Directions in Type I spaces

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Abstract

A direction in a Type I space $X = \bigcup_{\alpha<\omega_1} X_\alpha$ is a closed and unbounded subset $D$ of $X$ such that given any continuous $f : X \to \mathbb{L}_{\geq 0}$ (the closed long ray), if $f$ is unbounded on $D$ then $f$ is unbounded on each unbounded subset of $D$. A closed copy of $\omega_1$ is a direction in any Type I space. We study various aspects of directions and show some independence results. A sample: There is an $\omega$-bounded Type I space without direction; $\text{PFA}$ implies that a locally compact countably tight $\omega_1$-compact Type I space contains a direction; if there is a Suslin tree then there is an $\omega_1$-compact Type I manifold without direction; there are Type I first countable spaces which contain directions and whose closed unbounded subsets contain each a closed unbounded discrete subset. We also study a naturel order on the directions of a given space and show that we may obtain various classical ordered types with the space a manifold (often $\omega$-bounded).

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1 Introduction

In this paper ‘space’ means ‘topological space’. A space $X$ is Type I if $X = \bigcup_{\alpha < \omega_1} X_\alpha$ with $X_\alpha$ open, $\overline{X_\alpha}$ Lindelöf and $\overline{X_\alpha} \subset X_\beta$ whenever $\alpha < \beta$. We will assume that none of our Type I spaces are Lindelöf, so that $X \neq X_\alpha$ for any $\alpha$, although this is not a part of the usual definition. A subset of a Type I space $X$ is called bounded if it is contained in some $X_\alpha$, otherwise it is called unbounded (this definition does not depend on the particular sequence $\langle X_\alpha : \alpha \in \omega_1 \rangle$). Likewise, a function into a Type I space is unbounded iff its range is unbounded. We abbreviate ‘closed and unbounded’ by club. We use brackets $\langle \cdot, \cdot \rangle$ for ordered pairs to avoid confusion with open intervals. $\mathbb{L}_{\geq 0}$ denotes the closed long ray (with the point 0), i.e. the space $\omega_1 \times [0,1]$ endowed with the lexicographic order topology. We shall often view $\omega_1$ as a subspace of $\mathbb{L}_{\geq 0}$, identifying the point $\langle \alpha, 0 \rangle \in \mathbb{L}_{\geq 0}$ with $\alpha \in \omega_1$, and writing directly $\alpha \in \mathbb{L}_{\geq 0}$. Every function is assumed continuous unless specified, and the restriction of a function $f$ to a subset $A$ of its domain will be denoted by $f|_A$.

This paper is about the notion (introduced in [1]) of directions in Type I topological spaces. There are many equivalent definitions (see Section 3), one relies on the following partial order.

**Definition 1.1.** The unbounded function order (ufo) $\preceq$ on the subsets of a Type I space $X$ is defined as $A \preceq B$ whenever for all $f : X \rightarrow \mathbb{L}_{\geq 0}$, $f|_A$ unbounded implies $f|_B$ unbounded, or equivalently if $f|_B$ bounded implies $f|_A$ bounded. If $A \preceq B \preceq A$, we say that $A$ and $B$ are ufo-equivalent. We denote the strict order by $\prec$. 

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The ufo is actually a preorder and extends the order given by the inclusion (and its variant ‘inclusion outside of a bounded set’). Notice that $X$ itself is always ufo-maximal. While $\preceq$ is defined on all subsets of $X$, it is of interest only for unbounded subsets, since $f(A)$ is bounded whenever $A$ is. It is moreover easy to check that $A$ is ufo-equivalent to $\overline{A}$ (see Section 2.2). We will thus consider the restriction of $\preceq$ to the set of club subsets of $X$.

**Definition 1.2.** A club subset $D$ of a Type I space $X = \bigcup_{\alpha \in \omega_1} X_\alpha$ is a direction in $X$ iff it is ufo-equivalent to any of its club subsets. If $D = X$, we say that $X$ is a direction in itself. We denote by $(\mathcal{D}_X, \preceq)$ the poset of ufo-equivalent classes of directions in $X$.

In words: $D$ is a direction in $X$ iff for any function $f : X \to \mathbb{L}_{\geq 0}$, if an unbounded subset of $D$ has a bounded image then $f$ is bounded on all of $D$. The definition of direction is relative: the space $X$ consisting of two disjoint copies of $\mathbb{L}_{\geq 0}$ is not a direction in itself, but any copy of $X$ in $Y = \mathbb{L}_{\geq 0} \times S^1$ is a direction in it since $Y$ is a direction in itself by Propositions 1.3 and 6.8 below.

The concept of direction grew out of while investigating the properties of club subsets of finite powers of $\mathbb{L}_{\geq 0}$. The following elementary proposition (due to the author and D. Cimasoni) serves as a motivation for the questions we investigate in this paper.

The first octant $\bar{O}$ is the subspace of $\mathbb{L}_2^{\geq 0}$ given by \[
\{\langle x, y \rangle \in \mathbb{L}_2^{\geq 0} : y \leq x \}\.
\]

**Proposition 1.3** ([1,2]).

(a) $\omega_1$ endowed with the order topology is a direction in itself, and thus any closed copy of $\omega_1$ in a Type I space $X$ is a direction in $X$. The same holds for $\mathbb{L}_{\geq 0}$.

(b) $(\mathcal{D}_O, \preceq)$ is isomorphic to the ordered set $(\{0, 1\}, \leq)$. Indeed, any horizontal line in $O$ is $\prec$ the diagonal, any two horizontal lines are ufo-equivalent, the diagonal is ufo-maximal in $(\mathcal{D}_O, \preceq)$, and any direction in $O$ is ufo-equivalent to either an horizontal line or the diagonal.

(c) If $f : \mathbb{L}_2^{\geq 0} \to \mathbb{L}_{\geq 0}$ is unbounded, then $f$ restricted to the diagonal is unbounded.

Notice that a closed copy of $\omega_1$ in a Type I space has to be unbounded. Point (b) is the reason of the choice of terminology: in $O$ the (ufo equivalent classes of) directions correspond to the two possibilities of escaping: horizontally or diagonally.

In view of this proposition, the following (intentionally vaguely stated) questions felt natural:

1. Which properties do ensure that a Type I space possesses a direction ?
2. What does a direction ‘look like’? Can it be discrete? Can there be many discrete directions in a space?
3. Which properties do ensure that a space contains a direction in itself? How do directions in themselves behave in products and subspaces?
4. Given a Type I space $X$ containing many directions and an unbounded map $f : X \to \mathbb{L}_{\geq 0}$, is there some direction on which $f$ is unbounded?
5. What does $(\mathcal{D}_X, \preceq)$ look like? When does it have a maximal/minimal element (if non-empty)? Can it be ‘complicated’?
6. Do we gain or lose something if one replaces $L \geq 0$ by another Type I space $Y$ and define $Y$-directions in a similar manner?

By ‘many directions’ we mean the following:

**Definition 1.4.** We call an open subset of a Type I space large if it contains a club set. A Type I space is spanned by directions if any large open subset contains a direction.

This paper contains our findings about these questions in various classes of spaces. We have a bias towards manifolds, and we tried to find manifold examples of spaces with such or such property when possible. (By a manifold we mean a Hausdorff connected space each of whose points has a neighborhood homeomorphic to $\mathbb{R}^n$ for some $n$. In a manifold with boundary the boundary points have neighborhoods homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$.) Many of our results have the following format:

**Theorem prototype.** Let $\mathcal{P}$ be a property claiming that a Type I space contains some object related to directions. Then the following holds.

(A) There is an $\omega$-bounded space without the property $\mathcal{P}$.

(B) $\neg\text{SH}$ implies that there is an $\omega_1$-compact first countable space (often locally compact, often a manifold) which does not have property $\mathcal{P}$.

(C) $\text{PFA}$ implies that any $\omega_1$-compact countably tight locally compact space has property $\mathcal{P}$.

$\text{PFA}$ denotes the proper forcing axiom and $\neg\text{SH}$ the negation of the Suslin hypothesis, thus the existence of a Suslin tree. Recall that a space is $\omega_1$-compact iff it does not contain an uncountable closed discrete subset, and $\omega$-bounded iff the closure of any countable subset is compact. An $\omega$-bounded space is countably compact, and a Type I countably compact space $X$ is $\omega$-bounded. A space $X$ is countably tight if whenever $x$ is in the closure of some $A \subset X$, then there is a countable $B \subset A$ having $x$ in its closure.

It happens that for properties related to Questions 2 to 5, the way of getting a counterexample in (A) and (B) is to start with a space in the relevant class which possesses no direction and to feed it to a constructing procedure (adapted to the property) which gives a space that fails to satisfy $\mathcal{P}$ in the same class (or almost). Using a claim by Nyikos (whose proof has not been supplied to this date, see Section 3.2), we can even add the item:

(D) $\Diamond^+ \implies$ there is an $\omega$-bounded first countable space (often a manifold) which does not have property $\mathcal{P}$.

Since this point comes almost for free if Nyikos’ claim is indeed correct, we decided to include it but denoted each occurrence with an asterisk ‘∗’ until a proof of the claim is given.

The scheme (A)–(C) (and *(D)) is not completely accurate since some of the construction procedures either push us outside the class of spaces we are looking at or simply do not work in some cases, but it summarizes a good part of this paper and explains the organization of its sections which we now detail.
The generalities about Type I spaces and directions are gathered in Section 2.

Section 3 is related to Question 1: We show that there is a Type I \( \omega \)-bounded space containing no direction and obtain independence results in the format (B)–(C) above. We also show in Sections 3.3 and 3.4 that it is not possible to obtain \( \omega \)-bounded spaces containing no direction by looking at subspaces of the tree \( 2^{<\omega_1} \) (with a natural topology) or closed preimages of \( \omega_1 \) with ‘small’ fibers.

Section 4 contains the constructions, divided in two parts, and is the longest section of the paper. In 4.1 we build surfaces by gluing copies of the octant in many ways, which yield various examples of ufo-equivalent classes of directions. We show in particular that we can find surfaces \( S \) such that \( \langle D_S, \leq \rangle \) is order-isomorphic to the following: the Cantor set, an antichain of size the continuum, any ordinal \( \omega \), a compact Aronszajn or Suslin line. Some of these constructions can be generalized, yielding the construction procedures mentioned above. This is done in 4.2. We have chosen to gather all the constructions (and the main lemmas about them) in this section, but the reader may skip it until they are needed later.

In Section 5 we investigate Question 4: given a space spanned with directions and an unbounded map \( f : X \to \mathbb{L}_{\geq 0} \), is \( f \) unbounded on some direction? We obtain results of the (B)–(C) type, though (A) is not achieved. We use the spaces \( \text{Up}(X, s) \) of Section 4.2.

Section 6 deals with direction in themselves and Question 3. We first show in 6.1 that they have a close relationship with monoclinic and monolithic spaces (as defined by Nyikos), especially for normal spaces. In 6.2 we investigate what happens when we take the product of a direction in itself with some space. Then in 6.3 we look inside the directions to see whether they always contain a direction of itself. Our result takes the (A)–(C) (and \( * \)(D)) prototypical form above. We use the spaces \( \Sigma(X, s) \) of Section 4.2.

In Section 7 we analyse Question 2. We produce an example of a Type I locally metrizable locally separable space possessing directions and which has the property that any unbounded subset contains an uncountable closed discrete set. In particular, any direction contains a discrete one. If \( \diamondsuit^* \) holds, there is a collectionwise Hausdorff such space. Other similar spaces are given as well, all using spaces \( \Omega(X, A, s) \) of Section 4.2. On the other hand, we show that a regular strongly collectionwise Hausdorff Type I space cannot contain a discrete direction.

In Section 8 we investigate Question 5, in particular the existence of a ufo-maximal or minimal direction in a space. Using spaces \( S^{a,b,r} \) and \( \Xi(X, A, s) \) of Sections 4.1 and 4.2 we obtain a theorem of the form (A)–(C) (and \( *(D) \)), that is, the existence of spaces in various classes containing no ufo-minimal or maximal direction. The corresponding item (C) for maximal directions is missing, though. In 8.2 we introduce a partial order on maps \( X \to \mathbb{L}_{\geq 0} \) and show some relations with the ufo.

We briefly investigate in Section 9 a strengthening of the concept of direction called \( \text{locally dominant subsets} \), and show in \( \text{ZFC} \) (following ideas of P. Nyikos exposed in Section 4.1) the existence of an \( \omega \)-bounded manifold containing no locally dominant subsets and similar results.

Finally, we look at Question 6 in Section 10. It is easy to see that if the space \( Y \) is regular, then a direction in some space \( X \) is also a \( Y \)-direction. But this may fail
for Hausdorff spaces, and we shall provide simple counter-examples.

All sections use the small toolbox of lemmas in Section 2. Other than that, Sections 3, 4, 6.1, 6.2, 8.2 and 10 are (reasonably) self-contained, while the others depend on Sections 3.1–3.2 and 4.

It might be important to note that all of our uses of PFA are indirect, in the sense that we use some of its consequences obtained by various authors. Likewise, our use of ♦∗ is through a result of Devlin and Shelah. Actually, no purely ‘set-theoretic’ construction is done in this paper, and we shall mainly manipulate Type I surfaces and ω₁-trees, building more complicated spaces either by gluing simple parts together, or through inductive processes of length ω₁ (or both). A good knowledge of ω₁-trees and ordinal manipulation should be mostly sufficient to read this paper.

Before ending this introduction, a word about separation axioms. Every space is assumed to be Hausdorff, but most results assume regularity (which implies Hausdorff in our definition) or some weaker form of it. The reason is that regularity of a Type I space X ensures that it is Tychonoff, and that there is at least one unbounded map $X \rightarrow \mathbb{L}_{≥0}$, see Lemma 2.3. This is not true for Hausdorff spaces (as seen in Section 10), which might render the definition of direction somewhat vacuous.

Disclaimer. This paper was mainly undertaken as a solo free-time project after our time at the university. As a consequence, we might have a very partial knowledge of both the present state of the subject and the literature, which yields a tendency to over-cite our own work, and maybe to ignore other’s containing similar (or stronger) results. We apologize if it is indeed the case, and would appreciate any comments.

Acknowledgements. This work started through discussions with P. Nyikos and S. Greenwood in Auckland in June 2006. Some of the results presented here are either due to P. Nyikos (and quoted such) or were obtained during these discussions (and might be misquoted). We thank both of them as well as D. Gauld and the Auckland University for the kind invitation. We also thank D. Cimasoni.

2 Generalities

2.1 Type I spaces

Recall that every space is assumed to be Hausdorff and every function continuous, otherwise specified. We identify an ordinal with the set of its predecessors, ω is thus the set of natural numbers and ω₁ the set of countable ordinals.

Given a Type I space $X = \cup_{α ∈ ω₁} X_α$, the sequence $⟨X_α : α ∈ ω₁⟩$ is canonical iff $X_α = \cup_{β < α} X_β$ for all limit α. Any sequence can be made canonical by adding the missing $X_α$’s. When a Type I space X is given, we shall implicitly assume that it comes with a canonical sequence $⟨X_α : α ∈ ω₁⟩$. Two canonical sequences agree on a club subset of ω₁.

The sets $B_α(X) = \overline{X_α} - X_α$ are called the bones of X, and the union of the bones is the skeleton of X, denoted by $Sk(X)$. For notational convenience, we assume that $X_0$ is empty and define $B_0(X)$ to be any point in $X_1$. 
(This makes the definition of a slicer below more straightforward.) The next three lemmas are well known, and we will often use them (sometimes implicitly). We give them without proof.

**Lemma 2.1.** Let $C$ be club in a Type I space $X = \bigcup_{\alpha<\omega_1} X_\alpha$. Then the following holds.
(a) If $X$ is $\omega_1$-compact, then $C \cap B_\alpha(X) \neq \emptyset$ for a stationary set of $\alpha$.
(b) If $X$ is $\omega$-bounded, then $C \cap B_\alpha(X) \neq \emptyset$ for a club set of $\alpha$.

**Lemma 2.2.** Let $X$ be a Type I space. Then $X$ is countably compact iff $X$ is $\omega$-bounded.

**Lemma 2.3.** Let $X$ by Type I. Then $A \subset X$ is open (resp. closed) iff $A \cap X_\alpha$ is open (resp. $A \cap X_\alpha$ is closed) for each $\alpha$, and $f : X \rightarrow \mathbb{L}_{\geq 0}$ is continuous iff $f \big|_{X_\alpha}$ is continuous for each $\alpha$.

It will be convenient to slice Type I spaces with a continuous function.

**Definition 2.4.** Let $X$ be Type I. A slicer is a map $s : X \rightarrow \mathbb{L}_{\geq 0}$ such that $s(X_{\alpha+1} - X_\alpha) \subset [\alpha, \alpha+1]$.

Notice that $s(B_\alpha(X)) = \{\alpha\}$ if $B_\alpha(X)$ is non-empty.

**Lemma 2.5.** Let $X = \bigcup_{\alpha \in \omega_1} X_\alpha$ be a Type I regular space. Then $X$ is Tychonoff, each $X_\alpha$ is normal, and there is a slicer $s : X \rightarrow \mathbb{L}_{\geq 0}$.

**Proof.** Recall that a regular Lindelöf space is normal, so $X_\alpha$ is normal, and $X$ is Tychonoff. By induction, suppose that $f$ is defined on $X_\beta$ for all $\beta < \alpha$ such that $f(B_\beta(X)) = \{\beta\}$ if $B_\alpha(X) \neq \emptyset$. If $\alpha$ is limit define $f$ to be constant of value $\alpha$ on $B_\alpha(X)$, otherwise define $f$ on $X_{\alpha+1}$ that agrees with $f$ on $\bigcup_{\beta < \alpha} X_\beta$ and takes the value $\alpha + 1$ on $B_{\alpha+1}(X)$ using the normality of $X_{\alpha+1}$. Tietze’s extension theorem, and the fact that $B_{\alpha+1}(X) \cap \bigcup_{\beta < \alpha} X_\beta = \emptyset$. \hfill \Box

The next lemma will be useful in various places.

**Lemma 2.6.** Let $X$ be Type I and $\omega_1$-compact, and $f : X \rightarrow \mathbb{L}_{\geq 0}$ be unbounded on a club $E \subset X$. Then the set
$$C(f, E) = \{x \in E : \text{there is } \alpha \in \omega_1 \text{ with } x \in B_\alpha(X) \cap f^{-1}(\langle \alpha, \omega_1 \rangle)\}$$
is club in $X$.

**Proof.** It is clear that $C(f, E)$ is closed, we show that it is unbounded. Let $\alpha_0 \in \omega_1$, we define $\alpha_\gamma$ by induction. If $\gamma$ is limit, set $\alpha_\gamma = \sup_{\beta < \gamma} \alpha_\beta$. Given $\alpha_\gamma$, take some $\alpha_{\gamma+1} \in (X - X_{\alpha_\gamma+1}) \cap E$ such that $f(x_{\gamma+1}) \geq \alpha_\gamma$. Set $\alpha_{\gamma+1}$ be minimal such that $x_{\gamma+1} \in X_{\alpha_{\gamma+1}}$. Set $F \subset E$ to be the closure of $\{x_{\gamma+1} : \gamma \in \omega_1\}$, by Lemma 2.1 $F$ intersects a stationary subset of bones with index $\alpha_\gamma$ such that $\gamma$ is limit. But for such $\gamma$, a point $x$ in $F \cap B_{\alpha_\gamma}(X)$ is either equal to $x_{\gamma+1}$ or is an accumulation point of some $x_\beta$’s for smaller $\beta$’s converging to $\gamma$, and thus $f(x) \geq \alpha_\gamma$. \hfill \Box
2.2 Directions: Equivalent definitions and basic properties

**Lemma 2.7.** Let $X$ be Type I and $D \subset X$ be club. The following are equivalent:
(a) $D$ is a direction in $X$,
(b) for all maps $f : X \to \mathbb{L}_{\geq 0}$, either $f$ is bounded on $D$ or for all $\alpha \in \omega_1$, $f^{-1}([0, \alpha]) \cap D$ is bounded,
(c) there is no map $f : X \to \mathbb{L}_{\geq 0}$ with $f|_D$ unbounded and $f^{-1}(\{0\}) \cap D$ unbounded,
(d) for every $f : X \to \mathbb{L}_{\geq 0}$ with $f|_D$ unbounded and all $\alpha \in \omega_1$ there is $\gamma(\alpha) \in \omega_1$ such that $f(D - X_{\gamma(\alpha)}) \subset [\alpha, \omega_1)$,
(e) for every $f : X \to \mathbb{L}_{\geq 0}$ with $f|_D$ unbounded there is a club $C \subset \omega_1$ such that $f(D - X_\alpha) \subset [\alpha, \omega_1)$ for each $\alpha \in C$,
(f) for every $f : X \to \mathbb{L}_{\geq 0}$ and $\alpha \in \omega_1$, either $f^{-1}([0, \alpha]) \cap D$ is bounded, or $f^{-1}((\alpha, \omega_1)) \cap D$ is bounded.

**Proof.** (a) is easily seen equivalent to (b) and to (d), while (b) $\to$ (c) and (e) $\to$ (d) are immediate.
(c) $\to$ (b) If (b) does not hold for $D$, then there is a $g : X \to \mathbb{L}_{\geq 0}$ unbounded on $D$ and an $\alpha$ such that $g^{-1}([0, \alpha]) \cap D$ is unbounded. Let $p : \mathbb{L}_{\geq 0} \to \mathbb{L}_{\geq 0}$ be continuous such that $p([0, \alpha]) = 0$ and $p|_{[\alpha+1, \omega_1]} = id$, then $f = p \circ g$ yields a contradiction.
(d) $\to$ (e) By a leapfrog argument and continuity of $f$.
(b) $\to$ (f) If $f$ is unbounded on $D$, then $f^{-1}((\alpha, \omega_1)) \cap D$ is unbounded.
(f) $\to$ (d) If $f$ is unbounded on $D$, then $f^{-1}([0, \alpha]) \cap D \subset X_{\gamma(\alpha)}$ for some $\gamma(\alpha)$. □

An unbounded subset $A$ of a Type I space is a predirection if it satisfies the definition of direction except the requirement that $A$ is closed.

**Lemma 2.8.** $A$ is a predirection in $X$ iff $\overline{A}$ is a direction in $X$.

**Proof.** Let $f : X \to \mathbb{L}_{\geq 0}$, then $f$ is unbounded on $\overline{A}$ iff it is unbounded on $A$, $f^{-1}((0)) \cap A$ is bounded if $f^{-1}((0)) \cap \overline{A}$ is bounded, so $A$ is a predirection whenever $\overline{A}$ is a direction. If $A$ is a predirection and $f$ is unbounded on $A$ then $f^{-1}([0, 1)) \cap A$ is bounded, say by $\alpha$. If $x \in (\overline{A} - A) \cap f^{-1}((0))$ then $f^{-1}([0, 1))$, being a neighborhood of $x$, has a non-empty intersection with $A \cap f^{-1}([0, 1))$. Thus $f^{-1}((0)) \cap \overline{A} \subset \overline{X_\alpha}$, and $\overline{A}$ is a direction. □

It might be worth mentioning also the following simple lemma.

**Lemma 2.9.** Let $A \subset \omega_1$ be endowed with the subspace topology. Then $A$ is a direction in itself iff $A$ is stationary.

**Proof.** If $A$ is not stationary, it avoids a club subset of $\omega_1$, and it is easy to define a function witnessing that $A$ is not a direction in itself. If $A$ is stationary and $f : A \to \mathbb{L}_{\geq 0}$, then the closures of $f^{-1}((0))$ and $f^{-1}([1, \omega_1))$ in $\omega_1$ cannot be both unbounded, otherwise their common intersection with $A$ is non-empty, which contradicts their definition. Thus if $f$ is unbounded, $f^{-1}((0))$ must be bounded. □
3 Spaces with and without directions

3.1 A type I $\omega$-bounded subspace of $\beta\omega$ containing no direction

Recall that $\beta\omega$ is the Čech-Stone compactification of $\omega$, and that $\omega^* = \beta\omega - \omega$. Both are compact Hausdorff non-countably tight spaces.

**Theorem 3.1.** There is a Type I $\omega$-bounded space $Q$ (which is a subspace of $\beta\omega$) containing no direction.

We start with a lemma.

**Lemma 3.2.** There is an $\omega$-bounded Type I subspace $X$ of $\omega^*$ such that $X_\alpha$ is Lindelöf and $X_{\alpha+1}$ compact for each $\alpha$.

**Proof.** Let $p \in \omega^*$. It is shown in [20] Corollary 1.5.4] that the cover $U = \{U \subset \omega^* : U$ is clopen and $p \notin U\}$ of $\omega^* - \{p\}$ has no countable subfamily whose union is dense in $\omega^* - \{p\}$. We define $X_\alpha$ by induction on $\alpha$, each will be an at most countable union of members of $U$. Set $X_0 = \emptyset$. Given $X_\alpha$, take a finite family $F \subset U$ that covers the compact set $\overline{X_\alpha}$ and some $U \in U$ not included in $\bigcup F$. Set $X_{\alpha+1} = U \bigcup \bigcup F$. Then $X_{\alpha+1}$ is clopen (and thus compact). For limit $\alpha$ define $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$, then $X_\alpha$ is Lindelöf and not dense in $\omega^* - \{p\}$. Then $X = \bigcup_{\alpha < \omega_1} X_\alpha$ is a Type I $\omega$-bounded space. $\Box$

We will need the following classical result (see [20] Theorem 1.5.2] for the proof of a more general fact).

