Coloring the cube with rainbow cycles

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Abstract

For every even positive integer $k \geq 4$ let $f(n, k)$ denote the minimum number of colors required to color the edges of the $n$-dimensional cube $Q_n$, so that the edges of every copy of $k$-cycle $C_k$ receive $k$ distinct colors. Faudree, Gyárfás, Lesniak and Schelp proved that $f(n, 4) = n$ for $n = 4$ or $n > 5$. We consider larger $k$ and prove that if $k \equiv 0 \pmod{4}$, then there are positive constants $c_1, c_2$ depending only on $k$ such that

$$c_1 n^{k/4} < f(n, k) < c_2 n^{k/4}.$$

Our upper bound uses an old construction of Bose and Chowla of generalized Sidon sets. For $k \equiv 2 \pmod{4}$, the situation seems more complicated. For the smallest case $k = 6$ we show that

$$n \leq f(n, 6) < n^{1+o(1)}.$$

The upper bound is obtained from Behrend’s construction of a subset of the integers with no three term arithmetic progression.

1 Introduction

Given graphs $G$ and $H$, and an integer $q \leq |E(H)|$, a $(G, H, q)$-coloring is an edge-coloring of $G$ such that the edges of every copy of $H$ in $G$ receive at least $q$ colors. Let $f(G, H, q)$ be the minimum number of colors in a $(G, H, q)$-coloring. This general problem is hopeless in most cases, for example, when $G$ and $H$ are cliques, and $q = 2$, determining it is equivalent to determining the multicolor Ramsey number $R_k(p)$ which is a longstanding open problem. There has been more success in determining $f(G, H, q)$ when $G$ and $H$ are not cliques or when $q > 2$ (or both). Many Ramsey problems have received considerable attention when studied on the $n$-dimensional cube. The papers [2, 3] are examples where anti-Ramsey

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problems for subcubes in cubes and problems about monochromatic cycles in cubes are investigated. In [12], Offner found the exact value for the maximum number of colors for which it is possible to edge color the hypercube so that all subcubes of dimension $d$ contain all colors. Related Turán type problems for subcubes in cubes have been studied in [1].

Rainbow cycles have also been well studied as subgraphs of $K_n$. Erdős, Simonovits and Sós [13] introduced $AR(n, H)$, the maximum number of colors in an edge coloring of $K_n$ such that it contains no rainbow copy of $H$, and provided a conjecture when $H$ is a cycle and showed that their conjecture was true when $H = C_3$. Alon [14] proved their conjecture for cycles of length four and Montellano-Ballesteros and Neumann-Lara [11] proved the conjecture for all cycles in 2003. More recently, Choi [15] gave a shorter proof of the conjecture.

We continue this theme in the current note and let $G = Q_n$, the $n$-dimensional cube, and $H = C_k$, the cycle of length $k$. Our focus is on $q = |E(H)|$, in which case we will call a $(G, H, q)$-coloring an $H$-rainbow coloring, assuming that $G$ is obvious from context (in this paper $G = Q_n$ always).

**Definition 1** For $4 \leq k \leq 2^n$, let $f(n, k) = f(Q_n, C_k, k)$ be the minimum number of colors in a $C_k$-rainbow coloring of $Q_n$.

The smallest case $f(n, 4)$ was studied by Faudree, Gyárfás, Lesniak and Schelp [7] who proved that the trivial lower bound of $n$ is tight by providing, for all $n \geq 6$, a $C_4$-rainbow coloring with $n$ colors. We consider larger $k$. Our first result determines the order of magnitude of $f(n, k)$ for $k \equiv 0 \pmod{4}$.

**Theorem 2** Fix a positive $k \equiv 0 \pmod{4}$. There are constants $c_1, c_2 > 0$ depending only on $k$ such that

$$c_1 n^{k/4} < f(n, k) < c_2 n^{k/4}.$$  

The case $k \equiv 2 \pmod{4}$ seems more complicated. Our results imply that for such fixed $k$ there are positive constants $c_1', c_2'$ with

$$c_1' n^{[k/4]} < f(n, k) < c_2' n^{[k/4]}.$$  

We believe that the lower bound is closer to the truth. As evidence for this, we tackle the smallest case in this range, $k = 6$. As we will observe later, the lower bound $f(n, 6) \geq n$ is trivial for $n \geq 3$, and we obtain the following upper bound.

