text{conn}}^{n,k,s} \in \mathbb{Q} W_{\text{conn}}^{n,k,s}$ by the following recursion relation:

- $\beta_{\text{conn}}^{2,k,s} := \beta_{\text{biconn}}^{2,k,s}$,
- $\beta_{\text{conn}}^{n,k,s} := \beta_{\text{biconn}}^{n,k,s} + \frac{1}{k + n - 1}$.
\[ \sum_{k'=0}^{k} \sum_{n'=2}^{n-1} \sum_{j=1}^{n-n'+1} \left( (k' + n' - 1) \beta'_{biconn}^{n',n'-1+k'-k,s} \right), n > 2. \tag{8} \]

Then, for fixed values of \( n \) and \( k \), \( \beta^{n,k,s} = \sum_{G \in W^{n,k,s}} \alpha_G G \); \( \alpha_G \in \mathbb{Q} \) for all \( G \in W^{n,k,s} \). Moreover, given an arbitrary equivalence class \( \mathcal{C} \subseteq W^{n,k,s} \), the following holds: (i) There exists \( G \in \mathcal{C} \) so that \( \alpha_G > 0 \); (ii) \( \sum_{G \in \mathcal{C}} \alpha_G = 1/|\text{Aut}(\mathcal{C})| \).

**Proof.** Let \( G \) denote any connected graph with \( m \geq 1 \) internal edges. We proceed to show that an arbitrary biconnected component of the graph \( G \) with \( m' \geq 1 \) internal edges adds \( m'/|m \cdot |\text{Aut}(G)| \) to the sum of the coefficients of all graphs isomorphic to the graph \( G \). As expected, we thus conclude that every one of the \( m \) internal edges of the graph \( G \) contributes \( 1/|m \cdot |\text{Aut}(G)| \) to that sum.

**Lemma 6.** Fix integers \( s \geq 0 \), \( k \geq 0 \) and \( n > 1 \). Let \( \beta^{n,k,s} = \sum_{G \in W^{n,k,s}} \alpha_G G \in \mathbb{Q}W^{n,k,s} \) be defined by formula (8). Let \( \mathcal{C} \subseteq W^{n,k,s} \) denote an arbitrary equivalence class. Then, there exists \( G \in \mathcal{C} \) so that \( \alpha_G > 0 \).

**Proof.** The proof proceeds by induction on the number of biconnected components \( \mu \). By Theorem 4 the statement holds for all graphs in \( \beta^{n,k,s} \) with only one biconnected component. We assume the statement to hold for graphs in \( \beta^{n,k,s} \) with \( \mu - 1 \geq 1 \) biconnected components. Let \( G \) denote any graph in \( \mathcal{C} \subseteq W^{n,k,s} \) with \( \mu \) biconnected components. Let \( \beta^{n',k',0} = \sum_{G' \in W^{n',k',0}} \eta_{G'} G' \); \( \eta_{G'} \in \mathbb{Q} \) be given by equation (5). Recall that by equation (8) the maps \( \beta_{biconn}^{n',k',0} \) read as
\[ r_{i_{biconn}}^{n',k',0} := \sum_{G' \in W^{n',k',0}} \eta_{G'} r_{i_{biconn}}^{G'}. \tag{9} \]