**Lemma 3.3.** Let $Y \subset \beta\omega$ be Lindelöf. Then every function $g : Y \to [0,1]$ can be continuously extended to a function $\beta\omega \to [0,1]$.

**Proof of Theorem 3.1** Take $Q = X = \bigcup_{\alpha < \omega_1} X_\alpha$ given by Lemma 3.2. Let $C$ be club in $X$. Up to changing the canonical sequence, we may assume that $C \cap (X_{\alpha+1} - X_\alpha)$ contains at least two points. We shall define closed subsets $D_\alpha, E_\alpha \subset C \cap \overline{X_\alpha}$ and $f_\alpha : \beta\omega \to [0,\alpha] \subset L_{\geq 0}$ such that the following holds:

1. $f_\alpha(D_\alpha) = \{0\}$, $f_\alpha(E_\alpha) = \{0, 1, 2, \ldots, \alpha\} \subset L_{\geq 0}$,
2. $f_\alpha|_{X_\beta} = f_\beta|_{X_\beta}$ whenever $\beta \leq \alpha$,
3. $D_\alpha \cap X_\beta = D_\beta$ and $E_\alpha \cap X_\beta = E_\beta$ whenever $\beta \leq \alpha$.

Once these are defined, set $D = \bigcup_{\alpha < \omega_1} D_\alpha$, $E = \bigcup_{\alpha < \omega_1} E_\alpha$. Then the map $f : X \to L_{\geq 0}$ defined by $f(x) = f_\alpha(x)$ for $\alpha$ such that $x \in X_\alpha$ is continuous, unbounded on $E$ and constant on $0$ on $D$. Since $D, E \subset C$, this shows that $C$ is not a direction in $X$. So, start with $E_0 = F_0 = \emptyset$. Let now $\alpha$ be successor or $0$, then $X_\alpha = \overline{X_\alpha}$. Take $d, e \in C \cap (X_{\alpha+1} - X_\alpha)$. The map $g : X_\alpha \cup \{d, e\} \to [0, \alpha + 1] \subset L_{\geq 0}$ defined as

$$g(x) = \begin{cases} f_\alpha(x) & \text{if } x \in X_\alpha \\ 0 & \text{if } x = d \\ \alpha + 1 & \text{if } x = e \end{cases}$$

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is continuous. Since $X_\alpha \cup \{d, e\}$ is compact, there is some $f_\alpha : \beta \omega \to [0, \alpha + 1]$ extending $g$. Set $D_{\alpha + 1} = D_\alpha \cup \{d\}$, $E_{\alpha + 1} = E_\alpha \cup \{e\}$, by construction the above conditions are fulfilled.

Let now $\alpha$ be limit. The map $g : X_\alpha \to [0, \alpha]$ defined by $g|_{X_\beta} = f_\beta|_{X_\beta}$ for $\beta < \alpha$ is continuous. Since $X_\alpha$ is Lindelöf, $g$ extends to a function $f_\alpha : \beta \omega \to [0, \alpha + 1]$. Set $D_\alpha = \bigcup_{\beta < \alpha} D_\beta \cup (f_\alpha^{-1}(\{0\}) \cap C \cap B_\alpha(X))$, $E_\alpha = \bigcup_{\beta < \alpha} E_\beta \cup (f_\alpha^{-1}(\{\alpha\}) \cap C \cap B_\alpha(X))$.

By construction $D_\alpha$ and $E_\alpha$ are closed in $X$ and satisfy the above conditions (1)–(3). This finishes the proof. 

**3.2 $\omega_1$-compact and $\omega$-bounded countably tight spaces**

The aim of this section is to prove the following pair of theorems.

**Theorem 3.4** (PFA). Let $X$ be countably tight, locally compact, $\omega_1$-compact and Type I, and $C \subset X$ be club. Then $X$ contains a closed copy of $\omega_1$ and thus a direction.

**Theorem 3.5** ($\neg$SH). There is an $\omega_1$-compact Type I surface $S_T$ containing no direction.

We thank P. Nyikos for the ideas leading to these theorems. Theorem 3.4 is a simple consequence of two results by other authors.

**Theorem 3.6.** (Balogh, Eisworth and al.) (PFA) Any countably tight perfect preimage of $\omega_1$ contains a closed copy of $\omega_1$.

Recall that a map is perfect if it is closed and any point has a compact preimage. Balogh showed in [5] the result for spaces of character $\leq \aleph_1$ and Eisworth proved it for countably tight spaces in [8]. For first countable spaces, the conclusion holds in a model of $\text{ZFC} + \text{CH}$ without inaccessible cardinals by a result of Eisworth and Nyikos [9].

**Lemma 3.7.** (PFA) Let $X$ be Type I, countably tight and $\omega$-bounded. Then $X$ contains a closed copy of $\omega_1$.

**Proof.** $X$ is $\omega$-bounded and thus countably compact, so the bones $B_\alpha(X)$ are compact, and $\text{Sk}(X)$ is a perfect preimage of $\omega_1$. 

**Proof of Theorem 3.4** The First Trichotomy Theorem of Eisworth and Nyikos [10] shows that PFA implies that a locally compact space is either a countable union of $\omega$-bounded subspaces, or has a closed uncountable discrete space, or has a Lindelöf subset with a non-Lindelöf closure. The latter two are impossible if $X$ is Type I and $\omega_1$-compact, thus $X$ is a countable union of $\omega$-bounded subspaces, one of which must be unbounded. We may thus assume that $X$ is $\omega$-bounded and conclude with Lemma 3.7.

**Problem 1.** Is there a model of $\text{ZFC}$ where any countably tight (or first countable) $\omega$-bounded Type I space contains a direction but not necessarily a closed copy of $\omega_1$?
We now prove Theorem 3.5 in a sequence of easy lemmas. We refer to [13] for the definitions and basic properties of Aronszajn and Suslin trees.

**Lemma 3.8.** Let \( T \) be an Aronszajn tree, and \( A \subseteq T \) be unbounded. Then there are incomparable \( x, y \in T \) with \( s \cap \{ z \in T : z \leq x \} \) and \( A \cap \{ z \in T : z \geq y \} \) both unbounded.

**Proof.** Suppose that there do not exist \( x \) and \( y \) as in the statement. Therefore, letting \( E \) be the set of \( y \in T \) such that there are uncountably many points of \( S \) above it, \( E \) intersects each level of \( T \) in exactly one point. But \( E \) is then a chain of type \( \omega_1 \), a contradiction. \( \square \)

If \( X \) is a Type I manifold, recall from [14] that \( \Upsilon(X) \) is the tree defined by the non-metrizable component boundaries of \( X \), i.e. \( \sigma \in \Upsilon(X) \) if it is the boundary in \( X \) of a non-metrizable component of \( X - X_\alpha \) (for some \( \alpha \)), \( \sigma \leq \sigma' \) whenever \( \sigma, \sigma' \) bound the components \( S, S' \), respectively, and \( S' \subset S \).

**Lemma 3.9.** Let \( M \) be a Type I manifold such that \( \Upsilon(M) \) is an Aronszajn tree. Then, \( M \) contains no direction.

**Proof.** Let \( D \) be club, we show that there are \( D_1, D_2 \) disjoint club subsets of \( D \) and a map \( f : M \to L_{\geq 0} \) with \( f|_{D_1} \) bounded and \( f|_{D_2} \) unbounded. Let \( \Upsilon(D) \subseteq \Upsilon(M) \) be the subtree of boundaries \( \partial C \) such that \( C \) intersects \( D \). Since \( D \) is club, \( \Upsilon(D) \) is unbounded, so there must be two non-metrizable components \( C_1, C_2 \) of \( M \) such that \( D_1 = D \cap C_1 \) and \( D_2 = D \cap C_2 \) are unbounded by Lemma 3.8. Let \( s : M \to L_{\geq 0} \) be the slicer given by Lemma 2.5. Let \( U \) be a neighborhood of \( \partial C_1 \in C_1 \). Define \( g : M \to L_{\geq 0} \) such that \( g|_{M-C_1} = s|_{M-C_1} \), \( g|_{C_1-U} \equiv 0 \), using the fact that \( \partial C_1, \partial C_2 \) are disjoint and included in the normal subspace \( M_\alpha \) for some \( \alpha \). \( \square \)

**Remark 3.10.** The previous lemma also holds under the weaker following assumptions: \( X \) is regular and for each \( \alpha \in \omega_1 \), \( X - \overline{X_\alpha} = \bigcup_{j \in J(\alpha)} U_{\alpha,j} \), where \( \{ U_{\alpha,j} : j \in J(\alpha) \} \) is a family of open unbounded subsets which are 2-by-2 disjoints, such that \( \{ U_{\alpha,j} : \alpha \in \omega_1, j \in J(\alpha) \} \) forms an Aronszajn tree for the reverse inclusion \( \supseteq \).

It is easy to build a surface \( S_T \) having a prescribed Hausdorff \( \omega_1 \)-tree \( T \) as \( \Upsilon \)-tree. There are many constructions, let us quickly describe one. The construction is easier to describe when the tree is binary, we shall do this case since it is enough for our purposes. First, build the road space \( R_T \) by inserting a line segment (0,1) between each consecutive points in \( T \). We then ‘fatten’ \( R_T \) to obtain \( S_T \), which is the union of equilateral triangles (with deleted vertices) \( M_x \) for \( x \in T \) that are glued together according to \( T \), as in Figure 1(left, middle), where \( y_0, y_1 \) are the immediate successors of \( x \in T \), and similarly for \( z_{i,0}, z_{i,1} \) and \( y_i \) (\( i = 0,1 \)). Inside the triangles, the topology is the usual one, and neighborhoods of points in the boundaries at successor heights are clear. A neighborhood of a point in the bottom boundary of \( M_x \) for \( x \) at a limit height is a strip zigzagging the surface down to a smaller height, as the one partly depicted in darker grey in Figure 1(middle). Notice that each \( M_x \) is a countable union of compact larger and larger ‘wedges’ \( W_{n,x} \) (\( n \in \omega \)) shown in Figure 1(right). An uncountable discrete subset \( E \) of \( S_T \) has an uncountable subset that intersects
∪_{x \in T} W_{n,x} for some fixed n. If there is a sequence x_k ∈ T converging to x ∈ T such that E ∩ W_{n,x_k} ≠ ∅, then there is an accumulation point in the bottom boundary of E ∩ W_{n,x}. It follows that ST is ω₁-compact iff T is ω₁-compact. For more details, see [4] where a variant of this construction is given.

Proof of Theorem 3.5. Start with a Hausdorff binary Suslin tree. A closed discrete subset of a Hausdorff tree is a countable union of antichains (see Theorem 4.11 in [16]), so a Hausdorff Suslin tree is ω₁-compact. Now ST described above is ω₁-compact and satisfies Υ(ST) = T. We conclude via Lemma 3.9.

Notice that Lemma 3.9 shows that it is easy to obtain Type I surfaces without direction in ZFC. Without additional axioms, the surface cannot be shown to be ω₁-compact. This construction cannot be generalized to obtain an ω₂-bounded space, since ω₂-bounded trees contain a copy of ω₁. On the other hand, Nyikos has announced the following theorem some years ago, which would give a consistent example of an ω₂-bounded Type I surface without direction, but his construction is still unpublished.

Since some of our results rely on it, we quote it with a ‘∗’, and stress that all our results depending on it do hold only under the proviso that Nyikos construction is indeed correct.

∗Theorem 3.11. (Nyikos) (♦⁺) There is an ω₂-bounded surface N containing no direction.

The next two sections show that such an example cannot be too simple.

3.3 Closed preimages of ω₁

A closed preimage of ω₁ is a space X such that there is a closed map π : X → ω₁, with π⁻¹(\{α\}) ≠ ∅ for all α. Under some weak axioms, it is not hard to find such spaces with very small fibers that do not contain a closed copy of ω₁, for instance, it is consistent with MA+¬CH that there are closed 2-to-1 preimages of ω₁ (i.e. |π⁻¹(\{α\})| = 2 for all α) that do not contain a copy of ω₁ (this result of Nyikos is published in [11, 19.1], see also [17]). This cannot be shown in ZFC by Theorem 3.6.

We shall show here that the fibers must be bigger for a space to contain no direction.
Let \( \langle \mathcal{C}_X, \subseteq \rangle \) denote the poset of club subsets of a Type I space \( X \), ordered by the inclusion. The following lemma is well known, (see [9], Prop. 2.2, for instance).

**Lemma 3.12.** Let \( X \) be a closed preimage of \( \omega_1 \). Then the following holds.
(a) \( \langle \mathcal{C}_X, \subseteq \rangle \) is downward countably closed, i.e. if \( \{C_i\}_{i \in \omega} \) is a sequence of clubs of \( X \) with \( C_i \supseteq C_{i+1} \) for all \( i \), then there is a club \( C \subset X \) with \( C \subset C_i \forall i \in \omega \).
(b) If \( C \subset X \) is club, \( \{\alpha : C \cap \pi^{-1}(\{\alpha\}) \neq \emptyset\} \) is club in \( \omega_1 \).

**Remark 3.13.** Lemma 3.12 (a) also holds for countably compact Type I spaces.

**Theorem 3.14.** Let \( X \) be a closed preimage of \( \omega_1 \) and \( S \subset \omega_1 \) be stationary.
(a) If \( |\pi^{-1}(\{\alpha\})| \leq \omega \) for \( \alpha \in S \), then \( X \) contains a direction.
(b) Suppose that \( 2^\omega \geq \lambda > \omega_1 \), with \( \lambda \) regular, and that \( |\pi^{-1}(\{\alpha\})| \leq \lambda \) for \( \alpha \in S \subset \omega_1 \). Then \( X \) contains a direction.

This theorem is an easy consequence of the following lemma. It is convenient to introduce some notation. Recall that \( 2^{<\omega_1} \) is the set of sequences of 0 and 1 of length \( < \omega_1 \) ordered by the inclusion, which makes it a tree. We say that a Type I space \( X \) is \( 2^{<\omega_1} \)-spiked if there are clubs \( C_t \subset X \) for \( t \in 2^{<\omega_1} \) such that \( C_t \cap C_s \neq \emptyset \) iff \( s \) and \( t \) are non-comparable for the inclusion, and \( C_t \subset C_s \) whenever \( t \supset s \).

**Lemma 3.15.** Suppose that \( X \) is Type I with \( \langle \mathcal{C}_X, \subseteq \rangle \) countably closed. If \( X \) does not contain a direction, then any club in \( X \) is \( 2^{<\omega_1} \)-spiked.

**Proof.** We proceed by induction. Let \( C \) be a club in \( X \) and set \( C_\emptyset = C \). If \( t \in 2^{<\omega_1} \) is at a limit level, set \( C_t = \cap_{s \subset t} C_s \), which is club by assumption. Given \( C_t \) for \( t \in 2^{<\omega_1} \), choose \( f_t : X \to \mathbb{L}_{>0} \) such that \( f_{t}^{-1}(\{0\}) \cap C_t \) is club and \( \sup_{C_t} \) unbounded. Set \( C_{t,0} = f_t^{-1}(\{0\}) \cap C_t \) and \( C_{t,1} = f_t^{-1}(\{1, \omega_1\}) \cap C_t \).

**Proof of Theorem 3.14**. Let \( X \) be a closed preimage of \( \omega_1 \) containing no direction. By Lemmas 3.12 and 3.15 there is a collection \( C_t \) of pairwise disjoint club subsets of cardinality the continuum.

(a) Take a countably infinite subset \( \{C_i : i \in \omega\} \) of the \( C_t \), then there is some \( \alpha \in S \cap \bigcap_{i \in \omega} \pi(C_i) \). Then \( \pi^{-1}(\{\alpha\}) \) contains at least \( \omega \) points, a contradiction.
(b) Let \( K_t \subset \omega_1 \) be the set of \( \alpha \) such that \( C_t \cap \pi^{-1}(\{\alpha\}) \neq \emptyset \). By Lemma 3.12 (b), \( K_t \) is club, let thus \( \alpha_t \in S \cap K_t \). Since \( \lambda > \omega_1 \) is regular, there is some \( \alpha \in S \) such that \( \{t \in 2^{\omega_1} : \alpha_t = \alpha\} = \lambda > \omega_1 \). Thus, there are \( \lambda \) disjoint sets intersecting \( \pi^{-1}(\{\alpha\}) \), which contradicts \( |\pi^{-1}(\{\alpha\})| \leq \lambda \).

**Corollary 3.16.** If \( \text{cf}(2^\omega) > \omega_1 \), any closed preimage of \( \omega_1 \) without direction must satisfy \( |\pi^{-1}(\{\alpha\})| \geq 2^\omega \) for club many \( \alpha \).

**Proof.** Otherwise \( |\pi^{-1}(\{\alpha\})| < 2^\omega \) for stationary many \( \alpha \). Taking \( \kappa \) to be the supremum of \( |\pi^{-1}(\{\alpha\})| \) for these \( \alpha \), we have \( \kappa < 2^\omega \), and the result follows.
3.4 Subsets of $2^{<\omega_1}$

The aim of this section is to show that there is no hope of finding an $\omega$-bounded Type I space without direction by looking at subsets of the tree $2^{<\omega_1}$ endowed with the ‘accumulated product topology’, i.e. a neighborhood base for $t \in 2^\alpha$ is given by the sets of the form $\cup_{\beta<\gamma \leq \alpha} B(\gamma; \alpha_1, \ldots, \alpha_n)$ where $\alpha_1 < \cdots < \alpha_{n-1} < \alpha_n \leq \beta < \alpha$ and

$$B(\gamma; \alpha_1, \ldots, \alpha_n) = \{s \in 2^\gamma : s(\alpha_i) = t(\alpha_i) \text{ for } i = 1, \ldots, n\}.$$

Notice in passing that $2^\alpha$ is homeomorphic to the Cantor set in $2^{<\omega_1}$, which is $\omega$-bounded, locally metrizable, $2^{<\omega_1}$-spiked (set $C_t = \{s \in X : s \geq t\}$), but contains a lot of directions. We could hope that one of its unbounded subsets has a closure that does not contain any direction. Not surprisingly this is not the case.

**Proposition 3.17.** Let $2^{<\omega_1}$ be endowed with the above topology. The following holds.

(a) Let $D$ be a direction in $2^{<\omega_1}$. Then $D$ is contained in a branch of $2^{<\omega_1}$ outside of a compact subset.

(b) Let $A \subset 2^{<\omega_1}$ be unbounded. Then $\overline{A}$ contains a closed copy of $\omega_1$.

**Proof.**

(a) As in Lemma 3.9 if there are non-comparable $x, y \in 2^{<\omega_1}$ with $\{z \in D : z \geq x\}$ and $\{z \in D : z \geq y\}$ both unbounded then $D$ is not a direction. Therefore, for each $\alpha \in \omega_1$ there is a unique $x \in 2^\alpha \subset 2^{<\omega_1}$ with $\{z \in D : z \geq x\}$ unbounded.

(b) Suppose that there is some unbounded $A \subset 2^{<\omega_1}$ with $\overline{A}$ containing no copy of $\omega_1$.

We can assume that $A \cap 2^\alpha = \{x_\alpha\}$ for each $\alpha \in \omega_1$. Indeed, $\overline{A}$ must intersect $2^\alpha$ for a club $C = \{\alpha_\beta : \beta \in \omega_1\}$, we take only one point in $A$ (or in $\overline{A}$ if there is none available in $A$) in $2^{\alpha_\beta}$, and we can assume that $\alpha_0 = 0$. Any $\gamma \in \omega_1 - C$ belongs to a unique interval $(\alpha_{\beta(\gamma)}, \alpha_{\beta(\gamma)+1})$, set $x_\gamma = x_{\alpha_{\beta(\gamma)+1}}$. Then, setting $B = \{x_\beta : \beta \in \omega_1\}$, $y \in B \cap 2^\alpha$ if and only if $y = z_{\alpha_{\beta(\gamma)}}$ for some $z \in \{x_\beta : \beta \in C\}$.

Let us thus $A \subset \{x_\alpha : \alpha \in \omega_1\}$, with $x_\alpha \in 2^\alpha$ and $\overline{A}$ containing no copy of $\omega_1$. Then, $\overline{A}$ is bounded in every branch, i.e.: $\forall y \in 2^{<\omega_1}, \exists \alpha(y) \in \omega_1$ such that $\forall \beta \geq \alpha(y)$, $y|_\beta \notin \overline{A}$. This means that $\forall \beta \geq \alpha(y)$, we have:

- $x_\beta \neq y|_\beta$, if $\beta$ is successor,
- $\exists \gamma(y, \beta) < \beta$ and a finite $F(y, \beta) \subset \gamma(y, \beta)$

such that $\forall \xi$ with $\gamma(y, \beta) \leq \xi \leq \beta$, $x_\xi|_{F(y, \beta)} \neq y|_{F(y, \beta)}$;

if $\beta$ is limit.

Let us fix $y \in 2^{<\omega_1}$. Since $\gamma(y, \beta) < \beta$ if $\beta$ is limit, there is a stationary $S_0(y) \subset \omega_1$ such that $\gamma(y, \beta)$ is constant on $S_0(y)$, taking the value $\gamma(y)$. Since there are only countably many finite subsets of $\gamma(y)$, there is a stationary $S(y) \subset S_0(y)$ such that $F(y, \beta)$ takes the constant value $F(y)$ on $S(y)$. Thus, whenever $\beta \in S(y)$, given $\xi$ with $\gamma(y) \leq \xi \leq \beta$, we have that $x_\xi|_{F(y)} \neq y|_{F(y)}$. But since $S(y)$ is unbounded, this is true for all $\xi \geq \gamma(y)$. 

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Now, endow \(2^{\omega_1}\) with the product topology, and cover it with the open sets

\[ U_y = \{ z \in 2^{\omega_1} : z|_{F(y)} = y|_{F(y)} \}. \]

By compactness there are \(y_0, \ldots, y_n\) such that \(\{U_{y_i} : i = 0, \ldots, n\}\) cover \(2^{\omega_1}\). Letting \(\gamma\) be the max of the \(\gamma(y_i)\), this implies that for all \(\xi \geq \gamma\), for all \(i = 1, \ldots, n\), \(x_\xi|_{F(y_i)} \neq y_i|_{F(y_i)}\). But since for any \(z\) in \(2^{\omega_1}\) there is an \(i\) with \(y_i|_{F(y_i)} = z|_{F(y_i)}\), the \(x_\xi\) for \(\xi \geq \gamma\) take values different from any \(z\) in \(2^{\omega_1}\) on \(\gamma\), which is a contradiction. Thus, there is some branch \(y \in 2^{\omega_1}\) such that \(A\) intersects \(y\) on a club set, so \(A\) contains a copy of \(\omega_1\).

\[ \square \]

4 Constructions

4.1 Surfaces built with the first octant \(O\) and their ufo-equivalent classes of directions

Here we describe some constructions that yield different examples of \(\langle O_X, \preceq \rangle\). Some already appeared in [14] and [1, Appendix A]. All are surfaces (with or without boundary), most of them \(\omega\)-bounded. Our goal is to show that there are ‘fairly complicated’ examples even in this restricted case and to lay the ground for the more elaborate constructions of Section 4.2.

We will use the first octant \(O\), which we recall is \(\{\langle x, y \rangle \in L^2_{\geq 0} : y \leq x\}\), as a building brick. Let \(H = \{\langle x, 0 \rangle : x \in L_{\geq 0}\} \subset O\) and \(\Delta = \{\langle x, x \rangle : x \in L_{\geq 0}\} \subset O\) be respectively the horizontal and the diagonal sides of \(O\). (Strictly speaking, \(O\) has only one boundary component homeomorphic to the long line \(L\), so we call \(H\) and \(\Delta\) ‘sides’ of \(O\).) We picture \(O\) as on Figure 2 (left), with an arrow going from \(H\) to \(\Delta\) when the orientation is relevant.

![Figure 2: The octant (left), neighborhoods in Example 4.2 (middle, right)](image)

Recall Proposition 1.3 (b): any horizontal in \(O\) is ufo-equivalent to \(H\) and

\[ \langle O, \preceq \rangle \simeq \langle \{H, \Delta\}, H \prec \Delta \rangle \simeq \langle \{0, 1\}, \leq \rangle. \quad (1) \]

For convenience, we compile in the next lemma some facts that we shall often use (sometimes implicitly). We omit the proof since we will prove more general statements later.
Lemma 4.1. Let \( U \subset \mathbb{O} \) be open. If \( U \) contains a club subset of some horizontal line in \( \mathbb{O} \), it contains some strip \( [\alpha, \omega_1) \times [a, b] \). If \( U \) contains a club subset of \( \Delta \subset \mathbb{O} \), it contains some \( [\alpha, \omega_1)^2 \cap \mathbb{O} \). If \( C \subset \mathbb{O} \) is club, then it club-intersects \( \Delta \) or some horizontal line.