**Theorem 3** For every $\epsilon > 0$ there exists $n_0$ such that for $n > n_0$ we have $f(n, 6) \leq n^{1+\epsilon}$.

It is rather easy to see that $f(Q_n, Q_3, 12) = f(Q_n, C_6, 3)$. Indeed, if $Q_n$ is edge-colored so that every $Q_3$ is rainbow, then every $C_6$ is rainbow since each one is contained in a rainbow $Q_3$ and so $f(Q_n, Q_3, 12) \geq f(Q_n, C_6, 3)$. On the other hand, it is easy to see that any two edges of a $Q_3$ lie in some $C_6$ and therefore if $Q_n$ is edge-colored so that every $C_6$ is rainbow then every $Q_3$ must also be rainbow and so $f(Q_n, Q_3, 12) \leq f(Q_n, C_6, 3)$.

Since $C_4 = Q_2$, the following corollary can also be considered an extension of the result [7] to subcubes.
Corollary 4 As \( n \to \infty \), we have \( f(Q_n, Q, 12) = n^{1+o(1)} \).

We will consider the vertices of \( Q_n \) as binary vectors of length \( n \) or as subsets of \([n] = \{1, \ldots, n\}\), depending on the context (with the natural bijection \( \vec{v} \leftrightarrow v \) where \( \vec{v} \) is the incidence vector for \( v \subset [n] \), i.e. \( \vec{v}_i = 1 \) iff \( i \in v \)). In particular, whenever we write \( v - w \) we mean set theoretic difference, \( v \cup w \) or \( v \cap w \) we mean set union/intersection and when we write \( \vec{v} \pm \vec{w} \) we mean vector addition/subtraction modulo 2. We write \( e_i \) for the standard basis vector, so \( e_i \) is one in the \( i \)th coordinate and zero in all other coordinates. Given an edge \( f = uv \) of \( Q_n \) where \( \vec{v} = \vec{u} + e_s \) for some \( s \), we say that \( v \) is the top vertex of \( f \) and \( u \) is the bottom vertex. We will say an edge is on level \( i \) of \( Q_n \) if its bottom vertex corresponds to a vector with \( i - 1 \) ones and the top vertex to a vector with \( i \) ones.

2 Proof of Theorem 2

The lower bound in Theorem 2 follows from the easy observation that in a \( C_k \)-rainbow coloring all edges at level \( k/4 \) must receive distinct colors. Indeed, given any two such edges \( f_1 = vw \) and \( f_2 = xy \), where \( \vec{w} = \vec{v} + e_i \) and \( \vec{y} = \vec{x} + e_j \), it suffices to find a copy of \( C_k \) containing \( f_1 \) and \( f_2 \). If \( f_1 \) and \( f_2 \) are incident then it is clear that we can find a \( C_k \) containing them as long as \( n > k \) which we may clearly assume. The two cases are illustrated below where \( r = k/2 - 2 \) and \( s_i \notin w \cup y \) for all \( i \in \{1, \ldots, r\} \).

Now, suppose \( f_1 \) and \( f_2 \) are not incident. We know that \( |x \cup v| \leq k/2 - 2 \) since \( x \) and \( v \) are each sets of size \( k/4 - 1 \). By successively deleting elements of \( v \) and \( x \) in the appropriate order, we can obtain a \( v, x \)-path of length \( k/2 - 2 \). Then, since \( w \) and \( y \) are sets of size \( k/4 \), we may find a \( w, y \)-path of length \( k/2 \) between them by successively adding the elements of \( y \) to \( w \) and vice versa along with extra elements as needed. The two paths along with
the edges $vw$ and $xy$ form a cycle of length $k$. This is shown in the following diagram. Let $y - w = \{y_1, \ldots, y_m\}$, $w - y = \{w_1, \ldots, w_m\}$ and $w \cap y = \{z_1, \ldots, z_l\}$ where $m + l = k/4$. Let $\{s_1, \ldots, s_r\}$ again be a set such that $s_i \notin y \cup w$ with $r = k/4 - m$.

