Tangent Spaces of Orbit Closures for Representations of Dynkin Quivers of Type $\mathbb{D}$

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Abstract

Let $\mathbb{k}$ be an algebraically closed field, $Q$ a finite quiver, and denote by $\text{rep}_d^Q$ the affine $\mathbb{k}$-scheme of representations of $Q$ with a fixed dimension vector $d$. Given a representation $M$ of $Q$ with dimension vector $d$, the set $O_M$ of points in $\mathbb{k}$ isomorphic as representations to $M$ is an orbit under an action on $\text{rep}_d^Q(\mathbb{k})$ of a product of general linear groups. The orbit $O_M$ and its Zariski closure $\overline{O}_M$, considered as reduced subschemes of $\text{rep}_d^Q$, are contained in an affine scheme $C_M$ defined by suitable rank conditions associated to $M$. For all Dynkin and extended Dynkin quivers, the sets of points of $O_M$ and $C_M$ coincide, or equivalently, $O_M$ is the reduced scheme associated to $C_M$. Moreover, $O_M = C_M$ provided $Q$ is a Dynkin quiver of type $A$, and this equality is a conjecture for the remaining Dynkin quivers (of type $D$ and $E$). Let $Q$ be a Dynkin quiver of type $\mathbb{D}$ and $M$ a finite dimensional representation of $Q$. We show that the equality $T_N O_M = T_N C_M$ of Zariski tangent spaces holds for any closed point $N$ of $\overline{O}_M$. As a consequence, we describe the tangent spaces to $\overline{O}_M$ in representation theoretic terms.

Keywords Representations of quivers · Orbit closures · Zariski tangent spaces

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1 Introduction and the Main Results

Throughout the paper $\mathbb{k}$ denotes an algebraically closed field of arbitrary characteristic. We will identify a $\mathbb{k}$-scheme $\mathcal{X}$ with its functor of points, i.e. the functor from the
category of commutative \( k \)-algebras to the category of sets sending \( R \) to the set of morphisms \( \text{Spec}(R) \to X \). Let \( k[\varepsilon] \) denote the \( k \)-algebra of dual numbers and consider the map 
\[
X(\pi) : X(k[\varepsilon]) \to X(k),
\]
where \( \pi : k[\varepsilon] \to k \) is the canonical surjective homomorphism. Given a \( k \)-rational point \( x \) of \( X \), i.e. \( x \in X(k) \), the fiber \( X(\pi)^{-1}(x) \) is the Zariski tangent space \( T_x X \) to \( X \) at \( x \). We are mostly interested in affine schemes \( X \) of finite type over \( k \), i.e. the schemes of the form \( \text{Spec}(R) \), where \( R \) is a finitely generated commutative \( k \)-algebra. For such schemes \( X(k) \) is the set of closed points of \( X \).

Let \( Q = (Q_0, Q_1) \) be a finite quiver, i.e. a finite set \( Q_0 \) of vertices and a finite set \( Q_1 \) of arrows \( \alpha : sa \to ta \), where \( sa \) and \( ta \) denote the starting and the terminating vertex of \( \alpha \), respectively. A representation of \( Q \) over \( k \) is a collection \( M = (M_a, M_\alpha : a \in Q_0, \alpha \in Q_1) \) of \( k \)-vector spaces \( M_a \) and \( k \)-linear maps \( M_\alpha : M_{sa} \to M_{ta} \). A morphism \( f : M \to N \) between two representations is a collection \( f = (f_a : M_a \to N_a ; a \in Q_0) \) of \( k \)-linear maps such that 
\[
f_{ta} \circ M_\alpha = N_\alpha \circ f_{sa}, \quad \text{for all} \ a \in Q_1.
\]

We denote by \( \text{rep}(Q) \) the category of finite dimensional representations of \( Q \), i.e. the representations \( M \) such that all vector spaces \( M_a \) are finite dimensional. For a representation \( M \) in \( \text{rep}(Q) \) we define its dimension vector \( \text{dim} M = (\dim_k M_a) \in \mathbb{N}^{Q_0} \).

We denote by \( \mathbb{M}_{p,q} \) the \( k \)-scheme of \( p \times q \)-matrices and by \( \text{GL}_d \) the group \( k \)-scheme of invertible \( d \times d \)-matrices, for any positive integers \( p, q \) and \( d \). Given a dimension vector \( d = (d_a) \in \mathbb{N}^{Q_0} \) we have the affine scheme 
\[
\text{rep}^d Q = \prod_{a \in Q_1} \mathbb{M}_{d_{ta} \times d_{sa}}.
\]

Thus the points of \( \text{rep}^d Q(k) \) can be identified with the representations \( M \) of \( Q \) such that \( M_a = k^{d_a} \) for any \( a \in Q_0 \). The group scheme 
\[
\text{GL}_d = \prod_{a \in Q_0} \text{GL}_{d_a}
\]
acts on \( \text{rep}^d Q \) via 
\[
g \ast M = (g_{ta} \cdot M_\alpha \cdot g_{sa}^{-1}),
\]
for any \( g = (g_a) \in \text{GL}_d(R), M = (M_a) \in \text{rep}^d Q(R), \) and a commutative \( k \)-algebra \( R \). Given a representation \( M \) in \( \text{rep}(Q) \), we denote by \( \mathcal{O}_M \) the \( \text{GL}_d(k) \)-orbit in \( \text{rep}^d Q(k) \) which consists of the representations in \( \text{rep}^d Q(k) \) isomorphic to \( M \), where \( d = \text{dim} M \). By abuse of notation, we treat \( \mathcal{O}_M \) and its closure \( \overline{\mathcal{O}_M} \) as reduced subschemes of \( \text{rep}^d Q \). It is an open and interesting problem to describe the defining ideal of \( \overline{\mathcal{O}_M} \) or even to exhibit polynomials having \( \overline{\mathcal{O}_M} \) as their zero set. If \( M \) and \( N \) are representations satisfying \( \mathcal{O}_N \subseteq \overline{\mathcal{O}_M} \), then we say that \( M \) degenerates to \( N \). Note that \( \overline{\mathcal{O}_M} \) is the union of \( \mathcal{O}_N(k) \), where \( N \) runs through the representations to which \( M \) degenerates.

In case of an equioriented Dynkin quiver of type \( A \) or a loop degenerations can be described by simple rank conditions (see \([1, 9]\)). These rank conditions can be interpreted as inequalities between the dimensions of homomorphism spaces. Let \( [X, Y] = \dim_k \text{Hom}_Q(X, Y) \), for \( X, Y \in \text{rep}(Q) \). It is well-known that if \( M \) degenerates to \( N \) then
\[
[X, N] \geq [X, M] \quad \text{and} \quad [N, X] \geq [M, X], \quad \text{for any} \ X \in \text{rep}(Q) \quad (1.1)
\]
(see for instance [13] 2.1). Moreover, due to Bongartz [4, 5], the reverse implication holds under an additional assumption on \( Q \).

**Theorem 1.1** Let \( Q \) be a Dynkin or an extended Dynkin quiver. Assume \( M \) and \( N \) belong to \( \text{rep}(Q) \) and \( \dim M = \dim N \). Then \( M \) degenerates to \( N \) if and only if the condition (1.1) is satisfied.

Inspired by the above inequalities (see also [4] Proposition 1), a closed \( \text{GL}_d \)-subscheme \( C_M \) of \( \text{rep}_Q^d \) containing \( \overline{O}_M \) was defined in [14]. Let \( \mathbb{k} Q = \bigoplus_{a,b \in Q_0} \mathbb{k} Q(a, b) \) denote the path algebra of \( Q \), where \( \mathbb{k} Q(a, b) \) is the vector space with a \( \mathbb{k} \)-basis formed by the paths in \( Q \) starting at \( b \) and terminating at \( a \). For any commutative \( \mathbb{k} \)-algebra \( R \), \( X \in \text{rep}_Q^d(R) \) and \( \omega \in \mathbb{k} Q(a, b) \), the matrix \( X_{\omega} \in M_{d_a,d_b}(R) \) is defined in the obvious way.

Let \( p, q \in \mathbb{N} \), and consider two sequences \((a_1, \ldots, a_p)\) and \((b_1, \ldots, b_q)\) of vertices in \( Q_0 \) and a \( p \times q \)-matrix \( \omega = (\omega_{i,j}) \) such that each \( \omega_{i,j} \) belongs to \( \mathbb{k} Q(a_i, b_j) \). We define a regular morphism

\[
\Theta_\omega : \text{rep}_Q^d \to M_{p',q'}, \quad \Theta_\omega(N) = \begin{bmatrix}
N_{\omega_{1,1}} & \cdots & N_{\omega_{1,q}} \\
\vdots & \ddots & \vdots \\
N_{\omega_{p',1}} & \cdots & N_{\omega_{p',q'}}
\end{bmatrix},
\]

where \( p' = \sum d_{a_i} \) and \( q' = \sum d_{b_j} \). For \( M \) in \( \text{rep}(Q) \), put \( \mathbf{d} = \text{dim} M \) and fix \( M' \) in \( O_M \).

Let \( \mathcal{I}_{M,\omega} \) be the ideal in the coordinate algebra \( \mathbb{k}[\text{rep}_Q^d] \) of \( \text{rep}_Q^d \) generated by the images via \((\Theta_\omega)^*\) of the minors of size \( 1 + \text{rk } \Theta_\omega(M') \) in \( \mathbb{k}[M_{p',q'}] \). We set \( \mathcal{I}_M = \bigcap_{\omega} \mathcal{I}_{M,\omega} \), where \( \omega \) runs through all possible matrices of linear combinations of paths with all possible sequences of starting and terminating vertices. Then \( C_M = \text{Spec}(\mathbb{k}[\text{rep}_Q^d]/\mathcal{I}_M) \) is a closed \( \text{GL}_d \)-subscheme of \( \text{rep}_Q^d \) containing \( \overline{O}_M \). Moreover, \( C_M(\mathbb{k}) \) consists of the representations \( N \in \text{rep}_Q^d(\mathbb{k}) \) satisfying (1.1). Hence we can reformulate Theorem 1.1 by saying that \( C_M(\mathbb{k}) = \overline{O}_M(\mathbb{k}) \) provided \( Q \) is a Dynkin or an extended Dynkin quiver. Note that this equality does not hold for the representation

\[
M = \mathbb{k} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\mathbb{k}^2}{\mathbb{k}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{k}.
\]

Here \( C_M(\mathbb{k}) \) has two irreducible components of dimension 5, with one of them being \( \overline{O}_M(\mathbb{k}) \) (see [14] Example 8.9).

Lakshmibai and Magyar described in [10] generators of the defining ideal of \( \overline{O}_M \) in case \( Q \) is an equioriented Dynkin quiver of type \( A \). They used the Zelevinsky immersion of \( \text{rep}_Q^d \) in a Schubert variety of a flag variety, and applied a description of generators of the defining ideal of this Schubert variety. It turned out that they showed that the defining ideal of \( \overline{O}_M \) equals \( \mathcal{I}_M \). The result was generalized in [14] Theorem 6.4 to the Dynkin quivers of type \( A \) with an arbitrary orientation:

**Theorem 1.2** Let \( Q \) be a Dynkin quiver of type \( A \) and \( M \in \text{rep}(Q) \). Then \( C_M = \overline{O}_M \).
tangent spaces in these cases. Note that this is not true even for the simplest extended Dynkin quiver. Namely, consider representations

\[ M = k^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = k^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

Then \( N \in \mathcal{O}_M, T_N \mathcal{O}_M \) has dimension 3, while \( T_N \mathcal{C}_M \) has dimension 4 (the trace function does not belong to the ideal \( \mathcal{I}_M \), see [14] Example 3.7).

Our main result shows that \( \mathcal{C}_M \) and \( \mathcal{O}_M \) have identical tangent spaces in type \( D \):

**Theorem 1.3** Let \( Q \) be a Dynkin quiver of type \( D \) and \( M \in \text{rep}(Q) \). Then \( \mathcal{O}_M(\mathbb{k}[\varepsilon]) = \mathcal{C}_M(\mathbb{k}[\varepsilon]) \). In other words, \( T_N \mathcal{O}_M = T_N \mathcal{C}_M \) for any \( N \) in \( \mathcal{O}_M(\mathbb{k}) = \mathcal{C}_M(\mathbb{k}) \).

We conjecture that the similar result holds for the Dynkin quivers of type \( E \). However, in that case the combinatorics of short exact sequences with indecomposable end terms is more involved and the proof requires new techniques. We plan to tackle this problem in a forthcoming paper.