4.1.1 Obtaining the Cantor set

Example 4.2. An \( \omega \)-bounded surface \( S \) with \( \langle \mathbb{D}_S, \preceq \rangle = \langle K, \preceq \rangle \), the Cantor set embedded in the interval with the induced order.

In particular, \( \langle \mathbb{D}_S, \preceq \rangle \) has cardinality the continuum. The idea is to start with \([0, 1] \times \mathbb{L}_{\geq 0}\) and to replace each \( \{p\} \times \mathbb{L}_{\geq 0} \) for \( p \) dyadic with a copy of \( \mathbb{O} \). Denote the dyadic numbers by \( \mathbb{D} \). Let thus

\[
S = \left( \bigcup_{p \in [0,1] \cap \mathbb{D}} O_p \right) \cup \left( \bigcup_{q \in [0,1] \cap \overline{\mathbb{D}}} L_q \right),
\]

where each \( O_p \) is a copy of \( \mathbb{O} \) and \( L_q \) a copy of \( \mathbb{L}_{\geq 0} \). The neighborhoods of a point in the interior of some \( O_p \) are the usual neighborhoods. If \( E \) is a subset or a point of \( \mathbb{O} \), we write \( E_p \) for the corresponding subset (or point) of \( O_p \), and similarly for subsets of \( \mathbb{L}_{\geq 0} \). For \( a, b \in \mathbb{L}_{\geq 0}, a < b \), define \( B_{a,b} = \{ (x,y) \in \mathbb{O} : a < x < b \} \) (Figure 2 middle). If \( \delta = \langle x,0 \rangle \), a system of neighborhoods of \( z \) is given by the \( N_{p',a,b} \) with \( p' < p, a < x < b \), where \( N_{p',a,b} \) is the union of the \( (B_{a,b})_r \subset O_r \) and of \( (a,b)_r \subset L_r \) for all \( p' < r < p \) together with an open set in \( O_p \) whose intersection with \( H_p \) is the segment \( (a,b) \), see Figure 2 (right). If \( \delta = \langle x, \infty \rangle \), we define the neighborhoods similarly, taking \( p < r < p' \) and an open set intersecting \( \Delta_p \) as \( (B_{a,b})_p \) does. Finally, if \( x \in L_q \), take \( q' < q < q'' \), and define the neighborhoods in the same way again. It is then not difficult to check that \( S \) is an \( \omega \)-bounded surface. Of course, each \( L_q \) and each \( H_p \) is a direction in \( S \).

Lemma 4.3. A club subset of \( S \) must club-intersect one of the \( L_q \)'s or one of the \( O_p \)'s. In the latter case it intersects either \( \Delta_p \) or a horizontal line in \( O_p \).

Proof. Let \( k : S \to [0,1] \times \mathbb{L}_{\geq 0} \) be the map that collapses each \( O_p \) by sending \( \langle x, y \rangle \) to \( x \). Since \( k \) is perfect and \( S \) is locally compact, \( k \) is closed, so \( k(C) \) is club in \([0,1] \times \mathbb{L}_{\geq 0}\). The result follows from Lemma 4.1. \( \square \)

It follows that a direction in \( S \) must be ufo-equivalent to \( L_r, H_r \) or \( \Delta_r \) for some \( r \). For \( p \in [0,1] \cap \mathbb{D} \), define the map \( f_p : S \to \mathbb{L}_{\geq 0} \) by \( f_p|_{L_r} = f_p|_{O_r} = 0 \) for \( r < p \), \( f_p(\langle x, y \rangle)_p = y, f_p(\langle x, y \rangle)_q = x \) if \( q > p \) with \( q \in \mathbb{D} \), and \( f_p|_{L_q} = \text{id} \) if \( q > p \) with \( q \notin \mathbb{D} \). Then \( f_p \) is continuous and witnesses that

\[
\begin{array}{ccc}
\Delta_r & \subset & H_p & \subset & \Delta_p \\
L_r & \subset & \Delta_p & \text{for} & r < p < r'.
\end{array}
\]

It follows that \( \langle \mathbb{D}_S, \preceq \rangle \) is indeed the Cantor Set.
Example 4.4. An $\omega$-bounded surface $S$ with $\langle D_S, \preceq \rangle$ containing antichains of cardinality the continuum.

As in Example 4.2 except that one replaces the $\{p\} \times \mathbb{L}_{\geq 0}$ for $p$ dyadic by a copy $L_p^2$ of $\mathbb{L}_{\geq 0}$ instead of $\emptyset$. The antichain is given by the diagonals of each inserted $L_p^2$ and of the $\{q\} \times \mathbb{L}_{\geq 0}$ for $q$ non- dyadic. Note however that the diagonal in $L_p^2$ is $\Rightarrow$ the horizontal, so some members of $\langle D_S, \preceq \rangle$ are comparable.

One can also replace each $\{p\} \times \mathbb{L}_{\geq 0}$ for $p$ dyadic by a copy of $S'$, $S'$ being made of two copies of the surface of Example 4.2 with their top boundaries identified. Then, $\langle D_S, \preceq \rangle$ contains copies of the Cantor set $K$ and antichains of cardinality the continuum.

The next examples are slightly more complicated.

4.1.2 Obtaining an antichain of cardinality the continuum (and additional properties)

Example 4.5 (P. Nyikos [18]). An $\omega$-bounded surface $R$ with $\langle D_R, \preceq \rangle$ an antichain of cardinality the continuum such that if $D, D'$ are ufo-equivalent directions in $R$, then neither $D - D'$ nor $D' - D$ contain a club.

P. Nyikos used this example to prove Theorem 4.3 in Section 4.2 below. We give a variant of his construction, which is a ‘stratiﬁed’ version of Example 4.4. We built $R = R^{\omega_1}$ by induction, defining $R^\alpha$ for each $\alpha \in \omega_1$. Each is an $\omega$-bounded surface with the following properties:

(i) Whenever $\beta \leq \alpha$, there is a closed map $k_{\beta, \alpha} : R^\beta \to R^\alpha$ that is bone-preserving.
(ii) Each $R^\alpha$ contains so-called rays (i.e. copies of $\mathbb{L}_{\geq 0}$), some of which are added at stage $\alpha$, and some rays will be affected during the process. Each ray crosses each bone of $S^\alpha$ in exactly one point, but two rays may intersect on a bounded set.
(iii) The unaffected rays are dense in $R^\alpha$. Moreover, for $\beta > \alpha + 1$ the points that are crossed only by unaffected rays added at stage $\alpha$ are dense in the $\beta$-th bone of $S^{\alpha+1}$, and for limit $\alpha$ the same holds with unaffected rays added at stage $< \alpha$.

Start with $R^0 = \mathbb{L}_{\geq 0} \times S^1$, the rays in $R^0$ being $\mathbb{L}_{\geq 0} \times \{x\}$ for each $x \in S^1$, and none is affected. At stage $\alpha + 1$, take a countable set $D$ of unaffected rays whose intersection with the $\alpha + 1$-th bone of $S^\alpha$ is dense, and such that any point in this intersection is crossed only by unaffected rays that were inserted at the $\alpha$-th stage. Above height $\alpha + 1$, replace each ray in $D$ by a copy of $[\alpha + 1, \omega_1)^2 \subset \mathbb{L}^2$ similarly as in Example 4.4 as suggested in Figure 3 below.

Then the rays in $D$ become affected, and are thought to continue through the diagonal in the respective inserted pieces. New rays appear: the horizontal and vertical lines in the inserted $[\alpha + 1, \omega_1)^2$, which run along the affected ray before separating from it (see Figure 3). Each is unaffected, except the ones corresponding to $[\alpha + 1, \omega_1) \times [\alpha + 1)$ and $[\alpha + 1) \times [\alpha + 1, \omega_1)$. We use the generic notation $E(\alpha + 1)$ for any copy of $[\alpha + 1, \omega_1)^2$ inserted at this stage. Then, the map $k_{\alpha+1, \alpha} : S^{\alpha+1} \to S^\alpha$ is defined by simply collapsing each inserted $[\alpha + 1, \omega_1)^2$ to its diagonal, we check as in Lemma 4.3 that it is indeed closed, so (i) is verified, (ii) and (iii) follow by construction.

At a limit stage $\alpha$, take the inverse limit of the $R^\beta$ for $\beta < \alpha$, to which we add
unaffected ray

Figure 3: Affecting the rays.

continuum many copies of $[\alpha, \omega_1) \subset \mathbb{L}_{\geq 0}$ as follows. First, whenever $\beta \geq \alpha$, $E(\alpha)_{\beta}$ denotes the subspace of $R^\beta$ obtained by the expansion of $E(\alpha)$ inside $R^\beta$ until stage $\beta \leq \omega_1$. Fix a sequence $\alpha_n \nearrow \alpha$ ($n \in \omega$), and take a nested sequence of $E(\alpha_n + 1)$ (that is: $E(\alpha_n + k + 1)^{\alpha} \subset E(\alpha_n + k)^{\alpha}$). Then this sequence accumulate to a new copy of $[\alpha, \omega_1)$, as in Figure 4 below.

added copy of $[\alpha, \omega_1)$

Figure 4: Adding copies of $[\alpha, \omega_1)$.

This copy is attached to the rays below as indicated on the figure, and forms a new affected ray. We use the generic notation $L(\alpha)$ for those rays inserted at limit stage $\alpha$. The map $h_{\alpha, \beta}$ for $\beta \leq \alpha$ is obtained by collapsing all pieces inserted at stages above $\alpha$, and (ii)–(iii) hold by construction. We let then $R = R^{\omega_1}$ be the inverse limit of the $R^{\alpha}$, and define the map $k_{\omega_1, \alpha}: R \to R^{\alpha}$ accordingly. Then $k_{\omega_1, \alpha}$ is a closed map. Notice that $R^1$ is essentially the surface given in Example 4.4.
Lemma 4.6. Any club subset of $R$ intersects some ray on a club set.

Proof. Let $C \subset R$ be club. By construction, for each successor $\alpha$, the union of all $E(\alpha)^{\omega_1}$ is dense in $R$. Recall that there are countably many $E(\alpha)$ added at each stage. If $C$ club-intersects some $L(\alpha)$, we are done. If there is a nested sequence $E(\alpha_n) \ (n \in \omega)$ such that $C$ club-intersects each $E(\alpha_n)^{\omega_1}$, then by construction $C$ club-intersects the $L(\alpha)$ to which these accumulate. If $C$ does not club-intersect any $E(\alpha)^{\omega_1}$ or any $L(\alpha)$, then $k_{\omega_1,0}(C) \subset R^0 = \mathbb{L}_{\geq 0} \times S^1$ must club-intersect some ray which is unaffected, and thus $C$ club-intersects it as well.

We may thus assume that $C$ club-intersects some $E(\alpha)^{\omega_1}$ but that it does not club-intersect any $E(\beta)^{\omega_1} \subset E(\alpha)^{\omega_1}$ for any $\beta > \alpha$, and does not club-intersect any $L(\beta)$ either. Recall that the diagonal in $E(\alpha)$ is affected at stage $\alpha$. If $C$ does not club-intersect this diagonal, then for some $\gamma$, $k_{\omega_1,\alpha}(C)$ club-intersects $[\alpha, \gamma] \times [\alpha, \omega_1) \subset E(\alpha)$ or its mirror image by the diagonal by Lemma 4.4. But since $k_{\omega_1,\alpha}(C)$ does not club-intersect some affected ray in $E(\alpha)$, it must club-intersect some unaffected ray, and so must do $C$. This finishes the proof.

Lemma 4.7. Given two different rays $C, D \subset R$, there is a map $f_{C,D} : R \to \mathbb{L}_{\geq 0}$ that is bounded on $C$ and unbounded on $D$.

Proof. Above some height $\alpha$, $C$ and $D$ will be disjoint. Take a ray that is ‘in between’ them and that is affected at stage $\beta \geq \alpha$. In the corresponding $E(\beta)$, define $f_{C,D}$ as the projection on the second factor if $C$ is ‘to the right’ of $D$ and on the first factor if it is ‘to the left’. Then $f$ can be easily extended to all of $R$.

The claimed properties of $(\mathfrak{D}_R, \preceq)$ follow directly: a direction must club-intersect exactly one ray, and two rays are $\preceq$-incomparable. We notice in passing that under $\text{CH}$ a variant of this construction enables to affect all rays in the space. Since any ray becomes a diagonal in some $E(\alpha)$ and a direction is essentially a ray, it then is not difficult to find a map $f : S \to \mathbb{L}_{\geq 0}$ that is bounded on each direction except a given one. We do not know whether we can build such a surface without $\text{CH}$.

4.1.3 Obtaining $\alpha < \omega_2$

Given two copies of $\emptyset$, one can glue them together along either $H$ or $\Delta$, as described in Figure 5 (left). To each gluing corresponds a function $r : 2 \to \{\uparrow, \downarrow\}$ in the obvious way. We can also glue a copy of $\emptyset_{\geq \alpha} = \{(x, y) \in \mathbb{L}_{\geq 0}^2 : \alpha \leq y \leq x\}$ at height $\alpha$, as in Figure 6 (right).

If we pile up $\omega$ copies $O_n \ (n \in \omega)$ of $\emptyset$, it is then possible to add a copy $L$ of $\mathbb{L}_{\geq 0}$ that serves as a ‘top side’ to which the copies accumulate. The neighborhoods of points in this added side are defined similarly as in Example 4.2 (see Figure 7, left).

We can also pile up $\omega_1$ copies of $\emptyset$ the following way. We build the surface by induction. At each stage $\gamma < \omega_1$, the surface will have a free upper side homeomorphic to $\mathbb{L}_{\geq 0}$, that we will call $\Delta_\gamma$. During the constructions, we say that a point $x$ in some copy of $\emptyset$ already included is at height $z \in \mathbb{L}_{\geq 0}$ if $x = \langle z, y \rangle$ for some $y$, and similarly for points in copies of $\mathbb{L}_{\geq 0}$. Let $O_0 = \mathbb{L}_{\geq 0}$. If $\gamma = \beta + 1$, let $O_\gamma$ be a copy of $\emptyset_{\geq \gamma}$ and glue $O_\gamma$ on the free side of $O_\beta$ as in Figure 8 (right), the orientation of the arrow being
Figure 5: Gluings of the octant: ↑↑, ↑↓, ↓↑, ↓↓ (left), at height $\alpha$ (right).

Figure 6: Piling up $\omega$ octants (left), $\Delta_\omega$ (right).

given by $r(\gamma)$. The free side of the surface thus built is then the union of the part of
the free side at stage $\beta$, up to the height $\gamma = \beta + 1$, and of the side of $O_\gamma$ that was not
ghed. At limit ordinals $\gamma < \omega_1$, notice that there is a copy $I$ of $[0, \gamma]$ that remains
free, but it does not continue above the height $\gamma$ since we added countably many
octants 'above it' (see Figure 5). Attach to $I$ a copy of $[\gamma, \omega_1) \subset L_{\geq 0}$ as suggested on
Figure 5 (right) for $\gamma = \omega_1$ defining the neighborhoods as before. This defines $\Delta_{\gamma}$.
Proceeding up to $\omega_1$ yields an $\omega$-bounded surface. Notice then that when we glue
$O_{\beta+1}$ on $O_\beta$, the part of the free side of $O_\beta$ at height between 0 and $\beta + 1$ remains
free, the union of all these free sides give a copy of $L_{\geq 0}$ that we call $\Delta_{\omega_1}$.
We can generalize a bit more and pile up $\alpha$ copies of $0$ for any $\omega_1 \leq \alpha < \omega_2$ as
follows. First, fix a (non-continuous) 1-to-1 map $h : \alpha \to \omega_1$, which gives the height
at which the $\gamma$-th copy of $O$ is glued, and again a map $s : \alpha \to \{\uparrow, \downarrow\}$, which gives the
orientation of the gluing. (The previous construction is the particular case $\alpha = \omega_1$
and $h$ the identity.) That is, we glue $O_{\beta+1}$ on the free side of $O_\beta$ at height $h(\beta + 1)$
(see Figure 7), and define $\Delta_{\beta+1}$ similarly as before. If $\gamma \leq \alpha$ is limit, things are
a bit more complicated, but since $h$ is 1-to-1, at a countable height only countably
many copies of the octant appear. Let us look between height $\beta$ and $\beta + 1$. If for
some $\xi < \gamma$, $h(\xi') > \beta$ for each $\xi \leq \xi' < \gamma$, then the piece of free side at this height
will remain untouched after stage $\xi$, otherwise there will be no free side left at these
heights. (Notice that if \( \text{cf}(\gamma) = \omega_1 \), we are always in the former case.) If there is no free side left, we add a copy of \( [\beta, \beta + 1] \) to which the (countably many) octants below accumulate, as before. The union of the remaining parts of the free sides and the one added is then a copy of \( L_{\geq 0} \) that we call again \( \Delta_\gamma \). A particular case is whenever \( \gamma = \alpha \), which gives the upper side \( \Delta_\alpha \), as in the case \( \alpha = \omega_1 \). We denote by \( S^{\alpha, h, r} \) the \( \omega \)-bounded surface (with boundary) thus obtained. A canonical sequence \( \bigcup_{\xi \in \omega_1} S^{\alpha, h, r}_\xi \) for \( S^{\alpha, h, r} \) is given by open sets defined as the union of points at height \( \leq \xi \) for each \( \xi < \omega_1 \).

![Figure 7: Gluing according to \( h \).](image)

**Lemma 4.8.** Let \( \alpha, h \) and \( r \) be as above, and \( C \subset S^{\alpha, h, r} \) be club. Then either \( C \cap \Delta_\gamma \) is club for some \( \gamma \leq \alpha \), or \( C \cap O_\gamma \) is club for some successor \( \gamma \leq \alpha \).

**Proof.** For clarity, we omit the superscripts \( h \) and \( r \) when they are not absolutely necessary, and write \( S^{\alpha} = \bigcup_{\xi \in \omega_1} S^{\alpha, h, r}_\xi \). If \( \beta \leq \alpha \), we denote by \( S^{\beta} \) the subspace of \( S^{\alpha} \) given by \( S^{\beta, h', s'} \), where \( h', s' \) are the restrictions to \( \beta \) of \( h, s \).

The proof is by induction on \( \alpha \). If \( \alpha = 0 \), the result is immediate. If \( \alpha = \beta + 1 \) then either \( C \) club-intersects \( O_{\beta+1} \subset S^{\alpha} \) or it club-intersects \( S^{\beta} \). If \( \text{cf}(\alpha) = \omega \), either \( C \) club-intersects \( \Delta_\alpha \), or it club-intersects some \( S^\beta \) for a smaller \( \beta \). In both cases we are done by induction.

Assume thus that \( \text{cf}(\alpha) = \omega_1 \). Fix a closed increasing \( \omega_1 \)-sequence \( \alpha_\xi \) converging to \( \alpha \). Since \( h \) is 1-to-1, there is a club subset of \( \xi \) such that \( h(\alpha_\xi') \geq \xi \) whenever \( \xi' \geq \xi \). We can thus assume that this holds for each \( \xi \), up to taking a subsequence. This implies that any member of \( (S^{\alpha} - S^{\alpha_\xi}) \) has height at least \( \xi \). If \( C \cap S^{\alpha_\xi} \) is club for some \( \xi \) we are done by induction, so assume that this intersection is bounded by \( \mu(\xi) \in \omega_1 \). Again, this implies that for a club subset \( E \subset \omega_1 \), \( C \cap S^{\alpha_\xi} \subset S^\xi \) when \( \xi = \mu(\xi) \).

Let \( \xi(0) \in E \), we shall find a point at height at least \( \xi(0) \) in \( C \cap \Delta_\alpha \), this implies that this intersection is club. Given \( \xi(n - 1) \), let \( x_n \) be a point at height at least \( \xi(n - 1) + 1 \) in \( C \cap (S^{\alpha} - S^{\alpha_\xi(n-1)}) \). Let \( \xi(n) \) be the smallest point in \( E \) that is \( \geq \) the height of \( x \). By definition, the limit \( x \) of a subsequence of the \( x_n \) will be a member of \( C \cap (S^\xi - S^{\alpha_\xi}) \), for \( \xi = \sup_n \xi(n) \). But by construction, it belongs to \( \Delta_\xi \). Moreover, since \( h \) does not go below \( \mu \) after stage \( \alpha_\xi \), \( \Delta_\xi \cap S^{\alpha_\xi} \) is included in \( \Delta_\alpha \), so \( x \in \Delta_\alpha \) as well. \( \square \)
A corollary of Lemma 4.8 and Proposition 4.3 (b) is:

**Corollary 4.9.** Any direction in $S^\alpha,h,r$ is ufo-equivalent to some $\Delta_\beta$, for $\beta \leq \alpha$.

**Remark 4.10.** We can weaken the assumption that $h : \alpha \to \omega_1$ is 1-to-1 to $h$ being $\omega$-to-1 without affecting the construction.

Notice in passing:

**Lemma 4.11.** Let $\alpha < \omega_2$, $h : \alpha \to \omega_1$ be 1-to-1 (non-continuous) and $r : \alpha \to \{\uparrow, \downarrow\}$. Then $S^\alpha,h,r$ is spanned by directions.

**Proof.** By Lemma 4.8, a club in $S^\alpha,h,r$ either club-intersects some $\Delta_\gamma$, and thus contains a copy of $\omega_1$, or it club-intersects the interior of some $O_\gamma$. In the latter case, it must club-intersect some horizontal in $O_\gamma$ by Lemma 4.1, and thus also contains a copy of $\omega_1$.

**Example 4.12.** For each $\alpha < \omega_2$, there is a surface $S$ with $\langle O_S, \leq \rangle = \langle \alpha, \leq \rangle$. If $\alpha$ is successor, then $S$ can be made $\omega$-bounded.

If $\alpha = \beta + 1$, take any (non-continuous) 1-to-1 $h : \beta \to \omega_1$, and $s$ constant on $\uparrow$.

Set $S = S^\beta,h,s$. If $\alpha$ is limit, set $\alpha = \beta$, choose $h$ and $s$ as in the successor case, and set $S = S^\beta,h,s - \Delta_\alpha$. The claimed properties of $\langle O_S, \leq \rangle$ follow from Corollary 4.9 and the next lemma.

**Lemma 4.13.** Let $\alpha < \omega_2$, $h : \alpha \to \omega_1$ be 1-to-1 (non-continuous), and $r : \alpha \to \{\uparrow, \downarrow\}$ be constant on $\uparrow$. Then, $\Delta_{\gamma'} < \Delta_\gamma$ whenever $\gamma' < \gamma \leq \alpha$.

**Proof.** (As above, we omit the superscripts $h$ and $r$.) The fact that $\Delta_{\gamma'} \preceq \Delta_\gamma$ is proved by induction on $\gamma$. Proposition 4.3 (b) gives the successor steps. For the limit steps, one first shows that if some open set $U$ contains $\Delta_\gamma$, then there are $\eta < \gamma$ and $\xi \in \omega_1$ such that outside of $S^\alpha_\xi$, $U$ contains $O_{\gamma'}$ for each $\eta \leq \gamma' \leq \gamma$. Indeed, if it is not the case, the complement of $U$ is unbounded in $S^\gamma$ for each $\gamma' < \gamma$. By Lemma 4.8, this implies that this complement intersects $\Delta_\gamma$, a contradiction. Thus, if $f : S^\alpha \to \mathbb{L}_{\geq 0}$ is bounded by $\xi$ on $\Delta_\gamma$, there is some $\eta$ such that $f^{-1}([0, \xi + 1))$ contains the terminal part of each $O_{\gamma'}$ for $\eta \leq \gamma' \leq \gamma$. This shows that $f$ is bounded on $\Delta_{\gamma'}$ for these $\gamma'$, and by induction $f$ is bounded on each $\gamma' < \gamma$.

To see that the inequality is strict, let $\gamma < \alpha$. Define $f$ to take the value $h(\gamma + 1)$ on $S^\gamma$. By construction the octant $O_{\gamma + 1} = O_\omega h(\gamma + 1)$ is glued at height $h(\gamma + 1)$. We may thus define $f$ to be the projection on the second coordinate in $O_{\gamma + 1}$. Points in $O_\eta$ for $\eta > \gamma$ at height below $\gamma + 1$ are sent by $f$ to $h(\gamma + 1)$, and to their second coordinate if their height is above $\gamma + 1$. Then, $f$ is continuous, bounded on $\Delta_{\gamma'}$ if $\gamma' \leq \gamma$, and unbounded if $\gamma' > \gamma$.

It is important to notice that if $\alpha \geq \omega_1$, we do not obtain the reverse order $\alpha^*$ by simply reversing each $\uparrow$ to $\downarrow$, as the $\Delta_{\beta}$ for $\beta$ of cofinality $\omega_1$ will not be $\preceq$ those for smaller $\beta$, see Theorem 5.3.