For the upper bound we need a classical construction of generalized Sidon sets by Bose and Chowla. A $B_t$-set $S = \{s_1, \ldots, s_n\}$ is a set of integers such that if $1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq n$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_t \leq n$, then

$$s_{i_1} + \cdots + s_{i_t} \neq s_{j_1} + \cdots + s_{j_t}$$

unless $(i_1, \ldots, i_t) = (j_1, \ldots, j_t)$. A consequence of this is that if $P, Q$ are nonempty disjoint subsets of $[n]$ with $|P| = |Q| \leq t$, then

$$\sum_{i \in P} s_i \neq \sum_{j \in Q} s_j. \quad (1)$$

The result below is phrased in a form that is suitable for our use later.

**Theorem 5 (Bose-Chowla [5])** For each fixed $t \geq 2$, there is a constant $A > 1$ such that for all $n$, there is a $B_t$-set $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$.

Now we provide the upper bound construction for Theorem 2.

**Construction 1.** Let $t = k/4 - 1$ and $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$ be a $B_t$-set as above. For each $v \in V(Q_n)$, let

$$a(v) = \sum_{i=1}^n \vec{v_i}s_i = \sum_{i: \vec{v_i}=1} s_i.$$
Given \( vw \in E(Q_n) \) with \( \vec{w} = \vec{v} + e_j \), let \( M = \lceil kAn^t \rceil \), and let
\[
d(vw) = a(v) + Mj.
\]
Suppose further that \( vw \) is at level \( p \) and \( p' \) is the congruence class of \( p \) modulo \( k/2 \). Then the color of the edge \( vw \) is
\[
\chi(vw) = (d(vw), p') \quad \square
\]
Let us now argue that this construction yields the upper bound in Theorem 2. First, the number of colors is at most
\[
\max_{vw} d(vw) \times \frac{k}{2} \leq (n \cdot \max \{ s_i + Mn \}) \frac{k}{2} \leq \frac{nk}{2} An^t + \frac{nk}{2} M < k^2 An^{t+1} = k^2 An^{k/4}
\]
as desired. Now we show that this is a \( C_k \)-rainbow coloring. Suppose for contradiction that \( H \) is a copy of \( C_k \) in \( Q_n \) and \( f_1 = vw, f_2 = xy \) are distinct edges of \( H \) with \( \chi(f_1) = \chi(f_2) \). Since \( H \) spans at most \( k/2 \) levels, \( f_1 \) and \( f_2 \) cannot lie in levels that differ by more than \( k/2 \), so \( \chi(f_1) \neq \chi(f_2) \) unless \( f_1 \) and \( f_2 \) are in the same level which we may henceforth assume. Let \( v, x \) be the bottom vertices of \( f_1, f_2 \), and \( \vec{w} = \vec{v} + e_i, \vec{y} = \vec{x} + e_j \). Assume without loss of generality that \( i \leq j \). If \( v = x \), then
\[
a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(v) + Mj.
\]
This implies that \( i = j \) and contradicts the fact that \( f_1 \neq f_2 \). We may therefore assume that \( v \neq x \). Similarly, if \( w = y \), then \( i < j \) and
\[
a(w) - s_i + Mi = a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(y) - s_j + Mj = a(w) - s_j + Mj.
\]
This implies the contradiction \( s_j - s_i = M(j - i) \geq M > An^t > s_j - s_i \). Consequently, we may assume that \( vw \) and \( xy \) share no vertex.