We give now a representation theoretic interpretation of Theorem 1.3. The sets \( \text{rep}_d(Q)(\mathbb{k}) \) and \( \text{rep}_d(Q)(\mathbb{k}[\varepsilon]) \) have natural structures of vector spaces over \( \mathbb{k} \). Using the decomposition \( \mathbb{k}[\varepsilon] = \mathbb{k} \cdot 1 \oplus \mathbb{k} \cdot \varepsilon \) one can write each element of \( \text{rep}_d(Q)(\mathbb{k}[\varepsilon]) \) uniquely in the form \( N + \varepsilon \cdot Z \), where \( N \) and \( Z \) belong to \( \text{rep}_d(Q)(\mathbb{k}) \). Here \( N \) is a closed point of \( \text{rep}_d(Q) \) and \( Z \) is a tangent vector in \( T_N \text{rep}_d(Q) \). It is a simple but ingenious idea (see [16]) to associate to such element \( N + \varepsilon \cdot Z \) a representation in \( \text{rep}_2(Q)(\mathbb{k}) \) of the following block form

\[
\begin{bmatrix} N & Z \\ 0 & N \end{bmatrix} = \left( \begin{bmatrix} N_\alpha & Z_\alpha \\ 0 & N_\alpha \end{bmatrix} : \alpha \in Q_1 \right).
\]

By [14] Corollary 7.4, Theorem 1.3 implies that if \( N \in \mathcal{O}_M \), then \( T_N \mathcal{O}_M \) consists of the \( Z \) satisfying the following two equivalent conditions:

- \( \left[ X, \begin{bmatrix} N & Z \\ 0 & N \end{bmatrix} \right] = 2 \cdot [X, N] \) for all \( X \in \text{rep}(Q) \) with \( [X, N] = [X, M] \),
- \( \left[ \begin{bmatrix} N & Z \\ 0 & N \end{bmatrix}, X \right] = 2 \cdot [N, X] \) for all \( X \in \text{rep}(Q) \) with \( [N, X] = [M, X] \).

We remark that Theorem 1.3 should have applications in the problem of describing the singular locus of \( \mathcal{O}_M \) in representation theoretic terms, for the representations \( M \) of Dynkin quivers of type \( D \) (see [14] Section 8).

The paper is organized as follows: in Section 2 we prove necessary facts about exact sequences in \( \text{rep}(Q) \) (in fact we formulate them in a more general setup of triangles in the derived category of \( \text{rep}(Q) \)), while in Section 3 we apply results of Section 2 in a geometric context and prove the main result. For basic background on representation theory of quivers we refer to [2, 3, 15].

## 2 Derived Categories for Representations of Dynkin Quivers

In order to prove Theorem 1.3, we need a result about existence of short exact sequences in \( \text{rep}(Q) \) with some special properties, where \( Q \) is a Dynkin quiver of type \( D \) (Corollary 2.23). Our idea is to use the embedding of \( \text{rep}(Q) \) in its derived category \( D^b(Q) = D^b(\text{rep}(Q)) \), and to prove the existence of triangles in \( D^b(Q) \) satisfying similar properties (Proposition 2.19). An advantage of working with the derived category is that its structure, including Auslander-Reiten theory, is more “regular” than the structure of \( \text{rep}(Q) \). In particular, the
formulation of Corollaries 2.14 and 2.15 would be much more complicated if we worked in the category of representations. We refer to [8] as a general reference for this section.

2.1 Dynkin Graphs

Throughout this section $\Delta = (\Delta_0, \Delta_1)$ is a Dynkin graph of one of the types $A_n$, $n \geq 1$, $D_n$, $n \geq 4$, or $E_6$, $E_7$, $E_8$, where $\Delta_0$ is the set of $n$ vertices of $\Delta$, and $\Delta_1$ is its set of edges, i.e. two element subsets of $\Delta_0$. If $(a, b)$ is an edge we say that $a$ and $b$ are adjacent. We denote by $a^-$ the (open) neighbourhood of a vertex $a$, i.e. the set of vertices adjacent to $a$. The degree of $a$ equals, by definition, the cardinality of $a^-$. Let $\text{dist}(a, b)$ be the length of the shortest walk in $\Delta$ between $a$ and $b$.

We define an integer $n_\Delta$ as follows:

$$n_{A_n} = n + 1, \quad n_{D_n} = 2n - 2, \quad n_{E_6} = 12, \quad n_{E_7} = 18, \quad n_{E_8} = 30. \quad (2.1)$$

With $\Delta$ we associate (Tits) quadratic form

$$q_\Delta : \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}, \quad q_\Delta(d) = \sum_{a \in \Delta_0} d_a^2 - \sum_{\{a, b\} \in \Delta_1} d_a \cdot d_b, \quad (2.2)$$

which is positive definite, i.e. $q_\Delta(d) > 0$ for any non-zero $d$. If $q_\Delta(d) = 1$ we say $d$ is a root. Obviously, if $d$ is a root, then $-d$ is also a root. There are $n \cdot n_\Delta$ roots, half of them are positive, where a vector $d$ is called positive provided $d \neq 0$ and $d_a \geq 0$, for each $a$. There is a unique maximal root $h_\Delta$ (i.e. $h_\Delta - d$ is positive for any root $d \neq h_\Delta$) which equals

$A_n$ : 

```
1 1 ... 1
```

$D_n$ :

```
1
2 2 ...
```

$E_6$ :

```
1 2 3 2 1
```

$E_7$ :

```
2 3 4 3 2 1
```

$E_8$ :

```
2 4 6 5 3 2
```

2.2 Derived Category for Acyclic Quivers

Throughout this subsection $Q$ is a finite quiver without oriented cycles. We denote by $D^b(Q) = D^b(\text{rep}(Q))$ the derived category of the abelian category $\text{rep}(Q)$. The category $D^b(Q)$ is triangulated, hence there is an auto-equivalence [1] of $D^b(Q)$ called the shift functor (“the suspension functor” and “the translation functor” are alternative names used by other authors) and a class of triangles (the name “distinguished triangles” is commonly used), written in the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$. There is a canonical full embedding of $\text{rep}(Q)$ in $D^b(Q)$, and we shall identify $\text{rep}(Q)$ with its image in $D^b(Q)$. In particular,

$$\text{Hom}_Q(X, Y) = \text{Hom}_{D^b(Q)}(X, Y) \quad \text{and} \quad \text{Ext}^1_Q(X, Y) = \text{Hom}_{D^b(Q)}(X, Y[1]),$$
for all $X, Y \in \text{rep}(Q)$. Based on the latter equality, there is a strong relationship between the short exact sequences in $\text{rep}(Q)$ and triangles in $\mathcal{D}^b(Q)$. Namely, for each short exact sequence $\sigma : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\text{rep}(Q)$ there is a unique morphism $\gamma \in \text{Hom}_{\mathcal{D}^b(Q)}(C, A[1])$ such that $\hat{\sigma} : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle in $\mathcal{D}^b(Q)$. Conversely, if $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle in $\mathcal{D}^b(Q)$ with $A$ and $C$ in $\text{rep}(Q)$, then there is an isomorphism $g : B \to B'$ in $\mathcal{D}^b(Q)$ such that $\sigma : 0 \to A \xrightarrow{g \circ \alpha} B' \xrightarrow{\beta \circ g^{-1}} C \to 0$ is a short exact sequence in $\text{rep}(Q)$. Moreover, in the above situation $\hat{\sigma}$ has the form $A \xrightarrow{g \circ \alpha} B' \xrightarrow{\beta \circ g^{-1}} C \xrightarrow{\gamma} A[1]$.

We now generalize the notions of a split exact sequence and a pullback to triangles. Let $\sigma$ be a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ in $\mathcal{D}^b(Q)$. We say that $\sigma$ splits if one of the following equivalent conditions is satisfied: (i) $\alpha$ is a section, (ii) $\beta$ is a retraction, (iii) $\gamma = 0$, (iv) $B$ is isomorphic to $A \oplus C$. Observe that an exact sequence $\sigma$ splits if and only if the triangle $\hat{\sigma}$ splits.

Given a triangle $\sigma : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ and a morphism $h : C' \to C$ in $\mathcal{D}^b(Q)$ we get the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B' \\
\downarrow & & \downarrow \gamma \circ h \\
A & \xrightarrow{\beta} & C \\
\downarrow h & & \downarrow \\
A[1] & \xrightarrow{\gamma} & A[1],
\end{array}
\]

where the upper row, called the pullback of $\sigma$ along $h$, is a triangle (note that the pullback is unique up to isomorphism of triangles). One defines pushouts dually. Observe that if $\sigma'$ is the pullback of a short exact sequence $\sigma$ along a homomorphism $h : C' \to C$ in $\text{rep}(Q)$, then $\hat{\sigma}'$ is the pullback of $\hat{\sigma}$ along $h$ (viewed as a morphism in $\mathcal{D}^b(Q)$).

We say that a triangle $\sigma$ is almost split (or an Auslander-Reiten triangle) if $A$ and $C$ are indecomposable, $\sigma$ does not split, but its pullbacks split for all morphisms to $X, Y$ which are not sections. The last condition can be replaced by the requirement that the pushouts of $\sigma$ split for all morphisms from $A$ which are not sections.

An important fact about the category $\mathcal{D}^b(Q)$ is that it has a Serre duality, i.e. there is an auto-equivalence $\nu : \mathcal{D}^b(Q) \to \mathcal{D}^b(Q)$, called a Serre functor, such that there are isomorphisms

\[
\text{Hom}_{\mathcal{D}^b(Q)}(Y, \nu X) \simeq D \text{Hom}_{\mathcal{D}^b(Q)}(X, Y) \simeq \text{Hom}_{\mathcal{D}^b(Q)}(\nu^{-1} Y, X),
\]

(2.3)

which are natural in $X$ and $Y$, where $D$ is the duality $\text{Hom}_k(\_ , \_ )$ on $k$ (see [11] Section I). The Serre functor $\nu$ restricts to an equivalence between the subcategory $\mathcal{P}_Q$ of the projective representations in $\text{rep}(Q)$ and the subcategory $\mathcal{I}_Q$ of the injective representations in $\text{rep}(Q)$. This restriction is called a Nakayama functor. For each vertex $a$ of $Q$, we denote by $P_a$ and $I_a$ the indecomposable projective and injective representation in $\text{rep}(Q)$ at $a$, respectively. We note that up to isomorphism, these are the only indecomposable objects of $\mathcal{P}_Q$ and $\mathcal{I}_Q$, respectively.

The existence of a Serre functor $\nu$ is closely related to the existence of almost split triangles in $\mathcal{D}^b(Q)$. Namely, we consider the auto-equivalence $\tau = \nu \circ [-1] \simeq [-1] \circ \nu$ of $\mathcal{D}^b(Q)$, and call it the Auslander-Reiten translation. Then there is an almost split triangle of the form $\tau C \to B \to C \to (\tau C)[1]$ for any indecomposable object $C$ in $\mathcal{D}^b(Q)$, and there
is an almost split triangle of the form \( A \to B' \to \tau^{-1}A \to A[1] \) for any indecomposable object \( A \) in \( \mathcal{D}^b(Q) \).

One defines the Grothendieck group \( K_0(\text{rep}(Q)) \) of \( \text{rep}(Q) \) as the quotient of the free abelian group with basis formed by the isomorphism classes \([X]\) of objects \( X \) in \( \text{rep}(Q) \), modulo the subgroup generated by \([A] - [B] + [C]\) for all short exact sequences \( 0 \to A \to B \to C \to 0 \) in \( \text{rep}(Q) \). The group \( K_0(\text{rep}(Q)) \) is isomorphic with \( \mathbb{Z}Q_0 \) via the map sending the class of \([X]\) to \( \dim X \). We will treat this isomorphism as an identification.

The Grothendieck group \( K_0(\mathcal{D}^b(Q)) \) of the category \( \mathcal{D}^b(Q) \) is defined in a similar way, the only difference is that one takes \([A] - [B] + [C]\) for the triangles \( A \to B \to C \to A[1] \) when forming the quotient. The embedding of \( \text{rep}(Q) \) in \( \mathcal{D}^b(Q) \) induces a group isomorphism from \( K_0(\text{rep}(Q)) \) to \( K_0(\mathcal{D}^b(Q)) \), which we will treat as an identification. In particular we will use the notation \( \dim X \) for \( X \in \mathcal{D}^b(Q) \). Note that \( \dim X[i] = (-1)^i \cdot \dim X \) for any object \( X \in \mathcal{D}^b(Q) \) and any integer \( i \).