**Problem 2.** Is there a Type I (first countable) space (or manifold) $X$ with $\langle O_X, \leq \rangle \simeq \langle \omega_2, \leq \rangle$? One with $\langle O_X, \leq \rangle \simeq \langle \alpha^*, \leq \rangle$ for some $\alpha \geq \omega_1$?
4.1.4 Obtaining an Aronszajn or Suslin line

The same type of ideas can be used to obtain more ‘exotic’ ordered types for the ufo-equivalent classes of direction. Recall that $2^{<\omega_1}$ is the tree of $0,1$-sequences of length $< \omega_1$ ordered by the inclusion. Let $T \subset 2^{<\omega_1}$ be a subtree (in this example we assume that subtrees are downward closed in $2^{<\omega_1}$). We let $\tilde{T} \subset 2^{\leq \omega_1}$ be the tree of all maximal branches, that is, given a maximal branch in $T$ we add to $\tilde{T}$ the union of its members (which defines a map $\alpha \to 2$ for some $\alpha \leq \omega_1$). Given $x, y \in \tilde{T}$, we let $d(x, y)$ be the smallest $\alpha$ such that $x(\alpha) \neq y(\alpha)$, and write $x \triangleleft y$ if $x(d(x, y)) < y(d(x, y))$. This defines a total order on $\tilde{T}$.

Example 4.14. Given an Aronszajn subtree $T$ of $2^{<\omega_1}$ without maximal element, there is an $\omega$-bounded surface $A_T$ such that $\langle D_{A_T}, \triangleleft \rangle$ is isomorphic to the set of maximal branches of $T$ ordered as above.

So, if $T$ is Suslin, $D_{A_T}$ is in fact a (compact) Suslin line. We need $T$ to be binary (or at least that the number of immediate successors of each point of $T$ is uniformly bounded by some $n \in \omega$) for the surface to be $\omega$-bounded.

The construction works as follows and is a kind of mix between Examples 4.5 and 4.12. Start with a copy $L_0$ of $L_{\leq 0}$. To each $t \in T$ at successor level $\alpha$ will correspond a copy $O_t$ of $\mathbb{D}_{\geq \alpha}$. Then $L_{t,0}$ is the horizontal $\{ \langle x, \alpha \rangle : x \geq \alpha \} \subset O_t$ and $L_{t,1}$ the diagonal $\Delta \subset O_t$. This defines $L_t$ for $t$ at successor levels once $O_t$ is defined. We shall also define $L_x$ for $x \in \tilde{T}$ at limit level $\alpha$. Each will be a copy of $[\alpha, \omega_1) \subset L_{\geq 0}$. While the interior of each $O_t$ remains in the final construction as well as $L_x$ for $x \in \tilde{T} - T$, $L_t$ for $t \in T$ will be affected as in Example 4.12.

Given $t \in \text{Lev}\alpha(T)$, take out $L_t$ above height $\alpha$ and sew in $O_t$, such that the arrow $\uparrow$ points to the $L_x$ for $t \triangleleft x$ as in Figure 8 below. This defines $L_{t,0}$ and $L_{t,1}$. At a limit stage $\beta$, insert copies $L_x$ of $[\beta, \omega_1) \subset L_{\geq 0}$ for each $x \in \tilde{T}$ at height $\beta$. These copies are inserted according to $t : L_x$ will be exactly between the $O_t$ and $O_s$ ($s, t$ at lower levels) such that $t \triangleleft x \triangleleft s$, as in Example 4.2.

![Figure 8: Inserting $O_t$.](image_url)

As in Example 4.12, we call $A_T$ the space obtained after completing the induction up to each $t$ at level $\alpha$, and $A_T = A_T^{\omega_1}$ is the inverse limit of these. It should be clear that $A_T$ is indeed $\omega$-bounded.

Lemma 4.15. Any club $C$ in $A_T$ club-intersects either the interior of some $O_t$ or some $L_x$.
Proof. This is very similar to Lemma 4.6 (but easier). First, for each $\alpha$ there is a closed map $k_{\omega_{1}, \alpha} : A_{\omega_{1}} \to A_{\omega}$ that collapses the rays and octants introduced after stage $\alpha$. If $C$ does not club-intersect the interior of some $O_{i}$, then $k_{\omega_{1},0}(C)$ club-intersects $L_{0}$ or $L_{1}$, say that the former holds. Thus, $C$ club-intersects the union of the $L_{x}$ for $x \geq 0$ in $T$. By induction, define $t(\alpha) \in \text{Lev}_{\alpha}(T)$ such that $k_{\omega_{1}, \alpha}(C)$ club-intersects $L_{t(\alpha)}$. At successor levels this is as for $L_{0}$, at limit levels we use the fact that a countable nested intersection of clubs is club in $A_{T}$ by $\omega$-boundedness. Since $T$ is Aronszajn, $t(\alpha) \in T - T$ for some $\alpha$, and thus $C$ club-intersects this $L_{x}$. \qed

The claim about $(\mathcal{D}_{A_{T}}, \preceq)$ follows as in Example 4.12 so we do not provide details.

Notice that all the examples in Section 4.1 are surfaces with a top and a bottom side, which can be used in place of the octant to build $\omega$-bounded surfaces with more and more complicated ufo-equivalent classes of directions.

### 4.2 Generalizations

In this section we generalize some of the constructions done in Section 4.1

#### 4.2.1 $\Sigma(X, s)$

Let $X$ be a Type I space with a slicer $s$. We define $\Sigma(X, s)$ as the space given by $(X \times [0, 1]) \cup L$, $L$ being a copy of $\mathbb{L}_{\geq 0}$, where the points in $X \times [0, 1)$ have the usual topology, and neighborhoods of $x \in L$ are given by all $W(y, z, a)$ for $y < x < z \in L$, $0 \leq a < 1$, defined as the union of $(y, z) \subset L$ with $s^{-1}((y, z)) \times (a, 1) \subset X \times [0, 1)$. Then $\Sigma(X, s)$ is a Type I space with canonical sequence $\Sigma(X, s)_{\alpha} = X_{\alpha} \times [0, 1) \cup [0, \alpha)$, whose properties follow closely those of $X$, as the next few lemmas show.

**Remark 4.16.** $\Sigma(X, s)$ may have fewer open sets than the mapping cylinder (as defined in classical homotopy theory) of the scale $s : X \to \mathbb{L}_{\geq 0}$.

For brevity, we shall often omit $s$ and write $\Sigma(X)$ in place of $\Sigma(X, s)$.

**Lemma 4.17.** If $X$ is regular, so is $\Sigma(X)$.

Proof. Let $x \in U \subset \Sigma(X)$, with $U$ open. If $x \notin L$, by regularity of $X \times [0, 1)$ there is an open $V$ such that $x \in V \subset \overline{V} \subset U$. If $x \in L$, there are $y < x < z$ and $a < 1$ such that $W(y, z, a) \subset U$. Take $y', z'$ with $y < y' < x < z'$ and $a'$ between $a$ and 1, then the following inclusions hold:

$$x \in W(y', z', a') \subset \overline{W(y', z', a')} \subset \{s^{-1}([y', z']) \times [a', 1]) \cup [y', z'] \subset W(y, z, a) \subset U.$$ \qed

**Lemma 4.18.** If $X$ is locally second countable, then so is $\Sigma(X)_{\alpha}$ for each $\alpha \in \omega_{1}$.

Proof. $\overline{X_{\alpha}}$ is Lindelöf and locally second countable hence second countable, thus so is $X_{\alpha} \times [0, 1)$. Given a countable basis for $X_{\alpha} \times [0, 1)$, one obtains a countable basis for $\Sigma(X)_{\alpha}$ by adding the sets containing the $W(x, y, a)$ for rational $x, y, a$. \qed
Corollary 4.19. If $X$ is regular and locally second countable, then $X_\alpha$ and $\Sigma(X)_\alpha$ are metrizable for each $\alpha \in \omega_1$.

Proof. By Urysohn’s theorem and the previous lemma.

We also have:

Lemma 4.20. $X$ is $\omega$-bounded ($\omega_1$-compact, respectively) iff $\Sigma(X)$ is $\omega$-bounded ($\omega_1$-compact, respectively).

Proof. The reverse implications are immediate. If $X$ is $\omega_1$-compact, then $\Sigma(X)$ is a countable union of $\omega_1$-compact closed subspaces, and is thus $\omega_1$-compact as well. Suppose that $X$ is $\omega$-bounded, and let $E \subset \Sigma(X)$ be countable, thus $E \subset \Sigma(X)_\alpha$ for some $\alpha$. Let a cover of $\overline{E}$ be given. Since $\overline{E} \cap L \subset [0, \alpha]$, it is compact, take a finite subcover of it. Then for some $a < 1$ the union of its members covers $\overline{E} \cap X \times (a, 1)$. Since $X \times [0, a]$ is $\omega$-bounded, we take a finite subcover of $\overline{E} \cap (X \times [0, a])$ to conclude.

Notice however that $\Sigma(X)$ might be non-locally compact even if $X$ is. If $X$ contains no direction, then $\Sigma(X)$ contains club subsets (copies of $X$) that do not contain a direction in itself. Call horizontals the subsets of $\Sigma(X)$ of the form $X \times \{x\}$ or $L$, and call $L$ the core of $\Sigma(X)$.

Lemma 4.21. Let $X$ be Type I such that uncountably many bones of $X$ are non-empty. If $U$ is open and contains a club subset $C$ of the core of $\Sigma(X)$, then $U \supset (X - X_\alpha) \times (a, 1)$ for some $0 \leq a < 1$ and $\alpha \in \omega_1$.

Proof. Suppose that it is not the case. Since $C$ is club in $L$ it contains a club subset of $\omega_1$. Start with $\alpha_0 = 0$. Given $\alpha_n$, choose $\alpha_{n+1} \in C$ such that $\alpha_{n+1} > \alpha_n$ and there is some $x_n \in (B_{\alpha_{n+1}}(X) \times (1 - \frac{1}{n}, 1)) - U$. Then the $x_n$ converge to $\alpha = \sup_{\alpha \in \omega} \alpha_n \in C$, a contradiction.

4.2.2  $\text{Up}(X, s)$, $\text{Down}(X, s)$

$\text{Up}(X, s)$ and $\text{Down}(X, s)$ are generalized octants.

Definition 4.22. Let $X$ be a Type I space possessing a slicer $s : X \to \mathbb{L}_{\geq 0}$. We set:

$$\text{Up}(X, s) = \{x, y \in X \times \mathbb{L}_{\geq 0} : y \geq s(x)\},$$
$$\text{Down}(X, s) = \{x, y \in X \times \mathbb{L}_{\geq 0} : y \leq s(x)\},$$
$$\Delta(X, s) = \{x, y \in X \times \mathbb{L}_{\geq 0} : y = s(x)\}.$$

A slicer for any subspace of $X \times \mathbb{L}_{\geq 0}$ is given by $\tilde{s}(x, y) = \max\{s(x), y\}$.

They are depicted in Figure [9]. The righthand side shows a somewhat accurate picture when $X$ is a surface, while in the lefthandside we depicted $X$ as a strip, which may be less ‘realistic’ (whatever that means) but makes some constructions easier to describe graphically. As for $\Sigma(X, s)$, we shall write $\text{Up}(X), \text{Down}(X), \Delta(X)$ for brevity except when the choice of the slicer is relevant. Notice that $\Delta(X)$, being the graph of $s$, is homeomorphic to $X$ through the map $(x, y) \mapsto x$. If $X$ is a
Manifold (in particular, a surface), then $\text{Up}(X)$ is a manifold with boundary $\Delta(X)$, while $\text{Down}(X)$ is ‘almost’ a manifold with two boundaries $X \times \{0\}$ and $\Delta(X)$, both homeomorphic to $X$. The only point where the space fails to be a manifold is $(x_0, 0)$, where $x_0$ is the (only) $s$-preimage of 0. By either deleting it, or through a small local change, one obtains a manifold as well.

Lemma 4.23. Let $X$ be Type I with a slicer $s$. Then the following holds.
(a) If $X$ is countably tight, then the projection on the first factor $\pi : X \times \mathbb{L}_{\geq 0} \to X$ is a closed map. Moreover, $\pi \big| \text{Down}(X)$ and $\pi \big| \text{Up}(X)$ are closed maps.
(b) If $X$ is locally compact, then $\pi \big| \text{Down}(X) : \text{Down}(X) \to X$ is a closed map.

Proof.
(a) Let $C \subset \text{Down}(X)$ be closed, $x \in X$ be in the closure of $\pi(C)$ and let $A = \{x_n : n \in \omega\} \subset \pi(C)$ be such that $x \in \overline{A}$. There is thus a sequence $y_n \in \mathbb{L}_{\geq 0}$ such that $(x_n, y_n) \in C$. Let $y$ be an accumulation point of the $y_n$. Then $(x, y)$ is an accumulation point of the $(x_n, y_n)$, and thus in $C$, so $x \in \pi(C)$. The ‘moreover’ part follows from the fact that both $\text{Up}(X)$ and $\text{Down}(X)$ are closed in $X \times \mathbb{L}_{\geq 0}$.
(b) Notice that $\pi \big| \text{Down}(X)$ is proper, that is, the preimage of a compact subset is compact. Indeed, a compact subset $K \subset X$ is contained in some $X_\alpha$, so $\pi^{-1}(K)$ is a closed subset of $K \times [0, \alpha]$, and is thus compact. If $X$ is locally compact, so is $\text{Down}(X)$. A proper map from a locally compact space to a Hausdorff space is closed.

Lemma 4.24. If $X$ contains no direction and is either countably tight or locally compact, then $\text{Down}(X)$ contains no direction.

Proof. Let $C$ be club in $\text{Down}(X)$. Then $\pi(C)$ is club as well by Lemma 4.23 and let $f : X \to \mathbb{L}_{\geq 0}$ witness the fact that $\pi(C)$ is not a direction. So $f^{-1}(\{0\}) \cap \pi(C)$ is unbounded while $f$ is unbounded on $\pi(C)$. Then, $g : \text{Down}(X) \to \mathbb{L}_{\geq 0}, (x, y) \mapsto f(x)$ shows that $C$ is not a direction.
Lemma 4.25. Let $X$ be either countably tight or locally compact, and $C \subset \text{Down}(X)$. Then the following holds.
(a) If $\pi(C)$ contains a closed discrete subset, then so does $C$.
(b) If $C$ is closed discrete, then so is $\pi(C)$.

Proof.
(a) Let $D \subset \pi(C)$ be closed discrete. For each $x \in D$, take $y(x) \in \pi^{-1}(\{x\}) \cap C$. Then $E = \{y(x) : x \in D\}$ is closed: take any $z \notin E$, if $\pi(z) \notin D$ then $\pi^{-1}(X - D) \cap E = \emptyset$, and if $\pi(z) = \pi(x)$ for some $x \in D$, there is some open interval $(a, b) \subset L_{\geq 0}$ such that $\pi^{-1}(X - (D - \{x\})) \cup (\{\pi(x)\} \times (a, b))$ contains $z$ and does not intersect $E$.
(b) Similar to Lemma 4.23 we skip the proof.

Lemma 4.26. Let $X$ be a Type 1 space with a slicer $s$. Then the following holds.
(a) If $X$ is $\omega$-bounded, then so are $X \times L_{\geq 0}$, $\text{Up}(X)$ and $\text{Down}(X)$.
(b) If $X$ is $\omega_1$-compact and either locally compact or countably tight, then $\text{Down}(X)$ is $\omega_1$-compact.
(c) If $X$ is first countable and $\omega_1$-compact, then $X \times L_{\geq 0}$ and $\text{Up}(X)$ are $\omega_1$-compact.

Proof.
(a) $\omega$-boundedness is productive, hence $X \times L_{\geq 0}$ is $\omega$-bounded. Since $\text{Up}(X)$ and $\text{Down}(X)$ are closed subsets of $X \times L_{\geq 0}$ they inherit the property.
(b) By Lemmas 4.23 and 4.25.
(c) Let $D$ be a closed uncountable discrete subset of $X \times L_{\geq 0}$. By Lemma 4.23 (a) $\pi(D)$ is closed. If $\pi(D)$ is unbounded, choose by induction $x_\alpha \in D$ and $\beta_\alpha \in \omega_1$ such that $\pi(x_\alpha) \in X_{\beta_\alpha+1} - X_{\beta_\alpha}$. Then $\{x_\alpha : \alpha \in \omega_1\}$ is closed and its projection is a club discrete subset of $X$ by Lemma 4.23, which is impossible. Thus $D \subset X_\alpha \times L_{\geq 0}$. By the next lemma, $D$ has a club intersection with some $\{x\} \times L_{\geq 0}$ and cannot be discrete. Thus $X \times L_{\geq 0}$ and $\text{Up}(X)$ (as a closed subset) are $\omega_1$-compact.

The missing lemma is very similar to the sublemma after Lemma 2.2 in [3]. The proof goes in part as that of Lemma 4.21. We state a more general version than the one just needed since we will use it again later.

Lemma 4.27. Let $L$ be $\omega$-bounded and Type 1, and $Y$ be first countable and Lindelöf. Let $D \subset L$ be club. Then any open set $U \subset Y \times L$ containing $\{y\} \times D$ contains $V \times Y$ for some open $V \subset Y$, and any club in $Y \times L$ club-intersects $\{y\} \times L$ for some $y \in Y$.

Proof. Since $D$ is also an $\omega$-bounded Type 1 space, we show the first assertion with $D = L$. Let $U \supset \{y\} \times L$ and suppose it does not contain $V \times L$ for any open $V \subset Y$. Take a countable base of neighborhoods $V_n$ whose intersection is $\{y\}$ and choose a point $(y_n, x_n) \in (V_n \times L) - U$ for each $n$. Let $x$ be an accumulation point of the $x_n$, then $(y, x)$ is an accumulation point of the sequence $(y_n, x_n)$ and thus does not belong to $U$, a contradiction.
Let now $C \subset Y \times L$ be closed, if $C \cap \{y\} \times L$ is bounded for each $y \in Y$, then $L - C \supset V(y) \times (L - L_\alpha(y))$ for some open $V(y) \supset y$. Take a countable subcover $V(y_n)$, then $C \subset L_\alpha$ for $\alpha = \sup_n \alpha(y_n)$. □
Lemma 4.28. Let $X$ be a first countable $\omega_1$-compact Type I space possessing a slicer $s$, and let $E \subset \text{Up}(X)$ be club. Then either there is an $\alpha$ such that $E \subset (X_\alpha \times \mathbb{L}_{\geq 0})$, or $E \cap \Delta(X)$ is club.

Proof. $\text{Up}(X)$ is $\omega_1$-compact by Lemma 4.26 (c). Let $f : \text{Up}(X) \to \mathbb{L}_{\geq 0}$ be given by $f(x, y) = s(x)$. If $f$ is unbounded on $E$, the subset $C(f, E) \subset E$ given by Lemma 4.26 is club, and by definition if $(x, y) \in C(f, E)$ then $y = s(x)$ so this point belongs to $\Delta(X)$. If $f$ is bounded by $\alpha$ on $E$, then $E \subset X_\alpha \times \mathbb{L}_{\geq 0}$.

Lemma 4.29. Let $X$ be a first countable, $\omega_1$-compact Type I space possessing a slicer $s$. If $U \subset \text{Up}(X)$ is a large open set, it contains a copy of $\omega_1$ located in $X_\alpha \times \mathbb{L}_{\geq 0}$ for some $\alpha$.

Proof. Let $C \subset U$ be club. If $C \cap (X_\alpha \times \mathbb{L}_{\geq 0})$ is unbounded, by Lemma 4.27 $C$ club-intersects $\{x\} \times \mathbb{L}_{\geq 0}$ for some $x \in X_\alpha$. If $C \cap X_\alpha \times \mathbb{L}_{\geq 0}$ is bounded for each $\alpha$, then $C \cap \Delta(X)$ is club by Lemma 4.28. Now if $U \cap X_\alpha \times \mathbb{L}_{\geq 0}$ is bounded for each $\alpha$, build a sequence $x_\gamma \notin U$ as in Lemma 2.6 to obtain a point in $C \cap U$, a contradiction.

Lemma 4.30. Let $X$ be a first countable $\omega_1$-compact Type I space with a slicer $s$ and possessing no direction. If $D \subset \text{Up}(X)$ is a direction, it is contained in $X_\alpha \times \mathbb{L}_{\geq 0}$ for some $\alpha$.

Proof. Otherwise, by Lemma 4.28 its intersection with $\Delta(X)$ is club. Let $h : X \to \Delta(X)$ be the homeomorphism $h(x) = (x, s(x))$, let $g : X \to \mathbb{L}_{\geq 0}$ witness the fact that $h^{-1}(D \cap \Delta(X))$ is not a direction. Then $f : \text{Up}(X) \to \mathbb{L}_{\geq 0}$ given by $f(x, y) = g(x)$ shows that $D$ is not a direction.

4.2.3 $\Omega(X, A, s)$

We may pile up copies of $\text{Down}(X)$ like we did with the octant as on Figure 10. (As in the previous section, we often write $\text{Down}(X)$ instead of $\text{Down}(X, s)$.)

We define $X^{(\omega)}$ as the union of countably many copies $X^{(n)}$ ($n \in \omega$) of $\text{Down}(X)$, where we identify $(x, 0) \in X^{(n+1)}$ with $(x, s(x)) \in X^{(n)}$. By abuse of notation we still write $X^{(n)}$ for the images of these subspaces in $X^{(\omega)}$ (they are thus not disjoint). The map $s^{(\omega)}$ is defined as being equal to $\tilde{s}$ on each $X^{(n)}$, it is a slicer for $X^{(\omega)}$. We denote by $\Delta(X)^{(n)}$ the copies of $X$ given by the images of $X \times \{0\} \subset X^{(n)}$ (identified with $\Delta(X) \subset X^{(n-1)}$ if $n \geq 1$) in the resulting space. A canonical sequence $(X^{(\omega)})^\alpha$, follows naturally given one for $X$. If $X = \mathbb{L}_{\geq 0}$ with the slicer given by the identity, this construction yields exactly $(S^{\omega, h, r} - \Delta_\omega)$ with $h(n) = 0$ and $r(n) = \uparrow$ for each $n$.

Lemma 4.31. In $X^{(\omega)}$, $\Delta(X)^{(m)} \not\subset \Delta(X)^{(n)}$ if $n < m$.

Proof. Let $n \in \omega$. Define $g : X^{(\omega)} \to \mathbb{L}_{\geq 0}$ as being identically 0 on $X^{(k)}$ for $k < n$, equal to $(x, y) \mapsto y$ on $X^{(n)}$, and to $\tilde{s}$ on $X^{(\ell)}$ for $\ell > n$. Then $g$ is continuous, unbounded on each $\Delta(X)^{(\ell)}$ for $\ell > n$ and identically 0 on $\Delta(X)^{(k)}$ for $k \leq n$. 

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Now, set $\Omega(X)$ to be the disjoint union of $X^{(\omega)}$ and a copy $L$ of $\mathbb{L}_{\geq 0}$. The topology on $X^{(\omega)}$ is the one described above. Given $x < y$ in $\mathbb{L}_{\geq 0}$ and $n \in \omega$, set $V(x, y, n) = (s^{(\omega)})^{-1}(x, y)) - \cup_{k<n}X^{(k)}$.

The neighborhoods of $z \in L$ are then given by the union of $V(x, y, n)$ with $(x, y) \subset L$, for $x < z < y$ and $n \in \omega$. In words: points with the same slicer value $z$ in the $X^{(n)}$ accumulate to $z \in L$ when $n$ grows. Again, a canonical sequence $\Omega(X)_\alpha$ follows automatically from one for $X$. By definition the induced topology on $L$ is the usual one. It is easy to check that Lemmas 4.17–4.20 also hold for $\Omega(X)$, in fact $\Omega(X)_\alpha$ and $\Sigma(X)_\alpha$ are essentially homeomorphic (the only difference being at the preimage of 0 under the slicer). Given a subset $A \subset L$, the space $\Omega(X, A) \subset \Omega(X)$ is defined as the subspace $X^{(\omega)} \sqcup A$. When the choice of the slicer $s$ has some importance, we write more accurately $\Omega(X, A, s)$ or $\Omega(X, s)$.

**Lemma 4.32.** Let $X$ by a Type I space possessing a slicer, such that $B_\alpha(X) \neq \emptyset$ for a stationary set of $\alpha$. Let $A \subset L$ be unbounded. Then $A$ is a direction in $\Omega(X, A)$.