We proceed to show that a graph isomorphic to \( G \) is generated by applying the maps \( r_{i_{biconn}}^{n',k',0} \) to graphs with \( \mu - 1 \) biconnected components occurring in \( \beta^{n,n'+1,k-k'}_{biconn} = \sum_{G' \in W^{n',n+1,k-k'}} \beta_{biconn}^{n',n+1,k-k',s} G' \); \( \beta_{biconn}^{n,n'+1,k-k',s} \in \mathbb{Q} \) with non-zero coefficient. Let \( \hat{G} \) be an arbitrary biconnected component of the graph \( G \). Let \( \hat{V} = \{v_{i_1}, \ldots, v_{i_{n'}}\} \subset V \) be its vertex set, where \( i_1 < \ldots < i_{n'} \). Also, let \( k' \) denote its cyclomatic number. Contracting the graph \( \hat{G} \) to the vertex \( v_{i_1} \) yields a graph \( c_{\hat{G}}(G) \in W^{n,n'+1,k-k',s} \) with \( \mu = 1 \) biconnected components. Let \( \mathcal{B} \) denote the equivalence class containing the graph \( c_{\hat{G}}(G) \). Let \( G' \in W^{n',k',0} \) be a biconnected graph isomorphic to \( \psi(\hat{G}) \), where \( \psi(\hat{G}) \) is the graph obtained from \( \hat{G} \) by erasing all the external edges. By induction assumption, there exists a graph in \( \mathcal{B} \), say, \( H^* \), so that \( H^* \cong c_{\hat{G}}(G) \) and \( \beta_{H^*} > 0 \). Let \( v_j \) with \( j \in \{1, \ldots, n - n' + 1\} \) be the vertex of the graph \( H^* \) which is mapped to \( v_{i_1} \) of \( c_{\hat{G}}(G) \) by an isomorphism. Applying the map \( r_{j_{biconn}}^{H^*} \) to the graph \( H^* \) yields a linear combination of graphs, one of which, say, \( H \), is isomorphic to \( G \). That is, \( \alpha_H > 0 \) and \( H \cong G \).
Lemma 7. Fix integers \( k \geq 0, \ n \geq 1 \) and \( s \geq n \). Let \( \mathcal{C} \subseteq W_{\text{conn}}^{n,k,s} \) denote an equivalence class. Let \( G = (V,K,E); \ E = E_{\text{int}} \cup E_{\text{ext}}, \) together with the maps \( \varphi_{\text{int}} \) and \( \varphi_{\text{ext}} \) denote a graph in \( \mathcal{C} \). Assume that \( V \cap \varphi_{\text{int}}(E_{\text{ext}}) = V \). Let \( \beta_{\text{conn}}^{n,k,s} = \sum_{G \in W_{\text{conn}}^{n,k,s}} \alpha_{G} \in \mathcal{Q}W_{\text{conn}}^{n,k,s} \) be defined by formula \( \mathfrak{N} \). Then, \( \sum_{G \in \mathcal{C}} \alpha_{G} = 1/|\text{Aut}(\mathcal{C})| \).

Proof. The proof proceeds by induction on the number of biconnected components \( \mu \). By Theorem \( \mathfrak{H} \) the statement holds for all graphs in \( \beta_{\text{conn}}^{n,k,s} \) with only one biconnected component. We assume the statement to hold for graphs in \( \beta_{\text{conn}}^{n,k,s} \) with \( \mu - 1 \geq 1 \) biconnected components. Let the graph \( G \in \mathcal{C} \subseteq W_{\text{conn}}^{n,k,s} \) have \( \mu \) biconnected components. Let \( m = k + n - 1 \) denote its internal edge number. By Lemma \( \mathfrak{B} \) there exists a graph, say, \( H \in \mathcal{C} \) which occurs in \( \beta_{\text{conn}}^{n,k,s} \) with non-zero coefficient. That is, \( H \cong G \) and \( \alpha_{H} > 0 \). Moreover, the graph \( H \in \mathcal{C} \) is so that every one of its vertices has at least one (labeled) external edge. Hence, \( |\text{Aut}_{\text{vertex}}(\mathcal{C})| = 1 \) so that \( |\text{Aut}(\mathcal{C})| = |\text{Aut}_{\text{edge}}(\mathcal{C})| \). We proceed to show that \( \sum_{G \in \mathcal{C}} \alpha_{G} = 1/|\text{Aut}(\mathcal{C})| \). To this end, we check from which graphs with \( \mu - 1 \) biconnected components, the graphs in the equivalence class \( \mathcal{C} \) are generated by the recursion formula \( \mathfrak{S} \), and how many times they are generated.