Given two objects \( X \) and \( Y \) in \( \mathcal{D}^b(Q) \) we denote by \([X, Y]\) the dimension of \( \text{Hom}_{\mathcal{D}^b(Q)}(X, Y) \). Moreover, we set \([X, Y]^i = [X[-i], Y] = [X, Y[i]]\) for any integer \( i \). If \( X \) and \( Y \) belong to \( \text{rep}(Q) \) then \([X, Y]^i = 0 \) provided \( i \not\in \{0, 1\} \). We define the bilinear form \( b_Q : \mathbb{Z}Q_0 \times \mathbb{Z}Q_0 \to \mathbb{Z} \) by the formula

\[
b_Q(\mathbf{d}, \mathbf{e}) = \sum_{a \in Q_0} d_a \cdot e_a - \sum_{a \in Q_1} d_a \cdot e_{\alpha a},
\]

for all \( \mathbf{d}, \mathbf{e} \in \mathbb{Z}Q_0 \). Then

\[
b_Q(\dim X, \dim Y) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot [X, Y]^i,
\]

for all \( X, Y \in \mathcal{D}^b(Q) \) (see \[8\] III.1). In particular, if \( X, Y \in \text{rep}(Q) \), then

\[
b_Q(\dim X, \dim Y) = [X, Y]_0 - [X, Y]^1 = \dim_k \text{Hom}_Q(X, Y) - \dim_k \text{Ext}_Q^1(X, Y).
\]

Observe that the quadratic form associated with \( b_Q \) coincides with \( q_\Delta \), where \( \Delta \) is the underlying graph of \( Q \) and \( q_\Delta \) is the quadratic form introduced in Eq. 2.2. Recall that the Yoneda lemma states that if \( a \in Q_0 \) and \( M \in \text{rep}(Q) \), then \([P_a, M] \) is the \( a \)-th coordinate of \( \dim M \). This easily implies that \( b_Q(\dim P_a, \mathbf{d}) = d_a \), for each \( \mathbf{d} \in \mathbb{Z}Q_0 \).

We collect below a few facts concerning indecomposable objects in \( \mathcal{D}^b(Q) \) under the assumption that \( Q \) is a Dynkin quiver.

**Lemma 2.1** Let \( Q \) be a Dynkin quiver, and \( X \) and \( Y \) be indecomposable objects in \( \mathcal{D}^b(Q) \).

Then:

1. \( X \simeq \tau^n aX \) for a unique pair \((n_X, a_X) \in \mathbb{Z} \times Q_0\).
2. \([X, X] = 1 \) and \( q_\Delta(\dim X) = 1 \).
3. \([X, Y]^i \) is non-zero for at most one integer \( i \).
4. \([X, Y] \leq h^\Delta_{a_X} \).

**Proof** The first three properties are well known (see for example \[8\] Chapter I).

(4) We may assume that \([X, Y] > 0 \). Applying the automorphism \( \tau^{-n_X} \) we get

\[
[X, Y] = [\tau^{n_X}P_{a_X}, Y] = [P_{a_X}, \tau^{-n_X}Y] = b_Q(\dim P_{a_X}, \dim \tau^{-n_X}Y),
\]

where the last equality follows from (3) and the assumption \([X, Y] > 0 \). As we observed above \( b_Q(\dim P_{a_X}, \dim \tau^{-n_X}Y) \) is the \( a_X \)-th coordinate of the vector \( \dim \tau^{-n_X}Y \). By (2),
2.3 Mesh Categories for Dynkin Graphs

Throughout this subsection $\Delta = (\Delta_0, \Delta_1)$ is a Dynkin graph. We say that two elements $(p, a)$ and $(q, b)$ of the product $\mathbb{Z} \times \Delta_0$ are equivalent provided the integer $(q - p) + \text{dist}(a, b)$ is even. This is an equivalence relation since $\Delta$ is a tree, and thus $\mathbb{Z} \times \Delta_0$ is partitioned into two parts

$$\mathbb{Z} \times \Delta_0 = (\mathbb{Z} \times \Delta_0)^v \sqcup (\mathbb{Z} \times \Delta_0)^m,$$

where we use the former set to parameterize the vertices and the latter one to parameterize the meshes of the quiver $\mathbb{Z}\Delta$ defined below. In order to decide which part stands for $(\mathbb{Z} \times \Delta_0)^v$, we choose a base vertex $b_0 \in \Delta_0$ and require that $(0, b_0)$ belongs to $(\mathbb{Z} \times \Delta_0)^v$.

We define an infinite quiver $\mathbb{Z}\Delta$ without multiple arrows as follows. The set $(\mathbb{Z}\Delta)_0$ of vertices of $\mathbb{Z}\Delta$ consists of $v_{p,a}$, where $(p, a) \in (\mathbb{Z} \times \Delta_0)^v$. There is an arrow in $\mathbb{Z}\Delta$ starting at $v_{p,a}$ and terminating at $v_{q,b}$ if and only if $a$ and $b$ are adjacent in $\Delta$ and $q - p = 1$. For example, if $\Delta = A_4$:

$$\bullet \quad \bullet \quad \bullet \quad \bullet$$

and we choose $b_0 = b$, then $\mathbb{Z}\Delta$ has the form

For each pair $(p, a)$ in $(\mathbb{Z} \times \Delta_0)^m$, we consider the smallest subquiver $m_{p,a}$ of $\mathbb{Z}\Delta$, called a mesh, containing all paths (of length two) starting at $v_{p-1,a}$ and terminating at $v_{p+1,a}$:

$$v_{p,b_1} \quad v_{p,b_r} \quad \bullet \quad \bullet \quad v_{p-1,a} \quad v_{p+1,a}$$

where $\{b_1, \ldots, b_r\} = a^-$. We denote by $(\mathbb{Z}\Delta)_2$ the set of meshes $m_{p,a}$, where $(p, a) \in (\mathbb{Z} \times \Delta_0)^m$. With each mesh $m_{p,a}$ we associate its mesh relation, i.e. the sum of the paths starting at $v_{p-1,a}$ and terminating at $v_{p+1,a}$, considered as morphisms in the path category $\mathbb{k}[\mathbb{Z}\Delta]$ of $\mathbb{Z}\Delta$. The mesh category $\mathbb{k}(\mathbb{Z}\Delta)$ of $\mathbb{Z}\Delta$ is the quotient of $\mathbb{k}[\mathbb{Z}\Delta]$ modulo the ideal generated by all mesh relations.

Now let $Q$ be a Dynkin quiver with underlying graph $\Delta$. An important fact is that $\mathbb{k}(\mathbb{Z}\Delta)$ is equivalent as a $\mathbb{k}$-linear category to the category of indecomposable objects in
When Q is fixed, then we shall identify \((\mathbb{Z}\Delta)_0\) with a complete set of pairwise non-isomorphic indecomposable objects of \(\mathcal{D}^b(Q)\). Moreover, we may also assume that under this identification \(P_a = v_{p_a,a}\) with an appropriate integer \(p_a\), for each vertex \(a \in Q_0 = \Delta_0\). The three crucial auto-equivalences of \(\mathcal{D}^b(Q)\): the Auslander-Reiten translation \(\tau\), the Serre functor \(\nu\) and the shift functor \([1]\) act on the indecomposable objects by the formulas

\[
\tau(v_{p,a}) = v_{p-2,a}, \quad \nu(v_{p,a}) = v_{p+n_\Delta - 2, \phi_\Delta(a)}, \quad v_{p,a}[1] = v_{p+n_\Delta, \phi_\Delta(a)},
\]

where \(n_\Delta\) was defined in Eq. 2.1 and \(\phi_\Delta\) is the automorphism of \(\Delta\) defined as follows: \(\phi_\Delta\) is the unique non-trivial involution of \(\Delta\) provided \(\Delta\) is either of type \(A_n\) with \(n \geq 2\), or \(\mathbb{D}_n\) with \(n\) odd, or \(\mathbb{E}_6\); and \(\phi_\Delta\) is the identity on \(\Delta\) for the remaining Dynkin graphs. We note that the automorphism of \(\mathbb{Z}\Delta\) induced by \(\nu\) is sometimes called a Nakayama permutation ([7] 6.5). We also remark that the quiver \(\mathbb{Z}\Delta\) is isomorphic to the quiver \(\mathbb{Z}Q\) defined in [12].

The almost split triangles in \(\mathcal{D}^b(Q)\) are parameterized by the meshes in \(\mathbb{Z}\Delta\). More precisely, there is an almost split triangle of the form

\[
\text{AR}(m_{p,a}): v_{p-1,a} \to \bigoplus_{b \in a^-} v_{p,b} \to v_{p+1,a} \to v_{p-1,a}[1],
\]

where \(v_{p-1,a}[1] = v_{p+n_\Delta - 1, \phi_\Delta(a)}\), for any mesh \(m_{p,a} \in (\mathbb{Z}\Delta)_2\).

We note that the paths in \(\mathbb{Z}\Delta\) have the form

\[
\omega: v_{p,a} \to v_{p+1,a} \to \cdots \to v_{q-1,a} \to v_{q,a},
\]

where each two consecutive vertices in the sequence \((a_p, a_{p+1}, \ldots, a_{q-1}, a_q)\) are adjacent. Therefore \(\omega\) can be viewed as a lifting of a walk in \(\Delta\). In particular, \(q - p \geq \text{dist}(a_p, a_q)\).

The path \(\omega\) is called sectional if \(a_{i-1} \neq a_{i+1}\) for any integer \(i\) with \(p < i < q\). Since \(\Delta\) is a tree, this condition is equivalent to the fact that the vertices \(a_p, \ldots, a_q\) are pairwise different, and also equivalent to the equality \(q - p = \text{dist}(a_p, a_q)\).

**Lemma 2.2** Let \(v_{p,a}\) and \(v_{q,b}\) be vertices in \(\mathbb{Z}\Delta\). Then:

1. \([v_{p,a}, v_{p,a}] = 1\).
2. \([v_{p,a}, v_{q,b}] = [v_{q,b}, v_{p+n_\Delta - 2, \phi_\Delta(a)}]\).
3. If \([v_{p,a}, v_{q,b}] > 0\) then \(p + \text{dist}(a, b) \leq q \leq p + n_\Delta - 2 - \text{dist}(b, \phi_\Delta(a))\).
4. \([v_{p,a}, v_{q,b}] \leq \min(h^A_a, h^B_b)\).

**Proof** (1) follows from Lemma 2.1(2), but can also be derived directly from the definition of the mesh category \(\text{lk}(\mathbb{Z}\Delta)\). (2) is a consequence of the Serre duality (2.3).

(3) If \([v_{p,a}, v_{q,b}] > 0\) then also \([v_{q,b}, v_{p+n_\Delta - 2, \phi_\Delta(a)}] > 0\), by (2). Hence there are paths in \(\mathbb{Z}\Delta\) from \(v_{p,a}\) to \(v_{q,b}\) and from \(v_{q,b}\) to \(v_{p+n_\Delta - 2, \phi_\Delta(a)}\).

(4) Since \(P_a = v_{p,a}, v_{p,a} = \tau^r P_a\) for some integer \(r\). By Lemma 2.1(4), \([v_{p,a}, v_{q,b}] \leq h^A_a\). The other inequality \([v_{p,a}, v_{q,b}] \leq h^B_b\) follows from the first one and (2). \(\square\)

Let us explain how using the above lemma and almost split sequences, we can calculate the dimension \([v_{p,a}, v_{q,b}]\) for all vertices \(v_{p,a}\) and \(v_{q,b}\). Namely, if \(q \leq p\) or \(q \geq p + n_\Delta - 1\) then \([v_{p,a}, v_{q,b}] = 0\) except \([v_{p,a}, v_{p,a}] = 1\). We obtain formulas in the remaining cases by induction on \(q\), using the following lemma.

**Lemma 2.3** Let \(v_{p,a}\) and \(v_{q,b}\) be vertices in \(\mathbb{Z}\Delta\) such that \(p < q < p + n_\Delta\). Then

\[
[v_{p,a}, v_{q,b}] = \sum_{c \in b^-} [v_{p,a}, v_{q-1,c}] - [v_{p,a}, v_{q-2,b}].
\]
Proof Applying the functor $\text{Hom}_{D^b(Q)}(v_p,a,?)$ to the triangle $\text{AR}(m_{q-1,b})$ we get the exact sequence
\[
\cdots \rightarrow \text{Hom}(v_p,a,v_{q-n} \Delta, \phi(b)) \rightarrow \text{Hom}(v_p,a,v_{q-2}b) \rightarrow \bigoplus_{c \in b^-} \text{Hom}(v_p,a,v_{q-1}c) \rightarrow \text{Hom}(v_p,a,v_{q,b}) \rightarrow \text{Hom}(v_p,a,v_{q+n} \Delta, \phi(b)) \rightarrow \cdots
\]
The two extreme homomorphism spaces are zero by Lemma 2.2(3), and the claim follows.

Applying the above to sectional paths we get the following.

