Continuous variable tangle, monogamy inequality, and entanglement sharing in Gaussian states of continuous variable systems

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Abstract. For continuous-variable (CV) systems, we introduce a measure of entanglement, the CV tangle (contangle), with the purpose of quantifying the distributed (shared) entanglement in multimode, multipartite Gaussian states. This is achieved by a proper convex-roof extension of the squared logarithmic negativity. We prove that the contangle satisfies the Coffman–Kundu–Wootters monogamy inequality in all three-mode Gaussian states, and in all fully symmetric N-mode Gaussian states, for arbitrary N. For three-mode pure states, we prove that the residual entanglement is a genuine tripartite entanglement monotone under Gaussian local operations and classical communication. We show that pure, symmetric three-mode Gaussian states allow a promiscuous entanglement sharing, having both maximum tripartite residual entanglement and maximum couplewise entanglement between any pair of modes. These states are thus simultaneous CV analogues of both the GHZ and the W states of three qubits: in CV systems monogamy does not prevent promiscuity, and the inequivalence between different classes of maximally entangled states, holding for systems of three or more qubits, is removed.

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One of the main challenges in fundamental quantum theory as well as in quantum information and computation sciences lies in the characterization and quantification of bipartite entanglement for mixed states, and in the definition and interpretation of multipartite entanglement both for pure states and in the presence of mixedness [1, 2]. More intriguingly, a quantitative, physically significant, characterization of the entanglement of states shared by many parties can be attempted: this approach, introduced in a seminal paper by Coffman, Kundu and Wootters (CKW) [3], has led to the discovery of so-called ‘monogamy inequalities’, constraining the maximal entanglement distributed among different internal partitions of a multiparty system. Such inequalities are uprising as one of the fundamental guidelines on which proper multipartite entanglement measures have to be built [4].

While important insights have been gained on these issues in the context of qubit systems, a less satisfactory understanding has been achieved until recent times on higher-dimensional systems, as the structure of entangled states in Hilbert spaces of high dimensionality exhibits a formidable degree of complexity. However, and quite remarkably, in infinite-dimensional Hilbert spaces of continuous-variable (CV) systems, important progresses have been obtained in the understanding of the entanglement properties of a restricted but fundamental class of states, the so-called Gaussian states [5, 6]. These states, besides being of great importance both from a theoretical point of view and in practical applications, share peculiar features that make their structural properties amenable to accurate and detailed theoretical analysis [7].

In this paper, we address the problem of distributing entanglement among multiple modes of a CV system. We introduce the CV tangle (contangle) to quantify entanglement sharing in Gaussian states and we prove that it satisfies the CKW monogamy inequality [3]. Nevertheless, even in the basic instance of three modes, we show that pure, symmetric Gaussian states, at variance with their discrete-variable counterparts, allow a promiscuous sharing of quantum correlations, exhibiting both maximum tripartite residual entanglement and maximum couplewise entanglement between any pair of modes.

The paper is organized as follows: in section 1, we review the basic properties of Gaussian states of CV systems, and set up notations; in section 2, we address the quantification of entanglement sharing in such states, introducing a new entanglement monotone which is shown to generalize the tangle defined in discrete-variable systems; in section 3, we apply this measure to prove that all three-mode Gaussian states and all symmetric multimode Gaussian states
satisfy a monogamy inequality for CV entanglement, and that in the specific case of three-mode states the residual entanglement, emerging from the monogamy inequality, is a genuine tripartite entanglement monotone; in section 4, we exploit this result to investigate the sharing structure of tripartite entanglement in Gaussian states, unveiling striking differences with their discrete-variable counterparts; finally, in section 5, we summarize our results and outline possible roadmaps ahead.

1. Gaussian states: structural properties

In a CV system consisting of \( N \) canonical bosonic modes, associated to an infinite-dimensional Hilbert space, and described by the vector \( \hat{X} \) of the field quadrature operators, Gaussian states (such as coherent, squeezed, thermal and squeezed thermal states) are those states characterized by first and second statistical moments of the canonical quadrature operators. When addressing physical properties, like entanglement, that must be invariant under local unitary operations, first moments can be neglected and Gaussian states can then be fully described by the \( 2N \times 2N \) real covariance matrix (CM) \( \sigma \), whose entries are

\[
\sigma_{ij} = \frac{1}{2} \langle \{ \hat{X}_i, \hat{X}_j \} \rangle - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle.
\]

This allows, for Gaussian states, to indicate them indifferently by the density matrix \( \rho \) or by the CM \( \sigma \). A physical CM \( \sigma \) must fulfil the uncertainty relation

\[
\sigma + i\Omega \geq 0,
\]

with the symplectic form \( \Omega = \bigoplus_{i=1}^{2N} \omega \) and \( \omega = \delta_{i-1,j} - \delta_{i+1,j}, \) \( i, j = 1, 2 \). Symplectic operations (i.e. belonging to the group \( Sp(2N,\mathbb{R}) = \{ S \in SL(2N, \mathbb{R}) : S^T \Omega S = \Omega \} \) acting by congruence on CMs in phase space, amount to unitary operations on density matrices in Hilbert space. In phase space, any \( N \)-mode Gaussian state can be written as \( \sigma = S^T \nu S \), with \( \nu = \text{diag} \{ n_1, n_1, n_2, n_2, \ldots, n_N, n_N \} \). The set \( \Sigma = \{ n_i \} \) constitutes the symplectic spectrum of \( \sigma \) and its elements must fulfil the conditions \( n_i \geq 1 \), ensuring positivity of the density matrix \( \rho \) associated to \( \sigma \). The symplectic eigenvalues \( n_i \) can be computed as the eigenvalues of the matrix \( \Omega^2 \Omega \). The degree of purity \( \mu = \text{Tr} \rho^2 \) of a Gaussian state with CM \( \sigma \) is simply \( \mu = 1/\sqrt{\det \sigma} \).