Choose any one of the \( \mu \) biconnected components of the graph \( H \in \mathcal{C} \). Let this be a graph, say, \( \tilde{G} \), with vertex set \( \tilde{V} = \{ v_{1}, \ldots, v_{n} \} \subseteq V \), where \( i_{1} < \ldots < i_{n} \). Also, let \( k', m' = k' + n' - 1 \) and \( s' \geq n' \) denote its cyclomatic number, internal edge number and external edge number, respectively. Moreover, let \( \mathcal{A} \) denote the equivalence class containing the graph \( \tilde{G} \). Since this is a subgraph of the graph \( H \), \( |\text{Aut}(\mathcal{A})| = |\text{Aut}_{\text{edge}}(\mathcal{A})| \). Contracting the graph \( \tilde{G} \) to the vertex \( v_{i_{1}} \) yields a graph \( c_{\tilde{G}}(H) \in W_{\text{conn}}^{n-n'+1,k-k',s} \) with \( \mu - 1 \) biconnected components. Let \( \mathcal{B} \) denote the equivalence class containing \( c_{\tilde{G}}(H) \). The graphs in \( \mathcal{B} \) have no non-trivial vertex symmetries. Hence, the order of their automorphism group is related to that of the graph \( H \in \mathcal{C} \) via

\[
|\text{Aut}(\mathcal{B})| = \frac{|\text{Aut}(\mathcal{C})|}{|\text{Aut}(\mathcal{A})|}.
\]

Let \( \beta_{\text{conn}}^{n-n'+1,k-k',s} = \sum_{G' \in W_{\text{conn}}^{n-n'+1,k-k',s}} \beta_{G'} \cdot \beta_{G'}; \ \beta_{G'} \in \mathbb{Q} \). By induction assumption,

\[
\sum_{G' \in \mathcal{B}} \beta_{G'} = \frac{1}{|\text{Aut}(\mathcal{B})|}.
\]

Now, let \( \beta_{\text{biconn}}^{n',k',0} = \sum_{G' \in W_{\text{biconn}}^{n',k',0}} \eta_{G'} \cdot G' \in \mathcal{Q} \). Let \( G' \in W_{\text{biconn}}^{n',k',0} \) be a biconnected graph isomorphic to \( \psi(\tilde{G}) \). Let \( \mathcal{D} \subseteq W_{\text{biconn}}^{n',k',0} \) denote the equivalence class containing \( G' \). The order of the automorphism group of the graph \( G' \) is related to that of \( \tilde{G} \) via

\[
|\text{Aut}(\mathcal{D})| = |\text{Aut}(\mathcal{A})| \cdot |\text{Aut}_{\text{vertex}}(\mathcal{D})|
\]

for \( |\text{Aut}(\mathcal{A})| = |\text{Aut}_{\text{edge}}(\mathcal{A})| = |\text{Aut}_{\text{edge}}(\mathcal{D})| \). By Lemma \( \mathfrak{G} \) there exists a graph, say, \( H' \in \mathcal{D} \) so that \( H' \cong c_{\tilde{G}}(H) \) and \( \beta_{H'} > 0 \). Let \( v_{j} \) with \( j \in \mathbb{Z} \).
\{1, \ldots, n - n' + 1\} be the vertex of the graph \(H^*\) which is mapped to \(v_{i_1}\) of \(c_G(H)\) by an isomorphism. Apply the map \(r_j^{G'}\) to the graph \(H^*\). Notice that there are \(|\text{Aut}_{\text{vertex}}(\mathcal{D})|\) ways to distribute the \(s'\) external edges assigned to the vertex \(v_j\) of the graph \(H^*\) between all the vertices of the graph \(\mathcal{E}_j(G')\) so as to obtain a graph in the equivalence class \(\mathcal{C} \ni H\). Therefore, there are \(|\text{Aut}_{\text{vertex}}(\mathcal{D})|\) graphs in the linear combination \(r_j^{G'}(H^*)\) which are isomorphic to the graph \(G\). Clearly, the map \(r_j^{G'}\) produces a graph isomorphic to \(H\) from the graph \(H^*\) with coefficient \(\alpha_{H'} = \beta_{H'} \in \mathbb{Q}\). Now, formula (8) prescribes to apply the maps \(r_j^{G'}\) to the vertex which is mapped to \(v_{i_1}\) by an isomorphism of every graph in the equivalence class \(\mathcal{B}\) occurring in \(\beta_{\text{conn}}^{n-n'+1,k-k',s}\) with non-zero coefficient. Therefore,