Lemma 2.4 If $v_{p,a} \rightarrow v_{p+1,a+1} \rightarrow \cdots \rightarrow v_{q-1,a-1} \rightarrow v_{q,a}$ is a sectional path in $\mathbb{Z} \Delta$ then $[v_{p,a}, v_{q,a}] = 1$.

Proof The claim follows by induction on the length $(q - p)$ of the path, where the base step $q - p = 0$ follows from Lemma 2.2. For the induction step we apply Lemma 2.3 for $v_{p,a} = v_{p,a}$ and $v_{q,b} = v_{q,a}$, and use that there is no path in $\mathbb{Z} \Delta$ from $v_{p,a}$ to $v_{q-2,a}$, and if there is a path from $v_{p,a}$ to $v_{q-1,c}$ with $c \in (a_q)^-$ then $c = a_{q-1}$.

2.4 Defect Functions on Meshes

Throughout this subsection $Q$ is a Dynkin quiver with underlying graph $\Delta$. In particular, $Q_0 = \Delta_0$. Consider a triangle
\[
\sigma : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]
\]
and an object $X$ in $D^b(Q)$. There is a commutative diagram of the form
\[
\begin{array}{ccc}
\text{Hom}_{D^b(Q)}(X,C) & \xrightarrow{\text{Hom}_{D^b(Q)}(X,\gamma)} & \text{Hom}_{D^b(Q)}(X,A[1]) \\
D \text{Hom}_{D^b(Q)}(C[-1],\tau X) & & D \text{Hom}_{D^b(Q)}(A,\tau X)
\end{array}
\]
where the vertical arrows represent $k$-linear isomorphisms obtained by applying the Auslander-Reiten translation $\tau$ and the Serre duality (2.3). This motivates the definition of the following integer-valued function measuring how far a triangle is from being split.

Definition 2.5 Given a triangle $\sigma : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ in $D^b(Q)$ we define a non-negative function
\[
\delta_\sigma : (\mathbb{Z} \Delta)_2 \rightarrow \mathbb{Z}, \quad \delta_\sigma(m_{p,a}) = \text{rk}(\text{Hom}_{D^b(Q)}(v_{p+1,a}, \gamma)) = \text{rk}(\text{Hom}_{D^b(Q)}(\gamma[-1], v_{p-1,a})).
\]

Some fundamental properties of $\delta_\sigma$, easy consequences of the definition, are collected in the following lemma.

Lemma 2.6 Let $\sigma : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ be a triangle in $D^b(Q)$. Then the following hold:
\begin{itemize}
\item[(1)] $\sigma$ splits if and only if $\delta_\sigma = 0$.
\item[(2)] If $B = 0$, then $\delta_\sigma(m_{p,a}) = [A, v_{p-1,a}] = [v_{p+1,a}, C]$.
\end{itemize}
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(3) $\delta_\sigma(m_{p,a}) \leq [A, v_{p-1,a}]$ and $\delta_\sigma(m_{p,a}) \leq [v_{p+1,a}, C]$.

(4) $\delta_\sigma' \leq \delta_\sigma$ if $\sigma'$ is a pullback or a pushout of $\sigma$.

(5) If $\sigma = AR(m_{q,b})$, then

$$\delta_\sigma(m_{p,a}) = \begin{cases} 1, & (p, a) = (q, b), \\
0, & \text{otherwise.} \end{cases}$$

Example 2.7 Let $\Delta$ be the Dynkin graph $\mathbb{D}_6$:

Thus $n_\Delta = 10$ and the quiver $\mathbb{Z}\Delta$ has the form

We consider the triangle $\sigma : A \to 0 \to A[1] \to A[1]$ with $A = v_{3,b_1}$. Then $A[1] = v_{13,b_1}$ and $\delta_\sigma(m_{q,b}) = [v_{3,b_1}, v_{q-1,b}] = [v_{q+1,b}, v_{13,b_1}]$, by Lemma 2.6(2). We find $\delta_\sigma$ by calculating the dimensions $[v_{3,b_1}, v_{q-1,b}]$, which can be done by the method based on Lemma 2.3. We illustrate the function $\delta_\sigma$ by writing each non-zero value $\delta_\sigma(m_{p,a})$ between the vertices $v_{p-1,a}$ and $v_{p+1,a}$:

Here we replace the arrows by edges, and additionally draw dashed segments between $v_{p-1,a}$ and $v_{p+1,a}$ if the vertex $a$ has degree 1 in $\Delta$.

We set $\langle X, Y \rangle = \sum_{i \leq 0} (-1)^i \cdot [X, Y]^i$ for any objects $X$ and $Y$ in $\mathcal{D}^b(Q)$. Our next aim is to define integer-valued functions on the set of meshes, using $\langle X, Y \rangle$. We derive from Lemma 2.2(3) the following fact.

Corollary 2.8 Assume $(p, a)$ and $(q, b)$ belong to $(\mathbb{Z} \times \Delta_0)^\circ$. If $\langle v_{p,a}, v_{q,b} \rangle \neq 0$ then $q - p \geq \text{dist}(a, b)$.

Lemma 2.9 Let $X, M, N \in \mathcal{D}^b(Q)$ with $\text{dim}M = \text{dim}N$. Then

$$\langle X, N \rangle - \langle X, M \rangle = \langle N, \tau X \rangle - \langle M, \tau X \rangle.$$
Proof By Eq. 2.3, \([N, \tau X] = [N, \tau (X[i])] = [v^{-1}\tau (X[i]), N] = [X[i - 1], N] = [X, N]^{i-1}\), for any integer \(i\). Consequently, 
\[
\langle X, N \rangle - \langle N, \tau X \rangle = b_Q(\text{dim}X, \text{dim}N) = b_Q(\text{dim}X, \text{dim}M) = \langle X, M \rangle - \langle M, \tau X \rangle. \]

Definition 2.10 Let \(M, N \in \mathcal{D}^b(Q)\) be such that \(\text{dim}M = \text{dim}N\). We define the function \(\delta_{M,N}\) : \((\mathbb{Z}/\Delta_1)^2 \rightarrow \mathbb{Z}\) by

\[
\delta_{M,N}(m_{p,a}) = \langle v_{p+1,a}, N \rangle - \langle v_{p+1,a}, M \rangle = \langle N, v_{p-1,a} \rangle - \langle M, v_{p-1,a} \rangle,
\]

for any mesh \(m_{p,a} \in (\mathbb{Z}/\Delta_1)^2\).

By Corollary 2.8 we conclude the following fact.

Corollary 2.11 Let \(M, N \in \mathcal{D}^b(Q)\) with \(\text{dim}M = \text{dim}N\). Then \(\delta_{M,N}(m_{p,a}) \neq 0\) only for finitely many meshes \(m_{p,a} \in (\mathbb{Z}/\Delta_1)^2\).

Applying Hom functors we get the following.

Corollary 2.12 \(\delta_{\sigma} = \delta_{B,A@C}\) for any triangle \(\sigma : A \rightarrow B \rightarrow C \rightarrow A[1]\) in \(\mathcal{D}^b(Q)\).

Combining the above corollary and Lemma 2.6(5) we get the following fact.

Lemma 2.13 Let \((p, a)\) and \((q, b)\) belong to \((\mathbb{Z} \times \Delta_0)\). Then

\[
\langle v_{p,a} \oplus v_{p+2,a}, v_{q,b} \rangle - \bigoplus_{c \in a^-} v_{p+1,c} \oplus v_{q,b} = \langle v_{q,b}, v_{p-2,a} \oplus v_{p,a} \rangle - \bigoplus_{c \in a^-} v_{p-1,c} = \begin{cases} 1, & (q, b) = (p, a), \\ 0, & \text{otherwise}. \end{cases}
\]

Let \(N\) and \(X\) be objects of \(\mathcal{D}^b(Q)\) and assume that \(X\) is indecomposable. We denote by \(\text{mult}_X(N)\) the multiplicity of \(X\) as a direct summand of \(N\). In particular,

\[
N \simeq \bigoplus_{(p,a) \in (\mathbb{Z} \times \Delta_0)} (v_{p,a})^{\text{mult}_{p,a}(N)}.
\]

As an immediate consequence of Lemma 2.13 we get:

Corollary 2.14 For any object \(N\) of \(\mathcal{D}^b(Q)\) and \((p, a) \in (\mathbb{Z} \times \Delta_0)\)

\[
\text{mult}_{v_{p,a}}(N) = \langle v_{p,a}, N \rangle - \sum_{b \in a^-} \langle v_{p+1,b}, N \rangle + \langle v_{p+2,a}, N \rangle
\]

\[
= \langle N, v_{p-2,a} \rangle - \sum_{b \in a^-} \langle N, v_{p-1,b} \rangle + \langle N, v_{p,a} \rangle.
\]

Corollary 2.15 Let \(M, N \in \mathcal{D}^b(Q)\) with \(\text{dim}M = \text{dim}N\) and \(v_{p,a} \in (\mathbb{Z}/\Delta_0)\). Then

\[
\text{mult}_{v_{p,a}}(N) - \text{mult}_{v_{p,a}}(M) = \delta_{M,N}(m_{p-1,a}) - \sum_{b \in a^-} \delta_{M,N}(m_{p,b}) + \delta_{M,N}(m_{p+1,a}).
\]

Applying the above corollary for the vertices lying on a sectional path we obtain the following fact.
Corollary 2.16 Let $M, N \in D^b(Q)$ with $\dim M = \dim N$ and 
\[ v_{p,a_p} \rightarrow v_{p+1,a_{p+1}} \rightarrow \cdots \rightarrow v_{q-1,a_{q-1}} \rightarrow v_{q,a_q} \]
be a sectional path in $\mathbb{Z} \Delta$. Let $\mathcal{M}$ be the subset of $(\mathbb{Z} \times \Delta_0)^m$ consisting of the pairs $(j, b)$ such that $p \leq j \leq q$ and $b$ is adjacent to $a_j$, but does not belong to the set $\{a_p, \ldots, a_q\}$. Then
\[
\sum_{i=p}^{q} (\text{mult}_{v_{i,a_i}}(N) - \text{mult}_{v_{i,a_i}}(M)) = \delta_{M,N}(m_{p-1,a_p}) - \sum_{(j,b) \in \mathcal{M}} \delta_{M,N}(m_{j,b}) + \delta_{M,N}(m_{q+1,a_q}).
\]

Observe in the above situation that if $b$ is adjacent to $a_j$, then $b$ belongs to the set $\{a_p, \ldots, a_q\}$ if and only if either $j > p$ and $b = a_{j-1}$ or $j < q$ and $b = a_{j+1}$.

Given a non-negative function $\delta : (\mathbb{Z} \Delta)^2 \rightarrow \mathbb{Z}$, for instance $\delta_{\sigma}$ for a triangle $\sigma$, we define its support
\[ \text{supp}(\delta) = \{ m \in (\mathbb{Z} \Delta)^2 ; \delta(m) > 0 \}. \]

2.5 Application to Type $D$

Throughout this subsection $Q$ is a Dynkin quiver of type $D_n$, $n \geq 4$, with underlying graph $\Delta$:

\[ c' \quad b_0 \quad b_1 \quad \cdots \quad b_{n-5} \quad b_{n-4} \quad c \]

In particular, $h_a^c = 1$ if $a \in \{c, c', c''\}$, and $h_a^c = 2$ otherwise. Applying Lemmas 2.1(4) and 2.6(3), we get the following corollaries.

Corollary 2.17 We have $[v_{p,a}, v_{q,b}] \leq 2$. Moreover, if $a$ or $b$ belongs to $\{c, c', c''\}$, then $[v_{p,a}, v_{q,b}] \leq 1$.

Corollary 2.18 Let $\sigma : A \rightarrow B \rightarrow C \rightarrow A[1]$ be a triangle in $D^b(Q)$ with $A$ or $C$ indecomposable. Then $\delta_{\sigma}(m_{p,a}) \leq 2$ and the inequality is strict if $a$ belongs to $\{c, c', c''\}$.

The main aim of this subsection is to prove the following fact, which is the key to the proof of Theorem 1.3.

Proposition 2.19 Let $\sigma : A \rightarrow B \rightarrow C \rightarrow A[1]$ be a triangle in $D^b(Q)$ with $A$ and $C$ indecomposable. Assume that $M, N \in D^b(Q)$ satisfy $\dim M = \dim N$, $\delta_{M,N} \geq 0$, $\text{supp}(\delta_{\sigma}) \subseteq \text{supp}(\delta_{M,N})$, but the inequality $\delta_{M,N} \geq \delta_{\sigma}$ does not hold.