Concerning the entanglement, positivity of the partially transposed state \( \tilde{\rho} \) (from now on ‘\( \sim \’ \) will denote partial transposition), obtained by transposing the reduced state of only one of the subsystems, is a necessary and sufficient condition (PPT criterion) of separability for \((N+1)\)-mode Gaussian states of \((1 \times N)\)-mode bipartitions \([8, 9]\) and for \((M+N)\)-mode bisymmetric Gaussian states of \((M \times N)\)-mode bipartitions \([10]\). In phase space, partial transposition in a \((1 \times N)\)-mode bipartition amounts to a mirror reflection of one quadrature associated to the single-mode party \([8]\). If \( \{ \tilde{n}_i \} \) is the symplectic spectrum of the partially transposed CM \( \tilde{\sigma} \), then a \((N+1)\)-mode Gaussian state with CM \( \sigma \) is separable if and only if \( \tilde{n}_i \geq 1 \) \( \forall i \). This implies that a proper measure of CV entanglement is the logarithmic negativity \([11]\)

\[
E_N \equiv \ln \| \tilde{\rho} \|_1,
\]

where \( \| \cdot \|_1 \) denotes the trace norm \([12]\). The logarithmic negativity \( E_N \) is readily computed in terms of the symplectic spectrum \( \tilde{n}_i \) of \( \sigma \) as

\[
E_N = \sum_{i: \tilde{n}_i < 1} \ln \tilde{n}_i.
\]
Such a measure quantifies the extent to which the PPT condition is violated. For two-mode symmetric states, the logarithmic negativity is equivalent to the entanglement of formation (EoF) $E_F$ [13]:

$$E_F(\rho) \equiv \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle),$$  \hspace{1cm} (4)

where $E(|\psi_i\rangle)$ is the von Neumann entropy (or entropy of entanglement) of the pure state $|\psi_i\rangle$, and the infimum is taken over all possible pure states decompositions $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. In fact, the logarithmic negativity is positive defined, additive, monotone under local operations and classical communication (LOCC) [14], constitutes an upper bound to the distillable entanglement in a quantum state $\rho$, and is related to the entanglement cost under PPT preserving operations [15].

2. The contangle

Our aim is to analyse the distribution of entanglement between different (partitions of) modes in CV systems. In [3], CKW proved for system of three qubits, and conjectured for $N$ qubits (this conjecture has now been proven by Osborne and Verstraete [16]), that the bipartite entanglement $E$ (properly quantified) between, say, qubit A and the remaining two-qubits partition (BC) is never smaller than the sum of the A|B and A|C bipartite entanglements in the reduced states:

$$E_{A\mid(BC)} \geq E_{A\mid B} + E_{A\mid C}. \hspace{1cm} (5)$$

This statement quantifies the so-called monogamy of quantum entanglement [17], in opposition to the classical correlations, which are not constrained and can be freely shared. One would expect a similar inequality to hold for three-mode Gaussian states, namely

$$E^{ij(k)} - E^{ij} - E^{ik} \geq 0, \hspace{1cm} (6)$$

where $E$ is a proper measure of bipartite CV entanglement and the indexes $\{i, j, k\}$ label the three modes. However, the demonstration of such a property is plagued by subtle difficulties. Let us for instance consider the simplest conceivable instance of a pure three-mode Gaussian state completely invariant under mode permutations. These pure Gaussian states are named fully symmetric, and their standard form CM (see [18, 19]), for any number of modes, is only parametrized by the local mixedness $a_{\text{loc}} = 1/\mu_{\text{loc}}$, an increasing function of the single-mode squeezing $r_{\text{loc}}$, and $a_{\text{loc}} \to 1^+$ when $r_{\text{loc}} \to 0^+$. For these states, the inequality (6) can be violated for small values of the local squeezing factor, using either the logarithmic negativity $E_N$ or the EoF $E_F$ (which is computable in this case, because the two-mode reduced mixed states of a pure symmetric three-mode Gaussian states are again symmetric) to quantify the bipartite entanglement. This fact implies that none of these two measures is the proper candidate for approaching the task of quantifying entanglement sharing in CV systems. This situation is reminiscent of the case of qubit systems, for which the CKW inequality holds using the tangle $\tau$, defined as the square of the concurrence [20], but can fail if one chooses equivalent measures of bipartite entanglement such as the concurrence itself or the EoF [3].