\[
\sum_{G \in \mathcal{E}} \alpha_G^* = |\text{Aut}_{\text{vertex}}(\mathcal{D})| \cdot \sum_{G' \in \mathcal{B}} \beta_{G'}^* = \frac{|\text{Aut}_{\text{vertex}}(\mathcal{D})|}{|\text{Aut}(\mathcal{D})|} \cdot \frac{|\text{Aut}_{\text{vertex}}(\mathcal{D})| \cdot |\text{Aut}(\mathcal{D})|}{|\text{Aut}(\mathcal{D})|} = \frac{|\text{Aut}(\mathcal{D})|}{|\text{Aut}(\mathcal{D})|} \cdot \frac{|\text{Aut}(\mathcal{D})|}{|\text{Aut}(\mathcal{D})|} = \frac{|\text{Aut}(\mathcal{D})|}{|\text{Aut}(\mathcal{D})|},
\]

where the factor \(|\text{Aut}_{\text{vertex}}(\mathcal{D})|\) on the right hand side of the first equality, is due to the fact that every graph (with non-zero coefficient) in the equivalence class \(\mathcal{B}\) generates \(|\text{Aut}_{\text{vertex}}(\mathcal{D})|\) graphs in \(\mathcal{C}\). Hence, according to Theorem 4 and formulas (8) and (9), the contribution to \(\sum_{G \in \mathcal{E}} \alpha_G\) is \(m'/(m \cdot |\text{Aut}(\mathcal{D})|)\). Distributing this factor between the \(m'\) internal edges of the graph \(G\) yields \(1/(m \cdot |\text{Aut}(\mathcal{D})|)\) for each edge. Repeating the same consideration for every biconnected component of the graph \(G\) yields that every edge of each biconnected component adds \(1/(m \cdot |\text{Aut}(\mathcal{D})|)\) to \(\sum_{G \in \mathcal{E}} \alpha_G\).

We conclude that every one of the \(m\) internal edges of the graph \(G\) contributes \(1/(m \cdot |\text{Aut}(\mathcal{D})|)\) to \(\sum_{G \in \mathcal{E}} \alpha_G\). Hence, the overall contribution is exactly \(1/|\text{Aut}(\mathcal{D})|\). This completes the proof.

\(\beta_{\text{conn}}^{n,k,s}\) satisfies the following property.

**Lemma 8.** Fix integers \(s \geq 0, k \geq 0\) and \(n > 1\). Let \(\beta_{\text{conn}}^{n,k,s} = \sum_{G \in W^{n,k,s}_{\text{conn}}} \alpha_G G \in \mathcal{Q} W^{n,k,s}_{\text{conn}}\) be defined by formula (8). Moreover, let \(K' \subset K\) be a finite set so that \(K \cap K' = \emptyset\). Let \(E'_{\text{ext}} \subseteq [K']^2; s' := \text{card}(E'_{\text{ext}})\). Assume that the elements of \(E'_{\text{ext}}\) satisfy \(\{e_a, e_{a'}\} \cap \{e_b, e_{b'}\} = \emptyset\). Also, let \(L' = \{x_{s+1}, \ldots, x_{s+s'}\}\) be a label set so that \(L \cap L' = \emptyset\). Let \(l' : E'_{\text{ext}} \to [K', L']\) be a labeling of the elements of \(E'_{\text{ext}}\). Then, \(\beta_{\text{conn}}^{n,k,s+s'} = \xi_{E'_{\text{ext}} \cdot V}(\beta_{\text{conn}}^{n,k,s)}\).