**Proof.** Let $f: \Omega(X, A) \to \mathbb{L}_{\geq 0}$, and set $F_0 = f^{-1}([0, 1])$, $F_1 = f^{-1}([2, \omega_1))$. For $n \in \omega$ and $x \in \mathbb{L}_{\geq 0}$, set $N(x, n) = ((s^{(\omega)})^{-1}(x)) - \cup_{k<n}X^{(k)}$. Notice that if $\alpha \in \omega_1$, $N(\alpha, n) = B_\alpha(X) \times (X^{(\omega)} - \cup_{k<n}X^{(k)})$. If $F_0$ and $F_1$ are both unbounded in $A$, then for some $n$ the sets $C_i = \{x \in \mathbb{L}_{\geq 0} : N(x, n) \subset F_i \} (i = 0, 1)$ are both unbounded. Notice that if $\ell > n$ then $N(x, n) \supset \tilde{z}^{-1}(\{x\})$ in $X^{(\ell)}$. If $x_m \in \mathbb{L}_{\geq 0}$ is a sequence converging to $\alpha \in \omega_1$ and $N(x_m, \ell) \subset F_i$, then $N(\alpha, \ell) \subset F_i$ and when $\ell > n F_i \cap X^{(\ell)}$ contains $B_\alpha \times [0, \alpha]$. Thus, the sets of $\alpha$ such that either $B_\alpha(X) = \emptyset$ or $\alpha \in C_i \subset L$ is club for $i = 0, 1$, so their intersection must contain some $\alpha$ for which $\emptyset \neq N(\alpha, n) \subset C_i$ for both $i$, which is impossible.

We will show that a direction in $\Omega(X, A)$ is forced to be inside $A$ in good cases. Before, a small technical lemma.
Lemma 4.33. Let \( f : \text{Down}(X) \to \mathbb{L}_{\geq 0} \) satisfy \( f(x) \leq \bar{s}(x) \) for all \( x \in \text{Down}(X) \) and \( n \in \omega \) be given. Then there is \( g : \Omega(X,A) \to \mathbb{L}_{\geq 0} \) such that \( g|_{X^{(n)}} = f \) and \( g|_{A} = \text{id} \).

Proof. Set \( g|_{A} = \text{id} \) and define \( g|_{X^{(n)}} \) as follows. Let \( \langle x,y \rangle^{(k)} \) denote the point \( \langle x,y \rangle \in X^{(k)} \). If \( k > n+1 \), set \( g(\langle x,y \rangle^{(k)}) = \bar{s}(\langle x,y \rangle) \). If \( k < n-1 \), set \( g(\langle x,y \rangle^{(k)}) = 0 \). Then, set \( g(\langle x,y \rangle^{(n+1)}) = \max\{f(\langle x,s(x) \rangle),y\} \), \( g(\langle x,y \rangle^{(n)}) = f(\langle x,y \rangle) \) and finally \( g(\langle x,y \rangle^{(n-1)}) = \min\{f(\langle x,0 \rangle),y\} \). \( g \) is continuous and satisfies the required properties.

Lemma 4.34. Let \( X \) be a type I space which possesses a slicer and contains no direction. If \( X \) is locally compact or countably tight, then a direction in \( \Omega(X,A) \) is contained in \( A \) outside of some bounded set.

Proof. Let \( C \) be club in \( \Omega(X,A) \). Suppose that \( C \cap X^{(\omega)} \) is unbounded. Thus, for some \( n, C \cap X^{(n)} \) is unbounded as well. But by Lemma 4.24 there is some \( h : X^{(n)} \to \mathbb{L}_{\geq 0} \) witnessing that \( C \cap X^{(n)} \) is not a direction. Notice that \( f(x) = \min\{h(x),\bar{s}(x)\} \) is unbounded on \( C \cap X^{(n)} \) as well and is bounded on some unbounded subset of it. Then \( g : \Omega(X,A) \to \mathbb{L}_{\geq 0} \) given by the previous lemma shows that \( C \) is not a direction.

4.2.4 \( \Xi(X,A,s) \)

This is similar to the construction in Section 4.1.2 but done in \( \text{Up}(X,s) \) instead of \( \mathbb{L}_{\geq 0} \times [0,1] \).

Let \( X \) be Type I with a slicer \( s \). Let \( A \subset \mathbb{L}_{\geq 0} \) be discrete. \( \Xi(X,A,s) \) is the space obtained with the following procedure. Start with \( \text{Up}(X) \). For each \( a \in A \), take out \( s^{-1}\{a\} \times [a,\omega_{1}) \subset \text{Up}(X) \), and sew in the scar a copy \( E_{a} \) of \( s^{-1}\{a\} \times \omega_{2} \) exactly as in the construction of Example 4.5. The orientation of the added octants (times \( s^{-1}\{a\} \)) is chosen such that the arrow points 'in the direction of the diagonal \( \Delta(X)' \), as in Figure 11.

This can be made precise by specifying explicitly the neighborhoods of the points in the horizontals and diagonals of the octants sewed in, but this is quite tedious to describe in detail. As in Example 4.5 if \( a_{n} \) is an increasing sequence in \( A \) converging to some \( a \in \mathbb{L}_{\geq 0} \), thus \( s^{-1}\{a_{n}\} \times \mathbb{L}_{\geq a} \) accumulates to \( s^{-1}\{a\} \times \mathbb{L}_{\geq a} \), then the inserted \( E_{a} \) also accumulate to it. The \( \Xi(X,A,s)_{a} \) can be defined again as in Example 4.5 to give \( \Xi(X,A,s) \) a structure of Type I space. Notice that the 'diagonal' \( \Delta(X) \) has not been altered by the construction, and we write \( \Delta(X) \subset \Xi(X,A,s) \) by a slight abuse. Denote by \( p : \Xi(X,A,s) \to \text{Up}(X,s) \) the map which collapses the inserted \( E_{s} = s^{-1}\{s\} \times \omega_{2} \) by projection on the horizontal in \( \Omega_{\geq a} \). Then define the slicer \( s_{\Xi} : \Xi(X,A,s) \to \mathbb{L}_{\geq 0} \) by \( s_{\Xi} = \bar{s} \circ p \). If \( E \subset X \), we can build \( \Xi(E,A,s \vert_{E}) \) and may consider it as a subspace of \( \Xi(X,A) \). This holds also if \( E \) is bounded, as the construction uses only the continuity of the slicer \( s \) and no other property.

Lemma 4.35. The following holds.

(a) If \( X \) is \( \omega \)-bounded, then so is \( \Xi(X,A,s) \).
(b) If \( X \) is \( \omega_{1} \)-compact and first countable, then so then so is \( \Xi(X,A,s) \).
Proof. This is very similar to Lemma 4.26 so we do not provide much detail. (a) Let \( \{ x_n : n \in \omega \} \subset \Xi(X, A, s) \) be given. If infinitely many \( x_n \) belong to some \( E_a \), there is an accumulation point since \( E_a \) is \( \omega \)-bounded. Otherwise we may assume that there is at most one \( x_n \) in each \( E_a \). As in Lemma 4.23 (b) \( p \) is proper and thus closed whenever \( X \) is \( \omega \)-bounded, so \( \{ p(x_n) : n \in \omega \} \) has an accumulation point \( y \in \text{Up}(X, s) \). By construction the \( x_n \) must accumulate to some preimage of \( y \) under \( p \).

(b) By Lemma 4.26 (c), \( s^{-1}(\{ a \}) \times \mathbb{L}_{\geq a} \) is \( \omega_1 \)-compact and so is \( E_a \) (as a closed subset). Given \( D \subset \Xi(X, A, s) \) closed discrete, then either \( D \cap E_a \) is uncountable for some \( a \) and we are done, or there is an uncountable \( D' \subset D \) such that each \( |D \cap E_a| \leq 1 \) for each \( a \in A \). In the latter case \( p(D') \) is closed discrete, applying Lemma 4.26 (c) again we obtain an accumulation point \( y \) and as in (a) some preimage of \( y \) must be an accumulation point of \( D' \).

The next two lemmas are the \( \Xi \)-versions of Lemmas 4.28 and 4.29. Their proofs are similar and we skip them.

**Lemma 4.36.** Let \( X \) be a Type I space that is either \( \omega \)-bounded or \( \omega_1 \)-compact and first countable. Let \( A \subset \mathbb{L}_{\geq 0} \) be discrete. If \( D \subset \Xi(X, A, s) \) is club, then either it is contained in \( \Xi(X_\alpha, A, s \upharpoonright X_\alpha) \) for some \( \alpha \), or its intersection with \( \Delta(X) \) is club.

**Lemma 4.37.** Let \( X \) be a Type I \( \omega_1 \)-compact and first countable space. Let \( A \subset \mathbb{L}_{\geq 0} \) be discrete. If \( U \subset \Xi(X, A, s) \) is a large open set, it contains a copy of \( \omega_1 \) located in \( \Xi(X_\alpha, A, s \upharpoonright X_\alpha) \) for some \( \alpha \).

Finally, we have the following version of Lemma 4.30

**Lemma 4.38.** Let \( X \) be a Type I space that is either \( \omega \)-bounded or \( \omega_1 \)-compact and first countable. Let \( A \subset \mathbb{L}_{\geq 0} \) be unbounded and discrete. If \( X \) does not possess a direction, then there is no maximal direction in \( \Xi(X, A, s) \).
Proof. Let $D \subset \Xi(X, A, s)$ be a direction. If $D \cap \Delta(X, s)$ is club, as in Lemma 4.30 we may find a map witnessing that $D$ is not a direction, a contradiction. Thus by Lemma 4.36 $D \subset \Xi(X, A, s|_{X_\alpha})$ for some $\alpha$. Take $a \in A$ bigger than $\alpha$ and denote $s^{-1}([0, a])$ by $X_a$. Define $f : \Xi(X, A, s) \to \mathbb{L}_{\geq 0}$ as follows. On $\Xi(X, A, s|_{X_\alpha})$ it takes the value $a$. In $E_a = s^{-1}([a]) \times \mathbb{O}_{\geq a}$, the point $\langle x, y_1, y_2 \rangle$ is sent to $y_1$. Finally, a point $x$ elsewhere is sent to $s_\Xi(x)$. This defines a continuous map which is bounded on $D$ but unbounded on any ‘vertical’ $\{y\} \times \mathbb{L}_{\geq s(y)}$ whenever $s(y) > a$. Thus $D$ is not ufo-maximal. \qed

5 Bounded and unbounded maps

Let $X$ be Type I and $f : X \to \mathbb{L}_{\geq 0}$ be unbounded. Is there a direction on which $f$ is unbounded? Any space without direction provides a trivial counterexample, $X = \emptyset - \Delta$, $f$ being the projection on the second coordinate, is another. The question is more interesting for spaces spanned with directions.

Theorem 5.1 (PFA). Let $X$ be a Type I countably tight locally compact space which is spanned by directions, and let $f : X \to \mathbb{L}_{\geq 0}$ be an unbounded map. Then there is a direction in $X$ on which $f$ is unbounded.

Notice that $\omega_1$-compactness is not included in the assumptions.

Theorem 5.2. The following holds.

(a) Under $\neg$SH, there is an $\omega_1$-compact 3-manifold $M$ spanned by directions and an unbounded function $f : M \to \mathbb{L}_{\geq 0}$ bounded on each direction in $M$.

(b) $\left(\Diamond^+\right)$ There is an $\omega$-bounded 3-manifold as in (a).

Proof of Theorem 5.2. Let $f : X \to \mathbb{L}_{\geq 0}$ be unbounded. Let us suppose that there is some $\omega_1$-compact $Y \subset X$, closed in $X$, on which $f$ is unbounded. Then $Y$ is unbounded, therefore the set $C(f) = C(f, X)$ introduced in Lemma 2.6 is club. As before, by Theorem 3.4 there is a (closed) copy of $\omega_1$ in $C(f)$ on which $f$ is thus unbounded, and we are done.

Assume now that any club of $X$ on which $f$ is unbounded possesses a closed uncountable discrete subset. Up to changing the canonical sequence, we can assume that $f(X_\alpha) \subset [0, \alpha)$ for all $\alpha \in \omega_1$. Set $\alpha_0$ to be 0, choose by induction $x_\beta \in X$ and $\alpha_{\beta+1} \in \omega_1$ with $f(x_\beta) \in (\alpha_\beta, \alpha_{\beta+1})$, and set $\alpha_\beta = \sup_{\gamma < \beta} \alpha_\gamma$ when $\beta$ is limit. Let $Y$ be the closure in $X$ of $\{x_\beta : \beta \in \omega_1\}$. Since $f$ is unbounded on $Y$, there is a closed uncountable discrete subset $Z \subset Y$ which is thus unbounded (by Lindelöfness of each $X_\alpha$). Thus, $Z$ is club and $f$ is unbounded on each club subset of $Z$. Let $\gamma_0 = 0$. Given $\beta$, choose $y_{\beta+1} \in Z$ and $\gamma_\beta \in \omega_1$ such that $\gamma_\beta < f(y_{\beta+1}) < \gamma_{\beta+1}$ and $y_\beta \in X_{\gamma_{\beta+1}} - X_{\gamma_\beta}$. Set then $V = \bigcup_{\beta \in \omega_1} V_\beta$, where

$$V_\beta = (X_{\gamma_{\beta+1}} - X_{\gamma_\beta}) \cap f^{-1}((\gamma_\beta, \gamma_{\beta+1})).$$

Then $V$ is a large open set since it contains each $y_\beta$, and since $X$ is spanned by directions, one must be included in $V$. By definition of the $V_\beta$, $f$ must be unbounded on it. \qed

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For Theorem 5.2, we will use the spaces $\text{Up}(X)$ defined in Section 4.2.2.

Proof of Theorem 5.2.
(a) Let $S_T$ be a fattening of a Suslin tree $T$ as defined in Section 3.2. By Lemma 4.29, for any large open set $U$ of $\text{Up}(S_T)$ there is some $y \in S_T$ such that $U \cap \{y\} \times L_{\geq 0}$ contains a club, and thus a copy of $\omega_1$. Hence $\text{Up}(S_T)$ is spanned by directions. But the unbounded map $f : \text{Up}(S_T) \to L_{\geq 0}$ given by $f(x, y) = s(x)$ is bounded on any direction by Lemma 4.30. If one prefers a boundaryless example, take the double of $\text{Up}(S_T)$ (i.e. take two copies of $M$ and identify pointwise the boundaries).
(b) The same construction than in (a) starting with the $\omega$-bounded surface given by Theorem 3.11.

Remark 5.3. We can do the same construction starting with $Q$ given by Theorem 3.1. However, $\text{Up}(Q, s)$ might not be spanned by directions because Lemmas 4.29–4.30 may not hold. See Proposition 6.7 below and recall that $\beta \omega$ has P-points (consistently). $\text{Up}(Q, s)$ is then an $\omega$-bounded space which possesses directions but with an unbounded map which is bounded on each direction. However, the disjoint union of $L_{\geq 0}$ and $Q$ is a much simpler example.

6 Directions in themselves

6.1 Relations with monoclinic and monolithic spaces

Following Nyikos [15, 18], we say that a Type I space $X$ is monoclinic if any unbounded 0-set (i.e. inverse image of 0 for some real valued function) contains the bones $B_\alpha(X)$ (which are assumed to be non-empty) for a club set of $\alpha$. Notice that the inverse image of a closed subset of $\mathbb{R}$ is a 0-set. $X$ is monolithic if any club subset of $X$ contains the bones (non-empty again) on a club set. (Caution: the word ‘monolithic’ is sometimes used to denote a completely different property.) A real valued map on a Type I space $X$ is said to be eventually constant iff it is constant outside some $X_\alpha$. The following was partially stated in [18]:

Theorem 6.1. Let $X$ be a Type I space. The items below are related as follows.

\[
\begin{align*}
(a) & \longrightarrow (b) \longrightarrow (c) \\
(c) & \leftrightarrow (d) \leftrightarrow (e) \quad \text{and} \\
(f) & \longrightarrow (g) \longrightarrow (h)
\end{align*}
\]

Moreover, if $X$ possesses a slicer then $(f) \leftrightarrow (b)$ and $(g) \leftrightarrow (c)$ hold, and if $X$ is normal then items (b) to (g) are equivalent.
(a) $X$ is monolithic.
(b) If $A, B \subset X$ are club, then $A \cap B$ is club.
(c) $X$ is monoclinic.
(d) If $A, B \subset X$ are unbounded 0-sets, then $A \cap B$ is club.
(e) Every real valued function on $X$ is eventually constant.
(f) Any club set of $X$ is a direction in itself.

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(g) Any unbounded 0-set of $X$ is a direction in itself.
(h) $X$ is a direction in itself.

Proof. Let $X$ be Type I. Let us first prove (c) $\leftrightarrow$ (d) $\leftrightarrow$ (e).

(c) $\rightarrow$ (d) Immediate.
(d) $\rightarrow$ (e) (We phrased this proof so as to emphasize that we use only that $\mathbb{R}$ is regular, Lindelöf and has countable pseudocharacter.) Let $f : X \rightarrow \mathbb{R}$. For each $\alpha$ pick $x_\alpha \in X - X_\alpha$ and let $z_\alpha = f(x_\alpha)$, $Z = \{z_\alpha : \alpha \in \omega_1\}$. By Lindelöfness, some $z \in \mathbb{R}$ is a point of complete accumulation of $z$ (i.e. any neighborhood of $z$ has an intersection with $Z$ of cardinality $\aleph_1$). If $f$ is not eventually constant taking value $z$, choose $y_\alpha \in X - X_\alpha$ such that $f(y_\alpha) \neq z$. Take a sequence $\{U_\alpha : \alpha \in \omega\}$ of neighborhoods of $z$ whose intersection is $\{z\}$. Then for some $n$, $\mathbb{R} - U_n$ contains uncountably many $f(y_\alpha)$. By regularity there is some open $U \ni z$ such that $U \subset U_n$. Then $f^{-1}(U)$ and $f^{-1}(\mathbb{R} - U_n)$ are disjoint unbounded 0-sets, a contradiction.

(e) $\rightarrow$ (c) If $X$ is not monoclinic then for some real valued $f$, $f^{-1}(\{0\})$ is unbounded but does not contain all the bones, and thus $f$ is not eventually constant.

We now prove the other implications.

(a) $\rightarrow$ (b) $\rightarrow$ (d) and (f) $\rightarrow$ (g) $\rightarrow$ (h) are immediate. Let us first prove (e) $\rightarrow$ (h), which is not in the diagram but is useful for other implications.

(e) $\rightarrow$ (h) Given $f : X \rightarrow \mathbb{L}_{\geq 0}$ unbounded, set $g(x) = \min\{f(x), 1\}$, then for some $\beta$, $g$ is constant on $X - X_\beta$. Since $f$ is unbounded, $g$ must be constant on 1. Thus $f^{-1}(\{0\}) \subset X_\beta$ is bounded.

(b) $\rightarrow$ (f) Since (b) is hereditary for club subsets, a club in $X$ satisfies (e). We conclude with (e) $\rightarrow$ (h).

(c) $\rightarrow$ (g) We show (e) $\rightarrow$ (g). Let $C = f^{-1}(\{0\})$ be unbounded for some $f : X \rightarrow \mathbb{R}$. By (e), there is $\alpha$ such that $C \supset X - X_\alpha$. But $X - X_\alpha$ is a direction in itself by (e) $\rightarrow$ (h).

(f) $\rightarrow$ (b) if $X$ has a slicer. If $A, B \subset X$ are disjoint club sets, the map sending $A$ to 0 and which is equal to the restriction of a slicer in $B$ is continuous on $A \cup B$, which shows that $A \cup B$ is not a direction in itself.

(g) $\rightarrow$ (d) if $X$ has a slicer. As (f) $\rightarrow$ (b).

Equivalence of items (b) to (g) if $X$ is normal. Since $X$ is regular, (g) $\rightarrow$ (c) and it is enough to show (c) $\rightarrow$ (b). Assume that (c) holds. Let $A, B$ be disjoint club sets of $X$. By normality there is a continuous $f : X \rightarrow \mathbb{R}$ such that $f^{-1}(\{0\}) \supset A, f^{-1}(\{1\}) \supset B$, and thus $f^{-1}(\{0\}), f^{-1}(\{1\})$ are disjoint unbounded 0-sets, contradicting (c).

Notice that we also have:

**Lemma 6.2.** Let $X$ be a Type I space. Then (e) in Theorem 6.1 is equivalent to:

(e') Any map $f : X \rightarrow Y$ where $Y$ is regular Lindelöf and has countable pseudocharacter is eventually constant.

Proof. (e') $\rightarrow$ (e) is immediate. Let $f : X \rightarrow Y$ be given and assume that (e) and thus (d) above holds. The proof of (d) $\rightarrow$ (e) shows that there are disjoint closed $E, F \subset Y$ with unbounded preimages in $X$. Since $Y$ is regular and Lindelöf it is normal, so there is a map $g : Y \rightarrow \mathbb{R}$ which takes the value 0 on $E$ and 1 on $F$. Then $g \circ f : X \rightarrow \mathbb{R}$ is a non-eventually constant map, a contradiction. \qed
The implication charts for regular and normal spaces are thus reduced to:

\[
\begin{align*}
(a) & \rightarrow (b) \rightarrow (g) \rightarrow (h) & \text{regular spaces} \\
(a) & \rightarrow (g) \rightarrow (h) & \text{normal spaces}
\end{align*}
\]

None of the other implications holds in the relevant classes. Neither \((a) \rightarrow (b)\) nor \((g) \rightarrow (h)\) does reverse for normal spaces. For \((h) \not\rightarrow (g)\), notice that the normal space \(L_{\geq 0} \times [0,1]\) is a direction in itself while \(L_{\geq 0} \times \{0,1\}\) is not.

**Example 6.3.** A 2-to-1 closed preimage of \(\omega_1\) which is not monolithic but satisfies (b) of Theorem 6.1.

Such a space is thus locally countable, locally metrizable, locally compact, \(\omega\)-bounded and normal (even collectionwise) by Lemma 6.5 below. This is a simple modification of a 1979 construction by Nyikos [11, 19.1]. Let \(X = \omega_1 \times \{0,1\}\) and \(\pi\) be the projection on the first factor. Denote by \(\Lambda\) the set of limit ordinals in \(\omega_1\), and by \(\Lambda_1\) the set of limit of limits. Points \(\langle \alpha, i \rangle\) are isolated whenever \(\alpha \notin \Lambda_1\). The neighborhoods of points at levels in \(\Lambda_1\) are defined as:

\[
\langle \alpha, 0 \rangle : \{\langle \alpha, 0 \rangle\} \cup \{\langle \gamma, i \rangle : \beta < \gamma < \alpha, \gamma \in \Lambda, i = 0, 1\},
\]

\[
\langle \alpha, 1 \rangle : \{\langle \alpha, 1 \rangle\} \cup \{\langle \gamma, i \rangle : \beta < \gamma < \alpha, \gamma \notin \Lambda, i = 0, 1\}
\]

for \(\beta < \alpha\). It is immediate that \(X\) is Hausdorff, \(\omega\)-bounded and locally countable and that \(\pi\) is closed. Since \(\pi^{-1}([0,\alpha])\) is compact, by Urysohn’s theorem it is metrizable and \(X\) is locally compact.

Now, given two club subsets \(A, B \subset X\), by Lemma 2.1 (b) both have non-empty intersection with the bones of \(X\) on a club set \(C \subset \omega_1\). But then both contain \(\langle \alpha, 0 \rangle\) for any \(\alpha \in \Lambda_1 \cap C\), so \(X\) satisfies (b). The club subset \(\Lambda_1 \times \{0\}\) does not contain any bone (it is actually a copy of \(\omega_1\)), so \(X\) is not monolithic.

We now show that \((g) \not\rightarrow (b)\) for regular spaces.

**Example 6.4.** (Nyikos [15]) There is a regular Type I space satisfying \((g)\) but not (b) of Theorem 6.1.

We take \(X\) to be the positive part of a tangent bundle of \(L_+\) of class 7 as in [15, Section 6] where it is shown that \(X\) satisfies (e) and thus (g) but is not normal. The next lemma (whose proof is extracted from Lemma 9.5 and Theorem 4.22 in the same paper) shows that \(X\) does not satisfy (b). Recall in passing that a metrizable space is collectionwise normal.

**Lemma 6.5.** (Nyikos [15], in effect.) Let \(X\) be a regular Type I space satisfying (b) of Theorem 6.1. Then \(X\) is \(\omega_1\)-compact and normal. If moreover \(X_\alpha\) is collectionwise normal for each \(\alpha\), then so is \(X\).

We refer to [15] for the definition of collectionwise normality.

**Proof.** Given disjoint closed sets \(A, B\), one of them must be bounded, say \(A \subset X_\alpha\). By Lemma 2.3, \(X_\alpha+1\) is normal, so we can separate \(A\) and \(B\) inside it by \(U, V\). Then \(U, V \cup (X - X_\alpha)\) separate \(A\) and \(B\). This shows that \(X\) is normal. If \(C\) is uncountable closed discrete, then for each \(\alpha\), \(X_\alpha \not\supset C\) by Lindelöfness. Thus \(C\) is club. But by
discreteness we can write $C$ as the union of two disjoint clubs, a contradiction. It follows that $X$ is $\omega_1$-compact.