It is then necessary to define a proper measure of CV entanglement that specifically quantifies entanglement sharing according to a monogamy inequality of the form (6). A first important hint
toward this goal comes by observing that, when dealing with $1 \times N$ partitions of fully symmetric multimode pure Gaussian states together with their $1 \times 1$ reduced partitions, the desired measure should be a monotonically decreasing function $f$ of the smallest symplectic eigenvalue $\tilde{n}_-$ of the corresponding partially transposed CM $\tilde{\sigma}$. This requirement stems from the fact that $\tilde{n}_-$ is the only eigenvalue that can be smaller than 1 [18], violating the PPT criterion with respect to the selected bipartition. Moreover, for a pure symmetric three-mode Gaussian state, it is necessary to require that the bipartite entanglements $E^{i(jk)}$ and $E^{i|j}$ be respectively functions $f(\tilde{n}^{i(jk)}_{-})$ and $f(\tilde{n}^{i|j}_{-})$ of the associated smallest symplectic eigenvalues $\tilde{n}^{i(jk)}_{-}$ and $\tilde{n}^{i|j}_{-}$, in such a way that they become infinitesimal of the same order in the limit of vanishing local squeezing, together with their first derivatives:

$$f(\tilde{n}^{i(jk)}_{-})/2 f(\tilde{n}^{i|j}_{-}) \simeq f'(\tilde{n}^{i(jk)}_{-})/2 f'(\tilde{n}^{i|j}_{-}) \to 1 \text{ for } a_{\text{loc}} \to 1^+, \quad (7)$$

where the prime denotes differentiation with respect to the single-mode mixedness $a_{\text{loc}}$. The violation of the sharing inequality (6) exhibited by the logarithmic negativity can be in fact traced back to the divergence of its first derivative in the limit of vanishing squeezing. The above condition formalizes the physical requirement that in a symmetric state the quantum correlations should appear smoothly and be distributed uniformly among all the three modes. One can then see that the unknown function $f$ exhibiting the desired property is simply the squared logarithmic negativity\(^1\)

$$f(\tilde{n}_{-}) = [- \ln \tilde{n}_{-}]^2. \quad (8)$$

We remind again that for fully symmetric $(N+1)$-mode pure Gaussian states, the partially transposed CM with respect to any $1 \times N$ bipartition, or with respect to any reduced $1 \times 1$ bipartition, has only one symplectic eigenvalue that can drop below 1 [18]; hence the simple form of the logarithmic negativity (and, equivalently, of its square) in equation (8).

Equipped with this finding, one can give a formal definition of a bipartite entanglement monotone that, as we will soon show, can be regarded as the contangle $E_\tau$. For a generic pure state $|\psi\rangle$ of a $(1+N)$-mode CV system, we define the square of the logarithmic negativity:

$$E_\tau(\psi) \equiv \ln^2 \|\tilde{\rho}\|_1, \quad \rho = |\psi\rangle\langle\psi|, \quad (9)$$

This is a proper measure of bipartite entanglement, being a convex, increasing function of the logarithmic negativity $E_N$, which is equivalent to the entropy of entanglement for arbitrary pure states. For any pure multimode Gaussian state $|\psi\rangle$, with CM $\sigma^p$, of $N+1$ modes assigned in a generic bipartition $1 \times N$, explicit evaluation gives immediately that $E_\tau(\psi) \equiv E_\tau(\sigma^p)$ takes the form

$$E_\tau(\sigma^p) = \ln^2 \left(1/\mu_1 - \sqrt{1/\mu_1^2 - 1}\right), \quad (10)$$

\(^1\) Notice that an infinite number of functions satisfying equation (7) can be obtained by expanding $f(\tilde{n}_{-})$ around $\tilde{n}_{-} = 1$ at any even order. However, they are all convex functions of $f$. If the inequality (6) holds for $f$, it will hold as well for any monotonically increasing, convex function of $f$, such as the logarithmic negativity raised to any even power $k \geqslant 2$, but not for $k = 1$.\[12pt]
where \( \mu_1 = 1 / \sqrt{\det \sigma_1} \) is the local purity of the reduced state of mode 1 with CM \( \sigma_1 \). Definition (9) is naturally extended to generic mixed states \( \rho \) of \( (N + 1) \)-mode CV systems through the convex-roof formalism (see also [21] where a similar measure, the convex-roof extended negativity, is studied). Namely, we can introduce the quantity

\[
E_\tau(\rho) \equiv \inf \left\{ p_i, \psi_i \right\} \sum_i p_i E_\tau(\psi_i),
\]

where the infimum is taken over all convex decompositions of \( \rho \) in terms of pure states \( \{|\psi_i\rangle\} \), and if the index \( i \) is continuous, the sum in equation (11) is replaced by an integral, and the probabilities \( \{p_i\} \) by a probability distribution \( \pi(\psi) \). Let us now recall that, for two qubits, the tangle can be defined as the convex roof of the squared negativity [21] (the latter being equal to the concurrence [20] for pure two-qubit states). Here, equation (11) states that the convex roof of the squared logarithmic negativity properly defines the contangle, or, in short, the contangle \( E_\tau(\rho) \), in which the logarithm takes into account for the infinite dimensionality of the underlying Hilbert space.