**Proof.** Let \(V^* = \{v_i, v_{n-n'+2}, \ldots, v_n\} \subseteq V\) be the vertex set of all graphs in \(\mathcal{E}(\beta_{\text{biconn}}^{n',k',0}); i \in \{1, \ldots, n-n'+1\}\). Clearly, \(\xi_{E'_{\text{ext}} \cdot V} = \sum_{E'_{\text{ext}} \subseteq E'_{\text{ext}} \cap \mathcal{E}_{\text{ext}, V \setminus V^* \cdot o} \xi_{\mathcal{E}_{\text{ext}, V^*}}} \xi_{E'_{\text{ext}} \cap \mathcal{E}_{\text{ext}, V \setminus V^* \cdot o} \xi_{\mathcal{E}_{\text{ext}, V^*}}} \cdot v^*_{\text{biconn}}\). Furthermore, \(\xi_{E'_{\text{ext}} \cdot V} \circ r_{i}^{\text{biconn}} = r_{i}^{\text{biconn}} \circ \xi_{E'_{\text{ext}} \cdot V} = \xi_{E'_{\text{ext}} \cdot V} \circ r_{i}^{\text{biconn}}\).
\[ \xi_{\text{ext}, \{v_i\}} : \mathcal{Q}W_{\text{conn}}^{n-n'+1,k-k',s} \rightarrow \mathcal{Q}W_{\text{conn}}^{n,k,s+s'}, \] where \( s' = \text{card}(\mathcal{L}_{\text{ext}}) \). Therefore, the equality \( \beta_{\text{conn}}^{n,k,s+s'} = \xi_{E_{\text{ext}}^{'}}(\beta_{\text{conn}}^{n,k,s}) \) follows immediately from the recursive definition (8).

**Lemma 9.** Fix integers \( s \geq 0, k \geq 0 \) and \( n > 1 \). Let \( \mathcal{C} \subseteq \mathcal{W}_{\text{conn}}^{n,k,s} \) denote an arbitrary equivalence class. Let \( \beta_{\text{conn}}^{n,k,s} = \sum_{G \in \mathcal{W}_{\text{conn}}^{n,k,s}} \alpha_G G \in \mathcal{Q}W_{\text{conn}}^{n,k,s} \) be defined by formula (8). Then, \( \sum_{G \in \mathcal{C}} \alpha_G = 1/|\text{Aut}(\mathcal{C})| \).

**Proof.** The proof is the same as that of Lemma 10 of [7] (see also Lemma 10 and Theorem 10 of [9] and [10], respectively).

This completes the proof of Theorem 5.

### 4.2.1 Examples

The present section overlaps Section 5.3.3 of [7]. We show the result of computing all mutually non-isomorphic connected graphs without external edges as contributions to \( \beta_{\text{conn}}^{n,k,0} \) via formula (8) up to order \( 2 \leq n + k \leq 5 \). The coefficients in front of graphs are the inverses of the orders of their groups of automorphisms.

\[ n = 2, k = 0 \quad \frac{1}{2} \quad \bullet - \bullet \]

\[ n = 3, k = 0 \quad \frac{1}{2} \quad \bullet - \bullet - \bullet \]

\[ n = 2, k = 1 \quad \frac{1}{2^2} \quad \bullet - \bullet - \bullet \]

\[ n = 4, k = 0 \quad \frac{1}{2} \quad \bullet - \bullet - \bullet + \frac{1}{3!} \quad \text{star} \]

\[ n = 3, k = 1 \quad \frac{1}{3!} \quad \text{triangle} + \frac{1}{2} \quad \bullet - \bullet - \bullet \]

\[ n = 2, k = 2 \quad \frac{1}{2 \cdot 3!} \quad \bullet - \bullet \]
4.3 2-edge connected graphs

By Lemma 8, Theorem 5 generalizes straightforwardly to 2-edge connected graphs.