Then there is an indecomposable direct summand $C'$ of $N$ together with a morphism $h : C' \rightarrow C$ such that the pullback $\sigma' : A \rightarrow B' \rightarrow C' \rightarrow A[1]$ of $\sigma$ along $h$ does not split, $\delta_{\sigma'} \leq \delta_{M,N}$ and $\text{supp}(\delta_{\sigma} - \delta_{\sigma'}) \subseteq \text{supp}(\delta_{M,N} - \delta_{\sigma'})$.

We introduce two integer-valued functions $\varphi$ and $\psi$ on $(\mathbb{Z} \Delta)^2$ as the compositions of the canonical bijection $(\mathbb{Z} \Delta)^2 \rightarrow \mathbb{Z} \times \Delta_0$ followed by the maps
\[ (p, a) \mapsto p + \text{dist}(c, a) \quad \text{and} \quad (p, a) \mapsto p - \text{dist}(c, a), \]
respectively.
Lemma 2.20 Let $\sigma : A \rightarrow B \rightarrow C \rightarrow A[1]$ be a triangle in $D^b(Q)$ such that $A$ and $C$ are indecomposable. Let $m$ be a mesh in $\mathbb{Z}\Delta$ such that $\delta_\sigma(m) = 2$ (in particular, $m = m_{p_0,b_r}$ for some pair $(p_0,b_r)$ in $(\mathbb{Z} \times \Delta_0)m$). Then

$$\delta_\sigma(m_{p,a}) = h_a^\delta$$

for all meshes $m_{p,a}$ satisfying $\varphi(m_{p,a}) \geq \varphi(m)$ and $\psi(m_{p,a}) \leq \psi(m)$.

We illustrate the above statement about the function $\delta_\sigma$ for $r = 2$ (hence $n \geq 6$) by the following picture

![Diagram](image.png)

**Proof** The claim follows by induction on $r \geq 0$ from the following two properties of $\delta_\sigma$:

(i) If $r > 0$ then $\delta_\sigma(m_{p_0-1,b_r-1}) = \delta_\sigma(m_{p_0+1,b_r-1}) = 2$.

(ii) If $r = 0$ then $\delta_\sigma(m_{p_0-1,c_r}) = \delta_\sigma(m_{p_0+1,c_r}) = \delta_\sigma(m_{p_0+1,c_r}) = 1$.

Indeed, the base step follows from (ii) and the induction step from (i).

Combining the assumption $\delta_\sigma(m_{p_0,b_r}) = 2$ with Lemma 2.6(3) and Corollary 2.17 gives the equalities $[A, v_{p_0-1,b_r}] = 2 = [v_{p_0+1,b_r}, C]$. Since $A$ and $C$ are indecomposable, $[v_{p_0+1,b_r}, A] = 0 = [C, v_{p_0-1,b_r}]$, by Lemma 2.2(3).

Let $\mathcal{L}$ be the set of vertices lying on the following sectional path in $\mathbb{Z}\Delta$:

$$v_{p_0-n+r+2,c} \rightarrow v_{p_0-n+r+3,b_n-4} \rightarrow \cdots \rightarrow v_{p_0-1,b_r}.$$ 

Let $p' = p_0 - n + r + 1$. Applying Corollary 2.16 for $v \in \mathcal{L}$ we get

$$\sum_{v \in \mathcal{L}} (\text{mult}_v(A \oplus C) - \text{mult}_v(B)) = \delta_\sigma(m_{p',c_r}) + \delta_\sigma(m_{p_0,b_r}) - \begin{cases} \delta_\sigma(m_{p_0-1,b_r-1}), & r > 0, \\ \delta_\sigma(m_{p_0-1,c_r}) + \delta_\sigma(m_{p_0-1,c_r}), & r = 0. \end{cases}$$

For any $v \in \mathcal{L}$, $[v, v_{p_0-1,b_r}] = 1$, by Lemma 2.4, thus $v$ is isomorphic neither to $A$ nor to $C$, and consequently $\text{mult}_v(A \oplus C) = 0$. Remembering that $\delta_\sigma(m_{p_0,b_r}) = 2$, we get

$$2 \leq \begin{cases} \delta_\sigma(m_{p_0-1,b_r-1}), & r > 0, \\ \delta_\sigma(m_{p_0-1,c_r}) + \delta_\sigma(m_{p_0-1,c_r}), & r = 0. \end{cases}$$

On the other hand, by Corollary 2.18, $\delta_\sigma(m_{p_0-1,b_r-1}) \leq 2$, $\delta_\sigma(m_{p_0-1,c_r}) \leq 1$ and $\delta_\sigma(m_{p_0-1,c_r}) \leq 1$. Consequently, we get three equalities of the six equalities appearing in (i) and (ii).

Dual considerations for the following sectional path in $\mathbb{Z}\Delta$:

$$v_{p_0+1,b_r} \rightarrow \cdots \rightarrow v_{p_0+n-r-3,b_n-4} \rightarrow v_{p_0+n-r-2,c}$$

lead to the remaining three equalities in (i) and (ii). $\square$

**Proof of Proposition 2.19.** Let $m_{p_0,a_0}$ be a mesh satisfying $\delta_{M,N}(m_{p_0,a_0}) < \delta_\sigma(m_{p_0,a_0})$. By Corollary 2.11, we can choose $m_{p_0,a_0}$ such that the value $\psi(m_{p_0,a_0})$ is minimal. We conclude from the assumption $\text{supp}(\delta_\sigma) \subseteq \text{supp}(\delta_{M,N})$ and Corollary 2.18 that...
The key observation is that by Corollary 2.15 we get the formula
\[
\sum_{\nu_{p,a} \in \mathcal{R}} h_{\sigma}^\Delta \cdot (\text{mult}_{\nu_{p,a}}(N) - \text{mult}_{\nu_{p,a}}(M)) = -2 \cdot \delta_{M,N}(\mathbf{m}_{p_0,b_r})
\]
\[+ \delta_{M,N}(\mathbf{m}_{p_0-r-1,c'}) + \delta_{M,N}(\mathbf{m}_{p_0-r-1,c''}) + \delta_{M,N}(\mathbf{m}_{p_0+r-1,c'}) + \delta_{M,N}(\mathbf{m}_{p_0+r+1,c''}).\]

Combining Lemma 2.20 with the assumption supp($\delta_{\sigma}$) \(\subseteq\) supp($\delta_{M,N}$) and using $\delta_{M,N}(\mathbf{m}_{p_0,b_r}) = 1$, we get that the right-hand side is at least 2. Hence mult$_C(N) > 0$ for some $C' = \nu_{p',a'} \in \mathcal{R}$. Again by Lemma 2.20, $\delta_{\sigma}((\mathbf{m}_{p'-1,a'}) > 0$, which from the definition of $\delta_{\sigma}$ means that $\gamma \circ h \neq 0$ for some morphism $h: C' \to C$. Let $\sigma'$ be the pullback of $\sigma$ along $h$. We need to prove that $\delta_{\sigma'} \leq \delta_{M,N}$ and supp($\delta_{\sigma} - \delta_{\sigma'}$) \(\subseteq\) supp($\delta_{M,N} - \delta_{\sigma'}$).

Since $\delta_{\sigma'} \leq \delta_{\sigma}$ and supp($\delta_{\sigma}$) \(\subseteq\) supp($\delta_{M,N}$), it suffices to show that $\delta_{M,N}(\mathbf{m}_{p,a}) \geq \delta_{\sigma}(\mathbf{m}_{p,a})$ whenever $\delta_{\sigma}(\mathbf{m}_{p,a}) > 0$. Thus we assume that $\delta_{\sigma}(\mathbf{m}_{p,a}) > 0$. By Lemma 2.6(3), $[\nu_{p+1,a}, C'] > 0$, and from Lemma 2.2(3) we conclude that $\psi(\nu_{p+1,a}) \leq \psi(C')$. Using the fact that $C'$ belongs to $\mathcal{R}$ and how the latter was defined, we get the following sequence of inequalities
\[
\psi(\mathbf{m}_{p,a}) < \psi(\nu_{p+1,a}) \leq \psi(C') < \psi(\mathbf{m}_{p_0,b_r}).
\]
It follows from our choice of the mesh $\mathbf{m}_{p_0,b_r}$ that $\delta_{M,N}(\mathbf{m}_{p,a}) \geq \delta_{\sigma}(\mathbf{m}_{p,a})$, which finishes the proof.

### 2.6 Passage from $\mathcal{D}^b(Q)$ to rep($Q$)

The main aim of this subsection is to prove a result analogous to Proposition 2.19, concerning the category rep($Q$), where $Q$ is a Dynkin quiver of type $\mathbb{D}$. Throughout this subsection $Q$ is a Dynkin quiver with underlying graph $\Delta$.

As observed in Section 2.3, the $\mathbb{k}$-linear structure of the category $\mathcal{D}^b(Q)$ is fully described by the quiver $\mathbb{Z}\Delta$. Similarly, the category rep($Q$) is fully described by its Auslander-Reiten quiver $\Gamma_Q$. Moreover, the identification of rep($Q$) as a full subcategory of $\mathcal{D}^b(Q)$ corresponds to the identification of $\Gamma_Q$ as a full convex subquiver of $\mathbb{Z}\Delta$, which we are going to explain. We note that introducing the Auslander-Reiten quiver $\Gamma_Q$ as a subquiver of $\mathbb{Z}\Delta$ was done already in [7] 6.5.

By a slice in $\mathbb{Z}\Delta$ we mean a full convex subquiver containing exactly one vertex $\nu_{r,a}$ for each $a \in \Delta_0$. Thus $|r_a - r_b| = 1$ for any adjacent vertices $a$ and $b$. Recall that $P_a = \nu_{p_a,a}$ for any vertex $a \in Q_0 = \Delta_0$. The vertices $P_a, a \in Q_0, together with the arrows connecting them form a slice $\mathcal{S}$ isomorphic to $Q^{op}$, where $Q^{op}$ is the opposite quiver of $Q$ having the
same set of vertices, but with the arrows reversed. Consequently, the vertices \( I_a = v(P_a) = v_{p_{a + \Delta - 2, \Delta}(a)}, a \in Q_0 \), lie on a slice \( vS \), which is also isomorphic to \( Q^{\text{op}} \). Then the Auslander-Reiten quiver \( \Gamma_Q \) is the smallest full convex subquiver of \( Z \Delta \) containing \( S \) and \( vS \). We denote by \( (\Gamma_Q)_2 \) the set of all meshes in \( Z \Delta \) which are contained in \( \Gamma_Q \).

The shifts \( (\Gamma_Q)[i], i \in \mathbb{Z}, \) are pairwise disjoint subquivers of \( Z \Delta \), hence we have the following inclusion

\[ \bigcup_{i \in \mathbb{Z}} (\Gamma_Q)[i] \subseteq Z \Delta. \]

In fact, this inclusion is the equality on the sets of vertices, and only the arrows connecting \( vS[i] \) with \( S[i + 1], i \in \mathbb{Z}, \) are missing (see for instance [8] I.5.5). For example, if \( Q \) is the quiver

\[
\begin{array}{ccc}
  & b & \\
  a & \rightarrow & c & \leftarrow & d
\end{array}
\]

then the above embedding of quivers looks as follows.

\[
\begin{array}{ccc}
  & & b & \\
  a & \rightarrow & c & \leftarrow & d
\end{array}
\]

\[
\begin{array}{ccc}
  & \Gamma_Q[-1] & \\
  P_a & \rightarrow & P_b
\end{array}
\]

\[
\begin{array}{ccc}
  & \Gamma_Q & \\
  P_c & \rightarrow & P_d
\end{array}
\]

\[
\begin{array}{ccc}
  & \Gamma_Q[1] & \\
  I_a & \rightarrow & I_d
\end{array}
\]

Let \( X \) and \( Y \) be representations in \( \text{rep}(Q) \). Then \( [X, Y]^i = 0 \) for \( i < 0 \), and hence \( (X, Y) = [X, Y] \). Therefore the definition of the integer-valued function \( \delta_{M,N} \) simplifies if we restrict to the subcategory \( \text{rep}(Q) \) as the following result explains.

**Lemma 2.21** Let \( Q \) be a Dynkin quiver. Assume that \( M \) and \( N \) belong to \( \text{rep}(Q) \) and \( \text{dim}M = \text{dim}N \). Let \( m_{p,a} \) be a mesh in \( (Z \Delta)_2 \). Then

\[
\delta_{M,N}(m_{p,a}) = [v_{p+1,a}, N] - [v_{p+1,a}, M] = [N, v_{p-1,a}] - [M, v_{p-1,a}]
\]

if \( m_{p,a} \) belongs to \( (\Gamma_Q)_2 \), and \( \delta_{M,N}(m_{p,a}) = 0 \), otherwise.