From now on, we will restrict our attention to Gaussian states. For any multimode, mixed Gaussian states with CM \( \sigma \), we will then denote the contangle by \( E_\tau(\sigma) \), in analogy with the notation used for the contangle \( E_\tau(\sigma^p) \) of pure Gaussian states in equation (10). Any multimode mixed Gaussian state with CM \( \sigma \), admits at least one decomposition in terms of pure Gaussian states \( \sigma^p \) only. The infimum of the average contangle, taken over all pure Gaussian state decompositions, defines then the Gaussian contangle \( G_\tau \):

\[
G_\tau(\sigma) \equiv \inf \{ \pi(\sigma^p), \sigma^p \} \int \pi(d\sigma^p) E_\tau(\sigma^p).
\]

It follows from the convex-roof construction that the Gaussian contangle \( G_\tau(\sigma) \) is an upper bound to the true contangle \( E_\tau(\sigma) \) (as the latter can be in principle minimized over a non-Gaussian decomposition):

\[
E_\tau(\sigma) \leq G_\tau(\sigma)
\]

and it can be shown that \( G_\tau(\sigma) \) is a bipartite entanglement monotone under Gaussian LOCC (GLOCC) [22, 23]. In fact, for Gaussian states, the Gaussian contangle, similarly to the Gaussian EoF [22], takes the simple form

\[
G_\tau(\sigma) = \inf_{\sigma^p \leq \sigma} E_\tau(\sigma^p),
\]

where the infimum runs over all pure Gaussian states with CM \( \sigma^p \leq \sigma \). Let us remark that, if \( \sigma \) denotes a mixed symmetric two-mode Gaussian state, then the Gaussian decomposition is the optimal one [13] (it is currently an open question whether this is true for all Gaussian states [24]), and the optimal pure-state CM \( \sigma^p \) minimizing \( G_\tau(\sigma) \) is characterized by having \( \tilde{n}_-(\hat{\sigma}^p) = \tilde{n}_-(\hat{\sigma}) \) [22]. The fact that the smallest symplectic eigenvalue is the same for both partially transposed CMs entails that \( E_\tau(\sigma) = G_\tau(\sigma) = \left[ \max\{0, -\ln \tilde{n}_-(\sigma)\} \right]^2 \). We thus consistently retrieve for the contangle, in this specific case, the expression previously found for the mixed symmetric reductions of fully symmetric three-mode pure states, equation (8).
3. Monogamy inequalities and residual multipartite entanglement

3.1. Monogamy inequality for all three-mode Gaussian states

We are now in the position to prove the first main result of the present paper: all three-mode Gaussian states satisfy the monogamy inequality (6), using the Gaussian contangle $G_r$ (or even the true contangle $E_r$ for pure states) to quantify bipartite entanglement.

We start by considering pure Gaussian states $\sigma^p$ of three modes, each of the three reduced single-mode states being described respectively by the CMs $\sigma_i$, $\sigma_j$, $\sigma_k$. Due to the equality of the symplectic spectra across a bipartite cut, following from the Schmidt decomposition operated at the CM level [25], any one of the two-mode reduced CMs $\sigma_{ij}$, $\sigma_{ik}$, $\sigma_{jk}$ will have smallest symplectic eigenvalue of the associated partially transposed CM equal to 1, and will thus represent the maximum value of the sum of the 1 modes with $\mu_i = |\mu_j = 1\rangle$ and the remaining modes: $\sum_{i=1}^3 \mu_i = 2$. Moreover, because the contangle between mode 1 and modes (23) is a function of $\mu_i$ alone, see equation (10), all the entanglement properties of three-mode pure Gaussian states are completely determined by the three local purities $\mu_i$, $\mu_j$ and $\mu_k$, or by the associated local mixednesses $\sigma_i \equiv 1/\mu_i$, $\sigma_j \equiv 1/\mu_j$ and $\sigma_k \equiv 1/\mu_k$. The local mixednesses $\sigma_i$, $\sigma_j$ and $\sigma_k$ have then to vary constrained by the triangle inequality

$$|a_j - a_k| + 1 \leq a_i \leq a_j + a_k - 1,$$

in order for $\sigma^p$ to be a physical state. This is a straightforward consequence of the uncertainty relation inequality (1) applied to the reduced states of any two modes (see [23, 28] for further details). Notice that inequality (15) is a stronger requirement than the general Araki–Lieb inequality (1) applied to the reduced states of any two modes (see [23, 28] for further details). For ease of notation, let us rename the mode indices: $\{i, j, k\} \equiv \{1, 2, 3\}$. Without loss of generality, we can assume $a_1 > 1$ (if $a_1 = 1$ the first mode is not correlated with the other two and all terms in inequality (6) are trivially zero). Moreover, we can restrict to the case of $\sigma_{12}$ and $\sigma_{13}$ being both entangled. In fact, if e.g. $\sigma_{13}$ denotes a separable state, then $E_r(\sigma_{12}) \leq E_r^{(23)}(\sigma^p)$ because tracing out mode three is a LOCC (see [11]), and thus the sharing inequality is automatically satisfied.