Suppose now that $\{A_\beta : \beta \in \kappa\}$ is a discrete collection of closed subsets of $X$. If $\kappa > \omega_1$, for some $\alpha$ uncountably many $A_\beta$ have a non-empty intersection with $X_\alpha$ contradicting its Lindelöfness, so $\kappa \leq \omega_1$. If some $A_\beta$ is unbounded then the others are bounded and as above we may separate all of them using the collectionwise normality of $X_\alpha$ as in the proof of normality. The same holds if $A = \bigcup_{\beta \in \kappa} A_\beta$ is bounded. If $A$ is unbounded and each $A_\beta$ is bounded, then $\kappa = \omega_1$ and we may divide it in two disjoint sets such that the union of the $A_\beta$ for $\beta$ in each set is unbounded. But then $A$ is the union of two disjoint club subsets, a contradiction.

### 6.2 In products

If $X$ is a direction in itself, then it is easy to see that its product with $\mathbb{R}$ and $[0,1]$ is a direction in itself as well. Of course, its product with a non-connected space cannot be a direction in itself. Also, the product of two Type I spaces one of which possessing a slicer is never a direction in itself. Note also the following.

**Example 6.6.** A compact connected space $X$ such that $L_{\geq 0} \times X$ is not a direction in itself.

*Proof.* Set $X$ to be $L_{\geq 0} \cup \{\omega_1\}$ with the obvious topology (i.e. neighborhoods of $\omega_1$ are terminal parts of $L_{\geq 0}$). Then use the map $f(x,y) = \min\{x,y\}$. 

Recall that a subset $A \subset X$ is a P-set whenever the intersection of any countable family of neighborhoods of $A$ is a neighborhood of $A$. An easy generalization of Example 6.6 is the following, whose proof is very similar to that of [12, Theorem 3.1].

**Proposition 6.7.** Let $Y$ be a normal space containing a closed P-set. Then $L_{\geq 0} \times Y$ is not a direction in itself.

*Proof.* Let $A$ be a closed P-set. By normality of $Y$, one chooses a sequence $\{U_\alpha : \alpha \in \omega_1\}$ of open neighborhoods of $A$ such that $U_\alpha \supset \bigcup_{\beta \leq \alpha} U_\beta$ whenever $\alpha < \beta$, and as in Lemma 2.5 we define $r : Y \to L_{\geq 0} \cup \{\omega_1\}$ such that $r(\overline{U_\alpha} - U_{\alpha+1}) \subset [\alpha, \alpha + 1]$ and $r(\bigcap_{\alpha \in \omega_1} U_\alpha) = \{\omega_1\}$. Then, set $f : L_{\geq 0} \times Y \to L_{\geq 0}$ as $(x,y) \mapsto \min\{x,r(y)\}$. This provides a function which is unbounded on $L_{\geq 0} \times Y$ but bounded on $L_{\geq 0} \times \{y_0\}$ for each $y_0 \notin \bigcap_{\alpha \in \omega_1} U_\alpha$.

On the positive side, we have the following proposition.

**Proposition 6.8.** Let $X = \bigcup_{\alpha \in \omega_1} X_\alpha$ be a Type I space which is a direction in itself, and $Y$ connected and such that $\overline{X_\alpha \times Y}$ is Lindelöf for each $\alpha \in \omega_1$. Then $X \times Y$ with canonical sequence $\langle X_\alpha \times Y : \alpha \in \omega_1 \rangle$ is a direction in itself if either

(a) $Y$ is separable, or
(b) $X$ is $\omega$-bounded and $Y$ is first countable.

The assumption on $\overline{X_\alpha \times Y}$ implies that $Y$ is Lindelöf and is satisfied if either $Y$ or each $X_\alpha$ is productively Lindelöf (i.e. its product with any Lindelöf space is Lindelöf), for instance $\sigma$-compact spaces.
Proof. Given \( f : X \times Y \to \mathbb{L}_{\geq 0} \), set
\[
A_0 = \{ y \in Y : f \text{ is bounded on } X \times \{ y \} \}, \\
A_1 = \{ y \in Y : f \text{ is unbounded on } X \times \{ y \} \}.
\]

(a) Let \( D \subset Y \) be countable and dense and set \( \gamma = \sup_{y \in A_0 \cap D} \sup f |_{X \times \{ y \}} < \omega_1 \).

By continuity, \( f \) is bounded by \( \gamma \) on \( X \times D \cap A_0 \). By Lemma 2.27(e), since \( X \) is a direction in itself there is a club \( C \subset \omega_1 \) such that for all \( y \in A_1 \cap D \) and all \( \alpha \in C \),
\[
f((X - X_\alpha) \times \{ y \}) \subset [\alpha, \omega_1).
\]

If \( y \in D \cap A_0 \cap D \cap A_1 = \emptyset \), since \( D \) is dense and \( Y \) is connected it follows that one of \( D \cap A_0, D \cap A_1 \) is empty. If the latter is empty, then \( f \) is bounded by \( \gamma \) on \( X \times Y \). If \( f \) is unbounded, \( D \cap A_1 = Y \), and by continuity \( f((X - X_\alpha) \times Y) \subset [\alpha, \omega_1) \) whenever \( \alpha \in C \), which shows that \( X \times Y \) is a direction in itself.

(b) As in (a), one sees that \( \overline{A_0} = A_0 \) and \( \overline{A_1} = A_1 \), using the fact that if \( y \in \overline{A_0} \) there is a countable \( B \subset A_0 \) with \( y \in \overline{B} \). Thus either \( A_0 = \emptyset \) or \( A_1 = \emptyset \). In the latter case, for each \( y \in A_0 = Y \) there is \( \gamma(y) \) such that \( f^{-1}([0, \gamma(y)]) \supset X \times \{ y \} \).

By lemma 4.27 there is an open \( V(y) \supset y \) with \( f^{-1}([0, \gamma(y)]) \supset X \times V_y \). A countable family of \( V_y \) covers \( Y \), hence the supremum of the corresponding \( \gamma(y) \) is a bound for \( f \) on \( X \times Y \).

If \( f \) is unbounded then \( A_1 = Y \). Since \( X \) is a direction in itself, given \( \gamma \in \omega_1 \) for each \( y \in Y \) there is \( \gamma(y) \) such that \( f^{-1}((\gamma, \omega_1)) \supset (X - X_\gamma) \times \{ y \} \), and we conclude similarly.

\[ \square \]

6.3 No direction in itself

Does a direction in some space always contain a direction in itself? For \( \omega_1 \)-compact countably tight locally compact spaces, the answer is yes under \textbf{PFA} by Theorem 3.4 but consistently ‘no’ for a similar class of spaces.

\textbf{Theorem 6.9.} The following holds.

(a) There is an \( \omega \)-bounded Type I space which is a direction in itself and which contains a club subset containing no direction in itself.

(b) \((-\text{SH})\) There is a Type I locally metrizable locally separable \( \omega_1 \)-compact space which is a direction in itself and which contains a club subset containing no direction in itself.

\(*(c)\) \((\diamondsuit^+)\) There is an \( \omega \)-bounded space as in (b).

In (b) our example is unfortunately not locally compact.

\textit{Proof of Theorem 6.9.} We will use \( \Sigma(X) \) for spaces \( X \) without direction in the relevant three classes. Since \( X \times \{ 0 \} \subset \Sigma(X) \) contains no direction in itself, we just need to show that \( \Sigma(X) \) is a direction in itself.

(a) Let \( Q \) be given by Theorem 3.1. Suppose that \( f : \Sigma(Q) \to \mathbb{L}_{\geq 0} \) is given with \( f^{-1}([0]) \) unbounded. If \( f^{-1}([0]) \cap Q \times [0, 1] \) is bounded then by density \( f \) is bounded on all of \( \Sigma(Q) \).

We may thus assume that \( f^{-1}([0]) \cap Q \times [0, 1] \) is unbounded. By Lemma 4.27 \( f^{-1}([0]) \cap Q \times \{ a \} \) is club for some \( a \in [0, 1] \). Let \( Z = \{ z \in Q : (z, a) \in f^{-1}([0]) \cap Q \times \{ a \} \} \). By Lemma 4.27 again \( f^{-1}([0, 1]) \cap Q \times [0, 1] \) contains
that corresponds to the height in 

order is a Suslin tree. We now look at the function only in 

to conclude, using again a countable dense subset of $[0, 1)$.

By Lemma 4.21, this implies that $f^{-1}([0, \alpha)) \supset (Q - Q_{\gamma}) \times (c_0, 1) \cup L$ for some $c_0$. By induction again, we find a sequence $e_\beta$ that 'goes down', and for some $\beta, \gamma \in \omega_1, f^{-1}([0, \beta + 1)) \supset \Sigma(Q) - \Sigma(Q_{\gamma})$. Since $f$ is bounded on $\Sigma(Q_{\gamma})$, $f$ is bounded on all of $\Sigma(Q)$ which is thus a direction in itself.

(b) Let $T$ be a Suslin tree with slicer $s : T \to \omega_1 \subset L_{\geq 0}$ given by the height in $T$. $T$ is regular, locally second countable and $\omega_1$-compact, thus $\Sigma(T)$ is locally metrizable and $\omega_1$-compact by Corollary 4.19 and Lemma 4.20. It is moreover locally separable since $T \times [0, 1)$ is such. We now show with a sequence of lemmas that $\Sigma(T)$ is a direction in itself. The idea is similar to that in (a): go 'up and down' until you fill up the entire space, but since Lemma 4.27 may fail (at least the part about large open sets), we must work harder and use some properties of Suslin trees.

**Lemma 6.10.** Let $T$ be a Suslin tree, and $f : T \times \mathbb{R} \to L_{\geq 0}$. Let $a, b \in \mathbb{R}$, then $f$ is bounded on $T \times \{a\}$ iff $f$ is bounded on $T \times \{b\}$.

This lemma does not hold if $T$ is an $\mathbb{R}$-special tree, see [4].

**Proof.** This will follow from the fact that the road space of a Suslin tree is not contractible. We can assume that $a = 0, b = 1$, and that $f$ is constant on $0$ on $T \times \{0\}$. Suppose that $f$ is unbounded on $T \times \{1\}$. By Lemma 2.6, the subset $T' = \{u \in T : u \in \text{Lev}_\alpha(T) \text{ and } f(u, 1) = \alpha\}$ is club in $T$, thus $T'$ with the induced order is a Suslin tree. We now look at the function only in $T' \times [0, 1]$. Consider the road space $R_{T'}$ of $T'$. There is an obvious way to define a slicer $s : R_{T'} \to L_{\geq 0}$ that corresponds to the height in $T' \subset R_{T'}$, and to extend the domain of $f$ to $R_{T'} \times [0, 1]$. For $(u, t) \in R_{T'} \times [0, 1]$ set $h(u, t) \in R_{T'}$ to be the element $v \leq u$ such that $s(v) = f(u, t)$, then $h : R_{T'} \times [0, 1) \to R_{T'}$ is continuous, $h(u, 0)$ is the identity and $h(u, 0)$ is constant on $0$. But such a map does not exist because the road space of a Suslin tree is not contractible, by Corollary 6.3 in [4].

**Lemma 6.11.** Let $T$ be a Suslin tree, and $f : \Sigma(T) \to L_{\geq 0}$ be such that for some $\beta \in \omega_1, f^{-1}([0, \beta])$ intersected with some horizontal is unbounded. Then $f$ is bounded on $\Sigma(T)$.

**Proof.** Suppose that for some $a \in [0, 1)$ and an unbounded $T' \subset T$, $T' \times \{a\} = f^{-1}([0, \beta]) \cap (T \times \{a\})$. But $T'$ is thus a Suslin subtree of $T$, and by the previous lemma, $f$ is bounded on $T' \times \{y\}$ for each $y \in \mathbb{R}$. Take a countable dense subset of $[0, 1)$, then the supremum of the bounds for the corresponding horizontal gives a bound for $f$ on $T' \times [0, 1)$, and $f$ is thus bounded on an unbounded subset of the core $L$. Since $L$ is a direction in itself, it follows that $f$ is bounded on it, say by $\alpha$. By Lemma 4.21 this implies that $f^{-1}([0, \alpha + 1))$ contains the terminal part of some horizontal, and thus $f$ is bounded on this horizontal. The previous lemma enables to conclude, using again a countable dense subset of $[0, 1)$, that $f$ is bounded on $T \times [0, 1)$.
We now finish the proof of (b). Let \( f : \Sigma(T) \to \mathbb{L}_{\geq 0} \) be such that \( f^{-1}(\{0\}) \) is unbounded. For uncountably many \( \alpha \) there are \( a_\alpha, b_\alpha \in \mathbb{R} \) and \( x_\alpha \) at height \( \alpha \) in \( T \) such that \( \{x_\alpha\} \times (a_\alpha, b_\alpha) \subset f^{-1}([0,1)) \). Thus for some \( a, b \in \mathbb{Q} \) there are uncountably many \( \alpha \) with \( \{x_\alpha\} \times (a, b) \subset f^{-1}([0,1)) \). This implies that \( f^{-1}([0,1]) \) intersects the horizontals at height between \( a \) and \( b \) unboundedly. By Lemma 6.11 this implies that \( f \) is bounded on \( \Sigma(T) \).

(c) Let \( N \) be given by Theorem 3.11. As in (b), \( \Sigma(N) \) is locally metrizable locally separable and \( \omega \)-bounded by Corollary 4.19 and Lemma 4.20. The proof that it is a direction in itself is verbatim as in (a).

Let us make some remarks about \( \Sigma(X) \). If one starts with \( S_T \) (defined in Section 3.2) in place of \( T \), then \( S_T \times [0,1) \) is a 3-manifold, but \( \Sigma(S_T) \) is not. It is possible to obtain a manifold if one starts with an \( \omega \)-bounded surface \( S \) that possesses a slicer with good properties.

**Definition 6.12.** Let \( S \) be a Type I surface. Then \( s : S \to \mathbb{L}_{\geq 0} \) is a gentle slicer if \( s \) is a slicer and for each \( \alpha \in \omega_1 \) there is a continuous \( \theta : S_\alpha \to S^1 \) such that \( x \mapsto \langle \theta(x), s(x) \rangle \) is a homeomorphism from \( S - s^{-1}({\{0\}}) \) to \( S^1 \times (0, \alpha) \).

This ensures that the points in the added copy of \( \mathbb{L}_{\geq 0} \) do possess neighborhoods homeomorphic to the 3-ball, as suggested on Figure 12, which implies the following corollary:

**Corollary 6.13.** (\( \Diamond^+ \)) If the surface given by Theorem 3.11 possesses a gentle slicer, there is an \( \omega \)-bounded 3-manifold which is a direction in itself and which possesses a club subset that is not a direction in itself.

![Figure 12: Neighborhoods in \( \Sigma(S) \) for a \( S \) a surface with a gentle slicer.](image)

### 7 Discrete directions

If a club subset of a Type I space is discrete, then it contains a club subset disjoint from some skeleton of \( X \), as easily seen. So, can a direction be disjoint from the
skeleton (and thus contain a discrete direction)? In general the answer is positive but not for $\omega_1$-compact spaces:

**Lemma 7.1.** Let $X$ be $\omega_1$-compact and Type I, and $D \subset X$ be club. Then, $D \cap \text{Sk}(X)$ is a direction in $X$ iff $D$ is a direction in $X$.

**Proof.** Let $D$ be a direction. By Lemma 2.1 $D \cap \text{Sk}(X)$ is club and thus a direction. Conversely, let $f : X \to \mathbb{L}_{\geq 0}$ be continuous with $f\big|_D$ unbounded. If $f\big|_{D \cap \text{Sk}(X)}$ is bounded by $\alpha$, say, then $f^{-1}(\{\alpha+1, \omega_1\}) \cap D$ is a club that misses all the bones, which is impossible since $X$ is $\omega_1$-compact, so $f\big|_{D \cap \text{Sk}(X)}$ is unbounded. So $f^{-1}(\{0\}) \cap D \cap \text{Sk}(X)$ is bounded, say by $\beta$. Thus, $f^{-1}(\{0\}) \cap D$ must be bounded since it misses all the bones above $\beta+1$.

If the space is not $\omega_1$-compact, both implications may fail. The space given by two copies of $\omega_1$ with respectively the discrete topology and the order topology is not a direction in itself but its skeleton is. The other implication may fail badly, as shown on the simple Example 7.6 below.

While directions in a space can be discrete, and the space itself can be highly non-$\omega_1$-compact (see Theorem 7.7), the presence of a direction forces some kind of ‘indiscreteness’, at least for regular spaces. Recall that a collection of subsets of a space $X$ is said *discrete iff* any point of $X$ has a neighborhood intersecting at most one element of the collection. This implies that given any subcollection, the union of the closures of its members is closed. If the sets are singletons, the union is thus a closed discrete set.

**Lemma 7.2.** Let $D$ be a direction in a regular Type I space $X$. Let $U$ be a neighborhood of $D$, then the subset of $\alpha$ such that $\overline{U} \cap B_\alpha(X) \neq \emptyset$ is stationary.

**Proof.** If there is a neighborhood $U$ of $D$ whose closure misses a club set of bones, up to changing the canonical sequence we can assume that it misses all the bones. We can also assume that $U_\alpha \cap D$ is non-empty for each $\alpha$, where $U_\alpha = U \cap (X_{\alpha+1} - \overline{X_\alpha})$. Notice that the collection of the $U_\alpha$ is discrete. Choose $x_\alpha \in D \cap U_\alpha$. Partition $\omega_1$ into two disjoint uncountable subsets $E_0, E_1$, then $D_i = \{x_\alpha : \alpha \in E_i\} \subset D$ are club for $i = 0, 1$. By Lemma 2.2 $X$ is Tychonoff, thus for $\alpha \in E_1$ there is $f_\alpha : U_\alpha \to \mathbb{L}_{\geq 0}$ that takes the value 0 on $U_\alpha - U_\alpha$ and $\alpha$ on $x_\alpha$. Define then $f : X \to [0, \alpha] \subset \mathbb{L}_{\geq 0}$ as taking the value 0 outside of $\bigcup_{\alpha \in E_1} U_\alpha$, and being equal to $f_\alpha$ on $U_\alpha$. Then $f$ is continuous and unbounded on $D$ but $f^{-1}(\{0\}) \supset D_0$ is unbounded as well, so $C$ is not a direction.

A space is *(strongly) collectionwise Hausdorff* iff any closed discrete set $D$ can be expanded to a disjoint (resp. discrete) collection of open sets. That is, for each $x \in D$ there is an open $U_x \ni x$ such that the collection of the $U_x$ is disjoint (resp. discrete). We abbreviate collectionwise Hausdorff by cwH. It is well known that a normal cwH space is strongly cwH. The following is very similar to the previous lemma:

**Lemma 7.3.** Let $X$ be Type I, regular and strongly cwH. Then a direction in $X$ cannot be discrete.
Proof. Let $C \subset X$ be closed discrete (and thus unbounded). We may list $C = \{x_\alpha : \alpha \in \omega_1\}$. Expand $C$ to a collection of discrete open sets $U_\alpha \ni x_\alpha$. Then proceed as in Lemma 7.2 to show that $C$ is not a direction.

Example 7.6 shows that ‘strongly’ cannot be omitted in the statement. The first part of the following lemma is an easy corollary.

**Lemma 7.4.** The following holds.
(a) Let $X$ be Type I, regular, strongly cwH and spanned by directions. Then $X$ is $\omega_1$-compact.
(b) Let $X$ be Type I, regular, $\omega_1$-compact and such that each $\overline{X_\alpha}$ is cwH. Then $X$ is strongly cwH.

**Proof.**
(a) If $X$ is not $\omega_1$-compact, there is a club discrete subset which expands to a discrete collection of open sets. The union of these open sets must contain a direction and thus a discrete one, which is impossible by the previous lemma.
(b) A closed discrete subset must be countable and thus contained in some $\overline{X_\alpha}$. Since $\overline{X_{\alpha+1}}$ is normal, it is strongly cwH and we may expand the discrete set in it.

**Corollary 7.5.** (PFA) Let $X$ be Type I, regular, strongly cwH, locally compact, countably tight and spanned by directions. Then any club subset of $X$ contains a copy of $\omega_1$.

**Proof.** By Theorem 3.4 and the previous lemma.

**Example 7.6.** A locally metrizable locally separable cwH Type I space which is a direction in itself and is spanned by discrete directions.

**Proof.** Take $X = (L_{\geq 0} \times \mathbb{R}) - (\omega_1 \times \mathbb{Q})$. The proof that $X$ is a direction in itself is similar to that of Theorem 6.9 (a). Let $f : X \to L_{\geq 0}$ be given such that $f^{-1}(\{0\})$ is club. For $r \in \mathbb{R}$ set $E_r = f^{-1}(\{0\}) \cap (L_{\geq 0} \times \{r\})$. If $f^{-1}(\{0\}) \cap L_{\geq 0} \times (\mathbb{R} - \mathbb{Q})$ is unbounded, by Lemma 4.27 $E_r$ is club for some $r \in \mathbb{R} - \mathbb{Q}$, else $E_r$ is club for some $r \in \mathbb{Q}$. In both cases there are $p_1 < r < q_1$ such that $X \cap \{x\} \times (p_1, q_1) \subset f^{-1}([0, 1))$ for an unbounded set of $x$. Thus $f^{-1}([0, 1)) \supset [\beta_1, \omega_1) \times [p_1, q_1] \cap X$ for some $\beta_1 \in \omega_1$. Proceeding by induction as in Theorem 6.9 (a), we find $\alpha \in \omega_1$ such that $f^{-1}([0, \alpha]) \supset [\beta_0, \omega_1) \times \mathbb{R} \cap X$ and $f$ is thus bounded on $X$. The proof that $X$ is spanned with discrete directions is very similar and we skip it. To see that it is cwH, notice that if $D \subset X$ is closed discrete then by Lemma 4.27 again its intersection with $L_{\geq 0} \times (\mathbb{R} - \mathbb{Q})$ must be bounded, say by $\alpha$. A metrizable space is cwH, so inside $X_\alpha$ we may separate the points of $D$. The remaining points of $D$ lie in $\cup_{\beta \in \omega_1} (\beta, \beta + 1) \times \mathbb{R}$, and we may separate $D$ in each piece.

It is possible to go beyond this example and find spaces (in various classes) such that any direction contains a discrete one.
Theorem 7.7. The following holds:
(a) There is a Type I locally metrizable locally separable space that possesses a discrete
direction, and such that any club in \( X \) contains a club discrete subset.
(b) Under \( \diamondsuit^* \), there is a cwH space as in (a).
(c) \( (\neg \text{SH}) \) There is a Type I locally metrizable locally separable space
possessing a discrete direction and a dense \( \omega_1 \)-compact subspace, and such that any
direction in the space contains a club discrete subset.
\( *(d) \ (\diamondsuit^+ ) \) There is a Type I space as in (c), such that the dense \( \omega_1 \)-compact
subspace is a manifold which is a countable union of \( \omega \)-bounded submanifolds. If the
surface given by Theorem 3.11 possesses a gentle slicer, the entire space is actually
a 3-manifold.

For (a) and (b) a subset of \( \Sigma(X) \) for well chosen \( X \) would suffice, but something
else is needed for (c) and (d) since any \( \Sigma(X) \)-variant would yield a direction in itself,
hence any club in it is a direction, which would contradict the existence of a club
\( \omega_1 \)-compact subset of the space by Lemma 7.1. We shall use \( \Omega(X, A) \), which gives
also the result for (a) and (b).

Proof of Theorem 7.7.
(a) Let \( T \) be an \( R \)-special Aronszajn tree with a slicer given by the height in \( T \),
and \( A \) the subset of successor ordinals in \( L \). Any club subset of \( T \) contains a club
discrete subset (see for instance [6, Theorem 2]). Lemmas 4.32–4.34 show that \( A \)
is a direction in \( \Omega(T, A) \), we just need to show that any club subset in
\( C \subset \Omega(T, A) \) contains a closed uncountable discrete subset. If \( C \cap A \)
is unbounded, we are done.
If not, then \( C \cap T^{(n)} \) is club for some \( n \), and we apply Lemma 4.25 (recall that
\( T^{(n)} \) is a copy of \( \text{Down}(T) \)).
(b) This is a consequence of (a) and a result of Devlin and Shelah [7] who used \( \diamondsuit^* \)
to build an \( R \)-special cwH tree \( T \). We check that this implies that \( \Omega(T, A) \) is also cwH,
so let \( C \subset \Omega(T, A) \) be closed discrete. For each \( n \), by Lemma 4.25 the projection
\( \pi(C \cap T^{(n)}) \) on the first factor of \( T^{(n)} \) is discrete. Since \( T \) is cwH, \( \pi(C \cap T^{(n)}) \)
leaves a club set \( E_n \subset \omega_1 \) of bones. This follows easily by the pressing down lemma and
holds in any Type I cwH space \( Z \) such that \( Z_\alpha \) is second countable (or just ccc) for
each \( \alpha \). Then \( C \) avoids the bones in \( E = \cap_{n \in \omega} E_n \), up to changing the canonical
sequence assume \( E \) to be \( \omega_1 \). Since \( \Omega(T, A)_\alpha + 1 - \Omega(T, A)_\alpha \) is metrizable, it is cwH
and we may separate \( C \) in each piece, yielding the result.
(c) Start with \( T \) a Suslin tree and \( A \) as in (a). Then, by Lemma 4.26 \( T^{(\omega)} \) is \( \omega_1 \)-
compact, and the result follows.
(d) Take \( N \) given Theorem 3.11 and \( A = L_{\geq 0} - \omega_1 \) and proceed as in (c). If \( N \)
possesses a gentle slicer, then \( \Omega(S, A) \) is a 3-manifold (see the discussion after the
proof of Theorem 6.9).