If \( |v \triangle x| > k/2 \), then any \( v, x \)-path in \( Q_n \) has length more than \( k/2 \) so there can be no cycle of length \( k \) containing both \( v \) and \( x \), contradiction. So we may assume that \( |v \triangle x| \leq k/2 \). Now \( \chi(vw) = \chi(xy) \) implies that
\[
a(v) + Mi = d(vw) = d(xy) = a(x) + Mj
\]
and this yields
\[
M(j - i) = Mj - Mi = a(v) - a(x) = a(v - x) = a(x - v) \leq \frac{|v \triangle x|}{2} An^t \leq \frac{k}{4} An^t < M.
\]
Consequently, we may assume that \( i = j \), \( a(v) = a(x) \), \( a(v - x) = a(x - v) \) and \( |v \triangle x| = |v \triangle y| \). If \( |v - x| = |x - v| \leq k/4 - 1 \), then
\[
a(v - x) = \sum_{i \in e-x} s_i \neq \sum_{j \in e-x} s_j = a(x - v)
\]
due to \( \square \), the definition of \( S \) and \( t = k/4 - 1 \). So we may assume that \( |v - x| = |x - v| = k/4 \) and \( |w \triangle y| = |v \triangle x| = k/2 \). This implies that \( \text{dist}_{Q_n}(w, y) = \text{dist}_{Q_n}(v, x) = k/2 \). Together with edges \( f_1, f_2 \), we conclude that \( C \) must have at least \( k + 2 \) edges, contradiction. \( \square \)
3 Proof of Theorem 3

We will first show the lower bound \( f(n, 6) \geq n \) for \( n \geq 3 \). It is immediate that no two edges incident with 0 receive the same color, for otherwise there would be two edges of the same color on level 1 of \( Q_n \) which we could easily extend to a non-rainbow \( C_6 \). Indeed, let \( i, j, k \) be distinct and consider the following \( C_6 \):

\[
0 \quad e_i \quad e_i + e_k \quad e_k \quad e_j + e_k \quad e_j \quad 0.
\]

To obtain the upper bound, we will give an explicit coloring that makes use of a classical construction of Behrend on sets of integers with no arithmetic progression of size three. Let \( r_3(N) \) denote the maximum size of a subset of \( \{1, \ldots, N\} \) that contains no 3-term arithmetic progression.

**Theorem 6 (Behrend [4])** There is a \( c > 0 \) such that if \( N \) is sufficiently large, then

\[
r_3(N) > N^{1 - \frac{c}{\sqrt{\log N}}}.\]

Theorem 6 clearly implies that for \( \epsilon > 0 \) and sufficiently large \( N \) we have \( r_3(N) > N^{1-\epsilon} \). The error term \( \epsilon \) was improved recently by Elkin [6] (see [8] for a simpler proof) and using Elkin’s result would give corresponding improvements in our result.

**Construction 2.** Let \( \epsilon > 0 \) and \( n \) be sufficiently large. Put \( N = \lceil n^{1+\epsilon} \rceil \) and let \( S = \{s_1, \ldots, s_n\} \subset \{1, \ldots, N\} \) contain no 3-term arithmetic progression. Such a set exists by Theorem 6 since

\[
n > n^{1-\epsilon^2} = n^{(1-\epsilon)(1+\epsilon)} > N^{1-2\epsilon}.
\]

Let

\[
a(v) = \sum_{i=1}^{n} \bar{v}_i s_i.
\]

Consider the edge \( vw \), where \( \bar{w} = \bar{v} + e_k \). Let

\[
d(vw) = a(v) + 2s_k \in Z_{2N}.
\]

We emphasize here that we are computing \( d(vw) \) modulo \( 2N \). Suppose further that \( vw \) is at level \( p \) and \( p' \) is the congruence class of \( p \) modulo 3. Then the color of the edge \( vw \) is

\[
\chi(vw) = (d(vw), p'). \quad \Box
\]

The number of colors used is at most \( 6N < n^{1+2\epsilon} \) as required. We will now show that this is a \( C_6 \)-rainbow coloring. Due to the second coordinate, it suffices to show that any two edges \( f_1, f_2 \) of a \( C_6 \) which are on the same of level of \( Q_n \) receive different colors. If \( f_1 \) and \( f_2 \) are incident, then they meet either at their top vertices or bottom vertices.