We will now prove inequality (6) in general by using the Gaussian contangle, as it will imply the inequality for the true contangle; in fact $G_r^{(23)}(\sigma^p) = E_r^{(23)}(\sigma^p)$ but $G_r^{(l)}(\sigma) \geq E_r^{(l)}(\sigma)$, $l = 2, 3$. Our strategy will be to show that, at fixed $a_1$, i.e. at fixed entanglement between mode 1 and the remaining modes:

$$E_r^{(23)} = \ln^2 \left( a_1 - \sqrt{a_1^2 - 1} \right),$$

the maximum value of the sum of the 12 and 13 bipartite entanglements can never exceed $E_r^{(23)}$. Namely, $\max_{s_1, d_1} Q \leq \ln^2 (a_1 - \sqrt{a_1^2 - 1})$, where

$$Q \equiv G_r(\sigma_{12}) + G_r(\sigma_{13}).$$
and the variables $a_2$ and $a_3$ have been replaced by the variables $s_1 = (a_2 + a_3)/2$ and $d_1 = (a_2 - a_3)/2$. The latter are constrained to vary in the region

$$s_1 \geq \frac{a_1 + 1}{2}, \quad |d_1| \leq \frac{a_1^2 - 1}{4s_1},$$

(18)
defined by the triangle inequality (15) and by the condition of the reduced two-mode bipartitions being entangled [23]. We recall now that each $\sigma_{ll}, l = 2, 3$, is a two-mode state of partial minimum uncertainty. For this class of states the Gaussian measures of entanglement, including $G_\tau$, can all be determined and have been computed explicitly [23]. Skipping straightforward but tedious calculational details, and omitting from now on the subscript 1, we have that

$$Q = \ln^2 [m(a, s, d) - \sqrt{m^2(a, s, d) - 1}] + \ln^2 [m(a, s, -d) - \sqrt{m^2(a, s, d) - 1}],$$

(19)

where $m = m_-$ if $D \leq 0$, $m = m_+$ otherwise, and

$$m_- \equiv |k_-|/[(s - d)^2 - 1],$$

$$m_+ \equiv 2[2a^2(1 + 2s^2 + 2d^2) - (4s^2 - 1)(4d^2 - 1) - a^4 - \sqrt{\delta}]^{1/2}/[4(s - d)],$$

$$D = 2(s - d) - \sqrt{2[k_-^2 + 2k_+ + |k_-|(k_-^2 + 8k_+)^{1/2}]/k_+},$$

$$k_{\pm} = a^2 \pm m(s + d)^2,$$

$$\delta = (a - 2d - 1)(a - 2d + 1)(a + 2d - 1)(a + 2d + 1)(a - 2s - 1)$$

$$\times (a - 2s + 1)(a + 2s - 1)(a + 2s + 1).$$

Studying the derivative of $m_{\mp}$ with respect to $s$, it is analytically proven that, in the whole space of parameters $\{a, s, d\}$ given by equation (18), both $m_-$ and $m_+$ are monotonically decreasing functions of $s$. The quantity $Q$ is then maximized over $s$ for the limiting value $s = s'^{\min} \equiv (a + 1)/2$. This value corresponds to three-mode pure states in which the reduced partition $2|3$ is always separable, as it is intuitive because the bipartite entanglement is maximally concentrated in the $1|2$ and $1|3$ partitions. With the position $s = s'^{\min}$, $D$ can be easily shown to be always negative, so that, for both reduced CMs $\sigma_{12}$ and $\sigma_{13}$, the Gaussian contangle is defined in terms of $m_-$. The latter, in turn, acquires the simple form

$$m_-(a, s'^{\min}, d) = \frac{1 + 3a + 2d}{3 + a - 2d}.$$

Consequently, the quantity $Q$ is immediately seen to be an even and convex function of $d$, which entails that it is globally maximized at the boundary $|d| = d'^{\max} \equiv (a - 1)/2$. It turns out that

$$Q[a, s = s'^{\min}, d = \pm d'^{\max}] = \ln^2(a - \sqrt{a^2 - 1}),$$

(20)
which implies that in this case the sharing inequality (6) is saturated and the genuine tripartite
entanglement is exactly zero. In fact this case yields states with \( a_2 = a_1 \) and \( a_3 = 1 \) (if \( d = d_{\text{max}} \)),
or \( a_3 = a_1 \) and \( a_2 = 1 \) (if \( d = -d_{\text{max}} \)), i.e. tensor products of a two–mode squeezed state and a
single-mode uncorrelated vacuum. Being the above quantity equation (20) the global maximum
of \( Q \), inequality (6) holds true for any \( Q \), that is for any pure three-mode Gaussian state,
choosing either the Gaussian contangle \( G_\tau \) or the true contangle \( E_\tau \) as measures of bipartite
entanglement.

By convex-roof construction, the above proof immediately extends to all mixed three-mode
Gaussian states \( \sigma \), with the bipartite entanglement measured by \( G_\tau \). Let \( \{\pi(d\sigma^P_m), \sigma^P_m\} \) be the
ensemble of pure Gaussian states minimizing the Gaussian convex roof in equation (12); then,
we have

\[
G_i^{[jk]}(\sigma) = \int \pi(d\sigma^P_m) G_i^{[jk]}(\sigma^P_m) \\
\geq \int \pi(d\sigma^P_m)[G_i^{[j]}(\sigma^P_m) + G_i^{[k]}(\sigma^P_m)] \\
\geq G_i^{[j]}(\sigma) + G_i^{[k]}(\sigma),
\]

where we have exploited the fact that the Gaussian contangle is convex by construction. This
concludes the proof of the monogamy inequality (6) for all three-mode Gaussian states. \qed