Recall that for an exact sequence \( \sigma : 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \) in \( \text{rep}(Q) \) we have the corresponding triangle \( \tilde{\sigma} : A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1] \) in \( D^b(Q) \). We set \( \delta_\sigma = \delta_{\tilde{\sigma}} \). In this case Definition 2.5 and Corollary 2.12 simplify as shown in the following lemma. We also note that the function \( \delta_\sigma \) is closely related to the defect functors considered in [2] IV.4.

**Lemma 2.22** Let \( \sigma : 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \) be a short exact sequence in \( \text{rep}(Q) \) and \( m_{p,a} \) be any mesh in \( Z \Delta \). Then \( \delta_\sigma(m_{p,a}) = 0 \) if \( m_{p,a} \) does not belong to \( (\Gamma_Q)_2 \). Otherwise,

\[
\delta_\sigma(m_{p,a}) = [v_{p+1,a}, A \oplus C] - [v_{p+1,a}, B] = [A \oplus C, v_{p-1,a}] - [B, v_{p-1,a}]
\]

equals the ranks of the last \( \mathbb{K} \)-linear morphisms in the induced exact sequences:

\[
0 \rightarrow \text{Hom}_Q(v_{p+1,a}, A) \rightarrow \text{Hom}_Q(v_{p+1,a}, B) \rightarrow \text{Hom}_Q(v_{p+1,a}, C) \rightarrow \text{Ext}_Q^1(v_{p+1,a}, A)
\]

and

\[
0 \rightarrow \text{Hom}_Q(C, v_{p-1,a}) \rightarrow \text{Hom}_Q(B, v_{p-1,a}) \rightarrow \text{Hom}_Q(A, v_{p-1,a}) \rightarrow \text{Ext}_Q^1(C, v_{p-1,a}).
\]
We are ready to restrict Proposition 2.19 to the category $\text{rep}(Q)$.

**Corollary 2.23** Let $Q$ be a Dynkin quiver of type $\mathbb{D}$ and $\sigma : 0 \to A \to B \to C \to 0$ be a short exact sequence in $\text{rep}(Q)$ with $A$ and $C$ indecomposable. Assume that $M, N \in \text{rep}(Q)$ satisfy $\dim M = \dim N$, $\delta_{M,N} \geq 0$, $\text{supp}(\delta_{\sigma}) \subseteq \text{supp}(\delta_{M,N})$, but the inequality $\delta_{M,N} \geq \delta_{\sigma}$ does not hold.

Then there is an indecomposable direct summand $C'$ of $N$ together with a homomorphism $h : C' \to C$ such that the pullback $\sigma' : A \to B' \to C' \to 0$ of $\sigma$ along $h$ does not split, $\delta_{\sigma'} \leq \delta_{M,N}$ and $\text{supp}(\delta_{\sigma} - \delta_{\sigma'}) \subseteq \text{supp}(\delta_{M,N} - \delta_{\sigma'})$.

### 3 Schemes of Representations of Quivers

The set $\text{rep}^d_Q(R)$, where $R$ is a commutative $\mathbb{k}$-algebra, has a natural structure of a vector space over $\mathbb{k}$, and using the addition simplifies working with elements of this set. In particular, any element of $\text{rep}^d_Q(\mathbb{k}[\varepsilon])$ can be uniquely presented in the form $N + \varepsilon \cdot Z$, where $N$ and $Z$ belong to $\text{rep}^d_Q(\mathbb{k})$. At the same time $Z$ is viewed as a tangent vector in the Zariski tangent space $T_N \text{rep}^d_Q$. Hence $T_N \text{rep}^d_Q = \text{rep}^d_Q(\mathbb{k})$ for any $N \in \text{rep}^d_Q(\mathbb{k})$. We start this section with a more adequate representation-theoretic interpretation of elements of $T_N \text{rep}^d_Q$.

#### 3.1 Tangent Vectors and Short Exact Sequences

Throughout this subsection $Q$ is a finite quiver. If $U = (U_a; a \in Q_0)$ and $V = (V_a; a \in Q_0)$ are collections of finite dimensional vector spaces indexed by the vertices of $Q$, then we define $\mathbb{k}$-schemes $\mathcal{V}_V^U$ and $\mathcal{A}_V^U$ by

$$
\mathcal{V}_V^U(R) = \prod_{a \in Q_0} \text{Hom}_R(R \otimes U_a, R \otimes V_a) \quad \text{and} \quad \mathcal{A}_V^U(R) = \prod_{a \in Q_1} \text{Hom}_R(R \otimes U_{sa}, R \otimes V_{ta}),
$$

where all tensor products in this section are taken over $\mathbb{k}$. In particular, if $\mathbb{k}^d = (\mathbb{k}^d_a)$ and we identify the $R$-homomorphisms of the form $R \otimes \mathbb{k}^d \to R \otimes \mathbb{k}^e$ with the corresponding matrices in $M_{e \times d}(R)$, then

$$
\mathcal{A}_\mathbb{k}^d = \text{rep}^d_Q.
$$

Similarly $\text{GL}_d$ is an open subscheme of $\mathcal{V}_\mathbb{k}^d_a$.

If $h \in \mathcal{V}_V^U(R)$ and $Z \in \mathcal{A}_V^U(R)$, then we define $Z \circ h \in \mathcal{A}_V^U(R)$ by

$$
(Z \circ h)_a = Z_a \circ h_{sa},
$$

for any $a \in Q_1$. We define $\circ : \mathcal{V}_V^U(R) \times \mathcal{A}_V^U(R) \to \mathcal{A}_V^U(R)$ dually. Finally we have $\circ : \mathcal{V}_V^U(R) \times \mathcal{V}_V^U(R) \to \mathcal{V}_V^U(R)$ defined in the obvious way. We note that the action $\star$ of $\text{GL}_d \subseteq \mathcal{V}_\mathbb{k}^d_a$ on $\text{rep}^d_Q = \mathcal{A}_\mathbb{k}^d$ can be written as $g \star M = g \circ M \circ g^{-1}$.

If $U = \bigoplus_{s \in S} U_s^a$ (i.e. $U_a = \bigoplus_{s \in S} U_a^s$ for each $a \in Q_0$) and $V = \bigoplus_{s' \in S'} V_{s'}$, then

$$
\mathcal{V}_V^U = \prod_{s \in S, s' \in S'} \mathcal{V}_{V_{s'}}^{U_s} \quad \text{and} \quad \mathcal{A}_V^U = \prod_{s \in S, s' \in S'} \mathcal{A}_{V_{s'}}^{U_s}.
$$

For $h \in \mathcal{V}_V^U(R)$, $q \in S$ and $p \in S'$, we denote by $h^{p,q}$ the image of $h$ under the projection $\mathcal{V}_V^U(R) \to \mathcal{V}_{V_{p}}^{U_{q}}(R)$ induced by the former of the above decompositions. Conversely, given $h^{p,q} \in \mathcal{V}_{V_{p}}^{U_{q}}(R)$, then $h^{p,q}$ denotes the image of $h^{p,q}$ under the section $\mathcal{V}_{V_{p}}^{U_{q}}(R) \to \mathcal{V}_V^U(R)$. 

\[ \square \]
We define $Z^{p,q} \in \mathcal{A}_{V, p}^U(R)$, for $Z \in \mathcal{A}_V^U(R)$, and $\tilde{Z}^{p,q} \in \mathcal{A}_V^U(R)$, for $Z^{p,q} \in \mathcal{A}_{V, p}^U(R)$, analogously. In particular,

$$h = \sum_{s \in S, s' \in S'} h^{s, s'}$$

and

$$Z = \sum_{s \in S, s' \in S'} \tilde{Z}^{s, s'}.$$  \hspace{1cm} (3.3)

If $U = (U_a, U_\alpha)$ and $V = (V_a, V_\alpha)$ are representations in $\text{rep}(Q)$, then we put

$$\mathcal{V}_V^U = \mathcal{V}_{(V_a)}^U \quad \text{and} \quad \mathcal{A}_V^U = \mathcal{A}_{(V_a)}^U.$$  

Moreover, we define $Z_Q^1(U, V) = \mathcal{A}_V^U(\mathbb{k})$. Since $\text{Hom}_Q(U, V) \subseteq \mathcal{V}_V^U(\mathbb{k})$, we note that using the compositions defined above we obtain that $Z_Q^1(?, ?)$ is a functor from $\text{rep}(Q)^{\text{op}} \times \text{rep}(Q)$ to $\text{mod} \mathbb{k}$. Observe that if $N \in \text{rep}_Q^d(\mathbb{k})$, then

$$Z_Q^1(N, N) = \mathcal{A}_N^k(\mathbb{k}) = \text{rep}_Q^d(\mathbb{k}) = T_N \text{rep}_Q^d,$$  \hspace{1cm} (3.4)

and we will view the elements of $Z_Q^1(N, N)$ as tangent vectors.

For representations $U$ and $V$ in $\text{rep}(Q)$ and $Z \in Z_Q^1(V, U)$, let $W = W(U, Z, V)$ denote the representation such that $W_a = U_a \oplus V_a$ and

$$W_a: U_{\alpha a} \oplus V_{\alpha a} \xrightarrow{\begin{bmatrix} U_a Z_a \\ 0 \end{bmatrix}} U_{\alpha a} \oplus V_{\alpha a}.$$  

Let $\varphi: U \rightarrow W$ and $\psi: W \rightarrow V$ denote the homomorphisms such that $\varphi_a: U_a \rightarrow U_a \oplus V_a$ is the canonical section and $\psi_a: U_a \oplus V_a \rightarrow V_a$ is the canonical projection, for any $a \in Q_0$. Then

$$\sigma(U, Z, V): 0 \rightarrow U \xrightarrow{\varphi} W(U, Z, V) \xrightarrow{\psi} V \rightarrow 0$$

is a short exact sequence in $\text{rep}(Q)$. If it causes no confusion, we will present the sequence $\sigma(U, Z, V)$ symbolically in the form

$$0 \rightarrow U \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} U \\ Z \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} V \rightarrow 0.$$  

Moreover, if $h \in \text{Hom}_Q(V', V)$ then $\sigma(U, Z, h, V')$ is the pullback of $\sigma(U, Z, V)$ along $h$ leading to the following commutative diagram

$$\begin{array}{cccc}
0 & \rightarrow & U & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} U \\ Z \end{bmatrix} & \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} V' & \rightarrow & 0 \\
0 & \rightarrow & U & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \begin{bmatrix} U \\ Z \end{bmatrix} & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix} h} V & \rightarrow & 0
\end{array}$$

Dually $\sigma(U', h' \circ Z, V)$ is the pushout of $\sigma(U, Z, V)$ along $h'$, for any homomorphism $h': U \rightarrow U'$.

Let $U$ and $V$ be representations in $\text{rep}(Q)$. We denote by $\mathbb{B}_Q^1(V, U)$ the image of the map

$$\eta_{V, U}: \mathcal{V}_V^U(\mathbb{k}) \rightarrow \mathcal{A}_V^U(\mathbb{k}) = Z_Q^1(V, U), \quad \eta_{V, U}(h) = h \circ (V_a) - (U_a) \circ h.$$  

Then $\mathbb{B}_Q^1(?, ?)$ is a $\mathbb{k}$-linear subfunctor of $Z_Q^1(?, ?)$. Moreover, $Z \in Z_Q^1(V, U)$ belongs to $\mathbb{B}_Q^1(V, U)$ if and only if the short exact sequence $\sigma(U, Z, V)$ splits. The quotient of $Z_Q^1(V, U)$ by $\mathbb{B}_Q^1(V, U)$ can be identified with the extension group $\text{Ext}_Q^1(V, U)$ of $V$ by $U$, and we have the following exact sequence

$$0 \rightarrow \text{Hom}_Q(V, U) \rightarrow \mathcal{V}_V^U(\mathbb{k}) \xrightarrow{\eta_{V, U}} \mathcal{A}_V^U(\mathbb{k}) \rightarrow \text{Ext}_Q^1(V, U) \rightarrow 0.$$
Applying the above for \( U = N = V \), where \( N \in \text{rep}_Q^d(\mathbb{k}) \), we get the exact sequence

\[
0 \to \text{End}_Q(N) \to \mathcal{Y}^{\text{ld}}_d(\mathbb{k}) \xrightarrow{\eta_{N,N}} \mathcal{A}^{\text{ld}}_d(\mathbb{k}) \to \text{Ext}^1_Q(N, N) \to 0.
\]

The space \( \mathcal{Y}^{\text{ld}}_d(\mathbb{k}) \) can be identified with the tangent space \( T_1 \text{GL}_d \), the space \( \mathcal{A}^{\text{ld}}_d(\mathbb{k}) = Z^1_Q(N, N) \) with the tangent space \( T_{N} \text{rep}_Q^d \), and \( \eta_{N,N} \) with the tangent map induced by the orbit map

\[
\text{GL}_d \to \text{rep}_Q^d, \quad g \mapsto g \star N.
\]

Under this identification, \( T_N \mathcal{O}_N = \mathbb{B}^1_Q(N, N) \) and the normal space \( T_N \text{rep}_Q^d / T_N \mathcal{O}_N \) at \( N \) to \( \mathcal{O}_N \) in \( \text{rep}_Q^d \) coincides with \( \text{Ext}^1_Q(N, N) \), which is a famous result by Voigt [16].