Some remarks. If \( MA + \neg \text{CH} \) holds, then any Aronszajn tree is special by a
result of Baumgartner, and this easily implies that \( T \) (and thus \( \Omega(T, A) \)) is not cwH.

Problem 3. Does Point (b) of Theorem 7.7 hold in ZFC?

Notice that none of our examples is spanned with directions.
Problem 4. Is there a (first countable) Type I space spanned by directions, such that any direction in it contains a discrete club subset?

We believe that a construction mixing $\Omega(X, A)$ and Example $1.3$ might work for an appropriate $X$, but it remains to be checked. We notice that Example $6.4$ gives a related (though much weaker) result in ZFC: a Type I surface which is a direction in itself, such that no unbounded subset of $X$ is countably compact. Indeed, it is quasi-developable, and hence any countably compact subset is compact and thus bounded (see Lemma $6.8$ and the discussion p. $162$ in $[15]$).

8 More on the ufo $\langle \mathcal{D}_X, \preceq \rangle$

8.1 Ufo-maximal and ufo-minimal directions

This section treats the question: Given a Type I space $X$ spanned by directions, are there ufo-maximal or minimal directions in $\langle \mathcal{D}_X, \preceq \rangle$? Notice that Theorem $4.4$ implies trivially (under PFA) that an $\omega_1$-compact countably tight locally compact space is spanned by directions, and even better: the direction is located inside the club.

We are going to use the $S^{\alpha,h,r}$ defined in Section $4.1.3$ in various examples. Our first result is the following.

Theorem 8.1. There is an $\omega_1$-compact surface $S$ spanned with directions such that $\langle \mathcal{D}_S, \preceq \rangle$ is isomorphic to $\langle \mathbb{Z}, \preceq \rangle$. In particular, there is neither an ufo-min nor an ufo-max direction in $S$.

Proof. Take $T = S^{\omega,h,r} - \Delta_\omega$, with $h : \omega \to \omega_1$ the inclusion and $r(n) = \uparrow$ for all $n \in \omega$. Then $T$ is $\omega_1$-compact, because any uncountable discrete set has to be unbounded, and any club $C \subset T$ must intersect one of the (closed) $\mathcal{O}_n$ in a club way. By Lemma $4.12$ $T$ is spanned by directions. Since it is a particular case of Example $4.13$ $\langle \mathcal{D}_T, \preceq \rangle \simeq \langle \omega, \preceq \rangle$. Taking $r(n) = \downarrow$ for all $n$ yields $T'$ with $\langle \mathcal{D}_{T'}, \preceq \rangle \simeq \langle \omega^*, \preceq \rangle$. Finally, glue $T$ and $T'$ along their remaining free side to obtain the desired $S$. □

In the $\omega$-bounded framework, all countable chains do possess minimal and maximal elements for sequential spaces (in other words, $\langle \mathcal{D}_M, \preceq \rangle$ is up- and downwards $\omega_1$-closed):

Lemma 8.2. Let $X = \bigcup_{\alpha \in \omega_1} X_\alpha$ be Type I, $\omega$-bounded and sequential, $C_i \subset X$ $i \in \mathbb{Z}$ be directions with $C_i \preceq C_{i+1}$. Then there are directions $D, D' \subset X$ such that $D \preceq C_i \preceq D'$ for all $i \in \mathbb{Z}$.

Proof. If $A = \{ D_j : j \in \omega \}$ is a countable set of directions, we let $D_A$ be the set of accumulation points of sequences $x_{j_n}$, $j_n$ a strictly increasing sequence in $\omega$, with $x_{j_n} \in D_{j_n}$ for each $n \in \omega$. $D_A$ is closed since $X$ is sequential, and if $X$ is $\omega$-bounded, it is unbounded. Suppose that $D_A$ is not a direction, there is thus a function $f : X \to \mathbb{L}_{\geq 0}$ unbounded on $D_A$ with $f^{-1}\{0\} \cap D_A$ unbounded. For each $\alpha$ there are therefore $k(\alpha)$ and $\ell(\alpha)$ in $\omega$, and $x_\alpha \in D_{k(\alpha)} - X_\alpha$, $y_\alpha \in D_{\ell(\alpha)} - X_\alpha$ such that $f(x_\alpha) \in [0, 1)$ and $f(y_\alpha) \in [\alpha, \omega_1)$. Let $k, \ell$ be such that $k(\alpha) = k$ for
uncountably many $\alpha$, and $\ell(\beta) = \ell$ for uncountably many $\beta$. Then, $f$ is unbounded on $D_k$ and bounded on an unbounded subset of $D_\ell$, and thus, since $D_\ell$ is a direction, $f$ is bounded on $D_k$. Moreover, $k$ and $\ell$ can be arbitrarily large in $\omega$ since $D_A = D_{A'}$ with $A' = \{D_j : m \leq j < \omega\}$. Thus, if $D_j \preceq D_{j+1}$ for all $j$ or if $D_j \succeq D_{j+1}$ for all $j$, there will be $k_1, k_2 \in \omega$, with $D_{k_1} \preceq D_{k_2}$ and $f$ unbounded on $D_{k_1}$ and bounded on $D_{k_2}$, a contradiction. Taking $A = \{C_j : j \leq 0\}$, $A' = \{C_j : j \geq 0\}$, $D = D_A$, $D' = D_{A'}$ are thus directions.

We now show that $D \preceq C_i$ for all $i \in \mathbb{Z}$. Let $f : X \to \mathbb{L}_{\geq 0}$ be bounded on $C_i$, then it is bounded on all $C_j$ with $j \leq i$. Let $\alpha$ be a bound for $f$ on $\bigcup_{j \leq i} C_j$, then by continuity $\alpha$ is a bound for $f$ on $D$, proving that $D \preceq C_i$. If now $f$ is unbounded on some $C_i$, it is unbounded on all $C_j$ with $j \geq i$. Given $\alpha$, let $x_j \in C_j$ ($j \geq i$) be such that $f(x_j) \in [\alpha, \omega_1)$. Taking a convergent subsequence we obtain a $x \in D'$ with $f(x) \in [\alpha, \omega_1)$, proving that $f$ is unbounded on $D'$, hence that $C_i \preceq D'$.

It happens that it is not difficult to find an $\omega$-bounded manifold without minimal direction:

**Theorem 8.3.** There is an $\omega$-bounded surface $S$ spanned by directions for which $\langle \mathcal{D}_S, \preceq \rangle$ does not possess a minimal direction.

**Proof of Theorem 8.3.** Consider $S^{\omega_1, 2, h, r}$, with $r(\alpha) = \downarrow$ for all $\alpha \in \omega_1 \cdot 2$, and $h(\beta) = h(\omega_1 + \beta) = \beta$ for each $\beta < \omega_1$. Define $S$ by identifying $\Delta_0$ and $\Delta_{\omega_1, 2}$ (see Figure 13).

![Figure 13: Theorem 8.3](image)

Then, if $\alpha < \beta < \omega_1$, $\Delta_\alpha \preceq \Delta_\beta$ and $\Delta_{\omega_1 + \alpha} \succeq \Delta_{\omega_1 + \beta}$. But $\Delta_{\omega_1} \not\preceq \Delta_\alpha$ for any $\alpha < \omega_1$, and neither $\Delta_0 = \Delta_{\omega_1, 2}$ is $\preceq \Delta_{\omega_1 + \alpha}$. Indeed, for $\Delta_{\omega_1}$ it is easy to find a continuous $f : S \to \mathbb{L}_{\geq 0}$ such that $f|_{\Delta_{\omega_1}}$ is the identity, while $f|_{\Delta_0}$ is constant on $\alpha$ after height $\alpha$ for $\alpha < \omega_1$ and on 0 for $\omega_1 < \alpha \leq \omega_1 \cdot 2$. Since this would be the only possible lower bound, the chain $\langle \Delta_\alpha : \alpha < \omega_1 \rangle$ does not possess such an lower bound. We may find a similar map for $\Delta_{\omega_1, 2} = \Delta_0$. Moreover any direction in $S^{\omega_1, h, r}$ is ufo-equivalent to some $\Delta_\alpha$ by Corollary 4.9 and therefore $\langle \mathcal{D}_S, \preceq \rangle$ is isomorphic to two copies of $\omega_1^*$ such that the members lying in different copies are not comparable. 

\[44\]
The case of the maximum is trickier. The naive idea of taking $S^\omega_{h,r}$ with $r(\alpha) = \uparrow$ for all $\alpha$ does not work: $\Delta_{\omega_1}$ is ufo-maximal (and $\geq \Delta_\alpha$ for each $\alpha$). We suspect that PFA implies that an $\omega$-bounded manifold (or Type I locally compact first countable space) always contains an ufo-maximal direction, but we have no evidence for this claim. (Notice that in that case, a positive result for $\omega$-bounded spaces do not hold.) On the other hand, we have the following:

**Theorem 8.4.** The following holds.

(a) There is an $\omega$-bounded space $X$ containing directions without ufo-maximal direction.

(b) (¬SH) There is an $\omega_1$-compact 3-manifold $M$ spanned with directions without ufo-maximal direction, actually $\langle \mathcal{D}_M, \preceq \rangle$ is a Suslin tree.

(c) (H+) There is an $\omega$-bounded 3-manifold as in (b), with $\langle \mathcal{D}_M, \preceq \rangle \simeq \langle \omega_1, \preceq \rangle$.

**Proof.** The proof uses the spaces $\Xi(X,A,s)$ of Section 4.2.4. In each case $A$ will be the subset of successor ordinals in $\mathbb{L}_{\geq 0}$. For (a), consider the space $\Xi(Q,A,s)$ with $Q$ given by Theorem 3.1 and apply Lemma 4.38. The proof for (b) and (c) is very similar: we consider $\Xi(S_T,A,s)$ and $\Xi(N,A,s)$, where $T$ is a Suslin tree and $N$ is given by Theorem 3.11, with a discrete unbounded $A$. In both cases the space is spanned by directions by Lemma 4.37 and Lemma 4.38 shows that there is no ufo-maximal direction. With a bit more work, we will show that the ufo-equivalent classes of directions is precisely the tree $T$ in the former case and $\langle \omega_1, \preceq \rangle$ in the latter case. Let $M = \Xi(S_T,A,s)$. Call vertical rays the subsets of $\Xi(S_T,A,s)$ which are either $\{y\} \times \mathbb{L}_{\geq s(y)}$ whenever $s(y) \notin A$, or an horizontal or diagonal ray in some inserted $\mathbb{L}_{\geq \alpha + 1}$ (which end up being ‘vertical’ because of the first factor $s^{-1}(\{\alpha + 1\})$).

Notice that any direction must have a club intersection with a vertical ray: its intersection with $\Delta(S_T)$ cannot be club, then by Lemma 4.36 it is contained in $\Xi((S_T)_{\alpha}, A,s_{(S_T)_{\alpha}})$. If its intersection with one of the inserted $E_{\alpha}$ is club, we are done, otherwise we apply Lemma 4.27 to conclude.

We will show later that we can choose $s$ such that each component $\sigma$ of $s^{-1}(\{\alpha + 1\})$ is homeomorphic to $\mathbb{R}$, hence $\sigma \times \mathbb{L}_{\geq \alpha + 1}$ is a direction in itself. Figure 14 (left) below explains what happens ‘just above’ this component. The only vertical parts depicted are the vertical rays in the inserted pieces. All the rays which lie in zones of the same shade of grey are ufo-equivalent, and $\geq$ can be read by following the lines from left to right on the small piece of tree below. In particular, the three central vertical sides are ufo-equivalent (and so are the vertical rays between them). The right hand part picture shows the limit level $\alpha$, since no $E_{\alpha}$ is inserted we depicted the vertical rays at height $\alpha$ and the two immediate successors (where there are inserted pieces). We have thus constructed ‘jumps’ between the directions in a perfect emulation of the tree $T$, and thus the ufo-equivalent classes of direction are precisely $T$. The case of $\Xi(N,A,s)$ is similar but easier: we show below that $s^{-1}(\{\alpha + 1\})$ has only one component (homeomorphic to the circle), so the inserted ‘jumps’ between directions do form a chain of type $\omega_1$.

We now show that $\Xi(S_T,A,s)$ and $\Xi(N,A,s)$ are indeed 3-manifolds when $s$ is well chosen. To this end we will for once emulate the theory of foliations, and we thus start with some terminology (which is not standard and valid just for this section).
Definition 8.5. Let \( s : S \to \mathbb{L}_{\geq 0} \) be a slicer on a Type I surface \( S \) and \( A \subset \mathbb{L}_{\geq 0} \). Then \( s \) is an \( A \)-foliation of \( S \) if for all \( x \in s^{-1}(A) \), there are \( a, b \in \mathbb{L}_{\geq 0} \) with \( a < s(x) < b \), an open subset \( U \ni x \) and a map \( \varphi : s^{-1}((a, b)) \cap U \to \mathbb{R} \) such that

\[
\langle s, \varphi \rangle : s^{-1}((a, b)) \cap U \to (a, b) \times \mathbb{R}
\]

is an homeomorphism. If \( A = \mathbb{L}_{\geq 0} \) we say that \( s \) is a foliation of \( S \).

If \( s \) is an \( A \)-foliation and \( a \in A \), then \( s^{-1}(\{a\}) \) is an at most countable disjoint union of 1-dimensional metrizable submanifolds (called leaves) which are thus copies of \( \mathbb{S}^1 \) or \( \mathbb{R} \). It is convenient to have tubular neighborhoods (i.e. the space ‘between two leaves’) that behave well around such components.

Definition 8.6. An \( A \)-foliation on a Type I surface given by a slicer \( s : S \to \mathbb{L}_{\geq 0} \) is tame if for each \( a \in A \), \( s^{-1}(\{a\}) \) is a discrete collection of components, and for each component \( \sigma \subset s^{-1}(\{a\}) \), \( U \) can be chosen such that \( U \supset \sigma \) in Definition 8.5.

Lemma 8.7. The following holds.
(a) If \( T \) is a binary Aronszajn tree, then there is a tame foliation on \( S_T \).
(b) If \( S \) is an \( \omega \)-bounded Type I surface, there is a tame \( A \)-foliation on \( S \) for \( A \) the subset of successors ordinals.

Proof.
(a) Figure 14 below shows what the foliation looks like in each piece of \( S_T \) and should convince the reader that the foliation is indeed tame.
(b) Nyikos’s bagpipe theorem [14, Thm 5.14] shows that there is some \( \alpha \) such that
$S - S_\alpha$ is a finite union of longpipes, that is, Type I surfaces $P$ such that for each $\beta$, $P_{\beta+1}$ is homeomorphic to the cylinder $S^1 \times (0,1)$ and the boundary of $P_{\gamma+1}$ in it is homeomorphic to the circle whenever $\gamma < \beta$. We may thus take a small strip around these circles at successor height, define the scale accordingly in those strips, and then extend to the rest of the space as in Lemma 2.5.

Notice that we can assume that $N$ of Theorem 3.11 is a longpipe: take one of the longpipes given by the bagpipe theorem and sew in a disk to close it. This gives an $\omega$-bounded surface without direction.

Lemma 8.8. If $s$ is a tame $A$-foliation of a surface $S$ and $A$ is discrete, then $\Xi(X, A, s)$ is a 3-manifold (with boundary).

Proof. Recall that $\text{Up}(S,s)$ is a 3-manifold. Let $x \in \Xi(X,A,s)$. We shall find a neighborhood of $x$ homeomorphic to $\mathbb{R}^3$ (or to $\mathbb{R}^2 \times [0,\infty)$). If $x$ lies in the interior of some inserted $E_a$, we are done. If $x$ is on the boundary of some inserted $E_a$, by discreteness of $A$ we may easily describe such a neighborhood. Suppose that $x$ is not in some $E_a$. Consider $p(x) \in \text{Up}(S,s)$, then $p(x) \in \text{Up}(S,s)_\alpha$ for some $\alpha$. Notice that since $\text{Up}(S,s)$ is the subspace of $S \times \mathbb{R}_\geq 0$ which is above the graph of $s$, there is an homeomorphism from $\text{Up}(S,s)_\alpha$ to $S_{\alpha} \times [0,1)$ that leaves the first coordinate fixed, so we can assume that $p(x) \in S_{\alpha} \times [0,1)$. We choose some neighborhood $V$ of its projection on $S_{\alpha}$ which is homeomorphic to $\mathbb{R}^2$ and denote by $\phi : V \to \mathbb{R}^2$ an homeomorphism. Let us see how $V \times [0,1)$ changes when we insert the $E_a$ (for $a < \alpha$). In $S \times [0,1)$ these insertions boil down to taking out $s^{-1}(\{a\}) \times [0,1)$ and sewing in $s^{-1}(\{a\}) \times P$, where $P = \{(x,y) \in [0,1)^2 : y \leq x\}$.

Let $L$ be the set of leaves in $\cup_{a \in A \cap (0,\alpha)} s^{-1}(\{a\})$. Notice that each component in $s^{-1}(\{a\})$ is closed in $S$ since the $A$-foliation is tame. Thus, by discreteness of $A$, for each leaf $\sigma \in L$ we may choose a neighborhood $U_\sigma$ with an homeomorphism which sends $\sigma$ to a vertical line. We can ensure that the $U_\sigma$ are pairwise disjoint. It is then easy to change the homeomorphism $\phi \times id : V \times [0,1) \to \mathbb{R}^2 \times [0,1)$ inside each $(V \cap U_\sigma) \times [0,1)$ to ‘absorb’ the inserted $E_a$, as shown in Figure 16 below. The northwest picture shows a leaf $\sigma$ times $[0,1]$ together with a tubular neighborhood while the southwest picture shows the effect of $\phi \times id$. The northeast and southeast pictures show the situation after the insertion of $E_a$. 

\[\text{Figure 15: Foliating } S_T.\]
This finishes the proof of Theorem 8.4.

8.2 A partial order on maps $X \to \mathbb{L}_{\geq 0}$ and its relations with the ufo

**Definition 8.9.** Let $X$ be Type I and $f, g : X \to \mathbb{L}_{\geq 0}$ be both unbounded. We say that $f \preceq g$ iff for all clubs $C \in X$, ‘$f|_C$ is bounded’ implies ‘$g|_C$ is bounded’. If $f \preceq g \preceq f$, we say that $f$ and $g$ are club-equivalent. We denote the strict order by $\prec$ and by $\langle \mathfrak{F} X, \preceq \rangle$ the poset of club equivalent classes.

(As for $\preceq$, $\preceq$ is actually a preorder.) Notice that $\preceq$ goes the other way than $\succeq$. The idea is that a function which is unbounded only on directions that are ‘big’ for $\preceq$ must be big for $\succeq$. This is made more precise in Proposition 8.14 below.

**Examples 8.10.**

- $f : \mathbb{L}^2_{\geq 0} \to \mathbb{L}_{\geq 0}$, $(x, y) \mapsto \min\{x, y\}$ is $\preceq$-maximal in $\mathfrak{F}_{\mathbb{L}^2_{\geq 0}}$.
- A slicer $X \to \mathbb{L}_{\geq 0}$ is always $\preceq$-minimal.
- Let $X = S^{\omega, h, r}$, with $r(n) = \uparrow$ for all $n \in \omega$ and $h : \omega \to \omega_1$ be the inclusion. Then $\mathfrak{F} X$ does not possess a $\preceq$-maximal map. Indeed, an unbounded map $f$ must be
unbounded on $\Delta_\omega$, but then for some $n$ it will be unbounded on all $\Delta_k$ for $k \geq n$ and bounded on the others. One then easily finds a map $g$ that has the same properties but for $n+1$ so $f \prec g$.

- $X = S^{\omega_1,h,r}$, with $r(\alpha) = \uparrow$ for all $\alpha \in \omega_1$ and $h : \omega_1 \to \omega_1$ the identity. Then $g_X$ has a $\preceq$-maximal map that is the identity on $\Delta_{\omega_1}$ and bounded on all $\Delta_\alpha$ with $\alpha < \omega_1$ (recall Theorem 8.3).

The connections between $\preceq$ for clubs subset and $\preceq$ for maps may be quite evasive in general, but are relatively explicit in the $\omega_1$-compact framework, and we shall restrict our attention to this case. The first step is to find some kind of 'minimal' club set on which a function is unbounded. This is provided by the $\omega_1$-compact framework, and we shall restrict our attention to this case. The first step is to find some kind of 'minimal' club set on which a function is unbounded. This is provided by the $C(f,D)$ of Lemma 2.0: If $X$ is the entire space, we denote $C(f,X)$ simply by $C(f)$. An immediate consequence of the definition of $C(f)$ is:

**Lemma 8.11.** If $X$ is a Type I $\omega_1$-compact space, $f : X \to \mathbb{L}_{\geq 0}$ continuous, $D \subset X$ club, then $f\big|_D$ is unbounded iff $D \cap C(f)$ is club.

Whether $f \preceq g$ can be read directly in $C(f)$ and $C(g)$:

**Lemma 8.12.** Let $X$ be an $\omega_1$-compact Type I space and $f, g : X \to \mathbb{L}_{\geq 0}$ be continuous. Then the following holds.

(a) $f \preceq g$ iff there is no club $C$ in $C(g) - C(f)$,

(b) $f \prec g$ iff $f \preceq g$ and $C(f) - C(g)$ contains a club.

**Proof.**

(a) By Lemma 8.11, $f \preceq g$ iff for all clubs $C \subset X$, $C \cap C(g)$ is club implies that $C \cap C(f)$ is club, which is equivalent to saying that there is no club in $C(g) - C(f)$. Indeed, if there is a club $C \subset (C(g) - C(f))$, then $C \cap C(g)$ is club while $C \cap C(f) = \emptyset$. Conversely, suppose that there is no club in $C(g) - C(f)$ and $C \cap C(g)$ is club. Then if $C \cap C(f)$ is bounded, say included in $X_\alpha$, by considering $C' = C - X_\alpha$ we have $C' \cap C(f) = \emptyset$, but then $C(g) - C(f) \supset (C' \cap C(g))$. Therefore $C \cap C(f)$ must be unbounded.

(b) $f \prec g$ iff $f \preceq g$ and there is a club $C$ with $f\big|_C$ unbounded and $g\big|_C$ bounded, thus $C \cap (C(f) - C(g))$ is club and we conclude through Lemma 8.11.

**Lemma 8.13.** Let $X$ be an $\omega_1$-compact Type I space and $f, g : X \to \mathbb{L}_{\geq 0}$ be continuous. Then $h : X \to \mathbb{L}_{\geq 0}$ defined by $h(x) = \min\{f(x), g(x)\}$ (written $h = \min(f,g)$) is continuous and $C(h) = C(g) \cap C(f)$.

**Proof.** Continuity is immediate, and $C(h)$ is the union of the points in the bones $B_\alpha(X)$ that are sent in $[\alpha, \omega_1)$ by $h$, so the second claim follows.

**Proposition 8.14.** Let $X$ be an $\omega_1$-compact Type I space, and $f : X \to \mathbb{L}_{\geq 0}$ be continuous. Then the following holds.

(a) If $C(f)$ is a direction, then $f$ is $\preceq$-maximal, and $C(f)$ is a $\preceq$-maximal direction.

(b) If $f$ is $\preceq$-maximal, then $C(f)$ is a $\preceq$-maximal direction in $X$.

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Proof.
(a) Suppose that there is some $g$ with $f \prec g$. Then $C(f) - C(g) \supset C$ for some club $C$ and $g\upharpoonright C$ is bounded. But since $C(g) - C(f)$ contains no club, $C(g) \cap C(f)$ is club, so $g\upharpoonright C(f)$ is unbounded, which contradicts the fact that $C(f)$ is a direction. Suppose now that there is some direction $D$ with $C(f) \prec D$. Let $g : X \to \mathbb{L}_{\geq 0}$ satisfy $g\upharpoonright C(f)$ bounded and $g\upharpoonright D$ unbounded, hence $C(g) \cap D$ is club. Up to taking $C(g) \cap D$, we can assume that $D \subset C(g)$. Since $C(g) \cap C(f)$ is bounded by Lemma \ref{lem:14}, so is $D \cap C(f)$, which implies $f\upharpoonright D$ bounded. But this contradicts $C(f) \prec D$, since $f$ is unbounded on $C(f)$.

(b) We show that $C(f)$ is a direction, its maximality then follows from (a). Suppose it is not the case, there is therefore $g : X \to \mathbb{L}_{\geq 0}$ with $g^{-1}(\{0\}) \cap C(f)$ and $C(g) \cap C(f)$ both club in $X$. Set $h = \min(f,g)$, then $C(h) - C(f)$ contains no club by Lemma \ref{lem:13} and $h^{-1}(\{0\}) \supset g^{-1}(\{0\})$, therefore $h^{-1}(\{0\}) \cap C(f)$ is club in $X$, so $C(f) - C(h)$ contains a club. By Lemma \ref{lem:12} this means that $f \prec h$, contradicting the maximality of $f$.