### 3.2. Residual tripartite entanglement and monotonicity

The sharing constraint (6) leads naturally to the definition of the residual contangle as a quantifier
of genuine tripartite entanglement (arravogliamet). This is in complete analogy with the case
of qubit systems, except that, at variance with the three-qubit case (where the residual tangle of
pure states is invariant under qubit permutations), for CV systems of three modes the residual
contangle is partition-dependent according to the choice of the reference mode (but for the fully
symmetric case). Then, the \textit{bona fide} quantification of tripartite entanglement is provided by the
minimum residual contangle:

\[
E_i^{[jk]} = \min_{(i,j,k)} [E_i^{[jk]} - E_i^{[j]} - E_i^{[k]}],
\]

where \((i, j, k)\) denotes all the possible permutations of the three-mode indexes. This definition
ensures that \(E_i^{[jk]}\) is invariant under mode permutations and is thus a genuine three-way property
of any three-mode Gaussian state. One can verify that

\[
E_i^{[jk]} - E_i^{[j]} - (E_i^{[jk]} - E_i^{[jk]}) \geq 0
\]

if and only if \( a_i \geq a_j \), and therefore the absolute minimum in equation (22) is attained by
the decomposition realized with respect to the reference mode \( i \) of smallest local mixedness
\( a_i \), i.e. of largest local purity \( \mu_i \).

A crucial requirement for \( E_i^{[jk]} \) to be a proper measure of tripartite entanglement is that it be
non-increasing under LOCC. The monotonicity of the residual tangle was proven for three-qubit
pure states in [29]. In the CV setting, we will now prove that for pure three-mode Gaussian states
the residual Gaussian contangle \( G_i^{[jk]} \), defined in analogy with equation (22), is an \textit{entanglement
monotone} under tripartite GLOCC, and specifically that it is non-increasing even for probabilistic
operations, which is a stronger property than being only monotone on average [22]. We thus want to prove that

\[ G^{ijkl}_\tau(G_p(\sigma^p)) \leq G^{ijkl}_\tau(\sigma^p), \]

where \( G_p \) is a pure GLOCC mapping pure Gaussian states \( \sigma^p \) into pure Gaussian states [30, 31]. Every GLOCC protocol can be realized through a local operation on one party only. Assume that the minimum in equation (22) is realized for the reference mode \( i \); the output of a pure GLOCC \( G_p \) acting on mode \( i \) yields a pure-state CM with \( a'_i \leq a_i \), while \( a_j \) and \( a_k \) remain unchanged [30]. Then, the monotonicity of the residual Gaussian contangle \( G^{ijkl}_\tau \) under GLOCC is equivalent to proving that \( G^{ijkl}_\tau = G^{ijkl}_\tau(G_p(\sigma^p)) - G^{ijkl}_\tau \) is a monotonically increasing function of \( a_i \) for pure Gaussian states. One can indeed show that the first derivative of \( G^{ijkl}_\tau \) with respect to \( a_i \), under the further constraint \( a_i \leq a_{j,k} \), is globally minimized for \( a_i = a_j = a_k \equiv a_{\text{loc}} \), i.e. for a fully symmetric state. It is easy to verify that this minimum is always positive for any \( a_{\text{loc}} > 1 \), because in fully symmetric states the residual contangle is an increasing function of \( a_{\text{loc}} \). Therefore the monotonicity of \( G^{ijkl}_\tau \) under GLOCC for all pure three-mode Gaussian states is finally proven. □

3.3. Monogamy inequality for \( N \)-mode symmetric Gaussian states

We next want to investigate whether the monogamy inequality (6) can be generalized to Gaussian states with an arbitrary number \( N + 1 \) of modes, namely whether

\[ E^{i(j_1,\ldots,j_N)} - \sum_{l=1}^{N} E^{ij_l} \geq 0. \]  \( (23) \)

Establishing this result in general is a highly nontrivial task, but we can readily prove it for all symmetric multimode Gaussian states. As usual, due to the convexity of \( G_\tau \), it will suffice to prove it for pure states, for which the Gaussian contangle coincides with the true contangle in every bipartition. For any \( N \) and for \( a_{\text{loc}} > 1 \) (for \( a_{\text{loc}} = 1 \) we have a product state),

\[ E^{i(j_1,\ldots,j_N)}_\tau = \ln^2 \left( a_{\text{loc}} - \sqrt{a_{\text{loc}}^2 - 1} \right) \]  \( (24) \)

is independent of \( N \), while the total two-mode contangle

\[ NE^{ij}_\tau = \frac{N}{4} \ln^2 \left\{ \left[ a_{\text{loc}}^2(N+1) - 1 - \sqrt{(a_{\text{loc}}^2 - 1)(a_{\text{loc}}^2(N+1)^2 - (N-1)^2)} \right]/N \right\} \]  \( (25) \)

is a monotonically decreasing function of the integer \( N \) at fixed \( a_{\text{loc}} \). Because the sharing inequality trivially holds for \( N = 1 \), it is inductively proven for any \( N \). □

This result, together with extensive numerical evidence obtained for randomly generated non-symmetric four-mode Gaussian states (see figure 1), strongly supports the conjecture that the monogamy inequality be true for all multimode Gaussian state, using the (Gaussian) contangle as a measure of bipartite entanglement. It is currently under way to fully prove the conjecture analytically.
Figure 1. Behaviour of the Gaussian contangle $G_{t}^{(1)(234)}$ between mode 1 and all the other remaining three modes (horizontal axis), plotted versus the total bipartite Gaussian contangle $\sum_{j=2}^{4} G_{t}^{1j}$ (vertical axis). The two quantities have been evaluated numerically in 60 000, randomly generated, four-mode Gaussian states. States saturating the monogamy inequality equation (23) fall on the solid red line. No physical state has been found lying in the upper-left half of the plane, above the red line. This numerical result supports the conjecture that the Gaussian contangle is monogamous, and satisfies equation (23) in all multimode Gaussian states. It is important to notice that, for increasing entanglement between one mode and the others, the states tend to distribute farther away from the $45^\circ$ line boundary, hinting at the presence of genuine multipartite entanglement shared between all the modes.