### 3.2 Proof of the Main Result

Throughout this subsection \( Q \) is a Dynkin quiver, \( M \) a representation in \( \text{rep}(Q) \) and \( d = \dim M \). We will work with closed subschemes of \( \text{rep}_Q^d \) and hence we start with a few general remarks on subschemes. If \( \mathcal{X} \) is a subscheme of a \( \mathbb{k} \)-scheme \( \mathcal{Y} \), then the corresponding map \( \mathcal{X}(R) \to \mathcal{Y}(R) \) is injective, and we will identify \( \mathcal{X}(R) \) with its image in \( \mathcal{Y}(R) \), for any commutative \( \mathbb{k} \)-algebra \( R \). The following fact can be concluded from [6] I.2.6.1:

**Lemma 3.1** Let \( \mathcal{X} \) be a closed subscheme of a \( \mathbb{k} \)-scheme \( \mathcal{Y} \) and \( \varphi : R \to S \) an injective homomorphism of commutative \( \mathbb{k} \)-algebras. Then \( \mathcal{X}(R) = \mathcal{Y}(\varphi)^{-1}(\mathcal{X}(S)) \).

We will use the above lemma several times for a \( \mathbb{k} \)-scheme \( \mathcal{Y} \) which is affine (specifically for \( \mathcal{Y} = \text{rep}_Q^d \)). Then the closed embedding \( \mathcal{X} \subseteq \mathcal{Y} \) is isomorphic to \( \text{Spec}(\psi) : \text{Spec}(A/I) \to \text{Spec}(A) \), where \( \psi : A \to A/I \) is the canonical surjective homomorphism, for some commutative \( \mathbb{k} \)-algebra \( A \) and an ideal \( I \). Hence the claim translates to an obvious fact about the existence of a homomorphism completing a given commutative diagram in the category of \( \mathbb{k} \)-algebras to another commutative diagram, as follows:

\[
\begin{array}{ccc}
A & \longrightarrow & R \\
\downarrow \psi & & \downarrow \varphi \\
A/I & \longrightarrow & S
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
A & \longrightarrow & R \\
\downarrow \psi & & \downarrow \varphi \\
A/I & \longrightarrow & S
\end{array}
\]

If \( \varphi : R \to S \) is an injective \( \mathbb{k} \)-algebra homomorphism, \( \mathcal{X} \) is a \( \mathbb{k} \)-scheme and \( x \in \mathcal{X}(R) \), it will be convenient to denote the image of \( x \) under the map \( \mathcal{X}(\varphi) : \mathcal{X}(R) \to \mathcal{X}(S) \) also by \( x \).
Let $\pi : \mathbb{k}[\varepsilon] \to k$ denote the canonical surjective homomorphism. Since $\overline{O}_M$ is a closed subscheme of $C_M$ and the latter is a closed subscheme of $\text{rep}^d_Q$, we have the following commutative diagram with inclusions:

$$
\begin{array}{ccc}
\overline{O}_M(\mathbb{k}[\varepsilon]) & \subseteq & C_M(\mathbb{k}[\varepsilon]) \\
\downarrow & & \downarrow \\
\overline{O}_M(k) & = & C_M(k) \subseteq \text{rep}^d_Q(k)
\end{array}
$$

where the equality in the second row follows from (a reformulation of) Theorem 1.1. Observe that $\overline{O}_M(\mathbb{k}[\varepsilon]) = C_M(\mathbb{k}[\varepsilon])$ if and only if $T_N\overline{O}_M = T_NC_M$ for all points $N \in \overline{O}_M(k)$. It follows from Theorem 1.1 and [14] Corollary 7.4 that

$$
\overline{O}_M(k) = C_M(k) = \{ N \in \text{rep}^d_Q(k); \ \delta_{M,N} \geq 0 \},
$$

(3.5)

$C_M(\mathbb{k}[\varepsilon]) = \{ N + \varepsilon \cdot Z \in \text{rep}^d_Q(\mathbb{k}[\varepsilon]); \ \delta_{M,N} \geq 0 \text{ and } \text{supp}(\delta_{\sigma(N,Z,N)}) \subseteq \text{supp}(\delta_{M,N}) \}$.

(3.6)

Let $N \in C_M(\mathbb{k})$. Given $U$ and $V$ in $\text{rep}(Q)$ we denote by $\mathbb{Z}^1_{M,N}(V, U)$ the subset of $\mathbb{Z}^1_Q(V, U)$ consisting of the elements $Z$ such that $\text{supp}(\delta_{\sigma(U,Z,V)}) \subseteq \text{supp}(\delta_{M,N})$. In particular,

$$
T_NC_M = \mathbb{Z}^1_{M,N}(N, N).
$$

It follows from [14, Section 7] that $\mathbb{Z}^1_{M,N}(?, ?, ?, ?)$ is a $\mathbb{k}$-linear subfunctor of $\mathbb{Z}^1_Q(?, ?, ?, ?)$ containing $\mathbb{B}^1_Q(?, ?, ?)$.

Let $N \in \text{rep}^d_Q(\mathbb{k})$. Assume that we have a fixed decomposition $N = \bigoplus_{s \in S} N^s$ as a representation of $Q$. In particular, the collection $(N_\alpha) = k^d$ decomposes as $\bigoplus_{s \in S} (N^s_\alpha)$. If $p, q \in S$, then using the notation introduced after Eq. 3.2 we have $H^{p,q} \in \mathcal{A}^q_{N^p}(k)$ for each $H \in \mathcal{A}^q_{k^d}(k)$. Equality (3.4) implies that we can apply the above notation both when $H = L$ is a point of $\text{rep}^d_Q(k)$, and when $H = Z$ is viewed as tangent vector. Conversely, given $H^{p,q} \in \mathcal{A}^q_{N^p}(k)$ we have $\overline{H}^{p,q} \in \mathcal{A}^q_{k^d}(k)$. Observe that $N^{p,p}$ is the collection $(N^p_\alpha)$ and $N^{p,q} = 0$, for all $p \neq q \in S$, hence $N = \sum_{s \in S} \mathbb{N}^s$. Analogously, applying the notation introduced after Eq. 3.2 for the scheme $\mathcal{V}^s_{k^d}$ we also have $g^{p,q} \in \mathcal{V}^q_{N^p}(R)$ for $g \in \text{GL}_d(R)$. Finally, if $s \in S$, we set $1^s$ for the identity on $N^s$, which is an element of $\mathcal{V}^s_{N^s}(\mathbb{k})$. Consequently, $\sum_{s \in S} 1^s$ is the identity on $(N_\alpha) = k^d$.

**Lemma 3.2** Let $N$ be a point in $\text{rep}^d_Q(k)$ with a fixed decomposition $N = \bigoplus_{s \in S} N^s$. Consider a representation $L = N + Z^{p,q}$ in $\text{rep}^d_Q(\mathbb{k})$ for $Z^{p,q} \in \mathcal{A}^q_{N^p}(k)$, then the following conditions hold:

1. $\delta_{L,N} = \delta_{\sigma(N^{p,p}, Z^{p,q}, N^{q,q})}$.
2. $L$ degenerates to $N$, i.e. $N \in \overline{O}_L(k)$.
3. $L \simeq N$ if and only if $Z^{p,q}$ belongs to $\mathbb{B}^1_Q(\mathbb{N}^q, N^{p,p})$.

**Proof** Since $L = \sum_{s \in S} \mathbb{N}^s + Z^{p,q}$, we have the following isomorphism of representations

$$
L \simeq \bigoplus_{s \neq p,q} N^s \oplus \left[ \begin{array}{c} N^p \\ 0 \end{array} \right] Z^{p,q} \right] ,
$$

\[ \bigoplus_{s \neq p,q} N^s \oplus \left[ \begin{array}{c} N^p \\ 0 \end{array} \right] Z^{p,q} \right] .
and (1) follows. In particular, \( \delta_{L,N} \geq 0 \), hence (2) holds, by Eq. 3.5.

(3). We have from (1) that \( L \) is isomorphic to \( N \) if and only if the sequence \( \sigma(N^p, Z^{p,q}, N^q) \) splits. The latter means that \( Z^{p,q} \) belongs to \( \mathbb{B}^1_Q(N^q, N^p) \).

\[ \sum_{s \in S} N^s \] be a fixed decomposition of a point \( N \) in \( \mathcal{O}_M(\mathbb{k}) \). Let \( Z^{p,q} \in A^N_{N^p}(\mathbb{k}) \), for \( p \neq q \) in \( S \), be such that \( \delta_{\sigma(N^p, Z^{p,q}, N^q)} \leq \delta_{M,N} \). Then

\[ N + Z^{p,q} \in \mathcal{O}_M(\mathbb{k}) \quad \text{and} \quad N + \varepsilon \cdot Z^{p,q} \in \mathcal{O}_M(\mathbb{k}[\varepsilon]) \] (equivalently, \( Z^{p,q} \in T_N(\mathcal{O}_M) \)).

**Proof** Let \( L = N + Z^{p,q} \). By Lemma 3.2(1), \( \delta_{L,N} = \delta_{\sigma(N^p, Z^{p,q}, N^q)} \) and \( \delta_{M,L} = \delta_{M,N} - \delta_{\sigma(N^p, Z^{p,q}, N^q)} \geq 0 \). Hence \( L \in \mathcal{O}_M(\mathbb{k}) \), by Eq. 3.5.

We claim that the point \( N + t \cdot Z^{p,q} \) of \( \text{rep}^d_Q(\mathbb{k}[t]) \) belongs to \( \mathcal{O}_M(\mathbb{k}[t]) \). Consider the element \( g \) in \( \text{GL}_d(\mathbb{k}[t, t^{-1}]) \) given by \( g = \sum_{s \neq p} \hat{t}^s + t \cdot \hat{t}^p \). Obviously \( g^{-1} = \sum_{s \neq p} \hat{t}^s + t^{-1} \cdot \hat{t}^p \). Since \( L = \sum_{s \in S} \hat{N}^{s,s} + Z^{p,q} \),

\[ g \ast L = g \circ L \circ g^{-1} = \sum_{s \in S} \hat{N}^{s,s} + t \cdot Z^{p,q} = N + t \cdot Z^{p,q} \]

as elements of \( \text{rep}^d_Q(\mathbb{k}[t, t^{-1}]) \). Hence the claim follows from the fact that \( \mathcal{O}_M(\mathbb{k}[t]) \) is a \( \text{GL}_d \)-invariant subscheme of \( \text{rep}^d_Q(\mathbb{k}) \) and by Lemma 3.1 applied to the canonical injective homomorphism \( \mathbb{k}[t] \to \mathbb{k}[t, t^{-1}] \).

Applying the homomorphism \( \mathbb{k}[t] \to \mathbb{k}[\varepsilon] \) sending \( t \) to \( \varepsilon \) we get that \( N + \varepsilon \cdot Z^{p,q} \) belongs to \( \mathcal{O}_M(\mathbb{k}[\varepsilon]) \).

The above lemma gives a method for detecting vectors tangent to \( \mathcal{O}_M(\mathbb{k}) \). This method is sufficient for the representations of Dynkin quivers of type \( \mathbb{A} \) as the proposition below shows. Obviously this proposition follows also immediately from Theorem 1.2.

**Proposition 3.4** Let \( Q \) be a Dynkin quiver of type \( \mathbb{A} \) and \( M \in \text{rep}(Q) \). Then \( \mathcal{O}_M(\mathbb{k}[\varepsilon]) = \mathcal{C}_M(\mathbb{k}[\varepsilon]) \). In other words, \( T_N(\mathcal{O}_M) = T_N(\mathcal{C}_M) \) for any \( N \) in \( \mathcal{O}_M(\mathbb{k}) \).

**Proof** Let \( N \in \mathcal{O}_M(\mathbb{k}) \) and fix a decomposition \( N = \bigoplus N^s \) of \( N \) such that each \( N^s \) is indecomposable.