Of course, there are spaces that do contain directions (and even maximal ones) that do not contain maximal maps, as $S^{\omega, h, r}$ in Example \ref{ex:10}. More generally, if there is a sequence of maps $f_i$ ($i \in \omega$), with $f_j \prec f_i$ when $j < i$, such that there is no $g$ with $f_i \prec g$ for all $i$, while there is no $\leq$-maximal map, it can be thought that $\cap_{i \in \omega} C(f_i)$ is a direction (if non-empty). But there are easy examples where this is not true: take $S^{\omega+1, h, r}$, where $h$ is the inclusion, $r(n) = \uparrow$ for all $n \in \omega$ and $r(\omega) = \downarrow$. Take $f_i$ ($i \in \omega$) such that $f_i$ is unbounded on $\Delta_j$ for all $j \geq i$ (and on $\Delta_{\omega+1}$), and bounded on the others. It is easy to see that there is no $g$ bigger than each $f_i$, but that $\cap_{i \in \omega} C(f_i)$ intersects both $\Delta_\omega$ and $\Delta_{\omega+1}$ in a club way, and is therefore not a direction. The following lemma shows that we can obtain a direction if we impose more restrictions on the sequence $f_i$.

Lemma 8.15. Let $X$ be an $\omega$-bounded Type I space. Let $f_i : X \to \mathbb{L}_{\geq 0}$ ($i \in \omega$) be a sequence with $f_i \preceq f_j$ when $i \leq j$. Then the following holds.
(a) If $\cap_{i \in \omega} C(f_i)$ is a direction, then there is no $g : X \to \mathbb{L}_{\geq 0}$ with $f_i \prec g$ for all $i \in \omega$, and $\cap_{i \in \omega} C(f_i)$ is $\preceq$-maximal.
(b) Suppose that there is no $\leq$-increasing sequence $g_i$ ($i \in \omega$) satisfying: $f_i \prec g_i$ for all $i$, and for $j \geq i$ $g_i \nsubseteq f_j$. Then $\cap_{i \in \omega} C(f_i)$ is a $\preceq$-maximal direction.

It is not very clear however whether there are examples where condition (b) holds.

Proof. If the sequence of $f_i$ stagnates after some $i$ then Proposition \ref{prop:14} applies. Assume that $f_i \prec f_j$ when $i < j$. By Lemma \ref{lem:12} $\cap_{i \in \omega} C(f_i)$ is club.
(a) The proof is the same than that of Proposition \ref{prop:14} once one notes that if $g\upharpoonright \cap_{i \in \omega} C(f_i)$ is bounded, then $g\upharpoonright C(f_j)$ must be bounded for all big enough $i$.
(b) If $C = \cap_{i \in \omega} C(f_i)$ is not a direction, there is a $g$ unbounded on $C$ with $g^{-1}(\{0\}) \cap C$. Define $g_i = \min(g, f_i)$ and derive a contradiction as in Proposition \ref{prop:14}.

It is easy to see that a map that is club-equivalent to a slicer is indeed a slicer for another canonical sequence. Moreover, a slicer has the property that $f^{-1}(\{x\})$ is bounded for each $x \in \mathbb{L}_{\geq 0}$, but in general this property does not define slicers:
Take $X$ to be the disjoint union of $\omega_1$ with the usual topology and $\omega_1$ with the discrete topology. Then the map $f$ given by the identity on the first copy and by some 1-to-1 map $\omega_1 \to [0, 1]$ on the second copy has the alleged property but is not club-equivalent to a slicer. We shall see below that both notions are equivalent if the space is $\omega$-bounded.

**Definition 8.16.** Let $X$ be Type I, and $f : X \to \mathbb{L}_{\geq 0}$. We set

$$
\text{Bd}(f) = \{ x \in \mathbb{L}_{\geq 0} : f^{-1}(\{x\}) \text{ is bounded} \} \\
\text{Ubd}(f) = \{ x \in \mathbb{L}_{\geq 0} : f^{-1}(\{x\}) \text{ is unbounded} \}.
$$

First we note the following:

**Lemma 8.17.** If $X$ is Type I and $\omega$-bounded, and $f : X \to \mathbb{L}_{\geq 0}$, then $\text{Ubd}(f)$ is closed in $\mathbb{L}_{\geq 0}$.

**Proof.** Since $\mathbb{L}_{\geq 0}$ is first countable we need only to check that $\text{Ubd}(f)$ is sequentially closed. Fix an arbitrary $\alpha \in \omega_1$, and let $\{y_i : i \in \omega\}$ be a sequence in $\text{Ubd}(f) \subset \mathbb{L}_{\geq 0}$ converging to $y$. Let $x_i \in f^{-1}(\{y_i\}) - X_\alpha$. Then for any accumulation point $x$ of the $x_i$, $f(x) = y$, and $x \in X - X_\alpha$.

**Lemma 8.18.** Let $X$ be Type I, and $f : X \to \mathbb{L}_{\geq 0}$. Then, (a) and (b) below are equivalent and imply (c). If $X$ is $\omega$-bounded, then (c) implies (a) and (b) as well.

(a) $f$ is club-equivalent to a slicer,
(b) There is a club $C \subset \omega_1$ such that $f$ is a slicer for the canonical sequence $(X_\alpha : \alpha \in C)$,
(c) $\text{Ubd}(f) = \emptyset$.

**Proof.** Note that a map is club equivalent to a slicer if and only if it is unbounded on each club of $X$.

(a) $\rightarrow$ (b). If $f$ is club-equivalent to a slicer, then $f^{-1}([0, \alpha])$ is bounded for all $\alpha \in \omega_1$. By a leapfrog argument this implies that the set of $\alpha$ such that $f^{-1}([0, \alpha)) \subset [0, \alpha)$ is club. Since the set of $\alpha$ with $f(X_\alpha) \subset [0, \alpha]$ is club too, letting $C$ be their intersection yields that $f$ is a slicer for the canonical sequence given by $C$.

(b) $\rightarrow$ (a). Immediate.

(b) $\rightarrow$ (c). If $C = f^{-1}(\{x\})$ is unbounded, then $f$ cannot be a slicer for any canonical sequence, otherwise $f(C)$ would be unbounded.

(c) $\rightarrow$ (a) if $X$ is $\omega$-bounded. Suppose that there is some club $C \subset X$ with $f(C) \subset [0, \alpha]$ for some $\alpha$. We define inductively closed intervals $I_i \subset I_{i-1} \subset \cdots \subset I_0 = [0, \alpha]$ by bisecting $I_i$ and choosing $I_{i+1}$ as one of the two pieces such that $f^{-1}(I_i) \cap C$ is unbounded. We may choose $I_i$ such that $\bigcap_{i \in \omega} I_i = \{x\}$, and since $X$ is countably compact, $(\mathfrak{C}_X, \subseteq)$ is countably closed (cf Lemma 8.12), so $\bigcap_{i \in \omega} (f^{-1}(I_i) \cap C) = f^{-1}(\{x\}) \cap C$ is club, which shows that $x \in \text{Ubd}(X) \neq \emptyset$, a contradiction.
9 Dominant subsets

There is a natural strengthening of the notion of direction which is the following.

**Definition 9.1.** Let $X$ be Type I. A club $D \subset X$ is locally dominant if there is an open set $U \supset D$ such that for all $f : X \rightarrow \mathbb{L}_{\geq 0}$, all club $C \subset U$ and all club $D' \subset D$, $f|_{D'}$ bounded implies $f|_{C}$ bounded. If $U = X$, we say that $D$ is globally dominant.

The requirement about the club subsets $D' \subset D$ is here to prevent $X$ itself to be always globally dominant. A locally dominant subset is trivially a direction (take $C = D$), but not conversely. A locally dominant subset of a Type I manifold is not always ufo-maximal: the horizontals are locally dominant in $\mathbb{O}$ (take an open horizontal strip). The reader may check that $\Delta_{\omega_1}$ and $\Delta_0$ are both ufo-maximal in the surface $S^{\omega_1, 2, h, r}$ described in Theorem 8.3 but are not locally dominant (see Theorem 9.4 below for a proof in a similar case), so local dominance and ufo-maximality, though similar, are logically unrelated.

**Definition 9.2.** A Type I space is spanned by dominant subsets if any large open subset contains a locally dominant subset.

We note that the surface of Example 4.5 gives us the following.

**Theorem 9.3** (P. Nyikos [18]). There is an $\omega$-bounded surface spanned by directions which does not contain any locally dominant subset.

**Proof.** By Lemma 4.6 any large open set of $R$ contains the terminal part of a ray, and the rays are not locally dominant by construction.

Notice as well that given $\alpha < \omega_2$ and $h, r$ as in Lemma 4.11 the surfaces $S^{\alpha, h, r}$ are spanned by dominant subsets. Indeed, given a club $C$ inside an open subset $U$, either $C$ contains a locally dominant subset or $C$ club-intersects some $\Delta_\alpha$ for a limit $\alpha$. In the latter case, $U$ contains the terminal part of $\Delta_\beta$ for some (smaller) successor $\beta$ which is locally dominant (see the proof of Lemma 4.13). Thus, the surfaces given by Theorems 8.1 and 8.3 are in fact spanned by dominant subsets. The counterpart of Theorem 5.2 for dominant subsets is easy to prove in ZFC:

**Theorem 9.4.** There is an $\omega$-bounded surface $S$ spanned by dominant subsets such that there is an unbounded $f : S \rightarrow \mathbb{L}_{\geq 0}$ that is bounded on each locally dominant subset of $S$.

**Proof of Theorem 9.4.** Let $S$ be $S^{\omega_1, h, r}$ with $h : \omega_1 \rightarrow \omega_1$ the identity and $r : \omega_1 \rightarrow \{\uparrow, \downarrow\}$ being such that for uncountably many $\alpha$, $r(\alpha) = \uparrow$ and $r(\alpha + 1) = \downarrow$. We show that no club subset $C$ of $\Delta_{\omega_1}$ can be locally dominant. Given an open $U$ containing $C$, there is a compact $K$ and a $\beta < \omega_1$ such that $U$ contains $\Delta_\alpha - K$ whenever $\beta < \alpha \leq \omega_1$. If $r(\alpha) = \uparrow$ and $r(\alpha + 1) = \downarrow$ there is a map $f : S^{\omega_1, h, r} \rightarrow \mathbb{L}_{\geq 0}$ which is unbounded on $\Delta_\gamma$ iff $\gamma = \alpha + 1$, and thus $C$ is not locally dominant. It is now easy to find a map $f : S^{\omega_1, h, r} \rightarrow \mathbb{L}_{\geq 0}$ such that $f|_{\Delta_\alpha}$ is bounded by $\alpha$ for each $\alpha < \omega_1$, and the identity on $\Delta_{\omega_1}$, so $f$ is bounded on any locally dominant subset of $S$.

Notice in passing that $\Delta_{\omega_1}$, being not comparable with any other $\Delta_\alpha$, is ufo-maximal.
10 The role of $\mathbb{L}_{\geq 0}$ and a Hausdorff peculiarity

In this section we investigate the question of the ‘universality’ of $\mathbb{L}_{\geq 0}$ in the definition of directions. To be precise, let us define:

**Definition 10.1.** Let $X, Y$ be Type I spaces, and let $D \subset X$ be club. Then $D$ is a $Y$-direction in $X$ iff given $f : X \to Y$ with $f$ unbounded on $D$, then $f$ is unbounded any club subset of $D$ as well. If there exists an $f : X \to Y$ which is unbounded on $D$, the direction is said to be true.

Directions will thus be called $\mathbb{L}_{\geq 0}$-directions in this section. Almost everything done in Section 2.2 holds for $Y$-directions as well. In particular, the equivalences in Lemma 2.7 hold, except clause (c) which may fail if there is no map $f : Y \to Y$ with $f(Y_\alpha) = \{0\}$ and $f|_{Y \setminus Y_{\alpha+1}} = id$.

**Lemma 10.2.** Let $X, Y$ be Type I spaces, then the following holds.
(a) $\mathbb{L}_{\geq 0}$ and $\omega_1$ are $Y$-directions in themselves.
(b) If $X$ is regular and $D$ is an $\mathbb{L}_{\geq 0}$-direction in $X$, then $D$ is a true $\mathbb{L}_{\geq 0}$-direction in $X$.
(c) If there is a slicer on $Y$, then any $\mathbb{L}_{\geq 0}$-direction in $X$ is a $Y$-direction in $X$.
(d) If there is an unbounded map $g : \mathbb{L}_{\geq 0} \to Y$, then a $Y$-direction in $X$ is an $\mathbb{L}_{\geq 0}$-direction in $X$.

**Proof.**
(a) Given $f : \mathbb{L}_{\geq 0} \to Y$, if $f^{-1}(Y_\alpha)$ is unbounded then $f^{-1}(Y_{\alpha+1})$ is club. It follows that $f^{-1}(X \setminus Y_{\alpha+1})$ cannot be club, hence $f$ is bounded.
(b) By Lemma 2.5 there is a slicer $s : X \to \mathbb{L}_{\geq 0}$ which is thus unbounded on $D$.
(c) Given $f : X \to Y$ unbounded on $D$ and $\alpha \in \omega_1$, then $s \circ f(D \setminus X_{\gamma(\alpha)}) \subset [\alpha, \omega_1)$ for some $\gamma(\alpha)$, and thus $f(D \setminus X_{\gamma(\alpha)}) \subset Y \setminus Y_\alpha$.
(d) Let $f : X \to \mathbb{L}_{\geq 0}$ be unbounded on $D$. If $g \circ f$ is bounded on $D$, then $g$ sends an unbounded subset of $\mathbb{L}_{\geq 0}$ to a bounded subset of $Y$. By (a) $g$ must be bounded, a contradiction. Hence $g \circ f$ is unbounded. Take a club $C$ such that $g \circ f(X \setminus Y_\alpha) \subset Y \setminus Y_\alpha$ and $g([0, \alpha)) \subset Y_\alpha$ for each $\alpha \in C$. Then $f(X \setminus X_\alpha) \subset [\alpha, \omega_1)$.

Clause (c) shows that $\mathbb{L}_{\geq 0}$ is a kind universal target for checking the ‘directionality’ in regular spaces. Though, the $Y$-direction might not be true, even if it is a true $\mathbb{L}_{\geq 0}$-direction. For instance if $Y$ is the positive part of a tangent bundle of $\mathbb{L}_+$ as in Example 6.1 then no map $\mathbb{L}_{\geq 0} \to Y$ is unbounded. By quotienting by a suitable $Z$-action on the fibers, as explained in [15] and [11, Section 6], this example can be made $\omega$-bounded. If $Y$ is not (path) connected, then a $Y$-direction may well not be an $\mathbb{L}_{\geq 0}$-direction, for instance $\omega_1 \times \mathbb{L}_{\geq 0}$ is an $\omega_1$-direction in itself.

The aim of this section is to prove the following theorems, which show that the conclusions of (b), (c) above may fail in the class of Hausdorff connected spaces, while (d) may fail as well for normal connected spaces.

**Theorem 10.3.** There are Type I Hausdorff non-regular spaces $R, X$, with $R$ connected, such that the following holds:
Lemma 10.5. Let \( Z, Y \) be Type I. Suppose that for all \( f : Z^2 \to Y \) and \( q \in Z^2 \), if \( f(q) \in Y - Y_\alpha \), then \( f(Z^2) \subset Y - Y_\alpha \). Let \( D \subset \text{Sk}(Y) \) be a \( Y \)-direction in \( Y \). Then \( E_{\alpha} = \bigcup_{\alpha \leq \omega} (D \cap B_{\alpha}(Y)) \times Z_\alpha \) is a true \( Y \)-direction in \( Y \times Z^2 \), and not a \( Z \)-direction.

Proof. Write \( X = Y \times Z^2 \) and denote by \( \pi_i \) the projection on the \( i \)-th factor. Then \( \pi_3 : Y \times Z^2 \to Z \) shows that \( E \) is not a \( Z \)-direction in \( X \). Let \( f : X \to Y \) be unbounded on \( E \). If \( f(E \cap Y \times \{q\}) \) is bounded for each \( q \in Z^2 \), then \( f(E) \) is bounded. (Otherwise for all \( \alpha \) there are \( y_\alpha, q_\alpha \) with \( f(y_\alpha, q_\alpha) \in E \cap (Y - Y_\alpha) \), then \( f(\{y_\alpha \} \times Z^2) \subset E \cap (Y - Y_\alpha) \) and \( f(E \cap Y \times \{q\}) \) is unbounded for each \( q \in Z^2 \).) Hence there is some \( q \) for which \( f(E \cap Y \times \{q\}) \) is unbounded. Since \( D \) is a \( Y \)-direction in \( Y \), there is a club \( C \subset \omega_1 \) such that \( f((D - Y_\alpha) \times \{q\}) \subset Y - Y_\alpha \) for \( \alpha \in C \). Thus, \( f((D - Y_\alpha) \times Z^2) \subset Y - Y_\alpha \). Since \( E - X_\alpha = E \cap (D - Y_\alpha) \times Z^2 \), \( E \) is a \( Y \)-direction, which is true since the projection \( \pi_1 : X \to Y \) is unbounded on \( E \). \( \square \)

The attentive reader may have noted the similarity between \( E \) and the spaces \( \text{Down}(X) \) of Section 3. We shall now define \( R = \bigcup_{\alpha \in \omega_1} R_\alpha \) such that \( \overline{R_\alpha} \) is countable and connected for each limit \( \alpha \). Our construction is a simple variant on Roy’s lattice space (Example 126 in [19]). Denote by \( \Lambda \) the limit ordinals in \( \omega_1 \). We assume that \( 0 \in C_0 \). For each \( \alpha \in \omega_1 \), write \( \alpha = \beta + \varphi(\alpha) \) with \( \varphi(\alpha) \in \omega \) and \( \beta \in \Lambda \). We let then \( R \) be \( \{(r, \alpha) : \alpha \in \omega_1, r \in C_{\varphi(\alpha)}\} \). When \( \alpha \in \Lambda \), we denote by \( q_\alpha \) the point \((0, \alpha)\). The topology is the following. Given \( \langle r, \alpha \rangle \in R \) and \( \epsilon > 0 \), denote by \( U_{r, \alpha, \epsilon} \) the set \( \{(s, \alpha) \in R : |r - s| < \epsilon\} \). If \( \varphi(\alpha) \) is odd, then neighborhood of \( \langle r, \alpha \rangle \) is given by the \( U_{r, \alpha, \epsilon} \) for \( \epsilon > 0 \). A neighborhood of \( \langle r, \alpha \rangle \) when \( \varphi(\alpha) \) is even and \( \epsilon > 0 \) is a stack of 3 intervals \( U_{r, \alpha - 1, \epsilon} \cup U_{r, \alpha, \epsilon} \cup U_{r, \alpha + 1, \epsilon} \) for \( \epsilon > 0 \) (notice that \( \alpha - 1 \) makes sense when \( \varphi(\alpha) > 0 \)). If \( \varphi(\alpha) = 0 \) (and thus \( \alpha \in \Lambda \)), a neighborhood of a point different from \( q_\alpha \) is given by a stack of 2 intervals \( U_{r, \alpha, \epsilon} \cup U_{r, \alpha + 1, \epsilon} \) for \( \epsilon > 0 \). Finally, the neighborhoods of \( q_\alpha = (0, \alpha) \) are given by \( U_{0, \alpha, \epsilon} \cup U_{0, \alpha + 1, \epsilon} \cup V_\gamma \), where \( \epsilon > 0, \gamma < \alpha, \) and \( V_\gamma = \{(r, \beta) : \gamma < \beta < \alpha\} \). We let \( R_\alpha = \{(r, \beta) : \beta < \alpha\} \).

Lemma 10.6. \( R \) is Hausdorff, Type I and each \( \overline{R_\alpha} \) is connected (thus \( R \) as well).

Proof. Hausdorffness is immediate since the \( C_0 \) are disjoint. \( R = \bigcup_{\alpha \in \omega_1} R_\alpha \), and \( \overline{R_\alpha} \) is \( R_\alpha \) if \( \varphi(\alpha) \) is even and \( \epsilon > 0 \), \( R_{\alpha + 1} \) if \( \varphi(\alpha) \) is odd, and \( R_\alpha \cup \{q_\alpha\} \) if \( \alpha \in \Lambda \). In any case, it is countable and thus Lindelöf, so \( R \) is of Type I. A clopen set containing an entire level \( C_{\varphi(\alpha)} \times \{\alpha\} \) must contain \( C_{\varphi(\beta)} \times \{\beta\} \) for \( \beta \) the immediate predecessor.
of $\alpha$ if $\alpha$ is successor, or $V_\gamma$ for some $\gamma < \alpha$ if $\alpha \in \Lambda$. (See [19, Example 126] for details, notice that we switched the role of ‘odd’ and ‘even’ in our version.) Thus, any clopen set containing $q_\alpha$ for $\alpha \in \Lambda$ contains $\overline{R_\alpha}$, which is then connected. ☐

**Lemma 10.7.** For any $n, m \in \omega$, Any $f : \mathbb{L}_{\geq 0}^n \to R^m$ is constant, and any $g : R^n \to \mathbb{L}_{\geq 0}^m$ is constant.

*Proof.* A canonical sequence for $R^n$ is given by $R^n_\alpha$, and $\overline{R_\alpha} = \overline{R_\alpha}^m$. For a club set of $\alpha$, $f([0, \alpha)^n) \subset \overline{R_\alpha}$. Thus $[0, \alpha)^n = \bigcup_{x \in \overline{R_\alpha}} f^{-1}(\{x\})$ and is therefore a countable union of disjoint closed subsets. This is impossible except if all but one are empty. The function is thus constant on $[0, \alpha)^n$ for a club set of $\alpha$, and thus on all of $\mathbb{L}_{\geq 0}^n$. Notice that $\overline{R_\alpha}^m$ is connected, thus $g$ must send $\overline{R_\alpha}^m$ to a connected and countable subspace of $\mathbb{L}_{\geq 0}^m$, and thus be constant on $\overline{R_\alpha}^m$. The map is is thus constant on all of $R^n$. ☐

**Lemma 10.8.** The set $\{q_\alpha : \alpha \in \Lambda\}$ is a true $R$-direction.

*Proof.* $\{q_\alpha : \alpha \in \Lambda\}$ is a closed copy of $\omega_1$, and we apply Lemma 10.2 (a). ☐

**Proof of Theorem 10.3**

(a) By Lemma 10.7

(b) Let $X$ be $R^2 \times \mathbb{L}_{\geq 0}^2$. By Lemmas 10.6, 10.8 we may apply Lemma 10.5 with $Z = R^2, Y = \mathbb{L}_{\geq 0}^2$ to find a true $\mathbb{L}_{\geq 0}$-direction in $R^2 \times \mathbb{L}_{\geq 0}^2 \times \{0\}$ which is not an $R$-direction. Similarly, there is a true $R$-direction in $\{q\} \times R \times \mathbb{L}_{\geq 0}^2$ (for any $q \in R$) which is not an $\mathbb{L}_{\geq 0}$-direction, by applying Lemma 10.5 with $Z = \mathbb{L}_{\geq 0}^2, Y = R$. ☐

The proof of Theorem 10.3 is very similar. Let $T$ be the subset of $\mathbb{L}_{\geq 0}^2 \times [-1, 1]$ given by taking a closed double sided topologist sine curve inside each $[\alpha, \alpha + 1]$. That is, $T$ contains $\{\alpha\} \times [-1, 1]$ for each $\alpha$, and curves inside $(\alpha, \alpha + 1) \times [-1, 1]$ oscillating as $\sin(1/x)$ and accumulating on both $\{\alpha\} \times [-1, 1]$ and $\{\alpha + 1\} \times [-1, 1]$. Then $T$ is a closed subset of the $\omega$-bounded normal space $\mathbb{L}_{\geq 0} \times [-1, 1]$ and inherits these properties. Denote $T \cap [0, \alpha)$ by $T_\alpha$.

**Lemma 10.9.** $T$ is a true $T$- and $\mathbb{L}_{\geq 0}$-direction in itself. The image of any map $\mathbb{L}_{\geq 0}^2 \to T$ is contained in $T_{\alpha + 1} - T_\alpha$ for some $\alpha$. Given $f : T \to T$, then $f(\overline{T_{\beta + 1}} - T_\beta)$ is contained $T_{\alpha + 1} - T_\alpha$ for some $\alpha$.

*Proof.* It is easy to see that $T$ is an $\mathbb{L}_{\geq 0}$-direction, and thus a $T$-direction by Lemma 10.2 (c). The other claims are immediate. ☐

**Proof of Theorem 10.4** By Lemma 10.9 $Z = \mathbb{L}_{\geq 0}$ and $Y = T$ satisfy the assumptions of Lemma 10.5. ☐

**Problem 5.** Are there regular Type I spaces $X, Y$, with $Y$ path connected, and a club $D \subset X$ such that $D$ is an $Y$-direction but not an $\mathbb{L}_{\geq 0}$-direction in $X$?
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