4. Structure of entanglement sharing: the CV GHZ/W states

We are now in the position to analyse the sharing structure of CV entanglement by taking the residual contangle as a measure of tripartite entanglement, in analogy with the study of three-qubit entanglement [29]. Namely, we pose the problem of identifying the three-mode analogues of the two inequivalent classes of fully inseparable three-qubit states, the GHZ state [32] $|\psi_{\text{GHZ}}\rangle = (1/\sqrt{2}) (|000\rangle + |111\rangle)$, and the $W$ state [29] $|\psi_{\text{W}}\rangle = (1/\sqrt{3}) (|001\rangle + |010\rangle + |100\rangle)$. These states are pure and fully symmetric. On the one hand, the GHZ state possesses maximal tripartite entanglement, quantified by the residual tangle [3, 29], without any two-qubit entanglement. On the other hand, the $W$ state contains the maximal two-party entanglement between any pair of qubits [29] and its tripartite residual tangle is consequently zero.
Surprisingly enough, in symmetric three-mode Gaussian states, if one aims at maximizing (at given single-mode squeezing $r_{\text{loc}}$ or single-mode mixedness $a_{\text{loc}}$) either the two-mode contangle $E_i^{\|j}$ in any reduced state (i.e. aiming at finding the CV analogue of the $W$ state), or the genuine tripartite contangle (i.e. aiming at defining the CV analogue of the GHZ state), one finds the same, unique family of pure symmetric three-mode squeezed states. These states, previously named ‘GHZ-type’ states [19], can be defined for generic $N$-mode systems, and constitute an ideal test-ground for the study of multimode CV entanglement [10, 18]. The peculiar nature of entanglement sharing in this class of CV GHZ/W states is further confirmed by the following observation. If one requires maximization of the $1 \times 2$ bipartite contangle $E_i^{\|jk}$ under the constraint of separability of all two-mode reductions, one finds a class of symmetric mixed states whose tripartite residual contangle is strictly smaller than the one of the GHZ/W states, at fixed local squeezing. Therefore, in symmetric three-mode Gaussian states, when there is no two-mode entanglement, the three-party one is not enhanced, but frustrated. Instead, if a mode is maximally entangled with another, it can also achieve maximal quantum correlations in a three-party relation.

These results, unveiling a major difference between discrete-variable and CV systems, establish the promiscuous nature of CV entanglement sharing in symmetric Gaussian states. Being associated to degrees of freedom with continuous spectra, states of CV systems need not saturate the CKW inequality to achieve maximum couplewise correlations. In fact, without violating the monogamy inequality (6), pure symmetric three-mode Gaussian states are maximally three-way entangled and, at the same time, maximally robust against the loss of one of the modes due, for instance, to decoherence.

Furthermore, the residual contangle equation (22) in GHZ/W states acquires a clear operative interpretation in terms of the optimal fidelity in a three-party CV teleportation network [33]. This finding readily provides an experimental test, in terms of success of teleportation-network experiments [34, 35], to observe the promiscuous distribution of CV entanglement in symmetric, three-mode Gaussian states.

5. Concluding remarks

It is a central trait of quantum information theory that there exist limitations to the free sharing of quantum correlations among multiple parties [4]. This aspect can be quantified in terms of monogamy constraints, as first introduced by CKW for states of three qubits. In this paper, we have generalized these monogamy constraints to infinite-dimensional systems. This extension required the definition of a proper entanglement monotone, able to capture the trade-off between the couplewise entanglement and the genuine tripartite and, in general, multipartite entanglement in multimode Gaussian states. We proved analytically that the CV entanglement is monogamous in all three mode and in all symmetric multimode Gaussian states, and have numerically convincing evidence that this holds true in all $N$-mode Gaussian states as well.

Very remarkably, in the case of pure states of three modes, the residual entanglement emerging from the monogamy inequality turns out to be a tripartite entanglement monotone under Gaussian LOCC, representing the first bona fide measure of genuine multipartite entanglement for CV systems. This measure has been applied to investigate the concrete structure of the distributed entanglement in three-mode Gaussian states, leading to the discovery that there exists a special class of states (pure, symmetric and three-mode squeezed states) which simultaneously maximize the genuine tripartite entanglement and the bipartite entanglement in the reduced states of any
pair of modes. This property, which has no counterpart in finite-dimensional systems, has been labelled by us *promiscuous sharing* of CV entanglement.