If incident at their bottom vertices, the edges are colored as follows and thus are distinctly colored:
If incident at their top vertices, the edges lie on a $C_4$ and are therefore distinctly colored.

If $f_1$ and $f_2$ are not incident, then there must be a path of length two between their bottom vertices. For if not, then they could not lie on a $C_6$ as the shortest path between their top vertices has length at least two. Moreover, the top vertices of $f_1$ and $f_2$ have symmetric difference precisely two since there is a path of length two between them. With these conditions, there are three ways the edges may be colored.
In the first coloring, $s_i + 2s_j \neq s_k + 2s_j$ holds due to $i$ and $k$ being distinct. In the second and third colorings, $s_i + 2s_j \neq s_k + 2s_i$ and $s_i + 2s_j \neq s_j + 2s_k$ hold due to our set $S$ being free of three term arithmetic progressions.

\[\text{4 Concluding Remark}\]

Our results imply a tight connection between $C_k$-rainbow colorings in the cube and constructions of large generalized Sidon sets. When $k \equiv 0 \pmod{4}$ Construction 1 gives the correct order of magnitude, however for $k \equiv 2 \pmod{4}$ the same method does not work. In this case an approach similar to Construction 2 would work provided we can construct large sets that do not contains solutions to certain equations.

**Conjecture 7** Fix $4 \leq k \equiv 2 \pmod{4}$. Then $f(n, k) = n^{\lfloor k/4 \rfloor + o(1)}$.

For the first open case $k = 10$, we can show that $f(n, 10) = n^{2+o(1)}$ provided one can construct a set $S \subset [N]$ with $|S| > N^{1/2-o(1)}$ that contains no nontrivial solution to any of the following equations:

\[
\begin{align*}
    x_1 + x_2 &= x_3 + x_4 \\
    x_1 + x_2 + x_3 &= x_4 + 2x_5 \\
    x_1 + 2x_2 &= x_3 + 2x_4.
\end{align*}
\]
Ruzsa [9, 10] defined the genus $g(E)$ of an equation

$$E : \quad a_1 x_1 + \cdots + a_k x_k = 0$$

as the largest $m$ such that there is a partition $S_1 \cup \cdots \cup S_m$ of $[k]$ where the $S_i$ are disjoint, non-empty and for all $j$,

$$\sum_{i \in S_j} a_i = 0. \quad (2)$$

A solution $(x_1, \ldots, x_k)$ of $E$ is trivial if there are $l$ distinct numbers among $\{x_1, \ldots, x_k\}$ and (2) holds for a partition $S_1 \cup \cdots \cup S_l$ of $[k]$ into disjoint, non-empty parts such that $x_i = x_j$ if and only if $i, j \in S_v$ for some $v$.

Ruzsa showed that if $S \subset [n]$ has no nontrivial solutions to $E$ then $|S| \leq O(n^{1/g(E)})$. The question of whether there exists $S$ with $|S| = n^{1/g(E) - o(1)}$ remains open for most equations $E$. The set of equations above has genus two so it is plausible that one can prove Conjecture 7 for $k = 10$ using this approach.

For the general case, we can provide a rainbow coloring if our set $S$ contains no nontrivial solutions to any of the three equations below with $m = \lfloor k/4 \rfloor$.

\[
\begin{align*}
x_1 + \cdots + x_m &= x_{m+1} + \cdots + x_{2m} \\
x_1 + \cdots + x_m + x_{m+1} &= x_{m+2} + \cdots + x_{2m} + 2x_{2m+1} \\
x_1 + \cdots + x_{m-1} + 2x_m &= x_{m+1} + \cdots + x_{2m-1} + 2x_{2m}.
\end{align*}
\]

The set of equations above has genus $m = \lfloor k/4 \rfloor$, so if Ruzsa’s question has a positive answer, then we would be able to construct a set of the desired size.

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