Choose \( Z \in T_N(\mathcal{C}_M) = \mathbb{Z}^1_{M,N}(N, N) \). Since \( Z = \sum_{p,q \in S} Z^{p,q} \) (see Eq. 3.3), it is sufficient to show that \( Z^{p,q} \in T_N(\mathcal{O}_M) \), for all \( p, q \in S \). Fix \( p \) and \( q \). We may assume \( Z^{p,q} \notin \mathbb{B}^1_Q(N, N) \), as \( \mathbb{B}^1_Q(N, N) = T_N(\mathcal{O}_N) \subseteq T_N(\mathcal{O}_M) \). Since \( \mathbb{Z}^1_{M,N}(?, ?, N^q, N^p) \) are subfunctors of \( \mathbb{Z}^1_Q(?, ?) \), \( Z^{p,q} \) belongs to \( \mathbb{Z}^1_{M,N}(N^q, N^p) \) but not to \( \mathbb{B}^1_Q(N^q, N^p) \). In particular, \( \text{Ext}^1_Q(N^q, N^p) \) is non-zero. Since \( Q \) is a Dynkin quiver, \( N^q \) is not isomorphic to \( N^p \), hence \( q \neq p \).

It follows from the definition of \( \mathbb{Z}^1_{M,N}(?, ?) \) that \( \text{supp}(\delta_{\sigma(N^p, Z^{p,q}, N^q)}) \subseteq \text{supp}(\delta_{M,N}) \). Combining Lemmas 2.2(4) and 2.6(3) we get that the values of the function \( \delta_{\sigma(N^p, Z^{p,q}, N^q)} \) do not exceed 1. Hence we conclude the inequality \( \delta_{\sigma(N^p, Z^{p,q}, N^q)} \leq \delta_{M,N} \), thus \( Z^{p,q} \) belongs to \( T_N(\mathcal{O}_M) \), by Lemma 3.3.

The above method does not extend to Dynkin quivers of types \( \mathbb{D} \) and \( \mathbb{E} \). A reason for this is that for these quivers there exist short exact sequences \( \sigma \) with indecomposable end terms such that the functions \( \delta_{\sigma} \) attain values larger than 1.
Proposition 3.5 Let $Q$ be a Dynkin quiver of type $\mathbb{D}$ and $N = \bigoplus_{s \in S} N^s$ be a decomposition of $N \in \overline{\mathcal{O}}_M(k)$ such that each representation $N^s$ is indecomposable. Let $Z_{p,q} \in \mathbb{Z}_{M,N}(N^q,N^p)$, for $p \neq q$ in $S$, be such that the inequality $\delta_{\sigma(N^p,Z_{p,q},N^q)} \leq \delta_{M,N}$ does not hold.

Then there is an index $r$ in $S \setminus \{p,q\}$ and a homomorphism $h^{q,r}$ in $\text{Hom}_Q(N^r,N^q)$ such that for $Y^{p,r} = Z_{p,q} \circ h^{q,r}$ the following conditions hold:

1. The point $L = N + Y^{p,r}$ in $\text{rep}_Q^d(k)$ belongs to $\overline{\mathcal{O}}_M(k)$.
2. $N$ is a proper degeneration of $L$, i.e. $\mathcal{O}_N \subsetneq \overline{\mathcal{O}}_L$. In particular, $\dim \mathcal{O}_N < \dim \mathcal{O}_L$.
3. The point $L + \varepsilon \cdot \hat{Z}_{p,q}^r$ in $\text{rep}_Q^d(k[\varepsilon])$ belongs to $\mathcal{C}_M(k[\varepsilon])$ (equivalently, $\hat{Z}_{p,q}^r \in T_L \mathcal{C}_M$).

Proof We apply Corollary 2.23 for $\sigma = \sigma(N^p,Z_{p,q},N^q)$. Hence there is an index $r$ in $S$ and a homomorphism $h^{q,r}$ in $\text{Hom}_Q(N^r,N^q)$ such that for $Y^{p,r} = Z_{p,q} \circ h^{q,r}$ the following conditions hold:

1. The sequence $\sigma(N^p,Y^{p,r},N^r): 0 \to N^p \xrightarrow{[1 \ 0]} \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0 \to N^r \to 0$ does not split. Equivalently, $Y^{p,r}$ does not belong to $\mathbb{B}_Q^1(N^r,N^p)$.
2. $\delta_{\sigma(N^p,Y^{p,r},N^r)} \leq \delta_{M,N}$.
3. $\text{supp}(\delta_{\sigma(N^p,Z_{p,q},N^q)} - \delta_{\sigma(N^p,Y^{p,r},N^r)}) \subseteq \text{supp}(\delta_{M,N} - \delta_{\sigma(N^p,Y^{q,r},N^r)})$.

We claim that $r \neq p,q$. Note that (i) means that $Y^{p,q} = Z_{p,q} \circ h^{q,r}$ does not belong to $\mathbb{B}_Q^1(N^r,N^p)$. In particular $\text{Ext}_1^Q(N^r,N^p)$ is non-zero, hence $r \neq p$. Moreover $h^{q,r}$ is non-zero. Suppose that $q = r$. By Lemma 2.1(2), $\text{End}_Q(N^q) = k$, hence $h^{q,r}$ would be an isomorphism, thus $\delta_{\sigma(N^p,Y^{p,r},N^r)} = \delta_{\sigma(N^p,Z_{p,q},N^q)}$. Then (ii) contradicts the assumptions on $\delta_{\sigma(N^p,Z_{p,q},N^q)}$. Consequently, $r \neq q$, and the claim is proved.

Now (1) follows from (ii) and Lemma 3.3, and (2) is a consequence of (i) and Lemma 3.2. In order to prove (3) it suffices by Eq. 3.6 to show that

4. $\text{supp}(\delta_{\sigma(L,\hat{Z}_{p,q}^r,L)}) \subseteq \text{supp}(\delta_{M,L})$.

As in the proof of Lemma 3.2, we see that $L$ is isomorphic to $\bigoplus_{s \neq p,q} N^s \oplus \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0$. Observe that $\sigma(L,\hat{Z}_{p,q}^r,L)$ is the direct sum of the following split sequence

$$0 \to \bigoplus_{s \neq p,r} N^s \to \bigoplus_{s \neq p,r} N^s \oplus \bigoplus_{s \neq p,q,r} N^s \oplus \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0 \to N^s \oplus \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0 \to 0$$

and the sequence

$$0 \to \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0 \to \begin{bmatrix} N^p & Y^{p,r} & Z_{p,q} \end{bmatrix}_0 \to \begin{bmatrix} N^p & Y^{p,r} \end{bmatrix}_0 \to N^q \to 0.$$
Proof of Theorem 1.3 Let \( Q \) be a Dynkin quiver of type \( \mathbb{D} \). By decreasing induction on \( \dim O_N \) we prove that \( T_N \overline{O}_M = T_N C_M \) for all \( N \in \overline{O}_M(\mathbb{k}) = C_M(\mathbb{k}) \). Choose \( N \in \overline{O}_M(\mathbb{k}) \) and assume \( T_L \overline{O}_M = T_L C_M \) for all \( L \in \overline{O}_M \) with \( \dim O_L > \dim O_N \). We also fix a decomposition \( N = \bigoplus N^s \) of \( N \) such that each \( N^s \) is indecomposable.

Let \( Z \in \mathbb{Z}_{1,M,N}(N,N) \). Similarly as in the proof of Proposition 3.4 it is enough to show that \( \overline{Z}^{p,q} \in T_N \overline{O}_M \), for all \( p,q \in S \). By repeating arguments from that proof, we may assume \( p \neq q \) and the inequality \( \delta_O(N^p, Z^{p,q}, N^q) \leq \delta_M, N \) does not hold, thus we may apply Proposition 3.5. In particular, there exists \( Y^{p,r} \in \mathbb{Z}_Q(N^r, N^p) \) such that \( L = N + Y^{p,r} \) belongs to \( \overline{O}_M(\mathbb{k}) \), \( \dim O_L > \dim O_N \), and \( \overline{Z}^{p,q} \) belongs to \( T_L C_M \). But the latter equals \( T_L \overline{O}_M \) by induction hypothesis, hence \( N + Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q} \) belongs to \( \overline{O}_M(\mathbb{k}[\varepsilon]) \).

We claim that the point \( N + t \cdot Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q} \) of \( \text{rep}_d^d(\mathbb{k}[t] \otimes \mathbb{k}[\varepsilon]) \) belongs to \( \overline{O}_M(\mathbb{k}[\varepsilon]) \). Consider the element \( g \in \text{GL}_d(\mathbb{k}[t, t^{-1}] \otimes \mathbb{k}[\varepsilon]) \) given by \( g = \sum_{s \neq r} \hat{1}^s + t^{-1} \cdot \hat{1}^r \). Then \( g^{-1} = \sum_{s \neq r} \hat{1}^s + t \cdot \hat{1}^r \). Moreover,

\[
g \ast (N + Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q}) = g \circ \left( \sum_{s \in S} N^{s,s} + Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q} \right) \circ g^{-1} = \sum_{s \in S} N^{s,s} + t \cdot Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q} = N + t \cdot Y^{p,r} + \varepsilon \cdot \overline{Z}^{p,q}
\]

as elements of \( \text{rep}_d^d(\mathbb{k}[t, t^{-1}] \otimes \mathbb{k}[\varepsilon]) \). Now the claim follows from the fact that \( \overline{O}_M \) is a \( \text{GL}_d^d \)-invariant subscheme of \( \text{rep}_d^d \) and by Lemma 3.1 applied to the canonical injective homomorphism \( \mathbb{k}[t] \otimes \mathbb{k}[\varepsilon] \to \mathbb{k}[t, t^{-1}] \otimes \mathbb{k}[\varepsilon] \).

Applying the homomorphism \( \mathbb{k}[t] \otimes \mathbb{k}[\varepsilon] \to \mathbb{k}[\varepsilon] \) sending \( 1 \otimes \varepsilon \) to \( \varepsilon \) and \( t \otimes 1 \) to 0 we get that \( N + \varepsilon \cdot \overline{Z}^{p,q} \) belongs to \( \overline{O}_M(\mathbb{k}[\varepsilon]) \). Consequently, \( \overline{Z}^{p,q} \in T_N \overline{O}_M \), which finishes the proof. \( \square \)

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Declarations

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References

1. Abeasis, S., Del Fra, A.: Degenerations for the representations of an equioriented quiver of type \( A_m \). Boll. Un. Mat. Ital. Suppl. 2, 157–171 (1980)
2. Auslander, M., Reiten, I., Smalø, S.O.: Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36. Cambridge Univ. Press, Cambridge (1995)
3. Assem, I., Simson, D., Skowroński, A.: Elements of the Representation Theory of Associative Algebras. vol. 1, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, Cambridge (2006)
4. Bongartz, K.: Degenerations for representations of tame quivers. Ann. Sci. École Norm. Sup. (4) 28(5), 647–668 (1995)
5. Bongartz, K.: On degenerations and extensions of finite-dimensional modules. Adv. Math. 121(2), 245–287 (1996)
6. Demazure, M., Gabriel, P.: Introduction to Algebraic Geometry and Algebraic Groups, North-Holland Math. Stud. vol., 39, North-Holland, Amsterdam-New York (1980)
7. Gabriel, P.: Auslander-Reiten sequences and representation-finite algebras, Representation Theory, I, Lecture Notes in Math., vol. 831, pp. 1–71. Springer, Berlin (1980)
8. Happel, D.: Triangulated Categories in the Representation Theory of Finite-dimensional Algebras, London Math. Soc. Lecture Note Ser., vol. 119. Cambridge Univ. Press, Cambridge (1988)
9. Hesselink, W.: Singularities in the nilpotent scheme of a classical group. Trans. Amer. Math. Soc. 222, 1–32 (1976)
10. Lakshmibai, V., Magyar, P.: Degeneracy schemes, quiver schemes, and Schubert varieties. Internat. Math. Res. Notices 12, 627–640 (1998)
11. Reiten, I., Van den Bergh, M.: Noetherian hereditary abelian categories satisfying Serre duality. J. Amer. Math. Soc. 15(2), 295–366 (2002)
12. Riedtmann, Ch.: Algebren, Darstellungskörer, Überlagerungen und zurück. Comment. Math. Helv. 55(2), 199–224 (1980)
13. Riedtmann, Ch.: Degenerations for representations of quivers with relations. Ann. Sci. École Norm. Sup. (4) 19(2), 275–301 (1986)
14. Riedtmann, Ch., Zwara, G.: Orbit closures and rank schemes. Comment. Math. Helv. 88(1), 55–84 (2013)
15. Ringel, C.M.: Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math., vol. 1099. Springer, Berlin (1984)
16. Voigt, D.: Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen, Lecture Notes in Math., vol. 592. Springer, Berlin (1977)

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