The collection of results presented here is of basic importance for the understanding of quantum correlations among multiple parties in systems with infinitely many degrees of freedom. Several hints emerging from our results are worth being investigated. Among them, the analysis of the effect of decoherence on states with promiscuous entanglement sharing, to study the actual robustness of this correlation structure in the presence of noise and mixedness. From a more fundamental point of view, the extension of the present techniques to define a more general measure of multipartite entanglement for all multimode, Gaussian and non-Gaussian states is the most evident, although very challenging, task to aim at. Another important follow-up should concern the experimental implications of our findings, ranging from the verification of the promiscuity in terms of quantum communication experiments [33], to the preparation of special classes of multiparty entangled Gaussian states and the implementation of these resources for novel protocols of quantum information with CVs. All these issues are being currently and actively studied [28].

In a wider perspective, we wish to mention that monogamy inequalities should play a fundamental role in the understanding of entanglement and other structural properties of a wide class of complex quantum systems both in discrete and CVs [16], [36]–[39]. At this stage, in the CV setting, the most interesting and urgent open problems are perhaps the extension of the CKW-type sharing inequalities to non-Gaussian states and to systems in which the reference party is constituted by more than one mode. The latter problem is the CV analogue of extending the monogamy inequalities to systems in which the reference party is constituted by more than one qubit [16]. Finally, another very intriguing open issue is the generalization to systems of arbitrary finite dimension $D$ interpolating between the case $D = 2$ and the infinite-dimensional instance [4]. In all these problems, the challenge is to define easily computable measures of entanglement that allow concrete quantifications and bounds on the sharing structure and distribution of quantum correlations.

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**References**

[1] Cirac J I 2002 *Fundamentals of Quantum Information* ed D Heiss (Berlin: Springer)
[2] van Loock P and Furusawa A 2003 *Quantum Information Theory with Continuous Variables* ed S L Braunstein and A K Pati (Dordrecht: Kluwer)
[3] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev. A* 61 052306
[4] Adesso G and Illuminati F 2006 *Int. J. Quantum Inform.* at press Preprint quant-ph/0506213
[5] Braunstein S L and van Loock P 2005 *Rev. Mod. Phys.* 77 513
[6] Eisert J and Plenio M B 2003 *Int. J. Quantum Inform.* 1 479
[7] Cerf N, Leuchs G and Polzik E S (ed) 2006 *Quantum Information with Continuous Variables of Atoms and Light* (London: Imperial College Press) at press
[8] Simon R 2000 *Phys. Rev. Lett.* 84 2726
[9] Werner R F and Wolf M M 2001 *Phys. Rev. Lett.* 86 3658
[10] Serafini A, Adesso G and Illuminati F 2005 Phys. Rev. A 71 032349
[11] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
[12] Eisert J 2001 PhD Thesis University of Potsdam
[13] Giedke G, Wolf M M, Krüger O, Werner R F and Cirac J I 2003 Phys. Rev. Lett. 91 107901
[14] Plenio M B 2005 Phys. Rev. Lett. 95 090503
[15] Audenaert K, Plenio M B and Eisert J 2003 Phys. Rev. Lett. 90 027901
[16] Osborne T J and Verstraete F 2005 Preprint quant-ph/0502176
[17] See e.g. Terhal B M 2004 IBM J. Res. Dev. 48 71
[18] Adesso G, Serafini A and Illuminati F 2004 Phys. Rev. Lett. 93 220504
[19] van Loock P and Furusawa A 2003 Phys. Rev. A 67 052315
[20] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[21] Lee S, Chi D P, Oh S D and Kim J 2003 Phys. Rev. A 68 062304
[22] Wolf M M, Giedke G, Krüger O, Werner R F and Cirac J I 2004 Phys. Rev. A 69 052320
[23] Adesso G and Illuminati F 2005 Phys. Rev. A 72 032334
[24] Krüger and Werner R F (ed) 2005 Preprint quant-ph/0504166 http://www.imaph.tu-bs.de/qi/problems
[25] Holevo A S and Werner R F 2001 Phys. Rev. A 63 032312
[26] Adesso G, Serafini A and Illuminati F 2004 Phys. Rev. Lett. 92 087901
[27] Adesso G, Serafini A and Illuminati F 2004 Phys. Rev. A 70 022318
[28] Adesso G, Serafini A and Illuminati F 2005 Preprint quant-ph/0512124
[29] Dür W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[30] Giedke G, Eisert J, Cirac J I and Plenio M B 2003 Quantum Inform. Comput. 3 211
[31] Eisert J, Scheel S and Plenio M B 2002 Phys. Rev. Lett. 89 137903
[32] Greenberger D M, Horne M A, Shimony A and Zeilinger A 1990 Am. J. Phys. 58 1131
[33] Adesso G and Illuminati F 2005 Phys. Rev. Lett. 95 150503
[34] van Loock P and Braunstein S L 2000 Phys. Rev. Lett. 84 3482
[35] Yonezawa H, Aoki T and Furusawa A 2004 Nature 431 430
[36] Osterloh A, Amico L, Falci G and Fazio R 2002 Nature 416 608
[37] Osborne T J and Nielsen M A 2002 Phys. Rev. A 66 032110
[38] Roscilde T, Verrucchi P, Fubini A, Haas S and Tognetti V 2005 Phys. Rev. Lett. 94 147208
[39] Plenio M B, Hartley J and Eisert J 2004 New J. Phys. 6 36