Theorem 10. Fix an integer \( s \geq 0 \). For all integers \( k > 0 \) and \( n > 1 \), define \( \beta_{2\text{-edge}}^{n,k,s} \in \mathbb{Q}W_{2\text{-edge}}^{n,k,s} \) by the following recursion relation:

\[
\begin{align*}
\beta_{2\text{-edge}}^{2,k,s} &:= \beta_{\text{biconn}}^{2,k,s}; \\
\beta_{2\text{-edge}}^{n,k,s} &:= \beta_{\text{biconn}}^{n,k,s} + \frac{1}{k + n - 1} \\
&+ \sum_{k' = 1}^{k-1} \sum_{n' = 2}^{n-1} \sum_{i=1}^{n-n'+1} (k' + n' - 1) \alpha_i^{n',k',0} \beta_{2\text{-edge}}^{n-n'+1,k-k',s}, n > 2
\end{align*}
\]

Then, for fixed values of \( n \) and \( k \), \( \beta_{2\text{-edge}}^{n,k,s} = \sum_{G \in W_{2\text{-edge}}^{n,k,s}} \alpha_G G \); \( \alpha_G \in \mathbb{Q} \) for all \( G \in W_{2\text{-edge}}^{n,k,s} \). Moreover, given an arbitrary equivalence class \( \mathcal{C} \subseteq W_{2\text{-edge}}^{n,k,s} \), the
following holds: (i) There exists $G \in \mathcal{C}$ so that $\alpha_G > 0$; (ii) $\sum_{G \in \mathcal{C}} \alpha_G = \frac{1}{|\text{Aut}(\mathcal{C})|}$.

### 4.3.1 Examples

We show the result of computing all mutually non-isomorphic 2-edge connected graphs without external edges as contributions to $\beta_{n,k}^{n,k,0}$ via formula (10) up to order $3 \leq n + k \leq 6$. The coefficients in front of graphs are the inverses of the orders of their groups of automorphisms.

$n = 2, k = 1 \quad \frac{1}{2^2}$

$n = 3, k = 1 \quad \frac{1}{3!}$

$n = 2, k = 2 \quad \frac{1}{2 \cdot 3!}$

$n = 4, k = 1 \quad \frac{1}{8}$

$n = 3, k = 2 \quad \frac{1}{2^2} + \frac{1}{2^3}$

$n = 2, k = 3 \quad \frac{1}{2 \cdot 4!}$

$n = 5, k = 1 \quad \frac{1}{10}$
4.3.2 Algorithmic considerations

We briefly discuss some of the algorithmic implications of the recursive definition (10). In the present section, only graphs without external edges are considered for these may be added via the maps $\xi_{E_{ext}, V}$.

An important algorithmic aspect is to determine \textit{a priori} the nature of the biconnected components of the 2-edge connected graphs generated by formula (10). In this context, the most straightforward simplification is to restrict the formula to graphs whose biconnected components are cycles. Let $C_n$ denote a cycle with $n$ vertices. Clearly, from formula (5) $\beta^{n,1,0}_{\text{2-edge},C} = 1/(2n)C_n$, where for simplicity all graphs in the same equivalence class are identified as the same. Therefore, formula (10) specializes to 2-edge connected graphs with the aforesaid property as follows:

$$
\beta^{n,k,0}_{\text{2-edge},C} := \frac{1}{2n}C_n + \frac{1}{2(k + n - 1)} \sum_{n'=2}^{n-1} \sum_{i=1}^{n-n'+1} r_i C_{n'} (\beta^{n-n'+1,k-1,0}_{\text{2-edge},C}), n > 2,
$$

where $\beta^{n,k,0}_{\text{2-edge},C} \in \mathbb{Q}W^{n,k}_{\text{2-edge}}$ denotes the linear combination of all 2-edge connected graphs with $n$ vertices and cyclomatic number $k$ so that every biconnected component is a cycle. In addition, suppose that one is only interested in calculating all 2-edge connected graphs whose biconnected components have a minimum vertex or cyclomatic number, say, $n_{\text{min}}$ and $k_{\text{min}}$, respectively. This is clearly obtained by changing the lower and upper limits of the sums over $n'$ and $k'$ in formula (10) to $n_{\text{min}}$ and $n - n_{\text{min}} + 1$ or $k_{\text{min}}$ and $k-k_{\text{min}}$, respectively.

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